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Research Article

General Hardy-type operators on local generalized Morrey spaces

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ABSTRACT. This paper extends the mapping properties of the general Hardy-type operators to local generalized Morrey spaces built on ball quasi-Banach function spaces. As applications of the main result, we establish the two weight norm inequalities of the Hardy operators to the local generalized Morrey spaces, the mapping properties of the Riemann-Liouville integrals on local generalized Morrey spaces built on rearrangement-invariant quasi-Banach function spaces, the Hardy inequalities on the local generalized Morrey spaces with variable exponents.

Keywords: General Hardy-type operator, Hardy inequality, Riemann-Liouville integrals, local generalized Morrey spaces, ball Banach function spaces, rearrangement-invariant, variable exponents.

2020 Mathematics Subject Classification: 42B25, 42B35, 46E40.

This paper extends the mapping properties of the general Hardy-type operators to the local generalized Morrey spaces built on ball quasi-Banach function spaces.

The general Hardy-type operators [25, Definition 2.5] include a number of important operators in analysis. The most important example is the Hardy operator. It also includes the Riemann-Liouville integrals. The mapping properties of the general Hardy-type operators on Lebesgue spaces and extensions of Lebesgue spaces were investigated in [1, 2, 4, 11, 12, 15, 16, 25, 26, 29, 31, 34, 35, 36, 37].

The local generalized Morrey spaces are extensions of the Lebesgue spaces and Morrey spaces [28, 33]. The local generalized Morrey spaces are members of the ball quasi-Banach function spaces introduced in [32]. A number of results from the harmonic analysis, such as the mapping properties of the singular integral operators, the fractional integral operators, the maximal Carleson operators, the geometric maximal functions, the minimal functions and the spherical maximal functions had been extended to the local generalized Morrey spaces [5, 6, 7, 8, 9, 13, 14, 19, 20, 30, 38, 40].

It motivates us to investigate the mapping properties of the general Hardy-type operators on the local generalized Morrey spaces. We find that whenever a given general Hardy-type operator is bounded on a ball quasi-Banach function space, it can be extended to be a bounded operator on the local generalized Morrey space built on this ball quasi-Banach function space. As applications of this main result, we extend the mapping properties of the general Hardy-type operators with Oinarov kernel on the weighted local generalized Morrey spaces, the Riemann-Liouville integral on the local generalized Morrey spaces built on rearrangement-invariant quasi-Banach function spaces. We also obtain the Hardy-type inequalities on the local generalized Morrey spaces with variable exponents.

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This paper is organized as follows. The definition of the general Hardy-type operator is given in Section 1. This section also contains the mapping properties of the general Hardy-type operators on weighted Lebesgue spaces. The main result is given in Section 2. The definitions of the ball quasi-Banach function spaces and its corresponding local generalized Morrey spaces are also presented in Section 2. The applications of the main result on the weighted local generalized Morrey spaces, the local generalized Morrey spaces built on rearrangement-invariant quasi-Banach function spaces and the local generalized Morrey spaces with variable exponents are given in Section 3.

1. Preliminaries and Definitions

Let $\mathcal{M}$ denote the class of Lebesgue measurable functions on $(0,\infty)$. For any Lebesgue measurable set E on $(0,\infty)$, the Lebesgue measure of E is denoted by $|E|$. Define $I_0=\{(0,r): r>0\}$ and $I=\{(s,r): r>s\geq 0\}$.

Let $p\in(0,\infty)$ and $v:(0,\infty)\to[0,\infty)$, the weighted Lebesgue space $L^p(v)$ consists of all Lebesgue measurable functions f satisfying

$$\|f\|_{L^p(v)}=\left(\int_0^\infty |f(x)|^p v(x)dx\right)^{\frac{1}{p}}<\infty.$$

Let $k:(0,\infty)\times(0,\infty)\to\mathbb{R}$ be a Lebesgue measurable function satisfying $k(x,y)\geq 0$ when $0<y<x$. The general Hardy-type operator with kernel k is defined as

$$Kf(x)=\int_0^x k(x,y)f(y)dy, \quad x\in(0,\infty),$$

see [25, Definition 2.5].

When $k(x,y)\equiv 1$, K is the Hardy operator $Hf(x)=\int_0^x f(t)dt$. When $\alpha\in[0,\infty)$ and $k(x,y)=\frac{1}{\Gamma(\alpha)}(x-y)^{\alpha-1}$, K is the Riemann-Liouville operator $R_\alpha f(x)=\frac{1}{\Gamma(\alpha)}\int_0^x(x-y)^{\alpha-1}f(y)dy$, see [2, 37].

Definition 1.1. *Let $k:(0,\infty)\times(0,\infty)\to(0,\infty)$ be a Lebesgue measurable function. We say that k is an Oinarov kernel if it satisfies*

(1) $k(x,y)\geq 0$ when $0<y<x$,
(2) k is non-decreasing in x or non-increasing in y,
(3) there is a constant $D>0$ such that for any $0<z<y<x$,

$$D^{-1}(k(x,y)+k(y,z))\leq k(x,z)\leq D(k(x,y)+k(y,z)).$$

The reader is referred to [25, Example 2.7] for the examples of the Oinarov kernels.

We now recall some well known boundedness results for the general Hardy-type operators with Oinarov kernels in the following. For any $s\in[0,\infty)$, we write

$$K_s f(x)=\int_0^x k(x,y)^s f(y)dy, \quad \tilde{K}_s f(y)=\int_y^\infty k(x,y)^s f(x)dx.$$

We have the following result for the boundedness of general Hardy-type operators on the weighted Lebesgue space.

Theorem 1.1. *Let $1 < p \leq q < \infty$, $u, v : (0,\infty) \to [0,\infty)$ and $k : (0,\infty) \times (0,\infty) \to \mathbb{R}$ be an Oinarov kernel. If K, u and v satisfy*

$$\sup_{t>0} (\tilde{K}_q u)^{1/q} (K_0 v^{1-p'})^{1/p'}(t) < \infty \tag{1.1}$$

$$\sup_{t>0} (\tilde{K}_0 u)^{1/q}(t) (K_{p'} v^{1-p'})^{1/p'}(t) < \infty, \tag{1.2}$$

then there is a constant $C > 0$ such that for any $f \in L^p(v)$

$$\|Kf\|_{L^q(u)} \leq C\|f\|_{L^p(v)}.$$

For the proof of the above result, the reader is referred to [25, Theorem 2.10]. We have the following results from [25, Theorem 2.15].

Theorem 1.2. *Let $1 < q < p < \infty$ and $\frac{1}{r} = \frac{1}{q} - \frac{1}{p}$. If k is an Oinarov kernel and*

$$\left(\int_0^\infty \left((\tilde{K}_q u)^{1/q}(t) (K_0 v^{1-p'})^{1/q'}(t) \right)^r v^{1-p'}(t) dt \right)^{\frac{1}{r}} < \infty, \tag{1.3}$$

$$\left(\int_0^\infty \left((\tilde{K}_0 u)^{1/p}(t) (K_{p'} v^{1-p'})^{1/p'}(t) \right)^r u(t) dt \right)^{\frac{1}{r}} < \infty, \tag{1.4}$$

then there is a constant $C > 0$ such that for any $f \in L^p(v)$

$$\|Kf\|_{L^q(u)} \leq C\|f\|_{L^p(v)}.$$

The above theorems also give the results in [26] where $K(x, y) = \phi(y/x)$ and $\phi : (0,1) \to (0,\infty)$ is a Lebesgue measurable function. When $k(x, y) = g(x - y)$ for some Lebesgue measurable function $g : (0,\infty) \to (0,\infty)$, the above theorems extend the results in [36]. For the mapping properties of the general Hardy-type operators on weighted Herz spaces, the reader is referred to [24].

2. MAIN RESULTS

The main result of this paper is established in this section. We obtain the mapping properties of the general Hardy-type operators on the local generalized Morrey spaces built on ball quasi-Banach function spaces. Notice that the main result given in this section applies to a general kernel k, not necessary restricted to the Oinarov kernel.

We begin with the definition of the ball quasi-Banach function spaces introduced in [32].

Definition 2.2. *A quasi-Banach space $X \subset \mathcal{M}$ is a ball quasi-Banach function space if it satisfies*

(1) *there is a constant $C > 0$ such that for any $f, g \in X$, $\|f + g\|_X \leq C(\|f\|_X + \|g\|_X)$,*
(2) *$\|f\|_X = 0$ if and only if $f = 0$ a.e. on $(0,\infty)$,*
(3) *$0 \leq g \leq f$ and $f \in X$ implies $g \in X$ and $\|g\|_X \leq \|f\|_X$,*
(4) *$f_n \uparrow f$ and $f \in X$ implies $\|f_n\|_X \uparrow \|f\|_X$,*
(5) *for any $E \in I$, we have $\chi_E \in X$.*

Whenever $\|\cdot\|_X$ satisfies (1)-(3) and

$$\chi_E \in \mathcal{M},\ |E| < \infty \Rightarrow \chi_E \in X, \tag{2.5}$$

X is called as a quasi-Banach function space.

Whenever $\|\cdot\|_X$ is a norm and for any $E \in I$, we have a constant $C > 0$ such that for any $f \in X$, we have $\int_E |f(x)|dx < \infty$, X is a ball Banach function space.

The family of the ball quasi-Banach function spaces includes a number of well known function spaces. The weighted Lebesgue spaces, the rearrangement-invariant quasi-Banach function spaces and the Lebesgue spaces with variable exponents are members of the ball quasi-Banach function spaces.

We now give the definition of the local generalized Morrey spaces built on ball quasi-Banach function spaces.

Definition 2.3. *Let X be a ball quasi-Banach function space and $\omega : (0,\infty) \to (0,\infty)$. The local generalized Morrey space LM_ω^X consists of all $f \in \mathcal{M}$ satisfying*

$$\|f\|_{LM_\omega^X} = \sup_{r>0} \frac{1}{\omega(r)} \|\chi_{(0,r)} f\|_X < \infty.$$

Whenever X is the Lebesgue space L^p, $p \in (1,\infty)$, LM_ω^X becomes the classical local generalized Morrey space.

The following results identify the conditions that ensure that LM_ω^X is a ball quasi-Banach function space.

Proposition 2.1. *Let X be a ball quasi-Banach function space and $\omega : (0,\infty) \to (0,\infty)$. If ω and X satisfy*

$$1 \le C\omega(r), \quad r \in (1,\infty), \tag{2.6}$$

$$\|\chi_{(0,r)}\|_X \le C\omega(r), \quad r \in (0,1) \tag{2.7}$$

for some $C > 0$, then LM_ω^X is a ball quasi-Banach function space.

Proof. It is easy to see that LM_ω^X satisfies Items (1)-(3) in Definition 2.2. To obtain Item (4) of Definition 2.2, it suffices to show that for any $s > 0$, we have $\chi_{(0,s)} \in LM_\omega^X$.

When $r \in (1,\infty)$, (2.6) guarantees that

$$\frac{1}{\omega(r)} \|\chi_{(0,r)}\chi_{(0,s)}\|_X \le \frac{1}{\omega(r)} \|\chi_{(0,s)}\|_X \le C\|\chi_{(0,s)}\|_X.$$

When $r \in (0,1)$, (2.7) yields

$$\frac{1}{\omega(r)} \|\chi_{(0,r)}\chi_{(0,s)}\|_X \le \frac{1}{\omega(r)} \|\chi_{(0,r)}\|_X \le C.$$

The above inequalities assure that

$$\sup_{r>0} \frac{1}{\omega(r)} \|\chi_{(0,r)}\chi_{(0,s)}\|_X \le C + C\|\chi_{(0,s)}\|_X$$

and, hence, $\chi_{(0,r)} \in LM_\omega^X$. □

We also have the following result with the range for r replaced by $\|\chi_{(0,r)}\|_X$.

Proposition 2.2. *Let X be a ball quasi-Banach function space and $\omega : (0,\infty) \to (0,\infty)$. If ω and X satisfy*

$$1 \le C\omega(r), \quad 1 < \|\chi_{(0,r)}\|_X, \tag{2.8}$$

$$\|\chi_{(0,r)}\|_X \le C\omega(r), \quad 1 \ge \|\chi_{(0,r)}\|_X \tag{2.9}$$

for some $C > 0$, then LM_ω^X is a ball quasi-Banach function space.

Proof. It suffices to show that for any $s > 0$, we have $\chi_{(0,s)} \in LM^X_\omega$.

When r satisfies $1 \le \|\chi_{(0,r)}\|_X$, (2.8) guarantees that

$$\frac{1}{\omega(r)}\|\chi_{(0,r)}\chi_{(0,s)}\|_X \le \frac{1}{\omega(r)}\|\chi_{(0,s)}\|_X \le C\|\chi_{(0,s)}\|_X.$$

When r satisfies $1 \ge \|\chi_{(0,r)}\|_X$, (2.9) yields

$$\frac{1}{\omega(r)}\|\chi_{(0,r)}\chi_{(0,s)}\|_X \le \frac{1}{\omega(r)}\|\chi_{(0,r)}\|_X \le C.$$

The above inequalities assure that

$$\sup_{r>0}\frac{1}{\omega(r)}\|\chi_{(0,r)}\chi_{(0,s)}\|_X \le C + C\|\chi_{(0,s)}\|_X$$

and, hence, $\chi_{(0,r)} \in LM^X_\omega$. □

We write $(X,\omega) \in \mathcal{N}$ if LM^X_ω is nontrivial. The above propositions assure that $(X,\omega) \in \mathcal{N}$ whenever X and ω satisfy (2.6)-(2.7) or (2.8)-(2.9).

We now present the main result, the mapping properties of the general Hardy-type operators on the local generalized Morrey space LM^X_ω.

Theorem 2.3. *Let X and Y be ball quasi-Banach function spaces and $\omega : (0,\infty) \to (0,\infty)$. Let $k : (0,\infty)\times(0,\infty) \to \mathbb{R}$ be a Lebesgue measurable function satisfying $k(x,y) \ge 0$ when $0 < y < x$. If $(X,\omega) \in \mathcal{N}$ and there is a constant $C > 0$ such that for any $f \in X$*

$$\|Kf\|_Y \le C\|f\|_X,$$

then for any $f \in LM^X_\omega$

$$\|Kf\|_{LM^Y_\omega} \le C\|f\|_{LM^X_\omega}. \tag{2.10}$$

Proof. Let $r > 0$ and $f \in LM^X_\omega$. When $x > r$, we have

$$\chi_{(0,r)}(x)(K|f|)(x) = 0 \le \int_0^x \chi_{(0,r)}(y)k(x,y)|f(y)|dy. \tag{2.11}$$

When $x \in (0,r]$, we have

$$\chi_{(0,r)}(x)(K|f|)(x) = \int_0^x k(x,y)|f(y)|dy = \int_0^x \chi_{(0,r)}(y)k(x,y)|f(y)|dy \tag{2.12}$$

because for any $y \in (0,x)$, we have $y \in (0,r)$. Hence, $\chi_{(0,r)}(y) = 1$.

Consequently, (2.11) and (2.12) give

$$\chi_{(0,r)}(x)(K|f|)(x) \le \int_0^x \chi_{(0,r)}(y)k(x,y)|f(y)|dy = K(\chi_{(0,r)}|f|)(x). \tag{2.13}$$

By applying the quasi-norm $\|\cdot\|_Y$ on both sides of (2.13), item (2) of Definition 2.2 yields

$$\|\chi_{(0,r)}K|f|\|_Y \le \|K(\chi_{(0,r)}|f|)\|_Y.$$

The boundedness of $K : X \to Y$ and $|Kf| \le K|f|$ guarantee that

$$\|\chi_{(0,r)}Kf\|_Y \le C\|\chi_{(0,r)}f\|_X.$$

By multiplying $\frac{1}{\omega(r)}$ on both sides of the above inequality, we obtain

$$\frac{1}{\omega(r)}\|\chi_{(0,r)}Kf\|_Y \le C\frac{1}{\omega(r)}\|\chi_{(0,r)}f\|_X \le C\|f\|_{LM^X_\omega}.$$

Finally, by taking the supremum over $r > 0$, we have

$$\|Kf\|_{LM_\omega^Y} = \sup_{r>0} \frac{1}{\omega(r)} \|\chi_{(0,r)} Kf\|_Y \le C \|f\|_{LM_\omega^X}.$$

□

The condition $(X, \omega) \in \mathcal{N}$ ensures that (2.10) is meaningful. In addition, the above result asserts that

$$\|K\|_{LM_\omega^X \to LM_\omega^Y} \le \|K\|_{X \to Y},$$

where $\|K\|_{LM_\omega^X \to LM_\omega^Y}$ and $\|K\|_{X\to Y}$ are the operator norms of $K : LM_\omega^X \to LM_\omega^Y$ and $K : X \to Y$, respectively.

Furthermore, the above result does not assume that k is an Oinarov kernel.

3. Applications

We give applications for Theorem 2.3 on some concrete function spaces and general Hardy-type operators in this section. We study the general Hardy-type operators with Oinarov kernel on the weighted local generalized Morrey spaces, the Riemann-Liouville integral on the local generalized Morrey spaces built on rearrangement-invariant quasi-Banach function spaces. We also establish the Hardy-type inequalities on the local generalized Morrey spaces with variable exponents.

3.1. **Weighted local generalized Morrey spaces.** We extend the mapping properties of the general Hardy-type operators with Oinarov kernel on weighted local generalized Morrey spaces in this section.

Definition 3.4. *Let $p \in (0, \infty)$, v be a locally integrable function and $\omega : (0, \infty) \to (0, \infty)$ be a Lebesgue measurable function. The weighted local generalized Morrey space $LM_{v,\omega}^p$ consists of all $f \in \mathcal{M}$ satisfying*

$$\|f\|_{LM_{v,\omega}^p} = \sup_{r>0} \frac{1}{\omega(r)} \|\chi_{(0,r)} f\|_{L^p(v)} < \infty.$$

Proposition 3.3. *Let $p \in (0, \infty)$, v be a locally integrable function and $\omega : (0, \infty) \to (0, \infty)$ be a Lebesgue measurable function. If ω satisfies (2.6) and*

$$\left(\int_0^r v(x)dx \right)^{\frac{1}{p}} \le C\omega(r), \quad r \in (0, 1) \tag{3.14}$$

for some $C > 0$, then $LM_{v,\omega}^p$ is a ball quasi-Banach function space.

Proof. As v is a local integrable function, for any $E \in I$, $\int_I v(x)dx < \infty$, we see that $L^p(v)$ is a ball quasi-Banach function space. According to Proposition 2.1, as ω satisfies (2.6) and (3.14), $LM_{v,\omega}^p$ is also a ball quasi-Banach function space. □

Proposition 3.3 guarantees that when v and ω satisfy (2.6) and (3.14), the weighted local generalized Morrey space $LM_{v,\omega}^p$ is nontrivial.

In particular, if $\theta \in (0, 1)$ and $\omega_\theta(r) = \left(\int_0^r v(x)dx \right)^{\frac{\theta}{p}}$, then (3.14) is fulfilled and LM_{v,ω_θ}^p is a ball quasi-Banach function space.

Theorems 1.1, 1.2 and 2.3 give the mapping properties of the general Hardy-type operators on the weighted local generalized Morrey spaces.

Theorem 3.4. *Let $p, q \in (1, \infty)$, $u, v : (0, \infty) \to [0, \infty)$ be locally integrable functions, $\omega : (0, \infty) \to (0, \infty)$ satisfy (2.6) and (3.14) and k be a Oinarov kernel.*

(1) If $p \le q$ and K, u and v satisfy (1.1) and (1.2), then there is a constant $C > 0$ such that for any $f \in LM^p_{v,\omega}$, we have

$$\|Kf\|_{LM^q_{u,\omega}} \le C\|f\|_{LM^p_{v,\omega}}.$$

(2) If $q \le p$, $\frac{1}{r} = \frac{1}{q} - \frac{1}{p}$ and K, u and v satisfy (1.3) and (1.4), then there is a constant $C > 0$ such that for any $f \in LM^p_{v,\omega}$, we have

$$\|Kf\|_{LM^q_{u,\omega}} \le C\|f\|_{LM^p_{v,\omega}}.$$

We now apply the above theorem to establish the mapping properties of the Hardy operator H on the weighted local generalized Morrey spaces. Recall that

$$\tilde{H}f(x) = \int_x^\infty f(y)dy.$$

Theorem 3.5. *Let $p, q \in (1, \infty)$, $u, v : (0, \infty) \to [0, \infty)$ be locally integrable functions and $\omega : (0, \infty) \to (0, \infty)$ satisfy (2.6) and (3.14).*

(1) If $p \le q$ and

$$\sup_{t>0} \left(\int_t^\infty u(y)dy\right)^{\frac{1}{q}} \left(\int_0^t v(y)^{1-p'}dy\right)^{\frac{1}{p'}} < \infty, \tag{3.15}$$

then there is a constant $C > 0$ such that for any $f \in LM^p_{v,\omega}$, we have

$$\|Hf\|_{LM^q_{u,\omega}} \le C\|f\|_{LM^p_{v,\omega}}.$$

(2) If $q \le p$, $\frac{1}{r} = \frac{1}{q} - \frac{1}{p}$ and

$$\left(\int_0^\infty \left((\tilde{H}u)^{1/q}(t)(Hv^{1-p'})^{1/q'}(t)\right)^r v^{1-p'}(t)dt\right)^{\frac{1}{r}} < \infty, \tag{3.16}$$

$$\left(\int_0^\infty \left((\tilde{H}u)^{1/p}(t)(Hv^{1-p'})^{1/p'}(t)\right)^r u(t)dt\right)^{\frac{1}{r}} < \infty, \tag{3.17}$$

then there is a constant $C > 0$ such that for any $f \in LM^p_{v,\omega}$, we have

$$\|Hf\|_{LM^q_{u,\omega}} \le C\|f\|_{LM^p_{v,\omega}}.$$

The above results are extensions of the two weight norm inequalities of the Hardy operator to the local generalized Morrey spaces.

We consider the case $q \in (0, 1)$ in the following. We first recall the mapping properties of H on the weighted Lebesgue spaces from [35, Theorem 1 (3)].

Theorem 3.6. *Let $0 < q < 1 < p < \infty$ and $u, v : (0, \infty) \to (0, \infty)$ be Lebesgue measurable functions. If u, v satisfy*

$$\int_0^\infty \left(\int_0^t (u(y))^{1-p'}dy\right)^{\frac{r}{p'}} \left(\int_t^\infty v(y)dy\right)^{\frac{r}{p}} dt < \infty, \tag{3.18}$$

where $\frac{1}{r} = \frac{1}{q} - \frac{1}{p}$, then

$$\left(\int_0^\infty |Hf(t)|^q v(t)dt\right)^{\frac{1}{q}} \le C\left(\int_0^\infty |f(t)|^p u(t)dt\right)^{\frac{1}{p}}.$$

We now have the mapping properties of operator H on the weighted local generalized Morrey spaces.

Theorem 3.7. *Let $0 < q < 1 < p < \infty$ and $u, v, \omega : (0,\infty) \to (0,\infty)$ be Lebesgue measurable functions. If u, v satisfy (3.18) and ω satisfies (2.6) and (3.14), then there is a constant $C > 0$ such that for any $f \in LM^p_{u,\omega}$*

$$\|Hf\|_{L^q_{v,\omega}} \leq C\|f\|_{LM^p_{u,\omega}}. \tag{3.19}$$

Proof. As $Hf(x) = \int_0^x f(y)dy$, H is a general Hardy operator with kernel $k(x,y) \equiv 1$. The preceding theorem asserts that $H : L^p(u) \to L^q(v)$ is bounded. Moreover, $(L^p, \omega) \in \mathcal{N}$ because Proposition 3.3 assures that $LM^p_{u,\omega}$ is a ball quasi-Banach function space. Thus, Theorem 2.3 yields (3.19). □

The above result shows that our main result also applies to local generalized Morrey spaces built on quasi-Banach function space X.

3.2. **Local generalized Morrey spaces built on rearrangement-invariant quasi-Banach function spaces.** In this section, we apply Theorem 2.3 to establish the mapping properties of the Riemann-Liouville integral on the local generalized Morrey spaces built on rearrangement-invariant quasi-Banach function spaces.

We first recall some notations for defining of the rearrangement-invariant quasi-Banach function spaces. For any $f \in \mathcal{M}$ and $s > 0$, write

$$d_f(s) = |\{x \in (0,\infty) : |f(x)| > s\}|$$

and

$$f^*(t) = \inf\{s > 0 : d_f(s) \leq t\}, \quad t > 0.$$

We recall the definition of rearrangement-invariant quasi-Banach function space (r.i.q.B.f.s.) from [17, Definition 2.1].

Definition 3.5. *A quasi-Banach function space X is said to be a r.i.q.B.f.s. if there exists a quasi-norm ρ_X satisfying Items (1)-(3) of Definition 2.2 and (2.5) such that for any $f \in X$, we have*

$$\|f\|_X = \rho_X(f^*).$$

Next, we recall the definition of the Boyd indices. For any $s \geq 0$ and $f \in \mathcal{M}(0,\infty)$, define $(D_s f)(t) = f(st)$, $t \in (0,\infty)$. Let $\|D_s\|_{X\to X}$ be the operator norm of D_s on X. We recall the definition of Boyd indices for r.i.q.B.f.s. from [27].

Definition 3.6. *Let X be a r.i.q.B.f.s. Define the lower Boyd index of X, p_X, and the upper Boyd index of X, q_X, by*

$$p_X = \sup\{p > 0 : \exists C > 0 \text{ such that } \forall\, 0 \leq s < 1,\ \|D_s\|_{X\to X} \leq Cs^{-1/p}\},$$
$$q_X = \inf\{q > 0 : \exists C > 0 \text{ such that } \forall\, 1 \leq s,\ \|D_s\|_{X\to X} \leq Cs^{-1/q}\},$$

respectively.

It is easy to see that the Boyd indices for the Lebesgue space L^p, $0 < p < \infty$ is $\frac{1}{p}$.

Proposition 3.4. *Let X be a r.i.q.B.f.s. and $\omega : (0,\infty) \to (0,\infty)$ be a Lebesgue measurable function. If ω satisfies (2.6) and there exists a $q > q_X$ and $C > 0$ such that*

$$r^{\frac{1}{q}} < C\omega(r), \quad r \in (0,1), \tag{3.20}$$

then LM^X_ω is a ball quasi-Banach function space.

Proof. As $D_{1/r}\chi_{(0,1)}(t) = \chi_{(0,1)}(t/r) = \chi_{(0,r)}(t)$, for any $r \in (0,1)$, we find that for any $q > q_X$, we have a constant $C > 0$ such that

$$\|\chi_{(0,r)}\|_X = \|D_{1/r}\chi_{(0,1)}\|_X \le Cr^{1/q}.$$

The above inequality and (3.20) guarantee that (2.7) is satisfied. Therefore, Proposition 2.1 asserts that LM^X_ω is a ball quasi-Banach function space. □

We need the following function space for the studies of the Riemann-Liouville integral.

Definition 3.7. *Let $\alpha \ge 0$ and X be a r.i.q.B.f.s. X_α consists of all $f \in \mathcal{M}$ satisfying*

$$\|f\|_{X_\alpha} = \rho_X(t^{-\alpha} f^*(t)) < \infty.$$

For instance, when $X = L^p$, X_α is the Lorentz spaces $L^{\frac{p}{1-p\alpha},\alpha}$, see [17, p.901]. Moreover, X_α has been used in [17, 18] for the studies of the mapping properties of the convolution operators, the Fourier integral operators and the k-plane transform on r.i.q.B.f.s.

We have the following result from [17, Proposition 3.1].

Proposition 3.5. *Let $\alpha > 0$ and X be a r.i.q.B.f.s. If $0 < p_X \le q_X < \frac{1}{\alpha}$, then X_α is a r.i.q.B.f.s.*

Theorem 3.8. *Let $\alpha > 0$, $\omega : (0,\infty) \to (0,\infty)$ be a Lebesgue measurable function and X be a r.i.q.B.f.s. If $0 < p_X \le q_X < \frac{1}{\alpha}$ and ω satisfies (2.6) and (3.20) for some $q > q_X$, then there is a constant $C > 0$ such that for any $f \in LM^X_\omega$*

$$\|R_\alpha f\|_{LM^{X_\alpha}_\omega} \le C\|f\|_{LM^X_\omega}.$$

Proof. It is well known that for any $p \in (1, \frac{1}{\alpha})$, $R_\alpha L^p \to L^q$ is bounded where $\frac{1}{q} = \frac{1}{p} - \alpha$. By applying [18, Theorem 4.1], we find that $R_\alpha : X \to X_\alpha$ is bounded. Consequently, as Riemann-Liouville integral is a member of general Hardy-type operator, Theorem 2.3 yields the boundedness of $R_\alpha : LM^X_\omega \to LM^{X_\alpha}_\omega$. □

The above result is new even for the local generalized Morrey space LM^p_ω. Notice that we have the following inequality

$$|R_\alpha f(x)| \le \int_0^\infty \frac{|f(y)|}{|x-y|^{1-\alpha}} dy = (I_\alpha |f|)(x), \quad x \in (0,\infty),$$

where I_α is the fractional integral operator on $(0,\infty)$. Therefore, by using the idea in [21, Theorem 3.1], we can obtain the mapping properties I_α and hence, the mapping properties of R_α on LM^X_ω. While by using the idea in [21, Theorem 3.1], we need to impose a stronger condition on ω, a condition similar to [21, (2.10)].

We now give another concrete example for Theorem 3.8. A function $\Phi : [0,+\infty] \to [0,+\infty]$ is a Young function if there exists an increasing and left-continuous function ϕ satisfying $\phi(0) = 0$ and that ϕ is neither identically zero nor identically infinite such that

$$\Phi(s) = \int_0^s \phi(u) du, \quad s \ge 0.$$

Let ϕ be a Young function. The Orlicz space L_Φ consists of all Lebesgue measurable functions f satisfying

$$\|f\|_{L_\Phi} = \inf\left\{\lambda > 0 : \int_0^\infty \Phi\left(\frac{|f(x)|}{\lambda}\right) dx \le 1\right\} < \infty.$$

Let $\alpha \in \mathbb{R}$ and Φ be a Young function. The Lorentz-Orlicz space $L_{\Phi,\alpha}$ consists of all Lebesgue measurable functions f satisfying

$$\|f\|_{L_{\Phi,\alpha}} = \inf\left\{\lambda > 0 : \int_0^\infty \Phi(t^{-\alpha/n} f^*(t)/\lambda) dt \le 1\right\} < \infty.$$

In view of [3, Chapter 4, Theorem 8.18], the Boyd indices of L_Φ are given by

$$p_{L_\Phi} = \lim_{t\to\infty} \frac{\log t}{\log g(t)}, \quad \text{and} \quad q_{L_\Phi} = \lim_{t\to 0^+} \frac{\log t}{\log g(t)},$$

where

$$g(t) = \limsup_{s\to\infty} \frac{\Phi^{-1}(s)}{\Phi^{-1}(s/t)}.$$

Theorem 3.8 yields the following mapping properties of the Riemann-Liouville integral on the local Orlicz-Morrey space $M^{L_\Phi}_\omega$.

Corollary 3.1. *Let $\alpha > 0$, $\omega : (0,\infty) \to (0,\infty)$ be a Lebesgue measurable function and Φ be a Young function. If $0 < p_{L_\Phi} \le q_{L_\Phi} < \frac{1}{\alpha}$ and ω satisfies (2.6) and (3.20) for some $q > q_{L_\Phi}$, then there is a constant $C > 0$ such that for any $f \in LM^X_\omega$*

$$\|R_\alpha f\|_{LM^{L_\Phi,\alpha}_\omega} \le C\|f\|_{LM^{L_\Phi}_\omega}.$$

For the studies of boundedness of the Calderón-Zygmund operators, the nonlinear commutators of the Calderón-Zygmund operators, the oscillatory singular integral operators, the singular integral operators with rough kernels and the Marcinkiewicz integrals on the local Orlicz-Morrey spaces on the local Orlicz-Morrey spaces, the reader is referred to [39].

3.3. **Local generalized Morrey spaces with variable exponents.**

In this section, we extend the Hardy-type inequalities in [12] to the local generalized Morrey spaces with variable exponent. Notice that the kernel for the operators studied in this section is not necessarily an Oinarov kernel. Thus, the results in this section give examples for the use of Theorem 2.3 is not restricted to the general Hardy-type operators with Oinarov kernels.

We begin with the definition of the Lebesgue space with variable exponent.

Definition 3.8. *Let $p(\cdot) : (0,\infty) \to [1,\infty)$ be a Lebesgue measurable function. The Lebesgue space with variable exponent $L^{p(\cdot)}$ consists of all $f \in \mathcal{M}$ satisfying*

$$\|f\|_{L^{p(\cdot)}} = \inf\left\{\lambda > 0 : \int_0^\infty \left|\frac{f(x)}{\lambda}\right|^{p(x)} dx \le 1\right\} < \infty.$$

For any Lebesgue measurable function $p(\cdot) : (0,\infty) \to (0,\infty)$, define

$$p_- = \inf_{x\in(0,\infty)} p(x) \quad \text{and} \quad p_+ = \sup_{x\in(0,\infty)} p(x).$$

For any Lebesgue measurable function $p(\cdot) : (0,\infty) \to (0,\infty)$, we write $p(\cdot) \in \mathcal{M}_{0,\infty}$ if there exists a constant $C > 0$ such that

(1) $0 \le p_- \le p_+ < \infty$,
(2) the limit $\lim_{x\to 0} p(x)$ exists, $p(0) = \lim_{x\to 0} p(x)$ and

$$|p(x) - p(0)| \le \frac{C}{-\ln x}, \quad \forall x \in (0, 1/2],$$

(3) the limit $\lim_{x\to\infty} p(x) = p_\infty$ exists and

$$|p(x) - p_\infty| \le \frac{C}{\ln x} \quad \forall x \in [2,\infty).$$

We write $p(\cdot) \in \mathcal{P}_{0,\infty}$ if $p(\cdot) \in \mathcal{M}_{0,\infty}$ and $p_- \geq 1$.

We have the following result from [12, Theorems 3.1, 3.3 and Remark 3.2]. To simplify the presentation of the results in the following, for any $a \in \mathbb{R}$ and Lebesgue measurable functions $\alpha(\cdot)$ and $\mu(\cdot)$ on $(0, \infty)$, we write

$$\mathcal{H}^{\alpha(\cdot)} f(x) = x^{\alpha(x)-1} \int_0^x \frac{f(y)}{y^{\alpha(y)}} dy$$

$$\mathcal{H}^a_{\mu(\cdot)} f(x) = x^{a+\mu(x)-1} \int_0^x \frac{f(y)}{y^a} dy,$$

respectively.

Theorem 3.9. *Let $p(\cdot) \in \mathcal{P}_{0,\infty}$ and $\alpha(\cdot)$ be a bounded function on $(0, \infty)$ such that the limit $\lim_{x\to\infty} \alpha(x) = \alpha_\infty$ exists and satisfies*

$$\alpha(0) < 1 - \frac{1}{p(0)}, \quad \alpha_\infty < 1 - \frac{1}{p_\infty} \tag{3.21}$$

$$|\alpha(x) - \alpha(0)| \leq \frac{C}{|\ln x|}, \quad \forall x \in (0, 1/2], \tag{3.22}$$

$$|\alpha(x) - \alpha_\infty| \leq \frac{C}{\ln x}, \quad \forall x \in (2, \infty) \tag{3.23}$$

for some $C > 0$, then there exists a constant $D > 0$ such that for any $f \in L^{p(\cdot)}$, we have

$$\|\mathcal{H}^{\alpha(\cdot)} f\|_{L^{p(\cdot)}} \leq D\|f\|_{L^{p(\cdot)}}.$$

Theorem 3.10. *Let $a \in \mathbb{R}$, $p(\cdot), \mu(\cdot) : (0, \infty) \to [1, \infty)$ be Lebesgue measurable functions. If*

$$a < \min\left\{1 - \frac{1}{p(0)}, 1 - \frac{1}{p_\infty}\right\}, \tag{3.24}$$

$p(\cdot) \in \mathcal{P}_{0,\infty}$, $\mu(\cdot) \in \mathcal{M}_{0,\infty}$,

$$0 \leq \mu(0) < \frac{1}{p(0)} \quad \text{and} \quad 0 \leq \mu_\infty < \frac{1}{p_\infty}, \tag{3.25}$$

then for any $q(\cdot) \in \mathcal{P}_{0,\infty}$ satisfying

$$\frac{1}{q(0)} = \frac{1}{p(0)} - \mu(0) \quad \text{and} \quad \frac{1}{q_\infty} = \frac{1}{p_\infty} - \mu_\infty, \tag{3.26}$$

we have a constant $D > 0$ such that for any $f \in L^{p(\cdot)}$

$$\|\mathcal{H}^a_{\mu(\cdot)} f\|_{L^{q(\cdot)}} \leq D\|f\|_{L^{p(\cdot)}}$$

The above results are generalizations of the Hardy inequalities to the Lebesgue spaces with variable exponents. For the proofs of the above theorems, the reader is referred to [12, Sections 5 and 6].

Notice that the kernels of the operators $\mathcal{H}^{\alpha(\cdot)}$ and $\mathcal{H}^a_{\mu(\cdot)}$ are

$$K_1(x, y) = \frac{x^{\alpha(x)-1}}{y^{\alpha(y)}} \quad \text{and} \quad K_2(x, y) = \frac{x^{a+\mu(x)-1}}{y^a},$$

respectively. In general, they are not non-decreasing in x nor non-increasing in y, therefore, they do not satisfy Item (1) of Definition 1.1.

Even though the operators $\mathcal{H}^{\alpha(\cdot)}$ and $\mathcal{H}^a_{\mu(\cdot)}$ were not covered by the results in Theorems 1.1 and 1.2, our main result, Theorem 2.3 also yields the mapping properties of these operators on the local generalized Morrey spaces with variable exponents.

Definition 3.9. *Let $p(\cdot):(0,\infty)\to[1,\infty)$ be a Lebesgue measurable function. The local generalized Morrey space with variable exponent $LM^{p(\cdot)}_\omega$ consists of all $f\in\mathcal{M}$ satisfying*

$$\|f\|_{LM^{p(\cdot)}_\omega}=\sup_{r>0}\frac{1}{\omega(r)}\|\chi_{(0,r)}f\|_{L^{p(\cdot)}}<\infty.$$

When $\omega\equiv 1$, $LM^{p(\cdot)}_\omega$ becomes the Lebesgue space with variable exponent $L^{p(\cdot)}$. Moreover, the local generalized Morrey spaces with variable exponents are extensions of the local generalized Morrey spaces.

For the mapping properties of the fractional integral operators, the maximal Carleson operator, the spherical maximal functions, the geometric maximal functions and the minimal functions on the local generalized Morrey spaces with variable exponent and the Hardy local generalized Morrey spaces with variable exponents, the reader is referred to [19, 22, 23, 38, 40].

Let $p(\cdot)\in\mathcal{P}_{0,\infty}$. Whenever ω satisfies

$$1\le C\omega(r),\quad \|\chi_{(0,r)}\|_{L^{p(\cdot)}}>1, \tag{3.27}$$

$$r^{\frac{1}{p_+}}\le C\omega(r),\quad 1\ge\|\chi_{(0,r)}\|_{L^{p(\cdot)}} \tag{3.28}$$

for some $C>0$, Proposition 2.2 and [10, Corollary 2.23] guarantee that LM^X_ω is a ball quasi-Banach function space.

We now present the boundedness of $\mathcal{H}^{\alpha(\cdot)}$ on the local generalized Morrey spaces with variable exponents in the following.

Theorem 3.11. *Let $p(\cdot)\in\mathcal{P}_{0,\infty}$, $\omega:(0,\infty)\to(0,\infty)$ be Lebesgue measurable function and $\alpha(\cdot)$ be a bounded function. Suppose that ω satisfies (3.27) and (3.28). If $\alpha(\cdot)$ satisfies (3.21), (3.22) and (3.23), then there is a constant $C>0$ such that for any $f\in LM^{p(\cdot)}_\omega$*

$$\|\mathcal{H}^{\alpha(\cdot)}f\|_{LM^{p(\cdot)}_\omega}\le C\|f\|_{LM^{p(\cdot)}_\omega}.$$

The above result is a consequence of Theorems 2.3 and 3.9.

Next, we have the mapping properties of $\mathcal{H}^a_{\mu(\cdot)}$ in the local generalized Morrey spaces with variable exponents. The following theorem is guaranteed by Theorems 2.3 and 3.10.

Theorem 3.12. *Let $a\in\mathbb{R}$, $p(\cdot)\in\mathcal{P}_{0,\infty}$, $\mu(\cdot)\in\mathcal{M}_{0,\infty}$ and $\omega(\cdot):(0,\infty)\to(0,\infty)$ be Lebesgue measurable functions. If a, $p(\cdot)$, $\mu(\cdot)$ and $\omega(\cdot)$ satisfy (3.24), (3.25), (3.27) and (3.28), then for any $q(\cdot)\in\mathcal{P}_{0,\infty}$ satisfying (3.26), there is a constant $C>0$ such that for any $f\in LM^{p(\cdot)}_\omega$*

$$\|\mathcal{H}^a_{\mu(\cdot)}f\|_{LM^{q(\cdot)}_\omega}\le C\|f\|_{LM^{p(\cdot)}_\omega}.$$

Theorems 3.11 and 3.12 are extensions of [12, Theorems 3.1 and 3.3] from the Lebesgue spaces with variable exponents to the local generalized Morrey spaces with variable exponents.

We now give some concrete examples on ω such that (3.27) and (3.28) are fulfilled. For any $\theta\in(0,\frac{1}{p_+})$, define $\omega_\theta(r)=r^\theta$. In view of [10, Corollary 2.23], we have $\lim_{r\to\infty}\|\chi_{(0,r)}\|_{L^{p(\cdot)}}=\infty$, therefore, ω_θ fulfills (3.27). Moreover, as $p_+<\infty$, [10, Theorems 2.58 and 2.62] assure that $\|\cdot\|_{L^{p(\cdot)}}$ is an absolutely continuous norm. Thus, $\lim_{r\to0}\|\chi_{(0,r)}\|_{L^{p(\cdot)}}=0$. Consequently, (3.28) is fulfilled.

Whenever $p(\cdot)$ satisfies the conditions in Theorem 3.11, $\mathcal{H}^{\alpha(\cdot)}$ is bounded on $LM^{p(\cdot)}_{\omega_\theta}$. Similarly, whenever a, $p(\cdot)$, $q(\cdot)$ and $\mu(\cdot)$ satisfy the conditions in Theorem 3.12, $\mathcal{H}^a_{\mu(\cdot)}:LM^{p(\cdot)}_{\omega_\theta}\to LM^{q(\cdot)}_{\omega_\theta}$ is bounded.

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Research Article

Multidimensional quadratic-phase Fourier transform and its uncertainty principles

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ABSTRACT. The main aim of this article is to propose a multidimensional quadratic-phase Fourier transform (MQFT) that generalises the well-known and recently introduced quadratic-phase Fourier transform (as well as, of course, the Fourier transform itself) to higher dimensions. In addition to the definition itself, some crucial properties of this new integral transform will be deduced. These include a Riemann-Lebesgue lemma for the MQFT, a Plancherel lemma for the MQFT and a Hausdorff-Young inequality for the MQFT. A second central objective consists of obtaining different uncertainty principles for this MQFT. To this end, using techniques that include obtaining various auxiliary inequalities, the study culminates in the deduction of L^p-type Heisenberg-Pauli-Weyl uncertainty principles and L^p-type Donoho-Stark uncertainty principles for the MQFT.

Keywords: Multidimensional quadratic-phase Fourier transform, Donoho-Stark uncertainty principle, Heisenberg-Pauli-Weyl uncertainty principle, Riemann-Lebesgue lemma, Hausdorff-Young inequality.

2020 Mathematics Subject Classification: 42B10, 42A38, 26D15, 44A15, 47A05, 47B38.

1. INTRODUCTION

The main theme of this work is the "multidimensional quadratic-phase Fourier transform", which is introduced here for the first time, generalising the well-known (one-dimensional) quadratic-phase Fourier transform [2, 3]. This last quadratic-phase Fourier transform has proved to be an integral operator with substantial virtues in the field of applications, showing great potential in terms of the flexibility of the possibilities for choosing its five free parameters. This can be seen in several recent publications, such as [1, 7, 11, 13, 14, 15, 16, 17, 18, 19] (among many other papers). Now, with the current introduction of the multidimensional quadratic-phase Fourier transform, where the roles of these parameters are now various matrices, it is expected that this new operator will also be well received and used, especially in the field of applications (even outside the discipline of Mathematics).

To better understand the structure of the proposed multidimensional quadratic-phase Fourier transform, we will deduce some of its fundamental properties, exhibit some of its relationships with other existing transforms and operators, and then derive some uncertainty principles associated with such new multidimensional quadratic-phase Fourier transform.

On this last point, it should be noted that in the scientific community in general, of all scientific disciplines, the most famous notion of uncertainty principles is related to Quantum Mechanics and directly associated to the fact that Heisenberg concluded that "the position and the momentum of an electron in an atom cannot be both determined explicitly, but only probabilistically under a certain uncertainty". Already in the Harmonic Analysis and Signal Processing

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community, the classic Heisenberg uncertainty principle for the Fourier transform is that the product of the duration and bandwidth of a signal $f(x)$ has a lower bound (which depends on the square of the L^2-norm of f). This inequality has been explored in various contexts and for various integral transforms other than the Fourier transform, becoming commonly known as the Heisenberg-Pauli-Weyl [9, 10, 20, 21] uncertainty principle. Another uncertainty principle that we will consider here is called Donoho-Stark and involves different concepts and quantities, based in particular on the so-called ϵ-concentration and on the measures of certain subsets.

This article is organised as follows. Section 2 is devoted to the introduction of the multidimensional quadratic-phase Fourier transform and the deduction of its fundamental properties (such as a Riemann-Lebesgue lemma, a Plancherel type theorem, an inversion formula and a Hausdorff-Young inequality), which are also useful tools in the following sections. In section 3, we obtain sufficient conditions to guarantee an uncertainty principle of the Heisenberg-Pauli-Weyl type, in a framework of $L^p(\mathbb{R}^n)$ spaces (with $1 \leq p \leq 2$), for the multidimensional quadratic-phase Fourier transform. In the last section, we will study various structural inequalities related to the multidimensional quadratic-phase Fourier transform, which will culminate in obtaining $L^p(\mathbb{R}^n)$ type Donoho-Stark uncertainty principles (in a first subsection for $p = 2$ and then, in a second subsection, for any integrability exponent p between 1 and 2).

2. The multidimensional quadratic-phase Fourier transform

In this section, we will introduce the multidimensional quadratic-phase Fourier transform and deduce some of its fundamental properties.

As briefly mentioned in the previous section, our main motivation in this work has to do with the introduction of a new integral transform that conveniently generalises several well-known integral transforms. In this sense, our goal was to be able to generalise the Fourier transform, the fractional Fourier transform, the linear canonical transform, the offset linear canonical transform and the quadratic-phase Fourier transform to a multidimensional context, and to make this generalisation as global as possible using as few restrictions as possible. These restrictions are essentially related to the concern that the new transform continues to have good elementary and useful properties so that it has great potential for applicability (particularly in the fields of engineering and applied physics). So, in addition to the purely mathematical aspect of obtaining a new "object" that generalises various other existing mathematical concepts, care was also taken to frame the new definition with elements that would allow us to verify the existence of interesting and crucial properties that would enhance the use of this new mathematical tool in various contexts of applicability.

In particular, let us recall that the well-known linear canonical transform of a given function f is defined by

$$\mathcal{L}_{\{a,b,c,d\}}f(x) = \frac{1}{\sqrt{2\pi ib}} \int_{\mathbb{R}} e^{\frac{i}{2b}(ay^2-2yx+dx^2)} f(y)dy$$

for $b \neq 0$, and by $\sqrt{d}\, e^{\frac{i}{2}cdx^2} f(d\,x)$, if $b = 0$. The four real parameters a, b, c and d are restricted to $ad - bc = 1$ and so only three parameters are free, thus transforming the linear canonical transform into a three-parameter integral transform. Initially, this was proposed independently for reasons deeply associated with the canonical transforms of paraxial optics [5] and quantum mechanics [12]. In fact, as is now well-known, the discovery and development of the theory of linear canonical transforms in the early 1970s was motivated by independent work on two quite different physical models: paraxial optics and nuclear physics. In the first case, the integral kernel of the linear canonical transform was written as a descriptor of the propagation of light in the paraxial regime by Stuart A. Collins Jr. [5] and, in the second case, the linear canonical transform was identified by Marcos Moshinsky and Christiane Quesne [12] as a powerful tool while they were working on certain problems on the alpha clustering and decay of radioactive nuclei.

In addition, there is also a very natural generalisation of the linear canonical transform itself, called the offset linear canonical transform (OLCT) (or "special affine Fourier transform"), which has additional flexibility by additionally presenting a time-shifted and frequency-modulated. Indeed, having in mind a set of six real parameters $a, b, c, d, \tau, \eta \in \mathbb{R}$, such that $ad-bc = 1$, it is usual to denote $A = (a, b, c, d, \tau, \eta)$, and for a function f (e.g. in $L^2(\mathbb{R})$), the OLCT of f is defined by

$$\mathcal{O}_A f(x) = \int_{\mathbb{R}} f(y)\, K_A(y, x)\, dy,$$

with

$$K_A(y, x) = \frac{1}{\sqrt{i2\pi|b|}} e^{i\frac{d\tau^2}{2b}}\, e^{i\left[\frac{a}{2b}y^2+\frac{1}{b}y(\tau-x)-\frac{1}{b}x(d\tau-b\eta)+\frac{d}{2b}x^2\right]},$$

if $b \neq 0$, and by $\sqrt{d}e^{i\frac{cd}{2}(x-\tau)^2+i\eta x} f\left[d(x-\tau)\right]$ if $b = 0$ (i.e., in the case of $b = 0$, the OLCT is simply a chirp multiplication operator). This generalisation has revealed a wide range of important applications, particularly in the area of signal processing and the modelling of optical systems. Naturally, this wide applicability is closely linked to the flexibility of the OLCT and its wide range of generalisations of other integral transforms, such as the Fourier transform and the fractional Fourier transform, the Fresnel transform, the shifted fractional Fourier transform and the linear canonical transform itself.

Moreover, for parameters $a, b, c, d, e \in \mathbb{R}$ (with $b \neq 0$), and the quadratic-phase function

$$Q_{(a,b,c,d,e)}(x, y) := ax^2 + bxy + cy^2 + dx + ey, \tag{2.1}$$

in [2] it was introduced the so-called quadratic-phase Fourier transform $\mathbb{Q}$ given by

$$(\mathbb{Q}f)(x) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(y)\, e^{iQ_{(a,b,c,d,e)}(x,y)}\, dy, \tag{2.2}$$

where $f \in L^1(\mathbb{R})$ or $f \in L^2(\mathbb{R})$. Thus, we may observe that when $a = c = d = e = 0$ and $b = \pm 1$, $\mathbb{Q}$ is simply the Fourier and inverse Fourier integral transforms, respectively. Moreover, when $d = e = 0$, the kernel generated by (2.1) includes the kernel of the linear canonical transform as well as of the one of the fractional Fourier transform (up to the choice of some constant factors that do not change the properties of corresponding integral operators). Given the above definitions, it is also clear that the quadratic-phase Fourier transform encompasses the OLCT as a particular case.

It is in this framework that we propose to introduce a generalisation of the quadratic-phase Fourier transform (2.2) to the n-dimensional setting, thus performing several generations of the aforementioned integral transforms at once. To this end, the central idea of the proposed definition was to consider the most appropriate possible replacement of the real parameters that appear in the quadratic-phase function (cf. (2.1)) of the kernel of the quadratic-phase Fourier transform by matrices (with real entries) and to take sufficient care to ensure that these matrices were arranged appropriately (given the non-commutativity of their multiplication) and that, as a result, fundamental properties of this new integral operator could be demonstrated.

It is therefore in this context and expectation that we propose the following definition of the multidimensional quadratic-phase Fourier transform.

Definition 2.1. *Let A, B, C, D and E be $n \times n$ matrices with B being symmetric and $\det(B) \neq 0$. The multidimensional quadratic-phase Fourier transform (MQFT) of $f \in L^1(\mathbb{R}^n)$ is defined by*

$$[\mathcal{Q}_M(f)](x) := \int_{\mathbb{R}^n} f(y)\, \mathcal{K}_M^{\mathcal{Q}}(x,y)\, dy,$$

where

$$\mathcal{K}_M^{\mathcal{Q}}(x,y) := \Omega(B,n) e^{iQ_{(A-E)}(x,y)}$$

with $\Omega(B,n) := \left(\frac{i}{2\pi}\right)^{n/2} (\det(B))^{1/2}$, $Q_{(A-E)}(x,y) := x^T Ax + x^T By + y^T Cy + \vec{1} Dx + \vec{1} Ey$, *and* $\vec{1} := (1,1,\ldots,1)$, *and where the symbol* ^{T} *is denoting the transpose operator.*

Remark 2.1. *As previously announced this is a generalisation, for the multidimensional case, of several other operators (or integral transforms), as it is the case of the "Quadratic-Phase Fourier Transform" introduced in* [2] *(and also related with the framework of* [3]*).*

Remark 2.2. *The just introduced multidimensional quadratic-phase Fourier transform is also a generalisation of several other multidimensional integral transforms. Namely:*

(i) for $A = C = D = E = 0$ and $B = I$, we recover the multidimensional Fourier transform;

(ii) for $D = E = 0$,

$$A = C = \frac{1}{2}\mathrm{diag}(\cot(\alpha_1), \cot(\alpha_2), \ldots, \cot(\alpha_n))$$

and

$$B = -\mathrm{diag}(\csc(\alpha_1), \csc(\alpha_2), \ldots, \csc(\alpha_n)),$$

with $\alpha_p \neq k\pi$, for all $k \in \mathbb{N}_0$ and $p = 1, \ldots, n$, we obtain the multidimensional fractional Fourier transform;

(iii) considering the multidimensional LCT (MLCT) defined in [4] *and the corresponding matrix*

$$M = \begin{bmatrix} G & H \\ I & J \end{bmatrix},$$

we obtain this transform, through the MQFT, considering $D = E = 0$ and

$$\begin{aligned} A &= \frac{JH^{-1}}{2}, \\ B &= -\, H^{-T}, \\ C &= \frac{H^{-1}G}{2}, \end{aligned}$$

with A, C being symmetric matrices. In this way, the matrix M (that characterises the MLCT), in terms of the matrices that appear in the kernel of the MQFT, is given by

$$M = \begin{bmatrix} -2B^{-T}C & -B^{-T} \\ B - 4AB^{-T}C^T & -2AB^{-T} \end{bmatrix},$$

being this M a symplectic matrix (under the present conditions).

Moreover, note that we can rewrite the MQFT in terms of the Fourier transform $\mathcal{F}$, some variable transformations and also certain chirp functions, in the form

$$[\mathcal{Q}_M(f)](x) = i^{n/2} (\det(B))^{1/2} e^{i(x^T Ax + \vec{1} Dx)} \left[\mathcal{F}(f(y) e^{i(y^T Cy + \vec{1} Ey)})\right](B^T x), \tag{2.3}$$

where

$$(\mathcal{F}f)(x) = \left(\frac{1}{\sqrt{2\pi}}\right)^n \int_{\mathbb{R}^n} f(y) e^{ix^T y}\, dy.$$

In this work, we will often use the usual $L^p(\mathbb{R}^n)$ norms (for $p \in [1, \infty]$) and denote them by $\|\cdot\|_{L^p(\mathbb{R}^n)}$.

Lemma 2.1 (Riemann-Lebesgue Lemma for the MQFT). *$\mathcal{Q}_M$ is a bounded linear operator from $L^1(\mathbb{R}^n)$ into $C_0(\mathbb{R}^n)$. Namely, if $f \in L^1(\mathbb{R}^n)$, then $\mathcal{Q}_M(f) \in C_0(\mathbb{R}^n)$ and*

$$\|\mathcal{Q}_M(f)\|_{L^\infty(\mathbb{R}^n)} \leq \frac{|\det(B)|^{1/2}}{(2\pi)^{n/2}} \|f\|_{L^1(\mathbb{R}^n)}.$$

Proof. Using the identity (2.3), and the Riemann-Lebesgue Lemma for the Fourier transform, we see that $\mathcal{Q}_M(f) \in C_0(\mathbb{R}^n)$, provided $f \in L^1(\mathbb{R}^n)$. Moreover, from the definition of $\mathcal{Q}_M$, we have

$$\begin{aligned}
\|\mathcal{Q}_M(f)\|_{L^\infty(\mathbb{R}^n)} &= \sup_{x\in\mathbb{R}^n} \left| \frac{i^{n/2}(\det(B))^{1/2}}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{iQ_{(A-E)}(x,y)} f(y)\, dy \right| \\
&\leq \sup_{x\in\mathbb{R}^n} \frac{|\det(B)|^{1/2}}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \left| e^{iQ_{(A-E)}(x,y)} \right| |f(y)|\, dy \\
&= \frac{|\det(B)|^{1/2}}{(2\pi)^{n/2}} \|f\|_{L^1(\mathbb{R}^n)}.
\end{aligned}$$

□

We will continue with a result that shows the invertibility of the MQFT and presents a formula for its inverse.

Theorem 2.1. *If $f \in L^1(\mathbb{R}^n)$ and $\mathcal{Q}_M(f) \in L^1(\mathbb{R}^n) \cap C_0(\mathbb{R}^n)$, then*

$$f(x) = \int_{\mathbb{R}^n} \overline{\mathcal{K}_M^{\mathcal{Q}}(y,x)} [\mathcal{Q}_M(f)](y)\, dy \tag{2.4}$$

for almost every $x \in \mathbb{R}^n$, where

$$\overline{\mathcal{K}_M^{\mathcal{Q}}(y,x)} := \overline{\Omega(B,n)} e^{-iQ_{(A-E)}(y,x)}. \tag{2.5}$$

Proof. Using a substitution of variable in (2.3) allows us to rewrite the $\mathcal{Q}_M$ in the form

(2.6)

$$[\mathcal{Q}_M(f)](x) = i^{n/2}(\det(B))^{-1/2} e^{i(x^T Ax + \vec{1}Dx)} \left[\mathcal{F}(f(B^{-1}y) e^{i((B^{-1}y)^T C(B^{-1}y) + \vec{1}E(B^{-1}y))}) \right](x).$$

We shall make use of the operators τ_B and M_g, given by

$$(\tau_B f)(x) := f(Bx)$$

and

$$(M_g f)(x) := g(x) f(x)$$

for the matrix B (and its inverse), and any function g, respectively.

So, from (2.6), we can write

$$[\mathcal{Q}_M(f)](x) = [M_{ce^{w_1}}\, \mathcal{F}\, \tau_{B^{-1}}\, M_{e^{w_2}}(f)](x), \tag{2.7}$$

with

$$\begin{aligned}
c :=& i^{n/2}(\det(B))^{-1/2}; \\
w_1(x) :=& i(x^T Ax + \vec{1}Dx); \\
w_2(x) :=& i(x^T Cx + \vec{1}Ex).
\end{aligned}$$

It is clear that all the operators used in the right-hand side of (2.7) are invertible in the present framework, and therefore, from (2.7), we have

$$\left[\mathcal{Q}_M^{-1}(f)\right](x) = \left[M_{e^{-w_2}}\, \tau_B\, \mathcal{F}^{-1}\, M_{c^{-1}e^{-w_1}}(f)\right](x),$$

and so (2.4) is obtained. □

Lemma 2.2 (Plancherel type Lemma for the MQFT). *If $f \in L^2(\mathbb{R}^n)$, then*

$$\|\mathcal{Q}_M(f)\|_{L^2(\mathbb{R}^n)} = \frac{1}{(2\pi)^{n/2}}\|f\|_{L^2(\mathbb{R}^n)}. \tag{2.8}$$

Proof. Using (2.3) and having in mind the Plancherel theorem for the Fourier transform, we have

$$\begin{aligned}
\|\mathcal{Q}_M(f)\|_{L^2(\mathbb{R}^n)} =& \left\| i^{n/2}(\det(B))^{1/2} e^{i(x^TAx+\vec{1}Dx)} \left[\mathcal{F}(f(y)e^{i(y^TCy+\vec{1}Ey)})\right](B^Tx)\right\|_{L^2(\mathbb{R}^n)}\\
=& |\det(B)|^{1/2}|\det(B)|^{-1/2}\frac{1}{(2\pi)^{n/2}}\|f\|_{L^2(\mathbb{R}^n)}\\
=& \frac{1}{(2\pi)^{n/2}}\|f\|_{L^2(\mathbb{R}^n)}.
\end{aligned}$$

□

Remark 2.3. It is clear from the identity (2.8) that, although the MQFT defined here is not unitary (in $L^2(\mathbb{R}^n)$), a small modification of the definition, taking into account a different constant, can compensate for the constant now obtained in the identity (2.8), transforming it into the constant one. From this perspective, it is easy to redefine the MQFT (using a different constant) to make it a unitary operator.

We recall that for $1 < p < 2$, we have

$$L^p(\mathbb{R}^n) \subset L^1(\mathbb{R}^n) + L^2(\mathbb{R}^n) = \{f_1 + f_2 : f_1 \in L^1(\mathbb{R}^n), f_2 \in L^2(\mathbb{R}^n)\}.$$

Thus, a possible way to interpret the definition of $\mathcal{Q}_M$ in $L^p(\mathbb{R}^n)$, for $1 < p < 2$, is to consider $f \in L^p(\mathbb{R}^n)$ such that $f = f_1 + f_2$, with $f_1 \in L^1(\mathbb{R}^n)$, $f_2 \in L^2(\mathbb{R}^n)$, and then read off the MQFT of f in the form $\mathcal{Q}_M(f) = \mathcal{Q}_M(f_1) + \mathcal{Q}_M(f_2)$.

For the reader's benefit, let us now briefly recall the statement of Riesz-Thorin Interpolation Theorem that we will use in the next proof.

Theorem 2.2 (Riesz-Thorin Interpolation Theorem; cf., e.g., [8]). *Let (X, μ) and (Y, ν) be measure spaces and $1 \le p_0, p_1, q_0, q_1 \le \infty$ (and the measure ν on Y is also required to be semifinite when $q_0 = q_1 = \infty$).*

If $T : (L^{p_0}(X, \mu) + L^{p_1}(X, \mu)) \to (L^{q_0}(Y, \nu) + L^{q_1}(Y, \nu))$ is a linear operator such that

$$\|Tf\|_{L^{q_0}(Y,\nu)} \le M_0\|f\|_{L^{p_0}(X,\mu)}, \quad \|Tg\|_{L^{q_1}(Y,\nu)} \le M_1\|g\|_{L^{p_1}(X,\mu)}$$

for all $f \in L^{p_0}(X, \mu)$ and $g \in L^{p_1}(X, \mu)$, and we consider the interpolated exponents

$$\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad \frac{1}{q_\theta} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$$

for some $\theta \in [0, 1]$, then $T : L^{p_\theta}(X, \mu) \to L^{q_\theta}(Y, \nu)$ is bounded and

$$\|Tg\|_{L^{q_\theta}(Y,\nu)} \le M_0^{1-\theta} M_1^{\theta}\|g\|_{L^{p_\theta}(X,\mu)}$$

for all $f \in L^{p_\theta}(X, \mu)$.

Theorem 2.3 (Hausdorff-Young Inequality for $\mathcal{Q}_M$). *Let $1 \leq p \leq 2$ and take p' as the conjugate exponent of p (meaning that $p' \geq 2$ and $\frac{1}{p} + \frac{1}{p'} = 1$). If $f \in L^p(\mathbb{R}^n)$ then $\mathcal{Q}_M(f) \in L^{p'}(\mathbb{R}^n)$ and*

$$\|\mathcal{Q}_M(f)\|_{L^{p'}(\mathbb{R}^n)} \leq \frac{|\det(B)|^{1/p-1/2}}{(2\pi)^{n/2}} \|f\|_{L^p(\mathbb{R}^n)}.$$

Proof. We recall that from Lemma 2.2 we already know that for $p = 2$ it holds

$$\|\mathcal{Q}_M(f)\|_{L^2(\mathbb{R}^n)} = \frac{1}{(2\pi)^{n/2}} \|f\|_{L^2(\mathbb{R}^n)}, \quad f \in L^2(\mathbb{R}^n), \tag{2.9}$$

and from Lemma 2.1, for $p = 1$, we have

$$\|\mathcal{Q}_M(f)\|_{L^\infty(\mathbb{R}^n)} \leq \frac{|\det(B)|^{1/2}}{(2\pi)^{n/2}} \|f\|_{L^1(\mathbb{R}^n)}, \quad f \in L^1(\mathbb{R}^n). \tag{2.10}$$

Thus, using the Riesz-Thorin Interpolation Theorem, we obtain that $\mathcal{Q}_M(f) : L^p(\mathbb{R}^n) \to L^{p'}(\mathbb{R}^n)$ is a bounded operator for $p \in [1, 2]$ (with p' being the conjugate exponent of p). In addition, the interpolation exponent θ must satisfy

$$\frac{\theta}{1} + \frac{1-\theta}{2} = \frac{1}{p}.$$

Thus, $\theta = \frac{2}{p} - 1$ and so, again from (2.9) and (2.10), it follows

$$\|\mathcal{Q}_M(f)\|_{L^{p'}(\mathbb{R}^n)} \leq \frac{|\det(B)|^{\theta/2}}{(2\pi)^{n/2}} \|f\|_{L^p(\mathbb{R}^n)}, \quad f \in L^p(\mathbb{R}^n).$$

□

3. Heisenberg-Pauli-Weyl uncertainty principle

In this section, we present a L^p-type Heisenberg-Pauli-Weyl uncertainty principle associated with the MQFT.

Theorem 3.4. *If $1 \leq p \leq 2$, $f \in L^2(\mathbb{R}^n)$, $yf \in L^p(\mathbb{R}^n)$, $x\mathcal{Q}_M(f) \in L^p(\mathbb{R}^n)$, then*

$$\|yf\|_{L^p(\mathbb{R}^n)} \|x\mathcal{Q}_M(f)\|_{L^p(\mathbb{R}^n)} \geq \frac{|\det(B)|^{1/2-1/p}}{\sigma_{max}(B)} \frac{n\|f\|_{L^2(\mathbb{R}^n)}}{2}, \tag{3.11}$$

where $\sigma_{max}(B)$ is the maximum singular value of the matrix B. Moreover, the equality holds if and only if $p = 2$, $\lambda_{max}(BB^T) = \lambda_{min}(BB^T)$ (where $\lambda(BB^T)$ represents an eigenvalue of the matrix BB^T) and $f(y)e^{i(y^TCy+\vec{1}Ey)}$ is a Gaussian function.

Proof. From (2.3), we know that

$$[\mathcal{Q}_M(f)](x) = i^{n/2}(\det(B))^{1/2} e^{i(x^TAx+\vec{1}Dx)} \left[\mathcal{F}(f(y)e^{i(y^TCy+\vec{1}Ey)})\right](B^Tx).$$

Moreover, $\|yf\|_{L^p(\mathbb{R}^n)} = \|yf(y)e^{i(y^TCy+\vec{1}Ey)}\|_{L^p(\mathbb{R}^n)}$ and

$$\begin{aligned}
&\|(B^Tx)\,[\mathcal{Q}_M(f)]\,(x)\|_{L^p(\mathbb{R}^n)} \\
&= \left\|(B^Tx) i^{n/2}(\det(B))^{1/2} e^{i(x^TAx+\vec{1}Dx)} \left[\mathcal{F}(f(y)e^{i(y^TCy+\vec{1}Ey)})\right](B^Tx)\right\|_{L^p(\mathbb{R}^n)} \\
&= |\det(B)|^{1/2} \left\|(B^Tx) \left[\mathcal{F}(f(y)e^{i(y^TCy+\vec{1}Ey)})\right](B^Tx)\right\|_{L^p(\mathbb{R}^n)} \\
&= |\det(B)|^{1/2-1/p} \left\|x \left[\mathcal{F}(f(y)e^{i(y^TCy+\vec{1}Ey)})\right](x)\right\|_{L^p(\mathbb{R}^n)}.
\end{aligned}$$

If $f \in L^2(\mathbb{R}^n)$, then $f(y)e^{i(y^T Cy+\vec{1}Ey)} \in L^2(\mathbb{R}^n)$. Using the Heisenberg-Pauli-Weyl uncertainty principle for the multidimensional Fourier transform (cf. Lemma 5 of [4]), we have

$$\begin{aligned}
&\|yf(y)\|_{L^p(\mathbb{R}^n)}\|(B^Tx)\left[\mathcal{Q}_M(f)\right](x)\|_{L^p(\mathbb{R}^n)} \\
&\qquad = |\det(B)|^{1/2-1/p}\|yf(y)e^{i(y^TCy+\vec{1}Ey)}\|_{L^p(\mathbb{R}^n)}\left\|x\left[\mathcal{F}(f(y)e^{i(y^TCy+\vec{1}Ey)})\right](x)\right\|_{L^p(\mathbb{R}^n)} \\
&\qquad \geq |\det(B)|^{1/2-1/p}\frac{n\|f\|_{L^2(\mathbb{R}^n)}}{2}. \qquad (3.12)
\end{aligned}$$

Additionally, $|B^Tx|^2 = x^TBB^Tx$. We note that the matrix BB^T is a real and symmetric matrix, so there exists an orthogonal matrix U such that

$$U^T(BB^T)U = \mathrm{diag}[\lambda_1, \lambda_2, \ldots, \lambda_n],$$

where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are eigenvalues of BB^T.

We also have that

$$|B^Tx|^2 = x^TBB^Tx \leq \lambda_{max}(BB^T)x^TUIU^Tx = \lambda_{max}(BB^T)|x|^2. \qquad (3.13)$$

Therefore,

$$|B^Tx|^p \leq \left[\lambda_{max}(BB^T)\right]^{p/2}|x|^p.$$

So, considering also now (3.12), it follows

$$\begin{aligned}
&\left[\lambda_{\max}(BB^T)\right]^{1/2}\|yf(y)\|_{L^p(\mathbb{R}^n)}\|x\left[\mathcal{Q}_M(f)\right](x)\|_{L^p(\mathbb{R}^n)} \\
&\qquad \geq \|yf(y)\|_{L^p(\mathbb{R}^n)}\|(B^Tx)\left[\mathcal{Q}_M(f)\right](x)\|_{L^p(\mathbb{R}^n)} \\
&\qquad \geq |\det(B)|^{1/2-1/p}\frac{n\|f\|_{L^2(\mathbb{R}^n)}}{2}. \qquad (3.14)
\end{aligned}$$

As $\lambda_{max}(BB^T) = \sigma^2_{max}(B)$, then the inequality can be rewritten as

$$\|yf(y)\|_{L^p(\mathbb{R}^n)}\|x\left[\mathcal{Q}_M(f)\right](x)\|_{L^p(\mathbb{R}^n)} \geq \frac{|\det(B)|^{1/2-1/p}}{\sigma_{max}(B)}\frac{n\|f\|_{L^2(\mathbb{R}^n)}}{2}. \qquad (3.15)$$

From (3.13), we have that $|B^Tx|^2 = \lambda_{max}(BB^T)|x|^2$ if and only if

$$\lambda_{max}(BB^T) = \lambda_{min}(BB^T) = \sigma^2_{max}(B) = \sigma^2_{min}(B) = \sigma^2(B). \qquad (3.16)$$

According to Lemma 5 of [4] (and also [6], for the unidimensional case), the equality in (3.14) is attained if and only if $p = 2$ and $f(y)e^{i(y^TCy+\vec{1}Ey)}$ is a Gaussian function, that is,

$$f(y)e^{i(y^TCy+\vec{1}Ey)} = ce^{k|y|^2},$$

where c is a constant and $k < 0$. So, we have

$$\|yf\|_{L^p(\mathbb{R}^n)}\|x\mathcal{Q}_M(f)\|_{L^p(\mathbb{R}^n)} = \frac{1}{\sigma_{max}(B)}\frac{n\|f\|_{L^2(\mathbb{R}^n)}}{2}$$

if and only if B satisfies (3.16) and $f(y)e^{i(y^TCy+\vec{1}Ey)} = ce^{k|y|^2}$. □

4. Donoho-Stark uncertainty principles

In this section, we study the Donoho-Stark uncertainty principles of type L^p. In a first subsection, we will do so in the most standard framework of $p = 2$, and then, in a second subsection, we will consider the case of p between 1 and 2.

4.1. **L^2-type Donoho-Stark uncertainty principles.** We start by defining two operators on $L^2(\mathbb{R}^n)$:

$$P_\Lambda f = \chi_\Lambda f$$

and

$$Q_\Gamma f = \mathcal{Q}_M^{-1}\left[\chi_\Gamma \mathcal{Q}_M(f)\right],$$

where Λ and Γ are measurable sets on $\mathbb{R}^n$, and χ_Γ denotes the characteristic function on Γ.

Definition 4.2. *(i) Let Λ be a measurable set on $\mathbb{R}^n$, $0 < \varepsilon_\Lambda < 1$ and $f \in L^2(\mathbb{R}^n)$. f is called ε_Λ-concentrated on Λ if*

$$\|P_{\Lambda^c} f\|_{L^2(\mathbb{R}^n)} \leq \varepsilon_\Lambda \|f\|_{L^2(\mathbb{R}^n)}.$$

(ii) Let Γ be a measurable set on $\mathbb{R}^n$, $0 < \varepsilon_\Gamma < 1$ and $f \in L^2(\mathbb{R}^n)$. $\mathcal{Q}_M(f)$ is said to be ε_Γ-concentrated on Γ if

$$\|Q_{\Gamma^c} f\|_{L^2(\mathbb{R}^n)} \leq \varepsilon_\Gamma \|f\|_{L^2(\mathbb{R}^n)}.$$

We will make use of the usual operator norms of $P_\Lambda, Q_\Gamma : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ defined by

$$\|P_\Lambda\| := \sup_{f \in L^2(\mathbb{R}^n)} \frac{\|P_\Lambda f\|_{L^2(\mathbb{R}^n)}}{\|f\|_{L^2(\mathbb{R}^n)}}$$

and

$$\|Q_\Gamma\| := \sup_{f \in L^2(\mathbb{R}^n)} \frac{\|Q_\Gamma f\|_{L^2(\mathbb{R}^n)}}{\|f\|_{L^2(\mathbb{R}^n)}},$$

respectively.

In addition, we will also use the Hilbert-Schmidt norm of operators $\mathcal{L} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ of the form $(\mathcal{L}f)(x) = \int_{\mathbb{R}^n} f(y)K(x,y)\,dy$, where $f \in L^2(\mathbb{R}^n)$ and $K(x,y) \in L^2(\mathbb{R}^n \times \mathbb{R}^n)$. We recall that the Hilbert-Schmidt norm of $\mathcal{L}$ is given by

$$\|\mathcal{L}\|_{HS} := \left(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |K(x,y)|^2 dy dx\right)^{1/2}.$$

Lemma 4.3. *Let Λ and Γ be two measurable sets of $\mathbb{R}^n$ such that $0 < |\Lambda|, |\Gamma| < \infty$. Then,*

$$\|Q_\Gamma P_\Lambda\|_{HS} = (2\pi)^{n/2} |\Omega(B,n)| |\Lambda|^{1/2} |\Gamma|^{1/2}.$$

Proof. From the definitions of P_Λ and Q_Γ, we have

$$\begin{aligned}
[Q_\Gamma P_\Lambda f](t) =& \mathcal{Q}_M^{-1}[\chi_\Gamma \mathcal{Q}_M(\chi_\Lambda f)](t) \\
=& \int_\Gamma \int_{\mathbb{R}^n} (\chi_\Lambda f)(y) \mathcal{K}_M^{\mathcal{Q}}(x,y) \overline{\mathcal{K}_M^{\mathcal{Q}}(x,t)} dy dx \\
=& \int_{\mathbb{R}^n} (\chi_\Lambda f)(y) \int_\Gamma \mathcal{K}_M^{\mathcal{Q}}(x,y) \overline{\mathcal{K}_M^{\mathcal{Q}}(x,t)} dy dx \\
=& \int_{\mathbb{R}^n} f(y) \chi_\Lambda(y) K(t,y) dy,
\end{aligned}$$

with $h_t(x) := K(t,y) = \int_{\mathbb{R}^n} \chi_\Gamma(t)\mathcal{K}_M^{\mathcal{Q}}(x,y)\overline{\mathcal{K}_M^{\mathcal{Q}}(x,t)}dx$. Let us now compute

$$\begin{aligned}
[\mathcal{Q}_M(\chi_\Lambda h_t)](x_1) =& \int_{\mathbb{R}^n} \chi_\Lambda(y)\mathcal{K}_M^{\mathcal{Q}}(x_1,y)\left(\int_{\mathbb{R}^n} \chi_\Gamma(t)\mathcal{K}_M^{\mathcal{Q}}(x,y)\overline{\mathcal{K}_M^{\mathcal{Q}}(x,t)}dx\right)dy \\
=& \int_{\mathbb{R}^n} \chi_\Lambda(y)\left(\int_{\mathbb{R}^n} \chi_\Gamma(t)\mathcal{K}_M^{\mathcal{Q}}(x,y)\overline{\mathcal{K}_M^{\mathcal{Q}}(x,t)}dx\right)\mathcal{K}_M^{\mathcal{Q}}(x_1,y)\,dy \\
=&\mathcal{Q}_M[\chi_\Lambda(\mathcal{Q}_M^{-1}(\chi_\Gamma\mathcal{K}_M^{\mathcal{Q}}))(t)](x_1) \\
=&\chi_\Lambda(t)\chi_\Gamma(x_1)\mathcal{K}_M^{\mathcal{Q}}(x_1,t).
\end{aligned}$$

Note that $\chi_\Lambda(\lambda)h_t(\lambda) \in L^2(\mathbb{R}^n)$. Using the last identity and the Plancherel Theorem, we have

$$\begin{aligned}
\|Q_\Gamma P_\Lambda\|_{HS}^2 =& \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |\chi_\Lambda(y)K(t,y)|^2\,dydt \\
=& \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |\chi_\Lambda(y)h_t(y)|^2\,dydt \\
=&(2\pi)^n\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |[\mathcal{Q}_M(\chi_\Lambda h_t)](x)|^2\,dxdt \\
=&(2\pi)^n\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |\chi_\Lambda(t)\chi_\Gamma(x)\mathcal{K}_M^{\mathcal{Q}}(x,t)|^2\,dxdt \\
=&(2\pi)^n|\Omega(B,n)|^2|\Lambda||\Gamma|.
\end{aligned}$$

So, $\|Q_\Gamma P_\Lambda\|_{HS} = (2\pi)^{n/2}|\Omega(B,n)||\Lambda|^{1/2}|\Gamma|^{1/2}$. □

The next Lemma gives a relation between $\|P_\Lambda Q_\Gamma\|_{HS}$ and $\|Q_\Gamma P_\Lambda\|_{HS}$.

Lemma 4.4. *Let Λ and Γ be subsets of $\mathbb{R}^n$ with finite (nonzero) measure. Then,*

$$\|P_\Lambda Q_\Gamma\|_{HS} = \|Q_\Gamma P_\Lambda\|_{HS}.$$

Proof. Let $K(t,y) = \int_\Gamma \mathcal{K}_M^{\mathcal{Q}}(x,y)\overline{\mathcal{K}_M^{\mathcal{Q}}(x,t)}dx$. We have that $K(t,y) = \overline{K(y,t)} \in L^2(\mathbb{R}^n)$ with respect to y. Let $f \in L^2(\mathbb{R}^n)$ and $g \in C_c^\infty(\mathbb{R}^n)$. Then, we have

$$\begin{aligned}
&\left|\int_\Gamma [\mathcal{Q}_M(f)](x)\overline{\mathcal{K}_M^{\mathcal{Q}}(x,t)}dx - \int_{\mathbb{R}^n} f(y)K(t,y)\,dy\right| \\
=&\left|\int_\Gamma [\mathcal{Q}_M(f-g)](x)\overline{\mathcal{K}_M^{\mathcal{Q}}(x,t)}dx + \int_\Gamma [\mathcal{Q}_M(g)](x)\overline{\mathcal{K}_M^{\mathcal{Q}}(x,t)}dx - \int_{\mathbb{R}^n} f(y)K(t,y)\,dy\right| \\
\leq&\left|\int_{\mathbb{R}^n} [\mathcal{Q}_M(f-g)](x)\chi_\Gamma(x)\overline{\mathcal{K}_M^{\mathcal{Q}}(x,t)}dx\right| + \left|\int_\Gamma [\mathcal{Q}_M(g)](x)\overline{\mathcal{K}_M^{\mathcal{Q}}(x,t)}dx - \int_{\mathbb{R}^n} f(y)K(t,y)\,dy\right| \\
=&\left|\int_{\mathbb{R}^n} [\mathcal{Q}_M(f-g)](x)\chi_\Gamma(x)\overline{\mathcal{K}_M^{\mathcal{Q}}(x,t)}dx\right| + \left|\int_\Gamma g(y)K(t,y)dy - \int_{\mathbb{R}^n} f(y)K(t,y)\,dy\right| \\
\leq&|\Gamma|^{1/2}|\Omega(B,n)|\|\mathcal{Q}_M(f-g)\|_{L^2(\mathbb{R}^n)} + \left|\int_\Gamma (f-g)(y)K(t,y)dy\right| \\
\leq&\frac{1}{(2\pi)^{n/2}}|\Gamma|^{1/2}|\Omega(B,n)|\|f-g\|_{L^2(\mathbb{R}^n)} + \|f-g\|_{L^2(\mathbb{R}^n)}\|K(t,y)\|_{L^2(\mathbb{R}^n)} \\
<&c\varepsilon
\end{aligned}$$

for a constant c and an arbitrarily small positive ε. So, this allows us to conclude that

$$\begin{aligned}[P_\Lambda Q_\Gamma f](t) =&\chi_\Lambda \mathcal{Q}_M^{-1}[\chi_\Gamma \mathcal{Q}_M(f)](t)\\ =&\chi_\Lambda(t)\int_\Gamma [\mathcal{Q}_M(f)](x)\overline{\mathcal{K}_M^{\mathcal{Q}}(x,t)}dx\\ =&\chi_\Lambda(t)\int_{\mathbb{R}^n} f(y)K(t,y)\,dy.\end{aligned}$$

Now, the last information together with the Plancherel Theorem give us

$$\begin{aligned}\|P_\Lambda Q_\Gamma\|_{HS}^2 =& \int_{\mathbb{R}^n}\chi_\Lambda(t)\int_{\mathbb{R}^n}|K(t,y)|^2\,dydt\\ =&\int_{\mathbb{R}^n}\chi_\Lambda(t)\int_{\mathbb{R}^n}|h_t(y)|^2\,dydt\\ =&(2\pi)^n\int_{\mathbb{R}^n}\chi_\Lambda(t)\int_{\mathbb{R}^n}|\mathcal{Q}_M(h_t)(x)|^2\,dxdt\\ =&(2\pi)^n\int_{\mathbb{R}^n}\chi_\Lambda(t)\int_{\mathbb{R}^n}|\chi_\Gamma(x)\mathcal{K}_{\mathcal{Q}}(x,t)|^2\,dxdt\\ =&(2\pi)^n|\Omega(B,n)|^2|\Lambda||\Gamma|.\end{aligned}$$

Therefore, $\|P_\Lambda Q_\Gamma\|_{HS} = (2\pi)^{n/2}|\Omega(B,n)||\Lambda|^{1/2}|\Gamma|^{1/2}$. □

Corollary 4.1. *Suppose that f, Λ and Γ satisfy the conditions of Lemmas 4.3 and 4.4. Then,*

(i) $\|Q_\Gamma P_\Lambda\| \le \|Q_\Gamma P_\Lambda\|_{HS} = (2\pi)^{n/2}|\Omega(B,n)||\Lambda|^{1/2}|\Gamma|^{1/2}$,
(ii) $\|P_\Lambda Q_\Gamma\| \le \|P_\Lambda Q_\Gamma\|_{HS} = (2\pi)^{n/2}|\Omega(B,n)||\Lambda|^{1/2}|\Gamma|^{1/2}$.

This corollary follows directly from the definitions of $\|\cdot\|$ and $\|\cdot\|_{HS}$ and Lemmas 4.3 and 4.4.

Theorem 4.5. *Let Λ and Γ be two measurable sets of $\mathbb{R}^n$ such that $0 < |\Lambda|, |\Gamma| < \infty$, $f \in L^2(\mathbb{R}^n)$ and $\varepsilon_1 + \varepsilon_2 < 1$. If f is ε_Λ-concentrated on Λ and $\mathcal{Q}_M(f)$ is ε_Γ-concentrated on Γ, then*

$$|\Lambda||\Gamma| \ge \frac{1}{(2\pi)^n}\left(\frac{1-\varepsilon_\Lambda-\varepsilon_\Gamma}{|\Omega(B,n)|}\right)^2. \tag{4.17}$$

Proof. By Lemma 2.2, we have that $\|Q_\Gamma f\|_{L^2(\mathbb{R}^n)} \le (2\pi)^{n/2}\|\mathcal{Q}_M(f)\|_{L^2(\mathbb{R}^n)} = \|f\|_{L^2(\mathbb{R}^n)}$ and so

$$\|Q_\Gamma\| = \sup_{f\in L^2(\mathbb{R}^n)}\frac{\|Q_\Gamma f\|_{L^2(\mathbb{R}^n)}}{\|f\|_{L^2(\mathbb{R}^n)}} \le 1. \tag{4.18}$$

Now, we consider

$$\begin{aligned}\|f-Q_\Gamma P_\Lambda f\|_{L^2(\mathbb{R}^n)} =&\|f-Q_\Gamma f+Q_\Gamma f-Q_\Gamma P_\Lambda f\|_{L^2(\mathbb{R}^n)}\\ \le&\|f-Q_\Gamma f\|_{L^2(\mathbb{R}^n)}+\|Q_\Gamma f-Q_\Gamma P_\Lambda f\|_{L^2(\mathbb{R}^n)}.\end{aligned}$$

Since $\mathcal{Q}_M(f)$ is ε_Γ-concentrated on Γ, we have that $\|f - Q_\Gamma f\|_{L^2(\mathbb{R}^n)} \le \varepsilon_\Gamma\|f\|_{L^2(\mathbb{R}^n)}$. On the other hand, using (4.18), we have

$$\|Q_\Gamma f - Q_\Gamma P_\Lambda f\|_{L^2(\mathbb{R}^n)} \le \|Q_\Gamma\|\|f-P_\Lambda f\|_{L^2(\mathbb{R}^n)} \le \|f-P_\Lambda f\|_{L^2(\mathbb{R}^n)} \le \varepsilon_\Lambda\|f\|_{L^2(\mathbb{R}^n)},$$

since f is ε_Λ-concentrated on Λ.

In this way, we have $\|f - Q_\Gamma P_\Lambda f\|_{L^2(\mathbb{R}^n)} \leq (\varepsilon_\Gamma + \varepsilon_\Lambda)\|f\|_{L^2(\mathbb{R}^n)}$, which gives that

$$\begin{aligned}\|Q_\Gamma P_\Lambda\| \geq & \frac{\|Q_\Gamma P_\Lambda f\|_{L^2(\mathbb{R}^n)}}{\|f\|_{L^2(\mathbb{R}^n)}} \\ \geq & \frac{\|f\|_{L^2(\mathbb{R}^n)} - \|f - Q_\Gamma P_\Lambda f\|_{L^2(\mathbb{R}^n)}}{\|f\|_{L^2(\mathbb{R}^n)}} \\ \geq & 1 - \varepsilon_\Gamma - \varepsilon_\Lambda\end{aligned}$$

(where we have used the inequality

$$\|f\|_{L^2(\mathbb{R}^n)} = \|f - Q_\Gamma P_\Lambda f + Q_\Gamma P_\Lambda f\|_{L^2(\mathbb{R}^n)} \leq \|f - Q_\Gamma P_\Lambda f\|_{L^2(\mathbb{R}^n)} + \|Q_\Gamma P_\Lambda f\|_{L^2(\mathbb{R}^n)}).$$

By Corollary 4.1, we obtain $(2\pi)^{n/2}|\Omega(B,n)||\Gamma|^{1/2}|\Lambda|^{1/2} \geq 1 - \varepsilon_\Gamma - \varepsilon_\Lambda$, that is equivalent to (4.17). □

Theorem 4.6. *Let $\Lambda, \Gamma \subseteq \mathbb{R}^n$ be two measurable sets such that $0 < |\Lambda|, |\Gamma| < \infty$, and $f \in L^2(\mathbb{R}^n)$. Let $\varepsilon_\Lambda, \varepsilon_\Gamma > 0$ be such that $\varepsilon_\Lambda^2 + \varepsilon_\Gamma^2 < 1$. If f is ε_Λ-concentrated on Λ and $\mathcal{Q}_M(f)$ is ε_Γ-concentrated on Γ, then*

$$|\Lambda||\Gamma| \geq \frac{1}{(2\pi)^n}\left(\frac{1 - \sqrt{\varepsilon_\Lambda^2 + \varepsilon_\Gamma^2}}{|\Omega(B,n)|}\right)^2.$$

Proof. We have

$$I = P_\Lambda + P_{\Lambda^c} = P_\Lambda Q_\Gamma + P_\Lambda Q_{\Gamma^c} + P_{\Lambda^c},$$

where I is the identity operator. From this identity, we obtain

$$\|f - P_\Lambda Q_\Gamma f\|^2_{L^2(\mathbb{R}^n)} = \|P_\Lambda Q_{\Gamma^c} f + P_{\Lambda^c} f\|^2_{L^2(\mathbb{R}^n)}.$$

From the orthogonality between P_Λ and P_{Λ^c}, we have

$$\|f - P_\Lambda Q_\Gamma f\|^2_{L^2(\mathbb{R}^n)} = \|P_\Lambda Q_{\Gamma^c} f + P_{\Lambda^c} f\|^2_{L^2(\mathbb{R}^n)} \leq \|Q_{\Gamma^c} f\|^2_{L^2(\mathbb{R}^n)} + \|P_{\Lambda^c} f\|^2_{L^2(\mathbb{R}^n)}.$$

This implies that

$$\begin{aligned}\|f - P_\Lambda Q_\Gamma f\|_{L^2(\mathbb{R}^n)} &\leq \left(\|P_{\Lambda^c} f\|^2_{L^2(\mathbb{R}^n)} + \|Q_{\Gamma^c} f\|^2_{L^2(\mathbb{R}^n)}\right)^{1/2} \\ &\leq \left(\varepsilon_\Lambda^2\|f\|^2_{L^2(\mathbb{R}^n)} + \varepsilon_\Gamma^2\|f\|^2_{L^2(\mathbb{R}^n)}\right)^{1/2} \\ &\leq \left(\varepsilon_\Lambda^2 + \varepsilon_\Gamma^2\right)^{1/2}\|f\|_{L^2(\mathbb{R}^n)}.\end{aligned}$$

On the other hand, we have

$$\begin{aligned}\|f - P_\Lambda Q_\Gamma f\|_{L^2(\mathbb{R}^n)} \geq & \|f\|_{L^2(\mathbb{R}^n)} - \|P_\Lambda Q_\Gamma f\|_{L^2(\mathbb{R}^n)} \\ \geq & \|f\|_{L^2(\mathbb{R}^n)} - \|P_\Lambda Q_\Gamma\|\|f\|_{L^2(\mathbb{R}^n)} \\ = & \left(1 - \|P_\Lambda Q_\Gamma\|\right)\|f\|_{L^2(\mathbb{R}^n)}.\end{aligned}$$

Consequently, we have

$$\left(1 - \|P_\Lambda Q_\Gamma\|\right)\|f\|_{L^2(\mathbb{R}^n)} \leq \|f - P_\Lambda Q_\Gamma f\|_{L^2(\mathbb{R}^n)} \leq \left(\varepsilon_\Lambda^2 + \varepsilon_\Gamma^2\right)^{1/2}\|f\|_{L^2(\mathbb{R}^n)}.$$

Corollary 4.1 gives us that $\|P_\Lambda Q_\Gamma\| \leq (2\pi)^{n/2}|\Omega(B,n)||\Lambda|^{1/2}|\Gamma|^{1/2}$. Hence,

$$\begin{aligned}\left(1 - (2\pi)^{n/2}|\Omega(B,n)||\Lambda|^{1/2}|\Gamma|^{1/2}\right)\|f\|_{L^2(\mathbb{R}^n)} &\leq \left(1 - \|P_\Lambda Q_\Gamma\|\right)\|f\|_{L^2(\mathbb{R}^n)} \\ &\leq \left(\varepsilon_\Lambda^2 + \varepsilon_\Gamma^2\right)^{1/2}\|f\|_{L^2(\mathbb{R}^n)},\end{aligned}$$

i.e.,

$$(2\pi)^{n/2}|\Omega(B,n)||\Lambda|^{1/2}|\Gamma|^{1/2} \geq 1 - \sqrt{\varepsilon_\Lambda^2 + \varepsilon_\Gamma^2},$$

and so,

$$|\Lambda||\Gamma| \geq \left(\frac{1 - \sqrt{\varepsilon_\Lambda^2 + \varepsilon_\Gamma^2}}{(2\pi)^{n/2}|\Omega(B,n)|} \right)^2 .$$

□

4.2. **L^p-type Donoho-Stark uncertainty principles, with $1 \leq p \leq 2$.** In this subsection we will study certain Donoho-Stark uncertainty principles in the context of $L^p(\mathbb{R}^n)$ spaces, for which, as preparatory results, we will obtain new inequalities that can also be compared, in a certain sense, with the Hausdorff-Young inequality already obtained for $\mathcal{Q}_M$ in the previous section. Those inequalities will also involve the essential supports ("$\operatorname{ess\,supp}$") of $f \in L^p(\mathbb{R}^n)$ and its MQFT.

Proposition 4.1. *If $f \in L^1(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$, $1 \leq p \leq 2$, then*

$$\|\mathcal{Q}_M(f)\|_{L^{p'}(\mathbb{R}^n)} \leq \frac{|\det(B)|^{1/2}}{(2\pi)^{n/2}} \|f\|_{L^p(\mathbb{R}^n)} |\operatorname{ess\,supp} f|^{1/p'} |\operatorname{ess\,supp} \mathcal{Q}_M(f)|^{1/p'},$$

where $\frac{1}{p} + \frac{1}{p'} = 1$.

Proof. By the Riemann-Lebesgue Lemma and Hölder's inequality, we have

$$\begin{aligned}
\|\mathcal{Q}_M(f)\|_{L^{p'}(\mathbb{R}^n)} \leq & \|\mathcal{Q}_M(f)\|_{L^\infty(\mathbb{R}^n)} |\operatorname{ess\,supp} \mathcal{Q}_M(f)|^{1/p'} \\
\leq & \frac{|\det(B)|^{1/2}}{(2\pi)^{n/2}} \|f\|_{L^1(\mathbb{R}^n)} |\operatorname{ess\,supp} \mathcal{Q}_M(f)|^{1/p'} \\
\leq & \frac{|\det(B)|^{1/2}}{(2\pi)^{n/2}} \|f\|_{L^p(\mathbb{R}^n)} |\operatorname{ess\,supp} f|^{1/p'} |\operatorname{ess\,supp} \mathcal{Q}_M(f)|^{1/p'}.
\end{aligned}$$

□

Proposition 4.2. *If $f \in L^2(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$, $1 \leq p \leq 2$, with p' being such that $1/p + 1/p' = 1$, then*

(4.19)

$$\|\mathcal{Q}_M(f)\|_{L^2(\mathbb{R}^n)} \leq \frac{|\det(B)|^{1/p-1/2}}{(2\pi)^{n/2}} \|f\|_{L^2(\mathbb{R}^n)} |\operatorname{ess\,supp} f|^{(2-p)/2p} |\operatorname{ess\,supp} \mathcal{Q}_M(f)|^{(p'-2)/2p'}.$$

Proof. By the Hausdorff-Young inequality and generalised Hölder's inequality, we obtain

$$\begin{aligned}
\|\mathcal{Q}_M(f)\|_{L^2(\mathbb{R}^n)} \leq & \|\mathcal{Q}_M(f)\|_{L^{p'}(\mathbb{R}^n)} |\operatorname{ess\,supp} \mathcal{Q}_M(f)|^{(p'-2)/2p'} \\
\leq & \frac{|\det(B)|^{1/p-1/2}}{(2\pi)^{n/2}} \|f\|_{L^p(\mathbb{R}^n)} |\operatorname{ess\,supp} \mathcal{Q}_M(f)|^{(p'-2)/2p'} \\
\leq & \frac{|\det(B)|^{1/p-1/2}}{(2\pi)^{n/2}} \|f\|_{L^2(\mathbb{R}^n)} |\operatorname{ess\,supp} f|^{(2-p)/2p} |\operatorname{ess\,supp} \mathcal{Q}_M(f)|^{(p'-2)/2p'}.
\end{aligned}$$

□

Corollary 4.2. *If $f \in L^2(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$, $1 \leq p \leq 2$, then*

$$|\operatorname{ess\,supp} \mathcal{Q}_M(f)|^{(p'-2)/2p'} \geq |\det(B)|^{1/2-1/p} |\operatorname{ess\,supp} f|^{(p-2)/2p},$$

where p' is the conjugate exponent of p.

Proof. We only have to consider the Plancherel Theorem for the MQFT, together with (4.19), to obtain

$$|\det(B)|^{1/p-1/2}|\text{ess supp } f|^{(2-p)/2p}|\text{ess supp } \mathcal{Q}_M(f)|^{(p'-2)/2p'} \geq 1.$$

□

Lemma 4.5. *Let* Λ, Γ *be two measurable subsets of* $\mathbb{R}^n$ *such that* $0 < |\Lambda|, |\Gamma| < \infty$ *and* $f \in L^1(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$, $1 \leq p \leq 2$, *with* $1/p + 1/p' = 1$. *Then,*

(i) $\|\mathcal{Q}_M(Q_\Gamma f)\|_{L^{p'}(\mathbb{R}^n)} \leq \frac{|\det(B)|^{1/p-1/2}}{(2\pi)^{n/2}}\|f\|_{L^p(\mathbb{R}^n)}$*;*

(ii) $\|\mathcal{Q}_M(Q_\Gamma P_\Lambda f)\|_{L^{p'}(\mathbb{R}^n)} \leq |\Omega(B,n)||\Lambda|^{1/p'}|\Gamma|^{1/p'}\|f\|_{L^p(\mathbb{R}^n)}$.

Proof. By the Hausdorff-Young inequality for $\mathcal{Q}_M$, we have

(i)

$$\begin{aligned}\|\mathcal{Q}_M(Q_\Gamma f)\|_{L^{p'}(\mathbb{R}^n)} =& \left(\int_\Gamma |[\mathcal{Q}_M(f)](x)|^{p'}\,dx\right)^{1/p'} \\ \leq& \|\mathcal{Q}_M(f)\|_{L^{p'}(\mathbb{R}^n)} \\ \leq& \frac{|\det(B)|^{1/p-1/2}}{(2\pi)^{n/2}}\|f\|_{L^p(\mathbb{R}^n)};\end{aligned}$$

(ii)

$$\begin{aligned}\|\mathcal{Q}_M(Q_\Gamma P_\Lambda f)\|_{L^{p'}(\mathbb{R}^n)} =& \left(\int_\Gamma |[\mathcal{Q}_M(P_\Lambda f)](x)|^{p'}\,dx\right)^{1/p'} \\ =& \left(\int_\Gamma \left|\int_\Lambda f(y)\mathcal{K}_M^{\mathcal{Q}}(x,y)\,dy\right|^{p'}\,dx\right)^{1/p'}.\end{aligned}$$

In addition, it holds

$$\begin{aligned}\left|\int_\Lambda f(y)\mathcal{K}_M^{\mathcal{Q}}(x,y)\,dy\right| \leq& \left(\int_\Lambda |f(y)|^p\,dx\right)^{1/p}\left(\int_\Lambda |\mathcal{K}_M^{\mathcal{Q}}(x,y)|^{p'}\,dy\right)^{1/p'} \\ \leq& \|f\|_{L^p(\mathbb{R}^n)}|\Lambda|^{1/p'}|\Omega(B,n)|.\end{aligned}$$

So,

$$\begin{aligned}\|\mathcal{Q}_M(Q_\Gamma P_\Lambda f)\|_{L^{p'}(\mathbb{R}^n)} \leq& \left(\int_\Gamma \|f\|^{p'}_{L^p(\mathbb{R}^n)}|\Lambda|\,dx\right)^{1/p'}|\Omega(B,n)| \\ =& \|f\|_{L^p(\mathbb{R}^n)}|\Lambda|^{1/p'}|\Gamma|^{1/p'}|\Omega(B,n)|.\end{aligned}$$

□

Definition 4.3. *(i)* $f \in L^p(\mathbb{R}^n)$ *is said to be* ε_Λ*-concentrated on* Λ *in* L^p*-norm if*

$$\|P_{\Lambda^c}\|_{L^p(\mathbb{R}^n)} = \|f - P_\Lambda f\|_{L^p(\mathbb{R}^n)} \leq \varepsilon_\Lambda\|f\|_{L^p(\mathbb{R}^n)}.$$

(ii) $\mathcal{Q}_M(f)$ *is is called* ε_Γ*-concentrated on* Γ *in* L^p*-norm if*

$$\|\mathcal{Q}_M(Q_{\Gamma^c} f)\|_{L^p(\mathbb{R}^n)} = \|\mathcal{Q}_M(f) - \mathcal{Q}_M(Q_\Gamma f)\|_{L^p(\mathbb{R}^n)} \leq \varepsilon_\Gamma\|\mathcal{Q}_M(f)\|_{L^p(\mathbb{R}^n)}.$$

Theorem 4.7. *Let Λ, Γ be two measurable subsets of $\mathbb{R}^n$ such that $0 < |\Lambda|, |\Gamma| < \infty$, and $f \in L^1(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$, $1 \le p \le 2$. If f is ε_Λ-concentrated on Λ in L^p-norm and $\mathcal{Q}_M(f)$ is ε_Γ-concentrated on Γ in $L^{p'}$-norm, and $1/p + 1/p' = 1$, then*

$$\|\mathcal{Q}_M(f)\|_{L^{p'}(\mathbb{R}^n)} \le \left(\frac{\frac{|\det(B)|^{1/p-1/2}}{(2\pi)^{n/2}}\varepsilon_\Lambda + |\Omega(B,n)||\Lambda|^{1-1/p}|\Gamma|^{1-1/p}}{1-\varepsilon_\Gamma} \right) \|f\|_{L^p(\mathbb{R}^n)}.$$

Proof. Consider

$$\begin{aligned}
\|\mathcal{Q}_M(f)\|_{L^{p'}(\mathbb{R}^n)} =& \|\mathcal{Q}_M(f) - \mathcal{Q}_M(Q_\Gamma P_\Lambda f) + \mathcal{Q}_M(Q_\Gamma P_\Lambda f)\|_{L^{p'}(\mathbb{R}^n)} \\
\le& \|\mathcal{Q}_M(Q_\Gamma P_\Lambda f)\|_{L^{p'}(\mathbb{R}^n)} + \|\mathcal{Q}_M(f) - \mathcal{Q}_M(Q_\Gamma f)\|_{L^{p'}(\mathbb{R}^n)} \\
&+ \|\mathcal{Q}_M(Q_\Gamma f) - \mathcal{Q}_M(Q_\Gamma P_\Lambda f)\|_{L^{p'}(\mathbb{R}^n)} \\
\le& \|\mathcal{Q}_M(Q_\Gamma P_\Lambda f)\|_{L^{p'}(\mathbb{R}^n)} + \varepsilon_\Gamma \|\mathcal{Q}_M(f)\|_{L^{p'}(\mathbb{R}^n)} + \|\mathcal{Q}_M[Q_\Gamma(f - P_\Lambda f)]\|_{L^{p'}(\mathbb{R}^n)}.
\end{aligned}$$

By (i) in Lemma 4.5 and the fact that f is ε_Λ-concentrated on Λ in the L^p norm, we have

$$\|\mathcal{Q}_M[Q_\Gamma(f - P_\Lambda f)]\|_{L^{p'}(\mathbb{R}^n)} \le \frac{|\det(B)|^{1/p-1/2}}{(2\pi)^{n/2}} \|f - P_\Lambda f\|_{L^p(\mathbb{R}^n)} \le \frac{|\det(B)|^{1/p-1/2}}{(2\pi)^{n/2}} \varepsilon_\Lambda \|f\|_{L^p(\mathbb{R}^n)}.$$

By (ii) in Lemma 4.5, we obtain

$$\|\mathcal{Q}_M(Q_\Gamma P_\Lambda f)\|_{L^{p'}(\mathbb{R}^n)} \le |\Omega(B,n)||\Lambda|^{1/p'}|\Gamma|^{1/p'}\|f\|_{L^p(\mathbb{R}^n)}.$$

Consequently,

$$\begin{aligned}
\|\mathcal{Q}_M(f)\|_{L^{p'}(\mathbb{R}^n)} \le& \frac{|\det(B)|^{1/p-1/2}}{(2\pi)^{n/2}} \varepsilon_\Lambda \|f\|_{L^p(\mathbb{R}^n)} + \varepsilon_\Gamma \|\mathcal{Q}_M(f)\|_{L^{p'}(\mathbb{R}^n)} \\
&+ |\Omega(B,n)||\Lambda|^{1-1/p}|\Gamma|^{1-1/p}\|f\|_{L^p(\mathbb{R}^n)},
\end{aligned}$$

which implies that

$$(1-\varepsilon_\Gamma)\|\mathcal{Q}_M(f)\|_{L^{p'}(\mathbb{R}^n)} \le \left(\frac{|\det(B)|^{1/p-1/2}}{(2\pi)^{n/2}}\varepsilon_\Lambda + |\Omega(B,n)||\Lambda|^{1-1/p}|\Gamma|^{1-1/p} \right) \|f\|_{L^p(\mathbb{R}^n)}$$

and so,

$$\|\mathcal{Q}_M(f)\|_{L^{p'}(\mathbb{R}^n)} \le \left(\frac{\frac{|\det(B)|^{1/p-1/2}}{(2\pi)^{n/2}}\varepsilon_\Lambda + |\Omega(B,n)||\Lambda|^{1-1/p}|\Gamma|^{1-1/p}}{1-\varepsilon_\Gamma} \right) \|f\|_{L^p(\mathbb{R}^n)}.$$

□

Theorem 4.8. *Let Λ, Γ be two measurable subsets of $\mathbb{R}^n$ such that $0 < |\Lambda|, |\Gamma| < \infty$, and $f \in L^1(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$, $1 \le p \le 2$. If f is ε_Λ-concentrated on Λ in L^1-norm and $\mathcal{Q}_M(f)$ is ε_Γ-concentrated on Γ in $L^{p'}$-norm, with $1/p + 1/p' = 1$, then*

$$\|\mathcal{Q}_M(f)\|_{L^{p'}(\mathbb{R}^n)} \le \frac{|\Gamma|^{1/p'}|\Lambda|^{1/p'}|\det(B)|^{1/2}}{(1-\varepsilon_\Gamma)(1-\varepsilon_\Lambda)(2\pi)^{n/2}} \|f\|_{L^p(\mathbb{R}^n)}.$$

Proof. We have

$$\begin{aligned}
\|\mathcal{Q}_M(f)\|_{L^{p'}(\mathbb{R}^n)} \le& \|\mathcal{Q}_M(f) - \mathcal{Q}_M(Q_\Gamma f)\|_{L^{p'}(\mathbb{R}^n)} + \|\mathcal{Q}_M(Q_\Gamma f)\|_{L^{p'}(\mathbb{R}^n)} \\
\le& \varepsilon_\Gamma \|\mathcal{Q}_M(f)\|_{L^{p'}(\mathbb{R}^n)} + \left(\int_\Gamma |[\mathcal{Q}_M(f)](x)|^{p'} \, dx \right)^{1/p'} \\
\le& \varepsilon_\Gamma \|\mathcal{Q}_M(f)\|_{L^{p'}(\mathbb{R}^n)} + |\Gamma|^{1/p'} \|\mathcal{Q}_M(f)\|_{L^\infty(\mathbb{R}^n)}.
\end{aligned}$$

So, recalling that $0 < \varepsilon_\Gamma < 1$, we have

$$\|\mathcal{Q}_M(f)\|_{L^{p'}(\mathbb{R}^n)} \leq \frac{|\Gamma|^{1/p'}}{1-\varepsilon_\Gamma}\|\mathcal{Q}_M(f)\|_{L^\infty(\mathbb{R}^n)}$$

and, by the Riemann-Lebesgue Lemma for the MQFT, it follows

$$\|\mathcal{Q}_M(f)\|_{L^{p'}(\mathbb{R}^n)} \leq \frac{|\Gamma|^{1/p'}|\det(B)|^{1/2}}{(1-\varepsilon_\Gamma)(2\pi)^{n/2}}\|f\|_{L^1(\mathbb{R}^n)}.$$

Since f is ε_Λ-concentrated on Λ in L^1-norm, we obtain

$$\begin{aligned}\|f\|_{L^1(\mathbb{R}^n)} \leq&\|P_{\Lambda^c}f\|_{L^1(\mathbb{R}^n)} + \|P_\Lambda f\|_{L^1(\mathbb{R}^n)}\\ \leq&\varepsilon_\Lambda\|f\|_{L^1(\mathbb{R}^n)} + \int_\Lambda |f(x)|\,dx\\ \leq&\varepsilon_\Lambda\|f\|_{L^1(\mathbb{R}^n)} + |\Lambda|^{1/p'}\|f\|_{L^p(\mathbb{R}^n)},\end{aligned}$$

by Hölder's inequality. This is equivalent to

$$\|f\|_{L^1(\mathbb{R}^n)} \leq \frac{|\Lambda|^{1/p'}}{1-\varepsilon_\Lambda}\|f\|_{L^p(\mathbb{R}^n)}.$$

So, we obtain

$$\|\mathcal{Q}_M(f)\|_{L^{p'}(\mathbb{R}^n)} \leq \frac{|\Gamma|^{1/p'}|\Lambda|^{1/p'}|\det(B)|^{1/2}}{(1-\varepsilon_\Gamma)(1-\varepsilon_\Lambda)(2\pi)^{n/2}}\|f\|_{L^p(\mathbb{R}^n)}.$$

□

Remark 4.4. If $p = p' = 2$, the previous theorem reduces to the classical case

$$|\Gamma|^{1/2}|\Lambda|^{1/2} \geq \frac{(1-\varepsilon_\Gamma)(1-\varepsilon_\Lambda)}{|\det(B)|^{1/2}}.$$

Theorem 4.9. *Let Λ, Γ be two measurable subsets of $\mathbb{R}^n$ such that $0 < |\Lambda|, |\Gamma| < \infty$, and $f \in L^1(\mathbb{R}^n) \cap L^q(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$, $1 < q < p < 2$. If f is ε_Λ-concentrated on Λ in L^q-norm and $\mathcal{Q}_M(f)$ is ε_Γ-concentrated on Γ in $L^{p'}$-norm, with $1/p + 1/p' = 1$, then*

$$\|\mathcal{Q}_M(f)\|_{L^{p'}(\mathbb{R}^n)} \leq \frac{(|\Gamma||\Lambda|)^{1/q-1/p}|\det(B)|^{1/p-1/2}}{(2\pi)^{n/2}(1-\varepsilon_\Gamma)(1-\varepsilon_\Lambda)}\|f\|_{L^p(\mathbb{R}^n)}.$$

Proof. Since $\mathcal{Q}_M(f)$ is ε_Γ-concentrated on Γ in $L^{p'}$-norm, we have

$$\begin{aligned}\|\mathcal{Q}_M(f)\|_{L^{p'}(\mathbb{R}^n)} =&\|\mathcal{Q}_M(f) - \mathcal{Q}_M(Q_\Gamma f) + \mathcal{Q}_M(Q_\Gamma f)\|_{L^{p'}(\mathbb{R}^n)}\\ \leq&\|\mathcal{Q}_M(f) - \mathcal{Q}_M(Q_\Gamma f)\|_{L^{p'}(\mathbb{R}^n)} + \left(\int_\Gamma |[\mathcal{Q}_M(f)](x)|^{p'}\,dx\right)^{1/p'}\\ \leq&\varepsilon_\Gamma\|\mathcal{Q}_M(f)\|_{L^{p'}(\mathbb{R}^n)} + |\Gamma|^{1/p'-1/q'}\|\mathcal{Q}_M(f)\|_{L^{q'}(\mathbb{R}^n)}\\ \leq&\varepsilon_\Gamma\|\mathcal{Q}_M(f)\|_{L^{p'}(\mathbb{R}^n)} + |\Gamma|^{1/q-1/p}\frac{|\det(B)|^{1/p-1/2}}{(2\pi)^{n/2}}\|f\|_{L^q(\mathbb{R}^n)},\end{aligned}$$

by the Hausdorff-Young inequality with $1/q + 1/q' = 1$. So, since $0 < \varepsilon_\Gamma < 1$, we have

$$\|\mathcal{Q}_M(f)\|_{L^{p'}(\mathbb{R}^n)} \leq \frac{|\Gamma|^{1/q-1/p}|\det(B)|^{1/p-1/2}}{(2\pi)^{n/2}(1-\varepsilon_\Gamma)}\|f\|_{L^q(\mathbb{R}^n)}.$$

Since

$$\|f\|_{L^q(\mathbb{R}^n)} \leq \|f - P_\Lambda f\|_{L^q(\mathbb{R}^n)} + \|P_\Lambda f\|_{L^q(\mathbb{R}^n)} \leq \varepsilon_\Lambda\|f\|_{L^q(\mathbb{R}^n)} + |\Lambda|^{1/q-1/p}\|f\|_{L^p(\mathbb{R}^n)},$$

we have

$$\|f\|_{L^q(\mathbb{R}^n)} \leq \frac{|\Lambda|^{1/q-1/p}}{1-\varepsilon_\Lambda}\|f\|_{L^p(\mathbb{R}^n)}.$$

Consequently,

$$\|\mathcal{Q}_M(f)\|_{L^{p'}(\mathbb{R}^n)} \leq \frac{|\Gamma|^{1/q-1/p}|\det(B)|^{1/p-1/2}}{(2\pi)^{n/2}(1-\varepsilon_\Gamma)} \cdot \frac{|\Lambda|^{1/q-1/p}}{1-\varepsilon_\Lambda}\|f\|_{L^p(\mathbb{R}^n)}.$$

□

Let us now prepare to culminate with the last significant result, which will have to do with an uncertainty principle associated with bandlimited functions, in relation to a certain class of functions, invariant under Q_Γ, which we will now formalise. For $1 \leq p \leq 2$ we shall consider $\mathcal{B}^p_{Q_\Gamma}(\mathbb{R}^n) := \{h \in L^1(\mathbb{R}^n) \cap L^p(\mathbb{R}^n) : Q_\Gamma h = h\}$.

If $f \in L^p(\mathbb{R}^n)$ satisfies

$$\|f-h\|_{L^p(\mathbb{R}^n)} \leq \varepsilon_\Gamma \|f\|_{L^p(\mathbb{R}^n)}$$

for some $h \in \mathcal{B}^p_{Q_\Gamma}(\mathbb{R}^n)$, then f is said to be ε_Γ-bandlimited on Γ in L^p-norm.

Lemma 4.6. *Let Λ, Γ be two measurable subsets of $\mathbb{R}^n$ such that $0 < |\Lambda|, |\Gamma| < \infty$. If $h \in \mathcal{B}^p_{Q_\Gamma}(\mathbb{R}^n)$, $1 \leq p \leq 2$, then*

$$\|P_\Lambda h\|_{L^p(\mathbb{R}^n)} \leq \frac{(|\Gamma||\Lambda|)^{1/p}|\det(B)|^{1/p}}{(2\pi)^n}\|h\|_{L^p(\mathbb{R}^n)}.$$

Proof. By the Hölder inequality, the Hausdorff-Young inequality and the definition of the $\mathcal{B}^p_{Q_\Gamma}(\mathbb{R}^n)$ space, we have

$$\begin{aligned}
\|\mathcal{Q}_M(h)\|_{L^1(\mathbb{R}^n)} &= \|\mathcal{Q}_M(Q_\Gamma h)\|_{L^1(\mathbb{R}^n)} \\
&= \|\chi_\Gamma \mathcal{Q}_M(h)\|_{L^1(\mathbb{R}^n)} \\
&\leq |\Gamma|^{1/p}\|\mathcal{Q}_M(h)\|_{L^{p'}(\mathbb{R}^n)} \\
&\leq |\Gamma|^{1/p}\frac{|\det(B)|^{1/p-1/2}}{(2\pi)^{n/2}}\|h\|_{L^p(\mathbb{R}^n)}
\end{aligned}$$

and

$$\begin{aligned}
\|\mathcal{Q}_M(h)\|_{L^2(\mathbb{R}^n)} &= \|\mathcal{Q}_M(Q_\Gamma h)\|_{L^2(\mathbb{R}^n)} \\
&= \|\chi_\Gamma \mathcal{Q}_M(h)\|_{L^2(\mathbb{R}^n)} \\
&\leq |\Gamma|^{1/2-1/p'}\|\mathcal{Q}_M(h)\|_{L^{p'}(\mathbb{R}^n)} \\
&\leq |\Gamma|^{1/p-1/2}\frac{|\det(B)|^{1/p-1/2}}{(2\pi)^{n/2}}\|h\|_{L^p(\mathbb{R}^n)},
\end{aligned}$$

which implies that $\mathcal{Q}_M(h) \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$. Therefore, we have

$$h(t) = (Q_\Gamma h)(t) = \mathcal{Q}_M^{-1}[\chi_\Gamma \mathcal{Q}_M(h)](t) = \int_\Gamma [\mathcal{Q}_M(h)](x)\overline{\mathcal{K}^{\mathcal{Q}}_M(x,t)}\,dx.$$

Hence,

$$\begin{aligned}|h(t)| \leq& \int_\Gamma |[\mathcal{Q}_M(h)](x)||\overline{\mathcal{K}_M^{\mathcal{Q}}(x,t)}|\,dx\\ =&\frac{|\det(B)|^{1/2}}{(2\pi)^{n/2}}\int_\Gamma |[\mathcal{Q}_M(h)](x)|\,dx\\ \leq&\frac{|\det(B)|^{1/2}}{(2\pi)^{n/2}}|\Gamma|^{1/p}\|\mathcal{Q}_M(h)\|_{L^{p'}(\mathbb{R}^n)}\\ \leq&\frac{|\det(B)|^{1/p}}{(2\pi)^{n}}|\Gamma|^{1/p}\|h\|_{L^p(\mathbb{R}^n)}.\end{aligned}$$

Consequently,

$$\|P_\Lambda h\|_{L^p(\mathbb{R}^n)} \leq \frac{(|\Gamma||\Lambda|)^{1/p}|\det(B)|^{1/p}}{(2\pi)^n}\|h\|_{L^p(\mathbb{R}^n)}.$$

□

Theorem 4.10. *Let* Λ, Γ *be two measurable subsets of* $\mathbb{R}^n$ *such that* $0 < |\Lambda|, |\Gamma| < \infty$, *and* $f \in L^1(\mathbb{R}^n) \cap L^q(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$, $1 < q \leq p < 2$. *If* f *is* ε_Λ*-concentrated on* Λ *in* L^q*-norm and* ε_Γ*-bandlimited on* Γ *in* L^p*-norm, then*

$$\|f\|_{L^q(\mathbb{R}^n)} \leq \left(\frac{\varepsilon_\Gamma|\Lambda|^{1/q-1/p}}{1-\varepsilon_\Lambda} + \frac{|\Gamma|^{1/p}|\Lambda|^{1/q}|\det(B)|^{1/p}(1+\varepsilon_\Gamma)}{(2\pi)^n(1-\varepsilon_\Lambda)}\right)\|f\|_{L^p(\mathbb{R}^n)}.$$

Proof. Since f is ε_Λ-concentrated on Λ in L^q-norm, we obtain

$$\begin{aligned}\|f\|_{L^q(\mathbb{R}^n)} \leq& \|f - P_\Lambda f\|_{L^q(\mathbb{R}^n)} + \|P_\Lambda f\|_{L^q(\mathbb{R}^n)}\\ \leq&\varepsilon_\Lambda\|f\|_{L^q(\mathbb{R}^n)} + |\Lambda|^{1/q-1/p}\|P_\Lambda f\|_{L^p(\mathbb{R}^n)},\end{aligned}$$

which implies

$$\|f\|_{L^q(\mathbb{R}^n)} \leq \frac{|\Lambda|^{1/q-1/p}}{1-\varepsilon_\Lambda}\|P_\Lambda f\|_{L^p(\mathbb{R}^n)}. \tag{4.20}$$

As f is ε_Γ-bandlimited on Γ in L^p-norm and by the previous lemma, there exists a function $h \in \mathcal{B}^p(\Gamma)$ such that

$$\begin{aligned}\|P_\Lambda f\|_{L^p(\mathbb{R}^n)} \leq& \|P_\Lambda(f-h)\|_{L^p(\mathbb{R}^n)} + \|P_\Lambda h\|_{L^p(\mathbb{R}^n)}\\ \leq&\|f-h\|_{L^p(\mathbb{R}^n)} + \|P_\Lambda h\|_{L^p(\mathbb{R}^n)}\\ \leq&\varepsilon_\Gamma\|f\|_{L^p(\mathbb{R}^n)} + \frac{(|\Gamma||\Lambda|)^{1/p}|\det(B)|^{1/p}}{(2\pi)^n}\|h\|_{L^p(\mathbb{R}^n)}.\end{aligned}$$

Since

$$\|h\|_{L^p(\mathbb{R}^n)} - \|f\|_{L^p(\mathbb{R}^n)} \leq \|h-f\|_{L^p(\mathbb{R}^n)} \leq \varepsilon_\Gamma\|f\|_{L^p(\mathbb{R}^n)},$$

we have that

$$\|h\|_{L^p(\mathbb{R}^n)} \leq (1+\varepsilon_\Gamma)\|f\|_{L^p(\mathbb{R}^n)}.$$

So,

$$\|P_\Lambda f\|_{L^p(\mathbb{R}^n)} \leq \left(\varepsilon_\Gamma + \frac{(|\Gamma||\Lambda|)^{1/p}|\det(B)|^{1/p}(1+\varepsilon_\Gamma)}{(2\pi)^n}\right)\|f\|_{L^p(\mathbb{R}^n)}.$$

Consequently, recalling (4.20), we have

$$\|f\|_{L^q(\mathbb{R}^n)} \leq \left(\frac{\varepsilon_\Gamma|\Lambda|^{1/q-1/p}}{1-\varepsilon_\Lambda} + \frac{|\Gamma|^{1/p}|\Lambda|^{1/q}|\det(B)|^{1/p}(1+\varepsilon_\Gamma)}{(2\pi)^n(1-\varepsilon_\Lambda)}\right)\|f\|_{L^p(\mathbb{R}^n)}.$$

□

If $p = q$, then the last result allows us to directly write the following corollary.

Corollary 4.3. *Let* Λ, Γ *be two measurable subsets of* $\mathbb{R}^n$ *such that* $0 < |\Lambda|, |\Gamma| < \infty$, *and* $f \in L^1(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$, $1 \le p \le 2$. *If* f *is* ε_Λ*-concentrated on* Λ *and* ε_Γ*-bandlimited on* Γ *in* L^p*-norm, then*

$$|\Gamma||\Lambda| \geq \frac{(1-\varepsilon_\Lambda - \varepsilon_\Gamma)^p (2\pi)^{np}}{|\det(B)|(1+\varepsilon_\Gamma)^p}.$$

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Research Article

The injective hull of ideals of weighted holomorphic mappings

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ABSTRACT. We study the injectivity of normed ideals of weighted holomorphic mappings. To be more precise, the concept of injective hull of normed weighted holomorphic ideals is introduced and characterized in terms of a domination property. The injective hulls of those ideals – generated by the procedures of composition and dual – are described and these descriptions are applied to some examples of such ideals. A characterization of the closed injective hull of an operator ideal in terms of an Ehrling-type inequality – due to Jarchow and Pelczyński– is established for weighted holomorphic mappings.

Keywords: Weighted holomorphic mapping, injective hull, domination theorem, operator ideal, Ehrling inequality.

2020 Mathematics Subject Classification: 47A63, 47L20, 46E50, 46T25.

1. INTRODUCTION

Influenced by the concept of operator ideals (see the book [23] by Pietsch), the notion of ideals of weighted holomorphic mappings was introduced in [10], although also the ideals of bounded holomorphic mappings were analysed in [11]. In [10], the composition procedure to generate weighted holomorphic ideals was studied and some examples of such ideals were presented.

Our aim in this paper is to address the injective procedure in the context of weighted holomorphic mappings. In the linear setting, the concept of injective hull of an operator ideal was dealt by Pietsch [23], although some ingredients already appeared in the paper [24] by Stephani.

Given an open subset U of a complex Banach space E, a weight v on U is a (strictly) positive continuous function. For any complex Banach space F, let $\mathcal{H}(U,F)$ be the space of all holomorphic mappings from U into F. The space of weighted holomorphic mappings, $\mathcal{H}_v^\infty(U,F)$, is the Banach space of all mappings $f\in\mathcal{H}(U,F)$ so that

$$\|f\|_v := \sup\{v(x)\,\|f(x)\| : x\in U\} < \infty,$$

under the weighted supremum norm $\|\cdot\|_v$. We will write $\mathcal{H}_v^\infty(U)$ instead of $\mathcal{H}_v^\infty(U,\mathbb{C})$. About the theory of weighted holomorphic mappings, the interested reader can consult the papers [3] by Bierstedt and Summers, [5, 6] by Bonet, Domanski and Lindström, and [16] by Gupta and Baweja. See also the recent survey [4] by Bonet on these function spaces, and the references therein.

By definition, the injective hull of a normed weighted holomorphic ideal $[\mathcal{I}^{\mathcal{H}_v^\infty}, \|\cdot\|_{\mathcal{I}^{\mathcal{H}_v^\infty}}]$ is the smallest injective normed weighted holomorphic ideal containing $\mathcal{I}^{\mathcal{H}_v^\infty}$. In Subsection 2.1, we establish the existence of this injective hull. As an immediate consequence, a normed

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weighted holomorphic ideal is injective if and only if it coincides with its injective hull. In Subsection 2.2, a characterization of the injective hull of a normed weighted holomorphic ideal is stated by means of a domination property, and it is applied to describe the injectivity of a normed weighted holomorphic ideal in a form similar to those obtained in the linear and polynomial versions [8, 9].

Using the linearization of weighted holomorphic mappings, we describe in Subsection 2.3 the injective hull of composition ideals of weighted holomorphic mappings and apply this description to establish the injectivity of the normed weighted holomorphic ideals generated by composition with some distinguished classes of bounded linear operators such as finite-rank, compact, weakly compact, separable, Rosenthal and Asplund operators.

In Subsection 2.4, the concept of dual weighted holomorphic ideal of an operator ideal $\mathcal{I}$ is introduced and showed that it coincides with the weighted holomorphic ideal generated by composition with the dual operator ideal $\mathcal{I}^{\mathrm{dual}}$. Moreover, we study the injectivity of such dual weighted holomorphic ideals as well as the dual weighted holomorphic ideals of the ideals of p-compact and Cohen strongly p-summing operators for any $p \in (1,\infty)$. Subsection 2.5 presents a weighted holomorphic variant of a characterization –due to Jarchow and Pelczyński [17]– of the closed injective hull of an operator ideal by means of an Ehrling-type inequality [13].

It should be noted that different authors have studied these questions for ideals of functions in both linear settings (for classical p-compact operators [14], (p,q)-compact operators [18], weakly p-nuclear operators [19] and multilinear mappings [20]) as well as in non-linear contexts (for holomorphic mappings [15], polynomials [9] and Lipschitz operators [1, 2, 25, 26], among others.

2. Results

We will present the results of this paper in various subsections. From now on, unless otherwise stated, E will denote a complex Banach space, U an open subset of E, v a weight on U, and F a complex Banach space. Our notation is standard. $\mathcal{L}(E,F)$ denotes the Banach space of all bounded linear operators from E into F, equipped with the operator canonical norm. E^* and B_E represent the dual space and the closed unit ball of E, respectively. Given a set $A \subseteq E$, $\overline{\mathrm{lin}}(A)$ and $\overline{\mathrm{aco}}(A)$ stand for the norm closed linear hull and the norm closed absolutely convex hull of A in E.

2.1. **The injective hull of ideals of weighted holomorphic mappings.** In light of Definition 2.4 in [10], a normed (Banach) ideal of weighted holomorphic mappings – or, in short, a normed (Banach) weighted holomorphic ideal – is an assignment $[\mathcal{I}^{\mathcal{H}_v^\infty}, \|\cdot\|_{\mathcal{I}^{\mathcal{H}_v^\infty}}]$ which associates every pair (U,F), – where E is a complex Banach space, U is an open subset of E and F is a complex Banach space – to both a set $\mathcal{I}^{\mathcal{H}_v^\infty}(U,F) \subseteq \mathcal{H}_v^\infty(U,F)$ and a function $\|\cdot\|_{\mathcal{I}^{\mathcal{H}_v^\infty}} : \mathcal{I}^{\mathcal{H}_v^\infty}(U,F) \to \mathbb{R}$ satisfying

(P1) $(\mathcal{I}^{\mathcal{H}_v^\infty}(U,F), \|\cdot\|_{\mathcal{I}^{\mathcal{H}_v^\infty}})$ is a normed (Banach) space and $\|f\|_v \leq \|f\|_{\mathcal{I}^{\mathcal{H}_v^\infty}}$ if $f \in \mathcal{I}^{\mathcal{H}_v^\infty}(U,F)$,

(P2) Given $h \in \mathcal{H}_v^\infty(U)$ and $y \in F$, the map $h \cdot y \colon x \in U \mapsto h(x)y \in F$ is in $\mathcal{I}^{\mathcal{H}_v^\infty}(U,F)$ with $\|h \cdot y\|_{\mathcal{I}^{\mathcal{H}_v^\infty}} = \|h\|_v \|y\|$,

(P3) The ideal property: if V is an open subset of E such that $V \subseteq U$, $h \in \mathcal{H}(V,U)$ with

$$c_v(h) := \sup_{x \in V} \frac{v(x)}{v(h(x))} < \infty,$$

$f \in \mathcal{I}^{\mathcal{H}_v^\infty}(U,F)$ and $T \in \mathcal{L}(F,G)$, where G is a complex Banach space, then $T \circ f \circ h \in \mathcal{I}^{\mathcal{H}_v^\infty}(V,G)$ with $\|T \circ f \circ h\|_{\mathcal{I}^{\mathcal{H}_v^\infty}} \leq \|T\| \, \|f\|_{\mathcal{I}^{\mathcal{H}_v^\infty}} c_v(h)$.

According to Sections 4.6 and 8.4 in [23], an operator ideal $\mathcal{I}$ is said to be injective if for each Banach space G and each isometric linear embedding $\iota\colon F \to G$, an operator $T \in \mathcal{L}(E, F)$ belongs to $\mathcal{I}(E, F)$ whenever $\iota \circ T \in \mathcal{I}(E, G)$. A normed operator ideal $[\mathcal{I}, \|\cdot\|_{\mathcal{I}}]$ is called injective if, in addition, $\|T\|_{\mathcal{I}} = \|\iota \circ T\|_{\mathcal{I}}$. The adaptation of this notion to the weighted holomorphic setting is as follows.

A normed weighted holomorphic ideal $[\mathcal{I}^{\mathcal{H}^\infty_v}, \|\cdot\|_{\mathcal{I}^{\mathcal{H}^\infty_v}}]$ is called:

(I) injective if for any map $f \in \mathcal{H}^\infty_v(U, F)$, any complex Banach space G and any into linear isometry $\iota\colon F \to G$, one has $f \in \mathcal{I}^{\mathcal{H}^\infty_v}(U, F)$ with $\|f\|_{\mathcal{H}^\infty_v} = \|\iota \circ f\|_{\mathcal{H}^\infty_v}$ whenever $\iota \circ f \in \mathcal{I}^{\mathcal{H}^\infty_v}(U, G)$.

Given normed weighted holomorphic ideals $[\mathcal{I}^{\mathcal{H}^\infty_v}, \|\cdot\|_{\mathcal{I}^{\mathcal{H}^\infty_v}}]$ and $[\mathcal{J}^{\mathcal{H}^\infty_v}, \|\cdot\|_{\mathcal{J}^{\mathcal{H}^\infty_v}}]$, the relation

$$[\mathcal{I}^{\mathcal{H}^\infty_v}, \|\cdot\|_{\mathcal{I}^{\mathcal{H}^\infty_v}}] \leq [\mathcal{J}^{\mathcal{H}^\infty_v}, \|\cdot\|_{\mathcal{J}^{\mathcal{H}^\infty_v}}]$$

means that for any complex Banach space E, any open set $U \subseteq E$ and any complex Banach space F, one has $\mathcal{I}^{\mathcal{H}^\infty_v}(U, F) \subseteq \mathcal{J}^{\mathcal{H}^\infty_v}(U, F)$ with $\|f\|_{\mathcal{J}^{\mathcal{H}^\infty_v}} \leq \|f\|_{\mathcal{I}^{\mathcal{H}^\infty_v}}$ for $f \in \mathcal{I}^{\mathcal{H}^\infty_v}(U, F)$.

Motivated by the linear and polynomial versions (see [17, Proposition 19.2.2] and [9, Proposition 2.3]), we next address the existence of the injective hull of a normed weighted holomorphic ideal. Recall that the unique smallest injective operator ideal $\mathcal{I}^{inj}$ that contains an operator ideal $\mathcal{I}$ is called the injective hull of $\mathcal{I}$ and described as the set

$$\mathcal{I}^{inj}(E, F) = \{T \in \mathcal{L}(E, F) \colon \iota_F \circ T \in \mathcal{I}(E, \ell_\infty(B_{Y^*})\},$$

where $\iota_F\colon F \to \ell_\infty(B_{F^*})$ is the canonical isometric linear embedding defined by

$$\langle \iota_F(y), y^* \rangle = y^*(y) \qquad (y^* \in B_{F^*},\ y \in F).$$

Taking $\|T\|_{\mathcal{I}^{inj}} = \|\iota_F \circ T\|_{\mathcal{I}}$ for $T \in \mathcal{I}^{inj}(E, F)$, $[\mathcal{I}^{inj}, \|\cdot\|_{\mathcal{I}^{inj}}]$ is a normed (Banach) operator ideal whenever $[\mathcal{I}, \|\cdot\|_{\mathcal{I}}]$ is so.

We now present the closely related concept in the setting of weighted holomorphic maps.

Proposition 2.1. *Let $[\mathcal{I}^{\mathcal{H}^\infty_v}, \|\cdot\|_{\mathcal{I}^{\mathcal{H}^\infty_v}}]$ be a normed (Banach) weighted holomorphic ideal. Then there exists a unique smallest normed (Banach) injective weighted holomorphic ideal $[(\mathcal{I}^{\mathcal{H}^\infty_v})^{inj}, \|\cdot\|_{(\mathcal{I}^{\mathcal{H}^\infty_v})^{inj}}]$ such that*

$$[\mathcal{I}^{\mathcal{H}^\infty_v}, \|\cdot\|_{\mathcal{I}^{\mathcal{H}^\infty_v}}] \leq [(\mathcal{I}^{\mathcal{H}^\infty_v})^{inj}, \|\cdot\|_{(\mathcal{I}^{\mathcal{H}^\infty_v})^{inj}}].$$

In fact, for any complex Banach space F, we have

$$(\mathcal{I}^{\mathcal{H}^\infty_v})^{inj}(U, F) = \left\{ f \in \mathcal{H}^\infty_v(U, F) \colon \iota_F \circ f \in \mathcal{I}^{\mathcal{H}^\infty_v}(U, \ell_\infty(B_{F^*}) \right\},$$

where $\iota_F\colon F \to \ell_\infty(B_{F^*})$ is the canonical isometric linear embedding, and

$$\|f\|_{(\mathcal{I}^{\mathcal{H}^\infty_v})^{inj}} = \|\iota_F \circ f\|_{\mathcal{I}^{\mathcal{H}^\infty_v}} \qquad (f \in (\mathcal{I}^{\mathcal{H}^\infty_v})^{inj}(U, F)).$$

The normed (Banach) ideal $[(\mathcal{I}^{\mathcal{H}^\infty_v})^{inj}, \|\cdot\|_{(\mathcal{I}^{\mathcal{H}^\infty_v})^{inj}}]$) is called the injective hull of $[\mathcal{I}^{\mathcal{H}^\infty_v}, \|\cdot\|_{\mathcal{I}^{\mathcal{H}^\infty_v}}]$.

Proof. Defining the set $(\mathcal{I}^{\mathcal{H}^\infty_v})^{inj}(U, F)$ and the function $\|\cdot\|_{(\mathcal{I}^{\mathcal{H}^\infty_v})^{inj}} \colon (\mathcal{I}^{\mathcal{H}^\infty_v})^{inj}(U, F) \to \mathbb{R}^+_0$ as above, we first show that $[(\mathcal{I}^{\mathcal{H}^\infty_v})^{inj}, \|\cdot\|_{(\mathcal{I}^{\mathcal{H}^\infty_v})^{inj}}]$ is an injective normed (Banach) weighted holomorphic ideal.

(P1) Given $f \in (\mathcal{I}^{\mathcal{H}^\infty_v})^{inj}(U, F)$, for all $x \in U$, we have

$$v(x)\,\|f(x)\| = v(x)\,\|\iota_F(f(x))\| \leq \|\iota_F \circ f\|_v \leq \|\iota_F \circ f\|_{\mathcal{I}^{\mathcal{H}^\infty_v}} = \|f\|_{(\mathcal{I}^{\mathcal{H}^\infty_v})^{inj}},$$

and thus $\|f\|_v \leq \|f\|_{(\mathcal{I}^{\mathcal{H}_v^\infty})^{inj}}$. Hence $f = 0$ whenever $\|f\|_{(\mathcal{I}^{\mathcal{H}_v^\infty})^{inj}} = 0$. It is readily to prove that $(\mathcal{I}^{\mathcal{H}_v^\infty})^{inj}(U,F)$ is a linear subspace of $\mathcal{H}_v^\infty(U,F)$ on which $\|\cdot\|_{\mathcal{H}_v^\infty}$ is absolutely homogeneous and satisfies the triangle inequality.

(P2) Given $h \in \mathcal{H}_v^\infty(U)$ and $y \in F$, we have $\iota_F \circ (h \cdot y) = h \cdot \iota_F(y) \in \mathcal{I}^{\mathcal{H}_v^\infty}(U, \ell_\infty(B_{F^*}))$ and therefore $h \cdot y \in (\mathcal{I}^{\mathcal{H}_v^\infty})^{inj}(U,F)$ with

$$\|h \cdot y\|_{(\mathcal{I}^{\mathcal{H}_v^\infty})^{inj}} = \|\iota_F \circ (h \cdot y)\|_{\mathcal{I}^{\mathcal{H}_v^\infty}} = \|h \cdot \iota_F(y)\|_{\mathcal{I}^{\mathcal{H}_v^\infty}} = \|h\|_v \|\iota_F(y)\| = \|h\|_v \|y\|.$$

(P3) Let $V \subseteq E$ be an open set such that $V \subseteq U$, $h \in \mathcal{H}(V,U)$ with

$$c_v(h) := \sup_{x \in V} \frac{v(x)}{v(h(x))} < \infty,$$

$f \in \mathcal{I}^{\mathcal{H}_v^\infty}(U,F)$ and $T \in \mathcal{L}(F,G)$, where G is a complex Banach space. Clearly, $\iota_F \circ f \in \mathcal{I}^{\mathcal{H}_v^\infty}(U, \ell_\infty(B_{F^*}))$. Since ι_F is an into linear isometry, there exists $S \in \mathcal{L}(\ell_\infty(B_{F^*}), \ell_\infty(B_{G^*}))$ such that $S \circ \iota_F = \iota_G \circ T$ and $\|S\| = \|\iota_G \circ T\|$ by the metric extension property of $\ell_\infty(B_{F^*})$ (see, for example, [23, Proposition C.3.2.1]). From $\iota_G \circ (T \circ f \circ h) = S \circ (\iota_F \circ f) \circ h \in \mathcal{I}^{\mathcal{H}_v^\infty}(U, \ell_\infty(B_{G^*}))$, we infer that $T \circ f \circ h \in (\mathcal{I}^{\mathcal{H}_v^\infty})^{inj}(U,G)$ with

$$\begin{aligned}\|T \circ f \circ h\|_{(\mathcal{I}^{\mathcal{H}_v^\infty})^{inj}} &= \|\iota_G \circ T \circ f \circ h\|_{\mathcal{I}^{\mathcal{H}_v^\infty}} = \|S \circ \iota_F \circ f \circ h\|_{\mathcal{I}^{\mathcal{H}_v^\infty}} \\ &\leq \|S\| \, \|\iota_F \circ f\|_{\mathcal{I}^{\mathcal{H}_v^\infty}} c_v(h) = \|\iota_G \circ T\| \, \|f\|_{(\mathcal{I}^{\mathcal{H}_v^\infty})^{inj}} c_v(h) \\ &\leq \|T\| \, \|f\|_{(\mathcal{I}^{\mathcal{H}_v^\infty})^{inj}} c_v(h)\end{aligned}$$

(I) Let $f \in \mathcal{H}_v^\infty(U,F)$ so that $\iota \circ f \in (\mathcal{I}^{\mathcal{H}_v^\infty})^{inj}(U,G)$ for any into linear isometry $\iota \colon F \to G$. The metric extension property of $\ell_\infty(B_{F^*})$ provides a $P \in \mathcal{L}(\ell_\infty(B_{G^*}), \ell_\infty(B_{F^*}))$ so that $P \circ \iota_G \circ \iota = \iota_F$ and $\|P\| = \|\iota_F\| = 1$. The conditions $\iota_G \circ \iota \circ f \in \mathcal{I}^{\mathcal{H}_v^\infty}(U, \ell_\infty(B_{G^*}))$ and $\iota_F \circ f = P \circ \iota_G \circ \iota \circ f$ imply $\iota_F \circ f \in \mathcal{I}^{\mathcal{H}_v^\infty}(U, \ell_\infty(B_{F^*}))$, and so $f \in (\mathcal{I}^{\mathcal{H}_v^\infty})^{inj}(U,F)$ with

$$\begin{aligned}\|f\|_{(\mathcal{I}^{\mathcal{H}_v^\infty})^{inj}} &= \|\iota_F \circ f\|_{\mathcal{I}^{\mathcal{H}_v^\infty}} = \|P \circ \iota_G \circ \iota \circ f\|_{\mathcal{I}^{\mathcal{H}_v^\infty}} \\ &\leq \|P\| \, \|\iota_G \circ \iota \circ f\|_{\mathcal{I}^{\mathcal{H}_v^\infty}} = \|\iota \circ f\|_{(\mathcal{I}^{\mathcal{H}_v^\infty})^{inj}} \leq \|f\|_{(\mathcal{I}^{\mathcal{H}_v^\infty})^{inj}}.\end{aligned}$$

On a hand, the ideal property of $[\mathcal{I}^{\mathcal{H}_v^\infty}, \|\cdot\|_{\mathcal{I}^{\mathcal{H}_v^\infty}}]$ yields

$$[\mathcal{I}^{\mathcal{H}_v^\infty}, \|\cdot\|_{\mathcal{I}^{\mathcal{H}_v^\infty}}] \leq [(\mathcal{I}^{\mathcal{H}_v^\infty})^{inj}, \|\cdot\|_{(\mathcal{I}^{\mathcal{H}_v^\infty})^{inj}}].$$

On the other hand, suppose $[\mathcal{J}^{\mathcal{H}_v^\infty}, \|\cdot\|_{\mathcal{J}^{\mathcal{H}_v^\infty}}]$ is an injective normed weighted holomorphic ideal so that $[\mathcal{I}^{\mathcal{H}_v^\infty}, \|\cdot\|_{\mathcal{I}^{\mathcal{H}_v^\infty}}] \leq [\mathcal{J}^{\mathcal{H}_v^\infty}, \|\cdot\|_{\mathcal{J}^{\mathcal{H}_v^\infty}}]$. If $f \in (\mathcal{I}^{\mathcal{H}_v^\infty})^{inj}(U,F)$, one has that $\iota_F \circ f \in \mathcal{I}^{\mathcal{H}_v^\infty}(U, \ell_\infty(B_{F^*})) \subseteq \mathcal{J}^{\mathcal{H}_v^\infty}(U, \ell_\infty(B_{F^*}))$, hence $f \in \mathcal{J}^{\mathcal{H}_v^\infty}(U,F)$ with $\|f\|_{\mathcal{J}^{\mathcal{H}_v^\infty}} = \|\iota_F \circ f\|_{\mathcal{J}^{\mathcal{H}_v^\infty}}$ by the injectivity of $\mathcal{J}^{\mathcal{H}_v^\infty}$, and so $\|f\|_{\mathcal{J}^{\mathcal{H}_v^\infty}} = \|\iota_F \circ f\|_{\mathcal{J}^{\mathcal{H}_v^\infty}} \leq \|\iota_F \circ f\|_{\mathcal{I}^{\mathcal{H}_v^\infty}} = \|f\|_{(\mathcal{I}^{\mathcal{H}_v^\infty})^{inj}}$. The uniqueness of $[(\mathcal{I}^{\mathcal{H}_v^\infty})^{inj}, \|\cdot\|_{(\mathcal{I}^{\mathcal{H}_v^\infty})^{inj}}]$ follows easily and this completes the proof. □

Based on the linear and polynomial variants in [17, Proposition 19.2.2] and [9, Corollary 2.4], respectively, the injectivity of a normed weighted holomorphic ideal is characterized by the coincidence with its injective hull.

Corollary 2.1. *Let $[\mathcal{I}^{\mathcal{H}_v^\infty}, \|\cdot\|_{\mathcal{I}^{\mathcal{H}_v^\infty}}]$ be a normed weighted holomorphic ideal. The following are equivalent:*

(1) $[\mathcal{I}^{\mathcal{H}_v^\infty}, \|\cdot\|_{\mathcal{I}^{\mathcal{H}_v^\infty}}]$ is injective.
(2) $[\mathcal{I}^{\mathcal{H}_v^\infty}, \|\cdot\|_{\mathcal{I}^{\mathcal{H}_v^\infty}}] = [(\mathcal{I}^{\mathcal{H}_v^\infty})^{inj}, \|\cdot\|_{(\mathcal{I}^{\mathcal{H}_v^\infty})^{inj}}]$.

□

Influenced by the hull procedure for the family of normed operator ideals – introduced by Pietsch in [23, Section 8.1] –, we obtain that the correspondence $\mathcal{I}^{\mathcal{H}_v^\infty} \mapsto (\mathcal{I}^{\mathcal{H}_v^\infty})^{inj}$ is a hull procedure in the weighted holomorphic setting.

Proposition 2.2. *If $[\mathcal{I}^{\mathcal{H}_v^\infty}, \|\cdot\|_{\mathcal{I}^{\mathcal{H}_v^\infty}}]$ and $[\mathcal{J}^{\mathcal{H}_v^\infty}, \|\cdot\|_{\mathcal{J}^{\mathcal{H}_v^\infty}}]$ are normed (Banach) weighted holomorphic ideals, then:*

1. *$[(\mathcal{I}^{\mathcal{H}_v^\infty})^{inj}, \|\cdot\|_{(\mathcal{I}^{\mathcal{H}_v^\infty})^{inj}}]$ is a normed (Banach) weighted holomorphic ideal,*
2. *$[(\mathcal{I}^{\mathcal{H}_v^\infty})^{inj}, \|\cdot\|_{(\mathcal{I}^{\mathcal{H}_v^\infty})^{inj}}] \leq [(\mathcal{J}^{\mathcal{H}_v^\infty})^{inj}, \|\cdot\|_{(\mathcal{J}^{\mathcal{H}_v^\infty})^{inj}}]$ if $[\mathcal{I}^{\mathcal{H}_v^\infty}, \|\cdot\|_{\mathcal{I}^{\mathcal{H}_v^\infty}}] \leq [\mathcal{J}^{\mathcal{H}_v^\infty}, \|\cdot\|_{\mathcal{J}^{\mathcal{H}_v^\infty}}]$,*
3. *$[((\mathcal{I}^{\mathcal{H}_v^\infty})^{inj})^{inj}, \|\cdot\|_{((\mathcal{I}^{\mathcal{H}_v^\infty})^{inj})^{inj}}] = [(\mathcal{I}^{\mathcal{H}_v^\infty})^{inj}, \|\cdot\|_{(\mathcal{I}^{\mathcal{H}_v^\infty})^{inj}}]$,*
4. *$[\mathcal{I}^{\mathcal{H}_v^\infty}, \|\cdot\|_{\mathcal{I}^{\mathcal{H}_v^\infty}}] \leq [(\mathcal{I}^{\mathcal{H}_v^\infty})^{inj}, \|\cdot\|_{(\mathcal{I}^{\mathcal{H}_v^\infty})^{inj}}]$.*

Proof. (i) and (iv) are deduced from Proposition 2.1, (iii) from Corollary 2.1, and (ii) follows as in the last part of the proof of Proposition 2.1. □

2.2. **The domination property.** The injective hull of a normed weighted holomorphic ideal can be characterized by the following domination property. This result is based on both the linear and polynomial versions stated respectively in [8, Lemma 3.1] and [9, Theorem 3.4].

Theorem 2.1. *Let $[\mathcal{I}^{\mathcal{H}_v^\infty}, \|\cdot\|_{\mathcal{I}^{\mathcal{H}_v^\infty}}]$ be a normed weighted holomorphic ideal, let F be a complex Banach space and let $f \in \mathcal{H}_v^\infty(U, F)$. The following assertions are equivalent:*

1. *f belongs to $(\mathcal{I}^{\mathcal{H}_v^\infty})^{inj}(U, F)$.*
2. *There exists a complex normed space G and a mapping $g \in \mathcal{I}^{\mathcal{H}_v^\infty}(U, G)$ such that*
$$\left\|\sum_{i=1}^n \lambda_i v(x_i) f(x_i)\right\| \leq \left\|\sum_{i=1}^n \lambda_i v(x_i) g(x_i)\right\|$$
for all $n \in \mathbb{N}$, $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ and $x_1, \ldots, x_n \in U$.

In this case, $\|f\|_{(\mathcal{I}^{\mathcal{H}_v^\infty})^{inj}} = \inf\{\|g\|_{\mathcal{I}^{\mathcal{H}_v^\infty}}\}$, where the infimum is taken over all spaces G and all mappings $g \in \mathcal{I}^{\mathcal{H}_v^\infty}(U, G)$ as in (ii), and this infimum is attained.

Proof. $(i) \Rightarrow (ii)$: Suppose that $f \in (\mathcal{I}^{\mathcal{H}_v^\infty})^{inj}(U, F)$. Take $G = \ell_\infty(B_{F^*})$ and $g = \iota_F \circ f$. Clearly, $g \in \mathcal{I}^{\mathcal{H}_v^\infty}(U, G)$ with $\|g\|_{\mathcal{I}^{\mathcal{H}_v^\infty}} = \|\iota_F \circ f\|_{\mathcal{I}^{\mathcal{H}_v^\infty}} = \|f\|_{(\mathcal{I}^{\mathcal{H}_v^\infty})^{inj}}$. Set $n \in \mathbb{N}$, $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ and $x_1, \ldots, x_n \in U$. An application of Hahn–Banach Theorem yields
$$\begin{aligned}\left\|\sum_{i=1}^n \lambda_i v(x_i) f(x_i)\right\| &= \sup_{y^* \in B_{F^*}} \left|\left\langle \sum_{i=1}^n \lambda_i v(x_i) f(x_i), y^*\right\rangle\right| = \sup_{y^* \in B_{F^*}} \left|\sum_{i=1}^n \lambda_i v(x_i) y^*(f(x_i))\right| \\ &= \sup_{y^* \in B_{F^*}} \left|\sum_{i=1}^n \lambda_i v(x_i) \langle \iota_F(f(x_i)), y^*\rangle\right| = \sup_{y^* \in B_{F^*}} \left|\sum_{i=1}^n \lambda_i v(x_i) \langle g(x_i), y^*\rangle\right| \\ &= \sup_{y^* \in B_{F^*}} \left|\left\langle \sum_{i=1}^n \lambda_i v(x_i) g(x_i), y^*\right\rangle\right| = \left\|\sum_{i=1}^n \lambda_i v(x_i) g(x_i)\right\|.\end{aligned}$$

$(ii) \Rightarrow (i)$: Let G and g be as in (ii). Take $G_0 = \mathrm{lin}(g(U)) \subseteq G$ and $T_0 \colon G_0 \to F$ given by
$$T_0\left(\sum_{i=1}^n \lambda_i v(x_i) g(x_i)\right) = \sum_{i=1}^n \lambda_i v(x_i) f(x_i)$$

for all $n \in \mathbb{N}$, $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ and $x_1, \ldots, x_n \in U$. Note that T_0 is well defined since

$$\begin{aligned}\sum_{i=1}^{n} \lambda_i v(x_i) g(x_i) = \sum_{j=1}^{m} \alpha_j v(x_j) g(x_j) &\Rightarrow \left| \sum_{i=1}^{n} \lambda_i v(x_i) g(x_i) - \sum_{j=1}^{m} \alpha_j v(x_j) g(x_j) \right| = 0 \\ &\Rightarrow \left| \sum_{i=1}^{n} \lambda_i v(x_i) f(x_i) - \sum_{j=1}^{m} \alpha_j v(x_j) f(x_j) \right| = 0 \\ &\Rightarrow \sum_{i=1}^{n} \lambda_i v(x_i) f(x_i) = \sum_{j=1}^{m} \alpha_j v(x_j) f(x_j),\end{aligned}$$

by using the inequality in (ii). The linearity of T_0 is clear, and since

$$\left\| T_0 \left(\sum_{i=1}^{n} \lambda_i v(x_i) g(x_i) \right) \right\| = \left\| \sum_{i=1}^{n} \lambda_i v(x_i) f(x_i) \right\| \leq \left\| \sum_{i=1}^{n} \lambda_i v(x_i) g(x_i) \right\|$$

for all $n \in \mathbb{N}$, $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ and $x_1, \ldots, x_n \in U$, we deduce that T_0 is continuous with $\|T_0\| \leq 1$. There exists a unique operator $T \in \mathcal{L}(\overline{G_0}, F)$ such that $T|_{G_0} = T_0$ and $\|T\| = \|T_0\|$. If $\iota \colon \overline{G_0} \to G$ is the inclusion operator, the metric extension property of $\ell_\infty(B_{G^*})$ yields an operator $S \in \mathcal{L}(G, \ell_\infty(B_{F^*}))$ so that $\iota_F \circ T = S \circ \iota$ and $\|S\| = \|\iota_F \circ T\|$. Since

$$(T \circ g)(x) = T(g(x)) = T_0(g(x)) = f(x)$$

for all $x \in U$, we have $T \circ g = f$, and thus $\iota_F \circ f = \iota_F \circ T \circ g = S \circ \iota \circ g = S \circ g$. Since $g \in \mathcal{I}^{\mathcal{H}_v^\infty}(U, G)$, the ideal property of $\mathcal{I}^{\mathcal{H}_v^\infty}$ shows that $\iota_F \circ f \in \mathcal{I}^{\mathcal{H}_v^\infty}(U, \ell_\infty(B_{F^*}))$, that is, $f \in (\mathcal{I}^{\mathcal{H}_v^\infty})^{inj}(U, F)$ with

$$\|f\|_{(\mathcal{I}^{\mathcal{H}_v^\infty})^{inj}} = \|\iota_F \circ f\|_{\mathcal{I}^{\mathcal{H}_v^\infty}} \leq \|S\| \, \|g\|_{\mathcal{I}^{\mathcal{H}_v^\infty}} \leq \|g\|_{\mathcal{I}^{\mathcal{H}_v^\infty}} .$$

Taking the infimum over all G's and g's as in (ii) yields that $\|f\|_{(\mathcal{I}^{\mathcal{H}_v^\infty})^{inj}} \leq \inf\{\|g\|_{\mathcal{I}^{\mathcal{H}_v^\infty}}\}$. □

The combination of Corollary 2.1 and Theorem 2.1 immediately provides the next characterization of the injectivity of a normed weighted holomorphic ideal, that can be compared with its linear version [8, Lemma 3.1] and its polynomial version [9, Theorem 3.4].

Corollary 2.2. *Let $[\mathcal{I}^{\mathcal{H}_v^\infty}, \|\cdot\|_{\mathcal{I}^{\mathcal{H}_v^\infty}}]$ be a normed weighted holomorphic ideal. Then $[\mathcal{I}^{\mathcal{H}_v^\infty}, \|\cdot\|_{\mathcal{I}^{\mathcal{H}_v^\infty}}]$ is injective if, and only if, given complex Banach spaces F, G and mappings $f \in \mathcal{H}_v^\infty(U, F)$, $g \in \mathcal{I}^{\mathcal{H}_v^\infty}(U, G)$ such that*

$$\left\| \sum_{i=1}^{n} \lambda_i v(x_i) f(x_i) \right\| \leq \left\| \sum_{i=1}^{n} \lambda_i v(x_i) g(x_i) \right\|$$

for all $n \in \mathbb{N}$, $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ and $x_1, \ldots, x_n \in U$, then $f \in \mathcal{I}^{\mathcal{H}_v^\infty}(U, F)$ and $\|f\|_{\mathcal{I}^{\mathcal{H}_v^\infty}} = \inf\{\|g\|_{\mathcal{I}^{\mathcal{H}_v^\infty}}\}$, where the infimum is taken over all complex Banach spaces G and all such mappings g. □

2.3. **The injective hull of composition ideals of weighted holomorphic mappings.** According to [10, Definition 2.5], given a normed operator ideal $[\mathcal{I}, \|\cdot\|_{\mathcal{I}}]$, a map $f \in \mathcal{H}(U, F)$ belongs to the composition ideal $\mathcal{I} \circ \mathcal{H}_v^\infty(U, F)$ if there exist a complex Banach space G, an operator $T \in \mathcal{I}(G, F)$ and a map $g \in \mathcal{H}_v^\infty(U, G)$ such that $f = T \circ g$. For any $f \in \mathcal{I} \circ \mathcal{H}_v^\infty(U, F)$, define

$$\|f\|_{\mathcal{I} \circ \mathcal{H}_v^\infty} = \inf\left\{ \|T\|_{\mathcal{I}} \, \|g\|_v \right\},$$

where the infimum is extended over all such factorizations of f. By [10, Proposition 2.6], we have that $[\mathcal{I} \circ \mathcal{H}_v^\infty, \|\cdot\|_{\mathcal{I} \circ \mathcal{H}_v^\infty}]$ is a normed weighted holomorphic ideal.

We now describe the injective hull of this ideal $[\mathcal{I} \circ \mathcal{H}_v^\infty, \|\cdot\|_{\mathcal{I}\circ\mathcal{H}_v^\infty}]$. Our approach requires some preliminaries about the linearization of weighted holomorphic maps. Following [3, 16], $\mathcal{G}_v^\infty(U)$ is the space of all linear functionals on $\mathcal{H}_v^\infty(U)$ whose restriction to $B_{\mathcal{H}_v^\infty(U)}$ is continuous for the compact-open topology. The following result collects the properties of $\mathcal{G}_v^\infty(U)$ that we will need later.

Theorem 2.2. [3, 7, 16, 21] *Let U be an open set of a complex Banach space E and v be a weight on U.*

(1) $\mathcal{G}_v^\infty(U)$ is a closed subspace of $\mathcal{H}_v^\infty(U)^*$, and the evaluation mapping $J_v\colon \mathcal{H}_v^\infty(U) \to \mathcal{G}_v^\infty(U)^*$, given by $J_v(f)(\phi) = \phi(f)$ for $\phi \in \mathcal{G}_v^\infty(U)$ and $f \in \mathcal{H}_v^\infty(U)$, is an isometric isomorphism.

(2) For each $x \in U$, the evaluation functional $\delta_x\colon \mathcal{H}_v^\infty(U) \to \mathbb{C}$, defined by $\delta_x(f) = f(x)$ for $f \in \mathcal{H}_v^\infty(U)$, is in $\mathcal{G}_v^\infty(U)$.

(3) The mapping $\Delta_v\colon U \to \mathcal{G}_v^\infty(U)$ given by $\Delta_v(x) = \delta_x$ is in $\mathcal{H}_v^\infty(U, \mathcal{G}_v^\infty(U))$ with $\|\Delta_v\|_v \leq 1$.

(4) $B_{\mathcal{G}_v^\infty(U)} = \overline{\mathrm{aco}}(\mathrm{At}_{\mathcal{G}_v^\infty(U)}) \subseteq \mathcal{H}_v^\infty(U)^*$ and $\mathcal{G}_v^\infty(U) = \overline{\mathrm{lin}}(\mathrm{At}_{\mathcal{G}_v^\infty(U)}) \subseteq \mathcal{H}_v^\infty(U)^*$, where $\mathrm{At}_{\mathcal{G}_v^\infty(U)} = \{v(x)\delta_x \colon x \in U\}$.

(5) For each $\phi \in \mathrm{lin}(\mathrm{At}_{\mathcal{G}_v^\infty(U)})$, we have

$$\|\phi\| = \inf\left\{\sum_{i=1}^n |\lambda_i| : \phi = \sum_{i=1}^n \lambda_i v(x_i)\delta_{x_i}\right\}.$$

(6) For every complex Banach space F and every mapping $f \in \mathcal{H}_v^\infty(U, F)$, there exists a unique operator $T_f \in \mathcal{L}(\mathcal{G}_v^\infty(U), F)$ such that $T_f \circ \Delta_v = f$. Furthermore, $\|T_f\| = \|f\|_v$.

(7) For each $f \in \mathcal{H}_v^\infty(U, F)$, the mapping $f^t\colon F^* \to \mathcal{H}_v^\infty(U)$, defined by $f^t(y^*) = y^* \circ f$ for all $y^* \in F^*$, is in $\mathcal{L}(F^*, \mathcal{H}_v^\infty(U))$ with $\|f^t\| = \|f\|_v$ and $f^t = J_v^{-1} \circ (T_f)^*$, where $(T_f)^*\colon F^* \to \mathcal{G}_v^\infty(U)^*$ is the adjoint operator of T_f. □

For $v = 1_U$, where $1_U(x) = 1$ for all $x \in U$, it is usual to write $\mathcal{H}^\infty(U, F)$ (the Banach space of all bounded holomorphic mappings from U into F, under the supremum norm) instead of $\mathcal{H}_v^\infty(U, F)$, $\mathcal{H}^\infty(U)$ rather than $\mathcal{H}^\infty(U, \mathbb{C})$ and, following Mujica's notation in [21], $\mathcal{G}^\infty(U)$ instead of $\mathcal{G}_v^\infty(U)$.

Proposition 2.3. *Let $[\mathcal{I}, \|\cdot\|_{\mathcal{I}}]$ be an operator ideal. Then*

$$[(\mathcal{I} \circ \mathcal{H}_v^\infty)^{inj}, \|\cdot\|_{(\mathcal{I}\circ\mathcal{H}_v^\infty)^{inj}}] = [\mathcal{I}^{inj} \circ \mathcal{H}_v^\infty, \|\cdot\|_{\mathcal{I}^{inj}\circ\mathcal{H}_v^\infty}].$$

In particular, the weighted holomorphic ideal $[\mathcal{I}^{inj} \circ \mathcal{H}_v^\infty, \|\cdot\|_{\mathcal{I}^{inj}\circ\mathcal{H}_v^\infty}]$ is injective.

Proof. Let F be a complex Banach space and $f \in (\mathcal{I} \circ \mathcal{H}_v^\infty)^{inj}(U, F)$. Hence $\iota_F \circ f \in \mathcal{I} \circ \mathcal{H}_v^\infty(U, \ell_\infty(B_{F^*}))$, and so $\iota_F \circ f = T \circ g$ for some complex Banach space G, an operator $T \in \mathcal{I}(G, \ell_\infty(B_{F^*}))$ and a map $g \in \mathcal{H}_v^\infty(U, G)$. By Theorem 2.2, we can find two operators $T_f \in \mathcal{L}(\mathcal{G}_v^\infty(U), F)$ and $T_g \in \mathcal{L}(\mathcal{G}_v^\infty(U), G)$ with $\|T_f\| = \|f\|_v$ and $\|T_g\| = \|g\|$ such that $T_f \circ \Delta_v = f$ and $T_g \circ \Delta_v = g$. Since $\mathcal{G}_v^\infty(U) = \overline{\mathrm{lin}}(\Delta_v(U)) \subseteq \mathcal{H}_v^\infty(U)^*$ and

$$\iota_F \circ T_f \circ \Delta_v = \iota_F \circ f = T \circ g = T \circ T_g \circ \Delta_v,$$

it follows that $\iota_F \circ T_f = T \circ T_g$, and thus $\iota_F \circ T_f \in \mathcal{I}(\mathcal{G}_v^\infty(U), \ell_\infty(B_{F^*}))$, that is, $T_f \in \mathcal{I}^{inj}(\mathcal{G}_v^\infty(U), F)$. Hence $f = T_f \circ \Delta_v \in \mathcal{I}^{inj} \circ \mathcal{H}_v^\infty(U, F)$. Moreover,

$$\|f\|_{\mathcal{I}^{inj}\circ\mathcal{H}_v^\infty} \leq \|T_f\|_{\mathcal{I}^{inj}} \|\Delta_v\| \leq \|T_f\|_{\mathcal{I}^{inj}} = \|\iota_F \circ T_f\|_{\mathcal{I}} \leq \|T\|_{\mathcal{I}} \|T_g\| = \|T\|_{\mathcal{I}} \|g\|_v,$$

and passing to the infimum over all the factorizations of $\iota_F \circ f$ yields

$$\|f\|_{\mathcal{I}^{inj}\circ\mathcal{H}_v^\infty} \leq \|\iota_F \circ f\|_{\mathcal{I}\circ\mathcal{H}_v^\infty} = \|f\|_{(\mathcal{I}\circ\mathcal{H}_v^\infty)^{inj}}.$$

Conversely, let $f \in \mathcal{I}^{inj} \circ \mathcal{H}_v^\infty(U,F)$. Hence $f = T \circ g$ for some complex Banach space G, $T \in \mathcal{I}^{inj}(G,F)$ and $g \in \mathcal{H}_v^\infty(U,G)$. Therefore $\iota_F \circ f = (\iota_F \circ T) \circ g \in \mathcal{I} \circ \mathcal{H}_v^\infty(U, \ell_\infty(B_{F^*})$, and thus $f \in (\mathcal{I} \circ \mathcal{H}_v^\infty)^{inj}(U,F)$ with

$$\|f\|_{(\mathcal{I}\circ\mathcal{H}_v^\infty)^{inj}} = \|\iota_F \circ f\|_{\mathcal{I}\circ\mathcal{H}_v^\infty} = \|\iota_F \circ T \circ g\|_{\mathcal{I}\circ\mathcal{H}_v^\infty} \le \|\iota_F \circ T\|_{\mathcal{I}} \|g\|_v = \|T\|_{\mathcal{I}^{inj}} \|g\|_v.$$

Taking the infimum over all the factorizations of f, we conclude that

$$\|f\|_{(\mathcal{I}\circ\mathcal{H}_v^\infty)^{inj}} \le \|f\|_{\mathcal{I}^{inj}\circ\mathcal{H}_v^\infty}.$$

□

From Proposition 2.3 and Corollary 2.1, we deduce the following.

Corollary 2.3. *Let $[\mathcal{I}, \|\cdot\|_{\mathcal{I}}]$ be an injective normed operator ideal. Then $[\mathcal{I} \circ \mathcal{H}_v^\infty, \|\cdot\|_{\mathcal{I}\circ\mathcal{H}_v^\infty}]$ is an injective weighted holomorphic ideal.* □

For $\mathcal{I} = \mathcal{F}, \overline{\mathcal{F}}, \mathcal{K}, \mathcal{W}, \mathcal{S}, \mathcal{R}, \mathcal{AS}$ and Banach spaces E, F, we will denote by $\mathcal{I}(E,F)$ the linear space of all finite-rank (approximable, compact, weakly compact, separable, Rosenthal, Asplund) bounded linear operators from E to F, respectively. The components $\mathcal{I}(E,F)$, equipped with the operator canonical norm $\|\cdot\|$, generate a normed operator ideal (see [23]). For a map $f \in \mathcal{H}(U,F)$, the v-range of f is the set

$$(vf)(U) = \{v(x)f(x) \colon x \in U\} \subseteq F.$$

Note that f belongs to $\mathcal{H}_v^\infty(U,F)$ if and only if $(vf)(U)$ is a norm-bounded subset of F. This motivates the following concepts.

Definition 2.1. *Let U be an open set of a complex Banach space E, let v be a weight on U and let F be a complex Banach space.*

A mapping $f \in \mathcal{H}_v^\infty(U,F)$ is said to be v-compact (resp., v-weakly compact, v-separable, v-Rosenthal, v-Asplund) if $(vf)(U)$ is a relatively compact (resp., relatively weakly compact, separable, Rosenthal, Asplund) subset of F.

A mapping $f \in \mathcal{H}_v^\infty(U,F)$ is said to have finite dimensional v-rank if $(vf)(U)$ is a finite dimensional subspace of F, and f is said to be v-approximable if it is the limit in the v-norm of a sequence of finite v-rank weighted holomorphic mappings of $\mathcal{H}_v^\infty(U,F)$.

For $\mathcal{I} = \mathcal{F}, \overline{\mathcal{F}}, \mathcal{K}, \mathcal{W}, \mathcal{S}, \mathcal{R}, \mathcal{AS}$, $\mathcal{H}_{v\mathcal{I}}^\infty(U,F)$ stand for the linear space of all finite v-rank (resp., v-approximable, v-compact, v-weakly compact, v-separable, v-Rosenthal, v-Asplund) weighted holomorphic mappings from U into F.

The same proofs of Theorem 2.9 and Corollary 2.10 in [10] yield the following two results.

Theorem 2.3. *Let $f \in \mathcal{H}_v^\infty(U,F)$ and $\mathcal{I} = \mathcal{F}, \overline{\mathcal{F}}, \mathcal{K}, \mathcal{W}, \mathcal{S}, \mathcal{R}, \mathcal{AS}$. For the normed operator ideal $[\mathcal{I}, \|\cdot\|_{\mathcal{I}}]$, the following are equivalent:*

(1) f belongs to $\mathcal{H}_{v\mathcal{I}}^\infty(U,F)$.
(2) T_f belongs to $\mathcal{I}(\mathcal{G}_v^\infty(U),F)$.

In this case, $\|f\|_v = \|T_f\|_I$. Furthermore, the correspondence $f \mapsto T_f$ is an isometric isomorphism from $(\mathcal{H}_{v\mathcal{I}}^\infty(U,F), \|\cdot\|_v)$ onto $(\mathcal{I}(\mathcal{G}_v^\infty(U),F), \|\cdot\|_I)$. □

Corollary 2.4. *$[\mathcal{H}_{v\mathcal{I}}^\infty, \|\cdot\|_v] = [\mathcal{I} \circ \mathcal{H}_v^\infty, \|\cdot\|_{\mathcal{I}\circ\mathcal{H}_v^\infty}]$ for $\mathcal{I} = \mathcal{F}, \overline{\mathcal{F}}, \mathcal{K}, \mathcal{W}, \mathcal{S}, \mathcal{R}, \mathcal{AS}$. As a consequence,*

(1) $[\mathcal{H}_{v\mathcal{I}}^\infty, \|\cdot\|_v]$ is a Banach weighted holomorphic ideal for $\mathcal{I} = \overline{\mathcal{F}}, \mathcal{K}, \mathcal{W}, \mathcal{S}, \mathcal{R}, \mathcal{AS}$,
(2) $[\mathcal{H}_{v\mathcal{F}}^\infty, \|\cdot\|_v]$ is a normed weighted holomorphic ideal.

□

We are in a position to establish the injectivity of these ideals.

Corollary 2.5. *For $\mathcal{I} = \mathcal{F}, \mathcal{K}, \mathcal{W}, \mathcal{S}, \mathcal{R}, \mathcal{AS}$, the weighted holomorphic ideal $[\mathcal{H}^\infty_{v\mathcal{I}}, \|\cdot\|_v]$ is injective.*

Proof. Applying Corollary 2.4 for the first and fourth equalities, Proposition 2.3 for the second, and [15] for the third, one has

$$\begin{aligned}[(\mathcal{H}^\infty_{v\mathcal{I}})^{inj}, (\|\cdot\|_v)^{inj}] &= [(\mathcal{I}\circ\mathcal{H}^\infty_v)^{inj}, \|\cdot\|_{(\mathcal{I}\circ\mathcal{H}^\infty_v)^{inj}}] = [\mathcal{I}^{inj}\circ\mathcal{H}^\infty_v, \|\cdot\|_{\mathcal{I}^{inj}\circ\mathcal{H}^\infty_v}] \\ &= [\mathcal{I}\circ\mathcal{H}^\infty_v, \|\cdot\|_{\mathcal{I}\circ\mathcal{H}^\infty_v}] = [\mathcal{H}^\infty_{v\mathcal{I}}, \|\cdot\|_v].\end{aligned}$$

□

We now identify the injective hull of the ideal $\mathcal{H}^\infty_{v\overline{\mathcal{F}}}$.

Corollary 2.6. $[(\mathcal{H}^\infty_{v\overline{\mathcal{F}}})^{inj}, (\|\cdot\|_v)^{inj}] = [\mathcal{H}^\infty_{v\mathcal{K}}, \|\cdot\|_v]$.

Proof. As in the preceding proof, one now has

$$\begin{aligned}[(\mathcal{H}^\infty_{v\overline{\mathcal{F}}})^{inj}, (\|\cdot\|_v)^{inj}] &= [(\overline{\mathcal{F}}\circ\mathcal{H}^\infty_v)^{inj}, \|\cdot\|_{(\overline{\mathcal{F}}\circ\mathcal{H}^\infty_v)^{inj}}] = [(\overline{\mathcal{F}})^{inj}\circ\mathcal{H}^\infty_v, \|\cdot\|_{(\overline{\mathcal{F}})^{inj}\circ\mathcal{H}^\infty_v}] \\ &= [\mathcal{K}\circ\mathcal{H}^\infty_v, \|\cdot\|_{\mathcal{K}\circ\mathcal{H}^\infty_v}] = [\mathcal{H}^\infty_{v\mathcal{K}}, \|\cdot\|_v]\end{aligned}$$

by Corollary 2.4 for the first and fourth equalities, Proposition 2.3 for the second, and the equality $[(\overline{\mathcal{F}})^{inj}, \|\cdot\|_{inj}] = [\mathcal{K}, \|\cdot\|]$ by [23, Proposition 4.6.13] for the third. □

2.4. **The injective hull of dual ideals of weighted holomorphic mappings.** Following [23, Section 4.4], given a normed operator ideal $[\mathcal{I}, \|\cdot\|_\mathcal{I}]$, the components

$$\mathcal{I}^{\mathrm{dual}}(E,F) := \{T \in \mathcal{L}(E,F)\colon T^* \in \mathcal{I}(F^*, E^*)\}$$

for any normed spaces E and F, endowed with the norm

$$\|T\|_{\mathcal{I}^{\mathrm{dual}}} = \|T^*\|_\mathcal{I} \qquad (T \in \mathcal{I}^{\mathrm{dual}}(E,F)),$$

define a normed operator ideal, $[\mathcal{I}^{\mathrm{dual}}, \|\cdot\|_{\mathcal{I}^d}]$, called dual ideal of $\mathcal{I}$. Moreover, $[\mathcal{I}, \|\cdot\|_\mathcal{I}]$ is said to be symmetric and completely symmetric if $[\mathcal{I}, \|\cdot\|_\mathcal{I}] \leq [\mathcal{I}^{\mathrm{dual}}, \|\cdot\|_{\mathcal{I}^{\mathrm{dual}}}]$ and $[\mathcal{I}, \|\cdot\|_\mathcal{I}] = [\mathcal{I}^{\mathrm{dual}}, \|\cdot\|_{\mathcal{I}^{\mathrm{dual}}}]$, respectively.

Based on the notion of transpose of a weighted holomorphic map (see Theorem 2.2), we introduce the concept of dual weighted holomorphic ideal of an operator ideal $\mathcal{I}$.

Definition 2.2. *Let $\mathcal{I}$ be an operator ideal. For any open subset U of a complex Banach space E, any weight v on U and any complex Banach space F, we define*

$$\mathcal{I}^{\mathcal{H}^\infty_v\text{-dual}}(U,F) = \{f \in \mathcal{H}^\infty_v(U,F)\colon f^t \in \mathcal{I}(F^*, \mathcal{H}^\infty_v(U))\}$$

If $[\mathcal{I}, \|\cdot\|_\mathcal{I}]$ is a normed operator ideal, we set

$$\|f\|_{\mathcal{I}^{\mathcal{H}^\infty_v\text{-dual}}} = \|f^t\|_\mathcal{I} \qquad (f \in \mathcal{I}^{\mathcal{H}^\infty_v\text{-dual}}(U,F)).$$

We now show that $[\mathcal{I}^{\mathcal{H}^\infty_v\text{-dual}}, \|\cdot\|_{\mathcal{I}^{\mathcal{H}^\infty_v\text{-dual}}}]$ is in fact a normed weighted holomorphic ideal.

Theorem 2.4. *Let $\mathcal{I}$ be an operator ideal. The following statements about a mapping $f \in \mathcal{H}^\infty_v(U,F)$ are equivalent:*

(1) f belongs to $\mathcal{I}^{\mathcal{H}^\infty_v\text{-dual}}(U,F)$.
(2) f belongs to $\mathcal{I}^{\mathrm{dual}}\circ\mathcal{H}^\infty_v(U,F)$.

If in addition $(\mathcal{I}, \|\cdot\|_\mathcal{I})$ is a normed operator ideal, then

$$\|f\|_{\mathcal{I}^{\mathcal{H}^\infty_v\text{-dual}}} = \|f\|_{\mathcal{I}^{dual}\circ\mathcal{H}^\infty_v} \qquad (f \in \mathcal{I}^{\mathcal{H}^\infty_v\text{-dual}}(U,F)).$$

Proof. $(i) \Rightarrow (ii)$: Let $f \in \mathcal{I}^{\mathcal{H}_v^\infty\text{-dual}}(U,F)$. Then $f^t \in \mathcal{I}(F^*,\mathcal{H}_v^\infty(U))$. By Theorem 2.2, we can take $T_f \in \mathcal{L}(\mathcal{G}_v^\infty(U),F)$ such that $T_f \circ \Delta_v = f$ and also $(T_f)^* = J_v \circ f^t$. Hence $(T_f)^* \in \mathcal{I}(F^*,\mathcal{G}_v^\infty(U)^*)$ and therefore $T_f \in \mathcal{I}^{\mathrm{dual}}(\mathcal{G}_v^\infty(U),F)$. Thus, by [10, Theorem 2.7] we have $f \in \mathcal{I}^{\mathrm{dual}} \circ \mathcal{H}_v^\infty(U,F)$ with $\|f\|_{\mathcal{I}^{\mathrm{dual}}\circ\mathcal{H}_v^\infty} = \|T_f\|_{\mathcal{I}^{\mathrm{dual}}}$. Further,

$$\|f\|_{\mathcal{I}^{\mathrm{dual}}\circ\mathcal{H}_v^\infty} = \|T_f\|_{\mathcal{I}^{\mathrm{dual}}} = \|(T_f)^*\|_{\mathcal{I}} = \|J_v \circ f^t\|_{\mathcal{I}} \leq \|J_v\|\|f^t\|_{\mathcal{I}} = \|f\|_{\mathcal{I}^{\mathcal{H}_v^\infty\text{-dual}}}.$$

$(ii) \Rightarrow (i)$: Let $f \in \mathcal{I}^{\mathrm{dual}} \circ \mathcal{H}_v^\infty(U,F)$. Then there are a complex Banach space G, a map $g \in \mathcal{H}_v^\infty(U,G)$ and an operator $T \in \mathcal{I}^{\mathrm{dual}}(G,F)$ such that $f = T \circ g$. Given $y^* \in F^*$, we have

$$f^t(y^*) = (T\circ g)^t(y^*) = y^* \circ (T \circ g) = (y^* \circ T) \circ g = T^*(y^*) \circ g = g^t(T^*(y^*)) = (g^t \circ T^*)(y^*),$$

and thus $f^t = g^t \circ T^*$. Since $T^* \in \mathcal{I}(F^*,G^*)$ and $g^t \in \mathcal{L}(G^*,\mathcal{H}_v^\infty(U))$, we obtain that $f^t \in \mathcal{I}(F^*,\mathcal{H}_v^\infty(U))$. Hence $f \in \mathcal{I}^{\mathcal{H}_v^\infty\text{-dual}}(U,F)$, and since

$$\|f\|_{\mathcal{I}^{\mathcal{H}_v^\infty\text{-dual}}} = \left\|f^t\right\|_{\mathcal{I}} = \left\|g^t \circ T^*\right\|_{\mathcal{I}} \leq \left\|g^t\right\| \|T^*\|_{\mathcal{I}} = \|g\|_v \|T\|_{\mathcal{I}^{\mathrm{dual}}},$$

by taking the infimum over all representations $T \circ g$ of f gives $\|f\|_{\mathcal{I}^{\mathcal{H}_v^\infty\text{-dual}}} \leq \|f\|_{\mathcal{I}^{\mathrm{dual}}\circ\mathcal{H}_v^\infty}$. □

Theorem 2.4 enables us to include the following description of the dual weighted holomorphic ideal of a completely symmetric normed operator ideal.

Corollary 2.7. $[\mathcal{I}^{\mathcal{H}_v^\infty\text{-dual}}, \|\cdot\|_{\mathcal{I}^{\mathcal{H}_v^\infty\text{-dual}}}] = [\mathcal{I}\circ\mathcal{H}_v^\infty, \|\cdot\|_{\mathcal{I}\circ\mathcal{H}_v^\infty}]$ *whenever* $[\mathcal{I}, \|\cdot\|_{\mathcal{I}}]$ *is a completely symmetric normed operator ideal.* □

The operator ideal $\mathcal{I} = \mathcal{F}, \overline{\mathcal{F}}, \mathcal{K}, \mathcal{W}$ is completely symmetric by [23, Proposition 4.4.7]. Then Corollaries 2.7 and 2.4 give us the following identifications.

Corollary 2.8. $[\mathcal{I}^{\mathcal{H}_v^\infty\text{-dual}}, \|\cdot\|_{\mathcal{I}^{\mathcal{H}_v^\infty\text{-dual}}}] = [\mathcal{H}_{v\mathcal{I}}^\infty, \|\cdot\|_v]$ *for* $\mathcal{I} = \mathcal{F}, \overline{\mathcal{F}}, \mathcal{K}, \mathcal{W}$. □

On the injectivity property, we can now establish the following.

Corollary 2.9. *If* $[\mathcal{I}, \|\cdot\|_{\mathcal{I}}]$ *is a completely symmetric injective normed operator ideal, then the weighted holomorphic ideal* $[\mathcal{I}^{\mathcal{H}_v^\infty\text{-dual}}, \|\cdot\|_{\mathcal{I}^{\mathcal{H}_v^\infty\text{-dual}}}]$ *is injective.*

Proof. Applying Theorem 2.4, Proposition 2.3 and the properties of $[\mathcal{I}, \|\cdot\|_{\mathcal{I}}]$, we have

$$\begin{aligned}
[(\mathcal{I}^{\mathcal{H}_v^\infty\text{-dual}})^{inj}, \|\cdot\|_{(\mathcal{I}^{\mathcal{H}_v^\infty\text{-dual}})^{inj}}] &= [(\mathcal{I}^{dual}\circ\mathcal{H}_v^\infty)^{inj}, \|\cdot\|_{(\mathcal{I}^{dual}\circ\mathcal{H}_v^\infty)^{inj}}] \\
&= [(\mathcal{I}^{dual})^{inj}\circ\mathcal{H}_v^\infty, \|\cdot\|_{(\mathcal{I}^{dual})^{inj}\circ\mathcal{H}_v^\infty}] \\
&= [\mathcal{I}^{inj}\circ\mathcal{H}_v^\infty, \|\cdot\|_{\mathcal{I}^{inj}\circ\mathcal{H}_v^\infty}] = [\mathcal{I}\circ\mathcal{H}_v^\infty, \|\cdot\|_{\mathcal{I}\circ\mathcal{H}_v^\infty}] \\
&= [\mathcal{I}^{dual}\circ\mathcal{H}_v^\infty, \|\cdot\|_{\mathcal{I}^{dual}\circ\mathcal{H}_v^\infty}] = [\mathcal{I}^{\mathcal{H}_v^\infty\text{-dual}}, \|\cdot\|_{\mathcal{I}^{\mathcal{H}_v^\infty\text{-dual}}}],
\end{aligned}$$

and the result follows from Corollary 2.1. □

Now, we describe the dual weighted holomorphic ideals of both the ideal $\mathcal{K}_p$ of p-compact operators [22] and the ideal $\mathcal{D}_p$ of Cohen strongly p-summing operators [12]. As usual, $\mathcal{N}_p$ denotes the ideal of p-nuclear operators, $\mathcal{I}_p$ the ideal of p-integral operators, and Π_p the ideal of absolutely p-summing operators (see [23]).

Corollary 2.10. *Let* $\mathcal{I}$ *and* $\mathcal{J}$ *be Banach operator ideals such that* $\mathcal{I}^{\mathrm{dual}} = \mathcal{J}^{inj}$. *Then* $\mathcal{I}^{\mathcal{H}_v^\infty\text{-dual}} = (\mathcal{J}\circ\mathcal{H}_v^\infty)^{inj}$. *As a consequence,* $\mathcal{K}_p^{\mathcal{H}_v^\infty\text{-dual}} = (\mathcal{N}_p\circ\mathcal{H}_v^\infty)^{inj}$ *and* $\mathcal{D}_p^{\mathcal{H}_v^\infty\text{-dual}} = (\mathcal{I}_{p^*}\circ\mathcal{H}_v^\infty)^{inj}$ *for any* $p \in (1,\infty)$, *where* p^* *denotes the Hölder conjugate index of* p.

Proof. The combination of Theorem 2.4 and Proposition 2.3 gives

$$\mathcal{I}^{\mathcal{H}_v^\infty\text{-dual}} = \mathcal{I}^{dual} \circ \mathcal{H}_v^\infty = \mathcal{J}^{inj} \circ \mathcal{H}_v^\infty = (\mathcal{J} \circ \mathcal{H}_v^\infty)^{inj}.$$

This equality yields the consequence in view that $\mathcal{K}_p^{\text{dual}} = \mathcal{N}_p^{inj}$ by [22, Theorem 6], and $\mathcal{D}_p^{\text{dual}} = \Pi_{p^*} = \mathcal{I}_{p^*}^{inj}$ by [12] and [22, Theorem 2.9.7]. □

2.5. **The closed injective hull of ideals of weighted holomorphic mappings.** According to [23, Section 4.2.1], given an operator ideal $\mathcal{I}$ and Banach spaces E and F, an operator $T \in \mathcal{L}(E, F)$ is in the closure of $\mathcal{I}(E, F)$ in $(\mathcal{L}(E, F), \|\cdot\|)$, denoted by $\overline{\mathcal{I}}(E, F)$, if there exists a sequence (T_n) in $\mathcal{I}(E, F)$ such that $\lim_{n\to\infty} \|T_n - T\| = 0$. In this way, the components $\overline{\mathcal{I}}(E, F)$ define an operator ideal $\overline{\mathcal{I}}$. This concept motivates the following in the setting of weighted holomorphic maps.

Definition 2.3. *Let U be an open set of a complex Banach space E, let v be a weight on U and let F be a complex Banach space. Given a weighted holomorphic ideal $\mathcal{I}^{\mathcal{H}_v^\infty}$, a map $f \in \mathcal{H}_v^\infty(U, F)$ is said to belong to the closure of $\mathcal{I}^{\mathcal{H}_v^\infty}(U, F)$ in $(\mathcal{H}_v^\infty(U, F), \|\cdot\|_v)$, and it is denoted by $f \in \overline{\mathcal{I}^{\mathcal{H}_v^\infty}}(U, F)$, if there exists a sequence (f_n) in $\mathcal{I}^{\mathcal{H}_v^\infty}(U, F)$ such that $\lim_{n\to\infty} \|f_n - f\|_v = 0$.*

It is easy to prove the following result.

Proposition 2.4. *Let $\mathcal{I}^{\mathcal{H}_v^\infty}$ be a weighted holomorphic ideal. Then $\overline{\mathcal{I}^{\mathcal{H}_v^\infty}}$ is a weighted holomorphic ideal containing $\mathcal{I}^{\mathcal{H}_v^\infty}$, and it is called the closure of $\mathcal{I}^{\mathcal{H}_v^\infty}$. We say that $\mathcal{I}^{\mathcal{H}_v^\infty}$ is closed if $\mathcal{I}^{\mathcal{H}_v^\infty} = \overline{\mathcal{I}^{\mathcal{H}_v^\infty}}$, and we call closed injective hull of $\mathcal{I}^{\mathcal{H}_v^\infty}$ – denoted by $(\overline{\mathcal{I}^{\mathcal{H}_v^\infty}})^{inj}$ – to the injective hull of the ideal $\overline{\mathcal{I}^{\mathcal{H}_v^\infty}}$.* □

The closed injective hull of a weighted holomorphic ideal of composition type admits the following description.

Proposition 2.5. *Let $[\mathcal{I}, \|\cdot\|_{\mathcal{I}}]$ be an operator ideal. Then*

$$[(\overline{\mathcal{I} \circ \mathcal{H}_v^\infty})^{inj}, \|\cdot\|_{(\overline{\mathcal{I} \circ \mathcal{H}_v^\infty})^{inj}}] = [(\overline{\mathcal{I}})^{inj} \circ \mathcal{H}_v^\infty, \|\cdot\|_{(\overline{\mathcal{I}})^{inj} \circ \mathcal{H}_v^\infty}].$$

In particular, the weighted holomorphic ideal $[(\overline{\mathcal{I}})^{inj} \circ \mathcal{H}_v^\infty, \|\cdot\|_{(\overline{\mathcal{I}})^{inj} \circ \mathcal{H}_v^\infty}]$ is injective.

Proof. We claim that $\overline{\mathcal{I}} \circ \mathcal{H}_v^\infty(U, F) = \overline{\mathcal{I} \circ \mathcal{H}_v^\infty}(U, F)$. Indeed, note first that $\overline{\mathcal{I}} \circ \mathcal{H}_v^\infty(U, F)$ is closed: let $f \in \mathcal{H}_v^\infty(U, F)$ and assume that (f_n) is a sequence in $\overline{\mathcal{I}} \circ \mathcal{H}_v^\infty(U, F)$ such that $\|f_n - f\|_v \to 0$ as $n \to \infty$; since $T_{f_n} \in \overline{\mathcal{I}}(\mathcal{G}_v^\infty(U), F)$ by Theorem 2.3 and $\|T_{f_n} - T_f\| = \|f_n - f\|_v$ for all $n \in \mathbb{N}$ by Theorem 2.2, we have that $T_f \in \overline{\mathcal{I}}(\mathcal{G}_v^\infty(U), F)$, and thus $f \in \overline{\mathcal{I}} \circ \mathcal{H}_v^\infty(U, F)$ again by Theorem 2.3.

Now, from $\mathcal{I} \circ \mathcal{H}_v^\infty(U, F) \subseteq \overline{\mathcal{I}} \circ \mathcal{H}_v^\infty(U, F)$, we infer that $\overline{\mathcal{I} \circ \mathcal{H}_v^\infty}(U, F) \subseteq \overline{\mathcal{I}} \circ \mathcal{H}_v^\infty(U, F)$. For the converse, take $f \in \overline{\mathcal{I}} \circ \mathcal{H}_v^\infty(U, F)$; hence $T = T \circ g$ for some complex Banach space G, $T \in \overline{\mathcal{I}}(G, F)$ and $g \in \mathcal{H}_v^\infty(U, G)$; thus we can find a sequence (T_n) in $\mathcal{I}(G, F)$ such that $\|T_n - T\| \to 0$ as $n \to \infty$, and since $\|T_n \circ g - T \circ g\|_v = \|(T_n - T) \circ g\|_v \leq \|T_n - T\| \, \|g\|_v$ for all $n \in \mathbb{N}$, we deduce that $f \in \overline{\mathcal{I} \circ \mathcal{H}_v^\infty}(U, F)$, and this proves our claim. Now, using Proposition 2.3, we conclude that

$$[(\overline{\mathcal{I}})^{inj} \circ \mathcal{H}_v^\infty, \|\cdot\|_{(\overline{\mathcal{I}})^{inj} \circ \mathcal{H}_v^\infty}] = [(\overline{\mathcal{I}} \circ \mathcal{H}_v^\infty)^{inj}, \|\cdot\|_{(\overline{\mathcal{I}} \circ \mathcal{H}_v^\infty)^{inj}}] = [(\overline{\mathcal{I} \circ \mathcal{H}_v^\infty})^{inj}, \|\cdot\|_{(\overline{\mathcal{I} \circ \mathcal{H}_v^\infty})^{inj}}].$$

□

In terms of an Ehrling-type inequality [13], Jarchow and Pelczyński characterized the closed injective hull of a Banach operator ideal in [17, Theorem 20.7.3]. We now present a variant of this result for weighted holomorphic maps.

Theorem 2.5. *For a weighted holomorphic ideal $\mathcal{I}^{\mathcal{H}_v^\infty}$ and $f \in \mathcal{H}_v^\infty(U, F)$, the following are equivalent:*

(1) f *belongs to* $(\overline{\mathcal{I}^{\mathcal{H}_v^\infty}})^{inj}(U,F)$.

(2) *For each* $\varepsilon > 0$, *there are a complex normed space* G_ε *and a mapping* $g_\varepsilon \in \mathcal{I}^{\mathcal{H}_v^\infty}(U,G_\varepsilon)$ *such that*

$$\left\|\sum_{i=1}^n \lambda_i v(x_i) f(x_i)\right\| \leq \left\|\sum_{i=1}^n \lambda_i v(x_i) g_\varepsilon(x_i)\right\| + \varepsilon \sum_{i=1}^n |\lambda_i|$$

for all $n \in \mathbb{N}$, $\lambda_1,\ldots,\lambda_n \in \mathbb{C}$ *and* $x_1,\ldots,x_n \in U$.

Proof. $(i) \Rightarrow (ii)$: Let $f \in (\overline{\mathcal{I}^{\mathcal{H}_v^\infty}})^{inj}(U,F)$ and $\varepsilon > 0$. Hence $\iota_F \circ f \in \overline{\mathcal{I}^{\mathcal{H}_v^\infty}}(U,\ell_\infty(B_{F^*}))$ and so we can find a map $g_\varepsilon \in \mathcal{I}^{\mathcal{H}_v^\infty}(U,\ell_\infty(B_{F^*}))$ such that $\|\iota_F \circ f - g_\varepsilon\|_v < \varepsilon$. For any $n \in \mathbb{N}$, $\lambda_1,\ldots,\lambda_n \in \mathbb{C}$ and $x_1,\ldots,x_n \in U$, we obtain

$$\begin{aligned}\left\|\sum_{i=1}^n \lambda_i v(x_i)(\iota_F(f(x_i)) - g_\varepsilon(x_i))\right\| &\leq \sum_{i=1}^n |\lambda_i|\, v(x_i)\, \|(\iota_F \circ f - g_\varepsilon)(x_i)\| \\ &\leq \sum_{i=1}^n |\lambda_i|\, \|\iota_F \circ f - g_\varepsilon\|_v \leq \varepsilon \sum_{i=1}^n |\lambda_i|,\end{aligned}$$

and therefore

$$\begin{aligned}\left\|\sum_{i=1}^n \lambda_i v(x_i) f(x_i)\right\| &= \left\|\iota_F\left(\sum_{i=1}^n \lambda_i v(x_i) f(x_i)\right)\right\| \\ &\leq \left\|\sum_{i=1}^n \lambda_i v(x_i) g_\varepsilon(x_i)\right\| + \left\|\sum_{i=1}^n \lambda_i v(x_i)(\iota_F(f(x_i)) - g_\varepsilon(x_i))\right\| \\ &\leq \left\|\sum_{i=1}^n \lambda_i v(x_i) g_\varepsilon(x_i)\right\| + \varepsilon \sum_{i=1}^n |\lambda_i|.\end{aligned}$$

$(ii) \Rightarrow (i)$: Let $\varepsilon > 0$ and $\phi = \sum_{i=1}^n \lambda_i v(x_i)\delta_{x_i} \in \operatorname{lin}(\mathrm{At}_{\mathcal{G}_v^\infty(U)})$. By (ii), we have a complex normed space G_ε and a map $g_\varepsilon \in \mathcal{I}^{\mathcal{H}_v^\infty}(U,G_\varepsilon)$ satisfying that

$$\begin{aligned}\|T_f(\phi)\| &= \left\|\sum_{i=1}^n \lambda_i v(x_i) T_f(\delta_{x_i})\right\| = \left\|\sum_{i=1}^n \lambda_i v(x_i) f(x_i)\right\| \\ &\leq \left\|\sum_{i=1}^n \lambda_i v(x_i) g_\varepsilon(x_i)\right\| + \varepsilon \sum_{i=1}^n |\lambda_i| \\ &= \left\|\sum_{i=1}^n \lambda_i v(x_i) T_{g_\varepsilon}(\delta_{x_i})\right\| + \varepsilon \sum_{i=1}^n |\lambda_i| \\ &= \|T_{g_\varepsilon}(\phi)\| + \varepsilon \sum_{i=1}^n |\lambda_i|,\end{aligned}$$

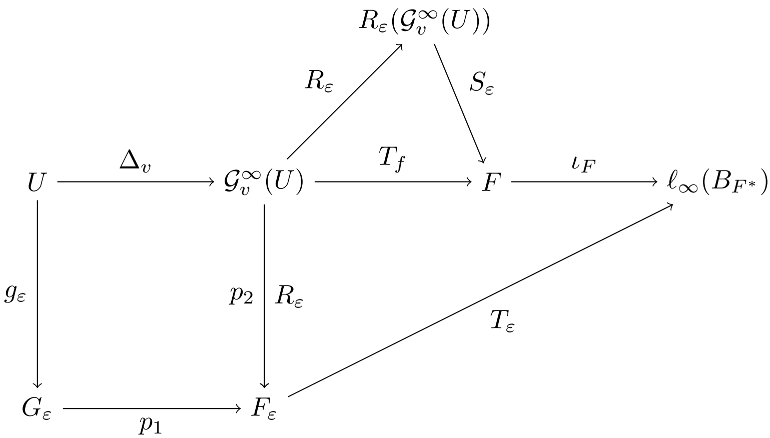
and taking the infimum over all the representations of ϕ, Theorem 2.2 gives

$$\|T_f(\phi)\| \leq \|T_{g_\varepsilon}(\phi)\| + \varepsilon\, \|\phi\|.$$

Consider the Banach space $F_\varepsilon = G_\varepsilon \oplus_{\ell_1} \mathcal{G}_v^\infty(U)$ and define the map $R_\varepsilon \colon \mathcal{G}_v^\infty(U) \to F_\varepsilon$ by $R_\varepsilon(\phi) = (T_{g_\varepsilon}(\phi), \varepsilon\phi)$. Clearly, R_ε is an injective continuous linear operator with $\|R_\varepsilon\| \leq \|g_\varepsilon\|_v + \varepsilon$. By the inequality above, the map $S_\varepsilon \colon R_\varepsilon(\mathcal{G}_v^\infty(U)) \to F$ given by $S_\varepsilon(R_\varepsilon(\phi)) = T_f(\phi)$ is well defined. Clearly, S_ε is linear and since

$$\|S_\varepsilon(R_\varepsilon(\phi))\| = \|T_f(\phi)\| \leq \|T_{g_\varepsilon}(\phi)\| + \varepsilon\, \|\phi\| = \|(T_{g_\varepsilon}(\phi), \varepsilon\phi)\| = \|R_\varepsilon(\phi)\|$$

for all $\phi \in \mathcal{G}_v^\infty(U)$, it is continuous with $\|S_\varepsilon\| \leq 1$. By the metric extension property of $\ell_\infty(B_{F^*})$, there exists an operator $T_\varepsilon \in \mathcal{L}(F_\varepsilon, \ell_\infty(B_{F^*}))$ such that $\iota_F \circ S_\varepsilon = T_\varepsilon|_{R_\varepsilon(\mathcal{G}_v^\infty(U))}$ and $\|T_\varepsilon\| = \|S_\varepsilon\|$.



Define now the maps $h_\varepsilon, k_\varepsilon \colon U \to \ell_\infty(B_{F^*})$ by $h_\varepsilon(x) = T_\varepsilon(g_\varepsilon(x), 0)$ and $k_\varepsilon(x) = T_\varepsilon(0, \varepsilon\Delta_v(x))$ for all $x \in U$. On a hand, $h_\varepsilon = T_\varepsilon \circ p_1 \circ g_\varepsilon \in \mathcal{I}^{\mathcal{H}_v^\infty}(U, \ell_\infty(B_{F^*}))$, where $p_1 \colon G_\varepsilon \to F_\varepsilon$ is the linear continuous map defined by $p_1(y) = (y, 0)$, and, on the other hand, $k_\varepsilon = T_\varepsilon \circ p_2 \circ \varepsilon\Delta_v \in \mathcal{H}_v^\infty(U, \ell_\infty(B_{F^*}))$, where $p_2 \colon \mathcal{G}_v^\infty(U) \to F_\varepsilon$ comes given by $p_2(\phi) = (0, \phi)$, with $\|k_\varepsilon\|_v \leq \varepsilon$ since

$$v(x)\,\|k_\varepsilon(x)\| = v(x)\,\|(T_\varepsilon \circ p_2 \circ \varepsilon\Delta_v)(x)\| \leq v(x)\,\|T_\varepsilon\|\,\varepsilon\,\|\Delta_v(x)\| \leq \varepsilon\,\|T_\varepsilon\| \leq \varepsilon$$

for all $x \in U$. We have

$$\begin{aligned}(h_\varepsilon + k_\varepsilon)(x) &= T_\varepsilon(g_\varepsilon(x), 0) + T_\varepsilon(0, \varepsilon\Delta_v(x)) = T_\varepsilon(T_{g_\varepsilon}(\delta_x), \varepsilon\delta_x) \\ &= (T_\varepsilon \circ R_\varepsilon)(\delta_x) = (\iota_F \circ S_\varepsilon \circ R_\varepsilon)(\delta_x) \\ &= (\iota_F \circ T_f)(\delta_x) = (\iota_F \circ f)(x)\end{aligned}$$

for all $x \in U$, and thus $h_\varepsilon + k_\varepsilon = \iota_F \circ f$. Hence $\|\iota_F \circ f - h_\varepsilon\|_v = \|k_\varepsilon\|_v \leq \varepsilon$, that is, $\iota_F \circ f \in \overline{\mathcal{I}^{\mathcal{H}_v^\infty}}(U, \ell_\infty(B_{F^*}))$ and thus $f \in (\overline{\mathcal{I}^{\mathcal{H}_v^\infty}})^{inj}(U, F)$. □

In the case that the weighted holomorphic ideal $\mathcal{I}^{\mathcal{H}_v^\infty}$ is equipped with a Banach ideal norm, Theorem 2.5 admits the following improvement.

Corollary 2.11. *Let $[\mathcal{I}^{\mathcal{H}_v^\infty}, \|\cdot\|_{\mathcal{I}^{\mathcal{H}_v^\infty}}]$ be a Banach weighted holomorphic ideal and let $f \in \mathcal{H}_v^\infty(U, F)$. The following are equivalent:*

(1) f belongs to $(\overline{\mathcal{I}^{\mathcal{H}_v^\infty}})^{inj}(U, F)$.

(2) There exists a complex Banach space G, a mapping $g \in \mathcal{I}^{\mathcal{H}_v^\infty}(U, G)$ and a function $N \colon \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\left\| \sum_{i=1}^{n} \lambda_i v(x_i) f(x_i) \right\| \leq N(\varepsilon) \left\| \sum_{i=1}^{n} \lambda_i v(x_i) g(x_i) \right\| + \varepsilon \sum_{i=1}^{n} |\lambda_i|$$

for all $n \in \mathbb{N}$, $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$, $x_1, \ldots, x_n \in U$, and $\varepsilon > 0$.

Proof. In view of Theorem 2.5, all we need to show is $(i) \Rightarrow (ii)$. Let $f \in (\overline{\mathcal{I}^{\mathcal{H}_v^\infty}})^{inj}(U, F)$. By Theorem 2.5, for each $m \in \mathbb{N}$, there are a complex Banach space G_m and a map $g_m \in$

$\mathcal{I}^{\mathcal{H}_v^\infty}(U, G_m)$ such that

$$\left\|\sum_{i=1}^{n} \lambda_i v(x_i) f(x_i)\right\| \leq \left\|\sum_{i=1}^{n} \lambda_i v(x_i) g_m(x_i)\right\| + \frac{1}{2^m} \sum_{i=1}^{n} |\lambda_i|$$

for all $n \in \mathbb{N}$, $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ and $x_1, \ldots, x_n \in U$. Take the Banach space $G = (\oplus_{m \in \mathbb{N}} G_m)_{\ell_1}$ and, for each $m \in \mathbb{N}$, the canonical inclusion $I_m \colon G_m \to G$. Then $I_m \circ g_m \in \mathcal{I}^{\mathcal{H}_v^\infty}(U, G)$, and because of

$$\sum_{k=1}^{m} \frac{\|I_k \circ g_k\|_{\mathcal{I}^{\mathcal{H}_v^\infty}}}{2^k \|g_k\|_{\mathcal{I}^{\mathcal{H}_v^\infty}}} \leq \sum_{k=1}^{m} \frac{\|I_k\| \, \|g_k\|_{\mathcal{I}^{\mathcal{H}_v^\infty}}}{2^k \|g_k\|_{\mathcal{I}^{\mathcal{H}_v^\infty}}} \leq \sum_{k=1}^{m} \frac{1}{2^k} \leq 1$$

for all $m \in \mathbb{N}$, the series $\sum_{m \geq 1} (I_m \circ g_m)/2^m \|g_m\|_{\mathcal{I}^{\mathcal{H}_v^\infty}}$ converges in $(\mathcal{I}^{\mathcal{H}_v^\infty}(U, G), \|\cdot\|_{\mathcal{I}^{\mathcal{H}_v^\infty}})$ to the weighted holomorphic map $g = \sum_{m=1}^{\infty} (I_m \circ g_m)/2^m \|g_m\|_{\mathcal{I}^{\mathcal{H}_v^\infty}} \in \mathcal{I}^{\mathcal{H}_v^\infty}(U, G)$. Using the inequality above, we deduce

$$\begin{aligned}
\left\|\sum_{i=1}^{n} \lambda_i v(x_i) f(x_i)\right\| &\leq 2^m \|g_m\|_{\mathcal{I}^{\mathcal{H}_v^\infty}} \left\|\sum_{i=1}^{n} \frac{\lambda_i v(x_i)}{2^m \|g_m\|_{\mathcal{I}^{\mathcal{H}_v^\infty}}} g_m(x_i)\right\| + \frac{1}{2^m} \sum_{i=1}^{n} |\lambda_i| \\
&\leq 2^m \|g_m\|_{\mathcal{I}^{\mathcal{H}_v^\infty}} \sum_{m=1}^{\infty} \left\|\sum_{i=1}^{n} \frac{\lambda_i v(x_i)}{2^m \|g_m\|_{\mathcal{I}^{\mathcal{H}_v^\infty}}} g_m(x_i)\right\| + \frac{1}{2^m} \sum_{i=1}^{n} |\lambda_i| \\
&= 2^m \|g_m\|_{\mathcal{I}^{\mathcal{H}_v^\infty}} \left\|\sum_{i=1}^{n} \lambda_i v(x_i) g(x_i)\right\| + \frac{1}{2^m} \sum_{i=1}^{n} |\lambda_i|
\end{aligned}$$

for all $n \in \mathbb{N}$, $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$, $x_1, \ldots, x_n \in U$ and $m \in \mathbb{N}$. Finally, this inequality yields the inequality in the statement defining $N \colon \mathbb{R}^+ \to \mathbb{R}^+$ by

$$N(\varepsilon) = \begin{cases} 2 \|g_1\|_{\mathcal{I}^{\mathcal{H}_v^\infty}} & \text{if } \varepsilon > 1, \\ 2^m \|g_m\|_{\mathcal{I}^{\mathcal{H}_v^\infty}} & \text{if } 2^{-m} < \varepsilon \leq 2^{-m+1}, \ m \in \mathbb{N}. \end{cases}$$

□

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