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Research Article

Halpern-type relaxed algorithms with alternated and multi-step inertia for split feasibility problems with applications in classification problems

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ABSTRACT. In this article, we construct two Halpern-type relaxed algorithms with alternated and multi-step inertial extrapolation steps for split feasibility problems in infinite-dimensional Hilbert spaces. The first is the most general inertial method that employs three inertial steps in a single algorithm, one of which is an alternated inertial step, while the others are multi-step inertial steps, representing the recent improvements over the classical inertial step. Besides the inertial steps, the second algorithm uses a three-term conjugate gradient-like direction, which accelerates the sequence of iterates toward a solution of the problem. In proving the convergence of the second algorithm, we dispense with some of the restrictive assumptions in some conjugate gradient-like methods. Both algorithms employ a self-adaptive and monotonic step-length criterion, which does not require a knowledge of the norm of the underlying operator or the use of any line search procedure. Moreover, we formulate and prove some strong convergence theorems for each of the algorithms based on the convergence theorem of an alternated inertial Halpern-type relaxed algorithm with perturbations in real Hilbert spaces. Further, we analyse their applications to classification problems for some real-world datasets based on the extreme learning machine (ELM) with the ℓ_1-regularization approach (that is, the Lasso model) and the $\ell_1 - \ell_2$ hybrid regularization approach. Furthermore, we investigate their performance in solving a constrained minimization problem in infinite-dimensional Hilbert spaces. Finally, the numerical results of all experiments show that our proposed methods are robust, computationally efficient and achieve better generalization performance and stability than some existing algorithms in the literature.

Keywords: Relaxed $\mathcal{CQ}$ method; Alternated inertial method; Multi-step inertial method; Conjugate gradient method, Split feasibility problem; Classification problem.

2020 Mathematics Subject Classification: 47H05, 47J20, 47J25, 47J30, 65K15, 90C25.

1. INTRODUCTION

Throughout this work, let $\mathcal{H}_1$ and $\mathcal{H}_2$ be real Hilbert spaces, $\mathcal{C}$ and $\mathcal{Q}$ denote nonempty closed and convex sets in $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively, and $\mathcal{B} : \mathcal{H}_1 \to \mathcal{H}_2$ be a bounded linear operator. The split feasibility problem, first introduced by Censor and Elfving [10], is the problem of finding a point $x^* \in \mathcal{C}$ such that

$$\mathcal{B}x^* \in \mathcal{Q}. \tag{1.1}$$

Most of the motivations for studying problem (1.1) stem from its usefulness is solving various inverse problems arising from many real-world applications, such as X-ray tomography [41], machine learning [50, 13], image and signal reconstruction and jointly constrained Nash equilibrium [20, 52], to mention but just a few. The primary task in studying problem (1.1) is

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to develop a robust and efficient numerical algorithm for its solution. Based on the following fixed point problem:

$$x = P_{\mathcal{C}}\big(I - \tau\mathcal{B}^*(I - P_{\mathcal{Q}})\mathcal{B}\big)x \tag{1.2}$$

and the particular case of a Fréchet differentiable real-valued function $g : \mathcal{H}_1 \to \mathbb{R}$ defined by

$$g(x) = \frac{1}{2}||(I - P_{\mathcal{Q}})\mathcal{B}x||^2, \tag{1.3}$$

the iterative algorithm called the $\mathcal{CQ}$ algorithm for solving problem (1.1) was firstly developed by Byrne [7], which is recursively generated for any initial point $x_0 \in \mathcal{H}_1$ by

$$x_{n+1} = P_{\mathcal{C}}\big(x_n - \tau\mathcal{B}^*(I - P_{\mathcal{Q}})\mathcal{B}x_n\big), \quad \forall n \geq 0, \tag{1.4}$$

where $P_{\mathcal{C}} : \mathcal{H}_1 \to \mathcal{C}$ and $P_{\mathcal{Q}} : \mathcal{H}_2 \to \mathcal{Q}$ are the metric (orthogonal) projection operators, I is the identity operator in $\mathcal{H}_1$, $\mathcal{B}^*$ is the adjoint of $\mathcal{B}$ and $\tau \in \left(0, \frac{2}{||\mathcal{B}||^2}\right)$ is the step-length. However, in many practical application, there are two major drawbacks in the implementations of Algorithm (1.4): the first is that, it requires in each iteration to computes two projections $P_{\mathcal{C}}$ and $P_{\mathcal{Q}}$, which depends heavily on the geometry of the sets $\mathcal{C}$ and $\mathcal{Q}$, these are extremely expensive operations and sometimes not even possible for a wide range of practical problems and the second is that, the step length depends on the information of the norm of $\mathcal{B}$, which is generally very hard to obtain in many practice.

By defining $\mathcal{C}$ and $\mathcal{Q}$ as the following sub level sets:

$$\mathcal{C} = \{x \in \mathcal{H}_1 : c(x) \leq 0\}, \quad \mathcal{Q} = \{t \in \mathcal{H}_2 : q(t) \leq 0\}, \tag{1.5}$$

where $c : \mathcal{H}_1 \to \mathbb{R}$ and $q : \mathcal{H}_2 \to \mathbb{R}$ are weakly lower semi-continuous and convex functions and the two half-spaces at points x_n by

$$\mathcal{C}_n = \{x \in \mathcal{H}_1 : c(x_n) \leq \langle \phi_n, x_n - x\rangle\}, \quad \mathcal{Q}_n = \{t \in \mathcal{H}_2 : q(\mathcal{B}x_n) \leq \langle \varphi_n, \mathcal{B}x_n - t\rangle\}, \tag{1.6}$$

with $\phi_n \in \partial c(x_n)$, $\varphi_n \in \partial q(\mathcal{B}x_n)$, $\mathcal{C} \subseteq \mathcal{C}_n$ and $\mathcal{Q} \subseteq \mathcal{Q}_n$ for each $n \geq 0$, Yang [59] proposed the relaxed version of the method (1.4), which suggests to replace the two arbitrary sets $\mathcal{C}$ and $\mathcal{Q}$ with the half-spaces $\mathcal{C}_n$ and $\mathcal{Q}_n$, respectively, so that the projections $P_{\mathcal{C}_n}$ and $P_{\mathcal{Q}_n}$ can easily be computed using their known closed-form expressions (see [5], Example 29.20).

On the other hand, some researchers have suggested some methods, which do not require the calculation of $||\mathcal{B}||$. One of such methods is that of Qu and Xiu [44], in which they adopted an Armijo-like step length and presented a modified version of the algorithm in [59]. In this light, the authors of the works in [18, 23, 49] subsequently proposed some algorithms with Armijo-like step lengths to solve problem (1.1). It has been noted that finding the step length that is appropriate in each iteration using Armijo-like step length involves multiple search procedures, which may leads to an inefficiency in the performance and computations of the algorithms. To mitigate this drawback, Dong et al. [21] proposed an adaptive relaxed algorithm for the problem (1.1), in which the authors adopted the simple ways of computing a monotonic step length in each iteration based on the information of the previous iterates. Similarly, very recently, Tan et al. [53] introduced another adaptive relaxed algorithm based on the non-monotonic step length technique.

However, various researchers attempt to construct some methods with fast convergence properties, since they are mostly required in various applications [12, 32]. In recent years, some authors developed various algorithms [50, 46, 4, 39, 45, 51, 42, 27, 58, 2] based on Polyak's inertial method [43], to improve their convergence rates. However, it has been noted in several instances that the speed of some methods with Polyak's one-step inertial term

$$x_n + \lambda(x_n - x_{n-1}), \quad \forall\lambda > 0,$$

appear to be slower than their corresponding non-inertial ones (see [6, 38] and the references therein). Thus some authors [37, 14, 19] suggested to employ the idea of the multi-step inertial technique, which could help to maintained the expected improvements in the speed of these schemes. Additionally, to improve the speed of the inertial algorithms, the idea of the general inertial technique with two inertial steps was introduced by Dong et al. [17], which includes the classical Polyak's inertial method as a special case. Some researchers incorporated the idea of the general inertial method to improve the performance of their methods with several real-world applications (see e.g., [35, 57]). Similarly, motivated by the idea of the multi-step inertial technique and that of the general inertial technique, Dong et al. [19] introduced the general multi-step inertial Krasnosel'skíí -Mann algorithm, which is formulated as follows:

$$\begin{cases} w_n = x_n + \sum_{k\in K_n} \gamma_{n,k}(x_{n-k} - x_{n-k-1}), \\ v_n = x_n + \sum_{k\in K_n} \delta_{n,k}(x_{n-k} - x_{n-k-1}), \\ x_{n+1} = (1-\alpha_n)w_n + \alpha_n T v_n, \quad \forall n \geq 1, \end{cases} \tag{1.7}$$

where $K_n \subseteq \{0, 1, 2, \cdots, n-1\}$, $\gamma_{n,k},\ \delta_{n,k} \in (-1, 2]$. They proved its weak convergence to a fixed point of a nonexpansive operator T based on the convergence of the Krasnosel'skíí-Mann algorithm with perturbations in a real Hilbert space. They numerically show that the scheme (1.7) is faster than some inertial methods in solving the problems considered in [19]. Additionally, for any two given points x_{n-1} and x_n for each $n \geq 1$, Mu and Peng [40] suggested the following alternated inertial term:

$$y_n = \begin{cases} x_n, & \text{if } n \text{ is even}, \\ x_n + \lambda_n(x_n - x_{n-1}), & \text{if } n \text{ is odd}, \end{cases} \tag{1.8}$$

which is a modification of the Polyak's inertial method. The advantage of the modified version in (1.8) is its ability to recover Fejér monotonicity property of its even subsequence in relation to the set of the solutions of a problem. This important property is usually lost in the case of the non-modified version. Very recently, some methods based on (1.8) for solving problem (1.1) were developed [21, 53, 48, 1]. Although the algorithms in [21, 53, 48] based on (1.8) were shown to achieve better computational efficiencies when their numerical results are compared with some existing methods on signal and image processing problems, but their weak convergence property was only obtained.

Additionally, in view of (1.3) and the fact that $\nabla g(x) = \mathcal{B}^*(I - P_{\mathcal{Q}})\mathcal{B}x$, it is not difficult to see that all the aforementioned methods for solving problem (1.1), such as those in [50, 7, 59, 44, 18, 23, 49, 21, 53, 46, 39, 45, 48, 1] are hybrid steepest-types with the directions $d_n = -\nabla g_n(x_n)$ at a point x_n. However, as noted from [33], the accelerated versions of these methods may be constructed when considered with the following conjugate gradient-like direction (1.9) or the three-term conjugate gradient-like direction (1.10) (see [31, 30]):

$$d_n = -\nabla g_n(x_n) + \varsigma_n^{(1)} d_{n-1} \tag{1.9}$$

and

$$d_n = -\nabla g_n(x_n) + \varsigma_n^{(1)} d_{n-1} - \varsigma_n^{(2)} s_n, \forall n \geq 1, \tag{1.10}$$

respectively, where, for each $i = 1, 2$, $\varsigma_n^{(i)} \in [0, \infty)$ and $s_n \in \mathcal{H}_1$ is an arbitrary point. As numerically shown in [33, 31, 30], provided that, for each $i = 1, 2$, $\lim_{n\to\infty} \varsigma_n^{(i)} = 0$ and $\{s_n\}$ is bounded, the hybrid gradient method with the direction (1.10) is faster than its variant with the direction (1.9). In the light of this, some authors improved their iterative methods by combining them with either of the directions (1.9) or (1.10) for different problems (see [26, 16, 3, 36] and the references therein). Recently, motivated by the self-adaptive relaxed algorithm [60], Polyak's one-step inertial method [43] and the conjugate gradient-like direction (1.9), Che et al. [11] proposed the accelerated relaxed algorithm for the problem (1.1) in finite-dimensional real Hilbert spaces. Although the proposed algorithm in [11] with the conjugate gradient-like direction (1.9) has achieved some good performance on signal and image restoration problems, but it is noted that its convergence results heavely rely on the conditions that, for any sequence $\{x_n\}$ generated by the their algorithm, the sequences $\{(I - P_{\mathcal{C}_n})x_n\}$ and $\{(I - P_{\mathcal{Q}_n})\mathcal{B}x_n\}$ are bounded. These are very restrictive assumptions and it would be of great interest to dispense them.

Motivated and inspired by the results in [21, 53, 37, 17, 40, 33], we first develop an alternated inertial Halpern-type relaxed $\mathcal{CQ}$ algorithm with perturbations (AiHRAP), which employs the monotonic self-adaptive step length criterion that does not require any information about the norm of the operator or the use of a line search procedure. Moreover, we establish its strong convergence to a minimum-norm solution of problem (1.1) in infinite-dimensional real Hilbert spaces. Further, we introduce two extensions of AiHRAP: the first is an alternated and multi-step inertial Halpern-type relaxed $\mathcal{CQ}$ algorithm (AMiHRA), which to the best of our knowledge is the most general inertial method in the literature that involves three steps of the recent improvements of the classical inertial method, one of which is the alternated inertial step [40], while the others are the multi-step inertial steps [37], and the second is an accelerated alternated and multi-step inertial Halpern-type relaxed algorithm (AAMiHRA) that combines the three term conjugate gradient-like direction [33] and two steps of the aforementioned improved versions of the inertial term with the monotonic self-adaptive step length criterion. Moreover, we analyse their applications on classification problems for some real-world datasets based on the extreme learning machine (ELM) with the ℓ_1-regularization approach (that is, the Lasso model) and the $\ell_1 - \ell_2$ hybrid regularization approach. Furthermore, we investigate their performance in solving constrained minimization problems in infinite-dimensional Hilbert spaces.

2. Preliminaries

In this work, we use $x_n \to x^*$ (resp., $x_n \rightharpoonup x^*$) to represent the strong (resp., weak) convergence of a sequence $\{x_n\}$ to a point x^*. For any $x, y \in \mathcal{H}$ and $\lambda \in [0,1]$, we require the following identities:

$$||x+y||^2 = ||x||^2 + ||y||^2 + 2\langle x, y\rangle \tag{2.11}$$

and

$$||\lambda x + (1-\lambda)y||^2 = \lambda||x||^2 + (1-\lambda)||y||^2 - \lambda(1-\lambda)||x-y||^2. \tag{2.12}$$

Definition 2.1 ([5]). *Let $\mathcal{T} : \mathcal{H} \to \mathcal{H}$ be a mapping. Then $\mathcal{T}$ is called*

(1) K-Lipschitz continuous with $K > 0$ if

$$||\mathcal{T}x - \mathcal{T}y|| \leq K||x-y||, \quad \forall x, y \in \mathcal{H}; \tag{2.13}$$

(2) nonexpansive if (2.13) holds with $K = 1$;

(3) firmly nonexpansive if

$$||\mathcal{T}x - \mathcal{T}y|| \leq \langle x-y, \mathcal{T}x - \mathcal{T}y\rangle, \quad \forall x, y \in \mathcal{H}. \tag{2.14}$$

For any $x \in \mathcal{H}$ and $y \in \mathcal{C}$, we have the following properties (see [25]):

$$\langle x - P_{\mathcal{C}}x, P_{\mathcal{C}}x - y\rangle \geq 0, \tag{2.15}$$

equivalently,

$$||x - P_{\mathcal{C}}x||^2 + ||y - P_{\mathcal{C}}x||^2 \leq ||x-y||^2. \tag{2.16}$$

Remark 2.1. *It is commonly known that $I - P_{\mathcal{C}}$ satisfies the inequality (2.14) (see [56]).*

Definition 2.2 ([5])**.** *Let $f : \mathcal{H} \to (-\infty, +\infty]$ be a convex and proper function. Then:*

(1) f is said to be (weakly) lower semi-continuous (w-lsc) if for any sequence $x_n \in \mathcal{H}$ such that $(x_n \rightharpoonup x^*)$ $x_n \to x^*$ as $n \to \infty$, we have

$$\liminf_{n\to\infty} f(x_n) \geq f(x^*). \tag{2.17}$$

(2) $\partial f(x)$ is known as the subdifferential of f at a point x, which is defined by

$$\partial f(x) := \{v \in \mathcal{H} : \langle v, y - x\rangle + f(x) \leq f(y), \forall y \in \mathcal{H}\}.$$

An element $v \in \partial f(x)$ is called a subgradient of f at x.

Lemma 2.1 ([56, 9])**.** *Let $\tau > 0$ and $x^* \in \mathcal{H}_1$. The point x^* solves problem (1.1) if and only if it solves the fixed point problem:*

$$x^* = P_{\mathcal{C}}(I - \tau\mathcal{B}^*(I - P_{\mathcal{Q}})\mathcal{B})x^*.$$

Lemma 2.2 ([28])**.** *Let $\{x_n\}$ be a sequence of nonnegative real numbers such that $\forall n \geq 1$,*

$$x_{n+1} \leq (1 - \beta_n)x_n + \beta_n\Gamma_n,$$

$$x_{n+1} \leq x_n - \chi_n + \Phi_n, \forall n \geq 1,$$

where $\beta_n \in (0,1)$, $\chi_n \in [0, +\infty)$ and Γ_n, $\Phi_n \in (-\infty, +\infty)$ such that

(B1) $\lim\limits_{n\to\infty} \beta_n = 0$ and $\sum_{n=1}^{\infty} \beta_n = \infty$;
(B2) $\lim\limits_{n\to\infty} \Phi_n = 0$;
(B3) $\lim\limits_{j\to\infty} \chi_{n_j} = 0$ implies that $\limsup\limits_{r\to\infty} \Gamma_{n_j} \leq 0$ for any subsequence $\{n_j\}$ of $\{n\}$,

Then $\lim\limits_{n\to\infty} x_n = 0$.

3. Main results

3.1. **Alternated Inertial Halpern-type Relaxed Algorithm with Perturbations.** In this part, we introduce the alternated inertial Halpern-type relaxed algorithm with perturbations and analyse its strong convergence to the minimum-norm solution of the problem (1.1) in real Hilbert spaces. For its construction, we define g_n, $\mathcal{C}$, $\mathcal{Q}$, $\mathcal{C}_n$ and $\mathcal{Q}_n$ as in the equations (1.3), (1.5) and (1.6), respectively. Moreover, to establish its convergence, we require the conditions in the following assumption:

Assumption 1:

$(A1)$ The solutions' set of problem (1.1) is denoted by $\Omega \neq \emptyset$.
$(A2)$ $c : \mathcal{H}_1 \to \mathbb{R}$ and $q : \mathcal{H}_2 \to \mathbb{R}$ are respectively convex, subdifferentiable and weakly lower semicontinuous functions on $\mathcal{H}_1$ and $\mathcal{H}_2$.
$(A3)$ For any $x \in \mathcal{H}_1$ and $y \in \mathcal{H}_2$, at least one subgradient $\phi \in \partial c(x)$ and $\varphi \in \partial q(y)$ are obtainable and the subdifferential operators ∂c and ∂q are bounded on bounded sets.
$(A4)$ Let $\tau_1 > 0$, $\varepsilon > 0$, $\rho \in (0, \frac{1}{\varepsilon})$, $\delta_n \in (0,1)$ such that $\lim\limits_{n\to\infty} \delta_n = 0$ and $\sum_{n=0}^{\infty} \delta_n = +\infty$.

Algorithm 1 Alternated inertial Halpern-type Relaxed $\mathcal{CQ}$ Algorithm with Perturbations (AiHRAP)

Initialization: Take $\tau_1,\ \varepsilon,\ \rho$ and $\{\delta_n\}$ such that the conditions $(A4)$ of Assumption 1 holds. Select $\lambda_n \in [0, +\infty)$, $u \in \mathcal{C}$, $x_0,\ x_1 \in \mathcal{H}_1$ and set $n = 1$.

Step 1. Compute

$$y_n = \begin{cases} x_n, & \text{if } n \text{ is even,} \\ \\ x_n + \lambda_n(x_n - x_{n-1}), & \text{if } n \text{ is odd.} \end{cases} \tag{3.18}$$

Step 2. Compute

$$h_n = P_{\mathcal{C}_n}(y_n - \rho\tau_n \nabla g_n(y_n) + e_1(y_n)).$$

If $h_n = y_n$, then, stop the iteration and $h_n \in \Omega$, else, go to Step 3.

Step 3. Compute

$$m_n = P_{\mathcal{C}_n}(y_n - \rho\tau_n \nabla g_n(h_n) + e_2(y_n)).$$

Step 4. Compute

$$x_{n+1} = \delta_n u + (1 - \delta_n) m_n$$

and update the step-length τ_{n+1} by

$$\tau_{n+1} = \begin{cases} \min\left\{\frac{\varepsilon||y_n - h_n||}{||\nabla g_n(y_n) - \nabla g_n(h_n)||},\ \tau_n\right\}, & \text{if } \nabla g_n(y_n) \neq \nabla g_n(h_n), \\ \\ \tau_n, & \text{otherwise.} \end{cases} \tag{3.19}$$

Set $n := n + 1$ and go back to Step 1.

Remark 3.2. *In Algorithm 1, for all $n \geq 1$, we select the inertial parameter λ_n as follows;*

$$\lambda_n = \begin{cases} \min\left\{\frac{\xi_n}{||x_n - x_{n-1}||^2},\ \eta_1\right\}, & \text{if } x_n \neq x_{n-1}, \\ \\ \eta_1, & \text{otherwise,} \end{cases} \tag{3.20}$$

where $\xi_n \in [0, +\infty)$ such that $\lim\limits_{n\to\infty} \frac{\xi_n}{\delta_n} = 0$ and $\eta_1 > 0$. Moreover, for the analysis of the convergence of Algorithm 1, we provide the following additional assumption:

Assumption 2: Assume that, for each $i = 1, 2$, the sequence of perturbations $\{e_i(y_n)\}$ satisfies

$$\lim_{n\to\infty} \frac{||e_i(y_n)||}{\delta_n} = 0. \tag{3.21}$$

Remark 3.3. *It appears from Algorithm 1 that*

$$m_n = P_{\mathcal{C}_n}(y_n - \rho\tau_n \nabla g_n(h_n)) + \bar{e}_2(y_n), \forall n \geq 1 \tag{3.22}$$

so that

$$\begin{aligned} ||\bar{e}_2(y_n)|| &= ||P_{\mathcal{C}_n}(y_n - \rho\tau_n \nabla g_n(h_n) + e_2(y_n)) - P_{\mathcal{C}_n}(y_n - \rho\tau_n \nabla g_n(h_n))|| \\ &\leq ||e_2(y_n)||. \end{aligned} \tag{3.23}$$

Combining (3.21) and (3.23), we have

$$\lim_{n\to\infty} \frac{||\bar{e}_2(y_n)||}{\delta_n} = 0. \tag{3.24}$$

In the first, we validate the stopping criterion of Algorithm 1 in the following remark.

Remark 3.4. *If we let $h_n = y_n$ in Algorithm 1, then we see that*

$$h_n = P_{\mathcal{C}_n}(h_n - \rho\tau_n \nabla g_n(h_n) + e_1(h_n)), \forall n \geq 1,$$

which implies that $h_n \in \mathcal{C}_n$. Thus, by the means of Lemma 2.1, we have $Bh_n \in \mathcal{Q}_n$. Together with (1.5) and (1.6), we obtain that $h_n \in \mathcal{C}$ and $Bh_n \in \mathcal{Q}$. Therefore, $h_n \in \Omega$.

Lemma 3.3. *Suppose that $\{\tau_n\}$ is a sequence of step lengths generated by (3.19). Then it is well defined and $\tau_n \geq \frac{\varepsilon}{||B||^2}$ for all $n \geq 1$.*

Proof. By the lipschitz contuinity of ∇g_n with constant $||B||^2$, we obtain

$$\frac{\varepsilon||y_n - h_n||}{||\nabla g_n(y_n) - \nabla g_n(h_n)||} \geq \frac{\varepsilon||y_n - h_n||}{||B||^2||y_n - h_n||} = \frac{\varepsilon}{||B||^2}.$$

In view of this and (3.19), one sees that $\tau_{n+1} \geq \min\{\tau_n, \frac{\varepsilon}{||B||^2}\}$. By induction, we obtain that $\tau_n \geq \min\{\tau_1, \frac{\varepsilon}{||B||^2}\}$. It is also seen from (3.19) that $\tau_{n+1} \leq \tau_n$ for all $n \in \mathbb{N}$. In view of the monotonicity and the existence of the lower bound of the sequence $\{\tau_n\}$, we obtain that $\lim\limits_{n\to\infty} \tau_n$ exists. Since $\min\{\tau_1, \frac{\varepsilon}{||B||^2}\}$ is a lower bound of the sequence $\{\tau_n\}$, we can find $\tau > 0$ such that $\lim\limits_{n\to\infty} \tau_n = \tau$. This completes the proof. □

Next, we establish that an even subsequence $\{x_{2n}\}$ of $\{x_n\}$ by Algorithm 1 is bounded.

Lemma 3.4. *Let $\{x_n\}$ be a sequence produced by Algorithm 1. Then, for any point $z \in \Omega$, an even subsequence $\{||x_{2n} - z||\}$ of $\{||x_n - z||\}$ is bounded.*

Proof. Let $z \in \Omega$. Then $Bz \in \mathcal{Q}_n$ and, consequently, $\nabla g_n(z) = B^*(I - P_{\mathcal{Q}_n})Bz = 0$. Therefore, together with the fact that $I - P_{\mathcal{Q}_n}$ satisfies (2.14), we have

$$\begin{aligned}
\langle \nabla g_n(h_n), h_n - z\rangle &= \langle B^*(I - P_{\mathcal{Q}_n})Bh_n - B^*(I - P_{\mathcal{Q}_n})Bz, h_n - z\rangle \\
&= \langle (I - P_{\mathcal{Q}_n})Bh_n - (I - P_{\mathcal{Q}_n})Bz, Bh_n - Bz\rangle \\
&\geq ||(I - P_{\mathcal{Q}_n})Bh_n||^2 \\
&= 2g_n(h_n), \forall n \geq 1. \qquad (3.25)
\end{aligned}$$

Letting $p_n = P_{\mathcal{C}_n}(y_n - \rho\tau_n \nabla g_n(h_n))$, it follows from the inequalities (2.16) and (3.25) that

$$\begin{aligned}
||p_n - z||^2 &\leq ||P_{\mathcal{C}_n}(y_n - \rho\tau_n \nabla g_n(h_n)) - z||^2 \\
&\leq ||y_n - \rho\tau_n \nabla g_n(h_n) - z||^2 - ||y_n - \rho\tau_n \nabla g_n(h_n) - p_n||^2 \\
&= ||y_n - z||^2 - ||y_n - p_n||^2 - 2\rho\tau_n \langle \nabla g_n(h_n), y_n - z\rangle \\
&+ 2\rho\tau_n \langle \nabla g_n(h_n), y_n - p_n\rangle \\
&\leq ||y_n - z||^2 - ||y_n - p_n||^2 - 4\rho\tau_n g_n(h_n) \\
&- 2\rho\tau_n \langle \nabla g_n(h_n), p_n - h_n\rangle. \qquad (3.26)
\end{aligned}$$

Now, we estimate the rightmost term of (3.26) as follows:
We noticed from (2.11) that

$$||y_n - h_n||^2 + ||h_n - p_n||^2 - ||y_n - p_n||^2 = 2\langle y_n - h_n, p_n - h_n\rangle. \qquad (3.27)$$

By the fact that $p_n \in \mathcal{C}_n$, we obtain from (3.19), the property (2.15) and the mean value inequality that

$$\begin{aligned}
2\langle y_n - h_n, p_n - h_n\rangle &= 2\langle y_n - \rho\tau_n \nabla g_n(y_n) + e_1(y_n) - h_n, p_n - h_n\rangle \\
&+ 2\rho\tau_n \langle \nabla g_n(y_n) - \nabla g_n(h_n), p_n - h_n\rangle \\
&+ 2\rho\tau_n \langle \nabla g_n(h_n), p_n - h_n\rangle \\
&- 2\langle e_1(y_n), p_n - h_n\rangle \\
&\leq 2\rho\tau_n ||\nabla g_n(y_n) - \nabla g_n(h_n)|| ||p_n - h_n|| \\
&+ 2\rho\tau_n \langle \nabla g_n(h_n), p_n - h_n\rangle \\
&+ 2||e_1(y_n)|| ||p_n - h_n|| \\
&\leq \Big(\frac{\varepsilon\rho\tau_n}{\tau_{n+1}} + ||e_1(y_n)||\Big)\big(||y_n - h_n||^2 + ||p_n - h_n||^2\big) \\
&+ ||e_1(y_n)|| + 2\rho\tau_n \langle \nabla g_n(h_n), p_n - h_n\rangle . \qquad (3.28)
\end{aligned}$$

Combining (3.27) and (3.28), we deduce that

$$\begin{aligned}
2\rho\tau_n \langle \nabla g_n(h_n), p_n - h_n\rangle &\geq \Big(1 - \Big(\frac{\varepsilon\rho\tau_n}{\tau_{n+1}} + ||e_1(y_n)||\Big)\Big)\big(||y_n - h_n||^2 + ||p_n - h_n||^2\big) \\
&- ||e_1(y_n)|| - ||y_n - p_n||^2. \qquad (3.29)
\end{aligned}$$

In view of the inequalities (3.26), (3.29) and Lemma 3.3, one sees that

$$\begin{aligned}
||p_n - z||^2 &\leq ||y_n - z||^2 - \frac{4\rho\varepsilon}{||B||^2} g_n(h_n) + ||e_1(y_n)|| \\
&- \rho_n\big(||y_n - h_n||^2 + ||p_n - h_n||^2\big), \qquad (3.30)
\end{aligned}$$

where

$$\rho_n = \Big(1 - \Big(\frac{\varepsilon\rho\tau_n}{\tau_{n+1}} + ||e_1(y_n)||\Big)\Big). \qquad (3.31)$$

Note that, for any $\varepsilon > 0$ and $\rho \in (0, \frac{1}{\varepsilon})$, we immediately see, from Lemma 3.3, Assumption 2 and equation (3.31), that there exists $\rho^* > 0$ such that $\lim_{n\to\infty} \rho_n = \rho^*$, where

$$\rho^* = (1 - \varepsilon\rho). \qquad (3.32)$$

Thus we can find a positive number R such that $\rho_n > 0$ for all $n \geq R$. Together with (3.30) and the definition of m_n in Algorithm 1, we see that

$$\begin{aligned}
||m_n - z||^2 &= ||p_n + \bar{e}_2(y_n) - z||^2 \\
&\leq (1 + ||\bar{e}_2(y_n)||)||p_n - z||^2 + ||\bar{e}_2(y_n)|| + ||\bar{e}_2(y_n)||^2 \\
&\leq \big(1 + ||\bar{e}_2(y_n)||\big)||y_n - z||^2 + \vartheta_n - \Theta_n \\
&\leq \big(1 + ||\bar{e}_2(y_n)||\big)||y_n - z||^2 + \vartheta_n, \forall n \geq R, \qquad (3.33)
\end{aligned}$$

where

$$\Theta_n = \big(1 + ||\bar{e}_2(y_n)||\big)\Big(\frac{4\rho\varepsilon}{||B||^2} g_n(h_n) + \rho_n\big(||y_n - h_n||^2 + ||p_n - h_n||^2\big)\Big)$$

and

$$\vartheta_n = \big(1 + ||\bar{e}_2(y_n)||\big)||e_1(y_n)|| + ||\bar{e}_2(y_n)|| + ||\bar{e}_2(y_n)||^2.$$

Using the convexity of $||\cdot||^2$, it follows from (3.33) that

$$||x_{n+1}-z||^2 \le \delta_n||u-z||^2 + (1-\delta_n)\big(1+||\bar{e}_2(y_n)||\big)||y_n-z||^2 + (1-\delta_n)(\vartheta_n-\Theta_n). \tag{3.34}$$

In view of (3.18) and taking $n+1=2n+1$ in (3.34), we see that

$$||x_{2n+1}-z||^2 \le \delta_{2n}||u-z||^2 + (1-\delta_{2n})\big(1+||\bar{e}_2(y_{2n})||\big)||x_{2n}-z||^2 + (1-\delta_{2n})(\vartheta_{2n}-\Theta_{2n}) \tag{3.35}$$

and

$$||y_{2n+1}-z||^2 \le (1+\lambda_{2n+1})||x_{2n+1}-z||^2 - \lambda_{2n+1}||x_{2n}-z||^2 + \lambda_{2n+1}(1+\lambda_{2n+1})||x_{2n+1}-x_{2n}||^2. \tag{3.36}$$

Combining (3.35) and (3.36) for $n+1=2n+2$ in (3.34), we deduce that

$$\begin{aligned}
||x_{2n+2}-z||^2 &\le \delta_{2n+1}||u-z||^2 + (1-\delta_{2n+1})\big(1+||\bar{e}_2(y_{2n+1})||\big)||y_{2n+1}-z||^2\\
&\quad + (1-\delta_{2n+1})(\vartheta_{2n+1}-\Theta_{2n+1})\\
&\le \delta_{2n+1}||u-z||^2 + \big(1+||\bar{e}_2(y_{2n+1})||\big)(1+\lambda_{2n+1})\Big(\delta_{2n}||u-z||^2\\
&\quad + (1-\delta_{2n})\big(1+||\bar{e}_2(y_{2n})||\big)||x_{2n}-z||^2 + (1-\delta_{2n})(\vartheta_{2n}-\Theta_{2n})\Big)\\
&\quad - \lambda_{2n+1}\big(1+||\bar{e}_2(y_{2n+1})||\big)||x_{2n}-z||^2 + (1-\delta_{2n+1})(\vartheta_{2n+1}-\Theta_{2n+1})\\
&\quad + \lambda_{2n+1}\big(1+||\bar{e}_2(y_{2n+1})||\big)(1+\lambda_{2n+1})||x_{2n+1}-x_{2n}||^2\\
&\le (1-\delta_{2n})\big(1+||\bar{e}_2(y_{2n})||\big)\big(1+||\bar{e}_2(y_{2n+1})||\big)(1+\lambda_{2n+1})||x_{2n}-z||^2\\
&\quad + \big(1+||\bar{e}_2(y_{2n+1})||\big)(1+\lambda_{2n+1})\big(2\delta_{2n}||u-z||^2 + (1-\delta_{2n})(\vartheta_{2n}-\Theta_{2n})\\
&\quad + (1-\delta_{2n+1})(\vartheta_{2n+1}-\Theta_{2n+1}) + \lambda_{2n+1}||x_{2n+1}-x_{2n}||^2\big).
\end{aligned} \tag{3.37}$$

Using (3.31), (3.32) and the fact that for any $\varepsilon>0$, $\rho\in(0,\frac{1}{\varepsilon})$, we find from (3.37) that

$$\begin{aligned}
||x_{2n+2}-z||^2 &\le (1-\delta_{2n})||x_{2n}-z||^2 + \frac{1}{\big(1+||\bar{e}_2(y_{2n})||\big)}\Big(2\delta_{2n}||u-z||^2 + (1-\delta_{2n})\vartheta_{2n}\\
&\quad + (1-\delta_{2n+1})\vartheta_{2n+1} + \lambda_{2n+1}||x_{2n+1}-x_{2n}||^2\Big),\ \forall\, n\ge R.
\end{aligned} \tag{3.38}$$

Taking

$$\begin{aligned}
M = \sup_{n\ge1}\frac{1}{\big(1+||\bar{e}_2(y_{2n})||\big)}&\Big(2||u-z||^2 + \frac{(1-\delta_{2n})}{\delta_{2n}}\vartheta_{2n} + \frac{\lambda_{2n+1}}{\delta_{2n}}||x_{2n+1}-x_{2n}||^2\\
&+ \frac{(1-\delta_{2n+1})}{\delta_{2n}}\vartheta_{2n+1}\Big),
\end{aligned}$$

then, by (3.38) and the condition $(A4)$, we obtain that

$$\begin{aligned}
||x_{2n+2}-z||^2 &\le (1-\delta_{2n})||x_{2n}-z||^2 + \delta_{2n}M\\
&\le \max\big\{||x_{2n}-z||^2, M\big\}\\
&\ \ \vdots\\
&\le \max\big\{||x_0-z||^2, M\big\}, \forall n\ge R.
\end{aligned} \tag{3.39}$$

By Remark 3.2, Assumption 2, the condition $(A4)$ and the inequality (3.39), we obtain that, for any $z\in\Omega$, the even subsequence $\{||x_{2n}-z||\}$ of $\{||x_n-z||\}$ produced by Algorithm 1 is bounded. Consequently, the even subsequence $\{x_{2n}\}$ of $\{x_n\}$ generated by Algorithm 1 is bounded. This completes the proof. $\square$

Next is to state and prove the following strong convergence theorem for Algorithm 1:

Theorem 3.1. *Let the conditions of Assumptions* 1, 2 *and Remark* 3.2 *hold, and* $\{x_n\}$ *be a sequence generated by Algorithm* 1*. Then,* $\{x_n\}$ *converges strongly to a point* $z^* \in \Omega$*, where* $z^* = P_\Omega 0$*.*

Proof. Let $z \in \Omega$. Then, by (2.11) and (3.33), we get

$$\begin{aligned}||x_{n+1} - z||^2 &= \delta_n^2||u - z||^2 + (1-\delta_n)^2||m_n - z||^2 + 2\delta_n(1-\delta_n)\langle m_n - z, u - z\rangle \\ &\leq (1-\delta_n)\big(1 + ||\bar{e}_2(y_n)||\big)||y_n - z||^2 + (1-\delta_n)\vartheta_n + \delta_n^2||u-z||^2 \\ &\quad + 2\delta_n(1-\delta_n)\langle m_n - z, u - z\rangle . \end{aligned} \tag{3.40}$$

Similar arguments used in deriving (3.35) lead to obtain from (3.40) that

$$\begin{aligned}||x_{2n+1} - z||^2 \leq{}& (1-\delta_{2n})\big(1 + ||\bar{e}_2(y_{2n})||\big)||x_{2n} - z||^2 + (1-\delta_{2n})\vartheta_{2n} \\ &+ \delta_{2n}^2||u-z||^2 + 2\delta_{2n}(1-\delta_{2n})\langle m_{2n} - z, u - z\rangle . \end{aligned} \tag{3.41}$$

Connecting (3.36) and (3.41) for $n+1 = 2n+2$ in (3.40) and following same lines of the proof of (3.38), one finds that

$$\begin{aligned}||x_{2n+2} - z||^2 \leq{}& \big(1 + ||\bar{e}_2(y_{2n+1})||\big)||y_{2n+1} - z||^2 + (1-\delta_{2n+1})\vartheta_{2n+1} \\ &+ \delta_{2n+1}^2||u-z||^2 + 2\delta_{2n+1}(1-\delta_{2n+1})\langle m_{2n+1} - z, u - z\rangle \\ \leq{}& (1-\delta_{2n})||x_n - z||^2 + \frac{1}{\big(1 + ||\bar{e}_2(y_n)||\big)}\Big(2\delta_{2n}^2||u-z||^2 + (1-\delta_{2n})\vartheta_{2n} \\ &+ \lambda_{2n+1}||x_{2n+1} - x_{2n}||^2 + 2\delta_{2n}(1-\delta_{2n})\langle m_{2n} - z, u - z\rangle\Big) \\ &+ \frac{2\delta_{2n+1}(1-\delta_{2n+1})}{\big(1 + ||\bar{e}_2(y_{2n})||\big)\big(1 + ||\bar{e}_2(y_{2n+1})||\big)(1+\lambda_{2n+1})}\langle m_{2n+1} - z, u - z\rangle \\ &+ (1-\delta_{2n+1})\vartheta_{2n+1}. \end{aligned} \tag{3.42}$$

Without loss of generality, using the condition $(A4)$ and Assumption 2, we assume that $r, s > 0$ exist such that, for all $n \geq 1$,

$$\frac{4(1-\delta_n)\rho\varepsilon}{\big(1 + ||\bar{e}_2(y_{n-1})||\big)(1+\lambda_n)||\mathcal{B}||^2} \geq r, \quad \frac{(1-\delta_n)\rho_n}{\big(1 + ||\bar{e}_2(y_{n-1})||\big)(1+\lambda_n)} \geq s.$$

In view of (3.37) and (3.42), one observes that

$$||x_{2n+2} - z||^2 \leq ||x_{2n} - z||^2 - \chi_{2n} + \Phi_{2n}$$

and

$$||x_{2n+2} - z||^2 \leq (1-\delta_{2n})||x_{2n} - z||^2 + \delta_{2n}\Gamma_{2n}, \tag{3.43}$$

where

$$\begin{aligned}\chi_{2n} ={}& p\big(g_{2n}(h_{2n}) + g_{2n+1}(h_{2n+1})\big) + q\big(||y_{2n} - h_{2n}||^2 + ||p_{2n} - h_{2n}||^2 \\ &+ ||y_{2n+1} - h_{2n+1}||^2 + ||p_{2n+1} - h_{2n+1}||^2\big),\end{aligned}$$

$$\begin{aligned}\Phi_{2n} ={}& \frac{1}{\big(1 + ||\bar{e}_2(y_{2n})||\big)}\big(2\delta_{2n}||u-z||^2 + (1-\delta_{2n})\vartheta_{2n} + (1-\delta_{2n+1})\vartheta_{2n+1} \\ &+ \lambda_{2n+1}||x_{2n+1} - x_{2n}||^2\big)\end{aligned}$$

and

$$\Gamma_{2n} = \frac{1}{(1+||\bar{e}_2(y_n)||)}\Big(2\delta_{2n}||u-z||^2 + \frac{(1-\delta_{2n})}{\delta_{2n}}\vartheta_{2n} + \frac{\lambda_{2n+1}}{\delta_{2n}}||x_{2n+1}-x_{2n}||^2$$
$$+ 2(1-\delta_{2n})\langle m_{2n}-z, u-z\rangle\Big)$$
$$+ \frac{2\delta_{2n+1}(1-\delta_{2n+1})}{\delta_{2n}(1+||\bar{e}_2(y_{2n})||)(1+||\bar{e}_2(y_{2n+1})||)(1+\lambda_{2n+1})}\langle m_{2n+1}-z, u-z\rangle$$
$$+ \frac{(1-\delta_{2n+1})}{\delta_{2n}}\vartheta_{2n+1}.$$

Using condition $(A4)$, Assumptions 2 and Remark 3.2, we find that $\lim_{n\to\infty}\Phi_{2n} = 0$. Thus, to apply Lemma 2.2, it remains only to show that, for any subsequence $\{\chi_{2n_j}\}$ of $\{\chi_{2n}\}$, the following is true:

$$\lim_{j\to\infty}\chi_{2n_j} = 0 \Longrightarrow \limsup_{j\to\infty}\Gamma_{2n_j} \le 0.$$

Now, suppose that $\{\chi_{2n_j}\}$ is a subsequence of $\{\chi_{2n}\}$ such that $\lim_{j\to\infty}\chi_{2n_j} = 0$. Then, in view of (3.31), the condition $(A4)$, Assumption 2 and the fact that $\lim_{n\to\infty}\rho_n = \rho^* > 0$, we obtain that

$$\lim_{j\to\infty}||x_{2n_j} - h_{2n_j}|| = 0, \quad \lim_{j\to\infty}||p_{2n_j} - h_{2n_j}|| = 0,$$
$$\lim_{j\to\infty}||y_{2n_j+1} - h_{2n_j+1}|| = 0, \quad \lim_{j\to\infty}||p_{2n_j+1} - h_{2n_j+1}|| = 0,$$
$$\lim_{j\to\infty}g_{2n_j}(h_{2n_j}) = 0 \Longleftrightarrow \lim_{j\to\infty}||(I-P_{\mathcal{Q}_{2n_j}})\mathcal{B}h_{2n_j}||^2 = 0$$

and

$$\lim_{j\to\infty}g_{2n_j+1}(h_{2n_j+1}) = 0 \Longleftrightarrow \lim_{j\to\infty}||(I-P_{\mathcal{Q}_{2n_j+1}})\mathcal{B}h_{2n_j+1}||^2 = 0. \tag{3.44}$$

Since an even subsequence $\{x_{2n}\}$ of $\{x_n\}$ is bounded, it follows that there exists a subsequence $\{x_{2n_j}\}$ of $\{x_{2n}\}$ converging weakly to a point say x^*. Then the condition $(A3)$ of Assumption 1 guarantees the existence of a constant $\varrho > 0$ such that $||\varphi_{2n_j}|| \le \varrho$. Together with the definition of $\mathcal{Q}_{2n_j}$, $P_{\mathcal{Q}_{2n_j}}\mathcal{B}h_{2n_j} \in \mathcal{Q}_{2n_j}$ and the results in (3.44), we have

$$\begin{aligned} q(\mathcal{B}h_{2n_j}) &\le \Big\langle \varphi_{2n_j}, \mathcal{B}h_{2n_j} - P_{\mathcal{Q}_{2n_j}}\mathcal{B}h_{2n_j}\Big\rangle \\ &\le ||\varphi_{2n_j}||||\mathcal{B}h_{2n_j} - P_{\mathcal{Q}_{2n_j}}\mathcal{B}h_{2n_j}|| \\ &\le \varrho||(I-P_{\mathcal{Q}_{2n_j}})\mathcal{B}h_{2n_j}||^2 \to 0 \ \ as\ \ j\to\infty. \end{aligned} \tag{3.45}$$

So, It is not difficult to see from the weakly lower semicontuinity of q and (3.45) that

$$q(\mathcal{B}x^*) \le \liminf_{j\to\infty} q(\mathcal{B}h_{2n_j}) \le 0, \tag{3.46}$$

which implies that $\mathcal{B}x^* \in \mathcal{Q}$.

Similarly, the boundedness of ∂c on bounded sets also implies the existence of $\sigma > 0$, such that $||\phi_{2n_j}|| \le \sigma$. From the definition of $\mathcal{C}_{2n_j}$, $p_{2n_j} \in \mathcal{C}_{2n_j}$ and (3.44), we see that

$$\begin{aligned} c(h_{2n_j}) &\le \langle \phi_{2n_j}, h_{2n_j} - p_{2n_j}\rangle \\ &\le ||\phi_{2n_j}||||h_{2n_j} - p_{2n_j}|| \\ &\le \sigma||h_{2n_j} - p_{2n_j}|| \to 0 \ \ as\ \ j\to\infty. \end{aligned} \tag{3.47}$$

Using similar arguments used in deriving (3.46), one obtains that $c(x^*) \le 0$, showing that $x^* \in \mathcal{C}$. Then the conclusion that $x^* \in \Omega$ is reached, which implies generally that $\omega_w(x_n) \subset \Omega$

since the choice of x^* was arbitrarily. From the definition of m_{2n_j} in Algorithm 1, (3.24) and (3.44), we, respectively, see that

$$\lim_{j\to\infty} ||m_{2n_j} - p_{2n_j}|| = 0 \tag{3.48}$$

and

$$||x_{2n_j} - p_{2n_j}|| \le ||x_{2n_j} - h_{2n_j}|| + ||h_{2n_j} - p_{2n_j}|| \to 0 \text{ as } j \to \infty. \tag{3.49}$$

Combining (3.48) and (3.49), one finds that

$$\lim_{j\to\infty} ||m_{2n_j} - x_{2n_j}|| = 0. \tag{3.50}$$

In view of the definition of x_{2n_j+1} in Algorithm 1, (3.50) and the condition $(A4)$, we deduce that

$$||x_{2n_j+1} - x_{2n_j}|| \le \delta_{2n_j}||u - x_{2n_j}|| + (1 - \delta_{2n_j})||m_{2n_j} - x_{2n_j}|| \to 0 \text{ as } j \to \infty. \tag{3.51}$$

Similarly, by Remark 3.2, we get

$$||y_{2n_j+1} - x_{2n_j+1}|| \le \lambda_{2n_j+1}||x_{2n_j+1} - x_{2n_j}|| \to 0 \text{ as } j \to \infty. \tag{3.52}$$

It is also, respectively, seen from (3.44), (3.22) and (3.24) that

$$||y_{2n_j+1} - p_{2n_j+1}|| \le ||y_{2n_j+1} - h_{2n_j+1}|| + ||h_{2n_j+1} - p_{2n_j+1}|| \to 0 \text{ as } j \to \infty \tag{3.53}$$

and

$$\lim_{j\to\infty} ||m_{2n_j+1} - p_{2n_j+1}|| = 0. \tag{3.54}$$

Therefore, by (3.50) - (3.54) and the metric projection property in (2.15), we find that

$$\limsup_{j\to\infty} \left\langle m_{2n_j} - z, u - z \right\rangle = \max_{x^*\in\omega_w(x_{2n})} \langle x^* - z, u - z\rangle \le 0$$

and

$$\limsup_{j\to\infty} \left\langle m_{2n_j+1} - z, u - z \right\rangle = \max_{x^*\in\omega_w(x_{2n})} \langle x^* - z, u - z\rangle \le 0. \tag{3.55}$$

Thus, by the condition $(A4)$, Assumption 2, Remark 3.2 and (3.55), we see that $\limsup_{j\to\infty} \Gamma_{2n_j} \le 0$. Therefore, it follows from Lemma 2.2 that $\lim_{n\to\infty} ||x_{2n} - z^*|| = 0$ and hence $x_{2n} \to z^* = P_\Omega 0$ as $n \to \infty$.

Finally, combining the fact that $\lim_{n\to\infty} ||x_{2n} - z^*|| = 0$ and (3.51), we see that $\lim_{n\to\infty} ||x_{2n+1} - z^*|| = 0$. Thus we conclude that the odd subsequence $\{x_{2n+1}\}$ of $\{x_n\}$ produced by Algorithm 1 converges strongly to $z^* \in \Omega$. Hence the whole sequence $\{x_n\}$ produced by Algorithm 1 strongly converges to $z^* \in \Omega$. This completes the proof. □

To obtain some extensions of Algorithm 1, we make the following assumption.

Assumption 3: Let $k \in K_n \subseteq \{0, 1, 2, \cdots, n-1\}$, y_{n-k} and y_{n-k-1} be arbitrary points in $\mathcal{H}_1$ for all $n \ge 1$. Choose $\varsigma_{n,k}$, $\sigma_{n,k} \in [0, +\infty)$ such that $\lim_{n\to\infty} \frac{\sum_{k\in K_n} \varsigma_{n,k}}{\delta_n} = 0$ and $\lim_{n\to\infty} \frac{\sum_{k\in K_n} \sigma_{n,k}}{\delta_n} = 0$. Select $\beta_{n,k} \in [0, \bar{\beta}_{n,k}]$, $\delta_{n,k} \in [0, \bar{\delta}_{n,k}]$ for all $n \ge 1$, $k \in K_n$ and any $\eta_2, \eta_3 > 0$ such that

$$\bar{\beta}_{n,k} := \begin{cases} \min\left\{\frac{\varsigma_{n,k}}{||y_{n-k} - y_{n-k-1}||},\ \eta_2\right\}, & \text{if } y_{n-k} \neq y_{n-k-1}, \\ \eta_2, & \text{otherwise} \end{cases} \tag{3.56}$$

and

$$\bar{\delta}_{n,k} := \begin{cases} \min\left\{\frac{\sigma_{n,k}}{||y_{n-k}-y_{n-k-1}||},\ \eta_3\right\}, & \text{if } y_{n-k} \neq y_{n-k-1}, \\ \eta_3, & \text{otherwise.} \end{cases} \tag{3.57}$$

Remark 3.5. *We can easily see from Assumption 3 that, for every $n \geq 1$ and $k \in K_n$,*

$$\beta_{n,k}||y_{n-k} - y_{n-k-1}|| \leq \varsigma_{n,k}, \quad \delta_{n,k}||y_{n-k} - y_{n-k-1}|| \leq \sigma_{n,k}.$$

Then, in view of the fact that, for every $k \in K_n$,

$$\lim_{n\to\infty} \frac{\sum_{k\in K_n} \varsigma_{n,k}}{\delta_n} = 0, \quad \lim_{n\to\infty} \frac{\sum_{k\in K_n} \sigma_{n,k}}{\delta_n} = 0,$$

we, respectively, obtain that

$$\lim_{n\to\infty} \frac{\sum_{k\in K_n} \beta_{n,k}||y_{n-k} - y_{n-k-1}||}{\delta_n} = 0, \quad \lim_{n\to\infty} \frac{\sum_{k\in K_n} \delta_{n,k}||y_{n-k} - y_{n-k-1}||}{\delta_n} = 0. \tag{3.58}$$

So, it is not difficult to observe that, for each $n \geq 1$, taking

$$e_1(y_n) = \sum_{k\in K_n} \beta_{n,k}(y_{n-k} - y_{n-k-1}) \tag{3.59}$$

and

$$e_2(y_n) = \sum_{k\in K_n} \delta_{n,k}(y_{n-k} - y_{n-k-1}), \tag{3.60}$$

then Algorithm 1 becomes the following alternated and multi-step inertial Halpern-type relaxed algorithm for the problem (1.1).

Algorithm 2 Alternated and Multi-step Inertial Halpern-type Relaxed Algorithm (AMiHRA)

Initialization: Take $\tau_1,\ \varepsilon,\ \rho$ and $\{\delta_n\}$ such that the condition $(A4)$ of Assumption 1 holds. Select $K_n,\ \beta_{n,k}$ and $\delta_{n,k}$ for all $k \in K_n$ as described in Assumption 3, λ_n as in Remark 3.2, $u \in \mathcal{C}$, $y_0,\ x_0,\ x_1 \in \mathcal{H}_1$ and set $n = 1$.

Step 1. Compute y_n by (3.18).

Step 2. Compute

$$w_n = y_n + \sum_{k\in K_n} \beta_{n,k}(y_{n-k} - y_{n-k-1})$$

and

$$h_n = P_{\mathcal{C}_n}(w_n - \rho\tau_n\nabla g_n(y_n)).$$

If $h_n = w_n = y_n$, then stop the iteration and $h_n \in \Omega$. Else, go to Step 3.

Step 3. Compute

$$u_n = y_n + \sum_{k\in K_n} \delta_{n,k}(y_{n-k} - y_{n-k-1})$$

and

$$x_{n+1} = \delta_n u + (1 - \delta_n)P_{\mathcal{C}_n}(u_n - \rho\tau_n\nabla g_n(h_n)),$$

update the step-length τ_{n+1} by (3.19), set $n := n + 1$ and go back to Step 1.

Theorem 3.2. *Let $\{x_n\}$ be a sequence produced by Algorithm 2 such that the conditions of Assumption 1, 3 and Remark 3.2 hold. Then the sequence $\{x_n\}$ strongly converges to a point $z^* \in \Omega$, where $z^* = P_\Omega 0$.*

Proof. In view of the choice of $\varsigma_{n,k},\ \sigma_{n,k},\ \beta_{n,k},\ \delta_{n,k}$ for all $n \geq 1$ and $k \in K_n$ in Assumption 3 and the equations (3.56), (3.57), (3.59) and (3.60), it is clear that the conditions of Assumption 2 are satisfied when Assumption 3 holds. Therefore, the complete proof of Theorem 3.2 follows from that of Theorem 3.1. This completes the proof. □

Remark 3.6. *in the following remarks, we consider some new and existing algorithms for solving the problem (1.1) related to Algorithm 2:*

(1) If $K_n = \{0\}, \varsigma_{n,k} = \varsigma_n,\ \sigma_{n,k} = \sigma_n,\ \beta_{n,k} = \beta_n$ and $\delta_{n,k} = \delta_n$ for all $n \geq 1$ in Assumption 3, then the AMiHRA becomes a relaxed $\mathcal{CQ}$ algorithm that combines an alternated inertial step and two classical Polyak's inertial steps.

(2) If $K_n = \{0\}$ and $\beta_{n,k} = \delta_{n,k} = 0$ for all $n \geq 1$, then the AMiHRA reduces to Halpern-type of Algorithm 3.1 in [53] with monotonic step-length criterion.

(3) If $\lambda_n = 0$ for all $n \geq 1$, then the AMiHRA becomes a general multi-step inertial Halpern-type relaxed $\mathcal{CQ}$ algorithm that combines two multi-step inertial terms for the problem (1.1).

We also consider the following as another extension of Algorithm 1:

Algorithm 3 Accelerated Alternated and Multi-step Inertial Halpern-type Relaxed Algorithm (AAMiHRA)

Initialization: Take $\tau_1,\ \varepsilon,\ \rho$ and $\{\delta_n\}$ such that the condition $(A4)$ holds. Select $\sigma > 0,\ \omega_n,\ \varsigma_n^{(2)} \in [0, +\infty)$ such that $\lim\limits_{n\to\infty} \frac{\omega_n}{\delta_n} = 0,\ \lim\limits_{n\to\infty} \frac{\varsigma_n^{(2)}}{\delta_n} = 0$, K_n, $\delta_{n,k}$ for all $k \in K_n$ as described in Assumption 3, λ_n as in Remark 3.2 and a bounded sequence $\{s_n\} \subset \mathcal{H}_1$. Choose $u \in \mathcal{C}$, $y_0,\ x_0,\ x_1 \in \mathcal{H}_1$ and set $n = 1$.

Step 1. Compute y_n by (3.18).

Step 2. Compute

$$\varsigma_n^{(1)} = \frac{\omega_n}{\max\{||d_n||, \sigma\}}, \tag{3.61}$$

$$d_{n+1} = \begin{cases} -\nabla g_n(y_n), & \text{if } n = 0, \\ -\frac{1}{\vartheta}\rho\tau_n\nabla g_n(y_n) + \varsigma_n^{(1)} d_n - \varsigma_n^{(2)} s_n, & \text{otherwise} \end{cases} \tag{3.62}$$

and

$$h_n = P_{\mathcal{C}_n}(y_n + \vartheta d_{n+1}).$$

If $h_n = y_n$, then stop the iteration and $h_n \in \Omega$. Else, go to Step 3.

Step 3. Compute

$$u_n = y_n + \sum_{k\in K_n} \delta_{n,k}(y_{n-k} - y_{n-k-1}),$$

and

$$x_{n+1} = \delta_n u + (1 - \delta_n)P_{\mathcal{C}_n}(u_n - \rho\tau_n\nabla g_n(h_n)),$$

update the step-length τ_{n+1} by (3.19), set $n := n + 1$ and go back to Step 1.

Remark 3.7. *Observe that taking $e_2(y_n)$ as in (3.60) and defining $e_1(y_n)$ as follows:*

$$e_1(y_n) = \vartheta\big(\varsigma_n^{(1)} d_n - \varsigma_n^{(2)} s_n\big), \tag{3.63}$$

with $\varsigma_n^{(1)}$ for all $n \geq 1$ to be obtained by (3.61), then, from the conditions on $\omega_n,\ \varsigma_n^{(2)}$ and the boundedness of the sequence $\{s_n\}$, Algorithm 1 becomes Algorithm 3. Thus we formulate and prove the following theorem:

Theorem 3.3. *Suppose that the conditions of Assumption* 1 *and* 3 *hold,* $\{e_1(y_n)\}$ *and* $\{e_2(y_n)\}$ *are the sequences generated by* (3.63) *and* (3.60)*, respectively,* $\{s_n\} \subset \mathcal{H}_1$ *is bounded and* $\{x_n\}$ *is a sequence produced by Algorithm* 3*. Then* $\{x_n\}$ *strongly converges to a point* $z^* \in \Omega$*, where* $z^* = P_\Omega 0$*.*

Proof. In view of the choice of ω_n, $\varsigma_n^{(2)}$ in Algorithm 3, $\sigma_{n,k}$ and $\delta_{n,k}$ for all $n \geq 1$ and $k \in K_n$ in Assumption 3, the boundedness of $\{s_n\}$ and the equations (3.57), (3.60) and (3.63), it is obvious that the conditions of Assumption 2 are satisfied when Assumption 3 and Remark 3.7 hold. Therefore, the complete proof of the Theorem 3.3 follows from that of Theorem 3.1. This completes the proof. □

Remark 3.8. *We provide some new brand of self-adaptive relaxed* $\mathcal{CQ}$ *Algorithms for solving the problem* (1.1) *based on Algorithm* 3*.*

(1) *If* $K_n = \{0\}$ *and* $\delta_{n,k} = \delta_n$ *for all* $n \geq 0$ *in Assumption* 3*, then the AAMiHRA becomes a general accelerated inertial Halpern-type relaxed algorithm, which combines an alternated inertial step, the classical Polyak's inertial step and a three-term conjugate-like direction in a single algorithm with monotonically decreasing step-length criterion.*
(2) *If* $K_n = \{0\}$ *and* $\delta_{n,k} = \delta_n = 0$ *for all* $n \geq 1$*, then the AAMiHRA reduces a Halpern-type of the alternated inertial algorithm* 3.1 *in* [53] *with three-term conjugate gradient-like direction and monotonic step-length criterion.*
(3) *If* $\varsigma_n^{(i)} = 0$ *for all* $i = 1, 2$ *and* $n \geq 1$*, then the AAMiHRA reduces an alternated and multi-step inertial Halpern-type relaxed algorithm with monotonic step-length criterion.*
(4) *If* $K_n = \{0\}$*,* $\delta_{n,k} = \delta_n = 0$ *and* $\varsigma_n^{(i)} = 0$ *for all* $i = 1, 2$ *and* $n \geq 1$*, then the AAMiHRA reduces a Halpern-type of the alternated inertial algorithm* 3.1 *in* [53] *with monotonic step-length criterion.*

4. Numerical Experiments

In this section, we investigate the performance and efficiency of the proposed algorithms (i.e., AMiHRA and AAMiHRA) in solving classification problems and constrained minimization problems. We conducted the experiments using R2023a Matlab in a PC with 12th Gen Intel(R) Core(TM)i5-124P 1.70 GHz processor and 16.0GB RAM.

4.1. **The Constrained Minimization Problem.**

In this part, we consider the following constrained minimization problem:

$$\min_{x \in \mathcal{C}} \frac{1}{2}||\mathcal{B}x - P_\mathcal{Q}\mathcal{B}x||^2, \tag{4.64}$$

where $\mathcal{C} = \{x \in L_2[0,1] : \langle x(t), 3t^2 \rangle = 0\}$ and $\mathcal{Q} = \{x \in L_2[0,1] : \langle x(t), \frac{t}{3} \rangle \geq -1\}$ are in $L_2[0,1]$.

Setting $g(x) = \frac{1}{2}||\mathcal{B}x - P_\mathcal{Q}\mathcal{B}x||^2$, it is not difficult to see that $\nabla g(x) = \mathcal{B}^*(I - P_\mathcal{Q})\mathcal{B}x$ is $||\mathcal{B}||^2$-Lipschitch continuous. Thus, problem (4.64) can be transformed into problem (1.1) with $\mathcal{H}_1 = \mathcal{H}_2 = L_2[0,1]$, where $||x|| = \left(\int_0^1 |x(t)|^2 dt\right)^{1/2}$ and $\langle x, y \rangle = \int_0^1 x(t)y(t)dt$ are, respectively, the norm and the inner product in $L_2[0,1]$. For all the experiments, we consider $\mathcal{B} = I$, where I is the identity mapping, i.e., $\mathcal{B}x = x$. Since $\mathcal{Q}$ and $\mathcal{C}$ are half-space and hyper-plane, respectively, to apply our proposed algorithms (i.e., AMiHRA and AAMiHRA) to solve problem (4.64), we take $\mathcal{Q}_n = \mathcal{Q}$ for all $n \geq 1$, define $c(x) = \langle x(t), 3t^2 \rangle$ for all $x \in L_2[0,1]$, so that $\mathcal{C}$ satisfies (1.5) and we consider g_n and $\mathcal{C}_n$ as described in (1.3) and (1.6), respectively, and set $\nabla g_n = \mathcal{B}^*(I - P_{\mathcal{Q}_n})\mathcal{B}$. We use the defined explicit projection formula in [24] to compute the projection

$P_{\mathcal{Q}_n}$ and the projection $P_{\mathcal{C}_n}$ by the following:

$$P_{\mathcal{C}_n}(x_n(t)) = \begin{cases} x_n(t), \text{ if } \langle 3t^2, t - x_n(t)\rangle \geq c(x_n(t)), \\ x_n(t) - \frac{c(x_n(t)) + \langle 3t^2, x_n(t) - t\rangle}{||3t^2||_{L_2}^2} 3t^2, \text{ otherwise}. \end{cases} \tag{4.65}$$

We compare the performance of our algorithms, the AMiHRA and the AAMiHRA with the algorithms of Tan et al. [53] and Dong et al. [21], which we abbreviated in this work as TQW Alg 3.1 and DLY Alg 4-II, respectively. For the experiments, we select the following parameters:

(1) We set $\tau_1 = 0.058,\ \eta_1 = \eta_3 = 5,\ \varepsilon = 0.1,\ \rho = \frac{1}{\varepsilon} - \frac{1}{3000\varepsilon},\ \delta_n = \frac{1}{10^5 n+1},\ \xi_n = \frac{1}{(n+10)^2}$, and $\sigma_{n,k} = \frac{1}{n^5 k^5}$ for the AMiHRA and the AAMiHRA. In particular, we select $\eta_2 = 5$ and $\varsigma_{n,k} = \frac{1}{n^3 k^3}$ for the AMiHRA and $\sigma = 0.1,\ \vartheta = 3,\ \omega_n = \frac{1}{(n+10)^5}$ and $\varsigma_n^{(2)} = \frac{1}{(10n+1)^3}$ for the AAMiHRA.
(2) In TQW Alg 3.1, we choose $\lambda_1 = 0.058,\ \mu = 0.1,\ \beta = 1.3,\ \alpha = 1,\ \theta_n = 0.2,\ \rho_n = \frac{10^{-1}}{(n+1)^2}$ and $\xi_n = 1 + \frac{10^{-1}}{(n+1)^2}$.
(3) In DLY Alg 4-II, we set $\tau_1 = 0.058,\ \varepsilon = 0.1,\ \rho = 0.2$ and $\lambda_n = \frac{1}{50n+1} - 1$.

For the implementations of the algorithms, we consider four different cases of the initial values of $x_0,\ x_1,\ y_0, u$ and s_n:
Case I: $x_0(t) = \sin(t),\ x_1(t) = t^2,\ y_0(t) = 0.5t^2,\ u(t) = t$ and $s_n(t) = 1.7t$;
Case II: $x_0(t) = e^t,\ x_1(t) = t^3,\ y_0(t) = 3\sin(t),\ u(t) = t$ and $s_n(t) = 10\sqrt{t}$;
Case III: $x_0(t) = \cos(t),\ x_1(t) = \tanh t,\ y_0(t) = 0.5t^2,\ u(t) = \frac{t}{100}$ and $s_n(t) = 5t^3$;
Case IV: $x_0(t) = e^{3t^2},\ x_1(t) = t^5,\ y_0(t) = \sin t^2,\ u(t) = \frac{\sqrt[3]{t}}{10}$ and $s_n(t) = \sqrt[4]{t}$.
We used the stopping rule

$$E_n = \frac{1}{2}\left(||x_n(t) - P_{\mathcal{C}_n}x_n(t)||_{L_2}^2 + ||x_n(t) - P_{\mathcal{Q}_n}x_n(t)||_{L_2}^2\right) < 10^{-10}$$

and the maximum number of iterations of 200 to terminate the iterations for all the algorithms. The performance results of all the algorithms, which include the execution times in second represented by "Time", the number of iterations denoted by "Iter." and the error E_n are reported in Table 1 and we plot the corresponding error results for the four cases in Figures 1, 2, 3 and 4.

TABLE 1. Compare the performance of the algorithms for the four cases

Cases	Algorithms	Iter.	Time(s)	E_n
Case I	AMiHRA	30	0.0531	4.98E-14
	AAMiHRA	30	0.0507	1.17E-13
	TQW Alg 3.1	98	0.0827	7.94E-11
	DLY Alg 4-II	115	0.0926	8.94E-11
Case II	AMiHRA	30	0.0482	2.64E-13
	AAMiHRA	30	0.0415	2.70E-12
	TQW Alg 3.1	98	0.0731	7.70E-11
	DLY Alg 4-II	115	0.0925	8.67E-11
Case III	AMiHRA	30	0.0439	2.45E-13
	AAMiHRA	30	0.0439	6.13E-14
	TQW Alg 3.1	98	0.0789	8.10E-11
	DLY Alg 4-II	115	0.0828	9.11E-11
Case IV	AMiHRA	30	0.0443	1.43E-13
	AAMiHRA	30	0.0389	2.10E-13
	TQW Alg 3.1	98	0.0733	7.43E-11
	DLY Alg 4-II	115	0.0903	8.36E-11

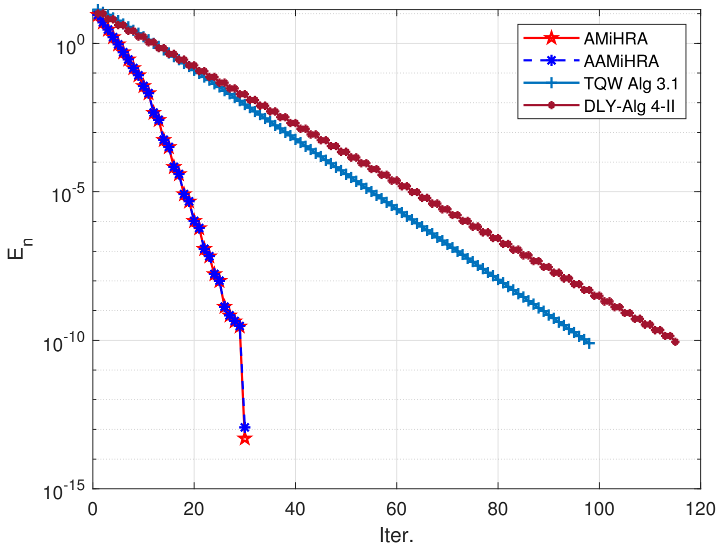

FIGURE 1. Error plotting of E_n of all the algorithms for Case I.

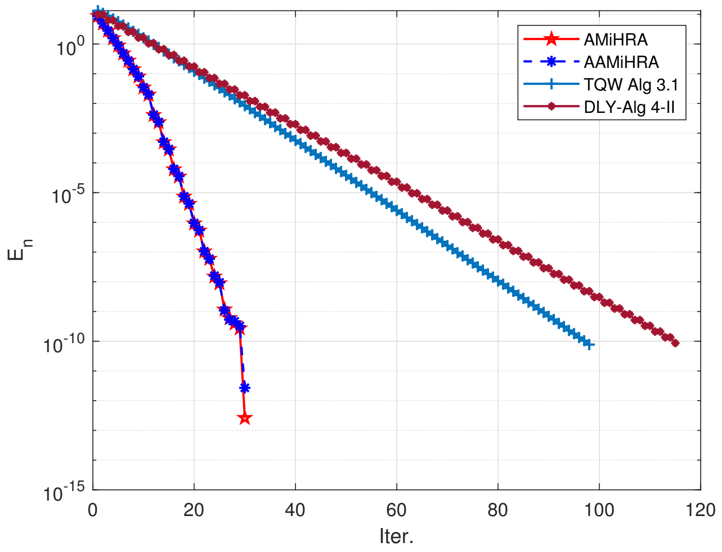

FIGURE 2. Error plotting of E_n of all the algorithms for Case II.

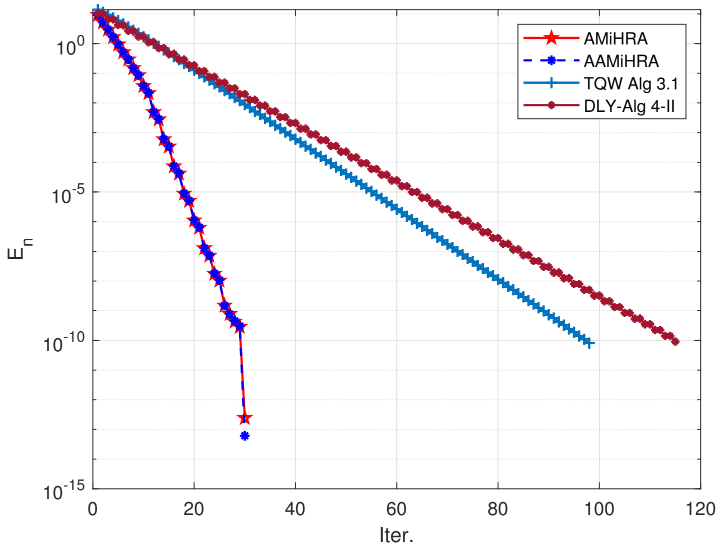

FIGURE 3. Error plotting of E_n of all the algorithms for Case III.

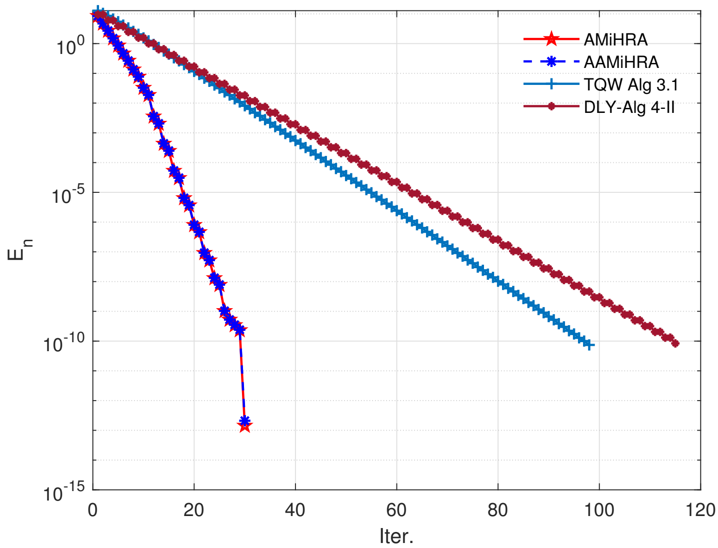

FIGURE 4. Error plotting of E_n of all the algorithms for Case IV.

Remark 4.9. *We observed from Table 1 and Figures 1, 2, 3 and 4 that the proposed algorithms outperform the compared algorithms in all the experiments. In particular, the AMiHRA achieves the fewest errors in most of the experiments, while AAMiHRA has the shortest execution times in all the experiments.*

4.2. **Classification Problems.** In this part, we conduct a series of experiments on some real-world benchmark datasets to investigate the performance of the suggested algorithms (i.e., AMiHRA and AAMiHRA). In all the experiments, we consider an efficient learning algorithm called extreme learning machine ELM for single-hidden layer feedforward neural networks SLFNs [29] and take $\mathcal{K} = \{(x_j, t_j) \in \mathbb{R}^k \times \mathbb{R}^m,\ j = 1, 2, \cdots, \mathcal{N}\}$ as an $\mathcal{N}$ distinct training data points set, where for each input point $x_j = \left[x_{j1}, x_{j2}, \cdots, x_{jk}\right]^T$, $t_j = \left[t_{j1}, t_{j2}, \cdots, t_{jm}\right]^T$ is its corresponding target. The SLFNs output function with $\mathcal{L}$ number of nodes in the hidden layer has the following formulation.

$$g_j = \sum_{i=1}^{\mathcal{L}} \beta_i f_i(x_j), \quad \forall j = 1, 2, \cdots, \mathcal{N}, \tag{4.66}$$

where $f_i(x_j) = \mathcal{F}\Big(\langle \omega_i, x_j\rangle + b_i\Big)$, $\mathcal{F}$ is an activation function, $\omega_i = (\omega_{i1}, \omega_{i2} \cdots, \omega_{ik})^T$ is an input weight vector linking the i^{th} hidden node and the input nodes, $\beta_i = (\beta_{i1}, \beta_{i2}, \cdots, \beta_{im})^T$ is an output weight vector linking the i^{th} hidden node and the output nodes and b_i is a bias. To train a SLFNs is to solve the linear system:

$$\mathcal{G}\beta = \mathsf{T}, \tag{4.67}$$

where the hidden layer output matrix $\mathcal{G}$ of order $\mathcal{N} \times \mathcal{L}$ is given by

$$\mathcal{G} = \left[f_1(x), f_2(x), \cdots, f_{\mathcal{L}}(x)\right],$$

$\beta = (\beta_1, \beta_2, \cdots, \beta_{\mathcal{L}})^T$ and $\mathsf{T} = (t_1, t_2, \cdots, t_{\mathcal{N}})^T$ are the output weights and the target data matrices, respectively and the i^{th} column of $\mathcal{G}$ is the i^{th} hidden node output based on $x_1, x_2 \cdots, x_{\mathcal{N}}$, which is defined by $f_i(x) = \left[f_i(x_1), f_i(x_2), \cdots, f_i(x_{\mathcal{N}})\right]^T$. To solve (4.67) by ELM is simply to find an optimal output weight $\hat{\beta} = \mathcal{G}^\dagger \mathsf{T}$, where $\mathcal{G}^\dagger$ represents the Moore-Penrose generalized inverse of the matrix $\mathcal{G}$ [47].

From the perspective of the sparsity of the output weight parameter β for some high-dimensional data, Cao et al. [8] proposed an ℓ_1-regularization approach to solve problem (4.67) based on the following Lasso model [54]:

$$\min_{\beta\in\mathbb{R}^{\mathcal{L}\times m}} \left\{\frac{1}{2}||\mathsf{T}-\mathcal{G}\beta||_2^2 \;:\; ||\beta||_1 \leq c\right\}, \tag{4.68}$$

where $c > 0$ is the regularization parameter. However, for better prediction accuracy, sparsity and stability, Ye et al. [61] unified both the ℓ_1 and the ℓ_2 penalties into a single model called the $\ell_1 - \ell_2$ hybrid regularization approach. Their model is described as follows.

$$\min_{\beta\in\mathbb{R}^{\mathcal{L}\times m}} \left\{\frac{1}{2}||\mathsf{T}-\mathcal{G}\beta||_2^2 \;:\; \lambda||s||_1 + \gamma||s||_2^2 \leq c\right\}, \tag{4.69}$$

where $\lambda,\ \gamma \geq 0$ and $c > 0$ are the regularization parameters. Suantai et al. [50] transformed the problem (4.68) into problem (1.1) by taking $\mathcal{C} = \{\beta \in \mathbb{R}^{\mathcal{L}\times m} \;:\; ||\beta||_1 \leq c\}$, $\mathcal{Q} = \{\mathsf{T}\} \subseteq \mathbb{R}^{\mathcal{K}\times m}$, $c(\beta) = ||\beta||_1 - c$, $q(x) = \frac{1}{2}||x-\mathsf{T}||^2$ and defined g_n, $\mathcal{C}_n$ and $\mathcal{Q}_n$ as in (1.3) and (1.6), respectively. They also used their proposed inertial relaxed $\mathcal{CQ}$ algorithm to solve the problem (4.67) based on the model (4.68).

Inspired by the sparsity, the stability and the generalization performance of (4.69), we observe that transforming problem (4.69) into problem (1.1) is of paramount important, which is possible by taking

$$\mathcal{C} = \{\beta \in \mathbb{R}^{\mathcal{L}\times m} \;:\; \lambda||\beta||_1 + \gamma||\beta||_2^2 \leq c\},$$
$$\mathcal{Q} = \{\mathsf{T}\} \subseteq \mathbb{R}^{\mathcal{K}\times m}, \quad q(x) = \frac{1}{2}||x-\mathsf{T}||^2.$$

Moreover, it is easily seen that the function $c(\beta) = \lambda||\beta||_1 + \gamma||\beta||_2^2 - c$ is strongly convex, so it is convex. We consider g_n, $\mathcal{C}_n$ and $\mathcal{Q}_n$ as defined in (1.3) and (1.6), respectively. Therefore, our proposed algorithms (i.e., AMiHRA and AAMiHRA) can be used to solve the problem (4.67) based on the both models (4.68) and (4.69).

To investigate the performance of the proposed algorithms, we employed them to solve problem (4.67) based on the models (4.68) and (4.69), for which we used the abbreviations AMiHRA - ℓ_1, AMiHRA - $\ell_1 - \ell_2$, AAMiHRA - ℓ_1 and AAMiHRA - $\ell_1 - \ell_2$ to denote them respectively. We compare their results with the algorithms of Tan et al. [53], Dong et al. [21] and Abubakar et al. [1] based on the model (4.68), which we respectively abbreviated in this work as TQW Alg 3.1 $-\ \ell_1$, DLY Alg 4-II - ℓ_1 and AKTIS Alg 1 $-\ \ell_1$. We carried out the experiments on three real-world classification datasets, including Breast Cancer Wisconsin (Breast Cancer W.) dataset [55], Heart disease dataset [34] and Glass identification dataset [22]. The detailed information on each of the datasets is provided in Table 2.

TABLE 2. Details of each dataset

Datasets	Instances	Classes	Features	Tasks
Breast Cancer W.	569	2	30	Classification
Heart disease	303	2	13	Classification
Glass Identification	214	6	9	Classification

In all the experiments, we fixedly choose 70% of each of the datasets for training and 30% for testing. We also set the following for the parameters:

(1) We set $\tau_1 = 3.03 \times 10^{-5},\ \eta_1 = \eta_2 = \eta_3 = 3,\ \varepsilon = 0.012,\ \rho = 0.13,\ \delta_n = \frac{1}{10^5 n+1},\ \xi_n = \frac{1}{(n+10)^{3.4}}$, $\varsigma_{n,k} = \frac{1}{n^3k^3}$ and $\sigma_{n,k} = \frac{1}{n^5k^5}$ for AMiHRA - ℓ_1, AMiHRA - $\ell_1 - \ell_2$, AAMiHRA - ℓ_1 and AAMiHRA - $\ell_1 - \ell_2$. In particular, we select $\sigma = 0.1,\ \vartheta = 3,\ \omega_n = \frac{1}{(n+10)^5}$ and $\varsigma_n^{(2)} = \frac{1}{(10n+1)^3}$ for AAMiHRA - ℓ_1 and AAMiHRA - $\ell_1 - \ell_2$.

(2) In TQW Alg 3.1$-\ell_1$, we choose $\lambda_1 = 3.03\times 10^{-5},\ \mu = 0.012,\ \beta = 0.13,\ \alpha = 0.997,\ \theta_n = 0.002,\ \rho_n = \frac{10^{-1}}{(n+1)^2}$ and $\xi_n = 1 + \frac{10^{-1}}{(n+1)^2}$.

(3) In DLY Alg 4-II - ℓ_1, we set $\tau_1 = 3.03 \times 10^{-5},\ \varepsilon = 0.012,\ \rho = 0.13$ and $\lambda_n = \frac{1}{50n+1} - 1$.

(4) In AKTIS Alg $1 - \ell_1$, we select $\varrho = 0.5,\ \varepsilon_n = \frac{1}{n^2},\ \zeta_n = \frac{1}{7500(n+5)}$ and $\vartheta_n = 0.8 - \zeta_n$.

We respectively calculate the accuracies and precisions by the following relations.

$$\text{Accuracy} = \frac{\text{TP} + \text{TN}}{\text{TP} + \text{FP} + \text{TN} + \text{FN}} \times 100\%, \tag{4.70}$$

$$\text{Precision} = \frac{\text{TP}}{\text{FP} + \text{FN}} \times 100\%, \tag{4.71}$$

where TP := True positive, TN := True negative, FP = False positive and FN = False negative, and estimate their averages as well as their standard deviations (SDs). We use these metrics and the number of iterations denoted by "Iter." to investigate the effectiveness and the stability of the suggested algorithms.

In the first part of the experiments, we set eC := ones$(\mathcal{L}, m)$, $x_0 = -1eC,\ x_1 = eC,\ u = y_0 = 2eC,\ s_n = 1.7eC$, $\mathcal{F}(x) = \tanh(x)$ as the activation function, $c = 0.061,\ \lambda = 0.9999,\ \gamma = 0.00505$ and used $||x_{n+1} - x_n|| < 10^{-3}$ and 500 as the Maximum iteration count to terminate the iterations for all the algorithms. We then analyzed the sensitivity of all the algorithms on the Breast Cancer W. dataset over different number of hidden nodes. The performance of all the algorithms are shown in Table 3 and we plot the corresponding results on training and testing accuracies, and training and testing precisions in Figures 5, 6, 7 and 8, respectively.

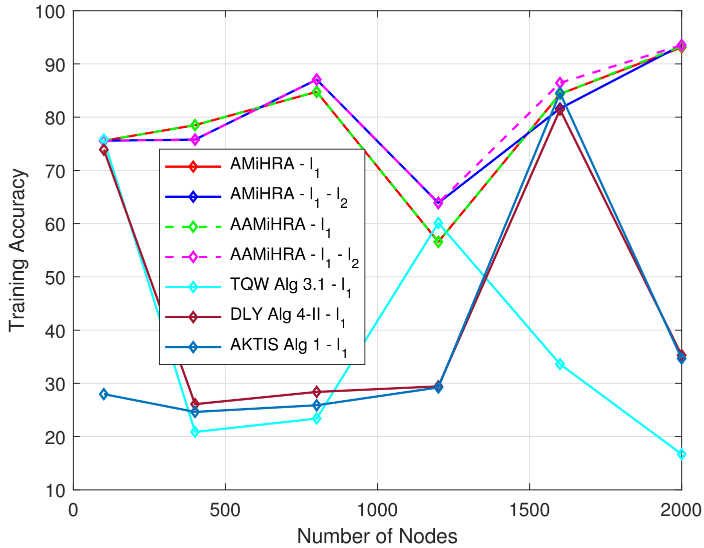

FIGURE 5. Compare the training accuracies of all the algorithms over different number of hidden nodes on the Breast Cancer W. dataset using the activation function $\mathcal{F}(x) = \tanh(x)$.

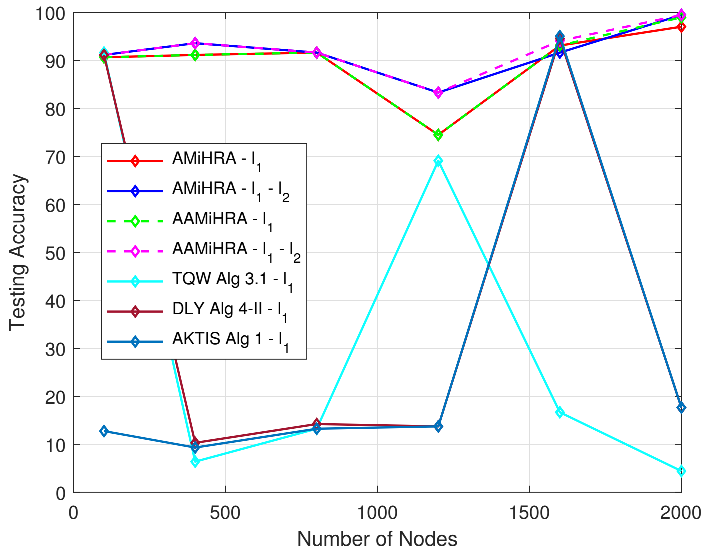

FIGURE 6. Compare the testing accuracies of all the algorithms over different number of hidden nodes on the Breast Cancer W. dataset using the activation function $\mathcal{F}(x) = \tanh(x)$.

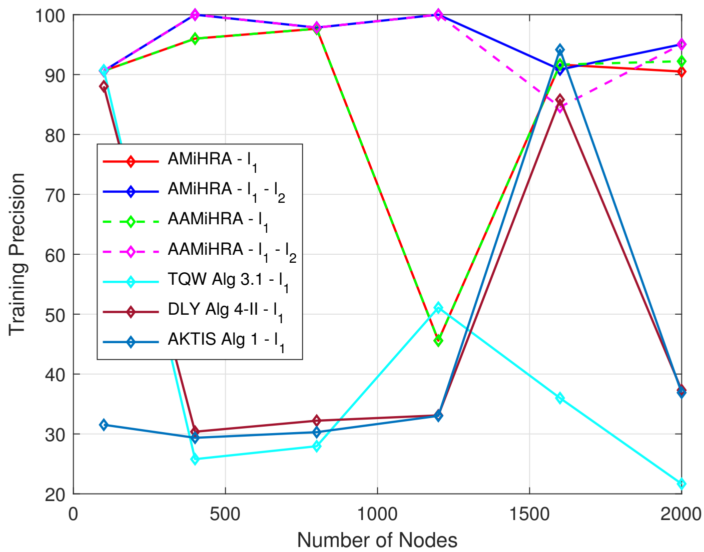

FIGURE 7. Compare the training precisions of all the algorithms over different number of hidden nodes on the Breast Cancer W. dataset using the activation function $\mathcal{F}(x) = \tanh(x)$.

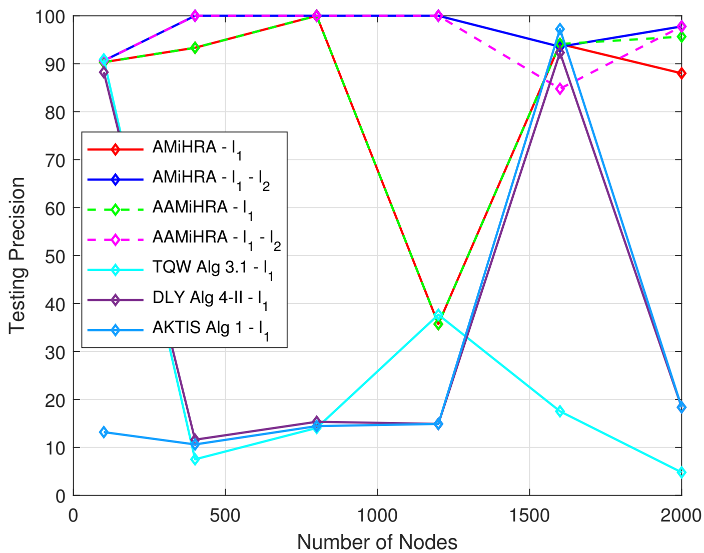

FIGURE 8. Compare the testing precisions of all the algorithms over different number of hidden nodes on the Breast Cancer W. dataset using the activation function $\mathcal{F}(x) = \tanh(x)$.

TABLE 3. Performance results of all the algorithms on the Breast Cancer W. dataset. The best and suboptimal results are highlighted in bold and underlined, respectively.

	Accuracy (%)													
Algorithms	AMiHRA - ℓ_1		AMiHRA - $\ell\_1 - \ell_2$		AAMiHRA - ℓ_1		AAMiHRA - $\ell_1 - \ell_2$		TQW Alg 3.1 $- \ell_1$		DLY Alg 4-II - ℓ_1		AKTIS Alg 1 $- \ell_1$	
Nodes	Training	Testing	Training	Testing	Training	Testing	Training	Testing	Training	Testing	Training	Testing	Training	Testing
100	75.5741	90.6863	75.5741	91.1765	75.5741	90.6863	75.5741	91.1765	**75.7829**	**91.6667**	73.904	91.1765	27.9749	12.7451
400	**78.4969**	91.1765	75.7829	**93.6275**	**78.4969**	91.1765	75.7829	**93.6275**	20.8768	6.3725	26.096	10.2941	24.6347	9.3137
800	84.7599	**91.6667**	**87.0564**	**91.6667**	84.7599	**91.6667**	**87.0564**	**91.6667**	23.382	13.2353	28.3925	14.2157	25.8873	13.2353
1200	56.5762	74.5098	**63.8831**	**83.3333**	56.5762	74.5098	**63.8831**	**83.3333**	60.1253	69.1176	29.4363	13.7255	29.2276	13.7255
1600	84.3424	93.1373	81.6284	91.6667	84.3424	93.1373	**86.4301**	94.1176	33.6117	16.6667	81.4196	94.6078	84.5511	**95.098**
2000	93.1106	97.0588	**93.5282**	**99.5098**	93.3194	99.0196	**93.5282**	**99.5098**	16.7015	4.4118	35.2818	17.6471	34.6555	17.6471
Aver. Acc.	78.81	89.7059	79.5755	91.8301	78.8448	90.0327	**80.3758**	**92.2386**	38.4134	33.5784	45.755	40.2778	37.8219	26.9608
SDs	12.4533	7.794	**10.3102**	**5.1939**	12.5015	8.1947	10.6811	5.2743	24.0658	37.2236	25.013	40.8359	23.1552	33.4858

	Precision (%)													
Algorithms	AMiHRA - ℓ_1		AMiHRA - $\ell_1 - \ell_2$		AAMiHRA - ℓ_1		AAMiHRA - $\ell_1 - \ell_2$		TQW Alg 3.1 $- \ell_1$		DLY Alg 4-II - ℓ_1		AKTIS Alg 1 $- \ell_1$	
Nodes	Training	Testing	Training	Testing	Training	Testing	Training	Testing	Training	Testing	Training	Testing	Training	Testing
100	90.625	90.3226	90.625	90.625	90.625	90.3226	90.625	90.625	**90.7216**	**90.9091**	88.0435	88.2353	31.5271	13.1868
400	96.00	93.3333	**100**	**100**	96.00	93.3333	**100**	**100**	25.7895	7.5145	30.3704	11.6022	29.3532	10.6145
800	97.6562	**100**	**97.8417**	**100**	97.6562	**100**	**97.8417**	**100**	27.9487	14.0541	32.2115	15.3439	30.2956	14.4385
1200	45.5782	35.7143	**100**	**100**	45.5782	35.7143	**100**	**100**	51.087	37.6623	33.0969	14.8936	33.0189	14.8936
1600	91.6667	94.1176	90.8397	93.5484	91.6667	94.1176	84.5745	84.7826	35.9909	17.5258	85.8108	92.3077	**94.1606**	**97.2222**
2000	90.50	88.00	**95.0549**	**97.7778**	92.228	95.6522	**95.0549**	**97.7778**	21.6667	4.7904	37.3068	18.3673	36.8889	18.3673
Aver. Prec.	85.3377	83.5813	**95.7269**	**96.9919**	85.6257	84.8567	94.6827	95.5309	42.2007	28.7427	51.14	40.125	42.5407	28.1205
SDs	19.7023	23.7988	**4.2745**	**4.004**	19.8052	24.2831	6.0878	6.3963	25.9457	32.5888	27.8227	38.9238	25.4259	33.9464

Remark 4.10. *Comparing the performance results of all the algorithms shown in Table 3 and Figures 5, 6, 7 and 8, we make the following remarks.*

(1) *It is easily seen that our proposed algorithms, the AMiHRA - ℓ_1 and the AAMiHRA - ℓ_1 comparatively achieve higher training and testing accuracies and precisions than TQW Alg $3.1-\ell_1$, DLY Alg 4-II - ℓ_1 and AKTIS Alg $1-\ell_1$. Meanwhile, the SDs of both the training and testing accuracies and precisions of the AMiHRA - ℓ_1 and the AAMiHRA - ℓ_1 are extremely smaller than those of TQW Alg $3.1-\ell_1$, DLY Alg 4-II - ℓ_1 and AKTIS Alg $1-\ell_1$. These illustrate that the AMiHRA - ℓ_1 and the AAMiHRA - ℓ_1 achieve better stability and generalization performance in the experiments.*

(2) *It is also noted that due to the presence of the ℓ_2 penalty, the AMiHRA - $\ell_1-\ell_2$ and the AAMiHRA - $\ell_1-\ell_2$ have higher training and testing accuracies and precisions in most of the results than their corresponding AMiHRA - ℓ_1 and AAMiHRA - ℓ_1, which demonstrate their ability to achieve better generalization performance. Additionally, the SDs of both the training and testing accuracies and precisions of the AMiHRA - $\ell_1-\ell_2$ and the AAMiHRA - $\ell_1-\ell_2$ are extremely smaller than those of the AMiHRA - ℓ_1 and the AAMiHRA - ℓ_1, which show that they are more stable.*

Though the effectiveness and stability of our proposed algorithms have been demonstrated in the aforementioned experiments, to further investigate their comparative performance in this practical applications, we still need to conduct more statistical analysis. In this regard, we used the three UCI datasets mentioned in our earlier discussion and four different activation functions to measure and compare the statistical performance of all the algorithms. In the second series of experiments, we set $eC := \text{ones}(\mathcal{L}, m), eQ := \text{randn}(\mathcal{L}, m)$, $x_0 = -1eQ,\ x_1 = eQ,\ y_0 = 2eQ,\ u = 10^{-5}eC,\ s_n = 1.7eC$. We choose $\mathcal{L} = 100$, and used $||x_{n+1} - x_n|| < 10^{-5}$ and 100 as the Maximum number of iterations to terminate the the process for all the algorithms. As depicted in Table 4, we set the parameters $c,\ \lambda$ and γ according to the dataset and the activation function. The training and testing accuracies as well as the number of iterations of all the algorithms are reported in Table 4. We further display the performance comparison results among the algorithms based on the number of wins, ties and looses in Tables 5.

Activation functions		Sigmoid				Radbas			
				Accuracy (%)				Accuracy (%)	
Datasets	Algorithms	$c,\ \lambda,\ \gamma$	Iter.	Training	Testing	$c,\ \lambda,\ \gamma$	Iter.	Training	Testing
	AMiHRA - ℓ_1	0.95	9	97.7011	**98.4615**	1.1	8	93.1034	90.7692
	AMiHRA - $\ell_1 - \ell_2$	0.95, 0.9999, 0.00505	9	**98.2759**	**98.4615**	1.1, 0.9999, 0.00505	9	**94.8276**	**95.3846**
	AAMiHRA - ℓ_1	0.95	9	97.7011	**98.4615**	1.1	8	93.1034	90.7692
Breast Cancer W.	AAMiHRA - $\ell_1 - \ell_2$	0.95, 0.9999, 0.00505	9	**98.2759**	**98.4615**	1.1, 0.9999, 0.00505	9	**94.8276**	**95.3846**
	TQW Alg 3.1 - ℓ_1	0.95	9	94.2529	95.3846	1.1	8	88.5057	80.00
	DLY Alg 4-II - ℓ_1	0.95	11	51.7241	58.4615	1.1	11	78.7356	78.4615
	AKTIS Alg 1 - ℓ_1	0.95	17	54.5977	63.0769	1.1	16	29.8851	26.1538
	AMiHRA - ℓ_1	2.7	7	**96.4912**	**93.4783**	2.7	7	**95.614**	**86.9565**
	AMiHRA - $\ell_1 - \ell_2$	2.7, 0.999, 0.001	7	95.614	**93.4783**	2.7, 0.999, 0.001	7	**95.614**	**86.9565**
	AAMiHRA - ℓ_1	2.7	7	**96.4912**	**93.4783**	2.7	7	**95.614**	**86.9565**
Heart Disease	AAMiHRA - $\ell_1 - \ell_2$	2.7, 0.999, 0.001	7	**96.4912**	**93.4783**	2.7, 0.999, 0.001	7	**95.614**	**86.9565**
	TQW Alg 3.1 - ℓ_1	2.7	7	33.3333	21.7391	2.7	7	42.9825	45.6522
	DLY Alg 4-II - ℓ_1	2.7	7	92.1053	91.3043	2.7	7	91.2281	**86.9565**
	AKTIS Alg 1 - ℓ_1	2.7	16	33.3333	30.4348	2.7	16	25.4386	26.087
	AMiHRA - ℓ_1	0.91	9	90.00	90.00	0.701	10	96.00	**100.00**
	AMiHRA - $\ell_1 - \ell_2$	0.91, 0.999, 0.002	9	**98.00**	**95.00**	0.701, 0.99, 0.0107	9	**98.00**	**100.00**
	AAMiHRA - ℓ_1	0.91	9	90.00	90.00	0.701	10	96.00	**100.00**
Glass Identification	AAMiHRA - $\ell_1 - \ell_2$	0.91, 0.999, 0.002	9	**98.00**	**95.00**	0.701, 0.99, 0.0107	9	**98.00**	**100.00**
	TQW Alg 3.1 - ℓ_1	0.91	9	42.00	35.00	0.701	10	68.00	90.00
	DLY Alg 4-II - ℓ_1	0.91	9	24.00	20.00	0.701	7	0.00	0.00
	AKTIS Alg 1 - ℓ_1	0.91	13	30.00	45.00	0.701	13	30.00	15.00

Activation functions		Tribas				Hardlim			
				Accuracy (%)				Accuracy (%)	
Datasets	Algorithms	$c,\ \lambda,\ \gamma$	Iter.	Training	Testing	$c,\ \lambda,\ \gamma$	Iter.	Training	Testing
	AMiHRA - ℓ_1	0.94	9	96.5517	95.3846	1.05	9	**96.5517**	96.9231
	AMiHRA - $\ell_1 - \ell_2$	0.94, 0.9999, 0.00505	9	**98.2759**	**96.9231**	1.05, 0.999, 0.0009	9	95.977	98.4615
	AAMiHRA - ℓ_1	0.94	9	96.5517	95.3846	1.05	9	**96.5517**	96.9231
Breast Cancer W.	AAMiHRA - $\ell_1 - \ell_2$	0.94, 0.9999, 0.00505	9	**98.2759**	**96.9231**	1.05, 0.999, 0.0009	9	95.977	**100.00**
	TQW Alg 3.1 - ℓ_1	0.94	9	63.2184	61.5385	1.05	9	92.5287	84.6154
	DLY Alg 4-II - ℓ_1	0.94	11	60.9195	53.8462	1.05	11	95.4023	98.4615
	AKTIS Alg 1 - ℓ_1	0.94	40	47.7011	40	1.05	16	58.046	61.5385
	AMiHRA - ℓ_1	2.7	7	**85.9649**	**80.4348**	2.7	7	**85.9649**	**80.4348**
	AMiHRA - $\ell_1 - \ell_2$	2.7, 0.999, 0.001	7	**85.9649**	**80.4348**	2.7, 0.999, 0.001	7	**85.9649**	**80.4348**
	AAMiHRA - ℓ_1	2.7	7	**85.9649**	**80.4348**	2.7	7	**85.9649**	**80.4348**
Heart Disease	AAMiHRA - $\ell_1 - \ell_2$	2.7, 0.999, 0.001	7	**85.9649**	**80.4348**	2.7, 0.999, 0.001	7	**85.9649**	**80.4348**
	TQW Alg 3.1 - ℓ_1	2.7	7	45.614	45.6522	2.7	7	30.7018	15.2174
	DLY Alg 4-II - ℓ_1	2.7	7	84.2105	76.087	2.7	7	77.193	67.3913
	AKTIS Alg 1 - ℓ_1	2.7	16	29.8246	28.2609	2.7	16	25.4386	21.7391
	AMiHRA - ℓ_1	0.65	12	80.00	**95.00**	0.885	10	**94.00**	90.00
	AMiHRA - $\ell_1 - \ell_2$	0.65, 0.91, 0.25	12	**98.00**	**95.00**	0.885, 0.999, 0.0017	10	**94.00**	**95.00**
	AAMiHRA - ℓ_1	0.65	12	80.00	**95.00**	0.885	10	**94.00**	90.00
Glass Identification	AAMiHRA - $\ell_1 - \ell_2$	0.65, 0.91, 0.25	12	**98.00**	**95.00**	0.885, 0.999, 0.0017	10	**94.00**	**95.00**
	TQW Alg 3.1 - ℓ_1	0.65	10	68.00	85.00	0.885	9	36.00	40.00
	DLY Alg 4-II - ℓ_1	0.65	9	0.00	0.00	0.885	9	32.00	25.00
	AKTIS Alg 1 - ℓ_1	0.65	13	16.00	5.00	0.885	13	18.00	15.00

TABLE 4. Performance results of all the algorithms on all the dataset and four activation functions. The best and suboptimal results are highlighted in bold and underlined, respectively

	wins / ties / looses		
	AMiHRA - ℓ_1 vs. TQW Alg 3.1 $- \ell_1$	AMiHRA - ℓ_1 vs. DLY Alg 4-II - ℓ_1	AMiHRA - ℓ_1 vs. AKTIS Alg 1 - ℓ_1
Training	12/0/0	12/0/0	12/0/0
Testing	12/0/0	10/1/1	12/0/0
	AAMiHRA - ℓ_1 vs. TQW Alg 3.1 $- \ell_1$	AAMiHRA - ℓ_1 vs. DLY Alg 4-II - ℓ_1	AAMiHRA - ℓ_1 vs. AKTIS Alg 1 $- \ell_1$
Training	12/0/0	12/0/0	12/0/0
Testing	12/0/0	10/1/1	12/0/0
	AMiHRA - $\ell_1 - \ell_2$ vs. AMiHRA - ℓ_1		AAMiHRA - $\ell_1 - \ell_2$ vs. AAMiHRA - ℓ_1
Training	6/4/2		6/5/1
Testing	5/7/0		5/7/0

TABLE 5. Number of wins, ties and looses of all the algorithms.

Remark 4.11. (1) *We adopted the Wilconxon signed-ranks and Sign test* [15] *as the statistical methods to compare the reported results of all the algorithms in Table 4. In accordance with the statistical analysis on these results with Wilconxon signed-ranks, it is noted from Table 5 that our proposed algorithms (i.e., AMiHRA - ℓ_1 and AAMiHRA - ℓ_1) considerably achieve better training and testing accuracies than TQW Alg 3.1 $- \ell_1$, DLY Alg 4-II - ℓ_1 and AKTIS Alg $1 - \ell_1$. It is also found from the same table that the presence of the ℓ_2 penalty in AMiHRA - $\ell_1 - \ell_2$ and AAMiHRA - $\ell_1 - \ell_2$ improves their ability to achieve better and robust generalization performance than their correspondings AMiHRA - ℓ_1 and AAMiHRA - ℓ_1 in these experiments.*

(2) *On the hand, based on the null-hypothesis in the sign test* [15], *it is discovered that the normal distribution $h\left(\frac{h}{2}, \frac{\sqrt{h}}{2}\right)$ is obeyed by the number of wins for an algorithm and $h = (b$ datasets $\times$ d activation functions$)$. For this test, we assert that an algorithm is significantly better than the other, when its number of wins, compared to other is at least $\frac{h}{2} + Z_{m/2} \times \frac{\sqrt{h}}{2}$, where m is the assigned significant level. In all the experiments, we assigned $h = 12$ and $m = 0.1$, then $8 < \frac{12}{2} + 1.645 \times \frac{\sqrt{12}}{2} < 9$. This implies that an algorithm will be said to significantly achieves better performance, if its number of wins reaches at least 9. So, based on these facts, it is noticed from Table 5 that AMiHRA-ℓ_1 and AAMiHRA - ℓ_1 significantly achieve better performance than TQW Alg $3.1 - \ell_1$, DLY Alg 4-II - ℓ_1 and AKTIS Alg $1 - \ell_1$.*

(3) *Meanwhile, the number of wins of AMiHRA - $\ell_1 - \ell_2$ and AAMiHRA - $\ell_1 - \ell_2$ with ℓ_2 penalty when compared with their corresponding AMiHRA - ℓ_1 and AAMiHRA - ℓ_1 as shown in Table 5 are less than the least number, however, we noticed that they considerably achieve the highest number of wins in the experiments.*

5. CONCLUSION

This paper introduces two efficient Halpern-type inertial methods. The first is the alternated and multi-step inertial Halpern-type relaxed algorithm (AMiHRA) that involves three improved versions of the inertial steps, one of which is the alternated inertial step (1.8), while the others are the multi-step inertial steps (1.7), and the second is the accelerated alternated and multi-step inertial Halpern-type relaxed algorithm (AAMiHRA) that combines the three term conjugate gradient-like direction (1.10), the alternated inertial step (1.8) and the multi-step inertial step (1.7). In each of the two proposed algorithms, the monotonic self-adaptive step length criteria is used, which do not require any information about the norm of the underlying operator or the use of any line search procedure. The strong convergence theorem for each of the algorithms to a solution of problem (1.1) is formulated and proved based on the convergence theorem of the alternated inertial Halpern-type relaxed algorithm with perturbations in real

Hilbert spaces. The applications of the proposed methods in solving constrained minimization problems and classification problems based on the extreme learning machine ELM are analysed and their numerical results have been compared with the algorithms in [21, 53, 1]. In all the experiments based on ℓ_1-regularization approach, that is model (4.68), the numerical results show that the proposed algorithms (i.e., AMiHRA and AAMiHRA) are robust, computationally efficients and achieve better generalisation performance and stability than the algorithms in [21, 53, 1]. It is also noted from the results of the experiments that the proposed algorithms achieve better accuracy and stability based on the $\ell_1 - \ell_2$ hybrid regularization model, (i.e., the model (4.69)) than with ℓ_1-regularization model, (i.e., the model (4.69)).

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Availability of data and materials The datasets analyzed in this study are available in `https://archive.ics.uci.edu/`.

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Research Article

Duffin–Schaeffer inequality revisited

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ABSTRACT. The classical Markov inequality asserts that the n-th Chebyshev polynomial $T_n(x) = \cos n \arccos x$, $x \in [-1, 1]$, has the largest $C[-1, 1]$-norm of its derivatives within the set of algebraic polynomials of degree at most n whose absolute value in $[-1, 1]$ does not exceed one. In 1941 R.J. Duffin and A.C. Schaeffer found a remarkable refinement of Markov inequality, showing that this extremal property of T_n persists in the wider class of polynomials whose modulus is bounded by one at the extreme points of T_n in $[-1, 1]$. Their result gives rise to the definition of DS-type inequalities, which are comparison-type theorems of the following nature: inequalities between the absolute values of two polynomials of degree not exceeding n on a given set of $n + 1$ points in $[-1, 1]$ induce inequalities between the $C[-1, 1]$-norms of their derivatives. Here we apply the approach from a 1992 paper of A. Shadrin to prove some DS-type inequalities where Jacobi polynomials are extremal. In particular, we obtain an extension of the result of Duffin and Schaeffer.

Keywords: Markov inequality, Duffin–Schaeffer inequality, Chebyshev polynomials, Jacobi polynomials, interlacing of zeros.

2020 Mathematics Subject Classification: 41A17.

1. INTRODUCTION

Throughout this paper, π_n stands for the set of real-valued algebraic polynomials of degree not exceeding n, and $\|\cdot\|$ is the uniform norm in $[-1, 1]$,

$$\|g\| := \max_{x\in[-1,1]} |g(x)|.$$

The classical inequality of the brothers Markov reads as follows:

Theorem 1.1. *If $f \in \pi_n$ satisfies*

$$\|f\| \le 1, \tag{1.1}$$

then

$$\|f^{(k)}\| \le \|T_n^{(k)}\|, \quad k = 1, \dots, n, \tag{1.2}$$

and the equality in (1.2) occurs if and only if $f = \pm T_n$.

Here and henceforth, T_n is the n-th Chebyshev polynomial of the first kind, defined by

$$T_n(x) = \cos(n \arccos x), \quad x \in [-1, 1] .$$

The case $k = 1$ is due to Andrei Markov [7], and his brother Vladimir Markov [8] proved the general case, $1 \le k \le n$. For the intriguing history of Markov inequality and some of its proofs the reader is referred to the survey paper [21] .

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In 1941 Duffin and Schaeffer [5] found the following remarkable extension of Theorem 1.1 (for a proof, see also [18, Theorem 2.24] or [19, Section 5.6]):

Theorem 1.2 ([5]). *Inequality (1.2) remains true if assumption (1.1) is replaced with*

$$\Big|f\Big(\cos\frac{\nu\pi}{n}\Big)\Big|\le 1,\qquad \nu=0,\dots,n. \tag{1.3}$$

Theorem 1.2 may be viewed as a comparison type result: the inequality $|f|\le|T_n|$ at the $n+1$ points in $[-1,1]$ where $|T_n|=1$ implies inequalities between the uniform norms of the derivatives of f and T_n. This observation motivated the author to formulate in [9] the following:

Definition 1.1. *Let Q be a polynomial of degree n, and $\Delta=\{t_\nu\}_{\nu=0}^n$, where $1\ge t_0>\dots>t_n\ge -1$. The pair $\{Q,\Delta\}$ is said to admit Duffin–Schaeffer–type inequality (in short, DS–inequality), if for any $f\in\pi_n$, the assumption*

$$|f|\le|Q|\quad \textit{at the points from}\ \ \Delta$$

implies

$$\|f^{(k)}\|\le\|Q^{(k)}\|,\qquad k=1,\dots,n.$$

In this definition Q (called henceforth as *majorant*) is mutually assumed to be an oscillating polynomial in $[-1,1]$ (i.e., having n distinct zeros in $(-1,1)$), however, Δ is not necessarily the set of its critical points.

The shortest ever given proof of Markov's inequality, which moreover captures the refinement of Duffin and Schaeffer, is due to Alexei Shadrin [20]. Its main ingredient is the following:

Theorem 1.3 ([20]). *Let $Q\in\pi_n$ have n distinct zeros, all located in $(-1,1)$. If $f\in\pi_n$ satisfies*

$$|f|\le|Q|\ \ \textit{at the zeros of}\ \ (x^2-1)Q'(x),$$

then for each $k\in\{1,\dots,n\}$ and for every $x\in[-1,1]$ there holds

$$|f^{(k)}(x)|\le\max\Big\{|Q^{(k)}(x)|,\Big|\frac{x^2-1}{k}Q^{(k+1)}(x)+xQ^{(k)}(x)\Big|\Big\}.$$

Theorem 1.3 was applied in [3] for the proof of DS-inequalities where Q is an ultraspherical polynomial $P_n^{(\lambda)}$, $\lambda\ge 0$ and Δ is the set of its extreme points in $[-1,1]$. As a matter of fact, Theorem 1.3 implies DS-inequality whenever Q is oscillating polynomial with positive expansion in Chebyshev polynomials of the first kind and Δ is the set of its extreme points. Using Shadrin's idea to the proof of Theorem 1.3, we established various DS-type inequalities in [9], where, typically, Q is an ultraspherical polynomial and Δ is formed by the zeros of another ultraspherical polynomial.

In the present paper, we apply the approach from [20] to obtain DS-inequalities, where some Jacobi polynomials are the extremisers. As a particular case, we prove the following extension of the inequality of Duffin and Schaeffer, given by Theorem 1.2:

Theorem 1.4. *Let $f\in\pi_n$ satisfy $|f(1)|\le 1+2nc$ for some $c\in[0,1]$ and*

$$\Big|f\Big(\cos\frac{\nu\pi}{n}\Big)\Big|\le 1,\qquad \nu=1,\dots,n\,.$$

Then

$$\|f^{(k)}\|\le\|Q_n^{(k)}\|\,,\qquad k=1,\dots,n, \tag{1.4}$$

where $Q_n(x)=(1-c)\,T_n(x)+c\,W_n(x)$, with T_n and W_n being the n-th Chebyshev polynomials of the first and the fourth kind,

$$T_n(x)=\cos(n\theta),\qquad W_n(x)=\frac{\sin\big(n+\frac12\big)\theta}{\sin\big(\frac12\theta\big)},\qquad x=\cos\theta\,.$$

The equality in (1.4) *is attained only for* $f = \pm Q_n$.

The rest of the paper is organized as follows. In Section 2 we prove Theorem 2.5, which provides pointwise estimates for the derivatives of a polynomial $f \in \pi_n$ whose modulus is bounded at a set of $n+1$ distinct points in $[-1, 1]$. In Section 3 we apply Theorem 2.5 to obtain some DS-inequalities where the majorants are Jacobi polynomials, Theorem 1.4 being a particular case of them. In Section 4 we discuss applications of DS-inequalities and the interlink between DS-inequalities and Markov-type inequalities for polynomials with a curved majorant.

2. POINTWISE ESTIMATES FOR DERIVATIVES OF A POLYNOMIAL

If p and q are algebraic polynomials with only real and simple zeros, we say that the zeros of p and q *interlace*, if one can trace all the zeros of both polynomials, switching alternatively from a zero of p to zero of q and vice versa and moving only in one direction. If, in addition, no zero of p coincides with a zero of q, then the zeros of p and q are said to *interlace strictly*.

Clearly, interlacing is only possible if p and q are polynomials of the same degree or of degrees which differ by one. In the latter case, if p is of degree $n+1$, q is of degree n and the zeros of p and q interlace strictly, we say shortly that *the zeros of* q *separate the zeros of* p. The following lemma, due to V. Markov [8], asserts that the interlacing property is inherited by the zeros of the derivatives:

Lemma 2.1. *If the zeros of polynomials* p *and* q *interlace, then the zeros of* p' *and* q' *interlace strictly.*

Proofs of Lemma 2.1 can be found, e.g., in [11, Lemma 4], [18, Lemma 2.7.1] and [20]. For the sake of brevity, we write in this section

$$p \prec q$$

to say that p and q are polynomials of the same degree with interlacing zeros, with relation "$\le$" between the corresponding zeros of p and q. The notation

$$p \prec q \prec p$$

means that the zeros of p and q interlace and p is of higher degree than q.

The following theorem provides pointwise bounds for derivatives of polynomials $f \in \pi_n$ satisfying $|f| \le |Q_n|$ on a set of $n+1$ points related in a specific way to the majorant Q_n.

Theorem 2.5. *Let* Q_n *be a polynomial of degree* n *with only real and distinct zeros, all located in* $(-1, 1)$*, and let* ω *be a polynomial of degree* $n-1$ *whose zeros separate the zeros of* Q_n*. Assume that for some* $k \in \{1, \dots, n\}$ *and constants* $a \ge 1$ *and* $b \le -1$*,*

$$Q_n^{(k)}(x) = \big[(a-b-2)x + a + b\big]\omega^{(k)}(x) + k(a-b)\omega^{(k-1)}(x)\,. \tag{2.5}$$

If $f \in \pi_n$ *satisfies*

$$|f| \le |Q_n| \quad \text{at the zeros of } (x^2-1)\omega(x), \tag{2.6}$$

then

$$|f^{(k)}(x)| \le \max\Big\{\, \Big|Q_n^{(k)}(x)\Big|, |Z_{n,k}(x)|\,\Big\} \quad \text{for all } x \in \mathbb{R},$$

where

$$Z_{n,k}(x) = \big[2x^2 - (a+b)x + b - a\big]\omega^{(k)}(x) + k(2x-a-b)\omega^{(k-1)}(x)\,. \tag{2.7}$$

Proof. We consider first the case $1 \le k \le n-1$. With the notation introduced above, the assumption for the zeros of Q_n and ω can be written shortly as

$$Q_n \prec \omega \prec Q_n\,. \tag{2.8}$$

If

$$\omega_0(x) = (x+1)\omega(x)\,, \quad \omega_n(x) = (x-1)\omega(x),$$

then obviously

$$\omega_0 \prec Q_n \prec \omega_n$$

and, by Lemma 2.1,

$$\omega_0^{(k)} \prec Q_n^{(k)} \prec \omega_n^{(k)}\,. \tag{2.9}$$

Denote by $\{\alpha_i^k\}_{i=1}^{n-k}$ and $\{\beta_i^k\}_{i=1}^{n-k}$ the zeros of $\omega_0^{(k)}$ and $\omega_n^{(k)}$, respectively, labeled in increasing order, then it follows from (2.9) that each interval (α_i^k, β_i^k), $i = 1, \dots, n-k$, contains exactly one zero of $Q_n^{(k)}$, hence the zeros of $Q_n^{(k)}$ belong to the set

$$J_{n,k} = J_{n,k}(\omega) = \bigcup_{i=1}^{n-k} (\alpha_i^k, \beta_i^k)$$

and each interval (α_i^k, β_i^k) contains one zero of $Q_n^{(k)}$. Consequently,

$$Q_n^{(k)}(\beta_i^k)\, Q_n^{(k)}(\alpha_{i+1}^k) > 0, \quad i = 1, \dots, n-k-1 \tag{2.10}$$

(this statement is void if $k = n-1$). Denote by $I_{n,k} = I_{n,k}(\omega)$ the complementary set $\mathbb{R} \setminus J_{n,k}$,

$$I_{n,k} = I_{n,k}(\omega) = (-\infty, \alpha_1^k] \cup \bigcup_{i=1}^{n-k-1} [\beta_i^k, \alpha_{i+1}^k] \cup [\beta_{n-k}^k, \infty)\,.$$

The sets $I_{n,k}$ and $J_{n,k}$ are referred to as *Chebyshev set* and *Zolotarev set*, respectively.

Let $t_1 < \cdots < t_{n-1}$ be the zeros of ω, and $t_0 = -1$, $t_n = 1$. If $f \in \pi_n$ satisfies $|f(t_i)| \le |Q_n(t_i)|$ for $i = 0, \dots, n$, then

$$\left|f^{(k)}(x)\right| \le \left|Q_n^{(k)}(x)\right|, \qquad x \in I_{n,k}\,. \tag{2.11}$$

Indeed, if $\{\ell_i\}_{i=0}^n$ are the fundamental polynomials for interpolation at $\{t_i\}_{i=0}^n$, then

$$\ell_n \prec \ell_{n-1} \prec \cdots \prec \ell_0$$

and, by Lemma 2.1,

$$\ell_n^{(k)} \prec \ell_{n-1}^{(k)} \prec \cdots \prec \ell_0^{(k)}\,.$$

This observation, combined with the fact that the sign of the leading coefficient of $\ell_i(x)$ is $(-1)^i$, $i = 0, 1, \dots, n$, implies that if x is an interior point of $I_{n,k}$, then the signs of $\{\ell_i^{(k)}(x)\}_{i=0}^n$ alternate. In view of (2.8), so do the signs of $\{Q_n(t_i)\}_{i=0}^n$, and using (2.6) we conclude that

$$\begin{aligned} \left|f^{(k)}(x)\right| &= \left|\sum_{i=0}^n \ell_i^{(k)}(x) f(t_i)\right| \le \sum_{i=0}^n \left|\ell_i^{(k)}(x) f(t_i)\right| \\ &\le \sum_{i=0}^n \left|\ell_i^{(k)}(x) Q_n(t_i)\right| = \left|\sum_{i=0}^n \ell_i^{(k)}(x) Q_n(t_i)\right| = \left|Q_n^{(k)}(x)\right|\,. \end{aligned} \tag{2.12}$$

Obviously, (2.12) remains true also when x is a boundary point of $I_{n,k}$, and hence (2.11) is proved. It is readily seen from (2.12) that in the case when x is an interior point of $I_{n,k}$, the inequality (2.11) is strict unless $f = \pm Q_n$.

Next, we show that on the Zolotarev set $J_{n,k}$, $\left|f^{(k)}\right|$ is bounded by $|Z_{n,k}|$, i.e.,

$$\left|f^{(k)}(x)\right| \le |Z_{n,k}(x)|\,, \qquad x \in J_{n,k}. \tag{2.13}$$

Using the representation of $Q_n^{(k)}$ and $Z_{n,k}$, given in (2.5) and (2.7), we find that

$$Z_{n,k}(x) - Q_n^{(k)}(x) = 2(x-a)\omega_0^{(k)}(x),$$
$$Z_{n,k}(x) + Q_n^{(k)}(x) = 2(x-b)\omega_n^{(k)}(x).$$

Hence,

$$Z_{n,k}(x) = \begin{cases} Q_n^{(k)}(x), & x \in \{\alpha_i^k\}_{i=1}^{n-k} \cup \{a\}, \\ \\ -Q_n^{(k)}(x), & x \in \{b\} \cup \{\beta_i^k\}_{i=1}^{n-k}. \end{cases} \tag{2.14}$$

In view of (2.11), for an arbitrary constant $c \in (-1,1)$, we have

$$\left|c\, f^{(k)}(x)\right| < \left|Q_n^{(k)}(x)\right| = |Z_{n,k}(x)|\ , \quad x \in \{b\} \cup \{\alpha_i^k\}_{i=1}^{n-k} \cup \{\beta_i^k\}_{i=1}^{n-k} \cup \{a\}. \tag{2.15}$$

It follows from (2.10), (2.11), (2.14) and (2.15) that $Z_{n,k} - c\, f^{(k)}$ has a zero in each interval $(\beta_i^k, \alpha_{i+1}^k)$, $1 \le i \le n-k-1$ (again, this statement is void if $k = n-1$). Moreover, $Z_{n,k} - c\, f^{(k)}$ has a zero in each of intervals (b, α_1^k) and (β_{n-k}^k, a). To see this, we observe from (2.5) and (2.7) that the leading coefficients of $Q_n^{(k)}(x)$ and $Z_{n,k}(x)$ have the same sign. Since $Q_n^{(k)}$ has no zeros outside the interval $(\alpha_1^k, \beta_{n-k}^k)$, we find from (2.14) and (2.15) that

$$\operatorname{sign}\{Z_{n,k}(x) - c\, f^{(k)}(x)\}_{|x=\alpha_1^k} = \operatorname{sign}\{Q_n^{(k)}(\alpha_1^k)\} = -\operatorname{sign}\{Z_{n,k}(x) - c\, f^{(k)}(x)\}_{|x=b},$$
$$\operatorname{sign}\{Z_{n,k}(x) - c\, f^{(k)}(x)\}_{|x=\beta_{n-k}^k} = -\operatorname{sign}\{Q_n^{(k)}(\beta_{n-k}^k)\} = -\operatorname{sign}\{Z_{n,k}(x) - c\, f^{(k)}(x)\}_{|x=a}\,,$$

thus concluding that $Z_{n,k} - c\, f^{(k)}$ has a zero in each of the intervals (b, α_1^k) and (β_{n-k}^k, a).

Hence, all $n-k+1$ zeros of $Z_{n,k} - c\, f^{(k)}$ belong to the Chebyshev set $I_{n,k}$, and consequently $Z_{n,k} - c\, f^{(k)} \ne 0$ on $J_{n,k}$. Since $c \in (-1,1)$ is arbitrary, it follows that $\left|c\, f^{(k)}(x)\right| \ne |Z_{n,k}(x)|$, $x \in J_{n,k}$. On the boundary points of $J_{n,k} = \mathbb{R} \setminus I_{n,k}$ we have $\left|c\, f^{(k)}(x)\right| < |Z_{n,k}(x)|$, hence the same inequality holds true on $J_{n,k}$. Therefore, $\left|f^{(k)}(x)\right| \le |Z_{n,k}(x)|$ on the Zolotarev set, i.e., (2.13) holds true. The proof of Theorem 2.5 in the case $1 \le k \le n-1$ follows from (2.11) and (2.13). The remaining case $k = n$ is readily verified: since $\{\ell_i^{(n)}(x)\}_{i=0}^n$ is a sequence of sign alternating constants, (2.12) holds true in this case, too. □

Remark 2.1. *When $1 \le k \le n-1$ and x is an interior point of $I_{n,k}$, (2.12) implies the strict inequality $\left|f^{(n)}(x)\right| < \left|Q_n^{(n)}(x)\right|$ unless $f = \pm Q_n$. The same conclusion follows from (2.12) in the case $k = n$, i.e., $\left|f^{(n)}\right| < \left|Q_n^{(n)}\right|$ unless $f = \pm Q_n$.*

Remark 2.2. *Theorem 1.3 can be obtained as a special case of Theorem 2.5 with $a = 1$, $b = -1$ and $\omega = \frac{1}{2k}\, Q_n'$.*

3. DS-INEQUALITIES WITH JACOBI POLYNOMIALS AS MAJORANTS

Theorem 2.5 is applicable when the majorant Q_n is a Jacobi polynomial. Recall that Jacobi polynomials $\{P_m^{(\alpha,\beta)}\}_{m\in\mathbb{N}_0}$ are the orthogonal polynomials in $[-1,1]$ with respect to the weight function $w_{\alpha,\beta}(x) = (1-x)^\alpha(1+x)^\beta$, $\alpha, \beta > -1$, see e.g, [22, Chapt. 4].

Jacobi polynomials $P_n^{(\alpha,\beta)}$, $P_n^{(\alpha,\beta+1)}$ and $P_{n-1}^{(\alpha+1,\beta+1)}$ are connected with the identity

$$P_n^{(\alpha,\beta+1)}(x) = P_n^{(\alpha,\beta)}(x) + \frac{1}{2}(x-1)\,P_{n-1}^{(\alpha+1,\beta+1)}(x), \tag{3.16}$$

which is a consequence of Gauss' contiguous relations (see, e.g. [1, Section 2.5]) and the representation of Jacobi polynomials as hypergeometric functions. It follows from (3.16) and

$$\frac{d}{dx}\{P_n^{(\alpha,\beta)}(x)\} = \frac{1}{2}(n+\alpha+\beta+1)P_{n-1}^{(\alpha+1,\beta+1)}(x) \tag{3.17}$$

(see [22, eqn. (4.21.7)]) that the zeros of $P_n^{(\alpha,\beta+1)}(x)$ and $\frac{d}{dx}\{P_n^{(\alpha,\beta)}(x)\}$ interlace.

Setting $Q_n := P_n^{(\alpha,\beta+1)}$ and $q := P_n^{(\alpha,\beta)}$, by k-fold differentiation of (3.16) we get

$$Q_n^{(k)}(x) = \frac{1}{n+\alpha+\beta+1}\big[(x-1)q^{(k+1)}(x) + (n+\alpha+\beta+k+1)\,q^{(k)}(x)\big]. \tag{3.18}$$

This representation of $Q_n^{(k)}$ provides relation (2.5) between Q_n and ω in Theorem 2.5 with

$$\omega(x) = \frac{1}{2k}\,q'(x), \quad a = 1, \quad b = -1 - \frac{2k}{n+\alpha+\beta+1}.$$

Replacing these quantities in (2.7), we find that in this particular case Theorem 2.5 reads as:

Theorem 3.6. *Let $Q_n = P_n^{(\alpha,\beta+1)}$ and $q = P_n^{(\alpha,\beta)}$. If $f \in \pi_n$ satisfies*

$$|f| \le |Q_n| \quad \text{at the zeros of } (x^2-1)q'(x),$$

then for $k = 1,\dots,n$ and for every $x \in \mathbb{R}$,

$$\left|f^{(k)}(x)\right| \le \max\left\{\left|Q_n^{(k)}(x)\right|, |Z_{n,k}(x)|\right\},$$

where

$$Z_{n,k}(x) = \frac{x^2-1}{k}q^{(k+1)}(x) + xq^{(k)}(x) + \frac{\big((x-1)q'(x)\big)^{(k)}}{n+\alpha+\beta+1}. \tag{3.19}$$

Theorem 3.6 enables us to prove the following DS-inequality:

Theorem 3.7. *Let $Q_n = P_n^{(\alpha,\beta+1)}$ and $q = P_n^{(\alpha,\beta)}$, where $-1/2 < \alpha \le \beta$. If $f \in \pi_n$ satisfies*

$$|f| \le |Q_n| \quad \text{at the zeros of } (x^2-1)q'(x),$$

then

$$\|f^{(k)}\| \le \|Q_n^{(k)}\|, \qquad k = 1,\dots,n\,. \tag{3.20}$$

The equality occurs in (3.20) if and only if $f = \pm Q_n$.

Proof of Theorem 3.7. We shall show that for $Z_{n,k}$ defined in (3.19) there holds

$$\|Z_{n,k}\| < \|Q_n^{(k)}\|, \qquad k = 1,\dots,n. \tag{3.21}$$

We need the following property of Jacobi polynomials (cf. [22, Theorem 5.32.1] or [1, p. 350, Problem 40]):

Lemma 3.2. *Let $\max\{\alpha,\beta\} \ge -1/2$. Then for every $m \in \mathbb{N}$,*

$$\|P_m^{(\alpha,\beta)}\| = \begin{cases} P_m^{(\alpha,\beta)}(1) & = \binom{m+\alpha}{m}, \ \text{if } \ \alpha \ge \beta, \\ \left|P_m^{(\alpha,\beta)}(-1)\right| & = \binom{m+\beta}{m}, \ \text{if } \ \beta \ge \alpha. \end{cases}$$

Unless $\alpha = \beta = -1/2$, the norm of $P_m^{(\alpha,\beta)}$ is attained only at end point of the interval $[-1,1]$.

In view of (3.17), apart from constant factors, the polynomials $Q_n^{(k)}$, $q^{(k)}$ and $q^{(k+1)}$ are equal respectively to $P_{n-k}^{(\alpha+k,\beta+k+1)}$, $P_{n-k}^{(\alpha+k,\beta+k)}$ and $P_{n-k-1}^{(\alpha+k+1,\beta+k+1)}$. Since $\beta \geq \alpha \geq -1/2$, we have $\beta+k+1 \geq \alpha+k+1 > \alpha+k \geq 1/2$, then (3.18) and Lemma 3.2 imply

$$\|Q_n^{(k)}\| = |Q_n^{(k)}(-1)| = \frac{2}{n+\alpha+\beta+1}\left|q^{(k+1)}(-1)\right| + \left(1+\frac{k}{n+\alpha+\beta+1}\right)\left|q^{(k)}(-1)\right|. \tag{3.22}$$

We represent the polynomial $Z_{n,k}$ defined in (3.19) in the form

$$Z_{n,k}(x) = \varphi(x) + \frac{1}{n+\alpha+\beta+1}\,\psi(x), \tag{3.23}$$

where

$$\begin{aligned}\varphi(x) &= \frac{x^2-1}{k}q^{(k+1)}(x) + xq^{(k)}(x),\\ \psi(x) &= (x-1)\,q^{(k+1)}(x) + kq^{(k)}(x).\end{aligned}$$

Lemma 3.3. *Let $q = P_n^{(\alpha,\beta)}$, where $\beta \geq \alpha \geq -1/2$. Then*

$$\|\psi\| = \left\|\left(x-1)q^{(k+1)}(x) + kq^{(k)}(x)\right)\right\| = 2\left|q^{(k+1)}(-1)\right| + k\left|q^{(k)}(-1)\right| = |\psi(-1)|$$

and, in addition, $\|\psi\|$ is attained only at $x=-1$.

Proof of Lemma 3.3. By triangle inequality and Lemma 3.2,

$$\begin{aligned}\|\psi\| &\leq \left\|(x-1)q^{(k+1)}(x)\right\| + k\left\|q^{(k)}(x)\right\| \leq 2\left\|q^{(k+1)}\right\| + k\left\|q^{(k)}\right\|\\ &= 2\left|q^{(k+1)}(-1)\right| + k\left|q^{(k)}(-1)\right| = \left|-2\,q^{(k+1)}(-1) + k\,q^{(k)}(-1)\right|\\ &= |\psi(-1)|.\end{aligned}$$

On account of the last claim of Lemma 3.2, one can readily see that $x=-1$ is the unique point $[-1,1]$ where the norm of ψ is attained. □

Next, we estimate the norm of φ.

Lemma 3.4. *Let $q = P_n^{(\alpha,\beta)}$, where $\beta \geq \alpha \geq -1/2$. Then*

$$\|\varphi\| = \left\|\frac{x^2-1}{k}q^{(k+1)}(x) + xq^{(k)}(x)\right\| = \left|q^{(k)}(-1)\right| = |\varphi(-1)|, \qquad k=1,\ldots,n.$$

Proof of Lemma 3.4. We consider separately three cases.

Case 1: $\alpha=\beta=-1/2$. This case, corresponding to $q=T_n$, has been proven by Shadrin in [20, Lemma 3], it reads as

$$\left\|\frac{x^2-1}{k}T_n^{(k+1)}(x) + x\,T_n^{(k)}(x)\right\| = \left|T_n^{(k)}(-1)\right| = T_n^{(k)}(1).$$

Case 2: $\alpha=\beta>-1/2$. We make use of the fact that $q = P_n^{(\alpha,\alpha)}$ admits non-negative expansion in the Chebyshev polynomials of the first kind (cf. [2, eq. (7.34)]):

$$q(x) = \sum_{\nu=0}^{n} c_\nu\,T_\nu(x), \qquad c_\nu \geq 0.$$

Using the result from *Case 1*, we find

$$\begin{aligned}\Big\|\frac{x^2-1}{k}q^{(k+1)}(x)+x\,q^{(k)}(x)\Big\| &= \Big\|\sum_{\nu=0}^{n}c_\nu\Big(\frac{x^2-1}{k}T_\nu^{(k+1)}(x)+x\,T_\nu^{(k)}(x)\Big)\Big\|\\ &\le \sum_{\nu=0}^{n}c_\nu\Big\|\frac{x^2-1}{k}T_\nu^{(k+1)}(x)+x\,T_\nu^{(k)}(x)\Big\|\\ &= \sum_{\nu=0}^{n}c_\nu T_\nu^{(k)}(1)=q^{(k)}(1)=\Big|q^{(k)}(-1)\Big|.\end{aligned}$$

Case 3: $\beta>\alpha\ge-1/2$. Set $r=P_n^{(\beta,\alpha)}$, then $r(x)$ admits representation in the basis of $\{P_\nu^{(\alpha,\alpha)}(x)\}_{\nu=0}^{n}=:\{P_\nu(x)\}_{\nu=0}^{n}$ with non-negative coefficients (cf. [2, eq. (7.33)]):

$$r(x)=\sum_{\nu=0}^{n}c_\nu\,P_\nu(x),\qquad c_\nu=c_\nu(n,\alpha,\beta)\ge0.$$

This representation and the result from *Case 2* imply

$$\begin{aligned}\Big\|\frac{x^2-1}{k}r^{(k+1)}(x)+x\,r^{(k)}(x)\Big\| &= \Big\|\sum_{\nu=0}^{n}c_\nu\Big(\frac{x^2-1}{k}P_\nu^{(k+1)}(x)+xP_\nu^{(k)}(x)\Big)\Big\|\\ &\le \sum_{\nu=0}^{n}c_\nu\Big\|\frac{x^2-1}{k}P_\nu^{(k+1)}(x)+xP_\nu^{(k)}(x)\Big\|\\ &= \sum_{\nu=0}^{n}c_\nu P_\nu^{(k)}(1)=r^{(k)}(1).\end{aligned}$$

Now using the symmetry property $P_n^{(\beta,\alpha)}(-x)=(-1)^nP_n^{(\alpha,\beta)}(x)$ (cf. [4, p. 144, eq. (2.8)], for $q(x)=P_n^{(\alpha,\beta)}(x)=(-1)^nr(-x)$ we obtain

$$\Big\|\frac{x^2-1}{k}q^{(k+1)}(x)+x\,q^{(k)}(x)\Big\|=\Big\|\frac{x^2-1}{k}r^{(k+1)}(x)+x\,r^{(k)}(x)\Big\|=r^{(k)}(1)=\Big|q^{(k)}(-1)\Big|.$$

Lemma 3.4 is proved. □

Lemma 3.3 and Lemma 3.4 imply

$$\begin{aligned}\|Z_{n,k}\| &= \Big\|\varphi+\frac{1}{n+\alpha+\beta+1}\,\psi\Big\|\\ &\le \|\varphi\|+\frac{1}{n+\alpha+\beta+1}\|\psi\|\\ &= |\varphi(-1)|+\frac{1}{n+\alpha+\beta+1}\,|\psi(-1)|\\ &= \Big|q^{(k)}(-1)\Big|+\frac{1}{n+\alpha+\beta+1}\Big(2\Big|q^{(k+1)}(-1)\Big|+k\Big|q^{(k)}(-1)\Big|\Big)\\ &= \frac{2}{n+\alpha+\beta+1}\,\Big|q^{(k+1)}(-1)\Big|+\Big(1+\frac{k}{n+\alpha+\beta+1}\Big)\Big|q^{(k)}(-1)\Big|.\end{aligned}\tag{3.24}$$

According to (3.22), the last expression is equal to $\|Q_n^{(k)}\|$, so we have proved the inequality $\|Z_{n,k}\|\le\|Q_n^{(k)}\|$, and now inequality (3.20) in Theorem 3.7 follows from Theorem 3.6.

For the last statement of Theorem 3.7, we need to prove the strict inequality (3.21). We observe that $\varphi(-1)$ and $\psi(-1)$ have opposite signs, namely,

$$\operatorname{sign}\varphi(-1) = -\operatorname{sign} q^{(k)}(-1) = (-1)^{n-1-k}, \qquad \operatorname{sign}\psi(-1) = \operatorname{sign} q^{(k)}(-1) = (-1)^{n-k}.$$

Therefore, the inequality in the second line of (3.24) is strict, and hence (3.21) holds true. We are ready now to prove the last claim of Theorem 3.7. The case $k = n$ is a direct consequence from Remark 2.1. The case $1 \le k \le n-1$ is also justified with Remark 2.1 as follows. We recall that if $f \in \pi_n$ satisfies the assumptions of Theorem 3.6, then $|Z_{n,k}(x)|$ (resp. $\left|Q_n^{(k)}(x)\right|$) furnishes upper bound for $\left|f^{(k)}(x)\right|$ when x belongs to the Zolotarev set $J_{n,k}$ (resp. Chebyshev set $I_{n,k}$). In view of (3.21), the equality $\|f^{(k)}\| = \|Q_n^{(k)}\|$ can happen only when the norm of $f^{(k)}$ is attained at a point x from the set $I_{n,k}\bigcap[-1,1]$. Since $\left|f^{(k)}(x)\right| \le \left|Q_n^{(k)}(x)\right|$ for $x \in I_{n,k}\bigcap[-1,1]$ and, by Lemma 3.2, $\|Q_n^{(k)}\| = \left|Q_n^{(k)}(-1)\right|$ with $x = -1$ being the unique point where the norm of $Q_n^{(k)}$ is attained, it follows that $\|f^{(k)}\| = \|Q_n^{(k)}\|$ is possible only when $\left|f^{(k)}(-1)\right| = \left|Q_n^{(k)}(-1)\right|$. Since $x = -1$ is an interior point for $I_{n,k}$, the last equality holds only if $f = \pm Q_n$, by virtue of Remark 2.1. $\square$

Let us consider the special case $\alpha = \beta = -1/2$. According to (3.17),

$$q'(x) = \frac{1}{2} n\, P_{n-1}^{(1/2,1/2)}(x)$$

and, apart from a constant factor, q' is equal to the $(n-1)^{\text{th}}$ Chebyshev polynomial of the second kind U_{n-1}, which is defined for $x \in [-1,1]$ by

$$U_{n-1}(x) = \frac{\sin n\theta}{\sin\theta}, \quad x = \cos\theta,$$

and whose zeros are

$$t_\nu = \cos\frac{\nu\pi}{n}, \qquad \nu = 1, \ldots, n-1.$$

On the other hand, apart from a constant multiplier, $Q_n = P_n^{(-1/2,1/2)}$ is equal to the Chebyshev polynomial of the third kind $V_n(x)$, which is defined for $x \in [-1,1]$ by

$$V_n(x) = \frac{\cos\left(n + \frac{1}{2}\right)\theta}{\cos\left(\frac{1}{2}\theta\right)}, \qquad x = \cos\theta\,. \tag{3.25}$$

Clearly,

$$\begin{aligned} V_n\left(\cos\frac{\nu\pi}{n}\right) &= (-1)^\nu, \quad \nu = 0, 1, \ldots, n-1,\\ V_n(-1) &= (-1)^n(2n+1)\,. \end{aligned}$$

Thus, in the case $\alpha = \beta = -1/2$, Theorem 3.7 comes down to the following:

Corollary 3.1. *Let $f \in \pi_n$ satisfy $|f(-1)| \le 2n+1$ and*

$$\left|f\left(\cos\frac{\nu\pi}{n}\right)\right| \le 1, \qquad \nu = 0, \ldots, n-1\,.$$

Then

$$\left\|f^{(k)}\right\| \le \left\|V_n^{(k)}\right\|, \qquad k = 1, \ldots, n, \tag{3.26}$$

where V_n is the n-th Chebyshev polynomial of the third kind (3.25). *The equality in* (3.26) *occurs only if $f = \pm V_n$.*

The n-th Chebyshev polynomial of the fourth kind $W_n(x) = (-1)^n V_n(-x)$ is defined for $x \in [-1,1]$ by

$$W_n(x) = \frac{\sin\left(n+\frac{1}{2}\right)\theta}{\sin\left(\frac{1}{2}\theta\right)}, \qquad x = \cos\theta\,. \tag{3.27}$$

Reflection of the variable in Corollary 3.1 (i.e., replacement of x with $-x$) yields another corollary of Theorem 3.7:

Corollary 3.2. *Let $f \in \pi_n$ satisfy $|f(1)| \le 2n+1$ and*

$$\left|f\left(\cos\frac{\nu\pi}{n}\right)\right| \le 1, \qquad \nu = 1,\dots,n\,,$$

then

$$\|f^{(k)}\| \le \|W_n^{(k)}\|\,, \qquad k = 1,\dots,n, \tag{3.28}$$

where W_n is the n-th Chebyshev polynomial of the fourth kind (3.27). The equality in (3.28) occurs only if $f = \pm W_n$.

Proof of Theorem 1.4. Theorem 1.4 is deduced as a convex combination of the result of Duffin and Schaeffer (Theorem 1.2) and Corollary 3.2. Assume that for some $c \in [0,1]$, the polynomial $f \in \pi_n$ satisfies

$$\begin{gathered}\left|f\left(\cos\frac{\nu\pi}{n}\right)\right| \le 1, \quad \nu = 1,\dots,n,\\ |f(1)| \le 1+2cn.\end{gathered}$$

Clearly, f can be represented as $f(x) = (1-c)\,g(x) + c\,h(x)$, where $f,\ h \in \pi_n$ obey the restrictions

$$\begin{gathered}\left|g\left(\cos\frac{\nu\pi}{n}\right)\right| \le 1, \qquad \nu = 0,\dots,n,\\ \left|h\left(\cos\frac{\nu\pi}{n}\right)\right| \le 1, \qquad \nu = 1,\dots,n,\\ |h(1)| \le 2n+1.\end{gathered}$$

Theorem 1.2 implies

$$\|g^{(k)}\| \le \|T_n^{(k)}\| = T_n^{(k)}(1)$$

while Corollary 3.2 yields

$$\|h^{(k)}\| \le \|W_n^{(k)}\| = W_n^{(k)}(1).$$

Consequently,

$$\begin{aligned}\|f^{(k)}\| &\le (1-c)\,\|g^{(k)}\| + c\,\|h^{(k)}\| \le (1-c)\,\|T_n^{(k)}\| + c\,\|W_n^{(k)}\|\\ &= (1-c)\,T_n^{(k)}(1) + c\,W_n^{(k)}(1) = \|(1-c)\,T_n^{(k)} + c\,W_n^{(k)}\|\\ &= \|Q_n^{(k)}\|\,,\end{aligned}$$

where $Q_n(x) = (1-c)\,T_n(x) + c\,W_n(x)$. Since the norm of $Q_n^{(k)}$ is attained at $x=1$, we apply Remark 2.1 to conclude that the equality $\|f^{(k)}\| = \|Q_n^{(k)}\|$ occurs only when $f = \pm Q_n$. □

4. Concluding remarks

DS-inequalities can find application to the estimation of the round-off error of interpolatory formulae for numerical differentiation (see [9, p. 174, Remark 2]). Also, DS-inequalities may serve as a useful tool for establishing Markov-type inequalities for polynomials with curved majorants. For the readers convenience, we provide below a brief information on this topic.

We call *majorant* a continuous positive (or non-negative) function $\mu(x)$ in $[-1,1]$. If there exists a polynomial $P \in \pi_n$, $P \neq 0$, such that $-\mu(x) \leq P(x) \leq \mu(x)$, $x \in [-1,1]$, then there exists a unique (up to orientation) polynomial $\omega_\mu \in \pi_n$ (*snake polynomial*) which oscillates most between $\pm\mu$. The n-th snake polynomial ω_μ, associated with the majorant μ, is uniquely determined by the following properties:

a) $|\omega_\mu(x)| \leq \mu(x) \quad x \in [-1,1]\,;$

b) There exists a set $\delta^* = (\tau_i^*)_{i=0}^n$, $1 \geq \tau_0^* > \cdots > \tau_n^* \geq -1$, such that

$$\omega_\mu(\tau_i^*) = (-1)^i \mu(\tau_i^*), \quad i = 0, \ldots, n.$$

The set δ^* is referred to as *the set of alternation points of* ω_μ.

Associated with a given a majorant $\mu(x)$, we have the following extremal problems (cf. [14]):

Problem 1: Markov inequality with a majorant. *Given* $n, k \in \mathbb{N}$, $1 \leq k \leq n$, *and a majorant* $\mu \geq 0$, *find*

$$M_{k,n}(\mu) := \sup\{\|p^{(k)}\| \,:\, p \in \pi_n,\ |p(x)| \leq \mu(x),\ x \in [-1,1]\}\,.$$

Problem 2: Duffin-Schaeffer inequality with a majorant. *Given* $n, k \in \mathbb{N}$, $1 \leq k \leq n$, *and a majorant* $\mu \geq 0$, *find*

$$D_{k,n}(\mu) := \sup\{\|p^{(k)}\| \,:\, p \in \pi_n,\ |p(x)| \leq \mu(x),\ x \in \delta^*\}\,.$$

Clearly, $M_{k,n}(\mu) \leq D_{k,n}(\mu)$, and the results of V. A. Markov and of R. J. Duffin and A. C. Schaeffer (Theorem 1.2) read as:

$$\mu(x) \equiv 1 \Rightarrow M_{k,n}(\mu) = D_{k,n}(\mu) = \|T_n^{(k)}\|\,, \quad 1 \leq k \leq n\,.$$

A natural question is: for which other majorants μ the snake-polynomial ω_μ is extremal to both Problems 1 and 2, i.e., when do we have the equalities

$$M_{k,n}(\mu) \stackrel{?}{=} D_{k,n}(\mu) \stackrel{?}{=} \|\omega_\mu^{(k)}\|\,?$$

A conjecture (belonging to mathematical folklore) states that the extremal polynomial to Problem 1 is the snake polynomial ω_μ. So far, no counterexample to this conjecture is found. On the contrary, ω_μ is not always the extremal polynomial to Problem 2, the following counterexamples are known:

$$\begin{aligned} &1) \quad \mu(x) = \sqrt{1-x^2}, \quad k = 1 \quad \text{(cf. [16]);} \\ &2) \quad \mu(x) = 1 - x^2, \quad k = 1, 2 \quad \text{(cf. [17]).} \end{aligned}$$

The difficulty with the proof of the above conjecture comes from the fact that only in some exceptional cases the snake polynomials are known explicitly (and the same applies to the associated sets of alternation points). Assuming the snake polynomial ω_μ is known, a possible approach to showing that ω_μ is the extreme polynomial to Problem 2 is to show that DS-inequality holds for any pair (ω_μ, Δ) such that the points from Δ are separated by the zeros of ω_μ. For $\mu \equiv 1$ (and $\omega_\mu = T_n$) this plan was realized by the author in [10] (for $k = 1$) and [12] (the general case $1 \leq k \leq n$), thus showing that whenever T_n is a snake polynomial associated with some majorant μ, then T_n is the extreme polynomial to Problem 2. Even more is true: in [14, 15] we proved that whenever the snake polynomial associated with a majorant μ possessses positive or sign-alternating expansion in the Chebyshev polynomials of the first kind, it is the extreme polynomial to Problem 2.

The DS-inequalities in this paper were announced without proof in [13]. Although they can be derived from the results in [14, 15], we decided to propose here a direct self-contained proof, emphasizing to the important particular case presented by Theorem 1.4.

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Research Article

Study of some new one-parameter modifications of the Hardy-Hilbert integral inequality

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ABSTRACT. One of the pillars of mathematical analysis is the Hardy-Hilbert integral inequality. In this article, we advance the theory by introducing several new modifications to this inequality. They have the property of incorporating an adjustable parameter and different power functions, allowing for greater flexibility and broader applicability. Notably, one modification has a logarithmic structure, offering a distinctive extension to the classical framework. For the main results, the optimality of the corresponding constant factors is shown. Additional integral inequalities of various forms and scopes are also established. Thus, this work contributes to the ongoing development of Hardy-Hilbert-type inequalities by presenting new generalizations and providing rigorous mathematical justifications for each result.

Keywords: Hardy-Hilbert integral inequality, power function, beta function, primitives, generalized Hölder integral inequality, Hardy integral inequality.

2020 Mathematics Subject Classification: 26D15, 33E20.

1. INTRODUCTION

The classical Hardy-Hilbert integral inequality is a fundamental result in real and harmonic analysis. It plays an important role in establishing bounds in various double integral estimates. A formal statement of this inequality is given below. Let $p > 1$, $q = p/(p-1)$ and $f, g : [0, +\infty) \mapsto [0, +\infty)$ (be two non-negative functions) such that

$$\int_0^{+\infty} f^p(x)dx < +\infty, \quad \int_0^{+\infty} g^q(y)dy < +\infty.$$

Then there exists a sharp constant $\Delta > 0$ such that

$$\int_0^{+\infty}\int_0^{+\infty} \frac{1}{x+y} f(x)g(y)dxdy \leq \Delta \left[\int_0^{+\infty} f^p(x)dx\right]^{1/p} \left[\int_0^{+\infty} g^q(y)dy\right]^{1/q}. \tag{1.1}$$

This inequality was studied in detail by Hardy, Littlewood and Polya in the early 20th century. See [7]. The optimal constant Λ in Equation (1.1) is given by

$$\Delta = \frac{\pi}{\sin(\pi/p)}.$$

In other words, Δ is the smallest possible constant such that the inequality in Equation (1.1) holds for all admissible functions f and g.

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Thanks to its flexibility with respect to f and g, the Hardy-Hilbert integral inequality has significant applications in analysis, partial differential equations, and related fields. Numerous generalizations and refinements have been proposed, often involving modifications to the integrand (kernel) structure, the integration domain, or the dependence on parameters. For a complete overview, see [23, 19, 20, 18, 21, 9, 22, 24, 17, 2, 1, 4, 5, 8, 12, 3, 13, 16, 15].

For the purposes of this article, we highlight three well-known results below. The technical details can be found in the two reference books of B.C. Yang: [23, 22].

First result: An integral norm variant of the Hardy-Hilbert integral inequality is presented below. Let $p > 1$, $q = p/(p-1)$ and $f, g : [0, +\infty) \mapsto [0, +\infty)$ such that

$$\int_0^{+\infty} x^{p/2-1} f^p(x)dx < +\infty, \quad \int_0^{+\infty} y^{q/2-1} g^q(y)dy < +\infty.$$

Then the following inequality holds:

(1.2)
$$\int_0^{+\infty} \int_0^{+\infty} \frac{1}{x+y} f(x)g(y)dxdy \leq \pi \left[\int_0^{+\infty} x^{p/2-1} f^p(x)dx\right]^{1/p} \left[\int_0^{+\infty} y^{q/2-1} g^q(y)dy\right]^{1/q}.$$

Compared to the Hardy-Hilbert integral inequality, weighted integral norms of f and g are considered in the upper bound, with the weight functions $\omega(x) = x^{p/2-1}$ and $\rho(y) = y^{q/2-1}$, respectively. The purpose of these weight functions is to emphasize the different growth rates of f and g. In this setting, the constant π is found to be optimal.

Second result: The second interesting result is in the same vein but deals with power functions depending on an adjustable parameter. Let $p > 1$, $q = p/(p-1)$, $\alpha > -1$ and $f, g : [0, +\infty) \mapsto [0, +\infty)$ such that

$$\int_0^{+\infty} x^{p/2-1} f^p(x)dx < +\infty, \quad \int_0^{+\infty} y^{q/2-1} g^q(y)dy < +\infty.$$

Then the following inequality holds:

$$\int_0^{+\infty} \int_0^{+\infty} \frac{1}{x^{\alpha+1}+y^{\alpha+1}} f(x)g(y)dxdy$$
(1.3)
$$\leq \frac{\pi}{\alpha+1} \left[\int_0^{+\infty} x^{p/2-1} f^p(x)dx\right]^{1/p} \left[\int_0^{+\infty} y^{q/2-1} g^q(y)dy\right]^{1/q}.$$

The case $\alpha = 0$ corresponds to the result in Equation (1.2). It is therefore a one-parameter generalization. In this case, the constant $\pi/(\alpha+1)$ is also optimal.

Third result: The third and last key result is different from the previous two. It has the feature of dealing with a logarithmic function in the integrand, which strongly modifies the functional structure of the original Hardy-Hilbert integral inequality. Let $p > 1$, $q = p/(p-1)$ and $f, g : [0, +\infty) \mapsto [0, +\infty)$ such that

$$\int_0^{+\infty} x^{p/2-1} f^p(x)dx < +\infty, \quad \int_0^{+\infty} y^{q/2-1} g^q(y)dy < +\infty.$$

Then the following inequality holds:

$$\int_0^{+\infty} \int_0^{+\infty} \frac{1}{x-y} \log\left(\frac{x}{y}\right) f(x)g(y)dxdy$$
$$\leq \pi^2 \left[\int_0^{+\infty} x^{p/2-1} f^p(x)dx\right]^{1/p} \left[\int_0^{+\infty} y^{q/2-1} g^q(y)dy\right]^{1/q},$$

noting that the function $[1/(x-y)]\log(x/y)$ is always non-negative. This inequality can be seen as a natural extension of [7, Formula 342]. In this case, the constant π^2 is also optimal.

Building on these results, the contributions of this article are divided into three complementary parts. In the first part, we present integral formulas, which are new to our knowledge. In particular, they are not presented in [6], which remains the largest collection of reference integrals.

These formulas serve as the basis for the second part, where we use them to develop new forms of Hardy-Hilbert-type integral inequalities. The first form involves multiple power functions depending on an adjustable parameter. Specifically, we determine the optimal constant $\Lambda > 0$ such that

$$\int_0^{+\infty}\int_0^{+\infty}\frac{x^\alpha+y^\alpha}{x^{\alpha+1}+y^{\alpha+1}}f(x)g(y)dxdy$$
$$\leq\Lambda\left[\int_0^{+\infty}x^{p/2-1}f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty}y^{q/2-1}g^q(y)dy\right]^{1/q},$$

where $\alpha > -1/2$ is the adjustable parameter. The case $\alpha = 0$ corresponds to the result given in Equation (1.2). In this sense, the established inequality is a valuable generalization. We then use this innovative Hardy-Hilbert-type integral inequality to derive several new variants. These include modified power integrands and also variants involving the primitives of f and g, drawing connections to the Hardy inequality. We again refer to [7] for more details on this classical inequality.

In the third part, we introduce a logarithmic structure to the analysis. Specifically, using a new logarithmic formula, we determine the optimal constant $\Omega > 0$ such that

$$\int_0^{+\infty}\int_0^{+\infty}\frac{(x-y)\log(x/y)}{(x^{\alpha+1}+y^{\alpha+1})^2}f(x)g(y)dxdy$$
$$\leq\Omega\left[\int_0^{+\infty}x^{-(\alpha-1/2)p-1}f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty}y^{-(\alpha-1/2)q-1}g^q(y)dy\right]^{1/q},$$

with the same restriction on α, i.e., $\alpha > -1/2$, and noting that the function $(x-y)\log(x/y)$ is always non-negative. To our knowledge, this inequality is new even in the special case $\alpha = 0$. We also use this result to derive additional modifications, including one that involves the primitives of f and g.

All proofs are given in full, without omitting intermediate steps.

The rest of the article is organized as follows: Section 2 presents the new integral formulas. Sections 3 and 4 develop the first and second forms of Hardy-Hilbert-type inequalities, respectively. Section 5 gives the complete proofs of the results. Finally, Section 6 contains concluding remarks and perspectives.

2. Integral formulas

Key integral formulas are detailed in this section.

2.1. **First formula.** Our first formula is an explicit evaluation of a one-parameter integral that involves multiple power functions. It will play a central role in the development of our first Hardy-Hilbert integral inequalities.

Proposition 2.1. *For any $\alpha > -1/2$, the following equality holds:*

$$\int_0^{+\infty} \frac{1+x^\alpha}{\sqrt{x}(1+x^{\alpha+1})}dx = \frac{2\pi}{\alpha+1}\left[\sin\left(\frac{\pi}{2(\alpha+1)}\right)\right]^{-1}.$$

The proof is based on a careful application of a classical integral result involving the beta function, i.e., [6, entry 3.241.2]. To our knowledge, the formula in Proposition 2.1 is new. It may serve as a valuable tool in various contexts involving singular integrals, weighted inequalities, or the analysis of fractional operators, beyond the purposes of this article.

In particular, the special cases below are of interest because of the simplicity of the result.

- For $\alpha = 0$, we have
$$\int_0^{+\infty} \frac{1}{\sqrt{x}(1+x)}dx = \pi.$$
This is a very classical result, associated with the primitive $2\arctan[\sqrt{x}]$.
- For $\alpha = 1/2$, we get
$$\int_0^{+\infty} \frac{1+\sqrt{x}}{\sqrt{x}(1+x^{3/2})}dx = \frac{8\pi}{3\sqrt{3}}.$$
- For $\alpha = 1$, we have
$$\int_0^{+\infty} \frac{1+x}{\sqrt{x}(1+x^2)}dx = \sqrt{2}\pi.$$
- For $\alpha = 2$, we obtain
$$\int_0^{+\infty} \frac{1+x^2}{\sqrt{x}(1+x^3)}dx = \frac{4\pi}{3}.$$

Some negative values of α are also allowed. In particular, if we take $\alpha = -1/4$, Proposition 2.1 gives

$$\int_0^{+\infty} \frac{1+x^{-1/4}}{\sqrt{x}(1+x^{3/4})}dx = \frac{16\pi}{3\sqrt{3}}.$$

In fact, for the values considered of α, we are able to find primitives associated with the integrand. However, for decimals of large value of α, the complexity increases significantly and our formula becomes essential.

2.2. **Second formula.** Our second formula is derived directly from Proposition 2.1. Its main originality lies in the presence of the positive function $(x-1)\log(x)$ in the integrand and still power functions depending on an adjustable parameter. Like the previous result, it will play a key role in establishing optimal constants in a main Hardy-Hilbert-type integral inequality.

Proposition 2.2. *For any $\alpha > -1/2$, the following equality holds:*

$$\int_0^{+\infty} x^{\alpha-1/2}\frac{(x-1)\log(x)}{(1+x^{\alpha+1})^2}dx$$
$$=\frac{\pi}{(\alpha+1)^3}\left[\sin\left(\frac{\pi}{2(\alpha+1)}\right)\right]^{-2}\left[2(1+\alpha)\sin\left(\frac{\pi}{2(\alpha+1)}\right)-\pi\cos\left(\frac{\pi}{2(\alpha+1)}\right)\right].$$

More precisely, the proof is based on the differentiation of the identity in Proposition 2.1 with respect to the parameter α. Again, to the best of our knowledge, this formula is new. In particular, it is not referred to in [6]. As a short numerical study, the results below illustrate the formula by considering specific values of α.

- For $\alpha = -1/4$, we have
$$\int_0^{+\infty} x^{-3/4}\frac{(x-1)\log(x)}{(1+x^{3/4})^2}dx = \frac{256}{81}\left(\frac{3\sqrt{3}}{4}+\frac{\pi}{2}\right)\pi \approx 28.4945.$$
- For $\alpha = 1/2$, we get
$$\int_0^{+\infty} \frac{(x-1)\log(x)}{(1+x^{3/2})^2}dx = \frac{32}{81}\left(\frac{3\sqrt{3}}{2}-\frac{\pi}{2}\right)\pi \approx 1.27498.$$
- For $\alpha = 1$, we obtain
$$\int_0^{+\infty} x^{1/2}\frac{(x-1)\log(x)}{(1+x^2)^2}dx = \frac{1}{4}\pi\left(2\sqrt{2}-\frac{\pi}{\sqrt{2}}\right) \approx 0.476725.$$

Numerical checks confirm the correspondence of the left-hand and right-hand sides.

3. First Hardy-Hilbert-type integral inequalities

We are now in a position to present our first Hardy-Hilbert-type integral inequalities.

3.1. **Main result.** The proposition below describes our first modification of the Hardy-Hilbert integral inequality. We emphasize the original form of the integrand with multiple power functions depending on an adjustable parameter and the expression of the constant factor with the sine function.

Proposition 3.3. *Let $p > 1$, $q = p/(p-1)$, $\alpha > -1/2$, and $f, g : [0,+\infty) \mapsto [0,+\infty)$ such that*
$$\int_0^{+\infty} x^{p/2-1}f^p(x)dx < +\infty, \quad \int_0^{+\infty} y^{q/2-1}g^q(y)dy < +\infty.$$
Then the following inequality holds:
$$\int_0^{+\infty}\int_0^{+\infty} \frac{x^\alpha + y^\alpha}{x^{\alpha+1}+y^{\alpha+1}}f(x)g(y)dxdy$$
$$\leq \Lambda_\alpha \left[\int_0^{+\infty} x^{p/2-1}f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} y^{q/2-1}g^q(y)dy\right]^{1/q},$$
where
$$\Lambda_\alpha = \frac{2\pi}{\alpha+1}\left[\sin\left(\frac{\pi}{2(\alpha+1)}\right)\right]^{-1}. \tag{3.4}$$

The proof is based mainly on an appropriate decomposition of the integrand, the generalized Hölder integral inequality, changes of variables, and Proposition 2.1.

The cases presented below are of interest, especially for the simplicity of the integrands and the expressions of the factor constants:

- For $\alpha = 0$, we have
$$\int_0^{+\infty}\int_0^{+\infty} \frac{1}{x+y}f(x)g(y)dxdy$$
$$\leq \pi \left[\int_0^{+\infty} x^{p/2-1}f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} y^{q/2-1}g^q(y)dy\right]^{1/q}.$$
We recognize the modification of the Hardy-Hilbert integral inequality as presented in Equation (1.2). In this sense, Proposition 3.3 can be viewed as a generalization.

- For $\alpha = 1/2$, Proposition 3.3 gives

$$\int_0^{+\infty}\int_0^{+\infty}\frac{\sqrt{x}+\sqrt{y}}{x^{3/2}+y^{3/2}}f(x)g(y)dxdy$$
$$\leq\frac{8\pi}{3\sqrt{3}}\left[\int_0^{+\infty}x^{p/2-1}f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty}y^{q/2-1}g^q(y)dy\right]^{1/q}.$$

- For $\alpha = 1$, we have

$$\int_0^{+\infty}\int_0^{+\infty}\frac{x+y}{x^2+y^2}f(x)g(y)dxdy$$
$$\leq\sqrt{2}\pi\left[\int_0^{+\infty}x^{p/2-1}f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty}y^{q/2-1}g^q(y)dy\right]^{1/q}.$$

- For $\alpha = 2$, we get

$$\int_0^{+\infty}\int_0^{+\infty}\frac{x^2+y^2}{x^3+y^3}f(x)g(y)dxdy$$
$$\leq\frac{4\pi}{3}\left[\int_0^{+\infty}x^{p/2-1}f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty}y^{q/2-1}g^q(y)dy\right]^{1/q}.$$

- As a example of negative value for α, if we take $\alpha = -1/4$, we have

$$\int_0^{+\infty}\int_0^{+\infty}\frac{x^{-1/4}+y^{-1/4}}{x^{3/4}+y^{3/4}}f(x)g(y)dxdy$$
$$\leq\frac{16\pi}{3\sqrt{3}}\left[\int_0^{+\infty}x^{p/2-1}f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty}y^{q/2-1}g^q(y)dy\right]^{1/q}.$$

To the best of our knowledge, these results and the intermediate cases are new to the literature on integral inequalities. They open up new perspectives in functional analysis and in operator theory in particular.

3.2. **Additional results.** As formalized below, it is important to note that the factor constant in Proposition 3.3 is optimal.

Proposition 3.4. *In the setting of Proposition 3.3, the constant Λ_α in Equation (3.4) is optimal.*

The proof is by contradiction reasoning in combination with well-chosen extremal functions.

A hybrid version between the inequalities in Proposition 3.3 and Equation (1.3) is suggested below.

Proposition 3.5. *Let $p > 1$, $q = p/(p-1)$, $\alpha > -1/2$, $\beta \in [0,1]$ and $f, g : [0,+\infty) \mapsto [0,+\infty)$ such that*

$$\int_0^{+\infty}x^{p/2-1}f^p(x)dx < +\infty, \qquad \int_0^{+\infty}y^{q/2-1}g^q(y)dy < +\infty.$$

Then the following inequality holds:

$$\int_0^{+\infty}\int_0^{+\infty}\frac{(x^\alpha+y^\alpha)^\beta}{x^{\alpha+1}+y^{\alpha+1}}f(x)g(y)dxdy$$
$$\leq\Upsilon_{\alpha,\beta}\left[\int_0^{+\infty}x^{p/2-1}f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty}y^{q/2-1}g^q(y)dy\right]^{1/q},$$

where

$$\Upsilon_{\alpha,\beta} = \frac{\pi}{\alpha+1}2^{\beta}\left[\sin\left(\frac{\pi}{2(\alpha+1)}\right)\right]^{-\beta}. \tag{3.5}$$

The proof is based on a thorough use of the Hölder integral inequality, combined with Proposition 3.3 and Equation (1.3). Clearly, if we take $\beta = 1$, then Proposition 3.5 reduces to Proposition 3.3, and if we take $\beta = 0$, it reduces to Equation (1.3), all the intermediary values giving a new case.

A modification of Proposition 3.3 is developed below. The main change is the numerator term of the form $(x+y)^{\alpha}$, where α is the same parameter used in the denominator.

Proposition 3.6. *Let $p > 1$, $q = p/(p-1)$, $\alpha > 0$ and $f, g : [0, +\infty) \mapsto [0, +\infty)$ such that*

$$\int_0^{+\infty} x^{p/2-1} f^p(x)dx < +\infty, \quad \int_0^{+\infty} y^{q/2-1} g^q(y)dy < +\infty.$$

Then the following inequality holds:

$$\begin{aligned}&\int_0^{+\infty}\int_0^{+\infty} \frac{(x+y)^{\alpha}}{x^{\alpha+1}+y^{\alpha+1}} f(x)g(y)dxdy\\ \leq&\Theta_{\alpha,\beta}\left[\int_0^{+\infty} x^{p/2-1} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} y^{q/2-1} g^q(y)dy\right]^{1/q},\end{aligned}$$

where

$$\Theta_{\alpha,\beta} = \max(2^{\alpha-1}, 1)\frac{2\pi}{\alpha+1}\left[\sin\left(\frac{\pi}{2(\alpha+1)}\right)\right]^{-1}. \tag{3.6}$$

The proof follows from a convexity inequality and Proposition 3.3.

A variant of Proposition 3.3 is proposed below. It includes the primitives of f and g, following the spirit of the Hardy integral inequality.

Proposition 3.7. *Let $p > 1$, $q = p/(p-1)$, $\alpha > -1/2$, $f, g : [0, +\infty) \mapsto [0, +\infty)$ such that*

$$\int_0^{+\infty} f^p(x)dx < +\infty, \quad \int_0^{+\infty} g^q(y)dy < +\infty$$

and $F, G : [0, +\infty) \mapsto [0, +\infty)$ be their respective primitives given by

$$F(x) = \int_0^x f(t)dt, \quad G(y) = \int_0^y g(t)dt,$$

assuming that there exist (converge).

Then the following inequality holds:

$$\begin{aligned}&\int_0^{+\infty}\int_0^{+\infty} \frac{x^{\alpha}+y^{\alpha}}{x^{\alpha+1}+y^{\alpha+1}} x^{1/p-3/2}y^{1/q-3/2}F(x)G(y)dxdy\\ \leq&\Phi_{\alpha,p}\left[\int_0^{+\infty} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} g^q(y)dy\right]^{1/q},\end{aligned}$$

where

$$\Phi_{\alpha,p} = \frac{2\pi}{\alpha+1}\left[\sin\left(\frac{\pi}{2(\alpha+1)}\right)\right]^{-1}\left(\frac{p}{p-1}\right)\left(\frac{q}{q-1}\right). \tag{3.7}$$

The proof relies on Proposition 3.3 and the Hardy integral inequality. Note that, unlike the previous results, we are dealing with the unweighted integral norms of f and g.

The rest of the article is devoted to the second Hardy-Hilbert-type integral inequalities.

4. Second Hardy-Hilbert-type integral inequalities

4.1. **Main result.** The proposition below is our second major contribution of the Hardy-Hilbert integral inequality. We emphasize the original form of the integrand with a logarithmic function and multiple power functions. The expression of the constant factor with the sine function is also singular. It will be shown later that it is optimal.

Proposition 4.8. *Let $p > 1$, $q = p/(p-1)$, $\alpha > -1/2$, and $f, g : [0,+\infty) \mapsto [0,+\infty)$ such that*

$$\int_0^{+\infty} x^{-(\alpha-1/2)p-1} f^p(x)dx < +\infty, \qquad \int_0^{+\infty} y^{-(\alpha-1/2)q-1} g^q(y)dy < +\infty.$$

Then the following inequality holds:

$$\begin{aligned}&\int_0^{+\infty}\int_0^{+\infty} \frac{(x-y)\log(x/y)}{(x^{\alpha+1}+y^{\alpha+1})^2} f(x)g(y)dxdy\\ \leq&\Omega_\alpha \left[\int_0^{+\infty} x^{-(\alpha-1/2)p-1} f^p(x)dx\right]^{1/p} \left[\int_0^{+\infty} y^{-(\alpha-1/2)q-1} g^q(y)dy\right]^{1/q},\end{aligned}$$

where

$$\Omega_\alpha = \frac{\pi}{(\alpha+1)^3}\left[\sin\left(\frac{\pi}{2(\alpha+1)}\right)\right]^{-2}\left[2(1+\alpha)\sin\left(\frac{\pi}{2(\alpha+1)}\right) - \pi\cos\left(\frac{\pi}{2(\alpha+1)}\right)\right]. \tag{4.8}$$

The proof relies on an appropriate decomposition of the integrand, the generalized Hölder integral inequality, changes of variables, and Proposition 2.2.

Some special cases are highlighted below.

- For $\alpha = -1/4$, we have

$$\begin{aligned}&\int_0^{+\infty}\int_0^{+\infty} \frac{(x-y)\log(x/y)}{(x^{3/4}+y^{3/4})^2} f(x)g(y)dxdy\\ \leq&\frac{256}{81}\left(\frac{3\sqrt{3}}{4}+\frac{\pi}{2}\right)\pi \left[\int_0^{+\infty} x^{-(\alpha-1/2)p-1} f^p(x)dx\right]^{1/p} \left[\int_0^{+\infty} y^{-(\alpha-1/2)q-1} g^q(y)dy\right]^{1/q}.\end{aligned}$$

- For $\alpha = 1/2$, we obtain

$$\begin{aligned}&\int_0^{+\infty}\int_0^{+\infty} \frac{(x-y)\log(x/y)}{(x^{3/2}+y^{3/2})^2} f(x)g(y)dxdy\\ \leq&\frac{32}{81}\left(\frac{3\sqrt{3}}{2}-\frac{\pi}{2}\right)\pi \left[\int_0^{+\infty} x^{-(\alpha-1/2)p-1} f^p(x)dx\right]^{1/p} \left[\int_0^{+\infty} y^{-(\alpha-1/2)q-1} g^q(y)dy\right]^{1/q}.\end{aligned}$$

- For $\alpha = 1$, we get

$$\begin{aligned}&\int_0^{+\infty}\int_0^{+\infty} \frac{(x-y)\log(x/y)}{(x^2+y^2)^2} f(x)g(y)dxdy\\ \leq&\frac{1}{4}\pi\left(2\sqrt{2}-\frac{\pi}{\sqrt{2}}\right) \left[\int_0^{+\infty} x^{-(\alpha-1/2)p-1} f^p(x)dx\right]^{1/p} \left[\int_0^{+\infty} y^{-(\alpha-1/2)q-1} g^q(y)dy\right]^{1/q}.\end{aligned}$$

All of them are new, to the best of our knowledge.

4.2. **Additional results.** The constant in Proposition 4.8 is optimal, as formalized in the proposition below.

Proposition 4.9. *In the setting of Proposition 4.8, the constant Ω_α in Equation (4.8) is optimal.*

A special version of Proposition 4.8 is given below. It deals with only one function, and with an integrand that can be negative or positive; there is a certain degree of complexity in this aspect.

Proposition 4.10. *Let $p > 1$, $q = p/(p-1)$, $\alpha > -1/2$, and $f : [0, +\infty) \mapsto [0, +\infty)$ be a function such that*

$$\int_0^{+\infty} x^{-(\alpha-1/2)p-1} f^p(x)dx < +\infty, \qquad \int_0^{+\infty} y^{-(\alpha-1/2)q-1} f^q(y)dy < +\infty.$$

Then the following inequality holds:

$$\int_0^{+\infty}\int_0^{+\infty} \frac{x\log(x/y)}{(x^{\alpha+1}+y^{\alpha+1})^2} f(x)f(y)dxdy$$
$$\le \chi_\alpha \left[\int_0^{+\infty} x^{-(\alpha-1/2)p-1} f^p(x)dx\right]^{1/p} \left[\int_0^{+\infty} y^{-(\alpha-1/2)q-1} f^q(y)dy\right]^{1/q},$$

where

$$\chi_\alpha = \frac{\pi}{2(\alpha+1)^3}\left[\sin\left(\frac{\pi}{2(\alpha+1)}\right)\right]^{-2}\left[2(1+\alpha)\sin\left(\frac{\pi}{2(\alpha+1)}\right) - \pi\cos\left(\frac{\pi}{2(\alpha+1)}\right)\right]. \tag{4.9}$$

The proof follows from a re-examination of the double integral in Proposition 4.8 in the special case $f = g$.

A modification of Proposition 4.8 is given below. It deals with a sophisticated power function depending on y and x, i.e., $(x/y)^{y-x}$.

Proposition 4.11. *Let $p > 1$, $q = p/(p-1)$, $\alpha > -1/2$, and $f, g : [0, +\infty) \mapsto [0, +\infty)$ such that*

$$\int_0^{+\infty} x^{-(\alpha-1/2)p-1} f^p(x)dx < +\infty, \qquad \int_0^{+\infty} y^{-(\alpha-1/2)q-1} g^q(y)dy < +\infty.$$

Then the following inequality holds:

$$\int_0^{+\infty}\int_0^{+\infty} \left[1-\left(\frac{x}{y}\right)^{y-x}\right] \frac{1}{(x^{\alpha+1}+y^{\alpha+1})^2} f(x)g(y)dxdy$$
$$\le \Omega_\alpha \left[\int_0^{+\infty} x^{-(\alpha-1/2)p-1} f^p(x)dx\right]^{1/p} \left[\int_0^{+\infty} y^{-(\alpha-1/2)q-1} g^q(y)dy\right]^{1/q},$$

where Ω_α is given by Equation (4.8).

The proof is mainly based on a well-known logarithmic inequality and Proposition 4.8.

A primitive version of Proposition 4.8 is presented below.

Proposition 4.12. *Let $p > 1$, $q = p/(p-1)$, $\alpha > -1/2$, $f, g : [0, +\infty) \mapsto [0, +\infty)$ such that*

$$\int_0^{+\infty} f^p(x)dx < +\infty, \qquad \int_0^{+\infty} g^q(y)dy < +\infty$$

and $F, G : [0, +\infty) \mapsto [0, +\infty)$ be their respective primitives given by

$$F(x) = \int_0^x f(t)dt, \quad G(y) = \int_0^y g(t)dt,$$

assuming that there exist (converge).

Then the following inequality holds:

$$\begin{aligned}&\int_0^{+\infty}\int_0^{+\infty}\frac{(x-y)\log(x/y)}{(x^{\alpha+1}+y^{\alpha+1})^2}x^{\alpha+1/p-3/2}y^{\alpha+1/q-3/2}F(x)G(y)dxdy\\ \leq&\Psi_{\alpha,p}\left[\int_0^{+\infty}f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty}g^q(y)dy\right]^{1/q},\end{aligned}$$

where

$$\begin{aligned}\Psi_{\alpha,p}=&\frac{\pi}{(\alpha+1)^3}\left[\sin\left(\frac{\pi}{2(\alpha+1)}\right)\right]^{-2}\left[2(1+\alpha)\sin\left(\frac{\pi}{2(\alpha+1)}\right)-\pi\cos\left(\frac{\pi}{2(\alpha+1)}\right)\right]\\ &\times\left(\frac{p}{p-1}\right)\left(\frac{q}{q-1}\right).\end{aligned}\tag{4.10}$$

The proof relies on Proposition 4.8 and the Hardy integral inequality.

5. Proofs

This section contains the proofs of all our results.

5.1. Proofs of the propositions in Section 2.

Proof of Proposition 2.1. We need the integral formula below, with reference to [6, Entry 3.2412]. For any $\nu>\mu>0$, we have

$$\int_0^{+\infty}\frac{x^{\mu-1}}{1+x^{\nu}}dx=\frac{\pi}{\nu}\left[\sin\left(\frac{\mu\pi}{\nu}\right)\right]^{-1}.\tag{5.11}$$

An integral decomposition gives

$$\begin{aligned}\int_0^{+\infty}\frac{1+x^{\alpha}}{\sqrt{x}(1+x^{\alpha+1})}dx&=\int_0^{+\infty}\frac{1}{\sqrt{x}(1+x^{\alpha+1})}dx+\int_0^{+\infty}\frac{x^{\alpha}}{\sqrt{x}(1+x^{\alpha+1})}dx\\&=\int_0^{+\infty}\frac{x^{1/2-1}}{1+x^{\alpha+1}}dx+\int_0^{+\infty}\frac{x^{(\alpha+1/2)-1}}{1+x^{\alpha+1}}dx.\end{aligned}$$

Let us determine the expression of each of these integrals. First, applying Equation (5.11) to $\nu=\alpha+1$ and $\mu=1/2$, we get

$$\int_0^{+\infty}\frac{x^{1/2-1}}{1+x^{\alpha+1}}dx=\frac{\pi}{\alpha+1}\left[\sin\left(\frac{\pi}{2(\alpha+1)}\right)\right]^{-1}.$$

Second, applying again Equation (5.11) to $\nu=\alpha+1$, but to $\mu=\alpha+1/2$, which obviously satisfies $\mu=\alpha+1/2<\alpha+1=\nu$, and using the basic trigonometric formula $\sin(\pi-x)=\sin(x)$, we get

$$\begin{aligned}\int_0^{+\infty}\frac{x^{(\alpha+1/2)-1}}{1+x^{\alpha+1}}dx&=\frac{\pi}{\alpha+1}\left[\sin\left(\frac{(\alpha+1/2)\pi}{\alpha+1}\right)\right]^{-1}=\frac{\pi}{\alpha+1}\left[\sin\left(\frac{(2\alpha+1)\pi}{2(\alpha+1)}\right)\right]^{-1}\\&=\frac{\pi}{\alpha+1}\left[\sin\left(\pi-\frac{\pi}{2(\alpha+1)}\right)\right]^{-1}=\frac{\pi}{\alpha+1}\left[\sin\left(\frac{\pi}{2(\alpha+1)}\right)\right]^{-1}.\end{aligned}$$

We therefore have

$$\begin{aligned}\int_0^{+\infty} \frac{1+x^\alpha}{\sqrt{x}(1+x^{\alpha+1})}dx &= \frac{\pi}{\alpha+1}\left[\sin\left(\frac{\pi}{2(\alpha+1)}\right)\right]^{-1} + \frac{\pi}{\alpha+1}\left[\sin\left(\frac{\pi}{2(\alpha+1)}\right)\right]^{-1}\\ &= \frac{2\pi}{\alpha+1}\left[\sin\left(\frac{\pi}{2(\alpha+1)}\right)\right]^{-1}.\end{aligned}$$

This ends the proof of Proposition 2.1. □

Proof of Proposition 2.2. It follows from Proposition 2.1 that

$$\int_0^{+\infty} \frac{1+x^\alpha}{\sqrt{x}(1+x^{\alpha+1})}dx = \frac{2\pi}{\alpha+1}\left[\sin\left(\frac{\pi}{2(\alpha+1)}\right)\right]^{-1},$$

with $\alpha > -1/2$. Taking the partial derivative with respect to α on both sides, and developing the right-hand side term, we get

$$\begin{aligned}&\frac{\partial}{\partial\alpha}\int_0^{+\infty} \frac{1+x^\alpha}{\sqrt{x}(1+x^{\alpha+1})}dx\\ =&\frac{\partial}{\partial\alpha}\left\{\frac{2\pi}{\alpha+1}\left[\sin\left(\frac{\pi}{2(\alpha+1)}\right)\right]^{-1}\right\}\\ =&-\frac{\pi}{(\alpha+1)^3}\left[\sin\left(\frac{\pi}{2(\alpha+1)}\right)\right]^{-2}\left[2(1+\alpha)\sin\left(\frac{\pi}{2(\alpha+1)}\right)-\pi\cos\left(\frac{\pi}{2(\alpha+1)}\right)\right].\end{aligned} \tag{5.12}$$

Thanks to the Leibnitz integral rule, we can exchange the integral and partial derivative sign for the left-hand side term, which gives

$$\begin{aligned}\frac{\partial}{\partial\alpha}\int_0^{+\infty} \frac{1+x^\alpha}{\sqrt{x}(1+x^{\alpha+1})}dx &= \int_0^{+\infty}\frac{\partial}{\partial\alpha}\left[\frac{1+x^\alpha}{\sqrt{x}(1+x^{\alpha+1})}\right]dx\\ &=-\int_0^{+\infty} x^{\alpha-1/2}\frac{(x-1)\log(x)}{(1+x^{\alpha+1})^2}dx.\end{aligned} \tag{5.13}$$

It follows from Equations (5.12) and (5.13) that

$$\begin{aligned}&\int_0^{+\infty} x^{\alpha-1/2}\frac{(x-1)\log(x)}{(1+x^{\alpha+1})^2}dx\\ =&\frac{\pi}{(\alpha+1)^3}\left[\sin\left(\frac{\pi}{2(\alpha+1)}\right)\right]^{-2}\left[2(1+\alpha)\sin\left(\frac{\pi}{2(\alpha+1)}\right)-\pi\cos\left(\frac{\pi}{2(\alpha+1)}\right)\right].\end{aligned}$$

This concludes the proof of Proposition 2.2. □

5.2. **Proofs of the propositions in Section 3.**

Proof of Proposition 3.3. Decomposing appropriately the integrand, taking into account the identity $1/p + 1/q = 1$ and applying the Hölder integral inequality, we get

$$\begin{aligned}
&\int_0^{+\infty}\int_0^{+\infty} \frac{x^\alpha + y^\alpha}{x^{\alpha+1}+y^{\alpha+1}} f(x)g(y)dxdy \\
&= \int_0^{+\infty}\int_0^{+\infty} x^{1/(2q)}y^{-1/(2p)}\left[\frac{x^\alpha + y^\alpha}{x^{\alpha+1}+y^{\alpha+1}}\right]^{1/p} f(x) \\
&\times x^{-1/(2q)}y^{1/(2p)}\left[\frac{x^\alpha + y^\alpha}{x^{\alpha+1}+y^{\alpha+1}}\right]^{1/q} g(y)dxdy \\
&\leq \mathfrak{A}^{1/p}\mathfrak{B}^{1/q}, \qquad (5.14)
\end{aligned}$$

where $\mathfrak{A}$ and $\mathfrak{B}$ are given by

$$\mathfrak{A} = \int_0^{+\infty}\int_0^{+\infty} x^{p/(2q)}y^{-1/2}\frac{x^\alpha + y^\alpha}{x^{\alpha+1}+y^{\alpha+1}} f^p(x)dxdy$$

and

$$\mathfrak{B} = \int_0^{+\infty}\int_0^{+\infty} x^{-1/2}y^{q/(2p)}\frac{x^\alpha + y^\alpha}{x^{\alpha+1}+y^{\alpha+1}} g^q(y)dxdy.$$

Let us investigate the expressions of these terms, starting with $\mathfrak{A}$.

Exchanging the order of integration by the Fubini-Tonelli integral theorem, using the change of variables $u = y/x$, applying Proposition 2.1 from which emerges the constant Λ_α in Equation (3.4) and noting that $p/q = p - 1$, we get

$$\begin{aligned}
\mathfrak{A} &= \int_0^{+\infty} x^{p/(2q)} f^p(x)\left[\int_0^{+\infty} y^{-1/2}\frac{x^\alpha + y^\alpha}{x^{\alpha+1}+y^{\alpha+1}}dy\right]dx \\
&= \int_0^{+\infty} x^{p/(2q)-1/2} f^p(x)\left[\int_0^{+\infty} \frac{1+(y/x)^\alpha}{\sqrt{y/x}\,[1+(y/x)^{\alpha+1}]}\times\frac{1}{x}dy\right]dx \\
&= \int_0^{+\infty} x^{p/(2q)-1/2} f^p(x)\left\{\int_0^{+\infty} \frac{1+u^\alpha}{\sqrt{u}(1+u^{\alpha+1})}du\right\}dx \\
&= \Lambda_\alpha\int_0^{+\infty} x^{p/(2q)-1/2} f^p(x)dx \\
&= \Lambda_\alpha\int_0^{+\infty} x^{p/2-1} f^p(x)dx. \qquad (5.15)
\end{aligned}$$

For the term $\mathfrak{B}$, we proceed in a similar way, but with the change of variables $v = x/y$. We thus obtain

$$\begin{aligned}
\mathfrak{B} &= \int_0^{+\infty} y^{q/(2p)} g^q(y) \left[\int_0^{+\infty} x^{-1/2} \frac{x^\alpha + y^\alpha}{x^{\alpha+1} + y^{\alpha+1}} dx\right] dy \\
&= \int_0^{+\infty} y^{q/(2p)-1/2} g^q(y) \left[\int_0^{+\infty} \frac{1 + (x/y)^\alpha}{\sqrt{x/y}\,[1 + (x/y)^{\alpha+1}]} \times \frac{1}{y} dx\right] dy \\
&= \int_0^{+\infty} y^{q/(2p)-1/2} g^q(y) \left\{\int_0^{+\infty} \frac{1 + v^\alpha}{\sqrt{v}(1 + v^{\alpha+1})} dv\right\} dy \\
&= \Lambda_\alpha \int_0^{+\infty} y^{q/(2p)-1/2} g^q(y) dy \\
&= \Lambda_\alpha \int_0^{+\infty} y^{q/2-1} g^q(y) dy. \qquad (5.16)
\end{aligned}$$

It follows from Equations (5.14), (5.15) and (5.16), and the identity $1/ + 1/q = 1$, that

$$\begin{aligned}
&\int_0^{+\infty} \int_0^{+\infty} \frac{x^\alpha + y^\alpha}{x^{\alpha+1} + y^{\alpha+1}} f(x) g(y) dx dy \\
\leq &\left\{\Lambda_\alpha \int_0^{+\infty} x^{p/2-1} f^p(x) dx\right\}^{1/p} \left\{\Lambda_\alpha \int_0^{+\infty} y^{q/2-1} g^q(y) dy\right\}^{1/q} \\
= &\Lambda_\alpha \left[\int_0^{+\infty} x^{p/2-1} f^p(x) dx\right]^{1/p} \left[\int_0^{+\infty} y^{q/2-1} g^q(y) dy\right]^{1/q}.
\end{aligned}$$

This ends the proof of Proposition 3.3. □

Proof of Proposition 3.4. Let us assume, for the sake of contradiction, that the constant Λ_α is not optimal, i.e., that there exists a better constant κ, i.e., $\kappa \in (0, \Lambda_\alpha)$, satisfying, for any $f, g : [0, +\infty) \mapsto [0, +\infty)$,

$$\begin{aligned}
&\int_0^{+\infty} \int_0^{+\infty} \frac{x^\alpha + y^\alpha}{x^{\alpha+1} + y^{\alpha+1}} f(x) g(y) dx dy \\
\leq &\kappa \left[\int_0^{+\infty} x^{p/2-1} f^p(x) dx\right]^{1/p} \left[\int_0^{+\infty} y^{q/2-1} g^q(y) dy\right]^{1/q}. \qquad (5.17)
\end{aligned}$$

To derive a contradiction, we need to define specific (extremal) functions for f and g. For any $\epsilon \in (0, +\infty)$, let $f_\epsilon : [0, +\infty) \mapsto [0, +\infty)$ be

$$f_\epsilon(x) = \begin{cases} 0 & \text{if } x \in [0, 1), \\ x^{-1/2-\epsilon/p} & \text{if } x \in [1, +\infty), \end{cases}$$

and, similarly, let $g_\epsilon : [0, +\infty) \mapsto [0, +\infty)$ be

$$g_\epsilon(y) = \begin{cases} 0 & \text{if } y \in [0, 1), \\ y^{-1/2-\epsilon/q} & \text{if } y \in [1, +\infty). \end{cases}$$

We then have

$$\int_0^{+\infty} x^{p/2-1} f_\epsilon^p(x)dx = \int_1^{+\infty} x^{p/2-1}(x^{-1/2-\epsilon/p})^p dx = \int_1^{+\infty} x^{-\epsilon-1}dx$$
$$= \left[-\frac{1}{\epsilon}x^{-\epsilon}\right]_{x=1}^{x\to+\infty} = \frac{1}{\epsilon}$$

and, similarly,

$$\int_0^{+\infty} y^{q/2-1} g_\epsilon^q(y)dy = \int_1^{+\infty} y^{q/2-1}(y^{-1/2-\epsilon/q})^q dy = \int_1^{+\infty} y^{-\epsilon-1}dy$$
$$= \left[-\frac{1}{\epsilon}x^{-\epsilon}\right]_{x=1}^{x\to+\infty} = \frac{1}{\epsilon}.$$

This, with the identity $1/p + 1/q = 1$ and Equation (5.17), gives

$$\kappa = \kappa\epsilon \times \frac{1}{\epsilon^{1/p}} \times \frac{1}{\epsilon^{1/q}} = \epsilon\left\{\kappa\left[\int_0^{+\infty} x^{p/2-1} f_\epsilon^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} y^{q/2-1} g_\epsilon^q(y)dy\right]^{1/q}\right\}$$
$$\geq \epsilon\int_0^{+\infty}\int_0^{+\infty} \frac{x^\alpha + y^\alpha}{x^{\alpha+1}+y^{\alpha+1}} f_\epsilon(x)g_\epsilon(y)dxdy. \tag{5.18}$$

Let us now work on this double integral. Making the change of variables $x = uy$, using the Fubini-Tonelli integral theorem and considering the identity $1/p + 1/q = 1$, we get

$$\int_0^{+\infty}\int_0^{+\infty} \frac{x^\alpha + y^\alpha}{x^{\alpha+1}+y^{\alpha+1}} f_\epsilon(x)g_\epsilon(y)dxdy$$
$$= \int_1^{+\infty}\int_1^{+\infty} \frac{x^\alpha + y^\alpha}{x^{\alpha+1}+y^{\alpha+1}} x^{-1/2-\epsilon/p}y^{-1/2-\epsilon/q}dxdy$$
$$= \int_1^{+\infty}\left[\int_1^{+\infty} \frac{x^\alpha + y^\alpha}{x^{\alpha+1}+y^{\alpha+1}} x^{-1/2-\epsilon/p}dx\right] y^{-1/2-\epsilon/q}dy$$
$$= \int_1^{+\infty}\left[\int_{1/y}^{+\infty} \frac{(uy)^\alpha + y^\alpha}{(uy)^{\alpha+1}+y^{\alpha+1}} (uy)^{-1/2-\epsilon/p}(ydu)\right] y^{-1/2-\epsilon/q}dy$$
$$= \int_1^{+\infty}\left[\int_{1/y}^{+\infty} \frac{1+u^\alpha}{\sqrt{u}(1+u^{\alpha+1})} u^{-\epsilon/p}du\right] y^{-(1+\epsilon)}dy. \tag{5.19}$$

Using the Chasles integral relation, the Fubini-Tonelli integral theorem and the identity $1/p + 1/q = 1$, we obtain

$$\begin{aligned}
&\int_1^{+\infty}\left[\int_{1/y}^{+\infty}\frac{1+u^{\alpha}}{\sqrt{u}(1+u^{\alpha+1})}u^{-\epsilon/p}du\right]y^{-(1+\epsilon)}dy\\
&=\int_1^{+\infty}\left[\int_{1/y}^{1}\frac{1+u^{\alpha}}{\sqrt{u}(1+u^{\alpha+1})}u^{-\epsilon/p}du\right]y^{-(1+\epsilon)}dy\\
&+\int_1^{+\infty}\left[\int_{1}^{+\infty}\frac{1+u^{\alpha}}{\sqrt{u}(1+u^{\alpha+1})}u^{-\epsilon/p}du\right]y^{-(1+\epsilon)}dy\\
&=\int_0^{1}\left[\int_{1/u}^{+\infty}y^{-(1+\epsilon)}dy\right]\frac{1+u^{\alpha}}{\sqrt{u}(1+u^{\alpha+1})}u^{-\epsilon/p}du\\
&+\left[\int_{1}^{+\infty}\frac{1+u^{\alpha}}{\sqrt{u}(1+u^{\alpha+1})}u^{-\epsilon/p}du\right]\left[\int_1^{+\infty}y^{-(1+\epsilon)}dy\right]\\
&=\int_0^{1}\left(\frac{1}{\epsilon}u^{\epsilon}\right)\frac{1+u^{\alpha}}{\sqrt{u}(1+u^{\alpha+1})}u^{-\epsilon/p}du+\frac{1}{\epsilon}\times\left[\int_{1}^{+\infty}\frac{1+u^{\alpha}}{\sqrt{u}(1+u^{\alpha+1})}u^{-\epsilon/p}du\right]
\end{aligned}$$

$$=\frac{1}{\epsilon}\times\left[\int_0^{1}\frac{1+u^{\alpha}}{\sqrt{u}(1+u^{\alpha+1})}u^{\epsilon/q}du+\int_{1}^{+\infty}\frac{1+u^{\alpha}}{\sqrt{u}(1+u^{\alpha+1})}u^{-\epsilon/p}du\right]. \tag{5.20}$$

Combining Equations (5.18), (5.19) and (5.20) together, we obtain

$$\kappa\geq\int_0^{1}\frac{1+u^{\alpha}}{\sqrt{u}(1+u^{\alpha+1})}u^{\epsilon/q}du+\int_{1}^{+\infty}\frac{1+u^{\alpha}}{\sqrt{u}(1+u^{\alpha+1})}u^{-\epsilon/p}du.$$

Considering the inferior limit with respect to ϵ with $\epsilon \to 0^+$, we can apply the Fatou integral lemma. Using this, $\liminf_{\epsilon\to 0^+} u^{\epsilon/q} = 1$ for $u \in (0, 1)$, $\liminf_{\epsilon\to 0^+} u^{-\epsilon/p} = 1$ for $u \in [1, +\infty)$, the Chasles integral relation and Proposition 2.1, we obtain

$$\begin{aligned}
\kappa&\geq\liminf_{\epsilon\to 0^+}\int_0^{1}\frac{1+u^{\alpha}}{\sqrt{u}(1+u^{\alpha+1})}u^{\epsilon/q}du+\liminf_{\epsilon\to 0^+}\int_{1}^{+\infty}\frac{1+u^{\alpha}}{\sqrt{u}(1+u^{\alpha+1})}u^{-\epsilon/p}du\\
&\geq\int_0^{1}\frac{1+u^{\alpha}}{\sqrt{u}(1+u^{\alpha+1})}\left[\liminf_{\epsilon\to 0^+}u^{\epsilon/q}\right]du+\int_{1}^{+\infty}\frac{1+u^{\alpha}}{\sqrt{u}(1+u^{\alpha+1})}\left[\liminf_{\epsilon\to 0^+}u^{-\epsilon/p}\right]du\\
&=\int_0^{1}\frac{1+u^{\alpha}}{\sqrt{u}(1+u^{\alpha+1})}du+\int_{1}^{+\infty}\frac{1+u^{\alpha}}{\sqrt{u}(1+u^{\alpha+1})}du=\int_{0}^{+\infty}\frac{1+u^{\alpha}}{\sqrt{u}(1+u^{\alpha+1})}du=\Lambda_{\alpha}.
\end{aligned}$$

As a result, we can not have $\kappa \in (0, \Lambda_\alpha)$ as initially assumed. This contradiction implies that Λ_α is optimal. This ends the proof of Proposition 3.4. □

Proof of Proposition 3.5. After working on the exponent β, we can write

$$\begin{aligned}
&\int_0^{+\infty}\int_0^{+\infty}\frac{(x^{\alpha}+y^{\alpha})^{\beta}}{x^{\alpha+1}+y^{\alpha+1}}f(x)g(y)dxdy\\
&=\int_0^{+\infty}\int_0^{+\infty}\left[\frac{x^{\alpha}+y^{\alpha}}{x^{\alpha+1}+y^{\alpha+1}}f(x)g(y)\right]^{\beta}\left[\frac{1}{x^{\alpha+1}+y^{\alpha+1}}f(x)g(y)\right]^{1-\beta}dxdy.
\end{aligned}$$

It follows from the Hölder integral inequality (applied to the parameter $1/\beta > 1$) that

$$\int_0^{+\infty}\int_0^{+\infty}\left[\frac{x^\alpha+y^\alpha}{x^{\alpha+1}+y^{\alpha+1}}f(x)g(y)\right]^\beta\left[\frac{1}{x^{\alpha+1}+y^{\alpha+1}}f(x)g(y)\right]^{1-\beta}dxdy$$
$$\leq\left[\int_0^{+\infty}\int_0^{+\infty}\frac{x^\alpha+y^\alpha}{x^{\alpha+1}+y^{\alpha+1}}f(x)g(y)dxdy\right]^\beta$$
$$\times\left[\int_0^{+\infty}\int_0^{+\infty}\frac{1}{x^{\alpha+1}+y^{\alpha+1}}f(x)g(y)dxdy\right]^{1-\beta}. \tag{5.21}$$

Proposition 3.3 ensures that

$$\int_0^{+\infty}\int_0^{+\infty}\frac{x^\alpha+y^\alpha}{x^{\alpha+1}+y^{\alpha+1}}f(x)g(y)dxdy$$
$$\leq\Lambda_\alpha\left[\int_0^{+\infty}x^{p/2-1}f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty}y^{q/2-1}g^q(y)dy\right]^{1/q}, \tag{5.22}$$

where Λ_α is given in Equation (3.4).

On the other hand, a well-known result of the Hardy-Hilbert type ensures that

$$\int_0^{+\infty}\int_0^{+\infty}\frac{1}{x^{\alpha+1}+y^{\alpha+1}}f(x)g(y)dxdy$$
$$\leq\frac{\pi}{\alpha+1}\left[\int_0^{+\infty}x^{p/2-1}f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty}y^{q/2-1}g^q(y)dy\right]^{1/q}. \tag{5.23}$$

See Equation (1.3). Combining Equations (5.21), (5.22) and (5.23), and using the identity $1/p+1/q=1$, we have

$$\int_0^{+\infty}\int_0^{+\infty}\left[\frac{x^\alpha+y^\alpha}{x^{\alpha+1}+y^{\alpha+1}}f(x)g(y)\right]^\beta\left[\frac{1}{x^{\alpha+1}+y^{\alpha+1}}f(x)g(y)\right]^{1-\beta}dxdy$$
$$\leq\left\{\Lambda_\alpha\left[\int_0^{+\infty}x^{p/2-1}f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty}y^{q/2-1}g^q(y)dy\right]^{1/q}\right\}^\beta$$
$$\times\left\{\frac{\pi}{\alpha+1}\left[\int_0^{+\infty}x^{p/2-1}f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty}y^{q/2-1}g^q(y)dy\right]^{1/q}\right\}^{1-\beta}$$
$$=\Lambda_\alpha^\beta\left(\frac{\pi}{\alpha+1}\right)^{1-\beta}\left[\int_0^{+\infty}x^{p/2-1}f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty}y^{q/2-1}g^q(y)dy\right]^{1/q}$$
$$=\Upsilon_{\alpha,\beta}\left[\int_0^{+\infty}x^{p/2-1}f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty}y^{q/2-1}g^q(y)dy\right]^{1/q},$$

where

$$\Upsilon_{\alpha,\beta}=\Lambda_\alpha^\beta\left(\frac{\pi}{\alpha+1}\right)^{1-\beta}=\frac{\pi}{\alpha+1}2^\beta\left[\sin\left(\frac{\pi}{2(\alpha+1)}\right)\right]^{-\beta},$$

as indicated in Equation (3.5). This concludes the proof of Proposition 3.5. □

Proof of Proposition 3.6. An well-known inequality of convexity gives

$$(x+y)^\alpha\leq\max(2^{\alpha-1},1)(x^\alpha+y^\alpha).$$

See [11, Chapter 1]. This and Proposition 3.3 yield

$$\begin{aligned}&\int_0^{+\infty}\int_0^{+\infty}\frac{(x+y)^{\alpha}}{x^{\alpha+1}+y^{\alpha+1}}f(x)g(y)dxdy\\ &\leq \max(2^{\alpha-1},1)\int_0^{+\infty}\int_0^{+\infty}\frac{x^{\alpha}+y^{\alpha}}{x^{\alpha+1}+y^{\alpha+1}}f(x)g(y)dxdy\\ &\leq \max(2^{\alpha-1},1)\Lambda_{\alpha}\left[\int_0^{+\infty}x^{p/2-1}f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty}y^{q/2-1}g^q(y)dy\right]^{1/q}\\ &\leq \Theta_{\alpha}\left[\int_0^{+\infty}x^{p/2-1}f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty}y^{q/2-1}g^q(y)dy\right]^{1/q},\end{aligned}$$

where

$$\Theta_{\alpha}=\max(2^{\alpha-1},1)\Lambda_{\alpha}=\max(2^{\alpha-1},1)\frac{2\pi}{\alpha+1}\left[\sin\left(\frac{\pi}{2(\alpha+1)}\right)\right]^{-1},$$

and indicated in Equation (3.6). This achieves the proof of Proposition 3.6. □

Proof of Proposition 3.7. We can write

$$\begin{aligned}&\int_0^{+\infty}\int_0^{+\infty}\frac{x^{\alpha}+y^{\alpha}}{x^{\alpha+1}+y^{\alpha+1}}x^{1/p-3/2}y^{1/q-3/2}F(x)G(y)dxdy\\ &=\int_0^{+\infty}\int_0^{+\infty}\frac{x^{\alpha}+y^{\alpha}}{x^{\alpha+1}+y^{\alpha+1}}f_{\diamond}(x)g_{\diamond}(y)dxdy,\end{aligned}$$

where

$$f_{\diamond}(x)=x^{1/p-3/2}F(x),\quad g_{\diamond}(y)=y^{1/q-3/2}G(y).$$

Applying Proposition 3.3 to these functions, we get

$$\begin{aligned}&\int_0^{+\infty}\int_0^{+\infty}\frac{x^{\alpha}+y^{\alpha}}{x^{\alpha+1}+y^{\alpha+1}}f_{\diamond}(x)g_{\diamond}(y)dxdy\\ &\leq\Lambda_{\alpha}\left[\int_0^{+\infty}x^{p/2-1}f_{\diamond}^p(x)dx\right]^{1/p}\left[\int_0^{+\infty}y^{q/2-1}g_{\diamond}^q(y)dy\right]^{1/q},\end{aligned}\tag{5.24}$$

where Λ_{α} is given in Equation (3.4). Developing the right-hand side term, we obtain

$$\begin{aligned}&\left[\int_0^{+\infty}x^{p/2-1}f_{\diamond}^p(x)dx\right]^{1/p}\left[\int_0^{+\infty}y^{q/2-1}g_{\diamond}^q(y)dy\right]^{1/q}\\ &=\left[\int_0^{+\infty}x^{p/2-1}[x^{1/p-3/2}F(x)]^pdx\right]^{1/p}\left[\int_0^{+\infty}y^{q/2-1}[y^{1/q-3/2}G(y)]^qdy\right]^{1/q}\\ &=\left[\int_0^{+\infty}\frac{1}{x^p}F^p(x)dx\right]^{1/p}\left[\int_0^{+\infty}\frac{1}{y^q}G^q(y)dy\right]^{1/q}.\end{aligned}\tag{5.25}$$

The standard Hardy integral inequality applied to f and g gives

$$\int_0^{+\infty}\frac{1}{x^p}F^p(x)dx\leq\left(\frac{p}{p-1}\right)^p\int_0^{+\infty}f^p(x)dx\tag{5.26}$$

and

$$\int_0^{+\infty}\frac{1}{y^q}G^q(y)dy\leq\left(\frac{q}{q-1}\right)^q\int_0^{+\infty}g^q(y)dy.\tag{5.27}$$

See [7]. It follows from Equations (5.24), (5.25), (5.26) and (5.27) that

$$\begin{aligned}
&\int_0^{+\infty}\int_0^{+\infty}\frac{x^\alpha+y^\alpha}{x^{\alpha+1}+y^{\alpha+1}}x^{1/p-3/2}y^{1/q-3/2}F(x)G(y)dxdy\\
\leq&\Lambda_\alpha\left[\left(\frac{p}{p-1}\right)^p\int_0^{+\infty}f^p(x)dx\right]^{1/p}\left[\left(\frac{q}{q-1}\right)^q\int_0^{+\infty}g^q(y)dy\right]^{1/q}\\
=&\Phi_{\alpha,p}\left[\int_0^{+\infty}f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty}g^q(y)dy\right]^{1/q},
\end{aligned}$$

where

$$\begin{aligned}
\Phi_{\alpha,p}=&\Lambda_\alpha\left(\frac{p}{p-1}\right)\left(\frac{q}{q-1}\right)\\
=&\frac{2\pi}{\alpha+1}\left[\sin\left(\frac{\pi}{2(\alpha+1)}\right)\right]^{-1}\left(\frac{p}{p-1}\right)\left(\frac{q}{q-1}\right),
\end{aligned}$$

as indicated in Equation (3.7). This concludes the proof of Proposition 3.7. □

5.3. **Proofs of the propositions in Section 4.**

Proof of Proposition 4.8. Decomposing appropriately the integrand, taking into account the identity $1/p+1/q=1$ and applying the Hölder integral inequality, we get

$$\begin{aligned}
&\int_0^{+\infty}\int_0^{+\infty}\frac{(x-y)\log(x/y)}{(x^{\alpha+1}+y^{\alpha+1})^2}f(x)g(y)dxdy\\
=&\int_0^{+\infty}\int_0^{+\infty}x^{-(\alpha-1/2)/q}y^{(\alpha-1/2)/p}\left[\frac{(x-y)\log(x/y)}{(x^{\alpha+1}+y^{\alpha+1})^2}\right]^{1/p}f(x)\\
&\times x^{(\alpha-1/2)/q}y^{-(\alpha-1/2)/p}\left[\frac{(x-y)\log(x/y)}{(x^{\alpha+1}+y^{\alpha+1})^2}\right]^{1/q}g(y)dxdy\\
\leq&\mathfrak{C}^{1/p}\mathfrak{D}^{1/q}, \qquad (5.28)
\end{aligned}$$

where $\mathfrak{C}$ and $\mathfrak{D}$ are given by

$$\mathfrak{C}=\int_0^{+\infty}\int_0^{+\infty}x^{-(\alpha-1/2)p/q}y^{\alpha-1/2}\frac{(x-y)\log(x/y)}{(x^{\alpha+1}+y^{\alpha+1})^2}f^p(x)dxdy$$

and

$$\mathfrak{D}=\int_0^{+\infty}\int_0^{+\infty}x^{\alpha-1/2}y^{-(\alpha-1/2)q/p}\frac{(x-y)\log(x/y)}{(x^{\alpha+1}+y^{\alpha+1})^2}g^q(y)dxdy.$$

Let us investigate the expressions of these terms, starting with $\mathfrak{C}$.

Exchanging the order of integration by the Fubini-Tonelli integral theorem, using the change of variables $u=y/x$, applying Proposition 2.2 from which emerges the constant Ω_α in Equation

(4.8) and noting that $p/q = p-1$, we get

$$\begin{aligned}
\mathfrak{C} &= \int_0^{+\infty} x^{-(\alpha-1/2)p/q} f^p(x) \left[\int_0^{+\infty} y^{\alpha-1/2}\frac{(x-y)\log(x/y)}{(x^{\alpha+1}+y^{\alpha+1})^2}dy\right]dx \\
&= \int_0^{+\infty} x^{-(\alpha-1/2)p/q-\alpha-1/2} f^p(x) \left[\int_0^{+\infty} \left(\frac{y}{x}\right)^{\alpha-1/2}\frac{(y/x-1)\log(y/x)}{[1+(y/x)^{\alpha+1}]^2}\times\frac{1}{x}dy\right]dx \\
&= \int_0^{+\infty} x^{-(\alpha-1/2)p/q-\alpha-1/2} f^p(x) \left\{\int_0^{+\infty} u^{\alpha-1/2}\frac{(u-1)\log(u)}{(1+u^{\alpha+1})^2}du\right\}dx \\
&=\Omega_\alpha \int_0^{+\infty} x^{-(\alpha-1/2)p/q-\alpha-1/2} f^p(x)dx \\
&=\Omega_\alpha \int_0^{+\infty} x^{-(\alpha-1/2)p-1} f^p(x)dx. \qquad (5.29)
\end{aligned}$$

For the term $\mathfrak{D}$, we proceed in a similar way, but with the change of variables $v = x/y$. We thus obtain

$$\begin{aligned}
\mathfrak{D} &= \int_0^{+\infty} y^{-(\alpha-1/2)q/p} g^q(y) \left[\int_0^{+\infty} x^{\alpha-1/2}\frac{(x-y)\log(x/y)}{(x^{\alpha+1}+y^{\alpha+1})^2}dx\right]dy \\
&= \int_0^{+\infty} y^{-(\alpha-1/2)p/q-\alpha-1/2} g^q(y) \left[\int_0^{+\infty} \left(\frac{x}{y}\right)^{\alpha-1/2}\frac{(x/y-1)\log(x/y)}{[1+(x/y)^{\alpha+1}]^2}\times\frac{1}{y}dx\right]dy \\
&= \int_0^{+\infty} y^{-(\alpha-1/2)q/p-\alpha-1/2} g^q(y) \left\{\int_0^{+\infty} v^{\alpha-1/2}\frac{(v-1)\log(v)}{(1+v^{\alpha+1})^2}dv\right\}dx \\
&= \Omega_\alpha \int_0^{+\infty} y^{-(\alpha-1/2)q/p-\alpha-1/2} g^q(y)dy \\
&= \Omega_\alpha \int_0^{+\infty} y^{-(\alpha-1/2)q-1} g^q(y)dy. \qquad (5.30)
\end{aligned}$$

It follows from Equations (5.28), (5.29) and (5.30), and the identity $1/+1/q=1$, that

$$\begin{aligned}
&\int_0^{+\infty}\int_0^{+\infty} \frac{(x-y)\log(x/y)}{(x^{\alpha+1}+y^{\alpha+1})^2} f(x)g(y)dxdy \\
&\leq \left[\Omega_\alpha \int_0^{+\infty} x^{-(\alpha-1/2)p-1} f^p(x)dx\right]^{1/p} \left[\Omega_\alpha \int_0^{+\infty} y^{-(\alpha-1/2)q-1} g^q(y)dy\right]^{1/q} \\
&=\Omega_\alpha \left[\int_0^{+\infty} x^{-(\alpha-1/2)p-1} f^p(x)dx\right]^{1/p} \left[\int_0^{+\infty} y^{-(\alpha-1/2)q-1} g^q(y)dy\right]^{1/q}.
\end{aligned}$$

This ends the proof of Proposition 4.8. □

Proof of Proposition 4.9. The proof follows the lines of that of Proposition 3.4, but adapted to the situation. Some calculus details will be then omitted to avoid redundancy. Let us assume, for the sake of contradiction, that the constant Ω_α is not optimal, i.e., that there exists a better constant τ, i.e., $\tau \in (0, \Omega_\alpha)$, satisfying, for any $f, g : [0,+\infty) \mapsto [0,+\infty)$,

$$\begin{aligned}
&\int_0^{+\infty}\int_0^{+\infty} \frac{(x-y)\log(x/y)}{(x^{\alpha+1}+y^{\alpha+1})^2} f(x)g(y)dxdy \\
&\leq\tau \left[\int_0^{+\infty} x^{-(\alpha-1/2)p-1} f^p(x)dx\right]^{1/p} \left[\int_0^{+\infty} y^{-(\alpha-1/2)q-1} g^q(y)dy\right]^{1/q}. \qquad (5.31)
\end{aligned}$$

To derive a contradiction, we need to consider special functions. For any $\epsilon \in (0,+\infty)$, let $f_\epsilon : [0,+\infty) \mapsto [0,+\infty)$ be

$$f_\epsilon(x) = \begin{cases} 0 & \text{if } x \in [0,1), \\ x^{\alpha-1/2-\epsilon/p} & \text{if } x \in [1,+\infty), \end{cases}$$

and, similarly, let $g_\epsilon : [0,+\infty) \mapsto [0,+\infty)$ be

$$g_\epsilon(y) = \begin{cases} 0 & \text{if } y \in [0,1), \\ y^{\alpha-1/2-\epsilon/q} & \text{if } y \in [1,+\infty). \end{cases}$$

We then have

$$\int_0^{+\infty} x^{-(\alpha-1/2)p-1} f_\epsilon^p(x)dx = \frac{1}{\epsilon}, \quad \int_0^{+\infty} y^{-(\alpha-1/2)q-1} g_\epsilon^q(y)dy = \frac{1}{\epsilon}.$$

This, with the identity $1/p + 1/q = 1$ and Equation (5.31), gives

$$\tau = \epsilon \left\{ \tau \left[\int_0^{+\infty} x^{-(\alpha-1/2)p-1} f_\epsilon^p(x)dx \right]^{1/p} \left[\int_0^{+\infty} y^{-(\alpha-1/2)q-1} g_\epsilon^q(y)dy \right]^{1/q} \right\}$$

$$\geq \epsilon \int_0^{+\infty} \int_0^{+\infty} \frac{(x-y)\log(x/y)}{(x^{\alpha+1}+y^{\alpha+1})^2} f_\epsilon(x) g_\epsilon(y) dxdy. \tag{5.32}$$

Let us now work on this double integral. Making the change of variables $x = uy$, using the Fubini-Tonelli integral theorem and considering the identity $1/p + 1/q = 1$, we get

$$\int_0^{+\infty} \int_0^{+\infty} \frac{(x-y)\log(x/y)}{(x^{\alpha+1}+y^{\alpha+1})^2} f_\epsilon(x) g_\epsilon(y) dxdy$$

$$= \int_1^{+\infty} \left[\int_1^{+\infty} \frac{(x-y)\log(x/y)}{(x^{\alpha+1}+y^{\alpha+1})^2} x^{\alpha-1/2-\epsilon/p} dx \right] y^{\alpha-1/2-\epsilon/q} dy$$

$$= \int_1^{+\infty} \left[\int_{1/y}^{+\infty} u^{\alpha-1/2} \frac{(u-1)\log(u)}{(1+u^{\alpha+1})^2} u^{-\epsilon/p} du \right] y^{-(1+\epsilon)} dy. \tag{5.33}$$

Using the Chasles integral relation, the Fubini-Tonelli integral theorem and the identity $1/p + 1/q = 1$, we obtain

$$\int_1^{+\infty} \left[\int_{1/y}^{+\infty} u^{\alpha-1/2} \frac{(u-1)\log(u)}{(1+u^{\alpha+1})^2} u^{-\epsilon/p} du \right] y^{-(1+\epsilon)} dy$$

$$= \int_1^{+\infty} \left[\int_{1/y}^{1} u^{\alpha-1/2} \frac{(u-1)\log(u)}{(1+u^{\alpha+1})^2} u^{-\epsilon/p} du \right] y^{-(1+\epsilon)} dy$$

$$+ \int_1^{+\infty} \left[\int_{1}^{+\infty} u^{\alpha-1/2} \frac{(u-1)\log(u)}{(1+u^{\alpha+1})^2} u^{-\epsilon/p} du \right] y^{-(1+\epsilon)} dy$$

$$= \int_0^{1} \left[\int_{1/u}^{+\infty} y^{-(1+\epsilon)} dy \right] u^{\alpha-1/2} \frac{(u-1)\log(u)}{(1+u^{\alpha+1})^2} u^{-\epsilon/p} du$$

$$+ \left[\int_{1}^{+\infty} u^{\alpha-1/2} \frac{(u-1)\log(u)}{(1+u^{\alpha+1})^2} u^{-\epsilon/p} du \right] \left[\int_1^{+\infty} y^{-(1+\epsilon)} dy \right]$$

$$= \frac{1}{\epsilon} \times \left[\int_0^{1} u^{\alpha-1/2} \frac{(u-1)\log(u)}{(1+u^{\alpha+1})^2} u^{\epsilon/q} du + \int_1^{+\infty} u^{\alpha-1/2} \frac{(u-1)\log(u)}{(1+u^{\alpha+1})^2} u^{-\epsilon/p} du \right]. \tag{5.34}$$

Combining Equations (5.32), (5.33) and (5.34) together, we obtain

$$\tau \geq \int_0^1 u^{\alpha-1/2}\frac{(u-1)\log(u)}{(1+u^{\alpha+1})^2}u^{\epsilon/q}du + \int_1^{+\infty} u^{\alpha-1/2}\frac{(u-1)\log(u)}{(1+u^{\alpha+1})^2}u^{-\epsilon/p}du.$$

Considering the inferior limit with respect to ϵ with $\epsilon \to 0^+$, we can apply the Fatou integral lemma. Using this, $\liminf_{\epsilon\to 0^+} u^{\epsilon/q} = 1$ for $u \in (0,1)$, $\liminf_{\epsilon\to 0^+} u^{-\epsilon/p} = 1$ for $u \in [1,+\infty)$, the Chasles integral relation and Proposition 2.2, we obtain

$$\begin{aligned}
\tau &\geq \liminf_{\epsilon\to 0^+}\int_0^1 u^{\alpha-1/2}\frac{(u-1)\log(u)}{(1+u^{\alpha+1})^2}u^{\epsilon/q}du + \liminf_{\epsilon\to 0^+}\int_1^{+\infty} u^{\alpha-1/2}\frac{(u-1)\log(u)}{(1+u^{\alpha+1})^2}u^{-\epsilon/p}du \\
&\geq \int_0^1 u^{\alpha-1/2}\frac{(u-1)\log(u)}{(1+u^{\alpha+1})^2}\left[\liminf_{\epsilon\to 0^+} u^{\epsilon/q}\right]du + \int_1^{+\infty} u^{\alpha-1/2}\frac{(u-1)\log(u)}{(1+u^{\alpha+1})^2}\left[\liminf_{\epsilon\to 0^+} u^{-\epsilon/p}\right]du \\
&= \int_0^1 u^{\alpha-1/2}\frac{(u-1)\log(u)}{(1+u^{\alpha+1})^2}du + \int_1^{+\infty}\frac{(u-1)\log(u)}{(1+u^{\alpha+1})^2}du \\
&= \int_0^{+\infty} u^{\alpha-1/2}\frac{(u-1)\log(u)}{(1+u^{\alpha+1})^2}du = \Omega_\alpha.
\end{aligned}$$

As a result, we can not have $\tau \in (0,\Omega_\alpha)$ as initially assumed. This contradiction implies that Ω_α is optimal. This ends the proof of Proposition 4.9. □

Proof of Proposition 4.10. Applying Proposition 4.8 to $f = g$ and using the identity $1/p+1/q = 1$, we get

$$\begin{aligned}
&\int_0^{+\infty}\int_0^{+\infty}\frac{(x-y)\log(x/y)}{(x^{\alpha+1}+y^{\alpha+1})^2}f(x)f(y)dxdy \\
&\leq \Omega_\alpha\left[\int_0^{+\infty} x^{-(\alpha-1/2)p-1}f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} y^{-(\alpha-1/2)q-1}f^q(y)dy\right]^{1/q}.
\end{aligned} \tag{5.35}$$

Decomposing the double integral, using $\log(x/y) = -\log(y/x)$, applying the Fubini-Tonelli integral theorem and identifying a crucial symmetry in x and y, we get

$$\begin{aligned}
&\int_0^{+\infty}\int_0^{+\infty}\frac{(x-y)\log(x/y)}{(x^{\alpha+1}+y^{\alpha+1})^2}f(x)f(y)dxdy \\
&= \int_0^{+\infty}\int_0^{+\infty}\frac{x\log(x/y)}{(x^{\alpha+1}+y^{\alpha+1})^2}f(x)f(y)dxdy - \int_0^{+\infty}\int_0^{+\infty}\frac{y\log(x/y)}{(x^{\alpha+1}+y^{\alpha+1})^2}f(x)f(y)dxdy \\
&= \int_0^{+\infty}\int_0^{+\infty}\frac{x\log(x/y)}{(x^{\alpha+1}+y^{\alpha+1})^2}f(x)f(y)dxdy + \int_0^{+\infty}\int_0^{+\infty}\frac{y\log(y/x)}{(y^{\alpha+1}+x^{\alpha+1})^2}f(y)f(x)dydx \\
&= 2\int_0^{+\infty}\int_0^{+\infty}\frac{x\log(x/y)}{(x^{\alpha+1}+y^{\alpha+1})^2}f(x)f(y)dxdy.
\end{aligned} \tag{5.36}$$

Combining Equations (5.35) and (5.36), we obtain

$$\begin{aligned}
&\int_0^{+\infty}\int_0^{+\infty}\frac{x\log(x/y)}{(x^{\alpha+1}+y^{\alpha+1})^2}f(x)f(y)dxdy \\
&\leq \chi_\alpha\left[\int_0^{+\infty} x^{-(\alpha-1/2)p-1}f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} y^{-(\alpha-1/2)q-1}f^q(y)dy\right]^{1/q},
\end{aligned}$$

where

$$\chi_\alpha = \frac{1}{2}\Omega_\alpha$$
$$= \frac{\pi}{2(\alpha+1)^3}\left[\sin\left(\frac{\pi}{2(\alpha+1)}\right)\right]^{-2}\left[2(1+\alpha)\sin\left(\frac{\pi}{2(\alpha+1)}\right) - \pi\cos\left(\frac{\pi}{2(\alpha+1)}\right)\right],$$

as indicated in Equation (4.9). This concludes the proof of Proposition 4.10. □

Proof of Proposition 4.11. A well-known logarithmic inequality ensures that, for any $u > 0$,

$$1 - \frac{1}{u} \leq \log(u).$$

Applying this to $u = (x/y)^{x-y}$, we get

$$1 - \left(\frac{x}{y}\right)^{y-x} \leq \log\left[\left(\frac{x}{y}\right)^{x-y}\right] = (x-y)\log\left(\frac{x}{y}\right).$$

Using this and Proposition 4.8, we get

$$\int_0^{+\infty}\int_0^{+\infty}\left[1 - \left(\frac{x}{y}\right)^{y-x}\right]\frac{1}{(x^{\alpha+1}+y^{\alpha+1})^2}f(x)g(y)dxdy$$
$$\leq \int_0^{+\infty}\int_0^{+\infty}\frac{(x-y)\log(x/y)}{(x^{\alpha+1}+y^{\alpha+1})^2}f(x)g(y)dxdy$$
$$\leq \Omega_\alpha\left[\int_0^{+\infty}x^{-(\alpha-1/2)p-1}f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty}y^{-(\alpha-1/2)q-1}g^q(y)dy\right]^{1/q},$$

where Ω_α is given by Equation (4.8). This concludes the proof of Proposition 4.11. □

Proof of Proposition 4.12. We can write

$$\int_0^{+\infty}\int_0^{+\infty}\frac{(x-y)\log(x/y)}{(x^{\alpha+1}+y^{\alpha+1})^2}x^{\alpha+1/p-3/2}y^{\alpha+1/q-3/2}F(x)G(y)dxdy$$
$$= \int_0^{+\infty}\int_0^{+\infty}\frac{(x-y)\log(x/y)}{(x^{\alpha+1}+y^{\alpha+1})^2}f_\diamond(x)g_\diamond(y)dxdy,$$

where

$$f_\diamond(x) = x^{\alpha+1/p-3/2}F(x), \quad g_\diamond(y) = y^{\alpha+1/q-3/2}G(y).$$

Applying Proposition 3.3 to these functions, we get

$$\int_0^{+\infty}\int_0^{+\infty}\frac{(x-y)\log(x/y)}{(x^{\alpha+1}+y^{\alpha+1})^2}f_\diamond(x)g_\diamond(y)dxdy$$
$$\leq \Omega_\alpha\left[\int_0^{+\infty}x^{-(\alpha-1/2)p-1}f_\diamond^p(x)dx\right]^{1/p}\left[\int_0^{+\infty}y^{-(\alpha-1/2)q-1}g_\diamond^q(y)dy\right]^{1/q}, \tag{5.37}$$

where Ω_α is given in Equation (4.8). Developing the right-hand side term, we obtain

$$\left[\int_0^{+\infty} x^{-(\alpha-1/2)p-1} f_\diamond^p(x)dx\right]^{1/p} \left[\int_0^{+\infty} y^{-(\alpha-1/2)q-1} g_\diamond^q(y)dy\right]^{1/q}$$
$$= \left[\int_0^{+\infty} x^{-(\alpha-1/2)p-1}[x^{\alpha+1/p-3/2}F(x)]^p dx\right]^{1/p} \left[\int_0^{+\infty} y^{-(\alpha-1/2)q-1}[y^{\alpha+1/q-3/2}G(y)]^q dy\right]^{1/q}$$

(5.38)

$$= \left[\int_0^{+\infty} \frac{1}{x^p}F^p(x)dx\right]^{1/p} \left[\int_0^{+\infty} \frac{1}{y^q}G^q(y)dy\right]^{1/q}.$$

The standard Hardy integral inequality applied to f and g gives

$$\int_0^{+\infty} \frac{1}{x^p}F^p(x)dx \leq \left(\frac{p}{p-1}\right)^p \int_0^{+\infty} f^p(x)dx \tag{5.39}$$

and

$$\int_0^{+\infty} \frac{1}{y^q}G^q(y)dy \leq \left(\frac{q}{q-1}\right)^q \int_0^{+\infty} g^q(y)dy. \tag{5.40}$$

It follows from Equations (5.37), (5.38), (5.39) and (5.40) that

$$\int_0^{+\infty}\int_0^{+\infty} \frac{(x-y)\log(x/y)}{(x^{\alpha+1}+y^{\alpha+1})^2} x^{\alpha+1/p-3/2}y^{\alpha+1/q-3/2}F(x)G(y)dxdy$$
$$\leq \Omega_\alpha \left[\left(\frac{p}{p-1}\right)^p \int_0^{+\infty} f^p(x)dx\right]^{1/p} \left[\left(\frac{q}{q-1}\right)^q \int_0^{+\infty} g^q(y)dy\right]^{1/q}$$
$$= \Psi_{\alpha,p} \left[\int_0^{+\infty} f^p(x)dx\right]^{1/p} \left[\int_0^{+\infty} g^q(y)dy\right]^{1/q},$$

where

$$\Psi_{\alpha,p} = \Omega_\alpha \left(\frac{p}{p-1}\right)\left(\frac{q}{q-1}\right)$$
$$= \frac{\pi}{(\alpha+1)^3}\left[\sin\left(\frac{\pi}{2(\alpha+1)}\right)\right]^{-2}\left[2(1+\alpha)\sin\left(\frac{\pi}{2(\alpha+1)}\right) - \pi\cos\left(\frac{\pi}{2(\alpha+1)}\right)\right]$$
$$\times \left(\frac{p}{p-1}\right)\left(\frac{q}{q-1}\right),$$

as indicated in Equation (4.10). This concludes the proof of Proposition 4.12. □

6. Conclusion

In conclusion, this article makes some advances to the theory of Hardy-Hilbert-type integral inequalities. From a theoretical point of view, the incorporation of adjustable parameters and new functional forms, in particular the logarithmic modification, shows a significant extension of the classical inequality. This provides versatile tools for dealing with a wider class of problems. From an applied point of view, the flexibility of these new forms may prove useful in various branches of mathematical analysis where precision and adaptability are essential. Furthermore, by establishing the optimality of the constant factors, the article ensures the rigor and tightness necessary for future studies. These results thus open the door to further refinements and interdisciplinary applications, confirming the continuing interest of the Hardy-Hilbert-type integral inequalities in modern mathematical research.

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Research Article

Solitons of mean curvature flow in certain warped products: nonexistence, rigidity, and Moser-Bernstein type results

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ABSTRACT. We apply suitable maximum principles to obtain nonexistence and rigidity results for complete mean curvature flow solitons in certain warped product spaces. We also provide applications to self-shrinkers in Euclidean space, as well as to mean curvature flow solitons in real projective, pseudo-hyperbolic, Schwarzschild, and Reissner-Nordström spaces. Furthermore, we establish new Moser-Bernstein type results for entire graphs constructed over the fiber of the ambient space that are mean curvature flow solitons.

Keywords: Warped products, Euclidean space, real projective space, pseudo-hyperbolic spaces, Schwarzschild and Reissner-Nordström spaces, mean curvature flow solitons, self-shrinkers.

2020 Mathematics Subject Classification: 53E10, 35C08, 53C42.

1. INTRODUCTION

Let $\psi : \Sigma^n \to \mathbb{R}^{n+1}$ be an n-dimensional hypersurface in the $(n+1)$-dimensional Euclidean space $\mathbb{R}^{n+1}$. If the position vector ψ evolves in the direction of the mean curvature vector $\vec{H}$, then it gives rise to a solution to mean curvature flow:

$$\Psi : [0, T) \times \Sigma^n \to \mathbb{R}^{n+1}$$

satisfying $\Psi(0, \cdot) = \psi(\cdot)$ and

$$\frac{\partial \Psi}{\partial t}(t, p) = \vec{H}(t, p),$$

where $\vec{H}(t, p)$ stands for the (non-normalized) mean curvature vector of the hypersurface $\Sigma_t^n = \Psi(t, \Sigma^n)$ at a point $\Psi(t, p)$. This equation is called the *mean curvature flow equation*. The study of the mean curvature flow from the perspective of partial differential equations was started with Huisken [24] on the flow of convex hypersurfaces. One of the most important problems in the mean curvature flow is to understand the possible singularities that the flow goes through. A key starting point for singularity analysis is Huisken's monotonicity formula [24] because the monotonicity implies that the flow is asymptotically self-similar near a given type I singularity and thus, is modeled by self-shrinking solutions of the flow.

An n-dimensional two-sided hypersurface $\psi : \Sigma^n \to \mathbb{R}^{n+1}$ is called a *self-shrinker* if it satisfies

$$H = -\langle \psi, N \rangle,$$

where H and N denote the (non-normalized) mean curvature function and the unit normal vector field of the hypersurface, respectively. It is known that self-shrinkers play an important role in the study of the mean curvature flow because they describe all possible blow up at a

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given singularity of the mean curvature flow and, as it was pointed out by Colding and Minicozzi in [14], self-shrinkers are critical hypersurfaces for the entropy functional. The subject experienced an increasing activity after the seminal paper by Colding and Minicozzi [14] that inspired an impressive amount of work on existence and classification problems, rigidity and gap results, stability and spectral properties, see for instance [7, 8, 10, 11, 12, 13, 15, 18, 19, 23, 26, 27, 31] and the references therein.

More recently, Alías, de Lira and Rigoli [4] extended these investigations introducing the general definition of self-similar mean curvature flow in a Riemannian manifold $\overline{M}^{n+1}$ endowed with a vector field K and establishing the corresponding notion of mean curvature soliton. In particular, when $\overline{M}^{n+1}$ is a warped product of the type $I \times_f M^n$ and $K = f(t)\partial_t$, they applied weak maximum principles to guarantee that a complete n-dimensional mean curvature flow soliton is a slice of $\overline{M}^{n+1}$. In [16], Colombo, Mari and Rigoli also studied some properties of mean curvature flow solitons in general Riemannian manifolds and in warped products, with emphasis on constant curvature and Schwarzschild type spaces. They focused on splitting and rigidity results under various geometric conditions, ranging from the stability of the soliton to the fact that the image of its Gauss map be contained in suitable regions of the sphere. Moreover, they also investigated the case of entire mean curvature flow graphs.

Proceeding with this picture, our purpose in this paper is to apply suitable maximum principles in order to obtain nonexistence and rigidity results concerning complete n-dimensional mean curvature flow solitons with respect to the conformal vector field $K = f(t)\partial_t$ of a warped product space of the type $I \times_f M^n$ (see Sections 3 and 4). Applications to self-shrinkers in the Euclidean space, as well as to mean curvature flow solitons in the real projective, pseudo-hyperbolic, Schwarzschild and Reissner-Nordström spaces are also given. Furthermore, we study entire graphs constructed over the fiber M^n and which are mean curvature flow solitons with respect to K, obtaining new Moser-Bernstein type results (see Section 5).

2. Preliminaries

2.1. **Two-sided hypersurfaces in a warped product.** Let (M^n, g_M) be an n-dimensional $(n \geq 2)$ connected Riemannian manifold and let $I \subset \mathbb{R}$ be an open interval in $\mathbb{R}$ endowed with the metric dt^2. The product manifold $\overline{M}^{n+1} = I \times M^n$ endowed with the Riemannian metric

$$\overline{g} = \pi_I^*(dt^2) + f(\pi_I)^2 \pi_M^*(g_M), \tag{2.1}$$

where f is a positive smooth function on I, the maps π_I and π_M denote the projections onto I and M^n, respectively, is called a warped product with fiber M^n, base I and warping function f. Along this work, we will simply write $\overline{M}^{n+1} = I \times_f M^n$.

In this setting, we will consider the conformal closed vector field $K = f(t)\partial_t$ globally defined on $\overline{M}$, where $\partial_t = \frac{\partial}{\partial t}$ stands for the unit coordinate vector field tangent to I. From the relationship between the Levi-Civita connections of $\overline{M}$ and those of the base and the fiber (see [30, Proposition 7.35]) , it follows that

$$\overline{\nabla}_X K = f'(t)X \tag{2.2}$$

for any $X \in \mathfrak{X}(\overline{M})$, where $\overline{\nabla}$ is the Levi-Civita connection of $\overline{g}$.

Along this work, we will deal with connected two-sided hypersurfaces $\psi : \Sigma^n \to \overline{M}^{n+1}$ immersed in $\overline{M}^{n+1} = I \times_f M^n$, which means that its normal bundle is trivial, that is, there is on it a globally defined unit normal vector field $N \in T\Sigma^{\perp}$. In this setting, we will denote by g the induced metric of Σ^n and we will consider its shape operator (or Weingarten endomorphism),

$A : \mathfrak{X}(\Sigma) \to \mathfrak{X}(\Sigma)$, which is given by $A(X) = -\overline{\nabla}_X N$. So, the (non-normalized) mean curvature function of Σ^n is defined as been $H = \mathrm{tr}(A)$.

In the warped product $\overline{M}^{n+1} = I \times_f M^n$ there exists a remarkable family of two-sided hypersurfaces: its slices $M_{t_*} = \{t_*\} \times M$, with $t_* \in I$. The shape operator and the mean curvature of M_{t_*} with respect to $N = \partial_t$ are, respectively, $A_{t_*} = -\dfrac{f'(t_*)}{f(t_*)} I$, where I denotes the identity operator, and $H_{t_*} = -n\dfrac{f'(t_*)}{f(t_*)}$.

We will deal with two particular functions naturally attached to a two-sided hypersurface $\psi : \Sigma^n \to \overline{M}^{n+1}$, namely, the (vertical) height function $h = \pi_I \circ \psi$ and the angle function $\Theta = \overline{g}(N, \partial_t)$. Let us denote by $\overline{\nabla}$ and ∇ the gradients with respect to the metrics $\overline{g}$ and g, respectively. With a straightforward computation we show that the gradient of π_I on M^n is given by

$$\overline{\nabla}\pi_I = \overline{g}(\overline{\nabla}\pi_I, \partial_t)\partial_t = \partial_t$$

so that the gradient of h on Σ^n is

$$\nabla h = (\overline{\nabla}\pi_I)^\top = \partial_t^\top, \tag{2.3}$$

where $\partial_t^\top = \partial_t - \Theta N$ is the tangential component of ∂_t along Σ^n. From (2.3) we deduce that

$$|\nabla h|^2 + \Theta^2 = 1, \tag{2.4}$$

where ∇h is the gradient of h in the metric g and $|X|^2 = g(X, X)$ for any $X \in \mathfrak{X}(\Sigma)$. Moreover, from (2.2) and (2.3) we deduce that the Hessian of h in the metric g is given by

$$\begin{aligned} \nabla^2 h(X, X) &= g(\nabla_X \partial_t^\top, X) \\ &= \overline{g}(\overline{\nabla}_X(\partial_t - \Theta N), X) \\ &= \frac{f'(h)}{f(h)}(|X|^2 - g(\nabla h, X)^2) + g(AX, X)\Theta \end{aligned} \tag{2.5}$$

for any $X \in \mathfrak{X}(\Sigma)$. Hence, from (2.5) we obtain that the Laplacian of h in the metric g is

$$\Delta h = \frac{f'(h)}{f(h)}\left(n - |\nabla h|^2\right) + H\Theta. \tag{2.6}$$

2.2. **Mean curvature flow solitons.** We recall that the mean curvature flow $\Psi : [0, T) \times \Sigma^n \to \overline{M}^{n+1}$ of an immersion $\psi : \Sigma^n \to \overline{M}^{n+1}$ in a $(n+1)$-dimensional Riemannian manifold $\overline{M}^{n+1}$, satisfying $\Psi(0, \cdot) = \psi(\cdot)$, looks for solutions of the equation

$$\frac{\partial \Psi}{\partial t} = \vec{H},$$

where $\vec{H}(t, \cdot)$ is the (non-normalized) mean curvature vector of $\Sigma_t^n = \Psi(t, \Sigma^n)$. In our context, according to [4, Definition (1.1)], a two-sided hypersurface $\psi : \Sigma^n \to \overline{M}^{n+1}$ immersed in a warped product $\overline{M}^{n+1} = I \times_f M^n$ is said a *mean curvature flow soliton* with respect to $K = f(t)\partial_t$ with *soliton constant* $c \in \mathbb{R}$ if its (non-normalized) mean curvature function satisfies

$$H = cf(h)\Theta. \tag{2.7}$$

Adopting the terminology introduced in [4], we will also consider the *soliton function*

$$\zeta_c(t) = nf'(t) + cf(t)^2.$$

As it was observed in [4], a slice $M_{t_*} = \{t_*\} \times M^n$ is a mean curvature flow soliton with respect to $K = f(t)\partial_t$ and with soliton constant c given by

$$c = -n\frac{f'(t_*)}{f(t_*)^2}. \tag{2.8}$$

Moreover, t_* is implicitly given by the condition $\zeta_c(t_*) = 0$.

2.3. **Standard examples.** In this subsection we quote important examples which will be addressed along the next two sections. In the first one, we consider a suitable warped product model for the Euclidean space minus a point.

Example 2.1. *Let $o = (0, \ldots, 0)$ be the origin of the $(n+1)$-dimensional Euclidean space $\mathbb{R}^{n+1}$. We have that $\mathbb{R}^{n+1} \setminus \{o\}$ is isometric to $\mathbb{R}_+ \times_t \mathbb{S}^n$ (see* [28, Section 4, Example 1]*), whose slices $\{t\} \times \mathbb{S}^n$ are isometric to n-dimensional Euclidean spheres $\mathbb{S}^n(t)$ of radius $t \in \mathbb{R}_+$. In this setting, the mean curvature flow solitons with respect to $K = t\partial_t$ with soliton constant $c = -1$ are just the self-shrinkers. So, from* (2.8) *we conclude that $\mathbb{S}^n(\sqrt{n}) \equiv \{\sqrt{n}\} \times \mathbb{S}^n$ is the only slice which is a self-shrinker.*

In our next example, we consider a suitable warped product model for the real projective space.

Example 2.2. *We recall that the $(n+1)$-dimensional real projective space is given by the quotient $\mathbb{RP}^{n+1} = \mathbb{S}^{n+1}/\{\pm 1\}$, where $\{\pm 1\}$ is the group of diffeomorphisms of $(n+1)$-dimensional unit Euclidean sphere $\mathbb{S}^{n+1}$ consisting of the identity map $q \mapsto q$ and the antipodal map $q \mapsto -q$. We consider the Riemannian metric in $\mathbb{RP}^{n+1}$ in such a way that the natural projection $\pi : \mathbb{S}^{n+1} \to \mathbb{RP}^{n+1}$ becomes a local isometry. If P stands for the north pole of $\mathbb{S}^{n+1}$, then we denote by Cut_P the cut locus of $\pi(P) \in \mathbb{RP}^{n+1}$. We have that Cut_P is the image of the equator of $\mathbb{S}^{n+1}$ orthogonal to P via the natural projection, namely, $Cut_P = \pi(\mathbb{S}^n) = \mathbb{RP}^n$. Moreover, as it was proved in* [6, Section 9.111]*, $\mathbb{RP}^{n+1} \setminus \{\pi(P) \cup Cut_P\}$ is isometric to the warped product $\left(0, \frac{\pi}{2}\right) \times_{\sin t} \mathbb{S}^n$. From* (2.8) *we conclude that the slice $\{\cos^{-1}(\frac{\sqrt{4c^2+n^2}-n}{2|c|})\} \times \mathbb{S}^n$ is the only one that is a mean curvature flow soliton with respect to $K = \sin t\partial_t$ with soliton constant $c < 0$.*

Proceeding, we consider the so-called pseudo-hyperbolic spaces.

Example 2.3. *According to* [32]*, warped products of the type $I \times_{e^t} M^n$ are called* pseudo-hyperbolic spaces*. This terminology is due to the fact that the $(n+1)$-dimensional hyperbolic space $\mathbb{H}^{n+1}$ is isometric to the warped product $\mathbb{R} \times_{e^t} \mathbb{R}^n$, where the slices constitute a family of horospheres sharing a same fixed point in the asymptotic boundary $\partial_\infty \mathbb{H}^{n+1}$ and giving a complete foliation of $\mathbb{H}^{n+1}$ (for more details about pseudo-hyperbolic spaces see, for instance,* [2, 28, 32]*). From* (2.8) *we conclude that the slice $\{\log(-\frac{n}{c})\} \times M^n$ is the only one that is a mean curvature flow soliton with respect to $K = e^t\partial_t$ with soliton constant $c < 0$.*

In our last examples, we deal with the Schwarzschild and Reissner-Nordström spaces.

Example 2.4. *Given a mass parameter $\mathfrak{m} > 0$, the* Schwarzschild space *is defined to be the product $\overline{M}^{n+1} = (r_0(\mathfrak{m}), +\infty) \times \mathbb{S}^n$ furnished with the metric $\bar{g} = V_{\mathfrak{m}}(r)^{-1}dr^2 + r^2 g_{\mathbb{S}^n}$, where $g_{\mathbb{S}^n}$ is the standard metric of $\mathbb{S}^n$, $V_{\mathfrak{m}}(r) = 1 - 2\mathfrak{m}r^{1-n}$ stands for its potential function and $r_0(\mathfrak{m}) = (2\mathfrak{m})^{1/(n-1)}$ is the unique positive root of $V_{\mathfrak{m}}(r) = 0$. Its importance lies in the fact that the manifold $\mathbb{R} \times \overline{M}^{n+1}$ equipped with the Lorentzian static metric $-V_{\mathfrak{m}}(r)dt^2 + \bar{g}$ is a solution of the Einstein field equation in vacuum with zero cosmological constant (see, for instance,* [30, Chapter 13] *for more details concerning Schwarzschild geometry).*

As it was observed in [16, Example 1.3], $\overline{M}^{n+1}$ *can be reduced in the form* $I \times_f \mathbb{S}^n$ *with metric* (2.1) *via the following change of variables:*

$$t = \int_{r_0(\mathfrak{m})}^{r} \frac{d\sigma}{\sqrt{V_\mathfrak{m}(\sigma)}}, \qquad f(t) = r(t), \qquad I = \mathbb{R}_+. \tag{2.9}$$

As it was noted in [16, Example 4.1], *since* $V_\mathfrak{m}(r)$ *is strictly increasing on* $(r_0(\mathfrak{m}), +\infty)$, *it follows from* (2.9) *that the warping function* f *satisfies:*

$$f'(t) = \frac{dr}{dt} = \sqrt{V_\mathfrak{m}(r(t))} > 0 \qquad \text{and} \qquad f''(t) = \frac{1}{2}\frac{dV_\mathfrak{m}}{dr}(r(t)) > 0. \tag{2.10}$$

Thus, from (2.8) *and* (2.10) *we can verify that a slice* $\{t_*\} \times \mathbb{S}^n$ *is a mean curvature flow soliton with respect to* $f(t)\partial_t = r\sqrt{V_\mathfrak{m}(r)}\partial_r$ *with soliton constant* $c < 0$ *when* $t_* = t(r_*)$ *with* $r_* > r_0(\mathfrak{m})$ *solving the following equation*

$$V_\mathfrak{m}(r) = \frac{c^2}{n^2} r^4. \tag{2.11}$$

We note that such a solution exists if and only if the function $\varphi_\mathfrak{m}(t) = \frac{c^2}{n^2}t^4 + \frac{2\mathfrak{m}}{t^{n-1}} - 1$ *has a zero on* $(r_0(\mathfrak{m}), +\infty)$. *Notice that* $\varphi_\mathfrak{m}$ *is a convex function which goes to infinity if* t *goes to* 0 *or* $+\infty$ *and so* $\varphi_\mathfrak{m}$ *has a unique minimal point in* $(0, \infty)$. *Such value* $\hat{r}$ *is given implicitly by* $\varphi'_\mathfrak{m}(\hat{r}) = 0$, *that is,*

$$\frac{4c^2}{n^2}\hat{r}^3 - \frac{2\mathfrak{m}(n-1)}{\hat{r}^n} = 0.$$

Therefore, the equation (2.11) *has a solution if and only if* $\hat{r} > r_0(\mathfrak{m})$ *and* $\varphi_\mathfrak{m}(\hat{r}) \le 0$. *The last condition can be rewritten in the following way:*

$$\hat{r} = \left(\frac{\mathfrak{m}(n-1)n^2}{2c^2}\right)^{1/(n+3)} \ge \left(\frac{\mathfrak{m}(n+3)}{2}\right)^{1/(n-1)}. \tag{2.12}$$

In particular, there are two solutions $r_0(\mathfrak{m}) < r_{*,-} < \hat{r} < r_{*,+}$ *if the strict inequality holds in* (2.12), *and a unique solution* $r_* = \hat{r}$ *if equality holds.*

Example 2.5. *Given a mass parameter* $\mathfrak{m} > 0$ *and an electric charge* $\mathfrak{q} \in \mathbb{R}$, *with* $|\mathfrak{q}| \le \mathfrak{m}$, *the* Reissner-Nordström space *is defined to be the product* $\overline{M}^{n+1} = (r_0(\mathfrak{m}, \mathfrak{q}), +\infty) \times \mathbb{S}^n$ *endowed with the metric* $\bar{g} = V_{\mathfrak{m},\mathfrak{q}}(r)^{-1}dr^2 + r^2 g_{\mathbb{S}^n}$, *where* $g_{\mathbb{S}^n}$ *is the standard metric of* $\mathbb{S}^n$, $V_{\mathfrak{m},\mathfrak{q}}(r) = 1 - 2\mathfrak{m}r^{1-n} + \mathfrak{q}^2 r^{2-2n}$ *stands for its potential function and* $r_0(\mathfrak{m}, \mathfrak{q}) = \left(\frac{\mathfrak{q}^2}{\mathfrak{m} - \sqrt{\mathfrak{m}^2 - \mathfrak{q}^2}}\right)^{1/(n-1)}$ *is the largest positive zero of* $V_{\mathfrak{m},\mathfrak{q}}(r)$. *The importance of this model lies in the fact that the manifold* $\mathbb{R} \times \overline{M}^{n+1}$ *equipped with the Lorentzian static metric* $-V_{\mathfrak{m},\mathfrak{q}}(r)dt^2 + \bar{g}$ *is a charged black-hole solution of the Einstein field equation in vacuum with zero cosmological constant.*

As in the previous example, $\overline{M}^{n+1}$ *can be reduced in the form* $I \times_f \mathbb{S}^n$ *with metric* (2.1) *via the same change of variables as in* (2.9). *Furthermore, following the same previous steps, the warping function* f *has positive first and second derivatives. Moreover, we can verify that a slice* $\{t_*\} \times \mathbb{S}^n$ *is a mean curvature flow soliton with respect to* $f(t)\partial_t = r\sqrt{V_{\mathfrak{m},\mathfrak{q}}(r)}\partial_r$ *with soliton constant* $c < 0$ *when* $t_* = t(r_*)$ *with* $r_* > r_0(\mathfrak{m}, \mathfrak{q})$ *solving the following equation*

$$V_{\mathfrak{m},\mathfrak{q}}(r) = \frac{c^2}{n^2} r^4. \tag{2.13}$$

We observe that such a case is more complicated to make all values explicit, but qualitatively we can say that such a solution of (2.13) *exists if and only if the function* $\varphi_{\mathfrak{m},\mathfrak{q}}(x) = \frac{c^2}{n^2}x^4 + \frac{2\mathfrak{m}}{x^{n-1}} - \frac{\mathfrak{q}^2}{x^{2n-2}} - 1$ *has a zero on* $(r_0(\mathfrak{m}), +\infty)$. *Note that* $\varphi_{\mathfrak{m},\mathfrak{q}}$ *goes to positive infinity if* x *goes to positive infinity and* $\varphi_{\mathfrak{m},\mathfrak{q}}$

goes to negative infinity if x goes to zero. So, $\varphi_{\mathfrak{m},\mathfrak{q}}$ has at least one root in $(0,+\infty)$ and if such roots are greater than $r_0(\mathfrak{m},\mathfrak{q})$ we get the desired solutions r_*.

3. Nonexistence of complete mean curvature flow solitons

3.1. **Auxiliary results.** In order to investigate the nonexistence of complete mean curvature flow solitons, initially we introduce the following definition:

Definition 3.1. *The Laplacian operator Δ on a Riemannian manifold (Σ, g) satisfies the Omori-Yau maximum principle if for any $u \in C^2$ bounded from above, there exists a sequence $(p_k)_{k\geq 1}$ in Σ^n such that*

$$\lim_k u(p_k) = \sup_\Sigma u = u^*, \qquad \lim_k |\nabla u(p_k)| = 0 \qquad \text{and} \qquad \limsup_k \Delta u(p_k) \leq 0.$$

Now we recall the maximum principle due to Omori [29] and Yau [34]. Such concept gives us conditions to the validity of a maximum principle for the hessian or the Laplacian on a Riemannian manifold. Specifically, we quote the following result for the Laplacian:

Lemma 3.1 (Yau, [34]). *Let Σ^n be an n-dimensional complete Riemannian manifold whose Ricci curvature is bounded from below. Then the Laplacian Δ satisfies the Omori-Yau maximum principle on Σ.*

Denoting by K_M the sectional curvature of the fiber M^n, we will consider warped product spaces $I \times_f M^n$ satisfying the convergence condition

$$K_M \geq \sup_I (f'^2 - ff''). \tag{3.14}$$

Warped products satisfying (3.14) have been studying, for instance, in [4, 5, 17, 21]. The case that this condition holds for the Ricci curvature instead of the sectional curvature is also well known (see, for instance, [1, 3, 28]). Furthermore, it is not difficult to verify that there exists a wide class of warped product satisfying (3.14), including, for instance, the Euclidean space minus a point $\mathbb{R}^{n+1} \setminus \{o\} = \mathbb{R}_+ \times_t \mathbb{S}^n$, the real projective space (minus a suitable point and its cut locus) $(0, \frac{\pi}{2}) \times_{\sin t} \mathbb{S}^n$, the pseudo-hyperbolic spaces $I \times_{e^t} M^n$ with fiber having nonnegative sectional curvature and the Schwarzschild and Reissner-Nordström spaces $I \times_f \mathbb{S}^n$ (see Examples 2.1, 2.2, 2.3, 2.4 and 2.5).

Indeed, this verification for the Euclidean, the real projective, the pseudo-hyperbolic and the Schwarzschild spaces is quite simple. In the case of the Reissner-Nordström space, with a straightforward computation we get that

$$f'(t)^2 - f(t)f''(t) = 1 - \mathfrak{m}r(t)^{1-n} - n\left\{\mathfrak{m} - \mathfrak{q}^2 r(t)^{1-n}\right\} r(t)^{1-n}. \tag{3.15}$$

But, since $r(t) > r_0(\mathfrak{m},\mathfrak{q}) = \left(\frac{\mathfrak{q}^2}{\mathfrak{m}-\sqrt{\mathfrak{m}^2-\mathfrak{q}^2}}\right)^{1/(n-1)}$, it is not difficult to verify that we must have

$$\mathfrak{q}^2 r(t)^{1-n} < \mathfrak{m}. \tag{3.16}$$

Consequently, from (3.15) and (3.16) we conclude that the convergence condition (3.14) is also satisfied in the Reissner-Nordström space.

We recall that a hypersurface Σ^n lies in a *slab* of a warped product $I \times_f M^n$ when Σ^n is contained in a region of the type

$$[t_1, t_2] \times M^n = \{(t,p) \in I \times_f M^n : t_1 \leq t \leq t_2 \text{ and } p \in M^n\}.$$

We also recall the first and second Newton transformations, which are given by $P_1 = HI - A$ and $P_2 = S_2 I - AP_1$, and here S_2 stands for the second mean curvature, that is, $S_2 = \sum_{i<j} k_i k_j$,

where k_i are the principal curvatures of Σ^n. Finally, we say that an operator T on Σ is *f-bounded* whether there are continuous functions $G, H : \mathbb{R}^2 \to \mathbb{R}$ such that

$$G(f, f') \circ h(p)|u|^2 \leq \langle Tu, u\rangle \leq H(f, f') \circ h(p)|u|^2$$

for all $u \in T_p\Sigma$ and $p \in \Sigma$.

Next, considering an immersed hypersurface Σ^n in a slab of a warped product space $I \times_f M^n$ satisfying (3.14), we will verify that the Omori-Yau maximum principle is satisfied.

Proposition 3.1. *Let $\overline{M}^{n+1} = I \times_f M^n$ be a warped product which satisfying the convergence condition (3.14), for $n \geq 3$, and, let $\psi : \Sigma^n \to \overline{M}^{n+1}$ be a complete hypersurface with second Newton transformation f-bounded and lying in a slab. Then, the Laplacian on Σ^n satisfies the Omori-Yau maximum principle.*

Proof. First, we recall that the curvature tensor R of Σ^n can be described in terms of its Weingarten operator A and the curvature tensor $\overline{R}$ of the ambient $I \times_f M^n$ by the so-called Gauss' equation given by

$$g(R(X,Y)Z,W) = \bar{g}(\overline{R}(X,Y)Z,W) + g(A(X,Z),A(Y,W)) - g(A(X,W),A(Y,Z)) \tag{3.17}$$

for every tangent vector fields $X, Y, Z, W \in \mathfrak{X}(\Sigma)$.

Let us consider $X \in \mathfrak{X}(\Sigma)$ and take a local orthonormal frame $\{E_1, \ldots, E_n\}$ of $\mathfrak{X}(\Sigma)$. Then, it follows from Gauss equation (3.17) that the Ricci curvature Ric of Σ^n with respect to the induced metric g is given by

$$\begin{aligned}\mathrm{Ric}(X,X) &= \sum_i \bar{g}(\overline{R}(X,E_i)X,E_i) + H\langle AX, X\rangle - |AX|^2\\ &= \sum_i \bar{g}(\overline{R}(X,E_i)X,E_i) - \langle (AP_1)X, X\rangle\\ &= \sum_i \bar{g}(\overline{R}(X,E_i)X,E_i) + S_2|X|^2 - \langle P_2X, X\rangle.\end{aligned} \tag{3.18}$$

Moreover, with a straightforward computation, we get

$$\begin{aligned}\overline{R}(X,E_i)X &= \overline{R}(X^*,E_i^*)X^* + \bar{g}(X,\partial_t)\overline{R}(X^*,E_i^*)\partial_t + \bar{g}(X,\partial_t)\bar{g}(E_i,\partial_t)\overline{R}(X^*,\partial_t)\partial_t\\ &\quad + \bar{g}(E_i,\partial_t)\overline{R}(X^*,\partial_t)X^* + \bar{g}(X,\partial_t)\overline{R}(\partial_t,E_i^*)X^* + \bar{g}(X,\partial_t)^2\overline{R}(\partial_t,E_i^*)\partial_t,\end{aligned} \tag{3.19}$$

where $X^* = X - \bar{g}(X,\partial_t)\partial_t$ and $E_i^* = E_i - \bar{g}(E_i,\partial_t)\partial_t$ are the projections of the tangent vector fields X and E_i onto the fiber M^n, respectively.

Thus, by repeated use of the formulas of [30, Proposition 7.42] and using equation (2.3), from (3.19) we get

$$\begin{aligned}&\sum_i \bar{g}(\overline{R}(X,E_i)X,E_i)\\ &= \sum_i \bar{g}(R_M(X^*,E_i^*)X^*,E_i^*) - (n-1)\frac{f'(h)^2}{f(h)^2}|X|^2\\ &+ \left(\frac{f'(h)^2 - f(h)f''(h)}{f(h)^2}\right)|\nabla h|^2|X|^2 + (n-2)\left(\frac{f'(h)^2 - f(h)f''(h)}{f(h)^2}\right)g(X,\nabla h)^2,\end{aligned} \tag{3.20}$$

As in [30], the curvature tensor R of the hypersurface Σ^n is given by $R(X,Y)Z = \nabla_{[X,Y]}Z - [\nabla_X, \nabla_Y]Z$, where [ ] denotes the Lie bracket and $X, Y, Z \in \mathfrak{X}(\Sigma)$.

where R_M denotes the curvature tensor of the fiber M^n. But, it is not difficult to verify that

$$\sum_i \bar{g}(R_M(X^*, E_i^*)X^*, E_i^*) = \frac{1}{f^2}\sum_i K_M(X^*, E_i^*)(|X|^2 - g(\nabla h, E_i)^2|X|^2 - g(X, \nabla h)^2 - g(X, E_i)^2 + 2g(X, \nabla h)g(X, E_i)g(\nabla h, E_i)).$$

Thus, by using the convergence condition (3.14) and a direct computation, from (3.20) we obtain

$$\sum_i \bar{g}(\overline{R}(X, E_i)X, E_i) \geq -(n-1)\frac{f''(h)}{f(h)}|X|^2. \tag{3.21}$$

Thus, inserting the estimate (3.21) into the equation (3.18), and using the f-boundedness of P_2, we deduce that

$$\mathrm{Ric}(X, X) \geq \left(-(n-1)\frac{f''(h)}{f(h)} + \frac{n}{n-2}G(f, f') - H(f, f')\right)|X|^2. \tag{3.22}$$

Therefore, taking into account that Σ^n lies in a slab of the ambient space, from (3.22) we conclude that the Ricci curvature is bounded from below and by Lemma 3.1 the Laplacian satisfies the desired property. □

Corollary 3.1. *Let $\overline{M}^{n+1} = I \times_f M^n$ be a warped product which satisfying the convergence condition (3.14), and let $\psi : \Sigma^n \to \overline{M}^{n+1}$ be a complete mean curvature flow soliton with respect to $K = f(t)\partial_t$ and soliton constant $c \neq 0$. If the second mean curvature is bounded from below and Σ lies in a slab, then the Laplacian on Σ^n satisfies the Omori-Yau maximum principle.*

Proof. Since the second mean curvature is bounded from below and $\psi(\Sigma)$ lies in a slab, notice that $k_i^2 \leq H^2 - 2S_2 \leq c^2 f(h)^2 + d$ is bounded on Σ for all i. So, P_2 is bounded and the result follows from Proposition 3.1. For $n = 2$, this result is immediate. □

3.2. **Nonexistence results via Omori-Yau maximum principle.** Into the scope of a warped product $I \times_f M^n$ we are in position to state and prove our first nonexistence result concerning mean curvature flow solitons immersed in a slab of a warped product.

Theorem 3.1. *Let $\overline{M}^{n+1} = I \times_f M^n$ be a warped product whose fiber M^n satisfies hypothesis (3.14). There is no complete mean curvature flow soliton $\psi : \Sigma^n \to \overline{M}^{n+1}$ with respect to $K = f(t)\partial_t$ and soliton constant $c \neq 0$, with second mean curvature bounded from below, lying in a slab $[t_1, t_2] \times M^n$ and $\zeta_c(t)$ having a strict sign on $[t_1, t_2]$.*

Proof. Let us suppose by contradiction the existence of such a mean curvature flow soliton $\psi : \Sigma^n \to \overline{M}^{n+1}$. From (2.6) we have

$$\begin{aligned} \Delta h &= n\frac{f'(h)}{f(h)} - \frac{f'(h)}{f(h)}|\nabla h|^2 + cf(h)\Theta^2 \\ &= n\frac{f'(h)}{f(h)}\Theta^2 + (n-1)\frac{f'(h)}{f(h)}|\nabla h|^2 + cf\Theta^2 \\ &= (n-1)\frac{f'(h)}{f(h)}|\nabla h|^2 + \frac{nf'(h) + cf^2(h)}{f}\Theta^2 \\ &= (n-1)\frac{f'(h)}{f(h)}|\nabla h|^2 + \frac{\zeta_c(h)}{f(h)}\Theta^2, \end{aligned} \tag{3.23}$$

where we used (2.4) in the second equality. Since the second mean curvature is bounded and the hypersurface is contained in a slab, from Corollary 3.1 we are able to apply the Omori-Yau maximum principle. Indeed, there are sequences $\{x_k\}$ and $\{p_k\}$ such that

$$\lim_k h(p_k) = \sup_\Sigma h = h^*, \qquad \lim_k |\nabla h(p_k)| = 0 \qquad \text{and} \qquad \limsup_k \Delta h(p_k) \leq 0,$$

and

$$\lim_k h(x_k) = \inf_\Sigma h = h_*, \qquad \lim_k |\nabla h(x_k)| = 0 \qquad \text{and} \qquad \liminf_k \Delta h(x_k) \geq 0,$$

and thus, using that Θ goes to 1 along the sequences $\{p_k\}$ and $\{x_k\}$, we deduce from equation (3.23) that

$$\zeta_c(h^*) \leq 0 \leq \zeta_c(h_*),$$

which contradict our hypothesis on the function ζ_c.

□

Remark 3.1. *It is worth to point out that complete mean curvature flow solitons immersed in a slab of a warped product $I \times_f M^n$ and with second mean curvature bounded from below constitute natural generalizations of the compact ones, and they have already been studied by Alías, de Lira and Rigoli in* [4].

Taking into account Example 2.1, it is not difficult to verify that we get from the proof of Theorem 3.1 the following result concerning the nonexistence of complete self-shrinkers:

Corollary 3.2. *There exists no complete n-dimensional self-shrinker of $\mathbb{R}^{n+1}$ with second mean curvature bounded from below and lying in the closure of an n-dimensional annulus with either inner radius $r_{ir} > \sqrt{n}$ or outer radius $r_{or} < \sqrt{n}$.*

Remark 3.2. *We point out that the sphere of radius $\sqrt{n}$ satisfies all the hypotheses if we allow the inner radius r_{ir} (or outer radius r_{or}) equal to $\sqrt{n}$. We also notice that the self-shrinkers $\mathbb{S}^k(\sqrt{k}) \times \mathbb{R}^{n-k}$, for $1 \leq k \leq n-1$, of $\mathbb{R}^{n+1}$ have bounded second mean curvature but they do not belong to any n-dimensional annuli.*

Considering the discussion made in Example 2.2, from Theorem 3.1 we have:

Corollary 3.3. *Let $\overline{M}^{n+1} = \left(0, \frac{\pi}{2}\right) \times_{\sin t} \mathbb{S}^n$ be the warped product model of $\mathbb{RP}^{n+1} \setminus \{\pi(P) \cup Cut_P\}$. There is no complete mean curvature flow soliton $\psi : \Sigma^n \to \overline{M}^{n+1}$ with respect to $K = \sin t\partial_t$ with soliton constant $c < 0$, having second mean curvature bounded from below and lying in a slab $[t_1, t_2] \times M^n$, with either $\cos^{-1}(\frac{\sqrt{4c^2+n^2}-n}{2|c|}) < t_1 < \frac{\pi}{2}$ or $0 < t_2 < \cos^{-1}(\frac{\sqrt{4c^2+n^2}-n}{2|c|})$.*

When the ambient space is a pseudo-hyperbolic space (see Example 2.3), from Theorem 3.1 we also obtain the following consequence:

Corollary 3.4. *Let $\overline{M}^{n+1} = I \times_{e^t} M^n$ be a pseudo-hyperbolic space whose fiber M^n has nonnegative sectional curvature. There is no complete mean curvature flow soliton $\psi : \Sigma^n \to \overline{M}^{n+1}$ with respect to $K = e^t\partial_t$ with soliton constant $c < 0$, having second mean curvature bounded from below and lying in a slab $[t_1, t_2] \times M^n$, with either $t_1 > \log\left(-\frac{n}{c}\right)$ or $t_2 < \log\left(-\frac{n}{c}\right)$.*

Considering the context of Example 2.4, from Theorem 3.1 we get:

Corollary 3.5. *Let $\overline{M}^{n+1} = I \times_f \mathbb{S}^n$ be the Schwarzschild space. There is no complete mean curvature flow soliton $\psi : \Sigma^n \to \overline{M}^{n+1}$ with respect to $K = f(t)\partial_t$ with soliton constant $c < 0$, having second mean curvature bounded from below and lying in a slab $[t_1, t_2] \times \mathbb{S}^n$, with $f(t_2) \geq \sqrt{-\frac{n}{c}}$.*

Proof. Using (2.10) and definition of ζ_c we have

$$n\sqrt{V_{\mathfrak{m}}(r(t_1))} + cr(t_1)^2 \le \zeta_c(t) \le n\sqrt{V_{\mathfrak{m}}(r(t_2))} + cr(t_2)^2.$$

Since $V_{\mathfrak{m}}(r(t)) < 1$ for all $t \in I$, $r(t_2) = f(t_2) \ge \sqrt{-\frac{n}{c}}$ implies

$$\zeta_c(t) = n\sqrt{V_{\mathfrak{m}}(r(t))} + cr(t)^2 < 0$$

for all $t \ge t_1$. Therefore, we can apply Theorem 3.1 to conclude our result. $\square$

In the setting of Example 2.5, we can reason as in the proof of Corollary 3.5 to obtain the following nonexistence result:

Corollary 3.6. *Let $\overline{M}^{n+1} = I \times_f \mathbb{S}^n$ be the Reissner-Nordström space. There is no complete mean curvature flow soliton $\psi : \Sigma^n \to \overline{M}^{n+1}$ with respect to $K = f(t)\partial_t$ with soliton constant $c < 0$, having second mean curvature bounded from below and lying in a slab $[t_1, t_2] \times \mathbb{S}^n$, with $V_{\mathfrak{m},\mathfrak{q}}(r(t)) < \frac{c^2}{n^2}r(t)^4$ for all $t \in [t_1, t_2]$.*

Remark 3.3. *In Corollaries 3.3, 3.4, 3.5 and 3.6, if we assume $c > 0$ the condition ζ_c positive is immediate and so the nonexistence results follows directly.*

4. Rigidity of mean curvature flow solitons

4.1. **Rigidity results via an extension of Hopf's maximum principle.** We initiate this section regarding an extension of Hopf's theorem on a complete Riemannian manifold (Σ^n, g) due to Yau in [35]. For this, let us consider $\mathcal{L}^1_g(\Sigma) := \{u : \Sigma^n \to \mathbb{R} : \int_\Sigma |u| d\Sigma < +\infty\}$, where $d\Sigma$ is the measure related to the metric g.

Lemma 4.2. *Let u be a smooth function defined on a complete Riemannian manifold (Σ^n, g), such that Δu does not change sign on Σ^n. If $|\nabla u| \in \mathcal{L}^1_g(\Sigma)$, then Δu vanishes identically on Σ^n.*

Using the previous lemma, we have the following result:

Theorem 4.2. *Let $\overline{M}^{n+1} = I \times_f M^n$ be a warped product. Let $\psi : \Sigma^n \to \overline{M}$ be a complete mean curvature flow soliton with respect to $K = f(t)\partial_t$ and soliton constant $c \neq 0$, that lies in a slab $[t_1, t_2] \times M^n$, and whose $\zeta_c(t)$ does not change the sign. If $|\nabla h| \in \mathcal{L}^1_g(\Sigma)$, then Σ^n is a slice M_{t_*}, for $t_* \in [t_1, t_2]$ given implicitly by $\zeta_c(t_*) = 0$.*

Proof. Considering $F(t) = \int_{t_0}^t f(v)^{1-n} dv$ and compute the Laplacian of $F(h)$ as follows:

$$\begin{aligned}
\Delta F(h) &= F'(h)\Delta h + F''(h)|\nabla h|^2 \\
&= \frac{1}{f(h)^{n-1}}\Delta h + (1-n)f(h)^{-n}f'(h)|\nabla h|^2 \\
&= \frac{\zeta_c(h)}{f(h)^n}\Theta^2 + (n-1)\frac{f'(h)}{f(h)^n}|\nabla h|^2 + (1-n)f(h)^{-n}f'(h)|\nabla h|^2 \\
&= f(h)^{-n}\zeta_c(h)\Theta^2,
\end{aligned}$$

where we used equation (2.6) in the third equality. Thus $F(h)$ is either subharmonic or superharmonic. Since Σ is contained is a slab and $|\nabla h| \in \mathcal{L}^1(\Sigma)$, we have that $|\nabla F(h)| = f(h)^{1-n}|\nabla h|$ belongs to the 1-Lebesgue space too.

Applying Lemma 4.2, we deduce that $\Delta F(h) = 0$ and thus $\zeta_c(h)\Theta^2 = 0$ along Σ. Next, note that

$$\Delta F(h)^2 = 2F(h)\Delta F(h) + 2|\nabla F(h)|^2 = 2f(h)^{2-2n}|\nabla h|^2 \ge 0.$$

Applying Lemma 4.2 again, we deduce that $\nabla h = 0$ on Σ and from (2.4) we have $\Theta = 1$. Thus, $\zeta_c(h)$ vanishes on Σ, as we claimed.

□

From Theorem 4.2 we get the following rigidity result:

Corollary 4.7. *The only complete n-dimensional self-shrinker of $\mathbb{R}^{n+1}$ that lies in the closure of an n-dimensional annulus with either inner radius $r_{ir} \geq \sqrt{n}$ or outer radius $r_{or} \leq \sqrt{n}$ and such that $|\nabla h| \in \mathcal{L}^1_g(\Sigma)$ is $\mathbb{S}^n(\sqrt{n})$.*

Remark 4.4. *Related to Corollary 4.7, it is worth to mention that Pigola and Rimoldi [31] studied geometric properties of complete non-compact bounded self-shrinkers obtaining natural restrictions that force these hypersurfaces to be compact. In particular, they proved that the only complete bounded self-shrinker of $\mathbb{R}^3$ with $|A| \leq 1$ is $\mathbb{S}^2(\sqrt{2})$. Afterwards, Cavalcante and Espinar [8] showed that the only complete self-shrinker of $\mathbb{R}^{n+1}$ properly immersed in a closed cylinder $\overline{\mathbb{B}^{k+1}(r)} \times \mathbb{R}^{n-k}$, for some $k \in \{1, \cdots, n\}$ and radius $r \leq \sqrt{k}$, is the cylinder $\mathbb{S}^k(\sqrt{k}) \times \mathbb{R}^{n-k}$.*

Considering the setting of Example 2.2, from Theorem 3.1 we have:

Corollary 4.8. *Let $\overline{M}^{n+1} = \left(0, \frac{\pi}{2}\right) \times_{\sin t} \mathbb{S}^n$ be the warped product model of $\mathbb{RP}^{n+1} \setminus \{\pi(P) \cup Cut_P\}$. Let $\psi : \Sigma^n \to \overline{M}^{n+1}$ be a complete mean curvature flow soliton with respect to $K = \sin t\partial_t$ and soliton constant $c < 0$, that lies in a slab $[t_1, t_2] \times \mathbb{S}^n$, and either $\cos^{-1}(\frac{\sqrt{4c^2+n^2}-n}{2|c|}) \leq t_1 < \frac{\pi}{2}$ or $0 < t_2 \leq \cos^{-1}(\frac{\sqrt{4c^2+n^2}-n}{2|c|})$. If $|\nabla h| \in \mathcal{L}^1_g(\Sigma)$, then Σ^n is the slice $\{\cos^{-1}(\frac{\sqrt{4c^2+n^2}-n}{2|c|})\} \times \mathbb{S}^n$.*

When the ambient space is a pseudo-hyperbolic space, Theorem 4.2 reads as follows:

Corollary 4.9. *Let $\overline{M}^{n+1} = I \times_{e^t} M^n$ be a pseudo-hyperbolic space whose fiber M^n is complete. Let $\psi : \Sigma^n \to \overline{M}^{n+1}$ be a complete mean curvature flow soliton with respect to $K = e^t\partial_t$ and soliton constant $c < 0$, that lies in a slab $[t_1, t_2] \times M^n$, and $t_1 \geq \log\left(-\frac{n}{c}\right)$. If $|\nabla h| \in \mathcal{L}^1_g(\Sigma)$, then Σ^n is the slice $\{\log\left(-\frac{n}{c}\right)\} \times M^n$.*

Taking into account again the context of Example 2.4 and Example 2.5, from Theorem 4.2 we also obtain:

Corollary 4.10. *Let $\overline{M}^{n+1} = I \times_f \mathbb{S}^n$ be the Schwarzschild space and suppose that inequality (2.12) is satisfied. Let $\psi : \Sigma^n \to \overline{M}^{n+1}$ be a complete mean curvature flow soliton with respect to $K = f(t)\partial_t$ and soliton constant $c < 0$, that lies in a slab $[t_1, t_2] \times \mathbb{S}^n$, and $V_{\mathfrak{m}}(r(t)) \leq \frac{c^2}{n^2} r(t)^4$ for all $t \in [t_1, t_2]$. If $|\nabla h| \in \mathcal{L}^1_g(\Sigma)$, then Σ^n is a slice $\{t_*\} \times \mathbb{S}^n$, where $t_* = t(r_*)$ is such that $r_* > r_0(\mathfrak{m})$ solves equation (2.11).*

and

Corollary 4.11. *Let $\overline{M}^{n+1} = I \times_f \mathbb{S}^n$ be the Reissner-Nordström space and suppose that there is $r_* > r_0(\mathfrak{m}, \mathfrak{q})$. Let $\psi : \Sigma^n \to \overline{M}^{n+1}$ be a complete mean curvature flow soliton with respect to $K = f(t)\partial_t$ and soliton constant $c < 0$, that lies in a slab $[t_1, t_2] \times \mathbb{S}^n$, and $V_{\mathfrak{m},\mathfrak{q}}(r(t)) \leq \frac{c^2}{n^2} r(t)^4$ for all $t \in [t_1, t_2]$. If $|\nabla h| \in \mathcal{L}^1_g(\Sigma)$, then Σ^n is a slice $\{t_*\} \times \mathbb{S}^n$, where $t_* = t(r_*)$ is such that $r_* > r_0(\mathfrak{m}, \mathfrak{q})$ solves equation (2.13).*

4.2. **Rigidity results via a parabolicity criterion.** We recall that a Riemannian manifold is said to be *parabolic* if the only subharmonic functions on it that are bounded from above are the constants. On the other hand, given two Riemannian manifolds (Σ, g) and $(\Sigma^{'}, g^{'})$, a diffeomorphism ϕ from Σ onto $\Sigma^{'}$ is called a *quasi-isometry* if there exists a constant $\kappa \geq 1$ such that

$$\kappa^{-1}|v|_g \leq |d\phi(v)|_{g'} \leq \kappa|v|_g$$

for all $v \in T_p\Sigma$, $p \in \Sigma$. From [25, Theorem 1] (see also [22, Corollary 5.3]) we have the following:

Lemma 4.3. *Let (Σ, g) and $(\Sigma^{'}, g^{'})$ be two complete Riemannian manifolds. If Σ and $\Sigma^{'}$ are quasi-isometric, then Σ and $\Sigma^{'}$ are both parabolic or neither is parabolic.*

We can use the previous lemma to get the following parabolicity criterion:

Lemma 4.4. *Let $\psi : \Sigma^n \to \overline{M}^{n+1}$ be a complete hypersurface immersed in a warped product $\overline{M}^{n+1} = I \times_f M^n$, whose fiber (M^n, g_M) is complete with parabolic universal covering. If Θ is bounded away from zero, then $(\Sigma^n, \hat{g})$, endowed with the conformal metric $\hat{g} = \frac{1}{f(h)^2} g$, is parabolic.*

Proof. Given $p \in \Sigma^n$ and $v \in T_p\Sigma^n$, from (2.1) and (2.4) we have

$$g(v, v) = g(v, \nabla h)^2 + f(h)^2 g_M(d\pi(v), d\pi(v)). \tag{4.24}$$

Thus, from (4.24) we get

$$\hat{g}(v, v) = \frac{1}{f(h)^2} g(v, v) \geq g_M(d\pi(v), d\pi(v)). \tag{4.25}$$

On the other hand, using (2.4) and the Cauchy-Schwarz inequality in (4.24), we also have

$$\Theta^2 g(v, v) \leq f(h)^2 g_M(d\pi(v), d\pi(v)). \tag{4.26}$$

Since Θ is bounded away from zero, there exists a positive constant β such that $\Theta^2 \geq \beta^2$. Consequently, from (4.26) we get

$$\beta^2 g(v, v) \leq \Theta^2 g(v, v) \leq f(h)^2 g_M(d\pi(v), d\pi(v)). \tag{4.27}$$

Thus, from (4.27) we have

$$\hat{g}(v, v) \leq \frac{1}{\beta^2} g_M(d\pi(v), d\pi(v)). \tag{4.28}$$

Hence, using inequalities (4.25) and (4.28), we get

$$g_M(d\pi(v), d\pi(v)) \leq \hat{g}(v, v) \leq \frac{1}{\beta^2} g_M(d\pi(v), d\pi(v)). \tag{4.29}$$

So, taking the constant $\kappa = \frac{1}{\beta^2} \geq 1$, from (4.29) we obtain

$$\kappa^{-1} g_M(d\pi(v), d\pi(v)) \leq \hat{g}(v, v) \leq \kappa g_M(d\pi(v), d\pi(v)), \tag{4.30}$$

which means that π is a quasi-isometry between Σ and M.

Let $\Sigma^{'}$ be the universal Riemannian covering of Σ with projection $\pi_\Sigma : \Sigma^{'} \to \Sigma$. Then, the map $\pi_0 = \pi \circ \pi_\Sigma : \Sigma^{'} \to M$ is a covering map. If $M^{'}$ is the universal Riemannian covering of M with projection $\pi^{'} : M^{'} \to M$, then there exists a diffeomorphism $\phi : \Sigma^{'} \to M^{'}$ such that $\pi^{'} \circ \phi = \pi_0$. Moreover, from (4.30) it is not difficult to verify that ϕ is also a quasi-isometry. Therefore, since the universal Riemannian covering of M is parabolic, it follows from Lemma 4.3 that the universal Riemannian covering of Σ is parabolic and, hence, Σ must be also parabolic with respect to the metric $\hat{g}$. $\square$

For the next result, let us establish one notation. Define the *modified soliton function* as being the function

$$\bar{\zeta}_c(t) := f'(t)\zeta_c(t). \tag{4.31}$$

Using Lemma 4.4, we obtain the following result:

Theorem 4.3. *Let $\overline{M}^{n+1} = I \times_f M^n$ be a warped product whose fiber M^n is complete with parabolic universal covering and such that its warping function f satisfies*

$$(\log f)'' \leq \gamma[(\log f)']^2 \tag{4.32}$$

for some constant $\gamma > -1$, holding the equality only at isolated points of I. Let $\psi : \Sigma^n \to \overline{M}^{n+1}$ be a complete mean curvature flow soliton with respect to $K = f(t)\partial_t$ and soliton constant $c \neq 0$, such that Θ is bounded away from zero and $\inf_\Sigma f(h) > 0$. If $\bar{\zeta}_c(h) \leq 0$, then Σ^n is a slice M_{t_*} for some $t_* \in [t_1, t_2]$ which is implicitly given by the condition $\zeta_c(t_*) = 0$.

Proof. Let us consider on Σ^n the metric $\hat{g} = \frac{1}{f(h)^2} g$, which is conformal to its induced metric g. If we denote by $\hat{\Delta}$ the Laplacian with respect to the metric $\hat{g}$, from (2.4) and (2.6) we get

$$\begin{aligned}\hat{\Delta} h &= f(h)^2 \Delta h - (n-2) f(h) f'(h) |\nabla h|^2 \\ &= n f(h) f'(h) \Theta^2 + f(h) f'(h) |\nabla h|^2 + H f(h)^2 \Theta.\end{aligned} \tag{4.33}$$

With a straightforward computation, from (4.33) we obtain

$$\begin{aligned}\hat{\Delta} f(h) &= f''(h) \hat{g}(\widehat{\nabla} h, \widehat{\nabla} h) + f'(h) \hat{\Delta} h \\ &= f''(h) f(h)^2 |\nabla h|^2 + f'(h) \left(n f(h) f'(h) \Theta^2 + f(h) f'(h) |\nabla h|^2 + H f(h) \Theta \right) \\ &= n f(h) f'(h)^2 + H f'(h) f(h)^2 \Theta + f(h)^3 \left((\log f)''(h) - (n-2) \frac{f'(h)^2}{f(h)^2} \right) |\nabla h|^2.\end{aligned} \tag{4.34}$$

Given a positive real number α, we have that

$$\hat{\Delta} f(h)^{-\alpha} = \alpha(\alpha+1) f(h)^{-\alpha-2} \hat{g}(\widehat{\nabla} f(h), \widehat{\nabla} f(h)) - \alpha f(h)^{-\alpha-1} \hat{\Delta} f(h). \tag{4.35}$$

Using (4.34) in (4.35), we get

$$\begin{aligned}\hat{\Delta} f(h)^{-\alpha} &= -\alpha n f(h)^{-\alpha} f'(h)^2 - \alpha H f'(h) f(h)^{-\alpha+1} \Theta + \alpha(\alpha+1) f(h)^{-\alpha} f'(h)^2 |\nabla h|^2 \\ &\quad - \alpha f(h)^{-\alpha+2} \left((\log f)''(h) - (n-2) \frac{f'(h)^2}{f(h)^2} \right) |\nabla h|^2.\end{aligned} \tag{4.36}$$

But, from (2.4) we have

$$-\alpha n f(h)^{-\alpha} f'(h)^2 = -\alpha n f(h)^{-\alpha} f'(h)^2 |\nabla h|^2 - \alpha n f(h)^{-\alpha} f'(h)^2 \Theta^2. \tag{4.37}$$

Thus, from (4.36), (4.37), (2.7) and (4.31) we obtain

$$\begin{aligned}\hat{\Delta} f(h)^{-\alpha} &= -\alpha f(h)^{-\alpha} \bar{\zeta}_c(h) \Theta^2 \\ &\quad - \alpha f(h)^{-\alpha+2} \left\{ (\log f)''(h) - (\alpha - 1)[(\log f)'(h)]^2 \right\} |\nabla h|^2.\end{aligned} \tag{4.38}$$

First, we note that Lemma 4.4 guarantees that $(\Sigma^n, \hat{g})$ is parabolic. Moreover, it follows from (4.38) that $f(h)^{-\alpha}$ (where $\alpha = 1 + \gamma$) is subharmonic on Σ^n. Thus, since the hypothesis $\inf_\Sigma f(h) > 0$ implies that $f(h)^{-\alpha}$ is bounded from above, it follows from the parabolicity of $(\Sigma^n, \hat{g})$ that $f(h)$ is constant on Σ^n. Consequently, since we are assuming that the equality holds in (4.32) only at isolated points of I, returning to (4.38) we conclude that $|\nabla h| = 0$ on Σ^n, which means that Σ^n is a slice. $\square$

In the following we present several half-space results. More precisely, in the context of self-shrinkers, Theorem 4.3 reads as follows:

Corollary 4.12. *The only complete n-dimensional self-shrinker of $\mathbb{R}^{n+1}$ that lies in the closure of the unbounded domain determined by $\mathbb{S}^n(\sqrt{n}) \subset \mathbb{R}^{n+1}$ and such that Θ is bounded away from zero, is $\mathbb{S}^n(\sqrt{n})$.*

Taking into account once more Example 2.2, from Theorem 4.3 we get:

Corollary 4.13. *Let $\overline{M}^{n+1} = \left(0, \frac{\pi}{2}\right) \times_{\sin t} \mathbb{S}^n$ be the warped product model of $\mathbb{RP}^{n+1} \backslash \{\pi(P) \cup Cut_P\}$. Let $\psi : \Sigma^n \to \overline{M}^{n+1}$ be a complete mean curvature flow soliton with respect to $K = \sin t \partial_t$ and soliton constant $c < 0$, such that Θ is bounded away from zero. If $\cos^{-1}(\frac{\sqrt{4c^2+n^2}-n}{2|c|}) \leq h < \frac{\pi}{2}$, then Σ^n is the slice $\{\cos^{-1}(\frac{\sqrt{4c^2+n^2}-n}{2|c|})\} \times \mathbb{S}^n$.*

From Theorem 4.3 we obtain the following result:

Corollary 4.14. *Let $\overline{M}^{n+1} = I \times_{e^t} M^n$ be a pseudo-hyperbolic space whose fiber M^n is complete with parabolic universal covering. Let $\psi : \Sigma^n \to \overline{M}^{n+1}$ be a complete mean curvature flow soliton with respect to $K = e^t \partial_t$ and soliton constant $c < 0$, such that Θ is bounded away from zero. If $h \geq \log(-\frac{n}{c})$, then Σ^n is the slice $\{\log\left(-\frac{n}{c}\right)\} \times M^n$.*

In the setting of Example 2.4 and Example 2.5, we also have the following consequence of Theorem 4.3:

Corollary 4.15. *Let $\overline{M}^{n+1} = I \times_f \mathbb{S}^n$ be the Schwarzschild space and suppose that inequality (2.12) is satisfied. Let $\psi : \Sigma^n \to \overline{M}^{n+1}$ be a complete mean curvature flow soliton with respect to $K = f(t)\partial_t$ and soliton constant $c < 0$, such that Θ is bounded away from zero. If $V_{\mathfrak{m}}(r(h)) \leq \frac{c^2}{n^2} r(h)^4$ on Σ^n, then Σ^n is a slice $\{t_*\} \times \mathbb{S}^n$, where $t_* = t(r_*)$ is such that $r_* > r_0(\mathfrak{m})$ solves equation (2.11).*

and

Corollary 4.16. *Let $\overline{M}^{n+1} = I \times_f \mathbb{S}^n$ be the Reissner-Nordström space and suppose that there is $r_* > r_0(\mathfrak{m}, \mathfrak{q})$. Let $\psi : \Sigma^n \to \overline{M}^{n+1}$ be a complete mean curvature flow soliton with respect to $K = f(t)\partial_t$ and soliton constant $c < 0$, such that Θ is bounded away from zero. If $V_{\mathfrak{m},\mathfrak{q}}(r(h)) \leq \frac{c^2}{n^2} r(h)^4$ on Σ^n, then Σ^n is a slice $\{t_*\} \times \mathbb{S}^n$, where $t_* = t(r_*)$ is such that $r_* > r_0(\mathfrak{m}, \mathfrak{q})$ solves equation (2.13).*

5. Entire mean curvature flow graphs

Ecker and Huisken [20] proved that if an entire graph with polynomial volume growth is a self-shrinker, then it is necessarily a hyperplane. Later on, Wang [33] removed the condition of polynomial volume growth in Ecker-Huisken's Theorem. More recently, Colombo, Mari and Rigoli [16] extended this study to the context of entire mean curvature flow graphs in warped products. Motivated by these works, the last section of this paper is devoted to establish new Moser-Bernstein type results concerning entire graphs constructed over the fiber M^n of a warped product $\overline{M}^{n+1} = I \times_f M^n$, which are mean curvature flow solitons with respect to $K = f(t)\partial_t$ with soliton constant $c \neq 0$.

5.1. **A key nonlinear differential equation.** Let $\Omega \subseteq M^n$ be a domain. A function $u \in C^\infty(\Omega)$ such that $u(\Omega) \subseteq I$ defines a vertical graph in the warped product $\overline{M}^{n+1} = I \times_f M^n$. In such a case, $\Sigma(u)$ will denote the graph over Ω determined by u, that is,

$$\Sigma(u) = \{(u(p), p) : p \in \Omega\} \subset \overline{M}^{n+1}.$$

The graph is said to be entire if $\Omega = M^n$. Observe that $h(u(p),p) = u(p), p \in \Omega$. Hence, h and u can be identified in a natural way. The metric induced on Ω from the Riemannian metric of the ambient space via $\Sigma(u)$ is

$$g_u = du^2 + f(u)^2 g_M.$$

If M^n is complete and $\inf_M f(u) > 0$, then $\Sigma(u)$ furnished with the metric g_u is also complete. The unit vector field

$$N(p) = -\frac{f(u(p))}{\sqrt{f(u(p))^2 + |Du(p)|_M^2}} \left(\partial_t|_{(u(p),p)} - \frac{Du(p)}{f(u(p))^2} \right), \quad p \in \Omega, \tag{5.39}$$

where Du stands for the gradient of u in M and $|Du|_M = g_M(Du, Du)^{1/2}$, gives an orientation of $\Sigma(u)$ with respect to which we have $\Theta = \overline{g}(N, \partial_t) < 0$. The corresponding shape operator is given by

$$\begin{aligned} AX = &- \frac{1}{f(u)\sqrt{f(u)^2 + |Du|_M^2}} D_X Du + \frac{f'(u)}{\sqrt{f(u)^2 + |Du|_M^2}} X \\ &- \left(\frac{-g_M(D_X Du, Du)}{f(u)\left(f(u)^2 + |Du|_M^2\right)^{3/2}} - \frac{f'(u) g_M(Du, X)}{\left(f(u)^2 + |Du|_M^2\right)^{3/2}} \right) Du \end{aligned} \tag{5.40}$$

for any vector field X tangent to Ω, where D is the Levi-Civita connection in M^n.

Consequently, being $\Sigma(u)$ a vertical graph over a domain $\Omega \subseteq M^n$ and denoting by div_M the divergence operator computed in the metric g_M, it is not difficult to verify from (5.40) that the mean curvature function $H(u)$ of $\Sigma(u)$ is given by:

$$H(u) = -\operatorname{div}_M \left(\frac{Du}{f(u)\sqrt{f(u)^2 + |Du|_M^2}} \right) + \frac{f'(u)}{\sqrt{f(u)^2 + |Du|_M^2}} \left(n - \frac{|Du|_M^2}{f(u)^2} \right). \tag{5.41}$$

Hence, from (2.7) and (5.41) we have that $\Sigma(u)$ is a mean curvature flow soliton with respect to $K = f(t)\partial_t$ with soliton constant c if, and only if, u is a solution of the following nonlinear differential equation:

$$\operatorname{div}_M \left(\frac{Du}{f(u)\sqrt{f(u)^2 + |Du|_M^2}} \right) = \frac{1}{\sqrt{f(u)^2 + |Du|_M^2}} \left\{ c f(u)^2 + f'(u) \left(n - \frac{|Du|_M^2}{f(u)^2} \right) \right\}. \tag{5.42}$$

5.2. **Moser-Bernstein type results.** We say that $u \in C^\infty(M)$ has finite C^2 norm when

$$||u||_{\mathcal{C}^2(M)} := \sup_{|\gamma| \le 2} |D^\gamma u|_{L^\infty(M)} < +\infty.$$

In this context, we establish our first Moser-Bernstein type result:

Theorem 5.4. *Let $\overline{M}^{n+1} = I \times_f M^n$ be a warped product whose fiber M^n is complete with sectional curvature obeying the convergence condition (3.14). Suppose in addition that $c \neq 0$ and $\zeta_c(t) \geq 0$. If $u \in C^\infty(M)$ is an entire solution of equation (5.42), with finite C^2 norm and such that $|Du|_M \leq \mathcal{C} \inf_M |\zeta_c(u)|$ for some positive constant $\mathcal{C}$, then $u \equiv t_*$ for some $t_* \in I$ which is implicitly given by the condition $\zeta_c(t_*) = 0$.*

Proof. Let $u \in C^\infty(M)$ be such a solution of equation (5.42). It follows from (5.40) that the shape operator A of $\Sigma(u)$ is bounded, provided that u has finite C^2 norm. We note also that the finiteness of the C^2 norm of u implies, in particular, that u is bounded, which, in turn, guarantees that $\inf_M f(u) > 0$. Hence, since we are assuming that M^n is complete, we get that $(\Sigma(u), g_u)$ must be also complete.

Therefore, we can reason as in the proof of Theorem 3.1 obtaining that $\inf_M |\zeta_c(u)| = 0$ and, hence, the result follows from our constraint on $|Du|_M$. □

From the proof of Theorem 5.4 we also get the following nonexistence result:

Corollary 5.17. *Let $\overline{M}^{n+1} = I \times_f M^n$ be a warped product whose fiber M^n is complete with sectional curvature obeying the convergence condition (3.14). Suppose in addition that $c \neq 0$ and $\inf_I \zeta_c(t) > 0$. There exists no entire solution with finite C^2 norm of the equation (5.42).*

Proceeding, Theorem 4.2 allows us to obtain our next result.

Theorem 5.5. *Let $\overline{M}^{n+1} = I \times_f M^n$ be a warped product whose fiber M^n is complete. Suppose in addition that $c \neq 0$ and $\zeta_c(t)$ does not change the sign. If $u \in C^\infty(M)$ is a bounded entire solution of equation (5.42) such that $|Du|_M \in \mathcal{L}^1_{g_M}(M)$, then $u \equiv t_*$ for some $t_* \in I$ which is implicitly given by the condition $\zeta_c(t_*) = 0$.*

Proof. Let $u \in C^\infty(M)$ be such a bounded entire solution of equation (5.42). Denoting by dM and $d\Sigma$ the Riemannian volume elements of (M^n, g_M) and $(\Sigma(u), g_u)$, respectively, from [1, Equation (3.7)] we have that

$$|\nabla h| d\Sigma = f(u)^{n-1} |Du|_M dM. \tag{5.43}$$

Hence, since we are assuming that u is bounded with $|Du|_M \in \mathcal{L}^1_{g_M}(M)$, from relation (5.43) we conclude that $|\nabla h| \in \mathcal{L}^1_g(\Sigma(u))$. Therefore, the result follows by applying Theorem 4.2. □

From (5.39) we see that the assumption Θ bounded away from zero is equivalent to $|Du|_M \leq \mathcal{C} f(u)$ for some positive constant $\mathcal{C}$. So, Theorem 4.3 allows us to obtain our last Moser-Bernstein type result:

Theorem 5.6. *Let $\overline{M}^{n+1} = I \times_f M^n$ be a warped product whose fiber M^n is complete with parabolic universal covering and such that its warping function f satisfies (4.32), holding the equality only at isolated points of I. Suppose in addition that $c \neq 0$ and $\zeta_c(t) \leq 0$. If $u \in C^\infty(M)$ is a bounded entire solution of equation (5.42) such that $|Du|_M \leq \mathcal{C} f(u)$ for some positive constant $\mathcal{C}$, then $u \equiv t_*$ for some $t_* \in I$ which is implicitly given by the condition $\zeta_c(t_*) = 0$.*

Remark 5.5. Regarding all the nonexistence, rigidity and Moser-Bernstein type results which were established along our manuscript, it remains an interesting open problem to infer what is the geometric behavior of the mean curvature flow solitons in the *unbounded case*, that is, when it is not contained in a slab of the ambient space. Furthermore, it is worth noting that a natural future prospect related to our work is to extend it to the context of *multiply warped product spaces* (for details on these spaces, see [9, Section 3.6]).

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