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Analytical and numerical study on the solutions of a new (2+1)-dimensional conformable shallow water wave equation

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ABSTRACT. The (2+1)-dimensional conformable nonlinear shallow water wave equation is examined in this work. Initially, definitions and properties of suitable derivatives are presented. Subsequently, exact solutions to this equation are derived using the $\exp(-\phi(\xi))$-expansion and the modified extended tanh-function methods. Then, a numerical method, namely the residual power series method, is utilized to obtain approximate solutions. The interplay between analytical and numerical approaches is explored to validate the solutions. This study fills a gap in the literature on fractional shallow water models, particularly in (2+1)-dimensions, and offers new insights into wave dynamics governed by fractional derivatives. The physical implications of the findings are illustrated through 3D and 2D contour surfaces of some obtained data, offering insight into the physical interpretation of geometric structures. A table is also presented to compare the obtained results. These solutions highlight the practical uses of the investigated model and other nonlinear models in applied sciences. These techniques can potentially yield significant results in solving various fractional differential equations.

2020 Mathematics Subject Classification. 35R11, 35A08, 35C08, 35A35.
Keywords. $\exp(-\phi(\xi))$-expansion method, modified extended tanh-function method, residual power series method, shallow water wave equation, conformable derivative.

1. Introduction

Fractional differential equations are essential in various fields spanning social and fundamental sciences and engineering disciplines. Recently, their importance has increased due to their indispensable contribution to understanding complex physical processes in areas such as control theory, electrical circuits, and wave propagation. In particular, fractional differential equations arise in various applications, including electrical circuits, chemical engineering, biostatistics and epidemiology, mechanical systems, computer science, optimization, drug development, social sciences, medicine and biology, weather and climate models, robotics and artificial intelligence, and signal processing.

These equations are valuable tools for modeling, analyzing, and designing solutions for numerous engineering problems. Their ability to vividly illustrate nonlinear physical features makes them an essential framework for guiding future work. Consequently, finding solutions to these equations is a remarkable achievement in related fields. Several authors utilized various techniques to compute these solutions and gain a deeper understanding of the essential features of material structures in various settings.

A variety of analytical methods have been employed to pursue solutions and nuanced comprehension of these equations. It has become evident that no single technique can universally address all types of nonlinear problems with precision. This realization has given rise to numerous methods, including the modified simplest equation method [31,32], the auxiliary equation method [18,19], the modified extended tanh-function method [16], the Bernoulli sub-equation function method [35,36], the $\exp(-\phi(\xi))$-expansion method [11], the sine-Gordon expansion method [37], the modified exponential function method [38], the rational sine-Gordon expansion method [39], the $(1/G')$-expansion method [13,40], the (G'/G^2)-expansion

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method [14], the modified (G'/G)-expansion method [5], the φ^6-model expansion method [20], and the homotopy perturbation method [15,25,26] etc.

The allure of high-dimensional fractional partial differential equations (FPDEs) has captivated the attention of academics in recent years. Their prevalence extends across biology, chemistry, physics, engineering, mechanics, and economics, among other branches. Various derivative definitions have been proposed for fractional differential equations, including the Riemann-Liouville [17], Caputo [34], and conformable [2] derivatives. The Riemann-Liouville derivative, stemming from the contributions of Riemann and Liouville, stands out for its frequent application in contemporary mathematical discourse. Additionally, the conformable fractional derivative approach has gained popularity among mathematicians due to its simplicity and reliability. The term "conformable" refers to the use of conformable fractional derivatives in the equation, which generalizes the classical shallow water wave equation to account for non-integer-order calculus. This fractional framework allows the model to better describe physical processes that exhibit memory, nonlocal effects, or complex dynamics, making the equation more adaptable to real-world phenomena.

As a well-known FPDE, shallow water wave equations model wave behavior in shallow bodies of water like seas, rivers, or coastal regions. The (2+1)-dimensions account for two spatial variables and time, which allows for more complex interactions like wave breaking, dispersion, and nonlinear effects. Here, we address the following shallow water wave problem in (2+1)-dimensions [3],

$$A\mathcal{D}_t^{\vartheta} u_x + a u_{xx} + b(u)^2_{xx} + c u_{xxxx} + d u_{yy} = 0. \tag{1}$$

where $\mathcal{D}_t^{\vartheta}$ denotes the conformable derivative, and A, a, b, c, d are arbitrary constants. This equation serves as a descriptive model for the propagation of gravity waves on a water surface, particularly in scenarios where oblique waves directly interact with the surface [22]. Besides, the conformable shallow water wave equation describes the behavior of shallow water waves, typically focusing on how waves propagate in fluids where the horizontal length scale is much larger than the vertical depth.

Although there is a body of work on integer-order (2+1)-dimensional shallow water wave equations, the fractional (conformable) extension in (2+1)-dimensions is less studied. Research is particularly limited in deriving exact solutions for this model. For instance, in [27], the authors have obtained multiple rogue wave solutions to the model using the Hirota bilinear transformation and the trial function method. Besides, in this research paper, innovative methodologies are employed to present exact traveling wave solutions as well as the numerical solutions to Eq. (1). The objective is to surmount the limitations associated with conventional methods and offer effective solutions to this intricate equation.

The paper is organized as follows. Basic definitions are given in Section 2. The $\exp(-\phi(\xi))$-expansion method is described in detail in Section 3. The modified extended tanh-function approach is detailed in Section 4. A numerical approach, the residual power series method (RPSM), is introduced in Section 5. Section 6 contains analytical and approximate solutions of the studied equation. In Section 7, the paper presents the results.

2. Conformable Derivative

The conformable derivative is a relatively recent approach to fractional calculus that preserves many properties of the standard derivative, making it easier to apply to physical systems. Conformable derivatives have already been applied to classical models, improving the flexibility of solutions to represent more realistic physical phenomena.

Definition 1. The conformable derivative of a function, $h : [0, \infty) \to \mathbb{R}$, $t > 0$, $\theta \in (0, 1)$ of order θ is as follows defined:

$$\mathcal{D}_t^{\theta}(h)(t) = \lim_{\gamma \to 0} \frac{h(t + \gamma t^{1-\theta}) - h(t)}{\gamma}. \tag{2}$$

Additionally, in the event that h is-differentiable within a given interval $(0, k)$, where $k > 0$, and the $\lim_{t\to 0^+} \mathcal{D}_t^{\theta}(h)(t)$ exists, then definition is formed

$$\mathcal{D}_t^{\theta}(h)(0) = \lim_{t \to 0^+} \mathcal{D}_t^{\theta}(h)(t). \tag{3}$$

Lemma 1. Let h_1 and h_2 be θ-differentiable at $t > 0$ for $0 < \theta \leq 1$ [12,24,28]. There after,

- $\mathcal{D}_t^\theta(t^{s_1}) = s_1 t^{s_1-\theta}$, $s_1 \in \mathbb{R}$,
- $\mathcal{D}_t^\theta(s_1 h_1 + s_2 h_2) = s_1 \mathcal{D}_t^\theta(h_1) + s_2 \mathcal{D}_t^\theta(h_2)$, $s_1, s_2 \in \mathbb{R}$,
- $\mathcal{D}_t^\theta(\frac{h_1}{h_2}) = \frac{h_2.\mathcal{D}_t^\theta(h_1) - h_1 \mathcal{T}_t^\theta(h_2)}{h_2^2}$,
- $\mathcal{D}_t^\theta(h_1.h_2) = h_1.\mathcal{D}_t^\theta(h_2) + h_2.\mathcal{D}_t^\theta(h_1)$,
- $\mathcal{D}_t^\theta(h_1)(t) = t^{1-\theta}\frac{dh_1(t)}{dt}$,
- $\mathcal{D}_t^\theta(C) = 0$, when C is a const.

Definition 2. Let the function h with n variables be defined as $(y_1, y_2, \ldots, y_n)$. The partial derivatives of h with respect to y_i of order $\theta \in (0, 1]$ are given as [29,33]:

$$\frac{d^\theta}{dy_i^\theta} h(y_1, y_2, \ldots, y_n) = \lim_{\gamma \to 0} \frac{h(y_1, y_2, \ldots, y_{i-1}, y_i + \gamma y_i^{1-\theta}, y_n) - h(y_1, y_2, \ldots, y_n)}{\gamma}.$$

The next section is reserved to introduce the $\exp(-\phi(\xi))$-expansion, the modified extended tanh-function, and the RPS methods.

3. The Exp$(-\phi(\xi))$-Expansion Method

Examine the nonlinear equation, which is presented as follows:

$$\mathcal{P}(u, \mathcal{D}_t^\theta u, \mathcal{D}_x u, \mathcal{D}_y u, \mathcal{D}_x^2 u, \mathcal{D}_y^2 u, \ldots) = 0. \tag{4}$$

In this case, $\mathcal{D}_t^\theta$ represents the conformable derivative operator of the function. When $\mathcal{P}$ is a polynomial of $u(x, y, \ldots, t)$ and its derivatives, and the subscripts signifying partial derivatives. During utilizing the $\exp(-\phi(\xi))$-expansion method [1,21,23] for obtaining wave solutions of Eq. (4), it is crucial to carry out the next procedures.

- The real variables $x, y, z, \ldots, t$ are combined using ξ as a compound variable.

$$\xi = kx + ly + \cdots + \frac{mt^\theta}{\theta}, \quad u(x, y, z, \ldots, t) = u(\xi). \tag{5}$$

where the $k, l, \ldots, m$ are arbitrary values to be determined later.

- The following ordinary differential equation (ODE) is what is left after reducing Eq. (4),

$$\mathcal{H}(u(\xi), u'(\xi), u''(\xi), \ldots) = 0. \tag{6}$$

- The following finite series can be used to construct the precise solutions:

$$u(\xi) = B_0 + \sum_{r=1}^{N} B_r(\exp(\xi(-\phi)))^r, B_N \neq 0. \tag{7}$$

- The following ODE is satisfied by $\phi = \phi(\xi)$.

$$\phi'(\xi) = exp(-\phi(\xi)) + \eta exp(\phi(\xi)) + \lambda. \tag{8}$$

- Eq. (8) shows the following solutions when $\eta \neq 0$ and $\lambda^2 - 4\eta > 0$, depending on certain parameters.

$$u_1(\xi) = \frac{\ln\left(-\sqrt{(\lambda^2 - 4\eta)}\tanh\left(\frac{\sqrt{(\lambda^2-4\eta)}}{2}(h + \xi)\right) - \lambda\right)}{2\eta}, \tag{9}$$

in the case of $\lambda^2 - 4\eta < 0$ and $\eta \neq 0$

$$u_2(\xi) = \frac{\ln\left(\sqrt{(4\eta-\lambda^2)}\tanh\left(\frac{\sqrt{(4\eta-\lambda^2)}}{2}(h+\xi)\right)-\lambda\right)}{2\eta}, \tag{10}$$

in the case of $\lambda^2 - 4\eta > 0$, $\lambda \neq 0$ and $\eta = 0$,

$$u_3(\xi) = -\ln\left(\frac{\lambda}{\sinh(\lambda(h+\xi)) + \cosh(\lambda(h+\xi)) - 1}\right), \tag{11}$$

in the case of $\lambda^2 - 4\eta = 0$, $\lambda \neq 0$ and $\eta \neq 0$,

$$u_4(\xi) = \ln\left(-\frac{2(\lambda(h+\xi)+2)}{\lambda^2(h+\xi)}\right), \tag{12}$$

in the case of $\lambda^2 - 4\eta = 0$, $\lambda = 0$ and $\eta = 0$,

$$u_5(\xi) = \ln(h+\xi), \tag{13}$$

where the constant for integration is h.

- The determination of the N value in Eq. (7) involves considering the balance principle between the largest nonlinear terms and the highest order derivatives of $u(\xi)$ as outlined in Eq. (6). Upon replacing Eq. (7) with Eq. (8) into Eq. (6) and consolidating terms with identical powers of $\exp(-\phi)$, the left-hand side of Eq. (6) undergoes a transformation into a polynomial. This transformation results in a system of algebraic equations involving variables B_r, $(r = 0, 1, 2, 3, \ldots, N), c, \lambda$, and η. The solution to Eq. (6) can be obtained by setting all the coefficients of this polynomial to zero, solving the resulting system of algebraic equations, and then substituting the solutions back into Eq. (7).

4. Modified Extended Tanh-Function Method

Let us explore a specific partial differential equation (PDE) to illustrate the core concept of the modified extended tanh-function method [4,30,41].

$$\mathcal{B}(v, \mathcal{D}_t^\theta v, \mathcal{D}_x v, \mathcal{D}_y v, \mathcal{D}_x^2 v, \mathcal{D}_y^2 v, \ldots) = 0, \tag{14}$$

where $\mathcal{B}$ is a polynomial in $v(x, y, z, \ldots, t)$ with nonlinear components in its partial derivatives. The transformation,

$$\xi = kx + ly + \ldots + \frac{mt^\theta}{\theta}, \quad v(x, y, z, \ldots, t) = v(\xi), \tag{15}$$

converts Eq. (14) into an ODE presented in the subsequent form,

$$\mathcal{B}(v(\xi), v'(\xi), v''(\xi), \ldots) = 0. \tag{16}$$

Assume that the solution to Eq. (16) takes on the following form,

$$v(\xi) = A_0 + \sum_{r=1}^{N}\left(A_r\phi^r(\xi) + B_r\phi^{-r}(\xi)\right). \tag{17}$$

Here, $\phi(\xi)$ satisfies the following Riccati equation,

$$\phi'(\xi) = \sigma + \phi(\xi)^2, \tag{18}$$

where σ is a constant that will be found out afterward. As may be seen below, Eq. (18) admits several different solutions as,

- If $\sigma < 0$

$$\phi(\xi) = -\sqrt{-\sigma}tanh(\sqrt{-\sigma}\xi) \quad or \quad \phi(\xi) = -\sqrt{-\sigma}coth(\sqrt{-\sigma}\xi).$$

- If $\sigma > 0$

$$\phi(\xi) = \sqrt{\sigma} tan(\sqrt{\sigma}\xi) \quad or \quad \phi(\xi) = -\sqrt{\sigma} cot(\sqrt{\sigma}\xi).$$

- If $\sigma = 0$

$$\phi(\xi) = -\frac{1}{\xi}. \tag{19}$$

Determining the positive integer N in Eq. (17) involves achieving a balance between the highest order derivatives and the nonlinear variables. Symbolic calculations can be used to find the values of A_r and B_r by replacing Eq. (17), and Eq. (18) in Eq. (16). Following this path, by gathering terms with the same power ϕ^r, where $(r = 0, 1, 2, \cdots, N)$, and setting them to zero, produces the unknown constants. The exact solutions to Eq. (14) can subsequently be derived by replacing the determined values, along with the Eq. (17).

5. Residual Power Series Method (RPSM)

To illustrate the principle of the RPSM [6–10] algorithm, examine the following nonlinear fractional differential equation (FDE).

$$\mathcal{D}_t^{\theta} u(x, y, t) + R[x, y]u(x, y, t) + N[x, y]u(x, y, t) = h(x, y, t). \tag{20}$$

where $R[x, y]$ is a linear and $N[x, y]$ is a nonlinear operator. The initial condition of the equation is expressed as

$$u(x, y, 0) = f_0(x, y) = f(x, y). \tag{21}$$

Subject to the constraint of Eq. (21), the approach entails expanding a fractional series at $t = 0$ to find the solution to Eq. (20),

$$f_{n-1}(x, y) = h(x, y) = \mathcal{D}_t^{(n-1)\theta} u(x, y, 0). \tag{22}$$

As seen below, the solution can be stated as a series expansion,

$$u(x, y, t) = \sum_{n=0}^{\infty} f_n(x, y) \frac{t^{n\theta}}{\theta^n n!}. \tag{23}$$

Thus, for $\mathbb{R}^{\frac{1}{v}}$ be the radius of convergence, $0 \leq t < \mathbb{R}^{\frac{1}{v}}$ and $0 < \theta \leq 1$, the $k - th$ truncated series of $u(x, y, t)$, represented as,

$$u_k(x, y, t) = f(x, y) + \sum_{n=1}^{k} f_n(x, y) \frac{t^{n\theta}}{\theta^n n!}, k = 1, 2, 3, \ldots \tag{24}$$

Therefore, the $k - th$ residual function's initial expression is

$$Resu_k(x, y, t) = \mathcal{D}_t^{\theta} u_k(x, y, t) + R[x, y]u_k(x, y, t) + N[x, y]u_k(x, y, t) - h(x, y, t). \tag{25}$$

It is evident that for $t \geq 0$, $Resu(x, y, t) = 0$ and
$\lim_{k\to\infty} Resu_k(x, y, z, t) = Resu(x, y, z, t)$.

Calculating out $Resu_1(x, y, z, 0) = 0$, yields the first unknown function, $f_1(x, y, z)$. The fractional derivative of a constant is 0 in the conformable sense, hence
$\mathcal{D}_t^{(n-1)\omega} Resu_k(x, y, z, t) = 0$ relative to $n = 1, 2, 3, ..., k$. The desired $f_n(x, y, z)$ coefficients are obtained by solving this equation for $t = 0$. Thus, $u_n(x, y, z, t)$ solutions may be determined, respectively.

The $\exp(-\phi(\xi))$-expansion and the modified extended tanh-function method can generate a variety of exact solutions, including exponential, solitons, periodic, and rational solutions. They are highly adaptable to different types of nonlinear equations. Besides, unlike many other techniques (such as perturbation methods), the RPSM does not require linearization or small parameter assumptions, making it suitable for strongly nonlinear PDEs. These methods are adaptable to a wide variety of nonlinear PDEs. This adaptability makes them suitable for models where other methods might fail or require substantial modification. For example, if your PDE includes fractional derivatives, nonlinear terms, or higher-order terms, these methods can often be extended to handle such complexities.

6. Application of the techniques

For the analytical methods, if we examine Eq. (1) in this context,

$$A\mathcal{D}_t^{\theta}u_x + au_{xx} + b(u)^2_{xx} + cu_{xxxx} + du_{yy} = 0. \tag{26}$$

Utilizing $u(x,y,t) = u(\xi)$ with $\xi = kx + ly + \frac{mt^{\theta}}{\theta}$ and performing the integration results in,

$$ak^2u(\xi) + Akmu(\xi) + bk^2u(\xi)^2 + ck^4u''(\xi) + dl^2u(\xi) = 0. \tag{27}$$

Balancing, $u^2 = 2N, u'' = N + 2$ results in $N = 2$. Upon substitution it into Eq. (7) and Eq. (17), the following exact solutions are derived.

6.1. Analytical solutions by exp$(-\phi(\xi))$-expansion method.

Given that $N = 2$, upon substituting Eq. (7), the series of sums is as follows:

$$u = B_0 + B_1\exp(-\phi(\xi)) + B_2\exp(-\phi(\xi))^2. \tag{28}$$

When combined with Eq. (8), the algebraic system that follows is created.

$$ak^2B_0 + dl^2B_0 + AkmB_0 + bk^2B_0^2 + ck^4\eta\lambda B_1 + 2ck^4\eta^2B_2 = 0,$$
$$ak^2B_1 + dl^2B_1 + AkmB_1 + 2ck^4\eta B_1 + ck^4\lambda^2B_1 + 2bk^2B_0B_1 + 6ck^4\eta\lambda B_2 = 0,$$
$$3ck^4\lambda B_1 + bk^2B_1^2 + ak^2B_2 + dl^2B_2 + AkmB_2 + 8ck^4\eta B_2 + 4ck^4\lambda^2B_2$$
$$+2bk^2B_0B_2 = 0,$$
$$2ck^4B_1 + 10ck^4\lambda B_2 + 2bk^2B_1B_2 = 0,$$
$$6ck^4B_2 + bk^2B_2^2 = 0.$$

Two cases and two sets of solutions for B_0, B_1, B_2, and m are obtained.

Case 1.

$$B_0 = -\frac{6c\eta k^2}{b},\ B_1 = -\frac{6ck^2\lambda}{b},\ B_2 = -\frac{6ck^2}{b},$$
$$m = -\frac{ak^2 + ck^4\left(\lambda^2 - 4\eta\right) + dl^2}{Ak},\ \xi = kx + ly + \frac{mt^{\theta}}{\theta}.$$

Set 1.
For $\lambda^2 - 4\eta > 0$, $\eta \neq 0$,

$$u_1(x,y,t) = -\frac{6c\eta\, k^2}{b} - \frac{12c\,\eta k^2\lambda}{b\left(-\sqrt{\lambda^2-4\eta}\tanh\left(\frac{1}{2}\sqrt{\lambda^2-4\eta}\Psi\right)-\lambda\right)} - \frac{24c\eta^2k^2}{b\left(-\sqrt{\lambda^2-4\eta}\tanh\left(\frac{1}{2}\sqrt{\lambda^2-4\eta}\Psi\right)-\lambda\right)^2}, \tag{29}$$

For $\lambda^2 - 4\eta < 0$ and $\eta \neq 0$,

$$u_2(x,y,t) = -\frac{6c\eta\, k^2}{b} - \frac{12c\,\eta\lambda k^2}{b\left(\sqrt{4\eta-\lambda^2}\tan\left(\frac{1}{2}\sqrt{4\eta-\lambda^2}\Psi\right)-\lambda\right)} - \frac{24c\eta^2k^2}{b\left(\sqrt{4\eta-\lambda^2}\tan\left(\frac{1}{2}\sqrt{4\eta-\lambda^2}\Psi\right)-\lambda\right)^2}, \tag{30}$$

where $\Psi = \left(-\frac{t^{\theta}\left(ak^2+ck^4\left(\lambda^2-4\eta\right)+dl^2\right)}{A\theta k} + h + kx + ly\right)$.

For $\lambda^2 - 4\eta > 0$, $\lambda \neq 0$ and $\eta = 0$,

$$u_3(x,y,t) = -\frac{6ck^2\lambda^2}{b\left(\sinh(\lambda\Omega)+\cosh(\lambda\Omega)-1\right)} - \frac{6ck^2\lambda^2}{b\left(\sinh(\ (\lambda\Omega)+\cosh(\lambda\Omega)-1\right)^2}, \tag{31}$$

where $\Omega = \left(-\frac{t^\theta\left(ak^2+c\lambda^2k^4+dl^2\right)}{A\theta k}+h+kx+ly\right)$.

For $\lambda^2-4\eta=0$, $\lambda\neq 0$ and $\eta\neq 0$,

$$u_4(x,y,t) = -\frac{6c\eta k^2}{b} + \frac{3ck^2\lambda^3\Lambda}{b(\lambda\Lambda+2)} - \frac{3ck^2\lambda^4\Lambda^2}{2b(\lambda\Lambda+2)^2}, \tag{32}$$

where $\Lambda = \left(-\frac{t^\theta\left(ak^2+dl^2\right)}{A\theta k}+h+kx+ly\right)$

For $\lambda^2-4\eta=0$, $\lambda=0$ and $\eta=0$,

$$u_5(x,y,t) = -\frac{6ck^2}{b\left(-\frac{t^\theta(ak^2+4c\eta k^4+dl^2)}{A\theta k}+h+kx+ly\right)^2}. \tag{33}$$

Case 2.

$$B_0 = -\frac{ck^2\left(2\eta+\lambda^2\right)}{b},\ B_1 = -\frac{6ck^2\lambda}{b},\ B_2 = -\frac{6ck^2}{b},$$
$$m = \frac{-ak^2+ck^4\left(\lambda^2-4\eta\right)-dl^2}{Ak},\ \xi = kx+ly+\frac{mt^\theta}{\theta}$$

Set 2.

For $\lambda^2-4\eta>0$, $\eta\neq 0$,

$$u_6(x,y,t) = -\frac{ck^2\left(2\eta+\lambda^2\right)}{b} - \frac{12c\eta k^2\lambda}{b\left(-\sqrt{\lambda^2-4\eta}\tanh\left(\frac{1}{2}\sqrt{\lambda^2-4\eta}\Upsilon\right)-\lambda\right)} - \frac{24c\eta^2k^2}{b\left(-\sqrt{\lambda^2-4\eta}\tanh\left(\frac{1}{2}\sqrt{\lambda^2-4\eta}\Upsilon\right)-\lambda\right)^2}, \tag{34}$$

For $\lambda^2-4\eta<0$ and $\eta\neq 0$,

$$u_7(x,y,t) = -\frac{ck^2\left(2\eta+\lambda^2\right)}{b} - \frac{12c\eta k^2\lambda}{b\left(\sqrt{4\eta-\lambda^2}\tan\left(\frac{1}{2}\sqrt{4\eta-\lambda^2}\Upsilon\right)-\lambda\right)} - \frac{24c\eta^2k^2}{b\left(\sqrt{4\eta-\lambda^2}\tan\left(\frac{1}{2}\sqrt{4\eta-\lambda^2}\Upsilon\right)-\lambda\right)^2}, \tag{35}$$

where $\Upsilon = \left(\frac{t^\theta\left(-ak^2+ck^4\left(\lambda^2-4\eta\right)-dl^2\right)}{A\theta k}+h+kx+ly\right)$

For $\lambda^2-4\eta>0$, $\lambda\neq 0$ and $\eta=0$,

$$u_8(x,y,t) = -\frac{6ck^2\lambda^2}{b\left(\sinh(\lambda\Phi)+\cosh(\lambda\Phi)-1\right)} - \frac{6ck^2\lambda^2}{b\left(\sinh(\lambda\Phi)+\cosh(\lambda\Phi)-1\right)^2} - \frac{ck^2\lambda^2}{b}, \tag{36}$$

where $\Phi = \left(\frac{t^\theta\left(-ak^2+c\lambda^2k^4-dl^2\right)}{A\theta k}+h+kx+ly\right)$

For $\lambda^2 - 4\eta = 0$, $\lambda \neq 0$ and $\eta \neq 0$,

$$u_9(x,y,t) = -\frac{6c\eta k^2}{b} + \frac{3ck^2\lambda^3\Xi}{b\left(\lambda\Xi+2\right)} - \frac{3ck^2\lambda^4\left(\frac{t^\theta\left(-ak^2-dl^2\right)}{A\theta k}+h+kx+ly\right)^2}{2b\left(\lambda\left(\frac{t^\theta\left(-ak^2-dl^2\right)}{A\theta k}+h+kx+ly\right)+2\right)^2}, \tag{37}$$

where $\Xi = \left(\frac{t^\theta\left(-ak^2-dl^2\right)}{A\theta k}+h+kx+ly\right)$

For $\lambda^2 - 4\eta = 0$, $\lambda = 0$ and $\eta = 0$,

$$u_{10}(x,y,t) = -\frac{4c\eta k^2}{b} - \frac{6ck^2}{b\left(\frac{t^\theta\left(-ak^2+4c\eta k^4-dl^2\right)}{A\theta k}+h+kx+ly\right)^2}. \tag{38}$$

6.2. **The modified extended tanh-function method solutions.** By taking $N = 2$, Eq. (17) becomes,

$$v = A_0 + A_1\phi(\xi) + B_1\phi(\xi)^{-1} + A_2\phi(\xi)^2 + B_2\phi(\xi)^{-2}, \tag{39}$$

and when considered together with the Eq. (18) here, the following algebraic system of equations is obtained,

$$\begin{gathered}
ak^2A_0 + dl^2A_0 + AkmA_0 + bk^2A_0^2 + 2ck^4\sigma^2A_2 + 2bk^2A_1B_1 + 2ck^4B_2 \\
+2bk^2A_2B_2 = 0, \\
bk^2A_1^2 + ak^2A_2 + dl^2A_2 + AkmA_2 + 8ck^4\sigma A_2 + 2bk^2A_0A_2 = 0, \\
ak^2A_1 + dl^2A_1 + AkmA_1 + 2ck^4\sigma A_1 + 2bk^2A_0A_1 + 2bk^2A_2B_1 = 0, \\
bk^2B_1^2 + ak^2B_2 + dl^2B_2 + AkmB_2 + 8ck^4\sigma B_2 + 2bk^2A_0B_2 = 0, \\
ak^2B_1 + dl^2B_1 + AkmB_1 + 2ck^4\sigma B_1 + 2bk^2A_0B_1 + 2bk^2A_1B_2 = 0, \\
2ck^4\sigma^2B_1 + 2bk^2B_1B_2 = 0, \\
6ck^4\sigma^2B_2 + bk^2B_2^2 = 0, \\
2ck^4A_1 + 2bk^2A_1A_2 = 0, \\
6ck^4A_2 + bk^2A_2^2 = 0.
\end{gathered}$$

Four cases and four sets of solutions for A_0, A_1 A_2, B_1, B_2 and m are obtained here.

Case 3.

$$\begin{aligned}
A_0 &= -\frac{12ck^2\sigma}{b},\ A_1 = 0,\ B_1 = 0,\ A_2 = -\frac{6ck^2}{b},\ B_2 = -\frac{6ck^2\sigma^2}{b}, \\
m &= \frac{-ak^2+16ck^4\sigma-dl^2}{Ak},\ \xi = kx+ly+\frac{mt^\theta}{\theta}.
\end{aligned}$$

Set 3.

For $\sigma < 0$,

$$\begin{aligned}
v_1(x,y,t) = -\frac{12ck^2\sigma}{b} &+ \frac{6ck^2\sigma\tanh\left(\sqrt{-\sigma}\left(\frac{t^\theta\left(-ak^2+16ck^4\sigma-dl^2\right)}{A\theta k}+kx+ly\right)\right)^2}{b} \\
&+ \frac{6ck^2\sigma\coth\left(\sqrt{-\sigma}\left(\frac{t^\theta\left(-ak^2+16ck^4\sigma-dl^2\right)}{A\theta k}+kx+ly\right)\right)^2}{b},
\end{aligned} \tag{40}$$

or

$$v_2(x,y,t) = -\frac{12ck^2\sigma}{b} + \frac{6ck^2\sigma\tanh\left(\sqrt{-\sigma}\left(\frac{t^\theta\left(-ak^2+16ck^4\sigma-dl^2\right)}{A\theta k}+kx+ly\right)\right)^2}{b}$$

$$+\frac{6ck^2\sigma \coth\left(\sqrt{-\sigma}\left(\frac{t^\theta\left(-ak^2+16ck^4\sigma-dl^2\right)}{A\theta k}+kx+ly\right)\right)^2}{b}, \tag{41}$$

For $\sigma > 0$,

$$v_3(x,y,t) = -\frac{12ck^2\sigma}{b} - \frac{6ck^2\sigma \tan\left(\sqrt{\sigma}\left(\frac{t^\theta\left(-ak^2+16ck^4\sigma-dl^2\right)}{A\theta k}+kx+ly\right)\right)^2}{b}$$
$$-\frac{6ck^2\sigma \cot\left(\sqrt{\sigma}\left(\frac{t^\theta\left(-ak^2+16ck^4\sigma-dl^2\right)}{A\theta k}+kx+ly\right)\right)^2}{b}, \tag{42}$$

or

$$v_4(x,y,t) = -\frac{12ck^2\sigma}{b} - \frac{6ck^2\sigma \tan\left(\sqrt{\sigma}\left(\frac{t^\theta\left(-ak^2+16ck^4\sigma-dl^2\right)}{A\theta k}+kx+ly\right)\right)^2}{b}$$
$$-\frac{6ck^2\sigma \cot\left(\sqrt{\sigma}\left(\frac{t^\theta\left(-ak^2+16ck^4\sigma-dl^2\right)}{A\theta k}+kx+ly\right)\right)^2}{b}, \tag{43}$$

For $\sigma = 0$,

$$v_5(x,y,t) = -\frac{6ck^2}{b\left(\frac{t^\theta(-ak^2-dl^2)}{A\theta k}+kx+ly\right)^2}. \tag{44}$$

Case 4.

$$\begin{aligned} A_0 &= -\frac{6ck^2\sigma}{b},\ A_1 = 0,\ B_1 = 0,\ A_2 = -\frac{6ck^2}{b},\ B_2 = 0, \\ m &= \frac{-ak^2+4ck^4\sigma-dl^2}{Ak},\ \xi = kx+ly+\frac{mt^\theta}{\theta}. \end{aligned}$$

Set 4.
For $\sigma < 0$,

$$v_6(x,y,t) = -\frac{6ck^2\sigma}{b} + \frac{6ck^2\sigma \tanh\left(\sqrt{-\sigma}\left(\frac{t^\theta\left(-ak^2+4ck^4\sigma-dl^2\right)}{A\theta k}+kx+ly\right)\right)^2}{b}, \tag{45}$$

or

$$v_7(x,y,t) = -\frac{6ck^2\sigma}{b} + \frac{6ck^2\sigma \coth\left(\sqrt{-\sigma}\left(\frac{t^\theta\left(-ak^2+4ck^4\sigma-dl^2\right)}{A\theta k}+kx+ly\right)\right)^2}{b}, \tag{46}$$

For $\sigma > 0$,

$$v_8(x,y,t) = -\frac{6ck^2\sigma}{b} - \frac{6ck^2\sigma \tan\left(\sqrt{\sigma}\left(\frac{t^\theta\left(-ak^2+4ck^4\sigma-dl^2\right)}{A\theta k}+kx+ly\right)\right)^2}{b}, \tag{47}$$

or

$$v_9(x,y,t) = -\frac{6ck^2\sigma}{b} - \frac{6ck^2\sigma \cot\left(\sqrt{\sigma}\left(\frac{t^\theta\left(-ak^2+4ck^4\sigma-dl^2\right)}{A\theta k}+kx+ly\right)\right)^2}{b}, \tag{48}$$

For $\sigma = 0$,

$$v_{10}(x,y,t) = -\frac{6ck^2}{b\left(\frac{t^\theta(-ak^2-dl^2)}{A\theta k}+kx+ly\right)^2}. \tag{49}$$

Case 5.

$$\begin{aligned} A_0 &= -\frac{2ck^2\sigma}{b},\ A_1=0,\ B_1=0,\ A_2=-\frac{6ck^2}{b},\ B_2=0,\\ m &= \frac{-ak^2-4ck^4\sigma-dl^2}{Ak},\ \xi=kx+ly+\frac{mt^\theta}{\theta}. \end{aligned}$$

Set 5.
For $\sigma<0$,

$$v_{11}(x,y,t)=-\frac{2ck^2\sigma}{b}+\frac{6ck^2\sigma\tanh\left(\sqrt{-\sigma}\left(\frac{t^\theta\left(-ak^2-4ck^4\sigma-dl^2\right)}{A\theta k}+kx+ly\right)\right)^2}{b}, \tag{50}$$

or

$$v_{12}(x,y,t)=-\frac{2ck^2\sigma}{b}+\frac{6ck^2\sigma\coth\left(\sqrt{-\sigma}\left(\frac{t^\theta\left(-ak^2-4ck^4\sigma-dl^2\right)}{A\theta k}+kx+ly\right)\right)^2}{b}, \tag{51}$$

For $\sigma>0$,

$$v_{13}(x,y,t)=-\frac{2ck^2\sigma}{b}-\frac{6ck^2\sigma\tan\left(\sqrt{\sigma}\left(\frac{t^\theta\left(-ak^2-4ck^4\sigma-dl^2\right)}{A\theta k}+kx+ly\right)\right)^2}{b}, \tag{52}$$

or

$$v_{14}(x,y,t)=-\frac{2ck^2\sigma}{b}-\frac{6ck^2\sigma\cot\left(\sqrt{\sigma}\left(\frac{t^\theta\left(-ak^2-4ck^4\sigma-dl^2\right)}{A\theta k}+kx+ly\right)\right)^2}{b}, \tag{53}$$

For $\sigma=0$,

$$v_{15}(x,y,t)=-\frac{6ck^2}{b\left(\frac{t^\theta\left(-ak^2-dl^2\right)}{A\theta k}+kx+ly\right)^2}. \tag{54}$$

Case 6.

$$\begin{aligned} A_0 &= \frac{4ck^2\sigma}{b},\ A_1=0, B_1=0,\ A_2=-\frac{6ck^2}{b},\ B_2=-\frac{6ck^2\sigma^2}{b},\\ m &= \frac{-ak^2-16ck^4\sigma-dl^2}{Ak},\ \xi=kx+ly+\frac{mt^\theta}{\theta}. \end{aligned}$$

Set 6.
For $\sigma<0$,

$$\begin{aligned} v_{16}(x,y,t)=\frac{4ck^2\sigma}{b}&+\frac{6ck^2\sigma\tanh\left(\sqrt{-\sigma}\left(\frac{t^\theta\left(-ak^2-16ck^4\sigma-dl^2\right)^2}{A\theta k}+kx+ly\right)\right)}{b}\\ &+\frac{6ck^2\sigma\coth\left(\sqrt{-\sigma}\left(\frac{t^\theta\left(-ak^2-16ck^4\sigma-dl^2\right)}{A\theta k}+kx+ly\right)\right)^2}{b}, \end{aligned} \tag{55}$$

or

$$\begin{aligned} v_{17}(x,y,t)=\frac{4ck^2\sigma}{b}&+\frac{6ck^2\sigma\tanh\left(\sqrt{-\sigma}\left(\frac{t^\theta\left(-ak^2-16ck^4\sigma-dl^2\right)}{A\theta k}+kx+ly\right)\right)^2}{b}\\ &+\frac{6ck^2\sigma\coth\left(\sqrt{-\sigma}\left(\frac{t^\theta\left(-ak^2-16ck^4\sigma-dl^2\right)}{A\theta k}+kx+ly\right)\right)^2}{b}, \end{aligned} \tag{56}$$

For $\sigma > 0$,

$$v_{18}(x,y,t) = \frac{4ck^2\sigma}{b} - \frac{6ck^2\sigma\tan\left(\sqrt{\sigma}\left(\frac{t^\theta\left(-ak^2-16ck^4\sigma-dl^2\right)}{A\theta k}+kx+ly\right)\right)^2}{b} - \frac{6ck^2\sigma\cot\left(\sqrt{\sigma}\left(\frac{t^\theta\left(-ak^2-16ck^4\sigma-dl^2\right)}{A\theta k}+kx+ly\right)\right)^2}{b}, \quad (57)$$

or

$$v_{19}(x,y,t) = \frac{4ck^2\sigma}{b} - \frac{6ck^2\sigma\tan\left(\sqrt{\sigma}\left(\frac{t^\theta\left(-ak^2-16ck^4\sigma-dl^2\right)}{A\theta k}+kx+ly\right)\right)^2}{b} - \frac{6ck^2\sigma\cot\left(\sqrt{\sigma}\left(\frac{t^\theta\left(-ak^2-16ck^4\sigma-dl^2\right)}{A\theta k}+kx+ly\right)\right)^2}{b}, \quad (58)$$

For $\sigma = 0$,

$$v_{20}(x,y,t) = -\frac{6ck^2}{b\left(\frac{t^\theta\left(-ak^2-dl^2\right)}{A\theta k}+kx+ly\right)^2}. \quad (59)$$

Next we present 3D, contour, and 2D plots of some of the obtained analytical solutions.

6.3. **Approximate solutions by RPSM.** First, we assume an initial condition for $t = 0$, using an exact solutions found previously. Thus, from Eq. (45), the initial condition is taken as

$$v_6(x,y,0) = -\frac{6ck^2\sigma}{b} + \frac{6ck^2\sigma\tanh\left(\sqrt{-\sigma}\,(kx+ly)\right)^2}{b}. \quad (60)$$

For the approximate solutions to the (2+1)-dimensional shallow water wave equation (26), where $t \geq 0$, $0 < \theta \leq 1$, the RPSM solution is in the form of Eq. (24). Thus, Eq. (25) can be written as,

$$Resu_k(x,y,t) = A\mathcal{D}_t^\theta(u_k)_x + a(u_k)_{xx} + b(u_k)_{xx}^2 + c(u_k)_{xxxx} + d(u_k)_{yy} = 0. \quad (61)$$

Hence, $Resu_1(x,y,t)$ is obtained as,

$$\begin{aligned} Resu_1(x,y,t) \;=\; & A\left(f_1\right)_x + a\left(f_{xx} + \frac{t^\theta\left(f_1\right)_{xx}}{\theta}\right) \\ & +b\left(2\left(f_x + \frac{t^\theta\left(f_1\right)_x}{\theta}\right)^2 + 2\left(f + \frac{t^\theta f_1}{\theta}\right)\left(f_{xx} + \frac{t^\theta\left(f_1\right)_{xx}}{\theta}\right)\right) \\ & +c\left(f_{xxxx} + \frac{t^\theta\left(f_1\right)_{xxxx}}{\theta}\right) + d\left(f_{yy} + \frac{t^\theta\left(f_1\right)_{yy}}{\theta}\right), \end{aligned} \quad (62)$$

where $f = f(x,y)$ and $f_1 = f_1(x,y)$. The first unknown parameter is obtained by setting $t = 0$ as,

$$f_1 = \frac{12ck\sigma^2\left(ak^2 - 4ck^4\sigma + dl^2\right)\tanh\left(\sqrt{-\sigma}(kx+ly)\right) \times \operatorname{sech}^2\left(\sqrt{-\sigma}(kx+ly)\right)}{Ab\sqrt{-\sigma}}, \quad (63)$$

is determined, and consequently, $u_1 = u_1(x,y,t)$ is obtained as

$$\begin{aligned} u_1 \;=\; & \frac{12ck\sigma^2t^\theta\left(ak^2 - 4ck^4\sigma + dl^2\right)\tanh\left(\sqrt{-\sigma}(kx+ly)\right)\operatorname{sech}^2\left(\sqrt{-\sigma}(kx+ly)\right)}{Ab\theta\sqrt{-\sigma}} \\ & + \frac{6ck^2\sigma\tanh^2\left(\sqrt{-\sigma}(kx+ly)\right)}{b} - \frac{6ck^2\sigma}{b}. \end{aligned} \quad (64)$$

Similarly, the next residual term is

$$\begin{aligned} Resu_2 &= d\left(f_{yy} + \frac{t^{\theta}(f_1)_{yy}}{\theta} + \frac{t^{2\theta}(f_2)_{yy}}{2\theta^2}\right) \\ &+ At^{1-\theta}\left(t^{\theta-1}(f_1)_x + \frac{t^{2\theta-1}(f_2)_x}{\theta}\right) \\ &+ a\left(f_{xx} + \frac{t^{\theta}(f_1)_{xx}}{\theta} + \frac{t^{2\theta}(f_2)_{xx}}{2\theta^2}\right) \\ &+ 2b\left(f_x + \frac{t^{\theta}(f_1)_x}{\theta} + \frac{t^{2\theta}(f_2)_x}{2\theta^2}\right)^2 \\ &+ 2b\left(f + \frac{t^{\theta}(f_1)}{\theta} + \frac{t^{2\theta}(f_2)}{2\theta^2}\right)\left(f_{xx} + \frac{t^{\theta}(f_1)_{xx}}{\theta} + \frac{t^{2\theta}(f_2)_{xx}}{2\theta^2}\right) \\ &+ c\left(f_{xxxx} + \frac{t^{\theta}(f_1)_{xxxx}}{\theta} + \frac{t^{2\theta}(f_2)_{xxxx}}{2\theta^2}\right), \end{aligned} \tag{65}$$

where $f_2 = f_2(x, y)$ is written. For $t = 0$, the second unknown parameter can be obtained as follows by taking the first order derivative,

$$f_2 = \frac{12c\sigma^2\left(ak^2 - 4ck^4\sigma + dl^2\right)^2\left(\cosh\left(2\sqrt{-\sigma}(kx+ly)\right) - 2\right) \times \operatorname{sech}^4\left(\sqrt{-\sigma}(kx+ly)\right)}{A^2 b}, \tag{66}$$

Thus, $u_2 = u_2(x, y, t)$ solution becomes

$$\begin{aligned} u_2 &= \frac{6c\sigma^2 t^{2\theta}\left(ak^2 - 4ck^4\sigma + dl^2\right)^2\left(\cosh\left(2\sqrt{-\sigma}(kx+ly)\right) - 2\right) \times \operatorname{sech}^4\left(\sqrt{-\sigma}(kx+ly)\right)}{A^2 b\theta^2} \\ &+ \frac{12ck\sigma^2 t^{\theta}\left(ak^2 - 4ck^4\sigma + dl^2\right)\tanh\left(\sqrt{-\sigma}(kx+ly)\right)\operatorname{sech}^2\left(\sqrt{-\sigma}(kx+ly)\right)}{Ab\theta\sqrt{-\sigma}} \\ &+ \frac{6ck^2\sigma\tanh^2\left(\sqrt{-\sigma}(kx+ly)\right)}{b} - \frac{6ck^2\sigma}{b}. \end{aligned} \tag{67}$$

Similarly, the other solution is calculated as

$$\begin{aligned} u_3 &= -\frac{4c\sigma^3 t^{3\theta}\left(ak^2 - 4ck^4\sigma + dl^2\right)^3\left(\cosh\left(2\sqrt{-\sigma}(kx+ly)\right) - 5\right) \times \tanh\left(\sqrt{-\sigma}(kx+ly)\right)\operatorname{sech}^4\left(\sqrt{-\sigma}(kx+ly)\right)}{A^3 b\theta^3 k\sqrt{-\sigma}} \\ &+ \frac{6c\sigma^2 t^{2\theta}\left(ak^2 - 4ck^4\sigma + dl^2\right)^2\left(\cosh\left(2\sqrt{-\sigma}(kx+ly)\right) - 2\right) \times \operatorname{sech}^4\left(\sqrt{-\sigma}(kx+ly)\right)}{A^2 b\theta^2} \\ &+ \frac{12ck\sigma^2 t^{\theta}\left(ak^2 - 4ck^4\sigma + dl^2\right)\tanh\left(\sqrt{-\sigma}(kx+ly)\right)\operatorname{sech}^2\left(\sqrt{-\sigma}(kx+ly)\right)}{Ab\theta\sqrt{-\sigma}} \\ &+ \frac{6ck^2\sigma\tanh^2\left(\sqrt{-\sigma}(kx+ly)\right)}{b} - \frac{6ck^2\sigma}{b}. \end{aligned} \tag{68}$$

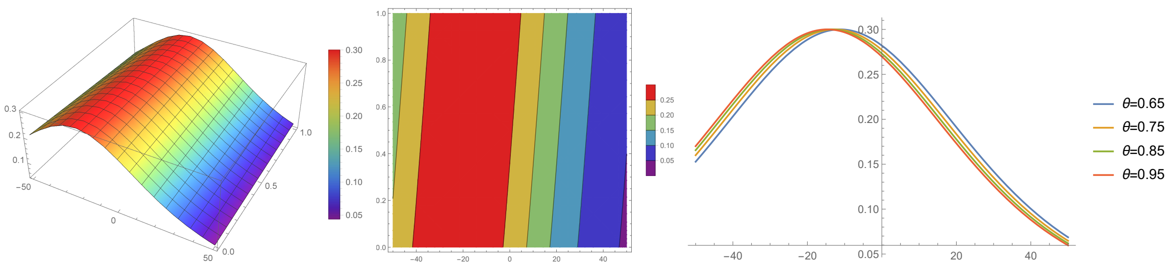
Next we present a comparison table and some 3D comparison plots with RPSM and the exact solutions.

Figure 1 and Figure 2 display the surface graphics of the analytical solutions, whereas Figures 3, 4, and 5 displays the surface graphics of the approximate solutions. Meanwhile, by taking the following values and ranges, approximate and exact solution were compared in Table 1.

TABLE 1. Comparing exact and RPSM solutions of Eq. (68) with exact solution of Eq. (45).

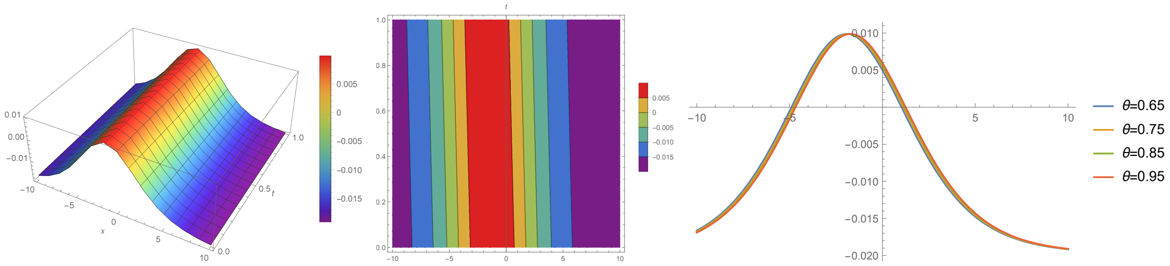
	$\theta = 0.75$			$\theta = 0.85$			$\theta = 0.95$		
t	*RPSM*	*Exact*	*Abs. Error*	*RPSM*	*Exact*	*Abs. Error*	*RPSM*	*Exact*	*Abs. Error*
0.0	0.575491	0.575491	0.00000	0.575491	0.575491	0.00000	0.575491	0.575491	0.00000
0.1	0.583687	0.583687	5.29179×10^{-9}	0.581233	0.581233	1.27760×10^{-9}	0.579571	0.579571	3.2607×10^{-10}
0.2	0.589284	0.589284	4.22828×10^{-8}	0.585848	0.585848	1.34733×10^{-8}	0.583378	0.583378	4.53833×10^{-9}
0.3	0.594195	0.594195	1.42543×10^{-7}	0.590117	0.590116	5.34286×10^{-8}	0.587089	0.587089	2.11688×10^{-8}
0.4	0.598709	0.598709	3.37514×10^{-7}	0.594176	0.594176	1.41958×10^{-7}	0.590740	0.590740	6.31116×10^{-8}
0.5	0.602949	0.602948	6.58507×10^{-7}	0.598087	0.598086	3.02863×10^{-7}	0.594348	0.594348	1.47233×10^{-7}
0.6	0.606981	0.606980	1.13671×10^{-6}	0.601883	0.601882	5.62409×10^{-7}	0.597921	0.597921	2.94118×10^{-7}
0.7	0.610850	0.610848	1.80317×10^{-6}	0.605586	0.605585	9.48980×10^{-7}	0.601467	0.601466	5.27882×10^{-7}
0.8	0.614582	0.614579	2.68881×10^{-6}	0.609212	0.609210	1.49283×10^{-6}	0.604988	0.604987	8.76021×10^{-7}
0.9	0.618200	0.618196	3.82442×10^{-6}	0.612770	0.612768	2.22590×10^{-6}	0.608488	0.608487	1.36928×10^{-6}
1.0	0.621718	0.621712	5.24064×10^{-6}	0.616270	0.616267	3.18163×10^{-6}	0.611970	0.611967	2.04154×10^{-6}

- Figure 1 $k = 0.2, c = 1$, $b = 0.01$, $y = 0.1$, $z = 0.5$, $h = 0.1$, $\eta = 0.05$, $\lambda = 0.5$, $d = 0.1$, $l = 0.5$, $a = 0.1$, $A = 0.1$ and $\theta = 0.95$, $-50 \leq x \leq 50$ for (A), (B), and $t = 0.99$ for (C).
- Figure 2 $c = 0.0001$, $k = 0.202$, $b = 0.001$, $\sigma = -1.21$, $l = 0.45$, $a = 0.221$, $d = 0.05$, $A = -1.01$, $y = 0.55$ and $\theta = 0.98$, $-10 \leq x \leq 10$ for (A), (B), and $t = 0.99$ for (C).
- Table 1 $x = 2$, $y = 1$, $c = 0.99$, $k = 0.22$, $b = 0.2$, $\sigma = -0.64$, $l = 0.45$, $a = 0.05$, $d = 0.006$, $A = 0.71$ and $0 \leq t \leq 1$.
- Figure 3 $x = -1$, $y = 1$, $c = 0.45$, $k = 0.01$, $b = 0.1$, $\sigma = 0.9$, $l = 0.01$, $a = 0.001$, $d = 0.6$, $A = 1$ and $\theta = 0.75$, $-50 \leq x \leq 50$ for (A) and (B) $0 \leq t \leq 1$.
- Figure 4 $x = 2$, $y = 1$, $c = 0.57$, $k = 0.12$, $b = 0.2$, $\sigma = -0.12$, $l = 0.01$, $a = 0.03$, $d = 0.01$, $A = 0.7$ and $\theta = 0.85$, $-50 \leq x \leq 50$ for (A) and (B) $0 \leq t \leq 1$.
- Figure 5 $x = 2$, $y = 1$, $c = 0.99$, $k = 0.22$, $b = 0.2$, $\sigma = -0.64$, $l = 0.45$, $a = 0.05$, $d = 0.006$, $A = 0.71$ and $\theta = 0.95$, $-50 \leq x \leq 50$ for (A) and (B) $0 \leq t \leq 1$.



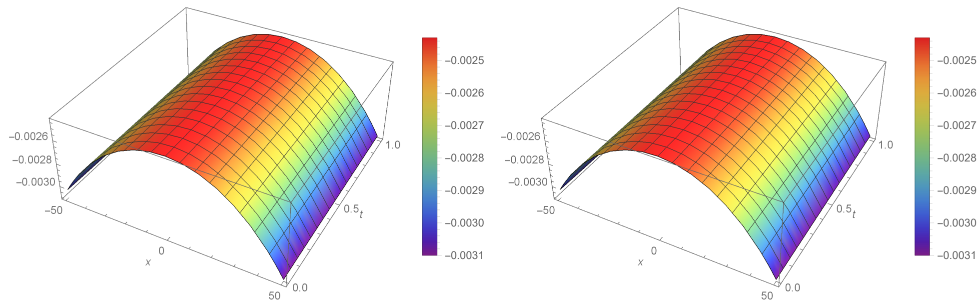
(A) (B) (C)

FIGURE 1. (A) 3D, (B) contour and (C) 2D plots of the $\exp(-\phi(\xi))$-expansion method solution $u_1(x, y, t)$ of Eq. (29).



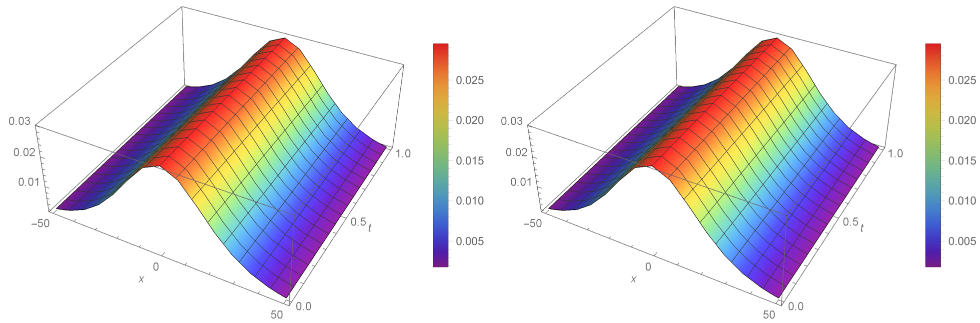
(A) (B) (C)

FIGURE 2. (A) 3D, (B) contour and (C) 2D plots of the modified extended tanh-function method solution $v_{11}(x, y, t)$ of Eq. (50).



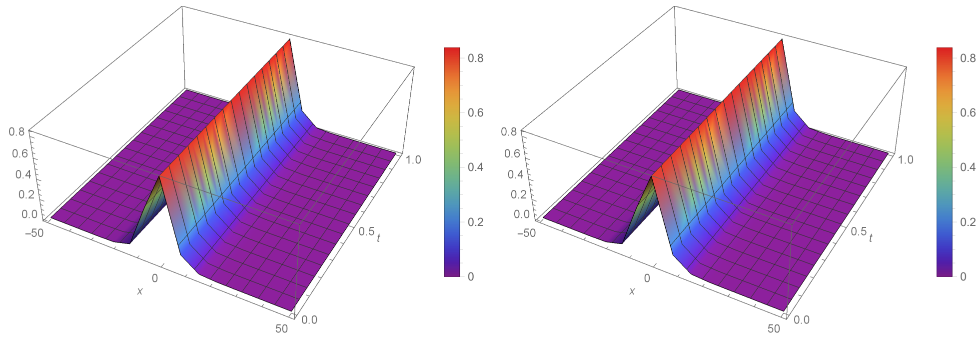
(A) RPSM solution (B) Exact solution

FIGURE 3. Comparison plots of the u_3 solution according to Eq. (68) with the exact solution.



(A) RPSM solution (B) Exact solution

FIGURE 4. Comparison plots of the u_3 solution according to Eq. (68) with the exact solution.



(A) RPSM solution (B) Exact solution

FIGURE 5. Comparison plots of the u_3 solution according to Eq. (68) with the exact solution.

Some new solutions to the present equation are displayed in the surface plots, which can be helpful in solving additional differential equations of arbitrary order.

7. CONCLUSION

This study investigated solutions to the (2+1)-dimensional shallow water wave equation with conformable derivative by use of the modified extended tanh-function and the $\exp(-\phi(\xi))$-expansion methods. Additionally, the RPSM was used to get approximations of the solutions. Many exact solutions with low computational complexity were obtained using the mentioned analytical approaches. Moreover, the RPSM is a straightforward method, and its independent calculation for each iteration step facilitates computations up to higher-order iterations. We also compared our analytical solutions with the numerical solutions to verify the validity of the results. This provide insights into the applicability of these methods for real-world modeling.

To visually represent the obtained solutions, 3D, contour, and 2D plots were generated. Analytical and approximate results, surface plots, and a comparison table illustrate the accuracy of the techniques. The solutions exhibit distinct features with important physical attributes not previously addressed before. In some interpretations of the figures, the physical behavior of the exact solutions is illustrated for specific numerical values. Understanding these applications is essential for their potential real-world implementations.

The accomplished solutions are crucial for comprehending the physical behavior of the problem. The suggested techniques are reliable and beneficial, providing light on the physical properties of various complicated non-linear models. This study contributes to understanding of higher-dimensional wave phenomena under fractional calculus, paving the way for future research on fractional fluid models.

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On some k-Oresme hybrid numbers including negative indices

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Abstract. In this study, we define k-Oresme hybrid numbers including negative indices and examine their properties, using the theory of number systems created by choosing the coefficients from unique number sets. Moreover, we derive some fundamental identities related to these numbers.

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Keywords. Oresme numbers, hybrid numbers, recurrence relations.

1. Introduction

In 1961, Nicole Oresme defined rational sequences and studied the powers of these rational numbers [10]. In 1974, A. F. Horadam reconsidered the sequence Nicole Oresme noticed while examining rational sequences, and named this sequence as Oresme sequence in memory of Oresme [7]. Horadam detailed many algebraic properties of this sequence, which is denoted by $\{O_n\}_{n\geq 1}$ and whose general term is $\left\{\frac{n}{2^n}\right\}$ [8]. The author also gave the recurrence relation of this sequence and different representations for this relation. In 2004, C. K. Cook variously presented the properties of Oresme numbers and their generalization [2]. He gave some identities similar to those in Horadam's work [4]. In 2019, G. Cerda-Morales studied a generalization of Oresme numbers with a new set of numbers called Oresme polynomials [1]. In 2019, T. Goy discussed some families of Toeplitz-Hessenberg determinants whose elements are Oresme numbers [3]. Since the sum formulas and generating function formulas of Oresme numbers in Horadam's study are the fundamental equations related to these numbers, these equations should be reminded.

$$\sum_{j=0}^{n-1} O_j = 4\left(\frac{1}{2} - O_{n+1}\right), \tag{1}$$

$$\sum_{j=0}^{n-1} O_{2j} = \frac{4}{9}\left(2 + O_{2n-1} - 5O_{2n}\right), \tag{2}$$

$$\sum_{j=0}^{n} O_{2j+1} = \frac{1}{9}\left(10 + 5O_{2n-1} - 16O_{2n}\right), \tag{3}$$

$$O_{n+1}O_{n-1} - (O_n)^2 = -\left(\frac{1}{4}\right)^n, \tag{4}$$

$$\sum_{n=0}^{\infty} \frac{n}{2^n}x^n = \frac{2x}{(2-x)^2}. \tag{5}$$

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k-Oresme sequence $\{O_n^{(k)}\}_{n\geq 2}$ which is a type of generalization of Oresme numbers is given by

$$O_n^{(k)} = O_{n-1}^{(k)} - \frac{1}{k^2}O_{n-2}^{(k)}, k \geq 2 \tag{6}$$

with initial conditions $O_1^{(k)} = \frac{1}{k}, O_0^{(k)} = 0$ [2]. In the case $k = 2$, this sequence is reduced to the classical Oresme sequence [2]. The characteristic equation of the recurrence relation in (6) is $x^2 - x + \frac{1}{k^2} = 0$. For $k^2 - 4 > 0$, the roots of this equation are

$$\alpha = \frac{k + \sqrt{k^2 - 4}}{2k}, \beta = \frac{k - \sqrt{k^2 - 4}}{2k},$$

respectively [8]. From the recurrence relation, the Binet's formula is given by

$$O_n^{(k)} = \frac{1}{\sqrt{k^2 - 4}}\left[\left(\frac{k + \sqrt{k^2 - 4}}{2k}\right)^n - \left(\frac{k - \sqrt{k^2 - 4}}{2k}\right)^n\right]. \tag{7}$$

In 2022, Halici et *al.* k-Oresme polynomials examined [5]. k-Oresme polynomials sequence $\{O_n^{(k)}(x)\}_{n\geq 2}$ which is a type of generalization of Oresme numbers is given by

$$O_{n+2}^{(k)}(x) = O_{n+1}^{(k)}(x) - \frac{1}{k^2x^2}O_n^{(k)}(x), x \geq 1 \tag{8}$$

with initial conditions $O_1^{(k)}(x) = \frac{1}{kx}, O_0^{(k)}(x) = 0$. In the case $k = 1$ and $x = 1$ this sequence is reduced to the classical Oresme sequence [5].
In 2021, Gurses et *al.* presented two new types of Oresme numbers [12]. And they investigated special linear recurrence relations and summation properties for DGC Oresme numbers of these types. The DGC numbers here are Dual-Hyperbolic Oresme numbers. In 2022, Halici et *al.* examined k-Oresme numbers with negative indices [4]. This sequence is denoted by $\{O_{-n}^{(k)}\}_{n\geq 0}$ and defined as

$$O_{-n}^{(k)} = k^2\left(O_{-n+1}^{(k)} - O_{-n+2}^{(k)}\right), k \geq 2 \tag{9}$$

with initial conditions $O_{-1}^{(k)} = -k, O_0^{(k)} = 0$. The nth term of this sequence is defined by

$$O_{-n}^{(k)} = -k^{2n}\frac{(\alpha^n - \beta^n)}{\sqrt{k^2 - 4}}. \tag{10}$$

On the other hand, in 2018, Ozdemir defined a non-commutative number system and called it hybrid numbers [11]. The author examined in detail the algebraic and geometrical properties of the new number system, which he described. In 2018, Szynal-Liana introduced Horadam hybrid numbers and examined their special cases [13]. Hybrid numbers and Horadam hybrid numbers are defined by

$$\mathbb{K} = \{z = a + b\mathbf{i} + c\boldsymbol{\epsilon} + d\mathbf{h}; a, b, c, d \in \mathbb{R}\} \tag{11}$$

and

$$H_n = Wn + W_{n+1}\mathbf{i} + W_{n+2}\boldsymbol{\epsilon} + W_{n+3}\mathbf{h}, \tag{12}$$

respectively [11, 13]. The relations provided between the three different base elements used in the set $\mathbb{K}$ are

$$\mathbf{i}^2 = -1, \boldsymbol{\epsilon}^{\mathbf{2}} = 0, \mathbf{h}^2 = 1, \mathbf{ih} = -\mathbf{hi} = \boldsymbol{\epsilon} + \mathbf{1}. \tag{13}$$

The conjugate of any hybrid number z is defined as $\overline{z} = a - b\mathbf{i} - c\boldsymbol{\epsilon} - d\mathbf{h}$.
The character value of element z, used in important identities, is

$$C(z) = z\overline{z} = \overline{z}z = a^2 + (b - c)^2 - c^2 - d^2. \tag{14}$$

This value of the hybrid number is often used to determine the generalized norm of a hybrid number. Depending on the selection of the coefficients a, b, c and d, different norms are obtained and they are examining.

$$\begin{array}{r|l} c = d = 0 & N(z) = \sqrt{a^2 + b^2}\ , \\ c = b = 0 & N(z) = \sqrt{|a^2 - d^2|}\ , \\ b = d = 0 & N(z) = |a|. \end{array}$$

In [14], Oresme hybrid numbers were defined and discussed by Syznal et *al.* For any positive number *n,* the *nth* Oresme hybrid number is

$$OH_n = O_n + O_{n+1}\mathbf{i} + O_{n+2}\boldsymbol{\epsilon} + O_{n+3}\mathbf{h} \tag{15}$$

and the k-Oresme hybrid number is

$$OH_n^{(k)} = O_n^{(k)} + O_{n+1}^{(k)}\mathbf{i} + O_{n+2}^{(k)}\boldsymbol{\epsilon} + O_{n+3}^{(k)}\mathbf{h} \tag{16}$$

[14]. In addition, in [14], the authors gave some fundamental identities with the help of iterative relation including k-Oresme hybrid numbers, but many important identities are not given in this study (see [14], Thr. 2.4). They also defined Oresme hybrationals for a nonzero real variable x and $n \geq 0$

$$OH_n(x) = On(x) + O_{n+1}(x)\mathbf{i} + O_{n+2}(x)\boldsymbol{\epsilon} + O_{n+3}(x)\mathbf{h}. \tag{17}$$

In [9], Gurses et *al.* defined Pentanacci and Pentanacci-Lucas hybrid numbers.
In this current study, we define and studied k-Oresme numbers with negative indices. For $k \geq 2$, $n \geq 0$ we give some important identities, such as the Cassini identity, which have various applications in the literature and include these numbers.

2. K-Oresme Hybrid Numbers Including Negative Indices

In [6], we introduced k-Oresme hybrid numbers and investigate their fundamental properties.
In this section, we constructed the theory of k-Oresme hybrid numbers with negative indices.

Definition 1. *By the aid of the hybrid numbers and the Oresme numbers, let us define k-Oresme hybrid numbers with negative indices as follows.*

$$OH_{-n}^{(k)} = O_{-n}^{(k)} + O_{-n+1}^{(k)}\mathbf{i} + O_{-n+2}^{(k)}\boldsymbol{\epsilon} + O_{-n+3}^{(k)}\mathbf{h}, n \geq 0. \tag{18}$$

The algebraic operations of the numbers we have just defined here are done by considering the algebraic operations of both Oresme numbers and hybrid numbers. Since we use the terms of this new number sequence $\{OH_{-n}^{(k)}\}_{n\geq 0}$ later, it is appropriate to write some of its terms.

$$\left\{..., \left(-k^3 - k\mathbf{i} + \frac{1}{k}\mathbf{h}\right), \left(-k + \frac{1}{k}\boldsymbol{\epsilon} + \frac{1}{k}\mathbf{h}\right), \left(\frac{1}{k} + \frac{1}{k}\boldsymbol{\epsilon} + \frac{(k^2-1)}{k^3}\mathbf{h}\right), ...\right\}.$$

In the following theorem, we give the Binet formula which provides the derivation of many important identities.

Theorem 1. *For the sequence $\{OH_{-n}^{(k)}\}_{n\geq 0}$, the Binet formula is*

$$OH_{-n}^{(k)} = \frac{-k^{2n}}{\sqrt{k^2-4}}\left(\alpha^n\tilde{\beta} - \beta^n\tilde{\alpha}\right), \tag{19}$$

where, $\tilde{\alpha} = 1 + \alpha\mathbf{i} + \alpha^2\boldsymbol{\epsilon} + \alpha^3\mathbf{h}$ *and* $\tilde{\beta} = 1 + \beta\mathbf{i} + \beta^2\boldsymbol{\epsilon} + \beta^3\mathbf{h}$.

Proof. From the equality (10),

$$OH_{-n}^{(k)} = -\frac{k^{2n}}{\sqrt{k^2-4}}\left[(\alpha^n - \beta^n)\right] - \frac{k^{2n}}{\sqrt{k^2-4}}\left[k^{-2}(\alpha^{n-1} - \beta^{n-1})\mathbf{i}\right]$$
$$-\frac{k^{2n}}{\sqrt{k^2-4}}\left[k^{-4}(\alpha^{n-2} - \beta^{n-2})\boldsymbol{\epsilon}\right] - \frac{k^{2n}}{\sqrt{k^2-4}}\left[k^{-6}(\alpha^{n-3} - \beta^{n-3})\mathbf{h}\right].$$

Then, we get

$$LHS = \frac{-k^{2n}}{\sqrt{k^2-4}}\left[\alpha^n\left(1 + \frac{1}{k^2\alpha}\mathbf{i} + \left(\frac{1}{k^2\alpha}\right)^2\boldsymbol{\epsilon} + \left(\frac{1}{k^2\alpha}\right)^3\mathbf{h}\right)\right]$$
$$+\frac{k^{2n}}{\sqrt{k^2-4}}\beta^n\left[\left(1 + \frac{1}{k^2\beta}\mathbf{i} + \left(\frac{1}{k^2\beta}\right)^2\boldsymbol{\epsilon} + \left(\frac{1}{k^2\beta}\right)^3\mathbf{h}\right)\right],$$
$$LHS = \frac{-k^{2n}}{\sqrt{k^2-4}}\left[\alpha^n\left(1 + \beta\mathbf{i} + \beta^2\boldsymbol{\epsilon} + \beta^3\mathbf{h}\right) - \beta^n\left(1 + \alpha\mathbf{i} + \alpha^2\boldsymbol{\epsilon} + \alpha^3\mathbf{h}\right)\right].$$

If we complete the necessary algebraic operations, then we obtain

$$OH_{-n}^{(k)} = \frac{-k^{2n}}{\sqrt{k^2-4}}\left(\alpha^n\tilde{\beta} - \beta^n\tilde{\alpha}\right)$$

which is desired result. Thus, the proof is completed. □

In the next theorem, we give the recurrence relation provided by the elements of the newly defined sequence.

Theorem 2. *For $n \in \mathbb{Z}$, the following equality is satisfied.*

$$OH_{-n+1}^{(k)} = k^2\left(OH_{-n+2}^{(k)} - OH_{-n+3}^{(k)}\right). \tag{20}$$

Proof. The proof of this equality is easily seen using induction. □

Theorem 3. *The character value for elements of the sequence $\{OH_{-n}^{(k)}\}_{n\geq 0}$ is*

$$C(z) = \left(O_{-n+1}^{(k)}\right)^2\left(\frac{k^8+k^4-1}{k^4}\right) - 2O_{-n+1}^{(k)}O_{-n+2}^{(k)}\left(\frac{k^6+k^2-1}{k^2}\right) \tag{21}$$

$$+\left(O_{-n+2}^{(k)}\right)^2\left(k^4-1\right).$$

Proof. From the definition in the equation (20), $C(z)$ is

$$C(z) = \left(O_{-n}^{(k)}\right)^2 + \left(O_{-n+1}^{(k)} - O_{-n+2}^{(k)}\right)^2 - \left(O_{-n+2}^{(k)}\right)^2 - \left(O_{-n+3}^{(k)}\right)^2,$$

$$LHS = \left[k^2\left(O_{-n+1}^{(k)} - O_{-n+2}^{(k)}\right)\right]^2 + \left(O_{-n+1}^{(k)}\right)^2$$
$$-2\left(O_{-n+1}^{(k)} - O_{-n+2}^{(k)}\right) - \left(O_{-n+2}^{(k)} - \frac{1}{k^2}O_{-n+1}^{(k)}\right)^2,$$

$$LHS = \left(O_{-n+1}^{(k)}\right)^2\left(\frac{k^8+k^4-1}{k^4}\right) - 2O_{-n+1}^{(k)}O_{-n+2}^{(k)}\left(\frac{k^6+k^2-1}{k^2}\right)$$
$$+\left(O_{-n+2}^{(k)}\right)^2\left(k^4-1\right).$$

So, the proof is completed. □

Theorem 4. *For elements of the sequence $\{OH_{-n}^{(k)}\}_{n\geq 0}$, we have*

$$i)\ OH_{-n}^{(k)} + \overline{OH_{-n}^{(k)}} = 2O_{-n}^{(k)}. \tag{22}$$

$$ii)\ C(OH_{-n}^{(k)}) = 2OH_{-n}^{(k)}.O_{-n}^{(k)} - \left(O_{-n}^{(k)}\right)^2. \tag{23}$$

Proof. Since the first equality in this theorem can be seen immediately from the definition (18), we consider the second equality.

$$\left(OH_{-n}^{(k)}\right)^2 = \left(O_{-n}^{(k)}\right)^2 - \left(O_{-n+1}^{(k)}\right)^2 + \left(O_{-n+3}^{(k)}\right)^2 + 2\left(O_{-n+1}^{(k)}O_{-n+2}^{(k)}\right)$$
$$+2\left(O_{-n}^{(k)}O_{-n+1}^{(k)}\mathbf{i} + O_{-n}^{(k)}O_{-n+2}^{(k)}\boldsymbol{\epsilon} + O_{-n}^{(k)}O_{-n+3}^{(k)}\mathbf{h}\right),$$

$$\left(OH_{-n}^{(k)}\right)^2 = 2O_{-n}^{(k)}OH_{-n}^{(k)} - \left(O_{-n}^{(k)}\right)^2 - \left(O_{-n+1}^{(k)}\right)^2 + \left(O_{-n+3}^{(k)}\right)^2 + 2O_{-n+1}^{(k)}O_{-n+2}^{(k)}.$$

Thus, we get,

$$\left(OH_{-n}^{(k)}\right)^2 = 2O_{-n}^{(k)}OH_{-n}^{(k)} - C(z),$$

$$C(z) = 2O_{-n}^{(k)}OH_{-n}^{(k)} - \left(OH_{-n}^{(k)}\right)^2.$$

□

In the next theorem, we give the Cassini identity which includes k-Oresme hybrid numbers with negative indices.

Theorem 5. *For elements of sequence* $\{OH_{-n}^{(k)}\}_{n\geq 0}$*, we have*

$$OH_{-n+1}^{(k)}OH_{-n-1}^{(k)}-\left(OH_{-n}^{(k)}\right)^2=-\frac{k^{2n}}{k^2-4}\left[\tilde{\beta}\tilde{\alpha}\left((k\alpha)^{-2}-2\right)+\tilde{\alpha}\tilde{\beta}\,(k\alpha)^2\right] \tag{24}$$

Proof. From, $OH_{-n+1}^{(k)}, OH_{-n-1}^{(k)}$ and $OH_{-n}^{(k)}$, we can write the left-hand side of the desired equation. Here, $\tilde{\alpha}\tilde{\beta}\neq\tilde{\beta}\tilde{\alpha}$. That is,

$$LHS=\left[\frac{-k^{2n-2}}{\sqrt{k^2-4}}\left(\alpha^{n-1}\tilde{\beta}-\beta^{n-1}\tilde{\alpha}\right)\right]\left[\frac{-k^{2n+2}}{\sqrt{k^2-4}}\left(\alpha^{n+1}\tilde{\beta}-\beta^{n+1}\tilde{\alpha}\right)\right]-\left[\frac{-k^{2n}}{\sqrt{k^2-4}}\left(\alpha^{n}\tilde{\beta}-\beta^{n}\tilde{\alpha}\right)\right]^2,$$

$$LHS=\frac{k^{4n}}{k^2-4}\left[\left(\alpha^{n-1}\tilde{\beta}-\beta^{n-1}\tilde{\alpha}\right)\left(\alpha^{n+1}\tilde{\beta}-\beta^{n+1}\tilde{\alpha}\right)-\left(\alpha^{n}\tilde{\beta}-\beta^{n}\tilde{\alpha}\right)^2\right],$$

$$LHS=-\frac{k^{4n}}{k^2-4}\left[\alpha^{n-1}\beta^{n+1}\tilde{\beta}\tilde{\alpha}+\alpha^{n+1}\beta^{n-1}\tilde{\alpha}\tilde{\beta}-2(\alpha\beta)^n\tilde{\beta}\tilde{\alpha}\right],$$

$$LHS=-\frac{k^{4n}}{k^2-4}(\alpha\beta)^n\left[\tilde{\beta}\tilde{\alpha}\left(\frac{\beta}{\alpha}-2\right)+\tilde{\alpha}\tilde{\beta}\left(\frac{\alpha}{\beta}\right)\right],$$

If we write the values $\alpha\beta$, $\frac{\beta}{\alpha}$ and $\frac{\alpha}{\beta}$, then we get

$$LHS=-\frac{k^{2n}}{k^2-4}\left[\tilde{\beta}\tilde{\alpha}\left((k\alpha)^{-2}-2\right)+\tilde{\alpha}\tilde{\beta}\,(k\alpha)^2\right].$$

Thus, we complete the proof. □

In the following theorem, we give the Catalan identity provided by k-Oresme hybrid numbers with the negative indices.

Theorem 6. *For* $n\geq r$*, the elements of sequence* $\{OH_{-n}^{(k)}\}_{n\geq 0}$ *provide.*

$$OH_{-n+r}^{(k)}OH_{-n-r}^{(k)}-\left(OH_{-n}^{(k)}\right)^2=-\frac{k^{2n}}{k^2-4}\left[\tilde{\beta}\tilde{\alpha}\left((k\alpha)^{-2r}-2\right)+\tilde{\alpha}\tilde{\beta}\,(k\alpha)^{2r}\right]. \tag{25}$$

Proof. $OH_{-n+r}^{(k)}OH_{-n-r}^{(k)}-\left(OH_{-n}^{(k)}\right)^2$ is equal to this:

$$LHS=\frac{k^{4n}}{k^2-4}\left[2(\alpha\beta)^n\tilde{\beta}\tilde{\alpha}-\alpha^{n-r}\beta^{n+r}\tilde{\beta}\tilde{\alpha}-\alpha^{n+r}\beta^{n-r}\tilde{\alpha}\tilde{\beta}\right],$$

$$LHS=-\frac{k^{4n}}{k^2-4}(\alpha\beta)^n\left[\tilde{\beta}\tilde{\alpha}\left(\left(\frac{\beta}{\alpha}\right)^r-2\right)+\tilde{\alpha}\tilde{\beta}\left(\frac{\alpha}{\beta}\right)^r\right],$$

If relations valid between the roots of characteristic equation of the sequence are used the relations

$$OH_{-n+r}^{(k)}OH_{-n-r}^{(k)}-\left(OH_{-n}^{(k)}\right)^2=-\frac{k^{2n}}{k^2-4}\left[\tilde{\beta}\tilde{\alpha}\left((k\alpha)^{-2r}-2\right)+\tilde{\alpha}\tilde{\beta}\,(k\alpha)^{2r}\right]$$

is obtained which is the desired result. □

In the case of $r=1$, it is obvious that this equation is reduced to the Cassini identity.
In the following theorem, we give the identity d'Ocagne containing the elements of the sequence $\{OH_{-n}^{(k)}\}_{n\geq 0}$.

Theorem 7. *For* $m,n\in\mathbb{Z}$*, the elements* $OH_{-n}^{(k)}$ *satisfy the following identity.*

$$OH_{-n+1}^{(k)}OH_{-m}^{(k)}-OH_{-n}^{(k)}OH_{-m+1}^{(k)}=\frac{k^{2m}}{\sqrt{k^2-4}}\left(\beta^{m-n}\tilde{\beta}\tilde{\alpha}-\alpha^{m-n}\tilde{\alpha}\tilde{\beta}\right). \tag{26}$$

Proof. If we use definition of the numbers $OH_{-n+1}^{(k)}, OH_{-m}^{(k)}, OH_{-n}^{(k)}$ and $OH_{-m+1}^{(k)}$, then we can write the left-hand side of the desired equation as follows.

$$OH_{-n+1}^{(k)} OH_{-m}^{(k)} - OH_{-n}^{(k)} OH_{-m+1}^{(k)} = \frac{k^{2(m+n-1)}}{k^2-4}\left(A+B\right),$$

where

$$A = \left[\alpha^{m+n-1}(\tilde{\beta})^2 - \alpha^{n-1}\beta^m \tilde{\beta}\tilde{\alpha} - \beta^{n-1}\alpha^m \tilde{\alpha}\tilde{\beta} + \beta^{m+n-1}(\tilde{\alpha})^2\right],$$

$$B = \left[-\alpha^{m+n-1}(\tilde{\beta})^2 + \alpha^{m-1}\beta^n \tilde{\alpha}\tilde{\beta} + \beta^{m-1}\alpha^n \tilde{\beta}\tilde{\alpha} - \beta^{m+n-1}(\tilde{\alpha})^2\right].$$

In the last equation we obtained, we can write the following equations as a result of simplification and some calculations.

$$OH_{-n+1}^{(k)} OH_{-m}^{(k)} - OH_{-n}^{(k)} OH_{-m+1}^{(k)} = \frac{k^{2(m+n-1)}}{k^2-4}\left[\tilde{\beta}\tilde{\alpha}\alpha^n\beta^m\left(\frac{1}{\beta}-\frac{1}{\alpha}\right) + \tilde{\alpha}\tilde{\beta}\alpha^m\beta^n\left(\frac{1}{\alpha}-\frac{1}{\beta}\right)\right],$$

$$OH_{-n+1}^{(k)} OH_{-m}^{(k)} - OH_{-n}^{(k)} OH_{-m+1}^{(k)} = \frac{k^{2(m+n-1)}}{k^2-4}\left(\frac{\alpha-\beta}{\alpha\beta}\right)\left(\alpha^n\beta^m\tilde{\beta}\tilde{\alpha} - \tilde{\alpha}\tilde{\beta}\alpha^m\beta^n\right),$$

$$OH_{-n+1}^{(k)} OH_{-m}^{(k)} - OH_{-n}^{(k)} OH_{-m+1}^{(k)} = \frac{k^{2m}}{\sqrt{k^2-4}}\left(\beta^{m-n}\tilde{\beta}\tilde{\alpha} - \alpha^{m-n}\tilde{\alpha}\tilde{\beta}\right).$$

Thus, the proof is completed. □

In the next theorem, we give the identity Honsberger's identity involving the elements of the sequence $\{OH_{-n}^{(k)}\}_{n\geq 0}$.

Theorem 8. *For $m, n \in \mathbb{Z}$, the following equation is true.*

$$OH_{-(m+n)}^{(k)} = kO_{-n}^{(k)} OH_{-m+1}^{(k)} - \frac{1}{k} O_{-n-1}^{(k)} OH_{-m}^{(k)}. \tag{27}$$

Proof. We write the following equation to see correctness of the desired equation.

$$OH_{-(m+n)}^{(k)} = k_1 O_{-n}^{(k)} - k_2 O_{-(n+1)}^{(k)}.$$

We should see that the equations $k_1 = kOH_{-m+1}^{(k)}$ and $k_2 = \frac{1}{k} OH_{-m}^{(k)}$ are true. From the Binet formula (19),

$$k^{2m}\left(\alpha^{m+n}\tilde{\beta} - \beta^{m+n}\tilde{\alpha}\right) = \beta^n\left(-k_1 + k_2k^2\alpha\right) + \alpha^n\left(k_1 - k_2k^2\beta\right),$$

can be written. Also, we get

$$-k^{2m}\beta^m\tilde{\alpha} = -k_1 + k_2k^2\alpha,$$

$$k^{2m}\alpha^m\tilde{\alpha} = k_1 - k_2k^2\beta.$$

And so, $k_2 = k^{2m-2}\frac{(\alpha^m\tilde{\beta}-\beta^m\tilde{\alpha})}{\alpha-\beta} = -\frac{1}{k}OH_{-m}$. If we substitute this value in the equation, then

$$k^{2m}\beta^m\tilde{\alpha} = -k_1 + \alpha k^{2m}\frac{\left(\alpha^m\tilde{\beta} - \beta^m\tilde{\alpha}\right)}{\alpha-\beta},$$

$$k^{2m}\beta^{m+1}\tilde{\alpha} - k^{2m}\alpha^{m+1}\tilde{\beta} = k_1(\alpha-\beta),$$

$$k_1 = -\frac{k^{2m}(\alpha^{m+1}\tilde{\beta} - \beta^{m+1}\tilde{\alpha})}{(\alpha-\beta} = kOH_{-m+1}.$$

Then, we obtain that

$$OH_{-(m+n)}^{(k)} = kO_{-n}^{(k)} OH_{-m+1}^{(k)} - \frac{1}{k} O_{-n-1}^{(k)} OH_{-m}^{(k)}.$$

□

In the following theorem, we give the generating function of k-Oresme hybrid numbers with negative indices.

Theorem 9. *For the sequence* $\{OH_{-n}^{(k)}\}_{n\geq 0}$*, the generating function is*

$$\sum_{i\geq 0} OH_i^{(k)} z^i = \frac{OH_0^{(k)}(1-zk^2) + zOH_{-1}^{(k)}}{1-zk^2+z^2k^2}. \tag{28}$$

Proof. The following equaitons are calculated respectively.

$$f(z) = OH_0^{(k)} + zOH_{-1}^{(k)} + z^2 OH_{-2}^{(k)} + z^3 OH_{-3}^{(k)} \cdots$$

$$zk^2 f(z) = zk^2 OH_0^{(k)} - z^2k^2 OH_{-1}^{(k)} - z^3k^2 OH_{-2}^{(k)} + \cdots$$

$$z^2k^2 f(z) = z^2k^2 OH_0^{(k)} + z^3k^2 OH_{-1}^{(k)} + z^4k^2 OH_{-2}^{(k)} + \cdots$$

$$f(z)\left(1 - zk^2 + z^2k^2\right) = OH_0^{(k)} + z\left(OH_{-1}^{(k)} - k^2 OH_0^{(k)}\right).$$

Then, we get

$$f(z) = \frac{OH_0^{(k)}(1-zk^2) + zOH_{-1}^{(k)}}{1-zk^2+z^2k^2}.$$

□

In the following theorem, we derive the formula for the sum of k-Oresme hybrid numbers with negative indices.

Theorem 10. *For the elements of sequence* $\{OH_{-n}^{(k)}\}_{n\geq 0}$*, the following is satisfied.*

$$\sum_{k=1}^{n} OH_{-n}^{(k)} = k^2\left(OH_{-n}^{(k)} - OH_1^{(k)}\right) - \left(OH_0^{(k)} + OH_{-n-1}^{(k)}\right). \tag{29}$$

Proof. If we use the Binet formula (19), then we write

$$\sum_{k=1}^{n} OH_n^{(k)} = \frac{-1}{\sqrt{k^2-4}}\left[\tilde{\beta}\sum_{k=1}^{n}\left(k^2\alpha\right)^n - \tilde{\alpha}\sum_{k=1}^{n}\left(k^2\beta\right)^n\right],$$

Here, $\tilde{\beta}\sum_{k=1}^{n}\left(k^2\alpha\right)^n$ and $\tilde{\alpha}\sum_{k=1}^{n}\left(k^2\beta\right)^n$ are

$$\tilde{\beta}\sum_{k=1}^{n}\left(k^2\alpha\right)^n = \tilde{\beta}\left(1 - k^2\beta - k^{2n+2}\alpha^{n+1} + k^{2n+4}\alpha^{n+1}\beta\right)$$

and

$$\tilde{\alpha}\sum_{k=1}^{n}\left(k^2\beta\right)^n = \tilde{\alpha}\left(1 - k^2\alpha - k^{2n+2}\beta^{n+1} + k^{2n+4}\beta^{n+1}\alpha\right),$$

respectively. If we substitute these calculated values, then we obtain

$$\sum_{k=1}^{n} OH_{-n}^{(k)} = -k^2 OH_1^{(k)} + k^2 OH_{-n}^{(k)} + OH_0^{(k)} - OH_{-n-1}^{(k)},$$

$$\sum_{k=1}^{n} OH_{-n}^{(k)} = k^2\left(OH_{-n}^{(k)} - OH_1^{(k)}\right) + \left(OH_0^{(k)} - OH_{-n-1}^{(k)}\right).$$

Thus, the proof is completed. □

Theorem 11. *For the elements of sequence* $\{OH_{-n}^{(k)}\}_{n\geq 0}$ *the following is satisfied.*

$$\sum_{j=1}^{n} OH_{-j}^{(k)} = -k^2\left(OH_2^{(k)} - OH_{-n+1}^{(k)}\right). \tag{30}$$

Proof. From the equality [4], $\sum_{i=1}^{n} O_{-i}^{(k)} = -k\left(1 - O_{-n+1}^{(k)}\right)$ and by induction

$$\sum_{j=0}^{n+1} OH_{-j}^{(k)} = OH_0^{(k)} + OH_{-1}^{(k)} + OH_{-2}^{(k)} + \cdots + OH_{-(n+1)}^{(k)},$$

can be written. Thus,

$$\sum_{j=0}^{n} OH_{-j}^{(k)} = \left(O_0^{(k)} + O_{-1}^{(k)}\mathbf{i} + O_{-2}^{(k)}\boldsymbol{\epsilon} + O_{-3}^{(k)}\mathbf{h}\right) + \left(O_{-1}^{(k)} + O_0^{(k)}\mathbf{i} + O_1^{(k)}\boldsymbol{\epsilon} + O_2^{(k)}\mathbf{h}\right) + \cdots,$$

$$\sum_{j=0}^{n} OH_{-j}^{(k)} = \left[-k - k\mathbf{i} + \frac{\mathbf{i}}{k} + \frac{2\boldsymbol{\epsilon}}{k} - \epsilon k + \frac{(k^2-1)}{k^3}\mathbf{h} + \frac{2\mathbf{h}}{k} - \mathbf{h}\right] + k^2 OH_{-n+1}^{(k)},$$

$$\sum_{j=0}^{n} OH_{-j}^{(k)} = \left[-k + \frac{(1-k^2)}{k}\mathbf{i} + \frac{(2-k^2)}{k}\boldsymbol{\epsilon} + \frac{(3k^2-k^4-1)}{k^3}\mathbf{h}\right] + k^2 OH_{-n+1}^{(k)},$$

$$\sum_{j=0}^{n} OH_{-j}^{(k)} = -k^2\left[\frac{1}{k} + \frac{(k^2-1)}{k^3}\mathbf{i} + \frac{(k^2-2)}{k^3}\boldsymbol{\epsilon} + \frac{(k^4-3k^2+1)}{k^5}\mathbf{h}\right] + k^2 OH_{-n+1}^{(k)}.$$

$$\sum_{j=0}^{n} OH_{-j}^{(k)} = -k^2\left(OH_2^{(k)} - OH_{-n+1}^{(k)}\right)$$

is obtained which is the proof is completed. □

Theorem 12. *For the elements of sequence* $\{OH_{-n}^{(k)}\}_{n\geq 0}$ *the following is satisfied.*

$$\sum_{j=1}^{n} (-1)^i OH_{-i}^{(k)} = \frac{1}{2k^2+1}\left(k + (-1)^n\left(k^2 OH_{-n}^{(k)} + OH_{-n-1}^{(k)}\right)\right). \tag{31}$$

Proof. It can be seen that the equality claimed in the statement of the theorem is true with the help of the induction method. □

Theorem 13. *For the elements of sequence* $\{OH_{-n}^{(k)}\}_{n\geq 0}$ *the following is satisfied.*

$$T = \sum_{n=1}^{t}\left(OH_{-n}^{(k)}\right)^2 = \frac{k^4\left(OH_1^{(k)} - \left(OH_{t+1}^{(k)}\right)^2\right) - OH_0^{(k)} + \left(OH_t^{(k)}\right)^2}{1+2k^2}$$

$$+\frac{2\left(\frac{1-x^t}{1-x}\right)\left[\frac{\tilde{\beta}\tilde{\alpha}\left((k\alpha)^{-2}-2\right)+\tilde{\alpha}\tilde{\beta}(k\alpha)^2}{k^2-4}\right]}{1+2k^2}. \tag{32}$$

Proof. We have $T = \left(OH_{-1}^{(k)}\right)^2 + \left(OH_{-2}^{(k)}\right)^2 + \left(OH_{-3}^{(k)}\right)^2 + \cdots$. From the equality (20),

$$OH_{-n}^{(k)} = \frac{k^2 OH_{-n-1}^{(k)} + OH_{-n+1}^{(k)}}{k^2}.$$

$$T = \sum_{n=1}^{t}\left(OH_{-n}^{(k)}\right)^2 = \sum_{n=1}^{t}\left[\frac{k^2 OH_{-n-1}^{(k)} + OH_{-n+1}^{(k)}}{k^2}\right]^2,$$

$$k^4 T = k^4\sum_{n=1}^{t}\left(OH_{-n+1}^{(k)}\right)^2 + \sum_{n=1}^{t}\left(OH_{-n-1}^{(k)}\right)^2 + 2k^2\sum_{n=1}^{t} OH_{-n+1}^{(k)} OH_{-n-1}^{(k)}.$$

If we also use the Cassini identity,
$OH_{n+1}^{(k)}OH_{n-1}^{(k)}=\left(OH_{-n}^{(k)}\right)^2-\frac{k^{2n}}{k^2-4}\left[\tilde{\beta}\tilde{\alpha}\left((k\alpha)^{-2}-2\right)+\tilde{\alpha}\tilde{\beta}\left(k\alpha\right)^2\right]$, then we get

$$k^4T=k^4\left(T-\left(OH_1^{(k)}\right)^2+\left(OH_{t+1}^{(k)}\right)^2\right)+\left(T-\left(OH_0^{(k)}\right)^2+\left(OH_t^{(k)}\right)^2\right)$$

$$+2k^2\sum_{n=1}^{t}\left(OH_n^{(k)}\right)^2-\frac{k^{2n}}{k^2-4}\left[\tilde{\beta}\tilde{\alpha}\left((k\alpha)^{-2}-2\right)+\tilde{\alpha}\tilde{\beta}\left(k\alpha\right)^2\right],$$

$$k^4T=k^4\left(T-\left(OH_1^{(k)}\right)^2+\left(OH_{t+1}^{(k)}\right)^2\right)+\left(T+\left(OH_0^{(k)}\right)^2-\left(OH_t^{(k)}\right)^2\right)$$

$$+2k^2T-2\sum_{n=1}^{t}\frac{k^{2n+2}}{k^2-4}\left[\tilde{\beta}\tilde{\alpha}\left((k\alpha)^{-2}-2\right)+\tilde{\alpha}\tilde{\beta}\left(k\alpha\right)^2\right],$$

$$k^4T=T(k^4+2k^2+1)-k^4\left(OH_1^{(k)}\right)^2+k^4\left(OH_{t+1}^{(k)}\right)^2+\left(OH_0^{(k)}\right)^2-\left(OH_t^{(k)}\right)^2$$

$$-2\left(\frac{1-x^t}{1-x}\right)\left[\frac{\tilde{\beta}\tilde{\alpha}\left((k\alpha)^{-2}-2\right)+\tilde{\alpha}\tilde{\beta}\left(k\alpha\right)^2}{k^2-4}\right].$$

Thus, we obtain

$$T=\sum_{n=1}^{t}\left(OH_{-n}^{(k)}\right)^2=\frac{k^4\left(OH_1^{(k)}-\left(OH_{t+1}^{(k)}\right)^2\right)-OH_0^{(k)}+\left(OH_t^{(k)}\right)^2}{1+2k^2}$$

$$+\frac{2\left(\frac{1-x^t}{1-x}\right)\left[\frac{\tilde{\beta}\tilde{\alpha}\left((k\alpha)^{-2}-2\right)+\tilde{\alpha}\tilde{\beta}(k\alpha)^2}{k^2-4}\right]}{1+2k^2}.$$

Which is the desired result. □

Theorem 14. *For the elements of sequence $\{OH_{-n}^{(k)}\}_{n\geq 0}$ the following is satisfied.*

$$i)\ \sum_{k=1}^{n}OH_{-2n}^{(k)}=\left(OH_0^{(k)}-OH_{-2n}^{(k)}\right)-k^4\left(OH_2^{(k)}-OH_{-(2n-2)}^{(k)}\right). \tag{33}$$

$$ii)\ \sum_{k=1}^{n}OH_{-2n-1}^{(k)}=\left(OH_0^{(k)}-OH_{-(2n-1)}^{(k)}\right)+k^2\left(OH_1^{(k)}-OH_{-(2n-2)}^{(k)}\right). \tag{34}$$

Proof.

$$i)\ \sum_{k=1}^{n}OH_{-2n}^{(k)}=\frac{-1}{\sqrt{k^2-4}}\left[\tilde{\beta}\sum_{k=1}^{n}(\alpha^2k^4)^n-\tilde{\alpha}\sum_{k=1}^{n}(\beta^2k^4)^n\right],$$

$$\sum_{k=1}^{n}OH_{-2n}^{(k)}=\frac{-1}{\sqrt{k^2-4}}\left[\tilde{\beta}\left(\frac{1-(k^4\alpha^2)^n}{1-k^4\alpha^2}\right)-\tilde{\alpha}\left(\frac{1-(k^4\beta^2)^n}{1-k^4\beta^2}\right)\right],$$

$$\sum_{k=1}^{n}OH_{-2n}^{(k)}=\frac{-1}{\sqrt{k^2-4}}\left[(\tilde{\beta}-\tilde{\alpha})-k^{4n}(\tilde{\beta}\alpha^{2n}-\tilde{\alpha}\beta^{2n})\right]$$

$$-\frac{1}{\sqrt{k^2-4}}\left[k^4(\tilde{\alpha}\alpha^2-\tilde{\beta}\beta^2)+k^{4n}(\tilde{\beta}\alpha^{2n-2}-\tilde{\alpha}\beta^{2n-2})\right],$$

$$\sum_{k=1}^{n} OH_{-2n}^{(k)} = \left(OH_0^{(k)} - OH_{-2n}^{(k)}\right) - k^4 \left(OH_2^{(k)} - OH_{-(2n-2)}^{(k)}\right).$$

Thus, the sum of k-Oresme hybrid numbers with the negative even indices is given. Similarly, the sum of terms with odd indices can be obtained. □

3. Conclusion

In this study, we inspired by the theory of number systems created by choosing coefficients from special number sets and defined at the negative indices k-Oresme hybrid numbers. We examined these newly identified numbers in detail. In particular, we obtained the fundamental and important identities provided by the elements of this sequence and frequently encountered in the literature.

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Some new results on quasi-ordered residuated systems

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Abstract. Quasi-ordered residuated system is a commutative residuated integral monoid ordered under a quasi-order was introduced in 2018 by Bonzio and Chajda as a generalization of commutative residuated lattices and hoop-algebras. This paper introduces the concept of atoms in these systems and analyzes its properties. Additionally, two extensions of the system $\mathfrak{A}$ to the system $\mathfrak{A} \cup \{w\}$ were designed so that the element w is an atom in $\mathfrak{A} \cup \{w\}$.

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Keywords. Quasi-ordered residuated system, atom in quasi-ordered residuated system, extension of quasi-ordered residuated system.

1. Introduction

The algebraic structure recognizable under the name quasi-ordered residuated system (QRS, by short) was introduced in 2018 by Bonzio and Chajda ([1]) as a generalization of hoop-algebras (in the sense of [3]) and commutative residuated lattices (in the sense of [7]). Quasi-ordered residuated system $\mathfrak{A} = \langle A, \cdot, \rightarrow, 1, \preccurlyeq \rangle$ is an integral monoid ordered by a quasi-order $\preccurlyeq$ at which the residuum $' \rightarrow '$ is associated with internal binary operation $' \cdot '$ in A by a special relationship

$$(\forall x, y, z \in A)(x \cdot y \preccurlyeq z \Longleftrightarrow x \preccurlyeq y \rightarrow z).$$

The results of the study of the internal structure of the QRS as well as its substructures were announced by by the author of this article in several of his reports (see, for example [11,12]). One of the specifics by which this algebraic structure differs from the commutative residuated lattice is that its residuum part $(A, \rightarrow, 1)$ is a BE-algebra with some additional features. Besides that, this algebraic structure, in the general case, does not satisfy the condition

$$(\forall x, y \in A)(x \cdot (x \rightarrow y) = y \cdot (y \rightarrow x)),$$

which is one of the hoop-algebras axioms.

The concept of atoms, as a specific phenomenon in many logical algebras, such as, for example, BCK/BCI/BCC-algebras, has been studied by several researchers (see, for example [4–6,8,10,14]). Atomic elements in residual lattices are also studied (see, for example, [9,15]).

This paper is a report on the properties of atoms in quasi-ordered residuated systems. However, due to the specificity of the quasi-order relation in QRSs, the method of defining the concept of atom used in the residual lattice and the mentioned logical algebra is not expedient for defining the concept of atom in QRSs. The definition of this concept in QRSs, which is used here, to be expedient must be specific. The paper is organized as follows: In the Preliminaries section, the necessary data and propositions on quasi-ordered residuated system for the comfonious monitoring of exposure in Section 3. Section 3 is central part of this report. The concept of atoms in the quasi-ordered residuated system was introduced. Some of the important features of this notion were registered. For example, the set $L(A)$ of all atoms in a quasi-ordered residuated system $\mathfrak{A}$ if not empty, is an anti-chain. However, this subset need not

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be a sub-semigroup of the monoid $(A,\cdot,1)$. Second, a criterion was found for determining whether an element $w \in A$ is an atom in $\mathfrak{A}$ or not. There are also several examples are included that illustrate the characteristics of this phenomenon in a quasi-ordered residuated system. Additionally, in the second subsection, two extensions of the system $\mathfrak{A}$ to the system $\mathfrak{A}\cup\{w\}$ were designed, so that the element w is an atom in $\mathfrak{A}\cup\{w\}$. The second of them is created so that $L(A\cup\{w\}) = L(A)\cup\{w\}$.

2. Preliminaries

It should be emphasized here that the formulas in this text are written in a standard way, as is usual in mathematical logic, with the standard use of labels for logical functions. Thus, the labels $\wedge$, $\vee$, $\Longrightarrow$, $\neg$, and so on, are labels for the logical functions of conjunction, disjunction, implication, negation, and so on. Brackets in formulas are used in the standard way, too. All formulas appearing in this paper are closed by some quantifier. If one of the formulas is open, then the variables that appear in it should be seen as free variables. In addition to the previous one, the sign $=:$, in the use of $A =: B$, should be understood in the sense that the letter A is the abbreviation for the formula B.

Recall that a *quasi-order relation* $'\preccurlyeq'$ on a set A is a binary relation which is reflexive and transitive.

Definition 1 ([1], Definition 2.1)**.** *A residuated relational system is a structure* $\mathfrak{A} = \langle A,\cdot,\rightarrow,1,R\rangle$, *where* $\langle A,\cdot,\rightarrow,1\rangle$ *is an algebra of type* $\langle 2,2,0\rangle$ *and* R *is a binary relation on* A *and satisfying the following properties:*

(1) $(A,\cdot,1)$ *is a commutative monoid;*
(2) $(\forall x \in A)((x,1) \in R)$*;*
(3) $(\forall x,y,z \in A)((x\cdot y,z) \in R \Longleftrightarrow (x,y\rightarrow z) \in R)$.

We will refer to the operation $\cdot$ *as multiplication, to* $\rightarrow$ *as its residuum and to condition (3) as residuation. A quasi-ordered residuated system (QRS, in short) is a residuated relational system* $\langle A,\cdot,\rightarrow,1,\preccurlyeq\rangle$, *where* $\preccurlyeq$ *is a quasi-order relation in the monoid* $(A,\cdot)$. *We denote this axiomatic system by* ***QRS***.

The following proposition shows the basic properties of quasi-ordered residuated systems.

Proposition 1 ([1], Proposition 3.1)**.** *Let* $\mathfrak{A}$ *be a quasi-ordered residuated system. Then*

(4) *The operation* $'\cdot'$ *preserves the pre-order in both positions;*

$$(\forall x,y,z \in A)(x \preccurlyeq y \Longrightarrow (x\cdot z \preccurlyeq y\cdot z \wedge z\cdot x \preccurlyeq z\cdot y));$$

(5) $(\forall x,y,z \in A)(x \preccurlyeq y \Longrightarrow (y\rightarrow z \preccurlyeq x\rightarrow z \wedge z\rightarrow x \preccurlyeq z\rightarrow y))$*;*
(6) $(\forall y,z \in A)(x\cdot(y\rightarrow z) \preccurlyeq y\rightarrow x\cdot z)$*;*
(7) $(\forall x,y,z \in A)(x\cdot y\rightarrow z \preccurlyeq x\rightarrow(y\rightarrow z))$*;*
(8) $(\forall x,y,z \in A)(x\rightarrow(y\rightarrow z) \preccurlyeq x\cdot y\rightarrow z)$*;*
(9) $(\forall x,y,z \in A)(x\rightarrow(y\rightarrow z) \preccurlyeq y\rightarrow(x\rightarrow z))$*;*
(10) $(\forall x,yz \in A)((x\rightarrow y)\cdot(y\rightarrow z) \preccurlyeq x\rightarrow z)$*;*
(11) $(\forall x,y \in A)((x\cdot y \preccurlyeq x) \wedge (x\cdot y \preccurlyeq y))$*;*
(12) $(\forall x,y,z \in A)(x\rightarrow y \preccurlyeq (y\rightarrow z)\rightarrow(x\rightarrow z))$*;*
(13) $(\forall x,y,z \in A)(y\rightarrow z \preccurlyeq (x\rightarrow y)\rightarrow(x\rightarrow z))$.

It is generally known that a quasi-order relation $\preccurlyeq$ on a set A generates an equivalence $\equiv_{\preccurlyeq} := \preccurlyeq \cap \preccurlyeq^{-1}$ on A. Due to properties (4) and (5), this equality relation is compatible with the operations in $\mathfrak{A}$. Thus, $\equiv_{\preccurlyeq}$ is a congruence on $\mathfrak{A}$. In what follows, we will sometime write these relations with $\equiv$ if there is no misunderstanding. In connection with the previous one, the quotient structure $\mathfrak{A}/\equiv =: \langle A/\equiv, *, \rightrightarrows, [1]_{\equiv}\rangle$ is a QRS, where the operations $*$ and $\rightrightarrows$ are determined as follows

$$(\forall x,y \in A)(([x]_{\equiv} * [y]_{\equiv} =: [x\cdot y]_{\equiv}) \wedge ([x]_{\equiv} \rightrightarrows [y]_{\equiv} =: [x\rightarrow y]_{\equiv})).$$

In the light of the previous note, it is easy to see that the following applies:
(7) and (8) give:

$$(\forall x,y,z \in A)(x\cdot y\rightarrow z \equiv_{\preccurlyeq} x\rightarrow(y\rightarrow z)).$$

Due to the universality of formula (9), we have:

(14) $(\forall x,y,z \in A)(x\rightarrow(y\rightarrow z) \equiv_{\preccurlyeq} y\rightarrow(x\rightarrow z))$.

In addition to the previous one it is easy to prove that

$$(\forall x,y \in A)(x \preccurlyeq y \Longleftrightarrow x\rightarrow y \equiv_{\preccurlyeq} 1).$$

Indeed, for any $x, y \in A$ the following holds $x \preccurlyeq y \Longleftrightarrow 1 \preccurlyeq x \rightarrow y \preccurlyeq 1$, relying on (3) and (2).

Definition 2. *By a hoop ([3]) we mean an algebra $(H, \cdot, \rightarrow, 1)$ in which $(H, \cdot, 1)$ is a commutative semigroup with the identity and the following assertions are valid:*

(H1) $(\forall x \in H)(x \rightarrow x = 1)$,
(H2) $(\forall x, y \in H)(x \cdot (x \rightarrow y) = y \cdot (y \rightarrow x))$ *and*
(H3) $(\forall x, y, z \in A)(x \cdot y \rightarrow z = x \rightarrow (y \rightarrow z))$.

A relation $\leqslant$ on hoop $(A, \cdot, 1)$ is defined ([3], pp. 62) by

$$(\forall x, y \in A)(x \leqslant y \Longleftrightarrow x \rightarrow y = 1).$$

The relation $\leqslant$ is a partial order on A compatible with the operation in the hoop $(A, \cdot, 1)$ in accordance with Proposition 2.2 in [3] (see [2], also). It is easy to see that every hoop is a (quasi-)ordered residuated system and vice versa does not have to be because, in the general case, the formula (H2) is not a valid formula in the **QRS** axiom system.

Example 1. *For a commutative monoid A, let $\mathfrak{P}(A)$ be denote the powerset of A ordered by set inclusion and '$\cdot$' the usual multiplication of subsets of A. Then $\langle \mathfrak{P}(A), \cdot, \rightarrow, A, \subseteq \rangle$ is a quasi-ordered residuated system in which the residuum is given by $(\forall X, Y \in \mathfrak{P}(A))(Y \rightarrow X =: \{z \in A : Yz \subseteq X\})$.* □

Examples 2, 3 and 4, included in this section, have an important application to Section 3 as well.

Example 2. *Let $A = \{1, 2, 3, 4\}$ and operations '$\cdot$' and '$\rightarrow$' defined on A as follows:*

·	1	a	b	c	d
1	1	a	b	c	d
a	a	a	a	a	a
b	b	a	b	b	b
c	c	a	b	c	b
d	d	a	b	b	d

and

→	1	a	b	c	d
1	1	a	b	c	d
a	1	1	1	1	1
b	1	a	1	1	1
c	1	a	d	1	d
d	1	a	c	c	1

Then $\mathfrak{A} =: \langle A, \cdot, \rightarrow, 1\rangle$ is a quasi-ordered residuated systems where the relation '$\preccurlyeq$' is defined as follows $\preccurlyeq =: \{(1,1), (a,1), (a,a), (a,b), (a,c), (a,d), (b,b), (b,c), (b,d), (b,1), (c,c), (c,1), (d,d), (d,1)\}$. □

Example 3. *Let $A = \{1, a, b, c, d, e\}$ and operations '$\cdot$' and '$\rightarrow$' defined on A as follows:*

·	1	a	b	c	d	e
1	1	a	b	c	d	e
a	a	a	a	a	a	a
b	b	a	b	a	b	a
c	c	a	a	a	a	c
d	d	a	b	a	b	c
e	e	a	a	c	c	e

and

→	1	a	b	c	d	e
1	1	a	b	c	d	e
a	1	1	1	1	1	1
b	1	e	1	e	1	e
c	1	b	b	1	1	1
d	1	a	b	e	1	e
e	1	b	b	d	d	1

Then $\mathfrak{A} =: \langle A, \cdot, \rightarrow, 1\rangle$ is a quasi-ordered residuated systems where the relation '$\preccurlyeq$' is defined as follows $\preccurlyeq =: \{(1,1), (a,1), (b,1), (c,1), (d,1), (e,1), (a,a), (a,b), (a,c), (a,d), (a,e), (b,b), (b,d), (c,c), (c,d), (c,e), (d,d), (e,e)\}$. By direct verification it can be proved that $\mathfrak{A}$ is a quasi-ordered residuated system. □

Definition 3 ([11], Definition 3.1)**.** *For a non-empty subset F of a quasi-ordered residuated system $\mathfrak{A}$ we say that it is a* filter *of $\mathfrak{A}$ if it satisfies conditions*

(F2) $(\forall u, v \in A)((u \in F \wedge u \preccurlyeq v) \Longrightarrow v \in F)$, *and*
(F3) $(\forall u, v \in A)((u \in F \wedge u \rightarrow v \in F) \Longrightarrow v \in F)$.

If the non-empty subset F of a quasi-ordered system $\mathfrak{A}$ satisfies the condition (F2), then it also satisfies the conditions:

(F0) $1 \in F$ and
(F1) $(\forall u, v \in A)((u \cdot v \in F \Longrightarrow (u \in F \wedge v \in F))$,

as it is shown ([11], Proposition 3.4 and Proposition 3.2).

Remark 1. *It is easy to see that the determination of filters in quasi-ordered residuated systems differs from the determination of filters either in residuated lattices or hoop-algebras.*

Example 4. *Let $A = \langle -\infty, 1] \subset \mathbb{R}$ (the real numbers field). If we define '$\cdot$' and '$\rightarrow$' as follows, $(\forall u, v \in A)(u \cdot v =: min\{u, v\})$ and $u \rightarrow v =: 1$ if $u \leqslant v$ and $u \rightarrow v =: v$ if $v < u$ for all $u, v \in A$, then $\mathfrak{A} =: \langle A, \cdot, \rightarrow, 1, < \rangle$ is a quasi-ordered residuated system. All filters in $\mathfrak{A}$ are in the form of $\langle x, 1]$, for $x \in \langle -\infty, 1]$.* □

Example 5. *Let $A = \{1, a, b, c, d\}$ as in Example 2. The subsets $F_0 = \{1\}$, $F_1 = \{1, d\}$, and $F_2 = \{1, b, c, d\}$ are filters in $\mathfrak{A}$. Subsets $\{1, a\}$, $\{1, b\}$, $\{1, c\}$, $\{1, a, b\}$, $\{1, a, c\}$, $\{1, a, d\}$, $\{1, b, c\}$, $\{1, b, d\}$, $\{1, c, d\}$, $\{1, a, b, c\}$, $\{1, a, b, d\}$ are not filters in $\mathfrak{A}$.* □

Example 6. *Let $A = \{1, a, b, c, d, e\}$ as in Example 3. Subsets $F_0 = \{1\}$, $F_1 = \{1, d\}$, $F_2 = \{1, e\}$, $F_3 = \{1, b, d\}$, $F_4 = \{1, c, d, e\}$ are non-trivial filters in $\mathfrak{A}$. Subsets $\{1, a\}$, $\{1, b\}$, $\{1, c\}$, $\{1, a, b\}$, $\{1, a, c\}$, $\{1, a, d\}$, $\{1, a, e\}$, $\{1, b, c\}$, $\{1, b, e\}$, $\{1, c, d\}$, $\{1, c, e\}$, $\{1, d, e\}$, $\{1, a, b, c\}$, $\{1, a, b, d\}$, $\{1, a, b, e\}$, $\{1, b, c, d\}$, $\{1, b, c, e\}$ are not filters in $\mathfrak{A}$.* □

3. The Main Results

This section is the central part of this report. In the first subsection, the concept of a quasi-ordered residuated system is introduced and its basic properties are registered. Also, a criterion was found that enables recognition of whether an element is an atom in a quasi-ordered residuated system or not. Several examples are given that illustrate this phenomenon and its characteristics. In the second subsection, it was shown that every quasi-ordered residuated system can be embedded in a quasi-ordered residuated system that has at least one atom.

3.1. Concept of atoms in quasi-ordered residuated system.

First, we will determine the concept of atoms in a quasi-ordered residuated system.

Definition 4. *Let $\mathfrak{A}$ be a quasi-ordered residuated system. An element $(1 \neq)a \in A$ is an* atom *in $\mathfrak{A}$ if*

(At) $(\forall x \in A)(a \preccurlyeq x \implies (x \equiv_{\preccurlyeq} a \vee x \equiv_{\preccurlyeq} 1))$

holds. The set of all atoms in $\mathfrak{A}$ is denoted by $L(A)$.

It can immediately be concluded that:

Theorem 1. *Elements of $L(A)$ are not comparable.*

Proof. Let $a, b \in L(A)$ be such that $a \neq b$. If we assume that $a \preccurlyeq b$, we would have $b \equiv_{\preccurlyeq} a$ or $b \equiv_{\preccurlyeq} 1$ because a is an atom in $\mathfrak{A}$. Since none of the obtained options is possible, we conclude that the elements a and b are not comparable. □

The following proposition gives a criterion for recognizing atoms in a quasi-ordered residuated system.

Proposition 2. *Let $\mathfrak{A}$ be a quasi-ordered residuated system and $a \in A$ such that $1 \neq a$. Then a is an atom in $\mathfrak{A}$ if the set $\{1, a\}$ is a filter in $\mathfrak{A}$.*

Proof. Let the subset $\{1, a\}$ be filter in $\mathfrak{A}$. Then holds

$$(\forall x \in A)((a \in \{1, a\} \wedge a \preccurlyeq x) \implies x \in \{1, a\})$$

according (F2). This means $x = 1$ or $x = a$. □

Formula (At) can be written in the form

$$(\forall x \in A)(a \rightarrow x \equiv_{\preccurlyeq} 1 \implies (x \equiv_{\preccurlyeq} a \vee x \equiv_{\preccurlyeq} 1)).$$

In this case, the proof of the previous proposition is demonstrated by referring to (F3) instead of (F2).

Example 7. *Let $A = \{1, a, b, c, d\}$ as in Examples 2 and 5. Then $\mathfrak{A} =: \langle A, \cdot, \rightarrow, 1 \rangle$ is a quasi-ordered residuated system. Since the subsets $\{1, c\}$ and $\{1, d\}$ are filters in $\mathfrak{A}$, then, by Proposition 2, elements c and d are atoms in $\mathfrak{A}$.* □

Example 8. *Let $A = \{1, a, b, c, d, e\}$ is as in Example 3 and example 6. Then $\mathfrak{A} =: \langle A, \cdot, \rightarrow, 1 \rangle$ is a quasi-ordered residuated system. Subsets $\{1, d\}$ and $\{1, e\}$ are filters in $\mathfrak{A}$ (see Example 6). Therefore the elements d and e are atoms in $\mathfrak{A}$.* □

Remark 2. *An additional explanation of the concept of atoms in a quasi-ordered residuated system can be given by using the concept of 'covering'. We say that z cover x if and only if $x \preccurlyeq z$ and there does not exists $y \in A$ such that $x \preccurlyeq y \preccurlyeq z$ and $x \neq y$. Thus, an atom in the quasi-ordered residuated system $\mathfrak{A} =: \langle A, \cdot, \rightarrow, 1\rangle$ is an element in A which is covered by the neutral 1.*

As the following example shows, not every quasi-ordered residuated system has to have atoms:

Example 9. *Let $\mathfrak{A}$ be as in Example 4. For an element a to be an atom in $\mathfrak{A}$, it must be $(\forall x \in A)(a < x \implies (x = a \vee x = 1))$ which is impossible, because A is not a discrete set. On the contrary, for $a < 1$, there are infinitely many elements x such that $a < x < 1$.* □

Remark 3. *In our effort to prove that the converse of Proposition 2 is valid, we encountered the following problem: Let $\mathfrak{A} =: \langle A, \cdot, \rightarrow, 1\rangle$ be a quasi-ordered residuated system and let a be an atom in $\mathfrak{A}$. Then the set $F = \{1, a\}$ is a filter in $\mathfrak{A}$. We need to prove that the set F satisfies the conditions (F2) and (F3):*

(F2)*: Let $x, y \in A$ be arbitrary elements such that $x \in F = \{1, a\}$ and $x \preccurlyeq y$. This means $x \equiv_{\preccurlyeq} a$ or $x \equiv_{\preccurlyeq} 1$. In the first case, we have that $a \preccurlyeq y$ implies $y \equiv_{\preccurlyeq} a$ or $y \equiv_{\preccurlyeq} 1$ because a is an atom in $\mathfrak{A}$. Thus $y \in F$. In the second case, we have $1 \equiv_{\preccurlyeq} x \preccurlyeq y$ so $y \equiv_{\preccurlyeq} 1 \in F$ as well. This shows that the set F satisfies the condition (F2).*

(F3)*: Let $x, y \in A$ be arbitrary elements such that $x \in F = \{1, a\}$ and $x \rightarrow y \in F$. Two options are possible:*

(i) *$x = 1$ and $1 \rightarrow y \in \{1, a\}$. If $1 \rightarrow y = 1$, then $1 \preccurlyeq y$ and, therefore, $y \equiv_{\preccurlyeq} 1 \in F$. If $a = 1 \rightarrow y$, then $a \preccurlyeq y$. From here it follows $y \equiv_{\preccurlyeq} a \in F$ or $y \equiv_{\preccurlyeq} 1 \in F$ because a is an atom in $\mathfrak{A}$.*

(ii) *$x = a$ and $a \rightarrow y \in \{1, a\}$. If $1 = a \rightarrow y$, then $a \preccurlyeq y$, so $y \equiv_{\preccurlyeq} 1 \in F$ or $y \equiv_{\preccurlyeq} a \in F$ because a is an atom in $\mathfrak{A}$. Let now, us assume, that $a = a \rightarrow y$. Then $a = a \rightarrow y \preccurlyeq 1$. From here we conclude that it is not $a \preccurlyeq y$. Indeed, if there were $a \preccurlyeq y$, according to (5) we would have $1 \equiv_{\preccurlyeq} a \rightarrow a \preccurlyeq a \rightarrow y = a$, which is impossible. So it must be $\neg(a \preccurlyeq y)$. The obtained conclusion does not allow us to demonstrate the implication $a \in F \wedge a \rightarrow y \in F \implies y \in F$.*

The following theorem says something more about the set $L(A)$ of all atoms for a given quasi-ordered residuated system $\mathfrak{A} = \langle A, \cdot, \rightarrow\rangle$.

Theorem 2. *Let $\mathfrak{A}$ be a quasi-ordered residuated system and $a, b \in L(A)$. Then:*

(a) *If $a \neq b$, then $a \rightarrow b = b$ and $b \rightarrow a = a$.*

(b) *$(\forall x \in A)((a \rightarrow x) \rightarrow x = a \vee (a \rightarrow x) \rightarrow x = 1)$.*

(c) *$(\forall x \in A)(x \rightarrow a = a \vee x \rightarrow a = 1)$.*

Proof. (a) Since $a \cdot b \preccurlyeq a$ according to (11), we conclude that $a \preccurlyeq b \rightarrow a$. From here it follows $b \rightarrow a = a$ or $b \rightarrow a = 1$ because a is an atom in A. Since the second option is not possible according to Theorem 1, we have $b \rightarrow a = a$.

The statement $a \rightarrow b = b$ can be proved analogously to the previous proof.

(b) Let A be a quasi-ordered residuated system, $a \in L(A)$ and $x \in A$ be an arbitrary element. From the valid formula $(a \rightarrow x) \rightarrow (a \rightarrow x) = 1$, it follows $a \rightarrow ((a \rightarrow x) \rightarrow x) = 1$ according to (14). This means $a \preccurlyeq (a \rightarrow x) \rightarrow x$. Since a is an atom in $\mathfrak{A}$, from here we get $a = (a \rightarrow x) \rightarrow x$ or $(a \rightarrow x) \rightarrow x = 1$. The second option gives $a \rightarrow x \preccurlyeq x$.

(c) For the elements a and $x \in A$, we have $a \cdot x \preccurlyeq a$ according to (11). Then $a \preccurlyeq x \rightarrow a$ by (3). Thus $x \rightarrow a = a$ or $x \rightarrow a = 1$ since a is an atom in $\mathfrak{A}$. The second option means $x \preccurlyeq a$. □

Example 10. *Let $A = \{1, a, b, c, d, e\}$ as in Examples 3, 6 and 8. Elements d and e are atoms in the quasi-ordered residuated system $\mathfrak{A}$. $d \rightarrow e = e$ and $e \rightarrow d = d$ hold for them, which illustrates the statement (a) in the previous theorem.*

To illustrate statement (b), we calculate:

$(d \rightarrow a) \rightarrow a = a \rightarrow a = 1$ *and* $d \rightarrow a = a \preccurlyeq a$,

$(d \rightarrow b) \rightarrow b = b \rightarrow b = 1$ *and* $d \rightarrow b = b \preccurlyeq b$,

$(d \rightarrow c) \rightarrow c = e \rightarrow c = d$ *and* $d \rightarrow c = e$ *and* $\neg(e \preccurlyeq c)$,

$(d \rightarrow e) \rightarrow e = e \rightarrow e = 1$ *and* $d \rightarrow e = e \preccurlyeq e$.

In this example, the following calculation illustrates statement (c):

$a \rightarrow d = 1$ *and* $a \preccurlyeq d$,

$b \to d = 1$ and $b \preccurlyeq d$,
$c \to d = 1$ and $c \preccurlyeq d$,
$e \to d = d$ and $\neg(e \preccurlyeq d)$. □

Remark 4. *Let $\mathfrak{A}$ be a quasi-ordered residuated system. For any $a \in A$ we define a subset $V(a)$ of A as follows $V(a) = \{x \in A : x \preccurlyeq a\}$. Note that $V(a)$ is non-empty, because $a \preccurlyeq a$ gives $a \in V(a)$. If $a \in L(A)$, then the set $V(a)$ is called a branch of $\mathfrak{A}$. A characteristic of the concept of branches $V(a) \cap V(b) = \emptyset$ in some logical algebras such as, for example, BCI-algebra ([6], Proposition 3.15) and weak BCC-algebra ([5], Corollary 3.18) in the case of quasi-ordered residuated systems is not present, as the following example shows. In Example 3, for atoms $d, e \in L(A)$ we have $V(d) = \{a, b, c, d\}$ and $V(e) = \{a, c, e\}$, so, therefore, is $V(d) \cap V(e) = \{a, c\} \neq \emptyset$.*

It seems that this tool in the case of quasi-ordered residuated systems is not of any use in studying the phenomenon of atoms in this algebraic structure. For the sake of illustration, in the Example 3, for atoms $d, e \in L(A)$ we have $e \cdot d = d \cdot e = c \notin L(A)$. Therefore, $L(A)$ is not a subsemigroup in A.

The converse of statement (b) in the Theorem 2 is valid:

Theorem 3. *Let $\mathfrak{A} =: \langle A, \cdot, \to \rangle$ be a quasi-ordered residuated system. If the element $a \in A$ satisfies the condition* (b)*, then a is an atom in $\mathfrak{A}$.*

Proof. Let $x \in A$ be such that $a \preccurlyeq x$. This means $a \to x = 1$. Then $(a \to x) \to x = 1 \to x = x$. If $(a \to x) \to x = a$, then $a = x$. If $(a \to x) \to x = 1$, we have $x = 1$. This proves that a is an atom in $\mathfrak{A}$. □

3.2. **Two types of extensions of quasi-ordered residuated systems.** Let $\mathfrak{A} =: \langle A, \cdot, \to, 1, \preccurlyeq \rangle$ be a quasi-ordered residuated system and $w \notin A$. We can extend the system $\mathfrak{A} =: \langle A, \cdot, \to, 1, \preccurlyeq \rangle$ to the system $\mathfrak{B} =: \langle A \cup \{w\}, *, \rightsquigarrow, 1, \preccurlyeq \rangle$ so that the element w is an atom in the system $\mathfrak{B}$.

Here we demonstrate two such extensions.

Theorem 4. *Let $\mathfrak{A} =: \langle A, \cdot, \to, 1 \rangle$ be a quasi-ordered residuated system and $w \notin A$. We can extend the system $\langle A, \cdot, \to, 1 \rangle$ to the system $\mathfrak{B} =: \langle A \cup \{w\}, *, \rightsquigarrow, 1 \rangle$ so that the element w is an atom in the system $\mathfrak{B}$.*

Proof. System $\mathfrak{B}$ can be created in the following way:

$$x * y = \begin{cases} x \cdot y & \text{for } \ x \in A \ \wedge \ y \in A, \\ x & \text{for } \ x \in A \ \wedge \ y = w, \\ y & \text{for } \ x = w \ \wedge \ y \in A, \\ w & \text{for } \ x = w \ \wedge \ y = w, \end{cases}$$

and

$$x \rightsquigarrow y = \begin{cases} x \to y & \text{for } \ x \in A \ \wedge \ y \in A, \\ 1 & \text{for } \ x \in A \ \wedge \ y = w, \\ y & \text{for } \ x = w \ \wedge \ y \in A, \\ 1 & \text{for } \ x = w \ \wedge \ y = w. \end{cases}$$

For elements $x, y \in A$ we have
$w * (x \cdot y) = x \cdot y$ and $(w * x) \cdot y = x \cdot y$ and
$w * (w * x) = w * x = x$ and $(w * w) * x = w * x = x$.
So, the set $B = A \cup \{w\}$ is a commutative monoid. Second, $w * x \preccurlyeq w \iff w \preccurlyeq x \rightsquigarrow a = 1$, $w * x \preccurlyeq x \iff x \preccurlyeq w \rightsquigarrow w = 1$ and $w * w = w \preccurlyeq w \iff w \preccurlyeq w \rightsquigarrow w = 1$. Therefore, $\mathfrak{B}$ is a quasi-ordered residuated system. It is immediately clear that w is an atom in $\mathfrak{B}$, because if $w \preccurlyeq x$ holds, then $1 = w \rightsquigarrow x = x$. □

The following example illustrates the first extension of a quasi-ordered residuated system.

Example 11. *Let $A = \{1, a, b, c, d\}$ as in Example 2. Then $\mathfrak{A} =: \langle A, \cdot, \to, 1 \rangle$ is a quasi-ordered residuated system. Here is $L(A) = \{c, d\}$. Let us put $B = A \cup \{w\} = \{1, a, b, c, d, w\}$ and define the operations in B as follows*

$*$	1	a	b	c	d	w
1	1	a	b	c	d	w
a	1	a	a	a	a	a
b	1	a	b	b	b	b
c	1	a	b	c	b	c
d	1	a	b	b	d	d
w	w	a	b	c	d	w

and

$\rightsquigarrow$	1	a	b	c	d	w
1	1	a	b	c	d	w
a	1	1	1	1	1	1
b	1	a	1	1	1	1
c	1	a	d	1	d	1
d	1	a	c	c	1	1
w	1	a	b	c	d	1

Then $\mathfrak{B} =: \langle A \cup \{w\}, *, \rightsquigarrow, 1\rangle$ *is a quasi-ordered residuated system. It is obvious that* w *is a single atom in the system* $\mathfrak{B}$*. Hence* $L(B) = \{w\}$*.*

Theorem 5. *The extension of a quasi-ordered residuated system* $\langle A, \cdot, \rightarrow, 1, \preccurlyeq\rangle$ *to system* $\mathfrak{B} =: \langle A \cup \{w\}, *, \rightsquigarrow, 1, \preccurlyeq\rangle$ *can also be realized so that the set* $L(A)$ *of all atoms of the system* $\mathfrak{A}$ *is expanded by one element, that is* $L(B) = L(A) \cup \{w\}$*.*

Proof. Let us take $w \notin A$. Let's form the set $B = A \cup \{w\}$ and design operations on B in the following way:

$$x * y = \begin{cases} x \cdot y & \text{for } x \in A \ \wedge \ y \in A, \\ x & \text{for } x \in A \setminus L(A) \ \wedge \ y = w, \\ y & \text{for } x = w \ \wedge \ y \in A \setminus L(A), \\ w & \text{for } x = w \ \wedge \ y = w, \\ max\{z \in A : z \preccurlyeq x \ \wedge \ z \preccurlyeq w\} & \text{for } x \in L(A) \ \wedge \ y = w \end{cases}$$

and

$$x \rightsquigarrow y = \begin{cases} x \rightarrow y & \text{for } x \in A \ \wedge \ y \in A, \\ 1 & \text{for } x \in A \setminus L(A) \ \wedge \ y = w, \\ y & \text{for } x = w \ \wedge \ y \in A \setminus L(A), \\ 1 & \text{for } x = w \ \wedge \ y = w, \\ w & \text{for } x \in L(A) \ \wedge \ y = w \\ y & \text{for } x = w \ \wedge \ y \in L(A). \end{cases}$$

It can be shown that $\mathfrak{B}$ is a quasi-ordered residuated system by patient and careful calculation. It is easy to conclude that a is an atom in $\mathfrak{B}$. Indeed, from $w \preccurlyeq x$ it follows $w \rightsquigarrow x = 1$ and $(w \rightsquigarrow x) \rightsquigarrow x = 1$, from which it follows that w is an atom in $\mathfrak{B}$ according to Theorem 3. □

The following example illustrates another way of extending the quasi-ordered residuated system mentioned above. The extension, described in the following example, is significantly different from the previous one.

Example 12. *Let* $A = \{1, a, b, c, d\}$ *as in Example 2. Elements* c *and* d *are atoms in this quasi-ordered residuated system:* $L(A) = \{c, d\}$*. Let us put* $B = A \cup \{w\}$ *and define the operations in* B *as follows*

$*$	1	a	b	c	d	w
1	1	a	b	c	d	w
a	1	a	a	a	a	a
b	1	a	b	b	b	b
c	1	a	b	c	b	b
d	1	a	b	b	d	b
w	w	a	b	c	d	w

and

$\rightsquigarrow$	1	a	b	c	d	w
1	1	a	b	c	d	w
a	1	1	1	1	1	1
b	1	a	1	1	1	1
c	1	a	d	1	d	w
d	1	a	c	c	1	w
w	1	a	b	c	d	1

From the second table, which determined the $\rightsquigarrow$ *operation, it can be seen that* $a \preccurlyeq c < 1 \wedge b \preccurlyeq c < 1$, $a \preccurlyeq d < 1 \wedge b \preccurlyeq d < 1$*, and* $a \preccurlyeq w < 1 \wedge b \preccurlyeq w < 1$*. So,* $L(B) = \{c, d, w\}$*.*

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Brouwerian almost distributive lattices

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Abstract. This paper presents the idea of a Brouwerian almost distributive lattice, a generalization of an almost distributive lattice, and a Brouwerian algebra. We also derive some properties on Brouwerian almost distributive lattices. A set of equivalent conditions is provided for a Brouwerian almost distributive lattice to transform into a Brouwerian algebra.

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1. Introduction

Lattice [1, 2] and [4] is a mathematical structure constructed on a set of elements associated with two binary operations $\sqcap_\star$ (greatest lower bound) and $\sqcup_\star$ (least upper bound). These operations satisfy specific properties, such as associativity, commutativity, and idempotence. A distributive lattice is a type of lattice where meet and join operations distribute over each other. The inclusion of another unary operation on a distributive lattice paved the way to study a new concept known as Boolean algebra [9]. Heyting algebras [3] generalize the idea of Boolean algebras, with the implication operation $\rightarrow_\star$ playing a central role. Brouwerian algebras, also known as Kripke or topological algebras, are a specific subclass of Heyting algebras that incorporate topological structures. In addition to the algebraic operations of a Heyting algebra, Brouwerian algebras [10] include topological constraints, often represented by topological spaces or partial orders with additional topological properties.

Swamy and Rao studied almost distributive lattice [11] to understand the behavior of lattices when distributivity is nearly satisfied. In an almost distributive lattice, the distributive law holds almost everywhere, but a few exceptions may exist. The connection between lattices and almost distributive lattices lies in their relationship to distributivity. While distributive lattices strictly adhere to the distributive law for all elements, almost distributive lattices relax this requirement by allowing a few exceptions. This relaxation allows for a broader class of structures to be studied while retaining some distributive lattices' properties.

Almost distributive lattices were first studied under two binary operations, $\sqcap_\star$ and $\sqcup_\star$. The inclusion of another binary operation $\rightarrow_\star$ laid the foundation for studying many more algebras on almost distributive lattices [5,6] and [7]. Till now, the study of all these algebras on an almost distributive lattice where with

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the inclusion of both the least element 0 and maximal element ν_1. In this paper, we initiated to define an algebra named a Brouwerian almost distributive lattice, which is a generalization of a Brouwerian algebra and an almost distributive lattice, with the exclusion of the least element 0. In this context, we will go through the fundamental definition of a Brouwerian almost distributive lattice and provide some examples demonstrating the independence of the axioms stated in the definition and some properties related to the structure. Finally, we present a collection of equivalence conditions that enable the Brouwerian almost distributive lattice to transform into a Brouwerian algebra.

2. Preliminaries

Let us recall some beneficial, necessary results on an almost distributive lattice, semi-Brouwerian algebra, and semi-Brouwerian almost distributive lattice, which are frequently used in the paper.

Definition 1. *[11] An algebra $(\mathcal{B}, \sqcup_\star, \sqcap_\star)$ of type (2,2) is called an almost distributive lattice (ADL), if it assures the subsequent axioms;*

1. $(\chi_1 \sqcup_\star \chi_2) \sqcap_\star \chi_3 = (\chi_1 \sqcap_\star \chi_3) \sqcup_\star (\chi_2 \sqcap_\star \chi_3)$
2. $\chi_1 \sqcap_\star (\chi_2 \sqcup_\star \chi_3) = (\chi_1 \sqcap_\star \chi_2) \sqcup_\star (\chi_1 \sqcap_\star \chi_3)$
3. $(\chi_1 \sqcup_\star \chi_2) \sqcap_\star \chi_2 = \chi_2$
4. $(\chi_1 \sqcup_\star \chi_2) \sqcap_\star \chi_1 = \chi_1$
5. $\chi_1 \sqcup_\star (\chi_1 \sqcap_\star \chi_2) = \chi_1$

for all $\chi_1, \chi_2, \chi_3 \in \mathcal{B}$.

Example 1. *[11] If $\mathcal{B}$ is a non-empty set, for any $\chi_1, \chi_2 \in \mathcal{B}$, define $\chi_1 \sqcap_\star \chi_2 = \chi_2, \chi_1 \sqcup_\star \chi_2 = \chi_1$, then $(\mathcal{B}, \sqcup_\star, \sqcap_\star)$ is an discrete ADL.*

Unless otherwise stated, $\mathcal{B}$ represents an almost distributive lattice $(\mathcal{B}, \sqcup_\star, \sqcap_\star)$ in this section. For any $\chi_1, \chi_2 \in \mathcal{B}$, $\chi_1 \leq_* \chi_2$ if $\chi_1 = \chi_1 \sqcap_\star \chi_2$ or equivalently $\chi_1 \sqcup_\star \chi_2 = \chi_2$, and it is noticed that $\leq_*$ is a partial order on $\mathcal{B}$.

Theorem 1. *[11] For any $\nu_1 \in S$, the following are equivalent,*

172. *ν_1 is a maximal element.*
173. *$\nu_1 \sqcup_\star \chi_1 = \nu_1$, for all $\chi_1 \in \mathcal{B}$.*
174. *$\nu_1 \sqcap_\star \chi_1 = \chi_1$, for all $\chi_1 \in \mathcal{B}$.*

Theorem 2. *[11] For any $\chi_1, \chi_2, \chi_3 \in \mathcal{B}$,*

172. $\chi_1 \sqcup_\star \chi_2 = \chi_1 \iff \chi_1 \sqcap_\star \chi_2 = \chi_1$.
173. $\chi_1 \sqcup_\star \chi_2 = \chi_2 \iff \chi_1 \sqcap_\star \chi_2 = \chi_1$.
174. $\chi_1 \sqcap_\star \chi_2 = \chi_2 \sqcap_\star \chi_1 = \chi_1$ *whenever* $\chi_1 \leq_* \chi_2$.
175. $\wedge_*$ *is associative.*
176. $\chi_1 \sqcap_\star \chi_2 \sqcap_\star \chi_3 = \chi_2 \sqcap_\star \chi_1 \sqcap_\star \chi_3$.
177. $(\chi_1 \sqcup_\star \chi_2) \sqcap_\star \chi_4 = (\chi_2 \sqcup_\star \chi_1) \wedge_* \chi_4$.
178. $\chi_1 \sqcap_\star \chi_2 \leq_* \chi_2$ *and* $\chi_1 \leq_* \chi_1 \vee_* \chi_2$.
179. $\chi_1 \sqcap_\star \chi_1 = \chi_1$ *and* $\chi_1 \sqcup_\star \chi_1 = \chi_1$.
180. *If* $\chi_1 \leq_* \chi_3$ *and* $\chi_2 \leq_* \chi_3$, *then* $\chi_1 \wedge_* \chi_2 = \chi_2 \sqcap_\star \chi_1$ *and* $\chi_1 \sqcup_\star \chi_2 = \chi_2 \sqcup_\star \chi_1$.

Theorem 3. *[11] Let $(\mathcal{B}, \sqcup_\star, \sqcap_\star, \nu_1)$ be an ADL. Then the following are equivalent;*

172. *$\mathcal{B}$ is a distributive lattice.*
173. *$(\mathcal{B}, \leq_*)$ is directed above.*
174. *$\sqcup_\star$ is commutative.*
175. *$\sqcap_\star$ is commutative.*
176. *$\sqcup_\star$ is right distributive over $\sqcap_\star$.*
177. *The relation $\theta = \{(\chi_1, \chi_2) \in \mathcal{B} \times \mathcal{B} \mid \chi_2 \sqcap_\star \chi_1 = \chi_1\}$ on $\mathcal{B}$ is antisymmetric.*

Definition 2. *[10] An algebra $(\mathcal{B}, \sqcup_\star, \sqcap_\star, \rightarrow_\star, 1)$ of type (2,2,2,0) is said to be a Brouwerian algebra, if it assures the subsequent axioms;*

172. *The system $(\mathcal{B}, \sqcup_\star, \sqcap_\star, 1)$ is a lattice with a greatest element 1.*
173. *For all $\chi_1, \chi_2, \chi_3 \in \mathcal{B}, \chi_1 \sqcap_\star \chi_3 \leq \chi_2$ if and only if $\chi_3 \leq \chi_1 \rightarrow_\star \chi_2$.*

Definition 3. *[8] $\mathcal{B}$ with a maximal element ν_1 is said to be a semi-Brouwerian almost distributive lattice (SBADL), if there is a binary operation $\rightarrow_\star$ on $\mathcal{B}$ with the subsequent axioms;*

(N_1) $(\chi_1 \rightarrow_\star \chi_1) \sqcap_\star \nu_1 = \nu_1$
(N_2) $\chi_1 \sqcap_\star (\chi_1 \rightarrow_\star \chi_2) = \chi_1 \sqcap_\star \chi_2 \sqcap_\star \nu_1$
(N_3) $\chi_1 \sqcap_\star (\chi_2 \rightarrow_\star \chi_3) = \chi_1 \sqcap_\star [(\chi_1 \sqcap_\star \chi_2) \rightarrow_\star (\chi_1 \sqcap_\star \chi_3)]$
(N_4) $(\chi_1 \rightarrow_\star \chi_2) \sqcap_\star \nu_1 = [(\chi_1 \sqcap_\star \nu_1) \rightarrow_\star (\chi_2 \sqcap_\star \nu_1)]$

for all $\chi_1, \chi_2, \chi_3 \in \mathcal{B}$.

3. Brouwerian Almost Distributive Lattices

In this section, we introduce Brouwerian almost distributive lattices and provide several counterexamples. We compare Brouwerian almost distributive lattices with semi-Brouwerian almost distributive lattices. We obtain several algebraic properties on Brouwerian almost distributive lattices. We derive some necessary and sufficient conditions for a Brouwerian almost distributive lattice to become a Brouwerian algebra.

Definition 4. *An almost distributive lattice $(\mathcal{B}, \sqcap_\star, \sqcup_\star)$ with a maximal element ν_1 is said to be a Brouwerian almost distributive lattice (abbreviated as BrADL), if there is a binary operation $\rightarrow_\star$ on $\mathcal{B}$, satisfying the following axioms;*

B_1. $(\chi_1 \rightarrow_\star \chi_1) \sqcap_\star \nu_1 = \nu_1$
B_2. $\chi_1 \sqcap_\star (\chi_1 \rightarrow_\star \chi_2) = \chi_1 \sqcap_\star \chi_2 \sqcap_\star \nu_1$
B_3. $\chi_2 \sqcap_\star (\chi_1 \rightarrow_\star \chi_2) = \chi_2 \sqcap_\star \nu_1$
B_4. $\chi_1 \rightarrow_\star (\chi_2 \sqcap_\star \chi_3) = (\chi_1 \rightarrow_\star \chi_2) \sqcap_\star (\chi_1 \rightarrow_\star \chi_3)$

for all $\chi_1, \chi_2, \chi_3 \in \mathcal{B}$.

In examples 2, 3, 4 and 5 we exhibit the independence of the axioms B_1,B_2,B_3 and B_4 of Definition 4.

Example 2. *Let $\mathcal{B} = \{1, 2, 3, 4, 5\}$ be a five-element chain with binary operation $\rightarrow_\star$ as illustrated in the following table;*

$\rightarrow_\star$	1	2	3	4	5
1	1	5	5	5	5
2	1	5	5	5	5
3	1	2	5	5	5
4	1	2	3	5	5
5	1	2	3	4	5

Clearly, $(\mathcal{B}, \sqcup_\star, \sqcap_\star)$ is an ADL with 5 as its maximal element and the binary operation $\rightarrow_\star$ satisfies the axioms B_2,B_3 and B_4 of Definition 4 but B_1 fails for the pair $(1, 1)$.

$(1 \rightarrow_\star 1) \sqcap_\star 5 = 5 \quad \Rightarrow \quad 1 \sqcap_\star 5 = 5$
$\qquad\qquad\qquad\qquad \Rightarrow \quad 1 \neq 5.$

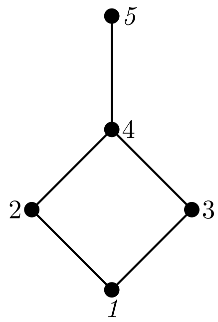
Example 3. *Let $\mathcal{B} = \{1, 2, 3, 4, 5\}$ be a five-element chain with binary operation $\rightarrow_\star$ as illustrated in the following table;*

$\rightarrow_\star$	1	2	3	4	5
1	5	5	5	5	5
2	5	5	5	5	5
3	1	2	5	5	5
4	1	2	3	5	5
5	1	2	3	4	5

Clearly, $(\mathcal{B}, \sqcup_\star, \sqcap_\star)$ is an ADL with 5 as its maximal element and the binary operation $\rightarrow_\star$ satisfies the axioms B_1,B_3 and B_4 of Definition 4 but B_2 fails for the pair $(2, 1)$.

$2 \sqcap_\star (2 \rightarrow_\star 1) = 2 \sqcap_\star 1 \sqcap_\star 5 \quad \Rightarrow \quad 2 \sqcap_\star 5 = 1 \sqcap_\star 5$
$\qquad\qquad\qquad\qquad\qquad\qquad \Rightarrow \quad 2 \neq 1.$

Example 4. *Let $\mathcal{B} = \{1, 2, 3, 4, 5\}$ be a set whose Hasse-diagram is*



with the binary operation $\rightarrow_\star$ *as illustrated in the following table;*

$\rightarrow_\star$	1	2	3	4	5
1	5	5	5	5	5
2	1	5	1	5	5
3	1	1	5	5	5
4	1	2	3	5	5
5	1	2	3	4	5

Clearly, $(\mathcal{B}, \sqcup_\star, \sqcap_\star)$ *is an ADL with* 5 *as its maximal element and the binary operation* $\rightarrow_\star$ *satisfies the axioms* B_1,B_2 *and* B_4 *of Definition 4 but* B_3 *fails for the pares* $(2, 3)$ *and* $(3, 2)$. *For the pair* $(2, 3)$

$$\begin{aligned} 3 \sqcap_\star (2 \rightarrow_\star 3) = 3 \sqcap_\star 5 \quad &\Rightarrow \quad 3 \sqcap_\star 1 = 3 \sqcap_\star 5 \\ &\Rightarrow \quad 1 \neq 3. \end{aligned}$$

Example 5. *Let* $\mathcal{B} = \{1, 2, 3, 4, 5\}$ *be a five-element chain with binary operation* $\rightarrow_\star$ *as illustrated in the following table;*

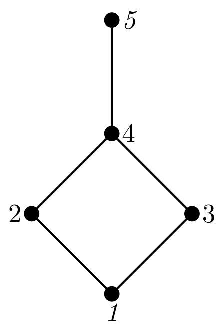
$\rightarrow_\star$	1	2	3	4	5
1	5	2	3	4	5
2	1	5	3	4	5
3	1	2	5	5	5
4	1	2	3	5	5
5	1	2	3	4	5

Clearly, $(\mathcal{B}, \sqcup_\star, \sqcap_\star)$ *is an ADL with* 5 *as its maximal element and the binary operation* $\rightarrow_\star$ *satisfies the axioms* B_1,B_2 *and* B_3 *of Definition 4 but* B_4 *fails for the triplets* $(1, 1, 2), (1, 1, 3), (1, 1, 4), (1, 2, 1), (1, 3, 1), (1, 4, 1), (2, 2, 3), (2, 2, 4), (2, 3, 2)$ *and* $(2, 4, 2)$.
For the triplet $(1, 1, 2)$

$$\begin{aligned} 1 \rightarrow_\star (1 \sqcap_\star 2) = (1 \rightarrow_\star 1) \sqcap_\star (1 \rightarrow_\star 2) \quad &\Rightarrow \quad 1 \rightarrow_\star 1 = 5 \sqcap_\star 2 \\ &\Rightarrow \quad 5 \neq 2. \end{aligned}$$

In examples 6 and 7 we define a binary operation $\rightarrow_\star$ on an ADL in such a way that it forms a Brouwerian almost distributive lattice.

Example 6. *Let* $\mathcal{B} = \{1, 2, 3, 4, 5\}$ *be a set whose Hasse-diagram is*



with the binary operation $\rightarrow_\star$ *as illustrated in the following table;*

$\rightarrow_\star$	1	2	3	4	5
1	5	5	5	5	5
2	3	5	3	5	5
3	2	2	3	5	5
4	1	2	3	5	5
5	1	2	3	4	5

Clearly, $(\mathcal{B}, \sqcup_\star, \sqcap_\star)$ *is an ADL with* 5 *as its maximal element and the binary operation* $\rightarrow_\star$ *satisfies all the axioms* B_1,B_2,B_3 *and* B_4 *of Definition 4. Therefore* $(\mathcal{B}, \sqcup_\star, \sqcap_\star, \rightarrow_\star, 5)$ *is a Brouwerian almost distributive lattice.*

Example 7. *Let* $\mathcal{B} = \{1, 2, 3, 4, 5\}$ *be a five-element chain with binary operation* $\rightarrow_\star$ *as illustrated in the following table;*

$\rightarrow_\star$	1	2	3	4	5
1	5	5	5	5	5
2	1	5	5	5	5
3	1	2	5	5	5
4	1	2	3	5	5
5	1	2	3	4	5

Clearly, $(\mathcal{B}, \sqcup_\star, \sqcap_\star)$ *is an ADL with* 5 *as its maximal element, and the binary operation* $\rightarrow_\star$ *satisfies all the axioms* B_1,B_2,B_3 *and* B_4 *of Definition 4. Therefore* $(\mathcal{B}, \sqcup_\star, \sqcap_\star, \rightarrow_\star, 5)$ *is a Brouwerian almost distributive lattice.*

In example 8 we demonstrate that the every binary operation $\rightarrow_\star$ defined on an ADL need not be a BrADL.

Example 8. *Let* $\mathcal{B} = \{1, 2, 3, 4, 5\}$ *be a five-element chain with binary operation* $\sqcup_\star, \sqcap_\star$ *and* $\rightarrow_\star$ *as illustrated in the following tables;*

$\sqcup_\star$	1	2	3	4	5
1	1	1	1	1	1
2	1	2	5	2	5
3	4	3	3	4	4
4	4	4	4	4	4
5	5	5	5	5	5

$\sqcap_\star$	1	2	3	4	5
1	1	2	3	4	5
2	2	2	3	3	2
3	2	2	3	3	2
4	1	2	3	4	5
5	1	2	3	4	5

$\rightarrow_\star$	1	2	3	4	5
1	1	2	5	2	5
2	5	5	5	5	5
3	5	2	5	2	5
4	5	1	2	5	5
5	1	2	3	4	5

Clearly, $(\mathcal{B}, \sqcup_\star, \sqcap_\star)$ *is an ADL with* 5 *as its maximal element, and the binary operation* $\rightarrow_\star$ *does not satisfy the axioms* B_1, B_2 ,B_3 *and* B_4 *of Definition 4.*
B_1 *for the pair* $(1, 2)$.
B_2 *for the pares* $(1, 3), (1, 4), (4, 1), (4, 2), (4, 3)$.
B_3 *for the pares* $(1, 4), (3, 4)$.
B_4 *for the triplets* $(3, 2, 3), (3, 2, 4), (3, 3, 1), (3, 3, 2), (3, 3, 4), (3, 3, 5), (3, 4, 1),$
$(3, 4, 3), (4, 2, 1), (4, 2, 4), (4, 2, 5), (4, 3, 1), (4, 3, 2), (4, 3, 5)$.
Therefore $(\mathcal{B}, \sqcup_\star, \sqcap_\star, \rightarrow_\star, 5)$ *is not a BrADL.*

Every Brouwerian algebra is a Brouwerian almost distributive lattice. Vice versa is not possible. For, see Example 9.

Example 9. *Let* $\mathcal{B} = \{1, 2, 3, 4, 5\}$ *be a five-element chain with binary operation* $\sqcup_\star, \sqcap_\star$ *and* $\rightarrow_\star$ *as illustrated in the following tables;*

$\sqcup_\star$	1	2	3	4	5
1	1	1	1	1	1
2	2	2	2	2	2
3	3	3	3	3	3
4	4	4	4	4	4
5	5	5	5	5	5

$\sqcap_\star$	1	2	3	4	5
1	1	2	3	4	5
2	1	2	3	4	5
3	1	2	3	4	5
4	1	2	3	4	5
5	1	2	3	4	5

$\rightarrow_\star$	1	2	3	4	5
1	5	5	5	5	5
2	1	5	3	4	5
3	5	2	5	5	5
4	1	2	3	5	5
5	1	2	3	4	5

Clearly, $(\mathcal{B}, \sqcup_\star, \sqcap_\star)$ is a discrete ADL also $(\mathcal{B}, \sqcup_\star, \sqcap_\star, \rightarrow_\star, 5)$ is a BrADL but not a BA (since it is not a lattice).

Example 10 shows that there is a binary operation $\rightarrow_\star$ on a five element chain which forms a BrADl but not a SBADL.

Example 10. *Let $\mathcal{B} = \{1, 2, 3, 4, 5\}$ be a five-element chain with binary operation $\rightarrow_\star$ as illustrated in the following table;*

$\rightarrow_\star$	1	2	3	4	5
1	5	5	5	5	5
2	1	5	5	5	5
3	1	2	5	5	5
4	1	2	5	5	5
5	1	2	3	4	5

Clearly, $(\mathcal{B}, \sqcup_\star, \sqcap_\star)$ is an ADL with 5 as its maximal element, and all the axioms B_1,B_2,B_3 and B_4 of Definition 4 are satisfied by the binary operation $\rightarrow_\star$. As a result $(\mathcal{B}, \sqcup_\star, \sqcap_\star, \rightarrow_\star, 5)$ is a BrADL. Furthermore, the triplet $(4, 5, 3)$ does not satisfy the axiom N_3 of Definition 3 when using the binary operation $\rightarrow_\star$. Therefore $(\mathcal{B}, \sqcup_\star, \sqcap_\star, \rightarrow_\star, 5)$ not a SBADL. Hence $(\mathcal{B}, \sqcup_\star, \sqcap_\star, \rightarrow_\star, 5)$ is a BrADL but not a SBADL.

Example 11 shows that there is a binary operation $\rightarrow_\star$ on a five element chain which forms a SBADL but not a BrADl.

Example 11. *Let $\mathcal{B} = \{1, 2, 3, 4, 5\}$ be a five-element chain with binary operation $\rightarrow_\star$ as illustrated in the following table;*

$\rightarrow_\star$	1	2	3	4	5
1	5	2	3	4	5
2	1	5	3	4	5
3	1	2	5	5	5
4	1	2	3	5	5
5	1	2	3	4	5

Clearly, $(\mathcal{B}, \sqcup_\star, \sqcap_\star)$ is an ADL with 5 as its maximal element and that all the axioms N_1,N_2,N_3 and N_4 of Definition 3 are satisfied by the binary operation $\rightarrow_\star$. Therefore $(\mathcal{B}, \sqcup_\star, \sqcap_\star, \rightarrow_\star, 5)$ is a SBADL. Here for the triplets $(1, 1, 2), (1, 1, 2), (1, 1, 3), (1, 1, 4), (1, 2, 1), (1, 3, 1), (1, 4, 1),$ $(2, 2, 3), (2, 2, 4), (2, 3, 2), (2, 4, 2)$, the binary operation $\rightarrow_\star$ fails to satisfy the axiom B_4 of Definition 4. Therefore $(\mathcal{B}, \sqcup_\star, \sqcap_\star, \rightarrow_\star, 5)$ is not a BrADL. Hence $(\mathcal{B}, \sqcup_\star, \sqcap_\star, \rightarrow_\star, 5)$ is not a BrADL but rather a SBADL.

Here, we derive the primary characteristics on BrADL that are crucial for advancing the theory's development. Unless otherwise stated, $\mathcal{B}$ denotes a Brouwerian almost distributive lattice $(\mathcal{B}, \sqcup_\star, \sqcap_\star, \rightarrow_\star, \nu_1)$, with ν_1 as its maximal element.

The properties that we derive in Theorem 4 and Theorem 5 plays a crucial role in developing the theory further.

Theorem 4. *For any $\chi_1, \chi_2 \in \mathcal{B}$, the following hold;*

172. $\nu_1 \rightarrow_\star \chi_1 = \chi_1 \sqcap_\star \nu_1$
173. $\chi_1 \rightarrow_\star \nu_1 = \nu_1$
174. $\chi_1 \rightarrow_\star (\chi_1 \sqcap_\star \chi_3) = \chi_1 \rightarrow_\star \chi_3$.

Proof. Let $\chi_1, \chi_2 \in \mathcal{B}$. Consider,

172. $\nu_1 \rightarrow_\star \chi_1 = \nu_1 \sqcap_\star (\nu_1 \rightarrow_\star \chi_1)$ (by 174 of Theorem 1.)
$= \nu_1 \sqcap_\star \chi_1 \sqcap_\star \nu_1$ (by B_2 of Definition 4.)
$= \chi_1 \sqcap_\star \nu_1$ (by 176 of Theorem 2.)

173. $\chi_1 \rightarrow_\star \nu_1 = \nu_1 \sqcap_\star (\chi_1 \rightarrow_\star \nu_1)$ (by 174 of Theorem 1.)
$= \nu_1 \sqcap_\star \nu_1$ (by B_3 of Definition 4.)
$= \nu_1$

174. $\chi_1 \rightarrow_\star (\chi_1 \sqcap_\star \chi_3) = (\chi_1 \rightarrow_\star \chi_1) \sqcap_\star (\chi_1 \rightarrow_\star \chi_3)$ (by B_4 of Definition 4.)
$= (\chi_1 \rightarrow_\star \chi_1) \sqcap_\star \nu_1 \sqcap_\star (\chi_1 \rightarrow_\star \chi_3)$
$= \nu_1 \sqcap_\star (\chi_1 \rightarrow_\star \chi_3)$ (by B_1 of Definition 4.)
$= \chi_1 \rightarrow_\star \chi_3$.

□

Theorem 5. *If $\chi_1 \leq \chi_2$ in $\mathcal{B}$ and $\chi_1, \chi_2, \chi_3 \in \mathcal{B}$, then the following holds;*

172. $\chi_3 \rightarrow_\star \chi_1 \leq \chi_3 \rightarrow_\star \chi_2$
173. $(\chi_1 \rightarrow_\star \chi_2) \sqcap_\star \nu_1 = \nu_1$

Proof. Let $\chi_1, \chi_2, \chi_3 \in \mathcal{B}$. Consider,

172. $(\chi_3 \rightarrow_\star \chi_1) \sqcap_\star (\chi_3 \rightarrow_\star \chi_2) = \chi_3 \rightarrow_\star (\chi_1 \sqcap_\star \chi_2)$ (by B_4 of Definition 4.)
$= \chi_3 \rightarrow_\star \chi_1$ (since $\chi_1 \leq \chi_2$).

Therefore $\chi_3 \rightarrow_\star \chi_1 \leq \chi_3 \rightarrow_\star \chi_2$.

173. $\chi_1 \leq \chi_2 \Rightarrow \chi_1 \rightarrow_\star \chi_1 \leq \chi_1 \rightarrow_\star \chi_2$ (by 172)
$\Rightarrow (\chi_1 \rightarrow_\star \chi_1) \sqcap_\star \nu_1 \leq (\chi_1 \rightarrow_\star \chi_2) \sqcap_\star \nu_1$
$\Rightarrow \nu_1 \leq (\chi_1 \rightarrow_\star \chi_2) \sqcap_\star \nu_1$ (by B_1 of Definition 4.)
$\Rightarrow \nu_1 \leq (\chi_1 \rightarrow_\star \chi_2) \sqcap_\star \nu_1 \leq \nu_1$
$\Rightarrow (\chi_1 \rightarrow_\star \chi_2) \sqcap_\star \nu_1 = \nu_1$

□

Corollary 3.15, is the consequence of B_2 and B_3 of Definition 4.

Corollary 1. *For any $\chi_1, \chi_2 \in \mathcal{B}$, the following holds;*

172. $\chi_1 \sqcap_\star \chi_2 \sqcap_\star \nu_1 \leq (\chi_1 \rightarrow_\star \chi_2) \sqcap_\star \nu_1$
173. $\chi_2 \sqcap_\star \nu_1 \leq (\chi_1 \rightarrow_\star \chi_2) \sqcap_\star \nu_1$.

Theorem 6. *For any $\chi_1, \chi_2, \chi_3 \in \mathcal{B}$, $\chi_1 \sqcap_\star \chi_3 \sqcap_\star \nu_1 \leq \chi_2 \sqcap_\star \nu_1$ if and only if $\chi_3 \sqcap_\star \nu_1 \leq (\chi_1 \rightarrow_\star \chi_2) \sqcap_\star \nu_1$.*

Proof. Let $\chi_1, \chi_2, \chi_3 \in \mathcal{B}$. Then, $\chi_1 \sqcap_\star \chi_3 \sqcap_\star \nu_1 \leq \chi_2 \sqcap_\star \nu_1$
$\Rightarrow \chi_1 \rightarrow_\star (\chi_1 \sqcap_\star \chi_3 \sqcap_\star \nu_1) \leq \chi_1 \rightarrow_\star (\chi_2 \sqcap_\star \nu_1)$
(by 172 of Theorem 5.)
$\Rightarrow (\chi_1 \rightarrow_\star \chi_1) \sqcap_\star (\chi_1 \rightarrow_\star \chi_3) \sqcap_\star (\chi_1 \rightarrow_\star \nu_1) \leq (\chi_1 \rightarrow_\star \chi_2) \sqcap_\star (\chi_1 \rightarrow_\star \nu_1)$
(by B_4 of Definition 4.)
$\Rightarrow (\chi_1 \rightarrow_\star \chi_1) \sqcap_\star \nu_1 \sqcap_\star (\chi_1 \rightarrow_\star \chi_3) \sqcap_\star \nu_1 \leq (\chi_1 \rightarrow_\star \chi_2) \sqcap_\star \nu_1$
(by 173 of Theorem 4.)
$\Rightarrow \nu_1 \sqcap_\star (\chi_1 \rightarrow_\star \chi_3) \sqcap_\star \nu_1 \leq (\chi_1 \rightarrow_\star \chi_2) \sqcap_\star \nu_1$
(by B_1 of Definition 4.)
$\Rightarrow (\chi_1 \rightarrow_\star \chi_3) \sqcap_\star \nu_1 \leq (\chi_1 \rightarrow_\star \chi_2) \sqcap_\star \nu_1$
$\Rightarrow \chi_3 \sqcap_\star \nu_1 \leq (\chi_1 \rightarrow_\star \chi_3) \sqcap_\star \nu_1 \leq (\chi_1 \rightarrow_\star \chi_2) \sqcap_\star \nu_1$
(by 173 of Corollary 1.)
$\Rightarrow \chi_3 \sqcap_\star \nu_1 \leq (\chi_1 \rightarrow_\star \chi_2) \sqcap_\star \nu_1$.

On the other hand,

$\chi_3 \sqcap_\star \nu_1 \leq (\chi_1 \rightarrow_\star \chi_2) \sqcap_\star \nu_1 \quad \Rightarrow \chi_1 \sqcap_\star \chi_3 \sqcap_\star \nu_1 \leq \chi_1 \sqcap_\star (\chi_1 \rightarrow_\star \chi_2) \sqcap_\star \nu_1$
$\Rightarrow \chi_1 \sqcap_\star \chi_3 \sqcap_\star \nu_1 \leq \chi_1 \sqcap_\star \chi_2 \sqcap_\star \nu_1$
(by B_2 of Definition 4.)
$\Rightarrow \chi_1 \sqcap_\star \chi_3 \sqcap_\star \nu_1 \leq \chi_2 \sqcap_\star \nu_1$.

□

Theorem 7. *For any* $\chi_1, \chi_2 \in \mathcal{B}$ *the following hold,*
$\chi_1 \sqcap_\star \nu_1 \leq [(\chi_1 \rightarrow_\star \chi_2) \rightarrow_\star \chi_2] \sqcap_\star \nu_1$.

Proof. Let $\chi_1, \chi_2 \in \mathcal{B}$. Then, by B_2 of Definition 4.
$\chi_1 \sqcap_\star (\chi_1 \rightarrow_\star \chi_2) = \chi_1 \sqcap_\star \chi_2 \sqcap_\star \nu_1$
$\Rightarrow \chi_1 \sqcap_\star (\chi_1 \rightarrow_\star \chi_2) \sqcap_\star \nu_1 = \chi_1 \sqcap_\star \chi_2 \sqcap_\star \nu_1 \leq \chi_2 \sqcap_\star \nu_1$
$\Rightarrow (\chi_1 \rightarrow_\star \chi_2) \sqcap_\star \chi_1 \sqcap_\star \nu_1 \leq \chi_2 \sqcap_\star \nu_1$ (by 176 of Theorem 2.)
$\Rightarrow \chi_1 \sqcap_\star \nu_1 \leq [(\chi_1 \rightarrow_\star \chi_2) \rightarrow_\star \chi_2] \sqcap_\star \nu_1$ (by Theorem 6.).

□

Theorem 8. *For any* $\chi_1, \chi_2, \chi_3 \in \mathcal{B}$*, the following holds;*

172. $\chi_1 \sqcap_\star \nu_1 \leq \chi_2 \sqcap_\star \nu_1$ *if and only if* $(\chi_1 \rightarrow_\star \chi_2) \sqcap_\star \nu_1 = \nu_1$.
173. $\chi_1 \sqcap_\star \nu_1 \leq (\chi_2 \rightarrow_\star \chi_3) \sqcap_\star \nu_1$ *if and only if* $\chi_2 \sqcap_\star \nu_1 \leq (\chi_1 \rightarrow_\star \chi_3) \sqcap_\star \nu_1$
174. $\chi_1 \sqcap_\star \chi_2 \sqcap_\star \nu_1 = \chi_1 \sqcap_\star \chi_3 \sqcap_\star \nu_1$ *if and only if* $(\chi_1 \rightarrow_\star \chi_2) \sqcap_\star \nu_1 = (\chi_1 \rightarrow_\star \chi_3) \sqcap_\star \nu_1$

Proof. Let $\chi_1, \chi_2, \nu_1 \in \mathcal{B}$. Consider,
172. $\chi_1 \sqcap_\star \nu_1 \leq \chi_2 \sqcap_\star \nu_1$
$\Rightarrow \chi_1 \rightarrow_\star (\chi_1 \sqcap_\star \nu_1) \leq \chi_1 \rightarrow_\star (\chi_2 \sqcap_\star \nu_1)$
(by 172 of Theorem 5.)
$\Rightarrow (\chi_1 \rightarrow_\star \chi_1) \sqcap_\star (\chi_1 \rightarrow_\star \nu_1) \leq (\chi_1 \rightarrow_\star \chi_2) \sqcap_\star (\chi_1 \rightarrow_\star \nu_1)$
(by B_4 of Definition 4.)
$\Rightarrow (\chi_1 \rightarrow_\star \chi_1) \sqcap_\star \nu_1 \leq (\chi_1 \rightarrow_\star \chi_2) \sqcap_\star \nu_1$
(by 173 of Theorem 4.)
$\Rightarrow \nu_1 \leq (\chi_1 \rightarrow_\star \chi_2) \sqcap_\star \nu_1 \leq \nu_1$
(by B_1 of Definition 4.)
$\Rightarrow (\chi_1 \rightarrow_\star \chi_2) \sqcap_\star \nu_1 = \nu_1$
On the other hand,
$(\chi_1 \rightarrow_\star \chi_2) \sqcap_\star \nu_1 = \nu_1 \quad \Rightarrow \chi_1 \sqcap_\star (\chi_1 \rightarrow_\star \chi_2) \sqcap_\star \nu_1 = \chi_1 \sqcap_\star \nu_1$
$\Rightarrow \chi_1 \sqcap_\star \chi_2 \sqcap_\star \nu_1 \sqcap_\star \nu_1 = \chi_1 \sqcap_\star \nu_1$
(by B_2 of Definition 4.)
$\Rightarrow \chi_1 \sqcap_\star \nu_1 \sqcap_\star \chi_2 \sqcap_\star \nu_1 = \chi_1 \sqcap_\star \nu_1$
(by 176 of Theorem 2.)
Therefore $\chi_1 \sqcap_\star \nu_1 \leq \chi_2 \sqcap_\star \nu_1$.
173. $\chi_1 \sqcap_\star \nu_1 \leq (\chi_2 \rightarrow_\star \chi_3) \sqcap_\star \nu_1$
$\Rightarrow \chi_2 \sqcap_\star \chi_1 \sqcap_\star \nu_1 \leq \chi_2 \sqcap_\star (\chi_2 \rightarrow_\star \chi_3) \sqcap_\star \nu_1$
$\Rightarrow \chi_2 \sqcap_\star \chi_1 \sqcap_\star \nu_1 \leq \chi_2 \sqcap_\star \chi_3 \sqcap_\star \nu_1$
(by B_2 of Definition 4.)
$\Rightarrow \chi_1 \sqcap_\star \chi_2 \sqcap_\star \nu_1 \leq \chi_2 \sqcap_\star \chi_3 \sqcap_\star \nu_1 \leq \chi_3 \sqcap_\star \nu_1$
(by 176 of Theorem 2.)
$\Rightarrow \chi_1 \sqcap_\star \chi_2 \sqcap_\star \nu_1 \leq \chi_3 \sqcap_\star \nu_1$
$\Rightarrow \chi_2 \sqcap_\star \nu_1 \leq (\chi_1 \rightarrow_\star \chi_3) \sqcap_\star \nu_1$
(by Theorem 6.)
On the other hand,
$\chi_2 \sqcap_\star \nu_1 \leq (\chi_1 \rightarrow_\star \chi_3) \sqcap_\star \nu_1$
$\Rightarrow \chi_1 \sqcap_\star \chi_2 \sqcap_\star \nu_1 \leq \chi_1 \sqcap_\star (\chi_1 \rightarrow_\star \chi_3) \sqcap_\star \nu_1$
$\Rightarrow \chi_1 \sqcap_\star \chi_2 \sqcap_\star \nu_1 \leq \chi_1 \sqcap_\star \chi_3 \sqcap_\star \nu_1$
(by B_2 of Definition 4.)
$\Rightarrow \chi_2 \sqcap_\star \chi_1 \sqcap_\star \nu_1 \leq \chi_1 \sqcap_\star \chi_3 \sqcap_\star \nu_1 \leq \chi_3 \sqcap_\star \nu_1$
(by 176 of Theorem 2.)
$\Rightarrow \chi_1 \sqcap_\star \nu_1 \leq (\chi_2 \rightarrow_\star \chi_3) \sqcap_\star \nu_1$
(by Theorem 6.)

174. $\chi_1 \sqcap_\star \chi_2 \sqcap_\star \nu_1 = \chi_1 \sqcap_\star \chi_3 \sqcap_\star \nu_1$
$\Rightarrow \chi_1 \rightarrow_\star (\chi_1 \sqcap_\star \chi_2 \sqcap_\star \nu_1) = \chi_1 \rightarrow_\star (\chi_1 \sqcap_\star \chi_3 \sqcap_\star \nu_1)$
(by 172 of Theorem 5.)
$\Rightarrow (\chi_1 \rightarrow_\star \chi_1) \sqcap_\star (\chi_1 \rightarrow_\star \chi_2) \sqcap_\star (\chi_1 \rightarrow_\star \nu_1)$
$= (\chi_1 \rightarrow_\star \chi_1) \sqcap_\star (\chi_1 \rightarrow_\star \chi_3) \sqcap_\star (\chi_1 \rightarrow_\star \nu_1)$
(by B_4 of Definition 4.)
$\Rightarrow (\chi_1 \rightarrow_\star \chi_1) \sqcap_\star (\chi_1 \rightarrow_\star \chi_2) \sqcap_\star \nu_1 = (\chi_1 \rightarrow_\star \chi_1) \sqcap_\star (\chi_1 \rightarrow_\star \chi_3) \sqcap_\star \nu_1$
(by 173 of Theorem 4.)
$\Rightarrow (\chi_1 \rightarrow_\star \chi_2) \sqcap_\star (\chi_1 \rightarrow_\star \chi_1) \sqcap_\star \nu_1 = (\chi_1 \rightarrow_\star \chi_3) \sqcap_\star (\chi_1 \rightarrow_\star \chi_1) \sqcap_\star \nu_1$
(by 176 of Theorem 2.)
$\Rightarrow (\chi_1 \rightarrow_\star \chi_2) \sqcap_\star \nu_1 = (\chi_1 \rightarrow_\star \chi_3) \sqcap_\star \nu_1$
(by B_1 of Definition 4.)
On the other hand, $(\chi_1 \rightarrow_\star \chi_2) \sqcap_\star \nu_1 = (\chi_1 \rightarrow_\star \chi_3) \sqcap_\star \nu_1$
$\Rightarrow \chi_1 \sqcap_\star (\chi_1 \rightarrow_\star \chi_2) \sqcap_\star \nu_1 = \chi_1 \sqcap_\star (\chi_1 \rightarrow_\star \chi_3) \sqcap_\star \nu_1$
$\Rightarrow \chi_1 \sqcap_\star \chi_2 \sqcap_\star \nu_1 = \chi_1 \sqcap_\star \chi_3 \sqcap_\star \nu_1$ (by B_2 of Definition 4.)

□

Theorem 9. *For any $\chi_1, \chi_2, \chi_3 \in \mathcal{B}$, the following holds;*

172. $[\chi_1 \rightarrow_\star (\chi_2 \rightarrow_\star \chi_3)] \sqcap_\star \nu_1 = [(\chi_1 \sqcap_\star \chi_2) \rightarrow_\star \chi_3] \sqcap_\star \nu_1$
173. $[(\chi_1 \sqcap_\star \chi_2) \rightarrow_\star \chi_3] \sqcap_\star \nu_1 = [(\chi_2 \sqcap_\star \chi_1) \rightarrow_\star \chi_3] \sqcap_\star \nu_1$
174. $[\chi_1 \rightarrow_\star (\chi_2 \rightarrow_\star \chi_3)] \sqcap_\star \nu_1 = [\chi_2 \rightarrow_\star (\chi_1 \rightarrow_\star \chi_3)] \sqcap_\star \nu_1$

Proof. Let $\chi_1, \chi_2, \chi_3 \in \mathcal{B}$.
172. $(\chi_1 \sqcap_\star \chi_2) \sqcap_\star [(\chi_1 \sqcap_\star \chi_2) \rightarrow_\star \chi_3) \sqcap_\star \nu_1$
$= \chi_1 \sqcap_\star \chi_2 \sqcap_\star \chi_3 \sqcap_\star \nu_1$ (by B_2 of Definition 4.)
$\Rightarrow \chi_2 \rightarrow_\star [\chi_1 \sqcap_\star \chi_2 \sqcap_\star [(\chi_1 \sqcap_\star \chi_2) \rightarrow_\star \chi_3] \sqcap_\star \nu_1]$
$= \chi_2 \rightarrow_\star (\chi_1 \sqcap_\star \chi_2 \sqcap_\star \chi_3 \sqcap_\star \nu_1)$ (by 172 of Theorem 5.)
$\Rightarrow \chi_2 \rightarrow_\star [\chi_2 \sqcap_\star [\chi_1 \sqcap_\star (\chi_1 \sqcap_\star \chi_2) \rightarrow_\star \chi_3) \sqcap_\star \nu_1]]$
$= \chi_2 \rightarrow_\star (\chi_2 \sqcap_\star \chi_1 \sqcap_\star \chi_3 \sqcap_\star \nu_1)$ (by 176 of Theorem 2.)
$\Rightarrow \chi_2 \rightarrow_\star [\chi_1 \sqcap_\star (\chi_1 \sqcap_\star \chi_2) \rightarrow_\star \chi_3) \sqcap_\star \nu_1]$
$= \chi_2 \rightarrow_\star (\chi_1 \sqcap_\star \chi_3 \sqcap_\star \nu_1) - - - -(I)$
(by 174 of Theorem 4.)
Consider,
$[\chi_1 \sqcap_\star (\chi_1 \sqcap_\star \chi_2) \rightarrow_\star \chi_3) \sqcap_\star \nu_1] \sqcap_\star [\chi_2 \rightarrow_\star (\chi_1 \sqcap_\star \chi_3 \sqcap_\star \nu_1)] \sqcap_\star \nu_1$
$= [\chi_1 \sqcap_\star (\chi_1 \sqcap_\star \chi_2) \rightarrow_\star \chi_3) \sqcap_\star \nu_1] \sqcap_\star [\chi_2 \rightarrow_\star [\chi_1 \sqcap_\star (\chi_1 \sqcap_\star \chi_2) \rightarrow_\star \chi_3) \sqcap_\star \nu_1]] \sqcap_\star \nu_1$
(by I)
$= [\chi_1 \sqcap_\star (\chi_1 \sqcap_\star \chi_2) \rightarrow_\star \chi_3) \sqcap_\star \nu_1] \sqcap_\star \nu_1.$ (by B_3 of Definition 4.)
Therefore $\chi_1 \sqcap_\star (\chi_1 \sqcap_\star \chi_2) \rightarrow_\star \chi_3) \sqcap_\star \nu_1 \leq [\chi_2 \rightarrow_\star (\chi_1 \sqcap_\star \chi_3 \sqcap_\star \nu_1)] \sqcap_\star \nu_1$.
Hence by B_4 of Definition 4,
$\chi_1 \sqcap_\star (\chi_1 \sqcap_\star \chi_2) \rightarrow_\star \chi_3) \sqcap_\star \nu_1 \leq (\chi_2 \rightarrow_\star \chi_3) \sqcap_\star \nu_1$.
Thus from Theorem 6,
$(\chi_1 \sqcap_\star \chi_2) \rightarrow_\star \chi_3) \sqcap_\star \nu_1 \leq [\chi_1 \rightarrow_\star (\chi_2 \rightarrow_\star \chi_3)] \sqcap_\star \nu_1$.
On the other hand,
$\chi_1 \sqcap_\star \chi_2 \sqcap_\star [\chi_1 \rightarrow_\star (\chi_2 \rightarrow_\star \chi_3)] \sqcap_\star \nu_1$
$= \chi_2 \sqcap_\star \chi_1 \sqcap_\star [\chi_1 \rightarrow_\star (\chi_2 \rightarrow_\star \chi_3)] \sqcap_\star \nu_1$
(by 176 of Theorem 2.)
$= \chi_2 \sqcap_\star \chi_1 \sqcap_\star (\chi_2 \rightarrow_\star \chi_3) \sqcap_\star \nu_1$
(by B_2 of Definition 4.)
$= \chi_1 \sqcap_\star \chi_2 \sqcap_\star (\chi_2 \rightarrow_\star \chi_3) \sqcap_\star \nu_1$
(by 176 of Theorem 2.)
$= \chi_1 \sqcap_\star \chi_2 \sqcap_\star \chi_3 \sqcap_\star \nu_1$
(by B_2 of Definition 4.).
Therefore $\chi_1 \sqcap_\star \chi_2 \sqcap_\star [\chi_1 \rightarrow_\star (\chi_2 \rightarrow_\star \chi_3)] \sqcap_\star \nu_1 = \chi_1 \sqcap_\star \chi_2 \sqcap_\star \chi_3 \sqcap_\star \nu_1$.

Now, $(\chi_1 \sqcap_\star \chi_2) \sqcap_\star [\chi_1 \to_\star (\chi_2 \to_\star \chi_3)] \sqcap_\star \nu_1 = \chi_1 \sqcap_\star \chi_2 \sqcap_\star \chi_3 \sqcap_\star \nu_1$
$\Rightarrow (\chi_1 \sqcap_\star \chi_2) \to_\star [(\chi_1 \sqcap_\star \chi_2) \sqcap_\star [\chi_1 \to_\star (\chi_2 \to_\star \chi_3)] \sqcap_\star \nu_1]$
$= (\chi_1 \sqcap_\star \chi_2) \to_\star [\chi_1 \sqcap_\star \chi_2 \sqcap_\star \chi_3 \sqcap_\star \nu_1]$
$\Rightarrow (\chi_1 \sqcap_\star \chi_2) \to_\star [(\chi_1 \sqcap_\star \chi_2) \sqcap_\star [\chi_1 \to_\star (\chi_2 \to_\star \chi_3)] \sqcap_\star \nu_1]$
$= (\chi_1 \sqcap_\star \chi_2) \to_\star (\chi_3 \sqcap_\star \nu_1)$
(by 174 of Theorem 4.)
$\Rightarrow [(\chi_1 \sqcap_\star \chi_2) \to_\star (\chi_1 \sqcap_\star \chi_2)] \sqcap_\star [(\chi_1 \sqcap_\star \chi_2) \to_\star (\chi_1 \to_\star (\chi_2 \to_\star \chi_3))] \sqcap_\star [(\chi_1 \sqcap_\star \chi_2) \to_\star \nu_1]$
$= [(\chi_1 \sqcap_\star \chi_2) \to_\star \chi_3] \sqcap_\star [(\chi_1 \sqcap_\star \chi_2) \to_\star \nu_1]$
(by B_4 of Definition 4.)
$\Rightarrow [(\chi_1 \sqcap_\star \chi_2) \to_\star (\chi_1 \sqcap_\star \chi_2)] \sqcap_\star [(\chi_1 \sqcap_\star \chi_2) \to_\star (\chi_1 \to_\star (\chi_2 \to_\star \chi_3))] \sqcap_\star \nu_1$
$= [(\chi_1 \sqcap_\star \chi_2) \to_\star \chi_3] \sqcap_\star \nu_1$
(by 174 of Theorem 4.)
$\Rightarrow [(\chi_1 \sqcap_\star \chi_2) \to_\star (\chi_1 \to_\star (\chi_2 \to_\star \chi_3))] \sqcap_\star \nu_1$
$= [(\chi_1 \sqcap_\star \chi_2) \to_\star \chi_3] \sqcap_\star \nu_1$
(by B_1 of Definition 4.)
$\Rightarrow [(\chi_1 \to_\star (\chi_2 \to_\star \chi_3))] \sqcap_\star \nu_1 \le [(\chi_1 \sqcap_\star \chi_2) \to_\star \chi_3] \sqcap_\star \nu_1$
(by 173 of Corollary 1.)
Therefore $[\chi_1 \to_\star (\chi_2 \to_\star \chi_3)] \sqcap_\star \nu_1 = [(\chi_1 \sqcap_\star \chi_2) \to_\star \chi_3] \sqcap_\star \nu_1$.
173. $(\chi_1 \sqcap_\star \chi_2) \sqcap_\star [(\chi_1 \sqcap_\star \chi_2) \to_\star \chi_3] \sqcap_\star \nu_1 = \chi_1 \sqcap_\star \chi_2 \sqcap_\star \chi_3 \sqcap_\star \nu_1$
(by B_2 of Definition 4.)
$\Rightarrow (\chi_2 \sqcap_\star \chi_1) \sqcap_\star [(\chi_1 \sqcap_\star \chi_2) \to_\star \chi_3] \sqcap_\star \nu_1 \le \chi_3 \sqcap_\star \nu_1$
(by 176 of Theorem 2.)
$\Rightarrow [(\chi_1 \sqcap_\star \chi_2) \to_\star \chi_3] \sqcap_\star \nu_1 \le [(\chi_2 \sqcap_\star \chi_1) \to_\star \chi_3] \sqcap_\star \nu_1$
(by Theorem 6.).
By symmetry, $[(\chi_1 \sqcap_\star \chi_2) \to_\star \chi_3] \sqcap_\star \nu_1 = [(\chi_2 \sqcap_\star \chi_1) \to_\star \chi_3] \sqcap_\star \nu_1$
174. Follows from 172 and 173. □

Theorem 10. *For any* $\chi_1, \chi_2, \chi_3 \in \mathcal{B}$, $(\chi_1 \to_\star \chi_2) \sqcap_\star \chi_3 =$
$[(\chi_1 \sqcap_\star \chi_3) \to_\star (\chi_2 \sqcap_\star \chi_3)] \sqcap_\star \chi_3$

Proof. Let $\chi_1, \chi_2, \chi_3 \in \mathcal{B}$. Then
$[(\chi_1 \sqcap_\star \chi_3) \to_\star (\chi_2 \sqcap_\star \chi_3)] \sqcap_\star \chi_3$
$= [((\chi_1 \sqcap_\star \chi_3) \to_\star \chi_2) \sqcap_\star ((\chi_1 \sqcap_\star \chi_3) \to_\star \chi_3)] \sqcap_\star \chi_3 \sqcap_\star \chi_3$
(by B_4 of Definition 4.)
$= [((\chi_1 \sqcap_\star \chi_3) \to_\star \chi_2) \sqcap_\star \chi_3 \sqcap_\star ((\chi_1 \sqcap_\star \chi_3) \to_\star \chi_3)] \sqcap_\star \chi_3$
(by 176 of Theorem 2.)
$= [((\chi_1 \sqcap_\star \chi_3) \to_\star \chi_2) \sqcap_\star \nu_1 \sqcap_\star \chi_3$
(by B_3 of Definition 4.)
$= [((\chi_3 \sqcap_\star \chi_1) \to_\star \chi_2) \sqcap_\star \nu_1 \sqcap_\star \chi_3$
(by 173 of Theorem 9.)
$= [\chi_3 \to_\star (\chi_1 \to_\star \chi_2)] \sqcap_\star \nu_1 \sqcap_\star \chi_3$
(by 172 of Theorem 9.)
$= \chi_3 \sqcap_\star [\chi_3 \to_\star (\chi_1 \to_\star \chi_2)] \sqcap_\star \nu_1 \sqcap_\star \chi_3$
(by 176 of Theorem 2.)
$= \chi_3 \sqcap_\star (\chi_1 \to_\star \chi_2) \sqcap_\star \nu_1 \sqcap_\star \chi_3$
(by B_2 of Definition 4.)
$= (\chi_1 \to_\star \chi_2) \sqcap_\star \chi_3$
(by 174 of Theorem 1 and 176 of Theorem 2.)
□

Corollary 2. *For any* $\chi_1, \chi_2, \chi_3 \in \mathcal{B}$, *if* $\chi_1 \sqcap_\star \chi_3 = \chi_2 \sqcap_\star \chi_3$, *then*

172. $(\chi_1 \to_\star \chi_3) \sqcap_\star \chi_3 = (\chi_2 \to_\star \chi_3) \sqcap_\star \chi_3$
173. $(\chi_3 \to_\star \chi_1) \sqcap_\star \chi_3 = (\chi_3 \to_\star \chi_2) \sqcap_\star \chi_3$.

Theorem 11. *If* $(\mathcal{B}, \sqcup_\star, \sqcap_\star, \to_\star, \nu_1)$ *is a BrADL, then for any maximal element* ν_2 *in* $\mathcal{B}$, $(\mathcal{B}, \sqcup_\star, \sqcap_\star, \to_{\nu_2}, \nu_2)$ *is a BrADL where* $\chi_1 \to_{\nu_2} \chi_2 = (\chi_1 \to_\star \chi_2) \sqcap_\star \nu_2$ *for* $\chi_1, \chi_2 \in \mathcal{B}$.

Proof. Let $(\mathcal{B}, \sqcup_\star, \sqcap_\star, \rightarrow_\star, \nu_1)$ be a BrADL and $\rightarrow_{\nu_2}$ is a maximal element in $\mathcal{B}$. For $\chi_1, \chi_2 \in \mathcal{B}$, define $\chi_1 \rightarrow_{\nu_2} \chi_2 = (\chi_1 \rightarrow_\star \chi_2) \wedge \nu_2$.
Then, for any $\chi_1, \chi_2 \in \mathcal{B}$,

172. $$\begin{aligned}(\chi_1 \rightarrow_{\nu_2} \chi_1) \sqcap_\star \nu_2 &= (\chi_1 \rightarrow_\star \chi_1) \sqcap_\star \nu_2 \sqcap_\star \nu_2 \\ &= \nu_1 \sqcap_\star \nu_2 \\ &= \nu_2.\end{aligned}$$

173. $$\begin{aligned}\chi_1 \sqcap_\star (\chi_1 \rightarrow_{\nu_2} \chi_2) &= \chi_1 \sqcap_\star (\chi_1 \rightarrow_\star \chi_2) \sqcap_\star \nu_2 \\ &= \chi_1 \sqcap_\star \chi_2 \sqcap_\star \nu_1 \sqcap_\star \nu_2 \\ &= \chi_1 \sqcap_\star \chi_2 \sqcap_\star \nu_2.\end{aligned}$$

174. $$\begin{aligned}\chi_2 \sqcap_\star (\chi_1 \rightarrow_{\nu_2} \chi_2) &= \chi_2 \sqcap_\star (\chi_1 \rightarrow_\star \chi_2) \sqcap_\star \nu_2 \\ &= \chi_2 \sqcap_\star \nu_1 \sqcap_\star \nu_2 \\ &= \chi_2 \sqcap_\star \nu_2.\end{aligned}$$

175. $$\begin{aligned}\chi_1 \rightarrow_{\nu_2} (\chi_2 \sqcap_\star \chi_3) &= [\chi_1 \rightarrow_\star (\chi_2 \sqcap_\star \chi_3)] \sqcap_\star \nu_2 \\ &= [(\chi_1 \rightarrow_\star \chi_2) \sqcap_\star (\chi_1 \rightarrow_\star \chi_3)] \sqcap_\star \nu_2 \\ &= [(\chi_1 \rightarrow_\star \chi_2) \sqcap_\star \nu_2] \sqcap_\star [(\chi_1 \rightarrow_\star \chi_3)] \sqcap_\star \nu_2] \\ &= (\chi_1 \rightarrow_{\nu_2} \chi_2) \sqcap_\star (\chi_1 \rightarrow_{\nu_2} \chi_3).\end{aligned}$$

Therefore $(\mathcal{B}, \sqcup_\star, \sqcap_\star, \rightarrow_{\nu_2}, \nu_2)$ is a BrADL. □

We give several equivalent conditions for a BrADL to become a Brouwerian algebra as we wrap up the paper.

Theorem 12. *[11] Let $(\mathcal{B}, \sqcup_\star, \sqcap_\star, \nu_1)$ be an ADL. Then the subsequent statements are comparable;*

172. *$\mathcal{B}$ is a Brouwerian algebra.*
173. *$(\mathcal{B}, \leq_*)$ is directed above.*
174. *$(\mathcal{B}, \sqcup_\star, \sqcap_\star)$ is a distributive lattice.*
175. *$\sqcup_\star$ is commutative.*
176. *$\sqcap_\star$ is commutative.*
177. *$\sqcup_\star$ is right distributive over $\sqcap_\star$.*
178. *The relation $\theta = \{(\chi_1, \chi_2) \in \mathcal{B} \times \mathcal{B} \mid \chi_2 \sqcap_\star \chi_1 = \chi_1\}$ is antisymmetric.*

4. Conclusion

The concept of a Brouwerian almost distributive lattice is presented in this paper with several examples and counter-examples, and some of its primary and necessary properties are studied. We derive a few identities and inequalities in a Brouwerian almost distributive lattice. Also, we provided a set of equivalence conditions required for transforming the Brouwerian almost distributive lattice into a Brouwerian algebra.

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Direction curves and construction of developable surfaces in Lorentz 3-space

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Abstract. In this work we investigate singularities for the three types of developable surfaces, introduced by Izumiya and Takeuchi, in Lorentz 3 space and give a local classification in terms of k-order frame. Moreover we search the necessary conditions of being a geodesic for principal direction curves of the rectifying developable surface.

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1. Introduction

A developable surface, useful application tool in cartographic projections and producing flat materials, is defined as a surface that can be developed into flat surfaces without changing the metric on the surface. There are several papers about developable surfaces and some of them are combined with the singularity theory. Zhao et al. investigated the geometric characteristics of developable surfaces with a single parameter that have regular curves [12]. The global behavior of singularities on flat surfaces in Euclidean 3-space is studied by Murata and Umehara [9]. Furthermore, the primary source of inspiration for this paper is the research on developable surface singularities in Euclidean 3-space presented by Izumiya and Takeuchi. They considered three types of developable surfaces named as rectifying developable of a curve, defined as the envelope of the set of rectifying planes along the curve, the second one called Darboux developable which has singularities at the terminal points of the curve's modified Darboux vectors and the third one is tangential Darboux developable which is determined by the space curve's tangent indicatrix's Darboux developable surface. They have shown that these developable surfaces are locally diffeomorphic to the swallowtail, the cuspidal edge or cuspidal cross cap [6,7].
Moreover Ishikawa and Yamashita provide a comprehensive response to the question of local diffeomorphism categorization in Euclidean 3-space and they give the following theorem;

Theorem 1. *Let ∇ be a torsion free affine connection on a manifold M. Let $\beta : I \longrightarrow M$ be a C^∞ curve from an open interval I.Let $\dim(M) = 3$.*

1) *If $(\nabla_\beta)(s_0)$,$(\nabla^2_\beta)(s_0)$,$(\nabla^3_\beta)(s_0)$ are linearly independent, then the $\nabla-$tangent surface is locally diffeomorphic to the cuspidal edge at $(s_0, 0)$.*
2) *If $(\nabla_\beta)(s_0)$,$(\nabla^2_\beta)(s_0)$,$(\nabla^3_\beta)(s_0)$ are linearly dependent and $(\nabla_\beta)(s_0)$,$(\nabla^2_\beta)(s_0)$,$(\nabla^4_\beta)(s_0)$ are linearly independent then $\nabla-$tangent surface is locally diffeomorphic to the cuspidal crosscap at $(s_0, 0)$.*
3) *If $(\nabla_\beta)(s_0) = 0$ and $(\nabla^2_\beta)(s_0)$,$(\nabla^3_\beta)(s_0)$,$(\nabla^4_\beta)(s_0)$ are linearly independent then $\nabla-$tangent surface is locally diffeomorphic to the swallowtail at $(s_0, 0)$ [5].*

The singular points that are mentioned above theorem can be examined in the following figures [6].

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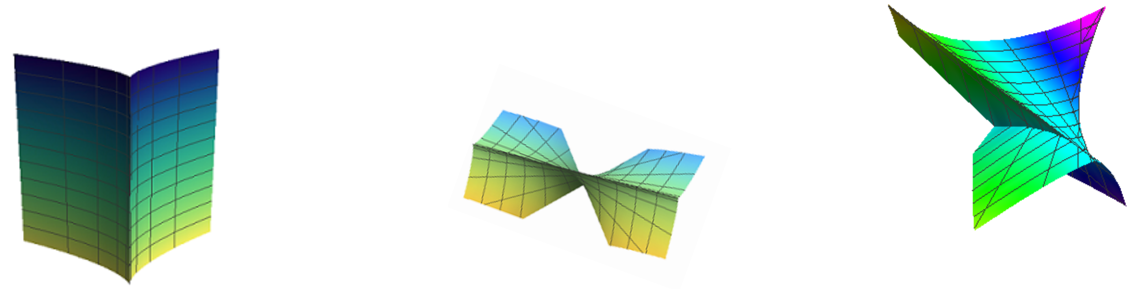


(A) Cuspidaledge (B) Cuspidal crosscap (C) Swallowtail

FIGURE 1. Types of singularities

Furthermore there are some papers about the singularities of surfaces in different spaces. Singularities of surfaces whose mean curvatures are constant, are studied by Brander in Lorentz 3-space [3], Fujimori et al. demonstrate that the cuspidal edges, swallowtails, and cuspidal cross caps are the general components of the singularities of spacelike maximum surfaces in Lorentz 3-space [4]. Kokubu et al. establish that only cuspidal edges and swallowtails are admissible in generic flat fronts in hyperbolic 3-space [8].

The primary aim of this study is to generalize the notion of developable surfaces provided by Izumiya and Takeuchi by utilizing the generalized alternative frame developed by [10] in Lorentz 3-space to include technical material of rulings. The alternative frame has the benefit of producing local classifications for the geometric structure of generalized developable surfaces in terms of k-slant helix, N_k-slant helix and conical surfaces. As a supplementary goal, we present the features of singularities when we explore the geometric properties of these generalized developable surfaces by combining the theory of Ishikawa and Yamashita [5]. Additionally, practical discussions on examples are held on the outcomes of theoretical investigations on generalized developable surfaces.

2. Basic Concepts and Notions

Consider the 3-dimensional Lorentz space E_1^3 provided with the following inner product: $\mathbb{R}^3$ endowed via the metric $\langle , \rangle$ as follows:

$$\begin{aligned} \langle , \rangle \quad : \quad & E^3 x E^3 \longrightarrow E \\ & (\xi, \zeta) \longrightarrow \langle \xi, \zeta \rangle = \xi_1\zeta_1 + \xi_2\zeta_2 - \xi_3\zeta_3 \end{aligned}$$

where $\xi = (\xi_1, \xi_2, \xi_3)$ and $\zeta = (\zeta_1, \zeta_2, \zeta_3) \in E^3$.

Let $\beta(s)$ be a differentiable curve of order (n+2) parametrized by arc-length, defined with the Frenet vector fileds $\{T, N, B\}$ and $\beta_0 = \int T ds$ be the tangential direction. In general the $k-$principal direction curve of β is determined as

$$\beta_k(s) = \int N_{k-1} ds, l \leqslant k \leqslant n$$

where the principal normal vector of β_{k-1} is N_{k-1}, β is named as the base curve of β_k. The Frenet vectors with the curvatures of β_k are defined as

$$T_k = N_{k-1}, \ N_k = \frac{N'_{k-1}}{\left\| N'_{k-1} \right\|}, \ B_k = N_k x T_k$$

$$\kappa_k = \sqrt{\left| \ \kappa_{k-1}^2 \pm \tau_{k-1}^2 \right|}, \tau_k = \sigma_{k-1}\kappa_k$$

where $\sigma_{k-1}, \kappa_{k-1}, \tau_{k-1}$ are the geodesic curvature, the first type of curvature and the second type of curvature of β_{k-1} respectively.

The Darboux vector W_k is the vector of angular velocity of the given frame of β_k and holds the following equations with regard to the frame apparatus:

i) Let β_k be a timelike curve, then

$$W_k = -\tau_k T_k - \kappa_k B_k$$

ii) Let β_k is a spacelike curve, then

$$W_k = -\tau_k T_k + \kappa_k B_k$$

Let $\{T_k, N_k, B_k, \kappa_k, \tau_k\}$ be the Frenet frame apparatus of the arc-length parametrized curve β_k in E_1^3. Then the equations below can be given:

$$\begin{pmatrix} T_k' \\ N_k' \\ B_k' \end{pmatrix} = \begin{pmatrix} 0 & \kappa_k & 0 \\ -\epsilon_0\epsilon_1\kappa_k & 0 & \tau_k \\ 0 & -\epsilon_1\epsilon_2\tau_k & 0 \end{pmatrix} \begin{pmatrix} T_k \\ N_k \\ B_k \end{pmatrix}$$

where $\langle T_k, T_k\rangle = \epsilon_0$, $\langle N_k, N_k\rangle = \epsilon_1$, $\langle B_k, B_k\rangle = \epsilon_2$, $\langle T_k, N_k\rangle = \langle T_k, B_k\rangle = \langle N_k, B_k\rangle = 0$ [9]

Definition 1. *Let $\beta(s) : I \longrightarrow \mathbb{R}^3$ be an arc-length parametrized curve with the Frenet frame apparatus $\{T, N, B, \kappa, \tau\}$. The curve β is called k-slant helix if the unit vector*

$$\beta_{k+1} = \frac{\beta_k\prime}{\|\beta_k\prime\|}$$

makes a constant angle with a fixed direction. Here $\beta_0 = \beta_s$ and $\beta_1 = \frac{\beta_0\prime}{\|\beta_0\prime\|}$ [1].

Definition 2. *Let $\beta(s) : I \longrightarrow \mathbb{R}^3$ be a differentiable curve of order (n+2) parametrized by arc-length, defined with the Frenet vector fields $\{T, N, B\}$ and $\beta_0 = \int T ds$ be the tangential direction. In general the k-principal direction curve of β is determined as*

$$\beta_k(s) = \int N_{k-1} ds, 1 \le k \le n$$

where the principal normal vector of β_{k-1} is N_{k-1}, $\beta_0(s) = \beta(s)$ and $N_0 = N$. Then the curve β is called N_k-slant helix which has the property that the principal normal vector N_k of β_k makes a constant angle with a fixed line. In other words β is called N_k-slant helix if β_k is a slant helix [10].

Thus we can give the following theorem:

Theorem 2. *$\beta(s) : I \longrightarrow \mathbb{R}^3$ is a (k+1)-slant helix if and only if β is a N_k-slant helix.*

Definition 3. *Let $\beta(s)$ be a non-degenerate, arc-length parametrized and differentiable curve of order (n+2), the $k-$principal direction curve of β is β_k in E_1^3. If the principal normal vector of β_k has steadily angle along a fixed axis, then β named as N_k slant helix [11].*

Note that, if k = 0 then the main curve β is named as slant helix, that is the principal normal vectors along β make a constant angle with an axis.

Theorem 3. *Let $\beta(s)$ be an arc-length parametrized, non-degenerate, differentiable curve of order (n+2) and β_k be the $k-$principal direction curve of β in E_1^3. Then β_k is a slant helix if and only if β is a N_k slant helix [11].*

To characterize a N_k slant helix in E_1^3 the following results are stated.

Theorem 4. ***i)*** *Assume that β is an arc-length parametrized timelike curve in E_1^3. Then β is a N_k slant helix if and only if either one of the next two functions is constant.*

$$\sigma_k(s) = \frac{\kappa_k^2}{(\tau_k^2 - \kappa_k^2)^{\frac{3}{2}}}\left(\frac{\tau_k}{\kappa_k}\right)' \quad or \quad \sigma_k(s) = \frac{\kappa_k^2}{(\kappa_k^2 - \tau_k^2)^{\frac{3}{2}}}\left(\frac{\tau_k}{\kappa_k}\right)'$$

where $\tau_k^2 - \kappa_k^2 \neq 0$.
ii) *Assume that β is an arc-length parametrized spacelike curve with the Frenet vectors $\{T, N, B, \kappa, \tau\}$ in E_1^3. Then there are two conditions for N_k slant helix case.*
a) *Assume that the unit normal vector of β is spacelike then β is a N_k slant helix if and only if either one of the below functions is constant where $\tau_k^2 - \kappa_k^2 \neq 0$*

$$\sigma_k(s) = \frac{\kappa_k^2}{(\tau_k^2 - \kappa_k^2)^{\frac{3}{2}}}\left(\frac{\tau_k}{\kappa_k}\right)' \quad or \quad \sigma_k(s) = \frac{\kappa_k^2}{(\kappa_k^2 - \tau_k^2)^{\frac{3}{2}}}\left(\frac{\tau_k}{\kappa_k}\right)'$$

b) *Assume that the unit normal vector of β is timelike. Then β is a N_k-slant helix if and only if the following function is constant*

$$\sigma_k(s) = \frac{\kappa_k^2}{(\tau_k^2 + \kappa_k^2)^{\frac{3}{2}}}\left(\frac{\tau_k}{\kappa_k}\right)'$$

[10].

Definition 4. *A ruled surface in* E_1^3 *is the transformation* $F_{(\gamma,\beta)}(s,u) = \gamma(s) + u\beta(s)$, *where* I *is an open interval and* $\gamma : I \longrightarrow E_1^3, \beta : I \longrightarrow E_1^3\backslash\{0\}$ *are smooth mappings.* γ *is called the base curve and* β *is called the director curve of the surface. The lines* $u \longrightarrow \gamma(s) + u\beta(s)$ *are named as rulings.*
A developable surface, also known as a flat surface, is a ruled surface where the Gaussian curvature K *is zero everywhere. Suppose that* γ *be an arc-length parametrized curve in* E_1^3 *with* $\kappa(s) \neq 0$. *We handle three types of developable surfaces associated to a non-degenerate space curve in Lorentz 3-space.*

1) A ruled surface $F_{(\gamma,\widetilde{W})}(s,u) = \gamma(s) + u\widetilde{W}(s)$ *is called the rectifying developable of* γ.

2) A ruled surface $F_{(B,T)}(s,u) = B(s) + uT(s)$ *is called the Darboux developable of* γ .

3) A ruled surface $F_{(\bar{W},N)}(s,u) = \bar{W}(s) + uN(s)$ *is called the tangential Darboux developable of* γ .

Here $\widetilde{W}(s)$=-$\dfrac{\tau}{\kappa}(s)T(s) + \epsilon_0 B(s)$ *is the modified Darboux vector field of* γ , *on condition that* $\kappa(s) \neq 0$.

Here $\epsilon_0 = -1$ *if* γ *is timelike and* $\epsilon_0 = 1$ *if* γ *is spacelike. Also* $\bar{W}(s)$ *is the unit Darboux vector field of* γ [6].

3. Construction of Developable Surfaces by Direction Curves

In this part we give a generalization of developable surfaces in terms of k-order frame in E_1^3 and obtain some results.

Definition 5. *Assume that* $\gamma(s)$ *is an arc-length parametrized non-degenerate, differentiable curve of order (n+2) and* γ_k *is the* $k-$*principal direction curve of* $\gamma, 1 \leqslant k \leqslant n$ *and the Darboux vector of* γ_k *is* W_k *in* E_1^3.

1) The k-rectifying developable of γ *is defined as the ruled surface given by* $F_{(\gamma,\widetilde{W}_k)}(s,u) = \gamma(s)+u\widetilde{W}_k(s)$
.

2) The k-Darboux developable of γ *is represented by the ruled surface* $F_{(B_k,T_k)}(s,u) = B_k(s) + uT_k(s)$.

3) The k-tangential Darboux developable of γ *is characterized by the ruled surface* $F_{(\bar{W}_k,N_k)}(s,u) = \bar{W}_k(s) + uN_k(s)$.

Here $\widetilde{W}_k(s)$=$-\dfrac{\tau_k}{\kappa_k}(s)T_k(s) + \epsilon_0 B_k(s)$ *is the modified Darboux vector field of* γ, *under the condition that* $\kappa_k(s) \neq 0$. *Here* ϵ_0 *is -1 if* γ *is timelike and 1 if* γ *is spacelike. And* $\bar{W}_k(s)$ *is the unit Darboux vector of* γ_k .

Theorem 5. *Assume that* $\gamma(s)$ *is an arc-length parametrized (n+2)-differentiable curve and* γ_k *and* γ_{k-1} *be the* $k-$*principal direction curve and* $(k-1)$ $-$*principal direction curve of* γ *respectively in* E_1^3.
i) *Let* γ_{k-1} *and* γ_k *be non-degenerate curves in* E_1^3. *Then there is a diffeomorphism between the k-Darboux developable of* γ *and the cuspidal edge at* $F_{(B_k,T_k)}(s_0,u_0)$ *if and only if* $\sigma_{k-1}(s_0) \neq 0, (\dfrac{\tau_k}{\kappa_k})' \neq 0$ *and* $u_0 = -\epsilon\dfrac{\tau_k}{\kappa_k}(s_0)$.
ii) *Let* γ_{k-1} *and* γ_k *be non-degenerate curves in* E_1^3. *Then the k-Darboux developable of* γ *is diffeomorphic to the swallowtail at* $F_{(B_k,T_k)}(s_0,u_0)$ *if and only if* $\sigma_{k-1}(s_0) \neq 0, (\dfrac{\tau_k}{\kappa_k})' = 0$ $(\dfrac{\tau_k}{\kappa_k})'' \neq 0$ *and* $u_0 = -\epsilon\dfrac{\tau_k}{\kappa_k}(s_0)$.
iii) *Let* γ_{k-1} *and* γ_k *be non-degenerate curves in* E_1^3. *Then a diffeomporhism exists between the Darboux developable of* γ *and the cuspidal crosscap at* $F_{(B_k,T_k)}(s_0,u_0)$ *if and only if* $\sigma_{k-1}(s_0) = 0, (\dfrac{\tau_k}{\kappa_k})' \neq 0$ *and* $u_0 = 0$.
Here $\epsilon = -1$ *if* γ_k *is timelike and* $\epsilon = 1$ *if* γ_k *is spacelike.*

Proof. Because of other cases are similar we only give the first proof.
Let γ_{k-1} be a timelike curve. Then $\gamma_k(s) = \int N_{k-1}ds$ is a spacelike curve and the Frenet frame apparatus

of γ_k is $\{T_k, N_k, B_k, \kappa_k, \tau_k\}$.Thus we have the Frenet formulas

$$\begin{bmatrix} T_k' \\ N_k' \\ B_k' \end{bmatrix} = \begin{bmatrix} 0 & \kappa_k & 0 \\ -\kappa_k & 0 & \tau_k \\ 0 & \tau_k & 0 \end{bmatrix} \begin{bmatrix} T_k \\ N_k \\ B_k \end{bmatrix}$$

The k-Darboux developable of γ is $F_{(B_k,T_k)}(s,u) = B_k + uT_k$ and a straight forward computation gives us the singular point of the k-Darboux developable as $u_0 = -\dfrac{\tau_k}{\kappa_k}(s_0)$.

The cuspidal edge singularities are obtained along points where $\gamma', \gamma'', \gamma'''$ are linearly independent. So if we do the necessary computations with the value of $u_0 = -\dfrac{\tau_k}{\kappa_k}(s_0)$, there is a diffeomorphism between the k-Darboux developable of γ and the cuspidal edge when $\sigma_{k-1}(s_0) \neq 0, (\dfrac{\tau_k}{\kappa_k})'(s_0) \neq 0$. □

As an extension of theorem 1, the following theorem yields a local charazterization for the k-tangential Darboux developable of a space curve.

Theorem 6. *Assume that $\gamma(s)$ is an arc-length parametrized, non-degenerate, differentiable curve of order (n+2) and γ_k and γ_{k-1} be the $k-$principal direction curve and $(k-1)$ $-$principal direction curve of γ respectively in E_1^3.*
i) *Let γ_{k-1} and γ_k be non-degenerate curves in E_1^3. Then there is a diffeomorphism between the k-tangential Darboux developable of γ and the cuspidal edge at $F_{(\bar{W}_k,N_k)}(s_0,u_0)$ if and only if $\sigma_k(s_0) \neq 0, \sigma_k'(s_0) \neq 0$ and $u_0 = \sigma_k(s_0)$.*
ii) *Let γ_{k-1} and γ_k be non-degenerate curves in E_1^3. Then there exists a diffeomorpism between k-tangential Darboux developable of γ and the swallowtail at $F_{(\bar{W}_k,N_k)}(s_0,u_0)$ if and only if $\sigma_k(s_0) \neq 0, \sigma_k''(s_0) \neq 0$ and $\sigma_k'(s_0) = 0$ and $u_0 = \sigma_k(s_0)$.*
iii) *Let γ_{k-1} and γ_k be non-degenerate curves in E_1^3. Then the k-tangential Darboux developable of γ is diffeomorphic to the cuspidal crosscap at $F_{(\bar{W}_k,N_k)}(s_0,u_0)$ if and only if $\sigma_k(s_0) = 0, \sigma_k'(s_0) \neq 0$ and $u_0 = 0$.*

Proof. Because of other cases are similar we only give the first proof.
Let γ_{k-1} be a timelike curve. Then $\gamma_k(s) = \int N_{k-1} ds$ is a spacelike curve and the Frenet frame apparatus of γ_k is $\{T_k, N_k, B_k, \kappa_k, \tau_k\}$. Thus we have the Frenet formulas

$$\begin{bmatrix} T_k' \\ N_k' \\ B_k' \end{bmatrix} = \begin{bmatrix} 0 & \kappa_k & 0 \\ -\kappa_k & 0 & \tau_k \\ 0 & \tau_k & 0 \end{bmatrix} \begin{bmatrix} T_k \\ N_k \\ B_k \end{bmatrix}$$

The k-tangential Darboux developable of γ is $F_{(\bar{W}_k,N_k)}(s,u) = \bar{W}_k(s) + uN_k(s)$ and a straightforward computation gives us the singular point of the k-tangential Darboux developable as $u_0 = \sigma_k(s_0)$.

As we mentioned before the cuspidal edge singularities are obtained at the points where $\gamma', \gamma'', \gamma'''$ are linearly independent. So if we do the necessary computations with the value of $u_0 = \sigma_k(s_0)$, the k-tangential Darboux developable of γ is diffeomorphic to the cuspidal edge when $\sigma_k(s_0) \neq 0, \sigma_k'(s_0) \neq 0$. □

Theorem 7. *Suppose that $\gamma(s)$ is an ac-length parametrized, non-degenerate, differentiable curve of order (n+2) and γ_k and γ_{k-1} be the $k-$principal direction curve and $(k-1)$ $-$principal direction curve of γ respectively in E_1^3.*
i) *Let γ_{k-1} and γ_k be non-degenerate curves in E_1^3. Then there is a diffeomorpism between k-rectifying developable of γ and the cuspidal edge at $F_{(\gamma,\tilde{W}_k)}(s_0,u_0)$ if and only if $(\dfrac{\tau_k}{\kappa_k})'(s_0) \neq 0, (\dfrac{\tau_k}{\kappa_k})''(s_0) \neq 0$ and $u_0 = \dfrac{1}{(\frac{\tau_k}{\kappa_k})'(s_0)}$.*
ii) *Let γ_{k-1} and γ_k be non-degenerate curves in E_1^3. Then the k-rectifying developable of γ is diffeomorphic to the swallowtail at $F_{(\gamma,\tilde{W}_k)}(s_0,u_0)$ if and only if $(\dfrac{\tau_k}{\kappa_k})'(s_0) \neq 0, (\dfrac{\tau_k}{\kappa_k})''(s_0) = 0$ and $u_0 = \dfrac{1}{(\frac{\tau_k}{\kappa_k})'(s_0)}$.*

Proof. With the use of Theorem 3, the proof can be obtained easily. □

Theorem 8. *Suppose that $\gamma(s)$ is an arc-length parametrized, non-degenerate, differentiable curve of order (n+2) and γ_k the $k-$principal direction curve of γ in E^3_1. The equivalent cases hold as follows:*
i) *The k-tangential Darboux developable of γ is a conical surface.*
ii) *γ_k is a slant helix.*
iii) *γ is a N_k slant helix.*
iv) *γ is a $(k+1)-$slant helix.*

Proof. The singular locus of the k-tangential Darboux developable is given by $\beta_k(s) = \bar{W}_k(s)+\sigma_k(s)N_k(s)$. Thus $F_{(\bar{W}_k,N_k)}$ is a conical surface if and only if $\beta'_k(s) = 0$. Let $F_{(\bar{W}_k,N_k)}$ be a conical surface. Then we have $\bar{W}'_k(s) - \sigma'_k(s)N_k(s) - \sigma_k(s)N'_k(s) = 0$ and we can easily see that $\bar{W}'_k(s) = \sigma_k(s)N'_k(s)$. Thus $\sigma'_k(s) = 0$ and γ_k is a slant helix and γ is a N_k slant helix.

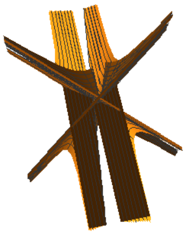
Conversely if γ is a N_k slant helix then γ_k is a slant helix. Thus $\sigma'_k(s) = 0$ and $\beta'_k(s) = \bar{W}'_k(s) - \sigma_k(s)N'_k(s)$. Since $\bar{W}'_k(s) = \sigma_k(s)N'_k(s)$ we have $\beta'_k(s) = 0$ and $F_{(\bar{W}_k,N_k)}$ is a conical surface. □

Theorem 9. *Assume that $\gamma(s)$ is a unit speed, differentiable curve of order (n+2) and γ_k is the $k-$principal direction curve of γ in E^3_1. Then the following cases are equivalent.*
i) *The k-rectifying developable of γ is a conical surface.*
ii) *γ_k is a conical geodesic curve.*

Proof. The singular locus of the k-rectifying developable of γ is given by $\beta_k(s) = \gamma_k + \frac{1}{(\frac{\tau_k}{\kappa_k})'}\widetilde{W}_k(s)$.
Let $F_{(\gamma,\widetilde{W}_k)}$ be a conical surface. According to the k-frames formulas we have $\beta'_k(s) = -\frac{(\frac{\tau_k}{\kappa_k})''}{(\frac{\tau_k}{\kappa_k})'}\widetilde{W}_k(s)$.
Therefore $\beta'_k(s) = 0$ if and only if $(\frac{\tau_k}{\kappa_k})'' = 0$. This completes the proof. □

Now let us explain the concepts of the present paper via some examples, thus we can show the connection between the role of the k order frame and determining the type of singularity.

Example 1. *Let $\alpha(s) = (\frac{1}{24}\sin 8s + \frac{2}{3}\sin 2s,\ -\frac{1}{24}\cos 8s + \frac{2}{3}\cos 2s,\ \frac{4}{15}\sin 5s) \in \mathbb{E}^3_1$ be an arc-length parametrized spacelike curve in $\mathbb{E}^3_1$. Then the developable surfaces of α and corresponding singular points are formed in terms of first and second order frames as follow:*
i) *The 1-Rectifying developable surface of α is $F_{(\alpha,\tilde{W}_1)} = \alpha + u\tilde{W}_1$ determined with the modified Darboux vector $\tilde{W}_1 = (\frac{5}{3}\cos 3s \csc 5s,\ (5+10\cos 2s)/(3+6\cos 2s+6\cos 4s),\ \frac{4}{3}\csc 5s)$ where $\left(\frac{\tau_1}{\kappa_1}\right)'(s_0) \neq 0$ and $\left(\frac{\tau_1}{\kappa_1}\right)''(s_0) \neq 0$ at $s_0 \neq \frac{1}{10}(2\pi n+\pi)$, $n \in \mathbb{Z}$, Theorem 7 explains that $F_{(\alpha,\tilde{W}_1)}$ is locally diffeomorphic to the cuspidal edge at the points $u_0 = \frac{1}{5}\sin[5s_0]^2$ for all $s_0 \neq \frac{1}{10}(2\pi n+\pi)$, and otherwise is locally diffeomorphic to the swallowtail. This implies the points are given by $u_0 = \frac{1}{5}\sin[5s_0]^2$ and $s_0 = \frac{1}{10}(2\pi n + \pi)$.*
ii) *The 1-Darboux developable surface of α, $F_{(B_1,T_1)} = B_1 + uT_1$ is obtained as where $\sigma = 5/4$ and $\left(\frac{\tau_1}{\kappa_1}\right)'(s_0) \neq 0$ for all s_0, then $F_{(B_1,T_1)}$ is just locally diffeomorphic to the cuspidal edge at the points $u_0 = -\cot[5s_0]$ and $\frac{5s_0}{\pi} \notin \mathbb{Z}$ from Theorem 5.*



(A) 1-Rectifying developable surface of α

(B) 1-Darboux developable surface of α

FIGURE 2. First order developable surfaces of a spacelike curve

iii) *The 1-Tangential Darboux developable surface of* α *is* $F_{(\overline{W}_1,N_1)} = \overline{W}_1 + uN_1$ *is obtained as where we have only one singular point at* $u_0 = 5$ *as can be seen in Figure 3.*

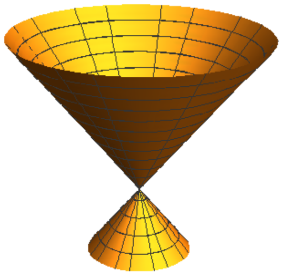


FIGURE 3. 1-Tangential Darboux developable surface of α

Now we observe the singular points of developable surfaces composed of the 2*-order frame* $\{T_2, N_2, B_2\}$ *of* α *with the curvatures* τ_2, κ_2*. Moreover, since the 2-tangent vector satisfies the equality such as* $\langle T_2, T_2\rangle < 0$, α *is a timelike curve with respect to the* 2*-order frame.*

(iv) *The 2-Rectifying developable surface of* α, $F_{(\alpha,\tilde{W}_2)} = \alpha + u\tilde{W}_2$ *is determined with the modified Darboux vector* $\tilde{W}_2 = (0, 0, -3/4)$ *, then we have* $\left(\frac{\tau_1}{\kappa_1}\right)'(s_0) = 0$ *and* $\left(\frac{\tau_1}{\kappa_1}\right)''(s_0) = 0$ *for all* $s_0 \in \mathbb{R}$*, thus 2-Rectifying developable surface of* α *has no singular points.*

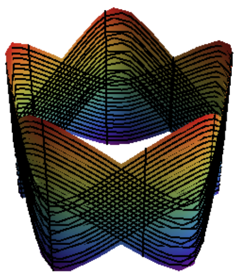


FIGURE 4. 2-Rectifying developable surface of α

(v) *The 2-Darboux developable surface of* α, $F_{(B_2,T_2)} = B_2 + uT_2$ *is can be obtained as where 2-Darboux developable surface of* α *is isometric to 1-Tangent Darboux developable surface of* α *then their points have the same type of singularity.*

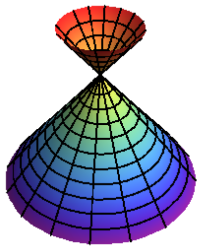


FIGURE 5. 2-Darboux developable surface of α

(vi) *The 2-Tangential Darboux developable surface of α is obtained as a plane defined by $(u\sin[3s], u\cos[3s], -1)$, hence there are no singular points and the trace of the points of the 2-Tangential Darboux developable surface $F_{(\overline{W}_2, N_2)}$ can be seen as follows: When we continue to express the spa-*

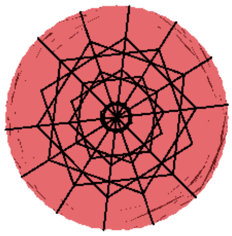


FIGURE 6. 2-Tangential Darboux developable surface of α

tial curve $\alpha \in \mathbb{E}_1^3$ with respect to the other vector triples of the $k-$order process, we can eliminate the singularity for each Darboux developable surface.

4. Conclusion

In differential geometry, developable surfaces are particular kind of surfaces that can be flattened onto a plane without distortion, that is, they can be unfurled into a flat shape without ripping or stretching. Particularly significant are these surfaces in the domains of computer graphics, manufacturing and architecture. An additional important tool in differential geometry are the alternative frames, which are the coordinate systems relative to a curve in the curve theory that provide different ways to describe the geometry and motion along the curve. In disciplines like computer graphics, robotics and physics these frames are especially helpful. They provide other viewpoints for interpreting and analyzing curves other than the widely used Serret-Frenet frame.

In this study we supply a generalized definition for developable surfaces by using an alternative frame produced via the direction curves through their rulings. A dynamic structure is resulted from the generalization and the theory of Ishikawa and Yamashita is utilized to characterize the singular points of these structures. The examples given in the article demonstrate the dynamic structure that arises from the alternative frame. Emotionally and raitonally, it is plainly observed that when the degree of the alternative frame utilized rise, the approach to the developable surfaces goes towards the light cone and the Lorentzian plane. This makes it easier to detect the evolution of singular points.

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Conformal semi-invariant Riemannian maps to Sasakian manifolds

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Abstract. The idea of conformal semi-invariant Riemannian maps to almost Hermitian manifolds was first put forward by Şahin and Akyol in [3]. In this paper, we expand this idea to Sasakian manifolds which are almost contact metric manifolds. Hereby, we present conformal semi-invariant Riemannian maps from Riemannian manifolds to Sasakian manifolds. Then, we prepare a illustrative example and investigate the geometry of the leaves of $D_1, D_2, \bar{D}_1$ and $\bar{D}_2$. We find necessary and sufficient conditions for conformal semi-invariant Riemannian maps to be totally geodesic. Also, we investigate the harmonicity of such maps.

2020 Mathematics Subject Classification. 53C15, 53C43.
Keywords. Semi-invariant Riemannian map, conformal semi-invariant Riemannian map, Sasakian manifolds.

1. Introduction

In [6], Fischer introduced Riemannian map between Riemannian manifolds as a generalization of an isometric immersion and Riemannian submersion that satisfies the well known generalized eikonal equation $\|\vartheta_*\|^2 = rank\vartheta$, which is a bridge between geometric optics and physical optics. Where ϑ is a Riemannian map and ϑ_* is its derivative map. Let $\vartheta : (S_1, g_{S_1}) \to (S_2, g_{S_2})$ be a smooth map between Riemannian manifolds such that $0 < rank\vartheta < \min\{\dim(S_1), \dim(S_2)\}$. We state the kernel space of ϑ_* by $V_q = \ker \vartheta_{*_q}$ at $q \in S_1$ and consider the orthogonal complementary space $H_q = (\ker \vartheta_{*_q})^\perp$ to $\ker \vartheta_{*_q}$ in T_qS_1. Then the tangent space T_qS_1 of S_1 at q has the decomposition $T_qS_1 = (\ker \vartheta_{*_q}) \oplus (\ker \vartheta_{*_q})^\perp = V_q \oplus H_q$. We state the range of ϑ_* by $range\vartheta_*$ at $q \in S_1$ and consider the orthogonal complementary space $(range\vartheta_{*_q})^\perp$ to $range\vartheta_{*_q}$ in the tangent space $T_{\vartheta(q)}S_2$ of S_2 at $\vartheta(q) \in S_2$. Since $rank\vartheta < \min\{\dim(S_1), \dim(S_2)\}$, we have $(\ker \vartheta_{*_q})^\perp \neq \{0\}$. Therefore the tangent space $T_{\vartheta(q)}S_2$ of S_2 at $\vartheta(q) \in S_2$ has the decomposition $T_{\vartheta(q)}S_2 = (range\vartheta_{*_q}) \oplus (range\vartheta_{*_q})^\perp$. Then ϑ is called Riemannian map at $q \in S_1$ if the horizontal restriction $\vartheta^h_{*_q} : (\ker \vartheta_{*_q})^\perp \to (range\vartheta_{*_q})$ is a linear isometry between the spaces $((\ker \vartheta_{*_q})^\perp, g_{S_1}|_{(\ker \vartheta_{*_q})^\perp})$ and $(range\vartheta_{*_q}, g_{S_2\vartheta(q)}|_{range\vartheta_{*_q}})$. In other words, ϑ satisfies

$$g_{S_2}(\vartheta_* A_1, \vartheta_* A_2) = g_{S_1}(A_1, A_2), \tag{1}$$

for all A_1, A_2 vector field tangent to $\Gamma(\ker \vartheta_{*_q})^\perp$.

Different features of Riemannian maps have been investigated extensively by many authors in [1, 7, 8, 10, 15–17, 20, 24, 25, 27–29]. Detailed development in the theory of Riemannian map can be found in [21].

Conformal Riemannian maps as a generalization of Riemannian maps and the harmonicity of such maps have been introduced in [22, 23]. Conformal anti-invariant Riemannian maps have been studied in [2]. In this article, we expand this concept to almost contact metric manifolds as a generalization of semi-invariant Riemannian maps and totally real submanifolds.

The paper is organized as follows. Section 2 contains preliminaries. Section 3 includes conformal semi-invariant Riemannian maps from Riemannian manifolds to Sasakian manifolds and provides this notion by non-trivial example. Then, we get a decomposition theorem by using the existence of conformal

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semi-invariant Riemannian maps. Moreover, conformal semi-invariant Riemannian maps allow us to obtain new conditions for a map to be harmonic. We also investigate the total geodesicity of conformal semi-invariant maps. In Section 4, we give scope for future studies.

2. Preliminaries

Let S be an odd-dimensional smooth manifold. Then, S has an almost contact structure [21] if there exist a tensor field F of type$-(1,1)$, a vector field ξ, and 1-form η on S such that

$$F^2E_1 = -E_1 + \eta(E_1)\xi, F\xi = 0, \eta \circ F = 0, \eta(\xi) = 1. \tag{2}$$

If there exists a Riemannian metric g_S on an almost contact manifold S satisfying:

$$\begin{aligned} g_S(FE_1, FE_2) &= g_S(E_1, E_2) - \eta(E_1)\eta(E_2), \\ g_S(E_1, FE_2) &= -g_S(FE_1, E_2), \end{aligned} \tag{3}$$

$$\eta(E_1) = g_S(E_1, \xi), \tag{4}$$

where E_1, E_2 are any vector fields on S, then S is called an almost contact metric manifold with an almost contact structure (F, ξ, η, g_S) and is symbolized by (S, F, ξ, η, g_S).

A manifold S with the structure (F, ξ, η, g_S) is said to be Sasakian structure given by [4]

$$(\nabla^S_{E_1} F)E_2 = g_S(E_1, E_2)\xi - \eta(E_2)E_1, \tag{5}$$

for any vector fields E_1, E_2 on S, where ∇ stands for the Riemannian connection of the metric g_S on S. For a Sasakian manifold, we get

$$\nabla^S_{E_1}\xi = -FE_1, \tag{6}$$

for any vector field E_1 on S.

ϑ_* can be considered as a part of bundle hom$(TS_1, \vartheta^{-1}TS_2) \to S_1$, where $\vartheta^{-1}TS_2$ is the pullback bundle. The bundle has a connection ∇ induced from the pullback connection $\overset{S_2}{\nabla}{}^{\vartheta}$ and the Levi-Civita connection ∇^{S_1}. Then the second fundamental form $(\nabla\vartheta_*)(A_1, A_2)$ of ϑ is given by [14]

$$(\nabla\vartheta_*)(A_1, A_2) = \overset{S_2}{\nabla}{}^{\vartheta}_{A_1}\vartheta_*A_2 - \vartheta_*(\nabla^{S_1}_{A_1}A_2), \tag{7}$$

for all $A_1, A_2 \in \Gamma(TS_1)$, where $\overset{S_2}{\nabla}{}^{\vartheta}_{A_1}\vartheta_*A_2 \circ \vartheta = \overset{S_2}{\nabla}{}^{\vartheta}_{\vartheta_*A_1}\vartheta_*A_2$. It is known that $(\nabla\vartheta_*)(A_1, A_2)$ is symmetric and $(\nabla\vartheta_*)(A_1, A_2)$ has no component in $range\vartheta_*$, for all $A_1, A_2 \in \Gamma(\ker\vartheta_*)^\perp$ [21]. It means that, we get

$$(\nabla\vartheta_*)(A_1, A_2) \in \Gamma(range\vartheta_*)^\perp.$$

The tension field of ϑ is defined to be the trace of the second fundamental form of ϑ, i.e. $\tau(\vartheta) = trace(\nabla\vartheta_*) = \sum_{i=1}^{m} (\nabla\vartheta_*)(e_i, e_i)$, where $m = \dim(S_1)$ and $\{e_1, e_2, ..., e_m\}$ is the orthonormal frame on S_1. Moreover, a map $\vartheta : (S_1, g_{S_1}) \to (S_2, g_{S2})$ is harmonic if and only if the tension field of ϑ vanishes at each point $q \in S_1$.

For any section B_1 of $(range\vartheta_*)^\perp$ and vector field A_1 on S_1, we get $\nabla^{\vartheta\perp}_{A_1}B_1$, which is the orthogonal projection of $\nabla^{S_2}_{A_1}B_1$ on $(range\vartheta_*)^\perp$, where $\nabla^{\vartheta\perp}$ is linear connection on $(range\vartheta_*)^\perp$ such that $\nabla^{\vartheta\perp}g_{S_2} = 0$. For a Riemannian map ϑ we describe S_{B_1} as ([21], p. 188)

$$\nabla^{S_2}_{\vartheta_*A_1}B_1 = -S_{B_1}\vartheta_*A_1 + \nabla^{\vartheta\perp}_{A_1}B_1, \tag{8}$$

where $S_{B_1}\vartheta_*A_1$ is the tangential component of $\nabla^{S_2}_{\vartheta_*A_1}B_1$ and $\nabla^{S_2}$is Levi-Civita connection on S_2. Therefore, we have $\nabla^{S_2}_{\vartheta_*A_1}B_1(q) \in T_{\vartheta(q)}S_2, S_{B_1}\vartheta_*A_1 \in \vartheta_{*q}(T_qS_1)$ and $\nabla^{\vartheta\perp}_{A_1}B_1 \in (\vartheta_{*q}(T_qS_1))^\perp$ at $q \in S_1$. We know that $S_{B_1}\vartheta_*A_1$ is bilinear in B_1, and ϑA_1 at q depends only on B_{1q} and $\vartheta_{*q}A_{1q}$. From here, using (7) and (8) we have

$$g_{S_2}(S_{B_1}\vartheta_*A_1, \vartheta_*A_2) = g_{S_2}(B_1, (\nabla\vartheta_*)(A_1, A_2)), \tag{9}$$

where S_{B_1} is self adjoint operator for $A_1, A_2 \in \Gamma(\ker\vartheta_*)^\perp$ and $B_1 \in \Gamma(range\vartheta_*)^\perp$.

For all $B_1, B_2 \in \Gamma(range\vartheta_*)^\perp$ we define

$$\nabla^{S_2}_{B_1}B_2 = R(\nabla^{S_2}_{B_1}B_2) + \nabla^{\vartheta\perp}_{B_1}B_2,$$

where $R(\nabla^{S_2}_{B_1}B_2)$) and $\nabla^{\vartheta\perp}_{B_1}B_2$ denote $range\vartheta_*$ and $(range\vartheta_*)^\perp$ part of $\nabla^{S_2}_{B_1}B_2$, respectively. Therefore $(range\vartheta_*)^\perp$ is totally geodesic if and only if

$$\nabla^{S_2}_{B_1}B_2 = \nabla^{\vartheta\perp}_{B_1}B_2. \tag{10}$$

3. Conformal Semi-Invariant Riemannian Maps to Sasakian Manifolds

Definition 1. *[23] Let $\vartheta : (S_1, g_{S_1}) \to (S_2, g_{S_2})$ be a conformal Riemannian map (CRM). Then, ϑ is a horizontally homothetic map if $H(grad\lambda) = 0$.*

Definition 2. *[22] Let $\vartheta : (S_1, g_{S_1}) \to (S_2, g_{S_2})$ be a smooth map between Riemannian manifolds. Then, ϑ is a CRM at $q \in S_1$ if $0 < rank\vartheta_{*q} \le \min\{\dim(S_1), \dim(S_2)\}$ and ϑ_{*q} maps the horizontal space $H(q) = (\ker\vartheta_{*q})^\perp$ conformally into $range\vartheta_{*q}$, it means that there exists a number $\lambda^2(q) \neq 0$ such that*

$$g_{S_2}(\vartheta_{*q}A_1, \vartheta_{*q}A_2) = \lambda^2(q) g_{S_1}(A_1, A_2),$$

for $A_1, A_2 \in \Gamma(\ker\vartheta_*)^\perp$. Moreover, if ϑ is CRM at any $q \in S_1$, then ϑ is called CRM.

Lastly, the second fundamental form of ϑ is given by [22]

$$(\nabla\vartheta_*)(A_1, A_2)^{range\vartheta_*} = A_1(\ln\lambda)\,\vartheta_*A_2 + A_2(\ln\lambda)\,\vartheta_*A_1 - g_{S_1}(A_1, A_2)\vartheta_*(grad\ln\lambda). \tag{11}$$

Therefore, if we state the $(range\vartheta_*)^\perp$ component of $(\nabla\vartheta_*)(A_1, A_2)$ by $(\nabla\vartheta_*)(A_1, A_2)^{(range\vartheta_*)^\perp}$, then we can write

$$(\nabla\vartheta_*)(A_1, A_2) = (\nabla\vartheta_*)(A_1, A_2)^{range\vartheta_*} + (\nabla\vartheta_*)(A_1, A_2)^{(range\vartheta_*)^\perp}, \tag{12}$$

for $A_1, A_2 \in \Gamma(\ker\vartheta_*)^\perp$. Therefore we get

$$\begin{aligned}(\nabla\vartheta_*)(A_1, A_2) &= A_1(\ln\lambda)\,\vartheta_*A_2 + A_2(\ln\lambda)\,\vartheta_*A_1 \\ &\quad - g_{S_1}(A_1, A_2)\vartheta_*(grad\ln\lambda) + (\nabla\vartheta_*)(A_1, A_2)^{(range\vartheta_*)^\perp}.\end{aligned} \tag{13}$$

Definition 3. *Let ϑ be a CRM from a Riemannian manifold (S_1, g_{S_1}) to an almost contact metric manifold $(S_2, F, \xi, \eta, g_{S_2})$. Then ϑ is a conformal semi-invariant Riemannian map (CSIRM) at $q \in S_1$ if there is a subbundle $D_1 \subseteq (range\vartheta_*)$ such that*

$$range\vartheta_{*q} = D_1 \oplus D_2, F(D_1) = D_1, F(D_2) \subseteq (range\vartheta_{*q})^\perp,$$

where D_2 is orthogonal complementary to D_1 in $range\vartheta_*$. If ϑ is a CSIRM for any $q \in S_1$, then ϑ is called a CSIRM.

For $\vartheta_*A_1 \in \Gamma(range\vartheta_*)$, then we write

$$F\vartheta_*A_1 = \phi\vartheta_*A_1 + \omega\vartheta_*A_1, \tag{14}$$

where $\phi\vartheta_*A_1 \in \Gamma(D_1)$ and $\omega\vartheta_*A_1 \in \Gamma(FD_2)$. Also, for $\vartheta_*A_1 \in \Gamma(D_1)$ and $\vartheta_*A_2 \in \Gamma(D_2)$, we have $g_{S_2}(\vartheta_*A_1, \vartheta_*A_2) = 0$. Thus we have two orthogonal distributions $\bar{D}_1$ and $\bar{D}_2$ such that

$$(\ker\vartheta_{*_q})^\perp = \bar{D}_1 \oplus \bar{D}_2.$$

On the other hand, for $B_1 \in \Gamma((range\vartheta_*)^\perp)$, then we have

$$FB_1 = \beta_1 B_1 + \alpha_1 B_1, \tag{15}$$

where $\beta_1 B_1 \in \Gamma(D_1)$ and $\alpha_1 B_1 \in \Gamma(\eta)$. Here η is the complementary orthogonal distribution to $\omega(D_2)$ in $(range\vartheta_*)^\perp$. It is easy to see that η is invariant with respect to F.

Example 1. *Let S_1 be an Euclidean space given by*

$$S_1 = \left\{(u_1, u_2, u_3, u_4, u_5) \in \mathbb{R}^5 : u_1 \neq 0, u_2 \neq 0, u_5 \neq 0\right\}.$$

We describe the Riemannian metric g_{S_1} on S_1 given by

$$g_{S_1} = du_1^2 + du_2^2 + du_3^2 + du_4^2 + du_5^2.$$

Let $S_2 = \left\{(v_1, v_2, v_3, v_4, v_5) \in \mathbb{R}^5\right\}$ be a Euclidean space with metric g_{S_2} on S_2 given by

$$g_{S_2} = e^{2u_1}dv_1^2 + e^{2u_1}dv_2^2 + e^{2u_1}dv_3^2 + dv_4^2 + dv_5^2.$$

Usual Sasakian structure (F, ξ, η) *on* (S_2, g_{S_2}) *can be choosen as* [5]

$$F(\frac{\partial}{\partial v_1}) = \frac{\partial}{\partial v_2}, F(\frac{\partial}{\partial v_2}) = -\frac{\partial}{\partial v_1}, F(\frac{\partial}{\partial v_3}) = \frac{\partial}{\partial v_4}, F(\frac{\partial}{\partial v_4}) = -\frac{\partial}{\partial v_3},$$
$$\eta = dv_5, \xi = \frac{\partial}{\partial v_5}, F(\xi) = 0.$$

Then a basis of $T_q S_1$ *is*

$$\left\{ e_i = e^{u_1}\frac{\partial}{\partial u_i} \text{ for } 1 \leq i \leq 5 \right\},$$

and a F*-basis on* $T_{\vartheta(q)}S_2$ *is*

$$\left\{ e_j^* = \frac{\partial}{\partial v_j} \text{ for } 1 \leq j \leq 4, e_4^* = e^{u_1}\frac{\partial}{\partial v_4}, \xi = e_5^* = \frac{\partial}{\partial v_5} \right\},$$

for all $q \in S_1$*. Now, we define a map* $\vartheta : (S_1, g_{S_1}) \to (S_2, g_{S_2}, F)$ *by*

$$\vartheta(u_1, u_2, u_3, u_4, u_5) = (u_1, u_2, u_5, 0, 0).$$

Then, we have

$$ker\vartheta_* = Span\{U_1 = e_3, U_2 = e_4\},$$
$$(ker\vartheta_*)^\perp = Span\{A_1 = e_1, A_2 = e_2, A_3 = e_5\}.$$

Hence it is easy to see that $\vartheta_* A_1 = e^{u_1}e_1^*, \vartheta_* A_2 = e^{u_1}e_2^*, \vartheta_* A_3 = e^{u_1}e_3^*$ *and* $g_{S_2}(\vartheta_*(A_{1i}), \vartheta_*(A_{1j})) = e^{2u_1}g_{S_1}(A_{1i}, A_{1j})$ *for* $i, j = 1, 2, 3$*. Thus* ϑ *is a CRM with* $\lambda = e^{2u_1}$ *and we get*

$$range\vartheta_* = Span\{e^{u_1}e_1^*, e^{u_1}e_2^*, e^{u_1}e_3^*\},$$
$$(range\vartheta_*)^\perp = Span\{e_4^*, \xi\},$$
$$D_1 = Span\{e^{u_1}e_1^*, e^{u_1}e_2^*\}, D_2 = Span\{e^{u_1}e_3^*\}.$$

Moreover it is easy to see that $F\vartheta_* A_1 = e^{u_1}e_2^*, F\vartheta_* A_2 = -e^{u_1}e_1^*, F\vartheta_* A_3 = e^{u_1}e_4^*$*. Thus* ϑ *is a CSIRM.*

Remark 1. *Throughout this article* $\xi \in (range\vartheta_*)^\perp$ *will be taken as the Reeb vector field.*

We obtain the following theorem for the geometry of the leaves of D_1.

Theorem 1. *Let* ϑ *be a CSIRM from a Riemannian manifold* (S_1, g_{S_1}) *to a Sasakian manifold* $(S_2, F, \xi, \eta, g_{S_2})$*. Then* D_1 *describes a totally geodesic foliation on* S_2 *if and only if*

(i).

$$g_{S_2}(A_1, \vartheta_* A_2)\eta(\alpha_1 B_1) = \eta(\nabla^{S_2}_{A_1}\vartheta_* A_2)\eta(B_1) + g_{S_2}(\beta_1 B_1(\ln\lambda)\vartheta_* A_1 + \vartheta_*(\nabla^{S_1}_{A_1}A_3), F\vartheta_* A_2) + g_{S_2}(S_{\alpha_1 B_1}\vartheta_* A_1, F\vartheta_* A_2)$$

(ii). $\phi S_{F\vartheta_* B_2}\vartheta_* A_1 - \eta(\omega\vartheta_* B_2)A_1$ *has no components in* $\Gamma(D_1)$*, for any* $A_1, A_2, A_3, B_2 \in \Gamma(ker\vartheta_*)^\perp$ *such that* $\vartheta_* A_1, \vartheta_* A_2 \in \Gamma(D_1), \vartheta_* B_2 \in \Gamma(D_2)$ *and* $B_1 \in \Gamma(range\vartheta_*)^\perp$ *such that* $\vartheta_* A_3 = \beta_1 B_1$*.*

Proof. For $\vartheta_* A_1, \vartheta_* A_2 \in \Gamma(D_1), B_1 \in \Gamma(range\vartheta_*)^\perp$ and $\vartheta_* B_2 \in \Gamma(D_2)$, since ϑ is a CRM, using (2) and (3) we have

$$g_{S_2}(\nabla^{S_2}_{A_1}\vartheta_* A_2, B_1) = g_{S_2}(F\nabla^{S_2}_{A_1}\vartheta_* A_2, FB_1) + \eta(\nabla^{S_2}_{A_1}\vartheta_* A_2)\eta(B_1).$$

From (4), (5) and (6) we get

$$g_{S_2}(\nabla^{S_2}_{A_1}\vartheta_* A_2, B_1) = -g_{S_2}(A_1, \vartheta_* A_2)\eta(FB_1) + g_{S_2}(A_1, FB_1)\underbrace{\eta(\vartheta_* A_2)}_{0} + \eta(\nabla^{S_2}_{A_1}\vartheta_* A_2)\eta(B_1) - g_{S_2}(\nabla^{S_2}_{A_1}FB_1, F\vartheta_* A_2)$$

From (15)

$$g_{S_2}(\nabla^{S_2}_{A_1}\vartheta_* A_2, B_1) = -g_{S_2}(A_1, \vartheta_* A_2)\eta(\alpha_1 B_1) + \eta(\nabla^{S_2}_{A_1}\vartheta_* A_2)\eta(B_1) - g_{S_2}(\nabla^{S_2}_{A_1}\beta_1 B_1, F\vartheta_* A_2) - g_{S_2}(\nabla^{S_2}_{A_1}\alpha_1 B_1, F\vartheta_* A_2).$$

Using (8), we have

$$g_{S_2}(\nabla^{S_2}_{A_1}\vartheta_* A_2, B_1) = -g_{S_2}(A_1, \vartheta_* A_2)\eta(\alpha_1 B_1) + \eta(\nabla^{S_2}_{A_1}\vartheta_* A_2)\eta(B_1)$$

$$-g_{S_2}(\nabla^{S_2}_{A_1}\vartheta_*A_3, F\vartheta_*A_2) - g_{S_2}(-S_{\alpha_1 B_1}\vartheta_*A_1 + \nabla^{\vartheta\perp}_{A_1}\vartheta_*A_1, F\vartheta_*A_2)$$

where $\beta_1 B_1 = \vartheta_* A_3 \in \Gamma(D_2)$ for $A_3 \in \Gamma(ker\vartheta_*)^\perp$. From (7) we have

$$\begin{aligned} g_{S_2}(\nabla^{S_2}_{A_1}\vartheta_*A_2, B_1) &= -g_{S_2}(A_1,\vartheta_*A_2)\eta(\alpha_1 B_1) + \eta(\nabla^{S_2}_{A_1}\vartheta_*A_2)\eta(B_1) \\ &\quad -g_{S_2}((\nabla^{S_2}\vartheta_*)(A_1,A_3) + \vartheta_*(\nabla^{S_1}_{A_1}A_3), F\vartheta_*A_2) \\ &\quad +g_{S_2}(S_{\alpha_1 B_1}\vartheta_*A_1, F\vartheta_*A_2) - \underbrace{g_{S_2}(\nabla^{\vartheta\perp}_{A_1}\vartheta_*A_1, F\vartheta_*A_2)}_{0}. \end{aligned}$$

Using (12) in the above equation we get

$$\begin{aligned} g_{S_2}(\nabla^{S_2}_{A_1}\vartheta_*A_2, B_1) &= -g_{S_2}(A_1,\vartheta_*A_2)\eta(\alpha_1 B_1) + \eta(\nabla^{S_2}_{A_1}\vartheta_*A_2)\eta(B_1) \\ &\quad -g_{S_2}((\nabla^{S_2}\vartheta_*)(A_1,A_3)^{range\vartheta_*} + \vartheta_*(\nabla^{S_1}_{A_1}A_3), F\vartheta_*A_2) \\ &\quad +g_{S_2}(S_{\alpha_1 B_1}\vartheta_*A_1, F\vartheta_*A_2). \end{aligned}$$

From (**??**)

$$\begin{aligned} g_{S_2}(\nabla^{S_2}_{A_1}\vartheta_*A_2, B_1) &= -g_{S_2}(A_1,\vartheta_*A_2)\eta(\alpha_1 B_1) + \eta(\nabla^{S_2}_{A_1}\vartheta_*A_2)\eta(B_1) \\ &\quad -g_{S_2}(A_1(\ln\lambda)\vartheta_*A_3 + A_3(\ln\lambda)\vartheta_*A_1 \\ &\quad -g_{S_1}(A_1,A_3)\vartheta_*(grad\ln\lambda) + \vartheta_*(\nabla^{S_1}_{A_1}A_3), F\vartheta_*A_2) \\ &\quad +g_{S_2}(S_{\alpha_1 B_1}\vartheta_*A_1, F\vartheta_*A_2). \end{aligned}$$

Since $grad(\ln\lambda) \in (range\vartheta_*)^\perp$, using (3) and $\vartheta_*A_3 = \beta_1 B_1$ we have

$$\begin{aligned} g_{S_2}(\nabla^{S_2}_{A_1}\vartheta_*A_2, B_1) &= -g_{S_2}(A_1,\vartheta_*A_2)\eta(\alpha_1 B_1) + \eta(\nabla^{S_2}_{A_1}\vartheta_*A_2)\eta(B_1) \\ &\quad -g_{S_2}(\beta_1 B_1(\ln\lambda)\vartheta_*A_1 + \vartheta_*(\nabla^{S_1}_{A_1}A_3), F\vartheta_*A_2) \\ &\quad +g_{S_2}(S_{\alpha_1 B_1}\vartheta_*A_1, F\vartheta_*A_2). \end{aligned}$$

This implies the proof of (i).
On the other hand, by using (3) we get

$$\begin{aligned} g_{S_2}(\nabla^{S_2}_{A_1}\vartheta_*A_2, \vartheta_*B_2) &= g_{S_2}(F\,\nabla^{S_2}_{A_1}\,\vartheta_*A_2, F\vartheta_*B_2) \\ &\quad +\eta(\nabla^{S_2}_{A_1}\vartheta_*A_2)\underbrace{\eta(\vartheta_*B_2)}_{0}. \end{aligned}$$

From (4), (5) and (6) we get

$$\begin{aligned} g_{S_2}(\nabla^{S_2}_{A_1}\vartheta_*A_2, \vartheta_*B_2) &= -g_{S_2}(A_1,\vartheta_*A_2)\eta(F\vartheta_*B_2) + g_{S_2}(A_1, F\vartheta_*B_2)\underbrace{\eta(\vartheta_*A_2)}_{0} \\ &\quad +g_{S_2}(F\,\nabla^{S_2}_{A_1}\,F\vartheta_*B_2, \vartheta_*A_2). \end{aligned}$$

From (14) and (8) we have

$$\begin{aligned} g_{S_2}(\nabla^{S_2}_{A_1}\vartheta_*A_2, \vartheta_*B_2) &= -g_{S_2}(A_1,\vartheta_*A_2)\eta(\omega\vartheta_*B_2) \\ &\quad +g_{S_2}(-\phi S_{F\vartheta_*B_2}\vartheta_*A_1 + \phi\,\nabla^{\vartheta\perp}_{A_1}\,F\vartheta_*B_2, \vartheta_*A_2) \\ &= -g_{S_2}(\eta(\omega\vartheta_*B_2)A_1 - \phi S_{F\vartheta_*B_2}\vartheta_*A_1, \vartheta_*A_2). \end{aligned}$$

This implies the proof of (ii). □

We obtain the following theorem for the geometry of the leaves of D_2.

Theorem 2. *Let ϑ be a CSIRM from a Riemannian manifold (S_1, g_{S_1}) to a Sasakian manifold $(S_2, F, \xi, \eta, g_{S_2})$. Then D_2 describes a totally geodesic foliation on S_2 if and only if*
(i).

$$\begin{aligned} \eta(\nabla^{S_2}_{B_3}\vartheta_*B_4)\eta(B_2) &= g_{S_2}((\nabla^{S_2}\vartheta_*)(B_3,A_4)^{(range\vartheta_*)^\perp} \\ &\quad +\nabla^{\varphi\perp}_{B_3}\,\alpha_1 B_2, F\vartheta_*B_4) + g_{S_2}(B_3,\vartheta_*B_4)\eta(\alpha_1 B_2), \end{aligned}$$

(ii). $\beta_1(\nabla^{S_2}\vartheta_*)(B_3,A_3)^{(range\vartheta_*)^\perp}$ *has no components in* $\Gamma(D_2)$*,*

for any $A_3, A_4, B_3, B_4 \in \Gamma(ker\vartheta_*)^\perp$ *such that* $\vartheta_* A_3, \vartheta_* B_3, \vartheta_* B_4 \in \Gamma(D_2), B_2 \in \Gamma(range\vartheta_*)^\perp$ *such that* $\vartheta_* A_4 = \beta_1 B_2$.

Proof. For $\vartheta_* A_3, \vartheta_* B_3, \vartheta_* B_4 \in \Gamma(D_2), B_2 \in \Gamma(range\vartheta_*)^\perp$, using (3),(5), (15) and since ϑ is a CRM, then we have

$$\begin{aligned} g_{S_2}(\nabla^{S_2}_{B_3}\vartheta_* B_4, B_2) &= g_{S_2}(F\nabla^{S_2}_{B_3}\vartheta_* B_4, FB_2) + \eta(\nabla^{S_2}_{B_3}\vartheta_* B_4)\eta(B_2) \\ &= g_{S_2}(-g_{S_2}(B_3, \vartheta_* B_4)\xi + \eta(\vartheta_* B_4)B_3 \\ &\quad + \nabla^{S_2}_{B_3} F\vartheta_* B_4, FB_2) + \eta(\nabla^{S_2}_{B_3}\vartheta_* B_4)\eta(B_2) \\ &= -g_{S_2}(B_3, \vartheta_* B_4)\eta(FB_2) + g_{S_2}(B_3, FB_2)\underbrace{\eta(\vartheta_* B_4)}_{0} \\ &\quad + g_{S_2}(\nabla^{S_2}_{B_3} F\vartheta_* B_4, FB_2) + \eta(\nabla^{S_2}_{B_3}\vartheta_* B_4)\eta(B_2) \\ &= -g_{S_2}(B_3, \vartheta_* B_4)\eta(\alpha_1 B_2) - g_{S_2}(\nabla^{S_2}_{B_3}\beta_1 B_2, F\vartheta_* B_4) \\ &\quad - g_{S_2}(\nabla^{S_2}_{B_3}\alpha_1 B_2, F\vartheta_* B_4) + \eta(\nabla^{S_2}_{B_3}\vartheta_* B_4)\eta(B_2) \end{aligned}$$

From (7), (8) and $\vartheta_* A_4 = \beta_1 B_2$ we have

$$\begin{aligned} g_{S_2}(\nabla^{S_2}_{B_3}\vartheta_* B_4, B_2) &= -g_{S_2}(B_3, \vartheta_* B_4)\eta(\alpha_1 B_2) - g_{S_2}((\nabla^{S_2}\vartheta_*)(B_3, A_4) + \vartheta_*(\nabla^{S_1}_{B_3} A_4) \\ &\quad - S_{\alpha_1 B_2}\vartheta_* B_3 + \nabla^{\vartheta\perp}_{B_3}\alpha_1 B_2, F\vartheta_* B_4) + \eta(\nabla^{S_2}_{B_3}\vartheta_* B_4)\eta(B_2). \end{aligned}$$

Since D_2 defines a totally geodesic foliation on S_2, using (12) we have

$$\begin{aligned} g_{S_2}(\nabla^{S_2}_{B_3}\vartheta_* B_4, B_2) &= -g_{S_2}((\nabla^{S_2}\vartheta_*)(B_3, A_4)^{(range\vartheta_*)^\perp} + \nabla^{\vartheta\perp}_{B_3}\alpha_1 B_2, F\vartheta_* B_4) \\ &\quad - g_{S_2}(B_3, \vartheta_* B_4)\eta(\alpha_1 B_2) + \eta(\nabla^{S_2}_{B_3}\vartheta_* B_4)\eta(B_2). \end{aligned}$$

This implies the proof of (i).

On the other hand, by the virtue of (3), (8), (12) and (15) we have

$$\begin{aligned} g_{S_2}(\nabla^{S_2}_{B_3}\vartheta_* A_3, \vartheta_* B_3) &= g_{S_2}(F(\nabla^{S_2}\vartheta_*)(B_3, A_3)^{(range\vartheta_*)^\perp}, \vartheta_* B_3) + \eta(\nabla^{S_2}_{B_3}\vartheta_* A_3)\underbrace{\eta(\vartheta_* B_3)}_{0} \\ &= g_{S_2}(\beta_1(\nabla^{S_2}\vartheta_*)(B_3, A_3)^{(range\vartheta_*)^\perp}, \vartheta_* B_3). \end{aligned}$$

Since D_2 defines a totally geodesic foliation on S_2 then we can say that $\beta_1(\nabla^{S_2}\vartheta_*)(B_3, A_3)^{(range\vartheta_*)^\perp}$ has no components in $\Gamma(D_2)$. This completes the proof of (ii). □

Theorem 3. *Let* ϑ *be a CSIRM from a Riemannian manifold* (S_1, g_{S_1}) *to a Sasakian manifold* $(S_2, F, \xi, \eta, g_{S_2})$. *If* $(range\vartheta_*)$ *defines a totally geodesic foliation on* S_2 *and* ϑ *is a horizontally homothetic CRM then we have*

$$\begin{aligned} g_{S_2}(S_{\alpha_1 B_1}\vartheta_* A_1, \phi\vartheta_* A_2) - g_{S_2}(\vartheta_*(\nabla^{S_1}_{A_1} A_3), \phi\vartheta_* A_2) &= g_{S_2}(S_{\omega\vartheta_* A_2}\vartheta_* A_1, \beta_1 B_1) - g_{S_2}(\nabla^{\vartheta\perp}_{A_1}\omega\vartheta_* A_2, \alpha_1 B_1) \\ &\quad - \eta(\nabla^{S_2}_{A_1}\vartheta_* A_2)\eta(B_1) + g_{S_2}(A_1, \vartheta_* A_2)\eta(\alpha_1 B_1) \quad (16) \end{aligned}$$

for any $A_1, A_2 \in \Gamma(ker\vartheta_*)^\perp$ *such that* $\vartheta_* A_1, \vartheta_* A_2 \in \Gamma(range\vartheta_*), B_1 \in \Gamma(range\vartheta_*)^\perp$ *such that* $\vartheta_* A_3 = \beta_1 B_1$.

Proof. For $A_1, A_2 \in \Gamma(ker\vartheta_*)^\perp$ and $B_1 \in \Gamma(range\vartheta_*)^\perp$, using (3) and (5) we get

$$g_{S_2}(\nabla^{S_2}_{A_1}\vartheta_* A_2, B_1) = g_{S_2}(\nabla^{S_2}_{A_1} F\vartheta_* A_2, FB_1) - g_{S_2}(A_1, \vartheta_* A_2)\eta(FB_1) + \eta(\nabla^{S_2}_{A_1}\vartheta_* A_2)\eta(B_1).$$

From (15) we have

$$\begin{aligned} g_{S_2}(\nabla^{S_2}_{A_1}\vartheta_* A_2, B_1) &= g_{S_2}(\nabla^{S_2}_{A_1} F\vartheta_* A_2, \beta_1 B_1) + g_{S_2}(\nabla^{S_2}_{A_1} F\vartheta_* A_2, \alpha_1 B_1) \\ &\quad - g_{S_2}(A_1, \vartheta_* A_2)\eta(\alpha_1 B_1) + \eta(\nabla^{S_2}_{A_1}\vartheta_* A_2)\eta(B_1) \\ &= -g_{S_2}(\nabla^{S_2}_{A_1}\beta_1 B_1, F\vartheta_* A_2) - g_{S_2}(\nabla^{S_2}_{A_1}\alpha_1 B_1, F\vartheta_* A_2) \\ &\quad - g_{S_2}(A_1, \vartheta_* A_2)\eta(\alpha_1 B_1) + \eta(\nabla^{S_2}_{A_1}\vartheta_* A_2)\eta(B_1). \end{aligned}$$

From (14) and $\vartheta_* A_3 = \beta_1 B_1$ then we have

$$g_{S_2}(\nabla^{S_2}_{A_1}\vartheta_* A_2, B_1) = -g_{S_2}(\nabla^{S_2}_{A_1}\vartheta_* A_3, \phi\vartheta_* A_2) + g_{S_2}(\nabla^{S_2}_{A_1}\omega\vartheta_* A_2, \vartheta_* A_3)$$

$$-g_{S_2}(\nabla^{S_2}_{A_1}\alpha_1 B_1,\phi\vartheta_* A_2)+g_{S_2}(\nabla^{S_2}_{A_1}\omega\vartheta_* A_2,\alpha_1 B_1)$$
$$-g_{S_2}(A_1,\vartheta_* A_2)\eta(\alpha_1 B_1)+\eta(\nabla^{S_2}_{A_1}\vartheta_* A_2)\eta(B_1).$$

Since ϑ is a CRM, using (7)

$$\begin{aligned} g_{S_2}(\nabla^{S_2}_{A_1}\vartheta_* A_2,B_1) &= -g_{S_2}((\nabla^{S_2}\vartheta_*)(A_1,A_3)+\vartheta_*(\nabla^{S_1}_{A_1}A_3),\phi\vartheta_* A_2)\\ &+g_{S_2}(\nabla^{S_2}_{A_1}\omega\vartheta_* A_2,\vartheta_* A_3)-g_{S_2}(\nabla^{S_2}_{A_1}\alpha_1 B_1,\phi\vartheta_* A_2)\\ &+g_{S_2}(\nabla^{S_2}_{A_1}\omega\vartheta_* A_2,\alpha_1 B_1)-g_{S_2}(A_1,\vartheta_* A_2)\eta(\alpha_1 B_1)\\ &+\eta(\nabla^{S_2}_{A_1}\vartheta_* A_2)\eta(B_1). \end{aligned}$$

Using (13), (8) and $\vartheta_* A_3=\beta_1 B_1$ we get

$$\begin{aligned} g_{S_2}(\nabla^{S_2}_{A_1}\vartheta_* A_2,B_1) &= -g_{S_2}(A_1(\ln\lambda)\,\vartheta_* A_3+A_3(\ln\lambda)\,\vartheta_* A_1\\ &-g_{S_1}(A_1,A_3)\vartheta_*(grad(\ln\lambda)),\phi\vartheta_* A_2)\\ &-g_{S_2}(\vartheta_*(\nabla^{S_1}_{A_1}A_3),\phi\vartheta_* A_2)-g_{S_2}(S_{\omega\vartheta_* A_2}\vartheta_* A_1,\vartheta_* A_3)\\ &+g_{S_2}(S_{\alpha_1 B_1}\vartheta_* A_1,\phi\vartheta_* A_2)+g_{S_2}(\nabla^{\vartheta\perp}_{A_1}\omega\vartheta_* A_2,\alpha_1 B_1)\\ &-g_{S_2}(A_1,\vartheta_* A_2)\eta(\alpha_1 B_1)+\eta(\nabla^{S_2}_{A_1}\vartheta_* A_2)\eta(B_1). \end{aligned}$$

If we take $A_1(\ln\lambda)=g_{S_1}(A_1,Hgrad(\ln\lambda))$ and $A_3(\ln\lambda)=g_{S_1}(A_3,Hgrad(\ln\lambda))$, then we obtain

$$\begin{aligned} g_{S_2}(\nabla^{S_2}_{A_1}\vartheta_* A_2,B_1) &= -g_{S_1}(A_1,Hgrad(\ln\lambda))g_{S_2}(\vartheta_* A_3,\phi\vartheta_* A_2)\\ &-g_{S_1}(A_3,Hgrad(\ln\lambda))g_{S_2}(\vartheta_* A_1,\phi\vartheta_* A_2)\\ &-g_{S_1}(A_1,A_3)g_{S_2}(\vartheta_*(grad(\ln\lambda)),\phi\vartheta_* A_2)\\ &-g_{S_2}(\vartheta_*(\nabla^{S_1}_{A_1}A_3),\phi\vartheta_* A_2)-g_{S_2}(S_{\omega\vartheta_* A_2}\vartheta_* A_1,\beta_1 B_1)\\ &+g_{S_2}(S_{\alpha_1 B_1}\vartheta_* A_1,\phi\vartheta_* A_2)+g_{S_2}(\nabla^{\vartheta\perp}_{A_1}\omega\vartheta_* A_2,\alpha_1 B_1)\\ &-g_{S_2}(A_1,\vartheta_* A_2)\eta(\alpha_1 B_1)+\eta(\nabla^{S_2}_{A_1}\vartheta_* A_2)\eta(B_1). \end{aligned} \tag{17}$$

Since $(range\vartheta_*)$ describes a totally geodesic foliation on S_2 and ϑ is a horizontally homothetic CRM, then from (17) we obtain (16). □

Theorem 4. *Let ϑ be a CSIRM from a Riemannian manifold (S_1,g_{S_1}) to a Sasakian manifold (S_2,F,ξ,η,g_{S_2}). Then $(range\vartheta_*)^\perp$ defines a totally geodesic foliation on S_2 if and only if*

$$\begin{aligned} g_{S_2}(\alpha_1 B_1,(\nabla\vartheta_*)(A_1,A_2)^{(range\vartheta_*)^\perp}) &= g_{S_2}(B_2,[B_1,\vartheta_* A_1]+\nabla^{\vartheta\perp}_{\vartheta_* A_1}F\beta_1 B_1\\ &+\ \alpha_1\nabla^{\vartheta\perp}_{\vartheta_* A_1}F\alpha_1 B_1)+B_1\eta(\omega\vartheta_* A_1), \end{aligned}$$

for any $B_1,B_2\in\Gamma(range\vartheta_*)^\perp$ and $A_1,A_2\in\Gamma(ker\vartheta_*)^\perp$ such that $\vartheta_* A_2=\beta_1 B_2$.

Proof. For any $B_1,B_2\in\Gamma(range\vartheta_*)^\perp$ and $A_1,A_2\in\Gamma(ker\vartheta_*)^\perp$, using (3), (5) and since S_2 is a Sasakian manifold,

$$\begin{aligned} g_{S_2}(\nabla^{S_2}_{B_1}B_2,\vartheta_* A_1) &= -g_{S_2}(B_2,[B_1,\vartheta_* A_1])-g_{S_2}(FB_2,\nabla^{S_2}_{\vartheta_* A_1}FB_1)\\ &-g_{S_2}(B_2,B_1)\eta(F\vartheta_* A_1)+\eta(\nabla^{S_2}_{B_1}B_2)\underbrace{\eta(\vartheta_* A_1)}_{0}. \end{aligned}$$

Then using (7), (8), (14) and (15) we have

$$\begin{aligned} g_{S_2}(\nabla^{S_2}_{B_1}B_2,\vartheta_* A_1) &= -g_{S_2}(B_2,[B_1,\vartheta_* A_1])-g_{S_2}(B_2,\nabla^{\vartheta\perp}_{\vartheta_* A_1}F\beta_1 B_1)+g_{S_2}(\alpha_1 B_1,(\nabla\vartheta_*)(A_1,A_2))\\ &-g_{S_2}(B_2,\alpha_1\nabla^{\vartheta\perp}_{\vartheta_* A_1}F\alpha_1 B_1)-g_{S_2}(B_2,B_1\eta(\omega\vartheta_* A_1)). \end{aligned}$$

From (12), (10) and since $(range\vartheta_*)^\perp$ defines a totally geodesic foliation we have

$$\begin{aligned} g_{S_2}(\alpha_1 B_1,(\nabla\vartheta_*)(A_1,A_2)^{(range\vartheta_*)^\perp}) &= g_{S_2}(B_2,[B_1,\vartheta_* A_1]+\nabla^{\vartheta\perp}_{\vartheta_* A_1}F\beta_1 B_1+\alpha_1\nabla^{\vartheta\perp}_{\vartheta_* A_1}F\alpha_1 B_1)\\ &+\ B_1\eta(\omega\vartheta_* A_1). \end{aligned}$$

This completes the proof. □

Remark 2. *Let ϑ be a CSIRM from a Riemannian manifold (S_1, g_{S_1}) to a Sasakian manifold $(S_2, F, \xi, \eta, g_{S_2})$. From the second fundamental form, one can easly see that $ker\vartheta_*$ and $(ker\vartheta_*)^\perp$ define a totally geodesic foliation on S_1.*

From the above fact we can state following theorem.

Theorem 5. *Let ϑ be a CSIRM from a Riemannian manifold (S_1, g_{S_1}) to a Sasakian manifold $(S_2, F, \xi, \eta, g_{S_2})$. Then ϑ is totally geodesic foliation if and only if*

$$\begin{aligned}\phi((\nabla\vartheta_*)(A_1,A_2)^{range\vartheta_*} - \vartheta_*(\nabla^{S_1}_{A_1}A_2) - S_{\omega\vartheta_*A_3}\vartheta_*A_1) &= -\beta_1((\nabla\vartheta_*)(A_1,A_2)^{(range\vartheta_*)^\perp}\\ &+ \nabla^{\vartheta\perp}_{A_1}\omega\vartheta_*A_3) - \vartheta_*(\nabla^{S_1}_{A_1}A_3), \quad (18)\\ \omega((\nabla\vartheta_*)(A_1,A_2)^{range\vartheta_*} - \vartheta_*(\nabla^{S_1}_{A_1}A_2) - S_{\omega\vartheta_*A_3}\vartheta_*A_1) &= -\alpha_1((\nabla\vartheta_*)(A_1,A_2)^{(range\vartheta_*)^\perp}\\ &+ + \nabla^{\vartheta\perp}_{A_1}\omega\vartheta_*A_3) + \eta(\nabla^{S_2}_{A_1}\vartheta_*A_3)\xi, \quad (19)\end{aligned}$$

for any $A_1, A_2, A_3 \in \Gamma(ker\vartheta_*)^\perp$ such that $\vartheta_*A_2 = \phi\vartheta_*A_3$.

Proof. For $A_1, A_3 \in \Gamma(ker\vartheta_*)^\perp$, using (2), (5), (7) and (14) we have

$$\begin{aligned}(\nabla^{S_2}\vartheta_*)(A_1,A_3) &= \nabla^{S_2}_{A_1}\vartheta_*A_3 - \vartheta_*(\nabla^{S_1}_{A_1}A_3)\\ &= -F(\nabla^{S_2}_{A_1}\phi\vartheta_*A_3 + \nabla^{S_2}_{A_1}\omega\vartheta_*A_3)\\ &\quad -\vartheta_*(\nabla^{S_1}_{A_1}A_3) + \eta(\nabla^{S_2}_{A_1}\vartheta_*A_3)\xi.\end{aligned}$$

From (8) and (12) we have

$$\begin{aligned}(\nabla^{S_2}\vartheta_*)(A_1,A_3) &= -F((\nabla\vartheta_*)(A_1,A_2)^{range\vartheta_*}) - F((\nabla\vartheta_*)(A_1,A_2)^{(range\vartheta_*)^\perp})\\ &\quad -F(\vartheta_*(\nabla^{S_1}_{A_1}A_2)) + F(S_{\omega\vartheta_*A_3}\vartheta_*A_1) - F(\nabla^{\vartheta\perp}_{A_1}\omega\vartheta_*A_3)\\ &\quad -\vartheta_*(\nabla^{S_1}_{A_1}A_3) + \eta(\nabla^{S_2}_{A_1}\vartheta_*A_3)\xi.\end{aligned}$$

Since ϑ is a CRM, from (14) and (15) we have

$$\begin{aligned}(\nabla^{S_2}\vartheta_*)(A_1,A_3) &= -\phi((\nabla\vartheta_*)(A_1,A_2)^{range\vartheta_*}) - \omega((\nabla\vartheta_*)(A_1,A_2)^{range\vartheta_*})\\ &\quad -\beta_1((\nabla\vartheta_*)(A_1,A_2)^{(range\vartheta_*)^\perp}) - \alpha_1((\nabla\vartheta_*)(A_1,A_2)^{(range\vartheta_*)^\perp})\\ &\quad -\phi(\vartheta_*(\nabla^{S_1}_{A_1}A_2)) - \omega(\vartheta_*(\nabla^{S_1}_{A_1}A_2)) + \phi(S_{\omega\vartheta_*A_3}\vartheta_*A_1) + \omega(S_{\omega\vartheta_*A_3}\vartheta_*A_1)\\ &\quad -\beta_1(\nabla^{\vartheta\perp}_{A_1}\omega\vartheta_*A_3) - \alpha_1(\nabla^{\vartheta\perp}_{A_1}\omega\vartheta_*A_3) - \vartheta_*(\nabla^{S_1}_{A_1}A_3) + \eta(\nabla^{S_2}_{A_1}\vartheta_*A_3)\xi.\end{aligned}$$

Taking $range\vartheta_*$ and $(range\vartheta_*)^\perp$ components we have

$$\begin{aligned}\phi((\nabla\vartheta_*)(A_1,A_3)^{range\vartheta_*}) &= -\phi((\nabla\vartheta_*)(A_1,A_2)^{range\vartheta_*}) + \vartheta_*(\nabla^{S_1}_{A_1}A_2) - S_{\omega\vartheta_*A_3}\vartheta_*A_1)\\ &\quad -\beta_1((\nabla\vartheta_*)(A_1,A_2)^{(range\vartheta_*)^\perp} + \nabla^{\vartheta\perp}_{A_1}\omega\vartheta_*A_3) - \vartheta_*(\nabla^{S_1}_{A_1}A_3)\end{aligned}$$

$$\begin{aligned}(\nabla\vartheta_*)(A_1,A_3)^{(range\vartheta_*)^\perp} &= -\omega((\nabla\vartheta_*)(A_1,A_2)^{range\vartheta_*} + \vartheta_*(\nabla^{S_1}_{A_1}A_2) - S_{\omega\vartheta_*A_3}\vartheta_*A_1)\\ &\quad -\alpha_1((\nabla\vartheta_*)(A_1,A_2)^{(range\vartheta_*)^\perp} + \nabla^{\vartheta\perp}_{A_1}\omega\vartheta_*A_3) + \eta(\nabla^{S_2}_{A_1}\vartheta_*A_3)\xi.\end{aligned}$$

Thus $(\nabla\vartheta_*)(A_1,A_3) = 0$ if and only if (18) and (19) are satisfied. This completes the proof. □

Proposition 1. *Let ϑ be a CSIRM from a Riemannian manifold (S_1, g_{S_1}) to a Sasakian manifold $(S_2, F, \xi, \eta, g_{S_2})$ such that $\dim(range\vartheta_*) > 1$. Then the following statements are true.*

(i). $\bar{D}_1$ defines a totally geodesic foliation if and only if $(\nabla\vartheta_*)(A_1, U_1)$ has no component in D_1 such that

$$g_{S_2}(-S_{F\vartheta_*A_1'}\vartheta_*A_1 + g_{S_2}(A_1', \xi)\vartheta_*A_1, F\vartheta_*A_2) = \eta(\nabla^{S_1}_{A_1}A_2)\eta(A_1')$$

for $A_1, A_2 \in \Gamma(\bar{D}_1), U_1 \in \Gamma(ker\vartheta_*)$ and $A_1' \in \Gamma(\bar{D}_2)$.

(ii). $\bar{D}_2$ defines a totally geodesic foliation if and only if $(\nabla\vartheta_*)(A_2, U_1)$ has no component in D_2 such that

$$g_{S_2}(S_{F\vartheta_*A_3'}\vartheta_*A_3, F\vartheta_*A_4) = g_{S_2}(g_{S_2}(A_3', \xi)\vartheta_*A_3, F\vartheta_*A_4) - \eta(\nabla^{S_1}_{\vartheta_*A_3}\vartheta_*A_4)\eta(\vartheta_*A_3')$$

for $A_2, A_3, A_4 \in \Gamma(\bar{D}_2), U_1 \in \Gamma(ker\vartheta_*)$ and $A_3' \in \Gamma(\bar{D}_2)$.

Proof. We know that $\bar{D}_1$ defines totally geodesic foliation if and only if $g_{S_1}(\nabla^{S_1}_{A_1} A_2, U_1) = 0$ and $g_{S_1}(\nabla^{S_1}_{A_1} A_2, A_1^{'}) = 0$ for $A_1, A_2 \in \Gamma(\bar{D}_1), U_1 \in \Gamma(ker\vartheta_*)$ and $A_1^{'} \in \Gamma(\bar{D}_2)$. Now, since ϑ is Riemannian map, using (1), (7) and (8) we have

$$\begin{aligned} g_{S_1}(\nabla^{S_1}_{A_1} A_2, U_1) &= -g_{S_1}(\nabla^{S_1}_{A_1} U_1, A_2) \\ &= -g_{S_2}(\vartheta_*(\nabla^{S_1}_{A_1} U_1), \vartheta_* A_2) \\ &= g_{S_2}((\nabla \vartheta_*)(A_1, U_1), \vartheta_* A_2), \end{aligned}$$

and similarly

$$\begin{aligned} g_{S_1}(\nabla^{S_1}_{A_1} A_2, A_1^{'}) &= -g_{S_1}(\nabla^{S_1}_{A_1} A_1^{'}, A_2) \\ &= -g_{S_2}(\vartheta_*(\nabla^{S_1}_{A_1} A_1^{'}), \vartheta_* A_2) \\ &= -g_{S_2}(\nabla^{\vartheta}_{A_1} \vartheta_* A_1^{'}, \vartheta_* A_2) \\ &= -g_{S_2}(\nabla^{S_2}_{\vartheta_* A_1} \vartheta_* A_1^{'}, \vartheta_* A_2). \end{aligned}$$

Since S_2 is Sasakian manifold, using (3), (5) and then (8), we have

$$g_{S_1}(\nabla^{S_1}_{A_1} A_2, A_1^{'}) = -g_{S_2}(-S_{F\vartheta_* A_1^{'}} \vartheta_* A_1, F\vartheta_* A_2) + \eta(\nabla^{S_1}_{A_1} A_2)\eta(A_1^{'}) - g_{S_2}(g_{S_2}(A_1^{'}, \xi)\vartheta_* A_1, F\vartheta_* A_2)$$

This completes the proof of (i).

On the other hand, we know that $\bar{D}_2$ defines a totally geodesic foliation if and only if $g_{S_1}(\nabla^{S_1}_{A_3} A_4, U_1) = 0$ and $g_{S_1}(\nabla^{S_1}_{A_3} A_4, A_3^{'}) = 0$ for $A_3, A_4 \in \Gamma(\bar{D}_2), U_1 \in \Gamma(ker\vartheta_*)$ and $A_1^{'} \in \Gamma(\bar{D}_1)$. Now, since ϑ is Riemannian map, using (1) and (7) we have

$$\begin{aligned} g_{S_1}(\nabla^{S_1}_{A_3} A_4, U_1) &= -g_{S_1}(\nabla^{S_1}_{A_3} U_1, A_4) \\ &= -g_{S_2}(\vartheta_*(\nabla^{S_1}_{A_3} U_1), \vartheta_* A_4) \\ &= g_{S_2}((\nabla^{S_2} \vartheta_*)(A_3, U_1), \vartheta_* A_4), \end{aligned}$$

and similarly

$$\begin{aligned} g_{S_1}(\nabla^{S_1}_{A_3} A_4, A_3^{'}) &= g_{S_2}(\vartheta_*(\nabla^{S_1}_{A_3} A_4), \vartheta_* A_3^{'}) \\ &= g_{S_2}(\nabla^{\vartheta}_{A_3} \vartheta_* A_4, \vartheta_* A_3^{'}) \\ &= g_{S_2}(\nabla^{S_2}_{\vartheta_* A_3} \vartheta_* A_4, \vartheta_* A_3^{'}). \end{aligned}$$

Since S_2 is Sasakian manifold, using (3),(5) and then (8), we have

$$g_{S_1}(\nabla^{S_1}_{A_3} A_4, A_3^{'}) = -g_{S_2}(-S_{F\vartheta_* A_3^{'}} \vartheta_* A_3, F\vartheta_* A_4) + \eta(\nabla^{S_1}_{\vartheta_* A_3} \vartheta_* A_4)\eta(\vartheta_* A_3^{'}) - g_{S_2}(g_{S_2}(A_3^{'}, \xi)\vartheta_* A_3, F\vartheta_* A_4).$$

This completes the proof of (ii). □

Definition 4. *[19] Let (S_1, g_{S_1}) be a Riemannian manifold and assume that the canonical foliations K_1 and K_2 such that $K_1 \cap K_2 = \{0\}$ everywhere. Then (S_1, g_{S_1}) is a locally product manifold if and only if K_1 and K_2 are totally geodesic foliations.*

Theorem 6. *Let ϑ be a CSIRM from a Riemannian manifold (S_1, g_{S_1}) to a Sasakian manifold $(S_2, F, \xi, \eta, g_{S_2})$ such that $\dim(range\vartheta_*) > 1$. Then $(ker\vartheta_*)^{\perp}$ is a locally product manifold of $\bar{D}_1$ and $\bar{D}_2$ if and only if*

(i). $(\nabla\vartheta_*)(A_1, U_1)$ has no component in D_1 such that –

$$g_{S_2}(S_{F\vartheta_* A_3^{'}} \vartheta_* A_1, F\vartheta_* A_2) = g_{S_2}(g_{S_2}(A_1^{'}, \xi)\vartheta_* A_1, F\vartheta_* A_4) - \eta(\nabla^{S_1}_{A_1} A_2)\eta(A_1^{'})$$

for $A_1, A_2 \in \Gamma(\bar{D}_1), U_1 \in \Gamma(ker\vartheta_*)$ and $A_1^{'} \in \Gamma(\bar{D}_2)$,

(ii). $(\nabla\vartheta_*)(A_3, U_1)$ has no component in D_2 such that

$$g_{S_2}(S_{F\vartheta_* A_3^{'}} \vartheta_* A_3, F\vartheta_* A_4) = g_{S_2}(g_{S_2}(A_3^{'}, \xi)\vartheta_* A_3, F\vartheta_* A_4) - \eta(\nabla^{S_1}_{\vartheta_* A_3} \vartheta_* A_4)\eta(A_3^{'})$$

for $A_2, A_3, A_4 \in \Gamma(\bar{D}_2), U_1 \in \Gamma(ker\vartheta_*)$ and $A_3^{'} \in \Gamma(\bar{D}_2)$.

Proof. The proof is clear by Proposition (1) and Definition (4). □

Theorem 7. *Let ϑ be a CSIRM from a Riemannian manifold (S_1, g_{S_1}) to a Sasakian manifold $(S_2, F, \xi, \eta, g_{S_2})$ such that $\dim(range\vartheta_*) > 1$. Then the base manifold is locally product manifold $S_2 \times S_2$ if and only if*

$$g_{S_2}(\vartheta_*(\nabla^{S_2}_{A_1}\phi\vartheta_*A_1), \beta_1 B_1) + g_{S_2}(\vartheta_*(\nabla^{\vartheta\perp}_{A_1}\omega\vartheta_*A_1), \alpha_1 B_1) + g_{S_2}(S_{\alpha_1 B_1}\vartheta_*A_1, \phi\vartheta_*A_1)$$
$$+\eta(\nabla^{S_2}_{\vartheta_*A_1}\vartheta_*A_1)\eta(B_1) = |\vartheta_*A_1|^2\,\Gamma(\alpha_1 B_1),$$

for $A_1 \in \Gamma(\bar{D}_1)$ and $B_1 \in \Gamma(range\vartheta_*)^\perp$.

Proof. Since S_2 is Sasakian manifold, using (3) and (3) we have

$$\begin{aligned} g_{S_2}(\nabla^{S_2}_{\vartheta_*A_1}\vartheta_*A_1, B_1) &= g_{S_2}(\nabla^{S_2}_{\vartheta_*A_1}F\vartheta_*A_1, FB_1) + \eta(\nabla^{S_2}_{\vartheta_*A_1}\vartheta_*A_1)\eta(B_1) \\ &\quad - |\vartheta_*A_1|^2\,\Gamma(\alpha_1 B_1) + \Gamma(\vartheta_*A_1)\underbrace{g_{S_2}(\vartheta_*A_1, FB_1)}_{0}, \end{aligned}$$

for $\vartheta_*A_1 \in \Gamma(range\vartheta_*)$ and $B_1 \in \Gamma(range\vartheta_*)^\perp$. Using (14) and (15) then we have

$$\begin{aligned} g_{S_2}(\nabla^{S_2}_{\vartheta_*A_1}\vartheta_*A_1, B_1) &= g_{S_2}(\nabla^{S_2}_{\vartheta_*A_1}\phi\vartheta_*A_1, \beta_1 B_1) + g_{S_2}(\nabla^{S_2}_{\vartheta_*A_1}\omega\vartheta_*A_1, \beta_1 B_1) + g_{S_2}(\nabla^{S_2}_{\vartheta_*A_1}\phi\vartheta_*A_1, \alpha_1 B_1) \\ &\quad + g_{S_2}(\nabla^{S_2}_{\vartheta_*A_1}\omega\vartheta_*A_1, \alpha_1 B_1) + \eta(\nabla^{S_2}_{\vartheta_*A_1}\vartheta_*A_1)\eta(B_1) - |\vartheta_*A_1|^2\,\Gamma(\alpha_1 B_1), \end{aligned}$$

using (8) in above equation, we get

$$\begin{aligned} g_{S_2}(\nabla^{S_2}_{\vartheta_*A_1}\vartheta_*A_1, B_1) &= g_{S_2}(\nabla^{S_2}_{\vartheta_*A_1}\phi\vartheta_*A_1, \beta_1 B_1) + g_{S_2}(S_{\alpha_1 B_1}\vartheta_*A_1, \phi\vartheta_*A_1) \\ &\quad + g_{S_2}(\nabla^{\vartheta\perp}_{A_1}\omega\vartheta_*A_1, \alpha_1 B_1) + \eta(\nabla^{S_2}_{\vartheta_*A_1}\vartheta_*A_1)\eta(B_1) - |\vartheta_*A_1|^2\,\Gamma(\alpha_1 B_1). \end{aligned}$$

Then, using (7) we obtain

$$\begin{aligned} g_{S_2}(\nabla^{S_2}_{\vartheta_*A_1}\vartheta_*A_1, B_1) &= g_{S_2}((\nabla\vartheta_*)(A_1, {}^*\vartheta_*(\phi\vartheta_*A_1)), \beta_1 B_1) + g_{S_2}(S_{\alpha_1 B_1}\vartheta_*A_1, \phi\vartheta_*A_1) \\ &\quad + g_{S_2}(\nabla^{\vartheta\perp}_{A_1}\omega\vartheta_*A_1, \alpha_1 B_1) + \eta(\nabla^{S_2}_{\vartheta_*A_1}\vartheta_*A_1)\eta(B_1) - |\vartheta_*A_1|^2\,\Gamma(\alpha_1 B_1), \end{aligned}$$

from Definition (4), the proof is completed. ∎

Now, we will examine the harmonicity of CSIRM from a Riemannian manifold (S_1, g_{S_1}) to Sasakian manifold $(S_2, F, \xi, \eta, g_{S_2})$ in the following theorem.

Theorem 8. *Let ϑ be a CSIRM from a Riemannian manifold (S_1, g_{S_1}) to a Sasakian manifold $(S_2, F, \xi, \eta, g_{S_2})$. Then ϑ is harmonic if and only if the following conditions are satisfied*

(i). The fibres are minimal,

(ii).

$$trace\phi S_{\omega\vartheta_*A_1}A_1 - \beta_1\,\nabla^{\vartheta\perp}_{A_1}\,\omega\vartheta_*A_1 - \vartheta_*(\nabla^{S_1}_{A_1}A_1) - (\nabla^{S_2}F\phi\vartheta_*A_1)^{range\vartheta_*} = 0,$$

(iii).

$$trace\omega S_{\omega\vartheta_*A_1}A_1 - \alpha_1\,\nabla^{\vartheta\perp}_{A_1}\,\omega\vartheta_*A_1 - (\nabla^{S_2}F\phi\vartheta_*A_1)^{(range\vartheta_*)^\perp} + \eta((\nabla\vartheta_*)(A_1, A_1)^{(range\vartheta_*)^\perp})\xi = 0$$

for $A_1 \in (ker\vartheta_*)^\perp$.

Proof. For $U_1 \in ker\vartheta_*$ using (7) we get

$$\begin{aligned} (\nabla\vartheta_*)(U_1, U_1) &= \nabla^{S_2}_{U_1}\vartheta_*U_1 - \vartheta_*(\nabla^{S_1}_{U_1}U_1) \\ &= -\vartheta_*(\nabla^{S_1}_{U_1}U_1), \end{aligned} \tag{20}$$

since $\vartheta_*U_1 = 0$. For $A_1 \in (ker\vartheta_*)^\perp$ using (3), (7), (15), (12) and (8) we have

$$\begin{aligned} (\nabla\vartheta_*)(A_1, A_1) &= \nabla^{S_2}_{A_1}\vartheta_*A_1 - \vartheta_*(\nabla^{S_1}_{A_1}A_1) \\ &= -\nabla^{S_2}_{A_1}\,F\phi\vartheta_*A_1 - F(\nabla^{S_2}_{A_1}\omega\vartheta_*A_1) - \vartheta_*(\nabla^{S_1}_{A_1}A_1) + \eta(\nabla^{S_2}_{A_1}\vartheta_*A_1)\xi \\ &= -\nabla^{S_2}_{A_1}\,F\phi\vartheta_*A_1 - F(-S_{\omega\vartheta_*A_1}\vartheta_*A_1 + \nabla^{\vartheta\perp}_{A_1}\omega\vartheta_*A_1) - \vartheta_*(\nabla^{S_1}_{A_1}A_1) + \eta(\nabla^{S_2}_{A_1}\vartheta_*A_1)\xi. \end{aligned}$$

Since ϑ is a CRM, from (14) and (15) we have

$$\begin{aligned} (\nabla\vartheta_*)(A_1, A_1) &= -\nabla^{S_2}_{A_1}\,F\phi\vartheta_*A_1 + \phi S_{\omega\vartheta_*A_1}\vartheta_*A_1 + \omega S_{\omega\vartheta_*\vartheta_*A_1}\vartheta_*A_1 \\ &\quad -\beta_1\,\nabla^{\vartheta\perp}_{A_1}\,\omega\vartheta_*A_1 - \alpha_1\,\nabla^{\vartheta\perp}_{A_1}\,\omega\vartheta_*A_1 - \vartheta_*(\nabla^{S_1}_{A_1}A_1) + \eta(\nabla^{S_2}_{A_1}\vartheta_*A_1)\xi. \end{aligned}$$

Taking $range\vartheta_*$ and $(range\vartheta_*)^\perp$ components we have

$$(\nabla\vartheta_*)(A_1,A_1)^{range\vartheta} = \phi S_{\omega\vartheta_* A_1}\vartheta_* A_1 - \beta_1 \nabla^{\vartheta\perp}_{A_1}\omega\vartheta_* A_1 - \vartheta_*(\nabla^{S_1}_{A_1}A_1) - (\nabla^{S_2}_{A_1}F\phi\vartheta_* A_1)^{range\vartheta} \tag{21}$$

and

$$\begin{aligned}(\nabla\vartheta_*)(A_1,A_1)^{(range\vartheta_*)^\perp} &= \omega S_{\omega\vartheta_*\vartheta_* A_1}\vartheta_* A_1 - \alpha_1 \nabla^{\vartheta\perp}_{A_1}\omega\vartheta_* A_1 - (\nabla^{S_2}_{A_1}F\phi\vartheta_* A_1)^{(range\vartheta_*)^\perp} \\ &+ \eta((\nabla\vartheta_*)(A_1,A_1)^{(range\vartheta_*)^\perp})\xi. \end{aligned} \tag{22}$$

Thus the proof is completed from (20), (21) and (22). □

4. Future Studies

The Clairaut Riemannian maps are particular Riemannian maps having important applications in the geometry [13,26]. The notions of invariant, anti-invariant, and semi-invariant Clairaut Riemannian maps with almost hermitian manifolds have been studied by the first author and other authors in [9,18,30,31]. Recently, the notions of Clairaut conformal submersions and Clairaut conformal Riemannian maps have been introduced in [11,12] and showed that these smooth maps generate a lot of interest due to their associated geometric properties. Our paper combined the notions of conformal Riemannian maps and semi-invariant Riemannian maps to Sasakian manifolds. Therefore in future it will be interesting to combine more notions of Clairaut Riemannian maps to these two notions and study Clairaut conformal semi-invariant Riemannian maps (and in particular Clairaut conformal semi-invariant submersions) to Kähler and/or to Sasakian manifolds.

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Fischer-Marsden conjecture on K-paracontact manifolds and quasi-para-Sasakian manifolds

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Abstract. The aim of this paper is to study of the non-trivial solutions of Fischer-Marsden conjecture on K-paracontact manifolds and 3-dimensional quasi-para-Sasakian manifolds. We prove that if a semi-Riemannian manifold of dimension $2n+1$ admits a non-trivial solution of Fischer-Marsden equation, then it has constant scalar curvature. We give a comprehensive classification for a $(2n+1)$-dimensional K-paracontact manifold which admits a non-trivial solution of Fischer-Marsden equation. We consider 3-dimensional quasi-para-Sasakian manifolds with β constant which admits Fischer-Marsden equation and prove that there are two possibilities. The first one is the scalar curvature $r=-6\beta^2$ and M^3 is Einstein. The second one is the manifold is paracosymplectic manifold and η-Einstein.

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Keywords. Fischer-Marsden equation, K-paracontact manifold, quasi-para-Sasakian manifold, gradient Ricci soliton.

1. Introduction

In modern physics, the general theory of relativity provides an interpretation of many cosmological events, from the expansion of the universe to black holes. A significant global solution of Einstein equation in general relativity is *static space-times.* A semi-Riemannian manifold (M^{2n+1}, g) and positive function λ, we say that $(\bar{M}^{2n+2}, \bar{g}) = M^{2n+1} \times_\lambda \mathbb{R}$ endowed with the metric $\bar{g} = g - \lambda^2 dt^2$ is a static space-time. In this case, the Einstein equation with perfect fluid as a matter field over $(\bar{M}^{2n+2}, \bar{g})$ is given by

$$S_{\bar{g}} - \frac{r_{\bar{g}}}{2}\bar{g} = T, \tag{1}$$

where $T = \mu\lambda^2 dt^2 + \rho g$ is the stress-energy-momentum tensor of perfect fluid, $S_{\bar{g}}$ and $r_{\bar{g}}$ denotes the Ricci tensor and scalar curvature for the metric $\bar{g}$, resp. Moreover, the smooth functions μ and ρ are *energy density* and *pressure* of the perfect fluid, resp. *Static perfect fluid space-times* is a generalization of the static vacuum spaces and solution of (1). Also, it provides models for black holes, galaxies and stellars [7, 9]. Fischer-Marsden equation can be considered as a special case of the static perfect fluid space-times [5, Remark 1.3].

On the other hand, Fischer-Marsden conjecture is closely related the conjecture that known as *Cosmic no-hair conjecture*. We recall the Cosmic no-hair conjecture as *"the hemisphere $\mathbb{S}^n_+$ is the only possible n-dimensional positive static triple with single-horizon and positive scalar curveture"* [9].

Let (M^{2n+1}, g) be a compact, orientable semi-Riemannian manifold. We denote the set of all unit volume semi-Riemannian metrics on (M^{2n+1}, g) by $\mathcal{M}$. The linearization of the scalar curvature $\mathcal{L}_g(g^*)$ is given by

$$\mathcal{L}_g g^* = -\Delta_g(tr_g g^*) + div(div(g^*)) - g(g^*, S),$$

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where Δ_g, div, g^* and S denotes the negative Laplacian of the semi-Riemannian metric g, divergence operator, symmetric $(0,2)$ tensor field on M and the Ricci tensor, resp. The formal L^2-adjoint $\mathcal{L}_g g^*$ of the linearized scalar curvature operator $\mathcal{L}_g$ is defined by

$$\mathcal{L}_g^*(\lambda) = -(\Delta_g \lambda) g + Hess_g \lambda - \lambda S, \tag{2}$$

where $Hess_g \lambda(U,V) = \nabla_g^2 \lambda(U,V) = g(\nabla_U D\lambda, V)$ is the Hessian operator of the smooth function λ on M and D is the gradient operator of g. We refer the equation $\mathcal{L}_g^*(\lambda) = 0$ as Fischer-Marsden equation (FME). The pair (g, λ) that satisfying $\mathcal{L}_g^*(\lambda) = 0$ is called a solution of Fischer-Marsden equation. A solution with $\lambda = 0$ is called a *trivial solution.* We note that a complete Riemannian manifold that admits a non-trivial solution of Fischer-Marsden equation ($\lambda \neq 0$) has constant scalar curvature [1,10]. Moreover, Corvino [8] proved that a non-trivial solution of FME implies the warped product metric $g^* = g - \lambda^2 dt^2$ is Einstein. Further, we recall Fischer-Marsden conjecture [10] as "*a compact Riemannian manifold that admits a non-trivial solution of the equation $\mathcal{L}_g^*(\lambda) = 0$ is necessarily an Einstein manifold*". In the case of g is conformally flat, counter examples of this conjecture are given by Kobayashi [12] and Lafontaine [16]. This conjecture is investigated by various authors [2–4, 19, 20].

A *Ricci soliton* is a generalization of an Einstein metric [11]. A semi-Riemannian metric g on a semi-Riemannian manifold M^{2n+1} is said to be Ricci soliton if there exist a real number μ and a vector field V on M^{2n+1} satisfying

$$\mathcal{L}_V g + 2S + 2\mu g = 0, \tag{3}$$

where $\mathcal{L}_V g$ and S denote the Lie derivative along the vector field V and the Ricci tensor of g, resp. The vector field V is also called the potential vector field. If soliton constant μ is zero, negative or positive, then the Ricci soliton is said to be *steady, shrinking* or *expanding*, resp. Furthermore, if V is a gradient of a smooth function f, namely, $V = Df$, then the Ricci soliton is called a *gradient Ricci soliton* and the equation (3) becomes

$$Hess(f) + S = \mu g, \tag{4}$$

where $Hess(f)$ is the Hessian of f. In semi-Riemannian manifold M^{2n+1}, the metric g is said to be *gradient η-Ricci soliton* if it satisfies

$$Hess(f) + S = \mu_1 g + \mu_2 \eta \otimes \eta, \tag{5}$$

where f is a smooth function and μ_1, μ_2 are constants [6].

All of the mentioned works motivate us to study Fischer-Marsden conjecture on K-paracontact manifolds and 3-dimensional quasi-para-Sasakian manifolds. This paper is organized in the following way. In section 2, we recall some notations required for this paper. In section 3, first, we prove the counter-part of the theorem which was proved in 1975 [1, 10], namely, we show that in a semi-Riemannian manifold which admits non-trivial solution of Fischer-Marsden equation, the scalar curvature is constant. After that, we gave a comprehensive classification for a $(2n+1)$-dimensional K-paracontact manifold which admits a non-trivial solution of Fischer-Marsden equation. With this Theorem, we have shown one of the difference between contact geometry and paracontact geometry. Also, we prove that if the Ricci operator commutes for a K-paracontact manifold M^{2n+1} with a non-trivial solution of Fischer-Marsden equation, then M^{2n+1} is an Einstein manifold. We show that if a $2n+1$-dimensional para-Sasakian manifold admits a non-trivial solution of Fischer-Marsden equation, then it is Einstein. Moreover, for $n = 1$, the Ricci tensor is parallel and the manifold is Ricci-semisymmetric. We also investigate the relation between Fischer-Marsden conjecture and gradient Ricci solitons on K-paracontact manifolds. In Section 4, we consider 3-dimensional quasi-para-Sasakian manifolds with β constant which admits Fischer-Marsden equation and prove that there are two possibilities. The first one is the scalar curvature $r = -6\beta^2$ and M^3 is Einstein. The second one is the manifold is paracosymplectic manifold which is locally a product of the real line $\mathbb{R}$ and a 2-dimensional para-Kaehlerian manifold, and η-Einstein. Finally, we give the relation between Fischer-Marsden conjecture and gradient Ricci solitons and gradient η-Ricci solitons on quasi-para-Sasakian manifolds M^3.

2. Preliminaries

A $(2n+1)-$ dimensional manifold M is called *almost paracontact manifold* if it admits triple (F, ξ, η) satisfying the followings:

$$\eta(\xi) = 1, \quad F^2 = I - \eta \otimes \xi \tag{6}$$

and F induces on almost paracomplex structure on each fiber of $\mathcal{D} = ker(\eta)$, where F, ξ and η are $(1,1)-$tensor field, vector field and $1-$form, resp. As a natural consequence, the tensor field F has rank $2n$, $F\xi = 0$ and $\eta \circ F = 0$. Here, ξ denotes a certain vector field (referred to as the *Reeb* or *characteristic vector field*) which is dual to η and satisfying $d\eta(\xi, U) = 0$ for all $U \in \chi(M)$. If the structure (M, F, ξ, η) admits a pseudo-Riemannian metric such that

$$g(FU, FV) = -g(U, V) + \eta(U)\eta(V), \tag{7}$$

for all $U, V \in \chi(M)$, then we say that (M, F, ξ, η, g) is an *almost paracontact metric manifold*. It should be noted that a pseudo-Riemannian metric with a given almost paracontact metric manifold structure always have a signature of $(n+1, n)$. On an almost paracontant metric manifold, there always exists an orthogonal basis $\{U_1, \ldots, U_n, V_1, \ldots, V_n, \xi\}$, namely $F-$basis, such that $g(U_i, U_j) = -g(V_i, V_j) = \delta_{ij}$ and $V_i = FU_i$, for any $i, j \in \{1, \ldots, n\}$. Moreover, it is possible to establish the definition of a skew-symmetric tensor field (a 2-form), commonly referred to as the fundamental form, denoted as Φ, by using the equation

$$\Phi(U, V) = g(U, FV).$$

Within the framework of almost paracontact manifolds, the tensor $N^{(1)}$ of type $(1,2)$ can be introduced by

$$N^{(1)}(U, V) = [F, F](U, V) - 2d\eta(U, V)\xi$$

where

$$[F, F](U, V) = F^2[U, V] + [FU, FV] - F[FU, V] - F[U, FV]$$

is the Nijenhuis torsion of F. The almost paracontact manifold is designated as *normal*, when $N^{(1)} = 0$ [23].

Furthermore, an almost paracontact metric manifold is referred to as a *paracontact metric manifold* if the following condition is satisfied for all vector fields $U, V \in \chi(M)$:

$$d\eta(U, V) = g(U, FV) = \Phi(U, V).$$

In a paracontact metric manifold, a symmetric, trace-free operator h is defined as $h := \frac{1}{2}\mathcal{L}_\xi F$, where $\mathcal{L}$ represents the Lie derivative. It is important to note that h equals zero if and only if the vector field ξ is a killing vector. When ξ is a Killing vector, the paracontact metric manifold is referred to as a *K-paracontact manifold*. A normal almost paracontact metric manifold is said to be *para-Sasakian manifold* if $\Phi = d\eta$. Furthermore, a para-Sasakian manifold is also K-paracontact, with the reverse holding true solely in a three-dimensional [23]. An almost paracontact metric manifold is called *quasi-para-Sasakian* when both the structure is normal and its fundamental 2-form is closed.

Actually, three dimensional quasi-para-Sasakian and para-Sasakian manifolds are normal almost paracontact metric manifold in the type of (α, β) with $(0, \beta)$ and $(0, -1)$, resp. In the case of $\alpha = \beta = 0$, the manifold is paracosymplectic [21].

An almost paracontact metric manifold is said to be *η-Einstein* if its Ricci tensor S is of the form

$$S = \mu_1 g + \mu_2 \eta \otimes \eta \tag{8}$$

where μ_1 and μ_2 are smooth functions on the manifold. If M is para-Sasakian, then μ_1 and μ_2 are constants ([23, Proposition 4.7]). If $\mu_2 = 0$, then the manifold is said to be *Einstein*.

In a K-paracontact manifold, we have the following relations [23]:

$$\nabla_U \xi = -FU, \tag{9}$$

$$Q\xi = -2n\xi, \tag{10}$$

$$R(\xi, U)V = (\nabla_U F)V, \tag{11}$$

$$(\nabla_{FU} F)FV - (\nabla_U F)V = 2g(U, V)\xi - (U + \eta(U)\xi)\eta(V), \tag{12}$$

for all $U, V \in \chi(M)$. On K-paracontact manifold, from (8) and (10), we have $\mu_1 + \mu_2 = -2n$. So K-paracontact manifold is Einstein if and only if $S(U, V) = -2ng(U, V)$ for all $U, V \in \chi(M)$. Moreover, the following curvature identities holds for a three-dimensional quasi-para-Sasakian manifold with β constant [14, 15]:

$$
\begin{aligned}
\nabla_U \xi =& \beta F U, & (13)\\
R(U,V)W =& (2\beta^2 + \frac{r}{2})(g(V,W)U - g(U,W)V) - (3\beta^2 + \frac{r}{2})(g(V,W)\eta(U)\xi \\
& - g(U,W)\eta(V)\xi + \eta(V)\eta(W)U - \eta(U)\eta(W)V), & (14)\\
S(U,V) =& (\beta^2 + \frac{r}{2})g(U,V) - (3\beta^2 + \frac{r}{2})\eta(U)\eta(V), & (15)\\
QU =& (\beta^2 + \frac{r}{2})U - (3\beta^2 + \frac{r}{2})\eta(U)\xi, & (16)\\
Q\xi =& -2\beta^2\xi, & (17)
\end{aligned}
$$

where R, S and r are respectively Riemannian curvature, Ricci tensor and scalar curvature of M.

3. K-Paracontact Manifolds Satisfying Fischer-Marsden Equation

Theorem 1. *If a semi-Riemannian manifold (M^n, g) admits a non-trivial solution (g, λ) of Fischer-Marsden equation, then it has constant scalar curvature.*

Proof. Let (M^n, g) be a semi-Riemannian manifold and $\{e_i | 1 \leq i \leq n\}$ be a local frame on a normal coordinate system at any point $p \in M$. Therefore, from [18, Proposition 33, p. 73], we have

$$\nabla_{e_i} e_j = 0 \tag{18}$$

and

$$\nabla_U e_i = \sum_{i=1}^{n} x_j \nabla_{e_j} e_i = 0 \tag{19}$$

for vector field $U = \sum_{i=1}^{n} x_j e_j$ on a neighborhood of $p \in M$. We also know that

$$div(Hess\lambda)(U) = \sum_{i=1}^{n} \varepsilon_i (\nabla_{e_i} Hess\lambda)(U, e_i), \tag{20}$$

where $\varepsilon_i = g(e_i, e_i)$. Computing this covariant derivative, using (18), we have

$$
\begin{aligned}
(\nabla_{e_i} Hess\lambda)(U, e_i) =& \nabla_{e_i} Hess\lambda(U, e_i) - Hess\lambda(\nabla_{e_i} U, e_i) - Hess\lambda(U, \nabla_{e_i} e_i)\\
=& \nabla_{e_i} g(\nabla_U D\lambda, e_i) - g(\nabla_{\nabla_{e_i} U} D\lambda, e_i)\\
=& g(\nabla_{e_i} \nabla_U D\lambda, e_i) - g(\nabla_{\nabla_{e_i} U} D\lambda, e_i). & (21)
\end{aligned}
$$

On the other hand, using the Riemannian curvature tensor and (19), we obtain

$$
\begin{aligned}
g(R(e_i, U)D\lambda, e_i) =& g(\nabla_{e_i} \nabla_U D\lambda, e_i) - g(\nabla_U \nabla_{e_i} D\lambda, e_i) - g(\nabla_{[e_i, U]} D\lambda, e_i)\\
=& g(\nabla_{e_i} \nabla_U D\lambda, e_i) - g(\nabla_U \nabla_{e_i} D\lambda, e_i) - g(\nabla_{\nabla_{e_i} U} D\lambda, e_i). & (22)
\end{aligned}
$$

Using (21) and (22), one can get

$$(\nabla_{e_i} Hess\lambda)(U, e_i) = g(R(e_i, U)D\lambda, e_i) + g(\nabla_U \nabla_{e_i} D\lambda, e_i). \tag{23}$$

By the help of (19) and writing (23) in (20), we derive

$$
\begin{aligned}
div(Hess\lambda)(U) =& \sum_{i=1}^{n} \varepsilon_i g(R(e_i, U)D\lambda, e_i) + \sum_{i=1}^{n} \varepsilon_i g(\nabla_U \nabla_{e_i} D\lambda, e_i)\\
=& \sum_{i=1}^{n} \varepsilon_i g(R(e_i, U)D\lambda, e_i) + \sum_{i=1}^{n} \varepsilon_i U(g(\nabla_{e_i} D\lambda, e_i))\\
=& S(U, D\lambda) + U(\Delta\lambda), & (24)
\end{aligned}
$$

for all vector field U. From (24), we have

$$div(Hess\lambda) = Q(D\lambda) + d(\Delta\lambda). \tag{25}$$

Again, computing the divergence of λS, we obtain

$$div(\lambda S)(U) = \sum_{i=1}^{n} \varepsilon_i (\nabla_{e_i} \lambda S)(U, e_i)$$

$$= \sum_{i=1}^{n} \varepsilon_i [e_i(\lambda) S(U, e_i) + \lambda (\nabla_{e_i} S)(U, e_i)],$$

which gives

$$div(\lambda S) = Q(D\lambda) + \frac{\lambda}{2} dr. \tag{26}$$

At the end, by the parallelity of the semi-Riemannian metric g, we get

$$\begin{aligned} div(\Delta\lambda.g)(U) &= \sum_{i=1}^{n} \varepsilon_i (\nabla_{e_i} \Delta\lambda.g)(U, e_i) \\ &= \sum_{i=1}^{n} \varepsilon_i [e_i(\Delta\lambda.g)(U, e_i) - \Delta\lambda.g(\nabla_{e_i} U, e_i) - \Delta\lambda.g(U, \nabla_{e_i} e_i)] \\ &= \sum_{i=1}^{n} \varepsilon [e_i(\Delta\lambda) g(U, e_i) + \Delta\lambda \{e_i g(U, e_i) - g(\nabla_{e_i} U, e_i) - g(U, \nabla_{e_i} e_i)\}] \\ &= \sum_{i=1}^{n} \varepsilon_i e_i(\Delta\lambda) g(U, e_i) \\ &= \sum_{i=1}^{n} \varepsilon_i g(U, e_i(\Delta\lambda) e_i) \\ &= g(U, d(\Delta\lambda)), \end{aligned}$$

which implies

$$div(\Delta\lambda.g) = d(\Delta\lambda). \tag{27}$$

If (g, λ) is a non-trivial solution of the Fischer-Marsden equation, i.e. $\lambda \neq 0$, then from (2), we have

$$-(\Delta_g \lambda) g + Hess_g \lambda - \lambda S = 0. \tag{28}$$

Taking the divergence in (28), and using (25), (26) and (27), we have

$$\frac{\lambda}{2} dr = 0. \tag{29}$$

Since $\lambda \neq 0$, from (29), the scalar curvature r is constant. □

Proposition 1. *[4] If (g, λ) is a non-trivial solution of the Fischer-Marsden equation on a $(2n+1)$-dimensional paracontact metric manifold M, then the Riemanian curvature tensor and Fischer-Marsden equation can be expressed as*

$$R(U, V) D\lambda = U(\lambda) QV - V(\lambda) QU + \lambda \{(\nabla_U Q)V - (\nabla_V Q)U\} + U(f)V - V(f)U, \tag{30}$$

and

$$\nabla_U D\lambda = \lambda QU + fU, \tag{31}$$

where $f = -\frac{\lambda r}{2n}$, λ is a function of Fischer-Marsden equation and $U, V \in \chi(M)$.

On a K-paracontact manifold, we have $L_\xi Q = 0$ [22]. Then using $L_\xi Q = 0$ and (9), we have the following result.

Lemma 1. *On a $(2n+1)$-dimensional K-paracontact manifold, we have*

$$\nabla_\xi Q = QF - FQ. \tag{32}$$

Theorem 2. *Let (g, λ) be a non-trivial solution of Fischer-Marsden equation on a K-paracontact manifold M of dimension $(2n+1)$. Then either*

(1) *$\xi(\lambda) = \pm\lambda$, or*
(2) *the manifold is an Einstein manifold, or*

(3) *the* $C \neq 0$ *tensor defined by* $C = Q + 2nI$ *and* $1 \leq rank(C_p) \leq n$ *for all* $p \in M$*, where* $C_p \neq 0$. *Further, there exists a basis* $\{U_1, V_1, \ldots, U_n, V_n, \xi\}$ *of* T_pM *such that*

$$g_p(\xi, \xi) = 1, g_p(U_i, V_i) = \pm 1$$

and

$$C_{|<U_i,V_i>} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad or \quad C_{|<U_i,V_i>} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

where there are exactly $rank(C_p)$ *submatrices of the first type.*

If $n = 1$*, such a basis* $\{\xi, U_1, V_1\}$ *satisfies that* $FU_1 = \pm U_1, FV_1 = \mp V_1$*, and the tensor* C *can be written as*

$$C_{|<U_i,V_i>} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Proof. From (9) and (10), we derive

$$(\nabla_U Q)\xi = 2nFU + QFU. \tag{33}$$

Letting $U = \xi$ in (30), we get

$$\begin{aligned} R(\xi, V)D\lambda =& \xi(\lambda)QV - V(\lambda)Q\xi + \lambda\{(\nabla_\xi Q)V - (\nabla_V Q)\xi\} \\ &+ \xi(f)V - V(f)\xi. \end{aligned}$$

In above equation, using (10), (32) and (33), we obtain

$$R(\xi, V)D\lambda = \xi(\lambda)QV + 2nV(\lambda)\xi - \lambda FQV - 2n\lambda FV + \xi(f)V - V(f)\xi. \tag{34}$$

Taking the inner product of (34) with the vector field U, we get

$$\begin{aligned} -g(R(V, \xi)D\lambda, U) =& \xi(\lambda)S(V, U) + 2nV(\lambda)\eta(U) + \lambda S(FU, V) \\ &- 2n\lambda g(FV, U) + \xi(f)g(V, U) - V(f)\eta(U). \end{aligned} \tag{35}$$

From (11) and (35), we have

$$\begin{aligned} &g((\nabla_V F)U, D\lambda) + \xi(\lambda)S(V, U) + [2nV(\lambda) - V(f)]\eta(U) \\ &- 2n\lambda g(FV, U) + \xi(f)g(V, U) + \lambda S(FU, V) = 0. \end{aligned} \tag{36}$$

Letting $U = FU$ and $V = FV$ in (36), we obtain

$$g((\nabla_{FV} F)FU, D\lambda) + \xi(\lambda)S(FV, FU) + \xi(f)g(FV, FU) - 2n\lambda g(F^2V, FU) + \lambda S(F^2U, FV) = 0. \tag{37}$$

By substracting (37) from (36) and using the equations (6), (7), (10) and (12), we get

$$\begin{aligned} &2\xi(\lambda - f)g(U, V) - V((2n + 1)\lambda - f)\eta(U) - \xi(\lambda - f)\eta(U)\eta(V) - \xi(\lambda)S(V, U) \\ &+ 4n\lambda g(FV, U) + \lambda g(U, QFV + FQV) + \xi(\lambda)g(QFV, FU) = 0. \end{aligned} \tag{38}$$

Since S is a symmetric tensor, we also have

$$\begin{aligned} &2\xi(\lambda - f)g(U, V) - U((2n + 1)\lambda - f)\eta(V) - \xi(\lambda - f)\eta(U)\eta(V) - \xi(\lambda)S(V, U) \\ &+ 4n\lambda g(FU, V) + \lambda g(V, QFU + FQU) + \xi(\lambda)g(QFU, FV) = 0. \end{aligned} \tag{39}$$

The equations (38) and (39) implies

$$0 = U((2n + 1)\lambda - f)\eta(V) - V((2n + 1)\lambda - f)\eta(U) + 8n\lambda g(FV, U) + 2\lambda g(U, QFV + FQV). \tag{40}$$

Putting $U = FU$ and $V = FV$ in (40), we obtain

$$4n\lambda g(FV, U) = -\lambda[g(U, QFV) + g(U, FQV)].$$

Since $\lambda \neq 0$ on M, we derive

$$-4nFV = (QF + FQ)V, \tag{41}$$

for all $V \in \chi(M)$. Let $\{e_i, Fe_i, \xi\}$, $(i = 1, 2, \ldots, n)$ be a local orthonormal F-basis. Using (7), we get

$$g(FQe_i, Fe_i) = -g(Qe_i, e_i). \tag{42}$$

By the definition of the scalar curvature, (41) and (42), we have

$$\begin{aligned} r &= S(\xi,\xi) + \sum_{i=1}^{n} \varepsilon_i \{S(e_i,e_i) + S(Fe_i,Fe_i)\} \\ &= g(Q\xi,\xi) + \sum_{i=1}^{n} \varepsilon_i \{g(QFe_i + FQe_i, Fe_i)\} \\ &= -2n(2n+1). \end{aligned} \tag{43}$$

Therefore, from the Proposition 1 the following equation is valid

$$f = (2n+1)\lambda. \tag{44}$$

Taking the inner product of (34) with $D\lambda$ and using in (44), we obtain

$$\xi(\lambda)[QD\lambda + 2nD\lambda] + \lambda[QFD\lambda + 2nFD\lambda] = 0. \tag{45}$$

Letting $D\lambda = V$ in (41) implies $QFD\lambda = -4nFD\lambda - FQD\lambda$. Hence, using the last equation, (45) becomes

$$\xi(\lambda)[QD\lambda + 2nD\lambda] + \lambda[-2nFD\lambda - FQD\lambda] = 0. \tag{46}$$

Finally, applying F to (46) and using (6), we have

$$\xi(\lambda)[FQD\lambda + 2nFD\lambda] + \lambda[-2nD\lambda - QD\lambda] = 0.$$

After some calculations, the last two equations imply

$$[(\xi(\lambda))^2 - \lambda^2][QD\lambda + 2nD\lambda] = 0.$$

Then, either $\xi(\lambda) = \pm\lambda$ or $QD\lambda + 2nD\lambda = 0$. Assume that $\xi(\lambda) \neq \pm\lambda$. Hence, $QD\lambda + 2nD\lambda = 0$. Taking the covariant derivative of $QD\lambda + 2nD\lambda = 0$ along the vector field U and using (31), we get

$$(\nabla_U Q)D\lambda + \lambda Q^2 U + (2n\lambda + f)QU + 2nfU = 0.$$

Contracting above equation over U with respect to a local orthonormal F-basis, we obtain

$$\sum_{i=1}^{n} \varepsilon_i [g((\nabla_{e_i} Q)D\lambda, e_i) + g((\nabla_{Fe_i} Q)D\lambda, Fe_i)] + g((\nabla_\xi Q)D\lambda, \xi) + \lambda |Q|^2 + (2n\lambda + f)r + 2n(2n+1)f = 0. \tag{47}$$

Using the well-known formula $divQ = \frac{1}{2}dr$ and (43), since $\lambda \neq 0$, from (47) we derive $|Q|^2 = 4n^2(2n+1)$. Finally, using the last equation and (43), we compute

$$|Q - \frac{r}{2n+1} I|^2 = |Q|^2 - \frac{2r^2}{2n+1} + \frac{r^2}{2n+1} = 0. \tag{48}$$

From (43) and (48), we have $|C|^2 = 0$, where the tensor $C = Q + 2nI$. Then, there are two possibilities. If $C = 0$, then $Q = -2nI$. In the case of $C \neq 0$, since C is self-adjoint and $Ker(\eta)$ is C-invariant we have from [18, p.260] that, at each point $p \in M$, $Ker(\eta_p) = W_1 \oplus \cdots \oplus W_l$ for some $(1 \leq l \leq 2n)$, where V_k are mutually orthogonal subspaces that are C-invariant and on $C_{|W_k}$ has matrix of either type:

$$\begin{pmatrix} \bar{\gamma} & & & & \\ 1 & \bar{\gamma} & & 0 & \\ & 1 & \bar{\gamma} & & \\ & & \ddots & \ddots & \\ & 0 & & 1 & \bar{\gamma} \end{pmatrix}$$

relative to a basis $U_1, \ldots, U_r$ of $W_k, r \geq 1$, such that the only non-zero products are $g_p(U_i, U_j) = \pm 1$ if $i + j = r + 1$, or of type

$$\begin{pmatrix} a & b & & & & & & & & \\ -b & a & & & & & & 0 & & \\ 1 & 0 & a & b & & & & & & \\ 0 & 1 & -b & a & & & & & & \\ & & 1 & 0 & a & b & & & & \\ & & 0 & 1 & -b & a & & & & \\ & & & & & \ddots & \ddots & & & \\ & & 0 & & & & & & & \\ & & & & & & 1 & 0 & a & b \\ & & & & & & 0 & 1 & -b & a \end{pmatrix} (b \neq 0)$$

relative to a basis $U_1, V_1, \ldots, U_m, V_m$ of W_k, such that the only non-zero products are $g_p(U_i, U_j) = 1 = -g_p(V_i, V_j)$ if $i + j = m + 1$. For $n = 1$, the rest of the proof is similar to the proof of Theorem 3.2 in [17]. This completes the proof. □

Theorem 3. *Let (g, λ) be a non-trivial solution of Fischer-Marsden equation on a K-paracontact manifold of dimension $(2n+1)$ that Ricci operator commutes, i.e. $QF = FQ$. Then the manifold is an Einstein manifold.*

Proof. From the assumption, (41) returns

$$-4nFV = 2FQV. \tag{49}$$

Applying F to the (49) and using (6) and (10), we obtain $QV = -2nV$. Hence, the manifold is an Einstein manifold. □

Remark 1. *In a $(2n+1)$-dimensional para-Sasakian manifold M satisfies the relation $S(FU, FV) = -S(U, V) - 2n\eta(U)\eta(V)$ [23, Lemma 3.15]. Letting $V = FV$ in the last equation, one can observe that the Ricci tensor commutes.*

With the help of the Theorem 4.1 in [13], Theorem 3 and Remark 1, we can state the following corollary.

Corollary 1. *If a $(2n+1)$-dimensional para-Sasakian manifold admits a non-trivial solution of Fischer-Marsden equation, then it is an Einstein manifold. Moreover, for $n = 1$, the Ricci tensor is parallel and the manifold is Ricci-semisymmetric.*

Corollary 2. *If (M^{2n+1}, g) is a K-paracontact manifold admitting a non-trivial solution of the Fischer-Marsden equation with $QF = FQ$, then g is a gradient Ricci soliton.*

Proof. Since the Ricci operator commutes with F, we have $Q = -2nI$ from Theorem 3. Then using this and (44), the equation (31) becomes

$$\nabla_U D\lambda = \lambda U,$$

which gives

$$Hess(\lambda)(U, V) = \lambda g(U, V). \tag{50}$$

In the view of (50) and $Q = -2nI$, we have

$$Hess\lambda + S - (\lambda - 2n)g = 0. \tag{51}$$

It follows from (4) and (51), g is a gradient Ricci soliton. □

4. 3-Dimensional Quasi-para-Sasakian Manifolds Admitting Fischer-Marsden Equation

In this section, we will consider 3-dimensional quasi-para-Sasakian manifolds with β constant which admits Fischer-Marsden equation. The general form of the following proposition is given in [14].

Proposition 2. *For a 3-dimensional quasi-para-Sasakian manifold M^3, the following equation holds*

$$(\nabla_V Q)\xi - (\nabla_\xi Q)V = -\beta(3\beta^2 + \frac{r}{2})FV,$$

for any vector field V.

Proof. Taking the covariant derivative of (17) along the vector field V and using the equations (13), (16) and (17), we get

$$(\nabla_V Q)\xi = -\beta(3\beta^2 + \frac{r}{2})FV. \tag{52}$$

Let $\{e, Fe, \xi\}$ be a local orthonormal F-basis. Using the well-known formula $divQ = \frac{dr}{2}$ and contracting (52) over V with respect to a local orthonormal F-basis, we obtain

$$\xi(r) = 0.$$

Similarly, taking the covariant derivative of (16) along ξ, we have

$$(\nabla_\xi Q)V = 0, \tag{53}$$

which completes the proof. □

Theorem 4. *Let (g, λ) be a non-trivial solution of Fischer-Marsden equation on a 3-dimensional quasi-para-Sasakian manifold M^3 with β constant. Then either*

(1) *the scalar curvature is $-6\beta^2$ and M^3 is Einstein, or*
(2) *M^3 is a paracosymplectic manifold which is locally a product of the real line $\mathbb{R}$ and a 2-dimensional para-Kaehlerian manifold, and η-Einstein.*

Proof. Letting $U = \xi$ in (30) and taking the inner product with E, we have

$$\begin{aligned} g(R(\xi, V)D\lambda, E) =& \xi(\lambda)S(V, E) - V(\lambda)S(\xi, E) + \lambda\{g((\nabla_\xi Q)V - (\nabla_V Q)\xi, E)\} \\ &+ \xi(f)g(V, E) - V(f)g(\xi, E). \end{aligned} \tag{54}$$

After some calculations, using the equations (16), (17) and (53) in (54), we get

$$g(R(\xi, V)D\lambda, E) =\xi(\lambda)S(V, E) + 2\beta^2 V(\lambda)\eta(E) + \lambda\beta(3\beta^2 + \frac{r}{2})g(FV, E) + \xi(f)g(V, E) - V(f)\eta(E). \tag{55}$$

We recall that the scalar curvature r is constant from Theorem 1. Putting $E = \xi$ in (55) and using the equation (15) and $f = -\frac{\lambda r}{2}$, we obtain

$$g(R(\xi, V)D\lambda, \xi) = (\frac{r}{2} + 2\beta^2)(V(\lambda) - \eta(V)\xi(\lambda)). \tag{56}$$

From (14), one can get

$$g(R(\xi, V)D\lambda, E) = -\beta^2(V(\lambda)\eta(E) - \xi(\lambda)g(V, E)). \tag{57}$$

For $E = \xi$ in (57), we obtain

$$g(R(\xi, V)D\lambda, \xi) = -\beta^2(V(\lambda) - \xi(\lambda)\eta(V)). \tag{58}$$

Therefore, the equations (56) and (58) imply

$$(\frac{r}{2} + 3\beta^2)(V(\lambda) - \eta(V)\xi(\lambda)) = 0. \tag{59}$$

From the above equation, two cases occur. We now check, case by case, whether (59) give rise to a local classification.

Case I: If $\frac{r}{2} + 3\beta^2 = 0$, then scalar curvature r is $-6\beta^2$.

Case II: If

$$V(\lambda) - \eta(V)\xi(\lambda) = 0, \tag{60}$$

then the gradient of λ is colinear with ξ, i.e. $D\lambda = \xi(\lambda)\xi$. Taking the covariant derivative of the last equation along the vector field U implies

$$\nabla_U D\lambda = \nabla_U(\xi(\lambda))\xi + \xi(\lambda)\nabla_U\xi. \tag{61}$$

Taking the inner product of (61) with V and using (13), we obtain

$$g(\nabla_U D\lambda, V) = U(\xi(\lambda))\eta(V) + \beta\xi(\lambda)g(FU, V). \tag{62}$$

Interchancing U and V in (62), we get

$$g(\nabla_V D\lambda, U) = V(\xi(\lambda))\eta(U) + \beta\xi(\lambda)g(FV, U). \tag{63}$$

Since the Hessian operator is symmetric, the equations (62) and (63) imply

$$U(\xi(\lambda))\eta(V) - V(\xi(\lambda))\eta(U) = 2\beta\xi(\lambda)g(U, FV). \tag{64}$$

Putting $U = FU$ and $V = FV$ in (64), we have

$$2\beta\xi(\lambda)g(FU, V) = 0.$$

If $\beta = 0$, then the manifold is paracosymplectic which is locally a product of the real line $\mathbb{R}$ and a 2-dimensional para-Kaehlerian manifold and η-Einstein from (15). Let $\xi(\lambda) = 0$ and $\beta \neq 0$. Then, from (60), λ is constant. Therefore, from (2), the Ricci operator S is zero. Hence, the manifold is Ricci flat. Using (15), we get $\beta = 0$, which is a contradiction of our assumption. So, this case does not occur. □

Corollary 3. *Let (M^3, g) is a quasi-para-Sasakian manifold that admitting non-trivial solution of Fischer-Marsden equation. Then either g is a gradient Ricci soliton or is a gradient η-Ricci soliton.*

Proof. From the assumption and Theorem 4, there are two possibilities.

Case I: If $r = -6\beta^2$, then the equations (16), (31) and $f = -\frac{\lambda r}{2}$ implies

$$\nabla_U D\lambda = \lambda\beta^2 U.$$

With similar idea in the proof of Corollary 2, we have

$$Hess\lambda + S - \beta^2(\lambda - 2)g = 0.$$

It means that g is a gradient Ricci soliton.

Case II: If $\beta = 0$, then we have $S(U, V) = \frac{r}{2}[g(U, V) - \eta(U)\eta(V)]$. On the other hand, using (16) and (31), we get $Hess\lambda = -\lambda\frac{r}{2}\eta \otimes \eta$. In the view of the last two equations, we obtain

$$Hess\lambda + S - \frac{r}{2}g + \frac{r}{2}(1 + \lambda)\eta \otimes \eta = 0,$$

which shows that from (5), g is a gradient η-Ricci soliton.

□

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A Sturm comparison criterion for impulsive hyperbolic equations on a rectangular prism

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Abstract. In this paper, new Sturmian comparison results and oscillatory properties of linear impulsive hyperbolic equations are obtained on a rectangular prism under fixed moment of impulse effects. Besides the Kreith's results [9,10], this paper represents an extension of earlier findings obtained on the rectangular domain in the plane to the results obtained in rectangular prism in space.

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Keywords. Impulsive hyperbolic equation, Sturm comparison, rectangular prism

1. Introduction

In 1969, Kreith [9] obtained a remarkable analogue of the Sturm comparison theorem between the pair of hyperbolic boundary value problems of the form

$$\begin{aligned} &u_{tt} - u_{xx} + p(x,t)u = 0 \\ &u_x(x_j,t) + (-1)^j r_j(t)u(x_j,t) = 0; \qquad (j = 1,2) \end{aligned} \tag{1}$$

and

$$\begin{aligned} &v_{tt} - v_{xx} + q(x,t)v = 0, \\ &v_x(x_j,t) + (-1)^j s_j(t)v(x_j,t) = 0; \qquad (j = 1,2) \end{aligned} \tag{2}$$

on the rectangular domain:

$$D = \{(x,t) : x_1 < x < x_2,\ t_1 < t < t_2)\}.$$

Theorem 1. *Let z_1 be a solution of problem (1) satisfying*

$$z_1(x,t_1) = z_1(x,t_2) = 0; \quad x_1 \le x \le x_2,$$

which is positive for $(x,t) \in [x_1,x_2] \times (t_1,t_2)$. If $q \ge p$ on D and $s_j \ge r_j$ $(j = 1,2)$ on $[t_1,t_2]$, then every solution z_2 of problem (2) has a zero in

$$\bar{D} = \{(x,t) : x_1 \le x \le x_2,\ t_1 \le t \le t_2)\}.$$

For the proof of Theorem 1, we address the readers [9, Theorem 1]. See also the monograph by Kreith [10, pp. 24–26].

Impulsive differential equations have been an interesting area for mathematics, physics, biology, chemistry, engineering, medicine etc. As far as impulsive ordinary differential equations are considered, there are many studies in terms of the existence of periodic solutions, asymptotic behavior, stability, Sturmian theory and oscillatory behavior of their solutions, see for example the book by Lakshmikantham, Bainov and Simeonov [11]. When partial differential equations under the impulse effect is considered, there are fewer publications compared to ordinary impulsive differential equations. Some of the noteworthy contributions have been made by Bainov et al. [1–4] for the first order impulsive partial differential inequalities,

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by Fu et al. [7] for the oscillation of impulsive hyperbolic systems, by Bainov and Simeonov [5] for the oscillatory behavior of impulsive differential equations, by Minchev [14] for the oscillation criteria of nonlinear hyperbolic differential and difference equations under impulse effect, by Cui et al. [6] for some problems on oscillation of impulsive hyperbolic differential systems with several retarded arguments, by Luo et al. [12] for oscillatory behavior of nonlinear impulsive partial functional differential equations, by Zhu et al. [18] for oscillation criteria of impulsive neutral hyperbolic equations, by Hernández et al. [8] for the existence of solutions of impulsive partial functional differential equations, by Ning et al. [15] for the oscillation of system of impulsive hyperbolic equations and by Luo et al. [13] for oscillatory solutions of impulsive quasilinear hyperbolic systems with delay. Oscillation theory for impulsive partial differential equations has received great attention and has been developing quite rapidly in recent years. As far as the Sturm theory is concerned, it seems there is only a single work [16] for impulsive hyperbolic equations in the literature. Recently, present authors [16] give some Sturm-type comparison criteria for impulsive hyperbolic equations on a rectangular domain. They attempted to give analogical comparison results for the couple of impulsive hyperbolic problems

$$\begin{cases} u_{tt}(x,t) - u_{xx}(x,t) + f(x,t)u(x,t) = 0; & (x,t) \in \Gamma \setminus \Gamma_{\text{imp}}, \\ \Delta u_t(x,t) + f_k(x,t)u(x,t) = 0; & (x,t) \in \Gamma_{\text{imp}} \end{cases} \tag{3}$$

satisfying the boundary conditions

$$u_x(x_j,t) + (-1)^j r_j(t) u(x_j,t) = 0; \qquad (j = 1,2) \tag{4}$$

and

$$\begin{cases} v_{tt}(x,t) - v_{xx}(x,t) + g(x,t)v(x,t) = 0; & (x,t) \in \Gamma \setminus \Gamma_{\text{imp}}, \\ \Delta v_t(x,t) + g_k(x,t)v(x,t) = 0; & (x,t) \in \Gamma_{\text{imp}} \end{cases} \tag{5}$$

satisfying the boundary conditions

$$v_x(x_j,t) + (-1)^j s_j(t) v(x_j,t) = 0; \quad (j = 1,2), \tag{6}$$

where

$$\begin{aligned} \Gamma :=& \{(x,t) : x \in (x_1,x_2),\ t \in (t_1,t_2)\} \quad \text{and} \\ \Gamma_{\text{imp}} :=& \{(x,t) \in \Gamma : t = \tau_k, k \in \mathbb{N}\}, \end{aligned}$$

$r_j, s_j \in \mathrm{C}([t_1,t_2],\mathbb{R})$ for $j = 1,2$, and $f, g, f_k, g_k \in \mathrm{C}(\Gamma,\mathbb{R})$ for $k \in \mathbb{N}$. Here $\{\tau_k\}$ is real-valued sequence such that

$$\tau_1 < \tau_2 < \cdots < \tau_k < \tau_{k+1} < \cdots \qquad (k \in \mathbb{N})$$

with $\lim_{n\to\infty} \tau_n = \infty$, and the operator Δ is the impulse operator defined as $\Delta\nu(x,\tau) = \nu(x,\tau^+) - \nu(x,\tau^-)$, where

$$\nu(x,\tau^{\pm}) = \lim_{(x,t)\to(x,\tau^{\pm})} \nu(x,\tau).$$

Theorem 2 ([16])**.** *Let u be a solution of problem (3)–(4) which is positive on Γ and satisfies $u(x,t_1) = u(x,t_2) = 0$ for all $x \in [x_1,x_2]$. If $g > f$ on Γ, $s_j > r_j$ $(j = 1,2)$ in $[t_1,t_2]$, and $g_k > f_k$ $(k \in \mathbb{N})$ on Γ_{imp}, then every solution v of problem (5)–(6) has a zero in closure $\bar{\Gamma}$ of Γ.*

Fix $x_0, y_0, t_0 \in \mathbb{R}$. Let $\mathcal{I} = (x_1,x_2) \subset [x_0,\infty)$, $\mathcal{J} = (y_1,y_2) \subset [y_0,\infty)$ and $\mathcal{K} = (t_1,t_2) \subset [t_0,\infty)$ be non-degenerate intervals.

Define the rectangular prism

$$\Omega = \mathcal{I} \times \mathcal{J} \times \mathcal{K},$$

and the domains

$$\begin{aligned} \mathcal{K}_{\text{imp}} &:= \{t \in \mathcal{K} : t = \tau_k, k \in \mathbb{N}\} \quad \text{and} \\ \Omega_{\text{imp}} &:= \mathcal{I} \times \mathcal{J} \times \mathcal{K}_{\text{imp}}, \end{aligned}$$

where $\{\tau_k\}$ is as defined previously.

Denote by $\mathrm{C}_{\text{imp}}\left(\bar{\Omega},\mathbb{R}\right)$ the set of functions $w : \bar{\Omega} \to \mathbb{R}$ satisfying the following properties:

(i) $w(x,y,t)$ is a continuous function for $(x,y,t) \in \bar{\Omega} \setminus \bar{\Omega}_{\text{imp}}$

(ii) There exist limits

$$\lim_{\substack{(x,y,t)\to(x,y,\tau_k)\\ t>\tau_k}} w(x,y,t) = w(x,y,\tau_k^+) \qquad (k\in\mathbb{N})$$

and

$$\lim_{\substack{(x,y,t)\to(x,y,\tau_k)\\ t<\tau_k}} w(x,y,t) = w(x,y,\tau_k^-) \qquad (k\in\mathbb{N})$$

for all $(x,y)\in\bar{\mathcal{I}}\times\bar{\mathcal{J}}$.

(iii) $\nu(x,y,t)$ is piecewise left continuous function at each τ_k, $k\in\mathbb{N}$, i.e.

$$\lim_{\substack{(x,y,t)\to(x,y,\tau_k)\\ t<\tau_k}} \nu(x,y,t) = \nu(x,y,\tau_k)$$

for each $k\in\mathbb{N}$ and $(x,y)\in\bar{\mathcal{I}}\times\bar{\mathcal{J}}$.

In this work, we give some Sturm-type comparison results for solutions of the couple of impulsive hyperbolic problems of the form

$$\begin{cases} u_{tt}-\triangle u+f(x,y,t)u=0; & (x,y,t)\in\Omega\setminus\Omega_{\text{imp}} \\ \Delta u_t+f_k(x,y,t)u=0; & (x,y,t)\in\Omega_{\text{imp}} \end{cases} \tag{7}$$

satisfying the boundary conditions

$$\begin{array}{ll} u_x(x_j,y,t)+(-1)^j r_j(t)u(x_j,y,t)=0; & (y,t)\in\bar{\mathcal{J}}\times\bar{\mathcal{K}}, \\ u_y(x,y_j,t)+(-1)^j r_{j+2}(t)u(x,y_j,t)=0; & (x,t)\in\bar{\mathcal{I}}\times\bar{\mathcal{K}}, \end{array} \tag{8}$$

and

$$\begin{cases} v_{tt}-\triangle v+g(x,y,t)v=0; & (x,y,t)\in\Omega\setminus\Omega_{\text{imp}} \\ \Delta v_t+g_k(x,y,t)v=0; & (x,y,t)\in\Omega_{\text{imp}} \end{cases} \tag{9}$$

satisfying the boundary conditions

$$\begin{array}{ll} v_x(x_j,y,t)+(-1)^j s_j(t)v(x_j,y,t)=0; & (y,t)\in\bar{\mathcal{J}}\times\bar{\mathcal{K}}, \\ v_y(x,y_j,t)+(-1)^j s_{j+2}(t)v(x,y_j,t)=0; & (x,t)\in\bar{\mathcal{I}}\times\bar{\mathcal{K}} \end{array} \tag{10}$$

for $j=1,2$, where $f,g:\bar{\Omega}\to\mathbb{R}$, $r_\ell,s_\ell:\bar{\mathcal{K}}\to\mathbb{R}$ are continuous functions for $\ell=1,2,3,4$,

$$\Delta w(x,y,t)=w(x,y,t^+)-w(x,y,t^-),$$

and $\triangle$ is the usual *Laplace operator*:

$$\triangle=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}.$$

A function $z\not\equiv 0$ is defined to be a solution of (7)–(8) (respectively (9)–(10)) if

- $z\in\mathrm{C}\left(\bar{\Omega},\mathbb{R}\right)$ (i.e., $\Delta z(x,y,\tau_k)=0$ for all $k\in\mathbb{N}$) and $z_t\in\mathrm{C}_{\text{imp}}\left(\bar{\Omega},\mathbb{R}\right)$;
- there exist second-order partial derivatives z_{tt}, z_{xx} and z_{yy} satisfying the first equation in (7) for each $(x,y,t)\in\Omega\setminus\Omega_{\text{imp}}$;
- z satisfies the second equation in (7) in Ω_{imp} and the boundary conditions given in (8).

Recently, present authors [17] considered the pair of Problems (7)–(8) and (9)–(10) without impulse effect, i.e. $f_k(x,y,t)\equiv 0\equiv g_k(x,y,t)$, and they obtained some Sturm-type comparison results between them.

Motivated by Theorems 1 and 2, and the results given in [17], we consider impulsive hyperbolic equations on a rectangular prism and their oscillatory properties. The results obtained in this work are conceivable as impulsive extension of those given in [17].

2. Linear Comparison Results

Based on the Kreith's comparison result obtained on the rectangular domain in the plane, we interfere to obtain an analogic result for the solutions of the couple of impulsive problems (7)–(8) and (9)–(10) on a *rectangular prism* in three-space.

Main result of the paper is the following.

Theorem 3 (Sturm comparison theorem)**.** *Let u be a solution of problem (7)–(8) satisfying the initial conditions*

$$u(x,y,t_1)=u(x,y,t_2)=0; \qquad (x,y)\in\bar{\mathcal{I}}\times\bar{\mathcal{J}}, \tag{11}$$

which is positive on Ω. If the inequalities

$$g(x,y,t)\geq f(x,y,t); \qquad (x,y,t)\in\Omega, \tag{12}$$

$$s_j(t)\geq r_j(t); \qquad t\in\bar{\mathcal{K}} \qquad (j=1,2,3,4), \tag{13}$$

and

$$g_k(x,y,t)\geq f_k(x,y,t); \qquad (x,y,t)\in\Omega_{\text{imp}} \qquad (k\in\mathbb{N}) \tag{14}$$

hold, then every solution v of problem (9)–(10) has a zero in $\bar{\Omega}$.

Proof. Suppose to contrary that v has no zero in $\bar{\Omega}$. Without loss of generality we may assume that $v>0$ in $\bar{\Omega}$. The proof of the case that $v<0$ in $\bar{\Omega}$ is similar.

Multiplying the first equations in (7) and (9) by v and u respectively, and subtracting, we see that the identity

$$\big[uv_x-vu_x\big]_x+\big[uv_y-vu_y\big]_y+\big[vu_t-uv_t\big]_t=\big[g(x,y,t)-f(x,y,t)\big]uv \tag{15}$$

holds for all $(x,y,t)\in\bar{\Omega}$. Integrating both sides of (15) over Ω, we obtain

$$\begin{aligned}&\iiint\limits_{\Omega}\big[g(x,y,t)-f(x,y,t)\big]uv\mathrm{d}V\\&\quad=\iiint\limits_{\Omega}\Big\{\big[uv_x-vu_x\big]_x+\big[uv_y-vu_y\big]_y+\big[vu_t-uv_t\big]_t\Big\}\mathrm{d}V,\end{aligned} \tag{16}$$

where $\mathrm{d}V$ is the volume element. The functions under integral signs have discontinuities of first kind at the jump points τ_k, so we divide the domain Ω into $(n+1)$ sub-domains in the following way:

$$\begin{aligned}\Omega_0&:=\{(x,y,t):(x,y)\in\mathcal{I}\times\mathcal{J},t\in(t_1,\tau_1]\},\\ \Omega_k&:=\{(x,y,t):(x,y)\in\mathcal{I}\times\mathcal{J},t\in(\tau_k,\tau_{k+1}]\}; \qquad k=1,2,\ldots,n-1,\\ \Omega_n&:=\{(x,y,t):(x,y)\in\mathcal{I}\times\mathcal{J},t\in(\tau_n,t_2)\}.\end{aligned}$$

This allows us to apply the divergence theorem to each triple integral

$$\iiint\limits_{\Omega_m}\Big\{\big[uv_x-vu_x\big]_x+\big[uv_y-vu_y\big]_y+\big[vu_t-uv_t\big]_t\Big\}\mathrm{d}V \tag{17}$$

for $m=0,1,\ldots,n$. We also note that each partition defined above satisfy

(i) $\bigcap\limits_{\ell=0}^{n}\Omega_\ell=\emptyset$;

(ii) $\Omega=\bigcup\limits_{\ell=0}^{n}\Omega_\ell$.

Clearly, we have from (i) and (ii) that

$$\begin{aligned}&\iiint\limits_{\Omega}\big[g(x,y,t)-f(x,y,t)\big]uv\mathrm{d}V\\&\quad=\iiint\limits_{\Omega_0}\big[g(x,y,t)-f(x,y,t)\big]uv\mathrm{d}V+\sum_{k=1}^{n-1}\iiint\limits_{\Omega_k}\big[g(x,y,t)-f(x,y,t)\big]uv\mathrm{d}V\\&\qquad+\iiint\limits_{\Omega_n}\big[g(x,y,t)-f(x,y,t)\big]uv\mathrm{d}V.\end{aligned} \tag{18}$$

We note that each Ω_m, $m=0,1,\ldots,n$, is a simple solid region with the piecewise smooth boundary $\mathcal{S}_m$. Applying divergence theorem to the smooth vector field

$$\mathbf{F}(x,y,t):=(uv_x-vu_x)\mathbf{i}+(uv_y-vu_y)\mathbf{j}+(vu_t-uv_t)\mathbf{k}, \tag{19}$$

on Ω_m, $m = 0, 1, \ldots, n$, the integral given in (17) turns out to be

$$\iiint\limits_{\Omega_m} \left\{ \left[uv_x - vu_x\right]_x + \left[uv_y - vu_y\right]_y + \left[vu_t - uv_t\right]_t \right\} \mathrm{d}V$$

$$= \iiint\limits_{\Omega_m} \operatorname{div} \mathbf{F} \mathrm{d}V \qquad \left(= \iiint\limits_{\Omega_m} \nabla \bullet \mathbf{F} \mathrm{d}V \right)$$

$$= \oiint\limits_{\mathcal{S}_m} \mathbf{F} \bullet \hat{\mathbf{N}} \mathrm{d}S \tag{20}$$

for $m = 0, 1, \ldots, n$, where $\hat{\mathbf{N}}$ is the unit outward normal to the surface $\mathcal{S}_m$ and the ∇ is the usual nabla (gradient) operator defined by

$$\nabla = \frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial t}\mathbf{k}.$$

Since $\mathcal{S}_m$ $(= \partial\Omega_m)$, $m = 0, 1, \ldots, n$, is the union of six regions, it can be expressed as

$$\mathcal{S}_m = \bigcup_{\mu=1}^{6} \mathcal{S}_{m\mu}, \tag{21}$$

where each $\mathcal{S}_{m\mu}$, $\mu = 1, \ldots, 6$, are disjoint, rectangular, oriented, closed surfaces. It follows from the fact (21) that, the integral on the right-hand side of (20) can be expressed as

$$\iiint\limits_{\Omega_m} \left\{ \left[uv_x - vu_x\right]_x + \left[uv_y - vu_y\right]_y + \left[vu_t - uv_t\right]_t \right\} \mathrm{d}V$$

$$= \sum_{\mu=1}^{6} \iint\limits_{\mathcal{S}_{m\mu}} \mathbf{F} \bullet \hat{\mathbf{N}}_{m\mu} \mathrm{d}S, \qquad (m = 0, 1, \ldots, n), \tag{22}$$

where each $\hat{\mathbf{N}}_{m\mu}$ are the unit outward normal vectors to each surface $\mathcal{S}_{m\mu}$ and defined by

$$\begin{array}{lll} \hat{\mathbf{N}}_{m1} = -\mathbf{i}, & \hat{\mathbf{N}}_{m2} = \mathbf{i}, & \hat{\mathbf{N}}_{m3} = -\mathbf{j} \\ \hat{\mathbf{N}}_{m4} = \mathbf{j}, & \hat{\mathbf{N}}_{m5} = -\mathbf{k}, & \hat{\mathbf{N}}_{m6} = \mathbf{k} \end{array} \tag{23}$$

for $m = 0, 1, \ldots, n$, and $\mathbf{F}$ is defined in (19).

Now, we start with the first integral in the right-hand side on (18). Taking $m = 0$ in (22) and using (15) and (23), it can be expressed as

$$\iiint\limits_{\Omega_0} \left[g(x,y,t) - f(x,y,t)\right] uv \mathrm{d}V = \oiint\limits_{\mathcal{S}_0} \mathbf{F} \bullet \hat{\mathbf{N}} \mathrm{d}S = \sum_{\mu=1}^{6} \iint\limits_{\mathcal{S}_{0\mu}} \mathbf{F} \bullet \hat{\mathbf{N}}_{0\mu} \mathrm{d}S, \tag{24}$$

where each surfaces $\mathcal{S}_{0\mu}$ are defined by

$$\begin{aligned} \mathcal{S}_{01} &= \{(x,y,t) : x = x_1, y \in \mathcal{J}, t \in (t_1, \tau_1]\}, \\ \mathcal{S}_{02} &= \{(x,y,t) : x = x_2, y \in \mathcal{J}, t \in (t_1, \tau_1]\}, \\ \mathcal{S}_{03} &= \{(x,y,t) : y = y_1, x \in \mathcal{I}, t \in (t_1, \tau_1]\}, \\ \mathcal{S}_{04} &= \{(x,y,t) : y = y_2, x \in \mathcal{I}, t \in (t_1, \tau_1]\}, \\ \mathcal{S}_{05} &= \{(x,y,t) : t = t_1, (x,y) \in \mathcal{I} \times \mathcal{J}\} \end{aligned}$$

and

$$\mathcal{S}_{06} = \{(x,y,t) : t = \tau_1, (x,y) \in \mathcal{I} \times \mathcal{J}\}.$$

Then by using the initial conditions (8) and (10), each integral on the right-hand side of (24) turn out to be

$$\iint\limits_{\mathcal{S}_{01}} \mathbf{F} \bullet \hat{\mathbf{N}}_{01} \mathrm{d}S = -\iint\limits_{\mathcal{S}_{01}} \mathbf{F} \bullet \mathbf{i} \mathrm{d}S$$

$$= -\int_{t_1}^{\tau_1} \int_{y_1}^{y_2} [uv_x - vu_x](x_1, y, t) \mathrm{d}y \mathrm{d}t$$

$$= \int_{t_1}^{\tau_1} \int_{y_1}^{y_2} [r_1(t) - s_1(t)]u(x_1, y, t)v(x_1, y, t)\mathrm{d}y\mathrm{d}t, \tag{25}$$

$$\begin{aligned}
\iint\limits_{\mathcal{S}_{02}} \mathbf{F} \bullet \hat{\mathbf{N}}_{02}\mathrm{d}S &= \iint\limits_{\mathcal{S}_{02}} \mathbf{F} \bullet \mathbf{i}\mathrm{d}S \\
&= \int_{t_1}^{\tau_1} \int_{y_1}^{y_2} [uv_x - vu_x](x_2, y, t)\mathrm{d}y\mathrm{d}t \\
&= \int_{t_1}^{\tau_1} \int_{y_1}^{y_2} [r_2(t) - s_2(t)]u(x_2, y, t)v(x_2, y, t)\mathrm{d}y\mathrm{d}t,
\end{aligned} \tag{26}$$

$$\begin{aligned}
\iint\limits_{\mathcal{S}_{03}} \mathbf{F} \bullet \hat{\mathbf{N}}_{03}\mathrm{d}S &= -\iint\limits_{\mathcal{S}_{03}} \mathbf{F} \bullet \mathbf{j}\mathrm{d}S \\
&= -\int_{t_1}^{\tau_1} \int_{x_1}^{x_2} [uv_y - vu_y](x, y_1, t)\mathrm{d}x\mathrm{d}t \\
&= \int_{t_1}^{\tau_1} \int_{x_1}^{x_2} [r_3(t) - s_3(t)]u(x, y_1, t)v(x, y_1, t)\mathrm{d}x\mathrm{d}t,
\end{aligned} \tag{27}$$

$$\begin{aligned}
\iint\limits_{\mathcal{S}_{04}} \mathbf{F} \bullet \hat{\mathbf{N}}_{04}\mathrm{d}S &= \iint\limits_{\mathcal{S}_{04}} \mathbf{F} \bullet \mathbf{j}\mathrm{d}S \\
&= \int_{t_1}^{\tau_1} \int_{x_1}^{x_2} [uv_y - vu_y](x, y_2, t)\mathrm{d}x\mathrm{d}t \\
&= \int_{t_1}^{\tau_1} \int_{x_1}^{x_2} [r_4(t) - s_4(t)]u(x, y_2, t)v(x, y_2, t)\mathrm{d}x\mathrm{d}t,
\end{aligned} \tag{28}$$

$$\iint\limits_{\mathcal{S}_{05}} \mathbf{F} \bullet \hat{\mathbf{N}}_{05}\mathrm{d}S = -\iint\limits_{\mathcal{S}_{05}} \mathbf{F} \bullet \mathbf{k}\mathrm{d}S = -\int_{y_1}^{y_2} \int_{x_1}^{x_2} [vu_t - uv_t](x, y, t_1)\mathrm{d}x\mathrm{d}y \tag{29}$$

and

$$\iint\limits_{\mathcal{S}_{06}} \mathbf{F} \bullet \hat{\mathbf{N}}_{06}\mathrm{d}S = \iint\limits_{\mathcal{S}_{06}} \mathbf{F} \bullet \mathbf{k}\mathrm{d}S = \int_{y_1}^{y_2} \int_{x_1}^{x_2} [vu_t - uv_t](x, y, \tau_1)\mathrm{d}x\mathrm{d}y. \tag{30}$$

Summing up equations (25)–(30), equation (24) turns out to be

$$\begin{aligned}
&\iiint\limits_{\Omega_0} [g(x, y, t) - f(x, y, t)]uv\mathrm{d}V \\
&= \int_{t_1}^{\tau_1} \int_{y_1}^{y_2} \Big\{[r_1(t) - s_1(t)]u(x_1, y, t)v(x_1, y, t) \\
&\quad + [r_2(t) - s_2(t)]u(x_2, y, t)v(x_2, y, t)\Big\}\mathrm{d}y\mathrm{d}t \\
&\quad + \int_{t_1}^{\tau_1} \int_{x_1}^{x_2} \Big\{[r_3(t) - s_3(t)]u(x, y_1, t)v(x, y_1, t) \\
&\quad + [r_4(t) - s_4(t)]u(x, y_2, t)v(x, y_2, t)\Big\}\mathrm{d}x\mathrm{d}t \\
&\quad + \int_{y_1}^{y_2} \int_{x_1}^{x_2} \Big\{[vu_t - uv_t](x, y, \tau_1) - [vu_t - uv_t](x, y, t_1)\Big\}\mathrm{d}x\mathrm{d}y.
\end{aligned} \tag{31}$$

Similarly, by taking $m = n$ in (22) and using (15) and (23), the last integral in the right-hand side on (18) can be expressed as

$$\iiint\limits_{\Omega_n} [g(x, y, t) - f(x, y, t)]uv\mathrm{d}V = \oiint\limits_{\mathcal{S}_n} \mathbf{F} \bullet \hat{\mathbf{N}}\mathrm{d}S = \sum_{\mu=1}^{6} \iint\limits_{\mathcal{S}_{n\mu}} \mathbf{F} \bullet \hat{\mathbf{N}}_{n\mu}\mathrm{d}S, \tag{32}$$

where

$$\begin{aligned}
\mathcal{S}_{n1} &= \{(x,y,t) : x = x_1, y \in \mathcal{J}, t \in (\tau_n, t_2]\},\\
\mathcal{S}_{n2} &= \{(x,y,t) : x = x_2, y \in \mathcal{J}, t \in (\tau_n, t_2]\},\\
\mathcal{S}_{n3} &= \{(x,y,t) : y = y_1, x \in \mathcal{I}, t \in (\tau_n, t_2]\},\\
\mathcal{S}_{n4} &= \{(x,y,t) : y = y_2, x \in \mathcal{I}, t \in (\tau_n, t_2]\},\\
\mathcal{S}_{n5} &= \{(x,y,t) : t = \tau_n, (x,y) \in \mathcal{I} \times \mathcal{J}\}
\end{aligned}$$

and

$$\mathcal{S}_{n6} = \{(x,y,t) : t = t_2, (x,y) \in \mathcal{I} \times \mathcal{J}\}.$$

Boundary conditions (8) and (10) imply that each integral on the right-hand side of (32) turn out to be

$$\begin{aligned}
\iint\limits_{\mathcal{S}_{n1}} \mathbf{F} \bullet \hat{\mathbf{N}}_{n1} \mathrm{d}S &= -\iint\limits_{\mathcal{S}_{n1}} \mathbf{F} \bullet \mathbf{i} \mathrm{d}S\\
&= -\int_{\tau_n^+}^{t_2} \int_{y_1}^{y_2} [uv_x - vu_x](x_1, y, t) \mathrm{d}y \mathrm{d}t\\
&= \int_{\tau_n^+}^{t_2} \int_{y_1}^{y_2} [r_1(t) - s_1(t)] u(x_1, y, t) v(x_1, y, t) \mathrm{d}y \mathrm{d}t, \qquad (33)
\end{aligned}$$

$$\begin{aligned}
\iint\limits_{\mathcal{S}_{n2}} \mathbf{F} \bullet \hat{\mathbf{N}}_{n2} \mathrm{d}S &= \iint\limits_{\mathcal{S}_{n2}} \mathbf{F} \bullet \mathbf{i} \mathrm{d}S\\
&= \int_{\tau_n^+}^{t_2} \int_{y_1}^{y_2} [uv_x - vu_x](x_2, y, t) \mathrm{d}y \mathrm{d}t\\
&= \int_{\tau_n^+}^{t_2} \int_{y_1}^{y_2} [r_2(t) - s_2(t)] u(x_2, y, t) v(x_2, y, t) \mathrm{d}y \mathrm{d}t, \qquad (34)
\end{aligned}$$

$$\begin{aligned}
\iint\limits_{\mathcal{S}_{n3}} \mathbf{F} \bullet \hat{\mathbf{N}}_{n3} \mathrm{d}S &= -\iint\limits_{\mathcal{S}_{n3}} \mathbf{F} \bullet \mathbf{j} \mathrm{d}S\\
&= -\int_{\tau_n^+}^{t_2} \int_{x_1}^{x_2} [uv_y - vu_y](x, y_1, t) \mathrm{d}x \mathrm{d}t\\
&= \int_{\tau_n^+}^{t_2} \int_{x_1}^{x_2} [r_3(t) - s_3(t)] u(x, y_1, t) v(x, y_1, t) \mathrm{d}x \mathrm{d}t, \qquad (35)
\end{aligned}$$

$$\begin{aligned}
\iint\limits_{\mathcal{S}_{n4}} \mathbf{F} \bullet \hat{\mathbf{N}}_{n4} \mathrm{d}S &= \iint\limits_{\mathcal{S}_{n4}} \mathbf{F} \bullet \mathbf{j} \mathrm{d}S\\
&= \int_{\tau_n^+}^{t_2} \int_{x_1}^{x_2} [uv_y - vu_y](x, y_2, t) \mathrm{d}x \mathrm{d}t\\
&= \int_{\tau_n^+}^{t_2} \int_{x_1}^{x_2} [r_4(t) - s_4(t)] u(x, y_2, t) v(x, y_2, t) \mathrm{d}x \mathrm{d}t, \qquad (36)
\end{aligned}$$

$$\iint\limits_{\mathcal{S}_{n5}} \mathbf{F} \bullet \hat{\mathbf{N}}_{n5} \mathrm{d}S = -\iint\limits_{\mathcal{S}_{n5}} \mathbf{F} \bullet \mathbf{k} \mathrm{d}S = -\int_{y_1}^{y_2} \int_{x_1}^{x_2} [vu_t - uv_t](x, y, \tau_n^+) \mathrm{d}x \mathrm{d}y \qquad (37)$$

and

$$\iint\limits_{\mathcal{S}_{n6}} \mathbf{F} \bullet \hat{\mathbf{N}}_{n6} \mathrm{d}S = \iint\limits_{\mathcal{S}_{n6}} \mathbf{F} \bullet \mathbf{k} \mathrm{d}S = \int_{y_1}^{y_2} \int_{x_1}^{x_2} [vu_t - uv_t](x, y, t_2) \mathrm{d}x \mathrm{d}y. \qquad (38)$$

By addition of integrals (33)–(38), equation (32) can be expressed as

$$\iiint\limits_{\Omega_n} \left[g(x,y,t) - f(x,y,t)\right] uv \mathrm{d}V$$

$$
\begin{aligned}
&= \int_{\tau_n^+}^{t_2}\int_{y_1}^{y_2}\Big\{[r_1(t)-s_1(t)]u(x_1,y,t)v(x_1,y,t)\\
&\quad + [r_2(t)-s_2(t)]u(x_2,y,t)v(x_2,y,t)\Big\}\mathrm{d}y\mathrm{d}t\\
&\quad + \int_{\tau_n^+}^{t_2}\int_{x_1}^{x_2}\Big\{[r_3(t)-s_3(t)]u(x,y_1,t)v(x,y_1,t)\\
&\quad + [r_4(t)-s_4(t)]u(x,y_2,t)v(x,y_2,t)\Big\}\mathrm{d}x\mathrm{d}t\\
&\quad + \int_{y_1}^{y_2}\int_{x_1}^{x_2}\Big\{[vu_t-uv_t](x,y,t_2)-[vu_t-uv_t](x,y,\tau_n^+)\Big\}\mathrm{d}x\mathrm{d}y. \qquad (39)
\end{aligned}
$$

Finally, we will examine the integrals in the mid part of (18), i.e.

$$
\iiint\limits_{\Omega_k}[g(x,y,t)-f(x,y,t)]uv\mathrm{d}V = \oiint\limits_{\mathcal{S}_k}\mathbf{F}\bullet\hat{\mathbf{N}}\mathrm{d}S = \sum_{\mu=1}^{6}\iint\limits_{\mathcal{S}_{k\mu}}\mathbf{F}\bullet\hat{\mathbf{N}}_{k\mu}\mathrm{d}S, \qquad (40)
$$

where

$$
\begin{aligned}
\mathcal{S}_{k1} &= \{(x,y,t): x=x_1, y\in\mathcal{J}, t\in(\tau_k,\tau_{k+1}]\},\\
\mathcal{S}_{k2} &= \{(x,y,t): x=x_2, y\in\mathcal{J}, t\in(\tau_k,\tau_{k+1}]\},\\
\mathcal{S}_{k3} &= \{(x,y,t): y=y_1, x\in\mathcal{I}, t\in(\tau_k,\tau_{k+1}]\},\\
\mathcal{S}_{k4} &= \{(x,y,t): y=y_2, x\in\mathcal{I}, t\in(\tau_k,\tau_{k+1}]\},\\
\mathcal{S}_{k5} &= \{(x,y,t): t=\tau_k, (x,y)\in\mathcal{I}\times\mathcal{J}\}
\end{aligned}
$$

and

$$
\mathcal{S}_{k6} = \{(x,y,t): t=\tau_{k+1}, (x,y)\in\mathcal{I}\times\mathcal{J}\}
$$

for $k=1,2,\ldots,n-1$.

Then integrals on the right-hand side of (40) become

$$
\begin{aligned}
\iint\limits_{\mathcal{S}_{k1}}\mathbf{F}\bullet\hat{\mathbf{N}}_{k1}\mathrm{d}S &= -\iint\limits_{\mathcal{S}_{k1}}\mathbf{F}\bullet\mathbf{i}\mathrm{d}S\\
&= -\int_{\tau_k^+}^{\tau_{k+1}}\int_{y_1}^{y_2}[uv_x-vu_x](x_1,y,t)\mathrm{d}y\mathrm{d}t\\
&= \int_{\tau_k^+}^{\tau_{k+1}}\int_{y_1}^{y_2}[r_1(t)-s_1(t)]u(x_1,y,t)v(x_1,y,t)\mathrm{d}y\mathrm{d}t, \qquad (41)
\end{aligned}
$$

$$
\begin{aligned}
\iint\limits_{\mathcal{S}_{k2}}\mathbf{F}\bullet\hat{\mathbf{N}}_{k2}\mathrm{d}S &= \iint\limits_{\mathcal{S}_{k2}}\mathbf{F}\bullet\mathbf{i}\mathrm{d}S\\
&= \int_{\tau_k^+}^{\tau_{k+1}}\int_{y_1}^{y_2}[uv_x-vu_x](x_2,y,t)\mathrm{d}y\mathrm{d}t\\
&= \int_{\tau_k^+}^{\tau_{k+1}}\int_{y_1}^{y_2}[r_2(t)-s_2(t)]u(x_2,y,t)v(x_2,y,t)\mathrm{d}y\mathrm{d}t, \qquad (42)
\end{aligned}
$$

$$
\begin{aligned}
\iint\limits_{\mathcal{S}_{k3}}\mathbf{F}\bullet\hat{\mathbf{N}}_{k3}\mathrm{d}S &= -\iint\limits_{\mathcal{S}_{k3}}\mathbf{F}\bullet\mathbf{j}\mathrm{d}S\\
&= -\int_{\tau_k^+}^{\tau_{k+1}}\int_{x_1}^{x_2}[uv_y-vu_y](x,y_1,t)\mathrm{d}x\mathrm{d}t\\
&= \int_{\tau_k^+}^{\tau_{k+1}}\int_{x_1}^{x_2}[r_3(t)-s_3(t)]u(x,y_1,t)v(x,y_1,t)\mathrm{d}x\mathrm{d}t, \qquad (43)
\end{aligned}
$$

$$\begin{aligned}\iint_{\mathcal{S}_{k4}} \mathbf{F}\bullet\hat{\mathbf{N}}_{k4}\mathrm{d}S &= \iint_{\mathcal{S}_{k4}} \mathbf{F}\bullet\mathbf{j}\mathrm{d}S \\ &= \int_{\tau_k^+}^{\tau_{k+1}}\int_{x_1}^{x_2}[uv_y - vu_y](x,y_2,t)\mathrm{d}x\mathrm{d}t \\ &= \int_{\tau_k^+}^{\tau_{k+1}}\int_{x_1}^{x_2}[r_4(t)-s_4(t)]u(x,y_2,t)v(x,y_2,t)\mathrm{d}x\mathrm{d}t \qquad (44)\end{aligned}$$

$$\iint_{\mathcal{S}_{k5}} \mathbf{F}\bullet\hat{\mathbf{N}}_{k5}\mathrm{d}S = -\iint_{\mathcal{S}_{k5}} \mathbf{F}\bullet\mathbf{k}\mathrm{d}S = -\int_{y_1}^{y_2}\int_{x_1}^{x_2}[vu_t - uv_t](x,y,\tau_k^+)\mathrm{d}x\mathrm{d}y \qquad (45)$$

and

$$\iint_{\mathcal{S}_{k6}} \mathbf{F}\bullet\hat{\mathbf{N}}_{k6}\mathrm{d}S = \iint_{\mathcal{S}_{k6}} \mathbf{F}\bullet\mathbf{k}\mathrm{d}S = \int_{y_1}^{y_2}\int_{x_1}^{x_2}[vu_t - uv_t](x,y,\tau_{k+1})\mathrm{d}x\mathrm{d}y \qquad (46)$$

for $k = 1, 2, \ldots, n-1$. Integrals (41)–(46) yield

$$\begin{aligned}&\iiint_{\Omega_k}\big[g(x,y,t)-f(x,y,t)\big]uv\mathrm{d}V \\ &= \int_{\tau_k^+}^{\tau_{k+1}}\int_{y_1}^{y_2}\Big\{[r_1(t)-s_1(t)]u(x_1,y,t)v(x_1,y,t) \\ &\quad + [r_2(t)-s_2(t)]u(x_2,y,t)v(x_2,y,t)\Big\}\mathrm{d}y\mathrm{d}t \\ &\quad + \int_{\tau_k^+}^{\tau_{k+1}}\int_{x_1}^{x_2}\Big\{[r_3(t)-s_3(t)]u(x,y_1,t)v(x,y_1,t) \\ &\quad + [r_4(t)-s_4(t)]u(x,y_2,t)v(x,y_2,t)\Big\}\mathrm{d}x\mathrm{d}t \\ &\quad + \int_{y_1}^{y_2}\int_{x_1}^{x_2}\Big\{-[vu_t-uv_t](x,y,\tau_k^+) + [vu_t-uv_t](x,y,\tau_{k+1})\Big\}\mathrm{d}x\mathrm{d}y \qquad (47)\end{aligned}$$

for $k = 1, 2, \ldots, n-1$.

Finally we add the integrals (31), (39) and (47) to obtain the main integral (18) as

$$\begin{aligned}&\iiint_{\Omega}\big[g(x,y,t)-f(x,y,t)\big]uv\mathrm{d}V \\ &= \left\{\int_{t_1}^{\tau_1}+\int_{\tau_1^+}^{\tau_2}+\cdots+\int_{\tau_{n-1}^+}^{\tau_n}+\int_{\tau_n^+}^{t_2}\right\}\int_{y_1}^{y_2}\Big\{[r_1(t)-s_1(t)]u(x_1,y,t)v(x_1,y,t) \\ &\quad + [r_2(t)-s_2(t)]u(x_2,y,t)v(x_2,y,t)\Big\}\mathrm{d}y\mathrm{d}t \\ &\quad + \left\{\int_{t_1}^{\tau_1}+\int_{\tau_1^+}^{\tau_2}+\cdots+\int_{\tau_{n-1}^+}^{\tau_n}+\int_{\tau_n^+}^{t_2}\right\}\int_{x_1}^{x_2}\Big\{[r_3(t)-s_3(t)]u(x,y_1,t)v(x,y_1,t) \\ &\quad + [r_4(t)-s_4(t)]u(x,y_2,t)v(x,y_2,t)\Big\}\mathrm{d}x\mathrm{d}t \\ &\quad + \int_{y_1}^{y_2}\int_{x_1}^{x_2}\Big\{[vu_t-uv_t](x,y,t_2) - [vu_t-uv_t](x,y,t_1) + [vu_t-uv_t](x,y,\tau_1) \\ &\quad + \sum_{k=1}^{n-1}\Big\{-[vu_t-uv_t](x,y,\tau_k^+) + [vu_t-uv_t](x,y,\tau_{k+1})\Big\} \\ &\quad - [vu_t-uv_t](x,y,\tau_n^+)\Big\}\mathrm{d}x\mathrm{d}y. \qquad (48)\end{aligned}$$

Noting that $\Delta u(x, y, \tau_k) = \Delta v(x, y, \tau_k) = 0$, $k \in \mathbb{N}$, the impulse conditions for the functions u_t and v_t in the second lines of (7) and (9), respectively, imply that the related impulse terms in (48) can be picked up as

$$\begin{aligned}
&-[vu_t - uv_t](x, y, \tau_1) + \sum_{k=1}^{n-1} \Big\{ [vu_t - uv_t](x, y, \tau_k^+) - [vu_t - uv_t](x, y, \tau_{k+1}) \Big\} \\
&\quad + [vu_t - uv_t](x, y, \tau_n^+) \\
&= \sum_{k=1}^{n} \Delta[vu_t - uv_t](x, y, \tau_k) \\
&= \sum_{t_1 \le \tau_k < t_2} \Delta[vu_t - uv_t](x, y, \tau_k) \\
&= \sum_{t_1 \le \tau_k < t_2} \Big\{ v(x, y, \tau_k)\Delta u_t(x, y, \tau_k) - u(x, y, \tau_k)\Delta v_t(x, y, \tau_k) \Big\} \\
&= \sum_{t_1 \le \tau_k < t_2} \big[g_k(x, y, \tau_k) - f_k(x, y, \tau_k)\big] u(x, y, \tau_k) v(x, y, \tau_k). \qquad (49)
\end{aligned}$$

Using initial conditions (11) and imposing impulse terms obtained in (49) into (48), we obtain the following handy identity

$$\begin{aligned}
&\iiint_{\Omega} \big[(g - f)uv\big](x, y, t)\mathrm{d}V + \sum_{t_1 \le \tau_k < t_2} \big[(g_k - f_k)uv\big](x, y, \tau_k) \\
&= \int_{t_1}^{t_2} \int_{y_1}^{y_2} \Big\{ [r_1(t) - s_1(t)]u(x_1, y, t)v(x_1, y, t) \\
&\quad + [r_2(t) - s_2(t)]u(x_2, y, t)v(x_2, y, t) \Big\} \mathrm{d}y\mathrm{d}t \\
&\quad + \int_{t_1}^{t_2} \int_{x_1}^{x_2} \Big\{ [r_3(t) - s_3(t)]u(x, y_1, t)v(x, y_1, t) \\
&\quad + [r_4(t) - s_4(t)]u(x, y_2, t)v(x, y_2, t) \Big\} \mathrm{d}x\mathrm{d}t \\
&\quad + \int_{y_1}^{y_2} \int_{x_1}^{x_2} \Big\{ (vu_t)(x, y, t_2) - (vu_t)(x, y, t_1) \Big\} \mathrm{d}x\mathrm{d}y. \qquad (50)
\end{aligned}$$

Conditions (12), (13) and (14) of Theorem 3 imply that left-hand side of (50) is nonnegative which is possible only when

$$\int_{y_1}^{y_2} \int_{x_1}^{x_2} \Big\{ v(x, y, t_1)u_t(x, y, t_1) - v(x, y, t_2)u_t(x, y, t_2) \Big\} \mathrm{d}x\mathrm{d}y \le 0 \qquad (51)$$

for all $x \in \bar{\mathcal{I}}$ and $y \in \bar{\mathcal{J}}$. However, (51) is not possible since $u(x, y, t_1) = u(x, y, t_2) = 0$ and $u(x, y, t) > 0$ for $(x, y, t) \in \bar{\Omega}$, $u_t(x, y, t_1) > 0$ and $u_t(x, y, t_2) < 0$. This contradiction yields that v can not be a positive solution of problem (9)–(10) on $\bar{\Omega}$.

The proof of the case that $v < 0$ in $\bar{\Omega}$, we let $v = -z$ in $\bar{\Omega}$. Then z becomes a positive solution of problem (9)–(10) in $\bar{\Omega}$. Repeating the same proof for z, we obtain that v can not be a negative solution of problem (9)–(10) on $\bar{\Omega}$. Therefore v must has a zero in $\bar{\Omega}$. The proof of Theorem 3 is complete. □

Remark 1. If the impulse effects are dropped from (7) and (9), i.e. $f_k(x, y, t) \equiv 0$ and $g_k(x, y, t) \equiv 0$, respectively, then Theorem 3 reduces to [17, Theorem 2.1].

Remark 2. If inequalities (12), (13) and (14) in Theorem 3 are replaced by the strict ones;

$$g(x, y, t) > f(x, y, t); \qquad (x, y, t) \in \Omega, \qquad (52)$$

$$s_j(t) > r_j(t); \qquad t \in \bar{\mathcal{K}} \qquad (j = 1, 2, 3, 4), \qquad (53)$$

and

$$g_k(x,y,t) > f_k(x,y,t); \qquad (x,y,t) \in \Omega_{\mathrm{imp}}, \qquad k \in \mathbb{N}, \tag{54}$$

Then it can be easily proved that v has a zero in interior of $\bar{\Omega}$.

Proposition 1 (Sturm comparison theorem)**.** *Let u be a positive solution of problem (7)–(8) on $\bar{\mathcal{I}}\times\bar{\mathcal{J}}\times\mathcal{K}$ satisfying the initial conditions (11). If inequalities (52), (53) and (54) hold, then every solution v of problem (9)–(10) has a zero in Ω.*

Proof. The proof is similar with those of Theorem 3 up to inequality (50). Under conditions (11), (52), (53) and (54) on $\bar{\mathcal{I}} \times \bar{\mathcal{J}} \times \mathcal{K}$, left-hand side of (50) is positive, and possible only when

$$\int_{y_1}^{y_2}\int_{x_1}^{x_2}\Big\{v(x,y,t_1)u_t(x,y,t_1) - v(x,y,t_2)u_t(x,y,t_2)\Big\}\mathrm{d}x\mathrm{d}y < 0 \tag{55}$$

for all $x \in \bar{\mathcal{I}}$ and $y \in \bar{\mathcal{J}}$. Then we have the analogous contradiction as in the proof of Theorem 3. Namely v must has a zero in Ω. □

Remark 3. Inequalities (52), (53) and (54) can be weakened and Proposition 1 can be commuted by the following result:

Proposition 2 (Sturm comparison theorem)**.** *Assume that inequalities (12), (13) and (14) hold. Let u be a positive solution of problem (7)–(8) on $\bar{\mathcal{I}} \times \bar{\mathcal{J}} \times \mathcal{K}$ satisfying the initial conditions (11). If either*

$$\{(x,y,t) \in \Omega : g(x,y,t) - f(x,y,t) > 0\} \neq \emptyset \tag{56}$$

or

$$\{t \in \bar{\mathcal{K}} : s_j(t) - r_j(t) > 0, j = 1,2,3,4\} \neq \emptyset, \tag{57}$$

or that

$$g_{k_0}(x,y,\tau_{k_0}) > f_{k_0}(x,y,\tau_{k_0}) \tag{58}$$

for some $k_0 \in \mathbb{N}$ for which $(x,y,\tau_{k_0}) \in \Omega_{\mathrm{imp}}$, then every solution v of problem (9)–(10) has a zero in Ω.

Proof. Clearly conditions (12)–(14) and (56)–(58) imply inequality (55). □

The following oscillation criterion is immediate.

Corollary 1 (Sturm oscillation theorem)**.** *If the inequalities*

$$g(x,y,t) \geq f(x,y,t); \qquad (x,y,t) \in \Omega^*, \tag{59}$$

$$s_j(t) \geq r_j(t); \qquad t \in [t_*,\infty) \qquad (j = 1,2,3,4) \tag{60}$$

and

$$g_k(x,y,t) \geq f_k(x,y,t); \qquad (x,y,t) \in \Omega^*_{\mathrm{imp}} \qquad (k \in \mathbb{N}), \tag{61}$$

hold for every $t_* \geq t_0$, then every solution of problem (9)–(10) is oscillatory whenever problem (7)–(8) is oscillatory, where

$$\Omega^* = \{(x,y,t) : x \in \mathcal{I}, y \in \mathcal{J},\ t \in [t^*,\infty)\} \tag{62}$$

and

$$\Omega^*_{\mathrm{imp}} = \{(x,y,t) : x \in \mathcal{I}, y \in \mathcal{J},\ t \in [t^*,\infty),\ t = \tau_k, k \in \mathbb{N}\}. \tag{63}$$

3. Nonlinear Comparison Results

The results obtained for linear equations in previous section can be extended to the nonlinear hyperbolic equations of the form

$$\begin{cases} u_{tt} - \triangle u + \mathcal{F}(u,x,y,t) = 0; & (x,y,t) \in \Omega, \\ \Delta u_t(x,y,t) + \mathcal{F}_k(u,x,y,t) = 0; & (x,y,t) \in \Omega_{\text{imp}} \end{cases} \tag{64}$$

and

$$\begin{cases} v_{tt} - \triangle v + \mathcal{G}(v,x,y,t) = 0; & (x,y,t) \in \Omega, \\ \Delta v_t(x,y,t) + \mathcal{G}_k(v,x,y,t) = 0; & (x,y,t) \in \Omega_{\text{imp}} \end{cases} \tag{65}$$

satisfying the boundary conditions (8) and (10), respectively. The functions $r_j(t)$ and $s_j(t)$, $j = 1,2,3,4$, are as previously defined. We assume without further mention that

(i) $u(x,y,t)$ and $v(x,y,t)$ are continuous functions for $(x,y,t) \in \bar{\Omega} \setminus \bar{\Omega}_{\text{imp}}$, and that $\mathcal{F}(u,x,y,t)$, $\mathcal{F}_k(u,x,y,t)$, $\mathcal{G}(v,x,y,t)$ and $\mathcal{G}_k(v,x,y,t)$, $k \in \mathbb{N}$ are real valued continuous functions defined on $\bar{\Omega} \setminus \bar{\Omega}_{\text{imp}}$;

(ii) $p(t), q(t) : \bar{\mathcal{K}} \to \mathbb{R}$ and $(\mu,x,y,t) \in \mathbb{R} \times \bar{\Omega}$ are continuous functions for which

$$\mu\mathcal{F}(\mu,x,y,t) \le p(t)\mu^2 \quad \text{and} \quad \mu\mathcal{G}(\mu,x,y,t) \ge q(t)\mu^2;$$

(iii) $\{p_k\}$ and $\{q_k\}$ are sequences of real numbers for which

$$\mu\mathcal{F}_k(\mu,x,y,t) \le p_k\mu^2 \quad \text{and} \quad \mu\mathcal{G}_k(\mu,x,y,t) \ge q_k\mu^2$$

for all $t \ge t_0$.

Now, we have the following nonlinear comparison result.

Theorem 4 (Sturm comparison theorem). *Let u be a positive solution of problem (64)–(8) on $\bar{\mathcal{I}} \times \bar{\mathcal{J}} \times \mathcal{K}$ satisfying the initial conditions (11). If the inequalities*

$$q(t) \ge p(t) \quad \textit{and} \quad s_j(t) \ge r_j(t) \qquad (j = 1,2,3,4) \tag{66}$$

hold for $t \in \bar{\mathcal{K}}$, and that

$$q_k \ge p_k \tag{67}$$

for all $k \in \mathbb{N}$ for which $\tau_k \in \bar{\mathcal{K}}$, then every solution v of problem (65)–(10) has a zero in $\bar{\Omega}$.

Proof. The proof is based on the inequality

$$\begin{aligned} \left[uv_x - vu_x\right]_x + \left[uv_y - vu_y\right]_y + \left[vu_t - uv_t\right]_t &= \left[u\mathcal{G}(v,x,y,t) - v\mathcal{F}(u,x,y,t)\right] \\ &\ge \left[q(t) - p(t)\right]uv \end{aligned}$$

for $u \in \mathrm{C}\left(\bar{\mathcal{I}} \times \bar{\mathcal{J}} \times \mathcal{K}, \mathbb{R}\right)$, $v \in \mathrm{C}\left(\bar{\Omega}, \mathbb{R}\right)$, and can be done following the same steps those in Theorem 3. Therefore it is left to the reader. □

Remark 4. If the inequalities given in (66) and (67) are replaced by the strict ones;

$$q(t) > p(t) \quad \text{and} \quad s_j(t) > r_j(t) \qquad (j = 1,2,3,4), \tag{68}$$

and that

$$q_k > p_k \tag{69}$$

for all $k \in \mathbb{N}$ for which $\tau_k \in \bar{\mathcal{K}}$, then we have the following comparison result.

Proposition 3 (Sturm comparison theorem). *Let u be a positive solution of problem (64)–(8) on $\bar{\mathcal{I}} \times \bar{\mathcal{J}} \times \mathcal{K}$ satisfying the initial conditions (11). If the inequalities in (68) and (69) hold for $t \in \bar{\mathcal{K}}$, then every solution v of problem (65)–(10) has a zero in Ω.*

Proposition 3 can be weakened by the following result.

Proposition 4 (Sturm comparison theorem)**.** *Assume that the inequalities in* (66) *and* (67) *hold for* $t \in \bar{\mathcal{K}}$*. Let* u *be a positive solution of problem* (64)–(8) *on* $\bar{\mathcal{I}} \times \bar{\mathcal{J}} \times \mathcal{K}$ *satisfying the initial conditions* (11)*. If either*

$$\left\{t \in \bar{\mathcal{K}} : q(t) - p(t) > 0\right\} \neq \emptyset$$

or

$$\left\{t \in \bar{\mathcal{K}} : s_j(t) - r_j(t) > 0, j = 1, 2, 3, 4\right\} \neq \emptyset$$

or that

$$q_{k_0} > p_{k_0}$$

for some $k_0 \in \mathbb{N}$*, then every solution* v *of problem* (65)–(10) *has a zero in* Ω*.*

The following oscillation criterion is immediate.

Corollary 2 (Sturm oscillation theorem)**.** *If the inequalities given in* (68) *and* (67) *are satisfied for* $t \in [t^*, \infty)$*, for every* $t^* \geq t_0$*, then every solution of problem* (65)–(10) *is oscillatory whenever problem* (64)–(8) *is oscillatory.*

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Approximation properties of convolution operators via statistical convergence based on a power series

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Abstract. In this study, our main goal is to obtain approximation properties of convolution operators for multivariables via a special method which is not included in any other methods given before, also known as P-statistical convergence. We present the P-statistical rate of this approximation and provide examples of convolution operators. It is noteworthy to express that one can not approximate f by earlier results for our examples. Therefore, our results fill an important gap in the existing literature. Furthermore, we also present a P-statistical approximation result in the space of periodic continuous functions of period 2π, for short C^*.

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1. Introduction and Background

In the theory of analysis, dealing with the approximation of a given function by other functions which are good and simple is an important problem. Consider a set consisting of functions with bad properties and a subset $Y \subset X$ where Y is a dense subset with good properties. Then one can write any function f of X as a limit of functions of Y. Actually this problem has been studied by Weierstrass [27] and he has shown that every continuous real valued function on $[a, b]$ can be written as a limit of polynomials, i.e., the set of all algebraic polynomials constructs a dense subset of $C[a, b]$. The proof of this theorem is long and hard to follow. Therefore giving a simpler alternative proof to this theorem turns out to be an attractive aim for other mathematicians. Bernstein [5] is the first who has given the simplest proof by using Bernstein polynomials. Then Bohman [6], Korovkin [19] and Popoviciu [20] have extended this by positive and linear operators, independently. The effect of positive and linear operators is well-known in approximation theory, functional analysis, statistics, computer engineering and image processing. One of the important class of such operators is convolution operators. Besides this, the limits used are classical limits in the setting of this theory. But if the classical limit fails, what can be done? The main goal of using summability theory is to still give a limit to a divergent sequence. Since it is effective in such cases, Gadjiev and Orhan [17] have combined summability and approximation theories. Then many results of approximation theory have been extended by statistical convergence, summation process, ideal convergence, power series, indeed general summability methods [2], [3], [11], [12], [22], [25]. In 2003, Srivastava and Gupta obtained approximation properties of operators of some summation-integral forms using classical convergence [23]. Later, in 2008, Duman obtained the approximation properties of convolution operators of integral form using A-statistical convergence defined by an infinite matrix $A = (a_{jn})$ instead of classical convergence [12]. Many mathematicians have also studied them with the use of different types of convergence in both single and multivariable cases, and they are still being researched nowadays. For example, in 2017, Atlıhan, Yurdakadim and Taş investigated the approximation

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properties of convolution operators in the multivariable case by using summation process [2]. In 2022, Çınar and Yıldız studied these operators via P-statistical summation process [8]. In this paper, our goal is to obtain approximation results for convolution operators both for one variable and for multivariables via P-statistical convergence defined by a power series. We have already known that neither statistical convergence nor P-statistical convergence implies each other [25]. We also present the P-statistical rate of this approximation. Furthermore we provide examples as an application of our results. Following the similar technique used here, we present a P-statistical approximation result in C^*, consisting of 2π periodic and continuous functions on $\mathbb{R}$.
Before recalling the basic concepts it is noteworthy to mention that approximation theory, studying positive linear operators, relaxing the positivity, linearity and assigning a limit when the classical limit fails is important since these results have applications in image processing, computer engineering, physics, statistics, computer aided geometric design, deep learning and 3D-modelling.

Now we pause to collect basic concepts which are the main tools of our study.
If

$$\delta(G) := \lim_{k} \frac{1}{k} \#\{n \leq k : n \in G\}$$

exists then it is called as the density of $G \subseteq \mathbb{N}$ where $\#$ is the number of the elements of enclosed set and $\mathbb{N}$ is natural numbers. If $\delta(G_\varepsilon) = 0$ for all $\varepsilon > 0$ where $G_\varepsilon = \{n \in \mathbb{N} : |u_n - l| \geq \varepsilon\}$ then it is said that $u = (u_n)$ statistically converges to l [14], [16], [21]. Let (p_n) be a real number sequence such that $p_1 > 0$, $p_n \geq 0$ for $n \geq 2$ and $p(t) := \sum_{n=1}^{\infty} p_n t^{n-1}$ with radius of convergence $R \in (0, \infty]$. Then we define the method of power series by the following:

Let

$$C_P := \left\{ f : (-R, R) \to \mathbb{R} | \lim_{0<t\to R^-} \frac{f(t)}{p(t)} \quad \text{exists} \right\},$$

$$C_{P_p} := \left\{ u = (u_n) | p_u(t) := \sum_{n=1}^{\infty} p_n t^{n-1} u_n \quad \text{with radius of convergence} \geq R \quad \text{and} \quad p_u \in C_p \right\}$$

and

$$P - \lim u = \lim_{0<t\to R^-} \frac{1}{p(t)} \sum_{n=1}^{\infty} p_n t^{n-1} u_n.$$

Here $P - \lim$ is a functional from C_{P_p} to $\mathbb{R}$ for short P and we say that u is $P-$convergent [7], [18].

The regularity of this method, i.e, if $P - \lim u$ is the same as the limit of u for every convergent sequence $u = (u_n)$, equals to

$$\lim_{t\to R^-} \frac{p_n t^{n-1}}{p(t)} = 0$$

for every $n \in \mathbb{N}$ [7].

Then combining the statistical convergence and power series, a novel concept of convergence known as P-statistical convergence has been introduced in [25] and some results have been investigated by this concept [9], [26].
Now we are ready to recall this concept of convergence.
If

$$\delta_P(G) := \lim_{0<t\to R^-} \frac{1}{p(t)} \sum_{n\in G} p_n t^{n-1}$$

exists then it is called as P-density of G where P is regular. It can be easily seen that $\delta_P(G) \in [0, 1]$ provided that it exists [25]. If $\delta_P(G_\varepsilon) = 0$ for every $\varepsilon > 0$, i.e,

$$\lim_{0<t\to R^-} \frac{1}{p(t)} \sum_{n\in G_\varepsilon} p_n t^{n-1} = 0$$

then we say that u P-statistically converges to l and denote by $st_P - \lim u = l$ where P is regular and u is a real number sequence [25].

Also, it is already known that $\left(C[a,b], ||.||_{C[a,b]}\right)$ is a Banach space where

$$||f|| := ||f||_{C[a,b]} = \sup_{x\in[a,b]} |f(x)|, \quad f \in C[a,b].$$

The following operators of convolution type for one variable:

$$T_n(f;x) = \int_a^b f(y)K_n(y-x)\,dy, \quad n \in \mathbb{N}, \quad x \in [a,b] \tag{1}$$

where $f \in C[a,b], a < b, a, b \in \mathbb{R}$ have been considered via P-statistical summation process in [8] .
One can easily see that T_n are linear, also suppose that K_n are continuous functions on $[a-b, b-a]$ and $K_n(u) \geq 0$ for every $u \in [a-b, b-a]$, for every $n \in \mathbb{N}$. Hence, T_n given by (1) are positive and linear. If one takes $a_{kj}^{(n)} = I$, identity matrix, for all $n \in \mathbb{N}$ in [8], the following results can be obtained immediately. For the completeness, we find it useful to recall the following:

Theorem 1. *[25] Let P be regular, L_n be positive and linear operators for each $n \in \mathbb{N}$ on $C[0,1]$ and $e_i(y) = y^i$, $i = 0, 1, 2$.*
Then

$$st_P - \lim_n ||L_n(e_i) - e_i|| = 0,$$

$i = 0, 1, 2$ implies that

$$st_P - \lim_n ||L_n(f) - f|| = 0$$

for any $f \in C[0,1]$.

It is worth for mentioning that this theorem is the P-statistical version of the well-known theorem given by Gadjiev and Orhan in 2002 [17].
Indeed, this result have been obtained in [22] as follows since $L_n(\varphi;x) = L_n(e_2;x) - 2xL_n(e_1;x) + x^2L_n(e_0;x)$ provided that L_n are positive and linear for every $n \in \mathbb{N}$ where $\varphi(y) := (y-x)^2$ for every $x \in [a,b]$.

Lemma 1. *[22] If*

$$st_P - \lim_n ||L_n(e_0) - e_0|| = 0$$

and

$$st_P - \lim_n ||L_n(\varphi)|| = 0$$

then

$$st_P - \lim_n ||L_n(f) - f|| = 0$$

holds for all $f \in C[a,b]$, where P is regular and L_n are positive and linear operators for each $n \in \mathbb{N}$.

Theorem 2. *[8] Let P be regular and (T_n) be given by (1). If*

$$st_P - \lim_n \int_{-\delta}^{\delta} K_n(y)\,dy = 1$$

and

$$st_P - \lim_n \Big(\sup_{|y|\geq\delta} K_n(y)\Big) = 0$$

hold for a fixed $\delta \in (0, \frac{b-a}{2})$ then

$$st_P - \lim_n ||T_n(f) - f||_\delta = 0$$

holds for every $f \in C[a,b]$ where

$$||f||_\delta := \sup_{a+\delta\leq x\leq b-\delta} |f(x)|.$$

Now, we provide examples such that the earlier results can not be used but we still have the opportunity to approximate f by the above results.

Example 1. *Define the sequences* (p_n) *and* (u_n) *as follows:*

$$p_n = \begin{cases} 1 & , \quad n = 2k \\ 0 & , \quad n = 2k+1 \end{cases}, \qquad u_n = \begin{cases} 0 & , \quad n = 2k \\ 1 & , \quad n = 2k+1 \end{cases}.$$

Notice that P *is regular and* $st_P - \lim u_n = 0$.
Then let T_n *on* $C[a,b]$ *be constructed by*

$$T_n(f;x) = \frac{n(1+u_n)}{\sqrt{\pi}} \int_a^b f(y) e^{-n^2(y-x)^2}\, dy. \tag{2}$$

Here

$$K_n(y) = \frac{n(1+u_n)}{\sqrt{\pi}} e^{-n^2y^2}$$

and T_n *defined by (2) is a convolution operator.*

Notice that Theorem 2.4 and Corollary 2.5 of [12] can not be applied for K_n since (u_n) is neither convergent nor statistically convergent. However, Theorem 2 can be applied to obtain that

$$st_P - \lim_n ||T_n(f) - f||_\delta = 0$$

for every $f \in C[a,b]$ and for a fixed $0 < \delta < \frac{b-a}{2}$.
The above example is standard to give but here, we construct another example which is extraordinary and is motivated by a result in [10].

Example 2. *Let* (u_n) *and the method* P *be constructed as below:*

$$u_n = \begin{cases} 0 & , \quad n = 2k \\ 1 & , \quad n = 2k+1 \end{cases}, \qquad p_n = \begin{cases} 0 & , \quad n = 2k \\ 1 & , \quad n = 2k+1 \end{cases}.$$

Notice that P *is regular and* (u_n) *is* P*-statistically convergent to 1. Then define* T_n *on* $C\left[\frac{-1}{2}, \frac{1}{2}\right]$ *as follows:*

$$T_n(f;x) = u_n \int_{\frac{-1}{2}}^{\frac{1}{2}} f(y)\lambda_n(y-x)dy = u_n L_n(f;x) \text{ with } \lambda_n(y) = c_n(1-y^2)^n,$$

$f \in C\left[\frac{-1}{2}, \frac{1}{2}\right]$ *and* (c_n) *chosen such that* $\int_{-1}^{1} \lambda_n(y)dy = 1$.
Since (u_n) *is neither convergent nor statistically convergent, one can not approximate* f *by the earlier theorems in the classical or statistical settings. But we still have the opportunity to approximate* f *since* (u_n) *is* P*-statistically convergent to 1 by using Theorem 2 and the uniform convergence of* $L_n(f;x)$ *to* f *on* $[\frac{-1}{2} + \delta, \frac{1}{2} + \delta]$ *for each* $0 < \delta < \frac{1}{2}$ *which is also known from [10].*

Furthermore, the rate of this approximation can be given as follows with the use of modulus of continuity and the concept of P-statistical convergence with the rate $o(a_n)$.
The other main tool of this study is P-statistical rate and it was introduced in [1] in light of [15] in 2023.

Definition 1. *[1] Let* (a_n) *be a non-increasing, positive real number sequence and* P *be regular. If*

$$\lim_{0<t\to R^-} \left[\frac{1}{p(t)} \sum_{n:|s_n - l| \geq \varepsilon a_n} p_n t^n \right] = 0$$

is true for every $\varepsilon > 0$ *then we say that* $s = (s_n)$ *is* P*-statistically convergent to* l *with the rate* $o(a_n)$ *and we denote by* $s_n - l = st_P - o(a_n)$, $(n \to \infty)$.

Here it is noteworthy to mention that the terms of the sequence (s_n) are controlling the rate.

Theorem 3. *Let* P *be regular and* (T_n) *be given by (1). Suppose also that* (a_n), (b_n) *are non-increasing sequences of positive numbers and* $0 < \delta < \frac{b-a}{2}$ *be fixed.*
If

$$||T_n(e_0) - e_0||_\delta = st_P - o(a_n), \quad (n \to \infty),$$

and

$$\omega(f, \lambda_n) = st_P - o(b_n), \quad (n \to \infty),$$

then

$$||T_n(f)-f||_\delta = st_P - o(\gamma_n), \quad (n\to\infty)$$

holds for all $f\in C[a,b]$. *Here* $\lambda_n := \sqrt{||T_n(\varphi)||_\delta}$ *and* $\gamma_n := \max\{a_n, b_n, a_n b_n\}$.

Proof. Since it is already shown that there exists $L>0$ such that

$$||T_n(f)-f||_\delta \le L\Big\{\omega(f,\lambda_n)+\omega(f,\lambda_n)||T_n(e_0)-e_0||_\delta + ||T_n(e_0)-e_0||_\delta\Big\}$$

holds for all $n\in\mathbb{N}$, we immediately obtain that

$$||T_n(f)-f||_\delta = st_P - o(\gamma_n)$$

where $\gamma_n := \max\{a_n, b_n, a_n b_n\}$. This gives the desired result. □

2. P-statistical approximation in C^*

Here, we present an approximation result of convolution type operators in the space of periodic functions of period 2π and continuous on $\mathbb{R}$, for short C^* by P-statistical convergence.
It is beneficial to recall $||f||_{C^*} = \sup_{x\in\mathbb{R}} |f(x)|$ is the standard norm on C^*.
Construct T_n for $f\in C^*$ and for each $n\in\mathbb{N}$ as follows:

$$T_n(f;x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)K_n(y-x)dy \tag{3}$$

where $K_n\in C^*$, $K_n(y)\ge 0$ for any $y\in[-\pi,\pi]$. Hence K_n is nonnegative on $\mathbb{R}$. Following a similar way as in earlier results, we can also conclude the next theorem.

Theorem 4. *Let* P *be regular and* (T_n) *be defined by (3). If*

$$\delta_P\Big(\{n\in\mathbb{N} : \frac{1}{2\pi}\int_{-\pi}^{\pi} K_n(y)\,dy = 1\}\Big) = 1$$

and

$$st_P - \lim_n \Big(\sup_{|y|\ge\delta} K_n(y)\Big) = 0$$

for any $\delta>0$ *then we get*

$$st_P - \lim_n ||T_n(f)-f||_{C^*} = 0$$

for any $f\in C^*$.

3. P-Statistical Approximation Of Convolution Operators For Multivariables

Korovkin type approximation theorems have been investigated in real m-dimensional space by statistical convergence and summation process in [4], [13]. In this section, we examine the approximation properties of the below convolution operators for multivariables:

$$T_n(f;x,y) = \int_c^d\int_a^b f(u,v)K_n(u-x,v-y)\,dudv \tag{4}$$

where $(x,y)\in J := [a,b]\times[c,d]$, $f\in C(J)$ and $C(J) = \{f | f : J \longrightarrow \mathbb{R} \quad \textit{continuous}\}$ with the norm $||f|| := \sup_{(x,y)\in J} |f(x,y)|$.
For the positivity we suppose for all $n\in\mathbb{N}$ that $K_n(t,z)$ are continuous and $K_n(t,z)\ge 0$ on $[a-b, b-a]\times[c-d, d-c]$. Hence, $T_n : C(J)\longrightarrow C(J)$ given by (4) are positive and linear operators.
Unfortunately, one variable is insufficient to give a model of real world problems therefore considering multivariable cases in approximation theory has great importance.
First, let us prove the following lemmas which lead us to our main theorem.

Lemma 2. *Let P be regular and $T_n : C(J) \longrightarrow C(J)$ be positive and linear operators for every $n \in \mathbb{N}$. If*

$$st_P - \lim_n ||T_n(f_i) - f_i|| = 0 \quad for \quad i = 0, 1, 2, 3$$

where $f_0(u, v) = 1,\ f_1(u, v) = u,\ f_2(u, v) = v,\ f_3(u, v) = u^2 + v^2$ then

$$st_P - \lim_n ||T_n(f) - f|| = 0$$

holds for all $f \in C(J)$.

Proof. Since $f \in C(J)$, we have $\delta > 0$ for every $\varepsilon > 0$ satisfying that $|f(u, v) - f(x, y)| < \varepsilon$ for every $(u, v) \in J$ such that $|u - x| < \delta$ and $|v - y| < \delta$. Then we can write that

$$\begin{aligned} |f(u,v) - f(x,y)| =& |f(u,v) - f(x,y)|\chi_{J_\delta}(u,v) + |f(u,v) - f(x,y)|\chi_{J\setminus J_\delta}(u,v) \\ \leq& \varepsilon + 2H\chi_{J\setminus J_\delta}(u,v) \end{aligned}$$

where χ_J is the characteristic function of J, $J_\delta = [x - \delta, x + \delta] \times [y - \delta, y + \delta] \cap J$ and $H := ||f||$. Also

$$\chi_{J\setminus J_\delta}(u,v) \leq \frac{(u-x)^2}{\delta^2} + \frac{(v-y)^2}{\delta^2}$$

holds and by combining the above inequalities, we have that

$$|f(u,v) - f(x,y)| \leq \varepsilon + \frac{2H}{\delta^2}\Big\{(u-x)^2 + (v-y)^2\Big\}$$

for every u, v, x, y.

Since T_n are positive and linear for all $n \in \mathbb{N}$, we can also obtain that

$$\begin{aligned} |T_n(f;x,y) - f(x,y)| \leq& T_n\Big(|f(u,v) - f(x,y)|; x, y\Big) \\ &+ |f(x,y)||T_n(f_0;x,y) - f_0(x,y)| \end{aligned}$$

and

$$\begin{aligned} |T_n(f;x,y) - f(x,y)| \leq& \varepsilon + \left(\varepsilon + H + \frac{(h_1^2 + h_2^2)2H}{\delta^2}|T_n(f_0;x,y) - f_0(x,y)|\right) \\ &+ \frac{4h_1H}{\delta^2}|T_n(f_1;x,y) - f_1(x,y)| \\ &+ \frac{4h_2H}{\delta^2}|T_n(f_2;x,y) - f_2(x,y)| \\ &+ \frac{2H}{\delta^2}|T_n(f_3;x,y) - f_3(x,y)| \end{aligned}$$

where $h_1 = \max\{|a|, |b|\}$, $h_2 = \max\{|c|, |d|\}$.

Thus by taking supremum over J, we have that

$$||T_n(f) - f|| \leq \varepsilon + K\Big\{||T_n(f_0) - f_0|| + ||T_n(f_1) - f_1|| + ||T_n(f_2) - f_2|| + ||T_n(f_3) - f_3||\Big\} \tag{5}$$

where

$$K := \max\left\{\varepsilon + H + \frac{(h_1^2 + h_2^2)2H}{\delta^2}, \frac{4h_1H}{\delta^2}, \frac{4h_2H}{\delta^2}, \frac{2h}{\delta^2}\right\}.$$

For a given $r > 0$ pick $\varepsilon > 0$ such that $\varepsilon < r$ and define the followings:

$$F = \{n : ||T_n(f) - f|| \geq r\}$$

$$F_1 = \{n : ||T_n(f_0) - f_0|| \geq \frac{r - \varepsilon}{4K}\}$$

$$F_2 = \{n : ||T_n(f_1) - f_1|| \geq \frac{r - \varepsilon}{4K}\}$$

$$F_3 = \{n : ||T_n(f_2) - f_2|| \geq \frac{r - \varepsilon}{4K}\}$$

$$F_4 = \{n : ||T_n(f_3) - f_3|| \geq \frac{r - \varepsilon}{4K}\}.$$

It is easy to notice that $F \subseteq F_1 \cup F_2 \cup F_3 \cup F_4$ by (5).
Then

$$\frac{1}{p(t)}\sum_{n\in F} p_n t^{n-1} \le \frac{1}{p(t)}\Big\{\sum_{n\in F_1} p_n t^{n-1} + \sum_{n\in F_2} p_n t^{n-1} + \sum_{n\in F_3} p_n t^{n-1} + \sum_{n\in F_4} p_n t^{n-1}\Big\}$$

holds and by taking limit in both sides we obtain that

$$\lim_{t\to R^-} \frac{1}{p(t)}\sum_{n\in F} p_n t^{n-1} = 0$$

then we complete the proof. □

By setting $\Gamma(u,v) = (u-x)^2 + (v-y)^2$, one can immediately get the following via a slight modification.

Lemma 3. *Let P be regular and $T_n : C(J) \longrightarrow C(J)$ be positive and linear operators for every $n \in \mathbb{N}$. If*

$$st_P - \lim_n ||T_n(f_0) - f_0|| = 0$$

and

$$st_P - \lim_n ||T_n(\Gamma)|| = 0$$

then we have

$$st_P - \lim_n ||T_n(f) - f|| = 0$$

for all $f \in C(J)$.

Let

$$||f||_\delta = \sup_{a+\delta\le x\le b-\delta, c+\delta\le y\le d-\delta} |f(x,y)|$$

where $0 < \delta < \min\{\frac{b-a}{2}, \frac{d-c}{2}\}$, $f \in C(J)$ and also let $B_\gamma := [a-b, b-a] \times [c-d, d-c] \backslash [-\gamma,\gamma] \times [-\gamma,\gamma]$ for any $\gamma > 0$ satisfying $\gamma < \min\{b-a, d-c\}$ along the paper.

Lemma 4. *Let P be regular, $\delta \in (0, \min\{\frac{b-a}{2}, \frac{d-c}{2}\})$ be fixed and consider the operators T_n given by (4). If*

$$st_P - \lim_n \int_{-\delta}^{\delta}\int_{-\delta}^{\delta} K_n(u,v)\, dudv = 1$$

and

$$st_P - \lim_n \Big(\sup_{(u,v)\in B_\gamma} K_n(u,v)\Big) = 0$$

for any $\gamma > 0$ then

$$st_P - \lim_n ||T_n(f_0) - f_0||_\delta = 0$$

holds.

Proof. Let $\delta \in (0, \min\{\frac{b-a}{2}, \frac{d-c}{2}\})$ be fixed and $(x,y) \in [a+\delta, b-\delta] \times [c+\delta, d-\delta]$.
We have that

$$\int_{-\delta}^{\delta}\int_{-\delta}^{\delta} K_n(u,v)\, dudv \le T_n(f_0; x, y) \le \int_{-(d-c)}^{d-c}\int_{-(b-a)}^{b-a} K_n(u,v)\, dudv$$

and

$$||T_n(f_0) - f_0||_\delta \le v_n$$

where

$$v_n := \max\Big\{\Big|\int_{-\delta}^{\delta}\int_{-\delta}^{\delta} K_n(u,v)\, dudv - 1\Big|, \Big|\int_{-(d-c)}^{d-c}\int_{-(b-a)}^{b-a} K_n(u,v)\, dudv - 1\Big|\Big\}.$$

By the hypothesis we get $st_P - \lim_n v_n = 0$.
We also have that

$$F := \{n : ||T_n(f_0) - f_0||_\delta \ge \varepsilon\} \subseteq \{n : v_n \ge \varepsilon\} =: F'$$

for a given $\varepsilon > 0$.
Then

$$\frac{1}{p(t)} \sum_{n \in F} p_n t^{n-1} \leq \frac{1}{p(t)} \sum_{n \in F'} p_n t^{n-1}$$

holds and by taking limit we obtain

$$\lim_{t \to R^-} \frac{1}{p(t)} \sum_{n \in F} p_n t^{n-1} = 0$$

which means that

$$st_P - \lim_n ||T_n(f_0) - f_0||_\delta = 0.$$

□

Lemma 5. *Let P be regular, $\delta \in (0, \min\{\frac{b-a}{2}, \frac{d-c}{2}\})$ be fixed and consider the operators T_n given by (4). If*

$$st_P - \lim_n \int_{-\delta}^{\delta} \int_{-\delta}^{\delta} K_n(u,v)\, dudv = 1$$

and

$$st_P - \lim_n \Big(\sup_{(u,v) \in B_\gamma} K_n(u,v) \Big) = 0$$

for any $\gamma > 0$ then we have

$$st_P - \lim_n ||T_n(\Gamma)||_\delta = 0.$$

Proof. Let $\delta \in (0, \min\{\frac{b-a}{2}, \frac{d-c}{2}\})$ be fixed and $(x,y) \in [a+\delta, b-\delta] \times [c+\delta, d-\delta]$. Since $\Gamma(u,v) = (u-x)^2 + (v-y)^2 \in C(J)$ we have that

$$\begin{aligned} T_n(\Gamma; x, y) &= \int_c^d \int_a^b [(u-x)^2 + (v-y)^2] K_n(u-x, v-y)\, dudv \\ &= \int_{c-y}^{d-y} \int_{a-x}^{b-x} (u^2+v^2) K_n(u,v)\, dudv \\ &\leq \int_{-(d-c)}^{d-c} \int_{-(b-a)}^{b-a} (u^2+v^2) K_n(u,v)\, dudv \end{aligned}$$

for every $n \in \mathbb{N}$.
Since Γ is continuous at (0,0), for sufficiently small $\varepsilon > 0$ ($0 < \sqrt{\varepsilon} < \delta$), $\Gamma(u,v) < 2\varepsilon$ holds whenever $|u| < \sqrt{\varepsilon}$, $|v| < \sqrt{\varepsilon}$.
Hence, we obtain that

$$\begin{aligned} T_n(\Gamma; x, y) =& 2\varepsilon \int_{-\sqrt{\varepsilon}}^{\sqrt{\varepsilon}} \int_{-\sqrt{\varepsilon}}^{\sqrt{\varepsilon}} K_n(u,v)\, dudv + \iint_{B_{\sqrt{\varepsilon}}} (u^2+v^2) K_n(u,v)\, du\, dv \\ \leq& 2\varepsilon \int_{-\delta}^{\delta} \int_{-\delta}^{\delta} K_n(u,v)\, dudv + \iint_{B_{\sqrt{\varepsilon}}} (u^2+v^2) K_n(u,v)\, du\, dv \\ \leq& 2\varepsilon \int_{-\delta}^{\delta} \int_{-\delta}^{\delta} K_n(u,v)\, dudv + \mathcal{R} \sup_{(u,v) \in B_{\sqrt{\varepsilon}}} K_n(u,v) \end{aligned}$$

where

$$\mathcal{R} = \int_{c-d}^{d-c} \int_{a-b}^{b-a} (u^2+v^2)\, dudv.$$

Following the similar ways in the earlier results and with the use of hypothesis, we conclude that

$$st_P - \lim_n ||T_n(\Gamma)||_\delta = 0.$$

□

Combining the above results, we can present the following approximation theorem for convolution operators in multivariable case.

Theorem 5. *Let P be regular, $\delta \in (0, \min\{\frac{b-a}{2}, \frac{d-c}{2}\})$ be fixed and the operators T_n given by (4). If*

$$st_P - \lim_n \int_{-\delta}^{\delta}\int_{-\delta}^{\delta} K_n(u,v)\, dudv = 1$$

and

$$st_P - \lim_n \Big(\sup_{(u,v)\in B_\gamma} K_n(u,v) \Big) = 0$$

for any $\gamma > 0$ then

$$st_P - \lim_n ||T_n(f) - f||_\delta = 0$$

holds for every $f \in C(J)$.

Now we can reorganize our Example 1 for multivariable case:

Example 3. *Let $T_n : C(J) \longrightarrow C(J)$ be constructed by*

$$T_n(f;x,y) = n^2 \frac{(1+u_n)}{\pi} \int_c^d \int_a^b f(u,v) e^{-n^2(u-x)^2} e^{-n^2(v-y)^2}\, dudv$$

where (u_n) and (p_n) defined as in Example 1,

$$K_n(u,v) = n^2 \frac{(1+u_n)}{\pi} e^{-n^2u^2} e^{-n^2v^2}.$$

For every $\delta \in (0, \min\{\frac{b-a}{2}, \frac{d-c}{2}\})$, one can have that

$$\int_{-\delta}^{\delta}\int_{-\delta}^{\delta} K_n(u,v)\, dudv = n^2 \frac{(1+u_n)}{\pi} \Big\{ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-n^2u^2} e^{-n^2v^2}\, dudv - \iint_{(u,v)\in \mathcal{B}_\delta} e^{-n^2u^2} e^{-n^2v^2}\, du\, dv \Big\}.$$

Here $\mathcal{B}_\delta := \{(u,v) : |u| \geq \delta \quad or \quad |v| \geq \delta\}$. Since $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-n^2u^2} e^{-n^2v^2}\, dudv = \frac{\pi}{n^2} < \infty$, we have

$$\lim_n \iint_{(u,v)\in \mathcal{B}_\delta} e^{-n^2u^2} e^{-n^2v^2}\, du\, dv = 0$$

which implies that

$$st_P - \lim_n \int_{-\delta}^{\delta}\int_{-\delta}^{\delta} K_n(u,v)\, dudv = 1.$$

One can also obtain that for any $\gamma > 0$

$$\sup_{(u,v)\in B_\gamma} K_n(u,v) \leq n^2 \frac{(1+u_n)}{\pi} \frac{1}{e^{n^2\gamma^2}}$$

and

$$\lim_n \frac{n^2}{e^{n^2\gamma^2}} = 0$$

which implies

$$st_P - \lim_n \Big(\sup_{(u,v)\in B_\gamma} K_n(u,v) \Big) = 0.$$

Therefore our theorem is satisfied for this example but the earlier results can not be applied since (u_n) is neither convergent nor statistically convergent.

In order to give the rate of this approximation we should recall full continuity modulus. Let $f : J \longrightarrow \mathbb{R}$ be continuous and $\lambda > 0$. The full continuity modulus of $f(x,y)$ is defined by

$$\omega(f,\lambda) = \max_{\sqrt{(x_1-x_2)^2+(y_1-y_2)^2} \leq \lambda} |f(x_1,y_1) - f(x_2,y_2)|.$$

It is known that $\lim_{\lambda\to 0} \omega(f,\lambda) = 0$ and for any $\lambda > 0$, $\omega(f,\lambda\Upsilon) \leq ([\Upsilon]+1)\omega(f,\lambda)$ [24].

Theorem 6. *Let P be regular and T_n given by (4). Assume also that $(a_n), (b_n)$ are non-increasing sequences of positive real numbers and $0 < \delta < \min\{\frac{b-a}{2}, \frac{d-c}{2}\}$.*
If

$$||T_n(e_0) - e_0||_\delta = st_P - o(a_n), \quad (n \to \infty),$$

and

$$\omega(f, \lambda_n) = st_P - o(b_n), \quad (n \to \infty),$$

then we have that

$$||T_n(f) - f||_\delta = st_P - o(\gamma_n), \quad (n \to \infty)$$

for every $f \in C(J)$. Here $\lambda_n := \sqrt{||T_n((u-x)^2 + (v-y)^2; x, y)||_\delta}$ and $\gamma_n = \max\{a_n, b_n, a_n b_n\}$.

Proof. Let $0 < \delta < \min\{\frac{b-a}{2}, \frac{d-c}{2}\}$, $f \in C(J)$ and $(x, y) \in [a+\delta, b-\delta] \times [c+\delta, d-\delta]$.
For any $\lambda > 0$ we have that

$$\begin{aligned}|T_n(f; x, y) - f(x, y)| \leq& T_n(|f(u, v) - f(x, y)|; x, y) + |f(x, y)||T_n(f_0) - f_0| \\ \leq& \omega(f, \lambda) T_n\left(1 + \frac{(u-x)^2 + (v-y)^2}{\lambda^2}; x, y\right) + |f(x, y)||T_n(f_0) - f_0| \\ \leq& \omega(f, \lambda)\left\{T_n(f_0) + \frac{1}{\lambda^2} T_n((u-x)^2 + (v-y)^2; x, y)\right\} + |f(x, y)||T_n(f_0) - f_0|\end{aligned}$$

since T_n are positive and linear operators.
This implies for all $n \in \mathbb{N}$, that

$$||T_n(f) - f||_\delta \leq \omega(f, \lambda)\left\{||T_n(f_0)||_\delta + \frac{1}{\lambda^2}||T_n((u-x)^2 + (v-y)^2; x, y)||_\delta\right\} + H_1||T_n(f_0) - f_0||_\delta$$

where $H_1 := ||f||_\delta$. Now letting $\lambda = \lambda_n = \sqrt{||T_n((u-x)^2 + (v-y)^2; x, y)||_\delta}$,

$$\begin{aligned}||T_n(f) - f||_\delta \leq& \omega(f, \lambda_n)\left\{||T_n(f_0)||_\delta + 1\right\} + H_1||T_n(f_0) - f_0||_\delta \\ \leq& 2\omega(f, \lambda_n) + \omega(f, \lambda_n)||T_n(f_0) - f_0||_\delta + H_1||T_n(f_0) - f_0||_\delta\end{aligned}$$

holds and also by letting $H := \max\{2, H_1\}$

$$||T_n(f) - f||_\delta \leq H\left\{\omega(f, \lambda_n) + \omega(f, \lambda_n)||T_n(f_0) - f_0||_\delta + ||T_n(f_0) - f_0||_\delta\right\}$$

for all $n \in \mathbb{N}$.
This implies

$$||T_n(f) - f||_\delta = st_P - o(\gamma_n)$$

where $\gamma_n = \max\{a_n, b_n, a_n b_n\}$. □

4. Concluding Remarks

The theory of approximation deals with the problem of expressing a given function by other functions which are good and simple. This problem goes back to Weierstrass and many mathematicians have studied on it after the well-known Weierstrass approximation theorem. Also after the simplest alternative proof of Bernstein to this theorem, Bohman, Korovkin and Popoviciu have extended this for positive and linear operators, independently. One of the important classes of such operators are convolution type operators and studying these operators as well as positive linear operators, relaxing positivity and linearity, assigning a limit when the classical limit fails have great importance since these type of results have applications in image processing, computer engineering, physics, statistics, computer aided geometric design, deep learning and 3D-modelling.
Here, we present approximation properties of convolution operators for multivariables via a special method called P-statistical convergence. It worths for mentioning that this method is not included in any other methods given before and unfortunately one variable is insufficient to give a model for real world problems. We also obtain the rate of this approximation and provide examples to support our results. Furthermore, an approximation result in the space of periodic functions of period 2π is presented by using similar techniques.

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Generalized Hukuhara diamond-alpha derivative of fuzzy valued functions on time scales

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Abstract. In the literature, the delta and nabla derivatives have been considered separately in the study of fuzzy number valued functions on time scales. In this paper, to unify these two derivatives for fuzzy number valued functions, we propose a new dynamic derivative called the diamond-alpha derivative, defined via the generalized Hukuhara difference. We establish several fundamental properties of the diamond-alpha derivative and investigate a particular class of fuzzy initial value problems on time scales with respect to this new derivative. Additionally, we provide numerical examples to illustrate our results.

2020 Mathematics Subject Classification. 34A07 and 34N05.
Keywords. Time scale, diamond-alpha derivative, generalized Hukuhara difference, fuzzy differential equations

1. Introduction

Dynamic equations on time scales theory is a relatively new field of study, and research in this area is expanding considerably in the last 35 years. In order to combine continuous and discrete structures, time scale theory was established. It enables simultaneous treatments to both difference and differential equations and expands the results to dynamic equations. Basics of time-scale calculus and some recent studies can be found in [1, 2, 6, 7, 12–15, 18, 21, 22, 25]. However, it's crucial to consider a lot of uncertain aspects while attempting to fully explore a real-world phenomenon. Zadeh [35] developed fuzzy set theory in order to define these ambiguous or inaccurate concepts. Kaleva [16] and Lakshmikantham and Mohapatra [17] established and explored the theory of fuzzy differential equations (FDEs) and its applications. One drawback of the Hukuhara differentiability-based methods is that the solution to an FDE only exists for longer support lengths. Bede et al. [3] investigated generalized Hukuhara differentiability in order to get over this drawback. And many authors [4, 20, 28] are enthusiastic about this new differentiability concept for fuzzy number valued functions because of this favored benefit. Fard and Bidgoli [10] investigated the calculus of fuzzy functions on time scales. In their study of fuzzy dynamic equations on time scales, Vasavi et al. [31–34], by implementing the Hukuhara difference, introduced the Hukuhara, 2nd type Hukuhara and generalized delta derivatives. The drawback of this derivative is that it only applies to fuzzy number valued functions on time scales where the diameter increases with length.

To the best of our knowledge, the delta and nabla derivatives have been used independently to study the derivatives of fuzzy number valued functions on time scales. The characteristics of generalized nabla differentiability for fuzzy number valued functions on time scales via Hukuhara difference were presented and examined by Leelavathi et al. [19]. Additionally, they acquired some generalized nabla differentiable fuzzy number valued function embedding results. Furthermore, under generalized nabla differentiability, they demonstrated a fundamental principle of a nabla integral calculus for fuzzy functions on time scales. Fuzzy differential equations on time scales under generalized delta derivative were examined by Vasali

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et al [31]. In order to achieve solutions for FDEs with decreasing length of support, they established the generalized delta derivative, which is based on four forms. These types of derivatives, in some cases (such as time scales with discrete points), can only describe the change of functions only on the left or right side of considered points. In order to provide a tool that can catch the change of functions on both sides of points in time scales, dynamic derivatives, called the diamond alpha derivative has been proposed by Sheng et al. [27]. This dynamic derivative is a convex linear combination of the delta and nabla derivatives. Later, Roger et. al [26] redefined the diamond-alpha derivative independently of the standard delta and nabla dynamic derivatives, and further examined its properties. In [30], they introduced a dynamic derivative called diamond-alpha derivative via generalized Hukuhara difference for interval valued functions on time scales. They furthermore studied a particular class of interval differential equations with respect to the diamond-alpha derivative.

In this work, motivated by [30], we introduce a dynamic derivative called as the diamond-alpha derivative, denoted as $\diamond^{\alpha}_{gH}$, for fuzzy number valued functions on time scales via generalized Hukuhara difference and Hausdorff metric for fuzzy sets and investigate its properties under different conditions on time scale $\mathbb{T}$. Through our main results, we establish foundational results concerning the existence and uniqueness of the $\diamond^{\alpha}_{gH}$-derivative for fuzzy functions. Additionally, we explore conditions under which fuzzy functions are $\diamond^{\alpha}_{gH}$-differentiable at both dense and isolated points on the time scale, providing criteria for the existence of limits in these contexts. The final results address the differentiability of the r-level sets of fuzzy functions, particularly under monotonicity "length conditions". These results enhance the understanding of $\diamond^{\alpha}_{gH}$-differentiability in fuzzy functions and its applications within fuzzy differential equations on time scales.

This paper's outline is as follows: We give some basic definitions and results relating to the calculus of time scales and fuzzy sets in Section 2. In Section 3, we present the main results and provide some examples to illustrate some of the results. In Section 4, we consider a particular class of fuzzy initial value problems on time scales and present some numerical examples.

2. Preliminaries

Definition 1. *[6] A nonempty closed subset of the real numbers $\mathbb{R}$ is called a time scale, often denoted by $\mathbb{T}$.*

Definition 2. *[6] The function $\sigma : \mathbb{T} \to \mathbb{R}$ defined by*

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}$$

is called the forward jump operator. Additionally, we set $\inf \varnothing := \sup \mathbb{T}$.

Definition 3. *[6] The function $\rho : \mathbb{T} \to \mathbb{R}$ defined by*

$$\rho(t) = \sup\{s \in \mathbb{T} : s < t\}$$

is called the backward jump operator. Additionally, we set $\sup \varnothing := \inf \mathbb{T}$.

Definition 4. *[6] If $\sigma(t) > t$, then $t \in \mathbb{T}$ is said to be a right-scattered point.*

Definition 5. *[6] If $\rho(t) < t$, then $t \in \mathbb{T}$ is said to be a left-scattered point.*

Definition 6. *[6] If $\sigma(t) = t$ and $t \neq \sup \mathbb{T}$, then $t \in \mathbb{T}$ is said to be a right-dense point.*

Definition 7. *[6] If $\rho(t) = t$ and $t \neq \inf \mathbb{T}$, then $t \in \mathbb{T}$ is said to be a left-dense point.*

Definition 8. *[6] The function $\mu : \mathbb{T} \to [0, \infty)$ defined by $\mu(t) = \sigma(t) - t$ is called the (forward) graininess.*

Definition 9. *[6] The function $\nu : \mathbb{T} \to [0, \infty)$ defined by $\nu(t) = t - \rho(t)$ is called the backward graininess.*

Additionally, we define the following notations for simplicity in the definitions and theorems throughout this paper: $\mu_{st} = \sigma(s) - t$ and $\nu_{st} = t - \rho(s)$.
The set $\mathbb{T}^{\kappa}$ is defined as follows: if $\mathbb{T}$ has a left-scattered maximum m, then $\mathbb{T}^{\kappa} := \mathbb{T} \setminus \{m\}$. If no such maximum exists, then $\mathbb{T}^{\kappa} := \mathbb{T}$. Similarly, the set $\mathbb{T}_{\kappa}$ is defined as follows: if $\mathbb{T}$ has a right-scattered minimum m, then $\mathbb{T}_{\kappa} := \mathbb{T} \setminus \{m\}$. If no such minimum exists, then $\mathbb{T}_{\kappa} := \mathbb{T}$.

Definition 10. *[6] Let $h : \mathbb{T} \to \mathbb{R}$ be a function and let $s \in \mathbb{T}^\kappa$. We define $(\Delta h)(s)$ as the number (if it exists) that satisfies the following property: for any $\epsilon > 0$, there exists a neighborhood $N_\mathbb{T}$ of s given by $N_\mathbb{T} := (s-\delta, s+\delta) \cap \mathbb{T}$ for some $\delta > 0$ such that*

$$|[h(\sigma(s)) - h(t)] - (\Delta h)(s)[\sigma(s) - t]| \leq \epsilon|\sigma(s) - t|$$

for all $t \in N_\mathbb{T}$. The value $(\Delta h)(s)$ is called the delta derivative of h at s.

Definition 11. *[6] Let $h : \mathbb{T} \to \mathbb{R}$ be a function, and let $s \in \mathbb{T}_\kappa$. We define $(\nabla h)(s)$ as the number (if it exists) that satisfies the following property: for any $\epsilon > 0$, there is a neighborhood $N_\mathbb{T}$ of s given by $N_\mathbb{T} := (s-\delta, s+\delta) \cap \mathbb{T}$ for some $\delta > 0$ such that*

$$|[h(\rho(s)) - h(t)] - (\nabla h)(s)[\rho(s) - t]| \leq \epsilon|\rho(s) - t|$$

for all $t \in N_\mathbb{T}$. The value $(\nabla h)(s)$ is referred to as the nabla derivative of h at s.

Definition 12. *[26] Let $h : \mathbb{T} \to \mathbb{R}$ be a function and $s \in \mathbb{T}^\kappa \cap \mathbb{T}_\kappa$. Then the $\diamond^\alpha$-derivative of h at the point $s \in \mathbb{T}^\kappa_\kappa$, denoted by $(\diamond^\alpha h)(s)$, is the number (provided it exists) that satisfies the following property: for any $\epsilon > 0$, there is a neighborhood $N_\mathbb{T}$ of s given by $N_\mathbb{T} := (s-\delta, s+\delta) \cap \mathbb{T}$ for some $\delta > 0$ such that*

$$|\alpha|h(\sigma(s)) - h(t)||\nu_{st}| + (1-\alpha)|h(\rho(s)) - h(t)||\mu_{st}| - (\diamond^\alpha h)(s)|\nu_{st}\mu_{st}| \leq \epsilon|\nu_{st}\mu_{st}|,$$

for any $t \in N_\mathbb{T}$. Here, $(\diamond^\alpha h)(s)$ referred to as the diamond-alpha derivative of h at s.

Definition 13. *[35] A fuzzy set u in a universe of discourse U is represented by a function $u : U \to [0,1]$, where $u(x)$ indicates the membership degree of x to the fuzzy set u.*

We use $F(U)$ to denote the set of all fuzzy subsets of U.

Definition 14. *[23] Let $u : U \to [0,1]$ be a fuzzy set. The r-level sets of u are defined as*

$$u_r = \{x \in U : u(x) \geq r\}$$

for $0 < r \leq 1$. The 0-level set of u

$$u_0 = cl\,\{x \in U : u(x) > 0\}$$

is called the support of the fuzzy set u. Here, cl denotes the closure of the set u.

Definition 15. *[23] Let $u : \mathbb{R} \to [0,1]$ be a fuzzy subset of the real numbers. Then, u is said to be a fuzzy number if it fulfills the following criteria:*

1. *u is normal, which means that there exists an $x_0 \in \mathbb{R}$ such that $u(x_0) = 1$.*
2. *u is quasi-concave, which means that for all $\lambda \in [0,1]$, the inequality $u(\lambda x + (1-\lambda)y) \geq \min\{u(x), u(y)\}$ holds.*
3. *u is upper semicontinuous on $\mathbb{R}$, which means that for any $\epsilon > 0$, there exists a $\delta > 0$ such that $u(x) - u(x_0) < \epsilon$ whenever $|x - x_0| < \delta$.*
4. *u is compactly supported, which means that the closure $cl\{x \in \mathbb{R} : u(x) > 0\}$ is compact.*

We use $F_N(\mathbb{R})$ to denote the set of all fuzzy numbers of $\mathbb{R}$.

Definition 16. *Let $a_1 \leq a_2 \leq a_3$ be real numbers. The fuzzy number denoted by $u = (a_1, a_2, a_3)$ is called a triangular fuzzy number whose membership function is*

$$u(x) = \begin{cases} \frac{x-a_1}{a_2-a_1}, & a_1 \leq x \leq a_2, \\ \frac{a_3-x}{a_3-a_2}, & a_2 \leq x \leq a_3, \\ 0, & otherwise. \end{cases}$$

Definition 17. *[29] Let $u, v \in F_N(\mathbb{R})$. The generalized Hukuhara difference (gH-difference) is the fuzzy number w, if it exists, such that*

$$u \ominus_{gH} v = w \iff u = v + w \text{ or } v = u + (-1)w.$$

Since level sets of a fuzzy number are closed and bounded intervals, we will denote r-level set of a fuzzy number u by $u_r = [u_r^-, u_r^+]$ and its length by $len(u_r) = u_r^+ - u_r^-$.

Remark 1. *The criteria for the existence of $w = u \ominus_{gH} v$ in $F_N(\mathbb{R})$ are as follows:*

Case (i):

- $w_r^- = u_r^- - v_r^-$ *and* $w_r^+ = u_r^+ - v_r^+$

- *Here, w_r^- must be increasing, w_r^+ must be decreasing, and it must hold that $w_r^- \leq w_r^+$ for all r in* $[0,1]$.

Case (ii):

- $w_r^- = u_r^+ - v_r^+$ *and* $w_r^+ = u_r^- - v_r^-$
- *Similarly, w_r^- must be increasing, w_r^+ must be decreasing, and $w_r^- \leq w_r^+$ must hold for all r in* $[0,1]$.

Theorem 1. *[5, 29] Let $u, v \in F_N(\mathbb{R})$. If gH-difference $u \ominus_{gH} v \in F_N(\mathbb{R})$ exists, then*

$$(u \ominus_{gH} v)_r = [\min\{u_r^- - v_r^-, u_r^+ - v_r^+\}, \max\{u_r^- - v_r^-, u_r^+ - v_r^+\}].$$

Definition 18. *[8] The metric $D_\infty : F_N(\mathbb{R}) \times F_N(\mathbb{R}) \to \mathbb{R}^+ \cup \{0\}$ defined by*

$$D_\infty(u,v) = \sup_{r\in[0,1]} \max\left\{\left|u_r^- - v_r^-\right|, \left|u_r^+ - v_r^+\right|\right\},$$

where $u_r = [u_r^-, u_r^+]$, $v_r = [v_r^-, v_r^+]$, is called Hausdorff metric for fuzzy numbers.

The Hausdorff metric provides a way to measure the distance between two fuzzy sets by considering their level sets. This metric allows researchers to compare the similarity or dissimilarity of fuzzy sets in a rigorous mathematical way. Specifically, it can be used to quantify how far apart two fuzzy sets are based on their support and their membership functions.

Theorem 2. *[8]*
Let a, b, c, $d \in F_N(\mathbb{R})$ and $m \in \mathbb{R}$. The Hausdorff metric satisfies the followings:

(1) $D_\infty(a+c, b+c) = D_\infty(a,b)$.
(2) $D_\infty(ma, mb) = |m|\, D_\infty(a,b)$.
(3) $D_\infty(a+b, c+d) \leq D_\infty(a,c) + D_\infty(b,d)$.

3. Generalized Hukuhara Diamond-Alpha Derivative of Fuzzy Valued Functions on Time Scales

Definition 19. *[31] Let $f : \mathbb{T} \to F_N(\mathbb{R})$ be a fuzzy function and let $s \in \mathbb{T}^\kappa$. The generalized Hukuhara delta derivative of f at s, if it exists, is a fuzzy number $(\Delta_{gH} f)(s) \in F_N(\mathbb{R})$ such that for any given $\epsilon > 0$, there exists a neighborhood $N_\mathbb{T}(s,\delta) = (s-\delta, s+\delta) \cap \mathbb{T}$ for some $\delta > 0$, such that for all $t \in N_\mathbb{T}$, $f(\sigma(s)) \ominus_{gH} f(t)$ exists and we have*

$$D_\infty(f(\sigma(s)) \ominus_{gH} f(t), (\Delta_{gH} f)(s)\mu_{st}) \leq \epsilon\,|\mu_{st}|.$$

Definition 20. *[19] Let $f : \mathbb{T} \to F_N(\mathbb{R})$ be a fuzzy function and let $s \in \mathbb{T}_\kappa$. The generalized Hukuhara nabla derivative of f at s, if it exists, is a fuzzy number $(\nabla_{gH} f)(s) \in F_N(\mathbb{R})$ such that for any given $\epsilon > 0$, there exists a neighborhood $N_\mathbb{T}(s,\delta) = (s-\delta, s+\delta) \cap \mathbb{T}$ for some $\delta > 0$, such that for all $t \in N_\mathbb{T}$, $f(t) \ominus_{gH} f(\rho(s))$ exists and we have*

$$D_\infty(f(t) \ominus_{gH} f(\rho(s)), (\nabla_{gH} f)(s)\nu_{st}) \leq \epsilon\,|\nu_{st}|.$$

Definition 21. *Let $f : \mathbb{T} \to F_N(\mathbb{R})$ be a fuzzy function and let $s \in \mathbb{T}_\kappa^\kappa$. The generalized Hukuhara diamond-alpha derivative of f at s, if it exists, is a fuzzy number $\left(\diamond_{gH}^\alpha f\right)(s) \in F_N(\mathbb{R})$ such that for any given $\epsilon > 0$, there exists a neighborhood $N_\mathbb{T}(s,\delta) = (s-\delta, s+\delta) \cap \mathbb{T}$ for some $\delta > 0$, such that for all $t \in N_\mathbb{T}$ $f(\sigma(s)) \ominus_{gH} f(t)$ and $f(t) \ominus_{gH} f(\rho(s))$ exist and we have*

$$D_\infty(\alpha[f(\sigma(s)) \ominus_{gH} f(t)]\nu_{st} + (1-\alpha)[f(t) \ominus_{gH} f(\rho(s))]\mu_{st}, (\diamond_{gH}^\alpha f)(s)\mu_{st}\nu_{st}) \leq \epsilon\,|\mu_{st}\nu_{st}|.$$

Theorem 3. *Let $f : \mathbb{T} \to F_N(\mathbb{R})$ be a fuzzy function and $s \in \mathbb{T}_\kappa^\kappa$. $\left(\diamond_{gH}^\alpha f\right)(s) \in F_N(\mathbb{R})$ is unique, if it exists.*

Proof. Let $s \in \mathbb{T}_\kappa^\kappa$. Assume $\left(\diamond_{gH}^\alpha f\right)_1(s)$ and $\left(\diamond_{gH}^\alpha f\right)_2(s)$ are $\diamond_{gH}^\alpha$-derivative of f at s. Let $\epsilon > 0$ be arbitrary. Then there exists a $\delta > 0$ such that for any $t \in N_\mathbb{T}(s,\delta)$ we have

$$D_\infty\big(\alpha\big[f(\sigma(s)) \ominus_{gH} f(t)\big]\,\nu_{st} + (1-\alpha)\big[f(t) \ominus_{gH} f(\rho(s))\big]\,\mu_{st},$$
$$\left(\diamond_{gH}^\alpha f\right)_1(s)\,\mu_{st}\,\nu_{st}\big) \leq \frac{\epsilon}{2}\,|\mu_{st}\,\nu_{st}|,$$
$$D_\infty\big(\alpha\big[f(\sigma(s)) \ominus_{gH} f(t)\big]\,\nu_{st} + (1-\alpha)\big[f(t) \ominus_{gH} f(\rho(s))\big]\,\mu_{st},$$

$$\left(\diamond_{gH}^{\alpha} f\right)_2(s)\,\mu_{st}\,\nu_{st}\right) \leq \frac{\epsilon}{2}\left|\mu_{st}\,\nu_{st}\right|.$$

$$\begin{aligned}
&D_\infty\big(\left(\diamond_{gH}^{\alpha} f\right)_1(s), \left(\diamond_{gH}^{\alpha} f\right)_2(s)\big)\\
&\quad= \frac{1}{|\mu_{st}\,\nu_{st}|} D_\infty\big(\left(\diamond_{gH}^{\alpha} f\right)_1(s)\,\mu_{st}\,\nu_{st}, \left(\diamond_{gH}^{\alpha} f\right)_2(s)\,\mu_{st}\,\nu_{st}\big)\\
&\quad= \frac{1}{|\mu_{st}\,\nu_{st}|} D_\infty\left(\begin{array}{c}\left(\diamond_{gH}^{\alpha} f\right)_1(s)\,\mu_{st}\,\nu_{st} + \alpha\big[f(\sigma(s)) \ominus_{gH} f(t)\big]\,\nu_{st}\\ +(1-\alpha)\big[f(t)\ominus_{gH} f(\rho(s))\big]\,\mu_{st},\\ \left(\diamond_{gH}^{\alpha} f\right)_2(s)\,\mu_{st}\,\nu_{st} + \alpha\big[f(\sigma(s)) \ominus_{gH} f(t)\big]\,\nu_{st}\\ +(1-\alpha)\big[f(t)\ominus_{gH} f(\rho(s))\big]\,\mu_{st}\end{array}\right)\\
&\quad\leq \frac{1}{|\mu_{st}\,\nu_{st}|} D_\infty\big(\left(\diamond_{gH}^{\alpha} f\right)_1(s)\,\mu_{st}\,\nu_{st}, \alpha\big[f(\sigma(s)) \ominus_{gH} f(t)\big]\,\nu_{st}\\
&\qquad + (1-\alpha)\big[f(t)\ominus_{gH} f(\rho(s))\big]\,\mu_{st}\big)\\
&\qquad + \frac{1}{|\mu_{st}\,\nu_{st}|} D_\infty\big(\left(\diamond_{gH}^{\alpha} f\right)_2(s)\,\mu_{ts}\,\nu_{ts}, \alpha\big[f(\sigma(s)) \ominus_{gH} f(t)\big]\,\nu_{st}\\
&\qquad + (1-\alpha)\big[f(t)\ominus_{gH} f(\rho(s))\big]\,\mu_{st}\big)\\
&\quad\leq \frac{1}{|\mu_{st}\,\nu_{st}|}\frac{\epsilon}{2}\left|\mu_{st}\,\nu_{st}\right| + \frac{1}{|\mu_{st}\,\nu_{st}|}\frac{\epsilon}{2}\left|\mu_{st}\,\nu_{st}\right|\\
&\quad\leq \epsilon.
\end{aligned}$$

Therefore, $\left(\diamond_{gH}^{\alpha} f\right)_1(s) = \left(\diamond_{gH}^{\alpha} f\right)_2(s)$. □

Theorem 4. *Let $f : \mathbb{T} \to F_N(\mathbb{R})$ be a function and $s \in \mathbb{T}_\kappa^\kappa$ a dense point. Then f is $\diamond_{gH}^{\alpha}$-differentiable at s if and only if the limit*

$$\lim_{t\to s} \frac{f(s)\ominus_{gH} f(t)}{s-t}$$

exists and

$$\left(\diamond_{gH}^{\alpha} f\right)(s) = \lim_{t\to s} \frac{f(s)\ominus_{gH} f(t)}{s-t}.$$

Proof. Since s is dense, $\sigma(s) = \rho(s) = s$. Hence, we obtain

$$\begin{aligned}
\alpha\frac{f(\sigma(s))\ominus_{gH} f(t)}{\mu_{st}} + (1-\alpha)\frac{f(t)\ominus_{gH} f(\rho(s))}{\nu_{st}} &= \alpha\frac{f(s)\ominus_{gH} f(t)}{\mu_{st}} + (1-\alpha)\frac{f(t)\ominus_{gH} f(s)}{\nu_{st}}\\
&= \alpha\frac{f(s)\ominus_{gH} f(t)}{\mu_{st}} + (1-\alpha)\frac{f(s)\ominus_{gH} f(t)}{\mu_{st}}\\
&= (\alpha+1-\alpha)\frac{f(s)\ominus_{gH} f(t)}{\mu_{st}}\\
&= \frac{f(s)\ominus_{gH} f(t)}{\mu_{st}}.
\end{aligned}$$

So, we have

$$D_\infty\left(\frac{f(s)\ominus_{gH} f(t)}{\mu_{st}}, \left(\diamond_{gH}^{\alpha} f\right)(s)\right) < \epsilon.$$

Therefore,

$$\left(\diamond_{gH}^{\alpha} f\right)(s) = \lim_{t\to s} \frac{f(s)\ominus_{gH} f(t)}{s-t}.$$

□

Theorem 5. *Let $f : \mathbb{T} \to F_N(\mathbb{R})$ be a function and $s \in \mathbb{T}_\kappa^\kappa$ be an isolated point. Then f is $\diamond_{gH}^{\alpha}$-differentiable at s and*

$$\left(\diamond_{gH}^{\alpha} f\right)(s) = \alpha\frac{f(\sigma(s))\ominus_{gH} f(s)}{\mu(s)} + (1-\alpha)\frac{f(s)\ominus_{gH} f(\rho(s))}{\nu(s)}.$$

Proof. Since s is an isolated point, we have

$$\lim_{t\to s}\left[\alpha\frac{f(\sigma(s))\ominus_{gH} f(t)}{\mu_{st}} + (1-\alpha)\frac{f(t)\ominus_{gH} f(\rho(s))}{\nu_{st}}\right]$$

$$= \quad \alpha\frac{f(\sigma(s))\ominus_{gH} f(s)}{\mu(s)} + (1-\alpha)\frac{f(s)\ominus_{gH} f(\rho(s))}{\nu(s)}.$$

Hence, we obtain that f is $\diamond^{\alpha}_{gH}$-differentiable at s and

$$\left(\diamond^{\alpha}_{gH} f\right)(s) = \alpha\frac{f(\sigma(s))\ominus_{gH} f(s)}{\mu(s)} + (1-\alpha)\frac{f(s)\ominus_{gH} f(\rho(s))}{\nu(s)}.$$

□

Theorem 6. *Let $f:\mathbb{T}\to F_N(\mathbb{R})$ be a fuzzy function and $s\in\mathbb{T}^{\kappa}_{\kappa}$. Assume f is Δ_{gH} and ∇_{gH} differentiable at s. Then, f is $\diamond^{\alpha}_{gH}$-differentiable at s and*

$$(\diamond^{\alpha}_{gH} f)(s) = \alpha(\Delta_{gH} f)(s) + (1-\alpha)(\nabla_{gH} f)(s).$$

Proof. Let $\epsilon>0$ be given. Since f is Δ_{gH} and ∇_{gH} differentiable at s, there exists $\delta>0$ such that for any $t\in N_{\mathbb{T}}(s,\delta)=(s-\delta,s+\delta)\cap\mathbb{T}$, we have

$$\begin{aligned} D_{\infty}(f(\sigma(s))\ominus_{gH} f(t), (\Delta_{gH} f)(s)\mu_{st}) &\leq \frac{\epsilon}{2}\left|\mu_{st}\right|, \\ D_{\infty}(f(t)\ominus_{gH} f(\rho(s)), (\nabla_{gH} f)(s)\nu_{st}) &\leq \frac{\epsilon}{2}\left|\nu_{st}\right|. \end{aligned}$$

It follows that

$$\begin{aligned} D\left(\alpha\left[f(\sigma(s))\ominus_{gH} f(t)\right]\nu_{st}, \alpha(\Delta_{gH} f)(s)\mu_{st}\nu_{st}\right) &\leq \frac{\epsilon\alpha}{2}\left|\mu_{st}\nu_{st}\right|, \\ D\left((1-\alpha)\left[f(t)\ominus_{gH} f(\rho(s))\right]\mu_{st}, (1-\alpha)(\nabla_{gH} f)(s)\mu_{st}\nu_{st}\right) &\leq \frac{\epsilon(1-\alpha)}{2}\left|\mu_{st}\nu_{st}\right|. \end{aligned}$$

We get

$$\begin{aligned} &D\Big(\alpha\big[f(\sigma(s))\ominus_{gH} f(t)\big]\,\nu_{st} + (1-\alpha)\big[f(t)\ominus_{gH} f(\rho(s))\big]\,\mu_{st}, (\alpha(\Delta_{gH} f)(s) + (1-\alpha)(\nabla_{gH} f)(s))\;\mu_{st}\,\nu_{st}\Big) \\ &\quad\leq D\big(\alpha\big[f(\sigma(s))\ominus_{gH} f(t)\big]\,\nu_{st}, \alpha(\Delta_{gH} f)(s)\,\mu_{st}\,\nu_{st}\big) \\ &\qquad + D\big((1-\alpha)\big[f(t)\ominus_{gH} f(\rho(s))\big]\,\mu_{st}, (1-\alpha)(\nabla_{gH} f)(s)\,\mu_{st}\,\nu_{st}\big) \\ &\quad\leq \frac{\epsilon\alpha}{2}\left|\mu_{st}\,\nu_{st}\right| + \frac{\epsilon(1-\alpha)}{2}\left|\mu_{st}\,\nu_{st}\right| \\ &\quad\leq \epsilon\left|\mu_{st}\,\nu_{st}\right|. \end{aligned}$$

Therefore, f is $\diamond^{\alpha}_{gH}$- differentiable at s and

$$(\diamond^{\alpha}_{gH} f)(s) = \alpha(\Delta_{gH} f)(s) + (1-\alpha)(\nabla_{gH} f)(s).$$

□

Theorem 7. *Let $f:\mathbb{T}\to F_N(\mathbb{R})$ be a function and the r-level sets of f be*

$$f_r(t) = \left[f_r^-(t), f_r^+(t)\right]$$

for any $t\in\mathbb{T}$ and $r\in[0,1]$, where $f_r^-:\mathbb{T}\to\mathbb{R}$ and $f_r^+:\mathbb{T}\to\mathbb{R}$ are the left and right end-points of the r-level sets. Assume $len(f_r(t)) := f_r^+(t) - f_r^-(t)$ is monotone on a neigborhood $N_{\mathbb{T}}(s,\delta)=(s-\delta,s+\delta)\cap\mathbb{T}$ for some $\delta>0$ and f is $\diamond^{\alpha}_{gH}$- differentiable at $s\in\mathbb{T}^{\kappa}_{\kappa}$. Then, f_r^- and f_r^+ are $\diamond^{\alpha}$-differentiable at s as well. Moreover,

(1) *if $len(f_r(t))$ is increasing on a neighborhood of $s\in\mathbb{T}^{\kappa}_{\kappa}$, then*

$$\left(\diamond^{\alpha}_{gH} f\right)_r(s) = \left[\left(\diamond^{\alpha} f_r^-\right)(s), \left(\diamond^{\alpha} f_r^+\right)(s)\right],$$

(2) *if $len(f_r(t))$ is decreasing on a neighborhood of $s\in\mathbb{T}^{\kappa}_{\kappa}$, then*

$$\left(\diamond^{\alpha}_{gH} f\right)_r(s) = \left[\left(\diamond^{\alpha} f_r^+\right)(s), \left(\diamond^{\alpha} f_r^-\right)(s)\right].$$

Proof. (1) Assume that $len(f_r(t)$ is increasing on $N_{\mathbb{T}}(s,\delta)=(s-\delta,s+\delta)\cap\mathbb{T}$ for some $\delta>0$, $\sigma(s)\neq t$ and $\rho(s)\neq t$ for any fixed $r\in[0,1]$. Let us consider the following cases.

Case 1: Let $\rho(s) < t < \sigma(s)$. Hence, $\mu_{st} = \sigma(s) - t > 0$ and $\nu_{st} = t - \rho(s) > 0$. Since $len(f_r(t))$ is increasing on $N_{\mathbb{T}}(s,\delta)$ we have $len(f(\sigma(s))) > len(f(t))$ and $len(f(t)) > len(f(\rho(s)))$ for any $t \in N_{\mathbb{T}}(s,\delta)$. Let

$$g(t) := \alpha \frac{f(\sigma(s)) \ominus_{gH} f(t)}{\mu_{st}} + (1-\alpha) \frac{f(t) \ominus_{gH} f(\rho(s))}{\nu_{st}}.$$

By Remark 1 and some interval arithmetics, we obtain

$$\left[g_r^-(t), g_r^+(t)\right] = \left[\alpha \frac{f_r^-(\sigma(s)) - f_r^-(t)}{\mu_{st}} + (1-\alpha)\frac{f_r^-(t) - f_r^-(\rho(s))}{\nu_{st}}, \right. \left. \alpha \frac{f_r^+(\sigma(s)) - f_r^+(t)}{\mu_{st}} + (1-\alpha)\frac{f_r^+(t) - f_r^+(\rho(s))}{\nu_{st}}\right].$$

Case 2: Let $t > \sigma(s)$. Hence, $\mu_{st} = \sigma(s) - t < 0$ and $\nu_{st} = t - \rho(s) > 0$. Since $len(f_r(t))$ is increasing on $N_{\mathbb{T}}(s,\delta)$ we have $len(f(\sigma(s))) < len(f(t))$ and $len(f(t)) > len(f(\rho(s)))$ for any $t \in N_{\mathbb{T}}(s,\delta)$. Let

$$g(t) := \alpha \frac{f(\sigma(s)) \ominus_{gH} f(t)}{\mu_{st}} + (1-\alpha) \frac{f(t) \ominus_{gH} f(\rho(s))}{\nu_{st}}.$$

By Remark 1 and some interval arithmetics, we obtain

$$\left[g_r^-(t), g_r^+(t)\right] = \left[\alpha \frac{f_r^-(\sigma(s)) - f_r^-(t)}{\mu_{st}} + (1-\alpha)\frac{f_r^-(t) - f_r^-(\rho(s))}{\nu_{st}}, \right. \left. \alpha \frac{f_r^+(\sigma(s)) - f_r^+(t)}{\mu_{st}} + (1-\alpha)\frac{f_r^+(t) - f_r^+(\rho(s))}{\nu_{st}}\right].$$

Case 3: Let $t < \rho(s)$. Hence, $\mu_{st} = \sigma(s) - t > 0$ and $\nu_{st} = t - \rho(s) < 0$. Since $len(f_r(t))$ is increasing on $N_{\mathbb{T}}(s,\delta)$ we have $len(f(\sigma(s))) > len(f(t))$ and $len(f(t)) < len(f(\rho(s)))$ for any $t \in N_{\mathbb{T}}(s,\delta)$. Let

$$g(t) := \alpha \frac{f(\sigma(s)) \ominus_{gH} f(t)}{\mu_{st}} + (1-\alpha) \frac{f(t) \ominus_{gH} f(\rho(s))}{\nu_{st}}.$$

Similarly, by Remark 1 and some interval arithmetics, we obtain

$$\left[g_r^-(t), g_r^+(t)\right] = \left[\sigma \frac{f_r^-(\sigma(s)) - f_r^-(t)}{\mu_{st}} + (1-\alpha)\frac{f_r^-(t) - f_r^-(\rho(s))}{\nu_{st}}, \right. \left. \alpha \frac{f_r^+(\sigma(s)) - f_r^+(t)}{\mu_{st}} + (1-\alpha)\frac{f_r^+(t) - f_r^+(\rho(s))}{\nu_{st}}\right].$$

Furthermore, since f is $\diamond_{gH}^{\alpha}$- differentiable at s, we derive

$$\lim_{t\to s}\left[\alpha \frac{f(\sigma(s)) \ominus_{gH} f(t)}{\mu_{st}} + (1-\alpha)\frac{f(t) \ominus_{gH} f(\rho(s))}{\nu_{st}}\right] = \left(\diamond_{gH}^{\alpha} f\right)(s) \in F_N(\mathbb{R}).$$

The proof of (2) can be done similarly. □

Definition 22. *Let $f : \mathbb{T} \to F_N(\mathbb{R})$ be a function and the r-level sets of f be*

$$f_r(t) = \left[f_r^-(t), f_r^+(t)\right]$$

for any $t \in \mathbb{T}$ and $r \in [0,1]$, where $f_r^- : \mathbb{T} \to \mathbb{R}$ and $f_r^+ : \mathbb{T} \to \mathbb{R}$ are the left and right end-points of the r-level sets. Assume that f is $\diamond_{gH}^{\alpha}$- differentiable at $s \in \mathbb{T}_{\kappa}^{\kappa}$. Then, f is said to be

(1) $\diamond_{gH1}^{\alpha}$- *differentiable at s if*

$$\left(\diamond_{gH}^{\alpha} f\right)_r(s) = \left[\left(\diamond^{\alpha} f_r^-\right)(s), \left(\diamond^{\alpha} f_r^+\right)(s)\right],$$

(2) $\diamond_{gH2}^{\alpha}$- *differentiable at s if*

$$\left(\diamond_{gH}^{\alpha} f\right)_r(s) = \left[\left(\diamond^{\alpha} f_r^+\right)(s), \left(\diamond^{\alpha} f_r^-\right)(s)\right].$$

Theorem 8. *Let $f : \mathbb{T} \to F_N(\mathbb{R})$ be a function and the r-level sets of f be*

$$f_r(t) = \left[f_r^-(t), f_r^+(t)\right]$$

for any $t \in \mathbb{T}$ and $r \in [0,1]$, where $f_r^- : \mathbb{T} \to \mathbb{R}$ and $f_r^+ : \mathbb{T} \to \mathbb{R}$ are left and right end-points of the r-level sets. Assume that f is $\diamond_{gH}^{\alpha}$- differentiable at $s \in \mathbb{T}_{\kappa}^{\kappa}$.

(1) *If f is $\diamond^{\alpha}_{gH1}$- differentiable on $N_T(s,\delta)$, then f has non-decreasing length of the closure of its support.*
(2) *If f is $\diamond^{\alpha}_{gH2}$- differentiable on $N_T(s,\delta)$, then f has non-increasing length of the closure of its support.*

Proof. (1) Assume f is $\diamond^{\alpha}_{gH1}$- differentiable on $N_{\mathbb{T}}(s,\delta)$ and $len(f_r(t))$ is decreasing length of the closure of its support for some $t \in N_{\mathbb{T}}(s,\delta)$ for any fixed $r \in [0,1]$. Then, by Theorem 7 we have

$$\left(\diamond^{\alpha}_{gH} f\right)_0 (t) = \left[\left(\diamond^{\alpha} f_0^{+}\right)(t), \left(\diamond^{\alpha} f_0^{-}\right)(t)\right],$$

which contradicts with the fact that f is $\diamond^{\alpha}_{gH1}$- differentiable on $N_{\mathbb{T}}(s,\dot{\delta})$. Hence, f has non-decreasing length of the closure of its support.

(2) Assume f is $\diamond^{\alpha}_{gH2}$- differentiable on $N_{\mathbb{T}}(s,\delta)$ and $len(f_r(t))$ is increasing length of the closure of its support for some $t \in N_{\mathbb{T}}(s,\delta)$ for any fixed $r \in [0,1]$. Then, by Theorem 7 we have

$$\left(\diamond^{\alpha}_{gH} f\right)_0 (t) = \left[\left(\diamond^{\alpha} f_0^{-}\right)(t), \left(\diamond^{\alpha} f_0^{+}\right)(t)\right].$$

which contradicts with the fact that f is $\diamond^{\alpha}_{gH2}$- differentiable on $N_{\mathbb{T}}(s,\dot{\delta})$. Hence, f has non-decreasing length of the closure of its support.

□

3.1. **Examples.**

Example 1. *Consider the time scale $\mathbb{T} = h\mathbb{Z} = \{hn : n \in \mathbb{Z}, h > 0\}$ and let $f : [0,\infty)_{\mathbb{T}} \to F_N(\mathbb{R})$ be a function such that $f(t) = (1,2,3)t$. The r-level sets of f are $f_r(t) = [1+r, 3-r]\,t$. By Theorem 5 , f is $\diamond^{\alpha}_{gH}$-differentiable at any $s \in [h,\infty)_{\mathbb{T}}$ such that*

$$\left(\diamond^{\alpha}_{gH} f\right)(s) = \alpha \frac{f(\sigma(s)) \ominus_{gH} f(s)}{\mu(s)} + (1-\alpha)\frac{f(s) \ominus_{gH} f(\rho(s))}{\nu(s)}.$$

Since $len(f_r(t)) = 2t(1-r)$, which is increasing for any fixed $r \in [0,1]$, we have

$$\left(\diamond^{\alpha}_{gH} f\right)_r (s) = \left[\left(\diamond^{\alpha} f_r^{-}\right)(s), \left(\diamond^{\alpha} f_r^{+}\right)(s)\right].$$

Let $\alpha = \frac{1}{2}$, then we have

$$\begin{aligned}\left(\diamond^{\frac{1}{2}} f_r^{-}\right)(s) &= \frac{1}{2}\frac{f_r^{-}(\sigma(s)) - f_r^{-}(s)}{\mu(s)} + \frac{1}{2}\frac{f_r^{-}(s) - f_r^{-}(\rho(s))}{\nu(s)} \\ &= \frac{1}{2}\left[\frac{f_r^{-}(s+h) - f_r^{-}(s)}{h} + \frac{f_r^{-}(s) - f_r^{-}(s-h)}{h}\right] \\ &= \frac{1}{2}\left[\frac{(1+r)(s+h) - (1+r)s}{h} + \frac{(1+r)s - (1+r)(s-h)}{h}\right] \\ &= 1 + r.\end{aligned}$$

Similarly we can obtain $\left(\diamond^{\frac{1}{2}} f_r^{+}\right)(s) = 3 - r$. Therefore, $\left(\diamond^{\frac{1}{2}}_{gH} f\right)_r (s) = [1+r, 3-r]$ and $\left(\diamond^{\frac{1}{2}}_{gH} f\right)(s) = (1,2,3)$.

Example 2. *Consider the time scale $\mathbb{T} = \{\sqrt{n} : n \in \mathbb{N}\}$ and let $f : \mathbb{T} \to F_N(\mathbb{R})$ be a function such that $f(t) = (1,2,3)\frac{1}{t^2}$. The r-level sets of f are $f_r(t) = [1+r, 3-r]\,\frac{1}{t^2}$ and*

$$\begin{aligned} len(f_r(t)) &= \frac{1}{t^2}(3 - r - 1 - r) \\ &= \frac{1}{t^2}(2 - 2r) \\ &= \frac{2}{t^2}(1-r),\end{aligned}$$

which is decreasing for any fixed $r \in [0,1]$. Hence, f is $\diamond^{\alpha}_{gH}$-differentiable at $s \in [\sqrt{2},\infty)_T$ with

$$\left(\diamond^{\alpha}_{gH} f\right)_r (s) = \left[\left(\diamond^{\alpha} f_r^{+}\right)(s), \left(\diamond^{\alpha} f_r^{-}\right)(s)\right],$$

where

$$
\begin{aligned}
\left(\diamond^{\alpha} f_r^{-}\right)(s) &= \alpha \frac{f_r^{-}(\sigma(s)) - f_r^{-}(s)}{\mu(s)} + (1-\alpha)\frac{f_r^{-}(s) - f_r^{-}(\rho(s))}{\nu(s)} \\
&= \alpha \frac{\frac{1+r}{\sigma^2(s)} - \frac{1+r}{s^2}}{\sigma(s) - s} + (1-\alpha)\frac{\frac{1+r}{s^2} - \frac{1+r}{\rho^2(s)}}{s - \rho(s)} \\
&= \alpha \frac{\frac{1+r}{s^2+1} - \frac{1+r}{s^2}}{\sqrt{s^2+1} - s} + (1-\alpha)\frac{\frac{1+r}{s^2} - \frac{1+r}{s^2-1}}{s - \sqrt{s^2-1}}.
\end{aligned}
$$

and

$$
\begin{aligned}
\left(\diamond^{\alpha} f_r^{+}\right)(s) &= \alpha \frac{f_r^{+}(\sigma(s)) - f_r^{+}(s)}{\mu(s)} + (1-\alpha)\frac{f_r^{+}(s) - f_r^{+}(\rho(s))}{\nu(s)} \\
&= \alpha \frac{\frac{3-r}{s^2+1} - \frac{3-r}{s^2}}{\sqrt{s^2+1} - s} + (1-\alpha)\frac{\frac{3-r}{s^2} - \frac{3-r}{s^2-1}}{s - \sqrt{s^2-1}}.
\end{aligned}
$$

4. Fuzzy Initial Value Problems on time scales with Generalized Hukuhara Diamond-Alpha Derivatives

In this section, we consider the following fuzzy initial value problem (FIVP):

$$
\begin{aligned}
(\diamond^{\alpha}_{gH} y)(t) &= f(t, y(t)), \qquad t \in (a,b)_T \subset \mathbb{T}^{\kappa}_{\kappa} \qquad (1)\\
y(t_0) &= y_0, \qquad (2)
\end{aligned}
$$

where $f : (a,b)_{\mathbb{T}} \times F_N(\mathbb{R}) \to F_N(\mathbb{R})$. Assume that r-level sets of y, f and $\diamond^{\alpha}_{gH} y$ are

$$
\begin{aligned}
y_r(t) &= \left[y_r^{-}(t), y_r^{+}(t)\right], \\
f_r(t) &= \left[f_r^{-}(t), f_r^{+}(t)\right], \\
(\diamond^{\alpha}_{gH} y)_r(t) &= [(\diamond^{\alpha} y_r^{-})(t), (\diamond^{\alpha} y_r^{+})(t)].
\end{aligned}
$$

There are two cases to be considered:

Case 1: $len(y_r)$ is increasing. By Theorem 7, we obtain

$$(\diamond^{\alpha}_{gH} y)_r(t) = [(\diamond^{\alpha} y_r^{-})(t), (\diamond^{\alpha} y_r^{+})(t)].$$

Therefore, FIVP (1)-(2) can be expressed by the system:

$$
\begin{aligned}
(\diamond^{\alpha} y_r^{-})(t) &= f_r^{-}(t, y_r^{-}(t), y_r^{+}(t)), \\
(\diamond^{\alpha} y_r^{+})(t) &= f_r^{+}(t, y_r^{-}(t), y_r^{+}(t)), \\
y_r^{-}(t_0) &= y_{0r}^{-}, \\
y_r^{+}(t_0) &= y_{0r}^{+},
\end{aligned}
$$

where $r \in [0,1]$ and $t \in (a,b)_{\mathbb{T}}$.

Case 2: $len(y_r)$ is decreasing. By Theorem 7, we obtain

$$(\diamond^{\alpha}_{gH} y)_r(t) = [(\diamond^{\alpha} y_r^{+})(t), (\diamond^{\alpha} y_r^{-})(t)].$$

Therefore, FIVP (1)-(2) can be expressed by

$$
\begin{aligned}
(\diamond^{\alpha} y_r^{-})(t) &= f_r^{+}(t, y_r^{-}(t), y_r^{+}(t)), \\
(\diamond^{\alpha} y_r^{+})(t) &= f_r^{-}(t, y_r^{-}(t), y_r^{+}(t)), \\
y_r^{-}(t_0) &= y_{0r}^{-}, \\
y_r^{+}(t_0) &= y_{0r}^{+},
\end{aligned}
$$

where $r \in [0,1]$ and $t \in (a,b)_{\mathbb{T}}$.

Assume $\mathbb{T} = \{t_0, t_1, t_2, ..., t_{N+1} : t_i < t_{i+1}, \forall i \in \overline{0,N}\}$ with $\mathbb{T}^{\kappa} = \mathbb{T} \backslash \{t_{N+1}\}$, $\mathbb{T}_{\kappa} = T \backslash \{t_0\}$ and $\mathbb{T}^{\kappa}_{\kappa} = \mathbb{T}^{\kappa} \cap \mathbb{T}_{\kappa}$. Since $\mathbb{T}$ is an isolated time scale, according to Theorem 5 we obtain

$$(\diamond^{\alpha} y_r^{-})(t) = \alpha \frac{y_r^{-}(t_{i+1}) - y_r^{-}(t_i)}{\mu(t_i)} + (1-\alpha)\frac{y_r^{-}(t_i) - y_r^{-}(t_{i-1})}{\nu(t_i)},$$

$$(\diamond^{\alpha} y_r^{+})(t) = \alpha \frac{y_r^{+}(t_{i+1}) - y_r^{+}(t_i)}{\mu(t_i)} + (1-\alpha)\frac{y_r^{+}(t_i) - y_r^{+}(t_{i-1})}{\nu(t_i)}.$$

Hence, Case 1 and Case 2 become

$$\begin{aligned}
\alpha \frac{y_r^{-}(t_{i+1}) - y_r^{-}(t_i)}{\mu(t_i)} + (1-\alpha)\frac{y_r^{-}(t_i) - y_r^{-}(t_{i-1})}{\nu(t_i)} &= f_r^{-}(t_i, y_r^{-}(t_i), y_r^{+}(t_i)),\\
\alpha \frac{y_r^{+}(t_{i+1}) - y_r^{+}(t_i)}{\mu(t_i)} + (1-\alpha)\frac{y_r^{+}(t_i) - y_r^{+}(t_{i-1})}{\nu(t_i)} &= f_r^{+}(t_i, y_r^{-}(t_i), y_r^{+}(t_i)),\\
y_r^{-}(t_0) &= y_{0r}^{-},\\
y_r^{+}(t_0) &= y_{0r}^{+},
\end{aligned}$$

and

$$\begin{aligned}
\alpha \frac{y_r^{-}(t_{i+1}) - y_r^{-}(t_i)}{\mu(t_i)} + (1-\alpha)\frac{y_r^{-}(t_i) - y_r^{-}(t_{i-1})}{\nu(t_i)} &= f_r^{+}(t_i, y_r^{-}(t_i), y_r^{+}(t_i)),\\
\alpha \frac{y_r^{+}(t_{i+1}) - y_r^{+}(t_i)}{\mu(t_i)} + (1-\alpha)\frac{y_r^{+}(t_i) - y_r^{+}(t_{i-1})}{\nu(t_i)} &= f_r^{-}(t_i, y_r^{-}(t_i), y_r^{+}(t_i)),\\
y_r^{-}(t_0) &= y_{0r}^{-},\\
y_r^{+}(t_0) &= y_{0r}^{+},
\end{aligned}$$

respectively.

4.1. **Numerical Examples.** Now we will give some numerical examples. Triangular fuzzy numbers are widely used in fuzzy applications. They offer a straightforward and efficient means of representing and handling uncertainty and vagueness in data. Therefore, in numerical examples, fuzzy constants and initial conditions will be represented as triangular fuzzy numbers.

Example 3. *Let $\mathbb{T} = h\mathbb{Z}$ and let us consider the following FIVP:*

$$(\diamond_{gH}^{\alpha} y)(t) = -y(t), t \in (0,5)_{h\mathbb{Z}}, \tag{3}$$

$$y(0) = y(\sigma(h)) = y(h) = (-1, 0, 1). \tag{4}$$

Assume r-level sets of y and $\diamond_{gH}^{\alpha} y$ are

$$\begin{aligned}
y_r(t) &= \left[y_r^{-}(t), y_r^{+}(t)\right],\\
(\diamond_{gH}^{\alpha} y)_r(t) &= [(\diamond^{\alpha} y_r^{-})(t), (\diamond^{\alpha} y_r^{+})(t)].
\end{aligned}$$

By using the method above, we obtain the following two systems:

Case1: *Under $\diamond_{gH1}^{\alpha}$-differentiability, the FIVP yields the following system:*

$$\begin{aligned}
\alpha \frac{y_r^{-}(t_{i+1}) - y_r^{-}(t_i)}{\mu(t_i)} + (1-\alpha)\frac{y_r^{-}(t_i) - y_r^{-}(t_{i-1})}{\nu(t_i)} &= -y_r^{+}(t_i),\\
\alpha \frac{y_r^{+}(t_{i+1}) - y_r^{+}(t_i)}{\mu(t_i)} + (1-\alpha)\frac{y_r^{+}(t_i) - y_r^{+}(t_{i-1})}{\nu(t_i)} &= -y_r^{-}(t_i),\\
y_r^{-}(0) = y_r^{-}(h) &= -1 + r,\\
y_r^{+}(0) = y_r^{+}(h) &= 1 - r.
\end{aligned}$$

Case2: *Under $\diamond_{gH2}^{\alpha}$-differentiability, the FIVP yields the following system:*

$$\begin{aligned}
\alpha \frac{y_r^{-}(t_{i+1}) - y_r^{-}(t_i)}{\mu(t_i)} + (1-\alpha)\frac{y_r^{-}(t_i) - y_r^{-}(t_{i-1})}{\nu(t_i)} &= -y_r^{-}(t_i),\\
\alpha \frac{y_r^{+}(t_{i+1}) - y_r^{+}(t_i)}{\mu(t_i)} + (1-\alpha)\frac{y_r^{+}(t_i) - y_r^{+}(t_{i-1})}{\nu(t_i)} &= -y_r^{+}(t_i),\\
y_r^{-}(0) = y_r^{-}(h) &= -1 + r,\\
y_r^{+}(0) = y_r^{+}(h) &= 1 - r.
\end{aligned}$$

The approximate and true solutions to these systems for $h = \frac{1}{10}$, $\alpha = 0.8$, and $r = 0$ are illustrated in Figure 1 and Figure 2. In these figures, since the end points of the solutions do not switch, these solutions *exist on* $[0,5]$ *in both cases. We observe that switching of the end points of the solution may occur, and the error in the approximate solution may increase as we change* α. *In Case 1, when we set* $\alpha = 0.43$, *switching occurs at* $t = \frac{27}{10}$, *which implies that* $\diamond^{\alpha}_{gH1}$*-differentiability does not exist after* $t = \frac{19}{5}$. *And in Case 2, when we set* $\alpha = 0.53$, *switching occurs at* $t = \frac{19}{5}$, *which implies that* $\diamond^{\alpha}_{gH2}$*-differentiability does not exist after* $t = \frac{19}{5}$; *also see Figure 3.*

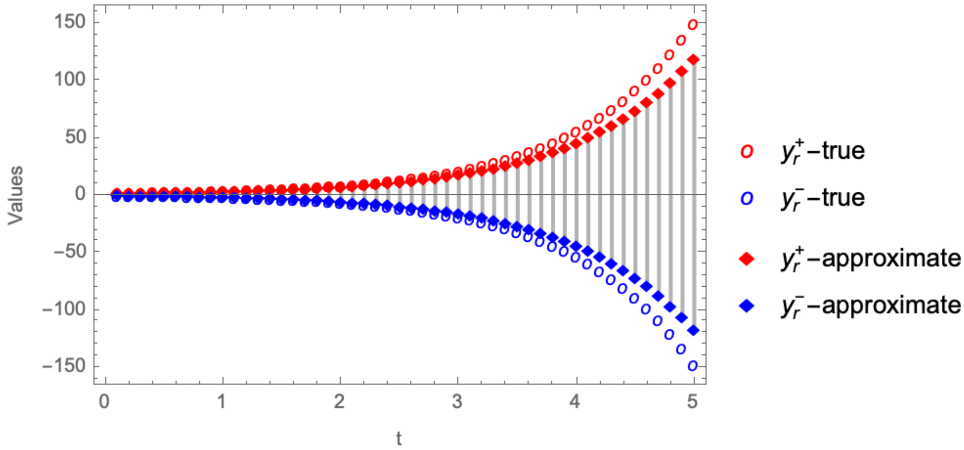

FIGURE 1. 0-level solutions to Case 1.

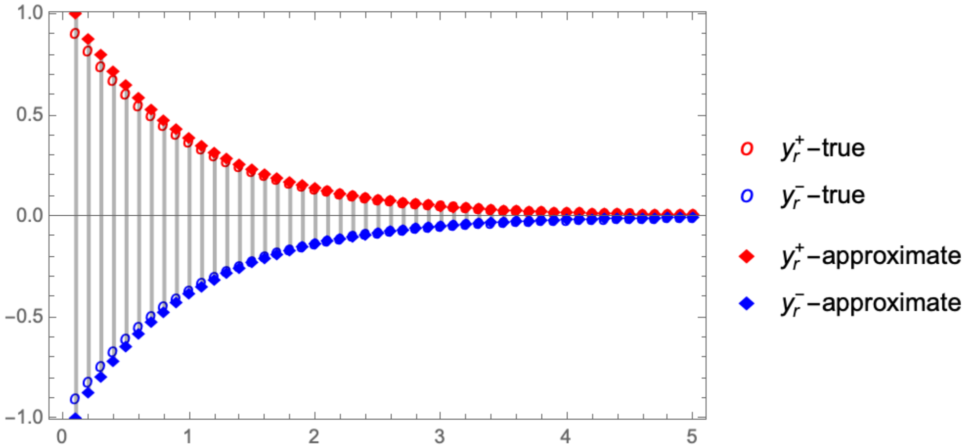

FIGURE 2. 0-level solutions to Case 2.

Example 4. *Let* $\mathbb{T} = h\mathbb{Z}$ *and let us consider the following FIVP:*

$$(\diamond^{\alpha}_{gH} y)(t) = -y(t) + (1,2,3)e^{-t}, t \in (0,5)_{h\mathbb{Z}}, \quad (5)$$

$$y(0) = y(\sigma(h)) = y(h) = (-2,0,2). \quad (6)$$

Assume r*-level sets of* y *and* $\diamond^{\alpha}_{gH} y$ *are*

$$y_r(t) = \left[y_r^-(t), y_r^+(t)\right],$$

$$(\diamond^{\alpha}_{gH} y)_r(t) = [(\diamond^{\alpha} y_r^-)(t), (\diamond^{\alpha} y_r^+)(t)].$$

By using the method above, we obtain the following two systems:

Case1: *Under* $\diamond^{\alpha}_{gH1}$*-differentiability, the FIVP yields the following system:*

$$\alpha \frac{y_r^-(t_{i+1}) - y_r^-(t_i)}{\mu(t_i)} + (1-\alpha)\frac{y_r^-(t_i) - y_r^-(t_{i-1})}{\nu(t_i)} = -y_r^+(t_i) + (1+r)e^{-t},$$

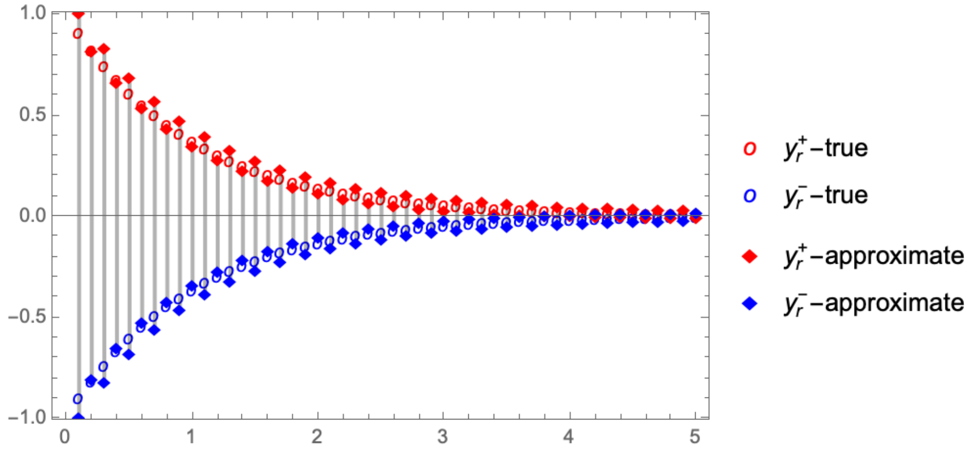

FIGURE 3. 0-level solutions to Case 2.

$$\begin{aligned}
\alpha\frac{y_r^+(t_{i+1})-y_r^+(t_i)}{\mu(t_i)}+(1-\alpha)\frac{y_r^+(t_i)-y_r^+(t_{i-1})}{\nu(t_i)} &= -y_r^-(t_i)+(3-r)e^{-t},\\
y_r^-(0)=y_r^-(h) &= -2+2r,\\
y_r^+(0)=y_r^+(h) &= 2-2r.
\end{aligned}$$

Case2: *Under $\diamond^{\alpha}_{gH2}$-differentiability, the FIVP yields the following system:*

$$\begin{aligned}
\alpha\frac{y_r^-(t_{i+1})-y_r^-(t_i)}{\mu(t_i)}+(1-\alpha)\frac{y_r^-(t_i)-y_r^-(t_{i-1})}{\nu(t_i)} &= -y_r^+(t_i)+(1+r)e^{-t},\\
\alpha\frac{y_r^+(t_{i+1})-y_r^+(t_i)}{\mu(t_i)}+(1-\alpha)\frac{y_r^+(t_i)-y_r^+(t_{i-1})}{\nu(t_i)} &= -y_r^-(t_i)+(3-r)e^{-t},\\
y_r^-(0)=y_r^-(h) &= -2+2r,\\
y_r^+(0)=y_r^+(h) &= 2-2r.
\end{aligned}$$

Figure 4 and Figure 5 illustrate the approximate and true solutions of these systems for $h=\frac{1}{15}$, $\alpha=0.6$, and $r=0$. In both figures, we have fuzzy solutions within the interval $[0,5]$ as there is no switching at the endpoints. In Case 1, setting $\alpha=0.45$ causes switching at $t=\frac{49}{15}$, indicating that $\diamond^{\alpha}_{gH1}$-differentiability in a neighborhood of $t=\frac{49}{15}$. In Case 2, setting $\alpha=0.53$ causes switching at $t=\frac{19}{5}$, indicating that $\diamond^{\alpha}_{gH2}$-differentiability in a neighborhood of $t=\frac{19}{5}$.

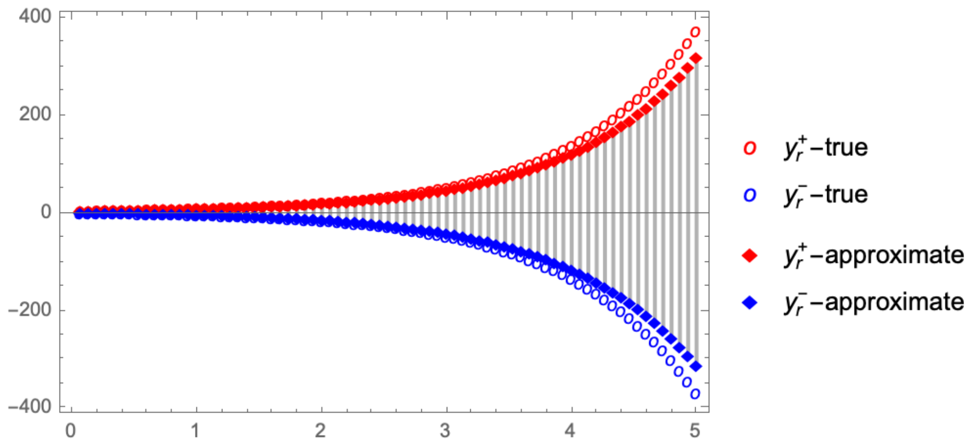

FIGURE 4. 0-level solutions to Case 1.

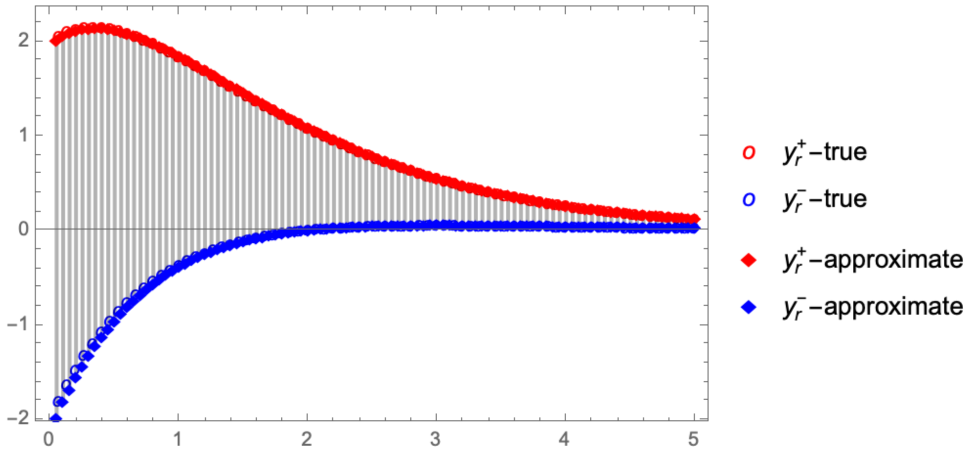

FIGURE 5. 0-level solutions to Case 2.

5. CONCLUSIONS

We have introduced the diamond-alpha derivative for fuzzy number valued functions on time scales by employing the generalized Hukuhara difference. Additionally, we have established some fundamental properties of this derivative and applied it to a specific class of fuzzy initial value problems on time scales. Numerical examples demonstrate the existence of approximate solutions under certain parameter settings with potential switching in the end points of the level sets of the solutions as the parameter α varies. Such switching can affect the accuracy of approximate solutions and the existence of $\diamond^{\alpha}_{gH1}$-differentiability or $\diamond^{\alpha}_{gH2}$-differentiability. These results enhance the understanding of the behavior of $\diamond^{\alpha}_{gH}$-differentiability in fuzzy functions and its applications within fuzzy differential equations on time scales.

Author Contribution Statements The authors equally contributed to the paper.

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Geometry of pointwise hemi-slant warped product submanifolds in para-contact manifolds

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Abstract. In this article, firstly we study pointwise slant, pointwise hemi-slant submanifolds whose ambient spaces are para-cosymplectic manifolds and we prove that there exist pointwise hemi-slant non-trivial warped product submanifolds whose ambient spaces are para- cosymplectic manifolds by giving some examples. We get several theorems and some characterizations. Later, we also obtain some inequalities.

2020 Mathematics Subject Classification. 53C43, 53C15, 53C40.
Keywords. Para-cosymplectic manifold, pointwise hemi-slant submanifold, warped product submanifold

1. Introduction

Slant submanifold was explained by B.Y. Chen in 1990 and he started the working in pseudo-Riemannian manifolds in 2012 [4]. Then, Almost contact manifold was indicated by I. Sato [10]. S. Zamkovoy researched almost para-contact metric manifolds [12] and An almost para-contact geometry is expressed as $(\mathcal{P}, \xi, \eta)$. Such that, $\mathcal{P}^2 = I - \eta \otimes \xi$ and $\eta(\xi) = 1$ on almost para-contact structure. Then, some researchers have been working Riemannian and semi-Riemannian manifolds in last years [1,2,5,8].

Bishop and O'Neill produced notion of warped product manifolds. Warped products are $\mathcal{N}_a$ and $\mathcal{N}_b$ be Riemannian manifolds with $\breve{g}_a$ and $\breve{g}_b$. Then, warped product manifold $\mathcal{N}_x = \mathcal{N}_a \times_k \mathcal{N}_b$ is a product manifold $\mathcal{N}_a \times \mathcal{N}_b$ equipped by $\breve{g}_x = \breve{g}_a + k^2 \breve{g}_b$ and k is a warping function of warped product manifold [3]. Warped products is generally used in differential geometry, theory of general relativity, theory of string, black holes. Warped product pseudo-slant submanifolds whose ambient spaces are Kaehler manifolds were worked by B. Sahin [9]. He proved that the warped product pseudo-slant $N_b^{\perp} \times_k N_a^{\theta}$ submanifold does not exist and he obtained a characterization and an inequality. Later S. Uddin and others worked warped product submanifolds whose ambient spaces are cosymplectic manifolds [11].

This article is organized as follows. In section 2, we introduce pointwise slant submanifolds of para-cosymlectic manifolds. Moreover, we give some definitions, examples and results. In section 3, we introduce proper pointwise hemi-slant submanifolds in para-cosymplectic manifolds and we give theorems, lemmas and examples. In section 4, we define pointwise hemi-slant non-trivial warped product submanifolds in para-cosymlectic manifolds. Also, we give some results and examples. In section 5, we obtain some inequalities.

2. Preliminaries

Let $\bar{\mathcal{N}}_x$ be a $(2\bar{n}+1)$-dimensional almost para-contact metric structure. If it is provided with structure $(\mathcal{P}, \xi, \eta, \breve{g}_1)$, that $\mathcal{P}$ is a tensor field of type $(1,1)$, η is a one form, ξ is a vector field and $\breve{g}_1$ is to expressed semi-Riemannian metric.

$$\mathcal{P}^2 = \mathcal{I} - \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \breve{g}_1(P\mathcal{X}_a, P\mathcal{Y}_b) = -\breve{g}_1(\mathcal{X}_a, \mathcal{Y}_b) + \eta(\mathcal{X}_a)\eta(\mathcal{Y}_b) \tag{1}$$

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These situations require that

$$\mathcal{P}\xi = 0, \quad \eta(\mathcal{P}\mathcal{X}_a) = 0, \quad \eta(\mathcal{X}_a) = \breve{g}_1(\mathcal{X}_a, \xi), \tag{2}$$

$$\breve{g}_1(\mathcal{P}\mathcal{X}_a, \mathcal{Y}_b) = -\breve{g}_1(\mathcal{X}_a, \mathcal{P}\mathcal{Y}_b). \tag{3}$$

An almost para-contact metric manifold is named para-cosymplectic manifold if the following relation is satisfied:

$$(\bar{\nabla}_{\mathcal{X}_a}\mathcal{P})\mathcal{Y}_b = 0, \quad \mathcal{P}\bar{\nabla}_{\mathcal{X}_a}\mathcal{Y}_b = \bar{\nabla}_{\mathcal{X}_a}\mathcal{P}\mathcal{Y}_b, \quad \bar{\nabla}_{\mathcal{X}_a}\xi = 0 \tag{4}$$

including any vector fields $\mathcal{X}_a, \mathcal{Y}_b$ on $\bar{\mathcal{N}_x}$.

Let currently, $\mathcal{N}_x$ is a submanifold of $(\mathcal{P}, \xi, \eta, \breve{g}_1)$. The Gauss and Weingarten equations are dedicated by

$$\bar{\nabla}_{\mathcal{X}_a}\mathcal{Y}_b = \nabla_{\mathcal{X}_a}\mathcal{Y}_b + h_1(\mathcal{X}_a, \mathcal{Y}_b), \tag{5}$$

$$\bar{\nabla}_{\mathcal{X}_a}V = -A_V\mathcal{X}_a + \nabla^{\perp}_{\mathcal{X}_a}V, \tag{6}$$

including $\mathcal{X}_a, \mathcal{Y}_b \in \Gamma(\mathcal{T}\mathcal{N}_x)$ and $V \in \Gamma(\mathcal{T}\mathcal{N}_b^{\perp})$, that h_1 is a second fundamental form of $\mathcal{N}_x$, A_V is the Weingarten endomorphism connected with V and $\nabla^{\perp}$ is the normal connection. A_V and h_1 are related by

$$\breve{g}_1(A_V\mathcal{X}_a, \mathcal{Y}_b) = \breve{g}_1(h_1(\mathcal{X}_a, \mathcal{Y}_b), V), \tag{7}$$

here $\breve{g}_1$ designates the semi-Riemannian metric on $\mathcal{N}_x$ with the one introduced on $\mathcal{N}_x$. For all tangent vector field $\mathcal{X}_a$, we denote

$$\mathcal{P}\mathcal{X}_a = R\mathcal{X}_a + S\mathcal{X}_a, \tag{8}$$

such that $R\mathcal{X}_a$ is the tangential component of $\mathcal{P}\mathcal{X}_a$ and $S\mathcal{X}_a$ is the normal one. For all normal vector field V,

$$\mathcal{P}V = rV + sV, \tag{9}$$

such that rV and sV are the tangential, normal components of $\mathcal{P}V$, respectively.
From the covariant derivative of the tensor fields R,S,r and s, we get

$$(\nabla_{\mathcal{X}_a}R)Y_b = \nabla_{\mathcal{X}_a}RY_b - R\nabla_{\mathcal{X}_a}Y_b, \tag{10}$$

$$(\nabla_{\mathcal{X}_a}S)Y_b = \nabla^{\perp}_{\mathcal{X}_a}S\mathcal{Y}_b - S\nabla_{\mathcal{X}_a}Y_b, \tag{11}$$

$$(\nabla_{\mathcal{X}_a}r)V = \nabla_{\mathcal{X}_a}rV - r\nabla^{\perp}_{\mathcal{X}_a}V, \tag{12}$$

$$(\nabla_{\mathcal{X}_a}s)V = \nabla^{\perp}_{\mathcal{X}_a}sV - s\nabla^{\perp}_{\mathcal{X}_a}V. \tag{13}$$

The mean curvature vector is indicated by

$$H = \frac{1}{n}traceh_1. \tag{14}$$

Definition 1. *We call that a submanifold $\mathcal{N}_x$ of almost para-contact metric structure $(\bar{\mathcal{N}_x}, \mathcal{P}, \xi, \eta, \breve{g}_1)$ is pointwise slant if for all time-like or space-like tangent vector field $\mathcal{X}_a$, the ratio $\breve{g}_1(R\mathcal{X}_a, R\mathcal{X}_a)/\breve{g}_1(\mathcal{P}\mathcal{X}_a, P\mathcal{X}_a)$ is a function. Moreover, a submanifold $\mathcal{N}_x$ of almost para-contact metric structure $\bar{\mathcal{N}_x}$ is named pointwise slant, if at each point $\mathtt{p} \in \mathcal{N}_x$, the Wirtinger angle $\theta(X)$ between $\mathcal{P}\mathcal{X}_a$ and $\mathcal{T}_p\mathcal{N}_x$ is dependent of the choice of the non-zero $\mathcal{X}_a \in \mathcal{T}_p\mathcal{N}_x$. In this instance, the Wirtinger angle causes a real-valued function $\theta : \mathcal{T}\mathcal{N}_x - 0 \to \mathcal{R}$ which is named the slant function or Wirtinger function of the pointwise slant submanifold.*

We express that a pointwise slant submanifold whose ambient spaces are almost para-contact manifold is named slant, if its Wirtinger function θ is globally constant. We state that all slant submanifold is a pointwise slant submanifold [9].

If $\mathcal{N}_x$ is a para-complex submanifold, in that case, $\mathcal{P}\mathcal{X}_a = R\mathcal{X}_a$ and the above ratio is equal to 1. Moreover if $\mathcal{N}_x$ is totaly real, then $R = 0$, so $\mathcal{P}\mathcal{X}_a = S\mathcal{X}_a$ and the above ratio equals 0. Hence, both para-complex submanifolds and totally real are the special situations of pointwise slant submanifolds.

Definition 2. *Let $\mathcal{N}_x$ be a proper pointwise slant submanifold of almost para-contact metric structure $(\bar{\mathcal{N}_x}, \mathcal{P}, \xi, \eta, \breve{g}_1)$. We call that it is of type-1 if for any spacelike(timelike) vector field $\mathcal{X}_a$, $R\mathcal{X}_a$ is timelike(spacelike), also $\frac{|R\mathcal{X}_a|}{|\mathcal{P}\mathcal{X}_a|} > 1$ and (For type-2) $\frac{|R\mathcal{X}_a|}{|\mathcal{P}\mathcal{X}_a|} < 1$.*

Theorem 1. *Let $\mathcal{N}_x$ be a pointwise slant submanifold in almost para-contact metric structure $(\bar{\mathcal{N}}_x, \mathcal{P}, \xi, \eta, \breve{g}_1)$. So that, for any spacelike (timelike) vector field $\mathcal{X}_a$, $R\mathcal{X}_a$ is timelike (spacelike) and $\mathcal{N}_x$ is the pointwise slant submanifold of type-1-2 necessary and sufficient condition*

$$\textbf{(a)} \quad \mu = R^2 = cosh^2\theta(I - \eta \otimes \xi), \quad \mu \in (1, +\infty) \quad (Type - 1), \tag{15}$$

$$\textbf{(b)} \quad \mu = R^2 = cos^2\theta(I - \eta \otimes \xi), \quad \mu \in (0, 1) \quad (Type - 2). \tag{16}$$

where θ denotes the slant function of N_x.

Proof. Firstly, if $\mathcal{N}_x$ is a pointwise slant submanifold of type-1 for any spacelike tangent vector field $\mathcal{X}_a$, $R\mathcal{X}_a$ is timelike. from the equation of (1), $\mathcal{P}\mathcal{X}_a$ also is. Furthermore, they supply $|R\mathcal{X}_a|/|\mathcal{P}\mathcal{X}_a| > 1$. So, there exists the slant function θ. Because of,

$$\cosh\theta = \frac{|R\mathcal{X}_a|}{|\mathcal{P}\mathcal{X}_a|} = \frac{\sqrt{-\breve{g}_1(R\mathcal{X}_a, R\mathcal{X}_a))}}{\sqrt{-\breve{g}_1(\mathcal{P}\mathcal{X}_a, \mathcal{P}\mathcal{X}_a)}} \tag{17}$$

and using (1) and (17) , we have

$$\breve{g}_1(R^2\mathcal{X}_a, \mathcal{X}_a) = \cosh^2\theta(I - \eta \otimes \xi)\breve{g}_1(\mathcal{X}_a, \mathcal{X}_a).$$

Thus, we get $R^2\mathcal{X}_a = cosh^2\theta(\mathcal{X}_a - \eta(\mathcal{X}_a)\xi)$. So, $\mu = R^2 = \cosh^2\theta(I - \eta \otimes \xi)$.
Also, for any time-like tangent vector field $\mathcal{Z}$, $R\mathcal{Z}$ and $\mathcal{P}\mathcal{Z}$ are spacelike. Therefore, in place of (17), we get

$$\cosh\theta = \frac{|R\mathcal{Z}|}{|\mathcal{P}\mathcal{Z}|} = \frac{\sqrt{\breve{g}_1(R\mathcal{Z}, R\mathcal{Z}))}}{\sqrt{\breve{g}_1(\mathcal{P}\mathcal{Z}, \mathcal{P}\mathcal{Z})}}$$

Because of $R^2\mathcal{X}_a = cosh^2\theta(\mathcal{X}_a - \eta(\mathcal{X}_a)\xi)$, for any spacelike and timelike $\mathcal{X}_a$ it further provides for lightlike vector fields and therefore we get $\mu = R^2 = \cosh^2\theta(I - \eta \otimes \xi)$. Thus, we get (a). In a similar way, we have (b) □

Corollary 1. *Let $\mathcal{N}_x$ be a pointwise slant submanifold of almost para-contact metric structure $(\bar{\mathcal{N}}_x, \mathcal{P}, \xi, \eta, \breve{g}_1)$ with the slant function θ. Later, for any non-null vector fields $\mathcal{X}_a, \mathcal{Y}_b \in \mathcal{T}\mathcal{N}_x - < \xi >$, If $\mathcal{N}_x$ is of type-1, type-2, we obtain:*

$$\begin{aligned} \breve{g}_1(R\mathcal{X}_a, R\mathcal{Y}_b) &= -\cosh^2\theta(\breve{g}_1(\mathcal{X}_a, \mathcal{Y}_b) - \eta(X)\eta(Y)), \\ \breve{g}_1(S\mathcal{X}_a, S\mathcal{Y}_b) &= \sinh^2\theta(\breve{g}_1(\mathcal{X}_a, \mathcal{Y}_b) - \eta(\mathcal{X}_a)\eta(\mathcal{Y}_b)), \end{aligned} \tag{18}$$

$$\begin{aligned} \breve{g}_1(R\mathcal{X}_a, R\mathcal{Y}_b) &= -\cos^2\theta(\breve{g}_1(\mathcal{X}_a, \mathcal{Y}_b) - \eta(X)\eta(Y)), \\ \breve{g}_1(S\mathcal{X}_a, S\mathcal{Y}_b) &= -\sin^2\theta(\breve{g}_1(\mathcal{X}_a, \mathcal{Y}_b) - \eta(\mathcal{X}_a)\eta(\mathcal{Y}_b)). \end{aligned} \tag{19}$$

Corollary 2. *Let $\mathcal{N}_x$ be a pointwise slant submanifold of an almost para-contact metric structure $(\bar{\mathcal{N}}_x, \mathcal{P}, \xi, \eta, \breve{g}_1)$. Later, let $\mathcal{N}_x$ be a pointwise slant submanifold of almost para-contact metric structure $\bar{\mathcal{N}}_x$. Therefore $\mathcal{N}_x$ is a pointwise slant submanifold of (type-1-2) necessary and sufficient condition,*
* $rS\mathcal{X}_a = -\sinh^2\theta(\mathcal{X}_a - \eta(\mathcal{X}_a)\xi)$ *and* $SRX = -sSX$ *(For type-1)*
* $rS\mathcal{X}_a = \sin^2\theta(\mathcal{X}_a - \eta(\mathcal{X}_a)\xi)$ *and* $SRX = -sSX$ *(For type-2)*
are satisfied for all timelike (spacelike) vector field $\mathcal{X}_a$.

3. Pointwise Hemi-Slant Submanifolds Whose Ambient Spaces are Para-Cosymplectic Manifolds

Definition 3. *A semi-Riemannian submanifold $\mathcal{N}_x$ of almost para-contact manifold $(\bar{\mathcal{N}}_x, \mathcal{P}, \xi, \eta, \breve{g}_1)$ is named to pointwise hemi-slant submanifold if there exist a two orthogonal distributions $D_t^{\perp}$, D_n^{α} with $\mathcal{N}_x$. Such that,*
1) $\mathcal{T}\mathcal{N}_x = \mathcal{D}_t^{\perp} \oplus \mathcal{D}_n^{\alpha} \oplus < \xi >$.
2) *The distribution $\mathcal{D}_t^{\perp}$ is an totally real distribution, $\mathcal{P}\mathcal{D}_t^{\perp} \subset \mathcal{T}^{\perp}\mathcal{N}_x$.*
3) *The distribution $\mathcal{D}_n^{\alpha}$ is a pointwise slant distribution.*
Then, we say θ as function.

Definition 4. *Let $\mathcal{N}_x$ be a pointwise hemi-slant submanifold of an almost para-contact metric structure $(\bar{\mathcal{N}}_x, \mathcal{P}, \xi, \eta, \breve{g}_1)$. Let $\mathcal{D}_n^\alpha$ be a pointwise slant distribution on $\mathcal{N}_x$. Then, we call that it is of (For type-1) if for any spacelike(timelike) vector field $\mathcal{X}_a$, $R\mathcal{X}_a$ is timelike(spacelike), also $\frac{|R\mathcal{X}_a|}{|\mathcal{P}\mathcal{X}_a|} > 1$ and (For type-2) $\frac{|R\mathcal{X}_a|}{|\mathcal{P}\mathcal{X}_a|} < 1$.*

Theorem 2. *Let $\mathcal{N}_x$ be a pointwise hemi-slant submanifold of almost para-contact metric structure $(\bar{\mathcal{N}}_x, \mathcal{P}, \xi, \eta, \breve{g}_1)$. $\mathcal{N}_x$ is the pointwise slant submanifold of type-1-2 necessary and sufficient condition*

$$\textbf{(a)} \quad \mu = R^2 = cosh^2\theta(I - \eta \otimes \xi), \quad \mu \in (1, +\infty), \quad (Type - 1). \tag{20}$$

$$\textbf{(b)} \quad \mu = R^2 = cos^2\theta(I - \eta \otimes \xi), \quad \mu \in (0, 1), \quad (Type - 2). \tag{21}$$

For any spacelike (timelike) vector field $\mathcal{X}_a$, $R\mathcal{X}_a$ is timelike (spacelike).

Proof. The proof is proved like the proof of Theorem 1. □

Corollary 3. *Let $\mathcal{N}_x$ be a pointwise hemi-slant submanifold of almost para-contact structure $(\bar{\mathcal{N}}_x, \mathcal{P}, \xi, \eta, \breve{g}_1)$. For non-null vector fields $\mathcal{X}_a, \mathcal{Y}_b \in \mathcal{T}\mathcal{N}_x - <\xi>$, if $\mathcal{D}_n^\alpha$ is of type-1 and type-2, then we obtain (respectively)*

$$\begin{aligned} \breve{g}_1(R\mathcal{X}_a, R\mathcal{Y}_b) &= -\cosh^2\theta(\breve{g}_1(\mathcal{X}_a, \mathcal{Y}_b) - \eta(X)\eta(Y)), \\ \breve{g}_1(S\mathcal{X}_a, S\mathcal{Y}_b) &= \sinh^2\theta(\breve{g}_1(\mathcal{X}_a, \mathcal{Y}_b) - \eta(\mathcal{X}_a)\eta(\mathcal{Y}_b)). \end{aligned} \tag{22}$$

and

$$\begin{aligned} \breve{g}_1(R\mathcal{X}_a, R\mathcal{Y}_b) &= -\cos^2\theta(\breve{g}_1(\mathcal{X}_a, \mathcal{Y}_b) - \eta(X)\eta(Y)), \\ \breve{g}_1(S\mathcal{X}_a, S\mathcal{Y}_b) &= -\sin^2\theta(\breve{g}_1(\mathcal{X}_a, \mathcal{Y}_b) - \eta(\mathcal{X}_a)\eta(\mathcal{Y}_b)). \end{aligned} \tag{23}$$

Lemma 1. *Let $\mathcal{N}_x$ be a pointwise hemi-slant type-1 and type-2 submanifold whose ambient space is para-cosymplectic manifold $\bar{\mathcal{N}}_x$. We get, $A_{\mathcal{P}\mathcal{Z}_a}\mathcal{W}_b = A_{\mathcal{P}\mathcal{W}_b}\mathcal{Z}_a$ is satisfied for any non-null vector fields $\mathcal{Z}_a, \mathcal{W}_b \in \mathcal{D}_t^\perp$.*

Proof. For type-*1*-*2* and for $\mathcal{Z}_a, \mathcal{W}_b \in \mathcal{D}_t^\perp$, $\mathcal{U}_a \in \Gamma(\mathcal{T}\mathcal{N}_x)$, we write $\mathcal{U}_a = \mathcal{P}_1\mathcal{U}_a + \mathcal{P}_2\mathcal{U}_a + \eta(\mathcal{U}_a)\xi$. Let be $\mathcal{T}N_x = \mathcal{D}_t^\perp \oplus \mathcal{D}_n^\alpha \oplus <\xi>$ and $\mathcal{T}^\perp N_x = \mathcal{P}\mathcal{D}_t^\perp \oplus S\mathcal{D}_n^\alpha \oplus \lambda$
Using (3),(4),(6) and (7), we obtain

$$\begin{aligned} \breve{g}_1(A_{\mathcal{P}\mathcal{W}_b}\mathcal{Z}_a, \mathcal{U}_a) &= \breve{g}_1(h_1(\mathcal{Z}_a, \mathcal{U}_a), \mathcal{P}\mathcal{W}_b) \\ &= -\breve{g}_1(-A_{\mathcal{P}\mathcal{Z}_a}\mathcal{U}_a + \nabla^\perp_{\mathcal{U}_a}\mathcal{P}\mathcal{Z}_a, \mathcal{W}_b) \\ &= \breve{g}_1(A_{\mathcal{P}\mathcal{Z}_a}\mathcal{W}_b, \mathcal{U}_a). \end{aligned}$$

□

Lemma 2. *Let $\mathcal{N}_x$ be a pointwise hemi-slant type-1 and type-2 submanifold whose ambient space is para-cosymplectic manifold $(\bar{\mathcal{N}}_x, \mathcal{P}, \xi, \eta, \breve{g}_1)$. In this case, the totally real distribution $\mathcal{D}_t^\perp$ is always integrable.*

Proof. For type-1, type-*2* and since $\bar{\mathcal{N}}_x$ is a para-cosymplectic manifold, using equations (1),(3),(4),(5),(6), (8) and from definition of projections for non-null vector fields $\mathcal{X}_a, \mathcal{Y}_b \in \mathcal{D}_t^\perp$ and $\mathcal{U}_a \in \mathcal{T}N_x$, we write

$$\begin{aligned} \breve{g}_1([\mathcal{X}_a, \mathcal{Y}_b], \mathcal{P}\mathcal{U}_a) &= -\breve{g}_1(\mathcal{P}[\mathcal{X}_a, \mathcal{Y}_b], \mathcal{U}_a) \\ &= -\breve{g}_1(\bar{\nabla}_{\mathcal{X}_a}\mathcal{P}\mathcal{Y}_b, \mathcal{U}_a) + \breve{g}_1(\bar{\nabla}_{\mathcal{Y}_b}\mathcal{P}\mathcal{X}_a, \mathcal{U}_a). \end{aligned}$$

The right hand side of the last equation should be zero. Thus, we derive

$$\begin{aligned} \breve{g}_1([\mathcal{X}_a, \mathcal{Y}_b], \mathcal{P}\mathcal{U}_a) &= 0, \\ \breve{g}_1([\mathcal{X}_a, \mathcal{Y}_b], R\mathcal{U}_a) + \breve{g}_1([\mathcal{X}_a, \mathcal{Y}_b], S\mathcal{U}_a) &= 0, \\ \breve{g}_1([\mathcal{X}_a, \mathcal{Y}_b], RP_2\mathcal{U}_a)) &= 0. \end{aligned}$$

From above equation, we have $[\mathcal{X}_a, \mathcal{Y}_b] = 0$. So, $\mathcal{D}_t^\perp$ is integrable. □

Lemma 3. *Let $\mathcal{N}_x$ be a pointwise hemi-slant type-1 and type-2 submanifold whose ambient space is para-cosymplectic manifold $(\bar{\mathcal{N}}_x, \mathcal{P}, \xi, \eta, \breve{g}_1)$. For $\mathcal{D}_n^\alpha \oplus <\xi>$ to be integrable, necessary and sufficient condition*
1) $\breve{g}_1(\nabla_{\mathcal{Y}_b}\mathcal{X}_a, \mathcal{Z}) = sech^2\theta(\breve{g}_1(h_1(\mathcal{X}_a, \mathcal{Z}), SR\mathcal{Y}_b) - \breve{g}_1(h_1(\mathcal{X}_a, R\mathcal{Y}_b), \mathcal{P}\mathcal{Z}))$ *(Tip-1)*

2) $\breve{g}_1(\nabla_{\mathcal{Y}_b}\mathcal{X}_a, Z) = sec^2\theta(\breve{g}_1(h_1(\mathcal{X}_a, Z), SR\mathcal{Y}_b) - \breve{g}_1(h_1(\mathcal{X}_a, R\mathcal{Y}_b), \mathcal{P}\mathcal{Z}))(Tip-2)$
for non-null vector fields $\mathcal{X}_a, \mathcal{Y}_b \in \mathcal{D}_n^\alpha \oplus <\xi>$ *and* $\mathcal{Z} \in \mathcal{D}_t^\perp$.

Proof. We demonstrate 1) and 2) in a similar method. We will give its proof when $\mathcal{D}_n^\alpha$ is type-1. $\bar{\mathcal{N}}_x$ is a para-cosymplectic manifold, using (1),(2),(3),(4),(5),(6),(7),(8) and Corollary 2, we write

$$\begin{aligned}
\breve{g}_1([\mathcal{X}_a, \mathcal{Y}_b], Z) &= \breve{g}_1(\bar{\nabla}_{\mathcal{X}_a}\mathcal{Y}_b - \bar{\nabla}_{\mathcal{Y}_b}\mathcal{X}_a, \mathcal{Z}) \\
&= -\breve{g}_1(\bar{\nabla}_{\mathcal{X}_a}\mathcal{P}\mathcal{Y}_b, \mathcal{P}\mathcal{Z}) - \eta(\bar{\nabla}_{\mathcal{X}_a}\mathcal{Y}_b)\eta(\mathcal{Z}) - \breve{g}_1(\bar{\nabla}_{\mathcal{Y}_b}\mathcal{X}_a, \mathcal{Z}) \\
&= -\breve{g}_1(\bar{\nabla}_{\mathcal{X}_a}R\mathcal{Y}_b, \mathcal{P}\mathcal{Z}) - \breve{g}_1(\bar{\nabla}_{\mathcal{X}_a}S\mathcal{Y}_b, \mathcal{P}\mathcal{Z}) - \breve{g}_1(\bar{\nabla}_{\mathcal{Y}_b}\mathcal{X}_a, \mathcal{Z}) \\
&= -\breve{g}_1(h_1(\mathcal{X}_a, R\mathcal{Y}_b), \mathcal{P}\mathcal{Z}) + \breve{g}_1(\bar{\nabla}_{\mathcal{X}_a}\mathcal{P}S\mathcal{Y}_b, \mathcal{Z}) - \breve{g}_1(\bar{\nabla}_{\mathcal{Y}_b}\mathcal{X}_a, \mathcal{Z}) \\
&= -\breve{g}_1(h_1(\mathcal{X}_a, R\mathcal{Y}_b), \mathcal{P}\mathcal{Z}) + \breve{g}_1(\bar{\nabla}_{\mathcal{X}_a}rS\mathcal{Y}_b, \mathcal{Z}) + \breve{g}_1(\bar{\nabla}_{\mathcal{X}_a}sS\mathcal{Y}_b, \mathcal{Z}) \\
&- \breve{g}_1(\bar{\nabla}_{\mathcal{Y}_b}\mathcal{X}_a, \mathcal{Z}) \\
&= -\breve{g}_1(h_1(\mathcal{X}_a, R\mathcal{Y}_b), \mathcal{P}\mathcal{Z}) - sinh^2\theta\breve{g}_1(\bar{\nabla}_{\mathcal{X}_a}\mathcal{Y}_b, \mathcal{Z}) \\
&+ \breve{g}_1(h_1(\mathcal{X}_a, \mathcal{Z}), SR\mathcal{Y}_b) - \breve{g}_1(\bar{\nabla}_{\mathcal{Y}_b}\mathcal{X}_a, \mathcal{Z})
\end{aligned}$$

making add subtract $sinh^2\theta\breve{g}_1(\bar{\nabla}_{\mathcal{Y}_b}\mathcal{X}_a, \mathcal{Z})$ above equation, we have

$$\begin{aligned}
cosh^2\theta\breve{g}_1(([\mathcal{X}_a, \mathcal{Y}_b], \mathcal{Z})) &= \breve{g}_1(h_1(\mathcal{X}_a, \mathcal{Z}), SR\mathcal{Y}_b) - \breve{g}_1(h_1(\mathcal{X}_a, R\mathcal{Y}_b), \mathcal{P}\mathcal{Z}) \\
&- cosh^2\theta\breve{g}_1(\bar{\nabla}_{\mathcal{Y}_b}\mathcal{X}_a, \mathcal{Z})
\end{aligned}$$

$$\begin{aligned}
\breve{g}_1(([\mathcal{X}_a, \mathcal{Y}_b], \mathcal{Z})) &= sech^2\theta(\breve{g}_1(h_1(\mathcal{X}_a, \mathcal{Z}), SR\mathcal{Y}_b) - \breve{g}_1(h_1(\mathcal{X}_a, R\mathcal{Y}_b), \mathcal{P}\mathcal{Z})) \\
&- \breve{g}_1(\bar{\nabla}_{\mathcal{Y}_b}\mathcal{X}_a, \mathcal{Z})
\end{aligned}$$

The right hand side of the last equation should be zero, proof is complete. □

Theorem 3. *Let* $\mathcal{N}_x$ *be a pointwise hemi-slant type1-2 submanifold whose ambient space is para-cosymplectic manifold* $(\bar{\mathcal{N}}_x, \mathcal{P}, \xi, \eta, \breve{g}_1)$. *In that case, totally real distribution* $\mathcal{D}_t^\perp$ *describes a totally geodesic foliation, necessary and sufficient condition*

$$\breve{g}_1(A_{\mathcal{P}\mathcal{W}_b}R\mathcal{X}a - A_{SR\mathcal{X}_a}\mathcal{W}_b, \mathcal{Z}_a) = 0 \tag{24}$$

is satisfied for non-null vector fields $\mathcal{Z}_a, \mathcal{W}_b \in \mathcal{D}_t^\perp$, $\mathcal{X}_a \in \mathcal{D}_n^\alpha \oplus <\xi>$.

Proof. For type-*1*, we obtain

$$\breve{g}_1(\nabla_{\mathcal{Z}_a}\mathcal{W}_b, \mathcal{X}_a) = \breve{g}_1(\bar{\nabla}_{\mathcal{Z}_a}\mathcal{W}_b, \mathcal{X}_a) - \breve{g}_1(h_1(\mathcal{Z}_a, \mathcal{W}_b), \mathcal{X}_a).$$

Using (1) and (5), we get

$$\begin{aligned}
\breve{g}_1(\nabla_{\mathcal{Z}_a}\mathcal{W}_b, \mathcal{X}_a) &= -\breve{g}_1(\mathcal{P}\bar{\nabla}_{\mathcal{Z}_a}\mathcal{W}_b, \mathcal{P}\mathcal{X}_a) + \eta(\bar{\nabla}_{\mathcal{Z}_a}\mathcal{W}_b)\eta(\mathcal{X}_a) \\
&= -\breve{g}_1(\mathcal{P}\bar{\nabla}_{\mathcal{Z}_a}\mathcal{W}_b, \mathcal{P}\mathcal{X}_a).
\end{aligned}$$

Using (6), (8), also from $\mathcal{P}\mathcal{W}$ and $S\mathcal{X}_a$ are orthogonally. We obtain

$$\begin{aligned}
\breve{g}_1(\nabla_{\mathcal{Z}_a}\mathcal{W}_b, \mathcal{X}_a) &= -\breve{g}_1(\bar{\nabla}_{\mathcal{Z}_a}\mathcal{P}\mathcal{W}_b, R\mathcal{X}_a) - \breve{g}_1(\bar{\nabla}_{\mathcal{Z}_a}\mathcal{P}\mathcal{W}_b, S\mathcal{X}_a) \\
&= \breve{g}_1(A_{\mathcal{P}\mathcal{W}_b}\mathcal{Z}_a, R\mathcal{X}_a) - \breve{g}_1(\nabla^\perp_{\mathcal{Z}_a}\mathcal{P}\mathcal{W}_b, R\mathcal{X}_a) + \breve{g}_1(\bar{\nabla}_{\mathcal{Z}_a}S\mathcal{X}_a, \mathcal{P}\mathcal{W}_b).
\end{aligned}$$

Using (1), (4) and (7). We obtain

$$\begin{aligned}
\breve{g}_1(\nabla_{\mathcal{Z}_a}\mathcal{W}_b, \mathcal{X}_a) &= \breve{g}_1(A_{\mathcal{P}\mathcal{W}_b}\mathcal{Z}_a, R\mathcal{X}_a) + \breve{g}_1(\bar{\nabla}_{\mathcal{Z}_a}S\mathcal{X}_a, \mathcal{P}\mathcal{W}_b) \\
&= \breve{g}_1(h_1(\mathcal{Z}_a, R\mathcal{X}_a), \mathcal{P}\mathcal{W}_b) - \breve{g}_1(\bar{\nabla}_{\mathcal{Z}_a}\mathcal{P}S\mathcal{X}_a, \mathcal{W}_b).
\end{aligned}$$

Using (9) and (Corollary 2 for type-1), we obtain

$$\begin{aligned}
\breve{g}_1(\nabla_{\mathcal{Z}_a}\mathcal{W}_b, \mathcal{X}_a) &= \breve{g}_1(h_1(\mathcal{Z}_a, R\mathcal{X}_a), \mathcal{P}\mathcal{W}_b) - \breve{g}_1(\bar{\nabla}_{\mathcal{Z}_a}rS\mathcal{X}_a, \mathcal{W}_b) \\
&- \breve{g}_1(\bar{\nabla}_{\mathcal{Z}_a}sS\mathcal{X}_a, \mathcal{W}_b) \\
&= \breve{g}_1(h_1(\mathcal{Z}_a, R\mathcal{X}_a), \mathcal{P}\mathcal{W}_b) + sinh^2\theta\breve{g}_1(\bar{\nabla}_{\mathcal{Z}_a}\mathcal{X}_a, \mathcal{W}_b) \\
&+ \breve{g}_1(\bar{\nabla}_{\mathcal{Z}_a}SR\mathcal{X}_a, \mathcal{W}_b).
\end{aligned}$$

Using (5), (6), (7) and because of $\mathcal{W}_b$ and $\mathcal{X}_a$ are orthogonally, we obtain

$$\breve{g}_1(\nabla_{\mathcal{Z}_a}\mathcal{W}_b, \mathcal{X}_a) = \breve{g}_1(h_1(\mathcal{Z}_a, R\mathcal{X}_a), \mathcal{P}\mathcal{W}_b) - sinh^2\theta\breve{g}_1(\nabla_{\mathcal{Z}_a}\mathcal{W}_b, \mathcal{X}_a)$$

$$
\begin{aligned}
& - \breve{g}_1(h_1(\mathcal{Z}_a,\mathcal{W}_b),SR\mathcal{X}_a) \\
cosh^2\theta\breve{g}_1(\nabla_{\mathcal{Z}_a}\mathcal{W}_b,\mathcal{X}_a) &= \breve{g}_1(h_1(R\mathcal{X}_a,\mathcal{Z}_a),\mathcal{P}\mathcal{W}_b) - \breve{g}_1(h_1(\mathcal{W}_b,\mathcal{Z}_a),SR\mathcal{X}_a) \\
cosh^2\theta\breve{g}_1(\nabla_{\mathcal{Z}_a}\mathcal{W}_b,\mathcal{X}_a) &= \breve{g}_1(A_{\mathcal{P}\mathcal{W}_b}R\mathcal{X}_a,\mathcal{Z}_a) - \breve{g}_1(A_{SR\mathcal{X}_a}\mathcal{W}_b,\mathcal{Z}_a).
\end{aligned}
$$

Thus, the proof is complete. In the same way, we get for type-2 □

Theorem 4. *Let $\mathcal{N}_x$ be a pointwise hemi-slant type1-2 submanifold whose ambient space is para-cosymplectic manifold $(\bar{\mathcal{N}}_x,\mathcal{P},\xi,\eta,\breve{g}_1)$. In that case, pointwise slant distribution $\mathcal{D}_n^\alpha\oplus<\xi>$ describes a totally geodesic foliation, necessary and sufficient condition*

$$\breve{g}_1(A_{SR\mathcal{X}_a}\mathcal{W}_b - A_{\mathcal{P}\mathcal{W}_b}R\mathcal{X}a,\mathcal{Y}_b) = 0 \tag{25}$$

is satisfied for non-null vector fields $\mathcal{W}_b\in\mathcal{D}_t^\perp$ and $\mathcal{X}_a,\mathcal{Y}_b\in\mathcal{D}_n^\alpha\oplus<\xi>$.

Proof. For type-1, using (1) and (5), we get

$$
\begin{aligned}
\breve{g}_1(\nabla_{\mathcal{Y}_b}\mathcal{X}_a,\mathcal{W}_b) &= \breve{g}_1(\bar{\nabla}_{\mathcal{Y}_b}\mathcal{X}_a,\mathcal{W}_b) \\
&= -\breve{g}_1(\bar{\nabla}_{\mathcal{Y}_b}\mathcal{P}\mathcal{X}_a,\mathcal{P}\mathcal{W}_b).
\end{aligned}
$$

Using (3), (5),(6), (8) and Corollary 2, we obtain

$$
\begin{aligned}
\breve{g}_1(\nabla_{\mathcal{Y}_b}\mathcal{X}_a,\mathcal{W}_b) &= -\breve{g}_1(\nabla_{\mathcal{Y}_b}R\mathcal{X}_a,\mathcal{P}\mathcal{W}_b) - \breve{g}_1(h_1(\mathcal{Y}_b,R\mathcal{X}_a),\mathcal{P}\mathcal{W}_b) \\
&+ \breve{g}_1(\mathcal{P}\bar{\nabla}_{\mathcal{Y}_b}S\mathcal{X}_a,\mathcal{W}_b) \\
&= -\breve{g}_1(h_1(\mathcal{Y}_b,R\mathcal{X}_a),\mathcal{P}\mathcal{W}_b) + \breve{g}_1(\bar{\nabla}_{\mathcal{Y}_b}rS\mathcal{X}_a,\mathcal{W}_b) \\
&+ \breve{g}_1(\bar{\nabla}_{\mathcal{Y}_b}sS\mathcal{X}_a,\mathcal{W}_b) \\
&- \breve{g}_1(\bar{\nabla}_{\mathcal{Y}_b}SR\mathcal{X}_a,\mathcal{W}_b). \\
(1+sinh^2\theta)\breve{g}_1(\nabla_{\mathcal{Y}_b}\mathcal{X}_a,\mathcal{W}_b) &= -\breve{g}_1(h_1(\mathcal{Y}_b,R\mathcal{X}_a),\mathcal{P}\mathcal{W}_b) - \breve{g}_1(\bar{\nabla}_{\mathcal{Y}_b}SR\mathcal{X}_a,\mathcal{W}_b). \\
(cosh^2\theta)\breve{g}_1(\nabla_{\mathcal{Y}_b}\mathcal{X}_a,\mathcal{W}_b) &= -\breve{g}_1(A_{\mathcal{P}\mathcal{W}_b}R\mathcal{X}_a,\mathcal{Y}_b) - \breve{g}_1(-A_{SR\mathcal{X}_a}\mathcal{Y}_b,\mathcal{W}_b) \\
&- \breve{g}_1(\nabla^\perp_{\mathcal{Y}_b}SR\mathcal{X}_a,\mathcal{W}_b) \\
&= \breve{g}_1(A_{SR\mathcal{X}_a}\mathcal{W}_b - A_{\mathcal{P}\mathcal{W}_b}R\mathcal{X}_a,\mathcal{Y}_b).
\end{aligned}
$$

So, the proof is completed. In the same way, we have for type-2 too. □

Corollary 4. *Let $\mathcal{N}_x$ be a pointwise hemi-slant submanifold type-1,2 submanifold whose ambient space is para-cosymplectic manifold $\bar{\mathcal{N}}_x$. Therefore $\mathcal{N}_x$ is a locally semi-Riemannian product structure, necessary and sufficient condition*

$$A_{\mathcal{P}\mathcal{Y}_b}R\mathcal{X}_a = A_{SR\mathcal{X}_a}\mathcal{Y}_b$$

is satisfied for $\mathcal{X}_a\in\mathcal{D}_n^\alpha\oplus<\xi>$ and $\mathcal{Y}_b\in\mathcal{D}_t^\perp$, that $\mathcal{N}_b^\perp$ is a anti-invariant submanifold and $\mathcal{N}_a^\theta$ is a pointwise slant submanifold of $\bar{\mathcal{N}}_x$.

4. Pointwise Hemi-Slant Non-Trivial Warped Product Submanifolds of Para-Cosymplectic Manifolds

Warped product manifolds were introduced by Bishop and O'Neill [3]. Projections of $\mathcal{N}_a\times\mathcal{N}_b$ are $\beta_1:\mathcal{N}_a\times\mathcal{N}_b\to\mathcal{N}_a$ and $\beta_2:\mathcal{N}_a\times\mathcal{N}_b\to\mathcal{N}_b$. Such that warped product manifold $\mathcal{N}_x=\mathcal{N}_a\times_k\mathcal{N}_b$ is the Riemannian manifold $\mathcal{N}_a\times\mathcal{N}_b=(\mathcal{N}_a\times\mathcal{N}_b,\breve{g})$ with the Riemannian structure. Therefore

$$\breve{g}(\mathcal{X}_a,\mathcal{Y}_b)=\breve{g}_1(\beta_{1*}\mathcal{X}_a,\beta_{1*}\mathcal{Y}_b)+(k\circ\beta_1)^2\breve{g}_1(\beta_{2*}\mathcal{X}_a,\beta_{2*}\mathcal{Y}_b)$$

for every vector fields $\mathcal{X}_a,\mathcal{Y}_b\in\Gamma(T\mathcal{N}_x)$, that * indicates the tangent map. The function k is named the warping function of the warped product manifold. Especially, if the warping function is non-constant, the manifold $\mathcal{N}_x$ is named to be non-trivial. $\mathcal{N}_a$ is totally geodesic and $\mathcal{N}_b$ is totally umbilical in $\mathcal{N}_x$.

Lemma 4. *Let $\mathcal{N}_x=\mathcal{N}_a\times_k\mathcal{N}_b$ be a warped product manifold with warping function k, therefore*
1) $\nabla_{\mathcal{X}_a}\mathcal{Y}_b\in\Gamma(\mathcal{T}\mathcal{N}_a)$ *is the lift of* $\nabla_{\mathcal{X}_a}\mathcal{Y}_b$ *on* $\mathcal{N}_a$*;*
2) $\nabla_{\mathcal{X}_a}\mathcal{Z}=\nabla_{\mathcal{Z}}\mathcal{X}_a=(\mathcal{X}_a lnk)\mathcal{Z}$ *;*
3) $\nabla_{\mathcal{Z}}\mathcal{W}=\bar{\nabla}^2_{\mathcal{Z}}\mathcal{W}-(\breve{g}(\mathcal{Z},\mathcal{W})\div k)$ *gradk ;*
are satisfied for non-null vector fields $\mathcal{X}_a,\mathcal{Y}_b\in\mathcal{T}\mathcal{N}_a$ and $\mathcal{Z},\mathcal{W}\in\mathcal{T}\mathcal{N}_b$, where gradk is the gradient of k

introduced as $\breve{g}_a(gradk, \mathcal{X}_a) = \mathcal{X}_a k$ *also* ∇ , $\bar{\nabla}^2$ *define the Levi-Civita connections on* $\mathcal{N}_x$ *and* $\mathcal{N}_b$ [3]. *As a result, we get*

$$||gradk||^2 = \sum_{v=1}^{s}(e_v(k))^2 \tag{26}$$

is satisfied for an orthonormal frame $(e_1, ..., e_s)$ *on* $\mathcal{N}_a$.

Theorem 5. *There does not exist a pointwise hemi-slant non-trivial warped product submanifolds* $\mathcal{N}_x = \mathcal{N}_b^{\perp} \times_k \mathcal{N}_a^{\theta}$ *whose ambient space is para-cosymplectic manifold* $\bar{\mathcal{N}_x}$ *and* $\xi \in \mathcal{T}\mathcal{N}_b^{\perp}$. *Such that* $\mathcal{N}_b^{\perp}$ *is totally real and* $\mathcal{N}_a^{\theta}$ *is pointwise slant submanifold of* $\bar{\mathcal{N}_x}$.

Proof. The non-existence of warped products pointwise semi-slant submanifolds whose ambient spaces are cosymplectic manifolds had proved by K.S. Park [7]. Similarly, we can demonstrate the non-existence of warped products pointwise hemi-slant submanifolds whose ambient spaces are para-cosymplectic manifolds. □

Let's consider para-cosymplectic structure on $\bar{R}_3^7$:

$$\mathcal{P}(\frac{\partial}{\partial x_i}) = \frac{\partial}{\partial y_i}, \;\; \mathcal{P}(\frac{\partial}{\partial y_i}) = \frac{\partial}{\partial x_i}, \;\;\; \mathcal{P}(\frac{\partial}{\partial z}) = 0, \;\; \xi = \frac{\partial}{\partial z}, \;\; \eta = dz.$$

Here, η is 1-form, ξ is vector field and $\breve{g}_1 = (+,-,+,-,+,-,+)$. $\breve{g}_1$ is pseudo-Riemannian metric. Also, $(\mathtt{x}_1, y_1, \mathtt{x}_2, y_2, \mathtt{x}_3, y_3, \mathtt{z})$ denotes the cartesian coordinates over $\bar{R}_3^7$. Then $(\bar{R}_3^7, \mathcal{P}, \xi, \eta, \breve{g}_1)$ is a para-cosymplectic manifold.
Let $\mathcal{N}_x$ be a semi-Riemannian submanifold of $\bar{\mathcal{R}_3^7}$ described by $\psi : \mathcal{N}_x \to \bar{\mathcal{R}_3^7}$.

Example 1. *For* $\mathtt{m} + \mathtt{n} > 0$ *and* $m + n \in \mathcal{R}$ *with*

$$\psi(m, \mathtt{n}, \mathtt{c}, t) = (cosh\mathtt{m}, \cosh n, \sinh n, \sinh m, c^3, \alpha, t),$$

$$\psi_{\mathtt{m}} = \sinh \mathtt{m}\frac{\partial}{\partial x_1} + \cosh \mathtt{m}\frac{\partial}{\partial y_2}, \quad \psi_{\mathtt{n}} = \sinh \mathtt{n}\frac{\partial}{\partial y_1} + \cosh \mathtt{n}\frac{\partial}{\partial x_2},$$

$$\psi_c = +3c^2\frac{\partial}{\partial x_3}, \quad \psi_t = \frac{\partial}{\partial z} = \xi.$$

Then, we get

$$\mathcal{P}\psi_{\mathtt{m}} = \sinh \mathtt{m}\frac{\partial}{\partial y_1} + \cosh \mathtt{m}\frac{\partial}{\partial x_2}, \quad \mathcal{P}\psi_{\mathtt{n}} = \sinh \mathtt{n}\frac{\partial}{\partial x_1} + \cosh \mathtt{n}\frac{\partial}{\partial y_2}, \quad \mathcal{P}\psi_c = 3c^2\frac{\partial}{\partial y_3}$$

describes a pointwise hemi-slant submanifold $\mathcal{N}_x^4$ *with type-*1 *whose ambient space is para-cosymplectic manifold* $(\bar{\mathcal{R}}_3^7, \mathcal{P}, \xi, \eta, \breve{g}_1)$ *with* $\mu = \mathcal{R}^2 = cosh^2(\mathtt{m}{+}\mathtt{n})(I - \eta \otimes \xi)$. *Actually* $D_n^{\alpha} = span\{\psi_{\mathtt{m}}, \psi_{\mathtt{n}}\}$ *is pointwise slant distribution with hemi-slant function and* $\mathcal{D}_t^{\perp} = span\{\psi_c\}$ *is anti-invariant distribution.*
It is easy to notice that D_n^{α}, $\mathcal{D}_t^{\perp}$ *are integrable. The induced metric tensor* $g_{\mathcal{N}_x}$ *on* $\mathcal{N}_x = \mathcal{N}_a^{\theta} \times_k \mathcal{N}_b^{\perp}$ *is given by* $g_{\mathcal{N}_x} = -{d_m}^2 + {d_n}^2 + (9c^4){d_c}^2 + {d_t}^2$.
Thus, $\mathcal{N}_x$ *is a pointwise hemi-slant non-trivial warped product type-1 submanifold whose ambient space is para-cosymplectic manifold* $\bar{\mathcal{R}_3^7}$ *with warping function* $k = 3c^2$.

Example 2. *For* $\mathtt{m} - \mathtt{n} \in (0, \frac{\pi}{2})$ *with*

$$\psi(\mathtt{m}, \mathtt{n}, c, t) = (\cos m, \cos n, \sin m, \sin n, \sin c, \pi, t),$$

$$\psi_{\mathtt{m}} = -\sin m\frac{\partial}{\partial x_1} + \cos m\frac{\partial}{\partial x_2}, \quad \psi_n = -\sin n\frac{\partial}{\partial y_1} + \cos \mathtt{n}\frac{\partial}{\partial y_2},$$

$$\psi_c = \cos c\frac{\partial}{\partial x_3}, \quad \psi_t = \frac{\partial}{\partial z} = \xi,$$

Then, we get

$$\mathcal{P}\psi_{\mathtt{m}} = -\sin \mathtt{m}\frac{\partial}{\partial y_1} + \cos \mathtt{m}\frac{\partial}{\partial y_2}, \quad \mathcal{P}\psi_{\mathtt{n}} = -\sin \mathtt{n}\frac{\partial}{\partial x_1} + \cos \mathtt{n}\frac{\partial}{\partial x_2}, \quad \mathcal{P}\psi_c = \cos c\frac{\partial}{\partial y_3}$$

describes a pointwise hemi-slant submanifold with type-2 in $(\bar{\mathcal{R}}_4^7, \mathcal{P}, \xi, \eta, \breve{g}_1)$, *with* $\mu = \mathcal{R}^2 = cos^2(\mathtt{m}{-}\mathtt{n})(I - \eta \otimes \xi)$. $D_n^{\alpha} = span\{\psi_{\mathtt{m}}, \psi_{\mathtt{n}}\}$ *is pointwise slant distribution with hemi-slant function and* $\mathcal{D}_t^{\perp} = span\{\psi_c\}$

is anti-invariant distribution and $\mathcal{P}\psi_c \perp T\mathcal{N}_x = span\{\psi_{\mathtt{m}}, \psi_{\mathtt{n}}, \psi_t\}$.
It is easy to notice that D_n^{α}, $\mathcal{D}_t^{\perp}$ *are integrable. The induced metric tensor* $g_{\mathcal{N}_x}$ *on* $\mathcal{N}_x = \mathcal{N}_a^{\theta} \times_k \mathcal{N}_b^{\perp}$ *is given by* $g_{\mathcal{N}_x} = {d_m}^2 - {d_n}^2 + (\cos^2 c){d_c}^2 + {d_t}^2$. *Thus,* $\mathcal{N}_x^4$ *is a pointwise hemi-slant non-trivial warped product type-2 submanifold whose ambient space is para-cosymplectic manifold* $\bar{\mathcal{R}}_3^7$ *with warping function* $k = cosc$.

Lemma 5. *Let* $\mathcal{N}_x = \mathcal{N}_a^{\theta} \times_k \mathcal{N}_b^{\perp}$ *be a pointwise hemi-slant non-trivial warped product type1-2 submanifold whose ambient space is para-cosymplectic manifold* $\bar{\mathcal{N}_x}$. *Such that* $\xi \in T\mathcal{N}_b^{\perp}$, *then*
1) $\xi(lnk) = 0$.
2) *For any* $\mathcal{X}_a, \mathcal{Y}_b \in T\mathcal{N}_a^{\theta}$ *and* $\mathcal{Z} \in T\mathcal{N}_b^{\perp}$,

$$\breve{g}_1(h_1(\mathcal{X}_a, \mathcal{Y}_b), \mathcal{P}\mathcal{Z}) = \breve{g}_1(h_1(\mathcal{X}_a, \mathcal{Z}), S\mathcal{Y}_b). \tag{27}$$

Proof. **1)** For any $\mathcal{X}_a \in T\mathcal{N}_a^{\theta}$ and $\xi \in T\mathcal{N}_b^{\perp}$, we obtain $\bar{\nabla}_{\mathcal{X}_a}\xi = 0$. Also Using (5),(6) and from Lemma 4 - (2), we obtain $\xi(lnk)\mathcal{X}_a = 0$ which means that $\xi(lnk) = 0$, for any non-zero vector field $\mathcal{X}_a \in T\mathcal{N}_a^{\theta}$ that proves 1).
2)Using (5),(3), (8),(6), (7), we derive

$$\begin{aligned}
\breve{g}_1(h_1(\mathcal{X}_a, \mathcal{Y}_b), \mathcal{P}\mathcal{Z}) &= \breve{g}_1(\bar{\nabla}_{\mathcal{X}_a}\mathcal{Y}_b - \nabla_{\mathcal{X}_a}\mathcal{Y}_b, \mathcal{P}\mathcal{Z}) \\
&= -\breve{g}_1(\bar{\nabla}_{\mathcal{X}_a}\mathcal{P}\mathcal{Y}_b, \mathcal{Z}) \\
&= -\breve{g}_1(\bar{\nabla}_{\mathcal{X}_a}S\mathcal{Y}_b, \mathcal{Z}) \\
&= -\breve{g}_1(-A_{S\mathcal{Y}_b}\mathcal{X}_a + \nabla^{\perp}_{\mathcal{X}_a}S\mathcal{Y}_b, \mathcal{Z}) \\
&= \breve{g}_1(h_1(\mathcal{X}_a, \mathcal{Z}), S\mathcal{Y}_b)
\end{aligned}$$

If we relocate $\mathcal{X}_a$ with $R\mathcal{X}_a$ and $\mathcal{Y}_b$ with $R\mathcal{Y}_b$ in (27) , then we get belove results

$$\breve{g}_1(h_1(R\mathcal{X}_a, \mathcal{Y}_b), \mathcal{P}\mathcal{Z}) = \breve{g}_1(h_1(R\mathcal{X}_a, \mathcal{Z}), S\mathcal{Y}_b), \tag{28}$$

$$\breve{g}_1(h_1(\mathcal{X}_a, R\mathcal{Y}_b), \mathcal{P}\mathcal{Z}) = \breve{g}_1(h_1(\mathcal{X}_a, \mathcal{Z}), SR\mathcal{Y}_b), \tag{29}$$

$$\breve{g}_1(h_1(R\mathcal{X}_a, R\mathcal{Y}_b), \mathcal{P}\mathcal{Z}) = \breve{g}_1(h_1(R\mathcal{X}_a, \mathcal{Z}), SR\mathcal{Y}_b). \tag{30}$$

□

Lemma 6. *Let* $\mathcal{N}_x = \mathcal{N}_a^{\theta} \times_k \mathcal{N}_b^{\perp}$ *be a pointwise hemi-slant non-trivial warped product type1-2 submanifold whose ambient space is para-cosymplectic manifold* $\bar{\mathcal{N}_x}$. *Such that* $\xi \in T\mathcal{N}_a^{\theta}$, *then*
1) $\breve{g}_1(h_1(\mathcal{Z}_a, \mathcal{W}_b), S\mathcal{X}_a) = \breve{g}_1(h_1(\mathcal{X}_a, \mathcal{Z}_a), \mathcal{P}, \mathcal{W}_b) + (R\mathcal{X}_a lnk)\breve{g}_1(\mathcal{Z}_a, \mathcal{W}_b)$,
2) a) *For type-1;*
$\breve{g}_1(h_1(\mathcal{Z}_a, \mathcal{W}_b), SR\mathcal{X}_a) = \breve{g}_1(h_1(R\mathcal{X}_a, \mathcal{Z}_a), \mathcal{P}\mathcal{W}_b) + (\mathcal{X}_a lnk)cosh^2\theta\breve{g}_1(\mathcal{Z}_a, \mathcal{W}_b)$,
b) *For type-2 ;*
$\breve{g}_1(h_1(\mathcal{Z}_a, \mathcal{W}_b), SR\mathcal{X}_a) = \breve{g}_1(h_1(R\mathcal{X}_a, \mathcal{Z}_a), \mathcal{P}\mathcal{W}_b) - (\mathcal{X}_a lnk)cos^2\theta\breve{g}_1(\mathcal{Z}_a, \mathcal{W}_b)$,
for any $\mathcal{Z}_a, \mathcal{W}_b \in T\mathcal{N}_b^{\perp}$ *and* $\mathcal{X}_a \in T\mathcal{N}_a^{\theta}$.

Proof. Using (8), (5) and Lemma 4-(2), we derive

$$\begin{aligned}
\breve{g}_1(h_1(\mathcal{Z}_a, \mathcal{W}_b), S\mathcal{X}_a) &= \breve{g}_1(h_1(\mathcal{X}_a, \mathcal{Z}_a), (\mathcal{P}\mathcal{X}_a - R\mathcal{X}_a)) \\
&= \breve{g}_1(\bar{\nabla}_{\mathcal{Z}_a}\mathcal{W}_b, \mathcal{P}\mathcal{X}_a) - \breve{g}_1(\nabla_{\mathcal{Z}_a}\mathcal{W}_b, \mathcal{P}\mathcal{X}_a) \\
&= \breve{g}_1(\bar{\nabla}_{\mathcal{Z}_a}\mathcal{P}\mathcal{W}_b, \mathcal{X}_a) - \breve{g}_1(\nabla_{\mathcal{Z}_a}\mathcal{W}, R\mathcal{X}_a).
\end{aligned}$$

By using (4) and from $\mathcal{W}_b$ and $R\mathcal{X}_a$ are orthogonality. Also later using (6), (7) and from Lemma 4-(2), we obtain

$$\begin{aligned}
\breve{g}_1(h_1(\mathcal{Z}_a, \mathcal{W}_b), S\mathcal{X}_a) &= -\breve{g}_1(\bar{\nabla}_{\mathcal{Z}_a}\mathcal{P}\mathcal{W}_b, \mathcal{X}_a) + \breve{g}_1(\mathcal{W}_b, \nabla_{\mathcal{Z}_a}R\mathcal{X}_a) \\
&= -\breve{g}_1(-A_{\mathcal{P}\mathcal{W}_b}\mathcal{Z}_a, \mathcal{X}_a) + \breve{g}_1(\nabla^{\perp}_{\mathcal{Z}_a}\mathcal{P}\mathcal{W}_b, \mathcal{X}_a) \\
&+ (R\mathcal{X}_a lnk)\breve{g}_1(\mathcal{Z}_a, \mathcal{W}_b) \\
&= \breve{g}_1((h_1(\mathcal{X}_a, \mathcal{Z}_a), \mathcal{P}\mathcal{W}_b) + (R\mathcal{X}_a lnk)\breve{g}_1(\mathcal{Z}_a, \mathcal{W}_b).
\end{aligned}$$

Therefore, Proof 1 is complete. Now, we will demonstrate proof 2(a) for type-1.
If we replace $\mathcal{X}_a$ and $R\mathcal{X}_a$ in the last equation and using (1), we derive

$$\begin{aligned}
\breve{g}_1(h_1(\mathcal{Z}_a, \mathcal{W}_b), S\mathcal{X}_a) &= \breve{g}_1((h_1(\mathcal{X}_a, \mathcal{Z}_a), \mathcal{P}\mathcal{W}_b) + (R\mathcal{X}_a lnk)\breve{g}_1(\mathcal{Z}_a, \mathcal{W}_b) \\
&= \breve{g}_1((h_1(R\mathcal{X}_a, \mathcal{Z}_a), \mathcal{P}\mathcal{W}_b) + R^2(\mathcal{X}_a lnk)\breve{g}_1(\mathcal{Z}_a, \mathcal{W}_b).
\end{aligned}$$

For type-1, (a);

$$\breve{g}_1(h_1(\mathcal{Z}_a,\mathcal{W}_b),SR\mathcal{X}_a) = \breve{g}_1((h_1(R\mathcal{X}_a,\mathcal{Z}_a),\mathcal{P}\mathcal{W}_b)+cosh^2\theta(\mathcal{X}_a lnk)\breve{g}_1(\mathcal{Z}_a,\mathcal{W}_b).$$

For type-2, (b);

$$\breve{g}_1(h_1(\mathcal{Z}_a,\mathcal{W}_b),SR\mathcal{X}_a) = \breve{g}_1((h_1(R\mathcal{X}_a,\mathcal{Z}_a),\mathcal{P}\mathcal{W}_b)+cos^2\theta(\mathcal{X}_a lnk)\breve{g}_1(\mathcal{Z}_a,\mathcal{W}_b).$$

□

Theorem 6. *Let $\mathcal{N}_x=\mathcal{N}_a^{\theta}\times_k\mathcal{N}_b^{\perp}$ be a pointwise hemi-slant non-trivial warped product type1-2 submanifold whose ambient space is para-cosymplectic manifold $\bar{\mathcal{N}}_x$. Then $\mathcal{N}_x$ is locally a mixed geodesic warped product pointwise submanifold $\mathcal{N}_a^{\theta}\times_k\mathcal{N}_b^{\perp}$ necessary and sufficient condition*

$$A_{\mathcal{P}\mathcal{Z}_a}\mathcal{X}_a=0,\quad A_{S\mathcal{X}_a}\mathcal{Z}_a=R\mathcal{X}_a(\varphi)\mathcal{Z}_a, A_{SR\mathcal{X}_a}\mathcal{Z}_a=cosh^2\theta\mathcal{X}_a(\varphi)\mathcal{Z}_a\quad (Type1), \tag{31}$$

$$A_{\mathcal{P}\mathcal{Z}_a}\mathcal{X}_a=0,\quad A_{S\mathcal{X}_a}\mathcal{Z}_a=R\mathcal{X}_a(\varphi)\mathcal{Z}_a A_{SR\mathcal{X}_a}\mathcal{Z}_a=cos^2\theta\mathcal{X}_a(\varphi)\mathcal{Z}_a\quad (Tip-2) \tag{32}$$

are satisfied for any $\mathcal{X}_a\in D_n^{\alpha}\oplus<\xi>$ and $\mathcal{Z}_a\in D_t^{\perp}$, that φ is a function on $\mathcal{N}_x$ and $\mathcal{W}_b(\varphi)=0$ is satisfied for any $\mathcal{W}_b\in D_t^{\perp}$.

Proof. Using advantage of Lemmas 4 and 5, we demonstrate that $\mathcal{N}_x$ is a mixed geodesic warped product pointwise submanifold. Let $\mathcal{N}_x$ be a hemi-slant submanifold with the slant distribution $\mathcal{D}_n^{\alpha}\oplus<\xi>$ and the anti-invariant distribution $\mathcal{D}_t^{\perp}$ with the cases shown in (31) and (32). Also using these conditions and Theorem 4, the distribution $\mathcal{D}_n^{\alpha}\oplus<\xi>$ describes a totally geodesic foliation and utilizing Lemma 2, $\mathcal{D}_t^{\perp}$ is integrable, imagine $h^{\perp}$ be the second fundamental form of the leaf $\mathcal{N}_b^{\perp}$ of $\mathcal{D}_t^{\perp}$ in $\mathcal{N}_x$, Also for any $\mathcal{X}_a\in\mathcal{D}_n^{\alpha}\oplus<\xi>$ and $\mathcal{W}_b,\mathcal{Z}_a\in\mathcal{D}_t^{\perp}$.
Utilizing (5), (1), (3), (4) and (8), we have

$$\begin{aligned}\breve{g}_1(h^{\perp}(\mathcal{Z}_a,\mathcal{W}_b),\mathcal{X}_a) &= \breve{g}_1((\bar{\nabla}_{\mathcal{Z}_a}\mathcal{W}_b,\mathcal{X}_a)\\ &= -\breve{g}_1((\bar{\nabla}_{\mathcal{Z}_a}\mathcal{P}\mathcal{W}_b,\mathcal{P}\mathcal{X}_a)+\eta(\bar{\nabla}_{\mathcal{Z}_a}\mathcal{P}\mathcal{W}_b)\eta(\mathcal{P}\mathcal{X}_a)\\ &= -\breve{g}_1((\bar{\nabla}_{\mathcal{Z}_a}\mathcal{P}\mathcal{W}_b,R\mathcal{X}_a)-\breve{g}_1((\bar{\nabla}_{\mathcal{Z}_a}\mathcal{P}\mathcal{W}_b,S\mathcal{X}_a).\end{aligned}$$

Utilizing (6) and therefore $\mathcal{P}\mathcal{W}_b$ and $S\mathcal{X}_a$ are orthogonality, we obtain

$$\breve{g}_1(h^{\perp}(\mathcal{Z}_a,\mathcal{W}_b),\mathcal{X}_a) = -\breve{g}_1((A_{\mathcal{P}\mathcal{W}_b}\mathcal{Z}_a,R\mathcal{X}_a)+\breve{g}_1(\mathcal{P}\mathcal{W}_b,\bar{\nabla}_{\mathcal{Z}_a}S\mathcal{X}_a).$$

Utilizing (1),(3),(4) and (9), we get

$$\begin{aligned}\breve{g}_1(h^{\perp}(\mathcal{Z}_a,\mathcal{W}_b),\mathcal{X}_a) &= -\breve{g}_1((A_{\mathcal{P}\mathcal{W}_b}R\mathcal{X}_a,\mathcal{Z}_a)-\breve{g}_1(\mathcal{W}_b,\bar{\nabla}_{\mathcal{Z}_a}rS\mathcal{X}_a)\\ &- \breve{g}_1(\mathcal{W}_b,\bar{\nabla}_{\mathcal{Z}_a}sS\mathcal{X}_a).\end{aligned}$$

Utilizing first condition of (31) and Corollary 2, we derive

$$\breve{g}_1(h^{\perp}(\mathcal{Z}_a,\mathcal{W}_b),\mathcal{X}_a) = \breve{g}_1(\mathcal{W}_b,\bar{\nabla}_{\mathcal{Z}_a}sinh^2\theta\mathcal{X}_a)+\breve{g}_1(\mathcal{W}_b,\bar{\nabla}_{\mathcal{Z}_a}SR\mathcal{X}_a).$$

Therefore, orthogonality of $\mathcal{W}_b$ with $\mathcal{X}_a$, using (5),(6) and (31), we derive

$$\begin{aligned}\breve{g}_1(h^{\perp}(\mathcal{Z}_a,\mathcal{W}_b),\mathcal{X}_a) &= -sinh^2\theta\breve{g}_1(\bar{\nabla}_{\mathcal{Z}_a}\mathcal{W}_b,\mathcal{X}_a)\\ &+ \breve{g}_1(\mathcal{W}_b,(-A_{SR\mathcal{X}_a}\mathcal{Z}_a+\nabla^{\perp}_{\mathcal{Z}_a}SR\mathcal{X}_a))\\ &= -sinh^2\theta\breve{g}_1(h^{\perp}(\mathcal{Z}_a,\mathcal{W}_b),\mathcal{X}_a)-\breve{g}_1(A_{SR\mathcal{X}_a}\mathcal{Z}_a,\mathcal{W}_b),\\ -cosh^2\theta\breve{g}_1(h^{\perp}(\mathcal{Z}_a,\mathcal{W}_b),\mathcal{X}_a) &= \breve{g}_1(A_{SR\mathcal{X}_a}\mathcal{Z}_a,\mathcal{W}_b)\\ &= cosh^2\theta\mathcal{X}_a(\varphi)\breve{g}_1(\mathcal{Z}_a,\mathcal{W}_b).\end{aligned}$$

From the describtion of gradient, we obtain

$$\breve{g}_1(h^{\perp}(\mathcal{Z}_a,\mathcal{W}_b),\mathcal{X}_a) = -\breve{g}_1(\mathcal{Z}_a,\mathcal{W}_b)\breve{g}_1(grad\varphi,\mathcal{X}_a).$$

So that, $h^{\perp}(\mathcal{Z}_a,\mathcal{W}_b)=-\breve{g}_1(\mathcal{Z}_a,\mathcal{W}_b)\breve{g}_1 grad\varphi$ for vectors $\mathcal{Z}_a,\mathcal{W}_b\in D_t^{\perp}$. $H=-grad\varphi$ and $\mathcal{N}_b^{\perp}$ is totally umbilical in $\mathcal{N}_x$
Now, we explain $grad\varphi$ is parallel suitable to the normal connection $\mathcal{D}_t^{\perp}$ of $\mathcal{N}_b^{\perp}$ in $\mathcal{N}_x$. For $\mathcal{X}_a\in\mathcal{D}_n^{\alpha}\oplus<\xi>$ and $\mathcal{W}_b\in D_t^{\perp}$, we derive

$$\begin{aligned}\breve{g}_1(D_{\mathcal{W}_b}grad\varphi,\mathcal{X}_a) &= \breve{g}_1(\nabla_{\mathcal{W}_b}grad\varphi,\mathcal{X}_a)\\ &= \mathcal{W}_b\breve{g}_1(grad\varphi,\mathcal{X}_a)-\breve{g}_1(grad\varphi,\nabla_{\mathcal{W}_b}\mathcal{X}_a)\\ &= \mathcal{W}_b(\mathcal{X}_a(\varphi))-\breve{g}_1(grad\varphi,[\mathcal{W}_b,\mathcal{X}_a])-\breve{g}_1(grad\varphi,\nabla_{\mathcal{X}_a}\mathcal{W}_b)\end{aligned}$$

$$= \quad \mathcal{X}_a(\mathcal{W}_b\varphi) + \breve{g}_1(\nabla_{\mathcal{X}_a} grad\varphi, \mathcal{W}_b) = 0.$$

So, $\mathcal{W}_b\varphi = 0$ is satisfied for every $\mathcal{W}_b \in \mathcal{D}_t^\perp$ also $\nabla_{\mathcal{X}_a} grad\varphi \in \mathcal{D}_n^\alpha \oplus < \xi >$ therefore $\mathcal{D}_n^\alpha \oplus < \xi >$ is totally geodesic. We understand that mean curvature of $\mathcal{N}_b^\perp$ is parallel. So that, the leaves of $\mathcal{D}_t^\perp$ are totally umbilical with parallel mean curvature $H = -grad\varphi$. So, $\mathcal{N}_b^\perp$ is called the extrinsic sphere in $\mathcal{N}_x$. By considering Hiepko ([6]), we attain that $\mathcal{N}_x$ is a warped product pointwise submanifold and the proof is completed for type-*1*.
In a similarly way, for type-2 is also proved. □

5. An Optimal Inequality

Let $\mathcal{N}_x = \mathcal{N}_a^\theta \times_k \mathcal{N}_b^\perp$ be a s-dimensional pointwise hemi-slant non-trivial warped product submanifold whose ambient space is $(2m+1)$-dimensional para-cosymplectic manifold $\bar{\mathcal{N}}_x$. Such that, $\mathcal{N}_b^\perp$ is dimension d_1 and $\mathcal{N}_a^\theta$ is dimension $d_2 = 2p + 1$ so ξ is tangent to $\mathcal{N}_a^\theta$. We take tangent spaces of $\mathcal{N}_b^\perp$ and $\mathcal{N}_a^\theta$ by $\mathcal{D}_t^\perp$ and $\mathcal{D}_n^\alpha \oplus < \xi >$. We create orthonormal frames according to type-1 and type-2. Firstly for type-1, the orthonormal frames of $\mathcal{D}_t^\perp$ and $\mathcal{D}_n^\alpha \oplus < \xi >$, respectively;
$\{\mathtt{E}_1, \mathtt{E}_2, .., \mathtt{E}_{d_1}\}$ and $\{\mathtt{E}_{d_1+1} = \mathtt{E}_1^*, ..., \mathtt{E}_{d_1+\mathtt{p}} = \mathtt{E}_\mathtt{p}^*, \mathtt{E}_{d_1+\mathtt{p}+1} = \mathtt{E}_{\mathtt{p}+1}^* = sech\theta R\mathtt{E}_1^*, ...$
$, \mathtt{E}_{d_1+2\mathtt{p}} = \mathtt{E}_{2\mathtt{p}}^* = sech\theta R\mathtt{E}_\mathtt{p}^*, \mathtt{E}_{d_1+2\mathtt{p}+1} = \mathtt{E}_{2\mathtt{p}+1}^* = \xi\}$ that θ is nonconstant.
At the moment, we will give orthonormal frames of the normal subbundles of $\mathcal{P}D_t^\perp$, SD_n^α and λ. This frames respectively are
$\{\mathtt{E}_{s+1} = \bar{\mathtt{E}}_1 = \mathcal{P}\mathtt{E}_1, \mathtt{E}_{s+2} = \bar{\mathtt{E}}_2 = \mathcal{P}\mathtt{E}_2, ..., \mathtt{E}_{s+d_1} = \bar{\mathtt{E}}_{d_1} = \mathcal{P}\mathtt{E}_{d_1}\}$,
$\{\mathtt{E}_{s+d_1+1} = \bar{\mathtt{E}}_{d_1+1} = csch\theta S\mathtt{E}_1^*, \mathtt{E}_{s+d_1+2} = \bar{\mathtt{E}}_{d_1+2} = csch\theta S\mathtt{E}_2^*, ..., \mathtt{E}_{s+d_1+p} = \bar{\mathtt{E}}_{d_1+p} = csch\theta S\mathtt{E}_p^*, \mathtt{E}_{s+d_1+p+1} = \bar{\mathtt{E}}_{d_1+p+1} = csch\theta sech\theta SR\mathtt{E}_1^*, ..., \mathtt{E}_{s+d_1+p+p} = \bar{\mathtt{E}}_{d_1+p+p} = csch\theta sech\theta SR\mathtt{E}_p^*\}$ and
$\{\mathtt{E}_{2s} = \bar{\mathtt{E}}_s, ..., \mathtt{E}_{2m+1} = \bar{\mathtt{E}}_{2(m-s+1)}\}$. where θ is the slant function.
Lets assume that
* on $\mathcal{D}_t^\perp$: orthonormal basis $\{\mathtt{E}_v\}_{v=1,...,d_1}$, where $d_1 = dim(\mathcal{D}_t^\perp)$; also, supposed that $\breve{g}_1(\mathtt{E}_v, \mathtt{E}_v) = 1$.
* on $\mathcal{D}_n^\alpha$: orthonormal basis $\{\mathtt{E}_w^*\}_{w=1,...,2p+1}$, where $2p + 1 = dim(\mathcal{D}_n^\alpha)$ also $\breve{g}_1(\mathtt{E}_w^*, \mathtt{E}_w^*) = \mp 1$.
* on $\mathcal{P}\mathcal{D}_t^\perp$: orthonormal basis $\{\mathtt{E}_v\}_{v=1,...,d_1}$, where $d_1 = dim\mathcal{P}(\mathcal{D}_t^\perp)$ also $\breve{g}_1(\mathcal{P}\mathtt{E}_v, \mathcal{P}\mathtt{E}_v) = -1$.
* on $\mathcal{S}\mathcal{D}_n^\alpha$: orthonormal basis $\{\mathtt{E}_w^*\}_{w=1,...,2p+1}$, where $2p + 1 = dim\mathcal{S}(\mathcal{D}_n^\alpha)$ also $\breve{g}_1(\mathtt{E}_w^*, \mathtt{E}_w^*) = \mp 1$.

Theorem 7. *Let $\mathcal{N}_x = \mathcal{N}_a^\theta \times_k \mathcal{N}_b^\perp$ be a s-dimensional mixed geodesic warped product pointwise hemi-slant of type-1 submanifold whose ambient space is $(2m+1)$- dimensional para-cosymplectic manifold $\bar{\mathcal{N}}_x$. So that $\mathcal{N}_a^\theta$ is a proper pointwise slant submanifold of dimension $2p+1$ and $\mathcal{N}_b^\perp$ is a totally real submanifold of dimension d_1 of $\bar{\mathcal{N}}_x$. So that $\mathcal{N}_b^\perp$ is spacelike. Then*
1) *The squared norm of the second fundamental form of $\mathcal{N}_x$ supplies*

$$||h_1||^2 \le d_1 coth^2\theta ||gradlnk||^2, \tag{33}$$

where $grad(lnk)$ is the gradient of lnk.

2) *If the equality sign of (33) holds the same way, then $\mathcal{N}_a^\theta$ is totally geodesic and $\mathcal{N}_b^\perp$ is totally umbilical in $\bar{\mathcal{N}}_x$.*

Proof. From description $||h_1||^2 = ||h_1(\mathcal{D}_m, \mathcal{D}_m)||^2 + 2||h_1(\mathcal{D}_m, \mathcal{D}_t^\perp)||^2 + ||h_1(\mathcal{D}_t^\perp, \mathcal{D}_t^\perp)||^2$, that $\mathcal{D}_m = \mathcal{D}_n^\alpha \oplus < \xi >$. Because of $\mathcal{N}_x$ is mixed geodesic, the middle term of the right-hand side should be zero. In that case, we obtain

$$||h_1||^2 = \sum_{r=s+1}^{2m+1} \sum_{v,w=1}^{2p+1} \breve{g}_1(h_1(\mathtt{E}_v^*, \mathtt{E}_w^*), \mathtt{E}_r)^2 + \sum_{r=s+1}^{2m+1} \sum_{l,b=1}^{d_1} \breve{g}_1(h_1(\mathtt{E}_l^*, \mathtt{E}_b^*), \mathtt{E}_r)^2$$

This equation can be seperated for the $\mathcal{P}D_t^\perp$, SD_n^α and λ components as follows

$$||h_1||^2 = \sum_{r=1}^{d_1} \sum_{v,w=1}^{2p+1} \breve{g}_1(h_1(\mathtt{E}_v^*, \mathtt{E}_w^*), \bar{\mathtt{E}}_r)^2$$
$$+ \sum_{r=d_1+1}^{2p+d_1} \sum_{v,w=1}^{2p+1} \breve{g}_1(h_1(\mathtt{E}_v^*, \mathtt{E}_w^*), \bar{\mathtt{E}}_r)^2$$

$$+\sum_{r=s}^{2(m-s+1)}\sum_{v,w=1}^{2p+1}\breve{g}_1(h_1(\mathtt{E}^*_v,\mathtt{E}^*_w),\bar{\mathtt{E}}_r)^2$$
$$+\sum_{r=1}^{d_1}\sum_{l,b=1}^{d_1}\breve{g}_1(h_1(\mathtt{E}_l,\mathtt{E}_b),\bar{\mathtt{E}}_r)^2$$
$$+\sum_{r=d_1+1}^{2p+1}\sum_{l,b=1}^{d_1}\breve{g}_1(h_1(\mathtt{E}_l,\mathtt{E}_b),\bar{\mathtt{E}}_r)^2$$
$$+\sum_{r=s}^{2(m-s+1)}\sum_{l,b=1}^{d_1}\breve{g}_1(h_1(\mathtt{E}_l,\mathtt{E}_b),\bar{\mathtt{E}}_r)^2 \tag{34}$$

Utilizing(27) and (30), the first term of right-hand side in the last equation vanishes same way and we should leave all the terms except the fifth term in the last equation, then we have
$||h_1||^2 \leq \sum_{r=d_1+1}^{2p+d_1}\sum_{l,b=1}^{d_1}\breve{g}_1(h_1(\mathtt{E}_l,\mathtt{E}_b),\bar{\mathtt{E}}_r)^2$
Using the frame of $S\mathcal{D}_n^\alpha$, we get,

$$\begin{aligned}||h_1||^2 &\leq \sum_{w=1}^{p}\sum_{l,b=1}^{d_1}\breve{g}_1(h_1(\mathtt{E}_l,\mathtt{E}_b),csch\theta S\mathtt{E}^*_w)^2\\ &+ \sum_{w=1}^{p}\sum_{l,b=1}^{d_1}\breve{g}_1(h_1(\mathtt{E}_l,\mathtt{E}_b),csch\theta sech\theta SR\mathtt{E}^*_w)^2\end{aligned}$$

Utilizing Lemma 5 and Lemma 6-1, we obtain

$$\begin{aligned}||h_1||^2 &\leq csch^2\theta\sum_{w=1}^{p}\sum_{l,b=1}^{d_1}(R\mathtt{E}^*_w lnk)^2\breve{g}_1(\mathtt{E}_l,\mathtt{E}_b)^2\\ &+ coth^2\theta\sum_{w=1}^{p}\sum_{l,b=1}^{d_1}(\mathtt{E}^*_w lnk)^2\breve{g}_1(\mathtt{E}_l,\mathtt{E}_b)^2\\ &= (csch^2\theta\sum_{w=1}^{p}(R\mathtt{E}^*_w lnk)^2 + coth^2\theta\sum_{w=1}^{p}(\mathtt{E}^*_w lnk)^2\\ &= d_1(csch^2\theta\sum_{w=1}^{p}(R\mathtt{E}^*_w lnk)^2 + coth^2\theta\sum_{w=1}^{p}(\mathtt{E}^*_w lnk)^2)\\ &= d_1(csch^2\theta\sum_{w=1}^{p}\breve{g}_1(\mathtt{E}^*_w,Rlnk)^2 + coth^2\theta\sum_{w=1}^{p}(\mathtt{E}^*_w lnk)^2)\end{aligned}$$

By using (26), the above equation will be simlified as

$$\begin{aligned}||h_1||^2 &\leq d_1[csch^2\theta(||Rgradlnk||^2 - \sum_{w=1}^{p}\breve{g}_1(\mathtt{E}^*_{p+w},Rgradlnk)^2)\\ &+ coth^2\theta\sum_{w=1}^{p}\sum_{l,b=1}^{d_1}(\mathtt{E}^*_w lnk)^2\breve{g}_1(\mathtt{E}^*_w lnk)^2]\\ &, \quad (for \quad Rgradlnk \in D_m \quad and \quad R\xi = 0 \quad)\\ &= d_1[csch^2\theta(||Rgradlnk||^2 - cosh^2\theta\sum_{w=1}^{p}\breve{g}_1(\mathtt{E}^*_w,gradlnk)^2)\\ &+ coth^2\theta\sum_{w=1}^{p}(\mathtt{E}^*_w lnk)^2]\end{aligned}$$

$$= \quad d_1[coth^2\theta||Rgradlnk||^2 - coth^2\theta\sum_{w=1}^{p}(\mathtt{E}^*_w lnk)^2 + coth^2\theta\sum_{w=1}^{p}(\mathtt{E}^*_w lnk)^2]$$

Last equation specifies in (33) and from the leaving terms in (34), we have the following connections from the second and the third leaving terms of (34).

$\breve{g}_1(h_1(\mathcal{D}_m,\mathcal{D}_m)S\mathcal{D}^\alpha_n) = 0$, $\breve{g}_1(h_1(\mathcal{D}_m,\mathcal{D}_m),\lambda) = 0$ that intend

$$h_1(\mathcal{D}_m,\mathcal{D}_m)\perp S\mathcal{D}^\alpha_n, \quad h_1(\mathcal{D}_m,\mathcal{D}_m)\perp\lambda \Rightarrow h_1(\mathcal{D}_m,\mathcal{D}_m)\in \mathcal{P}\mathcal{D}^\perp_t \tag{35}$$

Because of a mixed geodesic warped product pointwise submanifold and from Theorem 5 , we derive $\breve{g}_1(h_1(\mathcal{D}_m,\mathcal{D}_m),\mathcal{P}\mathcal{D}^\perp_t) = 0$. Such that

$$h_1(\mathcal{D}_m,\mathcal{D}_m)\perp\mathcal{P}\mathcal{D}^\perp_t \tag{36}$$

When we take into account (35) and (36), understand that $h_1(\mathcal{D}_m,\mathcal{D}_m) = 0$ using this connection with the fact that $\mathcal{N}^\theta_a$ is totaly geodesic in $\mathcal{N}_x$ ([3]).

From the leaving fourth and the sixth terms of (34) on the right side, we determine that $\breve{g}_1(h_1(\mathcal{D}^\perp_t,\mathcal{D}^\perp_t),\mathcal{P}\mathcal{D}^\perp_t) = 0$, $\breve{g}_1(h_1(\mathcal{D}^\perp_t,\mathcal{D}^\perp_t),\lambda) = 0$, we get

$$h_1(\mathcal{D}^\perp_t,\mathcal{D}^\perp_t)\perp\mathcal{P}\mathcal{D}^\perp_t, \quad h_1(\mathcal{D}^\perp_t,\mathcal{D}^\perp_t)\perp\lambda \Rightarrow h_1(\mathcal{D}^\perp_t,\mathcal{D}^\perp_t)\in S\mathcal{D}^\perp_t \tag{37}$$

For a mixed geodesic, from Lemma 5(1), we derive

$$\breve{g}_1(h_1(\mathcal{Z}_a,\mathcal{W}_b),S\mathcal{X}_a) = (R\mathcal{X}_a lnk)\breve{g}_1(\mathcal{Z}_a,\mathcal{W}_b) \tag{38}$$

for any $\mathcal{X}_a \in T\mathcal{N}^\theta_a$ and $\mathcal{Z}_a,\mathcal{W}_b \in T\mathcal{N}^\perp_b$.

Therefore, by the connections (37), (38) and substantially $\mathcal{N}^\perp_b$ is totally umbilical in $\mathcal{N}_x$ [3], we obtain that $\mathcal{N}^\perp_b$ is totally umbilical in $\bar{\mathcal{N}}_x$. □

Remark 1. *If $\mathcal{N}^\perp_b$ manifold of Theorem 7 is totally umbilical and timelike, equation (33) should be modified by*

$$||h_1||^2 \geq d_1 coth^2\theta||gradlnk||^2, \tag{39}$$

where $grad(lnk)$ is the gradient of lnk.

Theorem 8. *Let $\mathcal{N}_x = \mathcal{N}^\theta_a \times_k \mathcal{N}^\perp_b$ be a s-dimensional mixed geodesic warped product pointwise hemi-slant submanifold whose ambient space is $(2\mathtt{m}+1)$- dimensional para-cosymplectic manifold $\bar{\mathcal{N}}_x$. So that $\mathcal{N}^\theta_a$ is a pointwise slant submanifold and $\mathcal{N}^\perp_b$ is a totally real submanifold of dimension d_1 of $\bar{\mathcal{N}}_x$. Hence, $\mathcal{N}^\perp_b$ is spacelike and timelike. Then, (for type-2)*

$$||h_1||^2 \leq d_1 cot^2\theta||gradlnk||^2 (respectively, ||h_1||^2 \geq d_1 cot^2\theta||gradlnk||^2), \tag{40}$$

where $grad(lnk)$ is the gradient of lnk.

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Forced vibrations of a thin viscoelastic shell immersed in fluid under the effect of damping

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Abstract. The plane strain problem for low-frequency forced vibrations of a fluid-loaded thin viscoelastic shell is considered. A small structural damping is incorporated using the concept of a complex Young's modulus. The two-term asymptotic expansion is derived assuming that the structural damping is of the same order as the small thickness of the shell. It is demonstrated that the effect of the structural damping is remarkably greater than that of the radiation damping and the latter can be neglected in the vast majority of the problems.

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Keywords. Asymptotics, thin shell, complex Young's modulus, viscoelastic

1. Introduction

Structural damping plays a significant role in the dynamic analysis of mechanical systems since it governs the mechanism of energy dissipation which is crucial for various technical applications in civil, mechanical, naval and automotive engineering, e.g. see [1], [2], [3], [4] and references therein. There is a great number of publications on the subject. In particular, the vibrations of viscoelastic fluid-loaded shells were treated in numerical contributions, including [5], [6], [7], [8], [9], [10] to mention a few. At the same time, the asymptotic methods widely spread in the thin shell theory have been mainly applied within the purely elastic framework, e.g., see [11], [12], [13], [14], [15], [16].

The recent asymptotic analysis in [17] and [18] show that the radiation damping of low-frequency resonant vibrations of purely elastic cylindrical shells is remarkably small. It is natural to question, in this case, whether the formulations not taking into consideration structural damping may provide adequate evaluation of dynamic behaviour. This observation motivates to extend the framework of [17] to viscoelastic shells.

In this paper, the viscoelastic properties are incorporated using the simplest model of the structural damping based on the concept of a complex Young's modulus, see [19]. The imaginary part of the latter stands for energy dissipation. It is assumed to be of the same order that the relative thickness of the shell.

Instead of scattering problem tackled in [17], below we deal with a radiation problem. A time-harmonic load is assumed to be specified along the inner surface of the shell, while the outer face is subject to fluid loading. The developed asymptotic procedure is oriented to a coupled fluid-structure interaction problem similar to the above mentioned publications [17], [18], and also [20], studying a flat, fluid-loaded elastic layer. It was noted that for a long time, the asymptotic results for thin-walled bodies with traction free faces were readily adapted for modeling of fluid-structure interaction ignoring, in a sense, the effect of coupling, e.g., see [11].

We expand displacement, stresses and fluid pressure in the Fourier series across the polar angle prior to the asymptotic integration across the shell thickness. A two-term asymptotic solution is derived. As might be expected, a small term corresponding to the structural damping does not appear at leading order. However, it is shown that it is significantly greater than the contribution of the damping caused by radiation. The most important result of the presented analysis is that the latter may usually be neglected.

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2. Statement of the Problem

Consider a thin cylindrical shell with thickness $2h$ and a mid-surface radius R immersed in a compressible fluid for which $\eta = h/R \ll 1$ is a small geometric parameter, see Fig. 1. We specify curvilinear coordinates α_2 and α_3 for which $0 \leq \alpha_2 < 2\pi R$ and $-h \leq \alpha_3 \leq h$. The 2D plane strain equations

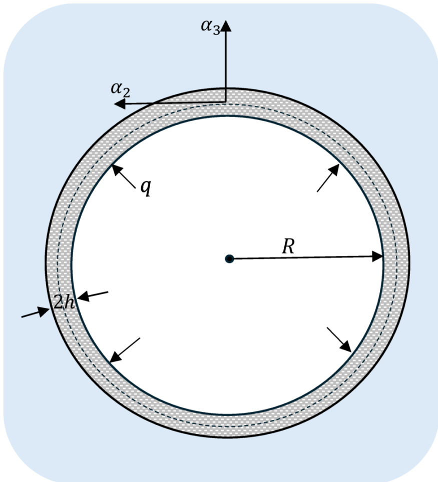

Figure 1. Schematic diagram of a thin cylindrical shell immersed in a fluid.

governing the time-harmonic vibrations of a shell, omitting the factor $\exp(-i\omega t)$, with ω representing the angular frequency and t denoting time, are given by, see [12],

$$\frac{R}{R+\alpha_3}\frac{\partial \sigma_{22}}{\partial \alpha_2} + \frac{\partial \sigma_{32}}{\partial \alpha_3} + \frac{2}{R+\alpha_3}\sigma_{32} + \rho\omega^2 v_2 = 0, \tag{1}$$

$$\frac{R}{R+\alpha_3}\frac{\partial \sigma_{32}}{\partial \alpha_2} + \frac{\partial \sigma_{33}}{\partial \alpha_3} - \frac{1}{R+\alpha_3}\sigma_{22} + \frac{1}{R+\alpha_3}\sigma_{33} + \rho\omega^2 v_3 = 0, \tag{2}$$

where σ_{ij} ($\sigma_{ij} = \sigma_{ji}$) and v_j, $i,j = 2,3$, are the stresses and displacements, respectively and ρ is the mass density of the shell. The corresponding stress-displacement relations are also presented as

$$\sigma_{22} = \frac{E}{1-\nu^2}\left(\frac{R}{R+\alpha_3}\frac{\partial v_2}{\partial \alpha_2} + \frac{1}{R+\alpha_3}v_3\right) + \frac{\nu}{1-\nu}\sigma_{33}, \tag{3}$$

$$E\frac{\partial v_3}{\partial \alpha_3} = (1-\nu^2)\sigma_{33} - \nu(1+\nu)\sigma_{22}, \tag{4}$$

$$\sigma_{32} = \frac{E}{2(1+\nu)}\left(\frac{R}{R+\alpha_3}\frac{\partial v_3}{\partial \alpha_2} + \frac{\partial v_2}{\partial \alpha_3} - \frac{1}{R+\alpha_3}v_2\right). \tag{5}$$

The mechanical parameters of the considered problem are the Young's modulus E and Poisson's ratio ν. To incorporate the effect of viscosity in the simplest manner, we define the Young's modulus in a complex form, e.g., see [19]

$$E = E_0(1 + i\alpha), \tag{6}$$

where E_0 and α are real constants. The fluid pressure is governed by the 2D Helmholtz equation

$$\Delta p + \frac{\omega^2}{c_f^2}p = 0 \tag{7}$$

where p is fluid pressure and c_f is the wave speed in the fluid. 2D Laplace operator Δ is given by

$$\Delta = \frac{R^2}{(R+\alpha_3)^2}\frac{\partial^2}{\partial \alpha_2^2} + \frac{1}{R+\alpha_3}\frac{\partial}{\partial \alpha_3} + \frac{\partial^2}{\partial \alpha_3^2}. \tag{8}$$

The boundary conditions along the shell faces are given by

$$\sigma_{32} = 0, \qquad \sigma_{33} = q \qquad \text{at} \qquad \alpha_3 = -h, \tag{9}$$

$$\sigma_{32} = 0, \qquad \sigma_{33} = -p, \qquad \text{and} \qquad v_3 = \frac{1}{\rho_f \omega^2} \frac{\partial p}{\partial \alpha_3} \quad \text{at} \quad \alpha_3 = h, \tag{10}$$

where q is the force applied at the inner surface of the shell and ρ_f is the fluid density.

In the dimensionless coordinates $\theta = \alpha_2/R$ and $\zeta = \alpha_3/h$ ($0 \leq \theta < 2\pi$ and $-1 \leq \zeta \leq 1$ inside the shell or $\zeta > 1$ outside the shell) the displacement and stress components of the shell, the acoustic pressure and the external force can be presented as

$$v_2(\theta, \zeta) = u_2(\zeta) \sin(n\theta), \qquad v_3(\theta, \zeta) = u_3(\zeta) \cos(n\theta), \tag{11}$$

$$\sigma_{22}(\theta, \zeta) = s_{22}(\zeta) \cos(n\theta), \quad \sigma_{32}(\theta, \zeta) = s_{32}(\zeta) \sin(n\theta), \quad \sigma_{33}(\theta, \zeta) = s_{33}(\zeta) \cos(n\theta), \tag{12}$$

$$p(\theta, \zeta) = P(\zeta) \cos(n\theta), \qquad q(\theta, \zeta) = Q(\zeta) \cos(n\theta). \tag{13}$$

3. Scaling

Let us now define the dimensionless equations in the previous section similar to those in [17] setting

$$u_2 = R u_2^*, \quad u_3 = R u_3^*, \tag{14}$$

$$s_{22} = E_0 \eta s_{22}^*, \quad s_{32} = E_0 \eta^2 s_{32}^*, \quad s_{33} = E_0 \eta^2 s_{33}^* \tag{15}$$

$$P = E_0 \eta^2 P^*, \quad Q = E_0 \eta^3 Q^*. \tag{16}$$

where the starred quantities are assumed to be of order unity. In addition, we assume that the viscosity coefficient (6) can be taken as

$$\alpha = \eta \alpha_0^*. \tag{17}$$

We also specify the dimensionless frequency by

$$\Omega = \eta^{-3/2} \omega R \sqrt{\frac{\rho}{E_0}}, \qquad \Omega \sim 1. \tag{18}$$

The fluid pressure, subject to the radiation condition, e.g., see [21], is found from equation (7) and is given by

$$p = p_0 H_n^{(2)} \left(\frac{\omega R (1 + \eta \zeta)}{c_f} \right). \tag{19}$$

where $H_n^{(2)}$ is the Hankel function of the second kind, see [22], and p_0 is an unknown constant.

Next, combining boundary conditions $(10)_2$ and $(10)_3$, and substituting (19) there, accounting $(11)_2$, $(12)_3$ and $(13)_1$, we have

$$u_3 + s_{33} \frac{\mathcal{H}}{c_f \rho_f} = 0 \tag{20}$$

where

$$\mathcal{H} = \frac{\left(H_n^{(2)}(z) \right)'}{H_n^{(2)}(z)} \qquad \text{at} \qquad z = \Omega \eta^{3/2} \frac{1}{c_f} \sqrt{\frac{E_0}{\rho}} (1 + \eta). \tag{21}$$

Inserting the dimensionless quantities (14), (15) and (17) into the equations of motion (1)–(2) and the relations (3)–(5), we obtain

$$\frac{\partial s_{32}^*}{\partial \zeta} - \frac{n}{1 + \eta \zeta} s_{22}^* + \frac{2\eta}{1 + \eta \zeta} s_{32}^* + \eta^2 \Omega^2 u_2^* = 0, \tag{22}$$

$$\frac{\partial s_{33}^*}{\partial \zeta} + \frac{n\eta}{1 + \eta \zeta} s_{32}^* - \frac{1}{1 + \eta \zeta} s_{22}^* + \frac{\eta}{1 + \eta \zeta} s_{33}^* + \eta^2 \Omega^2 u_3^* = 0 \tag{23}$$

and

$$\eta s_{22}^* = \frac{1 + i\eta\alpha_0}{1 - \nu^2} \frac{1}{1 + \eta \zeta} (n u_2^* + u_3^*) + \frac{\nu}{1 - \nu} \eta^2 s_{33}^*, \tag{24}$$

$$(1 + i\eta\alpha_0) \frac{\partial u_3^*}{\partial \zeta} = (1 - \nu^2) \eta^3 s_{33}^* - \nu (1 + \nu) \eta^2 s_{22}^*, \tag{25}$$

$$\eta^3 s_{32}^* = \frac{1+i\eta\alpha_0}{2(1+\nu)}\left(\frac{\partial u_2^*}{\partial\zeta} - \frac{\eta}{1+\eta\zeta}(u_2^* + n u_3^*)\right). \tag{26}$$

In addition, we set

$$\mathcal{H} = \eta^{-3/2}\mathcal{H}^*, \tag{27}$$

where, according to [22],

$$\mathcal{H}^* = -\frac{nc_f}{\Omega(1+\eta)}\sqrt{\frac{\rho}{E_0}}\left(1 + \frac{n-2}{4(n-1)}\Omega^2\eta^3(1+\eta)^2\frac{E_0}{c_f^2\rho} + \dots \right.$$
$$\left. \dots + i\frac{\pi}{2^{2n-1}c_f^{2n} n\left((n-1)!\right)^2}\Omega^{2n}\eta^{3n}(1+\eta)^{2n}\left(\frac{E_0}{\rho}\right)^n + \dots\right). \tag{28}$$

The contact conditions (9) and (10) become

$$s_{32}^* = 0, \quad \zeta = \pm 1 \qquad \text{and} \qquad s_{33}^* = \eta Q^*, \quad \zeta = -1, \tag{29}$$

$$\eta\Omega u_3^* + \frac{1}{c_f\rho_f}\sqrt{\frac{\rho}{E_0}}\mathcal{H}^* s_{33}^* = 0, \quad \zeta = 1. \tag{30}$$

In what follows, we expand all the started quantities in the asymptotic series as

$$f^* = f^{(0)} + \eta f^{(1)} + \eta^2 f^{(2)} + \dots. \tag{31}$$

4. Asymptotic Solution

Let us start by integrating (25) and (26) with respect to the thickness coordinate ζ to get at leading order

$$u_3^{(0)} = U_3^{(0)} \qquad \text{and} \qquad u_2^{(0)} = U_2^{(0)}, \tag{32}$$

where the unknown constants $U_3^{(0)}$ and $U_2^{(0)}$ are, due to (24), related by

$$U_2^{(0)} = -\frac{1}{n}U_3^{(0)}. \tag{33}$$

Formula (33) corresponds to the circumferential inextensibility of the mid-surface of a cylindrical shell, see [23].

Then, integrating (22) and (23) with respect to the thickness variable, we obtain

$$s_{3m}^{(0)} = -n^{3-m}\int_\zeta^1 s_{22}^{(0)}\,ds, \quad m = 2, 3. \tag{34}$$

Applying the conditions (29), we deduce

$$\int_{-1}^1 s_{22}^{(0)}\,ds = 0. \tag{35}$$

At next order, first, we integrate (25) in the thickness variable having

$$u_3^{(1)} = U_3^{(1)} \tag{36}$$

where $U_3^{(1)}$ is an unknown constant. In the same manner, integrating (26) in ζ and employing the relation (33), we get

$$u_2^{(1)} = -\frac{1-n^2}{n}\zeta U_3^{(0)} + U_2^{(1)} \tag{37}$$

and

$$U_2^{(1)} = -\frac{1}{n}U_3^{(1)}. \tag{38}$$

Now, integrating (24) and taking into account the latter relation, we arrive at

$$s_{22}^{(0)} = -\frac{(1-n^2)}{1-\nu^2}\zeta U_3^{(0)}. \tag{39}$$

As a result, formulae (34) may be rewritten as

$$s_{3m}^{(0)} = n^{3-m}\frac{(1-n^2)}{2(1-\nu^2)}(1-\zeta^2)U_3^{(0)}. \tag{40}$$

Next, we integrate (23) across the thickness and adopt formulae (39) and (40) together with the boundary condition $(29)_2$ to get

$$\int_{-1}^{1} s_{22}^{(1)} ds = \left(\frac{\rho_f}{\rho n}\Omega^2 + \frac{2n^2(1-n^2)}{3(1-\nu^2)}\right) U_3^{(0)} - Q^*. \tag{41}$$

We also integrate (22) in ζ and utilize formula (39). Then, we subject the resulting equation to condition $(29)_1$. As a result, we have

$$s_{32}^{(1)} = -n \int_{\zeta}^{1} s_{22}^{(1)} ds + \frac{2n(1-n^2)}{3(1-\nu^2)}. \tag{42}$$

Taking $\zeta = -1$ in the last equation, taking into consideration $(29)_1$ and (41), we derive

$$\left(\Omega^2 - \frac{2\rho(1-n^2)^2 n}{3\rho_f(1-\nu^2)}\right) U_3^{(0)} = \frac{\rho n}{\rho_f} Q^*. \tag{43}$$

It is clear that the effect of the material damping, i.e., the parameter α_0, on the stress components and the vertical displacement does not appear in this equation. To incorporate the effect of this parameter, we need to consider the next order approximation.

Following the same process carried out in the previous sections and omitting intermediate calculations, we get from (25)

$$u_3^{(2)} = \frac{\nu}{2(1-\nu)}(1-n^2)\zeta^2 U_3^{(0)} + U_3^{(2)}. \tag{44}$$

Similarly, it follows from (26) that

$$u_2^{(2)} = -\frac{1-n^2}{n}\zeta U_3^{(1)} + U_2^{(2)}. \tag{45}$$

Then, integration of equation (24), taking into consideration (32), (37), (38), (40), (42), (44) and (45), results in

$$U_2^{(2)} + \frac{1}{n}U_3^{(2)} = -\frac{\nu(1-n^2)}{2(1-\nu)n}U_3^{(0)}. \tag{46}$$

Inserting the last formula back into equation (23), we obtain

$$s_{22}^{(1)} = -\frac{1-n^2}{1-\nu^2}\zeta U_3^{(1)} + \frac{1-n^2}{1-\nu^2}\left(\zeta^2 - i\zeta\right) U_3^{(0)}. \tag{47}$$

Now, we revisit equation (23), using the dimensionless impenetrability condition (30) taken at first order, and also equations (39), (47) and (48). The result is

$$\begin{aligned} s_{33}^{(1)} =& \frac{(1-n^2)(1-\zeta^2)}{2(1-\nu^2)}U_3^{(1)} + \left(\frac{5\zeta^3 - 3\zeta - 2 + 4n^4(1-\zeta^3) - n^4(2-3\zeta+\zeta^3)}{6(1-\nu^2)}\right. \\ &\left. + \frac{\rho_f}{\rho n}\Omega^2 + i\alpha_0\frac{(1-\zeta^2)(1-n^2)}{2(1-\nu^2)}\right) U_3^{(0)}. \end{aligned} \tag{48}$$

This formula allows us to rewrite formula (42) as

$$s_{32}^{(1)} = \frac{n(1-n^2)}{2(1-\nu^2)}(1-\zeta^2)U_3^{(1)} - \frac{(1-n^2)(1-\zeta^2)}{1-\nu^2}\left(n\zeta - \frac{i\alpha_0}{2}\right) U_3^{(0)}. \tag{49}$$

Using (30) and integrating (22) and (23) along the thickness at second order, we derive, respectively,

$$\int_{-1}^{1} s_{22}^{(2)} d\zeta = \frac{2(1-n^2)}{3(1-\nu^2)}U_3^{(1)} - \left(\frac{2\Omega^2}{n^2} - i\alpha_0\frac{2(1-n^2)}{3(1-\nu^2)}\right) U_3^{(0)} \tag{50}$$

and

$$\begin{aligned} \int_{-1}^{1} s_{22}^{(2)} d\zeta = &\left(\frac{\rho_f}{\rho n}\Omega^2 + \frac{2n^2(1-n^2)}{3(1-\nu^2)}\right) U_3^{(1)} \\ &\left(\frac{2\rho n + 3\rho_f}{\rho n}\Omega^2 - \frac{2(1-n^2)}{3(1-\nu^2)}\left(1-n^2-i\alpha_0 n^2\right)\right) U_3^{(0)}. \end{aligned} \tag{51}$$

Comparing (50) and (51), we finally have

$$\left(\Omega^2 - \frac{2\rho(1-n^2)^2 n}{3\rho_f(1-\nu^2)}\right) U_3^{(1)} = -\left(\Omega^2 - \frac{2\rho(1-n^2)^2 n}{3\rho_f(1-\nu^2)}\right) U_3^{(0)} - 2\left(\Omega^2 + \frac{\rho}{\rho_f}\left(\frac{1+n^2}{n}\right)\Omega^2 - i\alpha_0 \frac{\rho(1-n^2)^2 n}{3\rho_f(1-\nu^2)}\right) U_3^{(0)}. \tag{52}$$

5. Discussion

Let us set $W = U_3^{(0)} + \eta U_3^{(1)}$. Then, we obtain from (43) and (52)

$$W = \frac{Q^*}{g(\Omega)} \frac{\rho n}{\rho_f}(1-\eta), \tag{53}$$

where

$$g(\Omega) = \Omega^2 - \frac{2\rho n(1-n^2)^2}{3\rho_f(1-\nu^2)} + \eta\left(2\Omega^2\left(1 + \frac{\rho(1+n^2)}{\rho_f n}\right) - i\alpha_0 \frac{2\rho n(1-n^2)^2}{3\rho_f(1-\nu^2)}\right). \tag{54}$$

The roots of the equation $g(\Omega) = 0$ correspond to the resonance frequencies. Let us adapt a two-term expansion $\Omega^2 = \Omega_0^2 + \eta\Omega_1^2 + \ldots$. In this case, we may rewrite (54) as

$$g(\Omega) = 2\Omega_0\left(\Omega - \Omega_0 + \eta\Omega_0\left(1 + \frac{\rho(1+n^2)}{\rho_f n} - \frac{i\alpha_0}{2}\right)\right) \tag{55}$$

in which

$$\Omega_0^2 = \frac{2\rho n(1-n^2)^2}{3\rho_f(1-\nu^2)}. \tag{56}$$

Thus, the approximate vertical displacement component takes the form

$$W = \frac{\rho n}{2\rho_f \Omega_0} \frac{Q^*}{\Omega - \Omega_0 + \eta\Omega_0\left(1 + \frac{\rho(1+n^2)}{\rho_f n} - \frac{i\alpha_0}{2}\right)} \tag{57}$$

predicting the complex resonance frequencies

$$\Omega = \Omega_0 - \eta\Omega_0\left(1 + \frac{\rho(1+n^2)}{\rho_f\, n} - \frac{i\alpha_0}{2}\right). \tag{58}$$

This formula demonstrates the role of the small viscosity of interest.

From the above derivation, it is clear that the damping due to the radiation corresponding to a small imaginary term in (28) is far beyond the accuracy of (57) taking into account the structural damping defined by the parameter α_0. As it was shown in [17], the order of the damping caused by the radiation is of order $O(\eta^{3n})$ is negligible compared with the considered structural damping which is of $O(\eta)$ as predicted by the asymptotic formulae above. Figures 2 and 3 illustrate the resonant behaviour of a thin cylindrical shell immersed in a fluid at $n = 2$ and $n = 3$ for the vertical displacement normalised by Q^* (see, equation (57)). In all numerical calculations, the problem parameters are $\rho = 2790$ kg/m^3, $\nu = 0.3$, and $\rho_f = 1000$ kg/m^3.

6. Concluding Remarks

An asymptotic procedure is developed for forced low-frequency vibrations of a thin viscoelastic cylindrical shell immersed in fluid. The effect of viscosity is accounted by adapting the concept of a complex Young's modulus.

Refined asymptotic formulae for the shell transverse displacement and the related complex resonance frequency are derived. They demonstrate that the incorporated effect of structural damping is much greater than the contribution of the damping due to the radiation of vibration energy into the fluid. As a result, the latter can be ignored in practical applications. This is also beneficial since its evaluation requires retaining extra higher order terms in the expansion (28), see also [17] for more details.

The proposed approach has a clear potential to be extended to more sophisticated models of viscoelastic behaviour as well as to a transversely inhomogeneous fluid-loaded shell, e.g., see [18]. The obtained results can also be readily generalized to scattering problems, including 3D ones.

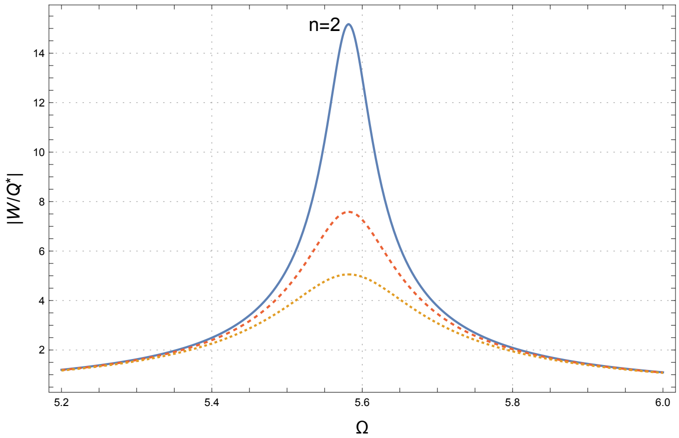

FIGURE 2. Displacement (57) for $n = 2$ with the Poisson ratio $\nu = 0.3$ and $\eta = 0.01$ with $\alpha_0 = 1$ (solid line), $\alpha_0 = 2$ (dashed red line) and $\alpha_0 = 3$ (dashed orange line).

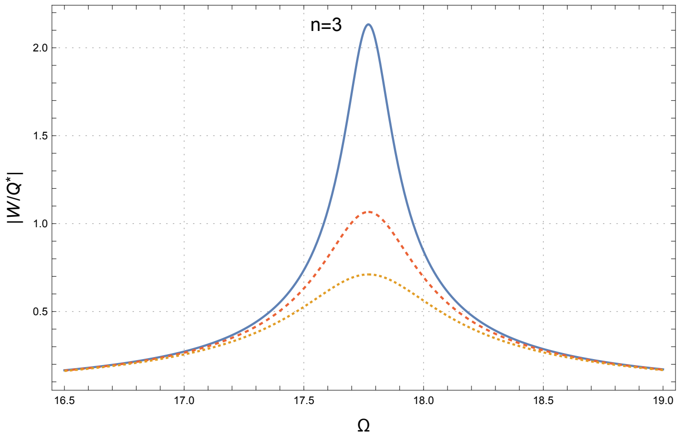

FIGURE 3. Displacement (57) for $n = 3$ with the Poisson ratio $\nu = 0.3$ and $\eta = 0.01$ with $\alpha_0 = 1$ (solid line), $\alpha_0 = 2$ (dashed red line) and $\alpha_0 = 3$ (dashed orange line).

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New integral inequalities involving p-convex and s-p-convex functions

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Abstract. In this study, new lemmas on p-convex and s-p-convex functions were derived utilizing the integral $\int_j^k \frac{(x^p-j^p)^f(k^p-x^p)^g m(x)}{x^{(f+g)p}}\,dx$. Through this equality, new integral inequalities were established, and novel upper bounds were obtained with the aid of Euler's beta and hypergeometric functions. The results provided new inequalities for the class of classical convex functions and the class of harmonic convex functions.

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Keywords. P-convex function, s-p-convex function, harmonically convex function, hypergeometric function.

1. Introduction

Recently, new and innovative approaches to classical convexity principles have been integrated to develop extended and generalised ideas in various fields. These developments include $p-$convex functions and s-p-second kind convex functions.

Definition 1. *A function $m : I \subset (0,\infty) \to \mathbb{R}$ is said to be $p-$convex, if*

$$m\left(\left[uj^p + (1-u)k^p\right]^{1/p}\right) \le um(j) + (1-u)m(k), \tag{1}$$

for all $j,k \in I$ and $u \in [0,1]$(see [8]).

For some new research, results and generalisations for the $p-$convex function (see [5], [6], [8], [9], [11], [12]). In definition 1, for $p = 1$, a $p-$ convex function reduces to a convex function, and for $p = -1$, a $p-$convex function reduces to a harmonically convex (HA) function.

Definition 2. *Let $s \in [0,1]$ and $p \in \mathbb{R}\backslash\{0\}$. A function $m : I \subset (0,\infty) \to [0,\infty)$ is said to be the $s-p$-convex function in second kind, if*

$$m\left(\left[uj^p + (1-u)k^p\right]^{\frac{1}{p}}\right) \le u^s m(j) + (1-u)^s m(k), \tag{2}$$

for all $j,k \in I$ and $u \in [0,1]$ (see [2]).

In inequality 2, for $s \in [0,1]$, if $p = 1$ and $s = 1$, it corresponds to the definition of convexity; if $p = -1$ and $s = 1$, it corresponds to the definition of harmonically convex function; if $p = 1$, it corresponds to the definition of $s-$convexity in the second kind; if $s = 1$, it corresponds to the definition of p-convexity. For some new research, results, and generalizations for the $s-p$-convex function. (see [2], [3].)

The Gauss-Jacobi typical generalised quadrature formula is a well-known mathematical inequality with an significant position in the literature and is defined as follows:

$$\int_j^k (x-j)^p(k-x)^q m(x)dx = \sum_{j=0}^{m} B_{m,j}\, m(\delta_j) + \Re_m[m] \tag{3}$$

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for certain $B_{m,j}$, δ_j and rest term $\Re_m[m]$ (see [4]).
In (see [7], [10], [13], [14]), the authors established several new integral inequalities concerning the left-hand side of equality 3 via several kinds of convexity.

Recall the following special functions, called beta functions and hypergeometric functions:
For $\text{Re}(j), \text{Re}(k) > 0$

$$\beta(j,k) = \int_0^1 u^{j-1}(1-u)^{k-1}du$$

The function is defined as the beta function. This integral is convergent for $j > 0$ and $k > 0$ (see [1]).
For $g > k > 0, |z| < 1$,

$${}_2F_1(j,k;g;z) = \frac{1}{\beta(k,g-k)} \int_0^1 u^{k-1}(1-u)^{g-k-1}(1-zu)^{-j}du$$

The function defined in the form of is called Hypergeometric function [1].
In their study, İ. İşcan et al. extended the generalized quadrature formula known in the literature as the Gauss-Jacobi integral equality to harmonic convex functions. Utilizing this lemma, they produced new integral inequalities and findings for harmonic convex functions. The fundamental lemma upon which their work is based is stated as follows:

Lemma 1. *[7] Let $m : [j,k] \subseteq (0,\infty) \to \mathbb{R}$ be a function integrable on the interval $[j,k]$ for fixed $f, g > 0$, then*

$$\int_j^k (x-j)^f (k-x)^g m(x)dx = j^{f+1}k^{g+1}(k-j)^{f+g+1} \int_0^1 \frac{t^f(1-t)^g}{A_t^{f+g+2}} m\left(\frac{jk}{A_t}\right) dt, \tag{4}$$

where $A_t = tj + (1-t)k$. Specifically, if $f = g$, the following equation is obtained:

$$\int_j^k (x-j)^f (k-x)^f m(x)dx = (jk)^{f+1}(k-j)^{2f+1} \int_0^1 \frac{t^f(1-t)^f}{A_t^{2f+2}} m\left(\frac{jk}{A_t}\right) dt.$$

The primary aim of this article was to consider the integral expression
$\int_j^k \frac{(x^p-j^p)^f(k^p-x^p)^g m(x)}{x^{(f+g)p}} dx$ as a new lemma for $p-$convex and $s-p-$convex functions. Subsequently, new theorems were contemplated in light of this lemma, leading to novel upper bounds for different classes of convex functions based on the Gauss-Jacobi expression. These upper bounds revealed new limits within the classes of classical convex functions and harmonic convex functions for varying values.

2. Main Result

We will use Lemmas 2 and 3 to obtain some new integral inequalities for $p-$convex and $s-p-$convex functions.

Lemma 2. *$m : [j,k] \subseteq [0,\infty) \to R$ be a function such that $m \in L[j,k]$. For $f, g > 0, p < 0$ the following equality holds.*

$$\int_j^k \frac{(x^p-j^p)^f (k^p-x^p)^g m(x)}{x^{(f+g)p}} dx = \frac{(k^p-j^p)^{f+g+1}}{p} \int_0^1 \frac{u^f(1-u)^g m\left((uk^p+(1-u)j^p)^{1/p}\right)}{(uk^p+(1-u)j^p)^{f+g+\frac{p-1}{p}}} du. \tag{5}$$

Proof. The intended result is easily calculated by taking $x = (uk^p+(1-u)j^p)^{1/p}$ and changing the variable. □

Conclusion 1. *If $p = -1$ is taken in Lemma 2, [7, Lemma 1] is obtained.*

Lemma 3. *$m : [j,k] \subseteq [0,\infty) \to R$ be a function such that $m \in L[j,k]$. For $f, g > 0, p > 0$ the following equality holds*

$$\int_j^k \frac{(x^p-j^p)^f (k^p-x^p)^g m(x)}{x^{(f+g)p}} dx = \frac{(k^p-j^p)^{f+g+1}}{p} \int_0^1 \frac{(1-u)^f u^g m\left((uj^p+(1-u)k^p)^{1/p}\right)}{(uj^p+(1-u)k^p)^{f+g+\frac{(p-1)}{p}}} du. \tag{6}$$

Proof. The intended result is easily calculated by taking $x = (uj^p + (1-u)k^p)^{1/p}$ and changing the variable. □

Conclusion 2. *If $p = 1$ then equality 6 in Lemma 3, then we get:*

$$\int_j^k \frac{(x-j)^f (k-x)^g m(x)}{x^{(f+g)}} dx = (k-j)^{f+g+1} \int_0^1 \frac{(1-u)^f u^g m(uj + (1-u)k)}{(uj + (1-u)k)^{f+g}} du$$

Theorem 1. *Let $m : [j,\ k] \subseteq [0, \infty) \to R$ be a function such that $m \in L[j,k]$. If m is $p-convex$ on $[j,k]$ for some fixed $f, g > 0$ then :*

a) *For $p < 0$*

$$\int_j^k \frac{(x^p - j^p)^f (k^p - x^p)^g m(x)}{x^{(f+g)p}} dx \tag{7}$$
$$\leq \frac{(k^p - j^p)^{f+g+1}}{pj^{(f+g+1)p-1}} \left[m(k)\beta(f+2, g+1)_2F_1\left(f+g+\frac{p-1}{p}, f+2; g+f+3; 1-\frac{k^p}{j^p}\right) \right.$$
$$\left. + m(j)\beta(f+1, g+2)_2F_1\left(f+g+\frac{p-1}{p}, f+1; g+f+3; 1-\frac{k^p}{j^p}\right)\right]$$

b) *For $p > 0$*

$$\int_j^k \frac{(x^p - j^p)^f (k^p - x^p)^g m(x)}{x^{(f+g)p}} dx \tag{8}$$
$$\leq \frac{(k^p - j^p)^{f+g+1}}{pk^{(f+g+1)p-1}} \left[m(j)\beta(g+2, f+1)_2F_1\left(f+g+\frac{(p-1)}{p}, g+2; g+f+3; 1-\frac{j^p}{k^p}\right) \right.$$
$$\left. + m(k)\beta(g+1, f+2)_2F_1\left(f+g+\frac{(p-1)}{p}, g+1; g+f+3; 1-\frac{j^p}{k^p}\right)\right]$$

Proof. Since m is $p-convex$ on $[j,\ k]$,using the lemma 2 for all $u \in [0,1]$ we have

$$\int_j^k \frac{(x^p - j^p)^f (k^p - x^p)^g m(x)}{x^{(f+g)p}} dx$$
$$= \frac{(k^p - j^p)^{f+g+1}}{p} \int_0^1 \frac{u^f (1-u)^g m\left((uk^p + (1-u)j^p)^{1/p}\right)}{(uk^p + (1-u)j^p)^{f+g+\frac{p-1}{p}}} du \tag{9}$$
$$\leq \frac{(k^p - j^p)^{f+g+1}}{p} \int_0^1 \frac{u^f (1-u)^g [um(k) + (1-u)m(j)]}{(uk^p + (1-u)j^p)^{f+g+\frac{p-1}{p}}} du$$
$$= \frac{(k^p - j^p)^{f+g+1}}{p} \int_0^1 \frac{u^f (1-u)^g [um(k) + (1-u)m(j)]}{\left(j^p\left(1-\left(1-\frac{k^p}{j^p}\right)u\right)\right)^{f+g+\frac{p-1}{p}}} du$$
$$= \frac{(k^p - j^p)^{f+g+1}}{p} \left(m(k) \int_0^1 \frac{u^{f+1}(1-u)^g}{\left(j^p\left(1-\left(1-\frac{k^p}{j^p}\right)u\right)\right)^{f+g+\frac{p-1}{p}}} du + m(j) \int_0^1 \frac{u^f (1-u)^{g+1}}{\left(j^p\left(1-\left(1-\frac{k^p}{j^p}\right)u\right)\right)^{f+g+\frac{p-1}{p}}} du \right)$$
$$= \frac{(k^p - j^p)^{f+g+1}}{pj^{(f+g+1)p-1}} \left(m(k) \int_0^1 \frac{u^{f+1}(1-u)^g}{\left(1-\left(1-\frac{k^p}{j^p}\right)u\right)^{f+g+\frac{p-1}{p}}} du + m(j) \int_0^1 \frac{u^f (1-u)^{g+1}}{\left(1-\left(1-\frac{k^p}{j^p}\right)u\right)^{f+g+\frac{p-1}{p}}} du \right)$$

where a simple calculation gives

$$\int_0^1 \frac{u^{f+1}(1-u)^g}{\left(1-\left(1-\frac{k^p}{j^p}\right)u\right)^{\frac{p-1}{p}}} du = \beta(f+2, g+1)_2F_1\left(f+g+\frac{p-1}{p}, f+2, g+f+3; 1-\frac{k^p}{j^p}\right) \tag{10}$$

and

$$\int_0^1 \frac{u^f (1-u)^{g+1}}{\left(1-\left(1-\frac{k^p}{j^p}\right)u\right)^{\frac{p-1}{p}}} du = \beta(f+1,g+2)_2F_1\left(f+g+\frac{p-1}{p}, f+1, g+f+3; 1-\frac{k^p}{j^p}\right) \tag{11}$$

Substituting equations 10 and 11 into the inequality 9, we obtain the required result. The proof is thus complete.

b) Using Lemma 3, inequality 8 is obtained by applying a similar proof method. □

Conclusion 3. *If $p=-1$ is taken in inequality 7, [7, Inequality of (2.2)] is obtained.*

Conclusion 4. *If it is taken $p=1$ in inequality 8, then the following inequality is obtained:*

$$\int_j^k \frac{(x-j)^f (k-x)^g m(x)}{x^{f+g}} dx \leq \frac{(k-j)^{f+g+1}}{k^{f+g}} [m(j)\beta(g+2,f+1) + m(k)\beta(g+1,f+2)]$$

Theorem 2. *Let $m:[j,\ k]\subseteq[0,\infty)\to R$ be a function such that $m\in L[j,k]$ and $\alpha\geq 1$ .If $|m|^\alpha$ is $p-convex$ on $[j,k]$ for some fixed $f,g>0$ then:*
a) *For $p<0$*

$$\begin{aligned} &\int_j^k \frac{(x^p-j^p)^f (k^p-x^p)^g m(x)}{x^{(f+g)p}} dx \\ &\leq \frac{(k^p-j^p)^{f+g+1}}{pj^{(f+g+1)p-1}} \left(\beta(f+1,g+1)_2F_1\left(f+g+\frac{p-1}{p}, f+1; f+g+2; 1-\frac{k^p}{j^p}\right)\right)^{1-\frac{1}{\alpha}} \\ &\times \left(\begin{array}{c} |m(k)|^\alpha \beta(f+2,g+1)_2F_1\left(f+g+\frac{p-1}{p}, f+2; f+g+3; 1-\frac{k^p}{j^p}\right) \\ + \\ |m(j)|^\alpha \beta(f+1,g+2)_2F_1\left(f+g+\frac{p-1}{p}, f+1; f+g+3; 1-\frac{k^p}{j^p}\right) \end{array} \right)^{\frac{1}{\alpha}} \end{aligned} \tag{12}$$

b) *For $p>0$*

$$\begin{aligned} &\int_j^k \frac{(x^p-j^p)^f (k^p-x^p)^g m(x)}{x^{(f+g)p}} dx \\ &\leq \frac{(k^p-j^p)^{f+g+1}}{pk^{(f+g)p-1}} \left(\beta(g+1,f+1)_2F_1\left(f+g+\frac{p-1}{p}, g+1; f+g+2; 1-\frac{j^p}{k^p}\right)\right)^{1-\frac{1}{\alpha}} \\ &\times \left(\begin{array}{c} |m(j)|^\alpha \beta(g+2,f+1)_2F_1\left(f+g+\frac{p-1}{p}, g+2; g+f+3; 1-\frac{j^p}{k^p}\right) \\ + \\ |m(k)|^\alpha \beta(g+1,f+2)_2F_1\left(f+g+\frac{p-1}{p}, g+1; g+f+3; 1-\frac{j^p}{k^p}\right) \end{array} \right)^{\frac{1}{\alpha}} \end{aligned} \tag{13}$$

Proof. Since $|m|^\alpha$ is $p-convex$ on $[j,k]$, using Lemma 2, by the power mean integral inequality for all $u\in[0,1]$ we have

$$\int_j^k \frac{(x^p-j^p)^f(k^p-x^p)^g m(x)}{x^{(f+g)p}}dx$$
$$\le \frac{(k^p-j^p)^{f+g+1}}{p}\int_0^1 \frac{u^f(1-u)^g\left|m\left((uk^p+(1-u)j^p)^{1/p}\right)\right|}{(uk^p+(1-u)j^p)^{f+g+\frac{p-1}{p}}}du$$
$$\le \frac{(k^p-j^p)^{f+g+1}}{p}\left(\int_0^1 \frac{u^f(1-u)^g}{(uk^p+(1-u)j^p)^{f+g+\frac{p-1}{p}}}du\right)^{1-\frac{1}{\alpha}}\left(\int_0^1 \frac{u^f(1-u)^g\left|m\left((uk^p+(1-u)j^p)^{1/p}\right)\right|^\alpha}{(uk^p+(1-u)j^p)^{f+g+\frac{p-1}{p}}}du\right)^{\frac{1}{\alpha}}$$
$$\le \frac{(k^p-j^p)^{f+g+1}}{p}\left(\int_0^1 \frac{u^f(1-u)^g}{(uk^p+(1-u)j^p)^{f+g+\frac{p-1}{p}}}du\right)^{1-\frac{1}{\alpha}}\left(\int_0^1 \frac{(u|m(k)|^\alpha+(1-u)|m(j)|^\alpha)}{(uk^p+(1-u)j^p)^{f+g+\frac{p-1}{p}}}du\right)^{\frac{1}{\alpha}}$$
$$= \frac{(k^p-j^p)^{f+g+1}}{p}\left(\int_0^1 \frac{u^f(1-u)^g}{j^{(g+f+1)p-1}\left(1-\left(1-\frac{k^p}{j^p}\right)u\right)^{f+g+\frac{p-1}{p}}}du\right)^{1-\frac{1}{\alpha}}\left(\int_0^1 \frac{u^f(1-u)^g(u|m(k)|^\alpha+(1-u)|m(j)|^\alpha)}{j^{(g+f+1)p-1}\left(1-\left(1-\frac{k^p}{j^p}\right)u\right)^{f+g+\frac{p-1}{p}}}du\right)^{\frac{1}{\alpha}}$$
$$= \frac{(k^p-j^p)^{f+g+1}}{pj^{(f+g+1)p-1}}\left(\beta(f+1,g+1)F_1\left(f+g+\frac{p-1}{p},f+1;f+g+2;1-\frac{k^p}{j^p}\right)\right)^{1-\frac{1}{\alpha}}$$
$$\times\left(\begin{array}{c}|m(k)|^\alpha\beta(f+2,g+1)_2F_1\left(f+g+\frac{p-1}{p},f+2;f+g+3;1-\frac{k^p}{j^p}\right)\\+\\|m(j)|^\alpha\beta(f+1,g+2)_2F_1\left(f+g+\frac{p-1}{p},f+1;f+g+3;1-\frac{k^p}{j^p}\right)\end{array}\right)^{\frac{1}{\alpha}}$$

which completes the proof.

b) Using Lemma 3, inequality 13 is obtained by applying a similar proof method. □

Conclusion 5. *If $p=-1$ is taken in inequality 12, [7, Inequality of (2.5)] is obtained.*

Conclusion 6. *If $p=1$ in inequality 13, the following inequality is obtained:*

$$\int_j^k \frac{(x^p-j^p)^f(k^p-x^p)^g m(x)}{x^{f+g}}dx \le (k-j)^{f+g+1}\beta(g+1,f+1)^{1-\frac{1}{\alpha}}\left[\begin{array}{c}|m(j)|^\alpha\beta(g+2,f+1)\\+\\|m(k)|^\alpha\beta(g+1,f+2)\end{array}\right]^{\frac{1}{\alpha}}$$

Theorem 3. *Let $m:[j,\ k]\subseteq[0,\infty)\to R$ be a function such that $m\in L[j,k]$ and $\alpha>1$ .If $|m|^\alpha$ is $p-convex$ on $[j,k]$ for some fixed $f,g>0$ then :*

a) *For $p<0$*

$$\int_j^k \frac{(x^p-j^p)^f(k^p-x^p)^g m(x)}{x^{(f+g)p}}dx \tag{14}$$
$$\le \frac{(k^p-j^p)^{f+g+1}}{pj^{(f+g+1)p-1}}\beta^{\frac{1}{\delta}}(f\delta+1,g\delta+1)_2F_1^{\frac{1}{\delta}}\left(\left(f+g+\frac{p-1}{p}\right)\delta,f\delta+1;(g+f)\delta+2;1-\frac{k^p}{j^p}\right)$$
$$\times\left(\frac{|m(k)|^\alpha+|m(j)|^\alpha}{2}\right)^{1/\alpha}$$

a) *For $p>0$*

$$\int_j^k \frac{(x^p-j^p)^f(k^p-x^p)^g m(x)}{x^{(f+g)p}}dx \tag{15}$$
$$\le \frac{(k^p-j^p)^{f+g+1}}{pk^{(f+g+1)p-1}}\beta^{\frac{1}{\delta}}(g\delta+1,f\delta+1)_2F_1^{\frac{1}{\delta}}\left(\left(f+g+\frac{p-1}{p}\right)\delta,g\delta+1;(f+g)\delta+2;1-\frac{j^p}{k^p}\right)$$
$$\times\left(\frac{|m(j)|^\alpha+|m(k)|^\alpha}{2}\right)^{1/\alpha}$$

where $1/\alpha + 1/\delta = 1$.

Proof.

a) Since $|m|^{\alpha}$ is $p-convex$ on $[j,k]$, using Lemma 2, by the Hölder integral inequality for all $u \in [0,1]$ we have

$$\int_j^k \frac{(x^p - j^p)^f (k^p - x^p)^g m(x)}{x^{(f+g)p}} dx$$
$$\leq \frac{(k^p - j^p)^{f+g+1}}{p} \int_0^1 \frac{u^f (1-u)^g}{(uk^p + (1-u)j^p)^{f+g+\frac{p-1}{p}}} \left| m\left((uk^p + (1-u)j^p)^{1/p} \right) \right| du$$
$$\leq \frac{(k^p - j^p)^{f+g+1}}{p} \left(\int_0^1 \frac{u^{f\delta}(1-u)^{g\delta}}{(uk^p + (1-u)j^p)^{\left(f+g+\frac{p-1}{p}\right)\delta}} du \right)^{\frac{1}{\delta}} \left(\int_0^1 \left| m\left((uk^p + (1-u)j^p)^{\frac{1}{p}} \right) \right|^{\alpha} du \right)^{\frac{1}{\alpha}}$$
$$\leq \frac{(k^p - j^p)^{f+g+1}}{p} \left(\int_0^1 \frac{u^{f\delta}(1-u)^{g\delta}}{(uk^p + (1-u)j^p)^{\left(f+g+\frac{p-1}{p}\right)\delta}} du \right)^{\frac{1}{\delta}} \left(\int_0^1 (u|m(k)|^{\alpha} + (1-u)|m(j)|^{\alpha})\, dt \right)^{\frac{1}{\alpha}}$$
$$= \frac{(k^p - j^p)^{f+g+1}}{p} \left(\int_0^1 \frac{u^{f\delta}(1-u)^{g\delta}}{j^{((f+g+1)p-1)\delta} \left(1 - \left(1 - \frac{k^P}{j^P}\right)u\right)^{\left(f+g+\frac{p-1}{p}\right)\delta}} du \right)^{\frac{1}{\delta}} \left(\int_0^1 (u|m(k)|^{\alpha} + (1-u)|m(j)|^{\alpha})\, d\boldsymbol{u} \right)^{\frac{1}{\alpha}}$$
$$= \frac{(k^p - j^p)^{f+g+1}}{pj^{(f+g+1)p-1}} \beta^{\frac{1}{\delta}}(f\delta+1, g\delta+1) {}_2F_1^{\frac{1}{\delta}} \left(\left(f + g + \frac{p-1}{p} \right) \delta, f\delta+1; (g+f)\delta+2; 1 - \frac{k^p}{j^p} \right)$$
$$\times \left(\frac{|m(k)|^{\alpha} + |m(j)|^{\alpha}}{2} \right)^{1/\alpha}$$

which completes the proof.

b) Using Lemma 3, inequality 15 is obtained by applying a similar proof method. □

Conclusion 7. *If* $p = -1$ *is taken in inequality 14, [7, Inequality of (2.6)] is obtained.*

Conclusion 8. *If* $p = 1$ *in inequality 15, the following inequality is obtained:*

$$\int_j^k \frac{(x-j)^f (k-x)^g m(x)}{x^{f+g}} dx$$
$$\leq \frac{(k-j)^{f+g+1}}{k^{f+g}} \beta^{\frac{1}{\delta}}(g\delta+1, f\delta+1) {}_2F_1^{\frac{1}{\delta}} \left((f+g)\delta, g\delta+1; (f+g)\delta+2; 1 - \frac{j}{k} \right)^{1/\alpha}$$
$$\times \left(\frac{|m(j)|^{\alpha} + |m(k)|^{\alpha}}{2} \right)^{1/\alpha}$$

Theorem 4. *Let* $m : [j,\ k] \subseteq [0, \infty) \to R$ *be a function such that* $m \in L[j,k]$ *and* $\alpha > 1$ *.If* $|m|^{\alpha}$ *is* $p-convex$ *on* $[j,k]$ *for some fixed* $f, g > 0$ *then :*

a) *For* $p < 0$

$$\int_j^k \frac{(x^p - j^p)^f (k^p - x^p)^g m(x)}{x^{(f+g)p}} dx \tag{16}$$
$$\leq \frac{(k^p - j^p)^{f+g+1}}{pj^{(f+g+1)p-1}} \left(\begin{array}{l} \beta^{\frac{1}{\delta}}(f\delta+1, g\delta+1) \left[{}_2F_1\left(\left(f+g+\frac{p-1}{p}\right)\alpha, 2; 3; 1 - \frac{k^p}{j^p} \right) \frac{|m(k)|^{\alpha}}{2} \right. \\ \left. + {}_2F_1\left(\left(f+g+\frac{p-1}{p}\right)\alpha, 1; 3; 1 - \frac{k^p}{j^p} \right) \frac{|m(j)|^{\alpha}}{2} \right] \end{array} \right)$$

b) *For* $p > 0$

$$\int_j^k \frac{(x^p - j^p)^f (k^p - x^p)^g m(x)}{x^{(f+g)p}} dx \tag{17}$$

$$\leq \frac{(k^p-j^p)^{f+g+1}}{pk^{((f+g+1)p-1)}}\left(\begin{array}{c}\beta^{\frac{1}{f}}(g\delta+1,f\delta+1)\left[{}_2F_1\left(\left(f+g+\frac{p-1}{p}\right)\alpha,2;3;1-\frac{j^p}{k^p}\right)\frac{|m(j)|^\alpha}{2}\right.\\ \left.+{}_2F_1\left((f+g+\frac{p-1}{p})\alpha,1;3;1-\frac{j^p}{k^p}\right)\frac{|m(k)|^\alpha}{2}\right]^{\frac{1}{\alpha}}\end{array}\right)$$

where $1/\alpha+1/\delta=1$.

Proof.

a) Since $|m|^\alpha$ is $p-convex$ on $[j,k]$, using Lemma 2, by the Hölder integral inequality for all $u\in[0,1]$ we have

$$\int_j^k \frac{(x^p-j^p)^f(k^p-x^p)^g m(x)}{x^{(f+g)p}}dx$$
$$\leq \frac{(k^p-j^p)^{f+g+1}}{p}\int_0^1 \frac{u^f(1-u)^g}{(uk^p+(1-u)j^p)^{f+g+\frac{p-1}{p}}}\left|m\left((uk^p+(1-u)j^p)^{1/p}\right)\right|du$$
$$\leq \frac{(k^p-j^p)^{f+g+1}}{p}\left(\int_0^1 u^{f\delta}(1-u)^{g\delta}du\right)^{\frac{1}{\delta}}$$
$$\times\left(\int_0^1 \frac{1}{(uk^p+(1-u)j^p)^{(f+g+\frac{p-1}{p})\alpha}}\left|m\left(uk^p+(1-u)j^p\right)^{1/p}\right|^\alpha du\right)^{\frac{1}{\alpha}}$$
$$=\frac{(k^p-j^p)^{f+g+1}}{p}\left[\left(\int_0^1 u^{f\delta}(1-u)^{g\delta}du\right)^{\frac{1}{\delta}}\left(\int_0^1 \frac{(u|m(k)|^\alpha+(1-u)|m(j)|^\alpha)}{(uk^p+(1-u)j^p)^{(f+g+\frac{p-1}{p})\alpha}}du\right)^{\frac{1}{\alpha}}\right]$$
$$=\frac{(k^p-j^p)^{f+g+1}}{p}\left[\left(\int_0^1 u^{f\delta}(1-u)^{g\delta}du\right)^{\frac{1}{\delta}}\left(\int_0^1 \frac{(u|m(k)|^\alpha+(1-u)|m(j)|^\alpha)}{j^{((f+g+1)p-1)\alpha}\left(1-\left(1-\frac{k^p}{j^p}\right)u\right)^{(f+g+\frac{p-1}{p})\alpha}}du\right)^{\frac{1}{\alpha}}\right]$$
$$=\frac{(k^p-j^p)^{f+g+1}}{pj^{(f+g+1)p-1}}\left(\begin{array}{c}\beta^{\frac{1}{\delta}}(f\delta+1,g\delta+1)\left[{}_2F_1\left(\left(f+g+\frac{p-1}{p}\right)\alpha,2;3;1-\frac{k^p}{j^p}\right)\frac{|m(k)|^\alpha}{2}\right.\\ \left.+{}_2F_1\left(\left(f+g+\frac{p-1}{p}\right)\alpha,1;3;1-\frac{k^p}{j^p}\right)\frac{|m(j)|^\alpha}{2}\right]\end{array}\right)$$

which completes the proof.

b) Using Lemma 3, inequality 17 is obtained by applying a similar proof method. □

Conclusion 9. *If* $p=-1$ *is taken in inequality 16, [7, Inequality of (2.7)] is obtained.*

Conclusion 10. *If* $p=1$ *in inequality 17, the following inequality is obtained:*

$$\int_j^k \frac{(x-j)^f(k-x)^g m(x)}{x^{f+g}}dx\leq\frac{(k-j)^{f+g+1}}{k^{f+g}}\left(\begin{array}{c}\beta^{\frac{1}{\delta}}(g\delta+1,f\delta+1)\left[{}_2F_1\left((f+g)\alpha,2;3;1-\frac{j}{k}\right)\frac{|m(j)|^\alpha}{2}\right.\\ \left.+{}_2F_1\left((f+g)\alpha,1;3;1-\frac{j}{k}\right)\frac{|m(k)|^\alpha}{2}\right]^{\frac{1}{\alpha}}\end{array}\right)$$

Theorem 5. *Let* $m:[j,k]\subseteq[0,\infty)\to R$ *be a function such that* $m\in L[j,k]$ *and* $\alpha\geq 1$*.If* $|m|^\alpha$ *is* $s-p-convex$ *in the second kind on* $[j,k]$ *for some fixed* $f,g>0$, $s\in[0,1]$ *then :*

a) *For* $p<0$

$$\int_j^k \frac{(x^p-j^p)^f(k^p-x^p)^g m(x)}{x^{(f+g)p}}dx \tag{18}$$
$$\leq\frac{(k^p-j^p)^{f+g+1}}{pj^{(f+g+1)p-1}}\left(\beta(f+1,g+1){}_2F_1\left(f+g+\frac{p-1}{p},f+1;f+g+2;1-\frac{k^p}{j^p}\right)\right)^{1-\frac{1}{\alpha}}$$
$$\times\left(\begin{array}{c}|m(k)|^\alpha\beta(f+s+1,g+1){}_2F_1\left(f+g+\frac{p-1}{p},f+s+1;f+g+s+2;1-\frac{k^p}{j^p}\right)\\+\\|m(j)|^\alpha\beta(f+1,g+s+1){}_2F_1\left(f+g+\frac{p-1}{p},f+1;f+g+s+2;1-\frac{k^p}{k^p}\right)\end{array}\right)^{\frac{1}{\alpha}}$$

b) *For $p > 0$*

$$\begin{aligned}
&\int_j^k \frac{(x^p - j^p)^f (k^p - x^p)^g m(x)}{x^{(f+g)p}} dx \\
&\le \frac{(k^p - j^p)^{f+g+1}}{pk^{((f+g+1)p-1)}} \left(\beta(g+1, f+1) F_1 \left(f + g + \frac{p-1}{p}, g+1; f+g+2; 1 - \frac{j^p}{k^p} \right) \right)^{1-\frac{1}{\alpha}} \\
&\times \left(\begin{array}{c} |m(j)|^\alpha \beta(g+s+1, f+1)_2 F_1 \left(f+g+\frac{p-1}{p}, g+s+1; f+g+s+2; 1-\frac{j^p}{k^p} \right) \\ + \\ |m(k)|^\alpha \beta(g+1, f+s+1)_2 F_1 \left(f+g+\frac{p-1}{p}, g+1; g+f+s+2; 1-\frac{j^p}{k^p} \right) \end{array} \right)^{\frac{1}{\alpha}}
\end{aligned} \tag{19}$$

Proof. Since $|m|^\alpha$ is $s-p-convex$ in the second kind on $[j,k]$, using Lemma 2, by the power mean integral inequality for all $u \in [0,1]$ we have

$$\begin{aligned}
&\int_j^k \frac{(x^p - j^p)^f (k^p - x^p)^g m(x)}{x^{(f+g)p}} dx \\
&\le \frac{(k^p - j^p)^{g+f+1}}{p} \int_0^1 \frac{u^f (1-u)^g \left| m\left((uk^p + (1-u) j^p)^{1/p} \right) \right|}{(uk^p + (1-u) j^p)^{f+g+\frac{p-1}{p}}} du \\
&\le \frac{(k^p - j^p)^{f+g+1}}{p} \left(\int_0^1 \frac{u^f (1-u)^g}{(uk^p + (1-u) j^p)^{f+g+\frac{p-1}{p}}} du \right)^{1-\frac{1}{\alpha}} \left(\int_0^1 \frac{u^f (1-u)^g \left| m\left((uk^p + (1-u) j^p)^{1/p} \right) \right|^\alpha}{(uk^p + (1-u) j^p)^{f+g+\frac{p-1}{p}}} du \right)^{\frac{1}{\alpha}} \\
&\le \frac{(k^p - j^p)^{g+f+1}}{p} \left(\int_0^1 \frac{u^f (1-u)^g}{(uk^p + (1-u) j^p)^{f+g+\frac{p-1}{p}}} du \right)^{1-\frac{1}{\alpha}} \left(\int_0^1 \frac{u^f (1-u)^g (u^s |m(k)|^\alpha + (1-u)^s |m(j)|^\alpha)}{(uk^p + (1-u) j^p)^{f+g+\frac{p-1}{p}}} du \right)^{\frac{1}{\alpha}} \\
&= \frac{(k^p - j^p)^{f+g+1}}{p} \left(\int_0^1 \frac{u^f (1-u)^g}{j^{(g+f+1)p-1} \left(1 - \left(1 - \frac{k^p}{j^p}\right) u\right)^{f+g+\frac{p-1}{P}}} du \right)^{1-\frac{1}{\alpha}} \left(\int_0^1 \frac{u^f (1-u)^g (u^s |m(k)|^\alpha + (1-u)^s |m(j)|^\alpha}{j^{(g+f+1)p-1} \left(1 - \left(1 - \frac{k^p}{j^p}\right) u\right)^{f+g+\frac{(p-1)}{p}}} \right)^{\frac{1}{\alpha}} \\
&= \frac{(k^p - j^p)^{f+g+1}}{pj^{(f+g+1)p-1}} \left(\beta(f+1, g+1) F_1 \left(f+g+\frac{p-1}{p}, f+1; f+g+2; 1-\frac{k^p}{j^p} \right) \right)^{1-\frac{1}{\alpha}} \\
&\times \left(\begin{array}{c} |m(k)|^\alpha \beta(f+s+1, g+1)_2 F_1 \left(f+g+\frac{p-1}{p}, f+s+1; f+g+s+2; 1-\frac{k^p}{j^p} \right) \\ + \\ |m(j)|^\alpha \beta(f+1, g+s+1)_2 F_1 \left(f+g+\frac{p-1}{p}, f+1; f+g+s+2; 1-\frac{k^p}{j^p} \right) \end{array} \right)^{\frac{1}{\alpha}}
\end{aligned}$$

which completes the proof.

b) Using Lemma 3, inequality 19 is obtained by applying a similar proof method. □

Conclusion 11. *If $p = -1$ in inequality 18, the following inequality is obtained:*

$$\begin{aligned}
&\int_j^k (x-j)^f (k-x)^g m(x) dx \\
&\le \left(\frac{j}{k} \right)^{f+1} (k-j)^{f+g+1} \left(\beta(f+1, g+1)_2 F_1 \left(f+g+2, f+1; f+g+2; 1-\frac{j}{k} \right) \right)^{1-\frac{1}{\alpha}} \\
&\times \left(\begin{array}{c} |m(k)|^\alpha \beta(f+s+1, g+1)_2 F_1 \left(f+g+2, f+s+1; f+g+s+2; 1-\frac{j}{k} \right) \\ + \\ |m(j)|^\alpha \beta(f+1, g+s+1)_2 F_1 \left(f+g+2, f+1; f+g+s+2; 1-\frac{j}{k} \right) \end{array} \right)^{\frac{1}{\alpha}}
\end{aligned} \tag{20}$$

Conclusion 12. *If $p = 1$ in inequality 19, the following inequality is obtained:*

$$\begin{aligned}
&\int_j^k \frac{(x-j)^f (k-x)^g m(x)}{x^{f+g}} dx \\
&\leq \frac{(k-j)^{f+g+1}}{k^{f+g}} \beta(g+1, f+1) {}_2F_1 \left(f+g, g+1; f+g+2; 1-\frac{j}{k} \right)^{1-\frac{1}{\alpha}} \\
&\times \left(\begin{array}{c} |m(j)|^\alpha \beta(g+s+1, f+1) {}_2F_1 \left(f+g, g+s+1; f+g+s+2; 1-\frac{j}{k} \right) \\ +|m(k)|^\alpha \beta(g+1, f+s+1) {}_2F_1 \left(f+g, g+1; f+g+s+2; 1-\frac{j}{k} \right) \end{array} \right)^{\frac{1}{\alpha}}
\end{aligned} \tag{21}$$

Theorem 6. *Let $m : [j,k] \subseteq [0,\infty) \to R$ be a function such that $m \in L[j,k]$ and $\alpha > 1$. If $|m|^\alpha$ is $s-p-convex$ in the second kind on $[j,k]$ for some fixed $f, g > 0$, $s \in [0,1]$ then:*

a) *For $p < 0$*

$$\begin{aligned}
&\int_j^k \frac{(x^p - j^p)^f (k^p - x^p)^g m(x)}{x^{(f+g)p}} dx \\
&\leq \frac{(k^p - j^p)^{f+g+1}}{pj^{(f+g+1)p-1}} \beta^{\frac{1}{\delta}}(f\delta+1, g\delta+1) {}_2F_1^{\frac{1}{\delta}} \left(\left(f+g+\frac{p-1}{p} \right) \delta, f\delta+1; (f+g)\delta+2; 1-\frac{k^p}{j^p} \right) \\
&\times \left(\frac{|m(k)|^\alpha + |m(j)|^\alpha}{s+1} \right)^{1/\alpha}
\end{aligned} \tag{22}$$

b) *For $p > 0$*

$$\begin{aligned}
&\int_j^k \frac{(x^p - j^p)^f (k^p - x^p)^g m(x)}{x^{(f+g)p}} dx \\
&\leq \frac{(k^p - j^p)^{f+g+1}}{pk^{(f+g+1)p-1}} \beta^{\frac{1}{\delta}}(\delta g+1, f\delta+1) {}_2F_1^{\frac{1}{\delta}} \left(\left(f+g+\frac{p-1}{p} \right) \delta, g\delta+1; (f+g)\delta+2; 1-\frac{j^p}{k^p} \right) \\
&\times \left(\frac{|m(j)|^\alpha + |m(k)|^\alpha}{s+1} \right)^{1/\alpha}
\end{aligned} \tag{23}$$

where $1/\alpha + 1/\delta = 1$.

Proof.

a) Since $|m|^\alpha$ is $s-p-convex$ in the second kind on $[j,k]$, using Lemma 2, by the Hölder integral inequality for all $u \in [0,1]$ we have

$$\begin{aligned}
&\int_j^k \frac{(x^p - j^p)^f (k^p - x^p)^g m(x)}{x^{(f+g)p}} dx \\
&\leq \frac{(k^p - j^p)^{f+g+1}}{p} \int_0^1 \frac{u^f (1-u)^g}{(uk^p + (1-u)j^p)^{\left(f+g+\frac{p-1}{p}\right)}} \left| m \left((uk^p + (1-u)j^p)^{1/p} \right) \right| du \\
&\leq \frac{(k^p - j^p)^{f+g+1}}{p} \left(\int_0^1 \frac{u^{f\delta}(1-u)^{\delta g}}{(uk^p + (1-u)j^p)^{\left(f+g+\frac{p-1}{p}\right)}} du \right)^{\frac{1}{\delta}} \left(\int_0^1 \left| m \left((uk^p + (1-u)j^p)^{\frac{1}{p}} \right) \right|^\alpha dt \right)^{\frac{1}{\alpha}} \\
&\leq \frac{(k^p - j^p)^{f+g+1}}{p} \left(\int_0^1 \frac{u^{f\delta}(1-u)^{\delta g}}{(uk^p + (1-u)j^p)^{\left(f+g+\frac{p-1}{p}\right)\delta}} du \right)^{\frac{1}{\delta}} \left(\int_0^1 (u^s |m(k)|^\alpha + (1-u)^s |m(j)|^\alpha) \, dt \right)^{\frac{1}{\alpha}} \\
&= \frac{(k^p - j^p)^{f+g+1}}{p} \left(\int_0^1 \frac{u^{f\delta}(1-u)^{\delta g}}{j^{((f+g+1)p-1)\delta} \left(1 - \left(1 - \frac{k^p}{j^p}\right) u\right)^{\left(f+g+\frac{p-1}{p}\right)\delta}} du \right)^{\frac{1}{\delta}} \left(\int_0^1 (u^s |m(k)|^\alpha + (1-u)^s |m(j)|^\alpha) \, dt \right)^{\frac{1}{\alpha}} \\
&= \frac{(k^p - j^p)^{f+g+1}}{pj^{(f+g+1)p-1}} \beta^{\frac{1}{\delta}}(f\delta+1, g\delta+1) {}_2F_1^{\frac{1}{\delta}} \left(\left(f+g+\frac{p-1}{p} \right) \delta, f\delta+1; (f+g)\delta+2; 1-\frac{k^p}{j^p} \right) \\
&\times \left(\frac{|m(k)|^\alpha + |m(j)|^\alpha}{s+1} \right)^{1/\alpha}
\end{aligned}$$

which completes the proof.

b) Using Lemma 3, inequality 23 is obtained by applying a similar proof method. □

Conclusion 13. *If $p = -1$ in inequality 22, the following inequality is obtained:*

$$\int_j^k (x-j)^f (k-x)^g m(x)dx$$
$$\leq \left(\frac{j}{k}\right)^{f+1} (k-j)^{f+g+1} \beta^{\frac{1}{\delta}}(f\delta+1, g\delta+1) {}_2F_1^{\frac{1}{\delta}} \left((f+g+2)\delta, f\delta+1; (g+f)\delta+2; 1-\frac{j}{k}\right)$$
$$\times \left(\frac{|m(k)|^\alpha + |m(j)|^\alpha}{s+1}\right)^{1/\alpha}$$

Conclusion 14. *If $p = 1$ in inequality 23, the following inequality is obtained:*

$$\int_j^k \frac{(x-j)^f (k-x)^g m(x)}{x^{(f+g)}} dx$$
$$\leq \frac{(k-j)^{f+g+1}}{k^{(f+g)}} \beta^{\frac{1}{\delta}}(g\delta+1, f\delta+1) {}_2F_1^{\frac{1}{\delta}} \left((f+g)\delta, g\delta+1; (f+g)\delta+2; 1-\frac{j}{k}\right)$$
$$\times \left(\frac{|m(j)|^\alpha + |m(k)|^\alpha}{s+1}\right)^{1/\alpha}$$

Theorem 7. *Let $m : [j,k] \subseteq [0,\infty) \to R$ be a function such that $m \in L[j,k]$ and $\alpha > 1$. If $|m|^\alpha$ is $s-p-$convex in the second kind on $[j,k]$ for some fixed $f, g > 0$, $s \in [0,1]$ then:*

a) *For $p < 0$*

$$\int_j^k \frac{(x^p-j^p)^f (k^p-x^p)^g m(x)}{x^{(f+g)p}} dx \tag{24}$$
$$\leq \frac{(k^p-j^p)^{f+g+1}}{pj^{(f+g+1)p-1}} \left(\begin{array}{c} \beta^{\frac{1}{\delta}}(f\delta+1, g\delta+1) \left[{}_2F_1\left(\left(f+g+\frac{p-1}{p}\right)\alpha, s+1; s+2; 1-\frac{k^p}{j^p}\right) \frac{|m(k)|^\alpha}{s+1} \right. \\ \left. + {}_2F_1\left(\left(f+g+\frac{p-1}{p}\right)\alpha, 1; s+1; 1-\frac{k^p}{j^p}\right) \frac{|m(j)|^\alpha}{s+1} \right]^{\frac{1}{\alpha}} \end{array} \right)$$

b) *For $p > 0$*

$$\int_j^k \frac{(x^p-j^p)^f (k^p-x^p)^g m(x)}{x^{(f+g)p}} dx \tag{25}$$
$$\leq \frac{(k^p-j^p)^{f+g+1}}{pk^{(f+g+1)p-1}} \left(\begin{array}{c} \beta^{\frac{1}{\delta}}(g\delta+1, f\delta+1) \left[{}_2F_1\left(\left(f+g+\frac{p-1}{p}\right)\alpha, s+1; s+2; 1-\frac{j^p}{k^p}\right) \frac{|m(j)|^\alpha}{s+1} \right. \\ \left. + {}_2F_1\left(\left(f+g+\frac{p-1}{p}\right)\alpha, 1; s+1; 1-\frac{j^p}{k^p}\right) \frac{|m(k)|^\alpha}{s+1} \right]^{\frac{1}{\alpha}} \end{array} \right)$$

where $1/\alpha + 1/\delta = 1$.

Proof.

a) Since $|m|^\alpha$ is $s-p-convex$ in the second kind on $[j,k]$, using Lemma 2, by the Hölder integral inequality for all $u \in [0,1]$ we have

$$\begin{aligned}
&\int_j^k \frac{(x^p-j^p)^f\,(k^p-x^p)^g\,m(x)}{x^{(f+g)p}}dx\\
&\le \frac{(k^p-j^p)^{f+g+1}}{p}\int_0^1 \frac{u^f(1-u)^g}{(uk^p+(1-u)j^p)^{f+g+\frac{p-1}{p}}}\left|m\left((uk^p+(1-u)j^p)^{1/p}\right)\right|du\\
&\le \frac{(k^p-j^p)^{f+g+1}}{p}\left(\int_0^1 u^{f\delta}(1-u)^{g\delta}du\right)^{\frac{1}{\delta}}\left(\int_0^1 \frac{1}{(uk^p+(1-u)j^p)^{\left(f+g+\frac{(p-1)}{p}\right)\alpha}}\left|\,m\left(uk^p+(1-u)j^p\right)^{1/p}\,\right|^{\alpha}du\right)^{\frac{1}{\alpha}}\\
&\le \frac{(k^p-j^p)^{f+g+1}}{p}\left[\left(\int_0^1 u^{f\delta}(1-u)^{g\delta}du\right)^{\frac{1}{\delta}}\left(\int_0^1 \frac{(u^s|m(k)|^\alpha+(1-u)^s|m(j)|^\alpha)}{(uk^p+(1-u)j^p)^{(f+g+\frac{p-1}{p})\alpha}}du\right)^{\frac{1}{\alpha}}\right]\\
&= \frac{(k^p-j^p)^{f+g+1}}{p}\left[\left(\int_0^1 u^{f\delta}(1-u)^{g\delta}du\right)^{\frac{1}{\delta}}\left(\int_0^1 \frac{(u^s|m(k)|^\alpha+(1-u)^s|m(j)|^\alpha)}{j^{((g+f+1)p-1)\alpha}\left(1-\left(1-\frac{k^p}{j^p}\right)u\right)^{(f+g+\frac{p-1}{p})\alpha}}du\right)^{\frac{1}{\alpha}}\right]\\
&= \frac{(\nu^p-j^p)^{f+g+1}}{pj^{(f+g+1)p-1}}\left(\begin{array}{c}\beta^{\frac{1}{m}}(f\delta+1,g\delta+1)\left[{}_2F_1\left(\left(f+g+\frac{p-1}{p}\right)\alpha,s+1;s+2;1-\frac{k^p}{j^p}\right)\frac{|m(k)|^\alpha}{s+1}\right.\\ \left.+{}_2F_1\left(\left(f+g+\frac{p-1}{p}\right)\alpha,1;s+2;1-\frac{k^p}{j^p}\right)\frac{|m(j)|^\alpha}{s+1}\right]^{\frac{1}{\alpha}}\end{array}\right)
\end{aligned}$$

which completes the proof.

b) Using Lemma 3, inequality 25 is obtained by applying a similar proof method. □

Conclusion 15. *If $p=-1$ in inequality 25, the following inequality is obtained:*

$$\begin{aligned}
&\int_j^k (x-j)^f(k-x)^g m(x)dx\\
&\le \left(\frac{j}{k}\right)^{f+1}(k-j)^{f+g+1}\left(\begin{array}{c}\beta^{\frac{1}{\delta}}(f\delta+1,g\delta+1)\left[{}_2F_1\left((f+g+2)\alpha,s+1;s+2;1-\frac{j}{k}\right)\frac{|m(k)|^\alpha}{s+1}\right.\\ \left.+{}_2F_1\left((f+g+2)\alpha,1;s+1;1-\frac{j}{k}\right)\frac{|m(j)|^\alpha}{s+1}\right]^{\frac{1}{\alpha}}\end{array}\right)
\end{aligned}$$

Conclusion 16. *If $p=1$ in inequality 25, the following inequality is obtained:*

$$\begin{aligned}
&\int_j^k \frac{(x-j)^f(k-x)^g m(x)}{x^{f+g}}dx\\
&\le \frac{(k-j)^{f+g+1}}{k^{(f+g)}}\left(\begin{array}{c}\beta^{\frac{1}{\delta}}(g\delta+1,f\delta+1)\left[{}_2F_1\left((f+g)\alpha,s+1;s+2;1-\frac{j}{k}\right)\frac{|m(j)|^\alpha}{s+1}\right.\\ \left.+{}_2F_1\left((f+g)\alpha,1;s+1;1-\frac{j^p}{k^p}\right)\frac{|m(k)|^\alpha}{s+1}\right]^{\frac{1}{\alpha}}\end{array}\right)
\end{aligned}$$

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An application of stochastic maximum principle for a constrained system with memory

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Abstract. In this research article, we study a stochastic control problem in a theoretical frame to solve a constrained task under memory impact. The nature of memory is modeled by Stochastic Differential Delay Equations and our state process evolves according to a jump-diffusion process with time-delay. We work on two specific types of constraints, which are described in the stochastic control problem as running gain components. We develop two theorems for corresponding deterministic and stochastic Lagrange multipliers. Furthermore, these theorems are applicable to a wide range of continuous-time stochastic optimal control problems in a diversified scientific area such as Operations Research, Biology, Computer Science, Engineering and Finance. Here, in this work, we apply our results to a financial application to investigate the optimal consumption process of a company via its wealth process with historical performance. We utilize the stochastic maximum principle, which is one of the main methods of continuous-time Stochastic Optimal Control theory. Moreover, we compute a real-valued Lagrange multiplier and clarify the relation between this value and the specified constraint.

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Keywords. Stochastic optimal control, stochastic maximum principle, stochastic differential delay equations, Lagrange multiplier, anticipated backward stochastic differential equations

1. Introduction and Unconstrained Control Problem

Stochastic Optimal Control theory is one of the main fields of sequential decision-making under uncertainty. Its fundamental goal is to determine the optimal control processes and the optimal value function for a specified control task, see [21,22,35]. The state process of a control problem is generally represented by a diffusion process, a jump-diffusion process or by a larger model such as a regime-switching process, see [2, 11, 15, 25, 28, 30, 33]. These processes meet specific mathematical requirements of each problem in a wide range of scientific disciplines such as finance, insurance, biology computer science, engineering etc. Whenever the uncertainty in an application can be expressed as a continuous-time process, diffusion processes can be used effectively. On the other side, in real-life applications, we usually require discontinuous formulations and in those cases, jump-diffusion processes and regime-switching models well-describe sudden changes in the process as well as in the environment.

Especially, in financial applications, the state processes may represent the price process of a risky asset, the wealth process of a company, the surplus process of an insurance policy, etc. Furthermore, since stochastic control theory provides quite strong tools to handle uncertainty and to develop optimal feedback controls, it is widely utilized in quantitative finance, see [6, 9, 12, 16, 26, 27, 34]. In this work, we use a jump-diffusion model to present the wealth process of a company and it is well known that such models efficiently describe the abrupt changes in the dynamics of a risky asset (for a broad literature, see [3]). The probabilistic literature for jump processes has been extensively developed and applied in financial mathematics so far, see also [1].

Moreover, in our article, we study a stochastic control problem with *memory* and *constraints*. The memory component is represented by a *time-delay* term, $\delta > 0$, in the dynamics of a Stochastic Differential Delay Equation (**SDDE**) (for a comprehensive theory of such equations, see [17]). Moreover, SDDEs express real-life financial phenomena more realistically with a meaning of historical performance of risky assets, economic inertia, time lag in financial operations. Hence, such systems have got significant

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attention from the researchers in the Stochastic Optimal Control field so far, see [7,8,14,20,23,25,28] and references therein.

Let us introduce the technical details and mathematical structure of our work:

As we stated, we use a jump-diffusion process with delay as the state process of our control task (for a detailed theory of continuous-time stochastic processes, see [1,13,19] and references therein).

Let $\mathbb{R}_0 := \mathbb{R} \setminus \{0\}$ be. $\mathcal{B}_0$ represents a Borel σ-field generated by the open subset O of $\mathbb{R}_0$, whose closure does not include the point 0.

Let $(N(dt, dz) : t \in [0, T], z \in \mathbb{R}_0)$ be a Poisson random measure on $([0, T] \times \mathbb{R}_0, \mathcal{B}([0, T]) \otimes \mathcal{B}_0)$. The Lévy measure of $N(\cdot, \cdot)$ is defined by ν and $\tilde{N}(dt, dz) := N(dt, dz) - \nu(dz)dt$ is a compensated Poisson random measure.
Let $(W(t) : t \in [0, T])$ be a Brownian motion. $(\Omega, \mathbb{F}, \mathcal{F}_t, \mathbb{P})$ represents a complete filtered probability space generated by the Brownian motion $W(\cdot)$ and the Poisson random measure $N(\cdot, \cdot)$. We define $\mathbb{F} = (\mathcal{F}_t : t \in [0, T])$ as a right-continuous, $\mathbb{P}$-completed filtration and assume that the Brownian motion and the Poisson random measure are independent of each other and adapted to $\mathbb{F}$.

We follow a controlled jump-diffusion model with a constant delay term $\delta > 0$, which is one of the most general representations of such systems and is introduced in [20] as follows:

$$\begin{aligned} dX(t) =& b(t, X(t), Y(t), A(t), u(t))dt + \sigma(t, X(t), Y(t), A(t), u(t))dW(t) \\ &+ \int_{\mathbb{R}_0} \eta(t, X(t), Y(t), A(t), u(t), z)\tilde{N}(dt, dz) \\ X(t) =& \theta(t), \qquad t \in [-\delta, 0], \end{aligned} \tag{1}$$

where for $t \in [0, T]$,

$$Y(t) = X(t - \delta), \qquad A(t) = \int_{t-\delta}^{t} e^{-\rho(t-r)} X(r)dr.$$

The coefficient functions of the model are defined as:

$$\begin{aligned} b &: [0, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathcal{U} \to \mathbb{R}, \\ \sigma &: [0, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathcal{U} \to \mathbb{R}, \\ \eta &: [0, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathcal{U} \times \mathbb{R}_0 \to \mathbb{R}, \end{aligned}$$

and generally, in financial applications, b, σ, and η represent appreciation rate, volatility and jump size of a risky asset correspondingly.

Moreover, for example, while Brownian motion $W(\cdot)$ catches little shocks in the price process of an asset, the Poisson random measure $N(\cdot, \cdot)$ captures the jumps of that process, which occur as a consequence of abrupt changes, sudden news or big sell/buy orders in the financial markets.

In this model, we observe the *memory* component in the dynamics of the system as $Y(\cdot)$ and $A(\cdot)$ terms. Note that for the systems described by SDDEs, rather than an initial value, we need an initial path. $\theta(\cdot)$ represents the initial path and is a continuous, deterministic function. Here, $\rho \geq 0$ is a constant averaging parameter.

We assume that $\mathcal{U}$ is a non-empty subset of $\mathbb{R}$ and represents a set of admissible control values $u(t)$, $t \in [0, T]$. We define an *admissible control process* $u(\cdot)$ as a $\mathcal{U}$-valued, $\mathcal{F}_t$-measurable and càdlàg process such that the Equation (1) has a unique solution $X(\cdot) \in L^2(\xi \times \mathbb{P})$, where ξ represents the Lebesgue measure on $[0, T]$. Let $\mathcal{A}$ denote a family of admissible control processes (for more detail, see [20]).

Moreover, we assume that

$$E\left[\int_0^T |u(t)|^2 \, dt\right] < \infty.$$

For all $u \in \mathcal{A}$, let us define the objective criterion in the classical sense (for a broad survey of the Stochastic Optimal Control theory, see [21,22,32] and references therein) as follows:

$$\begin{aligned} J(u) &= J(x, y, a, u) \\ &= E\left[\int_0^T f(t, X(t), Y(t), A(t), u(t))dt + g(X(T))\right], \end{aligned} \tag{2}$$

where $f : [0,T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathcal{U} \to \mathbb{R}$ represents the running gain and $g : \mathbb{R} \to \mathbb{R}$ corresponds to the terminal gain of the control task. Here, we assume that f and g are $\mathcal{C}^1$-functions with respect to x, y, a, u such that for all $x_i = x, y, a, u$,

$$E\Bigg[\int_0^T \Bigg(|f(t, X(t), Y(t), A(t), u(t))| + \left|\frac{\partial f}{\partial x_i}(t, X(t), Y(t), A(t), u(t))\right|^2 \Bigg) dt + |g(X(T))| + |g_x(X(T))|^2 \Bigg] < \infty.$$

Hence, in a classical *unconstrained* stochastic control problem, our goal is to find the optimal control $u^* \in \mathcal{A}$ such that

$$J(u^*) = \sup_{u \in \mathcal{A}} J(u). \tag{3}$$

On the other hand, in this work, we formulate the constraints inspired by Theorem 11.3.1 of [19] but with completely different constraints. In this theorem, the author presents an approach for the stochastic control tasks with a condition at the terminal time $T > 0$ for a diffusion process. Later, [4] gave an application of this theorem and [28] extended this theorem to the stochastic differential games with regimes.

Furthermore, [5] stated a version of Theorem 11.3.1 of [19] with constraint types (5) and (6) for a jump-diffusion process. These constraints describe deterministic and stochastic Lagrange multipliers, correspondingly and are different than the terminal conditions given in Theorem 11.3.1 of [19]. But the authors do not investigate the Lagrange multipliers however they claimed that their existence is a crucial condition to apply the proved theorems, see Theorem 5.2 and 5.4 of [5]. In our work, we study a stochastic control problem for a jump-diffusion process with the *memory* and the *constraints* defined with (5) and (6). Hence, our work extends the theorems of [5] to a *delayed model*. Moreover, we develop an application for which the corresponding Lagrange multiplier exists. In that sense, we should underline that our work is the first work that completes the desired task with the constraints (5)-(6) and also, by inserting a delay term, we study a larger model.

We do not prefer to define many technical conditions over b, σ, η in this section. In Section 2, we will develop two fundamental theorems to approach stochastic control problems with the constraints (5) and (6). These can be solved by both Stochastic Maximum Principle (**SMP**) and Dynamic Programming Principle (**DPP**). Thus, the technical assumptions have to be determined specifically depending on the preferred method. We will highlight them in Section 3, while we are studying an optimal consumption problem.

This article is organized as follows: In Section 2, we introduce the mathematical formulation of our constrained stochastic control problem and demonstrate the corresponding theorems in a Lagrangian environment. Section 3 is devoted to developing a financial application, which formulates the optimal consumption process of a company with memory. The final section gives a conclusion.

2. Reformulation of the Control Task within the Context of Constraints

In this section, we develop two theorems which describe the optimal control process and investigate the corresponding Lagrange multipliers for a time-delayed stochastic control system.

Firstly, let us state the value function of the constrained control problem:

$$\phi(x, y, a) = J(u^*) = \sup_{u \in \Theta} J(x, y, a, u). \tag{4}$$

Here, $J(\cdot)$ is defined by Equation (2) and the supremum is taken over Θ of all admissible controls $u : \mathbb{R} \to \mathcal{U} \subset \mathbb{R}$ such that

$$E\left[\int_0^T M(t, X(t), Y(t), A(t), u(t)) dt\right] = 0, \tag{5}$$

or

$$\int_0^T M(t, X(t), Y(t), A(t), u(t)) dt = 0 \ a.s.. \tag{6}$$

$M : [0,T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathcal{U} \to \mathbb{R}$ is a C^1 function with respect to x, y, and a such that for $x_i = x, y, a, u$:

$$E\left[\int_0^T \left(|M(t, X(t), Y(t), A(t), u(t))| + \left|\frac{\partial M}{\partial x_i}(t, X(t), Y(t), A(t), u(t))\right|^2 \right) dt\right] < \infty.$$

Here, we study two types of constraints: The constraint type (5) represents a real valued Lagrange multiplier and the type (6) discovers a stochastic one.

Thus, we should specify the set of stochastic Lagrange multipliers as in [27]:

$$\Delta = \left\{ \lambda : \Omega \to \mathbb{R} | \lambda \text{ is } \mathcal{F}_T - \text{measurable and } E[|\lambda|] < \infty \right\}.$$

Now, by observing the Equation (4) and the constraints (5) and (6), let us present the *unconstrained* stochastic control problem in the following way:

$$\begin{aligned} \phi^{\lambda}(x,y,a) = & \sup_{u \in \Theta} J(x,y,a,u) \\ = & \sup_{u \in \Theta} E^{x,y,a} \left[\int_0^T f(t, X(t), Y(t), A(t), u(t)) dt + g(X^u(T)) \right. \\ & \left. + \lambda \int_0^T M(t, X(t), Y(t), A(t), u(t)) dt \right], \end{aligned} \tag{7}$$

subject to the system (1).

First, we will prove the following theorem corresponding to the type (6):

Theorem 1. *Assume that for all $\lambda \in \Delta_1 \subset \Delta$, we can develop $\phi^{\lambda}(x,y,a)$ and the optimal control process $u^{*,\lambda}$, which solves the unconstrained stochastic control problem (7) subject to the system (1). Moreover, assume that there exists $\lambda_0 \in \Delta_1$, such that*

$$\int_0^T M(t, X_t^{u^{*,\lambda_0}}, Y_t^{u^{*,\lambda_0}}, A_t^{u^{*,\lambda_0}}, u_t^{*,\lambda_0}) dt = 0, \qquad a.s. \tag{8}$$

Then, $\phi(x,y,a) = \phi^{\lambda_0}(x,y,a)$ is obtained and $u^* = u^{*,\lambda_0}$ solves the constrained stochastic control problem (3) subject to (1) and (6).

Proof. The first inequality appears by definition of the optimal value function as follows:

$$\begin{aligned} \phi^{\lambda}(x,y,a) = & \ J(x,y,a,u^{*,\lambda}) \\ & = E^{x,y,a} \left[\int_0^T f(t, X_t^{u^{*,\lambda}}, Y_t^{u^{*,\lambda}}, A_t^{u^{*,\lambda}}, u^{*,\lambda}) dt \right. \\ & \quad \left. + \lambda \int_0^T M(t, X_t^{u^{*,\lambda}}, Y_t^{u^{*,\lambda}}, A_t^{u^{*,\lambda}}, u_t^{*,\lambda}) dt + g(X_T^{u^{*,\lambda}}) \right] \\ & \geq J(x,y,a,u^{\lambda}) \\ & = E^{x,y,a} \left[\int_0^T f(t, X_t^{u^{\lambda}}, Y_t^{u^{\lambda}}, A_t^{u^{\lambda}}, u^{\lambda}) dt \right. \\ & \quad \left. + \lambda \int_0^T M(X_t^{u^{\lambda}}, Y_t^{u^{\lambda}}, A_t^{u^{\lambda}}, u_t^{\lambda}) dt + g(X_T^{u^{\lambda}}) \right]. \end{aligned} \tag{9}$$

In particular, if $\lambda = \lambda_0$ exists and since $u_1 \in \Theta$ is feasible in the constrained control problem (3), then by (8):

$$\int_0^T M(t, X_t^{u^{*,\lambda_0}}, Y_t^{u^{*,\lambda_0}}, A_t^{u^{*,\lambda_0}}, u_t^{*,\lambda_0}) dt = \int_0^T M(X_t^{u^{\lambda}}, Y_t^{u^{\lambda}}, A_t^{u^{\lambda}}, u_t^{\lambda}) dt = 0 \tag{10}$$

Therefore, by (9) and (10):

$$\phi^{\lambda_0}(x,y,a) = J(u^{*,\lambda_0}) = J(x,y,a,u^{*,\lambda_0}) \geq J(x,y,a,u) = J(u),$$

for all $u \in \Theta$. Note that $u^{*,\lambda_0} \in \Theta$ and this completes the proof. □

The following theorem can be proved similarly for the constraint type (5).

Theorem 2. *Assume that for all $\lambda \in K \subset \mathbb{R}$, we can determine $\phi^{\lambda}(x,y,a)$ and the optimal control process $u^{*,\lambda}$ solving the unconstrained stochastic control problem (7) subject to (1). Furthermore, assume that there exists $\lambda_0 \in K$ such that*

$$E\left[\int_0^T M(t, X_t^{u^{*,\lambda_0}}, Y_t^{u^{*,\lambda_0}}, A_t^{u^{*,\lambda_0}}, u_t^{*,\lambda_0})dt\right] = 0.$$

Then, $\phi(x,y,a) = \phi^{\lambda_0}(x,y,a)$ and $u^* = u^{*,\lambda_0}$ solves the constrained stochastic control problem (3) subject to the model (1) and the constraint (5).

Remark 1. *Theorem 1 and 2 can be applied to a wide range of stochastic control problems by both **SMP** and **DPP** as long as it is possible to determine the corresponding Lagrange multipliers. If we prefer to apply DPP, we should be careful about Markov property. SDDEs provide a more realistic environment to interact but we loose Markov property. Moreover, since we have an initial path instead of an initial value for the system (1), our problem creates the corresponding partial differential equations so-called Hamilton-Jacobi-Bellman equations in an infinite dimensional space. Hence, a direct application of DPP is not mathematically possible (more details to handle such problems by DPP in [7, 8, 14] and reference therein).*

Remark 2. *To utilize SMP, we do not need any Markovian assumption different than DPP. Hence, in this work, we will combine the method described in Theorem 2 of our paper with Theorem 3.1 and Theorem 4.1 of [20] to find the optimal consumption process by SMP.*

Remark 3. *Our work is inspired from Theorem 11.3.1 of [19], but we should highlight that the constraint of Theorem 11.3.1 of [19] is defined at terminal time T as:*

$$E[M(X_T^u)] = 0, \tag{11}$$

which is completely different than our constraints (5)-(6). We put a condition over running gain component rather than the terminal gain. Moreover, we can see similar constraints in [5] but both [19] and [5] do not include memory impact.

Remark 4. *In [30], we studied memory impact within the framework of Lagrange multipliers similar to Equation 11, which is a different type of constraint as we stated in Remark 3. Furthermore, in [30], we focused on a dividend policy application in a regime-switching environment with a different control formulation. Our present work and [30] share a similar philosophy with completely different constraints and financial formulations.*

Now, let us present an application of Theorem 2 in finance.

3. Application to Finance

In this section, we will develop the formulation of an optimal consumption process that corresponds to the wealth process of a company with memory. This process evolves according to a time-delayed jump-diffusion model. The dynamics of the model carry past values of the wealth process in the form of $Y(t) = X(t-\delta)$, $t \in [0,T]$, where $\delta > 0$ is a constant. Our purpose is to develop a more realistic consumption policy, which depends on the information about the historical performance of the company as well.

$\mu(\cdot)$ is a deterministic function and represents the appreciation rate of the company. Furthermore, we suppose that $\sigma(t)$ and $\eta(t,z)$, $t \in [0,T]$, are given bounded, square integrable and adapted processes. $\mathcal{U}$ is a non-empty, closed and convex subset of $\mathbb{R}$. In this section, our problem formulation justifies the technical assumptions provided in [20] thus, we are allowed to apply Theorem 3.1 and Theorem 4.1 of that article.

The consumption process is a càdlàg, $\mathcal{F}_t$-adapted control process, which satisfies:

$$E\left[\int_0^T |c(t)|^2\, dt\right] < \infty.$$

Let us state the wealth process $X(t) = X^c(t)$, which is a special form of Equation (1) as follows:

$$
\begin{aligned}
dX(t) = &\Big(X(t-\delta)\mu(t) - c(t)\Big)dt + X(t-\delta)\Big(\sigma(t)dW(t) \\
&+ \int_{\mathbb{R}_0} \eta(t,z)\tilde{N}(dt,dz)\Big),\ t \in [0,T], \\
X(t) = &\ \theta(t), \quad t \in [-\delta, 0],
\end{aligned} \tag{12}
$$

where $\theta(\cdot)$ is a given nonnegative, deterministic and continuous function.

We assume that the company wants to maximize its wealth despite a quadratic running loss by balancing it corresponding to a constraint of linear running gain, which is described in terms of the control process. Moreover, the company aims to reach a level of a constant K times the terminal time $T > 0$. So we assume that the company takes into account time restrictions as well. We will develop and highlight the conditions over K at the end of our computations. Hence, our goal is to find the optimal consumption process $c^*(\cdot)$ by solving:

$$
\begin{aligned}
J(c^*) &= \sup_{c \in \Theta} J(c) \\
&= \sup_{c \in \Theta} E\Big[\int_0^T \alpha(t)c^2(t)dt + \beta X(T)\Big]
\end{aligned}
$$

subject to the system (12) and to the constraint:

$$
E\Big[\int_0^T \gamma(t)c(t)dt\Big] = TK,\ \ K \in \mathbb{R}, \tag{13}
$$

where $\alpha(\cdot) < 0$ and $\gamma(\cdot)$ are deterministic functions and $\beta \in \mathbb{R}$.

Now we can develop the Lagrangian form of this stochastic control problem as follows:

$$
\begin{aligned}
J(c^*) = &\ \sup_{c \in \Theta} J(c) \\
= &\sup_{c \in \Theta} E\Big[\int_0^T \alpha(t)c^2(t)dt + \lambda \int_0^T (\gamma(t)c(t) - K)dt + \beta X(T)\Big],
\end{aligned} \tag{14}
$$

for which we aim to find $c^* = c^{\lambda,*}$ and the real-valued Lagrange multiplier $\lambda = \lambda^0$ described in Theorem 2.

Since we apply SMP to solve the problem (14), first, we define the Hamiltonian corresponding to the wealth process (12):

$$
\begin{aligned}
H(t,x,y,a,c,p,q,r(\cdot)) = &\ \alpha(t)c^2 + \lambda(\gamma(t)c - K) + (\mu(t)y - c)p + y\sigma(t)q \\
&+ y\int_{\mathbb{R}_0} \eta(t,z)r(t,z)\nu(dz).
\end{aligned} \tag{15}
$$

Note that it is clearly seen that Hamiltonian H is a concave function of x, y, a and c, hence the concavity condition over H is satisfied, see Theorem 3.1 of [20] is justified.

Furthermore, we should present the corresponding Anticipated Backward Stochastic Differential Equation (**Anticipated BSDE**) and solve it for unknown $p(t),\ q(t)$, and $r(t,z)$.

For $t \in [0,T]$, let us introduce:

$$
\begin{aligned}
dp(t) = &-E\Big[\Big(\mu(t+\delta)p(t+\delta) + \sigma(t+\delta)q(t+\delta) \\
&+ \int_{\mathbb{R}_0} \eta(t+\delta,z)r(t+\delta,z)\nu(dz)\Big)\mathbf{1}_{[0,T-\delta]}(t)|\mathcal{F}_t\Big]dt \\
&+ q(t)dW(t) + \int_{\mathbb{R}_0} r(t,z)\tilde{N}(dt,dz)
\end{aligned} \tag{16}
$$

$$
p(T) = \beta. \tag{17}
$$

We call *Anticipated* to this type of BSDEs since as seen in $\mu, \sigma, \eta, p(\cdot), q(\cdot)$, and $r(\cdot,\cdot)$, the terms involve time-advanced values in the form of $t+\delta$ for $t \in [0,T]$. This type of BSDEs was first introduced and developed by Peng and Yang, see [23]. For technical definitions of the Hamiltonian (15) and the System

(16)-(17), please see Apendix 4 or Section 2 in [20]. Furthermore, see [25, 28] for the formulation of Anticipated BSDEs and their relation with SDDEs via different models.

We follow the technique described in [20] to find the solution for $p(\cdot), q(\cdot)$, and $r(\cdot,\cdot)$, which will be computed inductively in the following way:

Step 1: For $t \in [T-\delta, T]$, the corresponding adjoint equation becomes:

$$dp(t) = q(t)dW(t) + \int_{\mathbb{R}_0} r(t,z)\tilde{N}(dt,dz),$$
$$p(T) = \beta,$$

for which we have the solution:

$$p(t) = E[p(T)|\mathcal{F}_t] = \beta, \qquad t \in [T-\delta, T].$$

By martingale representation theorem, since the Lagrange multiplier is a real value, we choose $q = r = w = 0$. Hence, the Anticipated BSDE gets the form:

$$dp(t) = -\mu(t+\delta)p(t+\delta)\mathbf{1}_{[0,T-\delta]}(t)dt, \quad t \le T,$$
$$p(t) = \beta, \qquad t \in [T-\delta, T].$$

Step 2: We define:

$$h(t) = p(T-t), \qquad t \in [0, T]. \tag{18}$$

That way, we get a deterministic delay equation:

$$\begin{aligned} dh(t) &= -dp(T-t) = \mu(T-t+\delta)p(T-t+\delta)dt \\ &= \mu(T-t+\delta)h(t-\delta)dt, \qquad t \in [\delta, T], \\ h(t) &= p(T-t) = \beta, \quad t \in [0,\delta]. \end{aligned}$$

For such equations, again, we have an approach of solving inductively. Since we can compute $h(t)$ on $[(j-1)\delta, j\delta]$, we obtain:

$$\begin{aligned} h(t) &= h(j\delta) + \int_{j\delta}^{t} h'(s)ds \\ &= h(j\delta) + \int_{j\delta}^{t} \mu(T-s+\delta)h(s-\delta)ds \end{aligned} \tag{19}$$

for $t \in [j\delta, (j+1)\delta]$, $j = 1, 2, \ldots$.

Now, we should maximize the Hamiltonian (15) with respect to c to get:

$$c^*(t) = \frac{1}{2}\alpha(t)(p(t) - \lambda\gamma(t)), \ t \in [0, T]. \tag{20}$$

As a consequence of the nature of constrained stochastic control problems, we should compute the value of Lagrange multiplier λ^0 to use Theorem 2 properly.

Solving stochastic delay equations require special approaches different than usual stochastic differential equations. By the Equation (20), the wealth process becomes:

$$\begin{aligned} dX(t) &= (X(t-\delta)\mu(t) - \frac{1}{2}\alpha(t)(p(t) - \lambda\gamma(t)))dt + X(t-\delta)(\sigma(t)dW(t) \\ &\quad + \int_{\mathbb{R}_0} \eta(t,z)\tilde{N}(dt,dz)), \ t \in [0,T], \\ X(t) &= \theta(t), \quad t \in [-\delta, 0]. \end{aligned} \tag{21}$$

We know that the SDDE (21) can be solved by successive Itô integrations over steps of length δ (see Section 1, page 7 in [18]). Specifically, we assume that terminal time $T = 2\delta$. This assumption is just for the sake of simplicity and does not pretend to show the complete methodology of applying the technique. Thus, the total duration that we study is the interval of $[-\delta, 2\delta]$.

First, for $t \in [0, T]$, let us define:

$$dL(t) = \sigma(t)dW(t) + \int_{\mathbb{R}_0} \eta(t,z)\tilde{N}(dt,dz).$$

By also observing (18) and (19), we provide the following open form of the solution process:

$$X(t)=\theta(t),\ \textbf{if}\quad -\delta\le t\le 0,$$

$$X(t)=\theta(0)+\int_0^t\Big(\theta(s-\delta)\mu(s)-\frac{1}{2}\alpha(s)(h(T-s)-\lambda\gamma(s))\Big)ds$$
$$+\int_0^t\theta(s-\delta)dL(s)\qquad \textbf{if}\quad 0\le t\le\delta,$$

$$X(t)=X(\delta)+\int_\delta^t\Big(\Big\{\theta(0)+\int_0^{v-\delta}\Big(\theta(s-\delta)\mu(s)-\frac{1}{2}\alpha(s)(h(T-s)-\lambda\gamma(s))\Big)ds$$
$$+\int_0^{v-\delta}\theta(s-\delta)dL(s)\Big\}\mu(v)-\frac{1}{2}\alpha(v)(h(T-v)-\lambda\gamma(v))\Big)dv$$
$$+\int_\delta^t\Big\{\theta(0)+\int_0^{v-\delta}\Big(\theta(s-\delta)\mu(s)-\frac{1}{2}\alpha(s)(h(T-s)-\lambda\gamma(s))\Big)ds$$
$$+\int_0^{v-\delta}\theta(s-\delta)dL(s)\Big\}dL(v)\qquad \textbf{if}\quad \delta\le t\le 2\delta=T.$$

Now, the values of $h(T-t)$, $t\in[0,T]$ at the above integrals can be determined by following the boundary values of the integrals and their relation with t. Remember that $T=2\delta$.
Then, by (19)

$$\text{if},\quad 0\le s\le t\le\delta,\quad \text{then},\quad \delta\le T-s\le 2\delta,$$
$$h(2\delta-s)=h(\delta)+\int_\delta^{2\delta-s}\mu(3\delta-u)h(u-\delta)du,$$
$$h(2\delta-s)=\beta\Big(1+\int_\delta^{2\delta-s}\mu(3\delta-u)du\Big).$$

Moreover,

$$\text{if},\quad 0\le s\le v-\delta\quad\text{and}\quad \delta\le v\le t\le 2\delta,\ \text{then},\quad 0\le v-\delta\le t-\delta\le\delta,$$
$$\text{so},\quad 0\le s\le\delta,\quad \text{then}\quad \delta\le 2\delta-s\le 2\delta,\quad \text{then, by (19)},$$
$$h(2\delta-s)=\beta\Big(1+\int_\delta^{2\delta-s}\mu(3\delta-u)du\Big).$$

Finally,

$$\text{if},\quad \delta\le v\le t\le 2\delta,\quad \text{then},\quad 0\le 2\delta-v\le\delta,\quad \text{then, by 18}\quad h(2\delta-v)=\beta.$$

Firstly, we change the value of $h(\cdot)$ according to the relevant intervals in the above solution processes and integrate the Equation (21) from 0 to 2δ by following the above δ-length description of $X(\cdot)$. Then, we apply expectation to both sides of the Equation (21).

Now, let us introduce the following terms:

$$A=\int_0^{2\delta}\alpha(s)\gamma(s)ds+E\Big[\int_\delta^{2\delta}\Big(\int_0^{v-\delta}\alpha(s)\gamma(s)ds\Big)\Big\{\mu(v)dv+dL(v)\Big\}\Big]$$

and

$$B=\theta(0)-E\Big[X(2\delta)\Big]+\int_0^\delta\theta(s-\delta)\mu(s)ds$$
$$-\frac{1}{2}\beta\int_0^\delta\alpha(s)\Big(1+\int_\delta^{2\delta-s}\mu(3\delta-u)du\Big)ds+E\Big[\int_0^\delta\theta(s-\delta)dL(s)\Big]$$
$$+\int_\delta^{2\delta}\theta(0)\mu(v)dv+\int_\delta^{2\delta}\Big(\int_0^{v-\delta}\theta(s-\delta)\mu(s)ds\Big)\mu(v)dv$$
$$-\beta\int_\delta^{2\delta}\Big(\int_0^{v-\delta}\alpha(s)\Big(1+\int_\delta^{2\delta-s}\mu(3\delta-u)du\Big)ds\Big)\frac{1}{2}\mu(v)dv$$

$$+ E\Big[\int_\delta^{2\delta}\int_0^{v-\delta}\theta(s-\delta)dL(s)\Big)\mu(v)dv\Big]$$
$$-\frac{1}{2}\beta\int_\delta^{2\delta}\alpha(v)dv + E\Big[\int_\delta^{2\delta}\Big\{\theta(0)+\int_0^{v-\delta}\theta(s-\delta)\mu(s)ds$$
$$-\frac{1}{2}\beta\int_0^{v-\delta}\alpha(s)\Big(1+\int_\delta^{2\delta-s}\mu(3\delta-u)du\Big)ds + \int_0^{v-\delta}\theta(s-\delta)dL(s)\Big\}dL(v)\Big].$$

Then, we get:

$$\lambda = \frac{2B}{A} \qquad \text{on condition that} \quad A \neq 0. \tag{22}$$

Now, by (18) and (19), let us make some observations about the constraint (13):

$$\begin{aligned} E\Big[\int_0^T \gamma(t)c(t)dt\Big] =& \frac{1}{2}\int_0^\delta \gamma(t)\alpha(t)\Big[\beta\Big(1+\int_\delta^{2\delta-t}\mu(3\delta-u)du\Big)-\lambda\gamma(t)\Big]dt \\ &+ \frac{1}{2}\int_\delta^{2\delta}\gamma(t)\alpha(t)(\beta-\lambda\gamma(t))dt \\ =& 2\delta K. \end{aligned}$$

Then, let us utilize the above equality to clarify λ and define the following terms:

$$D = \frac{\beta}{2}\Big[\int_0^{2\delta}\gamma(t)\alpha(t)dt + \int_0^\delta \gamma(t)\alpha(t)\Big(\int_\delta^{2\delta-t}\mu(3\delta-u)du\Big)dt\Big] - 2\delta K,$$

and

$$C = \int_0^{2\delta}\gamma^2(t)\alpha(t)dt.$$

Then, we obtain:

$$\lambda = \frac{2D}{C} \qquad \text{on condition that} \quad C \neq 0. \tag{23}$$

Finally, by observations (22)-(23), we conclude that in order to use Theorem 2, we have to specify the K value in Equation (5) carefully such that

$$\frac{D}{C} = \frac{B}{A}.$$

By this final result, we determined explicitly the control process $c^*(\cdot)$, the Lagrange multiplier λ^0 and consequently, the solution for $p(\cdot)$ corresponding to the Anticipated BSDE (16)-(17), and all the technical assumptions required.

4. Conclusion and Future Work

In this work, we studied a constrained stochastic control problem and investigated the impact of delay term on Lagrange multipliers. We proved two theorems for two different types of constraints and gave an application in finance for the case of a real-valued Lagrange multiplier. We focused on the wealth process of a company, which evolves according to a jump-diffusion model with historical values in its dynamics. We observed that however the Theorems 1 and 2 are applicable for a wide range of control tasks by both SMP and DPP, determining the Lagrange multipliers remains as a challenge. It is not always easy to compute these parameters. Furthermore, the step of formulating these multipliers can not be ignored because the provided theorems are enforceable on the condition that there exists a Lagrange multiplier for which the constraint is justified. Despite this challenge, to the best of our knowledge, our article presents the first results for a delayed system with constraints in running gain of the control task and computes the corresponding Lagrange multiplier exactly. In our financial application, we clearly present the technical differences for solving a delayed SDE and a usual one by applying Itô's formula recursively.

Furthermore, since stochastic control theory is a discipline of sequential decision-making, we may encounter some challenges from the side of model selection. The decision maker may believe that her model is perfect. But in reality, generally, this is not the case. Especially, in finance, model misidentification can cause high financial losses. At this point, robust control designs different control or decision rules performing fare well across alternative models [11, 34]. Especially, in stochastic games, we handle model uncertainty in a relative entropy context as a penalty term [2, 6, 9]. It is known that Hansen and

Sargent [10] used a Lagrange multiplier theorem to convert the entropy constraint onto a penalty on perturbations from the model. Therefore, we would like to underline the potential of our work towards robust stochastic control and stochastic games.

Risk minimization and worst-case scenarios have significant value in quantitative finance and insurance since each action with uncertainty carries a potential for loss that cannot be underestimated. Therefore, as a further study, we aim to focus on the relation between Lagrange multipliers and robust control. These structures can be approached from the side of relative entropy as well as from the sides of Var and CVar concepts, see [9, 15, 16]. Furthermore, within the wide scope of *risk management*, Lagrange multipliers can be handled via computational methods such as deep learning and deep reinforcement learning, see [24, 31].

On the other hand, we strongly believe that however delay systems are demanding and challenging, they will be highlighted within the context of other hot fields such as Deep Learning. However, the aim of our research article is to provide theoretical and technical approaches, in [29], we present a collection of novel aspects within the intersection of computer science and stochastic optimal control under the memory component.

Declaration of Competing Interests The author declares no conflict of interest.

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Appendix

In order to apply SMP, we have to define corresponding Hamiltonian for a delayed system as follows: $H : [0,T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathcal{U} \times \mathbb{R} \times \mathbb{R} \times \mathcal{R} \to \mathbb{R}$,

$$\begin{aligned} H(t,x,y,a,u,p,q,r) =& f(t,x,y,a,u) + b(t,x,y,a,u)p + \sigma(t,x,y,a,u)q \\ &+ \int_{\mathbb{R}_0} \eta(t,x,y,a,u,z) r(t,z)\nu(dz) \end{aligned} \tag{24}$$

where $\mathcal{R}$ denotes the set of all functions
$r : [0,T] \times \mathbb{R}_0 \to \mathbb{R}$, for which the integral in (24) converges.

Associated to H, the adjoint, unknown and adapted processes $(p(t) \in \mathbb{R} : t \in [0,T])$, $(q(t) \in \mathbb{R} : t \in [0,T])$, and $(r(t,z) \in \mathcal{R} : t \in [0,T], z \in \mathbb{R}_0)$ are described by the following Anticipated BSDE with jumps:

$$\begin{aligned} dp(t) &= E[\mu(t)|\mathcal{F}_t]dt + q(t)dW(t) + \int_{\mathbb{R}_0} r(t,z)\tilde{N}(dt,dz) \\ p(T) &= g_x(X(T)), \end{aligned}$$

where

$$\begin{aligned} \mu(t) :=& -\frac{\partial H}{\partial x}(t, X(t), Y(t), A(t), u(t), p(t), q(t), r(t,\cdot)) \\ &- \frac{\partial H}{\partial y}(t+\delta, X(t+\delta), Y(t+\delta), A(t+\delta), u(t+\delta), p(t+\delta), q(t+\delta), r(t+\delta,\cdot)) \\ &\times \mathbf{1}_{[0,T-\delta]}(t) - e^{\rho t}\Big(\int_t^{t+\delta} \frac{\partial H}{\partial a}(s, X(s), Y(s), A(s), u(s), p(s), q(s), r(s,\cdot)) \end{aligned}$$

$$\times e^{-\rho s}\mathbf{1}_{[0,T]}(s)ds\Bigg). \tag{25}$$

As seen in $\mu(t)$, we have the future values of $X(s), u(s), p(s), q(s)$, and $r(s,\cdot)$ for $s \leq t+\delta$ in Equation (25), hence we call *Anticipated* to this type of BSDEs.

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Modified fibonomial graphs

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Abstract. In this paper, a new graph type similar to binomial graphs is constructed using fibonomial coefficients. The spectrum of this new graph was obtained, the energy of the graph and the sum of the Laplacian eigenvalues are calculated. In addition, the connectivity feature of the graph is examined and the properties of the vertices forming the graph are revealed.

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Keywords. Graph, adjacency matrix, characteristic polynomial, eigenvalue, eigenvector, graph energy, spectrum, Laplacian matrix, Laplacian eigenvalues, closed walk.

1. Introduction

Binomial graphs were introduced by Peter R. Christopher and John W. Kennedy [1]. For $n \geq 0$, the binomial graph B_n has vertex set V_n and edge set E_n, where $|V_n| = 2^n$ and $\{v_i, v_j\} \in E_n$ if $\binom{i+j}{j} \equiv 1 \,(\mathrm{mod}\, 2)$. The eigenvalues and eigenvectors of the adjancency matrices of the binomial graphs give important information about the closed walks in the binomial graphs. It was shown that the sum of the degrees is of the vertices in the binomial graph B_n is

$$\sum_{j=0}^{2^n-1} \deg(v_j) = \deg(v_0) + \sum_{j=1}^{2^n-1} \deg(v_j) = (2^n+1) + \sum_{j=0}^{n-1} \binom{n}{j} 2^j = 1 + \sum_{j=0}^{n} \binom{n}{j} 2^j = 3^n + 1.$$

Thus, the number of edges in binomial graph B_n is $\frac{1}{2}(3^n+1)$ [1].

The Fibonacci sequence, which has been widely studied, also holds an important place in graph theory. The Fibonacci sequence is defined by the $F_{n+1} = F_n + F_{n-1}$ relation, where $F_1 = 1$ and $F_2 = 1$. Similar to binomial coefficients, fibonomial coefficients are obtained with the help of Fibonacci numbers. Fibonomial coefficients are obtained as follows, for $1 \leq j \leq m$,

$$\begin{bmatrix} m \\ j \end{bmatrix}_F = \frac{F_m \; F_{m-1} \; \ldots \; F_{m-j+1}}{F_1 \; F_2 \; \ldots \; F_j}$$

where $\begin{bmatrix} m \\ 0 \end{bmatrix}_F = 1$ and $\begin{bmatrix} m \\ j \end{bmatrix}_F = 0$ for $m < j$.

If F_m in the numerator of the fraction is replaced by $F_j F_{m-j+1} + F_{j-1} F_{m-j}$, the following equation is obtained

$$\begin{bmatrix} m \\ j \end{bmatrix}_F = F_{m-j+1} \begin{bmatrix} m-1 \\ j-1 \end{bmatrix}_F + F_{j+1} \begin{bmatrix} m-1 \\ j \end{bmatrix}_F$$

[2].

In our work, we frequently use the Kronecker product operation of matrices to obtain the adjacency matrices. Kronecker product make it easier for us to calculate the eigenvalues of graphs. Let A be a

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$m \times n$ matrix and B be a $r \times s$ matrix. Then, the Kronecker product between A and B is the block matrix $A \otimes B = [a_{i,j}B]$, where $A \otimes B$ is a $mr \times ns$ matrix [6].

The $n \times n$ adjacency matrix $A\left(G\right)$ of a graph G with n vertices is a binary matrix. The non-diagonal entry $a_{i,j}$ of the adjacency matrix A is 1 if the i and j vertices are adjacent, and 0 otherwise. Also the loop at vertex v_i in the graph corresponds to the diagonal element a_{ii} in the adjacency matrix.

The eigenvalues of a graph are important in determining the algebraic properties of the graph. Additionally, the sequence of these eigenvalues gives the spectrum of the graph. We denote the spectrum of graph G by $\Lambda\left(G\right)$. Let $\lambda_1, \lambda_2, ..., \lambda_n$ denote the eigenvalues of the graph G, so that the spectrum of G is

$$\Lambda\left(G\right) = \{\lambda_1, \lambda_2, ..., \lambda_n\}$$

[7]. Let G be a graph with n vertices. The energy of the graph G is obtained by summing up all absolute values of the eigenvalues of the graph. Also, the energy of the graph G is denoted by $E(G)$. Gutman [3], who has worked on the energy of graphs for many years, gave the definition of the energy of non-simple graphs as follows:

$$E\left(G\right) = \sum_{i=1}^{n} \left|\lambda_i - \frac{S}{n}\right|,$$

where

$$S = tr\left(A(G)\right) = \sum_{i=1}^{n} \lambda_i.$$

We also want to talk about the sum of the Laplacian eigenvalues of a graph. For this we must first define the Laplacian matrix. If $D(G) = diag(d_1, d_2, ..., d_n)$ is the diagonal matrix associated to G, where $d_i = \deg(v_i)$ for all $i = 1, 2, 3, ..., n$ the matrix $L(G) = D(G) - A(G)$ is called the Laplacian matrix and its spectrum is called the Laplacian spectrum of the graph G [9]. If $\mu_1, \mu_2, ..., \mu_n$ denote the eigenvalues of $L\left(G\right)$, then the sum of the Laplacian eigenvalues of G is defined as

$$S(G) = tr\left(L\left(G\right)\right) = \sum_{i=1}^{n} \mu_i$$

[4]. Similar to the modified binomial coefficients defined by Shiro Ando [5], the following relationship is used to define modified fibonomial coefficients,

$$\left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle_F = F_{n+1} \cdot \begin{bmatrix} n \\ k \end{bmatrix}_F,$$

where

$$\left\langle \begin{matrix} n \\ 0 \end{matrix} \right\rangle_F = \left\langle \begin{matrix} n \\ n \end{matrix} \right\rangle_F = F_{n+1},$$

F_{n+1} is $n+1$ th Fibonacci number and

$$\left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle_F = \left\langle \begin{matrix} n \\ n-k \end{matrix} \right\rangle_F.$$

Modified fibonomial coefficients can be represented by a triangle similar to Pascal's triangle in Figure 1. Using the modified fibonomial coefficients we obtained, we construct a new type of graph and call it modified fibonomial graphs.

1
1 1
2 2 2
3 6 6 3
5 15 30 15 5
8 40 120 120 40 8
13 104 520 780 520 104 13
21 273 2184 5460 5460 2184 273 21

Figure 1. Modified fibonomial triangle

2. Modified Fibonomial Graphs

For each nonnegative integer n, we define the modified fibonomial graph $\mathcal{F}_n$ to has vertex set $V_n = \{v_j : j = 0, 1, 2, ..., 3.2^n - 1\}$ and the edge set

$$E_n = \left\{ \{v_i, v_j\} : \left\langle \begin{matrix} i+j \\ j \end{matrix} \right\rangle_F \equiv 1 \, (\mathrm{mod}\, 2) \right\}.$$

The adjacency matrix of $\mathcal{F}_n$ is defined as $A(\mathcal{F}_n) = [a_{i,j}]$, where

$$a_{i,j} \equiv \left\langle \begin{matrix} i+j \\ j \end{matrix} \right\rangle_F (\mathrm{mod}\, 2).$$

Since $\left\langle \begin{matrix} 0 \\ 0 \end{matrix} \right\rangle_F = 1$, the modified fibonomical graph F_n has only one loop at vertex v_0. $|V_n| = 3.2^n$ and for $j = 0, 1, ..., n$, F_n has $\binom{n+1}{j}$ vertices of degree 2^j and the vertex v_0 of degree $2^{n+1} + 1$. From here $|E_n| = \frac{1}{2} \left(3^{n+1} + 1\right)$.

$\mathcal{F}_0$, $\mathcal{F}_1$ and $\mathcal{F}_2$ modified fibonomial graphs and their adjacency matrices are given as follows.

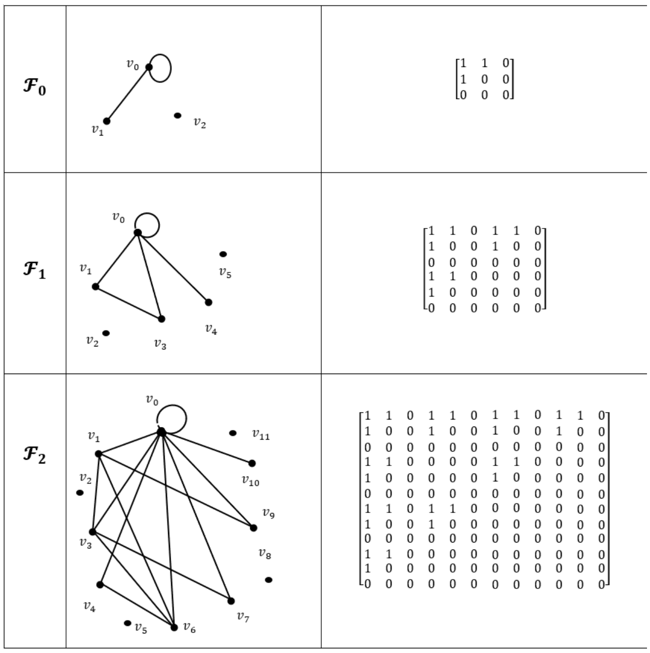

Figure 2. $\mathcal{F}_0$, $\mathcal{F}_1$ and $\mathcal{F}_2$ modified fibonomial graphs and their adjacency matrices

Here, by the Kronecker product, we obtain the adjacency matrices of the modified fibonomial graphs as follows. If we take $\mathcal{F} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$, then

$$A(F_0) = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$A(\mathcal{F}_1) = \begin{bmatrix} 1 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} = \mathcal{F} \otimes A(\mathcal{F}_0)$$

and

$$A\left(\mathcal{F}_2\right)=\left[\begin{array}{cccccccccccc} 1&1&0&1&1&0&1&1&0&1&1&0\\ 1&0&0&1&0&0&1&0&0&1&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 1&1&0&0&0&0&1&1&0&0&0&0\\ 1&0&0&0&0&0&1&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 1&1&0&1&1&0&0&0&0&0&0&0\\ 1&0&0&1&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 1&1&0&0&0&0&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0 \end{array}\right]=\mathcal{F}\otimes A\left(\mathcal{F}_1\right)$$

From here, for each $n \geq 1$, the adjacency matrix of the modified fibonomial graph $\mathcal{F}_n$ is

$$A\left(\mathcal{F}_n\right)=\left[\begin{array}{cc} A\left(\mathcal{F}_{n-1}\right) & A\left(\mathcal{F}_{n-1}\right)\\ A\left(\mathcal{F}_{n-1}\right) & 0 \end{array}\right]=\mathcal{F}\otimes A\left(\mathcal{F}_{n-1}\right).$$

3. Eigenvalues of Modified Fibonomial Graphs

The eigenvalues of a graph, which provide information about the spectral structure of the graph, are calculated with the help of the adjacency matrix of the graph. The eigenvalues of the adjacency matrix of a graph are defined as the eigenvalues of the graph and so they are just the roots of the equation $\wp(\mathcal{F}_n;x)=0$. Since $A\left(G\right)$ is symmetric, its eigenvalues are all real. We denote them by $\lambda_1,\lambda_2,...,\lambda_n$ and the set of all eigenvalues is the spectrum of G, denoted by $Spec(G)$ [8].

Lemma 1. *[1] Let matrix A be an $n\times n$ square matrix with eigenvalues $\lambda_1,\lambda_2,...,\lambda_n$ and matrix B be an $m\times m$ square matrix with eigenvalues $\mu_1,\mu_2,...,\mu_m$ then the eigenvalues of the $nm\times nm$ matrix $A\otimes B$ are $\lambda_i\mu_j$ for $1\leq i\leq n$ and $1\leq j\leq m$.*

Theorem 1. *Let $\varphi=\frac{1+\sqrt{5}}{2}$. For each nonnegative integer n, the modified fibonomial graphs $\mathcal{F}_n$ has 3.2^n eigenvalues. More precisely, it has eigenvalue 0 with multiplicity 2^n and $(-1)^j.\varphi^{n+1-2j}$ with multiplicity $\binom{n+1}{j}$ for each $j=0,1,2,...,n+1$. Then we can write the spectrum of the modified fibonomial graph $\mathcal{F}_n$ as follows,*

$$\Lambda(\mathcal{F}_n)=\{0^{2^n},\left((-1)^j.\varphi^{n+1-2j}\right)^{\binom{n+1}{j}}:j=0,1,2,...,n+1\}.$$

Proof. Since $\mathcal{F}=\left[\begin{array}{cc}1&1\\1&0\end{array}\right]$, the characteristic polynomial of $\mathcal{F}$ is

$$\wp(\mathcal{F};x)=x^2-x-1$$

so that $\Lambda(\mathcal{F})=\{\varphi,-\varphi^{-1}\}$. Additionally, since

$$A(\mathcal{F}_0)=\left[\begin{array}{ccc}1&1&0\\1&0&0\\0&0&0\end{array}\right],$$

the characteristic polynomial of $\mathcal{F}_0$ is

$$\wp(\mathcal{F}_0;x)=-x^3+x^2+1$$

so that $\Lambda(\mathcal{F}_0)=\{0,\varphi,-\varphi^{-1}\}$. Since

$$A(\mathcal{F}_1)=\mathcal{F}\otimes A(\mathcal{F}_0),\ \Lambda(\mathcal{F}_1)=\{0,0,-1,-1,\varphi^2,\varphi^{-2}\}$$

is obtained. We know that since $A(\mathcal{F}_n)=\mathcal{F}\otimes A(\mathcal{F}_{n-1})$, it is easy to see that the spectrum of $\mathcal{F}_n$ can be written in the form

$$\Lambda(\mathcal{F}_n)=\{0^{2^n},\left((-1)^j.\varphi^{n+1-2j}\right)^{\binom{n+1}{j}}:j=0,1,2,...,n+1\}.$$

□

4. Energy of the Modified Fibonomial Graphs

In this section, we calculate the energy of the non-simple the modified fibonomial graph of $\mathcal{F}_n$.

Theorem 2. *Let $\mathcal{F}_n$ be a modified fibonomial graph with 3.2^n vertices. If $E(\mathcal{F}_n)$ denotes the energy of the modified fibonomial graph $\mathcal{F}_n$, then*

$$E(\mathcal{F}_n) = \sum_{i=1}^{3.2^n} \left| \lambda_i - \frac{1}{3.2^n} \right|.$$

Proof. Gutman introduced the energy of a graph as

$$E(G) = \begin{cases} \sum_{i=1}^{n} |\lambda_i| & \text{if } G \text{ is a simple graph,} \\ \sum_{i=1}^{n} \left|\lambda_i - \frac{S}{n}\right| & \text{otherwise} \end{cases}$$

in his previous studies. Here we used the abbreviation $S = tr\,(A(G)) = \sum_{i=1}^{n} \lambda_i$. Summing up all the eigenvalues of $\mathcal{F}_0$ yields $S = 1$.

Since $A(\mathcal{F}_n) = \mathcal{F} \otimes A(\mathcal{F}_{n-1})$ and trace of the Kronecker product of matrices is the product of traces, sum of eigenvalues of $\mathcal{F}_n$ is

$$\sum_{i=1}^{3.2^n} \lambda_i = \sum_{j=0}^{n} (\frac{\varphi^2 - 1}{\varphi}).(-1)^j.\varphi^{n-2j} = 1.$$

□

Remark 1. *Since $\lim_{n\to\infty} \frac{1}{3.2^n} = 0$, the energy of $\mathcal{F}_n$ tends to*

$$\sum_{i=1}^{3.2^n} |\lambda_i|$$

when $n \to \infty$. Since all non-zero eigenvalues are in the form $(-1)^j.\varphi^{n+1-2j}$ with multiplicity $\binom{n+1}{j}$, we have

$$E(F_n) \cong \sum_{i=1}^{3.2^n} |\lambda_i| = \sum_{k=1}^{n+1} \left| \binom{n+1}{k} \varphi^{n+1-k} \varphi^{-k} \right| = \left(\varphi + \varphi^{-1}\right)^{n+1} = (\sqrt{5})^{n+1}$$

5. Connectivity of the Modified Fibonomial Graphs

Theorem 3. *In the modified fibonomial graphs, the vertex v_{3k+2} is isolated for all $k = 0, 1, 2, ..., 2^n - 1$.*

Proof. The adjancency matrix of $\mathcal{F}_n$ is defined as $A(\mathcal{F}_n) = [a_{i,j}]$, where

$$a_{i,j} = \left\langle \begin{matrix} i+j \\ j \end{matrix} \right\rangle_F (\mathrm{mod}\, 2)$$

Here we must show that

$$a_{3k+2,l} \equiv 0 \; for \; k = 0, 1, 2, ..., 2^n - 1 \; and \; l = 0, 1, 2, ..., 3.2^n - 1.$$

Since

$$\left\langle \begin{matrix} 3k+l+2 \\ l \end{matrix} \right\rangle_F = F_{3(k+1)+l} \cdot \left[\begin{matrix} 3k+l+2 \\ l \end{matrix} \right]_F,$$

we can write

$$\begin{aligned} \left\langle \begin{matrix} 3k+l+2 \\ l \end{matrix} \right\rangle_F =& (F_{l+1} \cdot F_{3(k+1)} + F_l \cdot F_{3k+2}) \left[\begin{matrix} 3k+l+2 \\ l \end{matrix} \right]_F \\ =& (F_{l+1} \cdot F_{3(k+1)} + F_l \cdot F_{3k+2}) \frac{F_{3k+l+2} \cdot F_{3k+l+1} \dots F_{3(k+1)}}{F_1 \cdot F_2 \dots F_l} \\ =& (F_{l+1}.F_{3(k+1)}) \frac{F_{3k+l+2} \cdot F_{3k+l+1} \dots F_{3(k+1)}}{F_1 \cdot F_2 \dots F_l} \\ & + F_l \cdot F_{3k+2} \frac{F_{3k+l+2} \cdot F_{3k+l+1} \dots F_{3(k+1)}}{F_1 \cdot F_2 \dots F_l} \end{aligned}$$

Since $F_{3(k+1)}$ is even, we have

$$\left\langle \begin{matrix} 3k+l+2 \\ l \end{matrix} \right\rangle_F \equiv 0 (\mathrm{mod}\, 2)$$

$$a_{3k,l} \equiv 1 \ and \ a_{3k+1,l} \equiv 1 \ for \ k = 0, 1, 2, ..., 2^n - 1 \ and \ l = 0, 1, 2, ..., 3.2^n - 1$$

$$\begin{aligned}
\left\langle \begin{matrix} 3k+l \\ l \end{matrix} \right\rangle_F &= F_{3k+l+1} \ . \ \left[\begin{matrix} 3k+l \\ l \end{matrix} \right]_F \\
&= (F_{3k} \ . \ F_{l+2} + F_{3k+1} \ . \ F_{l+1}) \left[\begin{matrix} 3k+l \\ l \end{matrix} \right]_F \\
&= (F_{3k} \ . \ F_{l+2} + F_{3k+1} \ . \ F_{l+1}) \frac{F_{3k+l} \ . \ F_{3k+l-1} \ ... \ F_{3k+1}}{F_1 \ . \ F_2 \ ... \ F_l} \\
&= F_{3k} F_{l+2} . \frac{F_{3k+l} \ . \ F_{3k+l-1} \ ... \ F_{3k+1}}{F_1 \ . \ F_2 \ ... \ F_l} + F_{3k+1} \ . \ F_{l+1} . \frac{F_{3k+l} \ . \ F_{3k+l-1} \ ... \ F_{3k+1}}{F_1 \ . \ F_2 \ ... \ F_l} \\
&\equiv 1 \, (\mathrm{mod}\, 2)
\end{aligned}$$

$$\begin{aligned}
\left\langle \begin{matrix} 3k+l+1 \\ l \end{matrix} \right\rangle_F &= F_{3k+l+2} \ . \ \left[\begin{matrix} 3k+l+1 \\ l \end{matrix} \right]_F \\
&= (F_{3k} \ . \ F_{l+3} + F_{3k+1} \ . \ F_{l+2}) \left[\begin{matrix} 3k+l+1 \\ l \end{matrix} \right]_F \\
&= (F_{3k} \ . \ F_{l+3} + F_{3k+1} \ . \ F_{l+2}) \frac{F_{3k+l+1} \ . \ F_{3k+l} \ ... \ F_{3k+2}}{F_1 \ . \ F_2 \ ... \ F_l} \\
&= F_{3k} F_{l+3} . \frac{F_{3k+l+1} \ . \ F_{3k+l} \ ... \ F_{3k+2}}{F_1 \ . \ F_2 \ ... \ F_l} + F_{3k+1} \ . \ F_{l+2} . \frac{F_{3k+l+1} \ . \ F_{3k+l+} \ ... \ F_{3k+2}}{F_1 \ . \ F_2 \ ... \ F_l} \\
&\equiv 1 \, (\mathrm{mod}\, 2)
\end{aligned}$$

Consequently the vertex v_{3k+2} is the isolated vertex for all $k = 0, 1, ..., 2^n - 1$. □

Theorem 4. *The modified fibonomial graph $\mathcal{F}_n$ contains exactly one loop, the one on the vertex v_0.*

Proof. It is clear that $a_{0,0} = 1$.

Now let's take $i \neq 0$. We must show that $a_{i,i} = 0$.

$$\left\langle \begin{matrix} 2i \\ i \end{matrix} \right\rangle_F = F_{2i+1} \left[\begin{matrix} 2i \\ i \end{matrix} \right]_F$$

where similar to the central binomial coefficient, $\left[\begin{matrix} 2n \\ n \end{matrix} \right]_F$ can be taken as the central fibonomial coefficient. Since the central fibonomical coefficients are always even,

$$\left\langle \begin{matrix} 2i \\ i \end{matrix} \right\rangle_F \equiv 0 \, (\mathrm{mod}\, 2) \, .$$

□

Corollary 1. *Let $\mathcal{F}_n$ be a modified fibonomial graphs with 3.2^n vertices, for each nonnegative integer n. The number of isolated vertices in this graph is 2^n.*

Proof. In the modified fibonomial graphs, the vertex v_{3k+2} is isolated for all $k = 0, 1, 2, ..., 2^n - 1$.

Then vertices $v_2, v_5, ..., v_{3.2^n - 1}$ are isolated.

Then the number of isolated vertices is obtained as 2^n. □

Corollary 2. *In the modified fibonomial graphs, the degree of vertex v_0 is $2^{n+1} + 1$.*

Proof. In the modified fibonomial graphs, the vertex v_0 is connected to

$$2^{n+1}-1$$

different vertices. The vertex v_0 is connected to every vertice except isolated vertices.

For all

$$k=0,1,2,...,2^n-1,\ a_{0,3k}=1 \ and\ a_{0,3k+1}=1$$

In that case,

$$\deg(v_0)=3.2^n-1-2^n+2=2^{n+1}+1.$$

□

6. Sum of the Laplacian Eigenvalues of the Modified Fibonomial Graphs

To obtain the sum of Laplacian eigenvalues of modified fibonomial graphs, we first examined the Laplacian matrices of modified fibonomial graphs. The Laplacian matrices of the modified fibonomial graphs $\mathcal{F}_0$, $\mathcal{F}_1$ and $\mathcal{F}_2$ are given below.

$$L(F_0)=\begin{bmatrix} 2 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$L(F_1)=\begin{bmatrix} 4 & -1 & 0 & -1 & -1 & 0 \\ -1 & 2 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 2 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$L(F_2)=\begin{bmatrix} 8 & -1 & 0 & -1 & -1 & 0 & -1 & -1 & 0 & -1 & -1 & 0 \\ -1 & 4 & 0 & -1 & -1 & 0 & -1 & -1 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 4 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 2 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & -1 & -1 & 0 & 4 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & -1 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Theorem 5. *Let L_n be an $n\times n$ Laplacian matrix with eigenvalues $\mu_1,\mu_2,...,\mu_n$ of the modified fibonomial graph $\mathcal{F}_n$. The sum of the Laplacian eigenvalues of the modified fibonomial graph $\mathcal{F}_n$ is*

$$S(\mathcal{F}_n)=\sum_{i=1}^{n}\mu_i=3^{n+1}$$

Proof. We know that the sum of the eigenvalues of the Laplacian matrix equal to sum of the diagonal etries of the Laplacian matrix.

$L(F_n)=D(F_n)-A(F_n)$ and the $a_{0,0}$ diagonal entry of the adjacency matrix is 1 and all other diagonal entry are 0. Also, $\mathcal{F}_n$ has $\binom{n+1}{j}$ vertices of degree 2^j, for $j=0,1,2,...,n$ and the vertex v_0 of degree $2^{n+1}+1$. Thus,

$$S(\mathcal{F}_n)=tr\left(L\left(F_n\right)\right)=\sum_{j=0}^{n+1}\binom{n+1}{j}2^j=3^{n+1}$$

□

7. Conclusion

In this article, we first examined the spectrum of the newly defined modified fibonomial graphs and tried to determine their relationships with similar graphs. Similar studies can be done on new graphs obtained using generalized Fibonacci numbers.

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