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Fixed Point Theorems for Almost α-ψ-Contractive Mappings in F-metric Spaces

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Abstract

In this paper, we introduce an almost α-ψ-contraction and a rational type α-ψ-contraction for α-admissible mappings in complete F-metric spaces which were introduced as a generalization of metric spaces. We prove the existence of a fixed point for these type contractions.

1. Introduction and preliminaries

The definition of metric spaces in 1905 and fixed point theory studies paved the way for important developments in both mathematics and the other sciences. Recently, some generalized metric spaces were introduced and fixed point theorems were proved. F-metric spaces were introduced as a generalization of metric spaces and Banach contraction principle were introduced in F-metric spaces in 2018 by Jleli *et al.* [1]. In F-metric spaces Hussain and Kanwal [2] proved coupled fixed point theorems, Mitrovic *et al.*[3], Laatefa and Ahmad [4] and Jahangir *et al.* [5] proved some generalized fixed point results. Altun and Erduran [6] proved fixed point results for single and multivalued mappings, Özturk [7] defined Ciric-Presic type contraction. Lateefa [8] and Zhou *et al.* [9] gave best proximity results in $F-$metric spaces. Alansari *et al.* [10] proved fuzzy fixed point theorems. Al-Mezel *et al.* [11] and Faraji *et al.*[12] defined $\alpha-\beta-$ admissible type contraction in $F-$metric spaces. Kanwal *et al.* [13] defined orthogonal $F-$metric spaces.

In recent years, several fundamental fixed point results have been extended and generalized by many authors in different directions. Samet *et al.* [14] introduced the concept of $\alpha-$admissible mappings on metric spaces. Many authors obtained some fixed point results using new concept and gave some applications [15, 16, 17, 18, 19, 20, 21, 22].

In this work, we define the concept of almost $\alpha-\psi-$contraction in $F-$metric spaces for $\alpha-$admissible mappings and we prove two main fixed point results.

Denoted by Ω the family of all functions $F:(0,\infty)\to\mathbb{R}$ satisfying following properties;

$F_1)$ F is increasing,
$F_2)$ For each sequence $\{u_s\}_{s\in\mathbb{N}}$, $\lim_{s\to\infty} u_s=0 \Leftrightarrow \lim_{s\to\infty} F(u_s)=-\infty$.

Definition 1.1. *[1] Let $L\neq\varnothing$ be a set and $(F,t)\in\Omega\times[0,+\infty)$. Assume that $\sigma: L^2\to[0,\infty)$ a function satisfying the following:*

(d_1) $\sigma(u_1,u_2)=0 \Leftrightarrow u_1=u_2$,
(d_2) $\sigma(u_1,u_2)=\sigma(u_2,u_1)$,

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(d_3) for all $s \geq 2$ ($s \in \mathbb{N}$) and for all $\{u_s\} \subset L$ such that if $\sigma(u_1,u_s) > 0$, $F(\sigma(u_1,u_s)) \leq F\left(\Sigma_{i=1}^{s}\sigma(u_i,u_{i+1})\right) + t$ for all $u_1,u_2 \in L$.
Then (L,σ) is named an $F-$metric space (shorty $F-ms$).

Definition 1.2. *[1] Let (L,σ) be an $F-ms$ and $\{u_s\}$ be a sequence in L.*

i. $\{u_s\}$ is named $F-$convergent if there is a $\mu \in L$ such that $\sigma(u_s,\mu) \to 0$ as $s \to \infty$.
ii. $\{u_s\}$ is named an $F-$Cauchy sequence if $\sigma(u_s,u_v) \to 0$ as $s,v \to \infty$.
iii. (L,σ) is named $F-$complete if each $F-$Cauchy sequence is $F-$convergent.

Denoted by Φ the family of all functions $\psi : [0,\infty) \to [0,\infty)$ satisfying following properties

($\psi1$) ψ is nondecreasing,
($\psi2$) $\sum_{s=1}^{\infty} \psi^s(u) < \infty$.

If ($\psi1$) and ($\psi2$) are satisfied, then

($\psi3$) $\psi(u) < u$ for $u > 0$

holds.

Example 1.3. *[6] Let $L = \mathbb{N}$ and $\sigma : L \times L \to [0,\infty)$ be defined by*

$$\sigma(u,\mu) = \begin{cases} (u-\mu)^2, & \text{if } u,\mu \in \{1,2,3\} \\ 2|u-\mu|, & \text{other} \end{cases}.$$

Then (L,σ) is an $F-$complete $F-ms$ with $F(u) = \frac{-1}{u}$ and $t = \ln 3$. But (L,σ) is not a metric space.

Example 1.4. *[23] Let $\Gamma = [0,\infty)$ and $\sigma : \Gamma \times \Gamma \to [0,\infty)$ be defined by*

$$\sigma(u,\mu) = \begin{cases} (u-\mu)^2, & \text{if } (u,\mu) \in [0,1] \times [0,1] \\ |u-\mu|, & \text{if } (u,\mu) \notin [0,1] \times [0,1] \end{cases}.$$

Then (Γ,σ) is an $F-$complete $F-ms$ with $F(u) = \ln u$ and $t = \frac{1}{2}$. But (Γ,σ) is not a metric space.

2. Almost α-$\psi-$ contractions

Definition 2.1. *Let (L,σ) be an $F-ms$. Let $T : L \to L$ be a self-mapping on L and $\alpha : L \times L \to [0,\infty)$ be a function. T is named an $\alpha-$admissible mapping if*

$$\alpha(u_1,u_2) \geq 1 \Longrightarrow \alpha(Tu_1,Tu_2) \geq 1$$

for all $u_1,u_2 \in L$.

Definition 2.2. *Let (L,σ) be an $F-ms$ and $T : L \to L$ be an $\alpha-$admissible mapping satisfying*

$$\alpha(u_1,u_2) \geq 1 \Longrightarrow \sigma(Tu_1,Tu_2) \leq \psi(K(u_1,u_2)) + a\min\{\sigma(u_1,Tu_2),\sigma(u_2,Tu_1)\} \tag{2.1}$$

for all $u_1,u_2 \in L$, where $\psi \in \Phi$, $a \geq 0$ and

$$K(u_1,u_2) = \max\{\sigma(u_1,u_2),\sigma(u_1,Tu_1),\sigma(u_2,Tu_2)\}.$$

Then, T is named to be an almost $\alpha-\psi-$contractive mapping.

Theorem 2.3. *Let (L,σ) be an $F-$complete $F-ms$ and $T : L \to L$ be an almost $\alpha-\psi-$contractive mapping. Assume that*

i. T is continuous or
ii. for a sequence $\{u_s\}$ in L if $u_s \to u$ and $\alpha(u_s,u_{s+1}) \geq 1$ for all s, then $\alpha(u,Tu) \geq 1$

holds and there exists $u_0 \in L$ such that $\alpha(u_0,Tu_0) \geq 1$. Then T has a fixed point (shortly FP).

Proof. Let $u_0 \in L$ such that $\alpha(u_0, Tu_0) \geq 1$. Define a sequence $\{u_s\} \subset L$ by $u_s = Tu_{s-1}$ for all $s \in \mathbb{N}$. If $u_s = u_{s+1}$ for any $s \in \mathbb{N}$, then u_s is an FP of T. We suppose that $T(u) \neq T(u+1)$ for all $u \in L$. Since T is an $\alpha-$admissible mapping and $\alpha(u_0, Tu_0) \geq 1$, we deduce that $\alpha(u_1, u_2) = \alpha(Tu_0, T^2u_0)$. Continuing this process, we get $\alpha(u_s, u_{s+1}) \geq 1$ for all $s \geq 0$. Now by (2.1) we get

$$\begin{aligned} \sigma(u_{s+1}, u_{s+2}) &= \sigma(Tu_s, Tu_{s+1}) \leq \psi(K(u_s, u_{s+1}) + a \min\{\sigma(u_{s+1}, Tu_s), \sigma(u_s, Tu_{s+1})\} \\ &= \psi(K(u_s, u_{s+1})) + a \min\{\sigma(u_{s+1}, u_{s+1}), \sigma(u_s, u_{s+2})\} \\ &= \psi(K(u_s, u_{s+1})) \end{aligned} \tag{2.2}$$

where

$$\begin{aligned} K(u_s, u_{s+1}) &= \max\{\sigma(u_s, u_{s+1}), \sigma(u_s, Tu_s), \sigma(u_{s+1}, Tu_{s+1})\} \\ &= \max\{\sigma(u_s, u_{s+1}), \sigma(u_s, u_{s+1}), \sigma(u_{s+1}, u_{s+2})\} \\ &= \max\{\sigma(u_s, u_{s+1}), \sigma(u_{s+1}, u_{s+2})\}. \end{aligned} \tag{2.3}$$

If $\sigma(u_{s+1}, u_{s+2}) \geq \sigma(u_s, u_{s+1})$, then from (2.2) and (2.3), we have

$$\sigma(u_{s+1}, u_{s+2}) \leq \psi(\sigma(u_{s+1}, u_{s+2})) < \sigma(u_{s+1}, u_{s+2})$$

which is a contradiction. Thus, $\sigma(u_{s+1}, u_{s+2}) < \sigma(u_s, u_{s+1})$ for all s and so from (2.2) and (2.3), we have

$$\sigma(u_{s+1}, u_{s+2}) \leq \psi(\sigma(u_{s+1}, u_{s+2})).$$

By induction, we have

$$\sigma(u_{s+1}, u_{s+2}) \leq \psi^{s+1}(\sigma(u_0, u_1))$$

for all $s \geq 0$. Let $k > 0$ be fixed and $(F, t) \in \Omega \times [0, +\infty)$ be such that

$$0 < j < l \text{ implies } F(j) < F(k) - t. \tag{2.4}$$

For $0 < F\left(\sum_{s \geq s(k)} \psi^s(\sigma(u_0, u_1))\right) < l$ and for each $s, v \in \mathbb{N}, s > v,$ using (2.4) and (F1) we have

$$F\left(\sum_{i=v}^{s-1} \psi^i(\sigma(u_0, u_1))\right) \leq F\left(\sum_{v \geq v(k)}^{s-1} \psi^v(\sigma(u_0, u_1))\right) < F(k) - t.$$

From (d_3), we have

$$\begin{aligned} F(\sigma(u_s, u_v)) &\leq F\left(\sum_{i=v}^{s-1} \sigma(u_i, u_{i+1})\right) + t \\ &\leq F\left(\sum_{i=v}^{s-1} \psi^i(\sigma(u_0, u_1))\right) + t \\ &< F(k). \end{aligned}$$

Therefore, $\{u_s\} \subset L$ is an $F-$Cauchy sequence. Since L is $F-$complete, there exists $u \in L$ such that $u_s \to u$. If T is continuous, then we have

$$\lim_{s \to \infty} \sigma(Tu_s, Tu) = \lim_{s \to \infty} \sigma(u_{s+1}, u) = 0.$$

Hence, u is an FP of T.

Now, suppose (ii) is satisfied. Since $\alpha(u_s, u_{s+1}) \geq 1$ for all s and $u_s \to u$ as $s \to \infty$, we have $\alpha(u, Tu) \geq 1$.

From (2.1), we get

$$\begin{aligned} \sigma(u_{s+1}, Tu) &= \sigma(Tu_s, Tu) \\ &\leq \psi(K(u_s, u) + a \min\{\sigma(u, Tu_s), \sigma(u_s, Tu)\}) \end{aligned}$$

where $a \geq 0$ and

$$\begin{aligned} K(u_s, u) &= \max\{\sigma(u_s, u), \sigma(u_s, Tu_s), \sigma(u, Tu)\} \\ &= \max\{\sigma(u_s, u), \sigma(u_s, u_{s+1}), \sigma(u, Tu)\}. \end{aligned}$$

If $K(u_s,u)=\sigma(u_s,u)$, then

$$\begin{aligned} F(\sigma(u_{s+1},Tu)) &\leq F(\psi(\sigma(u_s,u))+a\min\{\sigma(u,u_{s+1}),\sigma(u_s,Tu)\})+t \\ &\leq F(\sigma(u_s,u)+a\min\{\sigma(u,u_{s+1}),\sigma(u_s,Tu)\})+t. \end{aligned}$$

Taking limit as $s\to\infty$, we have $\lim_{s\to\infty}F(\sigma(u_s,u)+a\min\{\sigma(u,u_{s+1}),\sigma(u_s,Tu)\})+t=-\infty$. Hence, we get $\sigma(u,Tu)=0$.

If $K(u_s,u)=\sigma(u_s,u_{s+1})$, we have

$$\begin{aligned} F(\sigma(u_{s+1},Tu)) &\leq F(\psi(\sigma(u_s,u_{s+1}))+a\min\{\sigma(u,u_{s+1}),\sigma(u_s,Tu)\})+t \\ &\leq F(\sigma(u_s,u_{s+1})+a\min\{\sigma(u,u_{s+1}),\sigma(u_s,Tu)\})+t. \end{aligned}$$

Taking limit as $s\to\infty$ we have

$$\lim_{s\to\infty}F(\sigma(u_s,u_{s+1})+a\min\{\sigma(u,u_{s+1}),\sigma(u_s,Tu)\})+t=-\infty.$$

So, we get $\sigma(u,Tu)=0$.

If $K(u_s,u)=\sigma(u,Tu)$, then by (d_3),

$$\begin{aligned} F(\sigma(u_{s+1},Tu)) &\leq F(\psi(\sigma(u,Tu))+a\min\{\sigma(u,u_{s+1}),\sigma(u_s,Tu)\})+t \\ &\leq F(\sigma(u,Tu)+a\min\{\sigma(u,u_{s+1}),\sigma(u_s,Tu)\})+t \\ &\leq F(\sigma(u,u_s)+\sigma(u_s,Tu)+a\min\{\sigma(u,u_{s+1}),\sigma(u_s,Tu)\})+2t. \end{aligned}$$

Taking limit as $s\to\infty$, we have

$$\lim_{s\to\infty}F(\sigma(u,Tu)+a\min\{\sigma(u,u_{s+1}),\sigma(u_s,Tu)\})+2t=-\infty.$$

Thus, we have $F(\sigma(u,Tu))=0$. Therefore, $\sigma(u,Tu)=0$ and so T has an FP. □

Example 2.4. *Let $L=\mathbb{N}$ and $\sigma:L\times L\to[0,\infty)$ be defined by*

$$\sigma(u,\mu)=\begin{cases}(u-\mu)^2, & \text{if } u,\mu\in\{1,2,3\} \\ 2|u-\mu|, & \text{other}\end{cases}.$$

Let $T:L\to L$ be defined by

$$T(u)=\begin{cases}u/2, & \text{if } u \text{ is even} \\ (u+1)/2, & \text{if } u \text{ is odd}\end{cases}$$

and $\alpha:L\times L\to[0,\infty)$ be defined by $\alpha(u,\mu)=1$. Then, T is $\alpha-$admissible and continuous. Assume $\psi:[0,\infty)\to[0,\infty)$ be defined by $\psi(u)=\frac{u}{2}$ and $a>0$.

If $u=1$ and $\mu=2$, or u and μ are consecutive natural numbers with $\mu<u$, then $\sigma(Tu,T\mu)=0$. So, the proof is clear.

If u is even and μ is odd and $u,\mu\geq 4$, then

$$\begin{aligned} \sigma(Tu,T\mu) &= \sigma\left(\frac{u}{2},\frac{\mu+1}{2}\right)=2\left|\frac{u}{2}-\frac{\mu+1}{2}\right|=|u-\mu-1| \\ &\leq \frac{2|u-\mu|}{2}+a\min\{\sigma(u,T\mu),\sigma(\mu,Tu)\} \\ &\leq \psi(K(u,\mu))+a\min\{\sigma(u,T\mu),\sigma(\mu,Tu)\}. \end{aligned}$$

If u and μ are even, then

$$\begin{aligned} \sigma(Tu,T\mu) &= \sigma\left(\frac{u}{2},\frac{\mu}{2}\right)=2\left|\frac{u}{2}-\frac{\mu}{2}\right|=|u-\mu| \\ &\leq \frac{2|u-\mu|}{2}+a\min\{\sigma(u,T\mu),\sigma(\mu,Tu)\} \\ &\leq \psi(K(u,\mu))+a\min\{\sigma(u,T\mu),\sigma(Tu,\mu)\}. \end{aligned}$$

If u and μ are odd, then

$$\begin{aligned} \sigma(Tu,T\mu) &= \sigma\left(\frac{u+1}{2},\frac{\mu+1}{2}\right)=2\left|\frac{u}{2}-\frac{\mu}{2}\right|=|u-\mu| \\ &\leq \frac{2|u-\mu|}{2}+a\min\{\sigma(u,T\mu),\sigma(\mu,Tu)\} \\ &\leq \psi(K(u,\mu))+a\min\{\sigma(u,T\mu),\sigma(\mu,Tu)\}. \end{aligned}$$

Hence, all the conditions of Theorem 2.3 are satisfied. 1 is a unique FP of T.

In this example, if $u = 1$ and $\mu = 3$, then T is not an $\alpha - \psi$ contractive mapping. Therefore, T is an almost $\alpha - \psi$ weak contractive mapping for $a = 2$.

If we take $\psi(u) = u$ in Theorem 2.3, we have the following corollary.

Corollary 2.5. *Let (L,σ) be an $F-$complete $F-ms$ and $T : L \to L$ be an $\alpha-$admissible mapping satisfying*

$$\alpha(u_1,u_2) \geq 1 \Longrightarrow \sigma(Tu_1,Tu_2) \leq K(u_1,u_2) + a\min\{\sigma(u_1,Tu_2),\sigma(u_2,Tu_1)\}$$

for all $u_1,u_2 \in L$, where $a \geq 0$ and

$$K(u_1,u_2) = \max\{\sigma(u_1,Tu_1),\sigma(u_2,Tu_2),\sigma(u_1,u_2)\}.$$

Assume that

i. T is continuous or
ii. for a sequence $\{u_s\} \subset L$ if $u_s \to u$ and $\alpha(u_s,u_{s+1}) \geq 1$ for all $s \in \mathbb{N}$, then $\alpha(u,Tu) \geq 1$

holds and there exists $u_0 \in L$ such that $\alpha(u_0,Tu_0) \geq 1$. Then T has an FP.

Definition 2.6. *Let (L,σ) be an $F-ms$ and $T : L \to L$ be an $\alpha-$admissible mapping satisfying*

$$\alpha(u_1,u_2) \geq 1 \Longrightarrow \sigma(Tu_1,Tu_2) \leq \psi(K(u_1,u_2)) + a\min\{\sigma(u_1,Tu_2),\sigma(u_2,Tu_1)\} \tag{2.5}$$

for all $u_1,u_2 \in L$, where $\psi \in \Phi$, $a \geq 0$ and

$$K(u_1,u_2) = \max\left\{\frac{\sigma(u_1,Tu_1)\sigma(u_2,Tu_2)}{\sigma(u_1,u_2)+1},\sigma(u_1,u_2)\right\}.$$

Then T is said to be an almost $\alpha-\psi-$rational type contractive mapping.

Theorem 2.7. *Let (L,σ) be an $F-$complete $F-ms$ and $T : L \to L$ be an almost $\alpha-\psi-$rational type contractive mapping. Assume that*

i. T is continuous or
ii. for a sequence $\{u_s\} \subset L$ if $u_s \to u$ and $\alpha(u_s,u_{s+1}) \geq 1$ for all $s \in \mathbb{N}$, then $\alpha(u,Tu) \geq 1$

holds and there exists $u_0 \in L$ such that $\alpha(u_0,Tu_0) \geq 1$. Then T has an FP.

Proof. Let $u_0 \in L$ such that $\alpha(u_0,Tu_0) \geq 1$. Define a sequence $\{u_s\} \subset L$ by $u_s = Tu_{s-1}$ for all $s \in \mathbb{N}$. If $u_s = u_{s+1}$ for some $s \geq 0$, then u_s is an FP of T. We suppose that $u_s \neq u_{s+1}$ for all $s \in \mathbb{N}$. Since T is an $\alpha-$admissible mapping and $\alpha(u_0,Tu_0) \geq 1$, we deduce that $\alpha(u_1,u_2) = \alpha(Tu_0,T^2u_0)$. Continuing this process, we get $\alpha(u_s,u_{s+1}) \geq 1$ for all $s \in \mathbb{N}$. Now, by (2.5) we get

$$\begin{aligned}
\sigma(u_{s+1},u_{s+2}) &= \sigma(Tu_s,Tu_{s+1}) \leq \psi(K(u_s,u_{s+1}) + a\min\{\sigma(u_{s+1},Tu_s),\sigma(u_s,Tu_{s+1})\} \\
&= \psi(K(u_s,u_{s+1}) + a\min\{\sigma(u_{s+1},u_{s+1}),\sigma(u_s,u_{s+2})\} \\
&= \psi(K(u_s,u_{s+1}))
\end{aligned}$$

where

$$\begin{aligned}
K(u_s,u_{s+1}) &= \max\left\{\frac{\sigma(u_s,Tu_s)\sigma(u_{s+1},Tu_{s+1})}{D(u_s,u_{s+1})+1},D(u_s,u_{s+1})\right\} \\
&= \max\left\{\frac{\sigma(u_s,u_{s+1})\sigma(u_{s+1},u_{s+2})}{\sigma(u_s,u_{s+1})+1},\sigma(u_s,u_{s+1})\right\} \\
&= \max\{\sigma(u_{s+1},u_{s+2}),\sigma(u_s,u_{s+1})\}.
\end{aligned}$$

If $\sigma(u_{s+1},u_{s+2}) \geq \sigma(u_s,u_{s+1})$, then we have

$$\sigma(u_{s+1},u_{s+2}) \leq \psi(\sigma(u_{s+1},u_{s+2})) < \sigma(u_{s+1},u_{s+2})$$

which is a contradiction. Thus $\sigma(u_{s+1},u_{s+2}) < \sigma(u_s,u_{s+1})$ for all s. So, we have

$$\sigma(u_{s+1},u_{s+2}) \leq \psi(\sigma(u_s,u_{s+1})).$$

By induction, we get

$$\sigma(u_{s+1},u_{s+2}) \leq \psi^{s+1}(\sigma(u_0,u_1)).$$

Let $k>0$ be fixed and $(F,t)\in\Omega\times[0,+\infty)$ be such that

$$0<j<l \text{ implies } F(j)<F(k)-t.$$

For $0<F\left(\sum_{s\geq s(k)}\psi^s(\sigma(u_0,u_1))\right)<l$ and for each $s,v\in\mathbb{N}, s>v,$ using (2.5) and (F1) we have

$$F\left(\sum_{i=v}^{s-1}\psi^i(\sigma(u_0,u_1))\right)\leq F\left(\sum_{v\geq v(k)}^{s-1}\psi^s(\sigma(u_0,u_1))\right)<F(k)-t.$$

From (d_3) of $F-$metric,

$$\begin{aligned}F(\sigma(u_s,u_v) &\leq F\left(\sum_{i=v}^{s-1}\sigma(u_i,u_{i+1})\right)+t\\ &\leq F\left(\sum_{i=v}^{s-1}\psi^i((\sigma(u_0,u_1))\right)+t\\ &< F(k).\end{aligned}$$

Therefore, $\{u_s\}$ is an $F-$Cauchy sequence in L. Since L is $F-$complete, there exists $u\in L$ such that $u_s\to u$. If T is continuous, we have

$$\lim_{s\to\infty}\sigma(Tu_s,Tu)=\lim_{s\to\infty}\sigma(u_{s+1},u)=0.$$

So, u is an FP of T.

Now, suppose (ii) is satisfied. Since $\alpha(u_s,u_{s+1})\geq 1$ for all $s\geq 0$ and $u_s\to u$ as $u\to\infty$, we have $\alpha(u,Tu)\geq 1$.

From (2.5) we have

$$\begin{aligned}\sigma(u_{s+1},Tu) &= \sigma(Tu_s,Tu)\\ &\leq \psi(K(u_s,u))+a\min\{\sigma(u,Tu_s),\sigma(u_s,Tu)\}\end{aligned}$$

where $a\geq 0$ and

$$\begin{aligned}K(u_s,u) &= \max\left\{\frac{\sigma(u_s,Tu_s)\sigma(u,Tu)}{\sigma(u_s,u)+1},\sigma(u_s,u)\right\}\\ &= \max\left\{\frac{\sigma(u_s,u_{s+1})\sigma(u,Tu)}{\sigma(u_s,u)+1},\sigma(u_s,u)\right\}.\end{aligned}$$

If $K(u_s,u)=\sigma(u_s,u)$, then

$$\begin{aligned}F(\sigma(u_{s+1},Tu)) &\leq F(\psi(\sigma(u_s,u))+a\min\{\sigma(u,u_{s+1}),\sigma(u_s,Tu)\})+t\\ &\leq F(\sigma(u_s,u)+a\min\{\sigma(u,u_{s+1}),\sigma(u_s,Tu)\})+t.\end{aligned}$$

Taking limit as $s\to\infty$, we have

$$\lim_{s\to\infty}F(\sigma(u_s,u)+a\min\{\sigma(u,u_{s+1}),\sigma(u_s,Tu)\})+t=-\infty.$$

Thus, we get $F(\sigma(u,Tu))=0$. If $K(u_s,u)=\sigma(u_s,u_{s+1})$, we have

$$\begin{aligned}F(\sigma(u_{s+1},Tu)) &\leq F(\psi(\sigma(u_s,u_{s+1}))+a\min\{\sigma(u,u_{s+1}),\sigma(u_s,Tu)\})+t\\ &\leq F(\sigma(u_s,u_{s+1})+a\min\{\sigma(u,u_{s+1}),\sigma(u_s,Tu)\})+t.\end{aligned}$$

Taking limit as $s\to\infty$, we have

$$\lim_{s\to\infty}F(\sigma(u_s,u_{s+1})+a\min\{\sigma(u,u_{s+1}),\sigma(u_s,Tu)\})+t=-\infty.$$

Thus, using (F2) we get $F(\sigma(u,Tu))=0$. If $K(u_s,u)=\sigma(u,Tu)$, then by (d_3),

$$\begin{aligned}F(\sigma(u_{s+1},Tu)) &\leq F(\psi(\sigma(u,Tu))+a\min\{\sigma(u,u_{s+1}),\sigma(u_s,Tu)\})+t\\ &\leq F(\sigma(u,Tu)+a\min\{\sigma(u,u_{s+1}),\sigma(u_s,Tu)\})+t\\ &\leq F(\sigma(u,u_s)+\sigma(u_s,Tu)+a\min\{\sigma(u,u_{s+1}),\sigma(u_s,Tu)\})+2t.\end{aligned}$$

Taking limit as $s\to\infty$, we have

$$\lim_{s\to\infty}F(\sigma(u,Tu)+a\min\{\sigma(u,u_{s+1}),\sigma(u_s,Tu)\})+2t=-\infty.$$

Hence, we get $F(\sigma(u,Tu))=0$. Therefore, $\sigma(u,Tu)=0$ and so T has an FP. □

Example 2.8. *Let $\Gamma = [0,\infty)$ and $\sigma : \Gamma \times \Gamma \to [0,\infty)$ be defined by*

$$\sigma(u,\mu) = \begin{cases} (u-\mu)^2, & \text{if } (u,\mu) \in [0,1]\times[0,1] \\ |u-\mu|, & \text{if } (u,\mu) \notin [0,1]\times[0,1] \end{cases}.$$

Let $T : \Gamma \to \Gamma$ be defined by

$$T(u) = \begin{cases} 2u, & \text{if } u \in [0,1] \\ \frac{4u+2}{3}, & \text{if } u \in (1,\infty) \end{cases}$$

and
$\alpha : \Gamma \times \Gamma \to [0,\infty)$ be defined by

$$\alpha(u,\mu) = \begin{cases} 1, & \text{if } (u,\mu) \in [0,1]\times[0,1] \\ 0, & \text{if } (u,\mu) \notin [0,1]\times[0,1] \end{cases}.$$

Then, T is $\alpha-$admissible for $(u,\mu) \in [0,1]\times[0,1]$ and continuous. Assume $\psi : [0,\infty) \to [0,\infty)$ be defined by $\psi(u) = 2u$. If $u,\mu \leq \frac{1}{2}$ and $a = 100$, we have

$$\begin{aligned} \sigma(Tu,T\mu) &= \sigma(2u,2\mu) = |2u-2\mu| \\ &\leq 2|u-\mu|^2 + a\min\{|u-2\mu|^2, |\mu-2u|^2\} \\ &\leq \psi(\max\left\{\frac{\sigma(u,Tu),\sigma(\mu,T\mu)}{1+\sigma(u,\mu)}, \sigma(u,\mu)\right\}) + a\min\{\sigma(u,T\mu),\sigma(\mu,Tu)\}. \end{aligned}$$

If $1 \geq u,\mu > \frac{1}{2}$ and $a = 100$, we have

$$\begin{aligned} \sigma(Tu,T\mu) &= \sigma(2u,2\mu) = |2u-2\mu| \\ &\leq 2|u-\mu|^2 + a\min\{|u-2\mu|^2, |\mu-2u|^2\} \\ &\leq \psi(\max\left\{\frac{\sigma(u,Tu),\sigma(\mu,T\mu)}{1+\sigma(u,\mu)}, \sigma(u,\mu)\right\}) + a\min\{\sigma(u,T\mu),\sigma(\mu,Tu)\}. \end{aligned}$$

Hence, 0 is an FP of T.

Corollary 2.9. *Let (L,σ) be an $F-$complete $F-ms$ and $T : L \to L$ be an $\alpha-$admissible mapping satisfying*

$$\alpha(u_1,u_2) \geq 1 \Longrightarrow \sigma(Tu_1,Tu_2) \leq \psi(\sigma(u_1,u_2)) + a\min\{\sigma(u_1,Tu_2),\sigma(u_2,Tu_1)\}$$

for all $u_1,u_2 \in L$, where $\psi \in \Phi$, $a \geq 0$. Assume that

i. T is continuous or
ii. for a sequence $\{u_s\} \subset L$ if $u_s \to u$ and $\alpha(u_s,u_{s+1}) \geq 1$ for all $s \in \mathbb{N}$, then $\alpha(u,Tu) \geq 1$.

If there exists $u_0 \in L$ such that $\alpha(u_0,Tu_0) \geq 1$, then T has an FP.

If we take $\psi(u) = u$ in Theorem 2.7, we get the following corollary.

Corollary 2.10. *Let (L,σ) be an $F-$complete $F-ms$ and $T : L \to L$ be an α-admissible mapping satisfying*

$$\alpha(u_1,u_2) \geq 1 \Longrightarrow \sigma(Tu_1,Tu_2) \leq K(u_1,u_2) + a\min\{\sigma(u_1,Tu_2),\sigma(u_2,Tu_1)\}$$

for all $u_1,u_2 \in L$ and $a \geq 0$ and

$$K(u_1,u_2) = \max\left\{\frac{\sigma(u_1,Tu_1)\sigma(u_2,Tu_2)}{\sigma(u_1,u_2)+1}, \sigma(u_1,u_2)\right\}.$$

Assume that

i. T is continuous or
ii. for a sequence $\{u_s\} \subset L$ if $u_s \to u$ and $\alpha(u_s,u_{s+1}) \geq 1$ for all $s \in \mathbb{N}$, then $\alpha(u,Tu) \geq 1$

holds and there exists $u_0 \in L$ such that $\alpha(u_0,Tu_0) \geq 1$. Then T has an FP.

Declarations

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A Note on Statistical Continuity of Functions

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Abstract

In the present paper, we first recall the notion of statistical convergence of double sequences defined on topological spaces and reduced equivalent to the definition, along with some of its basic properties. Later, we define the concept of statistically continuous as a general case of the continuous function using the statistical convergence of double sequences. We define strong and weak statistically continuous functions as final definitions that arise as a direct consequence of statistically continuous functions. In the rest of the paper, we analyze the implications between the given definitions and investigate additional conditions for equality.

1. Introduction

The concepts of convergence and continuity, one of the joint research topics of mathematical analysis and topology, are fundamental issues. These concepts are used to characterize many properties in topological and metric spaces. The usual convergence studied in many topological spaces has been studied in detail to obtain new results and solve many topological problems in terms of convergence. From a more general point of view, different types of convergence can be defined to get new results. Based on this idea, new types of convergence, such as statistical convergence, have emerged.

The notion of statistical convergence, an extension of the usual convergence, was formerly named "almost convergence" by Zygmund in the first edition of his celebrated monograph published in Warsaw in 1935 [1]. The concept was formally introduced by Fast [2] and Steinhaus [3] and later was reintroduced by Schoenberg [4] and also independently by Buck [5]. Maio introduced and studied statistical convergence in topological spaces [6]. The concept of statistical convergence of double sequences was introduced by Muresaleen and Edely [7] using the double natural density. Later, Renukadevi and Vijayashanthi [8] applied this notion to topological spaces and showed many topological results. In recent years, many papers have been written using the idea of statistical and ideal convergence (see [9], [10], [11], [12], [13], [14], [15], [16], [17], [18], [19], [20], [21]). Specific to this presented work, we note that for the special case G=st-lim in [22], G-sequentially continuous function coincides with continuous in the ordinary sense since statistical limit is a subsequential method. Although statistical convergence was introduced over nearly the last ninety years, it has become an active area of research for forty years with the contributions by several authors, Salat [23], Fridy [24], Di Maio and Kočinac [6], Çakallı and Khan [25].

First, in the article, we present the notion of statistical convergence of double sequences defined on topological spaces and their equivalent case. Then, using the statistical convergence of double sequences, we define the notions of statistically continuous function, weakly statistically continuous function, and strongly statistically continuous function. The rest of the paper analyses the results between the given definitions.

2. Preliminaries

Basic definitions and theorems regarding statistical convergence of the double sequences in this section are given by Renukadevi et al. in [8].

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Throughout this paper, unless otherwise stated clearly, all spaces are assumed to be Hausdorff. For the real line with the natural topology, we use $\mathbb{R}$. We often say just space instead of topological space. A double sequence $(x_{nm})_{n,m\in\mathbb{N}}$ in a topological space X is said to converge to a point $x \in X$ in Pringsheims sense [26] if for every open set U containing x, there exists $l \in \mathbb{N}$ such that $x_{nm} \in U$ for all $m > l$ and $n > l$.

If $A \subset \mathbb{N}\times\mathbb{N}$, then A_{nm} denotes the set

$$A_{nm} = \{(k,l) \in A : k \le n, l \le m\}.$$

The double natural (or double asymptotic) density of A is given by

$$d(A) = \lim_{n,m\to\infty} \frac{|A_{nm}|}{nm},$$

if it exists. A subset A of $\mathbb{N}\times\mathbb{N}$ is statistically dense if $d(A) = 1$. We also recall that

$$d((\mathbb{N}\times\mathbb{N})\backslash A) = 1 - d(A)$$

for $A \subset \mathbb{N}\times\mathbb{N}$.

A double sequence $(x_{nm})_{n,m\in\mathbb{N}}$ in a space X is said to statistically convergent to $x \in X$, if for every neighborhood U of x,

$$d(\{(n,m) \in \mathbb{N}\times\mathbb{N} : x_{nm} \notin U\}) = 0.$$

We denote it by

$$x_{nm} \xrightarrow{st} x \text{ or } st - \lim_{n,m\to\infty} x_{nm} = x,$$

and we call x the statistical limit of (x_{nm}).

Theorem 2.1 ([8]). *If a double sequence $(x_{nm})_{n,m\in\mathbb{N}}$ is convergent, then it is statistically convergent.*

A double sequence $(x_{nm})_{n,m\in\mathbb{N}}$ is said to be st*-convergence [8] to $x \in X$ if there is $A \subset \mathbb{N}\times\mathbb{N}$ with $d(A) = 1$ such that

$$\lim_{n,m\to\infty,(n,m)\in A} x_{nm} = x.$$

We denote it by

$$x_{nm} \xrightarrow{st^*} x \text{ or } st^* - \lim_{n,m\to\infty} x_{nm} = x.$$

Theorem 2.2 ([8]). *If a double sequence $(x_{nm})_{n,m\in\mathbb{N}}$ st^*-convergent to $x \in X$, then $(x_{nm})_{n,m\in\mathbb{N}}$ st-convergent to x.*

The converse holds if the space X is first countable.

Theorem 2.3 ([8]). *Let X be a first countable space. If a double sequence $(x_{nm})_{n,m\in\mathbb{N}}$ st-convergent to $x \in X$, then (x_{nm}) st^*-convergence to x .*

Due to this theorem, the definition of statistical convergence of a double sequence is equivalently said that for the first countable space X, there exists a subset A of $\mathbb{N}\times\mathbb{N}$ with $d(A) = 1$ such that the double sequence $(x_{nm})_{(n,m)\in A}$ convergent to x, i.e. for every neighborhood V of x there is $n_0 \in \mathbb{N}$ such that $n,m \ge n_0$ and $(n,m) \in A$ imply $x_{nm} \in V$.

3. Main results

In this part of this study, we first define the concept of statistical continuity of functions, which is not available in the literature, by using the idea of statistically convergent double sequences.

Theorem 3.1. *If a double sequence $(x_{nm})_{n,m\in\mathbb{N}}$ is statistically convergent, then its statistical limit is unique.*

Proof. Suppose that

$$x_{nm} \xrightarrow{st} x_1 \text{ and } x_{nm} \xrightarrow{st} x_2$$

with $x_1 \neq x_2$. Let $x \in X$. Let U and V be neighborhood of x_1 and x_2, respectively, such that

$$U \cap V = \emptyset.$$

Since $x_{nm} \xrightarrow{st} x_1$ and $x_{nm} \xrightarrow{st} x_2$, then

$$d(K_1) = d(\{(n,m) \in \mathbb{N} \times \mathbb{N} : x_{nm} \notin U\}) = 0$$

and

$$d(K_2) = d(\{(n,m) \in \mathbb{N} \times \mathbb{N} : x_{nm} \notin V\}) = 0,$$

respectively. Let $K = K_1 \cup K_2$. Thus, $d(K) = 0$ which implies

$$d((\mathbb{N} \times \mathbb{N}) \backslash K) = 1.$$

If $(k,l) \in (\mathbb{N} \times \mathbb{N}) \backslash K$, then

$$x_{kl} \in U \text{ and } x_{kl} \in V.$$

This contradicts the fact that $U \cap V = \emptyset$. Hence $x_1 = x_2$. □

We denote by Y^X the set of all functions from a space X to a space Y.

Definition 3.2. *$f \in Y^X$ is called a statistically continuous function if for every double sequence $(x_{nm})_{n,m\in\mathbb{N}}$ in X statistically converging to x, $(f(x_{nm}))_{n,m\in\mathbb{N}}$ statistically convergent to $f(x)$.*

Proposition 3.3. *Every continuous function is statistically continuous.*

Proof. Let $f \in Y^X$ be continuous and $x_{nm} \xrightarrow{st} x$. Then, for every neighborhood U of x,

$$d(\{(n,m) \in \mathbb{N} \times \mathbb{N} : x_{nm} \notin U\}) = 0.$$

Since for every open neighbourhood V of $f(x)$, there exists an open neighbourhood U of x such that $U = f^{-1}(V)$ (due to continuity), hence

$$d(\{(n,m) \in \mathbb{N} \times \mathbb{N} : f(x_{nm}) \notin f(U) = V\}) = 0.$$

Therefore, $f(x_{nm}) \xrightarrow{st} f(x)$. □

The converse of Proposition 3.3 does not hold.

Example 3.4. *For the usual topology $\mathcal{U}$ and the countable complement topology τ on $\mathbb{R}$, the identity function $I : (\mathbb{R}, \tau) \to (\mathbb{R}, \mathcal{U})$ is statistically continuous, but not continuous.*

Now, we define the notion of weak statistical continuity, which is a general case of statistically continuous function for double sequences.

Definition 3.5. *$f \in Y^X$ is called a weakly statistically continuous function if for every double sequence $(x_{nm})_{n,m\in\mathbb{N}}$ in X converging to x, $(f(x_{nm}))_{n,m\in\mathbb{N}}$ statistically convergent to $f(x)$.*

Proposition 3.6. *Every statistically continuous function is weakly statistically continuous.*

Proof. Let $f \in Y^X$ be statistically continuous and $x_{nm} \longrightarrow x$. Then from Theorem 2.1

$$x_{nm} \xrightarrow{st} x.$$

Since f is statistically continuous, it follows from

$$f(x_{nm}) \xrightarrow{st} f(x).$$

Hence, f is weakly statistically continuous. □

Considering Proposition 3.3, it can give the following result.

Corollary 3.7. *Every continuous function is weakly statistically continuous.*

Theorem 3.8. *For the first countable spaces X and Y, a double sequence $(f_{nm})_{n,m\in\mathbb{N}}$ and a function $f \in Y^X$ be given. Then the following are equivalent:*

1. *f is continuous.*
2. *f is statistically continuous.*
3. *f is weakly statistically continuous.*

Proof. The implications $(1) \Rightarrow (2)$ and $(2) \Rightarrow (3)$ are given in Proposition 3.3 and Proposition 3.6, respectively.
$(3) \Rightarrow (1)$. Let K be a closed set in the space Y. It is enough to show that the set $f^{-1}(K)$ is closed in the space X. Take any $x \in \overline{f^{-1}(K)}$. Since X first countable space, there exists a double sequence $(x_{nm})_{n,m\in\mathbb{N}}$ in X such that

$$x_{nm} \longrightarrow x.$$

By using the weak statistical continuity of f, we get

$$f(x_{nm}) \xrightarrow{st} f(x).$$

From Theorem 2.3, there exists a subset A of $\mathbb{N}\times\mathbb{N}$ with $d(A) = 1$ such that

$$f(x_{n_k m_l}) \longrightarrow f(x).$$

Due to $f(x_{n_k m_l}) \in K$ and $f(x) \in \overline{K} = K$, we have $x \in f^{-1}(K)$ and as a result, $f^{-1}(K)$ is closed in the space X. As can be seen from here, f is continuous. □

Now, we define the notion of strong statistical continuity, which is a special case of statistically continuous functions for double sequences.

Definition 3.9. *$f \in Y^X$ is called a strongly statistically continuous function if for every double sequence $(x_{nm})_{n,m\in\mathbb{N}}$ in X statistically converging to x, $(f(x_{nm}))_{n,m\in\mathbb{N}}$ convergent to $f(x)$.*

Proposition 3.10. *Every strongly statistically continuous function is statistically continuous.*

Proof. Let $f \in Y^X$ be strongly statistically continuous and $x_{nm} \xrightarrow{st} x$. As f is strongly statistically continuous,

$$f(x_{nm}) \longrightarrow f(x).$$

Then from Theorem 2.1

$$f(x_{nm}) \xrightarrow{st} f(x).$$

This shows that f is statistically continuous. □

The converse of Proposition 3.10 does not hold.

Example 3.11. *Consider the statistically continuous function I in Example 3.4. it is not strongly statistically continuous because, when $x_{nm} \xrightarrow{st} x$, the convergence $x_{nm} \longrightarrow x$ does not satisfy.*

Theorem 3.12. *For the first countable spaces X and Y, every strongly statistically continuous function $f \in Y^X$ is continuous.*

Proof. Let f be astrongly statistically continuous function. Since $x_{nm} \longrightarrow x$, then we have $x_{nm} \xrightarrow{st} x$ from Theorem 2.1 and so, from the definition of strong statistical continuity,

$$f(x_{nm}) \longrightarrow f(x).$$

This means that f is sequentially continuous. We know that continuity and sequential continuity are equivalent in the first countable space. Therefore, f is continuous. □

Theorem 3.13. *Let $f \in Y^X$ and $g \in Z^Y$ be statistical continuous functions. Then $g \circ f$ is a statistical continuous function.*

Proof. Let $x_{nm} \xrightarrow{st} x$ in X. Since $f \in Y^X$ is statistical continuous function, then

$$f(x_{nm}) \xrightarrow{st} f(x)$$

in Y. Since $f(x_{nm}) \xrightarrow{st} f(x)$ in Y and $g \in Z^Y$ is a statistically continuous function, then we have

$$g(f(x_{nm})) \xrightarrow{st} g(f(x))$$

in Z. As a result, the $g \circ f$ is shown to be statistically continuous. □

Theorem 3.14. *Let $f, g \in \mathbb{R}^X$ be real-valued statistical continuous functions. Then the following is provided.*

1. *$g + f$ is a statistical continuous function.*
2. *$g.f$ is a statistical continuous function.*

Proof. (1). Let $x_{nm} \xrightarrow{st} x$ in X. Since f and g are real-valued statistical continuous functions,

$$f(x_{nm}) \xrightarrow{st} f(x) \text{ and } g(x_{nm}) \xrightarrow{st} g(x)$$

in $\mathbb{R}$, respectively. Therefore,

$$f(x_{nm}) + g(x_{nm}) \xrightarrow{st} f(x) + g(x)$$

and so

$$(f+g)(x_{nm}) \xrightarrow{st} (f+g)(x).$$

Hence, $g+f$ is a statistical continuous function.

(2). The proof is similar to (1) □

4. Conclusion

Continuous functions have an essential place in the study of topological spaces. Convergence can also be used to investigate the continuity of a function. Therefore, studies on these two concepts further research on topological spaces.

In this study, we investigated the continuity of topological valued functions using the statistical convergence of double sequences. A more general view of the continuity of real-valued functions is introduced by taking topological valued functions.

Since continuous functions are essential in mathematical studies, further research is required. Studies can be extended by using different convergence types and topological open-closed sets. As a continuation of this study, in future studies, uniform continuity with change in the value set can be defined and compared with this work.

Declarations

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Some Fixed Point Theorems for the New Generalizations of P-Contractive Maps

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Abstract

In this paper, we introduce the enriched P-contractive and the enriched Suzuki-type P-contractive maps, and for such maps, we establish the existence and uniqueness of fixed points (fps) in the setting of normed spaces. Also, we introduce the generalized Suzuki-type P-contractive map and prove some fp theorems for this map in compact metric spaces. These results unify, generalize, and complement various comparable results in the literature.

1. Introduction

Suppose that $\mathcal{H}$ is any nonempty set and $\varphi : \mathcal{H} \to \mathcal{H}$ is any map. If there is a point $x \in \mathcal{H}$ such that $\varphi(x) = x$, then x is known as a fixed point (fp) of φ. The set of all fps of φ is denoted by $F(\varphi)$, that is $F(\varphi) = \{x \in \mathcal{H} : \varphi(x) = x\}$. In mathematics, the fp theory, which examines the fps of maps, has a wide range of application areas: economics, physics, engineering, etc., as well as geometry, analysis, and topology, among other fields of mathematics (see [1, 2, 3, 4, 5, 6, 7, 8, 9, 10]). That is why this topic is essential to work and contribute.

In 1922, Banach [11] was the first who observed that if one takes a complete metric space, namely, $(\mathcal{H}, d)$, then any contraction map on $\mathcal{H}$, that is, a map $\varphi : \mathcal{H} \to \mathcal{H}$, which has the property that for all $x, y \in \mathcal{H}$, a constant $\theta \in [0, 1)$ exists with

$$d(\varphi(x), \varphi(y)) \leq \theta d(x, y)$$

admits a unique fp.

In 1962, Edelstein [12] generalized Banach's fp theorem as: any contractive map on a compact metric space $(\mathcal{H}, d)$, that is, a map $\varphi : \mathcal{H} \to \mathcal{H}$, which has the property that for all $x, y \in \mathcal{H}$ with $x \neq y$, one has

$$d(\varphi(x), \varphi(y)) < d(x, y)$$

admits a unique fp.

By weakening the concept of contractivity, in 2009, Suzuki [13] generalized Edelstein's fp theorem as: any Suzuki-type contractive map (SC) on a compact metric space $(\mathcal{H}, d)$, that is, a map $\varphi : \mathcal{H} \to \mathcal{H}$ such that for all $x, y \in \mathcal{H}$ with $x \neq y$, one has

$$\frac{1}{2} d(x, \varphi(x)) < d(x, y) \Longrightarrow d(\varphi(x), \varphi(y)) < d(x, y)$$

admits a unique fp.

In this direction, Popescu [14] introduced the concept of P-contraction maps as a generalization of contraction maps. A self-map φ on a metric space $(\mathcal{H}, d)$ is called P-contraction if there exists $\theta \in [0, 1)$ such that

$$d(\varphi(x), \varphi(y)) \leq \theta [d(x, y) + |d(x, \varphi(x)) - d(y, \varphi(y))|]$$

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for all $x, y \in \mathcal{H}$.

On the other hand, in 2020, Berinde and Păcurar [15] introduced the enriched version of contraction maps. A self-map φ on a nonempty subset $\mathcal{K}$ of a normed space $(\mathcal{H}, \|\cdot\|)$ is called enriched contraction if there exist $k \in [0, +\infty)$ and $\theta \in [0, k+1)$ such that

$$\|k(x-y)+\varphi(x)-\varphi(y)\| \leq \theta\|x-y\|$$

for all $x, y \in \mathcal{K}$. Also, we can conclude that if we take $k = 0$ in this inequality, we get the contraction map.

By following this, in 2022, Abbas, Anjum, and Rakočević [16] introduced the enriched version of SC maps and proved the following theorem.

Theorem 1.1. *[16, Theorem 6] Let $(\mathcal{H}, \|\cdot\|)$ be a compact normed space and $\varphi : \mathcal{H} \to \mathcal{H}$ be a map. If there exists $k \in [0, \infty)$ with $\lambda = \frac{1}{k+1}$ such that for all $x, y \in \mathcal{H}$,*

$$\frac{\lambda}{2}\|x-\varphi(x)\| < \|x-y\| \Longrightarrow \|k(x-y)+\varphi(x)-\varphi(y)\| < \|x-y\|.$$

Then, φ has a fp.

In 2024, Altun, Hançer, and Ateş [17] introduced the concept of enriched P-contraction maps as a generalization of P-contraction maps. A self-map φ on a nonempty subset $\mathcal{K}$ of a normed space $(\mathcal{H}, \|\cdot\|)$ is called enriched P-contraction if there exist $k \in [0, +\infty)$ and $\theta \in [0, k+1)$ such that

$$\|k(x-y)+\varphi(x)-\varphi(y)\| \leq \theta\|x-y\| + \frac{\theta}{k+1}\left|\|x-\varphi(x)\| - \|y-\varphi(y)\|\right|$$

for all $x, y \in \mathcal{K}$. It is clear that every P-contraction is an enriched P-contraction with $k = 0$.

Motivated by the above results, we first introduce the enriched P-contractive and the enriched Suzuki-type P-contractive maps and prove the generalizations of the results of [18, 19] in the setting of a normed space. Also, we introduce the generalized Suzuki-type P-contractive maps and prove the generalizations of the results of [19] in a compact metric space.

2. Preliminaries

In 2018, Altun, Durmaz and Olgun [18] introduced the following generalization of contractive maps.

Definition 2.1. *[18, Definition 2.2] Let $(\mathcal{H}, d)$ be a metric space and $\varphi : \mathcal{H} \to \mathcal{H}$ be a map. Then, the map φ is said to be P-contractive (PC) if*

$$d(\varphi(x), \varphi(y)) < d(x, y) + |d(x, \varphi(x)) - d(y, \varphi(y))|$$

for all $x, y \in \mathcal{H}$ with $x \neq y$.

Now, we give two examples of PC maps.

Example 2.2. *(see [20]) Let $\mathcal{H} = [0, 1]$ with the usual metric and $\varphi : \mathcal{H} \to \mathcal{H}$ be defined by*

$$\varphi(x) = \begin{cases} \frac{1}{2}, & \text{if } x = 0, \\ \frac{x}{2}, & \text{if } x \neq 0. \end{cases}$$

Then, the map φ is PC but not contractive.

Example 2.3. *(see [18])*

(i) Let $\mathcal{H} = [0, 2]$ with the usual metric and $\varphi : \mathcal{H} \to \mathcal{H}$ be defined by

$$\varphi(x) = \begin{cases} 1, & \text{if } x \leq 1, \\ 0, & \text{if } x > 1. \end{cases}$$

Then, the map φ is PC but not SC.

(ii) Let $\mathcal{H} = \{(0,0),(4,0),(0,4),(4,5),(5,4)\} \subset \mathbb{R}^2$ with the metric

$$d(x,y) = d((x_1,x_2),(y_1,y_2)) = |x_1 - y_1| + |x_2 - y_2|$$

for $x = (x_1,x_2), y = (y_1,y_2) \in \mathcal{H}$. Define a map $\varphi : \mathcal{H} \to \mathcal{H}$ by

$$\varphi = \begin{pmatrix} (0,0) & (4,0) & (0,4) & (4,5) & (5,4) \\ (0,0) & (0,0) & (0,0) & (4,0) & (0,4) \end{pmatrix}. \tag{2.1}$$

Then, the map φ is SC but not PC.

Moreover, the following fp theorem has been presented by Altun, Durmaz, and Olgun [18].

Theorem 2.4. *[18, Theorem 2.14] Let $(\mathcal{H},d)$ be a compact metric space and $\varphi : \mathcal{H} \to \mathcal{H}$ be a continuous PC map. Then, φ has a unique fp.*

Recently, in 2023, Altun [19] introduced a new class of map called Suzuki-type P-contractive as a generalization of SC and PC maps inspired by the previous maps we mentioned above.

Definition 2.5. *[19, Definition 2.1] Let $(\mathcal{H},d)$ be a metric space and $\varphi : \mathcal{H} \to \mathcal{H}$ be a map. Then φ is said to be Suzuki-type P-contractive (SPC) if*

$$\tfrac{1}{2}d(x,\varphi(x)) < d(x,y) \Longrightarrow d(\varphi(x),\varphi(y)) < d(x,y) + |d(x,\varphi(x)) - d(y,\varphi(y))|$$

for all $x,y \in \mathcal{H}$.

Remark 2.6. *The map φ in Example 2.3-(i) is SPC, but not SC. Also, the map φ in Example 2.3-(ii) is SPC, but not PC. Hence, we can see that the class of SPC maps generalizes the SC and PC maps.*

After that, he proved an fp theorem for the class of the new map above.

Theorem 2.7. *[19, Theorem 2.1] Let $(\mathcal{H},d)$ be a compact metric space and $\varphi : \mathcal{H} \to \mathcal{H}$ be a continuous SPC map. Then, φ has a unique fp.*

Lemma 2.8. *[21] Let $\mathcal{H}$ be a compact topological space and $f : \mathcal{H} \to \mathbb{R}$ be a lower semi-continuous function. Then, there exists an element $x_0 \in \mathcal{H}$ such that $f(x_0) = \inf\{f(x) : x \in \mathcal{H}\}$.*

Also, by Lemma 2.8, Altun [19] obtained the following result by assuming the lower semi-continuity of the function $f(x) = d(x,\varphi(x))$ instead of the continuity of φ.

Theorem 2.9. *[19, Theorem 2.2] Let $(\mathcal{H},d)$ be a compact metric space and $\varphi : \mathcal{H} \to \mathcal{H}$ be an SPC map. Then φ has a unique fp provided the function f defined by $f(x) = d(x,\varphi(x))$ is lower semi-continuous.*

3. Enriched versions of PC and SPC maps

First, we introduce the concept of an enriched P-contractive map as a generalization of PC maps.

Definition 3.1. *Let $\mathcal{K}$ be a nonempty subset of a normed space $(\mathcal{H},\|\cdot\|)$, $\varphi : \mathcal{K} \to \mathcal{K}$ be a map and $k \in [0,\infty)$. Then, the map φ is said to be enriched P-contractive (EPC) if*

$$\|k(x-y) + \varphi(x) - \varphi(y)\| < \|x-y\| + |\|x-\varphi(x)\| - \|y-\varphi(y)\||$$

for all $x,y \in \mathcal{K}$ with $x \neq y$.

Remark 3.2. *Obviously, we get the PC map if $k = 0$ in the above definition.*

Example 3.3. *Let $\mathcal{K} = \{(0,0),(4,0),(0,4),(4,5),(5,4)\}$ be a subset of the normed space $\mathbb{R}^2$ endowed with the norm $\|x\| = |x_1| + |x_2|$ for $x = (x_1,x_2) \in \mathbb{R}^2$ and $\varphi : \mathcal{K} \to \mathcal{K}$ be a map defined by (2.1). Since*

$$\|\varphi(x) - \varphi(y)\| = 8 > 2 = \|x-y\| + |\|x-\varphi(x)\| - \|y-\varphi(y)\||$$

for $x = (4,5)$ and $y = (5,4)$, we find that φ is not a PC map. On the other hand, we can see that, for $k = 4$,

$$\|k(x-y) + \varphi(x) - \varphi(y)\| = 0 < 2 = \|x-y\| + |\|x-\varphi(x)\| - \|y-\varphi(y)\||$$

for $x = (4,5)$ and $y = (5,4)$. Then, φ is an EPC map with $k = 4$.

We prove the following theorem, which is more general than Theorem 2.4, in the setting of normed spaces.

Theorem 3.4. *Let $\mathcal{K}$ be a nonempty, compact and convex subset of a normed space $(\mathcal{H}, \|\cdot\|)$ and $\varphi : \mathcal{K} \to \mathcal{K}$ be a continuous EPC map with $k \in [0,\infty)$. Then, φ has a unique fp.*

Proof. We consider $\varphi_\lambda(x) = (1-\lambda)x + \lambda\varphi(x)$, which is called the average map. Now, we may take $\lambda = \frac{1}{k+1}$. Then $k = \frac{1}{\lambda} - 1$. It is easy to see that $\lambda \in (0,1]$. Since φ is a EPC map, then we have

$$\|(\frac{1}{\lambda} - 1)(x-y) + \varphi(x) - \varphi(y)\| < \|x-y\| + |\|x-\varphi(x)\| - \|y-\varphi(y)\||.$$

By multiplying both sides of this inequality with λ, we obtain

$$\|(1-\lambda)(x-y) + \lambda(\varphi(x)-\varphi(y))\| < \lambda\|x-y\| + |\|\lambda x - \lambda\varphi(x)\| - \|\lambda y - \lambda\varphi(y)\||$$

or

$$\|[(1-\lambda)x + \lambda\varphi(x)] - [(1-\lambda)y + \lambda\varphi(y)]\| < \lambda\|x-y\| + |\|x - [(1-\lambda)x+\lambda\varphi(x)]\| - \|y - [(1-\lambda)y + \lambda\varphi(y)]\||.$$

From the definition of the average map, we get

$$\begin{aligned}\|\varphi_\lambda(x) - \varphi_\lambda(y)\| &< \lambda\|x-y\| + |\|x-\varphi_\lambda(x)\| - \|y-\varphi_\lambda(y)\|| \\ &\leq \|x-y\| + |\|x-\varphi_\lambda(x)\| - \|y-\varphi_\lambda(y)\||.\end{aligned}$$

Hence, we have

$$\|\varphi_\lambda(x) - \varphi_\lambda(y)\| < \|x-y\| + |\|x-\varphi_\lambda(x)\| - \|y-\varphi_\lambda(y)\||,$$

that is, φ_λ is a PC map. On the other hand, by the continuity of φ, it is easy to show that φ_λ is continuous. Since $\mathcal{K}$ is compact and φ_λ is continuous, then, by Lemma 2.8, there exists $u \in \mathcal{K}$ such that

$$\|u - \varphi_\lambda(u)\| = \inf\{\|x-\varphi_\lambda(x)\| : x \in \mathcal{K}\}. \tag{3.1}$$

We claim that $\|u-\varphi_\lambda(u)\| = 0$. On the contrary, we assume that $\|u-\varphi_\lambda(u)\| > 0$. By the equality (3.1) and the P-contractivity of φ_λ, we have

$$\begin{aligned}\|\varphi_\lambda(u) - \varphi_\lambda^2(u)\| &< \|u-\varphi_\lambda(u)\| + |\|u-\varphi_\lambda(u)\| - \|\varphi_\lambda(u) - \varphi_\lambda^2(u)\|| \\ &= \|u-\varphi_\lambda(u)\| + \|\varphi_\lambda(u) - \varphi_\lambda^2(u)\| - \|u-\varphi_\lambda(u)\| \\ &= \|\varphi_\lambda(u) - \varphi_\lambda^2(u)\|\end{aligned}$$

which is a contradiction. Therefore, we get $\|u-\varphi_\lambda(u)\| = 0$, that is, u is a fp of φ_λ. Now, assume that there is another fp of φ_λ, called v. Since both of u and v are fps of φ_λ, then we have

$$\begin{aligned}\|u-v\| &= \|\varphi_\lambda(u) - \varphi_\lambda(v)\| \\ &< \|u-v\| + |\|u-\varphi_\lambda(u)\| - \|v-\varphi_\lambda(v)\|| \\ &= \|u-v\|.\end{aligned}$$

This is a contradiction. Consequently, we get $F(\varphi_\lambda) = \{u\}$. Then

$$u = \varphi_\lambda(u) \Longleftrightarrow u = (1-\lambda)u + \lambda\varphi(u) \Longleftrightarrow u = \varphi(u).$$

Therefore, $F(\varphi) = \{u\}$. This completes the proof. □

If we take $k = 0$ in Theorem 3.4, we obtain $\lambda = 1$ and so $\varphi_1 = \varphi$. Then, we can derive Theorem 2.4 in the setting of normed spaces as follows.

Corollary 3.5. *Let $\mathcal{K}$ be a nonempty, compact subset of a normed space $(\mathcal{H}, \|\cdot\|)$ and $\varphi : \mathcal{K} \to \mathcal{K}$ be a continuous PC map. Then, φ has a unique fp.*

We introduce the concept of an enriched Suzuki-type P-contractive map as a generalization of SPC maps.

Definition 3.6. *Let $\mathcal{K}$ be a nonempty subset of a normed space $(\mathcal{H}, \|\cdot\|)$, $\varphi : \mathcal{K} \to \mathcal{K}$ be a map and $k \in [0,\infty)$. Then φ is said to be enriched Suzuki-type P-contractive (ESPC) if*

$$\frac{1}{2}\|x-\varphi(x)\| < (k+1)\|x-y\| \Longrightarrow \|k(x-y) + \varphi(x) - \varphi(y)\| < \|x-y\| + |\|x-\varphi(x)\| - \|y-\varphi(y)\||$$

for all $x, y \in \mathcal{K}$ with $x \neq y$.

Remark 3.7. *We can see that we get the SPC map if $k=0$ in the above definition.*

Remark 3.8. *Also, to observe the connections between the generalizations, we can give the following diagram:*

$$\begin{array}{ccccc} & & \Delta & & \\ & PC & \Longrightarrow & EPC & \\ \Omega & \Downarrow & & \Downarrow & \delta. \\ & SPC & \Longrightarrow & ESPC & \\ & & \nabla & & \end{array}$$

Here, Altun [19] showed the implication given by Ω. Also, the implications presented by Δ and ∇ can be seen from Remark 3.2 and Remark 3.7, respectively. Finally, it remains an open problem whether the converse of the implication given by δ is true.

We prove the following theorem, which is more general than Theorem 2.7, in the setting of normed spaces.

Theorem 3.9. *Let $\mathcal{K}$ be a nonempty, compact and convex subset of a normed space $(\mathcal{H},\|\cdot\|)$ and $\varphi:\mathcal{K}\to\mathcal{K}$ be a continuous ESPC map. Then, φ has a unique fp.*

Proof. We consider the average map $\varphi_\lambda(x)=(1-\lambda)x+\lambda\varphi(x)$. Now, we may take $\lambda=\dfrac{1}{k+1}$. Then $k=\dfrac{1}{\lambda}-1$. It is easy to see that $\lambda\in(0,1]$. Since φ is an ESPC map, then we have

$$\frac{1}{2}\|x-\varphi(x)\|<(k+1)\|x-y\| \Longrightarrow \|k(x-y)+\varphi(x)-\varphi(y)\|<\|x-y\|+|\|x-\varphi(x)\|-\|y-\varphi(y)\||.$$

From the left side of this implication, we have

$$\begin{aligned}\frac{1}{2}\|x-\varphi(x)\|<\frac{1}{\lambda}\|x-y\| &\Longrightarrow \frac{1}{2}\|\lambda x-\lambda\varphi(x)\|<\|x-y\|\\ &\Longrightarrow \frac{1}{2}\|x-\varphi_\lambda(x)\|<\|x-y\|.\end{aligned}$$

The right side of this implication becomes

$$\|(\frac{1}{\lambda}-1)(x-y)+\varphi(x)-\varphi(y)\|<\|x-y\|+|\|x-\varphi(x)\|-\|y-\varphi(y)\||.$$

By multiplying both sides of the above inequality with λ, we obtain

$$\|(1-\lambda)(x-y)+\lambda(\varphi(x)-\varphi(y))\|<\|\lambda x-\lambda y\|+|\|\lambda x-\lambda\varphi(x)\|-\|\lambda y-\lambda\varphi(y)\||.$$

From the definition of the average map, we get

$$\begin{aligned}\|\varphi_\lambda(x)-\varphi_\lambda(y)\| &< \lambda\|x-y\|+|\|x-\varphi_\lambda(x)\|-\|y-\varphi_\lambda(y)\||\\ &\le \|x-y\|+|\|x-\varphi_\lambda(x)\|-\|y-\varphi_\lambda(y)\||.\end{aligned}$$

Then, we have

$$\frac{1}{2}\|x-\varphi_\lambda(x)\|<\|x-y\| \Longrightarrow \|\varphi_\lambda(x)-\varphi_\lambda(y)\|<\|x-y\|+|\|x-\varphi_\lambda(x)\|-\|y-\varphi_\lambda(y)\||,$$

that is, φ_λ is an SPC map. Also, by the continuity of φ, it is easy to show that φ_λ is continuous. Similarly to the proof of Theorem 3.4, since $\mathcal{K}$ is compact and φ_λ is continuous, then, by Lemma 2.8, there exists $u\in\mathcal{K}$ such that satisfies the equality (3.1). We claim that $\|u-\varphi_\lambda(u)\|=0$. On the contrary, we assume that $\|u-\varphi_\lambda(u)\|>0$. In this case, since $0<\dfrac{1}{2}\|u-\varphi_\lambda(u)\|<\|u-\varphi_\lambda(u)\|$, then, by the equality (3.1), we have

$$\begin{aligned}\|\varphi_\lambda(u)-\varphi_\lambda^2(u)\| &< \|u-\varphi_\lambda(u)\|+|\|u-\varphi_\lambda(u)\|-\|\varphi_\lambda(u)-\varphi_\lambda^2(u)\||\\ &= \|u-\varphi_\lambda(u)\|+\|\varphi_\lambda(u)-\varphi_\lambda^2(u)\|-\|u-\varphi_\lambda(u)\|\\ &= \|\varphi_\lambda(u)-\varphi_\lambda^2(u)\|,\end{aligned}$$

which is a contradiction. Therefore, we get $\|u-\varphi_\lambda(u)\|=0$, that is, u is a fp of φ_λ. Now, assume that there is another fp of φ_λ, called v. Since $u\neq v$, then we have $0=\dfrac{1}{2}\|u-\varphi_\lambda(u)\|<\|u-v\|$. Hence, we get

$$\begin{aligned}\|u-v\| &= \|\varphi_\lambda(u)-\varphi_\lambda(v)\|\\ &< \|u-v\|+|\|u-\varphi_\lambda(u)\|-\|v-\varphi_\lambda(v)\||\\ &= \|u-v\|.\end{aligned}$$

This is a contradiction. Consequently, we obtain $F(\varphi_\lambda)=\{u\}$. Hence, $F(\varphi)=\{u\}$, that is, φ has a unique fp u. $\square$

If we take $k = 0$ in Theorem 3.9, we obtain $\lambda = 1$ and so $\varphi_1 = \varphi$. Then, we can derive Theorem 2.7 in the setting of normed spaces as described below.

Corollary 3.10. *Let $\mathcal{K}$ be a nonempty, compact subset of a normed space $(\mathcal{H}, \|\cdot\|)$ and $\varphi : \mathcal{K} \to \mathcal{K}$ be a continuous SPC map. Then, φ has a unique fp.*

4. A direct generalization of SPC maps

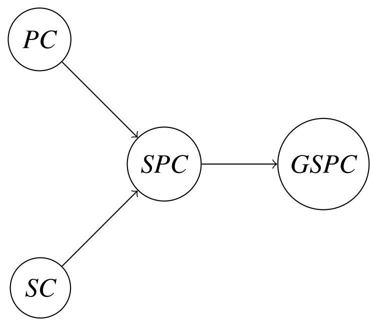
We begin by defining the generalized Suzuki-type P-contractive map, which is a direct generalization of SPC maps.

Definition 4.1. *Let $(\mathcal{H}, d)$ be a metric space and $\varphi : \mathcal{H} \to \mathcal{H}$ be a map. Then, the map φ is said to be generalized Suzuki-type P- contractive (GSPC) if*

$$\lambda d(x, \varphi(x)) < d(x,y) \Longrightarrow d(\varphi(x), \varphi(y)) < d(x,y) + |d(x, \varphi(x)) - d(y, \varphi(y))|$$

for all $x, y \in \mathcal{H}$ and $\lambda \in (0,1)$.

Remark 4.2. *We get the SPC map if we take $\lambda = \frac{1}{2}$ in the above definition. Hence, we generalize the SPC maps, as well as the SC and PC maps. We can express this in the below diagram:*



Here, Example 2.3 shows that the classes of SC and PC maps are distinct. Also, Altun [19] showed the implication between PC and SPC and the implication between SC and SPC. Finally, the last implication can be clearly seen when we take $\lambda = \frac{1}{2}$ in the definition of GSPC.

By Lemma 2.8, we prove the following theorem, which is a generalization of Theorem 2.9.

Theorem 4.3. *Let $(\mathcal{H}, d)$ be a compact metric space and $\varphi : \mathcal{H} \to \mathcal{H}$ be a GSPC map. Then φ has a unique fp in $\mathcal{H}$ provided the function $f(x) = d(x, \varphi(x))$ is lower semi-continuous.*

Proof. Since $\mathcal{H}$ is compact and $f : \mathcal{H} \to \mathbb{R}$ is lower semi-continuous, and by Lemma 2.8, then there exists $u \in \mathcal{H}$ such that $f(u) = \inf\{f(x) : x \in \mathcal{H}\}$, that is, we have

$$d(u, \varphi(u)) = \inf\{d(x, \varphi(x)) : x \in \mathcal{H}\}. \tag{4.1}$$

We now show that $d(u, \varphi(u)) = 0$. On the contrary, we suppose that $d(u, \varphi(u)) > 0$. In this case, since $\lambda \in (0,1)$, we get $0 < \lambda d(u, \varphi(u)) < d(u, \varphi(u))$. Then, by the equality (4.1), we have

$$\begin{aligned} d(\varphi(u), \varphi^2(u)) &< d(u, \varphi(u)) + |d(u, \varphi(u)) - d(\varphi(u), \varphi^2(u))| \\ &= d(u, \varphi(u)) + d(\varphi(u), \varphi^2(u)) - d(u, \varphi(u)) \\ &= d(\varphi(u), \varphi^2(u)), \end{aligned}$$

which is a contradiction. Therefore, we get $d(u, \varphi(u)) = 0$; that is, u is a fp of φ. Now, we need to show that the fp is unique. Assume that there is another fp of φ, called v. In this case, since $0 = \lambda d(u, \varphi(u)) < d(u, v)$, then we have

$$\begin{aligned} d(u, v) &= d(\varphi(u), \varphi(v)) \\ &< d(u, v) + |d(u, \varphi(u)) - d(v, \varphi(v))| \\ &= d(u, v), \end{aligned}$$

which leads to a contradiction. In conclusion, the fp of φ is unique. □

As a generalization of Theorem 2.7, we can give the following theorem using Theorem 4.3.

Theorem 4.4. *Let $(\mathcal{H}, d)$ be a compact metric space and $\varphi : \mathcal{H} \to \mathcal{H}$ be a continuous GSPC map. Then, φ has a unique fp.*

Proof. Since φ is continuous, then $f(x) = d(x, \varphi(x))$ is also continuous by the following inequality

$$|d(x, \varphi(x)) - d(y, \varphi(y))| \leq d(x,y) + d(\varphi(x), \varphi(y)).$$

So, it will be directly lower semi-continuous. Therefore, by Theorem 4.3, φ has a unique fp. □

Remark 4.5. *We can see that the constant, which is $\frac{1}{2}$ in the SPC maps, is to be $\lambda \in (0,1)$ in the GSPC maps. Consequently, when $\lambda = \frac{1}{2}$, Theorems 4.3 and 4.4 are reduced to Theorems 2.9 and 2.7, respectively.*

5. Conclusions

We define three new generalizations of PC maps and establish the existence theorems of a unique fp for these maps. We note that the EPC and ESPC maps include the PC and SPC maps, respectively, and the GSPC maps include the SPC maps. Our results improve and extend the corresponding results of [18, 19].

In forthcoming research, the existence of a unique fp for the multi-valued versions of the EPC and ESPC maps can be proved. Furthermore, the enriched version of GSPC maps may be introduced, and some fixed point theorems for this map might be offered.

Declarations

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Temperature-Dependent Parameters in Enzyme Kinetics: Impacts on Enzyme Denaturation

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Abstract

Enzymes are vital proteins in biological systems, responsible for regulating and coordinating numerous essential processes. The incorporation of denaturation rate accounts for the gradual loss of enzyme activity over time, which is particularly significant under experimental conditions where enzymes are susceptible to denaturation. It is noteworthy that adverse environmental conditions, such as high temperature or pH imbalance, can induce enzyme denaturation, leading to a loss of functionality over time. This structural disruption renders enzymes inactive, posing a crucial consideration in long-term enzyme kinetics studies. Furthermore, enzymes typically exhibit reduced catalytic activity at lower temperatures, which is pivotal for understanding their stability and efficacy in biological systems and industrial applications. Accordingly, we developed a mathematical model to investigate enzyme kinetics under varying temperature, aiming to analyse their respective impacts on both enzyme behaviour and product formation.

1. Introduction

Enzymes are proteins that play a vital role in biological systems by regulating and coordinating many essential processes. They reduce the activation energy required for chemical reactions and regulate cellular functions by increasing the rate of reactions. Basic metabolic processes such as food breakdown, energy production, and protein synthesis are also controlled by enzymes. Additionally, some enzymes play a role in intracellular and intercellular communication as well as in regulation of DNA, RNA and protein synthesis. Enzymes are also important for the immune system [1].

The effects of inhibitors at the cellular level are diverse and they are used to regulate many biological processes [2]. For example, many drugs act by inhibiting enzymes targeted in the treatment of diseases [3]. At the same time, in biochemical and cellular research, inhibitors are important tools for understanding the functions of certain enzymes or manipulating certain metabolic pathways [4]. Inhibitors are molecules that interfere with or halt the typical activities of enzymes. These molecules work by altering or blocking the catalytic activity of enzymes. Inhibitors are generally divided into two main categories: reversible inhibitors and irreversible inhibitors. Reversible inhibitors temporarily bind with the enzyme and this connection is reversible, meaning it can be broken easily. Reversible inhibitors generally bind to the enzyme's active site, preventing substrate binding or catalytic activity. If the inhibitor has the same or similar structure as the substrate and prevents the binding of the substrate by binding to the active site of the enzyme, it is called a competitive inhibitor. Uncompetitive inhibitor is a specific type of enzyme inhibitors that binds to the enzyme-substrate complex, rather than the free enzyme itself. Non-competitive inhibitors are molecules that bind to an enzyme at a site other than the active site, known as the allosteric site. This binding causes a change in the structure of the enzyme, which alters the active site and prevents the enzyme from catalyzing the reaction efficiently, even when the substrate is present in high concentrations [5].

Enzyme kinetics, the study of the rates at which enzyme-catalysed reactions proceed, is a cornerstone of biochemistry and molecular biology. Understanding these kinetics is crucial for a variety of applications, including drug development, metabolic engineering, and the elucidation of biochemical pathways. Mathematical modelling plays a pivotal role in interpreting

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and predicting the behaviour of enzyme systems, providing insights that are often difficult to obtain through experimental methods alone. The foundation of enzyme kinetics was laid by Michaelis and Menten with their pioneering work on the Michaelis-Menten mechanism [6]. This model introduced key concepts such as the Michaelis constant (K_{m}) and maximum reaction velocity ($V_{\max}$), which remain fundamental to enzyme kinetics today. Despite its simplicity, the Michaelis-Menten equation provides a robust framework for understanding single-substrate enzyme reactions under steady-state conditions.

Mendes et al. examined the use of computational techniques to enzyme kinetics [7]. Boehr et al. utilized molecular dynamics simulations to explore the structural basis of enzyme inhibition [8]. Other studies investigated mathematical modelling of uncompetitive inhibitor such as in bi-substrate enzymatic reactions [9]. Non-competitive inhibition is significant because it decreases the maximum rate of the reaction ($V_{\max}$) without affecting the affinity between the enzyme and the substrate (K_{m} remains constant). Cornish-Bowden provides a detailed derivation of the rate equations for linear mixed inhibition, emphasizing the importance of understanding the inhibitory constants for both competitive and noncompetitive components [10]. Cheng et al. investigated the kinetic parameters of enzyme inhibitors using a combination of experimental and computational approaches.

Temperature is key parameters for many organisms, and its effects and modelling have been studied in various biological research fields [11, 12, 13]. Each enzyme is affected differently by temperature, but ultimately, it begins to suffer damage at a certain temperature. This is an important factor, especially in long-term enzyme kinetics. As temperature increases, the kinetic energy of enzyme and substrate molecules also increases. This leads to faster movement and more frequent collisions between molecules. More frequent collisions result in faster enzyme-substrate reactions. Every chemical reaction has a specific activation energy. As temperature increases, more molecules gain enough energy to overcome this energy barrier, which speeds up the reaction.

However, there are also obstacles that slow down the reaction. Proteolytic enzymes can break down enzymes, particularly in cellular environments or in solutions containing proteases [14]. In industrial or laboratory conditions, enzymes can lose their activity over time when used continuously. Some enzymes require co-factors, such as metal ions or organic molecules, to function properly. The depletion of these co-factors can reduce the enzyme's effectiveness [15]. Including the denaturation rate accounts for the loss of active enzyme over time, which can be significant in experimental conditions where enzymes are prone to denaturation [16]. This addition makes the model more realistic and can be crucial for understanding the long-term behaviour of enzyme reactions, especially in industrial or biotechnological applications where enzyme stability is a key factor. Each enzyme is affected differently by temperature, but ultimately, it begins to suffer damage at a certain temperature. This is an important factor, especially in long-term enzyme kinetics.

Numerous scientific investigations have explored the influence of temperature on enzyme activity throughout history. For instance, Peterson et al., [17, 18], developed and validated the Equilibrium model, which elucidates how enzymes lose activity under high temperatures. Their research initially focused on alkaline phosphatase, demonstrating that enzyme activity rises with temperature until reaching an optimal point (approximately 57°C at time zero), but diminishes with time thereafter. Beyond this optimum, enzyme activity declines steeply due to combined effects of temperature and duration.

Additionally, their studies examined the temperature dependence of acid phosphatase's initial reaction rate. They found that this enzyme reaches peak reaction rates (around 1.1 mM) at 63°C, contrasting sharply with reaction rates estimated at 0.1 mM 22 °C and approximately 0.6 mM at 82°C. Furthermore, they extended their investigations to enzymes sourced from diverse thermal environments, including an Antarctic Sea bacterium, HK47 alkaline phosphatase (psychrotrophic), Bacillus cereus dihydrofolate reductase (mesophilic), and Caldicellulosiruptor saccharolyticus β-glucosidase (thermophilic) [19]. Their findings underscore the profound impact of temperature on enzymatic activities across a spectrum of biological systems.

Forsling and Widdas [20] conducted an investigation into the impact of temperature on Phenolphthalein, Phloretin, and Stilboestrol, revealing their competitive inhibition of the facilitated glucose transfer system in human red blood cells. Their findings indicated that the concentration necessary for 50% inhibition by Phenolphthalein decreases at lower temperatures and exhibits a gradual rise within the temperature range of $10-40$ °C. The Arrhenius plot derived from their data displayed a slope corresponding to $19,300$ calories per mole, highlighting the temperature sensitivity of these inhibitory effects.

Liu et al., investigated the impact of temperature on α-Amylase activity, determining both its optimal operating temperature and thermal stability [21]. They observed that the enzyme's relative activity initiates at approximately 65% at 30°C, reaches its peak activity at 100% around 50°C, and subsequently declines. At 80°C, the enzyme's activity decreases to approximately 50%.

In a related study, Coban et al., examined the temperature dependence of Sheep spleen tissue Glutathione Reductase (GR) enzyme activity [22]. They found that the enzyme exhibits an activity of 0.25 EU/ml from 0°C to 10°C. Beyond 10°C, its activity increases, reaching 0.55 EU/ml at its optimal temperature of 40°C, after which it diminishes sharply to 0 EU/ml at 80°C.

In this study, we will develop and analyze a comprehensive mathematical model to investigate the effects of temperature on reaction rates and enzyme denaturation. The model will incorporate various parameters, including the initial enzyme concentration, the rate of enzyme denaturation, and the influence of temperature on both the catalytic activity and stability of

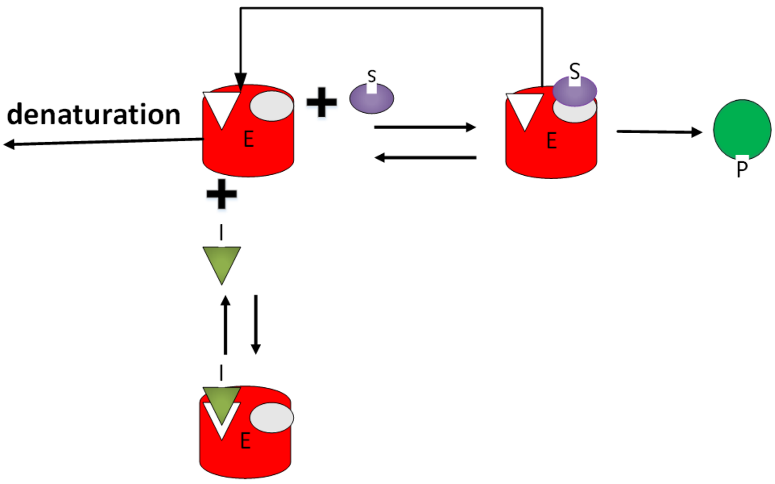


Figure 1: Enzyme reaction for competitive inhibition.

enzymes. By simulating different temperature conditions, we aim to provide a detailed understanding of how temperature variations impact enzymatic reactions over time. This will include examining the balance between the enhanced reaction rates at higher temperatures and the concomitant increase in denaturation rates that reduce enzyme activity. By accounting for these factors, our model will offer a more realistic and practical representation of enzyme kinetics, contributing to the optimization of enzymatic processes in various fields, such as biotechnology, pharmaceuticals, and metabolic engineering.

2. Methods

2.1. Developing mathematical models

The aim of our study is to model enzyme kinetics with different temperature: We consider the optimum temperature for enzyme reaction and apply these to enzyme-substrate interaction with competitive inhibition. Competitive inhibitors bind to the same site on the enzyme (active site) as the substrate, preventing the substrate from binding. In the presence of a competitive inhibitor, higher substrate concentrations can overcome the inhibitor's effect. The Michaelis constant (K_{m}) increases, but the maximum velocity ($V_{\max}$) remains unchanged. There are four main parts in the model: enzyme ($[E]=e$), substrate ($[S]=s$), inhibitor ($[I]=i$) and product ($[P]=p$), (see Figure 1). Furthermore, there are enzyme-substrate complex ($[ES]=x_1$) and enzyme-inhibitor complex ($[EI]=x_2$). The kinetic reactions of competitive inhibition can be shown in reaction scheme 2.1,

$$\begin{aligned} E+S \underset{k_{-1}}{\overset{k_1}{\leftrightarrow}} ES(x_1) \overset{k_2}{\rightarrow} E+P, \\ E+I \underset{k_{-3}}{\overset{k_3}{\leftrightarrow}} EI(x_2), \end{aligned} \tag{2.1}$$

and the model equations of this reaction figure as follows,

$$\frac{de}{dt} = -k_1es+k_{-1}x_1+k_2x_1-k_3ei+k_{-3}x_2-me, \tag{2.2}$$

$$\frac{ds}{dt} = -k_1es+k_{-1}x_1, \tag{2.3}$$

$$\frac{dx_1}{dt} = k_1es-k_{-1}x_1-k_2x_1, \tag{2.4}$$

$$\frac{dp}{dt} = k_2x_1, \tag{2.5}$$

$$\frac{di}{dt} = -k_3ei+k_{-3}x_2, \tag{2.6}$$

$$\frac{dx_2}{dt} = k_3ei-k_{-3}x_2. \tag{2.7}$$

Here the parameter k_1 represents the rate of forward formation of the enzyme-substrate complex ES from free enzyme E and substrate S, k_{-1} represents the rate of dissociation of the enzyme-substrate complex (ES), k_2 represents the rate of degradation of ES, k_3 the rate of formation of EI, k_{-3} the rate of dissociation of the enzyme-inhibitor complex (EI) and m the rate of denaturation of the free enzyme. We ignored the denaturation of enzyme complexes in the model because the reaction involving the substrate occurs much faster than the denaturation process, which primarily affects the free enzyme over time. The initial values of the variables are $e(0) = e_0$, $s(0) = s_0$, $i(0) = i_0$ and $x_1(0) = x_2(0) = p(0) = 0$. For simplicity, the sum of the equations 2.3, 2.4 and 2.5,

$$\frac{ds(t)}{dt} + \frac{dx_1(t)}{dt} + \frac{dp(t)}{dt} = 0,$$

and integrate with initial values, we obtain

$$s(t) + x_1(t) + p(t) = s_0 \Rightarrow x_1(t) = s_0 - s(t) - p(t).$$

Similarly, we obtain $x_2(t) = i_0 - i(t)$ from the equations 2.6 and 2.7. When we substitute these two variables into the model, then the model will have four main equations.

$$\begin{aligned} \frac{de(t)}{dt} &= -k_1(T)e(t)s(t) + (k_{-1}(T) + k_2(T))(s_0 - s(t) - p(t)) \\ &\quad - k_3(T)e(t)i(t) + k_{-3}(T)(i_0 - i(t)) - m(T)e(t), \\ \frac{ds(t)}{dt} &= -k_1(T)e(t)s(t) + k_{-1}(T)(s_0 - s(t) - p(t)), \\ \frac{dp(t)}{dt} &= k_2(T)(s_0 - s(t) - p(t)), \\ \frac{di(t)}{dt} &= -k_3(T)e(t)i(t) + k_{-3}(T)(i_0 - i(t)). \end{aligned} \tag{2.8}$$

2.2. Model parameterisation

This study aims to highlight the impact of temperature changes on the model, product concentration and enzyme denaturation. Each enzyme has an optimum temperature at which it reaches its maximum kinetic rate. At this temperature, the enzyme and substrate molecules collide with maximum efficiency, resulting in the highest reaction rate. In this study, the optimum temperature will be referred to as T_0. Beyond this temperature, the structure of the enzyme begins to break down, leading to denaturation. A denatured enzyme cannot bind to its substrate, causing the reaction rate to drop dramatically. At a certain temperature, the enzyme can become fully denatured and lose its activity completely. When the temperature falls slightly below T_0, the activity of the enzyme decreases because the kinetic energy of the molecules drops. The reaction rate slows down due to lower kinetic energy and fewer collisions. In this way, the velocity parameters will be defined as a function of T. We define three different models which will have different reaction rate functions. The first one is Arrhenius equation,

$$f_1(T) = A\exp\left(-\frac{E_\alpha}{R(T+273)}\right), \tag{2.9}$$

where A, E_a, R and T are pre-exponential factor, activation energy, universal gas constant and absolute temperature, respectively. In this study, the temperatures are reported in degrees Celsius, but they should be converted to Kelvin for use in the Arrhenius equation. Therefore, we convert the temperatures to Kelvin by adding 273. The default parameter values are given in Table 1, and the function is illustrated in Figure 2a. Using the function $f_1(T)$, the reaction velocity will be

$$k_i = k_{i_0} f_1(T), \qquad i = 1, -1, 2, 3, -3, \tag{2.10}$$

where k_{i_0} are reaction rate constant (they are generally optimum reaction rate for each i). Another parameter is the enzyme denaturation rate, which is defined by sigmoid function (see Figure 2d),

$$m(T) = \frac{1}{1+\exp\left(\frac{T_1 - T}{n}\right)}, \tag{2.11}$$

where T_1 controls the point at which the function reaches half of its maximum value, and n adjusts the steepness of the curve (slope). We will refer to it as Model I if the k_i parameters occur with the function $f_1(T)$ and the enzyme denaturation parameter $m(T)$ is as described in equation 2.11.

After exposure to high temperatures, enzymes denature, while at low temperatures, their activity slows down. Therefore, a single temperature function is used to encapsulate these conditions instead of employing two separate functions (the Arrhenius function and the $m(T)$ function). This approach ensures that the enzyme exhibits maximum reaction velocity at its optimum temperature, approximated by a Gaussian function

$$f_2(T) = \exp\left(-\frac{(T-T_0)^2}{2\sigma_1^2}\right), \tag{2.12}$$

Parameter	Symbol	Value	Unit
Reaction Rate to [ES] complex	k_{1_0}	1	$mM^{-1}s^{-1}$
Reaction Rate to [E] and [S]	k_{-1_0}	0.1	s^{-1}
Reaction Rate to [E] and [P]	k_{2_0}	0.5	s^{-1}
Reaction Rate to [EI] complex	k_{3_0}	1	$mM^{-1}s^{-1}$
Reaction Rate to [E] and [I]	k_{-3_0}	0.1	s^{-1}
Activation Energy	E_α	5×10^4	$J.mol^{-1}$
Universal Gas Constant	R	8.314	$J.mol^{-1}K^{-1}$
Pre-exponential Factor	A	16×10^6	–
Optimum Temperature	T_0	50	°C
Threshold Temperature	T_1	70	°C
Standard Deviation in Eq. 2.12 and 2.14	σ_1	20	°C
Standard Deviation in Eq. 2.14	σ_2	10	°C
Steepness Parameter see Eq. 2.11	n	20	°C

Table 1: Default parameter values and units. Detail for reaction rates direction see enzyme reaction scheme 2.1 and Figure 1.

where T_0 is optimum temperature and σ_1 is standard deviation which controls the width of the shape and, it is given in Figure 2b. Thereby, the rate function of the reaction will be

$$k_i(T) = f_2(T)k_{i_0}, \qquad i = 1,-1,2,3,-3. \tag{2.13}$$

System 2.8 with the reaction rates as in equations 2.13 and no denaturation rate ($m(T) = 0$) will be referred as Model II.

However, we observe that Model I does not account for the decreasing reaction velocity at temperatures higher than the optimum. According to the Arrhenius equation, the reaction rate continues to increase even above the optimum temperature. Similarly, Model II overlooks the decrease in effective enzyme concentration due to denaturation. Therefore, we introduce a new piecewise function to make the model more realistic as

$$f_3(T) = \begin{cases} \exp\left(-\dfrac{(T-T_0)^2}{2\sigma_1^2}\right) & T \le T_0, \\ 1 & T_0 < T \le T_0+10, \\ \exp\left(-\dfrac{(T-(T_0+10))^2}{2\sigma_2^2}\right) & T_0+10 < T, \end{cases} \tag{2.14}$$

where T_0 is optimum temperature, σ_1 and σ_2 are standard deviations. The new model (referred here as Model III) is possessed of new piecewise reaction rate function as

$$k_i(T) = f_3(T)k_{i_0} \qquad i = 1,-1,2,3,-3, \tag{2.15}$$

and the denaturation rate is $m(T)$ as given equation 2.11.

3. Results

Figure 3a illustrates the changes in product concentration over time. The curves depicted are generated using Model I, with temperatures set at 10°C, 30°C, 40°C, 50°C, 70°C and 90°C. In this system, where initial conditions are crucial, the substrate concentration must be significantly higher than the enzyme concentration.

As shown in Figure 3a, at lower temperatures, the reaction rates progresses very slowly. Despite this, the product concentration eventually reaches the expected equilibrium point, although it takes a longer duration. At the optimal temperature range, around 40 – 50°C, the equilibrium point is achieved at the fastest rate. However, at temperatures high enough to damage the enzyme, the reaction rate might initially be high, but the final product concentration may not reach the expected level due to a reduction in the enzyme's activity.

It is also examined the same conditions for Model II, as shown in Figure 3b. In this model, instead of enzyme denaturation, it is interpreted the slowing of the reaction rate as being due to temperatures below or above the optimum. In the same figure, it can be seen that the product dynamics in Model II yield almost the same results as Model I up to the optimum temperature. In this model, even if the ambient temperature is significantly above the optimum, the product eventually reaches the equilibrium point, albeit slowly, similar to low temperatures.

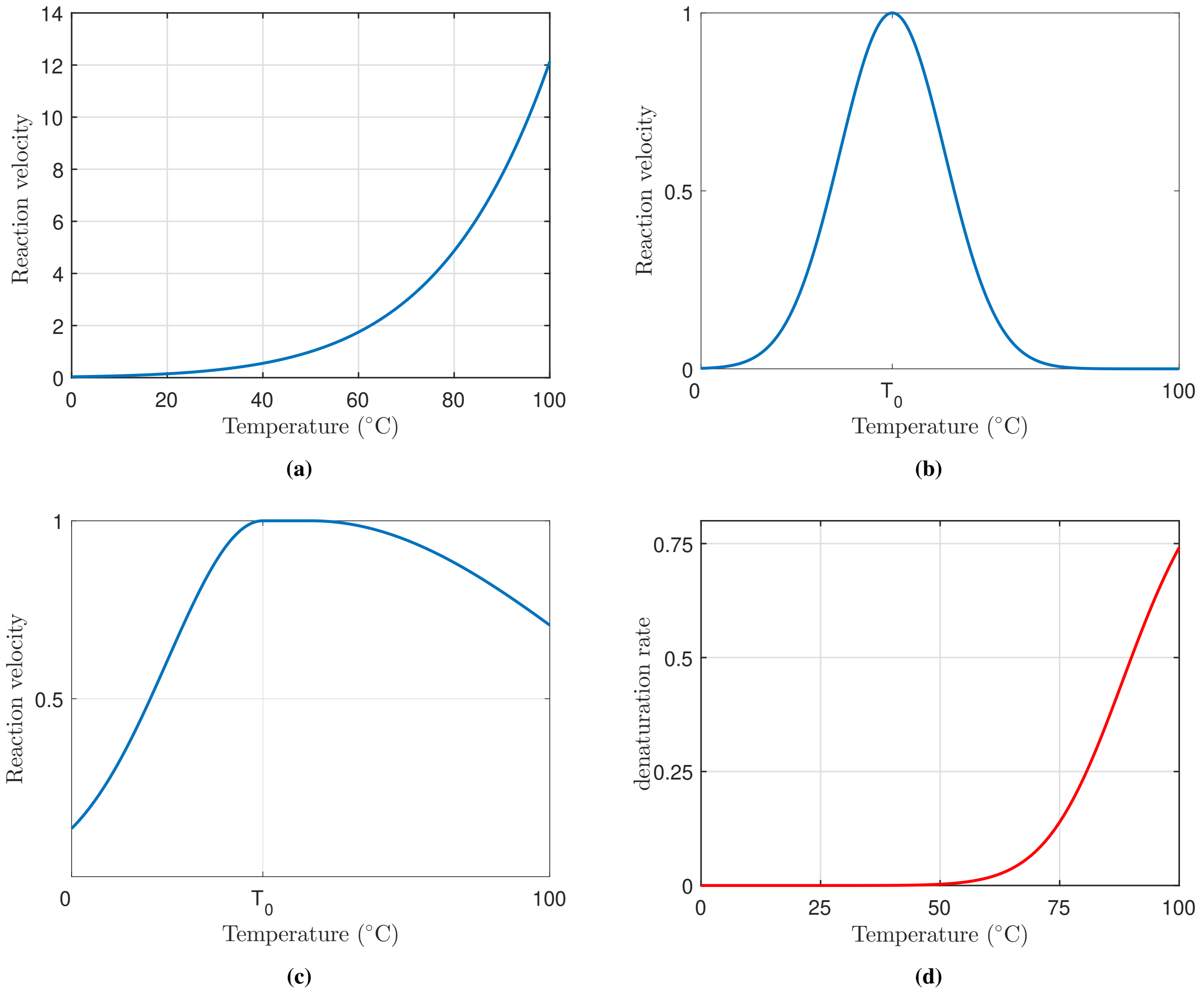


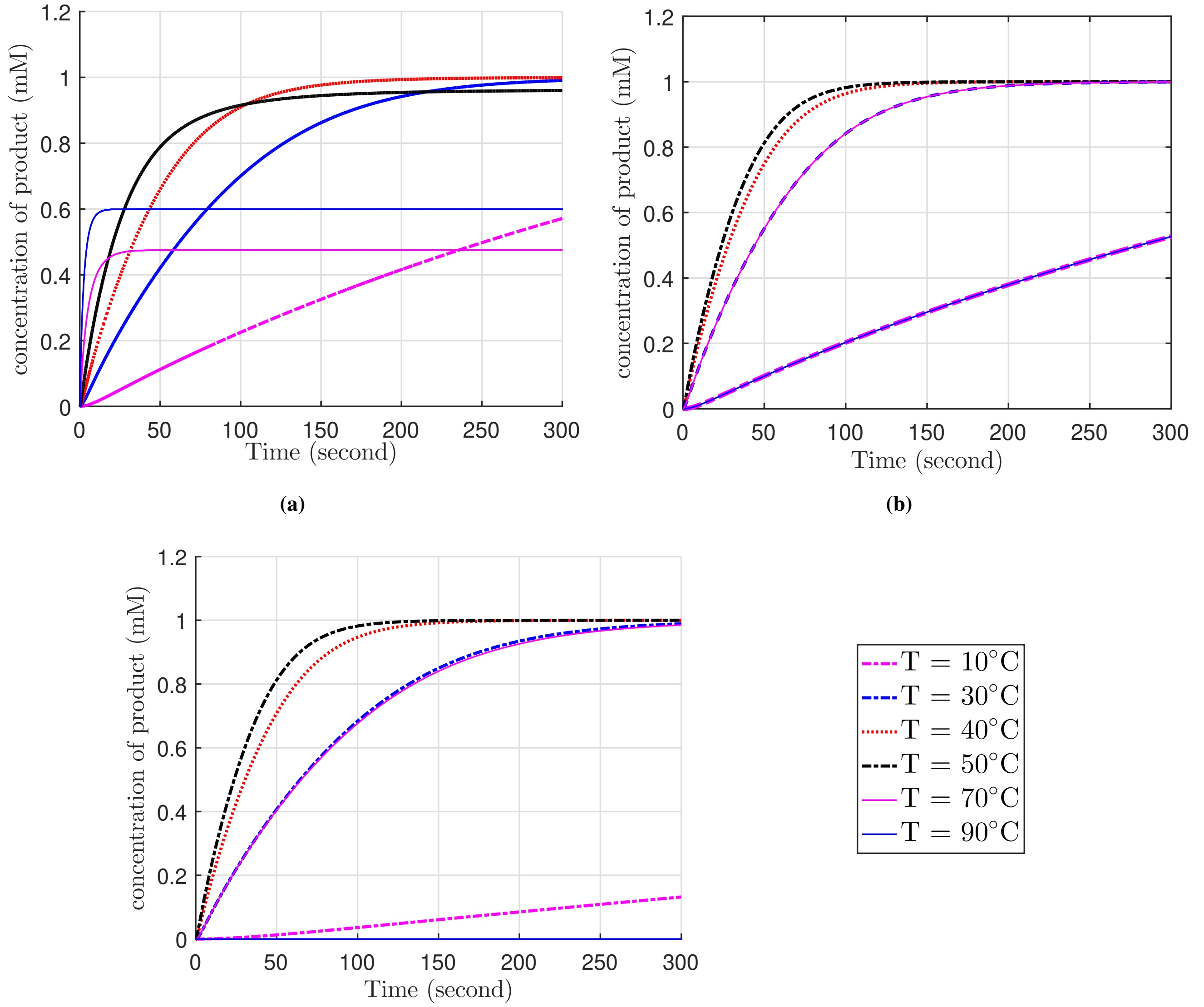
Figure 2: The graphs of the temperature-dependent parameter functions are shown above. The reaction rates depend on the function of temperature T as follows: (a) Arrhenius equation for Model I, (b) Gaussian function for Model II and (c) $f_3(T)$ function for Model III. Also, denaturation rate in Model I and Model III as in figure (d) and it is zero in Model II.

High temperatures increase the kinetic energy of the enzyme, leading to greater molecular motion. However, when this motion becomes excessive, it can hinder the proper binding between the enzyme and the substrate. Effective interaction between the enzyme and substrate requires a certain level of structural stability. Above the optimal temperature, the enzyme's active site may no longer properly bind to the substrate and catalyze the reaction efficiently. This leads to a reduction in the enzyme's effectiveness, making even the remaining active enzymes less efficient.

To address these issues, we combine the more realistic aspects of Model I and Model II to develop Model III. In this model, we retain the advantages of Model II for temperatures ranging from low to optimal, where the reaction rate behaves in accordance with the Gaussian function, effectively capturing the temperature dependence without the sharp decline seen in denaturation-prone conditions. This ensures that at lower temperatures, the reaction progresses, albeit slowly, until it reaches the equilibrium point, similar to what is observed in Model II.

For higher temperatures, Model III introduces a new function for reaction velocity that accounts for the decline in enzyme activity due to excessive molecular motion and structural instability. This new function provides a more accurate representation of the reaction kinetics at elevated temperatures, where the reaction rate initially rises but then decreases as the enzyme begins to denature. Additionally, Model III incorporates the denaturation rate, as illustrated in Figure 2d, to further enhance the model's realism by considering the actual loss of enzyme functionality over time at higher temperatures.

By combining these elements, Model III offers a more comprehensive and realistic portrayal of enzyme kinetics across a wider temperature range. The time-product concentration graph, shown in Figure 3c, reflects these improvements, demonstrating that while the product concentration reaches the equilibrium point rapidly at optimal temperatures, it takes longer at both lower and higher temperatures. At very high temperatures, the equilibrium concentration is reduced due to enzyme denaturation, providing a more accurate depiction of real-world biochemical processes.



(c)

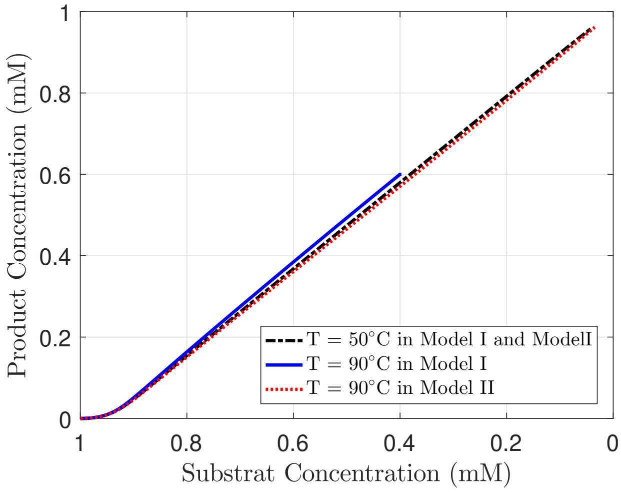
Figure 3: Time-product concentration graph at different temperatures with $e_0 = 0.1$ mM, $s_0 = 1$ mM, $i_0 = 0.1$ mM, $p_0 = 0$ and the default parameter values in the Table 1. Model I in 3a, Model II in 3b and Model III in 3c are showed with varying temperature values.

In summary, Model III builds on the strengths of Models I and II, offering a nuanced understanding of enzyme kinetics that takes into account the complex interplay between temperature, reaction rate, and enzyme stability. This model can serve as a valuable tool for predicting enzyme behavior in various industrial and research applications where temperature control is crucial for optimal performance.

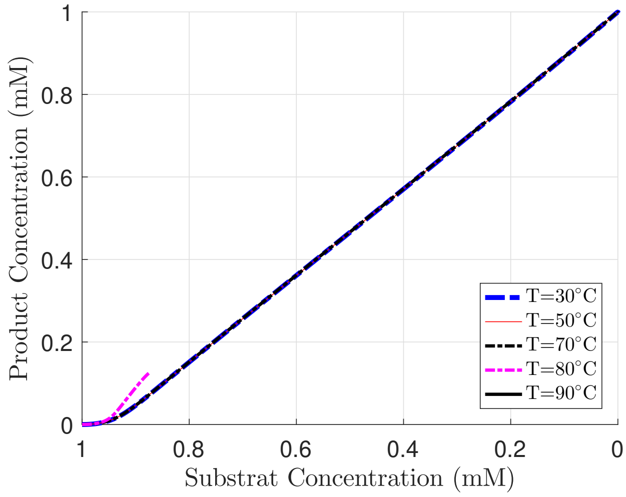
When comparing the substrate-product graphs for each model (see Figure 4), a linear inverse proportionality is observed. From low to optimum temperatures, all models behave similarly. However, at temperatures exceeding the enzyme's optimal range, the product concentration in Models I and III begins to decline due to enzyme denaturation. Additionally, in Model III, the product concentration is almost zero at 90°C. In contrast, in Model II, although the reaction rate slows down at these higher temperatures, the product concentration eventually reaches the maximum after an extended period.

This comparison highlights the distinct approaches of each model in handling extreme temperatures. Model I accounts for enzyme denaturation but does not let it affect the reaction rate. This leads to an unrealistic continuous increase in reaction rate as per the Arrhenius equation, failing to accurately predict the decline in product concentration at high temperatures due to denaturation effects. Model II, which uses a Gaussian function, better captures the temperature dependence of reaction rates but does not consider the loss of enzyme activity due to denaturation. This omission results in the model predicting a gradual achievement of maximum product concentration even at temperatures that would typically cause enzyme degradation.

Model III addresses these shortcomings by integrating a new reaction velocity function and considering the denaturation rate. This makes Model III more reflective of real biochemical processes where enzyme activity significantly decreases at high temperatures. The near-zero product concentration at 90°C in Model III underscores the impact of enzyme denaturation,



(a)



(b)

Figure 4: Substrate-product concentration graphs for model I and II in (a) and for Model III in (b) at different temperatures with $e_0 = 0.1$ mM, $s_0 = 1$ mM, $i_0 = 0.1$ mM, $p_0 = 0$ and the default parameter values in the Table 1. As seen in (b) product concentration is very low at 80°C and almost zero at 90°C.

providing a more realistic prediction for scenarios involving extreme temperatures.

These observations emphasize the importance of considering both reaction kinetics and enzyme stability in biochemical modeling. The inclusion of denaturation effects in Model III offers a more comprehensive understanding of enzyme behavior, which is crucial for applications requiring precise temperature control. For instance, in industrial processes involving enzyme catalysis, maintaining temperatures within the optimal range is essential to ensure maximum efficiency and product yield.

In conclusion, the substrate-product graphs reveal that while Models I and II have their respective strengths, they fall short in accurately depicting enzyme kinetics at high temperatures. Model III, by incorporating both reaction velocity adjustments and denaturation rates, provides a more accurate and practical representation, enhancing our ability to predict and optimize enzymatic reactions across a broader range of temperatures.

4. Discussion

Higher initial concentrations of enzymes generally correspond to increased reaction rates and product yields within a given timeframe. However, this relationship is contingent upon the enzyme's stability at elevated temperatures. Elevated temperatures can induce enzyme denaturation, a process wherein the enzyme loses its native structure and, consequently, its functional activity. Denaturation at high temperatures can nullify the benefits of high initial enzyme concentrations, thereby limiting the potential increase in product concentration.

Enzymes typically exhibit optimal activity within specific temperature ranges, known as their temperature optima. Beyond this range, enzymatic activity can decline rapidly due to decreased stability and increased susceptibility to denaturation. Increasing the initial enzyme concentration can initially enhance product yield by facilitating more frequent enzyme-substrate interactions before denaturation occurs. This phenomenon is advantageous up to the point where denaturation begins to significantly compromise enzyme activity. However, at temperatures causing complete enzyme denaturation, escalating initial enzyme concentrations ceases to be advantageous. Under such conditions, the enzyme loses its catalytic capability entirely, resulting in minimal or negligible product formation despite higher initial enzyme amounts.

Thus, while higher initial enzyme concentrations can potentially increase product yields under optimal or moderately elevated temperatures, the critical consideration remains the enzyme's thermal stability and susceptibility to denaturation, which ultimately govern its effectiveness in catalyzing reactions.

To investigate enzyme kinetics under varying temperatures, we developed a mathematical model aimed at analyzing their respective impacts on both enzyme behavior and product formation. Model III builds on the strengths of Models I and II, offering a nuanced understanding of enzyme kinetics that takes into account the complex interplay between temperature, reaction rate, and enzyme stability. This model serves as a valuable tool for predicting enzyme behavior in various industrial and research applications where temperature control is crucial for optimal performance.

The substrate-product graphs reveal that while Models I and II have their respective strengths, they fall short in accurately depicting enzyme kinetics at high temperatures. Model III, by incorporating both reaction velocity adjustments and denaturation rates, provides a more accurate and practical representation. This enhancement improves our ability to predict and optimize enzymatic reactions across a broader range of temperatures.

Furthermore, the article can accommodate uncompetitive, non-competitive, or mixed-inhibitor models in the same way. By including these different types of inhibition, the models can provide a comprehensive analysis of enzyme behavior under various conditions. Additionally, a pH variable can be added to the models alongside the temperature variable. This inclusion allows for a more accurate representation of enzyme activity, as both pH and temperature are critical factors influencing enzyme kinetics. While introducing more parameters can enhance the realism of the model, it also presents a challenge. The increased complexity may limit the model's adaptability to all types of enzymes, as each enzyme might respond differently to changes in pH, temperature, and inhibitor presence. Therefore, while more detailed models are beneficial for specific scenarios, a balance must be struck to maintain general applicability.

Declarations

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Trapezoid-type Inequalities Based on Generalized Conformable Integrals via Co-ordinated h-Convex Mappings

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Abstract

In this study, some new trapezoid type inequalities are generalized for $h-$ convex functions in coordinates by means of generalized conformable fractional integrals. For functions with $h-$convex absolute values of their partial derivatives, some new trapezoid type inequalities are obtained using the well-known Holder and Power Mean inequalities. In addition, some findings of this study include some results based on Riemann Liouville fractional integrals.

1. Introduction

Convexity theory represents a distinct and highly specialized field within the mathematical sciences. The field has attracted significant interest among researchers due to its extensive and diverse range of applications in fields such as engineering, optimization theory, energy systems, and physics. Over the years, many articles have been written on It is important to avoid making generalisations and to ensure that any new versions of existing theories are fully evidenced-based. some of its inequalities using different types of convex functions. This work aims to add a new one to the trapezoid type inequalities using $h-$convexity in co-ordinates. Some important definitions and theorems necessary for the main results of our work are given below:

Definition 1.1. *[1]Let I be convex set on $\mathbb{R}$. The function $\psi : I \to \mathbb{R}$ is said to be convex on I, if it is demonstrated that the following inequality is removed:*

$$\psi(\tau\chi + (1-\tau)\,\varphi) \leq \tau\psi(\chi) + (1-\tau)\psi(\varphi) \tag{1.1}$$

for all $(\chi,\varphi) \in I$ and $\tau \in [0,1]$. The mapping ψ is a concave on I if the inequality (1.1). In the event of a reversal, the aforementioned item should be held in the opposite direction for all $\tau \in [0,1]$ and $\chi,\varphi \in I$.

Let us begin by examining a rectangle positioned on a flat surface. $\Delta := [\sigma,\phi] \times [\varsigma,\rho]$ in $\mathbb{R}^2$. A mapping $\psi : \triangle \to \mathbb{R}$ the following definition of a mapping in coordinated convex is provided for the reader's convenience:

Definition 1.2. *[2]A function $\psi : \Delta \to \mathbb{R}$ The term "coordinated convex on" is used to describe this phenomenon. Δ, for all$(\chi,\varphi),(\upsilon,\omega) \in \Delta$ and $\tau,\xi \in [0,1]$. The following inequality must be satisfied:*

$$\psi(\tau\chi + (1-\tau)\,\varphi, \xi u + (1-\xi)\,\omega) \quad \leq \quad \tau\xi\;\psi(\chi,\upsilon) + \tau(1-\xi)\psi(\chi,\omega) + \xi(1-\tau)\psi(\varphi,\upsilon) + (1-\tau)(1-\xi)\psi(\varphi,\omega).$$

It is evident that all convex functions are convex with respect to the given coordinates. Nevertheless, it is not necessarily the case that every function which is convex in coordinates will necessarily be convex (see, [2]).

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Definition 1.3. *Let $h : I \subseteq \mathbb{R} \to \mathbb{R}$ be a non-negative function. We say that $\psi : I \to \mathbb{R}$ is an h-convex function, or that ψ belongs to the class $SX(h,I)$, if ψ non-negative and for all $\chi,\varphi \in I, \tau \in (0,1)$ we have,*

$$\psi(\tau\chi + (1-\tau)\ \varphi) \leq h(\tau)\psi(\chi) + h(1-\tau)\psi(\varphi).$$

If this inequality is reversed, then ψ is said to be $h-$concave [3].

Definition 1.4. *A function $\psi : I \to \mathbb{R}$ is said to be $h-$convex on the coordinates on Δ, if the following inequality holds:*

$$\begin{aligned} \psi(\tau\chi + (1-\tau)\ \varphi, \xi u + (1-\xi)\ \omega) \ &\leq\ h(\tau)h(\xi)\ \psi(\chi,\upsilon) + h(\tau)h(1-\xi)\psi(\chi,\omega) \\ &+\ h(\xi)h(1-\tau)\psi(\varphi,\upsilon) + h(1-\tau)h(1-\xi)\psi(\varphi,\omega) \end{aligned}$$

holds for all $\forall(\tau,\xi) \in [0,1]$ and $(\chi,\upsilon),(\chi,\omega),(\varphi,\upsilon),(\varphi,\omega) \in \Delta$ [4].

Definition 1.5. *The gamma function and beta function are defined by*

$$\Gamma(\chi) := \int_0^\infty \tau^{x-1} e^{-\tau} d\tau$$

and

$$B(\chi,\varphi) := \int_0^1 \tau^{x-1}(1-\tau)^{\varphi-1} d\tau,$$

respectively. Here, $0 < \chi,\varphi < \infty$.

In the analysis of mathematics in convex mappings, integral inequalities are the most frequently used field. These inequalities, which were formulated by C. Hermite and J. Hadamard, are widely cited in the literature (see, e.g., [1], [5, p.137],[6]). The aforementioned inequalities posit that if $\psi : I \to \mathbb{R}$ The function is convex on the interval. I of real numbers and $\sigma,\phi \in I$ with $\tau \leqslant \xi$ then,

$$\psi\left(\frac{\sigma+\phi}{2}\right) \leq \frac{1}{\phi-\sigma}\int\limits_{\tau}^{\xi} \psi(\chi)d\chi \leq \frac{\psi(\sigma)+\psi(\phi)}{2}.$$

The Hermite-Hadamard inequality is a celebrated result that elucidates the impact of a function's convexity on its mean value and integral value over a given interval. The inequality states that if a function ψ is convex on a real interval I and if a and b are two points in I, then the value of ψ at the midpoint of a and ϕ is less than or equal to the average value of f on the interval $[\sigma,\phi]$. Hermite Hadamard's inequality is employed to compare The mean value of a convex function is defined as the sum of the areas of the function's convex hulls, divided by the number of hulls. over a given interval with the values observed at the endpoints of the interval. This inequality can be extended to higher dimensions, such as the plane $\mathbb{R}^2$, where the interval is transformed into a rectangle. The inequality comprises two parts, one utilising the midpoint of the rectangle and the other employing the corners of the rectangle. The second part is referred to as the trapezoid-type inequality, due to its resemblance to the shape of a trapezoid. The trapezoid-type inequality, which forms part of the Hermite-Hadamard inequality, has been the subject of extensive research. The trapezoid-type inequalities for convex functions were initially formulated by Dragomir and Agarwal in [7]. In [8], Sarikaya et al. provided a generalisation of the inequalities for fractional integrals, and also demonstrated the validity of certain midpoint-type inequalities. The inequality has been the subject of extensive study and improvement, particularly in the context of fractional integrals, which represent a generalisation of ordinary integrals [2], [7], [8], [9], [10].

Theorem 1.6. *Suppose that $\psi : \Delta \to \mathbb{R}$ is co-ordinated convex. Then we have the following inequalities:*

$$\begin{aligned} \psi\left(\frac{\sigma+\phi}{2},\frac{\varsigma+\rho}{2}\right) \ &\leq\ \frac{1}{2}\left[\frac{1}{\sigma-\phi}\int\limits_{\sigma}^{\phi}\psi\left(\chi,\frac{\varsigma+\rho}{2}\right)d\chi + \frac{1}{\rho-c}\int\limits_{\varsigma}^{\rho}\psi\left(\frac{\tau+\xi}{2},\varphi\right)d\varphi\right] \\ &\leq\ \frac{1}{(\phi-\sigma)(\rho-\varsigma)}\int\limits_{\sigma}^{\phi}\int\limits_{\varsigma}^{\rho}\psi(\chi,\varphi)d\varphi d\chi \qquad (1.2) \\ &\leq\ \frac{1}{4}\left[\frac{1}{\phi-\sigma}\int\limits_{\tau}^{\xi}\psi(\chi,c)d\chi + \frac{1}{\phi-\sigma}\int\limits_{\tau}^{\xi}\psi(\chi,\rho)d\chi + \frac{1}{\rho-\varsigma}\int\limits_{\varsigma}^{\rho}\psi(\tau,\varphi)d\varphi + \frac{1}{\rho-\varsigma}\int\limits_{\varsigma}^{\rho}\psi(\xi,\varphi)d\varphi\right] \\ &\leq\ \frac{\psi(\tau,\varsigma)+\psi(\tau,\rho)+\psi(\xi,\varsigma)+\psi(\xi,\rho)}{4}. \end{aligned}$$

The above inequalities are sharp. The inequalities in (1.2) hold in reverse direction if the mapping ψ is a co-ordinated concave mapping.

Fractional calculus describes a type of calculus that is capable of utilising any real or complex number as the power of the derivative or the integral. There are a number of different approaches to defining fractional calculus, including the Caputo, Riemann-Liouville, and Grünwall-Letnikov definitions. While these definitions offer certain advantages, they also present certain problems. To illustrate, the Riemann-Liouville definition does not always yield a derivative of zero for a constant. In the Caputo definition, the function f must be differentiable [11], [12], [13], [14], [15], [16]. Furthermore, a number of definitions deviate from the conventional tenets of calculus, including the operations of division, multiplication and composition between two functions. To address these and other issues, Khalil et al. proposed a novel definition of fractional calculus, termed the compatible fractional derivative. Furthermore, the compatible fractional integral for powers between $(0<\sigma\leq 1)$ was also introduced. The researchers demonstrated several significant findings, including methods for multiplying two functions and calculating the mean value of a function. Additionally, they solved equations involving fractional calculus and exponential functions (see, [17], [18], [19], [20], [21], [22], [23], [24], [25]).

The definitions and mathematical foundations of the principles of conformable fractional calculus that are employed in this study are set forth below:

Definition 1.7. *[26]For $\psi\in L_1[\sigma,\phi]$, the Riemann-Liouville integrals of order $\tau>0$ are given by*

$$J^{\sigma}_{\sigma+}\psi(\chi)=\frac{1}{\Gamma(\sigma)}\int_{\sigma}^{\chi}(\chi-\tau)^{\sigma-1}\psi(\tau)d\tau,\ \ \chi>\sigma$$

and

$$J^{\sigma}_{\phi-}\psi(\chi)=\frac{1}{\Gamma(\sigma)}\int_{\chi}^{\phi}(\tau-\chi)^{\sigma-1}\psi(\tau)d\tau,\ \ \chi<\phi.$$

In accordance with the aforementioned criteria, the respective values are as follows: it can be demonstrated that the Riemann-Liouville integrals are equal to their classical counterparts when the requisite condition is $\theta=1$.

Definition 1.8. *[27]Let $\psi\in L_1([\sigma,\phi]\times[\varsigma,\rho])$. The Riemann-Liouville integrals $J^{\theta,\vartheta}_{\sigma+,\varsigma^+}, J^{\theta,\vartheta}_{\sigma+,\rho^-}, J^{\theta,\vartheta}_{\phi^-,\varsigma^+}$ and $J^{\theta,\vartheta}_{\phi^-,\rho^-}$ of order $\theta,\vartheta>0$ with $\sigma,\varsigma\geq 0$ are defined by*

$$J^{\theta,\vartheta}_{\sigma+,\varsigma^+}\psi(\chi,\varphi)=\frac{1}{\Gamma(\theta)\Gamma(\vartheta)}\int_{\sigma}^{\chi}\int_{\varsigma}^{\varphi}(\chi-\tau)^{\theta-1}(\varphi-\xi)^{\vartheta-1}\psi(\tau,\xi)d\xi d\tau,\ \ \chi>\sigma,\varphi>\varsigma, \tag{1.3}$$

$$J^{\theta,\vartheta}_{\sigma+,\rho^-}\psi(\chi,\varphi)=\frac{1}{\Gamma(\theta)\Gamma(\vartheta)}\int_{\sigma}^{\chi}\int_{\varphi}^{\rho}(\chi-\tau)^{\theta-1}(\xi-\varphi)^{\vartheta-1}\psi(\tau,\xi)d\xi d\tau,\ \ \chi>\sigma,\varphi<\rho, \tag{1.4}$$

$$J^{\theta,\vartheta}_{\phi-,\varsigma^+}\psi(\chi,\varphi)=\frac{1}{\Gamma(\theta)\Gamma(\vartheta)}\int_{\chi}^{\phi}\int_{\varsigma}^{\varphi}(\tau-\chi)^{\theta-1}(\varphi-\xi)^{\vartheta-1}\psi(\tau,\xi)d\xi d\tau,\ \ \chi<\phi,\varphi>\varsigma, \tag{1.5}$$

and

$$J^{\theta,\vartheta}_{\phi-,\rho^-}\psi(\chi,\varphi)=\frac{1}{\Gamma(\theta)\Gamma(\vartheta)}\int_{\chi}^{\phi}\int_{\varphi}^{\rho}(\tau-\chi)^{\theta-1}(\xi-\varphi)^{\vartheta-1}\psi(\tau,\xi)d\xi d\tau,\ \ \ \chi<\phi,\varphi<\rho, \tag{1.6}$$

respectively. Here, Γ is the Gama function.

Definition 1.9. *[28] For $\psi\in L_1[\sigma,\phi]$, the fractional conformable integral operator ${}^{\vartheta}I^{\theta}_{\sigma+}\psi$and ${}^{\vartheta}I^{\theta}_{\phi-}\psi$ of order $\vartheta>0$ and $\sigma\in(0,1]$ are presented by*

$${}^{\vartheta}\mathfrak{J}^{\theta}_{\sigma+}\psi(\chi)=\frac{1}{\Gamma(\vartheta)}\int_{\sigma}^{\chi}\left(\frac{(\chi-\sigma)^{\theta}-(\tau-\sigma)^{\theta}}{\theta}\right)^{\vartheta-1}\frac{\psi(\tau)}{(\tau-\sigma)^{1-\theta}}d\tau,\quad \tau>\sigma \tag{1.7}$$

and

$${}^{\vartheta}\mathfrak{J}^{\theta}_{\phi-}\psi(\chi)=\frac{1}{\Gamma(\vartheta)}\int_{\chi}^{\phi}\left(\frac{(\phi-\chi)^{\theta}-(\phi-\tau)^{\theta}}{\theta}\right)^{\vartheta-1}\frac{\psi(\tau)}{(\phi-\tau)^{1-\theta}}d\tau,\quad \tau<\phi, \tag{1.8}$$

respectively.

Definition 1.10. *[29]Let $\psi\in L_1([\sigma,\phi]\times[\varsigma,\rho])$ and let $\gamma_1\neq 0$, $\gamma_2\neq 0$, $\theta,\vartheta\in\mathbf{C}$, $Re(\theta)>0$ and $Re(\vartheta)>0$. The generalized conformable integral of order θ,ϑ of $\psi(\chi,\varphi)$ is defined by;*

$$\left({}^{\gamma_1\gamma_2}I^{\theta,\vartheta}_{\sigma^+,\varsigma^+}\psi\right)(\chi,\varphi)\ \ =\ \ \left[\frac{1}{\Gamma(\theta)\Gamma(\vartheta)}\int_{\sigma}^{\chi}\int_{\varsigma}^{\varphi}\left(\frac{(\chi-\sigma)^{\gamma_1}-(\tau-\sigma)^{\gamma_1}}{\gamma_1}\right)^{\theta-1}\right. \tag{1.9}$$

$$\times\left(\frac{(\varphi-\varsigma)^{\gamma_2}-(\xi-\varsigma)^{\gamma_2}}{\gamma_2}\right)^{\vartheta-1}\frac{\psi(\tau,\xi)}{(\tau-\sigma)^{1-\gamma_1}(\xi-\varsigma)^{1-\gamma_2}}d\xi d\tau\Bigg],$$

$$\left({}^{\gamma_1\gamma_2}I^{\theta,\vartheta}_{\phi^-,\varsigma^+}\psi\right)(\chi,\varphi) = \Bigg[\frac{1}{\Gamma(\theta)\Gamma(\vartheta)}\int_\chi^\phi\int_\varsigma^\varphi\left(\frac{(\phi-\chi)^{\gamma_1}-(\phi-\tau)^{\gamma_1}}{\gamma_1}\right)^{\theta-1} \times\left(\frac{(\varphi-\varsigma)^{\gamma_2}-(\xi-\varsigma)^{\gamma_2}}{\gamma_2}\right)^{\vartheta-1}\frac{\psi(\tau,\xi)}{(\phi-\tau)^{1-\gamma_1}(\xi-\varsigma)^{1-\gamma_2}}d\xi d\tau\Bigg], \tag{1.10}$$

$$\left({}^{\gamma_1\gamma_2}I^{\theta,\vartheta}_{\sigma^+,\rho^-}\psi\right)(\chi,\varphi) = \Bigg[\frac{1}{\Gamma(\theta)\Gamma(\vartheta)}\int_\sigma^\chi\int_\varphi^\rho\left(\frac{(\chi-\sigma)^{\gamma_1}-(\tau-\sigma)^{\gamma_1}}{\gamma_1}\right)^{\theta-1} \times\left(\frac{(\rho-\varphi)^{\gamma_2}-(\rho-\xi)^{\gamma_2}}{\gamma_2}\right)^{\vartheta-1}\frac{\psi(\tau,\xi)}{(\tau-\sigma)^{1-\gamma_1}(\rho-\xi)^{1-\gamma_2}}d\xi d\tau\Bigg], \tag{1.11}$$

and

$$\left({}^{\gamma_1\gamma_2}I^{\theta,\vartheta}_{\phi^-,\rho^-}\psi\right)(\chi,\varphi) = \Bigg[\frac{1}{\Gamma(\theta)\Gamma(\vartheta)}\int_\chi^\phi\int_\varphi^\rho\left(\frac{(\phi-\chi)^{\gamma_1}-(\phi-\tau)^{\gamma_1}}{\gamma_1}\right)^{\theta-1} \times\left(\frac{(\rho-\varphi)^{\gamma_2}-(\rho-\xi)^{\gamma_2}}{\gamma_2}\right)^{\vartheta-1}\frac{\psi(\tau,\xi)}{(\phi-\tau)^{1-\gamma_1}(\rho-\xi)^{1-\gamma_2}}d\xi d\tau\Bigg], \tag{1.12}$$

the generalized conformable integrals.

Remark 1.11. *[29] If $\gamma_1=\gamma_2=1$ in (1.9), (1.10),(1.11) and (1.12), we have (1.3)-(1.6) the Fractional integrals of the functions of two variables.*

Remark 1.12. *[29] If we consider $\theta=1$ and $\vartheta=1$ in (1.9), (1.10),(1.11) and (1.12), we have*

$$\left(I^{1,1}_{\sigma^+,\varsigma^+}\psi\right)(\chi,\varphi)=\int_\sigma^\chi\int_\varsigma^\varphi\frac{\psi(\tau,\xi)}{(\tau-\sigma)^{1-\gamma_1}(\xi-\varsigma)^{1-\gamma_2}}d\xi d\tau \tag{1.13}$$

$$\left(I^{1,1}_{\phi^-,\varsigma^+}\psi\right)(\chi,\varphi)=\int_\chi^\phi\int_\varsigma^\varphi\frac{\psi(\tau,\xi)}{(\phi-\tau)^{1-\gamma_1}(\xi-\varsigma)^{1-\gamma_2}}d\xi d\tau, \tag{1.14}$$

$$\left(I^{1,1}_{\sigma^+,\rho^-}\psi\right)(\chi,\varphi)=\int_\sigma^\chi\int_\varphi^\rho\frac{\psi(\tau,\xi)}{(\tau-\sigma)^{1-\gamma_1}(\rho-\xi)^{1-\gamma_2}}d\xi d\tau, \tag{1.15}$$

and

$$\left(I^{1,1}_{\phi^-,\rho^-}\psi\right)(\chi,\varphi)=\int_\chi^\phi\int_\varphi^\rho\frac{\psi(\tau,\xi)}{(\phi-\tau)^{1-\gamma_1}(\rho-\xi)^{1-\gamma_2}}d\xi d\tau. \tag{1.16}$$

the conformable fractional integrals for double integrals.

Theorem 1.13. *[30] In assume ψ is a co-ordinated convex function that goes from $[\sigma,\phi]\times[\varsigma,\rho]$ into $\mathbb{R}$ and let $\gamma_1\neq 0$, $\gamma_2\neq 0$, $\theta,\vartheta\in(0,1]$, $Re(\theta)>0$ and $Re(\vartheta)>0$. The following inequality holds for generalized conformable fractional integrals,*

$$\begin{aligned}\psi\left(\frac{\sigma+\phi}{2},\frac{\varsigma+\rho}{2}\right) &\leq \frac{2^{\gamma_1\theta-1}2^{\gamma_2\vartheta-1}\Gamma(\theta+1)\Gamma(\vartheta+1)\gamma_1^\theta\gamma_2^\vartheta}{(\phi-\sigma)^{\gamma_1\alpha}(\rho-\varsigma)^{\gamma_2\beta}}\Bigg[{}^{\gamma_1\gamma_2}I^{\theta,\vartheta}_{\sigma^+,\varsigma^+}\psi\left(\frac{\sigma+\phi}{2},\frac{\varsigma+\rho}{2}\right)\\ &+ {}^{\gamma_1\gamma_2}I^{\theta,\vartheta}_{\phi^-,\varsigma^+}\psi\left(\frac{\sigma+\phi}{2},\frac{\varsigma+\rho}{2}\right)+{}^{\gamma_1\gamma_2}I^{\theta,\vartheta}_{\sigma^+,\rho^-}\psi\left(\frac{\sigma+\phi}{2},\frac{\varsigma+\rho}{2}\right)+{}^{\gamma_1\gamma_2}I^{\theta,\vartheta}_{\phi^-,\rho^-}\psi\left(\frac{\sigma+\phi}{2},\frac{\varsigma+\rho}{2}\right)\Bigg]\\ &\leq \frac{\psi(\sigma,\varsigma)+\psi(\sigma,\rho)+\psi(\phi,\varsigma)+\psi(\phi,\rho)}{4}.\end{aligned} \tag{1.17}$$

2. Trapezoid type inequalities for co-ordinated h-convex functions

Lemma 2.1. *[31]Let $\psi:\triangle\subset\mathbb{R}^2\to\mathbb{R}$ be a partial differentiable mapping on $\Delta:=[\sigma,\phi]\times[\varsigma,\rho]$ in $\mathbb{R}^2$ with $0\le\sigma<\phi,\ 0\le\varsigma\le\rho$. If $\frac{\partial^2\psi}{\partial\tau\partial\xi}\in L_1(\Delta)$, then the following identity:*

$$
\begin{aligned}
&\frac{\psi(\sigma,\varsigma)+\psi(\sigma,\rho)+\psi(\phi,\varsigma)+\psi(\phi,\rho)}{4}+\frac{2^{\gamma_1\theta-1}2^{\gamma_2\vartheta-1}\Gamma(\theta+1)\Gamma(\vartheta+1)\gamma_1^{\theta}\gamma_2^{\vartheta}}{(\phi-\sigma)^{\gamma_1\theta}(\rho-\varsigma)^{\gamma_2\vartheta}} \qquad (2.1)\\
&\times\left[{}^{\gamma_1\gamma_2}I^{\theta,\vartheta}_{\sigma^+,\varsigma^+}\psi\left(\frac{\sigma+\phi}{2},\frac{\varsigma+\rho}{2}\right)+{}^{\gamma_1\gamma_2}I^{\theta,\vartheta}_{\phi^-,\varsigma^+}\psi\left(\frac{\sigma+\phi}{2},\frac{\varsigma+\rho}{2}\right)+{}^{\gamma_1\gamma_2}I^{\theta,\vartheta}_{\sigma^+,\rho^-}\psi\left(\frac{\sigma+\phi}{2},\frac{\varsigma+\rho}{2}\right)\right.\\
&\left.+{}^{\gamma_1\gamma_2}I^{\theta,\vartheta}_{\phi^-,\rho^-}\psi\left(\frac{\sigma+\phi}{2},\frac{\varsigma+\rho}{2}\right)\right]-A\\
=\;&\frac{\gamma_1^{\theta}\gamma_2^{\vartheta}(\phi-\sigma)(\rho-\varsigma)}{16}\\
&\times\left[\int_0^1\int_0^1\left(\frac{1-(1-\tau)^{\gamma_1}}{\gamma_1}\right)^{\theta}\left(\frac{1-(1-\xi)^{\gamma_2}}{\gamma_2}\right)^{\vartheta}\frac{\partial^2\psi}{\partial\tau\partial\xi}\left(\frac{1+\tau}{2}\sigma+\frac{1-\tau}{2}\phi,\frac{1+\xi}{2}\varsigma+\frac{1-\xi}{2}\rho\right)d\xi d\tau\right.\\
&-\int_0^1\int_0^1\left(\frac{1-(1-\tau)^{\gamma_1}}{\gamma_1}\right)^{\theta}\left(\frac{1-(1-\xi)^{\gamma_2}}{\gamma_2}\right)^{\vartheta}\frac{\partial^2\psi}{\partial\tau\partial\xi}\left(\frac{1+\tau}{2}\sigma+\frac{1-\tau}{2}\phi,\frac{1-\xi}{2}\varsigma+\frac{1+\xi}{2}\rho\right)d\xi d\tau\\
&-\int_0^1\int_0^1\left(\frac{1-(1-\tau)^{\gamma_1}}{\gamma_1}\right)^{\theta}\left(\frac{1-(1-\xi)^{\gamma_2}}{\gamma_2}\right)^{\vartheta}\frac{\partial^2\psi}{\partial\tau\partial\xi}\left(\frac{1-\tau}{2}\sigma+\frac{1+\tau}{2}\phi,\frac{1+\xi}{2}\varsigma+\frac{1-\xi}{2}\rho\right)d\xi d\tau\\
&\left.+\int_0^1\int_0^1\left(\frac{1-(1-\tau)^{\gamma_1}}{\gamma_1}\right)^{\theta}\left(\frac{1-(1-\xi)^{\gamma_2}}{\gamma_2}\right)^{\vartheta}\frac{\partial^2\psi}{\partial\tau\partial\xi}\left(\frac{1-\tau}{2}\sigma+\frac{1+\tau}{2}\phi,\frac{1-\xi}{2}\varsigma+\frac{1+\xi}{2}\rho\right)d\xi d\tau\right],
\end{aligned}
$$

where

$$
\begin{aligned}
A\;=\;&\frac{2^{\gamma_2\vartheta-2}\,\gamma_2^{\vartheta}\,\Gamma(\vartheta+1)}{(\rho-\varsigma)^{\gamma_2\vartheta}} \qquad (2.2)\\
&\left[{}^{\gamma_2}I^{\vartheta}_{\varsigma^+}\psi\left(\sigma,\frac{\varsigma+\rho}{2}\right)+{}^{\gamma_2}I^{\vartheta}_{\rho^-}\psi\left(\sigma,\frac{\varsigma+\rho}{2}\right)+{}^{\gamma_2}I^{\vartheta}_{\varsigma^+}\psi\left(\phi,\frac{\varsigma+\rho}{2}\right)+{}^{\gamma_2}I^{\vartheta}_{d^-}\psi\left(\phi,\frac{\varsigma+\rho}{2}\right)\right]\\
&+\frac{2^{\gamma_1\theta-2}\,\gamma_1^{\theta}\,\Gamma(\theta+1)}{(\phi-\sigma)^{\gamma_1\theta}}\\
&\left[{}^{\gamma_1}I^{\theta}_{\sigma^+}\psi\left(\frac{\sigma+\phi}{2},\varsigma\right)+{}^{\gamma_1}I^{\theta}_{\sigma^+}\psi\left(\frac{\sigma+\phi}{2},\rho\right){}^{\gamma_1}I^{\theta}_{\phi^-}\psi\left(\frac{\sigma+\phi}{2},\varsigma\right)+{}^{\gamma_1}I^{\theta}_{\phi^-}\psi\left(\frac{\sigma+\phi}{2},\rho\right)\right].
\end{aligned}
$$

Proof. By employing the technique of integration by parts, we obtain the following result:

$$
\begin{aligned}
I_1\;=\;&\int_0^1\int_0^1\left(\frac{1-(1-\tau)^{\gamma_1}}{\gamma_1}\right)^{\theta}\left(\frac{1-(1-\xi)^{\gamma_2}}{\gamma_2}\right)^{\vartheta}\frac{\partial^2\psi}{\partial\tau\partial\xi}\left(\frac{1+\tau}{2}\sigma+\frac{1-\tau}{2}\phi,\frac{1+\xi}{2}\varsigma+\frac{1-\xi}{2}\rho\right)d\xi d\tau\\
=\;&\int_0^1\left(\frac{1-(1-\xi)^{\gamma_2}}{\gamma_2}\right)^{\vartheta}\left\{\left(\frac{1-(1-\tau)^{\gamma_1}}{\gamma_1}\right)^{\theta}\frac{-2}{(\phi-\sigma)}\frac{\partial\psi}{\partial\xi}\left(\frac{1+\tau}{2}\sigma+\frac{1-\tau}{2}\phi,\frac{1+\xi}{2}\varsigma+\frac{1-\xi}{2}\rho\right)\Bigg|_0^1\right.\\
&\left.+\int_0^1\frac{2\theta}{(\phi-\sigma)}\left(\frac{1-(1-\tau)^{\gamma_1}}{\gamma_1}\right)^{\theta-1}(1-\tau)^{\gamma_1-1}\frac{\partial\psi}{\partial\xi}\left(\frac{1+\tau}{2}\sigma+\frac{1-\tau}{2}\phi,\frac{1+\xi}{2}\varsigma+\frac{1-\xi}{2}\rho\right)d\tau\right\}d\xi\\
=\;&\int_0^1\left(\frac{1-(1-\xi)^{\gamma_2}}{\gamma_2}\right)^{\vartheta}\left\{\left(\frac{1}{\gamma_1}\right)^{\theta}\left(\frac{-2}{\phi-\sigma}\right)\frac{\partial\psi}{\partial\xi}\left(\sigma,\frac{1+\xi}{2}\varsigma+\frac{1-\xi}{2}\rho\right)\right.\\
&\left.+\frac{2\theta}{(\phi-\sigma)}\int_0^1\left(\frac{1-(1-\tau)^{\gamma_1}}{\gamma_1}\right)^{\theta-1}(1-\tau)^{\gamma_1-1}\frac{\partial\psi}{\partial s}\left(\frac{1+\tau}{2}\sigma+\frac{1-\tau}{2}\phi,\frac{1+\xi}{2}\varsigma+\frac{1-\xi}{2}\rho\right)d\tau\right\}d\xi\\
=\;&\frac{-2}{(\phi-\sigma)\gamma_1^{\theta}}\int_0^1\left(\frac{1-(1-\xi)^{\gamma_2}}{\gamma_2}\right)^{\vartheta}\frac{\partial\psi}{\partial\xi}\left(\sigma,\frac{1+\xi}{2}\varsigma+\frac{1-\xi}{2}\rho\right)d\xi
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{2\theta}{(\phi-\sigma)}\left[\int_0^1\left(\frac{1-(1-\tau)^{\gamma_1}}{\gamma_1}\right)^{\theta-1}(1-\tau)^{\gamma_1-1}\right.\\
&\left.\times\left\{\int_0^1\left(\frac{1-(1-\xi)^{\gamma_2}}{\gamma_2}\right)^{\vartheta}\frac{\partial\psi}{\partial\varsigma}\left(\frac{1+\tau}{2}\sigma+\frac{1-\tau}{2}\phi,\frac{1+\xi}{2}\varsigma+\frac{1-\xi}{2}\rho\right)d\xi\right\}d\tau\right]\\
=\;&\frac{-2}{(\phi-\sigma)}\left(\frac{1}{\gamma_1}\right)^{\theta}\left[\left(\frac{1}{\gamma_2}\right)^{\vartheta}\frac{-2}{(\rho-\varsigma)}\psi(\sigma,\varsigma)\right.\\
&\left.+\frac{2\vartheta}{(\rho-\varsigma)}\int_0^1\left(\frac{1-(1-\xi)^{\gamma_2}}{\gamma_2}\right)^{\vartheta-1}(1-\xi)^{\gamma_2-1}\psi\left(\sigma,\frac{1+\xi}{2}\varsigma+\frac{1-\xi}{2}\rho\right)d\xi\right]\\
&+\frac{2\theta}{(\phi-\sigma)}\int_0^1\left(\frac{1-(1-\tau)^{\gamma_1}}{\gamma_1}\right)^{\theta-1}(1-\tau)^{\gamma_1-1}\left\{\left(\frac{1}{\gamma_2}\right)^{\beta}\frac{-2}{(\rho-\varsigma)}\psi\left(\frac{1+\tau}{2}+\frac{1-\tau}{2}\phi,\varsigma\right)\right.\\
&\left.+\frac{2\vartheta}{(\rho-\varsigma)}\int_0^1\left(\frac{1-(1-\xi)^{\gamma_2}}{\gamma_2}\right)^{\vartheta-1}(1-\xi)^{\gamma_2-1}\psi\left(\frac{1+\tau}{2}\sigma+\frac{1-\tau}{2}\phi,\frac{1+\xi}{2}\varsigma+\frac{1-\xi}{2}\rho\right)d\xi\right\}d\tau\\
=\;&\frac{4}{(\phi-\sigma)(\rho-\varsigma)}\frac{1}{\gamma_1^{\theta}\gamma_2^{\vartheta}}\psi(\sigma,\varsigma)\\
&-\frac{4\vartheta}{(\phi-\sigma)(\rho-\varsigma)}\left(\frac{1}{\gamma_1}\right)^{\theta}\int_0^1\left(\frac{1-(1-\xi)^{\gamma_2}}{\gamma_2}\right)^{\vartheta-1}(1-\xi)^{\gamma_2-1}\psi\left(\sigma,\frac{1+\xi}{2}\varsigma+\frac{1-\xi}{2}\rho\right)d\xi\\
&-\frac{4\theta}{(\phi-\sigma)(\rho-\varsigma)}\left(\frac{1}{\gamma_2}\right)^{\vartheta}\int_0^1\left(\frac{1-(1-\tau)^{\gamma_1}}{\gamma_1}\right)^{\theta-1}(1-\tau)^{\gamma_1-1}\psi\left(\frac{1+\tau}{2}\sigma+\frac{1-\tau}{2}\phi,\varsigma\right)d\tau\\
&+\frac{4\theta\vartheta}{(\phi-\sigma)(\rho-\varsigma)}\left[\int_0^1\int_0^1\left(\frac{1-(1-\tau)^{\gamma_1}}{\gamma_1}\right)^{\theta-1}(1-\tau)^{\gamma_1-1}\left(\frac{1-(1-\xi)^{\gamma_2}}{\gamma_2}\right)^{\vartheta-1}(1-\xi)^{\gamma_2-1}\right.\\
&\left.\times\,\psi\left(\frac{1+\tau}{2}\sigma+\frac{1-\tau}{2}\phi,\frac{1+\xi}{2}\varsigma+\frac{1-\xi}{2}\rho\right)d\xi d\tau\right].
\end{aligned}
\tag{2.3}
$$

In (2.3), using the change of the variables $u=\frac{1+\tau}{2}\sigma+\frac{1-\tau}{2}\phi$ and $v=\frac{1+\xi}{2}\varsigma+\frac{1-\xi}{2}\rho$, we can write,

$$
\begin{aligned}
I_1 =\;&\frac{4}{(\phi-\sigma)(\rho-\varsigma)}\frac{1}{\gamma_1^{\theta}\gamma_2^{\vartheta}}\psi(\sigma,\varsigma)-\left(\frac{2}{\rho-\varsigma}\right)^{\gamma_2\vartheta}\Gamma(\vartheta)\left({}^{\gamma_2}I^{\vartheta}_{\varsigma^+}\psi\right)\left(\sigma,\frac{\varsigma+\rho}{2}\right)-\left(\frac{2}{\phi-\sigma}\right)^{\gamma_1\theta}\Gamma(\theta)^{\gamma_1}I^{\theta}_{\sigma^+}\psi\left(\frac{\sigma+\phi}{2},\varsigma\right)\\
&+\frac{4\theta\vartheta}{(\phi-\sigma)(\rho-\varsigma)}\frac{2^{\gamma_1\theta}2^{\gamma_2\vartheta}\Gamma(\theta)\Gamma(\vartheta)}{(\phi-\sigma)^{\gamma_1\theta}(\rho-\varsigma)^{\gamma_2\vartheta}}\left({}^{\gamma_1\gamma_2}I^{\theta,\vartheta}_{\sigma^+,\varsigma^+}\psi\right)\left(\frac{\sigma+\phi}{2},\frac{\varsigma+\rho}{2}\right).
\end{aligned}
\tag{2.4}
$$

By employing the technique of integration by parts, the following result is obtained:

$$
\begin{aligned}
I_2 =\;&\int_0^1\int_0^1\left(\frac{1-(1-\tau)^{\gamma_1}}{\gamma_1}\right)^{\theta}\left(\frac{1-(1-\xi)^{\gamma_2}}{\gamma_2}\right)^{\vartheta}\frac{\partial^2\psi}{\partial\tau\partial\xi}\left(\frac{1+\tau}{2}\sigma+\frac{1-\tau}{2}\phi,\frac{1-\xi}{2}\varsigma+\frac{1+\xi}{2}\rho\right)d\xi d\tau\\
=\;&\frac{-4}{(\phi-\sigma)(\rho-\varsigma)}\frac{1}{\gamma_1^{\theta}\gamma_2^{\vartheta}}\psi(\sigma,\rho)+\left(\frac{2}{\rho-\varsigma}\right)^{\gamma_2\vartheta}\Gamma(\vartheta)\left({}^{\gamma_2}I^{\vartheta}_{\rho^-}\psi\right)\left(\sigma,\frac{\varsigma+\rho}{2}\right)+\left(\frac{2}{\phi-\sigma}\right)^{\gamma_1\theta}\Gamma(\theta)\left({}^{\gamma_1}I^{\theta}_{\sigma^+}\psi\right)\left(\frac{\sigma+\phi}{2},\rho\right)\\
&-\frac{4\theta\beta}{(\phi-\sigma)(\rho-\varsigma)}\frac{2^{\gamma_1\theta}2^{\gamma_2\vartheta}\Gamma(\theta)\Gamma(\vartheta)}{(\phi-\sigma)^{\gamma_1\theta}(\rho-\varsigma)^{\gamma_2\vartheta}}\left({}^{\gamma_1\gamma_2}I^{\theta,\vartheta}_{\phi^-,\varsigma^+}\psi\right)\left(\frac{\sigma+\phi}{2},\frac{\varsigma+\rho}{2}\right),
\end{aligned}
\tag{2.5}
$$

$$
\begin{aligned}
I_3 =\;&\int_0^1\int_0^1\left(\frac{1-(1-\tau)^{\gamma_1}}{\gamma_1}\right)^{\theta}\left(\frac{1-(1-\xi)^{\gamma_2}}{\gamma_2}\right)^{\vartheta}\frac{\partial^2\psi}{\partial\tau\partial\xi}\left(\frac{1-\tau}{2}\sigma+\frac{1+\tau}{2}\phi,\frac{1+\xi}{2}\varsigma+\frac{1-\xi}{2}\rho\right)d\xi d\tau\\
=\;&\frac{-4}{(\phi-\sigma)(\rho-\varsigma)}\frac{1}{\gamma_1^{\theta}\gamma_2^{\vartheta}}\psi(\phi,\varsigma)+\left(\frac{2}{\rho-\varsigma}\right)^{\gamma_2\vartheta}\Gamma(\vartheta)\left({}^{\gamma_2}I^{\vartheta}_{\varsigma^+}\psi\right)\left(\phi,\frac{\varsigma+\rho}{2}\right)+\left(\frac{2}{\phi-\sigma}\right)^{\gamma_1\theta}\Gamma(\theta)\left({}^{\gamma_1}I^{\theta}_{\phi^-}\psi\right)\left(\frac{\sigma+\phi}{2},\varsigma\right)\\
&-\frac{4\theta\vartheta}{(\phi-\sigma)(\rho-\varsigma)}\frac{2^{\gamma_1\theta}2^{\gamma_2\vartheta}\Gamma(\theta)\Gamma(\vartheta)}{(\phi-\sigma)^{\gamma_1\theta}(\rho-\varsigma)^{\gamma_2\vartheta}}\left({}^{\gamma_1\gamma_2}I^{\theta,\vartheta}_{\sigma^+,\rho^-}\psi\right)\left(\frac{\sigma+\phi}{2},\frac{\varsigma+\rho}{2}\right),
\end{aligned}
\tag{2.6}
$$

and

$$\begin{aligned}
I_4 &= \int_0^1\int_0^1 \left(\frac{1-(1-\tau)^{\gamma_1}}{\gamma_1}\right)^{\theta}\left(\frac{1-(1-\xi)^{\gamma_2}}{\gamma_2}\right)^{\vartheta}\frac{\partial^2\psi}{\partial\tau\partial\xi}\left(\frac{1-\tau}{2}\sigma+\frac{1+\tau}{2}\phi,\frac{1-\xi}{2}\varsigma+\frac{1+\xi}{2}\rho\right)d\xi d\tau \\
&= \frac{4}{(\phi-\sigma)(\rho-\varsigma)}\frac{1}{\gamma_1^{\theta}\gamma_2^{\vartheta}}\psi(\phi,\rho)-\left(\frac{2}{\rho-\varsigma}\right)^{\gamma_2\vartheta}\Gamma(\vartheta)\left({}^{\gamma_2}I^{\vartheta}_{\rho^-}\psi\right)\left(\phi,\frac{\varsigma+\rho}{2}\right)-\left(\frac{2}{\phi-\sigma}\right)^{\gamma_1\theta}\Gamma(\theta)\left({}^{\gamma_1}I^{\theta}_{\phi^-}\psi\right)\left(\frac{\sigma+\phi}{2},\rho\right) \\
&\quad+\frac{4\theta\vartheta}{(\phi-\sigma)(\rho-\varsigma)}\frac{2^{\gamma_1\theta}2^{\gamma_2\vartheta}\Gamma(\theta)\Gamma(\vartheta)}{(\phi-\sigma)^{\gamma_1\theta}(\rho-\varsigma)^{\gamma_2\vartheta}}\left({}^{\gamma_1\gamma_2}I^{\theta,\vartheta}_{\phi^-,\rho^-}\psi\right)\left(\frac{\sigma+\phi}{2},\frac{\varsigma+\rho}{2}\right). \qquad (2.7)
\end{aligned}$$

By equalities from (2.4)-(2.7), we obtain

$$\begin{aligned}
&\frac{\gamma_1^{\theta}\gamma_2^{\vartheta}(\phi-\sigma)(\rho-\varsigma)}{16}\left[I_1-I_2-I_3+I_4\right] \\
&= \frac{\psi(\sigma,\varsigma)+\psi(\sigma,\rho)+\psi(\phi,\varsigma)+\psi(\phi,\rho)}{4}+\frac{2^{\gamma_1\theta-1}2^{\gamma_2\vartheta-1}\Gamma(\theta+1)\Gamma(\vartheta+1)\gamma_1^{\theta}\gamma_2^{\vartheta}}{(\phi-\sigma)^{\gamma_1\theta}(\rho-\varsigma)^{\gamma_2\vartheta}} \\
&\quad\times\left[{}^{\gamma_1\gamma_2}I^{\theta,\vartheta}_{\sigma^+,\varsigma^+}\psi\left(\frac{\sigma+\phi}{2},\frac{\varsigma+\rho}{2}\right)+{}^{\gamma_1\gamma_2}I^{\theta,\vartheta}_{\phi^-,\varsigma^+}\psi\left(\frac{\sigma+\phi}{2},\frac{\varsigma+\rho}{2}\right)+{}^{\gamma_1\gamma_2}I^{\theta,\vartheta}_{\sigma^+,\rho^-}\psi\left(\frac{\sigma+\phi}{2},\frac{\varsigma+\rho}{2}\right)\right. \\
&\quad\left.+{}^{\gamma_1\gamma_2}I^{\theta,\vartheta}_{\phi^-,\rho^-}\psi\left(\frac{\sigma+\phi}{2},\frac{\varsigma+\rho}{2}\right)\right]-A.
\end{aligned}$$

□

This constitutes the proof.
The following section presents the initial Theorem, which encompasses the Hermite-Hadamard-type inequality for generalized conformable fractional integrals.

Theorem 2.2. *Let $\psi:\triangle\subset\mathbb{R}^2\to\mathbb{R}$ be a partial differentiable mapping on Δ with $0\le\sigma<\phi$, $0\le\varsigma<\rho$. If $\frac{\partial^2\psi}{\partial t\partial s}$ is a $h-$convex function on the coordinates on Δ, then the inequality below holds:*

$$\begin{aligned}
&\left|\frac{\psi(\sigma,\varsigma)+\psi(\sigma,\rho)+\psi(\phi,\varsigma)+\psi(\phi,\rho)}{4}+\frac{2^{\gamma_1\theta-1}2^{\gamma_2\vartheta-1}\Gamma(\theta+1)\Gamma(\vartheta+1)\gamma_1^{\theta}\gamma_2^{\vartheta}}{(\phi-\sigma)^{\gamma_1\theta}(\rho-\varsigma)^{\gamma_2\vartheta}}\right. \qquad (2.8) \\
&\times\left[{}^{\gamma_1\gamma_2}I^{\theta,\vartheta}_{\sigma^+,\varsigma^+}\psi\left(\frac{\sigma+\phi}{2},\frac{\varsigma+\rho}{2}\right)+{}^{\gamma_1\gamma_2}I^{\theta,\vartheta}_{\phi^-,\varsigma^+}\psi\left(\frac{\sigma+\phi}{2},\frac{\varsigma+\rho}{2}\right)+{}^{\gamma_1\gamma_2}I^{\theta,\vartheta}_{\sigma^+,\rho^-}\psi\left(\frac{\sigma+\phi}{2},\frac{\varsigma+\rho}{2}\right)\right. \\
&\left.\left.+{}^{\gamma_1\gamma_2}I^{\theta,\vartheta}_{\phi^-,\rho^-}\psi\left(\frac{\sigma+\phi}{2},\frac{\varsigma+\rho}{2}\right)\right]-A\right| \\
\le&\ \frac{(\phi-\sigma)(\rho-\varsigma)}{16\,\gamma_1\gamma_2}\left[\Psi(\gamma_1,\theta,h)\Psi(\gamma_2,\vartheta,h)+\Psi(\gamma_1,\theta,h)\Phi(\gamma_2,\vartheta,h)+\Phi(\gamma_1,\theta,h)\Psi(\gamma_2,\vartheta,h)+\Phi(\gamma_1,\theta,h)\Phi(\gamma_2,\vartheta,h)\right] \\
&\times\left[\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\sigma,\varsigma)\right|+\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\sigma,\rho)\right|+\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\phi,\varsigma)\right|+\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\phi,\rho)\right|\right],
\end{aligned}$$

where A is defined by (2.2) and $B(\cdot,\cdot)$ refers to the Beta function and

$$\begin{aligned}
\int_0^1\int_0^1\left(\frac{1-(1-\tau)^{\gamma_1}}{\gamma_1}\right)^{\theta}\left(\frac{1-(1-\xi)^{\gamma_2}}{\gamma_2}\right)^{\vartheta}h\left(\frac{1+\tau}{2}\right)h\left(\frac{1+\xi}{2}\right)d\xi d\tau &= \Psi(\gamma_1,\theta,h)\Psi(\gamma_2,\vartheta,h) \\
\int_0^1\int_0^1\left(\frac{1-(1-\tau)^{\gamma_1}}{\gamma_1}\right)^{\theta}\left(\frac{1-(1-\xi)^{\gamma_2}}{\gamma_2}\right)^{\vartheta}h\left(\frac{1+\tau}{2}\right)h\left(\frac{1-\xi}{2}\right)d\xi d\tau &= \Psi(\gamma_1,\theta,h)\Psi(\gamma_2,\vartheta,h) \\
\int_0^1\int_0^1\left(\frac{1-(1-\tau)^{\gamma_1}}{\gamma_1}\right)^{\theta}\left(\frac{1-(1-\xi)^{\gamma_2}}{\gamma_2}\right)^{\vartheta}h\left(\frac{1-\tau}{2}\right)h\left(\frac{1+\xi}{2}\right)d\xi d\tau &= \Psi(\gamma_1,\theta,h)\Psi(\gamma_2,\vartheta,h) \\
\int_0^1\int_0^1\left(\frac{1-(1-\tau)^{\gamma_1}}{\gamma_1}\right)^{\theta}\left(\frac{1-(1-\xi)^{\gamma_2}}{\gamma_2}\right)^{\vartheta}h\left(\frac{1-\tau}{2}\right)h\left(\frac{1-\xi}{2}\right)d\xi d\tau &= \Psi(\gamma_1,\theta,h)\Psi(\gamma_2,\vartheta,h)
\end{aligned}$$

equals were used.

Proof. From Lemma 1, we acquire

$$\begin{aligned}
&\left|\frac{\psi(\sigma,\varsigma)+\psi(\sigma,\rho)+\psi(\phi,\varsigma)+\psi(\phi,\rho)}{4}+\frac{2^{\gamma_1\theta-1}2^{\gamma_2\vartheta-1}\Gamma(\theta+1)\Gamma(\vartheta+1)\gamma_1^{\theta}\gamma_2^{\vartheta}}{(\phi-\sigma)^{\gamma_1\theta}(\rho-\varsigma)^{\gamma_2\vartheta}}\right. \\
&\times\left[{}^{\gamma_1\gamma_2}I^{\theta,\vartheta}_{\sigma^+,\varsigma^+}\psi\left(\frac{\sigma+\phi}{2},\frac{\varsigma+\rho}{2}\right)+{}^{\gamma_1\gamma_2}I^{\theta,\vartheta}_{\phi^-,\varsigma^+}\psi\left(\frac{\sigma+\phi}{2},\frac{\varsigma+\rho}{2}\right)+{}^{\gamma_1\gamma_2}I^{\theta,\vartheta}_{\sigma^+,\rho^-}\psi\left(\frac{\sigma+\phi}{2},\frac{\varsigma+\rho}{2}\right)\right. \\
&\left.\left.+\ {}^{\gamma_1\gamma_2}I^{\theta,\vartheta}_{\phi^-,\rho^-}\psi\left(\frac{\sigma+\phi}{2},\frac{\varsigma+\rho}{2}\right)\right]-A\right| \\
\leq\ &\frac{\gamma_1^{\theta}\gamma_2^{\vartheta}(\phi-\sigma)(\rho-\varsigma)}{16} \\
&\times\left[\int_0^1\int_0^1\left(\frac{1-(1-\tau)^{\gamma_1}}{\gamma_1}\right)^{\theta}\left(\frac{1-(1-\xi)^{\gamma_2}}{\gamma_2}\right)^{\vartheta}\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}\left(\frac{1+\tau}{2}\sigma+\frac{1-\tau}{2}\phi,\frac{1+\xi}{2}\varsigma+\frac{1-\xi}{2}\rho\right)\right|d\xi d\tau\right. \\
&+\int_0^1\int_0^1\left(\frac{1-(1-\tau)^{\gamma_1}}{\gamma_1}\right)^{\theta}\left(\frac{1-(1-\xi)^{\gamma_2}}{\gamma_2}\right)^{\vartheta}\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}\left(\frac{1+\tau}{2}\sigma+\frac{1-\tau}{2}\phi,\frac{1-\xi}{2}\varsigma+\frac{1+\xi}{2}\rho\right)\right|d\xi d\tau \\
&+\int_0^1\int_0^1\left(\frac{1-(1-\tau)^{\gamma_1}}{\gamma_1}\right)^{\theta}\left(\frac{1-(1-\xi)^{\gamma_2}}{\gamma_2}\right)^{\vartheta}\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}\left(\frac{1-\tau}{2}\sigma+\frac{1+\tau}{2}\phi,\frac{1+\xi}{2}\varsigma+\frac{1-\xi}{2}\rho\right)\right|d\xi d\tau \\
&\left.+\int_0^1\int_0^1\left(\frac{1-(1-\tau)^{\gamma_1}}{\gamma_1}\right)^{\theta}\left(\frac{1-(1-\xi)^{\gamma_2}}{\gamma_2}\right)^{\vartheta}\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}\left(\frac{1-\tau}{2}\sigma+\frac{1+\tau}{2}\phi,\frac{1-\xi}{2}\varsigma+\frac{1+\xi}{2}\rho\right)\right|d\xi d\tau\right]
\end{aligned} \tag{2.9}$$

Since $\frac{\partial^2\psi}{\partial\tau\partial\xi}$ is $h-$convex function on the co-ordinates on Δ, then one has:

$$\begin{aligned}
&\left|\frac{\psi(\sigma,\varsigma)+\psi(\sigma,\rho)+\psi(\phi,\varsigma)+\psi(\phi,\rho)}{4}+\frac{2^{\gamma_1\theta-1}2^{\gamma_2\vartheta-1}\Gamma(\theta+1)\Gamma(\vartheta+1)\gamma_1^{\theta}\gamma_2^{\vartheta}}{(\phi-\sigma)^{\gamma_1\theta}(\rho-\varsigma)^{\gamma_2\vartheta}}\right. \\
&\times\left[{}^{\gamma_1\gamma_2}I^{\theta,\vartheta}_{\sigma^+,\varsigma^+}\psi\left(\frac{\sigma+\phi}{2},\frac{\varsigma+\rho}{2}\right)+{}^{\gamma_1\gamma_2}I^{\theta,\vartheta}_{\phi^-,\varsigma^+}\psi\left(\frac{\sigma+\phi}{2},\frac{\varsigma+\rho}{2}\right)+\ {}^{\gamma_1\gamma_2}I^{\theta,\vartheta}_{\sigma^+,\rho^-}\psi\left(\frac{\sigma+\phi}{2},\frac{\varsigma+\rho}{2}\right)\right. \\
&\left.\left.+{}^{\gamma_1\gamma_2}I^{\theta,\vartheta}_{\phi^-,\rho^-}\psi\left(\frac{\sigma+\phi}{2},\frac{\varsigma+\rho}{2}\right)\right]-A\right| \\
\leq\ &\frac{\gamma_1^{\theta}\gamma_2^{\vartheta}(\phi-\sigma)(\rho-\varsigma)}{16} \\
&\times\left\{\int_0^1\int_0^1\left(\frac{1-(1-\tau)^{\gamma_1}}{\gamma_1}\right)^{\theta}\left(\frac{1-(1-\xi)^{\gamma_2}}{\gamma_2}\right)^{\vartheta}\right. \\
&\left[h\left(\frac{1+\tau}{2}\right)h\left(\frac{1+\xi}{2}\right)\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\sigma,\varsigma)\right|+h\left(\frac{1+\tau}{2}\right)h\left(\frac{1-\xi}{2}\right)\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\sigma,\rho)\right|\right. \\
&\left.+h\left(\frac{1-\tau}{2}\right)h\left(\frac{1+\xi}{2}\right)\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\phi,\varsigma)\right|+h\left(\frac{1-\tau}{2}\right)h\left(\frac{1-\xi}{2}\right)\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\phi,\rho)\right|\right]d\xi d\tau \\
&+\int_0^1\int_0^1\left(\frac{1-(1-\tau)^{\gamma_1}}{\gamma_1}\right)^{\theta}\left(\frac{1-(1-\xi)^{\gamma_2}}{\gamma_2}\right)^{\vartheta} \\
&\left[h\left(\frac{1+\tau}{2}\right)h\left(\frac{1-\xi}{2}\right)\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\sigma,\varsigma)\right|+h\left(\frac{1+\tau}{2}\right)h\left(\frac{1+\xi}{2}\right)\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\sigma,\rho)\right|\right. \\
&\left.+h\left(\frac{1-\tau}{2}\right)h\left(\frac{1-\xi}{2}\right)\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\phi,\varsigma)\right|+h\left(\frac{1-\tau}{2}\right)h\left(\frac{1+\xi}{2}\right)\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\phi,\rho)\right|\right]d\xi d\tau \\
&+\int_0^1\int_0^1\left(\frac{1-(1-\tau)^{\gamma_1}}{\gamma_1}\right)^{\theta}\left(\frac{1-(1-\xi)^{\gamma_2}}{\gamma_2}\right)^{\vartheta} \\
&\left[h\left(\frac{1-\tau}{2}\right)h\left(\frac{1+\xi}{2}\right)\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\sigma,\varsigma)\right|+h\left(\frac{1-\tau}{2}\right)h\left(\frac{1-\xi}{2}\right)\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\sigma,\rho)\right|\right. \\
&\left.+h\left(\frac{1+\tau}{2}\right)h\left(\frac{1+\xi}{2}\right)\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\phi,\varsigma)\right|+h\left(\frac{1+\tau}{2}\right)h\left(\frac{1-\xi}{2}\right)\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\phi,\rho)\right|\right]d\xi d\tau
\end{aligned}$$

$$+\int_0^1\int_0^1\left(\frac{1-(1-t)^{\gamma_1}}{\gamma_1}\right)^{\theta}\left(\frac{1-(1-\xi)^{\gamma_2}}{\gamma_2}\right)^{\vartheta}$$
$$\left[h\left(\frac{1-\tau}{2}\right)h\left(\frac{1-\xi}{2}\right)\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\sigma,\varsigma)\right|+h\left(\frac{1-\tau}{2}\right)h\left(\frac{1+\xi}{2}\right)\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\sigma,\rho)\right|\right.$$
$$\left.+h\left(\frac{1+\tau}{2}\right)h\left(\frac{1-\xi}{2}\right)\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\phi,\varsigma)\right|+h\left(\frac{1+\tau}{2}\right)h\left(\frac{1+\xi}{2}\right)\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\phi,\rho)\right|\right]d\xi d\tau\Big\}$$
$$=\frac{(\phi-\sigma)(\rho-\varsigma)}{16\,\gamma_1\gamma_2}\left[\Psi(\gamma_1,\theta,h)\Psi(\gamma_2,\vartheta,h)+\Psi(\gamma_1,\theta,h)\Phi(\gamma_2,\vartheta,h)\right.$$
$$\left.+\Phi(\gamma_1,\theta,h)\Psi(\gamma_2,\vartheta,h)+\Phi(\gamma_1,\theta,h)\Phi(\gamma_2,\vartheta,h)\right]\times\left[\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\sigma,\varsigma)\right|+\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\sigma,\rho)\right|+\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\phi,\varsigma)\right|+\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\phi,\rho)\right|\right],$$

which finishes the proof. □

Remark 2.3. *In Theorem 2.2, if we choose $h(t)=t$, then we have,*

$$\left|\frac{\psi(\sigma,\varsigma)+\psi(\sigma,\rho)+\psi(\phi,\varsigma)+\psi(\phi,\rho)}{4}+\frac{2^{\gamma_1\theta-1}2^{\gamma_2\vartheta-1}\Gamma(\theta+1)\Gamma(\vartheta+1)\gamma_1^{\theta}\gamma_2^{\vartheta}}{(\phi-\sigma)^{\gamma_1\theta}(\rho-\varsigma)^{\gamma_2\vartheta}}\right.\tag{2.10}$$
$$\times\left[{}^{\gamma_1\gamma_2}I^{\theta,\vartheta}_{\sigma^+,\varsigma^+}\psi\left(\frac{\sigma+\phi}{2},\frac{\varsigma+\rho}{2}\right)+{}^{\gamma_1\gamma_2}I^{\theta,\vartheta}_{\phi^-,\varsigma^+}\psi\left(\frac{\sigma+\phi}{2},\frac{\varsigma+\rho}{2}\right)+{}^{\gamma_1\gamma_2}I^{\theta,\vartheta}_{\sigma^+,\rho^-}\psi\left(\frac{\sigma+\phi}{2},\frac{\varsigma+\rho}{2}\right)\right.$$
$$\left.\left.+{}^{\gamma_1\gamma_2}I^{\theta,\vartheta}_{\phi^-,\rho^-}\psi\left(\frac{\sigma+\phi}{2},\frac{\varsigma+\rho}{2}\right)\right]-A\right|$$
$$\leq\frac{(\phi-\sigma)(\rho-\varsigma)}{16\,\gamma_1\gamma_2}B\left(\theta+1,\frac{1}{\gamma_1}\right)B\left(\vartheta+1,\frac{1}{\gamma_2}\right)\left[\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\sigma,\varsigma)\right|+\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\sigma,\rho)\right|+\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\phi,\varsigma)\right|+\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\phi,\rho)\right|\right],$$

which is given by Kiris et al. in [32]

Remark 2.4. *In Remark 3 , if we choose $\gamma_1=1$ and $\gamma_2=1$, the following inequalities are achieved [31]*

$$\left|\frac{\psi(\sigma,\varsigma)+\psi(\sigma,\rho)+\psi(\phi,\varsigma)+\psi(\phi,\rho)}{4}+\frac{2^{\theta-1}2^{\vartheta-1}\Gamma(\theta+1)\Gamma(\vartheta+1)\gamma_1^{\theta}\gamma_2^{\vartheta}}{(\phi-\sigma)^{\theta}(\rho-\varsigma)^{\vartheta}}\right.\tag{2.11}$$
$$\times\left[{}^{\gamma_1\gamma_2}I^{\theta,\vartheta}_{\sigma^+,\varsigma^+}\psi\left(\frac{\sigma+\phi}{2},\frac{\varsigma+\rho}{2}\right)+{}^{\gamma_1\gamma_2}I^{\theta,\vartheta}_{\phi^-,\varsigma^+}\psi\left(\frac{\sigma+\phi}{2},\frac{\varsigma+\rho}{2}\right)+{}^{\gamma_1\gamma_2}I^{\theta,\vartheta}_{\sigma^+,\rho^-}\psi\left(\frac{\sigma+\phi}{2},\frac{\varsigma+\rho}{2}\right)\right.$$
$$\left.\left.+{}^{\gamma_1\gamma_2}I^{\theta,\vartheta}_{\phi^-,\rho^-}\psi\left(\frac{\sigma+\phi}{2},\frac{\varsigma+\rho}{2}\right)\right]-A\right|$$
$$\leq\frac{(\phi-\sigma)(\rho-\varsigma)}{16}\frac{1}{\theta+1}\frac{1}{\vartheta+1}\left[\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\sigma,\varsigma)\right|+\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\sigma,\rho)\right|+\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\phi,\varsigma)\right|+\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\phi,\rho)\right|\right].$$

Theorem 2.5. *Let $\psi:\triangle\subset\mathbb{R}^2\to\mathbb{R}$ be a partial differentiable mapping on Δ with $0\leq\sigma<\phi,\ 0\leq\varsigma<\rho$. If $\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}\right|^q q>1$, is a $h-$convex function on the co-ordinates on Δ, then the inequality below holds.*

$$\left|\frac{\psi(\sigma,\varsigma)+\psi(\sigma,\rho)+\psi(\phi,\varsigma)+\psi(\phi,\rho)}{4}+\frac{2^{\gamma_1\theta-1}2^{\gamma_2\vartheta-1}\Gamma(\theta+1)\Gamma(\vartheta+1)\gamma_1^{\theta}\gamma_2^{\vartheta}}{(\phi-\sigma)^{\gamma_1\theta}(\rho-\varsigma)^{\gamma_2\vartheta}}\right.\tag{2.12}$$
$$\times\left[{}^{\gamma_1\gamma_2}I^{\theta,\vartheta}_{\sigma^+,\varsigma^+}\psi\left(\frac{\sigma+\phi}{2},\frac{\varsigma+\rho}{2}\right)+{}^{\gamma_1\gamma_2}I^{\theta,\vartheta}_{\phi^-,\varsigma^+}\psi\left(\frac{\sigma+\phi}{2},\frac{\varsigma+\rho}{2}\right)+{}^{\gamma_1\gamma_2}I^{\theta,\vartheta}_{\sigma^+,\rho^-}\psi\left(\frac{\sigma+\phi}{2},\frac{\varsigma+\rho}{2}\right)\right.$$
$$\left.\left.+{}^{\gamma_1\gamma_2}I^{\theta,\vartheta}_{\phi^-,\rho^-}\psi\left(\frac{\sigma+\phi}{2},\frac{\varsigma+\rho}{2}\right)\right]-A\right|$$
$$\leq\frac{(\phi-\sigma)(\rho-\varsigma)}{16}\left[\frac{4(h_1+h_2)^2}{\gamma_1\gamma_2}B\left(\theta p+1,\frac{1}{\gamma_1}\right)B\left(\vartheta p+1,\frac{1}{\gamma_2}\right)\right]^{\frac{1}{p}}$$
$$\times\left[\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\sigma,\varsigma)\right|^q+\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\sigma,\rho)\right|^q+\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\phi,\varsigma)\right|^q+\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\phi,\rho)\right|^q\right]^{\frac{1}{q}}$$

where A is defined by (2.2) and $B(\cdot,\cdot)$ refers to the Beta function and $\frac{1}{p}=1-\frac{1}{q}$ and

$$\int h\left(\frac{1+\tau}{2}\right)d\tau=\int h\left(\frac{1+\xi}{2}\right)d\xi=h_1$$

$$\int h\left(\frac{1-\tau}{2}\right)d\tau = \int h\left(\frac{1-\xi}{2}\right)d\xi = h_2$$

equals were used.

Proof. From Lemma, we have inequality (2.9). In order to employ the well-known Hölder's inequality for double integrals, in I_5 and since $\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}\right|^q$ is The application of $h-$convex functions to the coordinates of the triangle yields the following result: on $\triangle$

$$\begin{aligned}
I_5 &= \left\{\int_0^1\int_0^1\left(\frac{1-(1-\tau)^{\gamma_1}}{\gamma_1}\right)^{\theta}\left(\frac{1-(1-\xi)^{\gamma_2}}{\gamma_2}\right)^{\vartheta}\times\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}\left(\frac{1+\tau}{2}\sigma+\frac{1-\tau}{2}\phi,\frac{1+\xi}{2}\varsigma+\frac{1-\xi}{2}\rho\right)\right|d\xi d\tau\right\}\\
&\leq \left(\int_0^1\int_0^1\left(\frac{1-(1-\tau)^{\gamma_1}}{\gamma_1}\right)^{\theta p}\left(\frac{1-(1-\xi)^{\gamma_2}}{\gamma_2}\right)^{\vartheta p}d\xi d\tau\right)^{\frac{1}{p}} \qquad (2.13)\\
&\quad\times\left(\int_0^1\int_0^1\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}\left(\frac{1+\tau}{2}\sigma+\frac{1-\tau}{2}\phi,\frac{1+\xi}{2}\varsigma+\frac{1-\xi}{2}\rho\right)\right|^q d\xi d\tau\right)^{\frac{1}{q}}\\
&\leq \frac{1}{\gamma_1^{\theta}}\frac{1}{\gamma_2^{\vartheta}}\left(\int_0^1\int_0^1(1-(1-\tau)^{\gamma_1})^{\theta p}(1-(1-\xi)^{\gamma_2})^{\vartheta p}d\xi d\tau\right)^{\frac{1}{p}}\\
&\quad\times\left\{h\left(\frac{1+\tau}{2}\right)h\left(\frac{1+\xi}{2}\right)\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\sigma,\varsigma)\right|^q+h\left(\frac{1+\tau}{2}\right)h\left(\frac{1-\xi}{2}\right)\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\sigma,\rho)\right|^q\right.\\
&\quad\left.+h\left(\frac{1-\tau}{2}\right)h\left(\frac{1+\xi}{2}\right)\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\phi,\varsigma)\right|^q+h\left(\frac{1-\tau}{2}\right)h\left(\frac{1-\xi}{2}\right)\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\phi,\rho)\right|^q d\xi d\tau\right\}^{\frac{1}{q}}\\
&\leq \frac{1}{\gamma_1^{\theta}}\frac{1}{\gamma_2^{\vartheta}}\left(\frac{1}{\gamma_1}\frac{1}{\gamma_2}B\left(\theta p+1,\frac{1}{\gamma_1}\right)B\left(\vartheta p+1,\frac{1}{\gamma_2}\right)\right)^{\frac{1}{p}}\\
&\quad\times\left((h_1)^2\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\sigma,\varsigma)\right|^q+h_1h_2\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\sigma,\rho)\right|^q+h_1h_2\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\phi,\varsigma)\right|^q+(h_2)^2\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\phi,\rho)\right|^q\right)^{\frac{1}{q}}
\end{aligned}$$

Where we take advantage of the fact:

$$(\varpi-\sigma)^j \leq \varpi^j-\sigma^j,$$

for any $\varpi > \sigma \geq 0$ and $j \geq 1$. And similarly,

$$\begin{aligned}
I_6 &= \left\{\int_0^1\int_0^1\left(\frac{1-(1-\tau)^{\gamma_1}}{\gamma_1}\right)^{\theta}\left(\frac{1-(1-\xi)^{\gamma_2}}{\gamma_2}\right)^{\vartheta}\times\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}\left(\frac{1+\tau}{2}\sigma+\frac{1-\tau}{2}\phi,\frac{1-\xi}{2}\varsigma+\frac{1+\xi}{2}\rho\right)\right|d\xi d\tau\right\}\\
&\leq \left(\int_0^1\int_0^1\left(\frac{1-(1-\tau)^{\gamma_1}}{\gamma_1}\right)^{\theta p}\left(\frac{1-(1-\xi)^{\gamma_2}}{\gamma_2}\right)^{\vartheta p}d\xi d\tau\right)^{\frac{1}{p}} \qquad (2.14)\\
&\quad\times\left(\int_0^1\int_0^1\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}\left(\frac{1+\tau}{2}\sigma+\frac{1-\tau}{2}\phi,\frac{1-\xi}{2}\varsigma+\frac{1+\xi}{2}\rho\right)\right|^q d\xi d\tau\right)^{\frac{1}{q}}\\
&\leq \frac{1}{\gamma_1^{\theta}}\frac{1}{\gamma_2^{\vartheta}}\left(\frac{1}{\gamma_1}\frac{1}{\gamma_2}B\left(\theta p+1,\frac{1}{\gamma_1}\right)B\left(\vartheta p+1,\frac{1}{\gamma_2}\right)\right)^{\frac{1}{p}}\\
&\quad\times\left(h_1h_2\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\sigma,\varsigma)\right|^q+h_1^2\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\sigma,\rho)\right|^q+h_2^2\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\phi,\varsigma)\right|^q+h_1h_2\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\phi,\rho)\right|^q\right)^{\frac{1}{q}},
\end{aligned}$$

$$I_7 = \left\{\int_0^1\int_0^1\left(\frac{1-(1-\tau)^{\gamma_1}}{\gamma_1}\right)^{\theta}\left(\frac{1-(1-\xi)^{\gamma_2}}{\gamma_2}\right)^{\vartheta}\right. \qquad (2.15)$$

$$\times\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}\left(\frac{1-\tau}{2}\sigma+\frac{1+\tau}{2}\phi,\frac{1+\xi}{2}\varsigma+\frac{1-\xi}{2}\rho\right)\right|d\xi d\tau\Bigg\}$$

$$\leq\left(\int_0^1\int_0^1\left(\frac{1-(1-\tau)^{\gamma_1}}{\gamma_1}\right)^{\theta p}\left(\frac{1-(1-\xi)^{\gamma_2}}{\gamma_2}\right)^{\vartheta p}d\xi d\tau\right)^{\frac{1}{p}}$$

$$\times\left(\int_0^1\int_0^1\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}\left(\frac{1-\tau}{2}\sigma+\frac{1+\tau}{2}\phi,\frac{1+\xi}{2}\varsigma+\frac{1-\xi}{2}\rho\right)\right|^q d\xi d\tau\right)^{\frac{1}{q}}$$

$$\leq\frac{1}{\gamma_1^\theta}\frac{1}{\gamma_2^\vartheta}\left(\frac{1}{\gamma_1}\frac{1}{\gamma_2}B\left(\theta p+1,\frac{1}{\gamma_1}\right)B\left(\vartheta p+1,\frac{1}{\gamma_2}\right)\right)^{\frac{1}{p}}$$

$$\times\left(h_1h_2\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\sigma,\varsigma)\right|^q+h_2^2\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\sigma,\rho)\right|^q+h_1^2\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\phi,\varsigma)\right|^q+h_1h_2\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\phi,\rho)\right|^q\right)^{\frac{1}{q}}$$

and

$$I_8=\left\{\int_0^1\int_0^1\left(\frac{1-(1-\tau)^{\gamma_1}}{\gamma_1}\right)^{\theta}\left(\frac{1-(1-\xi)^{\gamma_2}}{\gamma_2}\right)^{\vartheta}\right. \tag{2.16}$$

$$\times\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}\left(\frac{1-\tau}{2}\sigma+\frac{1+\tau}{2}\phi,\frac{1-\xi}{2}\varsigma+\frac{1+\xi}{2}\rho\right)\right|d\xi d\tau\Bigg\}$$

$$\leq\left(\int_0^1\int_0^1\left(\frac{1-(1-t)^{\gamma_1}}{\gamma_1}\right)^{\theta p}\left(\frac{1-(1-\xi)^{\gamma_2}}{\gamma_2}\right)^{\vartheta p}d\xi d\tau\right)^{\frac{1}{p}}$$

$$\times\left(\int_0^1\int_0^1\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}\left(\frac{1-\tau}{2}\sigma+\frac{1+\tau}{2}\phi,\frac{1-\xi}{2}\varsigma+\frac{1+\xi}{2}\rho\right)\right|^q d\xi d\tau\right)^{\frac{1}{q}}$$

$$\leq\frac{1}{\gamma_1^\theta}\frac{1}{\gamma_2^\vartheta}\left(\frac{1}{\gamma_1}\frac{1}{\gamma_2}B\left(\theta p+1,\frac{1}{\gamma_1}\right)B\left(\vartheta p+1,\frac{1}{\gamma_2}\right)\right)^{\frac{1}{p}}$$

$$\times\left(h_2^2\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\sigma,\varsigma)\right|^q+h_1h_2\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\sigma,\rho)\right|^q+h_1h_2\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\phi,\varsigma)\right|^q+h_1^2\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\phi,\rho)\right|^q\right)^{\frac{1}{q}}.$$

If we substitute from (2.13)-(2.16) in (2.9), we obtain the first inequality of (2.12) is achieved. □

Remark 2.6. *If we set $h(t)=t$ in Theorem (2.5), then we have,*

$$\left|\frac{\psi(\sigma,\varsigma)+\psi(\sigma,\rho)+\psi(\phi,\varsigma)+\psi(\phi,\rho)}{4}+\frac{2^{\gamma_1\theta-1}2^{\gamma_2\vartheta-1}\Gamma(\theta+1)\Gamma(\vartheta+1)\gamma_1^\theta\gamma_2^\vartheta}{(\phi-\sigma)^{\gamma_1\theta}(\rho-\varsigma)^{\gamma_2\vartheta}}\right. \tag{2.17}$$

$$\times\left[\gamma_1\gamma_2 I^{\theta,\vartheta}_{\sigma^+,\varsigma^+}\psi\left(\frac{\sigma+\phi}{2},\frac{\varsigma+\rho}{2}\right)+\gamma_1\gamma_2 I^{\theta,\vartheta}_{\phi^-,\varsigma^+}\psi\left(\frac{\sigma+\phi}{2},\frac{\varsigma+\rho}{2}\right)+\gamma_1\gamma_2 I^{\theta,\vartheta}_{\phi^-,\varsigma^+}\psi\left(\frac{\sigma+\phi}{2},\frac{\varsigma+\rho}{2}\right)\right.$$

$$\left.\left.+\ \gamma_1\gamma_2 I^{\theta,\vartheta}_{\phi^-,\rho^-}\psi\left(\frac{\sigma+\phi}{2},\frac{\varsigma+\rho}{2}\right)\right]-A\right|$$

$$\leq\frac{(\phi-\sigma)(\rho-\varsigma)}{16}\left[\frac{16}{\gamma_1\gamma_2}B\left(\theta+1,\frac{1}{\gamma_1}\right)B\left(\vartheta p+1,\frac{1}{\gamma_2}\right)\right]^{\frac{1}{p}}$$

$$\times\left[\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\sigma,\varsigma)\right|^q+\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\sigma,\rho)\right|^q+\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\phi,\varsigma)\right|^q+\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\phi,\rho)\right|^q\right]^{\frac{1}{q}}$$

which is given by Kiris et al. in [32].

Remark 2.7. *If we take $h(t)=t$, $\gamma_1=1$ and $\gamma_2=1$ in Theorem (2.5), the following inequalities are achieved*

$$\left|\frac{\psi(\sigma,\varsigma)+\psi(\sigma,\rho)+\psi(\phi,\varsigma)+\psi(\phi,\rho)}{4}+\frac{2^{\theta-1}2^{\vartheta-1}\Gamma(\theta+1)\Gamma(\vartheta+1)}{(\phi-\sigma)^{\alpha}(\rho-\varsigma)^{\beta}}\right. \tag{2.18}$$

$$
\begin{aligned}
& \times\left[{}^{\gamma_1\gamma_2}I^{\theta,\vartheta}_{\sigma^+,\varsigma^+}\psi\left(\frac{\sigma+\phi}{2},\frac{\varsigma+\rho}{2}\right)+{}^{\gamma_1\gamma_2}I^{\theta,\vartheta}_{\phi^-,\varsigma^+}\psi\left(\frac{\sigma+\phi}{2},\frac{\varsigma+\rho}{2}\right)+{}^{\gamma_1\gamma_2}I^{\theta,\vartheta}_{\sigma^+,\rho^-}\psi\left(\frac{\sigma+\phi}{2},\frac{\varsigma+\rho}{2}\right)\right.\\
& \left.+{}^{\gamma_1\gamma_2}I^{\theta,\vartheta}_{\phi^-,\rho^-}\psi\left(\frac{\sigma+\phi}{2},\frac{\varsigma+\rho}{2}\right)\right]-A\Bigg|\\
\leq\; & \frac{(\phi-\sigma)(\rho-\varsigma)}{16}\left[\frac{16}{(\theta p+1)(\vartheta p+1)}\right]^{\frac{1}{p}}\\
& \times\left[\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\sigma,\varsigma)\right|^q+\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\sigma,\rho)\right|^q+\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\phi,\varsigma)\right|^q+\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\phi,\rho)\right|^q\right]^{\frac{1}{q}}
\end{aligned}
$$

which is proven by Hyder et al. in [31].

Theorem 2.8. *Assume $\psi:\triangle\subset\mathbb{R}^2\to\mathbb{R}$ The mapping is partially differentiable with respect to Δ with $0\leq\sigma<\phi,\ 0\leq\varsigma<\rho$. If $\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}\right|^q$ $q\geq 1$, is a h-convex function on the coordinates on Δ, then we have the following inequality:*

$$
\begin{aligned}
& \Bigg|\frac{\psi(\sigma,\varsigma)+\psi(\sigma,\rho)+\psi(\phi,\varsigma)+\psi(\phi,\rho)}{4}+\frac{2^{\gamma_1\theta-1}2^{\gamma_2\vartheta-1}\Gamma(\theta+1)\Gamma(\vartheta+1)\gamma_1^{\theta}\gamma_2^{\vartheta}}{(\phi-\sigma)^{\gamma_1\theta}(\rho-\varsigma)^{\gamma_2\vartheta}} \qquad (2.19)\\
& \times\left[{}^{\gamma_1\gamma_2}I^{\theta,\vartheta}_{\sigma^+,\varsigma^+}\psi\left(\frac{\sigma+\phi}{2},\frac{\varsigma+\rho}{2}\right)+{}^{\gamma_1\gamma_2}I^{\theta,\vartheta}_{\phi^-,\varsigma^+}\psi\left(\frac{\sigma+\phi}{2},\frac{\varsigma+\rho}{2}\right)+{}^{\gamma_1\gamma_2}I^{\theta,\vartheta}_{\sigma^+,\rho^-}\psi\left(\frac{\sigma+\phi}{2},\frac{\varsigma+\rho}{2}\right)\right.\\
& \left.+{}^{\gamma_1\gamma_2}I^{\theta,\vartheta}_{\phi^-,\rho^-}\psi\left(\frac{\sigma+\phi}{2},\frac{\varsigma+\rho}{2}\right)\right]-A\Bigg|\\
\leq\; & \frac{(\phi-\sigma)(\rho-\varsigma)}{16\gamma_1\gamma_2}\left(B\left(\theta+1,\frac{1}{\gamma_1}\right)B\left(\vartheta+1,\frac{1}{\gamma_2}\right)\right)^{1-\frac{1}{q}}\\
& \times\Bigg\{\Bigg(\Psi(\gamma_1,\theta,h)\Psi(\gamma_2,\vartheta,h)\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\sigma,\varsigma)\right|^q+\Psi(\gamma_1,\theta,h)\Psi(\gamma_2,\vartheta,h)\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\sigma,\rho)\right|^q\\
& \quad+\Psi(\gamma_1,\theta,h)\Psi(\gamma_2,\vartheta,h)\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\phi,\varsigma)\right|^q+\Psi(\gamma_1,\theta,h)\Psi(\gamma_2,\vartheta,h)\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\phi,\rho)\right|^q\Bigg)^{\frac{1}{q}}\\
& \quad+\Bigg(\Psi(\gamma_1,\theta,h)\Psi(\gamma_2,\vartheta,h)\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\sigma,\varsigma)\right|^q+\Psi(\gamma_1,\theta,h)\Psi(\gamma_2,\vartheta,h)\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\sigma,\rho)\right|^q\\
& \quad+\Psi(\gamma_1,\theta,h)\Psi(\gamma_2,\vartheta,h)\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\phi,\varsigma)\right|^q+\Psi(\gamma_1,\theta,h)\Psi(\gamma_2,\vartheta,h)\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\phi,\rho)\right|^q\Bigg)^{\frac{1}{q}}\\
& \quad+\Bigg(\Psi(\gamma_1,\theta,h)\Psi(\gamma_2,\vartheta,h)\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\sigma,\varsigma)\right|^q+\Psi(\gamma_1,\theta,h)\Psi(\gamma_2,\vartheta,h)\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\sigma,\rho)\right|^q\\
& \quad+\Psi(\gamma_1,\theta,h)\Psi(\gamma_2,\vartheta,h)\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\phi,\varsigma)\right|^q+\Psi(\gamma_1,\theta,h)\Psi(\gamma_2,\vartheta,h)\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\phi,\rho)\right|^q\Bigg)^{\frac{1}{q}}\\
& \quad+\Bigg(\Psi(\gamma_1,\theta,h)\Psi(\gamma_2,\vartheta,h)\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\sigma,\varsigma)\right|^q+\Psi(\gamma_1,\theta,h)\Psi(\gamma_2,\vartheta,h)\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\sigma,\rho)\right|^q\\
& \quad+\Psi(\gamma_1,\theta,h)\Psi(\gamma_2,\vartheta,h)\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\phi,\varsigma)\right|^q+\Psi(\gamma_1,\theta,h)\Psi(\gamma_2,\vartheta,h)\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\phi,\rho)\right|^q\Bigg)^{\frac{1}{q}}.
\end{aligned}
$$

Here A is defined as in (2.2) and

$$
\begin{aligned}
\int_0^1\int_0^1\left(\frac{1-(1-\tau)^{\gamma_1}}{\gamma_1}\right)^{\theta}\left(\frac{1-(1-\xi)^{\gamma_2}}{\gamma_2}\right)^{\vartheta}h\left(\frac{1+\tau}{2}\right)h\left(\frac{1+\xi}{2}\right)d\xi d\tau &= \Phi(\gamma_1,\theta,h)\Phi(\gamma_2,\vartheta,h)\\
\int_0^1\int_0^1\left(\frac{1-(1-\tau)^{\gamma_1}}{\gamma_1}\right)^{\theta}\left(\frac{1-(1-\xi)^{\gamma_2}}{\gamma_2}\right)^{\vartheta}h\left(\frac{1+\tau}{2}\right)h\left(\frac{1-\xi}{2}\right)d\xi d\tau &= \Phi(\gamma_1,\theta,h)\Phi(\gamma_2,\vartheta,h)\\
\int_0^1\int_0^1\left(\frac{1-(1-\tau)^{\gamma_1}}{\gamma_1}\right)^{\theta}\left(\frac{1-(1-\xi)^{\gamma_2}}{\gamma_2}\right)^{\vartheta}h\left(\frac{1-\tau}{2}\right)h\left(\frac{1+\xi}{2}\right)d\xi d\tau &= \Phi(\gamma_1,\theta,h)\Phi(\gamma_2,\vartheta,h)\\
\int_0^1\int_0^1\left(\frac{1-(1-\tau)^{\gamma_1}}{\gamma_1}\right)^{\theta}\left(\frac{1-(1-\xi)^{\gamma_2}}{\gamma_2}\right)^{\vartheta}h\left(\frac{1-\tau}{2}\right)h\left(\frac{1-\xi}{2}\right)d\xi d\tau &= \Phi(\gamma_1,\theta,h)\Phi(\gamma_2,\vartheta,h).
\end{aligned}
$$

Proof. By employing the principle of equality (2.9) and the Power-Mean inequality, in I_9, we get

$$\begin{aligned} I_9 = & \left[\int_0^1\int_0^1 \left(\frac{1-(1-\tau)^{\gamma_1}}{\gamma_1}\right)^{\theta}\left(\frac{1-(1-\xi)^{\gamma_2}}{\gamma_2}\right)^{\vartheta} \right. \\ & \left. \times\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}\left(\frac{1+\tau}{2}\sigma+\frac{1-\tau}{2}\phi,\frac{1+\xi}{2}\varsigma+\frac{1-\xi}{2}\rho\right)\right|d\xi d\tau\right] \\ \le & \left(\int_0^1\int_0^1 \left(\frac{1-(1-\tau)^{\gamma_1}}{\gamma_1}\right)^{\theta}\left(\frac{1-(1-\xi)^{\gamma_2}}{\gamma_2}\right)^{\vartheta} d\xi d\tau\right)^{1-\frac{1}{q}} \\ & \times\left(\int_0^1\int_0^1 \left(\frac{1-(1-\tau)^{\gamma_1}}{\gamma_1}\right)^{\theta}\left(\frac{1-(1-\xi)^{\gamma_2}}{\gamma_2}\right)^{\vartheta}\right. \\ & \left.\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}\left(\frac{1+\tau}{2}\sigma+\frac{1-\tau}{2}\phi,\frac{1+\xi}{2}\varsigma+\frac{1-\xi}{2}\rho\right)\right|^q d\xi d\tau\right)^{\frac{1}{q}} \end{aligned} \tag{2.20}$$

In light of the convexity observed in the h-convex function when expressed in co-ordinates, $\triangle$ of $\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}\right|^q$, then we acquire

$$\begin{aligned} \le & \left(\int_0^1\int_0^1 \left(\frac{1-(1-\tau)^{\gamma_1}}{\gamma_1}\right)^{\theta}\left(\frac{1-(1-\xi)^{\gamma_2}}{\gamma_2}\right)^{\vartheta} d\xi d\tau\right)^{1-\frac{1}{q}} \\ & \left(\int_0^1\int_0^1 \left(\frac{1-(1-\tau)^{\gamma_1}}{\gamma_1}\right)^{\theta}\left(\frac{1-(1-\xi)^{\gamma_2}}{\gamma_2}\right)^{\vartheta}\right. \\ & \times\left\{h\left(\frac{1+\tau}{2}\right)h\left(\frac{1+\xi}{2}\right)\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\sigma,\varsigma)\right|^q+h\left(\frac{1+\tau}{2}\right)h\left(\frac{1-\xi}{2}\right)\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\sigma,\rho)\right|^q\right. \\ & \left.+h\left(\frac{1-\tau}{2}\right)h\left(\frac{1+\xi}{2}\right)\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\phi,\varsigma)\right|^q+h\left(\frac{1-\tau}{2}\right)h\left(\frac{1-\xi}{2}\right)\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\phi,\rho)\right|^q d\xi d\tau\right\}^{\frac{1}{q}} \end{aligned}$$

In this inequality, the change of variables allows us to express it as follows:

$$\begin{aligned} & \int_0^1\int_0^1 \left(\frac{1-(1-\tau)^{\gamma_1}}{\gamma_1}\right)^{\theta}\left(\frac{1-(1-\xi)^{\gamma_2}}{\gamma_2}\right)^{\vartheta} \\ & \times\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}\left(\frac{1+\tau}{2}\sigma+\frac{1-\tau}{2}\phi,\frac{1+\xi}{2}\varsigma+\frac{1-\xi}{2}\rho\right)\right|d\xi d\tau \\ \le & \left(\frac{1}{\gamma_1^{\theta+1}}\frac{1}{\gamma_2^{\vartheta+1}}B\left(\theta+1,\frac{1}{\gamma_1}\right)B\left(\vartheta+1,\frac{1}{\gamma_2}\right)\right)^{1-\frac{1}{q}} \\ & \times\left\{\Psi(\gamma_1,\theta,h)\Psi(\gamma_2,\vartheta,h)\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(a,c)\right|^q+\Psi(\gamma_1,\theta,h)\Phi(\gamma_2,\vartheta,h)\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\sigma,\rho)\right|^q\right. \\ & \left.+\Phi(\gamma_1,\theta,h)\Psi(\gamma_2,\vartheta,h)\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\phi,\varsigma)\right|^q+\Phi(\gamma_1,\theta,h)\Phi(\gamma_2,\vartheta,h)\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\phi,\rho)\right|^q\right)\right\}^{\frac{1}{q}}. \end{aligned} \tag{2.21}$$

Similarly, we have

$$\begin{aligned} I_{10} = & \int_0^1\int_0^1 \left(\frac{1-(1-\tau)^{\gamma_1}}{\gamma_1}\right)^{\theta}\left(\frac{1-(1-\xi)^{\gamma_2}}{\gamma_2}\right)^{\vartheta} \\ & \times\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}\left(\frac{1+\tau}{2}\sigma+\frac{1-\tau}{2}\phi,\frac{1-\xi}{2}\varsigma+\frac{1+s}{2}\rho\right)\right|d\xi d\tau \\ & \times\left\{h\left(\frac{1+\tau}{2}\right)h\left(\frac{1-\xi}{2}\right)\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\sigma,\varsigma)\right|^q+h\left(\frac{1+\tau}{2}\right)h\left(\frac{1+\xi}{2}\right)\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\sigma,\rho)\right|^q\right. \\ & \left.+h\left(\frac{1-\tau}{2}\right)h\left(\frac{1-\xi}{2}\right)\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\phi,\varsigma)\right|^q+h\left(\frac{1-\tau}{2}\right)h\left(\frac{1+s}{2}\right)\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\phi,\rho)\right|^q d\xi d\tau\right\}^{\frac{1}{q}} \end{aligned} \tag{2.22}$$

$$\leq \left(\frac{1}{\gamma_1^{\theta+1}}\frac{1}{\gamma_2^{\vartheta+1}}B\left(\theta+1,\frac{1}{\gamma_1}\right)B\left(\vartheta+1,\frac{1}{\gamma_2}\right)\right)^{1-\frac{1}{q}}$$
$$\times\left\{\Psi(\gamma_1,\theta,h)\Phi(\gamma_2,\vartheta,h)\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\sigma,\varsigma)\right|^q+\Psi(\gamma_1,\theta,h)\Phi(\gamma_2,\vartheta,h)\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(a,d)\right|^q\right.$$
$$\left.+\ \Phi(\gamma_1,\theta,h)\Phi(\gamma_2,\vartheta,h)\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\phi,\varsigma)\right|^q+\Psi(\gamma_1,\theta,h)\Phi(\gamma_2,\vartheta,h)\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\phi,\rho)\right|^q)\right\}^{\frac{1}{q}}$$

and

$$I_{11}=\int_0^1\int_0^1\left(\frac{1-(1-\tau)^{\gamma_1}}{\gamma_1}\right)^{\theta}\left(\frac{1-(1-\xi)^{\gamma_2}}{\gamma_2}\right)^{\vartheta} \tag{2.23}$$
$$\times\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}\left(\frac{1-\tau}{2}\sigma+\frac{1+\tau}{2}\phi,\frac{1+\xi}{2}\varsigma+\frac{1-\xi}{2}\rho\right)\right|d\xi d\tau$$
$$\times\left\{h\left(\frac{1-\tau}{2}\right)h\left(\frac{1+\xi}{2}\right)\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\sigma,\varsigma)\right|^q+h\left(\frac{1-\tau}{2}\right)h\left(\frac{1-\xi}{2}\right)\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\sigma,\rho)\right|^q\right.$$
$$\left.+h\left(\frac{1+\tau}{2}\right)h\left(\frac{1+\xi}{2}\right)\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\phi,\varsigma)\right|^q+h\left(\frac{1+\tau}{2}\right)h\left(\frac{1-\xi}{2}\right)\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\phi,\rho)\right|^q d\xi d\tau\right\}^{\frac{1}{q}}$$
$$\leq \left(\frac{1}{\gamma_1^{\theta+1}}\frac{1}{\gamma_2^{\vartheta+1}}B\left(\theta+1,\frac{1}{\gamma_1}\right)B\left(\vartheta+1,\frac{1}{\gamma_2}\right)\right)^{1-\frac{1}{q}}$$
$$\times\left\{\Phi(\gamma_1,\theta,h)\Psi(\gamma_2,\vartheta,h)\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\sigma,\varsigma)\right|^q+\Phi(\gamma_1,\theta,h)\Psi(\gamma_2,\vartheta,h)\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\sigma,\rho)\right|^q\right.$$
$$\left.+\ \Phi(\gamma_1,\theta,h)\Psi(\gamma_2,\vartheta,h)\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\phi,\varsigma)\right|^q+\Phi(\gamma_1,\theta,h)\Psi(\gamma_2,\vartheta,h)\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\phi,\rho)\right|^q)\right\}^{\frac{1}{q}}.$$

Finally

$$I_{12}=\int_0^1\int_0^1\left(\frac{1-(1-\tau)^{\gamma_1}}{\gamma_1}\right)^{\theta}\left(\frac{1-(1-\xi)^{\gamma_2}}{\gamma_2}\right)^{\vartheta} \tag{2.24}$$
$$\times\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}\left(\frac{1-\tau}{2}\sigma+\frac{1+\tau}{2}\phi,\frac{1-\xi}{2}\varsigma+\frac{1+\xi}{2}\rho\right)\right|d\xi d\tau$$
$$\times\left\{h\left(\frac{1-\tau}{2}\right)h\left(\frac{1-\xi}{2}\right)\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\sigma,\varsigma)\right|^q+h\left(\frac{1-\tau}{2}\right)h\left(\frac{1+\xi}{2}\right)\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\sigma,\rho)\right|^q\right.$$
$$\left.+h\left(\frac{1+\tau}{2}\right)h\left(\frac{1-\xi}{2}\right)\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\phi,\varsigma)\right|^q+h\left(\frac{1+\tau}{2}\right)h\left(\frac{1+\xi}{2}\right)\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\phi,\rho)\right|^q d\xi d\tau\right\}^{\frac{1}{q}}$$
$$\leq \left(\frac{1}{\gamma_1^{\theta+1}}\frac{1}{\gamma_2^{\vartheta+1}}B\left(\theta+1,\frac{1}{\gamma_1}\right)B\left(\vartheta+1,\frac{1}{\gamma_2}\right)\right)^{1-\frac{1}{q}}\left(\frac{1}{4}\frac{1}{\gamma_1^{\theta+1}}\frac{1}{\gamma_2^{\vartheta+1}}\right.$$
$$\times\left\{\Phi(\gamma_1,\theta,h)\Phi(\gamma_2,\vartheta,h)\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\sigma,\varsigma)\right|^q+\Phi(\gamma_1,\theta,h)\Phi(\gamma_2,\vartheta,h)\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\sigma,\rho)\right|^q\right.$$
$$\left.+\ \Phi(\gamma_1,\theta,h)\Phi(\gamma_2,\vartheta,h)\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\phi,\varsigma)\right|^q+\Phi(\gamma_1,\theta,h)\Phi(\gamma_2,\vartheta,h)\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\phi,\rho)\right|^q\right\}^{\frac{1}{q}}.$$

By considering (2.21)-(2.24) in (2.9), The desired inequality is thus obtained. (2.19). □

Remark 2.9. *If we assign $h(t)=t$ in Theorem 4, then we have following inequality [32] :*

$$\left|\frac{\psi(\sigma,\varsigma)+\psi(\sigma,\rho)+\psi(\phi,\varsigma)+\psi(\phi,\rho)}{4}+\frac{2^{\theta-1}2^{\vartheta-1}\Gamma(\theta+1)\Gamma(\vartheta+1)}{(\phi-\sigma)^{\theta}(\rho-\varsigma)^{\vartheta}}\right. \tag{2.25}$$
$$\times\left[{}^{\gamma_1\gamma_2}I^{\theta,\vartheta}_{\sigma^+,\varsigma^+}\psi\left(\frac{\sigma+\phi}{2},\frac{\varsigma+\rho}{2}\right)+{}^{\gamma_1\gamma_2}I^{\theta,\beta}_{\phi^-,\varsigma^+}\psi\left(\frac{\sigma+\phi}{2},\frac{\varsigma+\rho}{2}\right)+{}^{\gamma_1\gamma_2}I^{\theta,\vartheta}_{\sigma^+,\rho^-}\psi\left(\frac{\sigma+\phi}{2},\frac{\varsigma+\rho}{2}\right)\right.$$

$$
\begin{aligned}
& + {}^{\gamma_1\gamma_2}I^{\theta,\vartheta}_{\phi^-,\rho^-}\psi\left(\frac{\sigma+\phi}{2},\frac{\varsigma+\rho}{2}\right)\Big]-A\Big| \\
\leq\ & \frac{(\phi-\sigma)(\rho-\varsigma)}{16}\left(\frac{1}{4}\right)^{\frac{1}{q}}\left(\frac{1}{\theta+1}\frac{1}{\vartheta+1}\right)^{1-\frac{1}{q}} \\
& \times\Bigg\{\Bigg(\left[2B\left(\theta+1,\frac{1}{\gamma_1}\right)-B\left(\theta+1,\frac{2}{\gamma_1}\right)\right]\left[2B\left(\vartheta+1,\frac{1}{\gamma_2}\right)-B\left(\vartheta+1,\frac{2}{\gamma_2}\right)\right]\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\sigma,\varsigma)\right|^q \\
& +\left[2B\left(\theta+1,\frac{1}{\gamma_1}\right)-B\left(\theta+1,\frac{2}{\gamma_1}\right)\right]\left[B\left(\vartheta+1,\frac{1}{\gamma_2}\right)\right]\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\sigma,\rho)\right|^q \\
& +\left[B\left(\theta+1,\frac{2}{\gamma_1}\right)\right]\left[2B\left(\vartheta+1,\frac{1}{\gamma_2}\right)-B\left(\vartheta+1,\frac{1}{\gamma_2}\right)\right]\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\phi,\varsigma)\right|^q \\
& +\left[B\left(\theta+1,\frac{2}{\gamma_1}\right)\right]\left[B\left(\vartheta+1,\frac{2}{\gamma_2}\right)\right]\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\phi,\rho)\right|^q\Bigg)^{\frac{1}{q}} \\
& +\Bigg(\left[2B\left(\theta+1,\frac{1}{\gamma_1}\right)-B\left(\theta+1,\frac{2}{\gamma_1}\right)\right]\left[B\left(\vartheta+1,\frac{2}{\gamma_2}\right)\right]\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\sigma,\varsigma)\right|^q \\
& +\left[2B\left(\theta+1,\frac{1}{\gamma_1}\right)-B\left(\theta+1,\frac{2}{\gamma_1}\right)\right] \\
& \times\left[2B\left(\vartheta+1,\frac{1}{\gamma_2}\right)-B\left(\vartheta+1,\frac{2}{\gamma_2}\right)\right]\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\sigma,\rho)\right|^q \quad +\left[B\left(\theta+1,\frac{2}{\gamma_1}\right)\right]\left[B\left(\vartheta+1,\frac{2}{\gamma_2}\right)\right]\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\phi,\varsigma)\right|^q \\
& +\left[B\left(\sigma+1,\frac{2}{\gamma_1}\right)\right]\left[2B\left(\vartheta+1,\frac{1}{\gamma_2}\right)-B\left(\vartheta+1,\frac{2}{\gamma_2}\right)\right]\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\phi,\rho)\right|^q\Bigg)^{\frac{1}{q}} \\
& +\Bigg(\left[B\left(\theta+1,\frac{2}{\gamma_1}\right)\right]\left[2B\left(\vartheta+1,\frac{1}{\gamma_2}\right)-B\left(\vartheta+1,\frac{2}{\gamma_2}\right)\right]\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\sigma,\varsigma)\right|^q \\
& +\left[B\left(\theta+1,\frac{2}{\gamma_1}\right)\right]\left[B\left(\vartheta+1,\frac{2}{\gamma_2}\right)\right]\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\sigma,d)\right|^q \\
& +\left[2B\left(\theta+1,\frac{1}{\gamma_1}\right)-B\left(\theta+1,\frac{2}{\gamma_1}\right)\right]\left[2B\left(\vartheta+1,\frac{1}{\gamma_2}\right)-B\left(\vartheta+1,\frac{2}{\gamma_2}\right)\right]\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\phi,\varsigma)\right|^q \\
& +\left[2B\left(\theta+1,\frac{1}{\gamma_1}\right)-B\left(\theta+1,\frac{2}{\gamma_1}\right)\right]\left[B\left(\vartheta+1,\frac{2}{\gamma_2}\right)\right]\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\phi,\rho)\right|^q\Bigg)^{\frac{1}{q}} \\
& +\Bigg(\left[B\left(\theta+1,\frac{2}{\gamma_1}\right)\right]\left[B\left(\vartheta+1,\frac{2}{\gamma_2}\right)\right]\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\sigma,\varsigma)\right|^q \\
& +\left[B\left(\theta+1,\frac{2}{\gamma_1}\right)\right]\left[2B\left(\vartheta+1,\frac{1}{\gamma_2}\right)-B\left(\vartheta+1,\frac{2}{\gamma_2}\right)\right]\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\sigma,d)\right|^q \\
& +\left[2B\left(\theta+1,\frac{1}{\gamma_1}\right)-B\left(\theta+1,\frac{2}{\gamma_1}\right)\right]\left[B\left(\vartheta+1,\frac{2}{\gamma_2}\right)\right]\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\phi,\varsigma)\right|^q \\
& +\left[2B\left(\theta+1,\frac{1}{\gamma_1}\right)-B\left(\theta+1,\frac{2}{\gamma_1}\right)\right]\left[2B\left(\vartheta+1,\frac{1}{\gamma_2}\right)-B\left(\vartheta+1,\frac{2}{\gamma_2}\right)\right]\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\phi,\rho)\right|^q\Bigg)^{\frac{1}{q}}.
\end{aligned}
$$

3. Conclusion

In this study, we derived some inequalities of trapezoid type for coordinated h-convex functions by means of conformable fractional integrals. To obtain new inequalities and to generalize the obtained inequalities, it would be useful for future work to use different types of convex maps or different types of fractional integral operators.

Declarations

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