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Research Article

The Mathematical Dynamics of the Caputo Fractional Order Social Media Addiction Design

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Abstract
The paper presents the mathematical dynamics and numerical simulations for a fractional-order social media addiction (FSMA) model. This addiction structure is replaced by involving the Caputo fractional (CF) derivative to get the FSMA model. In this study, our main goal is to understand how the fractional derivative impresses the dynamics of the model. Thus, the theoretical properties are first examined. Afterward, the stability properties of the mentioned model are discussed. Besides, the fractional backward differentiation formula (FBDF) displays numerical simulations of the model. Observing both theoretical and numerical results, the two equilibrium points' stability is not impacted by the order of fractional derivatives. However, each solution converges more quickly to its stationary state for higher values of the fractional-order derivative. Finally, we would like to say that the acquired numerical results are compatible with our theoretical outcomes.

Keywords: Fractional backward differentiation formula(FBDF), Social media addiction, Stability analysis
2020 AMS: 65L05, 34A08, 34A12, 34A34,34D23

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1. Introduction

Nowadays, one of the important real-life problems is the use of social media platforms. They have both advantages and disadvantages. It provides access to information in many parts of education, business, science, and health [1, 2], but intensive usage of social media can negatively affect both family relationships and people's daily lives. Besides using social media extensively can lead to addiction. This situation is one of the significant addictive problems such as alcoholism, smoking, game addiction, etc. There are many studies on this subject [3–9].

Mathematical frameworks for modeling infectious diseases have an essential role in constructing and understanding the dynamics of them. Many researchers have applied to models for alcohol and drug, gambling, smoking, social media, and other addictions [9–13]. Especially after it was noticed that fractional-order models can be better expressed in nonlinear physical problems than integer-order models, they are used to demonstrate the mathematical model of real-life issues such as epidemiology, science, economics, and engineering. There are many studies such as [14–18] etc. in the literature. Demirci proposed a fractional design of Hepatitis B transmission to discuss the analysis of the mathematical model under the effect of vaccination [14]. An approximate solution fractional structure for hepatitis B virus (HBV) infection was obtained in [15]. Then the global stability of the model was constructed in this study. Estimating the disease of COVID-19 in India was designated as a fractional-order SITR by Askar et. al. [16]. They demonstrated the existence and uniqueness of the solution. Also, boundedness and nonnegativity for the system were established. In [17], they developed a novel fractional-order COVID-19 model clarifying the spread of the disease. The generalized Adams-Bashforth-Moulton algorithm was utilized to get the numerical outcomes. In [18], the authors used the derivatives of type Liouville-Caputo and Conformable to get novel analytic solutions for the electric circuits.

One of the basic goals of the paper is to propose, analyze, and simulate the FSMA design in terms of CF derivative. The proposed design yet has not been solved numerically by using the FBDF. In the Matlab environment, we will utilize the code $flmm2$, which is developed by R. Garrappa (for details see [19, 20]). There are three optional techniques in this code, but we will use the present method to get approximate solutions to the system. Moreover, the stability properties of this present epidemiological design are discussed. Therefore, the global stability of the mentioned structure is examined utilizing the Lyapunov stability theorem. Moreover, the regular state's stability concerning both theoretical and numerical outcomes is not affected by the order of the fractional derivative. Additionally, as the order of the fractional derivative increases, the solutions converge to the regular states faster.

This study consists of five parts. The FSMA design firstly will be introduced in Section 2. The mathematical dynamics of the system will be established in Section 3. We will also search the stability properties in Section 4. In addition, FBDF will be used to obtain numerical solutions of the given structure in Section 5. We will give a summary in the last section 6.

2. The FSMA Design

The part describes the FSMA structure. Its integer-order model is formulated and analyzed by Alemneh and Alemu [7] follow as:

$$
\begin{aligned}
\frac{dS}{dt} &= \pi + \gamma\eta R - \beta\sigma AS - (k+\mu)S,\\
\frac{dE}{dt} &= \beta\sigma AS - (\delta+\mu)E,\\
\frac{dA}{dt} &= \theta\delta E - (\mu+\varepsilon+\rho)A,\\
\frac{dR}{dt} &= (1-\theta)\delta E + \varepsilon A - (\mu+\eta)R,\\
\frac{dQ}{dt} &= kS + (1-\gamma)\eta R - \mu Q.
\end{aligned}
\tag{2.1}
$$

There are five categories in the human population according to addiction status in the system (2.1). Individuals in the first category are not addicted but are susceptible to being addicted. They are represented by $S(t)$. Exposed classes are using social media less frequently but do not grow to an addicted level and are denoted by $E(t)$. The third category shows addicted people who spend most of their time on social media and are described by $A(t)$. Recovered populations are defined by $R(t)$. They recovered from their social media addiction. The last category includes the human populations who forever do not use and quit using social media and are specified by $Q(t)$. The total number of members of the population is $N = S+E+A+R+Q$. To depict a deterministic mathematical model, the following assumptions are considered by the authors [7]:

- The epidemic happens in a closed environment,
- The possibility of being addicted to social media is not attached to race, sex, and people's social position, members mix homogeneously, and social media addictive humans will spread to non-addictive when they get in touch with the compression of addiction.

Kongson et. al. [8] used the Atangana-Balenau-Caputo derivative for the fractional-order differentiation. In this research, we will consider an initial value problem of FSMA with a CF derivative.

$$
\begin{aligned}
D^{\alpha}S(t) &= \pi + \gamma\eta R - \beta\sigma AS - (k+\mu)S,\\
D^{\alpha}E(t) &= \beta\sigma AS - (\delta+\mu)E,\\
D^{\alpha}A(t) &= \theta\delta E - (\mu+\varepsilon+\rho)A,\\
D^{\alpha}R(t) &= (1-\theta)\delta E + \varepsilon A - (\mu+\eta)R,\\
D^{\alpha}Q(t) &= kS + (1-\gamma)\eta R - \mu Q,
\end{aligned}
\tag{2.2}
$$

with initial data $S(0) = S_0 > 0, E(0) = E_0 \geq 0, A(0) = A_0 \geq 0, R(0) = R_0 \geq 0, Q(0) = Q_0 \geq 0$.

3. Theoretical Results of the Structure

A detailed mathematical analysis of the present structure (2.2) is demonstrated in this part. First of all, the existence and uniqueness of the solution of (2.2) are discussed based on the given approach in [21].

Theorem 3.1. *Let the region be defined as* $\Phi \times [0, T_1]$*, where* $\Phi = \{(S,E,A,R,Q) \in \mathbb{R}^5 : \max(|S|,|E|,|A|,|R|,|Q| \leq \xi)\}$ *and* $T_1 < +\infty$*. There is a single solution* $Z(t) \in \Phi$ *of the recommended model (2.2) with an initial condition* $Z_0 = (S_0, E_0, A_0, R_0, Q_0)$*, which is described for all* $t \geq 0$.

Proof. Consider $H(Z) = (H_1(Z), H_2(Z), H_3(Z), H_4(Z), H_5(Z))$ and

$$\begin{aligned}
H_1(Z) &= \pi + \gamma\eta R - \beta\sigma AS - (k+\mu)S,\\
H_2(Z) &= \beta\sigma AS - (\delta+\mu)E,\\
H_3(Z) &= \theta\delta E - (\mu+\varepsilon+\rho)A,\\
H_4(Z) &= (1-\theta)\delta E + \varepsilon A - (\mu+\eta)R,\\
H_5(Z) &= kS + (1-\gamma)\eta R - \mu Q.
\end{aligned}$$

For any $Z, Z^* \in \Phi$ that

$$\begin{aligned}
||H(Z)-H(Z^*)|| &= |H_1(Z)-H_1(Z^*)| + |H_2(Z)-H_2(Z^*)| + |H_3(Z)-H_3(Z^*)|\\
&+|H_4(Z)-H_4(Z^*)| + |H_5(Z)-H_5(Z^*)|\\
&= |\pi+\gamma\eta R-\beta\sigma AS-(k+\mu)S-(\pi+\gamma\eta R^*-\beta\sigma A^*S^*-(k+\mu)S^*)|\\
&+|\beta\sigma AS-(\delta+\mu)E-(\beta\sigma A^*S^*-(\delta+\mu)E^*)|\\
&+|\theta\delta E-(\mu+\varepsilon+\rho)A-(\theta\delta E^*-(\mu+\varepsilon+\rho)A^*)|\\
&+|(1-\theta)\delta E+\varepsilon A-(\mu+\eta)R-((1-\theta)\delta E^*+\varepsilon A^*-(\mu+\eta)R^*)|\\
&+|kS+(1-\gamma)\eta R-\mu Q-(kS^*+(1-\gamma)\eta R^*-\mu Q^*)|\\
&\leq |\gamma\eta(R-R^*)| + |\beta\sigma(AS-A^*S^*)| + |(k+\mu)(S-S^*)|\\
&+|\beta\sigma(AS-A^*S^*)| + |(\delta+\mu)(E-E^*)| + |\theta\delta(E-E^*)|\\
&+|(\mu+\varepsilon+\rho)(A-A^*)| + |(1-\theta)\delta(E-E^*)| + |\varepsilon(A-A^*)|\\
&+|(\mu+\eta)(R-R^*)| + |k(S-S^*)| + |(1-\gamma)\eta(R-R^*)| + |\mu(Q-Q^*)|\\
&\leq (\mu+2\eta)|R-R^*| + (2\beta\sigma\xi+2k+\mu)|S-S^*|\\
&+(2\beta\sigma\xi+\mu+2\varepsilon+\rho)|A-A^*| + (\delta+\mu+1)|E-E^*| + \mu|Q-Q^*|\\
&\leq G||(S,E,A,R,Q)-(S^*,E^*,A^*,R^*,Q^*)||\\
&\leq G||Y-Y^*||,
\end{aligned}$$

where $G = \max\{(\mu+2\eta), (2\beta\sigma\xi+2k+\mu), (2\beta\sigma\xi+\mu+2\varepsilon+\rho), (\delta+\mu+1), \mu\}$. Then $H(Z)$ provides the Lipschitz condition. Therefore, we complete the proof. □

3.1 Positivity and boundedness of the solution

This part states the following theorem that ensures the solutions of (2.2) are non-negative and bounded.

Lemma 3.2. *[22] Let us assume that* $u \in C[a,b]$ *and* $D^\alpha u(t) \in C[a,b]$ *for* $\alpha \in (0,1]$*, then we have* $u(t) = u(a) + \frac{1}{\Gamma(\alpha)}(D^\beta u(\xi))(t-\xi)^\alpha$*, in here* $a \leq \xi \leq t$*,* $\forall t \in [a,b]$.

Lemma 3.3. *Let* $u \in C[a,b]$ *and* $D^\alpha u(t) \in C[a,b]$ *for* $\alpha \in (0,1]$*. If* $D^\alpha u(t) \geq 0$*,* $\forall t \in (a,b)$*, then* $u(t)$ *is nondecreasing* $\forall t \in [a,b]$*. If* $D^\alpha u(t) \leq 0$*,* $\forall t \in (a,b)$*, then* $u(t)$ *is nonincreasing on* $[a,b]$.

We will demonstrate uniform boundedness of the solution thanks to the next lemma.

Lemma 3.4. *[21] Asumme that* $v(t)$ *is a continuous function on* $[t_0, \infty)$*. If* $v(t)$ *provides*

$$D^\alpha v(t) \leq -\beta v(t) + \xi, \quad v(t_0) = v_0 \in \mathbb{R},$$

in here $0 < \alpha \leq 1$*,* $\alpha, \xi \in \mathbb{R}$ *and* $\beta \neq 0$*, and* $t_0 \geq 0$ *is an initial time. Then*

$$v(t) \leq \left(v_0 - \frac{\xi}{\alpha}\right) E_\alpha\left[-\beta(t-t_0)^\alpha\right] + \frac{\xi}{\beta},$$

where $E_\alpha(t)$ which is called the Mittag-Leffler function (MLF) with one parameter is described as follows

$$E_\alpha(t) = \sum_{j=0}^{\infty} \frac{t^j}{\Gamma(\alpha j+1)},$$

in here $\alpha > 0$.

Mittag-Leffler described the MLF in 1903 [23]. Since it is frequently utilized to calculate the solutions of fractional differential equations, it has great significance in fractional calculus. The advised books to readers are [24]- [26] for a more comprehensive introduction.

Theorem 3.5. *The solutions of our system (2.2) are uniformly bounded and non-negative.*

Proof. Let's describe a function $V(t) = S(t)+E(t)+A(t)+R(t)+Q(t)$. Then

$$\begin{aligned}
D^\alpha V(t) &= D^\alpha S(t)+D^\alpha E(t)+D^\alpha A(t)+D^\alpha R(t)+D^\alpha Q(t)\\
&= [\pi+\gamma\eta R-\beta\sigma AS-(k+\mu)S]+[\beta\sigma AS-(\delta+\mu)E]\\
&\quad +[\theta\delta E-(\mu+\varepsilon+\rho)A]+[(1-\theta)\delta E+\varepsilon A-(\mu+\eta)R]\\
&\quad +[kS+(1-\gamma)\eta R-\mu Q]\\
&= \pi-\mu S-\mu E-\mu R-\mu Q-(\mu+\rho)A\\
&= \pi-\mu(S+E+A+R+Q)-\rho A\\
&\le \pi-\mu V(t).
\end{aligned}$$

Now, let's apply Lemma (3.4), we obtain

$$\begin{aligned}
0 \le V(t) &\le V(0)E_\alpha(-\mu(t)^\alpha)+\pi(t)^\alpha E_{\alpha,\alpha+1}(-\mu(t)^\alpha))\\
&= V(0)E_\alpha(-\mu(t)^\alpha)+\tfrac{\pi}{\mu}(1-E_\alpha(-\mu(t)^\alpha)).
\end{aligned}$$

Utilizing Lemma 5 and Corollary 6 in [27], we conclude that $0 \le V(t) \le \frac{\pi}{\mu}, \quad t \to \infty$. Hence, the solutions of the system (2.2) starting in $\mathbb{R}_+{}^5$ are uniformly bounded within the region $\Omega_1 = \{(S,E,A,R,Q) \in \mathbb{R}_+{}^5 : V(t) \le \frac{\pi}{\mu}+\varepsilon_0, \varepsilon_0 > 0\}$.

Next, to show that the solutions are non-negative. From the model (2.2), we get

$$\begin{aligned}
D^\alpha S(t)|_{S=0} &= \pi+\gamma\eta R \ge 0,\\
D^\alpha E(t)|_{E=0} &= \beta\sigma AS \ge 0,\\
D^\alpha A(t)|_{A=0} &= \theta\delta E \ge 0,\\
D^\alpha R(t)|_{R=0} &= (1-\theta)\delta E+\varepsilon A \ge 0,\\
D^\alpha Q(t)|_{Q=0} &= kS+(1-\gamma)\eta R \ge 0.
\end{aligned}$$

The solutions of (2.2) are non-negative according to Lemma 1 and Lemma 2. □

Next, the equilibrium points and reproduction number of (2.2) are introduced. There are two possible equilibrium points in the present design which are called addiction-free and endemic equilibrium points.

The general form of a dynamical system involving the CF derivative is given as

$$D^\alpha z(t) = h(t,z), \quad z(0) = z_0 \tag{3.1}$$

in here $0 < \alpha \le 1$.

Definition 3.6. *[28] If $h(t,z^*) = 0$ for a point z^*, the point is called an equilibrium point of the system (3.1).*

When $E = A = 0$, then the addiction-free equilibrium point of (2.2) is obtained from Definition (3.6) following as: $P_0 = \left(\frac{\pi}{k+\mu}, 0, 0, 0, \frac{k\pi}{\mu(\mu+k)}\right)$. At this point, there is no addiction in the group.

The important tool, both mathematically and biologically meaningful, is the main reproduction number R_0 because it displays the spread of the addiction. Also, we will analyze the stability of the equilibrium points by using R_0. The next-generation matrix method [29] is used to obtain R_0. It is found as

$$R_0 = \frac{\beta\pi\sigma\theta\delta}{(k+\mu)(\delta+\mu)(\mu+\varepsilon+\rho)}.$$

If $R_0 > 1$, the endemic equilibrium point of the present model occurs. The point P_1 can be computed by setting the right hand side of the proposed design is equal to zero. $P_1 = (S_1, E_1, A_1, R_1, Q_1)$ is given by

$$\begin{aligned} S_1 &= \tfrac{(\delta+\mu)(\mu+\varepsilon+\rho)}{\theta\delta\beta\sigma}, \\ E_1 &= \tfrac{\kappa_1}{\kappa_2}, \\ A_1 &= \tfrac{\theta\delta E_1}{\mu+\varepsilon+\rho}, \\ R_1 &= \tfrac{\kappa_1+(\delta+\mu)E_1}{\gamma\eta}, \\ Q_1 &= \tfrac{kS_1+(1-\gamma)\eta R_1}{\mu}, \end{aligned}$$

in here $\kappa_1 = \frac{(k+\mu)(\delta+\mu)(\mu+\varepsilon+\rho)}{\beta\sigma\theta\delta} - \pi$ and $\kappa_2 = \frac{\gamma\eta\delta}{\mu+\eta}\left(1-\theta+\frac{\varepsilon\theta}{\mu+\varepsilon+\rho}\right) - \delta - \mu$.

4. Stability Analysis

The stability properties involving local and global asymptotic stability of the mentioned structure are presented.

Theorem 4.1. *The addiction-free equilibrium point, P_0, is locally asymptotically stable if $R_0 < 1$.*

Proof. Let's calculate the Jacobian matrix for the model (2.2) evaluated at P_0. We get

$$\begin{pmatrix} -k-\mu & 0 & -\frac{\beta\pi\sigma}{k+\mu} & \gamma\eta & 0 \\ 0 & -\delta-\mu & \frac{\beta\pi\sigma}{k+\mu} & 0 & 0 \\ 0 & \theta\delta & -\varepsilon-\rho-\mu & 0 & 0 \\ 0 & (1-\theta)\delta & \varepsilon & -\eta-\mu & 0 \\ k & 0 & 0 & (1-\gamma)\eta & -\mu \end{pmatrix} \tag{4.1}$$

Some of the negative eigenvalues of (4.1)

$$\lambda_1 = -\mu, \lambda_2 = -k-\mu, \lambda_3 = -\mu-\eta$$

and the other eigenvalues are derived from the quadratic equation

$$\lambda^2 + \Delta_1\lambda + \Delta_2 = 0$$

in here $\Delta_1 = \varepsilon+\rho+2\mu$ and $\Delta_2 = (\delta+\mu)(\varepsilon+\mu+\rho) - \frac{\beta\pi\sigma\theta\delta}{k+\mu}$. From Routh-Hurwitz criteria, the quadratic equation has strictly negative real root iff $\Delta_1 > 0$, $\Delta_2 > 0$ and $\Delta_1\Delta_2 > 0$. It is seen easily that $\Delta_1 > 0$ and Δ_2 can be rewritten as

$$\Delta_2 = (\delta+\mu)(\varepsilon+\mu+\rho)(1-\frac{\beta\pi\sigma\theta\delta}{k+\mu}) = (\delta+\mu)(\varepsilon+\mu+\rho)(1-R_0).$$

Thus, when $R_0 < 1$, all eigenvalues have the negative real parts. □

Theorem 4.2. *The endemic equilibrium point, P_1, is locally asymptotically stable if $R_0 > 1$.*

Proof. The proof can be done in a similar manner as in [7]. □

The most important concern for the fractional differential equation is about that the global stability of the solution. Now, we will use the Lyapunov functions to construct the stability of fractional systems. We will first give the following Lemma, which introduces the extended Volterra-type Lyapunov function to systems of fractional differential equations through an inequality that was defined by [30] for approximating the CF derivative of the function.

Lemma 4.3. *[30] Let's $v(t) \in \mathbb{R}^+$ be a continuous function. Then for any time $t \geq t_0$,*

$$D^\alpha\left[v(t) - v_* - v_*\ln\frac{v(t)}{v_*}\right] \leq \left(1-\frac{v_*}{v(t)}\right)D^\alpha v(t),\ v_* \in \mathbb{R}^+, \forall\alpha \in (0,1).$$

The upcoming result displays the solutions of (2.2) are uniformly continuous. The proof is done in the same manner as in [31].

Lemma 4.4. *The solutions S,E,A,R, and Q of system (2.2) are uniformly continuous functions on* $[0,\infty)$.

Theorem 4.5. *The addiction-free equilibrium point P_0 of the present design (2.2) is globally asymptotically stable if $R_0<1$ and unstable when $R_0>1$.*

Proof. Let's introduce a function $V(S,E,A,R,Q)=E+\frac{\delta+\mu}{\theta\delta}A$, which is called a Lyapunov function. Then,

$$\begin{aligned} D^\alpha V &= D^\alpha E+\tfrac{\delta+\mu}{\theta\delta}D^\alpha A \\ &= \beta\sigma AS-(\delta+\mu)E+\tfrac{\delta+\mu}{\theta\delta}[\theta\delta E-(\mu+\varepsilon+\rho)A] \\ &= \beta\sigma AS-\tfrac{(\mu+\varepsilon+\rho)(\delta+\mu)}{\theta\delta}A. \end{aligned} \tag{4.2}$$

When we use the addiction-free point of (2.2), $S_0=\frac{\pi}{k+\mu}$, we have from the Equation (4.2),

$$\begin{aligned} D^\alpha V &= \beta\sigma AS_0-\tfrac{(\mu+\varepsilon+\rho)(\delta+\mu)}{\theta\delta}A \\ &= \tfrac{(\mu+\varepsilon+\rho)(\delta+\mu)}{\theta\delta}[\tfrac{\beta\sigma\pi\theta\delta}{(k+\mu)(\mu+\varepsilon+\rho)(\delta+\mu)}-1] \\ &= \tfrac{(\mu+\varepsilon+\rho)(\delta+\mu)}{\theta\delta}[R_0-1]. \end{aligned}$$

Thus, $D^\alpha V\le 0$ when $R_0<1$. This conclusion gives that P_0 is globally asymptotically stable if $R_0<1$. □

Theorem 4.6. *The endemic equilibrium point P_1 of the present model (2.2) is globally asymptotically stable if $R_0>1$.*

Proof. We establish a Lyapunov function

$$\begin{aligned} W(S,E,A,R,Q) &= \left(S-S_1-S_1\ln\tfrac{S}{S_1}\right)+\left(E-E_1-E_1\ln\tfrac{E}{E_1}\right)+\left(A-A_1-A_1\ln\tfrac{A}{A_1}\right) \\ &+\left(R-R_1-R_1\ln\tfrac{R}{R_1}\right)+\left(Q-Q_1-Q_1\ln\tfrac{Q}{Q_1}\right). \end{aligned}$$

By using Lemma 4.3, we have

$$\begin{aligned} D^\alpha W &\le \left(1-\tfrac{S_1}{S}\right)D^\alpha S+\left(1-\tfrac{E_1}{E}\right)D^\alpha E+\left(1-\tfrac{A_1}{A}\right)D^\alpha A \\ &+\left(1-\tfrac{R1}{R}\right)D^\alpha R+\left(1-\tfrac{Q_1}{Q}\right)D^\alpha Q \\ &= \left(1-\tfrac{S_1}{S}\right)[\pi+\gamma\eta R-\beta\sigma AS-(k+\mu)S]+\left(1-\tfrac{E_1}{E}\right)[\beta\sigma AS-(\delta+\mu)E] \\ &+\left(1-\tfrac{A_1}{A}\right)[\theta\delta E-(\mu+\varepsilon+\rho)A]+\left(1-\tfrac{R_1}{R}\right)[(1-\alpha)\delta E+\varepsilon A-(\mu+\eta)R] \\ &+\left(1-\tfrac{Q_1}{Q}\right)[kS+(1-\gamma)\eta R-\mu Q]. \end{aligned} \tag{4.3}$$

From (2.2), we have that,

$$\begin{aligned} \pi &= \beta\sigma A_1S_1+(k+\mu)S_1-\gamma\eta R_1, \\ (\delta+\mu) &= \beta\sigma\tfrac{A_1S_1}{E_1}, \\ \theta\delta &= (\mu+\varepsilon+\rho)\tfrac{A_1}{E_1}, \\ \gamma\eta &= 1-\mu\tfrac{Q_1}{R_1}+k\tfrac{S_1}{R_1}, \\ (\mu+\eta) &= \delta\tfrac{E_1}{R_1}-(\mu+\varepsilon+\rho)\tfrac{A_1}{R_1}+\varepsilon\tfrac{A_1}{R_1}. \end{aligned} \tag{4.4}$$

Next, using the relations (4.4) into (4.3), we get

$$\begin{aligned} D^\alpha W &\le \beta\sigma A_1S_1\left[2-\tfrac{S_1}{S}+\tfrac{A}{A_1}-\tfrac{E}{E_1}-\tfrac{ASE_1}{A_1S_1E}\right]+(k+\mu)S_1\left[2-\tfrac{S_1}{S}-\tfrac{S}{S_1}\right] \\ &+\gamma\eta R_1\left[-1+\tfrac{R}{R_1}+\tfrac{S_1}{S}-\tfrac{RS_1}{R_1S}\right] \\ &+(\mu+\varepsilon+\rho)\left[-\tfrac{A_1}{A}-\tfrac{A_1E}{AE_1}+\tfrac{R}{R_1}+\tfrac{R_1E}{RE_1}\right] \\ &+\varepsilon A_1\left[1+\tfrac{A}{A_1}-\tfrac{R}{R_1}-\tfrac{AR_1}{A_1R}\right]+\mu Q_1\left[1-\tfrac{Q}{Q_1}-\tfrac{Q_1R}{QR_1}+\tfrac{R}{R_1}\right] \\ &+\delta E_1\left[1+\tfrac{E}{E_1}-\tfrac{R}{R_1}-\tfrac{ER_1}{E_1R}\right]+kS_1\left[\tfrac{S}{S_1}-\tfrac{R}{R_1}-\tfrac{SQ_1}{S_1Q}+\tfrac{Q_1R}{QR_1}\right]. \end{aligned}$$

Since the arithmetic mean exceeds the geometric mean, then

$$\left(2-\frac{S}{S_1}-\frac{S_1}{S}\right)\leq 0, \quad \left[2-\frac{S_1}{S}+\frac{A}{A_1}-\frac{E}{E_1}-\frac{ASE_1}{A_1S_1E}\right]\leq 0,$$

$$\left[-1+\frac{R}{R_1}+\frac{S_1}{S}-\frac{RS_1}{R_1S}\right]\leq 0, \quad \left[1+\frac{A}{A_1}-\frac{R}{R_1}-\frac{AR_1}{A_1R}\right]\leq 0,$$

$$\left[1-\frac{Q}{Q_1}-\frac{Q_1R}{QR_1}+\frac{R}{R_1}\right]\leq 0, \quad \left[1+\frac{E}{E_1}-\frac{R}{R_1}-\frac{ER_1}{E_1R}\right]\leq 0,$$

and

$$\left[\frac{S}{S_1}-\frac{R}{R_1}-\frac{SQ_1}{S_1Q}+\frac{Q_1R}{QR_1}\right]\leq 0.$$

Therefore, $D^\alpha W \leq 0$.

Let N_1 is the largest invariant set in $\{(S,E,A,R,Q);D^\alpha W=0\}$. Note that $D^\alpha W=0$ if and only if $S=S_1, E=E_1, A=A_1, R=R_1, Q=Q_1$ for any time t. Hence, it can be said that $N_1=\{P_1\}=\{(S_1,E_1,A_1,R_1,Q_1)\}$. When $R_0>1$, we obtain (2.2) is globally asymptotically stable at P_1 thanks to Lyapunov-LaSalle invariance principle. □

5. Numerical Results

Now, we will display the numerical solutions of the FSMA model. The approximate solution is demonstrated by using a fractional backward differentiation formula. For more detailed information, the readers can read the studies [20,32–34].

Nonnegative parameters are used to get the numerical outcomes. If we choose $\pi=0.5, \mu=0.05, \beta=0.3, \sigma=0.2, \theta=0.7, \rho=0.01, \delta=0.25, \varepsilon=0.7, \kappa=0.01, \gamma=0.35$ and $\eta=0.4$, then we get the reproduction number as $R_0=0.3838<1$ and the result of the numerical solution of the FSMA design as illustrated in Fig. (5.1). In this situation, $P_0=(8.3333,0,0,0,1.6667)$ is obtained. In Fig (5.1), it can be noted that while the number of exposed, addicted, quit-using, and recovered classes quickly increases, the number of susceptible populations decreases the first time. After, the number of exposed, addicted, and recovered populations decreases to zero over time. When exposed and addicted individuals in society heal, the number of recovering populations increases, and after the addict's transmission stops, the number of recovered populations decreases to zero. It should be noted that approximate solutions converge to the point P_0. The number of susceptible and quit-using social media populations is balanced and stable at 8.3333 and 1.6667, respectively.

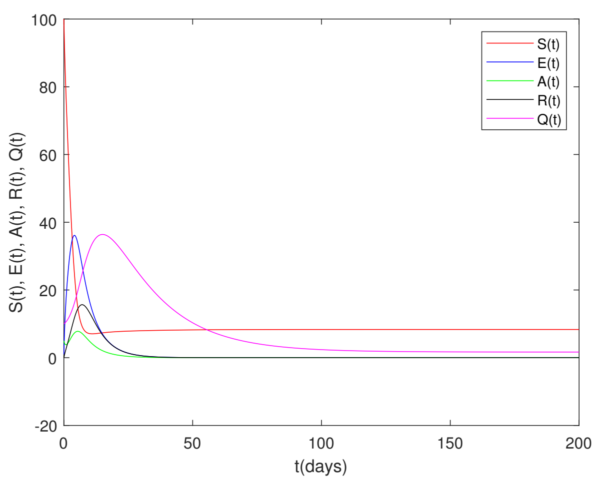

Figure 5.1. The Plot of the model for $\alpha=0.998$ in the case $R_0<1$.

When $\beta=0.8$, $R_0=1.0234>1$ is obtained. Also, the endemic equilibrium point is

$P_1=(8.1429,0.0505,0.0116,0.0265,1.7663)$

in this case. Additionally, the plot of the model is displayed in Fig. (5.2). In this step, we would like to say that approximate solutions converge to the endemic equilibrium point P_1. Moreover, each of the numerical solutions of the structure is displayed for various fractional orders $\alpha = 0.88, 0.92, 0.96, 1$ to understand the effect of fractional derivative orders in Fig. (5.3). Remark that the order of the fractional derivative α does not affect on the regular state's stability. All obtained solutions also converge more quickly to the regular states for higher values of α.

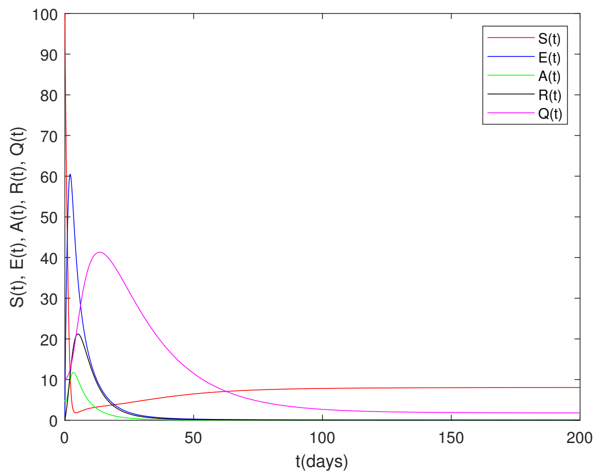

Figure 5.2. Plot of the system for $\alpha = 0.998$ in the case $R_0 > 1$.

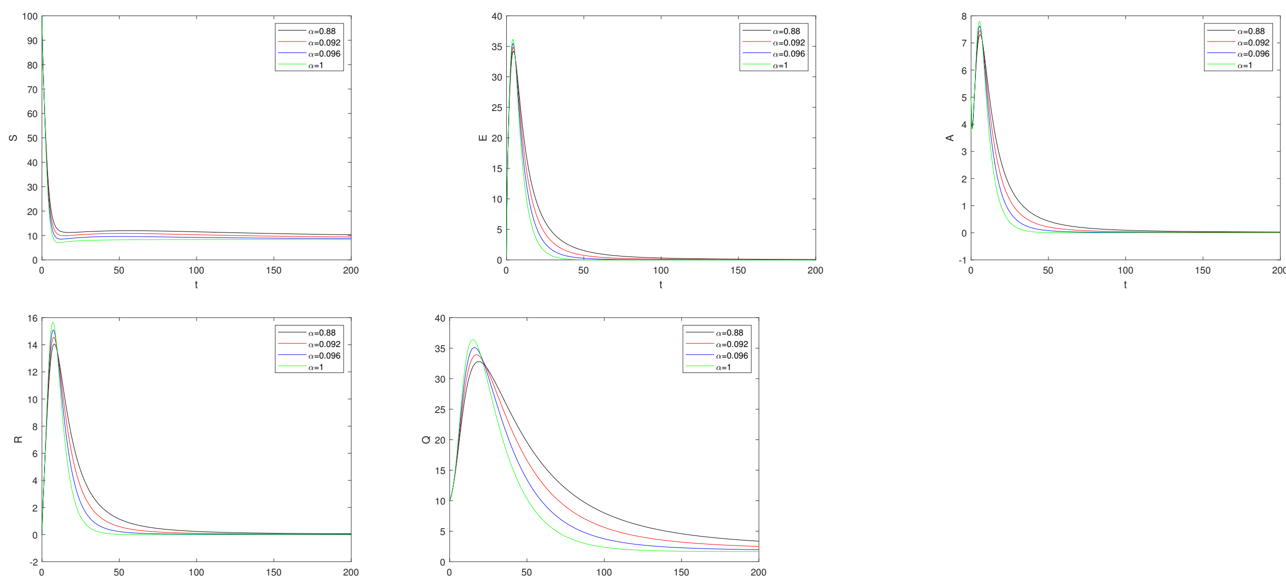

Figure 5.3. Plots of the infection as a function of time (days) with different values of fractional orders

6. Conclusion Remarks

This study focuses on the analyses of the FSMA design with CF derivative. Firstly, the mathematical analysis is examined. We verify the local and global analysis of the equilibria stability. FBDF is used to obtain approximate solutions. Numerical simulations display balance and stability at two equilibrium points P_0 and P_1 when $R_0 < 1$, and $R_0 > 1$, respectively. Furthermore, based on both theoretical and numerical outcomes, we notice that the order of fractional derivatives does not affect on the two equilibria' stability. However, each solution converges more quickly to its stationary state for higher values of the fractional-order derivative. Lastly, we would also like to say that the obtained numerical outcomes are compatible with our theoretical results.

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Research Article

The First Study of Mersenne–Leonardo Sequence

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Abstract
In this study, we introduce a new class of numbers, referred to as Modified Mersenne–Leonardo numbers. The aim of this paper is to define the Modified Mersenne–Leonardo sequence and investigate some of its properties, including the recurrence relation, summation formula, and generating function. Additionally, classical identities such as the Tagiuri–Vajda, Catalan, Cassini, and d'Ocagne identities are derived for the Modified Mersenne–Leonardo numbers.

Keywords: Binet's formula, Generating function, Leonardo numbers, Mersenne numbers, Mersenne–Leonardo numbers
2020 AMS: 11B39, 11B37

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1. Introduction

The Mersenne sequence is made up of non-negative integers in the form of a power of two minus one and is best known for some of the prime numbers that make it up, which are called Mersenne primes. These numbers are defined by the recurrence relation

$$M_n = 3M_{n-1} - 2M_{n-2}, \text{ for all } n \geq 2; \tag{1.1}$$

with initial terms $M_0 = 0$ and $M_1 = 1$. This recurrence serves as the foundation for their exploration and application in number theory, forming the sequence:

$$\{M_n\}_{n\geq 0} = \{0, 1, 3, 7, 15, 31, 63, 127, 255, \ldots\}.$$

which is referred as sequence A000225 in the OEIS [1]. Considering the initial values $m_0 = 2$ and $m_1 = 3$, with the identical recurrence relation $m_n = 3m_{n-1} - 2m_{n-2}$, for all $n \geq 2$ we have the Mersenne–Lucas numbers. The terms of this sequence are called Mersenne-Lucas numbers and are expressed in the form $m_n = 2^n + 1$, which is identified as sequence A000051 in OEIS [1]. These two classes of numbers are an indispensable concept in number theory, exhibiting significant implications in domains such as cryptography and the identification of large prime numbers.

In mathematical literature, there have been many studies of the sequences of Mersenne and Mersenne-Lucas numbers. For example, [2] offers a thorough and detailed examination of these two types of special numbers; in particular, they are intimately tied to classical problems in the theory of prime numbers, as seen in [3], [4] and [5]. It also looks at practical applications, and is particularly relevant in the specific context of cryptography, as seen in [6] and [7]. Some generalizations or extensions of the

Mersenne or Mersenne-Lucas sequences, generation functions, and several identities can be found in [8–10] and [11], among others.

The Leonardo sequence has similarities to the Fibonacci sequence, wherein each term is derived from the sum of the preceding two terms, with the addition of the constant value one. The recurrence for the Leonardo sequence is:

$$Le_n = Le_{n-1} + Le_{n-2} + 1, \text{ for all } n \geq 2;$$

with initial values $Le_0 = Le_1 = 1$. The first few Leonardo numbers are

$$\{Le_n\}_{n\geq 0} = \{1, 1, 3, 5, 9, 15, 25, 41, 67, 109, 177, 287, \dots\}.$$

This particular sequence is referenced as $A001595$ in the OEIS [1]. According [12] and [13, 14], this sequence is very similar to the Fibonacci sequence, including a relationship between Leonardo's numbers and Fibonacci's numbers that is

$$Le_n = 2F_{n+1} - 1.$$

The resurgence of interest in these Leonardo sequences can be attributed to the seminal paper by [15]. Primarily utilized in the domain of data structure and algorithmic analysis, it is particularly prevalent in the context of balanced trees, see [13] and [14], among others. Some generalizations or extensions of the Leonardo sequence, generating functions and several identities can be found at [16]- [23], among others.

In recent publications, [24] proposed the Pell–Leonardo sequence and presented a new sequence with a recurrence third-order, [25] in the section entitled *Generalized Bronze Leonardo sequences* deals with the sequences Bronze Leonardo, Bronze Leonardo–Lucas, and Modified Bronze Leonardo. Another work that is in line with the same logic of these one is [26]. These works motivated the formulation of our work about the Modified Mersenne-Leonardo numbers presented in Section 3 and the subsequent study.

The structure of the present paper is divided into two more sections, as follows. In Section 2, we briefly remember the Mersenne $\{M_n\}_{n\geq 0}$, and summarize the results of this sequence used in this study. In Section 3, we define the Modified Mersenne-Leonardo sequence; in particular, in the Subsection 3.1, we introduce the new sequence, detailing its characteristics and properties in connection with the classical Mersenne sequence. In Subsection 3.2 we present Binet's formula, which gives an explicit expression for the terms of the sequence, and the generating function associated with the sequence is presented. In Subsection 3.3 we present several fundamental identities for Modified Mersenne–Leonardo such as Tagiuri-Vajda, d'Ocagene and their consequences, accompanied by some numerical examples to illustrate their application. Finally, summation formulas involving the Modified Mersenne-Leonardo numbers are presented in Subsection 3.4. We conclude with some final considerations and state some future work on this topic.

2. Mersenne Numbers: A Background

Recall that the characteristic equation associated with equation (1.1) for the Mersenne sequence is given by $r^2 = 3r - 2$. This equation corresponds to the characteristic equation of a Horadam-type sequence, see [27] and [28]. Solving for r yields that the roots of the equation are $r_1 = 2$ and $r_2 = 1$.

The Binet formula provides a direct method to compute the n-th Mersenne number without having to iterate through the sequence. This formula offers an efficient way to calculate Mersenne-type numbers based on the sequence structure. Numerous identities, including Catalan's identity, Cassini's identity, and d'Ocagne's identity, related to Mersenne numbers are presented in classical literature.

As demonstrated in [8], the Binet formula for the Mersenne sequence is given in the next result:

Lemma 2.1. *Let $\{M_n\}_{n\geq 0}$ be the Mersenne sequence. Then:*

$$M_n = 2^n - 1\ . \tag{2.1}$$

Some of the most important properties that Mersenne numbers satisfy are summarized in the following result:

Lemma 2.2. *[8, Proposition 2.5] If M_j is the $j-$th term of the Mersenne sequence, then:*

(a) $M_j^2 = 4^j - M_{j+1}$

(b) $\sum_{j=0}^{n} M_j = M_{n+1} - (n+1) = 2M_n - n\ .$

According Soykan [11] the sequence $\{M_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$M_{-n} = \frac{3}{2}M_{-(n-1)} - \frac{1}{2}M_{-(n-2)} \tag{2.2}$$

for $n = 1,2,3,\ldots$. Therefore, the equation of the recurrence (1.1) holds for all integer n.

An examination of the Table 1, and applying the equation (2.2), it is possible to conclude that:

$$M_{-n} = -\frac{M_n}{2^n} \ ,$$

for all integers $n \geq 1$.

n	0	1	2	3	4	5	6	7	8	9	10
M_n	0	1	3	7	15	31	63	127	255	511	1023
M_{-n}	0	$-\frac{1}{2}$	$-\frac{3}{4}$	$-\frac{7}{8}$	$-\frac{15}{16}$	$-\frac{31}{32}$	$-\frac{63}{64}$	$-\frac{127}{128}$	$-\frac{255}{256}$	$-\frac{511}{512}$	$-\frac{1023}{1024}$

Table 2.1. The first few values of the Mersenne numbers with positive and negative subscripts

To facilitate the understanding of the subsequent result, it is first necessary to establish some preliminary auxiliary results.

Lemma 2.3. *Let n,s,k be an non-negative integers and $\{M_k\}_{k\geq 0}$ the Mersenne sequence. Then the following identity holds:*

$$2^{n+s+k} - 2^{n+s} - 2^{n+k} + 2^n = 2^n M_s M_k \ .$$

Proof. Note that

$$\begin{aligned} 2^{n+s+k} - 2^{n+s} - 2^{n+k} + 2^n &= 2^{n+s}(2^k-1) - 2^n(2^k-1) \\ &= 2^n M_s M_k \ . \end{aligned}$$

since $M_j = 2^j - 1$ the result follows. □

3. The Modified Mersenne–Leonardo Sequence

In this section, we introduce a new numerical sequence called the Modified Mersenne–Leonardo sequence. We begin with its definition and some of its properties. Next, we present the corresponding Binet formula and generating function. Finally, we conclude this section by stating several identities.

3.1 Introduction to Modified Mersenne–Leonardo sequence

As we have mentioned before, we found the motivation for our work mainly by the recent publications of [24] and [25], which introduced a new sequence of numbers with third-order recurrence, respectively, the Pell-Leonardo sequence and Generalized Bronze Leonardo sequences. In these works, various identities are established for these sequences. Following to the literature, now, let us define the Modified Mersenne-Leonardo sequence and explore its implications.

We define the Modified Mersenne–Leonardo numbers by using a recurrence relation, which it is stated in what follows:

Definition 3.1. *For all integer $n \geq 2$, the Modified Mersenne–Leonardo sequence $\{\mathcal{ML}_n\}_{n\geq 0}$ satisfies the following recurrence relation:*

$$\mathcal{ML}_{n+1} = 3\mathcal{ML}_n - 2\mathcal{ML}_{n-1} + 1, \tag{3.1}$$

with the initial values $\mathcal{ML}_0 = 0$ and $\mathcal{ML}_1 = 1$.

The first thirteen Modified Mersenne–Leonardo numbers are:

$$\{\mathcal{ML}_n\}_{n\geq 0} = \{0,1,4,11,26,57,120,247,512,1013,2036,4083,8178,\ldots\}.$$

We start with elementary observation that the sequence $\{\mathcal{ML}_n\}_{n\geq 0}$ satisfies the second non-homogeneous linear recurrence. An equivalent way to write equation (3.1) is

$$\mathcal{ML}_{n+1} = 3\mathcal{ML}_n - 2\mathcal{ML}_{n-1} + 1, \tag{3.2}$$

by subtracting equations (3.1) and (3.2), we obtain a homogeneous recurrence relation,

$$\mathcal{ML}_{n+1} = 4\mathcal{ML}_n - 5\mathcal{ML}_{n-1} + 2\mathcal{ML}_{n-2} \ .$$

The preceding discussion demonstrates that:

Proposition 3.2. *Let be $\{\mathcal{M}_n\}_{n\geq 0}$ the Modified Mersenne–Leonardo sequence that satisfies the homogeneous recurrence relation*

$$\mathcal{M}_{n+1} = 4\mathcal{M}_n - 5\mathcal{M}_{n-1} + 2\mathcal{M}_{n-2}\ , \tag{3.3}$$

with initial terms $\mathcal{M}_0 = 0$, $\mathcal{M}_1 = 1$, and $\mathcal{M}_2 = 4$.

The relationship between the Modified Mersenne–Leonardo numbers and the Mersenne numbers is expressed in the following proposition.

Proposition 3.3. *Let be $\{\mathcal{M}_n\}_{n\geq 0}$ the Modified Mersenne–Leonardo sequence and $\{M_n\}_{n\geq 0}$ the Mersenne sequence, then*

$$\mathcal{M}_{n+1} - \mathcal{M}_n = M_{n+1} \tag{3.4}$$

for all non-negative integer n.

Proof. We will prove this by induction on n. From the definition of Modified Mersenne–Leonardo numbers, we know that $\mathcal{M}_0 = M_0 = 0$ and $\mathcal{M}_1 = M_1 = 1$. Now, assume that equation (3.4) is true for all $1 < n \leq k$, and we will show that equation (3.4) also holds for $n = k+1$. Indeed, by applying the induction hypothesis and the homogeneous recurrence relation given by equation (3.3), we can express:

$$\begin{aligned}
\mathcal{M}_{k+1} &= 4\mathcal{M}_k - 5\mathcal{M}_{k-1} + 2\mathcal{M}_{k-2} \\
&= 3\mathcal{M}_k - 3\mathcal{M}_{k-1} - 2\mathcal{M}_{k-1} + 2\mathcal{M}_{k-2} + \mathcal{M}_k \\
&= 3(\mathcal{M}_k - \mathcal{M}_{k-1}) - 2(\mathcal{M}_{k-1} - \mathcal{M}_{k-2}) + \mathcal{M}_k \\
&\overset{\text{hip. ind.}}{=} 3M_k - 2M_{k-1} + \mathcal{M}_k\,.
\end{aligned}$$

As $M_{k+1} = 3M_k - 2M_{k-1}$, by equation (1.1), we obtain the result required. □

3.2 The Binet formula and generating functions

In this subsection, we introduce the Binet formula as well as the generating and exponential functions associated with the Modified Mersenne–Leonardo sequence. We also found the limit of the ratio $\mathcal{M}_{k+1}/\mathcal{M}_k$, for all $k \in \mathbb{N}$.

The characteristic equation associated with equation (3.3) for the Modified Mersenne–Leonardo sequence is given by $r^3 = 4r^2 - 5r + 2$. The roots are $r_1 = 1$ (the double root of the equation) and $r_2 = 2$. With these roots, the Binet formula provides a direct method to compute the n-th Modified Mersenne–Leonardo number without having to iterate through the sequence.

Now we will determine the Binet formula for Modified Mersenne–Leonardo sequence, and we obtain:

Proposition 3.4 (Binet's formula)**.** *Let $\{\mathcal{M}_n\}_{n\geq 0}$ be the Modified Mersenne–Leonardo sequence. Then:*

$$\mathcal{M}_n = 2^{n+1} - (n+2)\ . \tag{3.5}$$

Proof. Using equations (2.1) and (3.4), a straightforward calculation gives us:

$$\begin{aligned}
\mathcal{M}_1 - \mathcal{M}_0 &= 2 - 1 \\
\mathcal{M}_2 - \mathcal{M}_1 &= 2^2 - 1 \\
&\ \vdots \qquad \vdots \\
\mathcal{M}_n - \mathcal{M}_{n-1} &= 2^n - 1\,.
\end{aligned}$$

So,

$$\begin{aligned}
\mathcal{M}_n - \mathcal{M}_0 &= (2-1) + (2^2-1) + \ldots + (2^n - 1) \\
&= (1 + 2 + 2^2 + \ldots + 2^n) - (n+1) \\
&= 2^{n+1} - (n+2)\,.
\end{aligned}$$

Since $\mathcal{M}_0 = 0$, we arrive at the result. □

It follows directly from Proposition 3.4 , and by making use of Lemma 2.2(b), that:

Corollary 3.5. *Let $\{\mathbf{ML}_n\}_{n\geq 0}$ be the Modified Mersenne–Leonardo sequence. Then:*

$$\mathbf{ML}_n = M_{n+1} - (n+1) = 2M_n - n\ .$$

Which implies that

Corollary 3.6. *Let $\{\mathbf{ML}_n\}_{n\geq 0}$ be the Modified Mersenne–Leonardo sequence. Then:*

$$\mathbf{ML}_n = \sum_{j=0}^{n} M_j\ .$$

In the literature, it is important to note that the function $GF_{a_n}(x)$ is referred to as the ordinary generating function for the sequence $\{a_n\}_{n\geq 0}$, with

$$GF_{a_n}(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots + a_n x^n + \ldots \tag{3.6}$$

To make notation easier, let us denote $GF_{a_n}(x)$ by $L(x)$.

Additionally, making use of the homogeneous recurrence relation (Proposition 3.2), the following result states the ordinary generating function for the Modified Mersenne–Leonardo sequence.

Proposition 3.7. *The ordinary generating function for the Modified Mersenne–Leonardo sequence $\{\mathbf{ML}_n\}_{n\geq 0}$, denoted by $L(x)$, is given by*

$$L(x) = \frac{4x}{2x^3 - 5x^2 + 4x + 1}\ .$$

Proof. According to equation (3.6), the ordinary generating function for the Modified Mersenne–Leonardo sequence is $L(x) = \sum_{n=0}^{\infty} \mathbf{ML}_n x^n$; then using the equations $4x\cdot L(x)$, $-5x^2\cdot L(x)$ and $2x^3\cdot L(x)$, we obtain

$$\begin{array}{rccccc}
-L(x) = & -\mathbf{ML}_0 - & \mathbf{ML}_1 x - & \mathbf{ML}_2 x^2 - & \mathbf{ML}_3 x^3 & \ldots \\
 & & & & -\mathbf{ML}_n x^n - & \ldots \\
4x\cdot L(x) = & & 4\mathbf{ML}_0 x + & 4\mathbf{ML}_1 x^2 + & 4\mathbf{ML}_2 x^3 + & \ldots \\
 & & & & +4\mathbf{ML}_{n-1} x^n + & \ldots \\
-5x^2\cdot L(x) = & & & -5\mathbf{ML}_0 x^2 - & 5\mathbf{ML}_1 x^3 & \ldots \\
 & & & & -5\mathbf{ML}_{n-2} x^n - & \ldots \\
2x^3\cdot L(x) = & & & & 2\mathbf{ML}_0 x^3 + & \ldots \\
 & & & & +2\mathbf{ML}_{n-3} x^n + & \ldots
\end{array}$$

When we add both sides of these equations, we have:

$$\begin{aligned}
(2x^3 - 5x^2 + 4x - 1)L(x) &= -\mathbf{ML}_0 + (4\mathbf{ML}_1 - \mathbf{ML}_0)x + (-5\mathbf{ML}_0 + 4\mathbf{ML}_1 - \mathbf{ML}_2)x^2 + \\
&\quad (2\mathbf{ML}_0 - 5\mathbf{ML}_1 + 4\mathbf{ML}_2 - \mathbf{ML}_3)x^3 + \ldots \\
&\quad + (2\mathbf{ML}_{n-3} - 5\mathbf{ML}_{n-2} + 4\mathbf{ML}_{n-1} - \mathbf{ML}_n)x^n + \ldots \\
&= -\mathbf{ML}_0 + (4\mathbf{ML}_1 - \mathbf{ML}_0)x + (-5\mathbf{ML}_0 + 4\mathbf{ML}_1 - \mathbf{ML}_2)x^2 + 0 + 0\ldots
\end{aligned}$$

Since $\mathbf{ML}_0 = 0$, $\mathbf{ML}_1 = 1$, and $\mathbf{ML}_2 = 4$, the result follows easily. □

The exponential generating function $E_{a_n}(x)$ of a sequence $\{a_n\}_{n\geq 0}$ is defined as a power series of the form:

$$E_{a_n}(x) = a_0 + a_1 x + \frac{a_2 x^2}{2!} + \cdots + \frac{a_n x^n}{n!} + \cdots = \sum_{n=0}^{\infty} \frac{a_n x^n}{n!}\ .$$

In the next result, we consider $a_n = \mathbf{ML}_n$ and make use of equation (3.5), the Binet formula for Modified Mersenne–Leonardo sequence, and then we obtain the exponential generating function for this sequence.

Proposition 3.8. *For all $n \geq 0$ the exponential generating function for the Modified Mersenne–Leonardo sequence $\{\mathbf{ML}_n\}_{n\geq 0}$ is*

$$E_{\mathbf{ML}_n}(x) = 2e^{2x} - (x+2)e^x\ .$$

Proof. Note that

$$\begin{aligned} E_{\mathcal{M}_n}(x) &= \sum_{n=0}^{\infty} \frac{\mathcal{M}_n x^n}{n!} = \sum_{n=0}^{\infty} \frac{2^{n+1}-(n+2)}{n!} x^n \\ &= \sum_{n=0}^{\infty} \frac{2^{n+1}}{n!} x^n - \sum_{n=0}^{\infty} \frac{n+2}{n!} x^n = 2e^{2x} - (x+2)e^x. \end{aligned}$$

□

The Poisson generating function $P_{a_n}(x)$ for a sequence $\{a_n\}_{n\geq 0}$ is given by:

$$P_{a_n}(x) = \sum_{n=0}^{\infty} a_n \frac{x^n}{n!} e^{-x} .$$

This function generates the sequence $\{a_n\}_{n\geq 0}$ in terms of the parameter x. A relationship can be observed between exponential generation $E_{a_n}(x)$ and Poisson generation $P_{a_n}(x)$, which may be expressed by the following equation:

$$P_{a_n}(x) = e^{-x} E_{a_n}(x) .$$

As a consequence, the corresponding Poisson-generating function is obtained.

Corollary 3.9. *For all $n \geq 0$ the Poisson generating function for the Modified Mersenne–Leonardo sequence $\{\mathcal{M}_n\}_{n\geq 0}$ is*

$$P_{\mathcal{M}_n}(x) = 2e^x - (x+2) .$$

To conclude this section, we determine the limit of the ratio $\frac{\mathcal{M}_{n+1}}{\mathcal{M}_n}$, where $\mathcal{M}_n$ be the n-th term of Modified Mersenne–Leonardo sequence. Again, using Binet's formula, the equation (3.5), we get another property of Modified Mersenne-Leonardo sequences $\{\mathcal{M}_n\}_{n\in\mathbb{Z}}$, which is stated by the following proposition.

Proposition 3.10. *For all non-negative integer n, let $\mathcal{M}_n$ be the n-th term of Modified Mersenne–Leonardo sequence, then*

$$\lim_{n\to\infty} \frac{\mathcal{M}_{n+1}}{\mathcal{M}_n} = 2 . \tag{3.7}$$

Proof. We have that

$$\begin{aligned} \lim_{n\to\infty} \frac{\mathcal{M}_{n+1}}{\mathcal{M}_n} &= \lim_{n\to\infty} \frac{2^{n+2}-(n+3)}{2^{n+1}-(n+2)} \\ &= \lim_{n\to\infty} \frac{2}{1-\dfrac{n+2}{2^{n+1}}} - \lim_{n\to\infty} \frac{\frac{n+3}{2^{n+1}}}{1-\dfrac{n+2}{2^{n+1}}} = 2, \end{aligned}$$

since $\lim_{n\to\infty} \dfrac{n+k}{2^{n+1}} = 0$, for some integer k fixed. □

Furthermore, the following result can be demonstrated by using the basic techniques of the calculation of the limits and equation (3.7).

Corollary 3.11. *For all non-negative integers n, let $\mathcal{M}_n$ be the n-th term of Modified Mersenne–Leonardo sequence, then*

$$\lim_{n\to\infty} \frac{\mathcal{M}_{n-1}}{\mathcal{M}_n} = \frac{1}{2} .$$

3.3 Some identities

In this section, we derive and examine several identities associated with the Modified Mersenne–Leonardo sequence $\{\mathcal{M}_n\}_{n\geq 0}$. Through an exploration of these fundamental identities, our objective is to deepen the understanding of the structural properties and behavior of the Modified Mersenne–Leonardo sequence, clarifying its mathematical significance.

A direct calculation and employing the Binet formula, equation (3.5), yields the following result:

Proposition 3.12. *Let n and k be non-negative integers with $n \geq k$, and $\{\mathcal{M}_k\}_{k\geq 0}$ the Modified Mersenne–Leonardo sequence. Then the following identity holds:*

$$\mathcal{M}_{n+k}\mathcal{M}_{n-k} = 4^{n+1} + \Big((k(4^k-1)-(n+2)(4^k+1))2^{n-k+1}\Big) + \big(n^2+4n+4-k^2\big) \ . \tag{3.8}$$

Proof. Note that

$$\begin{aligned}\mathcal{M}_{n+k}\mathcal{M}_{n-k} &= (2^{n+k+1}-(n+k+2))(2^{n-k+1}-(n-k+2)) \\ &= 2^{2n+2}-(n-k+2)2^{n+k+1}-(n+k+2)2^{n-k+1}+(n+k+2)(n-k+2) \\ &= 4^{n+1}+\Big((k(4^k-1)-(n+2)(4^k+1))2^{n-k+1}\Big)+((n\cdot n)+(4\cdot n)+4-(k\cdot k)) \ ,\end{aligned}$$

as required. □

As an example, consider $n = 4$ and $k = 3$, so we have

$$\mathcal{M}_7\mathcal{M}_1 = 247\cdot 1 = 247,$$

and we get,

$$4^5+(3(4^3-1)-6(4^3+1))2^2+(4^2+4^2+4-9) = 1024+(189-390)4+27 = 247 \ .$$

From Proposition 3.12 follows the next result.

Corollary 3.13. *Let n be non-negative integers, and $\{\mathcal{M}_n\}_{n\geq 0}$ the Modified Mersenne–Leonardo sequence. Then the following identities hold:*

(a) $\mathcal{M}_{n+2}\mathcal{M}_{n-2} = n^2+\big(2^{n+3}-17n-4\big)2^{n-1}+4n$;

(b) $\mathcal{M}_{n+1}\mathcal{M}_{n-1} = n^2+\big(2^{n+2}-5n-7\big)2^{n}+4n+3$.

Proof. Setting $k = 2$ and $k = 1$, respectively, in equation (3.8). Since $M_2 = 3$ and $M_4 = 15$, we have the result. □

Other interesting result:

Proposition 3.14. *Let be non-negative the integers n and $\{\mathcal{M}_k\}_{k\geq 0}$ the Modified Mersenne–Leonardo sequence. Then the following identity holds:*

$$\mathcal{M}_{m+3}\mathcal{M}_{m+4}-\mathcal{M}_{m+1}\mathcal{M}_{m+6} = (84n+160)2^n+6 \ .$$

Proof. Note that

$$\begin{aligned}\mathcal{M}_{m+3}\mathcal{M}_{m+4}-\mathcal{M}_{m+1}\mathcal{M}_{m+6} &= [2^{n+4}-(n+5)][2^{n+5}-(n+6)]-[2^{n+2}-(n+3)][2^{n+7}-(n+9)] \\ &= 2^{n+2}[(n+8)-4(n+6)]+2^{n+5}[4(n+3)-(n+5)]+6 \\ &= (84n+160)2^n+6,\end{aligned}$$

as required. □

Take $n = 5$, and we present the following example, and we have

$$\mathcal{M}_8\mathcal{M}_9-\mathcal{M}_6\mathcal{M}_{11} = 502\cdot 1013-120\cdot 4083 = 18566 \ ;$$

on the other hand

$$(84\cdot 5+160)2^5+6 = 580\cdot 32+6 = 18566.$$

The following result demonstrates the efficacy of the Binet formula and helps to illustrate the convolution identity for the Modified Mersenne–Leonardo sequence.

Proposition 3.15 (Convolution's Identity)**.** *Let $\{\mathcal{M}_n\}_{n\geq 0}$ be the Modified Mersenne–Leonardo sequence, we have the following identities:*

$$\mathcal{M}_{m-1}\mathcal{M}_n+\mathcal{M}_m\mathcal{M}_{n+1}=10\cdot 2^{m+n}-(6m+10)2^n-(3n+8)2^m+(2mn+5m+3n+8)$$

for all m and n non-negative integers.

Proof. Applying the Binet formula for Modified Mersenne–Leonardo sequence, equation (1.1), we have

$$\begin{aligned}\mathcal{M}_{m-1}\mathcal{M}_n+\mathcal{M}_m\mathcal{M}_{n+1} &= [2^m-(m+1)][2^{n+1}-(n+2)]+[2^{m+1}-(m+2)][2^{n+2}-(n+3)]\\ &= [2^{m+n+1}-(n+2)2^m-(m+1)2^{n+1}+(m+1)(n+2)]+[2^{m+n+3}-(n+3)2^{m+1}-(m+2)2^{n+2}+(m+2)(n+3)]\\ &= 10\cdot 2^{m+n}-(6m+10)2^n-(3n+8)2^m+(2mn+5m+3n+8)\end{aligned}$$

as required. □

Now, the Tagiuri-Vajda's identity for the Modified Mersenne–Leonardo sequence $\{\mathcal{M}_n\}_{n\geq 0}$ is stated in as follows.

Theorem 3.16. *Let n,s,k be non-negative integers, and $\{\mathcal{M}_n\}_{n\geq 0}$ the Modified Mersenne–Leonardo sequence. We have*

$$\mathcal{M}_{n+s}\mathcal{M}_{n+k}-\mathcal{M}_n\mathcal{M}_{n+s+k}=2^{n+1}[(n+2)M_sM_k-kM_s-sM_k]+ks\ , \tag{3.9}$$

where $\{M_n\}_{n\geq 0}$ is the Mersenne sequence.

Proof. Using equation (3.5) again we obtain that

$$\begin{aligned}\mathcal{M}_{n+s}\mathcal{M}_{n+k}-\mathcal{M}_n\mathcal{M}_{n+s+k} &= [2^{n+s+1}-(n+s+2)][2^{n+k+1}-(n+k+2)]-[2^{n+1}-(n+2)][2^{n+s+k+1}-(n+s+k+2)]\\ &= (n+s+k+2)2^{n+1}-(n+k+2)2^{n+s+1}-(n+s+2)2^{n+k+1}+(n+2)2^{n+s+k+1}\\ &= [2n(2^{n+k+s}-2^{n+s}-2^{n+k}+2^n)]+[4(2^{n+k+s}-2^{n+s}-2^{n+k}+2^n)]-[2s(2^{n+k}-2^n]-[2k(2^{n+2}-2^n].\end{aligned}$$

By Lemma 2.3 we have

$$\begin{aligned}\mathcal{M}_{n+s}\mathcal{M}_{n+k}-\mathcal{M}_n\mathcal{M}_{n+s+k} &= 2n\cdot 2^nM_sM_k+4\cdot 2^nM_sM_k-2k\cdot 2^nM_s-2s\cdot 2^nM_k+ks\\ &= 2^n[(2n+4)M_sM_k-2kM_s-2sM_k]+ks\end{aligned}$$

and we have the validity of the result. □

In this example, take $n=4$, $s=1$ and $k=3$, and we have

$$\mathcal{M}_5\mathcal{M}_8-\mathcal{M}_4\mathcal{M}_8=57\cdot 247-26\cdot 502=1027\ ,$$

on the other hand

$$2^5[6M_1M_3-3M_1-1M_3]+3=32[6\cdot 7-3-7]+3=1027\ .$$

As a consequence of Tagiuri-Vajda's identity, the subsequent results of this section establish the respective identities of d'Ocagne, Catalan, and Cassini for the Mersenne-Leonard numbers.

First the d'Ocagne identity:

Proposition 3.17. *Let r,n be non-negative integers with $r\geq n$, and $\{\mathcal{M}_n\}_{n\geq 0}$ the Modified Mersenne–Leonardo sequence, then*

$$\mathcal{M}_{n+1}\mathcal{M}_r-\mathcal{M}_n\mathcal{M}_{r+1}=2^{n+1}[(n+1)M_{r-n}-(r-n)]+(r-n),$$

where $\{M_n\}_{n\geq 0}$ is the Mersenne sequence.

Proof. Consider $k=r-n$ and $s=1$ in equation (3.9), then

$$\mathcal{M}_{n+1}\mathcal{M}_r-\mathcal{M}_n\mathcal{M}_{r+1}=2^{n+1}[(n+2)M_1M_{r-n}-(r-n)M_1-M_{r-n}]+(r-n)$$

as $M_1=1$, we have

$$\mathcal{M}_{n+1}\mathcal{M}_r-\mathcal{M}_n\mathcal{M}_{r+1}=2^{n+1}[(n+1)M_{r-n}-(r-n)]+(r-n),$$

which proves the result. □

Now, the Catalan identity:

Proposition 3.18. *Let n, k be non-negative integers with $n \geq k$, and $\{\boldsymbol{M\!L}_n\}_{n\geq 0}$ the Modified Mersenne–Leonardo sequence, then*

$$\boldsymbol{M\!L}_{n+k}\boldsymbol{M\!L}_{n-k} - (\boldsymbol{M\!L}_n)^2 = 2^{n-k+1}[(1+2^k)kM_k - (n+2)M_k^2] - k^2 \ , \tag{3.10}$$

where $\{M_n\}_{n\geq 0}$ is the Mersenne sequence.

Proof. Let us assume that $s = -k$ in equation (3.9), and then

$$\boldsymbol{M\!L}_{n+k}\boldsymbol{M\!L}_{n-k} - \boldsymbol{M\!L}_n^2 = 2^{n+1}[(n+2)M_{-k}M_k - kM_{-k} - (-k)M_k] + k(-k).$$

Since $M_{-k} = \frac{-M_k}{2^k}$, the result follows. □

As a consequence of Catalan's identity is

Corollary 3.19. *For all non-negative integer n, we have*

$$\boldsymbol{M\!L}_n^2 - \boldsymbol{M\!L}_{n+2}\boldsymbol{M\!L}_{n-2} = (9n-12)2^{n-1} + 4,$$

where $\{\boldsymbol{M\!L}_n\}_{n\geq 0}$ is the Modified Mersenne–Leonardo sequence.

Proof. By doing $k = 2$ in equation (3.10), we have

$$\begin{aligned}\boldsymbol{M\!L}_{n+2}\boldsymbol{M\!L}_{n-2} - \boldsymbol{M\!L}_n^2 &= 2^{n-2+1}[(1+2^2)2M_2 - (n+2)M_2^2] - 2^2 \\ &= 2^{n-1}[10M_2 - (n+2)M_2^2] - 4,\end{aligned}$$

since $M_2 = 3$, we get

$$\boldsymbol{M\!L}_{n+2}\boldsymbol{M\!L}_{n-2} - \boldsymbol{M\!L}_n^2 = 2^{n-1}[30 - 9(n+2)] - 4,$$

and we have the result required. □

Another consequence of Catalan's identity, by doing $k = 1$ in equation (3.10) and since $M_1 = 1$, we have the following result.

Corollary 3.20. *[Cassini's identity] For all $n \in \mathbb{Z}$ then*

$$\boldsymbol{M\!L}_n^2 - \boldsymbol{M\!L}_{n+1}\boldsymbol{M\!L}_{n-1} = (n-1)2^n + 1,$$

where $\{\boldsymbol{M\!L}_n\}_{n\geq 0}$ is the Modified Mersenne–Leonardo sequence.

As a example, consider $n = 10$, so we have

$$\boldsymbol{M\!L}_{10}^2 - \boldsymbol{M\!L}_{11}\boldsymbol{M\!L}_9 = (2036)^2 - 4083 \cdot 1013 = 9217 \ ,$$

and we get,

$$9 \cdot 2^{10} + 1 = 9217 \ .$$

To finish this subsection, making the substitution of $n = 2m$ in Corollary 3.20, we obtain:

Corollary 3.21. *For all integer $m \geq 1$, we have*

$$\boldsymbol{M\!L}_{2m}^2 - \boldsymbol{M\!L}_{2m+1}\boldsymbol{M\!L}_{2m-1} = (2m-1)4^m + 1,$$

where $\{\boldsymbol{M\!L}_n\}_{n\geq 0}$ is the Modified Mersenne–Leonardo sequence.

This corollary is other Cassini's type identity where in this case, the first term on the left side of the equation is always considered with an even subscript.

3.4 Sum of terms involving the Modified Mersenne–Leonardo numbers

In this section, we present the results of our investigation into the partial sums of the Modified Mersenne–Leonardo numbers, considering a variable number of terms. Specifically, we analyze the sequence of partial sums, defined as the sum of the terms of the Modified Mersenne–Leonardo sequence for a given non-negative value of n,

$$\sum_{k=0}^{n} \mathcal{M}_k = \mathcal{M}_0 + \mathcal{M}_1 + \mathcal{M}_2 + \cdots + \mathcal{M}_n \ ,$$

for $n \geq 0$, and where $\{\mathcal{M}_n\}_{n\geq 0}$ is the Modified Mersenne–Leonardo sequence.

Proposition 3.22. *Let $\{\mathcal{M}_n\}_{n\geq 0}$ be the Modified Mersenne–Leonardo sequence, the following identities hold:*

$$\begin{aligned} &(a) \quad \textstyle\sum_{k=0}^{n} \mathcal{M}_k &&= M_{n+2} - \frac{(n+2)(n+3)}{2} \ , \\ &(b) \quad \textstyle\sum_{k=0}^{n} \mathcal{M}_{2k} &&= \frac{2}{3} M_{2n+1} - (n+1)(n+2) \ , \\ &(c) \quad \textstyle\sum_{k=0}^{n} \mathcal{M}_{2k+1} &&= \frac{8}{3} M_{2n+1} - (n+1)(n+3) \ . \end{aligned}$$

where $\{M_n\}_{n\geq 0}$ is the Mersenne sequence.

Proof. (a) Follows from the definition of partial sum of terms of the Modified Mersenne–Leonardo numbers, and making use of the Binet formula for Modified Mersenne–Leonardo sequence, the equation (3.5), we get

$$\begin{aligned} \sum_{k=0}^{n} \mathcal{M}_k &= \mathcal{M}_0 + \mathcal{M}_1 + \cdots + \mathcal{M}_n \\ &= (2-2) + (2^2-3) + (2^3-4) + \cdots + \big(2^{n+1} - (n+2)\big) \\ &= (1+2+2^2+\cdots+2^{n+1}) - \big(1+2+3+\cdots+(n+2)\big) \\ &= 2^{n+2} - 1 - \frac{(n+2)(n+3)}{2} \ , \end{aligned}$$

and we have the result required.

(b) See that

$$\begin{aligned} \sum_{k=0}^{n} \mathcal{M}_{2k} &= \mathcal{M}_0 + \mathcal{M}_2 + + \mathcal{M}_4 + \cdots + \mathcal{M}_{2n} \\ &= (2-2) + (2^3-4) + (2^5-6) + \cdots + \big(2^{2n+1} - (2n+2)\big) \\ &= 2(1+2^2+2^4 \cdots + 2^{2n}) - 2\big(1+2+3+\cdots+(n+1)\big) \\ &= 2\frac{(2^2)^{n+1}-1}{2^2-1} - (n+1)(n+2) \ , \end{aligned}$$

as required.

(c) Similarly, we have

$$\begin{aligned} \sum_{k=0}^{n} \mathcal{M}_{2k+1} &= \mathcal{M}_1 + \mathcal{M}_3 + \cdots + \mathcal{M}_{2n+1} \\ &= (2^2-3) + (2^4-5) + (2^6-7) + \cdots + \big(2^{2n+2} - (2n+3)\big) \ , \end{aligned}$$

making using of (b), which verifies the result. □

Remark 3.23. *Firstly, see that $\frac{(n+2)(n+3)}{2}$ is always integer, if n is even then $n+2$ is even; otherwise, $n+3$ is even. Now, it can be demonstrated that $3 = 2^2 - 1$ divides $M_{2n+1} = (2^2)^{n+1} - 1$ since that the condition $a - b$ divides $a^k - b^k$ for all integers a, b and k non-negative is satisfied.*

A direct consequence of the Proposition 3.22 is the next result.

Proposition 3.24. *Let be $\{\mathcal{M}_n\}_{n\geq 0}$ is the Modified Mersenne–Leonardo sequence and $\{M_n\}_{n\geq 0}$ is the Mersenne sequence. For $n \geq 0$, the following identities hold:*

$$\sum_{j=0}^{n}(-1)^k\mathcal{M}_k = (n+1) - \frac{6}{8}M_{2n+1};$$

if last term is negative, and

$$\sum_{j=0}^{n}(-1)^k\mathcal{M}_k = \frac{2}{3}(M_{2n+3} - 3M_{2n+1}) + (n+1);$$

if last term is positive.

Proof. (a) First, note that the last term is negative, which results in the following considerations:

$$\begin{aligned}\sum_{k=0}^{2n+1}(-1)^k\mathcal{M}_k &= \mathcal{M}_0 - \mathcal{M}_1 + \mathcal{M}_2 - \mathcal{M}_3 + \cdots + \mathcal{M}_{2n} - \mathcal{M}_{2n+1} \\ &= (\mathcal{M}_0 + \mathcal{M}_2 + \cdots + \mathcal{M}_{2n}) - (\mathcal{M}_1 + \mathcal{M}_3 + \cdots + \mathcal{M}_{2n+1}) \\ &= \sum_{k=0}^{n}\mathcal{M}_{2k} - \sum_{k=0}^{n}\mathcal{M}_{2k+1}\ .\end{aligned}$$

In accordance with Proposition 3.22, items (b) and (c), the result can be deduced.

(b) In which case that last term is positive, so

$$\begin{aligned}\sum_{k=0}^{2(n+1)}(-1)^k\mathcal{M}_k &= \mathcal{M}_0 - \mathcal{M}_1 + \mathcal{M}_2 - \mathcal{M}_3 + \cdots + \mathcal{M}_{2n} - \mathcal{M}_{2n+1} + \mathcal{M}_{2n+2} \\ &= \sum_{k=0}^{n+1}\mathcal{M}_{2k} - \sum_{k=0}^{n}\mathcal{M}_{(2k+1)}.\end{aligned}$$

Similarly, as in item (a), the result can be obtained by applying Proposition 3.22. □

Finally, in the context of sequences, the difference operator, denoted by Δ, is defined as $\Delta a_n = a_n - a_{n-1}$, where $\{a_n\}_{n\geq 0}$ is a sequence.

Making $S_n = \sum_{k=0}^{n}\mathcal{M}_k$ for all integer $n \geq 0$, consider the sequence $\{S_n\}_{n\geq 0}$, where $\{\mathcal{M}_n\}_{n\geq 0}$ is the Modified Mersenne–Leonardo sequence.

Proposition 3.25. *Let be $\{\mathcal{M}_n\}_{n\geq 0}$ be the Modified Mersenne–Leonardo sequence and $S_n = \sum_{k=0}^{n}\mathcal{M}_k$. For all integer $n \geq 1$, the following identities hold:*

(a) $\Delta S_n = \mathcal{M}_n$,

(b) $\Delta^2 S_n = M_{n-1}$;

(c) $\Delta^3 S_n = 2^{n-1}$.

where $\{M_n\}_{n\geq 0}$ is the Mersenne sequence.

Proof. (a) Using the Proposition 3.22, item (a), we have

$$\begin{aligned}\Delta S_n &= S_n - S_{n-1} \\ &= \left(M_{n+2} - \frac{(n+2)(n+3)}{2}\right) - \left(M_{n+1} - \frac{(n+1)(n+2)}{2}\right) \\ &= 2^{n+1} - (n+2)\ ,\end{aligned}$$

which verifies the result.

(b) By combining Proposition 3.3 and item (a).

(c) A straight calculation. □

4. Final Considerations

In this paper, we introduced the Modified Mersenne-Leonardo numbers and studied their properties. The aim of this work was to define the Modified Mersenne-Leonardo sequence as an extension of the Mersenne sequence and to examine some of its properties, particularly the recurrence relation, summation formula, and generating function.

We hope that this study will serve as motivation for further research, enabling a deeper exploration of the properties and applications of these sequences. We believe that they can be extended to the sets of complex numbers, quaternions, and hybrid numbers.

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Research Article

The New Hahn Sequence Space via (p,q)-Calculus

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Abstract
In this paper, a novel generalized Hahn sequence space, denoted as $h(C(p,q))$, is introduced by utilizing the (p,q)-Cesàro matrix. Fundamental properties of this sequence space, such as its completeness and inclusion relations with other well-known sequence spaces, are explored. The duals of this newly constructed sequence space are also determined, providing insights into its structural and functional characteristics. Furthermore, matrix mapping classes of the form $(h(C(p,q)):\mu)$ are characterized for various classical sequence spaces $\mu \in \{c_0, c, \ell_\infty, \ell_1, h\}$, extending the applicability of the proposed space to broader mathematical contexts. The results obtained contribute to the ongoing development of sequence space theory and its applications in functional analysis.

Keywords: Duals, Hahn sequence space, Matrix mappings, (p,q)-calculus, (p,q)-Cesàro matrix
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1. Introduction

The set containing all sequences of real or complex numbers is symbolized by ω. Each linear subspace of ω is referred to as a sequence space. Any complete metric sequence space Θ with continuous coordinates $f_s : \Theta \longrightarrow \mathbb{C}$, described by $f_s(u) = u_s$, is named as an FK-space for all $u = (u_s) \in \Theta$ and $s \in \mathbb{N}$, where $\mathbb{C}$ represents the complex field and $\mathbb{N} = \{0,1,2,3,\dots\}$. Furthermore, a BK-space is a normed FK-space. Some prominent examples of sequence spaces are c (the space of convergent sequences), c_0 (the space of null sequences), ℓ_∞ (the space of bounded sequences), and ℓ_p (the space of p-summable sequences).

The aforementioned spaces are BK-spaces due to the norms $\|u\|_{\ell_\infty} = \|u\|_c = \|u\|_{c_0} = \sup_{s\in\mathbb{N}} |u_s|$ and $\|u\|_{\ell_p} = \left(\sum_{s=0}^{\infty} |u_s|^p\right)^{1/p}$ for $1 \le p < \infty$.

Consider $\mathscr{D} = (d_{rs})_{\mathbb{N}\times\mathbb{N}}$ as an infinite matrix with real or complex elements. It will be denoted by $\mathscr{D}_r = (d_{rs})_{s=0}^{\infty}$ the sequence in the r^{th} row of $\mathscr{D}$ for every $r \in \mathbb{N}$. The $\mathscr{D}$-transform of a sequence $u = (u_s) \in \omega$, denoted by $(\mathscr{D}u)_r$, is described as $\sum_{s=0}^{\infty} d_{rs}u_s$, assuming that the series converges for every $r \in \mathbb{N}$.

Consider the sequence spaces Θ and Λ. A matrix $\mathscr{D}$ is called as a matrix mapping from Θ to Λ, if for all $u \in \Theta$, the image $\mathscr{D}u$ belongs to Λ. The class of all such matrices that defines a mapping from Θ to Λ is denoted by $(\Theta : \Lambda)$. Additionally, the notation $\Theta_{\mathscr{D}}$ is employed to represent the set of all sequences for which the $\mathscr{D}$-transform is contained in Θ, as expressed by:

$$\Theta_{\mathscr{D}} = \{u \in \omega : \mathscr{D}u \in \Theta\}.$$

In this case, $\Theta_{\mathscr{D}} \subset \omega$, too and $\Theta_{\mathscr{D}}$ is named as matrix domain of $\mathscr{D}$.

If $\mathscr{D}u \in c$ for every $u \in c$, the matrix $\mathscr{D}$ is known as conservative matrix. Moreover, the conservative matrix $\mathscr{D}$ that preserves the limit is known as regular.

In the presence of a linear bijection, which preserves the norm between Θ and Λ, these spaces are linearly isomorphic spaces, and this situation is denoted by $\Theta \cong \Lambda$.

When $u = (u_s) \in \Lambda$, if $v = (v_s) \in \Lambda$ for all vectors v that satisfy the condition $|v_s| \leq |u_s|$ for $s \in \mathbb{N}$, in that case the set $\Lambda \in \omega$ is said to be normal.

Consider that the sequence e^s whose s^{th} term is 1 and remaining terms are 0 and $e = (1,1,1,...)$. For an FK-space Λ, it can be given the following definitions:

1. [1] Λ is a wedge space if $e^s \to 0$ in Λ,
2. [2] Λ is a conservative space if $c \subset \Lambda$,
3. [2] Λ is a semi-conservative space if $\Lambda^{\mathscr{G}} \subset cs$ (equivalently $c_0 \subset \Lambda$) for $\Lambda^{\mathscr{G}} = \{(\mathscr{G}(e^s)) : \mathscr{G} \in \Lambda'\}$, where Λ' denotes the continuous dual of Λ.

Let the acronym ψ represents the set of sequences whose terms are all zero except for a finite number of them. For an FK-space $\Lambda \supset \psi$, the s^{th} section of $u \in \Lambda$ is denoted by $u^{[s]} = \sum_{s=1}^{r} u_s e^s$. If $u^{[s]} \to u$ $(s \to \infty)$ for all $u \in \Lambda$, it is said that the FK-space $\Lambda \supset \psi$ has AK. Moreover, if ψ is dense in Λ, in that case it is said that Λ has AD. It should be noted that if Λ has AK, then Λ has AD.

Studies examining new spaces obtained by the aid of special matrices and necessary basic concepts about sequence spaces can be found in studies [3, 4, 5, 6, 7, 8, 9, 10, 11].

It is known from [12], $[s]_{p,q}$, the (p,q)-integer number s is described as

$$[s]_{p,q} = \begin{cases} \frac{p^s - q^s}{p-q}, & s = 1,2,3,..., \\ 0, & s = 0, \end{cases}$$

for each $s \in \mathbb{N}$ and $0 < q < p \leq 1$.

Moreover, the q-integer number is described by

$$[s]_q = \frac{1-q^s}{1-q}, \ (s = 1,2,3,\dots), \ \ q \neq 1.$$

Based on the above discussion, by choosing $p = 1$, $[s]_{p,q}$ is reduced to $[s]_q$, and it is understood that $\lim_{q \to 1^-} \lim_{p \to 1^-} [s]_{p,q} = s$. Extensive information about q- and (p,q)-calculus can be obtained from studies [12, 13, 14].

The (p,q)-Cesàro matrix $C(p,q) = \left(c_{rs}^{p,q}\right)$ is described as

$$c_{rs}^{p,q} = \begin{cases} \dfrac{p^{r-s} q^s}{[r+1]_{p,q}}, & (0 \leq s \leq r), \\ 0, & (s > r) \end{cases}$$

for $0 < q < p \leq 1$ [15].

Due to the triangularity of $C(p,q)$, its inverse $C(p,q)^{-1} = \left(\left\{c_{rs}^{p,q}\right\}^{-1}\right)$ is expressed uniquely in the form

$$\{c_{rs}^{p,q}\}^{-1} = \begin{cases} (-1)^{r-s} \frac{p^{r-s}[s+1]_{p,q}}{q^r}, & (r-1 \leq s \leq r), \\ 0, & \text{otherwise}. \end{cases}$$

The q-analogue of C_1 (the first order Cesàro mean) is denoted by $C(q)$, while the (p,q)-analogue is denoted by $C(p,q)$. When $p = 1$, it is obvious that $C(p,q)$ simplifies to $C(q)$, which then reduces further to C_1 as $q \to 1$. As a result, $C(p,q)$ is a generalization of the matrices $C(q)$ and C_1.

The space bv, described as the domain of the forward difference operator Δ on ℓ_1, is in the form

$$bv = \left\{ u = (u_s) \in \omega : \sum_{s=1}^{\infty} |u_s - u_{s+1}| < \infty \right\}.$$

Furthermore, bv is a BK space with the norm

$$\|u\|_{bv} = \sum_{s=1}^{\infty} |u_s - u_{s+1}| \qquad (\forall u = (u_s) \in bv).$$

The Hahn sequence space h described in [16] is expressed by

$$h = \left\{ u = (u_s) \in \omega : \sum_{s=1}^{\infty} s|u_s - u_{s+1}| < \infty \right\} \cap c_0$$

and it is a BK-space with

$$\|u\| = \sum_{s=1}^{\infty} s|u_s - u_{s+1}| + \sup_s |u_s| \text{ for all } u = (u_s) \in h.$$

Furthermore, Rao [17] obtained that h is a BK space with AK with

$$\|u\|_h = \sum_{s=1}^{\infty} s|u_s - u_{s+1}| \text{ for all } u = (u_s) \in h.$$

After that, Goes [18] described the generalized Hahn space h^d expressed by

$$h^d = \{u = (u_s) \in \omega : \sum_{s=1}^{\infty} |d_s||u_s - u_{s+1}| < \infty\} \cap c_0$$

for $d = (d_s) \in \omega$ and $d_s \neq 0$.

A more general form of the Hahn sequence space is presented in [19] by

$$h_d = \{u = (u_s) \in \omega : \sum_{s=1}^{\infty} d_s|u_s - u_{s+1}| < \infty\} \cap c_0$$

for an unbounded and monotonically increasing sequence $d = (d_s)$ of positive real numbers. Studies examining Hahn sequence spaces and the necessary basic concepts about this field can be found in studies [17, 18, 19, 20, 21, 22, 23, 24, 25].

In this study, primarily, a new BK-space is described as the domain of $C(p,q)$ in the Hahn sequence space h, as an application of (p,q)-calculus to sequence spaces. After that, in order to specify the position of the mentioned space between the others, inclusion relations are incorporated, some algebraic and topological properties are examined, and its duals are calculated. At the end, some matrix transformations are presented.

2. Hahn Sequence Space $h(C(p,q))$

This section focuses on constructing a new Hahn sequence space $h(C(p,q))$, the relevant inclusion relations, some algebraic and topological properties of the aforementioned space, and its basis.

The sequence $v = (v_r)$, which is the $C(p,q)$-transform of any sequence u, is expressed as

$$v_r = (C(p,q)u)_r = \sum_{s=0}^{r} \frac{p^{r-s}q^s}{[r+1]_{p,q}} u_s. \tag{2.1}$$

Now, we construct the new generalized Hahn sequence space $h(C(p,q))$ by using (p,q)-Cesàro matrix as follows

$$h(C(p,q)) = \left\{ u = (u_r) \in \omega : \sum_{r=1}^{\infty} r|\Delta(C(p,q)u)_r| < \infty \text{ and } \lim_{r\to\infty} (C(p,q)u)_r = 0 \right\}$$

where

$$\begin{aligned} \Delta(C(p,q)u)_r &= (C(p,q)u)_r - (C(p,q)u)_{r+1} \\ &= \sum_{s=0}^{r} \left(\frac{p^{r-s}q^s}{[r+1]_{p,q}} - \frac{p^{r+1-s}q^s}{[r+2]_{p,q}} \right) u_s - \frac{q^{r+1}}{[r+2]_{p,q}} u_{r+1} \quad (r \in \mathbb{N}). \end{aligned} \tag{2.2}$$

We see that $h(C(p,q)) = h_{C(p,q)}$. In other words, $h(C(p,q))$ is domain of $C(p,q)$ in h. It can be noted that, as $p \to 1$, the space $h(C(p,q))$ is reduced to the space $h(C^q)$ presented by Yaying et al. [23].

On the other hand, it is possible to rewrite equation (2.1) as

$$u_r = \sum_{s=r-1}^{r} (-1)^{r-s} \frac{p^{r-s}[s+1]_{p,q}}{q^r} v_s \tag{2.3}$$

assuming that terms of sequences with negative indexes are 0.

Theorem 2.1. *$h(C(p,q))$ is a BK-space with*

$$\|u\|_{h(C(p,q))} = \sum_{r=1}^{\infty} r\,|\Delta(C(p,q)u)_r| < \infty. \tag{2.4}$$

Proof. It seems reasonable to suggest that since the matrix $C(p,q)$ is triangular and h is BK-space with $||\cdot||_h$, according to Theorem 4.3.2 of [2, p.61], $h(C(p,q))$ is BK-space with (2.4). □

Theorem 2.2. *$h(C(p,q)) \cong h$.*

Proof. For all u in $h(C(p,q))$, describe the mapping $\tau : h(C(p,q)) \to h$ as $\tau u = C(p,q)u = v$. In this case, τ is linear and one-to-one. Assuming that $u = (u_s)$ is defined as in (2.3), then $v = (v_r)$ can be any sequence in h.

Given that $v \in h$, by taking into consideration (2.2) and (2.3), it is reached that

$$\begin{aligned}
\|u\|_{h(C(p,q))} &= \sum_{r=1}^{\infty} r\,|\Delta(C(p,q)u)_r| \\
&= \sum_{r=1}^{\infty} r \left| \sum_{s=0}^{r} \left(\frac{p^{r-s}q^s}{[r+1]_{p,q}} - \frac{p^{r+1-s}q^s}{[r+2]_{p,q}} \right) u_s - \frac{q^{r+1}}{[r+2]_{p,q}} u_{r+1} \right| \\
&= \sum_{r=1}^{\infty} r \left| \sum_{s=0}^{r} \left(\frac{p^{r-s}q^s}{[r+1]_{p,q}} - \frac{p^{r+1-s}q^s}{[r+2]_{p,q}} \right) \left(\sum_{j=s-1}^{s} (-1)^{s-j} \frac{p^{s-j}[j+1]_{p,q}}{q^s} v_j \right) \right. \\
&\qquad \left. - \frac{q^{r+1}}{[r+2]_{p,q}} \left(\sum_{s=r}^{r+1} (-1)^{r+1-s} \frac{p^{r+1-s}[s+2]_{p,q}}{q^{r+1}} v_s \right) \right| \\
&= \sum_{r=1}^{\infty} r\,|\Delta v_r| = \|v\|_h < \infty.
\end{aligned}$$

Consequently, we understand that $u \in h(C(p,q))$ and τ is onto and preserves the norm. □

Theorem 2.3. *The following inclusion relations hold:*

1. $h \subset h(C(p,q))$
2. $h(C(p,q)) \subset \ell_1(C(p,q))$

Proof. 1. Let $0 < q < p \leq 1$. It is obvious that the inclusion $h \subset h(C(p,q))$ holds. Besides, let us consider the sequence $f = (f_s)_{s\in\mathbb{N}} = \left(\frac{q[s+1]_{p,q} - p[s]_{p,q}}{qp^{s+1}} \right)$. In that case,

$$\begin{aligned}
\lim_{s\to\infty} f_s &= \lim_{s\to\infty} \left(\frac{q[s+1]_{p,q} - p[s]_{p,q}}{qp^{s+1}} \right) \\
&= \lim_{s\to\infty} \left[\frac{1}{p-q} \left(1 - \left(\frac{q}{p} \right)^{s+1} \right) - \frac{1}{q} \left(1 - \left(\frac{q}{p} \right)^{s} \right) \right] \\
&= \frac{1}{p-q} - \frac{1}{q} = \frac{2q-p}{q(p-q)} \neq 0.
\end{aligned}$$

Thus f is not a sequence in h. On the other hand, $C(p,q)f = b = (b_s) = \left[\left(\frac{q}{p}\right)^s\right] \in h$. This follows from the following illustrations: We ensure that $\sum_s s|b_s - b_{s+1}| < \infty$, because $\left(\frac{q}{p}\right)^s \to 0$ for $s \to \infty$. We have

$$\begin{aligned}\sum_s s|b_s - b_{s+1}| &= \left|\frac{q}{p} - \frac{q^2}{p^2}\right| + 2\left|\frac{q^2}{p^2} - \frac{q^3}{p^3}\right| + 3\left|\frac{q^2}{p^2} - \frac{q^3}{p^3}\right| + \cdots \\ &= \frac{q}{p^2}|p-q| + 2\frac{q^2}{p^3}|p-q| + 3\frac{q^3}{p^4}|p-q| + \cdots \\ &= \frac{q}{p^2}(p-q)\left(1 + 2\frac{q}{p} + 3\frac{q^2}{p^2} + \cdots\right) \\ &\leq \frac{q}{p^2}(p-q)\frac{1}{\left(1-\frac{q}{p}\right)^2} \\ &= \frac{q}{p-q} < \infty.\end{aligned}$$

2. Consider the sequences $b_k = 2^k$ ($k \in \mathbb{N}$) and $v = (v_s)$ with

$$v_s = \begin{cases} 0, & s \neq 2^k, \\ \frac{1}{s}, & s = 2^k. \end{cases}$$

In that case, it is seen that the inclusion $h \subset \ell_1$ is strict. Consider that

$$u_s = \sum_{j=s-1}^{s} (-1)^{s-j} \frac{p^{s-j}[j+1]_{p,q}}{q^s} v_j$$

for each $s \in \mathbb{N}$. Since,

$$(C(p,q)u)_r = \sum_{s=0}^{r} \frac{p^{r-s}q^s}{[r+1]_{p,q}} u_s = \sum_{s=0}^{r} \frac{p^{r-s}q^s}{[r+1]_{p,q}} \sum_{j=s-1}^{s} (-1)^{s-j} \frac{p^{s-j}[j+1]_{p,q}}{q^s} v_j = v_r,$$

we obtain $C(p,q)u = v \in \ell_1 \backslash h$ and thus $v \in \ell_1(C(p,q)) \backslash h(C(p,q))$.

□

Theorem 2.4. *$h(C(p,q))$ has AK.*

Proof. Consider that $u = (u_r) \in h(C(p,q))$ with

$$(C(p,q)u)_r = \sum_{s=r}^{\infty} \left[(C(p,q)u)_s - (C(p,q)u)_{s+1}\right].$$

Then, it is reached that

$$r|(C(p,q)u)_r| \leq \sum_{s=r}^{\infty} s\left|(C(p,q)u)_s - (C(p,q)u)_{s+1}\right|$$

and consequently

$$\lim_{r\to\infty} r|(C(p,q)u)r| = 0. \tag{2.5}$$

By the relation (2.5), we obtain that

$$\|u - u^{[r]}\|_{C(p,q)} = r\left|(C(p,q)u)_{r+1}\right| + \sum_{s=r+1}^{\infty} s\left|(C(p,q)u)_s - (C(p,q)u)_{s+1}\right|$$

which tends to zero, as $r \to \infty$. □

Since every space that has AK also has AD, it can be given the next result:

Corollary 2.5. *$h(C(p,q))$ has AD.*

Theorem 2.6. *$h(C(p,q))$ is not normal.*

Proof. Let us take sequences $u=(u_r)=(1,-1,0,0,0,...)$ and $v=(v_r)=(1,1,0,0,0,...)$ such that $|u_r|\leq|v_r|$ for each positive integer r. Then, one can see that

$$\sum_{r=1}^{\infty} r|\Delta(C(p,q)u)_r| = \frac{q(p-q)^3}{p}\sum_{r=1}^{\infty}\frac{rp^r q^r}{(p^{r+1}-q^{r+1})(p^{r+2}-q^{r+2})} < \infty$$

that is, $u\in h(C(p,q))$ by D'Alembert's Ratio Test and

$$\sum_{r=1}^{\infty} r|\Delta(C(p,q)v)_r| = \sum_{r=1}^{\infty}\frac{rp^{r-1}2p^{r+2}-q^{r+2}-q^{r+1}p}{(p^{r+1}-q^{r+1})(p^{r+2}-q^{r+2})} = \infty.$$

Thus, it is obtained that $v\notin h(C(p,q))$. □

Theorem 2.7. *$h(C(p,q))$ is a wedge space.*

Proof. For $0<q<p\leq 1$, from the equation

$$\begin{aligned}
\|e^m-0\|_{C(p,q)} &= \sum_{r=1}^{\infty} r|\Delta(C(p,q)e^m)_r| \\
&= \frac{q^m(m-1)}{[m+1]_{p,q}} + \sum_{r=m}^{\infty} r|\Delta(C(p,q)e^m)_r| \\
&= \frac{q^m(m-1)}{[m+1]_{p,q}} + \sum_{r=0}^{\infty}(r+m)\left|\frac{p^r q^m}{[r+m+1]_{p,q}} - \frac{p^{r+1}q^m}{[r+m+2]_{p,q}}\right| \\
&= \frac{q^m(m-1)(p-q)}{p^{m+1}-q^{m+1}} + \sum_{r=0}^{\infty}(r+m)p^r q^m\left|\frac{[r+m+2]_{p,q}-p[r+m+1]_{p,q}}{[r+m+1]_{p,q}[r+m+2]_{p,q}}\right| \\
&= \frac{(m-1)(p-q)}{p\left(\frac{p}{q}\right)^m-q} + \sum_{r=0}^{\infty}(r+m)p^r q^m\left|\frac{q^{r+m+1}}{[r+m+1]_{p,q}[r+m+2]_{p,q}}\right| \\
&= \frac{(m-1)(p-q)}{p\left(\frac{p}{q}\right)^m-q} + \sum_{r=0}^{\infty}\frac{(r+m)p^r q^{r+2m+1}(p-q)^2}{(p^{r+m+1}-q^{r+m+1})(p^{r+m+2}-q^{r+m+2})},
\end{aligned}$$

we obtain that $e^m\to 0$ as $m\to\infty$ in $h(C(p,q))$, as desired. □

Theorem 2.8. *$h(C(p,q))$ isn't a conservative space.*

Proof. By choosing $u=e\in c$, we have

$$\lim_{r\to\infty}(C(p,q)u)_r = \lim_{r\to\infty}\frac{p^{r-s}q^s}{[r+1]_{p,q}} = \left(\frac{q}{p}\right)^s\frac{p-q}{p}\neq 0.$$

Consequently, $u\notin h(C(p,q))$. □

Theorem 2.9. *$h(C(p,q))$ isn't a semi-conservative space.*

Proof. Take the sequences $u=(u_r)=(1,1,0,0,0,...)\in c_0$ with the limit point 0. Then, one can see that

$$\sum_{r=1}^{\infty} r|\Delta(C(p,q)u)_r| = \sum_{r=1}^{\infty}\frac{rp^{r-1}2p^{r+2}-q^{r+2}-q^{r+1}p}{(p^{r+1}-q^{r+1})(p^{r+2}-q^{r+2})} = \infty$$

Thus, $u\notin h(C(p,q))$. □

A matrix domain $\Theta_{\mathscr{D}}$ has a basis iff Θ has a basis for a triangle $\mathscr{D}$ ([26]). It can be inferred that the Schauder basis of $h(C(p,q))$ is formed by the inverse image of the basis of h. This fact leads to the following outcomes:

Theorem 2.10. *Consider a sequence $b^{(s)} = \{b^{(s)}\}_{s\in\mathbb{N}}$ of the elements of the space $h(C(p,q))$ as*

$$b_r^{(s)} = \begin{cases} (-1)^{r-s}\dfrac{p^{r-s}[s+1]_{p,q}}{q^r} & , \quad s \le r \le s+1, \\ 0 & , \quad \text{otherwise.} \end{cases}$$

In this case, $\{b^{(s)}\}_{s\in\mathbb{N}}$ is a basis for $h(C(p,q))$, and any $u \in h(C(p,q))$ has a unique representation of the form

$$u = \sum_s \lambda_s b^{(s)}, \tag{2.6}$$

where $\lambda_s = (C(p,q)u)_s$ ($s \in \mathbb{N}$).

Proof. From the relation

$$C(p,q)b^{(s)} = e^s \in h, \tag{2.7}$$

we reach that $\{b^{(s)}\} \subset h(C(p,q))$. For $u \in h(C(p,q))$ and $n \in \mathbb{N}$, consider

$$u^{[n]} = \sum_s^n \lambda_s b^{(s)}. \tag{2.8}$$

In that case, it is obtained by applying $C(p,q)$ to (2.8) by the aid of (2.7) that

$$C(p,q)u^{[n]} = \sum_s^n \lambda_s C(p,q)b^{(s)} = \sum_s^n (C(p,q)u)_s e^s,$$

and

$$\left\{C(p,q)\left(u-u^{[n]}\right)\right\}_k = \begin{cases} 0, & 0 \le k \le n, \\ (C(p,q)u)_k, & k > n, \end{cases}$$

for all $k,n \in \mathbb{N}$. For an $\varepsilon > 0$, $\exists n_0 \in \mathbb{N} \ni$

$$|(C(p,q)u)_s| < \frac{\varepsilon}{2} \qquad (\forall n \ge n_0).$$

In that case,

$$\left\|u-u^{[n]}\right\|_{h(C(p,q))} = \sup_{r\ge n} |(C(p,q))_r| \le \sup_{r\ge n_0} |(C(p,q))_r| \le \frac{\varepsilon}{2} < \varepsilon$$

for all $n \ge n_0$, which proves that $u \in h(C(p,q))$ given by (2.6).

Consider another representation of u as $u = \sum_s \mu_s b^{(s)}$. From the continuity of the linear bijection τ described in the proof of the Theorem 2.2, it is obtained

$$(C(p,q)u)_r = \sum_s \mu \left[C(p,q)b^{(s)}\right]_r = \sum_k \mu e_r^{(s)} = \mu_r, (r \in \mathbb{N})$$

and this contradicts the situation $(C(p,q)u)_r = \lambda_r$. Hence, (2.6) is unique.

□

3. Dual Spaces

The aim of the current part is to ascertain duals of our novel sequence space. Hereafter, we refer to the set of all limited subsets of $\mathbb{N}$ as $\mathscr{N}$. Initially, let us provide a lemma which will be employed in the next results.

Lemma 3.1. *[17] The following claims are true:*

(i) $\mathscr{D}=(d_{rs})\in(h:\ell_1)$ *iff*

$$\sum_{r=1}^{\infty}|d_{rs}|<\infty,\ \ (s=1,2,...) \tag{3.1}$$

$$\sup_s\frac{1}{s}\sum_{r=1}^{\infty}\left|\sum_{j=1}^{s}d_{rj}\right|<\infty. \tag{3.2}$$

(ii) $\mathscr{D}=(d_{rs})\in(h:c)$ *iff*

$$\sup_{r,s}\frac{1}{s}\left|\sum_{j=1}^{s}d_{rj}\right|<\infty, \tag{3.3}$$

$$\lim_{r\to\infty}d_{rs}\ \textit{exists}\ (s=0,1,2,...). \tag{3.4}$$

(iii) $\mathscr{D}=(d_{rs})\in(h:c_0)$ *iff*

$$\lim_{r\to\infty}d_{rs}=0, \tag{3.5}$$

and (3.3) holds.

(iv) $\mathscr{D}=(d_{rs})\in(h:\ell_\infty)$ *iff (3.3) holds.*

(v) $\mathscr{D}=(d_{rs})\in(h:h)$ *iff* (3.5) *holds and*

$$\sum_{r=1}^{\infty}r|d_{rs}-d_{r+1,s}|<\infty,\ \ (s=1,2,...)$$

$$\sup_s\frac{1}{s}\sum_{r=1}^{\infty}r\left|\sum_{j=1}^{s}(d_{rj}-d_{r+1,j})\right|<\infty.$$

Theorem 3.2. *Define the sets Υ_1, Υ_2, Υ_3 and Υ_4, as follows:*

$$\Upsilon_1=\left\{u=(u_s)\in w:\sum_{r=1}^{\infty}\left|(-1)^{r-s}\frac{p^{r-s}[s+1]_{p,q}}{q^r}t_r\right|<\infty\right\},$$

$$\Upsilon_2=\left\{u=(u_s)\in w:\sup_s\frac{1}{s}\sum_{r=1}^{\infty}\left|\sum_{j=1}^{s}(-1)^{r-s}\frac{p^{r-s}[s+1]_{p,q}}{q^r}t_r\right|<\infty\right\},$$

$$\Upsilon_3=\left\{u=(u_s)\in w:\sup_{r,s}\frac{1}{s}\left|\sum_{i=1}^{s}\sum_{i=s}^{s+1}\frac{(-1)^{i-s}p^{i-s}[s+1]_{p,q}}{q^i}t_i\right|<\infty\right\},$$

$$\Upsilon_4=\left\{u=(u_s)\in w:\exists(\eta_s)\in\omega\ni\lim_{s\to\infty}\sum_{i=s}^{s+1}\frac{(-1)^{i-s}p^{i-s}[s+1]_{p,q}}{q^i}t_i=\eta_i\right\}$$

for all $i=1,2,\ldots$. Then the following statements hold:

1. $\{h(C(p,q)u))\}^{\alpha}=\Upsilon_1\cap\Upsilon_2,$
2. $\{h(C(p,q)u))\}^{\beta}=\Upsilon_3\cap\Upsilon_4,$
3. $\{h(C(p,q)u))\}^{\gamma}=\Upsilon_3.$

Proof. 1. Let us describe the matrix $G=(g_{rs})$ by aid of $t=(t_r)\in\omega$ by

$$g_{rs}=\begin{cases}(-1)^{r-s}\dfrac{p^{r-s}[s+1]_{p,q}}{q^r}t_r & , \quad (r-1\leqslant s\leqslant r)\\ 0 & , \quad (\text{otherwise})\end{cases}$$

for all $s,r\in\mathbb{N}$. By (2.3), it is reached that

$$t_ru_r=\sum_{s=r-1}^{r}(-1)^{r-s}\frac{p^{r-s}[s+1]_{p,q}}{q^r}t_rv_s=(Gv)_r,\ \ (r\in\mathbb{N}). \tag{3.6}$$

It follows from (3.6), $tu=(t_ru_r)\in\ell_1$ whenever $u\in h(C(p,q))$ iff $Gv\in\ell_1$ whenever $v\in h$. Hence, by (3.1) and (3.2), it is concluded that $\{h(C(p,q))\}^{\alpha}=\Upsilon_1\cap\Upsilon_2$.

2. Let us define the matrix $T=(t_{si})$ using the sequence $t=(t_s)$ by

$$t_{si}=\begin{cases}\displaystyle\sum_{i=s}^{s+1}\frac{(-1)^{i-s}p^{i-s}[s+1]_{p,q}}{q^i}t_i & , \quad (s\leqslant i\leqslant s+1),\\ 0 & , \quad (\text{otherwise}),\end{cases}$$

for all $r,s\in\mathbb{N}$. Assuming that $t=(t_s)\in\{h(C(p,q))\}^{\beta}$, the resulting sequence $tu=(t_su_s)\in cs$ converges for all $u=(u_s)\in\{h(C(p,q))\}$. To arrive at this conclusion, we examine the equality obtained by the r^{th} partial sum of the series $\sum_{s=0}^{r}t_su_s$ with (2.3)

$$\begin{aligned}\sum_{s=0}^{r}t_su_s&=\sum_{s=0}^{r}\left(\sum_{i=s-1}^{s}(-1)^{s-i}\frac{p^{s-i}[i+1]_{p,q}}{q^s}v_i\right)t_s\\ &=\sum_{s=0}^{r}\left(\frac{[s+1]_{p,q}}{q^s}v_s-\frac{p[s]_{p,q}}{q^s}v_{s-1}\right)t_s\\ &=\sum_{s=0}^{r-1}\left(\frac{[s+1]_{p,q}}{q^s}t_s-\frac{p[s+1]_{p,q}}{q^{s+1}}t_{s+1}\right)v_s+\frac{[r+1]_{p,q}}{q^r}t_rv_r\\ &=\sum_{s=0}^{r-1}\left(\sum_{i=s}^{s+1}\frac{(-1)^{i-s}p^{i-s}[s+1]_{p,q}}{q^i}t_i\right)v_s+\frac{[r+1]_{p,q}}{q^r}t_rv_r\end{aligned} \tag{3.7}$$

for any $r\in\mathbb{N}$. Recognizing that $h(C(p,q))\cong h$, we consider the limit that r approaches infinity in (3.7). Given that the series $\sum_{s=0}^{r}t_su_s$ is convergent, the series

$$\sum_{s=0}^{r-1}\left(\sum_{i=s}^{s+1}\frac{(-1)^{i-s}p^{i-s}[s+1]_{p,q}}{q^i}t_i\right)v_s$$

is also convergent and the term $\frac{[r+1]_{p,q}}{q^r}t_rv_r$ in the right side of (3.7) must tend to zero, as $r\to\infty$. Since $h\subset c_0$ this is achieved with $\frac{[r+1]_{p,q}}{q^r}t_rv_r\in\ell_\infty$, we therefore have

$$\sum_{s=0}^{\infty}t_su_s=\sum_{s=0}^{\infty}\left(\sum_{i=s}^{s+1}\frac{(-1)^{i-s}p^{i-s}[s+1]_{p,q}}{q^i}t_i\right)v_s=(Tv)_s \tag{3.8}$$

for any $s\in\mathbb{N}$. Hence, $T=(t_{si})\in(h:c)$. Thus, the conditions in (3.3) and (3.4) conditions are satisfied by the matrix T. Hence, $t=(t_s)\in\Upsilon_3\cap\Upsilon_4$.

Conversely, suppose that $t=(t_s)\in\Upsilon_3\cap\Upsilon_4$. Then, we again obtain the relation (3.8) by using (3.7). Therefore, since we have $T=(t_{si})\in(h:c)$ the series $\sum_{s=0}^{\infty}t_su_s$ is convergent for all $u=(u_s)\in h(C(p,q))$. Hence, $t=(t_s)\in\{h(C(p,q))\}^{\beta}$, that is, the conditions are sufficient.

3. We see from (3.3) that tu is an element of bs whenever u in $h(C(p,q))$ iff Tv is an element of ℓ_∞ for v in h. As a consequence, by (3.3), it is deduced that $\{h(C(p,q))\}^{\gamma}=\Upsilon_3$.

□

4. Matrix Mappings

Here, we provide some matrix mapping classes from $h(C(p,q))$ to $\mu \in \{c_0, c, \ell_\infty, \ell_1, h\}$. Define the infinite matrix $\mathscr{A}$ whose $(r,s)^{th}$ term a_{rs} is given by

$$a_{rs} = \sum_{i=s}^{s+1} \frac{(-1)^{i-s} p^{i-s} [s+1]_{p,q}}{q^i} d_{ri}$$

for all $r, s \in \mathbb{N}$.

Theorem 4.1. *$\mathscr{D} = (d_{rs}) \in (h(C(p,q)) : \mu)$ iff*

$$\mathscr{A} \in (h : \mu) \tag{4.1}$$

$$\left(\frac{[m+1]_{p,q}}{q^m} d_{rm} \right)_{m \in \mathbb{N}} \in \mu \tag{4.2}$$

for all $r, m \in \mathbb{N}$

Proof. Let $\mathscr{D} \in (h(C(p,q)) : \mu)$. Then, $\mathscr{D}u$ exists for all $u = (u_s) \in h(C(p,q))$, and belongs to the space μ. Thus, $\mathscr{D}_m \in \{h(C(p,q))\}^\beta$ which confirms the necessity of the conditions in (4.1) and (4.2).

Conversely, assume that the conditions in (4.1) and (4.2) hold. Let $u = (u_s) \in h(C(p,q))$. Then, $\mathscr{D}_m \in \{h(C(p,q))\}^\beta$ for each $m \in \mathbb{N}$, and $\mathscr{D}u$ exists. Therefore, we obtain the equality shown below:

$$\begin{aligned} \sum_{s=1}^{m} d_{rs} u_s &= \sum_{s=1}^{m} d_{rs} \left(\sum_{i=s-1}^{s} (-1)^{s-i} \frac{p^{s-i} [i+1]_{p,q}}{q^s} v_i \right) \\ &= \sum_{s=1}^{m-1} \left(\sum_{i=s}^{s+1} \frac{(-1)^{i-s} p^{i-s} [s+1]_{p,q}}{q^i} d_{ri} \right) v_s + \frac{[m+1]_{p,q}}{q^m} d_{rm} v_m \end{aligned} \tag{4.3}$$

for every $r, m \in \mathbb{N}$. In the light of the condition in (4.2) and passing to limits as $m \to \infty$ in (4.3), we deduce the following equality

$$\sum_{s=1}^{\infty} d_{rs} u_s = \sum_{s=1}^{\infty} a_{rs} v_s$$

for all $r, s \in \mathbb{N}$, where the matrix $A = (a_{rs})$ is defined as in (4.1). Thus A maps h into μ. This implies that $Av = \mathscr{D}u \in \mu$ is required. □

Now, combining Lemma 3.1 and Theorem 4.1, the following result is obtained:

Corollary 4.2. *The following claims are true:*

(i) *$\mathscr{D} \in (h(C(p,q)) : c_0)$ iff*

$$\sup_{r,s} \frac{1}{s} \left| \sum_{j=1}^{s} a_{rj} \right| < \infty, \tag{4.4}$$

$$\lim_{r \to \infty} a_{rs} \text{ exists } (s \in \mathbb{N}). \tag{4.5}$$

hold, and

$$\lim_{r \to \infty} a_{rs} = 0 \text{ for all } s \in \mathbb{N} \tag{4.6}$$

also holds.

(ii) *$\mathscr{D} \in (h(C(p,q)) : c)$ iff (4.4) and (4.5) hold, and*

$$\sup_{r,s} \frac{1}{s} \left| \sum_{j=1}^{s} a_{rj} \right| < \infty, \tag{4.7}$$

$$\lim_{r \to \infty} d_{rs} \text{ exists } (s \in \mathbb{N}).$$

also hold.

(iii) $\mathscr{D} \in (h(C(p,q)) : \ell_\infty)$ *iff* (4.4), (4.5) *and* (4.7) *hold.*

(iv) $\mathscr{D} \in (h(C(p,q)) : \ell_1)$ *iff* (4.4) *and* (4.5) *hold, and*

$$\sum_{r=1}^{\infty} |a_{rs}| < \infty, \quad (s = 1,2,\ldots)$$

$$\sup_s \frac{1}{s} \sum_{r=1}^{\infty} \left| \sum_{j=1}^{s} a_{rj} \right| < \infty.$$

(v) $\mathscr{D} \in (h(C(p,q)) : h)$ *iff* (4.4), (4.5) *and* (4.6) *hold, and*

$$\sum_{r=1}^{\infty} r\,|a_{rs} - a_{r+1,s}| < \infty, \quad (s = 1,2,\ldots)$$

$$\sup_s \frac{1}{s} \sum_{r=1}^{\infty} r \left| \sum_{j=1}^{s} (a_{rj} - a_{r+1,j}) \right| < \infty.$$

5. Conclusion

As an application of matrix summability methods to Banach sequence spaces, in this research, we presented a BK sequence space $h(C(p,q))$, which is the domain of the conservative (p,q)-Cesàro matrix $C(p,q)$ (the (p,q)-analogue of the first order Cesàro mean) on the Hahn sequence space. This work is an example of the broader application of (p,q)-calculus in the construction of Banach spaces.

As a future scope, we will study the normed and paranormed domains of the (p,q)-Cesàro matrix in some well-known spaces.

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Research Article

Investigations into Hermite-Hadamard-Fejér Inequalities within the Realm of Trigonometric Convexity

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Abstract
This study is predicated on the exploration of lemmas pertaining to the Hermite-Hadamard-Fejér type integral inequality, focusing on both trapezoidal and midpoint inequalities. It delves into the realm of trigonometrically convex functions and is structured around the foundational lemmas that govern these inequalities. Through rigorous analysis, the research has successfully derived novel theorems and garnered insightful results that enhance the understanding of trigonometric convexity. Further, the study has undertaken the application of these theorems to exemplify trigonometrically convex functions, thereby providing practical instances that underline the theoretical developments. These applications not only serve to demonstrate the utility of the newly formulated results but also contribute to the broader field of convex analysis by introducing innovative perspectives on integral inequalities. The synthesis of theory and application encapsulated in this research marks a significant stride in the advancement of mathematical inequalities and their relevance to the study of convex functions.

Keywords: Hermite-Hadamard-Fejer type inequality, h-convex functions, Trigonometrically convex function
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1. Introduction

Convex functions are utilized and extensively researched across a variety of fields, from physics to economics, mathematics to statistics, and even medicine. They are known to be among the most significant areas of study in the current century, with a vast body of literature dedicated to them. Amidst the importance of convex functions in the literature, many authors have identified and studied various types of convex functions. Research on these various types of convex functions has been and continues to be extensive and expanding.

One of the most significant classes of convex functions is the family of h-convex functions, originally explored and expanded in the foundational works of Varošanec [1] and Bombardelli and Varošanec [2]. This notion of h-convexity has proven remarkably versatile, giving rise to various specialized convexity classes such as s-convexity, (s,P)-convexity, trigonometric convexity, and exponential trigonometric convexity [3–5]. These generalized convexities have, in turn, facilitated the development of numerous refined inequalities of Hermite-Hadamard-Fejér type, as well as several new integral inequalities extending classical results [6–10]. For instance, Budak et al. [7, 11–13] obtained new trapezoid and midpoint-type inequalities for generalized quantum integrals, and additionally derived integral inequalities for conformable fractional integrals by employing the weight functions inherent in Fejér-type inequalities, whereas Çelik et al. [9] introduced generalized Milne-type inequalities under conformable fractional integrals. Demir [3] established novel Hermite-Hadamard-type inequalities for exponential trigonometric convex functions, while Demir et al. [5] derived Simpson's-type inequalities within the framework of trigonometric convexity. These advances build upon the classical insights of Hadamard [14] and Fejér [15], whose pioneering works laid the groundwork for modern research on convex functions and their associated integral inequalities. Later investigations by Dragomir and Pearce [8] offered a comprehensive survey of Hermite-Hadamard-type results, and Kadakal [16] further specialized these inequalities to trigonometrically convex functions. More recently, Turhan [4] presented novel generalizations of integral inequalities for trigonometrically-p functions, thereby highlighting the ongoing expansion and applicability of h-convexity in contemporary mathematical research.

Since the definition of convex functions inherently relies on an inequality condition, they are widely used in mathematics to find new lower or upper bounds, that is, for optimization. The well-known Hermite-Hadamard (H-H) inequality in the literature is stated for a continuous function $\xi : T \to \mathbb{R}$, for all $k,l \in T$ with $k < l$,

$$\xi\left(\frac{k+l}{2}\right) \leq \frac{1}{l-k}\int_k^l \xi(x)dx \leq \frac{\xi(k)+\xi(l)}{2}.$$

If ξ is a concave function, the inequality is reversed [14]. This inequality has been applied to many classes of convex functions; with the help of various lemmas, theorems on trapezoidal and midpoint type inequalities have been derived and results have been presented.

The introduction of the weighted version of the Hermite-Hadamard (H-H) inequality by Fejér in 1906 represents a significant evolution in the analysis of convex functions, culminating in what is now recognized as the H-H Fejér type inequality. This seminal development not only enriched the mathematical framework for examining convex functions but also facilitated the derivation of a broad spectrum of theorems and results tailored to various conditions of the weight function. Such advancements have had profound implications on both the left and right sides of different H-H inequalities, underscoring the historical importance and far-reaching impact of Fejér's work. Through this weighted form of the H-H inequality, Fejér's contribution has been pivotal in broadening the understanding and application of convex function inequalities, highlighting the intricate interplay between weight functions and the fundamental properties of these inequalities.

In this study, lemmas that yield trapezoidal and midpoint-type integral inequalities for trigonometric convex functions were investigated. While these lemmas are known for Hermite-Hadamard Fejer type integral inequalities and many studies, have produced trapezoidal type inequalities, new theorems, and results have also been obtained for midpoint type inequalities.

2. Preliminaries

In this section, we first present the foundational theorems and definitions that underpin this work. Following this, we introduce the pivotal lemmas that have not only inspired but also guided the development of the study, thereby establishing a robust conceptual framework for the ensuing analysis.

Theorem 2.1. *[15] Assume $\xi : [k,l] \to \mathbb{R}$ is a convex mapping. Then, the following inequality is satisfied:*

$$\xi\left(\frac{k+l}{2}\right)\int_k^l w(x)\,dx \leq \frac{1}{l-k}\int_k^l \xi(x)w(x)\,dx \leq \frac{\xi(k)+\xi(l)}{2}\int_k^l w(x)\,dx$$

where the function $w : [k,l] \to \mathbb{R}$ is nonnegative, integrable, and exhibits symmetry about $x = \frac{k+l}{2}$.

The domain of convex analysis has been significantly expanded with the introduction of h-convexity, as delineated by Varosanec. This sophisticated class of convexity, which is predicated upon a modulating function h that is both non-negative and distinct from zero, provides a more encompassing approach than classical convexity.

Definition 2.2. *[1] Let G and T be two intervals, and let $h : G \to \mathbb{R}$ be a non-negative function such that $h \neq 0$. A function $\xi : T \to \mathbb{R}$ is said to be an h-convex function if, for all $k,l \in T$ and for any $\kappa \in (0,1)$, the following inequality holds:*

$$\xi(\kappa k + (1-\kappa)l) \leq h(\kappa)\xi(k) + h(1-\kappa)\xi(l).$$

*Conversely, if the inequality holds in the opposite direction, then ξ is termed an *h-concave function*. Functions belonging to this class of convexity are referred to as members of the class $SX(h,K)$.*

This significant convex class has led to the emergence of numerous convexity classes. Among these, trigonometric convex functions expressed by H. Kadakal and the related H-H inequality and theorems pertaining to this convexity class are provided as follows:

Definition 2.3. *[16] Let $\xi : T \to \mathbb{R}$ be a non-negative function, where $k,l \in T$ and $\omega \in [0,1]$. In this case, if the following inequality is satisfied, the function ξ is referred to as a trigonometrically convex function:*

$$\xi(\omega k + (1-\omega)l) \leq \left(\sin\left(\frac{\pi\omega}{2}\right)\right)\xi(k) + \left(\cos\left(\frac{\pi\omega}{2}\right)\right)\xi(l).$$

The class of trigonometric convex functions is denoted by $TC(T)$. In the definition expressed, if $h(\omega) = \sin\left(\frac{\omega}{2}\right)$ is taken, then every trigonometric convex function becomes an h-convex function.

Theorem 2.4. *[16] Let T be an interval with $k,l \in T$ such that $k < l$. If the function $\xi : [k,l] \to \mathbb{R}$ is a trigonometrically convex function and $\xi \in L[k,l]$, then the following inequality is obtained:*

$$\xi\left(\frac{k+l}{2}\right) \leq \frac{\sqrt{2}}{l-k}\int_k^l \xi(\S)d\S.$$

Theorem 2.5. *[16] Let T be an interval with $k,l \in T$ such that $k < l$. If the function $\xi : [k,l] \to \mathbb{R}$ is trigonometrically convex and $\xi \in L[k,l]$, then the following inequality is obtained:*

$$\frac{1}{l-k}\int_k^l \xi(x)dx \leq \frac{2}{\pi}\left[\xi(k)+\xi(l)\right].$$

Theorem 2.6. *[16] Let T be an interval, $k,l \in T$ such that $k < l$, and $\xi : T \to \mathbb{R}$ be a continuously differentiable function. If $\xi' \in L[k,l]$ and $|\xi'|$ is a trigonometrically convex function, then the following inequality is obtained:*

$$\left|\frac{\xi(k)+\xi(l)}{2} - \frac{1}{l-k}\int_k^l \xi(x)dx\right| \leq \frac{2}{\pi}(l-k)\left[1-\frac{4}{\pi}(\sqrt{2}-1)\right]\left[\frac{|\xi'(k)|+|\xi'(l)|}{2}\right].$$

Theorem 2.7. *[16] Let T be an interval, $k,l \in T$ such that $k < l$, and $\xi : T \to \mathbb{R}$ be a continuously differentiable function. Let $\eta > 1$ and $\frac{1}{\eta}+\frac{1}{\theta} = 1$, for $|\xi'|^\eta$ being a trigonometrically convex function over the interval $[k,l]$, the following inequality is obtained:*

$$\left|\frac{\xi(k)+\xi(l)}{2} - \frac{1}{l-k}\int_k^l \xi(x)dx\right| \leq \frac{l-k}{2}\left(\frac{1}{\theta+1}\right)^{\frac{1}{\theta}} 2^{\frac{2}{\eta}}\pi^{\frac{-1}{\eta}}\left(\frac{k+l}{2}\right)^{\frac{1}{\eta}}\left[\frac{|\xi'(k)|^\eta+|\xi'(l)|^\eta}{2}\right]^{\frac{1}{\eta}}.$$

Theorem 2.8. *[16] Let T be an interval, $k,l \in T$ such that $k < l$, and $\xi : T \to \mathbb{R}$ be a continuously differentiable function. For $\eta \geq 1$, with $|\xi'|^\eta$ being a trigonometrically convex function over the interval $[k,l]$, the following inequality is obtained:*

$$\left|\frac{\xi(k)+\xi(l)}{2} - \frac{1}{l-k}\int_k^l \xi(x)dx\right| \leq \frac{l-k}{2}\left(\frac{1}{2}\right)^{1-\frac{3}{\eta}}\left[\frac{1}{\pi}-\frac{4(\sqrt{2}-1)}{\pi^2}\right]^{\frac{1}{\eta}}\left(\frac{k+l}{2}\right)^{\frac{1}{\eta}}\left[\frac{|\xi'(k)|^\eta+|\xi'(l)|^\eta}{2}\right]^{\frac{1}{\eta}}.$$

In this research, M. Z. Sarikaya introduced two significant lemmas that serve as the foundational basis for the investigation of H-H Fejér type inequalities and their applications to trapezoidal and midpoint-type inequalities. These lemmas, detailed below, are considered the cornerstone of the research:

Lemma 2.9. *[6] Let $\xi : K^\circ \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on K°, $k, l \in K^\circ$ with $k < l$, and $w : [k,l] \to [0,\infty)$ be a differentiable mapping. If $\xi' \in L[k,l]$, then the following equality holds:*

$$\frac{1}{l-k}\int_k^l \xi(x)w(x)dx - \frac{1}{l-k}\xi\left(\frac{k+l}{2}\right)\int_k^l w(x)dx = (l-k)\int_0^1 m(t)\xi'(\kappa k+(1-\kappa)l)d\kappa \tag{2.1}$$

for each $\kappa \in [0,1]$, where

$$m(\kappa) = \begin{cases} \int_0^\kappa w(ks+(1-s)l)ds, & \kappa \in [0,\frac{1}{2}] \\ -\int_\kappa^1 w(ks+(1-s)l)ds, & \kappa \in [\frac{1}{2},1]. \end{cases}$$

Lemma 2.10. *[6] Let $\xi : K^\circ \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on K°, $k, l \in K^\circ$ with $k < l$, and $w : [k,l] \to [0,\infty)$ be a differentiable mapping. If $\xi' \in L[k,l]$, then the following equality holds:*

$$\frac{1}{l-k}\left[\frac{\xi(k)+\xi(l)}{2} + \int_k^l w(x)dx\right] - \frac{1}{l-k}\int_k^l \xi(x)w(x)dx = \frac{(l-k)}{2}\int_0^1 p(\kappa)\xi'(\kappa k+(1-\kappa)l)d\kappa$$

for each $\kappa \in [0,1]$, where

$$p(\kappa) = \int_\kappa^1 w(as+(1-s)b)ds - \int_0^\kappa w(as+(1-s)b)ds.$$

3. Main Results

Theorem 3.1. *Let $\xi : I^\circ \subset \mathbb{R} \to \mathbb{R}$ be a differentiable function on I°, where $k, l \in I^\circ$ with $k < l$, and let $w : [k,l] \to \mathbb{R}$ be a differentiable function that is symmetric about $\frac{k+l}{2}$. Given that ξ' is trigonometrically convex over the interval $[k,l]$, the following inequality holds:*

$$\left|\frac{1}{l-k}\int_k^l \xi(x)w(x)dx - \frac{1}{l-k}\xi\left(\frac{k+l}{2}\right)\int_k^l w(x)dx\right| \le \frac{2\sqrt{2}}{\pi}\left[\int_{\frac{k+l}{2}}^l w(x)\sin\left(\frac{2x-k-l}{4(l-k)}\pi\right)dx\right]\left[\left|\xi'(k)\right| + \left|\xi'(l)\right|\right]. \tag{3.1}$$

Proof. Considering Lemma 2.9 and taking the absolute value of both sides, given that $|\xi'|$ is a trigonometrically convex function, we proceed as follows:

$$\begin{aligned}
&\left|\frac{1}{l-k}\int_k^l \xi(x)w(x)dx - \frac{1}{l-k}\xi\left(\frac{k+l}{2}\right)\int_k^l w(x)dx\right| \\
&\le (l-k)\left|\int_0^{1/2}\left(\int_0^t w(ks+(1-s)l)\,\mathrm{ds}\right)\xi'(tk+(1-t)l)\,\mathrm{dt} - \int_{1/2}^1\left(\int_t^1 w(ks+(1-s)l)\,\mathrm{ds}\right)\xi'(tk+(1-t)l)\,\mathrm{dt}\right| \\
&\le (l-k)\left[\int_0^{\frac{1}{2}}\left(\int_0^t w(ks+(1-s)l)\,\mathrm{ds}\right)\left|\xi'(tk+(1-t)l)\right|\mathrm{dt} + \int_{\frac{1}{2}}^1\left(\int_t^1 w(ks+(1-s)l)\,\mathrm{ds}\right)\left|\xi'(tk+(1-t)l)\right|\mathrm{dt}\right] \\
&\le (l-k)\left\{\int_0^{\frac{1}{2}}\left(\int_0^t w(ks+(1-s)l)\,\mathrm{ds}\right)\left[\begin{array}{c}\sin\left(\frac{\pi t}{2}\right)|\xi'(k)| \\ +\cos\left(\frac{\pi t}{2}\right)|\xi'(l)|\end{array}\right]dt + \int_{\frac{1}{2}}^1\left(\int_t^1 w(ks+(1-s)l)\,\mathrm{ds}\right)\left[\begin{array}{c}\sin\left(\frac{\pi t}{2}\right)|\xi'(k)| \\ +\cos\left(\frac{\pi t}{2}\right)|\xi'(l)|\end{array}\right]dt\right\}.
\end{aligned}$$

The following expressions are obtained by changing the order of integration in the integrals on the right side of the obtained final inequality, in accordance with Fubini's Theorem:

$$\left|\frac{1}{l-k}\int_k^l \xi(x)w(x)dx-\frac{1}{l-k}\xi\left(\frac{k+l}{2}\right)\int_k^l w(x)dx\right|$$
$$\leq (l-k)\left\{\int_0^{\frac{1}{2}} w(ks+(1-s)l)\left[\int_s^{\frac{1}{2}}\left(\begin{array}{c}\sin\left(\frac{\pi t}{2}\right)|\xi'(k)|\\ +\cos\left(\frac{\pi t}{2}\right)|\xi'(l)|\end{array}\right)dt\right]ds+\int_{\frac{1}{2}}^{1} w(ks+(1-s)l)\left[\int_{\frac{1}{2}}^{s}\left(\begin{array}{c}\sin\left(\frac{\pi t}{2}\right)|\xi'(k)|\\ +\cos\left(\frac{\pi t}{2}\right)|\xi'(l)|\end{array}\right)dt\right]ds\right\}$$

Upon resolving the inequalities on the right-hand side of the final inequality, the following inequality is obtained:

$$\left|\frac{1}{l-k}\int_k^l \xi(x)w(x)dx-\frac{1}{l-k}\xi\left(\frac{k+l}{2}\right)\int_k^l w(x)dx\right|$$

$$\leq (l-k)\left\{\int_0^{\frac{1}{2}} w(ks+(1-s)l)\left[\begin{array}{c}\left(-\frac{\sqrt{2}}{\pi}+\frac{2}{\pi}\cos\left(\frac{\pi s}{2}\right)\right)|\xi'(k)|\\ +\left(\frac{\sqrt{2}}{\pi}-\frac{2}{\pi}\sin\left(\frac{\pi s}{2}\right)\right)|\xi'(l)|\end{array}\right]ds+\int_{\frac{1}{2}}^{1} w(ks+(1-s)l)\left[\begin{array}{c}\left(\frac{\sqrt{2}}{\pi}-\frac{2}{\pi}\cos\left(\frac{\pi s}{2}\right)\right)|\xi'(k)|\\ +\left(\frac{2}{\pi}\sin\left(\frac{\pi s}{2}\right)-\frac{\sqrt{2}}{\pi}\right)|\xi'(l)|\end{array}\right]ds\right\}.$$

Following the variable transformation $x=ks+(1-s)l$ within this integral, employing the theorem's hypothesis that the function $w(x)$ is symmetric with respect to $x=\frac{k+l}{2}$ yields the sought inequality. □

Corollary 3.2. *If we take $w(x)=1$ in inequality of (3.1), we get*

$$\left|\frac{1}{l-k}\int_k^l \xi(x)dx-\xi\left(\frac{k+l}{2}\right)\right|\leq\frac{(4\sqrt{2}-4)(l-k)}{\pi^2}\left[|\xi'(k)|+|\xi'(l)|\right].$$

Theorem 3.3. *Let $\xi: I^\circ\subset\mathbb{R}\to\mathbb{R}$ be a function differentiable on I°, where $k,l\in I^\circ, k<l$, and let $w:[k,l]\to\mathbb{R}$ be a differentiable function that is symmetric with respect to $\frac{k+l}{2}$. Under the condition that $q\geq 1, \frac{1}{p}+\frac{1}{q}=1$, and given that the function $|\xi'|^q$ is trigonometrically convex on the interval $[k,l]$, it follows that:*

$$\left|\frac{1}{l-k}\int_k^l \xi(x)w(x)dx-\frac{1}{l-k}\xi\left(\frac{k+l}{2}\right)\int_k^l w(x)dx\right|$$
$$\leq (l-k)^{1-\frac{2}{p}}\left(\int_{\frac{k+l}{2}}^{l} w^p(x)\left(x-\frac{k+l}{2}\right)dx\right)^{\frac{1}{p}}\left[\begin{array}{c}\left(\frac{4\sqrt{2}-\pi\sqrt{2}}{2\pi^2}|\xi'(k)|^q+\frac{4\sqrt{2}+\pi\sqrt{2}-8}{2\pi^2}|\xi'(l)|^q\right)^{\frac{1}{q}}\\ +\frac{4\sqrt{2}+\pi\sqrt{2}-8}{2\pi^2}|\xi'(k)|^q+\left(\frac{4\sqrt{2}-\pi\sqrt{2}}{2\pi^2}|\xi'(l)|^q\right)^{\frac{1}{q}}\end{array}\right].$$

Proof. By taking the absolute value of both sides of equation (2.1) in Lemma 2.9, the following inequality is obtained:

$$\left|\frac{1}{l-k}\int_k^l \xi(x)w(x)dx-\frac{1}{l-k}\xi\left(\frac{k+l}{2}\right)\int_k^l w(x)dx\right|$$
$$\leq (l-k)\left[\int_0^{\frac{1}{2}}\int_s^{\frac{1}{2}} w(ks+(1-s)l)\left|\xi'(tk+(1-t)l)\right|dtds+\int_{\frac{1}{2}}^{1}\int_{\frac{1}{2}}^{s} w(ks+(1-s)l)\left|\xi'(tk+(1-t)l)\right|dtds\right].$$

By applying Hölder's inequality to each integral on the right-hand side of the resulting inequality, the following expression is obtained:

$$\left|\frac{1}{l-k}\int_k^l \xi(x)w(x)dx-\frac{1}{l-k}\xi\left(\frac{k+l}{2}\right)\int_k^l w(x)dx\right|$$
$$\leq (l-k)\left[\left(\int_0^{\frac{1}{2}}\int_s^{\frac{1}{2}} w^p(ks+(1-s)l)dtds\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}}\int_s^{\frac{1}{2}}\left|\xi'(tk+(1-t)l)\right|^q dtds\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\int_{\frac{1}{2}}^{1}\int_{\frac{1}{2}}^{s} w^p(ks+(1-s)l)dtds\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^{1}\int_{\frac{1}{2}}^{s}\left|\xi'(tk+(1-t)l)\right|^q dtds\right)^{\frac{1}{q}}\right]$$
$$\leq (l-k)\left[\left(\int_{\frac{k+l}{2}}^{l} w^p(x)\left(\frac{2x-l-k}{2(l-k)^2}\right)dx\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}}\int_s^{\frac{1}{2}}\left|\xi'(tk+(1-t)l)\right|^q dtds\right)^{\frac{1}{q}}\right.$$
$$\left.\left(\int_k^{\frac{k+l}{2}} w^p(x)\left(\frac{k+l-2x}{2(l-k)^2}\right)dx\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^{1}\int_{\frac{1}{2}}^{s}\left|\xi'(tk+(1-t)l)\right|^q dtds\right)^{\frac{1}{q}}\right].$$

Using the hypothesis that the function $w(x)$ is symmetric with respect to $x=\frac{k+l}{2}$ and the fact that the function $|\xi'|^q$ is trigonometrically convex, we can derive the following inequality:

$$\left|\frac{1}{l-k}\int_k^l \xi(x)w(x)dx-\frac{1}{l-k}\xi\left(\frac{k+l}{2}\right)\int_k^l w(x)dx\right|$$
$$\leq (l-k)^{1-\frac{2}{p}}\left(\int_{\frac{k+l}{2}}^{l} w^p(x)\left(\frac{2x-l-k}{2}\right)dx\right)^{\frac{1}{p}}\times\left[\left(\begin{array}{c}|\xi'(k)|^q\int_0^{\frac{1}{2}}\int_s^{\frac{1}{2}}\sin\left(\frac{\pi t}{2}\right)dtds\\ +|\xi'(l)|^q\int_0^{\frac{1}{2}}\int_s^{\frac{1}{2}}\cos\left(\frac{\pi t}{2}\right)dtds\end{array}\right)^{\frac{1}{q}}+\left(\begin{array}{c}|\xi'(k)|^q\int_{\frac{1}{2}}^{1}\int_{\frac{1}{2}}^{s}\sin\left(\frac{\pi t}{2}\right)dtds\\ +|\xi'(l)|^q\int_{\frac{1}{2}}^{1}\int_{\frac{1}{2}}^{s}\cos\left(\frac{\pi t}{2}\right)dtds\end{array}\right)^{\frac{1}{q}}\right].$$

Upon solving the final integrals here, the proof is completed. □

Corollary 3.4. *If $w(x)=1$ is taken in Theorem 3.3, the following inequality is obtained:*

$$\left|\frac{1}{l-k}\int_k^l \xi(x)dx-\xi\left(\frac{k+l}{2}\right)\right|\leq\frac{(l-k)}{2^{\frac{3}{p}}}\left[\begin{array}{c}\left(\frac{4\sqrt{2}-\pi\sqrt{2}}{2\pi^2}|\xi'(k)|^q+\frac{4\sqrt{2}+\pi\sqrt{2}-8}{2\pi^2}|\xi'(l)|^q\right)^{\frac{1}{q}}\\ +\frac{4\sqrt{2}+\pi\sqrt{2}-8}{2\pi^2}|\xi'(k)|^q+\left(\frac{4\sqrt{2}-\pi\sqrt{2}}{2\pi^2}|\xi'(l)|^q\right)^{\frac{1}{q}}\end{array}\right].$$

Theorem 3.5. *Let $\xi: I^{\circ}\subset\mathbb{R}\to\mathbb{R}$ be a function differentiable on I°, where $k,l\in I^{\circ}, k<l$, and let $w:[k,l]\to\mathbb{R}$ be a differentiable function that is symmetric with respect to $\frac{k+l}{2}$. Under the condition that $q>1$, and given that the function $|\xi'|^q$ is trigonometrically convex on the interval $[k,l]$, it follows that:*

$$\left|\frac{1}{l-k}\int_k^l \xi(x)w(x)dx-\frac{1}{l-k}\xi\left(\frac{k+l}{2}\right)\int_k^l w(x)dx\right|\leq\frac{1}{(l-k)^{1-\frac{1}{q}}}\left(\int_{\frac{k+l}{2}}^{l} w(x)\left(x-\frac{k+l}{2}\right)dx\right)^{1-\frac{1}{q}}$$
$$\left[\left(\begin{array}{c}|\xi'(k)|^q\int_{\frac{k+l}{2}}^{l} w(x)\left(\frac{-\sqrt{2}}{\pi}+\frac{2}{\pi}\cos\left(\frac{\pi(l-x)}{2(l-k)}\right)\right)dx\\ +|\xi'(l)|^q\int_{\frac{k+l}{2}}^{l} w(x)\left(\frac{\sqrt{2}}{\pi}-\frac{2}{\pi}\sin\left(\frac{\pi(l-x)}{2(l-k)}\right)\right)dx\end{array}\right)^{\frac{1}{q}}+\left(\begin{array}{c}|\xi'(k)|^q\int_{\frac{k+l}{2}}^{l} w(x)\left(\frac{-2}{\pi}\cos\left(\frac{\pi(x-k)}{2(l-k)}\right)+\frac{\sqrt{2}}{\pi}\right)dx\\ +|\xi'(l)|^q\int_{\frac{k+l}{2}}^{l} w(x)\left(\frac{2}{\pi}\sin\left(\frac{\pi(x-k)}{2(l-k)}\right)-\frac{\sqrt{2}}{\pi}\right)dx\end{array}\right)^{\frac{1}{q}}\right].$$

Proof. In proving the theorem, after taking the absolute value of equation (2.1) in Lemma 2.9, the following inequality is obtained.

$$\begin{aligned}&\left|\frac{1}{l-k}\int_k^l \xi(x)w(x)dx-\frac{1}{l-k}\xi\left(\frac{k+l}{2}\right)\int_k^l w(x)dx\right|\\&\leq (l-k)\left[\int_0^{\frac{1}{2}}\int_s^{\frac{1}{2}} w(ks+(1-s)l)\left|\xi'(tk+(1-t)l)\right|dtds+\int_{\frac{1}{2}}^{1}\int_{\frac{1}{2}}^{s} w(ks+(1-s)l)\left|\xi'(tk+(1-t)l)\right|dtds\right].\end{aligned}$$

Subsequently, by applying the Power Mean inequality to each integral on the right-hand side of the resulting expression, the following inequality is derived:

$$\begin{aligned}&\left|\frac{1}{l-k}\int_k^l \xi(x)w(x)dx-\frac{1}{l-k}\xi\left(\frac{k+l}{2}\right)\int_k^l w(x)dx\right|\\&\leq (l-k)\left[\left(\int_0^{\frac{1}{2}}\int_s^{\frac{1}{2}} w(ks+(1-s)l)dtds\right)^{1-\frac{1}{q}}\left(\int_0^{\frac{1}{2}}\int_s^{\frac{1}{2}} w(ks+(1-s)l)\left|\xi'(tk+(1-t)l)\right|^q dtds\right)^{\frac{1}{q}}\right.\\&\left.+\left(\int_{\frac{1}{2}}^{1}\int_{\frac{1}{2}}^{s} w(ks+(1-s)l)dtds\right)^{1-\frac{1}{q}}\left(\int_{\frac{1}{2}}^{1}\int_{\frac{1}{2}}^{s} w(ks+(1-s)l)\left|\xi'(tk+(1-t)l)\right|^q dtds\right)^{\frac{1}{q}}\right]\\&\leq (l-k)\left[\left(\int_0^{\frac{1}{2}} w(ks+(1-s)l)\left(\frac{1}{2}-s\right)ds\right)^{1-\frac{1}{q}}\left(\int_0^{\frac{1}{2}}\int_s^{\frac{1}{2}} w(ks+(1-s)l)\left|\xi'(tk+(1-t)l)\right|^q dtds\right)^{\frac{1}{q}}\right.\\&\left.+\left(\int_{\frac{1}{2}}^{1} w(ks+(1-s)l)\left(s-\frac{1}{2}\right)ds\right)^{1-\frac{1}{q}}\left(\int_{\frac{1}{2}}^{1}\int_{\frac{1}{2}}^{s} w(ks+(1-s)l)\left|\xi'(tk+(1-t)l)\right|^q dtds\right)^{\frac{1}{q}}\right].\end{aligned}$$

Since the function $|\xi'|^q$ is trigonometrically convex, the following inequality is obtained:

$$\begin{aligned}&\left|\frac{1}{l-k}\int_k^l \xi(x)w(x)dx-\frac{1}{l-k}\xi\left(\frac{k+l}{2}\right)\int_k^l w(x)dx\right|\\&\leq (l-k)\left[\left(\int_0^{\frac{1}{2}} w(ks+(1-s)l)\left(\frac{1}{2}-s\right)ds\right)^{1-\frac{1}{q}}\left(\int_0^{\frac{1}{2}}\int_s^{\frac{1}{2}} w(ks+(1-s)l)\left(\begin{array}{c}\sin\left(\frac{\pi t}{2}\right)|\xi'(k)|^q\\+\cos\left(\frac{\pi t}{2}\right)|\xi'(l)|^q\end{array}\right)dtds\right)^{\frac{1}{q}}\right.\\&\left.+\left(\int_{\frac{1}{2}}^{1} w(ks+(1-s)l)\left(s-\frac{1}{2}\right)ds\right)^{1-\frac{1}{q}}\left(\int_{\frac{1}{2}}^{1}\int_{\frac{1}{2}}^{s} w(ks+(1-s)l)\left(\begin{array}{c}\sin\left(\frac{\pi t}{2}\right)|\xi'(k)|^q\\+\cos\left(\frac{\pi t}{2}\right)|\xi'(l)|^q\end{array}\right)dtds\right)^{\frac{1}{q}}\right].\end{aligned}$$

If a change of variable is applied to the first integral on the right-hand side of the final inequality, the following expression is obtained:

$$\begin{aligned}
&\left|\frac{1}{l-k}\int_k^l \xi(x)w(x)dx-\frac{1}{l-k}\xi\left(\frac{k+l}{2}\right)\int_k^l w(x)dx\right| \\
&\leq \frac{1}{(l-k)^{1-\frac{2}{q}}}\left[\left(\int_{\frac{k+l}{2}}^{l} w(x)\left(\frac{2x-k-l}{2}\right)dx\right)^{1-\frac{1}{q}}\left(\int_0^{\frac{1}{2}}\int_s^{\frac{1}{2}} w(ks+(1-s)l)\left(\begin{array}{c}\sin\left(\frac{\pi t}{2}\right)|\xi'(k)|^q\\ +\cos\left(\frac{\pi t}{2}\right)|\xi'(l)|^q\end{array}\right)dtds\right)^{\frac{1}{q}}\right. \\
&\left.+\left(\int_k^{\frac{k+l}{2}} w(x)\left(\frac{k+l-2x}{2}\right)dx\right)^{1-\frac{1}{q}}\left(\int_{\frac{1}{2}}^{1}\int_{\frac{1}{2}}^{s} w(ks+(1-s)l)\left(\begin{array}{c}\sin\left(\frac{\pi t}{2}\right)|\xi'(k)|^q\\ +\cos\left(\frac{\pi t}{2}\right)|\xi'(l)|^q\end{array}\right)dtds\right)^{\frac{1}{q}}\right] \\
&\leq \frac{1}{(l-k)^{1-\frac{2}{q}}}\left[\left(\int_{\frac{k+l}{2}}^{l} w(x)\left(\frac{2x-k-l}{2}\right)dx\right)^{1-\frac{1}{q}}\left(\begin{array}{c}|\xi'(k)|^q\int_0^{\frac{1}{2}} w(ks+(1-s)l)\left(\frac{-\sqrt{2}}{\pi}+\frac{2}{\pi}\cos\left(\frac{\pi s}{2}\right)\right)ds\\ +|\xi'(l)|^q\int_0^{\frac{1}{2}} w(ks+(1-s)l)\left(\frac{\sqrt{2}}{\pi}-\frac{2}{\pi}\sin\left(\frac{\pi s}{2}\right)\right)ds\end{array}\right)^{\frac{1}{q}}\right. \\
&\left.+\left(\int_k^{\frac{k+l}{2}} w(x)\left(\frac{k+l-2x}{2}\right)dx\right)^{1-\frac{1}{q}}\left(\begin{array}{c}|\xi'(k)|^q\int_1^{\frac{1}{2}} w(ks+(1-s)l)\left(-\frac{2}{\pi}\cos\left(\frac{\pi s}{2}\right)+\frac{\sqrt{2}}{\pi}\right)ds\\ +|\xi'(l)|^q\int_{\frac{1}{2}}^{1} w(ks+(1-s)l)\left(\frac{2}{\pi}\sin\left(\frac{\pi s}{2}\right)-\frac{\sqrt{2}}{\pi}\right)ds\end{array}\right)^{\frac{1}{q}}\right].
\end{aligned}$$

In the final inequality, after taking the integrals on the right side and applying the variable change $x=ks+(1-s)l$, and then using the fact that the function $w(x)$ is symmetric with respect to $x=\frac{k+l}{2}$, the following inequality is obtained:

$$\begin{aligned}
&\left|\frac{1}{l-k}\int_k^l \xi(x)w(x)dx-\frac{1}{l-k}\xi\left(\frac{k+l}{2}\right)\int_k^l w(x)dx\right| \\
&\leq \frac{1}{(l-k)^{1-\frac{1}{q}}}\left[\left(\int_{\frac{k+l}{2}}^{l} w(x)\left(\frac{2x-k-l}{2}\right)dx\right)^{1-\frac{1}{q}}\left(\begin{array}{c}|\xi'(k)|^q\int_{\frac{k+l}{2}}^{l} w(x)\left(\frac{-\sqrt{2}}{\pi}+\frac{2}{\pi}\cos\left(\frac{\pi(l-x)}{2(l-k)}\right)\right)dx\\ +|\xi'(l)|^q\int_{\frac{k+l}{2}}^{l} w(x)\left(\frac{\sqrt{2}}{\pi}-\frac{2}{\pi}\sin\left(\frac{\pi(l-x)}{2(l-k)}\right)\right)dx\end{array}\right)^{\frac{1}{q}}\right. \\
&\left.+\left(\int_k^{\frac{k+l}{2}} w(x)\left(\frac{k+l-2x}{2}\right)dx\right)^{1-\frac{1}{q}}\left(\begin{array}{c}|\xi'(k)|^q\int_k^{\frac{k+l}{2}} w(x)\left(-\frac{2}{\pi}\cos\left(\frac{\pi(l-x)}{2(l-k)}\right)+\frac{\sqrt{2}}{\pi}\right)dx\\ +|\xi'(l)|^q\int_k^{\frac{k+l}{2}} w(x)\left(\frac{2}{\pi}\sin\left(\frac{\pi(l-x)}{2(l-k)}\right)-\frac{\sqrt{2}}{\pi}\right)dx\end{array}\right)^{\frac{1}{q}}\right] \\
&\leq \frac{1}{(l-k)^{1-\frac{1}{q}}}\left(\int_{\frac{k+l}{2}}^{l} w(x)\left(\frac{2x-k-l}{2}\right)dx\right)^{1-\frac{1}{q}}\left[\left(\begin{array}{c}|\xi'(k)|^q\int_{\frac{k+l}{2}}^{l} w(x)\left(\frac{-\sqrt{2}}{\pi}+\frac{2}{\pi}\cos\left(\frac{\pi(l-x)}{2(l-k)}\right)\right)dx\\ +|\xi'(l)|^q\int_{\frac{k+l}{2}}^{l} w(x)\left(\frac{\sqrt{2}}{\pi}-\frac{2}{\pi}\sin\left(\frac{\pi(l-x)}{2(l-k)}\right)\right)dx\end{array}\right)^{\frac{1}{q}}\right. \\
&\left.+\left(\begin{array}{c}|\xi'(k)|^q\int_{\frac{k+l}{2}}^{l} w(x)\left(-\frac{2}{\pi}\cos\left(\frac{\pi(x-k)}{2(l-k)}\right)+\frac{\sqrt{2}}{\pi}\right)dx\\ +|\xi'(l)|^q\int_{\frac{k+l}{2}}^{l} w(x)\left(\frac{2}{\pi}\sin\left(\frac{\pi(x-k)}{2(l-k)}\right)-\frac{\sqrt{2}}{\pi}\right)dx\end{array}\right)^{\frac{1}{q}}\right].
\end{aligned}$$

Thus, the proof is completed. □

Corollary 3.6. *If $w(x) = 1$ is assumed in Theorem 3.5, the following inequality is derived:*

$$\left|\frac{1}{l-k}\int_k^l \xi(x)dx - \xi\left(\frac{k+l}{2}\right)\right| \leq \frac{(l-k)}{2^{3-\frac{3}{q}}}\left(\frac{\sqrt{2}}{\pi}\right)^{\frac{1}{q}}$$

$$\times\left[\left(\frac{4-\pi}{2\pi}|\xi(k)|^q + \frac{\pi+4-4\sqrt{2}}{2\pi}|\xi(l)|^q\right)^{\frac{1}{q}} + \left(\frac{\pi+4-4\sqrt{2}}{2\pi}|\xi(k)|^q + \frac{4-\pi}{2\pi}|\xi(l)|^q\right)^{\frac{1}{q}}\right].$$

Theorem 3.7. *Let $\xi : I^\circ \subset \mathbb{R} \to \mathbb{R}$ be a function that is differentiable on I°, with $k,l \in I^\circ$ and $k < l$, and let $w : [k,l] \to \mathbb{R}$ be a differentiable function that is symmetric with respect to $\frac{k+l}{2}$. Given that $\frac{1}{p} + \frac{1}{q} = 1$ and $q \geq 1$, and considering that the function $|\xi'|^q$ is trigonometrically convex over the interval $[k,l]$, the following inequality is satisfied:*

$$\left|\frac{1}{l-k}\frac{\xi(k)+\xi(l)}{2}\int_k^l w(x)dx - \frac{1}{l-k}\int_k^l \xi(x)w(x)dx\right| \leq \frac{1}{\pi}\left(\int_0^1 h^p(t)dt\right)^{\frac{1}{p}}\left(\left|\xi'(k)\right|^q + \left|\xi'(l)\right|^q\right)^{\frac{1}{q}},$$

where

$$h(t) = \left|\int_{k+(l-k)t}^{l-(l-k)t} w(x)dx\right|.$$

Proof. If we start from Lemma 2.10, it is obtained

$$\left|\frac{1}{l-k}\frac{\xi(k)+\xi(l)}{2}\int_k^l w(x)dx - \frac{1}{l-k}\int_k^l \xi(x)w(x)dx\right|$$

$$\leq \frac{l-k}{2}\int_0^1\left|\int_t^1 w(ks+(1-s)l)ds - \int_0^t w(ks+(1-s)l)ds\right|\left|\xi'(kt+(1-t)l)\right|dt$$

$$= \frac{1}{2}\left|\int_k^{kt+(1-t)l} w(x)dx - \int_{kt+(1-t)l}^{l} w(x)dx\right|\left|\xi'(kt+(1-t)l)\right|dt. \tag{3.2}$$

Given that the function $w(x)$ is symmetric with respect to $x = \frac{k+l}{2}$,

1. Since $\forall x \in [0,\frac{1}{2}]$, $\int\limits_{tl+(1-t)k}^{l} \frac{w(x)dx}{l-k} - \int\limits_{tk+(1-t)l}^{l} \frac{w(x)dx}{l-k} = \int\limits_{k+(l-k)t}^{l-(l-k)t} \frac{w(x)dx}{l-k}$.
2. Since $\forall x \in [\frac{1}{2},1]$, $\int\limits_{tl+(1-t)k}^{l} \frac{w(x)dx}{l-k} - \int\limits_{tk+(1-t)l}^{l} \frac{w(x)dx}{l-k} = -\int\limits_{k+(l-k)t}^{l-(l-k)t} \frac{w(x)dx}{l-k}$,

it follows. From this point, for all $t \in [0,1]$, let $h(t) = \left|\int_{k+(l-k)t}^{l-(l-k)t} w(x)dx\right|$. Substituting this expression into inequality (3.2) and then applying Hölder's inequality, given that the function $|\xi'|^q$ is trigonometrically convex,

$$\left| \frac{1}{l-k} \frac{\xi(k)+\xi(l)}{2} \int_k^l w(x)dx - \frac{1}{l-k} \int_k^l \xi(x)w(x)dx \right|$$

$$\leq \frac{1}{2} \int_0^1 h(t) \left| \xi'(kt+(1-t)l) \right| dt \tag{3.3}$$

$$\leq \frac{1}{2} \left(\int_0^1 h^p(t)dt \right)^{\frac{1}{p}} \left(\int_0^1 \left| \xi'(kt+(1-t)l) \right|^q dt \right)^{\frac{1}{q}}$$

$$\leq \frac{1}{2} \left(\int_0^1 h^p(t)dt \right)^{\frac{1}{p}} \left(\left| \xi'(k) \right|^q \int_0^1 \sin\left(\frac{\pi t}{2}\right) dt + \left| \xi'(l) \right|^q \int_0^1 \cos\left(\frac{\pi t}{2}\right) dt \right)^{\frac{1}{q}}$$

it follows. When the simple integrals on the right side are solved, the proof is completed. □

Corollary 3.8. *If $w(x) = 1$ is assumed in Theorem 3.7, the following inequality is derived:*

$$\left| \frac{\xi(k)+\xi(l)}{2} - \frac{1}{l-k} \int_k^l \xi(x)dx \right| \leq \frac{l-k}{\pi(p+1)^{\frac{1}{p}}} \left(\left| \xi'(k) \right|^q + \left| \xi'(l) \right|^q \right)^{\frac{1}{q}}.$$

Proof. Given that $\int_0^1 h^p(t)dt = \left| \int_{k+(l-k)t}^{l-(l-k)t} dx \right|^p = (l-k)^p \left|1-2t\right|^p dt = (l-k)^p \left(\frac{1}{p+1}\right)$, the proof is completed. □

Theorem 3.9. *Let $\xi : I^\circ \subset \mathbb{R} \to \mathbb{R}$ be a function that is differentiable on I°, with $k,l \in I^\circ$ and $k < l$, and let $w : [k,l] \to \mathbb{R}$ be a differentiable function that is symmetric with respect to $\frac{k+l}{2}$. Under the condition $q > 1$, given that the function $|\xi'|^q$ is trigonometrically convex over the interval $[k,l]$, the following inequality is satisfied:*

$$\left| \frac{1}{l-k} \frac{\xi(k)+\xi(l)}{2} \int_k^l w(x)dx - \frac{1}{l-k} \int_k^l \xi(x)w(x)dx \right|$$

$$\leq \frac{1}{2} \left(\int_0^1 h(t)dt \right)^{1-\frac{1}{q}} \left(\left| \xi'(k) \right|^q \int_0^1 h(t)\sin\left(\frac{\pi t}{2}\right) dt + \left| \xi'(l) \right|^q \int_0^1 h(t)\cos\left(\frac{\pi t}{2}\right) dt \right)^{\frac{1}{q}}, \tag{3.4}$$

where

$$h(t) = \left| \int_{k+(l-k)t}^{l-(l-k)t} w(x)dx \right|$$

is defined.

Proof. By applying the Power Mean inequality together with the fact that the function $|\xi'|^q$ is trigonometrically convex in inequality (3.3), we obtain the following inequality:

$$\left| \frac{1}{l-k} \frac{\xi(k)+\xi(l)}{2} \int_k^l w(x)dx - \frac{1}{l-k} \int_k^l \xi(x)w(x)dx \right|$$

$$\leq \frac{1}{2} \left(\int_0^1 h(t)dt \right)^{1-\frac{1}{q}} \left(\int_0^1 h(t) \left| \xi'(kt+(1-t)l) \right|^q dt \right)^{\frac{1}{q}}$$

$$\leq \frac{1}{2} \left(\int_0^1 h(t)dt \right)^{1-\frac{1}{q}} \left(\left| \xi'(k) \right|^q \int_0^1 h(t)\sin\left(\frac{\pi t}{2}\right) dt + \left| \xi'(l) \right|^q \int_0^1 h(t)\cos\left(\frac{\pi t}{2}\right) dt \right)^{\frac{1}{q}}.$$

Thus, the proof is completed. □

Corollary 3.10. *If $w(x)=1$ is assumed in Theorem 3.9, the following inequality is obtained:*

$$\left|\frac{\xi(k)+\xi(l)}{2}-\frac{1}{l-k}\int_k^l \xi(x)dx\right| \leq \frac{(l-k)}{2^{2-\frac{1}{q}}}\left(\frac{2\pi+8-8\sqrt{2}}{\pi^2}\right)^{\frac{1}{q}}\left(\left|\xi'(k)\right|^q+\left|\xi'(l)\right|^q\right)^{\frac{1}{q}}.$$

Proof. Since $w(x)=1$ implies $h(t)=(l-k)|1-2t|$, the integrals obtained in inequality (3.4) can be found using Python as follows,

$$\begin{aligned}\int_0^1 |1-2t|\sin\left(\frac{\pi t}{2}\right)dt &= \frac{2\pi+8-8\sqrt{2}}{\pi^2}\\ \int_0^1 |1-2t|\cos\left(\frac{\pi t}{2}\right)dt &= \frac{2\pi+8-8\sqrt{2}}{\pi^2}.\end{aligned}$$

□

4. Application

Recent studies have emphasized the significant role of visualizing theoretical expressions through graphical representations. Inspired by this idea, certain results were generated in Python for specific data sets. One of the targeted outcomes was to demonstrate the impact of Hölder and Power Mean inequalities on the upper bound of an inequality, showcasing examples with variations across different values of p and q.

Example 4.1. *In Corollary 3.2, the function $\xi(x)=x^2$ was evaluated for randomly chosen values of k and l within the interval $[0,2]$ under the condition $k<l$, using a step size of 0.1. The left and right sides of the inequality were computed, and their graphs were illustrated:*

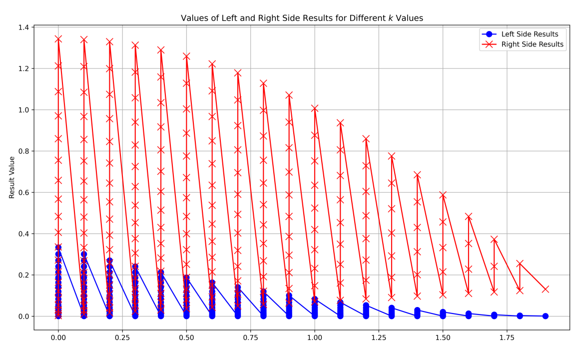

Figure 4.1. Graph of Corollary 3.2

Example 4.2. *In Corollary 3.4, for the function $\xi(x)=x^2$, with the left endpoint of the interval fixed at $k=0.5$ and randomly chosen values of l within $[0.6,2]$ under the condition $k<l$, graphs illustrating the effects on the upper bound of the Hölder inequality for various values of q and p were obtained.*

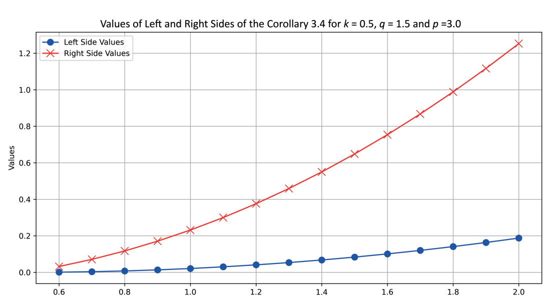

Figure 4.2. Graph of Corollary 3.4 with $p=3, q=1.5$

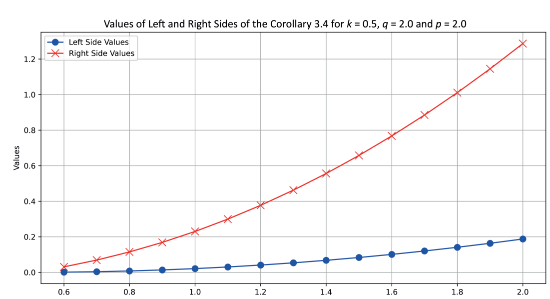

Figure 4.3. Graph of Corollary 3.4 with $p=q=2$

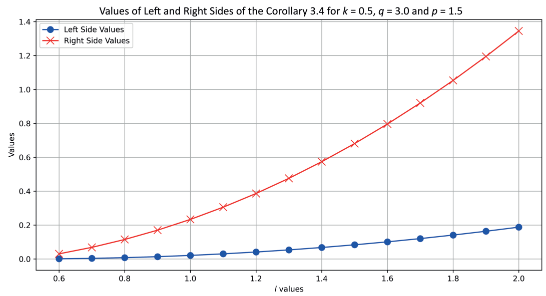

Figure 4.4. Graph of Corollary 3.4 with $p = 1.5, q = 3$

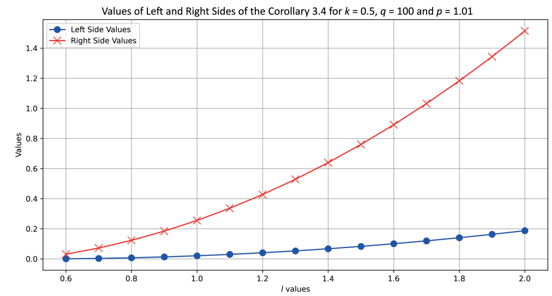

Figure 4.5. Graph of Corollary 3.4 with $p = 1.01, q = 100$

Example 4.3. *In corollary 3.6, with $\xi(x) = x^2$ and a fixed $k = 0.5$, randomly specific values of l were generated under the condition $k < l$. For these values, the fulfillment of the inequality for both the left and right sides, and for any values of q, was demonstrated, and the variations were obtained graphically:*

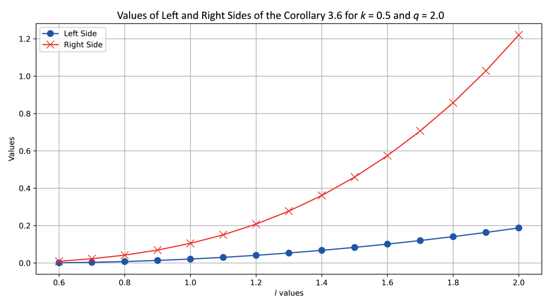

Figure 4.6. Graph of Corollary 3.6 with $q = 2$

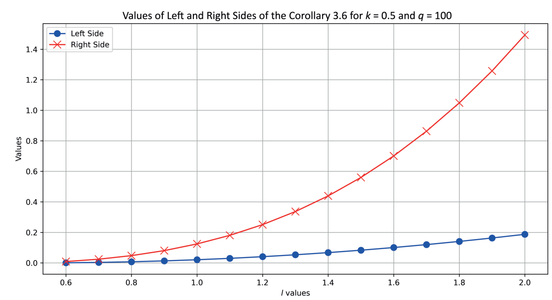

Figure 4.7. Graph of Corollary 3.6 with $q = 100$

Example 4.4. *In corollary 3.10, with $\xi(x) = x^2$ and a fixed $k = 0.5$, randomly specific values of l were generated under the condition $k < l$. For these values, it was demonstrated that the inequality was satisfied for both the left and right sides, and for any values of q, with the variations obtained graphically:*

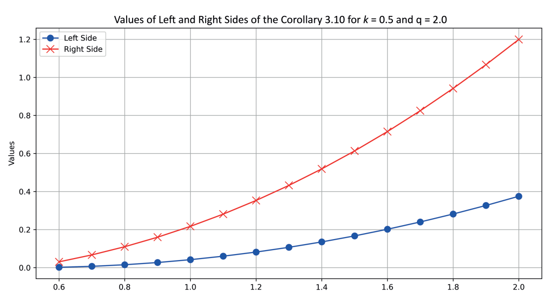

Figure 4.8. Graph of Corollary 3.10 with $q = 2$

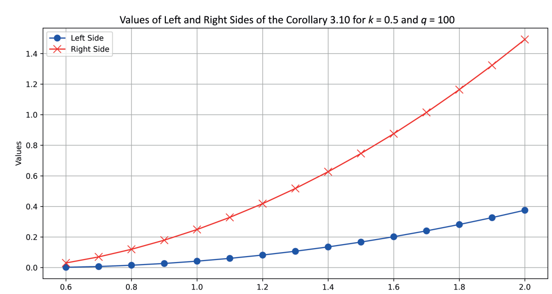

Figure 4.9. Graph of Corollary 3.10 with $q = 100$

5. Conclusion

In conclusion, the study has advanced the understanding of Hermite-Hadamard-Fejér type inequalities within the realm of trigonometrically convex functions. By employing Hölder's inequality and, consequently, the Power Mean inequality, novel upper bounds were derived and subsequently illustrated through graphical representations for various functions, thereby demonstrating their optimality across different parameter values.

Additionally, since the Fejér inequality is expressed through a weight function that can be transformed into various fractional integrals, the framework developed herein allows for the derivation of Hermite-Hadamard type inequalities for trigonometrically convex functions in the context of different fractional integrals. Essentially, this work constitutes a generalization of the classical Hermite-Hadamard midpoint and trapezoidal type inequalities, extending their applicability to fractional integral settings.

Future research could further explore these extensions by investigating broader classes of convex functions and their corresponding fractional integrals, as well as by examining potential applications in numerical analysis and optimization theory.

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Research Article

On the Solution of a Class of Discontinuous Sturm-Liouville Problems

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Abstract
This study examines boundary value problems consisting of a second-order differential equation with discontinuous coefficients and boundary conditions. Asymptotic formulas for the eigenvalues and eigenfunctions of the problem are derived, and an expansion formula is obtained based on the eigenfunctions.

Keywords: Asymptotic formulas, Discontinuous coefficient, Expansion formula, Sturm-Liouville problem
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1. Introduction

Boundary value problems involving discontinuities are prevalent in various fields, including mathematics, mechanics, physics, and other natural sciences. Applications of such boundary value problems in geophysics can be found in [1], [2]. Certain aspects of direct and inverse problems for differential operators with discontinuity conditions are discussed in [3]- [5]. The direct and inverse problems for the Sturm-Liouville operator are studied in [6] and [7]. In [8], an integral representation of the solution for the Sturm-Liouville operator is provided.

For a boundary value problem with a discontinuous coefficient, the direct and inverse problems concerning the Weyl function are examined in [9, 10]. For a similar problem, the fundamental equation, which plays a crucial role in solving the inverse problem, is formulated in [11]. The necessary and sufficient conditions for the solution of the inverse problem are analyzed in [12]. For a boundary value problem defined on a finite interval, consisting of an equation with discontinuous coefficients and boundary conditions with spectral parameters, both direct and inverse problems for the Sturm-Liouville and Dirac operators are considered in [13, 14].

Consider the following boundary value problem defined on the interval $[0,\pi]$:

$$-y''+q(x)y=\lambda^2\upsilon(x)y, \tag{1.1}$$

$$U_1(y):=y'(0)-hy(0)=0, \tag{1.2}$$

$$U_2(y):=y(\pi)=0, \tag{1.3}$$

where $q(x) \in L_2[0,\pi]$ is a real-valued function, λ is a spectral parameter, $h \neq 0$ is an arbitrary real number and

$$\upsilon(x) = \begin{cases} 1, & 0 \le x \le a, \\ \alpha^2, & a \le x \le \pi. \end{cases}$$

2. Preliminaries

Let us show the special solutions of equation (1.1) with $\phi(x,\lambda)$ and $\vartheta(x,\lambda)$ satisfying the conditions

$$\phi(0,\lambda) = 0, \quad \phi'(0,\lambda) = h, \tag{2.1}$$

$$\vartheta(\pi,\lambda) = 0, \quad \vartheta'(\pi,\lambda) = 1. \tag{2.2}$$

For the solution of the (1.1) equation, the following integral representation is obtained:

$$e(x,\lambda) = e_0(x,\lambda) + \int_{-\eta^+(x)}^{\eta^+(x)} K(x,t)e^{i\lambda t}dt,$$

where $\eta^{\pm}(x) = \pm x\sqrt{\upsilon(x)} + a(1 \mp \sqrt{\upsilon(x)})$, $K(x,.) \in L_1\left(-\eta^+(x), \eta^+(x)\right)$ and

$$e_0(x,\lambda) = \begin{cases} e^{i\lambda x}, & 0 \le x \le a, \\ \frac{1}{2}\left(1 + \frac{1}{\sqrt{\upsilon(x)}}\right)e^{i\lambda\eta^+(x)} + \frac{1}{2}\left(1 - \frac{1}{\sqrt{\upsilon(x)}}\right)e^{i\lambda\eta^-(x)}, & a \le x \le \pi. \end{cases}$$

In addition, the K_x derivative exists and provides the following properties

$$\frac{d}{dx}K\left(x,\eta^+(x)\right) = \frac{1}{4\sqrt{\upsilon(x)}}\left(1 + \frac{1}{\sqrt{\upsilon(x)}}\right)q(x), \tag{2.3}$$

$$\frac{d}{dx}\left\{K\left(x,\eta^-(x)+0\right) - K\left(x,\eta^+(x)-0\right)\right\} = \frac{1}{4\sqrt{\upsilon(x)}}\left(1 - \frac{1}{\sqrt{\upsilon(x)}}\right)q(x), \tag{2.4}$$

$$K\left(x,-\eta^+(x)\right) = 0. \tag{2.5}$$

Besides these properties, the following also apply if $q(x)$ is a differentiable function:

$$\upsilon(x)K''_{tt} - K''_{xx} + q(x)K = 0, \quad |t| < \eta^+(x), \tag{2.6}$$

$$\int_{-\eta^+(x)}^{\eta^+(x)} |K(x,t)|dt \le C\left(\exp\left\{\int_0^x |q(t)|dt\right\} - 1\right), \quad 0 < C \tag{2.7}$$

The special solution of equation (1.1) satisfying condition (2.1) is of the form

$$\phi(x,\lambda) = \phi_0(x,\lambda) + \int_0^{\eta^+(x)} A(x,t)\cos\lambda t dt + h\int_0^{\eta^+(x)} \tilde{A}(x,t)\frac{\sin\lambda t}{\lambda}dt, \tag{2.8}$$

and the kernel $A(x,t) = K(x,t) + K(x,-t)$ satisfies the conditions (2.3)-(2.7).

We define

$$\Gamma(\lambda) = \langle \phi(x,\lambda), \vartheta(x,\lambda) \rangle = \phi(x,\lambda)\vartheta'(x,\lambda) - \phi'(x,\lambda)\vartheta(x,\lambda). \tag{2.9}$$

The characteristic function $\Gamma(\lambda)$ is the Wronskian of the functions ϕ and ϑ. From Liouville's theorem, it can be seen that $\Gamma(\lambda)$ is independent of $x \in [0,\pi]$. From equation (2.9), if $x = 0$ and $x = \pi$ respectively, we obtain

$$\Gamma(\lambda) = U_2(\phi) = U_1(\vartheta).$$

Lemma 2.1. *The square of the zeros $\{\lambda_n\}_{n=0}^{\infty}$ of the characteristic function coincide with the eigenvalues of the boundary value problem (1.1)-(1.3). There is also a sequence $\{k_n\}_{n=0}^{\infty}$ such that $\vartheta(x,\lambda_n) = k_n\phi(x,\lambda_n)$ for each eigenvalue λ_n, where $\phi(x,\lambda_n)$ and $\vartheta(x,\lambda_n)$ are the eigenfunctions corresponding to the eigenvalue λ_n.*

Let us define the normalized numbers of the boundary value problem (1.1)-(1.3) as

$$\alpha_n := \int_0^{\pi} \phi^2(x,\lambda_n)\upsilon(x)dx.$$

Lemma 2.2. *The following relation holds*

$$-\dot{\Gamma}(\lambda_n) = 2\lambda_n k_n \alpha_n, \tag{2.10}$$

where $\dot{\Gamma}(\lambda) = \dfrac{d}{d\lambda}\Gamma(\lambda)$.

Proof. Since

$$-\phi''(x,\lambda_n) + q(x)\phi(x,\lambda_n) = \lambda_n^2\upsilon(x)\phi(x,\lambda_n),$$
$$-\vartheta''(x,\lambda) + q(x)\vartheta(x,\lambda) = \lambda^2\upsilon(x)\vartheta(x,\lambda),$$

we have

$$\frac{d}{dx}\langle\phi(x,\lambda_n),\vartheta(x,\lambda)\rangle = (\lambda_n^2-\lambda^2)\upsilon(x)\phi(x,\lambda_n)\vartheta(x,\lambda).$$

Integrating the last equation in the interval $[0,\pi]$ and considering the conditions (2.1), (2.2), we obtain

$$\Gamma(\lambda_n) - \Gamma(\lambda) = (\lambda_n^2-\lambda^2)\int_0^{\pi}\phi(x,\lambda_n)\vartheta(x,\lambda)\upsilon(x)dx.$$

The desired result is obtained by taking the limit for $\lambda \to \lambda_n$. □

3. Asymptotic Formulas of the Eigenvalues

Theorem 3.1. *The eigenvalues $\{\lambda_n\}$ and the eigenfuynctions $\phi(x,\lambda_n)$, $\vartheta(x,\lambda_n)$ are real. All zeros of $\Gamma(\lambda)$ are simple. Eigenfunctions related to different eigenvalues are orthogonal in $L_2(0,\pi)$.*

Lemma 3.2. *When $q(x)\equiv 0$, the eigenvalues of the boundary value problem (1.1)-(1.3) have the following asymptotic form:*

$$\left(\lambda_n^0\right)^2 = n + \vartheta(n), \quad \sup_n|\vartheta(n)| < +\infty.$$

Lemma 3.3. *The λ_n^0 roots of the function $\Gamma_0(\lambda)$ are discrete, i.e.*

$$\inf_{n\neq k}|\lambda_n^0 - \lambda_k^0| = \tau > 0.$$

Proof. Assume the opposite, that there are sequences $\{\lambda_k^{0'}\}$ and $\{\lambda_k^{0''}\}$ such that $\lambda_k^{0'} \neq \lambda_k^{0''}$, $\lambda_k^{0'} \to +\infty$, $\lambda_k^{0''} \to +\infty$ and

$$\lim_{k\to+\infty}\left[\lambda_k^{0'} - \lambda_k^{0''}\right]$$

for the zeros of the function $\Gamma_0(\lambda)$. Since the eigenfunctions of the boundary value problem (1.1)-(1.3) are orthogonal, we obtain

$$\begin{aligned}
0 =& \lambda_k^{0'}\lambda_k^{0''}\int_0^{\pi}\phi_0\left(x,\lambda_k^{0'}\right)\phi_0\left(x,\lambda_k^{0''}\right)\upsilon(x)dx\\
&= I_k + \int_0^{\pi}\left(\lambda_k^{0'}\right)^2\phi_0^2\left(x,\lambda_k^{0'}\right)\upsilon(x)dx\\
&\geq I_k + \int_0^{a}\left(\lambda_k^{0'}\right)^2\phi_0^2\left(x,\lambda_k^{0'}\right)\upsilon(x)dx\\
&= I_k + \frac{a}{2} - \frac{\sin 2\lambda_k^{0'}a}{4\lambda_k^{0'}},
\end{aligned} \tag{3.1}$$

where $I_k = \int_0^{\pi} \lambda_k^{0'} \phi_0\left(x, \lambda_k^{0'}\right) \left[\lambda_k^{0''} \phi_0\left(x, \lambda_k^{0''}\right) - \lambda_k^{0'} \phi_0\left(x, \lambda_k^{0'}\right)\right] \upsilon(x) dx$.

Now let us show that $I_k \to 0$ when $k \to +\infty$.

Since $|\lambda_k^{0''} \phi_0\left(x, \lambda_k^{0''}\right) - \lambda_k^{0'} \phi_0\left(x, \lambda_k^{0'}\right)| \le C|\lambda_k^{0''} - \lambda_k^{0'}|$, for $\forall x \in [0, \pi]$,

$$\lim_{k \to +\infty} |\lambda_k^{0''} \phi_0\left(x, \lambda_k^{0''}\right) - \lambda_k^{0'} \phi_0\left(x, \lambda_k^{0'}\right)| = 0.$$

In inequality (3.1), if the limit is taken when $k \to +\infty$, it can be shown that $0 \ge \frac{a}{2}$. This is in contradiction to the definition of the coefficient $\upsilon(x)$ in equation (1.1). It can, therefore, be concluded that the proof is complete. □

Lemma 3.4. *The set of eigenvalues of the boundary value problem (1.1)-(1.3) are countable and of the form*

$$\lambda_n = \lambda_n^0 + \frac{d_n}{\lambda_n^0} + \frac{\eta_n}{n}$$

where λ_n^0 is the zeros of the characteristic function $\Gamma_0(\lambda)$, d_n is a finite sequence and $\{\eta_n\} \in l_2$.

Proof. From the condition (1.3), $\phi(\pi, \lambda) = \Gamma(\lambda) \mid_{x=\pi}$ can be written. From the (2.8) representations of the function $\phi(\pi, \lambda)$, we get

$$\Gamma(\lambda) = \Gamma_0(\lambda) + \int_0^{\eta^+(\pi)} A(\pi, t) \cos \lambda t dt + h \int_0^{\eta^+(\pi)} \tilde{A}(\pi, t) \frac{\sin \lambda t}{\lambda} dt. \tag{3.2}$$

Let $\sigma < \frac{\tau}{2}$ be a sufficiently small positive number and let $G_\sigma = \{\lambda : |\lambda - \lambda_n^0| \ge \sigma\}$. From [15],

$$\mid \Gamma_0(\lambda) \mid \ge C_\sigma \frac{e^{|Im\lambda|\eta^+(\pi)}}{\lambda}, \quad \lambda \in G_\sigma.$$

On the other hand, if $f(x) \in L_1(0, \pi)$, from the expression

$$\lim_{|\lambda| \to \infty} e^{-|Im\lambda\pi|} \int_0^{\pi} f(x) \cos \lambda dx = \lim_{|\lambda| \to \infty} e^{-|Im\lambda\pi|} \int_o^{\pi} f(x) \sin \lambda dx$$

then

$$\Gamma(\lambda) - \Gamma_0(\lambda) = O\left(\frac{e^{|Im\lambda|\eta^+(\pi)}}{|\lambda|}\right), \quad |\lambda| \to \infty.$$

Therefore, for a sufficiently large n, the inequality
$\mid \Gamma(\lambda) - \Gamma_0(\lambda) \mid \le \mid \Gamma_0(\lambda) \mid$ is satisfied on the $\Omega_n = \{\lambda : |\lambda| = |\lambda_n^0| + \frac{\tau}{2}\}$ curves. Appliyng now Rouche's theorem to the curve $\omega_n(\sigma) = \{\lambda : |\lambda - \lambda_n^0| \le \sigma\}$, we conclude that for sufficiently large n, in $\omega_n(\sigma)$ there is exactly one zero of $\Gamma(\lambda)$, namely λ_n.

Now let's find the eigenvalues λ_n. For an arbitrary number $\sigma > 0$,

$$\lambda_n = \lambda_n^0 + \varepsilon_n, \quad \varepsilon_n = o(1), \quad n \to \infty \tag{3.3}$$

is obtained. Substituting (3.3) into (3.2) we get,

$$\Gamma(\lambda_n^0 + \varepsilon_n) = \Gamma_0(\lambda_n^0 + \varepsilon_n) + \int_0^{\eta^+(\pi)} A(\pi, t) \cos\left(\lambda_n^0 + \varepsilon_n\right) t dt + h \int_0^{\eta^+(\pi)} \tilde{A}(\pi, t) \frac{\sin\left(\lambda_n^0 + \varepsilon_n\right) t}{\lambda_n^0 + \varepsilon_n} dt = 0. \tag{3.4}$$

Considering the equations $\Gamma_0(\lambda_n^0) = 0$ and (3.4) together in relation $\Gamma_0(\lambda_n^0 + \varepsilon_n) = \dot{\Gamma}_0(\lambda_n^0)\varepsilon_n + o(\varepsilon_n^2)$, we find

$$\dot{\Gamma}_0(\lambda_n^0)\varepsilon_n + \int_0^{\eta^+(\pi)} A(\pi, t) \cos\left(\lambda_n^0 + \varepsilon_n\right) t dt + h \int_0^{\eta^+(\pi)} \tilde{A}(\pi, t) \frac{\sin\left(\lambda_n^0 + \varepsilon_n\right) t}{\lambda_n^0 + \varepsilon_n} dt \approx 0.$$

Using properties (2.4) and (2.5), since $\varepsilon_n = o(1)$ while $n \to \infty$ we find

$$\varepsilon_n \approx \frac{1}{\dot{\Gamma}_0(\lambda_n^0)\lambda_n^0}\left\{\frac{1}{4}\int_0^{\pi}\frac{1}{\sqrt{\upsilon(t)}}\left(1-\frac{1}{\sqrt{\upsilon(t)}}\right)\sin\left(\lambda_n^0\eta^-(\pi)\right)q(t)dt\right.$$
$$\left.-\frac{1}{4}\int_0^{\pi}\frac{1}{\sqrt{\upsilon(t)}}\left(1+\frac{1}{\sqrt{\upsilon(t)}}\right)\sin\left(\lambda_n^0\eta^-(\pi)\right)q(t)dt - \int_0^{\eta^+(\pi)}A_t'(\pi,t)\cos(\lambda_n^0 t)dt\right\}$$
$$+\frac{h}{\dot{\Gamma}_0(\lambda_n^0)\lambda_n^0}\left\{\frac{1}{4}\int_0^{\pi}\frac{1}{\sqrt{\upsilon(t)}}\left(1-\frac{1}{\sqrt{\upsilon(t)}}\right)\frac{\cos\left(\lambda_n^0\eta^-(\pi)\right)}{\lambda_n^0}q(t)dt\right.$$
$$\left.+\frac{1}{4}\int_0^{\pi}\frac{1}{\sqrt{\upsilon(t)}}\left(1+\frac{1}{\sqrt{\upsilon(t)}}\right)\frac{\cos\left(\lambda_n^0\eta^-(\pi)\right)}{\lambda_n^0}q(t)dt - \frac{1}{\lambda_n^0}\int_0^{\eta^+(\pi)}\tilde{A}_t'(\pi,t)\cos(\lambda_n^0 t)dt\right\}$$
$$=\frac{1}{\dot{\Gamma}_0(\lambda_n^0)\lambda_n^0}\left\{d_n+\eta_n+\frac{\tilde{\eta}_n}{\lambda_n^0}\right\}$$

where $\eta_n := \int_0^{\eta^+(\pi)}A_t'(\pi,t)\sin(\lambda_n^0 t)dt$, $\tilde{\eta}_n := \int_0^{\eta^+(\pi)}\tilde{A}_t'(\pi,t)\cos(\lambda_n^0 t)dt$ and $\eta_n, \tilde{\eta}_n \in l_2$. □

4. Expansion Formula with respect to Eigenfunctions

In this section, the completeness of the eigenfunctions of the boundary value problem (1.1)-(1.3) is shown, and the expansion formula for the eigenfunctions is obtained. Let

$$G(x,t;\lambda) := -\frac{1}{\Gamma(\lambda)}\begin{cases}\phi(t,\lambda)\vartheta(x,\lambda), & t \le x,\\ \vartheta(t,\lambda)\phi(x,\lambda), & t > x.\end{cases}$$

Consider the function

$$y(x,\lambda) = \int_0^{\pi}G(x,t;\lambda)f(t)\upsilon(t)dt. \tag{4.1}$$

Theorem 4.1. *The system of eigenfunctions $\{\phi(x,\lambda_n)\}_{n\ge 0}$ of the boundary value problem (1.1)-(1.3) is complete in $L_{2,\upsilon}[0,\pi]$.*

Proof. Using (2.10) and Lemma 2.1, we get

$$\vartheta(x,\lambda_n) = -\frac{\dot{\Gamma}(\lambda_n}{2\lambda_n\alpha_n}\phi(x,\lambda_n). \tag{4.2}$$

From (4.1) and (4.2), we have

$$Res_{\lambda=\lambda_n}y(x,\lambda) = \frac{1}{2\lambda_n\alpha_n}\phi(x,\lambda_n)\int_0^{\pi}\phi(t,\lambda_n)f(t)\upsilon(t)dt. \tag{4.3}$$

Let us assume $f(x) \in L_{2,\upsilon}[0,\pi]$ and $\int_0^{\pi}\phi(t,\lambda_n)\overline{f(t)}\upsilon(t)dt$. Then $Res_{\lambda=\lambda_n}y(x,\lambda) = 0$ is obtained. Thus, for each fixed $x \in [0,\pi]$, $y(x,\lambda)$ is entire with respect to λ. If $f(x) \in L_1(0,\pi)$, the equations

$$\lim_{|\lambda|\to\infty}\max_{x\in[0,\pi]}\left\{e^{-|Im\lambda|x}\left|\int_0^x f(t)\cos\lambda t dt\right|\right\} = 0$$

$$\lim_{|\lambda|\to\infty}\max_{x\in[0,\pi]}\left\{e^{-|Im\lambda|x}\left|\int_0^x f(t)\sin\lambda t dt\right|\right\} = 0$$

are satisfied. Also, for $|\lambda| \to \infty$, we get

$$\phi(x,\lambda) = O\left(\frac{1}{|\lambda|}e^{|Im\lambda|\eta^+(x)}\right),$$

$$\phi'(x,\lambda) = \phi_0'(x,\lambda) + O\left(\frac{1}{|\lambda|}e^{|Im\lambda|\eta^+(x)}\right) = O\left(e^{|Im\lambda|\eta^+(x)}\right),$$

$$\vartheta(x,\lambda) = O\left(\frac{1}{|\lambda|}e^{|Im\lambda|\left(\eta^+(\pi)-\eta^+(x)\right)}\right),$$

$$\vartheta'(x,\lambda) = \vartheta_0'(x,\lambda) + O\left(\frac{1}{|\lambda|}e^{|Im\lambda|\left(\eta^+(\pi)-\eta^+(x)\right)}\right) = O\left(e^{|Im\lambda|\left(\eta^+(\pi)-\eta^+(x)\right)}\right).$$

Then the following inequality is satisfied:

$$|\Gamma(\lambda)| \geq C_\sigma \frac{1}{|\lambda|}e^{|Im\lambda|\eta^+(\pi)}, \quad \lambda \in G_\sigma.$$

From equation (4.1) it follows that $|y(x,\lambda)| \leq \dfrac{C_\sigma}{|\lambda|}, \quad \lambda \in G_\sigma, \quad |\lambda| \geq \lambda^*$ for $\sigma > 0$ and sufficiently large $\lambda^* > 0$. Thus $f(x) \equiv 0$ is obtained almost everywhere in the interval $[0,\pi]$. Therefore, the proof is complete.

□

Theorem 4.2. *Let $f(x)$ be an absolutely continuous function on $[0,\pi]$. Then*

$$f(x) = \sum_{n=1}^{\infty} a_{,n}\phi(x,\lambda_n), \quad a_n = \frac{1}{\alpha_n}\int_0^\pi \phi(t,\lambda_n)f(t)\upsilon(t)dt, \tag{4.4}$$

and the series converges uniformly on $[0,\pi]$. For $f(x) \in L_{2,\upsilon}[0,\pi]$, the series (4.4) converces in $L_{2,\upsilon}[0,\pi]$ and the following Parseval's equality is satisfied:

$$\int_0^\pi |f(x)|^2\upsilon(x)dx = \sum_{n=1}^{\infty}\alpha_n|a_n|^2.$$

Proof. Let $f(x) \in AC[0,\pi]$. Since $\phi(x,\lambda)$ and $\vartheta(x,\lambda)$ are solutions of the boundary value problem (1.1)-(1.3), we get

$$y(x,\lambda) = -\frac{1}{\lambda^2\Gamma(\lambda)}\left\{\vartheta(x,\lambda)\int_0^x\left(-\phi''(t,\lambda)+q(t)\phi(t,\lambda)\right)f(t)dt \;+\phi(x,\lambda)\int_x^\pi\left(-\vartheta''(t,\lambda)+q(t)\vartheta(t,\lambda)\right)f(t)dt\right\}.$$

Integration of the terms containing second derivatives by parts yields in view of (1.2), (1.3)

$$y(x,\lambda) = -\frac{f(x)}{\lambda^2} - \frac{1}{\lambda^2}\left(Z_1(x,\lambda)+Z_2(x,\lambda)\right), \tag{4.5}$$

where

$$Z_1(x,\lambda) = \frac{1}{\Gamma(\lambda)}\left[\vartheta(x,\lambda)\int_0^x\phi'(t,\lambda)f'(t)dt + \phi(x,\lambda)\int_x^\pi\vartheta'(t,\lambda)f'(t)dt\right],$$

$$Z_2(x,\lambda) = \frac{1}{\Gamma(\lambda)}\left[\vartheta(x,\lambda)f(0) - \phi(x,\lambda)f(\pi)\right.$$
$$\left. +\vartheta(x,\lambda)\int_0^x\phi(t,\lambda)q(t)f(t)dt + \phi(x,\lambda)\int_x^\pi\vartheta(t,\lambda)q(t)f(t)dt\right].$$

Now consider the integral

$$I_n(x) = \frac{1}{2\pi i}\oint_{\Omega_n}\lambda y(x,\lambda)d\lambda,$$

where $\Omega_n = \left\{\lambda : |\lambda| = |\lambda_n^0| + \dfrac{\tau}{2}\right\}$ is a clockwise directed curve and n is a sufficiently large natural number. From (4.5), we get

$$\frac{1}{2\pi i}\oint_{\Omega_n}\lambda y(x,\lambda)d\lambda = \frac{1}{2\pi i}\oint_{\Omega_n}\frac{f(x)}{\lambda}d\lambda - \frac{1}{2\pi i}\oint_{\Omega_n}\frac{\{Z_1(x,\lambda)+Z_2(x,\lambda)\}}{\lambda}d\lambda. \tag{4.6}$$

Thus, we obtain

$$I_n(x) = 2\sum_{n=1}^{N}Res_{\lambda=\lambda_n}\left(\lambda y(x,\lambda)\right).$$

From the equations (4.3) and (4.6), we get

$$-f(x)+\varepsilon_n(x)=-\sum_{n=1}^{N}\frac{\phi(x,\lambda_n)}{\alpha_n}\left\{\int_0^{\pi}\phi(t,\lambda_n)f(t)\upsilon(t)dt\right\},$$

where

$$\varepsilon_n(x)=-\frac{1}{2\pi i}\oint_{\Omega_n}\{Z_1(x,\lambda)+Z_2(x,\lambda)\}d\lambda.$$

For fixed $\sigma>0$ and sufficiently large $\lambda^*>0$, the following relations hold:

$$\max_{x\in[0,\pi]}\|Z_2(x,\lambda)\|\leq\frac{C_2}{|\lambda|},\quad \lambda\in G_\sigma,\quad |\lambda|\geq\lambda^*, \tag{4.7}$$

$$\max_{x\in[0,\pi]}\|Z_1(x,\lambda)\|\leq\frac{C_1}{|\lambda|},\quad \lambda\in G_\sigma,\quad |\lambda|\geq\lambda^*. \tag{4.8}$$

From expressions (4.7) and (4.8) it follows that the equality

$$\lim_{n\to\infty}\max_{x\in[0,\pi]}|\varepsilon_n(x)|=0$$

is satisfied. The last equation gives the following expansion formula:

$$f(x)=\sum_{n=1}^{\infty}a_n\phi(x,\lambda_n),$$

where

$$a_n=\sum_{n=1}^{\infty}\frac{1}{\alpha_n}\left\{\int_0^{\pi}\phi(t,\lambda_n)f(t)\upsilon(t)dt\right\}.$$

Since the system $\{\phi(x,\lambda_n)\}$ forms an orthogonal base at $L_{2,\upsilon}[0,\pi]$, Parseval equality is satisfied. □

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