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Research Article

Risk Assessment for Breast Cancer with Integrated Group Decision-Making Method

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Abstract: The most prevalent invasive malignancy in women is breast cancer. The second most common cause of cancer deaths in women, behind lung cancer, is breast cancer. It begins with developing a tiny tumor or mass and spreads from breast cells, primarily in the milk ducts (ductal carcinoma) or glands (lobular carcinoma). Every woman needs to be aware of her risk of developing breast cancer to be proactive about risk reduction measures and for better care of the disease, even though the causes of breast cancer are not fully known. Numerous variables that can either raise or decrease the likelihood of acquiring breast cancer have been identified by independent investigations. By looking at these risk factors, it is feasible to determine a woman's estimated risk of acquiring a malignant breast illness. Fermatean fuzzy sets can adequately describe the uncertain data for determining breast cancer risk. The cumulative prospect theory is used to build the traditional Tomada de Decisão Iterativa Multicritério (TODIM) approach, which can be used to reflect the psychological behavior of the decision-maker. The Fermatean fuzzy cumulative prospect theory-TODIM approach is proposed in this paper to handle the problem of group decision-making. Using the entropy weight method with Fermatean fuzzy sets to obtain attribute weight information simultaneously improves rationality. This article applies the mentioned method to the risk assessment of breast cancer. It illustrates the risk assessment model based on the proposed method, concentrating on hot topics in contemporary culture.

Keywords: Breast cancer, Fermatean fuzzy environment, cumulative prospect theory, TODIM, group decision-making.

1. Introduction

Women of all ages are susceptible to breast cancer, which is a highly diverse disease. The breast comprises many tissues, including dense and fatty tissue that includes milk glands, lobes, and lobules. Breast cancer occurs when breast cells multiply uncontrollably, leading to tumor formation.

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Breast cancer is said to be metastatic if it spreads to other organs. There are two basic varieties of breast cancer: non-invasive breast cancer, which stays in the lobular part of the breasts, and invasive breast cancer, which spreads to nearby tissues or distant organs. In 2020, there were 2.3 million new instances of breast cancer in women and 685.000 deaths globally, according to the WHO data. For more targeted medicine and treatment, early prognosis prediction is therefore essential. Breast cancer is difficult to anticipate and treat since it is a complicated disease with a wide range of clinical outcomes. For instance, medical personnel face challenges with manual interpretation due to the high dimensionality of multimodal data. Therefore, the creation of computational algorithms becomes essential for precisely forecasting the prognosis of breast cancer. The importance of these methods in clinical decision-making is highlighted by the fact that these algorithms can help doctors choose the best course of action for their patients.

Artificial intelligence, cognitive science, psychology, philosophy, and other academic disciplines are all interested in how people reason and form opinions in practical situations. Various mathematical and statistical models usually describe these processes; decision-making (DM) becomes essential. Behavior management chooses which behavior patterns an individual or organization should use to accomplish a specific goal. Research indicates that while many decisions in daily life can be made without conscious thought, more thought and effort are needed to make complex and important decisions. In discrete situations with well-defined and limited options, multi-attribute decision-making, or MCDM, is employed. MCDM problems have a limited number of possible solutions. MCDM techniques are frequently used in decision-making processes like ranking, comparing, and selecting options. These methods are typically chosen because they enable quick DM without requiring intricate mathematical computations or sophisticated package software. The MCDM approach can only accomplish one objective. The goal is to solve the choice dilemma most cost-effectively and beneficially possible.

When there are several alternative outcomes of a particular event, but their likelihood is unknown, this is known as uncertainty. As a result, the DM must comprehend uncertainty. It takes time and effort to comprehend the likelihood that events will occur in reality. Consequently, there is uncertainty at every stage of the DM process. A strong basis for logical reasoning with vague and imperfect data is provided by fuzzy logic theory. Thanks to fuzzy logic theory, computers can understand human language and apply human knowledge. At this point, it starts employing symbols instead of numerical expressions. Fuzzy sets (FS) are symbolic expressions of this type. FSs are known to include choice variables, such as probability states.

1.1. Research Motivation

Quantifying the degrees of membership ($\mathcal{M}$) and non-membership ($\mathcal{N}$) in a single numerical number is only partially justified or technically sound in human cognitive and decision-making activities. When information must be provided as intervals rather than single-valued numbers, interval numbers may be used. The decision-maker can more easily convey his or her preference for $\mathcal{M}$ and $\mathcal{N}$ using intervals. Because of a lack of information, decision-makers may find it challenging to express their thoughts accurately with an exact number in specific real-world DM challenges. In Intuitionistic fuzzy set (IFS) theory, the $\mathcal{N}$ is defined in addition to the $\mathcal{M}$, whereas FS theory is designed only to reveal the $\mathcal{M}$ defined in the range. Pythagorean fuzzy sets (PFS), which Yager proposed [34] and in certain instances developed as an extension of IFSs since IFSs cannot adequately convey uncertainty PFSs employ the notion that the sum of the squares of $\mathcal{M}$ and $\mathcal{N}$ are less than or equal to 1 for circumstances when decision-making is impossible when $\mathcal{M}$ and $\mathcal{N}$ are added together. The Fermatean fuzzy set FFS is ascribed to Senapati and Yager [34]. The property is "the sum of the cubes of $\mathcal{M}$ and $\mathcal{N}$ are less than or equal to 1"attained by the $\mathcal{M}$ and $\mathcal{N}$ in the FFS.

The TODIM method ranks the alternatives in the MCDM problem. The TODIM method is a method used to make decisions under risky conditions. The form of the value function in the method is similar to the loss and gain function of the prospect theory. This function reflects the behavioral characteristics of decision-makers, such as risk aversion, and shows the degree of dominance of alternatives over each other. The global value function combines the gains and losses according to all decision criteria and allows the ranking of alternatives. The TODIM technique is a method that allows the use of qualitative data expressed with linguistic variables along with quantitative data.

The main advantage of the TODIM method compared to other behavioral decision techniques is that it considers the behavioral characteristics of DMs with limited rationality. The method includes gains and losses relative to the reference point in case of uncertainty, thus making DMs more sensitive to losses. In the case of decision-making based on full rationality, DMs only aim to maximize utility. In contrast, in the TODIM technique, DMs maximize total utility by considering losses. Therefore, the TODIM method can be considered a behavioral decision-making method based on partial rationality.

The primary finding of cumulative prospect theory (CPT) (and its precursor, prospect theory) is that people typically consider potential outcomes about a specific reference point, which is frequently the status quo rather than the ultimate status. We refer to this phenomenon as the framing effect. Additionally, their risk attitudes toward gains (i.e., outcomes above the reference point) and losses (i.e., outcomes below the reference point) differ, and they are typically more concerned with prospective losses than with potential profits (loss aversion). Lastly, people undervalue "average"events while overvalue extraordinary ones. Prospect Theory, which holds that people outweigh unexpected events regardless of their relative outcomes, is contrasted with the last statement.

1.2. Literature

Globally, breast cancer (BC) is the most common invasive cancer in women. After lung cancer, breast cancer is the second leading cause of death for women. It starts with the formation of a small tumor or mass and is brought on by breast cells, particularly those in the glands (lobular carcinoma) or milk ducts (ductal carcinoma) [38]. The border of a benign (non-cancerous) tumor is smooth and distinct. There may be irregularly bordered or hypothesized cancerous (malignant) lumps [31]. Even if the exact causes of BC are unknown, every woman should be aware of her risk of contracting the illness so that she can take proactive steps to reduce her risk and treat it successfully. Numerous factors that either increase or decrease the risk of getting breast cancer have been identified by independent studies [39–42]. It is possible to estimate a woman's predicted probability of developing malignant breast disease by evaluating these risk factors.

An annual breast cancer screening utilizing digital mammography is recommended for all women over the age of 40 to spot worrisome lesions early. Digital mammograms do not accurately indicate the post-screening risk of developing a malignant breast disease in those diagnosed with normal or benign findings, even though they are believed to be effective in detecting suspicious breast masses and lesions and grading the findings on a zero to six scale by BI-RADS [43] guidelines. Certain women should be treated separately by patients with lower risk factors since they may have genetic predispositions or other BC risk factors that put them in the high-risk category. In order to help women become more BC-conscious, it is imperative to create an integrated BC risk assessment model that incorporates the results of the initial screening study with the individual's demographic risk factors. This would make it possible for high-risk women to ask their doctors for sensible guidance on the best follow-up plan, increasing the likelihood that malignant tumors will be discovered early.

Zadeh's [35] concept of an FS highlighted the ambiguity and absurdity of a $\mathcal{M}$. Atanassov [2] then discovers the IFS, which could provide more detailed evaluation information by linking an item to a component's $\mathcal{N}$. However, because of their significant limitations in giving preference information, IFSs are designed to make it difficult for judgment specialists to make the proper assessments. Along with the $\mathcal{M}$, IFS also specifies the $\mathcal{N}$. IFS theory states that $\mathcal{M}$ and $\mathcal{N}$ fall within the $[0,1]$ range.

Imprecision must be taken into consideration in any DM procedure. Numerous methods and instruments have been put forth to address the unclear environment of collective DM. One of the most recent methods for dealing with uncertainty is FFS [23]. These sets provide a wider range of applications than the FS [35] extensions, the IFS [2], and the PFS [33]. Recently, FFs have inspired many studies [1, 5, 6, 10, 14, 24–26].

Problems in the real world are often very complicated. Complexity can be attributed to ambiguities, randomness, or limited understanding brought on by a lack of data or poor quality of information. Most tasks require identifying the variables in the problem statement using linguistic language. More precise forecasts and advantageous solutions will result from an understanding of decision-makers knowledge of confusing facts. Zadeh's [35] FS idea is a key component of fuzzy modeling, a mathematical approach that describes uncertainty in human systems. It is indisputable that FS theory cannot individually determine the satisfaction or dissatisfaction of human judgment, even though the study of partial membership required a significant divergence from conventional reasoning. Atanassov [2] created the intuitionistic fuzzy set (IFS) theory to overcome this limitation. Numerous academics in a range of optimization-related domains have since used IFSs. IFS is not designed to handle scenarios when the sum of $\mathcal{M}$ and $\mathcal{N}$ for some alternatives is more than one. To get around this restriction, Yager [33], [34] developed Pythagorean fuzzy sets (PFSs), which loosen it up so that the only condition at each option evaluation is that $\mathcal{M}$ and $\mathcal{N}$ sum of squares is less than 1. Senapati and Yager [23, 24] developed the FFS concept in response to the limitations imposed by PFSs. FFS theory was proposed by Senapati and Yager [23, 24] in response to the limitations imposed by IFSs and PFSs. The $\mathcal{M}$ and $\mathcal{M}$ cubic sum in an FFS must be less than or equal to 1. In addition, FFS-related applications are depicted in [5, 6, 8, 18, 24, 25, 27].

The $\mathcal{M}$'s ambiguity and vagueness were illustrated using [35]'s concept of an FS. Atanassov's intuitionistic FS (IFS) [2] links an element's $\mathcal{N}$ to an item, providing a more comprehensive explanation of assessment data. Yager [33, 34] developed the Pythagorean FS(PFS) idea to broaden the range of $\mathcal{M}$ and $\mathcal{N}$ so that $\mathcal{M}^2+\mathcal{N}^2 \leq 1$ in response to the IFS vulnerability previously described. Because of this, PFS offers professionals more evaluation opportunities to express their opinions on various objectives. The complexity of the DM framework increases the difficulties specialists have in producing reliable evaluation data. The development of IFS and PFS has addressed the ambiguity and vagueness caused by the complex subjectivity of human cognition. The FFS was the first to expand the scope of information assertions by adding the cubic sum of $\mathcal{M}$ and $\mathcal{N}$. Therefore, FFS manages ambiguous choice situations more efficiently and practically than IFS and PFS. Senapati and Yager started the FFS [23]. The fundamental characteristics of FFS were initially provided by Senapati and Yager [24, 25].

Garg et al. [5] have established general aggregation operators, based on Yager's t-norm and t-conorm, to cumulate the FF data in decision-making environments. In [17], a hybrid MCDM based on IVFF was proposed for risk analysis related to autonomous vehicle driving systems. Kirisci [8] defined new correlation coefficients based on the Fermatean hesitant fuzzy elements and interval-valued Fermatean hesitant fuzzy elements. The least common multiple expansion was used in the newly defined correlation coefficients. In [12], a three-way method for computing the correlation coefficients between FFSs has been given using the notions of variance and covariance. New distance and cosine similarity measures amongst FFSs have been defined [10]. A method was established to construct similarity measures between FFSs based on the cosine similarity and Euclidean distance measures. In [11], a new correlation coefficient and weighted correlation coefficient formularization have been proposed to evaluate the affair between two FFSs. In [14], an extended version of the ELECTRE-I model called the FF ELECTRE-I method for multi-criteria group decision-making with FF human assessments has been presented. Kirisci [15] defined the Fermatean hesitant fuzzy set and gave aggregation operations based on the Fermatean hesitant fuzzy set. The interval-valued Fermatean fuzzy linguistic Kernel Principal Component Analysis model has been given in [16]. The definition of FF soft sets and some properties were introduced [9]. Furthermore, the Fermatean fuzzy soft entropy and the formulas for standard distance measures, such as Hamming and Euclidean distance, were defined [9]. A new model for group decision-making methods in which experts' preferences can be expressed as incomplete FF-preference relations has been presented [27]. A multi-criteria decision-making strategy to evaluate the risk probabilities of autonomous vehicle driving systems by combining the AHP technique with interval-valued FFSs has been proposed in [28]. First, the interval-valued IFS was described in [3]. It represented the $\mathcal{M}$ and $\mathcal{N}$ by the closed subinterval of the interval [0,1]. The interval-valued PFS (IVPFS), whose $\mathcal{M}$ and $\mathcal{N}$ are represented by an interval number, was further proposed by [29]. Several operations and relations of IVPFS are also examined. Jeevaraj defined the IVFFS [6].

Gomes and Lima [7] provide the traditional TODIM approach for the first time due to the complexity of the decision environment. The TODIM approach is always used to evaluate some MADM conditions while considering the DMs confidence level to deliver a more equitable solution under risk. As a result, while the TODIM technique is an excellent MADM method, it has limitations; it does not have to provide an adequate mechanism for generating attribute weights and needs a comprehensive approach. As a result, Tian et al. [20] improved on the traditional TODIM technique. They used it with the CPT to change the weighting of attributes to make more reasonable decisions in practice. On the other hand, the risk assessment of science and technology projects could be considered a classic MAGDM issue. Some research is similar. Tüysüz and Kahraman [21] found the fuzzy analytic hierarchy process (AHP) to help analyze the risk of an information technology project. Kumar et al. [22] studied the risks of software projects and developed a new blended MCDM technique based on fuzzy DEMATEL, FMCDM, and TODIM knowledge. Suresh and Dillibabu [32] were also looking for a better model for software project evaluation and developed a framework for fuzzy DEMATEL, ANFIS MCDM, and IF-TODIM. Zhao et al. [36] proposed a CPT-TODIM method based on intuitionistic fuzzy sets for the MAGDM problem and used the CRITIC method to obtain the weight information of the attributes. In [37], the new CPT-TODIM approach based on PFSs has been implemented. Liao et al. [19] introduced the extended TODIM with CPT for probabilistic hesitant fuzzy multiple attributes group decision-making.

1.3. Necessity

The FFSs could effectively depict the imprecise or vague information of risk assessment issues of breast cancer. In light of this, the primary goal of this work is to offer a technique for evaluating breast cancer risk. To consider the limited rationality of physicians' thinking, we expand this unique TODIM method based on the CPT to the FFSs in this paper. We also use FFSs to transmit experts' appraisals of each alternative for each attribute. This combination has potential applications in related circumstances, which can strengthen and resupply the research. As a result, applying this research topic to MCDM for risk evaluation issues is intriguing.

Since the information description of breast cancer pre-diagnosis lays a solid foundation for later disease diagnosis, the current paper mainly focuses on the imprecision and incompleteness that existed in the problem modeling procedure.

A selectable method is required to reflect the psychological behaviors of physicians, and due to this requirement, the classical TODIM method based on cumulative prospect theory (CPT-TODIM) will be created.

To increase rationality, the weight information of the attributes will be obtained.

1.4. Originality

The literature has identified numerous hazards that can either increase or lower the chance of developing breast cancer. It is possible to evaluate a woman's likelihood of developing a malignant breast disease by examining these risk factors. Bridging the gap between FFSs and CPT-TODIM and investigating efficient models and ways with the aid of FFS CPT-TODIM in deficient information systems is essential, given that FFSs are expected to be a fundamental tool for breast cancer risks. This serves as the main driving force for the research in the paper.

1.5. Contribution

The following significant contributions can be eventually specified:

(1) FFS CPT-TODIM processes uncertain information in modeling breast cancer risks.

(2) The FFS CPT-TODIM method can more comprehensively address the bounded rationality of physicians regarding breast cancer risks.

(3) A comprehensive FF MCDM approach is constructed via FF CPT-TODIM. By using FFSs, physicians' evaluation of each alternative for each attribute can be captured more robustly.

2. Preliminaries

U, the initial universe set, will be used throughout the article.

For $\alpha_P : U \to [0,1]$ and $\beta_P : U \to [0,1]$, the FFS P is shown by $P = \{(u, \alpha_P(u), \beta_P(u)) : u \in U\}$, where the inequality $0 \leq \alpha_P^3(u) + \beta_P^3(u) \leq 1$ [23] is valid.

It is defined as $\gamma_P(u) = \sqrt[3]{1 - (\alpha_P^3(u) + \beta_P^3(u))}$ degree of indeterminacy of u to P.

Take three FFSs $P = \{\alpha_P, \beta_P\}$, $P_1 = \{\alpha_{P_1}, \beta_{P_1}\}$ and $P_2 = \{\alpha_{P_2}, \beta_{P_2}\}$. Then, some operations as follows [23]:

i. $P_1 \cap P_2 = \min\{\alpha_{P_1}, \alpha_{P_2}\}, \max\{\beta_{P_1}, \beta_{P_2}\}$,

ii. $P_1 \cup P_2 = \max \alpha_{P_1}, \alpha_{P_2}, \min \beta_{P_1}, \beta_{P_2}$,

iii. $P^t = \beta_P, \alpha_P$,

iv. $P_1 \boxplus P_2 = \left(\sqrt[3]{\alpha_{P_1}^3 + \alpha_{P_2}^3 - \alpha_{P_1}^3 \alpha_{P_2}^3}, \beta_{P_1}\beta_{P_2} \right)$,

v. $P_1 \boxtimes P_2 = \left(\alpha_{P_1}^3 \alpha_{P_2}^3, \sqrt[3]{\beta_{P_1}^3 + \beta_{P_2}^3 - \beta_{P_1}^3 \beta_{P_2}^3} \right)$,

vi. $\alpha P = \left(\sqrt[3]{1 - (1 - \alpha_P^3)^\lambda}, \alpha_P^\alpha \right), \quad \lambda > 0$,

vii. $P^\lambda = \left(\alpha_{P_1}^3, \sqrt[3]{1 - (1 - \beta_P^3)^\lambda} \right), \quad \lambda > 0$.

Let $P = \{\alpha_P, \beta_P\}$ be an FFN, then the score and accuracy functions of P are defined as:

$$SC(P) = \frac{1 + \alpha_P^3 - \beta_P^3}{2},$$

$$AC(P) = \alpha_P^3 + \beta_P^3$$

where $SC(P) \in [-1,1]$ and $AC(P) \in [0,1]$.

For any two FFNs $P_1 = \{\alpha_{P_1}, \beta_{P_1}\}$ and $P_2 = \{\alpha_{P_2}, \beta_{P_2}\}$,

(K1) If $SC(P_1) < SC(P_2)$, then $P_1 < P_2$,

(K2) If $SC(P_1) = SC(P_2)$, then

(A) If $AC(P_1) < AC(P_2)$, then $P_1 < P_2$,

(B) If $AC(P_1) = AC(P_2)$, then $P_1 \sim P_2$.

Let $P_i = \{\alpha_{P_i}, \beta_{P_i}\}$, $(i = 1, \cdots, n)$ be a collection of FFNs and $\omega = (\omega_1, \cdots, \omega_n)^T$ be the weight vector of P_i. Then, the Fermatean fuzzy weighted average (FFWA) operator is a mapping $FFWA : P^n \longrightarrow P$, where

$$FFWA(P_1, \cdots, P_n) = \left(\sum_{i=1}^{n} \omega_i \alpha_i, \sum_{i=1}^{n} \omega_i \beta_i \right).$$

3. CPT-TODIM based on FFSs

First, we will give the TODIM method based on CPT [20]. As follows, there is a decision matrix C, in which the schemes and attributes are provided by decision-makers:

$$C = (c_{ij})_{m\times n} \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1j} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2j} & \cdots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots & & \\ c_{i1} & c_{i2} & \cdots & c_{ij} & \cdots & c_{in} \\ \vdots & \vdots & \ddots & \vdots & & \\ c_{m1} & c_{m2} & \cdots & c_{mj} & \cdots & c_{mn} \end{bmatrix}. \tag{1}$$

The weighting vector of attributes is represented by $\varpi = (\varpi_1, \cdots, \varpi_n)^T$, which satisfies $\varpi_j \geq 0$ and $\sum_{j=1}^{n} \varpi_j = 1$.

Step 1: Figure out the modified weights $\Theta^*_{ikj}(\varpi_j)$ based on the original weighting vector of attributes and the weighting function by

$$\Theta_{ikj}(\varpi_j) = \begin{cases} \frac{((\varpi_j)^\lambda}{(\varpi_j)^\lambda + (1-\varpi_j)^\lambda)^{1/\lambda}} & , c_{ij} \geq c_{kj} \\ \frac{((\varpi_j)^\mu}{(\varpi_j)^\mu + (1-\varpi_j)^\mu)^{1/\mu}} & , c_{ij} < c_{kj}, \end{cases} \tag{2}$$

$$\Theta^*_{ikj}(\varpi_j) = \frac{\Theta_{ikj}(\varpi_j)}{\max\{\Theta_{ikj}(\varpi_l) : l \in n\}} \quad j \in n, \quad \forall (i,k), \tag{3}$$

where λ, μ are the parameters, which are used to describe the curvature of the weighting function.

Step 2: The relative predominance $\Delta_j(S_m, S_k)$ of scheme S_m compared with S_k in the attribute A_j can be computed by

$$\Delta_j(S_m, S_k) = \begin{cases} \frac{\Theta^*_{ikj}(\varpi_j).(c_{ij}-c_{kj})^\zeta}{\sum_{j=1}^{n} \Theta^*_{ikj}(\varpi_j)} & , c_{ij} > c_{kj} \\ 0 & , c_{ij} = c_{kj} \\ -\delta \cdot \frac{(\sum_{j=1}^{n} \Theta^*_{ikj}(\varpi_j)).(c_{ij}-c_{kj})^\eta}{\Theta^*_{ikj}(\varpi_j)} & , c_{ij} < c_{kj}, \end{cases} \tag{4}$$

where δ, ζ, η are the parameters.

Step 3: Equation (5) is applied to calculate the overall predominance $\Upsilon(S_m)$ of scheme S_m:

$$\Upsilon(S_m) = \sum_{k=1}^{m}\sum_{j=1}^{n}\Delta_j(S_m, S_k). \tag{5}$$

Step 4: Acquire the standard overall predominance

$$\Omega(S_m) = \frac{\Upsilon(S_m) - \min_m(\Upsilon(S_m))}{\max_m(\Upsilon(S_m)) - \min_m(\Upsilon(S_m))}. \tag{6}$$

Step 5: Rank the standard overall predominance $\Omega(S_m)$ to get the best scheme that has the biggest $\Omega(S_m)$ value.

Based on the above knowledge, a new model will be established to answer the multiple attribute group decision-making issue with Fermatean fuzzy information. There are three collections of information, which are named the set of alternatives $S = \{S_1, \cdots, S_m\}$, the set of attributes $A = \{A_1, \cdots, A_n\}$ and the set of decision makers $J = \{J_1, \cdots, J_p\}$. Through building the relation between the alternative and attribute, we can get the Fermatean fuzzy decision matrix $R^z = (r_{ij}^z)_{m\times n} = (\alpha_{ij}^z, \beta_{ij}^z)_{m\times n}$ provided by the decision maker J_z, where r_{ij}^z as well as α_{ij}^z respectively, indicate the membership degree and the non-membership degree about the alternative S_m keeping in line with the attribute A_j and satisfy $\alpha_{ij}^z, \beta_{ij}^z \in [0,1]$ and $(\alpha_{ij}^z)^3, (\beta_{ij}^z)^3 \leq 1$. Furthermore, the weighting vector of attribute $\varpi = (\varpi_1, \cdots, \varpi_n)^T$ and $\varrho = (\varrho_1, \cdots, \varrho_n)^T$ is the weighting vector of decision makers.

Algortihm of CPT-TODIM based on FFSs:

Stage 1: Process and Integrate the Information from Independent Decision Makers

1. Take advantage of Equation (7) to ensure the unification of all of the attributes:

$$\begin{aligned} M^z &= (m_{ij}^z)_{m\times n}, \\ m_{ij}^z &= (\phi_{ij}^z, \psi_{ij}^z) = \begin{cases} a_{ij}^z = (\alpha_{ij}^z, \beta_{ij}^z) & , A_j \textit{ is a positive attribute} \\ (m_{ij}^z)^c = (\beta_{ij}^z, \alpha_{ij}^z) & , A_j \textit{ is a negative attribute.} \end{cases} \end{aligned} \tag{7}$$

2. The Fermatean fuzzy power weighted averaging (FFPWA) operator makes the integration of Fermatean fuzzy decision matrices deriving from independent decision-makers come true. The specific process of calculation refers to Equations (8) - (11):

$$d(m_{ij}^z, m_{ij}^t) = \frac{\sqrt[3]{(\phi_{ij}^z - \phi_{ij}^t)^3 + (\psi_{ij}^z - \psi_{ij}^t)^3}}{\sqrt{2}} \tag{8}$$

$$sup(m_{ij}^z, m_{ij}^t) = 1 - d(m_{ij}^z, m_{ij}^t) \tag{9}$$

$$X(m_{ij}^z) = \sum_{t=1, t\neq z}^{s} \varphi_t sup(m_{ij}^z, m_{ij}^t), \quad z = 1, 2, \cdots, s \tag{10}$$

$$\begin{aligned} g_{ij} &= FFPWA_{\varphi}(m_{ij}^1, \cdots, m_{ij}^s) = \frac{\boxplus_{z=1}^s \left(\varphi_z(1 + X(m_{ij}^z)m_{ij}^z\right)}{\sum_{z=1}^s \varphi_z(X(m_{ij}^z))} \\ &= \left(\sqrt[3]{1 - \prod_{z=1}^{s} \left(1 - (\phi_{ij}^z)^3\right)^{\varphi_z(1+X(m_{ij}^z))/\sum_{z=1}^s \varphi_z(X(m_{ij}^z))}}, \right. \\ &\left. \prod_{z=1}^{s} \left((\psi_{ij}^z)^3\right)^{\varphi_z(1+X(m_{ij}^z))} \sum_{z=1}^{s} \varphi_z(X(m_{ij}^z)) \right) \end{aligned} \tag{11}$$

Stage 2: Acquire the Attribute Weights based on Existing Information

3. To get the original weighting vector of attributes ϖ, all related equations are sequentially listed: For $j, h = 1, 2, \cdots, n$,

$$\Lambda_{jh} = \frac{\sum_{i=1}^m \left(SC(g_{ij}) - \frac{1}{m}\sum_{i=1}^m SC(g_{ij})\right).\left(SC(g_{ih}) - \frac{1}{m}\sum_{i=1}^m SC(g_{ih})\right)}{\sqrt{\sum_{i=1}^m \left(SC(g_{ij}) - \frac{1}{m}\sum_{i=1}^m SC(g_{ij})\right)^2 . \left(SC(g_{ih}) - \frac{1}{m}\sum_{i=1}^m SC(g_{ih})\right)}}, \tag{12}$$

$$\Gamma_j = \sqrt{\frac{1}{m-1}\sum_{i=1}^{m}\left(SC(g_{ij}) - \frac{1}{m}\sum_{i=1}^{m} SC(g_{ij})\right)^2}, \tag{13}$$

$$\varpi_j = \frac{\Gamma_j . \sum_{h=1}^n (1 - \Lambda_{jh})}{\sum_{j=1}^n} \left(\Gamma_j . \sum_{h=1}^{n} (1 - \Lambda_{jh})\right). \tag{14}$$

4. Utilize the weighting function shown in Equations (15) and (16) to calculate the modified weights:

$$\Theta_{ikj}(\varpi_j) = \begin{cases} \frac{(\varpi_j)^\lambda}{((\varpi_j)^\lambda + (1-\varpi_j)^\lambda)^{1/\lambda}} & , g_{ij} \geq g_{kj}, \\ \frac{(\varpi_j)^\mu}{((\varpi_j)^\mu + (1-\varpi_j)^\mu)^{1/\mu}} & , g_{ij} < g_{kj}, \end{cases} \tag{15}$$

$$\Theta_{ikj}^*(\varpi_j) = \frac{\Theta_{ikj}(\varpi_j)}{\max\{\Theta_{ikj}(\varpi_l) : l \in n\}} \quad j \in n, \quad \forall (i, k), \tag{16}$$

where λ, μ are the parameters, which are used to describe the curvature of the weighting function.

Stage 3: Carry Through Pairwise Comparison for Any Alternative and Acquire the Eventual Standard of Ordering

5. Determine the relative predominance of alternative S_m compared with S_k underneath the attribute A_j:

$$d_{ikj} = \frac{\sqrt[3]{(\delta_{ij} - \vartheta_{ij})^3 + (\delta_{ij} - \vartheta_{ij})^3}}{\sqrt{2}}, \tag{17}$$

$$\Delta_j(S_m, S_k) = \begin{cases} \frac{\Theta^*_{ikj}(\varpi_j).(d_{ikj})^{\zeta}}{\sum_{j=1}^{n} \Theta^*_{ikj}(\varpi_j)} & , g_{ij} > g_{kj} \\ 0 & , g_{ij} = g_{kj} \\ -\delta \cdot \frac{\sum_{j=1}^{n} \Theta^*_{ikj}(\varpi_{ij})}{\Theta^*_{ikj}(\varpi_j)} \cdot (d_{ikj})^{\eta} & , c_{ij} < c_{kj}, \end{cases} \tag{18}$$

where δ, ζ, η are the parameters.

6. Determine the overall predominance $\Upsilon(S_m)$ and the standard overall predominance $\Omega(S_m)$ of the alternative S_m over all others in according to Equations (19) and (20):

$$\Upsilon(S_m) = \sum_{k=1}^{m} \sum_{j=1}^{n} \Delta_j(S_m, S_k), \tag{19}$$

$$\Omega(S_m) = \frac{\Upsilon(S_m) - \min_m(\Upsilon(S_m))}{\max_m(\Upsilon(S_m)) - \min_m(\Upsilon(S_m))}. \tag{20}$$

7. Rank the standard overall predominance $\Upsilon(S_m)$ and the bigger value of the standard overall predominance means the more excellent alternative.

4. Risk Analysis of Breast Cancer

Everyone wants to know how to lower their breast cancer risk. Although doctors do not know what causes breast cancer, they know there are factors linked to a higher-than-average risk of developing the disease. Some factors associated with increased breast cancer risk — being a woman, age, and genetics, for example — cannot be changed. Other factors — lack of exercise, smoking cigarettes, and eating certain foods — can be altered by lifestyle choices.

By choosing the healthiest lifestyle options, one can empower oneself and keep the risk of breast cancer as low as possible. If a factor cannot be changed (such as your genetics), you can learn about protective steps to help keep your risk as low as possible. We will classify a given individual's BC risk level into three different grades: S_1-Normal, S_2-Benign, and S_3-Malignant. The 14 main influencing personal risk factors related to the three main risk factors affecting BC

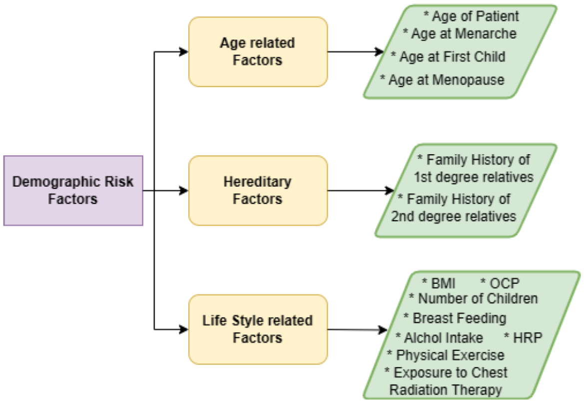

Figure 1: Demographic risk factors of breast cancer [30]

Table 1: Group decision matrix

	A_{11}	A_{12}	A_{13}	A_{14}	B_{11}	B_{22}	C_{31}	···
S_1	(0.6321, 0.4610)	(0.6357, 0.2744)	(0.3310, 0.5863)	(0.6485, 0.3762)	(0.8945, 0.2376)	(0.8637, 0.2551)	(0.7581, 0.3548)	···
S_2	(0.8577, 0.1646)	(0.7378, 0.2819)	(0.6974, 0.2119)	(0.6994, 0.3712)	(0.6513, 0.2487)	(0.3514, 0.6879)	(0.1542, 0.5897)	···
S_3	(0.4620, 0.3766)	(0.4925, 0.4526)	(0.5971, 0.2637)	(0.7038, 0.3100)	(0.6347, 0.4465)	(0.6405, 0.3542)	(0.5330, 0.4112)	···

	C_{35}	C_{36}	C_{37}	C_{38}
S_1	(0.6485, 0.2998)	(0.8214, 0.1258)	(0.2471, 0.7902)	(0.2416, 0.6548)
S_2	(0.6669, 0.3125)	(0.3126, 0.4121)	(0.3146, 0.3251)	(0.4011, 0.6175)
S_3	(0.7301, 0.2694)	(0.3082, 0.6812)	(0.3677, 0.5807)	(0.2103, 0.8899)

are shown in Figure 1 [30].

BC risk assessment could be regarded as a classical MCDM issue. Based on the above steps, in the following, we intend to apply the proposed FF-CPT TODIM method in this paper to the risk assessment of BC.

1. Uniform the positive and negative attributes by applying Equation (7) and concentrate a group decision matrix Q by utilizing Equations (8)-(11). The final results are shown in Table 1.

2. Take advantage of Equations (12)-(14) to obtain the original weighting vector of attributes.

3. Utilize the weighting function shown in Equations (15) and (16) to calculate the modified weights (Take $\lambda = 0.61$ and $\mu = 0.69$).

4. Determine the relative predominance of alternative S_m compared with S_k underneath the attribute A_j according to Equations (17) and (18) (Take $\delta = 0.91$, $\zeta = 0.88$, and $\eta = 2.25$). The original risk weights are denoted by Table 2. Modified weights tables for 14 risk criteria are not shown in the study as they would take up too much space.

5. Calculate the overall predominance and the standard overall predominance of the alternative S_m over all others according to Equations (19) and (20). The results are: $\Upsilon(S_1) = -16.6326$,

Table 2: The original risks weight

	A_{11}	A_{12}	A_{13}	A_{14}	B_{11}	B_{22}	C_{31}	C_{32}	C_{33}	C_{34}	C_{35}	C_{36}	C_{37}	C_{38}
ϖ	0.078	0.062	0.084	0.065	0.075	0.096	0.063	0.087	0.070	0.067	0.065	0.056	0.074	0.058

$\Upsilon(S_2) = -0.1402,\ \Upsilon(S_3) = -13.3205,\ \Omega(S_1) = 0.2843,\ \Omega(S_2) = 0.3927,\ \Omega(S_3) = 0.4585.$

6. Rank the standard overall predominance $\Omega(S_3)$ has biggest value that means the S_3 emerges as the most risky case.

5. Discussion

5.1. Comparative Analysis:

In this section, the proposed method is compared with the previously given IF CPT-TODIM [36], PF CPT-TODIM [37] and PHFS CPT-TODIM [19] methods. Table 3 presents the findings of a comparison between the suggested approach and established techniques. It was also seen from the rankings that the results obtained by the new method overlapped with the methods given, especially by PFS and PHFS. However, it was seen that S_1 was the third option in all methods.

Table 3: Ranking comparison

Method	S_1	S_2	S_3
IF CPT-TODIM [37]	3	1	2
PF CPT-TODIM [36]	3	2	1
PHFS CPT-TODIM [19]	3	2	1
Proposed Method	3	2	1

5.2. Superiority of Suggested Method

The FS, IFS, and PFS are combined to create the FFS. Total squares that are equal to or less than one, as well as member and nonmember satisfaction levels, are used to calculate PFS. The decision-maker rarely gives the $\mathcal{M}$ and $\mathcal{N}$ a specific attribute that would make the squares total greater than 1. As a result, the PFS cannot deal with this situation effectively. FFS, which can handle inconsistent and partially unknown data -both prevalent in real-world scenarios- is one of the most complete methods for getting over this restriction.

The results of the suggested strategy overlap with those of the available methods, according to the current and sensitivity assessments. The primary benefit of the suggested method over readily available DM solutions is that it incorporates extra data and tackles data uncertainty by accounting for aspects like $\mathcal{M}$ and $\mathcal{N}$ of criteria. The item's information may be examined more precisely and impartially. In the DM process, it is also a valuable tool for handling imprecise and erroneous data. As a result, the predicted information loss occurs since the reasoning for giving one parameter a score value has no bearing on the other values.

Conversely, there is no discernible loss of information with our suggested method. The intended methodology has an advantage over current approaches in that it can identify the degree of similarity and differentiation across data, avoiding incorrectly motivated judgments. The DM process can be aided by combining unclear and inaccurate information.

5.3. Limitations

This study still raises several issues. First, risk and uncertainty are not the same thing. This study primarily concerns the consequences of risk selection rather than promoting ambiguity. Given the complexity of evaluating women's potential for BC, risk aversion is crucial to uncertainty avoidance. Prospect theory was used in this work to operationalize risk choice. However, in order to identify possible hazards with BC, a more comprehensive evaluation could be necessary. Future research should concentrate on combining general risk choice criteria with particular BC risk markers.

Beyond the advantages of the suggested FF-based technique, its incapacity to fully assess the available possibilities limits its use in particular DM situations. When there are several criteria and options, creating FFSs is simpler. In order to overcome these constraints, we hope further to investigate the following topics in our upcoming work:

- The scope of the application can be expanded to include scenarios that can be obtained with different data.
- Extending the scope of outranking-based interval rough set theory methods -such as VIKOR, ELECTRE, DEMATEL, ANP, FMEA, BWM, and others- is another long-term objective.
- We aim to determine how various MCDM methods can be applied to the FF values.

Despite identifying and listing risks and sub-risks to BC, this article may need to locate and include further risks. Subjective weighting values were taken into consideration while applying the assessments. The results are, therefore, predicated on subjective weighting data.

6. Conclusion

The TODIM approach is always used to examine some MCDM circumstances by considering the DMs confidence level to provide a more reasonable option under risk. As a result, the TODIM method is an ideal MCDM method. However, it has limits. It is not required to give a suitable method for determining attribute weights, nor is it required to present a comprehensive methodology. FFS has emerged as a powerful extension of the FS that enables several degrees of truth connected with each preference information to express ambiguity and vagueness effectively. This article examines the difficulties of MCDM with FFSs. We propose the FF-CPT-TODIM technique, which exceptionally illustrates the actual state of mind for decision-makers based on the corresponding knowledge of FFSs and the classical TODIM method. In addition, an example of breast cancer risk assessment is shown to validate the applicability of the FF-CPT-TODIM method in handling MCDM problems.

As a result of the evaluation made by following the steps of the algorithm, it was seen that S_3-Malignant came first among the risks related to breast cancer. The parameter values may change the calculated result in the fourth section and there is no doubt that we need to select the perfect parameters to address the problem we are studying. The responsibility of this paper is not to analyze the parameters but to establish a brilliant PF-CPT-TODIM method for MCDM issues. In future studies, breast cancer risks can be evaluated using different risk analysis methods. Again, risk assessment can be done using different sets.

Declaration of Ethical Standards

The author declares that the materials and methods used in her study do not require ethical committee and/or legal special permission.

Conflicts of Interest

The author declares no conflict of interest.

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Research Article

On the Generalized Order-k Jacobsthal and Jacobtshal-Lucas Numbers

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Abstract: The classic Jacobsthal numbers were generalized to k sequences of the generalized order-k Jacobsthal numbers and then have been studied by several authors. In this paper, we explain that all of these studies used an incorrect version of order-k Jacobsthal numbers for reasons and give the correct definition of order-k Jacobsthal numbers. Further, we introduce the compatible generalized order-k Jacobsthal-Lucas numbers with the generalized order-k Jacobsthal numbers. Next, we give some properties of order-k Jacobsthal numbers and order-k Jacobsthal-Lucas numbers, including generating matrix, generalized Binet's formula, and elementary matrix identities. Further, we investigate specific examples for our results and give special identities, i.e., sum formula and interrelationships between these sequences.

Keywords: Generalized order-k sequence, Jacobsthal sequence, trace of matrix, Binet's formula, Jacobsthal-Lucas sequence.

1. Introduction

In modern science and daily mathematical practices, a great number of researchers have investigated many integer sequences and their generalizations for a long time, e.g., Fibonacci numbers or Lucas p-numbers. There are many papers and monographs devoted to the subject in the current literature. For example, the readers can read the references in [13, 20]. The main framework of the paper is carved out from the usual Pell and Jacobsthal numbers.

This paper deals with the well-known Jacobsthal $\{J_n\}_{n=0}^{\infty}$ and Jacobsthal-Lucas $\{j_n\}_{n=0}^{\infty}$ sequences, which are defined recursively as

$$J_0 = 0,\ J_1 = 1 \text{ and } J_{n+1} = J_n + 2J_{n-1} \text{ for } n \geqslant 2 \tag{1}$$

and

$$j_0 = 2,\ j_1 = 1 \text{ and } j_{n+1} = j_n + 2j_{n-1} \text{ for } n \geqslant 2, \tag{2}$$

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respectively. It should be noted that, in [10], Horadam investigated extensively and attracted attention to the mentioned sequences. These can be given in the following ways, named Binet's formulas:

$$J_n = \frac{2^n - (-1)^n}{3} \tag{3}$$

and

$$j_n = 2^n + (-1)^n, \tag{4}$$

respectively. In addition, the author of [10] presented some properties for these sequences as follows:

$$j_n J_n = J_{2n}, \tag{5}$$

$$j_n = J_{n+1} + 2J_{n-1}, \tag{6}$$

$$J_{n+1}J_{n-1} - {J_n}^2 = (-1)^n 2^{n-1}, \tag{7}$$

and

$$\sum_{i=1}^{n} j_i = \frac{j_{n+2} - 5}{2}. \tag{8}$$

In [11, 12], Koken and Bozkurt showed that the terms of the mentioned sequences can also be obtained via a generating matrix as follows:

$$F^n = \begin{bmatrix} J_{n+1} & 2J_n \\ J_n & 2J_{n-1} \end{bmatrix} \text{ and } E^n = \begin{cases} 3^n \begin{bmatrix} J_{n+1} & 2J_n \\ J_n & 2J_{n-1} \end{bmatrix} & if\ n\ even \\ 3^{n-1} \begin{bmatrix} j_{n+1} & 2j_n \\ j_n & 2j_{n-1} \end{bmatrix} & if\ n\ odd \end{cases}, \tag{9}$$

where $F = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}$ and $E = \begin{bmatrix} 5 & 2 \\ 1 & 4 \end{bmatrix}$. It should be noted that the following interesting property is satisfied:

$$tr\left(F^n\right) = J_{n+1} + 2J_{n-1}, \tag{10}$$

which is the right-hand side of Equation (6). Here, $tr\,(.)$ denotes the trace of an n-square matrix. Using Equation (1), we get

$$tr\left(F^n\right) = J_n + 4J_{n-1}. \tag{11}$$

Today, there are many systematic investigations regarding the Jacobsthal and Jacobsthal-Lucas sequences. The references given in [4, 5] can be read in this scope.

It should be noted that the main field of studies regarding the second-order sequences is to consider obtaining their generalized versions. These processes were made in various ways. Some of them can be summarized as follows. The second-order sequence can be defined with more general initial conditions, e.g., Horadam [8]; the coefficients of sequence can be chosen from more general

Table 1: Some values of the generalized order-4 Jacobsthal numbers

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
J_n^3	1	2	4	9	20	44	97	214	472	1041	2296	5064	11169	24634	54332
J_{-n}^3	0	1	0	-2	1	4	-4	-7	12	10	-31	-8	72	-15	-152
J_n^4	1	1	3	6	14	30	67	147	325	716	1580	3484	7685	16949	37383
J_{-n}^4	0	0	1	-1	-1	2	2	6	-1	13	-3	-28	20	52	-67

terms, e.g., Horadam [9]; each term of the sequence can be defined as a linear combination of the preceding arbitrary two terms e.g. Stakhov [16]; each term is a linear combination of k preceding terms with k initial conditions, e.g., Miles [15]; or the Binet's formula of the sequence can be considered in the general form, e.g., Stakhov and Rozin [17]. There are also many papers as in the references [1–3, 7, 18, 19] devoted to the subject.

In particular, we would like to mention a paper herein. In [21], Yilmaz and Bozkurt presented a new generalization of the second-order Jacobsthal numbers, inspired by Miles [15], as follows:

$$J_n^i = J_{n-1}^i + 2J_{n-2}^i + \cdots + J_{n-k+1}^i + J_{n-k}^i \tag{12}$$

for $n > 0$ and $1 \leqslant i \leqslant k$, with initial terms

$$J_n^i = \begin{cases} 1, & \text{if } i+n=1 \\ 0, & \text{otherwise} \end{cases} \quad \text{for } 1-k \leqslant n \leqslant 0, \tag{13}$$

where J_n^i is the nth term of the ith sequence. Clearly, when $k = 2$ and $i = 1$, this definition reduces to the the famous Jacobsthal sequence. The mentioned paper also displays some properties of the sequence, generalized Binet's formula and summation formula.

However, based on some reasons, we think that this definition, unfortunately, is not proper, namely not a generalization of the Jacobsthal numbers. It does not satisfy many chronic and genetic identities of the Jacobsthal sequence. For example, considering the definition in Equation (12) and initial conditions in (13), the terms of the generalized sequence with the negative subscripts are also an integer. But, as known from the results of Daşdemir [6], the terms with negative subscripts should not be integers. Moreover, one positive – one negative, consecutively, order of terms with negative subscripts is also not provided. Besides, there are also many issues similar to these situations. For a concrete example, letting $i = 3$ or $i = 4$ and $k = 4$, we can, thus, obtain the terms with negative subscripts of the generalized order-4 Jacobsthal numbers as in Table 1.

Based on the justification briefly explained above, in this paper, we give the true definition for generalization of the usual Jacobsthal sequence, i.e., generalized order-k Jacobsthal sequence, and then summarize elementary properties, including the generating matrix and generalized Binet's formula. One of the most important highlights of this study is that k-sequences of the generalized order-k Jacobsthal-Lucas sequence are defined. To do this, we will make use of the miscellaneous properties of matrices. Further, for this new definition, appropriate initial conditions that are of two different forms are also given. In particular, it is stated that the same integer sequences are obtained in both cases but the order of the sequences is permutationally changed.

2. On the Generalized Order-k Jacobsthal Numbers

In this section, we present the results regarding the generalized order-k Jacobsthal numbers. Note that since all the conclusions can be proved easily by using the known ways in the current literature, we will omit the proof processes in order not to bore the readers. For this purpose, the following definition is the starting point of the paper.

Definition 2.1 *k sequences of the generalized order-k Jacobsthal numbers are defined as*

$$J_n^i = J_{n-1}^i + J_{n-2}^i + \cdots + J_{n-k+1}^i + 2J_{n-k}^i \tag{14}$$

for $n > 0$ and $1 \leqslant i \leqslant k$, with initial terms

$$J_n^i = \begin{cases} 1, & \text{if } i+n=1 \\ 0, & \text{otherwise} \end{cases} \quad \text{for } 1-k \leqslant n \leqslant 0, \tag{15}$$

where J_n^i is the nth term of the ith sequence.

For the case where $i = 1$ and $k = 2$, our definition is reduced directly to the usual Jacobsthal numbers, i.e., $J_n^1 = J_n$, and when $i = k = 2$, it is reduced to two times of the Jacobsthal numbers, namely $J_n^2 = 2J_n$. In particular, the sequence J_n^k is called the generalized k-Jacobsthal numbers in the case of $i = k$. By the way, Table 2 displays some values of the generalized order-k Jacobsthal numbers, including the related initial conditions.

After this point, we will summarize the results regarding the generalized order-k Jacobsthal numbers. Let us define the following matrices:

$$\mathbf{A}_k = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & 2 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix} \text{ and } \mathbf{J}_{\tilde{k},n} = \begin{bmatrix} J_n^1 & J_n^2 & \cdots & J_n^{k-1} & J_n^k \\ J_{n-1}^1 & J_{n-1}^2 & \cdots & J_{n-1}^{k-1} & J_{n-1}^k \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ J_{n-k-2}^1 & J_{n-k-2}^2 & \cdots & J_{n-k-2}^{k-1} & J_{n-k-2}^k \\ J_{n-k-1}^1 & J_{n-k-1}^2 & \cdots & J_{n-k-1}^{k-1} & J_{n-k-1}^k \end{bmatrix}. \tag{16}$$

It is clear from Definition 2.1 that $\mathbf{J}_{\tilde{k},n} = \mathbf{A}_k \mathbf{J}_{\tilde{k},n-1}$. Expanding the right-hand side of this equation yields the following theorem.

Theorem 2.2 *The matrix equation*

$$\mathbf{J}_{\tilde{k},n} = {\mathbf{A}_k}^n \tag{17}$$

holds for a positive integer n.

Table 2: Some values of the generalized order-k Jacobsthal numbers

	$k=2$		$k=3$			$k=4$			
n/i	1	2	1	2	3	1	2	3	4
-7	$-\frac{21}{64}$	$\frac{43}{64}$	$-\frac{9}{32}$	$\frac{23}{32}$	$-\frac{9}{32}$	$-\frac{1}{16}$	$-\frac{1}{16}$	$-\frac{1}{16}$	$\frac{15}{16}$
-6	$\frac{11}{32}$	$-\frac{21}{32}$	$\frac{7}{16}$	$-\frac{9}{16}$	$-\frac{9}{16}$	$-\frac{1}{8}$	$-\frac{1}{8}$	$\frac{7}{8}$	$-\frac{1}{8}$
-5	$-\frac{5}{6}$	$\frac{11}{16}$	$-\frac{1}{8}$	$-\frac{1}{8}$	$\frac{7}{8}$	$-\frac{1}{4}$	$\frac{3}{4}$	$-\frac{1}{4}$	$-\frac{1}{4}$
-4	$\frac{3}{8}$	$-\frac{5}{8}$	$-\frac{1}{4}$	$\frac{3}{4}$	$-\frac{1}{4}$	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$
-3	$-\frac{1}{4}$	$\frac{3}{4}$	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	0	0	0	1
-2	$\frac{1}{2}$	$-\frac{1}{2}$	0	0	1	0	0	1	0
-1	0	1	0	1	0	0	1	0	0
0	1	0	1	0	0	1	0	0	0
1	1	2	1	1	2	1	1	1	2
2	3	2	2	3	2	2	2	3	2
3	5	6	5	4	4	4	5	4	4
4	11	10	9	9	10	9	8	8	8
5	21	22	18	19	18	17	17	17	18
6	43	42	37	36	36	34	34	35	34
7	85	86	73	73	74	68	69	68	68

	$k=5$					$k=6$					
n/i	1	2	3	4	5	1	2	3	4	5	6
-7	$-\frac{1}{8}$	$-\frac{1}{8}$	$\frac{7}{8}$	$-\frac{1}{8}$	$-\frac{1}{8}$	$-\frac{1}{4}$	$\frac{3}{4}$	$-\frac{1}{4}$	$-\frac{1}{4}$	$-\frac{1}{4}$	$-\frac{1}{4}$
-6	$-\frac{1}{4}$	$\frac{3}{4}$	$-\frac{1}{4}$	$-\frac{1}{4}$	$-\frac{1}{4}$	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$
-5	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	0	0	0	0	0	1
-4	0	0	0	0	1	0	0	0	0	1	0
-3	0	0	0	1	0	0	0	0	1	0	0
-2	0	0	1	0	0	0	0	1	0	0	0
-1	0	1	0	0	0	0	1	0	0	0	0
0	1	0	0	0	0	1	0	0	0	0	0
1	1	1	1	1	2	1	1	1	1	1	1
2	2	2	2	3	2	2	2	2	2	3	2
3	4	4	5	4	4	4	4	4	5	4	4
4	8	9	8	8	8	8	8	9	8	8	8
5	16	16	16	16	16	16	17	16	16	16	16
6	33	33	33	33	34	33	32	32	32	32	32
7	66	66	66	67	66	65	65	65	65	65	66

Next, the following results are satisfied.

Corollary 2.3 *Let n be any positive integer. Then, we have*

$$\det\left(\mathbf{J}_{k,n}^{\sim}\right) = \begin{cases} 2^n, & \text{if } k \text{ is odd} \\ (-2)^n, & \text{if } k \text{ is even} \end{cases}. \tag{18}$$

Lemma 2.4 *For a positive integer n, we have*

$$J_n^i = J_{n-1}^1 + J_{n-1}^{i+1} \text{ and } 2J_{n+1}^1 = J_n^k. \tag{19}$$

Theorem 2.2 says us that $\mathbf{J}_{k,n}^{\sim}$ is a generating matrix for the generalized order-k Jacobsthal numbers. This means that readers have a powerful tool for discovering their new identities. For instance, by taking the multiplication identities of matrices into account, we can write

$$\mathbf{J}_{k,n+m}^{\sim} = \mathbf{J}_{k,n}^{\sim}\mathbf{J}_{k,m}^{\sim} = \mathbf{J}_{k,m}^{\sim}\mathbf{J}_{k,n}^{\sim}. \tag{20}$$

We can, therefore, write the following strange result as an example.

Theorem 2.5 *Let n and m be any positive integers. Then,*

$$J_{n+m}^i = \sum_{j=1}^{k} J_n^j J_{m-j+1}^i. \tag{21}$$

Letting also

$$\mathbf{J} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & & & & \\ 0 & & & & \\ 0 & & & & \\ \vdots & & & & \\ 0 & & & & \mathbf{A}_k \end{bmatrix} \text{ and } \mathbf{T}_n = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ S_n & & & & \\ S_{n-1} & & & & \\ \vdots & & & & \\ S_{n-k+1} & & & & \mathbf{J}_{k,n}^{\sim} \end{bmatrix}, \tag{22}$$

where $S_n = \sum_{i=0}^{n-1} J_i^1$. Moreover, using the fact that $J_n^1 = \frac{1}{2}J_{n+1}^k$ yields $S_n = \frac{1}{2}\sum_{i=0}^{n} J_i^k$ and $S_n = J_{n-1}^1 + S_{n-1}$. Herein, according to all these explanations and since $S_{-i} = 0$ for $1 \le i \le k$ and $\mathbf{T}_1 = \mathbf{J}$, we can write

$$\mathbf{T}_{n+1} = \mathbf{T}_n\mathbf{J} = \mathbf{T}_{n-1}\mathbf{J}^2 = \ldots = \mathbf{T}_1\mathbf{J}^{n-1} = \mathbf{J}^n.$$

In this case, computing the right-hand side of the equation $\mathbf{T}_n = \mathbf{T}_1\mathbf{T}_{n-1}$, we can write

$$\begin{aligned} S_n = 1 + S_{n-1} + 2S_{n-2} &= 1 + S_n - J_{n-1}^1 + 2\left(S_{n-1} - J_{n-2}^1\right) \\ &= 1 + S_n - J_{n-1}^1 + 2\left(S_n - J_{n-1}^1 - J_{n-2}^1\right) \end{aligned}$$

and the following result can be given.

Theorem 2.6 *Let n be an positive integer. Then, we have*

$$S_n = \frac{1}{2}\left(3J_{n-1}^1 + 2J_{n-2}^1 - 1\right). \tag{23}$$

It should be noted that for $k = 2$, the summation in Equation (23) is reduced to the famous formula of the usual Jacobsthal numbers as

$$\sum_{i=1}^{n-1} J_i = \frac{J_{n+1} - 1}{2}.$$

It should be noted that Equation (14) is actually the k-order linear homogeneous difference equation, with constant coefficients, in the form of

$$x_n = x_{n-1} + x_{n-2} + \cdots + x_{n-k+1} + 2x_{n-k}.$$

We can then explore a solution to the last equation as $x_n = \lambda^n$, where λ is an unknown constant to be determined. On the substitution of this linear solution into our difference equation, we can, therefore, write

$$\lambda^n - \lambda^{n-1} - \lambda^{n-2} - \cdots - \lambda^{n-k+1} - 2\lambda^{n-k} = 0$$

or equally

$$(\lambda - 2)\left(\lambda^{n-1} + \lambda^{n-2} + \cdots + \lambda + 1\right) = 0.$$

Therefore, our characteristic equation has the distinct roots such as $\lambda_1 = 2$ and $\lambda_j = e^{\frac{2\pi i j}{k}}$ for $j = 2, 3, \cdots, k$, which are the eigenvalues of the matrix $\mathbf{A}_k$. Let $\mathbf{V}$ be Vandermonde matrix as follows:

$$\mathbf{V} = \begin{bmatrix} {\lambda_1}^{k-1} & {\lambda_2}^{k-1} & \cdots & {\lambda_k}^{k-1} \\ {\lambda_1}^{k-2} & {\lambda_2}^{k-2} & \cdots & {\lambda_k}^{k-2} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_1 & \lambda_2 & \cdots & \lambda_k \\ 1 & 1 & \cdots & 1 \end{bmatrix}.$$

Also, we consider the vector

$$\mathbf{b}_k{}^i = \left[\begin{array}{cccc} {\lambda_1}^{n+k-i} & {\lambda_2}^{n+k-i} & \ldots & {\lambda_k}^{n+k-i} \end{array}\right]^T.$$

Also, $\mathbf{V}_j{}^{(i)}$ be k-square matrix obtained from $\mathbf{V}$ by replacing the jth column of $\mathbf{V}$ with $\mathbf{b}_k{}^i$. Then, we have the generalized Binet's formula for the generalized order-k Jacobsthal numbers in the following theorem.

Theorem 2.7 *Let J_n^i be the nth term of ith Jacobsthal sequence for $1 \le i \le k$. Then, we have*

$$J_{n-i+1}^j = \frac{\det\left(\mathbf{V}_j{}^{(i)}\right)}{\det\left(\mathbf{V}\right)}.$$

We now focus on the generating functions for the generalized order-k Jacobsthal numbers. Fur this purpose, introduce the function

$$G^i{}_k(x) = \sum_{v=0}^{\infty} J_v{}^i x^v = J_0{}^i + J_1{}^i x + J_2{}^i x^2 + \cdots + J_k{}^i x^k + \cdots .$$

In contrast to the current literature, in our definition, the superscript i is arbitrary over its possible values, not fixed. In this case, the following important result can be given.

Theorem 2.8 *The generating functions for k sequences of the generalized order-k Jacobsthal numbers are*

$$G^i{}_k(x) = \frac{J_0{}^i + x\left(\sum_{v=0}^{k+n-1}{}^{*} x^v\right) J_n^i}{1 - x - x^2 - x^3 - \cdots - x^{k-1} - 2x^k}, \tag{24}$$

where $1-k \leqslant n \leqslant -1$ and the asterisk in summation denotes a protocol such that only the last term of the finite series is multiplied by 2.

Proof Summing $G^i{}_k(x)$, $-xG^i{}_k(x)$, $-x^2G^i{}_k(x)$, $\cdots$, $-x^{k-1}G^i{}_k(x)$, $-2x^kG^i{}_k(x)$ up, we can write

$$\left(1 - x - x^2 - x^3 - \cdots - x^{k-1} - 2x^k\right) G^i{}_k(x) = J_0{}^i + \left(J_1{}^i - J_0{}^i\right) x + \left(J_2{}^i - J_1{}^i - J_0{}^i\right) x^2$$
$$+ \left(J_3{}^i - J_2{}^i - J_1{}^i - J_0{}^i\right) x^3 + \cdots + \left(J_{k-1}{}^i - J_{k-2}{}^i - J_{k-3}{}^i - J_0{}^i\right) x^{k-1}$$

or in other situation,

$$\left(1 - x - x^2 - x^3 - \cdots - x^{k-1} - 2x^k\right) G^i{}_k(x) = J_0{}^i + \left(J_{-1}^i + \cdots + J_{-k+2}^i + 2J_{-k+1}^i\right) x$$
$$+ \left(J_{-1}^i + \cdots + J_{-k+3}^i + 2J_{-k+2}^i\right) x^2 + \left(J_{-1}^i + \cdots + J_{-k+4}^i + 2J_{-k+3}^i\right) x^3 + \cdots + 2J_{-1}^i x^{k-1}.$$

The coefficients of the last equation only consists of the initial conditions. Hence, the result follows. □

As an example, when $i = 3$ and $k = 7$, the generating function for the third sequence of the generalized order-7 Jacobsthal numbers is found as follows:

$$G^3{}_7(x) = \frac{x\left(1 + x + x^2 + x^3 + 2x^4\right)}{1 - x - x^2 - x^3 - x^4 - x^5 - x^6 - 2x^7}.$$

In addition, for the cases $i = 1, k = 2$ and $i = k = 2$, Equation (24) takes the shape

$$G^1{}_2(x) = \frac{1}{1 - x - 2x^2} \text{ and } G^2{}_2(x) = \frac{x}{1 - x - 2x^2},$$

respectively. In particular the second equation is in the well-known form.

3. On the Generalized Order-k Jacobsthal-Lucas Sequence

In this section, we consider the generalized order-k Jacobsthal-Lucas numbers. For this purpose, two special cases will be handled.

3.1. Constructing Generalized Jacobsthal-Lucas Sequences by Employing Trace Operator

As remembered, Equations (10) and (11) present a wonderful link between the Jacobsthal and Jacobsthal-Lucas numbers as follows:

$$j_n = tr\left(F^n\right) = J_n + 4J_{n-1} = J_n + 2\left(2J_{n-1}\right).$$

We can, therefore, use this idea to construct higher-ordered Jacobsthal-Lucas numbers. For this purpose, we, firstly, introduce tri-Jacobsthal numbers of third-order recurrence relation

$$J_0^3 = 0,\ J_1^3 = 0,\ J_2^3 = 1 \text{ and } J_{n+1}^3 = J_n^3 + J_{n-1}^3 + 2J_{n-2}^3 \text{ for } n \geqslant 0$$

and their generating matrix is

$$F_3 = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Note that these definitions can be obtained by simple vector-matrix operations. On the basis of the induction method, one can prove

$$F_3{}^n = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}^n = \begin{bmatrix} J_{n+2}^3 & J_{n+1}^3 + 2J_n^3 & 2J_{n+1}^3 \\ J_{n+1}^3 & J_n^3 + 2J_{n-1}^3 & 2J_n^3 \\ J_n^3 & J_{n-1}^3 + 2J_{n-2}^3 & 2J_{n-1}^3 \end{bmatrix}.$$

As a result, we get

$$tr\left(F_3{}^n\right) = J_{n+2}^3 + J_n^3 + 2J_{n-1}^3 + 2J_{n-1}^3 = J_{n+1}^3 + 2J_n^3 + 6J_{n-1}^3 = J_{n+1}^3 + 2J_n^3 + 2\left(3J_{n-1}^3\right).$$

After a similar process, one can write

$$tr\left(F_4{}^n\right) = J_{n+2}^4 + 2J_{n+1}^4 + 3J_n^4 + 2\left(4J_{n-1}^4\right)$$

and

$$tr\left(F_5{}^n\right) = J_{n+3}^4 + 2J_{n+2}^4 + 3J_{n+1}^4 + 4J_n^4 + 2\left(5J_{n-1}^4\right).$$

This process regularly continues as above with minor changes for increasing values of order. Hence, we can write this observation in a more general form as in the following.

Corollary 3.1 *Let J_n^k be a generalized Jacobsthal numbers of order-k in Definition 2.1. Then, the generalized Jacobsthal-Lucas numbers $\left\{j_n^k\right\}_{n=0}^{\infty}$ of order-k satisfies the interrelationship*

$$j_n^k = J_{n+k-2}^k + 2J_{n+k-3}^k + 3J_{n+k-4}^k + \cdots + (k-2)\,J_{n-3}^k + (k-1)\,J_{n-2}^k + 2kJ_{n-1}^k.$$

[h!]

Table 3: Some values of the generalized order-2, order-3 and order-4 Jacobsthal-Lucas numbers

	$k=2$		$k=3$			$k=4$			
n	$i=$ 1	2	1	2	3	1	2	3	4
-3						$-\frac{7}{8}$	$\frac{1}{8}$	$\frac{9}{8}$	$\frac{49}{8}$
-2			$-\frac{3}{4}$	$\frac{1}{4}$	$\frac{17}{4}$	$-\frac{3}{4}$	$\frac{1}{4}$	$\frac{21}{4}$	$-\frac{7}{4}$
-1	$-\frac{1}{2}$	$\frac{5}{2}$	$-\frac{1}{2}$	$\frac{7}{2}$	$-\frac{3}{2}$	$-\frac{1}{2}$	$\frac{9}{2}$	$-\frac{5}{2}$	$-\frac{3}{2}$
0	2	-1	3	-2	-1	4	-3	-2	-1
1	1	4	1	2	6	1	2	3	8
2	5	2	3	7	2	3	4	9	2
3	7	10	10	5	6	7	12	5	6
4	17	14	15	16	20	19	12	13	14
5	31	34	31	35	30	31	32	33	38

It should be noted that Corollary 3.1 is a general result. One can obtain the generalized Jacobsthal-Lucas numbers from the equation

$$j_n = J_{n+k-2} + 2J_{n+k-3} + 3J_{n+k-4} + \cdots + (k-2)\,J_{n-3} + (k-1)\,J_{n-2} + 2kJ_{n-1}.$$

These numbers satisfy the recurrence relation

$$j_n = j_{n-1} + j_{n-2} + j_{n-3} + j_{n-4} + \ldots + 2j_{n-k}$$

with the initial conditions $j_0 = k$ and for $1 \le i < k$, $j_{-k+i} = 2^{-k+i} - 1$.

Now, we obtain some terms of generalized order-k Jacobsthal-Lucas numbers. The easiest way of this purpose is to use the matrix $\mathbf{A}_k$ in Equation (16) with the matrix

$$\mathbf{M}_k = [1, 2, 3, \cdots, k-1, 2k].$$

Then, for any integer n, nth terms of the generalized order-k Jacobsthal-Lucas numbers can be found with the equation

$$[j_n^1, j_n^2, j_n^3, \cdots, j_n^k] = \mathbf{M}_k \times \mathbf{A}_k^{n-1}. \tag{25}$$

Using Equation (25), we can give some values of generalized order-k Jacobsthal-Lucas numbers which are given in the Table ??.

Let us define the following matrices to use matrix methods for the generalized order−k Jacobsthal-Lucas numbers

$$\mathbf{B}_k = \begin{bmatrix} k & 1-k & 2-k & 3-k & 4-k & \cdots & -1 \\ 2^{-1}-1 & 2^{-1}+k & 2^{-1}-k+1 & 2^{-1}-k+2 & 2^{-1}-k+3 & \cdots & 2^{-1}-2 \\ 2^{-2}-1 & 2^{-2} & 2^{-2}+k+1 & 2^{-2}-k+2 & 2^{-2}-k+3 & \cdots & 2^{-2}-2 \\ 2^{-3}-1 & 2^{-3} & 2^{-3}+1 & 2^{-3}+k+2 & 2^{-2}-k+3 & \cdots & 2^{-3}-2 \\ 2^{-4}-1 & 2^{-4} & 2^{-4}+1 & 2^{-4}+2 & \ddots & \cdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 2^{-k}-1 & 2^{1-k} & \cdots & \cdots & 2^{1-k}+3 & \cdots & 2^{1-k}+2k-2 \end{bmatrix} \tag{26}$$

and

$$\mathbf{j}_{k,n}^{\sim} = \begin{bmatrix} j_n^1 & J_n^2 & \cdots & j_n^{k-1} & j_n^k \\ j_{n-1}^1 & j_{n-1}^2 & \cdots & j_{n-1}^{k-1} & j_{n-1}^k \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ j_{n-k-2}^1 & j_{n-k-2}^2 & \cdots & j_{n-k-2}^{k-1} & j_{n-k-2}^k \\ j_{n-k-1}^1 & j_{n-k-1}^2 & \cdots & j_{n-k-1}^{k-1} & j_{n-k-1}^k \end{bmatrix}. \tag{27}$$

It is easy to prove the following theorem.

Theorem 3.2 *The matrix equation*

$$\mathbf{j}_{k,n}^{\sim} = \mathbf{B}_k \times \mathbf{A}_k^n \tag{28}$$

holds for a positive integer n*.*

This preparation leads us to the definition of the generalized order$-k$ Jacobsthal-Lucas numbers as follows.

Definition 3.3 *The* k*-sequences of the generalized order*$-k$ *Jacobsthal-Lucas numbers* $(KSOKJ-L)$ *for* $n > k$ *and* $1 \leqslant i \leqslant k$ *are defined as*

$$j_n^i = j_{n-1}^i + j_{n-2}^i + j_{n-3}^i + j_{n-4}^i + \ldots + 2j_{n-k}^i$$

with initial conditions

$$j_{k,n}^i = \begin{cases} k, & \text{if } (i,n) = (1,0) \\ -1-k+i, & if\ n = 0 \\ 2^n + i - 2, & \text{if } 2-k \leqslant i+n \leqslant 0 \\ 2^n + i + k - 2, & \text{if } i+n = 1,\ i > 1 \\ 2^n + i - k - 2, & \text{if } i+n \geqslant 2 \text{ and } n \neq 0 \end{cases},$$

We can find generating function and Binet's formula for the generalized order$-k$ Jacobsthal-Lucas numbers via Theorem 3.2 similar to the generalized order$-k$ Jacobsthal numbers mentioned above. But we don't give these calculations since these repetitions may be tedious for the readers.

3.2. Constructing Generalized Jacobsthal-Lucas Sequences by Derivative of Core Polynomial

Another method to obtain the generalized order$-k$ Jacobsthal-Lucas numbers is to use core polynomial. Let's define $P(x,t_1,t_2,\cdots,t_k) = x^k - t_1x^{k-1} - t_2x^{k-2} - \cdots - t_k$, where $t_1,t_2,\cdots,t_k$ are constants. So, its derivative is $P'(x,t_1,t_2,\cdots,t_k) = kx^{k-1} - t_1(k-1)x^{k-2} - \cdots - t_{k-1}$. It is obvious that if we take $t_1 = t_2 = \cdots = t_{k-1} = 1$ and $t_k = 2$, this polynomial reduces to the characteristic equation of order$-k$ Jacobsthal and Jacobsthal-Lucas numbers. We define the following matrix

$$\mathbf{C}_k = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 2 & 1 & 1 & \cdots & 1 \end{bmatrix}. \tag{29}$$

Then we can obtain the initial conditions of the generalized order$-k$ Jacobsthal-Lucas numbers by using the equation (see [14] for details)

$$\mathbf{j}'_{k,0} = -\mathbf{C}_k{}^{-k+1} - 2\mathbf{C}_k{}^{-k+2} - 3\mathbf{C}_k{}^{-k+3} - \cdots + k\mathbf{C}_k{}^{0}. \tag{30}$$

For example, if we take $k = 3$, then we have

$$\mathbf{C}_3 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & 1 & 1 \end{bmatrix} \tag{31}$$

and Equation (30) gives

$$\mathbf{j}'_{3,0} = \begin{bmatrix} \frac{17}{4} & \frac{1}{4} & -\frac{3}{4} \\ -\frac{3}{2} & \frac{7}{2} & -\frac{1}{2} \\ -1 & -2 & 3 \end{bmatrix}. \tag{32}$$

If we take $k = 4$, Equation (30) gives

$$\mathbf{j}'_{4,0} = \begin{bmatrix} \frac{49}{8} & \frac{9}{8} & \frac{1}{8} & -\frac{7}{8} \\ -\frac{7}{4} & \frac{21}{4} & \frac{1}{4} & -\frac{3}{4} \\ -\frac{3}{2} & -\frac{5}{2} & \frac{9}{2} & -\frac{1}{2} \\ -1 & -2 & -3 & 4 \end{bmatrix}. \tag{33}$$

Examining Equations (32) and (33) with Table 3, it is clear that matrices $\mathbf{j}^{\sim}_{k,0}$ and $\mathbf{j}'_{k,0}$ give same sequences in different order. Namely, let σ be the permutation $\begin{pmatrix} 1 & 2 & 3 & \cdots & k \\ k & k-1 & k-2 & \cdots & 1 \end{pmatrix} \in S_k$. Then nth column of the matrix $\mathbf{j}^{\sim}_{k,0}$ is the $\sigma(n)-$th column of the matrix $\mathbf{j}'_{k,0}$.

Finally, we are ready to define another form of generalized order$-k$ Jacobsthal-Lucas numbers by using the matrix $\mathbf{j}'_{k,0}$.

Definition 3.4 *The $k-$ sequences of the generalized order-k Jacobsthal-Lucas numbers satisfy the following recurrence relation for $n > k$ and $1 \leqslant i \leqslant k$*

$$j_n^i = j_{n-1}^i + j_{n-2}^i + j_{n-3}^i + j_{n-4}^i + \cdots\cdots\cdots + 2j_{n-k}^i$$

with initial conditions for $1 - k \leq i \leq 0$

$$j_{k,n}^i = \begin{cases} 2^n - i - 1, & \text{if } i - n < k \\ 2^n + 2k - i - 1, & \text{if } i - n = k \\ 2^n + k - i - 1, & \text{if } i - n > k \end{cases}.$$

Example 3.5 *For $k = 8$ and $i = 5$, Definition 3.3 gives the following sequence*

$$\{j_{8,n}^5\}_{n=-7}^{\infty} = \left\{\frac{385}{128}, \frac{193}{64}, \frac{97}{32}, -\frac{39}{8}, -\frac{19}{4}, -\frac{9}{2}, -4, 5, 7, 11, 27, 27, 59, 123, 251, 515, \cdots\right\}.$$

The same sequence can be obtained by taking $k = 8$ and $i = 4$ in Definition 3.4.

4. Conclusions

Due to an upward trend and scientific importance in mathematics and other branches, the integer sequences and their generalizations become indispensable to exploring wide usage areas and applications of the sequences under consideration. As the authors, while reviewing the current literature, we have caught our attention that the definition given for $k-$sequences of the generalized order-k Jacobsthal numbers is incorrect based on several reasons. To address this issue, at first, we presented the correct definition for the mentioned generalization. Unfortunately, our definition overrides all the results of the papers produced within the scope of the study by Yilmaz and Bozkurt [21]. Then, after the presentation of the definition, some fundamental identities of the generalization under consideration were performed, e.g., generating matrix, generating functions, and summation formula. Following, we took how to generalize the usual Jacobsthal-Lucas sequence in the framework of our new definition regarding the generalized Jacobsthal numbers into account. Instead of ordinary approaches in the literature, we developed combinatorial modeling to generalize the Jacobsthal-Lucas sequence to $k-$sequences of the generalized order-k numbers and gave two new definitions for that aim. Both definitions satisfy the following recurrence relation

$$j_n^i = j_{n-1}^i + j_{n-2}^i + j_{n-3}^i + j_{n-4}^i + \ldots + 2j_{n-k}^i \quad (\text{for } n > k \text{ and } 1 \leqslant i \leqslant k),$$

where two different initial conditions

$$j_{k,n}^i = \begin{cases} k, & if\ (i,n) = (1,0) \\ -1-k+i, & if\ n = 0 \\ 2^n + i - 2, & if\ 2-k \leqslant i+n \leqslant 0 \\ 2^n + i + k - 2, & if\ i+n = 1,\ i > 1 \\ 2^n + i - k - 2, & if\ i+n \geqslant 2 \text{ and } n \neq 0 \end{cases}$$

and

$$j_{k,n}^i = \begin{cases} 2^n - i - 1, & if\ i-n < k \\ 2^n + 2k - i - 1, & if\ i-n = k \\ 2^n + k - i - 1, & if\ i-n > k \end{cases}.$$

It should be noted that both definitions are, in fact, the permutationally same. Namely, ith sequence in the one generalization implies $(k-i+1)$-th sequence in the other one.

Another remarkable relation is the following relation between $k-$sequences of the generalized order-k Jacobsthal and Jacobsthal-Lucas numbers

$$j_n^k = J_{n+k-2}^k + 2J_{n+k-3}^k + 3J_{n+k-4}^k + \cdots + (k-2)\, J_{n-3}^k + (k-1)\, J_{n-2}^k + 2kJ_{n-1}^k.$$

As a concluding remark, we imply that the novelties of this study also provide researchers with many potential research opportunities.

Declaration of Ethical Standards

The authors declare that the materials and methods used in their study do not require ethical committee and/or legal special permission.

Authors Contributions

Author [Ahmet Daşdemir]: Evaluation of data, contributed to the research method, solving the problem, wrote the manuscript (%40).

Author [Göksal Bilgici]: Thought and designed the research/problem, solving the problem, wrote the manuscript (%35).

Author [Hossen Mohammed Mahdi Ahmed]: Collected the data, evaluation of data (%25).

Conflicts of Interest

The authors declare no conflict of interest.

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Research Article

On the Sufficient Conditions for the Univalence of Definite Integral Operators Involving Certain Functions in $\mathcal{S}$ Class

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Abstract: In general terms, integral operators play a very important role as a useful mathematical tool in order to reach the desired results and make different inferences by analyzing the relevant issues in mathematics and applied sciences. It is important to understand the conditions under which integral operators map certain analytic functions to starlike and convex functions and effectively characterizing and using them is of great importance for studies in this field. In present article, some integral operators preserving class $\mathcal{S}$ are examined from a different perspective and the relevant inequalities and equations for their univalence are determined and solved.

Keywords: Analytic function, convex function, normalized function, starlike function, univalent function.

1. Introduction

As the interaction of analysis and geometry, geometric function theory is a very interesting subbrunch of complex analysis. Perhaps the important reason for this interest is the image sets of complex functions to which certain conditions (such as being analytic, being normalized, being univalent, and being defined in the unit disc) exhibit very interesting geometric characterizations. In this sense, geometric function theory aims, in principle, to analysis the analytic properties of analytic functions depending on the geometric properties of their image sets. Moreover, geometric function theory also aims to classify functions with certain properties given above according to the common geometric characterizations exhibited by image sets (such as convex, starlike, close-to-convex, etc.). The arguments used in doing this are depends on Riemann mapping theorem in 1851 [16]. It is well known that, under certain conditions, the Riemann mapping theorem guarantees the existence of an analytic function that conformal maps a simply connected region of the complex plane to the open unit disc $|z| < 1, z \in \mathbb{C}$ (hereafter represented with $\mathcal{U}$). In more mathematical terms, where $\mathcal{D} \subset \mathbb{C}$ is a simply connected region with more than one boundary

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points, for any $z_0 \in \mathcal{D}$ there is a single function f that satisfies the conditions $f(z_0) > 0$ and $f'(z_0) > 0$ and conformally maps $\mathcal{D}$ to $\mathcal{U}$. Unfortunately, the Riemann mapping theorem in its current form creates a complicated situation for classifying analytic functions. The complicated situation is that it is very difficult or even impossible to classify the analytic functions defined on different domains according to the common geometric characterizations exhibited by the image sets. The complicated situation expressed was eliminated when Paul Koebe one of the intellectual scientists working in this field, took the open unit disc $\mathcal{U}$ as the domain in 1907, without losing generality. This idea is, in a sense, the inverse of Riemann's mapping theorem. Now, analytic functions with domains $\mathcal{U}$ can be classified [5, 6, 9].

As you may remember from the basic complex analysis information, if derivative

$$f'(z) = \lim_{\Delta \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} \tag{1}$$

exists for each $z \in \mathcal{D}$, the function $f(z)$ is said to be analytic in the set $\mathcal{D} \in \mathbb{C}$. Let us denote by $\mathcal{H}$ class formed by all complex functions thet are analytic in $\mathcal{U}$. In addition, as a subclass of the class $\mathcal{H}$, let's denote with $\mathcal{A}$ the class consisting of all functions in the class $\mathcal{H}$ that satify the conditions $f(0) = 0$ and $f'(0) = 1$, known as normalized conditions. Notice that the functions of class $\mathcal{A}$ consist of normalized analytic functions in $\mathcal{U}$. In addition to all these, if the condition of being one-to-one (that is, for all $z_1, z_2 \in \mathcal{U}, f(z_1) = f(z_2)$ implies $z_1 = z_2$(A.W. Goodman, 1983)) [7, 9] is imposed as a new condition on the functions in class $\mathcal{A}$ is formed, which is denotes by $\mathcal{S}$. In studies conducted in this field, a function that s both analytic and one-to-one in $\mathcal{U}$ is called a univalent function. Note that univalent implies being both analytic and one-to-one in $\mathcal{U}$. In the finall analysis, under the conditions given above, naturally any function $f(z)$ in the class $\mathcal{S}$ has a Taylor expansion given by

$$w = f(z) = z + \sum_{n=2}^{\infty} a_n z^n = z + a_2 z^2 + \ldots + a_n z^n + \ldots, z \in \mathcal{U} \text{ [9]}. \tag{2}$$

As stated above, geometric function theory focuses on the concept of univalence and analyticity. Riemann Mapping Theorem plays an important role in unifying both concepts. This combination interprets the geometric characterizations of sets of images in order to classify functions. It is well known that, $\mathcal{S}^*$ and $\mathcal{C}$ are the two usual subclasses of class $\mathcal{S}$ of starlike and convex functions, which geometric characterizations of image sets satisfy the inequalities

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > 0 \tag{3}$$

and

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0, \tag{4}$$

respectively [11, 17]. Therefore, these two classes can be given analytically as follows:

$$\mathcal{S}^* = \left\{ f(z) \in \mathcal{A} : \Re\left(\frac{zf'(z)}{f(z)} \right) > 0, z \in \mathcal{U} \right\} \tag{5}$$

and

$$\mathcal{C} = \left\{ f(z) \in \mathcal{A} : \Re\left(1 + \frac{zf''(z)}{f'(z)} \right) > 0, z \in \mathcal{U} \right\}, \tag{6}$$

respectively. For a better understanding of the study, the geometric definitions of starlike and convex functions can be given as follows, respectively.

Definition 1.1 *A domain $\mathcal{D} \subset \mathbb{C}$ is called starlike with respect to an interior point w_0 if the line segment connecting point w to any interior point of $\mathcal{D}$ lies entirely within $\mathcal{D}$. In this case, a function $f(z)$ is called starlike with respect to the interior point w_0 if it maps the open unit disc $\mathcal{U}$ to a region that is starlike with respect to w_0 [7].*

It is very important to know that in studies conducted in this field, starlike function expression (i.e., elements of class $\mathcal{S}^*$) are referred to functions that are starlike according to the origin (i.e., $w_0 = 0$).

Definition 1.2 *If the line segment connecting for every different pairs of points w_1 and w_2 of a region $\mathcal{D} \subset \mathbb{C}$ lies entirely in $\mathcal{D}$, $\mathcal{D}$ is called a convex region. In this case, f is called a convex function if the function f maps the open unit disc $\mathcal{U}$ to a convex region [7].*

Another well-known subclass of class $\mathcal{S}$ is the class of close-to-convex functions [8].

Definition 1.3 *A function $f \in \mathcal{A}$ is said to be close-to-convex in an open unit disc $\mathcal{U}$ if there is a function g in $\mathcal{U}$ such that*

$$\Re\left(\frac{f'(z)}{g'(z)} \right) > 0, z \in \mathcal{U}. \tag{7}$$

The class of close-to-convex functions is usually denoted by $\mathcal{K}$.

If $f = g$ is taken in (7), it can be easily seen that a function that is convex in $\mathcal{U}$ is close to convex. Similarly, it can be easily obtained that each starlike function is close to convex. For this, it will be sufficient to take a starlike function $h(z) = zg'(z)$, $z \in \mathcal{U}$.

Geometric function theory deals mostly with the study of the properties of functions belonging to class $\mathcal{S}$. As mentioned before, such functions were studied by Paul Koebe in 1907. In this sense, the function given by the

$$k(z) = \frac{z}{(1-z)^2} = z + 2z^2 + 3z^3 + \cdots = z + \sum_{z=2}^{\infty} nz^n \tag{8}$$

was first introduced by Koebe and is named after him, since this function is in class $\mathcal{S}$, it means that this function is analytic, normalized, and univalent in $\mathcal{U}$ which is simple to proven [8]. Firstly, the Koebe function $k(z)$ is analytic because it is complex differentiable at every point $z \in \mathcal{U}$. Secondly, the Koebe function $k(z)$ is normalized as it satisfies the normalization conditions $k(0) = 0$ and $k'(0) = 1$ in $\mathcal{U}$, where $k'(z) = 1 + \sum_{z=2}^{\infty} n^2 z^{n-1}$. On the other hand, if the necessary algebraic operations are performed, $z_1 = z_2$ is obtained when for all $z_1, z_2 \in \mathcal{U}, k(z_1) = k(z_2)$. As a result, the Koebe function $k(z)$ is univalent since it is analytic and one-to-one in $\mathcal{U}$. In geometric sense, under its properties, $k(z)$ Koebe function maps the open unit disk $\mathcal{U}$ conformally (i.e., preserves angles and orientation) on to the complex plane $\mathbb{C}$ excluding the slit along the negative real axis from $-\infty$ to $-1/4$. The existence of the Koebe function, which is vital in the analysis of class $\mathcal{S}$, naturally caused researchers to ask themselves different questions. In this sense, perhaps the most important problem that has attracted the attention of researchers and whose solution has been bothering them for a while is whether there is a relationship between the geometric feature of the image of a function belonging to the $\mathcal{S}$ class and the coefficients of the corresponding power series. Many researchers have struggled with this issue, known as the problem (or conjecture) of finding an upper bound for the coefficients of functions in the class $\mathcal{S}$. In 1916, Bieberbach stated and proved that a_2, the second coefficient of f functions in class $\mathcal{S}$, is bounded by 2 (that is, $|a_2| \leq 2$) and that equality within inequality is valid only for the Koebe function $k(z)$. He extended this further in his paper by assuming that all coefficients a_n of functions in class $\mathcal{S}$ are not larger than n with respect to their positions. Today, this conjectur is known as the Bieberbach conjecture [2].

Conjecture 1.4 *(Bieberbach Conjecture) All coefficients a_n of functions f in class $\mathcal{S}$ satisfy the inequality $|a_n| \leq n$ for each $n \geq 2$.*

This conjecture attracted a lot of attention because it remained unsolved for a long time. However, the methodological proof was made by Louis de Branges in 1984. In 1907, using Bieberbach conjecture $|a_2| \leq 2$ for $n = 2$, Koebe concluded that every function in class $\mathcal{S}$ contains $\{w : |w| \leq 1/4\}$ of the image set. Here again, equality within inequality is valid only for the Koebe function $k(z)$. The geometric result obtained by Koebe, also which is a reference for many other important results, is today known as the Koebe's $1/4$ Theorem or the Koebe-Bieberbach Theorem [6].

Theorem 1.5 *(Koebe's $1/4$ Theorem or Koebe-Bieberbach Theorem) The image of each function f in class $\mathcal{S}$ covers the disk $\{w : |w| \leq 1/4\}$ with center at the origin $w = f(0) = 0$ and radius $1/4$.*

Koebe's $1/4$ theorem, which is valid only for functions in $\mathcal{S}$ class, also guarantees the existence of the f^{-1} inverse of a function f in class $\mathcal{S}$, given by $f^{-1}(f(z)) = z$ $(z \in \mathcal{U})$, where $f^{-1}(w) = w - (a_2)w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \dots$.

In 1921, after the two important results given above, the Bieberbach conjecture for starlike ranges fo functions in class $\mathcal{S}^*$ was proven by Rolf Nevanlinna [11].

Theorem 1.6 *The power series coefficients of a function f in class $\mathcal{S}^*$ satisfy the inequality $|a_n| \leq n$ for $n = 2, 3, \dots$. Similarly, equality within inequality is valid only for the Koebe function $k(z)$.*

Corollary 1.7 *The power series coefficients of a function f in class $\mathcal{C}$ satisfy the inequality $|a_n| \leq 1$ for $n = 2, 3, \dots$. Equality within inequality is valid only for the Koebe function $f(z) = z(1-z)^{-2}$.*

Theorem 1.8 *The image of each function f in class $\mathcal{C}$ covers the disk $\{w : |w| \leq 1/2\}$ with center at the origin $w = f(0) = 0$ and radius $1/2$.*

At this stage, several important conclusions obtained from the given preliminary information are presented. In the light of the information given so far, naturally we can write $\mathcal{C} \subset \mathcal{S}^* \subset \mathcal{S} \subset \mathcal{A} \subset \mathcal{H}$ according to the subset relationship in the sets. If f is in class $\mathcal{S}$ then any function composed of scaling, translating, and/or rotating f is also in class $\mathcal{S}$. Then $k(z)$ Koebe function can be written as the composed of

$$w_0 = \frac{1+z}{1-z}, \ w_1 = z^2 \text{ and } w_2 = \frac{1}{4}\Big[z - 1\Big].$$

That is,

$$k(z) = (w_2 \circ w_1 \circ w_0)(z) = \frac{1}{4}\left[\left(\frac{1+z}{1-z}\right)^2 - 1\right].$$

According to the given composition operation, the graph of the $k(z)$ Koebe function can be easily drawn.

From previous section, we know that the image of Koebe function is the whole plane minus the part of the negative real axis from $1/4$ to negative infinity. This situation can be easily seen from Figure 1. Thus, it is clear that Koebe function is starlike with respect to origin and not convex.

Furthermore, in 1915, Alexander showed the existence of a very useful relationship between class $\mathcal{S}$ and class $\mathcal{C}$ [1, 10].

Theorem 1.9 *(Alexander's Theorem) Let f(z) be a function in class $\mathcal{S}$. Then, $f \in \mathcal{C}$ if and only if $zf'(z) \in \mathcal{S}^*$. So, if $f(z) \in \mathcal{S}^*$, then*

$$g(z) = \int_0^z \frac{f(z)}{z} dz \tag{9}$$

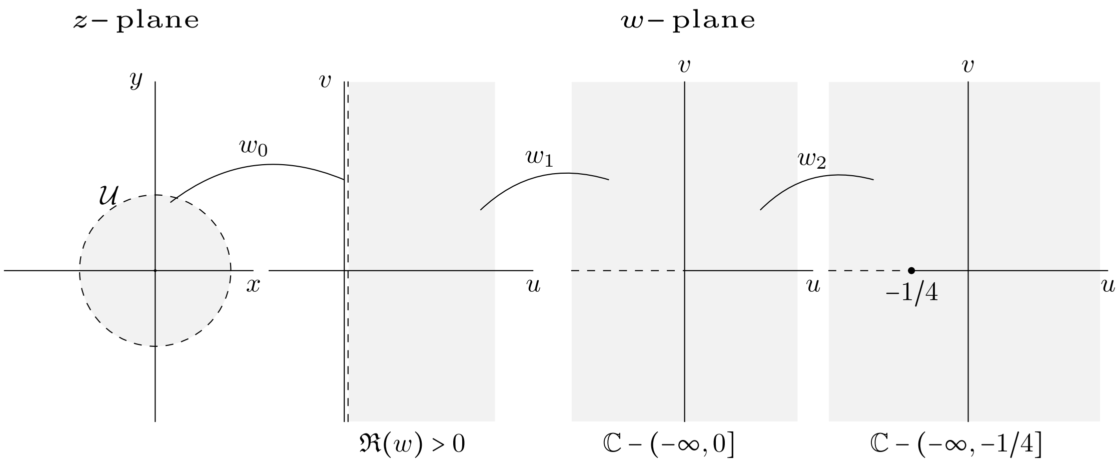

Figure 1: Image of open unit disc $\mathcal{U}$ under Koebe transform

is a convex function.

Notice that the Alexander integral operator maps functions from in class $\mathcal{S}^*$ to the class $\mathcal{C}$ of convex functions. This creative theorem, which is not difficult to prove, also accelerated the use of integral operators in geometric function theory. Some well-known integral operators in this sense are given below [3, 15].

- Alexander operator, 1915

$$g(z) = \int_0^z \frac{f(t)}{t} dt. \tag{10}$$

- Kim-Merkes operator (also atributed to Causey), 1963, α complex number

$$g(z) = \int_0^z \left(\frac{f(t)}{t}\right)^{\alpha} dt. \tag{11}$$

- Libera operator, 1965

$$g(z) = \frac{2}{z} \int_0^z f(t) dt. \tag{12}$$

- Bernardi operator, 1969, α complex number

$$g(z) = \frac{1+\alpha}{z^{\alpha}} \int_0^z f(t) t^{\alpha-1} dt. \tag{13}$$

- Pfaltzgraff operator, 1975, α complex number

$$g(z) = \int_0^z (f'(t))^{\alpha} dt. \tag{14}$$

Since 1907, many mathematicians have worked on integral operators that preserve class $\mathcal{S}$. In this sense, some important results can be found in [3, 12, 15]. The main purpose of these works is to determine the values of α which the functions

$$g(z) = \int_0^z \left(\frac{f(t)}{t}\right)^\alpha \text{ and } g(z) = \int_0^z (f'(t))^\alpha \, dt \tag{15}$$

when $f(z)$ function in class $\mathcal{S}$ defined by certain conditions related to univalence. Also, the theorems given below can be found in [4, 13, 14, 18, 19].

Theorem 1.10 *If $f(z) \in \mathcal{S}$ is close-to-convex, then*

$$g(z) = \int_0^z (f'(t))^\alpha \, dt \tag{16}$$

in class $\mathcal{S}$ for $\alpha \in [0,1]$.

Theorem 1.11 *If $f(z) \in \mathcal{S}$ is close-to-convex, then*

$$g(z) = \int_0^z \left(\frac{f(t)}{t}\right)^\alpha dt \tag{17}$$

then $g(z)$ in class $\mathcal{S}$ for $\alpha \in [0,1]$.

Theorem 1.12 *If $f(z) \in \mathcal{S}$ and*

$$g(z) = \int_0^z \left(\frac{f(t)}{t}\right)^\alpha dt, \tag{18}$$

in class $\mathcal{S}$ for $0 \le \alpha \le \left(\sqrt{1025} - 25\right)/100$.

Lemma 1.13 *Let $f(z) \in \mathcal{A}$. If $f(z)$ satisfies*

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > -\frac{1}{2} (z \in \mathcal{U}), \tag{19}$$

then $f(z)$ is in class $\mathcal{S}$.

2. Main Results

Theorem 2.1 *Let the function $f(z)$ given by (2) be a function in class $\mathcal{C}$, and*

$$g(z) = \int_0^z \left(\frac{f(t)}{t}\right)^\alpha dt. \tag{20}$$

Then, $g(z) \notin \mathcal{S}$ for $\alpha \in [0,3/2]$ but for $\alpha_0 < \alpha$, there exists a function $f(z) \in \mathcal{C}$, where α_0 is the smallest positive root of the $\alpha(\alpha+1)(\alpha+2) = 96$.

Proof It is follows from (20) that

$$1+\frac{zg''(z)}{g'(z)}=1+\frac{z\left\{\alpha\left(\frac{f(z)}{z}\right)^{\alpha-1}\left[\frac{zf'(z)-f(z)}{z^2}\right]\right\}}{\left(\frac{f(z)}{z}\right)^{\alpha}}$$

$$=1+\alpha\left(\frac{z}{f(z)}\right)\left[\frac{zf'(z)-f(z)}{z^2}\right]$$

$$=1+\alpha\frac{zf'(z)}{f(z)}-\alpha,$$

$$1+\Re\left(\frac{zg''(z)}{g'(z)}\right)=1-\alpha-\alpha\Re\left(\frac{zf''(z)}{f'(z)}\right)>1-\alpha\geq-0.5.$$

Thus, from Lemma 1.13, $g(z)\in\mathcal{S}$ is obtained for $\alpha\in[0,1.5]$. On the other hand, if we let $f(z)=z/(1-z)$ and $g(z)\in\mathcal{S}$, then we obtain

$$g'(z)=\left(\frac{\frac{z}{1-z}}{z}\right)^{\alpha}$$

$$=\frac{1}{(1-z)^{\alpha}}$$

$$=1+\alpha z+\frac{\alpha(\alpha+1)}{2!}z^2+\frac{\alpha(\alpha+1)(\alpha+2)}{3!}z^3+\dots.$$

Thus,

$$\left|\alpha\right|<4,\ \left|\frac{\alpha(\alpha+1)}{2!}\right|<9 \text{ and } \left|\frac{\alpha(\alpha+1)(\alpha+2)}{3!}\right|<16 \tag{21}$$

are obtained from the Conjecture 1.4. At this stage, with a simple algebraic calculation, the positive real number root of the second degree equation $\alpha^2+\alpha-18=0$ obtained as $\frac{-1+\sqrt{73}}{2}$. Letting α_0 be a positive real number, we must have the following inequqlity from (21):

$$0<\alpha\leq\alpha_0<\frac{-1+\sqrt{73}}{2}<4,$$

where α_0 is smallest positive real number root the equation(that is, $\alpha_0=3.65165$) $\alpha(\alpha+1)(\alpha+2)-96=0$. This result ends the proof of Theorem 2.1. □

Theorem 2.2 *Let the function $f(z)$ given by (2) be a function in class $\mathcal{S}^*$, and*

$$g(z)=\int_0^z\left(\frac{f(t)}{t}\right)^{\alpha}dt. \tag{22}$$

Then, $g(z)\in\mathcal{S}$ *for* $\alpha\in[0,3]$ *but for* $\alpha_0<\alpha$, *there exists a function* $f(z)\in\mathcal{S}^*$ *such that* $g(z)\notin\mathcal{S}$, *where* α_0 *is the smallest positive root of the* $\alpha(\alpha+1)(\alpha+2)=96$.

Proof When the same method as applied in the proof of Theorem 2.1 is applied,

$$\begin{aligned}1+\frac{zg''(z)}{g'(z)} &= 1+\frac{z\left\{\alpha\left(\frac{f(z)}{z}\right)^{\alpha-1}\left[\frac{zf'(z)-f(z)}{z^2}\right]\right\}}{\left(\frac{f(z)}{z}\right)^{\alpha}}\\ &= 1+\alpha\left(\frac{z}{f(z)}\right)\left[\frac{zf'(z)-f(z)}{z^2}\right]\\ &= 1+\alpha\frac{zf'(z)}{f(z)}-\alpha,\end{aligned}$$

$$1+\Re\left(\frac{zg''(z)}{g'(z)}\right)=1-\alpha-\alpha\Re\left(\frac{zf''(z)}{f'(z)}\right)>1-\alpha\geq-0.5,$$

$$\Re\left(\frac{zg''(z)}{g'(z)}\right)>0.5$$

is obtained for $\alpha\in[0,3]$. Letting $f(z)=z(1-z)^{-2}$ and $g(z)\in\mathcal{S}$, then we have

$$\begin{aligned}g'(z) &= \left(\frac{f(z)}{z}\right)^{\alpha}\\ &= 1+2\alpha z+\frac{2\alpha(2\alpha+1)}{2!}z^2+\frac{2\alpha(2\alpha+1)(2\alpha+2)}{3!}z^3+\ldots.\end{aligned}$$

Thus,

$$\left|2\alpha\right|<4,\quad\left|\frac{2\alpha(2\alpha+1)}{2!}\right|<9\ \text{and}\ \left|\frac{2\alpha(2\alpha+1)(2\alpha+2)}{3!}\right|<16 \tag{23}$$

are obtained from the Conjecture 1.4. At this stage, with a simple algebraic calculation, the positive real number root of the second degree equation $2\alpha^2+\alpha-9=0$ obtained as $\frac{-1+\sqrt{73}}{2}$. Letting α_0 be a positive real number, we must have the following inequqlity from (23):

$$0<\alpha\leq\alpha_0<\frac{-1+\sqrt{73}}{2}<4,$$

where α_0 is smallest positive real number root the equation (that is, $\alpha_0=3.15717$) $\alpha(2\alpha+1)(\alpha+1)-96=0$. This result ends the proof of Theorem 2.2. □

Theorem 2.3 *Let the function* $f(z)$ *given by (2) be a function in class* $\mathcal{C}$, *and*

$$g(z)=\int_0^z\left(\frac{f(t)}{t}\right)^{\alpha}dt. \tag{24}$$

Then, $g(z) \in \mathcal{S}$ for $\alpha \in [0, 1.5]$ but for $\alpha_0 < \alpha$, there exists a function $f(z) \in \mathcal{C}$ such that $g(z) \notin \mathcal{S}$, where α_0 is the smallest positive root of the $\alpha^2 + \alpha - 18 = 0$.

Proof If algebraic operations similar to those in Theorem 2.1 and Theorem 2.2 are performed,

$$1 + \Re\left(\frac{zg''(z)}{g'(z)}\right) = 1 - \alpha + \alpha\Re\left(\frac{zf''(z)}{f'(z)}\right) > 1 - \alpha \geq -0.5$$

obtained for $\alpha \in [0, 1.5]$. Letting $f(z) = z(1-z)^{-2}$ and $g(z) \in \mathcal{S}$, then we have

$$\begin{aligned} g'(z) &= \left(\frac{f(z)}{z}\right)^{\alpha} \\ &= 1 + \alpha z + \frac{\alpha(\alpha+1)}{2!}z^2 + \frac{\alpha(\alpha+1)(\alpha+2)}{3!}z^3 + \dots. \end{aligned}$$

Thus,

$$|\alpha| < 4, \left|\frac{\alpha(\alpha+1)}{2!}\right| < 9 \text{ and } \left|\frac{\alpha(\alpha+1)(\alpha+2)}{3!}\right| < 16 \tag{25}$$

are obtained from the Conjecture 1.4. At this stage, with a simple algebraic calculation, the positive real number root of the second degree equation $\alpha^2 + \alpha - 18 = 0$ obtained as $\frac{-1+\sqrt{73}}{2}$. Letting α_0 be a positive real number, we must have the following inequqlity from (25):

$$0 < \alpha \leq \alpha_0 < \frac{-1+\sqrt{73}}{2} < 4,$$

where α_0 is smallest positive real number root the equation (that is, $\alpha_0 = 3.65165$) $\alpha(\alpha+1)(\alpha+1) - 96 = 0$. This result ends the proof of Theorem 2.3. □

Theorem 2.4 *Let the function $f(z)$ given by (2) be a function in class $\mathcal{S}^*$, and*

$$g(z) = \int_0^z (f'(t))^{\alpha}\, dt. \tag{26}$$

Then, $g(z) \in \mathcal{S}$ for $\alpha \in [0, 1.5]$ but for $\alpha_0 < \alpha$, there exists a function $f(z) \in \mathcal{C}$ such that $g(z) \notin \mathcal{S}$, where α_0 is the smallest positive root of the $\alpha^2 + \alpha - 18 = 0$.

Proof

$$1+\frac{zg''(z)}{g'(z)} = 1+\frac{z\left\{\alpha\left(\frac{f(z)}{z}\right)^{\alpha-1}\left[\frac{zf'(z)-f(z)}{z^2}\right]\right\}}{\left(\frac{f(z)}{z}\right)^{\alpha}}$$

$$= 1+\alpha\left(\frac{z}{f(z)}\right)\left[\frac{zf'(z)-f(z)}{z^2}\right]$$

$$= 1+\alpha\frac{zf'(z)}{f(z)}-\alpha,$$

$$1+\Re\left(\frac{zg''(z)}{g'(z)}\right) = 1-\alpha-\alpha\Re\left(\frac{zf''(z)}{f'(z)}\right) > 1-\alpha \geq -0.5,$$

$$\Re\left(\frac{zg''(z)}{g'(z)}\right) > 0.5$$

is obtained for $\alpha \in [0,3]$. Letting $f(z) = z(1-z)^{-2}$ and $g(z) \in \mathcal{S}$, then we have

$$g'(z) = \left(\frac{f(z)}{z}\right)^{\alpha}$$

$$= 1+2\alpha z+\frac{2\alpha(2\alpha+1)}{2!}z^2+\frac{2\alpha(2\alpha+1)(2\alpha+2)}{3!}z^3+\ldots.$$

Thus,

$$\left|2\alpha\right| < 4,\quad \left|\frac{2\alpha(2\alpha+1)}{2!}\right| < 9 \text{ and } \left|\frac{2\alpha(2\alpha+1)(2\alpha+2)}{3!}\right| < 16 \tag{27}$$

are obtained from the Conjecture 1.4. At this stage, with a simple algebraic calculation, the positive real number root of the second degree equation $2\alpha^2+\alpha-9=0$ obtained as $\frac{-1+\sqrt{73}}{2}$. Letting α_0 be a positive real number, we must have the following inequqlity from (27):

$$0 < \alpha \leq \alpha_0 < \frac{-1+\sqrt{73}}{2} < 4,$$

where α_0 is smallest positive real number root the equation (that is, $\alpha_0 = 3.15717$) $\alpha(2\alpha+1)(\alpha+1)-96=0$. This result ends the proof of Theorem 2.4. □

3. Conclusion

The meaning of the derivative of a function $w = f(z)$ defined in the complex plane at a point given by (1) is different from its meaning in real analysis. In real analysis, the derivative of a function given by $y = f(x)$ is a measure of the ratio of the change in the independent variable x to the change in the dependent variable y of the function. As you may remember, this measure represents physical information such as flux, velocity or slope at a point. However, in complex variable functions, the main priority is whether or not there is a derivative. The existence of the derivative provides information about the analytical and geometric properties of the complex function. Does the existence of a derivative of a complex valued function f at a point z_0 mean that point z_0 is an interior point of the region, where the function is defined? Or is it a border point? It varies depending on what happened. To avoid this confusion, all analytic functions are defined on an open subset of the complex plane, that is, a region. In this case, differentiability in the complex sense refers to the limitation, size and shape of the image regions of the analytical functions $w = f(z)$ as geometric characterizations. These concepts are very important for classifying analytical functions. Integral and integral operators are very useful and of great importance in geometric function theory, especially in single-valued function theory. In this sense, it has been demonstrated through wonderful studies that the integral operators introduced help in the analytical classification of univalent functions. In the presented article, various inequalities and equations were obtained in addition to the existing studies.

Declaration of Ethical Standards

The author declares that the materials and methods used in her study do not require ethical committee and/or legal special permission.

Conflicts of Interest

The author declares no conflict of interest.

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Research Article

n-Dimensional Lattice Path Enumeration

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Abstract: Let $V_n := \{(h_1, h_2, \ldots, h_n) \ : h_i \in \{0,1\}, 1 \le i \le n\} - \{(0)\}$ be a set of integer vectors. We enumerate lattice paths that only uses vectors in V_n. Unlike most lattice path enumeration problems, the number of dimensions isn't fixed and the vector set is dependent on the dimension. This requires us to follow a different approach in explicitly expressing the number of lattice paths from origin to any point in n-dimensional space. We notice that a special case of this problem corresponds to Fubini numbers, which count the number of weak orderings of a set consisting of n elements. Then, we find the recursive relation of this sequence. Finally, we develop an algorithm that can be used to find the number of paths between any two points that do not touch the lattice points in $\mathbb{R}$. The crucial part of our algorithm is that it doesn't rely on finding all paths and checking each path for usage of restricted points.

Keywords: Lattice paths, forbidden paths, binary paths, enumeration in n dimensions.

1. Introduction

In the literature, lattice path is defined as; one of the shortest paths from one point to another in a model that consists of horizontal and vertical paths that intersect each other perpendicularly. Various researches have been carried out on the "lattice path" for many years. These studies gained momentum, especially after the 19th century and the most comprehensive studies on the subject have been made in recent years. We refer the reader [5] for a history of lattice path enumeration. A Hamiltonian path is a path that visits each vertex of a graph exactly once.

A Hamiltonian loop is a loop that visits each vertex exactly once. A graph containing a Hamilton cycle is also called a Hamiltonian graph. Determining whether such paths and loops exist in graphs is called the Hamiltonian path problem. In the study of E. Goodman and T.V. Narayana in 1969 [3], lattice paths were examined by including cross-steps. In 1976, B.R. Handa and S.G. Mohanty [4] conducted studies on lattice paths in high dimensions. Similarly, in a 1982

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article by A. Itai, C.H. Papadimitriou, and J.L. Szwarcfiter [6], the applications of Hamiltonian paths, cycles, and graphs in grid graphs were examined.

The Delannoy number refers to the paths used in mathematics to get from the south-west corner of a grid to the northeast corner in just simple steps (north, east and north-east). Most of Delannoy's work between 1886 and 1898 solved different mathematical problems using a chessboard. C. Krattenhaler and S.G. Mohanty [8], C. Krattenhaler [7], G. Mohanty [9] has done many studies on lattice paths. J.M. Autebert, M. Latapy and S.R. Schwer [1] brought their work "The lattice of Delannoy paths" to the literature in English and French. Later in 2003, J.M. Autebert and S.R. Schwer [2] expanded this concept (Delannoy path) to n-dimensional space and defined it over a particular type of alphabet (S-Alphabet) in their study called "On Generalized Delannoy Paths".

We enumerate lattice paths in an n-dimensional space for a fixed set of vectors $V_n := \{(h_1, h_2, \dots, h_n) \ : h_i \in \{0,1\}, 1 \le i \le n\} - \{(0)\}$. In [10, 11] lattice paths are studied in n-dimensions. Our goal is to find a formula that gives the number of paths from the origin to the point $(l_1, l_2, \dots, l_n)$ using only the vectors in V_n for $n \ge 2$. We usually refer to these vectors as *steps*. Figure 1 gives concrete examples of such lattice paths in 2 and 3-dimensional spaces. For example, when $n = 2$ we get the set of steps $V_2 = \{(1,0), (0,1), (1,1)\}$ which has been studied many times.

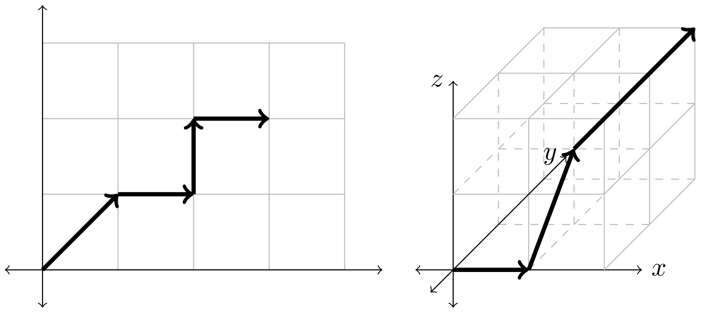

Figure 1: Left: A lattice path terminating at $(3,2)$ - Right: A lattice path terminating at $(2,2,2)$ consisting of vectors $(1,0,0)$, $(0,1,1)$ and $(1,1,1)$

The formula we found is

$$\sum_{j=1}^{2^n-1} \sum_{r_j=0}^{\infty} \frac{\left(\sum_{b=1}^{2^n-1} r_b\right)!}{\prod_{c=1}^{2^n-1} r_c!} \prod_{i=1}^{n} \left\lfloor x^{-\left[\left(\sum_{k=1}^{n-1} \sum_{p=\max(0,k+i-n)}^{\min(k,i-1)} f_n(i,k,p)\right) + r_i - l_i\right]^2} \right\rfloor \tag{1}$$

with

$$f_n(i,k,p) = \begin{cases} \sum_{v=1}^{\binom{n-i}{k-p}} r_{S+v}, & \text{if } i = p+1 \\ \sum_{z=1}^{i-p-1} \sum_{m_z=i-p-z}^{m_{z-1}-1} \left(\sum_{v=1}^{\binom{n-i}{k-p}} r_{S+\left[\sum_{t=1}^{i-p-1} \binom{n-m_t}{k-g_t}\right]+v} \right), & \text{if } i \neq p+1 \end{cases},$$

$S = \sum_{s=1}^{k} \binom{n}{s}$, $m_0 = i$, $g_0 = p$, $g_t = g_{t-1} - m_{t-1} + m_t + 1$ and $x > 1$.

First, we want to note that n is the number of dimensions in our space. This formula finds all possible paths (including the ones that do not terminate at the desired point) in n-dimensional space, then chooses the ones that terminate at the desired point $(l_1, l_2, \ldots, l_n)$.

2. Finding All Paths in n-Dimensional Space

There are $2^n - 1$ steps in V_n. Each a_j with $1 \leq j \leq 2^n - 1$ represents a different step in V_n. r_j is the number of a_j steps used in a path. A bundle of steps is an unordered group of steps that do not have to be different. Given all values of r_j for $1 \leq j \leq 2^n - 1$, we can form exactly one bundle of steps. For example for $n = 2$ and $(r_1, r_2, r_3) = (3, 1, 0)$, we get the bundle (a_1, a_1, a_1, a_3).

Lemma 2.1 *The number of all possible paths in n-dimensional space can be found with*

$$\sum_{j=1}^{2^n-1} \sum_{r_j=0}^{\infty} \frac{\left(\sum_{b=1}^{2^n-1} r_b\right)!}{\prod_{c=1}^{2^n-1} r_c!}. \tag{2}$$

Proof All possible bundle of steps comes from the sums $\sum_{j=1}^{2^n-1} \sum_{r_j=0}^{\infty}$. For a given bundle of steps, $\frac{\left(\sum_{b=1}^{2^n-1} r_b\right)!}{\prod_{c=1}^{2^n-1} r_c!}$ finds all possible permutations of that bundle. $\square$

3. Finding Paths That Terminate at the Desired Point

In this section, we find a formula that determines whether a path terminates at the desired point or not. Determining the terminal point of a path is the same as determining the terminal point of the bundle that the path was created. It follows because all arrangements of a bundle of steps terminate at the same point. The following part of our formula

$$\prod_{i=1}^{n} \left\lfloor e^{-\left[\left(\sum_{k=1}^{n-1} \sum_{p=\max(0,k+i-n)}^{\min(k,i-1)} f_n(i,k,p)\right)+r_i-l_i\right]^2} \right\rfloor$$

finds the distance traveled on all axes for a given bundle and multiplies (2) by 1 if the terminal point is the desired point, multiplies by 0 if not.

3.1. Arranging the Steps

We systematically assign vectors to the notations of the form a_j. First, we establish another notation for steps. Let $(h_1, h_2, \ldots, h_n)$ be a step in V_n. We write d_b to the notation of that step for every $h_b = 1$ with $1 \leq b \leq n$ and we have $u < y$ for $\ldots d_u d_y \ldots$. For example, the step $(1, 0, 0, 1)$ is given by the notation $d_1 d_4$. This notation tells which axes the steps move on.

Note that the notation $d_1 d_4$ represents $(1, 0, 0, 1)$ only if $n = 4$. For example, $d_1 d_4$ represents $(1, 0, 0, 1, 0)$ for $n = 5$.

The length of a step is the number of axes that step moves on (also the number of 1's in the vector $(h_1, h_2, \ldots, h_n)$ and the number of d_b terms in the notation of that step). We start sorting the steps by their length. The length of the steps ascends from 1 to n.

Lemma 3.1 *The notations a_j with $\sum_{s=1}^{k}\binom{n}{s} \le j \le \sum_{s=1}^{k+1}\binom{n}{s}$ represents steps with length of $k+1$.*

Proof It is easy to see that there are $\binom{n}{k}$ steps with length of k. The number of all steps with a length smaller than $k+1$ is $\sum_{s=1}^{k}\binom{n}{s}$. In our system of arranging steps, these steps that have length smaller than $k+1$ comes before (by comes before, we mean $y < u$ for a_y being a step with length less than $k+1$ and a_u being a step with a length of $k+1$) those with a length of $k+1$. □

Now we turn our attention to arranging steps of a fixed length. The arrangement of steps is very similar to an alphabetical arrangement. Assume d_b denotes the b-th letter in the alphabet. For example, d_1 denotes a, d_2 denotes b, d_3 denotes c and so on. We transform the notations consisting of d_b's to words. For example, $d_1d_3d_4d_8$ transforms into acdh. Next we do the classic alphabetical arrangement. The arrangement of steps for $n = 4$ is shown below:

$a_1 = d_1$	$a_5 = d_1d_2$	$a_{11} = d_1d_2d_3$	$a_{15} = d_1d_2d_3d_4$
$a_2 = d_2$	$a_6 = d_1d_3$	$a_{12} = d_1d_2d_4$	
$a_3 = d_3$	$a_7 = d_1d_4$	$a_{13} = d_1d_3d_4$	
$a_4 = d_4$	$a_8 = d_2d_3$	$a_{14} = d_2d_3d_4$	
	$a_9 = d_2d_4$		
	$a_{10} = d_3d_4$		

3.2. Distance Traveled on one Axis

We need to determine the distance traveled on a specific axis for a given bundle of steps. We call this axis the observed axis and represent it with d_i. We want to find all steps that have d_i in its notation. We can show such steps with

$$\underbrace{\overbrace{\ldots\ d_i}^{\text{p+1}}\overbrace{\ldots}^{\text{k-p}}}_{\text{k+1}}. \tag{3}$$

As shown in the notation, $k+1$ is the length of the steps and p is the number of d_b terms that are written before the observed axis. This tells that there are $k-p$ d_b terms written after the observed axis.

Lemma 3.2 *(i) The valid interval for i is $1 \le i \le n$.*

(ii) The valid interval for k *is* $0 \le k \le n-1$.

(iii) The valid interval for p *is* $\max(0, k+i-n) \le p \le \min(k, i-1)$.

Proof

(i) There are n axes in the n-dimensional space.

(ii) The minimum length of a step is 1 and the maximum length of a step is n. Hence, $0 \le k \le n-1$.

(iii) There are $i-1$ choices of axes before the observed axis and we choose p of them. Hence $p \le i-1$. There can be only k more terms other than the observed axis, since the length of a step is $k+1$. Hence, we get $p \le k$. It is easy to see that it is necessary to choose the smaller one of $i-1$ and k for the maximum valid value of p.

There are $n-i$ choices of axes after the observed axis and we choose $k-p$ of them. Hence, we get $p \le k+i-n$. On the other hand, we know $p \ge 0$. It is easy to see that it is necessary to choose the greater one of these values for the minimum valid value of p.

□

Lemma 3.3 *Let the part before the observed axis be fixed, more specifically* D. *Let* a_{q+1} *be the step* $Dd_i d_{i+1} d_{i+2} \dots d_{i+k-p-1} d_{i+k-p}$. *Then, all* a_j *steps with* $q+1 \le j \le j + \binom{n-i}{k-p}$ *travel on the* i*-th axis.*

Proof For a fixed part before the observed axis, there are $\binom{n-i}{k-p}$ steps. There are $n-i$ possible axes that can be written after the observed axis and we choose $k-p$ of them.

Next, we show that all of these steps are consecutive. Because of the alphabetical arrangement that we made, the observed axis and the part before it does not change until we go through all different $\binom{n-i}{k-p}$ combinations for the part after the observed axis. □

Lemma 3.4 *All different subsets of* $\{1, 2, 3, \dots, i-2, i-1\}$ *with* $i-p-1$ *elements are the sets* $\{m_1, m_2, m_3, \dots, m_{z-1}, m_z\}$ *with* $1 \le z \le i-p-1$, $m_{z-1} - 1 \ge m_z \ge i-p-z$, $m_0 = i$ *and* $m_z \in \mathbb{N}$.

Proof Consider all subsets of $\{1, 2, 3, \dots, i-2, i-1\}$ that consists of $i-p-1$ elements. Arrange each set in descending order. Let m_z be the z-th element from left of a subset, we get $1 \le z \le i-p-1$ and $m_{z-1} - 1 \ge m_z$. Next we show that $m_z \ge i-p-z$. There are $i-p-z-1$ elements to the right of m_z which are all smaller than m_z. This implies that $m_z \ge i-p-z$. Lastly we show that $m_0 = i$. We know $i-1 \ge m_1$ as m_1 is the greatest number in a subset. Thus, $m_0 = i$. □

We find all steps that travel on i-th axis for fixed values of i, k and p. We denote the function that finds the coefficients of all such steps in n-dimensional space by $f_n(i,k,p)$. $F_n(i,k,p)$ denotes the function that finds all steps with given i, k and p values. Note that if we find all such steps we can find the distance traveled by simply changing each a_j term with r_j. We say that all steps with given i, k and p values whose parts before the observed axis are the same are a section. For example, when observing d_3 in 6-dimensional space, $d_1d_3d_4$, $d_1d_3d_5$ and $d_1d_3d_6$ is a section.

Theorem 3.5

$$F_n(i,k,p) = \begin{cases} \sum_{v=1}^{\binom{n-i}{k-p}} a_{S+v}, & \text{if } i = p+1 \\ \sum_{z=1}^{i-p-1} \sum_{m_z=i-p-z}^{m_{z-1}-1} \left(\sum_{v=1}^{\binom{n-i}{k-p}} a_{S+\left[\sum_{t=1}^{i-p-1} \binom{n-m_t}{k-g_t}\right]+v} \right), & \text{if } i \neq p+1 \end{cases} \quad (4)$$

with $S = \sum_{s=1}^{k} \binom{n}{s}$, $m_0 = i$, $g_0 = p$ *and* $g_t = g_{t-1} - m_{t-1} + m_t + 1$.

Proof We first proof the case $i = p+1$. Every step with such i and p values can be denoted by $d_1d_2\dots d_{i-1}d_i\dots$. Because of the alphabetical arrangement, these steps precedes those of the same length. By Lemma 3.3, these steps are consecutive. All steps with a length smaller than $k+1$ is $S = \sum_{s=1}^{k} \binom{n}{s}$ and there are $\binom{n-i}{k-p}$ steps because we choose $k-p$ axes out of $n-i$ axis for the part after the observed axis.

Next we proof the case $i \neq p+1$. In the case $i = p+1$, we had to use all d_b with $b < i$ in the notation of a step. But for the case $i \neq p+1$ there is some d_b that is not used in the notation of a step. Out of $i-1$ axes we choose not to use $i-p-1$ of them. m_t terms represent these unused axes. For example, if $(m_1, m_2) = (3,1)$, d_3 and d_1 are not used in the notation of a step. Notice that for a fixed set of unused axes, all steps form a section. The number of steps in a section is $\binom{n-i}{k-p}$.

We split all steps with given i, k and p values into sections and for each section, we find the number of steps before that section. $\sum_{z=1}^{i-p-1} \sum_{m_z=i-p-z}^{m_{z-1}-1}$ generates m_t sets that forms the sections.

Consider not using $d_{m_{i-p-1}}$ but using all d_b with $b < m_{i-p-1}$. If $d_{m_{i-p-1}}$ is observed, the part before it will be fixed. Lemma 3.3 implies that if $d_{m_{i-p-1}}$ is not used, then we must have gone through all different combinations for the part after it. There are $m_{i-p-1}-1$ axes before $d_{m_{i-p-1}}$, thus there are $\binom{n-m_{i-p-1}}{k-g_{i-p-1}}$ different combinations for the part after $d_{m_{i-p-1}}$ with $g_{i-p-1} = m_{i-p-1}-1$. Because steps of the form $d_1d_2\dots d_{m_{ip-1}-1}d_{m_{ip-1}}$ precedes those of the same length, there are $S + \binom{n-m_{m_{i-p-1}}}{k-g_{i-p-1}}$ steps before the ones that do not use $d_{m_{i-p-1}}$. Note that first $m_{i-p-1}-1$ terms are fixed to $d_1d_2\dots d_{m_{i-p-1}-1}$ and we denote this by D.

After that, consider not using $d_{m_{i-p-2}}$ but using all d_b with $m_{i-p-1} < b < m_{i-p-2}$. $m_{i-p-2} - m_{i-p-1} - 1$ axes gets fixed after D. If $d_{m_{i-p-2}}$ is observed, there are $\binom{n-m_{m_{i-p-2}}}{k-g_{i-p-2}}$ with $g_{i-p-2} = g_{i-p-1} + m_{i-p-2} - m_{i-p-1} - 1$ different combinations for the part after it. There are $S + \binom{n-m_{i-p-1}}{k-g_{i-p-1}} +$

$\binom{n-m_{i-p-2}}{k-g_{i-p-2}}$ steps before the ones that do not use $d_{m_{i-p-1}}$ and $d_{m_{i-p-2}}$. The same idea applies to all other m_t terms and we get $g_t = g_{t-1} - m_{t-1} + m_t + 1$. If this equation is summed up for $1 \le t \le i-p-1$, we get $g_{i-p-1} = g_0 - m_0 + m_{i-p-1} + i - p - 1$ which implies $g_0 = p$. □

Example 3.6 *Let $n = 4$. The arrangement of these steps is made in Section 3. We can see that $F_4(3,1,1) = (a_6, a_8)$ since $F_4(3,1,1)$ is the steps with a length of 2 and 1 d_b term before d_3 in 4-dimensional space. Plugging the values into ((4)) we get*

$$F_4(3,1,1) = \sum_{z=1}^{1} \sum_{m_z=2-z}^{m_{z-1}-1} \left(\sum_{v=1}^{\binom{1}{0}} a_{S+\left[\sum_{t=1}^{1} \binom{4-m_t}{1-g_t}\right]+v} \right)$$

with $S = 4$, $m_0 = 3$ and $g_0 = 1$. We further simplify,

$$F_4(3,1,1) = \sum_{m_1=1}^{2} a_{\left(4+\binom{4-m_1}{1-g_1}+1\right)}.$$

1. $m_1 = 1$, $g_1 = 0$. We get a_8. 2. $m_1 = 2$, $g_1 = 1$. We get a_6.

Example 3.7 *Let $n = 5$. The arrangement of these steps is*

$a_1 = d_1$	$a_6 = d_1 d_2$	$a_{16} = d_1 d_2 d_3$	$a_{26} = d_1 d_2 d_3 d_4$	$a_{31} = d_1 d_2 d_3 d_4 d_5$
$a_2 = d_2$	$a_7 = d_1 d_3$	$a_{17} = d_1 d_2 d_4$	$a_{27} = d_1 d_2 d_3 d_5$	
$a_3 = d_3$	$a_8 = d_1 d_4$	$a_{18} = d_1 d_2 d_5$	$a_{28} = d_1 d_2 d_4 d_5$	
$a_4 = d_4$	$a_9 = d_1 d_5$	$a_{19} = d_1 d_3 d_4$	$a_{29} = d_1 d_3 d_4 d_5$	
$a_5 = d_5$	$a_{10} = d_2 d_3$	$a_{20} = d_1 d_3 d_5$	$a_{30} = d_2 d_3 d_4 d_5$	
	$a_{11} = d_2 d_4$	$a_{21} = d_1 d_4 d_5$		
	$a_{12} = d_2 d_5$	$a_{22} = d_2 d_3 d_4$		
	$a_{13} = d_3 d_4$	$a_{23} = d_2 d_3 d_5$		
	$a_{14} = d_3 d_5$	$a_{24} = d_2 d_4 d_5$		
	$a_{15} = d_4 d_5$	$a_{25} = d_3 d_4 d_5$		

Now, we show that $F_5(3,2,1) = (a_{21}, a_{24}, a_{25})$. Notice that even though a_{24} and a_{25} are consecutive, they do not form a section. Simplifying (4) gives

$$F_5(4,2,1) = \sum_{z=1}^{2} \sum_{m_z=3-z}^{m_{z-1}-1} a_{15+\left[\sum_{t=1}^{2} \binom{5-m_t}{2-g_t}\right]+1} = \sum_{m_1=2}^{3} \sum_{m_2=1}^{m_1-1} a_{15+\left[\sum_{t=1}^{2} \binom{5-m_t}{2-g_t}\right]+1}.$$

1. $m_1 = 2$, $m_2 = 1$, $g_1 = 0$, $g_2 = 0$. $a_{\left(15+\binom{3}{2}+\binom{4}{2}+1\right)} = a_{25}$.

2. $m_1 = 3$,

(a) $m_2 = 1$, $g_1 = 1$, $g_2 = 0$. $a_{\left(15+\binom{2}{1}+\binom{4}{2}+1\right)} = a_{24}$.

(b) $m_2 = 2$, $g_1 = 1$, $g_2 = 1$. $a_{\left(15+\binom{2}{1}+\binom{3}{1}+1\right)} = a_{22}$.

Corollary 3.8 *The distance traveled on i-th axis is*

$$\sum_{k=1}^{n-1} \sum_{p=\max(0,k+i-n)}^{\min(k,i-1)} f_n(i,k,p). \tag{5}$$

4. The Results

Let

$$K(\alpha) = \begin{cases} 1, & \text{if } \alpha = 0 \\ 0, & \text{if } \alpha \neq 0 \end{cases}. \tag{6}$$

We combine this function above with our results. $\sum_{k=1}^{n-1} \sum_{p=\max(0,k+i-n)}^{\min(k,i-1)} f_n(i,k,p) - l_i$ is the difference between the distance traveled and the distance wanted to travel on the i-th axis. We write $\alpha = \sum_{k=1}^{n-1} \sum_{p=\max(0,k+i-n)}^{\min(k,i-1)} f_n(i,k,p) - l_i$ in (6). For a given bundle, we multiply all results for $1 \leq i \leq n$. If the terminal point for that bundle is the desired point, the result will be 1.

Corollary 4.1 *In n-dimensional space, the number of paths from origin to $(l_1, l_2, \ldots, l_n)$ using only vectors in $V_n := \{(h_1, h_2, \ldots, h_n) \; : h_i \in \{0,1\}, 1 \leq i \leq n\}$ is*

$$\sum_{j=1}^{2^n-1} \sum_{r_j=0}^{\infty} \frac{\left(\sum_{b=1}^{2^n-1} r_b\right)!}{\prod_{c=1}^{2^n-1} r_c!} \prod_{i=1}^{n} K\left(\left(\sum_{k=1}^{n-1} \sum_{p=\max(0,k+i-n)}^{\min(k,i-1)} f_n(i,k,p)\right) + r_i - l_i\right)$$

with

$$f_n(i,k,p) = \begin{cases} \sum_{v=1}^{\binom{n-i}{k-p}} r_{S+v}, & \text{if } i = p+1 \\ \sum_{z=1}^{i-p-1} \sum_{m_z=i-p-z}^{m_{z-1}-1} \left(\sum_{v=1}^{\binom{n-i}{k-p}} r_{S+\left[\sum_{t=1}^{i-p-1} \binom{n-m_t}{k-g_t}\right]+v}\right), & \text{if } i \neq p+1 \end{cases},$$

$S = \sum_{s=1}^{k} \binom{n}{s}$, $m_0 = i$, $g_0 = p$, $g_t = g_{t-1} - m_{t-1} + m_t + 1$ *and* $x > 1$.

This formula can be generalized to counting the number of paths between any two lattice points in n-dimensional space.

Corollary 4.2 *In n-dimensional space, the number of paths from $(e_1, e_2, \ldots, e_n)$ to $(l_1, l_2, \ldots, l_n)$ using only vectors in $V_n := \{(h_1, h_2, \ldots, h_n) \; : h_i \in \{0,1\}, 1 \leq i \leq n\}$ is*

$$\sum_{j=1}^{2^n-1} \sum_{r_j=0}^{\infty} \frac{\left(\sum_{b=1}^{2^n-1} r_b\right)!}{\prod_{c=1}^{2^n-1} r_c!} \prod_{i=1}^{n} \left\lfloor x^{-\left[\left(\sum_{k=1}^{n-1} \sum_{p=\max(0,k+i-n)}^{\min(k,i-1)} f_n(i,k,p)\right) + r_i - (l_i - e_i)\right]^2} \right\rfloor$$

with

$$f_n(i,k,p)=\begin{cases}\sum_{v=1}^{\binom{n-i}{k-p}} r_{S+v}, & \text{if } i=p+1\\ \sum_{z=1}^{i-p-1}\sum_{m_z=i-p-z}^{m_{z-1}-1}\left(\sum_{v=1}^{\binom{n-i}{k-p}} r_{S+\left[\sum_{t=1}^{i-p-1}\binom{n-m_t}{k-g_t}\right]+v}\right), & \text{if } i\neq p+1\end{cases},$$

$S=\sum_{s=1}^{k}\binom{n}{s}$, $m_0=i$, $g_0=p$, $g_t=g_{t-1}-m_{t-1}+m_t+1$ *and* $x>1$.

We calculate the number of paths from origin to $(l_1,l_2,\dots,l_n)=(1,1,\dots,1)$ for $1\le n\le 6$ and we get the sequence 1, 3, 13, 75, 541, 4683. This sequence is OEIS sequence A000670, also called Fubini numbers. The formula for the n-th number in this sequence is $a_n=\sum_{i=1}^{n}\binom{n}{i}a_{n-i}$.

Corollary 4.3 *Let* $L(n)$ *be the number of lattice paths from origin to* $(l_1,l_2,\dots,l_n)=(1,1,\dots,1)$ *using steps in* V_n*. Then,*

$$L(n)=\sum_{i=1}^{n}\binom{n}{i}a_{n-i}. \tag{7}$$

Thus, we calculate the number of paths from origin to $(l_1,l_2,\dots,l_n)=(2,2,\dots,2)$ for $1\le n\le 5$. We get the numbers 1, 13, 409, 23917 and 2244361 which appears as OEIS sequence A055203.

l_2 \ l_1	0	1	2	3	4	5	6	7
7	575							
6	377	6287						
5	231	3417	25695					
4	129	1671	11049	50191				
3	63	705	4047	16081	50191			
2	25	239	1177	4047	11049	25695		
1	7	57	239	705	1671	3417	6287	
0	1	7	25	63	129	231	377	575

Figure 2: The number of paths from origin to $(l_1,l_2,3)$ using steps in V_3 for $l_1+l_2\le 7$

The numbers in Figure 2 was generated using Python.

5. Recursive Relation

Theorem 5.1 *Let $L(p)$ be the number of lattice paths from origin to p using steps in V_n. The recursive relation in this sequence is*

$$L(l_1, l_2, \dots, l_n) = \sum_{m=2}^{n-1} \sum_{b=2}^{n} \sum_{v_b=0}^{1} \sum_{v_1=1}^{1} \sum_{b=2}^{n} L(l_1 - v_1, l_2 - v_2, \dots, l_n - v_n). \tag{8}$$

Proof From $(l_1 - v_1, l_2 - v_2, \dots, l_n - v_n)$ with $v_i \in \{0, 1\}$ for all i, there is only 1 way of directly (without touching any other lattice points besides the one we want to reach) reaching $(l_1, l_2, \dots, l_n)$. If any $v_i \notin \{0, 1\}$, there will be no way of directly reaching $(l_1, l_2, \dots, l_n)$. Summing up the values of $L(l_1 - v_1, l_2 - v_2, \dots, l_n - v_n)$ gives us the number of path to $(l_1, l_2, \dots, l_n)$. But the case where $v_i = 0$ for all i is the point at which we are finding the recursive relation. Every time we fix one of v_i to 1 to solve this. □

In fact, the recursive relation in Theorem 5.1 can be generalized for any set of vectors.

Corollary 5.2 *Let K be a set of vectors and $L(l_1, l_2, \dots, l_n)$ be the number the number of lattice paths from origin to $(l_1, l_2, \dots, l_n)$. Then the recursive relation in this sequence is*

$$\sum L(l_1 - v_1, l_2 - v_2, \dots, l_n - v_n)$$

for $v_i \in v$ and $v \in K$.

6. An Algorithm for Lattice Paths with Restricted Points

We developed an algorithm that finds the number of paths from origin to any point without touching the points in $\mathbb{R}$.

Lemma 6.1 *Consider a set of lattice points. There is either one or no arrangements of these points such that i-th coordinate of a point is greater than or equal to the i-th coordinates of previous points for all i.*

Proof Consider two different lattice points $p_1 = (b_1, b_2, \dots, b_m)$ and $p_2 = (c_1, c_2, \dots, c_m)$. It is easy to see that there is either 1 or 0 arrangement such that $b_j \leq c_j$ for $1 \leq j \leq m$ or $c_j \leq b_j$ for $1 \leq j \leq m$. This means that 2 different points are not interchangable. If there is such an arrangement for a set of lattice points, it will be the only such arrangement. □

Corollary 6.2 *Let $L(p, p')$ denote the number of lattice paths from p to p' (without restrictions). The number of paths from p to p' that do not touch the lattice points in $\mathbb{R}$ can be computed by the following algorithm.*

1. *Let* r_m *be an* m*-element subset of* $\mathbb{R}$*.*

2. *For* $p_i \in r_m$*, calculate the quantity* $(-1)^n L(p,p_1)L(p_1,p_2)\ldots L(p_m,p')$ *for all permutations of* r_m*. There can not be more than one nonzero value and if there is one, note it down.*

3. *Do this for all subsets of* $\mathbb{R}$*.*

4. *The sum of the results is the number of paths from* p *to* p' *that do not touch the points in* $\mathbb{R}$*.*

Proof $L(p,p_1)L(p_1,p_2)\ldots L(p_m,p')$ gives the number of paths from p to p' that touch the points $(p_1,p_2,\ldots,p_m)$ in the given order. By Lemma 6, if there is such permutation, there will be only one. We multiply by $(-1)^n$ because of the inclusion exclusion principle. □

The efficiency of our algorithm lies on the fact that it doesn't compute all paths and check whether each path is using one of the restricted points or not. It utilizes the inclusion exclusion principle to avoid computing all paths.

7. Conclusion

A formula counting the number of paths from origin to the point $(l_1,l_2,\ldots,l_n)$ using steps in V_n has been found. The recursive relation between these numbers has been found and it has been observed that the technique used to find this recursive relation applies to general sets of vectors. The formula found can be generalized to find the number of paths between two lattice points. It has been observed that the numbers $L(1,1,\ldots,1)$ correspond to Fubini numbers which are the number of arrangements of n competitors. Lastly, an algorithm for lattice paths with restricted lattice points has been given.

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Declaration of Ethical Standards

The authors declare that the materials and methods used in their study do not require ethical committee and/or legal special permission.

Authors Contributions

Author [Alper Vural]: Collected the data, contributed to research method or evaluation of data, wrote the manuscript (%50).

Author [Cemil Karaçam]: Thought and designed the research/problem, contributed to completing the research and solving the problem (%50).

Conflicts of Interest

The authors declare no conflict of interest.

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Research Article

Clique Collocation Method to Solve the Third-Order Multisingular (MS) Functional Differential Equations

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Abstract: In this paper, a Clique collocation method is presented to numerically solve the third-order multisingular (MS) functional differential equation. This method convert this equation to a system of the algebraic equations via the collocation points and the matrix relations. Also, the error estimation technique is constituted for the third-order multisingular (MS) functional differential equation. Applications of the Clique collocation method and the error estimation technique are made for three examples. In addition, the comparison is made with another method in the literature. The obtained results are tabulated and visualized to demonstrate the effectiveness of the presented method. Applications of the method and graphics are made by using MATLAB. According to the applications, it is observed that the results have quite decent errors.

Keywords: Clique polynomials, collocation method, error estimation, functional differential equations, singular differential equations.

1. Introduction

Recently, studies on functional differential equations with singular points have been of great importance for researchers. Functional differential equations are used in many applications such as electrodynamics [12], models based on chemical kinetics [31], models of population growth [26], infection models of HIV-1 [27], models of tumor growth [37], B-virus infection hepatitis models [15] and many more [5, 8, 23, 33, 36]. Differential equations with singular points have been used in some important application areas such as oscillating magnetic fields [11], study of thermal explosions model [1], models of the stellar structure [38] and study of the model of isothermal gas spheres [7]. Many researchers have solved functional differential equations using many methods

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such as one-step implicit methods [3], Taylor polynomial method [35], homotopy analysis method [4], variational iteration method [10], Laguerre matrix method [42], a matrix-collocation method by using Müntz-Legendre polynomials [39], two novel memory-based root-finding approaches [30] and an iterative method [13]. In addition, singularly perturbed differential equations have been solved using many methods such as spline finite difference method [24], the finite difference methods [16, 22], the seventh order numerical method [9], B-spline collocation method [20, 21], Bessel collocation method [40] and Laguerre method [41]. Besides, an approach has been presented to study the existence, uniqueness and stability of the solutions of nonlinear differential equations with infinite delay [6]. The collocation methods are one of the methods that obtain effective results to calculate numerical solutions of differential equations. In the literature, the collocation method has been used to obtain approximate solutions of many differential equations such as the singular-perturbation problem [20], general linear differential-difference equations with variable coefficients [34], the generalized pantograph equations with linear functional argument [35] etc. [29, 40–42]. The clique polynomials were first introduced in [18] and associated with graph theory. Nevertheless, there are many studies in the literature on the numerical solutions of many differential equations using Clique polynomials [2, 14, 17, 19, 25, 28, 43]. Effective results are obtained from these studies. But there is no study in literature yet on the solutions of the third-order multisingular (MS) functional differential equations using Clique polynomials. Hence, the approximate solution of this equation is investigated based on Clique polynomials in this paper.

In this study, we consider the model based on the third-order multisingular (MS) functional differential equations with initial conditions [32]

$$\begin{cases} u^{'''}(s+\theta_1)+\frac{\beta_1}{s}\, u^{''}(s+\theta_2)+\frac{\beta_2}{s^2}\, u^{'}(s+\theta_3)+s\, u(s+\theta_4)=\alpha(s), \\ u(0)=k_1, \quad u^{'}(0)=k_2, \quad u^{''}(0)=k_3. \end{cases} \tag{1}$$

Here, the parameters $\beta_1, \beta_2, \theta_i \ (i=1,2,3,4),\ k_j \ (j=1,2,3)$ are the real constant values and $\alpha(s)$ is the continuous function.

Our aim is to obtain the approximate solution of (1) in form of the Clique polynomials

$$u_N(s)=\sum_{n=0}^{N} a_n C_n(s), \tag{2}$$

where $N>0$ is chosen to be any positive integer. Here, a_n and $C_n(s)$ are, respectively, the unknown coefficients and Clique polynomials described by [19]

$$C_n(s)=\sum_{k=0}^{n}\binom{n}{k}s^k. \tag{3}$$

The recursive formulation of the Clique polynomials is

$$C_{n+1}(s) = (1+s)C_n(s), \quad C_0(s) = 1, \quad C_1(s) = s+1. \tag{4}$$

Let's summarize rest of this paper as follows: The fundamental matrix relations are presented in Section 2. The Clique collocation method is presented in Section 3. The error estimation method is given in Section 4. In Section 5, the applications of the method are made. Also, a comparison is made with another method in the literature. Thus, the obtained results are interpreted. The results of the paper are summarized in Section 6.

2. Fundamental Matrix Relations

Let's start this section by writing the Clique polynomials in matrix form

$$\mathbf{C}_N(s) = \mathbf{S}_N(s)\mathbf{M}_N, \tag{5}$$

where $\mathbf{C}_N(s) = \begin{bmatrix} C_0(s) & C_1(s) & \cdots & C_N(s) \end{bmatrix}$, $\mathbf{S}_N(s) = \begin{bmatrix} 1 & s & s^2 & \cdots & s^N \end{bmatrix}$,

$$\mathbf{M}_N = \begin{bmatrix} \binom{0}{0} & \binom{1}{0} & \binom{2}{0} & \cdots & \binom{N}{0} \\ 0 & \binom{1}{1} & \binom{2}{1} & \cdots & \binom{N}{1} \\ 0 & 0 & \binom{2}{2} & \cdots & \binom{N}{2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \binom{N}{N} \end{bmatrix}.$$

Secondly, we can express the approximate solutions (2) as

$$u_N(s) = \mathbf{C}_N(s)\mathbf{A}_N, \tag{6}$$

where $\mathbf{C}_N(s) = \begin{bmatrix} C_0(s) & C_1(s) & \cdots & C_N(s) \end{bmatrix}$ and $\mathbf{A}_N = \begin{bmatrix} a_0 & a_1 & \cdots & a_N \end{bmatrix}^T$.

Using relation (5) in (6), we get

$$u_N(s) = \mathbf{S}_N(s)\mathbf{M}_N\mathbf{A}_N. \tag{7}$$

By taking the derivative of (7), we have

$$u_N^{'}(s) = \mathbf{S}_N(s)\mathbf{P}_N\mathbf{M}_N\mathbf{A}_N, \tag{8}$$

where

$$\mathbf{P}_N = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & N & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}.$$

Similarly, the second and the third derivative of (7) becomes

$$u_N^{''}(s) = \mathbf{S}_N(s)(\mathbf{P}_N)^2\mathbf{M}_N\mathbf{A}_N \tag{9}$$

and

$$u_N'''(s) = \mathbf{S}_N(s)(\mathbf{P}_N)^3\mathbf{M}_N\mathbf{A}_N. \tag{10}$$

By writing $s \to s+\theta_4$ in (7), we obtain the relation

$$u_N(s+\theta_4) = \mathbf{S}_N(s+\theta_4)\mathbf{M}_N\mathbf{A}_N$$

or since $\mathbf{S}_N(s+\theta_4) = \mathbf{S}_N(s)\mathbf{D}_N(\theta_4)$, we can also write it as

$$u_N(s+\theta_4) = \mathbf{S}_N(s)\mathbf{D}_N(\theta_4)\mathbf{M}_N\mathbf{A}_N, \tag{11}$$

where

$$\mathbf{D}_N(\theta_4) = \begin{bmatrix} \binom{0}{0}(\theta_4)^0 & \binom{1}{0}(\theta_4)^1 & \cdots & \binom{N}{0}(\theta_4)^N \\ 0 & \binom{1}{1}(\theta_4)^0 & \cdots & \binom{N}{1}(\theta_4)^{N-1} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \binom{N}{N}(\theta_4)^0 \end{bmatrix}.$$

Similarly, substituting $s \to s+\theta_3$, $s \to s+\theta_2$ and $s \to s+\theta_1$, respectively, into (8), (9) and (10), we have

$$u_N'(s+\theta_3) = \mathbf{S}_N(s)\mathbf{D}_N(\theta_3)\mathbf{P}_N\mathbf{M}_N\mathbf{A}_N, \tag{12}$$

$$u_N''(s+\theta_2) = \mathbf{S}_N(s)\mathbf{D}_N(\theta_2)(\mathbf{P}_N)^2\mathbf{M}_N\mathbf{A}_N \tag{13}$$

and

$$u_N'''(s+\theta_1) = \mathbf{S}_N(s)\mathbf{D}_N(\theta_1)(\mathbf{P}_N)^3\mathbf{M}_N\mathbf{A}_N, \tag{14}$$

where

$$\mathbf{D}_N(\theta_3) = \begin{bmatrix} \binom{0}{0}(\theta_3)^0 & \binom{1}{0}(\theta_3)^1 & \cdots & \binom{N}{0}(\theta_3)^N \\ 0 & \binom{1}{1}(\theta_3)^0 & \cdots & \binom{N}{1}(\theta_3)^{N-1} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \binom{N}{N}(\theta_3)^0 \end{bmatrix}, \quad \mathbf{D}_N(\theta_2) = \begin{bmatrix} \binom{0}{0}(\theta_2)^0 & \binom{1}{0}(\theta_2)^1 & \cdots & \binom{N}{0}(\theta_2)^N \\ 0 & \binom{1}{1}(\theta_2)^0 & \cdots & \binom{N}{1}(\theta_2)^{N-1} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \binom{N}{N}(\theta_2)^0 \end{bmatrix},$$

$$\mathbf{D}_N(\theta_1) = \begin{bmatrix} \binom{0}{0}(\theta_1)^0 & \binom{1}{0}(\theta_1)^1 & \cdots & \binom{N}{0}(\theta_1)^N \\ 0 & \binom{1}{1}(\theta_1)^0 & \cdots & \binom{N}{1}(\theta_1)^{N-1} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \binom{N}{N}(\theta_1)^0 \end{bmatrix}.$$

Finally in this section, by writing $s \to 0$ in (7), (8) and (9), we have, respectively

$$u_N(0) = \mathbf{S}_N(0)\mathbf{M}_N\mathbf{A}_N, \tag{15}$$

$$u_N'(0) = \mathbf{S}_N(0)\mathbf{P}_N\mathbf{M}_N\mathbf{A}_N \tag{16}$$

and

$$u_N''(0) = \mathbf{S}_N(0)(\mathbf{P}_N)^2\mathbf{M}_N\mathbf{A}_N. \tag{17}$$

3. Clique Collocation Method

Firstly, we write the relations (11) - (14) instead of (1) and so we have

$$\begin{aligned}\mathbf{S}_N(s)\mathbf{D}_N(\theta_1)(\mathbf{P}_N)^3\mathbf{M}_N\mathbf{A}_N + \tfrac{\beta_1}{s}\mathbf{S}_N(s)\mathbf{D}_N(\theta_2)(\mathbf{P}_N)^2\mathbf{M}_N\mathbf{A}_N + \tfrac{\beta_2}{s^2}\mathbf{S}_N(s)\mathbf{D}_N(\theta_3)\mathbf{P}_N\mathbf{M}_N\mathbf{A}_N \\ +s\,\mathbf{S}_N(s)\mathbf{D}_N(\theta_4)\mathbf{M}_N\mathbf{A}_N = \alpha(s).\end{aligned} \tag{18}$$

Secondly, we obtain

$$\begin{gathered}
\mathbf{S}_N(s_0)\mathbf{D}_N(\theta_1)(\mathbf{P}_N)^3\mathbf{M}_N\mathbf{A}_N + \tfrac{\beta_1}{s_0}\mathbf{S}_N(s_0)\mathbf{D}_N(\theta_2)(\mathbf{P}_N)^2\mathbf{M}_N\mathbf{A}_N + \tfrac{\beta_2}{(s_0)^2}\mathbf{S}_N(s_0)\mathbf{D}_N(\theta_3)\mathbf{P}_N\mathbf{M}_N\mathbf{A}_N + s_0\mathbf{S}_N(s_0)\mathbf{D}_N(\theta_4)\mathbf{M}_N\mathbf{A}_N = \alpha(s_0) \\
\mathbf{S}_N(s_1)\mathbf{D}_N(\theta_1)(\mathbf{P}_N)^3\mathbf{M}_N\mathbf{A}_N + \tfrac{\beta_1}{s_1}\mathbf{S}_N(s_1)\mathbf{D}_N(\theta_2)(\mathbf{P}_N)^2\mathbf{M}_N\mathbf{A}_N + \tfrac{\beta_2}{(s_1)^2}\mathbf{S}_N(s_1)\mathbf{D}_N(\theta_3)\mathbf{P}_N\mathbf{M}_N\mathbf{A}_N + s_1\mathbf{S}_N(s_1)\mathbf{D}_N(\theta_4)\mathbf{M}_N\mathbf{A}_N = \alpha(s_1) \\
\vdots \\
\mathbf{S}_N(s_N)\mathbf{D}_N(\theta_1)(\mathbf{P}_N)^3\mathbf{M}_N\mathbf{A}_N + \tfrac{\beta_1}{s_N}\mathbf{S}_N(s_N)\mathbf{D}_N(\theta_2)(\mathbf{P}_N)^2\mathbf{M}_N\mathbf{A}_N + \tfrac{\beta_2}{(s_N)^2}\mathbf{S}_N(s_N)\mathbf{D}_N(\theta_3)\mathbf{P}_N\mathbf{M}_N\mathbf{A}_N + s_N\mathbf{S}_N(s_N)\mathbf{D}_N(\theta_4)\mathbf{M}_N\mathbf{A}_N = \alpha(s_N)
\end{gathered} \tag{19}$$

by using the collocation points defined as

$$s_i = a + \frac{b-a}{N}i, \quad i = 0, 1, \ldots, N \tag{20}$$

in the range $[a, b]$, where a is a sufficiently small positive number in the range $0 < a < 1$.

Sytem (19) can also be written, briefly, as

$$\mathbf{W}\mathbf{A}_N = \mathbf{G}, \tag{21}$$

where

$$\mathbf{W} = \left(\mathbf{S}\mathbf{D}_N(\theta_1)(\mathbf{P}_N)^3 + \mathbf{E}_1\mathbf{S}\mathbf{D}_N(\theta_2)(\mathbf{P}_N)^2 + \mathbf{E}_2\mathbf{S}\mathbf{D}_N(\theta_3)\mathbf{P}_N + \mathbf{E}_3\mathbf{S}\mathbf{D}_N(\theta_4)\right)\mathbf{M}_N,$$

$$\mathbf{G} = \begin{bmatrix} \alpha(s_0) \\ \alpha(s_1) \\ \vdots \\ \alpha(s_N) \end{bmatrix}, \quad \mathbf{S} = \begin{bmatrix} \mathbf{S}_N(s_0) \\ \mathbf{S}_N(s_1) \\ \vdots \\ \mathbf{S}_N(s_N) \end{bmatrix}, \quad \mathbf{E}_1 = diag\left(\frac{\beta_1}{s_i}\right), \quad \mathbf{E}_2 = diag\left(\frac{\beta_2}{(s_i)^2}\right), \quad \mathbf{E}_3 = diag\,(s_i)\,.$$

As the next step, we write system (15)-(17) instead of any 3 rows of system (21). Thus, we get

$$\begin{gathered}
\mathbf{S}_N(s_0)\mathbf{D}_N(\theta_1)(\mathbf{P}_N)^3\mathbf{M}_N\mathbf{A}_N + \tfrac{\beta_1}{s_0}\mathbf{S}_N(s_0)\mathbf{D}_N(\theta_2)(\mathbf{P}_N)^2\mathbf{M}_N\mathbf{A}_N + \tfrac{\beta_2}{(s_0)^2}\mathbf{S}_N(s_0)\mathbf{D}_N(\theta_3)\mathbf{P}_N\mathbf{M}_N\mathbf{A}_N + s_0\mathbf{S}_N(s_0)\mathbf{D}_N(\theta_4)\mathbf{M}_N\mathbf{A}_N = \alpha(s_0) \\
\mathbf{S}_N(s_1)\mathbf{D}_N(\theta_1)(\mathbf{P}_N)^3\mathbf{M}_N\mathbf{A}_N + \tfrac{\beta_1}{s_1}\mathbf{S}_N(s_1)\mathbf{D}_N(\theta_2)(\mathbf{P}_N)^2\mathbf{M}_N\mathbf{A}_N + \tfrac{\beta_2}{(s_1)^2}\mathbf{S}_N(s_1)\mathbf{D}_N(\theta_3)\mathbf{P}_N\mathbf{M}_N\mathbf{A}_N + s_1\mathbf{S}_N(s_1)\mathbf{D}_N(\theta_4)\mathbf{M}_N\mathbf{A}_N = \alpha(s_1) \\
\vdots \\
\mathbf{S}_N(s_{N-3})\mathbf{D}_N(\theta_1)(\mathbf{P}_N)^3\mathbf{M}_N\mathbf{A}_N + \tfrac{\beta_1}{s_{N-3}}\mathbf{S}_N(s_{N-3})\mathbf{D}_N(\theta_2)(\mathbf{P}_N)^2\mathbf{M}_N\mathbf{A}_N + \tfrac{\beta_2}{(s_{N-3})^2}\mathbf{S}_N(s_{N-3})\mathbf{D}_N(\theta_3)\mathbf{P}_N\mathbf{M}_N\mathbf{A}_N + s_{N-3}\mathbf{S}_N(s_{N-3})\mathbf{D}_N(\theta_4)\mathbf{M}_N\mathbf{A}_N = \alpha(s_{N-3}) \\
\mathbf{S}_N(0)\mathbf{M}_N\mathbf{A}_N = k_1 \\
\mathbf{S}_N(0)\mathbf{P}_N\mathbf{M}_N\mathbf{A}_N = k_2 \\
\mathbf{S}_N(0)(\mathbf{P}_N)^2\mathbf{M}_N\mathbf{A}_N = k_3.
\end{gathered} \tag{22}$$

Let's note that we select the last 3 rows in the system (22). Finally, we solve the obtained new system and so we calculate the unknown Clique coefficients matrix $\mathbf{A}_N$. Hence, we achieve the Clique polynomial solutions $u_N(s)$ by putting the obtained matrix $\mathbf{A}_N$ into (6).

4. Error Estimation Technique

Let's start this section by defining residual function as

$$R_N(s) = L[u_N(s)] - \alpha(s). \tag{23}$$

Since the Clique polynomial solutions satisfy problem (1), we get

$$\begin{cases} u_N'''(s+\theta_1) + \frac{\beta_1}{s} u_N''(s+\theta_2) + \frac{\beta_2}{s^2} u_N'(s+\theta_3) + s\, u_N(s+\theta_4) = \alpha(s), \\ u_N(0) = k_1, \quad u_N'(0) = k_2, \quad u_N''(0) = k_3. \end{cases} \tag{24}$$

Secondly, we subtract (24) from (1) and so we gain the error problem

$$\begin{cases} e_N'''(s+\theta_1) + \frac{\beta_1}{s} e_N''(s+\theta_2) + \frac{\beta_2}{s^2} e_N'(s+\theta_3) + s\, e_N(s+\theta_4) = -R_N(s), \\ e_N(0) = 0, \quad e_N'(0) = 0, \quad e_N''(0) = 0. \end{cases} \tag{25}$$

Here, $u(s)$, $u_N(s)$ and $e_N(s)$ denote, respectively, the exact solution, the Clique polynomial solution and the actual error function. Also, let's note that $e_N(s) = u(s) - u_N(s)$.

Finally, we solve the system (25) according to Clique collocation method in previous section and thus we gain the estimated error function

$$e_{N,M}(s) = \sum_{n=0}^{M} a_n^* C_n(s), \tag{26}$$

where a_n^* is the unknown coefficients. The error estimation method is important. Because we can calculate the made error if the exact solution of the problem is not known.

5. Applications

In this section, the applications of methods in previous sections are made using MATLAB.

Example 5.1 *Firstly, we consider the model based on the the third-order multisingular (MS) functional differential equations with initial conditions [32] given as*

$$\begin{cases} u'''(s-1) + \frac{1}{s}u''(s+1) + \frac{2}{s^2}u'(s+2) + su(s) = e^{s-1} + \frac{1}{s}e^{s+1} + \frac{2}{s^2}e^{s+2} + se^s, \\ u(0) = 1, \quad u'(0) = 1, \quad u''(0) = 1. \end{cases} \tag{27}$$

Our aim is to obtain Clique polynomial solutions for $N = 3$ as:

$$u_3(s) = \sum_{i=0}^{3} a_i C_i(s), \tag{28}$$

or

$$u_3(s) = \mathbf{S}_3(s)\mathbf{M}_3\mathbf{A}_3, \tag{29}$$

where $\mathbf{S}_3(s) = \begin{bmatrix} 1 & s & s^2 & s^3 \end{bmatrix}$, $\mathbf{A}_3 = \begin{bmatrix} a_0 & a_1 & a_2 & a_3 \end{bmatrix}^T$ and $\mathbf{M}_3 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

If we use the relation (20), we write the collocation points for $a = 0.01$, $b = 1$ as $s_0 = \frac{1}{100}, s_1 = \frac{17}{50}, s_2 = \frac{67}{100}, s_3 = 1$. Hence, if we utilize the system (21), then we get

$$\mathbf{WA} = \mathbf{G}, \tag{30}$$

where

$$\mathbf{W} = \left(\mathbf{SD}_3(-1)(\mathbf{P}_3)^3 + \mathbf{E}_1\mathbf{SD}_3(1)(\mathbf{P}_3)^2 + \mathbf{E}_2\mathbf{SD}_3(2)\mathbf{P}_3 + \mathbf{E}_3\mathbf{SD}_3(0)\right)\mathbf{M}_3,$$

$$\mathbf{G} = \begin{bmatrix} e^{s_0-1} + \frac{1}{s_0}e^{s_0+1} + \frac{2}{s_0^2}e^{s_0+2} + s_0e^{s_0} \\ e^{s_1-1} + \frac{1}{s_1}e^{s_1+1} + \frac{2}{s_1^2}e^{s_1+2} + s_1e^{s_1} \\ e^{s_2-1} + \frac{1}{s_2}e^{s_2+1} + \frac{2}{s_2^2}e^{s_2+2} + s_2e^{s_2} \\ e^{s_3-1} + \frac{1}{s_3}e^{s_3+1} + \frac{2}{s_3^2}e^{s_3+2} + s_3e^{s_3} \end{bmatrix}, \quad \mathbf{S} = \begin{bmatrix} \mathbf{S}_3(s_0) \\ \mathbf{S}_3(s_1) \\ \mathbf{S}_3(s_2) \\ \mathbf{S}_3(s_3) \end{bmatrix}, \quad \mathbf{P}_3 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

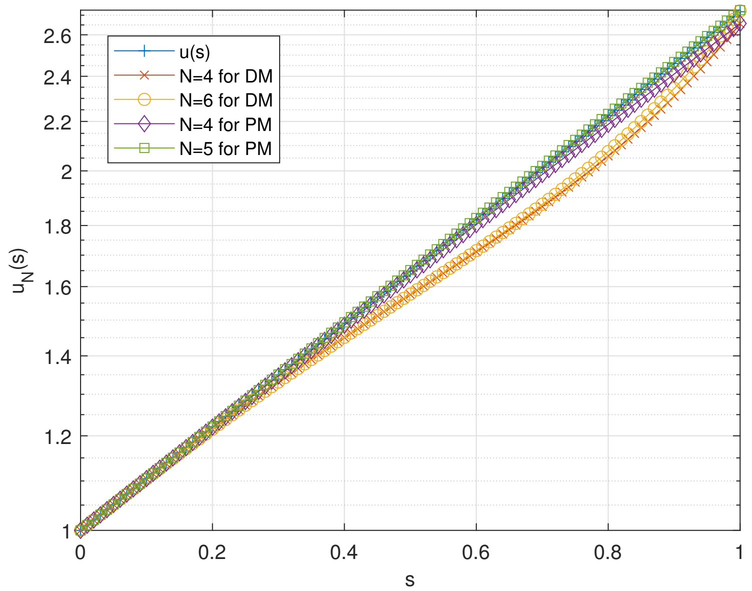
$$\mathbf{D}_3(-1) = \begin{bmatrix} 1 & -1 & 1 & -1 \\ 0 & 1 & -2 & 3 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{D}_3(1) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{D}_3(2) = \begin{bmatrix} 1 & 2 & 4 & 8 \\ 0 & 1 & 4 & 12 \\ 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\mathbf{D}_3(0) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{E}_1 = diag\left(\frac{1}{s_i}\right), \quad \mathbf{E}_2 = diag\left(\frac{2}{(s_i)^2}\right), \quad \mathbf{E}_3 = diag\left(s_i\right).$$

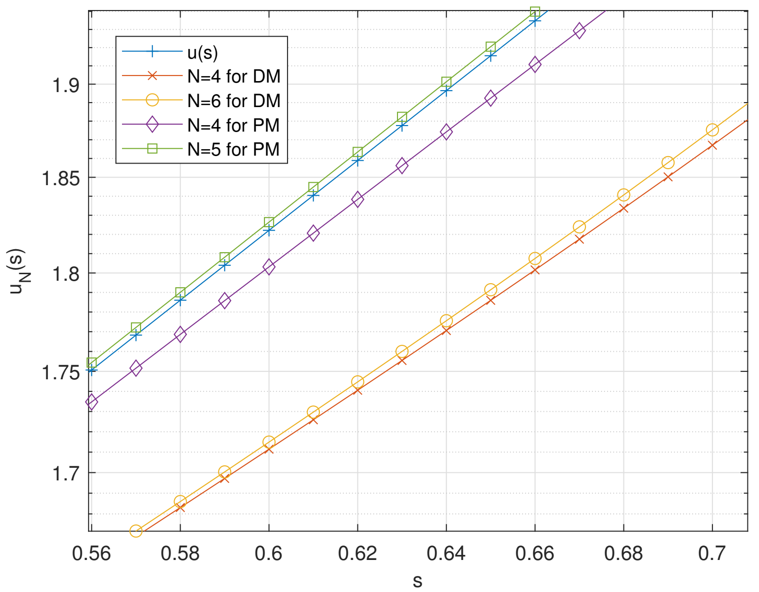
Figure 1 (a) shows the exact solution and the approximate solutions for $N = 4$ and $N = 5$. Also, this figure compares these solutions with the solutions of differential transform method (DM) [32] for $N = 4$ and $N = 6$. Figure 1 (b) shows these functions more closely. Accordingly, the closest result to the exact solution is obtained with our method. In addition, the values of these functions at some s points are compared with the solutions of DM [32] in Table 1.

Figure 2 (a) compares the actual absolute errors of Example 5.1 with the errors of DM [32] for $N = 4$ and $N = 5$. Accordingly, the results of DM [32] for $N = 4$ and $N = 6$ are the same. The result obtained with our method with $N = 4$ is better than these results.

The best result is obtained when $N = 5$ is chosen in our method. In other words, with our method, a more suitable result is obtained with a smaller N value. Figure 2 (b) visualizes the actual absolute errors of Example 5.1 for $N = 4$ and $N = 5$ and the estimated absolute errors of Example 5.1 for $(N, M) = (4, 5)$ and $(N, M) = (5, 6)$. This shows that the actual and estimated absolute error for $N = 4$ and $(N, M) = (4, 5)$ overlap. Also, it can be concluded that the actual



a) The exact solution and the approximate solutions



b) Close angle of solutions

Figure 1: Comparison of solutions of Example 5.1 with DM [32]

Table 1: Comparison of the solutions of Example 5.1 with DM [32]

s_i	Exact solution	DM [32] (N=4)	DM [32] (N=6)	Present Method (N=4)	Present Method (N=5)
0.01	1.01005	1.01005	1.01005	1.01005	1.01005
0.02	1.02020	1.02020	1.02020	1.0202	1.0202
0.03	1.03045	1.03045	1.03045	1.03045	1.03046
0.04	1.04081	1.04080	1.04080	1.0408	1.04081
0.05	1.05127	1.05125	1.05125	1.05125	1.05128
0.06	1.06183	1.06180	1.06180	1.06181	1.06185
0.07	1.07250	1.07246	1.07246	1.07246	1.07252
0.08	1.08328	1.08322	1.09408	1.08322	1.08331
0.09	1.09417	1.09408	1.09408	1.09408	1.09421
0.1	1.10517	1.10504	1.10504	1.10504	1.10521
0.2	1.22140	1.22044	1.22044	1.22046	1.2217
0.3	1.34985	1.34683	1.34685	1.34688	1.35074
0.4	1.49182	1.48515	1.48522	1.48526	1.49361
0.5	1.64872	1.63663	1.63686	1.63684	1.65168
0.6	1.82211	1.80282	1.80341	1.80316	1.82641
0.7	2.01375	1.98556	1.98688	1.98606	2.01938
0.8	2.22554	2.18698	2.18965	2.1877	2.23237
0.9	2.45960	2.40955	2.41451	2.41051	2.46733

and estimated absolute error for $N = 5$ and $(N, M) = (5, 6)$ are very close. Moreover, it can be seen from figures that the error decreases as N increases.

Example 5.2 *Our second model is [32]*

$$\begin{cases} u'''(s-1) + \frac{1}{s}u''(s+1) + \frac{2}{s^2}u'(s+2) + su(s) = s^5 + 45s + 48 + \frac{108}{s} + \frac{64}{s^2}, \\ u(0) = 1, \quad u'(0) = 0, \quad u''(0) = 0. \end{cases} \tag{31}$$

The exact solution of this problem is $1+s^4$. Our aim is to obtain Clique polynomial solutions for $N = 4$ as:

$$u_4(s) = \sum_{i=0}^{4} a_i C_i(s), \tag{32}$$

or

$$u_4(s) = \mathbf{S}_4(s)\mathbf{M}_4\mathbf{A}_4. \tag{33}$$

Utilizing the system (21), we get

$$\mathbf{WA} = \mathbf{G}, \tag{34}$$

where

$$\mathbf{W} = \left(\mathbf{SD}_4(-1)(\mathbf{P}_4)^3 + \mathbf{E}_1\mathbf{SD}_4(1)(\mathbf{P}_4)^2 + \mathbf{E}_2\mathbf{SD}_4(2)\mathbf{P}_4 + \mathbf{E}_4\mathbf{SD}_4(0)\right)\mathbf{M}_4,$$

$$\mathbf{G} = \begin{bmatrix} s_0^5 + 45s_0 + 48 + \frac{108}{s_0} + \frac{64}{s_0^2} \\ s_1^5 + 45s_1 + 48 + \frac{108}{s_1} + \frac{64}{s_1^2} \\ s_2^5 + 45s_2 + 48 + \frac{108}{s_2} + \frac{64}{s_2^2} \\ s_3^5 + 45s_3 + 48 + \frac{108}{s_3} + \frac{64}{s_3^2} \\ s_4^5 + 45s_4 + 48 + \frac{108}{s_4} + \frac{64}{s_4^2} \end{bmatrix}.$$

By writing $s \to 0$ in (7), (8) and (9), we have, respectively

$$u_4(0) = \mathbf{S}_4(0)\mathbf{M}_4\mathbf{A}_4, \tag{35}$$

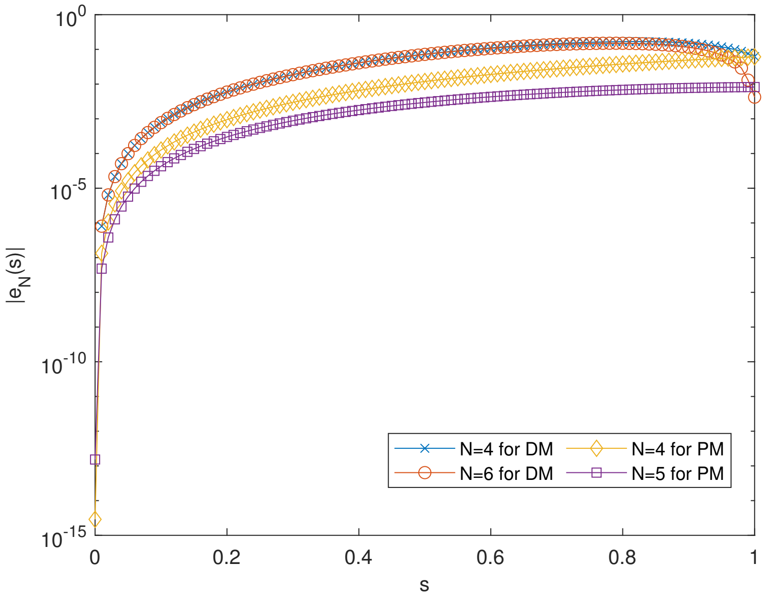
$$u_4'(0) = \mathbf{S}_4(0)\mathbf{P}_4\mathbf{M}_4\mathbf{A}_4 \tag{36}$$

and

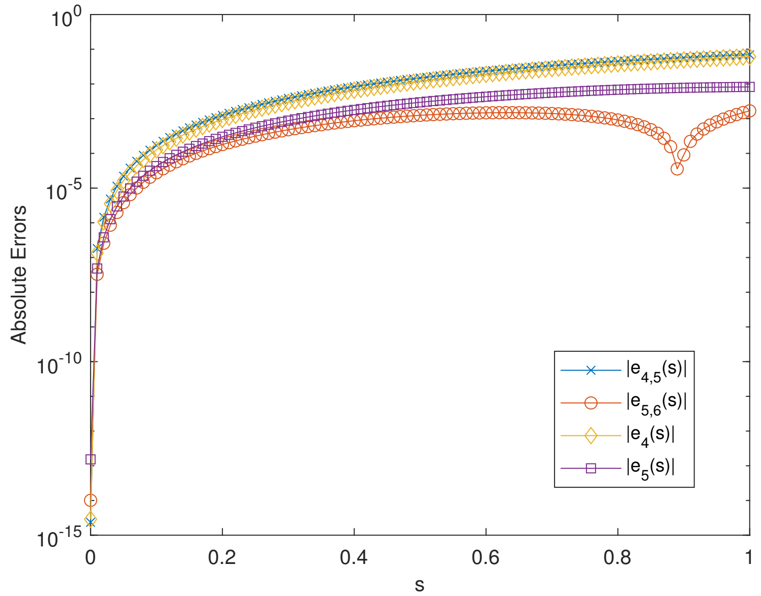
$$u_4'(0) = \mathbf{S}_4(0)(\mathbf{P}_4)^2\mathbf{M}_4\mathbf{A}_4, \tag{37}$$

where $\mathbf{S}_4(0) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \end{bmatrix}$.

Finally, the approximate solution is obtained $1+s^4$ by solving the system (34) with conditions (35), (36) and (37). This is the exact solution.



a) Actual absolute errors



b) The actual absolute errors and the estimated absolute errors

Figure 2: Comparison of absolute errors of Example 5.1 with DM [32]

Example 5.3 *Finally, we perform the model based on the the third-order multisingular (MS) functional differential equations with initial conditions [32]*

$$\begin{cases} u'''(s-1) + \frac{1}{s}u''(s+1) + \frac{2}{s^2}u'(s+2) + su(s) = s^4 + s + 18 + \frac{30}{s} + \frac{24}{s^2}, \\ u(0) = 1, \quad u'(0) = 0, \quad u''(0) = 0. \end{cases} \tag{38}$$

The exact solution of this problem is $1+s^3$. Our aim is to obtain Clique polynomial solutions for $N = 3$ as:

$$u_3(s) = \sum_{i=0}^{3} a_i C_i(s), \tag{39}$$

or

$$u_3(s) = \mathbf{S}_3(s)\mathbf{M}_3\mathbf{A}_3. \tag{40}$$

Utilizing the system (21), we obtain

$$\mathbf{WA} = \mathbf{G}, \tag{41}$$

where

$$\mathbf{W} = \left(\mathbf{SD}_3(-1)(\mathbf{P}_3)^3 + \mathbf{E}_1\mathbf{SD}_3(1)(\mathbf{P}_3)^2 + \mathbf{E}_2\mathbf{SD}_3(2)\mathbf{P}_3 + \mathbf{E}_3\mathbf{SD}_3(0)\right)\mathbf{M}_3,$$

$$\mathbf{G} = \begin{bmatrix} s_0^4 + s_0 + 48 + \frac{30}{s_0} + \frac{24}{s_0^2} \\ s_1^4 + s_1 + 48 + \frac{30}{s_1} + \frac{24}{s_1^2} \\ s_2^4 + s_2 + 48 + \frac{30}{s_2} + \frac{24}{s_2^2} \\ s_3^4 + s_3 + 48 + \frac{30}{s_3} + \frac{24}{s_3^2} \end{bmatrix}.$$

By writing $s \to 0$ in (7), (8) and (9), we have, respectively

$$u_3(0) = \mathbf{S}_3(0)\mathbf{M}_3\mathbf{A}_3, \tag{42}$$

$$u_3'(0) = \mathbf{S}_3(0)\mathbf{P}_3\mathbf{M}_3\mathbf{A}_3 \tag{43}$$

and

$$u_3'(0) = \mathbf{S}_3(0)(\mathbf{P}_3)^2\mathbf{M}_3\mathbf{A}_3, \tag{44}$$

where $\mathbf{S}_3(0) = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}$.

Finally, the approximate solution is obtained $1+s^4$ by solving the system (41) with conditions (42), (43) and (44). This is the exact solution.

6. Conclusions

In this paper, we investigate the approximate solution of the third-order multisingular (MS) functional differential equation via Clique collocation method. In addition to method, we constitute error estimation technique for the problem. Also, we make applications of the Clique collocation method and the error estimation technique for three examples by using MATLAB. Accordingly, we obtain the exact solution in Example 5.2 and Example 5.3. This result demonstrates the advantage of our method. In addition, we compare the results with differential transform method (DM) [32] for Example 5.1. Accordingly, the best result is obtained when $N = 5$ is chosen in our method. In other words, with our method, a more suitable result is obtained with a smaller N value. According to our method, the error decreases as N increases. Moreover, the estimated errors are close to the actual errors, which shows the importance of the error estimation technique. From all numerical results, we conclude that the presented method is efficient and reliable. The presented method can be improved for nonlinear multisingular functional differential equations or multisingular functional differential equations of fractional-order.

Declaration of Ethical Standards

The authors declare that the materials and methods used in their study do not require ethical committee and/or legal special permission.

Authors Contributions

Author [Gamze Yıldırım]: Collected the data, contributed to research method or evaluation of data, contributed to completing the research and solving the problem, wrote the manuscript (%50).

Author [Şuayip Yüzbaşı]: Thought and designed the research/problem, contributed to research method or evaluation of data, contributed to completing the research and solving the problem (%50).

Conflicts of Interest

The authors declare no conflict of interest.

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Research Article

A Quaternionic Product of Lines in the Plane $\mathbb{E}^2$

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Abstract: This note introduces a product of plane lines inspired by the product of quaternions. A technical condition is necessary for the existence of this product and some examples (squares, the axes and the bisectrices of the axes) are discussed.

Keywords: Line, quaternion, product, projective geometry.

1. Introduction

The quaternionic algebra $\mathbb{H}$ is a well-known setting of modern mathematics, together with the real algebra $\mathbb{R}$ and the complex algebra $\mathbb{C}$; since the bibliography on quaternions is huge we cite only the very first paper [8] of Sir William Rowan Hamilton. This algebraic structure was designed with the geometric goal of serving as a helpful tool for modelling the rotations of three-dimensional Euclidean space from the very beginning. The projections and involutions are expressed with quaternions in [1]. We note also that recently the applications of quaternions in the differential geometry are surveyed in [7].

The purpose of the present work is to use the product of $\mathbb{H}$ into another framework namely the set of lines of the Euclidean plane. The choice of the identification of a line with a quaternion is based on some previous papers of the author. We point out also that in order to obtain a suitable quaternionic product we introduce a technical condition in our Definition 2.1. These considerations yields a projective way to manage this product and we consider that the potential areas of applications are the incidence geometry [9].

This new product is discussed especially from the point of view of examples. In addition to squares we study some concrete examples by giving also numerical details.

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2. The Quaternionic Product of Two Distinguished Lines

Fix the set of all lines $\mathcal{L} := \{d : ax + by + c = 0; a^2 + b^2 > 0\}$ in the Euclidean plane $\mathbb{E}^2 := (\mathbb{R}^2, \langle \cdot, \cdot \rangle)$. The aim of this work is to introduce a product (inspired by quaternions) in $\mathcal{L}$ and hence the starting point of this paper is the identification of the given line

$$d = d(a, b, c) \in \mathbb{R}^3 \setminus \{(0, 0, c) : c \in \mathbb{R}\}$$

with the quaternion:

$$q(d) := \bar{k} + a\bar{i} + b\bar{j} + c = (c, a, b, 1) \in \mathbb{R}^4. \tag{1}$$

The quaternion $q(d)$ is pure imaginary if and only if the origin $O(0, 0) \in d$. We point out that although there are alternative ways to associate a quaternion to a given line, we choose the expression (1) according to our previous studies, namely (in the chronological order) [2, 4, 5].

From the real algebra structure $\mathbb{H}$ of the quaternions ([6, p. 89]), it follows a product of two lines:

$$d_1 \odot_q d_2 := q^{-1}(q(d_1) \cdot q(d_2)). \tag{2}$$

With the given parameters $(a_i, b_i, c_i), i = 1, 2$, we derive immediately:

$$q(d_1) \cdot q(d_2) = (a_1 b_2 - a_2 b_1 + c_1 + c_2)\bar{k} + (b_1 - b_2 + a_1 c_2 + a_2 c_1)\bar{i} + (a_2 - a_1 + b_1 c_2 + b_2 c_1)\bar{j} +$$

$$+ (c_1 c_2 - 1 - a_1 a_2 - b_1 b_2) =: D\bar{k} + A\bar{i} + B\bar{j} + C \tag{3}$$

and due to the expression of the coefficient of $\bar{k}$ in (1), we need a special condition for our approach.

Definition 2.1 *The given pair of lines $d_r = (a_r, b_r, c_r)$, $r = 1, 2$, is called q-distinguished if:*

$$D(d_1, d_2) := a_1 b_2 - a_2 b_1 + c_1 + c_2 = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} + c_1 + c_2 \neq 0. \tag{4}$$

Example 2.2 *i) For a fixed line $d(a, b, c)$ we have the square:*

$$q(d) \cdot q(d) = 2c\bar{k} + 2ac\bar{i} + 2bc\bar{j} + (c^2 - 1 - a^2 - b^2). \tag{5}$$

If $c \neq 0$, then the technical condition (4) is satisfied and then the pair (d, d) is q-distinguished. Also, $q(d) \in \mathbb{R}^4$ is a purely imaginary quaternion only for $c_{\pm} = \pm\sqrt{a^2 + b^2 + 1}$ with $c_- < -1$ and $c_+ > 1$.

ii) Let d_1 and d_2 be concurrent lines in O. Then, (4) means that their normals $N_1 = (a_1, b_1)$, $N_2 = (a_2, b_2)$ are linear independent vectors, i.e., the lines are different.

Let now $\mathcal{L}^2(q)$ be the set of q-distinguished pairs of lines. Working in a projective manner it follows a quaternionic product in $\mathcal{L}^2(q)$:

$$d_1 \odot_q d_2 := d\left(\frac{A}{D}, \frac{B}{D}, \frac{C}{D}\right) \tag{6}$$

supposing again that $A^2 + B^2 > 0$. It is worth to point out the combination of Euclidean and projective geometry of our approach; hence, $d_1 \odot_q d_2$ is also the line $d(A, B, C)$.

An important tool of the quaternionic theory is that of conjugate, which for our quaternion (1) means:

$$\overline{q(d)} := -\bar{k} - a\bar{i} - b\bar{j} + c = (c, -a, -b, -1) = -q(a, b, -c) \tag{7}$$

and our projective way of thinking allows the identification: $\overline{q(a,b,c)} = q(a, b, -c)$. The pair of parallel lines $(d(a,b,c), d(a,b,-c))$ is not q-distinguished. The real part C of the quaternion (5) is the Euclidean inner product $\langle \cdot, \cdot \rangle$ in $\mathbb{R}^4$ of the vectors $q(d_1)$ and $\overline{q(d_2)}$.

3. Concrete Examples

In the following we study this new product introduced in (6) through three large examples.

Example 3.1 *Revisiting the Example 2.2 (i) (recall that $c \neq 0$), we have immediately the square of a line d for which $O \notin d$ (recall that $\mathbb{RP}^1$ is the moduli space of lines that contain the origin):*

$$d^2_{\odot_q} : ax + by + \frac{c^2 - a^2 - b^2 - 1}{2c} = 0 \to d^2_{\odot_q} \neq d, \quad d^2_{\odot_q} \parallel d. \tag{8}$$

The expression above suggests as remarkable example the case of right triangle, $\triangle : c^2 = a^2 + b^2$, which gives the associated lines:

$$\begin{cases} d_{(\triangle,+)} : ax + by + \sqrt{a^2 + b^2} = 0, \quad a > 0, \quad b > 0, \\ (d_{(\triangle,+)})^2_{\odot_q} : ax + by - \frac{1}{2\sqrt{a^2+b^2}} = 0. \end{cases} \tag{9}$$

A particular case of the right triangle $\triangle$ is provided by the case when (a, b, c) is a Pythagorean triple and hence we know its parametrization ([3]):

$$a := \beta^2 - \alpha^2, \quad b := 2\alpha\beta, \quad c := \alpha^2 + \beta^2, \quad 0 < \alpha < \beta \in \mathbb{N}^*. \tag{10}$$

It follows the lines:

$$\begin{cases} d(Pythagorean) : (\beta^2 - \alpha^2)x + 2\alpha\beta y + (\alpha^2 + \beta^2) = 0, \\ (d(Pythagorean))^2_{\odot_q} : (\beta^2 - \alpha^2)x + 2\alpha\beta y - \frac{1}{2(\alpha^2+\beta^2)} = 0. \end{cases} \tag{11}$$

For a concrete example we choose the minimal pair $\alpha = 1 < \beta = 2$ *giving the minimal Pythagorean triple* $(a = 3, b = 4, c = 5)$ *with the associated lines:*

$$d(minimal) : 3x + 4y + 5 = 0, \quad (d(minimal))^2_{\odot_q} : 3x + 4y - \frac{1}{10} = 0. \tag{12}$$

This last situation suggests the general case:

$$d_t : (\cos t)x + (\sin t)y + 1 = 0, t \in \mathbb{R} \to (d_t)^2_{\odot_q} : (\cos t)x + (\sin t)y - \frac{1}{2} = 0. \tag{13}$$

Recall that for a given C^2 *periodic and convex function* $p = p(t) = p(t + 2\pi)$ *the convex envelope of the family of lines:*

$$d_p(t) : (\cos t)x + (\sin t)y = p(t)$$

is the oval $\mathcal{C}$ *parametrized by:*

$$\mathcal{C} : (x(t), y(t)) = (p(t)\cos t - p'(t)\sin t, p(t)\sin t + p'(t)\cos t). \tag{14}$$

Therefore, the oval generated by the family $(d_t)^2_{\odot_q}$ *is the Euclidean circle centered in* O *and of radius* $R = \frac{1}{2}$. *Remark also, that the scalar part of the line in the equation* (8) *suggests the real function:*

$$c \in (0, +\infty) \to f(c) := \frac{c^2 - a^2 - b^2 - 1}{2c}$$

which have as oblique asymptotic the line $y = \frac{1}{2}x$.

Example 3.2 *In this example, we will perform the quaternionic product of two different q-distinguished lines. The coordinates lines are so (conform the second Example 2.2) and then we have:*

$$Ox \odot_q Oy : x + y - 1 = 0, \quad Oy \odot_q Ox : x + y + 1 = 0 \tag{15}$$

and hence, generally speaking, the quaternionic product does not preserve the orthogonality nor the concurrency and is not commutative; we have only $Re(q(d_1) \cdot q(d_2)) = Re(q(d_2) \cdot q(d_1))$ *i.e.,* $C(d_1, d_2) = C(d_2, d_1)$. *In fact, if both lines contains the origin, i.e.,* $c_1 = c_2 = 0$, *then* $D(d_1, d_2) = -D(d_2, d_1)$ *and* $d_1 \odot_q d_2 \parallel d_2 \odot_q d_1$. *The bisectrices of the axes are also q-distinguished and their products are vertical lines:*

$$B_1 : x - y = 0, \quad B_2 : x + y = 0 \to B_1 \odot_q B_2 : x - \frac{1}{2} = 0, \quad B_2 \odot_q B_1 : x + \frac{1}{2} = 0. \tag{16}$$

Hence, the quaternionic product does not preserve the concurrence of the given lines.

Example 3.3 *Let three distinct points* $M_i(\alpha_i, \beta_i)$*,* $i = 1, 2, 3$ *and* d_1 *the line* M_0M_1 *respectively* d_2 *the line* M_0M_2*. Since for* d_1 *the coefficients are:*

$$a_1 = \beta_0 - \beta_1, \quad b_1 = \alpha_1 - \alpha_0, \quad c_1 = \alpha_0\beta_1 - \alpha_1\beta_0 \tag{17}$$

and similar relations hold for d_2 *the condition* (4) *reads:*

$$D(d_1, d_2) = \alpha_1\beta_2 - \alpha_2\beta_1 + 2(\alpha_0\beta_1 - \alpha_1\beta_0) \neq 0. \tag{18}$$

Since the translations are Euclidean isometries let us suppose that M_0 *is* O *and then* $D(d_1, d_2)$ *reduces to the scalar part* c *of the line* M_1M_2*; in the proper triangle* $M_0M_1M_2$ *the vertex* M_0 *does not belong to* M_1M_2 *and therefore* (4) *holds. We study now the possibility to introduce the notion of quaternionic triangle as one in which* $M_0M_1 \odot_q M_0M_2$ *is exactly* M_1M_2*. But, if we write explicitly this equality as the equalities of ratios:*

$$\frac{\beta_1 - \beta_2}{\alpha_1 - \alpha_2} = \frac{\alpha_2 - \alpha_1}{\beta_1 - \beta_2} = \frac{\alpha_1\beta_2 - \alpha_2\beta_1}{-1 - \alpha_1\alpha_2 - \beta_1\beta_2} \tag{19}$$

already the first equality yields the impossible $\alpha_1 - \alpha_2 = 0 = \beta_1 - \beta_2$*. This fact can be probably explained by the complex nature of* $\bar{i}$ *and* $\bar{j}$*.*

4. Conclusions

In this paper, we study the geometry of 2D Euclidean lines through an algebraic operation inspired by the product of quaternions. Some features of this new product are discussed directly on examples.

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Declaration of Ethical Standards

The author declares that the materials and methods used in her study do not require ethical committee and/or legal special permission.

Conflicts of Interest

The author declares no conflict of interest.

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Research Article

Some Results on Almost Contact Manifolds with B-Metric

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Abstract: In this work, almost contact B-metric manifolds and almost complex manifolds with Norden metric are considered. Almost complex manifolds with a Norden metric are obtained by the product of almost contact B-metric manifolds with $\mathbb{R}$, where almost complex structure and metric on the product manifold depend on two functions of $\mathbb{R}$. The relations between two classes of almost contact manifolds with B-metric (the classes $\mathcal{F}_4$ and $\mathcal{F}_5$) and classes of almost complex manifolds with a Norden metric are investigated.

Keywords: Almost complex manifold with a Norden metric, almost contact manifold, almost contact manifold with B-metric.

1. Introduction

Differentiable manifolds having special tensors are an object of interest in differential geometry. There are several studies on this area, for example, see [2, 4–8, 10, 11, 13–16, 19–21]. Differential manifolds having special tensor structure have been classified by considering the covariant derivative of their tensor structure [2, 4–8, 10, 11, 13, 21].

Manifolds with B-metric have been studied in the last 30 years by various researchers [7, 9, 10, 16, 20]. Recently, many differential geometers and theoretical physicists have been investigating Ricci solitons and η-Ricci solitons on manifolds with special structures, such as almost contact metric manifolds, almost paracontact metric manifolds, manifolds with B-metric, Norden manifolds, etc. [1, 3, 12, 17, 18]. In this investigations, classes of almost contact B-metric manifolds and almost complex manifolds with a Norden metric also gain importance.

In this study, we obtain an infinite number of Kaehlerian manifolds with a Norden metric in Theorem 3.3 and complex manifolds with a Norden metric (the class $W_1 \oplus W_2$) in Thoerem 3.5. In particular, we consider the classification of almost contact manifolds with B-metric and almost complex manifolds with a Norden metric given by [6, 7], respectively. We generalize the metric and

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the complex structure on the product manifold given in [9] by considering two functions. In [9], Sasaki-like manifolds which are subclasses of $\mathcal{F}_4$ of almost contact B-metric manifolds are studied. In this work, almost complex Norden metric manifolds are obtained from almost contact manifolds with B-metric M with product of $\mathbb{R}$ and an almost complex structure and a metric are defined on the product manifold $M \times \mathbb{R}$ depending on two functions σ and μ which are functions of t. Some relations between classes of almost complex manifolds with a Norden metric and the classes $\mathcal{F}_4$ and $\mathcal{F}_5$ of almost contact manifolds with B-metric are obtained.

2. Preliminaries

First, we introduce almost contact B-metric manifolds. A manifold M with odd dimension has an almost contact structure (φ, ξ, η), if it admits a vector field ξ, a map φ, and a 1-form η satisifying the following relations:

$$\eta(\xi) = 1, \quad \varphi^2 = -I + \eta \otimes \xi. \tag{1}$$

Here I is identity map. From (1),

$$\varphi(\xi) = 0, \quad \eta \circ \varphi = 0 \tag{2}$$

follow. In addition to an almost contact structure (φ, ξ, η), if there is a metric tensor g satisfying

$$g(\varphi(a), \varphi(b)) = -g(a, b) + \eta(a)\eta(b) \tag{3}$$

for all vector fields a, b, then M is said to be an almost contact manifold with B-metric. The Equation (3) yields

$$g(a, \xi) = \eta(a), g(\varphi(a), b) = g(a, \varphi(b)). \tag{4}$$

Assume ∇ is the Levi-Civita covariant derivative of g. We denote

$$\Gamma(a, b, c) = g((\nabla_a \varphi) b, c). \tag{5}$$

Γ has the following properties:

$$\begin{aligned} \Gamma(a,b,c) &= \Gamma(a,c,b), \\ \Gamma(a,\varphi(b),\varphi(c)) &= \Gamma(a,b,c) - \eta(b)\Gamma(a,\xi,c) - \eta(c)\Gamma(a,b,\xi), \\ \Gamma(a,\xi,\xi) &= 0 \end{aligned} \tag{6}$$

for all a, b, c vector fields. The 1-forms θ, θ^* and ω related with Γ are introduced as

$$\theta(a) = g^{ij}\Gamma(f_i, f_j, a), \quad \theta^*(a) = g^{ij}\Gamma(f_i, \varphi(f_j), a), \quad \omega(a) = \Gamma(\xi, \xi, a). \tag{7}$$

Here $\{f_1, \cdots, f_{2n}, \xi\}$ is a local frame, the inverse matrix of (g_{ij}) is denoted by (g^{ij}) and $a \in \chi(M)$ [7].

Using properties (6), the space of Levi-Civita connections of the endomorphism φ are defined as

$$\begin{aligned}\mathcal{F} = \big\{\Gamma \in \otimes_3^0 \ : \ \Gamma(a,b,c) &= \Gamma(a,c,b) \\ &= \Gamma(a,\varphi(b),\varphi(c)) + \eta(b)\Gamma(a,\xi,c) + \eta(c)\Gamma(a,b,\xi)\big\}.\end{aligned}$$

The space $\mathcal{F}$ is decomposed as

$$\mathcal{F} = \mathcal{F}_1 \oplus \cdots \oplus \mathcal{F}_{11}.$$

The subspaces $\mathcal{F}_i$ are invariant and orthogonal with respect to the action of $G \times I$, where $G = GL(n,\mathbb{C}) \cap O(n,n)$, i.e., G is the group of real matrices $\begin{pmatrix} A & B \\ -B & A \end{pmatrix}$ which belong to $O(n,n)$, A and B are $n \times n$ matrices [7].

Any almost contact manifold with B-metric belongs to a subclass $\mathcal{F}_{i_1} \oplus \cdots \oplus \mathcal{F}_{i_k}$ for $1 \le i_1 \le \cdots \le i_k \le 11$ of $\mathcal{F}$. The defining rules of classes we use are [7]:

$$\mathcal{F}_4 \ : \ \Gamma(a,b,c) = -\frac{\theta(\xi)}{2n}\left(\eta(b)g(\varphi(a),\varphi(c)) + \eta(c)g(\varphi(a),\varphi(b))\right), \tag{8}$$

$$\mathcal{F}_5 \ : \ \Gamma(a,b,c) = -\frac{\theta^*(\xi)}{2n}\left(\eta(b)g(\varphi(a),c) + \eta(c)g(\varphi(a),b)\right). \tag{9}$$

An even-dimensional semi-Riemannian manifold N having an almost complex structure J and a semi-Riemannian metric h such that $h(J(a),J(b)) = -h(a,b)$ is called an almost complex manifold with a Norden metric. $G = GL(n,\mathbb{C}) \cap O(n,n)$ is the structure group of N, where $GL(n,\mathbb{C}) \cap O(n,n)$ is the group of real matrices

$$\begin{pmatrix} A & B \\ -B & A \end{pmatrix}$$

which are in $O(n,n)$ (A and B are $n \times n$ matrices) [6].

Almost complex manifolds with Norden metric are classified by considering the Levi-Civita connection ∇J of J. The following notation is used

$$\Upsilon(a,b,c) := h\left(\left(\nabla_a J\right)b,c\right).$$

Υ satisfies

$$\Upsilon(a,b,c) = \Upsilon(a,c,b) \text{ and } \Upsilon(a,J(b),J(c)) = \Upsilon(a,b,c).$$

The 1-form Θ related with Υ is given by

$$\Theta(a) = h^{ij}\Upsilon(f_i,f_j,a) \tag{10}$$

for all $a \in \chi(N)$, where $\{f_1, f_2, \cdots, f_{2n}\}$ is a local frame on N and (h^{ij}) is the inverse matrix of h. The tensor Υ belongs to the space

$$W = \left\{ \Upsilon \in \otimes_3^0 \ : \ \Upsilon(a,b,c) = \Upsilon(a,c,b) = \Upsilon(a,J(b),J(c)) \right\},$$

which splits into a direct sum of three subspaces W_i, $i = 1,2,3$ [5]. Defining relations of almost complex manifolds with a Norden metric are:

1. Kaehlerian Norden metric manifolds: $\Upsilon(a,b,c) = 0$ for all $a,b,c \in \chi(N)$.

2. Class W_1 (Conformally Kaehlerian manifolds with a Norden metric):

$$\begin{aligned} \Upsilon(a,b,c) &= \frac{1}{2n}\left(h(a,b)\Theta(c) + h(a,c)\Theta(b)\right. \\ &\quad \left. + h(a,J(b))\Theta(J(c)) + h(a,J(c))\Theta(J(b))\right). \end{aligned} \tag{11}$$

3. Class W_2 (Special complex manifolds with a Norden metric):

$$\Upsilon(a,b,J(c)) + \Upsilon(b,c,J(a)) + \Upsilon(c,a,J(b)) = 0, \tag{12}$$

$$\Theta = 0. \tag{13}$$

4. Class W_3 (Quasi-Kaehlerian manifolds with a Norden metric):

$$\Upsilon(a,b,c) + \Upsilon(b,c,a) + \Upsilon(c,a,b) = 0. \tag{14}$$

5. Class $W_1 \oplus W_2$ (Complex manifolds with a Norden metric):

$$\Upsilon(a,b,J(c)) + \Upsilon(b,c,J(a)) + \Upsilon(c,a,J(b)) = 0.$$

6. Class $W_1 \oplus W_3$:

$$\begin{aligned} \Upsilon(a,b,c) + \Upsilon(b,c,a) + \Upsilon(c,a,b) &= \frac{1}{n}\left(h(a,b)\Theta(c) + h(a,c)\Theta(b)\right. \\ &+ h(b,c)\Theta(a) + h(a,J(b))\Theta(J(c)) \\ &+ \left. h(b,J(c))\Theta(J(a)) + h(c,J(a))\Theta(J(b))\right) \end{aligned} \tag{15}$$

7. Class $W_2 \oplus W_3$ (Semi-Kaehlerian manifolds with a Norden metric):

$$\Theta = 0.$$

8. Class $W_1 \oplus W_2 \oplus W_3$ (No relation):

Any $\Upsilon \in W$ can be written as $\Upsilon = \Upsilon_1 + \Upsilon_2 + \Upsilon_3 \in W$, where $\Upsilon_i \in W_i$. The projections Υ_i are given below [6]:

$$\begin{aligned} \Upsilon_1(a,b,c) &= \frac{1}{2n}\left(h(a,b)\Theta(c) + h(a,c)\Theta(b)\right. \\ &\quad \left. + h(a,J(b))\Theta(J(c)) + h(a,J(c))\Theta(J(b))\right), \end{aligned} \tag{16}$$

$$\begin{aligned}\Upsilon_2(a,b,c) &= -\frac{1}{2n}\left(h(a,b)\Theta(c)+h(a,c)\Theta(b)\right.\\&\quad +h(a,J(b))\Theta(J(c))+h(a,J(c))\Theta(J(b)))\\&\quad +\frac{1}{4}\left(2\Upsilon(a,b,c)+\Upsilon(b,c,a)+\Upsilon(c,a,b)\right.\\&\quad \left.-\Upsilon(J(b),c,J(a))+\Upsilon(J(c),a,J(b))\right),\end{aligned} \tag{17}$$

$$\begin{aligned}\Upsilon_3(a,b,c) &= \frac{1}{4}\left(2\Upsilon(a,b,c)-\Upsilon(b,c,a)-\Upsilon(c,a,b)\right.\\&\quad \left.+\Upsilon(J(b),c,J(a))-\Upsilon(J(c),a,J(b))\right).\end{aligned} \tag{18}$$

3. Almost Complex Manifolds with Norden Metric from Almost Contact Manifolds with B-Metric

Let (M,φ,ξ,η,g) be an almost contact manifold with B-metric, $\dim M = 2n+1$. Consider a vector field $\left(a,\alpha\frac{d}{dt}\right)$ on $M\times\mathbb{R}$, where a is a vector field on M, t is the coordinate of $\mathbb{R}$ and α is a C^∞ function on $M\times\mathbb{R}$. On $M\times\mathbb{R}$ we define an almost complex structure with a Norden metric $(\tilde{J},\tilde{h})$ with respect to the functions σ and μ on $M\times\mathbb{R}$, where σ and μ depend only on t as

$$\tilde{J}\left(a,\alpha\frac{d}{dt}\right) := \left(\varphi(a)-\alpha e^{-(\sigma+\mu)}\xi, e^{(\sigma+\mu)}\eta(a)\frac{d}{dt}\right), \tag{19}$$

$$\tilde{h}\left(\left(a,\alpha\frac{d}{dt}\right),\left(b,\beta\frac{d}{dt}\right)\right) := e^{2\sigma}g(a,b)+e^{2\sigma}(e^{2\mu}-1)\eta(a)\eta(b)-\alpha\beta. \tag{20}$$

In this study, we use the notation a,b,c for vector fields on M. In addition, we use A,B,C to denote vector fields on M such that $A,B,C\in Ker\eta$.

Using the Kozsul formula, we evaluate the components of Levi-Civita covariant derivative $\tilde{\nabla}$ of $\tilde{h}$ which are different than zero as

$$\begin{aligned}
\tilde{h}(\tilde{\nabla}_A B,C) &= e^{2\sigma}g(\nabla_A B,C),\\
\tilde{h}(\tilde{\nabla}_A B,\xi) &= e^{2\sigma}g(\nabla_A B,\xi)-e^{2\sigma}(e^{2\mu}-1)d\eta(A,B),\\
\tilde{h}(\tilde{\nabla}_A B,\frac{d}{dt}) &= -e^{2\sigma}\frac{d\sigma}{dt}g(A,B),\\
\tilde{h}(\tilde{\nabla}_A \xi,C) &= e^{2\sigma}g(\nabla_A\xi,C)+e^{2\sigma}(e^{2\mu}-1)d\eta(A,C),\\
\tilde{h}(\tilde{\nabla}_A \tfrac{d}{dt},C) &= e^{2\sigma}\frac{d\sigma}{dt}g(A,C),\\
\tilde{h}(\tilde{\nabla}_\xi B,C) &= e^{2\sigma}g(\nabla_\xi B,C)+e^{2\sigma}(e^{2\mu}-1)d\eta(B,C),\\
\tilde{h}(\tilde{\nabla}_\xi B,\xi) &= e^{2(\sigma+\mu)}g(\nabla_\xi B,\xi),\\
\tilde{h}(\tilde{\nabla}_\xi \xi,C) &= e^{2(\sigma+\mu)}g(\nabla_\xi\xi,C),\\
\tilde{h}(\tilde{\nabla}_\xi \xi,\frac{d}{dt}) &= -e^{2(\sigma+\mu)}(\frac{d\sigma}{dt}+\frac{d\mu}{dt}),\\
\tilde{h}(\tilde{\nabla}_\xi \tfrac{d}{dt},\xi) &= e^{2(\sigma+\mu)}(\frac{d\sigma}{dt}+\frac{d\mu}{dt}),\\
\tilde{h}(\tilde{\nabla}_{\frac{d}{dt}} B,C) &= e^{2\sigma}\frac{d\sigma}{dt}g(B,C),\\
\tilde{h}(\tilde{\nabla}_{\frac{d}{dt}} \xi,\xi) &= e^{2(\sigma+\mu)}(\frac{d\sigma}{dt}+\frac{d\mu}{dt}).
\end{aligned}$$

Then, we write down the non-zero components of $\tilde{\nabla}\tilde{J}$ as

$$\tilde{h}((\tilde{\nabla}_A\tilde{J})(B),C) = e^{2\sigma}g((\nabla_A\varphi)(B),C), \tag{21}$$

$$\begin{aligned}\tilde{h}((\tilde{\nabla}_A\tilde{J})(B),\xi) &= e^{2\sigma}\Big(g(\nabla_A\varphi(B),\xi) + e^{\sigma+\mu}\frac{d\sigma}{dt}g(A,B) \\ &\quad -(e^{2\mu}-1)d\eta(A,\varphi(B))\Big),\end{aligned} \tag{22}$$

$$\begin{aligned}\tilde{h}((\tilde{\nabla}_A\tilde{J})(B),\frac{d}{dt}) &= -e^{2\sigma}\frac{d\sigma}{dt}g(A,\varphi(B)) + e^{\sigma-\mu}g(\nabla_A B,\xi) \\ &\quad -e^{\sigma-\mu}(e^{2\mu}-1)d\eta(A,B),\end{aligned} \tag{23}$$

$$\begin{aligned}\tilde{h}((\tilde{\nabla}_A\tilde{J})(\xi),C) &= e^{3\sigma+\mu}\frac{d\sigma}{dt}g(A,C) - e^{2\sigma}g(\nabla_A\xi,\varphi(C)) \\ &\quad -e^{2\sigma}(e^{2\mu}-1)d\eta(A,\varphi(C)),\end{aligned} \tag{24}$$

$$\begin{aligned}\tilde{h}((\tilde{\nabla}_A\tilde{J})(\frac{d}{dt}),C) &= -e^{\sigma-\mu}g(\nabla_A\xi,C) - e^{\sigma-\mu}(e^{2\mu}-1)d\eta(A,C) \\ &\quad -e^{2\sigma}\frac{d\sigma}{dt}g(A,\varphi(C)),\end{aligned} \tag{25}$$

$$\begin{aligned}\tilde{h}((\tilde{\nabla}_\xi\tilde{J})(B),C) &= e^{2\sigma}g((\nabla_\xi\varphi)(B),C) \\ &\quad +e^{2\sigma}(e^{2\mu}-1)\left(d\eta(\varphi(B),C) - d\eta(B,\varphi(C))\right),\end{aligned} \tag{26}$$

$$\tilde{h}((\tilde{\nabla}_\xi\tilde{J})(B),\xi) = e^{2(\sigma+\mu)}g(\nabla_\xi\varphi(B),\xi), \tag{27}$$

$$\tilde{h}((\tilde{\nabla}_\xi\tilde{J})(B),\frac{d}{dt}) = e^{\sigma+\mu}g(\nabla_\xi B,\xi), \tag{28}$$

$$\tilde{h}((\tilde{\nabla}_\xi\tilde{J})(\xi),C) = e^{2(\sigma+\mu)}g(\nabla_\xi\xi,\varphi(C)), \tag{29}$$

$$\tilde{h}((\tilde{\nabla}_\xi\tilde{J})(\frac{d}{dt}),C) = -e^{\sigma+\mu}g(\nabla_\xi\xi,C), \tag{30}$$

$$\tilde{h}((\tilde{\nabla}_\xi\tilde{J})(\xi),\xi) = 2e^{3(\sigma+\mu)}\left(\frac{d\sigma}{dt}+\frac{d\mu}{dt}\right), \tag{31}$$

$$\tilde{h}\left((\tilde{\nabla}_\xi\tilde{J})\left(\frac{d}{dt}\right),\frac{d}{dt}\right) = 2e^{\sigma+\mu}\left(\frac{d\sigma}{dt}+\frac{d\mu}{dt}\right), \tag{32}$$

$$\tilde{h}\left((\tilde{\nabla}_{\frac{d}{dt}}\tilde{J})(\xi),\frac{d}{dt}\right) = e^{\sigma+\mu}\left(\frac{d\sigma}{dt}+\frac{d\mu}{dt}\right), \tag{33}$$

$$\tilde{h}\left((\tilde{\nabla}_{\frac{d}{dt}}\tilde{J})(\frac{d}{dt}),\xi\right) = -e^{\sigma+\mu}\left(\frac{d\sigma}{dt}+\frac{d\mu}{dt}\right). \tag{34}$$

Then, we have the Theorem 3.1.

Theorem 3.1 $\tilde{\nabla}\tilde{J} = 0$ *if and only if relations below are satisfied*

$$\Gamma(A,B,C) = \Gamma(\xi,\xi,C) = 0, \tag{35}$$

$$\frac{d\sigma}{dt} + \frac{d\mu}{dt} = 0, \tag{36}$$

$$\Gamma(\xi,B,C) = 0, \tag{37}$$

$$\Gamma(A,B,\xi) = -e^{\sigma+\mu}\frac{d\sigma}{dt}g(A,B) \tag{38}$$

for all $A,B,C \in Ker\eta$.

Proof Let $\tilde{\nabla}\tilde{J} = 0$. From Equations (21), (27)-(34), we get Equations (35), (36) and $\tilde{\nabla}_\xi\xi = 0$. Also, from Equation (25), we obtain

$$g\left(\nabla_A\xi,C\right) = -\left(e^{2\mu}-1\right)d\eta(A,C) - e^{\sigma+\mu}\frac{d\sigma}{dt}g(A,\varphi(C)). \tag{39}$$

Then, Equation (39) implies $d\eta = 0$. In addition, from Equation (26), we obtain $\beta(\xi,B,C) = 0$. Also, Equation (22) gives the relation (38). The converse of proof is clear. □

Now, we state Theorem 3.2 which is used to prove Theorem 3.3.

Theorem 3.2 *Assume* (M,φ,ξ,η,g) *is an almost contact manifold with B-metric. The followings are equivalent:*

(i) (M,φ,ξ,η,g) *satisfies the Equations (35), (37) and (38).*

(ii) (M,φ,ξ,η,g) *satisfies*

$$\Gamma(a,b,c) = e^{\sigma+\mu}\frac{d\sigma}{dt}\left(\eta(b)g(\varphi(a),\varphi(c)) + \eta(c)g(\varphi(a),\varphi(b))\right) \tag{40}$$

for all $a,b,c \in \chi(M)$.

Proof Let (M,φ,ξ,η,g) satisfy (35), (37) and (38). Take

$$\begin{array}{rcll} a &=& a-\eta(a)\xi+\eta(a)\xi = A+\eta(a)\xi, & A = a-\eta(a)\xi \\ b &=& b-\eta(b)\xi+\eta(b)\xi = B+\eta(b)\xi, & B = b-\eta(b)\xi \\ c &=& c-\eta(c)\xi+\eta(c)\xi = C+\eta(c)\xi, & C = c-\eta(c)\xi, \end{array}$$

where $A,B,C \in Ker\eta$. Then, we obtain

$$\begin{array}{rcl} \Gamma(a,b,c) &=& \Gamma\left(A+\eta(a)\xi, B+\eta(b)\xi, C+\eta(c)\xi\right) \\ &=& \Gamma(A,B,C)+\eta(c)\Gamma(A,B,\xi)+\eta(b)\Gamma(B,C,\xi) \\ && \eta(a)\Gamma(\xi,B,C)+\eta(a)\eta(c)\Gamma(\xi,\xi,B)+\eta(a)\eta(b)\Gamma(\xi,\xi,C) \\ &=& \eta(c)\Gamma(A,B,\xi)+\eta(b)\Gamma(A,C,\xi) \\ &=& -e^{\sigma+\mu}\frac{d\sigma}{dt}\left(\eta(c)g(A,B)+\eta(b)g(A,C)\right) \\ &=& e^{\sigma+\mu}\frac{d\sigma}{dt}\left(\eta(c)g(\varphi(a),\varphi(b))+\eta(b)g(\varphi(a),\varphi(c))\right). \end{array} \tag{41}$$

The proof of converse is trivial. □

Consider the defining relation of $\mathcal{F}_4$ of almost contact manifold with B-metric

$$\Gamma(a,b,c) = -\frac{\theta(\xi)}{2n}\left(\eta(b)g(\varphi(a),\varphi(c)) + \eta(c)g(\varphi(a),\varphi(b))\right).$$

Choose functions σ and μ so that

$$-\frac{\theta(\xi)}{2n} = e^{\sigma+\mu}\frac{d\sigma}{dt}. \tag{42}$$

Then, M is in $\mathcal{F}_4$. However, the Equation (42) has a solution if $\theta(\xi)$ is a constant real number. Consequently, the Theorem 3.3 is stated.

Theorem 3.3 *Let (M,φ,ξ,η,g) be an almost contact manifold with B-metric. $(M\times\mathbb{R},\tilde{J},\tilde{h})$ is Kaehlerian manifold with Norden metric iff the manifold M is of the class $\mathcal{F}_4$, $\theta(\xi)$ is a real number and following equalities are satisfied*

$$e^{\sigma+\mu}\frac{d\sigma}{dt} = -\frac{\theta(\xi)}{2n}, \qquad \frac{d\sigma}{dt} + \frac{d\mu}{dt} = 0. \tag{43}$$

Proof If $M\times\mathbb{R}$ is a Kaehlerian Norden metric manifold, from Theorem 3.1, we have Equations (35) - (38). Also from Theorem 3.2, we get the Equation (40). If functions σ and μ are chosen to satisfy

$$e^{\sigma+\mu}\frac{d\sigma}{dt} = -\frac{\theta(\xi)}{2n},$$

then M is of the class $\mathcal{F}_4$ since $\theta(\xi)$ is constant.

On the contrary, if M is of the class $\mathcal{F}_4$, $\theta(\xi)$ is constant and Equation (43) holds, then we have

$$\sigma(t) + \mu(t) = c, \quad c\in\mathbb{R}.$$

In addition, the differential equation $e^{\sigma+\mu}\frac{d\sigma}{dt} = -\frac{\theta(\xi)}{2n}$ has the solutions

$$\sigma(t) = -\frac{\theta(\xi)}{2n}e^{-c}t + c_1, \quad \mu(t) = c + \frac{\theta(\xi)}{2n}e^{-c}t - c_1, \quad c_1\in\mathbb{R}. \tag{44}$$

If σ and μ are chosen as in (44), then $(M\times\mathbb{R},\tilde{J},\tilde{h})$ is in trivial class. In fact, we obtain an infinite number of Kaehlerian manifolds with a Norden metric depending on c and c_1. □

Example 3.4 *Assume G is a five dimensional Lie group, take a basis $\{x_0,x_1,x_2,x_3,x_4\}$ of left-invariant vector fields such that the non-zero Lie brackets are*

$$[x_0,x_1] = \lambda x_2 + x_3 + \mu x_4, \quad [x_0,x_2] = -\lambda x_1 - \mu x_3 + x_4,$$

$$[x_0, x_3] = -x_1 - \mu x_2 + \lambda x_4, \quad [x_0, x_4] = \mu x_1 - x_2 - \lambda x_3,$$

where λ and μ are constants. Let g be the metric satisfying

$$g(x_0, x_0) = g(x_1, x_1) = g(x_2, x_2) = 1, \quad g(x_3, x_3) = g(x_4, x_4) = -1,$$

$$g(x_i, x_j) = 0, \quad i, j \in \{0, 1, \cdots, 4\}, i \neq j.$$

If we take $\xi = x_0$, $\varphi(x_1) = x_3$ and $\varphi(x_2) = x_4$, then (ξ, η, φ, g) is an almost contact structure with B-metric, where η is dual 1-form of x_0. From the Kozsul formula, we evaluate the non-zero Levi-Civita covariant derivative as

$$\nabla_{x_0} x_1 = \lambda x_2 + \mu x_4, \quad \nabla_{x_0} x_2 = -\lambda x_1 - \mu x_3,$$

$$\nabla_{x_0} x_3 = -\mu x_2 + \lambda x_4, \quad \nabla_{x_0} x_4 = \mu x_1 - \lambda x_3,$$

$$\lambda_{x_1} x_0 = -x_3, \quad \lambda_{x_2} x_0 = -x_4, \quad \lambda_{x_3} x_0 = x_1, \quad \lambda_{x_4} x_0 = x_2,$$

$$\lambda_{x_1} x_3 = \lambda_{x_2} x_4 = \lambda_{x_3} x_1 = \lambda_{x_4} x_2 = -x_0.$$

$(G, \varphi, \xi, \eta, g)$ is of class $\mathcal{F}_4$ with $\theta(\xi) = -2n$ [9]. If we take $\sigma(t) = e^{-c} t + c_1$, $\mu(t) = c - e^{-c} t - c_1$, where c and c_1 are arbitrary real numbers, then $G \times \mathbb{R}$ is a Kaehlerian manifold with a Norden metric.

Let $\{f_1, \cdots, f_n, \varphi(f_1), \cdots, \varphi(f_n), \xi\}$ be an orthonormal frame on open set U of M such that

$$g(f_i, f_i) = 1, \; g(\varphi(f_i), \varphi(f_i)) = -1, \quad g(\xi, \xi) = 1, \; 1 \leq i \leq n,$$

$$g(f_i, f_j) = g(\varphi(f_i), \varphi(f_j)) = g(f_i, \varphi(f_j)) = 0 \text{ for } i \neq j, \; 1 \leq i, j \leq n.$$

Then,

$$\left\{ (e^{-\sigma} f_1, 0), (e^{-\sigma} f_2, 0), \cdots, (e^{-\sigma} f_n, 0), (e^{-\sigma} \varphi(f_1), 0), \cdots, (e^{-\sigma} \varphi(f_n), 0), \left(e^{-(\sigma+\mu)} \xi, 0\right), \left(0, \frac{d}{dt}\right) \right\}$$

is an orthonormal frame of $\tilde{h}$ on the open subset $U \times \mathbb{R}$ of $M \times \mathbb{R}$. By using this frame, $\tilde{\Theta}\left(a, \alpha \frac{d}{dt}\right)$ is obtained by direct calculation:

$$\begin{aligned} \tilde{\Theta}\left(a, \alpha \frac{d}{dt}\right) &= \theta(a) - \alpha e^{-(\sigma+\mu)} \theta^*(\xi) + 2n e^{\sigma+\mu} \eta(a) \frac{d\sigma}{dt} \\ &\quad + 3 e^{\sigma+\mu} \left(\frac{d\sigma}{dt} + \frac{d\mu}{dt} \right) \eta(a) + g\left(\nabla_\xi \xi, \varphi(a)\right). \end{aligned} \tag{45}$$

Let M be in $\mathcal{F}_5$. We investigate the class of $M \times \mathbb{R}$.

Theorem 3.5 *If* $(M, \varphi, \xi, \eta, g)$ *is in* $\mathcal{F}_5$ *and* $\frac{d\sigma}{dt} + \frac{d\mu}{dt} = 0$, *then* $(M \times \mathbb{R}, \tilde{J}, \tilde{h})$ *belongs to* $W_1 \oplus W_2$.

Proof Since M is in $\mathcal{F}_5$, Equation (9) is satisfied. In the class $\mathcal{F}_5$, we have

$$\nabla_a \xi = -\frac{\theta^*(\xi)}{2n}\varphi^2(a), \quad d\eta = 0.$$

In addition, since $\frac{d\sigma}{dt} + \frac{d\mu}{dt} = 0$, the only components of Levi-Civita covariant derivative of $\tilde{J}$ which do not vanish are

$$\begin{aligned}
\tilde{g}\left((\tilde{\nabla}_A J)(B), \xi\right) &= -e^{2\sigma}\left(\frac{\theta^*(\xi)}{2n} g(A, \varphi(B)) - e^{\sigma+\mu}\frac{d\sigma}{dt} g(A, B)\right), \\
\tilde{g}\left((\tilde{\nabla}_A J)(B), \frac{d}{dt}\right) &= -e^{2\sigma}\left(\frac{d\sigma}{dt} g(A, \varphi(B)) + e^{-(\sigma+\mu)}\frac{\theta^*(\xi)}{2n} g(A, B)\right), \\
\tilde{g}\left((\tilde{\nabla}_A J)(\xi), C\right) &= e^{2\sigma}\left(e^{\sigma+\mu}\frac{d\sigma}{dt} g(A, C) - \frac{\theta^*(\xi)}{2n} g(A, \varphi(C))\right), \\
\tilde{g}\left((\tilde{\nabla}_A J)\left(\frac{d}{dt}\right), C\right) &= -e^{2\sigma}\left(e^{-(\sigma+\mu)}\frac{\theta^*(\xi)}{2n} g(A, C) + \frac{d\sigma}{dt} g(A, \varphi(C))\right).
\end{aligned}$$

Also, by direct calculation we have

$$\tilde{\Theta}\left(a, \alpha\frac{d}{dt}\right) = -\alpha e^{-(\sigma+\mu)}\theta^*(\xi) + 2ne^{\sigma+\mu}\eta(a)\frac{d\sigma}{dt}. \tag{46}$$

In addition, since

$$\Upsilon_1\left(\left(0, \frac{d}{dt}\right), (\xi, 0), (\xi, 0)\right) = \frac{1}{n}e^{\sigma+\mu}\theta^*(\xi) \neq 0 \tag{47}$$

and

$$\Upsilon_2\left(\left(0, \frac{d}{dt}\right), (\xi, 0), (\xi, 0)\right) = -\frac{1}{n}e^{\sigma+\mu}\theta^*(\xi) \neq 0, \tag{48}$$

the projections α_1, α_2 are non-zero. By direct calculation

$$\Upsilon_3\left(\left(a, \alpha\frac{d}{dt}\right), \left(b, \beta\frac{d}{dt}\right), \left(c, \gamma\frac{d}{dt}\right)\right) = 0. \tag{49}$$

Hence, $M \times \mathbb{R}$ is of the class $W_1 \oplus W_2$. □

Example 3.6 *Let* $\mathbb{R}^{2n+2} = \{(a_1, \cdots, a_{n+1}, b_1, \cdots, b_{n+1}) : a_i, b_i \in \mathbb{R}\}$. *Consider the canonical complex structure*

$$J\left(\frac{\partial}{\partial a_i}\right) = \frac{\partial}{\partial b_i}, \quad J\left(\frac{\partial}{\partial b_i}\right) = -\frac{\partial}{\partial a_i}, \quad 1 \leq i \leq n+1$$

and

$$g(u,u) = -\delta_{ij}x_i x_j + \delta_{ij}y_i y_j,$$

where $u = x_i\frac{\partial}{\partial a_i} + y_i\frac{\partial}{\partial b_i}$. *Identify the point* $p = (a_1, \cdots, a_{n+1}, b_1, \cdots, b_{n+1})$ *in* $\mathbb{R}^{2n+2}$ *with its position vector* P. *Let* M *be the hypersurface of* $\mathbb{R}^{2n+2}$ *determined by*

$$M = \left\{P \in \mathbb{R}^{2n+2} \ : \ g(P, J(P)) = 0, \ g(P,P) > 0\right\}.$$

Define vector field ξ *as*

$$\xi = -\frac{1}{\cosh t}P,$$

where $t \in (-\pi/2, \pi/2)$. *For any vector field* u, *we can define* φ *with regard to the unique decomposition*

$$J(u) = \varphi(u) + \frac{1}{\cosh t}\eta(u)J(P).$$

$(M, \varphi, \xi, \eta, g)$ *is in* $\mathcal{F}_5$ *[7]. From the Theorem 3.5, by choosing the functions* σ *and* μ *to satisfy* $\frac{d\sigma}{dt} + \frac{d\mu}{dt} = 0$, $M \times \mathbb{R}$ *is of the class* $W_1 \oplus W_2$.

Declaration of Ethical Standards

The authors declare that the materials and methods used in their study do not require ethical committee and/or legal special permission.

Authors Contributions

Author [Nülifer Özdemir]: Thought and designed the research/problem, contributed to research method or evaluation of data, collected the data, wrote the manuscript (%50).

Author [Elanur Eren]: Collected the data, contributed to completing the research and solving the problem (%50).

Conflicts of Interest

The authors declare no conflict of interest.

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