Journal of Universal Mathematics

Volume: 8 No:1 2025

http://dergipark.gov.tr/jum ISSN: 2618-5660

Journal of Universal Mathematics

Volume 8 Number 1 2021

JUM

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ISSN 2618-5660

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In this issue, we publish more valuable papers written with pleasure by our authors, carefully reviewed by our referees, despite all their busy time.

We thank our authors, reviewers, editors, and editing team for their contribution to this Volume.

We expect support from you, valuable researchers and writers, for our journal, which will be published in July 2025.

We wish you a successful scientific life.

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Assoc. Prof. Dr. Gökhan Çuvalcıoğlu
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ISSN 2618-5660

Journal of Universal Mathematics
http://dergipark.gov.tr/jum
Volume 8 Number 1 January 2025

This Journal is a peer reviewed journal and is indexed by Index Copernicus, Google Scholar, International Citation Index, Rootindexing, ResearchBib, ROAD, Directory of Indexing and Impact Factor, Eurasian Scientific Journal Index, DRJI, Journal Factor, World Catalogue of Scientific Journal, General Impact Factor.

ISSN 2618-5660

Journal of Universal Mathematics
http://dergipark.gov.tr/jum
Volume 8 Number 1 January 2025

Journal of Universal Mathematics
Vol.8 No.1 pp.1-19 (2025)
ISSN-2618-5660
DOI: 10.33773/jum.1501013

ON GENERALIZED CONFORMABLE FRACTIONAL OPERATORS

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Abstract. In this paper, we introduce the concepts of left and right generalized conformable fractional integrals, alongside the corresponding derivatives. Additionally, we extend our investigation to derive the generalized conformable derivatives for functions within specific spaces, elucidating their inherent properties.

1. Introduction

Fractional calculus, with its roots dating back to 1695, has evolved significantly over the years and garnered increasing significance, particularly in applied sciences. Its applications span various fields including physics, mechanics, electronics, chemistry, biology, and engineering $[2-7]$, [16]. Two commonly used approaches in fractional calculus are the Caputo and Riemann-Liouville derivatives.

The Riemann-Liouville approach entails iteratively applying the integral operator $n-$times, resulting in fractional integrals of non-integer order. This method has found widespread use due to its versatility across different disciplines. However, the standard fractional calculus framework may not always be sufficient for certain applications, necessitating the introduction of specialized kernels for a more unified approach to fractional derivatives.

The differentiation operator serves as a fundamental starting point for the iteration method in fractional calculus. By incorporating the required kernels, researchers aim to achieve a more comprehensive understanding and application of fractional derivatives across various scientific and engineering contexts $[8-10]$, $[19-20]$. In the present case, Abdeljawad defined as the following the left and right conformable derivatives, respectively[18],

$$\begin{aligned} {}_{\phi}T^{\alpha}f(\tau) &= (\tau-\phi)^{1-\alpha} f'(\tau), \\ T^{\alpha}_{\delta}f(\tau) &= (\delta-\tau)^{1-\alpha} f'(\tau). \end{aligned} \tag{1.1}$$

Date: **Received:** 2024-06-24; **Accepted:** 2025-01-28.

Key words and phrases. Conformable derivatives, Conformable integrals, Fractional derivatives, Fractional integrals.

In this context, assuming f is a differentiable function, we possess left and right conformable integrals as the following forms, respectively [1]

$$ {}^{\beta}_{\phi}J^{\alpha}f(\tau) = \frac{1}{\Gamma(\beta)} \int_{\phi}^{\tau} \left(\frac{(\tau-\phi)^{\alpha} - (\theta-\phi)^{\alpha}}{\alpha} \right)^{\beta-1} f(\theta) \frac{d\theta}{(\theta-\phi)^{1-\alpha}} \tag{1.2} $$

and

$$ {}^{\beta}J^{\alpha}_{\delta}f(\tau) = \frac{1}{\Gamma(\beta)} \int_{\tau}^{\delta} \left(\frac{(\delta-\tau)^{\alpha} - (\delta-\theta)^{\alpha}}{\alpha} \right)^{\beta-1} f(\theta) \frac{d\theta}{(\delta-\theta)^{1-\alpha}}. \tag{1.3} $$

In [1], authors introduced novel fractional operators characterized by two parameters, each with kernels distinct from conventional ones. Our paper closely examines the findings of [1], focusing on their implications and further developments. We extend upon their work by deriving new generalized fractional integrals and derivatives using the newly defined fractional operators.

Moreover, we provide a thorough exposition of basic definitions and tools essential to classical fractional calculus. These foundational concepts serve as the groundwork for our subsequent discussions and advancements.

Definition 1.1. ([17], [21]) Let $\gamma(\tau)$ be an increasing and positive monotone function on $[0, \infty)$. Furthermore, we'll consider γ as a monotonically increasing and positive function defined on the interval $[0, \infty)$, with its derivative γ' being continuous and $\gamma(0) = 0$. The space $X^{d}_{\gamma}(0, \infty)$ is the following form for $(1 \leq d < \infty)$,

$$ \|f\|_{X^{d}_{\gamma}} = \left(\int_{0}^{\infty} |f(\theta)|^{d} \gamma'(\tau) \, d\theta \right)^{\frac{1}{d}} < \infty \tag{1.4} $$

and if we choose $d = \infty$,

$$ \|f\|_{X^{\infty}_{\gamma}} = ess \sup_{1 \leq \theta < \infty} \left[f(\theta) \gamma'(\tau) \right]. \tag{1.5} $$

Additionally, if we take $\gamma(\tau) = \tau \ (1 \leq d < \infty)$ the space $X^{d}_{\gamma}(0, \infty)$, then we have the $L_{d}[0, \infty)-$space. Moreover, if we take $\gamma(\tau) = \frac{\tau^{k+1}}{k+1} \ (1 \leq d < \infty, \ k \geq 0)$ the space $X^{d}_{\gamma}(0, \infty)$, then we have the $L_{d,k}[0, \infty)-$space [17].

The authors derived the generalized left and right fractional integrals for β belonging to the complex numbers $(\beta \in \mathbb{C})$ with $Re(\beta) > 0$ in [8],

$$ \left({}_{\phi}I^{\beta,\alpha}f \right)(\theta) = \frac{1}{\Gamma(\beta)} \int_{\phi}^{\theta} \left(\frac{\theta^{\alpha} - y^{\alpha}}{\alpha} \right)^{\beta-1} f(y) \frac{dy}{y^{1-\alpha}} \tag{1.6} $$

and

$$ \left(I^{\beta,\alpha}_{\delta}f \right)(\theta) = \frac{1}{\Gamma(\beta)} \int_{\theta}^{\delta} \left(\frac{y^{\alpha} - \theta^{\alpha}}{\alpha} \right)^{\beta-1} f(y) \frac{dy}{y^{1-\alpha}}, \tag{1.7} $$

respectively.

The authors obtained left and right generalized fractional derivatives for β belonging to the complex numbers $(\beta \in \mathbb{C})$ with $Re(\beta) \geq 0$ in [9],

$$ \begin{aligned} \left({}_{\phi}D^{\beta,\alpha}f \right)(\theta) &= \zeta^{n} \left({}_{\phi}I^{n-\beta,\alpha}f \right)(\theta) \\ &= \frac{\zeta^{n}}{\Gamma(n-\beta)} \int_{\phi}^{\theta} \left(\frac{\theta^{\alpha} - y^{\alpha}}{\alpha} \right)^{n-\beta-1} f(y) \frac{dy}{y^{1-\alpha}} \end{aligned} \tag{1.8} $$

and

$$ \begin{aligned} \left(D^{\beta,\alpha}_{\delta}f \right)(\theta) &= (-\zeta)^{n} \left({}_{\phi}I^{n-\beta,\alpha}f \right)(\theta) \\ &= \frac{(-\zeta)^{n}}{\Gamma(n-\beta)} \int_{\theta}^{\delta} \left(\frac{y^{\alpha} - \theta^{\alpha}}{\alpha} \right)^{n-\beta-1} f(y) \frac{dy}{y^{1-\alpha}} \end{aligned} \tag{1.9} $$

respectively, where $\alpha > 0$ and where $\zeta = \theta^{1-\alpha}\frac{d}{d\theta}$.

The left and right generalized Caputo fractional derivatives, as defined by the authors in [15] through the utilization of [9], are expressed in the following forms, respectively,

$$\begin{aligned}\left({}_{\phi}^{C}D^{\beta,\alpha}f\right)(\theta) &= \left({}_{\phi}I^{n-\beta,\alpha}\left(\zeta\right)^{n}f\right)(\theta)\\ &= \frac{1}{\Gamma(n-\beta)}\int_{\phi}^{\theta}\left(\frac{\theta^{\alpha}-u^{\alpha}}{\alpha}\right)^{n-\beta-1}\frac{\zeta^{n}f(y)dy}{y^{1-\alpha}}\end{aligned} \tag{1.10}$$

and

$$\begin{aligned}\left({}^{C}D_{\delta}^{\beta,\alpha}f\right)(\theta) &= \left({}_{\phi}I^{n-\beta,\alpha}\left(-\zeta\right)^{n}f\right)(\theta)\\ &= \frac{1}{\Gamma(n-\beta)}\int_{\theta}^{\delta}\left(\frac{y^{\alpha}-\theta^{\alpha}}{\alpha}\right)^{n-\beta-1}\frac{(-\zeta)^{n}f(y)dy}{y^{1-\alpha}}.\end{aligned} \tag{1.11}$$

After introducing the generalized fractional conformable integral and derivative operators, we will highlight their significant implications and key characteristics. Additionally, we will delve into the properties of the defined generalized conformable derivative and extend our analysis to include the generalized conformable fractional derivatives within the Caputo framework. As a result, we will consolidate our findings and build upon the previously established consequences for both the generalized conformable derivatives and integrals.

2. THE GENERALIZED CONFORMABLE OPERATORS

In light of Abdeljawad's work on conformable integrals, which were extended to higher orders in reference [10], and Jarad et al.'s definition of fractional integrals in [1], we aim to establish a generalized conformable derivative. To achieve this, we'll consider γ as a monotonically increasing and positive function defined on the interval $[0,\infty)$, with its derivative γ' being continuous and $\gamma(0) = 0$. With these conditions in mind, our objective is to formulate the generalized conformable derivative based on the existing definitions of the conformable derivative

$${}_{\phi}^{\gamma}T^{\alpha}f(\theta) = \lim_{\varepsilon\to 0}\frac{f\left(\theta+\varepsilon\frac{(\gamma(\theta)-\gamma(\phi))^{1-\alpha}}{\gamma'(\theta)}\right)-f(\theta)}{\varepsilon}. \tag{2.1}$$

By taking into account equation (2.1). In here,

$$\Delta t = \varepsilon\frac{(\gamma(\theta)-\gamma(\phi))^{1-\alpha}}{\gamma'(\theta)} \Rightarrow \varepsilon = \frac{\Delta t.\gamma'(\theta)}{(\gamma(\theta)-\gamma(\phi))^{1-\alpha}} \tag{2.2}$$

we select Δt in the form. Then,

$${}_{\phi}^{\gamma}T^{\alpha}f(\theta) = \frac{(\gamma(\theta)-\gamma(\phi))^{1-\alpha}}{\gamma'(\theta)}f'(\theta). \tag{2.3}$$

We can assert formula of generalized conformable derivative, respectively,

$$\begin{aligned}{}_{\phi}^{\gamma}T^{\alpha}f(\theta) &= \frac{(\gamma(\theta)-\gamma(\phi))^{1-\alpha}}{\gamma'(\theta)}f'(\theta)\\ {}^{\gamma}T_{\delta}^{\alpha}f(\theta) &= \frac{(\gamma(\delta)-\gamma(\theta))^{1-\alpha}}{\gamma'(\theta)}f'(\theta).\end{aligned} \tag{2.4}$$

Additionally, we acquire generalized conformable integral operator. For this reason,

$$\int_{\phi}^{\tau}\frac{\gamma'(\theta_1)d\theta_1}{(\gamma(\theta_1)-\gamma(\phi))^{1-\alpha}}\int_{\phi}^{\theta_1}\frac{\gamma'(\theta_2)d\theta_2}{(\gamma(\theta_2)-\gamma(\phi))^{1-\alpha}}\cdots\int_{\phi}^{\theta_{n-1}}\frac{\gamma'(\theta_n)f(\theta_n)d\theta_n}{(\gamma(\theta_n)-\gamma(\phi))^{1-\alpha}}, \tag{2.5}$$

we should take $n-$times repeated integrals of the forms. Furthermore, If we employ a method akin to classical fractional integral techniques, then we write the equality

$$ {}^{\gamma}_{\phi}J^{n,\alpha}f(\tau) = \frac{1}{\Gamma(n)}\int_{\phi}^{\tau}\left[\frac{(\gamma(\tau)-\gamma(\phi))^{\alpha}-(\gamma(\theta)-\gamma(\phi))^{\alpha}}{\alpha}\right]^{n-1}\frac{\gamma'(\theta)f(\theta)d\theta}{(\gamma(\theta)-\gamma(\phi))^{1-\alpha}}. \tag{2.6}$$

Furthermore, we can acquire definition of the following for generalized conformable integrals drawing upon the equality presented in reference [2].

Definition 2.1. Let $f\in X_{\gamma}(0,\infty)$. Moreover, we will consider γ as a monotonically increasing and positive function defined on the interval $[0,\infty)$, with its derivative γ' being continuous and $\gamma(0)=0$. The left and right generalized conformable fractional integrals of order $n\in\mathbb{C}$, $Re(n)\geq 0$ and $\alpha>0$, respectively,

$$ {}^{\gamma}_{\phi}J^{n,\alpha}f(\tau) = \frac{1}{\Gamma(n)}\int_{\phi}^{\tau}\left[\frac{(\gamma(\tau)-\gamma(\phi))^{\alpha}-(\gamma(\theta)-\gamma(\phi))^{\alpha}}{\alpha}\right]^{n-1}\frac{\gamma'(\theta)f(\theta)d\theta}{(\gamma(\theta)-\gamma(\phi))^{1-\alpha}} \tag{2.7}$$

and

$$ {}^{\gamma}J^{n,\alpha}_{\delta}f(\tau) = \frac{1}{\Gamma(n)}\int_{\tau}^{\delta}\left[\frac{(\gamma(\delta)-\gamma(\tau))^{\alpha}-(\gamma(\delta)-\gamma(\theta))^{\alpha}}{\alpha}\right]^{n-1}\frac{\gamma'(\theta)f(\theta)d\theta}{(\gamma(\delta)-\gamma(\theta))^{1-\alpha}}. \tag{2.8}$$

Within this context, we introduce the subsequent definition, leveraging the framework provided by the generalized conformable derivative and integral operators.

Example 2.2. Let's calculate the result of the generalized conformable fractional integral ${}^{\gamma}_{0}J^{\frac{1}{2},1}f(\tau)$ for $f(\tau)=4\tau^{3}$.

Proof. If we choose $\gamma(x)=x$, $f(\theta)=4\theta^{3}$, $\alpha=1$, $n=\frac{1}{2}$ and $\phi=0$ in (2.7), we have

$$ {}^{\gamma}_{0}J^{\frac{1}{2},1}f(\tau) = \frac{1}{\Gamma\left(\frac{1}{2}\right)}\int_{0}^{\tau}(\tau-\theta)^{-\frac{1}{2}}4\theta^{3}d\theta.$$

Furhermore, by using variable change $\theta=\tau u$ and $d\theta=\tau du$, we acquire

$$\begin{aligned} {}^{\gamma}_{0}J^{\frac{1}{2},1}f(\tau) &= \frac{4}{\Gamma\left(\frac{1}{2}\right)}\theta^{\frac{7}{2}}\int_{0}^{1}(1-u)^{-\frac{1}{2}}u^{3}du \\ &= \frac{4.\theta^{\frac{7}{2}}}{\Gamma\left(\frac{1}{2}\right)}B\left(\frac{1}{2},4\right) \\ &= \frac{4.\theta^{\frac{7}{2}}\Gamma(4)}{\Gamma\left(\frac{9}{2}\right)}. \end{aligned}$$

The proof is done with Beta function and property of Beta function . ☐

Example 2.3. Let's calculate the result of the generalized conformable fractional integral ${}^{\gamma}_{0}J^{\frac{1}{2},1}\left({}^{\gamma}_{0}J^{\frac{1}{2},1}\left(4\tau^{3}\right)\right)$.

Proof. If we choose $\gamma(x)=x$, $f(\theta)=\frac{4\theta^{\frac{7}{2}}\Gamma(4)}{\Gamma\left(\frac{9}{2}\right)}$, $\alpha=1$, $n=\frac{1}{2}$ and $\phi=0$ in (2.7), we get

$$ {}^{\gamma}_{0}J^{\frac{1}{2},1}\left({}^{\gamma}_{0}J^{\frac{1}{2},1}\left(4\tau^{3}\right)\right) = \frac{1}{\Gamma\left(\frac{1}{2}\right)}\int_{0}^{\tau}(\tau-\theta)^{-\frac{1}{2}}\frac{4\theta^{\frac{7}{2}}\Gamma(4)}{\Gamma\left(\frac{9}{2}\right)}d\theta.$$

Moreover, by utilizing variable change $\theta=\tau u$ and $d\theta=\tau du$, we take

$$\begin{aligned} {}^{\gamma}_{0}J^{\frac{1}{2},1}\left(\frac{4\theta^{\frac{7}{2}}\Gamma(4)}{\Gamma\left(\frac{9}{2}\right)}\right) &= \frac{1}{\Gamma\left(\frac{1}{2}\right)}.\frac{4\Gamma(4)}{\Gamma\left(\frac{9}{2}\right)}\theta^{4}\int_{0}^{1}(1-u)^{-\frac{1}{2}}u^{\frac{7}{2}}du \\ &= \frac{1}{\Gamma\left(\frac{1}{2}\right)}.\frac{4\Gamma(4)}{\Gamma\left(\frac{9}{2}\right)}\theta^{4}B\left(\frac{9}{2},\frac{1}{2}\right) \\ &= \theta^{4}. \end{aligned}$$

The proof is done with Beta function and property of Beta function . ☐

Definition 2.4. Let $f \in X_{\gamma}(0,\infty)$. Furthermore, we will consider γ as a monotonically increasing and positive function defined on the interval $[0,\infty)$, with its derivative γ' being continuous and $\gamma(0)=0$. The left and right generalized conformable fractional derivatives of order $\beta \in \mathbb{C}$ and $Re(\beta) \geq 0$,

$$
\begin{aligned}
{}_{\phi}^{\gamma}D^{\beta,\alpha} f(\tau) &= {}_{\phi}^{\gamma}T^{n,\alpha}\left({}_{\phi}^{\gamma}J^{n-\beta,\alpha}\right) f(\tau) \\
&= \frac{{}_{\phi}^{\gamma}T^{n,\alpha}}{\Gamma(n-\beta)} \int_{\phi}^{\tau} \left[\frac{(\gamma(\tau)-\gamma(\phi))^{\alpha}-(\gamma(\theta)-\gamma(\phi))^{\alpha}}{\alpha}\right]^{n-\beta-1} \frac{\gamma'(\theta) f(\theta) d\theta}{(\gamma(\theta)-\gamma(\phi))^{1-\alpha}}
\end{aligned}
\tag{2.9}
$$

and

$$
\begin{aligned}
{}^{\gamma}D_{\delta}^{\beta,\alpha} f(\tau) &= {}^{\gamma}T_{\delta}^{n,\alpha}\left({}^{\gamma}J_{\delta}^{n-\beta,\alpha}\right) f(\tau) \\
&= \frac{{}^{\gamma}T_{\delta}^{n,\alpha}(-1)^{n}}{\Gamma(n-\beta)} \int_{\tau}^{\delta} \left[\frac{(\gamma(\delta)-\gamma(\tau))^{\alpha}-(\gamma(\delta)-\gamma(\theta))^{\alpha}}{\alpha}\right]^{n-\beta-1} \frac{\gamma'(\theta) f(\theta) d\theta}{(\gamma(\delta)-\gamma(\theta))^{1-\alpha}},
\end{aligned}
\tag{2.10}
$$

where $n = [Re(\beta)] + 1$,

$$
\begin{aligned}
{}_{\phi}^{\gamma}T^{n,\alpha} &= \underbrace{{}_{\phi}^{\gamma}T^{\alpha}\ \ {}_{\phi}^{\gamma}T^{\alpha} \cdots {}_{\phi}^{\gamma}T^{\alpha}}_{n-times}, \\
{}^{\gamma}T_{\delta}^{n,\alpha} &= \underbrace{{}^{\gamma}T_{\delta}^{\alpha}\ \ {}^{\gamma}T_{\delta}^{\alpha} \cdots {}^{\gamma}T_{\delta}^{\alpha}}_{n-times},
\end{aligned}
\tag{2.11}
$$

${}_{\phi}^{\gamma}T^{\alpha}$ and ${}^{\gamma}T_{\delta}^{\alpha}$ are the left and right generalized conformable differential operators.

Example 2.5. Let's calculate the result of the generalized conformable fractional derivative ${}_{0}^{\gamma}D^{\frac{1}{2},1} f(\tau)$ for $f(\tau) = \tau^{4}$.

Proof. If we choose $\gamma(x) = x$, $f(\theta) = \theta^{4}$, $\alpha = 1$, $n = 1$, $\beta = \frac{1}{2}$ and $\phi = 0$ in (2.9), we get

$$
{}_{0}^{\gamma}D^{\frac{1}{2},1} f(\tau) = \frac{{}_{0}^{\gamma}T^{1,1}}{\Gamma\left(\frac{1}{2}\right)} \int_{0}^{\tau} (\tau-\theta)^{-\frac{1}{2}} \theta^{4} d\theta.
$$

Furhermore, by using variable change $\theta = \tau u$ and $d\theta = \tau du$, we have

$$
\begin{aligned}
{}_{0}^{\gamma}D^{\frac{1}{2},1} f(\tau) &= \frac{\frac{d}{d\theta}}{\Gamma\left(\frac{1}{2}\right)} \int_{0}^{1} (1-u)^{-\frac{1}{2}} u^{4} \theta^{\frac{9}{2}} du \\
&= \frac{1}{\Gamma\left(\frac{1}{2}\right)} \frac{d}{d\theta}\left(\theta^{\frac{9}{2}}\right) \int_{0}^{1} u^{4} (1-u)^{-\frac{1}{2}} du \\
&= \frac{1}{\Gamma\left(\frac{1}{2}\right)} \frac{9}{2} \theta^{\frac{7}{2}} \frac{\Gamma(5)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{11}{2}\right)} \\
&= \frac{\Gamma(5)\theta^{\frac{7}{2}}}{\Gamma\left(\frac{9}{2}\right)}.
\end{aligned}
$$

The proof is done with Beta function and property of Beta function . ☐

Example 2.6. Let's calculate the result of the generalized conformable fractional derivative ${}_{0}^{\gamma}D^{\frac{1}{2},1}\left({}_{0}^{\gamma}D^{\frac{1}{2},1} f(\tau)\right)$ for $f(\tau) = \frac{\Gamma(5)\tau^{\frac{7}{2}}}{\Gamma\left(\frac{9}{2}\right)}$.

Proof. If we choose $\gamma(x) = x$, $f(\theta) = \frac{\Gamma(5)\theta^{\frac{7}{2}}}{\Gamma\left(\frac{9}{2}\right)}$, $\alpha = 1$, $n = 1$, $\beta = \frac{1}{2}$ and $\phi = 0$ in (2.9), we get

$$
{}_{0}^{\gamma}D^{\frac{1}{2},1} f(\tau) = \frac{{}_{0}^{\gamma}T^{1,1}}{\Gamma\left(\frac{1}{2}\right)} \int_{0}^{\tau} (\tau-\theta)^{-\frac{1}{2}} \frac{\Gamma(5)\theta^{\frac{7}{2}}}{\Gamma\left(\frac{9}{2}\right)} d\theta.
$$

Furhermore, by using variable change $\theta = \tau u$ and $d\theta = \tau du$, we have

$$\begin{aligned}
{}_0^\gamma D^{\frac{1}{2},1}\left(\frac{\Gamma(5)\tau^{\frac{7}{2}}}{\Gamma\left(\frac{9}{2}\right)}\right) &= \frac{\frac{d}{d\theta}}{\Gamma\left(\frac{1}{2}\right)}\frac{\Gamma(5)}{\Gamma\left(\frac{9}{2}\right)}\int_0^1 (1-u)^{-\frac{1}{2}}u^{\frac{7}{2}}\theta^4 du \\
&= \frac{1}{\Gamma\left(\frac{1}{2}\right)}\frac{\Gamma(5)}{\Gamma\left(\frac{9}{2}\right)}\frac{d}{d\theta}\left(\theta^4\right)\int_0^1 u^{\frac{7}{2}}(1-u)^{-\frac{1}{2}}du \\
&= \frac{1}{\Gamma\left(\frac{1}{2}\right)}\frac{\Gamma(5)}{\Gamma\left(\frac{9}{2}\right)}4\theta^3 B\left(\frac{9}{2},\frac{1}{2}\right) \\
&= \frac{1}{\Gamma\left(\frac{1}{2}\right)}\frac{\Gamma(5)}{\Gamma\left(\frac{9}{2}\right)}4\theta^3\frac{\Gamma\left(\frac{9}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma(5)} \\
&= 4\theta^3.
\end{aligned}$$

The proof is done with Beta function and property of Beta function . □

Theorem 2.7. *Let $f \in X_\gamma(0,\infty)$. Moreover, we will consider γ as a monotonically increasing and positive function defined on the interval $[0,\infty)$, with its derivative γ' being continuous and $\gamma(0) = 0$. Then, we have fractional integrals for Re $(\beta) > 0$ and Re $(\varsigma) > 0$,*

$$\begin{aligned}
{}_\phi^\gamma J^{\beta,\alpha}\left({}_\phi^\gamma J^{\varsigma,\alpha}\right)f(\tau) &= {}_\phi^\gamma J^{(\beta+\varsigma),\alpha}f(\tau), \\
{}^\gamma J_\delta^{\beta,\alpha}\left({}^\gamma J_\delta^{\varsigma,\alpha}\right)f(\tau) &= {}^\gamma J_\delta^{(\beta+\varsigma),\alpha}f(\tau).
\end{aligned} \tag{2.12}$$

Proof. With the assistance of equation (2.7) , we obtain

(2.13)

$$\begin{aligned}
&{}_\phi^\gamma J^{\beta,\alpha}\left({}_\phi^\gamma J^{\varsigma,\alpha}\right)f(\tau) \\
&= \frac{1}{\Gamma(\beta)}\int_\phi^\tau\left[\frac{(\gamma(\tau)-\gamma(\phi))^\alpha-(\gamma(\theta)-\gamma(\phi))^\alpha}{\alpha}\right]^{\beta-1}\frac{\left({}_\phi^\gamma J^{\varsigma,\alpha}\right)\gamma'(\theta)f(\theta)d\theta}{(\gamma(\theta)-\gamma(\phi))^{1-\alpha}} \\
&= \frac{1}{\Gamma(\beta)\Gamma(\varsigma)}\int_\phi^\tau\left[\frac{(\gamma(\tau)-\gamma(\phi))^\alpha-(\gamma(\theta)-\gamma(\phi))^\alpha}{\alpha}\right]^{\beta-1} \\
&\times\left(\int_\phi^\theta\left[\frac{(\gamma(\theta)-\gamma(\phi))^\alpha-(\gamma(u)-\gamma(\phi))^\alpha}{\alpha}\right]^{\varsigma-1}\frac{\gamma'(u)f(u)du}{(\gamma(u)-\gamma(\phi))^{1-\alpha}}\right)\frac{\gamma'(\theta)f(\theta)d\theta}{(\gamma(\theta)-\gamma(\phi))^{1-\alpha}} \\
&= \frac{1}{\Gamma(\beta)\Gamma(\varsigma)}\int_\phi^\tau\left[\frac{(\gamma(\tau)-\gamma(\phi))^\alpha-(\gamma(u)-\gamma(\phi))^\alpha}{\alpha}\right]^{\beta+\varsigma-1}\left(\int_0^1(1-z)^{\beta-1}z^{\varsigma+1}dz\right)\frac{\gamma'(u)f(u)du}{(\gamma(u)-\gamma(\phi))^{1-\alpha}} \\
&= \frac{1}{\Gamma(\beta+\varsigma)}\int_\phi^\tau\left[\frac{(\gamma(\tau)-\gamma(\phi))^\alpha-(\gamma(u)-\gamma(\phi))^\alpha}{\alpha}\right]^{\beta+\varsigma-1}\frac{\gamma'(u)f(u)du}{(\gamma(u)-\gamma(\phi))^{1-\alpha}} \\
&= {}_\phi^\gamma J^{(\beta+\varsigma),\alpha}f(\tau).
\end{aligned}$$

In here, we employed the change of variable,

(2.14)

$$(\gamma(\theta)-\gamma(\phi))^\alpha = (\gamma(u)-\gamma(\phi))^\alpha + z\left[(\gamma(\tau)-\gamma(\phi))^\alpha-(\gamma(u)-\gamma(\phi))^\alpha\right].$$

The proof of the second formula can similarly be illustrated using the similar approach. □

Lemma 2.8. *Let $f \in X_\gamma(0,\infty)$. Furthermore, we will consider γ as a monotonically increasing and positive function defined on the interval $[0,\infty)$, with its derivative γ' being continuous and $\gamma(0) = 0$. We possess for Re $(r) > 0$,*

$$\begin{aligned}
{}_\phi^\gamma J^{\beta,\alpha}(\gamma(\theta)-\gamma(\phi))^{\alpha(r-1)}(\tau) &= \frac{\Gamma(r)}{\Gamma(\beta+r)}\frac{\left[(\gamma(\tau)-\gamma(\phi))^\alpha\right]^{\beta+r-1}}{\alpha^\beta}, \\
{}^\gamma J_\delta^{\beta,\alpha}(\gamma(\delta)-\gamma(\theta))^{\alpha(r-1)}(\tau) &= \frac{\Gamma(r)}{\Gamma(\beta+r)}\frac{\left[(\gamma(\delta)-\gamma(\tau))^\alpha\right]^{\beta+r-1}}{\alpha^\beta}.
\end{aligned} \tag{2.15}$$

Proof. With the assistance of (2.7), we hold

$$\begin{aligned} &{}_{\phi}^{\gamma}J^{\beta,\alpha}\left(\gamma\left(\theta\right)-\gamma\left(\phi\right)\right)^{\alpha(r-1)}\left(\tau\right) \\ &= \frac{1}{\Gamma(\beta)}\int_{\phi}^{\tau}\left[\frac{(\gamma(\tau)-\gamma(\phi))^{\alpha}-(\gamma(\theta)-\gamma(\phi))^{\alpha}}{\alpha}\right]^{\beta-1}\frac{[(\gamma(\theta)-\gamma(\phi))^{\alpha}]^{r-1}\gamma'(\theta)d\theta}{(\gamma(\theta)-\gamma(\phi))^{1-\alpha}} \\ &= \frac{[(\gamma(\tau)-\gamma(\phi))^{\alpha}]^{\beta+r-1}}{\Gamma(\beta)\alpha^{\beta-1}}\int_{0}^{1}(1-z)^{\beta-1}z^{r-1}dz \\ &= \frac{\Gamma(r)}{\Gamma(\beta+r)}\frac{[(\gamma(\tau)-\gamma(\phi))^{\alpha}]^{\beta+r-1}}{\alpha^{\beta}}. \end{aligned} \tag{2.16}$$

Moreover, we employed the change of variable,

$$\left(\gamma\left(\theta\right)-\gamma\left(\phi\right)\right)^{\alpha}=z\left(\gamma\left(\tau\right)-\gamma\left(\phi\right)\right)^{\alpha}. \tag{2.17}$$

The proof of the second formula can similarly be illustrated using the similar approach. □

Lemma 2.9. *Let $f\in X_{\gamma}(0,\infty)$. Moreover, we will consider γ as a monotonically increasing and positive function defined on the interval $[0,\infty)$, with its derivative γ' being continuous and $\gamma(0)=0$. We possess for $Re\,(n-\alpha)>0$,*

$$\begin{aligned} \left[{}_{\phi}^{\gamma}D^{\beta,\alpha}\left(\gamma\left(\theta\right)-\gamma\left(\phi\right)\right)^{\alpha(r-1)}\right](\tau) &= \frac{\alpha^{\beta}\Gamma(r)}{\Gamma(r-\beta)}\left[\left(\gamma\left(\tau\right)-\gamma\left(\phi\right)\right)^{\alpha}\right]^{r-\beta-1}, \\ \left[{}^{\gamma}D_{\delta}^{\beta,\alpha}\left(\gamma\left(\delta\right)-\gamma\left(\theta\right)\right)^{\alpha(r-1)}\right](\tau) &= \frac{\alpha^{\beta}\Gamma(r)}{\Gamma(r-\beta)}\left[\left(\gamma\left(\delta\right)-\gamma\left(\tau\right)\right)^{\alpha}\right]^{r-\beta-1}. \end{aligned} \tag{2.18}$$

Proof. With the assistance of (2.9), we hold

$$\begin{aligned} &\left[{}_{\phi}^{\gamma}D^{\beta,\alpha}\left(\gamma\left(\theta\right)-\gamma\left(\phi\right)\right)^{\alpha(r-1)}\right](\tau) \\ &= \frac{{}_{\phi}^{\gamma}T^{n,\alpha}}{\Gamma(n-\beta)}\int_{\phi}^{\tau}\left[\frac{(\gamma(\tau)-\gamma(\phi))^{\alpha}-(\gamma(\theta)-\gamma(\phi))^{\alpha}}{\alpha}\right]^{n-\beta-1}\frac{[(\gamma(\theta)-\gamma(\phi))^{\alpha}]^{r-1}.\gamma'(\theta)d\theta}{(\gamma(\theta)-\gamma(\phi))^{1-\alpha}} \\ &= \frac{{}_{\phi}^{\gamma}T^{n,\alpha}[(\gamma(\tau)-\gamma(\phi))^{\alpha}]^{n+r-\beta-1}}{\Gamma(n-\beta)\alpha^{n-\beta}}\int_{0}^{1}(1-z)^{n-\beta-1}z^{r-1}dz \\ &= \frac{\alpha^{\beta}\Gamma(r)}{\Gamma(r-\beta)}\left[\left(\gamma\left(\tau\right)-\gamma\left(\phi\right)\right)^{\alpha}\right]^{r-\beta-1}. \end{aligned} \tag{2.19}$$

In here, we employed the change of variable,

$$\left(\gamma\left(\theta\right)-\gamma\left(\phi\right)\right)^{\alpha}=z\left(\gamma\left(\tau\right)-\gamma\left(\phi\right)\right)^{\alpha}. \tag{2.20}$$

The proof of the second formula can similarly be illustrated using the same approach. □

Remark 2.10. It can be illustrated that

$$\begin{aligned} {}_{\phi}^{\gamma}D^{\beta,\alpha}f &= {}_{\phi}^{\gamma}J^{\beta,-\alpha} \\ {}^{\gamma}D_{\delta}^{\beta,\alpha}f &= {}^{\gamma}J_{\delta}^{\beta,-\alpha}. \end{aligned} \tag{2.21}$$

3. GENERALIZED CONFORMABLE FRACTIONAL DERIVATIVES ON THE SPECIFIC SPACES

In this section, we will introduce several definitions pertaining to lemmas and theorems. Furthermore, we will showcase the significant outcomes of the generalized conformable fractional derivatives within the spaces $C_{\alpha,\phi}^{n}$ and $C_{\alpha,\delta}^{n}$.

Definition 3.1. [18] For $0<\alpha\le 1$ and an interval $[\phi,\delta]$, we describe

$${}^{\gamma}I_{\alpha}\left([\phi,\delta]\right)=\left\{f:[\phi,\delta]\to\mathbb{R}:f\left(\tau\right)=\left({}_{\phi}^{\gamma}I^{\beta,\alpha}\varphi\right)(\tau)+f\left(\phi\right)\ \text{for some }\ \varphi\in{}^{\gamma}L_{\alpha}\left(\phi\right)\right\} \tag{3.1}$$

and

$$ {}^{\gamma}_{\alpha}I\left([\phi,\delta]\right)= \left\{g:[\phi,\delta]\to\mathbb{R}:g\left(\tau\right)=\left({}^{\gamma}I_{\delta}^{\beta,\alpha}\varphi\right)\left(\tau\right)+g\left(\delta\right) \text{ for some } \varphi\in {}^{\gamma}L_{\alpha}\left(\delta\right)\right\}. \tag{3.2} $$

Where

$$ {}^{\gamma}L_{\alpha}\left(\phi\right)=\left\{\varphi:[\phi,\delta]\to\mathbb{R},\ \left({}^{\gamma}_{\phi}I^{\alpha,\beta}\varphi\right)\left(\tau\right) \text{ exists } \forall\ \tau\in[\phi,\delta]\right\} \tag{3.3} $$

and

$$ {}^{\gamma}L_{\alpha}\left(\delta\right)=\left\{\varphi:[\phi,\delta]\to\mathbb{R},\ \left({}^{\gamma}I_{\delta}^{\alpha,\beta}\varphi\right)\left(\tau\right) \text{ exists } \forall\ \tau\in[\phi,\delta]\right\}. \tag{3.4} $$

Definition 3.2. We can define for $\alpha\in(0,1]$ and $n=1,2,3,...,$

$$ \begin{aligned} C^{n}_{\alpha,\phi}\left([\phi,\delta]\right)&=\left\{f:[\phi,\delta]\to\mathbb{R} \textit{ such that } {}^{\gamma}_{\phi}T^{n-1,\alpha}f\in {}^{\gamma}I_{\alpha}\left([\phi,\delta]\right)\right\},\\ C^{n}_{\alpha,\delta}\left([\phi,\delta]\right)&=\left\{f:[\phi,\delta]\to\mathbb{R} \textit{ suc}\gamma \textit{ that } {}^{\gamma}T_{\delta}^{n-1,\alpha}f\in {}^{\gamma}_{\alpha}I\left([\phi,\delta]\right)\right\}. \end{aligned} \tag{3.5} $$

Lemma 3.3. *Let $f\in C^{n}_{\alpha,\phi}\left([\phi,\delta]\right)$ for $\alpha>0$. Moreover, we will consider γ as a monotonically increasing and positive function defined on the interval $[0,\infty)$, with its derivative γ' being continuous and $\gamma\left(0\right)=0$. Then f is expressed as the following form,*

$$ \begin{aligned} f\left(\tau\right)=&\frac{1}{(n-1)!}\int_{\phi}^{\tau}\left[\frac{(\gamma(\tau)-\gamma(\phi))^{\alpha}-(\gamma(\theta)-\gamma(\phi))^{\alpha}}{\alpha}\right]^{n-1}\frac{\varphi(\theta)\gamma^{'}(\theta)d\theta}{(\gamma(\theta)-\gamma(\phi))^{1-\alpha}}\\ &+\sum_{s=0}^{n-1}\left[\frac{(\gamma(\tau)-\gamma(\phi))^{\alpha}}{\alpha}\right]^{s}\frac{1}{s!}{}^{\gamma}_{\phi}T^{s,\alpha}f\left(\phi\right). \end{aligned} \tag{3.6} $$

In this place is $\varphi\left(\theta\right)=\left({}^{\gamma}_{\phi}T^{s,\alpha}f\right)\left(\theta\right)$.

Proof. If we take $f\in C^{n}_{\alpha,\phi}\left([\phi,\delta]\right)$, ${}^{\gamma}_{\phi}T^{n-1,\alpha}f\in {}^{\gamma}I_{\alpha}\left([\phi,\delta]\right)$ and φ is continuous function, then we acquire,

$$ \begin{aligned} {}^{\gamma}_{\phi}T^{n-1,\alpha}f\left(\tau\right)&=\int_{\phi}^{\tau}\frac{\varphi(\theta)\gamma^{'}(\theta)d\theta}{(\gamma(\theta)-\gamma(\phi))^{1-\alpha}}+{}^{\gamma}_{\phi}T^{n-1,\alpha}f\left(\phi\right)\\ \frac{(\gamma(\tau)-\gamma(\phi))^{1-\alpha}}{\gamma^{'}(\tau)}\frac{d}{d\tau}\left({}^{\gamma}_{\phi}T^{n-2,\alpha}f\left(\tau\right)\right)&=\int_{\phi}^{\tau}\frac{\varphi(\theta)\gamma^{'}(\theta)d\theta}{(\gamma(\theta)-\gamma(\phi))^{1-\alpha}}+{}^{\gamma}_{\phi}T^{n-1,\alpha}f\left(\phi\right)\\ \frac{d}{d\tau}\left({}^{\gamma}_{\phi}T^{n-2,\alpha}f\left(\tau\right)\right)&=\frac{\gamma^{'}(\tau)}{(\gamma(\tau)-\gamma(\phi))^{1-\alpha}}\left[\int_{\phi}^{\tau}\frac{\varphi(\theta)\gamma^{'}(\theta)d\theta}{(\gamma(\theta)-\gamma(\phi))^{1-\alpha}}\right.\\ &\left.+\frac{\gamma^{'}(\tau)}{(\gamma(\tau)-\gamma(\phi))^{1-\alpha}}.{}^{\gamma}_{\phi}T^{n-1,\alpha}f\left(\phi\right)\right]. \end{aligned} \tag{3.7} $$

If we integrate both of parties (3.7) from ϕ to τ, substituting τ with θ and θ with s on the both side of the equation, then we have

$$ \begin{aligned} {}^{\gamma}_{\phi}T^{n-2,\alpha}f\left(\tau\right)=&\int_{\phi}^{\tau}\left[\frac{(\gamma(\tau)-\gamma(\phi))^{\alpha}-(\gamma(s)-\gamma(\phi))^{\alpha}}{\alpha}\right]\frac{\varphi(s)\gamma^{'}(s)ds}{(\gamma(s)-\gamma(\phi))^{1-\alpha}}\\ &+\frac{(\gamma(\tau)-\gamma(\phi))^{\alpha}}{\alpha}.{}^{\gamma}_{\phi}T^{n-1,\alpha}f\left(\phi\right)+{}^{\gamma}_{\phi}T^{n-2,\alpha}f\left(\phi\right). \end{aligned} \tag{3.8} $$

By applying the equation (3.8) again same method, we get

$$ \begin{aligned} {}^{\gamma}_{\phi}T^{n-3,\alpha}f\left(\tau\right)=&\int_{\phi}^{\tau}\frac{1}{2}\left[\frac{(\gamma(\tau)-\gamma(\phi))^{\alpha}-(\gamma(s)-\gamma(\phi))^{\alpha}}{\alpha}\right]^{2}\frac{\varphi(s)\gamma^{'}(s)ds}{(\gamma(s)-\gamma(\phi))^{1-\alpha}}\\ &+\frac{1}{2}\left[\frac{(\gamma(\tau)-\gamma(\phi))^{\alpha}}{\alpha}\right]^{2}.{}^{\gamma}_{\phi}T^{n-1,\alpha}f\left(\phi\right)\\ &+\frac{(\gamma(\tau)-\gamma(\phi))^{\alpha}}{\alpha}.{}^{\gamma}_{\phi}T^{n-2,\alpha}f\left(\phi\right)+{}^{\gamma}_{\phi}T^{n-3,\alpha}f\left(\phi\right). \end{aligned} \tag{3.9} $$

By applying the same method iteratively $n-3$ times, then we have,

$$\begin{aligned} f(\tau) = & \frac{1}{(n-1)!}\int_{\phi}^{\tau}\left[\frac{(\gamma(\tau)-\gamma(\phi))^{\alpha}-(\gamma(\theta)-\gamma(\phi))^{\alpha}}{\alpha}\right]^{n-1}\frac{\varphi(\theta)\gamma^{'}(\theta)d\theta}{(\gamma(\theta)-\gamma(\phi))^{1-\alpha}} \\ & +\sum_{s=0}^{n-1}\left[\frac{(\gamma(\tau)-\gamma(\phi))^{\alpha}}{\alpha}\right]^{s}\frac{1}{s!}.{}_{\phi}^{\gamma}T^{s,\alpha}f(\phi). \end{aligned} \tag{3.10}$$

For $\varphi(\theta) ={}_{\phi}^{\gamma} T^{n,\alpha}f(\theta)$. It is evident that an analogous lemma holds for right generalized conformable fractional derivatives. □

Lemma 3.4. *Let $f\in C^{n}_{\alpha,\phi}([\phi,\delta])$ for $\alpha>0$. Furthermore, we will consider γ as a monotonically increasing and positive function defined on the interval $[0,\infty)$, with its derivative γ' being continuous and $\gamma(0)=0$. Then, f is expressed in form,*

$$\begin{aligned} f(\tau) &= \frac{1}{(n-1)!}\int_{\tau}^{\delta}\left[\frac{(\gamma(\delta)-\gamma(\tau))^{\alpha}-(\gamma(\delta)-\gamma(\theta))^{\alpha}}{\alpha}\right]^{n-1}\frac{\varphi(\theta)\gamma^{'}(\theta)d\theta}{(\gamma(\delta)-\gamma(\theta))^{1-\alpha}} \\ & \quad +\sum_{s=0}^{n-1}\left[\frac{(\gamma(\delta)-\gamma(\tau))^{\alpha}}{\alpha}\right]^{s}\frac{(-1)^{s}.{}^{\gamma}T_{\delta}^{s,\alpha}f(\delta)}{s!}. \end{aligned} \tag{3.11}$$

In this place is $\varphi(\theta)=({}^{\gamma}T_{\delta}^{s,\alpha}f)(\theta)$.

Proof. The proof follows a similar structure to *lemma* 3. □

In the *Theorem* 2, we will establish the generalized conformable fractional derivatives within the spaces $C^{n}_{\alpha,\phi}$ and $C^{n}_{\alpha,\delta}$.

Theorem 3.5. *Let $\beta\in\mathbb{C}$, $Re(\beta)>0$ and $n=[\beta]+1$. Moreover, we will consider γ as a monotonically increasing and positive function defined on the interval $[0,\infty)$, with its derivative γ' being continuous and $\gamma(0)=0$. The left and right generalized conformable fractional derivative are illustrated in the form for $f\in C^{n}_{\alpha,\phi}$ and $f\in C^{n}_{\alpha,\delta}$. Then, we write*

$$\begin{aligned} {}_{\phi}^{\gamma}D^{\beta,\alpha}f(\tau) = & \left({}_{\phi}^{\gamma}J^{n-\beta}\left({}_{\phi}^{\gamma}T^{n,\alpha}f\right)\right)(\tau) \\ & +\sum_{m=0}^{n-1}\frac{{}_{\phi}^{\gamma}T^{n,\alpha}f(\phi)}{\Gamma(m-\beta+1)}\left[\frac{(\gamma(\tau)-\gamma(\phi))^{\alpha}}{\alpha}\right]^{m-\beta} \end{aligned} \tag{3.12}$$

and

$$\begin{aligned} {}^{\gamma}D_{\delta}^{\beta,\alpha}f(\tau) = & \left({}^{\gamma}J_{\delta}^{n-\beta}\left({}^{\gamma}T_{\delta}^{n,\alpha}f\right)\right)(\tau) \\ & +\sum_{m=0}^{n-1}\frac{(-1)^{n}.{}^{\gamma}T_{\delta}^{n,\alpha}f(\delta)}{\Gamma(m-\beta+1)}\left[\frac{(\gamma(\delta)-\gamma(\tau))^{\alpha}}{\alpha}\right]^{m-\beta}. \end{aligned} \tag{3.13}$$

Proof. By using $f\in C^{n}_{\alpha,\phi}([\phi,\delta])$, we should select $f(\tau)$ in the *Lemma* 3, substituting τ with θ and θ with s that is as following form

$$\begin{aligned} f(\theta) &= \frac{1}{(n-1)!}\int_{\phi}^{\theta}\left[\frac{(\gamma(\theta)-\gamma(\phi))^{\alpha}-(\gamma(s)-\gamma(\phi))^{\alpha}}{\alpha}\right]^{n-1}\frac{{}_{\phi}^{\gamma}T^{n,\alpha}f(s)\gamma^{'}(s)ds}{(\gamma(s)-\gamma(\phi))^{1-\alpha}} \\ & \quad +\sum_{m=0}^{n-1}\left[\frac{(\gamma(\theta)-\gamma(\phi))^{\alpha}}{\alpha}\right]^{m}\frac{1}{m!}.{}_{\phi}^{\gamma}T^{m,\alpha}f(\phi). \end{aligned} \tag{3.14}$$

In here, we can state the following equality by using (2.9) for (3.14),

(3.15)
$$\begin{aligned}&{}_{\phi}^{\gamma}D^{\beta,\alpha}f(\tau)\\&=\frac{{}_{\phi}^{\gamma}T^{n,\alpha}}{\Gamma(n-\beta)}\int_{\phi}^{\tau}\left[\frac{(\gamma(\tau)-\gamma(\phi))^{\alpha}-(\gamma(\theta)-\gamma(\phi))^{\alpha}}{\alpha}\right]^{n-\beta-1}\frac{\gamma'(\theta)f(\theta)d\theta}{(\gamma(\theta)-\gamma(\phi))^{1-\alpha}}\\&=\frac{{}_{\phi}^{\gamma}T^{n,\alpha}}{\Gamma(n-\beta)}\int_{\phi}^{\tau}\left[\frac{(\gamma(\tau)-\gamma(\phi))^{\alpha}-(\gamma(\theta)-\gamma(\phi))^{\alpha}}{\alpha}\right]^{n-\beta-1}\\&\times\left(\frac{1}{(n-1)!}\int_{\phi}^{\theta}\left[\frac{(\gamma(\theta)-\gamma(\phi))^{\alpha}-(\gamma(s)-\gamma(\phi))^{\alpha}}{\alpha}\right]^{n-1}\frac{{}_{\phi}^{\gamma}T^{n,\alpha}f(s)\gamma'(s)ds}{(\gamma(s)-\gamma(\phi))^{1-\alpha}}\right)\frac{\gamma'(\theta)d\theta}{(\gamma(\theta)-\gamma(\phi))^{1-\alpha}}\\&+\frac{{}_{\phi}^{\gamma}T^{n,\alpha}}{\Gamma(n-\beta)}\int_{\phi}^{\tau}\left[\frac{(\gamma(\tau)-\gamma(\phi))^{\alpha}-(\gamma(\theta)-\gamma(\phi))^{\alpha}}{\alpha}\right]^{n-\beta-1}\\&\times\left(\sum_{m=0}^{n-1}\left[\frac{(\gamma(\theta)-\gamma(\phi))^{\alpha}}{\alpha}\right]^{m}\frac{1}{m!}.{}_{\phi}^{\gamma}T^{m,\alpha}f(\phi)\right)\frac{\gamma'(\theta)d\theta}{(\gamma(\theta)-\gamma(\phi))^{1-\alpha}}.\end{aligned}$$

By employing techniques such as changing the order of integration, the gamma function, and the beta function, along with the utilization of the following equations

(3.16)
$$(\gamma(\theta)-\gamma(\phi))^{\alpha}=(\gamma(s)-\gamma(\phi))^{\alpha}+z\left[(\gamma(\tau)-\gamma(\phi))^{\alpha}-(\gamma(s)-\gamma(\phi))^{\alpha}\right]$$

and

$$(\gamma(\theta)-\gamma(\phi))^{\alpha}=u(\gamma(\tau)-\gamma(\phi))^{\alpha}.$$

Then we obtain following form

$$\begin{aligned}&{}_{\phi}^{\gamma}D^{\beta,\alpha}f(\tau)\\&=\frac{{}_{\phi}^{\gamma}T^{n,\alpha}}{\Gamma(n-\beta)(n-1)!}\int_{\phi}^{\tau}\frac{{}_{\phi}^{\gamma}T^{n,\alpha}f(s)\gamma'(s)ds}{(\gamma(s)-\gamma(\phi))^{1-\alpha}}\\&\times\left(\int_0^1(1-z)^{n-\beta-1}(z)^{n-1}dz\right)\left[\frac{(\gamma(\tau)-\gamma(\phi))^{\alpha}-(\gamma(s)-\gamma(\phi))^{\alpha}}{\alpha}\right]^{2n-\beta-1}\\&+\sum_{m=0}^{n-1}\frac{{}_{\phi}^{\gamma}T^{n,\alpha}.{}_{\phi}^{\gamma}T^{n,\alpha}f(\phi)}{\Gamma(n-\beta).m!}\\&\times\left(\int_0^1(1-u)^{n-\beta-1}(u)^{m}du\right)\left[\frac{(\gamma(\tau)-\gamma(\phi))^{\alpha}}{\alpha}\right]^{n-\beta+m}.\end{aligned}\tag{3.17}$$

In here, we obtain by means of the operator ${}_{\phi}^{\gamma}T^{n,\alpha}$,

(3.18)
$$\begin{aligned}{}_{\phi}^{\gamma}D^{\beta,\alpha}f(\tau)&=\frac{1}{\Gamma(n-\beta)}\int_{\phi}^{\tau}\left[\frac{(\gamma(\tau)-\gamma(\phi))^{\alpha}-(\gamma(s)-\gamma(\phi))^{\alpha}}{\alpha}\right]^{n-\beta-1}\frac{\gamma'(s){}_{\phi}^{n}T^{\alpha}f(s)ds}{(\gamma(s)-\gamma(\phi))^{1-\alpha}}\\&+\sum_{m=0}^{n-1}\frac{{}_{\phi}^{\gamma}T^{n,\alpha}f(\phi)}{\Gamma(m-\beta+1)}\left[\frac{(\gamma(\tau)-\gamma(\phi))^{\alpha}}{\alpha}\right]^{m-\beta}.\end{aligned}$$

We have successfully concluded the proof. The proof of the right generalized conformable fractional derivative can be conducted in a similar manner. □

Theorem 3.6. *We assume that is* $Re(\beta)>m>0$ *for* $m\in\mathbb{N}$. *Furthermore, we will consider* γ *as a monotonically increasing and positive function defined on the interval* $[0,\infty)$, *with its derivative* γ' *being continuous and* $\gamma(0)=0$. *Then, we have*

$$\begin{aligned}{}_{\phi}^{\gamma}T^{m,\alpha}\left({}_{\phi}^{\gamma}J^{\beta,\alpha}f(\tau)\right)&={}_{\phi}^{\gamma}J^{\beta-m,\alpha}f(\tau),\\{}^{\gamma}T_{\delta}^{m,\alpha}\left({}^{\gamma}J_{\delta}^{\beta,\alpha}f(\tau)\right)&={}^{\gamma}J_{\delta}^{\beta-m,\alpha}f(\tau).\end{aligned}\tag{3.19}$$

Proof. We have by using (2.7),

(3.20)
$${}_{\phi}^{\gamma}T^{m,\alpha}\left({}_{\phi}^{\gamma}J^{\beta,\alpha}f(\tau)\right)={}_{\phi}^{\gamma}T^{m,\alpha}\left[\frac{1}{\Gamma(\beta)}\int_{\phi}^{\tau}\left[\frac{(\gamma(\tau)-\gamma(\phi))^{\alpha}-(\gamma(\theta)-\gamma(\phi))^{\alpha}}{\alpha}\right]^{\beta-1}\frac{\gamma'(\theta)f(\theta)d\theta}{(\gamma(\theta)-\gamma(\phi))^{1-\alpha}}\right].$$

By utilizing Leibniz rule for integrals, we get

$$
\begin{aligned}
&{}_{\phi}^{\gamma}T^{m,\alpha}\left({}_{\phi}^{\gamma}J^{\beta,\alpha}f(\tau)\right)\\
&= {}_{\phi}^{\gamma}T^{m-1,\alpha}\left[\frac{1}{\Gamma(\beta-1)}\int_{\phi}^{\tau}\left[\frac{(\gamma(\tau)-\gamma(\phi))^{\alpha}-(\gamma(\theta)-\gamma(\phi))^{\alpha}}{\alpha}\right]^{\beta-2}\frac{\gamma'(\theta)f(\theta)d\theta}{(\gamma(\theta)-\gamma(\phi))^{1-\alpha}}\right]\\
&= {}_{\phi}^{\gamma}T^{m-2,\alpha}\left[\frac{1}{\Gamma(\beta-2)}\int_{\phi}^{\tau}\left[\frac{(\gamma(\tau)-\gamma(\phi))^{\alpha}-(\gamma(\theta)-\gamma(\phi))^{\alpha}}{\alpha}\right]^{\beta-3}\frac{\gamma'(\theta)f(\theta)d\theta}{(\gamma(\theta)-\gamma(\phi))^{1-\alpha}}\right]\\
&\vdots\\
&= \left[\frac{1}{\Gamma(\beta-m)}\int_{\phi}^{\tau}\left[\frac{(\gamma(\tau)-\gamma(\phi))^{\alpha}-(\gamma(\theta)-\gamma(\phi))^{\alpha}}{\alpha}\right]^{\beta-m-1}\frac{\gamma'(\theta)f(\theta)d\theta}{(\gamma(\theta)-\gamma(\phi))^{1-\alpha}}\right]\\
&= {}_{\phi}^{\gamma}J^{\beta-m,\alpha}f(\tau).
\end{aligned}
\tag{3.21}
$$

The poof is successfully completed. The proof of the second formula can be similarly illustrated. □

Corollary 3.6.1. *We will consider γ as a monotonically increasing and positive function defined on the interval $[0,\infty)$, with its derivative γ' being continuous and $\gamma(0)=0$. If we take $Re(\varsigma) < Re(\beta)$, then we write*

$$
\begin{aligned}
&{}_{\phi}^{\gamma}D^{\varsigma,\alpha}\left({}_{\phi}^{\gamma}J^{\beta,\alpha}f(\tau)\right) = {}_{\phi}^{\gamma}J^{\beta-\varsigma,\alpha}f(\tau),\\
&{}^{\gamma}D_{\delta}^{\varsigma,\alpha}\left({}^{\gamma}J_{\delta}^{\beta,\alpha}f(\tau)\right) = {}^{\gamma}J_{\delta}^{\beta-\varsigma,\alpha}f(\tau).
\end{aligned}
\tag{3.22}
$$

Proof. By employing $Theorem$ 1 and $Theorem$ 3, we acquire

$$
\begin{aligned}
{}_{\phi}^{\gamma}D^{\varsigma,\alpha}\left({}_{\phi}^{\gamma}J^{\beta,\alpha}f(\tau)\right) &= {}_{\phi}^{\gamma}T^{m,\alpha}\left({}_{\phi}^{\gamma}J^{m-\varsigma,\alpha}\left({}_{\phi}^{\gamma}J^{\beta,\alpha}f(\tau)\right)\right)\\
&= {}_{\phi}^{\gamma}T^{m,\alpha}\left({}_{\phi}^{\gamma}J^{\beta+m-\varsigma,\alpha}f(\tau)\right)\\
&= {}_{\phi}^{\gamma}J^{\beta-\varsigma,\alpha}f(\tau).
\end{aligned}
\tag{3.23}
$$

The poof is successfully completed. The proof of the second formula can be similarly illustrated. □

Theorem 3.7. *Let $\beta > 0$ and $f \in C_{\alpha,\phi}^{n}[\phi,\delta]\left(f \in C_{\alpha,\delta}^{n}[\phi,\delta]\right)$. Moreover, we will consider γ as a monotonically increasing and positive function defined on the interval $[0,\infty)$, with its derivative γ' being continuous and $\gamma(0)=0$. Then, we have*

$$
\begin{aligned}
&{}_{\phi}^{\gamma}D^{\beta,\alpha}\left({}_{\phi}^{\gamma}J^{\beta,\alpha}f(\tau)\right) = f(\tau),\\
&{}^{\gamma}D_{\delta}^{\beta,\alpha}\left({}^{\gamma}J_{\delta}^{\beta,\alpha}f(\tau)\right) = f(\tau).
\end{aligned}
\tag{3.24}
$$

Proof. If we possess by using (2.7) and (2.9) , then we have

$$
\begin{aligned}
&{}_{\phi}^{\gamma}D^{\beta,\alpha}\left({}_{\phi}^{\gamma}J^{\beta,\alpha}f\left(\tau\right)\right)\\
&=\frac{{}_{\phi}^{\gamma}T^{n,\alpha}}{\Gamma(n-\beta)\Gamma(\beta)}\int_{\phi}^{\tau}\int_{\phi}^{\theta}\left[\frac{(\gamma(\tau)-\gamma(\phi))^{\alpha}-(\gamma(\theta)-\gamma(\phi))^{\alpha}}{\alpha}\right]^{n-\beta-1}\\
&\times\left[\frac{(\gamma(\theta)-\gamma(\phi))^{\alpha}-(\gamma(u)-\gamma(\phi))^{\alpha}}{\alpha}\right]^{\beta-1}\frac{\gamma'(u)f(u)du}{(\gamma(u)-\gamma(\phi))^{1-\alpha}}\frac{\gamma'(\theta)f(\theta)d\theta}{(\gamma(\theta)-\gamma(\phi))^{1-\alpha}}\\
&=\frac{{}_{\phi}^{\gamma}T^{n,\alpha}}{\Gamma(n-\beta)\Gamma(\beta)[\alpha]^{n-2}}\int_{\phi}^{\tau}\int_{\phi}^{\theta}\left[\left(\gamma\left(\tau\right)-\gamma\left(\phi\right)\right)^{\alpha}-\left(\gamma\left(\theta\right)-\gamma\left(\phi\right)\right)^{\alpha}\right]^{n-\beta-1}\\
&\times\left[\left(\gamma\left(\theta\right)-\gamma\left(\phi\right)\right)^{\alpha}-\left(\gamma\left(u\right)-\gamma\left(\phi\right)\right)^{\alpha}\right]^{\beta-1}\frac{\gamma'(u)f(u)du}{(\gamma(u)-\gamma(\phi))^{1-\alpha}}\frac{\gamma'(\theta)f(\theta)d\theta}{(\gamma(\theta)-\gamma(\phi))^{1-\alpha}}\\
&=\frac{{}_{\phi}^{\gamma}T^{n,\alpha}}{\Gamma(n-\beta)\Gamma(\beta)}\int_{\phi}^{\tau}\frac{f(u)\gamma'(u)du}{(\gamma(u)-\gamma(\phi))^{1-\alpha}}\\
&\times\left(\int_{0}^{1}\left(1-y\right)^{n-\beta-1}\left(y\right)^{\beta-1}dy\right)\left[\frac{(\gamma(\tau)-\gamma(\phi))^{\alpha}-(\gamma(u)-\gamma(\phi))^{\alpha}}{\alpha}\right]^{n-1}\\
&=\frac{{}_{\phi}^{\gamma}T^{n,\alpha}}{\Gamma(n-\beta)\Gamma(\beta)}\frac{\Gamma(n-\beta)\Gamma(\beta)}{\Gamma(n)}\int_{\phi}^{\tau}\left[\frac{(\gamma(\tau)-\gamma(\phi))^{\alpha}-(\gamma(u)-\gamma(\phi))^{\alpha}}{\alpha}\right]^{n-1}\frac{f(u)\gamma'(u)du}{(\gamma(u)-\gamma(\phi))^{1-\alpha}}\\
&={}_{\phi}^{\gamma}T^{n,\alpha}\left({}_{\phi}^{\gamma}J^{n,\alpha}f\right)\left(\tau\right)\\
&=f\left(\tau\right).
\end{aligned}
\tag{3.25}
$$

We complete the proof . □

Theorem 3.8. *Let $Re(\beta)>0,\ n=[Re(\beta)],\ f\in X_{\gamma}$ and ${}_{\phi}^{\gamma}J^{\beta,\alpha}f\in C_{\alpha,\phi}^{n}\left[\phi,\delta\right]\left({}^{\gamma}J_{\delta}^{\beta,\alpha}f\in C_{\alpha,\delta}^{n}\left[\phi,\delta\right]\right)$. Furthermore, we will consider γ as a monotonically increasing and positive function defined on the interval $[0,\infty)$, with its derivative γ' being continuous and $\gamma\left(0\right)=0$. Then, we have*

$$
{}_{\phi}^{\gamma}J^{\beta,\alpha}\left({}_{\phi}^{\gamma}D^{\beta,\alpha}f\left(\tau\right)\right)=f\left(\tau\right)-\sum_{j=0}^{n}\frac{{}_{\phi}^{\gamma}D^{\beta-j,\alpha}f\left(\phi\right)}{\Gamma\left(\beta-j+1\right)}\left[\frac{\left(\gamma\left(\tau\right)-\gamma\left(\phi\right)\right)^{\alpha}}{\alpha}\right]^{\beta-j}
\tag{3.26}
$$

and

$$
{}^{\gamma}J_{\delta}^{\beta,\alpha}\left({}^{\gamma}D_{\delta}^{\beta,\alpha}f\left(\tau\right)\right)=f\left(\tau\right)-\sum_{j=0}^{n}\frac{\left(-1\right)^{n-j}.{}^{\gamma}D_{\delta}^{\beta-j,\alpha}f\left(\delta\right)}{\Gamma\left(\beta-j+1\right)}\left[\frac{\left(\gamma\left(\delta\right)-\gamma\left(\tau\right)\right)^{\alpha}}{\alpha}\right]^{\beta-j}.
\tag{3.27}
$$

Proof. We can write by using (2.7) and (2.9) ,

$$
{}_{\phi}^{\gamma}J^{\beta,\alpha}\left({}_{\phi}^{\gamma}D^{\beta,\alpha}f\left(\tau\right)\right)=\frac{1}{\Gamma(\beta)}\int_{\phi}^{\tau}\left[\frac{(\gamma(\tau)-\gamma(\phi))^{\alpha}-(\gamma(\theta)-\gamma(\phi))^{\alpha}}{\alpha}\right]^{\beta-1}\frac{{}_{\phi}^{\gamma}T^{n,\alpha}\left({}_{\phi}^{\gamma}J^{n-\beta,\alpha}f(\theta)\right)\gamma'(\theta)d\theta}{(\gamma(\theta)-\gamma(\phi))^{1-\alpha}}.
\tag{3.28}
$$

By using the integration by parts once, we get

$$
\begin{aligned}
{}_{\phi}^{\gamma}J^{\beta,\alpha}\left({}_{\phi}^{\gamma}D^{\beta,\alpha}f\left(\tau\right)\right)&=\frac{{}_{\phi}^{\gamma}T^{1,\alpha}}{\Gamma(\beta+1)}\int_{\phi}^{\tau}\left[\frac{(\gamma(\tau)-\gamma(\phi))^{\alpha}-(\gamma(\theta)-\gamma(\phi))^{\alpha}}{\alpha}\right]^{\beta}\frac{{}_{\phi}^{\gamma}T^{n,\alpha}\left({}_{\phi}^{\gamma}J^{n-\beta,\alpha}f(\theta)\right)\gamma'(\theta)d\theta}{(\gamma(\theta)-\gamma(\phi))^{1-\alpha}}\\
&\quad-\frac{1}{\Gamma(\beta+1)}.{}_{\phi}^{\gamma}T^{n,\alpha}\left({}_{\phi}^{\gamma}J^{n-\beta,\alpha}f\left(\theta\right)\right).\left[\frac{(\gamma(\tau)-\gamma(\phi))^{\alpha}}{\alpha}\right]^{\beta}.
\end{aligned}
\tag{3.29}
$$

By using the integration by parts $n-$times, we obtain

(3.30)

$$
\begin{aligned}
&{}_{\phi}^{\gamma}J^{\beta,\alpha}\left({}_{\phi}^{\gamma}D^{\beta,\alpha}f(\tau)\right)\\
&= \frac{{}_{\phi}^{\gamma}T^{1,\alpha}}{\Gamma(\beta-n+1)}\int_{\phi}^{\tau}\left[\frac{(\gamma(\tau)-\gamma(\phi))^{\alpha}-(\gamma(\theta)-\gamma(\phi))^{\alpha}}{\alpha}\right]^{\beta-n}\frac{\left({}_{\phi}^{\gamma}J^{n-\beta,\alpha}f(\theta)\right)\gamma'(\theta)d\theta}{(\gamma(\theta)-\gamma(\phi))^{1-\alpha}}\\
&-\sum_{j=1}^{n}\frac{{}_{\phi}^{\gamma}T^{n-j,\alpha}\left({}_{\phi}^{\gamma}J^{n-\beta,\alpha}f(\phi)\right)}{\Gamma(\beta+2-j)}\left[\frac{(\gamma(\tau)-\gamma(\phi))^{\alpha}}{\alpha}\right]^{\beta-j+1}\\
&={}_{\phi}^{\gamma}T^{1,\alpha}\left[{}_{\phi}^{\gamma}J^{\beta-n+1,\alpha}\left({}_{\phi}^{\gamma}J^{n-\beta,\alpha}f(\tau)\right)-\sum_{j=1}^{n}\frac{{}_{\phi}^{\gamma}T^{n-j,\alpha}\left({}_{\phi}^{\gamma}J^{n-\beta,\alpha}f(\phi)\right)}{\Gamma(\beta+2-j)}\left[\frac{(\gamma(\tau)-\gamma(\phi))^{\alpha}}{\alpha}\right]^{\beta-j+1}\right]\\
&={}_{\phi}^{\gamma}T^{1,\alpha}\left[\left({}_{\phi}^{\gamma}J^{1,\alpha}f(\tau)\right)-\sum_{j=1}^{n}\frac{{}_{\phi}^{\gamma}T^{n-j,\alpha}\left({}_{\phi}^{\gamma}J^{n-\beta,\alpha}f(\phi)\right)}{\Gamma(\beta+2-j)}\left[\frac{(\gamma(\tau)-\gamma(\phi))^{\alpha}}{\alpha}\right]^{\beta-j+1}\right]\\
&= f(\tau)-\sum_{j=1}^{n}\frac{{}_{\phi}^{\gamma}D^{\beta-j,\alpha}f(\phi)}{\Gamma(\beta+1-j)}\left[\frac{(\gamma(\tau)-\gamma(\phi))^{\alpha}}{\alpha}\right]^{\beta-j}.
\end{aligned}
$$

The proof is successfully completed. The proof of the second formula can be illustrated in a similar fashion. □

4. GENERALIZED CONFORMABLE FRACTIONAL DERIVATIVES WITHIN CAPUTO FRAMEWORK

In this section, we will introduce several definitions relevant to the theorem, while also elucidating some properties of the generalized conformable derivative within the Caputo setting.

Definition 4.1. Let $\alpha > 0$, $Re(\beta) \geq 0$ and $n = [Re(\beta)] + 1$. Moreover, we will consider γ as a monotonically increasing and positive function defined on the interval $[0, \infty)$, with its derivative γ' being continuous and $\gamma(0) = 0$. If we get $f \in C^{n}_{\alpha,\phi}\left(f \in C^{n}_{\alpha,\delta}\right)$, then, we acquire the left and right generalized Caputo conformable fractional derivatives, respectively.

(4.1)

$$
\left({}_{\phi}^{\gamma,C}D^{\beta,\alpha}f(\tau)\right) = {}_{\phi}^{\gamma}D^{\beta,\alpha}\left[f(\theta)-\sum_{m=0}^{n-1}\frac{{}_{\phi}^{\gamma}T^{m,\alpha}f(\phi)}{m!}\left(\frac{(\gamma(\theta)-\gamma(\phi))^{\alpha}}{\alpha}\right)^{m}\right](\tau)
$$

and

(4.2)

$$
\left({}_{\phi}^{\gamma,C}D^{\beta,\alpha}f(\tau)\right) = {}^{\gamma}D_{\delta}^{\beta,\alpha}\left[f(\theta)-\sum_{m=0}^{n-1}\frac{(-1)^{m}.{}^{\gamma}T_{\delta}^{m,\alpha}f(\delta)}{m!}\left(\frac{(\gamma(\delta)-\gamma(\theta))^{\alpha}}{\alpha}\right)^{m}\right](\tau).
$$

Theorem 4.2. *Let $Re(\beta) \geq 0$ and $n = [Re(\beta)] + 1$. Furthermore, we will consider γ as a monotonically increasing and positive function defined on the interval $[0, \infty)$, with its derivative γ' being continuous and $\gamma(0) = 0$. If we take $f \in C^{n}_{\alpha,\phi}\left(f \in C^{n}_{\alpha,\delta}\right)$, then we obtain the left and right generalized Caputo fractional conformable derivatives in Caputo setting, respectively.*

$$
\left({}_{\phi}^{\gamma,C}D^{\beta,\alpha}f(\tau)\right) = {}_{\phi}^{\gamma}J^{n-\beta,\alpha}\left({}_{\phi}^{\gamma}T^{n,\alpha}f(\tau)\right) \quad (4.3)
$$

and

$$
\left({}^{\gamma,C}D_{\delta}^{\beta,\alpha}f(\tau)\right) = {}^{\gamma}J_{\delta}^{n-\beta,\alpha}\left({}^{\gamma}T_{\delta}^{n,\alpha}f(\tau)\right). \quad (4.4)
$$

Proof. By considering *Definition* 5, we possess

$$\begin{aligned}
&\left({}^{\gamma,C}_{\phi}D^{\beta,\alpha}f(\tau)\right) \\
&={}^{\gamma}_{\phi}D^{\beta,\alpha}\left[f(\theta)-\sum_{m=0}^{n-1}\frac{{}^{\gamma}_{\phi}T^{m,\alpha}f(\phi)}{m!}\left(\frac{(\gamma(\theta)-\gamma(\phi))^{\alpha}}{\alpha}\right)^{m}\right](\tau) \\
&={}^{\gamma}_{\phi}D^{\beta,\alpha}f(\tau)-\sum_{m=0}^{n-1}\frac{{}^{\gamma}_{\phi}T^{m,\alpha}f(\phi)}{m!}\frac{{}^{\gamma}_{\phi}T^{n,\alpha}}{\Gamma(n-\beta)}\left[\frac{(\gamma(\tau)-\gamma(\phi))^{\alpha}}{\alpha}\right]^{n-\beta+m}\frac{\Gamma(n-\beta)\Gamma(m+1)}{\Gamma(n-\beta+m+1)} \\
&={}^{\gamma}_{\phi}D^{\beta,\alpha}f(\tau)-\sum_{m=0}^{n-1}\frac{{}^{\gamma}_{\phi}T^{m,\alpha}f(\phi)}{\Gamma(m-\beta+1)}\left[\frac{(\gamma(\tau)-\gamma(\phi))^{\alpha}}{\alpha}\right]^{m-\beta}.
\end{aligned} \tag{4.5}$$

The proof is done. □

Lemma 4.3. *Let $\alpha>0$, $Re(\beta)\geq 0$, $n=[Re(\beta)]+1$ and $Re(\beta)\notin\mathbb{N}$. Moreover, we will consider γ as a monotonically increasing and positive function defined on the interval $[0,\infty)$, with its derivative γ' being continuous and $\gamma(0)=0$. If $f\in C^{n}_{\alpha,\phi}([\phi,\delta])\,(f\in C^{n}_{\alpha,\delta}([\phi,\delta]))$ then we have*

$$\left.\begin{aligned}
&{}^{\gamma}_{\phi}J^{\beta-s,\alpha}f(\phi)=0,\\
&{}^{\gamma}_{\phi}J^{\beta-s,\alpha}f(\delta)=0
\end{aligned}\right\}\text{for } s=0,1,...,n-1. \tag{4.6}$$

Proof. We hold

$$\begin{aligned}
{}^{\gamma}_{\phi}J^{\beta-s,\alpha}f(\tau) &={}^{\gamma}_{\phi}D^{s,\alpha}\left({}^{\gamma}_{\phi}J^{\beta,\alpha}f(\tau)\right)\\
&=\frac{1}{\Gamma(\beta-s)}\int_{\phi}^{\tau}\left[\frac{(\gamma(\tau)-\gamma(\phi))^{\alpha}-(\gamma(\theta)-\gamma(\phi))^{\alpha}}{\alpha}\right]^{\beta-s-1}\frac{f(\theta)\gamma'(\theta)d\theta}{(\gamma(\theta)-\gamma(\phi))^{1-\alpha}}.
\end{aligned} \tag{4.7}$$

In here, we can express through Hölder's inequality,

$$\begin{aligned}
&\left|{}^{\gamma}_{\phi}J^{\beta-s,\alpha}f(\tau)\right|\\
&\leq\frac{1}{\Gamma(\beta-s)}\left(\int_{\phi}^{\tau}|f(\theta)|^{p}\gamma'(\theta)\right)^{\frac{1}{p}}\left(\int_{\phi}^{\tau}\left(\left[\frac{(\gamma(\tau)-\gamma(\phi))^{\alpha}-(\gamma(\theta)-\gamma(\phi))^{\alpha}}{\alpha}\right]^{\beta-s-1}\frac{f(\theta)\gamma'(\theta)d\theta}{(\gamma(\theta)-\gamma(\phi))^{1-\alpha}}\right)^{q}\right)^{\frac{1}{q}}\\
&\leq\frac{\|f\|_{\tau_{\gamma}}}{(re(\beta)-s)\Gamma(\beta-s)}\left(\frac{(\gamma(\tau)-\gamma(\phi))^{\alpha}}{\alpha}\right)^{(re(\beta)-s)}.
\end{aligned} \tag{4.8}$$

For $\tau=\phi$, we say that

$$ {}^{\gamma}_{\phi}J^{\beta-s,\alpha}f(\phi)=0. \tag{4.9}$$

The proof is done. □

Lemma 4.4. *Let $\alpha>0$, $Re(\beta)\geq 0$ and $n=[Re(\beta)]+1$. Furthermore, we will consider γ as a monotonically increasing and positive function defined on the interval $[0,\infty)$, with its derivative γ' being continuous and $\gamma(0)=0$. If we get ${}^{\gamma}_{\phi}T^{n,\alpha}\in C^{n}_{\alpha,\phi}[\phi,\delta]$ $\left({}^{\gamma}T^{n,\alpha}_{\delta}\in C^{n}_{\alpha,\delta}[\phi,\delta]\right)$, then we obtain*

$$\begin{aligned}
&{}^{\gamma,C}_{\phi}D^{\beta,\alpha}f(\phi)=0,\\
&{}^{\gamma,C}D^{\beta,\alpha}_{\delta}f(\delta)=0.
\end{aligned} \tag{4.10}$$

Proof. It is clearly seen that

$$\left|{}^{\gamma,C}_{\phi}D^{\beta,\alpha}f(\tau)\right|\leq\frac{\left\|{}^{\gamma}_{\phi}T^{n,\alpha}\right\|_{X_{\gamma}}}{(n-re(\beta))\Gamma(n-\beta)}\left(\frac{(\gamma(\tau)-\gamma(\phi))^{\alpha}}{\alpha}\right)^{(n-re(\beta))} \tag{4.11}$$

and

$$\left|{}^{\gamma,C}D_{\delta}^{\beta,\alpha}f\left(\tau\right)\right| \leq \frac{\left\|{}^{\gamma}T_{\delta}^{n,\alpha}\right\|_{X_\gamma}}{(n-re(\beta))\Gamma(n-\beta)}\left(\frac{(\gamma(\delta)-\gamma(\tau))^{\alpha}}{\alpha}\right)^{(n-re(\beta))}. \tag{4.12}$$

The proof is done. □

Theorem 4.5. *Let* $Re\left(\beta\right) \geq 0$, $n = [Re\left(\beta\right)]+1$ *and* $f \in C_{\alpha,\phi}^{n}\left[\phi,\delta\right]$ $(f \in C_{\alpha,\delta}^{n}\left[\phi,\delta\right])$. *Moreover, we will consider* γ *as a monotonically increasing and positive function defined on the interval* $[0,\infty)$, *with its derivative* γ' *being continuous and* $\gamma\left(0\right) = 0$. *We can say that*

(1) If we take $Re\left(\beta\right) \notin \mathbb{N}$ *or* $\beta \in \mathbb{N}$, *then we acquire*

$$\begin{aligned} {}_{\phi}^{\gamma,C}D^{\beta,\alpha}\left({}_{\phi}^{\gamma}J^{\beta,\alpha}f\left(\tau\right)\right) &= f\left(\tau\right), \\ {}^{\gamma,C}D_{\delta}^{\beta,\alpha}\left({}^{\gamma}J_{\delta}^{\beta,\alpha}f\left(\tau\right)\right) &= f\left(\tau\right). \end{aligned} \tag{4.13}$$

(2) If we take $Re\left(\beta\right) \neq 0$ *or* $Re\left(\beta\right) \in \mathbb{N}$, *then we get*

$$\begin{aligned} {}_{\phi}^{\gamma,C}D^{\beta,\alpha}\left({}_{\phi}^{\gamma}J^{\beta,\alpha}f\left(\tau\right)\right) &= f\left(\tau\right) - \frac{{}_{\phi}^{\gamma}J^{\beta-n+1,\alpha}f\left(\phi\right)}{\Gamma\left(n-\beta\right)}\left[\frac{(\gamma(\tau)-\gamma(\phi))^{\alpha}}{\alpha}\right]^{n-\beta}, \\ {}^{\gamma,C}D_{\delta}^{\beta,\alpha}\left({}^{\gamma}J_{\delta}^{\beta,\alpha}f\left(\tau\right)\right) &= f\left(\tau\right) - \frac{{}^{\gamma}J_{\delta}^{\beta-n+1,\alpha}f\left(\phi\right)}{\Gamma\left(n-\beta\right)}\left[\frac{(\gamma(\delta)-\gamma(\tau))^{\alpha}}{\alpha}\right]^{n-\beta}. \end{aligned} \tag{4.14}$$

Proof. By using $Definition$ 6, we have,

$$\begin{aligned} &{}_{\phi}^{\gamma,C}D^{\beta,\alpha}\left({}_{\phi}^{\gamma}J^{\beta,\alpha}f\left(\tau\right)\right) \\ &= f(\tau) - \frac{{}_{\phi}^{\gamma}T^{n,\alpha}}{\Gamma(n-\beta)}\int_{\phi}^{\tau}\left[\frac{(\gamma(\tau)-\gamma(\phi))^{\alpha}-(\gamma(\theta)-\gamma(\phi))^{\alpha}}{\alpha}\right]^{n-\beta-1} \\ &\times\left(\sum_{m=0}^{n-1}\frac{{}_{\phi}^{\gamma}J^{m-\beta,\alpha}f(\phi)}{m!}\left(\frac{(\gamma(\theta)-\gamma(\phi))^{\alpha}}{\alpha}\right)^{m}\right)\frac{\gamma'(\theta)d\theta}{(\gamma(\theta)-\gamma(\phi))^{1-\alpha}} \\ &= f(\tau) - \sum_{m=0}^{n-1}\frac{{}_{\phi}^{\gamma}J^{m-\beta,\alpha}f(\phi)}{m!}\cdot\frac{{}_{\phi}^{\gamma}T^{n,\alpha}}{\Gamma(n-\beta)} \\ &\times\int_{\phi}^{\tau}\left[\frac{(\gamma(\tau)-\gamma(\phi))^{\alpha}-(\gamma(\theta)-\gamma(\phi))^{\alpha}}{\alpha}\right]^{n-\beta-1}\left[\frac{(\gamma(\theta)-\gamma(\phi))^{\alpha}}{\alpha}\right]^{m}\frac{\gamma'(\theta)d\theta}{(\gamma(\theta)-\gamma(\phi))^{1-\alpha}}. \end{aligned} \tag{4.15}$$

In here, by using the following the change of variable,

$$\left(\gamma\left(\theta\right)-\gamma\left(\phi\right)\right)^{\alpha} = z\left(\gamma\left(\tau\right)-\gamma\left(\phi\right)\right)^{\alpha} \tag{4.16}$$

we can hold

$${}_{\phi}^{\gamma,C}D^{\beta,\alpha}\left({}_{\phi}^{\gamma}J^{\beta,\alpha}f\left(\tau\right)\right) = f(\tau) - \sum_{m=0}^{n-1}\frac{{}_{\phi}^{\gamma}J^{m-\beta,\alpha}f(\phi)}{\Gamma(m-\beta+1)}\cdot\left[\frac{(\gamma(\tau)-\gamma(\phi))^{\alpha}}{\alpha}\right]^{m-\beta}. \tag{4.17}$$

In here, we establish ${}_{\phi}^{\gamma}J^{\beta-s,\alpha}f\left(\phi\right) = 0$ and ${}^{\gamma}J_{\delta}^{\beta-s,\alpha}f\left(\delta\right) = 0$ for $Re\left(\beta\right) \notin \mathbb{N}$ by using $Lemma$ 4. The case $\beta \in \mathbb{N}$ is inconsequential. Additionaly, if $Re\left(\beta\right) \in \mathbb{N}$, then we assert that ${}_{\phi}^{\gamma}J^{\beta-s,\alpha}f\left(\phi\right) = 0$ and ${}^{\gamma}J_{\delta}^{\beta-s,\alpha}f\left(\delta\right) = 0$ for $s = 0, 1, ..., n-2$ by using $Lemma$ 4. □

Theorem 4.6. *Let* $\beta \in \mathbb{C}$ *and* $f \in C_{\alpha,\phi}^{n}\left[\phi,\delta\right]\left(f \in C_{\alpha,\delta}^{n}\left[\phi,\delta\right]\right)$. *Furthermore, we will consider* γ *as a monotonically increasing and positive function defined on the* *interval* $[0,\infty)$, *with its derivative* γ' *being continuous and* $\gamma(0)=0$. *We have*

$$(4.18)\quad \begin{aligned} {}_{\phi}^{\gamma}J^{\beta,\alpha}\left({}_{\phi}^{\gamma,C}D^{\beta,\alpha}f(\tau)\right) &= f(\tau)-\sum_{m=0}^{n-1}\frac{{}_{\phi}^{\gamma}T^{m,\alpha}f(\phi)}{\Gamma(m+1)}\left[\frac{(\gamma(\tau)-\gamma(\phi))^{\alpha}}{\alpha}\right]^{m},\\ {}^{\gamma}J_{\delta}^{\beta,\alpha}\left({}^{\gamma,C}D_{\delta}^{\beta,\alpha}f(\tau)\right) &= f(\tau)-\sum_{m=0}^{n-1}\frac{{}^{\gamma}T_{\delta}^{m,\alpha}f(\delta)}{\Gamma(m+1)}\left[\frac{(\gamma(\delta)-\gamma(\tau))^{\alpha}}{\alpha}\right]^{m}. \end{aligned}$$

Proof. In here, we can write the following as,

$$(4.19)\quad \begin{aligned} {}_{\phi}^{\gamma}J^{\beta,\alpha}\left({}_{\phi}^{\gamma,C}D^{\beta,\alpha}f(\tau)\right) &= {}_{\phi}^{\gamma}J^{\beta,\alpha}\left({}_{\phi}^{\gamma}J^{n-\beta,\alpha}\left({}_{\phi}^{\gamma}T^{n,\alpha}f(\tau)\right)\right)\\ &= {}_{\phi}^{\gamma}J^{n,\alpha}\left({}_{\phi}^{\gamma}T^{n,\alpha}f(\tau)\right)\\ &= f(\tau)-\frac{{}_{\phi}^{\gamma}D^{\beta-j,\alpha}f(\phi)}{\Gamma(\beta-j+1)}\left[\frac{(\gamma(\tau)-\gamma(\phi))^{\alpha}}{\alpha}\right]^{\beta-j}\\ &= f(\tau)-\frac{{}_{\phi}^{\gamma}T^{m,\alpha}f(\phi)}{\Gamma(m+1)}\left[\frac{(\gamma(\tau)-\gamma(\phi))^{\alpha}}{\alpha}\right]^{m}. \end{aligned}$$

The proof is done. □

Theorem 4.7. *Let* $f\in C_{\alpha,\phi}^{p+r}[\phi,\delta]\left(f\in C_{\alpha,\delta}^{p+r}[\phi,\delta]\right)$, $Re(\beta)\geq 0$, $Re(\mu)\geq 0$, $r-1<[Re(\beta)]\leq r$ *and* $p-1<[Re(\beta)]\leq p$. *Moreover, we will consider* γ *as a monotonically increasing and positive function defined on the interval* $[0,\infty)$, *with its derivative* γ' *being continuous and* $\gamma(0)=0$. *Then, we write*

$$(4.20)\quad \begin{aligned} {}_{\phi}^{\gamma,C}D^{\beta,\alpha}\left({}_{\phi}^{\gamma,C}D^{\mu,\alpha}(f(\tau))\right) &= {}_{\phi}^{\gamma,C}D^{\beta+\mu,\alpha}f(\tau),\\ {}^{\gamma,C}D_{\delta}^{\beta,\alpha}\left({}^{\gamma,C}D_{\delta}^{\mu,\alpha}(f(\tau))\right) &= {}^{\gamma,C}D_{\delta}^{\beta+\mu,\alpha}f(\tau). \end{aligned}$$

Proof. The proof can be successfully completed by using $Theorem$ 1, $Theorem$ 4, $Theorem$ 6 and $Lemma$ 5. □

5. FRACTIONAL INTEGRALS CLASS

1. Taking $\gamma(\tau)=\tau$ in $Definition$ 2,

$$ {}_{\phi}^{\beta}J^{\alpha}f(\tau) = \frac{1}{\Gamma(\beta)}\int_{\phi}^{\tau}\left[\frac{(\tau-\phi)^{\alpha}-(\theta-\phi)^{\alpha}}{\alpha}\right]^{\beta-1}\frac{f(\theta)d\theta}{(\theta-\phi)^{1-\alpha}}.$$

We acquire the left fractional conformable integrals in [1].

2. Taking $\gamma(\tau)=\tau$ and $\alpha=1$ in $Definition$ 2,

$$ {}_{\phi}^{\gamma}J^{\beta,\alpha}f(\tau) = \frac{1}{\Gamma(\beta)}\int_{\phi}^{\tau}(\tau-\theta)^{\beta-1}f(\theta)\,d\theta.$$

We obtain the left Riemann-Lioville fractional integrals.

3. Taking $\gamma(\tau)=\tau$, $\alpha=1$ and $\phi=-\infty$ in $Definition$ 2,

$$ {}_{\phi}^{\gamma}J^{\beta,\alpha}f(\tau) = \frac{1}{\Gamma(\beta)}\int_{-\infty}^{\tau}(\tau-\theta)^{\beta-1}f(\theta)\,d\theta.$$

We get the left Liouville fractional integrals.

4. Taking $\gamma(\tau)=\tau$, $\phi=0$ and $\alpha=1$ in $Definition$ 2,

$$ {}_{\phi}^{\gamma}J^{\beta,\alpha}f(\tau) = \frac{1}{\Gamma(\beta)}\int_{0}^{\tau}(\tau-\theta)^{\beta-1}f(\theta)\,d\theta.$$

We have the left Riemann fractional integrals.

5. Taking $\gamma(\tau)=ln\tau$ and $\alpha=1$ in $Definition$ 2,

$$ {}_{\phi}^{\gamma}J^{\beta,\alpha}f(\tau) = \frac{1}{\Gamma(\beta)}\int_{\phi}^{\tau}\left(\ln\frac{\tau}{\theta}\right)^{\beta-1}\frac{f(\theta)}{\theta}d\theta.$$

We achieve the left Hadamard fractional integrals [11].

6. Taking $\gamma(\tau)=\tau^m$, $g(\tau)=\tau^{m\eta}f(\tau)$ and $\alpha=1$ in $Definition$ 2,

$$\tau^{-m(\beta+\eta)}.{}_{\phi}^{\gamma}J^{\beta,\alpha}g(\tau) = \frac{m\tau^{-m(\beta+\eta)}}{\Gamma(\beta)}\int_{\phi}^{\tau}\theta^{m\eta+m-1}\left(\tau^m-\theta^m\right)^{\beta-1}f(\theta)\,d\theta.$$

We acquire the left Erdélyi-Kober fractional integrals.

7. Taking $\gamma(\tau)=\tau^m$, $g(\tau)=\tau^{m\eta}f(\tau)$, $\phi=0$ and $\alpha=1$ in $Definition$ 2,

$$\tau^{-m(\beta+\eta)}.{}_{0}^{\gamma}J^{\beta,\alpha}g(\tau) = \frac{m\tau^{-m(\beta+\eta)}}{\Gamma(\beta)}\int_{o}^{\tau}\theta^{m\eta+m-1}\left(\tau^m-\theta^m\right)^{\beta-1}f(\theta)\,d\theta.$$

We obtain the left Erdélyi fractional integrals.

8. Taking $\gamma(\tau)=\tau$, $g(\tau)=\tau^{\eta}f(\tau)$, $\phi=0$ and $\alpha=1$ in $Definition$ 2,

$$\tau^{-(\beta+\eta)}.{}_{0}^{\gamma}J^{\beta,\alpha}g(\tau) = \frac{\tau^{-(\beta+\eta)}}{\Gamma(\beta)}\int_{o}^{\tau}\theta^{\eta}\left(\tau-\theta\right)^{\beta-1}f(\theta)\,d\theta.$$

We get the left Kober fractional integrals.

9. Taking $\gamma(\tau)=\tau^m$, $g(\tau)=\tau^{m\eta}f(\tau)$ and $\alpha=1$ in $Definition$ 2,

$$\frac{\tau^K}{m^\beta}.{}_{\phi}^{\gamma}J^{\beta,\alpha}g(\tau) = \frac{\tau^K m^{1-\beta}}{\Gamma(\beta)}\int_{\phi}^{\tau}\theta^{m\eta+m-1}\left(\tau^m-\theta^m\right)^{\beta-1}f(\theta)\,d\theta.$$

We have the left generalized fractional integrals that unify another six fractional integrals.

10. Taking $\gamma(\tau)=\tau^m$ and $\alpha=1$ in $Definition$ 2,

$$\frac{1}{m^\beta}.{}_{\phi}^{\gamma}J^{\beta,\alpha}f(\tau) = \frac{m^{1-\beta}}{\Gamma(\beta)}\int_{\phi}^{\tau}\theta^{m-1}\left(\tau^m-\theta^m\right)^{\beta-1}f(\theta)\,d\theta.$$

We achieve the left Katugampola fractional integrals.

6. FRACTIONAL DERIVATIVES CLASS

1. Taking $\phi=0$, $\alpha=1$ and $\gamma(\tau)=\tau$ in $Definition$ 3, we acquire Riemann-liouville fractional derivative

$$\begin{aligned}{}_{0}^{\gamma}D^{\beta,\alpha}f(\tau) &= {}_{0}^{\gamma}T^{n}\left({}_{0}^{\gamma}J^{n-\beta,\alpha}\right)f(\tau)\\ &= \frac{{}_{0}^{\gamma}T^{n}}{\Gamma(n-\beta)}\int_{0}^{\tau}\left[\tau-\theta\right]^{n-\beta-1}f(\theta)\,d\theta.\end{aligned}$$

2. Taking $\phi=0$ and $\alpha=1$ in $Definition$ 3, we obtain the $\gamma-$Riemann-liouville fractional derivative

$$\begin{aligned}{}_{0}^{\gamma}D^{\beta,\alpha}f(\tau) &= {}_{0}^{\gamma}T^{n}\left({}_{0}^{\gamma}J^{n-\beta,\alpha}\right)f(\tau)\\ &= \frac{{}_{0}^{\gamma}T^{n}}{\Gamma(n-\beta)}\int_{0}^{\tau}\left[\gamma(\tau)-\gamma(\theta)\right]^{n-\beta-1}\gamma'(\theta)\,f(\theta)\,d\theta.\end{aligned}$$

3. Taking $\gamma(\tau)=\tau$ and $\alpha=1$ in $Definition$ 3, we get the Caputo fractional derivative

$$\begin{aligned}{}_{\phi}^{\gamma}D^{\beta,\alpha}f(\tau) &= {}_{\phi}^{\gamma}J^{n-\beta,\alpha}\left({}_{\phi}^{\gamma}T^{n}\right)f(\tau)\\ &= \frac{1}{\Gamma(n-\beta)}\int_{\phi}^{\tau}\left[\tau-\theta\right]^{n-\beta-1}\left({}_{\phi}^{\gamma}T^{n}\right)f(\theta)\,d\theta.\end{aligned}$$

4. Taking $\alpha=1$,in $Definition$ 3 we have the left $\gamma-$Caputo fractional derivatives

$$\begin{aligned}{}_{\phi}^{\gamma}D^{\beta,\alpha}f(\tau) &= {}_{\phi}^{\gamma}J^{n-\beta,\alpha}\left({}_{\phi}^{\gamma}T^{n}\right)f(\tau)\\ &= \frac{1}{\Gamma(n-\beta)}\int_{\phi}^{\tau}\left[\gamma(\tau)-\gamma(\theta)\right]^{n-\beta-1}\gamma'(\theta){}_{\phi}^{\gamma}T^{n}f(\theta)\,d\theta.\end{aligned}$$

5. Taking $\gamma(\tau)=\tau$ in $Definition$ 3, we achieve the left fractional conformable derivatives in [1] ,

$$\begin{aligned}{}_{\phi}^{\gamma}D^{\beta,\alpha}f(\tau) &= {}_{\phi}^{\gamma}T^{n,\alpha}\left({}_{\phi}^{\gamma}J^{n-\beta,\alpha}\right)f(\tau)\\ &= \frac{{}_{\phi}^{\gamma}T^{n,\alpha}}{\Gamma(n-\beta)}\int_{\phi}^{\tau}\left[\frac{(\tau-\phi)^\alpha-(\theta-\phi)^\alpha}{\alpha}\right]^{n-\beta-1}\frac{f(\theta)d\theta}{(\theta-\phi)^{1-\alpha}}.\end{aligned}$$

6. Taking $\gamma(\tau)=\tau^{\rho}$ $\beta=0$ and $\alpha=1$,in $Definition$ 3, we acquire the Katugampola fractional derivative

$$\rho^{\beta}\cdot{}_{\phi}^{\gamma}D^{\beta,\alpha}f(\tau) \quad =\rho^{\beta}\left(\frac{1}{\rho\tau^{\rho-1}}\frac{d}{d\tau}\right)^{n}\cdot{}_{\phi}^{\gamma}J^{n-\beta,\alpha}f(\tau).$$

7. Taking $\gamma(\tau)=\tau$ and $\alpha=1$ in $Definition$ 3, we obtain the Riemann-Liouville fractional derivative

$${}_{\phi}^{\gamma}D^{\beta,\alpha}f(\tau) \quad =\left(\frac{d}{d\tau}\right)^{n}\frac{1}{\Gamma(n-\beta)}\int_{\phi}^{\tau}[\tau-\theta]^{n-\beta-1}f(\theta)\,d\theta.$$

8. Taking $\gamma(\tau)=\tau^{\rho}$ and $\alpha=1$ in $definition$ 3, we get the Caputo−Katugampola fractional derivative

$$\rho^{\beta}\cdot{}_{\phi}^{\gamma}D^{\beta,\alpha}f(\tau) \quad =\rho^{\beta}\cdot{}_{\phi}^{\gamma}J^{n-\beta,\alpha}\left(\frac{1}{\rho\tau^{\rho-1}}\frac{d}{d\tau}\right)^{n}f(\tau).$$

9. Taking $\gamma(\tau)=\ln\tau$ and $\alpha=1$ in $Definition$ 3, we have the Caputo−Hadamard fractional derivative in $[12-14]$

$${}_{\phi}^{\gamma}D^{\beta,\alpha}f(\tau) \quad =\frac{1}{\Gamma(n-\beta)}\int_{\phi}^{\tau}\left[\ln\frac{\tau}{\theta}\right]^{n-\beta-1}\left(\theta\frac{d}{d\theta}\right)^{n}f(\theta)\,\frac{d\theta}{\theta}.$$

10. Taking $\gamma(\tau)=\ln\tau$ and $\alpha=1$ in $Definition$ 3, we achiev the hadamard fractional derivative

$${}_{\phi}^{\gamma}D^{\beta,\alpha}f(\tau) \quad =\left(\tau\frac{d}{d\tau}\right)^{n}\frac{1}{\Gamma(n-\beta)}\int_{\phi}^{\tau}\left[\ln\frac{\tau}{\theta}\right]^{n-\beta-1}f(\theta)\,\frac{d\theta}{\theta}.$$

7. CONCLUSION

In this study, we introduced the left and right generalized conformable fractional integrals and derivatives. We explored significant implications and fundamental properties of these operators. Additionally, we derived the generalized conformable fractional derivatives within the Caputo framework. Ultimately, we presented classical consequences in the context of generalized conformable derivatives and integrals.

8. ACKNOWLEDGMENTS

The authors would like to thank the reviewers and editors of Journal of Universal Mathematics.

Funding
The authors declared that has not received any financial support for the research, authorship or publication of this study.

The Declaration of Conflict of Interest/ Common Interest
The author(s) declared that no conflict of interest or common interest

The Declaration of Ethics Committee Approval
This study does not be necessary ethical committee permission or any special permission.

The Declaration of Research and Publication Ethics
The author(s) declared that they comply with the scientific, ethical, and citation rules of Journal of Universal Mathematics in all processes of the study and that they do not make any falsification on the data collected. Besides, the author(s)

declared that Journal of Universal Mathematics and its editorial board have no responsibility for any ethical violations that may be encountered and this study has not been evaluated in any academic publication environment other than Journal of Universal Mathematics.

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Journal of Universal Mathematics
Vol.8 No.1 pp.20-32 (2025)
ISSN-2618-5660
DOI: 10.33773/jum.1555200

RE-VISIT $\mathcal{I}^*$-SEQUENTIAL TOPOLOGY

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Abstract. In this paper, $\mathcal{I}^*$-sequential topology is defined on a topological space (X, τ) by considering any ideal $\mathcal{I}$ which is a family of subset of natural numbers $\mathbb{N}$. It has been proven that $\mathcal{I}^*$-sequential topology is finer than $\mathcal{I}$-sequential topology. In connection with this fact, the notions $\mathcal{I}^*$-continuity and $\mathcal{I}^*$-sequential continuity are shown to be coincided. Additionally, $\mathcal{I}^*$-sequential compactness and related notions are defined and investigated.

1. Introduction and Preliminaries

Examining convergence of sequences is one of the main and famous problem in mathematical analysis. Especially, taking into consider different type convergence methods has led to a better understanding of the geometric and algebraic structure of the studied space. Statistical convergence, which is the most interesting method in terms of how it is defined, was introduced by Fast [6] and Steinhouse [23] in the year 1951, independently. Over the years, many studies on statistical convergence have been conducted and many application in different field of mathematics like, summability theory [21], number theory [5], trigonometric series [26], optimization and approximation theory [8] and etc. were given.

Recall the notion of statistical convergence in a topological space. For any subset A in $\mathbb{N}$, the asymptotic density of A is defined by

$$\delta(A) := \lim_{n\to\infty} \frac{1}{n} |\{k \in \mathbb{N} : k \leq n\}|$$

when the limit exists.

A sequence $\tilde{x} = (x_n)$ in the topological space (X, τ) is said to be statistically convergent to a point $x \in X$ if

$$\delta(\{n \in \mathbb{N}, x_n \notin U\}) = 0,$$

holds for any neighborhood U of x. It is denoted by $st - \lim_{x\to\infty} x_n = x$.

Date: **Received:** 2024-09-24; **Accepted:** 2024-12-05.
Key words and phrases. $\mathcal{I}$- convergence, $\mathcal{I}^*$-convergence, Sequentially $\mathcal{I}$-topology, Sequentially $\mathcal{I}^*$-topology.
Mathematics Subject Classification: 54A20; 54B15; 54C08; 54D55; 26A03; 40A05.

A subset $F \subseteq X$ is called sequentially closed if for each sequence $\tilde{x} = (x_n)$ in F with $x_n \to x \in X$ then $x \in F$ holds. A space (X, τ) is called sequential topological space if each sequentially closed subset of X is closed.

A sequence $\tilde{x} = (x_n) \subset X$ is said to be eventually in an open subset U of X, if there exists $n_0(U) \in \mathbb{N}$ such that $x_n \in U$ for all $n > n_0$. A subset $G \subseteq X$ is said to be sequentially open if $X - G$ is sequentially closed. Then, it is obvious that, a subset $U \subseteq X$ is sequentially open if and only if for each sequence $\tilde{x} = (x_n)$ converging to a point x in U, then $\tilde{x} = (x_n)$ is eventually in U.

After that in 2000, P. Kostyrko, et al. in [12] introduced the notion of ideal convergence which is completely different classical convergence but only its particular case coincides with classical and statistical convergence. Because of the flexibility of the ideal concept, several results in different spaces were given in [7, 9, 10, 11, 14, 18, 19, 20, 24]. Between the years 2012-2019, authors of the papers [2, 3, 4, 13, 16, 25] extended the notion of $\mathcal{I}$-convergence of a sequence to any topological space and proved several properties of this concept in a topological space. And very recently, the idea of $\mathcal{I}$-convergent is generalized and $\mathcal{I}^*$-convergent is defined.

Definition 1. [12] *Let S be a set and $\mathcal{I}$ be a sub family of $P(S)$. $\mathcal{I}$ is called an ideal on S if (i) For all $A, B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$ and (ii) If $A \in \mathcal{I}$ and $B \subseteq A$ then $B \in \mathcal{I}$ hold.*

The ideal $\mathcal{I}$ is called an admissible ideal if $\{s\} \in \mathcal{I}$ holds for all $s \in S$; and it is called proper ideal if $S \notin \mathcal{I}$. A proper ideal is called maximal ideal if it is maximal element ordered by inclusion in the set of all proper ideals defined on S.

An ideal $\mathcal{I}$ is called non trivial if $\mathcal{I} \neq \phi$ and $S \notin \mathcal{I}$.

Example 1. *$\mathcal{I}_{Fin} := \{A \subset \mathbb{N} : A \text{ is finite set}\}$ and $\mathcal{I}_\delta := \{A \subset \mathbb{N} : \delta(A) = 0\}$ are admissible and proper ideal on the set of natural numbers.*

Example 2. [11] *Let $\mathbb{N} = \bigcup_{i=1}^{\infty} \Delta_i$ be a decomposition of $\mathbb{N}$ such that for all $i \in \mathbb{N}$ the set Δ_i are infinite subsets of $\mathbb{N}$ and $\Delta_i \cap \Delta j = \phi$ holds for all $i \neq j$. Let*

$$\mathcal{I} := \{B \subset \mathbb{N} : B \text{ intersect at most finite number of } \Delta_j' s\}.$$

Then, $\mathcal{I}$ is an admissible and nontrivial ideal.

Definition 2. *Let $\mathcal{I}$ be an ideal and $K \subset S$ be any set. The set K is said*
(i) $\mathcal{I}$-thin if $K \in \mathcal{I}$,
(ii) $\mathcal{I}$-non thin if $K \notin \mathcal{I}$,
(iii) relatively $\mathcal{I}$-non thin if there exist $A \in \mathcal{I}$ such that $A \in K$.

The set of $\mathcal{I}$-thin, $\mathcal{I}$-non-thin and relatively $\mathcal{I}$-non-thin sets are denoted by $\mathcal{I}_t$, $\mathcal{I}_{nt}$ and $\mathcal{I}_{rnt}$, respectively.

The dual notion of ideal is called filter and defined as follows:

Definition 3. [19] *A family $\mathscr{F} \subseteq \mathscr{P}(S)$ is said to be filter if (i) $A \cap B \in \mathscr{F}$ for all $A, B \in \mathscr{F}$, and (ii) If $A \in \mathscr{F} \wedge A \subseteq B$, then $B \in \mathscr{F}$ hold.*

A filter $\mathscr{F}$ is called proper if $\phi \notin \mathscr{F}$. For every non-trivial ideal $\mathcal{I}$ defines a filter associated by $\mathcal{I}$ as $\mathscr{F}(\mathcal{I}) := \{A \subseteq S : S - A \in \mathcal{I}\}$ on the set S.

Remark 1. *Let $\mathcal{I}$ be an ideal and $K \subset S$ be a set. Then, $K \in \mathcal{I}_{rnt}$ if and only if there exists a set $B \in \mathscr{F}(\mathcal{I})$ such that $K \subset B$.*

Proof. It can be obtained from definition. So, it is omitted here. □

Remark 2. *If we consider $\mathcal{I} = Fin$, then $\mathcal{I}_t = \mathcal{I}$ and $\mathcal{I}_{nt} = \mathcal{I}_{rnt} = \mathcal{F}(\mathcal{I})$ holds.*

Remark 3. *If $\mathcal{I}$ is an admissible ideal, then $\mathcal{I}_{nt} \subset \mathcal{I}_{rnt}$.*

Proof. Let $\mathcal{I}$ be an admissible ideal and $A \subset \mathbb{N}$ be a an $\mathcal{I}$ non-thin subset. Hence, the set A is not finite set because of ideal $\mathcal{I}$ is admissible. From the set theory, it is well known that A contains a finite subset B which is belongs to $\mathcal{I}$. This implies that A is in $\mathcal{I}_{rnt}$. □

Lemma 1. *Let $\mathcal{I}$ be an ideal and A be a relatively $\mathcal{I}$ non-thin sub set of $\mathbb{N}$. Then, there exists a maximal set $B \in \mathcal{I}$ such that $B \subset A$ holds.*

Proof. Denote the set

$$\mathcal{A}^* = \{B \in \mathcal{I} : B \subset A\}.$$

$\mathcal{A}^*$ is partial order family with respect to inclusion. If we consider complete order sub family $\mathcal{A}$ of $\mathcal{A}^*$, then

$$\bigcup\{B : B \in \mathcal{A}\}$$

is the upper bound of $\mathcal{A}$. Then, Zorn's Lemma says that $\mathcal{A}^*$ has a maximal element. So, proof is ended. □

Thorough the paper, we are going to consider $S = \mathbb{N}$ set of natural numbers, $\mathcal{I}$ is an arbitrary ideal and (X, τ) is a topological space. Unless otherwise stated this triple X, τ and $\mathcal{I}$ will be displayed in $(X, \tau, \mathcal{I})$ format.

Definition 4. [25] *A sequence $\tilde{x} = (x_n)$ in a topological space $(X, \tau, \mathcal{I})$ is said to be $\mathcal{I}$-convergent to a point $x \in X$, if $\{n \in \mathbb{N} : x_n \notin U\} \in \mathcal{I}$ holds for any neighborhood U of x and it is denoted by $\mathcal{I} - lim x_n = x$.*

Remark 4. *If we consider $\mathcal{I}_\delta$ or $\mathcal{I}_{Fin}$, then ideal convergence is coincide with statistical or classical convergence, respectively.*

If $\mathcal{I}$ is an admissible ideal, then classical convergence implies $\mathcal{I}$-convergence. The converse statement is not true if X has at least two point, in generally. Let x and y be two different elements of X and $A \in \mathcal{I}$ be any set and consider a sequence $\tilde{x} = (x_n) \subset X$ with $x_n = x$ when $n \in A$ and $x_n = y$ when $n \notin A$. It is clear that the sequence $\tilde{x}$ is $\mathcal{I}$ convergent to y but not usual convergent.

Furthermore, the set of ideal convergent sequences and the set of convergent sequences are not comparable with respect to set inclusion for non-admissible ideal. To see this let us consider non-admissible ideal $\mathcal{I} = \mathcal{P}(2\mathbb{N})$. The real valued sequence $(x_n) = (\frac{1}{n})$ convergent to 0 in $\mathbb{R}$ with usual topology τ_e. Let $\varepsilon > 0$ be an arbitrary real number such that there exists $n_0 \in \mathbb{N}$ such that $\frac{1}{n_0} < \varepsilon < \frac{1}{n_0 - 1}$ holds. Then, following inclusion $\{1, 2, ..., n_0\} \subset \{n : |\frac{1}{n} - 0| > \varepsilon\}$ is satisfied. Since the set $\{1, 2, ..., n_0\} \notin I$, then $\{n : |\frac{1}{n} - 0| > \varepsilon\} \notin I$ holds. This implies that the sequence $(x_n) = (\frac{1}{n})$ is not $\mathcal{I}$ convergent to zero.

Similarly, if we consider a sequence (x_n) as follows:

$$x_n = \begin{cases} 0, & n = 2^{2k}, k \in K, \\ 1, & otherwise. \end{cases}$$

It is clear that this sequence ideal convergent to 1 but it is not convergent to any point in $\mathbb{R}$.

Definition 5. [1] *Let $\mathcal{I}$ be an ideal and X be a topological space. Then,*
(i) For a subset $A \subseteq X$, $\mathcal{I}$-closure of A is defined by

$$\bar{A}^{\mathcal{I}} := \{x \in X : \exists\ (x_n) \subset A : x_n \xrightarrow{\mathcal{I}} x\}.$$

(ii) A subset $F \subseteq X$ is said to be $\mathcal{I}$-closed if $\bar{F}^{\mathcal{I}} = F$ holds.
(iii) A subset $A \subseteq X$ is said to be $\mathcal{I}$-open if $X - A$ is $\mathcal{I}$-closed.

It is clear that $\bar{\phi}^{\mathcal{I}} = \phi$ and $A \subseteq \bar{A}^{\mathcal{I}}$ hold. Also, it can be easily seen that any open subset of topological space $(X, \tau, \mathcal{I})$ is also $\mathcal{I}$-open.

In the paper [22], $\mathcal{I}$-closure and $\mathcal{I}^*$-closure of a set A was defined by using $\mathcal{I}$ non-thin sequences. Let us recall it: A sequence $\tilde{x} = (x_n)_{n\in M}$ is called $\mathcal{I}$-thin if $M \in \mathcal{I}$, otherwise it is called $\mathcal{I}$-non-thin. Then, $\mathcal{I}$-closure of a set A is

$$\bar{A}^{\mathcal{I}} := \{x \in X : \exists\ (x_n)_{n\in M} \subset A : (x_n)_{n\in M} \xrightarrow{\mathcal{I}_M} x\}$$

where $\mathcal{I}_M := \{M \cap A : A \in \mathcal{I}\}$.

It is clear that $\mathcal{I}_M$ is an ideal $\mathcal{I}_M \subset \mathcal{I}$ for any subset $M \subset \mathbb{N}$.

Remark 5. *It is clear that $\mathcal{I}_M$ is an (admissible) ideal for any (admissible) ideal and $\mathcal{I}_M \subset \mathcal{I}$ holds for any subset $M \subset \mathbb{N}$.*

Theorem 1. *Let (X, τ) be a topological space, $\mathcal{I}$ be an ideal and $M \notin \mathcal{I}$. Then, $(x_n)_{n\in M} \xrightarrow{\mathcal{I}_M} x$ if and only if $(x_n)_{n\in M} \xrightarrow{\mathcal{I}} x$*

Proof. From the definitions, proof can be obtained easily. So it is omitted here. □

2. Further properties of $\mathcal{I}^*$- sequential topological space

Through the paper, we consider any ideal unless said otherwise. Let's remember the definition of $\mathcal{I}^*$-convergence of sequences for any ideal $\mathcal{I}$.

Definition 6. [13] *Let $(X, \tau, \mathcal{I})$ be a topological space. A sequence $\tilde{x} = (x_n)$ in X is said to be $\mathcal{I}^*$-convergent to a point $x \in X$ if there exist a set $M \in \mathscr{F}(\mathcal{I})$ where*

$$M = \{m_1 < m_2 < \cdots < m_k < \cdots\}$$

such that for any neighborhood U of x, there exists $N(U) \in \mathbb{N}$ such that $x_{m_k} \in U$ holds for all $m_k > N(U)$.

If X has an algebraic structure, then the Definition 6 can be reformulated in the following form as called decomposition theorem:

Theorem 2. *A sequence $\tilde{x} = (x_n)$ in $(X, \tau, \mathcal{I})$ is $\mathcal{I}^*$-convergent to $x \in X$ if and only if it can be written as $x_n = t_n + s_n$ for all $n \in \mathbb{N}$ such that $\tilde{t} = (t_n) \subset X$ is a $\mathcal{I}_{Fin}$-convergent to x and $\tilde{s} = (s_n) \subset X$ is non zero only in a set of $\mathcal{I}$.*

Proof. Assume that $x_n := t_n + s_n$ is satisfied for all $n \in \mathbb{N}$ where $t_n \to x(\mathcal{I}_{Fin})$ and (s_n) is non zero only in a set from ideal $\mathcal{I}$. Since $t_n \to x(\mathcal{I}_{Fin})$, then for any neighborhood U of x

$$\{n \in \mathbb{N} :\ t_n \notin U\} \in \mathcal{I}_{Fin}$$

holds. Let $M := \mathbb{N} - \{n \in \mathbb{N} :\ t_n \notin U\}$. Then, $s_n = 0$ for all $n \in M$. So, $x_n = t_n$ and this implies that for any neighborhood U of x $x_n \in U$ holds for all $n \in M$. Hence, $x_n \xrightarrow{\mathcal{I}^*} x$.

Conversely, let $x_n \overset{\mathcal{I}^*}{\to} x$. Then, there exists $M \in F(\mathcal{I})$ such that $(x_n)_{n\in M}$ convergent to x. Take into consider sequences $\tilde{t} = (t_n)$ and $\tilde{s} = (s_n)$ as follow

$$t_n := \begin{cases} x_n, & n \in M, \\ x, & n \notin M, \end{cases} \text{ and } s_n := \begin{cases} 0, & n \in M, \\ x_n - x, & n \notin M. \end{cases}$$

It is clear that $t_n \to x(\mathcal{I}_{Fin})$ and (s_n) is nonzero only on a set from the ideal $\mathcal{I}$ and $x_n = t_n + s_n$ holds for all $n \in \mathbb{N}$. □

In [13], it was pointed out that $\mathcal{I}^*$-convergence implies that $\mathcal{I}$-convergence. In the following example, we will show that the converse statement is not true, in generally.

Example 3. *Let $(\mathbb{R}, \tau_e)$ be an Euclidean topological space and let $B_n(0) := (-\frac{1}{2n}, \frac{1}{2n})$ for $n \in \mathbb{N}$ be a monotonically decreasing open base at zero. Define a real valued sequence $\tilde{x} = (x_n)$ such that*

$$x_n \in B_n(0) - B_{n+1}(0)$$

where $x_n = \frac{2n+1}{4n^2+4n}$. It is clear that $x_n \to 0$, $n \to \infty$.

Consider the ideal given in Example 2 and let us note that any Δ_i is a member of $\mathcal{I}$.

Let $\tilde{y} = (y_n)$ be a sequence defined by $y_n = x_j$ if $n \in \Delta j$. Let U be any open set containing zero. Choose a positive integer m such that $B_n(0) \subset U$ holds for all $n > m$. Then,

$$\{n : y_n \notin U\} \subset \Delta_1 \cup \Delta_2 \cup \Delta_3 ... \cup \Delta_m \in \mathcal{I}$$

implies that $y_n \overset{\mathcal{I}}{\to} 0$ satisfies.

Now, suppose that $y_n \overset{\mathcal{I}^*}{\to} 0$ holds. Hence, there exists a set

$$M := \mathbb{N} - H = \{m_1 < m_2 < ... < m_k ...\} \in \mathscr{F}(\mathcal{I})$$

where $H \in \mathcal{I}$ such that for any neighborhood U of zero there exists $N \in \mathbb{N}$ such that $x_{m_k} \in U$ for all $m_k > N$.

Let $l \in \mathbb{N}$ be a fixed number and assume that

$$H \subset \Delta_1 \cup \Delta_2 \cup \Delta_3 ... \cup \Delta_l$$

then $\Delta_i \subset \mathbb{N} - H$ holds for all $i > l+1$. Therefore, for each $i > l+1$, there is infinitely many k's such that $y_{m_k} = x_i$. But, the limit $lim y_{n_k}$ doesn't exists because of $x_i \neq x_j$ for all $i \neq j$.

Theorem 3. *Let (X, τ) be a topological space, and $\mathcal{I}$ be a finite ideal. Then, $\mathcal{I}$-convergence and $\mathcal{I}^*$-convergence are coincided.*

Proof. We already know that if $x_n \overset{\mathcal{I}^*}{\to} x$ then $x_n \overset{\mathcal{I}}{\to} x$ for any ideal. Let a sequence $x_n \overset{\mathcal{I}}{\to} x$, then for any neighborhood U of x, we have $A := \{n \in \mathbb{N} : x_n \notin U\} \in \mathcal{I}$.

Consider $M = \mathbb{N} - A \in \mathscr{F}(\mathcal{I})$ and arrange M as

$$M = \{m_1 < m_2 < \cdots < m_k < \cdots\}.$$

Since the set A is finite, then there exists $N \in \mathbb{N}$ such that $x_{m_k} \in U$ holds for all $m > N$. Therefore, $x_n \overset{\mathcal{I}^*}{\to} x$, hods. □

Theorem 4. *Let $(X, \tau, \mathcal{I})$ be a topological space. If every sub-sequence (x_{n_k}) of $(x_n) \subseteq X$ is $\mathcal{I}^*$- convergent to a point $x_0 \in X$, then (x_n) is $\mathcal{I}^*$- convergent to x_0.*

Proof. Let us assume that $(x_{n)}$ is not $\mathcal{I}^*$-convergent to point x_0. Then, for all $M \in \mathscr{F}(\mathcal{I})$ and for all $N \in \mathbb{N}$ there exists $n_k > N$ such that $x_{n_k} \notin U$, where U is any neighborhood of x_0. If we take $N = 1$, then there exists the sub-sequence $(x_{n_k}) \notin U$, for all $n_k > 1$. This means that there exists a sub-sequence of (x_n) which is not converging to the point of x_0 which is contradiction. □

Now, let's see with the following example that the converse of Theorem 4 is not true, in generally.

Example 4. *Let $(\mathbb{R}, \tau_e)$ be a topological space, $\mathcal{I}$ be any ideal and $K \in \mathscr{F}(\mathcal{I})$ be an arbitrary set. Define a sequence as*

$$y_n = \begin{cases} 2^n, & n \notin K, \\ \frac{1}{n}, & n \in K. \end{cases}$$

The sequence (y_n) is $\mathcal{I}^*$-convergent to zero but its subsequence $(y_{n_k}) = (2^{n_k})$ for $n_k \notin K$ is not $\mathcal{I}^*$-convergent.

Lemma 2. *Let $\mathcal{I}$ and $\mathcal{J}$ be two ideals of $\mathbb{N}$ such that $\mathcal{I} \subseteq \mathcal{J}$ and $\tilde{x} = (x_n)$ be a sequence in a topological space (X, τ). Then, $x_n \xrightarrow{\mathcal{I}^*} x$ implies $x_n \xrightarrow{\mathcal{J}^*} x$.*

Proof. Let $(x_n) \xrightarrow{\mathcal{I}^*} x$ holds. That is, there exists $M \in \mathscr{F}(\mathcal{I})$ as

$$M = \{m_1 < m_2 < \ldots, < m_k < \ldots\}$$

such that for any neighborhood U of x, there exists $N \in \mathbb{N}$ such that $x_{m_k} \in U$ holds for all $m_k > N$ holds. Since $\mathbb{N} - M \in \mathcal{I}$, then from the assumption $\mathbb{N} - M \in \mathcal{J}$ is satisfied. So, $(x_n) \xrightarrow{J^*} x$. □

It is stated in (Lemma 2 in [1]) that every subsequence of $\mathcal{I}$-convergent sequence in a topological space (X, τ) is also $\mathcal{I}$-convergent. Moreover, Example 4 shows that this statement is not true for $\mathcal{I}^*$-convergence. Because of this reason, when defining $\mathcal{I}^*$-closure of a set A, the sequence itself will be considered instead of its subsequences.

Definition 7. *Let $(X, \tau, \mathcal{I})$ be a topological space. Then,*
(i) $\mathcal{I}^*$-Closure of a set A is defined by

$$\overline{A}^{\mathcal{I}^*} := \{x \in X : \exists (x_n) \subset A \text{ such that } x_n \xrightarrow{\mathcal{I}^*} x\}$$

(ii) A subset $F \subseteq X$ is said to be $\mathcal{I}^*$-closed if $\overline{F}^{\mathcal{I}^*} = F$ holds.
(iii) A subset $U \subseteq X$ is said to be $\mathcal{I}^*$-open if $X - U$ is $\mathcal{I}^*$-closed.

Remark 6. *It is clear that $\overline{\phi}^{\mathcal{I}^*} = \phi$ and $A \subset \overline{A}^{\mathcal{I}^*}$ are true for any $A \subseteq X$.*

Theorem 5. *Let (X, τ) be a topological space and $\mathcal{I}$ is an admissible ideal. Then, every $\mathcal{I}$-open subset is $\mathcal{I}^*$- open.*

Proof. Let U be an $\mathcal{I}$-open subset of X. Then, $X - U$ is $\mathcal{I}$-closed such that $X - U = \overline{X - U}^{\mathcal{I}}$ holds. To prove $X - U = \overline{X - U}^{\mathcal{I}^*}$ it is sufficient to show that $\overline{X - U}^{\mathcal{I}^*} \subset X - U$ holds. Let $x \in \overline{X - U}^{\mathcal{I}^*}$ be an arbitrary element. Then, there exists a sequence $\tilde{x} = (x_n) \subset X - U$ such that $x_n \xrightarrow{\mathcal{I}^*} x$ holds. Therefore, Theorem 1 gives that $x_n \xrightarrow{\mathcal{I}} x$ holds. This implies that $x \in \overline{X - U}^{\mathcal{I}} = X - U$. Hence, the proof is completed. □

Corollary 1. *Let (X,τ) be a topological space and $\mathcal{I}$ be a finite ideal. Then, $A\subset X$ is $\mathcal{I}$-open if and only if A is $\mathcal{I}^*$-open.*

Theorem 6. *Let $\mathcal{I}$ and $\mathcal{J}$ be two ideal such that $\mathcal{I}\subset\mathcal{J}$ and X be a topological space. If $U\subset X$ is $\mathcal{J}^*$- open then it is $\mathcal{I}^*$-open.*

Proof. Let U be $\mathcal{J}^*$-open then $X-U$ is $\mathcal{J}^*$-closed and $X-U=\overline{X-U}^{\mathcal{J}^*}$ holds.

We must to prove $\overline{X-U}^{\mathcal{I}^*}\subset X-U$. Let $x\in\overline{X-U}^{\mathcal{I}^*}$ be an arbitrary point, then there exists a sequence $(x_n)\subset X-U$ such that (x_n) is $\mathcal{I}^*$-convergent to a point $x\in X-U$. Then by Theorem 2 the sequence (x_n), $\mathcal{J}^*$ converges to x. Hence, $x\in\overline{X-U}^{\mathcal{J}}=X-U$ this implies that $x\in X-U$ and U is $\mathcal{J}^*$-open. □

Definition 8. *Let A be a subset of topological space $(X,\tau,\mathcal{I})$. Then, $\mathcal{I}^*$ interior of A is defined as*

$$A^{o^{\mathcal{I}^*}}:=A-\overline{(X-A)}^{\mathcal{I}^*}.$$

Lemma 3. *Let A be a subset of topological space $(X,\tau,\mathcal{I})$. Then, the set A is $\mathcal{I}^*$-open if and only if $A^{o^{\mathcal{I}^*}}=A$.*

Proof. Let A be $\mathcal{I}^*$-open subset of topological space $(X,\tau,\mathcal{I})$. Then, $X-A$ is $\mathcal{I}^*$-closed and $X-A=\overline{X-A}^{\mathcal{I}^*}$ holds. This implies that

$$A^{o^{\mathcal{I}^*}}=A-\overline{(X-A)}^{\mathcal{I}^*}=A-(X-A)=A.$$

Conversely assume that $A=A^{o^{\mathcal{I}^*}}$ holds. Considering the definition, the equality $A=A-\overline{(X-A)}^{\mathcal{I}^*}$ is obtained. This implies that $A\cap\overline{(X-A)}^{\mathcal{I}^*}=\phi$ holds. Therefore, $\overline{(X-A)}^{\mathcal{I}^*}\subset X-A$. Hence, $X-A$ is $\mathcal{I}^*$- closed and the set A is $\mathcal{I}^*$-open. □

Theorem 7. *Let A be a subset of topological space $(X,\tau,\mathcal{I})$. Then, the following statements are equivalent:*

(i) A is $\mathcal{I}^*$- closed.

(ii) $A=\bigcap\{F:F \text{ is } \mathcal{I}^*-\text{ closed and } A\subset F\}$.

Proof. From the definitions it is obvious that $(i)\Rightarrow(ii)$. So, we are going to prove $(ii)\Rightarrow(i)$. To show that $\bar{A}^{\mathcal{I}^*}=A$ holds it is sufficient to prove that $\bar{A}^{\mathcal{I}^*}\subseteq A$ holds. Let $x_0\in\bar{A}^{\mathcal{I}^*}$ is an arbitrary point, then there exists a sequence $(x_n)\subset A$ such that (x_n) is $\mathcal{I}^*$-convergent to x_0. Assume that $x_0\notin A$. So, (ii) implies that

$$x_0\notin\bigcap\{F:F\ is\ \mathcal{I}^*-\text{ closed and } A\subset F\}.$$

Hence, there exists an $\mathcal{I}^*$-closed set F such that $A\subset F$ and $x_0\notin F$. Since $(x_n)\subset A\subset F$, then $x_0\in F$ which is a contradiction to assumption. □

Theorem 8. *Let A be a subset of topological space $(X,\tau,\mathcal{I})$. Then, the following statements are equivalent:*

(i) A is $\mathcal{I}^*$- open.

(ii) $A=\bigcup\{U:U \text{ is } \mathcal{I}^*-\text{ open and } U\subset A\}$.

Proof. From the definitions $(i)\Rightarrow(ii)$ is obvious. So, we are going to prove inverse of this case. Let us consider $A=\bigcup\{U:U\ is\ \mathcal{I}^*-\text{open and } U\subset A\}$. To prove A is $\mathcal{I}^*$- open subset of X, we must to show that $A=A^{o^{\mathcal{I}^*}}$ holds. It is known that $A^{o^{\mathcal{I}^*}}$ always subset of A. So, it is sufficient to show that $A\subset A^{o^{\mathcal{I}^*}}$ holds.

Let $x_0 \in A$ be an arbitrary point, then there is an open subset U of A such that $x_0 \in U$. Since $U \subset A$ then $x_0 \in A^{o^{\mathcal{I}^*}}$ and this implies that $A \subset A^{o^{\mathcal{I}^*}}$ holds. □

Definition 9. *Let $\mathcal{I}$ be an ideal and U be a subset of topological space X. A sequence $\tilde{x} = (x_n) \subset X$ is $\mathcal{I}^*$-eventually in U if there exists $M \in \mathscr{F}(\mathcal{I})$ such that $x_m \in U$ holds for all $m \in M$.*

In the following a new characterization will be given for the $\mathcal{I}^*$-open set.

Proposition 1. *Let $\mathcal{I}$ be a maximal ideal and (X, τ) be a topological space. Then, a subset $U \subseteq X$ is $\mathcal{I}^*$-open if and only if each $\mathcal{I}^*$-convergent sequence to a point $x \in U$ in X is $\mathcal{I}^*$-eventually in U.*

Proof. Let us assume that U be an $\mathcal{I}^*$ open subset of (X, τ). Consider an arbitrary sequence $\tilde{x} = (x_n) \subset X$ which is $\mathcal{I}^*$-convergent to a point $x \in U$. Since U is $\mathcal{I}^*$-open, then it is neighborhood of the point x. So, $E := \{n : x_n \notin U\} \in \mathcal{I}$ and $M(= \mathbb{N} - E) = \{n : x_n \in U\} \in \mathscr{F}(\mathcal{I})$ holds. Hence, for all $m \in M$ such that $x_m \in U$ and this implies that $\tilde{x}$ is $\mathcal{I}^*$-eventually in U.

Let us assume each $\mathcal{I}^*$-convergent sequence to a point $x_0 \in U$ is $\mathcal{I}^*$-eventually in U. That is, if $\tilde{x}$ is a sequence which is $\mathcal{I}^*$-convergent to $x_0 \in U$, then there exists $M \in \mathscr{F}(\mathcal{I})$ such that $x_m \in U$ holds for all $m \in M$. Now, we are going to show that U is $\mathcal{I}^*$-open. It is enough to prove $X - U$ is $\mathcal{I}^*$-closed. To do this we will focus the inclusion $\overline{X - U}^{\mathcal{I}^*} \subseteq (X - U)$ is satisfied. Let $x \in \overline{X - U}^{\mathcal{I}^*}$ be an arbitrary point. Then, there exists a sequence $(x_n) \subset (X - U)$ such that (x_n) is $\mathcal{I}^*$-convergent to x. Assume that $x \in U$. From the assumption there exists $M \in \mathscr{F}(\mathcal{I})$ such that $x_m \in U$ for all $m \in M$, but we have $x_n \in X - U$, for all n which is contradiction. Hence $x \in X - U$ and U is $\mathcal{I}^*$-open. □

Lemma 4. *Let $\mathcal{I}$ be an admissible ideal and (X, τ) be a topological space. If U and V are $\mathcal{I}^*$-open subsets of X, then $U \cap V$ is $\mathcal{I}^*$-open.*

Proof. Let $\tilde{x} = (x_n)$ be an $\mathcal{I}^*$-convergent sequence in X which convergent to a point $x \in U \cap V$. Since U and V are $\mathcal{I}^*$-open sets and the sequence $\tilde{x}$ is $\mathcal{I}^*$-converging to a point x in U also in V. So, by the help of Proposition 1, the sequence $\tilde{x}$ is $\mathcal{I}^*$-eventually in U and also in V. Then, there exists $M_1, M_2 \in \mathscr{F}(\mathcal{I})$ such that $x_m \in U$ for all $m \in M_1$ and $x_m \in V$ for all $m \in M_2$. If we consider $M = M_1 \cap M_2 \in \mathscr{F}(\mathcal{I})$, then $x_m \in U \cap V$ holds for all $m \in M$. This shows that $U \cap V$ is $\mathcal{I}^*$-open subset of X. □

Theorem 9. *Let $\mathcal{I}$ be a maximal ideal and (X, τ) be a topological space. A sequence $\tilde{x} = (x_n) \subset X$ is $\mathcal{I}^*$-convergent to an element $x \in X$ if and only if for any $\mathcal{I}^*$-open subset U of X with $x \in U$, there exists $M \in \mathscr{F}(\mathcal{I})$ such that $x_m \in U$, for all $m \in M$.*

Proof. Let $\mathcal{I}$ be a maximal ideal and $\tilde{x} = (x_n)$ be an $\mathcal{I}^*$-convergent sequence to $x \in X$. Let U be an $\mathcal{I}^*$-open subset of X with $x \in U$. Then, $\tilde{x}$ will be $\mathcal{I}^*$-eventually in U. Hence, there exists a set $M \in \mathscr{F}(\mathcal{I})$ such that $x_m \in U$, for all $m \in M$.

The converse statement is clear from the definition of $\mathcal{I}^*$-convergence. So, it is omitted here. □

Theorem 10. *($\mathcal{I}^*$-sequential topology) Let $(X, \tau, \mathcal{I})$ be a topological space. Then, the family*

$$\tau_{\mathcal{I}^*} := \{U \in P(X) : U \text{ is } \mathcal{I}^* - \text{open set}\}$$

is a topology on X.

Proof. It is obvious that X and ϕ are $\mathcal{I}^*$-open sets. By Lemma 4, we can say that finite intersection of $\mathcal{I}^*$-open sets is $\mathcal{I}^*$-open.

Let $(U_\alpha)_{\alpha\in\Lambda}$ be an arbitrary family of elements of $\tau_{\mathcal{I}^*}$. We are going to show that their union belongs to $\tau_{\mathcal{I}^*}$. Since

$$X - \bigcup_{\alpha\in\Lambda} U_\alpha = \bigcap_{\alpha\in\Lambda} (X - U_\alpha),$$

then it is sufficient to show that $\bigcap_{\alpha\in\Lambda}(X - U_\alpha)$ is $\mathcal{I}^*$- closed. That is,

$$\overline{\bigcap_{\alpha\in\Lambda} (X - U_\alpha)}^{\mathcal{I}^*} = \bigcap_{\alpha\in\Lambda} (X - U_\alpha).$$

Let $x \in \overline{\bigcap_{\alpha\in\Lambda}(X - U_\alpha)}^{\mathcal{I}^*}$ be an arbitrary point. Then, there exists a sequence $(x_n) \subset \bigcap_{\alpha\in\Lambda}(X - U_\alpha)$ such that $x_n \xrightarrow{\mathcal{I}^*} x$ holds. Therefore, for all $\alpha \in \Lambda$ the sequence $(x_n) \subseteq (X - U_\alpha)$ and $x_n \xrightarrow{\mathcal{I}^*} x$. Since the set $X - U_\alpha$ is $\mathcal{I}^*$- closed for all $\alpha \in \Lambda$, then $x \in X - U_\alpha$. Hence, $x \in \bigcap_{\alpha\in\Lambda}(X - U_\alpha)$ thus $\bigcap_{\alpha\in\Lambda} X - U_\alpha$ is $\mathcal{I}^*$-closed. □

Theorem 11. *If $\mathcal{I}$ is admissible ideal and the topological space (X, τ) has no limit point, then every $\mathcal{I}^*$-open set is $\mathcal{I}$-open set.*

Proof. Let U be an $\mathcal{I}^*$-open set, i.e. $X - U$ is $\mathcal{I}^*$-closed. To prove U is $\mathcal{I}$-open, it is enough to show that $X - U$ is $\mathcal{I}$-closed set. It is clear that $X - U \subseteq \overline{X - U}^{\mathcal{I}}$ holds. Let $x \in \overline{X - U}^{\mathcal{I}}$ be an arbitrary point. Then, there exists a sequence $(x_n) \subset X - U$ such that (x_n) is $\mathcal{I}$-convergent to x.

Since $\mathcal{I}$ is admissible and X has no limit point, then by [13] the sequence (x_n) will be $\mathcal{I}^*$-convergent to x. Therefore, $x \in \overline{X - U}^{\mathcal{I}^*}$. This implies that $x \in X - U$ holds. □

Corollary 2. *Under the assumption of Theorem 12, $\mathcal{I}$-sequentially and $\mathcal{I}^*$-sequentially topology are coincide.*

Definition 10. [13] *Let $\mathcal{I}$ be an ideal of $\mathbb{N}$, it is said that the ideal $\mathcal{I}$ satisfies additive property (AP) if for every countable family $(A_i)_{i\in\mathbb{N}} \subset I$, there exists a countable family $(B_i)_{i\in\mathbb{N}}$ of sets such that $A_i \triangle B_i \in F(\mathcal{I})$ for all $i \in \mathbb{N}$ and $B = \cup B_i \in I$.*

Theorem 12. *Let $\mathcal{I}$ be an admissible ideal which has the (AP)-property, and (X, τ) is first countable topological space. Then, every $\mathcal{I}^*$-open subset of X is $\mathcal{I}$-open.*

Proof. Let U be an arbitrary $\mathcal{I}^*$-open subset of X. Then, $X - U$ is $\mathcal{I}^*$-closed, so $X - U = \overline{X - U}^{\mathcal{I}^*}$ holds. To proof U is $\mathcal{I}$-open, we must show that it's complement is $\mathcal{I}$-closed. Let $x \in \overline{X - U}^{\mathcal{I}}$ be an arbitrary point. Then, there exists a sequence $x_n \subset X - U$ such that it is $\mathcal{I}$-converging to the point x. As the ideal $\mathcal{I}$ has (AP)-property and the space X is first countable by [13] it is $\mathcal{I}^*$-converging to x. So $x \in \overline{X - U}^{\mathcal{I}^*}$. So, $x \in X - U$. Hence, this fact implies that $X - U$ is $\mathcal{I}$-closed and U is $\mathcal{I}$-open. □

Corollary 3. *Under the assumption of Theorem 12, it can be say that $\mathcal{I}$-sequential and $\mathcal{I}^*$-sequential topology are coincide.*

Definition 11. *Let $(X, \tau_1, \mathcal{I})$ and $(Y, \tau_2, \mathcal{I})$ be two topological space and $f : X \to Y$ be a function. The function f is said to be (i) $\mathcal{I}^*$-continuous if $f^{-1}(U)$ is $\mathcal{I}^*$-open subset of X for every $\mathcal{I}^*$-open subset U of Y.*

(ii) sequentially $\mathcal{I}^*$-continuous if $f(x_n)$ is $\mathcal{I}^*$-convergent to $f(x)$ for each sequence (x_n) in X which (x_n) is $\mathcal{I}^*$ convergent to x.

It is well known that the definitions given above are not necessarily equivalent in classical topological spaces. In the following theorem we will show that they are equivalent notions for typologies produced with the help of ideals.

Theorem 13. *Let $(X, \tau_1, \mathcal{I})$ and $(Y, \tau_2, \mathcal{I})$ be two topological space and $f : X \to Y$ be a function. Then, f is sequentially $\mathcal{I}^*$-continuous if and only if f is $\mathcal{I}^*$-continuous function.*

Proof. Let f be a sequentially $\mathcal{I}^*$-continuous function and U be any $\mathcal{I}^*$-open set in Y. Assume that $f^{-1}(U)$ is not $\mathcal{I}^*$- open in X, equivalently $X - f^{-1}(U)$ is not $\mathcal{I}^*$- closed. We conclude from the assumption that $\overline{X - f^{-1}(U)}^{\mathcal{I}^*}$ is not subset of $X - f^{-1}(U)$. So, there exists a point $x \in \overline{X - f^{-1}(U)}^{\mathcal{I}^*}$ such that $x \notin X - f^{-1}(U)$. This means that there exists a sequence $(x_n) \subset X - f^{-1}(U)$ such that it is $\mathcal{I}^*$-converging to x and $x \in f^{-1}(U)$. Since f is sequentially continuous, then the sequence $f(x_n)$ is I^*-converging to $f(x)$. This implies that $f(x_n) \subset Y - U$ which is not in case so $f^{-1}(U)$ is $\mathcal{I}^*$-open subset of X.

Let $f : X \to Y$ be an $\mathcal{I}^*$-continuous mapping and assume that $x_n \xrightarrow{\mathcal{I}^*} x$. Then, for any neighborhood U of x, there exists $N \in \mathbb{N}$ and $M \in \mathscr{F}(\mathcal{I})$ such that $x_{m_k} \in U$ for all $m_k \in M$. Let V be any $\mathcal{I}^*$-open neighborhood of $f(x)$, then $f^{-1}(V) \subset X$ is $\mathcal{I}^*$-open and contain x. Hence, there exists $N \in \mathbb{N}$ and $M \in \mathscr{F}(\mathcal{I})$ such that $x_{m_k} \in f^{-1}(V)$. As a result of this discussion, it can easily be seen that $f(x_{m_k}) \in V$ hence $f(x_n) \xrightarrow{\mathcal{I}^*} f(x)$. □

3. Sequentially $\mathcal{I}^*$-compactness

The notion of compactness which is one of the most significant topological properties of the sets was formally introduced by M. Frechet in 1906. There are many different type of compactness introduced by mathematicians over time. Recently, using the concept of ideal the concept of $\mathcal{I}$-compactness was defined by Newcomb in [15] and studied by Rancin in the paper [17]. In this section, we will go one step further and define the concept of $\mathcal{I}^*$-sequentially compactness and examine some of its basic properties.

Let's start with the concept of boundedness in normed space which is directly related to compactness.

Definition 12. [20] *Let $(X, \|.\|)$ be a normed space and $\mathcal{I}$ be an ideal of $\mathbb{N}$. A sequence $\tilde{x} = (x_n)$ in X is called (i) $\mathcal{I}$-bounded if there exist $K > 0$ such that $\{n \in \mathbb{N} : \|x_n\| > K\} \in \mathcal{I}$ holds.*

(ii) $\mathcal{I}^*$-bounded if there exists $M \in \mathscr{F}(\mathcal{I})$ such that $(x_n)_{n \in M}$ is bounded.

Remark 7. *Let $(X, \|.\|)$ be a normed space and $\mathcal{I}$ be an ideal of $\mathbb{N}$. Then, every $\mathcal{I}$-bounded sequence is $\mathcal{I}^*$-bounded.*

Proof. Assume that $(x_n) \subset X$ is $\mathcal{I}$-bounded sequence in X. Then, there exists $K > 0$ such that $\{n : \|x_n\| > K\} \in \mathcal{I}$ holds. If we denote $M := \{m : \|x_n\| < K\}$.

Then, $M \in \mathscr{F}(\mathcal{I})$ and $\|x_n\| < K$ holds for all $n \in M$. Hence, (x_n) is $\mathcal{I}^*$- bounded sequence. □

Corollary 4. *Let X be a normed space and $\mathcal{I}$ be an ideal of $\mathbb{N}$. Then, every bounded sequence is $\mathcal{I}^*$-bounded.*

Proof. Let X be a normed space and $(x_n) \subset X$ be a bounded sequence in X. Then, the sequence (x_n) is $\mathcal{I}$-bounded which is given in [1] and by Remark 7 it is $\mathcal{I}^*$-bounded. □

Definition 13. *Let $(X, \tau, \mathcal{I})$ be a topological space. A subset $F \subset X$ is said to sequentially $\mathcal{I}^*$-compact if any sequence $(x_n) \subset F$ has an $\mathcal{I}^*$-convergent subsequence (x_{n_k}) such that $x_{n_k} \xrightarrow{I^*} x \in F$.*

Theorem 14. *Let $(X, \tau, \mathcal{I})$ be a topological space and $f : X \to \mathbb{R}$ be a sequentially $\mathcal{I}^*$-continuous function. If A is sequentially $\mathcal{I}^*$-compact subset of X, then $f(A)$ is $\mathcal{I}^*$-bounded.*

Proof. On the contrary assume that $f(A)$ is not $\mathcal{I}^*$-bounded. Then, there exists a sequence (y_n) in $f(A)$ such that it is not $\mathcal{I}^*$-bounded. That is

$$\{n \in \mathbb{N} : |y_n| < M\} \notin \mathscr{F}(\mathcal{I})$$

holds for all positive $M > 0$. Also, there exists a sequence (x_n) in A such that $f(x_n) =: y_n$ holds for all $n \in \mathbb{N}$. Since A is sequentially $\mathcal{I}^*$- compact, then there exists a subsequence (x_{n_k}) of (x_n) which is $\mathcal{I}^*$-convergent to a point x_0 in A. Moreover, f is sequentially $\mathcal{I}^*$-continuous function then $f(x_n)$ is $\mathcal{I}^*$-convergent to $f(x_0)$. So, there exists $E \in \mathscr{F}(\mathcal{I})$ where

$$E = \{m_1 < m_2 < \cdots < m_k < \cdots\}$$

such that for any neighborhood U of $f(x_0)$, there exists $N \in \mathbb{N}$ such that $f(x_{n_{m_k}}) \in U$ holds for all $m_k > N$. As a result of this analysis, it can be say that $(y_n) = f(x_{n_k})$ is $\mathcal{I}$-convergent to $f(x_0)$. Then, $\{n \in \mathbb{N} : |f(x_n)| > M\} \in \mathcal{I}$ holds for any neighborhood U of $f(x_0)$. So, we have $\{n \in \mathbb{N} : |x_n| < M\} \in \mathscr{F}(\mathcal{I})$ which is not in case so $f(A)$ is $\mathcal{I}^*$-bounded. □

Lemma 5. *Let $(X, \tau, \mathcal{I})$ and $(Y, \tau, \mathcal{I})$ be topological spaces. If X is sequentially $\mathcal{I}^*$-compact and $f : X \to Y$ is sequentially $\mathcal{I}^*$-continuous function, then $f(X)$ is sequentially $\mathcal{I}^*$-compact.*

Proof. It can be proved easily. So it is omitted. □

4. CONCLUSIONS AND SOME REMARKS

In the paper, we defined the $\mathcal{I}^*$-sequential topology on a topological space (X, τ) and proved that $\mathcal{I}^*$-sequential topology is finer then $\mathcal{I}$-sequential topology. Also, we observed that under the conditions of if the space X has no limit point and $\mathcal{I}$ be an admissible ideal then, the $\mathcal{I}$-sequentially topology and the $\mathcal{I}^*$-sequentially topology are coincide, i.e $\tau_{\mathcal{I}} = \tau_{\mathcal{I}^*}$. Also, If $\mathcal{I}$ is an admissible ideal with (AP)-property, and (X, τ) is a first countable topological space, then $\mathcal{I}$-sequentially topology and $\mathcal{I}^*$-sequentially topology are coincide, i.e $\tau_{\mathcal{I}} = \tau_{\mathcal{I}^*}$. Interestingly, it has been proven that the concepts of $\mathcal{I}^*$-continuity and $\mathcal{I}^*$-sequential continuity of a function are equivalent. As a continuation of this study, some questions can be asked:

$Q1$: Is there any topology (different from discrete topology) over X that are finer than the $\mathcal{I}^*$-sequentially topology?

$Q2$: Is there any sequential type topology between $\mathcal{I}$-sequential topology and $\mathcal{I}^*$-sequential topology on topological space X?

$Q3$: Can $\mathcal{I}$-sequential topology (or $\mathcal{I}^*$-sequential topology) be metrizable?

5. ACKNOWLEDGMENTS

The authors would like to thank the reviewers and editors of Journal of Universal Mathematics.

Funding
The authors declared that has not received any financial support for the research, authorship or publication of this study.

The Declaration of Conflict of Interest/ Common Interest
The author(s) declared that no conflict of interest or common interest

The Declaration of Ethics Committee Approval
This study does not be necessary ethical committee permission or any special permission.

The Declaration of Research and Publication Ethics
The author(s) declared that they comply with the scientific, ethical, and citation rules of Journal of Universal Mathematics in all processes of the study and that they do not make any falsification on the data collected. Besides, the author(s) declared that Journal of Universal Mathematics and its editorial board have no responsibility for any ethical violations that may be encountered and this study has not been evaluated in any academic publication environment other than Journal of Universal Mathematics.

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Journal of Universal Mathematics
Vol.8 No.1 pp.33-39 (2025)
ISSN-2618-5660
DOI: 10.33773/jum.1446824

FIXED POINT THEOREMS IN SOME FUZZY METRIC SPACES VIA INTERPOLATIVE CONTRACTIONS

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Abstract. In this article, an interpolative contraction existing in the literature is adapted to different fuzzy metric spaces. Using this contraction, a fixed point theorem in two fuzzy metric spaces is proven and an example is presented. Thus, a more general form of some concepts and theorems existing in the literature has been obtained.

1. Introduction

In our daily lives, situations that are uncertain are often faced. For each scenario encountered,determining what is "right" or "wrong" using the logic-based approach relied upon by modern computers is difficult. Many events in nature involve uncertainty, and the concept of "fuzziness" provides the flexibility needed to accurately describe such situations. This idea was introduced by Lotfi Zadeh [12], allowing phenomena that were once considered unknowable to be explained.

In recent years, various generalizations of the metric concept, which is key in fixed point theory, have been developed. One such generalization was initially introduced in [9] and later modified in [2], leading to the development of the fuzzy metric space.

Following the work of Stefan Banach [1], who laid the foundation for the fixed point theorem, adaptations of this theorem to different types of spaces have been made, contributing to research in many scientific fields. It has become a crucial tool, not only in functional analysis but also in general topology and other disciplines.

After the contributions of Grabiec [3], significant progress has been made on this theorem in the context of two spaces ([6], [10]). The type of space being studied and the contraction mapping used are the two main aspects that need to be considered.

2. Preliminaries

After defining the $t-norm$, which is considered the basic operator of fuzzy logic, some concepts to be used in this article will be presented.

Date: **Received:** 2024-07-30; **Accepted:** 2024-12-02.
Key words and phrases. Fuzzy metric space, Fixed-point, Interpolative contraction.
Mathematics Subject Classification: 03E72, 47H10, 54H25, 54E50.

Definition 1. [11] *Let* $* : [0,1] \times [0,1] \to [0,1]$ *be a binary operation that called a continuous t-norm if the conditions hold; for all* $\acute{y}, \breve{u}, \check{z}, \dot{g} \in [0,1]$ $\acute{y} * 1 = \acute{y}$ *and* $\acute{y} * \breve{u} < \check{z} * \dot{g}$, *whenever* $\acute{y} < \check{z}$ *and* $\breve{u} < \dot{g}$ *and in addition associative, commutative and continuous.*

After KM [9] and GV [2], a lot of definitions and theorems were created for fuzzy metric space (FMS). So these important discoveries attracted the attention of many writers.

Definition 2. [2] $(\hat{W}, \hat{Y}, *)$, $\hat{W}$ $(\neq \varnothing)$, *is called a FMS; provided that* $*$ *is a continuous* $t - norm$, $\hat{Y}$ *is a fuzzy set on* $\hat{W}^2 \times (0, \infty)$ *satisfying the conditions* $\forall$ $\gamma, \rho, \eta \in \hat{W}$ *and* $\acute{s}, \acute{r} > 0$;

(FM₁) $\hat{Y}(\gamma, \rho, \acute{s}) > 0$;
(FM₂) $\hat{Y}(\gamma, \rho, \acute{s}) = 1 \iff \gamma = \rho$;
(FM₃) $\hat{Y}(\gamma, \rho, \acute{s}) = \hat{Y}(\rho, \gamma, \acute{s})$;
(FM₄) $\hat{Y}(\gamma, \rho, \acute{s}) * \hat{Y}(\rho, \eta, \acute{r}) \leq \hat{Y}(\gamma, \eta, \acute{s} + \acute{r})$;
(FM₅) $\hat{Y}(\gamma, \rho, \cdot) : (0, \infty) \to [0,1]$ *is continuous*,

When (FM₄) is replaced by (NA),

$$(NA) = \hat{Y}(\gamma, \rho, \acute{s}) * \hat{Y}(\rho, \eta, \acute{r}) \leq \hat{Y}(\gamma, \eta, \max\{\acute{s}, \acute{r}\})$$

or

$$\hat{Y}(\gamma, \rho, \acute{s}) * \hat{Y}(\rho, \eta, \acute{s}) \leq \hat{Y}(\gamma, \eta, \acute{s})$$

then $(\hat{W}, \hat{Y}, *)$ *is named Non-Archimedean* (NA) *FMS* [7] .

Metrics that do not depend on "t" are called stationary fuzzy metrics. When examined from this aspect; it is clearly seen that these fuzzy metrics are the most similar to classical ones.

Definition 3. [5] $(\hat{W}, \hat{Y}, *)$, $\hat{W}$ $(\neq \varnothing)$, *is called a stationary FMS (SFMS); If* $*$ *is a continuous t-norm,* $\hat{Y}$ *is a fuzzy set on* $\hat{W}^2$ *satisfying the conditions* $\forall$ $\gamma, \rho \in \hat{W}$;

(SF_1) $\hat{Y}(\gamma, \rho) > 0$;
(SF_2) $\hat{Y}(\gamma, \rho) = 1 \iff \gamma = \rho$;
(SF_3) $\hat{Y}(\gamma, \rho) = \hat{Y}(\rho, \gamma)$;
(SF_4) $\hat{Y}(\gamma, \rho) * \hat{Y}(\rho, \eta) \leq \hat{Y}(\gamma, \eta)$.

$(\gamma_i)_{i \in \mathbb{N}}$ *in this space* $(\hat{W}, \hat{Y})$ *is Cauchy if* $\lim_{i,j \to \infty} \hat{Y}(\gamma_i, \gamma_j) = 1$;

$(\gamma_i)_{i \in \mathbb{N}} \to \gamma \in \hat{W}$ *if* $\lim_{i \to \infty} \hat{Y}(\gamma_i, \gamma) = 1$.

Now a newly fuzzy metrics defined in [4] is presented below that in the study " $\wedge_{t>0} \hat{Y}(\gamma, \rho, t) > 0$ on $\hat{W}$" were examined.

Definition 4. [4] $(\hat{W}, \hat{Y}^0, *)$, $\hat{W}$ $(\neq \varnothing)$,*is called an extended FMS (EFMS);If* $*$ *is a continuous t-norm,* $\hat{Y}^0$ *is a fuzzy set on* $\hat{W}^2 \times$ $[0, \infty)$ *satisfying the conditions* $\forall$ $\gamma, \rho, \eta \in \hat{W}$ *and* $\acute{s}, \acute{r} \geq 0$;

(EF₁) $\hat{Y}^0(\gamma, \rho, \acute{s}) > 0$;
(EF₂) $\hat{Y}^0(\gamma, \rho, \acute{s}) = 1 \iff \gamma = \rho$;
(EF₃) $\hat{Y}^0(\gamma, \rho, \acute{s}) = \hat{Y}^0(\rho, \gamma, \acute{s})$;
(EF₄) $\hat{Y}^0(\gamma, \rho, \acute{s}) * \hat{Y}^0(\rho, \eta, \acute{r}) \leq \hat{Y}^0(\gamma, \eta, \acute{s} + \acute{r})$;
(EF₅) $\hat{Y}^0_{\gamma,\rho} : [0, \infty) \to (0,1]$ *is continuous.*

Similarly replacing (EF4) by $(NA)^* = \hat{Y}^0(\gamma,\rho,\acute{s}) * \hat{Y}^0(\rho,\eta,\acute{r}) \leq \hat{Y}^0(\gamma,\eta,\max\{\acute{s},\acute{r}\})$ *or* $\hat{Y}^0(\gamma,\rho,\acute{s}) * \hat{Y}^0(\rho,\eta,\acute{s}) \leq \hat{Y}^0(\gamma,\eta,\acute{s})$ *for* $\forall\ \gamma,\rho,\eta \in \hat{W}$ *and* $\acute{s},\acute{r} \geq 0$*; then* $(\hat{W},\hat{Y}^0,*)$ *is named* $NA\ EFMS$.

Theorem 1. [4] *Let* $\hat{Y}$ *and its extension set* $\hat{Y}^0$ *be defined on* $\hat{W}^2 \times (0,\infty)$*, and* $\hat{W}^2\times\ [0,\infty)$ *respectively.*

$$\hat{Y}^0(\gamma,\rho,\acute{s}) = \hat{Y}(\gamma,\rho,\acute{s}) \text{ for all } \gamma,\rho \in \hat{W},\ \acute{s} > 0 \quad \text{and}$$
$$\hat{Y}^0(\gamma,\rho,0) = \wedge_{t>0}\hat{Y}(\gamma,\rho,\acute{s}).$$

So, $(\hat{W},\hat{Y}^0,*)$ *is an EFMS if and only if* $(\hat{W}.\hat{Y},*)$ *is a FMS satisfying* $\forall\ \gamma,\rho \in \hat{W}$ *the condition* $\wedge_{\acute{s}>0}\hat{Y}(\gamma,\rho,\acute{s}) > 0$.

Proposition 1. [4] $(\hat{W},N_{\hat{Y}},*)$ *is a SFMS on X if and only if* $\wedge_{\acute{s}>0}\hat{Y}(\gamma,\rho,\acute{s}) > 0$ $\forall\ \gamma,\rho \in \hat{W}$*. That is,*

$$\hat{Y}^0(\gamma,\rho,0) = \wedge_{\acute{s}>0}\hat{Y}(\gamma,\rho,\acute{s}) = N_{\hat{Y}}(\gamma,\rho) \tag{2.1}$$

Proposition 2. [4] $(\hat{W},\hat{Y}^0,*)$ *is complete if and only if* $(\hat{W},N_{\hat{Y}},*)$ *is complete.*

In the literature, the concepts of completeness and Caucy have been defined in various ways and used in fuzzy metric spaces ([2], [3]). One of them is adapted to EFMS in [4]. It is presented below;

Definition 5. [4] $\{\gamma_n\}$ *in* $\hat{W}$ *is named Cauchy sequence if given* $\delta \in (0,1)$*, it can be find* $n_\delta \in \mathbb{N}$ *such that* $\hat{Y}^0(\gamma_n,\gamma_m,0) > 1-\delta$ *for all* $n,m \geq n_\delta$ $.\{\gamma_n\}$ *is Cauchy sequence*$\Longleftrightarrow \lim_{m,n}\hat{Y}^0(\gamma_n,\gamma_m,0) = 1$.

Since the spaces to which every Cauchy sequence converges are complete, the same situation is valid in EFMS.

An interpolative type contraction was studied in [8] in partial metric space *(PMS)*;

Definition 6. [8] *Let* $(\hat{W},p)$ *be a PMS,* $\Im : X \longrightarrow X$ *is named an interpolative Reich-Rus-Ciric type contraction, if there exist constants* $\lambda \in [0,1)$ *and* $\alpha,\beta \in (0,1)$ *such that*

$$p(\Im\gamma,\Im\rho) \leq \lambda\,[p(\gamma,\rho)]^\beta\,[p(\gamma,\Im\rho)]^\alpha\,.\,[d(\rho,\Im\rho)]^{1-\alpha-\beta}$$

for all $\gamma,\rho \in X/Fix(\Im)$.

Theorem 2. [8] *In the framework of a PMS* $(\hat{W},p)$*, if* $\Im : \hat{W} \longrightarrow \hat{W}$ *is an interpolative Reich-Rus-Ciric type contraction, then* $\Im$ *posseses a fixed point in* $\hat{W}$.

In this article, it is intended to obtain generalized versions inspired the contraction obtained by interpolative approach and to adapt this contraction first to fuzzy metrics and then to extended ones.

3. Main Result

Definition 7. $\Omega : \hat{W} \longrightarrow \hat{W}$ *is called a fuzzy-interpolative Reich-Rus-Ciric type contraction; If* $(\hat{W},\hat{Y},*)$ *is a FMS and there exist constants* $\lambda \in [0,1)$ *and* $\alpha,\beta \in (0,1)$;

$$\left[1-\hat{Y}(\Omega\gamma,\Omega\rho,\acute{s})\right] \geq \lambda\left[1-\hat{Y}(\gamma,\rho,\acute{s})\right]^\beta\left[1-\hat{Y}(\gamma,\Omega\gamma,\acute{s})\right]^\alpha.\left[1-\hat{Y}(\rho,\Omega\rho,\acute{s})\right]^{1-\alpha-\beta} \tag{3.1}$$

for all $\gamma, \rho \in \hat{W}/Fix(\Omega)$.

Theorem 3. *Let* $(\hat{W}, \hat{Y}, *)$ *be a complete NA FMS. Provided that* $\Omega : \hat{W} \longrightarrow \hat{W}$ *is a fuzzy-interpolative Reich-Rus-Ciric type contraction, then* Ω *has a fixed point in* $\hat{W}$.

Proof. Let $\rho_0 \in \hat{W}$. $(\rho_n)_{n\in\mathbb{N}} \in \hat{W}$ is a sequence with $\rho_{n+1} = \Omega\rho_n$.

Here, by examining the cases where $\gamma_{n+1} = \gamma_n$ and $\gamma_n \neq \gamma_{n+1}$; it will be obtain that γ^* is the fixed point in the both cases.

Let be $\rho_{n+1} = \rho_n$ (for some $n \in \mathbb{N}$), $\gamma^* = \gamma_n$.

Let be $\rho_n \neq \rho_{n+1}$ ($\forall\ n \in \mathbb{N}$);

By replacing the values such as $\gamma = \rho_{n-1}$, $\rho = \rho_n$,

$$\begin{aligned}
\left[1-\hat{Y}(\Omega\rho_{n-1},\Omega\rho_n,\acute{s})\right] &\geq \lambda\left[1-\hat{Y}(\rho_{n-1},\rho_n,\acute{s})\right]^{\beta}\left[1-\hat{Y}(\rho_{n-1},\Omega\rho_{n-1},\acute{s})\right]^{\alpha}.\left[1-\hat{Y}(\rho_n,\Omega\rho_n,\acute{s})\right]^{1-\alpha-\beta}\\
\left[1-\hat{Y}(\rho_n,\Omega\rho_n,\acute{s})\right]^{\alpha+\beta} &\geq \lambda\left[1-\hat{Y}(\rho_n,\rho_{n-1},\acute{s})\right]^{\beta}.\left[1-\hat{Y}(\rho_{n-1},\Omega\rho_{n-1},\acute{s})\right]^{\alpha}\\
&= \lambda.\left[1-\hat{Y}(\rho_{n-1},\Omega\rho_{n-1},\acute{s})\right]^{\alpha+\beta}
\end{aligned}$$

and

$$\left[1-\hat{Y}(\rho_n,\Omega\rho_n,\acute{s})\right]^{\alpha+\beta} \geq \lambda.\left[1-\hat{Y}(\rho_{n-1},\Omega\rho_{n-1},\acute{s})\right]^{\alpha+\beta}$$

so $\left\{\hat{Y}(\rho_{n-1},\Omega\rho_{n-1},\acute{s})\right\}$ is non-increasing;

$$\left[1-\hat{Y}(\rho_n,\Omega\rho_n,\acute{s})\right] \geq \lambda.\left[1-\hat{Y}(\rho_{n-1},\Omega\rho_{n-1},\acute{s})\right]$$

this implies that,

$$\left[1-\hat{Y}(\rho_n,\Omega\rho_n,\acute{s})\right] \geq \lambda^n.\left[1-\hat{Y}(\rho_0,\rho_1,\acute{s})\right]$$

as $n \to \infty$,

$$\lim_{n\to\infty}\left[1-\hat{Y}(\rho_n,\Omega\rho_n,\acute{s})\right] \geq \lambda^n.\lim_{n\to\infty}\left[1-\hat{Y}(\rho_0,\rho_1,\acute{s}\right]$$

$\lambda^n \to 0$ we obtain,

$$\lim_{n\to\infty}\left[1-\hat{Y}(\rho_n,\Omega\rho_n,\acute{s})\right] = 0 \Longrightarrow \hat{Y}(\rho_n,\Omega\rho_n,\acute{s}) = 1.$$

Using Def.4 with (NA), for $n < m$;

$$\hat{Y}(\rho_n,\rho_m,\acute{s}) \geq \hat{Y}(\rho_n,\rho_{n+1},\acute{s}) * \hat{Y}(\rho_{n+1},\rho_{n+2},\acute{s}) * ... * \hat{Y}(\rho_{m-1},\rho_m,\acute{s})$$

and as $n, m \to \infty$,

$$\begin{aligned}
\lim_{n,m\to\infty}\hat{Y}(\rho_n,\rho_m,\acute{s}) &\geq \lim_{n\to\infty}\hat{Y}(\rho_n,\rho_{n+1},\acute{s}) * \lim_{n\to\infty}\hat{Y}(\rho_{n+1},\rho_{n+2},\acute{s}) * ... * \lim_{n\to\infty}\hat{Y}(\rho_{m-1},\rho_m,\acute{s})\\
&\geq 1 * 1 * ... * 1\\
&\geq 1
\end{aligned}$$

and

$$\lim_{n\to\infty}\hat{Y}(\rho_n,\rho_m,\acute{s}) = 1.$$

Because $\hat{Y}$ is complete and $\{\rho_n\}$ is a Cauchy, $\exists\ \rho^* \in \hat{Y}$: as $n \to \infty$ and $\rho_n \to \rho^*$.

Assuming $\Omega\rho^* \neq \rho^*$and implementing (3.1) with $\gamma = \rho_n$, $\rho = \rho^*$,

$$\left[1-\hat{Y}(\Omega\rho_n,\Omega\rho^*,\acute{s})\right] \geq \lambda\left[1-\hat{Y}(\rho_n,\rho^*,\acute{s})\right]^{\beta}\left[1-\hat{Y}(\rho_n,\Omega\rho_n,\acute{s})\right]^{\alpha}.\left[1-\hat{Y}(\rho^*,\Omega\rho^*,\acute{s})\right]^{1-\alpha-\beta}$$

and as $n \to \infty$,

$$\left[1-\hat{Y}(\Omega\rho^*,\Omega\rho^*,\acute{s})\right] \geq \lambda\left[1-\hat{Y}(\rho^*,\rho^*,\acute{s})\right]^{\beta}\left[1-\hat{Y}(\rho^*,\Omega\rho^*,\acute{s})\right]^{\alpha}.\left[1-\hat{Y}(\rho^*,\Omega\rho^*,\acute{s})\right]^{1-\alpha-\beta}$$

so $1-\hat{Y}(\rho^*,\Omega\rho^*,\acute{s})=0 \Longrightarrow \Omega\rho^*=\rho^*$. It is a contradiction.
That is $\Omega\rho^*=\rho^*$ and ρ^* is a fixed point of Ω. □

Definition 8. *Let $(\hat{W},\hat{Y}^0,*)$ be an EFMS. $\Omega:\hat{W}\longrightarrow\hat{W}$ is a fuzzy$-\hat{Y}^0-$interpolative Reich-Rus-Ciric type contraction, provided that (3.1) is satisfied for all $\acute{s}\geq 0$. Particularly, Ω is called $fuzzy-0-$ interpolative Reich-Rus-Ciric type contraction, provided that (3.1) is satisfied for $\acute{s}=0$.*

Theorem 4. *Let $(\hat{W},\hat{Y}^0,*)$ be a complete NA EFMS. Provided that $\Omega:\hat{W}\longrightarrow\hat{W}$ is a $fuzzy-\hat{Y}^0-$ interpolative Reich-Rus-Ciric type contraction, then Ω has a fixed point in $\hat{W}$.*

Proof. It will be examine two cases.
I. $\acute{s}>0$;
The situation where $\hat{Y}^0(\gamma,\rho,\acute{s})=\hat{Y}(\gamma,\rho,\acute{s})\ \forall\ \gamma,\rho\in\hat{W}$ is actually the same as the case proven in Theorem3.1.
II. $\acute{s}=0$;
Let $\gamma_0\in\hat{W}$. $(\gamma_n)_{n\in\mathbb{N}}\in\hat{W}$ is a sequence with $\gamma_{n+1}=\Omega\gamma_n$
Here, by examining the cases where $\gamma_{n+1}=\gamma_n$ and $\gamma_n\neq\gamma_{n+1}$, it will be obtain that γ^* is a fixed point of Ω.
Let be $\gamma_{n+1}=\gamma_n$ (for some $n\in\mathbb{N}$), $\gamma^*=\gamma_n$.
Let be $\gamma_n\neq\gamma_{n+1}$ ($\forall\ n\in\mathbb{N}$)
Using (2.1) and (3.1) with $\gamma=\rho_{n-1}$, $\rho=\rho_n$, $\acute{s}=0$

$$\left[1-\hat{Y}^0(\Omega\rho_{n-1},\Omega\rho_n,0)\right]\geq\lambda\left[1-N_{\hat{Y}}(\rho_{n-1},\rho_n)\right]^{\beta}\left[1-N_{\hat{Y}}(\rho_{n-1},\Omega\rho_{n-1})\right]^{\alpha}.\left[1-N_{\hat{Y}}(\rho_n,\Omega\rho_n)\right]^{1-\alpha-\beta}$$

and so

$$\left[1-N_{\hat{Y}}(\rho_n,\Omega\rho_n)\right]\geq\lambda.\left[1-N_{\hat{Y}}(\rho_{n-1},\Omega\rho_{n-1})\right]$$

$\left\{N_{\hat{Y}}(\rho_n,\rho_{n+1})\right\}$ is non-increasing and by iterating

$$\left[1-N_{\hat{Y}}(\rho_n,\rho_{n+1})\right]\geq\lambda^n\left[1-N_{\hat{Y}}(\rho_0,\rho_1)\right].$$

Since, as $n\to\infty$ and $\lambda^n\to 0$,

$$N_{\hat{Y}}(\rho_n,\rho_{n+1})\to 1.$$

Using (3.1) with $\gamma=\rho_n$, $\rho=\rho_m$, $\acute{s}=0$ $(n<m)$,

$$\begin{aligned}\lim_{n\to\infty}N_{\hat{Y}}(\rho_n,\rho_m) &\geq \lim_{n\to\infty}N_{\hat{Y}}(\rho_n,\rho_{n+1})*\lim_{n\to\infty}N_{\hat{Y}}(\rho_{n+1},\rho_{n+2})*...*\lim_{n\to\infty}N_{\hat{Y}}(\rho_{m-1},\rho_m)\\ &\geq 1*1*...*1=1\end{aligned}$$

it is obtained that

$$\lim_{n\to\infty}N_{\hat{Y}}(\rho_n,\rho_m)=1.$$

$\{\rho_n\}$ is a Cauchy and $\hat{W}$ is complete,then $\exists\ \rho^*\in\hat{W}$: as $n\to\infty$ and $\rho_n\to\rho^*$.
Because of Ω is continuous, $\Omega\rho_n\to\Omega\rho^*$ and by using (2.1),

$$\lim_{n\to\infty}N_{\hat{Y}}(\Omega\rho_n,\Omega\rho^*)=1.$$

the limit is unique and so $\rho^*=\Omega\rho^*$.So the proof is completed. □

Example 1. *Let $\hat{W} = \{1,2,3,4\}$ be a set, $*$ is product $t-norm$, $\hat{Y}^0$ is an EFMS on $\hat{W}$ and for $\forall\ \gamma,\rho \in \hat{W}$;*

$$\hat{Y}^0(\gamma,\rho,t) = e^{-\frac{|\gamma-\rho|}{t+1}} .$$

$(\hat{W},\hat{Y}^0,*)$ is a complete Non-Archimedean EFMS and we define a self mapping $\Omega = \binom{1\ 2\ 3\ 4}{3\ 1\ 2\ 4}$ on $\hat{W}$.

Ω is a $fuzzy-\hat{Y}^0-$ interpolative Reich-Rus-Ciric type contraction for all $\zeta,\rho \in \hat{W}$ and $\lambda = \alpha = \beta = \frac{1}{2}$ such that;

I.*for $\gamma = 1,\ \rho = 2$*

$$\begin{aligned}
1-\hat{Y}(3,1.t) &= 1-e^{-\frac{|\gamma-\rho|}{\acute{s}+1}} \\
&= 1-e^{-\frac{2}{\acute{s}+1}} \\
&= \left(1-e^{-\frac{1}{\acute{s}+1}}\right)\left(1+e^{-\frac{1}{\acute{s}+1}}\right) \\
&> \left(1-e^{-\frac{1}{\acute{s}+1}}\right) \\
&= \sqrt{1-e^{-\frac{1}{\acute{s}+1}}}\sqrt{1+e^{-\frac{1}{\acute{s}+1}}}\left(1-e^{-\frac{1}{\acute{s}+1}}\right)^0 \\
&> \sqrt{1-e^{-\frac{1}{\acute{s}+1}}}\sqrt{1-e^{-\frac{2}{\acute{s}+1}}}\left(1-e^{-\frac{1}{\acute{s}+1}}\right)^0 \\
&> \frac{1}{2}\left(1-e^{-\frac{1}{\acute{s}+1}}\right)^{\frac{1}{2}}\left(1-e^{-\frac{2}{\acute{s}+1}}\right)^{\frac{1}{2}}\left(1-e^{-\frac{1}{\acute{s}+1}}\right)^0 \\
&= \lambda\left[1-\hat{Y}(1,2,t)\right]^{\beta}\left[1-\hat{Y}(1,3,t)\right]^{\alpha}.\left[1-\hat{Y}(2,1,t)\right]^{1-\alpha-\beta}
\end{aligned}$$

Similarly, it can be shown to be true for **II**.*($\gamma = 1, \rho = 3$) and for* **III**.*($\gamma = 2$, $\rho = 3$).*

So the conditions of Theo4. are satisfied. "4" is unique fixed point of Ω.

4. Conclusion

In the literature, many contraction mappings defined in metric spaces have been adapted to fuzzy metric spaces. However, the contraction used in this study is hybrid, that is, a contraction obtained by the interpolative approach. The contraction is first transferred to a fuzzy metric space and then adapted to an extended fuzzy metric space. In this way, many contraction mappings can be redefined by the interpolative approach and transferred to different fuzzy metric spaces.

5. ACKNOWLEDGMENTS

The author would like to thank the reviewers and editors of Journal of Universal Mathematics.

Funding
The author declared that has not received any financial support for the research, authorship or publication of this study.

The Declaration of Conflict of Interest/ Common Interest
The author declared that no conflict of interest or common interest

The Declaration of Ethics Committee Approval
This study does not be necessary ethical committee permission or any special permission.

The Declaration of Research and Publication Ethics
The author declared that they comply with the scientific, ethical, and citation rules of Journal of Universal Mathematics in all processes of the study and that they do not make any falsification on the data collected. Besides, the author declared that Journal of Universal Mathematics and its editorial board have no responsibility for any ethical violations that may be encountered and this study has not been evaluated in any academic publication environment other than Journal of Universal Mathematics.

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Journal of Universal Mathematics
Vol.8 No.1 pp.40-51 (2025)
ISSN-2618-5660
DOI: 10.33773/jum.1551436

FRACTIONAL ORDER LORENZ CHAOS MODEL AND NUMERICAL APPLICATION

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Abstract. The most complex steady-state behaviour known in dynamical systems is that which is characterised as "chaos". The three-dimensional Lorenz system, which is linear and non-periodic, is a chaotic system that is used to study the properties of a two-dimensional liquid layer that is homogeneously heated from below and cooled from above. In this study, the fractional order Lorenz Chaos model is considered and mathematically analysed. This model consists of three compartments: x orbit, y orbit and z orbit. The fractional derivative is used in the sense of Caputo. The numerical results for the fractional Lorenz Chaos model are obtained with the help of the Euler method, and graphs are drawn.

1. Introduction

Chaos is science that helps to explain non-linear phenomena, defined, in its shortest definition, as the order of disorder. It is a complex process, but one with its own internal order. It is important to note that chaos is not randomness. Chaos is a unique "order" that shows complex behavior. The most complex steady-state behavior known in dynamical systems is "chaos". The study of chaos is part of the theory of nonlinear dynamical systems [1].
Chaos and chaotic signals are characterized by irregularity in the time dimension, sensitive dependence on initial conditions, an unlimited number of different periodic oscillations, a wide noise-like power spectrum, a fractal dimension of the limit set, and signals whose amplitude and frequency cannot be determined but vary in a limited area [2].
The scientific term "chaos" speaks of an interconnectedness that exists within and underlies seemingly random events. Chaos science focuses on hidden patterns of form, subtle differences, the "sensitivity" of things and the "rules" of how the unpredictable gives rise to the new. Chaos is a science that seeks to understand the movements that create the complex patterns of form, from lightning storms, foaming rivers, hurricanes, jagged mountain peaks, jagged coastlines and river deltas to the nerves and blood vessels in our bodies. Chaos is a pattern of behavior that reaches a regular state or repeats itself endlessly. In phase space, the state of all the information of a dynamic

Date: **Received:** 2024-09-17; **Accepted:** 2025-01-31.
Key words and phrases. Fractional Lorenz Chaos Model, Mathematical Modeling, Euler Method, Caputo Derivative.
Mathematics Subject Classification: 34A08, 92B05.

system at a given moment in time is reduced to a single point. This point is the dynamical system itself at that exact moment. In contrast, in the next state following this moment, the system will change, albeit very slightly, and the point will be displaced. The strange attractor occurs in phase space, one of the most important discoveries of modern science. Edward Lorenz was a pioneer of chaos science. In 1963, M.I.T. scientist E. N. Lorenz, while simulating fluid heat-radiation in the atmosphere to predict the weather, observed a new type of irregular oscillations and proposed a model. The mathematics used by Lorenz in his model of the atmosphere was widely investigated in the 1970s, and over time it became known that a fundamental property of a chaotic system is that the smallest difference in two different sets of initial conditions can lead to large differences in the state of the system [3].

The existence of chaos in various branches of engineering and other sciences such as nuclear physics, solid state physics, laser optics, chemistry, biology, medicine, ecology, astronomy, sociology, economics, international relations, history, hydraulics, atmospherics, electricity, electronics, machinery, etc., intensive studies on the subject and the developments in the field have led to the emergence of many application areas related to chaos and chaotic systems. The application areas related to chaos and chaotic systems include; chaotic parallel distributed processing, deterministic nonlinear prediction, identification and modeling of nonlinear systems, nonlinear filtering, biomedical and medical applications, dynamic information compression and coding, chaotic reliable communication, precise pattern recognition, use of chaotic dynamics for music and art, artificial generation of chaotic oscillations, realization of chaotic systems electronically, optically and optoelectronically, detection and control of chaotic vibrations and oscillations, control of lasers, turbulence control, control of crane and ship oscillations, weather forecasting [4].

For numerical modeling and simulation of a physical system with block diagrams, a mathematical model including one or more differential equations and initial conditions on the variables is required. The system can be of linear or nonlinear type. Block diagrams can be modeled and simulated with electronic circuit programs using analog operational elements. Again, the same simulation results can be obtained by setting up the real electronic circuit of the electronic circuit that is numerically modeled and simulated. The system resulting from the implementation of block diagrams as electronic circuits can also be called an "analog computer". The mathematical model of the analog computer created to model a specific physical system is identical to the mathematical model of the system [5, 6].

This paper consists of four parts. In the first part, information about Chaos science and its application area is given. In the second part, the formation of the fractional Lorenz Chaos model, mathematical analysis of the existence, uniqueness and non-negativity of the system and the Generalized Euler method are presented. In the third section, the fractional Lorenz model is applied with the Generalised Euler method and numerical results are obtained and graphs are drawn. In the fourth section, conclusions are given.

2. Fractional Derivation and Fractional Lorenz Chaos Model

The most commonly used definitions of the fractional derivative are Riemann-Liouville, Caputo, Atangana-Baleanu and the Conformable derivative. In this study, because the classical initial conditions are easily applicable and provide ease of calculation, the Caputo derivative operator was preferred and modeling was created. The definition of the Caputo fractional derivative is given below.

Definition 2.1. ([4]) Let $f(t)$ be a function. It can be continuously differentiable n times. The value of the function $f(t)$ for the value of α that satisfies the condition $n-1<\alpha<n$. The Caputo

fractional derivative of $\alpha-$th order $f(t)$ is defined by $D_t^{\alpha} f(t)=\frac{1}{\Gamma(n-\alpha)}\int_a^t (t-x)^{(n-\alpha-1)} f^n(x)dx$. These comparisons show that the Caputo fractional-order model presented is more representative of the system than its integer-ordered form. Mathematical modelling based on enhanced models naturally leads to differential equations of fractional order and to the necessity of the formulation of initial conditions to such equations. The main advantage of Caputo's approach is that the initial conditions for fractional differential equations with Caputo derivatives take on the same form as for integer-order differential equations, contain the limit values of integer-order derivatives of unknown functions at the terminal $t=\alpha$.

Definition 2.2. [4] The Riemann-Liouville (RL) fractional-order integral of a function $A(t) \in C_n$ $(n \geq -1)$ is given by

$$\mathrm{J}^{\gamma} A(t)=\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{(\gamma-1)} A(s)ds,\ J^0 A(t)=A(t). \tag{2.1}$$

Definition 2.3. [4] The series expansion of two-parametrized form of Mittag-Leffler function for $a, b > 0$ is given by

$$\mathrm{E}_{a,b}(t) = \sum_{t=0}^{\infty} \frac{t^i}{\Gamma(ai+b)}. \tag{2.2}$$

2.1. **The Fractional Lorenz Chaos Model.** The chaotic Lorenz system is the most famous chaotic system for two-dimensional fluid behavior. The chaotic Lorenz system is described by the following system of equations:

$$\begin{aligned} \frac{d^{\alpha}X}{dt^{\alpha}} &= \delta(X-Y) \\ \frac{d^{\alpha}Y}{dt^{\alpha}} &= X(\gamma-Z)-Y \\ \frac{d^{\alpha}Z}{dt^{\alpha}} &= XY-\epsilon Z. \end{aligned} \tag{2.3}$$

Here $\frac{d^{\alpha}}{dt^{\alpha}}$ is the Caputo fractional derivative of α-th order with respect. The initial values are defined as,

$$\mathrm{X}(0)=\mathrm{X}_0,\ Y(0)=Y_0,\ Z(0)=Z_0$$

$0<\alpha\leq 1$ time t.

Since fractional order models have a memory feature in time-dependent events, they produce more realistic and accurate results than integer order models. For this reason, the established model was created as fractional order. By taking α=1 in system (2.3), the differential equation of fractional order is reduced to a differential equation of full order.

Here δ, γ and ϵ are system parameters, X, Y and Z are dynamic variables. As can be seen from the equations, this chaotic system is a 3rd order system where nonlinearity is ensured by linear product terms. The system is characterized by the generation of non-periodic oscillations whose spectrum is spread over a wide frequency region. Since these oscillations resemble noise and depend on initial conditions in an unpredictable way, it has been realized that they can be used in covert communications [5-23]. Chaotic systems are characterized by "extreme sensitivity to initial conditions". If two chaotic systems of similar structure start to operate with a small difference in initial values, they will soon drift apart.

2.2. **Existence, Uniqueness and Non-Negativity of the System.** We investigate the exintence and uniqueness of the solutions of the fractional-order system (2.3) in the region $B \times [t_0, T]$ where

$$B = \{(X, Y, Z) \in R^3_+ : max\{\mid X \mid, \mid Y \mid, \mid Z \mid\} \leq \Psi, min\{\mid X \mid, \mid Y \mid, \mid Z \mid \geq \Psi_0\} \tag{2.4}$$

and $T < +\infty$.

Theorem 2.4. *For each $H_0 = (X_0, Y_0, Z_0) \in B$ there exists a unique solution $H(t) \in B$ of the fractional-order system (2.3) with intial condition H_0, which is defined for all $t \geq 0$.*

Proof: We denote $H = (X, Y, Z)$ and $\bar{H} = (\bar{X}, \bar{Y}, \bar{Z})$. Consider a mapping $M(H) = (M_1(H), M_2(H), M_3(H))$

$$\begin{aligned} M_1(H) &= \delta(X - Y) \\ M_2(H) &= X(\gamma - Z) - Y \\ M_3(H) &= XY - \epsilon Z. \end{aligned} \tag{2.5}$$

For any $H, \bar{H} \in B$ it follows from equation (2.5) that

$$\parallel M(H) - M(\bar{H}) \parallel = \mid M_1(H) - M_1(\bar{H}) \mid + \mid M_2(H) - M_2(\bar{H}) \mid + \mid M_3(H) - M_3(\bar{H}) \mid \tag{2.6}$$

$$\begin{aligned} \mid M_1(H) - M_1(\bar{H}) \mid &= \mid \delta(X - Y) - \delta(\bar{X} - \bar{Y}) \mid \\ &= \mid \delta(X - \bar{X}) - \delta(Y - \bar{Y}) \mid \\ &\leq \delta \mid X - \bar{X} \mid + \delta \mid Y - \bar{Y} \mid \end{aligned}$$

$$\begin{aligned} \mid M_2(H) - M_2(\bar{H}) \mid &= \mid X(\gamma - Z) - Y - \bar{X}(\gamma - \bar{Z}) + \bar{Y} \mid \\ &= \mid \gamma(X - \bar{X}) - (XZ - \bar{X}\bar{Z}) - (Y - \bar{Y}) \mid \\ &\leq \gamma \mid X - \bar{X} \mid + \Psi \mid X - \bar{X} \mid + \Psi \mid Z - \bar{Z} \mid + \mid Y - \bar{Y} \mid \end{aligned}$$

$$\begin{aligned} \mid M_3(H) - M_3(\bar{H}) \mid &= \mid XY - \epsilon Z - \bar{X}\bar{Y} + \epsilon \bar{Z} \mid \\ &= \mid (XY - \bar{X}\bar{Y}) - \epsilon(Z - \bar{Z}) \mid \\ &\leq \Psi \mid X - \bar{X} \mid + \Psi \mid Y - \bar{Y} \mid + \epsilon \mid Z - \bar{Z} \mid \end{aligned}$$

Then equation (2.6) becomes,

$$\parallel M(H) - M(\bar{H}) \parallel \leq \delta \mid X - \bar{X} \mid + \delta \mid Y - \bar{Y} \mid + \gamma \mid X - \bar{X} \mid + \Psi \mid X - \bar{X} \mid$$

$+\Psi \mid Z-\bar{Z}\mid + \mid Y-\bar{Y}\mid +\Psi \mid X-\bar{X}\mid +\Psi \mid Y-\bar{Y}\mid +\epsilon \mid Z-\bar{Z}\mid$

$\leq (\delta+\gamma+2\Psi) \mid X-\bar{X}\mid +(1+\delta+\Psi) \mid Y-\bar{Y}\mid +(\Psi+\epsilon) \mid Z-\bar{Z}\mid$

$\parallel M(H)-M(\bar{H}) \parallel \leq L \parallel H-\bar{H} \parallel$

where $L = max(\delta+\gamma+2\Psi, 1+\delta+\Psi, \Psi+\epsilon)$.

Therefore $M(H)$ obeys Lipschitz condition which implies the existence and uniqueness of solution of the fractional-order system (2.3).

Theorem 2.5. $\forall\ t \geq 0,\ X(0) = X_0 \geq 0,\ Y(0) = Y_0 \geq 0,\ Z(0) = Z_0 \geq 0,$ *the solutions of the system in (2.3) with initial conditions* $(X(t), Y(t), Z(t)) \in R^3_+$ *are not negative.*

Proof: (Generalized Mean Value Theorem) Let $f(x) \in C[a,b]$ and $D^\alpha f(x) \in C[a,b]$ for $0 < \alpha \leq 1$. Then we have

$$f(x) = f(\alpha) + \frac{1}{\Gamma(\alpha)} D^\alpha f(\epsilon)(x-a)^\alpha \tag{2.7}$$

with $0 \leq \epsilon \leq x$, $\forall x \in (a,b]$.
The existence and uniqueness of the solution (2.3) in $(0,\infty)$ can be obtained via Generalized Mean Value Theorem. We need to show that the domain R^3_+ is positively invariant. Since

$$\begin{aligned} D^\alpha X &= \delta(X-Y) \geq 0 \\ D^\alpha Y &= X(\gamma - Z) - Y \geq 0 \\ D^\alpha Z &= XY - \epsilon Z \geq 0 \end{aligned}$$

on each hyperplane bounding the nonnegative orthant, the vector field points into R^3_+.

2.3. **Generalized Euler Method.**

In this paper, we used the Generalized Euler method to solve the initial value problem with the Caputo fractional derivative. Many of the mathematical models consist of nonlinear systems, and finding solutions to these systems can be quite difficult. In most cases, analytical solutions cannot be found and a numerical approach should be considered for this. One of these approaches is the Generalized Euler method [15].
$D^\alpha y(t) = f(t, y(t)), y(0) = y_0, 0 < \alpha \leq 1, 0 < t < \alpha$ for the initial value problem, $h = \frac{a}{n}$ impending $[t_j, t_{j+1}]$ is divided into n sub with $j = 0, 1, .., n-1$. Suppose that $y(t), D^\alpha y(t)$ and $D^{2\alpha} y(t)$ are continuous in range $[0,a]$ and using the generalized Taylor's formula, the following equation is obtained [15].

$$y(t_1) = y(t_0) + \frac{h^\alpha}{\Gamma(\alpha+1)} f(t_0, y(t_0)).$$

This process will be repeated to create an array. Let $t_j = t_{j+1} + h$ such that

$$y(t_{j+1}) = y(t_j) + \frac{h^\alpha}{\Gamma(\alpha+1)} f(t_j, y(t_j)$$

$$
\begin{aligned}
D^{\alpha}X(t) &= \delta(X(0) - Y(0)) \\
D^{\alpha}Y(t) &= X(0)(\gamma - Z(0)) - Y(0) \\
D^{\alpha}Z(t) &= X(0)Y(0) - \epsilon Z(0).
\end{aligned} \tag{2.8}
$$

$j = 0, 1, .., n-1$ the generalized formula in the form is obtained. For each $k = 0, 1, ..., n-1$ with step size h. For $t \in [0, h)$, $\frac{t}{h} \in [0, 1)$ we have

$$
\begin{aligned}
D^{\alpha}X(t) &= \delta(X(0) - Y(0)) \\
D^{\alpha}Y(t) &= X(0)(\gamma - Z(0)) - Y(0) \\
D^{\alpha}Z(t) &= X(0)Y(0) - \epsilon Z(0).
\end{aligned} \tag{2.9}
$$

The solution of (2.9) reduces to

$$
\begin{aligned}
X(1) &= X(0) + \frac{h^{\alpha}}{\Gamma(\alpha+1)}(\delta(X(0) - Y(0))) \\
Y(1) &= Y(0) + \frac{h^{\alpha}}{\Gamma(\alpha+1)}(X(0)(\gamma - Z(0)) - Y(0)) \\
Z(1) &= Z(0) + \frac{h^{\alpha}}{\Gamma(\alpha+1)}(X(0)Y(0) - \epsilon Z(0)).
\end{aligned} \tag{2.10}
$$

For $t \in [h, 2h)$, $\frac{t}{h} \in [1, 2)$, we get

$$
\begin{aligned}
X(2) &= X(1) + \frac{h^{\alpha}}{\Gamma(\alpha+1)}(\delta(X(1) - Y(1))) \\
Y(2) &= Y(1) + \frac{h^{\alpha}}{\Gamma(\alpha+1)}(X(1)(\gamma - Z(1)) - Y(1)) \\
Z(2) &= Z(1) + \frac{h^{\alpha}}{\Gamma(\alpha+1)}(X(1)Y(1) - \epsilon Z(1)).
\end{aligned} \tag{2.11}
$$

Repeating the process n times, we obtain

$$
\begin{aligned}
X(n+1) &= X(n) + \frac{h^{\alpha}}{\Gamma(\alpha+1)}(\delta(X(n) - Y(n))) \\
Y(n+1) &= Y(n) + \frac{h^{\alpha}}{\Gamma(\alpha+1)}(X(n)(\gamma - Z(n)) - Y(n)) \\
Z(n+1) &= Z(n) + \frac{h^{\alpha}}{\Gamma(\alpha+1)}(X(n)Y(n) - \epsilon Z(n)).
\end{aligned} \tag{2.12}
$$

3. Numerical Simulation of Fractional Lorenz Chaos Model

In the chaotic Lorenz system, the weather at a given instant is represented by a point in the three-dimensional phase space and the course of the weather over time is represented by a trajectory passing through these points. This trajectory represents the history of the dynamical system.

Since chaotic systems are nonlinear, their trajectories are very complex but not random. As time progresses, trajectories begin to fill the phase space and never close over; they repeat. This kind of behavior is a sign of chaos. The set of possible weather states obtained by running the system is called the Lorenz attractor. The Lorenz attractor does not occupy any volume in three-dimensional space.

Let $X = 0,001, Y = 0,0, Z = 0,0, \gamma = 28, \delta = 10, \epsilon = \frac{8}{3}$ and let's take size of step $h = 0.1$. Hence we get the following results and tables. Using the Euler method, we obtain the following tables.

TABLE 1. The values of X, Y and Z at the moment t for $\alpha = 1$.

t	$X(t)$	$Y(t)$	$Z(t)$
0	0,001	0,00	0,00
1	0,0015	0,0028	0,00
2	0,0028	0,00252	0,00
3	0,0025	0,0101	0,0000007
4	0,0101	0,0161	0,000003
5	0,0161	0,0428	0,0000182
6	0,0428	0,0837	0,0000828
7	0,0837	0,1950	0,000419
8	0,1950	0,4100	0,0019
9	0,4100	0,9160	0,00944
10	0,9160	1,9730	0,0445
11	1,9730	4,3370	0,2130
12	4,3370	9,3870	1,0120
13	9,3870	20,1550	4,8150
14	20,1500	39,9000	22,4500

TABLE 2. The values of X, Y and Z at the moment t for $\alpha = 0.9$.

t	$X(t)$	$Y(t)$	$Z(t)$
0	0,001	0,00	0,00
1	0,0020	0,00366	0,00
2	0,00489	0,00205	-0,0000001
3	0,00117	0,0197	0,00000121
4	0,0254	0,0214	0,00000382
5	0,0202	0,1110	0,0000739
6	0,1402	0,1710	0,000344
7	0,1809	0,6620	0,00336
8	0,8110	1,2390	0,01780
9	1,3710	4,0500	0,1430
10	4,8770	8,5190	0,8200
11	9,6440	24,7500	5,9740
12	29,4280	49,3200	35,1500
13	55,4700	15,3100	212,910
14	2,9100	-132,4000	250,00

TABLE 3. The values of X, Y and Z at the moment t for $\alpha = 0.8$.

t	$X(t)$	$Y(t)$	$Z(t)$
0	0,001	0,00	0,00
1	-0,0007	0,00476	0,00
2	0,0086	0,00061	0,000000568
3	-0,00499	0,0414	0,000000582
4	0,0740	0,0106	-0,0000349
5	-0,0330	0,3610	0,000114
6	0,6390	0,1380	-0,00212
7	-0,2120	3,1620	0,0139
8	5,5300	1,6110	-0,106
9	-1,1380	27,7800	1,458
10	48,0800	17,9100	-4,585
11	-3,2480	281,500	144,110
12	481,300	297,790	-76,740
13	169,030	882,770	243,470
14	148,800	-692,170	267,184

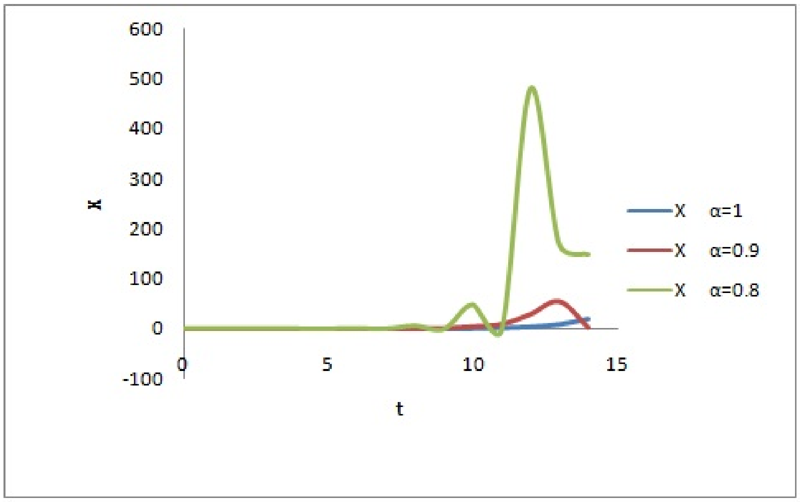

Figure 1. Graph of X phase plane change with time.

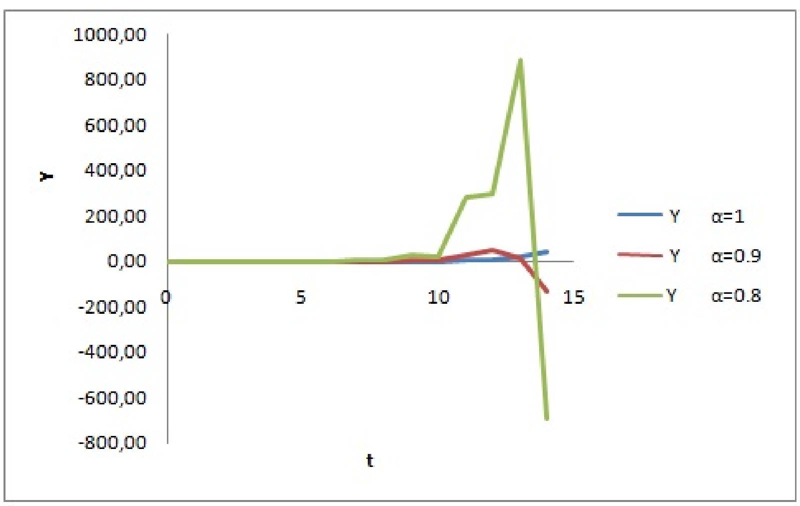

Figure 2. Graph of Y phase plane change with time.

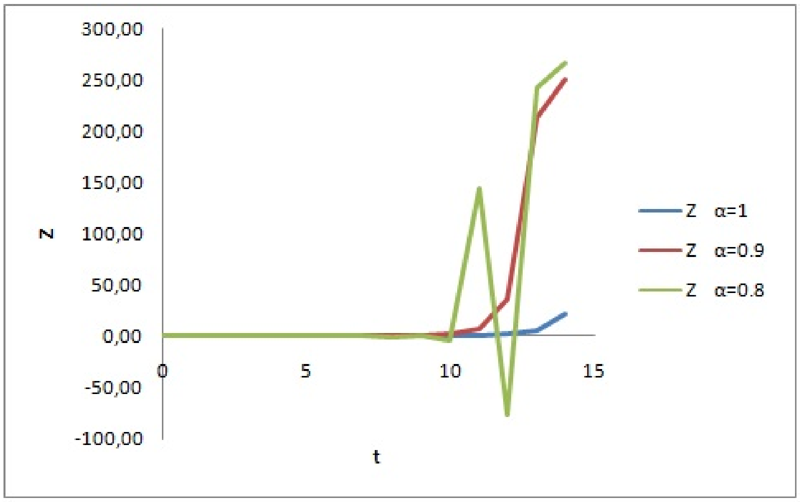

Figure 3. Graph of Z phase plane change with time.

Table 3, Table 4 and Table 5 show the changes of x, y and z for different cases of α .

4. Conclusions and Comments

Chaos-based reliable communication systems have become an alternative to the standard spread spectrum communication systems in the literature because they can spread the spectrum of information signals over a wide area, have a noise-like structure and can be realized with simple, inexpensive chaotic circuitry. In this study, the existence, uniqueness and non-negativity of the fractional order Lorenz Chaos model system were mathematically analysed. In the obtained graphs, it is observed that while the x phase plane is constant for α=1 and α=0.9, it starts to decrease after reaching a maximum value at a certain point for α=0.8. While the Y phase plane is constant for α=1 and α=0.9, it is observed that for α=0.8 it starts to decrease rapidly after taking the maximum value at a certain point. In the Z phase plane, it is observed that it progresses steadily for α=1, increases rapidly after a certain point for α=0.9, and increases rapidly after taking the minimum value at a certain point for α=0.8.

5. ACKNOWLEDGMENTS

The author would like to thank the reviewers and editors of Journal of Universal Mathematics.

Funding
The author declared that has not received any financial support for the research, authorship or publication of this study.

The Declaration of Conflict of Interest/ Common Interest
The author declared that no conflict of interest or common interest

The Declaration of Ethics Committee Approval
This study does not be necessary ethical committee permission or any special permission.

The Declaration of Research and Publication Ethics
The author declared that they comply with the scientific, ethical, and citation rules of Journal of Universal Mathematics in all processes of the study and that they do not make any falsification on the data collected. Besides, the author declared that Journal of Universal Mathematics and its editorial board have no responsibility for any ethical violations that may be encountered and this study has not been evaluated in any academic publication environment other than Journal of Universal Mathematics.

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JOURNAL OF UNIVERSAL MATHEMATICS
VOL.8 NO.1 PP.52-69 (2025)
ISSN-2618-5660
DOI: 10.33773/jum.1551816

PERFORMANCE COMPARISON OF FIXED-POINT ITERATION METHOD AND TEACHING-LEARNING BASED OPTIMISATION: A STUDY ON NONLINEAR EQUATION SYSTEMS

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ABSTRACT. This study focuses on two main objectives. Firstly, the similarities and differences between the mathematically based fixed point iteration method and the metaheuristic teaching-learning based optimisation method are presented. Secondly, the performance of these two methods in finding solutions of a complex system of linear equations is compared. In this way, other researchers will be able to make a comparison between the results previously discussed by the authors in [2] and [3], respectively, and have an idea about choosing the required optimisation method using these results in their future research.

1. INTRODUCTION

Root-finding problems are one of the most frequently encountered and critically important topics in mathematics and engineering. Finding solutions to nonlinear equations plays an important role in both theoretical studies and practical applications [4-6]. However, since analytical solutions are not possible in many cases, iterative methods come into play. These methods use an iterative process to find the roots of complex equations and are evaluated by performance criteria such as convergence rates and accuracy levels.

Traditional optimization methods usually involve mathematical modelling, using knowledge of derivatives as well as various techniques such as linear programming, integer programming, genetic algorithms. These methods seek solutions to optimize a given objective function under a set of constraints. However, these methods may not be sufficient for some problems. For example, in complex dynamic systems, the problem structure and constraints may change over time or be uncertain. Also, traditional optimization methods may be limited in terms of computational power and data processing capabilities when dealing with large datasets. Different optimization methods have been developed to overcome the limitations of traditional approaches and produce more efficient solutions [7-10]. These methods include data collection, analysis and learning processes. One of these methods is the Teaching-Learning Based Optimization (TLBO) algorithm, which uses the information obtained from past data in the teaching process to support future decisions [11].

In this paper, we investigate the performance of two different iterative methods - the Fixed-Point Iterative Method and the Teaching-Learning Optimization Algorithm (TLBO) - on Capra and Canale's (2002) system of nonlinear equations given in [1]. The Fixed-Point Iterative Method is a classical and widely used technique, based on a simple iterative process to find the root of the equation. On the other

Date: **Received**: 2024-09-17; **Accepted**: 2024-12-02.
Key words and phrases. Root finding, Fixed point iteration method, Teaching-learning algorithm, Optimization techniques.

hand, the TLBO algorithm is a more modern optimization method inspired by the teaching and learning processes in nature.

The aim of this study is to compare the performance of these two methods, to reveal their common and different aspects and to determine which method is more effective in which situations. For this purpose, various tests were performed on Capra and Canale's system of nonlinear equations and the results obtained were analysed graphically. This study provides important findings for understanding the performance of different iterative methods on nonlinear equations and sheds light on future research.

1.1 Fixed-Point Iteration Method

The fixed-point iteration method was first used by the German mathematician L.E.J. Brouwer in the early 1900s and is used in many areas of mathematics, especially in numerical analysis. This method is used to find approximate solutions of linear equations as well as approximate solutions of nonlinear systems of equations.

In this method, which is used to solve an equation of the form $f(x)=0$, let the given equation be expressed by the function $x=g(x)$. Let the point x_0 be the first estimated point and the point $x=x_0$ be chosen such that $|g'(x)|<1$. By this we mean that convergence is absolute, i.e. it always converges towards the root. In this case, with successive iteration

$$
\begin{aligned}
x_1 &= g(x_0)\\
x_2 &= g(x_1)\\
&\vdots\\
x_n &= g(x_{n-1})
\end{aligned}
$$

iterative method is obtained.

The absolute difference between the root found and the previous root gives the absolute error, $E_0 = |x_1 - x_0|$, $E_1 = |x_2 - x_1|, \dots, E_n = |x_{n+1} - x_n|$ be defined as the zeroth, first and nth absolute errors respectively. In this case, one can see the following

$$
\begin{aligned}
\frac{E_1}{E_0} &= \frac{|x_2 - x_1|}{|x_1 - x_0|} = \frac{|g(x_1) - g(x_0)|}{|x_1 - x_0|}\\
\frac{E_2}{E_1} &= \frac{|x_3 - x_2|}{|x_2 - x_1|} = \frac{|g(x_2) - g(x_1)|}{|x_2 - x_1|}\\
&\vdots\\
\frac{E_{n+1}}{E_n} &= \frac{|x_{n+2} - x_{n+1}|}{|x_{n+1} - x_n|} = \frac{|g(x_{n+1}) - g(x_n)|}{|x_{n+1} - x_n|}
\end{aligned}
$$

Therefore

$$\lim_{x_1 \to x_0} \frac{E_1}{E_0} = \lim_{x_1 \to x_0} \left| \frac{g(x_1) - g(x_0)}{x_1 - x_0} \right| = |g'(x_0)|$$
$$\lim_{x_2 \to x_1} \frac{E_2}{E_1} = \lim_{x_2 \to x_1} \left| \frac{g(x_2) - g(x_1)}{x_2 - x_1} \right| = |g'(x_1)|$$
$$\vdots \qquad \vdots \qquad \vdots$$
$$\lim_{x_{n+1} \to x_n} \frac{E_{n+1}}{E_n} = \lim_{x_{n+1} \to x_n} \left| \frac{g(x_{n+1}) - g(x_n)}{x_{n+1} - x_n} \right| = |g'(x_n)|$$

can be written. It can be seen that for a given iteration number n, if $|g'(x_n)| < 1$ while $n \to \infty$, then x_n converges to real root. In particular, the fixed-point iteration method also gives an idea that if $|g'(x_0)| < 1$ for $x_1 \to x_0$ then the initial solution can be used to reach the conclusion.

The main idea behind the choice of fixed-point iteration functions is to decompose the equation $f(x) = 0$ appropriately and replace it with two equations of the form $y_1 = g(x)$ and $y_2 = h(x)$. The generated system is solved sequentially. Here, the following equation can be written for $g(x)$ and $h(x)$, which are parts of the equation:

$$f(x) = g(x) - h(x) = 0.$$

By doing this, the number of equations to be solved is doubled, but the equations are simplified. One of them can even be directly equal to x or solved with respect to x. In the application of the method, iteration starts with an initial value that is assumed to be close to the root. The first equation is either equal to x_0 or x_1 is found by substituting x_0. In the second equation, x_2 is calculated using x_1 and this process is continued until the desired approximate root value is reached. For this, the following algorithm is applied.

Step 1. An initial value x_0 close to the root is estimated.
Step 2. The equation $f(x) = 0$ is rearranged in the form of $x = g(x)$.
Step 3. A new value for the root is calculated in the equation $x_{i+1} = g(x_i)$.
Step 4. If $\varepsilon_a = \left| \frac{x_{i+1} - x_i}{x_{i+1}} \right| .100 \le \varepsilon_s$ then stop, otherwise go to Step 3 by taking $x_i = x_{i+1}$.

In the fixed-point method, there is always the possibility of divergence as well as convergence. Convergence and divergence are shown graphically in Figure 1 [13].

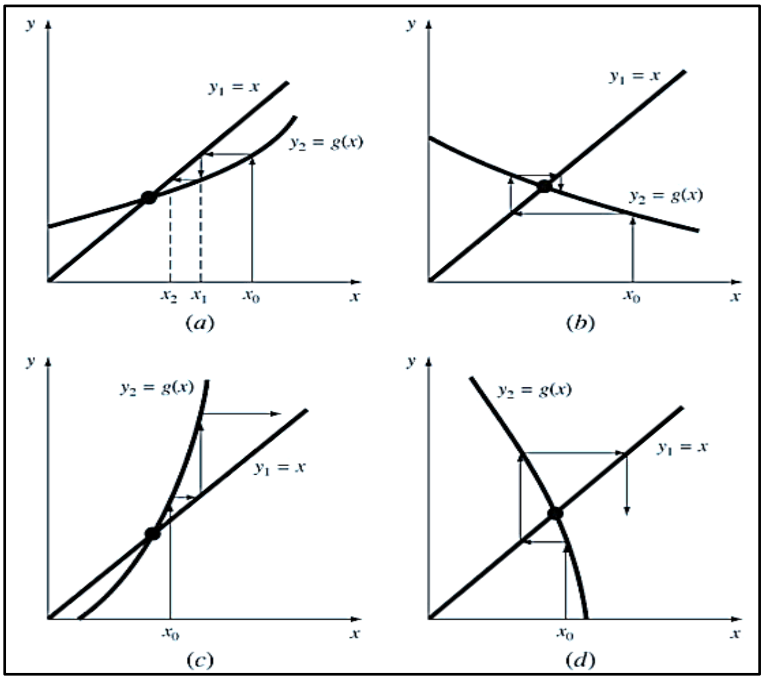

Figure 1. The fixed-point method's a) convergence case, b) convergence case c) divergence case, d) divergence case

Figure 1 (a) and Figure 1 (b) are graphical representations of the convergence of fixed-point iteration and Figure 1 (c) and Figure 1 (d) are graphical representations of the divergence of fixed-point iteration, where (a) and (b) are called monotonic graphs and (c) and (d) are called oscillating or spiral graphs. Convergence can be realized under the condition $|g'(x)| < 1$.

In Figure 1, the point x_0 is used as the initial value. In the graph (d), when moving from the point x_0 to the line $y_1 = x$ from the point where the curve $y_2 = g(x)$ is reached, it seems to be approached to the root value, but then when the iteration is continued, in other words, when trying to approach the intersection point of the curves using the newly found approximate root value, it is seen that it moves away from this point. These situations can be encountered from time to time in the constant iteration method. Similarly, in graph (b), each iteration gets closer and closer to the root and the error value decreases with each step.

Note that convergence occurs when the absolute value of the slope of the function $y_2 = g(x)$ is smaller than the slope of the function $y_1 = x$. If convergence occurs, the error at each step is the same or smaller than the error at the previous step. Therefore, fixed point iteration has linear convergence.

1.2 Teaching-Learning Based Optimization Algorithm (TLBO)

Learning and teaching based optimization (TLBO) can be defined as an approach that combines learning and teaching components related to the optimization problem. In the learning phase, TLBO analyses data relevant to problem solving and extracts knowledge and patterns by learning from this data. The learning process can converge to the optimal solution by using strategies, constraints or other factors to solve the problem with necessary updates to the teacher's experience and results. The learning process often involves statistical analysis, machine learning or artificial intelligence techniques.

A teaching-learning based optimization method is considered by Rao et al. [12]. As the solution population, the operations take place with classes and students as its members. The aim is to increase the knowledge level of the students in the class in order to obtain the optimum solution. Basically, it is realized in two phases such as teaching and learning. It is represented as a matrix representing the classes and the students in the classes. Each row in the matrix corresponds to a student. The rows represent the design proposal. The analysis starts with the random assignment of sections from a pre-prepared list of profiles [12].

Learning Phase. The student who gives the best solution in the class is considered as the teacher. Accordingly, the other students are updated according to the following relationships by utilizing the teacher's knowledge. If the updated student gives a better solution than the old one, he/she replaces the old student.

Teaching Phase. The process in this phase is very similar to the previous phase. There is interaction between the students in the class. There is a process of transferring knowledge from one student with a better solution and a higher level of knowledge to another student. If the new student finds a better solution than the current student, he/she will take his/her place.

With teaching and learning based optimization, if the teaching and learning steps are considered as the interaction between teachers and students in a classroom, first the population (class size) dimensions to be evaluated are determined. Then the objective function is determined. In line with the determined objective function, the best individual (x) in the population is assigned as a teacher. The mean of the population (class) is calculated. Interaction between teacher and student is ensured. At this stage, a teacher tries to transfer information between students and increase the average result of the class. In the next stage, students try to increase their knowledge level through interaction among themselves. Students can also gain knowledge by discussing and interacting with other students. A student standing in the center of the class can communicate with those in the next row and across. The interaction will be provided in such a way that a student will learn new information if the other student has more information about him/her.

$$x_{new} = x + r.(x_{best} - T_f.x_{arithmetic-mean}) \quad (1)$$

In Equation (1), T_f is a constant that takes the value 1 or 2. *r* represents a random number in the closed interval [0,1]. x_{new} is the new student, *x* is the best student from the previous iteration, x_{best} is the best student and $x_{arithmetic-mean}$ is the arithmetic mean of the population. With the formula given in Equation (1), the knowledge level of the population (students) is determined after the interaction between the population individuals. The best individual is then selected as the teacher. The cycle continues until the determined learning level is achieved. Learning and teaching based optimization (TLBO) is defined as an algorithm that can model the effect of learning on students in the classroom [11].

1.3. Comparison of Common and Differences between Fixed Point Iteration Method and TLBO Algorithm

While the fixed-point iteration method and the TLBO algorithm differ in their applications and specific methodologies, they undoubtedly share the following common features, especially in the context of iterative and optimisation processes.

Fixed point iteration and TLBO algorithms aim to solve different types of problems with iterative approaches. Starting with an initial guess, FPI iteratively applies a function and converges to a fixed point satisfying the condition $f(x)=x$. In contrast, TLBO is based on improving a population of solutions. TLBO consists of two phases: teaching and learning: In the teaching phase the best solution guides the process, while in the learning phase the solutions are improved by learning from each other.

Both algorithms have different convergence goals and dependencies. While fixed point iteration method focuses on fixed point finding problems, TLBO is designed to solve direct optimisation problems. In fixed point iteration method, the initial guess affects both the convergence speed and the final solution, while in TLBO the quality of the initial population determines the performance of the algorithm and the quality of the solution obtained.

The stopping criteria also differ between the two methods. Fixed point iteration stops when the difference between consecutive iterations falls below a certain threshold. TLBO usually stops when it reaches a certain number of iterations, when convergence reaches a threshold, or when the improvement rate becomes negligible.

Fixed point iteration method focuses on a single solution point by providing a mathematical approach. TLBO is a heuristic metaheuristic that iteratively evolves a population of solutions. While fixed point iteration method is mostly used in areas such as numerical analysis, equation solving and mathematical modelling, TLBO has a wide range of applications in engineering, economics and scientific optimisation problems.

As a result, fixed point iteration method has a simpler and mathematical structure, while TLBO is a complex and powerful optimisation technique inspired by the teaching-learning process. The nature, objectives and application areas of the two determine their suitability for different types of problems.

1.4. Optimization Approach for Finding the Roots

When the optimization process is used to find the roots of algebraic equations, the problem of finding the unknown values in each equation becomes an optimization problem to be solved by numerical methods. Optimization is the process of obtaining the best value of an objective function according to specified criteria. Since the numerical approach for finding roots in algebraic equations usually involves an iterative process, similarly, in finding roots with an optimization algorithm, starting from a given starting point, candidate root values are iteratively updated and reach a minimum or maximum value when the objective function is sufficiently close or a certain tolerance value is reached.

In this section, Theorem 1.4.2 is used as a generalization of Theorem 1.4.1 for equations in one variable for finding roots in algebraic equations.

Theorem 1.4.1. (Root Search in Optimization Algorithm)
For I=[a,b] and $I \subset \mathbb{R}$, if the function $f: I \to \mathbb{R}$ is continuous, then it has at least one minima on this interval and if $\left|f\left(x_i\right)\right| = 0$ then there exists at least one $x_i \in I$, $\left(i \in \mathbb{N}\right)$ satisfying this equality (Köse et al., [2]).

Theorem 1.4.2. (Root Finding Algorithm for Nonlinear Equation Systems)

Let I=[a,b] and $I \subset □$, If the functions $f_i : I^n \to □$ are continuous, then for each $1 \le i \le n$ the functions f_i have at least one minimum value in this interval and have at least one point $x = (x_1, x_2, ...x_n) \in I$ that satisfies the equality $\sum_{i=1}^{n} \left| f_i(x_i) \right| = 0$ (Köse et al., [2]).

1.5. Numerical Example

In [1], Canale and Capra considered a system of equations consisting of functions of two variables $f_1(x,y)$ and $f_2(x,y)$

$$\begin{aligned} f_1(x,y) &= x^2 + xy - 10 = 0 \\ f_2(x,y) &= y + 3xy^2 - 57 = 0 \end{aligned} \qquad (2)$$

Since the real roots of this system of equations are x = 2 and y = 3, he used the fixed-point iteration method and the Newton-Raphson method to solve the system of equations, starting with initial guesses x = 1.5 and y = 3.5.

In this study, the same problem will be addressed using a mathematics-based fixed-point iteration method and a meta-heuristic, the teaching-learning algorithm. Throughout the paper, $f_1(x,y)$ and $f_2(x,y)$ will be replaced by f_1 and f_2 , respectively, in the equation system given by (2).

2. APPLICATION OF METHODS AND ALGORITHMS

In Section 1.5, the success of the approximate solution of the equation system given by (2), which consists of nonlinear equations in two variables, will be measured first by the fixed-point iteration method and then by teaching-learning algorithm.

2.1. Fixed Point Iteration Method Application

Fixed point iteration functions in two variables associated with the functions f_1 and f_2, will be considered

$$\begin{aligned} &g_1(x,y) = \frac{10}{x+y}, g_2(x,y) = \frac{57}{1+3xy}, g_3(x,y) = \frac{10}{x} - x, \\ &g_4(x,y) = \frac{57-y}{3y^2}, g_5 = \sqrt{10-xy}, g_6(x,y) = \sqrt{\frac{57-y}{3x}} \end{aligned} \qquad (3)$$

These functions will be denoted as g_1, g_2, g_3, g_4, g_5, g_6 for short. In this study, we have created three different sets of iteration functions for the functions f_1 and f_2. The fixed point iteration functions related to the function f_1 are g_1, g_3, g_5 and fixed point iteration functions related to the function f_2 are g_2, g_4, g_6 . The iteration steps will be performed by taking $g_1 = x$, $g_2 = y$, $g_3 = y$, $g_4 = x$, g_5=x, $g_6 = y$ and by choosing initial conditions as $x_0 = 1{,}5$ and $y_0 = 3{,}5$. The calculations were performed for all three iteration function sets by taking the maximum number of iterations as 50 and the tolerance value as 0.01 in the MATLAB program.

Capra and Canale also discussed iteration function sets in their book as in the following forms [1].

$$g_3^*(x,y)=\frac{10-x^2}{y}=x,\ \ g_4^*(x,y)=57-3xy^2=y \qquad (4)$$

$$g_5(x,y)=\sqrt{10-xy}=x,\ \ g_6(x,y)=\sqrt{\frac{57-y}{3x}}=y \qquad (5)$$

In addition to the iteration sets considered by Capra and Canale, it can be seen that the iteration sets g_1 and g_2 given in (3) are also considered in this study. The iteration set given by (4) considered by Capra and Canale is in the form $g_3^*(x,y)=x, g_4^*(x,y)=y$, but in this work, unlike the previous one, $g_3=y,\ g_4=x$ is taken. These iteration steps can be practical and fast, depending on the experience of the mathematician solving the system in the normal method. But when we ask the Artificial Intelligent (AI) to generate these functions, it immediately suggests the convergent iteration function from Capra and Canale's book as the iteration function. But it does not suggest that there may be other functions and how they can be selected when a problem arises. We form the equation in mathematical theory about this. When we take the first derivative of the iteration function and set the initial condition in the first derivative, we claim that it can converge if the result is less than 1.

This example illustrates the most serious shortcoming of fixed-point iteration, namely that convergence often depends on the way the equations are formulated. Moreover, even in cases where convergence is possible, divergence can occur if the initial guesses are not close enough to the true solution. Using simple reasoning, it can be seen that sufficient conditions for convergence are of the form

$$\left|\frac{\partial f_1}{\partial x}\right|+\left|\frac{\partial f_1}{\partial y}\right|<1$$

and

$$\left|\frac{\partial f_2}{\partial x}\right|+\left|\frac{\partial f_2}{\partial y}\right|<1$$

for the case with two equations. These criteria are so restrictive that fixed point iteration can be considered of limited utility in solving nonlinear systems. However, it can be seen that the contribution of this method is greater when solving linear systems.

For each iteration function set considered in this study, the fact that $\left|g_i'(x_0)\right|=\left|\frac{\partial g_i}{\partial x}(x_0)\right|+\left|\frac{\partial g_i}{\partial y}(x_0)\right|<1$ for $1\le i\le 6$ also gives an idea about the result under the initial condition x_0.

Using the iteration function set that satisfies this condition is more appropriate to ensure convergence, otherwise a divergence from the true solution will occur. Let us now give the implementation steps of both algorithms below.

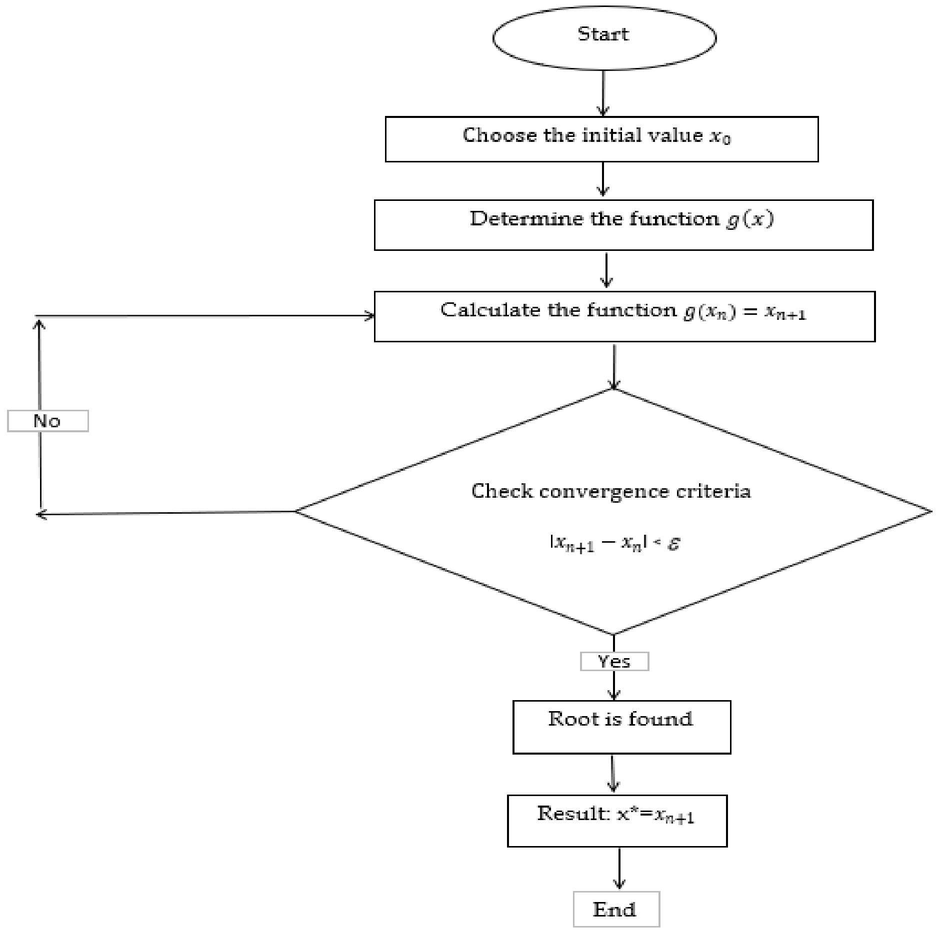

Figure 2. Basic flow diagram of fixed-point iteration method

2.2. Teaching-Learning Based Optimization Algorithm Application

In order to solve the system of equations given by (2), TLBO method is used by taking the number of populations 40, the number of variables in the population 2, the upper bound [10, 10] and the lower bound [-8, -8].

In each iteration, the best result x and y result and its value in the function are shown.

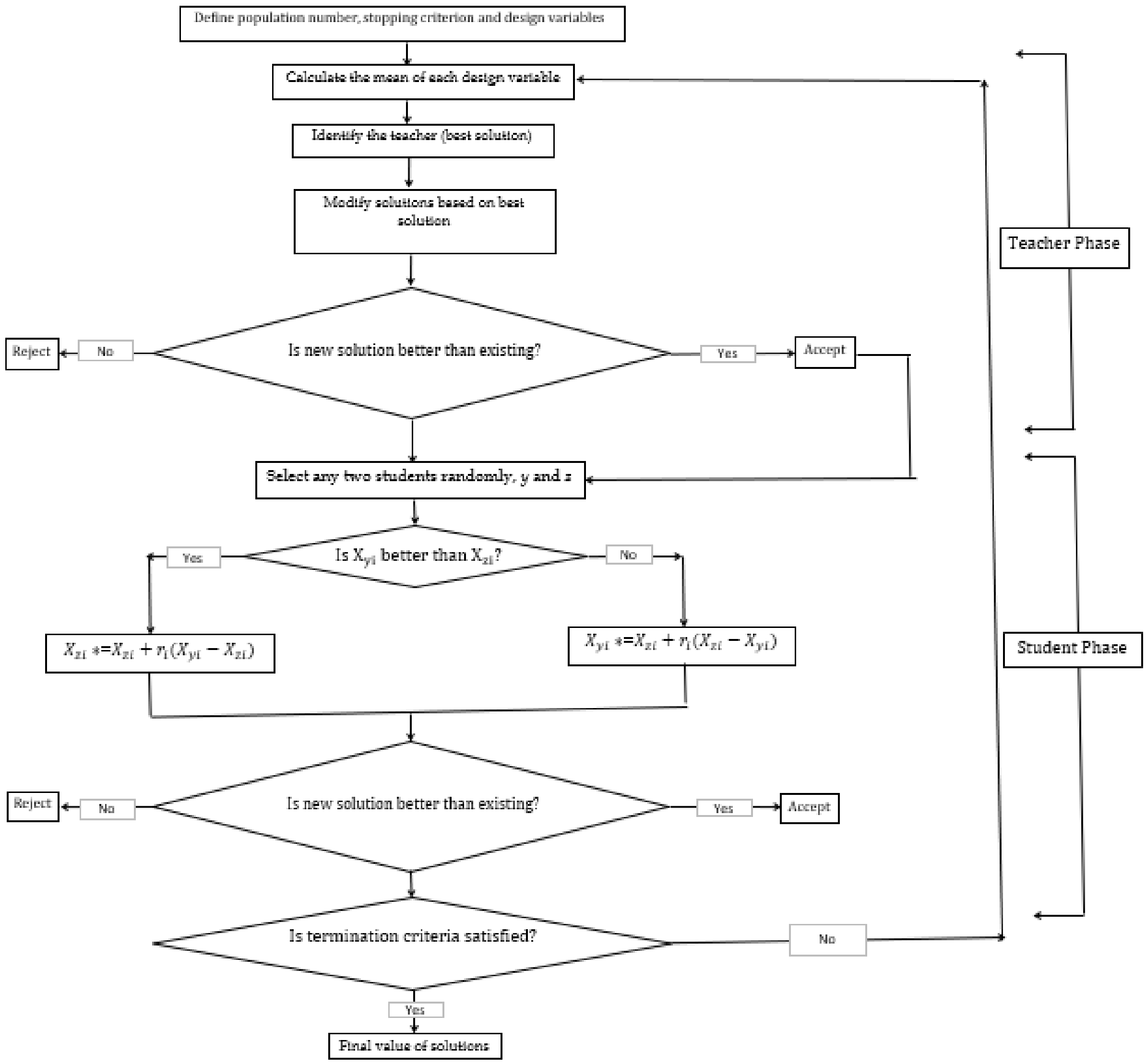

Figure 3. Flow diagram of learning-teaching based optimization algorithm

3. PERFORMANCE RESULTS

In this section, firstly, the convergence performance results of the system of equations given by (2) on three different iteration function sets obtained by the fixed-point iteration method are compared. Then, the performance results of teaching-learning based optimization algorithms for finding approximate solutions of the same equation system are obtained.

The following table, shows the convergence tables for $g_1, g_2, g_3, g_4, g_5, g_6$ fixed point iteration function sets, where x and y are solutions, the error of x is E_x and the error of y is E_y .

	Data for function set g_1, g_2				Data for function set g_3, g_4				Data for function set g_5, g_6			
Number of iterations	x	y	E_x	E_y	x	y	E_x	E_y	x	y	E_x	E_y

1	2,0000	3,4030	0,5000	0,0970	5,1667	1,4558	3,6667	2,0442	2.1794	3.4480	0,6794	0,0520
2	1,8508	2,6613	0,1492	0,7417	-3,2312	8,7362	8,3978	7,2805	1.5764	2.8619	0,6030	0,5861
3	2,2162	3,6129	0,3654	0,9515	0,1363	0.2108	3,3675	8,5255	2.3427	3.3834	0,7663	0,5215
4	1,7155	2,2781	0,5007	1,3348	73,2092	426,0335	73,0729	425,8227	1.4400	2.7620	0,9027	0,6213
5	2,5040	4,4796	0,7885	2,2015	-73,0726	-0,0007	146,2819	426,0342	2.4541	3.5433	1,0141	0,7813
6	1,4320	1,6450	1,0721	2,8346	72,9358	4,1366*10^7	146,0084	4,1366*10^7	1.1421	2.6946	1,3120	0,8487
7	3,2500	7,0662	1,8180	5,4212	-72,987	0	145,7345	4,1366*10^7	2.6311	3.9811	1,4889	1,2865
8	0,9693	0,8155	2,2806	6,2506	72,6613	2,9261*10^17	145,4600	2,9261*10^17	0	2.5917	2,6311	1,3894
9	5,6027	16,9061	4,6333	16,0906	-72,5237	-0,0001	145,1850	2,9261*10^17	3.1747	3.6281	3,2486	1,0364
10	0,4443	0,1999	5,1584	16,7062	72,3858	1,4641*10^37	144,9095	1,4641*10^37	2.4546	2.3581	0,7731	3,8440
11	15524	45,0089	15,0798	44,8090	-72,24767	0,0000	144,6335	1,4641*10^37	2.5174	2.0847	2,5549	0,3299
12	0,1652	0,0272	15,3589	44,9817	72,1092	3,6655*10^76	144,3569	3,6655*10^76	2.6063	2.4692	1,2893	0,9714
13	51981	56,2424	51,8156	56,2152	-71,9706	0,0000	144,0798	3,6655*10^76	2.2209	2.5194	1,0208	0,6176
14	0,0924	0,0065	51,8884	56,2359	71,8316	2,2975*10^155	143,8022	2,2975*10^155	2.3038	2.7040	0,8905	0,4933
15	101,1123	56,8975	101,0199	56,8910	-71,6924	0,0000	143,5240	2,2975*10^155	2.1611	2.7086	0,7196	0,5316
16	0,0632	0,0033	101,0490	56,8942	71,5529	Inf	143,2453	Inf	2.2072	2.7792	0,7261	0,4248
17	150,1736	56,9643	150,1103	56,9610	-71,4132	NaN	142,9661	NaN	2.1454	2.7723	0,6526	0,4646
18	0,0483	0,0022	150,1253	56,9621	71,2731	NaN	142,6863	NaN	2.1712	2.8047	0,6603	0,4129
19	198,0277	56,9817	197,9794	56,9795	-71,1328	NaN	142,4060	NaN	2.1392	2.7985	0,6254	0,4325
20	0,0392	0,0017	197,9885	56,9800	70,9922	NaN	142,1251	NaN	2.1543	2.8160	0,6291	0,4063
21	244,5107	56,9887	244,4715	56,9870	-70,8514	NaN	141,8436	NaN	2.1359	2.8117	0,6103	0,4162
22	0,0331	0,0014	244,4776	56,9873	70,7102	NaN	141,5616	NaN	2.1451	2.8221	0,6121	0,4016
23	289,5943	56,9923	289,5612	56,9909	-70,5688	NaN	141,2791	NaN	2.1338	2.8193	0,6011	0,4070
24	0,0288	0,0012	289,5655	56,9911	70,4271	NaN	140,9960	NaN	2.1396	2.8257	0,6020	0,3983
25	333,2890	56,9943	333,2602	56,9932	-70,2851	NaN	140,7123	NaN	2.1326	2.8239	0,5952	0,4015
26	0,0256	0,0010	333,2634	56,9933	70,14286	NaN	140,4280	NaN	2.1363	2.8280	0,5958	0,3961
27	375,6203	56,9956	375,5948	56,9946	-70,0003	NaN	140,1431	NaN	2.1318	2.8268	0,5915	0,3981
28	0,0231	0,0009	375,5973	56,9947	69,8574	NaN	139,8577	NaN	2.1341	2.8294	0,5918	0,3947
29	416,6204	56,9965	416,5973	56,9956	-69,7143	NaN	139,5717	NaN	2.1312	2.8286	0,5891	0,3960
30	0,0211	0,0008	416,5993	56,9957	69,5708	NaN	139,2851	NaN	2.1328	2.8303	0,5893	0,3937
31	456,3244	56,9971	456,3032	56,9963	-69,4271	NaN	138,9979	NaN	2.1309	2.8298	0,5875	0,3946
32	0,0195	0,0007	456,3049	56,9964	69,2831	NaN	138,7102	NaN	2.1319	2.8309	0,5876	0,3931
33	494,7685	56,9976	494,7490	56,9968	-69,1387	NaN	138,4218	NaN	2.1306	2.8306	0,5865	0,3937
34	0,0181	0,0007	494,7504	56,9969	68,9941	NaN	138,1328	NaN	2.1313	2.8313	0,5866	0,3927
35	531,9896	56,9979	531,9715	56,9972	-68,8492	NaN	137,8433	NaN	2.1305	2.8311	0,5858	0,3931
36	0,0170	0,0006	531,9727	56,9973	68,7039	NaN	137,5531	NaN	2.1309	2.8316	0,5859	0,3924
37	568,0242	56,9982	568,0073	56,9976	-68,5584	NaN	137,2623	NaN	2.1304	2.8314	0,5854	0,3927
38	0,0160	0,0006	568,0082	56,9976	68,4125	NaN	136,9709	NaN	2.1307	2.8317	0,5854	0,3923
39	602,9084	56,9984	602,8924	56,9978	-68,2663	NaN	136,6788	NaN	2.1303	2.8316	0,5851	0,3924
40	0,0152	0,0006	602,8932	56,9978	68,1198	NaN	136,3862	NaN	2.1305	2.8318	0,5851	0,3922
41	636,6774	56,9986	636,6623	56,9980	-67,9730	NaN	136,0929	NaN	2.1303	2.8318	0,5849	0,3923
42	0,0144	0,0005	636,6630	56,9980	67,8259	NaN	135,7990	NaN	2.1304	2.8319	0,5849	0,3921
43	669,3659	56,9987	669,3515	56,9982	-67,6785	NaN	135,5044	NaN	2.1303	2.8319	0,5848	0,3921
44	0,0138	0,0005	669,3521	56,9982	67,5307	NaN	135,2092	NaN	2.1303	2.8320	0,5848	0,3920
45	701,0076	56,9988	700,9938	56,9983	-67,3826	NaN	134,9134	NaN	2.1302	2.8319	0,5847	0,3921
46	0,0132	0,0005	700,9944	56,9984	67,2342	NaN	134,6169	NaN	2.1303	2.8320	0,5847	0,3920
47	731,6352	56,9989	731,6221	56,9985	-67,0855	NaN	134,3198	NaN	2.1302	2.8320	0,5846	0,3920
48	0,0127	0,0005	731,6226	56,9985	66,9364	NaN	134,0220	NaN	2.1303	2.8320	0,5846	0,3920
49	761,2809	56,9990	761,2682	56,9986	-66,7870	NaN	133,7235	NaN	2.1302	2.8320	0,5846	0,3920
50	0,0122	0,0004	761,2687	56,9986	66,6373	NaN	133,4244	NaN	2.1303	2.8320	0,5846	0,3920

Table 1. Convergence table for function sets g_1, g_2, g_3, g_4, g_5, g_6

It is seen from Table 1 that although the fixed-point iteration function set is close to the true root in the first iteration, it is observed that the roots and oscillate and do not converge to the true root as the number of iterations increases. It is also observed that the errors E_x and E_y increase continuously with the number of iterations.

It is also seen that the roots x and y in the fixed-point iteration function set are quite far from the true root, i.e. diverging. Even at the 16th iteration the error E_y goes to infinity.

For the fixed-point iteration functions g_5, g_6 it is seen that it converges to the true root with $x = 2{,}1302$ and $y = 2{,}8320$ values at the 47th iteration. When the number of iterations is further increased in this step, it can be seen that it will get closer to the true root.

Now let us present the performance results obtained with the TLBO algorithm in the table below.

Iteration Number	**Best Result** *x*	**Best Result** *y*	$f_{\min} = \lvert f_1 \rvert + \lvert f_2 \rvert$	**Iteration Number**	**Best Result** *x*	**Best Result** *y*	$f_{\min} = \lvert f_1 \rvert + \lvert f_2 \rvert$
1	5,5081	-1,8522	12,3035	**26**	1,9908	3,0072	0,0657
2	3,0623	2,4998	9,9395	**27**	1,9908	3,0072	0,0657
3	0,8620	4,4977	5,5669	**28**	1,9908	3,0072	0,0657
4	1,6075	3,2514	4,9562	**29**	1,9908	3,0072	0,0657
5	1,6075	3,2514	4,9562	**30**	1,9908	3,0072	0,0657
6	2,2156	2,7944	3,4030	**31**	1,9908	3,0072	0,0657
7	2,2156	2,7944	3,4030	**32**	1,9908	3,0072	0,0657
8	1,9962	3,0389	1,3935	**33**	1,9908	3,0072	0,0657
9	1,9962	3,0389	1,3935	**34**	1,9908	3,0072	0,0657
10	1,9978	2,9936	0,3242	**35**	1,9908	3,0072	0,0657
11	1,9978	2,9936	0,3242	**36**	1,9908	3,0072	0,0657
12	2,0144	2,9876	0,1481	**37**	1,9908	3,0072	0,0657
13	2,0144	2,9876	0,1481	**38**	2,0042	2,9963	0,0458
14	2,0144	2,9876	0,1481	**39**	2,0036	2,9975	0,0268
15	2,0144	2,9876	0,1481	**40**	2,0036	2,9975	0,0268
16	2,0144	2,9876	0,1481	**41**	2,0036	2,9975	0,0268
17	2,0144	2,9876	0,1481	**42**	2,0026	2,9981	0,0150
18	2,0144	2,9876	0,1481	**43**	1,9982	3,0013	0,0109
19	2,0144	2,9876	0,1481	**44**	1,9982	3,0013	0,0109
20	2,0144	2,9876	0,1481	**45**	1,9982	3,0013	0,0109
21	2,0144	2,9876	0,1481	**46**	1,9982	3,0013	0,0109
22	2,0142	2,9882	0,1305	**47**	1,9982	3,0013	0,0109
23	2,0142	2,9882	0,1305	**48**	1,9982	3,0013	0,0109
24	2,0142	2,9882	0,1305	**49**	1,9995	3,0005	0,0072
25	2,0142	2,9882	0,1305	**50**	1,9995	3,0005	0,0072

Table 2. Convergence data with TLBO algorithm

It can be seen from the table above that the best solution with the TLBO algorithm is found as $f_{\min} = 0{,}0072$ at the 49th iteration with *x=1.9995* and *y=3.0005*. This shows that the TLBO algorithm can be used as a successful approach that is very close to the real solution.

3.1. Convergence Graphs

The graphs showing the convergence speed of both algorithms in the root finding process according to the number of iterations are given below.

3.1.1. Convergence Graphs for Fixed Point Iteration Function Sets

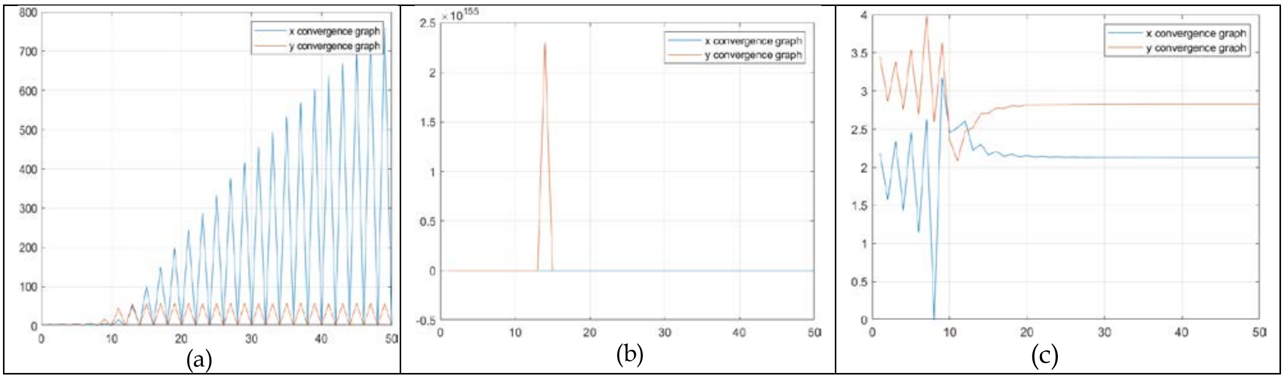

Figure 4. (a) Convergence graphs of fixed-point iteration set $g_1 = x, g_2 = y$

(b) Convergence graphs of fixed-point iteration set $g_3 = y, g_4 = x$

(c) Convergence graphs of fixed-point iteration set $g_5 = x, g_6 = y$

Figure 4 (a) shows that in the fixed-point iteration function set $x = g_1, y = g_2$, variable x converges to 1.25 but does not converge to the true root 2, variable y converges to 0.9 but does not converge to the true root 3. Therefore, $x = g_1, y = g_2$ fixed point iteration function set is divergent to the true root.

Figure 4 (b) shows that the x and y roots oscillate and do not converge to the true root in the fixed-point iteration function set g_3, g_4 .

Figure 4 (c) shows that in the $g_5 = x, g_6 = y$ fixed point iteration function set, x converges to 2.1 and y converges to 2.8, and as the number of iterations increases, it converges to the true root x=2 and y=3.

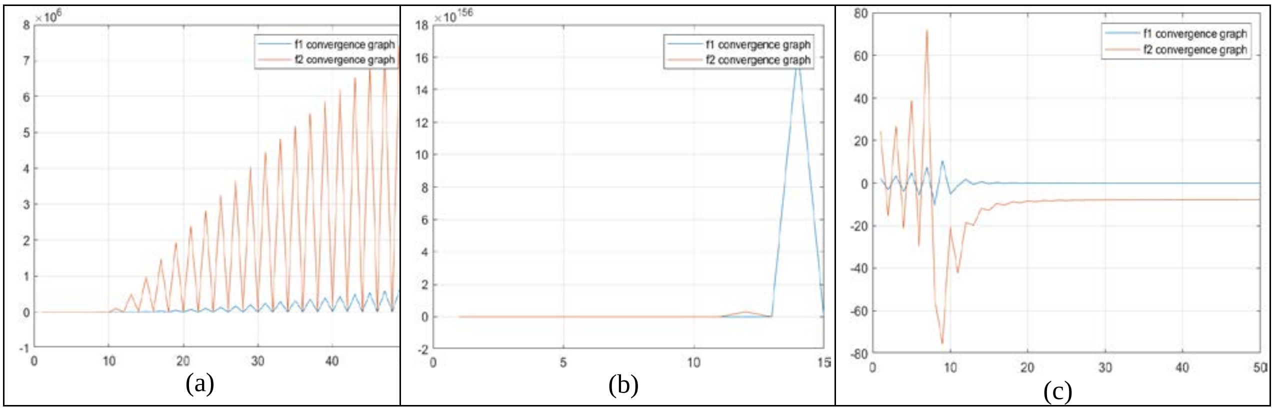

Figure 5. (a) Convergence graph for fixed point iteration function sets $x = g_1, y = g_2$ of functions f_1 ve f_2

(b) Convergence graph for fixed point iteration function sets $x = g_4, y = g_3$ of functions f_1 ve f_2

(c) Convergence graph for fixed point iteration function sets $x = g_5, y = g_6$ of functions f_1 ve f_2

According to Figure 5 (a) and (b), it can be seen that the functions f_1 and f_2 do not converge to zero for the values $x = g_1, y = g_2$ and $x = g_4, y = g_3$ obtained from the fixed point iteration function sets and therefore diverges. Figure 5 (c) shows that the functions of f_1 and f_2 converge to zero for the fixed-point iteration function set $x = g_5, y = g_6$. When the x and y values obtained from the functions $x = g_5, y = g_6$ are substituted into the functions f_1 and f_2 , f_1 approaches zero, that is, the true root, while f_2 approaches a value close to zero.

In order to find the root with Heuristic Optimization algorithms, the convergence to the root is checked by taking the sum of the absolute values of the objective functions using Theorem 1.4.2. Therefore, there is no need for derivatives or generating extra functions to solve the root finding problem. The objective function must converge to the minimum value. Here, the convergence to zero of the sum of the absolute values of the objective functions for the approximate roots of the fixed point iteration and the approximate roots of the heuristic method is graphically compared and given in Figures 6 and 7.

As a result of Theorem 1.4.2, now let us present the graphs obtained from the absolute sums of f_1 and f_2 values in Figure 5. The approach of the sum of $|f_1|+|f_2|$ to zero depending on the iteration number is given in Figure 6.

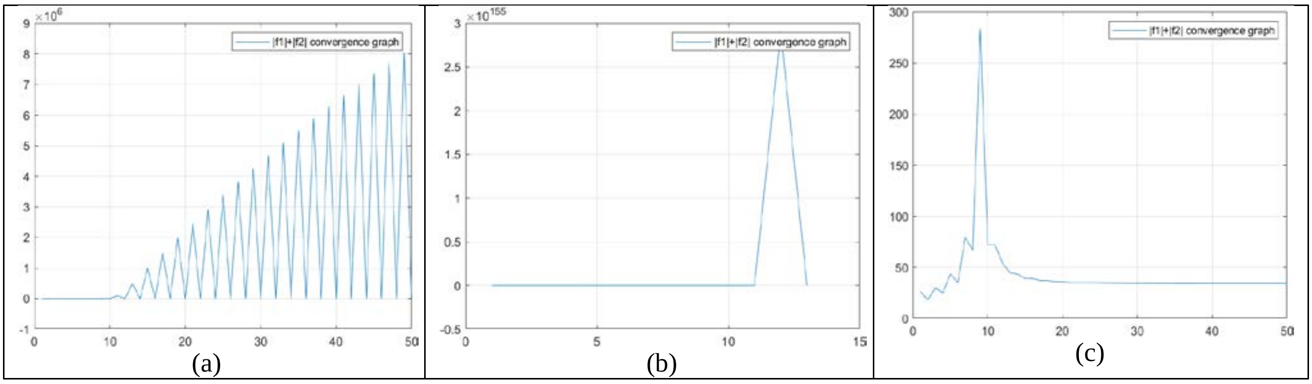

Figure 6. (a) Convergence graphs of fixed-point iteration functions $x = g_1$, $y = g_2$ of functions $|f_1|+|f_2|$
(b) Convergence graphs of fixed-point iteration functions $y = g_3$, $x = g_4$ of functions $|f_1|+|f_2|$
(c) Convergence graphs for fixed point iteration functions $x = g_5$, $y = g_6$ of functions $|f_1|+|f_2|$

In Figure 6 (a), it can be seen that the sum of $|f_1|+|f_2|$ for the first two selected iteration formulas does not approach zero and makes a fluctuating search for fixed point iteration function sets $x = g_1$, $y = g_2$. Figure 6 (b) shows that the sum of $|f_1|+|f_2|$ does not converge to zero for fixed point iteration function sets $x = g_4$, $y = g_3$, while Figure 6 (c) shows that the sum of $|f_1|+|f_2|$ converges to zero for fixed point iteration functions $x = g_5$, $y = g_6$.

3.1.2. Convergence Graphs for the Teaching-Learning Algorithm

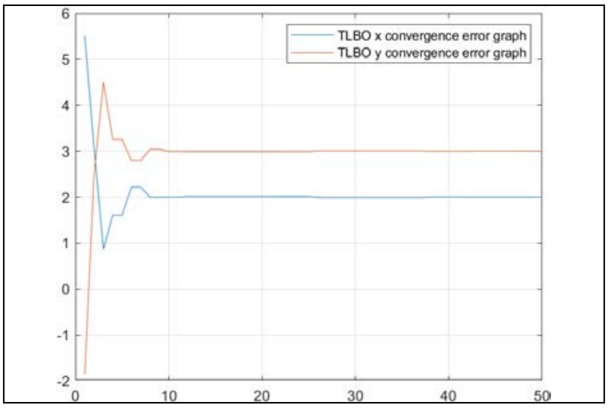

Figure 7. Convergence error graph of TLBO algorithm

According to Figure 8, it is seen that the error margins in x and ye values are very close to 0.01 until the 10th iteration with the TLBO algorithm. In this respect, it can be said that the TLBO algorithm approaches the actual x and y values with very little error.

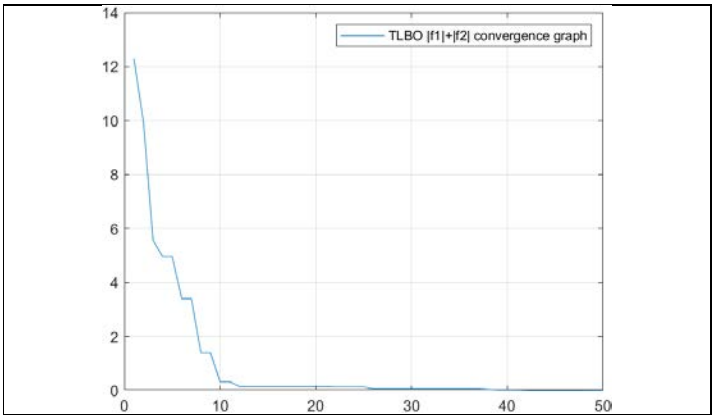

Figure 8. Zero convergence graph of $f_{\min} = |f_1| + |f_2|$ for TLBO algorithm.

Figure 8 shows that the value of $f_{\min} = |f_1| + |f_2|$ starts to approach zero after the 10th iteration and reaches its closest value to zero at the 49th iteration.

3.2. Error Analysis Graphs

The graphs comparing the error values at each iteration step for the fixed-point iteration method and the teaching-learning based optimization methods that we used to find approximate solutions of the system of equations given by (2) will be given below.

3.2.1 Convergence Error Graphs for Fixed Point Iteration Functions

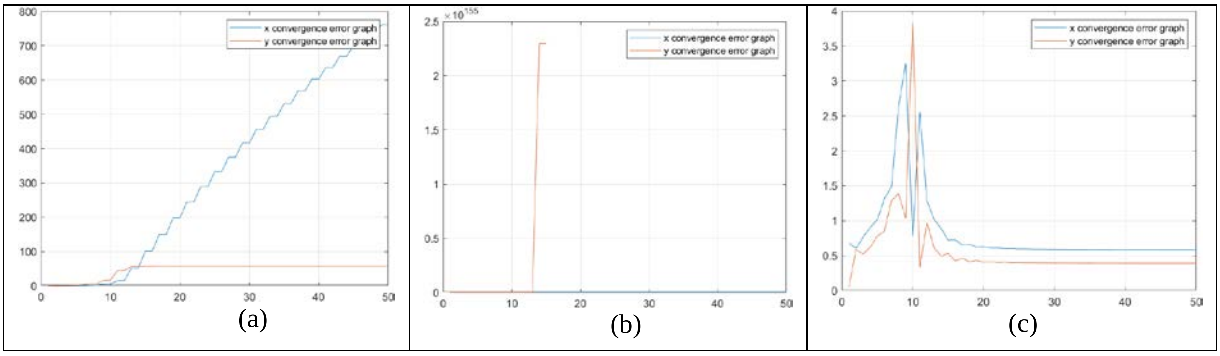

Figure 9. (a) Convergence error graphs of fixed-point iteration function set $x = g_1,\ y = g_2$
(b) Convergence error graphs of fixed-point iteration function set $x = g_4,\ y = g_3$
(c) Convergence error graphs of fixed-point iteration function set $x = g_5, y = g_6$

The convergence error graphs of fixed-point iteration functions are presented in Figure 9. Figure 9 (a) and Figure 9 (b) illustrate that the errors obtained for the fixed-point iteration functions exhibit fluctuations, whereas Figure 9 (c) demonstrates that the errors for the fixed-point iteration functions converge to zero.

Convergence Error Graphs for the TLBO algorithm are given in Figure 10. As illustrated in Figure 10, the convergence error graph for the TLBO algorithm exhibits a similar pattern to the graph (Figure 8) of the function converging to the minimum value. The TLBO algorithm demonstrates a consistent reduction in the approximation error as it approaches the true root. To summarize; the convergence error graphs for the TLBO algorithm, display a consistent reduction in approximation error, similar to the function's graph in Figure 8, as the algorithm steadily converges to the true root for variables (x) and (y).

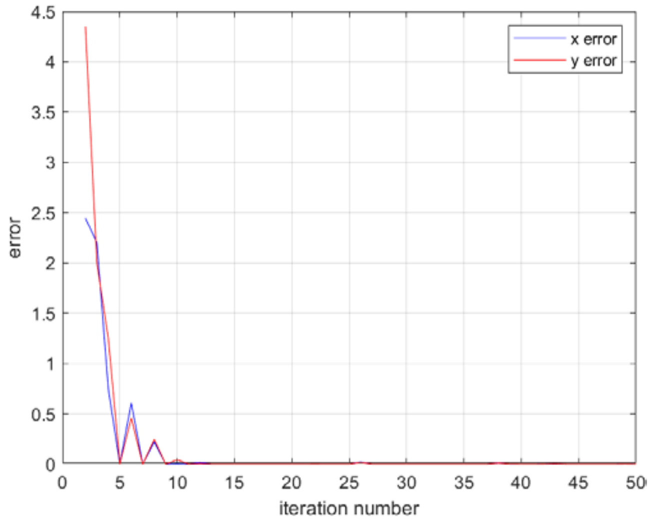

Figure 10. Convergence error graphs of the TLBO algorithm x, y

4. CONCLUSION

It is seen that TLBO converges better to the true root according to the number of iterations. Since the fixed-point iteration method aims to approach the best solution by considering different iteration sets, the success to be achieved here varies according to the choice of iteration sets. Even from this point of view, the fixed-point iteration method is an optimization method that requires more operations and cannot be said to be more successful than the TLBO algorithm in terms of convergence in the problem considered.

Since the problem considered in this paper consists of only two nonlinear equations, the analysis of computation times does not make a significant difference, since current computers are quite powerful and therefore the total computation times of the algorithms differ by milliseconds. In more complex systems, with more equations, the time difference can be more discriminating.

Choosing different functions can lead to better results, but there's a risk of non-convergence due to dependency on function creation and initial values. The functions from Canale and Capra's book [1] are used here as they are standard references, helping those interested in the field to understand the topic and make comparisons.

Heuristic optimization techniques, like numeric methods, don't provide exact solutions but can get close to the real solution. By setting a maximum number of iterations or acceptable error margins, we can achieve a good approximation. In the teaching and learning algorithm, iterations are capped at 50 steps to avoid repetition, usually resulting in a stable approximation despite further iterations.

5. ACKNOWLEDGEMENTS

The authors would like to thank the reviewers and editors of Journal of Universal Mathematics.

Funding
The authors declared that has not received any financial support for the research, authorship or publication of this study.

The Declaration of Conflict of Interest/ Common Interest
The authors declared that no conflict of interest or common interest.

The Declaration of Ethics Committee Approval
This study does not be necessary ethical committee permission or any special permission.

The Declaration of Research and Publication Ethics
The authors declared that they comply with the scientific, ethical, and citation rules of Journal of Universal Mathematics in all processes of the study and that they do not make any falsification on the data collected. Besides, the author declared that Journal of Universal Mathematics and its editorial board have no responsibility for any ethical violations that may be encountered, and this study has not been evaluated in any academic publication environment other than Journal of Universal Mathematics.

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