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Research Article

A Study of Caputo Sequential Fractional Differential Equations with Mixed Boundary Conditions

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Abstract

In this paper, we investigate the existence of solutions for a sequential fractional differential equation involving Caputo-type derivative subject to mixed boundary conditions. The core results are derived by employing Krasnoselskii's fixed point theorem and the Leray-Schauder fixed point theorem. We end this study by two illustrative numerical examples, which validate the applicability of our obtained results.

1. Introduction

Fractional calculus has emerged as a critical field in mathematics, generalizing traditional differentiation and integration to non-integer orders. This extension provides a powerful framework for modeling complex phenomena and systems across diverse disciplines such as physics, engineering, biology, engineering, mechanics, economics, and other fields [1–5].

Boundary value problems (BVPs) for fractional differential equations (FDEs) that emerge and describe linear and nonlinear phenomena have obtained much attention in the scientific community and specially in engineering. Recently, there are several researchers [6–10] that have used FDEs to model natural phenomena. In this regard, due to this exponential growth of the fractional calculus added to differential equations, many researchers have focused their attention on the investigation of existence, uniqueness and stability of solutions of FDEs under different types of boundary conditions by using a set of fixed point theories, such as Banach's, the Leray- Schauder alternative, Darbo's theorem and Mönch's fixed point theorem [11–15] and the references therein. Sequential FDEs have also received considerable attention for instance see [16–22].

It is worth mentioning that Mahmudov et al. [23] establish the existence of solutions for the following nonlinear sequential fractional differential equation subject to nonlocal fractional integral conditions:

$$\begin{cases} \left({}^CD^{\nu}+\omega {}^CD^{\nu-1}\right)\kappa(\tau)=f\left(\tau,\kappa(\tau),{}^CD^{\nu-1}\kappa(\tau)\right),\ 1<\nu<2, 0\leq\tau\leq T,\\ \alpha_1\kappa(\eta)+\beta_1\kappa(T)=\gamma_1\int_0^{\xi}\kappa(s)\,ds+\varepsilon_1,\\ \alpha_2{}^CD^{\nu-1}\kappa(\eta)+\beta_2{}^CD^{\nu-1}\kappa(T)=\gamma_2\int_{\zeta}^{\xi}\kappa(s)\,ds+\varepsilon_2,\end{cases}$$

where ${}^CD^{\nu}$ is the standard Caputo fractional derivative of order ν, $0\leq\eta\leq T$, $0<\xi<\zeta<T$, $\alpha_1,\alpha_2,\beta_1,\beta_2,\gamma_1,\gamma_2,\varepsilon_1,\varepsilon_2\in\mathbb{R}$.

In [24], Awadalla et al. studied the following nonlinear sequential FDE to nonseparated nonlocal integral fractional boundary conditions:

$$\begin{cases} \left({}^CD^{\nu}+\omega {}^CD^{\nu-1}\right)\kappa(\tau)=f(\tau,\kappa(\tau)),\ 1\leq\nu\leq2, 0\leq\tau\leq T,\\ \omega_1\kappa(\sigma)+\rho_1\kappa(T)=\upsilon_1\int_0^{\eta}\kappa(\zeta)\,d\zeta,\\ \omega_2{}^CD^{\nu-1}\kappa(\sigma)+\rho_2{}^CD^{\nu-1}\kappa(T)=\upsilon_2\int_{\varsigma}^{T}\kappa(\zeta)\,d\zeta,\end{cases}$$

where $0\leq\sigma\leq T$, $0<\eta<\varsigma<T$, $\omega\in\mathbb{R}_+$, $\omega_1,\omega_2,\rho_1,\rho_2,\upsilon_1,\upsilon_2\in\mathbb{R}$.

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In [25], Yan investigated the existence and uniqueness of solutions to the boundary value problem of a nonlinear FDEs:

$$\begin{cases} {}^{C}D^{\nu}\mu(\tau)+{}^{C}D^{\nu-1}[p(\tau)\mu(\tau)]=h(\tau,\mu(\tau)),\, 0<\tau<1, \\ \mu(0)=\mu'(0)=\mu'(1)=0, \end{cases}$$

where $2<\mu\leq 3$, $h\in C[0,1]$ and $p\in C^{3}([0,1],\mathbb{R})$.

Inspired by the works mentioned above, we investigate the existence results for a sequential FDEs of the form

$$\begin{cases} ({}^{c}D^{\nu+1}+\omega{}^{c}D^{\nu})\kappa(\tau)=f(\tau,\kappa(\tau),{}^{c}D^{\nu-1}\kappa(\tau)), \quad 1<\nu\leq 2, \omega>0, \tau\in[0,1], \\ \kappa(0)+\beta\kappa(1)=I^{\nu-1}\kappa(\mu)+I^{\nu}\kappa(\mu), \quad 0<\mu<1, \\ \kappa'(0)+\gamma\kappa'(1)={}^{c}D^{\nu-1}\kappa(\mu)+{}^{c}D^{\nu}\kappa(\mu), \quad \beta,\gamma\in\mathbb{R}, \\ \kappa''(0)=0. \end{cases} \tag{1.1}$$

Here ${}^{c}D^{\nu+1}$, ${}^{c}D^{\nu}$, ${}^{c}D^{\nu-1}$ are the Caputo fractional derivatives of order $\nu+1$, ν, and $\nu-1$ respectively, $f:[0,1]\times\mathbb{R}^{2}\longrightarrow\mathbb{R}$ is a continuous function, and $1+\gamma-\frac{\mu^{2-\nu}}{\Gamma(3-\nu)}\neq 0$, $1+\beta-\frac{\nu\mu^{\nu-1}+\mu^{\nu}}{\Gamma(\nu+1)}\neq 0$.

The rest of this paper is organised as follows. Section 2 presents definitions and preliminary concepts. Section 3 investigates the existence of solutions for the problem (1.1) by using Krasnoselskii's fixed point theorem and the Leray-Schauder fixed point theorem. Section 4 gives examples, and conclusion section is deticated to summarizing our obtained results.

2. Preliminaries

In this section, it is essential to present some basic concepts and important lemmas. For more details, the interested readers can consult [3].

Definition 2.1. *The Riemann-Liouville fractional integral of order $\nu>0$ for a function $f:(0,+\infty)\to\mathbb{R}$ is defined as*

$$I^{\nu}f(\tau)=\frac{1}{\Gamma(\nu)}\int_{0}^{\tau}(\tau-s)^{\nu-1}f(s)ds,$$

provided the right side is pointwise defined on $(0,+\infty)$ where $\Gamma(.)$ is the Gamma function.

Definition 2.2. *Let a function $f:[0,+\infty)\to\mathbb{R}$, the Caputo derivative of fractional order $\nu>0$ is defined as*

$${}^{c}D^{\nu}f(\tau)=\frac{1}{\Gamma(n-\nu)}\int_{0}^{\tau}(\tau-s)^{n-\nu-1}f^{(n)}(s)ds,\ n=[\nu]+1,$$

where $[\nu]$ denotes the integer part of the real number ν, provided the right side is pointwise defined on $(0,+\infty)$.

Lemma 2.3. *Let $\nu>0$ and $f\in AC^{N}[0,1]$. Then the equation*

$${}^{c}D^{\nu}f(\tau)=0,$$

has a unique solution

$$f(\tau)=\sum_{i=0}^{N-1}a_{i}\tau^{i},$$

and

$$I^{\nu c}D^{\nu}f(\tau)=f(\tau)+\sum_{i=0}^{N-1}a_{i}\tau^{i}$$

for some $a_{i}\in\mathbb{R}$, $i=0,1,2,...,N-1$, $N=[\nu]+1$.

Lemma 2.4. *Let $\nu>\sigma>0$ and $f\in L^{p}(0,1)\subset L^{1}(0,1), 0\leq p\leq+\infty$. Then the next formulas hold.*

(i) $({}^{c}D^{\sigma}I^{\nu}f)(\tau)=I^{\nu-\sigma}f(\tau)$,
(ii) $({}^{c}D^{\nu}I^{\nu}f)(\tau)=f(\tau)$.

Definition 2.5. *([26]) The sequential fractional derivative for a function f can be written as*

$$D^{\nu}f(\tau)=D^{\nu_{1}}D^{\nu_{2}}...D^{\nu_{m}}f(\tau),$$

where $\nu=(\nu_{1},\nu_{2},...,\nu_{m})$ is a multi-index.

Lemma 2.6. *For a given $\xi\in C([0,1],\mathbb{R})$, the unique solution of the problem*

$$\begin{cases} ({}^{c}D^{\nu+1}+\omega{}^{c}D^{\nu})\kappa(\tau)=\xi(\tau), \quad \tau\in[0,1], \\ \kappa(0)+\beta\kappa(1)=I^{\nu-1}\kappa(\mu)+I^{\nu}\kappa(\mu), \\ \kappa'(0)+\gamma\kappa'(1)={}^{c}D^{\nu-1}\kappa(\mu)+{}^{c}D^{\nu}\kappa(\mu), \\ \kappa''(0)=0, \end{cases}$$

is expressed as

$$\begin{aligned}\kappa(\tau) = &\int_0^\tau e^{-\omega(\tau-\sigma)} I^\nu \xi(\sigma) d\sigma + \frac{\tau}{\Theta_1}\Bigg[(\omega^2-\omega)\int_0^\mu \frac{(\mu-\sigma)^{1-\nu}}{\Gamma(2-\nu)} \times \left(\int_0^\sigma e^{-\omega(\sigma-\vartheta)} I^\nu \xi(\vartheta) d\vartheta\right) d\sigma \\ &+ \int_0^\mu \frac{(\mu-\sigma)^{1-\nu}}{\Gamma(2-\nu)}[(1-\omega)I^\nu\xi(\sigma)+I^{\nu-1}\xi(\sigma)]d\sigma + \gamma\omega\int_0^1 e^{-\omega(1-\sigma)} I^\nu\xi(\sigma)d\sigma - \gamma I^\nu\xi(1)\Bigg] \\ &+ \frac{1}{\Theta_2}\int_0^\mu \left[\frac{(\mu-\sigma)^{\nu-2}}{\Gamma(\nu-1)} + \frac{(\mu-\sigma)^{\nu-1}}{\Gamma(\nu)}\right]\left(\int_0^\sigma e^{-\omega(\sigma-\vartheta)} I^\nu\xi(\vartheta)d\vartheta\right)d\sigma \\ &+ \frac{\Theta_3}{\Theta_1\Theta_2}\Bigg[(\omega^2-\omega)\int_0^\mu \frac{(\mu-\sigma)^{1-\nu}}{\Gamma(2-\nu)}\left(\int_0^\sigma e^{-\omega(\sigma-\vartheta)} I^\nu\xi(\vartheta)d\vartheta\right)d\sigma \\ &+ \int_0^\mu \frac{(\mu-\sigma)^{1-\nu}}{\Gamma(2-\nu)}[(1-\omega)I^\nu\xi(\sigma)+I^{\nu-1}\xi(\sigma)]d\sigma + \gamma\omega\int_0^1 e^{-\omega(1-\sigma)} I^\nu\xi(\sigma)d\sigma - \gamma I^\nu\xi(1)\Bigg],\end{aligned} \tag{2.1}$$

where

$$\Theta_1 = 1+\gamma - \frac{\mu^{2-\nu}}{\Gamma(3-\nu)}, \quad \Theta_2 = 1+\beta - \frac{\nu\mu^{\nu-1}+\mu^\nu}{\Gamma(\nu+1)}, \quad \Theta_3 = \frac{\mu^\nu(\mu+\nu+1)}{\Gamma(\nu+2)} - \beta. \tag{2.2}$$

Proof. By Lemma 2.3, we find

$$(D+\omega)\kappa(\tau) = I^\nu\xi(\tau) + a_0 + a_1\tau, \tag{2.3}$$

where $a_0, a_1 \in \mathbb{R}$. Then, (2.3) is equivalent to

$$D(e^{\omega\tau}\kappa(\tau)) = e^{\omega\tau}(I^\nu\xi(\tau) + a_0 + a_1\tau),$$

and integrating this expression from 0 to τ, we have

$$e^{\omega\tau}\kappa(\tau) = \int_0^\tau e^{\omega\sigma} I^\nu\xi(\sigma)d\sigma + \left(\frac{a_0}{\omega} - \frac{a_1}{\omega^2}\right)e^{\omega\tau} + \frac{a_1\tau}{\omega}e^{\omega\tau} + \left(\frac{a_1}{\omega^2} - \frac{a_0}{\omega} + \kappa(0)\right).$$

Therefore we deduce that

$$\kappa(\tau) = \mathscr{A} + \mathscr{B}\tau + \mathscr{C}e^{-\omega\tau} + \int_0^\tau e^{-\omega(\tau-\sigma)} I^\nu\xi(\sigma)d\sigma, \tag{2.4}$$

where $\mathscr{A} = \frac{a_0}{\omega} - \frac{a_1}{\omega^2}$, $\mathscr{B} = \frac{a_1}{\omega}$ and $\mathscr{C} = \frac{a_1}{\omega^2} - \frac{a_0}{\omega} + \kappa(0)$.
Then, the second derivative of function κ with respect to τ is given by

$$\kappa''(\tau) = \mathscr{C}\omega^2 e^{-\omega\tau} + \omega^2\int_0^\tau e^{-\omega(\tau-\sigma)} I^\nu\xi(\sigma)d\sigma - \omega I^\nu\xi(\tau) + I^{\nu-1}\xi(\tau).$$

By condition $\kappa''(0) = 0$, we have $\mathscr{C} = 0$.
From (2.4), we get

$${}^cD^{\nu-1}\kappa(\mu) + {}^cD^\nu\kappa(\mu) = \int_0^\mu \frac{(\mu-\sigma)^{1-\nu}}{\Gamma(2-\nu)}\left[(\omega^2-\omega)\int_0^\sigma e^{-\omega(\sigma-\vartheta)} I^\nu\xi(\vartheta)d\vartheta + (1-\omega)I^\nu\xi(\sigma) + I^{\nu-1}\xi(\sigma) + \mathscr{B}\right]d\sigma,$$

and

$$\begin{aligned}I^{\nu-1}\kappa(\mu) + I^\nu\kappa(\mu) = &\mathscr{A}\int_0^\mu\left(\frac{(\mu-\sigma)^{\nu-2}}{\Gamma(\nu-1)} + \frac{(\mu-\sigma)^{\nu-1}}{\Gamma(\nu)}\right)d\sigma + \mathscr{B}\int_0^\mu\left(\frac{(\mu-\sigma)^{\nu-2}}{\Gamma(\nu-1)} + \frac{(\mu-\sigma)^{\nu-1}}{\Gamma(\nu)}\right)\sigma d\sigma \\ &+ \int_0^\mu\left(\frac{(\mu-\sigma)^{\nu-2}}{\Gamma(\nu-1)} + \frac{(\mu-\sigma)^{\nu-1}}{\Gamma(\nu)}\right)\left(\int_0^\sigma e^{-\omega(\sigma-\vartheta)} I^\nu\xi(\vartheta)d\vartheta\right)d\sigma.\end{aligned}$$

The condition $\kappa(0) + \beta\kappa(1) = I^{\nu-1}\kappa(\mu) + I^\nu\kappa(\mu)$ gives

$$\begin{aligned}&\mathscr{A}\left(1+\beta - \int_0^\mu\left(\frac{(\mu-\sigma)^{\nu-2}}{\Gamma(\nu-1)} + \frac{(\mu-\sigma)^{\nu-1}}{\Gamma(\nu)}\right)d\sigma\right) + \mathscr{B}\left(\beta - \int_0^\mu\left(\frac{(\mu-\sigma)^{\nu-2}}{\Gamma(\nu-1)} + \frac{(\mu-\sigma)^{\nu-1}}{\Gamma(\nu)}\right)\sigma d\sigma\right) \\ &= \int_0^\mu\left(\frac{(\mu-\sigma)^{\nu-2}}{\Gamma(\nu-1)} + \frac{(\mu-\sigma)^{\nu-1}}{\Gamma(\nu)}\right)\left(\int_0^\sigma e^{-\omega(\sigma-\vartheta)} I^\nu\xi(\vartheta)d\vartheta\right)d\sigma,\end{aligned} \tag{2.5}$$

and the condition $\kappa'(0) + \gamma\kappa'(1) = {}^cD^{\nu-1}\kappa(\mu) + {}^cD^\nu\kappa(\mu)$ gives

$$\begin{aligned}\mathscr{B}\left(1+\gamma - \frac{\mu^{2-\nu}}{\Gamma(3-\nu)}\right) = &(\omega^2-\omega)\int_0^\mu \frac{(\mu-\sigma)^{1-\nu}}{\Gamma(2-\nu)}\left(\int_0^\sigma e^{-\omega(\sigma-\vartheta)} I^\nu\xi(\vartheta)d\vartheta\right)d\sigma \\ &+ \int_0^\mu \frac{(\mu-\sigma)^{1-\nu}}{\Gamma(2-\nu)}[(1-\omega)I^\nu\xi(\sigma) + I^{\nu-1}\xi(\sigma)]d\sigma + \gamma\omega\int_0^1 e^{-\omega(1-\sigma)} I^\nu\xi(\sigma)d\sigma - \gamma I^\nu\xi(1).\end{aligned} \tag{2.6}$$

A simultaneous solution of (2.5) and (2.6) yields to

$$\mathscr{A} = \frac{1}{\Theta_2}\int_0^{\mu}\left[\frac{(\mu-\sigma)^{\nu-2}}{\Gamma(\nu-1)}+\frac{(\mu-\sigma)^{\nu-1}}{\Gamma(\nu)}\right]\left(\int_0^{\sigma}e^{-\omega(\sigma-\vartheta)}I^{\nu}\xi(\vartheta)d\vartheta\right)d\sigma$$
$$+\frac{\Theta_3}{\Theta_1\Theta_2}\left[(\omega^2-\omega)\int_0^{\mu}\frac{(\mu-\sigma)^{1-\nu}}{\Gamma(2-\nu)}\left(\int_0^{\sigma}e^{-\omega(\sigma-\vartheta)}I^{\nu}\xi(\vartheta)d\vartheta\right)d\sigma\right.$$
$$\left.+\int_0^{\mu}\frac{(\mu-\sigma)^{1-\nu}}{\Gamma(2-\nu)}[(1-\omega)I^{\nu}\xi(\sigma)+I^{\nu-1}\xi(\sigma)]d\sigma+\gamma\omega\int_0^{1}e^{-\omega(1-\sigma)}I^{\nu}\xi(\sigma)d\sigma-\gamma I^{\nu}\xi(1)\right],$$

$$\mathscr{B} = \frac{1}{\Theta_1}\left[(\omega^2-\omega)\int_0^{\mu}\frac{(\mu-\sigma)^{1-\nu}}{\Gamma(2-\nu)}\left(\int_0^{\sigma}e^{-\omega(\sigma-\vartheta)}I^{\nu}\xi(\vartheta)d\vartheta\right)d\sigma\right.$$
$$\left.+\int_0^{\mu}\frac{(\mu-\sigma)^{1-\nu}}{\Gamma(2-\nu)}[(1-\omega)I^{\nu}\xi(\sigma)+I^{\nu-1}\xi(\sigma)]d\sigma+\gamma\omega\int_0^{1}e^{-\omega(1-\sigma)}I^{\nu}\xi(\sigma)d\sigma-\gamma I^{\nu}\xi(1)\right].$$

Inserting the values of $\mathscr{A}$, $\mathscr{B}$ and $\mathscr{C}$ into (2.4), we get (2.1). □

Lemma 2.7. *For $\xi \in C([0,1],\mathbb{R})$ with $\|\xi\| = \sup_{\kappa\in[0,1]}|\xi(\tau)|$, we have*

i) $|I^{\nu}\xi(\tau)| \le \frac{1}{\Gamma(\nu+1)}\|\xi\|$.

ii) $\left|\int_0^{\tau}e^{-\omega(\tau-\sigma)}I^{\nu}\xi(\sigma)d\sigma\right| \le \frac{1-e^{-\omega}}{\omega\Gamma(\nu+1)}\|\xi\|$.

iii) $\left|\int_0^{\mu}\frac{(\mu-\sigma)^{1-\nu}}{\Gamma(2-\nu)}\left(\int_0^{\sigma}e^{-\omega(\sigma-\vartheta)}I^{\nu}\xi(\vartheta)d\vartheta\right)d\sigma\right| \le \frac{\mu(\omega\mu+e^{-\omega\mu}-1)}{\omega^2\Gamma(2-\nu)\Gamma(\nu+1)}\|\xi\|$.

iv) $\left|\int_0^{\mu}\frac{(\mu-\sigma)^{1-\nu}}{\Gamma(2-\nu)}[(1-\omega)I^{\nu}\xi(\sigma)+I^{\nu-1}\xi(\sigma)]d\sigma\right| \le \frac{|1-\omega|\mu^2+\nu\mu}{\Gamma(\nu+1)\Gamma(3-\nu)}\|\xi\|$.

v) $\left|\int_0^{\mu}\left[\frac{(\mu-\sigma)^{\nu-2}}{\Gamma(\nu-1)}+\frac{(\mu-\sigma)^{\nu-1}}{\Gamma(\nu)}\right]\left(\int_0^{\sigma}e^{-\omega(\sigma-\vartheta)}I^{\nu}\xi(\vartheta)d\vartheta\right)d\sigma\right| \le \frac{(1-e^{-\omega\mu})(\nu\mu^{2\nu-1}+\mu^{2\nu})}{\omega[\Gamma(\nu+1)]^2}\|\xi\|$,

where $I^{\nu}\xi(\tau) = \int_0^{\tau}\frac{(\tau-\eta)^{\nu-1}}{\Gamma(\nu)}\xi(\eta)d\eta, \quad I^{\nu-1}\xi(\tau) = \int_0^{\tau}\frac{(\tau-\eta)^{\nu-2}}{\Gamma(\nu-1)}\xi(\eta)d\eta$.

Proof. For $\xi \in C([0,1],\mathbb{R})$ with $\|\xi\| = \sup_{\kappa\in[0,1]}|\xi(\tau)|$, we have

i)

$$\begin{aligned}|I^{\nu}\xi(\tau)| &= \left|\int_0^{\tau}\frac{(\tau-\sigma)^{\nu-1}}{\Gamma(\nu)}\xi(\sigma)d\sigma\right| \\ &\le \int_0^{\tau}\frac{(\tau-\sigma)^{\nu-1}}{\Gamma(\nu)}|\xi(\sigma)|d\sigma \\ &\le \frac{\tau^{\nu}}{\Gamma(\nu+1)}\|\xi\| \\ &\le \frac{1}{\Gamma(\nu+1)}\|\xi\|.\end{aligned}$$

ii)

$$\begin{aligned}\left|\int_0^{\tau}e^{-\omega(\tau-\sigma)}I^{\nu}\xi(\sigma)d\sigma\right| &\le \int_0^{\tau}e^{-\omega(\tau-\sigma)}\left|I^{\nu}\xi(\sigma)\right|d\sigma \\ &\le \frac{\|\xi\|}{\Gamma(\nu+1)}\int_0^{\tau}e^{-\omega(\tau-\sigma)}d\sigma \\ &\le \frac{1-e^{-\omega}}{\omega\Gamma(\nu+1)}\|\xi\|.\end{aligned}$$

iii)

$$\int_0^{\vartheta}\frac{(\vartheta-\eta)^{\nu-1}}{\Gamma(\nu)}d\eta = \frac{\vartheta^{\nu}}{\Gamma(\nu+1)}$$

and

$$\int_0^{\sigma}e^{-\omega(\sigma-\vartheta)}\frac{\vartheta^{\nu}}{\Gamma(\nu+1)}d\vartheta \le \frac{\sigma^{\nu}}{\Gamma(\nu+1)}\int_0^{\sigma}e^{-\omega(\sigma-\vartheta)}d\vartheta = \frac{\sigma^{\nu}(1-e^{-\omega\sigma})}{\omega\Gamma(\nu+1)}.$$

Hence

$$\left|\int_0^\mu \frac{(\mu-\sigma)^{1-\nu}}{\Gamma(2-\nu)}\left(\int_0^\sigma e^{-\omega(\sigma-\vartheta)}I^\nu\xi(\vartheta)d\vartheta\right)d\sigma\right|$$

$$=\left|\int_0^\mu \frac{(\mu-\sigma)^{1-\nu}}{\Gamma(2-\nu)}\left(\int_0^\sigma e^{-\omega(\sigma-\vartheta)}\left(\int_0^\vartheta \frac{(\vartheta-\eta)^{\nu-1}}{\Gamma(\nu)}\xi(\eta)d\eta\right)d\vartheta\right)d\sigma\right|$$

$$\leq \|\xi\|\int_0^\mu \frac{(\mu-\sigma)^{1-\nu}}{\Gamma(2-\nu)}\frac{\sigma^\nu(1-e^{-\omega\sigma})}{\omega\Gamma(\nu+1)}d\sigma$$

$$\leq \|\xi\|\frac{\mu^{1-\nu}}{\Gamma(2-\nu)}\frac{\mu^\nu}{\omega\Gamma(\nu+1)}\int_0^\mu (1-e^{-\omega\sigma})d\sigma$$

$$\leq \frac{\mu(\omega\mu+e^{-\omega\mu}-1)}{\omega^2\Gamma(2-\nu)\Gamma(\nu+1)}\|\xi\|.$$

iv)

$$\int_0^\mu \frac{(\mu-\sigma)^{1-\nu}}{\Gamma(2-\nu)}d\sigma \leq \int_0^\mu \frac{\mu^{1-\nu}}{\Gamma(2-\nu)}d\sigma,$$

and

$$\left|I^{\nu-1}\xi(\sigma)\right| = \left|\int_0^\sigma \frac{(\sigma-\eta)^{\nu-2}}{\Gamma(\nu-1)}\xi(\eta d\eta\right| \leq \frac{\sigma^\nu}{\Gamma(\nu+1)}\|\xi\|.$$

Hence

$$\left|\int_0^\mu \frac{(\mu-\sigma)^{1-\nu}}{\Gamma(2-\nu)}[(1-\omega)I^\nu\xi(\sigma)+I^{\nu-1}\xi(\sigma)]d\sigma\right| \leq \int_0^\mu \frac{(\mu-\sigma)^{1-\nu}}{\Gamma(2-\nu)}\left[|1-\omega|\left|I^\nu\xi(\sigma)\right|+\left|I^{\nu-1}\xi(\sigma)\right|\right]d\sigma$$

$$\leq \int_0^\mu \frac{(\mu-\sigma)^{1-\nu}}{\Gamma(2-\nu)}\left[\frac{|1-\omega|\sigma^\nu+\nu\sigma^{\nu-1}}{\Gamma(\nu+1)}\right]\|\xi\|d\sigma$$

$$\leq \frac{|1-\omega|\mu^\nu+\nu\mu^{\nu-1}}{\Gamma(\nu+1)}\|\xi\|\int_0^\mu \frac{(\mu-\sigma)^{1-\nu}}{\Gamma(2-\nu)}d\sigma$$

$$\leq \frac{|1-\omega|\mu^2+\nu\mu}{\Gamma(\nu+1)\Gamma(3-\nu)}\|\xi\|.$$

v)

$$\int_0^\mu \frac{(\mu-\sigma)^{\nu-2}}{\Gamma(\nu-1)}d\sigma = \frac{\mu^{\nu-1}}{\Gamma(\nu)}, \quad \int_0^\mu \frac{(\mu-\sigma)^{\nu-1}}{\Gamma(\nu)}d\sigma = \frac{\mu^\nu}{\Gamma(\nu+1)}$$

and

$$\left|\int_0^\sigma e^{-\omega(\sigma-\vartheta)}I^\nu\xi(\vartheta)d\vartheta\right| \leq \frac{\sigma^\nu(1-e^{-\omega\sigma})}{\omega\Gamma(\nu+1)}\|\xi\|.$$

Hence

$$\left|\int_0^\mu \left[\frac{(\mu-\sigma)^{\nu-2}}{\Gamma(\nu-1)}+\frac{(\mu-\sigma)^{\nu-1}}{\Gamma(\nu)}\right]\left(\int_0^\sigma e^{-\omega(\sigma-\vartheta)}I^\nu\xi(\vartheta)d\vartheta\right)d\sigma\right|$$

$$\leq \int_0^\mu \left[\frac{(\mu-\sigma)^{\nu-2}}{\Gamma(\nu-1)}+\frac{(\mu-\sigma)^{\nu-1}}{\Gamma(\nu)}\right]\left|\int_0^\sigma e^{-\omega(\sigma-\vartheta)}I^\nu\xi(\vartheta)d\vartheta\right|d\sigma$$

$$\leq \frac{\mu^\nu(1-e^{-\omega})}{\omega\Gamma(\nu+1)}\|\xi\|\int_0^\mu \left[\frac{(\mu-\sigma)^{\nu-2}}{\Gamma(\nu-1)}+\frac{(\mu-\sigma)^{\nu-1}}{\Gamma(\nu)}\right]d\sigma \;\leq\; \frac{(1-e^{-\omega\mu})(\nu\mu^{2\nu-1}+\mu^{2\nu})}{\omega[\Gamma(\nu+1)]^2}\|\xi\|.$$

□

Lemma 2.8. *(Krasnoselskii's fixed point theorem [27]). Let $\mathfrak{X}$ be a Banach space, $Y \subseteq \mathfrak{X}$ be nonempty, bounded, closed and convex. Let $\mathfrak{T}_1, \mathfrak{T}_2$ be two maps and satisfy:*

(i) $\mathfrak{T}_1 y_1 + \mathfrak{T}_2 y_2 \in Y$, $\forall y_1, y_2 \in Y$;
(ii) $\mathfrak{T}_1$ is compact and continuous;
(iii) $\mathfrak{T}_2$ is a contraction mapping.

Then there exists $y_3 \in Y$ such that $y_3 = \mathfrak{T}_1 y_3 + \mathfrak{T}_2 y_3$.

Lemma 2.9. *(Leray-Schauder fixed point theorem [28]) Let $\mathfrak{X}$ be a Banach space, $Y \subseteq \mathfrak{X}$ be nonempty, bounded and convex, H be an open subset of Y with $0 \in H$. Let map $\mathfrak{G} : \overline{H} \longrightarrow Y$ be continuous and compact (that is, $\mathfrak{G}(\overline{H})$ is a relatively compact subset of Y). Then, one of the following representations is true:*

(i) there exist $z \in \partial H$ and $\varepsilon \in (0,1)$ such that $z = \varepsilon\mathfrak{G}(z)$;
(ii) $\mathfrak{G}$ has a fixed point $z \in H$.

3. Existence Results

This section deals with the existence results for problem (1.1).
Let $\mathfrak{X}=\left\{\kappa \mid \kappa \in C([0,1],\mathbb{R}) \text{ and } {}^cD^{\nu-1}\kappa \in C([0,1],\mathbb{R})\right\}$ denote the Banach space of all continuous functions from $[0,1]$ into $\mathbb{R}$ endowed with the usual norm defined by

$$\|\kappa\|_{\mathfrak{X}}=\|\kappa\|+\left\|{}^cD^{\nu-1}\kappa\right\|=\sup_{\tau\in[0,1]}|\kappa(\tau)|+\sup_{\tau\in[0,1]}\left|{}^cD^{\nu-1}\kappa(\tau)\right|,$$

where $1<\nu\le 2$.
In view of Lemma 2.6, we transform problem (1.1) to an equivalent fixed point problem as

$$\kappa=\mathfrak{G}\kappa,$$

$$\begin{aligned}
(\mathfrak{G}\kappa)(\tau)&=\int_0^\tau e^{-\omega(\tau-\sigma)}I^\nu f(\sigma,\kappa(\sigma),{}^cD^{\nu-1}\kappa(\sigma))d\sigma\\
&+\frac{\tau}{\Theta_1}\left[(\omega^2-\omega)\int_0^\mu\frac{(\mu-\sigma)^{1-\nu}}{\Gamma(2-\nu)}\left(\int_0^\sigma e^{-\omega(\sigma-\vartheta)}I^\nu f(\vartheta,\kappa(\vartheta),{}^cD^{\nu-1}\kappa(\vartheta))d\vartheta\right)d\sigma\right.\\
&+\int_0^\mu\frac{(\mu-\sigma)^{1-\nu}}{\Gamma(2-\nu)}[(1-\omega)I^\nu f(\sigma,\kappa(\sigma),{}^cD^{\nu-1}\kappa(\sigma))+I^{\nu-1}f(\sigma,\kappa(\sigma),{}^cD^{\nu-1}\kappa(\sigma))]d\sigma\\
&\left.+\gamma\omega\int_0^1 e^{-\omega(1-\sigma)}I^\nu f(\sigma,\kappa(\sigma),{}^cD^{\nu-1}\kappa(\sigma))d\sigma-\gamma I^\nu f(1,\kappa(1),{}^cD^{\nu-1}\kappa(1))\right]\\
&+\frac{1}{\Theta_2}\int_0^\mu\left[\frac{(\mu-\sigma)^{\nu-2}}{\Gamma(\nu-1)}+\frac{(\mu-\sigma)^{\nu-1}}{\Gamma(\nu)}\right]\left(\int_0^\sigma e^{-\omega(\sigma-\vartheta)}I^\nu f(\vartheta,\kappa(\vartheta),{}^cD^{\nu-1}\kappa(\vartheta))d\vartheta\right)d\sigma\\
&+\frac{\Theta_3}{\Theta_1\Theta_2}\left[(\omega^2-\omega)\int_0^\mu\frac{(\mu-\sigma)^{1-\nu}}{\Gamma(2-\nu)}\left(\int_0^\sigma e^{-\omega(\sigma-\vartheta)}I^\nu f(\vartheta,\kappa(\vartheta),{}^cD^{\nu-1}\kappa(\vartheta))d\vartheta\right)d\sigma\right.\\
&+\int_0^\mu\frac{(\mu-\sigma)^{1-\nu}}{\Gamma(2-\nu)}[(1-\omega)I^\nu f(\sigma,\kappa(\sigma),{}^cD^{\nu-1}\kappa(\sigma))+I^{\nu-1}f(\sigma,\kappa(\sigma),{}^cD^{\nu-1}\kappa(\sigma))]d\sigma\\
&\left.+\gamma\omega\int_0^1 e^{-\omega(1-\sigma)}I^\nu f(\sigma,\kappa(\sigma),{}^cD^{\nu-1}\kappa(\sigma))d\sigma-\gamma I^\nu f(1,\kappa(1),{}^cD^{\nu-1}\kappa(1))\right].
\end{aligned} \tag{3.1}$$

For convenience, we let

$$\begin{aligned}
\Pi_1&=\frac{1-e^{-\omega}}{\omega\Gamma(\nu+1)}+\frac{\mu(|\Theta_2|+|\Theta_3|)(e^{-\omega\mu}+\omega\mu-1)\left|\omega^2-\omega\right|}{|\Theta_1||\Theta_2|\omega^2\Gamma(2-\nu)\Gamma(\nu+1)}\\
&+\frac{\gamma(|\Theta_2|+|\Theta_3|)(2-e^{-\omega})}{|\Theta_1||\Theta_2|\Gamma(\nu+1)}+\frac{(|\Theta_2|+|\Theta_3|)(|1-\omega|\mu^2+\nu\mu)}{|\Theta_1||\Theta_2|\Gamma(\nu+1)\Gamma(3-\nu)}+\frac{(1-e^{-\omega\mu})(\nu\mu^{2\nu-1}+\mu^{2\nu})}{\omega|\Theta_2|[\Gamma(\nu+1)]^2}.
\end{aligned} \tag{3.2}$$

$$\Pi_2=\frac{(|\Theta_1|+|\gamma|)(2-e^{-\omega})}{|\Theta_1|\Gamma(\nu+1)}+\frac{\mu(e^{-\omega\mu}+\omega\mu-1)\left|\omega^2-\omega\right|}{|\Theta_1|\omega^2\Gamma(2-\nu)\Gamma(\nu+1)}+\frac{|1-\omega|\mu^2+\nu\mu}{|\Theta_1||\Gamma(\nu+1)\Gamma(3-\nu)}.$$

$$\begin{aligned}
\widetilde{\Pi}_1&=\Pi_1-\frac{1-e^{-\omega}}{\omega\Gamma(\nu+1)}.\\
\widetilde{\Pi}_2&=\Pi_2-\frac{2-e^{-\omega}}{\Gamma(\nu+1)}.
\end{aligned} \tag{3.3}$$

Theorem 3.1. *Assume $f:[0,1]\times\mathbb{R}^2\to\mathbb{R}$ be a continuous function, which satisfies the following conditions:*
$(\mathfrak{B}_1)$ *There exists a constant $q>0$ such that*

$$|f(\tau,\kappa_1,\kappa_2)-f(\tau,\widetilde{\kappa}_1,\widetilde{\kappa}_2)|\le q\left(\|\kappa_1-\widetilde{\kappa}_1\|+\|\kappa_2-\widetilde{\kappa}_2\|\right),$$

$\forall\tau\in[0,1],\ \kappa_i,\widetilde{\kappa}_i\in\mathbb{R}, i=1,2.$
$(\mathfrak{B}_2)$ $\forall\tau\in[0,1],\ \forall\,\kappa_1,\kappa_2\in\mathbb{R},\exists\theta\in C([0,1],\mathbb{R}^+):\ |f(\tau,\kappa_1,\kappa_2)|\le\theta(\tau).$
Then the problem (1.1) has at least one solution on $[0,1]$ if

$$q\left(\widetilde{\Pi}_1+\frac{\widetilde{\Pi}_2}{\Gamma(3-\nu)}\right)<1,$$

where $\widetilde{\Pi}_1,\widetilde{\Pi}_2$ are given by (3.3).

Proof. Set $\sup_{\kappa\in[0,1]}|\kappa(\tau)| = \|\kappa\|$, we fix

$$\rho \geq \left(\Pi_1 + \frac{\Pi_2}{\Gamma(3-\nu)}\right)\|\kappa\|,$$

where Π_1, Π_2 given by (3.2) and define the ball $\mathscr{S}_\rho = \{\kappa \in \mathfrak{X} : \|\kappa\|_{\mathfrak{X}} \leq \rho\}$. Consider the operators $\mathfrak{G}_1$ and $\mathfrak{G}_2$ on $\mathscr{S}_\rho$

$$\begin{aligned}
(\mathfrak{G}_1\kappa)(\tau) &= \int_0^\tau e^{-\omega(\tau-\sigma)} I^\nu f\big(\sigma,\kappa(\sigma),{}^cD^{\nu-1}\kappa(\sigma)\big)\,d\sigma,\\
(\mathfrak{G}_2\kappa)(\tau) &= \frac{\tau}{\Theta_1}\Big[(\omega^2-\omega)\int_0^\mu \frac{(\mu-\sigma)^{1-\nu}}{\Gamma(2-\nu)}\big(\int_0^\sigma e^{-\omega(\sigma-\vartheta)} I^\nu f(\vartheta,\kappa(\vartheta),{}^cD^{\nu-1}\kappa(\vartheta))\,d\vartheta\big)\,d\sigma\\
&\quad+\int_0^\mu \frac{(\mu-\sigma)^{1-\nu}}{\Gamma(2-\nu)}\big[(1-\omega)I^\nu f(\sigma,\kappa(\sigma),{}^cD^{\nu-1}\kappa(\sigma)) + I^{\nu-1} f(\sigma,\kappa(\sigma),{}^cD^{\nu-1}\kappa(\sigma))\big]\,d\sigma\\
&\quad+\gamma\omega\int_0^1 e^{-\omega(1-\sigma)} I^\nu f(\sigma,\kappa(\sigma),{}^cD^{\nu-1}\kappa(\sigma))\,d\sigma - \gamma I^\nu f\big(1,\kappa(1),{}^cD^{\nu-1}\kappa(1)\big)\Big]\\
&\quad+\frac{1}{\Theta_2}\int_0^\mu\Big[\frac{(\mu-\sigma)^{\nu-2}}{\Gamma(\nu-1)} + \frac{(\mu-\sigma)^{\nu-1}}{\Gamma(\nu)}\Big]\big(\int_0^\sigma e^{-\omega(\sigma-\vartheta)} I^\nu f(\vartheta,\kappa(\vartheta),{}^cD^{\nu-1}\kappa(\vartheta))\,d\vartheta\big)\,d\sigma\\
&\quad+\frac{\Theta_3}{\Theta_1\Theta_2}\Big[(\omega^2-\omega)\int_0^\mu \frac{(\mu-\sigma)^{1-\nu}}{\Gamma(2-\nu)}\big(\int_0^\sigma e^{-\omega(\sigma-\vartheta)} I^\nu f(\vartheta,\kappa(\vartheta),{}^cD^{\nu-1}\kappa(\vartheta))\,d\vartheta\big)\,d\sigma\\
&\quad+\int_0^\mu \frac{(\mu-\sigma)^{1-\nu}}{\Gamma(2-\nu)}\big[(1-\omega)I^\nu f(\sigma,\kappa(\sigma),{}^cD^{\nu-1}\kappa(\sigma)) + I^{\nu-1} f(\sigma,\kappa(\sigma),{}^cD^{\nu-1}\kappa(\sigma))\big]\,d\sigma\\
&\quad+\gamma\omega\int_0^1 e^{-\omega(1-\sigma)} I^\nu f(\sigma,\kappa(\sigma),{}^cD^{\nu-1}\kappa(\sigma))\,d\sigma - \gamma I^\nu f\big(1,\kappa(1),{}^cD^{\nu-1}\kappa(1)\big)\Big].
\end{aligned}$$

In what follows, we use three steps to complete the proof of the theorem.

Step 1. $\forall\kappa_1,\kappa_2 \in \mathscr{S}_\rho$, $(\mathfrak{G}_1\kappa_1)(\tau) + (\mathfrak{G}_2\kappa_2)(\tau) \in \mathscr{S}_\rho$.
From Lemma 2.7 and by the use of condition $(\mathfrak{B}_2)$, for each $\kappa_1,\kappa_2 \in \mathscr{S}_\rho$

$$\begin{aligned}
&|(\mathfrak{G}_1\kappa_1)(\tau) + (\mathfrak{G}_2\kappa_2)(\tau)|\\
&\leq \left|\int_0^\tau e^{-\omega(\tau-\sigma)} I^\nu f(\sigma,\kappa_1(\sigma),{}^cD^{\nu-1}\kappa_1(\sigma))d\sigma\right|\\
&\quad+\left|\tfrac{\tau}{\Theta_1}\right|\Big[\left|\omega^2-\omega\right|\left|\int_0^\mu \tfrac{(\mu-\sigma)^{1-\nu}}{\Gamma(2-\nu)}\left(\int_0^\sigma e^{-\omega(\sigma-\vartheta)} I^\nu f(\vartheta,\kappa_2(\vartheta),{}^cD^{\nu-1}\kappa_2(\vartheta))d\vartheta\right)d\sigma\right|\\
&\quad+\left|\int_0^\mu \tfrac{(\mu-\sigma)^{1-\nu}}{\Gamma(2-\nu)}[(1-\omega)I^\nu f(\sigma,\kappa_2(\sigma),{}^cD^{\nu-1}\kappa_2(\sigma)) + I^{\nu-1} f(\sigma,\kappa_2(\sigma),{}^cD^{\nu-1}\kappa_2(\sigma))]d\sigma\right|\\
&\quad+|\gamma\omega|\left|\int_0^1 e^{-\omega(1-\sigma)} I^\nu f(\sigma,\kappa_2(\sigma),{}^cD^{\nu-1}\kappa_2(\sigma))d\sigma\right| + |\gamma|\left|I^\nu f(1,\kappa_2(1),{}^cD^{\nu-1}\kappa_2(1))\right|\Big]\\
&\quad+\tfrac{1}{|\Theta_2|}\left|\int_0^\mu\left[\tfrac{(\mu-\sigma)^{\nu-2}}{\Gamma(\nu-1)} + \tfrac{(\mu-\sigma)^{\nu-1}}{\Gamma(\nu)}\right]\left(\int_0^\sigma e^{-\omega(\sigma-\vartheta)} I^\nu f(\vartheta,\kappa_2(\vartheta),{}^cD^{\nu-1}\kappa_2(\vartheta))d\vartheta\right)d\sigma\right|\\
&\quad+\left|\tfrac{\Theta_3}{\Theta_1\Theta_2}\right|\Big[\left|\omega^2-\omega\right|\left|\int_0^\mu \tfrac{(\mu-\sigma)^{1-\nu}}{\Gamma(2-\nu)}\left(\int_0^\sigma e^{-\omega(\sigma-\vartheta)} I^\nu f(\vartheta,\kappa_2(\vartheta),{}^cD^{\nu-1}\kappa_2(\vartheta))d\vartheta\right)d\sigma\right|\\
&\quad+\left|\int_0^\mu \tfrac{(\mu-\sigma)^{1-\nu}}{\Gamma(2-\nu)}[(1-\omega)I^\nu f(\sigma,\kappa_2(\sigma),{}^cD^{\nu-1}\kappa_2(\sigma)) + I^{\nu-1} f(\sigma,\kappa_2(\sigma),{}^cD^{\nu-1}\kappa_2(\sigma))]d\sigma\right|\\
&\quad+|\gamma\omega|\left|\int_0^1 e^{-\omega(1-\sigma)} I^\nu f(\sigma,\kappa_2(\sigma),{}^cD^{\nu-1}\kappa_2(\sigma))d\sigma\right| + |\gamma|\left|I^\nu f(1,\kappa_2(1),{}^cD^{\nu-1}\kappa_2(1))\right|\Big]\\
&\leq \tfrac{1-e^{-\omega}}{\omega\Gamma(\nu+1)}\|\theta\| + \tfrac{1}{|\Theta_1|}\left(\tfrac{\mu(e^{-\omega\mu}+\omega\mu-1)\left|\omega^2-\omega\right|}{\omega^2\Gamma(2-\nu)\Gamma(\nu+1)}\|\theta\| + \tfrac{|1-\omega|\mu^2+\nu\mu}{\Gamma(\nu+1)\Gamma(3-\nu)}\|\theta\|\right.\\
&\quad\left.+\tfrac{|\gamma|(1-e^{-\omega})}{\Gamma(\nu+1)}\|\theta\| + \tfrac{|\gamma|}{\Gamma(\nu+1)}\|\theta\|\right) + \tfrac{(1-e^{-\omega\mu})(\nu\mu^{2\nu-1}+\mu^{2\nu})}{\omega|\Theta_2|[\Gamma(\nu+1)]^2}\|\theta\|\\
&\quad+\tfrac{|\Theta_3|}{|\Theta_1||\Theta_2|}\left(\tfrac{\mu(e^{-\omega\mu}+\omega\mu-1)\left|\omega^2-\omega\right|}{\omega^2\Gamma(2-\nu)\Gamma(\nu+1)}\|\theta\| + \tfrac{|1-\omega|\mu^2+\nu\mu}{\Gamma(\nu+1)\Gamma(3-\nu)}\|\theta\|\right.\\
&\quad\left.+\tfrac{|\gamma|(1-e^{-\omega})}{\Gamma(\nu+1)}\|\theta\| + \tfrac{|\gamma|}{\Gamma(\nu+1)}\|\theta\|\right)\\
&\leq \left(\tfrac{1-e^{-\omega}}{\omega\Gamma(\nu+1)} + \tfrac{\mu(|\Theta_2|+|\Theta_3|)(e^{-\omega\mu}+\omega\mu-1)\left|\omega^2-\omega\right|}{|\Theta_1||\Theta_2|\omega^2\Gamma(2-\nu)\Gamma(\nu+1)}\right.\\
&\quad+\tfrac{\gamma(|\Theta_2|+|\Theta_3|)(2-e^{-\omega})}{|\Theta_1||\Theta_2|\Gamma(\nu+1)} + \tfrac{(|\Theta_2|+|\Theta_3|)(|1-\omega|\mu^2+\nu\mu)}{|\Theta_1||\Theta_2|\Gamma(\nu+1)\Gamma(3-\nu)}\\
&\quad\left.+\tfrac{(1-e^{-\omega\mu})(\nu\mu^{2\nu-1}+\mu^{2\nu})}{\omega|\Theta_2|[\Gamma(\nu+1)]^2}\right)\|\theta\|
\end{aligned}$$

$$\leq \Pi_1 \|\theta\|.$$

Thus

$$\|(\mathfrak{G}_1\kappa_1)+(\mathfrak{G}_2\kappa_2)\| \leq \Pi_1 \|\theta\|.$$

Also we have

$$(\mathfrak{G}'_1\kappa_1)(\tau) = -\omega\int_0^\tau e^{-\omega(\tau-\sigma)} I^\nu f(\sigma,\kappa_1(\sigma),{}^cD^{\nu-1}\kappa_1(\sigma))d\sigma + I^\nu f(\tau,\kappa_1(\tau),{}^cD^{\nu-1}\kappa_1(\tau)).$$

$$\begin{aligned}(\mathfrak{G}'_2\kappa_2)(\tau) = \frac{1}{\Theta_1}\Bigg[&(\omega^2-\omega)\int_0^\mu \frac{(\mu-\sigma)^{1-\nu}}{\Gamma(2-\nu)}\left(\int_0^\sigma e^{-\omega(\sigma-\vartheta)} I^\nu f(\vartheta,\kappa_2(\vartheta),{}^cD^{\nu-1}\kappa_2(\vartheta))d\vartheta\right)d\sigma \\ &+\int_0^\mu \frac{(\mu-\sigma)^{1-\nu}}{\Gamma(2-\nu)}[(1-\omega)I^\nu f(\sigma,\kappa_2(\sigma),{}^cD^{\nu-1}\kappa_2(\sigma)) + I^{\nu-1} f(\sigma,\kappa_2(\sigma),{}^cD^{\nu-1}\kappa_2(\sigma))]d\sigma \\ &+\gamma\omega\int_0^1 e^{-\omega(1-\sigma)} I^{\nu-1} f(\sigma,\kappa_2(\sigma),{}^cD^{\nu-1}\kappa_2(\sigma))d\sigma - \gamma I^\nu f(1,\kappa_2(1),{}^cD^{\nu-1}\kappa_2(1))\Bigg].\end{aligned}$$

Hence

$$\begin{aligned}&\left|(\mathfrak{G}'_1\kappa_1)(\tau)+(\mathfrak{G}'_2\kappa_2)(\tau)\right| \\ &\leq \omega\left|\int_0^\tau e^{-\omega(\tau-\sigma)} I^\nu f(\sigma,\kappa_1(\sigma),{}^cD^{\nu-1}\kappa_1(\sigma))d\sigma\right| + \left|I^\nu f(\tau,\kappa_1(\tau),{}^cD^{\nu-1}\kappa_1(\tau))\right| \\ &+\frac{1}{|\Theta_1|}\Bigg[\left|\omega^2-\omega\right|\left|\int_0^\mu \frac{(\mu-\sigma)^{1-\nu}}{\Gamma(2-\nu)}\left(\int_0^\sigma e^{-\omega(\sigma-\vartheta)} I^\nu f(\vartheta,\kappa_2(\vartheta),{}^cD^{\nu-1}\kappa_2(\vartheta))d\vartheta\right)d\sigma\right| \\ &+\left|\int_0^\mu \frac{(\mu-\sigma)^{1-\nu}}{\Gamma(2-\nu)}[(1-\omega)I^\nu f(\sigma,\kappa_2(\sigma),{}^cD^{\nu-1}\kappa_2(\sigma)) + I^{\nu-1} f(\sigma,\kappa_2(\sigma),{}^cD^{\nu-1}\kappa_2(\sigma))]d\sigma\right| \\ &+|\gamma|\,\omega\left|\int_0^1 e^{-\omega(1-\sigma)} I^{\nu-1} f(\sigma,\kappa_2(\sigma),{}^cD^{\nu-1}\kappa_2(\sigma))d\sigma\right| + |\gamma|\left|I^\nu f(1,\kappa_2(1),{}^cD^{\nu-1}\kappa_2(1))\right|\Bigg] \\ &\leq \frac{1-e^{-\omega}}{\Gamma(\nu+1)}\|\theta\| + \frac{1}{\Gamma(\nu+1)}\|\theta\| + \frac{\mu(e^{-\omega\mu}+\omega\mu-1)\left|\omega^2-\omega\right|}{|\Theta_1|\,\omega^2\Gamma(2-\nu)\Gamma(\nu+1)}\|\theta\| \\ &+\frac{|1-\omega|\,\mu^2+\nu\mu}{|\Theta_1|\,|\Gamma(\nu+1)\Gamma(3-\nu)}\|\theta\| + \frac{|\gamma|\,(1-e^{-\omega})}{|\Theta_1|\Gamma(\nu+1)}\|\theta\| + \frac{|\gamma|}{|\Theta_1|\Gamma(\nu+1)}\|\theta\| \\ &\leq \Bigg(\frac{(|\Theta_1|+|\gamma|)(2-e^{-\omega})}{|\Theta_1|\Gamma(\nu+1)} + \frac{\mu(e^{-\omega\mu}+\omega\mu-1)\left|\omega^2-\omega\right|}{|\Theta_1|\,\omega^2\Gamma(2-\nu)\Gamma(\nu+1)} \\ &\quad+\frac{|1-\omega|\,\mu^2+\nu\mu}{|\Theta_1|\,|\Gamma(\nu+1)\Gamma(3-\nu)}\Bigg)\|\theta\| \\ &\leq \Pi_2\|\theta\|.\end{aligned}$$

From Definition 2.2 with $1<\nu\leq 2$, we get

$$\begin{aligned}\left|{}^cD^{\nu-1}(\mathfrak{G}_1\kappa_1+\mathfrak{G}_2\kappa_2)(\tau)\right| &\leq \int_0^\tau \frac{(\tau-\sigma)^{1-\nu}}{\Gamma(2-\nu)}\left|(\mathfrak{G}'_1\kappa_1+\mathfrak{G}'_2\kappa_2)(\sigma)\right|d\sigma \\ &\leq \Pi_2\|\theta\|\int_0^\tau \frac{(\tau-\sigma)^{1-\nu}}{\Gamma(2-\nu)}d\sigma \\ &\leq \Pi_2\|\theta\|\frac{\tau^{2-\nu}}{\Gamma(3-\nu)} \\ &\leq \frac{\Pi_2}{\Gamma(3-\nu)}\|\theta\|.\end{aligned}$$

From the above inequalities, we get

$$\begin{aligned}\|\mathfrak{G}_1\kappa_1+\mathfrak{G}_2\kappa_2\|_{\mathfrak{X}} &= \|\mathfrak{G}_1\kappa_1+\mathfrak{G}_2\kappa_2\| + \left\|{}^cD^{\nu-1}(\mathfrak{G}_1\kappa_1+\mathfrak{G}_2\kappa_2)\right\| \\ &\leq \left(\Pi_1+\frac{\Pi_2}{\Gamma(3-\nu)}\right)\|\theta\| \\ &\leq \rho.\end{aligned}$$

Thus, $\mathfrak{G}_1\kappa_1+\mathfrak{G}_2\kappa_2\in\mathscr{S}_\rho$.

Step 2. $\mathfrak{G}_1 : C([0,1],\mathbb{R}) \to C([0,1],\mathbb{R})$ is continuous and compact.
Let $\tau_1,\tau_2 \in [0,1]$ with $\tau_1 < \tau_2$ and $\kappa \in \mathscr{S}_\rho$. By using Lemma 2.7 and condition $(\mathfrak{B}_2)$, one can find

$$\begin{aligned}\left|(\mathfrak{G}_1\kappa)(\tau_2)-(\mathfrak{G}_1\kappa)(\tau_1)\right| &= \left|\int_0^{\tau_2} e^{-\omega(\tau_2-\sigma)} I^\nu f\big(\sigma,\kappa(\sigma),{}^cD^{\nu-1}\kappa(\sigma)\big)\,d\sigma - \int_0^{\tau_1} e^{-\omega(\tau_1-\sigma)} I^\nu f\big(\sigma,\kappa(\sigma),{}^cD^{\nu-1}\kappa(\sigma)\big)\,d\sigma\right| \\ &= \left|\int_0^{\tau_1} e^{\omega\sigma}(e^{-\omega\tau_2}-e^{-\omega\tau_1}) I^\nu f\big(\sigma,\kappa(\sigma),{}^cD^{\nu-1}\kappa(\sigma)\big)\,d\sigma + \int_{\tau_1}^{\tau_2} e^{-\omega(\tau_2-\sigma)} I^\nu f\big(\sigma,\kappa(\sigma),{}^cD^{\nu-1}\kappa(\sigma)\big)\,d\sigma\right| \\ &\le \int_0^{\tau_1} e^{\omega\sigma}\left|e^{-\omega\tau_2}-e^{-\omega\tau_1}\right|\left|I^\nu f(\sigma,\kappa(\sigma),{}^cD^{\nu-1}\kappa(\sigma))\right|d\sigma + \int_{\tau_1}^{\tau_2} e^{-\omega(\tau_2-\sigma)}\left|I^\nu f(\sigma,\kappa(\sigma),{}^cD^{\nu-1}\kappa(\sigma))\right|d\sigma \\ &\le \frac{1}{\Gamma(\nu+1)}\left(\int_0^{\tau_1} e^{\omega\sigma}\left|e^{-\omega\tau_2}-e^{-\omega\tau_1}\right|d\sigma + \int_{\tau_1}^{\tau_2} e^{-\omega(\tau_2-\sigma)}\,d\sigma\right)\|\theta\|.\end{aligned}$$

and

$$\begin{aligned}\left|{}^cD^{\nu-1}(\mathfrak{G}_1\kappa)(\tau_2)-{}^cD^{\nu-1}(\mathfrak{G}_1\kappa)(\tau_1)\right| &= \left|\int_0^{\tau_2}\frac{(\tau_2-\sigma)^{1-\nu}}{\Gamma(2-\nu)}(\mathfrak{G}_1'\kappa)(\sigma)\,d\sigma - \int_0^{\tau_1}\frac{(\tau_1-\sigma)^{1-\nu}}{\Gamma(2-\nu)}(\mathfrak{G}_1'\kappa)(\sigma)\,d\sigma\right| \\ &= \left|\int_0^{\tau_1}\frac{(\tau_2-\sigma)^{1-\nu}-(\tau_1-\sigma)^{1-\nu}}{\Gamma(2-\nu)}(\mathfrak{G}_1'\kappa)(\sigma)\,d\sigma + \int_{\tau_1}^{\tau_2}\frac{(\tau_2-\sigma)^{1-\nu}}{\Gamma(2-\nu)}(\mathfrak{G}_1'\kappa)(\sigma)\,d\sigma\right| \\ &\le \frac{1}{\Gamma(2-\nu)}\bigg(\int_0^{\tau_1}\left|(\tau_2-\sigma)^{1-\nu}-(\tau_1-\sigma)^{1-\nu}\right|\left|(\mathfrak{G}_1'\kappa)(\sigma)\right|d\sigma \\ &\quad + \int_{\tau_1}^{\tau_2}(\tau_2-\sigma)^{1-\nu}\left|(\mathfrak{G}_1'\kappa)(\sigma)\right|d\sigma\bigg) \\ &\le \frac{2-e^{-\omega}}{\Gamma(2-\nu)\Gamma(\nu+1)}\bigg(\int_0^{\tau_1}\left|(\tau_2-\sigma)^{1-\nu}-(\tau_1-\sigma)^{1-\nu}\right|d\sigma \\ &\quad + \int_{\tau_1}^{\tau_2}(\tau_2-\sigma)^{1-\nu}\,d\sigma\bigg)\|\theta\|.\end{aligned}$$

Clearly, $|(\mathfrak{G}_1\kappa)(\tau_2)-(\mathfrak{G}_1\kappa)(\tau_1)| \to 0$ and $\left|{}^cD^{\nu-1}(\mathfrak{G}_1\kappa)(\tau_2)-{}^cD^{\nu-1}(\mathfrak{G}_1\kappa)(\tau_1)\right| \to 0$ independent of κ as $\tau_1 \to \tau_2$. Thus $\mathfrak{G}_1$ is relatively compact on S_ρ. Then, by the Arzelá-Ascoli theorem, $\mathfrak{G}_1$ is compact on $\mathscr{S}_\rho$.

Step 3. $\mathfrak{G}_2 : C([0,1],\mathbb{R}) \to C([0,1],\mathbb{R})$ is contraction.
From Lemma 2.7 and the use of condition $(\mathfrak{B}_1)$, for $\tau \in [0,1]$, $\kappa_1,\kappa_2 \in \mathscr{S}_\rho$, we can derive

$$\begin{aligned}&|(\mathfrak{G}_2\kappa_1)(\tau)-(\mathfrak{G}_2\kappa_2)(\tau)| \le \left|\frac{\tau}{\Theta_1}\right|\Bigg[\left|\omega^2-\omega\right|\left|\int_0^\mu\frac{(\mu-\sigma)^{1-\nu}}{\Gamma(2-\nu)}\bigg(\int_0^\sigma e^{-\omega(\sigma-\vartheta)}I^\nu[f(\vartheta,\kappa_1(\vartheta),{}^cD^{\nu-1}\kappa_1(\vartheta)) \\ &-f(\vartheta,\kappa_2(\vartheta),{}^cD^{\nu-1}\kappa_2(\vartheta))]d\vartheta\bigg)d\sigma\right| + \left|\int_0^\mu\frac{(\mu-\sigma)^{1-\nu}}{\Gamma(2-\nu)}\Big[(1-\omega)I^\nu[f(\sigma,\kappa_1(\sigma),{}^cD^{\nu-1}\kappa_1(\sigma)) - f(\sigma,\kappa_2(\sigma),{}^cD^{\nu-1}\kappa_2(\sigma))]\right. \\ &\left.+\ I^{\nu-1}\Big[f(\sigma,\kappa_1(\sigma),{}^cD^{\nu-1}\kappa_1(\sigma))-f(\sigma,\kappa_2(\sigma),{}^cD^{\nu-1}\kappa_2(\sigma))\Big]\Big]d\sigma\right| \\ &+|\gamma\omega|\left|\int_0^1 e^{-\omega(1-\sigma)}I^\nu\Big[f(\sigma,\kappa_1(\sigma),{}^cD^{\nu-1}\kappa_1(\sigma)) - f(\sigma,\kappa_2(\sigma),{}^cD^{\nu-1}\kappa_2(\sigma))\Big]d\sigma\right| \\ &+|\gamma|\left|I^\nu[f(1,\kappa_1(1),{}^cD^{\nu-1}\kappa_1(1))-f(1,\kappa_2(1),{}^cD^{\nu-1}\kappa_2(1))]\right|\Bigg] \\ &+\frac{1}{|\Theta_2|}\left|\int_0^\mu\left[\frac{(\mu-\sigma)^{\nu-2}}{\Gamma(\nu-1)}+\frac{(\mu-\sigma)^{\nu-1}}{\Gamma(\nu)}\right]\bigg(\int_0^\sigma e^{-\omega(\sigma-\vartheta)}I^\nu[f(\vartheta,\kappa_1(\vartheta),{}^cD^{\nu-1}\kappa_1(\vartheta)) - f(\vartheta,\kappa_2(\vartheta),{}^cD^{\nu-1}\kappa_2(\vartheta))]d\vartheta\bigg)d\sigma\right| \\ &+\left|\frac{\Theta_3}{\Theta_1\Theta_2}\right|\Bigg[\left|\omega^2-\omega\right|\left|\int_0^\mu\frac{(\mu-\sigma)^{1-\nu}}{\Gamma(2-\nu)}\bigg(\int_0^\sigma e^{-\omega(\sigma-\vartheta)}I^\nu\Big[f(\vartheta,\kappa_1(\vartheta),{}^cD^{\nu-1}\kappa_1(\vartheta)) - f(\vartheta,\kappa_2(\vartheta),{}^cD^{\nu-1}\kappa_2(\vartheta))\Big]d\vartheta\bigg)d\sigma\right|\end{aligned}$$

$$+\left|\int_0^{\mu}\frac{(\mu-\sigma)^{1-\nu}}{\Gamma(2-\nu)}\left[(1-\omega)I^{\nu}[f(\sigma,\kappa_1(\sigma),{}^cD^{\nu-1}\kappa_1(\sigma))-f(\sigma,\kappa_2(\sigma),{}^cD^{\nu-1}\kappa_2(\sigma))]\right.\right.$$

$$\left.\left.+I^{\nu-1}[f(\sigma,\kappa_1(\sigma),{}^cD^{\nu-1}\kappa_1(\sigma))-f(\sigma,\kappa_2(\sigma),{}^cD^{\nu-1}\kappa_2(\sigma))]\right]d\sigma\right|$$

$$+|\gamma\omega|\left|\int_0^1 e^{-\omega(1-\sigma)}I^{\nu}\left[f(\sigma,\kappa_1(\sigma),{}^cD^{\nu-1}\kappa_1(\sigma))-f(\sigma,\kappa_2(\sigma),{}^cD^{\nu-1}\kappa_2(\sigma))\right]d\sigma\right|$$

$$+|\gamma|\left|I^{\nu}[f(1,\kappa_1(1),{}^cD^{\nu-1}\kappa_1(1))-f(1,\kappa_2(1),{}^cD^{\nu-1}\kappa_2(1))]\right|\Big]$$

$$\leq\left(\frac{\mu(|\Theta_2|+|\Theta_3|)(e^{-\omega\mu}+\omega\mu-1)\left|\omega^2-\omega\right|}{|\Theta_1||\Theta_2|\omega^2\Gamma(2-\nu)\Gamma(\nu+1)}+\frac{\gamma(|\Theta_2|+|\Theta_3|)(2-e^{-\omega})}{|\Theta_1||\Theta_2|\Gamma(\nu+1)}+\frac{(|\Theta_2|+|\Theta_3|)(|1-\omega|\mu^2+\nu\mu)}{|\Theta_1||\Theta_2|\Gamma(\nu+1)\Gamma(3-\nu)}+\frac{(1-e^{-\omega\mu})(\nu\mu^{2\nu-1}+\mu^{2\nu})}{\omega|\Theta_2|[\Gamma(\nu+1)]^2}\right)$$

$$.q(\|\kappa_1-\kappa_2\|+\left\|{}^cD^{\nu-1}\kappa_1-{}^cD^{\nu-1}\kappa_2\right\|)$$

$$\leq q\widetilde{\Pi_1}(\|\kappa_1-\kappa_2\|+\left\|{}^cD^{\nu-1}\kappa_1-{}^cD^{\nu-1}\kappa_2\right\|).$$

Also

$$\left|(\mathfrak{G}'_2\kappa_1)(\tau)-(\mathfrak{G}'_2\kappa_2)(\tau)\right|\leq q\widetilde{\Pi_2}(\|\kappa_1-\kappa_2\|+\left\|{}^cD^{\nu-1}\kappa_1-{}^cD^{\nu-1}\kappa_2\right\|).$$

Which implies that

$$\begin{aligned}\left|{}^cD^{\nu-1}(\mathfrak{G}_2\kappa_1)(\tau)-{}^cD^{\nu-1}(\mathfrak{G}_2\kappa_2)(\tau)\right| &\leq \int_0^{\tau}\frac{(\tau-\sigma)^{1-\nu}}{\Gamma(2-\nu)}\left|(\mathfrak{G}'_2\kappa_1)(\sigma)-(\mathfrak{G}'_2\kappa_2)(\sigma)\right|d\sigma\\ &\leq q\widetilde{\Pi_2}(\|\kappa_1-\kappa_2\|+\left\|{}^cD^{\nu-1}\kappa_1-{}^cD^{\nu-1}\kappa_2\right\|)\int_0^{\tau}\frac{(\tau-\sigma)^{1-\nu}}{\Gamma(2-\nu)}d\sigma\\ &\leq q\widetilde{\Pi_2}\left(\|\kappa_1-\kappa_2\|+\left\|{}^cD^{\nu-1}\kappa_1-{}^cD^{\nu-1}\kappa_2\right\|\right)\frac{\tau^{2-\nu}}{\Gamma(3-\nu)}\\ &\leq \frac{q\widetilde{\Pi_2}}{\Gamma(3-\nu)}\left(\|\kappa_1-\kappa_2\|+\left\|{}^cD^{\nu-1}\kappa_1-{}^cD^{\nu-1}\kappa_2\right\|\right).\end{aligned}$$

From the above inequalities, we have

$$\begin{aligned}\|\mathfrak{G}_2\kappa_1-\mathfrak{G}_2\kappa_2\|_{\mathfrak{X}} &= \|\mathfrak{G}_2\kappa_1-\mathfrak{G}_2\kappa_2\|+\left\|{}^cD^{\nu-1}(\mathfrak{G}_2\kappa_1)-{}^cD^{\nu-1}(\mathfrak{G}_2\kappa_2)\right\|\\ &\leq q\left(\widetilde{\Pi_1}+\frac{\widetilde{\Pi_2}}{\Gamma(3-\nu)}\right)\left(\|\kappa_1-\kappa_2\|+\left\|{}^cD^{\nu-1}\kappa_1-{}^cD^{\nu-1}\kappa_2\right\|\right)\\ &\leq q\left(\widetilde{\Pi_1}+\frac{\widetilde{\Pi_2}}{\Gamma(3-\nu)}\right)\|\kappa_1-\kappa_2\|_{\mathfrak{X}}.\end{aligned}$$

As $q\left(\widetilde{\Pi_1}+\frac{\widetilde{\Pi_2}}{\Gamma(3-\nu)}\right)<1$, $\mathfrak{G}_2$ is contraction. From Lemma 2.8, there exists $\kappa\in\mathscr{S}_{\rho}$ such that $\kappa(\tau)=(\mathfrak{G}_1\kappa)(\tau)+(\mathfrak{G}_2\kappa)(\tau)=(\mathfrak{G}\kappa)(\tau)$, which means that κ is the solution of problem (1.1). □

The prove of the next result is based on Lemma 2.9.

Theorem 3.2. *Let $f\in C\left([0,1]\times\mathbb{R}^2,\mathbb{R}\right)$ and assume that*
($\mathfrak{B}_3$) *For all $(\tau,\kappa_1,\kappa_2)\in[0,1]\times\mathbb{R}^2$, there exist a function $\mathscr{J}\in C([0,1],\mathbb{R}^+)$, and a nondecreasing continuous function $\mathscr{R}:[0,\infty)\to[0,\infty)$ such that*

$$|f(\tau,\kappa_1,\kappa_2)|\leq\mathscr{J}(\tau)\mathscr{R}(\|\kappa_1\|+\|\kappa_2\|);$$

($\mathfrak{B}_4$) *there exists a constant $N>0$ such that*

$$\frac{N}{\|\mathscr{J}\|\mathscr{R}(N)}>\Pi_1+\frac{\Pi_2}{\Gamma(3-\nu)},$$

where Π_1,Π_2 are given by (3.2).

Then problem (1.1) has at least one solution on $[0,1]$.

Proof. Consider the operator $\mathfrak{G}:\mathfrak{X}\to\mathfrak{X}$ defined in (3.1). At first, we show that $\mathfrak{G}$ maps bounded sets into bounded sets in $C([0,1],\mathbb{R})$. For $\rho>0$, let $\mathfrak{D}_\rho=\{\kappa\in C([0,1],\mathbb{R}):\|\kappa\|_{\mathfrak{X}}\leq\rho\}$ be a bounded set in $C([0,1],\mathbb{R})$. Then

$$\begin{aligned}
|(\mathfrak{G}\kappa)(\tau)| \leq & \left|\int_0^\tau e^{-\omega(\tau-\sigma)} I^\nu f(\sigma,\kappa(\sigma),{}^cD^{\nu-1}\kappa(\sigma))d\sigma\right| \\
&+\left|\frac{\tau}{\Theta_1}\right|\left[\left|\omega^2-\omega\right|\left|\int_0^\mu \frac{(\mu-\sigma)^{1-\nu}}{\Gamma(2-\nu)}\left(\int_0^\sigma e^{-\omega(\sigma-\vartheta)} I^\nu f(\vartheta,\kappa(\vartheta),{}^cD^{\nu-1}\kappa(\vartheta))d\vartheta\right)d\sigma\right|\right. \\
&+\left|\int_0^\mu \frac{(\mu-\sigma)^{1-\nu}}{\Gamma(2-\nu)}[(1-\omega)I^\nu f(\sigma,\kappa(\sigma),{}^cD^{\nu-1}\kappa(\sigma))+I^{\nu-1} f(\sigma,\kappa(\sigma),{}^cD^{\nu-1}\kappa(\sigma))]d\sigma\right| \\
&\left.+|\gamma\omega|\left|\int_0^1 e^{-\omega(1-\sigma)} I^\nu f(\sigma,\kappa(\sigma),{}^cD^{\nu-1}\kappa(\sigma))d\sigma\right|+|\gamma|\left|I^\nu f(1,\kappa(1),{}^cD^{\nu-1}\kappa(1))\right|\right] \\
&+\frac{1}{|\Theta_2|}\left|\int_0^\mu\left[\frac{(\mu-\sigma)^{\nu-2}}{\Gamma(\nu-1)}+\frac{(\mu-\sigma)^{\nu-1}}{\Gamma(\nu)}\right]\left(\int_0^\sigma e^{-\omega(\sigma-\vartheta)} I^\nu f(\vartheta,\kappa(\vartheta),{}^cD^{\nu-1}\kappa(\vartheta))d\vartheta\right)d\sigma\right| \\
&+\left|\frac{\Theta_3}{\Theta_1\Theta_2}\right|\left[\left|\omega^2-\omega\right|\left|\int_0^\mu \frac{(\mu-\sigma)^{1-\nu}}{\Gamma(2-\nu)}\left(\int_0^\sigma e^{-\omega(\sigma-\vartheta)} I^\nu f(\vartheta,\kappa(\vartheta),{}^cD^{\nu-1}\kappa(\vartheta))d\vartheta\right)d\sigma\right|\right. \\
&+\left|\int_0^\mu \frac{(\mu-\sigma)^{1-\nu}}{\Gamma(2-\nu)}[(1-\omega)I^\nu f(\sigma,\kappa(\sigma),{}^cD^{\nu-1}\kappa(\sigma))\right. \\
&\left.+I^{\nu-1} f(\sigma,\kappa(\sigma),{}^cD^{\nu-1}\kappa(\sigma))]d\sigma\right|+|\gamma\omega|\left|\int_0^1 e^{-\omega(1-\sigma)} I^\nu f(\sigma,\kappa(\sigma),{}^cD^{\nu-1}\kappa(\sigma))d\sigma\right| \\
&\left.+|\gamma|\left|I^\nu f(1,\kappa(1),{}^cD^{\nu-1}\kappa(1))\right|\right] \\
\leq & \left(\frac{1-e^{-\omega}}{\omega\Gamma(\nu+1)}+\frac{\mu(|\Theta_2|+|\Theta_3|)(e^{-\omega\mu}+\omega\mu-1)\left|\omega^2-\omega\right|}{|\Theta_1|\,|\Theta_2|\,\omega^2\Gamma(2-\nu)\Gamma(\nu+1)}\right. \\
&+\frac{\gamma(|\Theta_2|+|\Theta_3|)(2-e^{-\omega})}{|\Theta_1|\,|\Theta_2|\,\Gamma(\nu+1)}+\frac{(|\Theta_2|+|\Theta_3|)(|1-\omega|\,\mu^2+\nu\mu)}{|\Theta_1|\,|\Theta_2|\,\Gamma(\nu+1)\Gamma(3-\nu)} \\
&\left.+\frac{(1-e^{-\omega\mu})(\nu\mu^{2\nu-1}+\mu^{2\nu})}{\omega\,|\Theta_2|\,[\Gamma(\nu+1)]^2}\right)\|\mathscr{J}\|\mathscr{R}\left(\|\kappa\|+\left\|{}^cD^{\nu-1}\kappa\right\|\right) \\
\leq & \Pi_1\|\mathscr{J}\|\mathscr{R}\left(\|\kappa\|+\left\|{}^cD^{\nu-1}\kappa\right\|\right)=\Pi_1\|\mathscr{J}\|\mathscr{R}(\|\kappa\|_{\mathfrak{X}}),
\end{aligned}$$

where Π_1 are given in (3.2).
Hence

$$\|\mathfrak{G}\kappa\|\leq\Pi_1\|\mathscr{J}\|\mathscr{R}(\|\kappa\|_{\mathfrak{X}}).$$

Also we have

$$\begin{aligned}
\left|(\mathfrak{G}'\kappa)(\tau)\right| \leq & \,\omega\left|\int_0^\tau e^{-\omega(\tau-\sigma)} I^\nu f(\sigma,\kappa(\sigma),{}^cD^{\nu-1}\kappa(\sigma))d\sigma\right|+\left|I^\nu f(\tau,\kappa(\tau),{}^cD^{\nu-1}\kappa(\tau))\right| \\
&+\frac{1}{|\Theta_1|}\left[\left|\omega^2-\omega\right|\left|\int_0^\mu \frac{(\mu-\sigma)^{1-\nu}}{\Gamma(2-\nu)}\left(\int_0^\sigma e^{-\omega(\sigma-\vartheta)} I^\nu f(\vartheta,\kappa(\vartheta),{}^cD^{\nu-1}\kappa(\vartheta))d\vartheta\right)d\sigma\right|\right. \\
&+\left|\int_0^\mu \frac{(\mu-\sigma)^{1-\nu}}{\Gamma(2-\nu)}\left[(1-\omega)I^\nu f(\sigma,\kappa(\sigma),{}^cD^{\nu-1}\kappa(\sigma))+I^{\nu-1} f(\sigma,\kappa(\sigma),{}^cD^{\nu-1}\kappa(\sigma))\right]d\sigma\right| \\
&\left.+|\gamma|\,\omega\left|\int_0^1 e^{-\omega(1-\sigma)} I^{\nu-1} f(\sigma,\kappa(\sigma),{}^cD^{\nu-1}\kappa(\sigma))d\sigma\right|+|\gamma|\left|I^\nu f(1,\kappa(1),{}^cD^{\nu-1}\kappa(1))\right|\right] \\
\leq & \left(\frac{(|\Theta_1|+|\gamma|)(2-e^{-\omega})}{|\Theta_1|\,\Gamma(\nu+1)}+\frac{\mu(e^{-\omega\mu}+\omega\mu-1)\left|\omega^2-\omega\right|}{|\Theta_1|\,\omega^2\Gamma(2-\nu)\Gamma(\nu+1)}\right. \\
&\left.+\frac{|1-\omega|\,\mu^2+\nu\mu}{|\Theta_1|\,|\Gamma(\nu+1)\Gamma(3-\nu)}\right)\|\mathscr{J}\|\mathscr{R}\left(\|\kappa\|+\left\|{}^cD^{\nu-1}\kappa\right\|\right) \\
\leq & \Pi_2\|\mathscr{J}\|\mathscr{R}\left(\|\kappa\|+\left\|{}^cD^{\nu-1}\kappa\right\|\right)=\Pi_2\|\mathscr{J}\|\mathscr{R}(\|\kappa\|_{\mathfrak{X}}),
\end{aligned}$$

where Π_2 given by (3.2).

By Definition 2.2 for $\nu \in (1,2]$, we get

$$\begin{aligned}\left|{}^cD^{\nu-1}(\mathfrak{G}\kappa)(\tau)\right| &\leq \int_0^\tau \frac{(\tau-\sigma)^{1-\nu}}{\Gamma(2-\nu)}\left|(\mathfrak{G}'\kappa)(\sigma)\right|d\sigma \\ &\leq \Pi_2\,\|\mathscr{J}\|\mathscr{R}(\|\kappa\|_{\mathfrak{X}})\int_0^\tau \frac{(\tau-\sigma)^{1-\nu}}{\Gamma(2-\nu)}d\sigma \\ &\leq \Pi_2\,\|\mathscr{J}\|\mathscr{R}(\|\kappa\|_{\mathfrak{X}})\frac{\tau^{2-\nu}}{\Gamma(3-\nu)} \\ &\leq \frac{\Pi_2}{\Gamma(3-\nu)}\|\mathscr{J}\|\mathscr{R}(\|\kappa\|_{\mathfrak{X}}).\end{aligned}$$

Hence

$$\begin{aligned}\|\mathfrak{G}\kappa\|_{\mathfrak{X}} = \|\mathfrak{G}\kappa\| + \left\|{}^cD^{\nu-1}\mathfrak{G}\kappa\right\| &\leq \left(\Pi_1 + \frac{\Pi_2}{\Gamma(3-\nu)}\right)\|\mathscr{J}\|\mathscr{R}(\|\kappa\|_{\mathfrak{X}}) \\ &\leq \left(\Pi_1 + \frac{\Pi_2}{\Gamma(3-\nu)}\right)\|\mathscr{J}\|\mathscr{R}(\rho).\end{aligned}$$

Next we show that $\mathfrak{G}$ maps bounded sets into equicontinuous sets of $C([0,1],\mathbb{R})$. Let $\tau_1,\tau_2 \in [0,1]$ with $\tau_1 < \tau_2$ and $\kappa \in \mathfrak{D}_\rho$, where $\mathfrak{D}_\rho$ is a bounded set of $C([0,1],\mathbb{R})$. Then we obtain

$$\begin{aligned}|(\mathfrak{G}\kappa)(\tau_2)-(\mathfrak{G}\kappa)(\tau_1)| \leq& \left|\int_0^{\tau_2} e^{-\omega(\tau_2-\sigma)}I^\nu f(\sigma,\kappa(\sigma),{}^cD^{\nu-1}\kappa(\sigma))d\sigma\right. \\ &\left. - \int_0^{\tau_1} e^{-\omega(\tau_1-\sigma)}I^\nu f(\sigma,\kappa(\sigma),{}^cD^{\nu-1}\kappa(\sigma))d\sigma\right| \\ &+\left|\frac{\tau_2-\tau_1}{\Theta_1}\right|\left|(\omega^2-\omega)\int_0^\mu \frac{(\mu-\sigma)^{1-\nu}}{\Gamma(2-\nu)}\left(\int_0^\sigma e^{-\omega(\sigma-\vartheta)}I^\nu f(\vartheta,\kappa(\vartheta),{}^cD^{\nu-1}\kappa(\vartheta))d\vartheta\right)d\sigma\right. \\ &+\int_0^\mu \frac{(\mu-\sigma)^{1-\nu}}{\Gamma(2-\nu)}[(1-\omega)I^\nu f(\sigma,\kappa(\sigma),{}^cD^{\nu-1}\kappa(\sigma))+I^{\nu-1}f(\sigma,\kappa(\sigma),{}^cD^{\nu-1}\kappa(\sigma))]d\sigma \\ &\left.+\gamma\omega\int_0^1 e^{-\omega(1-\sigma)}I^\nu f(\sigma,\kappa(\sigma),{}^cD^{\nu-1}\kappa(\sigma))d\sigma-\gamma I^\nu f(1,\kappa(1),{}^cD^{\nu-1}\kappa(1))\right| \\ \leq& \int_0^{\tau_1} e^{\omega\sigma}\left|e^{-\omega\tau_2}-e^{-\omega\tau_1}\right|\left|I^\nu f(\sigma,\kappa(\sigma),{}^cD^{\nu-1}\kappa(\sigma))\right|d\sigma \\ &+\int_{\tau_1}^{\tau_2} e^{-\omega(\tau_2-\sigma)}\left|I^\nu f(\sigma,\kappa(\sigma),{}^cD^{\nu-1}\kappa(\sigma))\right|d\sigma \\ &+\left|\frac{\tau_2-\tau_1}{\Theta_1}\right|\left|(\omega^2-\omega)\int_0^\mu \frac{(\mu-\sigma)^{1-\nu}}{\Gamma(2-\nu)}\left(\int_0^\sigma e^{-\omega(\sigma-\vartheta)}I^\nu f(\vartheta,\kappa(\vartheta),{}^cD^{\nu-1}\kappa(\vartheta))d\vartheta\right)d\sigma\right. \\ &+\int_0^\mu \frac{(\mu-\sigma)^{1-\nu}}{\Gamma(2-\nu)}[(1-\omega)I^\nu f(\sigma,\kappa(\sigma),{}^cD^{\nu-1}\kappa(\sigma))+I^{\nu-1}f(\sigma,\kappa(\sigma),{}^cD^{\nu-1}\kappa(\sigma))]d\sigma \\ &\left.+\gamma\omega\int_0^1 e^{-\omega(1-\sigma)}I^\nu f(\sigma,\kappa(\sigma),{}^cD^{\nu-1}\kappa(\sigma))d\sigma-\gamma I^\nu f(1,\kappa(1),{}^cD^{\nu-1}\kappa(1))\right| \\ \leq& \left(\int_0^{\tau_1} e^{\omega\sigma}\left|e^{-\omega\tau_2}-e^{-\omega\tau_1}\right|d\sigma+\int_{\tau_1}^{\tau_2} e^{-\omega(\tau_2-\sigma)}d\sigma\right)\frac{\|\mathscr{J}\|\mathscr{R}(\|\kappa\|_{\mathfrak{X}})}{\Gamma(\nu+1)} \\ &+\left|\frac{\tau_2-\tau_1}{\Theta_1}\right|\left|(\omega^2-\omega)\int_0^\mu \frac{(\mu-\sigma)^{1-\nu}}{\Gamma(2-\nu)}\left(\int_0^\sigma e^{-\omega(\sigma-\vartheta)}I^\nu f(\vartheta,\kappa(\vartheta),{}^cD^{\nu-1}\kappa(\vartheta))d\vartheta\right)d\sigma\right. \\ &+\int_0^\mu \frac{(\mu-\sigma)^{1-\nu}}{\Gamma(2-\nu)}[(1-\omega)I^\nu f(\sigma,\kappa(\sigma),{}^cD^{\nu-1}\kappa(\sigma))+I^{\nu-1}f(\sigma,\kappa(\sigma),{}^cD^{\nu-1}\kappa(\sigma))]d\sigma \\ &\left.+\gamma\omega\int_0^1 e^{-\omega(1-\sigma)}I^\nu f(\sigma,\kappa(\sigma),{}^cD^{\nu-1}\kappa(\sigma))d\sigma-\gamma I^\nu f(1,\kappa(1),{}^cD^{\nu-1}\kappa(1))\right|.\end{aligned}$$

Also

$$\begin{aligned}\left|{}^cD^{\nu-1}(\mathfrak{G}\kappa)(\tau_2)-{}^cD^{\nu-1}(\mathfrak{G}\kappa)(\tau_1)\right| &= \left|\int_0^{\tau_2}\frac{(\tau_2-\sigma)^{1-\nu}}{\Gamma(2-\nu)}(\mathfrak{G}'\kappa)(\sigma)d\sigma-\int_0^{\tau_1}\frac{(\tau_1-\sigma)^{1-\nu}}{\Gamma(2-\nu)}(\mathfrak{G}'\kappa)(\sigma)d\sigma\right| \\ &\leq \int_0^{\tau_1}\frac{\left|(\tau_2-\sigma)^{1-\nu}-(\tau_1-\sigma)^{1-\nu}\right|}{\Gamma(2-\nu)}\left|(\mathfrak{G}'\kappa)(\sigma)\right|d\sigma+\int_{\tau_1}^{\tau_2}\frac{\left|(\tau_2-\sigma)^{1-\nu}\right|}{\Gamma(2-\nu)}\left|(\mathfrak{G}'\kappa)(\sigma)\right|d\sigma \\ &\leq \left(\int_0^{\tau_1}\frac{\left|(\tau_2-\sigma)^{1-\nu}-(\tau_1-\sigma)^{1-\nu}\right|}{\Gamma(2-\nu)}d\sigma+\int_{\tau_1}^{\tau_2}\frac{\left|(\tau_2-\sigma)^{1-\nu}\right|}{\Gamma(2-\nu)}d\sigma\right)\Pi_2\,\|\mathscr{J}\|\mathscr{R}(\|\kappa\|_{\mathfrak{X}}).\end{aligned}$$

Obviously, the right-hand side of the above inequality tends to zero independently of $\kappa \in \mathfrak{D}_\rho$ as $\tau_2-\tau_1 \to 0$. As $\mathfrak{G}$ verifies the above assumptions, then by the use of Arzelá-Ascoli theorem, we claim that $\mathfrak{G}: C([0,1],\mathbb{R}) \to C([0,1],\mathbb{R})$ is completely continuous.
To achieve the satisfaction of the hypotheses of the Leray-Schauder nonlinear alternative theorem, is to show the boundedness of the set

of all solutions to equation $\kappa = \lambda \mathfrak{G}\kappa$ for $\lambda \in [0,1]$. Assume that κ is a solution, then in the same manner as we show the operator $\mathfrak{G}$ is bounded, we can obtain

$$|\kappa(\tau)| = |\lambda(\mathfrak{G}\kappa)(\tau)| \leq \varPi_1 \|\mathscr{J}\| \mathscr{R}(\|\kappa\|_{\mathfrak{X}}).$$

Also we have

$$\left|\kappa'(\tau)\right| = \left|\lambda(\mathfrak{G}'\kappa)(\tau)\right| \leq \varPi_2 \|\mathscr{J}\| \mathscr{R}(\|\kappa\|_{\mathfrak{X}}).$$

By Definition 2.2 for $1 < \nu \leq 2$, we get

$$\left|{}^cD^{\nu-1}\kappa(\tau)\right| = \left|\lambda\, {}^cD^{\nu-1}(\mathfrak{G}\kappa)(\tau)\right| \leq \frac{\varPi_2}{\Gamma(3-\nu)} \|\mathscr{J}\| \mathscr{R}(\|\kappa\|_{\mathfrak{X}}).$$

Hence

$$\|\kappa\|_{\mathfrak{X}} = \|\kappa\| + \left\|{}^cD^{\nu-1}\kappa\right\| \leq \left(\varPi_1 + \frac{\varPi_2}{\Gamma(3-\nu)}\right) \|\mathscr{J}\| \mathscr{R}(\|\kappa\|_{\mathfrak{X}}).$$

Consequently, we have

$$\frac{\|\kappa\|_{\mathfrak{X}}}{\|\mathscr{J}\| \mathscr{R}(\|\kappa\|_{\mathfrak{X}})} \leq \varPi_1 + \frac{\varPi_2}{\Gamma(3-\nu)}.$$

In view of $(\mathfrak{B}_4)$, there exists N such that $\|\kappa\|_{\mathfrak{X}} \neq N$.
Let us set

$$\mathscr{V} = \{\kappa \in C([0,1],\mathbb{R}) : \|\kappa\|_{\mathfrak{X}} < N\}.$$

Note that the operator $\mathfrak{G} : \overline{\mathscr{V}} \to C([0,1],\mathbb{R})$ is continuous and completely continuous. From the choice of $\mathscr{V}$, there is no $\kappa \in \partial\mathscr{V}$ with $\kappa = \lambda\mathfrak{G}\kappa$ for some $\lambda \in [0,1]$. Consequently, by Lemma 2.9, we conclude that $\mathfrak{G}$ has a fixed point $\kappa \in \overline{\mathscr{V}}$ which is a solution of the problem (1.1). □

4. Examples

Example 4.1. *Consider the following sequential fractional boundary value problem involving Caputo-type derivative :*

$$\begin{cases} ({}^cD^{\frac{5}{2}} + 2\,{}^cD^{\frac{3}{2}})\kappa(\tau) = f(\tau,\kappa(\tau),{}^cD^{\frac{1}{2}}\kappa(\tau)), \quad \tau \in [0,1], \\ \kappa(0) + \kappa(1) = I^{\frac{1}{2}}\kappa(\frac{1}{2}) + I^{\frac{3}{2}}\kappa(\frac{1}{2}), \\ \kappa'(0) + \kappa'(1) = {}^cD^{\frac{1}{2}}\kappa(\frac{1}{2}) + {}^cD^{\frac{3}{2}}\kappa(\frac{1}{2}), \\ \kappa''(0) = 0. \end{cases} \tag{4.1}$$

Here $\nu = \frac{3}{2}$, $\omega = 2$, $\beta = \gamma = 1$, $\mu = \frac{1}{2}$.
With the given values, it is found that
$\Theta_1 \simeq 1.202115439, \Theta_2 \simeq 0.936153918, \Theta_3 \simeq -0.202115439$, *where* Θ_1, Θ_2 *and* Θ_3 *defined by (2.2).*
We take
$f(\tau,\kappa(\tau),{}^cD^{\frac{1}{2}}\kappa(\tau)) = \frac{2}{5(\tau^2+42)}\left(\cos(\kappa(\tau)+1) + \frac{3\left|{}^cD^{\frac{1}{2}}\kappa(\tau)\right|}{3+{}^cD^{\frac{1}{2}}\kappa(\tau)} + e^{-\tau}\sin\tau\right)$ *in (4.1). Then*

$$\begin{aligned} &\left|f(\tau,\kappa_1(\tau),{}^cD^{\frac{1}{2}}\kappa_1(\tau)) - f(\tau,\kappa_2(\tau),{}^cD^{\frac{1}{2}}\kappa_2(\tau))\right| \\ &\leq \frac{2}{5(\tau^2+42)}\left(|\cos(\kappa_1(\tau)+1) - \cos(\kappa_2(\tau)+1)| + \left|\frac{3\left|{}^cD^{\frac{1}{2}}\kappa_1(\tau)\right|}{3+{}^cD^{\frac{1}{2}}\kappa_1(\tau)} - \frac{3\left|{}^cD^{\frac{1}{2}}\kappa_2(\tau)\right|}{3+{}^cD^{\frac{1}{2}}\kappa_2(\tau)}\right|\right) \\ &\leq \frac{2}{5(\tau^2+42)}\left(|\kappa_1(\tau) - \kappa_2(\tau)| + \left|{}^cD^{\frac{1}{2}}\kappa_1(\tau) - {}^cD^{\frac{1}{2}}\kappa_2(\tau)\right|\right) \\ &\leq q\left(\|\kappa_1 - \kappa_2\| + \left\|{}^cD^{\frac{1}{2}}\kappa_1 - {}^cD^{\frac{1}{2}}\kappa_2\right\|\right), \end{aligned}$$

with $q = \frac{1}{105}$, *and*

$$\begin{aligned} \left|f(\tau,\kappa(\tau),{}^cD^{\frac{1}{2}}\kappa(\tau))\right| &= \left|\frac{2}{5(\tau^2+42)}\right|\left|\cos(\kappa(\tau)+1) + \frac{3\left|{}^cD^{\frac{1}{2}}\kappa(\tau)\right|}{3+{}^cD^{\frac{1}{2}}\kappa(\tau)} + e^{-\tau}\sin\tau\right| \\ &\leq \frac{2(4+e^{-\tau}\sin\tau)}{5(\tau^2+42)} = \theta(\tau). \end{aligned}$$

We found $\widetilde{\varPi_1} \simeq 2.412349768$ *and* $\widetilde{\varPi_2} \simeq 1.905439962$ ($\widetilde{\varPi_1}, \widetilde{\varPi_2}$ *defined by (3.3)). Further* $q\left(\widetilde{\varPi_1} + \frac{\widetilde{\varPi_2}}{\Gamma(3-\nu)}\right) \simeq 0.043451509 < 1$. *Thus, all the conditions of Theorem 3.1 are fulfilled. Hence, the problem (4.1) has a solution on* $[0,1]$.

Example 4.2. *Consider the problem (4.1) and*

$$f(\tau,\kappa(\tau),{}^{c}D^{\frac{1}{2}}\kappa(\tau))=\frac{1}{2\sqrt{\tau+729}}\left(\tan^{-1}(\kappa(\tau)+1)+\ln(\left|{}^{c}D^{\frac{1}{2}}\kappa(\tau)\right|+2)\right).$$

Clearly, we get

$$\begin{aligned}\left|f(\tau,\kappa(\tau),{}^{c}D^{\frac{1}{2}}\kappa(\tau))\right| &\le \left|\frac{1}{2\sqrt{\tau+729}}\right|\left|\tan^{-1}(\kappa(\tau)+1)+\ln(\left|{}^{c}D^{\frac{1}{2}}\kappa(\tau)\right|+2)\right|\\ &\le \frac{1}{2\sqrt{\tau+729}}\left(|\kappa(\tau)|+\left|{}^{c}D^{\frac{1}{2}}\kappa(\tau)\right|+3\right)\\ &\le \mathscr{J}(\tau)\mathscr{R}(\|\kappa\|_{\mathfrak{X}}),\end{aligned}$$

where $\mathscr{J}(\tau)=\frac{1}{2\sqrt{\tau+729}}$, $\mathscr{R}(\|\kappa\|_{\mathfrak{X}})=\|\kappa\|_{\mathfrak{X}}+3$.
With the obove assumption, we can obtain $\Pi_1\simeq 2.737572985$, $\Pi_2\simeq 3.308139177$ *(*Π_1,Π_2 *defined by (3.2)),* $\|\mathscr{J}\|=\frac{1}{54}$.
By the use of condition ($\mathfrak{B}_4$*), we find* $N>0.408402939$. *Hence by Theorem 3.2, the problem (4.1) has a solution on* $[0,1]$.

5. Conclusion

In this paper, we investigate the existence of solutions for a sequential FDEs with integro-differential bounday conditions. Our study is based on Krasnoselskii's fixed point theorem and the Leray-Schauder fixed point theorem under some suitable conditions.

Our research can be extented to the inclusion form of our considerd problem by applying the multivalued fixed point theorems such as the nonlinear alternative of Leray-Schauder type for Kakutani maps, the fixed point theorem contraction multivalued maps due to Covitz and Nadler.

For future works, we plan to investigate the existence results of these equations involving other fractional derivatives, such as Caputo-Hadamard and Hilfer. The stochastic versions of the sequential FDEs will be among the aim of our forthcoming studies. Furthermore, we will study the systems of nonlinear sequential FDEs with deviated arguments by employing numerical methods to approximate their solutions.

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Research Article

On New Sequences of p-Binomial and Catalan Transforms of the k-Mersenne Numbers and Associated Generating Functions

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Abstract

In this work, we investigate the binomial transforms and Catalan transform of the k-Mersenne and k-Mersenne-Lucas numbers and examine the new integer sequences. We apply the p-binomial, rising p-binomial, and falling p-binomial transforms to the k-Mersenne sequences and present the associated generating and exponential generating functions for these transforms. Lastly, we provide the corresponding Binet-type formulas and recurrence relations for binomial transforms of the k-Mersenne (k-Mersenne-Lucas) numbers. These results are supported by numerical illustrations.

1. Introduction and Preliminaries

Number sequences represent ordered sets of numbers that reveal underlying mathematical patterns with significant implications for problem-solving. Various types of number sequences exist, including Fibonacci, arithmetic, and geometric progressions, each exhibiting unique characteristics. Notably, the Fibonacci sequence $(0,1,1,2,3,5,8,13...)$, where each term is the sum of the two preceding terms, appears in nature, art, and architecture, showcasing the inherent beauty of mathematical patterns in the world around us.

Fibonacci numbers have attracted considerable attention among number theorists due to their fascinating properties, leading to extensive research on their characteristics, extensions, generalizations, and applications. For properties and application of Fibonacci like numbers one can see [1–3], the journals 'Fibonacci Quarterly', 'Journal of Integer Sequences', 'INTEGERS', etc. Among others, one of these generalizations is Mersenne numbers which are of the kind 2^n-1, $n\in\mathbb{N}$ and one of the generalizations of Mersenne numbers is termed as k-Mersenne numbers. By this study, we examine various new integer sequences and their generating functions through binomial, p-binomial, and Catalan transforms with the k-Mersenne numbers.

For $n\geq 0$, the sequence $a_{n+1}=3a_n-2a_{n-1}$ generates Mersenne numbers $\{M_n\}$ [4] when $a_0=0,\ a_1=1$, and Mersenne-Lucas numbers $\{m_n\}$ [5] when $a_0=2,\ a_1=3$. For recent developments, generalizations and applications of the Mersenne numbers, one can see [5–12]. Let us restate the definitions and some useful results of the Mersenne numbers.

Definition 1.1. *[5, 11] Let $k\in\mathbb{R}^+$ and $n\in\mathbb{N}$, then the k-Mersenne numbers $\{M_{k,n}\}$ and k-Mersenne-Lucas numbers $\{m_{k,n}\}$ are given as*

$$M_{k,n+1}=3kM_{k,n}-2M_{k,n-1},\quad \textit{with } M_{k,0}=0,\ M_{k,1}=1 \tag{1.1}$$
$$\textit{and}\qquad m_{k,n+1}=3km_{k,n}-2m_{k,n-1},\quad \textit{with } m_{k,0}=2,\ m_{k,1}=3k.$$

For (1.1) the characteristic equation is $x^2-3kx+2=0$ and its roots are $r_1=(3k+\sqrt{9k^2-8})/2$ and $r_2=(3k-\sqrt{9k^2-8})/2$ which satisfy the following relations:

$$r_1+r_2=3k,\quad r_1-r_2=\sqrt{9k^2-8},\quad r_1r_2=2. \tag{1.2}$$

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The Binet's formula for these numbers are

$$M_{k,n} = \frac{r_1^n - r_2^n}{r_1 - r_2} \quad \text{and} \quad m_{k,n} = r_1^n + r_2^n.$$

The first few terms of these numbers are as follows:

n	$M_{k,n}$	$m_{k,n}$
0	0	2
1	1	3k
2	$3k$	$9k^2-4$
3	$9k^2-2$	$27k^3-18k$
4	$27k^3-12k$	$81k^4-72k^2+8$
5	$81k^4-54k^2+4$	$243k^5-270k^3+60k$
6	$243k^5-216k^3+36k$	$729k^6-972k^4+324k^2-16$
7	$729k^6-810k^4+216k^2-8$	$2187k^7-3402k^5+1512k^3-168k$
8	$2187k^7-2916k^5+1080k^3-96k$	$6561k^8-11664k^6+6480k^4-1152k^2+32$
$\vdots$	$\vdots$	$\vdots$

At instance, for $k=1$, the k-Mersenne sequence gives the classic Mersenne sequence $\{1,3,7,15,31,63,127,255,...\}$: [A001595], for $k=2$, $\{1,6,34,192,1084,6120,34552,195072,...\}$: [A154244] and for $k=3$, $\{1,9,79,693,6079,53325,467767,4103253,...\}$, etc. where sequence [A154244] is the binomial transform of [A126473].
Similarly in the k-Mersenne-Lucas sequence, $k=1$ gives the classic Mersenne-Lucas numbers i.e. $\{2,3,5,9,17,33,65,129,...\}$: [A000051], for $k=2$, $\{2,6,32,180,1016,5736,32384,182832,...\}$ and for $k=3$, $\{2,9,77,675,5921,51939,455609,3996603,...\}$, etc.

1.1. Binomial and Catalan transforms

In literature, there are various transforms that performs on number sequences. For instance, the Binomial Transform (BT) [13, 14], Catalan Transform (CT) [15, 16], Hankel Transform (HT), Discrete Fourier Transform (DFT), etc. The overall estimation of BT and CT is based on integers, not based on the floating numbers, make them faster and more reliable method of transform unlikely DFT, DCT (Discrete Cosine Transform), etc.
The Catalan numbers $\{C_n\}$ [A000108] (due to Eugene Charles Catalan) are defined by

$$C_n = \frac{1}{1+n}\binom{2n}{n} = \frac{2n!}{(1+n)!(n)!}.$$

Thus, $\{1,1,2,5,14,42,132,429,...\}$ are first few terms of Catalan sequence. The Catalan numbers C_n also satisfy the following relation:

$$\frac{C_{n+1}}{C_n} = \frac{2(2n+1)}{n+2}.$$

Barry [15] give the Catalan transform $\{CA_n\}$ corresponding to a number sequence $\{A_n\}$ as follows:

$$CA_n = \sum_{r=0}^{n} \frac{r}{2n-r}\binom{2n-r}{n-r} A_r,\ n \geq 1, \text{ where } CA_0 \text{ is proviced.} \tag{1.3}$$

We should note that for the Catalan numbers, the generating function $c(x)$ is given as

$$c(x) = \frac{1-\sqrt{1-4x}}{2x}. \tag{1.4}$$

Also, the binomial transform $\{b_n\}$ for an integer sequence $\{a_n\}$ is given as

$$b_n = \sum_{r=0}^{n} \binom{n}{r} a_r,$$

which is an invertible transformation and inverse transformation is given as

$$a_n = \sum_{r=0}^{n} \binom{n}{r} (-1)^{n-r} b_r.$$

Recently, Falcon and Plaza [13] obtained the binomial transforms for k-Fibonacci numbers and then Falcon [16] obtained the Catalan transform of this sequence. Tastan and Ozkan [17] and Prasad et al. [18] obtained the Catalan transform of the k-Pell and extended k-Horadam sequences, respectively. Yilmaz and Taskara [19] studied the recurrent binomial transforms for a sequence of matrices associated with the Padovan and Perrin numbers. Özkan et al. [20] studied the Catalan transform for the Jacobsthal numbers and polynomials. The binomial transform, Catalan transform and the Catalan triangle are versatile mathematical tools that have applications in various fields, particularly in combinatorics, number theory, Encoding/Decoding, various counting problems, applied fields like signal and image processing, etc.
Some recent developments on binomial and Catalan transforms with a number sequence can be seen in [13, 15, 16, 21–25].

2. Binomial and Catalan Transforms on k-Mersenne Sequences

Here, we start by applying the binomial and Catalan transform to the k-Mersenne numbers and investigate the newly generated integer sequences.
Let us define the binomial transform $B_k = \{BM_{k,n}\}$ of k-Mersenne sequence and $C_k = \{Bm_{k,n}\}$ of k-Mersenne-Lucas sequence, where

$$BM_{k,n} = \sum_{r=0}^{n} \binom{n}{r} M_{k,r} \quad \text{and} \quad Bm_{k,n} = \sum_{r=0}^{n} \binom{n}{r} m_{k,r}. \tag{2.1}$$

On running n over $\mathbb{N} \cup \{0\}$, the binomial transform for both the sequences are, respectively:

$$B_k = \{0, 1,\ 3k+2,\ 9k^2+9k+1,\ 27k^3+36k^2+6k-4, \ldots\}, \tag{2.2}$$
$$C_k = \{2, 2+3k, 9k^2+6k-2,\ 27k^3+27k^2-9k-10,\ 81k^4+108k^3-18k^2-60k-14, \ldots\}. \tag{2.3}$$

Thus, setting $k = 1,2,3,4,5$ in (2.2) and (2.3), the binomial transforms are:

$B_1 = \{0,1,5,19,65,211,665,\ldots\} : A001047$
$B_2 = \{0,1,8,55,368,2449,16280,\ldots\}$
$B_3 = \{0,1,11,109,1067,10429,101915,\ldots\}$
$B_4 = \{0,1,14,181,2324,29821,382634,\ldots\}$
$B_5 = \{0,1,17,271,4301,68239,1082645,\ldots\}$

$C_1 = \{2,5,13,35,97,275,793,\ldots\} : A007689$
$C_2 = \{2,8,46,296,1954,12968,86158,\ldots\}$
$C_3 = \{2,11,97,935,9121,89111,870769\ldots\}$
$C_4 = \{2,14,166,2114,27106,347774,4462246\ldots\}$
$C_5 = \{2,17,253,3995,63361,1005227,15948361,\ldots\}$.

2.1. The p-binomial, Rising p-binomial, and Falling p-binomial transforms

Analogous to binomial transform, Spivey and Steil [14] introduced three kinds of the binomial transform with two inputs: an integer sequence A_n and a fixed quantity (scalar) p and referred them as the p-binomial, rising p-binomial and falling p-binomial transform. Falcon and Plaza [13] studied these binomial transforms for k-Fibonacci sequences. Building upon the work of Spivey [14], this section examines transformations for k-Mersenne and k-Mersenne-Lucas sequences. Closed-form definitions for these transformations are presented, along with several specific examples.
The p-binomial transforms $\mathscr{B}M_p$ and $\mathscr{B}m_p$ for sequences $M_{k,n}$ and $m_{k,n}$, respectively, are the sequences $\mathscr{B}M_p = \{x_{p,n}\}_{n\geq 0}$ and $\mathscr{B}m_p = \{y_{p,n}\}_{n\geq 0}$, where $x_{p,n}$ and $y_{p,n}$ are given as

$$x_{p,n} = \begin{cases} \sum_{i=0}^{n} \binom{n}{i} (p)^n M_{k,i} & \text{for } p \neq 0,\ n \neq 0, \\ M_{k,0} & \text{for } p = 0 \text{ or } n = 0 \end{cases} \tag{2.4}$$

and

$$y_{p,n} = \begin{cases} \sum_{i=0}^{n} \binom{n}{i} (p)^n m_{k,i} & \text{for } p \neq 0, n \neq 0, \\ m_{k,0} & \text{for } p = 0 \text{ or } n = 0. \end{cases} \tag{2.5}$$

Let $n \geq 0$ then for the k-Mersenne sequence $\{M_{k,n}\}$, the rising p-binomial transform $\mathscr{R}M_p$ and falling p-binomial transform $\mathscr{F}M_p$ are the sequence $\mathscr{R}M_p = \{r_{p,n}\}$ and $\mathscr{F}M_p = \{f_{p,n}\}$, respectively, where $r_{p,n}$ and $f_{p,n}$ are defined as

$$r_{p,n} = \begin{cases} \sum_{i=0}^{n} \binom{n}{i} (p)^i M_{k,i} & \text{for } p \neq 0, \\ M_{k,0} & \text{for } p = 0, \end{cases} \quad \text{and} \quad f_{p,n} = \begin{cases} \sum_{i=0}^{n} \binom{n}{i} (p)^{n-i} M_{k,i} & \text{for } p \neq 0, \\ M_{k,0} & \text{for } p = 0. \end{cases} \tag{2.6}$$

Similarly for k-Mersenne-Lucas sequence, $\mathscr{R}m_p = \{s_{p,n}\}_{n\geq 0}$ and $\mathscr{F}m_p = \{t_{p,n}\}_{n\geq 0}$, where $s_{p,n}$ and $t_{p,n}$ are given as

$$s_{p,n} = \begin{cases} \sum_{i=0}^{n} \binom{n}{i} (p)^i m_{k,i} & \text{for } p \neq 0, \\ m_{k,0} & \text{for } p = 0, \end{cases} \quad \text{and} \quad t_{p,n} = \begin{cases} k\sum_{i=0}^{n} \binom{n}{i} (p)^{n-i} m_{k,i} & \text{for } p \neq 0, \\ m_{k,0} & \text{for } p = 0. \end{cases} \tag{2.7}$$

We should note that for $p = 1$, the above three p-binomial transforms overlap with the binomial transform B_k and C_k.
From (2.4) and (2.5), after performing the necessary calculations, we have

$$\mathscr{B}M_p = \{0, p, p^2(3k+2), p^3(9k^2+9k+1), p^4(27k^3+36k^2+6k-4), \ldots\},$$
$$\mathscr{B}m_p = \{2, p(2+3k), p^2(9k^2+6k-2), p^3(27k^3+27k^2-9k-10), p^4(81k^4+108k^3-18k^2-60k-14), \ldots\}.$$

Setting $p = 1,2,3,4$ in the above p-binomial transforms $\mathscr{B}M_p$ and $\mathscr{B}m_p$, we get

$$\mathscr{B}M_1 = \{0, 1, (3k+2),\ (9k^2+9k+1),\ (27k^3+36k^2+6k-4), \ldots\},$$
$$\mathscr{B}M_2 = \{0, 2, 4(3k+2), 8(9k^2+9k+1), 16(27k^3+36k^2+6k-4), \ldots\},$$
$$\mathscr{B}M_3 = \{0, 3, 9(3k+2), 27(9k^2+9k+1), 81(27k^3+36k^2+6k-4), \ldots\},$$
$$\mathscr{B}M_4 = \{0, 4, 16(3k+2), 64(9k^2+9k+1), 256(27k^3+36k^2+6k-4), \ldots\},$$

and

$$\mathscr{B}m_1 = \{2,\ (2+3k),\ (9k^2+6k-2),\ (27k^3+27k^2-9k-10),\ (81k^4+108k^3-18k^2-60k-14),...\},$$
$$\mathscr{B}m_2 = \{2, 2(2+3k), 4(9k^2+6k-2), 8(27k^3+27k^2-9k-10), 16(81k^4+108k^3-18k^2-60k-14),...\},$$
$$\mathscr{B}m_3 = \{2, 3(2+3k), 9(9k^2+6k-2), 27(27k^3+27k^2-9k-10), 81(81k^4+108k^3-18k^2-60k-14),...\},$$
$$\mathscr{B}m_4 = \{2, 4(2+3k), 16(9k^2+6k-2), 64(27k^3+27k^2-9k-10), 256(81k^4+108k^3-18k^2-60k-14),...\}.$$

Similarly, on performing the necessary calculations with the rising p-binomial transforms given in (2.6) and (2.7), we have

$$\mathscr{R}M_p = \{0, p, 2p+3kp^2, 3p+9kp^2+(9k^2-2)p^3, 4p+18kp^2+(36k^2-8)p^3+(27k^3-12k)p^4,...\},$$
$$\mathscr{R}m_p = \{2,\ 2+3kp,\ 2+6kp+9k^2p^2-4p^2,\ 2+9kp+(27k^2-12)p^2+(27k^3-18k)p^3,...\}.$$

Thus, on setting $p=1,2,3,4$ in the above rising p-binomial transforms $\mathscr{R}M_p$ and $\mathscr{R}m_p$, we get

$$\mathscr{R}M_1 = \{0, 1, 2+3k, 1+9k+9k^2, -4+6k+36k^2+27k^3,...\},$$
$$\mathscr{R}M_2 = \{0, 2, 4+12k, -10+36k+72k^2, -56-120k+288k^2+432k^3,...\},$$
$$\mathscr{R}M_3 = \{0, 3, 6+27k, -45+81k+243k^2, -204-810k+972k^2+2187k^3,...\},$$
$$\mathscr{R}M_4 = \{0, 4, 8+48k, -116+144k+576k^2, -496-2784k+2304k^2+6912k^3,...\},$$

and

$$\mathscr{R}m_1 = \{2, 2+3k, -2+6k+9k^2, -10-9k+27k^2+27k^3,...\},$$
$$\mathscr{R}m_2 = \{2, 2+6k, -14+12k+36k^2, -46-126k+108k^2+216k^3,...\},$$
$$\mathscr{R}m_3 = \{2, 2+9k, -34+18k+81k^2, -106-459k+243k^2+729k^3,...\},$$
$$\mathscr{R}m_4 = \{2, 2+12k, -62+24k+144k^2, -190-1116k+432k^2+1728k^3,...\}.$$

Finally, performing the necessary calculations with the falling p-binomial transform for $\{M_{k,n}\}$ and $\{m_{k,n}\}$ gives the following sequences in p:

$$\mathscr{F}M_p = \{0, 1, 2p+3k, 3p^2+9kp+(9k^2-2), 4p^3+18kp^2+(36k^2-8)p+(27k^3-12k),...\},$$
$$\mathscr{F}m_p = \{2, 2p+3k, 2p^2+6kp+9k^2-4, 2p^3+9kp^2+(27k^2-12)p+(27k^3-18k),...\}.$$

Thus, first few falling p-binomial transforms are:

$$\mathscr{F}M_1 = \{0, 1, 2+3k, 1+9k+9k^2, -4+6k+36k^2+27k^3,...\},$$
$$\mathscr{F}M_2 = \{0, 1, 4+3k, 10+18k+9k^2, 16+60k+72k^2+27k^3,...\},$$
$$\mathscr{F}M_3 = \{0, 1, 6+3k, 25+27k+27k^2, 84+150k+108k^2+27k^3,...\},$$
$$\mathscr{F}M_4 = \{0, 1, 8+3k, 46+36k+9k^2, 224+276k+144k^2+27k^3,...\},$$

and

$$\mathscr{F}m_1 = \{2, 2+3k, -2+6k+9k^2, -10-9k+27k^2+27k^3,...\},$$
$$\mathscr{F}m_2 = \{2, 4+3k, 4+12k+9k^2, -8+18k+54k^2+27k^3,...\},$$
$$\mathscr{F}m_3 = \{2, 6+3k, 14+18k+9k^2, 18+63k+81k^2+27k^3,...\},$$
$$\mathscr{F}m_4 = \{2, 8+3k, 28+24k+9k^2, 80+126k+108k^2+27k^3,...\}.$$

2.2. Catalan transform and Catalan triangle

The Catalan transform and the Catalan triangle are mathematical tools that have applications in various fields, particularly in combinatorics, number theory, Encoding/Decoding, various counting problems, etc. More recently, Özkan, et al. [26] studied the k-Mersenne sequences where they examined the Catalan transform and other properties. Some recent developments on Catalan transforms with a number sequence can be seen in [15, 16, 20, 23].

For $n \geq 0$, let us define the Catalan transform $CW_k = \{CW_{k,n}\}_{n\geq 0}$ of the sequence $\{W_{k,n}\}$ following (1.3), where $W_{k,n} = M_{k,n}$ or $m_{k,n}$, as follows:

$$CW_{k,n} = \sum_{r=0}^{n} \frac{r}{2n-r}\binom{2n-r}{n-r} W_{k,r} \quad \text{with} \;\; CW_{k,0} = 0. \tag{2.8}$$

Thus, we have

$$CW_{k,1} = \sum_{r=0}^{1} \frac{r}{2-r}\binom{2-r}{1-r} W_{k,r} = 0W_{k,0} + 1W_{k,1} = W_{k,1},$$
$$CW_{k,2} = \sum_{r=0}^{2} \frac{r}{4-r}\binom{4-r}{2-r} W_{k,r} = \frac{1}{3}\binom{3}{1}W_{k,1} + \frac{2}{2}\binom{4}{0}W_{k,2} = W_{k,1} + W_{k,2},$$
$$CW_{k,3} = \sum_{r=0}^{3} \frac{r}{6-r}\binom{6-r}{3-r} W_{k,r} = \frac{1}{5}\binom{5}{2}W_{k,1} + \frac{2}{4}\binom{4}{1}W_{k,2} + \frac{3}{3}\binom{3}{0}W_{k,3} = 2W_{k,1} + 2W_{k,2} + W_{k,3},$$

Similarly,

$$CW_{k,4}=\sum_{r=0}^{4}\frac{r}{8-r}\binom{8-r}{4-r}W_{k,r}=5W_{k,1}+5W_{k,2}+3W_{k,3}+W_{k,4},$$

$$CW_{k,5}=\sum_{r=0}^{5}\frac{r}{10-r}\binom{10-r}{5-r}W_{k,r}=14W_{k,1}+14W_{k,2}+9W_{k,3}+4W_{k,4}+W_{k,5},$$

$$CW_{k,6}=\sum_{r=0}^{6}\frac{r}{12-r}\binom{12-r}{6-r}W_{k,r}=42W_{k,1}+42W_{k,2}+28W_{k,3}+14W_{k,4}+5W_{k,5}+W_{k,6},$$

$$CW_{k,7}=132W_{k,1}+132W_{k,2}+90W_{k,3}+48W_{k,4}+20W_{k,5}+6W_{k,6}+W_{k,7}.$$

The above transforms can be represented in matrix form as $\mathscr{G}W_k^T=LX^T$ with $X=[W_{k,1},W_{k,2},W_{k,3},...]$ and $\mathscr{G}W_k=[CW_{k,1},CW_{k,2},CW_{k,3},...]$, where $L=[a_{ij}]_{i,j\geq 1}$ is given as

$$L=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 2 & 2 & 1 & 0 & 0 & 0 & 0 & \dots \\ 5 & 5 & 3 & 1 & 0 & 0 & 0 & \dots \\ 14 & 14 & 9 & 4 & 1 & 0 & 0 & \dots \\ 42 & 42 & 28 & 14 & 5 & 1 & 0 & \dots \\ 132 & 132 & 90 & 48 & 20 & 6 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

We should note that from matrix L, we have
(a). entries of the first column is the Catalan numbers and for $i\geq 2$ the second column is equal to first column.
(b). for $i\geq j>1$, $a_{i,j}=\sum_{r=j-1}^{i-1}a_{i-1,r}$.
The lower triangular matrix $L_{n,n-i}$ gives the Catalan triangle whose entries are defined by relation

$$a_{n+1,n-i}=\frac{(i+1)(2n-i)!}{(n+1)!(n-i)!},\quad 0\leq i\leq n$$

and under the assumption $n-i=k$, it becomes $a_{n+1,k}=\binom{n+k}{k}-\binom{n+k}{k-1}=\frac{n+1-k}{n+1}\binom{n+k}{k}$, where $k=n,n-1,...,0$.
Thus the Catalan triangles associated with the Catalan transform for sequences $M_{k,n}$ and $m_{k,n}$ are shown in the following tables:

CM_0	0						
CM_1	0	1					
CM_2	0	3	1				
CM_3	0	9	6	0			
CM_4	0	27	27	3	-1		
CM_5	0	81	108	27	-6	0	
CM_6	0	243	405	162	-18	-6	6

Table 2.1: Catalan triangle of the k-Mersenne sequence

Cm_0	0						
Cm_1	3	0					
Cm_2	9	3	-4				
Cm_3	27	18	-12	-8			
Cm_4	81	81	-27	-39	28		
Cm_5	243	324	-27	-162	-60	-24	
Cm_6	729	1215	162	-594	-306	-78	-72

Table 2.2: Catalan triangle of the k-Mersenne-Lucas sequence

Example 2.1. *Let $k=1,2,3,4,5,6$, then using (2.8) the Catalan transforms for sequences $M_{k,n}$ and $m_{k,n}$ are:*

$\mathscr{G}M_1=\{0,1,4,15,56,...\}$
$\mathscr{G}M_2=\{0,1,7,48,329,...\}$
$\mathscr{G}M_3=\{0,1,10,99,980,...\}$
$\mathscr{G}M_4=\{0,1,13,168,2171,...\}$
$\mathscr{G}M_5=\{0,1,16,255,4064,...\}$
$\mathscr{G}M_6=\{0,1,19,360,6821,...\}$

$\mathscr{G}m_1=\{0,3,8,25,124,...\}$
$\mathscr{G}m_2=\{0,6,38,256,1786,...\}$
$\mathscr{G}m_3=\{0,9,86,847,8416,...\}$
$\mathscr{G}m_4=\{0,12,152,1960,25360,...\}$
$\mathscr{G}m_5=\{0,15,236,3757,59908,...\}$
$\mathscr{G}m_6=\{0,18,338,6400,121294,...\}$.

3. Generating Functions of the Binomial and Catalan Transforms

From [11] and [5], we should note that the generating functions $M(t)$ and $m(t)$ for the k-Mersenne and k-Mersenne-Lucas sequences, respectively, are:

$$M(t)=\frac{t}{1-3kt+2t^2}\quad\text{and}\quad m(t)=\frac{2-3kt}{1-3kt+2t^2}. \tag{3.1}$$

By the virtue of [15], note that if A_n is any sequence and $A(t)$ is its generating function then the generating function $B(t)$ of the binomial transform of A_n is given by

$$B(t)=\frac{1}{1-t}A\left(\frac{t}{1-t}\right). \tag{3.2}$$

Also, if $c(t)$ is the ordinary generating function (see (1.4)) for Catalan numbers then the generating function for the corresponding Catalan transform of the sequence A_n is given by $A(tc(t))$.

Theorem 3.1. *For the binomial and Catalan transforms of the k-Mersenne sequence, the generating functions $M_B(t)$ and $M_C(t)$ are given by*

$$M_B(t)=\frac{t}{1-t(2+3k)+3t^2(1+k)}$$

and

$$M_C(t)=\frac{1-\sqrt{1-4t}}{4-3k-4t+(3k-2)\sqrt{1-4t}}.$$

Proof. Using generating function $M(t)$ (See (3.1)) of the k-Mersenne sequence, the generating function $M_B(t)$ of the corresponding binomial transform is given by (3.2) as

$$\begin{aligned}M_B(t)&=\frac{1}{1-t}M\left(\frac{t}{1-t}\right)\\&=\frac{1}{1-t}\left(\frac{\frac{t}{1-t}}{1-3k(\frac{t}{1-t})+2(\frac{t}{1-t})^2}\right)\quad\text{(using (3.1))}\\&=\frac{1}{1-t}\left(\frac{(1-t)t}{(1-t)^2-3kt(1-t)+2t^2}\right).\end{aligned}$$

After some necessary calculations, we get

$$M_B(t)=\frac{t}{1-t(2+3k)+3t^2(1+k)}.$$

In a similar fashion, the generating function $M_C(t)$ is obtained by simplifying $M_C(t)=M(tc(t))$. □

Theorem 3.2. *For the binomial and Catalan transforms of the k-Mersenne-Lucas sequence, the generating functions $m_B(t)$ and $m_C(t)$ are given as*

$$m_B(t)=\frac{2-t(2+3k)}{1-t(2+3k)+3t^2(1+k)}$$

and

$$m_C(t)=\frac{4-3k(1-\sqrt{1-4t})}{4-3k-4t+(3k-2)\sqrt{1-4t}}.$$

Proof. Combining (3.1)) and (3.2) for the k-Mersenne-Lucas sequence, we have $m_B(t)=\frac{1}{1-t}m\left(\frac{t}{1-t}\right)$, thus

$$\begin{aligned}m_B(t)&=\frac{1}{1-t}\left(\frac{2-3k\left(\frac{t}{1-t}\right)}{1-3k(\frac{t}{1-t})+2(\frac{t}{1-t})^2}\right)\quad\text{(using (3.1))}\\&=\frac{1}{1-t}\left(\frac{2(1-t)^2-3k(1-t)t}{(1-t)^2-3kt(1-t)+2t^2}\right)\\&=\frac{2(1-t)-3kt}{1-t(2+3k)+3t^2(1+k)}.\end{aligned}$$

In a similar fashion, the generating function $m_C(t)$ is obtained by simplifying $m_C(t)=m(tc(t))$. □

Theorem 3.3. *For the p-binomial transforms of the sequences $M_{k,n}$ and $m_{k,n}$, the generating functions $w_M(p,t)$ and $w_m(p,t)$ are given by*

$$w_M(p,t)=\frac{pt}{1-pt(2+3k)+3p^2t^2(1+k)}$$

and

$$w_m(p,t)=\frac{2-pt(2+3k)}{1-pt(2+3k)+3p^2t^2(1+k)}.$$

Proof. To prove the result, we use the fact from [14] that if $M(t)$ is the generating function for $\{M_{k,n}\}$ then for the associated p-binomial transform, the generating function is given by

$$\begin{aligned} w_M(p,t) &= \frac{1}{1-pt} M\Big(\frac{pt}{1-pt}\Big) \\ &= \frac{1}{1-pt}\left[\frac{\frac{pt}{1-pt}}{1-3k(\frac{pt}{1-pt})+2(\frac{pt}{1-pt})^2}\right] \\ &= \frac{1}{1-pt}\left[\frac{(1-pt)pt}{(1-pt)^2-3kpt(1-pt)+2p^2t^2}\right]. \end{aligned}$$

After some necessary calculations, we get

$$w_M(p,t) = \frac{pt}{1-pt(2+3k)+3p^2t^2(1+k)}.$$

Similarly the second identity holds. □

Theorem 3.4. *The exponential generating function $E_M(t)$ and $E_m(t)$ for sequences $M_{k,n}$ and $m_{k,n}$ are given by*

$$E_M(t) = \frac{e^{r_1 t} - e^{r_2 t}}{\sqrt{9k^2-8}} \qquad \text{and} \qquad E_m(t) = e^{r_1 t} + e^{r_2 t}.$$

Proof. It can be easily proved using Binet's formula of the respective sequences. □

Theorem 3.5. *For p-binomial transforms $\mathscr{B}M_p$ and $\mathscr{B}m_p$, rising p-binomial transforms $\mathscr{R}M_p$ and $\mathscr{R}m_p$ and falling p-binomial transforms $\mathscr{F}M_p$ and $\mathscr{F}m_p$, the exponential generating functions are given by*

1. $E_{\mathscr{B}M}(p,t) = \dfrac{e^{pt}(e^{r_1 pt} - e^{r_2 pt})}{\sqrt{9k^2-8}}$,
2. $E_{\mathscr{R}M}(p,t) = \dfrac{e^{t}(e^{r_1 pt} - e^{r_2 pt})}{\sqrt{9k^2-8}}$,
3. $E_{\mathscr{F}M}(p,t) = \dfrac{e^{pt}(e^{r_1 t} - e^{r_2 t})}{\sqrt{9k^2-8}}$,
4. $E_{\mathscr{B}m}(p,t) = e^{pt}(e^{r_1 pt} + e^{r_2 pt})$,
5. $E_{\mathscr{R}m}(p,t) = e^{t}(e^{r_1 pt} + e^{r_2 pt})$,
6. $E_{\mathscr{F}m}(p,t) = e^{pt}(e^{r_1 t} + e^{r_2 t})$.

Proof. In accordance with [[14], Theorem 5.1], we should note that "if $E(t)$ is the exponential generating function of a sequence A_n then the exponential generating functions for the p-binomial transforms, rising p-binomial transforms and falling p-binomial transforms of sequence A_n are given by, respectively, $e^{pt}E(pt)$, $e^{t}E(pt)$ and $e^{pt}E(t)$ respectively".
Thus, the results follows from the above fact. □

4. New Recurrences From the Binomial Transforms

Now we establish a recurrence relation for the above obtained binomial transforms. Then, we obtain their Binet type formula which help us to establish several identities and results.

Theorem 4.1. *For the binomial transforms $\{BM_{k,n}\}$ and $\{Bm_{k,n}\}$, we have*

1. $BM_{k,n+1} - BM_{k,n} = \displaystyle\sum_{a=0}^{n} \binom{n}{a} M_{k,a+1}$,
2. $Bm_{k,n+1} - Bm_{k,n} = \displaystyle\sum_{a=0}^{n} \binom{n}{a} m_{k,a+1}$.

Proof. Since, from the binomial theorem we have

$$\binom{n+1}{a} = \binom{n}{a} + \binom{n}{a-1}.$$

Thus, using (2.1), we have

$$\begin{aligned} BM_{k,n+1} &= \sum_{a=0}^{n+1}\binom{n+1}{a} M_{k,a} = \sum_{a=0}^{n+1}\left[\binom{n}{a} + \binom{n}{a-1}\right] M_{k,a} \\ &= \sum_{a=0}^{n+1}\binom{n}{a} M_{k,a} + \sum_{a=0}^{n+1}\binom{n}{a-1} M_{k,a} \\ &= \sum_{a=0}^{n}\binom{n}{a} M_{k,a} + \sum_{a=0}^{n}\binom{n}{a} M_{k,a+1} \qquad \left(\text{Since, } \binom{n}{n+1} = 0 \text{ and } \binom{n}{-1} = 0\right) \\ &= BM_{k,n} + \sum_{a=0}^{n}\binom{n}{a} M_{k,a+1} \qquad (\text{from } (2.1)). \end{aligned}$$

Thus, the required result. Similarly the second identity holds. □

From Theorem 4.1, we deduce that

$$BM_{k,n+1}=\sum_{a=0}^{n}\binom{n}{a}\left[M_{k,a}+M_{k,a+1}\right] \quad \text{and} \quad Bm_{k,n+1}=\sum_{a=0}^{n}\binom{n}{a}\left[m_{k,a}+m_{k,a+1}\right]. \tag{4.1}$$

Theorem 4.2. *For $n\geq 0$, the binomial transforms $\{BM_{k,n}\}$ and $\{Bm_{k,n}\}$ posses the following recurrence relations:*

$$\begin{aligned} BM_{k,n+2}&=(2+3k)BM_{k,n+1}-(3+3k)BM_{k,n} \quad \textit{with} \quad BM_{k,0}=0,\ BM_{k,1}=1,\\ Bm_{k,n+2}&=(2+3k)Bm_{k,n+1}-(3+3k)Bm_{k,n} \quad \textit{with} \quad Bm_{k,0}=2,\ Bm_{k,1}=2+3k. \end{aligned} \tag{4.2}$$

Proof. From (4.1), we can write

$$\begin{aligned} BM_{k,n+1}&=\sum_{a=1}^{n}\binom{n}{a}\left[M_{k,a}+M_{k,a+1}\right]+M_{k,0}+M_{k,1}\\ &=\sum_{a=1}^{n}\binom{n}{a}\left[M_{k,a}+3kM_{k,a}-2M_{k,a-1}\right]+1 \quad (\text{using } (1.1))\\ &=(1+3k)\sum_{a=1}^{n}\binom{n}{a}M_{k,a}-2\sum_{a=1}^{n}\binom{n}{a}M_{k,a-1}+1\\ &=(1+3k)BM_{k,n}-2\sum_{a=1}^{n}\binom{n}{a}M_{k,a-1}+1 \quad (\text{using } (2.1)). \end{aligned} \tag{4.3}$$

Now replacing n by $n+1$ in the above equation, we get

$$BM_{k,n+2}=(1+3k)BM_{k,n+1}-2\sum_{a=1}^{n+1}\binom{n+1}{a}M_{k,a-1}+1.$$

On some elementary calculations and simplification, we obtain

$$BM_{k,n+2}=(1+3k)BM_{k,n+1}-2\sum_{a=0}^{n}\binom{n}{a}M_{k,a}-2BM_{k,n}+1$$

or,

$$BM_{k,n+2}-(1+3k)BM_{k,n+1}+2BM_{k,n}=-2\sum_{a=0}^{n}\binom{n}{a}M_{k,a}+1.$$

Thus, on substituting from (4.3), we have

$$BM_{k,n+2}=(2+3k)BM_{k,n+1}-(3+3k)BM_{k,n}.$$

Similarly the second recurrence relation can be achieved. □

Some terms of the binomial transforms corresponding to the k-Mersenne and k-Mersenne-Lucas numbers are

$$\begin{aligned} &BM_{k,0}=0,\ BM_{k,1}=1,\ BM_{k,2}=3k+2,\ BM_{k,3}=9k^2+9k+1,\\ &BM_{k,4}=27k^3+36k^2+6k-4,\ldots, \end{aligned}$$

and

$$\begin{aligned} &Bm_{k,0}=2,\ Bm_{k,1}=2+3k,\ Bm_{k,2}=9k^2+6k-2,\\ &Bm_{k,3}=27k^3+27k^2-9k-10,\ Bm_{k,4}=81k^4+108k^3-18k^2-60k-14,\ldots. \end{aligned}$$

The characteristic equation corresponding to recurrence relation (4.2) is $x^2-(2+3k)x+(3+3k)=0$ which has the following two roots

$$\lambda_1=\frac{(2+3k)+\sqrt{9k^2-8}}{2} \quad \text{and} \quad \lambda_2=\frac{(2+3k)-\sqrt{9k^2-8}}{2} \tag{4.4}$$

$$\text{i.e.} \quad \lambda_1=1+r_1 \quad \text{and} \quad \lambda_2=1+r_2, \tag{4.5}$$

that plays an important role in setting the explicit formula for a sequence.

Theorem 4.3 (Binet type formula)**.** *For the binomial transforms $\{BM_{k,n}\}$ and $\{Bm_{k,n}\}$, we have*

$$BM_{k,n}=\frac{\lambda_1^n-\lambda_2^n}{\sqrt{9k^2-8}} \quad \textit{and} \quad Bm_{k,n}=\lambda_1^n+\lambda_2^n.$$

Proof. Using (4.4) and performing some necessary calculations proves the result. □

Theorem 4.4. *If $M_{k,n}=\frac{r_1^n-r_2^n}{\sqrt{9k^2-8}}$ and $m_{k,n}=r_1^n+r_2^n$, then the nth term of the corresponding binomials transform are given by*

$$BM_{k,n}=\frac{(r_1+1)^n-(r_2+1)^n}{\sqrt{9k^2-8}} \quad \textit{and} \quad Bm_{k,n}=(r_1+1)^n+(r_2+1)^n.$$

Proof. Using Theorem 4.3 and relation (4.5), the result can be established. □

Thus, after getting the Binet type formula for a sequence, one can easily derive and prove several identities. For example, for sequences $M_{k,n}$ and $m_{k,n}$ sum of the first n terms of the corresponding binomial transform are given as the follows.

Theorem 4.5. *For $n \geq 0$, we have*

$$\sum_{a=0}^{n} BM_{k,a} = \frac{1}{2}\Big[(3+3k)BM_{k,n} - BM_{k,n+1} + 1\Big]$$

and

$$\sum_{a=0}^{n} Bm_{k,a} = \frac{1}{2}\Big[(3+3k)Bm_{k,n} - Bm_{k,n+1} - 3k\Big].$$

Proof. Using Theorem 4.3, we have

$$\begin{aligned}
\sum_{a=0}^{n} BM_{k,a} &= \frac{1}{\sqrt{9k^2-8}}\Big(\sum_{a=0}^{n}\lambda_1^a - \sum_{a=0}^{n}\lambda_2^a\Big)\\
&= \frac{1}{\sqrt{9k^2-8}}\Big(\frac{\lambda_1^{n+1}-1}{\lambda_1-1} - \frac{\lambda_2^{n+1}-1}{\lambda_2-1}\Big)\\
&= \frac{1}{\sqrt{9k^2-8}}\Big[\frac{(\lambda_2-1)(\lambda_1^{n+1}-1)-(\lambda_1-1)(\lambda_2^{n+1}-1)}{(\lambda_1-1)(\lambda_2-1)}\Big]\\
&= \frac{\lambda_2\lambda_1^{n+1}-\lambda_1\lambda_2^{n+1}-\lambda_1^{n+1}+\lambda_2^{n+1}}{2\sqrt{9k^2-8}} + \frac{1}{2} \qquad \text{(using (1.2) and (4.4))}\\
&= \frac{\lambda_1\lambda_2(\lambda_1^n-\lambda_2^n)-(\lambda_1^{n+1}-\lambda_2^{n+1})}{2\sqrt{9k^2-8}} + \frac{1}{2}\\
&= \frac{1}{2}\Big[(3+3k)BM_{k,n} - BM_{k,n+1} + 1\Big].
\end{aligned}$$

Argument for the second identity is very similar to the above. □

5. Conclusion

This study demonstrates the transformative effect of various binomial and Catalan transforms on k-Mersenne and k-Mersenne-Lucas numbers, generating novel integer sequences, some of which are identified in OEIS. Various generating and exponential generating functions, alongside Binet-type formulas and new recurrence relations are obtained for associated binomial transforms, which provides a foundation for further investigation. This is useful because number patterns show up in many places – like in computer science for writing code, in understanding natural patterns, or even in designing secure systems. Finding these new patterns and their rules gives scientists more tools to use in these areas.

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Research Article

Fuzzy Solutions of Fuzzy Fractional Parabolic Integro Differential Equations

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Abstract

This work primarily investigates the numerical solution of fuzzy fractional parabolic integro-differential equations of the Volterra type with the time derivative defined in the Caputo sense using the fuzzy Adomian decomposition method. Fuzzy fractional partial integro-differential equations pose significant mathematical challenges due to the interplay between fuzziness and fractional-order dynamics, while at the same time, there is a growing need for accurate and efficient methods to model real-world phenomena involving uncertainty in physics, biology, and engineering. The fuzzy Adomian decomposition method provides an alternative approach for obtaining approximate fuzzy solutions, and its applicability to such equations has not been studied in detail previously in the literature. Furthermore, existence and uniqueness theorems for the fuzzy fractional partial integro-differential equation are established by considering the differentiability type of the solution. The accuracy and efficiency of the proposed method are demonstrated through a series of numerical experiments.

1. Introduction

Fractional differential equations and fractional integral equations are powerful tools for modeling and describing the hereditary properties of various materials and processes. In recent years, the widespread use of Fractional differential equations in engineering and scientific domains has motivated researchers to pursue advancements in both theoretical and applied research methods. Many researchers have focused on establishing existence results to confirm that the mathematical models accurately describe real-world phenomena [1–3], while other research has concentrated on finding explicit or approximate solutions to these models. [4–6]. When modeling real-world phenomena using fractional differential equations, the behavior of dynamical systems can be complex and affected by errors and uncertainty. To address this, some researchers have introduced approaches that define parameters and initial conditions within a fuzzy fractional framework. Early contributions to the study of fuzzy fractional differential equations were made by Agarwal et al. [7] and Arshad et al. [8], who applied the Riemann–Liouville derivative with fuzzy initial conditions. This approach extends the classical Riemann–Liouville derivative using the Hukuhara difference ($\mathtt{H}$-difference) [9]. However, a limitation of the $\mathtt{H}$-difference is that the support of fuzzy solutions tends to increase over time (see [10–12]). Moreover, the Riemann–Liouville derivative requires knowledge of the fractional derivative of the unknown solution at the initial point, which is often difficult to measure or may not exist. To overcome these challenges, several studies have combined Caputo derivatives with generalized Hukuhara differentiability ($g\mathtt{H}$-differentiability), leading to the concept of Caputo gH-differentiability, as discussed in works by Salahshour et al. [13], Long et al. [14], Alqudah et. al. [15] and Saeed et. al. [16].
Recently, numerous authors have developed and analyzed various numerical techniques of fuzzy fractional differential equations. These include studies on the existence of global solutions using upper and lower solutions method [17], integro-differential equations with generalized Caputo differentiability [18], the fractional differential transform method [19], the Adomain decomposition method [20, 21], the Jacobi polynomial operational matrix [22], the two-dimensional Legendre wavelet method [23], the power series method [24, 25], homotopy perturbation transform method [26] and the optimal homotopy asymptotic method [27].

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The main objective of this work is to prove the uniqueness of solution of fuzzy fractional partial integro-differential equation. We also investigate the numerical solution of fuzzy fractional parabolic integro-differential equations under Caputo generalized Hukuhara differentiability by fuzzy Adomian decomposition method. To achieve this, we convert a fuzzy fractional parabolic integro-differential equation into a system of crisp equations that can be solved bya standard numerical method.
The significance of this study from the theoretical point of view is that the current fuzzy Adomian decomposition method is developed for a general form of fuzzy fractional partial integro-differential equation under Caputo generalized Hukuhara differentiability. This can greatly help the numerical study of fuzzy fractional partial integro-differential equations and other equations in this form due to the difficulty of solving these equations analytically.
The paper is organized as follows: Section 2 introduces essential definitions and notations related to fuzzy fractional calculus. In Section 3, we present a fuzzy fractional partial integro-differential equation under Caputo generalized Hukuhara differentiability and examine the existence and uniqueness of its solutions. Section 4 discusses the convergence of the Fuzzy Adomian Decomposition Method for determining approximate solutions to the fuzzy fractional parabolic integro-differential equation (fuzzy fractional parabolic IDEs). Additionally, we explore solutions of fuzzy fractional parabolic IDEs under different differentiability types. Section 5 provides examples to illustrate the effectiveness of the proposed method.

2. Preliminary Concepts

In this section, we recall some of the basic preliminaries of fuzzy fractional calculus.
Let $\mathfrak{C}[\mathtt{I},\mathbb{R}]$ be the Banach space of all real-valued continuous functions from $\mathtt{I}=[0,a]$ into $\mathbb{R}$. For measurable real-valued function $f:\mathtt{I}\to\mathbb{R}$, define the norm $\|f\|_{L^p(\mathtt{I},\mathbb{R})}=\left(\int_{\mathtt{I}}|f(\varkappa)|^p\right)^{\frac{1}{p}}<\infty,\ 1\le p<\infty$, where $L^p(\mathtt{I},\mathbb{R})$ denote the Banach space of all Lebesgue measurable real-valued functions f. Also, we use the notations listed below:

$\mathscr{F}_R$ is the set of all fuzzy numbers on $\mathbb{R}$.
$\mathfrak{C}[\mathtt{J},\mathscr{F}_R]$ is a space of all continuous fuzzy-valued functions which are on $\mathtt{J}=[0,a]\times[0,b]\subset\mathbb{R}^2$.
$\mathfrak{L}[\mathtt{J},\mathscr{F}_R]$ is the set of Lebesque integrable for fuzzy-valued functions on $\mathbf{B}$, where $\mathbf{B}\subset\mathbb{R}^m, m\in\mathbb{N}$.

Definition 2.1. *[28] A fuzzy number is a mapping $\alpha:\mathbb{R}\to[0,1]$ with the following features:*

(1) *For $\varkappa_0\in\mathbb{R},\alpha$ is normal. It means, $\alpha(\varkappa_0)=1$.*
(2) *For $\varkappa_1,\varkappa_2,\in\mathbb{R}$ and $\mathbf{t}\in[0,1]$, α is convex such that*

$$\alpha(\mathbf{t}\varkappa_1+(1-\mathbf{t})\varkappa_2)\ge\min\{\alpha(\varkappa_1),\alpha(\varkappa_2)\}.$$

(3) *α is upper semicontinuous.*
(4) *$cl\{\varkappa\in\mathbb{R},\alpha(\varkappa)>0\}$ is compact.*

The set of a fuzzy number $\alpha(\varkappa)\in\mathscr{F}_R$ in the ς-level form is denoted by $[\alpha]^\varsigma$ and defined as:

$$\begin{cases}\{\varkappa\in\mathbb{R}\mid\alpha(\varkappa)\ge\varsigma\} & \text{if } 0<\varsigma\le1,\\ \text{cl}(\text{supp}\,\alpha(\varkappa)) & \text{if } \varsigma=0.\end{cases}$$

It is clear that the set of a fuzzy number $\varkappa$ in ς-level form is a closed and bounded interval $[\underline{\alpha_\varsigma},\overline{\alpha_\varsigma}]$, where $\underline{\alpha_\varsigma}$ is the left end point and $\overline{\alpha_\varsigma}$ is the right end point.
For any arbitrary elements $\alpha,\beta\in\mathscr{F}_R$ and scalar $k\in\mathbb{R}$, the operations of addition and scalar multiplication are respectively defined by their ς-level sets as follows:

$$[\alpha+\beta]^\varsigma=(\underline{\alpha_\varsigma}+\underline{\beta_\varsigma},\overline{\alpha_\varsigma}+\overline{\beta_\varsigma}),$$

$$[k\alpha]^\varsigma=\begin{cases}(k\underline{\alpha_\varsigma},k\overline{\alpha_\varsigma}) & \text{if } k\ge0,\\ (k\overline{\alpha_\varsigma},k\underline{\alpha_\varsigma}) & \text{if } k<0.\end{cases}$$

A *triangular fuzzy number* is characterized as a fuzzy set in $\mathbb{R}_F$, represented by an ordered triple $\alpha=(a,b,c)\in\mathbb{R}^3$ where $a\le b\le c$. The ς-level set of α is given by the endpoints:

$$\underline{\alpha_\varsigma}=a+(b-a)\varsigma,\quad \overline{\alpha_\varsigma}=c-(c-b)\varsigma,$$

for all $\varsigma\in[0,1]$.

Definition 2.2. *[14] Let $\mathbb{D}:\mathscr{F}_R\times\mathscr{F}_R\longrightarrow\mathbb{R}$ be the Hausdorff distance between two fuzzy numbers α,β and defined as*

$$\begin{aligned}\mathbb{D}(\alpha,\beta)&=\sup_{0\le\varsigma\le1}d_H\{[\alpha]^\varsigma,[\beta]^\varsigma\}\\&=\sup_{0\le\varsigma\le1}max\{|\underline{\alpha_\varsigma}-\underline{\beta_\varsigma}|,|\overline{\alpha_\varsigma}-\overline{\beta_\varsigma}|\},\end{aligned}$$

where the metric space $(\mathscr{F}_R,\mathbb{D})$ is complete, separable and locally compact. The supremum metric D^* on $\mathfrak{C}[\mathtt{J},\mathscr{F}_R]$ is considered as

$$D^*(\alpha,\beta)=\sup_{(\varkappa,\mathbf{t})\in\mathtt{J}}\{\mathbb{D}(\alpha(\varkappa,\mathbf{t}),\beta(\varkappa,\mathbf{t}))\}. \tag{2.1}$$

Definition 2.3. *[28] The Hukuhara difference (H-difference) between two fuzzy numbers α and β is defined as*

$$\alpha \ominus \beta = w \quad \Leftrightarrow \quad \alpha = \beta + w,$$

where $+$ denotes the standard fuzzy addition. Moreover, if $\alpha \ominus \beta$ exists, then $\alpha \ominus \alpha = 0$.

In [14] authors have given some properties of the metric $\mathbb{D}$ in $\mathscr{F}_R$ and Hukuhara difference as:

Lemma 2.4. *For all $\alpha, \beta, l, \gamma, \varpi \in \mathscr{F}_R$ we have*

(1) $\mathbb{D}(\alpha + l, \beta + l) = \mathbb{D}(\alpha, \beta)$.
(2) $\mathbb{D}(\alpha + \beta, \gamma + \varpi) \leq \mathbb{D}(\alpha, \gamma) + \mathbb{D}(q, \varpi)$.
(3) $\mathbb{D}(\alpha + \beta, 0) = \mathbb{D}(\alpha, 0) + \mathbb{D}(\beta, 0)$.
(4) *If $\alpha \ominus \beta$ exists then $(-1)\alpha \ominus (-1)\beta$ exists and $(-1)(\alpha \ominus \beta) = (-1)\alpha \ominus (-1)\beta$.*
(5) *If $\alpha \ominus \beta$ and $\gamma \ominus \varpi$ exist then $\mathbb{D}(\alpha \ominus \beta, \gamma \ominus \varpi) \leq \mathbb{D}(\alpha, \gamma) + D(\beta, \varpi)$.*

Definition 2.5. *[28] The generalized Hukuhara difference(gH-difference for short) of two fuzzy numbers $\alpha, \beta \in \mathscr{F}_R$ is defined as follows:*

$$\alpha \ominus_{gH} \beta = w \quad \Leftrightarrow \quad \begin{cases} (i)\ \alpha = \beta + w, \\ or \\ (ii)\ \beta = \alpha + (-1)w. \end{cases}$$

It is easy to show that (i) and (ii) are both valid if and only if w is a crisp number.
For the ς-levels, the generalized Hukuhara difference (gH-difference) between α and β is given by:

$$[\alpha \ominus_{gH} \beta]^{\varsigma} = \left[\min\{\underline{\alpha_\varsigma} - \underline{\beta_\varsigma}, \overline{\alpha_\varsigma} - \overline{\beta_\varsigma}\}, \max\{\underline{\alpha_\varsigma} - \underline{\beta_\varsigma}, \overline{\alpha_\varsigma} - \overline{\beta_\varsigma}\}\right].$$

If the Hukuhara difference (H-difference) exists, then $\alpha \ominus \beta = \alpha \ominus_{gH} \beta$. The conditions for the existence of $\alpha \ominus_{gH} \beta \in \mathscr{F}_R$ are shown in [10, 29].

Remark 2.6. *Throughout the remainder of this paper, we assume that $\alpha \ominus_{gH} \beta \in \mathscr{F}_R$.*

Definition 2.7. *[30] A fuzzy number α can be represented in parametric form as $[\underline{\alpha}(\varsigma), \overline{\alpha}(\varsigma)]$, for $0 \leq \varsigma \leq 1$, if and only if*

(i) *$\underline{\alpha}(\varsigma)$ is increasing bounded function and left continuous over (0,1].*
(ii) *$\overline{\alpha}(\varsigma)$ is decreasing bounded function and right continuous over (0,1].*
(iii) *$\underline{\alpha}(\varsigma) \leq \overline{\alpha}(\varsigma)$.*

Allahviranloo [28] introduced the definition of fuzzy partial derivative as follows:

Definition 2.8. *Let $\nu : \mathtt{J} \longrightarrow \mathscr{F}_R$, then gH-partial derivative of first order at the point $(\varkappa_0, \mathbf{t}_0) \in \mathtt{J}$ with respect to variables $\varkappa$, $\mathbf{t}$ are denoted by $\frac{\partial \nu(\varkappa_0, \mathbf{t}_0)}{\partial \varkappa}$, $\frac{\partial \nu(\varkappa_0, \mathbf{t}_0)}{\partial \mathbf{t}}$ and given by*

$$\frac{\partial \nu(\varkappa_0, \mathbf{t}_0)}{\partial \varkappa} = \lim_{h \to 0} \frac{\nu(\varkappa_0 + h, \mathbf{t}_0) \ominus_{gH} \nu(\varkappa_0, \mathbf{t}_0)}{h},$$

$$\frac{\partial \nu(\varkappa_0, \mathbf{t}_0)}{\partial \mathbf{t}} = \lim_{k \to 0} \frac{\nu(\varkappa_0, \mathbf{t}_0 + k) \ominus_{gH} \nu(\varkappa_0, \mathbf{t}_0)}{k},$$

provided that $\frac{\partial \nu(\varkappa_0, \mathbf{t}_0)}{\partial \varkappa}$ and $\frac{\partial \nu(\varkappa_0, \mathbf{t}_0)}{\partial \mathbf{t}} \in \mathscr{F}_R$.

Definition 2.9. *Let $\nu : \mathtt{J} \longrightarrow \mathscr{F}_R$ be gH-partial differentiable with respect to $\varkappa$ at $(\varkappa_0, \mathbf{t}_0) \in \mathtt{J}$. We say that*

(1) ν is (i) gH-partial differentiable with respect to $\varkappa$ at $(\varkappa_0, \mathbf{t}_0) \in \mathtt{J}$. If

$$\left[\frac{\partial \nu(\varkappa_0, \mathbf{t}_0, \varsigma)}{\partial \varkappa}\right] = \left[\frac{\partial \underline{\nu}(\varkappa_0, \mathbf{t}_0, \varsigma)}{\partial \varkappa}, \frac{\partial \overline{\nu}(\varkappa_0, \mathbf{t}_0, \varsigma)}{\partial \varkappa}\right], \quad \forall \varsigma \in [0,1].$$

(2) ν is (ii) gH-partial differentiable with respect to $\varkappa$ at $(\varkappa_0, \mathbf{t}_0) \in \mathtt{J}$. If

$$\left[\frac{\partial \nu(\varkappa_0, \mathbf{t}_0, \varsigma)}{\partial \varkappa}\right] = \left[\frac{\partial \overline{\nu}(\varkappa_0, \mathbf{t}_0, \varsigma)}{\partial \varkappa}, \frac{\partial \underline{\nu}(\varkappa_0, \mathbf{t}_0, \varsigma)}{\partial \varkappa}\right], \quad \forall \varsigma \in [0,1].$$

Definition 2.10. *[14] For a fixed $\varkappa_0$, the point $(\varkappa_0, \mathbf{t}) \in \mathtt{J}$ is called a* switching point for the differentiability of ν with respect to $\varkappa_0$ *if, in every neighborhood V of $(\varkappa_0, \mathbf{t})$, there exist points $A_1(\varkappa_1, \mathbf{t})$ and $A_2(\varkappa_2, \mathbf{t})$ with $\varkappa_1 < \varkappa_0 < \varkappa_2$ such that either:*

1. *ν is (i)-gH differentiable at A_1 and (ii)-gH differentiable at A_2 for all $\mathbf{t}$, or*
2. *ν is (i)-gH differentiable at A_2 and (ii)-gH differentiable at A_1 for all $\mathbf{t}$.*

Lemma 2.11. *[14] (Newton–Leibniz formula) Let $\nu \in \mathfrak{C}(\mathbb{R}^2, \mathscr{F}_R)$.*

(1) If ν is (i)-gH differentiable with respect to $\mathbf{t}$, with no switching point on $\mathbb{R} \times [b, \mathbf{t}]$, then

$$\int_b^{\mathbf{t}} \frac{\partial \nu(\varkappa, \delta)}{\partial \delta} d\delta = \nu(\varkappa, \mathbf{t}) \ominus \nu(\varkappa, b).$$

(2) If ν is (ii)-gH differentiable with respect to $\mathbf{t}$, with no switching point on $\mathbb{R}\times[b,\mathbf{t}]$, then

$$\int_b^{\mathbf{t}} \frac{\partial \nu(\varkappa,\delta)}{\partial \delta}\, d\delta = (-1)\nu(\varkappa,b)\ominus(-1)\nu(\varkappa,\mathbf{t}).$$

In [13, 28], authors have defined the concepts of Riemann-Liouville integral and Caputo's gH-derivative of fuzzy valued functions as follows:

Definition 2.12. *Let $\nu(\varkappa)\in\mathfrak{C}[\mathscr{I},\mathscr{F}_R]\cap\mathfrak{L}[\mathscr{I},\mathscr{F}_R]$, $\mathscr{I}\in\mathbb{R}$. The fuzzy fractional integral in Riemann-Liouville sense of order $\theta>0$ is defined as*

$$\mathfrak{I}^{\theta}\nu(\varkappa,\varsigma)=[\mathfrak{I}^{\theta}\underline{\nu}(\varkappa,\varsigma),\mathfrak{I}^{\theta}\overline{\nu}(\varkappa,\varsigma)],\ \ \varsigma\in[0,1],$$

where

$$\mathfrak{I}^{\theta}\underline{\nu}(\varkappa,\varsigma)=\frac{1}{\Gamma(\theta)}\int_0^{\varkappa}(\varkappa-\tau)^{\theta-1}\underline{\nu}(\tau,\varsigma)d\tau,\ \varkappa>0,$$

$$\mathfrak{I}^{\theta}\overline{\nu}(\varkappa,\varsigma)=\frac{1}{\Gamma(\theta)}\int_0^{\varkappa}(\varkappa-\tau)^{\theta-1}\overline{\nu}(\tau,\varsigma)d\tau,\ \varkappa>0.$$

Definition 2.13. *Let $\nu(\varkappa)\in\mathfrak{C}[\mathscr{I},\mathscr{F}_R]\cap\mathfrak{L}[\mathscr{I},\mathscr{F}_R]$. Then the fuzzy fractional Caputo's gH-derivative under (i) gH-differentiability is defined as*

$${}^{c}_{gH}\mathfrak{D}^{\theta}_{\varkappa}\nu(\varkappa,\varsigma)=[\,{}^{c}\mathfrak{D}^{\theta}_{\varkappa}\underline{\nu}(\varkappa,\varsigma),\ {}^{c}\mathfrak{D}^{\theta}_{\varkappa}\overline{\nu}(\varkappa,\varsigma)]$$

and under (ii) gH-differentiability is given as:

$${}^{c}_{gH}\mathfrak{D}^{\theta}_{\varkappa}\nu(\varkappa,\varsigma)=[\,{}^{c}\mathfrak{D}^{\theta}_{\varkappa}\overline{\nu}(\varkappa,\varsigma),\ {}^{c}\mathfrak{D}^{\theta}_{\varkappa}\underline{\nu}(\varkappa,\varsigma)],$$

where

$${}^{c}\mathfrak{D}^{\theta}_{\varkappa}\underline{\nu}(\varkappa,\varsigma)]=\frac{1}{\Gamma(\mathtt{m}-\theta)}\int_0^{\varkappa}(\varkappa-\tau)^{\mathtt{m}-\theta-1}\ \underline{\nu}^{(\mathtt{m})}(\tau,\varsigma)d\tau,$$

$${}^{c}\mathfrak{D}^{\theta}_{\varkappa}\overline{\nu}(\varkappa,\varsigma)=\frac{1}{\Gamma(\mathtt{m}-\theta)}\int_0^{\varkappa}(\varkappa-\tau)^{\mathtt{m}-\theta-1}\ \overline{\nu}^{(\mathtt{m})}(\tau,\varsigma)d\tau.$$

Proposition 2.14. *[28] If $\nu(\varkappa):[0,a]\to E_f$ is an integrable fuzzy function and $\theta_1>0$, $\theta_2>0$. Then,*

$$(\mathfrak{I}^{\theta_1})(\mathfrak{I}^{\theta_2})\nu(\varkappa)=(\mathfrak{I}^{\theta_1+\theta_2})\nu(\varkappa),\ \ \varkappa\in[0,a].$$

Theorem 2.15. *[31](Holder's Inequality) If q_1 and q_2 are positive numbers satisfying the relation $\frac{1}{q_1}+\frac{1}{q_2}=1$ and if $\mathtt{f}\in L^{q_1}(0,a)$, $\mathtt{g}\in L^{q_2}(0,a)$, then $\mathtt{fg}\in L(0,a)$ and*

$$\int_0^a|\mathtt{f}(\varkappa)\mathtt{g}(\varkappa)|d\varkappa\le\left(\int_0^a|\mathtt{f}(\varkappa)|^{q_1}\right)^{\frac{1}{q_1}}\left(\int_0^a|\mathtt{g}(\varkappa)|^{q_2}\right)^{\frac{1}{q_2}}.$$

3. Fuzzy Fractional Partial Integro-Differential Equations (FFPIDEs)

In the current section, we establish that the following FFPIDEs of Volterra type have a unique solution in $\mathfrak{C}(\mathtt{J},\mathscr{F}_R)$.

$$\begin{aligned}&{}^{c}\mathfrak{D}^{\theta}_{\mathbf{t}}\nu(\varkappa,\mathbf{t})=\Upsilon(\varkappa,\mathbf{t},\nu,\nu_{\varkappa},\nu_{\varkappa\varkappa},S\nu),\\&\nu(\varkappa,0)=\pounds(\varkappa),\end{aligned}\tag{3.1}$$

where ${}^{c}\mathfrak{D}^{\theta}_{\mathbf{t}}$ is the fuzzy Caputo derivative with respect to $\mathbf{t}$, $0<\theta<1,(\varkappa,\mathbf{t})\in\mathtt{J}$, $\Upsilon:\mathfrak{I}\times\mathscr{F}_R\times\mathscr{F}_R\times\mathscr{F}_R\times\mathscr{F}_R\to\mathscr{F}_R$ and S is a linear integral operator given by:

$$S\nu=\int_0^{\mathbf{t}}k(\varkappa,\mathbf{t},s)\nu(s,\mathbf{t})ds,$$

where k is a sufficiently smooth crisp function.
The following lemma provides the equivalent formulations to equation (3.1).

Lemma 3.1. *The fuzzy initial value problem (3.1) is equivalent to one of the following integrals equations:*

Case (I): *If ν is $(i)-$gH differentiable, then*

$$\nu(\varkappa,\mathbf{t})=\pounds(\varkappa)+\mathfrak{I}^{\theta}_{\mathbf{t}}\left[\Upsilon(\varkappa,\mathbf{t},\nu,\nu_{\varkappa},\nu_{\varkappa\varkappa},S\nu)\right].\tag{3.2}$$

Case (II): *If ν is $(ii)-$gH differentiable, then*

$$\nu(\varkappa,\mathbf{t})=\pounds(\varkappa,)\ominus(-1)\mathfrak{I}^{\theta}_{\mathbf{t}}\left[\Upsilon(\varkappa,\mathbf{t},\nu,\nu_{\varkappa},\nu_{\varkappa\varkappa},S\nu)\right].\tag{3.3}$$

Proof. Applying integral operator $\mathfrak{I}^{\theta}_{\mathbf{t}}$ on both the sides of equation (3.1) and from the Proposition 2.14 and the Lemma 2.11 we get (3.2) and (3.3). Thus (3.1) and (3.2) - (3.3) are equivalent. □

Now we establish the existence and uniqueness of the fuzzy solution to the problem (3.1) using the Banach contraction principle.

Theorem 3.2. *Let $\mathfrak{C}(\mathtt{J},\mathscr{F}_R)$ be a Banach space of all continuous fuzzy-valued functions. Assume that the following hypotheses are fulfilled*

- $H1$: *For any $\nu,\omega \in \mathfrak{C}(\mathtt{J},\mathscr{F}_R)$, there exists a constant $\theta_1 \in (0,\theta)$ and real-valued positive functions $\mathfrak{K}_1(\varkappa,\mathbf{t}),\mathfrak{K}_2(\varkappa,\mathbf{t}) \in L^{\frac{1}{\theta}}(\mathtt{J},\mathbb{R}^+)$ such that*

$$\mathbb{D}\Big(\Upsilon(\varkappa,\mathbf{t},\nu,\nu_\varkappa,\nu_{\varkappa\varkappa},S\nu),\Upsilon(\omega,\mathbf{t},\omega,\omega_\varkappa,\omega_{\varkappa\varkappa},S\omega)\Big)\le \mathfrak{K}_1(\varkappa,\mathbf{t})\mathbb{D}\Big(\nu(\varkappa,\mathbf{t}),\omega(\varkappa,\mathbf{t})\Big)$$
$$+\mathfrak{K}_2(\varkappa,\mathbf{t})\mathbb{D}\Big(S\nu(\varkappa,\mathbf{t}),S\omega(\varkappa,\mathbf{t},)\Big).$$

- $H2$: *For the set of all non negative continuous function on $I=\{(\varkappa,\mathbf{t},s)\in\mathbb{R}\times\mathbb{R}\times\mathbb{R}: 0\le s\le t\le b\}$ there exist η_o such that*

$$\eta_o=\sup_{(\varkappa,\mathbf{t})\in J}\int_0^{\mathbf{t}}|k(\varkappa,\mathbf{t},s)|ds. <+\infty$$

and $\mathfrak{M}=\{\mathfrak{K}_1(\varkappa,\mathbf{t},s)+\eta_0 d\mathfrak{K}_2(\varkappa,\mathbf{t},s)\}_{L^{\frac{1}{\theta_1}}(\mathtt{J},\mathbb{R}^+)}$.

If

$$l^*=\frac{\mathfrak{M}d^{\theta-\theta_1}}{\Gamma(\theta)(\frac{\theta-\theta_1}{1-\theta_1})^{1-\theta_1}}<1,$$

then the problem (3.1) has a unique solution defined on $\mathtt{J}$.

Proof. We define the operator $\Xi:\mathfrak{C}(\mathtt{J},\mathscr{F}_R)\to\mathfrak{C}(\mathtt{J},\mathscr{F}_R)$ by

$$\Xi(\nu(\varkappa,\mathbf{t}))=\pounds(\varkappa)+\frac{1}{\Gamma(\theta)}\int_0^{\mathbf{t}}(\mathbf{t}-\rho)^{\theta-1}\Upsilon(\varkappa,\rho,\nu,\nu_\varkappa,\nu_{\varkappa\varkappa},S\nu)d\rho,$$

for all $(\varkappa,\mathbf{t})\in\mathtt{J}$.
Assume that $\nu\in\mathfrak{C}(\mathtt{J},\mathscr{F}_R)$. First we show that Ξ is a fuzzy continuous operator. Let us assume that $\{\nu_n\}$ be a sequence such that $\nu_n\to\nu$ as $n\to\infty$ in $\mathfrak{C}[\mathtt{J}]$. Then for each $(\varkappa,\mathbf{t})\in\mathtt{J}$. We have

$$\mathbb{D}\Big((\Xi\nu_n)(\varkappa,\mathbf{t}),(\Xi\nu)(\varkappa,\mathbf{t})\Big)$$

$$\le\frac{1}{\Gamma(\theta)}\int_0^{\mathbf{t}}(\mathbf{t}-\rho)^{\theta-1}\mathbb{D}\Big(\Upsilon(\varkappa,\rho,\nu_n,\nu_{n\varkappa},\nu_{n\varkappa\varkappa},S\nu_n),\Upsilon(\varkappa,\rho,\nu,\nu_\varkappa,\nu_{\varkappa\varkappa},S\nu)\Big)d\rho.$$

From (2.1) and $H1$ we have

$$\mathbb{D}\Big((\Xi\nu_n)(\varkappa,\mathbf{t}),(\Xi\nu)(\varkappa,\mathbf{t})\Big)$$

$$\le\frac{D^*\Big(\Upsilon(\varkappa,\rho,\nu_n,\nu_{n\varkappa},\nu_{n\varkappa\varkappa},S\nu_n),\Upsilon(\varkappa,\rho,\nu,\nu_\varkappa,\nu_{\varkappa\varkappa},S\nu)\Big)}{\Gamma(\theta)}\int_0^{\mathbf{t}}(\mathbf{t}-\rho)^{\theta-1}d\rho.$$

$$\le\frac{\mathbf{t}^\theta D^*\Big(\Upsilon(\varkappa,\rho,\nu_n,\nu_{n\varkappa},\nu_{n\varkappa\varkappa},S\nu_n),\Upsilon(\varkappa,\rho,\nu,\nu_\varkappa,\nu_{\varkappa\varkappa},S\nu)\Big)}{\Gamma(\theta+1)}.$$

Since ν is a fuzzy continuous function, we have

$$\mathbb{D}\Big((\Xi\nu_n)(\varkappa,\mathbf{t}),(\Xi\nu)(\varkappa,\mathbf{t})\Big)$$

$$\le\frac{\mathbf{t}^\theta D^*\Big(\Upsilon(\varkappa,\rho,\nu_n,\nu_{n\varkappa},\nu_{n\varkappa\varkappa},S\nu_n),\Upsilon(\varkappa,\rho,\nu,\nu_\varkappa,\nu_{\varkappa\varkappa},S\nu)\Big)}{\Gamma(\theta+1)}$$
$$\to 0 \text{ as } n\to\infty.$$

Hence, N^* is a fuzzy continuous operator.
Now, we transform the problem (3.1) into a fixed-point problem. Suppose that $\nu(\varkappa,\mathbf{t})$ is a (i)-gH differentiable. We shall prove that Ξ is a contraction mapping using Banach contraction principle theorem. For this, let $\nu,\omega\in\mathfrak{C}(\mathtt{J},\mathscr{F}_R)$ and $(\varkappa,\mathbf{t})\in\mathtt{J}$. Using Lemma 2.11, $H1$ and Theorem 2.15 we have that

$$\mathbb{D}\Big((\Xi\nu)(\varkappa,\mathbf{t}),(\Xi\omega)(\varkappa,\mathbf{t})\Big)$$
$$\le\frac{1}{\Gamma(\theta_1)}\int_0^{\mathbf{t}}(\mathbf{t}-\rho)^{\theta-1}\mathbb{D}\Big(\Upsilon(\varkappa,\mathbf{t},\nu,\nu_\varkappa,\nu_{\varkappa\varkappa},S\nu),\Upsilon(\omega,\mathbf{t},\omega,\omega_\varkappa,\omega_{\varkappa\varkappa},S\omega)\Big)d\rho$$

$$\le\frac{1}{\Gamma(\theta)}\int_0^{\mathbf{t}}(\mathbf{t}-\rho)^{\theta-1}$$
$$\Big[\mathfrak{K}_1(\varkappa,\mathbf{t})\mathbb{D}\Big(\nu(\varkappa,\mathbf{t}),\omega(\varkappa,\mathbf{t})\Big)+\mathfrak{K}_2(\varkappa,\mathbf{t})\mathbb{D}\Big(S\nu(\varkappa,\mathbf{t}),S\omega(\varkappa,\mathbf{t},)\Big)\Big]d\rho$$

$$\leq \frac{D^*\Big(\nu(\varkappa,\mathbf{t}),\omega(\varkappa,\mathbf{t})\Big)}{\Gamma(\theta)}\int_0^{\mathbf{t}}|(\mathbf{t}-\rho)|^{\theta-1}\big[\mathfrak{K}_1(\varkappa,\mathbf{t})+\eta_0 d\mathfrak{K}_2(\varkappa,t)\big]d\rho$$

$$\leq \frac{D^*\Big(\nu(\varkappa,\mathbf{t}),\omega(\varkappa,\mathbf{t})\Big)}{\Gamma(\theta)}\left(\int_0^{\mathbf{t}}(\mathbf{t}-\rho)^{\frac{\theta-1}{1-\theta_1}}d\rho\right)^{1-\theta_1}\left(\int_0^{\mathbf{t}}\big[\mathfrak{K}_1(\varkappa,\mathbf{t})+\eta_0 d\mathfrak{K}_2(\varkappa,\mathbf{t})\big]^{\frac{1}{\theta_1}}d\rho\right)^{\theta_1}.$$

This implies that

$$D^*\Big((\Xi\nu)(\varkappa,\mathbf{t}),(\Xi\omega)(\varkappa,\mathbf{t})\Big)\leq\frac{\mathfrak{M}d^{\theta-\theta_1}}{\Gamma(\theta)(\frac{\theta-\theta_1}{1-\theta_1})^{1-\theta_1}}D^*\Big(\nu(\varkappa,\mathbf{t}),\omega(\varkappa,\mathbf{t})\Big)$$

$$\leq l^*D^*\Big(\nu(\varkappa,\mathbf{t}),\omega(\varkappa,\mathbf{t})\Big).$$

Since $l^*<1$, the operator Ξ is a contraction mapping. Thus, according to Banach fixed point theorem, the problem (3.1) has a unique fuzzy solution ν defined on $\mathtt{J}$ which is the unique $(i)-g\mathtt{H}$ differentiable solution of the problem (3.1).
Let $\nu(\varkappa,\mathbf{t})$ be $(ii)-g\mathtt{H}$ differentiable. In this case, we define the operator $\Xi:\mathfrak{C}(\mathtt{J},\mathscr{F}_R)\to\mathfrak{C}(\mathtt{J},\mathscr{F}_R)$ by

$$\Xi(\nu(\varkappa,\mathbf{t}))=\pounds(\varkappa)\ominus(-1)\frac{1}{\Gamma(\theta)}\int_0^{\mathbf{t}}(\mathbf{t}-\rho)^{\theta-1}\Upsilon(\varkappa,\rho,\nu,\nu_\varkappa,\nu_{\varkappa\varkappa},S\nu)d\rho.$$

Similarly, this type of differentiability can be demonstrated and therefore, it is not included in the proof. □

4. Fuzzy Adomian Decomposition Method (FADM)

The Adomian decomposition method, introduced by G. Adomian in 1984 [32], is a straightforward and effective approach for solving both linear and nonlinear differential equations. This method serves as a powerful tool for approximating solutions to fuzzy differential equations by representing the solution as an infinite series, often converging to the exact solution. Although the Adomian decomposition method may have some limitations, such as being computationally intensive for complex problems, it is particularly valuable for problems that are challenging or unsolvable by other means. Recently, several researchers have employed this method to solve various linear and nonlinear systems within fuzzy frameworks. For instance, Pandit et al studied a population dynamic model of two species and solved it using the FADM [33]. Saeed et al. [34] applied FADM to solve some nonlinear FFPDE. Further we can see [35–39].
Consider the following FFPIDE:

$$^c\mathfrak{D}_{\mathbf{t}}^{\theta}\nu(\varkappa,\mathbf{t})=\Upsilon(\varkappa,\mathbf{t},\nu,\nu_\varkappa,\nu_{\varkappa\varkappa},S\nu)=L\nu(\varkappa,\mathbf{t})+A\nu(\varkappa,\mathbf{t})+I\nu(\varkappa,\mathbf{t}), \tag{4.1}$$

with fuzzy initial condition $\nu(\varkappa,0)=\pounds(\varkappa)$. Where L is a linear operator, A represents the nonlinear operator and I is an integral operator. The Adomian supposes that the unknown function $\nu(\varkappa,\mathbf{t})$ can be written by a series as

$$\nu(\varkappa,\mathbf{t})=\sum_{k=o}^{\infty}\nu_k(\varkappa,\mathbf{t}). \tag{4.2}$$

The nonlinear operator is represented by an infinite series as

$$A\nu=\sum_{k=0}^{\infty}M_k,$$

where M_k is Adomian polynomials given by

$$M_k=\frac{1}{k!}\frac{\partial^k}{\partial\beta^k}\left[A(\sum_i^{\infty}\beta^i\nu_i)\right]_{\beta=0}.$$

Finally, to compute the terms of the series $\sum_{k=o}^{\infty}\nu_k$, we use the following iterated scheme

- If ν is $(i)-g\mathtt{H}$ differentiable, then

$$\begin{aligned}&\nu_0(\varkappa,\mathbf{t})=\pounds(\varkappa),\\&\nu_{k+1}(\varkappa,\mathbf{t})=\mathfrak{I}_{\mathbf{t}}^{\theta}\big(L\nu_k+\sum_{k=0}^{\infty}M_k+I\nu_k\big).\end{aligned} \tag{4.3}$$

- If ν is $(ii)-g\mathtt{H}$ differentiable, then

$$\begin{aligned}&\nu_0(\varkappa,\mathbf{t})=\pounds(\varkappa),\\&\nu_{k+1}(\varkappa,\mathbf{t})=\ominus(-1)\mathfrak{I}_t^{\theta}\big(L\nu_k+\sum_{k=0}^{\infty}M_k+I\nu_k\big).\end{aligned} \tag{4.4}$$

4.1. Convergence of FADM

Here, we aim to thoroughly demonstrate the convergence of the series solution that is obtained from equation (4.2), by analyzing its structure and applying appropriate mathematical techniques to ensure that the solution behaves as expected under the given conditions.

Theorem 4.1. *Assume that the operators L, N and I defined in Equation (4.1) satisfy the following Lipschitz conditions with constants L_1, L_2 and L_3.*

$$\mathbb{D}(Lv_k(\varkappa,\mathbf{t}),Lv_{k-1}(\varkappa,\mathbf{t}))\le L_1,$$
$$\mathbb{D}(Av_k(\varkappa,\mathbf{t}),Av_{k-1}(\varkappa,\mathbf{t}))\le L_2,$$
$$\mathbb{D}(Iv_k(\varkappa,\mathbf{t}),Iv_{k-1}(\varkappa,\mathbf{t}))\le L_3.$$

The series solution (4.2) of Equations (4.1) converges to the exact solution if $0<L_1+L_2+L_3<1$ and $D(v_k,0)<\infty, k\ge 0$, where v_k are given by (4.3) and (4.4).

Proof. Here we will prove the theorem for case v is $(i)-g\mathtt{H}$ differentiable. The proof of case v $(ii)-g\mathtt{H}$ differentiable is similar, so it will omitted.
Let S_n be the partial sum of the series $S_n=\sum_{k=o}^{n}v_k(\varkappa,t)$. We prove that S_n is a Cauchy sequence in the Banach space $\mathfrak{C}(\mathtt{J},\mathscr{F}_R)$. By hypothesis, we get

$$\begin{aligned}
&D^*(S_n(\varkappa,\mathbf{t}),S_m(\varkappa,\mathbf{t}))\\
&=\sup_{(\varkappa,\mathbf{t})\in\mathtt{J}}\mathbb{D}\left(\sum_{k=m+1}^{n}v_k(\varkappa,\mathbf{t}),0\right)\\
&=\sup_{(\varkappa,\mathbf{t})\in\mathtt{J}}\mathbb{D}\left(\frac{1}{\Gamma(\theta)}\int_0^{\mathbf{t}}(\mathbf{t}-\rho)^{\theta-1}\big(LS_{n-1}(\varkappa,\rho)+AS_{n-1}(\varkappa,\rho)\right.\\
&\quad+IS_{n-1}(\varkappa,\rho)\big)d\rho,\frac{1}{\Gamma(\theta)}\int_0^{\mathbf{t}}(\mathbf{t}-\rho)^{\theta-1}\big(LS_{m-1}(\varkappa,\rho)+AS_{m-1}(\varkappa,\rho)\\
&\quad\left.+IS_{m-1}(\varkappa,\rho)\big)d\rho\right)\\
&\le\frac{1}{\Gamma(\theta)}\sup_{(\varkappa,\mathbf{t})\in\mathtt{J}}\Big(\mathbb{D}\big(LS_{n-1}(\varkappa,\mathbf{t}),LS_{m-1}(\varkappa,\mathbf{t})\big)+\mathbb{D}\big(AS_{n-1}(\varkappa,\mathbf{t}),AS_{m-1}(\varkappa,\mathbf{t})\big)\\
&\quad+\mathbb{D}\big(IS_{n-1}(\varkappa,\mathbf{t}),IS_{m-1}(\varkappa,\mathbf{t})\big)\Big)\int_0^{\mathbf{t}}|\mathbf{t}-\rho|^{\theta-1}d\rho\\
&\le CD^*\big(S_{n-1}(\varkappa,\mathbf{t}),S_{m-1}(\varkappa,\mathbf{t})\big),
\end{aligned}$$

where $C=(L_1+L_2+L_3)\frac{b^{\theta}}{\Gamma(\theta+1)}$.

If $n=m+1$. We get

$$\begin{aligned}
D^*(S_n(\varkappa,\mathbf{t}),S_m(\varkappa,\mathbf{t})\le&CD^*\big(S_m(\varkappa,\mathbf{t}),S_{m-1}(\varkappa,\mathbf{t})\big)\\
\le&C^2D^*\big(S_{m-1}(\varkappa,\mathbf{t}),S_{m-2}(\varkappa,\mathbf{t})\big)\\
&\vdots\\
\le&C^mD^*\big(S_1(\varkappa,\mathbf{t}),S_0(\varkappa,\mathbf{t})\big).
\end{aligned}$$

Now, for $n>m$, we have

$$\begin{aligned}
D^*(S_n(\varkappa,\mathbf{t}),S_m(\varkappa,\mathbf{t}))\le&D^*\big(S_m(\varkappa,\mathbf{t}),S_{m+1}(\varkappa,\mathbf{t})\big)+\cdots+D^*\big(S_n(\varkappa,\mathbf{t}),S_{n+1}(\varkappa,\mathbf{t})\big)\\
\le&\frac{C^m}{1-C}D^*\big(v_1(\varkappa,\mathbf{t}),0\big).
\end{aligned}$$

Since v is bounded, as $m\to\infty$, then $D^*(S_n(\varkappa,\mathbf{t}),S_m(\varkappa,\mathbf{t}))\to\infty$. Hence, S_n is a Cauchy sequence in $\mathfrak{C}(\mathfrak{I},\mathscr{F}_R)$ and therefore, the series converges and the proof is complete. □

4.2. FADM for solving fuzzy fractional parabolic IDEs

Now, we employ the FADM to analyze the following fuzzy fractional parabolic IDEs

$$\begin{aligned}
&{}^{c}\mathfrak{D}_{\mathbf{t}}^{\theta}v(\varkappa,\mathbf{t})=v_{\varkappa\varkappa}(\varkappa,\mathbf{t})+\int_0^{\mathbf{t}}k(\varkappa,\mathbf{t},s)v(s,\mathbf{t})ds+\mathfrak{h}(\varkappa,\mathbf{t}),\\
&v(\varkappa,0)=\pounds(\varkappa),
\end{aligned}\tag{4.5}$$

where ${}^{c}\mathfrak{D}_{\mathbf{t}}^{\theta}$ is the fuzzy Caputo derivative with respect to $\mathbf{t}$, $0<\theta<1,(\varkappa,t)\in\mathtt{J}$, k is a crisp function whose sign does not change in $\mathtt{J}$ and $\pounds,\mathfrak{h}$ are known crisp or fuzzy valued functions.
By applying the operator $\mathfrak{I}_{\mathbf{t}}^{\theta}$ to both side of equation (4.5), we get

$$v(\varkappa,\mathbf{t})\ominus v(\varkappa,0)=\mathfrak{I}_{\mathbf{t}}^{\theta}\left[v_{\varkappa\varkappa}(\varkappa,\mathbf{t})+\int_0^{\mathbf{t}}k(\varkappa,\mathbf{t},s)v(s,\mathbf{t})ds\oplus\mathfrak{h}(\varkappa,\mathbf{t})\right].$$

If ν is $(i)-g\mathtt{H}$ differentiable, then

$$\nu(\varkappa,\mathbf{t})=\nu(\varkappa,0)+\mathfrak{I}_{\mathbf{t}}^{\theta}\left[\nu_{\varkappa\varkappa}(\varkappa,\mathbf{t})+\int_0^{\mathbf{t}}k(\varkappa,\mathbf{t},s)\nu(s,\mathbf{t})ds\oplus\mathfrak{h}(\varkappa,\mathbf{t})\right] \tag{4.6}$$

and if ν is $(ii)-g\mathtt{H}$ differentiable, then

$$\nu(\varkappa,\mathbf{t})=\nu(\varkappa,0)\ominus(-1)\mathfrak{I}_{\mathbf{t}}^{\theta}\left[\nu_{\varkappa\varkappa}(\varkappa,\mathbf{t})+\int_0^{\mathbf{t}}k(\varkappa,\mathbf{t},s)\nu(s,\mathbf{t})ds\oplus\mathfrak{h}(\varkappa,\mathbf{t})\right]. \tag{4.7}$$

Now we study four cases to find the numerical solution:

Case (1): Let ν be $(i)-g\mathtt{H}$ differentiable and $k(\varkappa,\mathbf{t},s)$ be a positive real function, then the parametric form of equation (4.6) is:

$$\underline{\nu}(\varkappa,\mathbf{t},\varsigma)=\underline{\pounds}(\varkappa,\varsigma)+\mathfrak{I}_{\mathbf{t}}^{\theta}\left[\underline{\nu}_{\varkappa\varkappa}(\varkappa,\mathbf{t},\varsigma)+\int_0^{\mathbf{t}}k(\varkappa,\mathbf{t},s)\underline{\nu}(s,\mathbf{t},\varsigma)ds+\underline{\mathfrak{h}}(\varkappa,\mathbf{t},\varsigma)\right],$$

$$\overline{\nu}(\varkappa,\mathbf{t},\varsigma)=\overline{\pounds}(\varkappa,\varsigma)+\mathfrak{I}_{\mathbf{t}}^{\theta}\left[\overline{\nu}_{\varkappa\varkappa}(\varkappa,\mathbf{t},\varsigma)+\int_0^{\mathbf{t}}k(\varkappa,\mathbf{t},s)\overline{\nu}(s,\mathbf{t},\varsigma)ds+\overline{\mathfrak{h}}(\varkappa,\mathbf{t},\varsigma)\right].$$

The standard Adomian Method assumes that the solution $\nu(\varkappa,\mathbf{t},\varsigma)$ can be written as the following series

$$\underline{\nu}(\varkappa,\mathbf{t},\varsigma)=\sum_{k=o}^{\infty}\underline{\nu}_k(\varkappa,\mathbf{t},\varsigma),$$

$$\overline{\nu}(\varkappa,\mathbf{t},\varsigma)=\sum_{k=o}^{\infty}\overline{\nu}_k(\varkappa,\mathbf{t}\varsigma).$$

Finally, to calculate the terms of the above series, we use the following iterated scheme

$$\underline{\nu}_0(\varkappa,\mathbf{t},\varsigma)=\underline{\pounds}(\varkappa,\varsigma)+\mathfrak{I}_{\mathbf{t}}^{\theta}\underline{\mathfrak{h}}(\varkappa,\mathbf{t},\varsigma),$$

$$\underline{\nu}_{k+1}(\varkappa,\mathbf{t},\varsigma)=\mathfrak{I}_{\mathbf{t}}^{\theta}\left[\underline{\nu}_{\varkappa\varkappa}(\varkappa,\mathbf{t},\varsigma)+\int_0^{\mathbf{t}}k(\varkappa,\mathbf{t},s)\underline{\nu}(s,\mathbf{t},\varsigma)ds\right]$$

and

$$\overline{\nu}_0(\varkappa,\mathbf{t},\varsigma)=\overline{\pounds}(\varkappa,\varsigma)+\mathfrak{I}_{\mathbf{t}}^{\theta}\overline{\mathfrak{h}}(\varkappa,\mathbf{t},\varsigma),$$

$$\overline{\nu}_{k+1}(\varkappa,\mathbf{t},\varsigma)=\mathfrak{I}_{\mathbf{t}}^{\theta}\left[\overline{\nu}_{\varkappa\varkappa}(\varkappa,\mathbf{t},\varsigma)+\int_0^{\mathbf{t}}k(\varkappa,\mathbf{t},s)\overline{\nu}(s,\mathbf{t},\varsigma)ds\right].$$

Case (2): Let ν be $(i)-g\mathtt{H}$ differentiable and $k(\varkappa,\mathbf{t},s)$ be a negative real function, then the parametric form of equation (4.6) is:

$$\underline{\nu}(\varkappa,\mathbf{t},\varsigma)=\underline{\pounds}(\varkappa,\varsigma)+\mathfrak{I}_{\mathbf{t}}^{\theta}\left[\underline{\nu}_{\varkappa\varkappa}(\varkappa,\mathbf{t},\varsigma)+\int_0^{\mathbf{t}}k(\varkappa,\mathbf{t},s)\overline{\nu}(s,\mathbf{t},\varsigma)ds+\underline{\mathfrak{h}}(\varkappa,\mathbf{t},\varsigma)\right],$$

$$\overline{\nu}(\varkappa,\mathbf{t},\varsigma)=\overline{\pounds}(\varkappa,\varsigma)+\mathfrak{I}_{\mathbf{t}}^{\theta}\left[\overline{\nu}_{\varkappa\varkappa}(\varkappa,\mathbf{t},\varsigma)+\int_0^{\mathbf{t}}k(\varkappa,\mathbf{t},s)\underline{\nu}(s,\mathbf{t},\varsigma)ds+\overline{\mathfrak{h}}(\varkappa,\mathbf{t},\varsigma)\right].$$

According to the above process , we get the solutions $\underline{\nu}(\varkappa,\mathbf{t},\varsigma)$ and $\overline{\nu}(\varkappa,\mathbf{t},\varsigma)$.

Case (3): Let ν be $(ii)-g\mathtt{H}$ differentiable and $k(\varkappa,\mathbf{t},s)$ be positive real function, then the parametric form of equation (4.7) is:

$$\underline{\nu}(\varkappa,\mathbf{t},\varsigma)=\underline{\pounds}(\varkappa,\varsigma)+\mathfrak{I}_{\mathbf{t}}^{\theta}\left[\overline{\nu}_{\varkappa\varkappa}(\varkappa,\mathbf{t},\varsigma)+\int_0^{\mathbf{t}}k(\varkappa,\mathbf{t},s)\overline{\nu}(s,\mathbf{t},\varsigma)ds+\overline{\mathfrak{h}}(\varkappa,\mathbf{t},\varsigma)\right],$$

$$\overline{\nu}(\varkappa,\mathbf{t},\varsigma)=\overline{\pounds}(\varkappa,\varsigma)+\mathfrak{I}_{\mathbf{t}}^{\theta}\left[\underline{\nu}_{\varkappa\varkappa}(\varkappa,\mathbf{t},\varsigma)+\int_0^{\mathbf{t}}k(\varkappa,\mathbf{t},s)\underline{\nu}(s,\mathbf{t},\varsigma)ds+\underline{\mathfrak{h}}(\varkappa,\mathbf{t},\varsigma)\right].$$

Then in the same way to previous case, we obtain the solutions $\underline{\nu}(\varkappa,\mathbf{t},\varsigma)$ and $\overline{\nu}(\varkappa,\mathbf{t},\varsigma)$.

Case (4): Let ν be $(ii)-g\mathtt{H}$ differentiable and $k(\varkappa,\mathbf{t},s)$ be negative real function, then the parametric form of equation (4.7) is:

$$\underline{\nu}(\varkappa,\mathbf{t},\varsigma)=\underline{\pounds}(\varkappa,\varsigma)+\mathfrak{I}_{\mathbf{t}}^{\theta}\left[\overline{\nu}_{\varkappa\varkappa}(\varkappa,\mathbf{t},\varsigma)+\int_0^{\mathbf{t}}k(\varkappa,\mathbf{t},s)\underline{\nu}(s,\mathbf{t},\varsigma)ds+\overline{\mathfrak{h}}(\varkappa,\mathbf{t},\varsigma)\right],$$

$$\overline{\nu}(\varkappa,\mathbf{t},\varsigma)=\overline{\pounds}(\varkappa,\varsigma)+\mathfrak{I}_{\mathbf{t}}^{\theta}\left[\underline{\nu}_{\varkappa\varkappa}(\varkappa,\mathbf{t},\varsigma)+\int_0^{\mathbf{t}}k(\varkappa,\mathbf{t},s)\overline{\nu}(s,\mathbf{t},\varsigma)ds+\underline{\mathfrak{h}}(\varkappa,\mathbf{t},\varsigma)\right].$$

Therefore, by applying the method discussed in detail in the previous case, we get the solutions $\underline{\nu}(\varkappa,\mathbf{t},\varsigma)$ and $\overline{\nu}(\varkappa,\mathbf{t},\varsigma)$.

Remark 4.2. *In this subsection, the fuzzy fractional parabolic IDEs were converted into a system of scalar differential equations via ς-level representations and subsequently solved using the FADM. While the scalarized problems yield unique solutions under standard conditions, it is important to note that this does not necessarily guarantee the uniqueness of the original fuzzy solution. This discrepancy arises due to the inherent properties of fuzzy arithmetic, particularly in the multiplication of fuzzy numbers, which may result in non-uniqueness or bifurcation of solutions. In such cases, multiple fuzzy-valued functions can correspond to the same scalar ς-level solutions. A similar observation has been made in [40], where a numerical scheme for fuzzy fractional models resulted in bifurcated solutions depending on the nature of fuzzy multiplication. To address this challenge, we adopt the concept of maximal solutions as proposed in [29]. A maximal fuzzy solution is one that dominates all other admissible fuzzy solutions pointwise, providing an upper bound to the solution set. This approach not only accommodates the possibility of non-uniqueness but also strengthens the interpretation and reliability of the obtained fuzzy solution.*
Hence, while our method guarantees the uniqueness of the scalar components, we emphasize that the full fuzzy solution may admit multiple interpretations. The incorporation of maximal solutions allows for a well-defined framework within which these solutions can be understood and compared.

5. Applications and Simulations

In this chapter, we provide a series of numerical examples corresponding to each of the cases discussed in the previous chapter, with the aim of illustrating the applicability and effectiveness of the proposed methods.

Example 5.1. *Consider the following fuzzy fractional parabolic IDEs:*

$$\begin{cases} {}^{c}\mathfrak{D}_{\mathbf{t}}^{\theta} \nu(\varkappa,\mathbf{t}) = \nu_{\varkappa\varkappa}(\varkappa,\mathbf{t}) + \int_0^{\mathbf{t}} (\mathbf{t}-s)\nu(\varkappa,s)ds + \mathfrak{h}(\varkappa,\mathbf{t}), \ \ 0 \leq \varkappa, \mathbf{t} \leq 1, \\ \nu(\varkappa,0) = 0, \end{cases} \tag{5.1}$$

where

$$\mathfrak{h}(\varkappa,\mathbf{t}) = [\varsigma+1, 5-3\varsigma]cos\varkappa[\frac{\mathbf{t}^{1-\theta}}{\Gamma(2-\theta)} + \mathbf{t} - \frac{\mathbf{t}^3}{6}].$$

The exact solution of Equation (5.1) is $\nu(\varkappa,\mathbf{t}) = [\varsigma+1, 5-3\varsigma]\mathbf{t}cos\varkappa$. Figure 5.1 represents (a) the exact solutions and its ${}^{c}\mathfrak{D}_{\mathbf{t}}^{\theta}\nu(\varkappa,t)$ plotted in (b) with $\theta = \frac{3}{4}$ and different values of uncertainty $\varsigma \in [0,1]$. As it is seen, $\nu(\varkappa,t)$ is $(i)-g$H-differentiable, this problem has been solved by FADM for $k = 2$. Figure 5.2 clearly shows that the numerical solution converges to the exact solution. This demonstrates that the proposed method is highly efficient for obtaining numerical solutions to these problems.

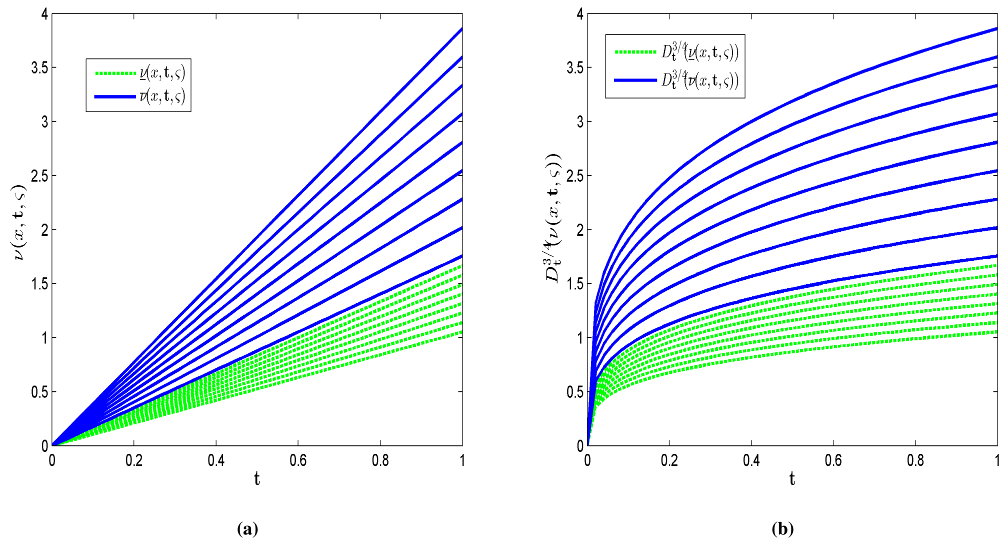


Figure 5.1: 2D plots of $\nu(\varkappa,\mathbf{t})$ and ${}^{c}\mathfrak{D}_{\mathbf{t}}^{\frac{3}{4}}\nu(\varkappa,\mathbf{t})$ of ς-level of Example 5.1 at $\varkappa = \frac{1}{2}$.

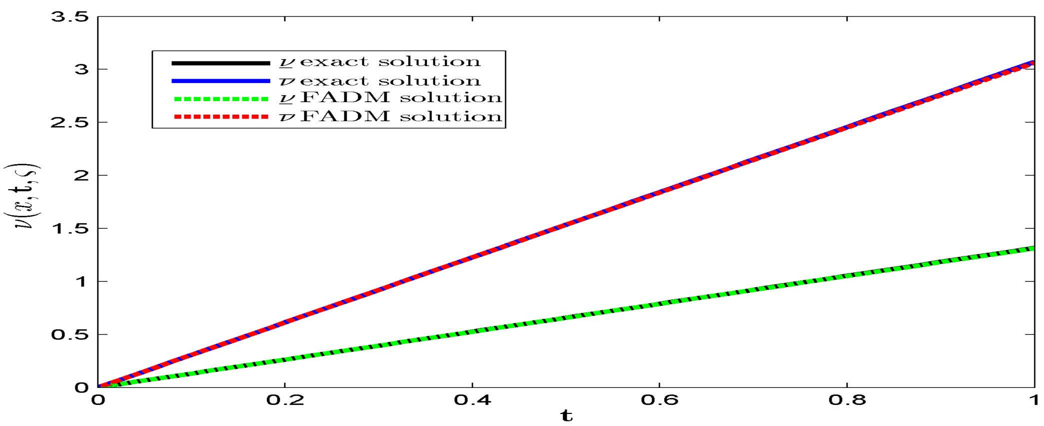


Figure 5.2: 2D plot of ς-level representations of exact and FADM solution of Example 5.1 at $\varkappa = \frac{1}{2}$ and $\varsigma = \frac{1}{2}$.

Example 5.2. *Let us Consider the fuzzy parabolic IDEs of fractional order as:*

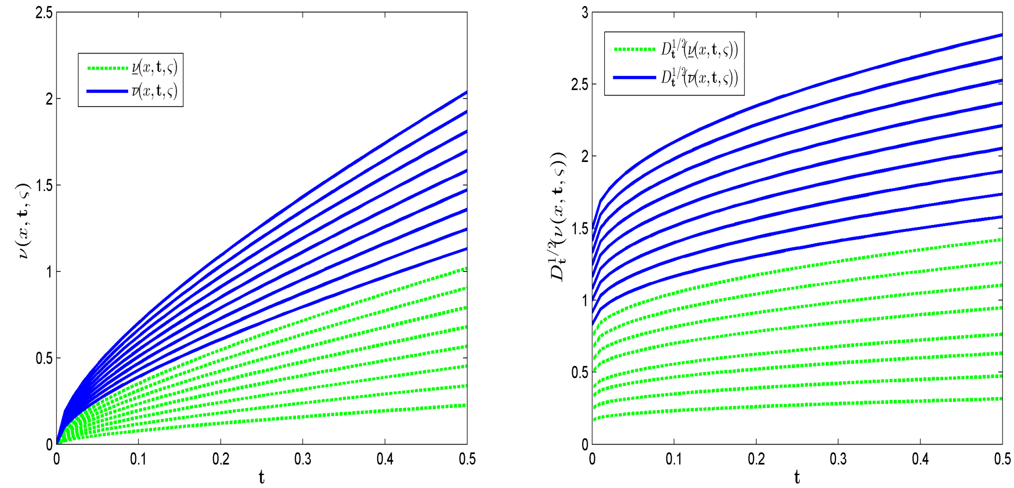
$$\begin{cases} {}^{c}\mathfrak{D}_{\mathbf{t}}^{\theta} \nu(\varkappa,\mathbf{t}) = \nu_{\varkappa\varkappa}(\varkappa,\mathbf{t}) + \int_0^{\mathbf{t}} -\varkappa(\mathbf{t}-s)\nu(\varkappa,s)ds + \mathfrak{h}(\varkappa,\mathbf{t}), \ \ 0 \leq \varkappa \leq 1, 0 \leq \mathbf{t} \leq 0.5, \\ \nu(\varkappa,0) = 0, \end{cases} \tag{5.2}$$

where

$$\mathfrak{h}(\varkappa,\mathbf{t}) = B_1(1-\varkappa^2)\left[\frac{\mathbf{t}^{1-\theta}}{\Gamma(2-\theta)} + \Gamma(\theta+1)\right] + B_1(\mathbf{t}+\mathbf{t}^{\theta}) + B_2\left[\frac{\mathbf{t}^3}{6} + \frac{\mathbf{t}^{2+\theta}}{(1+\theta)(2+\theta)}\right]$$

and $B_1 = [\varsigma, 2-\varsigma]$, $B_2 = [2-\varsigma, \varsigma]$.

The exact solution of Equation (5.2) is $\nu(\varkappa,\mathbf{t}) = B_1(1-\varkappa^2)(\mathbf{t}+\mathbf{t}^{\theta})$. Figure 5.3 represents (a) the exact solutions and its ${}^{c}\mathfrak{D}_{\mathbf{t}}^{\theta}\nu(\varkappa,\mathbf{t})$ plotted in (b) with $\theta = \frac{1}{2}$ and different values of uncertainty $\varsigma \in [0,1]$. From Figure 5.3, $\nu(\varkappa,t)$ is $(i)-g\mathrm{H}$-differentiable. By applying the FADM for $k=2$, we get the numerical results shown in Figure 5.4.



(a) **(b)**

Figure 5.3: 2D plots of $\nu(\varkappa,\mathbf{t})$ and ${}^{c}\mathfrak{D}_{\mathbf{t}}^{\frac{1}{2}}\nu(\varkappa,\mathbf{t})$ of ς-level of Example 5.2 at $\varkappa = \frac{1}{4}$.

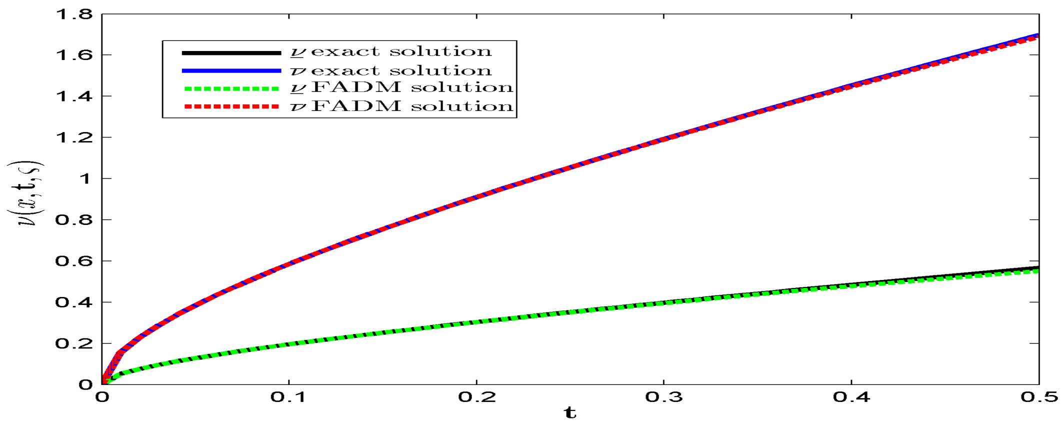


Figure 5.4: 2D graph of ς-level representations of exact and FADM solution of Example 5.2 at $\varkappa = \frac{1}{4}$ and $\varsigma = \frac{1}{2}$.

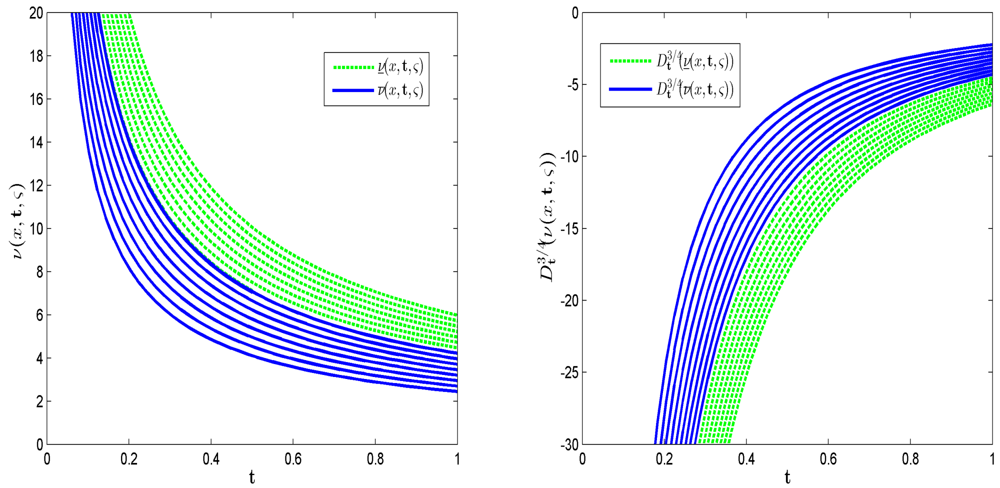
Example 5.3. *Now we consider another fuzzy fractional parabolic IDEs under the initialcondition as:*

$$\begin{cases} {}^{c}\mathfrak{D}_{\mathbf{t}}^{\theta}\nu(\varkappa,\mathbf{t}) = \nu_{\varkappa\varkappa}(\varkappa,\mathbf{t}) + \int_0^{\mathbf{t}} \mathbf{t}\nu(\varkappa,s)ds + \mathfrak{h}(\varkappa,\mathbf{t}), \quad 0 \le \varkappa,\mathbf{t} \le 1, \\ \nu(\varkappa,0) = 0, \end{cases} \tag{5.3}$$

where

$$\mathfrak{h}(\varkappa,\mathbf{t}) = [2+3\varsigma, 8-3\varsigma]sin\varkappa\left[\frac{\Gamma(1-\theta)}{\Gamma(2-\theta)}\mathbf{t}^{-2\theta} + \mathbf{t}^{-\theta} - \frac{\mathbf{t}^{2-\theta}}{1-\theta}\right].$$

The exact solution of Equation (5.3) is $\nu(\varkappa,\mathbf{t}) = [2+3\varsigma, 8-3\varsigma]sin\varkappa\mathbf{t}^{-\theta}$. Figure 5.5 represents (a) the exact solutions and its ${}^{c}\mathfrak{D}_{\mathbf{t}}^{\theta}\nu(\varkappa,\mathbf{t})$ plotted in (b) with $\theta = \frac{3}{4}$ and different values of uncertainty $\varsigma \in [0,1]$. As it is seen, $\nu(\varkappa,\mathbf{t})$ is $(ii)-g\mathrm{H}$-differentiable, so by applying the FADM discussed in detail in Subsection 4.2, with $k=3$, we have the numerical results shown in Figure 5.6.



(a) (b)

Figure 5.5: 2D plots of $\nu(\varkappa,\mathbf{t})$ and ${}^{c}\mathfrak{D}_{\mathbf{t}}^{\frac{3}{4}}\nu(\varkappa,\mathbf{t})$ of ς-level of Example 5.3 at $\varkappa = 1$.

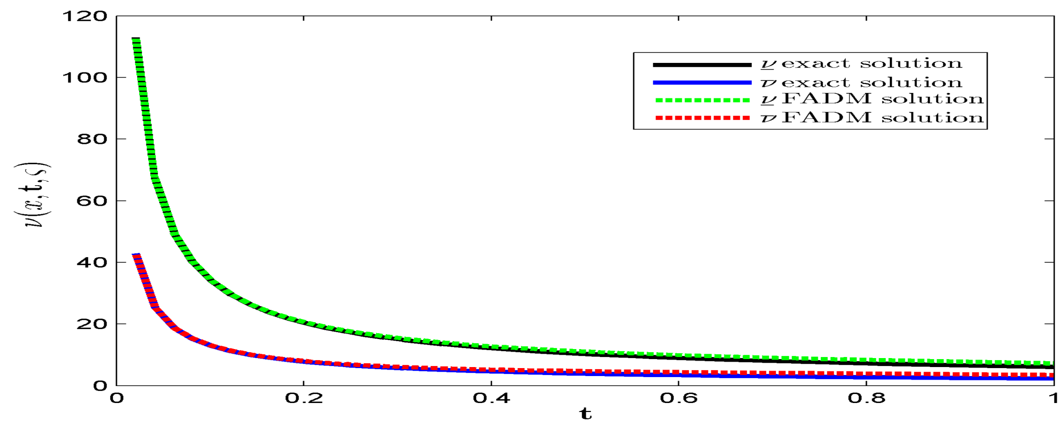


Figure 5.6: 2D graph of ς-level representations of exact and FADM solution of Example 5.3 at $\varkappa = 1$ and $\varsigma = \frac{1}{4}$.

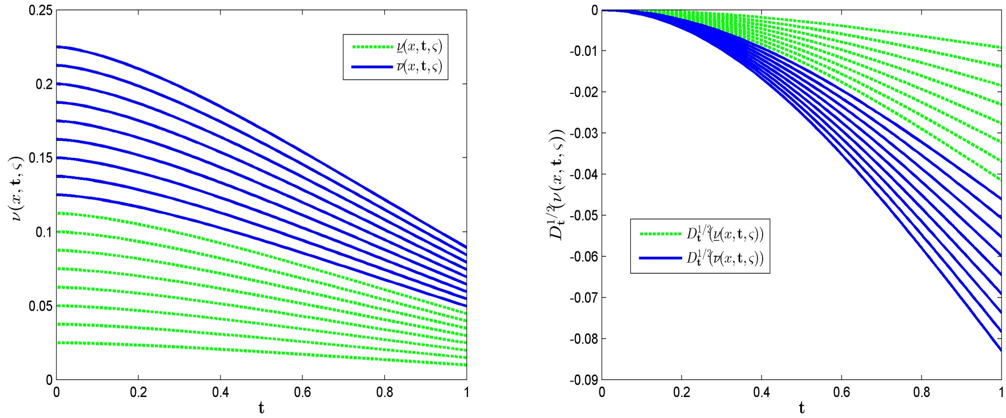
Example 5.4. *Consider the following fuzzy fractional parabolic IDEs:*

$$\begin{cases} {}^{c}\mathfrak{D}_{\mathbf{t}}^{\theta}\nu(\varkappa,\mathbf{t}) = \nu_{\varkappa\varkappa}(\varkappa,\mathbf{t}) + \int_0^{\mathbf{t}} -\nu(\varkappa,s)ds, \quad 0 \le \varkappa,\mathbf{t} \le 1, \\ \nu(\varkappa,0) = [0.5\varsigma, 1-0.5\varsigma]\varkappa. \end{cases} \tag{5.4}$$

The exact solution of Equation (5.4) is given by $\nu(\varkappa,\mathbf{t}) = [0.5\varsigma, 1-0.5\varsigma]\varkappa E_{\theta+1}(-\mathbf{t}^{\theta+1})$. Figure 5.7 represents (a) the exact solutions and its ${}^{c}\mathfrak{D}_{\mathbf{t}}^{\theta}\nu(\varkappa,\mathbf{t})$ plotted in (b) with $\theta = \frac{1}{2}$ and different values of uncertainty $\varsigma \in [0,1]$. Figure 5.7 shows that $\nu(\varkappa,\mathbf{t})$ is $(ii)-g\mathrm{H}$-differentiable. Thus, by applying the FADM discussed in detail in case 4, for $k = 4$. Figure 5.8 shows the exact and approximate results.
The results in Figure 5.8 show that the numerical solution converges to the exact solution. This confirms that the proposed method is highly efficient for obtaining numerical solutions to such problems.

6. Conclusion

In this article, we have established sufficient conditions for the existence and uniqueness of solutions to FFPIDEs. Additionally, we applied the FADM to obtain approximate solutions for the problem taking into account the type of differentiability. Also, the convergence of FADM to the exact solution is proved. Four illustrative examples of fuzzy fractional parabolic IDEs are provided to validate the effectiveness and performance of our method. The proposed method provides reliable series solutions with continuity depending on the fuzzy fractional derivative. As the number of decomposed terms increases, the numerical solution converges. As a future extension, this method could be applied to two-dimensional fuzzy fractional parabolic IDEs with both constant and variable coefficients and could also be expanded to address nonlinear problems.



(a) **(b)**

Figure 5.7: 2D plots of $\nu(\varkappa,\mathbf{t})$ and ${}^{c}\mathfrak{D}_{\mathbf{t}}^{\frac{1}{2}}\nu(\varkappa,\mathbf{t})$ of ς-level of Example 5.4 at $\varkappa=\frac{1}{4}$.

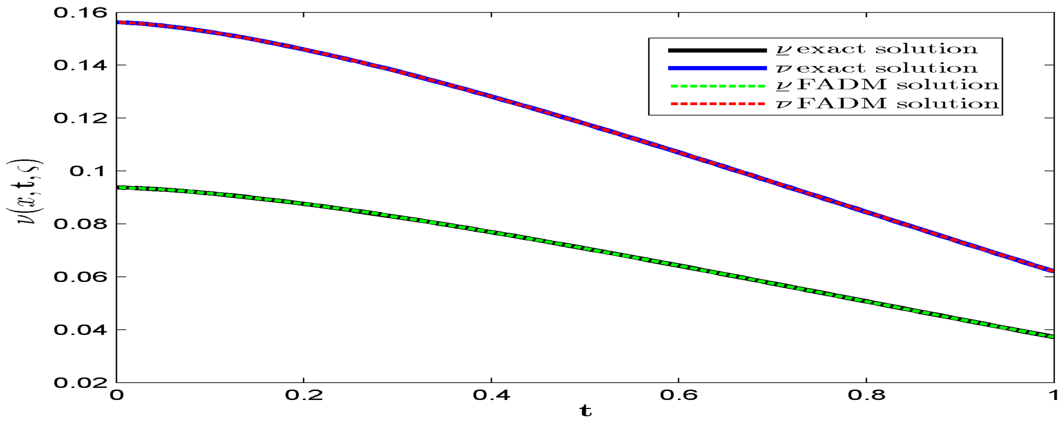


Figure 5.8: 2D graph of ς-level representations of exact and FADM solution of Example 5.4 at $\varkappa=\frac{1}{4}$ and $\varsigma=\frac{3}{4}$.

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Research Article

Newton-type Inequalities for Fractional Integrals by Various Function Classes

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Abstract

The authors of the paper examine some Newton-type inequalities for various function classes using Riemann-Liouville fractional integrals. Namely, we establish some Newton-type inequalities for bounded functions by fractional integrals. In addition, we construct some fractional Newton-type inequalities for Lipschitzian functions. Furthermore, we offer some Newton-type inequalities by fractional integrals of bounded variation. Finally, we provide our results by using special cases of theorems and obtained examples.

1. Introduction & Preliminaries

Inequality theory is a crucial subject in many branches of mathematics with numerous number of applications. Many mathematicians have established the Hermite-Hadamard, Simpson, and Newton-type inequalities and they are very interested in generalizing and extending it to the case of various classes of functions, including s-convex functions, quasi-convex functions, log-convex functions, etc. In recent years, fractional calculus has increased interest because of the its demonstrated applications in a range of the inequality theory on convex functions. It can be obtained the bounds of new formulas by using the Hermite-Hadamard-type, Simpson-type inequality, and Newton-type inequality. Simpson-type inequalities are derived from Simpson's rules and take the following form of inequalities:

i. Simpson's $1/3$ rule, or Simpson's quadrature formula:

$$\int_a^b f(x)\,dx \approx \frac{b-a}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right].$$

ii. The Simpson's second formula, often known as the Simpson's 3/8 rule, or the Newton-Cotes quadrature formula:

$$\int_a^b f(x)\,dx \approx \frac{b-a}{8}\left[f(a)+3f\left(\frac{2a+b}{3}\right)+3f\left(\frac{a+2b}{3}\right)+f(b)\right].$$

The most popular Newton-Cotes quadrature using a three-point Simpson-type inequality is as follows:

Theorem 1.1. *If $f:[a,b]\to\mathbb{R}$ is a four times differentiable and continuous function on (a,b), and let $\left\|f^{(4)}\right\|_\infty = \sup_{x\in(a,b)}\left|f^{(4)}(x)\right| < \infty$, then the following inequality holds:*

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right| \le \frac{1}{2880}\left\|f^{(4)}\right\|_\infty (b-a)^4.$$

According to the Simpson $3/8$ inequality, the Simpson $3/8$ rule is a classical closed type quadrature rule is as follows:

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Theorem 1.2. *Note that $f:[a,b]\to\mathbb{R}$ is a four times differentiable and continuous function on (a,b), and $\left\|f^{(4)}\right\|_{\infty}=\sup_{x\in(a,b)}\left|f^{(4)}(x)\right|<\infty$. Then, one has the inequality*

$$\left|\frac{1}{8}\left[f(a)+3f\left(\frac{2a+b}{3}\right)+3f\left(\frac{a+2b}{3}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right|\leq\frac{1}{6480}\left\|f^{(4)}\right\|_{\infty}(b-a)^4.$$

Definition 1.3 (See [1]). *Suppose that I is an interval of real numbers. Then, a function $f:I\to\mathbb{R}$ is said to be convex, if*

$$f(tx+(1-t)y)\leq tf(x)+(1-t)f(y)$$

is valid $\forall x,y\in I$ and $\forall t\in[0,1]$.

The three-point Newton-Cotes quadrature rule is the basis for Simpson's second rule. Results for three-step quadratic kernel computations are commonly referred to as Newton-type results. It is known from the literature that these results are Newton-type inequalities. There have been several mathematicians who have been considered to Newton-type inequalities. For example, in paper [2], some Newton-type inequalities are proved for the case of functions whose second derivatives are convex. In paper [3], several Newton-type inequalities are constructed by post-quantum integrals. Noor et al. proved Newton-type inequalities connected with harmonic convex and p-harmonic convex functions in [4] and [5], respectively. Moreover, in paper [6], some Newton-type inequalities were considered for the case of quantum-differentiable convex functions. Furthermore, in paper [7], several error estimates of the Newton-type quadrature formula were presented by bounded variation and Lipschitzian mappings. For some recent results on Newton-type inequalities, see [8–10] and the references therein.

Definition 1.4 (See [11, 12]). *Let us consider $f\in L_1[a,b]$, $a,b\in\mathbb{R}$ with $a<b$. The Riemann-Liouville fractional integrals $J^{\alpha}_{a+}f$ and $J^{\alpha}_{b-}f$ of order $\alpha>0$ are given by*

$$J^{\alpha}_{a+}f(x)=\frac{1}{\Gamma(\alpha)}\int_a^x(x-t)^{\alpha-1}f(t)dt,\quad x>a$$

and

$$J^{\alpha}_{b-}f(x)=\frac{1}{\Gamma(\alpha)}\int_x^b(t-x)^{\alpha-1}f(t)dt,\quad x<b,$$

respectively. Here, Γ denotes the Gamma function defined by

$$\Gamma(\alpha)=\int_0^{\infty}e^{-u}u^{\alpha-1}du.$$

Note that $J^0_{a+}f(x)=J^0_{b-}f(x)=f(x)$.

By means of the well-known Riemann-liouville fractional integrals for differentiable convex functions, some Newton-type inequalities are given as follows:

Theorem 1.5 (See [13]). *Let $f:[a,b]\to\mathbb{R}$ be an absolutely continuous mapping (a,b) so that $f'\in L_1([a,b])$. Let us also consider that the function $|f'|$ is convex on $[a,b]$. Then, the following inequality holds:*

$$\left|\frac{1}{8}\left[f(a)+3f\left(\frac{2a+b}{3}\right)+3f\left(\frac{a+2b}{3}\right)+f(b)\right]-\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J^{\alpha}_{a+}f(b)+J^{\alpha}_{b-}f(a)\right]\right|$$
$$\leq\frac{(b-a)}{2}\left(\Omega_1(\alpha)+\Omega_2(\alpha)+\Omega_3(\alpha)\right)\left[\left|f'(a)\right|+\left|f'(b)\right|\right],$$

where

$$\begin{cases}\Omega_1(\alpha)=\frac{2\alpha}{\alpha+1}\left(\frac{1}{8}\right)^{1+\frac{1}{\alpha}}+\frac{1}{(\alpha+1)3^{\alpha+1}}-\frac{1}{24},\\[2ex]\Omega_2(\alpha)=\frac{2\alpha}{\alpha+1}\left(\frac{1}{2}\right)^{1+\frac{1}{\alpha}}+\frac{1+2^{\alpha+1}}{(\alpha+1)3^{\alpha+1}}-\frac{1}{2},\\[2ex]\Omega_3(\alpha)=\frac{2\alpha}{\alpha+1}\left(\frac{7}{8}\right)^{1+\frac{1}{\alpha}}+\frac{2^{\alpha+1}+3^{\alpha+1}}{(\alpha+1)3^{\alpha+1}}-\frac{35}{24}.\end{cases}$$

The popularity of fractional calculus has increased in recent years because of its wide range of applications in various fields of science. Given the importance of fractional calculus, several operators for fractional integrals can be taken into account. For example, in paper [14], sundry Newton-type inequalities are acquired for the case of functions whose first derivative in absolute value at certain power are arithmetically-harmonically convex. In addition, in paper [15], some Newton-type inequalities are proved using Riemann-Liouville fractional integrals for differentiable convex functions and several Riemann-Liouville fractional Newton-type inequalities are presented for functions of bounded variation. Please refer to the [16–22] articles for further information and topics that are not explained.

The structure of the paper is divided into four parts, starting with an overview of the introduction and preliminaries. The fundamental definitions of fractional calculus and other relevant research in this discipline are given above. In Section 2, we will demonstrate an integral equality that is essential to establish the main findings. The authors of the paper will be presented some Newton-type inequalities for various function classes using Riemann-Liouville fractional integrals in Section 3. More precisely, in subsection 3.1, some Newton-type inequalities will be presented for differentiable convex functions by using Riemann-Liouville fractional integrals. Moreover, we will provide several graphical examples in order to demonstrate the accuracy of the newly established inequalities. In subsection 3.2, we will give several Newton-type for bounded functions by fractional integrals. In subsection 3.3, some fractional Newton-type inequalities will be established for Lipschitzian functions. Furthermore, some Newton-type inequalities will be proved by fractional integrals of bounded variation in subsection 3.4. Finally, we will discuss our opinions on Newton-type inequalities and their potential consequences for future research areas in Section 4.

2. Main Result

In this section, we establish an integral equality involving Riemann-Liouville fractional integrals.

Lemma 2.1. *If $f:[a,b]\to\mathbb{R}$ is an absolutely continuous function (a,b) such that $f'\in L_1[a,b]$, then the following equality holds:*

$$\frac{1}{8}\left[f(a)+3f\left(\frac{2a+b}{3}\right)+3f\left(\frac{a+2b}{3}\right)+f(b)\right]-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J^{\alpha}_{a+}f\left(\frac{a+b}{2}\right)+J^{\alpha}_{b-}f\left(\frac{a+b}{2}\right)\right]$$
$$=\frac{b-a}{4}\left[I_1+I_2\right].$$

Here,

$$\begin{cases} I_1=\int\limits_0^{\frac{1}{3}} t^{\alpha}\left[f'\left(\frac{1+t}{2}b+\frac{1-t}{2}a\right)-f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right]dt,\\ I_2=\int\limits_{\frac{1}{3}}^{1}\left(t^{\alpha}-\frac{3}{4}\right)\left[f'\left(\frac{1+t}{2}b+\frac{1-t}{2}a\right)-f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right]dt.\end{cases}$$

Proof. With the help of the integration by parts, we can quickly acquire

$$\begin{aligned} I_1&=\int\limits_0^{\frac{1}{3}} t^{\alpha}\left[f'\left(\frac{1+t}{2}b+\frac{1-t}{2}a\right)-f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right]dt \qquad (2.1)\\ &=\frac{2}{b-a}t^{\alpha}\left[f\left(\frac{1+t}{2}b+\frac{1-t}{2}a\right)+f\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right]\Bigg|_0^{\frac{1}{3}}\\ &\quad-\frac{2\alpha}{b-a}\int\limits_0^{\frac{1}{3}} t^{\alpha-1}\left[f\left(\frac{1+t}{2}b+\frac{1-t}{2}a\right)+f\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right]dt\\ &=\frac{2}{b-a}\left(\frac{1}{3}\right)^{\alpha}\left[f\left(\frac{a+2b}{3}\right)+f\left(\frac{2a+b}{3}\right)\right]\\ &\quad-\frac{2\alpha}{b-a}\int\limits_0^{\frac{1}{3}} t^{\alpha-1}\left[f\left(\frac{1+t}{2}b+\frac{1-t}{2}a\right)+f\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right]dt.\end{aligned}$$

If we apply a similar process above, then we have

$$\begin{aligned} I_2&=\int\limits_{\frac{1}{3}}^{1}\left(t^{\alpha}-\frac{3}{4}\right)\left[f'\left(\frac{1+t}{2}b+\frac{1-t}{2}a\right)-f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right]dt \qquad (2.2)\\ &=\frac{1}{2(b-a)}\left[f(a)+f(b)\right]-\frac{2}{b-a}\left(\left(\frac{1}{3}\right)^{\alpha}-\frac{3}{4}\right)\left[f\left(\frac{a+2b}{3}\right)+f\left(\frac{2a+b}{3}\right)\right]\\ &\quad-\frac{2\alpha}{b-a}\int\limits_{\frac{1}{3}}^{1} t^{\alpha-1}\left[f\left(\frac{1+t}{2}b+\frac{1-t}{2}a\right)+f\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right]dt.\end{aligned}$$

If we combine (2.1) and (2.2), then we readily get

$$\begin{aligned} I_1+I_2&=\frac{1}{2(b-a)}\left[f(a)+3f\left(\frac{2a+b}{3}\right)+3f\left(\frac{a+2b}{3}\right)+f(b)\right] \qquad (2.3)\\ &\quad-\frac{2\alpha}{b-a}\int\limits_0^{1} t^{\alpha-1}\left[f\left(\frac{1+t}{2}b+\frac{1-t}{2}a\right)+f\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right]dt.\end{aligned}$$

Let us consider the change of the variables $x=\frac{1+t}{2}b+\frac{1-t}{2}a$ and $y=\frac{1+t}{2}a+\frac{1-t}{2}b$ for $t\in[0,1]$. Then, the equality (2.3) can be rewritten as follows

$$I_1+I_2=\frac{1}{2(b-a)}\left[f(a)+3f\left(\frac{2a+b}{3}\right)+3f\left(\frac{a+2b}{3}\right)+f(b)\right]-\frac{2^{\alpha+1}\Gamma(\alpha+1)}{(b-a)^{\alpha+1}}\left[J^{\alpha}_{a+}f\left(\frac{a+b}{2}\right)+J^{\alpha}_{b-}f\left(\frac{a+b}{2}\right)\right]. \qquad (2.4)$$

Multiplying both sides of (2.4) by $\frac{b-a}{4}$, we arrive the proof of Lemma 2.1. □

3. Inequalities for various function classes

In this section, we prove several Newton-type inequalities for various function classes using Riemann-Liouville fractional integrals. To be more precise, some Newton-type inequalities established for differentiable convex functions by using Riemann-Liouville fractional integrals. In addition, we acquire several graphical examples in order to demonstrate the accuracy of the newly established inequalities. Moreover, we present some Newton-type inequalities for bounded functions by fractional integrals. Afterwards, several fractional Newton-type inequalities are obtained for Lipschitzian functions. Furthermore, some Newton-type inequalities are proved by fractional integrals of bounded variation.

3.1. Fractional Newton-type inequalities for convex functions

Theorem 3.1. *Assume that the assumptions of Lemma 2.1 hold and the function $|f'|$ is convex on the interval $[a,b]$. Then, one can prove fractional Newton-type inequality*

$$\left|\frac{1}{8}\left[f(a)+3f\left(\frac{2a+b}{3}\right)+3f\left(\frac{a+2b}{3}\right)+f(b)\right]-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J_{a+}^{\alpha}f\left(\frac{a+b}{2}\right)+J_{b-}^{\alpha}f\left(\frac{a+b}{2}\right)\right]\right| \le \frac{b-a}{4}\left(\Omega_1(\alpha)+\Omega_2(\alpha)\right)\left[|f'(a)|+|f'(b)|\right]. \tag{3.1}$$

Here,

$$\Omega_1(\alpha)=\int_0^{\frac{1}{3}} t^{\alpha}dt=\frac{1}{\alpha+1}\left(\frac{1}{3}\right)^{\alpha+1},$$

and

$$\Omega_2(\alpha)=\int_{\frac{1}{3}}^{1}\left|t^{\alpha}-\frac{3}{4}\right|dt=\begin{cases}\frac{1}{\alpha+1}\left(1-\left(\frac{1}{3}\right)^{\alpha+1}\right)-\frac{1}{2}, & \frac{\ln\left(\frac{3}{4}\right)}{\ln\left(\frac{1}{3}\right)}<\alpha,\\ \frac{1}{\alpha+1}\left[2\alpha\left(\frac{3}{4}\right)^{1+\frac{1}{\alpha}}+\left(\frac{1}{3}\right)^{\alpha+1}+1\right]-1, & 0<\alpha\le\frac{\ln\left(\frac{3}{4}\right)}{\ln\left(\frac{1}{3}\right)}.\end{cases}$$

Proof. By taking into account the absolute value of Lemma 2.1, we can directly have

$$\begin{aligned}&\left|\frac{1}{8}\left[f(a)+3f\left(\frac{2a+b}{3}\right)+3f\left(\frac{a+2b}{3}\right)+f(b)\right]-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J_{a+}^{\alpha}f\left(\frac{a+b}{2}\right)+J_{b-}^{\alpha}f\left(\frac{a+b}{2}\right)\right]\right| \\ &\le \frac{b-a}{4}\left\{\int_0^{\frac{1}{3}}|t^{\alpha}|\left[\left|f'\left(\frac{1+t}{2}b+\frac{1-t}{2}a\right)\right|+\left|f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|\right]dt\right. \\ &\left.+\int_{\frac{1}{3}}^{1}\left|t^{\alpha}-\frac{3}{4}\right|\left[\left|f'\left(\frac{1+t}{2}b+\frac{1-t}{2}a\right)\right|+\left|f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|\right]dt\right\}.\end{aligned} \tag{3.2}$$

Since $|f'|$ is convex, we have

$$\begin{aligned}&\left|\frac{1}{8}\left[f(a)+3f\left(\frac{2a+b}{3}\right)+3f\left(\frac{a+2b}{3}\right)+f(b)\right]-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J_{a+}^{\alpha}f\left(\frac{a+b}{2}\right)+J_{b-}^{\alpha}f\left(\frac{a+b}{2}\right)\right]\right| \\ &\le \frac{b-a}{4}\left\{\int_0^{\frac{1}{3}}t^{\alpha}\left[\left(\frac{1+t}{2}\right)|f'(b)|+\left(\frac{1-t}{2}\right)|f'(a)|+\left(\frac{1+t}{2}\right)|f'(a)|+\left(\frac{1-t}{2}\right)|f'(b)|\right]dt\right. \\ &\left.+\int_{\frac{1}{3}}^{1}\left|t^{\alpha}-\frac{3}{4}\right|\left[\left(\frac{1+t}{2}\right)|f'(b)|+\left(\frac{1-t}{2}\right)|f'(a)|+\left(\frac{1+t}{2}\right)|f'(a)|+\left(\frac{1-t}{2}\right)|f'(b)|\right]dt\right\} \\ &=\frac{b-a}{4}\left(\Omega_1(\alpha)+\Omega_2(\alpha)\right)\left[|f'(a)|+|f'(b)|\right].\end{aligned}$$

This finishes the proof of Theorem 3.1. □

Remark 3.2. *If we choose $\alpha=1$ in Theorem 3.1, then we can obtain Newton-type inequality*

$$\left|\frac{1}{8}\left[f(a)+3f\left(\frac{2a+b}{3}\right)+3f\left(\frac{a+2b}{3}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\le\frac{25(b-a)}{576}\left[|f'(a)|+|f'(b)|\right],$$

which is established by Sitthiwirattham et al. in paper [15, Remark 3].

Example 3.3. *If a function $f:[a,b]=[0,4]\to\mathbb{R}$ is described by $f(x)=\frac{x^2}{2}$ with $\alpha\in(0,15]$, then the left-hand side of (3.1) reduces to*

$$\begin{aligned}&\left|\frac{1}{8}\left[f(a)+3f\left(\frac{2a+b}{3}\right)+3f\left(\frac{a+2b}{3}\right)+f(b)\right]-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J_{a+}^{\alpha}f\left(\frac{a+b}{2}\right)+J_{b-}^{\alpha}f\left(\frac{a+b}{2}\right)\right]\right| \\ &=\left|\frac{1}{8}\left[f(0)+3f\left(\frac{4}{3}\right)+3f\left(\frac{8}{3}\right)+f(4)\right]-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{4^{\alpha}}\left[J_{0+}^{\alpha}f(2)+J_{4-}^{\alpha}f(2)\right]\right|=\frac{4}{3}\left|\frac{1-\alpha}{\alpha+2}\right|.\end{aligned} \tag{3.3}$$

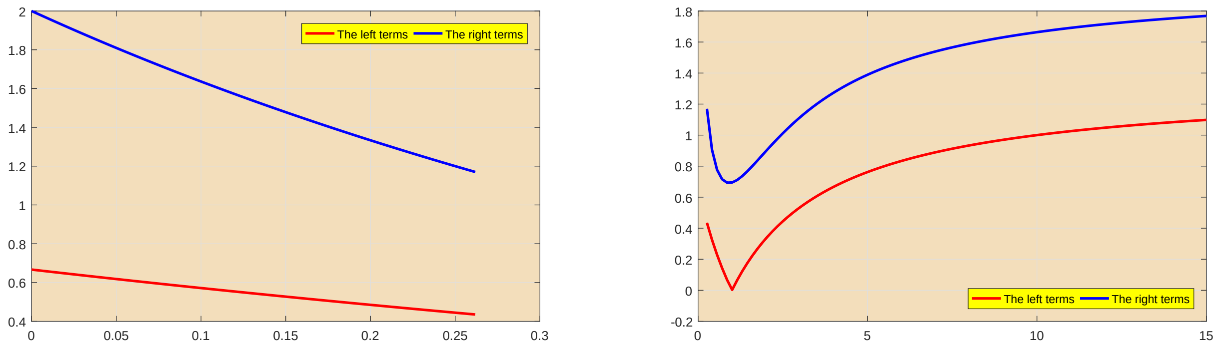
The right hand-side of (3.1) coincides with

$$\frac{b-a}{4}\left(\Omega_1(\alpha)+\Omega_2(\alpha)\right)\left[\left|f'(a)\right|+\left|f'(b)\right|\right]=4\left(\Omega_1(\alpha)+\Omega_2(\alpha)\right)$$

$$=\begin{cases}\frac{2(1-\alpha)}{\alpha+1}, & 0<\alpha\leq\frac{\ln\left(\frac{3}{4}\right)}{\ln\left(\frac{1}{3}\right)},\\ 4\left[\frac{2\alpha}{\alpha+1}\left(\frac{3}{4}\right)^{1+\frac{1}{\alpha}}+\frac{2}{\alpha+1}\left(\frac{1}{3}\right)^{\alpha+1}+\frac{1}{\alpha+1}-1\right], & \frac{\ln\left(\frac{3}{4}\right)}{\ln\left(\frac{1}{3}\right)}<\alpha\leq 15.\end{cases}$$

Finally, we have

$$\begin{cases}\frac{4}{3}\left|\frac{1-\alpha}{\alpha+2}\right|\leq\frac{2(1-\alpha)}{\alpha+1}, & 0<\alpha\leq\frac{\ln\left(\frac{3}{4}\right)}{\ln\left(\frac{1}{3}\right)},\\ \frac{4}{3}\left|\frac{1-\alpha}{\alpha+2}\right|\leq 4\left[\frac{2\alpha}{\alpha+1}\left(\frac{3}{4}\right)^{1+\frac{1}{\alpha}}+\frac{2}{\alpha+1}\left(\frac{1}{3}\right)^{\alpha+1}+\frac{1}{\alpha+1}-1\right], & \frac{\ln\left(\frac{3}{4}\right)}{\ln\left(\frac{1}{3}\right)}<\alpha\leq 15.\end{cases}$$



(a) Graph based on the interval $0<\alpha\leq\frac{\ln\left(\frac{3}{4}\right)}{\ln\left(\frac{1}{3}\right)}$.

(b) Graph based on the interval $\frac{\ln\left(\frac{3}{4}\right)}{\ln\left(\frac{1}{3}\right)}<\alpha\leq 15$.

Figure 3.1: The left-hand side of (3.1) is consistently below the right-hand side of this inequality for all values of $\alpha\in(0,15]$ in Example 3.3.

Theorem 3.4. *Let us consider that the assumptions in Lemma 2.1 hold and the function $|f'|^q$, $q>1$ is convex on $[a,b]$. Then, the Newton-type inequality*

$$\left|\frac{1}{8}\left[f(a)+3f\left(\frac{2a+b}{3}\right)+3f\left(\frac{a+2b}{3}\right)+f(b)\right]-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J_{a+}^{\alpha}f\left(\frac{a+b}{2}\right)+J_{b-}^{\alpha}f\left(\frac{a+b}{2}\right)\right]\right| \tag{3.4}$$

$$\leq\frac{b-a}{4}\left\{\left(\frac{1}{(\alpha p+1)}\left(\frac{1}{3}\right)^{\alpha p+1}\right)^{\frac{1}{p}}\left[\left(\frac{7|f'(b)|^q+5|f'(a)|^q}{36}\right)^{\frac{1}{q}}+\left(\frac{7|f'(a)|^q+5|f'(b)|^q}{36}\right)^{\frac{1}{q}}\right]\right.$$

$$\left.+\left(\int_{\frac{1}{3}}^{1}\left|t^{\alpha}-\frac{3}{4}\right|^{p}dt\right)^{\frac{1}{p}}\left[\left(\frac{5|f'(b)|^q+|f'(a)|^q}{9}\right)^{\frac{1}{q}}+\left(\frac{5|f'(a)|^q+|f'(b)|^q}{9}\right)^{\frac{1}{q}}\right]\right\}$$

is valid. Here, $\frac{1}{p}+\frac{1}{q}=1$.

Proof. By applying Hölder's inequality to (3.2), we obtain

$$\left|\frac{1}{8}\left[f(a)+3f\left(\frac{2a+b}{3}\right)+3f\left(\frac{a+2b}{3}\right)+f(b)\right]-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J_{a+}^{\alpha}f\left(\frac{a+b}{2}\right)+J_{b-}^{\alpha}f\left(\frac{a+b}{2}\right)\right]\right|$$

$$\leq\frac{b-a}{4}\left\{\left(\int_{0}^{\frac{1}{3}}|t^{\alpha}|^{p}dt\right)^{\frac{1}{p}}\left(\int_{0}^{\frac{1}{3}}\left|f'\left(\frac{1+t}{2}b+\frac{1-t}{2}a\right)\right|^{q}dt\right)^{\frac{1}{q}}+\left(\int_{0}^{\frac{1}{3}}|t^{\alpha}|^{p}dt\right)^{\frac{1}{p}}\left(\int_{0}^{\frac{1}{3}}\left|f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|^{q}dt\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\int_{\frac{1}{3}}^{1}\left|t^{\alpha}-\frac{3}{4}\right|^{p}dt\right)^{\frac{1}{p}}\left(\int_{\frac{1}{3}}^{1}\left|f'\left(\frac{1+t}{2}b+\frac{1-t}{2}a\right)\right|^{q}dt\right)^{\frac{1}{q}}+\left(\int_{\frac{1}{3}}^{1}\left|t^{\alpha}-\frac{3}{4}\right|^{p}dt\right)^{\frac{1}{p}}\left(\int_{\frac{1}{3}}^{1}\left|f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|^{q}dt\right)^{\frac{1}{q}}\right\}.$$

Taking advantage of the convexity $|f'|^q$, we get

$$\left|\frac{1}{8}\left[f(a)+3f\left(\frac{2a+b}{3}\right)+3f\left(\frac{a+2b}{3}\right)+f(b)\right]-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J_{a+}^{\alpha}f\left(\frac{a+b}{2}\right)+J_{b-}^{\alpha}f\left(\frac{a+b}{2}\right)\right]\right|$$

$$\leq \frac{b-a}{4}\left\{\left(\int_0^{\frac{1}{3}} t^{\alpha p}dt\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{3}}\left(\frac{1+t}{2}|f'(b)|^q+\frac{1-t}{2}|f'(a)|^q\right)dt\right)^{\frac{1}{q}}+\left(\int_0^{\frac{1}{3}} t^{\alpha p}dt\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{3}}\left(\frac{1+t}{2}|f'(a)|^q+\frac{1-t}{2}|f'(b)|^q\right)dt\right)^{\frac{1}{q}}\right.$$

$$+\left(\int_{\frac{1}{3}}^{1}\left|t^{\alpha}-\frac{3}{4}\right|^p dt\right)^{\frac{1}{p}}\left(\int_{\frac{1}{3}}^{1}\left(\frac{1+t}{2}|f'(b)|^q+\frac{1-t}{2}|f'(a)|^q\right)dt\right)^{\frac{1}{q}}$$

$$\left.+\left(\int_{\frac{1}{3}}^{1}\left|t^{\alpha}-\frac{3}{4}\right|^p dt\right)^{\frac{1}{p}}\left(\int_{\frac{1}{3}}^{1}\left(\frac{1+t}{2}|f'(a)|^q+\frac{1-t}{2}|f'(b)|^q\right)dt\right)^{\frac{1}{q}}\right\}$$

$$=\frac{b-a}{4}\left\{\left(\frac{1}{(\alpha p+1)}\left(\frac{1}{3}\right)^{\alpha p+1}\right)^{\frac{1}{p}}\left[\left(\frac{7|f'(b)|^q+5|f'(a)|^q}{36}\right)^{\frac{1}{q}}+\left(\frac{7|f'(a)|^q+5|f'(b)|^q}{36}\right)^{\frac{1}{q}}\right]\right.$$

$$\left.+\left(\int_{\frac{1}{3}}^{1}\left|t^{\alpha}-\frac{3}{4}\right|^p dt\right)^{\frac{1}{p}}\left[\left(\frac{5|f'(b)|^q+|f'(a)|^q}{9}\right)^{\frac{1}{q}}+\left(\frac{5|f'(a)|^q+|f'(b)|^q}{9}\right)^{\frac{1}{q}}\right]\right\},$$

which completes the proof of Theorem 3.4. □

Corollary 3.5. *If we assign $\alpha=1$ in Theorem 3.4, then we obtain*

$$\left|\frac{1}{8}\left[f(a)+3f\left(\frac{2a+b}{3}\right)+3f\left(\frac{a+2b}{3}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(t)dt\right|$$

$$\leq\frac{b-a}{4}\left\{\left(\frac{1}{p+1}\left[\left(\frac{1}{4}\right)^{p+1}+\left(\frac{5}{12}\right)^{p+1}\right]\right)^{\frac{1}{p}}\times\left[\left(\frac{5|f'(b)|^q+|f'(a)|^q}{9}\right)^{\frac{1}{q}}+\left(\frac{5|f'(a)|^q+|f'(b)|^q}{9}\right)^{\frac{1}{q}}\right]\right.$$

$$\left.+\left(\frac{1}{p+1}\left(\frac{1}{3}\right)^{p+1}\right)^{\frac{1}{p}}\left[\left(\frac{7|f'(b)|^q+5|f'(a)|^q}{36}\right)^{\frac{1}{q}}+\left(\frac{7|f'(a)|^q+5|f'(b)|^q}{36}\right)^{\frac{1}{q}}\right]\right\}.$$

Example 3.6. *Let us consider a function $f:[a,b]=[0,4]\to\mathbb{R}$ given by $f(x)=\frac{x^2}{2}$. From Theorem 3.4 with $\alpha\in(0,15]$ and $p=q=2$, the left-hand side of (3.4) reduces to equality (3.3) and the right hand-side of (3.4) is equal to*

$$\left[\frac{1}{2\alpha+1}\left(\frac{1}{3}\right)^{2\alpha+1}\right]^{\frac{1}{2}}\left(\frac{2\sqrt{7}+2\sqrt{5}}{3}\right)+\left[\frac{3}{2(\alpha+1)}\left(\frac{1}{3}\right)^{\alpha+1}-\frac{1}{2\alpha+1}\left(\frac{1}{3}\right)^{2\alpha+1}+\frac{1}{2\alpha+1}-\frac{3}{2(\alpha+1)}+\frac{3}{8}\right]^{\frac{1}{2}}\left(2+\sqrt{2}\right).$$

Consequently, we have the inequality

$$\frac{4}{3}\left|\frac{1-\alpha}{\alpha+2}\right|\leq\left[\frac{1}{2\alpha+1}\left(\frac{1}{3}\right)^{2\alpha+1}\right]^{\frac{1}{2}}\left(\frac{2\sqrt{7}+2\sqrt{5}}{3}\right)+\left[\frac{3}{2(\alpha+1)}\left(\frac{1}{3}\right)^{\alpha+1}\right.$$

$$\left.-\frac{1}{2\alpha+1}\left(\frac{1}{3}\right)^{2\alpha+1}+\frac{1}{2\alpha+1}-\frac{3}{2(\alpha+1)}+\frac{3}{8}\right]^{\frac{1}{2}}\left(2+\sqrt{2}\right).$$

By using MATLAB software, as one can see in Example 3.6, it is easy to confirm that the left-hand side of (3.4) is always lower than the right-hand side of (3.4) in Figure 3.2 for all values of $\alpha\in(0,15]$.

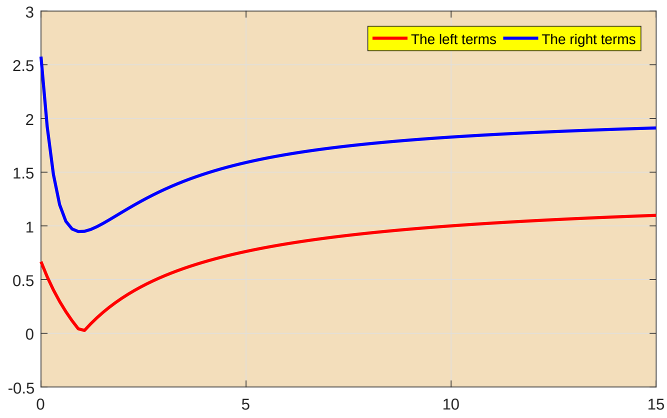


Figure 3.2: MATLAB has been evaluated and ploted the graph of both sides of (3.4) in Example 3.6.

Theorem 3.7. *Suppose that the assumptions of Lemma 2.1 hold and the function $|f'|^q$, $q \geq 1$ is convex on $[a,b]$. Then, one can obtain the Newton-type inequality*

$$\left|\frac{1}{8}\left[f(a)+3f\left(\frac{2a+b}{3}\right)+3f\left(\frac{a+2b}{3}\right)+f(b)\right]-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J_{a+}^{\alpha}f\left(\frac{a+b}{2}\right)+J_{b-}^{\alpha}f\left(\frac{a+b}{2}\right)\right]\right| \tag{3.5}$$

$$\leq \frac{b-a}{4}\left\{(\Omega_1(\alpha))^{1-\frac{1}{q}}\left[\left[\Omega_3(\alpha)|f'(b)|^q+\Omega_4(\alpha)|f'(a)|^q\right]^{\frac{1}{q}}+\left[\Omega_3(\alpha)|f'(a)|^q+\Omega_4(\alpha)|f'(b)|^q\right]^{\frac{1}{q}}\right]\right.$$

$$\left.+(\Omega_2(\alpha))^{1-\frac{1}{q}}\left[\left[\Omega_5(\alpha)|f'(b)|^q+\Omega_6(\alpha)|f'(a)|^q\right]^{\frac{1}{q}}+\left[\Omega_5(\alpha)|f'(a)|^q+\Omega_6(\alpha)|f'(b)|^q\right]^{\frac{1}{q}}\right]\right\}.$$

Here, $\Omega_1(\alpha)$ and $\Omega_2(\alpha)$ are specified in Theorem 3.1 and

$$\Omega_3(\alpha)=\int_0^{\frac{1}{3}}\left(\frac{1+t}{2}\right)t^{\alpha}dt=\frac{4\alpha+7}{6(\alpha+1)(\alpha+2)}\left(\frac{1}{3}\right)^{\alpha+1},$$

$$\Omega_4(\alpha)=\int_0^{\frac{1}{3}}\left(\frac{1-t}{2}\right)t^{\alpha}dt=\frac{2\alpha+5}{6(\alpha+1)(\alpha+2)}\left(\frac{1}{3}\right)^{\alpha+1},$$

$$\Omega_5(\alpha)=\int_{\frac{1}{3}}^{1}\left(\frac{1+t}{2}\right)\left|t^{\alpha}-\frac{3}{4}\right|dt=\begin{cases}\frac{1}{2(\alpha+1)}\left(1-\left(\frac{1}{3}\right)^{\alpha+1}\right)+\frac{1}{2(\alpha+2)}\left(\left(\frac{1}{3}\right)^{\alpha+2}-1\right)-\frac{1}{12}, & \frac{\ln\left(\frac{3}{4}\right)}{\ln\left(\frac{1}{3}\right)}<\alpha,\\ \frac{1}{2(\alpha+1)}\left[2\alpha\left(\frac{3}{4}\right)^{1+\frac{1}{\alpha}}+\left(\frac{1}{3}\right)^{\alpha+1}+1\right] & \\ \quad+\frac{1}{2(\alpha+2)}\left[\alpha\left(\frac{3}{4}\right)^{1+\frac{2}{\alpha}}+\left(\frac{1}{3}\right)^{\alpha+2}+1\right]-\frac{17}{24}, & 0<\alpha\leq\frac{\ln\left(\frac{3}{4}\right)}{\ln\left(\frac{1}{3}\right)}.\end{cases}$$

$$\Omega_6(\alpha)=\int_{\frac{1}{3}}^{1}\left(\frac{1-t}{2}\right)\left|t^{\alpha}-\frac{3}{4}\right|dt=\begin{cases}\frac{1}{2(\alpha+1)}\left(1-\left(\frac{1}{3}\right)^{\alpha+1}\right)-\frac{1}{2(\alpha+2)}\left(\left(\frac{1}{3}\right)^{\alpha+2}-1\right)-\frac{5}{12}, & \frac{\ln\left(\frac{3}{4}\right)}{\ln\left(\frac{1}{3}\right)}<\alpha,\\ \frac{1}{2(\alpha+1)}\left[2\alpha\left(\frac{3}{4}\right)^{1+\frac{1}{\alpha}}+\left(\frac{1}{3}\right)^{\alpha+1}+1\right] & \\ \quad-\frac{1}{2(\alpha+2)}\left[\alpha\left(\frac{3}{4}\right)^{1+\frac{2}{\alpha}}+\left(\frac{1}{3}\right)^{\alpha+2}+1\right]-\frac{7}{24}, & 0<\alpha\leq\frac{\ln\left(\frac{3}{4}\right)}{\ln\left(\frac{1}{3}\right)}.\end{cases}$$

Proof. When we first apply (3.2) to the power-mean inequality, one can obtain

$$\left|\frac{1}{8}\left[f(a)+3f\left(\frac{2a+b}{3}\right)+3f\left(\frac{a+2b}{3}\right)+f(b)\right]-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J_{a+}^{\alpha}f\left(\frac{a+b}{2}\right)+J_{b-}^{\alpha}f\left(\frac{a+b}{2}\right)\right]\right|$$

$$\leq \frac{b-a}{4}\left\{\left(\int_0^{\frac{1}{3}}|t^{\alpha}|dt\right)^{1-\frac{1}{q}}\left(\int_0^{\frac{1}{3}}|t^{\alpha}|\left|f'\left(\frac{1+t}{2}b+\frac{1-t}{2}a\right)\right|^q dt\right)^{\frac{1}{q}}\right.$$

$$+\left(\int_0^{\frac{1}{3}}|t^\alpha|dt\right)^{1-\frac{1}{q}}\left(\int_0^{\frac{1}{3}}|t^\alpha|\left|f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|^q dt\right)^{\frac{1}{q}}$$
$$+\left(\int_{\frac{1}{3}}^{1}\left|t^\alpha-\frac{3}{4}\right|dt\right)^{1-\frac{1}{q}}\left(\int_{\frac{1}{3}}^{1}\left|t^\alpha-\frac{3}{4}\right|\left|f'\left(\frac{1+t}{2}b+\frac{1-t}{2}a\right)\right|^q dt\right)^{\frac{1}{q}}$$
$$+\left(\int_{\frac{1}{3}}^{1}\left|t^\alpha-\frac{3}{4}\right|dt\right)^{1-\frac{1}{q}}\left(\int_{\frac{1}{3}}^{1}\left|t^\alpha-\frac{3}{4}\right|\left|f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|^q dt\right)^{\frac{1}{q}}\Bigg\}.$$

With the help of the convexity of $|f'|^q$, we get

$$\left|\frac{1}{8}\left[f(a)+3f\left(\frac{2a+b}{3}\right)+3f\left(\frac{a+2b}{3}\right)+f(b)\right]-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha}\left[J^\alpha_{a+}f\left(\frac{a+b}{2}\right)+J^\alpha_{b-}f\left(\frac{a+b}{2}\right)\right]\right|$$
$$\leq\frac{b-a}{4}\left\{\left(\int_0^{\frac{1}{3}}t^\alpha dt\right)^{1-\frac{1}{q}}\left(\int_0^{\frac{1}{3}}t^\alpha\left[\left(\frac{1+t}{2}\right)|f'(b)|^q+\left(\frac{1-t}{2}\right)|f'(a)|^q\right]dt\right)^{\frac{1}{q}}\right.$$
$$+\left(\int_0^{\frac{1}{3}}t^\alpha dt\right)^{1-\frac{1}{q}}\left(\int_0^{\frac{1}{3}}t^\alpha\left[\left(\frac{1+t}{2}\right)|f'(a)|^q+\left(\frac{1-t}{2}\right)|f'(b)|^q\right]dt\right)^{\frac{1}{q}}$$
$$+\left(\int_{\frac{1}{3}}^{1}\left|t^\alpha-\frac{3}{4}\right|dt\right)^{1-\frac{1}{q}}\left(\int_{\frac{1}{3}}^{1}\left|t^\alpha-\frac{3}{4}\right|\left[\left(\frac{1+t}{2}\right)|f'(b)|^q+\left(\frac{1-t}{2}\right)|f'(a)|^q\right]dt\right)^{\frac{1}{q}}$$
$$\left.+\left(\int_{\frac{1}{3}}^{1}\left|t^\alpha-\frac{3}{4}\right|dt\right)^{1-\frac{1}{q}}\left(\int_{\frac{1}{3}}^{1}\left|t^\alpha-\frac{3}{4}\right|\left[\left(\frac{1+t}{2}\right)|f'(a)|^q+\left(\frac{1-t}{2}\right)|f'(b)|^q\right]dt\right)^{\frac{1}{q}}\right\}$$
$$=\frac{b-a}{4}\left\{(\Omega_1(\alpha))^{1-\frac{1}{q}}\left[\left[\Omega_3(\alpha)|f'(b)|^q+\Omega_4(\alpha)|f'(a)|^q\right]^{\frac{1}{q}}+\left[\Omega_3(\alpha)|f'(a)|^q+\Omega_4(\alpha)|f'(b)|^q\right]^{\frac{1}{q}}\right]\right.$$
$$\left.+(\Omega_2(\alpha))^{1-\frac{1}{q}}\left[\left[\Omega_5(\alpha)|f'(b)|^q+\Omega_6(\alpha)|f'(a)|^q\right]^{\frac{1}{q}}+\left[\Omega_5(\alpha)|f'(a)|^q+\Omega_6(\alpha)|f'(b)|^q\right]^{\frac{1}{q}}\right]\right\}.$$

This ends the proof of Theorem 3.7. □

Corollary 3.8. *If we select $\alpha=1$ in Theorem 3.7, then we have the Newton-type inequality*

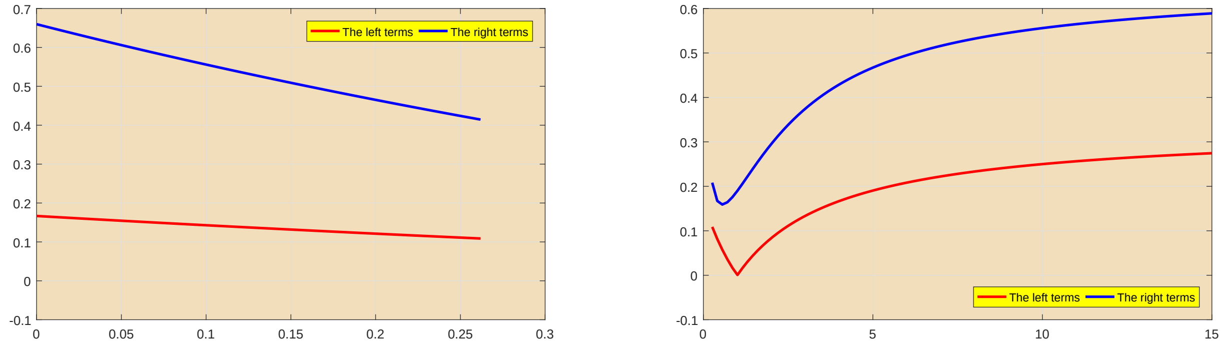
$$\left|\frac{1}{8}\left[f(a)+3f\left(\frac{2a+b}{3}\right)+3f\left(\frac{a+2b}{3}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(t)dt\right|$$
$$\leq\frac{b-a}{72}\left\{\left[\left(\frac{11|f'(b)|^q+7|f'(a)|^q}{18}\right)^{\frac{1}{q}}+\left(\frac{11|f'(a)|^q+7|f'(b)|^q}{18}\right)^{\frac{1}{q}}\right]\right.$$
$$\left.+\left(\frac{17}{8}\right)^{1-\frac{1}{q}}\left[\left(\frac{973|f'(b)|^q+251|f'(a)|^q}{576}\right)^{\frac{1}{q}}+\left(\frac{973|f'(a)|^q+251|f'(b)|^q}{576}\right)^{\frac{1}{q}}\right]\right\}.$$

Example 3.9. *A function $f:[a,b]=[0,4]\to\mathbb{R}$ is presented by $f(x)=\frac{x^2}{2}$. From Theorem 3.7 with $\alpha\in(0,15]$ and $q=2$, the left-hand side of (3.5) reduces to an equality (3.3) and the right hand-side of (3.5) is*

$$4\left\{(\Omega_1(\alpha))^{\frac{1}{2}}\left[[\Omega_3(\alpha)]^{\frac{1}{2}}+[\Omega_4(\alpha)]^{\frac{1}{2}}\right]+(\Omega_2(\alpha))^{\frac{1}{2}}\left[[\Omega_5(\alpha)]^{\frac{1}{2}}+[\Omega_6(\alpha)]^{\frac{1}{2}}\right]\right\}.$$

Finally, we have the inequality

$$\frac{1}{3}\left|\frac{1-\alpha}{\alpha+2}\right|\leq(\Omega_1(\alpha))^{\frac{1}{2}}\left[[\Omega_3(\alpha)]^{\frac{1}{2}}+[\Omega_4(\alpha)]^{\frac{1}{2}}\right]+(\Omega_2(\alpha))^{\frac{1}{2}}\left[[\Omega_5(\alpha)]^{\frac{1}{2}}+[\Omega_6(\alpha)]^{\frac{1}{2}}\right].$$



(a) Graph based on the interval $0 < \alpha \leq \frac{\ln\left(\frac{3}{4}\right)}{\ln\left(\frac{1}{3}\right)}$.

(b) Graph based on the interval $\frac{\ln\left(\frac{3}{4}\right)}{\ln\left(\frac{1}{3}\right)} < \alpha \leq 15$.

Figure 3.3: As one can see in Example 3.9 that the left-hand side of (3.5) constantly stays below the right-hand side.

3.2. Fractional Newton-type inequalities for bounded functions

Theorem 3.10. *Consider that the conditions of Lemma 2.1 hold. If there exist $m, M \in \mathbb{R}$ such that $m \leq f'(t) \leq M$ for $t \in [a,b]$, then it follows*

$$\left|\frac{1}{8}\left[f(a)+3f\left(\frac{2a+b}{3}\right)+3f\left(\frac{a+2b}{3}\right)+f(b)\right]-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J_{a+}^{\alpha}f\left(\frac{a+b}{2}\right)+J_{b-}^{\alpha}f\left(\frac{a+b}{2}\right)\right]\right| \leq \frac{b-a}{4}\left\{\Omega_1(\alpha)+\Omega_2(\alpha)\right\}(M-m). \tag{3.6}$$

Proof. By using the Lemma 2.1, we have

$$\begin{aligned}&\frac{1}{8}\left[f(a)+3f\left(\frac{2a+b}{3}\right)+3f\left(\frac{a+2b}{3}\right)+f(b)\right]-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J_{a+}^{\alpha}f\left(\frac{a+b}{2}\right)+J_{b-}^{\alpha}f\left(\frac{a+b}{2}\right)\right] \\ &= \frac{b-a}{4}\left\{\int_0^{\frac{1}{3}} t^{\alpha}\left[f'\left(\frac{1+t}{2}b+\frac{1-t}{2}a\right)-\frac{m+M}{2}\right]dt+\int_0^{\frac{1}{3}} t^{\alpha}\left[\frac{m+M}{2}-f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right]dt\right. \\ &\left.+\int_{\frac{1}{3}}^{1}\left(t^{\alpha}-\frac{3}{4}\right)\left[f'\left(\frac{1+t}{2}b+\frac{1-t}{2}a\right)-\frac{m+M}{2}\right]dt+\int_{\frac{1}{3}}^{1}\left(t^{\alpha}-\frac{3}{4}\right)\left[\frac{m+M}{2}-f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right]dt\right\}.\end{aligned} \tag{3.7}$$

When we use the absolute value of (3.7), we have

$$\begin{aligned}&\left|\frac{1}{8}\left[f(a)+3f\left(\frac{2a+b}{3}\right)+3f\left(\frac{a+2b}{3}\right)+f(b)\right]-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J_{a+}^{\alpha}f\left(\frac{a+b}{2}\right)+J_{b-}^{\alpha}f\left(\frac{a+b}{2}\right)\right]\right| \\ &\leq \frac{b-a}{4}\left\{\int_0^{\frac{1}{3}} |t^{\alpha}|\left|f'\left(\frac{1+t}{2}b+\frac{1-t}{2}a\right)-\frac{m+M}{2}\right|dt+\int_0^{\frac{1}{3}} |t^{\alpha}|\left|\frac{m+M}{2}-f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|dt\right. \\ &\left.+\int_{\frac{1}{3}}^{1}\left|t^{\alpha}-\frac{3}{4}\right|\left|f'\left(\frac{1+t}{2}b+\frac{1-t}{2}a\right)-\frac{m+M}{2}\right|dt+\int_{\frac{1}{3}}^{1}\left|t^{\alpha}-\frac{3}{4}\right|\left|\frac{m+M}{2}-f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|dt\right\}.\end{aligned}$$

If we use $m \leq f'(t) \leq M$ for $t \in [a,b]$, then we get

$$\left|f'\left(\frac{1+t}{2}b+\frac{1-t}{2}a\right)-\frac{m+M}{2}\right| \leq \frac{M-m}{2} \tag{3.8}$$

and

$$\left|\frac{m+M}{2}-f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right| \leq \frac{M-m}{2}. \tag{3.9}$$

With the help of (3.8) and (3.9), we obtain

$$\left|\frac{1}{8}\left[f(a)+3f\left(\frac{a+2b}{3}\right)+3f\left(\frac{2a+b}{3}\right)+f(b)\right]-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J_{a+}^{\alpha}f\left(\frac{a+b}{2}\right)+J_{b-}^{\alpha}f\left(\frac{a+b}{2}\right)\right]\right|$$
$$\leq \frac{b-a}{4}(M-m)\left\{\int_0^{\frac{1}{3}} t^{\alpha}dt+\int_{\frac{1}{3}}^{1}\left|t^{\alpha}-\frac{3}{4}\right|dt\right\}=\frac{b-a}{4}\left\{\Omega_1(\alpha)+\Omega_2(\alpha)\right\}(M-m).$$

□

Corollary 3.11. *If we choose $\alpha=1$ in Theorem 3.10, then one can obtain*

$$\left|\frac{1}{8}\left[f(a)+3f\left(\frac{2a+b}{3}\right)+3f\left(\frac{a+2b}{3}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(t)dt\right|\leq\frac{25(b-a)}{576}(M-m).$$

Corollary 3.12. *Under assumptions of Theorem 3.10, if there exists $M\in\mathbb{R}^+$ such that $|f'(t)|\leq M$ for all $t\in[a,b]$, then it follows*

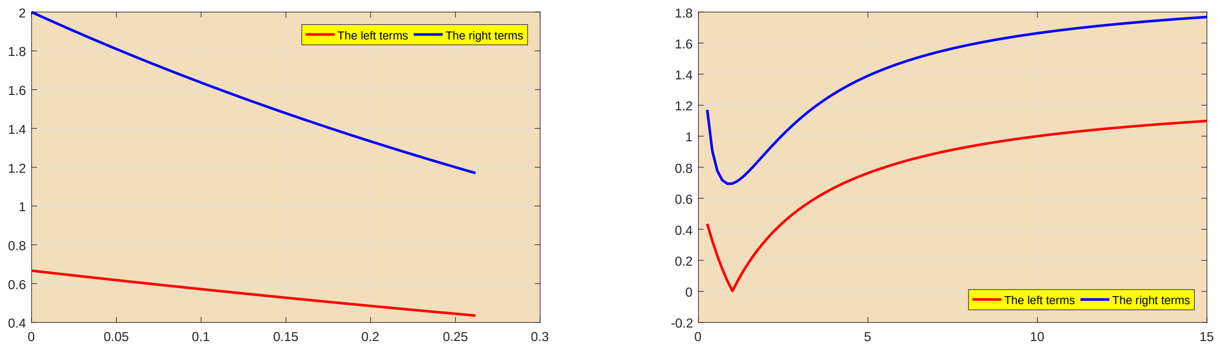
$$\left|\frac{1}{8}\left[f(a)+3f\left(\frac{2a+b}{3}\right)+3f\left(\frac{a+2b}{3}\right)+f(b)\right]-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J_{a+}^{\alpha}f\left(\frac{a+b}{2}\right)+J_{b-}^{\alpha}f\left(\frac{a+b}{2}\right)\right]\right|$$
$$\leq\frac{b-a}{2}\left\{\Omega_1(\alpha)+\Omega_2(\alpha)\right\}M.$$

Corollary 3.13. *Let us consider $\alpha=1$ in Corollary 3.12. Then, the following inequality holds:*

$$\left|\frac{1}{8}\left[f(a)+3f\left(\frac{2a+b}{3}\right)+3f\left(\frac{a+2b}{3}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(t)dt\right|\leq\frac{25(b-a)}{288}M.$$

Example 3.14. *A function $f:[a,b]=[0,4]\to\mathbb{R}$ is given by $f(x)=\frac{x^2}{2}$. From Theorem 3.10 with $\alpha\in(0,15]$ and $0\leq f'(t)\leq 4$, the left-hand side of (3.6) becomes to equality (3.3) and the right hand-side of (3.6) is*

$$4\left\{\Omega_1(\alpha)+\Omega_2(\alpha)\right\}$$
$$=\begin{cases}\frac{2(1-\alpha)}{\alpha+1}, & 0<\alpha\leq\frac{\ln\left(\frac{3}{4}\right)}{\ln\left(\frac{1}{3}\right)},\\ 4\left[\frac{2\alpha}{\alpha+1}\left(\frac{3}{4}\right)^{1+\frac{1}{\alpha}}+\frac{2}{\alpha+1}\left(\frac{1}{3}\right)^{\alpha+1}+\frac{1}{\alpha+1}-1\right], & \frac{\ln\left(\frac{3}{4}\right)}{\ln\left(\frac{1}{3}\right)}<\alpha\leq 15.\end{cases}$$



(a) Graph based on the interval $0<\alpha\leq\frac{\ln\left(\frac{3}{4}\right)}{\ln\left(\frac{1}{3}\right)}$. **(b)** Graph based on the interval $\frac{\ln\left(\frac{3}{4}\right)}{\ln\left(\frac{1}{3}\right)}<\alpha\leq 15$.

Figure 3.4: Example 3.14 illustrates how the left side of (3.6) consistently remains lower than the right side.

3.3. Fractional Newton-type inequalities for Lipschitzian functions

Theorem 3.15. *Suppose that the assumptions of Lemma 2.1 are valid. If f' is a L-Lipschitzian function on $[a,b]$, then the following inequality*

$$\frac{1}{8}\left[f(a)+3f\left(\frac{2a+b}{3}\right)+3f\left(\frac{a+2b}{3}\right)+f(b)\right]-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J_{a+}^{\alpha}f\left(\frac{a+b}{2}\right)+J_{b-}^{\alpha}f\left(\frac{a+b}{2}\right)\right]$$
$$\leq\frac{(b-a)^2}{4}L\left\{\Omega_7(\alpha)+\Omega_8(\alpha)\right\}$$

is valid. Here,

$$\Omega_7(\alpha)=\int_0^{\frac{1}{3}} t^{\alpha+1}dt=\frac{1}{\alpha+2}\left(\frac{1}{3}\right)^{\alpha+2}$$

and

$$\Omega_8(\alpha)=\int_{\frac{1}{3}}^{1} t\left|t^{\alpha}-\frac{3}{4}\right|dt=\begin{cases}\frac{1}{\alpha+2}\left(1-\left(\frac{1}{3}\right)^{\alpha+2}\right)-\frac{1}{3}, & 0<\alpha\leq\frac{\ln\left(\frac{3}{4}\right)}{\ln\left(\frac{1}{3}\right)},\\ \frac{1}{\alpha+2}\left[\alpha\left(\frac{3}{4}\right)^{1+\frac{2}{\alpha}}+\left(\frac{1}{3}\right)^{\alpha+2}+1\right]-\frac{5}{12}, & \frac{\ln\left(\frac{3}{4}\right)}{\ln\left(\frac{1}{3}\right)}<\alpha.\end{cases}$$

Proof. With the help of Lemma 2.1 and since f' is L-Lipschitzian function, we have

$$\begin{aligned}&\left|\frac{1}{8}\left[f(a)+3f\left(\frac{2a+b}{3}\right)+3f\left(\frac{a+2b}{3}\right)+f(b)\right]-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J_{a+}^{\alpha}f\left(\frac{a+b}{2}\right)+J_{b-}^{\alpha}f\left(\frac{a+b}{2}\right)\right]\right|\\
&\leq\frac{b-a}{4}\left\{\int_0^{\frac{1}{3}}|t^{\alpha}|\left|f'\left(\frac{1+t}{2}b+\frac{1-t}{2}a\right)-f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|dt\right.\\
&\quad\left.+\int_{\frac{1}{3}}^{1}\left|t^{\alpha}-\frac{3}{4}\right|\left|f'\left(\frac{1+t}{2}b+\frac{1-t}{2}a\right)-f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|dt\right\}\\
&\leq\frac{b-a}{4}\left\{\int_0^{\frac{1}{3}}t^{\alpha}Lt(b-a)dt+\int_{\frac{1}{3}}^{1}\left|t^{\alpha}-\frac{3}{4}\right|Lt(b-a)dt\right\}\\
&=\frac{(b-a)^2}{4}L\{\Omega_7(\alpha)+\Omega_8(\alpha)\}.\end{aligned}$$

□

Corollary 3.16. *Consider $\alpha=1$ in Theorem 3.15. Then, the following Newton-type inequality holds:*

$$\left|\frac{1}{8}\left[f(a)+3f\left(\frac{2a+b}{3}\right)+3f\left(\frac{a+2b}{3}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(t)dt\right|\leq\frac{425(b-a)^2}{20736}L.$$

3.4. Newton-type inequalities for functions of bounded variation

Theorem 3.17. *Let $f:[a,b]\to\mathbb{R}$ be a function of bounded variation on $[a,b]$. Then, we obtain*

$$\begin{aligned}&\left|\frac{1}{8}\left[f(a)+3f\left(\frac{2a+b}{3}\right)+3f\left(\frac{a+2b}{3}\right)+f(b)\right]-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J_{a+}^{\alpha}f\left(\frac{a+b}{2}\right)+J_{b-}^{\alpha}f\left(\frac{a+b}{2}\right)\right]\right|\\
&\leq\frac{1}{2}\max\left\{\left|\frac{3}{4}-\left(\frac{1}{3}\right)^{\alpha}\right|,\frac{1}{4},\left(\frac{1}{3}\right)^{\alpha}\right\}\bigvee_a^b(f),\end{aligned}$$

where $\bigvee_a^b(f)$ denotes the total variation of f on $[a,b]$.

Proof. Define the function $K_{\alpha}(x)$ by

$$K_{\alpha}(x)=\begin{cases}\frac{3}{4}\left(\frac{b-a}{2}\right)^{\alpha}-\left(\frac{a+b}{2}-x\right)^{\alpha}, & a\leq x<\frac{2a+b}{3},\\ \left(x-\frac{a+b}{2}\right)^{\alpha}, & \frac{2a+b}{3}\leq x<\frac{a+2b}{3},\\ \left(x-\frac{a+b}{2}\right)^{\alpha}-\frac{3}{4}\left(\frac{b-a}{2}\right)^{\alpha}, & \frac{a+2b}{3}\leq x\leq b.\end{cases}$$

By using the integrating by parts, we have

$$\int_a^b K_\alpha(x)df(x)$$
$$= \int_a^{\frac{2a+b}{3}} \left[\frac{3}{4}\left(\frac{b-a}{2}\right)^\alpha - \left(\frac{a+b}{2}-x\right)^\alpha\right] df(x) + \int_{\frac{2a+b}{3}}^{\frac{a+2b}{3}} \left(x-\frac{a+b}{2}\right)^\alpha df(x)$$
$$+ \int_{\frac{a+2b}{3}}^{b} \left[\left(x-\frac{a+b}{2}\right)^\alpha - \frac{3}{4}\left(\frac{b-a}{2}\right)^\alpha\right] df(x)$$
$$= \left[\frac{3}{4}\left(\frac{b-a}{2}\right)^\alpha - \left(\frac{a+b}{2}-x\right)^\alpha\right] f(x)\Big|_a^{\frac{2a+b}{3}} - \alpha \int_a^{\frac{2a+b}{3}} \left(\frac{a+b}{2}-x\right)^{\alpha-1} f(x)dx$$
$$+ \left(x-\frac{a+b}{2}\right)^\alpha f(x)\Big|_{\frac{2a+b}{3}}^{\frac{a+2b}{3}} - \alpha \int_{\frac{2a+b}{3}}^{\frac{a+2b}{3}} \left(x-\frac{a+b}{2}\right)^{\alpha-1} f(x)dx$$
$$+ \left[\left(x-\frac{a+b}{2}\right)^\alpha - \frac{3}{4}\left(\frac{b-a}{2}\right)^\alpha\right] f(x)\Big|_{\frac{a+2b}{3}}^{b} - \alpha \int_{\frac{a+2b}{3}}^{b} \left(x-\frac{a+b}{2}\right)^{\alpha-1} f(x)dx$$
$$= \frac{1}{4}\left(\frac{b-a}{2}\right)^\alpha f(a) + \frac{3}{4}\left(\frac{b-a}{2}\right)^\alpha f\left(\frac{2a+b}{3}\right) + \frac{3}{4}\left(\frac{b-a}{2}\right)^\alpha f\left(\frac{a+2b}{3}\right) + \frac{1}{4}\left(\frac{b-a}{2}\right)^\alpha f(b)$$
$$-\alpha \int_a^{\frac{2a+b}{3}} \left(\frac{a+b}{2}-x\right)^{\alpha-1} f(x)dx - \alpha \int_{\frac{2a+b}{3}}^{b} \left(x-\frac{a+b}{2}\right)^{\alpha-1} f(x)dx$$
$$= \frac{(b-a)^\alpha}{2^{\alpha-1}}\frac{1}{8}\left[f(a)+3f\left(\frac{2a+b}{3}\right)+3f\left(\frac{a+2b}{3}\right)+f(b)\right] - \Gamma(\alpha+1)\left[J^\alpha_{a+}f\left(\frac{a+b}{2}\right)+J^\alpha_{b-}f\left(\frac{a+b}{2}\right)\right].$$

In other words, one can get

$$\frac{1}{8}\left[f(a)+3f\left(\frac{2a+b}{3}\right)+3f\left(\frac{a+2b}{3}\right)+f(b)\right] - \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha}\left[J^\alpha_{a+}f\left(\frac{a+b}{2}\right)+J^\alpha_{b-}f\left(\frac{a+b}{2}\right)\right]$$
$$= \frac{2^{\alpha-1}}{(b-a)^\alpha}\int_a^b K_\alpha(x)df(x).$$

It is known that if $g,f:[a,b]\to\mathbb{R}$ are such that g is continuous on $[a,b]$ and f is of bounded variation on $[a,b]$, then $\int_a^b g(t)df(t)$ exists and

$$\left|\int_a^b g(t)df(t)\right| \le \sup_{t\in[a,b]}|g(t)|\bigvee_a^b(f). \tag{3.10}$$

By using (3.10), it yields

$$\left|\frac{1}{8}\left[f(a)+3f\left(\frac{2a+b}{3}\right)+3f\left(\frac{a+2b}{3}\right)+f(b)\right] - \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha}\left[J^\alpha_{a+}f\left(\frac{a+b}{2}\right)+J^\alpha_{b-}f\left(\frac{a+b}{2}\right)\right]\right|$$
$$= \frac{2^{\alpha-1}}{(b-a)^\alpha}\left|\int_a^b K_\alpha(x)df(x)\right|$$
$$\le \frac{2^{\alpha-1}}{(b-a)^\alpha}\left\{\left|\int_a^{\frac{2a+b}{3}} \left[\frac{3}{4}\left(\frac{b-a}{2}\right)^\alpha - \left(\frac{a+b}{2}-x\right)^\alpha\right] df(x)\right| + \left|\int_{\frac{2a+b}{3}}^{\frac{a+2b}{3}} \left(x-\frac{a+b}{2}\right)^\alpha df(x)\right|\right.$$
$$\left. + \left|\int_{\frac{a+2b}{3}}^{b} \left[\left(x-\frac{a+b}{2}\right)^\alpha - \frac{3}{4}\left(\frac{b-a}{2}\right)^\alpha\right] df(x)\right|\right\}$$
$$\le \frac{2^{\alpha-1}}{(b-a)^\alpha}\left\{\sup_{x\in\left[a,\frac{2a+b}{3}\right]}\left|\frac{3}{4}\left(\frac{b-a}{2}\right)^\alpha - \left(\frac{a+b}{2}-x\right)^\alpha\right|\bigvee_a^{\frac{2a+b}{3}}(f) + \sup_{x\in\left[\frac{2a+b}{3},\frac{a+2b}{3}\right]}\left|\left(x-\frac{a+b}{2}\right)^\alpha\right|\bigvee_{\frac{2a+b}{3}}^{\frac{a+2b}{3}}(f)\right.$$

$$
\begin{aligned}
& + \sup_{x\in\left[\frac{a+2b}{3},b\right]} \left|\left(x-\frac{a+b}{2}\right)^{\alpha}-\frac{3}{4}\left(\frac{b-a}{2}\right)^{\alpha}\right| \bigvee_{\frac{a+2b}{3}}^{b}(f)\Bigg\} \\
&= \frac{2^{\alpha-1}}{(b-a)^{\alpha}}\left\{\max\left[\left|\frac{3}{4}\left(\frac{b-a}{2}\right)^{\alpha}-\left(\frac{b-a}{6}\right)^{\alpha}\right|,\frac{1}{4}\left(\frac{b-a}{2}\right)^{\alpha}\right]\bigvee_{a}^{\frac{2a+b}{3}}(f)+\left(\frac{b-a}{6}\right)^{\alpha}\bigvee_{\frac{2a+b}{3}}^{\frac{a+2b}{3}}(f)\right. \\
& \left. +\max\left[\left|\frac{3}{4}\left(\frac{b-a}{2}\right)^{\alpha}-\left(\frac{b-a}{6}\right)^{\alpha}\right|,\frac{1}{4}\left(\frac{b-a}{2}\right)^{\alpha}\right]\bigvee_{\frac{a+2b}{3}}^{b}(f)\right\} \\
&= \frac{1}{2}\left\{\max\left[\left|\frac{3}{4}-\left(\frac{1}{3}\right)^{\alpha}\right|,\frac{1}{4}\right]\bigvee_{a}^{\frac{2a+b}{3}}(f)+\left(\frac{1}{3}\right)^{\alpha}\bigvee_{\frac{2a+b}{3}}^{\frac{a+2b}{3}}(f)+\max\left[\left|\frac{3}{4}-\left(\frac{1}{3}\right)^{\alpha}\right|,\frac{1}{4}\right]\bigvee_{\frac{a+2b}{3}}^{b}(f)\right\} \\
&\leq \frac{1}{2}\max\left\{\left|\frac{3}{4}-\left(\frac{1}{3}\right)^{\alpha}\right|,\frac{1}{4},\left(\frac{1}{3}\right)^{\alpha}\right\}\bigvee_{a}^{b}(f).
\end{aligned}
$$

□

Remark 3.18. *Let us consider $\alpha = 1$ in Theorem 3.17. Then, the following inequality holds:*

$$\left|\frac{1}{8}\left[f(a)+3f\left(\frac{2a+b}{3}\right)+3f\left(\frac{a+2b}{3}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(t)dt\right| \leq \frac{5}{24}\bigvee_{a}^{b}(f).$$

which is given by Alomari in [23].

4. Conclusion

In this paper, some Newton-type inequalities are establish for various function classes involving Riemann-Liouville fractional integrals. First of all, we present an integral identity that is necessary in order to prove the main findings of the paper. Subsequently several Newton-type inequalities are investigated for differentiable convex functions by using the Riemann-Liouville fractional integrals. In addition to this, we give several examples using graphs in order to show that our main result is correct. Moreover we prove sundry Newton-type for bounded functions by fractional integrals. Furthermore, several fractional Newton-type inequalities are obtained for Lipschitzian functions. Finally, some Newton-type inequalities are acquired by fractional integrals of bounded variation.

The concepts and approaches for our findings about Newton-type inequalities using Riemann-Liouville fractional integrals could clear the way for additional studies in this area in subsequent publications. Improvements or generalizations of our results can be investigated by using different kinds of convex function classes or other types of fractional integral operators. Finally, one can acquire several Newton-type inequalities for various function classes with the help of the quantum calculus.

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Research Article

Neutrosophic $\mathscr{I}$-Statistical Convergence of a Sequence of Neutrosophic Random Variables In Probability

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Abstract

This paper presents a novel perspective on established neutrosophic statistical convergence by utilizing ideals and proposing new ideas. Specifically, we explore the neutrosophic $\mathscr{I}$-statistical convergence of sequences of neutrosophic random variables (briefly, *NRVs*) in probability, as well as the neutrosophic $\mathscr{I}$-lacunary statistical convergence and neutrosophic $\mathscr{I}$-λ-statistical convergence of such sequences in probability. Additionally, we investigate their interconnections and examine some fundamental properties of these concepts.

1. Introduction

Smarandache [1] proposed a novel philosophical perspective that examines the origin, nature, and extent of neutralities, along with their interactions in diverse contexts. The central tenet of neutrosophy asserts that any concept possesses not only a degree of truth—as typically considered in many-valued logic systems—but also independent degrees of falsity and indeterminacy. In this approach, Smarandache appears to interpret indeterminacy from both subjective and objective standpoints, including aspects such as uncertainty, imprecision, vagueness, and error. Neutrosophy constitutes a recent mathematical theory that extends classical logic and fuzzy logic, encompassing constructs such as neutrosophic set theory, neutrosophic probability, neutrosophic statistics, and neutrosophic logic.

Recently, Bisher and Hatip in 2020 [2] employed the concepts of random variables and the indeterminacy associated with neutrosophic sets, offering an initial formulation of NRVs by introducing several fundamental notions. Subsequently, in 2021, Granados [3] established additional theoretical developments related to NRVs, and later, Granados and Sanabria [4] investigated the concept of independence within the context of NRVs. Furthermore, in 2020, Granados et al. [5, 6] examined certain neutrosophic probability distributions, both discrete [5] and continuous [6], based on the structure of NRVs.

On the other hand, the concept of statistical convergence can be traced back to A. Zygmund in 1935, and it gained further attention following its reintroduction by Steinhaus [7] and Fast [8] in 1951 for sequences of real numbers. Since then, various generalizations and applications have been explored. These developments have also been employed in neutrosophic theory. For instance, Kirişci and Şimşek [9] proposed the concept of neutrosophic normed space (briefly NNS), where they investigated statistical convergence and statistically Cauchy sequences, along with statistical completeness. Granados and Dhital [10] extended these results to double sequences in NNS, defining notions such as double statistically Cauchy sequences and their associated completeness. Kişi [11] explored the notions of ideal convergence and ideal Cauchy sequences in NNS. Khan et al. [12] defined lacunary statistically Cauchy sequences and examined the relationship between statistical completeness and classical completeness in NNS. In a subsequent work, Khan et al. [13] utilized λ-statistical convergence to generalize these concepts, presenting λ-statistically Cauchy sequences and completeness results in NNS, along with relevant inclusion relations. Ali et al. [14] investigated statistical convergence and statistically Cauchy sequences in neutrosophic metric spaces, inspired by analogous definitions in fuzzy metric spaces, and provided several characterizations. Al-Hamido [15] proposed a new generalized neutrosophic topological space that goes beyond both classical and crisp neutrosophic topologies, introducing novel types of neutrosophic sets and

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related concepts. Granados and Choudhury [16] formulated the notion of quasi-statistical convergence for triple sequences in NNS as an extension of statistical convergence, investigated key properties, and demonstrated that quasi-statistical Cauchy sequences are equivalent to quasi-statistical convergent sequences in this setting.
Inspired by the aforementioned studies and the growing interest in the investigation of neutrosophic statistical convergence, this paper aims to introduce three novel types of statistical convergence for sequences of NRVs, which are as follows:

1. Neutrosophic $\mathscr{I}$-statistical convergence in probability.
2. Neutrosophic $\mathscr{I}$-lacunary statistical convergence in probability.
3. Neutrosophic $\mathscr{I}$-λ-statistical convergence in probability.

In the present study, all the existing results provided in [17–20] are generalized and refined.

2. Preliminaries

In this section, we outline key concepts that are essential for the progression of the study.

Definition 2.1. *(see [21]) Let $\mathscr{T}$ denote a non-empty, fixed set. A neutrosophic set $\mathscr{A}$ is characterized by the expression $\{t,(\mu\mathscr{A}(t),\delta\mathscr{A}(t),\gamma\mathscr{A}(t)):t\in\mathscr{T}\}$, where $\mu\mathscr{A}(t),\delta\mathscr{A}(t)$ and $\gamma\mathscr{A}(t)$ represent the respective degrees of membership, indeterminacy, and non-membership of each element $t\in\mathscr{T}$ within the set $\mathscr{A}$.*

Definition 2.2. *(see [22]) Let $\mathscr{K}$ represent a field. The neutrosophic field associated with $\mathscr{K}$ and I is represented as $\langle\mathscr{K}\cup I\rangle$, where the operations are those of $\mathscr{K}$, and I is the neutrosophic element satisfying the property $I^2=I$.*

Definition 2.3. *(see [23]) A classical neutrosophic number takes the form $a+bI$, where a and b are real or complex numbers and I represents the indeterminacy satisfying $0.I=0$ and $I^2=I$, which implies that $I^n=I$ for all positive integers n.*

Definition 2.4. *(see [23]) The neutrosophic probability associated with the occurrence of event $\mathscr{A}$ is given by*

$$NP(\mathscr{A})=(ch(\mathscr{A}),ch(neut\mathscr{A}),ch(anti\mathscr{A}))=(T,I,F)$$

where T,I,F signify standard or non-standard subsets of the non-standard unitary interval $]^{-}0,1^{+}[$.

Now, we put forward some notions of NRVs [2].

Definition 2.5. *Let $\mathscr{T}$ be a real-valued deterministic random variable, with the mapping:*
$\mathscr{T}:\Omega\to\mathbb{R}$
where Ω is the event space. An NRV $\mathscr{T}_U$ is defined by:
$\mathscr{T}_U:\Omega\to\mathbb{R}(I)$
and
$\mathscr{T}_U=\mathscr{T}+I$
where I denotes indeterminacy.

Theorem 2.6. *Consider the NRV $\mathscr{T}_U=\mathscr{T}+I$, with the cumulative distribution function of $\mathscr{T}_U$ given by $F_{\mathscr{T}_U}(x)=P(\mathscr{T}_U\leq t)$. The following expressions hold:*

1. $F_{\mathscr{T}_U}(t)=F_{\mathscr{T}}(t-I)$,
2. $f_{\mathscr{T}_U}(t)=f_{\mathscr{T}}(t-I)$.

In these, $F_{\mathscr{T}_U}$ and $f_{\mathscr{T}_U}$ denote the cumulative distribution function and the probability density function of $\mathscr{T}_U$, respectively.

Theorem 2.7. *Consider the NRV $\mathscr{T}_U=\mathscr{T}+I$. The expected value is given by:*
$E(\mathscr{T}_U)=E(\mathscr{T})+I$.

Next, we provide some definitions concerning ideal spaces, as defined by Kuratowski [24]:

Definition 2.8. *A ideal I on a set X, as defined by [24] is a collection of non-empty subsets of X that meets the following requirements.*

1. *When $A\subset B$ and $B\in I$, then $A\in I$.*
2. *When $A\in I$ and $B\in I$, then $A\cup B\in I$.*

3. Neutrosophic $\mathscr{I}$-Statistical Convergence in Probability

Definition 3.1. *Consider a sequence $\{\mathscr{T}_{U_\alpha}\}_{\alpha\in\mathbb{N}}$ of NRVs, where each $\mathscr{T}_{U_\alpha}$ is constructed on a same event space $\mathscr{S}$, along with a specified class Λ of subsets of $\mathscr{S}$, and a probability function $\mathscr{P}:\Lambda\to\mathbb{R}$. This sequence is regarded as neutrosophic $\mathscr{I}$-statistically convergent (N-$\mathscr{I}$-stat-convergent) in probability to an NRV $\mathscr{T}_U$, where $\mathscr{T}:\mathscr{S}\to\mathbb{R}$, if for any $\rho,\varsigma,\rho>0$,*

$$\left\{\alpha\in\mathbb{N}:\frac{1}{\alpha}|\{\beta\leq\alpha:\mathscr{P}(|\mathscr{T}_{U_\beta}-\mathscr{T}_U|\geq\rho)\geq\varsigma\}|\geq\rho\right\}\in\mathscr{I},$$

or equivalently,

$$\left\{\alpha\in\mathbb{N}:\frac{1}{\alpha}|\{\beta\leq\alpha:1-\mathscr{P}(|\mathscr{T}_{U_\beta}-\mathscr{T}_U|<\rho)\geq\varsigma\}|\geq\rho\right\}\in\mathscr{I}.$$

This convergence is represented by $\mathscr{T}_{U_\alpha}\xrightarrow{S^{P_U}(\mathscr{I})}\mathscr{T}_U$. The collection of all sequences of NRVs that are N-$\mathscr{I}$-stat-convergent in probability is referred to as $S^{P_U}(\mathscr{I})$.

Theorem 3.2. *If $\mathscr{T}_{U_\alpha} \overset{S^{p_U}(\mathscr{I})}{\to} \mathscr{T}_U$ and $\mathscr{T}_{U_\alpha} \overset{S^{p_U}(\mathscr{I})}{\to} \mathscr{Y}_U$, then $\mathscr{P}\{\mathscr{T}_U = \mathscr{Y}_U\} = 1$.*

Proof. Let $\rho, \varsigma > 0$ and $0 < \rho < 1$, then

$$\mathscr{U} = \left\{\alpha \in \mathbb{N} : \frac{1}{\alpha}\left|\left\{\beta \leq \alpha : \mathscr{P}\left(|\mathscr{T}_{U_\beta} - \mathscr{T}_U| \geq \frac{\rho}{2}\right) \geq \frac{\varsigma}{2}\right\}\right| < \frac{\rho}{3}\right\} \in \mathscr{F}(\mathscr{I}),$$

and

$$\mathscr{V} = \left\{\alpha \in \mathbb{N} : \frac{1}{\alpha}\left|\left\{\beta \leq \alpha : \mathscr{P}\left(|\mathscr{T}_{U_\beta} - \mathscr{Y}_U| \geq \frac{\rho}{2}\right) \geq \frac{\varsigma}{2}\right\}\right| < \frac{\rho}{3}\right\} \in \mathscr{F}(\mathscr{I}).$$

Since $\mathscr{U} \cap \mathscr{V} \in \mathscr{F}(\mathscr{I})$ and $\emptyset \notin \mathscr{F}(\mathscr{I})$ implies that $\mathscr{U} \cap \mathscr{V} \neq \emptyset$. Now, let $\gamma \in \mathscr{U} \cap \mathscr{V}$. Then,

$$\frac{1}{\gamma}\left|\left\{\beta \leq \gamma : \mathscr{P}\left(|\mathscr{T}_{U_\beta} - \mathscr{T}_U| \geq \frac{\rho}{2}\right) \geq \frac{\varsigma}{2}\right\}\right| < \frac{\rho}{3}$$

and

$$\frac{1}{\gamma}\left|\left\{\beta \leq \gamma : \mathscr{P}\left(|\mathscr{T}_{U_\beta} - \mathscr{Y}_U| \geq \frac{\rho}{2}\right) \geq \frac{\varsigma}{2}\right\}\right| < \frac{\rho}{3}.$$

This implies,

$$\frac{1}{\gamma}\left|\left\{\beta \leq \gamma : \mathscr{P}\left(|\mathscr{T}_{U_\beta} - \mathscr{T}_U| \geq \frac{\rho}{2}\right) \geq \frac{\varsigma}{2} \text{ or } \mathscr{P}\left(|\mathscr{T}_{U_\beta} - \mathscr{Y}_U| \geq \frac{\rho}{2}\right) \geq \frac{\varsigma}{2}\right\}\right| < \rho < 1.$$

Therefore, there exists any $\beta \leq \gamma$ such that $\mathscr{P}\left(|\mathscr{T}_{U_\beta} - \mathscr{T}_U| \geq \frac{\rho}{2}\right) < \frac{\varsigma}{2}$ and $\mathscr{P}\left(|\mathscr{T}_{U_\beta} - \mathscr{Y}_U| \geq \frac{\rho}{2}\right) < \frac{\varsigma}{2}$. Hence,

$$\mathscr{P}(|\mathscr{T}_U - \mathscr{Y}_U| \geq \rho) \leq \mathscr{P}\left(|\mathscr{T}_{U_\beta} - \mathscr{T}_U| \geq \frac{\rho}{2}\right) < \frac{\varsigma}{2} + \mathscr{P}\left(|\mathscr{T}_{U_\beta} - \mathscr{Y}_U| \geq \frac{\rho}{2}\right) < \varsigma.$$

□

Theorem 3.3. *If a sequence of constants $\mathscr{T}_{U_\alpha} \overset{S^{p_U}(\mathscr{I})}{\to} \mathscr{T}_U$, we can treat each constant as an NRV with a one-point distribution at that specific value, thus expressing the convergence as $\mathscr{T}_{U_\alpha} \overset{S^{p_U}(\mathscr{I})}{\to} \mathscr{T}_U$.*

Proof. The proof is succeeded by the definition. □

The reverse of the above theorem is not always valid, as demonstrated in the following example:

Example 3.4. *Consider $\mathscr{K} = \{\alpha^2 + \mathscr{I}\}$, where $\alpha = 1, 2, 3, \ldots$ and let the neutrosophic density $\mathscr{T}_{U_\alpha}$ be defined as $f_\alpha(x - \mathscr{I}) = 1 + \mathscr{I}$ for $\mathscr{I} < x < 1 + \mathscr{I}$, and equal to $\mathscr{I}$ otherwise. If $\alpha \in \mathscr{K}$, then $f_\alpha(x - \mathscr{I}) = \frac{\alpha(x-\mathscr{I})^{\alpha-1}}{2^\alpha}$, for $\mathscr{I} < x < 2 + \mathscr{I}$; and $f_\alpha(x - \mathscr{I}) = \mathscr{I}$ if $\alpha \in \mathbb{N} - \mathscr{K}$. Now, for $\rho \in (0, 1)$, we have*
(i) $\mathscr{P}(|\mathscr{T}_{U_\alpha} - 2| \geq \rho) = 1 + I$ if $\alpha \in \mathscr{K}$ and
(ii) $\mathscr{P}(|\mathscr{T}_{U_\alpha} - 2| \geq \rho) = \left(1 - \frac{\rho}{2} + \mathscr{I}\right)^n$ if $\alpha \in \mathbb{N} - \mathscr{K}$.
Thus, $\mathscr{T}_{U_\alpha} \overset{S^{p_U}(\mathscr{I}_d)}{\to} 2 + \mathscr{I}$.

Theorem 3.5. *The properties listed below hold for N-$\mathscr{I}$-stat-convergence in probability:*

1. $\mathscr{T}_{U_\alpha} \overset{S^{p_U}(\mathscr{I})}{\to} \mathscr{T}_U$ *if and only if* $\mathscr{T}_{U_\alpha} - \mathscr{T}_U \overset{S^{p_U}(\mathscr{I})}{\to} 0$,
2. $\mathscr{T}_{U_\alpha} \overset{S^{p_U}(\mathscr{I})}{\to} \mathscr{T}_U$, *then* $p\mathscr{T}_{U_\alpha} \overset{S^{p_U}(\mathscr{I})}{\to} p\mathscr{T}_U$, *where* $p \in \mathbb{R}$,
3. $\mathscr{T}_{U_\alpha} \overset{S^{p_U}(\mathscr{I})}{\to} \mathscr{T}_U$ *and* $\mathscr{W}_{U_\alpha} \overset{S^{p_U}(\mathscr{I})}{\to} \mathscr{W}_U$, *then* $\mathscr{T}_{U_\alpha} + \mathscr{W}_{U_\alpha} \overset{S^{p_U}(\mathscr{I})}{\to} \mathscr{T}_U + \mathscr{W}_U$,
4. $\mathscr{T}_{U_\alpha} \overset{S^{p_U}(\mathscr{I})}{\to} \mathscr{T}_U$ *and* $\mathscr{W}_{U_\alpha} \overset{S^{p_U}(\mathscr{I})}{\to} \mathscr{W}_U$, *then* $\mathscr{T}_{U_\alpha} - \mathscr{W}_{U_\alpha} \overset{S^{p_U}(\mathscr{I})}{\to} \mathscr{T}_U - \mathscr{W}_U$,
5. $\mathscr{T}_{U_\alpha} \overset{S^{p_U}(\mathscr{I})}{\to} t + I$, *then* $\mathscr{T}^2_{U_\alpha} \overset{S^{p_U}(\mathscr{I})}{\to} (t + I)^2$
6. $\mathscr{T}_{U_\alpha} \overset{S^{p_U}(\mathscr{I})}{\to} t + I_1$ *and* $\mathscr{W}_{U_\alpha} \overset{S^{p_U}(\mathscr{I})}{\to} w + I_2$, *then* $\mathscr{T}_{U_\alpha}\mathscr{W}_{U_\alpha} \overset{S^{p_U}(\mathscr{I})}{\to} (t + I_1)(w + I_2)$ *where* $I_1 \neq I_2$,
7. $\mathscr{T}_{U_\alpha} \overset{S^{p_U}(\mathscr{I})}{\to} t + I_1$ *and* $\mathscr{W}_{U_\alpha} \overset{S^{p_U}(\mathscr{I})}{\to} w + I_2$, *then* $\mathscr{T}_{U_\alpha}\mathscr{W}_{U_\alpha} \overset{S^{p_U}(\mathscr{I})}{\to} [tw + I(1 + t + w)]$ *where* $I_1 = I_2 = I$,
8. $\mathscr{T}_{U_\alpha} \overset{S^{p_U}(\mathscr{I})}{\to} t + I_1$ *and* $\mathscr{W}_{U_\alpha} \overset{S^{p_U}(\mathscr{I})}{\to} w + I_2$, *then* $\mathscr{T}_{U_\alpha}/\mathscr{W}_{U_\alpha} \overset{S^{p_U}(\mathscr{I})}{\to} (t + I_1)/(w + I_2)$ *provided* $w \neq -I_2$.
9. $\mathscr{T}_{U_\alpha} \overset{S^{p_U}(\mathscr{I})}{\to} \mathscr{T}_U$ *and* $\mathscr{W}_{U_\alpha} \overset{S^{p_U}(\mathscr{I})}{\to} \mathscr{W}_U$, *then* $\mathscr{T}_{U_\alpha}\mathscr{W}_{U_\alpha} \overset{S^{p_U}(\mathscr{I})}{\to} \mathscr{T}_U\mathscr{W}_U$,
10. *If* $0 \leq \mathscr{T}_{U_\alpha} \leq \mathscr{W}_{U_\alpha}$ *and* $\mathscr{W}_{U_\alpha} \overset{S^{p_U}(\mathscr{I})}{\to} 0$, *then* $\mathscr{T}_{U_\alpha} \overset{S^{p_U}(\mathscr{I})}{\to} 0$,
11. *If* $\mathscr{T}_{U_\alpha} \overset{S^{p_U}(\mathscr{I})}{\to} \mathscr{T}_U$, *then for each* $\rho, \varsigma > 0$, *there exists* $\beta \in \mathbb{N}$ *so that any* $\rho > 0$

$$\left\{\alpha \in \mathbb{N} : \frac{1}{\alpha}|\{\beta \leq \alpha : \mathscr{P}(|\mathscr{T}_{U_\beta} - \mathscr{T}_U| \geq \rho) \geq \varsigma\}| \geq \rho\right\} \in \mathscr{I}.$$

This will be called the neutrosophic $\mathscr{I}$-statistical Cauchy condition in probability.

Proof. Let ρ, ς, ρ denote arbitrary positive real values. Then,

Properties (1), (2), (3), and (4) directly follow from the definitions, so their proofs are omitted.
(5) If $\mathscr{Z}_{U_\alpha} \overset{S^{\rho_U}(\mathscr{I})}{\to} 0$, then $\mathscr{Z}^2_{U_\alpha} \overset{S^{\rho_U}(\mathscr{I})}{\to} 0$ for

$$\{\beta \le \alpha : \mathscr{P}(|\mathscr{Z}^2_{U_\beta} - 0| \ge \rho) \ge \varsigma\} = \{\beta \le \alpha : \mathscr{Z}_{U_\beta} - 0| \ge \sqrt{\rho}) \ge \varsigma\}.$$

Next, let

$$\mathscr{T}^2_{U_\alpha} = (\mathscr{T}_{U_\alpha} - (t+I))^2 + 2(t+I)(\mathscr{T}_{U_\alpha} - (t+I)) + (t+I)^2 \overset{S^{\rho_U}(\mathscr{I})}{\to} (t+I)^2.$$

(6) We get

$$\begin{aligned} \mathscr{T}_{U_\alpha}\mathscr{W}_{U_\alpha} &= \tfrac{1}{4}\{(\mathscr{T}_{U_\alpha} + \mathscr{W}_{U_\alpha})^2 - (\mathscr{T}_{U_\alpha} - \mathscr{W}_{U_\alpha})^2\} \\ &\overset{S^{\rho_U}(\mathscr{I})}{\to} \tfrac{1}{4}\{(t+I_1+w+I_2)^2 - (t+I_1-(w+I_2))^2 = (t+I_1)(w+I_2). \end{aligned}$$

(7) We have

$$\begin{aligned} \mathscr{T}_{U_\alpha}\mathscr{W}_{U_\alpha} &= \tfrac{1}{4}\{(\mathscr{T}_{U_\alpha} + \mathscr{W}_{U_\alpha})^2 - (\mathscr{T}_{U_\alpha} - \mathscr{W}_{U_\alpha})^2\} \\ &\overset{S^{\rho_U}(\mathscr{I})}{\to} \tfrac{1}{4}\{(t+w+2I)^2 - (t-w)^2 = [tw + I(1+t+w)]. \end{aligned}$$

(8) Let $\mathscr{U}$ and $\mathscr{V}$ be events of $|\mathscr{W}_{U_\alpha} - (w+I_2)| < |w+I_2|$, $\left|\frac{1}{\mathscr{W}_{U_\alpha}} - \frac{1}{w+I_2}\right| \ge \rho$, respectively. Now,

$$\begin{aligned} \left|\frac{1}{\mathscr{W}_{U_\alpha}} - \frac{1}{w+I_2}\right| &= \frac{|\mathscr{W}_{U_\alpha} - (w+I_2)|}{|(w+I_2)\mathscr{W}_{U_\alpha}|} \\ &= \frac{|\mathscr{W}_{U_\alpha} - (w+I_2)}{|w+I_2||(w+I_2) + (\mathscr{W}_{U_\alpha} - (w+I_2)|} \\ &\le \frac{|\mathscr{W}_{U_\alpha} - (w+I_2)}{|w+I_2||(|w+I_2| - |\mathscr{W}_{U_\alpha} - (w+I_2)|)|}. \end{aligned}$$

If $\mathscr{U}$ and $\mathscr{V}$ occur simultaneously, then

$$|\mathscr{W}_{U_\alpha} - (w+I_2)| \ge \frac{\rho|w+I_2|^2}{1+\rho|w+I_2|},$$

that follows from the above inequality. Next, let $\rho_0 = \dfrac{\rho|w+I_2|^2}{1+\rho|w+I_2|}$ and let $\mathscr{G}$ be the event $|\mathscr{W}_{U_\alpha} - (w+I_2)| \ge \rho_0$. This implies $\mathscr{U}\mathscr{V} \subset \mathscr{G}$, then $\mathscr{P}(\mathscr{B}) \le \mathscr{C} + \mathscr{P}(\overline{\mathscr{A}})$, where the bar represents the set of complement. This implies

$$\begin{aligned} &\left\{\beta \le \alpha : \mathscr{P}(\left|\tfrac{1}{\mathscr{W}_{U_\alpha}} - \frac{1}{w+I_2}\right| \ge \rho) \ge \varsigma\right\} \\ &\subset \{\beta \le \alpha : \mathscr{P}(|\mathscr{W}_{U_\alpha} - (w+I_2)| \ge \rho_0) \\ &\ge \tfrac{1}{2}\varsigma \cup \{\beta \le \alpha : \mathscr{P}(\mathscr{W}_{U_\alpha} - (w+I_2)| \ge |w+I_2|) \ge \tfrac{1}{2}\varsigma\}. \end{aligned}$$

Therefore, $\frac{1}{\mathscr{W}_{U_\alpha}} \overset{S^{\rho_U}(\mathscr{I})}{\to} \dfrac{1}{w+I_2}$ provided $w \ne I_2$. Consequently, $\frac{\mathscr{T}_{U_\alpha}}{\mathscr{W}_{U_\alpha}} \overset{S^{\rho_U}(\mathscr{I})}{\to} \dfrac{t+I_1}{w+I_2}$ provided $w \ne I_2$. If $I_2 = I_1 = I$, it can be seen that $\frac{\mathscr{T}_{U_\alpha}}{\mathscr{W}_{U_\alpha}} \overset{S^{\rho_U}(\mathscr{I})}{\to} \frac{t+I}{w+I}$, provided $w \ne I$.
(10) First at all, we should prove that if $\mathscr{T}_{U_\alpha} \overset{S^{\rho_U}(\mathscr{I})}{\to} \mathscr{T}_U$ and $\mathscr{Z}_U$ is a neutrosophic random variable then $\mathscr{T}_{U_\alpha}\mathscr{Z}_U \overset{S^{\rho_U}(\mathscr{I})}{\to} \mathscr{T}_U\mathscr{Z}_U$. Since $\mathscr{Z}_U$ is a neutrosophic random variable, given $\varsigma > 0$, there exists an $\kappa > 0$ such that $\mathscr{P}(|\mathscr{Z}_N| > \kappa) \le \frac{1}{2}\varsigma$. Then, for any $\rho > 0$,

$$\begin{aligned} \mathscr{P}(\mathscr{T}_{U_\alpha}\mathscr{Z}_U - \mathscr{T}_U\mathscr{Z}_U| \ge \rho) &= \mathscr{P}(|\mathscr{T}_{U_\alpha} - \mathscr{T}_U||\mathscr{Z}_U| \ge \rho, |\mathscr{Z}_U > \kappa) \\ &+ \mathscr{P}(|\mathscr{T}_{U_\alpha} - \mathscr{T}_U||\mathscr{Z}_U| \ge \rho, |\mathscr{Z}_U \le \kappa) \\ &\le \tfrac{1}{2}\varsigma + \mathscr{P}(|\mathscr{T}_{U_\alpha} - \mathscr{T}_U| \ge \rho/\kappa). \end{aligned}$$

This implies,

$$\begin{aligned} &\{\beta \le \alpha : \mathscr{P}(|\mathscr{T}_{U_\beta}\mathscr{Z}_U - \mathscr{T}_U\mathscr{Z}_U| \ge \rho) \ge \varsigma\} \\ &\subset \{\beta \le \alpha : \mathbb{P}(|\mathscr{T}_{U_\beta} - (t+I)| \ge \rho/\kappa) \ge \tfrac{1}{2}\varsigma\} \in \mathscr{I}. \end{aligned}$$

Thus, $(\mathscr{T}_{U_\alpha} - \mathscr{T}_U)(\mathscr{W}_{U_\alpha} - \mathscr{W}_U) \overset{S^{\rho_U}(\mathscr{I})}{\to} 0$. Therefore, this implies $\mathscr{T}_{U_\alpha}\mathscr{W}_{U_\alpha} \overset{S^{\rho_U}(\mathscr{I})}{\to} \mathscr{T}_U\mathscr{W}_U$.
(11) Proof is straightforward and hence omitted.
(12) Take $\beta \in \mathbb{N}$ such that $\mathscr{P}(|\mathscr{T}_{U_\beta} - \mathscr{T}_U| \ge \frac{1}{2}\rho) < \frac{1}{2}\varsigma$. Then,

$$\begin{aligned} &\{\beta \le \alpha : \mathscr{P}(|\mathscr{T}_{U_\alpha} - \mathscr{T}_{U_\beta}| \ge \rho) \ge \varsigma\} \ge \rho\} \\ &\subset \{\beta \le \alpha : \mathscr{P}(|\mathscr{T}_{U_\beta} - \mathscr{T}_U| \ge \tfrac{1}{2}\rho) \ge \tfrac{1}{2}\varsigma\} \ge \rho\} \in \mathscr{I}. \end{aligned}$$

□

4. Neutrosophic $\mathscr{I}$-Lacunary Statistical Convergence in Probability

Fridy [25] defined a lacunary sequence is an increasing integer sequence $\theta = \{\mathfrak{s}_\upsilon\}_{\upsilon\in\mathbb{N}\cup\{0\}}$ such that $\mathfrak{s}_0 = 0$ and $\mathfrak{h}_\upsilon = \mathfrak{s}_\upsilon - \mathfrak{s}_{\upsilon-1} \to \infty$, as $\upsilon \to \infty$; and $I_\upsilon = (\mathfrak{s}_{\upsilon-1}, \mathfrak{s}_\upsilon]$ and $\mathfrak{q}_\upsilon = \frac{\mathfrak{s}_\upsilon}{\mathfrak{s}_{\upsilon-1}}$.
Next, we define neutrosophic $\mathscr{I}$-lacunary statistical convergence in probability:

Definition 4.1. *The sequence $\{\mathscr{T}_{U_\alpha}\}_{\alpha\in\mathbb{N}}$ is regarded as neutrosophic $\mathscr{I}$-lacunary statistically (N-$\mathscr{I}$-S_θ) convergent in probability to a NRV $\mathscr{T}_U$, where $\mathscr{T} : \mathscr{S} \to \mathbb{R}$, if for any $\rho, \varsigma, \rho > 0$,*

$$\left\{\upsilon \in \mathbb{N} : \frac{1}{\mathfrak{h}_\upsilon}|\{\beta \in I_\upsilon : \mathscr{P}(|\mathscr{T}_{U_\beta} - \mathscr{T}_U| \ge \rho) \ge \varsigma\}| \ge \rho\right\} \in \mathscr{I}.$$

This convergence is represented by $\mathscr{T}_{U_\alpha} \stackrel{S_\theta^{P_N}(\mathscr{I})}{\to} \mathscr{T}_U$. The collection of all sequences of NRVs that are N-$\mathscr{I}$-S_θ-convergent in probability is referred to as $S_\theta^{P_N}(\mathscr{I})$.

Example 4.2. *Let $\mathscr{T}_{U_\alpha}(\omega) = \omega + \frac{(-1)^\alpha}{\alpha}$, where ω is a random variable uniformly distributed over $[0,1]$, and define $\mathscr{T}_U(\omega) = \omega$. Let the lacunary sequence $\theta = \{k_r\}_{r\in\mathbb{N}}$ be given by $k_r = 2^r$, so that the lacunary intervals are $I_r = (k_{r-1}, k_r] = (2^{r-1}, 2^r]$, and the interval length is $\mathfrak{h}_r = k_r - k_{r-1} = 2^{r-1}$. Let the ideal $\mathscr{I}$ be the family of subsets of $\mathbb{N}$ with natural density zero. Now consider the sequence $\{\mathscr{T}_{U_\alpha}\}$. For each α, we have*

$$|\mathscr{T}_{U_\alpha}(\omega) - \mathscr{T}_U(\omega)| = \left|\frac{(-1)^\alpha}{\alpha}\right| = \frac{1}{\alpha}.$$

Then the probability that the absolute difference exceeds any fixed $\rho > 0$ is

$$\mathscr{P}(|\mathscr{T}_{U_\alpha} - \mathscr{T}_U| \ge \rho) = \begin{cases} 1, & \text{if } \frac{1}{\alpha} \ge \rho, \\ 0, & \text{otherwise}. \end{cases}$$

Let us fix $\rho = 0.1$, $\varsigma = 0.5$, and $\rho = 0.25$. In each lacunary interval I_r, the number of indices β such that $\mathscr{P}(|\mathscr{T}_{U_\beta} - \mathscr{T}_U| \ge \rho) \ge \varsigma$ is finite and gets smaller as β increases. Specifically, for large enough r, most $\beta \in I_r$ satisfy $\frac{1}{\beta} < \rho$, so the corresponding probability is zero. Therefore, the proportion

$$\frac{1}{\mathfrak{h}_r}\left|\left\{\beta \in I_r : \mathscr{P}(|\mathscr{T}_{U_\beta} - \mathscr{T}_U| \ge \rho) \ge \varsigma\right\}\right|$$

becomes less than any $\rho > 0$ for large r. Hence, the set

$$\left\{r \in \mathbb{N} : \frac{1}{\mathfrak{h}_r}\left|\left\{\beta \in I_r : \mathscr{P}(|\mathscr{T}_{U_\beta} - \mathscr{T}_U| \ge \rho) \ge \varsigma\right\}\right| \ge \rho\right\}$$

has natural density zero, so it belongs to $\mathscr{I}$.
Thus, we conclude that $\mathscr{T}_{U_\alpha} \stackrel{S_\theta^{P_N}(\mathscr{I})}{\to} \mathscr{T}_U$, i.e., the sequence is neutrosophic $\mathscr{I}$-lacunary statistically convergent in probability to $\mathscr{T}_U$.

Theorem 4.3. *If $\mathscr{T}_{U_\alpha} \stackrel{S_\theta^{P_N}(\mathscr{I})}{\to} \mathscr{T}_U$ and $\mathscr{T}_{U_\alpha} \stackrel{S_\theta^{P_N}(\mathscr{I})}{\to} \mathscr{Y}_U$, then $\mathscr{P}\{\mathscr{T}_U = \mathscr{Y}_U\} = 1$.*

Proof. Let $\mathscr{P}(|\mathscr{T}_U - \mathscr{Y}_U| \ge \rho) = \varsigma > 0$, where for some $\rho > 0$. Then,

$$\begin{aligned}\mathscr{P}(|\mathscr{T}_U - \mathscr{Y}_U| \ge \rho) \le \mathscr{P}\left(|\mathscr{T}_{U_\beta} - \mathscr{T}_U| \ge \frac{\rho}{2}\right) \\ + \mathscr{P}\left(|\mathscr{T}_{U_\beta} - \mathscr{Y}_U| \ge \frac{\rho}{2}\right).\end{aligned}$$

Thus,

$$\begin{aligned}&\left\{\upsilon \in \mathbb{N} : \frac{1}{\mathfrak{h}_\upsilon}|\{U \in I_\upsilon : \mathscr{P}(|\mathscr{T}_U - \mathscr{Y}_U| \ge \rho) \ge \varsigma\}| \ge \tfrac{1}{2}\right\} \\ &\subset \left\{\upsilon \in \mathbb{N} : \frac{1}{\mathfrak{h}_\upsilon}|\{\beta \in I_\upsilon : \mathscr{P}(|\mathscr{T}_{U_\beta} - \mathscr{T}_U| \ge \tfrac{\rho}{2}) \ge \tfrac{\varsigma}{2}\}| \ge \tfrac{1}{4}\right\} \\ &\cup \left\{\upsilon \in \mathbb{N} : \frac{1}{\mathfrak{h}_\upsilon}|\{\beta \in I_\upsilon : \mathscr{P}(|\mathscr{T}_{U_\beta} - \mathscr{Y}_U| \ge \tfrac{\rho}{2}) \ge \tfrac{\varsigma}{2}\}| \ge \tfrac{1}{4}\right\} \in \mathscr{I},\end{aligned}$$

where $\mathscr{N}$ is a neutrosophic set. □

Theorem 4.4. *The following statements are equivalent:*

1. $\mathscr{T}_{U_\alpha} \stackrel{S_\theta^{P_N}(\mathscr{I})}{\to} \mathscr{T}_U$.
2. *For all $\rho, \varsigma > 0$,*

$$\left\{\upsilon \in \mathbb{N} : \frac{1}{\mathfrak{h}_\upsilon}\sum_{\alpha\in I_\upsilon} \mathscr{P}(|\mathscr{T}_{U_\alpha} - \mathscr{T}_U| \ge \rho) \ge \varsigma\right\} \in \mathscr{I}.$$

Proof. We begin proving $(1) \Rightarrow (2)$: First, let's consider that $\mathscr{T}_{U_\alpha} \overset{S_\theta^{PN}(\mathscr{I})}{\rightarrow} \mathscr{T}_U$, then we have

$$\frac{1}{\mathfrak{h}_\upsilon}\sum_{\alpha\in I_\upsilon}\mathscr{P}(|\mathscr{T}_{U_\alpha}-\mathscr{T}_U|\geq\rho)\leq\frac{1}{\mathfrak{h}_\upsilon}|\{\beta\in I_\upsilon:\mathscr{P}(|\mathscr{T}_{U_\beta}-\mathscr{T}_U|\geq\rho)\geq\frac{\varsigma}{2}\}|+\frac{\varsigma}{2}.$$

Consequently, we obtain

$$\left\{\upsilon\in\mathbb{N}:\frac{1}{\mathfrak{h}_\upsilon}\Sigma_{\alpha\in I_\upsilon}\mathscr{P}(|\mathscr{T}_{U_\alpha}-\mathscr{T}_U|\geq\rho)\geq\varsigma\right\}$$
$$\subset\left\{\upsilon\in\mathbb{N}:\frac{1}{\mathfrak{h}_\upsilon}|\{\beta\in I_\upsilon:\mathscr{P}(|\mathscr{T}_{U_\beta}-\mathscr{Y}_U|\geq\rho)\geq\tfrac{\varsigma}{2}\}|\geq\tfrac{\varsigma}{2}\right\}\in\mathscr{I}.$$

Next, we prove $(2) \Rightarrow (1)$: Let's consider that for all $\rho, \varsigma > 0$,

$$\left\{\upsilon\in\mathbb{N}:\frac{1}{\mathfrak{h}_\upsilon}\sum_{\alpha\in I_\upsilon}\mathscr{P}(|\mathscr{T}_{U_\alpha}-\mathscr{T}_U|\geq\rho)\geq\varsigma\right\}\in\mathscr{I}.$$

supplies. Then,

$$\frac{1}{\varsigma\mathfrak{h}_\upsilon}\sum_{\alpha\in I_\upsilon}\mathscr{P}(|\mathscr{T}_{U_\alpha}-\mathscr{T}_U|\geq\rho)\geq\frac{1}{\mathfrak{h}_\upsilon}|\{\beta\in I_\upsilon:\mathscr{P}(|\mathscr{T}_{U_\beta}-\mathscr{Y}_U|\geq\rho)\geq\varsigma\}|.$$

Then, for any $\rho > 0$,

$$\left\{\upsilon\in\mathbb{N}:\frac{1}{\mathfrak{h}_\upsilon}|\{\beta\in I_\upsilon:\mathscr{P}(|\mathscr{T}_{U_\beta}-\mathscr{Y}_U|\geq\rho)\geq\varsigma\}|\geq\rho\right\}$$
$$\subset\left\{\upsilon\in\mathbb{N}:\frac{1}{\varsigma\mathfrak{h}_\upsilon}\Sigma_{\alpha\in I_\upsilon}\mathscr{P}(|\mathscr{T}_{U_\alpha}-\mathscr{T}_U|\geq\rho)\geq\varsigma\rho\right\}\in\mathscr{I}.$$

Therefore, we have $\mathscr{T}_{U_\alpha} \overset{S_\theta^{PN}(\mathscr{I})}{\rightarrow} \mathscr{T}_U$. □

5. Neutrosophic $\mathscr{I}$- λ-Statistical Convergence in Probability

In [17], Ghosal formulated the following concepts. Let $\lambda = \{\lambda_\alpha\}_{\alpha\in\mathbb{N}}$ be a non-decreasing sequence of positive real numbers such that $\lambda_1 = 1$, $\lambda_{\alpha+1} \leq \lambda_\alpha + 1$ and $\lambda_\alpha \to \infty$ as $\alpha \to \infty$. For a given sequence of real numbers $(w_\beta)_{\beta\in\mathbb{N}}$, the generalised De la Valeé-Pousin mean is demonstrated by

$$t_\alpha(x)=\frac{1}{\lambda_\alpha}\sum_{\beta\in Q_\alpha}w_\beta$$

where $Q_\alpha = [\alpha-\lambda_\alpha+1,\alpha]$.
Next, we present the definition of neutrosophic $\mathscr{I}$-λ-statistical convergence in probability:

Definition 5.1. *The sequence* $\{\mathscr{T}_{U_\alpha}\}_{\alpha\in\mathbb{N}}$ *is termed neutrosophic* $\mathscr{I}$-λ*-statistical convergent in probability to a NRV* $\mathscr{T}_U$*, where* $\mathscr{T}:\mathscr{S}\to\mathbb{R}$*, if for any* $\rho, \varsigma, \rho > 0$,

$$\left\{\alpha\in\mathbb{N}:\frac{1}{\lambda_\alpha}|\{\beta\in Q_\alpha:\mathscr{P}(|\mathscr{T}_{U_\beta}-\mathscr{T}_U|\geq\rho)\geq\varsigma\}|\geq\rho\right\}\in\mathscr{I}.$$

This type of convergence will be written as $\mathscr{T}_{U_\alpha} \overset{S_\lambda^{PU}(\mathscr{I})}{\rightarrow} \mathscr{T}_U$.

Example 5.2. *Let* $\mathscr{T}_{U_\alpha}(\omega) = \omega + \frac{1}{\sqrt{\alpha}}$ *for each* $\alpha\in\mathbb{N}$*, where* ω *is a random variable uniformly distributed over* $[0,1]$*, and define* $\mathscr{T}_U(\omega) = \omega$*. Then*

$$|\mathscr{T}_{U_\alpha}(\omega)-\mathscr{T}_U(\omega)|=\frac{1}{\sqrt{\alpha}}.$$

Thus,

$$\mathscr{P}(|\mathscr{T}_{U_\alpha}-\mathscr{T}_U|\geq\rho)=\begin{cases}1, & \text{if } \frac{1}{\sqrt{\alpha}}\geq\rho,\\ 0, & \text{otherwise}.\end{cases}$$

Now, define the non-decreasing sequence $\lambda = \{\lambda_\alpha\}_{\alpha\in\mathbb{N}}$ *as* $\lambda_\alpha = \alpha$ *and the intervals* $Q_\alpha = [\alpha-\lambda_\alpha+1,\alpha] = [1,\alpha]$ *for all* $\alpha\in\mathbb{N}$*. Let the ideal* $\mathscr{I}$ *be the family of subsets of* $\mathbb{N}$ *with natural density zero.*
Fix $\rho = 0.05$*,* $\varsigma = 0.5$*, and* $\rho = 0.2$*. Then for large* α*,* $\frac{1}{\sqrt{\alpha}} < \rho$*, so*

$$\mathscr{P}(|\mathscr{T}_{U_\beta}-\mathscr{T}_U|\geq\rho)<\varsigma\quad\text{for most } \beta\in Q_\alpha.$$

Therefore, the number of $\beta\in Q_\alpha$ *for which the probability exceeds* ς *becomes negligible compared to* λ_α*, i.e.,*

$$\frac{1}{\lambda_\alpha}\left|\left\{\beta\in Q_\alpha:\mathscr{P}(|\mathscr{T}_{U_\beta}-\mathscr{T}_U|\geq\rho)\geq\varsigma\right\}\right|<\rho$$

for all sufficiently large α.
Hence, the set

$$\left\{\alpha \in \mathbb{N} : \frac{1}{\lambda_\alpha}\left|\left\{\beta \in Q_\alpha : \mathscr{P}(|\mathscr{T}_{U_\beta} - \mathscr{T}_U| \geq \rho) \geq \varsigma\right\}\right| \geq \rho\right\}$$

is finite, and so belongs to the ideal $\mathscr{I}$.
Thus, we conclude that $\mathscr{T}_{U_\alpha} \overset{S_\lambda^{PU}(\mathscr{I})}{\rightarrow} \mathscr{T}_U$, *i.e., the sequence is neutrosophic* $\mathscr{I}$-λ-*statistically convergent in probability to* $\mathscr{T}_U$.

Definition 5.3. *The sequence* $\{\mathscr{T}_{U_\alpha}\}_{\alpha\in\mathbb{N}}$ *is regarded as neutrosophic* $[V,\lambda]$-$\mathscr{I}$-*summability in probability to a NRV* $\mathscr{T}_U$, *where* $\mathscr{X} : \mathscr{S} \to \mathbb{R}$, *provided that for any* $\rho, \varsigma > 0$,

$$\left\{\alpha \in \mathbb{N} : \frac{1}{\lambda_\alpha}\sum_{\beta\in Q_\alpha} \mathscr{P}(|\mathscr{T}_{U_\beta} - \mathscr{T}_U| \geq \rho) \geq \varsigma\right\} \in \mathscr{I}.$$

We denote this summability as $\mathscr{T}_{U_\alpha} \overset{[V,\lambda]^{PU}(\mathscr{I})}{\rightarrow} \mathscr{T}_U$.

Example 5.4. *Let* $\mathscr{T}_{U_\alpha}(\omega) = \omega + \frac{1}{\alpha}$ *for each* $\alpha \in \mathbb{N}$, *where* ω *is a random variable uniformly distributed on* $[0,1]$, *and define* $\mathscr{T}_U(\omega) = \omega$. *Then,*

$$|\mathscr{T}_{U_\alpha}(\omega) - \mathscr{T}_U(\omega)| = \frac{1}{\alpha},$$

so that

$$\mathscr{P}(|\mathscr{T}_{U_\alpha} - \mathscr{T}_U| \geq \rho) = \begin{cases} 1, & \text{if } \frac{1}{\alpha} \geq \rho, \\ 0, & \text{otherwise}. \end{cases}$$

Now, define $\lambda = \{\lambda_\alpha\}_{\alpha\in\mathbb{N}}$ *by* $\lambda_\alpha = \alpha$, *and* $Q_\alpha = [\alpha - \lambda_\alpha + 1, \alpha] = [1, \alpha]$. *Let* $\mathscr{I}$ *be the family of all subsets of* $\mathbb{N}$ *with natural density zero. Fix* $\rho = 0.01$ *and* $\varsigma = 0.1$. *For sufficiently large* α, *we have* $\frac{1}{\alpha} < \rho$, *so* $\mathscr{P}(|\mathscr{T}_{U_\beta} - \mathscr{T}_U| \geq \rho) = 0$ *for most* $\beta \in Q_\alpha$. *Therefore,*

$$\frac{1}{\lambda_\alpha}\sum_{\beta\in Q_\alpha} \mathscr{P}(|\mathscr{T}_{U_\beta} - \mathscr{T}_U| \geq \rho) \to 0 \quad \text{as } \alpha \to \infty.$$

Hence, the set

$$\left\{\alpha \in \mathbb{N} : \frac{1}{\lambda_\alpha}\sum_{\beta\in Q_\alpha} \mathscr{P}(|\mathscr{T}_{U_\beta} - \mathscr{T}_U| \geq \rho) \geq \varsigma\right\}$$

is finite and therefore belongs to the ideal $\mathscr{I}$.
Consequently, $\mathscr{T}_{U_\alpha} \overset{[V,\lambda]^{PU}(\mathscr{I})}{\rightarrow} \mathscr{T}_U$, *i.e., the sequence is neutrosophic* $[V,\lambda]$-$\mathscr{I}$-*summable in probability to* $\mathscr{T}_U$.

Theorem 5.5. *For any sequence of NRVs* $\{\mathscr{X}_{N_n}\}_{n\in\mathbb{N}}$, *The statements listed below are equivalent:*

1. $\mathscr{T}_{U_\alpha} \overset{S_\lambda^{PU}(\mathscr{I})}{\rightarrow} \mathscr{T}_U$.
2. $\mathscr{T}_{U_\alpha} \overset{[V,\lambda]^{PU}(\mathscr{I})}{\rightarrow} \mathscr{T}_U$.

Proof. The proof proceeds in a similar manner to that of Theorem 4.4 and is therefore omitted. □

Theorem 5.6. *If* $\mathscr{T}_{U_\alpha} \overset{S_\lambda^{PU}(\mathscr{I})}{\rightarrow} \mathscr{T}_U$ *and* $\mathscr{T}_{U_\alpha} \overset{S_\lambda^{PU}(\mathscr{I})}{\rightarrow} \mathscr{Y}_U$, *then* $\mathscr{P}\{\mathscr{T}_N = \mathscr{Y}_N\} = 1$.

Proof. Since the reasoning parallels that of Theorem 3.2, we omit the detailed proof. □

6. Conclusion

In this paper, we have introduced certain notions of statistical convergence in probability for sequences of NRVs. We also established some fundamental properties and examined their interrelations. Furthermore, it can be observed that the results presented here extend the classical framework developed in [17–19]. This aligns with the viewpoint expressed by Smarandache in [1, 20], which suggests that the neutrosophic statistical framework provides a broader generalization than its classical counterpart. For future studies, we recommend extending these notions by using [26–30].

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