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Two New General Integral Results Related to the Hilbert Integral Inequality

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Abstract

In this article, we generalize two integral results from the literature. The first result concerns a flexible double integral inequality, considering a specific form for the integrated function and a double integral as a lower or upper bound. Several examples are discussed, as well as some of its indirect connections with the Hilbert integral inequality. The second result also gives a double integral inequality, but with the product of the square root of simple integrals, following the spirit of the Hilbert integral inequality. Several theoretical and numerical examples are discussed. Both of our results have the property of being dependent on several adjustable functions and parameters, thus offering a wide range of applications.

1. Introduction

Historically, integral inequalities have attracted attention in almost all areas of mathematics. Some of the most famous are the Cauchy-Schwarz integral inequality, the Jensen integral inequality, the Hölder integral inequality, the Minkowski integral inequality, the Hardy-Littlewood-Sobolev integral inequality, the Hilbert integral inequality, the Sobolev integral inequality, the Gagliardo-Nirenberg integral inequality, the Poincaré integral inequality, the Grönwall integral inequality, the Young integral inequality, the logarithmic Sobolev integral inequality, the Chebyshev integral inequality, the Steffensen integral inequality and the Grüss integral inequality. They are widely used in fields as diverse as calculus, functional analysis, probability theory, numerical analysis, mathematical physics, and partial differential equations. For a comprehensive introduction to these inequalities, see [1, 2, 3, 4, 5]. In recent research, the study of integral inequalities has taken on considerable importance. For some contemporary references, i.e., in 2024 at the time of writing, see, for example, [6, 7, 8, 9].

In this article, we focus on the framework of the Hilbert integral inequality. It plays an important role in applications involving double integrals, where certain types of product and ratio functions are present. This is particularly the case in analysis, approximation theory, probability theory and partial differential equations. Mathematically, the Hilbert integral inequality is expressed as follows:

$$\int_0^{+\infty}\int_0^{+\infty}\frac{f(x)g(y)}{x+y}dxdy \leq \pi\sqrt{\int_0^{+\infty} f^2(x)dx}\sqrt{\int_0^{+\infty} g^2(x)dx}, \tag{1.1}$$

where $f,g:[0,+\infty)\to[0,+\infty)$ are quadratic integrable functions. The upper bound is thus of the form constant multiplied by the L_2 norms of f and g. The constant π is optimal and cannot be improved, as shown in [1, 5]. Note that, in the special case $g=f$, the Hilbert integral inequality reduces to

$$\int_0^{+\infty}\int_0^{+\infty}\frac{f(x)f(y)}{x+y}dxdy \leq \pi\int_0^{+\infty} f^2(x)dx. \tag{1.2}$$

This simplified version will have some focus for the purposes of this article. The importance of the Hilbert integral inequality has led to numerous variants and extensions, with applications in both pure and applied mathematics. These variants have been the subject of extensive research, as can be seen in the studies in [10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24]. In

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addition, the survey in [25] provides a comprehensive overview of these developments, including the various techniques used to improve or generalize the inequality. It also gives examples of how these inequalities are used in different contexts. For some recent references on the topic, i.e., in 2024 at the time of writing, see [26, 27, 28, 29].

In this article, we demonstrate two general integral inequalities that extend some results established in [17], which in turn extend those in [16]. In particular, the following formula is discussed in [17, Lemma 2.1]:

$$\int_0^{+\infty}\int_0^{+\infty}\frac{f(x)f(y)}{h(x,y)}F(x,y)dxdy=\int_0^{+\infty}\int_0^{+\infty}\frac{f(x)f(y)}{h(x,y)}dxdy, \tag{1.3}$$

where $h:[0,+\infty)^2\mapsto[0,+\infty)$ is a symmetric bivariate function, and $F:[0,+\infty)^2\mapsto\mathbb{R}$ is a bivariate function depending on an intermediate univariate function $k:[0,+\infty)\mapsto[0,+\infty)$, of the form $F(x,y)=1+k(x)-k(y)$ (or, without loss of generality, $F(x,y)=1+k(y)-k(x)$). In the first result of this article, we show how to extend Equation (1.3), with a more general function F depending on two intermediate univariate functions. In particular, inequalities come naturally depending on the monotonicity of these functions. It is worth noting that the lower or upper bound obtained is expressed as a double integral, similar to the right term in Equation (1.3).

In the second result, still based on our extended function F, we generalize [17, Part of the proof of Theorem 3.1] by demonstrating a new variant of the Hilbert integral inequality. It is innovative in its use of two adjustable univariate functions and parameters. More specifically, we demonstrate an integral inequality of the following form:

$$\int_0^{+\infty}\int_0^{+\infty}\frac{f(x)f(y)}{h(x,y)}F(x,y)dxdy\le\sqrt{\int_0^{+\infty}p(x)f^2(x)dx}\sqrt{\int_0^{+\infty}q(x)f^2(x)dx},$$

where $p,q:[0,+\infty)\mapsto[0,+\infty)$ are explicitly determined. In a sense, it extends the special Hilbert integral inequality presented in Equation (1.2); when F reduces to the constant 1, it is expected that p and q reduce to the constant π. Some consequences of this result are discussed and a new precise variant of the Hilbert integral inequality is established.

The rest of the article is divided into three sections: Section 2 presents the first general integral inequality result, including the detailed proofs and some examples. A connection with the Hilbert integral inequality is also made. Section 3 deals with the second general integral inequality result. It also gives detailed proof, discussion and some examples. Section 4 contains a conclusion.

2. First general integral inequality result

The proposition below is our first general result on integral inequalities, which significantly extends the scope of [17, Lemma 2.1]. A double integral is obtained as a lower or upper bound.

Proposition 2.1. *Let $f:[0,+\infty)\mapsto[0,+\infty)$ and $u,v:[0,+\infty)\mapsto\mathbb{R}$ be univariate functions, and $h:[0,+\infty)^2\mapsto[0,+\infty)$ be a bivariate function. We suppose that h is symmetric, i.e., $h(x,y)=h(y,x)$ for any $(x,y)\in[0,+\infty)^2$. Based on u and v, let $F:[0,+\infty)^2\mapsto\mathbb{R}$ be the bivariate function defined by*

$$F(x,y)=1+u(x)[v(x)-v(y)].$$

Then, distinguishing four cases of assumptions on u and v, the results below hold.

Case 1: *If v is constant and u is an arbitrary function, we have*

$$\int_0^{+\infty}\int_0^{+\infty}\frac{f(x)f(y)}{h(x,y)}F(x,y)dxdy=\int_0^{+\infty}\int_0^{+\infty}\frac{f(x)f(y)}{h(x,y)}dxdy,$$

provided that the integrals involved converge.

Case 2: *If u is constant and v is an arbitrary function, we have*

$$\int_0^{+\infty}\int_0^{+\infty}\frac{f(x)f(y)}{h(x,y)}F(x,y)dxdy=\int_0^{+\infty}\int_0^{+\infty}\frac{f(x)f(y)}{h(x,y)}dxdy,$$

provided that the integrals involved converge and

$$\int_0^{+\infty}\int_0^{+\infty}\frac{f(x)f(y)}{h(x,y)}|v(y)|dydx<+\infty. \tag{2.1}$$

Case 3: *If u and v are both increasing, or both decreasing, we have*

$$\int_0^{+\infty}\int_0^{+\infty}\frac{f(x)f(y)}{h(x,y)}F(x,y)dxdy \geq \int_0^{+\infty}\int_0^{+\infty}\frac{f(x)f(y)}{h(x,y)}dxdy,$$

provided that the integrals involved converge and

$$\int_0^{+\infty}\int_0^{+\infty}\frac{f(x)f(y)}{h(x,y)}|u(y)||v(x)-v(y)|dydx < +\infty. \tag{2.2}$$

Case 4: *If u is increasing and v is decreasing, or if u is decreasing and v is increasing, we have*

$$\int_0^{+\infty}\int_0^{+\infty}\frac{f(x)f(y)}{h(x,y)}F(x,y)dxdy \leq \int_0^{+\infty}\int_0^{+\infty}\frac{f(x)f(y)}{h(x,y)}dxdy,$$

provided that the integrals involved converge and the assumption in Equation (2.2) holds.

Proof. Let us prove the four cases, one after the other.

Case 1: If v is constant and u is an arbitrary function, we have $F(x,y)=1$, so that

$$\int_0^{+\infty}\int_0^{+\infty}\frac{f(x)f(y)}{h(x,y)}F(x,y)dxdy = \int_0^{+\infty}\int_0^{+\infty}\frac{f(x)f(y)}{h(x,y)}dxdy.$$

Case 2: If u is constant, say $u(x)=c$ for any $x\in[0,+\infty)$ and v is an arbitrary function, we have

$$\begin{aligned}\int_0^{+\infty}\int_0^{+\infty}\frac{f(x)f(y)}{h(x,y)}F(x,y)dxdy &= \int_0^{+\infty}\int_0^{+\infty}\frac{f(x)f(y)}{h(x,y)}\{1+c[v(x)-v(y)]\}dxdy\\ &= \int_0^{+\infty}\int_0^{+\infty}\frac{f(x)f(y)}{h(x,y)}dxdy\\ &+ c\left[\int_0^{+\infty}\int_0^{+\infty}\frac{f(x)f(y)}{h(x,y)}v(x)dxdy - \int_0^{+\infty}\int_0^{+\infty}\frac{f(x)f(y)}{h(x,y)}v(y)dxdy\right].\end{aligned}$$

Let us focus on the last integral term (without the constant factor). Changing the notations x and y, using the symmetry of h and the Fubini theorem thanks to Equation (2.1) to justify the change of the order of integration, it can be expressed as

$$\begin{aligned}\int_0^{+\infty}\int_0^{+\infty}\frac{f(x)f(y)}{h(x,y)}v(y)dxdy &= \int_0^{+\infty}\int_0^{+\infty}\frac{f(y)f(x)}{h(y,x)}v(x)dydx\\ &= \int_0^{+\infty}\int_0^{+\infty}\frac{f(x)f(y)}{h(x,y)}v(x)dxdy.\end{aligned}$$

So we have

$$\begin{aligned}\int_0^{+\infty}\int_0^{+\infty}\frac{f(x)f(y)}{h(x,y)}F(x,y)dxdy &= \int_0^{+\infty}\int_0^{+\infty}\frac{f(x)f(y)}{h(x,y)}dxdy + c\times 0\\ &= \int_0^{+\infty}\int_0^{+\infty}\frac{f(x)f(y)}{h(x,y)}dxdy.\end{aligned}$$

The desired result is obtained.

Case 3: Let us now suppose that u and v are both increasing, or both decreasing. The following decomposition holds:

$$\begin{aligned}\int_0^{+\infty}\int_0^{+\infty}\frac{f(x)f(y)}{h(x,y)}F(x,y)dxdy &= \int_0^{+\infty}\int_0^{+\infty}\frac{f(x)f(y)}{h(x,y)}\{1+u(x)[v(x)-v(y)]\}dxdy\\ &= \int_0^{+\infty}\int_0^{+\infty}\frac{f(x)f(y)}{h(x,y)}dxdy + \frac{1}{2}\int_0^{+\infty}\int_0^{+\infty}\frac{f(x)f(y)}{h(x,y)}u(x)[v(x)-v(y)]dxdy\\ &+ \frac{1}{2}\int_0^{+\infty}\int_0^{+\infty}\frac{f(x)f(y)}{h(x,y)}u(x)[v(x)-v(y)]dxdy.\end{aligned}$$

Let us focus on the last integral term (without the constant factor). Changing the notations x and y, using the symmetry of h and the Fubini theorem thanks to Equation (2.2) to justify the change of the order of integration, it can be expressed as

$$\begin{aligned}\int_0^{+\infty}\int_0^{+\infty}\frac{f(x)f(y)}{h(x,y)}u(x)[v(x)-v(y)]dxdy &= \int_0^{+\infty}\int_0^{+\infty}\frac{f(y)f(x)}{h(y,x)}u(y)[v(y)-v(x)]dydx \\ &= -\int_0^{+\infty}\int_0^{+\infty}\frac{f(x)f(y)}{h(x,y)}u(y)[v(x)-v(y)]dxdy.\end{aligned}$$

We therefore have

$$\begin{aligned}\int_0^{+\infty}\int_0^{+\infty}\frac{f(x)f(y)}{h(x,y)}F(x,y)dxdy &= \int_0^{+\infty}\int_0^{+\infty}\frac{f(x)f(y)}{h(x,y)}dxdy \\ &+ \frac{1}{2}\int_0^{+\infty}\int_0^{+\infty}\frac{f(x)f(y)}{h(x,y)}[u(x)-u(y)][v(x)-v(y)]dxdy. \qquad (2.3)\end{aligned}$$

If u and v are both increasing, for any $x\geq y$, we have $u(x)\geq u(y)$ and $v(x)\geq v(y)$, implying that $[u(x)-u(y)][v(x)-v(y)]\geq 0$, and, for any $y\geq x$, we have $u(y)\geq u(x)$ and $v(y)\geq v(x)$, still implying that $[u(x)-u(y)][v(x)-v(y)]\geq 0$.

On the other hand, if u and v are both decreasing, for any $x\geq y$, we have $u(y)\geq u(x)$ and $v(y)\geq v(x)$, implying again that $[u(x)-u(y)][v(x)-v(y)]\geq 0$, and, for any $y\geq x$, we have $u(x)\geq u(y)$ and $v(x)\geq v(y)$, still implying that $[u(x)-u(y)][v(x)-v(y)]\geq 0$. Since f and h are positive, we have

$$\frac{1}{2}\int_0^{+\infty}\int_0^{+\infty}\frac{f(x)f(y)}{h(x,y)}[u(x)-u(y)][v(x)-v(y)]dxdy\geq 0.$$

This and Equation (2.3) imply that

$$\int_0^{+\infty}\int_0^{+\infty}\frac{f(x)f(y)}{h(x,y)}F(x,y)dxdy\geq\int_0^{+\infty}\int_0^{+\infty}\frac{f(x)f(y)}{h(x,y)}dxdy.$$

The desired result is obtained.

Case 4: Let us now suppose that u is increasing and v is decreasing, or u is decreasing and v is increasing. Applying Equation (2.3), we still can write

$$\begin{aligned}\int_0^{+\infty}\int_0^{+\infty}\frac{f(x)f(y)}{h(x,y)}F(x,y)dxdy &= \int_0^{+\infty}\int_0^{+\infty}\frac{f(x)f(y)}{h(x,y)}dxdy \\ &+ \frac{1}{2}\int_0^{+\infty}\int_0^{+\infty}\frac{f(x)f(y)}{h(x,y)}[u(x)-u(y)][v(x)-v(y)]dxdy. \qquad (2.4)\end{aligned}$$

If u is increasing and v is decreasing, for any $x\geq y$, we have $u(x)\geq u(y)$ and $v(y)\geq v(x)$, implying that $[u(x)-u(y)][v(x)-v(y)]\leq 0$, and, for any $y\geq x$, we have $u(y)\geq u(x)$ and $v(x)\geq v(y)$, still implying that $[u(x)-u(y)][v(x)-v(y)]\leq 0$.

On the other hand, if u is decreasing and v is increasing, for any $x\geq y$, we have $u(y)\geq u(x)$ and $v(x)\geq v(y)$, implying again that $[u(x)-u(y)][v(x)-v(y)]\leq 0$, and, for any $y\geq x$, we have $u(x)\geq u(y)$ and $v(y)\geq v(x)$, still implying that $[u(x)-u(y)][v(x)-v(y)]\leq 0$. Since f and h are positive, we have

$$\frac{1}{2}\int_0^{+\infty}\int_0^{+\infty}\frac{f(x)f(y)}{h(x,y)}[u(x)-u(y)][v(x)-v(y)]dxdy\leq 0.$$

The combination of this with Equation (2.4) gives

$$\int_0^{+\infty}\int_0^{+\infty}\frac{f(x)f(y)}{h(x,y)}F(x,y)dxdy\leq\int_0^{+\infty}\int_0^{+\infty}\frac{f(x)f(y)}{h(x,y)}dxdy.$$

The desired result is obtained.

This concludes the proof of Proposition 2.1. □

The interest of Proposition 2.1 is that the double integral under consideration is very general in form, and lower and upper bounds can be derived under a simple monotonicity analysis of only two intermediate functions. However, if there is no monotonicity (or no constant constant function), it cannot be applied.

Taking u as the constant equal to 1 (and v arbitrary), Case 2 in Proposition 2.1 becomes [17, Lemma 2.1], recalled in Equation (1.3) (with $k = v$). It also extends [16, Lemma 1.3], which considers u as the constant equal to 1 and $v(x) = 1/(1+x)$. The other cases give new perspectives of applications.

As a direct consequence, if u is increasing and v is decreasing, or if u is decreasing and v is increasing, applying Case 4 of Proposition 2.1 with $h(x,y) = x+y$ and the Hilbert integral inequality, we get

$$\int_0^{+\infty}\int_0^{+\infty}\frac{f(x)f(y)}{x+y}F(x,y)dxdy \leq \int_0^{+\infty}\int_0^{+\infty}\frac{f(x)f(y)}{x+y}dxdy \leq \pi\int_0^{+\infty} f^2(x)dx,$$

provided that the integrals involved converge and the assumption in Equation (2.2) holds. Some numerical examples are now proposed to illustrate the results in Proposition 2.1, starting with Case 2. We take $f(x) = e^{-x}$ and $h(x,y) = x+y$, so that

$$\int_0^{+\infty}\int_0^{+\infty}\frac{f(x)f(y)}{h(x,y)}dxdy = \int_0^{+\infty}\int_0^{+\infty}\frac{e^{-x-y}}{x+y}dxdy = 1,$$

to work with a manageable benchmark.

Illustration of Case 2: Taking $u(x) = 4$ and $v(x) = \log(x)$, so that u is constant and v is an arbitrary selected function, we have $F(x,y) = 1+u(x)[v(x)-v(y)] = 1+4\log(x/y)$, and

$$\begin{aligned}\int_0^{+\infty}\int_0^{+\infty}\frac{f(x)f(y)}{h(x,y)}F(x,y)dxdy &= \int_0^{+\infty}\int_0^{+\infty}\frac{e^{-x-y}}{x+y}\left[1+4\log\left(\frac{x}{y}\right)\right]dxdy \\ &= 1 \\ &= \int_0^{+\infty}\int_0^{+\infty}\frac{e^{-x-y}}{x+y}dxdy \\ &= \int_0^{+\infty}\int_0^{+\infty}\frac{f(x)f(y)}{h(x,y)}dxdy.\end{aligned}$$

As expected, the desired double integrals are equal.

Illustration of Case 3: Taking $u(x) = x$ and $v(x) = x^2$, so that u and v are both increasing, we have $F(x,y) = 1+u(x)[v(x)-v(y)] = 1+x(x^2-y^2)$, and

$$\begin{aligned}\int_0^{+\infty}\int_0^{+\infty}\frac{f(x)f(y)}{h(x,y)}F(x,y)dxdy &= \int_0^{+\infty}\int_0^{+\infty}\frac{e^{-x-y}}{x+y}\left[1+x(x^2-y^2)\right]dxdy \\ &= 2 \\ &\geq 1 \\ &= \int_0^{+\infty}\int_0^{+\infty}\frac{e^{-x-y}}{x+y}dxdy \\ &= \int_0^{+\infty}\int_0^{+\infty}\frac{f(x)f(y)}{h(x,y)}dxdy.\end{aligned}$$

As another example for this case, taking $u(x) = e^{-x}$ and $v(x) = 1/(1+x)$, so that u and v are both decreasing, we have $F(x,y) = 1+u(x)[v(x)-v(y)] = 1+e^{-x}[1/(1+x)-1/(1+y)]$, and

$$\begin{aligned}\int_0^{+\infty}\int_0^{+\infty}\frac{f(x)f(y)}{h(x,y)}F(x,y)dxdy &= \int_0^{+\infty}\int_0^{+\infty}\frac{e^{-x-y}}{x+y}\left[1+e^{-x}\left(\frac{1}{1+x}-\frac{1}{1+y}\right)\right]dxdy \\ &\approx 1.03772 \\ &\geq 1 \\ &= \int_0^{+\infty}\int_0^{+\infty}\frac{e^{-x-y}}{x+y}dxdy = \int_0^{+\infty}\int_0^{+\infty}\frac{f(x)f(y)}{h(x,y)}dxdy.\end{aligned}$$

As expected, the desired inequality is obtained for both examples.

Illustration of Case 4: Taking $u(x)=\sqrt{x}$ and $v(x)=1/(1+x^2)$, so that u is increasing and v is decreasing, we have $F(x,y)=1+u(x)[v(x)-v(y)]=1+\sqrt{x}[1/(1+x^2)-1/(1+y^2)]$, and

$$\begin{aligned}\int_0^{+\infty}\int_0^{+\infty}\frac{f(x)f(y)}{h(x,y)}F(x,y)dxdy &= \int_0^{+\infty}\int_0^{+\infty}\frac{e^{-x-y}}{x+y}\left[1+\sqrt{x}\left(\frac{1}{1+x^2}-\frac{1}{1+y^2}\right)\right]dxdy\\ &\approx 0.931516\\ &\leq 1\\ &= \int_0^{+\infty}\int_0^{+\infty}\frac{e^{-x-y}}{x+y}dxdy\\ &= \int_0^{+\infty}\int_0^{+\infty}\frac{f(x)f(y)}{h(x,y)}dxdy.\end{aligned}$$

As another example for this case, taking $u(x)=e^{-x^2}$ and $v(x)=\log(x)$, so that u is decreasing and v is increasing, we have $F(x,y)=1+u(x)[v(x)-v(y)]=1+e^{-x^2}\log(x/y)$, and

$$\begin{aligned}\int_0^{+\infty}\int_0^{+\infty}\frac{f(x)f(y)}{h(x,y)}F(x,y)dxdy &= \int_0^{+\infty}\int_0^{+\infty}\frac{e^{-x-y}}{x+y}\left[1+e^{-x^2}\log\left(\frac{x}{y}\right)\right]dxdy\\ &\approx 0.752483\\ &\leq 1\\ &= \int_0^{+\infty}\int_0^{+\infty}\frac{e^{-x-y}}{x+y}dxdy\\ &= \int_0^{+\infty}\int_0^{+\infty}\frac{f(x)f(y)}{h(x,y)}dxdy.\end{aligned}$$

As expected, the desired inequality is obtained for both examples.

The next section is devoted to a general variant of the Hilbert integral inequality, with some connection to the main double integral in Proposition 2.1. Additional assumptions are made on F and h, including the positivity of F.

3. Second general integral inequality result

Inspired by [17, Theorem 3.1] and in the light of the functional configuration in Proposition 2.1, the result below shows a generalized variant of the Hilbert integral inequality. Upper bounds are obtained through various weighted L_2 norms of f.

Proposition 3.1. *Let $f:[0,+\infty)\mapsto[0,+\infty)$ and $u,v:[0,+\infty)\mapsto\mathbb{R}$ be univariate functions, and $h:[0,+\infty)^2\mapsto[0,+\infty)$ be a bivariate function. Based on u and v, let $F:[0,+\infty)^2\mapsto\mathbb{R}$ be the bivariate function defined by*

$$F(x,y)=1+u(x)[v(x)-v(y)].$$

The assumptions below are made for F and h.

A1: *F is positive, i.e., for any $(x,y)\in[0,+\infty)^2$, $F(x,y)\geq 0$.*

A2: *h is symmetric, i.e., $h(x,y)=h(y,x)$ for any $(x,y)\in[0,+\infty)^2$, and homogeneous in the sense that there exists $\lambda\in\mathbb{R}$ satisfying, for any $(x,y,z)\in[0,+\infty)^3$,*

$$h(zx,zy)=z^{\lambda}h(x,y).$$

Then, for any $\alpha\in\mathbb{R}$, the following integral inequality holds:

$$\begin{aligned}\int_0^{+\infty}\int_0^{+\infty}\frac{f(x)f(y)}{h(x,y)}F(x,y)dxdy &\leq \sqrt{\int_0^{+\infty}x^{1-\lambda}\left\{[1+u(x)v(x)]c_{\alpha}-u(x)T_{\alpha}[v](x)\right\}f^2(x)dx}\\ &\times \sqrt{\int_0^{+\infty}x^{1-\lambda}\left\{c_{\alpha}+T_{\alpha}[uv](x)-v(x)T_{\alpha}[u](x)\right\}f^2(x)dx},\end{aligned}$$

where

$$c_{\alpha}=\int_0^{+\infty}\frac{r^{\alpha}}{h(1,r)}dr$$

and, for any function $k : [0,+\infty) \mapsto \mathbb{R}$, $T_\alpha[k]$ is the following integral operator:

$$T_\alpha[k](x) = \int_0^{+\infty} \frac{r^\alpha}{h(1,r)} k(rx)dr,$$

provided that the integrals involved converge. Taking k as the constant equal to 1*, we can note that $T_\alpha[k](x) = c_\alpha$.*

Proof. Using the positivity of F described in A1, the decomposition $(y/x)^{\alpha/2}(x/y)^{\alpha/2} = 1$ and applying the Cauchy-Schwarz integral inequality according to the variables x and y, we get

$$\begin{aligned}
\int_0^{+\infty}\int_0^{+\infty} \frac{f(x)f(y)}{h(x,y)} F(x,y)dxdy &= \int_0^{+\infty}\int_0^{+\infty} \frac{f(x)}{\sqrt{h(x,y)}}\sqrt{F(x,y)}\left(\frac{y}{x}\right)^{\alpha/2} \times \frac{f(y)}{\sqrt{h(x,y)}}\sqrt{F(x,y)}\left(\frac{x}{y}\right)^{\alpha/2} dxdy \\
&\leq \sqrt{\int_0^{+\infty}\int_0^{+\infty} \frac{f^2(x)}{h(x,y)} F(x,y)\left(\frac{y}{x}\right)^{\alpha} dxdy}\sqrt{\int_0^{+\infty}\int_0^{+\infty} \frac{f^2(y)}{h(x,y)} F(x,y)\left(\frac{x}{y}\right)^{\alpha} dxdy} \\
&= \sqrt{\int_0^{+\infty} p(x)f^2(x)dx}\sqrt{\int_0^{+\infty} q(y)f^2(y)dy}, \qquad (3.1)
\end{aligned}$$

where

$$p(x) = \int_0^{+\infty} \frac{1}{h(x,y)} F(x,y)\left(\frac{y}{x}\right)^{\alpha} dy$$

and

$$q(y) = \int_0^{+\infty} \frac{1}{h(x,y)} F(x,y)\left(\frac{x}{y}\right)^{\alpha} dx.$$

Let us now express $p(x)$ and $q(y)$, one after the other. Using the change of variables $y = rx$ and the homogeneous property of h in A2, we get

$$\begin{aligned}
p(x) &= x\int_0^{+\infty} \frac{1}{h(x,rx)} F(x,rx)r^\alpha dr = x^{1-\lambda}\int_0^{+\infty} \frac{1}{h(1,r)} F(x,rx)r^\alpha dr \\
&= x^{1-\lambda}\int_0^{+\infty} \frac{1}{h(1,r)}\{1+u(x)[v(x)-v(rx)]\}r^\alpha dr \\
&= x^{1-\lambda}\left\{[1+u(x)v(x)]\int_0^{+\infty} \frac{r^\alpha}{h(1,r)}dr - u(x)\int_0^{+\infty} \frac{r^\alpha}{h(1,r)}v(rx)dr\right\} \\
&= x^{1-\lambda}\{[1+u(x)v(x)]c_\alpha - u(x)T_\alpha[v](x)\}. \qquad (3.2)
\end{aligned}$$

On the other hand, for $q(y)$, using the change of variables $x = ry$, the symmetry and the homogeneous property of h in A2, we get

$$\begin{aligned}
q(y) &= y\int_0^{+\infty} \frac{1}{h(ry,y)} F(ry,y)r^\alpha dr = y^{1-\lambda}\int_0^{+\infty} \frac{1}{h(r,1)} F(ry,y)r^\alpha dr \\
&= y^{1-\lambda}\int_0^{+\infty} \frac{1}{h(1,r)}\{1+u(ry)[v(ry)-v(y)]\}r^\alpha dr \\
&= y^{1-\lambda}\left\{\int_0^{+\infty} \frac{r^\alpha}{h(1,r)}dr + \int_0^{+\infty} \frac{r^\alpha}{h(1,r)}u(ry)v(ry)dr - v(y)\int_0^{+\infty} \frac{r^\alpha}{h(1,r)}u(ry)dr\right\} \\
&= y^{1-\lambda}\{c_\alpha + T_\alpha[uv](y) - v(y)T_\alpha[u](y)\}. \qquad (3.3)
\end{aligned}$$

Combining Equations (3.1), (3.2) and (3.3), and standardizing the notation x and y, we obtain

$$\begin{aligned}
\int_0^{+\infty}\int_0^{+\infty} \frac{f(x)f(y)}{h(x,y)} F(x,y)dxdy &\leq \sqrt{\int_0^{+\infty} x^{1-\lambda}\{[1+u(x)v(x)]c_\alpha - u(x)T_\alpha[v](x)\}f^2(x)dx} \\
&\times \sqrt{\int_0^{+\infty} x^{1-\lambda}\{c_\alpha + T_\alpha[uv](x) - v(x)T_\alpha[u](x)\}f^2(x)dx},
\end{aligned}$$

which is the desired inequality. This concludes the proof. □

The interest of Proposition 3.1 lies in its generality and the form of the upper bound obtained; it is typical of those appearing in some variants of the Hilbert integral inequality, i.e., with the product of two weighted L_2 norms of f.

In fact, if we analyze the proof of Proposition 3.1, it can be easily extended to two functions, $f,g:[0,+\infty)\mapsto[0,+\infty)$, as follows:

$$\begin{aligned}\int_0^{+\infty}\int_0^{+\infty}\frac{f(x)g(y)}{h(x,y)}F(x,y)dxdy &\leq \sqrt{\int_0^{+\infty}x^{1-\lambda}\left\{[1+u(x)v(x)]c_\alpha-u(x)T_\alpha[v](x)\right\}f^2(x)dx}\\ &\times\sqrt{\int_0^{+\infty}x^{1-\lambda}\left\{c_\alpha+T_\alpha[uv](x)-v(x)T_\alpha[u](x)\right\}g^2(x)dx}.\end{aligned}$$

We have concentrated on the case $f=g$ mainly to make some connections with Proposition 2.1.

Let now discuss A1. If, for any $x\in[0,+\infty)$, $u(x)\in[0,1]$ and $v(x)\in[0,1]$, then, for any $(x,y)\in[0,+\infty)^2$, we have $u(x)v(x)\geq 0$ and $u(x)v(y)\leq 1$, so that

$$F(x,y)=u(x)v(x)+[1-u(x)v(y)]\geq 0.$$

The assumption A1 is thus satisfied.

In the context of Case 3 in Proposition 2.1, i.e., if u and v are both increasing, or both decreasing, under some integrability assumptions, if A1 and A2 of Proposition 3.1 are satisfied, then this result gives

$$\begin{aligned}\int_0^{+\infty}\int_0^{+\infty}\frac{f(x)f(y)}{h(x,y)}dxdy &\leq \int_0^{+\infty}\int_0^{+\infty}\frac{f(x)f(y)}{h(x,y)}F(x,y)dxdy\\ &\leq \sqrt{\int_0^{+\infty}x^{1-\lambda}\left\{[1+u(x)v(x)]c_\alpha-u(x)T_\alpha[v](x)\right\}f^2(x)dx}\\ &\times\sqrt{\int_0^{+\infty}x^{1-\lambda}\left\{c_\alpha+T_\alpha[uv](x)-v(x)T_\alpha[u](x)\right\}f^2(x)dx}.\end{aligned} \tag{3.4}$$

As noted in [17], the choices $\alpha=-1/2$, $h(x,y)=x+y$, $u(x)=1$ and $v(x)=1/(1+x)$ give the improved Hilbert integral inequality demonstrated in [16, Theorem 2.1].

With this in mind, let us illustrate Proposition 3.1 with a new example activating the function u. We consider $\alpha=-1/2$, $h(x,y)=x+y$, $u(x)=1/(1+x)$ and $v(x)=1/(1+x)$. So we have

$$F(x,y)=1+\frac{1}{1+x}\left(\frac{1}{1+x}-\frac{1}{1+y}\right)=1+\frac{y-x}{(1+x)^2(1+y)}.$$

Since $u(x)\in[0,1]$ and $v(x)\in[0,1]$, the assumption A1 holds. Furthermore, with the selected function h, the assumption A2 is obviously satisfied with $\lambda=1$. Let now remark that

$$\begin{aligned}c_\alpha &= \int_0^{+\infty}\frac{r^\alpha}{h(1,r)}dr\\ &= \int_0^{+\infty}\frac{1}{\sqrt{r}(1+r)}dr\\ &= \left\{2\arctan[\sqrt{r}]\right\}_{r=0}^{r\to+\infty}\\ &= \pi,\end{aligned}$$

$$\begin{aligned}T_\alpha[u](x) &= \int_0^{+\infty}\frac{r^\alpha}{h(1,r)}u(rx)dr=\int_0^{+\infty}\frac{1}{\sqrt{r}(1+r)(1+rx)}dr\\ &= \left\{\frac{2}{x-1}\left[\sqrt{x}\arctan[\sqrt{xr}]-\arctan[\sqrt{r}]\right]\right\}_{r=0}^{r\to+\infty}\\ &= \frac{\pi}{1+\sqrt{x}},\end{aligned}$$

$$T_\alpha[v](x)=T_\alpha[u](x)=\frac{\pi}{1+\sqrt{x}}$$

and

$$\begin{aligned}
T_\alpha[uv](x) &= \int_0^{+\infty} \frac{r^\alpha}{h(1,r)} u(rx)v(rx)dr \\
&= \int_0^{+\infty} \frac{1}{\sqrt{r}(1+r)(1+rx)^2} dr \\
&= \left\{ \frac{1}{(x-1)^2} \left[\frac{(x-1)x\sqrt{r}}{1+rx} + 2\arctan[\sqrt{r}] + (x-3)\sqrt{x}\arctan[\sqrt{xr}] \right] \right\}_{r=0}^{r\to+\infty} \\
&= \frac{[2+\sqrt{x}]\,\pi}{2\,[1+\sqrt{x}]^2}.
\end{aligned}$$

It follows from Proposition 3.1 that

$$\begin{aligned}
\int_0^{+\infty}\int_0^{+\infty} \frac{f(x)f(y)}{x+y}\left[1+\frac{y-x}{(1+x)^2(1+y)}\right]dxdy &= \int_0^{+\infty}\int_0^{+\infty} \frac{f(x)f(y)}{h(x,y)} F(x,y)dxdy \\
&\leq \sqrt{\int_0^{+\infty} x^{1-\lambda}\left\{[1+u(x)v(x)]c_\alpha - u(x)T_\alpha[v](x)\right\} f^2(x)dx} \\
&\times \sqrt{\int_0^{+\infty} x^{1-\lambda}\left\{c_\alpha + T_\alpha[uv](x) - v(x)T_\alpha[u](x)\right\} f^2(x)dx} \\
&= \sqrt{\int_0^{+\infty} \left\{\left[1+\frac{1}{(1+x)^2}\right]\pi - \frac{\pi}{(1+x)[1+\sqrt{x}]}\right\} f^2(x)dx} \\
&\times \sqrt{\int_0^{+\infty} \left\{\pi + \frac{[2+\sqrt{x}]\,\pi}{2\,[1+\sqrt{x}]^2} - \frac{\pi}{(1+x)[1+\sqrt{x}]}\right\} f^2(x)dx} \\
&= \pi\sqrt{\int_0^{+\infty} \frac{x^{5/2}+2x^{3/2}+x^2+x+2\sqrt{x}+1}{[1+\sqrt{x}](1+x)^2} f^2(x)dx} \\
&\times \sqrt{\int_0^{+\infty} \frac{5x^{3/2}+2x^2+6x+3\sqrt{x}+2}{2[1+\sqrt{x}]^2(1+x)} f^2(x)dx.}
\end{aligned}$$

Also, since u and v are both decreasing, based on Equation (3.4), we have

$$\begin{aligned}
\int_0^{+\infty}\int_0^{+\infty} \frac{f(x)f(y)}{x+y}dxdy &\leq \int_0^{+\infty}\int_0^{+\infty} \frac{f(x)f(y)}{x+y}\left[1+\frac{y-x}{(1+x)^2(1+y)}\right]dxdy \\
&\leq \pi\sqrt{\int_0^{+\infty} \frac{x^{5/2}+2x^{3/2}+x^2+x+2\sqrt{x}+1}{[1+\sqrt{x}](1+x)^2} f^2(x)dx} \\
&\times \sqrt{\int_0^{+\infty} \frac{5x^{3/2}+2x^2+6x+3\sqrt{x}+2}{2[1+\sqrt{x}]^2(1+x)} f^2(x)dx.}
\end{aligned}$$

Let us verify these inequalities with a numerical example. Considering $f(x) = e^{-x}$, we have

$$\int_0^{+\infty}\int_0^{+\infty} \frac{e^{-x-y}}{x+y}dxdy = 1,$$

$$\int_0^{+\infty}\int_0^{+\infty} \frac{e^{-x-y}}{x+y}\left[1+\frac{y-x}{(1+x)^2(1+y)}\right]dxdy \approx 1.02897$$

$$\int_0^{+\infty} \frac{x^{5/2}+2x^{3/2}+x^2+x+2\sqrt{x}+1}{[1+\sqrt{x}](1+x)^2} e^{-2x}dx \approx 0.535435,$$

$$\int_0^{+\infty} \frac{5x^{3/2}+2x^2+6x+3\sqrt{x}+2}{2[1+\sqrt{x}]^2(1+x)} e^{-2x}dx \approx 0.523919,$$

and we check that $1 \leq 1.02897 \leq \pi\sqrt{0.535435}\sqrt{0.523919} \approx 1.66393$. So many more examples can be formulated on a similar basis of analysis.

4. Conclusion

In this article, we have established two new integral inequalities that extend some key results in [17, 16]. Both are centered on the following double integral:

$$\int_0^{+\infty}\int_0^{+\infty}\frac{f(x)f(y)}{h(x,y)}F(x,y)dxdy$$

where the novelty lies in the general definition of F of the following form: $F(x,y)=1+u(x)[v(x)-v(y)]$. The first result is adaptable and gives lower and upper bounds for this double integral. The second result is related to the setting of the Hilbert integral inequality, where some new upper bounds are obtained involving weighted L_2 norms of f. The perspectives of our results make them important in several mathematical areas where challenging double integrals (involving certain product and ratio functions) need to be bounded in order to draw conclusions.

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Lower and Upper Bounds for Some Degree-Based Indices of Graphs

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Abstract

Topological indices are mathematical measurements regarding the chemical structures of any simple finite graph. These are used for QSAR and QSPR studies. We get bounds for some degree based topological indices of a graph using solely the vertex degrees. We obtain upper and lower bounds for these indices and investigate for the complete graphs, path graphs and Fibonacci-sum graphs.

1. Introduction

Topological indices are important for the graph theory studies. Several significant topological indices such as Zagreb index, Randic index and Wiener index has been introduced to measure the characters of graphs.

Now, we recall the definitions of some topological indices we used in this study:

The multiplicative Randic index is defined in [1] as

$$MR(G) = \prod_{uv\in E(G)} \sqrt{\frac{1}{\deg(u)\deg(v)}}.$$

The reduced reciprocal Randic index was described in [1] as

$$RRR(G) = \sum_{uv\in E(G)} \sqrt{(\deg(u)-1)(\deg(v)-1)}.$$

The Narumi-Katayama index was introduced in [2] as

$$NK(G) = \prod_{i=1}^{n} \deg(v_i).$$

The symmetric division deg index was described in [3] as

$$SD(G) = \sum_{uv\in E(G)} \frac{\deg(u)^2+\deg(v)^2}{\deg(u)\deg(v)}.$$

In literature, there are some studies including these indices such as [4, 5, 6, 7].

In [8], a Fibonacci-sum graph was defined as $G_n = (V,E)$, where $V = [n] = \{F_2 = 1, F_3 = 2, F_4 = 3, 4, 5, \cdots, n\}$ is the vertex set and $E = \{\{i, j\} : i, j \in V,\ i \neq j,\ i + j$ is a Fibonacci number$\}$ is the edge set.

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It is obvious from the definition that G_n is a simple graph.

Also, some properties of the Fibonacci-sum graphs were obtained in the following theorems [9]:

Lemma 1.1. *G_n is connected for each $n \geq 1$.*

Lemma 1.2. *Let $n \geq 2$, and t be any positive integer satisfy that $F_t \leq n < F_{t+1}$. Then the only neighbour of the vertex F_t is F_{t-1}.*

Lemma 1.3. *Let $n \geq 1$ and let $y \in [1,n]$. Let for $t \geq 2$, $F_t \leq y < F_{t+1}$ and for $l \geq t$, $F_l \leq y+n < F_{l+1}$. Then the degree of y is*

$$\deg_{G_n}(y) = \begin{cases} l-t, & \text{if } 2y \text{ is not a Fibonacci number,} \\ l-t-1, & \text{if } 2y \text{ is a Fibonacci number.} \end{cases}$$

Theorem 1.4. *Vertex 2 has maximum degree in the Fibonacci-sum graph G_n (for any $n \geq 2$). Also, if $n+2$ is a Fibonacci number, then $\deg_{G_n}(1) = \deg_{G_n}(2) - 1$; otherwise, $\deg_{G_n}(1) = \deg_{G_n}(2)$.*

As a result of the above theorem, in the Fibonacci-sum graph G_n, 2 has the maximum degree and one of the vertices with maximum degree less than the degree of 2 is 1. Also, by Lemma 1.2 $d(F_k) = 1$ for $F_k \leq n < F_{k+1}$. Thus, for any $i \in V(G_n)$, we have

$$d(2) \geq d(1) \geq d(i) \geq d(F_k) \tag{1.1}$$

where $F_k \leq n < F_{k+1}$. In this case, we get

$$F_{l_1} \leq 2+n < F_{l_1+1}, \text{ then } \deg(2) = l_1 - 3, \tag{1.2}$$

$$F_{l_2} \leq 1+n < F_{l_2+1}, \text{ then } \deg(1) = l_2 - 3. \tag{1.3}$$

In [10], the spectral properties of Fibonacci-sum and Lucas-sum graphs were examined and some bounds were obtained. Also, in [11] another type of graphs associated with Fibonacci numbers was studied.

The aim of this study is obtain upper and lower bounds of multiplicative Randic index, reduced reciprocal Randic index, Narumi-Katayama index and symmetric division index for the general graphs using vertex degree. Then, we obtain upper and lower bounds for these indices for some special graphs and Fibonacci-sum graphs. Finally, we compared the bounds on these indices for some graphs.

2. Main results

In this section all of the theorems are given for $n \geq 3$.

Theorem 2.1. *Let G be a simple connected graph with n vertices, k pendant vertices and m edges. Then we get*

$$\left(\frac{1}{n-1}\right)^m \leq MR(G) \leq \left(\frac{1}{2}\right)^{\frac{2m-k}{2}}.$$

The lower bound holds for $G \cong K_n$ and the upper bound holds for $G \cong P_n$.

Proof. Since the graph has k pendant vertices and the other vertices is of at least degree 2, we get the upper bound for the multiplicative Randic index of G as

$$MR(G) \leq \left(\frac{1}{\sqrt{2}}\right)^k \left(\frac{1}{2}\right)^{n-1-k} = \left(\frac{1}{2}\right)^{\frac{2m-k}{2}}.$$

Also, since the vertices have the maximum degree at most $n-1$, we have the lower bound for the multiplicative Randic index of G as

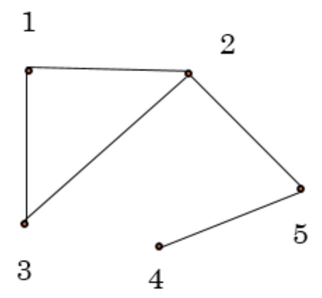
$$\left(\frac{1}{n-1}\right)^m \leq MR(G).$$

As a conclusion, we obtain

$$\left(\frac{1}{n-1}\right)^m \leq MR(G) \leq \left(\frac{1}{2}\right)^{\frac{2m-k}{2}}.$$

□

Figure 1: Simple connected graph



Example 2.2. *For the given graph in Figure 1 the bounds for the multiplicative Randic index are*

$$0.0009 \leq MR(G) = 0.044 \leq 0.024.$$

Corollary 2.3. *Let $G = K_{p,q}$. If $p < q$, then*

$$\left(\frac{1}{q}\right)^{pq} \leq MR(K_{p,q}) \leq \left(\frac{1}{p}\right)^{pq}.$$

If $p = q$, then

$$MR(K_{p,q}) = \left(\frac{1}{p}\right)^{p^2}.$$

Proof. Since the $K_{p,q}$ graph has pq edges, the proof can be seen easily. □

Theorem 2.4. *If G_n is a Fibonacci-sum graph, then*

$$\left(\frac{1}{\sqrt{(l_1-3)(l_2-3)}}\right)^{n-1} \leq MR(G_n) \leq \left(\frac{1}{\sqrt{2}}\right)^{n-r}$$

where l_1,l_2 are integers in (1.2), (1.3), respectively and r is the number of the vertices with degree 1 in G_n.

Proof. Since r is the number of the vertices with degree 1 in G_n, the degrees of the other vertices are at least 2. Thus, there are r vertices with degree 1 and $n-r$ vertices with degree at least 2. Hence, we get the upper bound for the multiplicative Randic index of G_n as

$$MR(G_n) \leq \left(\frac{1}{\sqrt{2}}\right)^{n-r}.$$

Also, since by Theorem 1.4, 2 has the maximum degree and one of the vertices with maximum degree less than the degree of 2 is 1, we have the lower bound for the multiplicative Randic index of G_n as

$$\left(\frac{1}{\sqrt{\deg(2)\deg(1)}}\right)^{n-1} \leq MR(G_n).$$

As a conclusion, we obtain

$$\left(\frac{1}{\sqrt{(l_1-3)(l_2-3)}}\right)^{n-1} \leq MR(G_n) \leq \left(\frac{1}{\sqrt{2}}\right)^{n-r}.$$

□

Theorem 2.5. *Let G be a simple connected graph with n vertices, k pendant vertices and m edges, then*

$$m-k \leq RRR(G) \leq m(n-2).$$

The lower bound holds for $G \cong P_n$ and the upper bound holds for $G \cong K_n$.

Proof. Since the graph has k pendant vertices and the other vertices is of at least degree 2, we have the lower bound as

$$m-k \leq RRR(G).$$

Also, since the vertices have the maximum degree at most $n-1$, we have the upper bound as

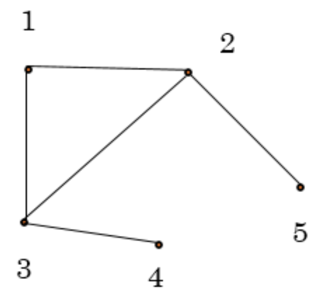
$$RRR(G) \leq m(n-2).$$

As a conclusion, we obtain

$$m-k \leq RRR(G) \leq m(n-2).$$

□

Figure 2: Simple connected graph



Example 2.6. *For the given graph in Figure 2 the bounds for the reduced reciprocal Randic index are*

$$3 \leq RRR(G) = 4.82 \leq 15.$$

Corollary 2.7. *Let $G = K_{p,q}$. If $p < q$, then*

$$RRR(K_{p,q}) = pq\sqrt{(p-1)(q-1)}.$$

If $p = q$

$$RRR(K_{p,q}) = p^2(p-1).$$

Proof. Since $m = pq$ in $K_{p,q}$,the proof is trivial. □

Theorem 2.8. *If G_n is a Fibonacci-sum graph, then*

$$m \leq RRR(G_n) \leq m\sqrt{(l_1-4)(l_2-4)}$$

where l_1,l_2 are the integers in (1.2), (1.3), respectively, and $m = |E(G_n)|$.

Proof. By Lemma 1.2, in the Fibonacci-sum graph G_n, F_t is adjacent to only F_{t-1} for $F_t \leq n < F_{t+1}$. Also, since the other neighbour of F_{t-1} is F_{t-2}, $\deg(F_{t-1}) = 2$. By the same way, $\deg(F_{t-2}) \geq 2$. Thus, we get the lower bound for the reduced reciprocal Randic index of G_n as

$$m\sqrt{\deg(F_{t-1}-1)\deg(F_{t-2}-1)} = m \leq RRR(G_n).$$

Since $1 \sim 2$ and by using (1.1), we get the upper bound for the reduced reciprocal Randic index of G_n as

$$RRR(G_n) \leq m\sqrt{(\deg(1)-1)(\deg(2)-1)}.$$

Hence, we obtain

$$m \leq RRR(G_n) \leq m\sqrt{(l_1-4)(l_2-4)}.$$

□

Theorem 2.9. *Let G be a simple connected graph with n vertices and k pendant vertices then*

$$2^{n-k} \leq NK(G) \leq (n-1)^n$$

The lower bound holds for $G \cong P_n$ and the upper bound holds for $G \cong K_n$.

Proof. Since the graph has k pendant vertices and the other vertices is of at least degree 2, we obtain the lower bound as

$$2^{n-k} \leq NK(G).$$

Also, since the vertices have the maximum degree at most $n-1$, we get the upper bound as

$$NK(G) \leq (n-1)^n.$$

□

Example 2.10. *For the given graph in Figure 2 the bounds for the Narumi-Katayama index are*

$$8 \leq NK(G) = 18 \leq 1024.$$

Corollary 2.11. *Let $G = K_{p,q}$ then*

$$NK(K_{p,q}) = p^q q^p.$$

Proof. Since there are q points of degree p and p points of degree q in the graph $K_{p,q}$, we obtain $NK(K_{p,q}) = p^q q^p$. □

Theorem 2.12. *For the Narumi-Katayama index of the Fibonacci-sum graph G_n, the following inequality holds:*

$$2^{n-r} \leq NK(G_n) \leq (l_1-3)(l_2-3)^{n-1}$$

where l_1,l_2 are the integers in (1.2), (1.3), respectively and r is the number of the vertices with degree 1 in G.

Proof. Since r is the number of the vertices with degree 1 in G_n, then the degrees of the other vertices are at least 2. Thus, there are r vertices with degree 1 and $n-r$ vertices with degree at least 2. Hence, we get the lower bound for the Narumi-Katayama index of G_n as

$$2^{n-r} \leq NK(G_n).$$

Also, since by Theorem 1.4, 2 has the maximum degree and one of the vertices with maximum degree less than the degree of 2 is 1, we have the upper bound for the Narumi-Katayama index of G_n as

$$NK(G_n) \leq \deg(2)(\deg(1))^{n-1}.$$

As a result, we obtain

$$2^{n-r} \leq NK(G_n) \leq (l_1-3)(l_2-3)^{n-1}.$$

□

Theorem 2.13. *Let G be a simple connected graph with n vertices and m edges, then*

$$2m \leq SD(G) \leq m\frac{(n-1)^2+1}{n-1}.$$

Proof. If $\deg(u)$ is maximum and $\deg(v)$ is minimum, then the expression

$$\frac{\deg(u)^2+\deg(v)^2}{\deg(u)\deg(v)} \tag{2.1}$$

takes its maximum value. In G, $n-1$ is the maximum degree and if we take the pendant vertex which is adjacent to $n-1$, then the expression (2.1) takes its maximum value. Thus, we get

$$SD(G) = \frac{\deg(u)^2+\deg(v)^2}{\deg(u)\deg(v)} \leq m\frac{(n-1)^2+1}{n-1}.$$

In other way, when $\deg(u)$ and $\deg(v)$ are equal, then the expression (2.1) takes its minimum value. Thus, we get

$$2m \leq SD(G).$$

Hence, we obtain

$$2m \leq SD(G) \leq m\frac{(n-1)^2+1}{n-1}.$$

□

Example 2.14. *For the given graph in Figure 1 the bounds for the symmetric division index are*

$$10 \leq SD(G) = 11 \leq 21.25.$$

Corollary 2.15. *Let $G = K_{p,q}$ then*

$$SD(K_{p,q}) = p^2 + q^2.$$

Proof. Since $m = pq$ in $K_{p,q}$, the proof is trivial. □

Theorem 2.16. *If G_n is a Fibonacci-sum graph, then*

$$2m \leq SD(G_n) \leq m\,(l_1 - 2)$$

where l_1 is the integer in (1.2) and $m = |E(G_n)|$.

Proof. If $\deg(u)$ is maximum and $\deg(v)$ is minimum, then the expression

$$\frac{\deg(u)^2 + \deg(v)^2}{\deg(u)\deg(v)} \tag{2.2}$$

takes its maximum value. In G_n, 2 has the maximum degree and if we take the 1 degreed vertex which is adjacent to 2, then the expression (2.2) takes its maximum value. Thus we have

$$\frac{\deg(u)^2 + \deg(v)^2}{\deg(u)\deg(v)} \leq \deg(2) + 1.$$

Hence, we get the upper bound for the symmetric division index of G_n as

$$SD(G_n) \leq m\,(l_1 - 2).$$

In other way, when $\deg(u)$ and $\deg(v)$ are equal, then the expression (2.2) takes its minimum value. Thus we have

$$2 \leq \frac{\deg(u)^2 + \deg(v)^2}{\deg(u)\deg(v)}.$$

Hence, we get

$$2m \leq SD(G_n).$$

In conclusion, we obtain

$$2m \leq SD(G_n) \leq m\,(l_1 - 2).$$

□

Declarations

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A Study on Fourth-Order Coupled Boundary Value Problems: Existence, Uniqueness and Approximations

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Abstract

This study examines the existence and approximation of solutions for a coupled system of fourth-order boundary value problems (4th-BVPs), which model the interactions between two distinct but interrelated physical systems. These coupled boundary value problems arise in various applications in engineering and physics, including the analysis of bending behaviors in beams and vibrations in interconnected structural components. By leveraging Green's functions and building upon prior research in fourth-order differential equations, we derive sufficient conditions for the existence and uniqueness of solutions to the system. Additionally, we provide a numerical framework for approximating these solutions, offering practical insights for real-world applications.

1. Introduction

4th-BVPs play a crucial role in many engineering and physics applications, such as analyzing the bending behavior of elastic beams, the stability of mechanical systems, fluid dynamics, biomechanical processes, and vibration models. These problems enable the mathematical modeling and analysis of system behaviors by incorporating higher-order derivatives, which are essential for capturing complex physical phenomena.

In real-world scenarios, physical systems rarely function in isolation; they often involve intricate interactions among multiple structural elements, variables, or external forces. Modeling and analyzing such systems, particularly those with multiple degrees of freedom or coupled dynamics, necessitate the formulation of systems of interdependent differential equations. These systems provide a robust framework for understanding how the behavior of one component affects the entire system.

In this study, we focus on a coupled system of two fourth-order differential equations that represent two distinct yet interrelated physical systems. The coupled system is described as follows:

$$\left.\begin{array}{c}\varphi_1''''(x)+\beta_1^2\varphi_1''(x)=\Gamma_1(x,\varphi_1(x),\varphi_2(x))\\ \varphi_2''''(x)+\beta_2^2\varphi_2''(x)=\Gamma_2(x,\varphi_1(x),\varphi_2(x))\\ \varphi_1(0)=\varphi_2(0)=\varphi_1(L)=\varphi_2(L)=0\\ \varphi_1'(0)=\varphi_2'(0)=\varphi_1'(L)=\varphi_2'(L)=0\end{array}\quad, \ x\in[0,L]\right\} \tag{1.1}$$

Here, φ_1 and φ_2 represent the solutions corresponding to two distinct yet interacting physical systems, such as the bending behaviors of two beams or the vibrations of two structural components. The functions Γ_1 and Γ_2 model the mutual interactions between the two systems.

Such coupled systems are particularly significant in engineering disciplines, where the analysis of interconnected structures is critical. They provide insight into how individual components influence the overall system behavior, enabling more effective designs and analyses of complex structures.

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The motivation for this study stems from the need to model and analyze singular systems frequently encountered in engineering and physics. Such systems are characterized by interdependent components, often described by a complex network of equations due to their inherent interactions. For instance, in structural mechanics, beam systems or load-bearing elements interact with one another in ways that cannot be adequately captured by isolated models. To address these challenges, systems of coupled equations, such as those considered here, are essential for understanding the interplay between different components. The analysis and solutions of such systems are of paramount importance for designing and optimizing physical systems.

4th-BVPs, in particular, pose significant challenges due to their nonlinearity and complex boundary conditions. These difficulties make the investigation of existence, uniqueness, and approximation of solutions critical. The importance of such analyses is underscored by their wide-ranging applications in engineering and physics, where understanding system behaviors requires accurate mathematical modeling and solution methodologies.

Previous studies have substantially advanced the understanding of 4th-BVPs. For example, Agarwal [1] explored the existence and uniqueness of solutions to 4th-BVPs in the context of elastic beam bending. Kaufmann and Kosmatov [2] and Habib [3] extended this work to other applications. More recently, Almuthaybiri and Tisdell [4] established stricter conditions for the existence and uniqueness of solutions, while Chen and Cui [5] investigated the continuity of derivatives for solutions to 4th-BVPs.

Despite this progress, studies addressing coupled systems of dependent differential equations remain relatively rare. Interest in this area has grown in recent years, as seen in the work of Zhai and Anderson [6], who established existence and uniqueness results for doubly dependent differential equation systems. Granas and Guenther [7] contributed analytical techniques for solving more general systems of this type.

The objective of this work is to analyze the coupled system of 4th-BVPs defined by (1.1), focusing on the conditions for the existence and uniqueness of solutions. Additionally, we aim to develop iterative methods for approximating solutions when they exist, providing a comprehensive understanding of the system and its numerical treatment.

In a related study, Rao and Jagan [8] investigated the following boundary value problem (BVP):

$$\left.\begin{array}{l} \varphi''''(x)+\beta^2\varphi''(x)=\Gamma(x,\varphi(x)) \\ \varphi(0)=\varphi'(0)=\varphi'(L)=\varphi(L)=0 \end{array} , x\in[0,L]\right\} \tag{1.2}$$

Using Green's method, they demonstrated the existence of a solution for this equation, thereby contributing to the growing body of work on 4th-BVPs.

Proposition 1.1. *(see [8]) Let $\Gamma(x,\varphi(x))$ be a continuous function on $[0,L]\times\mathbb{R}$ and Lipschitz with a Lipschitz constant K with respect to the second variable. Assume that $\omega=2-\beta L\sin(\beta L)-2\cos(\beta L)\neq 0$, $\Gamma(x,0)\neq 0$, and*

$$M<\frac{1}{K}$$

where $M=\frac{L^3}{6}k_1+\frac{L^4}{24}(1+k_2)$ with

$$k_1=\left\|\frac{(\sin(\beta t)-\beta t))(1-\cos(\beta L)+(1-\cos(\beta t))(\beta L-\sin(\beta L))}{\beta\omega}\right\|_\infty,$$

and

$$k_2=\left\|\frac{(\cos(\beta t)-1))(1-\cos(\beta L)+(\beta t-\sin(\beta t))\sin(\beta L)}{\omega}\right\|_\infty.$$

Then the equation (1.2) has a unique solution, and

$$\int_0^L |G(x,t)|dt\leq M.$$

is satisfied, where the Green's function associated with (1.2) is defined as follows

$$G(x,\xi)=\begin{cases} G_1(x,\xi), & 0\leq\xi\leq x\leq L, \\ G_2(x,\xi), & 0\leq x\leq\xi\leq L. \end{cases}$$

where

$$\begin{aligned}
K(x,\xi) &= \frac{1}{\beta^3}\left[\beta(x-\xi)-\sin\beta(x-\xi)\right],\\
K_x(x,\xi) &= \frac{1}{\beta^2}\left[1-\cos\beta(x-\xi)\right],\\
G_1(x,\xi) &= \frac{K_x(L,\xi)\left[(\beta L-\sin\beta L)(1-\cos\beta x)\right]}{\beta(2-2\cos\beta L-\beta L\sin\beta L)}+\frac{K_x(L,\xi)\left[(1-\cos\beta L)(\sin\beta x-\beta x)\right]}{\beta(2-2\cos\beta L-\beta L\sin\beta L)}\\
&+ \frac{K(L,\xi)\left[\sin\beta L(\beta x-\sin\beta x)\right]}{(2-2\cos\beta L-\beta L\sin\beta L)}+\frac{K(L,\xi)\left[(1-\cos\beta L)(\cos\beta x-1)\right]}{(2-2\cos\beta L-\beta L\sin\beta L)}+K(x,\xi),\\
G_2(x,\xi) &= \frac{K_x(L,\xi)\left[(\beta L-\sin\beta L)(1-\cos\beta x)\right]}{\beta(2-2\cos\beta L-\beta L\sin\beta L)}+\frac{K_x(L,\xi)\left[(1-\cos\beta L)(\sin\beta x-\beta x)\right]}{\beta(2-2\cos\beta L-\beta L\sin\beta L)}\\
&+ \frac{K(L,\xi)\left[\sin\beta L(\beta x-\sin\beta x)\right]}{(2-2\cos\beta L-\beta L\sin\beta L)}+\frac{K(L,\xi)\left[(1-\cos\beta L)(\cos\beta x-1)\right]}{(2-2\cos\beta L-\beta L\sin\beta L)}.
\end{aligned}$$

From now on, let X denote the space of all functions that are four times differentiable, $C^{(4)}[0,L]$ where the norm $||\varphi||_\infty$ on X is the supremum norm. Additionally, the norm $||(\varphi_1,\varphi_2)||$ on X^2 is defined by $||(\varphi_1,\varphi_2)||=||\varphi_1||_\infty+||\varphi_2||_\infty$.

2. Main Results

Building on Proposition 1, we present our first result concerning the existence of solutions and their approximation for the 4th-BVPs system (1.1) in the following theorem.

Theorem 2.1. *If*

$$\begin{aligned}
\|\Gamma_i(x,\varphi_1(x),\varphi_2(x))-\Gamma_i(x,\widetilde{\varphi}_1(x),\varphi_2(x)\|_\infty &\le K_i||\varphi_1(x)-\widetilde{\varphi}_1(x)||_\infty\\
||\Gamma_i(x,\varphi_1(x),\varphi_2(x))-\Gamma_i(x,\varphi_1(x),\widetilde{\varphi}_2(x))||_\infty &\le L_i||\varphi_2(x)-\widetilde{\varphi}_2(x)||_\infty
\end{aligned}$$

for $i=1,2$, $\Gamma_1(x,0,\varphi_2(x))\neq 0$, $\Gamma_2(x,\varphi_1(x),0)\neq 0$, *and*

$$\theta=\max\{K_1+K_2,L_1+L_2\}M<1$$

where $M=\max\{M_1,M_2\}$, *and* M_1,M_2 *are given as in Proposition 1.1 for the first and second equations, respectively, then the system (1.1) has a solution which is unique. Furthermore, the iteration* $\{(\varphi_{1,n},\varphi_{2,n})\}_{n\ge 0}$ *defined by*

$$\begin{aligned}
\varphi_{1,n+1}(x) &= \int_0^L G(x,t)\Gamma_1(t,\varphi_{1,n}(t),\varphi_{2,n}(t))dt\\
\varphi_{2,n+1}(x) &= \int_0^L G(x,t)\Gamma_2(t,\varphi_{1,n}(t),\varphi_{2,n}(t))dt
\end{aligned} \tag{2.1}$$

where $(\varphi_{1,0},\varphi_{2,0})\in X^2$, *is convergent to the solution.*

Proof. Let $T(\varphi_1,\varphi_2)=\left(\int_0^L G(x,t)\Gamma_1(t,\varphi_1(t),\varphi_2(t))dt,\int_0^L G(x,t)\Gamma_2(t,\varphi_1(t),\varphi_2(t))dt\right)$. Since we have

$$\begin{aligned}
\|T(\varphi_1,\varphi_2)-T(\widetilde{\varphi}_1,\widetilde{\varphi}_2)\| &= \left\|\int_0^L G(x,t)\Gamma_1(t,\varphi_1(t),\varphi_2(t))dt-\int_0^L G(x,t)\Gamma_1(t,\widetilde{\varphi}_1(t),\widetilde{\varphi}_2(t))dt\right\|_\infty\\
&+\left\|\int_0^L G(x,t)\Gamma_2(t,\varphi_1(t),\varphi_2(t))dt-\int_0^L G(x,t)\Gamma_2(t,\widetilde{\varphi}_1(t),\widetilde{\varphi}_2(t))dt\right\|_\infty\\
&\le M_1\|\Gamma_1(x,\varphi_1(x),\varphi_2(x))-\Gamma_1(x,\widetilde{\varphi}_1(x),\widetilde{\varphi}_2(x))\|_\infty+M_2\|\Gamma_2(x,\varphi_1(x),\varphi_2(x))-\Gamma_2(x,\widetilde{\varphi}_1(x),\widetilde{\varphi}_2(x))\|_\infty\\
&\le K_1M||\varphi_1-\widetilde{\varphi}_1||_\infty+L_1M||\varphi_2-\widetilde{\varphi}_2||_\infty+K_2M||\varphi_1-\widetilde{\varphi}_1||_\infty+L_2M||\varphi_2-\widetilde{\varphi}_2||_\infty\\
&\le M\max\{K_1+K_1,L_1+L_2\}||(\varphi_1,\varphi_2)-(\widetilde{\varphi}_1,\widetilde{\varphi}_2)||\\
&= \theta||(\varphi_1,\varphi_2)-(\widetilde{\varphi}_1,\widetilde{\varphi}_2)||,
\end{aligned}$$

for any $\varphi_1, \varphi_2, \widetilde{\varphi}_1, \widetilde{\varphi}_2 \in X$, T is contraction and by Banach contraction principle, T has a unique fixed point which is also the solution of (1.1) . Let $(\varphi_{1,p}, \varphi_{2,p})$ be the fixed point of T. Then, we have

$$\begin{aligned} \left\|(\varphi_{1,n+1}, \varphi_{2,n+1}) - (\varphi_{1,p}, \varphi_{2,p})\right\| &= \left\|T(\varphi_{1,n}, \varphi_{2,n}) - T(\varphi_{1,p}, \varphi_{2,p})\right\| \\ &\leq \theta||(\varphi_{1,n}, \varphi_{2,n}) - (\varphi_{1,p}, \varphi_{2,p})|| \\ &\leq \theta^2||(\varphi_{1,n-1}, \varphi_{2,n-1}) - (\varphi_{1,p}, \varphi_{2,p})|| \\ &\quad \ldots \\ &\leq \theta^{n+1}||(\varphi_{1,0}, \varphi_{2,0}) - (\varphi_{1,p}, \varphi_{2,p})||. \end{aligned}$$

Since $\theta < 1$, we conclude that $\lim_{n\to\infty}||(\varphi_{1,n+1}, \varphi_{2,n+1}) - (\varphi_{1,p}, \varphi_{2,p})|| = 0$. □

Example 2.2. *Let $X = C^{(4)}[0,1]$ and consider the following system of BVPs*

$$\left.\begin{array}{c} \varphi_1''''(x) + 2^2\varphi_1''(x) = 2\varphi_1(x) - \frac{2}{3}\varphi_2(x) + 1 \\ \varphi_2''''(x) + 3^2\varphi_2''(x) = \frac{6}{5}\varphi_1(x) - 4\varphi_2(x) + 1 \\ \varphi_1(0) = \varphi_2(0) = \varphi_1(1) = \varphi_2(1) = 0 \\ \varphi_1'(0) = \varphi_2'(0) = \varphi_1'(1) = \varphi_2'(1) = 0 \end{array} \quad, \ x \in [0,1] \right\} \tag{2.2}$$

Then $M_1 = 1.104e-01$ and $M_2 = 6.985e-02$. Since $\Gamma_1(x, \varphi_1(x), \varphi_2(x)) = 2\varphi_1(x) - \frac{2}{3}\varphi_2(x) + 1$ and $\Gamma_2(x, \varphi_1(x), \varphi_2(x)) = \frac{6}{5}\varphi_1(x) - 4\varphi_2(x) + 1$,it is also satisfied that

$$\begin{aligned} ||\Gamma_1(x, \varphi_1(x), \varphi_2(x)) - \Gamma_1(x, \widetilde{\varphi}_1(x), \varphi_2(x))||_\infty &= \left\|2\varphi_1(x) - \frac{2}{3}\varphi_2(x) + 1 - (2\widetilde{\varphi}_1(x) - \frac{2}{3}\varphi_2(x) + 1)\right\|_\infty \\ &\leq 2\left\|\varphi_1(x) - \widetilde{\varphi}_1(x)\right\|_\infty, \ K_1 = 2, \\ ||\Gamma_2(x, \varphi_1(x), \varphi_2(x)) - \Gamma_2(x, \widetilde{\varphi}_1(x), \varphi_2(x))||_\infty &= \left\|\frac{6}{5}\varphi_1(x) - 4\varphi_2(x) + 1 - \left(\frac{6}{5}\widetilde{\varphi}_1(x) - 4\varphi_2(x) + 1\right)\right\|_\infty \\ &\leq \frac{6}{5}||\varphi_1(x) - \widetilde{\varphi}_1(x)||_\infty, \ K_2 = \frac{6}{5}, \\ ||\Gamma_1(x, \varphi_1(x), \varphi_2(x)) - \Gamma_1(x, \varphi_1(x), \widetilde{\varphi}_2(x))||_\infty &= \left\|2\varphi_1(x) - \frac{2}{3}\varphi_2(x) + 1 - (2\varphi_1(x) - \frac{2}{3}\widetilde{\varphi}_2(x) + 1)\right\|_\infty \\ &\leq \frac{2}{3}||\varphi_2(x) - \widetilde{\varphi}_2(x)||_\infty, \ L_1 = \frac{2}{3}, \\ ||\Gamma_2(x, \varphi_1(x), \varphi_2(x)) - \Gamma_2(x, \varphi_1(x), \widetilde{\varphi}_2(x))||_\infty &= \left\|\frac{6}{5}\varphi_1(x) - 4\varphi_2(x) + 1 - \left(\frac{6}{5}\varphi_1(x) - 4\widetilde{\varphi}_2(x) + 1\right)\right\|_\infty \\ &\leq 4||\varphi_2(x) - \widetilde{\varphi}_2(x)||_\infty, \ L_2 = 4, \end{aligned}$$

for all $\varphi_1, \varphi_2, \widetilde{\varphi}_1, \widetilde{\varphi}_2 \in X$. Obviously, since $K_1M_1 < 1$ and $K_2M_2 < 1$, by Proposition 1.1,

$$\varphi_1(x) = \int_0^1 G(x,t)\Gamma_1(t, \varphi_1(t), \varphi_2(t))dt$$

has a solution for fixed $\varphi_2 \in X$ and

$$\varphi_2(x) = \int_0^1 G(x,t)\Gamma_2(t, \varphi_1(t), \varphi_2(t))dt$$

has a solution for fixed $\varphi_1 \in X$. Let

$$\begin{aligned} T(\varphi_1, \varphi_2) &= \left(\int_0^1 G(x,t)\Gamma_1(t, \varphi_1(t), \varphi_2(t))dt, \int_0^1 G(x,t)\Gamma_2(t, \varphi_1(t), \varphi_2(t))dt\right) \\ &= \left(\int_0^1 G(x,t)\left(\frac{4}{7}\varphi_1(t) + \frac{1}{4}\varphi_2(t) + 1\right)dt, \int_0^1 G(x,t)\left(\frac{2}{3}\varphi_1(t) - \frac{1}{2}\varphi_2(t) + 1\right)dt\right). \end{aligned}$$

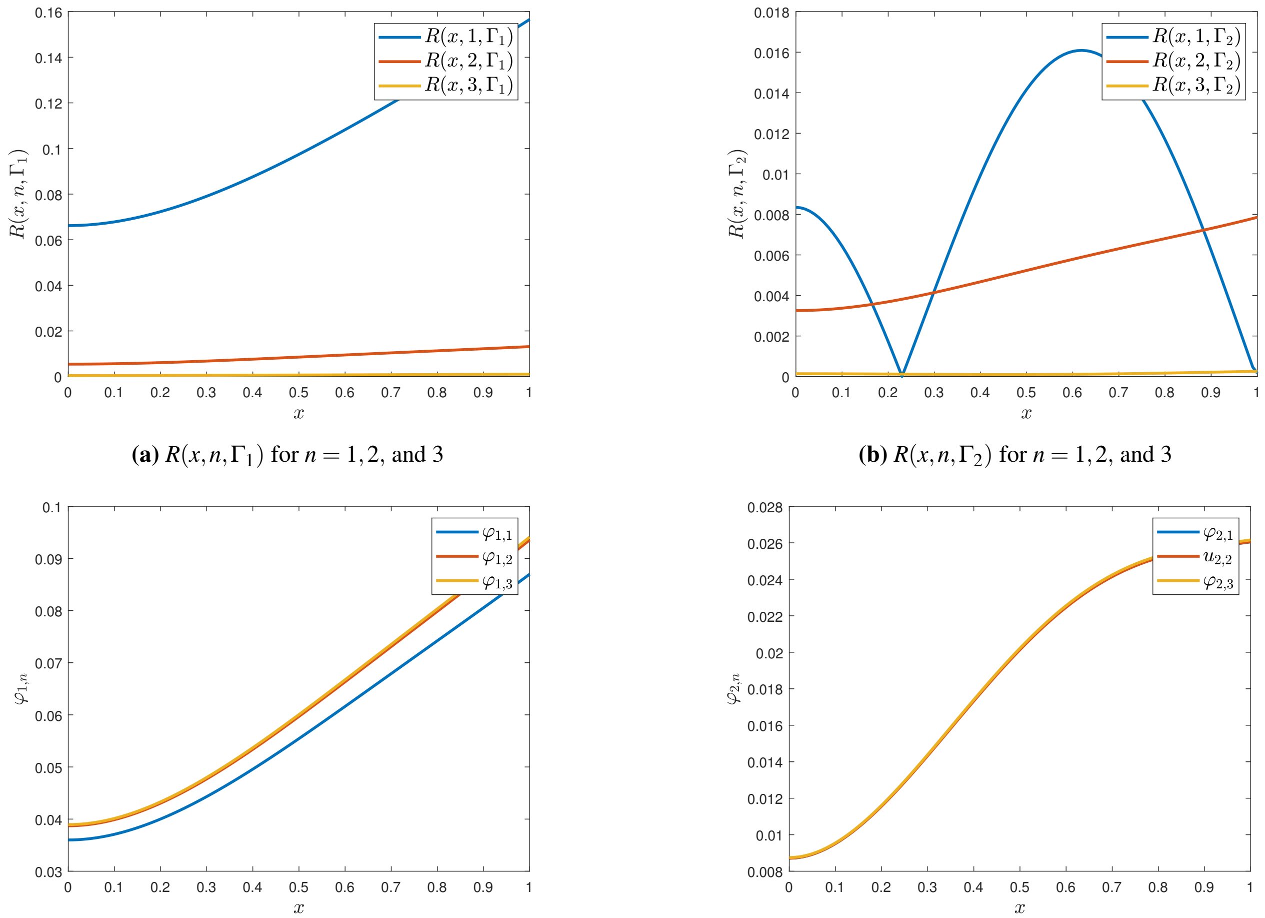
Since

$$\begin{aligned}\theta &= \max\{K_1+K_2, L_1+L_2\}M \\ &= \max\left\{2+\frac{6}{5}, \frac{2}{3}+4\right\} 1.104e-01 \\ &= 5.155e-01 < 1,\end{aligned}$$

the system (2.2) has the solution by Theorem 2.1. In addition, the iteration $\{(\varphi_{1,n}(x), \varphi_{2,n}(x))\}_{n\geq 0}$ *defined by*

$$\begin{aligned}\varphi_{1,n}(x) &= \int_0^1 G(x,t)\left(\frac{4}{7}\varphi_{1,n}(t)+\frac{1}{4}\varphi_{2,n}(t)+1\right)dt \\ \varphi_{2,n}(x) &= \int_0^1 G(x,t)\left(\frac{2}{3}\varphi_{1,n}(t)-\frac{1}{2}\varphi_{2,n}(t)+1\right)dt\end{aligned} \tag{2.3}$$

is convergent to the solution staring with $(\varphi_{1,0}, \varphi_{2,0}) = (x,x)$. *Let* $R(x,n,\Gamma_i) = |\varphi''''_{i,n}(x)+\beta^2\varphi''_{i,n}(x)-\Gamma_i(x,\varphi_{1,n}(x),\varphi_{2,n}(x))|$ *be the residual error for* $i = 1,2$ *and* $n > 0$. *The Residual errors for* $n = 1,2,3$ *are shown in Figure 1 and Table 1.*



(a) $R(x,n,\Gamma_1)$ for $n = 1,2$, and 3

(b) $R(x,n,\Gamma_2)$ for $n = 1,2$, and 3

(c) $\varphi_{1,n}$ for $n = 1,2$, and 3

(d) $\varphi_{2,n}$ for $n = 1,2$, and 3

Figure 1

	n=1		n=2		n=3	
	$R(x,1,\Gamma_1)$	$R(x,1,\Gamma_2)$	$R(x,2,\Gamma_1)$	$R(x,2,\Gamma_2)$	$R(x,3,\Gamma_1)$	$R(x,3,\Gamma_2)$
0	6,619E-02	8,347E-03	5,435E-03	3,253E-03	4,366E-04	1,394E-04
0.1	6,782E-02	6,436E-03	5,613E-03	3,372E-03	4,492E-04	1,343E-04
0.2	7,231E-02	1,722E-03	6,091E-03	3,689E-03	4,834E-04	1,222E-04
0.3	7,906E-02	4,229E-03	6,787E-03	4,143E-03	5,339E-04	1,088E-04
0.4	8,759E-02	9,935E-03	7,622E-03	4,674E-03	5,957E-04	9,953E-05
0.5	9,743E-02	1,416E-02	8,530E-03	5,233E-03	6,646E-04	9,895E-05
0.6	1,082E-01	1,604E-02	9,461E-03	5,783E-03	7,373E-04	1,102E-04
0.7	1,197E-01	1,518E-02	1,038E-02	6,305E-03	8,115E-04	1,342E-04
0.8	1,316E-01	1,169E-02	1,128E-02	6,805E-03	8,862E-04	1,697E-04
0.9	1,439E-01	6,216E-03	1,219E-02	7,306E-03	9,623E-04	2,131E-04
1.0	1,566E-01	1,674E-04	1,313E-02	7,856E-03	1,042E-03	2,587E-04

Table 1: Residual errors for $n = 1,2,$ and 3

Theorem 2.3. *Let Γ_i for $i = 1,2$ and θ be as in Theorem 2.1 and assume that there exist $\widetilde{\Gamma}_i(x,\varphi_1(x),\varphi_2(x))$ functions on $[0,L]\times X^2$ such that*

$$||\Gamma_i(x,\varphi_1(x),\varphi_2(x)) - \widetilde{\Gamma}_i(x,\varphi_1(x),\varphi_2(x))||_\infty \leq \xi_i$$

for $i = 1,2$, and the following system

$$\left.\begin{array}{r}\varphi_1''''(x)+\beta^2\varphi_1''(x)=\widetilde{\Gamma}_1(x,\varphi_1(x),\varphi_2(x))\\ \varphi_2''''(x)+\beta_1^2\varphi_2''(x)=\widetilde{\Gamma}_2(x,\varphi_1(x),\varphi_2(x))\\ \varphi_1(0)=\varphi_2(0)=\varphi_1(L)=\varphi_2(L)=0\\ \varphi_1'(0)=\varphi_2'(0)=\varphi_1'(L)=\varphi_2'(L)=0\end{array}\quad, x\in[0,L]\right\} \tag{2.4}$$

has a solution. Then

$$||(\varphi_{1,p},\varphi_{2,p}) - (\widetilde{\varphi}_{1,p},\widetilde{\varphi}_{2,p})|| \leq M\frac{\xi_1+\xi_2}{1-\theta}$$

holds for $(\varphi_{1,p},\varphi_{2,p})$ and $(\widetilde{\varphi}_{1,p},\widetilde{\varphi}_{2,p})$, where $(\varphi_{1,p},\varphi_{2,p})$ and $(\widetilde{\varphi}_{1,p},\widetilde{\varphi}_{2,p})$ are the the solutions of systems (1.1) and (2.4),respectively, and $M = \max\{M_1,M_2\}$, and M_1,M_2 are given as in Propositon 1.1 for first and second equation, respectively.

Proof. Let

$$T(\varphi_1,\varphi_2) = \left(\int_0^L G(x,t)\Gamma_1(t,\varphi_1(t),\varphi_2(t))dt, \int_0^L G(x,t)\Gamma_2(t,\varphi_1(t),\varphi_2(t))dt\right)$$

and

$$S(\varphi_1,\varphi_2) = \left(\int_0^L G(x,t)\widetilde{\Gamma}_1(t,\varphi_1(t),\varphi_2(t))dt, \int_0^L G(x,t)\widetilde{\Gamma}_2(t,\varphi_1(t),\varphi_2(t))dt\right).$$

Then, by Theorem 2.1, T has a fixed point $(\varphi_{1,p},\varphi_{2,p})$ which is the unique solution of system (1.1). Let $(\varphi_{1,0},\varphi_{2,0}) = (\widetilde{\varphi}_{1,p},\widetilde{\varphi}_{2,p})$ be a fixed point of S, which is also a solution of system (2.4), and define $(\varphi_{1,n+1},\varphi_{2,n+1}) = T(\varphi_{1,n},\varphi_{2,n})$. Then, $\{(\varphi_{1,n+1},\varphi_{2,n+1})\}_{n\geq 0}$ converges to $(\varphi_{1,p},\varphi_{2,p})$ by Theorem 2.1. Since

$$\begin{aligned}||(\varphi_{1,n+1},\varphi_{2,n+1}) - (\varphi_{1,n},\varphi_{2,n})|| &= ||T(\varphi_{1,n},\varphi_{2,n}) - T(\varphi_{1,n-1},\varphi_{2,n-1})||\\ &\leq \theta||(\varphi_{1,n},\varphi_{2,n}) - (\varphi_{1,n-1},\varphi_{2,n-1})||\\ &\quad\cdots\\ &\leq \theta^n||(\varphi_{1,1},\varphi_{2,1}) - (\varphi_{1,0},\varphi_{2,0})||,\end{aligned}$$

then

$$\begin{aligned}
||(\varphi_{1,n},\varphi_{2,n})-(\varphi_{1,0},\varphi_{2,0})|| &\leq \sum_{k=1}^{n}||(\varphi_{1,k},\varphi_{2,k})-(\varphi_{1,k-1},\varphi_{2,k-1})|| \\
&\leq \sum_{k=1}^{n}\theta^{k-1}||(\varphi_{1,1},\varphi_{2,1})-(\varphi_{1,0},\varphi_{2,0})|| \\
&\leq \frac{1}{1-\theta}||(\varphi_{1,1},\varphi_{2,1})-(\varphi_{1,0},\varphi_{2,0})|| \\
&= \frac{1}{1-\theta}||T(\varphi_{1,0},\varphi_{2,0})-S(\varphi_{1,0},\varphi_{2,0})|| \\
&\leq \frac{1}{1-\theta}\left(\begin{array}{l} \left\| \int_0^L G(x,t)\Gamma_1(t,\varphi_{1,0}(t),\varphi_{2,0}(t))dt-\int_0^L G(x,t)\widetilde{\Gamma}_1(t,\varphi_{1,0}(t),\varphi_{2,0}(t))dt \right\|_\infty \\ +\left\| \int_0^L G(x,t)\Gamma_2(t,\varphi_{1,0}(t),\varphi_{2,0}(t))dt-\int_0^L G(x,t)\widetilde{\Gamma}_2(t,\varphi_{1,0}(t),\varphi_{2,0}(t))dt \right\|_\infty \end{array}\right) \\
&\leq \frac{1}{1-\theta}\left(\begin{array}{l} \left\|\int_0^L G(x,t)dt\right\|_\infty \left\| \Gamma_1(x,\varphi_{1,0}(x),\varphi_{2,0}(x))-\widetilde{\Gamma}_1(x,\varphi_{1,0}(x),\varphi_{2,0}(x)) \right\|_\infty \\ +\left\|\int_0^L G(x,t)dt\right\|_\infty \left\| \Gamma_2(x,\varphi_{1,0}(x),\varphi_{2,0}(x))-\widetilde{\Gamma}_2(x,\varphi_{1,0}(x),\varphi_{2,0}(x)) \right\|_\infty \end{array}\right) \\
&\leq M\frac{\xi_1+\xi_2}{1-\theta}
\end{aligned}$$

which implies that

$$||(\varphi_{1,p},\varphi_{2,p})-(\widetilde{\varphi}_{1,p},\widetilde{\varphi}_{2,p})|| \leq M\frac{\xi_1+\xi_2}{1-\theta}.$$

□

Example 2.4. *Consider the following system of BVP*

$$\left.\begin{array}{l} \varphi_1''''(x)+2^2\varphi_1''(x)=2\varphi_1^{\frac{9}{10}}(x)-\frac{2}{3}\varphi_2^{\frac{4}{3}}(x)+\frac{x+9}{x+10} \\ \varphi_2''''(x)+3^2\varphi_2''(x)=\frac{6}{5}\varphi_1^{\cos(\varphi_1(x))}(x)-4\varphi_2^{\frac{\sin(\varphi_2(x))}{\varphi_2(x)}}(x)+1 \quad ,\ x\in[0,1] \\ \varphi_1(0)=\varphi_2(0)=\varphi_1(1)=\varphi_2(1)=0 \\ \varphi_1'(0)=\varphi_2'(0)=\varphi_1'(1)=\varphi_2'(1)=0 \end{array}\right\} \tag{2.5}$$

Solving the this system of BVP directly is highly challenging or even infeasible due to the nonlinear functions involved in. However, thanks to Theorem 2.3, approximate solutions close to the exact one can be obtained without directly solving the equation.

Let X, Γ_1, Γ_2, β, and β_2 be as defined in Example 2.2. Additionally, let $\overline{X}=\{\varphi\in X: 0\leq\varphi(x)\leq 1\}$. It can be observed from Figure 1 that the solution of system (2.2) belongs to $\overline{X}\times\overline{X}$. Then the functions $\widetilde{\Gamma}_1(x,\varphi_1(x),\varphi_2(x))=2\varphi_1^{\frac{9}{10}}(x)-\frac{2}{3}\varphi_2^{\frac{4}{3}}(x)+\frac{x+9}{x+10}$ and $\widetilde{\Gamma}_2(x,\varphi_1(x),\varphi_2(x))=\frac{6}{5}\varphi_1^{\cos(\varphi_1(x))}-4\varphi_2^{\frac{\sin(\varphi_2(x))}{\varphi_2(x)}}+1$ satisfy the following

$$\begin{aligned}
||\Gamma_1(x,\varphi_1(x),\varphi_2(x))-\widetilde{\Gamma}_1(x,\varphi_1(x),\varphi_2(x))||_\infty &= \left\| 2\varphi_1(x)-\tfrac{2}{3}\varphi_2(x)+1-\left(2\varphi_1^{\frac{9}{10}}(x)-\tfrac{2}{3}\varphi_2^{\frac{4}{3}}(x)+\frac{x+9}{x+10}\right) \right\|_\infty \\
&\leq \left\|2\varphi_1(x)-2\varphi_1(x)^{\frac{9}{10}}\right\|_\infty+\left\|\frac{2}{3}\varphi_2(x)-\frac{2}{3}\varphi_1^{\frac{4}{3}}(x)\right\|_\infty+\left\|1-\frac{x+9}{x+10}\right\|_\infty \\
&\leq 2.43e-01=\xi_1, \\
||\Gamma_2(x,\varphi_1(x),\varphi_2(x))-\widetilde{\Gamma}_2(x,\varphi_1(x),\varphi_2(x))||_\infty &= \left\| \tfrac{6}{5}\varphi_1(x)-4\varphi_2(x)+1-\left(\tfrac{6}{5}\varphi_1^{\cos(\varphi_1(x))}(x)-4\varphi_2^{\frac{\sin(\varphi_2(x))}{\varphi_2(x)}}(x)+1\right) \right\|_\infty \\
&\leq \left\|\frac{6}{5}\varphi_1(x)-\frac{6}{5}\varphi_1^{\cos(\varphi_1(x))}(x)\right\|_\infty+\left\|4\varphi_2(x)-4\varphi_2^{\frac{\sin(\varphi_2(x))}{\varphi_2(x)}}(x)\right\|_\infty \\
&\leq 1.52e-01=\xi_2,
\end{aligned}$$

for all $\varphi_1, \varphi_2 \in X$*.Since* $M_1 = 1.104e-01$ *and* $M_2 = 6.985e-02$*, we have* $M = \max\{M_1, M_2\} = 1.104e-01$*. Then, by Theorem 2.3, we have:*

$$\begin{aligned} ||(\varphi_{1,p}, \varphi_{2,p}) - (\widetilde{\varphi}_{1,p}, \widetilde{\varphi}_{2,p})|| &\leq M\frac{\xi_1+\xi_2}{1-\theta} \\ &= 9.02e-02 \end{aligned}$$

where $(\varphi_{1,p}, \varphi_{2,p})$ *is the solution of the system (2.2) and* $(\widetilde{\varphi}_{1,p}, \widetilde{\varphi}_{2,p})$ *is the solution of the system (2.5). As a result, without solving the system (2.5) which is more challenging to solve, it is possible to approximate the solution by solving the simpler system (2.2), which closely resembles the original system (2.5).*

Theorem 2.5. *Let* M *be as in Proposition 1.1. If*

$$||\Gamma(x, \varphi_1(x)) - \Gamma(x, \varphi_2(x))||_\infty \leq K||\varphi_1(x) - \varphi_2(x)||_\infty$$

and $\theta = KM < 1$*, then the iteration defined by*

$$\varphi_{n+1}(x) = \int_0^L G(x,t)\Gamma(t, \varphi_n(t))dt \tag{2.6}$$

is convergent to the solution of the following BVP problem

$$\left.\begin{array}{c} \varphi''''(x) + \beta^2\varphi''(x) = \Gamma(x, \varphi(x)) \\ \varphi(0) = \varphi'(0) = \varphi(L) = \varphi'(L) = 0 \end{array} , x \in [0, L]\right\}. \tag{2.7}$$

Proof. Let $T(\varphi) = \int_0^L G(x,t)\Gamma(t, \varphi(t))dt$. Then T is a contraction, indeed,

$$\begin{aligned} \|T(\varphi_1) - T(\varphi_2)\|_\infty &= \left\|\int_0^L G(x,t)\Gamma(t, \varphi_1(t))dt - \int_0^L G(x,t)\Gamma(t, \varphi_2(t))dt\right\|_\infty \\ &\leq M\|\Gamma(x, \varphi_1(x)) - \Gamma(x, \varphi_2(x))\|_\infty \\ &\leq \theta||\varphi_1 - \varphi_2||_\infty, \end{aligned}$$

for any $\varphi_1, \varphi_2 \in X$, and thus, T has a unique solution by Proposition 1.1. Let $\varphi_p = T(\varphi_p) = \int_0^L G(x,t)\Gamma(t, \varphi_p(t))dt$ be the unique fixed point of T. Then, we have

$$\begin{aligned} \|\varphi_{n+1} - \varphi_p\|_\infty &= \|T(\varphi_n) - T(\varphi_p)\|_\infty \\ &\leq \theta||\varphi_n - \varphi_p||_\infty \\ &\leq \theta^2||\varphi_{n-1} - \varphi_p||_\infty \\ &\dots \\ &\leq \theta^{n+1}||\varphi_0 - \varphi_p||_\infty \end{aligned}$$

which gives $\lim_{n\to\infty}||\varphi_{n+1} - \varphi_p|| = 0$, since $\theta < 1$. □

Example 2.6. *Let* $X = C^{(4)}[0,1]$ *and consider the following BVP*

$$\left.\begin{array}{c} \varphi''''(x) + 2^2\varphi''(x) = 2\varphi(x) + x^2 + 1 \\ \varphi(0) = \varphi(0) = \varphi(1) = \varphi(1) = 0 \end{array} , x \in [0, 1]\right\} \tag{2.8}$$

Then $M = 2.209e-01$*. Since* $\Gamma(x, \varphi(x)) = 2\varphi(x) + x^2 + 1$*,*

$$\begin{aligned} ||\Gamma(x, \varphi_1(x)) - \Gamma(x, \varphi_2(x))||_\infty &= ||2\varphi_1(x) + x^2 + 1 - (2\varphi_2(x) + x^2 + 1)||_\infty \\ &\leq 2||\varphi_1(x) - \varphi_2(x)||_\infty, K = 2, \end{aligned}$$

for all $\varphi_1, \varphi_2 \in X$*. Obviously, since* $KM < 1$*, by Proposition 1.1, we have*

$$\varphi_p(x) = \int_0^1 G(x,t)\Gamma(t, \varphi_p(t))dt$$

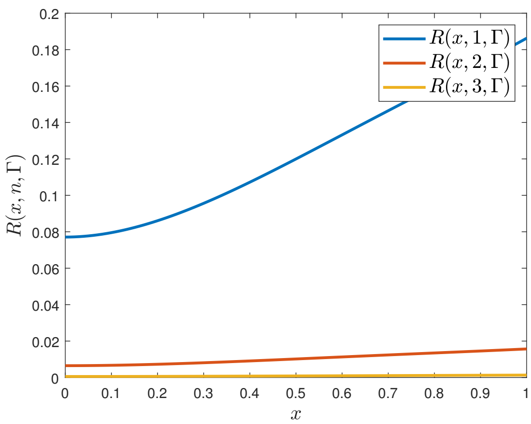
	n=1	**n=2**	**n=3**
	$R(x,1,f)$	$R(x,2,f)$	$R(x,3,f)$
0	7,710E-02	6,489E-03	5,481E-04
0.1	7,953E-02	6,701E-03	5,659E-04
0.2	8,607E-02	7,269E-03	6,139E-04
0.3	9,564E-02	8,095E-03	6,838E-04
0.4	1,072E-01	9,089E-03	7,678E-04
0.5	1,200E-01	1,017E-02	8,593E-04
0.6	1,332E-01	1,128E-02	9,531E-04
0.7	1,465E-01	1,239E-02	1,046E-03
0.8	1,596E-01	1,347E-02	1,138E-03
0.9	1,727E-01	1,455E-02	1,229E-03
1.0	1,863E-01	1,568E-02	1,324E-03

Table 2: Residual errors for $n = 1,2,$and 3

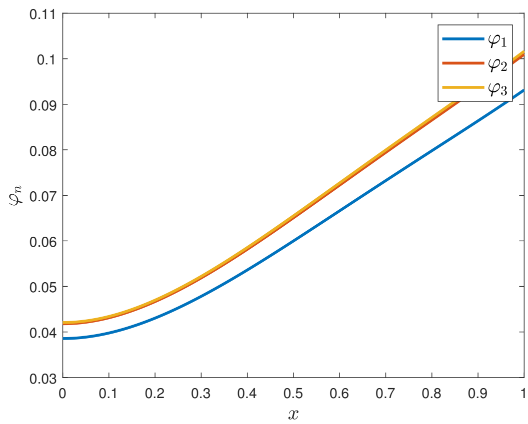
which is the solution of equation (2.8). Also,

$$\varphi_{n+1}(x) = \int_0^1 G(x,t)(2\varphi_n(t) + t^2 + 1)dt \tag{2.9}$$

is convergent to the solution φ_p. *Let* $R(x,n,\Gamma) = |\varphi_n''''(x) + \beta^2\varphi_n''(x) - \Gamma(x,\varphi_n(x))|$ *be the residual error for* $n > 0$. *Residual errors for* $n = 1,2,$*and* 3 *are shown in Figure 2 and Table 2.*



(a) $R(x,n,f)$ for $n = 1,2,$ and 3



(b) φ_n for $n = 1,2,$ and 3

Figure 2

Theorem 2.7. *Let* Γ *and* θ *be as in the Theorem 2.5 and* M *be as in Proposition 1.1. Assume that there exist* $\widetilde{\Gamma}(x,\varphi(x))$ *functions on* $[0,L] \times X$ *such that*

$$||\Gamma(x,\varphi(x)) - \widetilde{\Gamma}(x,\varphi(x))||_\infty \leq \xi.$$

If

$$\left.\begin{array}{c} \varphi''''(x) + \beta^2\varphi''(x) = \widetilde{\Gamma}(x,\varphi(x)) \\ \varphi(0) = \varphi'(0) = \varphi(L) = \varphi'(L) = 0 \end{array}, \ x \in [0,L] \right\} \tag{2.10}$$

has a solution, then

$$||\varphi_p - \widetilde{\varphi}_p||_\infty \leq M\frac{\xi_1}{1-\theta}$$

holds for $\varphi_p, \widetilde{\varphi}_p$ *which are the the solutions of BVPs (2.7) and (2.10),respectively.*

Proof. Let $T(\varphi)=\int\limits_0^L G(x,t)\Gamma(t,\varphi(t))dt$ and $S(\varphi)=\int\limits_0^L G(x,t)\widetilde{\Gamma}(t,\varphi(t))dt$. Then by Theorem 2.1, T has a fixed point φ_p which is the unique solution of system (1.2). Let $\varphi_0=\widetilde{\varphi}_p$ be the fixed point of S which is also the solution of system (2.10). Define $\varphi_{n+1}=T(\varphi_n)$. Then $\{\varphi_n\}_{n\geq 0}$ converges to φ_p by Theorem 2.5. Since

$$
\begin{aligned}
||\varphi_{n+1}-\varphi_n||_\infty &= ||T(\varphi_n)-T(\varphi_{n-1})||_\infty \\
&\leq \theta||\varphi_n-\varphi_{n-1}||_\infty \\
&\ \ \cdots \\
&\leq \theta^n||\varphi_1-\varphi_0||_\infty,
\end{aligned}
$$

$$
\begin{aligned}
||\varphi_n-\varphi_0||_\infty &\leq \sum_{k=1}^n ||\varphi_k-\varphi_{k-1}||_\infty \\
&\leq \sum_{k=1}^n \theta^{k-1}||\varphi_k-\varphi_{k-1}||_\infty \\
&\leq \frac{1}{1-\theta}||\varphi_k-\varphi_{k-1}||_\infty \\
&= \frac{1}{1-\theta}||T(\varphi_0)-S(\varphi_0)||_\infty \\
&= \frac{1}{1-\theta}\left(\left\|\int_0^L G(x,t)\Gamma(t,\varphi_0(t))dt-\int_0^L G(x,t)\widetilde{\Gamma}(t,\varphi_0(t))dt\right\|_\infty\right) \\
&\leq \frac{1}{1-\theta}\left(\left\|\int_0^L G(x,t)dt\right\|_\infty \left\|\Gamma(x,\varphi_0(x))-\widetilde{\Gamma}(x,\varphi_0(x))\right\|_\infty\right) \\
&= M\frac{\xi}{1-\theta}
\end{aligned}
$$

which implies that

$$||\varphi_p-\widetilde{\varphi}_p||_\infty \leq M\frac{\xi}{1-\theta}$$

□

Example 2.8. *Consider the following BVP:*

$$
\left.
\begin{aligned}
\varphi''''(x)+2^2\varphi''(x) = 2\varphi^{\varphi^2(x)+\cos(\varphi(x))}(x)+x^2+1 \\
\varphi(0)=\varphi'(0)=\varphi(1)=\varphi'(1)=0
\end{aligned}
\ , x\in[0,1]\right\} \tag{2.11}
$$

Solving this BVP directly is highly challenging or even infeasible due to the nonlinear functions involved in. However, thanks to Theorem 2.3, approximate solutions close to the exact one can be obtained without directly solving the equation.

Let Γ and β be as defined in Example 2.6 and $\overline{X}=\{\varphi\in X: 0\leq\varphi(x)\leq 1\}$. It can be observed from figure 2 that the solution of equation (2.5) belongs to $\overline{X}\times\overline{X}$. Then $\widetilde{\Gamma}(x,\varphi(x))=2\varphi^{\varphi^2(x)+\cos(\varphi(x))}(x)+x^2+1$ satisfy the following

$$
\begin{aligned}
||\Gamma(x,\varphi(x))-\widetilde{\Gamma}(x,\varphi(x))||_\infty &= \left\|2\varphi(x)+t^2+1-(2\varphi^{\varphi^2(x)+\cos(\varphi(x))}(x)+x^2+1)\right\|_\infty \\
&\leq \left\|2\varphi(x)-2\varphi^{\varphi^2(x)+\cos(\varphi(x))}(x)\right\|_\infty \\
&\leq 0.61e-01=\xi
\end{aligned}
$$

for all $\varphi\in X$ and since $M=2.209e-01$ and $\theta=KM=4.419e-01$.Then, by Theorem 2.3 we have the following estimate for the solution of the system (2.5)

$$
\begin{aligned}
||\varphi_p-\widetilde{\varphi}_p||_\infty &\leq M\frac{\xi_1}{1-\theta} \\
&= 2.42e-02
\end{aligned}
$$

in which φ_p is the solution of the equation (2.2) and $\widetilde{\varphi}_p$ is the solution of the equation (2.11). As a result, without solving the equation (2.11) which is more challenging to solve, it is possible to approximate the solution of the equation (2.11) by solving the simpler equation (2.2) , which closely resembles the original equation.

3. Conclusion

In this study, we have analyzed a system of interdependent fourth-order differential equations that model coupled physical phenomena, such as the bending of elastic beams and the vibrations of structural elements. By establishing conditions for the existence and uniqueness of solutions, we have provided a rigorous mathematical framework for addressing higher-order boundary value problems. Furthermore, our application of iterative methods not only demonstrates the solvability of such systems but also offers practical tools for engineers and scientists working on related applications.

Our findings contribute significantly to the literature by extending classical results on fourth-order boundary value problems and complementing prior works. Beyond the theoretical advancements, our results open several promising directions for future research. One key extension involves exploring more generalized nonlinear coupled systems and their numerical solutions. Additionally, investigating the stability and convergence properties of iterative methods in different boundary conditions could enhance their applicability.

In conclusion, this study underscores the importance of coupled fourth-order differential equation systems in mathematical modeling and highlights the need for advanced analytical and numerical techniques for their solution. The broader impact of this work lies in its potential to bridge theoretical insights with practical applications across multiple scientific and engineering domains.

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Conformable Fractional Milne-Type Inequalities Through Twice-Differentiable Convex Functions

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Abstract

In this article, we present a new integral identity based on conformable fractional integral operators with the help of twice-differentiable functions. Then, using this newly derived identity, we propose several Milne-type inequalities for twice-differentiable convex functions by means of conformable fractional integral operators and offer an example with an associated graph. Also, we note that the obtained results improve and expand some of the previous discoveries in the field of integral inequalities. Moreover, along with expanding on previous results, our results suggest effective approaches and methods for dealing with a variety of mathematical and scientific issues.

1. Introduction

In mathematics, the concept of convexity emerges as a fundamental idea, supported by extensive research and numerous practical applications, with a significant impact in various disciplines. Besides, convexity, by providing an essential framework for analyzing the geometric properties of sets and functions, forms the basis of various theories such as optimization theory, measure theory, approximation theory, and information theory, as well as their applications in science and engineering [1, 2, 3]. The formal definition of a convex function is given by the following:

Definition 1.1. *A function $\Lambda : I \to \mathbb{R}$ is said to be convex if for all $x, y \in I$ and $\lambda \in (0,1)$, we have*

$$\Lambda(\lambda x + (1-\lambda)y) \leq \lambda\Lambda(x) + (1-\lambda)\Lambda(y), \tag{1.1}$$

where I is an interval of the real numbers. In case the inequality (1.1) is reversed, Λ is known as concave.

Integral inequalities, used to determine the error bounds of numerical integration formulas, are an indispensable tool. For this reason, their applications have increased and impacted many contemporary areas of mathematics. The Hermite-Hadamard inequality [4], when expressed as below, is a fundamental inequality related to the concept of convexity:

$$\Lambda\left(\frac{\theta+\upsilon}{2}\right) \leq \frac{1}{\upsilon-\theta}\int_{\theta}^{\upsilon}\Lambda(x)dx \leq \frac{\Lambda(\theta)+\Lambda(\upsilon)}{2} \tag{1.2}$$

where $\Lambda : I \to \mathbb{R}$ is a convex function on I and $\theta, \upsilon \in I$ with $\theta < \upsilon$. When Λ is concave, both inequalities in the statement hold in the opposite direction. For a deeper exploration of the historical context of inequality (1.2), we suggest [5, 6, 7], and the sources they reference.

The Milne inequality, among integral inequalities, is the most prominent and widely cited inequality and it is formulated as follows:

$$\left|\frac{1}{3}\left[2\Lambda(\theta) - \Lambda\left(\frac{\theta+\upsilon}{2}\right) + 2\Lambda(\upsilon)\right] - \frac{1}{\upsilon-\theta}\int_{\theta}^{\upsilon}\Lambda(x)dx\right| \leq \frac{7(\upsilon-\theta)^4}{23040}\left\|\Lambda^{(4)}\right\|_{\infty},$$

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where $\Lambda:[\theta,\upsilon]\to\mathbb{R}$ is a four times differentiable mapping on (θ,υ) and $\left\|\Lambda^{(4)}\right\|_{\infty}=\sup_{x\in(\theta,\upsilon)}\left|\Lambda^{(4)}(x)\right|<\infty$.

This inequality, which determines the error bound of the integral value using the Milne rule, is extremely important. Therefore, it ensures the accuracy and reliability of numerical integration in various applications. For this reason, there has been a notable increase in research focusing on Milne inequality. In 2013, Alomari and Liu [8] conducted a study to predict the bounds of Milne's quadrature rule using reduced derivatives and convex functions. In [9], Rom*á*n-Flores et al. derived several Milne-type inequalities for interval-valued functions and explained their connections with other classical inequalities. In 2022, Djenaoui and Meftah [10] developed some new methods for Milne's quadrature rule applying the functions whose first derivative is π-convex. Focusing on strong multiplicative convex functions, Umar et al.[11] proved various Milne-type and Hermite-Hadamard-type integral inequalities.

On the other hand, fractional calculus, which extends the concepts of derivatives and integrals to non-integer orders, has increasingly become an effective tool in scientific fields such as physics, engineering, and chemistry [12, 13]. Since the beginning of fractional calculus, various fractional derivative and integral operators have been developed. Some notable examples include the Riemann-Liouville, conformable, Caputo, and Hadamard fractional integral operators, each of which plays a critical role in solving problems in applied mathematics and analysis.

Kilbas et al. [14] introduced the Riemann-Liouville fractional integral operators using the following approach:

Definition 1.2 ([14]). *The Riemann-Liouville integrals $I^{\varepsilon}_{\theta+}\Lambda(x)$ and $I^{\varepsilon}_{\upsilon-}\Lambda(x)$ of order $\varepsilon>0$ are given by*

$$I^{\varepsilon}_{\theta+}\Lambda(x)=\frac{1}{\Gamma(\varepsilon)}\int_{\theta}^{x}(x-\mu)^{\varepsilon-1}\Lambda(\mu)d\mu,\quad x>\theta, \tag{1.3}$$

and

$$I^{\varepsilon}_{\upsilon-}\Lambda(x)=\frac{1}{\Gamma(\varepsilon)}\int_{x}^{\upsilon}(\mu-x)^{\varepsilon-1}\Lambda(\mu)d\mu,\quad x<\upsilon, \tag{1.4}$$

respectively, where $\Lambda\in L_1[\theta,\upsilon]$. Here, Γ is the Gamma function defined by

$$\Gamma(\varepsilon):=\int_{0}^{\infty}\mu^{\varepsilon-1}e^{-\mu}d\mu.$$

Riemann-Liouville integrals are equal to the classical integrals for the case of $\varepsilon=1$.

Through these integral operators, several scientists conducted the studies to develop various integral inequalities. With the use of these operators, the studies focusing on the Milne inequality has gradually gained more importance through the years. Significant contributions can be found in [15, 16, 17, 18, 19] and further references therein.

New operators have been proposed to better define certain situations that classical fractional integral operators struggle to model effectively [20, 21]. Especially, conformable fractional integral operators, specified by Jarad et al. [22] as presented follows, not only come closer to the classical integral and differentiation principles but also generalize a range of fractional integral operators like Riemann-Liouville and Hadamard.

Definition 1.3. *The fractional conformable integral operator ${}^{\varepsilon}\mathfrak{J}^{\sigma}_{\theta+}\Lambda(x)$ and ${}^{\varepsilon}\mathfrak{J}^{\sigma}_{\upsilon-}\Lambda(x)$ of order $\varepsilon\in\mathbb{R}^{+}$ and $\sigma\in(0,1]$ are presented by*

$${}^{\varepsilon}\mathfrak{J}^{\sigma}_{\theta+}\Lambda(x)=\frac{1}{\Gamma(\varepsilon)}\int_{\theta}^{x}\left(\frac{(x-\theta)^{\sigma}-(\mu-\theta)^{\sigma}}{\sigma}\right)^{\varepsilon-1}\frac{\Lambda(\mu)}{(\mu-\theta)^{1-\sigma}}d\mu,\quad \mu>\theta, \tag{1.5}$$

and

$${}^{\varepsilon}\mathfrak{J}^{\sigma}_{\upsilon-}\Lambda(x)=\frac{1}{\Gamma(\varepsilon)}\int_{x}^{\upsilon}\left(\frac{(\upsilon-x)^{\sigma}-(\upsilon-\mu)^{\sigma}}{\sigma}\right)^{\varepsilon-1}\frac{\Lambda(\mu)}{(\upsilon-\mu)^{1-\sigma}}d\mu,\quad \mu<\upsilon, \tag{1.6}$$

respectively, where $\Lambda\in L_1[\theta,\upsilon]$.

Take notice that the fractional integral in (1.5) reduces to the Riemann-Liouville fractional integral in (1.3) if $\sigma=1$. Additionally, the fractional integral in (1.6) simplifies to the Riemann-Liouville fractional integral in (1.4) if $\sigma=1$.

Following the discovery of these innovative operators, remarkable research has been carried out to formulate inequalities based on such integral operators. For example, the aim of Set et al. [23] was to prove an identity for convex functions using fractional conformable integral operators and two types of Hermite-Hadamard inequalities. By utilizing conformable fractional integrals, in 2023, Hezenci et al. [24] developed new inequalities for the left and right sides of the Hermite-Hadamard inequality for twice-differentiable mappings. In 2024, Ying et al. [25], who investigated conformable fractional Milne-type inequalities, presented a comprehensive example with graphical representations that provide numerical support and visual confirmation of the established inequalities. Hezenci and Budak [26], with the aid of conformable fractional integrals, proved various trapezoid-type inequalities with π-convex functions. Moreover, they [27] developed many Bullen-type inequalities for twice-differentiable functions. In [28], Çelik et al. created new Milne type inequalities with the help of these operators for bounded functions, Lipschitzian functions and functions of bounded variation. For more information on the inequalities derived through this fractional integral operators with various functions, readers are referred to [29, 30, 31] and the references mentioned there.

In the sequel, the following definition will be utilized.

Definition 1.4. *Let $\sigma, \varepsilon > 0$. Then, the beta function is defined by*

$$\mathcal{B}(\sigma,\varepsilon) := \int_0^1 \mu^{\sigma-1}(1-\mu)^{\varepsilon-1}\,d\mu.$$

Also, let $0 \leq x \leq 1$. The incomplete beta function, a generalization of the beta function, is defined as

$$\mathcal{B}_x(\sigma,\varepsilon) := \int_0^x \mu^{\sigma-1}(1-\mu)^{\varepsilon-1}\,d\mu.$$

Meanwhile, the development of integral inequalities, has often been dependent on classical techniques like Hölder inequality and its alternative form, the power mean inequality.

Theorem 1.5 (Hölder inequality)**.** *Let $p,q>1$, $\frac{1}{p}+\frac{1}{q}=1$ and $\Lambda, g : [\theta,\upsilon] \to \mathbb{R}$. If $|\Lambda|^p$ and $|g|^q$ are integrable functions on $[\theta,\upsilon]$, then*

$$\int_\theta^\upsilon |\Lambda(\mu)g(\mu)|d\mu \leq \left(\int_\theta^\upsilon |\Lambda(\mu)|^p d\mu\right)^{\frac{1}{p}} \left(\int_\theta^\upsilon |g(\mu)|^q d\mu\right)^{\frac{1}{q}}.$$

Theorem 1.6 (Power mean inequality)**.** *Let $q \geq 1$ and $\Lambda, g : [\theta,\upsilon] \to \mathbb{R}$. If $|\Lambda|$ and $|\Lambda||g|^q$ are integrable functions on $[\theta,\upsilon]$, then*

$$\int_\theta^\upsilon |\Lambda(\mu)g(\mu)|d\mu \leq \left(\int_\theta^\upsilon |\Lambda(\mu)|d\mu\right)^{1-\frac{1}{q}} \left(\int_\theta^\upsilon |\Lambda(\mu)||g(\mu)|^q d\mu\right)^{\frac{1}{q}}.$$

In line with ongoing research and the articles mentioned above, this article aims to present similar versions of Milne-type inequalities in the context of Riemann integrals through the use of conformable fractional integral operators. To achieve this goal, we will first present an identity for twice-differentiable functions using conformable fractional integral operators. Then, we derive some important Milne-type inequalities by utilizing convexity, the Hölder inequality, and the power mean inequality. Given the proper assumptions on σ and ε, these results advance and generalize the inequalities obtained in prior studies.

2. Main results

This section focuses on deriving Milne-type inequalities for twice-differentiable convex functions within the framework of conformable fractional integrals. To achieve this, we begin by establishing the following identity, which serves as a foundation for obtaining conformable fractional forms of Milne-type inequalities.

Lemma 2.1. *If $\Lambda : [\theta,\upsilon] \to \mathbb{R}$ is a twice-differentiable function on (θ,υ) and $\Lambda'' \in L_1[\theta,\upsilon]$, then the following equality holds:*

$$\frac{1}{3}\left[2\Lambda(\theta) - \Lambda\left(\frac{\theta+\upsilon}{2}\right) + 2\Lambda(\upsilon)\right] - \frac{2^{\sigma\varepsilon-1}\sigma^\varepsilon\Gamma(\varepsilon+1)}{(\upsilon-\theta)^{\sigma\varepsilon}}\left[{}^{\varepsilon}\mathcal{J}^{\sigma}_{\theta+}\Lambda\left(\frac{\theta+\upsilon}{2}\right) + {}^{\varepsilon}\mathcal{J}^{\sigma}_{\upsilon-}\Lambda\left(\frac{\theta+\upsilon}{2}\right)\right] = \frac{(\upsilon-\theta)^2\sigma^\varepsilon}{8}[I_1+I_2], \tag{2.1}$$

where

$$\begin{cases} I_1 = \int\limits_0^1 \left(\int\limits_\mu^1 \left[\left(\frac{1-(1-\pi)^\sigma}{\sigma}\right)^\varepsilon + \frac{1}{3\sigma^\varepsilon}\right] d\pi\right) \Lambda''\left(\frac{1-\mu}{2}\theta + \frac{1+\mu}{2}\upsilon\right) d\mu, \\ I_2 = \int\limits_0^1 \left(\int\limits_\mu^1 \left[\left(\frac{1-(1-\pi)^\sigma}{\sigma}\right)^\varepsilon + \frac{1}{3\sigma^\varepsilon}\right] d\pi\right) \Lambda''\left(\frac{1+\mu}{2}\theta + \frac{1-\mu}{2}\upsilon\right) d\mu. \end{cases}$$

Proof. From the application of integration by parts, it follows that

$$\begin{aligned}
I_1 &= \int_0^1\left(\int_\mu^1\left[\left(\frac{1-(1-\pi)^\sigma}{\sigma}\right)^\varepsilon+\frac{1}{3\sigma^\varepsilon}\right]d\pi\right)\Lambda''\left(\frac{1-\mu}{2}\theta+\frac{1+\mu}{2}\upsilon\right)d\mu \\
&= \frac{2}{\upsilon-\theta}\left(\int_\mu^1\left[\left(\frac{1-(1-\pi)^\sigma}{\sigma}\right)^\varepsilon+\frac{1}{3\sigma^\varepsilon}\right]d\pi\right)\Lambda'\left(\frac{1-\mu}{2}\theta+\frac{1+\mu}{2}\upsilon\right)\Bigg|_0^1 \\
&\quad+\frac{2}{\upsilon-\theta}\int_0^1\left[\left(\frac{1-(1-\mu)^\sigma}{\sigma}\right)^\varepsilon+\frac{1}{3\sigma^\varepsilon}\right]\Lambda'\left(\frac{1-\mu}{2}\theta+\frac{1+\mu}{2}\upsilon\right)d\mu \\
&= -\frac{2}{\upsilon-\theta}\left(\int_0^1\left[\left(\frac{1-(1-\pi)^\sigma}{\sigma}\right)^\varepsilon+\frac{1}{3\sigma^\varepsilon}\right]d\pi\right)\Lambda'\left(\frac{\theta+\upsilon}{2}\right) \\
&\quad+\frac{2}{\upsilon-\theta}\left\{\frac{2}{\upsilon-\theta}\left[\left(\frac{1-(1-\mu)^\sigma}{\sigma}\right)^\varepsilon+\frac{1}{3\sigma^\varepsilon}\right]\Lambda\left(\frac{1-\mu}{2}\theta+\frac{1+\mu}{2}\upsilon\right)\Bigg|_0^1\right. \\
&\quad\left.-\frac{2\varepsilon}{\upsilon-\theta}\int_0^1\left(\frac{1-(1-\mu)^\sigma}{\sigma}\right)^{\varepsilon-1}(1-\mu)^{\sigma-1}\Lambda\left(\frac{1-\mu}{2}\theta+\frac{1+\mu}{2}\upsilon\right)d\mu\right\}.
\end{aligned}$$

If we utilize the change of variables $x=\frac{1-\mu}{2}\theta+\frac{1+\mu}{2}\upsilon$, we obtain

$$\begin{aligned}
I_1 &= -\frac{2}{\upsilon-\theta}\left(\int_0^1\left[\left(\frac{1-(1-\pi)^\sigma}{\sigma}\right)^\varepsilon+\frac{1}{3\sigma^\varepsilon}\right]d\pi\right)\Lambda'\left(\frac{\theta+\upsilon}{2}\right) \qquad (2.2)\\
&\quad+\frac{4}{(\upsilon-\theta)^2\sigma^\varepsilon}\left[\frac{4}{3}\Lambda(\upsilon)-\frac{1}{3}\Lambda\left(\frac{\theta+\upsilon}{2}\right)\right]-\left(\frac{2}{\upsilon-\theta}\right)^{\sigma\varepsilon+2}\varepsilon\int_{\frac{\theta+\upsilon}{2}}^{\upsilon}\left(\frac{\left(\frac{\upsilon-\theta}{2}\right)^\sigma-(\upsilon-x)^\sigma}{\sigma}\right)^{\varepsilon-1}\frac{\Lambda(x)}{(\upsilon-x)^{1-\sigma}}dx \\
&= -\frac{2}{\upsilon-\theta}\left(\int_0^1\left[\left(\frac{1-(1-\pi)^\sigma}{\sigma}\right)^\varepsilon+\frac{1}{3\sigma^\varepsilon}\right]d\pi\right)\Lambda'\left(\frac{\theta+\upsilon}{2}\right)+\frac{4}{(\upsilon-\theta)^2\sigma^\varepsilon}\left[\frac{4}{3}\Lambda(\upsilon)-\frac{1}{3}\Lambda\left(\frac{\theta+\upsilon}{2}\right)\right] \\
&\quad-\left(\frac{2}{\upsilon-\theta}\right)^{\sigma\varepsilon+2}\Gamma(\varepsilon+1)\ {}^{\varepsilon}\mathfrak{J}^{\sigma}_{\upsilon-}\Lambda\left(\frac{\theta+\upsilon}{2}\right).
\end{aligned}$$

By following a similar approach, the following result is achieved:

$$\begin{aligned}
I_2 &= \int_0^1\left(\int_\mu^1\left[\left(\frac{1-(1-\pi)^\sigma}{\sigma}\right)^\varepsilon+\frac{1}{3\sigma^\varepsilon}\right]d\pi\right)\Lambda''\left(\frac{1+\mu}{2}\theta+\frac{1-\mu}{2}\upsilon\right)d\mu \qquad (2.3)\\
&= \frac{2}{\upsilon-\theta}\left(\int_0^1\left[\left(\frac{1-(1-\pi)^\sigma}{\sigma}\right)^\varepsilon+\frac{1}{3\sigma^\varepsilon}\right]d\pi\right)\Lambda'\left(\frac{\theta+\upsilon}{2}\right)+\frac{4}{(\upsilon-\theta)^2\sigma^\varepsilon}\left[\frac{4}{3}\Lambda(\theta)-\frac{1}{3}\Lambda\left(\frac{\theta+\upsilon}{2}\right)\right] \\
&\quad-\left(\frac{2}{\upsilon-\theta}\right)^{\sigma\varepsilon+2}\Gamma(\varepsilon+1)\ {}^{\varepsilon}\mathfrak{J}^{\sigma}_{\theta+}\Lambda\left(\frac{\theta+\upsilon}{2}\right).
\end{aligned}$$

As a result, by merging the findings in (2.2) and (2.3) and multiplying it with $\frac{(\upsilon-\theta)^2\sigma^\varepsilon}{8}$, the equality given in (2.1) is established. □

Remark 2.2. *Let us choose $\sigma=1$ in Lemma 2.1. From this, we get the identity*

$$\begin{aligned}
&\frac{1}{3}\left[2\Lambda(\theta)-\Lambda\left(\frac{\theta+\upsilon}{2}\right)+2\Lambda(\upsilon)\right]-\frac{2^{\varepsilon-1}}{(\upsilon-\theta)^\varepsilon}\Gamma(\varepsilon+1)\left[I^\varepsilon_{\theta+}\Lambda\left(\frac{\theta+\upsilon}{2}\right)+I^\varepsilon_{\upsilon-}\Lambda\left(\frac{\theta+\upsilon}{2}\right)\right] \\
&= \frac{(\upsilon-\theta)^2}{24(\varepsilon+1)}\left[\int_0^1\left(\varepsilon+4-\mu(\varepsilon+1)-3\mu^{\varepsilon+1}\right)\left[\Lambda''\left(\frac{1-\mu}{\mu}\theta+\frac{1+\mu}{2}\upsilon\right)+\Lambda''\left(\frac{1+\mu}{2}\theta+\frac{1-\mu}{2}\upsilon\right)\right]d\mu\right],
\end{aligned}$$

which was presented by Budak et al. in [32].

Corollary 2.3. *By taking $\sigma=\varepsilon=1$ in Lemma 2.1, we derive*

$$\begin{aligned}&\frac{1}{3}\left[2\Lambda(\theta)-\Lambda\left(\frac{\theta+\upsilon}{2}\right)+2\Lambda(\upsilon)\right]-\frac{1}{\upsilon-\theta}\int_{\theta}^{\upsilon}\Lambda(\mu)d\mu\\ =\;&\frac{(\upsilon-\theta)^2}{48}\left[\int_0^1(3\mu+5)(1-\mu)\left[\Lambda''\left(\frac{1-\mu}{\mu}\theta+\frac{1+\mu}{2}\upsilon\right)+\Lambda''\left(\frac{1+\mu}{2}\theta+\frac{1-\mu}{2}\upsilon\right)\right]d\mu\right].\end{aligned}$$

Theorem 2.4. *Let $\Lambda:[\theta,\upsilon]\to\mathbb{R}$ be a twice-differentiable function on (θ,υ) such that $\Lambda''\in L_1[\theta,\upsilon]$. If $|\Lambda''|$ is a convex function on $[\theta,\upsilon]$, then the following inequality holds:*

$$\begin{aligned}&\left|\frac{1}{3}\left[2\Lambda(\theta)-\Lambda\left(\frac{\theta+\upsilon}{2}\right)+2\Lambda(\upsilon)\right]-\frac{2^{\sigma\varepsilon-1}\sigma^{\varepsilon}\Gamma(\varepsilon+1)}{(\upsilon-\theta)^{\sigma\varepsilon}}\left[{}^{\varepsilon}\mathcal{J}^{\sigma}_{\theta^+}\Lambda\left(\frac{\theta+\upsilon}{2}\right)+{}^{\varepsilon}\mathcal{J}^{\sigma}_{\upsilon^-}\Lambda\left(\frac{\theta+\upsilon}{2}\right)\right]\right| \qquad (2.4)\\ \leq\;&\frac{(\upsilon-\theta)^2\sigma^{\varepsilon}}{8}\psi_1(\sigma,\varepsilon)\left[\left|\Lambda''(\theta)\right|+\left|\Lambda''(\upsilon)\right|\right],\end{aligned}$$

where

$$\begin{aligned}\psi_1(\sigma,\varepsilon)&=\int_0^1\left|\int_{\mu}^1\left[\left(\frac{1-(1-\pi)^{\sigma}}{\sigma}\right)^{\varepsilon}+\frac{1}{3\sigma^{\varepsilon}}\right]d\pi\right|d\mu\\ &=\frac{1}{\sigma^{\varepsilon}}\int_0^1\left(\frac{1}{\sigma}\left(\mathcal{B}_{(1-\mu)^{\sigma}}\left(\frac{1}{\sigma},\varepsilon+1\right)\right)+\frac{1-\mu}{3}\right)d\mu.\end{aligned}$$

Proof. If we take the absolute value of the identity (2.1), we get

$$\begin{aligned}&\left|\frac{1}{3}\left[2\Lambda(\theta)-\Lambda\left(\frac{\theta+\upsilon}{2}\right)+2\Lambda(\upsilon)\right]-\frac{2^{\sigma\varepsilon-1}\sigma^{\varepsilon}\Gamma(\varepsilon+1)}{(\upsilon-\theta)^{\sigma\varepsilon}}\left[{}^{\varepsilon}\mathcal{J}^{\sigma}_{\theta^+}\Lambda\left(\frac{\theta+\upsilon}{2}\right)+{}^{\varepsilon}\mathcal{J}^{\sigma}_{\upsilon^-}\Lambda\left(\frac{\theta+\upsilon}{2}\right)\right]\right|\\ \leq\;&\frac{(\upsilon-\theta)^2\sigma^{\varepsilon}}{8}\left[\int_0^1\left|\int_{\mu}^1\left[\left(\frac{1-(1-\pi)^{\sigma}}{\sigma}\right)^{\varepsilon}+\frac{1}{3\sigma^{\varepsilon}}\right]d\pi\right|\left|\Lambda''\left(\frac{1-\mu}{2}\theta+\frac{1+\mu}{2}\upsilon\right)\right|d\mu\right.\\ &\left.+\int_0^1\left|\int_{\mu}^1\left[\left(\frac{1-(1-\pi)^{\sigma}}{\sigma}\right)^{\varepsilon}+\frac{1}{3\sigma^{\varepsilon}}\right]d\pi\right|\left|\Lambda''\left(\frac{1+\mu}{2}\theta+\frac{1-\mu}{2}\upsilon\right)\right|d\mu\right].\end{aligned}$$

Considering that the function $|\Lambda''|$ is convex, we obtain

$$\begin{aligned}&\left|\frac{1}{3}\left[2\Lambda(\theta)-\Lambda\left(\frac{\theta+\upsilon}{2}\right)+2\Lambda(\upsilon)\right]-\frac{2^{\sigma\varepsilon-1}\sigma^{\varepsilon}\Gamma(\varepsilon+1)}{(\upsilon-\theta)^{\sigma\varepsilon}}\left[{}^{\varepsilon}\mathcal{J}^{\sigma}_{\theta^+}\Lambda\left(\frac{\theta+\upsilon}{2}\right)+{}^{\varepsilon}\mathcal{J}^{\sigma}_{\upsilon^-}\Lambda\left(\frac{\theta+\upsilon}{2}\right)\right]\right|\\ \leq\;&\frac{(\upsilon-\theta)^2\sigma^{\varepsilon}}{8}\left[\int_0^1\left|\int_{\mu}^1\left[\left(\frac{1-(1-\pi)^{\sigma}}{\sigma}\right)^{\varepsilon}+\frac{1}{3\sigma^{\varepsilon}}\right]d\pi\right|\left(\frac{1-\mu}{2}\left|\Lambda''(\theta)\right|+\frac{1+\mu}{2}\left|\Lambda''(\upsilon)\right|\right)d\mu\right.\\ &\left.+\int_0^1\left|\int_{\mu}^1\left[\left(\frac{1-(1-\pi)^{\sigma}}{\sigma}\right)^{\varepsilon}+\frac{1}{3\sigma^{\varepsilon}}\right]d\pi\right|\left(\frac{1+\mu}{2}\left|\Lambda''(\theta)\right|+\frac{1-\mu}{2}\left|\Lambda''(\upsilon)\right|\right)d\mu\right]\\ =\;&\frac{(\upsilon-\theta)^2\sigma^{\varepsilon}}{8}\left(\int_0^1\left|\int_{\mu}^1\left[\left(\frac{1-(1-\pi)^{\sigma}}{\sigma}\right)^{\varepsilon}+\frac{1}{3\sigma^{\varepsilon}}\right]d\pi\right|d\mu\right)\left[\left|\Lambda''(\theta)\right|+\left|\Lambda''(\upsilon)\right|\right].\end{aligned}$$

Thus, we reach at the result (2.4). □

Remark 2.5. *Consider $\sigma = 1$ in Theorem 2.4. In this case, the inequality (2.4) is the Milne-type inequality for twice-differentiable convex functions, involving the Riemann-Liouville fractional integral operators:*

$$\left|\frac{1}{3}\left[2\Lambda(\theta)-\Lambda\left(\frac{\theta+\upsilon}{2}\right)+2\Lambda(\upsilon)\right]-\frac{2^{\varepsilon-1}}{(\upsilon-\theta)^{\varepsilon}}\Gamma(\varepsilon+1)\left[I^{\varepsilon}_{\theta^+}\Lambda\left(\frac{\theta+\upsilon}{2}\right)+I^{\varepsilon}_{\upsilon^-}\Lambda\left(\frac{\theta+\upsilon}{2}\right)\right]\right|$$

$$\leq \frac{(\upsilon-\theta)^2}{8}\psi_1(1,\varepsilon)\left[\left|\Lambda''(\theta)\right|+\left|\Lambda''(\upsilon)\right|\right],$$

satisfying

$$\psi_1(1,\varepsilon)=\int_0^1\left|\int_\mu^1\left(\pi^{\varepsilon}+\frac{1}{3}\right)d\pi\right|d\mu=\int_0^1\left(\frac{1}{\varepsilon+1}+\frac{1}{3}-\frac{\mu^{\varepsilon+1}}{\varepsilon+1}-\frac{\mu}{3}\right)d\mu=\frac{\varepsilon+8}{6(\varepsilon+2)},$$

which was given by Budak et al. in [32].

Remark 2.6. *By taking $\sigma = 1$ and $\varepsilon = 1$, we get,*

$$\left|\frac{1}{3}\left[2\Lambda(\theta)-\Lambda\left(\frac{\theta+\upsilon}{2}\right)+2\Lambda(\upsilon)\right]-\frac{1}{\upsilon-\theta}\int_\theta^\upsilon\Lambda(\mu)d\mu\right|\leq\frac{(\upsilon-\theta)^2}{16}\left[\left|\Lambda''(\theta)\right|+\left|\Lambda''(\upsilon)\right|\right],$$

which was provided by Budak et al. in [32].

Example 2.7. *Considering the function $\Lambda(x) = x^4$ on the interval $[0,1]$, we proceed to calculate the right-hand side of inequality (2.4) as follows:*

$$\frac{3}{2}\int_0^1\left[\frac{1}{\sigma}\mathcal{B}_{(1-\mu)^{\sigma}}\left(\frac{1}{\sigma},\varepsilon+1\right)+\frac{1-\mu}{3}\right]d\mu:=\Psi_1.$$

In addition, it is apparent that

$$\left|\frac{1}{3}\left[2\Lambda(\theta)-\Lambda\left(\frac{\theta+\upsilon}{2}\right)+2\Lambda(\upsilon)\right]-\frac{2^{\sigma\varepsilon-1}\sigma^{\varepsilon}\Gamma(\varepsilon+1)}{(\upsilon-\theta)^{\sigma\varepsilon}}\left[{}^{\varepsilon}\mathcal{J}^{\sigma}_{\theta^+}\Lambda\left(\frac{\theta+\upsilon}{2}\right)+{}^{\varepsilon}\mathcal{J}^{\sigma}_{\upsilon^-}\Lambda\left(\frac{\theta+\upsilon}{2}\right)\right]\right|$$

$$=\left|\frac{31}{48}-\frac{\varepsilon}{2}\left(\frac{1}{\varepsilon}-2\mathcal{B}\left(\frac{1}{\sigma}+1,\varepsilon\right)+\frac{3}{2}\mathcal{B}\left(\frac{2}{\sigma}+1,\varepsilon\right)-\frac{1}{2}\mathcal{B}\left(\frac{3}{\sigma}+1,\varepsilon\right)+\frac{1}{8}\mathcal{B}\left(\frac{4}{\sigma}+1,\varepsilon\right)\right)\right|:=\Psi_2.$$

Thus, as Figure 1 illustrates, the left side of the inequality (2.4) is always situated beneath the right side of this inequality for all $0<\sigma<1$ and $0<\varepsilon<10$.

Theorem 2.8. *Let $\Lambda:[\theta,\upsilon]\to\mathbb{R}$ be a twice-differentiable function on (θ,υ) such that $\Lambda''\in L_1[\theta,\upsilon]$. If $|\Lambda''|^q$ is a convex function on $[\theta,\upsilon]$ with $q>1$, then the following inequalities hold:*

$$\left|\frac{1}{3}\left[2\Lambda(\theta)-\Lambda\left(\frac{\theta+\upsilon}{2}\right)+2\Lambda(\upsilon)\right]-\frac{2^{\sigma\varepsilon-1}\sigma^{\varepsilon}\Gamma(\varepsilon+1)}{(\upsilon-\theta)^{\sigma\varepsilon}}\left[{}^{\varepsilon}\mathcal{J}^{\sigma}_{\theta^+}\Lambda\left(\frac{\theta+\upsilon}{2}\right)+{}^{\varepsilon}\mathcal{J}^{\sigma}_{\upsilon^-}\Lambda\left(\frac{\theta+\upsilon}{2}\right)\right]\right|$$

$$\leq\frac{(\upsilon-\theta)^2\sigma^{\varepsilon}}{8}\varphi(\sigma,\varepsilon,p)\left[\left(\frac{|\Lambda''(\theta)|^q+3|\Lambda''(\upsilon)|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{3|\Lambda''(\theta)|^q+|\Lambda''(\upsilon)|^q}{4}\right)^{\frac{1}{q}}\right]$$

$$\leq\frac{(\upsilon-\theta)^2\sigma^{\varepsilon}}{8}4^{\frac{1}{p}}\varphi(\sigma,\varepsilon,p)\left[|\Lambda''(\theta)|+|\Lambda''(\upsilon)|\right],$$

where $\frac{1}{p}+\frac{1}{q}=1$ and

$$\begin{aligned}\varphi(\sigma,\varepsilon,p)&=\left(\int_0^1\left|\int_\mu^1\left[\left(\frac{1-(1-\pi)^{\sigma}}{\sigma}\right)^{\varepsilon}+\frac{1}{3\sigma^{\varepsilon}}\right]d\pi\right|^p d\mu\right)^{\frac{1}{p}}\\&=\frac{1}{\sigma^{\varepsilon}}\left(\int_0^1\left[\frac{1}{\sigma}\mathcal{B}_{(1-\mu)^{\sigma}}\left(\frac{1}{\sigma},\varepsilon+1\right)+\frac{1-\mu}{3}\right]^p d\mu\right)^{\frac{1}{p}}.\end{aligned}$$

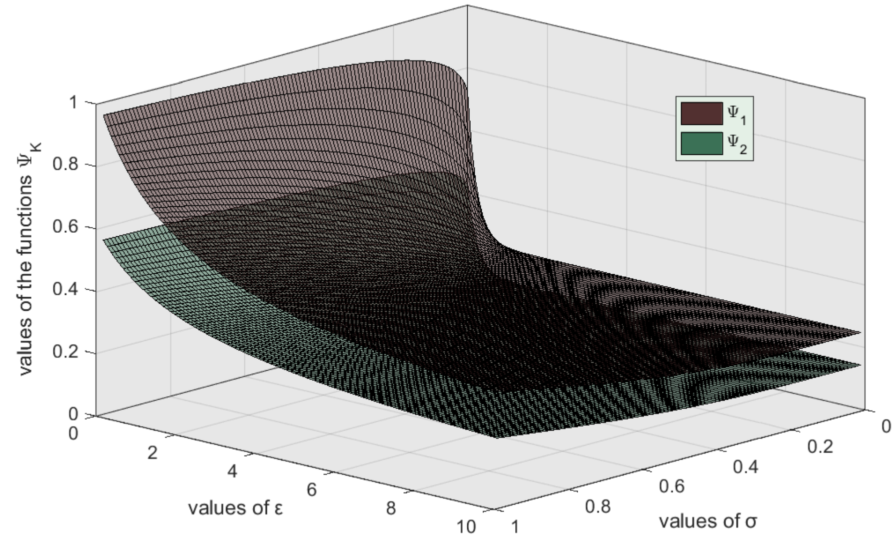

Figure 1: The graph of both sides of inequality (2.4) according to Example 1, which is computed and drawn by MATLAB program, depending on $\sigma \in (0,1)$ and $\varepsilon \in (0,10)$.

Proof. By utilizing the well-known Hölder's inequality, according to Lemma 2.1, we establish

$$\left|\frac{1}{3}\left[2\Lambda(\theta)-\Lambda\left(\frac{\theta+\upsilon}{2}\right)+2\Lambda(\upsilon)\right]-\frac{2^{\sigma\varepsilon-1}\sigma^{\varepsilon}\Gamma(\varepsilon+1)}{(\upsilon-\theta)^{\sigma\varepsilon}}\left[{}^{\varepsilon}\mathfrak{J}^{\sigma}_{\theta^{+}}\Lambda\left(\frac{\theta+\upsilon}{2}\right)+{}^{\varepsilon}\mathfrak{J}^{\sigma}_{\upsilon^{-}}\Lambda\left(\frac{\theta+\upsilon}{2}\right)\right]\right|$$

$$\leq \frac{(\upsilon-\theta)^{2}\sigma^{\varepsilon}}{8}\left[\left(\int_{0}^{1}\left|\int_{\mu}^{1}\left[\left(\frac{1-(1-\pi)^{\sigma}}{\sigma}\right)^{\varepsilon}+\frac{1}{3\sigma^{\varepsilon}}\right]d\pi\right|^{p}d\mu\right)^{\frac{1}{p}}\left(\int_{0}^{1}\left|\Lambda''\left(\frac{1-\mu}{2}\theta+\frac{1+\mu}{2}\upsilon\right)\right|^{q}d\mu\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\int_{0}^{1}\left|\int_{\mu}^{1}\left[\left(\frac{1-(1-\pi)^{\sigma}}{\sigma}\right)^{\varepsilon}+\frac{1}{3\sigma^{\varepsilon}}\right]d\pi\right|^{p}d\mu\right)^{\frac{1}{p}}\left(\int_{0}^{1}\left|\Lambda''\left(\frac{1+\mu}{2}\theta+\frac{1-\mu}{2}\upsilon\right)\right|^{q}d\mu\right)^{\frac{1}{q}}\right].$$

Since $|\Lambda''|^{q}$ is a convex function, we have

$$\left|\frac{1}{3}\left[2\Lambda(\theta)-\Lambda\left(\frac{\theta+\upsilon}{2}\right)+2\Lambda(\upsilon)\right]-\frac{2^{\sigma\varepsilon-1}\sigma^{\varepsilon}\Gamma(\varepsilon+1)}{(\upsilon-\theta)^{\sigma\varepsilon}}\left[{}^{\varepsilon}\mathfrak{J}^{\sigma}_{\theta^{+}}\Lambda\left(\frac{\theta+\upsilon}{2}\right)+{}^{\varepsilon}\mathfrak{J}^{\sigma}_{\upsilon^{-}}\Lambda\left(\frac{\theta+\upsilon}{2}\right)\right]\right|$$

$$\leq \frac{(\upsilon-\theta)^{2}\sigma^{\varepsilon}}{8}\left(\int_{0}^{1}\left|\int_{\mu}^{1}\left[\left(\frac{1-(1-\pi)^{\sigma}}{\sigma}\right)^{\varepsilon}+\frac{1}{3\sigma^{\varepsilon}}\right]d\pi\right|^{p}d\mu\right)^{\frac{1}{p}}$$

$$\times\left[\left(\int_{0}^{1}\left(\frac{1-\mu}{2}\left|\Lambda''(\theta)\right|^{q}+\frac{1+\mu}{2}\left|\Lambda''(\upsilon)\right|^{q}\right)d\mu\right)^{\frac{1}{q}}+\left(\int_{0}^{1}\left(\frac{1+\mu}{2}\left|\Lambda''(\theta)\right|^{q}+\frac{1-\mu}{2}\left|\Lambda''(\upsilon)\right|^{q}\right)d\mu\right)^{\frac{1}{q}}\right]$$

$$= \frac{(\upsilon-\theta)^{2}\sigma^{\varepsilon}}{8}\left(\int_{0}^{1}\left|\int_{\mu}^{1}\left[\left(\frac{1-(1-\pi)^{\sigma}}{\sigma}\right)^{\varepsilon}+\frac{1}{3\sigma^{\varepsilon}}\right]d\pi\right|^{p}d\mu\right)^{\frac{1}{p}}$$

$$\times\left[\left(\frac{\left|\Lambda''(\theta)\right|^{q}+3\left|\Lambda''(\upsilon)\right|^{q}}{4}\right)^{\frac{1}{q}}+\left(\frac{3\left|\Lambda''(\theta)\right|^{q}+\left|\Lambda''(\upsilon)\right|^{q}}{4}\right)^{\frac{1}{q}}\right].$$

Moreover, for $0\leq\pi<1$ and $\eta_k,\rho_k\geq 0$ with $k\in\{1,2,...,n\}$, the inequality

$$\sum_{k=1}^{n}(\eta_k+\rho_k)^{\pi}\leq\sum_{k=1}^{n}\eta_k^{\pi}+\sum_{k=1}^{n}\rho_k^{\pi}$$

is a widely acknowledged property. Therefore, the proof of the second inequality follows easily by choosing $\eta_1 = 3|\Lambda''(\theta)|^q$, $\rho_1 = |\Lambda''(\upsilon)|^q$, $\eta_2 = |\Lambda''(\theta)|^q$ and $\rho_2 = 3|\Lambda''(\upsilon)|^q$, under the assumption that $1+3^{\frac{1}{q}} \leq 4$. □

Remark 2.9. *By specifying $\sigma = 1$ in Theorem 2.8, we get a Milne-type inequality for twice-differentiable convex functions, incorporating the Riemann-Liouville fractional integral operators:*

$$\left|\frac{1}{3}\left[2\Lambda(\theta)-\Lambda\left(\frac{\theta+\upsilon}{2}\right)+2\Lambda(\upsilon)\right]-\frac{2^{\varepsilon-1}}{(\upsilon-\theta)^{\varepsilon}}\Gamma(\varepsilon+1)\left[I^{\varepsilon}_{\theta^+}\Lambda\left(\frac{\theta+\upsilon}{2}\right)+I^{\varepsilon}_{\upsilon^-}\Lambda\left(\frac{\theta+\upsilon}{2}\right)\right]\right|$$

$$\leq \frac{(\upsilon-\theta)^2}{8}\varphi(1,\varepsilon,p)\left[\left(\frac{|\Lambda''(\theta)|^q+3|\Lambda''(\upsilon)|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{3|\Lambda''(\theta)|^q+|\Lambda''(\upsilon)|^q}{4}\right)^{\frac{1}{q}}\right]$$

$$\leq \frac{(\upsilon-\theta)^2}{8}4^{\frac{1}{p}}\varphi(1,\varepsilon,p)\left[|\Lambda''(\theta)|+|\Lambda''(\upsilon)|\right],$$

where

$$\varphi(1,\varepsilon,p)=\left(\int_0^1\left|\int_\mu^1\left(\pi^{\varepsilon}+\frac{1}{3}\right)^p d\pi\right|d\mu\right)^{\frac{1}{p}}=\frac{1}{3(\varepsilon+1)}\left(\int_0^1\left(\varepsilon+4-\mu(\varepsilon+1)-3\mu^{\varepsilon+1}\right)^p d\mu\right)^{\frac{1}{p}}.$$

This finding was established by Budak et al. [32].

Remark 2.10. *Setting $\sigma = 1$ and $\varepsilon = 1$ in Theorem 2.8 yields a Milne-type inequality for twice-differentiable convex functions, based on classical Riemann integral operator:*

$$\left|\frac{1}{3}\left[2\Lambda(\theta)-\Lambda\left(\frac{\theta+\upsilon}{2}\right)+2\Lambda(\upsilon)\right]-\frac{1}{\upsilon-\theta}\int_\theta^\upsilon\Lambda(x)\,dx\right|$$

$$\leq \frac{(\upsilon-\theta)^2}{8}\left(\int_0^1\left(\int_\mu^1\left(\pi+\frac{1}{3}\right)d\pi\right)^p d\mu\right)^{\frac{1}{p}}\left[\left(\frac{|\Lambda''(\theta)|^q+3|\Lambda''(\upsilon)|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{3|\Lambda''(\theta)|^q+|\Lambda''(\upsilon)|^q}{4}\right)^{\frac{1}{q}}\right]$$

$$\leq \frac{(\upsilon-\theta)^2}{48}\left(\int_0^1[(5+3\mu)(1-\mu)]^p d\mu\right)^{\frac{1}{p}}\left[|\Lambda''(\theta)|+|\Lambda''(\upsilon)|\right].$$

The validity of this result was confirmed by Budak et al. [32].

Theorem 2.11. *Let $\Lambda : [\theta,\upsilon] \to \mathbb{R}$ be a twice-differentiable function on (θ,υ) such that $\Lambda'' \in L_1([\theta,\upsilon])$. If $|\Lambda''|^q$ is a convex function on $[\theta,\upsilon]$ with $q > 1$, then the following inequality holds:*

$$\left|\frac{1}{3}\left[2\Lambda(\theta)-\Lambda\left(\frac{\theta+\upsilon}{2}\right)+2\Lambda(\upsilon)\right]-\frac{2^{\sigma\varepsilon-1}\sigma^{\varepsilon}\Gamma(\varepsilon+1)}{(\upsilon-\theta)^{\sigma\varepsilon}}\left[{}^{\varepsilon}\mathfrak{J}^{\sigma}_{\theta^+}\Lambda\left(\frac{\theta+\upsilon}{2}\right)+{}^{\varepsilon}\mathfrak{J}^{\sigma}_{\upsilon^-}\Lambda\left(\frac{\theta+\upsilon}{2}\right)\right]\right|$$

$$\leq \frac{(\upsilon-\theta)^2\sigma^{\varepsilon}}{8}(\psi_1(\sigma,\varepsilon))^{1-\frac{1}{q}}\left[\left(\left(\frac{\psi_1(\sigma,\varepsilon)}{2}-\psi_2(\sigma,\varepsilon)\right)|\Lambda''(\theta)|^q+\left(\frac{\psi_1(\sigma,\varepsilon)}{2}+\psi_2(\sigma,\varepsilon)\right)|\Lambda''(\upsilon)|^q\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\left(\frac{\psi_1(\sigma,\varepsilon)}{2}+\psi_2(\sigma,\varepsilon)\right)|\Lambda''(\theta)|^q+\left(\frac{\psi_1(\sigma,\varepsilon)}{2}-\psi_2(\sigma,\varepsilon)\right)|\Lambda''(\upsilon)|^q\right)^{\frac{1}{q}}\right],$$

where $\psi_1(\sigma,\varepsilon)$ is expressed as in Theorem 2.4 and

$$\psi_2(\sigma,\varepsilon) = \int_0^1\frac{\mu}{2}\left|\int_0^\mu\left[\left(\frac{1-(1-\pi)^{\sigma}}{\sigma}\right)^{\varepsilon}+\frac{1}{3\sigma^{\varepsilon}}\right]d\pi\right|d\mu$$

$$= \frac{1}{2\sigma^{\varepsilon}}\int_0^1\mu\left(\frac{1}{\sigma}\mathcal{B}_{(1-\mu)^{\sigma}}\left(\frac{1}{\sigma},\varepsilon+1\right)+\frac{1-\mu}{3}\right)d\mu.$$

Proof. By applying the absolute value to the identity (2.1), by the power mean inequality, we have

$$\left|\frac{1}{3}\left[2\Lambda(\theta)-\Lambda\left(\frac{\theta+\upsilon}{2}\right)+2\Lambda(\upsilon)\right]-\frac{2^{\sigma\varepsilon-1}\sigma^{\varepsilon}\Gamma(\varepsilon+1)}{(\upsilon-\theta)^{\sigma\varepsilon}}\left[{}^{\varepsilon}\mathcal{J}^{\sigma}_{\theta^+}\Lambda\left(\frac{\theta+\upsilon}{2}\right)+{}^{\varepsilon}\mathcal{J}^{\sigma}_{\upsilon^-}\Lambda\left(\frac{\theta+\upsilon}{2}\right)\right]\right|$$

$$\leq \frac{(\upsilon-\theta)^2\sigma^{\varepsilon}}{8}\left[\left(\int_0^1\left|\int_\mu^1\left[\left(\frac{1-(1-\pi)^{\sigma}}{\sigma}\right)^{\varepsilon}+\frac{1}{3\sigma^{\varepsilon}}\right]d\pi\right|d\mu\right)^{1-\frac{1}{q}}\right.$$

$$\times\left(\int_0^1\left|\int_\mu^1\left[\left(\frac{1-(1-\pi)^{\sigma}}{\sigma}\right)^{\varepsilon}+\frac{1}{3\sigma^{\varepsilon}}\right]d\pi\right|\left|\Lambda''\left(\frac{1-\mu}{2}\theta+\frac{1+\mu}{2}\upsilon\right)\right|^q d\mu\right)^{\frac{1}{q}}$$

$$+\left(\int_0^1\left|\int_\mu^1\left[\left(\frac{1-(1-\pi)^{\sigma}}{\sigma}\right)^{\varepsilon}+\frac{1}{3\sigma^{\varepsilon}}\right]d\pi\right|d\mu\right)^{1-\frac{1}{q}}$$

$$\left.\times\left(\int_0^1\left|\int_\mu^1\left[\left(\frac{1-(1-\pi)^{\sigma}}{\sigma}\right)^{\varepsilon}+\frac{1}{3\sigma^{\varepsilon}}\right]d\pi\right|\left|\Lambda''\left(\frac{1+\mu}{2}\theta+\frac{1-\mu}{2}\upsilon\right)\right|^q d\mu\right)^{\frac{1}{q}}\right].$$

Taking into account the convexity of the $|\Lambda''|^q$, we obtain

$$\left|\frac{1}{3}\left[2\Lambda(\theta)-\Lambda\left(\frac{\theta+\upsilon}{2}\right)+2\Lambda(\upsilon)\right]-\frac{2^{\sigma\varepsilon-1}\sigma^{\varepsilon}\Gamma(\varepsilon+1)}{(\upsilon-\theta)^{\sigma\varepsilon}}\left[{}^{\varepsilon}\mathcal{J}^{\sigma}_{\theta^+}\Lambda\left(\frac{\theta+\upsilon}{2}\right)+{}^{\varepsilon}\mathcal{J}^{\sigma}_{\upsilon^-}\Lambda\left(\frac{\theta+\upsilon}{2}\right)\right]\right|$$

$$\leq \frac{(\upsilon-\theta)^2\sigma^{\varepsilon}}{8}\left(\int_0^1\left|\int_\mu^1\left[\left(\frac{1-(1-\pi)^{\sigma}}{\sigma}\right)^{\varepsilon}+\frac{1}{3\sigma^{\varepsilon}}\right]d\pi\right|d\mu\right)^{1-\frac{1}{q}}$$

$$\times\left[\left(\int_0^1\left|\int_\mu^1\left[\left(\frac{1-(1-\pi)^{\sigma}}{\sigma}\right)^{\varepsilon}+\frac{1}{3\sigma^{\varepsilon}}\right]d\pi\right|\left[\frac{1-\mu}{2}\left|\Lambda''(\theta)\right|^q+\frac{1+\mu}{2}\left|\Lambda''(\upsilon)\right|^q\right]d\mu\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\int_0^1\left|\int_\mu^1\left[\left(\frac{1-(1-\pi)^{\sigma}}{\sigma}\right)^{\varepsilon}+\frac{1}{3\sigma^{\varepsilon}}\right]d\pi\right|\left[\frac{1+\mu}{2}\left|\Lambda''(\theta)\right|^q+\frac{1-\mu}{2}\left|\Lambda''(\upsilon)\right|^q\right]d\mu\right)^{\frac{1}{q}}\right].$$

Therefore, it is inferred that

$$\left|\frac{1}{3}\left[2\Lambda(\theta)-\Lambda\left(\frac{\theta+\upsilon}{2}\right)+2\Lambda(\upsilon)\right]-\frac{2^{\sigma\varepsilon-1}\sigma^{\varepsilon}\Gamma(\varepsilon+1)}{(\upsilon-\theta)^{\sigma\varepsilon}}\left[{}^{\varepsilon}\mathcal{J}^{\sigma}_{\theta^+}\Lambda\left(\frac{\theta+\upsilon}{2}\right)+{}^{\varepsilon}\mathcal{J}^{\sigma}_{\upsilon^-}\Lambda\left(\frac{\theta+\upsilon}{2}\right)\right]\right|$$

$$\leq \frac{(\upsilon-\theta)^2\sigma^{\varepsilon}}{8}\left(\psi_1(\sigma,\varepsilon)\right)^{1-\frac{1}{q}}\left[\left(\left(\frac{\psi_1(\sigma,\varepsilon)}{2}-\psi_2(\sigma,\varepsilon)\right)\left|\Lambda''(\theta)\right|^q+\left(\frac{\psi_1(\sigma,\varepsilon)}{2}+\psi_2(\sigma,\varepsilon)\right)\left|\Lambda''(\upsilon)\right|^q\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\left(\frac{\psi_1(\sigma,\varepsilon)}{2}+\psi_2(\sigma,\varepsilon)\right)\left|\Lambda''(\theta)\right|^q+\left(\frac{\psi_1(\sigma,\varepsilon)}{2}-\psi_2(\sigma,\varepsilon)\right)\left|\Lambda''(\upsilon)\right|^q\right)^{\frac{1}{q}}\right].$$

□

Remark 2.12. *Under the assumption $\sigma = 1$ in Theorem 2.11, we arrive at the following inequality:*

$$\left|\frac{1}{3}\left[2\Lambda(\theta)-\Lambda\left(\frac{\theta+\upsilon}{2}\right)+2\Lambda(\upsilon)\right]-\frac{2^{\varepsilon-1}}{(\upsilon-\theta)^{\varepsilon}}\Gamma(\varepsilon+1)\left[I^{\varepsilon}_{\theta^+}\Lambda\left(\frac{\theta+\upsilon}{2}\right)+I^{\varepsilon}_{\upsilon^-}\Lambda\left(\frac{\theta+\upsilon}{2}\right)\right]\right|$$

$$\leq \frac{(\upsilon-\theta)^2}{8}\left(\psi_1(1,\varepsilon)\right)^{1-\frac{1}{q}}\left[\left(\left(\frac{\psi_1(1,\varepsilon)}{2}-\psi_2(1,\varepsilon)\right)\left|\Lambda''(\theta)\right|^q+\left(\frac{\psi_1(1,\varepsilon)}{2}+\psi_2(1,\varepsilon)\right)\left|\Lambda''(\upsilon)\right|^q\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\left(\frac{\psi_1(1,\varepsilon)}{2}+\psi_2(1,\varepsilon)\right)\left|\Lambda''(\theta)\right|^q+\left(\frac{\psi_1(1,\varepsilon)}{2}-\psi_2(1,\varepsilon)\right)\left|\Lambda''(\upsilon)\right|^q\right)^{\frac{1}{q}}\right]$$

$$= \frac{(\upsilon-\theta)^2}{48}\left(\frac{\varepsilon+8}{\varepsilon+2}\right)^{1-\frac{1}{q}}\left[\left(\left(\frac{2\varepsilon^2+19\varepsilon+48}{6(\varepsilon+2)(\varepsilon+3)}\right)\left|\Lambda''(\theta)\right|^q+\left(\frac{4\varepsilon^2+47\varepsilon+96}{6(\varepsilon+2)(\varepsilon+3)}\right)\left|\Lambda''(\upsilon)\right|^q\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\left(\frac{4\varepsilon^2+47\varepsilon+96}{6(\varepsilon+2)(\varepsilon+3)}\right)\left|\Lambda''(\theta)\right|^q+\left(\frac{2\varepsilon^2+19\varepsilon+48}{6(\varepsilon+2)(\varepsilon+3)}\right)\left|\Lambda''(\upsilon)\right|^q\right)^{\frac{1}{q}}\right].$$

Here, $\psi_1(1,\varepsilon)$ is presented in Remark 2.5 and also

$$\psi_2(1,\varepsilon)=\frac{1}{2}\int_0^1\left[\int_\mu^1\mu\left(\pi^{\varepsilon}+\frac{1}{3}\right)d\pi\right]d\mu=\frac{1}{2}\int_0^1\left[\int_0^\pi\mu\left(\pi^{\varepsilon}+\frac{1}{3}\right)d\mu\right]d\pi=\frac{\varepsilon+12}{36(\varepsilon+3)}.$$

As shown by Budak et al. [32], this result holds true.

Remark 2.13. *By taking $\sigma = 1$ and $\varepsilon = 1$ in Theorem 2.11, we get*

$$\left|\frac{1}{3}\left[2\Lambda(\theta)-\Lambda\left(\frac{\theta+\upsilon}{2}\right)+2\Lambda(\upsilon)\right]-\frac{1}{\upsilon-\theta}\int_\theta^\upsilon\Lambda(x)\,dx\right|$$

$$\leq \frac{(\upsilon-\theta)^2}{16}\left[\left(\frac{23\left|\Lambda''(\theta)\right|^q+49\left|\Lambda''(\upsilon)\right|^q}{72}\right)^{\frac{1}{q}}+\left(\frac{49\left|\Lambda''(\theta)\right|^q+23\left|\Lambda''(\upsilon)\right|^q}{72}\right)^{\frac{1}{q}}\right].$$

This outcome was established by Budak et al. [32].

Declarations

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Global Behavior of a Nonlinear System of Difference Equations

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Abstract

In this paper, we study the admissible solutions of the nonlinear system of difference equations

$$x_{n+1}=\frac{y_n}{x_n},\quad y_{n+1}=\frac{y_n}{\check{a}x_n+\check{b}y_n},\quad n=0,1,\ldots,$$

where $\check{a}$, $\check{b}$ are real numbers and the initial values x_0, y_0 are nonzero real numbers. In case $\check{b}<0$ and $\check{a}^2<-4\check{b}$, we show that there are eventually periodic solutions when either $tan^{-1}\frac{\sqrt{-4\check{b}-\check{a}^2}}{\check{a}}\in]\frac{\pi}{2},\pi[$ (with $\check{a}<0$) is a rational multiple of π or $tan^{-1}\frac{\sqrt{-4\check{b}-\check{a}^2}}{\check{a}}\in]0,\frac{\pi}{2}[$ (with $\check{a}>0$) as well.

1. Introduction

Difference equations and systems of difference equations occur in the applications of mathematics in growth and decay models, physics, economics, biology, circuit analysis, dynamical systems and other fields. It can be appeared as an approximation to solutions of differential equations. To study the behavior of the solutions to systems of difference equations, we may be able derive its solutions otherwise, we can investigate its long-term behaviors via the stability of its equilibrium points.

In [1], Kudlak et al. studied the existence of unbounded solutions of the system of difference equations

$$x_{n+1}=\frac{x_n}{y_n},\ y_{n+1}=x_n+\gamma_n y_n,\ n=0,1,\ldots,$$

where $0<\gamma_n<1$ and the initial values are positive real numbers.

Camouzis et al. [2], studied the global behavior of the system of difference equations

$$x_{n+1}=\frac{\alpha_1+\gamma_1 y_n}{x_n},\ y_{n+1}=\frac{\beta_2 x_n+\gamma_2 y_n}{B_2 x_n+C_2 y_n},\ n=0,1,\ldots, \tag{1.1}$$

with nonnegative parameters and positive initial conditions. They studied the boundedness character of the system (1.1) in its special cases.

In [3], Camouzis et al. studied the solutions of the system

$$x_{n+1}=\frac{y_n}{x_n},\ y_{n+1}=\frac{\gamma_2 y_n}{A_2+B_2 x_n+y_n},\ n=0,1,\ldots,$$

with nonnegative parameters and positive initial conditions.

Cinar [4], studied the positive solutions of the system of difference equations

$$x_{n+1}=\frac{1}{y_n},\ y_{n+1}=\frac{y_n}{x_{n-1}y_{n-1}},\ n=0,1,\ldots,$$

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where the initial values x_0, y_0, x_{-1} and y_{-1} are positive real numbers.

Clark and Kulenovic [5], studied the global stability properties and asymptotic behavior of solutions of the system of difference equation

$$x_{n+1} = \frac{x_n}{a+cy_n},\ y_{n+1} = \frac{y_n}{b+dx_n},\ n = 0,1,\dots,$$

where a,b,c,d are positive real numbers and the initial values x_0, y_0 are nonnegative real numbers. For more on difference equations, see [6]-[27] and the references therein. For more on systems of difference equations that are solved in closed form, see [28]-[33] and the references therein.

In this paper, we study the admissible solutions of the nonlinear system of difference equations

$$x_{n+1} = \frac{y_n}{x_n}, \quad y_{n+1} = \frac{y_n}{\breve{a}x_n + \breve{b}y_n}, \quad n = 0,1,\dots, \tag{1.2}$$

where $\breve{a}$, $\breve{b}$ are real numbers and the initial values x_0, y_0 are nonzero real numbers.

Consider the k^{th}-order difference equation

$$x_{n+1} = h(x_n, x_{n-1}, \dots, x_{n-k+1}),\ n = 0,1,\dots. \tag{1.3}$$

where the initial values $x_0, x_{-1}, \dots,$ and x_{-k+1} are real numbers. The set

$$H = \{(x_0, x_{-1}, \dots, x_{-k+1}) \in \mathbb{R}^k : x_n \text{ is undefined for some } n \in \mathbb{N}\},$$

is called the Forbidden set to Equation (1.3). The complement of the Forbidden set is called the Good set. Any solution $\{x_n\}_{n=-k+1}^{\infty}$ to Equation (1.3) with initial values belongs to the Good set is well-defined or admissible solution to Equation (1.3).

2. Case $\breve{a}\breve{b} = 0$

In this section, we shall investigate the case $\breve{a}\breve{b} = 0$.

Assume that $\breve{a} = 0$. Then the solution of system (1.2) is

$$\begin{cases} x_{2n} = \dfrac{x_0}{\breve{b}y_0} & ,n = 1,2,\dots, \\ x_{2n+1} = \dfrac{y_0}{x_0} & ,n = 1,2,\dots, \\ y_n = \dfrac{1}{\breve{b}} & ,n = 1,2.\dots \end{cases} \tag{2.1}$$

It is clear that, every admissible solution of system (1.2) is eventually 2-periodic.

In fact, for any admissible solution $\{(x_n, y_n)\}_{n=0}^{\infty}$ of system (1.2), we have

$$(x_{2n+1}, y_{2n+1}) = (x_{2n-1}, y_{2n-1}) = \left(\frac{y_0}{x_0}, \frac{1}{\breve{b}}\right),\ n = 1,2,\dots,$$

and

$$(x_{2n+2}, y_{2n+2}) = (x_{2n}, y_{2n}) = \left(\frac{x_0}{\breve{b}y_0}, \frac{1}{\breve{b}}\right),\ n = 1,2,\dots.$$

Now assume that $\breve{b} = 0$. Then the solution of system (1.2) is

$$\begin{cases} x_n = \dfrac{1}{\breve{a}} & ,n = 2,3,\dots, \\ y_n = \dfrac{1}{\breve{a}^2} & ,n = 2,3,\dots. \end{cases} \tag{2.2}$$

In this case, every admissible solution $\{(x_n, y_n)\}_{n=0}^{\infty}$ of system (1.2) converges to $\left(\dfrac{1}{\breve{a}}, \dfrac{1}{\breve{a}^2}\right)$.

3. Case $\breve{a}\breve{b} > 0$

In this section, we shall derive the admissible solutions of system (1.2) and investigate the global stability of its equilibrium points when $\breve{a}\breve{b} > 0$.

3.1. Case $\breve{a} > 0$ and $\breve{b} > 0$

Assume that $\breve{a}$ and $\breve{b}$ are positive real numbers. For system (1.2), we can write

$$u_{n+1} = \breve{a} + \frac{\breve{b}}{u_n}, \ n = 0, 1, ..., \tag{3.1}$$

where

$$u_n = \frac{x_n}{y_n}, \text{ with } u_0 = \frac{x_0}{y_0}.$$

Solving Equation (3.1) and substituting in system (1.2), we can write the admissible solution of system (1.2) as

$$\begin{cases} x_n = \dfrac{\breve{b}y_0\theta_{n-2} + x_0\theta_{n-1}}{\breve{b}y_0\theta_{n-1} + x_0\theta_n} & , n = 1, 2, ..., \\ \\ y_n = \dfrac{\breve{b}y_0\theta_{n-2} + x_0\theta_{n-1}}{\breve{b}y_0\theta_n + x_0\theta_{n+1}} & , n = 1, 2..., \end{cases} \tag{3.2}$$

where $\theta_j = \dfrac{t_1^j - t_2^j}{\sqrt{\breve{a}^2 + 4\breve{b}}}$, $t_1 = \dfrac{\breve{a} + \sqrt{\breve{a}^2 + 4\breve{b}}}{2}$ and $t_2 = \dfrac{\breve{a} - \sqrt{\breve{a}^2 + 4\breve{b}}}{2}$, $j = -1, 0, ...$
The forbidden set for system (1.2) can be written as

$$F_1 = \bigcup_{j=1}^{2} \{(v_1, v_2) \in \mathbb{R}^2 : v_j = 0\} \cup \bigcup_{n=1}^{\infty} \{(v_1, v_2) \in \mathbb{R}^2 : v_1 = -\frac{\theta_n}{\theta_{n+1}}\breve{b}v_2\}.$$

The equilibrium points of system (1.2) satisfy the equations

$$\bar{x} = \frac{\bar{y}}{\bar{x}} \text{ and } \bar{y} = \frac{\bar{y}}{\breve{a}\bar{x} + \breve{b}\bar{y}}.$$

Then we have two equilibrium points $E_1(\bar{x}_1, \bar{y}_1)$ and $E_2(\bar{x}_2, \bar{y}_2)$, where $\bar{x}_1$ and $\bar{x}_2$ are the solutions of the equation

$$\breve{b}x^2 + \breve{a}x - 1 = 0.$$

Consider the associated system of system (1.2)

$$G_1(x, y) = (y/x, y/(\breve{a}x + \breve{b}y)). \tag{3.3}$$

The Jacobian matrix corresponding to system (3.3) at an equilibrium point of system (1.2) is

$$J_{G_1}(\bar{x}, \bar{y}) = \begin{pmatrix} -1 & \frac{1}{\bar{x}} \\ -\breve{a}\bar{y} & \breve{a}\bar{x} \end{pmatrix}.$$

For more results on the stability of difference equations, see [24].

Theorem 3.1. *The following statements are true:*

1. *The equilibrium point $E_1(\bar{x}_1, \bar{y}_1)$ of system (1.2) is locally asymptotically stable.*
2. *The equilibrium point $E_2(\bar{x}_2, \bar{y}_2)$ of system (1.2) is unstable (saddle point).*

Proof. The eigenvalues of the Jacobian matrix $J_{G_1}(\bar{x}, \bar{y})$ are $\lambda_1 = 0$ and $\lambda_2 = -\breve{b}\bar{y}$. Then $|\lambda_2| = \breve{b}\bar{y} = 1 - \breve{a}\bar{x}$.

1. For the equilibrium point $E_1(\bar{x}_1, \bar{y}_1)$ of system (1.2) we have that

$$0 < \bar{x}_1 = -\frac{\breve{a}}{2\breve{b}} + \frac{\sqrt{\breve{a}^2 + 4\breve{b}}}{2\breve{b}} < \frac{1}{\breve{a}}.$$

This implies that

$$0 < \lambda_2 = 1 - \breve{a}\bar{x}_1 < 1,$$

and the result follows.

2. For the equilibrium point $E_2(\bar{x}_2,\bar{y}_2)$ of system (1.2) there is nothing to say, since $\bar{x}_2 = -\frac{\breve{a}}{2\breve{b}} - \frac{\sqrt{\breve{a}^2+4\breve{b}}}{2\breve{b}} < 0.$

□

Theorem 3.2. *The equilibrium point $E_1(\bar{x}_1,\bar{y}_1)$ of system (1.2) is globally asymptotically stable.*

Proof. Let $\{(x_n,y_n)\}_{n=0}^{\infty}$ be an admissible solution for system (1.2). Then using the solution form (3.2) we get

$$\begin{aligned} x_n &= \frac{\breve{b}y_0\theta_{n-2}+x_0\theta_{n-1}}{\breve{b}y_0\theta_{n-1}+x_0\theta_n} \\ &= \frac{\theta_{n-2}}{\theta_{n-1}}\frac{\breve{b}y_0+x_0\frac{\theta_{n-1}}{\theta_{n-2}}}{\breve{b}y_0+x_0\frac{\theta_n}{\theta_{n-1}}} \qquad \to \bar{x}_1 \text{ as } n\to\infty, \end{aligned}$$

where $\frac{\theta_n}{\theta_{n-1}} \to t_1$ as $n\to\infty$. Similarly,

$$\begin{aligned} y_n &= \frac{\breve{b}y_0\theta_{n-2}+x_0\theta_{n-1}}{\breve{b}y_0\theta_n+x_0\theta_{n+1}} \\ &= \frac{\theta_{n-2}}{\theta_n}\frac{\breve{b}y_0+x_0\frac{\theta_{n-1}}{\theta_{n-2}}}{\breve{b}y_0+x_0\frac{\theta_{n+1}}{\theta_n}} \qquad \to \bar{y}_1 \text{ as } n\to\infty. \end{aligned}$$

Then the equilibrium point $E_1(\bar{x}_1,\bar{y}_1)$ of system (1.2) is a global attractor of all admissible solutions of system (1.2). In view of Theorem (3.1), we conclude that the equilibrium point $E_1(\bar{x}_1,\bar{y}_1)$ of system (1.2) is globally asymptotically stable. □

3.2. Case $\breve{a} < 0$ and $\breve{b} < 0$

Assume that $\breve{a}$ and $\breve{b}$ are negative real numbers. We can write $\breve{a} = -a$ and $\breve{b} = -b$ for some positive reals a and b.
For system (1.2), we can write

$$u_{n+1} = -a - \frac{b}{u_n},\ n = 0,1,..., \tag{3.4}$$

where

$$u_n = \frac{x_n}{y_n},\ \text{with } u_0 = \frac{x_0}{y_0}.$$

We shall consider three cases:
Case $a^2 > 4b$
Solving Equation (3.4) and substituting in system (1.2), we can write the solution of system (1.2) as

$$\begin{cases} x_n = \dfrac{by_0\psi_{n-2}-x_0\psi_{n-1}}{by_0\psi_{n-1}-x_0\psi_n} & ,n=1,2,..., \\ y_n = \dfrac{by_0\psi_{n-2}-x_0\psi_{n-1}}{by_0\psi_n-x_0\psi_{n+1}} & ,n=1,2..., \end{cases} \tag{3.5}$$

where $\psi_j = \frac{t_+^j - t_-^j}{\sqrt{a^2-4b}}$, $t_+ = \frac{-a+\sqrt{a^2-4b}}{2}$ and $t_- = \frac{-a-\sqrt{a^2-4b}}{2}$, $j = -1,0,....$
The equilibrium points of system (1.2) satisfy the equations

$$\bar{x} = \frac{\bar{y}}{\bar{x}} \text{ and } \bar{y} = -\frac{\bar{y}}{a\bar{x}+b\bar{y}}.$$

Then we have two equilibrium points $L_+(\bar{x}_+,\bar{y}_+)$ and $L_-(\bar{x}_-,\bar{y}_-)$, where $\bar{x}_+$ and $\bar{x}_-$ are the solutions of the equation

$$bx^2+ax+1=0.$$

Theorem 3.3. *The following statements are true:*

1. *The equilibrium point $L_+(\bar{x}_+,\bar{y}_+)$ of system (1.2) is locally asymptotically stable.*

2. *The equilibrium point $L_-(\bar{x}_-, \bar{y_-})$ of system (1.2) is unstable (saddle point).*

Proof. Consider the associated system of system (1.2)

$$G_2(x,y) = (y/x, -y/(ax+by)). \tag{3.6}$$

The Jacobian matrix corresponding to system (3.6) at an equilibrium point of system (1.2) is

$$J_{G_2}(\bar{x},\bar{y}) = \begin{pmatrix} -1 & \frac{1}{\bar{x}} \\ -a\bar{y} & -a\bar{x} \end{pmatrix}. \tag{3.7}$$

The eigenvalues of the Jacobian matrix $J_{G_2}(\bar{x},\bar{y})$ are $\lambda_1 = 0$ and $\lambda_2 = -1-a\bar{x}$.

1. For the equilibrium point $L_+(\bar{x}_+, \bar{y_+})$ of system (1.2) we have that

$$-\frac{2}{a} < \bar{x}_+ = -\frac{a}{2b} + \frac{\sqrt{a^2-4b}}{2b} < -\frac{1}{a}.$$

   This implies that

$$0 < \lambda_2 = -1 - a\bar{x}_+ < 1,$$

   and the result follows.
2. For the equilibrium point $L_-(\bar{x}_-, \bar{y}_-)$ of system (1.2), we have

$$\bar{x}_- = -\frac{a}{2b} - \frac{\sqrt{a^2-4b}}{2b} < -\frac{2}{a}.$$

   Then

$$\lambda_2 = -1 - a\bar{x}_- > 1.$$

   Therefore, the equilibrium point $L_-(\bar{x}_-, \bar{y}_-)$ of system (1.2) is unstable (saddle point).

□

Theorem 3.4. *The equilibrium point $L_+(\bar{x}_+, \bar{y_+})$ of system (1.2) is globally asymptotically stable.*

Proof. Let $\{(x_n,y_n)\}_{n=0}^{\infty}$ be an admissible solution for system (1.2). For the global attractivity of the equilibrium point $L_+(\bar{x}_+, \bar{y_+})$, it is sufficient to see that

$$\frac{\psi_n}{\psi_{n-1}} \to t_- \text{ as } n \to \infty.$$

In view of Theorem (3.3), we conclude that the equilibrium point $L_+(\bar{x}_+, \bar{y_+})$ of system (1.2) is globally asymptotically stable. □

Case $a^2 = 4b$

Suppose that $a^2 = 4b$. Solving Equation (3.4) and substituting in system (1.2), we can write the solution of system (1.2) as

$$\begin{cases} x_n = -\frac{2}{a}\frac{ay_0(n-2)+2x_0(n-1)}{ay_0(n-1)+2x_0 n} & ,n = 1,2,\ldots, \\ y_n = \left(-\frac{2}{a}\right)^2 \frac{ay_0(n-2)+2x_0(n-1)}{ay_0 n+2x_0(n+1)} & ,n = 1,2.\ldots \end{cases} \tag{3.8}$$

Theorem 3.5. *The unique equilibrium point $L\left(-\frac{2}{a}, \frac{4}{a^2}\right)$ of system (1.2) is nonhyperbolic point.*

Proof. There is nothing to say except that, the eigenvalues of the Jacobian matrix (4.8) are

$$\lambda_1 = 0 \text{ and } \lambda_2 = -1 - a\bar{x} = -1 - a\left(-\frac{2}{a}\right) = 1.$$

□

From the solution form (3.8), we conclude that, every admissible solution for system (1.2) converges to the unique equilibrium point $L\left(-\frac{2}{a},\frac{4}{a^2}\right)$.

Case $a^2 < 4b$

Suppose that $a^2 < 4b$. Solving Equation (3.4) and substituting in system (1.2), we can write the solution of system (1.2) as

$$\begin{cases} x_n = \frac{1}{\sqrt{b}}\frac{\sqrt{b}y_0\sin(n-2)\alpha - x_0\sin(n-1)\alpha}{\sqrt{b}y_0\sin(n-1)\alpha - x_0\sin n\alpha} & ,n=1,2,...,\\ y_n = \frac{1}{b}\frac{\sqrt{b}y_0\sin(n-2)\alpha - x_0\sin(n-1)\alpha}{\sqrt{b}y_0\sin n\alpha - x_0\sin(n+1)\alpha} & ,n=1,2..., \end{cases} \tag{3.9}$$

where $\alpha = tan^{-1}\frac{-\sqrt{4b-a^2}}{a} \in]\frac{\pi}{2},\pi[$.

Theorem 3.6. *Assume that $a^2 < 4b$. If $\alpha = \frac{l}{k}\pi$ is a rational multiple of π (l and k are relatively positive prime integers) such that $\frac{k}{2} < l < k$. Then every admissible solution $\{(x_n,y_n)\}_{n=0}^{\infty}$ of system (1.2) is eventually k-periodic.*

Proof. Assume that $\alpha = \frac{l}{k}\pi$ is a rational multiple of π (l and k are relatively positive prime integers) such that $\frac{k}{2} < l < k$ and let $\{(x_n,y_n)\}_{n=0}^{\infty}$ of system (1.2). Then for $n \geq 1$, we have

$$\begin{aligned} x_{n+k} &= \frac{1}{\sqrt{b}}\frac{\sqrt{b}y_0\sin(n+k-2)\alpha - x_0\sin(n+k-1)\alpha}{\sqrt{b}y_0\sin(n+k-1)\alpha - x_0\sin n+k\alpha}\\ &= \frac{1}{\sqrt{b}}\frac{\sqrt{b}y_0(-1)^l\sin(n+k-2)\alpha - x_0(-1)^l\sin(n+k-1)\alpha}{\sqrt{b}y_0(-1)^l\sin(n+k-1)\alpha - x_0(-1)^l\sin n+k\alpha}\\ &= \frac{1}{\sqrt{b}}\frac{\sqrt{b}y_0\sin(n-2)\alpha - x_0\sin(n-1)\alpha}{\sqrt{b}y_0\sin(n-1)\alpha - x_0\sin n\alpha}\\ &= x_n. \end{aligned}$$

Similarly, we can see that $y_{n+k} = y_n$ for all $n \geq 1$.
Therefore, the admissible solution $\{(x_n,y_n)\}_{n=0}^{\infty}$ of system (1.2) is eventually k-periodic (in fact except for the initial point (x_0,y_0)). □

The forbidden set for system (1.2) depends on the relation between a and b. For system (1.2) we have the following:

1. If $a^2 > 4b$, then the forbidden set of system (1.2) is

$$F_2 = \bigcup_{j=1}^{2}\{(v_1,v_2)\in\mathbb{R}^2 : v_j = 0\} \cup \bigcup_{n=1}^{\infty}\{(v_1,v_2)\in\mathbb{R}^2 : v_1 = \frac{\theta_n}{\theta_{n+1}}bv_2\}.$$

2. If $a^2 = 4b$, then the forbidden set of system (1.2) is

$$F_3 = \bigcup_{j=1}^{2}\{(v_1,v_2)\in\mathbb{R}^2 : v_j = 0\} \cup \bigcup_{n=1}^{\infty}\{(v_1,v_2)\in\mathbb{R}^2 : v_1 = -\frac{n}{n+1}(\frac{a}{2})v_2\}.$$

3. If $a^2 < 4b$, then the forbidden set of system (1.2) is

$$F_4 = \bigcup_{j=1}^{2}\{(v_1,v_2)\in\mathbb{R}^2 : v_j = 0\} \cup \bigcup_{n=1}^{\infty}\{(v_1,v_2)\in\mathbb{R}^2 : v_1 = \sqrt{b}\frac{\sin n\alpha}{\sin(n+1)\alpha}v_2\}.$$

4. Case $\breve{a}\breve{b} < 0$

In this section, we shall derive the solution of system (1.2) and investigate the global stability of its equilibrium points when $\breve{a}\breve{b} < 0$.

4.1. Case $\check{a} < 0$ and $\check{b} > 0$

Assume that $\check{a} = -a < 0$ and $\check{b} = b > 0$. Then we can write system (1.2) as

$$u_{n+1} = -a + \frac{b}{u_n},\ n = 0, 1, ..., \tag{4.1}$$

where

$$u_n = \frac{x_n}{y_n},\ \text{with}\ u_0 = \frac{x_0}{y_0}.$$

Solving Equation (4.1) and substituting in system (1.2), we can write the solution of system (1.2) as

$$\begin{cases} x_n = \dfrac{by_0\acute{\theta}_{n-2} + x_0\acute{\theta}_{n-1}}{by_0\acute{\theta}_{n-1} + x_0\acute{\theta}_n} & ,n = 1, 2, ..., \\ \\ y_n = \dfrac{by_0\acute{\theta}_{n-2} + x_0\acute{\theta}_{n-1}}{by_0\acute{\theta}_n + x_0\acute{\theta}_{n+1}} & ,n = 1, 2..., \end{cases} \tag{4.2}$$

where $\acute{\theta}_j = \dfrac{\acute{t}_1^j - \acute{t}_2^j}{\sqrt{a^2+4b}}$, $\acute{t}_1 = \dfrac{-a+\sqrt{a^2+4b}}{2}$ and $\acute{t}_2 = \dfrac{-a-\sqrt{a^2+4b}}{2}$, $j = -1, 0, ...$

The forbidden set of system (1.2) can be written as

$$F_5 = \bigcup_{j=1}^{2}\{(v_1, v_2) \in \mathbb{R}^2 : v_j = 0\} \cup \bigcup_{n=1}^{\infty}\{(v_1, v_2) \in \mathbb{R}^2 : v_1 = -\frac{\acute{\theta}_n}{\acute{\theta}_{n+1}} bv_2\}.$$

The equilibrium points of system (1.2) satisfy the equations

$$\bar{x} = \frac{\bar{y}}{\bar{x}}\ \text{and}\ \bar{y} = \frac{\bar{y}}{-a\bar{x} + b\bar{y}}.$$

Then we have two equilibrium points $\acute{E}_1(\bar{x}_1, \bar{y}_1)$ and $\acute{E}_2(\bar{x}_2, \bar{y}_2)$, where $\bar{x}_1$ and $\bar{x}_2$ are the solutions of the equation

$$bx^2 - ax - 1 = 0.$$

Theorem 4.1. *The following statements are true:*

1. *The equilibrium point $\acute{E}_1(\bar{x}_1, \bar{y}_1)$ of system (1.2) is unstable (saddle point).*
2. *The equilibrium point $\acute{E}_2(\bar{x}_2, \bar{y}_2)$ of system (1.2)is locally asymptotically stable.*

Proof. Consider the associated system of system (1.2)

$$G_3(x, y) = (y/x, y/(-ax + by)). \tag{4.3}$$

The Jacobian matrix corresponding to system (4.3) at an equilibrium point of system (1.2) is

$$J_{G_3}(\bar{x}, \bar{y}) = \begin{pmatrix} -1 & \frac{1}{\bar{x}} \\ a\bar{y} & -a\bar{x} \end{pmatrix}. \tag{4.4}$$

The eigenvalues of the Jacobian matrix $J_{G_3}(\bar{x}, \bar{y})$ are $\acute{\lambda}_1 = 0$ and $\acute{\lambda}_2 = -1 - a\bar{x}$.

1. For the equilibrium point $\acute{E}_1(\bar{x}_1, \bar{y}_1)$ of system (1.2), we have

$$1 + a\bar{x}_1 = 1 + a(\frac{a}{2b} + \frac{\sqrt{a^2+4b}}{2b}) > 1.$$

Then

$$|\acute{\lambda}_2| = 1 + a\bar{x}_1 > 1.$$

Therefore, the equilibrium point $\acute{E}_1(\bar{x}_1, \bar{y}_1)$ of system (1.2) is unstable (saddle point).

2. For the equilibrium point $\acute{E}_2(\bar{x}_2, \bar{y}_2)$ of system (1.2) we have that
$$-\frac{2}{a} < \bar{x}_2 = \frac{a}{2b} - \frac{\sqrt{a^2+4b}}{2b}.$$
This implies that
$$0 < |\acute{\lambda}_2| = |-1 - a\bar{x}_2| < 1,$$
and the result follows.

□

Theorem 4.2. *The equilibrium point $\acute{E}_2(\bar{x}_2, \bar{y}_2)$ of system (1.2) is globally asymptotically stable.*

Proof. Let $\{(x_n, y_n)\}_{n=0}^{\infty}$ be an admissible solution for system (1.2). For the global attractivity of the equilibrium point $\acute{E}_2(\bar{x}_2, \bar{y}_2)$, it is sufficient to see that $\dfrac{\acute{\theta}_n}{\acute{\theta}_{n-1}} \to \acute{t}_2$ as $n \to \infty$.
In view of Theorem (4.1), we conclude that the equilibrium point $\acute{E}_2(\bar{x}_2, \bar{y}_2)$ of system (1.2) is globally asymptotically stable. □

4.2. Case $\breve{a} > 0$ and $\breve{b} < 0$

Assume that $\breve{a} = a > 0$ and $\breve{b} = -b < 0$. Then we can write system (1.2) as
$$u_{n+1} = a - \frac{b}{u_n}, \; n = 0, 1, ..., \tag{4.5}$$
where
$$u_n = \frac{x_n}{y_n}, \text{ with } u_0 = \frac{x_0}{y_0}.$$
We shall consider three cases:
Case $a^2 > 4b$
Solving Equation (4.5) and substituting in system (1.2), we can write the solution of system (1.2) as
$$\begin{cases} x_n = \dfrac{by_0\acute{\psi}_{n-2} - x_0\acute{\psi}_{n-1}}{by_0\acute{\psi}_{n-1} - x_0\acute{\psi}_n} & , n = 1, 2, ..., \\ y_n = \dfrac{by_0\acute{\psi}_{n-2} - x_0\acute{\psi}_{n-1}}{by_0\acute{\psi}_n - x_0\acute{\psi}_{n+1}} & , n = 1, 2..., \end{cases} \tag{4.6}$$
where $\acute{\psi}_j = \dfrac{\acute{t}_+^j - \acute{t}_-^j}{\sqrt{a^2-4b}}$, $\acute{t}_+ = \dfrac{a+\sqrt{a^2-4b}}{2}$ and $\acute{t}_- = \dfrac{a-\sqrt{a^2-4b}}{2}$, $j = -1, 0, ...$
The equilibrium points of system (1.2) satisfy the equations
$$\bar{x} = \frac{\bar{y}}{\bar{x}} \text{ and } \bar{y} = \frac{\bar{y}}{a\bar{x} - b\bar{y}}.$$
Then we have two equilibrium points $\acute{L}_+(\bar{x}_+, \bar{y}_+)$ and $\acute{L}_-(\bar{x}_-, \bar{y}_-)$, where $\bar{x}_+$ and $\bar{x}_-$ are the admissible solutions of the equation
$$bx^2 - ax + 1 = 0.$$

Theorem 4.3. *The following statements are true:*

1. *The equilibrium point $\acute{L}_+(\bar{x}_+, \bar{y}_+)$ of system (1.2) is unstable (saddle point).*
2. *The equilibrium point $\acute{L}_-(\bar{x}_-, \bar{y}_-)$ of system (1.2) is locally asymptotically stable.*

Proof. Consider the associated system of system (1.2)
$$G_4(x, y) = (y/x, y/(ax - by)). \tag{4.7}$$
The Jacobian matrix corresponding to system (4.7) at an equilibrium point of system (1.2) is
$$J_{G_4}(\bar{x}, \bar{y}) = \begin{pmatrix} -1 & \dfrac{1}{\bar{x}} \\ -a\bar{y} & a\bar{x} \end{pmatrix}. \tag{4.8}$$
The eigenvalues of the Jacobian matrix $J_{G_4}(\bar{x}, \bar{y})$ are $|\lambda_1| = 0$ and $|\lambda_2| = a\bar{x} - 1$.

1. For the equilibrium point $\acute{L}_{+}(\bar{x}_{+},\bar{y}_{+})$ of system (1.2) we have that

$$\bar{x}_{+}=\frac{a}{2b}+\frac{\sqrt{a^2-4b}}{2b}>\frac{a}{2b}>\frac{2}{a}.$$

   This implies that

$$|\lambda_2|=a\bar{x}_{+}-1>1,$$

   and the result follows.

2. For the equilibrium point $\acute{L}_{-}(\bar{x}_{-},\bar{y}_{-})$ of system (1.2), we have $\frac{1}{a}<\bar{x}_{-}=\frac{a}{2b}-\frac{\sqrt{a^2-4b}}{2b}<\frac{2}{a}$.
   Then

$$|\lambda_2|=a\bar{x}_{-}-1<1.$$

   Therefore, the equilibrium point $\acute{L}_{-}(\bar{x}_{-},\bar{y}_{-})$ of system (1.2) is locally asymptotically stable.

□

Theorem 4.4. *The equilibrium point $\acute{L}_{-}(\bar{x}_{-},\bar{y}_{-})$ of system (1.2) is globally asymptotically stable.*

Proof. Let $\{(x_n,y_n)\}_{n=0}^{\infty}$ be an admissible solution for system (1.2). For the global attractivity of the equilibrium point $\acute{L}_{-}(\bar{x}_{-},\bar{y}_{-})$, it is sufficient to see that $\frac{\psi_n}{\psi_{n-1}}\to \acute{t}_{+}$ as $n\to\infty$.
In view of Theorem (4.3), we conclude that the equilibrium point $\acute{L}_{-}(\bar{x}_{-},\bar{y}_{-})$ of system (1.2) is globally asymptotically stable. □

Case $a^2=4b$
Suppose that $a^2=4b$. Solving Equation (4.5) and substituting in system (1.2), we can write the admissible solution of system (1.2) as

$$\begin{cases} x_n=\dfrac{2}{a}\dfrac{ay_0(n-2)-2x_0(n-1)}{ay_0(n-1)-2x_0n} & ,n=1,2,...,\\[2ex] y_n=\left(\dfrac{2}{a}\right)^2\dfrac{ay_0(n-2)-2x_0(n-1)}{ay_0n-2x_0(n+1)} & ,n=1,2.... \end{cases} \tag{4.9}$$

Theorem 4.5. *The unique equilibrium point $\acute{L}\left(\frac{2}{a},\frac{4}{a^2}\right)$ of system (1.2) is nonhyperbolic point.*

Proof. There is nothing to say except that, the eigenvalues of the Jacobian matrix (4.8) are

$$\lambda_1=0 \text{ and } \lambda_2=a\bar{x}-1=a(\frac{2}{a})-1=1.$$

□

From the admissible solution form (4.9), we conclude that, every admissible solution for system (1.2) converges to the unique equilibrium point $\acute{L}(\frac{2}{a},\frac{4}{a^2})$.

Case $a^2<4b$
Suppose that $a^2<4b$. Solving Equation (4.5) and substituting in system (1.2), we can write the solution of system (1.2) as

$$\begin{cases} x_n=\dfrac{1}{\sqrt{b}}\dfrac{\sqrt{b}y_0\sin(n-2)\beta-x_0\sin(n-1)\beta}{\sqrt{b}y_0\sin(n-1)\beta-x_0\sin n\beta} & ,n=1,2,...,\\[2ex] y_n=\dfrac{1}{b}\dfrac{\sqrt{b}y_0\sin(n-2)\beta-x_0\sin(n-1)\beta}{\sqrt{b}y_0\sin n\beta-x_0\sin(n+1)\beta} & ,n=1,2..., \end{cases} \tag{4.10}$$

where $\beta=tan^{-1}\frac{\sqrt{4b-a^2}}{a}\in]0,\frac{\pi}{2}[$.

Theorem 4.6. *Assume that $a^2<4b$. If $\beta=\frac{l}{k}\pi$ is a rational multiple of π (l and k are relatively positive prime integers) such that $0<l<\frac{k}{2}$. Then every admissible solution $\{(x_n,y_n)\}_{n=0}^{\infty}$ of system (1.2) is eventually k-periodic.*

Proof. The proof is similar to that of Theorem (3.6) and is omitted. □

We end this subsection by introducing the forbidden set for system (1.2), which depends on the relation between a and b. For system (1.2) we have the following:

1. If $a^2 > 4b$, then the forbidden set of system (1.2) is

$$F_6 = \bigcup_{j=1}^{2}\{(v_1,v_2)\in\mathbb{R}^2 : v_j = 0\}\cup\bigcup_{n=1}^{\infty}\{(v_1,v_2)\in\mathbb{R}^2 : v_1 = \frac{\acute{\psi}_n}{\acute{\psi}_{n+1}}bv_2\}.$$

2. If $a^2 = 4b$, then the forbidden set of system (1.2) is

$$F_7 = \bigcup_{j=1}^{2}\{(v_1,v_2)\in\mathbb{R}^2 : v_j = 0\}\cup\bigcup_{n=1}^{\infty}\{(v_1,v_2)\in\mathbb{R}^2 : v_1 = \frac{n}{n+1}(\frac{a}{2})v_2\}.$$

3. If $a^2 < 4b$, then the forbidden set of system (1.2) is

$$F_8 = \bigcup_{j=1}^{2}\{(v_1,v_2)\in\mathbb{R}^2 : v_j = 0\}\cup\bigcup_{n=1}^{\infty}\{(v_1,v_2)\in\mathbb{R}^2 : v_1 = \sqrt{b}\frac{\sin n\beta}{\sin(n+1)\beta}v_2\}.$$

Conclusion

In this work, we derived and studied the admissible solutions of the nonlinear system of difference equations

$$x_{n+1} = \frac{y_n}{x_n}, \quad y_{n+1} = \frac{y_n}{\breve{a}x_n + \breve{b}y_n}, \quad n = 0,1,\ldots,$$

where $\breve{a}$, $\breve{b}$ are real numbers and the initial values x_0, y_0 are nonzero real numbers.

We discussed the linearized and global stability of the solutions for all nontrivial values of $\breve{a}$ and $\breve{b}$ as well as introduced the forbidden sets.

We showed under certain conditions that, there exist eventually periodic solutions when $\breve{a} < 0$ and $\breve{b} < 0$ as well as when $\breve{a} > 0$ and $\breve{b} < 0$.

We conjecture that the same results can be obtained for the system

$$x_{n+1} = \frac{y_{n-k}}{x_{n-k}}, \quad y_{n+1} = \frac{y_{n-k}}{\breve{a}x_{n-k} + \breve{b}y_{n-k}}, \quad n = 0,1,\ldots,$$

where $\breve{a}$, $\breve{b}$ are real numbers and the initial points (x_{-i}, y_{-i}), where $i = 0,1,\ldots,k$ are nonzero real numbers.

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