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Production of potassium hydroxide-activated biochar and its use as a filler in polylactic acid for food packaging

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Abstract — Petroleum-containing packaging materials of the past and present have created serious ecological problems for the environment due to their resistance to biodegradation. In this context, researches have been conducted to promote the use of biodegradable films as an alternative to packaging materials. Among various biopolymers, poly(lactide) (PLA) has found application in the food industry owing to its promising properties and is currently one of the most industrially produced bioplastics. In this study, biomasses of olive pruning wastes, which are abundant in the Çanakkale region, were converted into biochar (BC) by slow pyrolysis, and their characterization was examined by adding them to PLA at different rates (5%, 10%, 15%, 20% by mass). Specific surface area analysis (BET), scanning electron microscopy (SEM) analysis, biochar yield, ash content, surface contact angles, and antimicrobial activity of film depending on the BC concentration were evaluated. As a result, potassium hydroxide (KOH) activated BC was successfully synthesized with a surface area of 1022 m^2/g. The hydrophobicity of films was improved with increasing BC ratio. Also, the film shows good antimicrobial activity toward gram-negative bacteria.

Keywords: *Biochar from olive branches, polylactic acid film, antimicrobial activity*

1. Introduction

Because petroleum-containing packing materials are resistant to biodegradation, they have caused significant ecological difficulties for the environment. The use and development of biodegradable packaging has been the general trend in the food packaging industry in recent years. Research has been done in this regard to encourage the use of biodegradable films in place of traditional packaging materials. Environmentally friendly plastics that can be substituted for petroleum-based plastics are biodegradable films [1]. Biopolymers are naturally formed by biomass and can be decomposed into their components by microorganisms in the environment. They are also defined as green polymers [2]. One of the most widely produced bioplastics today, poly(lactide) (PLA) is a biopolymer that has found use in the food sector due to its promising qualities [3]. The comparative characteristics of commercial and biobased polymers are displayed in Table 1.

Biomass has gained popularity as a renewable resource and organic solid waste in recent years. Lignocellulosic biomass can be thermochemically transformed into solid biochar. It is composed of aromatic and carbohydrate polymers, such as lignin and hemicellulose. Biochar is a porous, carbon-containing solid that is very resistant to breakdown and has a high degree of aromatization. It is created when biomass from plant or animal waste is thermally broken down without oxygen or in an atmosphere with low oxygen levels [4]. Biochar is made from biomass via a variety of thermochemical conversion processes, including carbonization, torrefaction, and

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pyrolysis. The slow pyrolysis process is the most often used of these. The biomass is broken down and transformed at a high temperature in an inert atmosphere during the pyrolysis process. Thus, the biomass forms a porous solid structure. Additionally, volatile condensable and non-condensable products are created. Liquid biofuels can be made from the condensable volatile compounds generated during the manufacture of biochar. Liquid bio-oils can be transformed into useful compounds or utilized as energy carriers in an inert atmosphere [5].

Table 1. Comparison of petroleum-derived plastic packaging and bioplastic packaging [6]

Petroleum-Based Plastics	Bio-plastic
It is usually produced from fossil fuels and petrochemicals.	Produced from natural resources.
Causes high amounts of greenhouse gases.	Causes a small amount of greenhouse gases.
Cause environmental pollution.	Environmentally friendly.
It takes centuries to break down in nature.	Full biodegradation in nature takes 3-6 months after the end of use.
Used in the production of products such as construction, textiles, bottles, shoes, food packaging, and grocery bags.	Used in areas such as biodegradable food packaging, disposable biomedical instruments, carpets, and bags.
They consist of non-renewable resources.	They consist of renewable resources.
Some of the Traditional Plastics are high-density polyethylene, low-density polyethylene, polystyrene, polyethylene terephthalate, etc.	Some of the bioplastics are polylactic acid, polyamide11, starch, and cellulose-based protein-lipid-based biopolymers.

Generally, biochar stability and properties depend on feedstock and pyrolysis conditions. Biochar produced at high temperatures has been reported to contain more fixed carbon, have predominantly stable carbon bonding, and have high thermal stability [7]. In recent years, biochar applications have received considerable attention in the fields of environment, agriculture, and industry. Biochar acts as a carbon sink in the soil, slowing down the chemical oxidation and reduction of biomass and preventing carbon emission to the atmosphere [8]. Especially activated biochar has a high adsorption capacity. Therefore, they can be used for the disposal of pollutants in wastewater [9]. There are numerous studies on the use of biochar for different applications. Sharma et al. [10] concluded that biochar can be used as activated charcoal and adsorbent in various applications by applying different activation methods to biochar. Singh et al. [9] reported that activated biochar with a high surface area can be used for phenol removal from refinery wastewater. Li et al. [11] reported that biochar produced from wood with high pyrolysis temperatures can be useful for organic carbon and nutrient retention in soil. They also stated that the biochar produced from corn husks and leaves at high pyrolysis temperatures will have a high effect on the improvement of acidic soils since the pH value is high. Mehdi et al. [12] stated that carbonaceous materials produced from biochar are suitable materials for super-capacitors due to their physical, chemical, and physicochemical properties and conducted studies on this subject.

Biochar is also effective in the use of filler components in polymers for various applications. It was reported that the addition of BC enhanced the hydrophobicity, mechanical, and sorption properties of packaging materials. Various amounts of biochar (0, 2, 4, 6, and 8% by mass) were added to the grta percha matrix to create black and biodegradable biochar/grta percha composite films. In comparison to the control film, the results demonstrated that biochar significantly enhanced the hydrophobicity and mechanical characteristics. High strength and barrier qualities were also demonstrated by the composite films. Specifically, the water vapor permeability value was lowest, and the tensile strength was highest (18.3 MPa) for the 2% BC film. After 60 days in the soil, the composite films outgrew the polyethylene (PE) film in terms of biodegradability [13]. The impact of incorporating BC (derived from pine wood) into polypropylene composites including cellulosic fibers (such as rice husk, coffee husk, coarse wool, and waste wood) was examined by Das et al. [14]. By adding BC, the flame-retardant qualities were enhanced. They claimed that using biochar from a feedstock equivalent to biomass produced the greatest outcomes [14]. According to Bartoli et al. [15], several mechanical properties of epoxy composites are tunable as a result of the production parameters of olive-based BC. Through the examination of BCs produced by various pyrolysis procedures, they were able to determine the correlation between mechanical qualities and morphology [15]. In a study performed by Idrees et al. [16], BC was mixed with polyethylene terephthalate (PET) at different ratios and used as packaging material. It was observed that a composite filament appropriate for 3D printing applications was created by melting down BC that was formed from the carbonization of starch-based packaging material. In comparison to the undoped polymer matrix, the inclusion of BC particles increased the matrix's processability, producing composite products with better mechanical and thermal properties [16]. In their 2017 study, Moustafa et al. [17] created BC particles by grinding coffee, which they then combined with a biodegradable poly (butylene adipate-co-terephthalate) (PBAT) matrix. Applications for food packaging were made of it. Despite its hydrophobicity, BC was found to enhance the thermo-mechanical characteristics of the polymer matrix [17]. Arrigo et al. [18] developed PLA-based biocomposites using BC and two distinct processing techniques: melt mixing and solvent casting. Morphological and rheological analyses were used to determine the distribution of BC particles and the degree of polymer-filler interactions. Additionally, the thermal and mechanical behavior of the biocomposites was examined in order to determine whether the process actually had an impact on the PLA's actual properties [18]. Sobhan et al. [19] developed a packaging material that works as an ammonia sensor in packaging by producing activated BC from corn cobs and adding it to the PLA matrix at different ratios. Here, the percentage of BC was increased up to 85%. The mechanical properties and natural degradation of the desired material were not examined [19].

Literature studies have shown that biochar improves many properties of films. In this study, for the first time in the literature, biochar was produced from olive branches by KOH activation and its effect on some characteristic properties of PLA film was determined.

2. Materials and Methods

2.1. Biochar Production

The BC production procedure is shown in Figure 1. For the preparation of biochar; olive branches were first dried in an oven at 105 ºC for 1 day and cut into 2 mm lengths (Figure 1a). Biochar activation follows the following steps provided in [9]:

i. mixing and grinding KOH fractions with biochar in a 3: 1 ratio biochar

ii. thermal activation of the mixture in a tube furnace, in an inert environment, at 650℃ for 1 hour at a temperature rise rate of 10℃/min (Figure 1b)

iii. soaking the mixture in 0.5 N HCl solution at room temperature overnight to remove K_2CO_3, excess KOH and other impurities

iv. removal of the mixture from the HCl solution by filtration and washing with distilled water until the pH value is stabilized

v. drying of the activated biochar in an oven.

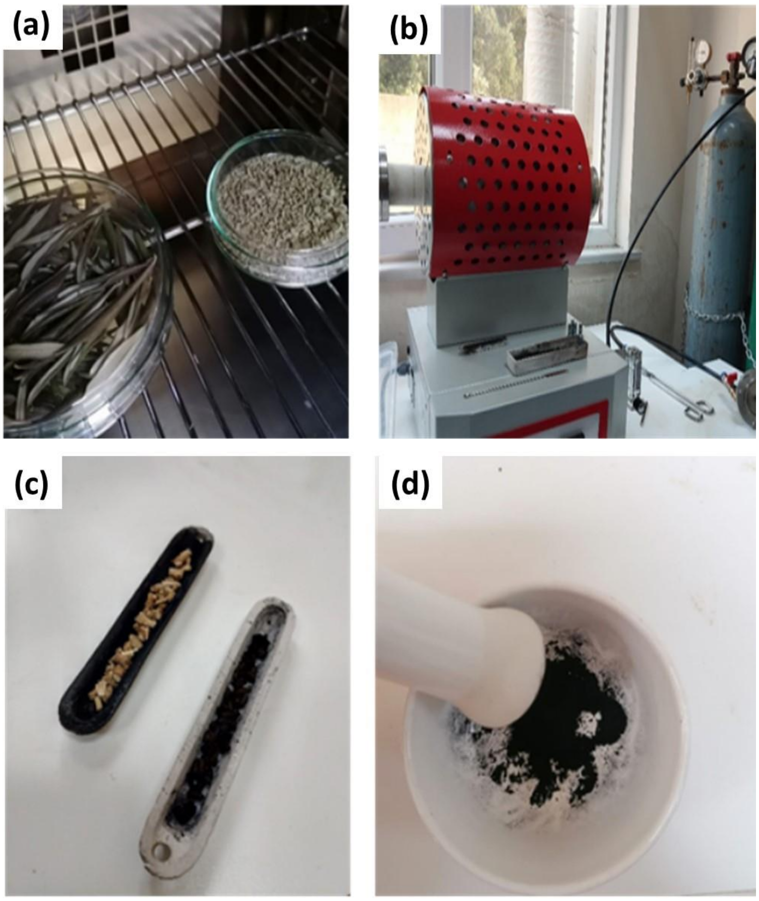

Figure 1. Biochar preparation stages

Figure 1c shows the olives branches before and after pyrolysis and Figure 1d shows the grounded particles below 10 µm.

The main reaction is given as (2.1) [20]. Potassium metals enable pore formation in biochar layers [9]. Thus, the biochar surface area is increased. Compared to physical activation, chemical activation using KOH gives biochar a higher surface area [21].

$$6KOH + C \leftrightarrow 2K + 2K_2CO_3 + 3H_2 \tag{2.1}$$

BC yield and ash content were calculated as shown in (2.2) and (2.3), respectively [22].

$$\text{BC Yield (\%)} = \frac{M_f(\text{g})}{M_i(\text{g})}\text{x}100 \tag{2.2}$$

$$\text{Ash content (\%)} = \frac{M_r(\text{g})}{M_f(\text{g})}\text{x}100 \tag{2.3}$$

where M_i and M_f are the weights of biomass and biochar before and after pyrolysis, respectively. M_r is the ash content of BC after it was kept at 850 ºC in an ash oven for 4 hours.

2.2. Film Preparation

10% PLA by mass, 90% chloroform, and 10% DMF by volume formed the polymer solution, which underwent stirring at 40 °C until it was dissolved. In a separate experiment, 0–20% biochar (by mass of polymer) was dissolved in 5 ml of DMF and then mixed for 30 minutes using a homogenizer to distribute. Following the homogenizer, the BC solution was added to the PLA solution and mixed for two hours at room temperature. The mixture was transferred onto a glass Petri dish, and the films were submerged in water to finish the phase separation. The films were then taken out and heated to dry at 60 °C.

The film preparation stage is shown in Figure 2.

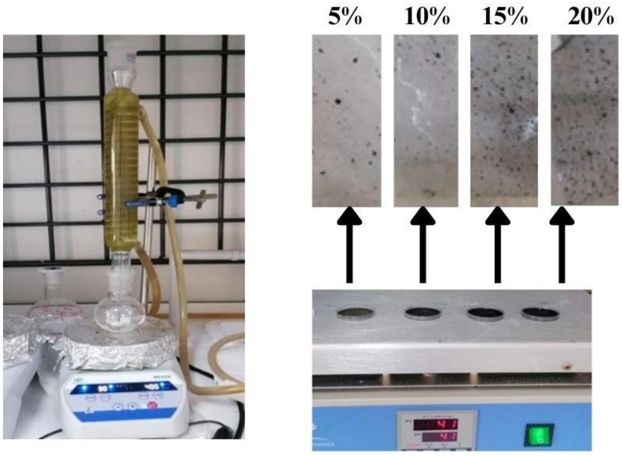

Figure 2. Film preparation stages

2.3. Characterization

The morphological analysis of the biochar produced was determined by scanning electron microscopy (SEM, JEOL JSM-7100-F). The surface hydrophobicity of the produced films was determined by contact angle tests. BC's pores were characterized using the Quadrasorb SI Brunauer-Emmett-Teller (BET) apparatus. The samples underwent an hour-long vacuum sealing and degassing process at 200°C. Adsorption of nitrogen gas was employed.

2.4. Antimicrobial Activity

Escherichia coli strain, a gram (-) bacterial strain, is used to determine antimicrobial activity. By the disk diffusion method, the antimicrobial activity of film samples against *E. coli* bacteria is observed. The bacteria to be tested are inoculated in a tryptic soy broth medium and the density is adjusted according to the 0.5 McFarland turbidity standard (108 microorganisms/ml). Spread 100 µm of the bacterial suspension on a petri dish containing Mueller Hilton agar medium with a sterile swab. 10 mm diameter samples were cut from the films, placed on the petri dishes, and incubated at 37°C for 24 hours. The test was repeated for three times.

3. Results and Discussion

3.1. Characterization

Figure 3 shows SEM analysis of biochar with and without KOH activation. As seen in Figure 3a, BC has macro pores. Activation is applied to increase the porosity and surface area of biochar. Activation is divided into two main groups: physical and chemical activation. Chemical activation can be carried out with suitable acids, bases or metal salts. In this study KOH base was used for activation. The main purpose of activation using base is to increase the surface area of the biochar and the functional groups it contains. As seen in Figure 3b, KOH activation significantly increased the porosity. This result was also supported by BET analysis.

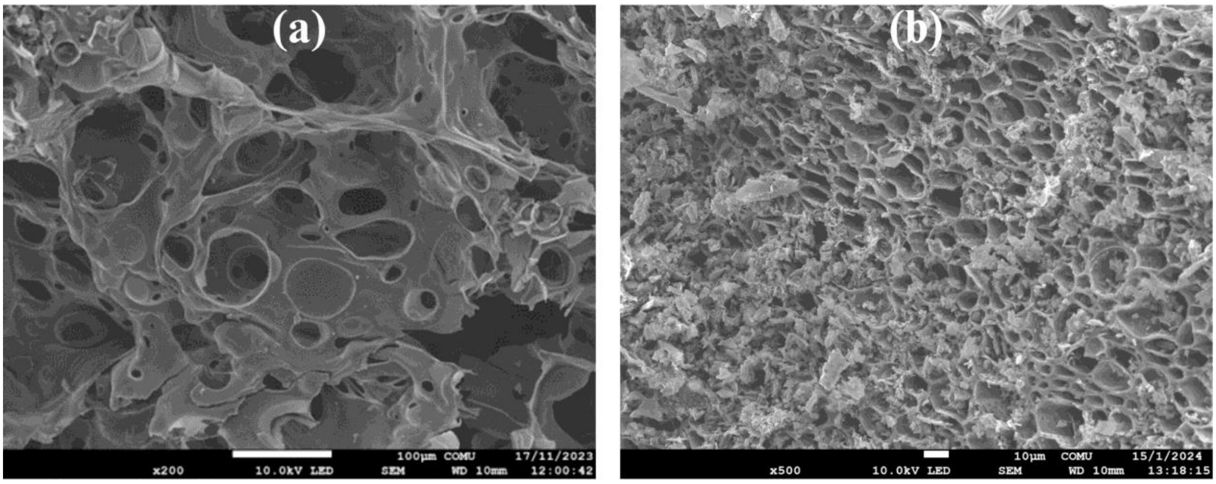

Figure 3. SEM micrographs of BC without (a) and with KOH activation (b)

Table 2 shows that the specific surface area of biochar increased from 1.477 m^2/g to 1022.201 m^2/g as a result of solid KOH activation. According to the literature, KOH provides a higher surface area to biochar compared to non-activated BC. Activation with KOH increases the specific surface area of biochar by increasing the number of mesopores [21]. There are many biochar and activated biochar studies in the literature. In order to compare the specific surface area results, biochar studies were collected as a result of a literature search (Table 2). As seen in the table, although different biomasses were used, the activation process at the same temperature increased the surface area. However, in tests with or without the same activation, the surface area increases again as the temperature increases. However, uncontrolled pyrolysis temperature increase decreases the yield. In this study, high surface area was obtained with KOH activation at relatively low temperature.

Table 2. Comparison of surface area

Raw material	Pyrolysis temperature (°C)	Activation Method	Surface area (m^2/g)	Reference
Olive Branch	**600**	**-**	**1.477**	**This study**
Olive Branch	**650**	**KOH**	**1022.201**	**This study**
Corn husk	600	-	86.750	[21]
Corn husk	600	K_2CO_3	541.910	[21]
Bamboo	600	-	307.100	[8]
Spruce Wood	600	-	465.140	[5]
Eucalyptus Branch	700	KOH	1754.000	[9]
Cone Cone	800	KOH	1714.500	[23]
Douglas Fir	700	KOH	1050	[24]

Biochar yield varies according to feedstock and pyrolysis temperature. Biochar yield is inversely proportional to the amount of volatile matter in the biomass content and directly proportional to the amount of lignin in the biomass content [8]. For pyrolysis applied to the same feedstock, biochar yield decreases as the pyrolysis temperature increases [21]. As a result of biochar yield analysis, the yield of biochar produced at 650°C pyrolysis temperature, 1 hour cooking time, 10°C/min temperature increase rate, and 1L/min N_2 flow rate was calculated as 29.75%. As a result of the literature review, Table 3 shows the comparison of the achieved yield with the literature. As can be seen in the table, the efficiency decreased in pyrolysis at higher temperature because it is a known fact that at higher temperature, more components are separated from the medium and porosity increases. In this study, lower yields were obtained from the pyrolysis process at lower temperature since it was performed at 650 ºC.

Table 3. Comparison of the yield

Raw Material	Pyrolysis Temperature (°C)	Yield (%)	Reference
Olive Branch	650	29.7	This study
Rice Husk	500	34.85	[22]
Bamboo	600	27	[8]
Eucalyptus	500	30.2	[25]
Spruce Wood	600	25.6	[5]
Wood	700	22.71	[26]

As a result of biochar ash determination, the ash content of biochar (BC-650-1) was calculated as 3.37%. As a result of the literature review, Table 4 shows the ash content ratio comparisons.

Table 4. Comparison of ash contents

Raw Material	Pyrolysis Temperature (°C)	Ash Content (%)	Reference
Olive Branch	**650**	**3.37**	**This study**
Corn husk	600	4.1	[21]
Bamboo	600	4.65	[8]
Spruce Wood	600	1.5	[5]
Duglas Fir	700	2.89	[24]

The ash contained in lignocellulosic materials consists of calcium, magnesium, and potassium carbonates and oxides [6]. Biochar produced at higher pyrolysis temperatures has a higher ash content [8]. When Table 4 is investigated, it is seen that the ash ratios of biochar produced at similar temperatures are close to each other but different. The difference in ash content is due to the variability of the inorganic matter content of different raw materials [26]. Additionally, just like the yield, high-temperature pyrolysis results in less residue because a greater proportion of the ingredients are transformed. As can be seen in the table, in this study, less residue remained from the treatment at 600 ºC, while more residue remained from the treatment at 700 ºC.

The angle formed by the tangent line at the liquid’s contact surface point and the film surface’s baseline is known as the contact angle. It is frequently employed to gauge a film’s water resistance [27]. When food is coated and exposed to water during storage, the ability of the edible coating to withstand water damage is crucial [28]. The contact angle serves as a gauge for a surface’s wettability and establishes its level of hydrophobicity. It is well known that when the contact angle rises, surface hydrophobicity does as well [29]. The Sessile Drop method was utilized to measure the contact angles of the films. Results are shown in Figure 4. As can be seen in the figure, the contact angles decreased as the BC ratio increased.

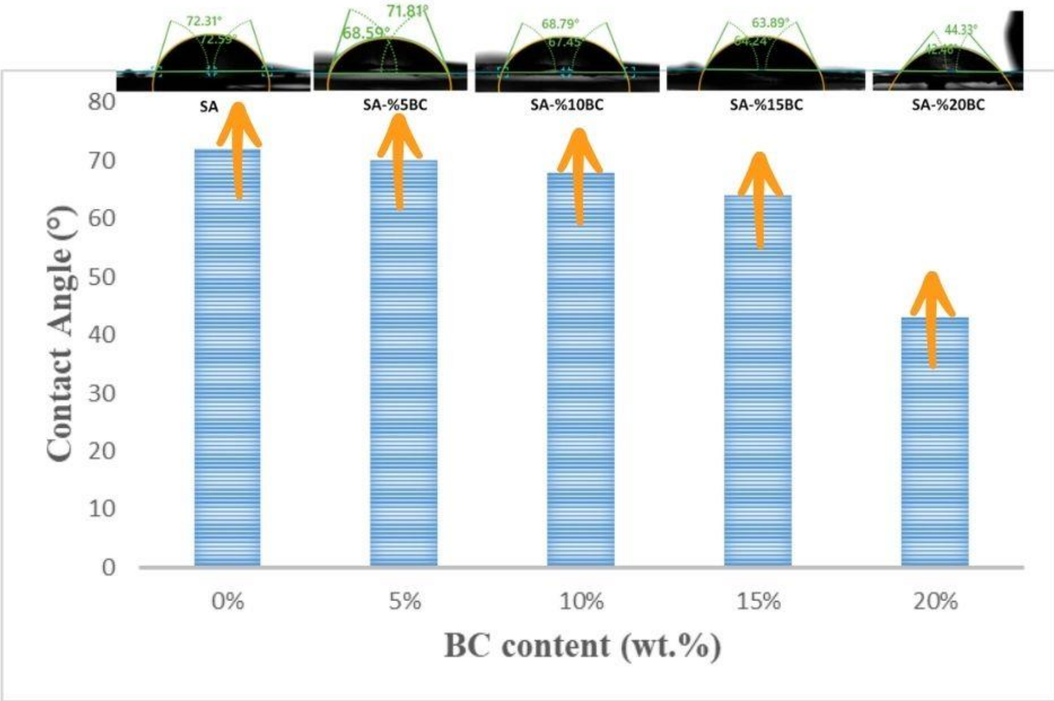

Figure 4. Effect of BC content on contact angle

According to a study by Vogler [30], hydrophobic surfaces show a contact angle greater than 65°, while hydrophilic surfaces show less than 65°. It has become widely acknowledged in recent years that a substance is considered hydrophilic when the contact angle is less than 90° and hydrophobic when it is larger than 90°. Surface chemistry and roughness are connected to surface wettability [31]. As seen in the figure, the contact angle values of the film samples were found to be in the range of 42.46-72.59˚. It is thought that PLA films containing 15% and 20% BC show hydrophilic properties [32].

The antimicrobial activity of the prepared samples was determined using the disk diffusion method. Figure 5 shows the petri dish after 24 hours of inoculation. There is no visible bacterial growth was observed when the bacteria planted on packaging films with different BC concentrations were kept in an oven under appropriate conditions. As shown in Figure 5, In films doped with 5% and 10% BC, microbial growth occurred around the film but not on it, indicating that the films are antimicrobially effective. In films doped with 15% and 20% BC, no microbial growth was observed, indicating that BC doping improves the antimicrobial properties of the films. The contribution of carbon-based materials to antimicrobial activity was described by Nishshankage et al. (2024). These processes include adsorption to bacterial and fungal cell walls via diffusion and electrostatic interactions. When BC pierces through the membranes and cell walls of bacteria and fungi, they cause cytoplasm to seep out of the cells. Additionally, when BC binds to DNA and RNA, they destroy the nucleic acid structures, which effectively stops bacteria from growing [33]. If bacterial growth had been observed, discoloration would have been observed on the parts of the packaging films where bacteria grew [34].

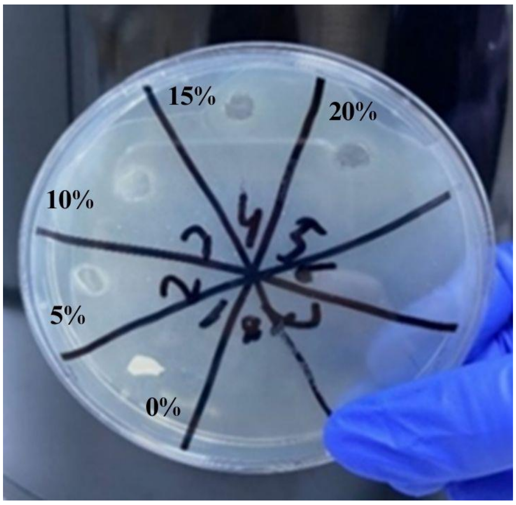

Figure 5. Antimicrobial test results after 24 hours

4. Conclusion

Avoiding plastic waste has become a necessity for a sustainable environment. The packaging industry accounts for more than 20% of the total plastics industry. Therefore, it is of great importance that food packaging is done with bio-based packaging instead of petroleum-derived packaging. In this study, biochar with high pore size was produced by KOH activation and added to the PLA matrix to form bio-based packaging film samples.

i. The results showed that the addition of BC improved the hydrophilicity of the film but reduced its resistance to water at overloading.

ii. The yield of the produced films was higher than those of the BC particles produced at the same conditions.

iii. The contact angle values of the samples were found to be in the range of 42.46-72.59˚. PLA films containing 15% and 20% BC showed hydrophilic properties, which is thought to be due to the fact that biochar has a water-affinity structure.

iv. According to the results of the antimicrobial activity test, no visible bacterial growth was observed when the bacteria planted on the BC loaded PLA packaging films. This proves that the films are antimicrobial. Antimicrobial packaging films are also extremely important for human health.

In the future, it is recommended to investigate the properties of KOH-activated films by performing basic tests necessary for food packaging.

Author Contributions

All the authors equally contributed to this work. They all read and approved the final version of the paper.

Conflict of Interest

All the authors declare no conflict of interest.

Ethical Review and Approval

No approval from the Board of Ethics is required.

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An efficient method for solving Schlömilch-type integral equations

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Abstract — Schlömilch integral equations have many applications in terrestrial physics and serve as useful tools for various ionospheric problems. Recently, researchers have investigated Schlömilch-type integral equations. Unlike Schlömilch integral equations, there are only a few works in the literature that discuss the classification and solution methods for Schlömilch -type equations. In this study, we mainly focus on introducing an efficient method based on a modified homotopy approach for solving certain Schlömilch-type equations. To demonstrate the efficiency and simplicity of the proposed algorithm, we also present some extensions that enable the solution of important application-related problems.

Keywords: *Fredholm integral equations, homotopy, perturbation theory, Rayleigh equation*

Subject Classification (2020): 45B05, 65H20

1. Introduction

The Schlömilch's integral equation plays an important role in the quasi-transverse approximations. It is considered to be one of the important equations of mathematical physics. Many researchers from various fields have been studying the equation and its solution [1–3]. There is a sufficient number of works that have been published recently on the theoretical and computational aspect of the equation [4–7]. The standard Schlömilch's integral equation admits the following form:

$$\psi(x) = \frac{2}{\pi}\int_0^{\pi/2} \phi(x\sin\theta)\,d\theta, \quad -\pi \le x \le \pi$$

where ψ is known and ϕ is the desired function. A solution for this equation has been shown to be of the following form:

$$\phi(x) = \psi(0) + x\int_0^{\pi/2} \psi'(x\sin\theta)\,d\theta$$

where the derivative is taken with respect to the argument $\eta = x\sin\theta$.

There are some other types of Schlömilch's integral equations, such as generalized Schlömilch's integral equations and nonlinear Schlömilch's integral equations. However, we will not consider them here. Instead, we refer the interested readers to [8–11].

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In this study, we aim to study Schlömilch-type integral equations. The standard linear Schlömilch-type integral equation is defined as

$$\psi(x) = \frac{2}{\pi}\int_0^{\pi/2} \phi(x\cos\theta)\,d\theta, \quad -\pi \le x \le \pi \tag{1.1}$$

where ψ is known and ϕ is the desired function [9]. Similar to what has been done in standard Schlömilch's integral equation theory, we define and examine two other forms of this equation. These are the *generalized Schlömilch-type integral equation* and the *nonlinear Schlömilch-type integral equation*. They are listed in the following, respectively:

$$\psi(x) = \frac{2}{\pi}\int_0^{\pi/2} \phi(x\cos^r\theta)\,d\theta, \quad -\pi \le x \le \pi \quad \text{and} \quad r \ge 1 \tag{1.2}$$

and

$$\psi(x) = \frac{2}{\pi}\int_0^{\pi/2} N(\phi(x\cos\theta))\,d\theta, \quad -\pi \le x \le \pi \tag{1.3}$$

where $N(\phi(x\cos\theta))$ is a nonlinear function of $\phi(x\cos\theta)$.

In addition, we also consider the equation of the form:

$$\psi(x) = \frac{2}{\pi}\int_0^{\pi/2} f(\cos(\theta))\phi(x\cos\theta)\,d\theta, \quad -\pi \le x \le \pi \tag{1.4}$$

This is an important equation in fluid dynamics. We see that it is related to the Rayleigh equation, which is considered to be one of the most important equations in fluid dynamics [12, 13].

The rest of the paper is organized as follows: Section 2 reviews the homotopy perturbation method and provides its basic properties. Section 3 considers a modification to homotopy that works for Schlömilch's integral equations. This section shows that it also works for the Schlömilch-type integral equations. In addition, the section introduces and solves a type of equation that produces important results that lead to an easy solution of the Rayleigh equation. Moreover, it applies a method based on a modification of homotopy and obtains an algorithm for the solution. Section 4 considers a special case of (1.4) to get an equation that leads to the Rayleigh equation. The final section discusses the conclusion of this study.

2. Homotopy Perturbation Method

He [14] proposed the homotopy perturbation method in 1999. Since then, it has been used for many different kinds of linear and nonlinear problems. The method is a combination of homotopy in topology and perturbation method. This method is useful in the sense that there is no need to use small parameters or any kind of linearization. The solution algorithm is very simple, and a few iterations lead to highly accurate solutions [15–17]. The homotopy perturbation method has been successfully applied to integral equations. We present recent studies showcasing various applications of homotopy methods, along with other numerical approaches to integral equations, for comparison [18–20].

We explain the method by showing its application to (1.1). Let

$$L(\phi) = \frac{2}{\pi}\int_0^{\pi/2} \phi(x\cos\theta)\,dt - \psi(x) = 0$$

with solution $\phi(x)$. The most important part is to construct a suitable homotopy with an embedding parameter $p \in [0, 1]$, which will also be used as an expanding parameter.

$$\mathcal{H}(\phi, p) = (1 - p)F(\phi) + pL(\phi)$$

where $F(\phi)$ is a functional operator with known solution, say ϕ_0. Setting the homotopy equation to zero and observing the equation as the parameter goes from zero to 1, one can easily interpret this as the trivial problem deforms to the original problem. Another important aspect of the equation is when considering the parameter as an expanding parameter as

$$\phi = \phi_0 + p\phi_1 + p^2\phi_2 + p^3\phi_3 + \cdots$$

As $p \to 1$, the series becomes an approximate solution of (1.1).

3. Modified Homotopy Perturbation Method

The main purpose and main idea for a modification to a homotopy are to accelerate the rate of convergence of the obtained series as a result of the application of the homotopy perturbation method. Thus, if a modification involves introducing a new term to a homotopy, it is expected that this new term will contribute to the rate of convergence of the obtained series [21–23]. Before delving into a modification to the homotopy, we state the following theorem.

Theorem 3.1. ψ is a polynomial function if and only if the solution of (1.1) is a polynomial function of the same degree.

The proof of this theorem is similar to the proof of a similar theorem provided in [5] for standard Schlömilch's integral equation.

Assuming ψ is a polynomial function, we define a homotopy

$$\mathcal{H}(\phi, p) = (1-p)F(\phi) + pL(\phi) + p(1-p)\sum_{k=0}^{n} q_k x^k \tag{3.1}$$

where F is a functional operator with a known solution, L is a linear operator, and q_k's are constants to be determined. We want to make a note that this homotopy works well for the Schlömilch's integral equations. We claim that it also works for the Schlömilch-type integral equations. We assume ψ is a polynomial function throughout the paper unless otherwise stated.

3.1. Schlömilch-type Integral Equation

In the subsection, consider (1.1) with ψ being a polynomial function of degree n. We start with setting the homotopy equation (3.1) equal to zero with

$$F(\phi) = \phi \text{ and } L(\phi) = \frac{2}{\pi}\int_0^{\pi/2} \phi(x\cos\theta)\, dt - \psi(x)$$

We then replace ϕ with that $\phi = \phi_0 + p\phi_1 + p^2\phi_2 + \cdots$. Equating the like terms with respect to orders of p along with setting that $\phi_2 = 0$, we end up with the following formula for q_k's.

$$q_k = a_k - \sqrt{\pi}\frac{\Gamma(\frac{k}{2}+1)}{\Gamma(\frac{k+1}{2})}a_k, \quad k = 0, 1, 2, \cdots, n$$

where $\psi(x) = \sum_{k=0}^{n} a_k x^k$. This setup produces the solution, namely

$$\phi(x) = \sum_{k=0}^{n}(a_k - q_k)x^k \tag{3.2}$$

Example 3.2. [9, 11] Consider the following Schlömilch-type integral equation

$$1 + 2x = \frac{2}{\pi}\int_0^{\pi/2} \phi(x\cos\theta)\, d\theta, \quad -\pi \le x \le \pi$$

We have $a_0 = 1$ and $a_1 = 2$. These values provide $q_0 = 0$ and $q_1 = 2 - \pi$. Using (3.2), we get the solution, which is $\phi(x) = 1 + \pi x$. It is easy to verify that this is a solution. Besides, a comparison with the other methods shows how efficient and simple it is to use this algorithm.

Example 3.3. Consider

$$x - \frac{x^3}{3} = \frac{2}{\pi}\int_0^{\pi/2} \phi(x\cos\theta)\,d\theta, \quad -\pi \le x \le \pi$$

We have $a_0 = a_2 = 0$, $a_1 = 1$, and $a_3 = -\frac{1}{3}$. Corresponding q_k values are $q_0 = q_2 = 0$, $q_1 = 1 - \frac{\pi}{2}$, and $q_3 = \frac{\pi}{4} - \frac{1}{3}$. We obtain that $\phi(x) = \frac{\pi}{2}x - \frac{\pi}{4}x^3$. By a simple substitution, one can easily verify that this is a solution.

3.2. Generalized Schlömilch-type Integral Equation

In the subsection, we consider (1.2). Since the procedure is similar to what we did in the preceding case, We will not go into details; rather, we will provide only the results. Following the similar steps done in the preceding case, we have

$$q_k = a_k - \sqrt{\pi}\frac{\Gamma(\frac{rk}{2}+1)}{\Gamma(\frac{rk+1}{2})}a_k, \quad k = 0, 1, 2, \cdots, n$$

where $\psi(x) = \sum_{k=0}^{n} a_k x^k$. The solution is

$$\phi(x) = \sum_{k=0}^{n}(a_k - q_k)x^k$$

Example 3.4. Consider

$$x + 3x^2 = \frac{2}{\pi}\int_0^{\pi/2} \phi(x\cos^2\theta)\,d\theta, \quad -\pi \le x \le \pi$$

We have $a_0 = 0$, $a_1 = 1$, and $a_2 = 3$. Corresponding q_k values are $q_0 = 0$, $q_1 = -1$, and $q_2 = -5$. We obtain that $\phi(x) = 2x + 8x^2$, which is the exact solution.

3.3. Nonlinear Schlömilch-type Integral Equation

In the subsection, we consider (1.3). Again, we will not go into details; rather, we will provide only the results by applying to an example. We require the nonlinear function N to be invertible.

Example 3.5. [9] Consider

$$\frac{4}{3\pi}x^3 = \frac{2}{\pi}\int_0^{\pi/2} \phi^3(x\cos(\theta))\,d\theta, \quad -\pi \le x \le \pi \tag{3.3}$$

We first set $N(\phi(x\cos(\theta))) = \phi^3(x\cos(\theta)) = \Omega(x\cos(\theta))$ to transform (3.3) to the standard form. In other words, (3.3) becomes

$$\frac{4}{3\pi}x^3 = \frac{2}{\pi}\int_0^{\pi/2} \Omega(x\cos(\theta))\,d\theta, \quad -\pi \le x \le \pi$$

This takes us to (1.1). Following the procedure provided for (1.1), we get $\Omega(x) = x^3$. Since $\phi^3 = \Omega$, we get $\phi(x) = x$, which is a solution to (3.3). This is exactly the same solution obtained in [9] using the Regularization-Adomian method.

Equation of the form: $\psi(x) = \frac{2}{\pi}\int_0^{\pi/2} f(\cos(\theta))\phi(x\cos\theta)\,d\theta$

We now consider (1.4)

$$\psi(x)=\frac{2}{\pi}\int_0^{\pi/2} f(\cos(\theta))\phi(x\cos\theta)\,d\theta, \quad -\pi\le x\le\pi$$

We apply the homotopy

$$\mathcal{H}(\phi,p)=(1-p)F(\phi)+pL(\phi)+p(1-p)\sum_{k=0}^{n} q_k x^k$$

with

$$F(\phi)=\phi \text{ and } L(\phi)=\frac{2}{\pi}\int_0^{\pi/2} f(\cos(\theta))\phi(x\cos\theta)\,d\theta-\psi(x)$$

Following the steps explained above, we first set $\mathcal{H}(\phi,p)=0$. This amounts

$$\mathcal{H}(f,p)=(1-p)F(\phi)+pL(\phi)+p(1-p)\sum_{k=0}^{n} q_k x^k=0 \tag{3.4}$$

where $F(\phi)=\phi$ and $L(\phi)=\frac{2}{\pi}\int_0^{\pi/2} f(\cos(\theta))\phi(x\cos\theta)\,d\theta-\psi(x)$.
Using p as an expanding parameter, we have $\phi=\phi_0+p\phi_1+p^2\phi_2+\cdots$. Substituting this into (3.4) and equating the like terms, we have

$$\begin{aligned}
\phi_0&=0\\
\phi_1&=\sum_{k=0}^{n}(a_k-q_k)x^k\\
\phi_2&=\phi_1-\frac{2}{\pi}\int_0^{\pi/2} f(\cos(\theta))\phi_1(x\cos\theta)\,d\theta+\sum_{k=0}^{n} q_k x^k\\
&=\sum_{k=0}^{n}(a_k-q_k)(1-c_k)x^k\\
\phi_{i+1}&=\phi_i-\frac{2}{\pi}\int_0^{\pi/2} f(\cos(\theta))\phi_i(x\cos\theta)\,d\theta
\end{aligned}$$

where $c_k=\frac{2}{\pi}\int_0^{\pi/2} f(\cos(\theta))\cos^k(\theta)\,d\theta$. Setting that $\phi_2=0$, we first get the q_k values and these are

$$q_k=\frac{a_k(c_k-1)}{c_k}, \quad k=1,2,\cdots,n$$

where $c_k\neq 0$. Once the q_k values are known, then the solution is provided by

$$\begin{aligned}
\phi(x)&=\sum_{k=0}^{n}(a_k-q_k)x^k\\
&=\sum_{k=0}^{n}\frac{a_k}{c_k}x^k
\end{aligned} \tag{3.5}$$

4. An Application Problem

Consider the following equation

$$\psi(x)=\frac{2}{\pi}\int_0^{\pi/2} \cos(\theta)\phi(x\cos(\theta))\,d\theta \tag{4.1}$$

This will lead to one of the most important equations in fluid dynamics. In particular, for $\psi(x)=\frac{x}{2}-\frac{x^3}{8}$, (4.1) provides the Rayleigh equation [12, 13].

We apply the algorithm that we described in the previous section. To be more precise, we assume that $f(\cos(\theta)) = \cos(\theta)$ and apply the algorithm to (4.1). It follows that the formula for $c_k = \dfrac{2}{\pi}\displaystyle\int_0^{\pi/2} f(\cos(\theta))\cos^k(\theta)\,d\theta$ becomes

$$c_k = \frac{2}{\pi}\int_0^{\pi/2} \cos^{k+1}(\theta)\,d\theta$$

This implies that

$$c_0 = \frac{2}{\pi}, \quad c_1 = \frac{1}{2}, \quad c_2 = \frac{4}{3\pi}, \quad \text{and} \quad c_3 = \frac{3}{8}$$

The solution is then

$$\phi(x) = x - \frac{x^3}{3}$$

which is obtained from (3.5). We want to point out that this is exactly the same solution of the Rayleigh equation obtained in [11] with two different methods, namely, the Homotopy analysis method and the Variational iteration method.

5. Conclusion

This study contains a set of results for solutions of the Schlömilch-type integral equations. We showed that a modification to homotopy that works for the standard Schlömilch's integral equations works well for Schlömilch-type integral equations as well. In addition, we also considered an important case that led to solving some important application problems. As a future direction, we think the algorithm introduced here can be extended and applied to solve some other important application problems. A modification to homotopy can also be redefined or improved to get better algorithms.

Author Contributions

The author read and approved the final version of the paper.

Conflicts of Interest

The author declares no conflict of interest.

Ethical Review and Approval

No approval from the Board of Ethics is required.

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Approximation theorems using the method of $\mathcal{I}_2$-statistical convergence

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Abstract — In this study, we utilize the concept of $\mathcal{I}$-statistical convergence for double sequences to establish a general approximation theorem of Korovkin-type for double sequences of positive linear operators ($PLOs$) mapping from $H_\omega(X)$ to $C_B(X)$ where $X = [0,\infty) \times [0,\infty)$. We then present an example that demonstrates the applicability of our new main result in cases where classical and statistical approaches are not sufficient. Furthermore, we compute the convergence rate of these double sequences of positive linear operators by employing the modulus of smoothness.

Keywords: *Double sequence, statistical convergence, $\mathcal{I}_2$-statistical convergence, Korovkin theorem, Bleimann-Butzer-Hahn operators*

Subject Classification (2020): 41A25, 41A36

1. Introduction

The investigation of the approximation properties exhibited by positive linear operators has emerged as a critical area of research that continues to capture the interest of scholars working in functional analysis and approximation theory. These operators play a pivotal role in extending classical results related to approximation theory and have proven to be precious tools across a wide range of mathematical disciplines. This includes computational mathematics, stochastic analysis, and studying abstract function spaces. One of the most notable contributions in this area is the development of Korovkin-type approximation theorems, which provide a robust and elegant framework for establishing criteria for convergence [1–13]. These theorems have become essential instruments in studying the behavior of sequences and families of operators, offering deep insights into the structural features of different function spaces and their topological properties.

The field of approximation theory has experienced considerable growth, particularly with the introduction of more sophisticated convergence concepts. Among these advancements is the rise of statistical convergence and, more recently, the generalization of this notion known as $\mathcal{I}$-statistical convergence. These newer frameworks provide a more comprehensive and flexible foundation for analyzing convergence phenomena, particularly in scenarios where traditional convergence methods

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may not be applicable. The strength of $\mathcal{I}$-statistical convergence lies in its ability to handle situations where classical forms of convergence fail to offer useful results. Given this enhanced flexibility in analysis, we explore the approximation properties of double sequences generated by positive linear operators through the lens of $\mathcal{I}$-statistical convergence. Specifically, we establish a novel Korovkin-type approximation theorem tailored to operators that map functions from the weighted space $H_\omega(X)$ to the space $C_B(X)$ of bounded continuous functions. Here, X denotes the unbounded domain $[0,\infty)\times[0,\infty)$, which provides the context for our study.

This study extends Korovkin-type approximation theorems to the domain of double sequences via $\mathcal{I}$-statistical convergence, establishing convergence properties beyond the scope of classical and statistical approaches. We present an analysis of convergence rates for these operators through the modulus of smoothness approximation technique. To demonstrate the practical significance of our theoretical results, we construct an illustrative example that highlights the effectiveness of the $\mathcal{I}$-statistical framework, particularly in cases where traditional methods are insufficient.

2. Preliminaries

This section provides some basic notions to be needed in the following sections. Revisit the concepts of convergence in the sense of Pringsheim, statistical convergence, and $\mathcal{I}$-statistical convergence for double sequences.

Throughout this paper, let $u=\{u_{mn}\}$ be a double sequence with real terms.

Definition 2.1. [14] A sequence $u=\{u_{mn}\}$ is called to be convergent in Pringsheim's sense if, for every $\epsilon>0$, there exists $M=M(\epsilon)\in\mathbb{N}$ such that for all $m,n>M$, the inequality $|u_{mn}-\ell|<\epsilon$ holds, where ℓ is referred as the Pringsheim limit of the sequence, denoted by $P-\lim u_{mn}=\ell$.

We shall refer to such a u as P-convergent for brevity. A double sequence is termed bounded if there is a positive constant N such that $|u_{mn}|\le N$ for every $(m,n)\in\mathbb{N}^2=\mathbb{N}\times\mathbb{N}$. It is important to highlight that, on the contrary, single sequences, a convergent double sequence is not necessarily bounded.

Definition 2.2. [15] If $G\subseteq\mathbb{N}^2$, then G_{jk} denotes the set $\{(m,n)\in G: m\le j, n\le k\}$. The double natural density of G is defined as

$$\delta_2(G):=P-\lim_{j,k}\frac{1}{jk}|G_{jk}|$$

if it exists and assume that the symbol $|.|$ indicates the cardinality of the set. The sequence $u=\{u_{mn}\}$ is statistically convergent to ℓ on the condition that for all $\epsilon>0$, the set $G:=G_\epsilon:=\{m\le j, n\le k:\ |u_{mn}-\ell|\ge\epsilon\}$ has natural density zero; i.e.,

$$P-\lim_{j,k}\frac{1}{jk}|\{m\le j,n\le k:\ |u_{mn}-\ell|\ge\epsilon\}|=0$$

in this case we indicate with $st_2-\lim u_{mn}=\ell$.

Kostyrko et al. have defined $\mathcal{I}$-convergence using the ideal $\mathcal{I}$ [16]. This type of convergence can be seen as a general form of statistical convergence.

Definition 2.3. [16] Let a class $\mathcal{I}$ of subsets of U, a non-empty set, is called an *ideal* in U iff (i) $\emptyset\in\mathcal{I}$, (ii) $E,F\in\mathcal{I}$ implies $E\cup F\in\mathcal{I}$ (additive) and (iii) for each $E\in\mathcal{I}$ and $F\subseteq E$ we have $F\in\mathcal{I}$ (hereditary).

If $\{u\}\in\mathcal{I}$ for each $u\in U$ then an ideal called *admissible.* If $\mathcal{I}$ is a *non-trivial ideal* in U (i.e. $U\notin\mathcal{I}$, $\mathcal{I}\ne\{\emptyset\}$) then the family of sets $\mathcal{F}=\{X\subseteq U:(\exists E\in\mathcal{I})(X=U\setminus E)\}$ is a *filter* in U and we call such a filter, the filter linking with the ideal $\mathcal{I}$. A non-trivial ideal $\mathcal{I}_2$ of $\mathbb{N}^2$ is called s*trongly admissible* if $\{j\} \times \mathbb{N}$ and $\mathbb{N} \times \{j\}$ belong to $\mathcal{I}_2$ for every $j \in \mathbb{N}$. It seems obvious that a strongly admissible ideal is also admissible. Let

$$\mathcal{I}_2^0 = \left\{F \subseteq \mathbb{N}^2 : (\exists m(F) \in \mathbb{N})\,(j, k \geq m(F) \Rightarrow (j,k) \notin F)\right\}$$

then $\mathcal{I}_2^0$ is a non-trivial strongly admissible ideal [17] and clearly $\mathcal{I}_2$ is strongly admissible iff $\mathcal{I}_2^0 \subseteq \mathcal{I}_2$.

Remark 2.4. Note that if $\mathcal{I}_2$ is the ideal $\mathcal{I}_2^0$ then $\mathcal{I}_2$-convergence coincides with Pringsheim convergence and if we take $\mathcal{I}_2^\delta := \left\{G \subseteq \mathbb{N}^2 : \delta_2(G) = 0\right\}$ then $\mathcal{I}_2^\delta$-convergence becomes statistical convergence.

Definition 2.5. [18] A sequence $u = \{u_m\}$ is called $\mathcal{I}-$statistically convergent to $L \in X$, if for every $\epsilon > 0$, and every $\eta > 0$,

$$\left\{j \in \mathbb{N} : \frac{1}{j}\left|\{m \leq j : |u_m - L| \geq \epsilon\}\right| \geq \eta\right\} \in \mathcal{I}$$

Definition 2.6. [2] A sequence u is called $\mathcal{I}_2$-statistically convergent to θ if for all $\epsilon > 0$ and $\eta > 0$,

$$\left\{(j,k) \in \mathbb{N}^2 : \frac{1}{jk}\ \left|\{m \leq j, n \leq k : \ |u_{mn} - \theta| \geq \epsilon\}\right| \geq \eta\right\} \in \mathcal{I}_2$$

symbolically, $\mathcal{I}_{stat}^2-\lim u = \theta$.

For the remainder of the paper, we will denote $\mathcal{I}_2$ as a non-trivial strongly admissible ideal on $\mathbb{N}^2$.

3. New Approximation Theorem

In this part, we provide an approximation theorem of Korovkin-type for double sequences of PLOs acting on two variables, mapping from $H_\omega(X)$ to $C_B(X)$ over the domain $X = [0,\infty) \times [0,\infty)$ via $\mathcal{I}_2$-statistical convergence. Furthermore, we provide an illustrative example demonstrating that our new main result holds in cases where its classical and statistical counterparts are inapplicable.

We show by $C_B(X)$ the space of all bounded and continuous real-valued functions on X. This space is equipped with the supremum norm

$$\|f\|_X = \sup_{(x,y)\in X} |f(x,y)|, \ \ (f \in C_B(X))$$

Consider the space $H_\omega(X)$ consisting of all real valued functions g on X and providing

$$|g(u,t) - g(x,y)| \leq \omega\left(\left|\frac{u}{1+u} - \frac{x}{1+x}\right|, \left|\frac{t}{1+t} - \frac{y}{1+y}\right|\right)$$

In this context, ω represents a function of the modulus of continuity type, as it fulfil the conditions, for $\delta, \delta_1, \delta_2 > 0$, as follows:

i. ω is nonnegative, increasing function on X with regard to δ_1, δ_2

ii. $\omega(\delta, \delta_1 + \delta_2) \leq \omega(\delta, \delta_1) + \omega(\delta, \delta_2)$

iii. $\omega(\delta_1 + \delta_2, \delta) \leq \omega(\delta_1, \delta) + \omega(\delta_2, \delta)$

iv. $\lim\limits_{\delta_1,\delta_2 \to 0} \omega(\delta_1, \delta_2) = 0$

Then, it is clear from (iv) that all functions belonging to $H_\omega(X)$ are continuous on X. Moreover, all functions $g \in H_\omega(X)$ fulfil the inequality

$$|g(u,t)| \leq |g(0,0)| + \omega(1,1), \ \ x, y \geq 0$$

and hence the function g is bounded on X. As a result, $H_\omega(X) \subseteq C_B(X)$.

Furthermore, we use the test functions below

$$g_0(u,t) = 1,\ g_1(u,t) = \frac{u}{1+u},\ g_2(u,t) = \frac{t}{1+t} \text{ and } g_3(u,t) = \left(\frac{u}{1+u}\right)^2 + \left(\frac{t}{1+t}\right)^2$$

The main result of the relevant section is given in the following theorem:

Theorem 3.1. Let $\{S_{mn}\}$ be a double sequence of PLOs moving from $H_\omega(X)$ into $C_B(X)$. Suppose that the following conditions are valid:

$$\mathcal{I}^2_{stat} - \lim \|S_{mn}(g_l) - g_l\|_X = 0, l \in \{0,1,2,3\} \tag{3.1}$$

Then, for any $g \in H_\omega(X)$,

$$\mathcal{I}^2_{stat} - \lim \|S_{mn}(g) - g\|_X = 0 \tag{3.2}$$

Proof. Suppose that (3.1) holds. Let $g \in H_\omega(X)$ and $(x,y) \in X$ be fixed. Since $g \in H_\omega(X)$, for all $(u,t) \in X$, we write

$$|g(u,t) - g(x,y)| \le \epsilon + \frac{2K}{\delta^2}\left\{\left(\frac{u}{1+u} - \frac{x}{1+x}\right)^2 + \left(\frac{t}{1+t} - \frac{y}{1+y}\right)^2\right\}$$

where $K := \|g\|_X$. After some easy calculations, since S_{mn}, is PLOs, we obtain

$$\begin{aligned}|S_{mn}(g;x,y) - g(x,y)| \le\ & \epsilon + \tau\{|S_{mn}(g_0;x,y) - g_0(x,y)| \\ & + |S_{mn}(g_1;x,y) - g_1(x,y)| + |S_{mn}(g_2;x,y) - g_2(x,y)| \\ & + |S_{mn}(g_3;x,y) - g_3(x,y)|\}\end{aligned}$$

where $\tau := \max\left\{\epsilon + K + \frac{2K}{\delta^2}, \frac{4K}{\delta^2}, \frac{2K}{\delta^2}\right\}$. Now, taking supremum over $(x,y) \in X$ we have

$$\|S_{mn}(g) - g\|_X \le \epsilon + \tau\sum_{l=0}^{3} \|S_{mn}(g_l) - g_l\|_X \tag{3.3}$$

For a given $\beta > 0$, choose $\epsilon > 0$ such that $\epsilon < \beta$. Then, setting

$$U := \{m \le j, n \le k : \|S_{mn}(g) - g\|_X \ge \beta\}$$

$$U_l := \left\{m \le j, n \le k : \|S_{mn}(g_l) - g_l\|_X \ge \frac{\beta - \epsilon}{4\tau}\right\}, l \in \{0,1,2,3\}$$

From (3.3), we obtain

$$U \subseteq \bigcup_{l=0}^{3} U_l$$

which gives

$$\frac{|U|}{jk} \le \sum_{l=0}^{3} \frac{|U_l|}{jk}$$

For every $\delta > 0$, we have

$$\left\{(m,n) : \frac{\left|\left\{m \le j, n \le k : \|S_{mn}(g) - g\|_X \ge \beta\right\}\right|}{jk} \ge \delta\right\} \subseteq \bigcup_{l=0}^{3}\left\{(m,n) : \frac{\left|\left\{m \le j, n \le k : \|S_{mn}(g_l) - g_l\|_X \ge \frac{\beta-\epsilon}{4\tau}\right\}\right|}{jk} \ge \frac{\delta}{3}\right\}$$

Since the ideal $\mathcal{I}_2$ possesses the properties of additivity and heredity, the proof of the theorem is thereby completed. □

By making appropriate choices, as indicated in Remark 2.4, we derive the following statistical of Theorem 3.1.

Corollary 3.2. Let $\{S_{mn}\}$ be a double sequence of PLOs moving from $H_\omega(X)$ into $C_B(X)$. Suppose that the following conditions apply:

$$st_2-\lim\left\|S_{mn}\left(g_l\right)-g_l\right\|_X=0,\ l\in\{0,1,2,3\}$$

Then, for any $g\in H_\omega\left(X\right)$,

$$st_2-\lim\left\|S_{mn}\left(g\right)-g\right\|_X=0$$

4. Application of Approximation Theorem

In the present section, we construct a sequence of PLOs that demonstrates the power of Theorem 3.1. Specifically, this sequence satisfies the hypotheses of Theorem 3.1 while failing to satisfy the conditions required by Corollary 3.2.

Example 4.1. Let us take the following Bleimann-Butzer-Hahn operators of two variables [10]:

$$L_{mn}(g;x,y)=\frac{1}{(1+x)^m(1+y)^n}\sum_{k=0}^{m}\sum_{s=0}^{n}f\left(\frac{k}{m-k+1},\frac{s}{n-s+1}\right)\binom{m}{k}\binom{n}{s}x^ky^s \tag{4.1}$$

where $g\in H_\omega\left(X\right)$, and $X=[0,\infty)\times[0,\infty)$. It is known that

$$L_{mn}(g_0;x,y)=1$$

$$L_{mn}(g_1;x,y)=\frac{m}{m+1}\frac{x}{1+x}$$

$$L_{mn}(g_2;x,y)=\frac{n}{n+1}\frac{y}{1+y}$$

$$L_{mn}(g_3;x,y)=\frac{m\left(m-1\right)}{\left(m+1\right)^2}\left(\frac{x}{1+x}\right)^2+\frac{m}{\left(m+1\right)^2}\frac{x}{1+x}+\frac{n\left(n-1\right)}{\left(n+1\right)^2}\left(\frac{y}{1+y}\right)^2+\frac{n}{\left(n+1\right)^2}\frac{y}{1+y}$$

Besides, let $E\in\mathcal{I}_2$ be infinite set and define $u=\{u_{mn}\}$ by

$$u_{mn}=\begin{cases}\frac{1}{mn}, & j-\sqrt{j}+1\le m\le j,\ k-\sqrt{k}+1\le n\le k,\ (j,k)\notin E\\ mn, & 1\le m\le j,\ 1\le n\le k,\ (j,k)\in E\\ 0, & \text{otherwise}\end{cases} \tag{4.2}$$

For every $\epsilon>0$, since $v_{jk}:=\frac{1}{jk}\left|\{(m,n):|u_{mn}-0|\ge\epsilon\}\right|\le\frac{\sqrt{j}\sqrt{k}}{jk}$ tends to zero in Pringsheim's sense and $(j,k)\notin E$, so for every $\delta>0$, we get

$$\{(i,j):v_{jk}\ge\delta\}\subseteq E\cup\left\{\left(\mathbb{N}^2\backslash E\right)\cap\left(\left(\{1,2,...,j_1\}\times\mathbb{N}\right)\cup\left(\mathbb{N}\times\{1,2,...,j_1\}\right)\right)\right\}\in\mathcal{I}_2$$

for some $j_1\in\mathbb{N}$. Hence, we obtain $\mathcal{I}^2_{stat}-\lim u=0$. Note that, $\{u_{mn}\}$ is neither statistically convergent nor classically convergent to zero. Using (4.1) and (4.2), we now define the following PLOs on $H_\omega\left(X\right)$:

$$S_{mn}\left(g;x,y\right)=\left(1+u_{mn}\right)L_{mn}\left(g;x,y\right) \tag{4.3}$$

Since $\mathcal{I}^2_{stat}-\lim u=0$, we can show that the sequence $\{S_{mn}\}$, as defined by (4.3) satisfies all conditions of Theorem 3.1. Consequently, for all $g\in H_\omega\left(X\right)$, we have

$$\mathcal{I}^2_{stat}-\lim\left\|S_{mn}\left(g\right)-g\right\|_X=0$$

Since u is neither classically convergent nor statistically convergent to zero, the sequence $\{S_{mn}(g)\}$ cannot convergence to g on X in the usual or statistical sense.

5. Rate of $\mathcal{I}_2$-statistical Convergence

The rate of convergence for double sequences of PLOs, in terms of $\mathcal{I}_2$-statistical convergence, is described by the modulus of smoothness. Define the following modulus of smoothness for the bivariate case, similarly to the one in [19], (see [20]):

$$\overline{\omega}_2(g;\delta_1,\delta_2)=\sup\left\{|g(u,t)-g(x,y)|:(u,t),(x,y)\in X \text{ and } \left|\frac{u}{1+u}-\frac{x}{1+x}\right|\leq\delta_1,\left|\frac{t}{1+t}-\frac{y}{1+y}\right|\leq\delta_2\right\} \quad (5.1)$$

where $\delta_1,\delta_2>0$. It is clear that if $g\in H_\omega(K)$ then, we have

i. $\lim\limits_{\delta_1,\delta_2\to 0}\bar{\omega}_2(g;\delta_1,\delta_2)=0$

ii. $|g(u,t)-g(x,y)|\leq\bar{\omega}_2(g;\delta_1,\delta_2)\left(1+\frac{\left|\frac{u}{1+u}-\frac{x}{1+x}\right|}{\delta_1}\right)\left(1+\frac{\left|\frac{t}{1+t}-\frac{y}{1+y}\right|}{\delta_2}\right)$

Theorem 5.1. Let $\{S_{mn}\}$ be a double sequence of PLOs moving from $H_\omega(X)$ into $C_B(X)$. Suppose that the following conditions are valid:

i. $\mathcal{I}^2_{stat}-\lim\|S_{mn}(g_0)-g_0\|_X=0$

ii. $\mathcal{I}^2_{stat}-\lim\bar{\omega}_2(g;\gamma_{mn},\eta_{mn})=0$

where $\gamma_{mn}:=\sqrt{\left\|S_{mn}\left(\left(\frac{u}{1+u}-\frac{\cdot}{1+\cdot}\right)^2\right)\right\|_X}$ and $\eta_{mn}:=\sqrt{\left\|S_{mn}\left(\left(\frac{v}{1+v}-\frac{\cdot}{1+\cdot}\right)^2\right)\right\|_X}$

Then, for any $g\in H_\omega(X)$,

$$\mathcal{I}^2_{stat}-\lim\|S_{mn}(g)-g\|_X=0$$

Proof. Let $g\in H_\omega(X)$ and $(x,y)\in X$ be fixed. Since S_{mn} is linear and positive and also, thanks to (5.1),

$$|S_{mn}(g;x,y)-g(x,y)|\leq\tau|S_{mn}(g_0;x,y)-g_0(x,y)|+S_{mn}(|g(u,t)-g(x,y)|;x)$$

where $\tau:=\|g\|_X$. We get, with the help of the Cauchy-Schwartz inequality,

$$\begin{aligned}
|S_{mn}(g;x,y)-g(x,y)| &\leq \tau|S_{mn}(g_0;x,y)-g_0(x,y)| \\
&\quad +S_{mn}\left(\bar{\omega}_2(g;\delta_1,\delta_2)\left(1+\frac{\left|\frac{u}{1+u}-\frac{x}{1+x}\right|}{\delta_1}\right)\left(1+\frac{\left|\frac{t}{1+t}-\frac{y}{1+y}\right|}{\delta_2}\right);x,y\right) \\
&\leq \tau|S_{mn}(g_0;x,y)-g_0(x,y)| \\
&\quad +\bar{\omega}_2(g;\delta_1,\delta_2)\left\{S_{mn}(g_0;x,y)+\frac{1}{\delta_1}S_{mn}\left(\left|\frac{u}{1+u}-\frac{x}{1+x}\right|;x,y\right)\right. \\
&\quad +\frac{1}{\delta_2}S_{mn}\left(\left|\frac{t}{1+t}-\frac{y}{1+y}\right|;x,y\right) \\
&\quad \left.+\frac{1}{\delta_1\delta_2}S_{mn}\left(\left|\frac{u}{1+u}-\frac{x}{1+x}\right|\left|\frac{t}{1+t}-\frac{y}{1+y}\right|;x,y\right)\right\} \\
&\leq \bar{\omega}_2(g;\delta_1,\delta_2)\left\{S_{mn}(g_0;x,y)+\frac{1}{\delta_1}\sqrt{S_{mn}\left(\left(\frac{u}{1+u}-\frac{x}{1+x}\right)^2;x,y\right)}\sqrt{S_{mn}(g_0;x,y)}\right\} \\
&\quad +\frac{1}{\delta_2}\sqrt{S_{mn}\left(\left(\frac{t}{1+t}-\frac{y}{1+y}\right)^2;x,y\right)}\sqrt{S_{mn}(g_0;x,y)} \\
&\quad +\frac{1}{\delta_1\delta_2}\sqrt{S_{mn}\left(\left(\frac{u}{1+u}-\frac{x}{1+x}\right)^2;x,y\right)}\sqrt{S_{mn}\left(\left(\frac{t}{1+t}-\frac{y}{1+y}\right)^2;x,y\right)} \\
&\quad +\tau|S_{mn}(g_0;x,y)-g_0(x,y)|
\end{aligned}$$

Taking supremum over $(x,y)\in X$, we obtain

$$\begin{aligned}\left\|S_{mn}(g)-g\right\|_X \leq\ & \bar{\omega}_2(f;\delta_1,\delta_2)\left\|S_{m,n}(g_0)-g_0\right\|_X \\ & +\bar{\omega}_2(f;\delta_1,\delta_2)\frac{1}{\delta_1}\sqrt{\left\|S_{mn}\left(\left(\frac{u}{1+u}-\frac{\cdot}{1+\cdot}\right)^2\right)\right\|_X}\sqrt{\left\|S_{mn}(g_0)\right\|_X} \\ & +\bar{\omega}_2(f;\delta_1,\delta_2)\frac{1}{\delta_2}\sqrt{\left\|S_{mn}\left(\left(\frac{v}{1+v}-\frac{\cdot}{1+\cdot}\right)^2\right)\right\|_X}\sqrt{\left\|S_{mn}(g_0)\right\|_X} \\ & +\bar{\omega}_2(f;\delta_1,\delta_2)\frac{1}{\delta_1\delta_2}\sqrt{\left\|S_{mn}\left(\left(\frac{u}{1+u}-\frac{\cdot}{1+\cdot}\right)^2\right)\right\|_X}\sqrt{\left\|S_{mn}\left(\left(\frac{v}{1+v}-\frac{\cdot}{1+\cdot}\right)^2\right)\right\|_X} \\ & +\tau\left\|S_{mn}(g_0)-g_0\right\|_X+\bar{\omega}_2(f;\delta_1,\delta_2)\end{aligned}$$

Let $\delta_1 := \gamma_{mn} := \sqrt{\left\|S_{mn}\left(\left(\frac{u}{1+u}-\frac{\cdot}{1+\cdot}\right)^2\right)\right\|_X}$ and $\delta_2 := \eta_{mn} := \sqrt{\left\|S_{mn}\left(\left(\frac{v}{1+v}-\frac{\cdot}{1+\cdot}\right)^2\right)\right\|_X}$. Thus,

$$\begin{aligned}\left\|S_{mn}(g)-g\right\|_X \leq\ & \bar{\omega}_2(f;\delta_1,\delta_2)\left\|S_{mn}(g_0)-g_0\right\|_X \\ & +2\bar{\omega}_2(f;\delta_1,\delta_2)\sqrt{\left\|S_{mn}(g_0)\right\|_X} \\ & +\tau\left\|S_{mn}(g_0)-g_0\right\|_X+2\bar{\omega}_2(f;\delta_1,\delta_2)\end{aligned}$$

As a result, by the conditions of the theorem, for any $g\in H_\omega(X)$,

$$\mathcal{I}^2_{stat}-\lim\left\|S_{mn}(g)-g\right\|_X=0$$

□

6. Conclusion

This paper introduces a novel perspective on Korovkin-type approximation in $H_\omega(X)$. We develop a new approximation theorem by combining $\mathcal{I}$-statistical convergence with test functions such as $g_0(u,v)=1$, $g_1(u,v)=\frac{u}{1+u}$, $g_2(u,v)=\frac{v}{1+v}$ and $g_3(u,v)=\left(\frac{u}{1+u}\right)^2+\left(\frac{v}{1+v}\right)^2$. This approach significantly improves over conventional methods, particularly in its enhanced ability to handle approximations in unbounded domains. The theoretical advancements are further demonstrated through a concrete example, highlighting the practical relevance of the proposed framework. Later, we focus on the speed of convergence, utilizing the modulus of smoothness to achieve this. Our findings suggest that this method opens new avenues for approximation theory, potentially expanding the reach of Korovkin-type theorems and facilitating their application to more complex operators and diverse settings. Future research could build on these results by exploring the method's applicability to a broader range of function spaces and examining its impact in other areas of analysis.

Author Contributions

All the authors equally contributed to this work. They all read and approved the final version of the paper.

Conflicts of Interest

All the authors declare no conflict of interest.

Ethical Review and Approval

No approval from the Board of Ethics is required.

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Selection of third-party logistics provider based on extended multimoora technique under double hierarchy linguistic single-valued neutrosophic set

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Abstract — The demand for third-party logistics (3PL) providers becomes an increasingly important issue for corporations seeking improved customer service and cost reduction. Currently, there is no way to select the appropriate method for selecting 3PL. Therefore, this paper develops a new extended multi-objective optimization ratio analysis plus full multiplicative form (MULTIMOORA) method under double hierarchy linguistic single-valued neutrosophic sets (DHLSVNSs). For this, we first develop a new mathematical tool, i.e., DHLSVNSs, by studying single-valued neutrosophic set (SVN) and double hierarchy linguistic term set (DHLTSs), which is very effective for solving uncertainty in decision-making problems. A list of Einstein aggregation operators and their fundamental aspects for DHLSVNSs are presented based on Einstein's norms, as aggregation operators play an essential role in decision-making. A step-by-step algorithm of the Extended DHLSVN-MULTIMOORA approach is designed to tackle ambiguous and uncertain data during decision-making problems. The algorithm developed for the suggested technique is illustrated with a numerical example relevant to 3PL. A comparison of the proposed methods with various existing methodologies is carried out to demonstrate the superiority of the suggested algorithms.

Keywords: *Double hierarchy linguistic single-valued neutroshopic set, multimoora technique, multi criteria group decision making problems (MCGDM), aggregation operators, third-party logistics provider*

1. Introduction

Logistics plays an important role in establishing an industry's supply chain. However, with the market becoming increasingly global, industries now see logistics as a critical area where they may reduce costs and raise the standard of their customer service [1]. Logistics outsourcing, often known as third-party logistics (3PL), is a growing trend in the global business sector [2]. According to [3,4], suppliers can provide enterprises with the necessary services, including professional logistics and transportation, warehousing, logistics information systems, product return services, and inventory management. As a result, 3PL plays an important part in the logistical activities between the outsourced firm, the marketplace, and the customers. The key advantages of logistics alliances are that they allow the outsourced firm to focus on its core competencies, increase efficiency, improve service, eliminate transportation costs, restructure supply chains, and build market credibility [5–7].

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Consequently, selecting a suitable 3PL provider who can meet various requirements is critical for an enterprise's growth and competency [8–10]. In the provider selection process, the outsourcing firm frequently faces difficulties working with many logistics suppliers. Analysts are faced with a challenging task in selecting suitable suppliers. To address this challenge, Multi-Attribute Group Decision Making (MAGDM) is a process that plays an important role in selecting the best solution. There are two key goals in this method. While the primary goal is to create an environment in which the value of some basic criteria can be easily assessed, the second goal is to analyze data that is often unclear or ambiguous. To manage these related data, researchers developed the Fuzzy Set (FS) [11] theory, which provides information for managing the imprecision and inaccuracy of information by assigning membership degrees to each element of a fixed set. However, this idea has limitations due to the lack of non-membership; therefore, Atanassov [12] extended FS by including non-membership and developing a new theory called intuitionistic fuzzy set (IFS). Many scholars have used this theory to solve various DM problems. However, it has been suggested that they cannot handle the ambiguous and contradictory information that occurs in reality. Therefore, Smarandache [13] developed a new theory of neutrosophic sets (NS), which describes uncertain data by considering three mutually independent functions, namely true, uncertain, and false lying in $]0^-, 1^+[$.

NS theory can represent unclear data better than FS, IFS, and uncertainty speculation because it is consistent with human instinct judgments and feelings. The NS theory deals with indeterminate information, but it isn't easy to enforce in practical situations. Thus Wang et al. [14] developed a special type of NS, namely a single-valued neutrosophic set (SVNS), to handle real-world problems easily. SVNS is a helpful tool for representing situations with incomplete, uncertain, and inconsistent information. Some scholars have studied SVNS and defined various aggregation operators (AO) for SVNS. Aggregating data from several sources into a single AO is important. As a result, Li et al. [15] introduced the innovative concept of generalized simplified neutrosophic Einstein AOs. For SVNSs, Liu [16] proposed AOs based on the Archimedean t-norm and t-conorm and applied them to decision-making problems. Ji et al. [17] concentrated on the Frank operations of SVNNs and created the SVN prioritized Bonferroni mean (SVNFNPBM) operator according to the Frank aggregation function. Nancy and Garg [18] established SVNN operations as Frank-weighted aggregation operators and suggested a decision-making framework. Biswas et al. [19, 20] utilized the TOPSIS method for decision-making problems under the SVN environment. Considering the neutrosophic set, Zhang et al. [21] developed the general cloud method and other related ideas, such as backward cloud generators, two aggregation operators, and NNC distance measure. Lu and Ye [22] introduced hybrid weighted arithmetic and geometric aggregation functions under SVN information and used these operators to build decision-making challenges. Baser and Uluçay [23] defined an effective Q-neutrosophic soft sets and Its Application in Decision Making. Baser and Uluçay [24] also defined the applications of neutrosophic soft set decision-making problems. Reseacher [25] designed a TOPSIS-Based Entropy Measure for N-Valued Neutrosophic Trapezoidal Numbers. Uluçay et al. [26] defined a prioritized aggregation operators for the Evaluation of renewable energy sources.

Later, Uluçay and Deli [27] invented a novel vikor method for generalized trapezoidal hesitant fuzzy numbers. Uluçay et al. [28] proposed a N-valued neutrosophic trapezoidal numbers with similarity measures. The knowledge evaluated in the neutrosophic environment is quantitative and is represented numerically. However, in practice, most unclear or ambiguous data examined by the decision-maker (DMs) have qualitative aspects, for example, extremely poor, poor, fair, slightly good, very good, outstanding. The linguistic variable [29] is essential to access information and process qualitative data in these situations. Therefore, Li et al. [30] considered three membership degrees, such as truth, indeterminacy, and falsity, in the form of linguistic variables and developed the linguistic neutrosophic sets (LNS). Since LNS is particularly suitable for representing more complex linguistic information predicted by humans. Later, Gou et al. [31,32] extended the LTS and developed a new theory, namely double hierarchy linguistic term set (DHLTS), for more robust modeling of expert expressions. In most real-world problems, getting the correct reflection of attributes is very difficult for DMs. Besides that, most DMs find it more suitable to conduct qualitative evaluations of attributes. DHLTS conveys appropriate data more conveniently in complex expressions than single LTSs. DHLTS consists of two components, the first and second hierarchy linguistic term sets, allowing more flexibility in describing uncertainty and ambiguity. Many researchers successfully applied this concept [33].

Saleem et al. [34] proposed double hierarchy hesitant linguistic Einstein aggregation operators to solve real word problems. Multi-attribute group decision-making (MAGDM) is a procedure in which a panel of decision experts assesses the most advantageous alternative that supports certain features. To solve the MAGDM problem, Brauers and Zavadskas developed MOORA and MULTIMOORA techniques [35]. The traditional MOORA technique uses precise data to assess the preferred alternative based on the relative significance of many criteria. Later, Brauers and Zavadskas [36] invented the MULTIMOORA technique by considering crisp set theory, which is based on three approaches: (1) ratio system approach, (2) reference point approach, and (3) full multiplicative form approach. The MULTIMOORA technique is one of the most important techniques for handling real-world problems more significantly. Thus, Brauers et al. [37] extended the MULTIMOORA method fuzzy set. In the context of FSs, Hafezalkotob et al. [38] perform a summary of the MULTIMOORA approach. Alkan et al. [39] used the fuzzy MULTIMOORA approach to rank renewable energy sources. Liang et al. [40] investigated the MULTIMOORA approach in picking mining methods. Fattahi and Khalilzadeh expanded the MULTIMOORA approach for risk evaluation in a fuzzy environment [41]. Based on the objective weighting technique, Dahooie [42] enhanced the fuzzy MULTIMOORA technique. Zhang et al. [43] integrated the suggested intuitionistic fuzzy MULTIMOORA method and applied it to energy storage technologies selection. To assess solid waste management strategies, Garg and Rani [44] presented a MULTIMOORA method involving aggregation operators under IFS. Later, Chen et al. [45] extended the MULTIMOORA method to linguistic evaluations. Zhang [46] considered the MULTIMOORA method to solve decision-making problems in a linguistic intuitionistic fuzzy environment. Ding and Zhong [15] introduced the MULTIMOORA approach using two-dimensional uncertain linguistic variables (TDULVs). Balezentis and Balezentis [47] suggested a 2-tuple linguistic fuzzy MULTIMOORA technique. Wei [48] proposed the 2-tuple linguistic intuitionistic FSs (2TLIFSs). Akram et al. [49] recently created the 2TLPF-MULTIMOORA technique.

In light of the above literature, it is analyzed that there is no application and detail about the combined study of SVNSs and DHLTSs for handling the uncertainty and fuzziness under the MULTIMOORA technique. So, Inspired by the above discussion in this study, we define a new theory, namely double hierarchy linguistic single-valued neutrosophic sets (DHLSVNSs), to evaluate decision-making problems more accurately. The main motivations for this work are as follows:

i. To develop a novel notion of double hierarchy linguistic single-valued neutrosophic sets (DHLSVNSs) by extending the SVNs to DHLTSs. DHLSVNSs is a more generalized version that effectively resolves ambiguity in decision-making problems. They are adaptable tools that allow DMs to provide assessments in the form of DHLSVNSs.

ii. To define new operational laws for DHLSVNSs based on Einstein t-norm and t-conorms.

iii. To develop a list of Einstein aggregation operators and discuss the related properties.

iv. To create an extended MULTIMOORA technique for solving decision-making problems. Because the MULTIMOORA approach has several standout qualities, including shorter computation times, more thorough mathematical calculations, simplicity, and consistency of outcomes. It comprises of three MOORA techniques utilize full multiplicative form, reference point, and ratio analysis. It uses aggregation algorithms to incorporate the subordinate ranks of the alternatives and more correctly reflect the results.

v. In order to show the applicability and dependability, we applied the provided methodology, a numerical case study selecting a third party logistic service provider.

vi. To demonstrate the stability and validity of our developed work, we compare our proposed technique to previous methods.

From literature the existence idea are very helpful to solve decision making problems, but this ideas only handle the decision making problems qualitatively. Therefore, the novelty of this paper is to developed a novel idea for solving decision making qualitatively. The main focus of this study is to find the best third party logistic service provider that can assist in real-life problems. This research study has contributed to the analysis of MCGDM under ambiguity in the following manner:

i. We design a new operation for DHLSVNSs to handle the decision issues more accurately.

ii. Einstein t-norm and t-conorm have great significance as they incorporate the properties of several others. Therefore, to aggregate the DM process, we define the aggregation operators and basic operation of DHLSVNSs based on the Einstein t-norm . We introduce a variety of aggregation operators such as the Einstein weighted averaging and geometric aggregation operator and thier basic properties.

iii. A new DHLSVN-Multimoora method is developed to handle complex decision making problems under DHLSVNSs.

iv. A novel distance measure and score function is proposed for finding the the ranking and distance between tow different double hierarchy linguistic neutrosophic numbers.

The summary of this article is as follows: The basic concepts related to SVNSs, LTSs and DHLTSs are given in Section 2. Section 3 includes the novel notion of DHLSVNSs and score function, which can help the DM process. Section 4 includes the distance measures and Einstein aggregation operators of DHLSVNSs. Section 5 presents a step-wise algorithm for Extended MULTIMOORA method under a double hierarchy linguistic single-valued neutrosophic context. Section 6 describes a numerical application related to third-party logistic selection. Section 7 compares the proposed method with existing techniques to demonstrate its applicability. Section 8 concludes this article. The list of abbreviations and symbols are given in Table 1.

Table 1. List of abbreviations and symbols

Description	**Abbreviation**
Truth membership degree	$\mu(a)$
Indeterminacy membership degree	$\eta(a)$
Falsity membership degree	$\nu(a)$
Linguistic single-valued neutrosophic number	LSVNNs
Linguistic truth membership degree	$\beth^L_\mu(a)$
Linguistic Indeterminacy membership degree	$\beth^L_\eta(a)$
Linguistic Falsity membership degree	$\beth^L_\nu(a)$

Table 1. (Continued) List of abbreviations and symbols

Description	**Abbreviation**
Single-valued neutrosophic set	SVNS
Neutrosophic set	NS
First hierarchy linguistic term	FHLT
Second hierarchy linguistic term	SHLT
Double hierarchy linguistic term sets	DHLTSs
Double hierarchy linguistic single-valued neutrosophic set	DHLSVNS
Double hierarchy linguistic single-valued neutrosophic numbers	DHLSVNNs
Double hierarchy linguistic single-valued neutrosophic Einstein weighted averaging	DHLSVNEWA
Double hierarchy linguistic single-valued neutrosophic Einstein arithmetic	DHLSVNEA
Double hierarchy linguistic single-valued neutrosophic Einstein weighted geometric	DHLSVNEWG
Double hierarchy linguistic single-valued neutrosophic Einstein geometric	DHLSVNEG

2. Preliminaries

This section provides some basic notions to be required in the following sections.

Definition 2.1. [13] Let $X \neq \phi$. The structure $N = \{a, \langle \mu(a), \eta(a), \nu(a)|a \in X\rangle\}$ is called neutrosophic set (NS), where for each $a \in X$, $\mu(a), \eta(a), \nu(a) : X \longrightarrow \,]0^-, 1^+[$ is the truth, indeterminacy and falsity membership degree respectively, with conditions $0^- \leq (\mu(a)) + (\eta(a)) + (\nu(a)) \leq 3^+$.

Definition 2.2. [14] For a non empty set X. The single-valued neutrosophic set (SVNS) is mathematically denoted by $V = \{a, \langle \mu(a), \eta(a), \nu(a)|a \in X\rangle\}$, where $a \in X$, $\mu(a), \eta(a), \nu(a) : X \longrightarrow [0, 1]$ represents the truth, indeterminacy and falsity membership degree respectively subject to the conditions $0 \leq (\mu(a)) + (\eta(a)) + (\nu(a)) \leq 3$,

Definition 2.3. [50] Let $\beth = \{\beth_\gamma|\gamma = 0, 1, \cdots, \tau\}$, is the linguistic term with odd cardinality and $\beth^{[0,\tau]} = \{\beth_\gamma|\beth_0 \leq \beth_\gamma \leq \beth_\tau, \gamma \in [0, \tau]\}$ is the continuous linguistic term set. Then the structure $\beth = \{a, \beth^L_\mu(a), \beth^L_\eta(a), \beth^L_\nu(a)|a \in X\}$ is known as linguistic single-valued neutrosophic number (LSVNNs), where for each $a \in X$, $\beth^L_\mu(a), \beth^L_\eta(a), \beth^L_\nu(a) \in \beth^{[0,\tau]}$ represent the truth, indeterminacy and falsity linguistic degree respectively, such that $0 \leq \mu(a) + \eta(a) + \nu(a) \leq 3\tau$. The tripled $\langle \beth_\mu, \beth_\eta, \beth_\nu\rangle$ is said to LSVNNs and denoted by $\beth = \langle \beth_\mu, \beth_\eta, \beth_\nu\rangle$. If $\beth_\mu, \beth_\eta, \beth_\nu \in \beth$, then tripled $\langle \beth_\mu, \beth_\eta, \beth_\nu\rangle$ are the original LSVNNs.

Definition 2.4. [31] Let $\beth = \{\beth_\gamma|\gamma = -\tau, \cdots, -1, 0, 1, \cdots, \tau\}$ be the first (FHLT) and $\daleth = \{\daleth_\varphi|\varphi = -\delta, \cdots, -1, 0, 1, \cdots, \delta\}$ be the second hierarchy linguistic term (SHLT) sets, then the double hierarchy linguistic term sets is symbolically denoted by

$$\beth\daleth = \left\{\beth_{\gamma\langle\daleth_\varphi\rangle}|\theta = -\tau, \cdots, -1, 0, 1, \cdots, \tau; \varphi = -\delta, \cdots, -1, 0, 1, \cdots, \delta\right\}$$

where $\beth_\gamma$ is the first hierarchy and $\daleth_\varphi$ represent the second hierarchy linguistic terms, respectively.

3. Formation of Double Hierarchy Linguistic Single-Valued Neutrosophic Sets

This section explores the novel notion of double hierarchy linguistic single-valued neutrosophic sets on the base of [31, 32].

Definition 3.1. Let $\gimel = \left\{ \left\langle \gimel^L_\mu(a), \gimel^L_\eta(a), \gimel^L_\nu(a) | \mu, \eta, \nu = 0, 1, \cdots, \tau \right\rangle \right\}$ be the first hierarchy linguistic single-valued neutrosophic sets and

$$\daleth = \left\{ \left\langle \daleth^L_w(a), \daleth^L_x(a), \daleth^L_y(a) | w, x, y = 0, 1, \cdots, \delta \right\rangle \right\}$$

be the second hierarchy linguistic single-valued neutrosophic sets, then the double hierarchy linguistic single-valued neutrosophic sets (DHLSVNSs) is defined as:

$$\gimel\daleth = \left\{ \left\langle \gimel^L_{\mu\langle \daleth^L_w \rangle}, \gimel^L_{\eta\langle \daleth^L_x \rangle}, \gimel^L_{\nu\langle \daleth^L_y \rangle} \right\rangle | \mu, \eta, \nu \in [0, \tau] ; w, x, y \in [0, \delta] \right\} \tag{3.1}$$

Where $\gimel^L_\mu, \gimel^L_\eta, \gimel^L_\nu \in \gimel$ represents the truth, indeterminacy and falsity degree of first hierarchy linguistic term sets and $\daleth^L_w, \daleth^L_x, \daleth^L_y \in \daleth$ is the truth, indeterminacy and falsity degree of second hierarchy linguistic term sets, such that $0 \leq \mu + \eta + \nu \leq 3\tau$ and $0 \leq w + x + y \leq 3\delta$. Simply it can be represented as

$$\gimel\daleth = \left\langle \gimel_{\mu\langle \daleth_w \rangle}, \gimel_{\eta\langle \daleth_x \rangle}, \gimel_{\nu\langle \daleth_x \rangle} \right\rangle$$

Definition 3.2. Let $\gimel\daleth_i = \left\langle \gimel_{\mu_i\langle \daleth_{w_i} \rangle}, \gimel_{\eta_i\langle \daleth_{x_i} \rangle}, \gimel_{\nu_i\langle \daleth_{y_i} \rangle} \right\rangle (i \in N)$ be a DHLSVNSs. Then mathematically the score are denoted and defined by

$$S_C = \left(\left(2 + \gimel_{\left(\frac{\mu_i}{\tau}\right) - \left(\frac{\eta_i}{\tau}\right) - \left(\frac{v_i}{\tau}\right)} \right) + \left(2 + \gimel_{\left(\frac{w_i}{\delta}\right) - \left(\frac{x_i}{\delta}\right) - \left(\frac{y_i}{\delta}\right)} \right) \right) / 2 \in [0, 1] \tag{3.2}$$

4. Einstein Operation

Since the inception of fuzzy set theory, the set theoretical operators have played an essential role. A variety of special operators have been incorporated in the general notions of the t-norms and t-conorms, which meet the needs of the conjunction and disjunction operators, accordingly. There are numerous t-norms and t-conorms types that can be employed to execute the corresponding intersections and unions. The Einstein product and Einstein sum are examples of t-norms and t-conorms, which are defined as follows.

Definition 4.1. [51] Let $a, b \in R$. Then, the family of Einstein t-norms are mathematically defined as

$$a \oplus_e b = \frac{a + b}{1 + ab} \tag{4.1}$$

Definition 4.2. [51] Let $a, b \in R$. Then, the family of Einstein t-conorms are mathematically defined as

$$a \otimes_e b = \frac{ab}{1 + (1 - a)(1 - b)} \tag{4.2}$$

for all $a, b \in [0, 1]^2$.

We introduce the Einstein operations for DHLSVNSs and examine some of their desirable characteristics.

Definition 4.3. Let $\gimel_{\daleth_1} = \left\langle \gimel_{\mu_1}\langle \daleth_{w_1}\rangle, \gimel_{\eta_1}\langle \daleth_{x_1}\rangle, \gimel_{\nu_1}\langle \daleth_{y_1}\rangle \right\rangle$ and $\gimel_{\daleth_2} = \left\langle \gimel_{\mu_2}\langle \daleth_{w_2}\rangle,\ \gimel_{\eta_2}\langle \daleth_{x_2}\rangle, \gimel_{\nu_2}\langle \daleth_{y_2}\rangle \right\rangle$ be the two double hierarchy linguistic single-valued neutrosophic sets. Then, Einstein's operational laws for (DHLSVNSs) are as follows:

i.

$$\gimel_{\daleth_1} \oplus \gimel_{\daleth_2} = \begin{pmatrix} \left(\gimel_{\tau\left(\frac{\frac{\mu_1}{\tau}+\frac{\mu_2}{\tau}}{1+\frac{\mu_1}{\tau}\frac{\mu_2}{\tau}}\right)} \left\langle \daleth_{\delta\left(\frac{\frac{w_1}{\delta}+\frac{w_2}{\delta}}{1+\frac{w_1}{\delta}\frac{w_2}{\delta}}\right)} \right\rangle \right), \\ \left(\gimel_{\tau\left(\frac{\frac{\eta_1}{\tau}\frac{\eta_2}{\tau}}{1+\left(1-\frac{\eta_1}{\tau}\right)\left(1-\frac{\eta_2}{\tau}\right)}\right)} \left\langle \daleth_{\delta\left(\frac{\frac{x_1}{\delta}\frac{x_2}{\delta}}{1+\left(1-\frac{x_1}{\delta}\right)\left(1-\frac{x_2}{\delta}\right)}\right)} \right\rangle \right), \\ \left(\gimel_{\tau\left(\frac{\frac{\nu_1}{\tau}\frac{v_2}{\tau}}{1+\left(1-\frac{\nu_1}{\tau}\right)\left(1-\frac{v_2}{\tau}\right)}\right)} \left\langle \daleth_{\delta\left(\frac{\frac{y}{\delta}\frac{y_2}{\delta}}{1+\left(1-\frac{y}{\delta}\right)\left(1-\frac{y_2}{\delta}\right)}\right)} \right\rangle \right) \end{pmatrix} \tag{4.3}$$

ii.

$$\gimel_{\daleth_1} \otimes \gimel_{\daleth_2} = \begin{pmatrix} \left(\gimel_{\tau\left(\frac{\frac{\mu_1}{\tau}\frac{\mu_2}{\tau}}{1+\left(1-\frac{\mu_1}{\tau}\right)\left(1-\frac{\mu_2}{\tau}\right)}\right)} \left\langle \daleth_{\delta\left(\frac{\frac{w_1}{\delta}\frac{w_2}{\delta}}{1+\left(1-\frac{w_1}{\delta}\right)\left(1-\frac{w_2}{\delta}\right)}\right)} \right\rangle \right), \\ \left(\gimel_{\tau\left(\frac{\frac{\eta_1}{\tau}+\frac{\eta_2}{\tau}}{1+\frac{\eta_1}{\tau}\frac{\eta_2}{\tau}}\right)} \left\langle \daleth_{\delta\left(\frac{\frac{x_1}{\delta}+\frac{x_2}{\delta}}{1+\frac{x_1}{\delta}\frac{x_2}{\delta}}\right)} \right\rangle \right), \\ \left(\gimel_{\tau\left(\frac{\frac{v_1}{\tau}+\frac{v_2}{\tau}}{1+\frac{v_1}{\tau}\frac{v_2}{\tau}}\right)} \left\langle \daleth_{\delta\left(\frac{\frac{y_1}{\delta}+\frac{y_2}{\delta}}{1+\frac{y_1}{\delta}\frac{y_2}{\delta}}\right)} \right\rangle \right) \end{pmatrix} \tag{4.4}$$

iii.

$$k.\beth_{\daleth_1} = \left(\begin{array}{c} \left(\beth_{\tau\left(\frac{\left(1+\frac{\mu_1}{\tau}\right)^k - \left(1-\frac{\mu_2}{\tau}\right)^k}{\left(1+\frac{\mu_1}{\tau}\right)^k + \left(1-\frac{\mu_2}{\tau}\right)^k}\right)} \left\langle \daleth_{\delta\left(\frac{\left(1+\frac{w_1}{\delta}\right)^k - \left(1-\frac{w_2}{\delta}\right)^k}{\left(1+\frac{w_1}{\delta}\right)^k + \left(1-\frac{w_2}{\delta}\right)^k}\right)} \right\rangle \right), \\ \left(\beth_{\tau\left(\frac{2\left(\frac{\eta_1}{\tau}\right)^k}{\left(2-\frac{\eta_1}{\tau}\right)^k + \left(\frac{\eta_1}{\tau}\right)^k}\right)} \left\langle \daleth_{\delta\left(\frac{2\left(\frac{x_1}{\delta}\right)^k}{\left(2-\frac{x_1}{\delta}\right)^k + \left(\frac{x_1}{\delta}\right)^k}\right)} \right\rangle \right), \\ \left(\beth_{\tau\left(\frac{2\left(\frac{v_1}{\tau}\right)^k}{\left(2-\frac{v_1}{\tau}\right)^k + \left(\frac{v_1}{\tau}\right)^k}\right)} \left\langle \daleth_{\delta\left(\frac{2\left(\frac{y_1}{\delta}\right)^k}{\left(2-\frac{y_1}{\delta}\right)^k + \left(\frac{y_1}{\delta}\right)^k}\right)} \right\rangle \right) \end{array} \right) \tag{4.5}$$

iv.

$$\beth_{\daleth_1}^k = \left(\begin{array}{c} \left(\beth_{\tau\left(\frac{2\left(\frac{\mu_1}{\tau}\right)^k}{\left(2-\frac{\mu_1}{\tau}\right)^k + \left(\frac{\mu_1}{\tau}\right)^k}\right)} \left\langle \daleth_{\delta\left(\frac{2\left(\frac{w_1}{\delta}\right)^k}{\left(2-\frac{w_1}{\delta}\right)^k + \left(\frac{w_1}{\delta}\right)^k}\right)} \right\rangle \right), \\ \left(\beth_{\tau\left(\frac{\left(1+\frac{\eta_1}{\tau}\right)^k - \left(1-\frac{\eta_2}{\tau}\right)^k}{\left(1+\frac{\eta_1}{\tau}\right)^k + \left(1-\frac{\eta_2}{\tau}\right)^k}\right)} \left\langle \daleth_{\delta\left(\frac{\left(1+\frac{x_1}{\delta}\right)^k - \left(1-\frac{x_2}{\delta}\right)^k}{\left(1+\frac{x_1}{\delta}\right)^k + \left(1-\frac{x_2}{\delta}\right)^k}\right)} \right\rangle \right), \\ \left(\beth_{\tau\left(\frac{\left(1+\frac{v_1}{\tau}\right)^k - \left(1-\frac{v_2}{\tau}\right)^k}{\left(1+\frac{v_1}{\tau}\right)^k + \left(1-\frac{v_2}{\tau}\right)^k}\right)} \left\langle \daleth_{\delta\left(\frac{\left(1+\frac{y_1}{\delta}\right)^k - \left(1-\frac{y_2}{\delta}\right)^k}{\left(1+\frac{y_1}{\delta}\right)^k + \left(1-\frac{y_2}{\delta}\right)^k}\right)} \right\rangle \right) \end{array} \right) \tag{4.6}$$

4.1. Double Hierarchy Linguistic Single-Valued Neutrosophic Einstein Averaging Aggregation Information

This section devoted a list of Einstein weighted averaging aggregation operators and Einstein averaging aggregation operators for DHLSVNSs also describes its basic properties as follows:

Definition 4.4. Suppose we have a family $\beth_{\daleth_i} = \left\langle \beth_{\mu_i}\langle\daleth_{w_i}\rangle, \beth_{\eta_i}\langle\daleth_{x_i}\rangle, \beth_{\nu_i}\langle\daleth_{y_i}\rangle \right\rangle$ $(i \in \mathbb{N})$ of single-valued neutrosophic double hierarchy linguistic numbers (SVNDHLNs) and $\omega = (\omega_1, \omega_2, \cdots, \omega_n)^T$ represent weight vectors of given family restricted to $\omega_i > 0$, $\sum_{i=1}^{n} \omega_i = 1$. Then based on above operational laws the double hierarchy linguistic single-valued neutrosophic Einstein weighted averaging (DHLSVNEWA) operator are defined as:

$$DHLSVNEWA(\gimel_{\daleth_1}, \gimel_{\daleth_2}, \cdots, \gimel_{\daleth_n}) = \omega_1.\gimel_{\daleth_1} \oplus \omega_2.\gimel_{\daleth_2}, \cdots, \oplus\omega_n.\gimel_{\daleth_n}$$

$$= \left(\begin{array}{c} \left(\gimel_{\tau\left(\frac{\prod_{i=1}^{n}\left(1+\frac{\mu_i}{\tau}\right)^{\omega_i} - \prod_{i=1}^{n}\left(1-\frac{\mu_i}{\tau}\right)^{\omega_i}}{\prod_{i=1}^{n}\left(1+\frac{\mu_i}{\tau}\right)^{\omega_i} + \prod_{i=1}^{n}\left(1-\frac{\mu_i}{\tau}\right)^{\omega_i}}\right)} \left\langle \daleth_{\delta\left(\frac{\prod_{i=1}^{n}\left(1+\frac{\mu_i}{\delta}\right)^{\omega_i} - \prod_{i=1}^{n}\left(1-\frac{\mu_i}{\delta}\right)^{\omega_i}}{\prod_{i=1}^{n}\left(1+\frac{\mu_i}{\delta}\right)^{\omega_i} + \prod_{i=1}^{n}\left(1-\frac{\mu_i}{\delta}\right)^{\omega_i}}\right)} \right\rangle \right), \\ \left(\gimel_{\tau\left(\frac{2\prod_{i=1}^{n}\left(\frac{\eta_i}{\tau}\right)^{\omega_i}}{\left(2-\frac{\eta_1}{\tau}\right)^{\omega_i} + \left(\frac{\eta_1}{\tau}\right)^{\omega_i}}\right)} \left\langle \daleth_{\delta\left(\frac{2\prod_{i=1}^{n}\left(\frac{\eta_i}{\delta}\right)^{\omega_i}}{\left(2-\frac{x_1}{\delta}\right)^{\omega_i} + \left(\frac{x_1}{\delta}\right)^{\omega_i}}\right)} \right\rangle \right), \\ \left(\gimel_{\tau\left(\frac{2\prod_{i=1}^{n}\left(\frac{v_1}{\tau}\right)^{\omega_i}}{\prod_{i=1}^{n}\left(2-\frac{v_1}{\tau}\right)^{\omega_i} + \prod_{i=1}^{n}\left(\frac{v_1}{\tau}\right)^{\omega_i}}\right)} \left\langle \daleth_{\delta\left(\frac{2\prod_{i=1}^{n}\left(\frac{y_1}{\delta}\right)^{\omega_i}}{\prod_{i=1}^{n}\left(2-\frac{y_1}{\delta}\right)^{\omega_i} + \prod_{i=1}^{n}\left(\frac{y_1}{\delta}\right)^{\omega_i}}\right)} \right\rangle \right) \end{array} \right) \tag{4.7}$$

Theorem 4.5. Suppose we have a family $\gimel_{\daleth_i} = \left\langle \gimel_{\mu_i\langle \daleth_{w_i}\rangle}, \gimel_{\eta_i\langle \daleth_{x_i}\rangle}, \gimel_{\nu_i\langle \daleth_{y_i}\rangle} \right\rangle (i \in \mathbb{N})$ of double hierarchy linguistic single-valued neutrosophic numbers and $\omega = (\omega_1, \omega_2, \cdots, \omega_n)^T$ represent weight vectors of given family restricted to $\omega_i > 0, \sum_{i=1}^{n} \omega_i = 1$. Then, DHLSVNEWA operators satisfies the following properties as:

i. (Idempotency) Suppose for all $\gimel_{\daleth_i}(i \in \mathbb{N})$, is equal i.e $\gimel_{\daleth_i} = \gimel_{\daleth}$, then

$$DHLSVNEWA\left(\gimel_{\daleth_1}, \gimel_{\daleth_2}, \cdots, \gimel_{\daleth_n}\right) = \gimel_{\daleth}$$

ii. (Monotonicity) Consider $\gimel_{\daleth_i}$, $\gimel^*_{\daleth_i}$ be two sets of DHLEs, $\gimel_{\daleth_i} \le \gimel^*_{\daleth_i}$ for all i; then:

$$DHLSVNEWA\left(\gimel_{\daleth_1}, \gimel_{\daleth_2}, \cdots, \gimel_{\daleth_n}\right) \le DHLSVNEWA\left(\gimel^*_{\daleth_1}, \gimel^*_{\daleth_2}, \cdots, \gimel^*_{\daleth_3}\right)$$

iii. (Boundedness) Consider $\gimel^-_{\daleth} = \min_{1\le i\le n}\left\{\gimel_{\daleth_i}\right\}$, $\gimel^+_{\daleth} = \max_{1\le i\le n}\left\{\gimel_{\daleth_i}\right\}$. Then,

$$\gimel^-_{\daleth} \le DHLSVNEWA\left(\gimel_{\daleth_1}, \gimel_{\daleth_2}, \cdots, \gimel_{\daleth_n}\right) \le \gimel^+_{\daleth}$$

Proof. Given that $\beth\daleth_i = \beth\daleth$ for each i, then

$$DHLSVNEWA(\beth\daleth_1, \beth\daleth_2, \cdots, \beth\daleth_n) = \omega_1.\beth\daleth_1 \oplus \omega_2.\beth\daleth_2, \cdots, \oplus\omega_n.\beth\daleth_n$$

$$= \left(\begin{array}{c} \left(\beth_{\tau\left(\frac{\prod_{i=1}^{n}\left(1+\frac{\mu_i}{\tau}\right)^{\omega_i} - \prod_{i=1}^{n}\left(1-\frac{\mu_i}{\tau}\right)^{\omega_i}}{\prod_{i=1}^{n}\left(1+\frac{\mu_i}{\tau}\right)^{\omega_i} + \prod_{i=1}^{n}\left(1-\frac{\mu_i}{\tau}\right)^{\omega_i}}\right)} \left\langle \daleth_{\delta\left(\frac{\prod_{i=1}^{n}\left(1+\frac{\mu_i}{\delta}\right)^{\omega_i} - \prod_{i=1}^{n}\left(1-\frac{\mu_i}{\delta}\right)^{\omega_i}}{\prod_{i=1}^{n}\left(1+\frac{\mu_i}{\delta}\right)^{\omega_i} + \prod_{i=1}^{n}\left(1-\frac{\mu_i}{\delta}\right)^{\omega_i}}\right)} \right\rangle \right), \\ \left(\beth_{\tau\left(\frac{2\prod_{i=1}^{n}\left(\frac{\eta_i}{\tau}\right)^{\omega_i}}{\left(2-\frac{\eta_1}{\tau}\right)^{\omega_i} + \left(\frac{\eta_1}{\tau}\right)^{\omega_i}}\right)} \left\langle \daleth_{\delta\left(\frac{2\prod_{i=1}^{n}\left(\frac{\eta_i}{\delta}\right)^{\omega_i}}{\left(2-\frac{x_1}{\delta}\right)^{\omega_i} + \left(\frac{x_1}{\delta}\right)^{\omega_i}}\right)} \right\rangle \right), \\ \left(\beth_{\tau\left(\frac{2\prod_{i=1}^{n}\left(\frac{v_1}{\tau}\right)^{\omega_i}}{\prod_{i=1}^{n}\left(2-\frac{v_1}{\tau}\right)^{\omega_i} + \prod_{i=1}^{n}\left(\frac{v_1}{\tau}\right)^{\omega_i}}\right)} \left\langle \daleth_{\delta\left(\frac{2\prod_{i=1}^{n}\left(\frac{y_1}{\delta}\right)^{\omega_i}}{\prod_{i=1}^{n}\left(2-\frac{y_1}{\delta}\right)^{\omega_i} + \prod_{i=1}^{n}\left(\frac{y_1}{\delta}\right)^{\omega_i}}\right)} \right\rangle \right) \end{array} \right)$$

$$
= \left(\begin{array}{c}
\left(\beth_{\tau\left(\frac{\left(1+\frac{\mu_i}{\tau}\right)^{\sum_{i=1}^{n}} - \left(1-\frac{\mu_i}{\tau}\right)^{\sum_{i=1}^{n}}}{\left(1+\frac{\mu_i}{\tau}\right)^{\sum_{i=1}^{n}} + \left(1-\frac{\mu_i}{\tau}\right)^{\sum_{i=1}^{n}}}\right)} \left\langle \daleth_{\delta\left(\frac{\left(1+\frac{\mu_i}{\delta}\right)^{\sum_{i=1}^{n}} - \left(1-\frac{\mu_i}{\delta}\right)^{\sum_{i=1}^{n}}}{\left(1+\frac{\mu_i}{\delta}\right)^{\sum_{i=1}^{n}} + \left(1-\frac{\mu_i}{\delta}\right)^{\sum_{i=1}^{n}}}\right)} \right\rangle \right), \\
\left(\beth_{\tau\left(\frac{2\left(\frac{\eta_i}{\tau}\right)^{\sum_{i=1}^{n}}}{\left(2-\frac{\eta_1}{\tau}\right)^{\sum_{i=1}^{n}} + \left(\frac{\eta_1}{\tau}\right)^{\sum_{i=1}^{n}}}\right)} \left\langle \daleth_{\delta\left(\frac{2\left(\frac{\eta_i}{\delta}\right)^{\sum_{i=1}^{n}}}{\left(2-\frac{x_1}{\delta}\right)^{\sum_{i=1}^{n}} + \left(\frac{x_1}{\delta}\right)^{\sum_{i=1}^{n}}}\right)} \right\rangle \right), \\
\left(\beth_{\tau\left(\frac{2\left(\frac{v_1}{\tau}\right)^{\sum_{i=1}^{n}}}{\left(2-\frac{v_1}{\tau}\right)^{\sum_{i=1}^{n}} + \left(\frac{v_1}{\tau}\right)^{\sum_{i=1}^{n}}}\right)} \left\langle \daleth_{\delta\left(\frac{2\left(\frac{y_1}{\delta}\right)^{\sum_{i=1}^{n}}}{\left(2-\frac{y_1}{\delta}\right)^{\sum_{i=1}^{n}} + \left(\frac{y_1}{\delta}\right)^{\sum_{i=1}^{n}}}\right)} \right\rangle \right)
\end{array} \right)
$$

$$
= \left\langle \beth_{\mu\langle\daleth_w\rangle}, \beth_{\eta\langle\daleth_x\rangle}, \beth_{\nu\langle\daleth_y\rangle} \right\rangle = \beth_{\daleth}
$$

□

Proof. Since $\beth_{\daleth_i} \leq \beth^*_{\daleth_i}$ then, $\omega_i \beth_{\daleth_i} \leq \omega_i \beth^*_{\daleth_i}$, accordingly we deduce that $\oplus_{i=1}^{n} \omega_i . \beth_{\daleth_i} \leq \oplus_{i=1}^{n} \omega_i \beth^*_{\daleth_i}$. Hence, $DHLSVNEWA(\beth_{\daleth_1}, \beth_{\daleth_2}, \cdots, \beth_{\daleth_n}) = \oplus_{i=1}^{n} \omega_i \beth_{\daleth_i}$ and $DHLSVNEWA(\beth^*_{\daleth_1}, \beth^*_{\daleth_2}, \cdots, \beth^*_{\daleth_n}) = \oplus_{i=1}^{n} \omega_i \beth^*_{\daleth_i}$, we can generate

$$
DHLSVNEWA(\beth_{\daleth_1}, \beth_{\daleth_2}, \cdots, \beth_{\daleth_n}) \leq DHLSVNEWA(\beth^*_{\daleth_1}, \beth^*_{\daleth_2}, \cdots, \beth^*_{\daleth_n})
$$

□

Proof. Since $\beth^-_{\daleth} = \min_{1 \leq i \leq n} \{\beth_{\daleth_i}\}$ and $\beth^+_{\daleth} = \max_{1 \leq i \leq n} \{\beth_{\daleth_i}\}$, then according to the monotonicity properties

$$
DHLSVNEWA(\beth^-_{\daleth_1}, \beth^-_{\daleth_2}, \cdots, \beth^-_{\daleth_n}) \leq DHLSVNEWA(\beth_{\daleth_1}, \beth_{\daleth_2}, \cdots, \beth_{\daleth_n}) \leq DHLSVNEWA(\beth^+_{\daleth_1}, \beth^+_{\daleth_2}, \cdots, \beth^+_{\daleth_n})
$$

Furthermore, by mean of idempotency properties, we have

$$DHLSVNEWA(\gimel\daleth^-_{1}, \gimel\daleth^-_{2}, \cdots, \gimel\daleth^-_{n}) = \gimel\daleth^-_{1}, \quad DHLSVNEWA(\gimel\daleth^+_{1}, \gimel\daleth^+_{2}, \cdots, \gimel\daleth^+_{n}) = \gimel\daleth^+$$

Accordingly, we can deduce that

$$\gimel\daleth^- \leq DHLSVNEWA\left(\gimel\daleth_1, \gimel\daleth_2, \cdots, \gimel\daleth_n\right) \leq \gimel\daleth^+$$

□

Definition 4.6. Suppose we have a family $\gimel\daleth_i = \left\langle \gimel_{\mu_i}\langle\daleth_{w_i}\rangle, \gimel_{\eta_i}\langle\daleth_{x_i}\rangle, \gimel_{\nu_i}\langle\daleth_{y_i}\rangle \right\rangle$ $(i \in \mathbb{N})$ of double hierarchy linguistic single-valued neutrosophic numbers, then the double hierarchy linguistic single-valued neutrosophic Einstein arithmetic (DHLSVNEA) mean operators are as follows:

$$SVNDHLEA(\gimel\daleth_1, \gimel\daleth_2, \cdots, \gimel\daleth_n) = \gimel\daleth_1 \oplus \gimel\daleth_2, \cdots, \oplus\gimel\daleth_n$$

$$= \left(\begin{array}{c} \left(\gimel_{\tau\left(\frac{\prod_{i=1}^{n}\left(1+\frac{\mu_i}{\tau}\right)-\prod_{i=1}^{n}\left(1-\frac{\mu_i}{\tau}\right)}{\prod_{i=1}^{n}\left(1+\frac{\mu_i}{\tau}\right)+\prod_{i=1}^{n}\left(1-\frac{\mu_i}{\tau}\right)}\right)} \left\langle \daleth_{\delta\left(\frac{\prod_{i=1}^{n}\left(1+\frac{\mu_i}{\delta}\right)-\prod_{i=1}^{n}\left(1-\frac{\mu_i}{\delta}\right)}{\prod_{i=1}^{n}\left(1+\frac{\mu_i}{\delta}\right)+\prod_{i=1}^{n}\left(1-\frac{\mu_i}{\delta}\right)}\right)} \right\rangle \right), \\ \left(\gimel_{\tau\left(\frac{2\prod_{i=1}^{n}\left(\frac{\eta_i}{\tau}\right)}{\left(2-\frac{\eta_1}{\tau}\right)+\left(\frac{\eta_1}{\tau}\right)}\right)} \left\langle \daleth_{\delta\left(\frac{2\prod_{i=1}^{n}\left(\frac{\eta_i}{\delta}\right)}{\left(2-\frac{x_1}{\delta}\right)+\left(\frac{x_1}{\delta}\right)}\right)} \right\rangle \right), \\ \left(\gimel_{\tau\left(\frac{2\prod_{i=1}^{n}\left(\frac{v_1}{\tau}\right)}{\prod_{i=1}^{n}\left(2-\frac{v_1}{\tau}\right)+\prod_{i=1}^{n}\left(\frac{v_1}{\tau}\right)}\right)} \left\langle \daleth_{\delta\left(\frac{2\prod_{i=1}^{n}\left(\frac{y_1}{\delta}\right)}{\prod_{i=1}^{n}\left(2-\frac{y_1}{\delta}\right)+\prod_{i=1}^{n}\left(\frac{y_1}{\delta}\right)}\right)} \right\rangle \right) \end{array} \right) \tag{4.8}$$

Theorem 4.7. Suppose we have a family $\gimel\daleth_i = \left\langle \gimel_{\mu_i}\langle\daleth_{w_i}\rangle, \gimel_{\eta_i}\langle\daleth_{x_i}\rangle, \gimel_{\nu_i}\langle\daleth_{y_i}\rangle \right\rangle$ $(i \in \mathbb{N})$ of double hierarchy linguistic single-valued neutrosophic numbers. Then, DHLSVNEA operators satisfies the following properties as:

i. (Idempotency) Suppose for all $i \in \mathbb{N}$, $\gimel\daleth_i = \gimel\daleth$. Then,

$$DHLSVNEA\left(\gimel\daleth_1, \gimel\daleth_2, \cdots, \gimel\daleth_n\right) = \gimel\daleth$$

ii. (Boundedness) Consider $\beth_{\neg}^{-} = \min_{1\leq i\leq n}\{\beth_{\neg_i}\}$ and $\beth_{\neg}^{+} = \max_{1\leq i\leq n}\{\beth_{\neg_i}\}$. Then,

$$\beth_{\neg}^{-} \leq DHLSVNEA\left(\beth_{\neg_1}, \beth_{\neg_2}, \cdots, \beth_{\neg_n}\right) \leq \beth_{\neg}^{+}$$

iii. (Monotonicity) Consider $\beth_{\neg_i}$, $\beth^{*}_{\neg_i}$ be two sets of DHLEs, $\beth_{\neg_i} \leq \beth^{*}_{\neg_i}$ for all i, then,

$$DHLSVNEA\left(\beth_{\neg_1}, \beth_{\neg_2}, \cdots, \beth_{\neg_n}\right) \leq DHLSVNEA\left(\beth^{*}_{\neg_1}, \beth^{*}_{\neg_2}, \cdots, \beth^{*}_{\neg_3}\right)$$

Proof. It is clear from Theorem 4.5. □

4.2. Double Hierarchy Linguistic Single-Valued Neutrosophic Einstein Geometric Aggregation Information

This section devoted a list of Einstein geometric aggregation operators such as double hierarchy linguistic single-valued neutrosophic Einstein Weighted geometric (DHLSVNEWG) and double hierarchy linguistic single-valued neutrosophic Einstein geometric (DHLSVNEG) operators also describes its basic properties as follows:

Definition 4.8. Suppose we have a family $\beth_{\neg_i} = \left\langle \beth_{\mu_i}\langle \neg_{w_i}\rangle, \beth_{\eta_i}\langle \neg_{x_i}\rangle, \beth_{\nu_i}\langle \neg_{y_i}\rangle \right\rangle (i \in \mathbb{N})$ of double hierarchy linguistic single-valued neutrosophic numbers (DHLSVNNs), then the double hierarchy linguistic single-valued neutrosophic Einstein geometric (DHLSVNEG) mean operators are as follows:

$$DHLSVNEG(\beth_{\neg_1}, \beth_{\neg_2}, \cdots, \beth_{\neg_n}) = (\beth_{\neg_1}) \otimes (\beth_{\neg_2}), \cdots, \otimes (\beth_{\neg_n})$$
$$\left(\begin{array}{c} \left(\beth_{\tau\left(\frac{2\prod_{i=1}^{n}\left(\frac{\mu_i}{\tau}\right)}{\left(2-\frac{\mu_i}{\tau}\right)^k+\left(\frac{\mu_i}{\tau}\right)}\right)} \left\langle \neg_{\delta\left(\frac{2\prod_{i=1}^{n}\left(\frac{w_i}{\delta}\right)}{\left(2-\frac{w_i}{\delta}\right)+\left(\frac{w_i}{\delta}\right)}\right)} \right\rangle \right), \\ \left(\beth_{\tau\left(\frac{\prod_{i=1}^{n}\left(1+\frac{\eta_i}{\tau}\right)-\prod_{i=1}^{n}\left(1-\frac{\eta_i}{\tau}\right)}{\prod_{i=1}^{n}\left(1+\frac{\eta_i}{\tau}\right)+\prod_{i=1}^{n}\left(1-\frac{\eta_i}{\tau}\right)}\right)} \left\langle \neg_{\delta\left(\frac{\prod_{i=1}^{n}\left(1+\frac{x_i}{\delta}\right)-\prod_{i=1}^{n}\left(1-\frac{x_i}{\delta}\right)}{\prod_{i=1}^{n}\left(1+\frac{x_i}{\delta}\right)+\prod_{i=1}^{n}\left(1-\frac{x_i}{\delta}\right)}\right)} \right\rangle \right), \\ \left(\beth_{\tau\left(\frac{\prod_{i=1}^{n}\left(1+\frac{v_i}{\tau}\right)-\prod_{i=1}^{n}\left(1-\frac{v_i}{\tau}\right)}{\prod_{i=1}^{n}\left(1+\frac{v_i}{\tau}\right)+\prod_{i=1}^{n}\left(1-\frac{v_i}{\tau}\right)}\right)} \left\langle \neg_{\delta\left(\frac{\prod_{i=1}^{n}\left(1+\frac{y_i}{\delta}\right)-\prod_{i=1}^{n}\left(1-\frac{y_i}{\delta}\right)}{\prod_{i=1}^{n}\left(1+\frac{y_i}{\delta}\right)+\prod_{i=1}^{n}\left(1-\frac{y_i}{\delta}\right)}\right)} \right\rangle \right) \end{array}\right) \tag{4.9}$$

Theorem 4.9. Suppose we have a family $\gimel\daleth_i = \left\langle \gimel_{\mu_i\langle \daleth_{w_i}\rangle}, \gimel_{\eta_i\langle \daleth_{x_i}\rangle}, \gimel_{\nu_i\langle \daleth_{y_i}\rangle} \right\rangle (i \in \mathbb{N})$ of double hierarchy linguistic single-valued neutrosophic numbers. Then DHLSVNEG satisfies the following properties as:

i. (Idempotency) Suppose for all $i \in \mathbb{N}$, $\gimel\daleth_i = \gimel\daleth$. Then,

$$DHLSVNEG\left(\gimel\daleth_1, \gimel\daleth_2, \cdots, \gimel\daleth_n\right) = \gimel\daleth$$

ii. (Boundedness) Consider $\gimel\daleth^- = \min_{1\leq i\leq n}\{\gimel\daleth_i\}$, $\gimel\daleth^+ = \max_{1\leq i\leq n}\{\gimel\daleth_i\}$. Then,

$$\gimel\daleth^- \leq DHLSVNEG\left(\gimel\daleth_1, \gimel\daleth_2, \cdots, \gimel\daleth_n\right) \leq \gimel\daleth^+$$

iii. (Monotonicity) Consider $\gimel\daleth_i$, $\gimel\daleth^*_i$ be two sets of DHLEs, $\gimel\daleth_i \leq \gimel\daleth^*_i$ for all i, then

$$DHLSVNEG\left(\gimel\daleth_1, \gimel\daleth_2, \cdots, \gimel\daleth_n\right) \leq DHLSVNEG\left(\gimel\daleth^*_1, \gimel\daleth^*_2, \cdots, \gimel\daleth^*_3\right)$$

Proof. It is clear from Theorem 4.5. □

Definition 4.10. Suppose we have a family $\gimel\daleth_i = \left\langle \gimel_{\mu_i\langle \daleth_{w_i}\rangle}, \gimel_{\eta_i\langle \daleth_{x_i}\rangle}, \gimel_{\nu_i\langle \daleth_{y_i}\rangle} \right\rangle (i \in \mathbb{N})$ of double hierarchy linguistic single-valued neutrosophic numbers and $\omega = (\omega_1, \omega_2, \cdots, \omega_n)^T$ represent weight vectors of given family restricted to $\omega_i > 0$, $\sum_{i=1}^{n} \omega_i = 1$. Then based on above operational laws the double hierarchy linguistic single-valued neutrosophic Einstein weighted geometric (DHLSVNEWG) operator are defined by

$$DHLSVNEWG(\gimel\daleth_1, \gimel\daleth_2, \cdots, \gimel\daleth_n) = \left(\gimel\daleth_1\right)^{\omega_1} \otimes \left(\gimel\daleth_2\right)^{\omega_2}, \cdots, \otimes \left(\gimel\daleth_n\right)^{\omega_n}$$

$$\left(\begin{array}{c} \left(\gimel_{\tau\left(\frac{2\prod_{i=1}^{n}\left(\frac{\mu_i}{\tau}\right)^{\omega_i}}{\left(2-\frac{\mu_i}{\tau}\right)^{\omega_i}+\left(\frac{\mu_i}{\tau}\right)^{\omega_i}}\right)\left\langle \daleth_{\delta\left(\frac{2\prod_{i=1}^{n}\left(\frac{w_i}{\delta}\right)^{\omega_i}}{\left(2-\frac{w_i}{\delta}\right)^{\omega_i}+\left(\frac{w_i}{\delta}\right)^{\omega_i}}\right)}\right\rangle} \right), \\ \left(\gimel_{\tau\left(\frac{\prod_{i=1}^{n}\left(1+\frac{\eta_i}{\tau}\right)^{\omega_i}-\prod_{i=1}^{n}\left(1-\frac{\eta_i}{\tau}\right)^{\omega_i}}{\prod_{i=1}^{n}\left(1+\frac{\eta_i}{\tau}\right)^{\omega_i}+\prod_{i=1}^{n}\left(1-\frac{\eta_i}{\tau}\right)^{\omega_i}}\right)\left\langle \daleth_{\delta\left(\frac{\prod_{i=1}^{n}\left(1+\frac{x_i}{\delta}\right)^{\omega_i}-\prod_{i=1}^{n}\left(1-\frac{x_i}{\delta}\right)^{\omega_i}}{\prod_{i=1}^{n}\left(1+\frac{x_i}{\delta}\right)^{\omega_i}+\prod_{i=1}^{n}\left(1-\frac{x_i}{\delta}\right)^{\omega_i}}\right)}\right\rangle} \right), \\ \left(\gimel_{\tau\left(\frac{\prod_{i=1}^{n}\left(1+\frac{v_i}{\tau}\right)^{\omega_i}-\prod_{i=1}^{n}\left(1-\frac{v_i}{\tau}\right)^{\omega_i}}{\prod_{i=1}^{n}\left(1+\frac{v_i}{\tau}\right)^{\omega_i}+\prod_{i=1}^{n}\left(1-\frac{v_i}{\tau}\right)^{\omega_i}}\right)\left\langle \daleth_{\delta\left(\frac{\prod_{i=1}^{n}\left(1+\frac{y_i}{\delta}\right)^{\omega_i}-\prod_{i=1}^{n}\left(1-\frac{y_i}{\delta}\right)^{\omega_i}}{\prod_{i=1}^{n}\left(1+\frac{y_i}{\delta}\right)^{\omega_i}+\prod_{i=1}^{n}\left(1-\frac{y_i}{\delta}\right)^{\omega_i}}\right)}\right\rangle} \right) \end{array}\right) \tag{4.10}$$

Theorem 4.11. Suppose we have a family $\beth_{\daleth_i} = \left\langle \beth_{\mu_i\langle\daleth_{w_i}\rangle}, \beth_{\eta_i\langle\daleth_{x_i}\rangle}, \beth_{\nu_i\langle\daleth_{y_i}\rangle} \right\rangle$ $(i \in \mathbb{N})$ of double hierarchy linguistic single-valued neutrosophic numbers (DHLSVNNs) and $\omega = (\omega_1, \omega_2, \cdots, \omega_n)^T$ represent weight vectors of given family restricted to $\omega_i > 0$, $\sum\limits_{i=1}^{n} \omega_i = 1$. Then DHLSVNEWG operator satisfies the following properties as:

i. (Idempotency) Suppose for all $i \in \mathbb{N}$, $\beth_{\daleth_i} = \beth_{\daleth}$. Then,

$$DHLSVNEWG\left(\beth_{\daleth_1}, \beth_{\daleth_2}, \cdots, \beth_{\daleth_n}\right) = \beth_{\daleth}$$

ii. (Boundedness) Consider $\beth_{\daleth}^{-} = \min_{1\le i\le n}\{\beth_{\daleth_i}\}$, $\beth_{\daleth}^{+} = \max_{1\le i\le n}\{\beth_{\daleth_i}\}$. Then,

$$\beth_{\daleth}^{-} \le DHLSVNEWG\left(\beth_{\daleth_1}, \beth_{\daleth_2}, \cdots, \beth_{\daleth_n}\right) \le \beth_{\daleth}^{+}$$

iii. (Monotonicity) Consider $\beth_{\daleth_i}$, $\beth_{\daleth_i}^{*}$ be two sets of DHLEs, $\beth_{\daleth_i} \le \beth_{\daleth_i}^{*}$ for all i, then

$$DHLSVNEWG\left(\beth_{\daleth_1}, \beth_{\daleth_2}, \cdots, \beth_{\daleth_n}\right) \le DHLSVNEWG\left(\beth_{\daleth_1}^{*}, \beth_{\daleth_2}^{*}, \cdots, \beth_{\daleth_3}^{*}\right)$$

PROOF. It is clear from Theorem 4.5. □

Definition 4.12. Let $\beth_{\daleth_1} = \left\langle \beth_{\mu_1\langle\daleth_{w_1}\rangle}, \beth_{\eta_1\langle\daleth_{x_1}\rangle}, \beth_{\nu_1\langle\daleth_{y_1}\rangle} \right\rangle$ and $\beth_{\daleth_2} = \left\langle \beth_{\mu_2\langle\daleth_{w_2}\rangle}, \beth_{\eta_2\langle\daleth_{x_2}\rangle}, \beth_{\nu_2\langle\daleth_{y_2}\rangle} \right\rangle$ be the two double hierarchy linguistic single-valued neutrosophic numbers (DHLSVNNs). Then, for any $\Delta > 0$, the distance of two DHLSVNNs $\beth_{\daleth_1}$ and $\beth_{\daleth_2}$ is mathematically defined by

$$d\left(\beth_{\daleth_1}, \beth_{\daleth_2}\right) = \frac{1}{6}\left[\begin{array}{c} \left|\beth_{\left(\frac{\mu_1}{\tau}\right)} - \beth_{\left(\frac{\mu_1}{\tau}\right)}\right|^{\Delta} + \left|\daleth_{\left(\frac{w_1}{\delta}\right)} - \daleth_{\left(\frac{w_2}{\delta}\right)}\right|^{\Delta} + \left|\beth_{\left(\frac{\eta_1}{\tau}\right)} - \beth_{\left(\frac{\eta_2}{\tau}\right)}\right|^{\Delta} \\ \left|\daleth_{\left(\frac{x_1}{\delta}\right)} - \daleth_{\left(\frac{x_1}{\delta}\right)}\right|^{\Delta} + \left|\beth_{\left(\frac{\nu_1}{\tau}\right)} - \beth_{\left(\frac{\nu_2}{\tau}\right)}\right|^{\Delta} + \left|\daleth_{\left(\frac{y_1}{\tau}\right)} - \daleth_{\left(\frac{y_1}{\tau}\right)}\right|^{\Delta} \end{array}\right]^{\frac{1}{\Delta}}$$

5. Extended DHLSVN-MULTIMOORA Technique

In this section, we present an extended version of the MULTIMOORA approach to handle MAGDM in the DHLSVN environment and evaluate the best choice in decision making. The DHLSVN-MULTIMOORA method consists of three methods: the double hierarchy single-valued neutrosophic ratio system (DHLSVNRS) approach, the double hierarchy linguistic single-valued neutrosophic reference point (DHLSVNRP) approach and the double hierarchy linguistic single-valued neutrosophic full multiplicative form (DHLSVNFMF) approach. The first four steps are the same in all three approaches. A decision matrix describes the values of alternatives supporting specific criteria in the MAGDM issue under an DHLSVNS context.

Suppose two sets $A = \{A_1, A_2, \cdots, A_m\}$ and $B = \{B_1, B_2, \cdots, B_n\}$ represent m no of alternatives and n no of criteria respectively. Let $\omega = \{\omega_1, \omega_2, \cdots, \omega_n\}$ be the unknown weight vectors such that $\sum_{j=1}^{n} \omega_j = 1$ assign to corresponding criteria by a set $DM = \{DM_1, DM_2, \cdots, DM_g\}$ of decision makers. Our goal is to select the superior alternative among the possible alternatives that meet specific criteria using the extended DHLSVN-MULIMOORA approach, which is classified in the following section.

Step 1. Construction of DHLSVN decision matrix

Each DM examines the criteria for selecting alternatives. Decision makers (DMs) analyse the capabilities of alternatives that meet specific criteria and allocate the DHLSVNNs to each alternative that meets those criteria in the form of LT number. The DHLSVNNs decision matrix offered by g no of DMs is as follows:

$$Q=[Q^p_{ij}]_{m\times n}=\begin{array}{cc} & \begin{array}{cccc} B_1 & B_2 & \cdots & B_n \end{array} \\ \begin{array}{c} A_1 \\ A_2 \\ \vdots \\ A_m \end{array} & \begin{pmatrix} Q^p_{11} & Q^p_{12} & \cdots & Q^p_{1n} \\ Q^p_{21} & Q^p_{22} & \cdots & Q^p_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ Q^p_{m1} & Q^p_{m2} & \cdots & Q^p_{mn} \end{pmatrix} \end{array}$$

where each entry of the individual decision matrix is of the form $Q^p_{ij}=\left\langle \gimel^h_{\mu_{ij}\langle\daleth_{w_{ij}}\rangle}, \gimel^h_{\eta_{ij}\langle\daleth_{x_{ij}}\rangle}, \gimel^h_{\nu_{ij}\langle\daleth_{y_{ij}}\rangle} \right\rangle$, $(i\in\{1,2,\cdots,m\})\,(j\in\{1,2,\cdots,n\})$ and $(p\in\{1,2,\cdots,g\})$.

Step 2. Normalization of DHLSVN decision matrix

If there are cost criteria in MAGDM problems, they must be normalized. The the following equation converts non-economical criteria to beneficial criteria:

$$N^c=\begin{cases} \gimel^h_{\mu_{ij}\langle\daleth_{w_{ij}}\rangle}, \gimel^h_{\eta_{ij}\langle\daleth_{x_{ij}}\rangle}, \gimel^h_{\nu_{ij}\langle\daleth_{y_{ij}}\rangle}, & \text{for benefit} \\ \gimel^h_{\nu_{ij}\langle\daleth_{y_{ij}}\rangle}, \gimel^h_{\eta_{ij}\langle\daleth_{x_{ij}}\rangle}, \gimel^h_{\mu_{ij}\langle\daleth_{w_{ij}}\rangle}, & \text{for cost} \end{cases}$$

Step 3. Construction of aggregated DHLSVN decision matrix

In the decision-making process, the aggregated DHLSVN decision matrix is created to determine the group decision of DMs by aggregating the individual judgement of DMs. By applying the double hierarchy linguistic single-valued neutrosophic Einstein weighted aggregation operator (4.7), and construct the double hierarchy linguistic single-valued neutrosophic aggregated matrix as follows:

$$Q=[Q_{ij}]_{m\times n}=\begin{array}{cc} & \begin{array}{cccc} B_1 & B_2 & \cdots & B_n \end{array} \\ \begin{array}{c} A_1 \\ A_2 \\ \vdots \\ A_m \end{array} & \begin{pmatrix} Q_{11} & Q_{12} & \cdots & Q_{1n} \\ Q_{21} & Q_{22} & \cdots & Q_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ Q_{m1} & Q_{m2} & \cdots & Q_{mn} \end{pmatrix} \end{array}$$

Step 4. DHLSVN weighted aggregated decision matrix (DHLSVNWA)

By applying the weight vector of criteria and Definition 4.3, calculate the double hierarchy linguistic single-valued neutrosophic weighted aggregated decision matrix as follows:

$$\Re=[r_{ij}]_{m\times n}=\begin{array}{cc} & \begin{array}{cccc} C_1 & C_2 & \cdots & C_n \end{array} \\ \begin{array}{c} A_1 \\ A_2 \\ \vdots \\ A_m \end{array} & \begin{pmatrix} r_{11} & r_{12} & \cdots & r_{1n} \\ r_{21} & r_{22} & \cdots & r_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ r_{m1} & r_{m2} & \cdots & r_{mn} \end{pmatrix} \end{array}$$

Step 5. DHLSVN Ratio system approach (DHLSVNRSP)

i. Calculate the Ratio Y_i^+ by using the double hierarchy linguistic single-valued neutrosophic Einstein arithmetic mean operators defined in (4.8).

ii. Calculate score value of Y_i^+ by Definition 3.2, of score function denoted by $Sc(Y_i^+)$.

iii. Arrange the score of Y_i^+ in increasing order. The maximum score of Y_i^+ will be the best alternatives.

Step 6. DHLSVN reference point approach (DHLSVNRP)

i. Again calculate the score of each entries of weighted aggregated decision matrix by using Definition 3.2.

ii. Calculate the reference point (Q_j^*) of alternatives is as follows:

$$Q_j^* = \max_j \left(sc[Q_{ij}]_{m\times n} \right)$$

iii. Calculate distance from each alternatives to reference point and weighted aggregated decision matrix by Definition 4.12.

iv. Rank the alternatives.

The conclusions are evaluated based on the values determined from the reference point. In this approach, we rank the alternatives based on maximum distance $\max_j \left(d\left(Q_{ij}, Q_j^* \right) \right)$in decreasing order and the best alternative has the lowest value that is $\min_i \left(\max_j \left(d\left(Q_{ij}, Q_j^* \right) \right) \right)$.

Step 7. DHLSVN full multiplicative approach

i. Utilizing (4.9), the defintion of double hierarchy linguistic single-valued neutrosophic Einstein geometric mean operator to calculate A_i^+.

ii. Calculate score of A_i^+ by Definition 3.2.

iii. The alternatives are ranked based on score value of A_i^+ in descending order. The alternative having maximum score value is the best results.

Step 8. Final Ranking

The final results of alternatives are determined from the ranking of the above three approaches. According to the ranking of all three approaches, alternatives are arranged in descending order, and select the best alternative. The graphical flowchart of the above-proposed method is provided in Figure 1.

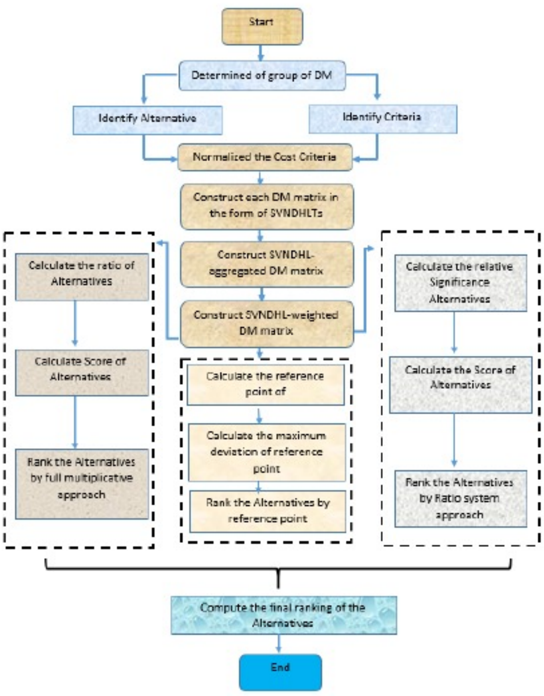

Figure 1. Graphical framework of proposed method

6. Application of Proposed Method

As with the expansion of the airport and the increase in air cargo business, the existing 3PL providers have been unable to meet the needs of the airport's transportation. This section provides a practical example concerning the selection of 3PL providers for the airport's transportation to validate the applicability and practicality of the developed methodology.

Therefore, we consider an illustrative case study in which an airport company wants to evaluate and select four (alternatives) potential 3PL providers, denoted as $\{A_1, A_2, A_3, A_4\}$ represent different airline companies. The four companies are passenger and cargo airlines, accounting for a major share of the air logistics market. For the selection of For 3PL providers, most research has focused on time rate, total assets, customer satisfaction, and personalized service. The definition of the most critical criteria of 3PL provider selection is shown as:

B_1: Total assets. All assets owned by a logistics enterprise
B_2: Time rate. Logistics delivery on time rate
B_3: Customer satisfaction. Matching degree of customer expectation and customer experience
B_4: Personalized service. Diversification degree in logistics products and services

On the based of above four defined criteria $B_j(j \in \{1,2,3,4\})$, we have four 3PL providers as alternatives $A_i(i \in \{1,2,3,4\})$ from which we will select the best 3PL. The following evaluation steps can solve the process of 3PL provider selection.

6.1. Evaluation Steps

In the process of the four 3PL provider selections, let the group of three expert DM_1, DM_2 and DM_3 are invited having weight vector $\omega_i = \{0.4, 0.5, 0.1\}$ to evaluate four alternatives based on the above criteria. The step-wise extended MULTIMOORA method within DHLSVNSs for selecting 3PL is as under:

Step 1. Construction of DHLSVN decision matrix

Construct the decision maker evaluation matrix in the from of DHLSVNSs , so the linguistic term set are denoted by $\gimel = \{\gimel_0 = \textit{medium}, \gimel_1 = \textit{low}, \gimel_2 = \textit{sightly low}, \gimel_3 = \textit{very low}, \gimel_4 = \textit{high}\ , \gimel_5 = \textit{slightly high}, \gimel_6 = \textit{very high}\}$ and $\daleth = \{\daleth_0 = \textit{right}, \daleth_1 = \textit{only right}, \daleth_2 = \textit{much}, \daleth_3 = \textit{very much}, \daleth_4 = \textit{little}\ , \daleth_5 = \textit{just little}\ , \daleth_6 = \textit{extermely little}\}$ are defined on the basis of following set as follows in Table 2-4

Table 2. Decision maker evaluation matrix DM_1

A_i	B_1	B_2	B_3	B_4
A_1	$\left\langle \gimel_{1\langle\daleth_1\rangle}, \gimel_{3\langle\daleth_4\rangle}, \gimel_{2\langle\daleth_1\rangle} \right\rangle$	$\left\langle \gimel_{4\langle\daleth_2\rangle}, \gimel_{2\langle\daleth_1\rangle}, \gimel_{0\langle\daleth_3\rangle} \right\rangle$	$\left\langle \gimel_{6\langle\daleth_1\rangle}, \gimel_{2\langle\daleth_1\rangle}, \gimel_{0\langle\daleth_2\rangle} \right\rangle$	$\left\langle \gimel_{1\langle\daleth_3\rangle}, \gimel_{5\langle\daleth_4\rangle}, \gimel_{3\langle\daleth_0\rangle} \right\rangle$
A_2	$\left\langle \gimel_{6\langle\daleth_3\rangle}, \gimel_{6\langle\daleth_4\rangle}, \gimel_{6\langle\daleth_2\rangle} \right\rangle$	$\left\langle \gimel_{5\langle\daleth_3\rangle}, \gimel_{3\langle\daleth_2\rangle}, \gimel_{1\langle\daleth_2\rangle} \right\rangle$	$\left\langle \gimel_{2\langle\daleth_3\rangle}, \gimel_{4\langle\daleth_2\rangle}, \gimel_{3\langle\daleth_1\rangle} \right\rangle$	$\left\langle \gimel_{3\langle\daleth_4\rangle}, \gimel_{1\langle\daleth_3\rangle}, \gimel_{2\langle\daleth_2\rangle} \right\rangle$
A_3	$\left\langle \gimel_{4\langle\daleth_2\rangle}, \gimel_{2\langle\daleth_1\rangle}, \gimel_{2\langle\daleth_3\rangle} \right\rangle$	$\left\langle \gimel_{2\langle\daleth_4\rangle}, \gimel_{4\langle\daleth_1\rangle}, \gimel_{2\langle\daleth_1\rangle} \right\rangle$	$\left\langle \gimel_{3\langle\daleth_0\rangle}, \gimel_{2\langle\daleth_0\rangle}, \gimel_{1\langle\daleth_4\rangle} \right\rangle$	$\left\langle \gimel_{4\langle\daleth_3\rangle}, \gimel_{2\langle\daleth_1\rangle}, \gimel_{1\langle\daleth_0\rangle} \right\rangle$
A_4	$\left\langle \gimel_{6\langle\daleth_4\rangle}, \gimel_{0\langle\daleth_0\rangle}, \gimel_{2\langle\daleth_1\rangle} \right\rangle$	$\left\langle \gimel_{4\langle\daleth_2\rangle}, \gimel_{5\langle\daleth_3\rangle}, \gimel_{1\langle\daleth_0\rangle} \right\rangle$	$\left\langle \gimel_{5\langle\daleth_4\rangle}, \gimel_{2\langle\daleth_1\rangle}, \gimel_{1\langle\daleth_3\rangle} \right\rangle$	$\left\langle \gimel_{0\langle\daleth_2\rangle}, \gimel_{3\langle\daleth_4\rangle}, \gimel_{0\langle\daleth_2\rangle} \right\rangle$

Table 3. Decision maker evaluation matrix DM_2

A_i	B_1	B_2	B_3	B_4
A_1	$\left\langle \gimel_{0\langle\daleth_{3}\rangle}, \gimel_{4\langle\daleth_{2}\rangle}, \gimel_{3\langle\daleth_{2}\rangle} \right\rangle$	$\left\langle \gimel_{1\langle\daleth_{1}\rangle}, \gimel_{3\langle\daleth_{2}\rangle}, \gimel_{0\langle\daleth_{2}\rangle} \right\rangle$	$\left\langle \gimel_{2\langle\daleth_{1}\rangle}, \gimel_{2\langle\daleth_{3}\rangle}, \gimel_{0\langle\daleth_{3}\rangle} \right\rangle$	$\left\langle \gimel_{1\langle\daleth_{2}\rangle}, \gimel_{4\langle\daleth_{3}\rangle}, \gimel_{1\langle\daleth_{0}\rangle} \right\rangle$
A_2	$\left\langle \gimel_{0\langle\daleth_{1}\rangle}, \gimel_{0\langle\daleth_{2}\rangle}, \gimel_{0\langle\daleth_{2}\rangle} \right\rangle$	$\left\langle \gimel_{1\langle\daleth_{2}\rangle}, \gimel_{2\langle\daleth_{1}\rangle}, \gimel_{4\langle\daleth_{3}\rangle} \right\rangle$	$\left\langle \gimel_{1\langle\daleth_{3}\rangle}, \gimel_{2\langle\daleth_{1}\rangle}, \gimel_{1\langle\daleth_{3}\rangle} \right\rangle$	$\left\langle \gimel_{2\langle\daleth_{3}\rangle}, \gimel_{1\langle\daleth_{2}\rangle}, \gimel_{3\langle\daleth_{4}\rangle} \right\rangle$
A_3	$\left\langle \gimel_{3\langle\daleth_{1}\rangle}, \gimel_{2\langle\daleth_{2}\rangle}, \gimel_{1\langle\daleth_{4}\rangle} \right\rangle$	$\left\langle \gimel_{2\langle\daleth_{3}\rangle}, \gimel_{3\langle\daleth_{4}\rangle}, \gimel_{2\langle\daleth_{2}\rangle} \right\rangle$	$\left\langle \gimel_{1\langle\daleth_{2}\rangle}, \gimel_{3\langle\daleth_{1}\rangle}, \gimel_{2\langle\daleth_{1}\rangle} \right\rangle$	$\left\langle \gimel_{0\langle\daleth_{3}\rangle}, \gimel_{1\langle\daleth_{6}\rangle}, \gimel_{5\langle\daleth_{1}\rangle} \right\rangle$
A_4	$\left\langle \gimel_{5\langle\daleth_{1}\rangle}, \gimel_{3\langle\daleth_{0}\rangle}, \gimel_{3\langle\daleth_{4}\rangle} \right\rangle$	$\left\langle \gimel_{1\langle\daleth_{0}\rangle}, \gimel_{2\langle\daleth_{1}\rangle}, \gimel_{2\langle\daleth_{4}\rangle} \right\rangle$	$\left\langle \gimel_{2\langle\daleth_{2}\rangle}, \gimel_{3\langle\daleth_{1}\rangle}, \gimel_{1\langle\daleth_{3}\rangle} \right\rangle$	$\left\langle \gimel_{1\langle\daleth_{2}\rangle}, \gimel_{3\langle\daleth_{2}\rangle}, \gimel_{4\langle\daleth_{1}\rangle} \right\rangle$

Table 4. Decision maker evaluation matrix DM_3

A_i	B_1	B_2	B_3	B_4
A_1	$\left\langle \gimel_{2\langle\daleth_{1}\rangle}, \gimel_{2\langle\daleth_{2}\rangle}, \gimel_{4\langle\daleth_{3}\rangle} \right\rangle$	$\left\langle \gimel_{0\langle\daleth_{1}\rangle}, \gimel_{1\langle\daleth_{5}\rangle}, \gimel_{2\langle\daleth_{1}\rangle} \right\rangle$	$\left\langle \gimel_{1\langle\daleth_{0}\rangle}, \gimel_{2\langle\daleth_{2}\rangle}, \gimel_{3\langle\daleth_{2}\rangle} \right\rangle$	$\left\langle \gimel_{0\langle\daleth_{2}\rangle}, \gimel_{3\langle\daleth_{2}\rangle}, \gimel_{1\langle\daleth_{2}\rangle} \right\rangle$
A_2	$\left\langle \gimel_{3\langle\daleth_{4}\rangle}, \gimel_{1\langle\daleth_{3}\rangle}, \gimel_{2\langle\daleth_{2}\rangle} \right\rangle$	$\left\langle \gimel_{2\langle\daleth_{2}\rangle}, \gimel_{3\langle\daleth_{2}\rangle}, \gimel_{1\langle\daleth_{2}\rangle} \right\rangle$	$\left\langle \gimel_{3\langle\daleth_{0}\rangle}, \gimel_{4\langle\daleth_{0}\rangle}, \gimel_{2\langle\daleth_{1}\rangle} \right\rangle$	$\left\langle \gimel_{3\langle\daleth_{1}\rangle}, \gimel_{4\langle\daleth_{1}\rangle}, \gimel_{3\langle\daleth_{0}\rangle} \right\rangle$
A_3	$\left\langle \gimel_{2\langle\daleth_{3}\rangle}, \gimel_{3\langle\daleth_{2}\rangle}, \gimel_{1\langle\daleth_{1}\rangle} \right\rangle$	$\left\langle \gimel_{3\langle\daleth_{3}\rangle}, \gimel_{2\langle\daleth_{3}\rangle}, \gimel_{3\langle\daleth_{2}\rangle} \right\rangle$	$\left\langle \gimel_{2\langle\daleth_{1}\rangle}, \gimel_{3\langle\daleth_{1}\rangle}, \gimel_{2\langle\daleth_{0}\rangle} \right\rangle$	$\left\langle \gimel_{2\langle\daleth_{3}\rangle}, \gimel_{6\langle\daleth_{2}\rangle}, \gimel_{2\langle\daleth_{0}\rangle} \right\rangle$
A_4	$\left\langle \gimel_{1\langle\daleth_{3}\rangle}, \gimel_{4\langle\daleth_{1}\rangle}, \gimel_{3\langle\daleth_{2}\rangle} \right\rangle$	$\left\langle \gimel_{3\langle\daleth_{4}\rangle}, \gimel_{4\langle\daleth_{1}\rangle}, \gimel_{1\langle\daleth_{2}\rangle} \right\rangle$	$\left\langle \gimel_{4\langle\daleth_{4}\rangle}, \gimel_{2\langle\daleth_{0}\rangle}, \gimel_{1\langle\daleth_{0}\rangle} \right\rangle$	$\left\langle \gimel_{3\langle\daleth_{1}\rangle}, \gimel_{4\langle\daleth_{3}\rangle}, \gimel_{4\langle\daleth_{1}\rangle} \right\rangle$

Step 2. Normalization of DHLSVN decision matrix

In this example all the criteria benefits; hence, here we skip the the normalization DHLSVN matrix.

Step 3. Construction of aggregated DHLSVN decision matrix

The aggregated DHLSVN decision matrix is constructed by using double hierarchy linguistic single-valued neutrosophic Einstein weighted averaging aggregation operator (4.7), and weight vector $\omega_i = \{0.4, 0.5, 0.1\}$ of decision maker in Table 5.

Table 5. Aggregated DHLSVN decision matrix

A_i	B_1	B_2	B_3	B_4
A_1	$\left\langle \gimel_{1.10\langle\daleth_{2.06}\rangle}, \gimel_{3.35\langle\daleth_{2.68}\rangle}, \gimel_{2.64\langle\daleth_{1.59}\rangle} \right\rangle$	$\left\langle \gimel_{2.31\langle\daleth_{1.41}\rangle}, \gimel_{2.31\langle\daleth_{1.70}\rangle}, \gimel_{0.00\langle\daleth_{2.21}\rangle} \right\rangle$	$\left\langle \gimel_{6.00\langle\daleth_{0.90}\rangle}, \gimel_{2.00\langle\daleth_{1.90}\rangle}, \gimel_{0.00\langle\daleth_{2.46}\rangle} \right\rangle$	$\left\langle \gimel_{0.90\langle\daleth_{2.42}\rangle}, \gimel_{4.27\langle\daleth_{3.25}\rangle}, \gimel_{1.59\langle\daleth_{0.00}\rangle} \right\rangle$
A_2	$\left\langle \gimel_{6.00\langle\daleth_{2.19}\rangle}, \gimel_{0.00\langle\daleth_{2.79}\rangle}, \gimel_{0.00\langle\daleth_{2.00}\rangle} \right\rangle$	$\left\langle \gimel_{3.21\langle\daleth_{2.42}\rangle}, \gimel_{2.46\langle\daleth_{1.42}\rangle}, \gimel_{2.10\langle\daleth_{2.46}\rangle} \right\rangle$	$\left\langle \gimel_{1.62\langle\daleth_{2.74}\rangle}, \gimel_{2.88\langle\daleth_{0.00}\rangle}, \gimel_{1.70\langle\daleth_{1.77}\rangle} \right\rangle$	$\left\langle \gimel_{2.52\langle\daleth_{3.27}\rangle}, \gimel_{1.16\langle\daleth_{2.21}\rangle}, \gimel_{2.56\langle\daleth_{0.00}\rangle} \right\rangle$
A_3	$\left\langle \gimel_{3.35\langle\daleth_{1.62}\rangle}, \gimel_{2.08\langle\daleth_{1.52}\rangle}, \gimel_{1.32\langle 3.16\rangle} \right\rangle$	$\left\langle \gimel_{2.10\langle\daleth_{3.43}\rangle}, \gimel_{3.25\langle\daleth_{2.34}\rangle}, \gimel_{2.08\langle\daleth_{1.52}\rangle} \right\rangle$	$\left\langle \gimel_{1.95\langle\daleth_{1.12}\rangle}, \gimel_{2.56\langle\daleth_{0.00}\rangle}, \gimel_{1.52\langle\daleth_{0.00}\rangle} \right\rangle$	$\left\langle \gimel_{2.05\langle\daleth_{3.00}\rangle}, \gimel_{1.64\langle\daleth_{2.95}\rangle}, \gimel_{2.59\langle\daleth_{0.00}\rangle} \right\rangle$
A_4	$\left\langle \gimel_{6.00\langle\daleth_{2.58}\rangle}, \gimel_{0.00\langle\daleth_{0.00}\rangle}, \gimel_{2.56\langle\daleth_{2.24}\rangle} \right\rangle$	$\left\langle \gimel_{2.58\langle\daleth_{1.29}\rangle}, \gimel_{3.20\langle\daleth_{1.59}\rangle}, \gimel_{1.42\langle\daleth_{0.00}\rangle} \right\rangle$	$\left\langle \gimel_{3.75\langle\daleth_{3.11}\rangle}, \gimel_{2.46\langle\daleth_{0.00}\rangle}, \gimel_{1.00\langle\daleth_{0.00}\rangle} \right\rangle$	$\left\langle \gimel_{0.82\langle\daleth_{1.90}\rangle}, \gimel_{3.09\langle\daleth_{2.79}\rangle}, \gimel_{0.00\langle\daleth_{1.32}\rangle} \right\rangle$

Step 4. Construction of weighted aggregated DHLSVN decision matrix

By considering weight vectors of criteria (0.2,0.3,0.4,0.1), and Definition 4.3, the DHLSVN weighted the aggregated matrix is determined in Table 6, as follows:

Table 6. DHLSVN weighted the aggregated matrix

A_i	B_1	B_2	B_3	B_4
A_1	$\left\langle \gimel_{0.22\langle\daleth_{0.42}\rangle}, \gimel_{5.21\langle\daleth_{4.99}\rangle}, \gimel_{4.98\langle\daleth_{4.57}\rangle} \right\rangle$	$\left\langle \gimel_{0.72\langle\daleth_{0.43}\rangle}, \gimel_{4.32\langle\daleth_{3.97}\rangle}, \gimel_{0.00\langle\daleth_{4.27}\rangle} \right\rangle$	$\left\langle \gimel_{6.00\langle\daleth_{0.36}\rangle}, \gimel_{3.59\langle\daleth_{3.52}\rangle}, \gimel_{0.00\langle\daleth_{3.91}\rangle} \right\rangle$	$\left\langle \gimel_{0.09\langle\daleth_{0.25}\rangle}, \gimel_{5.74\langle\daleth_{5.59}\rangle}, \gimel_{5.27\langle\daleth_{0.00}\rangle} \right\rangle$
A_2	$\left\langle \gimel_{6.00\langle\daleth_{0.46}\rangle}, \gimel_{0.00\langle\daleth_{5.03}\rangle}, \gimel_{0.00\langle\daleth_{4.74}\rangle} \right\rangle$	$\left\langle \gimel_{1.06\langle\daleth_{0.76}\rangle}, \gimel_{4.40\langle\daleth_{3.79}\rangle}, \gimel_{4.21\langle\daleth_{4.40}\rangle} \right\rangle$	$\left\langle \gimel_{0.66\langle\daleth_{1.17}\rangle}, \gimel_{4.17\langle\daleth_{0.00}\rangle}, \gimel_{3.38\langle\daleth_{3.43}\rangle} \right\rangle$	$\left\langle \gimel_{0.26\langle\daleth_{0.36}\rangle}, \gimel_{5.16\langle\daleth_{5.41}\rangle}, \gimel_{5.47\langle\daleth_{0.00}\rangle} \right\rangle$
A_3	$\left\langle \gimel_{0.75\langle\daleth_{0.33}\rangle}, \gimel_{4.78\langle\daleth_{4.54}\rangle}, \gimel_{4.44\langle\daleth_{5.15}\rangle} \right\rangle$	$\left\langle \gimel_{0.65\langle\daleth_{1.15}\rangle}, \gimel_{4.78\langle\daleth_{4.34}\rangle}, \gimel_{4.20\langle\daleth_{3.86}\rangle} \right\rangle$	$\left\langle \gimel_{0.80\langle\daleth_{0.45}\rangle}, \gimel_{3.97\langle\daleth_{0.00}\rangle}, \gimel_{3.24\langle\daleth_{0.00}\rangle} \right\rangle$	$\left\langle \gimel_{0.21\langle\daleth_{0.32}\rangle}, \gimel_{5.28\langle\daleth_{5.54}\rangle}, \gimel_{5.47\langle\daleth_{0.00}\rangle} \right\rangle$
A_4	$\left\langle \gimel_{6.00\langle\daleth_{0.55}\rangle}, \gimel_{0.00\langle\daleth_{0.00}\rangle}, \gimel_{4.95\langle\daleth_{4.84}\rangle} \right\rangle$	$\left\langle \gimel_{0.82\langle\daleth_{0.39}\rangle}, \gimel_{4.76\langle\daleth_{3.90}\rangle}, \gimel_{3.79\langle\daleth_{0.00}\rangle} \right\rangle$	$\left\langle \gimel_{1.71\langle\daleth_{1.35}\rangle}, \gimel_{3.91\langle\daleth_{0.00}\rangle}, \gimel_{2.77\langle\daleth_{0.00}\rangle} \right\rangle$	$\left\langle \gimel_{0.19\langle\daleth_{5.56}\rangle}, \gimel_{5.51\langle\daleth_{4.84}\rangle}, \gimel_{0.00\langle\daleth_{5.20}\rangle} \right\rangle$

Step 5. Ratio System Approach of SVNDHL (SVNDHLRSP)

i. In ratio system approach the Y_i^+ are calculated by using the Definition 4.6 of double hierarchy linguistic single-valued neutrosophic Einstein arithmetic (DHLSVNDEA) mean operators given in Table 6**.**

ii. According to definition the score calculate the score values of Y_i^+ given Table 7.

iii. According to score value Y_i^+ the alternatives are ranked in Table 7.

Table 7. Ranking of alternative based on ratio system approach

A_i	Y_i^+	$Score\left(Y_i^+\right)$	Ranking
A_1	$\left\langle \gimel_{6.00\langle \daleth_{1.24}\rangle}, \gimel_{1.74\langle \daleth_{1.36}\rangle}, \gimel_{0.00\langle \daleth_{0.00}\rangle} \right\rangle$	0.7814	3
A_2	$\left\langle \gimel_{6.00\langle \daleth_{2.37}\rangle}, \gimel_{0.00\langle \daleth_{0.00}\rangle}, \gimel_{0.00\langle \daleth_{0.00}\rangle} \right\rangle$	0.8994	1
A_3	$\left\langle \gimel_{2.18\langle \daleth_{1.96}\rangle}, \gimel_{1.75\langle \daleth_{0.00}\rangle}, \gimel_{1.077\langle \daleth_{0.00}\rangle} \right\rangle$	0.7030	4
A_4	$\left\langle \gimel_{6.00\langle \daleth_{2.26}\rangle}, \gimel_{0.00\langle \daleth_{0.00}\rangle}, \gimel_{0.00\langle \daleth_{0.00}\rangle} \right\rangle$	0.8962	2

Step 6. DHLSVN reference point approach (DHLSVRP)

i. In this approach, the reference points $\left(Q_j^*\right)$ are calculated by evaluating the score of each entries of DHLSVN weighted the aggregated matrix in Table 8.

Table 8. Reference points

B_1	B_2	B_3	B_4
$\left\langle \gimel_{6.00\langle \daleth_{0.46}\rangle}, \gimel_{0.00\langle \daleth_{5.03}\rangle}, \gimel_{0.00\langle \daleth_{4.74}\rangle} \right\rangle$	$\left\langle \gimel_{0.82\langle \daleth_{0.39}\rangle}, \gimel_{4.76\langle \daleth_{3.90}\rangle}, \gimel_{3.79\langle \daleth_{0.00}\rangle} \right\rangle$	$\left\langle \gimel_{1.71\langle \daleth_{1.35}\rangle}, \gimel_{3.91\langle \daleth_{0.00}\rangle}, \gimel_{2.77\langle \daleth_{0.00}\rangle} \right\rangle$	$\left\langle \gimel_{0.26\langle \daleth_{0.36}\rangle}, \gimel_{5.16\langle \daleth_{5.41}\rangle}, \gimel_{5.47\langle \daleth_{0.00}\rangle} \right\rangle$

ii. The distance from each alternatives to reference point and weighted aggregated decision matrix are calculated in Table 9.

Table 9. Distance of each alternatives and reference poin

	$d\left(Q_{1j}, Q_j^*\right)$	$d\left(Q_{2j}, Q_j^*\right)$	$d\left(Q_{3j}, Q_j^*\right)$	$d\left(Q_{4j}, Q_j^*\right)$	$\max_j\left(d\left(Q_{1j}, Q_j^*\right)\right)$	Ranking
A_1	0.450	0.242	0.439	0.034	0.450	4
A_2	0.000	0.164	0.154	0.000	0.164	1
A_3	0.430	0.157	0.065	0.009	0.430	3
A_4	0.282	0.000	0.000	0.320	0.320	2

Step 7. DHLSVN full multiplicative approach

i. In this step the A_i^+ are computed by using the double hierarchy linguistic single-valued neutrosophic Einstein geometric mean operator equation in Table in Table 9

ii. The score of full multiplicative A_i^+ and ranking of alternatives are computed in Table 10.

Table 10. Full multiplicative

Alternative	A_i^+	$Sc\left(A_i^+\right)$	Ranking
A_1	$\left\langle \gimel_{6.00\langle \daleth_{1.24}\rangle}, \gimel_{5.98\langle \daleth_{5.97}\rangle}, \gimel_{5.41\langle \daleth_{5.94}\rangle} \right\rangle$	0.220	3
A_2	$\left\langle \gimel_{6.00\langle \daleth_{2.37}\rangle}, \gimel_{5.82\langle \daleth_{5.87}\rangle}, \gimel_{5.69\langle \daleth_{5.94}\rangle} \right\rangle$	0.251	2
A_3	$\left\langle \gimel_{2.18\langle \daleth_{1.96}\rangle}, \gimel_{5.98\langle \daleth_{5.86}\rangle}, \gimel_{5.95\langle \daleth_{5.80}\rangle} \right\rangle$	0.126	4
A_4	$\left\langle \gimel_{6.00\langle \daleth_{2.64}\rangle}, \gimel_{5.84\langle \daleth_{4.68}\rangle}, \gimel_{5.90\langle \daleth_{5.31}\rangle} \right\rangle$	0.291	1

Step 8. Final Ranking

The overall ranking of alternatives based on the above three approaches are given in Table 11.

Table 11. Overall ranking of alternatives

Alternative	Ratio system	Reference point	Full multiplicative	Ranking
A_1	3	4	3	3
A_2	1	1	2	1
A_3	4	3	4	4
A_4	2	2	1	2

The graphical ranking of alternatives based on Extended DHLSVN-MULTIMOORA Techniques are given in Figure 2:

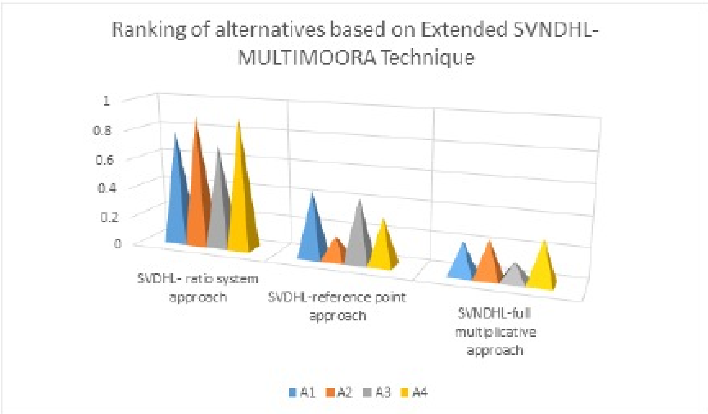

Figure 2. Graphically ranking of alternatives

7. Comparison Analysis

To verify the validity and significant effect of our developed strategy, we solve the problem by utilizing other approaches, including linguistic neutrosophic number weighted averaging (LNNWA) operator [50], TOPSIS method under linguistic neutrosophic number [52], generalized single valued neutrosophic linguistic weighted averaging (GSVNLWA) operator [53] and Single-valued neutrosophic linguistic TOPSIS [54]. The results are shown in Table 12.

Table 12. Comparative analysis

Existence Methods	$\mathbf{A_1}$	$\mathbf{A_2}$	$\mathbf{A_3}$	$\mathbf{A_4}$
LNNWA [50]	0.75	0.77	0.76	0.80
LNN-TOPSIS [52]	0.11	0.52	0.34	0.55
GSVNLWA [53]	0.73	0.78	0.76	0.79
SVNL-Extended TOPSIS [54]	0.86	0.91	0.88	0.92

To demonstrate the effectiveness of the proposed technique, the proposed approaches are compared with the existing operator LNNWA and LNN-TOPSIS in order to defend its dominance in DM problems. To achieve this, we first transform DHLSVN to LNN by second term equal to zero for comparison with the existing theory. By taking the same example, applying the existing LNNWA and LNN-TOPSIS method, the best result is A_4, which is similar to our proposed method, which shows the practicability of our proposed method.

To compare the proposed technique with the SVNL-TOPSIS and GSVNLWA operators we convert the DHLSVN to single-valued neutrosophic linguistic number and apply the existing method to same example the result are A_4, same to that of our method. Hence the same results indicate that our proposed method is an effective way to solve the decision-making problem.

8. Conclusion

In the current paper, the Extended DHLSVN-MULTIMOORA method consists of three parts: the ratio system approach, the reference point approach, and the complete multiplicative approach, which is developed to solve the MAGDM problem with vague information. DHLSVNSs is a more generalized tool that incorporates first and second hierarchy linguistic term sets with three mutually independent functions, namely true, uncertain, and false, to handle uncertain data more freely. Furthermore, the suggested research offered a list of new operation rules and Einstein aggregation operators by utilizing Einstein norms for DHLSVNSs to handle uncertainty in real-world decision-making problems. To handle multi-criteria group decision-making problems (MAGDM) A step-wise algorithm is given that is useful for DHLSVNSs. Finally, the proposed method is applied to third-party logistic service providers and also compared with other existing methods to show their effectiveness and applicability. The developed research has a variety of applications in real-world problems. In the future, many different MCGDM based on DHLSVNSs can be extended to various research areas, such as decision-making, medical diagnosis, pattern recognition, and image processing.

Author Contributions

All the authors equally contributed to this work. They all read and approved the final version of the paper.

Conflicts of Interest

All the authors declare no conflict of interest.

Ethical Review and Approval

No approval from the Board of Ethics is required.

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Soft difference-product: A new product for soft sets with its decision-making

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Abstract ─ A thorough mathematical foundation for handling uncertainty is provided by the concept of soft sets. Soft set operations are key concepts in soft set theory since they offer novel approaches to problems requiring parametric data. The "soft difference-product" a new product operation for soft sets, is proposed in this study along with all of its algebraic properties concerning different types of soft equalities and subsets. Additionally, we explore the connections between this product and other soft set operations by investigating the distributions of soft difference-product over other soft set operations. Using the *uni-int* operator and the uni-int decision function for the soft-difference product, we apply the *uni-int* decision-making method, which selects a set of optimal elements from the alternatives by giving an example that shows how the approach may be conducted effectively in various areas. Since the theoretical underpinnings of soft computing techniques are drawn from purely mathematical concepts, this study is crucial to the literature on soft sets.

Keywords: *Soft set, soft difference-product, soft subset, soft equal relations*

Subject Classification (2020): 03E20, 03E72

1. Introduction

Numerous mathematicians have developed a variety of mathematical tools to solve and model complex problems involving ambiguity, vagueness, and uncertainty in a variety of domains, including the social sciences, engineering, economics, and the medical sciences. Molodtsov [1] showed that these theories have inherent challenges, which are related to the potential for identifying a membership function in the case of fuzzy set theory [2] and the need to investigate the existence of the mean by conducting a large number of trials in the case of probability theory.

Therefore, Molodtsov [1] proposed the soft set, a novel mathematical technique, and looked into its uses in several fields, including probability theory, operations research, and game theory. Molodstov's soft set theory differs greatly from traditional ideas since it does not impose any limitations on the approximate description. After Maji et al. [3] used soft set theory in a decision-making problem, several researchers [4–10] developed some innovative soft set-based decision-making solutions and methods such as parameterization reduction of soft sets, soft information based on the theory of soft sets, texture classification using a novel, soft-set theory based classification algorithm, soft decision making for patients suspected influenza, and soft set-based decision making for patients suspected influenza-like illness, respectively. The trumpeted soft set-based decision-making method known as "*uni-int* decision-making" was proposed by Çağman and Enginoğlu [11].

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Additionally, the soft matrix was introduced by Çağman and Enginoğlu [12], who also developed decision-making techniques for the OR, AND, AND-NOT, and OR-NOT products of the soft matrices. They then applied these techniques to resolve uncertainties and other real-world problems. Soft set theory has been widely and successfully used to handle decision-making problems [13–24] via bijective soft set, exclusive disjunctive soft sets, generalized uni-int decision making schemes, soft discernibility matrix, soft approximations and uni-int decision making, the role of operators on soft set in decision making problems, reduced soft matrices and generalized products, cardinality inverse soft matrix theory, semantics of soft sets, the mean operators and generalized products of fuzzy soft matrices, and soft set-valued mappings, respectively.

In recent years, several researchers have investigated the underlying principles of soft set theory. A thorough theoretical study of soft sets, including soft subsets and supersets, equality of soft sets, and soft set operations like union, intersection, AND-product, and OR-product, was provided by Maji et al. [25]. Pei and Miao [26] investigated the relationship between soft sets and information systems and redefined intersection and soft set subsets. New soft set operations including the restricted union, restricted intersection, restricted difference, and extended intersection were proposed and studied by Ali et al. [27]. After that, the authors [28-41] examined the operations of the soft sets and the algebraic structures of the collection of the soft sets, proposed improved and novel methods, and identified several conceptual errors regarding the underlying assumptions of soft set theory that were presented in the published papers. Over the past several years, there has been a major advancement in the research of soft sets. Eren and Çalışıcı [42] defined a new kind of difference operation of soft sets. Stojanovic [43] described the extended symmetric difference of soft sets and investigated its fundamental properties. Many new types of soft set operations have been proposed and thoroughly examined in [44-49] such as soft binary piecewise difference operation, complementary soft binary piecewise theta, difference, union, intersection, and star operation of soft sets, respectively.

Two core concepts in soft set theory are soft equal relations and soft subsets. The first to use a somewhat accurate notion of soft subsets was Maji et al. [25]. One may consider the concept of soft subsets, which was established by Pei and Miao [26] and Feng et al. [29], to be an extension of Maji's earlier definitions [25]. Qin and Hong [50] introduced two new types of congruence relations and soft equal relations on soft sets. To modify Maji's soft distributive laws, Jun and Yang [51] used a wider range of soft subsets and extended soft equal relations, which we call J-soft equal relations for consistency's sake. Jun and Yang [51] conducted more research on the generalized soft distributive principles of soft product operations. Liu et al.[52] published a brief research note on soft L-subsets and soft L-equal relations, motivated by the novel ideas of Jun and Yang [51]. One significant result in [52] is that distributive rules do not hold for all of the soft equality described in the literature.

Thus, Feng et al. [53] extended the study reported in [52] by focusing on soft subsets and the soft products proposed in [24]. Feng et al. [53] focused on the different types of soft subsets and the algebraic properties of soft product operations. Along with commutative laws, association rules, and other crucial features, they also covered distributional laws, which were extensively researched by several academics. Besides, they provided theoretical research on the soft products, including the AND-product and OR-product using soft L-subsets, in addition to other relevant subjects. They completed some unfinished discoveries on soft product operations that had previously been reported in the literature and thoroughly investigated the algebraic characteristics of soft product operations in terms of J-equality and L-equality. Soft L-equal relations were shown to be congruent on free soft algebras and their associated quotient structures, which are commutative semigroups. For further information on soft equal connections such as generalized soft equality and soft lattice structure, generalized operations in soft set theory via relaxed conditions on parameters, g-soft equality and gf-soft equality relations, and T-soft equality relation, we refer to [54–58], respectively.

Çağman and Enginoğlu [11] revised the idea and workings of Molodtsov's soft sets to make them more useful. Moreover, they proposed four types of products in soft set theory: AND-product, OR-product, AND-NOT-product, OR-NOT-product, and uni-int decision function. Using these new definitions, they proposed a uniform decision-making procedure that chooses the best components from the range of options. Finally, they provided an example that demonstrates how the approach may be effectively used for a range of issues, including uncertainty. The AND-product of soft sets, which has long served as the foundation and a tool used by decision-makers in decision-making problems, was examined theoretically by Sezgin et al. [59]. Even though many scholars have studied the AND-product and its features concerning different types of soft equalities, such as soft L-equality and soft J-equality, the authors of [59] thoroughly examined the entire algebraic properties of the AND-product, including idempotent laws, commutative laws, associative laws, and other fundamental properties and compared them to previously obtained properties in terms of soft F-subsets, soft M-equality, soft L-equality, and soft J-equality. By establishing the distributive characteristics of AND-product over restricted, extended, and soft binary piecewise soft set operations, they also showed that the set of all soft sets over the universe is a commutative hemiring with identity in the sense of soft L-equality when combined with restricted/extended union and AND-product and that the set of all soft sets over the universe combined with restricted/extended symmetric difference and AND-product is also a commutative hemiring with identity in the sense of soft L-equality. Çağman and Enginoğlu [11] defined AND-product for soft sets, the domain of the approximation function of which is ExE, where E is the set of parameters. Furthermore, they show that this product is not commutative and associative under M-equality, but holds De Morgan Laws.

In this study, we first propose a new product for soft sets, which we call the "soft difference-product", using Molodtsov's concept of soft sets. Unlike the AND-NOT-product for soft sets defined in [11], the domain of the approximation function of the soft difference-product is the cartesian product of the parameter sets of the soft sets. We give an example of a soft difference-product and study its algebraic properties in detail regarding several soft subsets and soft equality types, such as M-subset/equality, F-subset/equality, L-subset/equality, and J-subset/equality. Moreover, we derive the distributions of the soft difference-product over several types of certain soft set operations. Finally, we apply the *uni-int* decision-making method proposed by Çağman and Enginoğlu [11] on soft difference-product to choose the best elements from the possibilities and provide an example that demonstrates how the approach may be effectively applied for many areas. This study aims to add to the literature on soft sets, as soft sets are a useful mathematical tool for identifying uncertainty and the theoretical foundations of soft computing approaches are derived from purely mathematical principles. This paper is organized as follows. In Section 2, we remind the basic concepts of soft set theory. Section 3 proposes the soft difference-product and discusses its whole algebraic properties in terms of several types of soft equalities and soft subsets. In Section 4, we examine the distributions of the soft difference-product over several types of soft set operations. In Section 5, the *uni-int* decision operators and function for soft difference-product are applied to a decision-making problem. The conclusion section has a deduction.

2. Preliminaries

This section presents some basic concepts to be needed in the following sections.

Definition 2.1. [1] Let U be the universal set, E be the parameter, $P(U)$ be the power set of U and $\mathcal{M} \subseteq E$. A pair $(\mho, \mathcal{M})$ is called a soft set over U where $\mho$ is a set-valued function such that $\mho: \mathcal{M} \rightarrow P(U)$.

Although Çağman and Enginoğlu [11] modified Molodstov's concept of soft sets, we continue to use the original definition of the soft set in our study. Throughout this paper, the collection of all the soft sets defined over U is designated as $S_E(U)$. Let $\mathcal{M}$ be a fixed subset of E and $S_{\mathcal{M}}(U)$ be the collection of all those soft sets over U with the fixed parameter set $\mathcal{M}$. That is, while in the set $S_{\mathcal{M}}(U)$, there are only soft sets whose parameter sets are $\mathcal{M}$; in the set $S_E(U)$, there are soft sets whose parameter sets may be any set. From now on, while soft sets will be designated by SS and parameter set by PS; soft sets will be designated by SSs and parameter sets by PSs for the sake of ease.

Definition 2.2. [27] Let $(\mho, \mathcal{M})$ be an SS over U. $(\mho, \mathcal{M})$ is called a relative null SS (with respect to the PS $\mathcal{M}$), denoted by $\emptyset_{\mathcal{M}}$, if $\mho(m) = \emptyset$ for all $m \in \mathcal{M}$ and $(\mho, \mathcal{M})$ is called a relative whole SS (with respect to the PS $\mathcal{M}$), denoted by $U_{\mathcal{M}}$ if $\mho(m) = U$ for all $m \in \mathcal{M}$. The relative whole SS U_E with respect to the universe set of parameters E is called the absolute SS over U.

The empty SS over U is the unique SS over U with an empty PS, represented by $\emptyset_{\emptyset}$. Note $\emptyset_{\emptyset}$ and $\emptyset_{\mathcal{M}}$ are different [31]. In the following, we always consider SSs with non-empty PSs in the universe U, unless otherwise stated.

The concept of soft subset, which we refer to here as soft M-subset to prevent confusion, was initially defined by Maji et al. [25] in the following extremely strict way:

Definition 2.3. [25] Let $(\mho, \mathcal{M})$ and $(\mathfrak{F}, \mathcal{D})$ be two SSs over U. $(\mho, \mathcal{M})$ is called a soft M-subset of $(\mathfrak{F}, \mathcal{D})$ denoted by $(\mho, \mathcal{M}) \tilde{\subseteq}_{\mathrm{M}} (\mathfrak{F}, \mathcal{D})$ if $\mathcal{M} \subseteq \mathcal{D}$ and $\mho(m) = \mathfrak{F}(m)$ for all $m \in \mathcal{M}$. Two SSs $(\mho, \mathcal{M})$ and $(\mathfrak{F}, \mathcal{D})$ are said to be soft M-equal, denoted by $(\mho, \mathcal{M}) =_{\mathrm{M}} (\mathfrak{F}, \mathcal{D})$, if $(\mho, \mathcal{M}) \tilde{\subseteq}_{\mathrm{M}} (\mathfrak{F}, \mathcal{D})$ and $(\mathfrak{F}, \mathcal{D}) \tilde{\subseteq}_{\mathrm{M}} (\mho, \mathcal{M})$.

Definition 2.4. [26] Let $(\mho, \mathcal{M})$ and $(\mathfrak{F}, \mathcal{D})$ be two SSs over U. $(\mho, \mathcal{M})$ is called a soft F-subset of $(\mathfrak{F}, \mathcal{D})$ denoted by $(\mho, \mathcal{M}) \tilde{\subseteq}_{\mathrm{F}} (\mathfrak{F}, \mathcal{D})$ if $\mathcal{M} \subseteq \mathcal{D}$ and $\mho(m) \subseteq \mathfrak{F}(m)$ for all $m \in \mathcal{M}$. Two SSs $(\mho, \mathcal{M})$ and $(\mathfrak{F}, \mathcal{D})$ are said to be soft F-equal, denoted by $(\mho, \mathcal{M}) =_{\mathrm{F}} (\mathfrak{F}, \mathcal{D})$, if $(\mho, \mathcal{M}) \tilde{\subseteq}_{\mathrm{F}} (\mathfrak{F}, \mathcal{D})$ and $(\mathfrak{F}, \mathcal{D}) \tilde{\subseteq}_{\mathrm{F}} (\mho, \mathcal{M})$.

It should be noted that the definitions of soft F-subset and soft F-equal were initially provided by Pei and Miao in [26]. However, some SS papers regarding soft subsets and soft equalities claimed that Feng et al. provided these definitions first in [29]. As a result, the letter "F" is used to denote this connection.

It was demonstrated in [52] that the soft equal relations $=_{\mathrm{M}}$ and $=_{\mathrm{F}}$ coincide. In other words, $(\mho, \mathcal{M}) =_{\mathrm{M}} (\mathfrak{F}, \mathcal{D}) \Leftrightarrow (\mho, \mathcal{M}) =_{\mathrm{F}} (\mathfrak{F}, \mathcal{D})$. Since they share the same set of parameters and approximation function, two SSs that meet this soft equivalence are truly identical [52], hence $(\mho, \mathcal{M}) =_{\mathrm{M}} (\mathfrak{F}, \mathcal{D})$ means, in fact, $(\mho, \mathcal{M}) = (\mathfrak{F}, \mathcal{D})$.

Jun and Yang [51] extended the ideas of F-soft subsets and soft F-equal relations by loosening the restrictions on PSs. We refer to them as soft J-subsets and soft J-equal relations, the initial letter of Jun, even though in [51] they are named generalized soft subset and generalized soft equal relation.

Definition 2.5 [51] Let $(\mho, \mathcal{M})$ and $(\mathfrak{F}, \mathcal{D})$ be two SSs over U. $(\mho, \mathcal{M})$ is called a soft J-subset of $(\mathfrak{F}, \mathcal{D})$ denoted by $(\mho, \mathcal{M}) \tilde{\subseteq}_{\mathrm{J}} (\mathfrak{F}, \mathcal{D})$ if for all $m \in \mathcal{M}$, there exists $d \in \mathcal{D}$ such that $\mho(m) \subseteq \mathfrak{F}(d)$. Two SSs $(\mho, \mathcal{M})$ and $(\mathfrak{F}, \mathcal{D})$ are said to be soft J-equal, denoted by $(\mho, \mathcal{M}) =_{\mathrm{J}} (\mathfrak{F}, \mathcal{D})$, if $(\mho, \mathcal{M}) \tilde{\subseteq}_{\mathrm{J}} (\mathfrak{F}, \mathcal{D})$ and $(\mathfrak{F}, \mathcal{D}) \tilde{\subseteq}_{\mathrm{J}} (\mho, \mathcal{M})$.

In [52] and [53], it was shown that $(\mho, \mathcal{M}) \tilde{\subseteq}_{\mathrm{M}} (\mathfrak{F}, \mathcal{D}) \Rightarrow (\mho, \mathcal{M}) \tilde{\subseteq}_{\mathrm{F}} (\mathfrak{F}, \mathcal{D}) \Rightarrow (\mho, \mathcal{M}) \tilde{\subseteq}_{\mathrm{J}} (\mathfrak{F}, \mathcal{D})$, but the converse may not be true.

Besides, Liu et al. [52] presented the following new kind of soft subsets (henceforth referred to as soft L-subsets and soft L-equality) that generalize both soft M-subsets and ontology-based soft subsets, inspired by the ideas of soft J-subset [51] and ontology-based soft subsets [30]:

Definition 2.6 [52] Let $(\mho, \mathcal{M})$ and $(\mathfrak{F}, \mathcal{D})$ be two SSs over U. $(\mho, \mathcal{M})$ is called a soft L-subset of $(\mathfrak{F}, \mathcal{D})$ denoted by $(\mho, \mathcal{M}) \tilde{\subseteq}_{\mathrm{L}} (\mathfrak{F}, \mathcal{D})$ if for all $m \in \mathcal{M}$, there exists $d \in \mathcal{D}$ such that $\mho(m) = \mathfrak{F}(d)$. Two SSs $(\mho, \mathcal{M})$ and $(\mathfrak{F}, \mathcal{D})$ are said to be soft L-equal, denoted by $(\mho, \mathcal{M}) =_{\mathrm{L}} (\mathfrak{F}, \mathcal{D})$, if $(\mho, \mathcal{M}) \tilde{\subseteq}_{\mathrm{L}} (\mathfrak{F}, \mathcal{D})$ and $(\mathfrak{F}, \mathcal{D}) \tilde{\subseteq}_{\mathrm{J}} (\mho, \mathcal{M})$.

As regards the relations between certain types of soft subsets and soft equalities, $(\mho, \mathcal{M}) \tilde{\subseteq}_{\mathrm{M}} (\mathfrak{F}, \mathcal{D}) \Rightarrow (\mho, \mathcal{M}) \tilde{\subseteq}_{\mathrm{L}} (\mathfrak{F}, \mathcal{D}) \Rightarrow (\mho, \mathcal{M}) \tilde{\subseteq}_{\mathrm{J}} (\mathfrak{F}, \mathcal{D})$ and $(\mho, \mathcal{M}) =_{\mathrm{M}} (\mathfrak{F}, \mathcal{D}) \Rightarrow (\mho, \mathcal{M}) =_{\mathrm{L}} (\mathfrak{F}, \mathcal{D}) \Rightarrow (\mho, \mathcal{M}) =_{\mathrm{J}} (\mathfrak{F}, \mathcal{D})$ [52]. However, the converses may not be true. Moreover, it is well-known that $(\mho, \mathcal{M}) =_{\mathrm{M}} (\mathfrak{F}, \mathcal{D})$ if and only if $(\mho, \mathcal{M}) =_{\mathrm{F}} (\mathfrak{F}, \mathcal{D})$.

We may thus deduce that soft M-equality (and so soft F-equality) is the strictest sense, whereas soft J-equality is the weakest soft equal connection. In the middle of these is the idea of the soft L-equal connection [52].

Example 2.7. Let $E = \{c_1, c_2, c_3, c_4, c_5\}$ be the PS, $\mathcal{M} = \{c_1, c_4\}$ and $\mathcal{D} = \{c_1, c_4, c_5\}$ be the subsets of E, and $U = \{z_1, z_2, z_3, z_4, z_5\}$ be the initial universe set. Let

$$(\mho, \mathcal{M}) = \{(c_1, \{z_1, z_3\}), (c_4, \{z_2, z_3, z_5\})\},$$

$$(\mathfrak{F}, \mathcal{D}) = \{(c_1, \{z_1, z_3\}), (c_4, \{z_2, z_3\}), (c_5, \{z_1, z_2, z_3, z_5\})\},$$

and

$$(\mathcal{V}, \mathcal{D}) = \{(c_1, \{z_2, z_3, z_5\}), (c_4, \{z_1, z_3\}), (c_5, \{z_1, z_2, z_3, z_5\})\}$$

Since $\mho(c_1) \subseteq \mathfrak{F}(c_1)$ (and also $\mho(c_1) \subseteq \mathfrak{F}(c_5)$) and $\mho(c_4) \subseteq \mathfrak{F}(c_5)$, it is obvious that $(\mho, \mathcal{M}) \mathrel{\tilde{\subseteq}_{\mathrm{J}}} (\mathfrak{F}, \mathcal{D})$. However, since $\mho(c_4) \neq \mathfrak{F}(c_1)$, $\mho(c_4) \neq \mathfrak{F}(c_4)$, and $\mho(c_4) \neq \mathfrak{F}(c_5)$, we can deduce that $(\mho, \mathcal{M})$ is not a soft L-subset of $(\mathfrak{F}, \mathcal{D})$. Moreover, as $\mho(c_4) \neq \mathfrak{F}(c_4)$, $(\mho, \mathcal{M})$ is not a soft M-subset of $(\mathfrak{F}, \mathcal{D})$. Moreover as

$\mho(c_1) = \mathcal{V}(c_4)$ and $\mho(c_4) = \mathcal{V}(c_1)$, it is obvious that $(\mho, \mathcal{M}) \mathrel{\tilde{\subseteq}_{\mathrm{L}}} (\mathcal{V}, \mathcal{D})$. However, as $\mho(c_1) \neq \mathcal{V}(c_1)$, $\mho(c_4) \neq \mathcal{V}(c_4)$, $(\mho, \mathcal{M})$ is not again a soft M-subset of $(\mathcal{V}, \mathcal{D})$.

Example 2.8. Let $E = \{c_1, c_2, c_3, c_4, c_5\}$ be the PS, $\mathcal{M} = \{c_1, c_4\}$ and $\mathcal{D} = \{c_1, c_4, c_5\}$ be the subsets of E, and $U = \{z_1, z_2, z_3, z_4, z_5\}$ be the initial universe set. Let $(\mho, \mathcal{M}) = \{(c_1, \{z_1, z_3\}), (c_4, \{z_1, z_2, z_3, z_5\})\}$ and $(\mathfrak{F}, \mathcal{D}) = \{(c_1, \{z_1, z_2, z_3\}), (c_4, \{z_1, z_2, z_3, z_5\}), (c_5, \{z_1\})\}$. Since $\mho(c_1) \neq \mathfrak{F}(c_1)$, $\mho(c_1) \neq \mathfrak{F}(c_4)$, and $\mho(c_1) \neq \mathfrak{F}(c_5)$, it is obvious that $(\mho, \mathcal{M}) \neq_{\mathrm{L}} (\mathfrak{F}, \mathcal{D})$. However, since $\mho(c_1) \subseteq \mathfrak{F}(c_1)$ (moreover $\mho(c_1) \subseteq \mathfrak{F}(c_4)$) and $\mho(c_4) \subseteq \mathfrak{F}(c_4)$, we can deduce that $(\mho, \mathcal{M}) \mathrel{\tilde{\subseteq}_{\mathrm{J}}} (\mathfrak{F}, \mathcal{D})$. Moreover, since $\mathfrak{F}(c_1) \subseteq \mho(c_4)$ and $\mathfrak{F}(c_4) \subseteq \mho(c_4)$, and $\mathfrak{F}(c_5) \subseteq \mho(c_1)$, we can deduce that $(\mathfrak{F}, \mathcal{D}) \mathrel{\tilde{\subseteq}_{\mathrm{J}}} (\mho, \mathcal{M})$. Therefore, $(\mho, \mathcal{M}) =_{\mathrm{J}} (\mathfrak{F}, \mathcal{D})$. As $\mho(c_1) \neq \mathfrak{F}(c_1)$ and $\mho(c_4) \neq \mathfrak{F}(c_4)$, it is obvious that $(\mho, \mathcal{M})$ is not a soft M-subset of $(\mathfrak{F}, \mathcal{D})$.

For more on soft F-equality, soft M-equality, soft J-equality, soft L-equality, and some other existing definitions of soft subsets and soft equal relations in the literature, we refer to [50-58].

Definition 2.9. [27] Let $(\mho, \mathcal{M})$ be an SS over U. The relative complement of an SS Let $(\mho, \mathcal{M})$, denoted by $(\mho, \mathcal{M})^{\mathrm{r}}$, is defined by $(\mho, \mathcal{M})^{\mathrm{r}} = (\mho^{\mathrm{r}}, \mathcal{M})$, where $\mho^{\mathrm{r}}: \mathcal{M} \to P(U)$ is a mapping given by $\mho^{\mathrm{r}}(m) = \mathrm{U} \setminus \mho(m)$, for all $m \in \mathcal{M}$. From now on, $\mathrm{U} \setminus \mho(m) = [\mho(m)]'$ is designated by $\mho'(m)$ for the sake of designation.

Definition 2.10. [25] Let $(\mho, \mathcal{M})$ and $(\mathfrak{F}, \mathcal{D})$ be two SSs over U. The AND-product ($\wedge$-product) of $(\mho, \mathcal{M})$ and $(\mathfrak{F}, \mathcal{D})$ is denoted by $(\mho, \mathcal{M}) \wedge (\mathfrak{F}, \mathcal{D})$, and is defined by $(\mho, \mathcal{M}) \wedge (\mathfrak{F}, \mathcal{D}) = (\sigma, \mathcal{M} \mathrm{x} \mathcal{D})$, where for all $(m, d) \in \mathcal{M} \mathrm{x} \mathcal{D}$, $\sigma(m, d) = \mho(m) \cap \mathfrak{F}(d)$.

Definition 2.11. [25] Let $(\mho, \mathcal{M})$ and $(\mathfrak{F}, \mathcal{D})$ be two SSs over U. The OR-product ($\vee$-product) of $(\mho, \mathcal{M})$ and $(\mathfrak{F}, \mathcal{D})$ is denoted by $(\mho, \mathcal{M}) \vee (\mathfrak{F}, \mathcal{D})$, and is defined by $(\mho, \mathcal{M}) \vee (\mathfrak{F}, \mathcal{D}) = (\sigma, \mathcal{M} \mathrm{x} \mathcal{D})$, where for all $(m, d) \in \mathcal{M} \mathrm{x} \mathcal{D}$, $\sigma(m, d) = \mho(m) \cup \mathfrak{F}(d)$.

Let "$\circledast$" to stand for set operations like $\cap, \cup, \setminus, \triangle$. The following definitions are for restricted, extended, and soft binary piecewise operations of soft sets.

Definition 2.12. [27] Let $(\mho, \mathcal{M})$ and $(\mathfrak{F}, \mathcal{D})$ be two SSs over U. The restricted $\circledast$ operation of $(\mho, \mathcal{M})$ and $(\mathfrak{F}, \mathcal{D})$, denoted by $(\mho, \mathcal{M}) \circledast_{\mathrm{R}} (\mathfrak{F}, \mathcal{D})$ is defined by $(\mho, \mathcal{M}) \circledast_{\mathrm{R}} (\mathfrak{F}, \mathcal{D}) = (\sigma, \mathcal{J})$, where $\mathcal{J} = \mathcal{M} \cap \mathcal{D}$ and if $\mathcal{J} \neq \emptyset$, then for all $\mathrm{j} \in \mathcal{J}$, $\sigma(\mathrm{j}) = \mho(\mathrm{j}) \circledast \mathfrak{F}(\mathrm{j})$; if $\mathcal{J} = \emptyset$, then $(\mho, \mathcal{M}) \circledast_{\mathrm{R}} (\mathfrak{F}, \mathcal{D}) = \emptyset_{\emptyset}$.

Definition 2.13. [27,43] Let $(\mho, \mathcal{M})$ and $(\mathfrak{F}, \mathcal{D})$ be two SSs over U. The extended $\circledast$ operation of $(\mho, \mathcal{M})$ and $(\mathfrak{F}, \mathcal{D})$, denoted by $(\mho, \mathcal{M}) \circledast_{\varepsilon} (\mathfrak{F}, \mathcal{D})$ is defined by $(\mho, \mathcal{M}) \circledast_{\varepsilon} (\mathfrak{F}, \mathcal{D}) = (\sigma, \mathcal{J})$, where $\mathcal{J} = \mathcal{M} \cup \mathcal{D}$ and then for all $\mathrm{j} \in \mathcal{J}$,

$$\sigma(\mathrm{j}) = \begin{cases} \mho(\mathrm{j}), & \mathrm{j} \in \mathcal{M} \setminus \mathcal{D} \\ \mathfrak{F}(\mathrm{j}), & \mathrm{j} \in \mathcal{D} \setminus \mathcal{M} \\ \mho(\mathrm{j}) \circledast \mathfrak{F}(\mathrm{j}), & \mathrm{j} \in \mathcal{M} \cap \mathcal{D} \end{cases}$$

Definition 2.14. [44] Let $(\mho,\mathcal{M})$ and $(\mathfrak{F},\mathcal{D})$ be two SSs over U. The soft binary piecewise $\circledast$ operation of $(\mho,\mathcal{M})$ and $(\mathfrak{F},\mathcal{D})$, denoted by $(\mho,\mathcal{M})\,\widetilde{\circledast}\,(\mathfrak{F},\mathcal{D})$, is defined by $(\mho,\mathcal{M})\,\widetilde{\circledast}\,(\mathfrak{F},\mathcal{D})=(\mho,\mathcal{M})$, where for all $\mathrm{j}\in\mathcal{M}$,

$$\mho(\mathrm{j})=\begin{cases}\mho(\mathrm{j}), & \mathrm{j}\in\mathcal{M}\setminus\mathcal{D},\\ \mho(\mathrm{j})\circledast\mathfrak{F}(\mathrm{j}), & \mathrm{j}\in\mathcal{M}\cap\mathcal{D}\end{cases}$$

For more about soft sets and picture fuzzy soft sets, we refer to [60-81].

3. Soft Difference-Product and Its Algebraic Properties

Çağman and Enginoğlu [11] defined AND-NOT-product for soft sets as Definition 3.1. In this subsection, we introduce a new product for soft sets, called soft difference-product in a similar way to the AND-NOT-product for soft sets. We give its example and examine its algebraic properties in detail depth in terms of specific kinds of soft equalities and soft subsets.

Definition 3.1. [11] Let $(\mho,\mathcal{M})$ and $(\mathfrak{F},\mathcal{D})$ be SSs over U. The $\overline{\wedge}$-product (AND-NOT-product) of $(\mho,\mathcal{M})$ and $(\mathfrak{F},\mathcal{D})$, denoted by $(\mho,\mathcal{M})\ \overline{\wedge}\ (\mathfrak{F},\mathcal{D})$, is defined by $(\mho,\mathcal{M})\ \overline{\wedge}\ (\mathfrak{F},\mathcal{D})=(\mho,ExE)$, where for all $(m,d)\in$ ExE, $\mho(m,d)=\mho(m)\setminus\mathfrak{F}(d)=\mho(m)\cap\mathfrak{F}'(d)$.

Definition 3.2. Let $(\mho,\mathcal{M})$ and $(\mathfrak{F},\mathcal{D})$ be SSs over U. The soft difference-product of $(\mho,\mathcal{M})$ and $(\mathfrak{F},\mathcal{D})$, denoted by $(\mho,\mathcal{M})\wedge_{\setminus}(\mathfrak{F},\mathcal{D})$, is defined by $(\mho,\mathcal{M})\wedge_{\setminus}(\mathfrak{F},\mathcal{D})=(\mho,\mathcal{M}\mathrm{x}\mathcal{D})$, where for all $(m,d)\in\mathcal{M}\mathrm{x}\mathcal{D}$, $\mho(m,d)=\mho(m)\setminus\mathfrak{F}(d)=\mho(m)\cap\mathfrak{F}'(d)$.

It is observed that while the domain of the approximation function of AND-NOT-product of $(\mho,\mathcal{M})$ and $(\mathfrak{F},\mathcal{D})$ defined by Çağman and Enginoğlu [11] is ExE, the domain of the approximation function of soft difference-product of $(\mho,\mathcal{M})$ and $(\mathfrak{F},\mathcal{D})$ is $\mathcal{M}\mathrm{x}\mathcal{D}$, leading to $(\mho,\mathcal{M})\wedge_{\setminus}(\mathfrak{F},\mathcal{D})\neq(\mho,\mathcal{M})\ \overline{\wedge}\ (\mathfrak{F},\mathcal{D})$. Since every input value (from the domain) must be associated with exactly one output value (in the range) in order for a function to be defined, this case giving rise to the resulting soft sets of AND-NOT-product and soft difference-product differ from each other as seen in the following example.

Example 3.3. Assume that $E=\{e_1,e_2,e_3,e_4,e_5\}$ be the PS, $\mathcal{M}=\{e_1,e_2,e_3\}$ and $\mathcal{D}=\{e_1,e_4,\mathrm{e}_5\}$ be the subsets of E, $U=\{h_1,h_2,h_3,h_4,h_5,h_6\}$ be the universal set, the SSs $(\mho,\mathcal{M})$ and $(\mathfrak{F},\mathcal{D})$ be over U such that

$$(\mho,\mathcal{M})=\{(e_1,\{h_1,h_2,h_3,h_5\}),(e_2,\{h_1,h_2,h_3\}),(e_3,\{h_4,h_5,h_6\})\}$$

and

$$(\mathfrak{F},\mathcal{D})=\{(e_1,\{h_6\}),(e_4,\{h_2,h_3,h_5\}),(e_5,\{h_2\})\}$$

Let $(\mho,\mathcal{M})\wedge_{\setminus}(\mathfrak{F},\mathcal{D})=(\mho,\mathcal{M}\mathrm{x}\mathcal{D})$. Then,

$$(\mho,\mathcal{M}\mathrm{x}\mathcal{D})=\{((e_1,e_1),\{h_1,h_2,h_3,h_5\}),((e_1,e_4),\{h_1\}),((e_1,e_5),\{h_1,h_3,h_5\}),((e_2,e_1),\{h_1,h_2,h_3\}),((e_2,e_4),\{h_1\}),$$
$$((e_2,e_5),\{h_1,h_3\}),((e_3,e_1),\{h_4,h_5\}),((e_3,e_4),\{h_4,h_6\}),((e_3,e_5),\{h_4,h_5,h_6\})\}$$

Assume that $(\mho,\mathcal{M})\ \overline{\wedge}\ (\mathfrak{F},\mathcal{D})=(\daleth,ExE)$. Then,

$$(\daleth,ExE)=\{((e_1,e_1),\{h_1,h_2,h_3,h_5\}),((e_1,e_2),\{h_1,h_2,h_3,h_5\}),((e_1,e_3),\{h_1,h_2,h_3,h_5\}),((e_1,e_4),\{h_1\}),$$
$$((e_1,e_5),\{h_1,h_3,h_5\}),((e_2,e_1),\{h_1,h_2,h_3\}),((e_2,e_2),\{h_1,h_2,h_3\}),((e_2,e_3),\{h_1,h_2,h_3\}),((e_2,e_4),\{h_1\}),((e_2,e_5),\{h_1,h_3\}),$$
$$((e_3,e_1),\{h_4,h_5\}),((e_3,e_2),\{h_4,h_5,h_6\}),((e_3,e_3),\{h_4,h_5,h_6\}),((e_3,e_4),\{h_4,h_6\}),((e_3,e_5),\{h_4,h_5,h_6\})\}$$

Here, note that since $\daleth(e_4,e_1)=\daleth(e_4,e_2)=\daleth(e_4,e_3)=\daleth(e_4,e_4)=\daleth(e_4,e_5)=\daleth(e_5,e_1)=\daleth(e_5,e_2)=\daleth(e_5,e_3)=\daleth(e_5,e_4)=\daleth(e_5,e_5)=\emptyset$, they are not designated in the soft set $(\daleth,ExE)$. It can be easily observed that $(\mho,\mathcal{M}\mathrm{x}\mathcal{D})\neq(\daleth,ExE)$.

It is more convenient to use the table method to write the result of the soft difference-product than writing it in the list method.

Table 1. The table designation of the soft difference-product's result of the soft sets in Example 3.3

$(℧,\mathcal{M})\Lambda_{\backslash}(\mathfrak{F},\mathcal{D})$	e_1	e_4	e_5
e_1	$\{h_1,h_2,h_3,h_5\}$	$\{h_1\}$	$\{h_1,h_3,h_5\}$
e_2	$\{h_1,h_2,h_3\}$	$\{h_1\}$	$\{h_1,h_3\}$
e_3	$\{h_4,h_5\}$	$\{h_4,h_6\}$	$\{h_4,h_5,h_6\}$

Proposition 3.4. $\Lambda_{\backslash}$-product is closed in $S_E(U)$.

PROOF. It is obvious that $\Lambda_{\backslash}$-product is a binary operation in $S_E(U)$. In fact, let $(℧,\mathcal{M})$ and $(\mathfrak{F},\mathcal{D})$ be SSs over U. Then,

$$\Lambda_{\backslash}: S_E(U)\ x\ S_E(U) \longrightarrow S_E(U)$$

$$\big((℧,\mathcal{M}),(\mathfrak{F},\mathcal{D})\big) \longrightarrow (℧,\mathcal{M})\Lambda_{\backslash}(\mathfrak{F},\mathcal{D}) = (\mathcal{V},\mathcal{M}\mathrm{x}\mathcal{D}) = (\mathcal{V},\mathcal{J})$$

Since the set $S_E(U)$ contains all the SS over U, $(\mathcal{V},\mathcal{J}) \in S_E(U)$. Here, note that the set $S_{\mathcal{M}}(U)$ is not closed under $\Lambda_{\backslash}$-product. That is, when $(℧,\mathcal{M})$ and $(\mathfrak{F},\mathcal{M})$ are the elements of $S_{\mathcal{M}}(U)$, $(℧,\mathcal{M})\Lambda_{\backslash}(\mathfrak{F},\mathcal{M})$ is an element of $S_{\mathcal{M}x\mathcal{M}}(U)$ not $S_{\mathcal{M}}(U)$. □

Proposition 3.5. Let $(℧,\mathcal{M})$, $(\mathfrak{F},\mathcal{D})$, and $(\mathcal{V},\mathcal{J})$ be SSs over U. Then,

$$(℧,\mathcal{M})\Lambda_{\backslash}\big[(\mathfrak{F},\mathcal{D})\Lambda_{\backslash}(\mathcal{V},\mathcal{J})\big] \neq_{\mathrm{M}} \big[(℧,\mathcal{M})\Lambda_{\backslash}(\mathfrak{F},\mathcal{D})\big]\Lambda_{\backslash}(\mathcal{V},\mathcal{J})$$

Thus, $\Lambda_{\backslash}$ -product is not associative in $S_E(U)$.

PROOF. In order to show that $\Lambda_{\backslash}$-product is not associative in $S_E(U)$, we provided an example: Let $E = \{e_1,e_2,e_3,e_4\}$ be the PS, $\mathcal{M} = \{e_2,e_3\}$, $\mathcal{D} = \{e_1\}$, and $\mathcal{J} = \{e_4\}$ be the subsets of E , $U = \{h_1,h_2,h_3,h_4,h_5\}$ be the universal set, and $(℧,\mathcal{M})$, $(\mathfrak{F},\mathcal{D})$ ve $(\mathcal{V},\mathcal{J})$ be SSs over U such that $(℧,\mathcal{M}) = \{(e_2,\{h_3,h_4\}),(e_3,\{h_1\})\}$, $(\mathfrak{F},\mathcal{D}) = \{(e_1,\emptyset)\}$, and $(\mathcal{V},\mathcal{J}) = \{(e_4,\{h_1,h_3,h_5\})\}$. We show that

$$(℧,\mathcal{M})\Lambda_{\backslash}\big[(\mathfrak{F},\mathcal{D})\Lambda_{\backslash}(\mathcal{V},\mathcal{J})\big] \neq_{\mathrm{M}} \big[(℧,\mathcal{M})\Lambda_{\backslash}(\mathfrak{F},\mathcal{D})\big]\Lambda_{\backslash}(\mathcal{V},\mathcal{J})$$

Let $(\mathfrak{F},\mathcal{D})\Lambda_{\backslash}(\mathcal{V},\mathcal{J}) = (\mho,\mathcal{D}\mathrm{x}\mathcal{J})$, Then,

$$(\mathfrak{F},\mathcal{D})\Lambda_{\backslash}(\mathcal{V},\mathcal{J}) = (\mho,\mathcal{D}\mathrm{x}\mathcal{J}) = \{((e_1,e_4),\emptyset)\}$$

and let $(℧,\mathcal{M})\ \Lambda_{\backslash}\ (\mho,\mathcal{D}\mathrm{x}\mathcal{J}) = \big(\mathfrak{X},\mathcal{M}\mathrm{x}(\mathcal{D}\mathrm{x}\mathcal{J})\big)$. Thus,

$$(℧,\mathcal{M})\ \Lambda_{\backslash}\ (\mho,\mathcal{D}\mathrm{x}\mathcal{J}) = \big(\mathfrak{X},\mathcal{M}\mathrm{x}(\mathcal{D}\mathrm{x}\mathcal{J})\big) = \{((e_2,(e_1,e_4)),\{h_3,h_4\}),((e_3,(e_1,e_4)),\{h_1\})\}$$

Assume that $(℧,\mathcal{M})\Lambda_{\backslash}(\mathfrak{F},\mathcal{D}) = (\mathfrak{Z},\mathcal{M}\mathrm{x}\mathcal{D})$. Thereby,

$$(℧,\mathcal{M})\Lambda_{\backslash}(\mathfrak{F},\mathcal{D}) = (\mathfrak{Z},\mathcal{M}\mathrm{x}\mathcal{D}) = \{\big((e_2,e_1),\{h_3,h_4\}\big),((e_3,e_1),\{h_1\})\}$$

Suppose that $(\mathfrak{Z},\mathcal{M}\mathrm{x}\mathcal{D})\ \Lambda_{\backslash}\ (\mathcal{V},\mathcal{J}) = \big(\wp,(\mathcal{M}\mathrm{x}\mathcal{D})\mathrm{x}\mathcal{J}\big)$. Therefore,

$$(\mathfrak{Z},\mathcal{M}\mathrm{x}\mathcal{D})\ \Lambda_{\backslash}\ (\mathcal{V},\mathcal{J}) = \big(\wp,(\mathcal{M}\mathrm{x}\mathcal{D})\mathrm{x}\mathcal{J}\big) = \{(((e_2,e_1),e_4),\{h_4\}),(((e_3,e_1),e_4),\emptyset)\}$$

It is seen that $\big(\mathfrak{X},\mathcal{M}\mathrm{x}(\mathcal{D}\mathrm{x}\mathcal{J})\big) \neq_{\mathrm{M}} \big(\wp,(\mathcal{M}\mathrm{x}\mathcal{D})\mathrm{x}\mathcal{J}\big)$. It is also seen that $\big(\mathfrak{X},\mathcal{M}\mathrm{x}(\mathcal{D}\mathrm{x}\mathcal{J})\big) \neq_{\mathrm{L}} \big(\wp,(\mathcal{M}\mathrm{x}\mathcal{D})\mathrm{x}\mathcal{J}\big)$ and $\big(\mathfrak{X},\mathcal{M}\mathrm{x}(\mathcal{D}\mathrm{x}\mathcal{J})\big) \neq_{\mathrm{J}} \big(\wp,(\mathcal{M}\mathrm{x}\mathcal{D})\mathrm{x}\mathcal{J}\big)$. □

Proposition 3.6. Let $(\mathfrak{F},\mathcal{D})$ and $(\mathcal{V},\mathcal{J})$ be SSs over U. Then, $(\mathfrak{F},\mathcal{D})\Lambda_{\backslash}(\mathcal{V},\mathcal{J}) \neq_{\mathrm{M}} (\mathcal{V},\mathcal{J})\Lambda_{\backslash}(\mathfrak{F},\mathcal{D})$. That is, $\Lambda_{\backslash}$-product is not commutative in $S_E(U)$.

Proof. Let $(\mathfrak{F},\mathcal{D})\Lambda_{\backslash}(\mathcal{V},\mathcal{J}) = (\varsigma,\mathcal{D}\mathrm{x}\mathcal{J})$ and $(\mathcal{V},\mathcal{J})\Lambda_{\backslash}(\mathfrak{F},\mathcal{D})$=$(\mathfrak{X},\mathcal{J}\mathrm{x}\mathcal{D})$. Since $\mathcal{D}\mathrm{x}\mathcal{J} \neq \mathcal{J}\mathrm{x}\mathcal{D}$, the rest of the proof is obvious. □

Proposition 3.7. Let $(\mathfrak{F},\mathcal{D})$ and $(\mathcal{V},\mathcal{J})$ be SSs over U. Then, $(\mathfrak{F},\mathcal{D})\Lambda_{\backslash}(\mathcal{V},\mathcal{J}) \neq_{\mathrm{J}} (\mathcal{V},\mathcal{J})\Lambda_{\backslash}(\mathfrak{F},\mathcal{D})$. That is, $\Lambda_{\backslash}$-product is not commutative in $S_E(U)$ under J-equlity.

Proof. In order to show that $\Lambda_{\backslash}$-product is not commutative in $S_E(U)$ under J-equality, we provide an example. Let $E = \{\mathrm{e}_1, e_2, e_3, e_4, e_5\}$ be the PS, $\mathcal{D} = \{e_3, e_5\}$ and $\mathcal{J} = \{e_1\}$ be the subsets of E, $U = \{h_1, h_2, h_3, h_4, h_5\}$ be the universal set, $(\mathfrak{F},\mathcal{D})$ and $(\mathcal{V},\mathcal{J})$ be SSs over U such that $(\mathfrak{F},\mathcal{D}) = \{(e_3,\{h_1,h_2\}),(e_5,U)\}$ and $(\mathcal{V},\mathcal{J}) = \{(e_1,\{h_5\})\}$. We show that $(\mathfrak{F},\mathcal{D})\Lambda_{\backslash}(\mathcal{V},\mathcal{J}) \neq_{\mathrm{J}} (\mathcal{V},\mathcal{J})\Lambda_{\backslash}(\mathfrak{F},\mathcal{D})$. Let $(\mathfrak{F},\mathcal{D})\Lambda_{\backslash}(\mathcal{V},\mathcal{J}) = (\varsigma,\mathcal{D}\mathrm{x}\mathcal{J})$. Then,

$$(\mathfrak{F},\mathcal{D})\Lambda_{\backslash}(\mathcal{V},\mathcal{J}) = (\varsigma,\mathcal{D}\mathrm{x}\mathcal{J}) = \{((e_3,e_1),\{h_1,h_2\}),((e_5,e_1),\{h_1,h_2,h_3,h_4\})\}$$

Suppose that $(\mathcal{V},\mathcal{J})\Lambda_{\backslash}(\mathfrak{F},\mathcal{D}) = (\mathfrak{z},\mathcal{J}\mathrm{x}\mathcal{D})$. Then,

$$(\mathcal{V},\mathcal{J})\Lambda_{\backslash}(\mathfrak{F},\mathcal{D}) = (\mathfrak{z},\mathcal{J}\mathrm{x}\mathcal{D}) = \{\big((e_1,e_3),\{h_5\}\big),((e_1,e_5),\emptyset)\}$$

Thus, $(\mathfrak{F},\mathcal{D})\Lambda_{\backslash}(\mathcal{V},\mathcal{J}) \neq_{\mathrm{J}} (\mathcal{V},\mathcal{J})\Lambda_{\backslash}(\mathfrak{F},\mathcal{D})$.

Moreover, it is obvious that $(\mathfrak{F},\mathcal{D})\Lambda_{\backslash}(\mathcal{V},\mathcal{J}) \neq_{\mathrm{L}} (\mathcal{V},\mathcal{J})\Lambda_{\backslash}(\mathfrak{F},\mathcal{D})$. □

Proposition 3.8. Let $(\mathfrak{F},\mathcal{D})$ be an SS over U. Then, $(\mathfrak{F},\mathcal{D})\Lambda_{\backslash}\emptyset_{\emptyset} =_{\mathrm{M}} \emptyset_{\emptyset}\Lambda_{\backslash}(\mathfrak{F},\mathcal{D}) =_{\mathrm{M}} \emptyset_{\emptyset}$. That is, $\emptyset_{\emptyset}$ (the empty SS) is the absorbing element of $\Lambda_{\backslash}$-product in $S_E(U)$ under M-equality.

Proof. Let $\emptyset_{\emptyset} = (\varsigma,\emptyset)$ and $(\mathfrak{F},\mathcal{D})\Lambda_{\backslash}\emptyset_{\emptyset} = (\mathfrak{F},\mathcal{D})\Lambda_{\backslash}(\varsigma,\emptyset) = \big(\wp,\mathcal{D}\mathrm{x}\emptyset\big) = (\wp,\emptyset)$. Since the only SS whose PS is $\emptyset_{\emptyset}$, $\big(\wp,\emptyset\big) = \emptyset_{\emptyset}$. One can similarly show that $\emptyset_{\emptyset}\Lambda_{\theta}(\mathfrak{F},\mathcal{D}) =_{\mathrm{M}} \emptyset_{\emptyset}$. □

Proposition 3.9. Let $(\mho,\mathcal{M})$ be an SS over U. Then, $\emptyset_{\mathcal{M}}\Lambda_{\backslash}(\mho,\mathcal{M}) =_{\mathrm{L}} \emptyset_{\mathcal{M}}$. That is, $\emptyset_{\mathcal{M}}$ is the left absorbing element of $\Lambda_{\backslash}$-product in $S_{\mathcal{M}}(U)$ under L-equality.

Proof. Let $\emptyset_{\mathcal{M}} = (\mathcal{V},\mathcal{M})$ and $(\mathcal{V},\mathcal{M})\Lambda_{\backslash}(\mho,\mathcal{M}) = (\mathfrak{z},\mathcal{M}\mathrm{x}\mathcal{M})$. Then, for all $m \in \mathcal{M}$, $\mathcal{V}(m) = \emptyset$ and for all $(m,d) \in \mathcal{M}\mathrm{x}\mathcal{M}$, $\mathfrak{z}(m,d) = \mathcal{V}(m) \cap \mho'(d)$=$\emptyset \cap \mho'(d) = \emptyset$. Since, for all $(m,d) \in \mathcal{M}\mathrm{x}\mathcal{M}$, there exists $m \in \mathcal{M}$ such that $\mathfrak{z}(m,d) = \emptyset = \mathcal{V}(m)$, $\emptyset_{\mathcal{M}}\Lambda_{\backslash}(\mho,\mathcal{M}) \widetilde{\subseteq}_{\mathrm{L}} \emptyset_{\mathcal{M}}$. Moreover, for all $m \in \mathcal{M}$, there exists $(m,d) \in \mathcal{M}\mathrm{x}\mathcal{M}$ such that $\mathcal{V}(m) = \emptyset = \mathfrak{z}(m,d)$, implying that $\emptyset_{\mathcal{M}} \widetilde{\subseteq}_{\mathrm{L}} \emptyset_{\mathcal{M}}\Lambda_{\backslash}(\mho,\mathcal{M})$ Thereby, $\emptyset_{\mathcal{M}}\Lambda_{\backslash}(\mho,\mathcal{M}) =_{\mathrm{L}} \emptyset_{\mathcal{M}}$. □

Proposition 3.10. Let $(\mho,\mathcal{M})$ be an SS over U. Then, $(\mho,\mathcal{M})\Lambda_{\backslash}\emptyset_{\mathcal{M}} =_{\mathrm{L}} (\mho,\mathcal{M})$. That is, $\emptyset_{\mathcal{M}}$ is the right identity element of $\Lambda_{\backslash}$-product in $S_{\mathcal{M}}(U)$ under L-equality.

Proof. Let $\emptyset_{\mathcal{M}} = (\mathcal{V},\mathcal{M})$ and $(\mho,\mathcal{M})\Lambda_{\backslash}(\mathcal{V},\mathcal{M}) = (\mathfrak{z},\mathcal{M}\mathrm{x}\mathcal{M})$. Then, for all $m \in \mathcal{M}$, $\mathcal{V}(m) = \emptyset$ and for all $(m,d) \in \mathcal{M}\mathrm{x}\mathcal{M}$, $\mathfrak{z}(m,d) = \mho(m) \cap \emptyset' = \mho(d) \cap U = \mho(d)$. Since, for all $(m,d) \in \mathcal{M}\mathrm{x}\mathcal{M}$, there exists $d \in \mathcal{M}$ such that $\mathfrak{z}(m,d) = \mho(d)$, $(\mho,\mathcal{M})\Lambda_{\backslash}\emptyset_{\mathcal{M}} \widetilde{\subseteq}_{\mathrm{L}} (\mho,\mathcal{M})$. Moreover, for all $d \in \mathcal{M}$, there exists $(m,d) \in \mathcal{M}\mathrm{x}\mathcal{M}$ such that $\mho(d) = \mathfrak{z}(m,d)$, implying that $(\mho,\mathcal{M})_{\mathrm{L}} \widetilde{\subseteq}_{\mathrm{L}} (\mho,\mathcal{M})\Lambda_{\backslash}\emptyset_{\mathcal{M}}$. Thereby, $(\mho,\mathcal{M})\Lambda_{\backslash}\emptyset_{\mathcal{M}} =_{\mathrm{L}} (\mho,\mathcal{M})$. □

Proposition 3.11. Let $(\mathfrak{F},\mathcal{D})$ be an SS over U. Then, $(\mathfrak{F},\mathcal{D})\Lambda_{\backslash}\emptyset_{\mathcal{D}} =_{\mathrm{M}} (\mathfrak{F},\mathcal{D}\mathrm{x}\mathcal{D})$ and $\emptyset_{\mathcal{D}}\Lambda_{\backslash}(\mathfrak{F},\mathcal{D}) =_{\mathrm{M}} \emptyset_{\mathcal{D}\mathrm{x}\mathcal{D}}$.

Proof. Let $\emptyset_{\mathcal{D}} = (\varsigma,\mathcal{D})$. Then, for all $d \in \mathcal{D}, \varsigma(d) = \emptyset$. Let $(\mathfrak{F},\mathcal{D})\Lambda_{\backslash}\emptyset_{\mathcal{D}} = (\mathfrak{F},\mathcal{D})\Lambda_{\backslash}(\varsigma,\mathcal{D}) = \big(\wp,\mathcal{D}\mathrm{x}\mathcal{D}\big)$. Thus, for all $(d,m) \in \mathcal{D}\mathrm{x}\mathcal{D}, \wp(d,m) = \mathfrak{F}(d) \cap \varsigma'(m) = \mathfrak{F}(d) \cap \emptyset' = \mathfrak{F}(d) \cap \mathrm{U} = \mathfrak{F}(d)$, implying that $\big(\wp,\mathcal{D}\mathrm{x}\mathcal{D}\big) = (\mathfrak{F},\mathcal{D}\mathrm{x}\mathcal{D})$. Let $\emptyset_{\mathcal{D}}\Lambda_{\backslash}(\mathfrak{F},\mathcal{D}) = (\varsigma,\mathcal{D})\Lambda_{\backslash}(\mathfrak{F},\mathcal{D}) = (\mathcal{V},\mathcal{D}\mathrm{x}\mathcal{D})$. Then, for all $(d,m) \in \mathcal{D}\mathrm{x}\mathcal{D}, \mathcal{V}(d,m) = \varsigma(d) \cap \mathfrak{F}'(m) = \emptyset \cap \mathfrak{F}'(m) = \emptyset$, hence $(\mathcal{V},\mathcal{D}\mathrm{x}\mathcal{D}) = \emptyset_{\mathcal{D}\mathrm{x}\mathcal{D}}$. □

Proposition 3.12. Let $(\mathfrak{F},\mathcal{D})$ be an SS over U. Then, $(\mathfrak{F},\mathcal{D})\Lambda_{\backslash}\mathrm{U}_{\mathcal{D}} =_{\mathrm{M}} \emptyset_{\mathcal{D}\mathrm{x}\mathcal{D}}$ and $\mathrm{U}_{\mathcal{D}}\Lambda_{\backslash}(\mathfrak{F},\mathcal{D}) =_{\mathrm{M}} (\mathfrak{F},\mathcal{D}\mathrm{x}\mathcal{D})^{\mathrm{r}}$.

Proof. Let $\mathrm{U}_{\mathcal{D}} = (℣,\mathcal{D})$. Then, for all $d \in \mathcal{D}$, $℣(d) = \mathrm{U}$. Let $(\mathfrak{F},\mathcal{D})\Lambda_{\backslash}\mathrm{U}_{\mathcal{D}} = (\mathfrak{F},\mathcal{D})\Lambda_{\backslash}(℣,\mathcal{D}) = (\mathfrak{X},\mathcal{D}\mathrm{x}\mathcal{D})$. Thus, for all $(d,m) \in \mathcal{D}\mathrm{x}\mathcal{D}$, $\mathfrak{X}(d,m) = \mathfrak{F}(d) \cap ℣'(m) = \mathfrak{F}(d) \cap \mathrm{U}' = \mathfrak{F}(d) \cap \emptyset = \emptyset$, implying that $(\mathfrak{X},\mathcal{D}\mathrm{x}\mathcal{D}) = \emptyset_{\mathcal{D}\mathrm{x}\mathcal{D}}$. Let $\mathrm{U}_{\mathcal{D}}\Lambda_{\backslash}(\mathfrak{F},\mathcal{D}) = (℣,\mathcal{D})\Lambda_{\backslash}(\mathfrak{F},\mathcal{D}) = (ლ,\mathcal{D}\mathrm{x}\mathcal{D})$. Then, for all $(d,m) \in \mathcal{D}\mathrm{x}\mathcal{D}$, $ლ(d,m) = ℣(d) \cap \mathfrak{F}'(m) = \mathrm{U} \cap \mathfrak{F}'(m) = \mathfrak{F}'(m)$, implying that $(ლ,\mathcal{D}\mathrm{x}\mathcal{D}) = (\mathfrak{F},\mathcal{D}\mathrm{x}\mathcal{D})^{\mathrm{r}}$. □

Proposition 3.13. Let $(℧,\mathcal{M})$ be an SS over U. Then, $(℧,\mathcal{M})\Lambda_{\backslash}(℧,\mathcal{M}) \tilde{\subseteq}_{\mathrm{J}} (℧,\mathcal{M})$. That is, $\Lambda_{\backslash}$-product is not idempotent in $S_E(U)$ under J-equality.

Proof. Let $(℧,\mathcal{M})\Lambda_{\backslash}(℧,\mathcal{M}) = (\mathfrak{F},\mathcal{M}\mathrm{x}\mathcal{M})$. Then, for all $(m,d) \in \mathcal{M}\mathrm{x}\mathcal{M}$, $\mathfrak{F}(m,d) = ℧(m) \cap ℧'(d)$. Since for all $(m,d) \in \mathcal{M}\mathrm{x}\mathcal{M}$, there exists $m \in \mathcal{M}$ such that $\mathfrak{F}(m,d) = ℧(m) \cap ℧'(d) \subseteq ℧(m)$, $(\mathfrak{F},\mathcal{M}\mathrm{x}\mathcal{M}) \tilde{\subseteq}_{\mathrm{J}} (℧,\mathcal{M})$ is obtained. □

Proposition 3.14. Let $(℧,\mathcal{M})$ and $(\mathfrak{F},\mathcal{D})$ be SSs over U, Then, $(℧,\mathcal{M})\Lambda_{\backslash}(\mathfrak{F},\mathcal{D}) \tilde{\subseteq}_{\mathrm{J}} (\mathfrak{F},\mathcal{D})^{\mathrm{r}}$ and $(℧,\mathcal{M})\Lambda_{\backslash}(\mathfrak{F},\mathcal{D}) \tilde{\subseteq}_{\mathrm{J}} (℧,\mathcal{M})$. □

Proof. Let $(℧,\mathcal{M})\Lambda_{\backslash}(\mathfrak{F},\mathcal{D}) = (℣,\mathcal{M}\mathrm{x}\mathcal{D})$. Then, for all $(m,d) \in \mathcal{M}\mathrm{x}\mathcal{D}$, $℣(m,d) = ℧(m) \cap \mathfrak{F}'(d)$. Since for all $(m,d) \in \mathcal{M}\mathrm{x}\mathcal{D}$, there exists $d \in \mathcal{D}$ such that $℧(m) \cap \mathfrak{F}'(d) \subseteq \mathfrak{F}'(d)$, $(℧,\mathcal{M})\Lambda_{\theta}(\mathfrak{F},\mathcal{D}) \tilde{\subseteq}_{\mathrm{J}} (\mathfrak{F},\mathcal{D})^{\mathrm{r}}$. Similarly, since for all $(m,d) \in \mathcal{M}\mathrm{x}\mathcal{D}$, there exists $m \in \mathcal{M}$ such that $℧(m) \cap \mathfrak{F}'(d) \subseteq ℧(m)$, $(℧,\mathcal{M})\Lambda_{\backslash}(\mathfrak{F},\mathcal{D}) \tilde{\subseteq}_{\mathrm{J}} (℧,\mathcal{M})$ is obtained.

Proposition 3.15. Let $(℧,\mathcal{M})$ and $(\mathfrak{F},\mathcal{D})$ be SSs over U. Then, $[(℧,\mathcal{M})\Lambda_{\backslash}(\mathfrak{F},\mathcal{D})]^{\mathrm{r}} = (℧,\mathcal{M})^{\mathrm{r}} \veebar (\mathfrak{F},\mathcal{D})^{\mathrm{r}}$.

Proof. Let $(℧,\mathcal{M})\Lambda_{\backslash}(\mathfrak{F},\mathcal{D}) = (℣,\mathcal{M}\mathrm{x}\mathcal{D})$. Then, for all $(m,d) \in \mathcal{M}\mathrm{x}\mathcal{D}$, $℣(m,d) = ℧(m) \cap \mathfrak{F}'(d)$. Thus, $℣'(m,d) = ℧'(m) \cup \mathfrak{F}(d) = ℧'(m) \cup (\mathfrak{F}')'(d)$. Hence, $(℣',\mathcal{M}X\mathcal{D}) = (℧,\mathcal{M})^{\mathrm{r}} \veebar (\mathfrak{F},\mathcal{D})^{\mathrm{r}}$. (For $\veebar$-product, please see [11]). □

Proposition 3.16. Let $(℧,\mathcal{M})$ and $(\mathfrak{F},\mathcal{D})$ be SSs over U. Then, $(℧,\mathcal{M})\Lambda_{\backslash}(\mathfrak{F},\mathcal{D}) \tilde{\subseteq}_{\mathrm{F}} (℧,\mathcal{M}) \veebar (\mathfrak{F},\mathcal{D})$.

Proof. Let $(℧,\mathcal{M})\Lambda_{\backslash}(\mathfrak{F},\mathcal{D}) = (℣,\mathcal{M}\mathrm{x}\mathcal{D})$ and $(℧,\mathcal{M}) \veebar (\mathfrak{F},\mathcal{D}) = (Ʊ,\mathcal{M}\mathrm{x}\mathcal{D})$. Then, for all $(m,d) \in \mathcal{M}\mathrm{x}\mathcal{D}$, $℣(m,d) = ℧(m) \cap \mathfrak{F}'(d)$ and for all $(m,d) \in \mathcal{M}\mathrm{x}\mathcal{D}$, $Ʊ(m,d) = ℧(m) \cup \mathfrak{F}'(d)$. Thus, for all $(m,d) \in \mathcal{M}\mathrm{x}\mathcal{D}$, $℣(m,d) = ℧(m) \cap \mathfrak{F}'(d) \subseteq ℧(m) \cup \mathfrak{F}'(d) = Ʊ(m,d)$. □

Proposition 3.17. Let $(℧,\mathcal{M})$, $(\mathfrak{F},\mathcal{D})$, and $(℣,\mathcal{J})$ be SSs over U. If $(℧,\mathcal{M}) \tilde{\subseteq}_{\mathrm{F}} (\mathfrak{F},\mathcal{D})$, then $(℧,\mathcal{M})\Lambda_{\backslash}(℣,\mathcal{J}) \tilde{\subseteq}_{\mathrm{F}} (\mathfrak{F},\mathcal{D})\Lambda_{\backslash}(℣,\mathcal{J})$.

Proof. Let $(℧,\mathcal{M}) \tilde{\subseteq}_{\mathrm{F}} (\mathfrak{F},\mathcal{D})$. Then, $\mathcal{M} \subseteq \mathcal{D}$ and for all $m \in \mathcal{M}$, $℧(m) \subseteq \mathfrak{F}(m)$. Thus, $\mathcal{M} \mathrm{x} \mathcal{J} \subseteq \mathcal{D} \mathrm{x} \mathcal{J}$ and for all $(m,\mathrm{j}) \in \mathcal{M} \mathrm{x} \mathcal{J}$, $℧(m) \cap ℣'(\mathrm{j}) \subseteq \mathfrak{F}(m) \cap ℣'(\mathrm{j})$. □

Proposition 3.18. Let $(℧,\mathcal{M})$, $(\mathfrak{F},\mathcal{D})$, $(℣,\mathcal{J})$, and $(Ʊ,\mathfrak{X})$ be SSs over U. If $(℧,\mathcal{M}) \tilde{\subseteq}_{\mathrm{F}} (\mathfrak{F},\mathcal{D})$ and $(℣,\mathcal{J})^{\mathrm{r}} \tilde{\subseteq}_{\mathrm{F}} (Ʊ,\mathfrak{X})^{\mathrm{r}}$, then $(℧,\mathcal{M})\Lambda_{\backslash}(℣,\mathcal{J}) \tilde{\subseteq}_{\mathrm{F}} (\mathfrak{F},\mathcal{D})\Lambda_{\backslash}(Ʊ,\mathfrak{X})$.

Proof. Let $(℧,\mathcal{M}) \tilde{\subseteq}_{\mathrm{F}} (\mathfrak{F},\mathcal{D})$ and $(℣,\mathcal{J})^{\mathrm{r}} \tilde{\subseteq}_{\mathrm{F}} (Ʊ,\mathfrak{X})^{\mathrm{r}}$. Then, $\mathcal{M} \subseteq \mathcal{D}$, $\mathcal{J} \subseteq \mathfrak{X}$, for all $m \in \mathcal{M}$, $℧(m) \subseteq \mathfrak{F}(m)$ and for all $\mathrm{j} \in \mathcal{J}$, $℣'(\mathrm{j}) \subseteq Ʊ'(\mathrm{j})$. Thus, $\mathcal{M}\mathrm{x}\mathcal{J} \subseteq \mathcal{D}\mathrm{x}\mathfrak{X}$, for all $(m,\mathrm{j}) \in \mathcal{M}\mathrm{x}\mathcal{J}$, $℧(m) \cap ℣'(\mathrm{j}) \subseteq \mathfrak{F}(m) \cap Ʊ'(\mathrm{j})$. □

Proposition 3.19. Let $(℧,\mathcal{M})$, $(\mathfrak{F},\mathcal{M})$, $(Ʊ,\mathcal{M})$, and $(℣,\mathcal{M})$ be SSs over U. If $(℧,\mathcal{M}) \tilde{\subseteq}_{\mathrm{F}} (\mathfrak{F},\mathcal{M})$ and $(℣,\mathcal{M}) \tilde{\subseteq}_{\mathrm{F}} (Ʊ,\mathcal{M})$, then $(℧,\mathcal{M})\Lambda_{\backslash}(Ʊ,\mathcal{M}) \tilde{\subseteq}_{\mathrm{F}} (\mathfrak{F},\mathcal{M})\Lambda_{\backslash}(℣,\mathcal{M})$.

Proof. Let $(℧,\mathcal{M}) \tilde{\subseteq}_{\mathrm{F}} (\mathfrak{F},\mathcal{M})$ and $(℣,\mathcal{M}) \tilde{\subseteq}_{\mathrm{F}} (Ʊ,\mathcal{M})$. Thus, for all $m \in \mathcal{M}$, $℧(m) \subseteq \mathfrak{F}(m)$ and for all $\mathrm{j} \in \mathcal{M}$, $℣(\mathrm{j}) \subseteq Ʊ(\mathrm{j})$. Hence, for all $(m,\mathrm{j}) \in \mathcal{M}\mathrm{x}\mathcal{M}$, $℧(m) \cap Ʊ'(\mathrm{j}) \subseteq \mathfrak{F}(m) \cap ℣'(\mathrm{j})$. □

Proposition 3.20. Let $(℧,\mathcal{M})$ and $(\mathfrak{F},\mathcal{D})$ be SSs over U. Then, $\emptyset_{\mathcal{M}\mathrm{x}\mathcal{D}} \tilde{\subseteq}_{\mathrm{F}} (℧,\mathcal{M})\Lambda_{\backslash}(\mathfrak{F},\mathcal{D})$ and $\emptyset_{\mathcal{D}\mathrm{x}\mathcal{M}} \tilde{\subseteq}_{\mathrm{F}} (\mathfrak{F},\mathcal{D})\Lambda_{\backslash}(℧,\mathcal{M})$.

Proof. Let $\emptyset_{\mathcal{M}\mathrm{x}\mathcal{D}} = (\mathcal{V}, \mathcal{M}\mathrm{x}\mathcal{D})$ and $(\mho, \mathcal{M})\Lambda_{\backslash}(\mathfrak{F}, \mathcal{D}) = (\text{ƻ}, \mathcal{M}\mathrm{x}\mathcal{D})$. Then, for $(m, d) \in \mathcal{M}\mathrm{x}\mathcal{D}$, $\mathcal{V}(m, d) = \emptyset$ and for all $(m, d) \in \mathcal{M}\mathrm{x}\mathcal{D}$, $\text{ƻ}(m, d) = \mho(m) \cap \mathfrak{F}'(d)$. Since $\mathcal{M}\mathrm{x}\mathcal{D} \subseteq \mathcal{M}\mathrm{x}\mathcal{D}$ and for all $(m, d) \in \mathcal{M}\mathrm{x}\mathcal{D}$, $\mathcal{V}(m, d) = \emptyset \subseteq \mho(m) \cap \mathfrak{F}'(d) = \text{ƻ}(m, d)$, $\emptyset_{\mathcal{M}\mathrm{x}\mathcal{D}} \tilde{\subseteq}_{\mathrm{F}} (\mho, \mathcal{M})\Lambda_{\backslash}(\mathfrak{F}, \mathcal{D})$ is obtained. Similarly, $\emptyset_{\mathcal{D}\mathrm{x}\mathcal{M}} \tilde{\subseteq}_{\mathrm{F}} (\mathfrak{F}, \mathcal{D})\Lambda_{\backslash}(\mho, \mathcal{M})$ can be illustrated. □

Proposition 3.21. Let $(\mho, \mathcal{M})$ and $(\mathfrak{F}, \mathcal{D})$ be SSs over U. Then, $\emptyset_{\mathcal{M}} \tilde{\subseteq}_{\mathrm{J}} (\mho, \mathcal{M})\Lambda_{\backslash}(\mathfrak{F}, \mathcal{D})$, $\emptyset_{\mathcal{D}} \tilde{\subseteq}_{\mathrm{J}} (\mho, \mathcal{M})\Lambda_{\backslash}(\mathfrak{F}, \mathcal{D})$ and $\emptyset_{\mathrm{E}} \tilde{\subseteq}_{\mathrm{J}} (\mho, \mathcal{M})\Lambda_{\backslash}(\mathfrak{F}, \mathcal{D})$.

Proof. Let $\emptyset_{\mathcal{M}} = (\mathcal{V}, \mathcal{M})$ and $(\mho, \mathcal{M})\Lambda_{\backslash}(\mathfrak{F}, \mathcal{D}) = (\text{ƻ}, \mathcal{M}\mathrm{x}\mathcal{D})$. Then, for all $m \in \mathcal{M}$, $\mathcal{V}(m) = \emptyset$ and for all $(m, d) \in \mathcal{M}\mathrm{x}\mathcal{D}$, $\text{ƻ}(m, d) = \mho(m) \cap \mathfrak{F}'(d)$. Since for all $m \in \mathcal{M}$, there exist $(m, d) \in \mathcal{M}\mathrm{x}\mathcal{D}$ such that $\mathcal{V}(m) = \emptyset \subseteq \mho(m) \cap \mathfrak{F}'(d) = \text{ƻ}(m, d)$, $\emptyset_{\mathcal{M}} \tilde{\subseteq}_{\mathrm{J}} (\mho, \mathcal{M})\Lambda_{\backslash}(\mathfrak{F}, \mathcal{D})$ is obtained. One can similarly show that $\emptyset_{\mathcal{D}} \tilde{\subseteq}_{\mathrm{J}} (\mho, \mathcal{M})\Lambda_{\backslash}(\mathfrak{F}, \mathcal{D})$ and $\emptyset_{\mathrm{E}} \tilde{\subseteq}_{\mathrm{J}} (\mho, \mathcal{M})\Lambda_{\backslash}(\mathfrak{F}, \mathcal{D})$. □

Proposition 3.22. Let $(\mho, \mathcal{M})$ and $(\mathfrak{F}, \mathcal{D})$ be SSs over U. Then, $(\mho, \mathcal{M})\Lambda_{\backslash}(\mathfrak{F}, \mathcal{D}) \tilde{\subseteq}_{\mathrm{F}} \mathrm{U}_{\mathcal{M}\mathrm{x}\mathcal{D}}$ and $(\mathfrak{F}, \mathcal{D})\Lambda_{\backslash}(\mho, \mathcal{M}) \tilde{\subseteq}_{\mathrm{F}} \mathrm{U}_{\mathcal{D}\mathrm{x}\mathcal{M}}$.

Proof. Let $\mathrm{U}_{\mathcal{M}\mathrm{x}\mathcal{D}} = (\mathcal{V}, \mathcal{M}\mathrm{x}\mathcal{D})$ and $(\mho, \mathcal{M})\Lambda_{\backslash}(\mathfrak{F}, \mathcal{D}) = (\text{ƻ}, \mathcal{M}\mathrm{x}\mathcal{D})$. Then, for all $(m, d) \in \mathcal{M}\mathrm{x}\mathcal{D}$, $\mathcal{V}(m, d) = U$ and for all $(m, d) \in \mathcal{M}\mathrm{x}\mathcal{D}$, $\text{ƻ}(m, d) = \mho(m) \cap \mathfrak{F}'(d)$. Since $\mathcal{M}\mathrm{x}\mathcal{D} \subseteq \mathcal{M}\mathrm{x}\mathcal{D}$ and for all $(m, d) \in \mathcal{M}\mathrm{x}\mathcal{D}$, $\text{ƻ}(m, d) = \mho(m) \cap \mathfrak{F}'(d) \subseteq U$, $(\mho, \mathcal{M})\Lambda_{\backslash}(\mathfrak{F}, \mathcal{D}) \tilde{\subseteq}_{\mathrm{F}} \mathrm{U}_{\mathcal{M}\mathrm{x}\mathcal{D}}$ is obtained. One can similarly show that $(\mathfrak{F}, \mathcal{D})\Lambda_{\backslash}(\mho, \mathcal{M}) \tilde{\subseteq}_{\mathrm{F}} \mathrm{U}_{\mathcal{D}\mathrm{x}\mathcal{M}}$. □

Proposition 3.23. Let $(\mho, \mathcal{M})$ and $(\mathfrak{F}, \mathcal{D})$ be SSs over U. Then, $(\mho, \mathcal{M})\Lambda_{\backslash}(\mathfrak{F}, \mathcal{D}) \tilde{\subseteq}_{\mathrm{J}} \mathrm{U}_{\mathcal{M}}$, $(\mho, \mathcal{M})\Lambda_{\backslash}(\mathfrak{F}, \mathcal{D}) \tilde{\subseteq}_{\mathrm{J}} \mathrm{U}_{\mathcal{D}}$.

Proof. Let $\mathrm{U}_{\mathcal{M}} = (\mathcal{V}, \mathcal{M})$ and $(\mho, \mathcal{M})\Lambda_{\backslash}(\mathfrak{F}, \mathcal{D}) = (\text{ƻ}, \mathcal{M}\mathrm{x}\mathcal{D})$. Then, for all $m \in \mathcal{M}$, $\mathcal{V}(m) = U$ and for all $(m, d) \in \mathcal{M}\mathrm{x}\mathcal{D}$, $\text{ƻ}(m, d) = \mho(m) \cap \mathfrak{F}'(d)$. Since for all $(m, d) \in \mathcal{M}\mathrm{x}\mathcal{D}$, there exist $m \in \mathcal{M}$ such that $\text{ƻ}(m, d) = \mho(m) \cap \mathfrak{F}'(d) \subseteq U = \mathcal{V}(m)$, $(\mho, \mathcal{M})\Lambda_{\backslash}(\mathfrak{F}, \mathcal{D}) \tilde{\subseteq}_{\mathrm{J}} \mathrm{U}_{\mathcal{M}}$ is obtained. Similarly, $(\mho, \mathcal{M})\Lambda_{\backslash}(\mathfrak{F}, \mathcal{D}) \tilde{\subseteq}_{\mathrm{J}} \mathrm{U}_{\mathcal{D}}$ can be observed. □

Proposition 3.24. Let $(\mho, \mathcal{M})$ and $(\mathfrak{F}, \mathcal{D})$ be SSs over U. Then, $(\mho, \mathcal{M})\Lambda_{\theta}(\mathfrak{F}, \mathcal{D}) =_{\mathrm{M}} \mathrm{U}_{\mathcal{M}\mathrm{x}\mathcal{D}}$ if and only if $(\mho, \mathcal{M}) =_{\mathrm{M}} \mathrm{U}_{\mathcal{M}}$ and $(\mathfrak{F}, \mathcal{D}) =_{\mathrm{M}} \emptyset_{\mathcal{D}}$.

Proof. Let $\mathrm{U}_{\mathcal{M}\mathrm{x}\mathcal{D}} = (\text{Ʊ}, \mathcal{M}\mathrm{x}\mathcal{D})$ and $(\mho, \mathcal{M})\Lambda_{\backslash}(\mathfrak{F}, \mathcal{D}) = (\mathcal{V}, \mathcal{M}\mathrm{x}\mathcal{D})$. Then, for all $(m, d) \in \mathcal{M}\mathrm{x}\mathcal{D}$, $\text{Ʊ}(m, d) = U$ and for all $(m, d) \in \mathcal{M}\mathrm{x}\mathcal{D}$, $\mathcal{V}(m, d) = \mho(m) \cap \mathfrak{F}'(d)$. Let $(\text{Ʊ}, \mathcal{M}\mathrm{x}\mathcal{D}) = (\mathcal{V}, \mathcal{M}\mathrm{x}\mathcal{D})$. Then, for all $(m, d) \in \mathcal{M}\mathrm{x}\mathcal{D}$, $\mho(m) \cap \mathfrak{F}'(d) = U$. Thus, for all $m \in \mathcal{M}$, $\mho(m) = U$ and for all $d \in \mathcal{D}$, $\mathfrak{F}'(d) =$ U. Thereby, $(\mho, \mathcal{M}) = \mathrm{U}_{\mathcal{M}}$ and $(\mathfrak{F}, \mathcal{D}) = \emptyset_{\mathcal{D}}$.

Conversely, let $(\mho, \mathcal{M}) =_{\mathrm{M}} \mathrm{U}_{\mathcal{M}}$ and $(\mathfrak{F}, \mathcal{D}) =_{\mathrm{M}} \emptyset_{\mathcal{D}}$. Then, for all $m \in \mathcal{M}$, $\mho(m) = U$ and for all $d \in \mathcal{D}$, $\mathfrak{F}(d) = \emptyset$. Thus, for all $(m, d) \in \mathcal{M}\mathrm{x}\mathcal{D}$, $\mathcal{V}(m, d) = \mho(m) \cap \mathfrak{F}'(d) = \mathrm{U} \cap \mathrm{U} = \mathrm{U}$, implying that $(\mho, \mathcal{M})\Lambda_{\backslash}(\mathfrak{F}, \mathcal{D}) =_{\mathrm{M}} \mathrm{U}_{\mathcal{M}\mathrm{x}\mathcal{D}}$. □

Proposition 3.25. Let $(\mho, \mathcal{M})$ and $(\mathfrak{F}, \mathcal{D})$ be SSs over U. Then, $(\mho, \mathcal{M})\Lambda_{\backslash}(\mathfrak{F}, \mathcal{D}) =_{\mathrm{M}} \emptyset_{\emptyset}$ if and only if $(\mho, \mathcal{M}) =_{\mathrm{M}} \emptyset_{\emptyset}$ or $(\mathfrak{F}, \mathcal{D}) =_{\mathrm{M}} \emptyset_{\emptyset}$.

Proof. Let $(\mho, \mathcal{M})\Lambda_{\backslash}(\mathfrak{F}, \mathcal{D}) =_{\mathrm{M}} \emptyset_{\emptyset}$. Then, $\mathcal{M}\mathrm{x}\mathcal{D} = \emptyset$, and so $\mathcal{M} = \emptyset$ or $\mathcal{D} = \emptyset$ Since $\emptyset_{\emptyset}$ is the only SS with the empty PS, $(\mho, \mathcal{M}) =_{\mathrm{M}} \emptyset_{\emptyset}$ or $(\mathfrak{F}, \mathcal{D}) =_{\mathrm{M}} \emptyset_{\emptyset}$.

Conversely, let $(\mho, \mathcal{M}) =_{\mathrm{M}} \emptyset_{\emptyset}$ or $(\mathfrak{F}, \mathcal{D}) =_{\mathrm{M}} \emptyset_{\emptyset}$. Thus, $\mathcal{M} = \emptyset$ or $\mathcal{D} = \emptyset$, implying that $\mathcal{M}\mathrm{x}\mathcal{D} = \emptyset$ and $(\mho, \mathcal{M})\Lambda_{\backslash}(\mathfrak{F}, \mathcal{D}) =_{\mathrm{M}} \emptyset_{\emptyset}$. □

4. Distributions of Soft Difference-Product over Certain Types of Soft Set Operations

In this section, we explore the distributions of soft difference-product over restricted, extended, soft binary piecewise intersection and union operations, AND-product and OR-product.

Theorem 4.1. Let $(℧,\mathcal{M})$, $(𝔉,\mathcal{D})$, and $(Ꮙ,\mathcal{J})$ be SSs over U. Then, we have the following distributions of soft difference-product over restricted intersection and union operations:

i. $(℧,\mathcal{M})\Lambda_{\backslash}[(𝔉,\mathcal{D}) \cup_{\mathrm{R}} (Ꮙ,\mathcal{J})] =_{\mathrm{M}} [(℧,\mathcal{M})\Lambda_{\backslash}(𝔉,\mathcal{D})] \cap_{\mathrm{R}} [(℧,\mathcal{M})\Lambda_{\backslash}(Ꮙ,\mathcal{J})]$
ii. $(℧,\mathcal{M})\Lambda_{\backslash}[(𝔉,\mathcal{D}) \cap_{\mathrm{R}} (Ꮙ,\mathcal{J})] =_{\mathrm{M}} [(℧,\mathcal{M})\Lambda_{\backslash}(𝔉,\mathcal{D})] \cup_{\mathrm{R}} [(℧,\mathcal{M})\Lambda_{\backslash}(Ꮙ,\mathcal{J})]$
iii. $[(℧,\mathcal{M}) \cap_{\mathrm{R}} (𝔉,\mathcal{D})]\Lambda_{\backslash}(Ꮙ,\mathcal{J}) =_{\mathrm{M}} [(℧,\mathcal{M})\Lambda_{\backslash}(Ꮙ,\mathcal{J})] \cap_{\mathrm{R}} [(𝔉,\mathcal{D})\Lambda_{\backslash}(Ꮙ,\mathcal{J})]$
iv. $[(℧,\mathcal{M}) \cup_{\mathrm{R}} (𝔉,\mathcal{D})]\Lambda_{\backslash}(Ꮙ,\mathcal{J}) =_{\mathrm{M}} [(℧,\mathcal{M})\Lambda_{\backslash}(Ꮙ,\mathcal{J})] \cup_{\mathrm{R}} [(𝔉,\mathcal{D})\Lambda_{\backslash}(Ꮙ,\mathcal{J})]$

PROOF.

i. The PS of the (left-hand side) LHS is $\mathcal{M}\mathrm{x}(\mathcal{D} \cap \mathcal{J})$, and the PS of the right-hand side (RHS) is $(\mathcal{M}\mathrm{x}\mathcal{D}) \cap (\mathcal{M}\mathrm{x}\mathcal{J})$. Since $\mathcal{M}\mathrm{x}(\mathcal{D} \cap \mathcal{J}) = (\mathcal{M}\mathrm{x}\mathcal{D}) \cap (\mathcal{M}\mathrm{x}\mathcal{J})$, the first condition of the M-equality is satisfied. Let $(𝔉,\mathcal{D}) \cup_{\mathrm{R}} (Ꮙ,\mathcal{J}) = (Ӿ,\mathcal{D} \cap \mathcal{J})$, where for all $\varphi \in \mathcal{D} \cap \mathcal{J}$, $Ӿ(\varphi) = 𝔉(\varphi) \cup Ꮙ(\varphi)$. Let $(𝔉,\mathcal{D}) \cup_{\mathrm{R}} (Ꮙ,\mathcal{J}) = (Ӿ,\mathcal{D} \cap \mathcal{J})$, where for all $\varphi \in \mathcal{D} \cap \mathcal{J}$, $Ӿ(\varphi) = 𝔉(\varphi) \cup Ꮙ(\varphi)$ and $(℧,\mathcal{M})\Lambda_{\backslash}(Ӿ,\mathcal{D} \cap \mathcal{J}) = (₴,\mathcal{M}\mathrm{x}(\mathcal{D} \cap \mathcal{J}))$, where for all $(m,\varphi) \in \mathcal{M}\mathrm{x}(\mathcal{D} \cap \mathcal{J})$, $₴(m,\varphi) = ℧(m) \cap Ӿ'(\varphi)$. Thus

$$₴(m,\varphi) = ℧(m) \cap [𝔉(\varphi) \cup Ꮙ(\varphi)]' = ℧(m) \cap [𝔉'(\varphi) \cap Ꮙ'(\varphi)]$$

Let $(℧,\mathcal{M})\Lambda_{\backslash}(𝔉,\mathcal{D}) = (Ŧ,\mathcal{M}\mathrm{x}\mathcal{D})$ and $(℧,\mathcal{M})\Lambda_{\backslash}(Ꮙ,\mathcal{J}) = (ღ,\mathcal{M}\mathrm{x}\mathcal{J})$, where for all $(m,d) \in \mathcal{M}\mathrm{x}\mathcal{D}$, $Ŧ(m,d) = ℧(m) \cap 𝔉'(d)$ and for all $(m,\mathrm{j}) \in \mathcal{M}\mathrm{x}\mathcal{J}$, $ღ(m,\mathrm{j}) = ℧(m) \cap Ꮙ'(\mathrm{j})$. Suppose that $(Ŧ,\mathcal{M}\mathrm{x}\mathcal{D}) \cap_{\mathrm{R}} \left(ღ,\mathcal{M}\mathrm{x}\mathcal{J}\right) = (☼,(\mathcal{M}\mathrm{x}\mathcal{D}) \cap (\mathcal{M}\mathrm{x}\mathcal{J}))$, where for all $(m,\varphi) \in (\mathcal{M}\mathrm{x}\mathcal{D}) \cap (\mathcal{M}\mathrm{x}\mathcal{J}) = \mathcal{M}\mathrm{x}(\mathcal{D} \cap \mathcal{J})$,

$$☼(m,\varphi) = Ŧ(m,\varphi) \cap ღ(m,\varphi) = [℧(m) \cap 𝔉'(\varphi)] \cap [℧(m) \cap Ꮙ'(\varphi)]$$

Thereby, $(℧,\mathcal{M})\Lambda_{\backslash}[(𝔉,\mathcal{D}) \cup_{\mathrm{R}} (Ꮙ,\mathcal{J})] =_{\mathrm{M}} [(℧,\mathcal{M})\Lambda_{\backslash}(𝔉,\mathcal{D})] \cap_{\mathrm{R}} [(℧,\mathcal{M})\Lambda_{\backslash}(Ꮙ,\mathcal{J})]$.

Here, if $\mathcal{D} \cap \mathcal{J} = \emptyset$, then $\mathcal{M}\mathrm{x}(\mathcal{D} \cap \mathcal{J}) = (\mathcal{M}\mathrm{x}\mathcal{D}) \cap (\mathcal{M}\mathrm{x}\mathcal{J}) = \emptyset$. Since the only soft set with an empty PS is $\emptyset_{\emptyset}$, then both sides are $\emptyset_{\emptyset}$. Since $(\mathcal{M}\mathrm{x}\mathcal{D}) \cap (\mathcal{M}\mathrm{x}\mathcal{J}) = \mathcal{M}\mathrm{x}(\mathcal{D} \cap \mathcal{J})$, if $(\mathcal{M}\mathrm{x}\mathcal{D}) \cap (\mathcal{M}\mathrm{x}\mathcal{J})=\emptyset$, then $\mathcal{M}=\emptyset$ or $\mathcal{D} \cap \mathcal{J} = \emptyset$. By assumption, $\mathcal{M} \neq \emptyset$. Thus, $(\mathcal{M}\mathrm{x}\mathcal{D}) \cap (\mathcal{M}\mathrm{x}\mathcal{J})=\emptyset$ implies that $\mathcal{D} \cap \mathcal{J} = \emptyset$. Therefore, under this condition, both sides are again $\emptyset_{\emptyset}$. □

iii. The PS of the LHS is $(\mathcal{M} \cap \mathcal{D})\mathrm{x}\mathcal{J}$, the PS of the RHS is $(\mathcal{M}\mathrm{x}\mathcal{J}) \cap (\mathcal{D}\mathrm{x}\mathcal{J})$, and since $(\mathcal{M} \cap \mathcal{D})\mathrm{x}\mathcal{J} = (\mathcal{M}\mathrm{x}\mathcal{J}) \cap (\mathcal{D}\mathrm{x}\mathcal{J})$, the first condition of M-equality is satisfied. Let $(℧,\mathcal{M}) \cap_{\mathrm{R}} (𝔉,\mathcal{D}) = (Ӿ,\mathcal{M} \cap \mathcal{D})$, where for all $\varphi \in \mathcal{M} \cap \mathcal{D}$, $Ӿ(\varphi) = ℧(\varphi) \cap 𝔉(\varphi)$ and $(Ӿ,\mathcal{M} \cap \mathcal{D})\Lambda_{\backslash}(Ꮙ,\mathcal{J}) = (₴,(\mathcal{M} \cap \mathcal{D})\mathrm{x}\mathcal{J})$, where for all $(\varphi,\mathrm{j}) \in (\mathcal{M} \cap \mathcal{D})\mathrm{x}\mathcal{J}$, $₴(\varphi,\mathrm{j}) = Ӿ(\varphi) \cap Ꮙ'(\mathrm{j})$. Thus,

$$₴(\varphi,\mathrm{j}) = [℧(\varphi) \cap 𝔉(\varphi)] \cap Ꮙ'(\mathrm{j})$$

Let $(℧,\mathcal{M})\Lambda_{\backslash}(Ꮙ,\mathcal{J}) = (Ŧ,\mathcal{M}\mathrm{x}\mathcal{J})$ and $(𝔉,\mathcal{D})\Lambda_{\backslash}(Ꮙ,\mathcal{J}) = (ღ,\mathcal{D}\mathrm{x}\mathcal{J})$, where for all $(m,\mathrm{j}) \in \mathcal{M}\mathrm{x}\mathcal{J}$, $Ŧ(m,\mathrm{j}) = ℧(m) \cap Ꮙ'(\mathrm{j})$ and for all $(d,\mathrm{j}) \in \mathcal{D}\mathrm{x}\mathcal{J}$, $ღ(d,\mathrm{j}) = 𝔉(d) \cap Ꮙ'(\mathrm{j})$. Assume that $(Ŧ,\mathcal{M}\mathrm{x}\mathcal{J}) \cap_{\mathrm{R}} \left(ღ,\mathcal{D}\mathrm{x}\mathcal{J}\right) = (☼,(\mathcal{M}\mathrm{x}\mathcal{J}) \cap (\mathcal{D}\mathrm{x}\mathcal{J}))$, where for all $(\varphi,\mathrm{j}) \in (\mathcal{M}\mathrm{x}\mathcal{J}) \cap (\mathcal{D}\mathrm{x}\mathcal{J}) = (\mathcal{M} \cap \mathcal{D})\mathrm{x}\mathcal{J}$,

$$☼(\varphi,\mathrm{j}) = Ŧ(\varphi,\mathrm{j}) \cap ღ(\varphi,\mathrm{j}) = [℧(\varphi) \cap Ꮙ'(\mathrm{j})] \cap [𝔉(\varphi) \cap Ꮙ'(\mathrm{j})]$$

Thus, $[(℧,\mathcal{M}) \cap_{\mathrm{R}} (𝔉,\mathcal{D})]\Lambda_{\backslash}(Ꮙ,\mathcal{J}) =_{\mathrm{M}} [(℧,\mathcal{M})\Lambda_{\backslash}(Ꮙ,\mathcal{J})] \cap_{\mathrm{R}} [(𝔉,\mathcal{D})\Lambda_{\backslash}(Ꮙ,\mathcal{J})]$.

Here, if $\mathcal{M} \cap \mathcal{D} = \emptyset$, then $(\mathcal{M} \cap \mathcal{D})\mathrm{x}\mathcal{J} = (\mathcal{M}\mathrm{x}\mathcal{J}) \cap (\mathcal{D}\mathrm{x}\mathcal{J}) = \emptyset$. Since the only soft set with the empty parameter set is $\emptyset_{\emptyset}$, both sides of the equality are $\emptyset_{\emptyset}$. Moreover, since $(\mathcal{M}\mathrm{x}\mathcal{J}) \cap (\mathcal{D}\mathrm{x}\mathcal{J}) = (\mathcal{M} \cap \mathcal{D})\mathrm{x}\mathcal{J}$, if

$(\mathcal{M}\mathrm{x}\mathcal{J}) \cap (\mathcal{D}\mathrm{x}\mathcal{J})=\emptyset$, then $\mathcal{M} \cap \mathcal{D} = \emptyset$ or $\mathcal{J}=\emptyset$. By assumption, $\mathcal{J} \neq \emptyset$. Thus, $(\mathcal{M}\mathrm{x}\mathcal{J}) \cap (\mathcal{D}\mathrm{x}\mathcal{J}) = \emptyset$ implies that $\mathcal{M} \cap \mathcal{D} = \emptyset$. Hence, under this condition, both sides of the equality are again $\emptyset_{\emptyset}$. □

Note 4.2. The restricted soft set operation can not distribute over soft difference-product as the intersection does not distribute over cartesian product and it is compulsory for two SSs to be M-equal that their PS should be the same.

Theorem 4.3. Let $(Ʊ,\mathcal{M})$, $(𝔉,\mathcal{D})$, and $(Ꮙ,\mathcal{J})$ be SSs over U. Then, we have the following distributions of soft difference-product over extended intersection and union operations:

i. $(Ʊ,\mathcal{M})\Lambda_{\backslash}[(𝔉,\mathcal{D}) \cap_{\varepsilon} (Ꮙ,\mathcal{J})] =_{\mathrm{M}} [(Ʊ,\mathcal{M})\Lambda_{\backslash}(𝔉,\mathcal{D})] \cup_{\varepsilon} [(Ʊ,\mathcal{M})\Lambda_{\backslash}(Ꮙ,\mathcal{J})]$
ii. $(Ʊ,\mathcal{M})\Lambda_{\backslash}[(𝔉,\mathcal{D}) \cup_{\varepsilon} (Ꮙ,\mathcal{J})] =_{\mathrm{M}} [(Ʊ,\mathcal{M})\Lambda_{\backslash}(𝔉,\mathcal{D})] \cap_{\varepsilon} [(Ʊ,\mathcal{M})\Lambda_{\backslash}(Ꮙ,\mathcal{J})]$
iii. $[(Ʊ,\mathcal{M}) \cup_{\varepsilon} (𝔉,\mathcal{D})]\Lambda_{\backslash}(Ꮙ,\mathcal{J}) =_{\mathrm{M}} [(Ʊ,\mathcal{M})\Lambda_{\backslash}(Ꮙ,\mathcal{J})] \cup_{\varepsilon} [(𝔉,\mathcal{D})\Lambda_{\backslash}(Ꮙ,\mathcal{J})]$
iv. $[(Ʊ,\mathcal{M}) \cap_{\varepsilon} (𝔉,\mathcal{D})]\Lambda_{\backslash}(Ꮙ,\mathcal{J}) =_{\mathrm{M}} [(Ʊ,\mathcal{M})\Lambda_{\backslash}(Ꮙ,\mathcal{J})] \cap_{\varepsilon} [(𝔉,\mathcal{D})\Lambda_{\backslash}(Ꮙ,\mathcal{J})]$

Proof.

i. The PS of the LHS is $\mathcal{M}\mathrm{x}(\mathcal{D} \cup \mathcal{J})$, and the PS of the RHS is $(\mathcal{M}\mathrm{x}\mathcal{D}) \cup (\mathcal{M}\mathrm{x}\mathcal{J})$. Since $\mathcal{M}\mathrm{x}(\mathcal{D} \cup \mathcal{J}) = (\mathcal{M}\mathrm{x}\mathcal{D}) \cup (\mathcal{M}\mathrm{x}\mathcal{J})$, the first condition of the M-equality is satisfied. As $\mathcal{M} \neq \emptyset$, $\mathcal{D} \neq \emptyset$ and $\mathcal{J} \neq \emptyset$, $\mathcal{M}\mathrm{x}(\mathcal{D} \cup \mathcal{J}) \neq \emptyset$ and $(\mathcal{M}\mathrm{x}\mathcal{D}) \cup (\mathcal{M}\mathrm{x}\mathcal{J}) \neq \emptyset$. Thus, no side may be equal to an empty soft set. Let $(𝔉,\mathcal{D}) \cap_{\varepsilon} (Ꮙ,\mathcal{J}) = (Ӿ,\mathcal{D} \cup \mathcal{J})$, where for all $\varphi \in \mathcal{D} \cup \mathcal{J}$,

$$Ӿ(\varphi)=\begin{cases} 𝔉(\varphi), & \varphi\in\mathcal{D}\text{-}\mathcal{J} \\ Ꮙ(\varphi), & \varphi\in\mathcal{J}\text{-}\mathcal{D} \\ 𝔉(\varphi) \cap Ꮙ(\varphi), & \varphi\in\mathcal{D} \cap \mathcal{J} \end{cases}$$

Let $(Ʊ,\mathcal{M})\Lambda_{\backslash}(Ӿ,\mathcal{D} \cup \mathcal{J}) = \big(ƻ,\mathcal{M}\mathrm{x}(\mathcal{D} \cup \mathcal{J})\big)$, where for all $(m,\varphi) \in \mathcal{M}\mathrm{x}(\mathcal{D} \cup \mathcal{J})$, $ƻ(m,\varphi) = Ʊ(m) \cap Ӿ'(\varphi)$. Thus, for all $(m,\varphi) \in \mathcal{M}\mathrm{x}(\mathcal{D} \cup \mathcal{J})$,

$$ƻ(m,\varphi)=\begin{cases} Ʊ(m) \cap 𝔉'(\varphi), & (m,\varphi)\in\mathcal{M}\mathrm{x}(\mathcal{D}\text{-}\mathcal{J}) \\ Ʊ(m) \cap Ꮙ'(\varphi), & (m,\varphi)\in\mathcal{M}\mathrm{x}(\mathcal{J}\text{-}\mathcal{D}) \\ Ʊ(m) \cap [𝔉'(\varphi) \cup Ꮙ'(\varphi)], & (m,\varphi)\in\mathcal{M}\mathrm{x}(\mathcal{D} \cap \mathcal{J}) \end{cases}$$

Assume that $(Ʊ,\mathcal{M})\Lambda_{\backslash}(𝔉,\mathcal{D}) = (Ŧ,\mathcal{M}\mathrm{x}\mathcal{D})$ and $(Ʊ,\mathcal{M})\Lambda_{\backslash}(Ꮙ,\mathcal{J}) = (ღ,\mathcal{M}\mathrm{x}\mathcal{J})$, where for all $(m,d) \in \mathcal{M}\mathrm{x}\mathcal{D}$, $Ŧ(m,d) = Ʊ(m) \cap 𝔉'(d)$ and for all $(m,\mathrm{j}) \in \mathcal{M}\mathrm{x}\mathcal{J}$, $ღ(m,\mathrm{j}) = Ʊ(m) \cap Ꮙ'(\mathrm{j})$. Suppose that $(Ŧ,\mathcal{M}\mathrm{x}\mathcal{D}) \cup_{\varepsilon} \big(ღ,\mathcal{M}\mathrm{x}\mathcal{J}\big) = (☼,(\mathcal{M}\mathrm{x}\mathcal{D}) \cup (\mathcal{M}\mathrm{x}\mathcal{J}))$, where for al $(m,\varphi) \in (\mathcal{M}\mathrm{x}\mathcal{D}) \cup (\mathcal{M}\mathrm{x}\mathcal{J}) = \mathcal{M}\mathrm{x}(\mathcal{D} \cup \mathcal{J})$,

$$☼(m,\varphi)=\begin{cases} Ŧ(m,\varphi), & (m,\varphi)\in(\mathcal{M}\mathrm{x}\mathcal{D})\text{-}(\mathcal{M}\mathrm{x}\mathcal{J}) = \mathcal{M}\mathrm{x}(\mathcal{D}\text{-}\mathcal{J}) \\ ღ(m,\varphi), & (m,\varphi)\in(\mathcal{M}\mathrm{x}\mathcal{J})\text{-}(\mathcal{M}\mathrm{x}\mathcal{D}) = \mathcal{M}\mathrm{x}(\mathcal{J}\text{-}\mathcal{D}) \\ Ŧ(m,\varphi) \cup ღ(m,\varphi), & (m,\varphi)\in(\mathcal{M}\mathrm{x}\mathcal{D}) \cap (\mathcal{M}\mathrm{x}\mathcal{J}) = \mathcal{M}\mathrm{x}(\mathcal{D} \cap \mathcal{J}) \end{cases}$$

Thereby,

$$☼(m,\varphi)=\begin{cases} Ʊ(m) \cap 𝔉'(\varphi), & (m,\varphi)\in(\mathcal{M}\mathrm{x}\mathcal{D})\text{-}(\mathcal{M}\mathrm{x}\mathcal{J}) = \mathcal{M}\mathrm{x}(\mathcal{D}\text{-}\mathcal{J}) \\ Ʊ(m) \cap Ꮙ'(\varphi), & (m,\varphi)\in(\mathcal{M}\mathrm{x}\mathcal{J})\text{-}(\mathcal{M}\mathrm{x}\mathcal{D}) = \mathcal{M}\mathrm{x}(\mathcal{J}\text{-}\mathcal{D}) \\ [Ʊ(m) \cap 𝔉'(\varphi)] \cup [Ʊ(m) \cap Ꮙ'(\varphi)], & (m,\varphi)\in(\mathcal{M}\mathrm{x}\mathcal{D}) \cap (\mathcal{M}\mathrm{x}\mathcal{J}) = \mathcal{M}\mathrm{x}(\mathcal{D} \cap \mathcal{J}) \end{cases}$$

Hence, $(Ʊ,\mathcal{M})\Lambda_{\backslash}[(𝔉,\mathcal{D}) \cap_{\varepsilon} (Ꮙ,\mathcal{J})] =_{\mathrm{M}} [(Ʊ,\mathcal{M})\Lambda_{\backslash}(𝔉,\mathcal{D})] \cup_{\varepsilon} [(Ʊ,\mathcal{M})\Lambda_{\backslash}(Ꮙ,\mathcal{J})]$. □

iii. The PS of the LHS is $(\mathcal{M} \cup \mathcal{D})\mathrm{x}\mathcal{J}$, and the PS of the RHS is $(\mathcal{M}\mathrm{x}\mathcal{J}) \cup (\mathcal{D}\mathrm{x}\mathcal{J})$. Since $(\mathcal{M} \cup \mathcal{D})\mathrm{x}\mathcal{J} = (\mathcal{M}\mathrm{x}\mathcal{J}) \cup (\mathcal{D}\mathrm{x}\mathcal{J})$, the first condition of the M-equality is satisfied. By assumption, $\mathcal{M} \neq \emptyset$, $\mathcal{D} \neq \emptyset$, and $\mathcal{J} \neq$

$\emptyset$. Thus, $(\mathcal{M}\cup\mathcal{D})\mathrm{x}\mathcal{J}\neq\emptyset$ and $(\mathcal{M}\mathrm{x}\mathcal{J})\cup(\mathcal{D}\mathrm{x}\mathcal{J})\neq\emptyset$. Thereby, no side may be equal to an empty soft set. Let $(\text{Ʊ},\mathcal{M})\cup_\varepsilon(\mathfrak{F},\mathcal{D})=(\mathfrak{X},\mathcal{M}\cup\mathcal{D})$, where for all $\varphi\in\mathcal{M}\cup\mathcal{D}$,

$$\mathfrak{X}(\varphi)=\begin{cases}\text{Ʊ}(\varphi), & \varphi\in\mathcal{M}\text{-}\mathcal{D}\\ \mathfrak{F}(\varphi), & \varphi\in\mathcal{D}\text{-}\mathcal{M}\\ \text{Ʊ}(\varphi)\cup\mathfrak{F}(\varphi), & \varphi\in\mathcal{M}\cap\mathcal{D}\end{cases}$$

Let $(\mathfrak{X},\mathcal{M}\cup\mathcal{D})\Lambda_\backslash(\text{℣},\mathcal{J})=(\text{ƻ},(\mathcal{M}\cup\mathcal{D})\mathrm{x}\mathcal{J})$, where for all $(\varphi,\mathrm{j})\in(\mathcal{M}\cup\mathcal{D})\mathrm{x}\mathcal{J}$, $\text{ƻ}(\varphi,\mathrm{j})=\mathfrak{X}(\varphi)\cap\text{℣}'(\mathrm{j})$. Thus, for all $(\varphi,\mathrm{j})\in(\mathcal{M}\cup\mathcal{D})\mathrm{x}\mathcal{J}$,

$$\text{ƻ}(\varphi,\mathrm{j})=\begin{cases}\text{Ʊ}(\varphi)\cap\text{℣}'(\mathrm{j}), & (\varphi,\mathrm{j})\in(\mathcal{M}\text{-}\mathcal{D})\mathrm{x}\mathcal{J}\\ \mathfrak{F}(\varphi)\cap\text{℣}'(\mathrm{j}), & (\varphi,\mathrm{j})\in(\mathcal{D}\text{-}\mathcal{M})\mathrm{x}\mathcal{J}\\ [\text{Ʊ}(\varphi)\cup\mathfrak{F}(\varphi)]\cap\text{℣}'(\mathrm{j}), & (\varphi,\mathrm{j})\in(\mathcal{M}\cap\mathcal{D})\mathrm{x}\mathcal{J}\end{cases}$$

Suppose that $(\text{Ʊ},\mathcal{M})\Lambda_\backslash(\text{℣},\mathcal{J})=(\text{₮},\mathcal{M}\mathrm{x}\mathcal{J})$ and $(\mathfrak{F},\mathcal{D})\Lambda_\backslash(\text{℣},\mathcal{J})=(\text{ღ},\mathcal{D}\mathrm{x}\mathcal{J})$, where for all $(m,\mathrm{j})\in\mathcal{M}\mathrm{x}\mathcal{J}$, $\text{₮}(m,\mathrm{j})=\text{Ʊ}(m)\cap\text{℣}'(\mathrm{j})$ and for all $(d,\mathrm{j})\in\mathcal{D}\mathrm{x}\mathcal{J}$, $\text{ღ}(d,\mathrm{j})=\mathfrak{F}(d)\cap\text{℣}'(\mathrm{j})$. Let $(\text{₮},\mathcal{M}\mathrm{x}\mathcal{J})\cup_\varepsilon\left(\text{ღ},\mathcal{D}\mathrm{x}\mathcal{J}\right)=(\text{☼},(\mathcal{M}\mathrm{x}\mathcal{J})\cup(\mathcal{D}\mathrm{x}\mathcal{J}))$, where for all $(\varphi,\mathrm{j})\in(\mathcal{M}\mathrm{x}\mathcal{J})\cup(\mathcal{D}\mathrm{x}\mathcal{J})=(\mathcal{M}\cup\mathcal{D})\mathrm{x}\mathcal{J}$,

$$\text{☼}(\varphi,\mathrm{j})=\begin{cases}\text{₮}(\varphi,\mathrm{j}), & (\varphi,\mathrm{j})\in(\mathcal{M}\mathrm{x}\mathcal{J})\text{-}(\mathcal{D}\mathrm{x}\mathcal{J})=(\mathcal{M}\text{-}\mathcal{D})\mathrm{x}\mathcal{J}\\ \text{ღ}(\varphi,\mathrm{j}), & (\varphi,\mathrm{j})\in(\mathcal{D}\mathrm{x}\mathcal{J})\text{-}(\mathcal{M}\mathrm{x}\mathcal{J})=(\mathcal{D}\text{-}\mathcal{M})\mathrm{x}\mathcal{J}\\ \text{₮}(\varphi,\mathrm{j})\cup\text{ღ}(\varphi,\mathrm{j}), & (\varphi,\mathrm{j})\in(\mathcal{M}\mathrm{x}\mathcal{J})\cap(\mathcal{D}\mathrm{x}\mathcal{J})=(\mathcal{M}\cap\mathcal{D})\mathrm{x}\mathcal{J}\end{cases}$$

Thereby,

$$\text{☼}(\varphi,\mathrm{j})=\begin{cases}\text{Ʊ}(\varphi)\cap\text{℣}'(\mathrm{j}), & (\varphi,\mathrm{j})\in(\mathcal{M}\mathrm{x}\mathcal{J})\text{-}(\mathcal{D}\mathrm{x}\mathcal{J})=(\mathcal{M}\text{-}\mathcal{D})\mathrm{x}\mathcal{J}\\ \mathfrak{F}(\varphi)\cap\text{℣}'(\mathrm{j}), & (\varphi,\mathrm{j})\in(\mathcal{D}\mathrm{x}\mathcal{J})\text{-}(\mathcal{M}\mathrm{x}\mathcal{J})=(\mathcal{D}\text{-}\mathcal{M})\mathrm{x}\mathcal{J}\\ [\text{Ʊ}(\varphi)\cap\text{℣}'(\mathrm{j})]\cup[\mathfrak{F}(\varphi)\cap\text{℣}'(\mathrm{j})], & (\varphi,\mathrm{j})\in(\mathcal{M}\mathrm{x}\mathcal{J})\cap(\mathcal{D}\mathrm{x}\mathcal{J})=(\mathcal{M}\cap\mathcal{D})\mathrm{x}\mathcal{J}\end{cases}$$

Hence, $[(\text{Ʊ},\mathcal{M})\cup_\varepsilon(\mathfrak{F},\mathcal{D})]\Lambda_\backslash(\text{℣},\mathcal{J})=_{\mathrm{M}}[(\text{Ʊ},\mathcal{M})\Lambda_\backslash(\text{℣},\mathcal{J})]\cup_\varepsilon[(\mathfrak{F},\mathcal{D})\Lambda_\backslash(\text{℣},\mathcal{J})]$. □

Note 4.4. The extended soft set operation can not distribute over soft difference-product as the union operation does not distribute over cartesian product and it is compulsory for two SSs to be M-equal that their PS should be the same.

Theorem 4.5. Let $(\text{Ʊ},\mathcal{M})$, $(\mathfrak{F},\mathcal{D})$, and $(\text{℣},\mathcal{J})$ be SSs over U. Then, we have the following distributions of soft difference-product over soft binary piecewise intersection and union operations:

i. $(\text{Ʊ},\mathcal{M})\Lambda_\backslash[(\mathfrak{F},\mathcal{D})\,\tilde{\cap}\,(\text{℣},\mathcal{J})]=_{\mathrm{M}}[(\text{Ʊ},\mathcal{M})\Lambda_\backslash(\mathfrak{F},\mathcal{D})]\,\tilde{\cup}\,[(\text{Ʊ},\mathcal{M})\Lambda_\backslash(\text{℣},\mathcal{J})]$
ii. $(\text{Ʊ},\mathcal{M})\Lambda_\backslash[(\mathfrak{F},\mathcal{D})\,\tilde{\cup}\,(\text{℣},\mathcal{J})]=_{\mathrm{M}}[(\text{Ʊ},\mathcal{M})\Lambda_\backslash(\mathfrak{F},\mathcal{D})]\,\tilde{\cap}\,[(\text{Ʊ},\mathcal{M})\Lambda_\backslash(\text{℣},\mathcal{J})]$
iii. $[(\text{Ʊ},\mathcal{M})\,\tilde{\cup}\,(\mathfrak{F},\mathcal{D})]\Lambda_\backslash(\text{℣},\mathcal{J})=_{\mathrm{M}}[(\text{Ʊ},\mathcal{M})\Lambda_\backslash(\text{℣},\mathcal{J})]\,\tilde{\cup}\,[(\mathfrak{F},\mathcal{D})\Lambda_\backslash(\text{℣},\mathcal{J})]$
iv. $[(\text{Ʊ},\mathcal{M})\,\tilde{\cap}\,(\mathfrak{F},\mathcal{D})]\Lambda_\backslash(\text{℣},\mathcal{J})=_{\mathrm{M}}[(\text{Ʊ},\mathcal{M})\Lambda_\backslash(\text{℣},\mathcal{J})]\,\tilde{\cap}\,[(\mathfrak{F},\mathcal{D})\Lambda_\backslash(\text{℣},\mathcal{J})]$

PROOF.

i. Since the PS of the SSs of both sides is $\mathcal{M}\mathrm{x}\mathcal{D}$, and the first condition of the M-equality is satisfied. Moreover since $\mathcal{M}\neq\emptyset$ and $\mathcal{D}\neq\emptyset$ by assumption, $\mathcal{M}\mathrm{x}\mathcal{D}\neq\emptyset$. Thus, no side may be equal to an empty soft set. Let $(\mathfrak{F},\mathcal{D})\,\tilde{\cap}\,(\text{℣},\mathcal{J})=(\mathfrak{X},\mathcal{D})$, where for all $d\in\mathcal{D}$,

$$\mathfrak{X}(d)=\begin{cases}\mathfrak{F}(d), & d\in\mathcal{D}\text{-}\mathcal{J}\\ \mathfrak{F}(d)\cap\text{℣}(d), & d\in\mathcal{D}\cap\mathcal{J}\end{cases}$$

Let $(\text{Ʊ},\mathcal{M})\Lambda_\backslash(\mathfrak{X},\mathcal{D})=\left(\text{ღ},\mathcal{M}\mathrm{x}\mathcal{D}\right)$, where for all $(m,d)\in\mathcal{M}\mathrm{x}\mathcal{D}$, $\text{ღ}(m,d)=\text{Ʊ}(m)\cap\mathfrak{X}'(d)$. Thus,

$$\text{ღ}(m,d)=\begin{cases}\mho(m)\cap\mathfrak{F}'(d), & (m,d)\in\mathcal{M}\text{x}(\mathcal{D}\text{-}\mathcal{J})\\ \mho(m)\cap[\mathfrak{F}'(d)\cup\mathcal{V}'(d)], & (m,d)\in\mathcal{M}\text{x}(\mathcal{D}\cap\mathcal{J})\end{cases}$$

Suppose that $(\mho,\mathcal{M})\wedge_{\setminus}(\mathfrak{F},\mathcal{D})=(\text{Ŧ},\mathcal{M}\text{x}\mathcal{D})$ and $(\mho,\mathcal{M})\wedge_{\setminus}(\mathcal{V},\mathcal{J})=(\text{₴},\mathcal{M}\text{x}\mathcal{J})$, where for all $(m,d)\in\mathcal{M}\text{x}\mathcal{D}$, $\text{Ŧ}(m,d)=\mho(m)\cap\mathfrak{F}'(d)$ and for all $(m,\mathrm{j})\in\mathcal{M}\text{x}\mathcal{J}$, $\text{₴}(m,\mathrm{j})=\mho(m)\cap\mathcal{V}'(\mathrm{j})$. Let $(\text{Ŧ},\mathcal{M}\text{x}\mathcal{D})\,\tilde{\cup}\,(\text{₴},\mathcal{M}\text{x}\mathcal{J})=(\text{☼},\mathcal{M}\text{x}\mathcal{D})$, where for all $(m,d)\in\mathcal{M}\text{x}\mathcal{D}$,

$$\text{☼}(m,d)=\begin{cases}\text{Ŧ}(m,d), & (m,d)\in(\mathcal{M}\text{x}\mathcal{D})\text{-}(\mathcal{M}\text{x}\mathcal{J})=\mathcal{M}\text{x}(\mathcal{D}\text{-}\mathcal{J})\\ \text{Ŧ}(m,d)\cup\text{₴}(m,d), & (m,d)\in(\mathcal{M}\text{x}\mathcal{D})\cap(\mathcal{M}\text{x}\mathcal{J})=\mathcal{M}\text{x}(\mathcal{D}\cap\mathcal{J})\end{cases}$$

Hence,

$$\text{☼}(m,d)=\begin{cases}\mho(m)\cap\mathfrak{F}'(d), & (m,d)\in(\mathcal{M}\text{x}\mathcal{D})\text{-}(\mathcal{M}\text{x}\mathcal{J})=\mathcal{M}\text{x}(\mathcal{D}\text{-}\mathcal{J})\\ [\mho(m)\cap\mathfrak{F}'(d)]\cup[\mho(m)\cap\mathcal{V}'(d)], & (m,d)\in(\mathcal{M}\text{x}\mathcal{D})\cap(\mathcal{M}\text{x}\mathcal{J})=\mathcal{M}\text{x}(\mathcal{D}\cap\mathcal{J})\end{cases}$$

Thus, $(\mho,\mathcal{M})\wedge_{\setminus}[(\mathfrak{F},\mathcal{D})\,\tilde{\cap}\,(\mathcal{V},\mathcal{J})]=_{\mathrm{M}}[(\mho,\mathcal{M})\wedge_{\setminus}(\mathfrak{F},\mathcal{D})]\,\tilde{\cup}\,[(\mho,\mathcal{M})\wedge_{\setminus}(\mathcal{V},\mathcal{J})]$. Since $\mathcal{M}\neq\mathcal{M}\text{x}\mathcal{M}$, the soft binary piecewise operations do not distribute over soft plus-product operations. ◻

iii. Since the PS of the SSs of both sides is $\mathcal{M}\text{x}\mathcal{J}$, the first condition of the M-equality is satisfied. Moreover since $\mathcal{M}\neq\emptyset$ and $\mathcal{J}\neq\emptyset$ by assumption, $\mathcal{M}\text{x}\mathcal{J}\neq\emptyset$. Thus, it is impossible that any side is equal to empty soft set. Let $(\mho,\mathcal{M})\,\tilde{\cup}\,(\mathfrak{F},\mathcal{D})=(\text{Ӿ},\mathcal{M})$, where for all $m\in\mathcal{M}$,

$$\text{Ӿ}(m)=\begin{cases}\mho(m), & m\in\mathcal{M}\text{-}\mathcal{D}\\ \mho(m)\cup\mathfrak{F}(m), & m\in\mathcal{M}\cap\mathcal{D}\end{cases}$$

Let $(\text{Ӿ},\mathcal{M})\wedge_{\setminus}(\mathcal{V},\mathcal{J})=(\text{ღ},\mathcal{M}\text{x}\mathcal{J})$, where for all $(m,\mathrm{j})\in\mathcal{M}\text{x}\mathcal{J}$, $\text{ღ}(m,\mathrm{j})=\text{Ӿ}(m)\cap\mathcal{V}'(\mathrm{j})$. Thus,

$$\text{Ӿ}(m,\mathrm{j})=\begin{cases}\mho(m)\cap\mathcal{V}'(\mathrm{j}), & (m,\mathrm{j})\in(\mathcal{M}\text{-}\mathcal{D})\text{x}\mathcal{J}\\ [\mho(m)\cup\mathfrak{F}(m)]\cap\mathcal{V}'(\mathrm{j}), & (m,\mathrm{j})\in(\mathcal{M}\cap\mathcal{D})\text{x}\mathcal{J}\end{cases}$$

Assume that $(\mho,\mathcal{M})\wedge_{\setminus}(\mathcal{V},\mathcal{J})=(\text{Ŧ},\mathcal{M}\text{x}\mathcal{J})$ and $(\mathfrak{F},\mathcal{D})\wedge_{\setminus}(\mathcal{V},\mathcal{J})=(\text{₴},\mathcal{D}\text{x}\mathcal{J})$, where for all $(m,\mathrm{j})\in\mathcal{M}\text{x}\mathcal{J}$ $\text{Ŧ}(m,\mathrm{j})=\mho(m)\cap\mathcal{V}'(\mathrm{j})$ and for all $(d,\mathrm{j})\in\mathcal{D}\text{x}\mathcal{J}$, $\text{₴}(d,\mathrm{j})=\mathfrak{F}(d)\cap\mathcal{V}'(\mathrm{j})$. Let $(\text{Ŧ},\mathcal{M}\text{x}\mathcal{J})\,\tilde{\cup}\,(\text{₴},\mathcal{D}\text{x}\mathcal{J})=(\text{☼},\mathcal{M}\text{x}\mathcal{J})$, where for all $(m,\mathrm{j})\in\mathcal{M}\text{x}\mathcal{J}$,

$$\text{☼}(m,\mathrm{j})=\begin{cases}\text{Ŧ}(m,\mathrm{j}), & (m,\mathrm{j})\in(\mathcal{M}\text{x}\mathcal{J})\text{-}(\mathcal{D}\text{x}\mathcal{J})=(\mathcal{M}\text{-}\mathcal{D})\text{x}\mathcal{J}\\ \text{Ŧ}(m,\mathrm{j})\cup\text{₴}(m,\mathrm{j}), & (m,\mathrm{j})\in(\mathcal{M}\text{x}\mathcal{J})\cap(\mathcal{D}\text{x}\mathcal{J})=(\mathcal{M}\cap\mathcal{D})\text{x}\mathcal{J}\end{cases}$$

Thus,

$$\text{☼}(m,\mathrm{j})=\begin{cases}\mho(m)\cap\mathcal{V}'(\mathrm{j}), & (m,\mathrm{j})\in(\mathcal{M}\text{x}\mathcal{J})\text{-}(\mathcal{D}\text{x}\mathcal{J})=(\mathcal{M}\text{-}\mathcal{D})\text{x}\mathcal{J}\\ [\mho(m)\cap\mathcal{V}'(\mathrm{j})]\cup[\mathfrak{F}(m)\cap\mathcal{V}'(\mathrm{j})], & (m,\mathrm{j})\in(\mathcal{M}\text{x}\mathcal{J})\cap(\mathcal{D}\text{x}\mathcal{J})=(\mathcal{M}\cap\mathcal{D})\text{x}\mathcal{J}\end{cases}$$

Thereby, $[(\mho,\mathcal{M})\,\tilde{\cup}\,(\mathfrak{F},\mathcal{D})]\wedge_{\setminus}(\mathcal{V},\mathcal{J})=_{\mathrm{M}}[(\mho,\mathcal{M})\wedge_{\setminus}(\mathcal{V},\mathcal{J})]\,\tilde{\cup}\,[(\mathfrak{F},\mathcal{D})\wedge_{\setminus}(\mathcal{V},\mathcal{J})]$. ◻

Proposition 4.6. Let $(\mho,\mathcal{M})$, $(\mathfrak{F},\mathcal{D})$ and $(\mathcal{V},\mathcal{J})$ be SSs over U. Then,

i. $(\mho,\mathcal{M})\wedge_{\setminus}[(\mathfrak{F},\mathcal{D})\wedge(\mathcal{V},\mathcal{J})]\subseteq_{\mathrm{L}}[(\mho,\mathcal{M})\wedge_{\setminus}(\mathfrak{F},\mathcal{D})]\vee[(\mho,\mathcal{M})\wedge_{\setminus}(\mathcal{V},\mathcal{J})]$
ii. $(\mho,\mathcal{M})\wedge_{\setminus}[(\mathfrak{F},\mathcal{D})\vee(\mathcal{V},\mathcal{J})]\subseteq_{\mathrm{L}}[(\mho,\mathcal{M})\wedge_{\setminus}(\mathfrak{F},\mathcal{D})]\wedge[(\mho,\mathcal{M})\wedge_{\setminus}(\mathcal{V},\mathcal{J})]$

Proof.

i. Let $(\mathfrak{F},\mathcal{D})\wedge(\mathcal{V},\mathcal{J})=(\text{Ծ},\mathcal{D}\text{x}\mathcal{J})$, where for all $(d,\mathrm{j})\in\mathcal{D}\text{x}\mathcal{J}$, $\text{Ծ}(d,\mathrm{j})=\mathfrak{F}(d)\cap\mathcal{V}(\mathrm{j})$ and $(\mho,\mathcal{M})\wedge_{\setminus}(\text{Ծ},\mathcal{D}\text{x}\mathcal{J})=(\text{Ӿ},\mathcal{M}\text{x}(\mathcal{D}\text{x}\mathcal{J}))$, where for all $(m,(d,\mathrm{j}))\in\mathcal{M}\text{x}(\mathcal{D}\text{x}\mathcal{J})$,

$$\text{Ӿ}\big(m,(d,\mathrm{j})\big)=\mho(m)\cap[\mathfrak{F}(d)\cap\mathcal{V}(\mathrm{j})]'=\mho(m)\cap[\mathfrak{F}'(d)\cup\mathcal{V}'(\mathrm{j})]$$

Suppose that $(\mho,\mathcal{M})\wedge_{\setminus}(\mathfrak{F},\mathcal{D})=(\text{☼},\mathcal{M}\text{x}\mathcal{D})$ and $(\mho,\mathcal{M})\wedge_{\setminus}(\mathcal{V},\mathcal{J})=(\text{₴},\mathcal{M}\text{x}\mathcal{J})$, where for all $(m,d)\in\mathcal{M}\text{x}\mathcal{D}$, $\text{☼}(m,d)=\mho(m)\cap\mathfrak{F}'(d)$ and for all $(m,\mathrm{j})\in\mathcal{M}\text{x}\mathcal{J}$, $\text{₴}(m,\mathrm{j})=\mho(m)\cap\mathcal{V}'(\mathrm{j})$. Suppose that $(\text{☼},\mathcal{M}\text{x}\mathcal{D})\vee(\text{₴},\mathcal{M}\text{x}\mathcal{J})=(\text{ღ},(\mathcal{M}\text{x}\mathcal{D})\text{x}(\mathcal{M}\text{x}\mathcal{J}))$, where for all $((m,d),(m,\mathrm{j}))\in(\mathcal{M}\text{x}\mathcal{D})\text{x}(\mathcal{M}\text{x}\mathcal{J})$,

$$\text{ღ}\big((m,d),(m,\text{j})\big) = [\text{Ʊ}(m) \cap \mathfrak{F}'(d)] \cup [\text{Ʊ}(m) \cap \text{Ꮙ}'(\text{j})]$$

Thus, for all $(m,(d,\text{j})) \in \mathcal{M}\text{x}(\mathcal{D}\text{x}\mathcal{J})$, there exists $((m,d),(m,\text{j})) \in (\mathcal{M}\text{x}\mathcal{D})\text{x}(\mathcal{M}\text{x}\mathcal{J})$ such that

$$\mathfrak{X}\big(m,(d,\text{j})\big) = \text{Ʊ}(m) \cap [\mathfrak{F}'(d) \cup \text{Ꮙ}'(\text{j})] = [\text{Ʊ}(m) \cap \mathfrak{F}'(d)] \cup [\text{Ʊ}(m) \cap \text{Ꮙ}'(\text{j})] = \text{ღ}\big((m,d),(m,\text{j})\big) \quad \square$$

It is obvious that the L-subset in Proposition 4.6. cannot be L-equality with the following example:

Example 4.7. Let $\text{E} = \{e_1,e_2,e_3,e_4,e_5\}$ be the parameter set, $\mathcal{M} = \{e_1,e_5\}$, $\mathcal{D} = \{e_3\}$ and $\mathcal{J} = \{e_2\}$ be the subsets of E, $U = \{h_1,h_2,h_3,h_4,h_5,h_6\}$ be the universal set and $(\text{Ʊ},\mathcal{M})$, $(\mathfrak{F},\mathcal{D})$, and $(\text{Ꮙ},\mathcal{J})$ be SSs over U as follows:

$$(\text{Ʊ},\mathcal{M}) = \{(e_1,\{h_1,h_6\}),(e_5,\{h_2,h_4,h_5\})\},$$

$$(\mathfrak{F},\mathcal{D}) = \{(e_3,\{h_1,h_3,h_4\})\},$$

and

$$(\text{Ꮙ},\mathcal{J}) = \{(e_2,\{h_1,h_4,h_5\})\}$$

We show that

$$(\text{Ʊ},\mathcal{M})\Lambda_{\backslash}[(\mathfrak{F},\mathcal{D})\Lambda(\text{Ꮙ},\mathcal{J})] \neq_{\text{L}} [(\text{Ʊ},\mathcal{M})\Lambda_{\backslash}(\mathfrak{F},\mathcal{D})]\text{V}[(\text{Ʊ},\mathcal{M})\Lambda_{\backslash}(\text{Ꮙ},\mathcal{J})]$$

Let $(\mathfrak{F},\mathcal{D})\Lambda(\text{Ꮙ},\mathcal{J}) = (\text{Ʊ̃},\mathcal{D}\text{x}\mathcal{J})$, where

$$(\mathfrak{F},\mathcal{D})\Lambda(\text{Ꮙ},\mathcal{J}) = (\text{Ʊ̃},\mathcal{D}\text{x}\mathcal{J}) = \{((e_3,e_2),\{h_1,h_4\})\}$$

Assume that $(\text{Ʊ},\mathcal{M})\Lambda_{\backslash}(\text{Ʊ̃},\mathcal{D}\text{x}\mathcal{J}) = \big(\mathfrak{X},\mathcal{M}\text{x}(\mathcal{D}\text{x}\mathcal{J})\big)$, where

$$(\text{Ʊ},\mathcal{M})\Lambda_{\backslash}(\text{Ʊ̃},\mathcal{D}\text{x}\mathcal{J}) = \big(\mathfrak{X},\mathcal{M}\text{x}(\mathcal{D}\text{x}\mathcal{J})\big) = \{((e_1,(e_3,e_2)),\{h_6\}),((e_5,(e_3,e_2)),\{h_2,h_5\})\}$$

Let $(\text{Ʊ},\mathcal{M})\Lambda_{\backslash}(\mathfrak{F},\mathcal{D}) = (\text{₴},\mathcal{M}\text{x}\mathcal{D})$, where

$$(\text{Ʊ},\mathcal{M})\Lambda_{\backslash}(\mathfrak{F},\mathcal{D}) = (\text{₴},\mathcal{M}\text{x}\mathcal{D}) = \{\big((e_1,e_3),\{h_6\}\big),((e_5,e_3),\{h_2,h_5\})\}$$

Suppose that $(\text{Ʊ},\mathcal{M})\Lambda_{\backslash}(\text{Ꮙ},\mathcal{J}) = (\text{☼},\mathcal{M}\text{x}\mathcal{J})$, where

$$(\text{Ʊ},\mathcal{M})\Lambda_{\backslash}(\text{Ꮙ},\mathcal{J}) = (\text{☼},\mathcal{M}\text{x}\mathcal{J}) = \{\big((e_1,e_2),\{h_6\}\big),((e_5,e_2),\{h_2\})\}$$

Let $(\text{₴},\mathcal{M}\text{x}\mathcal{D})\text{V}(\text{☼},\mathcal{M}\text{x}\mathcal{J}) = (\text{ღ},(\mathcal{M}\text{x}\mathcal{D})\text{x}(\mathcal{M}\text{x}\mathcal{J}))$. Then,

$$\begin{aligned}(\text{₴},\mathcal{M}\text{x}\mathcal{D})\text{V}(\text{☼},\mathcal{M}\text{x}\mathcal{J} &= \big(\text{ღ},(\mathcal{M}\text{x}\mathcal{D})\text{x}(\mathcal{M}\text{x}\mathcal{J})\big)\\ &= \Big\{\Big(\big((e_1,e_3),(e_1,e_2)\big),\{h_6\}\Big),\Big(\big((e_1,e_3),(e_5,e_2)\big),\{h_2,h_6\}\Big),\Big(\big((e_5,e_3),(e_1,e_2)\big),\{h_2\\ &\quad \Big(\big((e_5,e_3),(e_5,e_2)\big),\{h_2,h_5\}\Big)\Big\}\end{aligned}$$

Thus, $\text{ღ}\big((e_1,e_3),(e_5,e_2)\big) \neq \mathfrak{X}(e_1,(e_3,e_2))$, $\text{ღ}\big((e_1,e_3),(e_5,e_2)\big) \neq \mathfrak{X}((e_5,(e_3,e_2))$, $\text{ღ}\big((e_5,e_3),(e_1,e_2)\big) \neq \mathfrak{X}(e_1,(e_3,e_2))$, and $\text{ღ}\big((e_5,e_3),(e_1,e_2)\big) \neq \mathfrak{X}((e_5,(e_3,e_2))$, implying that $(\text{ღ},(\mathcal{M}\text{x}\mathcal{D})\text{x}(\mathcal{M}\text{x}\mathcal{J})) \nsubseteq_{\text{L}} \big(\mathfrak{X},\mathcal{M}\text{x}(\mathcal{D}\text{x}\mathcal{J})\big)$.

5. *uni-int* Decision-Making Method Applied to Soft Difference-Product

In this section, the *uni-int* operator and *uni-int* decision function defined by Çağman and Enginoğlu [11] are applied for the soft difference-product for the *uni-int* decision-making method. This method reduces a set to its subset according to the criteria given by the decision-makers. Therefore, instead of considering a large number of possibilities, decision-makers concentrate on a small number.

Throughout this section, all the soft difference-products ($\Lambda_{\setminus}$) of the SSs over U are assumed to be contained in the set $\Lambda_{\setminus}(U)$, and the approximation function of the soft difference-product of $(\mho,\mathcal{M})$ and $(\mathfrak{F},\mathcal{D})$, that is, $(\mho,\mathcal{M})\Lambda_{\setminus}(\mathfrak{F},\mathcal{D})$ is

$$\mho_{\mathcal{M}}\Lambda_{\setminus}\mathfrak{F}_{\mathcal{D}}: \mathcal{M}\mathrm{x}\mathcal{D} \to P(U)$$

where $\mho_{\mathcal{M}}\Lambda_{\setminus}\mathfrak{F}_{\mathcal{D}}(m,d) = \mho(m) \cap \mathfrak{F}'(d)$ for all $(m,d) \in \mathcal{M}\mathrm{x}\mathcal{D}$.

Definition 5.1. Let $(\mho,\mathcal{M})$ and $(\mathfrak{F},\mathcal{D})$ be SS over U. Then, *uni-int* operators for soft difference-product, denoted by $uni_x int_y$ and $uni_y int_x$ are defined respectively as

$uni_x int_y$: $\Lambda_{\setminus} \to P(U)$, $\quad uni_x int_y(\mho_{\mathcal{M}}\Lambda_{\setminus}\mathfrak{F}_{\mathcal{D}}) = \bigcup_{m\in\mathcal{M}}(\bigcap_{d\in\mathcal{D}}(\mho_{\mathcal{M}}\Lambda_{\setminus}\mathfrak{F}_{\mathcal{D}}(m,d)))$

$uni_y int_x$: $\Lambda_{\setminus} \to P(U)$, $\quad uni_y int_x(\mho_{\mathcal{M}}\Lambda_{\setminus}\mathfrak{F}_{\mathcal{D}}) = \bigcup_{d\in\mathcal{D}}(\bigcap_{m\in\mathcal{M}}(\mho_{\mathcal{M}}\Lambda_{\setminus}\mathfrak{F}_{\mathcal{D}}(m,d)))$

Definition 5.2. [11] Let $(\mho,\mathcal{M})\Lambda_{\setminus}(\mathfrak{F},\mathcal{D}) \in \Lambda_{\setminus}(U)$. Then, *uni-int* decision function for soft difference-product, denoted by *uni-int* are defined by

uni-int: $\Lambda_{\setminus} \to P(U)$, $\quad$ *uni-int*$(\mho_{\mathcal{M}}\Lambda_{\setminus}\mathfrak{F}_{\mathcal{D}}) = uni_x int_y(\mho_{\mathcal{M}}\Lambda_{\setminus}\mathfrak{F}_{\mathcal{D}}) \cup uni_y int_x(\mho_{\mathcal{M}}\Lambda_{\setminus}\mathfrak{F}_{\mathcal{D}})$

that reduces the size of the universe U. Thus, the values *uni-int*$(\mho_{\mathcal{M}}\Lambda_{\setminus}\mathfrak{F}_{\mathcal{D}})$ is a subset of U called *uni-int* decision set of $\mho_{\mathcal{M}}\Lambda_{\setminus}\mathfrak{F}_{\mathcal{D}}$.

Presume that a set of choices and a set of parameters are given. Then, taking into account the problem, a set of ideal alternatives is selected utilizing the *uni-int* decision-making technique, which is organized as follows:

Step 1: Select the feasible subsets from the parameter collection.

Step 2: Construct the SSs for every parameter set.

Step 3: Determine the SSs' soft difference-product.

Step 4: Determine the product's *uni-int* decision set.

The application of soft set theory to the *uni-int* decision-making for the soft difference-products is demonstrated as follows:

Example 5.3. A private club has announced a recruitment notice for contracted coaches, and candidates will be selected based on the success in oral and practical exams from among those invited, which will be up to three (3) times the number of available positions. The selected candidates will then undergo a comprehensive training program, and those who successfully complete it will qualify to join the private club professional caoch team. Applications will first undergo a preliminary elimination to check compliance with the required qualifications for the applied position, and applications found to not meet any of the requirements will be canceled. Among the candidates whose applications are considered valid, Mrs Çam, the club's Human Resources manager, will carry out a selection process based on the results of the oral and practical exams administered to the candidates. Taking into account the characteristics that should be present in a coach candidate, Mrs Çam will first determine the parameters he wants in the selected coach candidates, and then the parameters he definitely does not want. Based on this, he will make his decision using the *uni-int* decision-making method of the soft difference-product. Assume that the coach candidate whose application is considered valid is as follows: $U = \{z_1, z_2, \cdots, z_{25}\}$. Let the set of parameters to be used for determining the selected coaches be $\{c_1, c_2, \ldots, c_8\}$. Here, c_i are parameters, where $i \in \{1,2,\cdots,8\}$ correspond to the following descriptions, respectively:

c_1 = "having the professional knowledge required by the position,"

c_2 = "having a low level of general culture and general ability",

c_3 = "having practical experience in the professional field required by the position",

c_4 = “having strong communication and reasoning abilities”,

c_5 = “having low physical endurance and conditioning”,

c_6 = “being closed to innovations, scientific and technological developments”,

c_7 = “being motivated and determined”

c_8 = “having low self-confidence, persuasion skills, and credibility”.

To solve this coach recruitment process problem, we can apply the *uni-int* method to soft difference-product as follows:

Step 1: Determining the sets of parameters

The decision maker, Mrs Çam from the existing set of parameters, first selects the parameters he would like to have in the selected coach candidates (*i*), and then selects the parameters he DOES NOT want (*ii*): That is,

(*i*) Parameters that must absolutely be present in selected candidates: These represent traits or skills that are essential and desired in a coach, and their absence disqualifies a candidate.

(*ii*) Parameters that are preferred NOT to be present in candidates to be selected: These represent undesirable traits or deficiencies that make a candidate unsuitable for selection.

By categorizing these parameters into two sets, the selection process ensures clarity and alignment with the decision-maker’s priorities. Let these parameter sets be $\mathcal{M}=\{c_1,c_3,c_4,c_7\}$ and $\mathcal{D}=\{c_2,c_5,c_6,c_8\}$, respectively.

Step 2: Construct the soft sets by using the parameter sets determined in Step 1.

Using these parameter sets, the decision-maker constructs the soft sets $(\mho,\mathcal{M})$ and $(\mho,\mathcal{D})$, respectively:

$$(\mho,\mathcal{M})=\{(c_1,\{z_1,z_3,z_4,z_7,z_9,z_{13},z_{15},z_{17},z_{22},z_{24}\}),(c_3,\{z_2,z_3,z_5,z_8,z_{11},z_{17},z_{21},z_{23},z_{25}\}),$$
$$(c_4,\{z_6,z_9,z_{13},z_{16},z_{18},z_{19},z_{21},z_{22},z_{24}\}),(c_7,\{z_1,z_3,z_6,z_{10},z_{11},z_{13},z_{17},z_{22},z_{23},z_{25}\})\}$$

and

$$(\mho,\mathcal{D})=\{(c_2,\{z_8,z_9,z_{12},z_{14},z_{16},z_{17},z_{22},z_{25}\}),(c_5,\{z_1,z_3,z_5,z_7,z_{11},z_{15},z_{19},z_{20},z_{21}\}),$$
$$(c_6,\{z_3,z_4,z_6,z_9,z_{13},z_{18},z_{19},z_{25}\}),(c_8,\{z_7,z_{10},z_{11},z_{14},z_{19},z_{20},z_{21},z_{22},z_{23},z_{25}\})\}$$

While $(\mho,\mathcal{M})$ is a soft set representing candidates close to the ideal by possessing the desired parameters in $\mathcal{M}$, $(\mho,\mathcal{D})$ is a soft set representing candidates to be eliminated due to undesirable parameters in $\mathcal{D}$.

Step 3: Determine the $\Lambda_\backslash$-product of soft sets.

$$\begin{aligned}\mho_{\mathcal{M}}\Lambda_\backslash\mho_{\mathcal{D}} = &\{((c_1,c_2),\{z_1,z_3,z_4,z_7,z_{13},z_{15},z_{24}\}),((c_1,c_5),\{z_4,z_9,z_{13},z_{17},z_{22},z_{24}\}),\\ &((c_1,c_6),\{z_1,z_7,z_{15},z_{17},z_{22},z_{24}\}),((c_1,c_8),\{z_1,z_3,z_4,z_9,z_{13},z_{15},z_{17},z_{24}\}),\\ &((c_3,c_2),\{z_2,z_3,z_5,z_{11},z_{21},z_{23}\}),((c_3,c_5),\{z_2,z_8,z_{17},z_{23},z_{25}\}),\\ &((c_3,c_6),\{z_2,z_5,z_8,z_{11},z_{17},z_{21},z_{23}\}),((c_3,c_8),\{z_2,z_3,z_5,z_8,z_{17}\}),\\ &((c_4,c_2),\{z_6,z_{13},z_{18},z_{19},z_{21},z_{24}\}),((c_4,c_5),\{z_6,z_9,z_{13},z_{16},z_{18},z_{22},z_{24}\}),\\ &((c_4,c_6),\{z_{16},z_{21},z_{22},z_{24}\}),((c_4,c_8),\{z_6,z_9,z_{13},z_{16},z_{18},z_{24}\}),\end{aligned}$$

$$\big((c_7,c_2),\{z_1,z_3,z_6,z_{10},z_{11},z_{13},z_{23}\}\big),\big((c_7,c_5),\{z_6,z_{10},z_{13},z_{17},z_{22},z_{23},z_{25}\}\big),$$

$$\big((c_7,c_6),\{z_1,z_{10},z_{11},z_{17},z_{22},z_{23}\}\big),\big((c_7,c_8),\{z_1,z_3,z_6,z_{13},z_{17}\}\big)\}$$

Step 4: Determine the set of *uni-int*($\mho_{\mathcal{M}}\Lambda_{\backslash}\mho_{\mathcal{D}}$):

$$uni_m-int_d(\mho_{\mathcal{M}}\Lambda_{\backslash}\mho_{\mathcal{D}})=\cup_{m\in\mathcal{M}}(\cap_{d\in\mathcal{D}}(\mho_{\mathcal{M}}\Lambda_{\backslash}\mho_{\mathcal{D}})(m,d)))$$

We determine first $\cap_{d\in\mathcal{D}}\Big((\mho_{\mathcal{M}}\Lambda_{\backslash}\mho_{\mathcal{D}})(m,d)\Big)$:

$$(\mho_{\mathcal{M}}\Lambda_{\backslash}\mho_{\mathcal{D}})(c_1,c_2)\cap(\mho_{\mathcal{M}}\Lambda_{\backslash}\mho_{\mathcal{D}})(c_1,c_5)\cap(\mho_{\mathcal{M}}\Lambda_{\backslash}\mho_{\mathcal{D}})(c_1,c_6)\cap(\mho_{\mathcal{M}}\Lambda_{\backslash}\mho_{\mathcal{D}})(c_1,c_8)$$
$$=\{z_1,z_3,z_4,z_7,z_{13},z_{15},z_{24}\}\cap\{z_4,z_9,z_{13},z_{17},z_{22},z_{24}\}\cap\{z_1,z_7,z_{15},z_{17},z_{22},z_{24}\}$$
$$\cap\{z_1,z_3,z_4,z_9,z_{13},z_{15},z_{17},z_{24}\}=\{z_{24}\}$$

$$(\mho_{\mathcal{M}}\Lambda_{\backslash}\mho_{\mathcal{D}})(c_3,c_2)\cap(\mho_{\mathcal{M}}\Lambda_{\backslash}\mho_{\mathcal{D}})(c_3,c_5)\cap(\mho_{\mathcal{M}}\Lambda_{\backslash}\mho_{\mathcal{D}})(c_3,c_6)\cap(\mho_{\mathcal{M}}\Lambda_{\backslash}\mho_{\mathcal{D}})(c_3,c_8)$$
$$=\{z_2,z_3,z_5,z_{11},z_{21},z_{23}\}\cap\{z_2,z_8,z_{17},z_{23},z_{25}\}\cap\{z_2,z_5,z_8,z_{11},z_{17},z_{21},z_{23}\}$$
$$\cap\{z_2,z_3,z_5,z_8,z_{17}\}=\{z_2\}$$

$$(\mho_{\mathcal{M}}\Lambda_{\backslash}\mho_{\mathcal{D}})(c_4,c_2)\cap(\mho_{\mathcal{M}}\Lambda_{\backslash}\mho_{\mathcal{D}})(c_4,c_5)\cap(\mho_{\mathcal{M}}\Lambda_{\backslash}\mho)(c_4,c_6)\cap(\mho_{\mathcal{M}}\Lambda_{\backslash}\mho_{\mathcal{D}})(c_4,c_8)$$
$$=\{z_6,z_{13},z_{18},z_{19},z_{21},z_{24}\}\cap\{z_6,z_9,z_{13},z_{16},z_{18},z_{22},z_{24}\}\cap\{z_{16},z_{21},z_{22},z_{24}\}$$
$$\cap\{z_6,z_9,z_{13},z_{16},z_{18},z_{24}\}=\{z_{24}\}$$

$$(\mho_{\mathcal{M}}\Lambda_{\backslash}\mho_{\mathcal{D}})(c_7,c_2)\cap(\mho_{\mathcal{M}}\Lambda_{\backslash}\mho_{\mathcal{D}})(c_7,c_5)\cap(\mho_{\mathcal{M}}\Lambda_{\backslash}\mho_{\mathcal{D}})(c_7,c_6)\cap(\mho_{\mathcal{M}}\Lambda_{\backslash}\mho_{\mathcal{D}})(c_7,c_8)$$
$$=\{z_1,z_3,z_6,z_{10},z_{11},z_{13},z_{23}\}\cap\{z_6,z_{10},z_{13},z_{17},z_{22},z_{23},z_{25}\}\cap\{z_1,z_{10},z_{11},z_{17},z_{22},z_{23}\}$$
$$\cap\{z_1,z_3,z_6,z_{13},z_{17}\}=\emptyset$$

Thus,

$$uni_m-int_d(\mho_{\mathcal{M}}\Lambda_{\backslash}\mho_{\mathcal{D}})=\cup_{m\in\mathcal{M}}\Big(\cap_{d\in\mathcal{D}}\Big(\mho_{\mathcal{M}}\Lambda_{\backslash}\mho_{\mathcal{D}}(m,d)\Big)\Big)=\{z_{24}\}\cup\{z_2\}\cup\{z_{24}\}\cup\emptyset=\{z_2,z_{24}\}$$

$$uni_d-int_m(\mho_{\mathcal{M}}\Lambda_{\backslash}\mho_{\mathcal{D}})=\cup_{d\in\mathcal{D}}(\cap_{m\in\mathcal{M}}(\mho_{\mathcal{M}}\Lambda_{\backslash}\mho_{\mathcal{D}})(m,d)))$$

We determine first $\cap_{m\in\mathcal{M}}\Big((\mho_{\mathcal{M}}\Lambda_{\backslash}\mho_{\mathcal{D}})(m,d)\Big)$:

$$(\mho_{\mathcal{M}}\Lambda_{\backslash}\mho_{\mathcal{D}})(c_1,c_2)\cap(\mho_{\mathcal{M}}\Lambda_{\backslash}\mho_{\mathcal{D}})(c_3,c_2)\cap(\mho_{\mathcal{M}}\Lambda_{\backslash}\mho_{\mathcal{D}})(c_4,c_2)\cap(\mho_{\mathcal{M}}\Lambda_{\backslash}\mho_{\mathcal{D}})(c_7,c_2)$$
$$=\{z_1,z_3,z_4,z_7,z_{13},z_{15},z_{24}\}\cap\{z_2,z_3,z_5,z_{11},z_{21},z_{23}\}\cap\{z_6,z_{13},z_{18},z_{19},z_{21},z_{24}\}$$
$$\cap\{z_1,z_3,z_6,z_{10},z_{11},z_{13},z_{23}\}=\emptyset$$

$$(\mho_{\mathcal{M}}\Lambda_{\backslash}\mho_{\mathcal{D}})(c_1,c_5)\cap(\mho_{\mathcal{M}}\Lambda_{\backslash}\mho_{\mathcal{D}})(c_3,c_5)\cap(\mho_{\mathcal{M}}\Lambda_{\backslash}\mho_{\mathcal{D}})(c_4,c_5)\cap(\mho_{\mathcal{M}}\Lambda_{\backslash}\mho_{\mathcal{D}})(c_7,c_5)$$
$$=\{z_4,z_9,z_{13},z_{17},z_{22},z_{24}\}\cap\{z_2,z_8,z_{17},z_{23},z_{25}\}\cap\{z_6,z_9,z_{13},z_{16},z_{18},z_{22},z_{24}\}$$
$$\cap\{z_6,z_{10},z_{13},z_{17},z_{22},z_{23},z_{25}\}=\emptyset$$

$$(\mho_{\mathcal{M}}\Lambda_{\backslash}\mho_{\mathcal{D}})(c_1,c_6)\cap(\mho_{\mathcal{M}}\Lambda_{\backslash}\mho_{\mathcal{D}})(c_3,c_6)\cap(\mho_{\mathcal{M}}\Lambda_{\backslash}\mho_{\mathcal{D}})(c_4,c_6)\cap(\mho_{\mathcal{M}}\Lambda_{\backslash}\mho_{\mathcal{D}})(c_7,c_6)$$
$$=\{z_1,z_7,z_{15},z_{17},z_{22},z_{24}\}\cap\{z_2,z_5,z_8,z_{11},z_{17},z_{21},z_{23}\}\cap\{z_{16},z_{21},z_{22},z_{24}\}$$
$$\cap\{z_1,z_{10},z_{11},z_{17},z_{22},z_{23}\}=\emptyset$$

$$(\mho_{\mathcal{M}}\Lambda_{\backslash}\mho_{\mathcal{D}})(c_1,c_8)\cap(\mho_{\mathcal{M}}\Lambda_{\backslash}\mho_{\mathcal{D}})(c_3,c_8)\cap(\mho_{\mathcal{M}}\Lambda_{\backslash}\mho_{\mathcal{D}})(c_4,c_8)\cap(\mho_{\mathcal{M}}\Lambda_{\backslash}\mho_{\mathcal{D}})(c_7,c_8)$$
$$=\{z_1,z_3,z_4,z_9,z_{13},z_{15},z_{17},z_{24}\}\cap\{z_2,z_3,z_5,z_8,z_{17}\}\cap\{z_6,z_9,z_{13},z_{16},z_{18},z_{24}\}$$
$$\cap\{z_1,z_3,z_6,z_{13},z_{17}\}=\emptyset$$

Thus,

$$uni_d-int_m(\mho_{\mathcal{M}}\Lambda_{\backslash}\mho_{\mathcal{D}})=\cup_{d\in\mathcal{D}}\left(\cap_{m\in\mathcal{M}}\left((\mho_{\mathcal{M}}\Lambda_{\backslash}\mho_{\mathcal{D}})(m,d)\right)\right)=\emptyset\cup\emptyset\cup\emptyset\cup\emptyset=\emptyset$$

Hence,

$$\textit{uni-int}\big(\mho_{\mathcal{M}}\Lambda_{\backslash}\mathfrak{F}_{\mathcal{D}}\big)=[uni_m-int_d(\mho_{\mathcal{M}}\Lambda_{\backslash}\mho_{\mathcal{D}})]\cup[uni_d-int_m(\mho_{\mathcal{M}}\Lambda_{\backslash}\mho_{\mathcal{D}})]=\{z_2,z_{24}\}\cup\emptyset=\{z_2,z_{24}\}$$

From this, it can be concluded that out of the 25 applicants whose applications are accepted for the private club's coach recruitment process, the candidates z_2 and z_{24} earn the right to undergo a comprehensive training program of this club to join the private club professional coach team.

6. Conclusion

In this study, we first presented a new type of soft set product that we term the "soft difference-product" using Molodtsov's soft set. We provided its example and closely analyzed its algebraic characteristics in terms of different types of soft subset and soft equality, including M-subset/equality, F-subset/equality, L-subset/equality, and J-subset/equality. Furthermore, the distributions of soft difference-product over various kinds of soft set operations were obtained. Finally, we applied the *uni-int* soft decision-making method that selects the optimal elements among potential options without the need of rough or fuzzy soft sets. Additionally, we included an example that shows how the method may be used successfully for a real-world scenario problem. This work may enable several applications, such as new approaches to decision-making and novel soft set-based cryptography techniques. Future research may propose some more new types of soft product operations and further examine fundamental characteristics associated with different kinds of soft equal relations to contribute to the soft set literature from a theoretical and practical standpoint.

Author Contributions

All the authors equally contributed to this work. This paper is derived from the second author's master's thesis supervised by the first author. They all read and approved the final version of the paper.

Conflict of Interest

All the authors declare no conflict of interest.

Ethical Review and Approval

No approval from the Board of Ethics is required.

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