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Central Bell-Based Type 2 Bernoulli Polynomials of Order β

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Abstract

In recent years, Hermite-based special polynomials, Bell-based special polynomials, and Laguerre-based special polynomials have been explored, and numerous properties and applications have been investigated by many mathematicians. Here, we consider the central Bell-based type 2 Bernoulli polynomials of order β that extend the concepts of central Bell polynomials and type 2 Bernoulli polynomials. Then, we derive diverse formulas, relations, and identities, such as some summation formulas, an addition formula, two partial derivative properties, a recurrence relation, two explicit formulas, and two summation formulas covering central Bell polynomials and central factorial numbers of the second kind. Moreover, we investigate an implicit summation formula for central Bell-based type 2 Bernoulli polynomials of order β utilizing some series manipulation methods. Also, we developed three useful symmetric identities for the central Bell-based type 2 Bernoulli polynomials of order β.

1. Introduction

Special numbers and polynomials are essential in many scientific fields, including mathematics, applied sciences, engineering, physics, and related research areas. These areas include functional analysis, ordinary, and partial differential equations, elementary and analytic number theory, mathematical physics, mathematical analysis, and quantum mechanics. For example, Bernoulli and Bell polynomials and numbers (cf. [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16]) have been extensively examined in diverse fields of mathematics, such as, for instance, combinatorics, numerical analysis, probability theory, quantum physics (quantum groups), homotopy theory (stable homotopy groups of spheres), p-adic analytic number theory, and differential topology (differential structures on spheres). The Bell, Bernoulli, central Bell, type 2 Bernoulli, and central factorial polynomials are among the most critical polynomials in the theory of special polynomials. Numerous physicists and mathematicians have recently extensively examined and studied the polynomials above and their various generalizations, cf. [1, 17, 18, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 19, 20, 13, 14, 15, 21, 22, 23, 24, 25, 16]

Recently, Duran et al. [4] considered the Bell-based Stirling polynomials of the second kind given by

$$\sum_{r=0}^{\infty} {}_{\phi}S_2\left(r,p:\omega,\gamma\right)\frac{z^r}{r!}=\frac{\left(e^z-1\right)^p}{p!}e^{\omega z+\gamma\left(e^z-1\right)}$$

and investigated some helpful properties and relations, covering several summation formulas associated with the Stirling numbers of the second kind and the Bell polynomials. Then, they also defined Bell-based Bernoulli polynomials of order β given by

$$\sum_{r=0}^{\infty} {}_{\phi}B_r^{(\beta)}\left(\omega;\gamma\right)\frac{z^r}{r!}=\left(\frac{z}{e^z-1}\right)^{\beta}e^{\omega z+\gamma\left(e^z-1\right)}$$

and investigated diverse formulas and correlations, covering several derivative properties, summation formulas, implicit summation formulas, and symmetric identities for Bell-based Bernoulli polynomials of order β. Moreover, they investigated

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some new formulas for these polynomials arising from umbral calculus. After considering the aforementioned study, many versions of Bell-based special polynomials have been defined, and some of their properties and applications have been investigated, cf. [3, 4, 5, 6, 9].

Inspired and motivated by the definitions of the Bell-based Stirling polynomials of the second kind and Bell-based Bernoulli polynomials of higher order by Duran et al. [4], in this work, we consider the central Bell-based type 2 Bernoulli polynomials of order β. Then, we investigate diverse relations, identities, and formulas, including five summation formulas in Theorems 2.6, 2.13, and 2.14; an addition formula in Theorem 2.7; two partial derivative properties in Theorem 2.8; a recurrence relation in Theorem 2.9, and two explicit formulas in Theorems 2.11 and 2.12. Also, we acquire an implicit summation formula in Theorem 2.15 and three symmetric identities for the central Bell-based type 2 Bernoulli polynomials of order β in Theorems 2.17, 2.18, and 2.19.

The Stirling numbers $S_2(r,q)$ of the second kind are provided as follows (*cf.* [3, 4, 9, 23]):

$$\sum_{r=0}^{\infty} S_2(r,q)\frac{z^r}{r!}=\frac{(e^z-1)^q}{q!}, \tag{1.1}$$

which means, for $r\in\mathbb{N}_0=\mathbb{N}\cup\{0\}$,

$$\omega^r=\sum_{q=0}^{r} S_2(r,q)(\omega)_q, \tag{1.2}$$

where $(\omega)_0=1$ and $(\omega)_r=\omega(\omega-1)(\omega-2)\cdots(\omega-(r-1))$ for $r\in\mathbb{N}$ (cf. [1, 17, 2, 5, 7, 8, 11, 16]).

For $q\in\mathbb{N}_0$, the central factorial polynomials $T(r,q:\omega)$ and numbers $T(r,q)$ of the second kind are defined as follows (cf. [1, 17, 18, 2, 11, 12, 19, 20, 13, 14, 15, 21, 22, 24, 25, 16]):

$$\sum_{r=0}^{\infty} T(r,q:\omega)\frac{z^r}{r!}=\frac{\left(e^{\frac{z}{2}}-e^{-\frac{z}{2}}\right)^q}{q!}e^{\omega z} \text{ and } \sum_{r=0}^{\infty} T(r,q)\frac{z^r}{r!}=\frac{\left(e^{\frac{z}{2}}-e^{-\frac{z}{2}}\right)^q}{q!}. \tag{1.3}$$

The numbers $T(r,q)$ are computed by, for $r\in\mathbb{N}_0$:

$$\omega^r=\sum_{q=0}^{r} T(r,q)\,\omega^{[q]}, \tag{1.4}$$

where the notation $\omega^{[q]}$ termed as the central factorial equals to $\omega\left(\omega+\frac{q}{2}-1\right)\left(\omega+\frac{q}{2}-2\right)\cdots\left(\omega-\frac{q}{2}+1\right)$ with $\omega^{[0]}=1$, cf. [1, 17, 18, 2, 11, 12, 19, 20, 13, 14, 15, 21, 22, 24, 25, 16]

The classical Bell polynomials $\phi_r(\omega)$ and central Bell polynomials $\phi_r^{(c)}(\omega)$ are provided by:

$$\sum_{r=0}^{\infty}\phi_r(\omega)\frac{z^r}{r!}=e^{\omega(e^z-1)} \tag{1.5}$$

and

$$\sum_{r=0}^{\infty}\phi_r^{(c)}(\omega)\frac{z^r}{r!}=e^{\omega\left(e^{\frac{z}{2}}-e^{-\frac{z}{2}}\right)}. \tag{1.6}$$

The classical Bell numbers ϕ_r (cf. [3, 4, 5, 6, 9]) and usual central Bell numbers $\phi_r^{(c)}$ (cf. [1, 2, 10, 11, 12, 14, 15, 16]) are obtained by taking $\omega=1$ in (1.5) and (1.6), that is $\phi_r(1):=\phi_r$ and $\phi_r^{(c)}(1):=\phi_r^{(c)}$, provided by

$$\sum_{r=0}^{\infty}\phi_r\frac{z^r}{r!}=e^{(e^z-1)} \text{ and } \sum_{r=0}^{\infty}\phi_r^{(c)}\frac{z^r}{r!}=e^{\left(e^{\frac{z}{2}}-e^{-\frac{z}{2}}\right)}. \tag{1.7}$$

Also, the bivariate central Bell polynomials $\phi_r^{(c)}(\omega;\gamma)$ are defined by

$$\sum_{r=0}^{\infty}\phi_r^{(c)}(\omega;\gamma)\frac{z^r}{r!}=e^{\omega z+\gamma\left(e^{\frac{z}{2}}-e^{-\frac{z}{2}}\right)}. \tag{1.8}$$

We observe from (1.1), (1.3), (1.5), and (1.6) that

$$\phi_r(\omega)=\sum_{q=0}^{r} S(r,q)\,\omega^q,$$

(cf. [3, 4, 5, 6, 9]) and

$$\phi_r^{(c)}(\omega)=\sum_{q=0}^{r}T(r,q)\,\omega^q, \tag{1.9}$$

(cf. [1, 2, 10, 11, 12, 14, 15, 16]).

The type 2 Bernoulli polynomials $b_r^{(\beta)}(\omega)$ of order β are defined by (*cf.* [7, 10, 19, 13]):

$$\sum_{r=0}^{\infty}b_r^{(\beta)}(\omega)\frac{z^r}{r!}=\left(\frac{z}{e^{\frac{z}{2}}-e^{-\frac{z}{2}}}\right)^{\beta}e^{\omega z}. \tag{1.10}$$

Setting $\omega=0$ in (1.10), we get $b_r^{(\beta)}(0):=b_r^{(\beta)}$ known as the type 2 Bernoulli numbers of order β. The numbers $b_r^{(1)}:=b_r$ and the polynomials $b_r^{(1)}(\omega):=b_r(\omega)$ are termed the classical type 2 Bernoulli numbers and polynomials, respectively. For $q\in\mathbb{N}_0$, let $S_q(r)=\sum_{\upsilon=0}^{r}\upsilon^q$ that also generated by (*cf.* [4]):

$$\frac{e^{(r+1)z}-1}{e^z-1}=\sum_{q=0}^{\infty}S_q(r)\frac{z^q}{q!}. \tag{1.11}$$

2. Central Bell-Based Type 2 Bernoulli Polynomials of Order β

In this part, we define the central Bell-based type 2 Bernoulli polynomials of higher order and analyze several relations and formulas covering addition formulas, partial derivation rules, summation formulas, and correlations with the central Bell polynomials and the central factorial numbers of the second kind.

We now define central Bell-based type 2 Bernoulli polynomials of order β as given below.

Definition 2.1. *The central Bell-based type 2 Bernoulli polynomials of order β are defined as follows:*

$$\sum_{r=0}^{\infty}{}_{CB}b_r^{(\beta)}(\omega;\gamma)\frac{z^r}{r!}=\left(\frac{z}{e^{\frac{z}{2}}-e^{-\frac{z}{2}}}\right)^{\beta}e^{\omega z+\gamma\left(e^{\frac{z}{2}}-e^{-\frac{z}{2}}\right)}. \tag{2.1}$$

Several particular circumstances of the central Bell-based type 2 Bernoulli polynomials of order β are examined as follows.

Remark 2.2. *When $\omega=0$ in (2.1), we get type 2 central Bell-Bernoulli polynomials ${}_{CB}b_r^{(\beta)}(\gamma)$ of order β, which are also novel extensions of the type 2 Bernoulli numbers of order β in (1.10), given below:*

$$\sum_{r=0}^{\infty}{}_{CB}b_r^{(\beta)}(\gamma)\frac{z^r}{r!}=\left(\frac{z}{e^{\frac{z}{2}}-e^{-\frac{z}{2}}}\right)^{\beta}e^{\gamma\left(e^{\frac{z}{2}}-e^{-\frac{z}{2}}\right)}. \tag{2.2}$$

Remark 2.3. *Upon letting $\gamma=0$ in (2.1), the central Bell-based type 2 Bernoulli polynomials ${}_{CB}b_r^{(\beta)}(\omega;\gamma)$ of order β become type 2 Bernoulli polynomials $b_r^{(\beta)}(\omega)$ of order β in (1.10).*

Remark 2.4. *When $\gamma=0$ and $\beta=1$, the polynomials ${}_{CB}b_r^{(\beta)}(\omega;\gamma)$ become the type 2 Bernoulli polynomials $b_r(\omega)$ in (1.10).*

Remark 2.5. *For $\beta=1$ in (2.1), we get ${}_{CB}b_r^{(1)}(\omega;\gamma):={}_{CB}b_r(\omega;\gamma)$ that are termed the central Bell-based type 2 Bernoulli polynomials.*

The following theorems analyze multifarious properties of ${}_{CB}b_r^{(\beta)}(\omega;\gamma)$.

Theorem 2.6. *The following equalities*

$$ {}_{CB}b_r^{(\beta)}(\omega;\gamma)=\sum_{p=0}^{r}\binom{r}{p}b_p^{(\beta)}\phi_{r-p}^{(c)}(\omega;\gamma), \tag{2.3}$$

$$ {}_{CB}b_r^{(\beta)}(\omega;\gamma)=\sum_{p=0}^{r}\binom{r}{p}b_p^{(\beta)}(\omega)\,\phi_{r-p}^{(c)}(\gamma), \tag{2.4}$$

$$ {}_{CB}b_r^{(\beta)}(\omega;\gamma)=\sum_{p=0}^{r}\binom{r}{p}{}_{CB}b_p^{(\beta)}(\gamma)\,\omega^{r-p}, \tag{2.5}$$

hold for $r\in\mathbb{N}_0$.

Proof. We compute using (1.10), (1.8), (2.1) and (2.2) that

$$\begin{aligned}\sum_{r=0}^{\infty} {}_{CB}b_r^{(\beta)}(\omega;\gamma)\frac{z^r}{r!} &= \left(\frac{z}{e^{\frac{z}{2}}-e^{-\frac{z}{2}}}\right)^{\beta} e^{\omega z+\gamma\left(e^{\frac{z}{2}}-e^{-\frac{z}{2}}\right)} \\ &= \sum_{r=0}^{\infty} b_r^{(\beta)}\frac{z^r}{r!}\sum_{r=0}^{\infty}\phi_r^{(c)}(\omega;\gamma)\frac{z^r}{r!} \\ &= \sum_{r=0}^{\infty}\left[\sum_{p=0}^{r}\binom{r}{p} b_p^{(\beta)}\phi_{r-p}^{(c)}(\omega;\gamma)\right]\frac{z^r}{r!},\end{aligned}$$

which implies the formula in (2.3). The proof of (2.4) and (2.5) can be done similarly. □

Theorem 2.7. *The following relationship*

$${}_{CB}b_r^{(\beta_1+\beta_2)}(\omega_1+\omega_2;\gamma_1+\gamma_2)=\sum_{p=0}^{r}\binom{r}{p}\,{}_{CB}b_p^{(\beta_1)}(\omega_1;\gamma_1)\,{}_{CB}b_{r-p}^{(\beta_2)}(\omega_2;\gamma_2) \tag{2.6}$$

holds for $r\in\mathbb{N}_0$.

Proof. We compute from (2.1) that

$$\begin{aligned}\sum_{r=0}^{\infty} {}_{CB}b_r^{(\beta_1+\beta_2)}(\omega_1+\omega_2;\gamma_1+\gamma_2)\frac{z^r}{r!} &= \left(\frac{z}{e^{\frac{z}{2}}-e^{-\frac{z}{2}}}\right)^{\beta_1+\beta_2} e^{(\omega_1+\omega_2)z+(\gamma_1+\gamma_2)\left(e^{\frac{z}{2}}-e^{-\frac{z}{2}}\right)} \\ &= \left(\frac{z}{e^{\frac{z}{2}}-e^{-\frac{z}{2}}}\right)^{\beta_1} e^{\omega_1 z+\gamma_1\left(e^{\frac{z}{2}}-e^{-\frac{z}{2}}\right)}\left(\frac{z}{e^{\frac{z}{2}}-e^{-\frac{z}{2}}}\right)^{\beta_2} e^{\omega_2 z+\gamma_2\left(e^{\frac{z}{2}}-e^{-\frac{z}{2}}\right)} \\ &= \sum_{r=0}^{\infty} {}_{CB}b_r^{(\beta_1)}(\omega_1;\gamma_1)\frac{z^r}{r!}\sum_{r=0}^{\infty} {}_{CB}b_r^{(\beta_2)}(\omega_2;\gamma_2)\frac{z^r}{r!} \\ &= \sum_{r=0}^{\infty}\sum_{p=0}^{r}\binom{r}{p}\,{}_{CB}b_p^{(\beta_1)}(\omega_1;\gamma_1)\,{}_{CB}b_{r-p}^{(\beta_2)}(\omega_2;\gamma_2)\frac{z^r}{r!},\end{aligned}$$

which means the claimed equality (2.6). □

Some of the particular circumstances of Theorem 2.7 are provided as follows:

$$\begin{aligned}{}_{CB}b_r^{(\beta)}(\omega+1;\gamma) &= \sum_{p=0}^{r}\binom{r}{p}\,{}_{CB}b_p^{(\beta)}(\omega;\gamma), \\ {}_{CB}b_r^{(\beta)}(\omega;\gamma+1) &= \sum_{p=0}^{r}\binom{r}{p}\,{}_{CB}b_r^{(\beta)}(\omega;\gamma_1)\,\phi_r^{(c)}, \\ {}_{CB}b_r^{(\beta_1+\beta_2)}(\omega;\gamma) &= \sum_{p=0}^{r}\binom{r}{p}\,{}_{CB}b_p^{(\beta_1)}(\omega;\gamma)\,b_{r-p}^{(\beta_2)},\end{aligned}$$

where the first formula is an extension of the formula for type 2 Bernoulli polynomials provided by (*cf.* [15]):

$$b_r(\omega+1)=\sum_{p=0}^{r}\binom{r}{p}b_p(\omega).$$

Theorem 2.8. *The difference operator formulas for* ${}_{CB}b_r^{(\beta)}(\omega;\gamma)$*:*

$$\frac{\partial}{\partial\omega}\,{}_{CB}b_r^{(\beta)}(\omega;\gamma)=r\,{}_{CB}b_{r-1}^{(\beta)}(\omega;\gamma) \tag{2.7}$$

and

$$\frac{\partial}{\partial\gamma}\,{}_{CB}b_r^{(\beta)}(\omega;\gamma)={}_{CB}b_r^{(\beta)}\left(\omega+\frac{1}{2};\gamma\right)-{}_{CB}b_r^{(\beta)}\left(\omega-\frac{1}{2};\gamma\right) \tag{2.8}$$

are valid for $r\in\mathbb{N}$.

Proof. Based on the following properties

$$\frac{\partial}{\partial\omega}e^{\omega z+\gamma\left(e^{\frac{z}{2}}-e^{-\frac{z}{2}}\right)}=ze^{\omega z+\gamma\left(e^{\frac{z}{2}}-e^{-\frac{z}{2}}\right)}\text{ and }\frac{\partial}{\partial\omega}e^{\omega z+\gamma\left(e^{\frac{z}{2}}-e^{-\frac{z}{2}}\right)}=\left(e^{\frac{z}{2}}-e^{-\frac{z}{2}}\right)e^{\omega z+\gamma\left(e^{\frac{z}{2}}-e^{-\frac{z}{2}}\right)},$$

the proofs are easily done. We omit the details. □

A recurrence relation for ${}_{CB}b_r^{(\beta)}(\omega;\gamma)$ is given as follows.

Theorem 2.9. *The following equality*

$$\begin{aligned}\phi_r^{(\beta)}(\omega;\gamma) &= \frac{{}_{CB}b_{r+1}\left(\omega+\frac{1}{2};\gamma\right)-{}_{CB}b_{r+1}\left(\omega-\frac{1}{2};\gamma\right)}{r+1} \\ &= \frac{1}{r+1}\sum_{p=0}^{r+1}\binom{r+1}{p}{}_{CB}b_{r+1-p}(\omega;\gamma)\left(\frac{1}{2^p}-\frac{1}{(-2)^p}\right)\end{aligned} \tag{2.9}$$

is valid for $r\in\mathbb{N}_0$.

Proof. Based on the following relation

$$e^{\omega z+\gamma\left(e^{\frac{z}{2}}-e^{-\frac{z}{2}}\right)} = \frac{e^{\frac{z}{2}}-e^{-\frac{z}{2}}}{z}\sum_{r=0}^{\infty}{}_{CB}b_r(\omega;\gamma)\frac{z^r}{r!},$$

the proof is readily completed, using (2.1). We omit the details. □

Remark 2.10. *The result (2.9) is a generalization of the identity for $b_r(\omega)$ provided by*

$$\omega^r = \frac{b_{r+1}\left(\omega+\frac{1}{2}\right)-b_{r+1}\left(\omega-\frac{1}{2}\right)}{r+1}.$$

A formula for ${}_{CB}b_r(\omega;\gamma)$ is given as follows.

Theorem 2.11. *The following equality*

$${}_{CB}b_r(\omega;\gamma) = r\sum_{p=0}^{\infty}\sum_{\upsilon=0}^{p-1}\frac{\gamma^p}{p!}\binom{p-1}{\upsilon}(-1)^{p-\upsilon-1}\left(\omega+\upsilon+\frac{1-p}{2}\right)^{r-1}$$

holds for $r\in\mathbb{N}_0$.

Proof. Utilizing Definition 2.1, we obtain

$$\begin{aligned}\sum_{r=0}^{\infty}{}_{CB}b_r(\omega;\gamma)\frac{z^r}{r!} &= \frac{ze^{\omega z}}{e^{\frac{z}{2}}-e^{-\frac{z}{2}}}e^{\gamma\left(e^{\frac{z}{2}}-e^{-\frac{z}{2}}\right)} \\ &= ze^{\omega z}\sum_{p=0}^{\infty}\frac{\gamma^p}{p!}\left(e^{\frac{z}{2}}-e^{-\frac{z}{2}}\right)^{p-1} \\ &= z\sum_{p=0}^{\infty}\frac{\gamma^p}{p!}\sum_{\upsilon=0}^{p-1}\binom{p-1}{\upsilon}(-1)^{p-\upsilon-1}e^{(2\upsilon-p+1+2\omega)\frac{z}{2}} \\ &= \sum_{r=0}^{\infty}\sum_{p=0}^{\infty}\sum_{\upsilon=0}^{p-1}\frac{\gamma^p}{p!}\binom{p-1}{\upsilon}(-1)^{p-\upsilon-1}\left(\omega+\upsilon+\frac{1-p}{2}\right)^r\frac{z^{r+1}}{r!},\end{aligned}$$

which means the claimed equality. □

Theorem 2.12. *The following equality*

$${}_{CB}b_r^{(q)}(\omega;\gamma) = \sum_{p=q}^{r-q}\frac{(r)_q}{(p)_q}\gamma^p T(r-q,p-q:\omega)$$

holds for $r,q\in\mathbb{N}_0$.

Proof. By means of Definition 2.1, we derive

$$\begin{aligned}\sum_{r=0}^{\infty}{}_{CB}b_r^{(q)}(\omega;\gamma)\frac{z^r}{r!} &= \left(\frac{z}{e^{\frac{z}{2}}-e^{-\frac{z}{2}}}\right)^q e^{\gamma\left(e^{\frac{z}{2}}-e^{-\frac{z}{2}}\right)}e^{\omega z} \\ &= z^q\sum_{p=0}^{\infty}\frac{\gamma^p}{(p)_q}\frac{\left(e^{\frac{z}{2}}-e^{-\frac{z}{2}}\right)^{p-q}}{(p-q)!}e^{\omega z} \\ &= z^q\sum_{p=0}^{\infty}\frac{\gamma^p}{(p)_q}\sum_{r=0}^{\infty}T(r,p-q:\omega)\frac{z^r}{r!} \\ &= \sum_{r=0}^{\infty}\sum_{p=q}^{r}\frac{\gamma^p}{(p)_q}T(r,p-q:\omega)\frac{z^{r+q}}{r!},\end{aligned}$$

which gives the claimed formula. □

The direct consequent of Theorem 2.12 is as follows:

$$ {}_{CB}b_r(\omega;\gamma) = r\sum_{p=1}^{r-1}\frac{\gamma^p}{p}T(r-1,p-q:\omega). $$

Theorem 2.13. *The following formula*

$$ \phi_r^{(c)}(\omega;\gamma) = \frac{1}{\binom{r+q}{q}}\sum_{\upsilon=0}^{r+q}\binom{r+q}{\upsilon}T(r+q-\upsilon,q)\ {}_{CB}b_{\upsilon}^{(q)}(\omega;\gamma) \tag{2.10} $$

holds for $r\in\mathbb{N}_0$ *and* $q\in\mathbb{Z}$.

Proof. Utilizing Definition 2.1 and (1.8), based on the following computations

$$ \begin{aligned} e^{\omega z+\gamma\left(e^{\frac{z}{2}}-e^{-\frac{z}{2}}\right)} &= q!z^{-q}\frac{\left(e^{\frac{z}{2}}-e^{-\frac{z}{2}}\right)^q}{q!}\sum_{r=0}^{\infty}{}_{CB}b_r^{(q)}(\omega;\gamma)\frac{z^r}{r!} \\ &= q!z^{-q}\sum_{r=0}^{\infty}T(r,q)\frac{z^r}{r!}\sum_{r=0}^{\infty}{}_{CB}b_r^{(q)}(\omega;\gamma)\frac{z^r}{r!} \\ &= q!\sum_{r=0}^{\infty}\sum_{\upsilon=0}^{r}\binom{r}{\upsilon}T(r-\upsilon,q)\ {}_{CB}b_{\upsilon}^{(q)}(\omega;\gamma)\frac{z^{r-q}}{r!}, \end{aligned} $$

the proof is completed. □

Theorem 2.14. *The following correlation*

$$ {}_{CB}b_r^{(\beta)}(\omega;\gamma) = \sum_{\upsilon=0}^{r}\sum_{p=0}^{\upsilon}\binom{r}{\upsilon}(2\omega)_p T\left(\upsilon,p:\frac{p-2\omega}{2}\right)\ {}_{CB}b_{r-\upsilon}^{(\beta)}(\gamma) \tag{2.11} $$

holds for $r\in\mathbb{N}_0$.

Proof. Utilizing Definition 2.1 and using (1.1) and (2.2), we obtain

$$ \begin{aligned} \sum_{r=0}^{\infty}{}_{CB}b_r^{(\beta)}(\omega;\gamma)\frac{z^r}{r!} &= \left(\frac{z}{e^{\frac{z}{2}}-e^{-\frac{z}{2}}}\right)^{\beta}e^{\gamma\left(e^{\frac{z}{2}}-e^{-\frac{z}{2}}\right)}\left(e^{\frac{z}{2}}-e^{-\frac{z}{2}}+e^{-\frac{z}{2}}\right)^{2\omega} \\ &= \left(\frac{z}{e^{\frac{z}{2}}-e^{-\frac{z}{2}}}\right)^{\beta}e^{\gamma\left(e^{\frac{z}{2}}-e^{-\frac{z}{2}}\right)}\sum_{p=0}^{\infty}(2\omega)_p\frac{\left(e^{\frac{z}{2}}-e^{-\frac{z}{2}}\right)^p}{p!}e^{z\frac{(p-2\omega)}{2}} \\ &= \sum_{r=0}^{\infty}{}_{CB}b_r^{(\beta)}(\gamma)\frac{z^r}{r!}\sum_{r=0}^{\infty}\sum_{p=0}^{r}(2\omega)_p T\left(r,p:\frac{p-2\omega}{2}\right)\frac{z^r}{r!} \\ &= \sum_{r=0}^{\infty}\sum_{\upsilon=0}^{r}\sum_{p=0}^{\upsilon}\binom{r}{\upsilon}(2\omega)_p T\left(\upsilon,p:\frac{p-2\omega}{2}\right)\ {}_{CB}b_{r-\upsilon}^{(\beta)}(\gamma)\frac{z^r}{r!}, \end{aligned} $$

which implies the desired equality (2.11). □

We give the following series equalities (*cf.* [4]):

$$ \sum_{r,q=0}^{\infty}f(r+q)\frac{\omega^r}{r!}\frac{\gamma^q}{q!} = \sum_{N=0}^{\infty}f(N)\frac{(\omega+\gamma)^N}{N!} \tag{2.12} $$

and

$$ \sum_{p=0}^{\infty}\sum_{\upsilon=0}^{p}A(\upsilon,p-\upsilon) = \sum_{p,\upsilon=0}^{\infty}A(\upsilon,p). \tag{2.13} $$

Theorem 2.15. *The following equality holds:*

$$ {}_{CB}b_{p+\upsilon}^{(\beta)}(\omega;\gamma) = \sum_{r,q=0}^{p,\upsilon}\binom{p}{r}\binom{\upsilon}{q}(\omega-\kappa)^{r+q}\ {}_{CB}b_{p+\upsilon-r-q}^{(\beta)}(\omega;\gamma). \tag{2.14} $$

Proof. We get by changing z by $z+u$ in (2.1) and utilizing (2.12) that

$$e^{-\kappa(z+u)}\sum_{p,\upsilon=0}^{\infty}{}_{CB}b_{p+\upsilon}^{(\beta)}(\kappa;\gamma)\frac{z^p}{p!}\frac{u^\upsilon}{\upsilon!}=\left(\frac{z+u}{e^{\frac{z+u}{2}}-e^{-\frac{z+u}{2}}}\right)^{\beta}e^{\gamma\left(e^{\frac{z+u}{2}}-e^{-\frac{z+u}{2}}\right)},$$

and

$$e^{-\omega(z+u)}\sum_{p,\upsilon=0}^{\infty}{}_{CB}b_{p+\upsilon}^{(\beta)}(\omega;\gamma)\frac{z^p}{p!}\frac{u^\upsilon}{\upsilon!}=\left(\frac{z+u}{e^{\frac{z+u}{2}}-e^{-\frac{z+u}{2}}}\right)^{\beta}e^{\gamma\left(e^{\frac{z+u}{2}}-e^{-\frac{z+u}{2}}\right)},$$

which means the following equality

$$e^{(\omega-\kappa)(z+u)}\sum_{p,\upsilon=0}^{\infty}{}_{CB}b_{p+\upsilon}^{(\beta)}(\kappa;\gamma)\frac{z^p}{p!}\frac{u^\upsilon}{\upsilon!}=\sum_{p,\upsilon=0}^{\infty}{}_{CB}b_{p+\upsilon}^{(\beta)}(\omega;\gamma)\frac{z^p}{p!}\frac{u^\upsilon}{\upsilon!}.$$

Therefore, utilizing (2.13), we have

$$\begin{aligned}\sum_{p,\upsilon=0}^{\infty}{}_{CB}b_{p+\upsilon}^{(\beta)}(\omega;\gamma)\frac{z^p}{p!}\frac{u^\upsilon}{\upsilon!} &= \sum_{r,q=0}^{\infty}(\omega-\kappa)^{r+q}\frac{z^r}{r!}\frac{u^q}{q!}\sum_{p,\upsilon=0}^{\infty}{}_{CB}b_{p+\upsilon}^{(\beta)}(\kappa;\gamma)\frac{z^p}{p!}\frac{u^\upsilon}{\upsilon!}\\ &= \sum_{p,\upsilon=0}^{\infty}\sum_{r,q=0}^{p,\upsilon}\frac{(\omega-\kappa)^{r+q}\,{}_{CB}b_{p+\upsilon-r-q}^{(\beta)}(\omega;\gamma)}{r!q!(p-\upsilon)!(\upsilon-q)!}z^pu^\upsilon,\end{aligned}$$

which means the claimed formula (2.14). □

Corollary 2.16. *Putting $\upsilon=0$ in (2.14), we have*

$${}_{CB}b_{p}^{(\beta)}(\omega;\gamma)=\sum_{r=0}^{p}\binom{p}{r}{}_{CB}b_{p-r}^{(\beta)}(\kappa;\gamma)(\omega-\kappa)^r.$$

Theorem 2.17. *The following identity holds for $a,b\in\mathbb{R}$ and $r\geq 0$:*

$$\sum_{p=0}^{r}\binom{r}{p}{}_{CB}b_{r-p}^{(\beta)}(b\omega;\gamma)\,{}_{CB}b_{p}^{(\beta)}(a\omega;\gamma)a^{r-p}b^p=\sum_{p=0}^{r}\binom{r}{p}{}_{CB}b_{p}^{(\beta)}(b\omega;\gamma)\,{}_{CB}b_{r-p}^{(\beta)}(a\omega;\gamma)a^pb^{r-p}. \tag{2.15}$$

Proof. Choose

$$\Upsilon=\left(\frac{az}{e^{\frac{az}{2}}-e^{-\frac{az}{2}}}\frac{bz}{e^{\frac{bz}{2}}-e^{-\frac{bz}{2}}}\right)^{\beta}e^{2ab\omega z+\gamma\left(e^{\frac{az}{2}}-e^{-\frac{az}{2}}\right)+\gamma\left(e^{\frac{bz}{2}}-e^{-\frac{bz}{2}}\right)}.$$

We compute two expansions of Υ:

$$\begin{aligned}\Upsilon &= \sum_{r=0}^{\infty}{}_{CB}b_{r}^{(\beta)}(b\omega;\gamma)\frac{(az)^r}{r!}\sum_{r=0}^{\infty}{}_{CB}b_{r}^{(\beta)}(a\omega;\gamma)\frac{(bz)^r}{r!}\\ &= \sum_{r=0}^{\infty}\sum_{p=0}^{r}\binom{r}{p}{}_{CB}b_{r-p}^{(\beta)}(b\omega;\gamma)\,{}_{CB}b_{p}^{(\beta)}(a\omega;\gamma)a^{r-p}b^p\frac{z^r}{r!}\end{aligned}$$

and similarly

$$\Upsilon=\sum_{r=0}^{\infty}\sum_{p=0}^{r}\binom{r}{p}{}_{CB}b_{p}^{(\beta)}(b\omega;\gamma)\,{}_{CB}b_{r-p}^{(\beta)}(a\omega;\gamma)a^pb^{r-p}\frac{z^r}{r!},$$

which means the claimed identity (2.15).

□

Here is another symmetric identity for ${}_{CB}b_{r}^{(\beta)}(\omega;\gamma)$ given below.

Theorem 2.18. *The following identity holds for $a,b\in\mathbb{R}$ and $r\geq 0$:*

$$\begin{aligned}&\sum_{p=0}^{r}\sum_{i=0}^{b-1}\sum_{j=0}^{a-1}\binom{r}{p}{}_{CB}b_{p}^{(\beta)}\left(b\omega_1-b+i+\frac{1}{2}+\frac{b}{a}\left(j+\frac{1}{2}\right);\gamma\right){}_{CB}b_{r-p}^{(\beta)}(a\omega_2;\gamma)a^pb^{r-p} \qquad (2.16)\\ =&\sum_{p=0}^{r}\sum_{i=0}^{b-1}\sum_{j=0}^{a-1}\binom{r}{p}{}_{CB}b_{r-p}^{(\beta)}(b\omega_1;\gamma){}_{CB}b_{p}^{(\beta)}\left(a\omega_2-a+j+\frac{1}{2}+\frac{a}{b}\left(i+\frac{1}{2}\right);\gamma\right)a^{r-p}b^p.\end{aligned}$$

Proof. Let

$$\begin{aligned}\Psi &= \frac{(az)^{\beta}(bz)^{\beta}}{\left(e^{\frac{az}{2}}-e^{-\frac{az}{2}}\right)^{\beta+1}\left(e^{\frac{bz}{2}}-e^{-\frac{bz}{2}}\right)^{\beta+1}}\left(e^{\frac{abz}{2}}-e^{-\frac{abz}{2}}\right)^{2}e^{ab(\omega_1+\omega_2)z+\gamma\left(e^{\frac{az}{2}}-e^{-\frac{az}{2}}\right)+\gamma\left(e^{\frac{bz}{2}}-e^{-\frac{bz}{2}}\right)}\\
&= \left(\frac{az}{e^{\frac{az}{2}}-e^{-\frac{az}{2}}}\right)^{\beta}e^{ab\omega_1 z+\gamma\left(e^{\frac{az}{2}}-e^{-\frac{az}{2}}\right)}\left(\frac{e^{\frac{abz}{2}}-e^{-\frac{abz}{2}}}{e^{\frac{az}{2}}-e^{-\frac{az}{2}}}\right)\\
&\quad\times\left(\frac{bz}{e^{\frac{bz}{2}}-e^{-\frac{bz}{2}}}\right)^{\beta}e^{ab\omega_2 z+\gamma\left(e^{\frac{bz}{2}}-e^{-\frac{bz}{2}}\right)}\left(\frac{e^{\frac{abz}{2}}-e^{-\frac{abz}{2}}}{e^{\frac{bz}{2}}-e^{-\frac{bz}{2}}}\right).\end{aligned}$$

Utilizing

$$\frac{e^{\frac{abz}{2}}-e^{-\frac{abz}{2}}}{e^{\frac{az}{2}}-e^{-\frac{az}{2}}}=\sum_{i=0}^{b-1}e^{az\left(i+\frac{1-b}{2}\right)}\text{ and }\frac{e^{\frac{abz}{2}}-e^{-\frac{abz}{2}}}{e^{\frac{bz}{2}}-e^{-\frac{bz}{2}}}=\sum_{j=0}^{a-1}e^{bz\left(j+\frac{1-a}{2}\right)},$$

we observe that

$$\begin{aligned}\Psi &= \left(\frac{az}{e^{\frac{az}{2}}-e^{-\frac{az}{2}}}\right)^{\beta}e^{ab\omega_1 z+\gamma\left(e^{\frac{az}{2}}-e^{-\frac{az}{2}}\right)}\sum_{i=0}^{b-1}e^{az\left(i+\frac{1-b}{2}\right)}\left(\frac{bz}{e^{\frac{bz}{2}}-e^{-\frac{bz}{2}}}\right)^{\beta}e^{ab\omega_2 z+\gamma\left(e^{\frac{bz}{2}}-e^{-\frac{bz}{2}}\right)}\sum_{j=0}^{a-1}e^{bz\left(j+\frac{1-a}{2}\right)}\\
&= \sum_{i=0}^{b-1}\sum_{j=0}^{a-1}\left(\frac{az}{e^{\frac{az}{2}}-e^{-\frac{az}{2}}}\right)^{\beta}e^{\left(i+\frac{b}{a}(j+1)+b\omega_1-b+\frac{1}{2}\right)az+\gamma\left(e^{\frac{az}{2}}-e^{-\frac{az}{2}}\right)}\left(\frac{bz}{e^{\frac{bz}{2}}-e^{-\frac{bz}{2}}}\right)^{\beta}e^{ab\omega_2 z+\gamma\left(e^{\frac{bz}{2}}-e^{-\frac{bz}{2}}\right)}\\
&= \sum_{r=0}^{\infty}\sum_{p=0}^{r}\sum_{i=0}^{b-1}\sum_{j=0}^{a-1}\binom{r}{p}\,{}_{CB}b_p^{(\beta)}\left(b\omega_1-b+i+\frac{1}{2}+\frac{b}{a}\left(j+\frac{1}{2}\right);\gamma\right)\,{}_{CB}b_{r-p}^{(\beta)}(a\omega_2;\gamma)\,a^p b^{r-p}\frac{z^r}{r!},\end{aligned}$$

and in the same way,

$$\Psi=\sum_{r=0}^{\infty}\sum_{p=0}^{r}\sum_{i=0}^{b-1}\sum_{j=0}^{a-1}\binom{r}{p}\,{}_{CB}b_{r-p}^{(\beta)}(b\omega_1;\gamma)\,{}_{CB}b_p^{(\beta)}\left(a\omega_2-a+j+\frac{1}{2}+\frac{a}{b}\left(i+\frac{1}{2}\right);\gamma\right)a^{r-p}b^{p}\frac{z^r}{r!},$$

which imply the desired identity (2.16).

□

Finally, we give our last symmetric identity.

Theorem 2.19. *The following equality holds for $a,b\in\mathbb{Z}$ and $r\geq 0$:*

$$\begin{aligned}&\sum_{\upsilon=0}^{r}\sum_{p=0}^{\upsilon}\binom{r}{\upsilon}\binom{\upsilon}{p}S_{r-\upsilon}(b-1)\,{}_{CB}b_p^{(\beta)}\left(b\omega_1+\frac{1-b}{2};\gamma\right){}_{CB}b_{\upsilon-p}^{(\beta+1)}(a\omega_2;\gamma)\,a^{r+p+1-\upsilon}b^{\upsilon-p}\\
&=\sum_{\upsilon=0}^{r}\sum_{p=0}^{\upsilon}\binom{r}{\upsilon}\binom{\upsilon}{p}S_{r-\upsilon}(a-1)\,{}_{CB}b_{\upsilon-p}^{(\beta+1)}(b\omega_1;\gamma)\,{}_{CB}b_{\upsilon-p}^{(\beta)}\left(a\omega_2+\frac{1-a}{2};\gamma\right)a^{\upsilon-p}b^{r+p+1-\upsilon}.\end{aligned}\tag{2.17}$$

Proof. Let

$$\Omega=\frac{(az)^{\beta+1}(bz)^{\beta+1}}{\left(e^{\frac{az}{2}}-e^{-\frac{az}{2}}\right)^{\beta+1}\left(e^{\frac{bz}{2}}-e^{-\frac{bz}{2}}\right)^{\beta+1}}\left(e^{\frac{abz}{2}}-e^{-\frac{abz}{2}}\right)e^{ab(\omega_1+\omega_2)z+\gamma\left(e^{\frac{az}{2}}-e^{-\frac{az}{2}}\right)+\gamma\left(e^{\frac{bz}{2}}-e^{-\frac{bz}{2}}\right)}.$$

By (1.11) and (2.1), we observe that

$$\begin{aligned}\Omega &= az\left(\frac{e^{\frac{abz}{2}}-e^{-\frac{abz}{2}}}{e^{\frac{az}{2}}-e^{-\frac{az}{2}}}\right)\left(\frac{az}{e^{\frac{az}{2}}-e^{-\frac{az}{2}}}\right)^{\beta}e^{ab\omega_1 z+\gamma\left(e^{\frac{az}{2}}-e^{-\frac{az}{2}}\right)}\left(\frac{bz}{e^{\frac{bz}{2}}-e^{-\frac{bz}{2}}}\right)^{\beta+1}e^{ab\omega_2 z+\gamma\left(e^{\frac{bz}{2}}-e^{-\frac{bz}{2}}\right)}\\
&= aze^{az\left(\frac{1-b}{2}\right)}\sum_{r=0}^{\infty}S_r(b-1)\frac{(az)^r}{r!}\left(\frac{az}{e^{\frac{az}{2}}-e^{-\frac{az}{2}}}\right)^{\beta}e^{ab\omega_1 z+\gamma\left(e^{\frac{az}{2}}-e^{-\frac{az}{2}}\right)}\left(\frac{bz}{e^{\frac{bz}{2}}-e^{-\frac{bz}{2}}}\right)^{\beta+1}e^{ab\omega_2 z+\gamma\left(e^{\frac{bz}{2}}-e^{-\frac{bz}{2}}\right)}\\
&= az\sum_{r=0}^{\infty}S_r(b-1)\frac{(az)^r}{r!}\sum_{r=0}^{\infty}{}_{CB}b_r^{(\beta)}\left(b\omega_1+\frac{1-b}{2};\gamma\right)\frac{(az)^r}{r!}\sum_{r=0}^{\infty}{}_{CB}b_r^{(\beta+1)}(a\omega_2;\gamma)\frac{(bz)^r}{r!}\\
&= \sum_{r=0}^{\infty}\sum_{\upsilon=0}^{r}\sum_{p=0}^{\upsilon}\binom{r}{\upsilon}\binom{\upsilon}{p}S_{r-\upsilon}(b-1)\,{}_{CB}b_p^{(\beta)}\left(b\omega_1+\frac{1-b}{2};\gamma\right){}_{CB}b_{\upsilon-p}^{(\beta+1)}(a\omega_2;\gamma)\,a^{r+p+1-\upsilon}b^{\upsilon-p}\frac{z^{r-1}}{r!},\end{aligned}$$

and also

$$\Omega=\sum_{r=0}^{\infty}\sum_{\upsilon=0}^{r}\sum_{p=0}^{\upsilon}\binom{r}{\upsilon}\binom{\upsilon}{p}S_{r-\upsilon}(a-1)\ {}_{CB}b_{\upsilon-p}^{(\beta+1)}(b\omega_1;\gamma)\ {}_{CB}b_{\upsilon-p}^{(\beta)}\left(a\omega_2+\frac{1-a}{2};\gamma\right)a^{\upsilon-p}b^{r+p+1-\upsilon}\frac{z^{r-1}}{r!},$$

which mean the asserted identity (2.17).

□

3. Conclusions

In recent years, Duran, Araci, and Acikgoz [4] considered the Bell-based Bernoulli polynomials of order β given below

$$\left(\frac{z}{e^z-1}\right)^{\beta}e^{\omega z+\gamma(e^z-1)}=\sum_{r=0}^{\infty}{}_{\phi}B_r^{(\beta)}(\omega;\gamma)\frac{z^r}{r!}$$

and derived many formulas and relations, covering several symmetric properties, derivative properties, summation formulas, and addition formulas. Inspired and motivated by the aforesaid study, in this paper, we have defined the central Bell-based type 2 Bernoulli polynomials of order β provided below

$$\left(\frac{z}{e^{\frac{z}{2}}-e^{-\frac{z}{2}}}\right)^{\beta}e^{\omega z+\gamma\left(e^{\frac{z}{2}}-e^{-\frac{z}{2}}\right)}=\sum_{r=0}^{\infty}{}_{CB}b_r^{(\beta)}(\omega;\gamma)\frac{z^r}{r!}$$

and we have derived diverse formulas and properties covering several derivative properties and summation equalities. In addition, we have obtained three symmetric identities and an implicit summation formula for the mentioned polynomials. Additionally, we have examined several particular circumstances of the obtained results that are extensions of the many previous results, some of which are included in [1, 17, 18, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 19, 20, 13, 14, 15, 21, 22, 23, 24, 25, 16]. We will consider the possibility of analyzing the polynomials discussed in this paper in the context of umbral calculus and the monomiality principle for future directions.

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Generalized Bullen Type Inequalities and Their Applications

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Abstract

This paper presents a novel extension of Bullen-type inequalities for convex functions by leveraging recently established generalized identities. Through rigorous proofs, we derive new inequalities that exhibit strong connections to both the left- and right-hand sides of the Hermite-Hadamard inequalities for Riemann-integrable functions. Additionally, we apply these results to various special means of two positive numbers.

1. Introduction

Definition 1.1. *A function* $f:[a,b]\subset\mathbb{R}\to\mathbb{R}$ *is said to be* convex *if the following inequality holds:*

$$f(\lambda x+(1-\lambda)y)\leq\lambda f(x)+(1-\lambda)f(y)$$

for all $x,y\in[a,b]$ *and* $\lambda\in[0,1]$. *The function* f *is called* concave *if* $(-f)$ *is convex.*

The theory of convex functions is a fundamental area of mathematics with applications across a wide range of fields, including optimization theory, control theory, operations research, geometry, functional analysis, and information theory. It is also highly relevant in other scientific disciplines such as economics, finance, engineering, and management sciences.

One of the most well-known results in this area is the Hermite–Hadamard integral inequality (see [1]), which serves as a fundamental tool for studying the behaviour of convex functions. This inequality has far-reaching implications and has been the subject of extensive research in recent years, giving rise to new and powerful mathematical techniques for addressing a broad spectrum of problems. The literature contains numerous extensions and refinements of this inequality (see [2]–[13]).

$$f\left(\frac{a+b}{2}\right)\leq\frac{1}{b-a}\int_a^b f(x)\,dx\leq\frac{f(a)+f(b)}{2}\tag{1.1}$$

where $f:I\subset\mathbb{R}\to\mathbb{R}$ is a convex function on an interval I and $a,b\in I$ with $a<b$.
Suppose that $f:[a,b]\to\mathbb{R}$ is convex on $[a,b]$. Then the following chain of inequalities holds:

$$\begin{aligned}f\left(\frac{a+b}{2}\right)&\leq\frac{1}{2}\left[f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)\right]\\&\leq\frac{1}{b-a}\int_a^b f(x)\,dx\\&\leq\frac{1}{2}\left[f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2}\right]\leq\frac{f(a)+f(b)}{2}.\end{aligned}\tag{1.2}$$

The third inequality in (1.2) is commonly known as *Bullen's inequality*.

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These inequalities were first introduced independently by Charles Hermite and Jacques Hadamard in the late 19th century, and they have since found numerous applications in analysis, geometry, and probability theory. The Hermite–Hadamard inequalities state that if a function is convex on a closed interval, then the average value of the function over that interval lies between its value at the midpoint and the average of its values at the endpoints.

These inequalities serve as powerful tools for estimating integrals and are foundational results in the theory of convex functions. They have been applied in a variety of contexts, including integral calculus, probability theory, statistics, optimization, and number theory. Moreover, they are instrumental in solving physical and engineering problems where determining the average value of a function is required.

Hadamard's inequality, in particular, is widely applied and carries significant geometric interpretations. Bullen's inequality, on the other hand, can be interpreted as a convex combination of the midpoint and trapezoidal rules for numerical integration. This inequality has also been extensively investigated in the literature, leading to various generalizations and a rich body of related research (see [14]–[25]).

In this paper, we present a new extension of Bullen-type inequalities for convex functions by utilizing recently established generalized identities. Through rigorous proofs, we establish inequalities that are strongly connected to both sides of the Hermite-Hadamard inequalities for Riemann-integrable functions. Furthermore, we demonstrate the applicability of these inequalities to various special means of two positive numbers.

2. Main Results

To prove our main results, we require the following lemma:

Lemma 2.1. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable function on I°, the interior of the interval I, where $a,b \in I^\circ$ with $a < b$, and suppose that $f' \in L[a,b]$. Then the following identity holds:*

$$\frac{1}{2}\int_a^b K(x,t)f''(t)\,dt = f(x) + \frac{f(a)+f(b)}{2} - \left[\frac{1}{x-a}\int_a^x f(t)\,dt + \frac{1}{b-x}\int_x^b f(t)\,dt\right] \tag{2.1}$$

where

$$K(x,t) = \begin{cases} \frac{1}{x-a}(x-t)(t-a) & \text{for } a \le t < x, \\ \\ \frac{1}{b-x}(t-x)(b-t) & \text{for } x \le t \le b. \end{cases}$$

Proof. By integration by parts, we have

$$\begin{aligned}
\int_a^b K(x,t)f''(t)\,dt &= \frac{1}{x-a}\int_a^x (x-t)(t-a)f''(t)\,dt + \frac{1}{b-x}\int_x^b (t-x)(b-t)f''(t)\,dt \\
&= \frac{1}{x-a}\left[(x-t)(t-a)f'(t)\right]_a^x - \frac{1}{x-a}\int_a^x (a+x-2t)f'(t)\,dt \\
&\quad + \frac{1}{b-x}\left[(t-x)(b-t)f'(t)\right]_x^b - \frac{1}{b-x}\int_x^b (b+x-2t)f'(t)\,dt \\
&= -\frac{1}{x-a}\left[(a+x-2t)f(t)\right]_a^x - \frac{2}{x-a}\int_a^x f(t)\,dt \\
&\quad - \frac{1}{b-x}\left[(b+x-2t)f(t)\right]_x^b - \frac{2}{b-x}\int_x^b f(t)\,dt \\
&= 2f(x)+f(a)+f(b) - \frac{2}{x-a}\int_a^x f(t)\,dt - \frac{2}{b-x}\int_x^b f(t)\,dt.
\end{aligned}$$

Multiplying both sides by $\frac{1}{2}$ yields the desired identity (2.1). □

Remark 2.2. *In Lemma 2.1, if we choose $x = \frac{a+b}{2}$, then identity (2.1) becomes:*

$$\frac{1}{2(b-a)}\int_a^b K(t)f''(t)\,dt = \frac{1}{2}\left[f\left(\frac{a+b}{2}\right) + \frac{f(a)+f(b)}{2}\right] - \frac{1}{b-a}\int_a^b f(t)\,dt$$

where

$$K(t) = \begin{cases} \left(\frac{a+b}{2} - t\right)(t-a) & \text{for } a \le t < \frac{a+b}{2}, \\ \\ \left(t - \frac{a+b}{2}\right)(b-t) & \text{for } \frac{a+b}{2} \le t \le b. \end{cases}$$

Theorem 2.3. *Under the assumptions of Lemma 2.1, if $|f''|$ is convex on $[a,b]$, then the following inequality holds:*

$$\left|f(x)+\frac{f(a)+f(b)}{2}-\left[\frac{1}{x-a}\int_a^x f(t)\,dt+\frac{1}{b-x}\int_x^b f(t)\,dt\right]\right| \tag{2.2}$$

$$\leq \frac{(x-a)^2+(b-x)^2}{24}\left|f''(x)\right|+\frac{(x-a)^2\,|f''(a)|+(b-x)^2\,|f''(b)|}{24},$$

for all $x\in(a,b)$.

Proof. Since $|f''|$ is convex on $[a,b]$, we have

$$\left|f''(t)\right|\leq\frac{t-a}{x-a}\left|f''(x)\right|+\frac{x-t}{x-a}\left|f''(a)\right|,\quad \text{for } t\in[a,x],$$

and

$$\left|f''(t)\right|\leq\frac{b-t}{b-x}\left|f''(x)\right|+\frac{t-x}{b-x}\left|f''(b)\right|,\quad \text{for } t\in[x,b].$$

Taking the absolute value of both sides in equation (2.1), and applying the convexity estimates, we obtain:

$$\begin{aligned}&\left|f(x)+\frac{f(a)+f(b)}{2}-\left[\frac{1}{x-a}\int_a^x f(t)\,dt+\frac{1}{b-x}\int_x^b f(t)\,dt\right]\right|\\ &\leq\frac{1}{2(x-a)}\int_a^x(x-t)(t-a)\left|f''(t)\right|dt+\frac{1}{2(b-x)}\int_x^b(t-x)(b-t)\left|f''(t)\right|dt\\ &\leq\frac{|f''(x)|}{2(x-a)^2}\int_a^x(x-t)(t-a)^2dt+\frac{|f''(a)|}{2(x-a)^2}\int_a^x(x-t)^2(t-a)dt\\ &\quad+\frac{|f''(x)|}{2(b-x)^2}\int_x^b(t-x)(b-t)^2dt+\frac{|f''(b)|}{2(b-x)^2}\int_x^b(t-x)^2(b-t)dt.\end{aligned}$$

Evaluating the integrals and simplifying yields:

$$\left|f(x)+\frac{f(a)+f(b)}{2}-\left[\frac{1}{x-a}\int_a^x f(t)\,dt+\frac{1}{b-x}\int_x^b f(t)\,dt\right]\right|\leq\frac{(x-a)^2+(b-x)^2}{24}\left|f''(x)\right|+\frac{(x-a)^2\,|f''(a)|+(b-x)^2\,|f''(b)|}{24}.$$

□

Remark 2.4. *If we choose $x=\frac{a+b}{2}$ in Theorem 2.3, then inequality (2.2) becomes:*

$$\begin{aligned}\left|\frac{1}{2}\left[f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2}\right]-\frac{1}{b-a}\int_a^b f(t)\,dt\right| &\leq \frac{(b-a)^2}{2}\left[\frac{1}{48}\left|f''\left(\frac{a+b}{2}\right)\right|+\frac{|f''(a)|+|f''(b)|}{96}\right]\\ &\leq \frac{(b-a)^2}{48}\left[\frac{|f''(a)|+|f''(b)|}{2}\right],\end{aligned}$$

which coincides with the result previously obtained by Sarikaya and Aktan in [23].

Theorem 2.5. *Under the assumptions of Lemma 2.1, if $|f''|^q$ is convex on $[a,b]$ for some $q>1$, then the following inequality holds:*

$$\left|f(x)+\frac{f(a)+f(b)}{2}-\left[\frac{1}{x-a}\int_a^x f(t)\,dt+\frac{1}{b-x}\int_x^b f(t)\,dt\right]\right| \tag{2.3}$$

$$\leq \frac{1}{2}\left[(x-a)^{p+1}+(b-x)^{p+1}\right]^{\frac{1}{p}}\cdot B^{\frac{1}{p}}(1+p,1+p)\cdot\left(\frac{|f''(a)|^q+|f''(b)|^q}{2}\right)^{\frac{1}{q}},$$

for all $x\in(a,b)$.

Proof. Taking the absolute value of identity (2.1) and applying Hölder's integral inequality, together with the convexity of $|f''|^q$, we obtain:

$$\begin{aligned}&\left|f(x)+\frac{f(a)+f(b)}{2}-\left[\frac{1}{x-a}\int_a^x f(t)\,dt+\frac{1}{b-x}\int_x^b f(t)\,dt\right]\right|\\ &\leq\frac{1}{2}\left(\int_a^b|K(x,t)|^p\,dt\right)^{\frac{1}{p}}\left(\int_a^b\left|f''(t)\right|^q dt\right)^{\frac{1}{q}},\quad \text{where } \frac{1}{p}+\frac{1}{q}=1.\\ &\leq\frac{1}{2}\left(\frac{1}{(x-a)^p}\int_a^x(x-t)^p(t-a)^p\,dt+\frac{1}{(b-x)^p}\int_x^b(t-x)^p(b-t)^p\,dt\right)^{\frac{1}{p}}\times\left(\int_a^b\left[\frac{t-a}{b-a}\left|f''(b)\right|^q+\frac{b-t}{b-a}\left|f''(a)\right|^q\right]dt\right)^{\frac{1}{q}}\\ &=\frac{1}{2}\left[(x-a)^{p+1}+(b-x)^{p+1}\right]^{\frac{1}{p}}B^{\frac{1}{p}}(1+p,1+p)\left(\frac{|f''(a)|^q+|f''(b)|^q}{2}\right)^{\frac{1}{q}}.\end{aligned}$$

This completes the proof. □

Corollary 2.6. *Under the assumptions of Theorem 2.5, if we take $x=\frac{a+b}{2}$ in inequality (2.3), then we obtain:*

$$\left|\frac{1}{2}\left[f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2}\right]-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \leq \frac{(b-a)^{\frac{1}{p}+1}}{8}B^{\frac{1}{p}}(1+p,1+p)\left(\frac{|f''(a)|^q+|f''(b)|^q}{2}\right)^{\frac{1}{q}}. \tag{2.4}$$

3. Applications

As in [1], we consider the means for arbitrary real numbers $a,b\in\mathbb{R}^+$ with $a\neq b$. The following classical means are defined as:

- $A(a,b)=\dfrac{a+b}{2}$ (Arithmetic Mean)
- $H(a,b)=\dfrac{2ab}{a+b}$ (Harmonic Mean)
- $K(a,b)=\sqrt{\dfrac{a^2+b^2}{2}}$ (Quadratic Mean)
- $G(a,b)=\sqrt{ab}$ (Geometric Mean)
- $L(a,b)=\dfrac{b-a}{\ln b-\ln a}$ (Logarithmic Mean)
- $I(a,b)=\dfrac{1}{e}\left(\dfrac{b^b}{a^a}\right)^{1/(b-a)}$ (Identric Mean)
- $L_n(a,b)=\left(\dfrac{b^{n+1}-a^{n+1}}{(b-a)(n+1)}\right)^{1/n}, n\in\mathbb{R}\setminus\{-1,0\}$ (Generalized Logarithmic Mean)

The following inequality among classical means is well known in the literature:

$$H(a,b)\leq G(a,b)\leq L(a,b)\leq I(a,b)\leq A(a,b).$$

Proposition 3.1. *Assume that $n>3$ and $b>a>0$. Then the following inequality holds:*

$$\left|\frac{A^n(a,b)+A(a^n,b^n)}{2}-L_n^n(a,b)\right|\leq n(n-1)\frac{(b-a)^2}{48}\cdot\frac{A^{n-2}(a,b)+A(a^{n-2},b^{n-2})}{2}\leq n(n-1)\frac{(b-a)^2}{48}A(a^{n-2},b^{n-2}).$$

Proof. The result follows from Theorem 2.3 by choosing $x=\dfrac{a+b}{2}$ and $f(t)=t^n$ for $t>0$. Then we have:

$$f''(t)=n(n-1)t^{n-2}.$$

Since

$$\left(\left|f''(t)\right|\right)''=n(n-1)(n-2)(n-3)t^{n-4},$$

it follows that $|f''(t)|=n(n-1)t^{n-2}$ is convex on $[a,b]$ for $n>3$, and the inequality is obtained directly from Theorem 2.3. □

Proposition 3.2. *Let $b>a>0$. Then the following inequality holds:*

$$\left|\frac{\ln A(a,b)+A(\ln a,\ln b)}{2}-\ln I(a,b)\right|\leq\frac{(b-a)^2}{96}\left[A^{-2}(a,b)+H^{-1}(a^2,b^2)\right]\leq\frac{(b-a)^2}{48}\cdot H^{-1}(a^2,b^2).$$

Proof. This result also follows from Theorem 2.3 with $x=\dfrac{a+b}{2}$ and $f(t)=\ln t$ for $t>0$. Then,

$$f''(t)=-\frac{1}{t^2},\quad\text{and hence}\quad\left|f''(t)\right|=\frac{1}{t^2},$$

which is convex on $[a,b]$.
From Theorem 2.3, we obtain:

$$\left|\frac{1}{2}\left[\ln\left(\frac{a+b}{2}\right)+\frac{\ln a+\ln b}{2}\right]-\frac{1}{b-a}\ln\left(\frac{b^b}{a^a}\right)+1\right|\leq\frac{(b-a)^2}{96}\left[\left(\frac{2}{a+b}\right)^{-2}+\frac{a^2+b^2}{2a^2b^2}\right]\leq\frac{(b-a)^2}{48}\cdot\frac{a^2+b^2}{2a^2b^2}.$$

The desired result follows immediately from simplification of the terms on the right-hand side. □

Proposition 3.3. *Let $b > a > 0$. Then*

$$\left|\frac{A^{-1}(a,b)+H^{-1}(a,b)}{2}-L^{-1}(a,b)\right| \leq \frac{(b-a)^2}{48}\left[A^{-3}(a,b)+H^{-1}(a^3,b^3)\right] \leq \frac{(b-a)^2}{24}H^{-1}(a^3,b^3).$$

Proof. The result is derived from Theorem 2.3 by choosing $x=\frac{a+b}{2}$ and setting $f(t)=\frac{1}{t}$ for $t>0$. Then,

$$f''(t)=\frac{2}{t^3}, \quad \text{so} \quad |f''(t)|=\frac{2}{t^3},$$

which is convex on $[a,b]$.
Thus,

$$\left|\frac{1}{2}\left[\frac{2}{a+b}+\frac{a+b}{2ab}\right]-\frac{\ln b-\ln a}{b-a}\right| \leq \frac{(b-a)^2}{48}\left[\left(\frac{a+b}{2}\right)^{-3}+\frac{a^3+b^3}{2a^3b^3}\right] \leq \frac{(b-a)^2}{24}\cdot\frac{a^3+b^3}{2a^3b^3}.$$

This completes the proof. □

Proposition 3.4. *Assume $n>2$, $q>1$, and $(n-2)q>1$ with $b>a>0$. Then*

$$\left|\frac{A^n(a,b)+A(a^n,b^n)}{2}-L_n^n(a,b)\right| \leq n(n-1)\cdot\frac{(b-a)^{\frac{1}{p}+1}}{8}\cdot B^{\frac{1}{p}}(1+p,1+p)\cdot A^{\frac{1}{q}}(a^{(n-2)q},b^{(n-2)q}).$$

Proof. This result follows from Corollary 2.6 by taking $f(t)=t^n$ for $t>0$, so that

$$f''(t)=n(n-1)t^{n-2}.$$

Then

$$\left(\left|f''(t)\right|^q\right)''=|n(n-1)|^q\cdot(n-2)q\cdot((n-2)q-1)t^{(n-2)q-2},$$

which shows that $|f''(t)|^q=|n(n-1)|^q\cdot t^{(n-2)q}$ is convex on $[a,b]$ under the assumption $(n-2)q>1$. The result then follows directly from Corollary 2.6. □

Proposition 3.5. *Let $q>1$ and $b>a>0$. Then*

$$\left|\frac{\ln A(a,b)+A(\ln a,\ln b)}{2}-\ln I(a,b)\right| \leq \frac{(b-a)^{\frac{1}{q}+1}}{8}\cdot B^{\frac{1}{q}}(1+q,1+q)\cdot H^{-\frac{1}{q}}(a^q,b^q).$$

Proof. The result follows from Corollary 2.6 by choosing $f(t)=\ln t$, so that

$$f''(t)=-\frac{1}{t^2}, \quad \text{and} \quad |f''^q=\frac{1}{t^{2q}}.$$

Since$|f''(t)|^q$ is convex on $[a,b]$ for $q>1$, the inequality

$$\left|\frac{1}{2}\left[\ln\left(\frac{a+b}{2}\right)+\frac{\ln a+\ln b}{2}\right]-\frac{1}{b-a}\ln\left(\frac{b^b}{a^a}\right)+1\right| \leq \left(\frac{a^{2q}+b^{2q}}{2a^{2q}b^{2q}}\right)^{1/q}$$

holds. The desired result is then obtained using the harmonic mean representation:

$$\left(\frac{a^{2q}+b^{2q}}{2a^{2q}b^{2q}}\right)^{1/q}=H^{-\frac{1}{q}}(a^q,b^q).$$

□

4. Conclusion

In conclusion, this paper presents novel extensions of Bullen-type inequalities and establishes their applicability to functions whose absolute value of the first derivative is convex. Our contributions build upon existing research, offering refined insights and analytical techniques that can be utilized across a broad range of mathematical and scientific problems. Future work may focus on further exploring the implications and potential applications of these extensions, which hold promise for advancing theoretical knowledge and fostering innovation in various disciplines. for future directions.

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More Efficient Solutions for Numerical Analysis of the Nonlinear Generalized Regularized Long Wave (Grlw) Using the Operator Splitting Method

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Abstract

Through the use of two numerical techniques, the purpose of this study is to examine the approximate outcomes of the (GRLW) equation. The utilized methods are the collocation method with quintic B-spline, which is based on finite elements and yields good results for nonlinear evolution equations, and the strang splitting technique, which is simple to apply, practical, and quick. In order to provide approximate solutions for the main problem, the collocation method is combined with the Strang splitting method for this study. Three examples—the formation of the Maxwellian initial condition, the interaction of two solitary waves, and a single solitary wave—are taken into consideration in order to assess the accuracy of these algorithms. To demonstrate how closely the exact solutions close to numerical results and to contrast them with other solutions in the literature, error norms, and conservation quantities are computed. Tables and graphs are used to illustrate the solutions that have generated. Based on the results obtained and the practical, easy-to-use, and current features of the methodologies, this article stands out from the rest.

1. Introduction

Analytical solutions of nonlinear evolution equations, especially those containing nonlinear terms, which play an important role in various fields of science such as physics, applied mathematics and engineering problems, may not generally be obtained. Therefore, due to the existence of limited boundary and initial conditions in obtaining analytical solutions, approximate solutions of such equations have become quite suitable for the study of physical phenomena. The regularized long wave (RLW) equation, which was first proposed by Peregrine [1] and forms the basis of the generalized regular long wave (GRLW) equation discussed in this study, is one of the significant patterns in the physics environment due to the fact that it describes phenomena with weak nonlinearity and dispersion waves. Later, the RLW equation was investigated by Benjamin *et al* [2], who disputed it as an improved pattern of the KdV equation, which describes long waves by presuming a small wave amplitude and a large wave length in nonlinear dispersion and great number of physical systems. GRLW equation is connected with the generalized Korteweg-de Vries (GKdV) equation presented as

$$U_t + \varepsilon U^p U_x + \mu U_{xxx} = 0. \tag{1.1}$$

These generally expressed equations are non-linear wave equations with $(p+1)$th non-linearity and they have solitary wave solutions with pulse-like properties. The GRLW equation designed to obtain approximate solutions in this study is described by the form

$$U_t + U_x + p(p+1)U^p U_x - \mu U_{xxt} = 0 \tag{1.2}$$

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with the initial-boundary conditions presented as follows

$$\begin{aligned} U(x,0) &= g(x), \quad x_L \leq x \leq x_R, \\ U(x_L,t) &= U(x_R,t) = 0, \\ U_x(x_L,t) &= U_x(x_R,t) = 0. \end{aligned} \tag{1.3}$$

where physical boundary conditions of this equation are expressed as $U \to 0$ when $x \to \pm\infty$ and here t and x are subscripts that indicate variations in time and space and p is a non-negative integer and μ is a positive constant. $f(x)$ refers to a localized disturbance within the range $[x_L, x_R]$, while U refers to the vertical displacement of the water surface or similar physical quantity. Many scientists have tried to obtain solutions of the (GRLW) equation numerically and analytically. Zhang [3] considered a finite difference method for the (GRLW) equation. Both Karakoç and Bhowmik [4] and Roshan [5] approximated the solutions of the equation using the Petrov–Galerkin method. The Galerkin approximation with cubic B-splines was constructed to acquire the approximate solution of the (GRLW) equation by Zeybek and Karakoç [6]. Zeybek and Karakoç [7] and Karakoç and Zeybek [8] used collocation method with the help of quintic and septic B-splines, respectively, for solitary-wave solutions of the equation. A new compact finite difference method (CFDM) was proposed by [9] for equation. Mokhtari and Mohammadi [10] utilized Sinc-collocation method to the equation. Recently, Karakoç *et al* [11] applied to the equation an exact method named Riccati–Bernoulli sub-ODE method and a numerical method named Subdomain finite element method. By taking $p = 1$ in the (GRLW) equation, the (RLW) equation, which is a special case of this equation, is obtained. Solutions to this equation have been obtained by many methods. One can easily refer to refs. [2], [12]–[26]. If $p = 2$ is taken into account in the (GRLW) equation, the (MRLW) equation, which is a special case of this equation, is gotten. The reader can examine refs. [27, 28] for the solutions of this equation, which have been obtained by many methods.

The aim of this study is to investigate approximate solutions of the equation (1.2). The GRLW equation has been previously solved by the Collocation method. However, in this article, the solutions have been obtained by combining the collocation method with the Strang splitting technique. This method is simple, practical and fast to implement, so it can be preferred more in the literature.The Strang splitting technique, which is one of the Operator splitting techniques that is very practical and produces accurate results, is used to obtain solutions. Two numerical schemes are created for the main equation via the splitting technique. These schemes are applied the collocation method with the help of quintic B-spline. The results obtained are illustrated with tables and graphs.

2. Operator Splitting Method

Operator splitting is an effective technique for solving coupled systems of partial differential equations. Because one obtains a series of equations by dividing a complex equation into simpler and easier parts. Operator splitting means that the spatial differential operator contained in the equations is divided into the sum of different sub-operators with simpler forms, so that the corresponding equations be able to solve more easily. Then, as per the procedure of the splitting technique, a series of sub-equations are solved instead of the main equation. There are operator splitting techniques that include different algorithms such as Lie-Trotter, strang and higher order splitting techniques. In this study, the second order Strang splitting technique, which is one of the easy and convenient splitting techniques used to obtain faster results, will be used. Let's consider a complex problem that has the following form.

$$\frac{dU(t)}{dt} = (\omega_1 + \omega_2)U(t), \;\; U(0) = U_0, t \in [0,T]. \tag{2.1}$$

The problem (2.1) can be split into the following subequations in one dimensional form

$$\begin{aligned} \frac{dU^*(t)}{dt} &= \omega_1 U^*(t), \;\; U^*(t_n) = U_{sp}^n = U_0 \quad , \quad t \in [t_n, t_{n+1}], \\ \frac{dU^{**}(t)}{dt} &= \omega_2 U^{**}(t), \;\; U^{**}(t_n) = U^*(t_{n+1}) \quad , \quad t \in [t_n, t_{n+1}] \end{aligned}$$

in which $U_{sp}^n = U_0$ is known and $(U_{sp})_{t_{n+1}} = U_{t_{n+1}}^{**}$ is the approximate solution at $t_n = t_{n+1}$. Here, ω_1 and ω_2 differential operators. $[0,T]$ is a time interval for arbitrary $T \geq 0$, and this interval can be divided into M subintervals $[t_n, t_{n+1}]$, $(n = 0,1,2,...,M-1)$ that satisfy the condition $0 = \leq t_0 \leq t_1 \leq t_2 ... \leq t_M = T$ and each interval is of length $\Delta t = t_{n+1} - t_n$.
Second order strang splitting technique can be presented with the following algorithm

$$\begin{aligned} \frac{dU^*(t)}{dt} &= \omega_1 U^*(t), \;\; U^*(t_n) = U^{***}(t_n) \quad , \quad t \in [t_n, t_{n+1/2}], \\ \frac{dU^{**}(t)}{dt} &= \omega_2 U^{**}(t), \;\; U^{**}(t_n) = U^*(t_{n+1/2}) \quad , \quad t \in [t_n, t_{n+1}] \\ \frac{dU^{***}(t)}{dt} &= \omega_1 U^{***}(t), \;\; U^{***}(t_{n+1/2}) = U^{**}(t_{n+1}) \quad , \quad t \in [t_{n+1/2}, t_{n+1}] \end{aligned} \tag{2.2}$$

in which $(U^*)_0 = U_0$ and $U^{***}(t_{n+1})$ are the approximate solution at $t_n = t_{n+1}$ [29, 30]. As it is known, solutions of equation (2.1) can be found over the entire time interval. However, instead of doing this, according to the procedure of this algorithm, the first equation of (2.2) is solved with half the time step, then the second equation of (2.2) is solved with the whole time step, and then the first equation of (2.2) is solved again with half the time step. Thus, the process is completed. For the solutions of (2.1) in [31], Taylor series expansion up to the first order and the second order have been used. It has been obtained that the approach has a first-order accuracy of $(O(\Delta t))$ for Lie-Trotter splitting technique and a second-order accuracy $(O(\Delta t^2))$ for Strang splitting technique.

3. The Construction of the Collocation Method

Let the solution range of the main problem $[x_L, x_R]$ be divided into N finite elements of equal length $h = x_{j+1} - x_j$ for the nodes x_j, $j = 0(1)$ such that $x_L = x_0 \leq x_1 \leq \ldots \leq x_N = x_R$. Quintic B-splines $\varphi_{-2}(x), \varphi_{-1}(x), \ldots \varphi_{N+2}(x)$ for nodes x_j can be defined on the interval $[x_L, x_R]$ as follows by [33]

$$\varphi_j(x) = \frac{1}{h^5}\begin{cases} p_0 = (x-x_{j-3})^5, & x \in [x_{j-3}, x_{j-2}] \\ p_1 = p_0 - 6(x-x_{j-2})^5, & x \in [x_{j-2}, x_{j-1}] \\ p_2 = p_1 - 6(x-x_{j-2})^5 + 15(x-x_{j-1})^5, & x \in [x_{j-1}, x_j] \\ p_3 = p_2 - 6(x-x_{j-2})^5 - 20(x-x_j)^5, & x \in [x_j, x_{j+1}] \\ p_4 = p_3 - 6(x-x_{j-2})^5 + 15(x-x_{j+1})^5, & x \in [x_{j+1}, x_{j+2}] \\ p_5 = p_4 - 6(x-x_{j-2})^5 - 6(x-x_{j+2})^5, & x \in [x_{j+2}, x_{mj3}] \\ 0, & \text{otherwise.} \end{cases} \tag{3.1}$$

The numerical solution, $U_N(x,t)$, is defined in terms of quintic B-spline functions with form:

$$U_N(x,t) = \sum_{j=-2}^{N+2} \varphi_j(x)\delta_j(t) \tag{3.2}$$

in which $\delta_j(t)$ is the unknown time-dependent quantity and it is found from the boundary and quintic B-spline collocation conditions. When written instead of B-spline functions (3.1) in the approximate function (3.2), the nodal values U_j, U_j', U_j'' are written as follows depending on $\delta_j(t)$

$$\begin{aligned} U_j &= \delta_{j-2} + 26\delta_{j-1} + 66\delta_j + 26\delta_{j+1} + \delta_{j+2}, \\ U_j' &= \frac{5}{h}(-\delta_{j-2} - 10\delta_{j-1} + 10\delta_{j+1} + \delta_{j+2}), \\ U_j'' &= \frac{20}{h^2}(\delta_{j-2} + 2\delta_{j-1} - 6\delta_j + 2\delta_{j+1} + \delta_{j+2}), \end{aligned} \tag{3.3}$$

and the variation of U with the interval $[x_j, x_{j+1}]$ can be obtained with form

$$U = \sum_{j=-2}^{N+2} \varphi_j \delta_j. \tag{3.4}$$

Now, let's split the GRLW equation as follows:

$$U_t - \mu U_{xxt} = 0, \tag{3.5}$$

$$\tag{3.6}$$

$$U_t - \mu U_{xxt} + U_x + p(p+1)U^p U_x = 0. \tag{3.7}$$

When the nodal values and space derivatives of U_j in (3.3) are used in the (3.5) and (3.7) equations, two ordinary differential equations are obtained as follows

$$\begin{aligned} \dot{\delta}_{j-2} + 26\dot{\delta}_{j-1} + 66\dot{\delta}_j + 26\dot{\delta}_{j+1} + \dot{\delta}_{j+2} - \frac{20\mu}{h^2}(\dot{\delta}_{j-2} + 2\dot{\delta}_{j-1} - 6\dot{\delta}_j + 2\dot{\delta}_{j+1} + \dot{\delta}_{j+2}) \\ + \frac{5}{h}(-\delta_{j-2} - 10\delta_{j-1} + 10\delta_{j+1} + \delta_{j+2}) = 0, \end{aligned} \tag{3.8}$$

$$\begin{aligned} \dot{\delta}_{j-2} + 26\dot{\delta}_{j-1} + 66\dot{\delta}_j + 26\dot{\delta}_{j+1} + \dot{\delta}_{j+2} - \frac{20\mu}{h^2}(\dot{\delta}_{j-2} + 2\dot{\delta}_{j-1} - 6\dot{\delta}_j + 2\dot{\delta}_{j+1} + \dot{\delta}_{j+2}) \\ + \frac{5z_j}{h}(-\delta_{j-2} - 10\delta_{j-1} + 10\delta_{j+1} + \delta_{j+2}) = 0, \end{aligned} \tag{3.9}$$

in which symbol $''.''$ is derivative according to time t and z_j is linearization operation by

$$z_j = p(p+1)(\delta_{j-2}+26\delta_{j-1}+66\delta_j+26\delta_{j+1}+\delta_{j+2})^p.$$

If it is written instead of $\dfrac{\delta_j^{n+1}+\delta_j^n}{2}$ for the quantity δ_j and $\dfrac{\delta_j^{n+1}-\delta_j^n}{\Delta t}$ for the quantity $\dot{\delta}_j$ in Eqs.(3.8) and (3.9), two numerical system presented in the following are acquired ,

$$k_1\delta_{j-2}^{n+1}+k_2\delta_{j-1}^{n+1}+k_3\delta_j^{n+1}+k_4\delta_{j+1}^{n+1}+k_5\delta_{j+2}^{n+1}=k_5\delta_{j-2}^n+k_4\delta_{j-1}^n+k_3\delta_j^n+k_2\delta_{j+1}^n+k_1\delta_{j+2}^n \tag{3.10}$$

$$l_1\delta_{j-2}^{n+1}+l_2\delta_{j-1}^{n+1}+l_3\delta_j^{n+1}+l_4\delta_{j+1}^{n+1}+l_5\delta_{j+2}^{n+1}=l_5\delta_{j-2}^n+l_4\delta_{j-1}^n+l_3\delta_j^n+l_2\delta_{j+1}^n+l_1\delta_{j+2}^n \tag{3.11}$$

in which $k_j, l_j (j=1(1)5)$,and z_j are $z_j = p(p+1)U^p$

$$k_1 = 1-\frac{20\mu}{h^2}-\frac{5\Delta t}{2h}, k_2 = 26-\frac{40\mu}{h^2}-\frac{25\Delta t}{h}, k_3 = 66+\frac{120\mu}{h^2},$$
$$k_4 = 26-\frac{40\mu}{h^2}+\frac{25\Delta t}{h}, k_5 = 1-\frac{20\mu}{h^2}+\frac{5\Delta t}{h}$$

$$l_1 = 1-\frac{20\mu}{h^2}-\frac{5z_j\Delta t}{2h}, l_2 = 26-\frac{40\mu}{h^2}-\frac{25z_j\Delta t}{h}, l_3 = 66+\frac{120\mu}{h^2},$$
$$l_4 = 26-\frac{40\mu}{h^2}+\frac{25z_j\Delta t}{h}, l_5 = 1-\frac{20\mu}{h^2}+\frac{5z_j\Delta t}{2h}.$$

Systems (3.10) and (3.11) contain unknown quantities $(N+5)$, while $(N+1)$ consist of linear equations. However, only one solution for these systems must be obtained. While doing this, since the virtual parameters are not in the solution region, these parameters are eliminated by using U and U' in Equation(3.3) and the boundary conditions $U(x_L,t)=U(x_R,t)=0$ and $U_x(x_L,t)=U_x(x_R,t)=0$. In this way, the matrix system $(N+1)$ x $(N+1)$ for the $(N+1)$ unknowns quantities is obtained for the systems (3.10) and (3.11).

The closed form of the matrix systems (3.10) and (3.11) above can be expressed as

$$A_1\delta^{n+1} = A_1^T\delta^n$$
$$B_1\lambda^{n+1} = B_1^T\lambda^n$$

for the unknown time dependent quantities $\delta^T = [\delta_0\delta_1...\delta_N]$ and $\lambda^T = [\lambda_0\lambda_1...\lambda_N]$ to be calculated and A_1 and B_1 are coefficient matrices with the form

$$A_1 = \begin{bmatrix}
\bar{k}_3 & \bar{k}_4 & \bar{k}_5 & & & & & & & & \\
k_2 & k_3 & k_4 & k_5 & & & & & & & \\
k_1 & k_2 & k_3 & k_4 & k_5 & & & & & & \\
 & & \ddots & \ddots & \ddots & \ddots & & & & & \\
 & & & k_1 & k_2 & k_3 & k_4 & & & & \\
 & & & & k_5 & 1 & & & & & \\
 & & & & \bar{k}_4 & & \bar{k}_1 & & & & \\
 & & & & \bar{k}_3 & & k_1 & \bar{k}_2 & \bar{k}_3 & & \\
 & & & & & & & k_1 & k_2 & k_3 & k_4 \\
 & & & & & & & & k_1 & \bar{k}_2 & \bar{k}_3 \\
 & & & & & & & & & \bar{k}_1 & \bar{k}_2
\end{bmatrix}$$

$$B_1 = \begin{bmatrix}
\bar{l}_3 & \bar{l}_4 & \bar{l}_5 & & & & & & & & \\
l_2 & \bar{l}_3 & l_4 & l_5 & & & & & & & \\
l_1 & l_2 & l_3 & l_4 & l_5 & & & & & & \\
 & & \ddots & \ddots & \ddots & \ddots & & & & & \\
 & & & l_1 & l_2 & l_3 & l_4 & & & & \\
 & & & & l_5 & 1 & & & & & \\
 & & & & \bar{l}_4 & & \bar{l}_1 & & & & \\
 & & & & \bar{l}_3 & & l_1 & \bar{l}_2 & \bar{l}_3 & & \\
 & & & & & & & & l_1 & l_2 & l_3 \\
 & & & & & & & & & \bar{l}_2 & \bar{l}_3 \\
 & & & & & & & & & & \bar{l}_2
\end{bmatrix}$$

$$\bar{k}_3 = \frac{165}{4}k_1 - \frac{33}{8}k_2 + k_3, \bar{k}_4 = \frac{65}{2}k_1 - \frac{9}{4}k_2 + k_4, \bar{k}_5 = \frac{9}{4}k_1 - \frac{1}{8}k_2 + k_5,$$

$$\bar{k}_2 = -\frac{33}{8}k_1 + k_2, \bar{k}_3 = -\frac{9}{4}k_1 + k_3, \bar{k}_4 = -\frac{1}{8}k_1 + k_4,$$

$$\bar{\bar{k}}_2 = -\frac{1}{8}k_5 + k_2, \bar{\bar{k}}_3 = -\frac{9}{4}k_5 + k_3, \bar{\bar{k}}_4 = -\frac{33}{8}k_5 + k_4,$$

$$\bar{\bar{k}}_1 = \frac{9}{4}k_5 - \frac{1}{8}k_4 + k_1, \bar{\bar{k}}_2 = \frac{65}{2}k_5 - \frac{9}{4}k_4 + k_2, \bar{\bar{k}}_3 = \frac{165}{4}k_5 - \frac{33}{8}k_4 + k_3,$$

$$\bar{l}_3 = \frac{165}{4}l_1 - \frac{33}{8}l_2 + l_3, \bar{l}_4 = \frac{65}{2}l_1 - \frac{9}{4}l_2 + l_4, \bar{l}_5 = \frac{9}{4}l_1 - \frac{1}{8}l_2 + l_5,$$

$$\bar{l}_2 = -\frac{33}{8}l_1 + l_2, \bar{l}_3 = -\frac{9}{4}l_1 + l_3, \bar{l}_4 = -\frac{1}{8}l_1 + l_4,$$

$$\bar{\bar{l}}_2 = -\frac{1}{8}l_5 + l_2, \bar{\bar{l}}_3 = -\frac{9}{4}l_5 + l_3, \bar{\bar{l}}_4 = -\frac{33}{8}l_5 + l_4,$$

$$\bar{\bar{l}}_1 = \frac{9}{4}l_5 - \frac{1}{8}l_4 + l_1, \bar{\bar{l}}_2 = \frac{65}{2}l_5 - \frac{9}{4}l_4 + l_2, \bar{\bar{l}}_3 = \frac{165}{4}l_5 - \frac{33}{8}l_4 + l_3.$$

In order to produce more attractive, effective and accurate results for each time step, the internal iteration formula presented as follows is applied 3 or 5 times to z_j in Eq.(3.11)

$$(\delta^*)^n = \delta^n + \frac{1}{2}(\delta^n - \delta^{n-1}).$$

4. The Initial Vector δ_j^0

To start the iteration process for the systems (3.10) and (3.11), it is necessary to determine the initial vector δ_j^0. For this, initial parameters are computeded utilizing initial condition $U(x_j,0) = U_N(x_j,0) = g_0(x_j), j = 0(1)N$ and 1st and 2nd order derivatives on the boundaries presented with the main problem. In other words, these vectors to be calculated are computed from the system of algebraic equations presented as follows

$$\begin{aligned} \delta_{m-2}^0 + 26\delta_{m-1}^0 + 66\delta_m^0 + 26\delta_{m+1}^0 + \delta_{m+2}^0 &= g_0(x_m), m = 0(1)N \\ -\delta_{-2}^0 - 10\delta_{-1}^0 + 10\delta_1^0 + \delta_2^0 &= g_0'(x_L), \\ \delta_{-2}^0 + 2\delta_{-1}^0 - 6\delta_0^0 + 2\delta_1^0 + \delta_2^0 &= g_0''(x_L), \\ \delta_{N-2}^0 + 2\delta_{N-1}^0 - 6\delta_N^0 + 2\delta_{N+1}^0 + \delta_{N+2}^0 &= g_0''(x_R), \\ -\delta_{N-2}^0 - 10\delta_{N-1}^0 + 10\delta_{N+1}^0 + \delta_{N+2}^0 &= g_0'(x_R). \end{aligned} \tag{4.1}$$

In conclusion, the matrix equation for the initial vector δ^0 is acquired by

$$\begin{bmatrix} 54 & 60 & 6 & & & & \\ 25.25 & 67.5 & 26.25 & 1 & & & \\ 1 & 26 & 66 & 26 & 1 & & \\ & & & \ddots & & & \\ & & 1 & 26 & 66 & 26 & 1 \\ & & & 1 & 26.25 & 67.5 & 25.25 \\ & & & & 6 & 60 & 54 \end{bmatrix} \begin{bmatrix} {\delta^0}_0 \\ {\delta^0}_1 \\ {\delta^0}_2 \\ . \\ . \\ . \\ {\delta^0}_{N-2} \\ {\delta^0}_{N-1} \\ {\delta^0}_N \end{bmatrix} = \begin{bmatrix} U_0 \\ U_1 \\ U_2 \\ . \\ . \\ . \\ U_{N-2} \\ U_{N-1} \\ U_N \end{bmatrix}.$$

With the current symbolic programming languages, calculating such matrices is fairly simple and useful. These features of the schemes that are being presented are indicative of their dependability and resilience.

5. Stability Analysis of Numerical Algorithm

Von Neumann theory is used to analyze the stability of the Strang splitting method applied to the GRLW equation. Let the growth factors of a typical Fourier mode be described as follows for stability analysis based on Von Neumann theory of systems (3.10) and (3.11)

$$\delta_j^n = \rho_1^n e^{ij\gamma h}, \tag{5.1}$$

$$\Psi_j^n = \rho_2^n e^{ij\gamma h}. \tag{5.2}$$

Here, γ represents the mode number and h denotes the element size. The Fourier mode (5.1) is substituted for (3.10) and the Fourier mode (5.2) is substituted for (3.11). The Fourier mode method cannot be applied to the system (3.11) because it contains a nonlinear term $p(p+1)U^pU_x$. Instead, the system must first be linearized and then the Von Neumann method is applied, assuming that the amount of $p(p+1)U^p$ in the nonlinear term is taken as a local constant like z_j. One of the most popular methods for analyzing the stability analysis of approximation systems for linear or linearized partial differential equations is Von Neumann analysis. Using the Euler formula $e^{i\Phi} = cos\Phi + isin\Phi$, the following growth factors are obtained: ρ_1 and ρ_2

$$\rho_1 = \frac{A_1 - iB_1}{A_1 + iB_1}, \quad \rho_2 = \frac{A_1 - iC_1}{A_1 + iC_1}, \tag{5.3}$$

$$A_1 = \left(2 - \frac{40\mu}{h^2}\right) cos(2\gamma h) + \left(52 - \frac{80\mu}{h^2}\right) cos(\gamma h) + \left(66 + \frac{120\mu}{h^2}\right),$$

$$B = \frac{5\Delta t}{h} sin(2\gamma h) + \frac{50\Delta t}{h} sin(\gamma h),$$

and

$$C = \frac{5z_m\Delta t}{h} sin(2\gamma h) + \frac{50z_m\Delta t}{h} sin(\gamma h).$$

For $k_1, k_2, ..., k_9, k_{10}$ and $l_1, l_2, ..., l_9, l_{10}$ founded in section 3. It can be written $|\rho_1|.|\rho_2| = 1$. For the entire system with the Strang Splitting algorithm because $|\rho_1| \leq 1$, and $|\rho_2| \leq 1$ according to the von Neumann theory, which are satisfied. This makes it obvious that the systems (3.10) and (3.11) are unconditionally stable. Equation (5.3) yields $|\rho_1| = |\rho_2| = 1$, which explains this.

6. Numerical Experiments and Discussion

The error norms L_2 and L_∞ to demonstrate the perfection of numerical schemes in terms of accuracy and at the same time, invariants I_1, I_2 and I_3 such as mass, momentum and energy are examined to report how well numerical schemes preserve physical quantities. These are given in the following format

$$L_2 = ||U - U_N||_2 = \sqrt{h \sum_{j=0}^{N} (U - U_N)^2},$$

$$L_\infty = ||U - U_N||_\infty = \max_j |U - U_N|,$$

$$I_1 = \int_{x_L}^{x_R} U dx,$$
$$I_2 = \int_{x_L}^{x_R} [U^2 + \mu(U_x)^2] dx,$$
$$I_3 = \int_{x_L}^{x_R} [U^4 - \mu(U_x)^2] dx.$$

The analytical solution of the GRLW equation is presented as follows in [7]

$$U(x,t) = \left(\frac{c(p+2)}{2p} sech^2 \left[\frac{p}{2}\sqrt{\frac{c}{\mu(c+1)}}(x - (c+1)t - x_0)\right]\right)^{1/p}$$

in which $\frac{c(p+2)}{2p}$ is the amplitude, $c+1$ is the wave speed in the direction diffusion and x_0 is an arbitrary constant. In this study, it would be good to mention that these calculations are obtained for the problems of single solitary wave and intersection of two solitary waves and the growth of the Maxwellian initial condition.

Table 1: The error norms at $t = 20$ for $\mu = 1$ of the single solitary wave

		$p=2$			$p=3$			$p=4$		
$c \rightarrow$		0.03	0.1	0.3	0.03	0.1	0.3	0.03	0.1	0.3
$amp. \rightarrow$		0.17	0.31	0.54	0.29	0.43	0.62	0.38	0.52	0.68
h	Δt									
L_2x10^3										
0.1	0.01	0.99567	0.01186	0.00785	1.33411	0.01308	0.02711	1.57429	0.01429	0.07221
0.2	0.01	0.87463	0.01082	0.00954	1.17194	0.01199	0.03564	1.38292	0.01357	0.10329
0.1	0.025	0.99567	0.01240	0.04850	1.33411	0.01588	0.16665	1.57430	0.02532	0.44136
0.2	0.025	0.87463	0.01143	0.05016	1.17194	0.01521	0.17511	1.38292	0.02592	0.47230
L_∞x10^3										
0.1	0.01	0.41622	0.00668	0.00353	0.55769	0.00732	0.01290	0.65810	0.00782	0.03584
0.2	0.01	0.41622	0.00668	0.00446	0.55769	0.00732	0.01750	0.65810	0.00782	0.05238
0.1	0.025	0.41622	0.00668	0.02178	0.55769	0.00732	0.07910	0.65810	0.00908	0.21863
0.2	0.025	0.41622	0.00668	0.02268	0.55769	0.00732	0.08369	0.65810	0.00980	0.23440
		$p=6$			$p=8$			$p=10$		
$c \rightarrow$		0.03	0.1	0.3	0.03	0.1	0.3	0.03	0.1	0.3
$amp. \rightarrow$		0.17	0.31	0.54	0.29	0.43	0.62	0.38	0.52	0.68
h	Δt									
L_2x10^3										
0.1	0.01	1.88672	0.02196	0.33489	2.07926	0.06272	1.21542	2.20933	0.20864	4.16765
0.2	0.01	1.65737	0.03011	0.58326	1.82651	0.11858	2.68428	1.94078	0.46826	1.21572
0.1	0.025	1.88673	0.09767	2.02469	2.07930	0.36196	7.39167	2.20953	1.22335	27.2734
0.2	0.025	1.65738	0.10812	2.27201	1.82656	0.41870	8.84774	1.94105	1.48236	35.0904
L_∞x10^3										
0.1	0.01	0.78870	0.00848	0.17647	0.86919	0.02863	0.66880	0.92356	0.10223	2.38050
0.2	0.01	0.78870	0.01243	0.31129	0.86919	0.05636	1.48646	0.92356	0.23179	6.92021
0.1	0.025	0.78870	0.04350	1.06550	0.86919	0.170238	4.06219	0.92356	0.60052	15.5740
0.2	0.025	0.78870	0.04869	1.19985	0.86919	0.19792	4.87227	0.92356	0.72958	19.9888

6.1. First example: A single solitary wave

In the first example, to compare numerical solutions, the parameters in the studies [5, 8, 27, 28, 11, 4, 32, 3, 6, 7] are taken into consideration. As in these studies, the solution region $[0, 100]]$,and $x_0 = 40$, $\mu = 1$ are selected. Calculations are performed for different values $h, \Delta t, p$ and c until time $t = 20$. First, for different values of $\Delta t, h$ and p , the situation with solitary waves with amplitudes of $0.17, 0.31$ and 0.54 for speeds $c = 0.03, 0.1$ and 0.3, respectively, is considered and the solutions are found at time $t = 20$. The results of the error norms L_2 and L_∞ that provide the solutions are depicted in Table 1. This table shows that the error norms L_2 and L_∞ produce results that are as small as intended. Secondly, conservation constants and error norms are calculated at $t = 10$ with different values of $\Delta t, h$ and c for $p = 2, 3$ and 4. The data of these calculations are depicted in Tables $2, 3, 5, 7, 9$ and 11 and based on the results, it is concluded that the conservation quantities are well preserved and the error norms are small enough. Thirdly, the datas of conservation quantities I_1, I_2 and I_3 and error norms L_2 and L_∞ in Tables $3, 5, 7, 9$ and 11 are compared with those obtained by different methods in the literature. The results of the comparison are listed in Tables $4, 6, 8, 10$ and 12. It can be seen from these tables that the solutions obtained thanks to the collocation method combined with the Strang splitting algorithm proposed in this study are as perfect as they are promising. Figure 1 shows the motion of single solitary wave at various times t and with different values of p. From this figure, it is possible to see that the solitary wave, traveling at a constant speed, moves towards the right and still maintains its shape and and increases the energy of this wave with increasing p values.

6.2. Second example: The interaction of two solitary waves

In the second example, the GRLW equation with initial condition presented in the following form, written as the linear sum of two well-separated solitary waves traveling in the same direction and having different amplitudes, is targeted. Numerical calculations are performed with conditions $\Delta t = 0.025, h = 0.2, c_1 = 4, c_2 = 1, x_1 = 25, x_2 = 55, \mu = 1$ for $p = 2$ on the region $[0, 250]$ at $t = 0(4)20$, $\Delta t = 0.01, h = 0.1, c_1 = 48/5, c_2 = 6/5, x_1 = 20, x_2 = 50, \mu = 1$ for $p = 3$ on the region $[0, 120]$ at $t = 0(1)6$, and $\Delta t = 0.01, h = 0.125, c_1 = 64/3, c_2 = 4/3, x_1 = 20, x_2 = 80, \mu = 1$ for $p = 4$ on the region $[0, 200]$ at $t = 0(1)6$. For this purpose, conservation quantities are computed. The solutions of all calculations are reported in Tables 13-15, comparing with those in [7]. As can be observed from these tables, the conservation quantities calculated with the collocation

Table 2: Invariants and errors for single solitary wave with $\Delta t = 0.01, h = 0.1$, $\mu = 1$, and $c = 0.3$ on the region $[0,100]$ for $p = 2$

t	I_1	I_2	I_3	L_2	L_∞
0	3.58196673	1.34507649	0.15372303	0.000000000000	0.000000000000
2	3.58196673	1.34507640	0.15372312	0.000001861902	0.000001007019
4	3.58196673	1.34507629	0.15372323	0.000003499876	0.000001725688
6	3.58196673	1.34507621	0.15372331	0.000005003949	0.000002352101
8	3.58196673	1.34507616	0.15372336	0.000006443838	0.000002949496
10	3.58196673	1.34507612	0.15372340	0.000007851673	0.000003535734

Table 3: Invariants and errors for single solitary wave with $\Delta t = 0.025, h = 0.2$, $\mu = 1$, and $c = 1$ on the region $[0,100]$ from 0 to 10 in increments of 2 for $p = 2$

t	I_1	I_2	I_3	L_2	L_∞
0	4.44288294	3.29983161	1.41421360	0.0000000000	0.0000000000
2	4.44288294	3.29979589	1.41424932	0.0002939377	0.0001776334
4	4.44288294	3.29977191	1.41427330	0.0005531879	0.0003079917
6	4.44288294	3.29976284	1.41428237	0.0007998396	0.0004328585
8	4.44288294	3.29975926	1.41428595	0.00010430543	0.0005568057
10	4.44288294	3.29975778	1.41428743	0.0012853426	0.0006805326

Table 4: The error norms and invariants of the single solitary wave with $\Delta t = 0.025, h = 0.2$, $\mu = 1$, and $c = 1$ on the region $[0,100]$ for $p = 2$ at $t = 10$

method	I_1	I_2	I_3	L_2	L_∞
present	4.4428829	3.2997577	1.41428743	0.0012853426	0.0006805326
[11]	4.4428679	3.2998244	1.4142061	0.009619	0.004971
[5]	4.44288	3.29981	1.41416	0.00300533	0.00168749
[8]first approach	4.442866	3.299822	1.414204	0.002632463	0.001393064
[8]second approach	4.442866	3.299715	1.414312	0.002571481	0.001340210
[6]	4.4431	3.3003	1.4146	0.0024175	0.0010809
[27] B-spline coll-CN	4.442	3.299	1.413	0.01639	0.00924
[27] B-spline coll + PA-CN	4.440	3.296	1.411	0.0203	0.0112
[28]	4.44288	3.29983	1.41420	0.00930196	0.00543718
[7]	4.4428	3.2997	1.4143	0.0025893	0.0013518
[4]	4.443175	3.300302	1.414692	0.002415468	0.001079686
[32]	4.4431	3.3003	1.4146	0.0024155	0.0010797
[10]	4.4428	3.2998	1.4141	0.0030053	0.0016874

Table 5: Invariants and errors for single solitary wave with $\Delta t = 0.01, h = 0.1$, $\mu = 1$, and $c = 0.3$ on the region $[0,100]$ from 0 to 10 in increments of 2 for $p = 3$

t	I_1	I_2	I_3	L_2	L_∞
0	3.67755181	1.56574088	0.22683850	0.00000000	0.00000000
2	3.67755181	1.56574072	0.22683866	0.00000297	0.00000183
4	3.67755181	1.56574048	0.22683891	0.00000581	0.00000324
6	3.67755181	1.56574027	0.22683912	0.00000852	0.00000449
8	3.67755181	1.56574010	0.22683928	0.00001119	0.00000570
10	3.67755181	1.56573997	0.22683942	0.00001383	0.00000689

Table 6: Comparation of invariants and errors for single solitary wave with $\Delta t = 0.01, h = 0.1$, $\mu = 1$, and $c = 0.3$ on the region $[0,100]$ for $p = 3$

method	I_1	I_2	I_3	L_2	L_∞
method	3.67755181	1.56573997	0.22683942	0.0000138	0.00000689
[5]	3.67755000	1.56574000	0.22683700	0.0000719	0.0000377
[8]second approach	3.67760690	1.56576200	0.22684460	0.0000785	0.0000365
[6]	3.6776	1.5657	0.2268	0.0001913	0.0000779

Table 7: Invariants and errors for single solitary wave with $\Delta t = 0.01, h = 0.1$, $\mu = 1$, and $c = 0.3$ on the region $[0,100]$ from 0 to 10 in increments of 2 for $p = 4$

t	I_1	I_2	I_3	L_2	L_∞
0	3.7592300	1.7300029	0.2894090	0.0000000	0.0000000
2	3.7592300	1.7300024	0.2894095	0.0000069	0.0000044
4	3.7592300	1.7300017	0.2894101	0.0000138	0.0000078
6	3.75923000	1.7300012	0.2894107	0.0000206	0.0000111
8	3.75923000	1.7300008	0.2894111	0.0000276	0.0000144
10	3.75923000	1.7300004	0.2894114	0.0000347	0.0000178

Table 8: Comparation of invariants and errors for single solitary wave with $\Delta t = 0.01, h = 0.1$, $\mu = 1$, and $c = 0.3$ on the region $[0,100]$ for $p = 4$

method	I_1	I_2	I_3	L_2	L_∞
method	3.7592300	1.7300004	0.2894114	0.0000347	0.0000178
[5]	3.7592300	1.7299900	0.2894060	0.0001225	0.0000662
[8]second approach	3.7592863	1.7300259	0.2894169	0.0000980	0.0000480
[6]	3.7592	1.7300	0.2894	0.0003089	0.0001444

Table 9: Invariants and errors for single solitary wave with $\Delta t = 0.025, h = 0.1$, $\mu = 1$, and $c = 6/5$ on the region $[0,100]$ from 0 to 10 in increments of 2 for $p = 3$

t	I_1	I_2	I_3	L_2	L_∞
0	3.79712709	2.88122489	0.97293454	0.000000000	0.000000000
2	3.79712709	2.88110865	0.97305079	0.000117031	0.000719921
4	3.79712709	2.88105895	0.97310049	0.000227035	0.000133748
6	3.79712709	2.88104403	0.97311540	0.000033573	0.000195366
8	3.79712709	2.88103884	0.973120604	0.000444397	0.000257147
10	3.79712709	2.88103667	0.97312277	0.000553257	0.000319084

Table 10: Comparation of invariants and errors for the single solitary wave with $\Delta t = 0.025, h = 0.1$, $\mu = 1$, and $c = 6/5$ on the region $[0,100]$ at $t = 10$ for $p = 3$

method	I_1	I_2	I_3	L_2	L_∞
present	3.797127	2.881036	0.973122	0.005532	0.003190
[11]	3.797185	2.881252	0.973157	0.011026	0.006355
[5]	3.79713	2.88123	0.972243	0.007767	0.004708
[8]first approach	3.797185	2.881252	0.973145	0.008972	0.005175
[8]second approach	3.797133	2.881089	0.973128	0.007778	0.004441
[6]	3.801670	2.888066	0.979294	0.013291	0.008478
[4]	3.797282	2.881293	0..973446	0.006128	0.003722

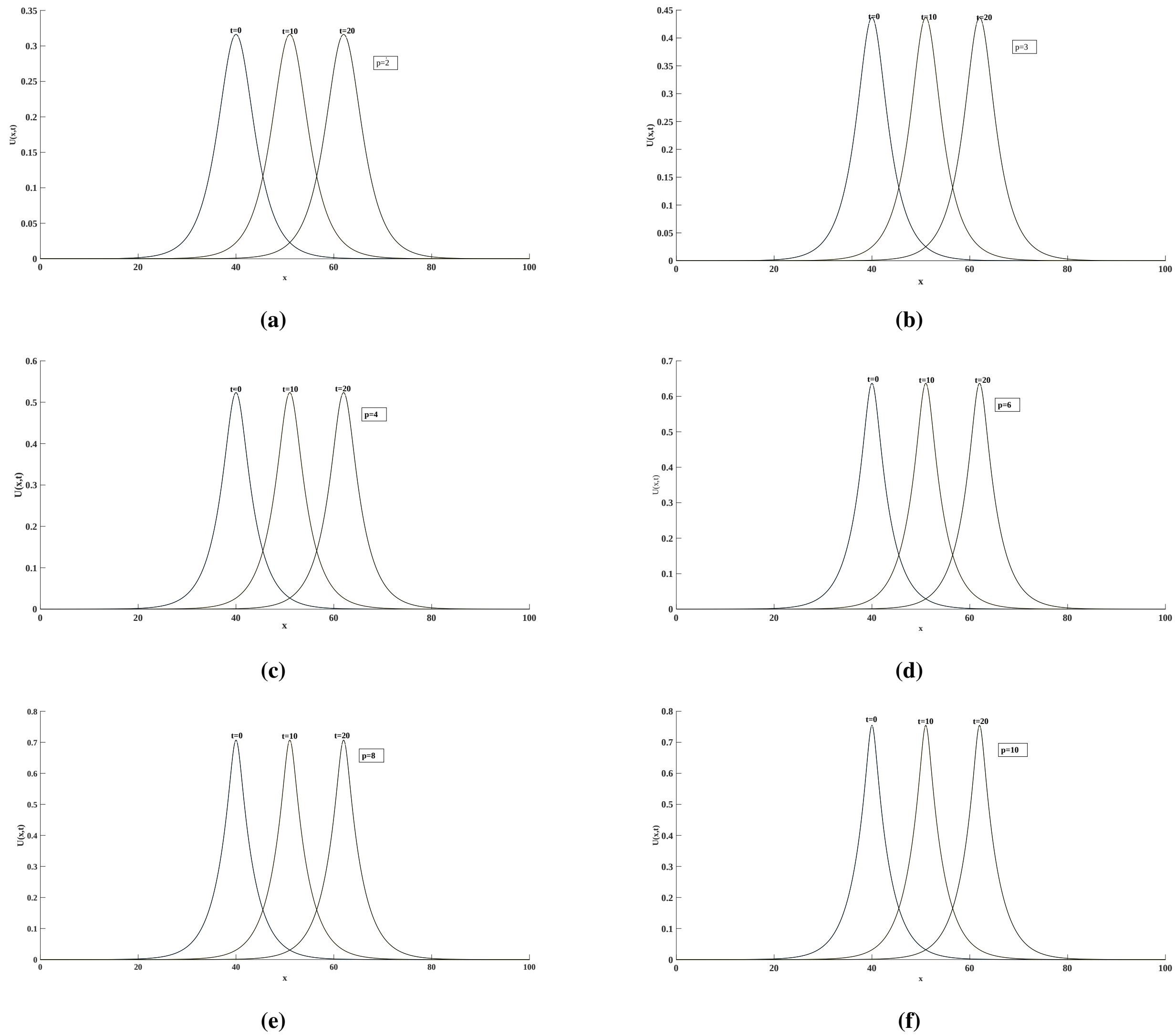

Figure 1: A single solitary wave movement at $[0,100]$ for $c = 0,1$ and $x_0 = 40$

Table 11: Invariants and errors for single solitary wave with $\Delta t = 0.01, h = 0.1$, $\mu = 1$, and $c = 4/3$ on the region $[0,100]$ from 0 to 10 in increments of 2 for $p = 4$

t	I_1	I_2	I_3	L_2	L_∞
0	3.46865611	2.67167341	0.72917047	0.000000000	0.000000000
2	3.46865611	2.67163139	0.72921249	0.000481813	0.000307699
4	3.46865611	2.67161752	0.72922636	0.000949370	0.000588519
6	3.46865611	2.67161379	0.72923009	0.000141623	0.000872966
8	3.46865611	2.67161260	0.72923128	0.000188397	0.000115795
10	3.46865611	2.67161212	0.72923176	0.000235269	0.000144089

Table 12: Comparation of invariants and errors for the single solitary wave with $\Delta t = 0.01, h = 0.1$, $\mu = 1$, and $c = 4/3$ on the region $[0,100]$ at $t = 10$ for $p = 4$

method	I_1	I_2	I_3	L_2	L_∞
present	3.46865611	2.67161212	0.72923176	0.002352	0.001440
[11]	3.468709	2.671696	0.729303	0.008696	0.005314
[5]	3.46866	2.67168	0.728881	0.002460	0.001566
[8]first approach	3.468709	2.671696	0.729258	0.003351	0.002049
[8]second approach	3.468671	2.671658	0.729237	0.002698	0.001656
[6]	3.470439	2.674445	0.731987	0.001511	0.000857
[4]	3.468799	2.671742	0.730001	0.001283	0.000821

method combined with the Strang splitting algorithm are compatible with those in ref.[7] presented with the quintic B-spline collocation method. Figures 2 and 3 depict the action of interaction of two solitary waves for various times. It can be clearly seen from these figures that at $t = 0$, the wave with lower energy is located to the right of the wave with larger energy. Later, the wave with greater energy catches up with the smaller one and leaves it behind.

6.3. Last example: The Maxwellian initial condition

In the last example, the problem of how the Maxwell pulse presented as follows, which appears as the initial condition, turns into a solitary waves is examined.

$$U(x,0) = exp(-(x-40)^2).$$

Here, the value of μ determines how the solution behaves [4]. As a result, for $p = 2,3,4$, with values of $\mu = 0.025, 0.05$, and $\mu = 0.1$, numerical calculations are completed until time $t = 0.05$. Table 16 displays the calculated numerical invariants at various t values and this table shows that the invariants are quite compatible among themselves. Figure 4 illustrates how the Maxwellian initial condition developed into solitary waves.

Table 13: Comparison of invariants of two solitary waves with values $\Delta t = 0.025, h = 0.2$, for $x_1 = 25, x_2 = 55, c_1 = 4, c_2 = 1$ on the region $[0,250]$ at $t = 0(4)20$ for $p = 2$ with those in [7]

	method			[7]		
t	I_1	I_2	I_3	I_1	I_2	I_3
0	11.46769767	14.62924187	22.88046714	11.4676	14.6292	22.8803
4	11.46769767	14.62560599	22.88410302	11.4676	14.6277	22.8818
8	11.46769767	14.13410877	23.37560024	11.4676	14.1399	23.3695
12	11.46769767	14.67865616	22.83105285	11.4676	14.6803	22.8292
16	11.46769767	14.64185929	22.86784972	11.4676	14.6442	22.8653
20	11.46769767	14.62835609	22.88135292	11.4676	14.6309	22.8786

Table 14: Comparison of invariants of two solitary waves with values $\Delta t = 0.01, h = 0.1$, for $x_1 = 20, x_2 = 50, c_1 = 48/5, c_2 = 6/5$ on the region $[0,120]$ at $t = 0(1)6$ for $p = 3$ with those in [7]

	method			[7]		
t	I_1	I_2	I_3	I_1	I_2	I_3
0	9.69074161	12.94438041	17.01872563	9.6907	12.9443	17.0186
1	9.69074161	12.93790956	17.02519648	9.6894	12.9433	17.0197
2	9.69074161	12.93262436	17.03048168	9.6881	12.9391	17.0239
3	9.69074161	12.31072166	17.65238437	9.6851	12.3044	17.6586
4	9.69074161	12.96109129	17.00201474	9.6860	12.9704	16.9926
5	9.69074161	13.04585327	16.91725276	9.6848	13.0539	16.9091
6	9.69074161	12.99335590	16.96975014	9.6835	13.0028	16.9601

Table 15: Comparison of invariants of two solitary waves with values $\Delta t = 0.01, h = 0.125$, for $x_1 = 20, x_2 = 50, c_1 = 64/3, c_2 = 4/3$ on the region $[0,200]$ at $t = 0(1)6$ for $p = 4$ with those in [7]

	method			[7]		
t	I_1	I_2	I_3	I_1	I_2	I_3
0	8.83427261	12.17088582	14.02942463	8.8342	12.1708	14.0294
1	8.83427261	11.47188138	14.72842908	8.6650	11.9332	14.2670
2	8.83427261	11.33376433	14.86654612	8.5662	11.7919	14.4083
3	8.83427261	11.25540256	14.94490789	8.4965	11.6913	14.5090
4	8.83427261	11.20082492	14.99948554	8.4529	11.4644	14.7358
5	8.83427261	11.08672895	14.97358150	8.4089	11.7254	14.4748
6	8.83427261	11.00520465	14.98510581	8.3702	11.5990	14.6012

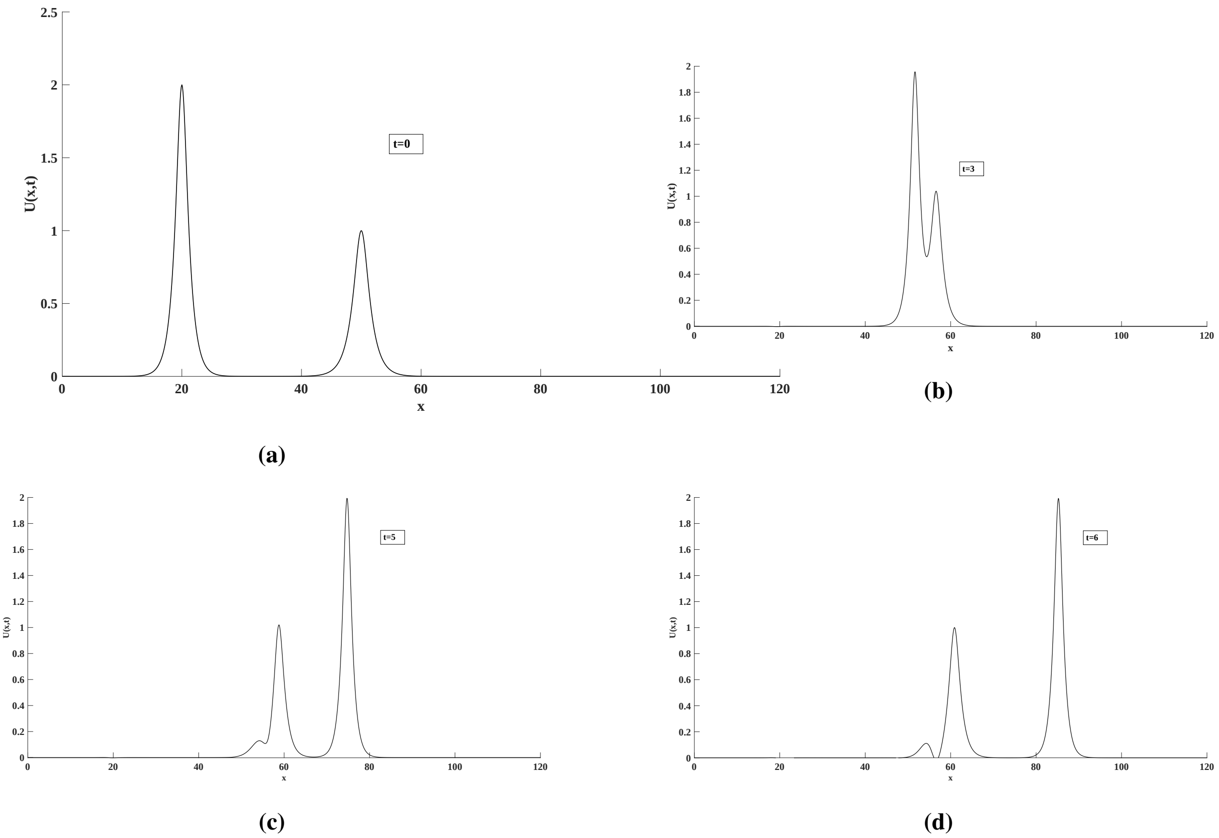

Figure 2: The interactions of two solitary waves at $p = 3$

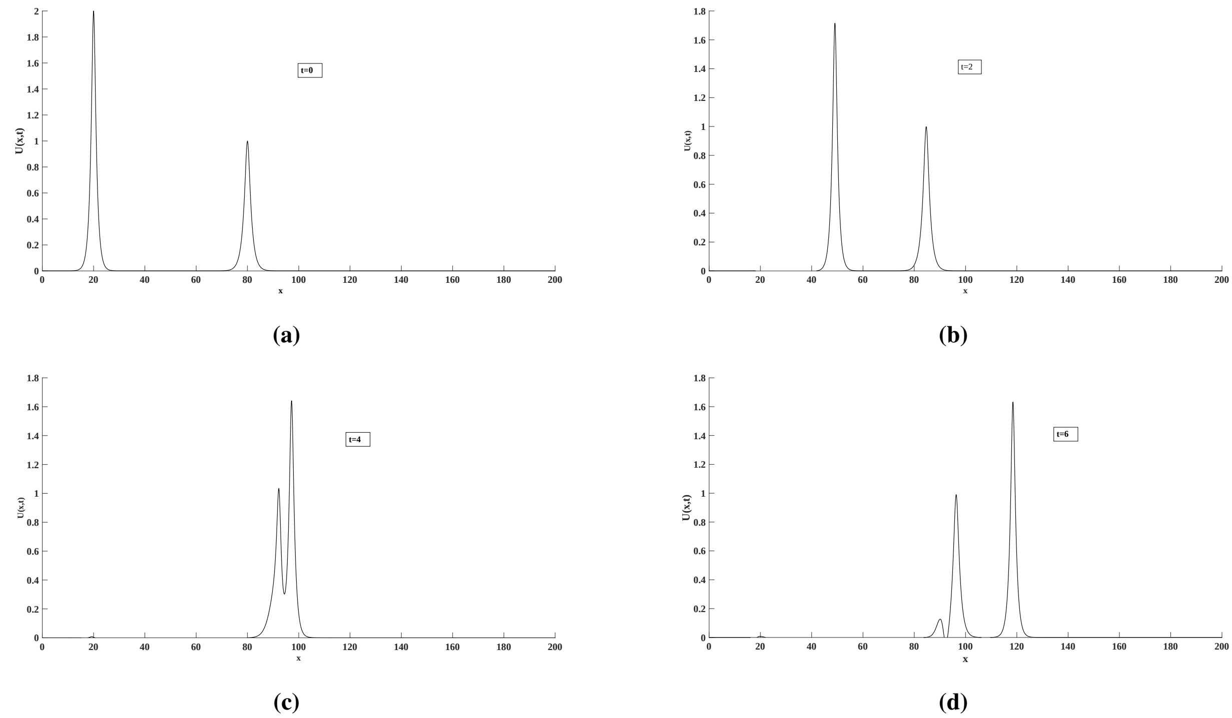

Figure 3: The interactions of two solitary waves at $p = 4$

Table 16: Maxwellian initial condition for various μ values

		$p=2$			$p=3$			$p=4$		
μ	t	I_1	I_2	I_3	I_1	I_2	I_3	I_1	I_2	I_3
0.025	0.01	1.77245	1.28464	0.85485	1.77245	1.28464	0.85475	1.77245	1.28464	0.85454
	0.03	1.77245	1.28464	0.85457	1.77245	1.28464	0.85361	1.77245	1.28464	0.85165
	0.05	1.77245	1.28464	0.85399	1.77245	1.28464	0.85125	1.77245	1.28464	0.84541
0.05	0.01	1.77245	1.31597	0.82352	1.77245	1.31597	0.82341	1.77245	1.31597	0.82320
	0.03	1.77245	1.31597	0.82322	1.77245	1.31597	0.82224	1.77245	1.31597	0.82034
	0.05	1.77245	1.31597	0.82261	1.77245	1.31597	0.81988	1.77245	1.31597	0.81455
0.1	0.01	1.77245	1.37864	0.76087	1.77245	1.37864	0.76078	1.77245	1.37864	0.76062
	0.03	1.77245	1.37864	0.76065	1.77245	1.37864	0.75989	1.77245	1.37864	0.75844
	0.05	1.77245	1.37864	0.76021	1.77245	1.37864	0.75809	1.77245	1.37864	0.75410

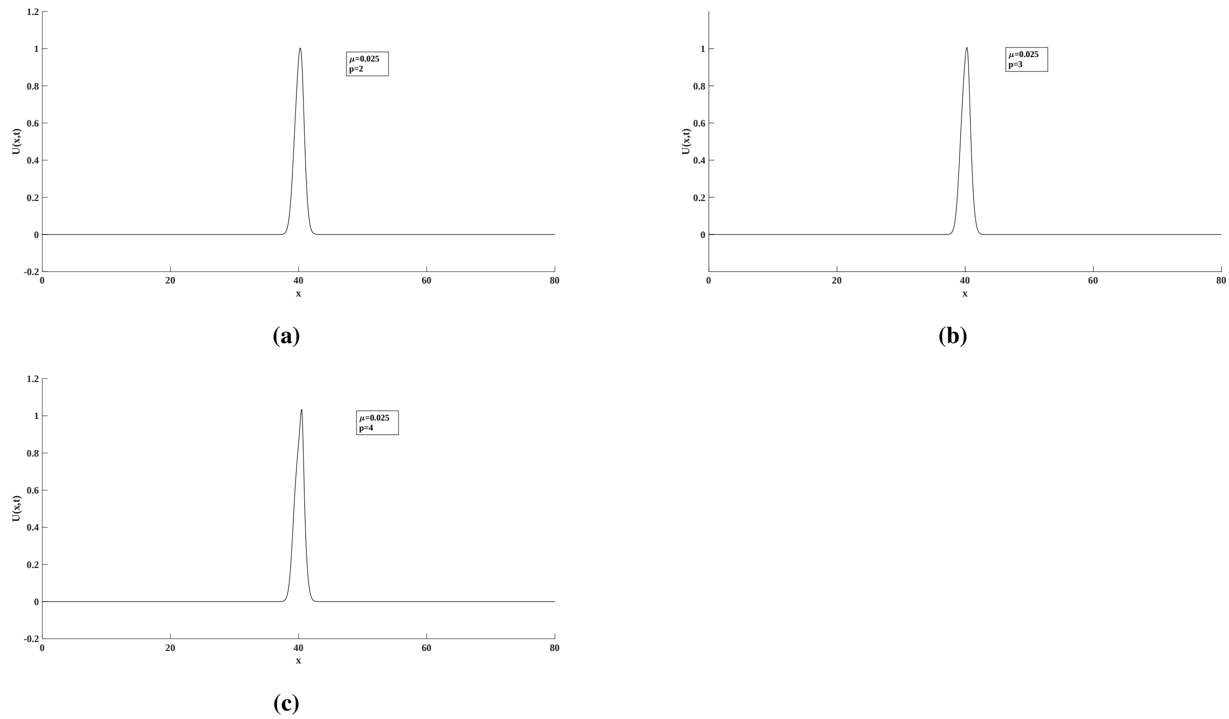

Figure 4: Graphics of Maxwell initial condition for different p values at $t = 0.05$

7. Conclusion

To obtain the solitary-wave solutions of the GRLW problem, this paper establishes two different linearization techniques, the collocation method and the Strang splitting algorithm. In order to achieve this, the collocation method is combined with the Strang splitting algorithm to perform numerical calculations, and the collocation method is applied to each scheme. In particular, the error norms L_2, L_∞ and the invariants I_1, I_2, and I_3 have been calculated for each of the three examples: A single solitary wave, the interaction of two solitary waves and the Maxwellian initial condition. The results obtained are listed in tables and figures. These tables show how the invariant values agree with other findings and the variations of the invariants are quite small. The figures show that the method applied in the article is compatible with the figures in similar examples in the literature. Compared to previous numerical methods, smaller error norms are obtained. The outcome of the error norms obtained are superior to those from earlier numerical techniques. As a result, based on the results produced in this study, it can be said with certainty that the numerical scheme that has been presented is more preferred and trustworthy for improving the numerical solutions of the physically significant nonlinear partial differential equations.

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Dynamical Behavior of Solutions to Higher-Order System of Fuzzy Difference Equations

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Abstract

In this paper, we concentrate on the global behavior of the fuzzy difference equations system with higher order

$$\alpha_{n+1} = \tau_1 + \frac{\alpha_n}{\sum_{i=1}^{m} \beta_{n-i}}, \beta_{n+1} = \tau_2 + \frac{\beta_n}{\sum_{i=1}^{m} \alpha_{n-i}}, \quad n \in \mathbb{N}_0,$$

where α_n, β_n are positive fuzzy number sequences, parameters τ_1, τ_2 and the initial values $\alpha_{-i}, \beta_{-i}, i \in \{0, 1, \ldots, m\}$, are positive fuzzy numbers. Firstly, we show the existence and uniqueness of the positive fuzzy solution to the mentioned system. Furthermore, we are searching for the boundedness, persistence and convergence of the positive solution to the given system. Finally, we give some numerical examples to show the efficiency of our results.

1. Introduction

Difference equations has many applications in the real world to many areas such as economics, biology, psychology, sociology, computer sciences etc. That's why, much more attention is given to this area. There are many data in our natural world. Collecting and establishing discrete mathematical models to figure out their behaviors is crucial. A discrete dynamical models of systems are generally established by using difference equations approach. These difference equation models can be seen simple. But, it is really important to comprehend the behaviors of their solutions in the cases generating general solution expressions is difficult.

DeVault et al., in [1], showed that every positive solution of the equation

$$x_{n+1} = \frac{A}{x_n} + \frac{1}{x_{n-2}}, \quad n \in \mathbb{N}_0,$$

where the parameter $A \in (0, \infty)$, converges to a period of two solutions. Later, Abu-Saris et al., in [2], studied the global asymptotic stability of the unique equilibrium point $\tilde{y} = 1 + A$ of the following equation

$$y_{n+1} = A + \frac{y_n}{y_{n-k}}, \quad n \in \mathbb{N}_0,$$

where the parameter A and the initial conditions $y_0, y_{-1}, \ldots, y_{-k}$, are positive real numbers.

Papaschinopoulos and Schinas, in [3], studied the oscillatory behavior, the boundedness of the solutions and global asymptotic stability of the positive equilibrium point of the difference equations system

$$x_{n+1} = A + \frac{y_n}{x_{n-p}}, y_{n+1} = A + \frac{x_n}{y_{n-q}}, \quad n \in \mathbb{N}_0,$$

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where p,q are positive integers, the parameter A and the initial conditions $x_{-p},\dots,x_0,y_{-q},\dots,y_0$, are positive real numbers. Also, Zhang et al., in [4], investigated the boundedness, persistence, and global asymptotic behavior of positive solution for the rational difference equations system

$$x_{n+1}=A+\frac{x_n}{\sum_{i=1}^{k}y_{n-i}},y_{n+1}=B+\frac{y_n}{\sum_{i=1}^{k}x_{n-i}},\quad n\in\mathbb{N}_0, \tag{1.1}$$

where the parameters A,B and the initial conditions $x_{-i},y_{-i},i\in\{0,1,\dots,k\}$, are positive real numbers. For more information, see [5, 6, 7]. Further studies about difference equations or difference equation systems can be found in [1, 2, 3, 8, 9, 10, 11, 12, 13] and references therein.

Fuzzy set theory is a mathematical paradigm that deals with sets with indefinite or uncertain bounds. It provides for partial membership of an element in a set. This ambiguous or uncertain data idea is important in modern analytics. The data is frequently incomplete, unclear, or subject to change. Fuzzy set theory allows analysts to model and manipulate such data effectively. It leads to more educated decision-making and improved analytics outcomes.

Zadeh, in [14], introduced the concept of fuzzy sets as a technique of dealing with unclear or imprecise data in engineering and computer science in 1965. Since then, the fuzzy set theory has grown significantly, and its applications have spread across a variety of disciplines such as decision-making, pattern recognition, image processing, natural language processing, and control systems. There are more information about fuzzy set theory at [7, 15, 16, 17].

Deeba et al., in [18], studied fuzzy analog of the first order difference equation

$$x_{n+1}=wx_n+q,\quad n\in\mathbb{N}_0,$$

where x_n is a fuzzy number sequence and the initials w,q,x_0 are fuzzy numbers. Deeba and Korvin [19] considered a model

$$C_{n+1}=C_n-c_1C_{n-1}+c_2,\quad n\in\mathbb{N}_0,$$

where c_1,c_2, are the fuzzy parameters, C_0,C_1, are the fuzzy initial conditions which determines the level of CO_2 in blood. There are also many researches which study qualitative behaviors of positive solutions of fuzzy difference equations and FDEs. For example, Papaschinopoulos and Papadopoulos, in [20], investigated the existence, boundedness, oscillatory and asymptotic behaviors of the positive solutions of the fuzzy difference equation

$$x_{n+1}=A+\frac{B}{x_n},\quad n\in\mathbb{N}_0,$$

with positive fuzzy parameters A,B and positive fuzzy initial condition x_0. They also studied the fuzzy difference equation

$$x_{n+1}=A+\frac{x_n}{x_{n-m}},\quad n\in\mathbb{N}_0,$$

where x_n is a positive fuzzy number sequence and $A,x_0,x_{-1},\dots,x_{-m}$ are positive fuzzy numbers. Yalcinkaya et al., in [21], investigated qualitative behavior of the fuzzy difference equation

$$z_{n+1}=\frac{Az_{n-s}}{B+C\prod_{i=0}^{s}z_{n-i}^{p}},\quad n\in\mathbb{N}_0,$$

with positive integer s, positive parameters A,B,C and positive initial conditions $z_{-i},i\in\{0,1,\dots,s\}$. Zhang et al., in [22], investigated dynamical behavior of the second-order exponential type fuzzy difference equation

$$x_{n+1}=\frac{A+Be^{-x_n}}{C+x_{n-1}},\quad n\in\mathbb{N}_0,$$

with positive fuzzy parameters A,B,C and positive fuzzy initial conditions x_{-1},x_0. Moreover, Atpinar and Yazlik, in [23], analyzed the existence, uniqueness and the qualitative behavior of the two-dimensional exponential FDEs

$$x_{n+1}=\frac{\alpha_1+\beta_1e^{-x_{n-1}}}{\gamma_1+y_n},\quad y_{n+1}=\frac{\alpha_2+\beta_2e^{-y_{n-1}}}{\gamma_2+x_n},\quad n\in\mathbb{N}_0,$$

where the parameters $\alpha_1,\alpha_2,\beta_1,\beta_2,\gamma_1,\gamma_2$ and the initial conditions x_{-1},x_0,y_{-1},y_0 are positive fuzzy numbers. There are more studies about fuzzy difference equations [22, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33] and references therein. The fuzzy difference equations and fuzzy difference equations system, briefly FDEs, have not been studied extensively, yet. Inspired by the aforementioned studies, we concentrate on the FDEs

$$\alpha_{n+1}=\tau_1+\frac{\alpha_n}{\sum_{i=1}^{m}\beta_{n-i}},\beta_{n+1}=\tau_2+\frac{\beta_n}{\sum_{i=1}^{m}\alpha_{n-i}},\quad n\in\mathbb{N}_0, \tag{1.2}$$

where α_n,β_n are positive fuzzy number sequences, the parameters τ_1,τ_2 and the initial values $\alpha_{-i},\beta_{-i},i\in\{0,1,\dots,m\}$, are positive fuzzy numbers.

2. Preliminaries

In this section, we briefly give some definitions, lemmas and theorems which are used throughout the paper. For more information and details can be found in [7, 15, 16, 34].

Let $\mathbb{R}_f$ represent the space of all fuzzy numbers and $w \in \mathbb{R}_f$. For all $\gamma \in (0,1]$ $[w]^\gamma = \{x \in \mathbb{R} : w(x) \geq \gamma\}$ and $[w]^0 = \overline{\bigcup_{\gamma \in (0,1]}[w]^\gamma} = \overline{\{x \in \mathbb{R} : w(x) > 0\}}$. Here, we say that $[w]^0$ is the support of the fuzzy number w and show it by $supp(w)$. w is called a positive fuzzy number if $supp(w) \subset (0,\infty)$. $\mathbb{R}_f^+$ denotes the space of all fuzzy numbers. Let $x, y \in \mathbb{R}_f$, $\lambda \in \mathbb{R}$ and $[x]^\gamma = [L_{x_n}^\gamma, R_{x_n}^\gamma]$, $[y]^\gamma = [L_{y_n}^\gamma, R_{y_n}^\gamma]$. For $\gamma \in (0,1]$ the operations scalar multiplication, addition, multiplication and division on fuzzy numbers are defined as follows:

$$
\begin{aligned}
[\lambda x]^\gamma &= \lambda [x]^\gamma, \\
[x+y]^\gamma &= [x]^\gamma + [y]^\gamma, \\
[xy]^\gamma &= \left[\min\{L_x^\gamma L_y^\gamma, L_x^\gamma R_y^\gamma, R_x^\gamma L_y^\gamma, R_x^\gamma R_y^\gamma\}, \max\{L_x^\gamma L_y^\gamma, L_x^\gamma R_y^\gamma, R_x^\gamma L_y^\gamma, R_x^\gamma R_y^\gamma\}\right], \\
[\frac{x}{y}]^\gamma &= \left[\min\{\frac{L_x^\gamma}{L_y^\gamma}, \frac{L_x^\gamma}{R_y^\gamma}, \frac{R_x^\gamma}{L_y^\gamma}, \frac{R_x^\gamma}{R_y^\gamma}\}, \max\{\frac{L_x^\gamma}{L_y^\gamma}, \frac{L_x^\gamma}{R_y^\gamma}, \frac{R_x^\gamma}{L_y^\gamma}, \frac{R_x^\gamma}{R_y^\gamma}\}\right], 0 \notin [y]^\gamma,
\end{aligned}
$$

respectively.

Definition 2.1. *Consider a fuzzy subset of the real line $w : \mathbb{R} \to (0,1]$ and suppose that the following properties hold:*

(a) w is normal, i. e., there exists $x_0 \in \mathbb{R}$ such that $w(x_0) = 1$,
(b) w is convex, i. e., $\forall \lambda \in (0,1]$ and $x_1, x_2 \in \mathbb{R}$, $w(\lambda x_1 + (1-\lambda)x_2) \geq \min\{w(x_1), w(x_2)\}$
(c) w is upper semi-continuous on $\mathbb{R}$,
(d) w is compactly supported, i. e., $\overline{\bigcup_{\gamma \in (0,1]}[w]^\gamma} = \overline{\{x \in \mathbb{R} : w(x) > 0\}}$ is compact,

we say that w is a fuzzy number.

Lemma 2.2. *Let $x \in R_f^+$ and $[x]^\gamma = [L_x^\gamma, R_x^\gamma]$ for $\gamma \in (0,1]$. Then, for $[L_x^\gamma, R_x^\gamma]$ the following conditions hold:*

1. *L_x^γ is non-decreasing and left continuous,*
2. *R_x^γ is non-increasing and right continuous,*
3. *$L_x^\gamma \leq R_x^\gamma$.*

Lemma 2.3. *Let f be a continuous function from $\mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+$ into $\mathbb{R}^+$. For any $x, y, z, t \in \mathbb{R}_f^+$ and $\gamma \in (0,1]$,*

$$[f(x,y,z,t)]^\gamma = f([x]^\gamma, [y]^\gamma, [z]^\gamma, [t]^\gamma).$$

Definition 2.4. *Let $\{x_n\}$ be a positive fuzzy number sequence. If there exist positive real numbers m, M such that $supp(x_n) \subset [m, M]$, then we say that positive fuzzy sequence (x_n) is bounded and persistent.*

Theorem 2.5. *Let $[x]^\gamma \in R_f^+$ be a fuzzy number. Then,*

1. *$[x]^\gamma$ is a closed interval $\forall \gamma \in (0,1]$,*
2. *For $\gamma_1, \gamma_2 \in (0,1]$, if $\gamma_1 \leq \gamma_2$, then $x^{\gamma_2} \subseteq x^{\gamma_1}$,*
3. *For any sequence γ_n converging to $\gamma \in (0,1]$ from below, $\cap_{n=1}^\infty x^{\gamma_n} = x^\gamma$,*
4. *For any sequence γ_n converging to 0 from above, $\overline{\cup_{n=1}^\infty [x]^{\gamma_n}} = [x]^0$.*

Definition 2.6. *Let x, y be fuzzy numbers with $[x]^\gamma = [L_x^\gamma, R_x^\gamma]$ and $[y]^\gamma = [L_y^\gamma, R_y^\gamma]$ for $\gamma \in (0,1]$. Then, the metric on fuzzy number space is defined as follows:*

$$D(x,y) = \sup_{\gamma \in (0,1]} \max\{|L_x^\gamma - L_y^\gamma|, |R_x^\gamma - R_y^\gamma|\}. \tag{2.1}$$

Moreover, the norm on fuzzy number space is defined by

$$||X|| = \sup_{\gamma \in (0,1]} \max\{|L_x^\gamma|, |R_x^\gamma|\}.$$

3. Main Results

In this section, we study FDEs (1.2) for positive initial fuzzy numbers. Firstly, we investigate the existence and uniqueness of positive solutions of (1.2) in the following theorem.

Theorem 3.1. *Consider the system (1.2) for positive fuzzy numbers* τ_1, τ_2. *Then, for given any positive fuzzy numbers* $\alpha_{-i}, \beta_{-i}, i \in \{0,1,\ldots,m\}$, *the system has a unique positive solution.*

Proof. Let the parameters τ_1, τ_2 and the initial conditions $\alpha_{-i}, \beta_{-i}, i \in \{0,1,\ldots,m\}$, be positive fuzzy numbers. Suppose that there exist fuzzy number sequences which satisfy (1.2). Consider their γ-cuts for $\gamma \in (0,1]$;

$$\begin{cases} [\alpha_n]^\gamma & = [L^\gamma_{\alpha_n}, R^\gamma_{\alpha_n}], [\beta_n]^\gamma = [L^\gamma_{\beta_n}, R^\gamma_{\beta_n}], \\ [\alpha_{n-i}]^\gamma & = [L^\gamma_{\alpha_{n-i}}, R^\gamma_{\alpha_{n-i}}], [\beta_{n-i}]^\gamma = [L^\gamma_{\beta_{n-i}}, R^\gamma_{\beta_{n-i}}], \\ [\tau_1]^\gamma & = [\tau^\gamma_{1,l}, \tau^\gamma_{1,r}], [\tau_2]^\gamma = [\tau^\gamma_{2,l}, \tau^\gamma_{2,r}]. \end{cases} \tag{3.1}$$

By using (1.2), (3.1) and Lemma (2.3),

$$\begin{aligned} [\alpha_{n+1}]^\gamma &= [L^\gamma_{\alpha_{n+1}}, R^\gamma_{\alpha_{n+1}}], \\ &= \left[\tau_1 + \frac{\alpha_n}{\sum_{i=1}^m \beta_{n-i}}\right]^\gamma, \\ &= [\tau_1]^\gamma + \frac{[\alpha_n]^\gamma}{\sum_{i=1}^m [\beta_{n-i}]^\gamma}, \\ &= [\tau^\gamma_{1,l}, \tau^\gamma_{1,r}] + \frac{[L^\gamma_{\alpha_n}, R^\gamma_{\alpha_n}]}{\sum_{i=1}^m [L^\gamma_{\beta_{n-i}}, R^\gamma_{\beta_{n-i}}]}, \\ &= \left[\tau^\gamma_{1,l} + \frac{L^\gamma_{\alpha_n}}{\sum_{i=1}^m R^\gamma_{\beta_{n-i}}}, \tau^\gamma_{1,r} + \frac{R^\gamma_{\alpha_n}}{\sum_{i=1}^m L^\gamma_{\beta_{n-i}}}\right], \end{aligned} \tag{3.2}$$

and similarly

$$\begin{aligned} [\beta_{n+1}]^\gamma &= [L^\gamma_{\beta_{n+1}}, R^\gamma_{\beta_{n+1}}], \\ &= \left[\tau_2 + \frac{\beta_n}{\sum_{i=1}^m \alpha_{n-i}}\right]^\gamma, \\ &= [\tau_2]^\gamma + \frac{[\beta_n]^\gamma}{\sum_{i=1}^m [\alpha_{n-i}]^\gamma}, \\ &= [\tau^\gamma_{2,l}, \tau^\gamma_{2,r}] + \frac{[L^\gamma_{\beta_n}, R^\gamma_{\beta_n}]}{\sum_{i=1}^m [L^\gamma_{\alpha_{n-i}}, R^\gamma_{\alpha_{n-i}}]}, \\ &= \left[\tau^\gamma_{2,l} + \frac{L^\gamma_{\beta_n}}{\sum_{i=1}^m R^\gamma_{\alpha_{n-i}}}, \tau^\gamma_{2,r} + \frac{R^\gamma_{\beta_n}}{\sum_{i=1}^m L^\gamma_{\alpha_{n-i}}}\right]. \end{aligned} \tag{3.3}$$

So, we obtained the following equations system:

$$\begin{aligned} L^\gamma_{\alpha_{n+1}} &= \tau^\gamma_{1,l} + \frac{L^\gamma_{\alpha_n}}{\sum_{i=1}^m R^\gamma_{\beta_{n-i}}}, \ R^\gamma_{\alpha_{n+1}} = \tau^\gamma_{1,r} + \frac{R^\gamma_{\alpha_n}}{\sum_{i=1}^m L^\gamma_{\beta_{n-i}}}, \\ L^\gamma_{\beta_{n+1}} &= \tau^\gamma_{2,l} + \frac{L^\gamma_{\beta_n}}{\sum_{i=1}^m R^\gamma_{\alpha_{n-i}}}, \ R^\gamma_{\beta_{n+1}} = \tau^\gamma_{2,r} + \frac{R^\gamma_{\beta_n}}{\sum_{i=1}^m L^\gamma_{\alpha_{n-i}}}. \end{aligned} \tag{3.4}$$

Let $0 \le \gamma_1 \le \gamma_2 \le 1$. From Lemma (2.2),

$$\begin{aligned} 0 < \tau^{\gamma_1}_{1,l} \le \tau^{\gamma_2}_{1,l} \le \tau^{\gamma_2}_{1,r} \le \tau^{\gamma_1}_{1,r}, \\ 0 < \tau^{\gamma_1}_{2,l} \le \tau^{\gamma_2}_{2,l} \le \tau^{\gamma_2}_{2,r} \le \tau^{\gamma_1}_{2,r}, \end{aligned} \tag{3.5}$$

$$\begin{aligned} 0 < L^{\gamma_1}_{\alpha_{n-i}} \le L^{\gamma_2}_{\alpha_{n-i}} \le R^{\gamma_2}_{\alpha_{n-i}} \le R^{\gamma_1}_{\alpha_{n-i}}, \\ 0 < L^{\gamma_1}_{\beta_{n-i}} \le L^{\gamma_2}_{\beta_{n-i}} \le R^{\gamma_2}_{\beta_{n-i}} \le R^{\gamma_1}_{\beta_{n-i}}, \end{aligned} \tag{3.6}$$

for $i \in \{0,1,\ldots,m\}$ and

$$\begin{aligned} 0 < L^{\gamma_1}_{\alpha_n} \le L^{\gamma_2}_{\alpha_n} \le R^{\gamma_2}_{\alpha_n} \le R^{\gamma_1}_{\alpha_n}, \\ 0 < L^{\gamma_1}_{\beta_n} \le L^{\gamma_2}_{\beta_n} \le R^{\gamma_2}_{\beta_n} \le R^{\gamma_1}_{\beta_n}, \end{aligned} \tag{3.7}$$

where $n \in \mathbb{N}_0$. By using inequalities in (3.5), (3.6) and (3.7) and keeping in mind that $\gamma_1 \leq \gamma_2$,

$$L^{\gamma_1}_{\alpha_{n+1}} = \tau^{\gamma_1}_{1,l} + \frac{L^{\gamma_1}_{\alpha_n}}{\sum_{i=1}^m R^{\gamma_1}_{\beta_{n-i}}} \leq \tau^{\gamma_2}_{1,l} + \frac{L^{\gamma_2}_{\alpha_n}}{\sum_{i=1}^m R^{\gamma_2}_{\beta_{n-i}}} \leq L^{\gamma_2}_{\alpha_{n+1}}$$
$$\leq \tau^{\gamma_2}_{1,r} + \frac{R^{\gamma_2}_{\alpha_n}}{\sum_{i=1}^m L^{\gamma_2}_{\beta_{n-i}}} \leq R^{\gamma_2}_{\alpha_{n+1}} \leq \tau^{\gamma_1}_{1,r} + \frac{R^{\gamma_1}_{\alpha_n}}{\sum_{i=1}^m L^{\gamma_1}_{\beta_{n-i}}} = R^{\gamma_1}_{\alpha_{n+1}}$$

and

$$L^{\gamma_1}_{\beta_{n+1}} = \tau^{\gamma_1}_{2,l} + \frac{L^{\gamma_1}_{\beta_n}}{\sum_{i=1}^m R^{\gamma_1}_{\alpha_{n-i}}} \leq \tau^{\gamma_2}_{2,l} + \frac{L^{\gamma_1}_{\beta_n}}{\sum_{i=1}^m R^{\gamma_1}_{\alpha_{n-i}}} \leq L^{\gamma_2}_{\beta_{n+1}}$$
$$\leq \tau^{\gamma_2}_{2,r} + \frac{R^{\gamma_2}_{\beta_n}}{\sum_{i=1}^m L^{\gamma_2}_{\alpha_{n-i}}} \leq R^{\gamma_2}_{\beta_{n+1}} \leq \tau^{\gamma_1}_{2,r} + \frac{R^{\gamma_1}_{\beta_n}}{\sum_{i=1}^m L^{\gamma_1}_{\alpha_{n-i}}} = R^{\gamma_1}_{\beta_{n+1}}.$$

Next, by using induction, we will show positive fuzzy solution of the FDEs (1.2) exists. For $n = 0$,

$$[L^{\gamma}_{\alpha_1}, R^{\gamma}_{\alpha_1}] = \left[\tau^{\gamma}_{1,l} + \frac{L^{\gamma}_{\alpha_0}}{\sum_{i=1}^m R^{\gamma}_{\beta_{-i}}}, \tau^{\gamma}_{1,r} + \frac{R^{\gamma}_{\alpha_0}}{\sum_{i=1}^m L^{\gamma}_{\beta_{-i}}}\right],$$
$$[L^{\gamma}_{\beta_1}, R^{\gamma}_{\beta_1}] = \left[\tau^{\gamma}_{2,l} + \frac{L^{\gamma}_{\beta_0}}{\sum_{i=1}^m R^{\gamma}_{\alpha_{-i}}}, \tau^{\gamma}_{2,r} + \frac{R^{\gamma}_{\beta_0}}{\sum_{i=1}^m L^{\gamma}_{\alpha_{-i}}}\right].$$

Since, τ_1, τ_2 and α_i, β_i for $i \in \{0, 1, \ldots, m\}$ are positive fuzzy numbers, for $\gamma \in (0,1]$, $[L^{\gamma}_{\alpha_1}, R^{\gamma}_{\alpha_1}]$ and $[L^{\gamma}_{\beta_1}, R^{\gamma}_{\beta_1}]$ are γ-cuts of $\alpha_1 = \tau^{\gamma}_1 + \frac{\alpha^{\gamma}_0}{\sum_{i=1}^m \beta^{\gamma}_{-i}}$ and $\beta_1 = \tau^{\gamma}_2 + \frac{\beta^{\gamma}_0}{\sum_{i=1}^m \alpha^{\gamma}_{-i}}$. Moreover, $\tau_{1,l}, \tau_{1,r}, \tau_{2,l}, \tau_{2,r}$ and for $i \in \{0, 1, \ldots, m\}$ $L^{\gamma}_{\alpha_{-i}}, R^{\gamma}_{\alpha_{-i}}, L^{\gamma}_{\beta_{-i}}, R^{\gamma}_{\beta_{-i}}$ are left continuous, then so are $L^{\gamma}_{\alpha_1}, R^{\gamma}_{\alpha_1}, L^{\gamma}_{\beta_1}, R^{\gamma}_{\beta_1}$.
Now, assume that for $j \in \{1, 2, \ldots, k\}$, $[L^{\gamma}_{\alpha_j}, R^{\gamma}_{\alpha_j}]$ and $[L^{\gamma}_{\beta_j}, R^{\gamma}_{\beta_j}]$ are the γ-cuts of $\alpha_j = \tau_1 + \frac{\alpha_{j-1}}{\sum_{i=1}^m \beta^{\gamma}_{j-i-1}}$ and $\beta_j = \tau_1 + \frac{\beta_{j-1}}{\sum_{i=1}^m \alpha^{\gamma}_{j-i-1}}$. For $n = k+1$, we have

$$[\alpha_{k+1}]^{\gamma} = \left[\tau^{\gamma}_{1,l} + \frac{L^{\gamma}_{\alpha_k}}{\sum_{i=1}^m R^{\gamma}_{\beta_{k-i}}}, \tau^{\gamma}_{1,r} + \frac{R^{\gamma}_{\alpha_k}}{\sum_{i=1}^m L^{\gamma}_{\beta_{k-i}}}\right] = \left[\tau_1 + \frac{\alpha_k}{\sum_{i=1}^m \beta_{k-i}}\right]^{\gamma},$$
$$[\beta_{k+1}]^{\gamma} = \left[\tau^{\gamma}_{2,l} + \frac{L^{\gamma}_{\beta_k}}{\sum_{i=1}^m R^{\gamma}_{\alpha_{k-i}}}, \tau^{\gamma}_{2,r} + \frac{R^{\gamma}_{\beta_k}}{\sum_{i=1}^m L^{\gamma}_{\alpha_{k-i}}}\right] = \left[\tau_2 + \frac{\beta_k}{\sum_{i=1}^m \alpha_{k-i}}\right]^{\gamma}.$$

Therefore, $[L^{\gamma}_{\alpha_{k+1}}, R^{\gamma}_{\alpha_{k+1}}]$ and $[L^{\gamma}_{\beta_{k+1}}, R^{\gamma}_{\beta_{k+1}}]$ are the γ-cuts of the fuzzy numbers $\alpha_{k+1} = \tau_1 + \frac{\alpha_k}{\sum_{i=1}^m \beta_{k-i}}$ and $\beta_{k+1} = \tau_2 + \frac{\beta_k}{\sum_{i=1}^m \alpha_{k-i}}$.
Hence, for $\forall n \in \mathbb{N}$ and $\forall \gamma \in (0,1]$, $[L^{\gamma}_{\alpha_n}, R^{\gamma}_{\alpha_n}]$ and $[L^{\gamma}_{\beta_n}, R^{\gamma}_{\beta_n}]$ are the γ-cuts of the fuzzy numbers α_n and β_n, by induction.

Now, we claim that supports of both α_n and β_n, $\operatorname{supp}\alpha_n = \overline{\bigcup_{\gamma \in (0,1]} [L^{\gamma}_{\alpha_n}, R^{\gamma}_{\alpha_n}]}$ and $\operatorname{supp}\beta_n = \overline{\bigcup_{\gamma \in (0,1]} [L^{\gamma}_{\beta_n}, R^{\gamma}_{\beta_n}]}$ are compact by induction. For $n = 1$, since τ_1, τ_2 and $\alpha_{-i}, \beta_{-i}, i \in \{0, 1, \ldots, m\}$ are positive fuzzy numbers, there exist $M_{\tau_1}, N_{\tau_1}, M_{\tau_2}, N_{\tau_2}, M_{\alpha_{-i}}, N_{\alpha_{-i}}, M_{\beta_{-i}}, N_{\beta_{-i}} \in \{0, 1, \ldots, m\}$ such that for all $\gamma \in (0,1]$,

$$\begin{cases} [\tau^{\gamma}_{1,l}, \tau^{\gamma}_{1,r}] & \subseteq \overline{\bigcup_{\gamma \in (0,1]} [\tau^{\gamma}_{1,l}, \tau^{\gamma}_{1,r}]} \subseteq [M_{\tau_1}, N_{\tau_1}] \\ [\tau^{\gamma}_{2,l}, \tau^{\gamma}_{2,r}] & \subseteq \overline{\bigcup_{\gamma \in (0,1]} [\tau^{\gamma}_{2,l}, \tau^{\gamma}_{2,r}]} \subseteq [M_{\tau_2}, N_{\tau_2}] \\ [L^{\gamma}_{\alpha_1}, R^{\gamma}_{\alpha_1}] & \subseteq \overline{\bigcup_{\gamma \in (0,1]} [L^{\gamma}_{\alpha_1}, R^{\gamma}_{\alpha_1}]} \subseteq [M_{\alpha_1}, N_{\alpha_1}] \\ [L^{\gamma}_{\beta_1}, R^{\gamma}_{\beta_1}] & \subseteq \overline{\bigcup_{\gamma \in (0,1]} [L^{\gamma}_{\beta_1}, R^{\gamma}_{\beta_1}]} \subseteq [M_{\beta_1}, N_{\beta_1}]. \end{cases}$$

By using induction, we obtain $\overline{\bigcup_{\gamma \in (0,1]} [\alpha^{\gamma}_{n,l}, \alpha^{\gamma}_{n,r}]}$ and $\overline{\bigcup_{\gamma \in (0,1]} [\beta^{\gamma}_{n,l}, \beta^{\gamma}_{n,r}]}$ are compact and $\overline{\bigcup_{\gamma \in (0,1]} [\alpha^{\gamma}_{n,l}, \alpha^{\gamma}_{n,r}]}$ and $\overline{\bigcup_{\gamma \in (0,1]} [\beta^{\gamma}_{n,l}, \beta^{\gamma}_{n,r}]} \subseteq (0, +\infty)$ for $n \in \mathbb{N}_0$. Hence, $\alpha_n = [\alpha_n]^{\gamma} = [L^{\gamma}_{\alpha_n}, R^{\gamma}_{\alpha_n}]$ and $\beta_n = [\beta_n]^{\gamma} = [L^{\gamma}_{\beta_n}, R^{\gamma}_{\beta_n}]$ are also positive fuzzy number sequences.

Finally, we will show uniqueness of positive solutions of FDEs (1.2) by using contradiction method. Assume that there exist other solutions α'_n and β'_n to the given system (1.2) with the same initial initial values τ_1, τ_2 and $\alpha_{-i}, \beta_{-i}, i \in \{0, 1, \ldots, m\}$.

Then, for $\alpha \in (0,1]$;

$$[\alpha_n]^{\gamma} = [\alpha'_n]^{\gamma}, [\beta_n]^{\gamma} = [\beta'_n]^{\gamma}.$$

Hence, there exists a unique solution of (1.2) for given initial conditions τ_1, τ_2 and $\alpha_{-i}, \beta_{-i}, i \in \{0, 1, \ldots, m\}$, which is desired. □

Now, we will investigate boundedness and persistence of positive solutions of (1.2).

Let u_n, v_n, w_n, t_n represent $L^{\gamma}_{\alpha_n}, R^{\gamma}_{\alpha_n}, L^{\gamma}_{\beta_n}, R^{\gamma}_{\beta_n}$ respectively. Then, from (3.4), we can write the following system as

$$\begin{cases} L_{\alpha_{n+1}} & = u_{n+1} = \lambda_1 + \frac{u_n}{\sum_{i=1}^{m} t_{n-i}}, \\ R_{\alpha_{n+1}} & = v_{n+1} = \lambda_2 + \frac{v_n}{\sum_{i=1}^{m} w_{n-i}}, \\ L_{\beta_{n+1}} & = w_{n+1} = \lambda_3 + \frac{w_n}{\sum_{i=1}^{m} v_{n-i}}, \\ R_{\beta_{n+1}} & = t_{n+1} = \lambda_4 + \frac{t_n}{\sum_{i=1}^{m} u_{n-i}}, \end{cases} \quad n \in \mathbb{N}_0, \tag{3.8}$$

where the parameters $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ are positive real numbers.

Theorem 3.2. *Consider system* (3.8) *and suppose that*

$$\frac{1}{m} < \min\{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}. \tag{3.9}$$

If (3.9) *is satisfied, then for every positive solutions (u_n, v_n, w_n, t_n) of* (3.8) *for $n > m$ the following inequalities hold:*

$$\begin{aligned} \lambda_1 \le u_n \le \frac{1}{(m\lambda_4)^{n-m}} \left(u_m - \frac{m\lambda_1\lambda_4}{m\lambda_4 - 1} \right) + \frac{m\lambda_1\lambda_4}{m\lambda_4 - 1}, \\ \lambda_2 \le v_n \le \frac{1}{(m\lambda_3)^{n-m}} \left(v_m - \frac{m\lambda_2\lambda_3}{m\lambda_3 - 1} \right) + \frac{m\lambda_2\lambda_3}{m\lambda_3 - 1}, \\ \lambda_3 \le w_n \le \frac{1}{(m\lambda_2)^{n-m}} \left(w_m - \frac{m\lambda_2\lambda_3}{m\lambda_2 - 1} \right) + \frac{m\lambda_2\lambda_3}{m\lambda_2 - 1}, \\ \lambda_4 \le t_n \le \frac{1}{(m\lambda_1)^{n-m}} \left(t_m - \frac{m\lambda_1\lambda_4}{m\lambda_1 - 1} \right) + \frac{m\lambda_1\lambda_4}{m\lambda_1 - 1}, \end{aligned} \tag{3.10}$$

which shows the boundedness and persistence of (u_n, v_n, w_n, t_n).

Proof. Let (u_n, v_n, w_n, t_n) be positive solution of system (3.8). Since, u_n, v_n, w_n, t_n, for all $n \ge 1$, are positive,

$$\lambda_1 \le u_n, \quad \lambda_2 \le v_n, \quad \lambda_3 \le w_n, \quad \lambda_4 \le t_n. \tag{3.11}$$

Furthermore, by (3.8) and (3.11), we get

$$\begin{cases} u_n = \lambda_1 + \frac{u_n}{\sum_{i=2}^{m+1} t_{n-i}} \le \lambda_1 + \frac{1}{m\lambda_4} u_{n-1}, \\ v_n = \lambda_2 + \frac{v_n}{\sum_{i=2}^{m+1} w_{n-i}} \le \lambda_2 + \frac{1}{m\lambda_3} v_{n-1}, \\ w_n = \lambda_3 + \frac{w_n}{\sum_{i=2}^{m+1} v_{n-i}} \le \lambda_3 + \frac{1}{m\lambda_2} w_{n-1}, \\ t_n = \lambda_4 + \frac{t_n}{\sum_{i=2}^{m+1} u_{n-i}} \le \lambda_4 + \frac{1}{m\lambda_1} t_{n-1}, \end{cases} \tag{3.12}$$

for $n > m$. On this part of the proof, we just show boundedness for u_n. Since, proofs for v_n, w_n, t_n are similar, we omit them. Define $\tilde{u}_n = \lambda_1 + \frac{1}{m\lambda_4}\tilde{u}_{n-1}$ for $n > m$ and $\tilde{u}_n = u_n$ for $n = 1, 2, \dots, m$. Our claim is

$$u_n \le \tilde{u}_n, n \in \mathbb{N}. \tag{3.13}$$

We show satisfying the inequality in (3.13) by induction. It is obvious that $u_n \le \tilde{u}_n$ for $n \in \{1, 2, \dots, m\}$. Suppose that (3.13) holds for any $k = n \ge m+1$. Then, from (3.12), we have

$$u_{n+1} \le \lambda_1 + \frac{1}{m\lambda_4} u_n \le \lambda_1 + \frac{1}{m\lambda_4} \tilde{u}_n = \tilde{u}_{n+1}. \tag{3.14}$$

Therefore, $u_n \le \tilde{u}_n$ for $n \in \mathbb{N}$, by induction. Then,

$$\begin{aligned} \tilde{u}_n &= \lambda_1 + \frac{1}{m\lambda_4} \tilde{u}_{n-1} \\ \tilde{u}_n &= \lambda_1 + \frac{1}{m\lambda_4} \left(\lambda_1 + \frac{1}{m\lambda_4} \tilde{u}_{n-2} \right) \\ &\vdots \\ \tilde{u}_n &= \frac{1}{(m\lambda_4)^{n-m}} \left(u_m - \frac{m\lambda_1\lambda_4}{m\lambda_4 - 1} \right) + \frac{m\lambda_1\lambda_4}{m\lambda_4 - 1}. \end{aligned}$$

Hence, u_n is bounded. So proof for u_n is finished. Similarly, it can be shown that v_n, w_n, t_n are also bounded. □

Theorem 3.3. *Consider system (3.8). If the condition (3.10) holds, then (3.8) has a unique equilibrium point $(\bar{u},\bar{v},\bar{w},\bar{t})$ given by*

$$\bar{u}=\frac{m^2\lambda_1\lambda_4-1}{m(m\lambda_4-1)},\ \bar{v}=\frac{m^2\lambda_2\lambda_3-1}{m(m\lambda_3-1)},\ \bar{w}=\frac{m^2\lambda_2\lambda_3-1}{m(m\lambda_2-1)},\ \bar{t}=\frac{m^2\lambda_1\lambda_4-1}{m(m\lambda_1-1)}, \tag{3.15}$$

and every positive solution tends to given equilibrium point as $n\to\infty$.

Proof. From equilibrium point definition, we can simply obtain the equilibrium point given as

$$\Gamma=\left(\frac{m^2\lambda_1\lambda_4-1}{m(m\lambda_4-1)},\frac{m^2\lambda_2\lambda_3-1}{m(m\lambda_3-1)},\frac{m^2\lambda_2\lambda_3-1}{m(m\lambda_2-1)},\frac{m^2\lambda_1\lambda_4-1}{m(m\lambda_1-1)}\right). \tag{3.16}$$

Since every positive solution of system (3.8) is bounded and persistent from Theorem (3.2), it can be written that

$$\begin{aligned}
\lim_{n\to\infty}\inf u_n=l_1,\ \lim_{n\to\infty}\sup u_n=L_1,\\
\lim_{n\to\infty}\inf v_n=l_2,\ \lim_{n\to\infty}\sup v_n=L_2,\\
\lim_{n\to\infty}\inf w_n=l_3,\ \lim_{n\to\infty}\sup w_n=L_3,\\
\lim_{n\to\infty}\inf t_n=l_4,\ \lim_{n\to\infty}\sup t_n=L_4,
\end{aligned} \tag{3.17}$$

where $l_i,L_i\in(0,\infty)$, for $i\in\{1,2,3,4\}$. Then, by using Theorem (3.2) and (3.17),

$$\begin{aligned}
\lambda_1+\frac{l_1}{mL_4}\le l_1,\ L_1\le\lambda_1+\frac{L_1}{ml_4},\\
\lambda_2+\frac{l_2}{mL_3}\le l_2,\ L_2\le\lambda_2+\frac{L_2}{ml_3},\\
\lambda_3+\frac{l_3}{mL_2}\le l_3,\ L_3\le\lambda_3+\frac{L_3}{ml_2},\\
\lambda_4+\frac{l_4}{mL_1}\le l_4,\ L_4\le\lambda_4+\frac{L_4}{ml_1}.
\end{aligned}$$

Next, after arranging the inequalities, we obtain

$$\begin{aligned}
\frac{m\lambda_4L_1+l_4}{m}\le L_1l_4\le\frac{m\lambda_1l_4+L_1}{m},\\
\frac{m\lambda_3L_2+l_3}{m}\le L_2l_3\le\frac{m\lambda_2l_3+L_2}{m},\\
\frac{m\lambda_2L_3+l_2}{m}\le L_3l_2\le\frac{m\lambda_3l_2+L_3}{m},\\
\frac{m\lambda_1L_4+l_1}{m}\le L_4l_1\le\frac{m\lambda_4l_1+L_4}{m},
\end{aligned} \tag{3.18}$$

from which it follows that

$$\begin{aligned}
L_1(m\lambda_4-1)\le l_4(m\lambda_1-1),\\
L_2(m\lambda_3-1)\le l_3(m\lambda_2-1),\\
L_3(m\lambda_2-1)\le l_2(m\lambda_3-1),\\
L_4(m\lambda_1-1)\le l_1(m\lambda_4-1).
\end{aligned} \tag{3.19}$$

It is obvious from hypothesis of the Theorem (3.2) that $1<m\lambda_i$ for $i\in\{1,2,3,4\}$. Multiplying the first and the fourth inequalities in (3.19) and the second and the third inequalities in (3.19) gives

$$L_1L_4\le l_1l_4,\quad L_2L_3\le l_2l_3. \tag{3.20}$$

So,

$$L_1L_4=l_1l_4,\quad L_1L_4=l_1l_4. \tag{3.21}$$

Our claim is

$$L_1=l_1,\quad L_2=l_2,\quad L_3=l_3,\quad L_4=l_4.$$

Assume that $l_1 < L_1, l_2 < L_2, l_3 < L_3, l_4 < L_4$. Then by using (3.21), we get

$$\begin{aligned} L_1L_4 &= l_1l_4 < l_1L_4, \\ L_1L_4 &= l_1l_4 < L_1l_4, \\ L_2L_3 &= l_2l_3 < l_2L_3, \\ L_2L_3 &= l_2l_3 < L_2l_3, \end{aligned}$$

gives us

$$\begin{aligned} L_1 &< l_1, \\ L_2 &< l_2, \\ L_3 &< l_3, \\ L_4 &< l_4, \end{aligned}$$

which is a contradiction. Therefore,

$$l_1 = L_1,\ l_2 = L_2,\ l_3 = L_3,\ l_4 = L_4. \tag{3.22}$$

Hence, by using (3.8) and (3.22), it follows that

$$\lim_{n\to\infty} u_n = \tilde{u},\ \lim_{n\to\infty} v_n = \tilde{v},\ \lim_{n\to\infty} w_n = \tilde{w},\ \lim_{n\to\infty} t_n = \tilde{t}.$$

Thus, proof is completed. □

Theorem 3.4. *Consider system* (3.8)*. If both* (3.9) *and the following inequalities*

$$\frac{m^2\lambda_1\lambda_4 - 1}{m\lambda_1 - 1} + \frac{m^2\lambda_1\lambda_4 - 1}{m\lambda_4 - 1} < 1, \quad \frac{m^2\lambda_2\lambda_3 - 1}{m\lambda_2 - 1} + \frac{m^2\lambda_2\lambda_3 - 1}{m\lambda_3 - 1} < 1, \tag{3.23}$$

are satisfied, then the unique positive equilibrium point given in (3.15) *is locally asymptotically stable.*

Proof. From Theorem (3.3), the system (3.8) has a unique equilibrium point (3.15). The linearized equation of system (3.8) about the equilibrium point is

$$\Omega_{n+1} = \mathcal{P}\Omega_n$$

where $\Omega_n = (u_n, u_{n-1}, \ldots, u_{n-m}, v_n, v_{n-1}, \ldots, v_{n-m}, w_n, w_{n-1}, , \ldots, w_{n-m}, t_n, t_{n-1}, \ldots, t_{n-m})^T$ and $\mathcal{P} = (\rho_{ij}), 1 \le i, j \le 4m+4$ is a $(4m+4) \times (4m+4)$ matrix such that

$$\mathcal{P} = \begin{bmatrix} \mathcal{P}_{\tilde{t}} & \mathcal{P}_0 & \mathcal{P}_0 & \mathcal{P}_1 \\ \mathcal{P}_0 & \mathcal{P}_{\tilde{w}} & \mathcal{P}_2 & \mathcal{P}_0 \\ \mathcal{P}_0 & \mathcal{P}_3 & \mathcal{P}_{\tilde{v}} & \mathcal{P}_0 \\ \mathcal{P}_4 & \mathcal{P}_0 & \mathcal{P}_0 & \mathcal{P}_{\tilde{u}} \end{bmatrix}_{(4m+4)\times(4m+4)}, \tag{3.24}$$

where $\mathcal{P}_{\tilde{u}}, \mathcal{P}_{\tilde{v}}, \mathcal{P}_{\tilde{w}}, \mathcal{P}_{\tilde{t}}, \mathcal{P}_0, \mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3, \mathcal{P}_4$ are $(m+1)\times(m+1)$ matrices are defined as follows:

$$\mathcal{P}_{\tilde{u}} = \begin{bmatrix} \frac{1}{m\tilde{u}} & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}, \mathcal{P}_{\tilde{v}} = \begin{bmatrix} \frac{1}{m\tilde{v}} & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}, \mathcal{P}_{\tilde{w}} = \begin{bmatrix} \frac{1}{m\tilde{w}} & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix},$$

$$\mathcal{P}_{\tilde{t}} = \begin{bmatrix} \frac{1}{m\tilde{t}} & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}, \mathcal{P}_0 = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix}, \mathcal{P}_1 = \begin{bmatrix} 0 & -\frac{\tilde{u}}{m^2\tilde{t}^2} & \cdots & -\frac{\tilde{u}}{m^2\tilde{t}^2} & -\frac{\tilde{u}}{m^2\tilde{t}^2} \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix},$$

$$\mathcal{P}_2 = \begin{bmatrix} 0 & -\frac{\tilde{v}}{m^2\tilde{w}^2} & \cdots & -\frac{\tilde{v}}{m^2\tilde{w}^2} & -\frac{\tilde{v}}{m^2\tilde{w}^2} \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix}, \mathcal{P}_3 = \begin{bmatrix} 0 & -\frac{\tilde{w}}{m^2\tilde{v}^2} & \cdots & -\frac{\tilde{w}}{m^2\tilde{v}^2} & -\frac{\tilde{w}}{m^2\tilde{v}^2} \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix},$$

$$\mathcal{P}_4 = \begin{bmatrix} 0 & -\frac{\tilde{t}}{m^2\tilde{u}^2} & \cdots & -\frac{\tilde{t}}{m^2\tilde{u}^2} & -\frac{\tilde{t}}{m^2\tilde{u}^2} \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix}.$$

Let $\sigma_1, \sigma_2, \ldots, \sigma_{4m+4}$ be the eigenvalues of the matrix $\mathcal{P}$ and D be the diagonal matrix $(d_1, d_2, \ldots, d_{4m+4})$ such that $d_1 = d_{m+2} = d_{2m+3} = d_{3m+4} = 1$ and $d_j = d_{m+1+j} = d_{2m+2+j} = d_{3m+3+j} = 1 - j\varepsilon$, for $j \in \{2, 3, \ldots, m+1\}$, where

$$0 < \varepsilon < \frac{1}{m+1} min\big\{(1 - \frac{\bar{u}+\bar{t}}{m\bar{u}^2}), (1 - \frac{\bar{u}+\bar{t}}{m\bar{t}^2}), (1 - \frac{\bar{v}+\bar{w}}{m\bar{v}^2}), (1 - \frac{\bar{v}+\bar{w}}{m\bar{w}^2})\big\}. \tag{3.25}$$

It is obvious that D is invertible. Computing $D\mathcal{P}D^{-1}$ gives us the matrix

$$\mathcal{P}^{(1)} = \begin{bmatrix} \mathcal{P}^{(1)}_{\bar{t}} & \mathcal{P}^{(1)}_0 & \mathcal{P}^{(1)}_0 & \mathcal{P}^{(1)}_1 \\ \mathcal{P}^{(1)}_0 & \mathcal{P}^{(1)}_{\bar{w}} & \mathcal{P}^{(1)}_2 & \mathcal{P}^{(1)}_0 \\ \mathcal{P}^{(1)}_0 & \mathcal{P}^{(1)}_3 & \mathcal{P}^{(1)}_{\bar{v}} & \mathcal{P}^{(1)}_0 \\ \mathcal{P}^{(1)}_4 & \mathcal{P}^{(1)}_0 & \mathcal{P}^{(1)}_0 & \mathcal{P}^{(1)}_{\bar{u}} \end{bmatrix}_{(4m+4)\times(4m+4)}, \tag{3.26}$$

where

$$\mathcal{P}^{(1)}_{\bar{u}} = \begin{bmatrix} \frac{1}{m\bar{u}} & 0 & \cdots & 0 & 0 \\ d_{3m+5}d^{-1}_{3m+4} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & d_{4m+4}d^{-1}_{4m+3} & 0 \end{bmatrix}, \mathcal{P}^{(1)}_{\bar{v}} = \begin{bmatrix} \frac{1}{m\bar{v}} & 0 & \cdots & 0 & 0 \\ d_{2m+4}d^{-1}_{2m+3} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & d_{3m+3}d^{-1}_{3m+2} & 0 \end{bmatrix},$$

$$\mathcal{P}^{(1)}_{\bar{w}} = \begin{bmatrix} \frac{1}{m\bar{w}} & 0 & \cdots & 0 & 0 \\ d_{m+3}d^{-1}_{m+2} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & d_{2m+2}d^{-1}_{2m+1} & 0 \end{bmatrix}, \mathcal{P}^{(1)}_{\bar{t}} = \begin{bmatrix} \frac{1}{m\bar{t}} & 0 & \cdots & 0 & 0 \\ d_2 d^{-1}_1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & d_{m+1}d^{-1}_m & 0 \end{bmatrix},$$

$$\mathcal{P}^{(1)}_0 = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix},$$

$$\mathcal{P}^{(1)}_1 = \begin{bmatrix} 0 & \frac{-\bar{u}}{m^2\bar{t}^2} d_1 d^{-1}_{3m+5} & \cdots & \frac{-\bar{u}}{m^2\bar{t}^2} d_1 d^{-1}_{4m+3} & \frac{-\bar{u}}{m^2\bar{t}^2} d_1 d^{-1}_{4m+4} \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix},$$

$$\mathcal{P}^{(1)}_2 = \begin{bmatrix} 0 & \frac{-\bar{v}}{m^2\bar{w}^2} d_{m+2} d^{-1}_{2m+4} & \cdots & \frac{\bar{v}}{m^2\bar{w}^2} d_{m+2} d^{-1}_{3m+2} & \frac{-\bar{v}}{m^2\bar{w}^2} d_{m+2} d^{-1}_{3m+3} \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix},$$

$$\mathcal{P}^{(1)}_3 = \begin{bmatrix} 0 & \frac{-\bar{w}}{m^2\bar{v}^2} d_{2m+3} d^{-1}_{m+3} & \cdots & \frac{-\bar{w}}{m^2\bar{v}^2} d_{2m+3} d^{-1}_{2m+1} & \frac{-\bar{w}}{m^2\bar{v}^2} d_{2m+3} d^{-1}_{2m+2} \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix},$$

$$\mathcal{P}^{(1)}_4 = \begin{bmatrix} 0 & \frac{-\bar{t}}{m^2\bar{u}^2} d_{3m+4} d^{-1}_2 & \cdots & \frac{-\bar{t}}{m^2\bar{u}^2} d_{3m+4} d^{-1}_m & \frac{-\bar{t}}{m^2\bar{u}^2} d_{3m+4} d^{-1}_{m+1} \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix},$$

are $(m+1)\times(m+1)$ matrices. Also,

$$\begin{aligned} &0 < d_{m+1} < d_m < \cdots < d_1, \\ &0 < d_{2m+2} < d_{2m+1} < \cdots < d_{m+2}, \\ &0 < d_{3m+3} < d_{3m+2} < \cdots < d_{2m+3}, \\ &0 < d_{4m+4} < d_{4m+3} < \cdots < d_{3m+4}, \end{aligned}$$

implies that

$$\begin{aligned} d_2 d_1^{-1} &< 1, \\ d_3 d_2^{-1} &< 1, \\ &\vdots \\ d_{m+1} d_m^{-1} &< 1, \\ d_{m+3} d_{m+2}^{-1} &< 1, \\ &\vdots \\ d_{2m+2} d_{2m+1}^{-1} &< 1, \\ &. \end{aligned}$$

and

$$\begin{aligned} d_{2m+4} d_{2m+3}^{-1} &< 1, \\ &\vdots \\ d_{3m+3} d_{3m+2}^{-1} &< 1, \\ d_{3m+5} d_{3m+4}^{-1} &< 1, \\ &\vdots \\ d_{4m+4} d_{4m+3}^{-1} &< 1. \end{aligned}$$

Moreover, by using (3.9), (3.23) and (3.25), we obtain

$$\begin{aligned} \frac{1}{m\bar{t}} + \frac{\bar{u}}{m^2\bar{t}^2} d_1 d_{3m+5}^{-1} + \cdots + \frac{\bar{u}}{m^2\bar{t}^2} d_1 d_{4m+4}^{-1} &= \frac{1}{m\bar{t}} + \Big(\frac{1}{1-2\varepsilon} + \cdots + \frac{1}{1-(m+1)\varepsilon}\Big) \frac{\bar{u}}{m^2\bar{t}^2} \\ &< \frac{1}{m\bar{t}} + \frac{1}{1-(m+1)\varepsilon} \frac{\bar{u}}{m\bar{t}^2} \\ &< \frac{1}{1-(m+1)\varepsilon} \Big(\frac{1}{m\bar{t}} + \frac{\bar{u}}{m\bar{t}^2}\Big) \\ &< 1, \end{aligned}$$

$$\begin{aligned} \frac{1}{m\bar{w}} + \frac{\bar{v}}{m^2\bar{w}^2} d_{m+2} d_{2m+4}^{-1} + \cdots + \frac{\bar{v}}{m^2\bar{w}^2} d_{m+2} d_{3m+3}^{-1} &= \frac{1}{m\bar{w}} + \Big(\frac{1}{1-2\varepsilon} + \cdots + \frac{1}{1-(m+1)\varepsilon}\Big) \frac{\bar{v}}{m^2\bar{w}^2} \\ &< \frac{1}{m\bar{w}} + \frac{1}{1-(m+1)\varepsilon} \frac{\bar{v}}{m\bar{w}^2} \\ &< \frac{1}{1-(m+1)\varepsilon} \Big(\frac{1}{m\bar{w}} + \frac{\bar{v}}{m\bar{w}^2}\Big) \\ &< 1, \end{aligned}$$

$$\begin{aligned} \frac{1}{m\bar{v}} + \frac{\bar{w}}{m^2\bar{v}^2} d_{2m+3} d_{m+3}^{-1} + \cdots + \frac{\bar{w}}{m^2\bar{v}^2} d_{2m+3} d_{2m+2}^{-1} &= \frac{1}{m\bar{v}} + \Big(\frac{1}{1-2\varepsilon} + \cdots + \frac{1}{1-(m+1)\varepsilon}\Big) \frac{\bar{w}}{m^2\bar{v}^2} \\ &< \frac{1}{m\bar{v}} + \frac{1}{1-(m+1)\varepsilon} \frac{\bar{w}}{m\bar{v}^2} \\ &< \frac{1}{1-(m+1)\varepsilon} \Big(\frac{1}{m\bar{v}} + \frac{\bar{w}}{m\bar{v}^2}\Big) \\ &< 1 \end{aligned}$$

and

$$\begin{aligned} \frac{1}{m\bar{u}} + \frac{\bar{t}}{m^2\bar{u}^2} d_{3m+4} d_2^{-1} + \cdots + \frac{\bar{t}}{m^2\bar{w}^2} d_{3m+4} d_{m+1}^{-1} &= \frac{1}{m\bar{u}} + \Big(\frac{1}{1-2\varepsilon} + \cdots + \frac{1}{1-(m+1)\varepsilon}\Big) \frac{\bar{t}}{m^2\bar{u}^2} \\ &< \frac{1}{m\bar{u}} + \frac{1}{1-(m+1)\varepsilon} \frac{\bar{t}}{m\bar{u}^2} \\ &< \frac{1}{1-(m+1)\varepsilon} \Big(\frac{1}{m\bar{t}} + \frac{\bar{u}}{m\bar{u}^2}\Big) \\ &< 1. \end{aligned}$$

Since $\mathcal{P}$ and $D\mathcal{P}D^{-1}$ has the same eigenvalues, for $j \in \{1,2,\dots,4m+4\}$, we can write the following inequality as

$$\begin{aligned}\max|\mathcal{P}_j| \le ||D\mathcal{P}D^{-1}||_\infty &= \max\left\{\begin{array}{l} d_2d_1^{-1},...,d_{m+1}d_m^{-1},d_{m+3}d_{m+2}^{-1},\dots,d_{2m+2}d_{2m+1}^{-1},\\ d_{2m+4}d_{2m+3}^{-1},\dots,d_{3m+3}d_{3m+2}^{-1},d_{3m+5}d_{3m+4}^{-1},\dots,d_{4m+4}d_{4m+3}^{-1},\\ \frac{1}{m\bar{t}}+\left(\frac{1}{1-2\varepsilon}+\cdots+\frac{1}{1-(m+1)\varepsilon}\right)\frac{\bar{u}}{m^2\bar{t}^2},\\ \frac{1}{m\bar{w}}+\left(\frac{1}{1-2\varepsilon}+\cdots+\frac{1}{1-(m+1)\varepsilon}\right)\frac{\bar{v}}{m^2\bar{w}^2},\\ \frac{1}{m\bar{v}}+\left(\frac{1}{1-2\varepsilon}+\cdots+\frac{1}{1-(m+1)\varepsilon}\right)\frac{\bar{w}}{m^2\bar{v}^2},\\ \frac{1}{m\bar{u}}+\left(\frac{1}{1-2\varepsilon}+\cdots+\frac{1}{1-(m+1)\varepsilon}\right)\frac{\bar{t}}{m^2\bar{u}^2}\end{array}\right\}\\ &< 1.\end{aligned}$$

Therefore, the equilibrium point given in (3.15) is locally asymptotically stable. □

Theorem 3.5. *If the conditions (3.9) and (3.23) are satisfied, then the unique equilibrium point given in (3.15) of the system (3.8) is globally asymptotically stable.*

Theorem 3.6. *Consider the FDEs (1.2) for all $\gamma \in (0,1]$. If*

$$\frac{1}{m} < \min\{\tau_{1,l},\tau_{1,r},\tau_{2,l},\tau_{2,r}\}, \tag{3.27}$$

then every positive solution (α_n,β_n) of the FDEs (1.2) is bounded and persistent.

Proof. Let (α_n,β_n) be a solution of (1.2) and satisfy (3.27). Then, we have

$$\begin{cases}[\alpha_n]^\gamma &= [L^\gamma_{\alpha_n},R^\gamma_{\alpha_n}], \quad [\beta_n]^\gamma = [L^\gamma_{\beta_n},R^\gamma_{\beta_n}],\\ [\tau_1]^\gamma &= [\tau^\gamma_{1,l},\tau^\gamma_{1,r}], \quad [\tau_2]^\gamma = [\tau^\gamma_{2,l},\tau^\gamma_{2,r}].\end{cases} \tag{3.28}$$

From (3.4) and Theorem (3.2), we have

$$\begin{aligned}\tau_{1,l} \le L_{\alpha_n} &\le \frac{1}{(m\tau_{2,r})^{n-m}}\left(L_{\alpha_m}-\frac{m\tau_{1,l}\tau_{2,r}}{m\tau_{2,r}-1}\right)+\frac{m\tau_{1,l}\tau_{2,r}}{m\tau_{2,r}-1},\\ \tau_{1,r} \le R_{\alpha_n} &\le \frac{1}{(m\tau_{2,l})^{n-m}}\left(R_{\alpha_m}-\frac{m\tau_{1,r}\tau_{2,l}}{m\tau_{2,l}-1}\right)+\frac{m\tau_{1,r}\tau_{2,l}}{m\tau_{2,l}-1},\\ \tau_{2,l} \le L_{\beta_n} &\le \frac{1}{(m\tau_{1,r})^{n-m}}\left(L_{\beta_m}-\frac{m\tau_{1,r}\tau_{2,l}}{m\tau_{1,r}-1}\right)+\frac{m\tau_{1,r}\tau_{2,l}}{m\tau_{1,r}-1},\\ \tau_{2,r} \le R_{\beta_n} &\le \frac{1}{(m\tau_{1,l})^{n-m}}\left(R_{\beta_m}-\frac{m\tau_{1,l}\tau_{2,r}}{m\tau_{1,l}-1}\right)+\frac{m\tau_{1,l}\tau_{2,r}}{m\tau_{1,l}-1}.\end{aligned} \tag{3.29}$$

Also, for all $\gamma \in (0,1]$, the support sets of τ_1,τ_2 are

$$\begin{aligned}&[\tau^\gamma_{1,l},\tau^\gamma_{1,r}] \subseteq [M_{\tau_1},N_{\tau_1}],\\ &[\tau^\gamma_{2,l},\tau^\gamma_{2,r}] \subseteq [M_{\tau_2},N_{\tau_2}].\end{aligned} \tag{3.30}$$

Moreover, left and right components of γ - cuts of $M_{\tau_1},N_{\tau_1},M_{\tau_2},N_{\tau_2}$ are positive real numbers. So, by using (3.29) and (3.30), for $\gamma \in (0,1]$, we obtain

$$\begin{aligned}[L^\gamma_{\alpha_n},R^\gamma_{\alpha_n}] &\subseteq \left[M_{\tau_1},\frac{1}{(mN_{\tau_{2,r}})^{n-m}}\left(M_{\alpha_m}-\frac{mM_{\tau_{1,l}}N_{\tau_{2,r}}}{mN_{\tau_{2,r}}-1}\right)+\frac{mM_{\tau_{1,l}}N_{\tau_{2,r}}}{mN_{\tau_{2,r}}-1}\right],\\ [L^\gamma_{\beta_n},R^\gamma_{\beta_n}] &\subseteq \left[N_{\tau_2},\frac{1}{(mM_{\tau_{1,l}})^{n-m}}\left(N_{\beta_m}-\frac{mM_{\tau_{1,l}}N_{\tau_{2,r}}}{mN_{\tau_{2,r}}-1}\right)+\frac{mM_{\tau_{1,l}}N_{\tau_{2,r}}}{mM_{\tau_{1,l}}-1}\right].\end{aligned} \tag{3.31}$$

from which along with there exist m_1,m_2,M_1,M_2 such that $m_1 \le M_{\tau_1}$,$m_2 \le N_{\tau_2}$, $\frac{1}{(mN_{\tau_{2,r}})^{n-m}}\left(M_{\alpha_m}-\frac{mM_{\tau_{1,l}}N_{\tau_{2,r}}}{mN_{\tau_{2,r}}-1}\right)+\frac{mM_{\tau_{1,l}}N_{\tau_{2,r}}}{mN_{\tau_{2,r}}-1} \le M_1$, and $\frac{1}{(mM_{\tau_{1,l}})^{n-m}}\left(N_{\beta_m}-\frac{mM_{\tau_{1,l}}N_{\tau_{2,r}}}{mN_{\tau_{2,r}}-1}\right)+\frac{mM_{\tau_{1,l}}N_{\tau_{2,r}}}{mM_{\tau_{1,l}}-1} \le M_2$.

Therefore,

$$[L^{\gamma}_{\alpha_{n,l}}, R^{\gamma}_{\alpha_{n,r}}] \subseteq \overline{\bigcup_{\gamma\in(0,1]} [L^{\gamma}_{\alpha_{n,l}}, R^{\gamma}_{\alpha_{n,r}}]} \subseteq [m_1, M_1],$$
$$[L^{\gamma}_{\beta_{n,l}}, R^{\gamma}_{\beta_{n,r}}] \subseteq \overline{\bigcup_{\gamma\in(0,1]} [L^{\gamma}_{\beta_{n,l}}, R^{\gamma}_{\beta_{n,r}}]} \subseteq [m_2, M_2].$$

Hence, every positive solution of FDEs system (1.2) is persistent and bounded. This completes proof. □

Theorem 3.7. *Let us consider the FDEs (1.2). If (3.27) holds, then the positive solution (α_n, β_n) of (1.2) converges to a unique equilibrium point $(\bar{\alpha}, \bar{\beta})$ as $n \to \infty$, where*

$$\bar{\alpha} = \left[\frac{m^2\tau_{1,l}\tau_{2,r}-1}{m(m\tau_{2,r}-1)}, \frac{m^2\tau_{1,r}\tau_{2,l}-1}{m(m\tau_{2,l}-1)}\right], \bar{\beta} = \left[\frac{m^2\tau_{1,r}\tau_{2,l}-1}{m(m\tau_{1,r}-1)}, \frac{m^2\tau_{1,l}\tau_{2,r}-1}{m(m\tau_{1,l}-1)}\right]. \tag{3.32}$$

Proof. Since (3.2) and (3.3) hold and also from Theorem (3.3), we can write

$$\lim_{n\to\infty} L^{\gamma}_{\alpha_n} = l_{\alpha_n} = \frac{m^2\tau_{1,l}\tau_{2,r}-1}{m(m\tau_{2,r}-1)}, \lim_{n\to\infty} R^{\gamma}_{\alpha_n} = r_{\alpha_n} = \frac{m^2\tau_{1,r}\tau_{2,l}-1}{m(m\tau_{2,l}-1)},$$
$$\lim_{n\to\infty} L^{\gamma}_{\beta_n} = l_{\beta_n} = \frac{m^2\tau_{1,r}\tau_{2,l}-1}{m(m\tau_{1,r}-1)}, \lim_{n\to\infty} R^{\gamma}_{\beta_n} = r_{\beta_n} = \frac{m^2\tau_{1,l}\tau_{2,r}-1}{m(m\tau_{1,l}-1)}.$$

Thus, we get

$$\lim_{n\to\infty} D\big(\alpha_n, \bar{\alpha}\big) = \lim_{n\to\infty} D(\alpha_n - [l_{\alpha_n}, r_{\alpha_n}]) = \lim_{n\to\infty} \sup\max\{|L^{\gamma}_{\alpha_n} - l_{\alpha_n}|, |R^{\gamma}_{\alpha_n} - r_{\alpha_n}|\} = 0,$$
$$\lim_{n\to\infty} D\big(\beta_n, \bar{\beta}\big) = \lim_{n\to\infty} D(\beta_n - [l_{\beta_n}, r_{\beta_n}]) = \lim_{n\to\infty} \sup\max\{|L^{\gamma}_{\beta_n} - l_{\beta_n}|, |R^{\gamma}_{\beta_n} - r_{\beta_n}|\} = 0.$$

So, $\lim_{n\to\infty} \alpha_n = \bar{\alpha}$ and $\lim_{n\to\infty} \beta_n = \bar{\beta}$ means that every positive solution of equation (1.2) converges to equilibrium point $\big(\bar{\alpha}, \bar{\beta}\big)$ as $n \to \infty$. □

4. Numerical Results

In this section we will give some numerical examples in order to verify the efficiency of the results.

Example 4.1. *Consider following system when $m = 4$ for system (1.2):*

$$\alpha_{n+1} = \tau_1 + \frac{\alpha_n}{\beta_{n-1}+\beta_{n-2}+\beta_{n-3}+\beta_{n-4}}, \beta_{n+1} = \tau_2 + \frac{\beta_n}{\alpha_{n-1}+\alpha_{n-2}+\alpha_{n-3}+\alpha_{n-4}}.$$

Also the parameters τ_1, τ_2 and the initial conditions α_{-i}, β_{-i}, for $i = \{0,1,2,3,4\}$, are triangular fuzzy numbers, respectively,

$$\tau_1(x) = \begin{cases} \frac{5x}{2} - 1, & 0.4 \le x \le 0.8 \\ -\frac{5x}{2} + 3, & 0.8 \le x \le 1.2 \end{cases}, \qquad \tau_2(x) = \begin{cases} \frac{10x}{3} - 1, & 0.3 \le x \le 0.6 \\ \frac{-10x}{3} + 3, & 0.6 \le x \le 0.9 \end{cases}, \tag{4.1}$$

$$\alpha_{-i}(x) = \begin{cases} 5x - \frac{13}{2}, & 1.3 \le x \le 1.5 \\ -5x + \frac{17}{2}, & 1.5 \le x \le 1.7 \end{cases}, \quad \beta_{-i}(x) = \begin{cases} 5x - \frac{13}{2}, & 1.3 \le x \le 1.5 \\ -5x + \frac{17}{2}, & 1.5 \le x \le 1.7 \end{cases}. \tag{4.2}$$

By using (4.1) and (4.2), the bounded support sets for $\gamma \in (0,1]$ are as follows

$$\begin{cases} supp\tau_1 \subseteq [0.4, 1.2], & supp\tau_2 \subseteq [0.3, 0.9], \\ supp\alpha_{-i} \subseteq [1.3, 1.7], & supp\beta_{-i} \subseteq [1.3, 1.7]. \end{cases} \tag{4.3}$$

This example shows persistence and boundedness of FDEs system (1.2) if condition $\frac{1}{m} < \min\{\tau_{1,l}, \tau_{1,r}, \tau_{2,l}, \tau_{2,r}\}$ is satisfied. Moreover, note that as $n \to \infty$, every positive solution of FDEs system (1.2) converges to a unique equilibrium point $(\tilde{\alpha}, \tilde{\beta})$ in given (3.32) as it can be seen in Figure (1). Figure (2) shows the attractors of system (1.2) for $\gamma = 0.2$, $\gamma = 0.5$, $\gamma = 0.8$ and $\gamma = 1$.

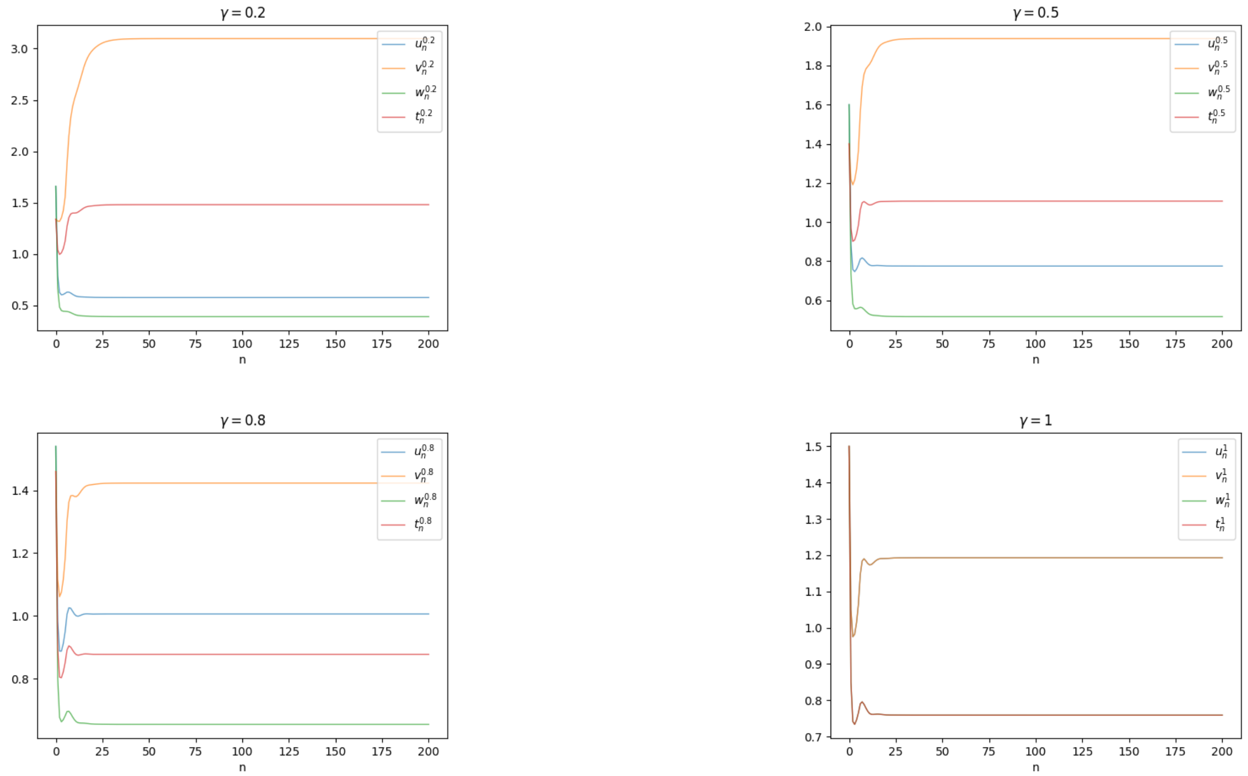

Figure 1: The solution of FDEs system (1.2) at $\gamma = 0.2$, $\gamma = 0.5$, $\gamma = 0.8$, $\gamma = 1$.

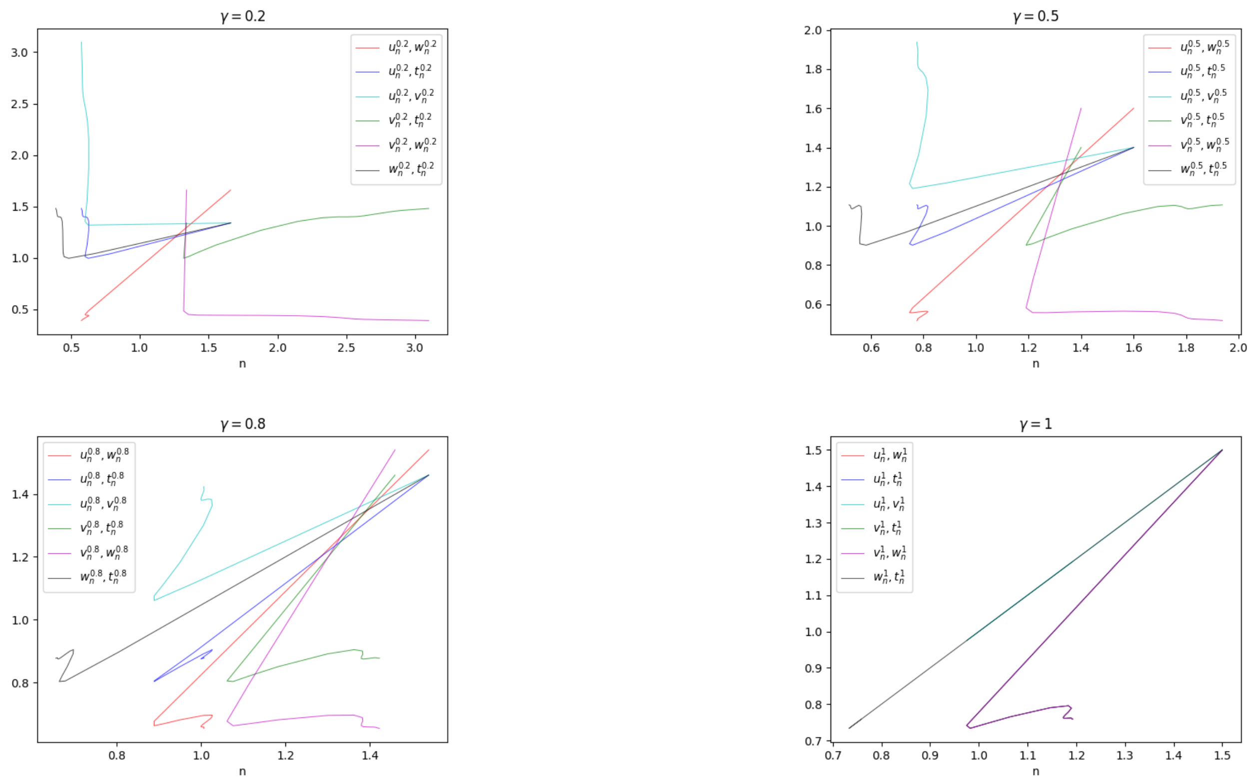

Figure 2: The attractors of FDEs system (1.2) at $\gamma = 0.2$, $\gamma = 0.5$, $\gamma = 0.8$, $\gamma = 1$.

Example 4.2. *Consider following system when $m = 3$ for* (1.2).

$$\alpha_{n+1} = \tau_1 + \frac{\alpha_n}{\beta_{n-1} + \beta_{n-2} + \beta_{n-3}}, \beta_{n+1} = \tau_2 + \frac{\beta_n}{\alpha_{n-1} + \alpha_{n-2} + \alpha_{n-3}}. \tag{4.4}$$

Also the parameters τ_1, τ_2 *and the initial conditions* α_{-i}, β_{-i}, *for* $i = \{0,1,2,3\}$, *are triangular fuzzy numbers, respectively,*

$$\tau_1(x) = \begin{cases} 5x-3, & 0.6 \le x \le 0.8 \\ -5x+5, & 0.8 \le x \le 1 \end{cases}, \qquad \tau_2(x) = \begin{cases} 5x-2, & 0.4 \le x \le 0.6 \\ -5x+4, & 0.6 \le x \le 0.8 \end{cases}, \tag{4.5}$$

$$\alpha_{-i}(x) = \begin{cases} 5x-\frac{3}{2}, & 0.3 \le x \le 0.5 \\ -5x+\frac{7}{2}, & 0.5 \le x \le 0.7 \end{cases}, \quad \beta_{-i}(x) = \begin{cases} 5x-\frac{3}{2}, & 0.3 \le x \le 0.5 \\ -5x+\frac{7}{2}, & 0.5 \le x \le 0.7 \end{cases}. \tag{4.6}$$

By using (4.5) *and* (4.6), *the bounded support sets for* $\gamma \in (0,1]$ *are as follows*

$$\begin{cases} supp\tau_1 \subseteq [0.6,1], & supp\tau_2 \subseteq [0.4,0.8], \\ supp\alpha_{-i} \subseteq [0.3,0.7], & supp\beta_{-i} \subseteq [0.3,0.7]. \end{cases} \tag{4.7}$$

This example shows persistence and boundedness of FDEs system (1.2) *if condition* $\frac{1}{m} < \min\{\tau_{1,l}, \tau_{1,r}, \tau_{2,l}, \tau_{2,r}\}$ *is satisfied. Moreover, note that as* $n \to \infty$, *every positive solution of FDEs system* (1.2) *converges to a unique equilibrium point* $(\tilde{\alpha}, \tilde{\beta})$ *in given* (3.32) *as it can be seen in Figure* (3). *Figure* (3) *shows the attractors of system* (1.2) *for* $\gamma = 0.2$, $\gamma = 0.5$, $\gamma = 0.8$ *and* $\gamma = 1$.

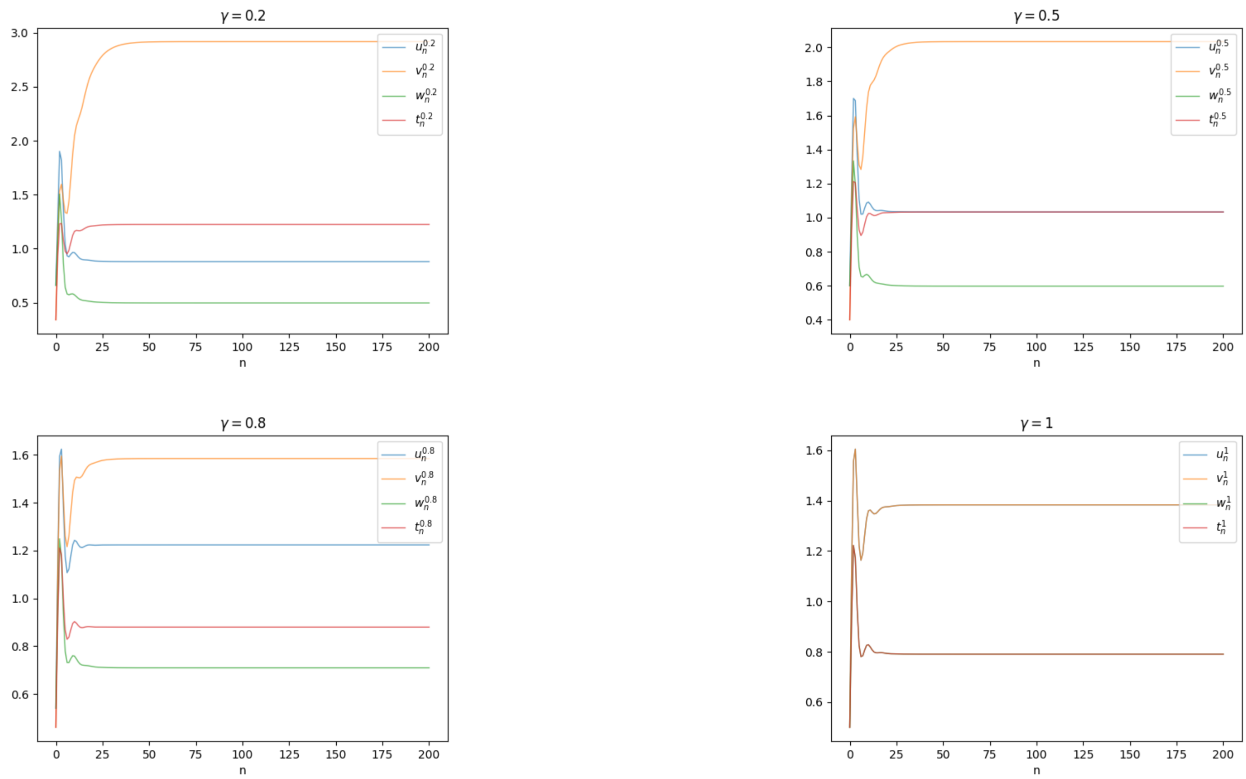

Figure 3: The solution of FDEs system (1.2) at $\gamma = 0.2$, $\gamma = 0.5$, $\gamma = 0.8$, $\gamma = 1$.

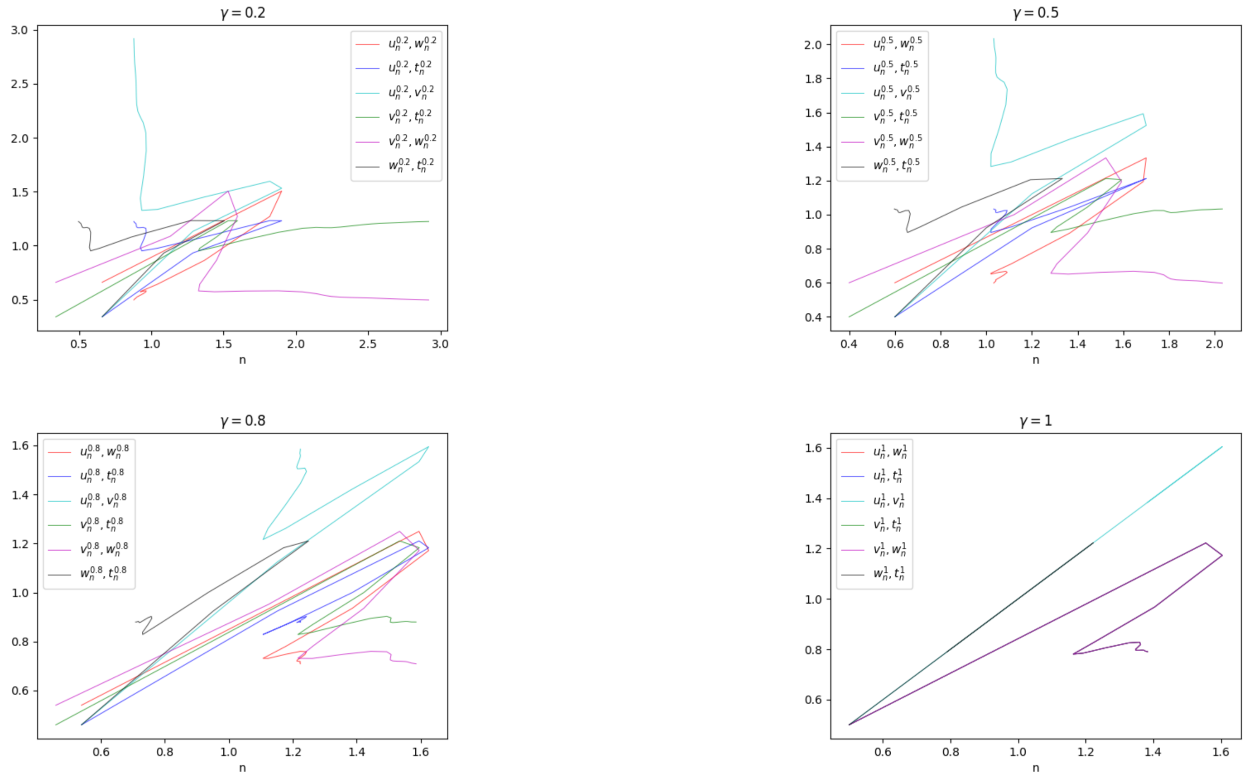

Figure 4: The attractors of FDEs system (1.2) at $\gamma = 0.2$, $\gamma = 0.5$, $\gamma = 0.8$, $\gamma = 1$.

Declarations

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Properties of a Subclass of Harmonic Univalent Functions Using the Al-Oboudi $q-$Differential Operator

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Abstract

In this paper, we introduce the Al-Oboudi $q-$differential operator, a generalized Sălăgean operator, for harmonic functions and define a new subclass of harmonic univalent functions using this operator. We investigate several fundamental properties of this subclass, including coefficient conditions, extreme points, distortion bounds, convex combination, and radii of convexity.

1. Introduction

Let $\mathbb{C}$ denote the complex plane and consider the open unit disk $\mathbb{E} = \{z \in \mathbb{C} : |z| < 1\}$. For a harmonic function $\mathfrak{f} = \mathfrak{u} + \overline{\mathfrak{v}}$ to be sense-preserving and locally univalent in the open unit disk $\mathbb{E}$, it is necessary and sufficient that the inequality $|\mathfrak{v}'(z)| < |\mathfrak{u}'(z)|$ holds in $\mathbb{E}$ (see [1]).

The class of functions that are harmonic, sense-preserving, univalent, and normalized by $\mathfrak{f}(0) = \mathfrak{f}_z(0) - 1 = 0$ in the open unit disk $\mathbb{E}$ is denoted by $\mathcal{SH}$. Within this class, the subclass of functions $\mathfrak{f} \in \mathcal{SH}$ that additionally satisfy $\mathfrak{v}'(0) = b_1 = 0$ is denoted by $\mathcal{SH}^0$. The functions $\mathfrak{u}$ and $\mathfrak{v}$ are analytic in the open unit disk $\mathbb{E}$ and have series expansions:

$$\mathfrak{u}(z) = z + \sum_{s=2}^{\infty} a_s z^s, \quad \mathfrak{v}(z) = \sum_{s=2}^{\infty} b_s z^s. \tag{1.1}$$

A function $\mathfrak{f} \in \mathcal{SH}^0$ can be expressed as $\mathfrak{f} = \mathfrak{u} + \overline{\mathfrak{v}}$. If we choose $\mathfrak{v}(z) = 0$, we obtain the class $\mathcal{S}$, which consists of analytic, univalent and normalized functions in $\mathbb{E}$. The relationships $\mathcal{S} \subset \mathcal{SH}^0 \subset \mathcal{SH}$ hold for the function classes $\mathcal{S}$, $\mathcal{SH}$, and $\mathcal{SH}^0$.

The subclasses $\mathcal{K}$ and $\mathcal{S}^*$ of $\mathcal{S}$ are characterized by their mappings of the unit disk $\mathbb{E}$ onto convex and starlike domains, respectively. Similarly, the subclasses of $\mathcal{SH}^0$ that map the unit disk $\mathbb{E}$ onto corresponding domains are denoted by $\mathcal{SH}^{0,*}$ and $\mathcal{KH}^0$. For a more detailed discussion, see [1, 2].

Jackson's q-derivative for a function $\psi \in \mathcal{S}$, where $0 < q < 1$, is defined as follows [3]:

$$D_q\psi(z) = \begin{cases} \dfrac{\psi(z) - \psi(qz)}{(1-q)z}, & \text{if } z \neq 0, \\ \psi'(0), & \text{if } z = 0. \end{cases} \tag{1.2}$$

Note that if ψ is differentiable at z, then as $q \to 1^-$, we have $D_q\psi(z) \to \psi'(z)$.

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Jackson also defined the q-integral as follows [4]:

$$\int_0^z \psi(\zeta)\, d_q\zeta = z(1-q)\sum_{k=0}^{\infty} q^k \psi(zq^k), \tag{1.3}$$

provided that the series on the right-hand side converges.

Jahangiri et al. [5] introduced the *modified Sălăgean q-differential operator* for harmonic functions of the form $\mathfrak{f} = \mathfrak{u} + \overline{\mathfrak{v}}$, where $\mathfrak{m} \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}$. This operator is defined as

$$D_q^{\mathfrak{m}} \mathfrak{f}(z) = D_q^{\mathfrak{m}} \mathfrak{u}(z) + (-1)^{\mathfrak{m}} \overline{D_q^{\mathfrak{m}} \mathfrak{v}(z)}, \tag{1.4}$$

where

$$D_q^{\mathfrak{m}} \mathfrak{u}(z) = z + \sum_{s=2}^{\infty} [s]_q^{\mathfrak{m}} a_s z^s \quad \text{and} \quad D_q^{\mathfrak{m}} \mathfrak{v}(z) = \sum_{s=2}^{\infty} [s]_q^{\mathfrak{m}} b_s z^s. \tag{1.5}$$

For a harmonic function $\mathfrak{f} = \mathfrak{u} + \overline{\mathfrak{v}}$, where $\mathfrak{m} \in \mathbb{N}_0$ and $\delta \geq 0$, we define the *modified Al-Oboudi q-differential operator* $D_{\delta,q}^{\mathfrak{m}} \mathfrak{f}(z)$ as follows:

$$D_{\delta,q}^{0} \mathfrak{f}(z) = D_q^{0} \mathfrak{f}(z) = \mathfrak{u}(z) + \overline{\mathfrak{v}(z)}, \tag{1.6}$$

$$D_{\delta,q}^{1} \mathfrak{f}(z) = (1-\delta) D_q^{0} \mathfrak{f}(z) + \delta D_q^{1} \mathfrak{f}(z), \tag{1.7}$$

$$\vdots$$

$$D_{\delta,q}^{\mathfrak{m}} \mathfrak{f}(z) = D_{\delta,q}^{1} \left(D_{\delta,q}^{\mathfrak{m}-1} \mathfrak{f}(z) \right). \tag{1.8}$$

Using the expression of $\mathfrak{f}$ given in (1.1), it follows from (1.7) and (1.8) that

$$D_{\delta,q}^{\mathfrak{m}} \mathfrak{f}(z) = z + \sum_{s=2}^{\infty} \left[\delta([s]_q - 1) + 1\right]^{\mathfrak{m}} a_s z^s + (-1)^{\mathfrak{m}} \sum_{s=2}^{\infty} \left[\delta([s]_q + 1) - 1\right]^{\mathfrak{m}} \overline{b_s z^s}. \tag{1.9}$$

We note that the operator $D_{\delta,q}^{\mathfrak{m}} \mathfrak{f}(z)$ reduces to several known differential operators for specific choices of the parameters δ, q. More precisely:

- For $\delta = 1$, the operator coincides with the $q-$analogue of the modified Sălăgean operator studied by Jahangiri et al. [5].
- As $q \to 1^-$, the operator becomes the generalization of the modified Sălăgean operator investigated by Yaşar and Yalçın [6].
- For $\delta = 1$ and $q \to 1^-$, we recover the modified Sălăgean differential operator defined by Jahangiri et al. [7].
- If $\mathfrak{v}(z) \equiv 0$, the operator reduces to the generalized $q-$Sălăgean operator introduced by Aouf et al. [8].
- If $\mathfrak{v}(z) \equiv 0$ and $q \to 1^-$, the operator reduces to the Al-Oboudi differential operator [9].
- For $\mathfrak{v}(z) \equiv 0$, $q \to 1^-$, and $\delta = 1$, we obtain the classical Sălăgean differential operator [10].

In 2019, Ahuja and Çetinkaya [11] introduced the class of $q-$harmonic, sense-preserving, and univalent functions $\mathfrak{f} = \mathfrak{u} + \overline{\mathfrak{v}}$, denoted by $\mathcal{SH}_q$. For a function $\mathfrak{f}$ to be included in class $\mathcal{SH}_q$, it must meet the following requirements:

$$\omega(z) = \left| \frac{D_q \mathfrak{v}(z)}{D_q \mathfrak{u}(z)} \right| < 1.$$

Additionally, as $q \to 1^-$, the class $\mathcal{SH}$ is recovered.

For $0 \leq \alpha < 1$, the class of harmonic functions $f = \mathfrak{u} + \overline{\mathfrak{v}} \in \mathcal{SH}_q$ that satisfy the inequality

$$\operatorname{Re}\left\{ \frac{z D_q \mathfrak{u}(z) - \overline{z D_q \mathfrak{v}(z)}}{\mathfrak{u}(z) + \overline{\mathfrak{v}}(z)} \right\} > \alpha$$

is denoted by $\mathcal{SH}_q^*(\alpha)$. Functions in this class are referred to as $q-$starlike harmonic functions of order α. Similarly, for $0 \leq \alpha < 1$, the class of harmonic functions $f = \mathfrak{u} + \overline{\mathfrak{v}} \in \mathcal{SH}_q$ satisfying the inequality

$$\operatorname{Re}\left\{ \frac{z D_q (z D_q \mathfrak{u}(z)) - \overline{z D_q (z D_q \mathfrak{v}(z))}}{z D_q \mathfrak{u}(z) - \overline{z D_q \mathfrak{v}(z)}} \right\} > \alpha$$

is denoted by $\mathcal{KH}_q(\alpha)$. The functions in this class are called $q-$convex harmonic functions of order α. For a more detailed discussion, see [12, 13, 14].

Let

$$\mathfrak{f}_k(z) = z + \sum_{s=2}^{\infty} a_{k,s} z^s + \sum_{s=2}^{\infty} \overline{b_{k,s} z^s} \quad (z \in \mathbb{E}, k = 1,2),$$

then the functions $\mathfrak{f}_1$ and $\mathfrak{f}_2$ have the following Hadamard product (or convolution):

$$(\mathfrak{f}_1 * \mathfrak{f}_2)(z) = z + \sum_{s=2}^{\infty} a_{1,s} a_{2,s} z^s + \sum_{s=2}^{\infty} \overline{b_{1,s} b_{2,s} z^s} \quad (z \in \mathbb{E}).$$

Furthermore, if $\mathfrak{f} \in \mathcal{SH}_q$, we obtain

$$\begin{aligned} D_{\delta,q}^{\mathfrak{m}} \mathfrak{f}(z) &= \mathfrak{f}(z) * \underbrace{\left(\chi_1(z) + \overline{\chi_2(z)}\right) * \cdots * \left(\chi_1(z) + \overline{\chi_2(z)}\right)}_{\mathfrak{m} \text{ times}}, \\ &= \mathfrak{u}(z) * \underbrace{\chi_1(z) * \cdots * \chi_1(z)}_{\mathfrak{m} \text{ times}} + \mathfrak{v}(z) * \underbrace{\chi_2(z) * \cdots * \chi_2(z)}_{\mathfrak{m} \text{ times}}. \end{aligned}$$

where

$$\chi_1(z) = \frac{(\delta-1)qz^2 + z}{(1-z)(1-qz)}, \quad \chi_2(z) = \frac{(\delta-1)qz^2 + (1-2\delta)z^2}{(1-z)(1-qz)}.$$

A function $\mathfrak{f} : \mathbb{E} \to \mathbb{C}$ is said to be subordinate to another function $\mathfrak{g} : \mathbb{E} \to \mathbb{C}$, denoted by $\mathfrak{f}(z) \prec \mathfrak{g}(z)$, if there exists a complex-valued function ω mapping $\mathbb{E}$ into itself with $\omega(0) = 0$, such that $\mathfrak{f}(z) = \mathfrak{g}(\omega(z))$ (see [15]).

Denote $\mathcal{SH}_q^0(\delta, \mathfrak{m}, \eta, \mu)$ as the subclass of $\mathcal{SH}_q^0$ consisting of functions $\mathfrak{f}$ of the form (1.1) that satisfy the condition:

$$\frac{D_{\delta,q}^{\mathfrak{m}+1} \mathfrak{f}(z)}{D_{\delta,q}^{\mathfrak{m}} \mathfrak{f}(z)} \prec \frac{1+\eta z}{1+\mu z}, \quad -\mu \leq \eta < \mu \leq 1. \tag{1.10}$$

As $q \to 1^-$, this class converges to $\mathcal{SH}^0(\delta, \mathfrak{m}, \eta, \mu)$ introduced by Çakmak et al. [16]. Additionally, for $q \to 1^-$ and with the choices of specific parameters, the following classes are obtained, which have been previously studied:

(i) $\mathcal{SH}^0(1, \delta, \eta, \mu) = H_\delta(\eta, \mu), \quad \delta \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ ([17]),

(ii) $\mathcal{SH}^0(1, 1, \eta, \mu) = S_H^*(\eta, \mu) \cap \mathcal{SH}^0$ ([18]),

(iii) $\mathcal{SH}^0(\delta, \mathfrak{m}, 2\alpha - 1, 1) = \mathcal{SH}(\delta, \mathfrak{m}, \alpha) \cap \mathcal{SH}^0$ ([6]),

(iv) $\mathcal{SH}^0(1, \mathfrak{m}, 2\alpha - 1, 1) = H^0(\mathfrak{m}, \alpha)$ ([7]),

(v) $\mathcal{SH}^0(1, 0, 2\alpha - 1, 1) = S_{H^0}^*(\alpha)$ ([19], [20], [21]),

(vi) $\mathcal{SH}^0(1, 1, 2\alpha - 1, 1) = S_{H^0}^c(\alpha)$ ([19]),

(vii) $\mathcal{SH}^0(\delta, \mathfrak{m}, 2\alpha - 1, 1) = \overline{\mathcal{SH}}(\delta, 1-\delta, \mathfrak{m}, \alpha)$ ([22]).

Further details on these classes and their properties can be found in the works of Jahangiri et al. [23], Murugusundaramoorthy et al. [24] and Canbulat et al. [25].

The aim of this paper is to advance the study of harmonic functions by introducing the Al-Oboudi $q-$differential operator, an extension of the well-known Sălăgean operator. We define a new subclass of harmonic univalent functions using this generalized operator, which allows us to explore several key properties of these functions. Building on techniques and methodologies from Dziok ([18], [26]), and Dziok et al. ([17]), we analyze fundamental aspects of this subclass, including coefficient conditions, extreme points, distortion bounds and radii of convexity. Through these investigations, our aim is to enhance the understanding of harmonic function theory and provide new insights into the geometric behavior of these functions.

2. Main Theorems and Results

First, we establish a necessary and sufficient condition involving convolution for harmonic functions in $\mathcal{SH}_q^0(\delta,\mathfrak{m},\eta,\mu)$.

Theorem 2.1. *Let $z\in\mathbb{E}\setminus\{0\}$ and suppose that $\mathfrak{f}$ belongs to $\mathcal{SH}_q^0$. The function $\mathfrak{f}$ is an element of $\mathcal{SH}_q^0(\delta,\mathfrak{m},\eta,\mu)$ if and only if the following condition is satisfied:*

$$D_{\delta,q}^{\mathfrak{m}}\mathfrak{f}(z)*\chi(z;\zeta)\neq 0 \text{ for all } (\zeta\in\mathbb{C} \text{ with } |\zeta|=1),$$

where

$$\chi(z;\zeta)=\frac{[(\eta-\mu)\zeta+\delta(1+\mu\zeta)]qz^2+(\mu-\eta)\zeta z}{(1-qz)(1-z)}-(-1)^{\mathfrak{m}}\frac{[-\delta(1+\mu\zeta)+(\mu-\eta)\zeta]q\bar{z}^2+[2\delta(1+\mu\zeta)-(\mu-\eta)\zeta]\bar{z}}{(1-q\bar{z})(1-\bar{z})}.$$

Proof. Let $\mathfrak{f}\in\mathcal{SH}_q^0$. The condition $\mathfrak{f}\in\mathcal{SH}_q^0(\delta,\mathfrak{m},\eta,\mu)$ is satisfied if and only if the condition (1.10) holds, which is equivalent to

$$\frac{D_{\delta,q}^{\mathfrak{m}+1}\mathfrak{f}(z)}{D_{\delta,q}^{\mathfrak{m}}\mathfrak{f}(z)}\neq\frac{1+\eta\zeta}{1+\mu\zeta} \text{ for } (\zeta\in\mathbb{C},|\zeta|=1). \tag{2.1}$$

Consider

$$D_{\delta,q}^{\mathfrak{m}}\mathfrak{f}(z)=D_{\delta,q}^{\mathfrak{m}}\mathfrak{f}(z)*\left(\frac{z}{1-z}+\frac{\bar{z}}{1-\bar{z}}\right),$$

and

$$D_{\delta,q}^{\mathfrak{m}+1}\mathfrak{f}(z)=D_{\delta,q}^{\mathfrak{m}}\mathfrak{f}(z)*\left(\chi_1(z)+\overline{\chi_2(z)}\right),$$

then the inequality (2.1) leads to

$$\begin{aligned}(1+\mu\zeta)D_{\delta,q}^{\mathfrak{m}+1}\mathfrak{f}(z)-(1+\eta\zeta)D_{\delta,q}^{\mathfrak{m}}\mathfrak{f}(z) &= D_{\delta,q}^{\mathfrak{m}}\mathfrak{f}(z)*\left\{\frac{(1+\mu\zeta)\left[(\delta-1)qz^2+z\right]}{(1-qz)(1-z)}+\frac{(1+\mu\zeta)\left[(\delta-1)q\bar{z}^2+(1-2\delta)\bar{z}\right]}{(1-q\bar{z})(1-\bar{z})}\right\}\\ &\quad -D_{\delta,q}^{\mathfrak{m}}\mathfrak{f}(z)*\left\{\frac{(1+\eta\zeta)z}{1-z}+\frac{(1+\eta\zeta)\bar{z}}{1-\bar{z}}\right\}\\ &= D_{\delta,q}^{\mathfrak{m}}\mathfrak{f}(z)*\left\{\frac{[(\eta-\mu)\zeta+\delta(1+\mu\zeta)]qz^2+(\mu-\eta)\zeta z}{(1-qz)(1-z)}\right.\\ &\quad\left.-\frac{[-\delta(1+\mu\zeta)+(\mu-\eta)\zeta]q\bar{z}^2+[2\delta(1+\mu\zeta)-(\mu-\eta)\zeta]\bar{z}}{(1-q\bar{z})(1-\bar{z})}\right\}\\ &= D_{\delta,q}^{\mathfrak{m}}\mathfrak{f}(z)*\chi(z;\zeta)\neq 0.\end{aligned}$$

□

Theorem 2.2. *Let $\mathfrak{f}=\mathfrak{u}+\overline{\mathfrak{v}}\in\mathcal{SH}_q^0$, where $\mathfrak{u}$ and $\mathfrak{v}$ are represented as in (1.1). Then, $\mathfrak{f}\in\mathcal{SH}_q^0(\delta,\mathfrak{m},\eta,\mu)$ if the following inequality is satisfied:*

$$\sum_{k=2}^{\infty}\left(P_k|a_k|+Q_k|b_k|\right)\leq\mu-\eta, \tag{2.2}$$

where the sequences P_s and Q_s are given by:

$$P_s=\left[1+\delta\left([s]_q-1\right)\right]^{\mathfrak{m}}\left[\delta\left([s]_q-1\right)(\mu+1)+\mu-\eta\right], \tag{2.3}$$

and

$$Q_s=\left[-1+\delta\left([s]_q+1\right)\right]^{\mathfrak{m}}\left[\delta\left([s]_q+1\right)(\mu+1)+\eta-\mu\right]. \tag{2.4}$$

Proof. The theorem is evidently valid for $\mathfrak{f}(z)=z$. Now, consider the case where $a_s\neq 0$ or $b_s\neq 0$ for $s\geq 2$. Since $P_s\geq [s]_q(\mu-\eta)$ and $Q_s\geq [s]_q(\mu-\eta)$, from (2.2), we obtain:

$$\begin{aligned}\left|D_q\mathfrak{u}(z)\right|-\left|D_q\mathfrak{v}(z)\right| &\geq 1-\sum_{s=2}^{\infty}[s]_q|a_s||z|^{s-1}-\sum_{s=2}^{\infty}[s]_q|b_s||z|^{s-1}\\ &\geq 1-|z|\sum_{s=2}^{\infty}[s]_q(|a_s|+|b_s|)\\ &\geq 1-\frac{|z|}{\mu-\eta}\sum_{s=2}^{\infty}(P_s|a_s|+Q_s|b_s|)\\ &\geq 1-|z|>0.\end{aligned}$$

Thus, $\mathfrak{f}$ belongs to $\mathcal{SH}_q^0$.

A function $\mathfrak{f}$ belongs to the class $\mathcal{SH}_q^0(\delta,\mathfrak{m},\eta,\mu)$ if there exists a complex-valued function ω such that $\omega(0)=0$ and $|\omega(z)|<1$ for all $z\in U$. This condition is met if and only if the following holds:

$$\frac{D_{\delta,q}^{\mathfrak{m}+1}\mathfrak{f}(z)}{D_{\delta,q}^{\mathfrak{m}}\mathfrak{f}(z)}=\frac{1+\eta\omega(z)}{1+\mu\omega(z)},$$

which is equivalent to the inequality:

$$\left|\frac{D_{\delta,q}^{\mathfrak{m}+1}\mathfrak{f}(z)-D_{\delta,q}^{\mathfrak{m}}\mathfrak{f}(z)}{\mu D_{\delta,q}^{\mathfrak{m}+1}\mathfrak{f}(z)-\eta D_{\delta,q}^{\mathfrak{m}}\mathfrak{f}(z)}\right|<1,\quad z\in\mathbb{E}. \tag{2.5}$$

The inequality in (2.5) holds because for $|z|=r$ with $0<r<1$, we have:

$$\begin{aligned}&\left|D_{\delta,q}^{\mathfrak{m}+1}\mathfrak{f}(z)-D_{\delta,q}^{\mathfrak{m}}\mathfrak{f}(z)\right|-\left|\mu D_{\delta,q}^{\mathfrak{m}+1}\mathfrak{f}(z)-\eta D_{\delta,q}^{\mathfrak{m}}\mathfrak{f}(z)\right|\\ =&\left|\sum_{s=2}^{\infty}[\delta([s]_q-1)+1]^{\mathfrak{m}}\delta([s]_q-1)a_sz^s-(-1)^{\mathfrak{m}}\sum_{s=2}^{\infty}[\delta([s]_q+1)-1]^{\mathfrak{m}}\delta([s]_q+1)\overline{b_sz^s}\right|\\ -&\left|(\mu-\eta)z+\sum_{s=2}^{\infty}[\delta([s]_q-1)+1]^{\mathfrak{m}}[\delta\mu([s]_q-1)+\mu-\eta]a_sz^s\right.\\ &\left.-(-1)^{\mathfrak{m}}\sum_{s=2}^{\infty}[\delta([s]_q+1)-1]^{\mathfrak{m}}[\delta\mu([s]_q+1)-\mu+\eta]\overline{b_sz^s}\right|\\ \leq&\sum_{s=2}^{\infty}[\delta([s]_q-1)+1]^{\mathfrak{m}}\delta([s]_q-1)|a_s|r^s+\sum_{s=2}^{\infty}[\delta([s]_q+1)-1]^{\mathfrak{m}}\delta([s]_q+1)|b_s|r^s\\ -&(\mu-\eta)r+\sum_{s=2}^{\infty}[\delta([s]_q-1)+1]^{\mathfrak{m}}[\delta\mu([s]_q-1)+\mu-\eta]|a_s|r^s\\ +&\sum_{s=2}^{\infty}[\delta([s]_q+1)-1]^{\mathfrak{m}}[\delta\mu([s]_q+1)-\mu+\eta]|b_s|r^s\\ <&\ 1.\end{aligned}$$

Therefore, $\mathfrak{f}\in\mathcal{SH}_q^0(\delta,\mathfrak{m},\eta,\mu)$, completing the proof. □

Next, we demonstrate that the condition given in (2.2) is also a necessary criterion for a function $\mathfrak{f}\in\mathcal{SH}_q^0$ to belong to the class $\mathcal{TSH}_q^0(\delta,\mathfrak{m},\eta,\mu)=\mathcal{T}^{\mathfrak{m}}\cap\mathcal{SH}^0(\delta,\mathfrak{m},\eta,\mu)$, where $\mathcal{T}^{\mathfrak{m}}$ represents the set of functions $\mathfrak{f}=\mathfrak{u}+\overline{\mathfrak{v}}\in\mathcal{SH}_q^0$ such that

$$\mathfrak{f}(z)=\mathfrak{u}(z)+\overline{\mathfrak{v}(z)}=z-\sum_{s=2}^{\infty}|a_s|z^s+(-1)^{\mathfrak{m}}\sum_{s=2}^{\infty}|b_s|\bar{z}^s,\quad z\in\mathbb{E}. \tag{2.6}$$

Theorem 2.3. *Consider the definition of $\mathfrak{f}=\mathfrak{u}+\overline{\mathfrak{v}}$ in (2.6). Then, $\mathfrak{f}\in\mathcal{TSH}_q^0(\delta,\mathfrak{m},\eta,\mu)$ if and only if condition (2.2) is satisfied.*

Proof. The sufficiency of this condition follows directly from Theorem 2.2. To prove necessity, suppose $\mathfrak{f}\in\mathcal{TSH}_q^0(\delta,\mathfrak{m},\eta,\mu)$. Using (2.5), we can write

$$\left| \frac{\sum\limits_{s=2}^{\infty} ([s]_q - 1)\,\delta\,[([s]_q - 1)\,\delta + 1]^{\mathfrak{m}}\,|a_s|\,z^s + ([s]_q + 1)\,\delta\,[([s]_q + 1)\,\delta - 1]^{\mathfrak{m}}\,|b_s|\,\bar{z}^s}{(\mu - \eta)z - \sum\limits_{s=2}^{\infty} [([s]_q - 1)\,\delta\mu + \mu - \eta]\,[([s]_q - 1)\,\delta + 1]^{\mathfrak{m}}\,|a_s|\,z^s + [([s]_q + 1)\,\delta\mu + \eta - \mu]\,[([s]_q + 1)\,\delta - 1]^{\mathfrak{m}}\,|b_s|\,\bar{z}^s} \right| < 1.$$

For $z = r < 1$, this simplifies to

$$\frac{\sum\limits_{s=2}^{\infty} \left\{([s]_q - 1)\,\delta\,[([s]_q - 1)\,\delta + 1]^{\mathfrak{m}}\,|a_s| + ([s]_q + 1)\,\delta\,[([s]_q + 1)\,\delta - 1]^{\mathfrak{m}}\,|b_s|\right\} r^{s-1}}{\mu - \eta - \sum\limits_{s=2}^{\infty} \left\{[([s]_q - 1)\,\delta\mu + \mu - \eta]\,[([s]_q - 1)\,\delta + 1]^{\mathfrak{m}}\,|a_s| + [([s]_q + 1)\,\delta\mu + \eta - \mu]\,[([s]_q + 1)\,\delta - 1]^{\mathfrak{m}}\,|b_s|\right\} r^{s-1}} < 1.$$

Therefore, for the terms P_s and Q_s as defined in (2.3) and (2.4), we have the inequality

$$\sum_{s=2}^{\infty} [P_s\,|a_s| + Q_s\,|b_s|]\, r^{s-1} < \mu - \eta \quad (0 \le r < 1). \tag{2.7}$$

Let $\{\sigma_s\}$ be the sequence defined by the partial sums of the series given by

$$\sum_{s=2}^{\infty} [P_s\,|a_s| + Q_s\,|b_s|]\,.$$

Since $\{\sigma_s\}$ is non-decreasing and bounded above by $\mu - \eta$, it must converge, and hence

$$\sum_{s=2}^{\infty} [P_s\,|a_s| + Q_s\,|b_s|] = \lim_{s\to\infty} \sigma_s \le \mu - \eta.$$

This establishes condition (2.2). □

In the following, we demonstrate that the function class given in equation (2.6) is both convex and compact.

Theorem 2.4. *The class $\mathcal{TSH}_q^0(\delta, \mathfrak{m}, \eta, \mu)$ is convex and compact within the space $\mathcal{SH}_q^0$.*

Proof. Consider a sequence $\mathfrak{f}_k \in \mathcal{TSH}_q^0(\delta, \mathfrak{m}, \eta, \mu)$, where

$$\mathfrak{f}_k(z) = z - \sum_{s=2}^{\infty} \left|a_{k,s}\right| z^s + (-1)^{\mathfrak{m}} \sum_{s=2}^{\infty} \left|b_{k,s}\right| \overline{z^s}, \quad z \in \mathbb{E},\ k \in \mathbb{N}. \tag{2.8}$$

To prove convexity, let $0 \le \lambda \le 1$, and suppose $\mathfrak{f}_1$ and $\mathfrak{f}_2$ belong to the class $\mathcal{TSH}_q^0(\delta, \mathfrak{m}, \eta, \mu)$, with each defined as in (2.8). Define a new function as

$$\begin{aligned} \kappa(z) &= \lambda \mathfrak{f}_1(z) + (1-\lambda)\mathfrak{f}_2(z) \\ &= z - \sum_{s=2}^{\infty} (\lambda\,|a_{1,s}| + (1-\lambda)\,|a_{2,s}|)\, z^s + (-1)^{\mathfrak{m}} \sum_{s=2}^{\infty} (\lambda\,|b_{1,s}| + (1-\lambda)\,|b_{2,s}|)\, \overline{z^s}. \end{aligned}$$

Next, we verify that $\kappa(z)$ also belongs to the class $\mathcal{TSH}_q^0(\delta, \mathfrak{m}, \eta, \mu)$. To achieve this, we examine the following condition:

$$\begin{aligned} \sum_{s=2}^{\infty} \{P_s\,[\lambda\,|a_{1,s}| + (1-\lambda)\,|a_{2,s}|] + Q_s\,[\lambda\,|b_{1,s}| + (1-\lambda)\,|b_{2,s}|]\} &= \lambda \sum_{s=2}^{\infty} \{P_s\,|a_{1,s}| + Q_s\,|b_{1,s}|\} + (1-\lambda) \sum_{s=2}^{\infty} \{P_s\,|a_{2,s}| + Q_s\,|b_{2,s}|\} \\ &\le \lambda(\mu - \eta) + (1-\lambda)(\mu - \eta) = \mu - \eta. \end{aligned}$$

Hence, $\kappa(z)$ remains within the class $\mathcal{TSH}_q^0(\delta, \mathfrak{m}, \eta, \mu)$, establishing the convexity of the class.

To demonstrate compactness, consider any function $\mathfrak{f}_k \in \mathcal{TSH}_q^0(\delta, \mathfrak{m}, \eta, \mu)$. We derive the following inequality for $|z| \le r$ $(0 < r < 1)$:

$$\begin{aligned} |\mathfrak{f}_k(z)| &\le r + \sum_{s=2}^{\infty} \left\{\left|a_{k,s}\right| + \left|b_{k,s}\right|\right\} r^s \\ &\le r + \sum_{s=2}^{\infty} \left\{P_s \left|a_{k,s}\right| + Q_s \left|b_{k,s}\right|\right\} r^s \\ &\le r + (\mu - \eta) r^2. \end{aligned}$$

This confirms that the class $\mathcal{TSH}_q^0(\delta,\mathfrak{m},\eta,\mu)$ is locally uniformly bounded.

Now, consider the sequence $\mathfrak{f}_k(z)$ given by $z-\sum\limits_{s=2}^{\infty}\left|a_{k,s}\right|z^s+(-1)^{\mathfrak{m}}\sum\limits_{s=2}^{\infty}\left|b_{k,s}\right|\overline{z^s}$. Let $\mathfrak{f}=\mathfrak{u}+\overline{\mathfrak{v}}$, where $\mathfrak{u}$ and $\mathfrak{v}$ are as described in equation (1.1). From Theorem 2.3, we have the inequality

$$\sum_{s=2}^{\infty}\{P_s\left|a_s\right|+Q_s\left|b_s\right|\}\leq\mu-\eta. \tag{2.9}$$

If $\mathfrak{f}_k\to\mathfrak{f}$, it follows that $\left|a_{k,s}\right|\to\left|a_s\right|$ and $\left|b_{k,s}\right|\to\left|b_s\right|$ as $k\to\infty$. The sequence $\{\sigma_s\}$, which represents the partial sums of the series $\sum_{s=2}^{\infty}\{P_s\left|a_s\right|+Q_s\left|b_s\right|\}$, is both monotonic and upper-bounded by $\mu-\eta$. Consequently, it is convergent. Therefore, we have

$$\sum_{s=2}^{\infty}\{P_s\left|a_s\right|+Q_s\left|b_s\right|\}=\lim_{s\to\infty}\sigma_s\leq\mu-\eta.$$

Thus, $\mathfrak{f}$ belongs to the class $\mathcal{TSH}_q^0(\delta,\mathfrak{m},\eta,\mu)$, and it follows that this class is closed. Consequently, the class $\mathcal{TSH}_q^0(\delta,\mathfrak{m},\eta,\mu)$ is compact within $\mathcal{SH}_q^0$. □

We now present the following result, originally established by Jahangiri [5].

Lemma 2.5 ([5])**.** *Consider the $q-$harmonic mapping $\mathfrak{f}=\mathfrak{u}+\overline{\mathfrak{v}}$, where $\mathfrak{u}$ and $\mathfrak{v}$ are defined as in (1.1). Suppose that the following condition is satisfied:*

$$\sum_{s=2}^{\infty}\left\{\frac{[s]_q-\alpha}{1-\alpha}\left|a_s\right|+\frac{[s]_q+\alpha}{1-\alpha}\left|b_s\right|\right\}\leq 1 \quad (z\in\mathbb{E}),$$

where $0\leq\alpha<1$. Consequently, the function $\mathfrak{f}$ belongs to the class $\mathcal{SH}_q^{0,*}(\alpha)$.

For functions belonging to the class $\mathcal{TSH}_q^0(\delta,\mathfrak{m},\eta,\mu)$, the radii of starlikeness and convexity are given by the following theorems.

Theorem 2.6. *Let $0\leq\alpha<1$, and let P_s and Q_s be defined by equations (2.3) and (2.4), respectively. Then*

$$r_\alpha^*(\mathcal{TSH}_q^0(\delta,\mathfrak{m},\eta,\mu))=\inf_{k\geq 2}\left[\frac{1-\alpha}{\mu-\eta}\min\left\{\frac{P_s}{[s]_q-\alpha},\frac{Q_s}{[s]_q+\alpha}\right\}\right]^{\frac{1}{s-1}}. \tag{2.10}$$

Proof. Let $\mathfrak{f}\in\mathcal{TSH}_q^0(\delta,\mathfrak{m},\eta,\mu)$ be represented by the form in (2.6). For $|z|=r<1$, the following holds:

$$\begin{aligned}\left|\frac{D_{1,q}\mathfrak{f}(z)-(1+\alpha)\mathfrak{f}(z)}{D_{1,q}\mathfrak{f}(z)+(1-\alpha)\mathfrak{f}(z)}\right| &= \left|\frac{-\alpha z-\sum_{s=2}^{\infty}([s]_q-1-\alpha)\left|a_s\right|z^s-(-1)^{\mathfrak{m}}\sum_{s=2}^{\infty}([s]_q+1+\alpha)\left|b_s\right|\overline{z}^s}{(2-\alpha)z-\sum_{s=2}^{\infty}([s]_q+1-\alpha)\left|a_s\right|z^s-(-1)^{\mathfrak{m}}\sum_{s=2}^{\infty}([s]_q-1+\alpha)\left|b_s\right|\overline{z}^s}\right| \\ &\leq \frac{\alpha+\sum_{s=2}^{\infty}\{([s]_q-1-\alpha)|a_s|+([s]_q+1+\alpha)|b_s|\}r^{s-1}}{2-\alpha-\sum_{s=2}^{\infty}\{([s]_q+1-\alpha)|a_s|+([s]_q-1+\alpha)|b_s|\}r^{s-1}}.\end{aligned}$$

According to Lemma 2.5, the function $\mathfrak{f}$ is $q-$starlike of order α in $\mathbb{E}_r$ if and only if

$$\left|\frac{D_{1,q}\mathfrak{f}(z)-(1+\alpha)\mathfrak{f}(z)}{D_{1,q}\mathfrak{f}(z)+(1-\alpha)\mathfrak{f}(z)}\right|<1,\ z\in\mathbb{E}_r$$

which is equivalent to:

$$\sum_{s=2}^{\infty}\left\{\frac{[s]_q-\alpha}{1-\alpha}\left|a_s\right|+\frac{[s]_q+\alpha}{1-\alpha}\left|b_s\right|\right\}r^{s-1}\leq 1. \tag{2.11}$$

Furthermore, by Theorem 2.2, the following condition must be satisfied:

$$\sum_{s=2}^{\infty}\left\{\frac{P_s}{\mu-\eta}\left|a_s\right|+\frac{Q_s}{\mu-\eta}\left|b_s\right|\right\}r^{s-1}\leq 1.$$

The inequality in (2.11) holds if:

$$\frac{[s]_q-\alpha}{1-\alpha}r^{s-1}\leq\frac{P_s}{\mu-\eta}r^{s-1},$$

$$\frac{[s]_q+\alpha}{1-\alpha}r^{s-1} \leq \frac{Q_s}{\mu-\eta}r^{s-1} \quad (s=2,3,\ldots),$$

or equivalently:

$$r \leq \frac{1-\alpha}{\mu-\eta}\min\left\{\frac{P_s}{[s]_q-\alpha},\frac{Q_s}{[s]_q+\alpha}\right\}^{\frac{1}{s-1}} \quad (s=2,3,\ldots).$$

Therefore, the function $\mathfrak{f}$ is $q-$starlike of order α in the disk $\mathbb{E}_{r_\alpha^*}$, where:

$$r_\alpha^* := \inf_{s\geq 2}\left[\frac{1-\alpha}{\mu-\eta}\min\left\{\frac{P_s}{[s]_q-\alpha},\frac{Q_s}{[s]_q+\alpha}\right\}\right]^{\frac{1}{s-1}}.$$

Finally, the extremal function:

$$\mathfrak{f}_s(z)=\mathfrak{u}_s(z)+\overline{\mathfrak{v}_s(z)}=z-\frac{\mu-\eta}{P_s}z^s+(-1)^{\mathfrak{m}}\frac{\mu-\eta}{Q_s}\overline{z}^s$$

shows that the radius r_α^* cannot be increased. Thus, we obtain the result (2.10). □

Using a similar approach, we derive the following result.

Theorem 2.7. *Let $0\leq\alpha<1$, and let P_s and Q_s be defined as in (2.3) and (2.4). Then, we have*

$$r_\alpha^c(\mathcal{TSH}_q^0(\delta,\mathfrak{m},\eta,\mu))=\inf_{s\geq 2}\left[\frac{1-\alpha}{\mu-\eta}\min\left\{\frac{P_s}{[s]_q([s]_q-\alpha)},\frac{Q_s}{[s]_q([s]_q+\alpha)}\right\}\right]^{\frac{1}{s-1}}.$$

Our next result concerns the extreme points of the class $\mathcal{TSH}_q^0(\delta,\mathfrak{m},\eta,\mu)$.

Theorem 2.8. *The functions $\mathfrak{u}=\mathfrak{u}_s$ and $\mathfrak{v}=\mathfrak{v}_s$ are defined as follows:*

$$\begin{aligned}\mathfrak{u}_1(z)&=z,\\ \mathfrak{u}_s(z)&=z-\tfrac{\mu-\eta}{P_s}z^s,\\ \mathfrak{v}_s(z)&=(-1)^{\mathfrak{m}}\tfrac{\mu-\eta}{Q_s}\overline{z^s} \quad (z\in\mathbb{E},\ s\geq 2).\end{aligned} \tag{2.12}$$

The functions $\mathfrak{f}=\mathfrak{u}+\overline{\mathfrak{v}}$, which are represented by the series expansion given in 1.1, are the extreme points of class $\mathcal{TSH}_q^0(\delta,\mathfrak{m},\eta,\mu)$.

Proof. Consider the function $\mathfrak{v}_s$ defined by

$$\mathfrak{v}_s=\lambda\mathfrak{f}_1+(1-\lambda)\mathfrak{f}_2,$$

where $0<\lambda<1$ and $\mathfrak{f}_1$ and $\mathfrak{f}_2$ are functions in the class $\mathcal{TSH}_q^0(\delta,\mathfrak{m},\eta,\mu)$. Each function $\mathfrak{f}_k$ is given by

$$\mathfrak{f}_k(z)=z-\sum_{s=2}^{\infty}|a_{k,s}|z^s+(-1)^{\mathfrak{m}}\sum_{s=2}^{\infty}|b_{k,s}|\overline{z^s},$$

where z is in $\mathbb{E}$ and k is either 1 or 2.
By (2.12), it follows that

$$|b_{1,s}|=|b_{2,s}|=\frac{\mu-\eta}{Q_s},$$

which implies $a_{1,k}=a_{2,k}=0$ for $k\in\{2,3,\ldots\}$ and $b_{1,k}=b_{2,k}=0$ for $k\in\{2,3,\ldots\}\setminus\{s\}$. Consequently, $\mathfrak{v}_s(z)=\mathfrak{f}_1(z)=\mathfrak{f}_2(z)$, and $\mathfrak{v}_s$ lies in the class of extreme points of $\mathcal{SH}_T^0(\delta,n,\eta,\mu)$. Similarly, the functions $\mathfrak{u}_s(z)$ can be verified as the extreme points of $\mathcal{TSH}_q^0(\delta,\mathfrak{m},\eta,\mu)$.
Now, assume that a function $\mathfrak{f}$ of the form (1.1) is an extreme point of $\mathcal{TSH}_q^0(\delta,\mathfrak{m},\eta,\mu)$ and that $\mathfrak{f}$ does not match the form (2.12). Then, there exists $n\in\{2,3,\ldots\}$ such that

$$0<|u_n|<\frac{\mu-\eta}{[([n]_q-1)\delta+1]^{\mathfrak{m}}[\delta([n]_q-1)(\mu+1)+\mu-\eta]}$$

or

$$0<|v_n|<\frac{\mu-\eta}{\left[\left([n]_q+1\right)\delta-1\right]^{\mathfrak{m}}\left[\delta\left([n]_q+1\right)(\mu+1)+\eta-\mu\right]}.$$

If

$$0<|u_n|<\frac{\mu-\eta}{\left[\left([n]_q-1\right)\delta+1\right]^{\mathfrak{m}}\left[\delta\left([n]_q-1\right)(\mu+1)+\mu-\eta\right]},$$

then setting

$$\lambda=\frac{|u_n|\left[\left([n]_q-1\right)\delta+1\right]^{\mathfrak{m}}\left[\delta\left([n]_q-1\right)(\mu+1)+\mu-\eta\right]}{\mu-\eta}$$

and

$$\psi=\frac{\mathfrak{f}-\lambda\mathfrak{u}_n}{1-\lambda},$$

we obtain $0<\lambda<1$ and $\mathfrak{u}_s\neq\psi$. Hence, $\mathfrak{f}$ is not an extreme point of $\mathcal{TSH}_q^0(\delta,\mathfrak{m},\eta,\mu)$.
Similarly, if

$$0<|v_n|<\frac{\mu-\eta}{\left[\left([n]_q+1\right)\delta-1\right]^{\mathfrak{m}}\left[\delta\left([n]_q+1\right)(\mu+1)+\eta-\mu\right]},$$

then setting

$$\lambda=\frac{|v_n|\left[\left([n]_q+1\right)\delta-1\right]^{\mathfrak{m}}\left[\delta\left([n]_q+1\right)(\mu+1)+\eta-\mu\right]}{\mu-\eta}$$

and

$$\psi=\frac{\mathfrak{f}-\lambda\mathfrak{v}_n}{1-\lambda},$$

results in $0<\lambda<1$ and $\mathfrak{v}_n\neq\psi$.
Therefore, $\mathfrak{f}$ is not an element of the set of extreme points in $\mathcal{TSH}_q^0(\delta,\mathfrak{m},\eta,\mu)$, thereby completing the proof. □

Consequently, according to Theorem 2.8, we obtain the following corollary.

Corollary 2.9. *Let $\mathfrak{f}$ be an element of $\mathcal{TSH}_q^0(\delta,\mathfrak{m},\eta,\mu)$, and let $|z|=r<1$. Then*

$$r-\frac{\mu-\eta}{(q\delta+1)^{\mathfrak{m}}\left[q\delta(\mu+1)+\eta-\mu\right]}r^2\leq|\mathfrak{f}(z)|\leq r+\frac{\mu-\eta}{(q\delta+1)^{\mathfrak{m}}\left[q\delta(\mu+1)+\eta-\mu\right]}r^2.$$

From Corollary 2.9, we can derive the following covering result.

Corollary 2.10. *If $\mathfrak{f}$ belongs to $\mathcal{TSH}_q^0(\delta,n,\eta,\mu)$, then $\mathbb{E}_r\subset\mathfrak{f}(\mathbb{E})$, where*

$$r=1-\frac{\mu-\eta}{(q\delta+1)^{\mathfrak{m}}\left[q\delta(\mu+1)+\eta-\mu\right]}.$$

3. Conclusion

In this paper, we introduced a new subclass of harmonic univalent functions by utilizing the generalized Al-Oboudi $q-$differential operator, which extends the classical Sălăgean operator within the framework of $q-$calculus. We derived several important results concerning the analytic and geometric properties of this subclass, including coefficient bounds, subordination conditions, extreme points, convolution characterizations, distortion theorems, and radii of starlikeness and convexity.

Furthermore, we demonstrate the compactness and convexity of the subclass $TSH_q^0(\delta,m,\eta,\mu)$, and established sharp bounds using extremal functions. The operator-theoretic approach adopted in this study not only generalizes many existing results in the literature but also provides a flexible framework for investigating broader families of harmonic mappings.

These findings contribute to the geometric function theory by enriching the structure of harmonic univalent function classes via $q-$calculus and highlight the potential of differential operators in unifying various known subclasses. Future work may focus on applying other classes of quantum differential operators or extending the current results to more general domains and functional settings.

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