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INDEXES AND DATABASES

IDUNAS is indexed by 4 national and international databases.

INDEXED DATABASES	
ACARINDEX	https://www.acarindex.com/search?type=article&q=Natural+and+Applied+Sciences+Journal
Academic Resource Index ResearchBib	https://journalseeker.researchbib.com/view/issn/2645-9000
Google Scholar	https://scholar.google.com.tr/scholar?q=izmir+democracy+university+natural+and+applied+sciences+journal&hl=tr&as_sdt=0&as_vis=1&oi=scholart
Ideal Online	https://www.idealonline.com.tr/IdealOnline/
APPLIED DATABASES	
Directory of Research Journals Indexing	http://olddrji.lbp.world/
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ABOUT

Natural & Applied Sciences Journal is an international open access, peer-reviewed, free of cost academic journal that includes articles and reviews on natural and applied sciences. It is published twice a year in June and December. Our journal accepts only English content from the second issue.

AIM & SCOPE

It aims to contribute to the knowledge of the field of natural and applied sciences by publishing qualified scientific studies in the field.

Natural & Applied Sciences Journal is an international open access, peer-reviewed, free of cost academic journal that includes articles and reviews on natural and applied sciences published by İzmir Democracy University. It is published twice a year in June and December. Our journal accepts only English content from the second issue.

PERIODS

June and December

ETHICAL PRINCIPLES AND PUBLICATION POLICY

All articles submitted for publication in the journal must conform to the ethical rules of scientific research. The authors of each article are required to sign the Copyright Form confirming that they have granted permission for their work to be published in Natural and Applied Science Journal. Publication will not take place, even if the manuscript is accepted without this form. Authors are solely responsible for the content of their articles and any responsibilities they may incur regarding copyrights. The work submitted to the journal should not have been published in any language, in any journal, or in the process of being evaluated in any other publication.

Articles should be prepared in accordance with the general ethical rules specified by DOI and DOAJ. Plagiarism Control All articles submitted to Natural and Applied Science Journal are checked using the iThenticate plagiarism detection software. Based on the similarity report generated by the software, the editorial board determines whether the article should be submitted to peer review or rejected.

PRICE POLICY

No fee is charged from the author or institution under any name.

INTERNATIONAL STANDARDS FOR AUTHORS

RESPONSIBLE RESEARCH PUBLICATION

A position statement developed at the 2nd World Conference on Research Integrity, Singapore, July 22-24, 2010.

Elizabeth Wager & Sabine Kleinert

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SUMMARY

- The research being reported should have been conducted in an ethical and responsible manner and should comply with all relevant legislation.

- Researchers should present their results clearly, honestly, and without fabrication, falsification, or inappropriate data manipulation.

- Researchers should strive to describe their methods clearly and unambiguously so that their findings can be confirmed by others.

- Researchers should adhere to publication requirements that submitted work is original, is not plagiarized, and has not been published elsewhere.

- Authors should take collective responsibility for submitted and published work.

- The authorship of research publications should accurately reflect individuals' contributions to the work and its reporting.

• Funding sources and relevant conflicts of interest should be disclosed.

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INTRODUCTION

Publication is the final stage of research and therefore a responsibility for all researchers. Scholarly publications are expected to provide a detailed and permanent record of research. Because publications form the basis for both new research and the application of findings, they can affect not only the research community but also, indirectly, society at large. Researchers therefore have a responsibility to ensure that their publications are honest, clear, accurate, complete and balanced, and should avoid misleading, selective or ambiguous reporting. Journal editors also have responsibilities for ensuring the integrity of the research literature and these are set out in companion guidelines.

This document aims to establish international standards for authors of scholarly research publications and to describe responsible research reporting practice. We hope these standards will be endorsed by research institutions, funders, and professional societies; promoted by editors and publishers; and will aid in research integrity training.

Responsible research publication

1 Soundness and reliability

1.1 The research being reported should have been conducted in an ethical and responsible manner and follow all relevant legislation. [See also the Singapore Statement on Research Integrity, www.singaporestatement.org]

1.2 The research being reported should be sound and carefully executed.

1.3 Researchers should use appropriate methods of data analysis and display (and, if needed, seek, and follow specialist advice on this).

1.4 Authors should take collective responsibility for their work and for the content of their publications. Researchers should check their publications carefully at all stages to ensure methods and findings are reported accurately. Authors should carefully check calculations, data presentations, typescripts/submissions, and proofs.

2 Honesty

2.1 Researchers should present their results honestly and without fabrication, falsification, or inappropriate data manipulation. Research images (e.g. micrographs, X-rays, pictures of electrophoresis gels) should not be modified in a misleading way.

2.2 Researchers should strive to describe their methods and to present their findings clearly and unambiguously. Researchers should follow applicable reporting guidelines. Publications should provide sufficient detail to permit experiments to be repeated by other researchers.

2.3 Reports of research should be complete. They should not omit inconvenient, inconsistent, or inexplicable findings or results that do not support the authors' or sponsors' hypothesis or interpretation.

2.4 Research funders and sponsors should not be able to veto publication of findings that do not favor their product or position. Researchers should not enter agreements that permit the research sponsor to veto or control the publication of the findings (unless there are exceptional circumstances, such as research classified by governments because of security implications).

2.5 Authors should alert the editor promptly if they discover an error in any submitted, accepted or published work. Authors should cooperate with editors in issuing corrections or retractions when required.

2.6 Authors should represent the work of others accurately in citations and quotations.

2.7 Authors should not copy references from other publications if they have not read the cited work.

3 Originality

3.1 Authors should adhere to publication requirements that submitted work is original and has not been published elsewhere in any language. Work should not be submitted concurrently to more than one publication unless the editors have agreed to co-publication. If articles are co-published this fact should be made clear to readers.

3.2 Applicable copyright laws and conventions should be followed. Copyright material (e.g. tables, figures, or extensive quotations) should be reproduced only with appropriate permission and acknowledgement.

3.3 Relevant previous work and publications, both by other researchers and the authors' own, should be properly acknowledged, and referenced. The primary literature should be cited where possible.

3.4 Data, text, figures, or ideas originated by other researchers should be properly acknowledged and should not be presented as if they were the authors' own. Original wording taken directly from publications by other researchers should appear in quotation marks with the appropriate citations.

3.5 Authors should inform editors if findings have been published previously or if multiple reports or multiple analyses of a single data set are under consideration for publication elsewhere. Authors should provide copies of related publications or work submitted to other journals.

3.6 Multiple publications arising from a single research project should be clearly identified as such and the primary publication should be referenced. Translations and adaptations for different audiences should be clearly identified as such, should acknowledge the original source, and should respect relevant copyright conventions and permission requirements. If in doubt, authors should seek permission from the original publisher before republishing any work.

4 Appropriate authorship and acknowledgement

4.1 The research literature serves as a record not only of what has been discovered but also of who made the discovery. The authorship of research publications should therefore accurately reflect individuals' contributions to the work and its reporting.

4.2 In cases where major contributors are listed as authors while those who made less substantial, or purely technical, contributions to the research or to the publication are listed in an acknowledgement section, the criteria for authorship and acknowledgement should be agreed at the start of the project. Ideally, authorship criteria within a particular field should be agreed, published and consistently applied by research institutions, professional and academic societies, and funders. While journal editors should publish and promote accepted authorship criteria appropriate to their field, they cannot be expected to adjudicate in authorship disputes. Responsibility for the correct attribution of authorship lies with authors themselves working under the guidance of their institution. Research institutions should promote and uphold fair and accepted standards of authorship and acknowledgement. When required, institutions should adjudicate in authorship disputes and should ensure that due process is followed.

4.3 Researchers should ensure that only those individuals who meet authorship criteria (i.e. made a substantial contribution to the work) are rewarded with authorship and that deserving authors are not omitted. Institutions and journal editors should encourage practices that prevent guest, gift, and ghost authorship.

Note:

• Guest authors are those who do not

• Gift authors are those who do meet accepted authorship criteria but are listed because of their seniority, reputation or supposed influence not

• Ghost authors are those who meet authorship criteria but are not listed meet accepted authorship criteria but are listed as a personal favor or in return for payment

4.4 All authors should agree to be listed and should approve the submitted and accepted versions of the publication. Any change to the author list should be approved by all authors including any who have been removed from the list. The corresponding author should act as a point of contact between the editor and the other authors and should keep co-authors informed and involve them in major decisions about the publication (e.g. responding to reviewers' comments).

4.5 Authors should not use acknowledgements misleadingly to imply a contribution or endorsement by individuals who have not, in fact, been involved with the work or given an endorsement.

5 Accountability and responsibility

5.1 All authors should have read and be familiar with the reported work and should ensure that publications follow the principles set out in these guidelines. In most cases, authors will be expected to take joint responsibility for the integrity of the research and its reporting. However, if authors take responsibility only for certain aspects of the research and its reporting, this should be specified in the publication.

5.2 Authors should work with the editor or publisher to correct their work promptly if errors or omissions are discovered after publication.

5.3 Authors should abide by relevant conventions, requirements, and regulations to make materials, reagents, software, or datasets available to other researchers who request them. Researchers, institutions, and funders should have clear policies for handling such requests. Authors must also follow relevant journal standards. While proper acknowledgement is expected, researchers should not demand authorship as a condition for sharing materials.

5.4 Authors should respond appropriately to post-publication comments and published correspondence. They should attempt to answer correspondents' questions and supply clarification, or additional details where needed.

6 Adherence to peer review and publication conventions

6.1 Authors should follow publishers' requirements that work is not submitted to more than one publication for consideration at the same time.

6.2 Authors should inform the editor if they withdraw their work from review or choose not to respond to reviewer comments after receiving a conditional acceptance.

6.3 Authors should respond to reviewers' comments in a professional and timely manner.

6.4 Authors should respect publishers' requests for press embargos and should not generally allow their findings to be reported in the press if they have been accepted for publication (but not yet published) in a scholarly publication. Authors and their institutions should liaise and cooperate with publishers to coordinate media activity (e.g. press releases and press conferences) around publication. Press releases should accurately reflect the work and should not include statements that go further than the research findings.

7 Responsible reporting of research involving humans or animals

7.1 Appropriate approval, licensing or registration should be obtained before the research begins and details should be provided in the report (e.g. Institutional Review Board, Research Ethics Committee approval, national licensing authorities for the use of animals).

7.2 If requested by editors, authors should supply evidence that reported research received the appropriate approval and was carried out ethically (e.g. copies of approvals, licenses, participant consent forms).

7.3 Researchers should not generally publish or share identifiable individual data collected in the course of research without specific consent from the individual (or their representative). Researchers should remember that many scholarly journals are now freely available on the internet and should therefore be mindful of the risk of causing danger or upset to unintended readers (e.g. research participants or their families who recognize themselves from case studies, descriptions, images, or pedigrees).

7.4 The appropriate statistical analyses should be determined at the start of the study and a data analysis plan for the prespecified outcomes should be prepared and followed. Secondary or post hoc analyses should be distinguished from primary analyses and those set out in the data analysis plan.

7.5 Researchers should publish all meaningful research results that might contribute to understanding. In particular, there is an ethical responsibility to publish the findings of all clinical trials. The publication of unsuccessful studies or experiments that reject a hypothesis may help prevent others from wasting time and resources on similar projects. If findings from small studies and those that fail to reach statistically significant results can be combined to produce more useful information (e.g. by meta-analysis) then such findings should be published.

7.6 Authors should supply research protocols to journal editors if requested (e.g. for clinical trials) so that reviewers and editors can compare the research report to the protocol to check that it was carried out as planned and that no relevant details have been omitted. Researchers should follow relevant requirements for clinical trial registration and should include the trial registration number in all publications arising from the trial.

INTERNATIONAL STANDARDS FOR EDITORS

RESPONSIBLE RESEARCH PUBLICATION

A position statement developed at the 2nd World Conference on Research Integrity, Singapore, July 22-24, 2010.

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Summary

• Editors are accountable and should take responsibility for everything they publish

- Editors should make fair and unbiased decisions independent from commercial consideration and ensure a fair and appropriate peer review process

- Editors should adopt editorial policies that encourage maximum transparency and complete, honest reporting

- Editors should guard the integrity of the published record by issuing corrections and retractions when needed and pursuing suspected or alleged research and publication misconduct

- Editors should pursue reviewer and editorial misconduct

- Editors should critically assess the ethical conduct of studies in humans and animals

- Peer reviewers and authors should be told what is expected of them

- Editors should have appropriate policies in place for handling editorial conflicts of interest

Introduction

As guardians and stewards of the research record, editors should encourage authors to strive for, and adhere themselves to, the highest standards of publication ethics. Furthermore, editors are in a unique position to indirectly foster responsible conduct of research through their policies and processes. To achieve the maximum effect within the research community, ideally all editors should adhere to universal standards and good practices. While there are important differences between different fields and not all areas covered are relevant to each research community, there are important common editorial policies, processes, and principles that editors should follow to ensure the integrity of the research record.

These guidelines are a starting point and are aimed at journal editors in particular. While books and monographs are important and relevant research records in many fields, guidelines for book editors are beyond the scope of these recommendations. It is hoped that in due course such guidelines can be added to this document.

Editors should regard themselves as part of the wider professional editorial community, keep themselves abreast of relevant policies and developments, and ensure their editorial staff is trained and kept informed of relevant issues.

To be a good editor requires many more principles than are covered here. These suggested principles, policies, and processes are particularly aimed at fostering research and publication integrity.

Editorial Principles

1. Accountability and responsibility for journal content

Editors have to take responsibility for everything they publish and should have procedures and policies in place to ensure the quality of the material they publish and maintain the integrity of the published record (see paragraphs 4-8).

2. Editorial independence and integrity

An important part of the responsibility to make fair and unbiased decisions is the upholding of the principle of editorial independence and integrity.

2.1 Separating decision-making from commercial considerations

Editors should make decisions on academic merit alone and take full responsibility for their decisions. Processes must be in place to separate commercial activities within a journal from editorial processes and decisions. Editors should take an active interest in the publisher's pricing policies and strive for wide and affordable accessibility of the material they publish.

Sponsored supplements must undergo the same rigorous quality control and peer review as any other content for the journal. Decisions on such material must be made in the same way as any other journal content. The sponsorship and role of the sponsor must be clearly declared to readers.

Advertisements need to be checked so that they follow journal guidelines, should be clearly distinguishable from other content, and should not in any way be linked to scholarly content.

2.2 Editors' relationship to the journal publisher or owner

Editors should ideally have a written contract setting out the terms and conditions of their appointment with the journal publisher or owner. The principle of editorial independence should be clearly stated in this contract. Journal publishers and owners should not have any role in decisions on content for commercial or political reasons. Publishers should not dismiss an editor because of any journal content unless there was gross editorial misconduct, or an independent investigation has concluded that the editor's decision to publish was against the journal's scholarly mission.

2.3 Journal metrics and decision-making

Editors should not attempt to inappropriately influence their journal's ranking by artificially increasing any journal metric. For example, it is inappropriate to demand that references to that journal's articles are included except for genuine scholarly reasons. In general, editors should ensure that papers are reviewed on purely scholarly grounds and that authors are not pressured to cite specific publications for non- scholarly reasons.

3. Editorial confidentiality

3.1 Authors' material

If a journal operates a system where peer reviewers are chosen by editors (rather than posting papers for all to comment as a pre-print version), editors must protect the confidentiality of authors' material and remind reviewers to do so as well. In general, editors should not share submitted papers with editors of other journals, unless with the authors' agreement or in cases of alleged misconduct (see below). Editors are generally under no obligation to provide material to lawyers for court cases. Editors should not give any indication of a paper's status with the journal to anyone other than the authors. Web-based submission systems must be run in a way that prevents unauthorized access.

In the case of a misconduct investigation, it may be necessary to disclose material to third parties (e.g., an institutional investigation committee or other editors).

3.2 Reviewers

Editors should protect reviewers' identities unless operating an open peer review system. However, if reviewers wish to disclose their names, this should be permitted.

If there is alleged or suspected reviewer misconduct it may be necessary to disclose a reviewer's name to a third party.

General editorial policies

4. Encourage maximum transparency and complete and honest reporting

To advance knowledge in scholarly fields, it is important to understand why particular work was done, how it was planned and conducted and by whom, and what it adds to current knowledge. To achieve this understanding, maximum transparency and complete and honest reporting are crucial.

4.1 Authorship and responsibility

Journals should have a clear policy on authorship that follows the standards within the relevant field. They should give guidance in their information for authors on what is expected of an author and, if there are different authorship conventions within a field, they should state which they adhere to.

For multidisciplinary and collaborative research, it should be apparent to readers who has done what and who takes responsibility for the conduct and validity of which aspect of the research. Each part of the work should have at least one author who takes responsibility for its validity. For example, individual contributions and responsibilities could be stated in a contributor section. All authors are expected to have contributed significantly to the paper and to be familiar with its entire content and ideally, this should be declared in an authorship statement submitted to the journal.

When there are undisputed changes in authorship for appropriate reasons, editors should require that all authors (including any whose names are being removed from an author list) agree with these in writing. Authorship disputes (i.e., disagreements on who should or should not be an author before or after publication) cannot be adjudicated by editors and should be resolved at institutional level or through other appropriate independent bodies for both published and unpublished papers. Editors should then act on the findings, for example by correcting authorship in published papers.

Journals should have a publicly declared policy on how papers submitted by editors or editorial board members are handled (see paragraph on editorial conflicts of interest: 8.2).

4.2 Conflicts of interest and role of the funding source

Editors should have policies that require all authors to declare any relevant financial and non-financial conflicts of interest and publish at least those that might influence a reader's perception of a paper, alongside the paper. The funding source of the research should be declared and published, and the role of the funding source in the conception, conduct, analysis, and reporting of the research should be stated and published.

Editors should make it clear in their information for authors if in certain sections of the journal (e.g., commissioned commentaries or review articles) certain conflicts of interest preclude authorship.

4.3 Full and honest reporting and adherence to reporting guidelines

Among the most important responsibilities of editors is to maintain a high standard in scholarly literature. Although the standards differ among journals, editors should work to ensure that all published papers make a substantial new contribution to their field. Editors should discourage so-called 'salami publications' (i.e., publication of the minimum publishable unit of research), avoid duplicate or redundant publication unless it is fully declared and acceptable to all (e.g., publication in a different language with cross-referencing), and encourage authors to place their work in the context of previous work (i.e., to state why this work was necessary/done, what this work adds or why a replication of previous work was required, and what readers should take away from it).

Journals should adopt policies that encourage full and honest reporting, for example, by requiring authors in fields where it is standard to submit protocols or study plans, and, where they exist, to provide evidence of adherence to relevant reporting guidelines. Although devised to improve reporting, adherence to reporting guidelines also makes it easier for editors, reviewers, and readers to judge the actual conduct of the research.

Digital image files, figures, and tables should adhere to the appropriate standards in the field. Images should not be inappropriately altered from the original or present findings in a misleading way.

Editors might also consider screening for plagiarism, duplicate or redundant publication by using anti-plagiarism software, or for image manipulation. If plagiarism or fraudulent image manipulation is detected, this should be pursued with the authors and relevant institutions (see paragraph on how to handle misconduct: 5.2)

5. Responding to criticisms and concerns

Reaction and response to published research by other researchers is an important part of scholarly debate in most fields and should generally be encouraged. In some fields, journals can facilitate this debate by publishing readers' responses. Criticism may be part of a general scholarly debate but can also highlight transgressions of research or publication integrity.

5.1 Ensuring integrity of the published record - corrections

When genuine errors in published work are pointed out by readers, authors, or editors, which do not render the work invalid, a correction (or erratum) should be published as soon as possible. The online version of the paper may be corrected with a date of correction and a link to the printed erratum. If the error renders the work or substantial parts of it invalid, the paper should be retracted with an explanation as to the reason for retraction (i.e., honest error).

5.2 Ensuring the integrity of the published record – suspected research or publication misconduct

If serious concerns are raised by readers, reviewers, or others, about the conduct, validity, or reporting of academic work, editors should initially contact the authors (ideally all authors) and allow them to respond to the concerns. If that response is unsatisfactory, editors should take this to the institutional level (see below). In rare cases, mostly in the biomedical field, when concerns are very serious and the published work is likely to influence clinical practice or public health, editors should consider informing readers about these concerns, for example by issuing an 'expression of concern', while the investigation is ongoing. Once an investigation is concluded, the appropriate action needs to be taken by editors with an accompanying comment that explains the findings of the investigation. Editors should also respond to findings from national research integrity organizations that indicate misconduct relating to a paper published in their journal. Editors can themselves decide to retract a paper if they are convinced that serious misconduct has happened even if an investigation by an institution or national body does not recommend it.

Editors should respond to all allegations or suspicions of research or publication misconduct raised by readers, reviewers, or other editors. Editors are often the first recipients of information about such concerns and should act, even in the case of a paper that has not been accepted or has already been rejected. Beyond the specific responsibility for their journal's publications, editors have a collective responsibility for the research record and should act whenever they become aware of potential misconduct if possible. Cases of possible plagiarism or duplicate/redundant publication can be assessed by editors themselves. However, in most other cases, editors should request an investigation by the institution or other appropriate bodies (after seeking an explanation from the authors first and if that explanation is unsatisfactory).

Retracted papers should be retained online, and they should be prominently marked as a retraction in all online versions, including the PDF, for the benefit of future readers.

For further guidance on specific allegations and suggested actions, such as retractions, see the COPE flowcharts and retraction guidelines (http://publicationethics.org/flowcharts; http://publicationethics.org/files/u661/Retractions_COPE_gline_final_3_Sept_09__2_.pdf).

5.3 Encourage scholarly debate

All journals should consider the best mechanism by which readers can discuss papers, voice criticisms, and add to the debate (in many fields this is done via a print or on-line correspondence section). Authors may contribute to the debate by being allowed to respond to comments and criticisms where relevant. Such a scholarly debate about published work should happen in a timely manner. Editors should clearly distinguish between criticisms of the limitations of a study and criticisms that raise the possibility of research misconduct. Any criticisms that raise the possibility of misconduct should not just be published but should be further investigated even if they are received a long time after publication. Editorial policies relevant only to journals that publish research in humans or animals.

6. Critically assess and require a high standard of ethical conduct of research

Especially in biomedical research but also in social sciences and humanities, ethical conduct of research is paramount in the protection of humans and animals. Ethical oversight, appropriate consent procedures, and adherence to relevant laws are required by authors. Editors need to be vigilant to concerns in this area.

6.1 Ethics approval and ethical conduct

Editors generally require approval of a study by an ethics committee (or institutional review board) and the assurance that it was conducted according to the Declaration of Helsinki for medical research in humans but, in addition, should be alert to areas of concern in the ethical conduct of research. This may mean that a paper is sent to peer reviewers with particular expertise in this area, to the journal's ethics committee if there is one, or that editors require further reassurances or evidence from authors or their institutions.

Papers may be rejected on ethical grounds even if the research had ethics committee approval.

6.2 Consent (to take part in research)

If research is done in humans, editors should ensure that a statement on the consent procedure is included in the paper. In most cases, written informed consent is the required norm. If there is any concern about the consent procedure, if the research is done in vulnerable groups, or if there are doubts about ethical conduct, editors should ask to see the consent form and enquire further from authors, exactly how consent was obtained.

6.3 Consent (for publication)

For all case reports, small case series, and images of people, editors should require the authors to have obtained explicit consent for publication (which is different from consent to take part in research). This consent should inform participants which journal the work will be published in, make it clear that, although all efforts will be made to remove unnecessary identifiers, complete anonymity is not possible, and ideally state that the person described has seen and agreed with the submitted paper.

The signed consent form should be kept with the patient file rather than sent to the journal (to maximize data protection and confidentiality, see paragraph 6.4). There may be exceptions where it is not possible to obtain consent, for example when the person has died. In such cases, careful consideration about possible harm is needed and out of courtesy attempts should be made to obtain assent from relatives. In very rare cases, an important public health message may justify publication without consent if it is not possible despite all efforts to obtain consent and the benefit of publication outweighs the possible harm.

6.4 Data protection and confidentiality

Editors should critically assess any potential breaches of data protection and patient confidentiality. This includes requiring properly informed consent for the actual research presented consent for publication where applicable (see paragraph 6.3) and having editorial policies that comply with guidelines on patient confidentiality.

6.5 Adherence to relevant laws and best practice guidelines for ethical conduct

Editors should require authors to adhere to relevant national and international laws and best practice guidelines where applicable, for example when undertaking animal research. Editors should encourage registration of clinical trials.

Editorial Processes

7. Ensuring a fair and appropriate peer review process

One of the most important responsibilities of editors is organizing and using peer review fairly and wisely. Editors should explain their peer review processes in the information for authors and also indicate which parts of the journal are peer reviewed.

7.1 Decision whether to review

Editors may reject a paper without peer review when it is deemed unsuitable for the journal's readers or is of poor quality. This decision should be made in a fair and unbiased way. The criteria used to make this decision should be made explicit. The decision not to send a paper for peer review should only be based on the academic content of the paper and should not be influenced by the nature of the authors or the host institution.

7.2 Interaction with peer reviewers

Editors should use appropriate peer reviewers for papers that are considered for publication by selecting people with sufficient expertise and avoiding those with conflicts of interest. Editors should ensure that reviews are received in a timely manner.

Peer reviewers should be told what is expected of them and should be informed about any changes in editorial policies. In particular, peer reviewers should be asked to assess research and publication ethics issues (i.e., whether they think the research was done and reported ethically, or if they have any suspicions of plagiarism, fabrication, falsification, or redundant publication). Editors should have a policy to request a formal conflict of interest declaration from peer reviewers and should ask peer reviewers to inform them about any such conflict of interest at the earliest opportunity so that they can make a decision on whether an unbiased review is possible. Certain conflicts of interest may disqualify a peer reviewer. Editors should stress confidentiality of the material to peer reviewers and should require peer reviewers to inform them when they ask a colleague for help with a review or if they mentor a more junior colleague in conducting peer review. Editors should ideally have a mechanism to monitor the quality and timeliness of peer review and to provide feedback to reviewers.

7.3 Reviewer misconduct

Editors must take reviewer misconduct seriously and pursue any allegation of breach of confidentiality, non-declaration of conflicts of interest (financial or non-financial), inappropriate use of confidential material, or delay of peer review for competitive advantage. Allegations of serious reviewer misconduct, such as plagiarism, should be taken to the institutional level (for further guidance see: http://publicationethics.org/files/u2/07_Reviewer_misconduct.pdf).

7.4 Interaction with authors

Editors should make it clear to authors what the role of the peer reviewer is because this may vary from journal to journal. Some editors regard peer reviewers as advisors and may not necessarily follow (or even ask for) reviewers' recommendations on acceptance or rejection. Correspondence from editors is usually with the corresponding author, who should guarantee to involve co-authors at all stages. Communicating with all authors at first submission and at final acceptance stage can be helpful to ensure all authors are aware of the submission and have approved the publication. Normally, editors should pass on all peer reviewers' comments in their entirety. However, in exceptional cases, it may be necessary to exclude parts of a review, if it, for example, contains libelous or offensive remarks. It is important, however, that such editorial discretion is not inappropriately used to suppress inconvenient comments.

There should always be good reasons, which are clearly communicated to authors, if additional reviewers are sought at a late stage in the process.

The final editorial decision and reasons for this should be clearly communicated to authors and reviewers. If a paper is rejected, editors should ideally have an appeals process. Editors, however, are not obliged to overturn their decision.

8. Editorial decision-making

Editors are in a powerful position by making decisions on publications, which makes it very important that this process is as fair and unbiased as possible, and is in accordance with the academic vision of the particular journal.

8.1 Editorial and journal processes

All editorial processes should be made clear in the information for authors. In particular, it should be stated what is expected of authors, which types of papers are published, and how papers are handled by the journal. All editors should be fully familiar with the journal policies, vision, and scope. The final responsibility for all decisions rests with the editor-in-chief.

8.2 Editorial conflicts of interest

Editors should not be involved in decisions about papers in which they have a conflict of interest, for example if they work or have worked in the same institution and collaborated with the authors, if they own stock in a particular company, or if they have a personal relationship with the authors. Journals should have a defined process for handling such papers. Journals should also have a process in place to handle papers submitted by editors or editorial board members to ensure unbiased and independent handling of such papers. This process should be stated in the information for authors. Editorial conflicts of interest should be declared, ideally publicly.

TABLE OF CONTENTS

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IDUNAS	NATURAL & APPLIED SCIENCES JOURNAL	2025 Vol. 8 No. 1 (1-17)

Phytochemicals, Nutrients and Anti-Nutrients Composition of the Aqueous Roots and Stem Extracts of *Typha domingensis*

Research Article

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Abstract

Medicinal plants contain various phytoconstituents and have nutritional benefits, medicinal properties, and pharmacological activities. *Typha domingensis* has been used as a source of food and in the treatment of many diseases including wounds, anxiety, depression, and bleeding disorders. The aim of this study is to evaluate the phytochemicals, nutrients and anti-nutrients content of the aqueous roots and stem extracts of *T. domingensis*. Phytochemicals, proximate, minerals and anti-nutrients composition were determined using Association of Official Analytical Chemists (AOAC) and Atomic Absorption Spectrometric (AAS) methods. The results revealed that aqueous roots and stem extracts of *T. domingensis* demonstrated significant (p< 0.05) amounts of glycosides (8.50 and 22.05 %), saponins (5.32 and 6.01 %), alkaloids (2.87 and 6.80 %), flavonoids (2.81 and 5.63 %), cardiac glycosides (1.73 and 7.41 %), steroids (5.44 and 1.90 %), and tannins (2.29 and 2.50 %), respectively. A significant (p< 0.05) amounts moisture (55.30 and 28.21%), fiber (34.77 and 19.50 %), carbohydrates (25.99 and 20.29), ash (20.41 and 11.40 %), lipids (7.82and 5.06 %) and protein (4.96 and 2.88%) were found in the aqueous roots and stem extracts of *T. domingensis*, respectively. The aqueous roots and stem extracts of *T. domingensis* demonstrated higher significant (p< 0.05) levels of calcium, sodium, iron, potassium, magnesium, aluminium, zinc, copper, manganese, cobalt with trace amount of nickel, lead, and cadmium. However, the amount of the phytochemicals and the nutrients was significantly (p< 0.05) higher in the aqueous roots extract of *T. domingensis* compared to the stem extract. A trace amount of oxalate, tannins, phytate, and cyanide was found in the aqueous roots and stem extracts of *T. domingensis*. The aqueous roots and stem extracts of *T.*

Domingensis contain a significant amount of phytochemicals and nutrients which could be attributed to its nutritional value and medicinal properties.

Keywords: Anti-nutrients; Minerals; Nutrients; Phytochemicals; *Typha domingensis*

1. INTRODUCTION

Research interest in medicinal plants has been increased because of their availability, easy accessibility, phytoconstituents contents, nutritional benefits, medicinal properties, and pharmacological activities of their constituents. About 80% of the world's population relies on plants and herbs for remedies (Khan and Ahmad, 2019). It was estimated that 95% of the people in developing countries depend on plants and herbs for therapeutic uses (Khan and Ahmad, 2019). Plants and herbs are less expensive and more easily accessible and have few side effects. Plants have been used for treatment of many diseases in many local communities in the world which could be attributed to the presence of phytochemicals (Olivia *et al.,* 2021; Anand *et al.,* 2019). Nutrients and phytochemicals demonstrate vital biological and biochemical functions and pharmacological properties. Nutritional value of plants is determined by their nutrients and anti-nutrients contents (Abubakar *et al.,* 2022). Phytochemicals in plants are responsible for their medicinal uses and pharmacological activities. Anti-nutrients in plants reduce nutrients bioavailability by preventing the absorption of essential nutrients (Abubakar *et al.,* 2022). Many bioactive phytochemicals extracted from medicinal plants have been used for drugs discovery and development (Kumar *et al.,* 2021).

Typha domingensis is an aquatic plant that belongs to the family *Typhaceae* and the genus *Typha* consisting of many species (Pandey and Verma, 2018; Xu and Chang, 2017). *T. domingensis* is abundantly available in tropical and subtropical areas. The plant is commonly found in moist soil, swamps, marshes, lakeshores, roadsides, and manmade reservoirs (He *et al.,* 2015; CABI, 2013). Aquatic plants including *T. domingensis* have been used by animals for foods and nutrition (Elsken, 2020; Tham and Udén, 2013). *T. domingensis* has been used locally for management of wounds, anxiety, depression and neuro and bleeding disorders (Chai *et al.,* 2014; Qin and Sun, 2005). The stem part of the plant has diuretic and astringent properties. The leaves of the plant demonstrated analgesic, antioxidant, and diuretic activities (Lopes *et al.,* 2017). The roots of the plant exhibited anti-inflammatory, antioxidant, astringent, cytotoxic, and diuretic properties (Bandaranayake, 1998). Study showed that extracts of *T. domingensis* demonstrated wound healing properties in rat models (Akkol *et al.,* 2011). It has been reported that hydroethanolic extract of *T. domingensis* demonstrated spasmolytic, bronchodilator, and vasodilating properties (Imran *et al.,* 2020). Rootstocks and rhizomes of Typha species including *T. domingensis* have been consumed as a source of nutritious foods (Zeng *et al.,* 2020). *Typha* species are important source of powder flour which is used in the manufacture of many food products including bread, cakes, and biscuits (Aljazy *et al.,* 2021).

In Nigeria *T. domingensis* is found in the northern region of the country and it is locally called Kachalla or Geron tsuntsu in Hausa. Different parts of the *T. domingensis* have been used in many local communities in Nigeria as source of foods and fodder for livestock, and raw materials for making house screens, boats, andstuffing pillows. *T. domingensis* has been abundantly grown in Shagari dam, in Shagari local government area, Sokoto state, Nigeria. Many people especially local herbalists in and outside the Sokoto community use the *T. domingensis* from this dam as source of food and for treatment of certain diseases. This study aims at evaluating the phytochemicals, nutrients and anti-nutrients contents of the aqueous roots and stem extract of *T. domingensis*.

2. MATERIALS AND METHODS

2.1 Chemicals and Reagents

Chemicals and reagents including Folin-Denis reagent, Wagner's reagent, Baljet's reagent, ethyl acetate, ethanol, chloroform, HCl, H_2SO_4, NaOH, $FeCl_2$, $FeCl_3$, and NH_3 were used in this study. The levels of purity of the chemicals were of standards (≥ 95%) set by American Chemical Society (ACS) grade. The chemicals and reagents purchased were manufactured by Reidel-de Haem (Merck, Germany), Sigma-Aldrich (St. Louis, MO, USA), and BDH Chemical Limited Poole (England, UK).

2.3 Plant Samples

Fresh roots and stem of *T. domingensis* L. were obtained from Shagari dam, Shagari local government area, Sokoto state, Nigeria with help of local herbalists. The samples were collected in early January 2023 from a moist site of the dam. The plant samples were identified at Taxonomy Unit, Department of Biological Sciences, Usmanu Danfodiyo University, Sokoto.

2.4 Preparation of the Samples Extract

The plant samples were thoroughly washed with distilled water, cut into pieces and then air dried at room temperature for 21 days. The dried samples were mechanically pulverized to fine powder using a grinding machine. The powdered samples were stored in a clean container at room temperature for further analysis. The extract preparation followed the method of Abubakar et al. (2021) with modifications. Five hundred grams (500 g) of each powdered sample was soaked in two liters of distilled water for three days with constant stirring at one-hour interval. Each sample extract was filtered using Whatman filter paper No 1 and concentrated using rotary evaporator under reduced pressure at 40°C for three hours. Each sample extract was weighed using analytical weighing balance and the percentage yield of samples extract was calculated. The weight and the percentage yield of the roots and stem extracts obtained was 34.2 g and 41.8 g and 6.8 % and 8.4 %, respectively. The samples extracts were stored in clean desiccators until further analysis.

2.5 Qualitative Phytochemical Screening

2.5.1 Test for Alkaloids

Qualitative determination of alkaloids in the aqueous roots and stem extracts of *T. domingensis* was carried out using Wagner's test according to the method described by Mosa et al. (2012), Abubakar et al. (2022; 2020) and Trease and Evans (1989). Each sample extract (3 mL) was added to the test tubes containing 3 mL of 1% HCl, heated for 20 minutes and then allowed to cool. The Wagner's reagent (1 mL) was added in drops into the test tube. The presence of alkaloids in the samples extract was observed by the formation of a reddish-brown precipitate.

2.5.2 Tests for Glycosides

Glycosides present in the aqueous roots and stem extracts of *T. domingensis* were qualitatively determined by Salkowski's test as described by Ibrahim et al. (2024) and Abubakar et al. (2022; 2020). Each sample extract (5 mL) was transferred into a test tube followed by the addition of 5 mL of 1 % H_2SO_4 solution, boiling for 15 minutes and then allowed to cool. The mixture was neutralized with 10% NaOH solution followed by addition of 5 mL of Fehling's solution A and B. The presence of glycosides was observed by the formation of brick red precipitate of reducing sugars.

2.5.3 Test for Tannins

Ferric chloride test was employed for qualitative determination of tannins in the aqueous roots and stem extracts of *T. domingensis* according to the method described by Trease and Evans (1989) and Ibrahim et al. (2024). Each sample extract (1 mL) was treated with 2 mL of 5% $FeCl_3$ solution. Tannins in the samples extract was identified by the formation of black or blue-green colour.

2.5.4 Test for Saponins

The presence of saponins in the aqueous roots and stem extracts of *T. domingensis* was determined using Froth test according to the method described by Mosa et al. (2012), Abubakar et al. (2022; 2020) and Trease and Evans (1989). Each sample extract (3 mL) was added into the test tube containing 3 mL of distilled water. The test tubes were vigorously shaken for 30 sec and allowed to stand for 30 min at room temperature. The formation of stable persistent froth indicated the presence of saponins.

2.5.5 Test for Flavonoids

The presence of flavonoids in the aqueous roots and stem extracts of *T. domingensis* was evaluated using sodium hydroxide test as described by Mosa et al. (2012) and Ibrahim et al. (2024). Each sample extract (3 mL) was transferred into the test tubes and then 1 mL of 10% NaOH solution was added into the test tubes. Flavonoids in the samples extract were identified by the formation of an intense yellow colour which became colourless after the addition of dilute HCl solution.

2.5.6 Test for Steroids

The aqueous roots and stem extracts of *T. domingensis* were analyzed for the presence of steroids using the method of Trease and Evans (1989) and Ibrahim et al. (2024). Each sample extract (500 µL) was treated with 5 mL of chloroform and 5 mL of H_2SO_4 solution. The presence of steroids in the samples extract was observed by the formation of violet colour which changed to blue-green.

2.5.7 Test for Terpenoids

Terpenoids in the aqueous roots and stem extract of *T. domingensis* were Qualitative determined using the method of Trease and Evans (1989) and Ibrahim et al. (2024). Each sample extract was treated with 1 mL of ethanol and acetic anhydride followed by addition of 10 mL of concentrated H_2SO_4 solution. Formation of pink colour indicated the presence of terpenoids.

2.5.8 Test for Cardiac Glycosides

Qualitative determination of alkaloid cardiac glycosides in the aqueous roots and stem extracts of *T. domingensis* was performed by Keller-Killani test using the method of Mosa et al. (2012) and Trease and Evans (1989). The samples extracts (5 mL) were transferred into test tubes and then treated with 2 mL of glacial acetic acid containing one drop of $FeCl_2$ solution. One mile of concentrated H_2SO_4 solution was added into the test tubes. The formation of brown ring at the interface indicated the presence of deoxysugar, a characteristic of cardenolides. The presence of cardiac glycosides in the samples extract was observed by the appearance of violet colour below the brown ring.

2.5.9 Test for Anthraquinones

The experimental identification of anthraquinones in the aqueous roots and stem extracts of *T. domingensis* was carried out according to the method described by Trease and Evans (1989) with little modifications. The powdered samples (0.2 g) were transferred into the test tubes followed by addition of 10 cm^3 of chloroform and vigorous shaking for 5 minutes. Each extract was filtered using Whatman filter paper and each filtrate was treated with equal volume of ammonia and then shaken for 5 minutes. The presence of anthraquinones was observed by the formation of bright pink colour in the upper aqueous layer.

2.6 Quantitative Determination of Phytochemicals

2.6.1 Determination of Alkaloids

Alkaloids in the aqueous roots and stem extracts of *T. domingensis* were quantitatively determined using the method of Trease and Evans (1989) and Ibrahim et al. (2024). Five grams of the dried samples extracts were dissolved in 100 mL of methanol followed by evaporation of the solvent. The residues obtained were treated with 20 mL of 2 mM sulphuric acid and the contents were thoroughly mixed, and then partitioned with ether. The aqueous portions were basified with strong ammonia solution and then extracted with excess chloroform for several times. The extracts were concentrated to dryness and the final alkaloid residues were weighed. The alkaloids content was calculated using the following equation:

$$\text{Alkaloids Content (\%)} = \frac{\text{Weight of alkaloids residue}}{\text{Weight of extract}} \text{X } 100$$

2.6.2 Determination of Flavonoids

The quantitative determination of flavonoids in the aqueous roots and stem extracts of *T. domingensis* was conducted according to the method described by Harborne (1973) and Ibrahim et al. (2024). Each dried sample extract (5 mg) was added into a test tube containing 50 mL of 2M HCl solution and then heated at 100 °C for 25 min under reflux. The contents were cool and then filtered using Whatman filter paper. The mixture was treated with 50 mL of ethyl acetate solution, filtered using filter paper and then concentrated to dryness. The weight of dried flavonoids residues were measured using analytical weighing balance. The flavonoids content was obtained using the following formula:

$$\text{Flavonoids Content (\%)} = \frac{\text{Weight of flavonoids residue}}{\text{Weight of extract}} \text{X } 100$$

2.6.3 Determination of Tannins

The quantitative determination of tannins content in the aqueous roots and stem extracts of *T. domingensis* was carried out using the method of AOAC (1999) and Ibrahim et al. (2024). Tannic acid standard solution was obtained by dissolving 10 mg of tannic acid in 100 mL of distilled water. The preparation of tannic acid standards (0 – 2.5 ml aliquots) was performed in 25 mL volumetric flasks. Folin-Denis reagent (2.5 mL) and sodium carbonate solution (1.25 mL) were added into the flask and then the contents were made up to the mark. The contents were incubated at room temperature for 30 minutes and then the absorbance was measured using spectrophotometer at 760 nm wavelength. The dried powder extract (1 g) was boiled in 80 ml of water for 30 minutes. The tannin content in the samples extract was obtained from the tannic acid standard curve.

2.6.4 Determination of Glycosides

Each sample extract (10 mL) was transferred into a 250 ml conical flask containing 50 mL of chloroform and the contents were shaken for 60 minutes. The mixture was filtered using Whatman filter paper followed by addition of 10 mL of pyridine and 2 mL of 2% sodium nitroprusside with vigorous shaking for 10 minutes. Three mil of 20% NaOH was then added to develop a brownish yellow colour. The absorbance of sample and standard was measured spectrophotometerically at 510 nm wavelength. Glycosides content in percentage was obtained using the formula:

$$\text{lycosides} \quad \text{Content (\%)} = \frac{\text{A x A} \quad \text{x DF}}{\text{Weight of extract}} \text{X } 10000$$

Where; A =Absorbance of sample; AG = Average gradient; DF = Dilution factor

2.6.5 Determination of Saponins

The aqueous roots and stem extracts of *T. domingensis* were quantitatively analyzed for saponins content using the method of El-Olemyl et al. (1994) and Ibrahim et al. (2024). Five grams of each dried sample extract was transferred into 250 mL conical flasks followed by addition of 150 mL of 50% alcohol. The mixture was heated at 100ºC for 30 minutes and then filtered using Whatman filter paper. The charcoal (1 g) was added to the filtrate and the contents were boiled for 30 minutes. The hot mixture was filtered and then allowed to cool at room temperature. To achieve total saponins precipitation, 150 mL of acetone was added into the filtrate. The mixture was filtered, and the filter paper was immediately transferred into the desiccator containing anhydrous $CaCl_2$ solution. The saponins residues were dried in oven and then weighed using analytical weighing balance. Saponins content in the samples extract was obtained using the following formula:

$$\text{Saponins Content (\%)} = \frac{\text{Weight of saponins residue}}{\text{Weight of extract}} \text{X } 100$$

2.6.6 Determination of Steroids

Total steroids in the aqueous roots and stem extracts of *T. domingensis* were quantitative determined using the method of Evans (1996) and Ibrahim et al. (2024). One mile of each sample extract was treated with 2 mL of $H_2SO_4$and 2 mL of $FeCl_2$ solution. Potassium hexacynoferrate (III) solution (2 mL) was added into the mixture and then incubated at 70 °C for 3o minutes with constant shaking. The absorbance of the samples against the blank was read at 780 nm wavelength using spectrophotometer. The following formula was used to obtain the steroids content in the samples extracts.

Steroids Content (%) = Absorbance of sample × 100

2.6.7 Determination of Cardiac Glycosides

Quantitative estimation of cardiac glycosides in the aqueous roots and stem extracts of *T. domingensis* was conducted using the method of Solich et al. (1992) with little modifications Each sample extract (10 mL) was treated with 10 mL of the prepared Baljet's reagent (95 mL of 1% picric acid + 5 mL of 10% NaOH). The mixture was diluted with 20 mL of distilled water and then allowed to stand for 60 minutes for colour development. The absorbance of samples and standard was read at 495 nm using spectrophotometer. The standard curve was prepared at different concentrations (12.5-100 mg/L) and the cardiac glycosides concentration was obtained and expressed in percentage.

2.7 Proximate Analysis

The crude lipid, moisture, carbohydrate, crude protein, ash and fiber contents of the aqueous roots and stem extracts of *T. domingensis* were determined using the method of AOAC (2010). All the tests were carried out in triplicate and the results were analyzed and expressed in percentage as the mean and standard error of mean.

2.8 Determination of Minerals Content

Atomic absorption spectrophotometeric technique was employed for determination of concentration of Ca, Fe, Zn, Mg, Cu, Co, Mn, Ni, Cd, and Pb in the aqueous roots and stem extracts of *T. domingensis* using the method of AOAC (1990; 2005). The level of K, Na and Al in the aqueous roots and stem extracts of *T. domingensis* was determined by flame photometeric technique according to the method described by AOAC (1990; 2005).

2.9 Determination of Anti-nutrients Contents

The level of phytate in the aqueous roots and stem extracts of *T. domingensis* was estimated according to the method described by Reddy and Love (1999). The concentration of cyanide in the aqueous roots and stem extract of *T. domingensis* was determined using the method of AOAC (1990). The aqueous roots and stem extract of *T. domingensis* was analyzed for oxalate content using the method described by Gupta et al. (2005). The AOAC (2005) method was employed for determination of tannins level in the aqueous roots and stem extract of *T. domingensis*.

2.10 Statistical Analysis

All the tests were carried out in triplicate and the results were expressed as mean ± standard error of mean. Statistical Package for Social Sciences (SPSS) Statistics version 22 software was used for data analysis. Differences among the average values were significantly computed using One-way analysis of variance (ANOVA) at confidence level (95%) and Tukey-Kramer multiple comparisons test. Significance was considered by two-tailed ($p< 0.05$) values.

3. RESULTS

3.1 Phytochemicals Composition of Aqueous Roots and Stem Extracts of *T. domingensis*

Table 1 shows the presence of phytochemical constituents in the aqueous roots and stem extracts of T. domingensis. Alkaloids, flavonoids, and cardiac glycosides were moderately and slightly present in the aqueous roots and stem extracts of *T. domingensis*, respectively. High and moderate amounts of glycosides were respectively observed in the aqueous roots and stem extracts of *T. domingensis*. All the extracts demonstrated moderate and slight amount of saponins and tannins, respectively. Anthraquinones and terpenoids were not detected in the extracts (Table 1).

Table 1. Qualitative Phytochemicals Screening of Aqueous Roots and Stem Extracts of *T. domingensis*

Phytochemical	Roots Extract	Stem Extract
Alkaloids	+	++
Glycosides	++	+++
Tannins	+	+
Saponins	++	++
Flavonoids	+	++
Steroids	++	+
Terpenoids	ND	ND
Cardiac glycosides	+	++
Anthraquinones	ND	ND

+++ = Highly present, ++ = Moderately present, + = Slightly present, ND = Not detected

The quantitative phytochemicals composition of aqueous roots and stem extracts of *T. domingensis* is shown in Table 2. Significant amounts of the phytochemicals was observed in the aqueous roots and stem extracts of *T. domingensis*. The aqueous roots extract of *T. domingensis* demonstrated higher and low significant ($p< 0.05$) amount of glycosides and cardiac glycosides, respectively. The aqueous stem extract of *T. domingensis* exhibited higher and low significant ($p< 0.05$) amount of glycosides and steroids, respectively. However, the aqueous stem extract of *T. domingensis* demonstrated higher significant ($p< 0.05$) amount of alkaloids, flavonoids, glycosides, and cardiac glycosides compared to the aqueous roots extract of *T. domingensis* (Table 2).

Table 2. Quantitative Phytochemicals Composition of Aqueous Roots and Stem Extracts of *T. domingensis*

Phytochemical	Composition (%) Roots Extract	Stem Extract
Alkaloids	2.87 ± 0.031	6.80 ± 0.023
Flavonoids	2.81 ± 0.023	5.63 ± 0.017
Tannins	2.29 ± 0.014	2.50 ± 0.018
Glycosides	8.50 ± 0.016	22.05 ± 0.043
Saponins	5.32 ± 0.138	6.01 ± 0.081
Steroids	5.44 ± 0.239	1.90 ± 0.020
Cardiac glycosides	1.73 ± 0.108	7.41 ± 0.031

Values are expressed as mean ± SEM (n = 3)

3.2 Proximate Composition of the Aqueous Roots and Stem Extract of *T. domingensis*

Figure 1 shows the proximate composition of the aqueous roots and stem extracts of *T. domingensis*. The aqueous roots and stem extracts of *T. domingensis* demonstrated higher (55.30 and 28.21%) and low (4.96 and 2.88%) significant ($p < 0.05$) amount of moisture and protein content, respectively. The amount of all the proximate parameters was significantly ($p < 0.05$) higher in the aqueous roots extract of *T. domingensis* compared to the stem extract of *T. domingensis* (Figure 1).

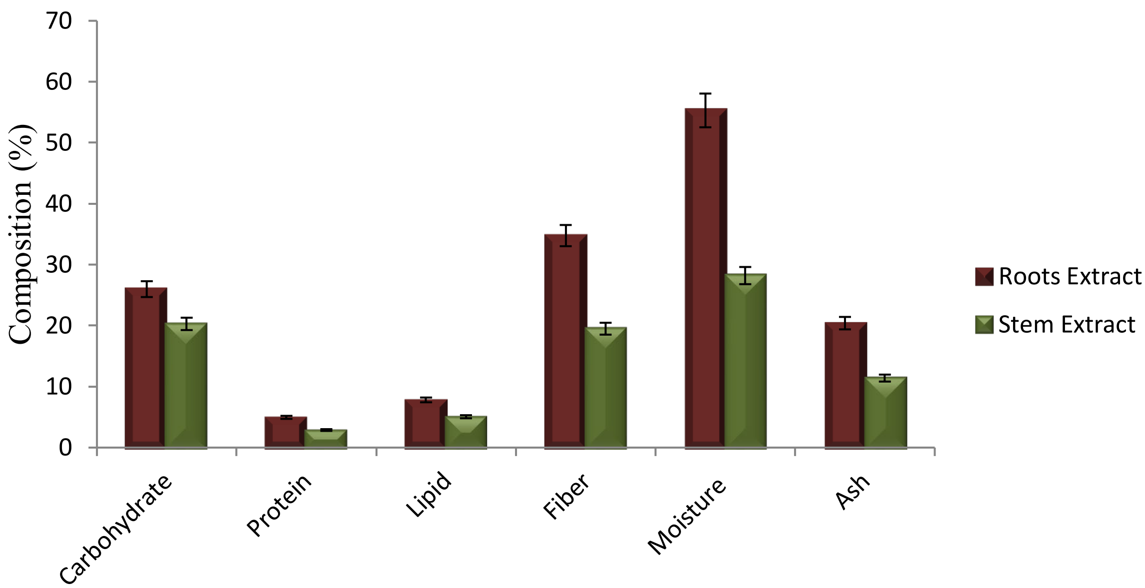

Figure 1. Proximate Composition of the Aqueous Roots and Stem Extracts of *T. domingensis*
Data are expressed as mean ± SEM (n = 3)

3.3 Minerals Composition of the Aqueous Roots and Stem Extracts of *T. domingensis*

Table 3 shows the minerals composition of the aqueous roots and stem extracts of *T. domingensis*. The aqueous roots and stem extracts of *T. domingensis* demonstrated higher significant ($p< 0.05$) levels of calcium, sodium, iron, potassium, and magnesium compared with the other minerals. The extracts contain moderate levels of aluminium, zinc, and copper. Also, trace level of nickel, manganese, cobalt, lead, and cadmium were observed in the extracts (Table 3). However, the aqueous roots extract of T. domingensis exhibited higher levels of calcium, sodium, iron, potassium, magnesium, aluminum, zinc, and copper compared to the stem extract (Table 3).

Table 3. Minerals Composition of the Aqueous Roots and Stem Extracts of *T. domingensis*

Element	**Concentration (mg/100g)**	
	Roots Extract	**Stem Extract**
Na	6.70 ± 0.069	4.80 ± 0.317
K	1.91 ± 0.040	1.23 ± 0.115
Ca	13.16 ± 0.069	8.65 ± 0.069
Fe	1.94 ± 0.023	2.81 ± 0.051
Zn	0.27 ± 0.028	0.14 ± 0.017
Mg	1.28 ± 0.017	1.04 ± 0.034
Cu	0.21 ± 0.028	0.13 ± 0.011
Al	0.91 ± 0.046	0.59 ± 0.034
Co	0.01 ± 0.003	0.02 ±0.004
Mn	0.03 ± 0.005	0.06 ± 0.023
Ni	0.09 ± 0.003	0.07 ± 0.004
Cd	0.02 ± 0.002	0.03 ± 0.001
Pb	0.02 ± 0.001	0.01 ± 0.001

Values are expressed as mean ± SEM (n = 3)

3.4 Anti-nutrients Composition of the Aqueous Roots and Stem Extracts of *T. domingensis*

The concentration of anti-nutrients in the aqueous roots and stem extracts of *T. domingensis* is shown in Figure 2. The aqueous roots and stem extracts of *T. domingensis* contain trace amount of oxalate, tannins, phytate, and cyanide. The aqueous roots extract of *T. domingensis* demonstrated high significant ($p< 0.05$) level of oxalate, tannins, phytate, and cyanide compared to the aqueous stem extract (Figure2).

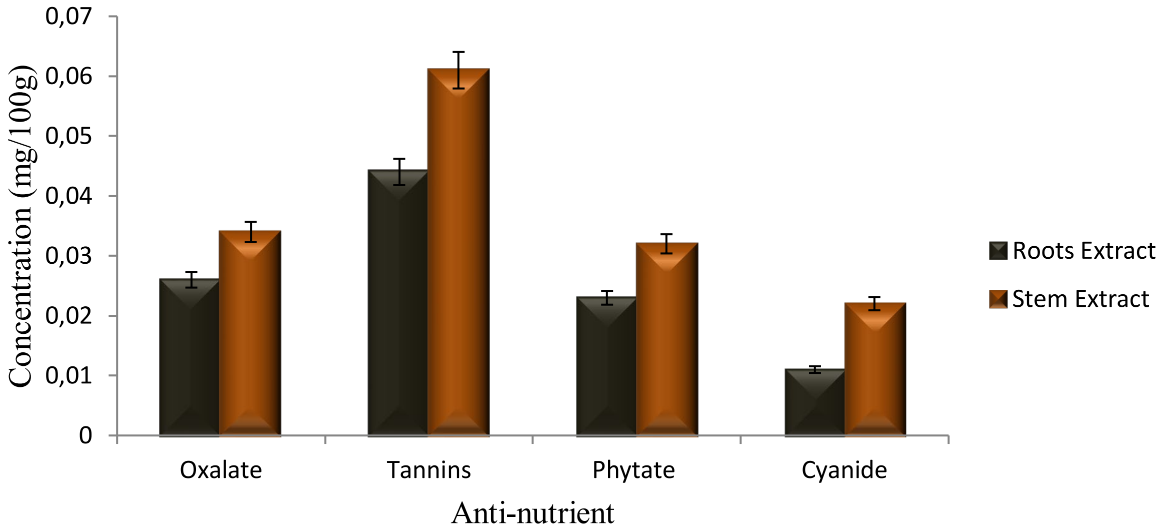

Figure 2. Anti-nutrients Composition of the Aqueous Roots and Stem Extracts of *T. domingensis*
Values are expressed as mean ± SEM (n = 3)

4. DISCUSSION

The results of this study revealed that aqueous roots and stem extracts of *T. domingensis* contain significant amount of many phytochemicals. This finding aligns with the study by Ndanyi et al. (2021) who reported the presence of many phytochemicals in some selected plants. Phytochemicals from many medicinal plants demonstrate several pharmacological activities and medicinal properties (Oghenejobo *et al.,* 2017). Alkaloids from different plants extract demonstrated analgesic activity (Brewer, 2011). Study by Mamta et al. (2013) showed that tannins from many plants extracts demonstrated anti-inflammatory and anti-cancer activities. It has been reported that tannins isolated from many plants demonstrated wound healing and astringents properties, and anti-ulcer activities (Abubakar *et al.,* 2022; Kar, 2007). Steroids from various plants have been used for synthesis of sex hormones and steroidal drugs (Majeed *et al.,* 2004). Saponins serve as an important source of steroidal hormones and have blood cholesterol lowering properties (Kar, 2007). Study showed that plants flavonoids demonstrated antioxidant activity scavenging properties, anti-cancer, anti-malarial, antihypertensive and anti-ulcer activities (Ballard and Marostica, 2018). Plants cardiac glycosides have been used in the management of coronary heart diseases and their complications (Denwick, 2002).

This study revealed that the aqueous roots and stem extracts of *T. domingensis* contain high significant amount of moisture, fiber, carbohydrate, and ash content. A relevant study by Grosshans (2014) showed that fresh Typha species demonstrated high moisture content. Also it has been reported that *Typha angustifolia* contains high amount of ash (Gravalos, 2010). The high amount of moisture and carbohydrate in the plant extracts indicated that the plant extracts have short shelf life and high caloric value, respectively. Dietary fiber plays a vital role in lowering high risk of coronary heart diseases, obesity, diabetes and cancer (Lattimer and Haub, 2010). Minerals content of plants is determined by their ash content (Onwuka, 2005). High ash content is an indicator of essential minerals that play vital roles in blood coagulation and management of certain haematological disorders (Okaka, 2001). The high ash content of the plant extracts indicated that *T. domingensis* could be a vital source of minerals. However, this finding revealed that the aqueous roots and stem extracts of *T. domingensis* contain reasonable amount of proteins and lipids. Proteins are important source of nutrients in breast feeding and have many biological and biochemical functions in synthesis and activities of enzymes and hormones (Wadhave *et al.,* 2014). Lipids are high energy nutrients and important source of fat-soluble vitamins that aids nutrients absorption (Ogbuagu *et al.,* 2011).

In this study the aqueous roots and stem extracts of *T. domingensis* demonstrated a high and low level of sodium and potassium, respectively. This finding is in agreement with the study by Grosshans (2014) who reported moderate and higher level of potassium and sodium in other Typha species. Sodium and potassium have many biological and biochemical functions which include regulation of acid-base balance, muscles contraction and nerve impulses transmission and maintenance of osmotic pressure and membrane potentials (Murray *et al.,* 2000). Potassium has an important role in the normal functioning of heart and skeletal muscle and regulation of many enzyme activities (Weaver, 2013). A significant amount of calcium, magnesium, zinc, iron, copper, manganese, and cobalt was found in the aqueous roots and stem extract of *T. domingensis* in the present study. Calcium plays an important role in regulation of vasodilatation and vascular contraction, nerve transmission, muscle function, hormonal secretion, and intracellular signaling (Catharine *et al.,* 2018). It is an important agent for blood clotting, formation of bone and teeth and function as co-factor in some enzymatic reactions (Abubakar *et al.,* 2022; Robert *et al.,* 2003). Magnesium plays a vital role in protein synthesis, release of muscle storage energy, formation of bones, maintenance of normal heart function and body temperature regulation (Akram *et al.,* 2020). It aids growth and integrity of bone, muscles and nerves functions, and regulation of the cardiac cycle (Allen and Sharma, 2019; Gragossian and Friede, 2019). Magnesium stimulates the activities of many enzymes (Vincente *et al.,* 2014). Zinc is an essential component of many enzymes and enables cell growth and proliferation, sexual maturity, and fertility (Akram *et al.,* 2020; Baltaci *et al.,* 2018). It plays a vital role in tissues formation, immune cell proliferation and maturation, wound repair, hair growth, signal transductor activation in postsynaptic neurons, regulation of oxidative stress, and gene expression (Baltaci *et al.,* 2018; Kimura and Kambe, 2016). Iron is a component of certain enzymes including those involves in oxidation-reduction reactions and proteins such as haemoglobin and myoglobin (Akram *et al.,* 2020). Iron regulate the activities of these enzymes and proteins including synthesis of hemoglobin, transport of oxygen, oxidative processes, cellular growth and catalytic reactions (Akram *et al.,* 2020; Yiannikourides and Latunde-Dada, 2019). Copper is a component of some enzymes including ferro-oxidase, catalase, cytochrome oxidase and tyrosinase and has an important role in bone formation and hematopoiesis (Leone *et al.,* 2006). It is required for red blood cell formation and contributes to the iron absorption in the gastrointestinal tract (Akram *et al.,* 2020). Manganese is essential for the activities of certain enzymes which include succinate dehydrogenase, arginase, and glucosyltransferase (Zeece, 2020; Tuschl *et al.,* 2013). It is required for the synthesis of chondroitin sulphate which is required for cartilage formation (Tuschl *et al.,* 2013). Cobalt serves an important function in the formation of vitamin B12 (Akram *et al.,* 2020). However, a very little amount of lead, nickel and cadmium observed in the samples extracts. These heavy metals demonstrate toxic effects on many organs and tissues in the body.

Results of this study showed that aqueous roots and stem extracts of *T. domingensis* contain very low amount of phytates, oxalate, tannins, and saponins. Anti-nutritional factors produce many adverse effects in food substances including reduction intake, digestion, and utilization of nutrients. Phytates present in diets can affect the bioavailability of minerals, solubility, functionality and digestibility of proteins and carbohydrates (Salunkhe *et al.,* 1990). Phytates can interfere with the absorption of many important minerals such as iron, zinc, magnesium and calcium (Masum *et al.,* 2011). This leads to high level of insoluble salts which are poorly absorbed by the gastrointestinal tract consequently reducing the bioavailability of minerals. Phytates also inhibit digestive enzymes like pepsin, trypsin and amylase (Kumar *et al.,* 2010). High levels of insoluble calcium oxalate in the kidneys form calcium oxalate crystals which contribute to the formation of kidney stones (Nachbar *et al.,* 2000). Oxalate hinders the absorption of calcium ion resulting to unavailability of the calcium for various functions in the body (Unuofin *et al.,* 2017; Ola and Oboh, 2000). It has been reported that high levels of oxalate in foods causes irritation in the mouth and the lining of the gut (Gemede and Ratta, 2014). Tannins demonstrated anti-nutritional effects due to their capability to impair the digestion of many nutrients and preventing the body from absorbing essential bioavailable compounds (Hendek and Ertop, 2018). Tannins produce many adverse effects in food substances such as inhibiting the activities of many enzymes, decreasing the protein quality of foods and interfering with dietary iron absorption (Felix and Mello, 2000). Study showed that tannins could be responsible for decreased feed intake, growth rate, feed efficiency and protein digestibility in animals (Aletor, 2005). High concentration of tannins in diets might cause decrease in microbial enzyme activities including cellulose and intestinal digestion (Aletor, 2005). Saponins can impair the protein digestion, uptake vitamins and minerals in the gut, as well as lead to the development of a leaky gut (Johnson *et al.,* 1986). Saponins have been reported to reduce the bioavailability of nutrients and decrease activities various enzymes including trypsin and chymotrypsin (Liener, 2003). Also, it has been reported that saponins demonstrated strong hypocholesterolemic effect (Ikewuchi, 2012).

5. CONCLUSION

The aqueous roots and stem extracts of *T. domingensis* demonstrated significant amounts of various phytochemicals, essential minerals, and proximate parameters. The presence of these essential nutrients and the phytoconstituents in the plant samples extracts justified that roots and stem of *T. domingensis* have nutritional value and medicinal properties.

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Sentiment Analysis and Automatic Response Generation for E-Commerce Comments

Research Article

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Abstract

This study addresses the use of machine learning techniques for automatic classification of product reviews on e-commerce sites and generating appropriate responses. It was carried out with approximately 15,000 data labeled as positive, negative and neutral obtained from the "E-Commerce Product Reviews" data set. The TF-IDF vectorization method, which is a text mining technique, was used in the study. Multinomial Naive Bayes, Support Vector Machine, Random Forest, Logistic Regression techniques were used for sentiment analysis. As a result of the studies, the accuracy values of Multinomial Naive Bayes, Support Vector Machine, Random Forest, Logistic Regression algorithms were 87%, 88%, 85% and 88%, respectively. In conclusion, automatic comment analysis tools can significantly improve customer relations for e-commerce sellers.

Keywords: Sentiment Analysis, Text Mining, Automatic Classification

1. INTRODUCTION

Product reviews on e-commerce sites are one of the critical elements that influence customers' shopping decisions. These reviews are of great importance not only for potential buyers but also for sellers in terms of product improvement, marketing and sales strategies, customer relations, credibility and reputation. In this context, studies on categorizing online user reviews with sentiment analysis have made valuable contributions to research in the field of e-commerce.

In particular, the study by Li et al. (2020) made a significant contribution to sentiment analysis in e-commerce reviews. In this study, a new model for sentiment analysis, SLCABG, is proposed based on the real book reviews of "dangdang.com", a Chinese e-commerce website. This model is based on sentiment dictionary. The SLCABG model combines the advantages of sentiment dictionary and deep learning technology and overcomes the shortcomings of the sentiment analysis model of product reviews. In addition, Tuzcu's (2020) applied Multilayer Perceptron (MLP) algorithm using Python programming language for sentiment analysis of online user comments and obtained successful results. There are many studies about this issue.

This study aims to apply machine learning techniques to e-commerce comments, building upon the methods of previous research. However, manually analyzing and responding to comments in large data sets is a time-consuming and challenging task. To alleviate this challenge, this study aims to automatically classify e-commerce comments and generate meaningful responses. After preprocessing steps on text data, a sentiment analysis model was built using TF-IDF vectorization and supervised learning models and then evaluated on test data. The model is used to generate automatic responses for new comments.

2. MATERIALS AND METHODS

Sentiment analysis is an artificial intelligence technology for understanding human emotions from data such as text, audio or image. Text-based sentiment analysis is widely used. This technology is used to identify emotional content in a text document. Sentiment analysis usually classifies emotional content in texts as positive, negative or neutral. It is usually performed with "Natural Language Processing" techniques. Machine learning algorithms are trained to associate certain words, phrases or sentences with certain emotions. Natural Language Processing (NLP) is a machine learning technology that gives computers the ability to interpret, process and understand language. It enables computers to comprehend human language.

As Çelik and Koç (2021) mention that the TF-IDF vectorization method is a feature engineering technique frequently used in text mining. For machine learning or deep learning, verbal expressions need to be made meaningful and expressed numerically. TF-IDF is a statistical measure that indicates how representative a word is within a document. "TF" stands for term frequency. It shows the frequency of the word in the document, i.e. the number of times the word occurs in the document. "DF" stands for document frequency, i.e. the number of documents divided by the number of occurrences of the word. "IDF" is the inverse document frequency, which is the logarithm of DF.

Supervised learning is a type of machine learning technique. Supervised learning, the data set used to train the algorithm contains labelled data (Onan and Korukoğlu, 2016). It is based on a machine learning model trying to learn the input and output relationships in the training data set. For each example, the training dataset contains inputs (features) and target outputs (labels or responses) corresponding to these inputs. Supervised learning uses various algorithms to model the relationship between inputs and outputs.

Multinomial Naive Bayes is a text classification algorithm in which features are classified and correlated with word counts or frequencies in documents (Calis et al., 2013). It aims to determine the class and category of the data presented. It is a supervised learning algorithm used in text classification. It is used in areas such as spam filtering and sentiment analysis. It uses Bayes theorem. Figure 1 shows the naive bayes classification.

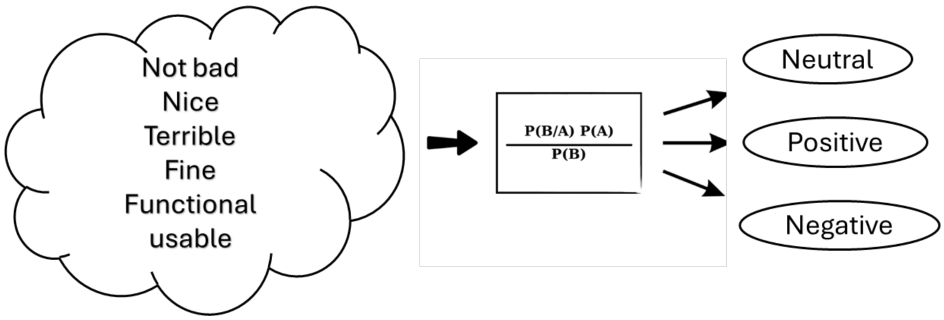

Figure 1. Naive Bayes Classification

Support Vector Machine is a supervised learning method used in classification problems (Ülgen et al., 2017). It draws a line to separate points placed on a plane. It aims for this line to be at the maximum distance for the points of both classes. It is a suitable algorithm for small and medium data sets. Kernel is an important parameter for Support Vector Machine and can take values such as 'linear', 'poly', 'rbf', 'sigmoid', 'precomputed'.

Random Forest is an ensemble learning method that generates a large number of decision trees in the training phase and predicts classes or numbers according to the problem. It is effective on complex and variable data sets. It is based on the principle that many decision trees come together to form a forest. Each tree is trained on data subsets generated by random sampling. Since the trees are trained independently on each of these subsets, they can focus on different features, which increases the generalization ability of the model.

Logistic Regression is a supervised learning algorithm used to predict the probability of a categorical dependent variable. It is a statistical model commonly used to solve classification problems. It focuses on predicting the probability of a dependent variable using a combination of independent variables.

3. DATA SET CHARACTERISTICS

For this study, which aims to automatically classify e-commerce comments and generate meaningful responses using artificial intelligence techniques, the "E-Commerce Product Comments" dataset was used (Çabuk, 2022).

As can be seen in Figure 2, the dataset consists of data classified as negative, positive and neutral. The label distribution helps us to better understand the success rates of the model. This figure shows that there is a balanced distribution of positive and negative labels, but there are not enough examples of neutral data, which may affect the performance of the model.

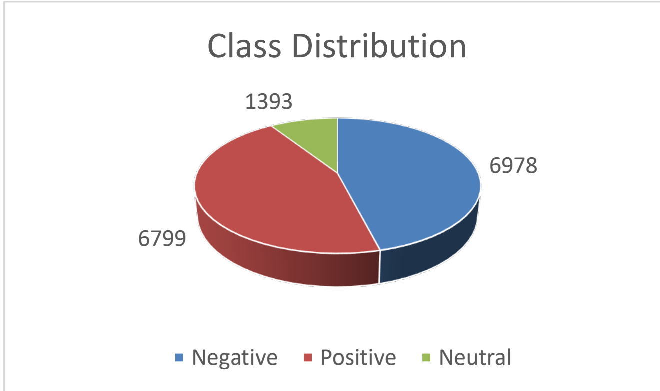

Figure 2. Data Set Class Distribution

In addition to the numerical distribution of the dataset, the textual content is also very important for the study. The structure of the text data may impact the results; therefore, the dataset was preprocessed to improve text processing and make it suitable for the model, ensuring accurate results.

The following steps were applied to the dataset (Ulusoy, 2022) ; Converting Text Data to Lowercase, all text data is converted to lower case, removing special characters, punctuation and special symbols (such as punctuation marks) have been removed from the text. Removing Numbers, removing stop words, extracting the Word root with stemmer.

4. RESULTS AND DISCUSSION

After modifying the dataset, the modeling and testing phase began, using the TF-IDF vectorization method and classifiers such as Multinomial Naive Bayes, Support Vector Machine, Random Forest, and Logistic Regression. The data set was divided into 20% test set and 80% training set. Figure 3 shows the accuracy values of the algorithms used in the project. The Support Vector Machine and Logistic Regression algorithms produced similar results.

Accuracy is calculated with the mathematical expression is shown in equation 1. TP means instances correctly predicted as positive, FP means instances incorrectly predicted as positive, FN means instances incorrectly predicted as negative, TN means instances correctly predicted as negative [9].

$$(TP+TN) / (FP+TP+TN+FN) \tag{1}$$

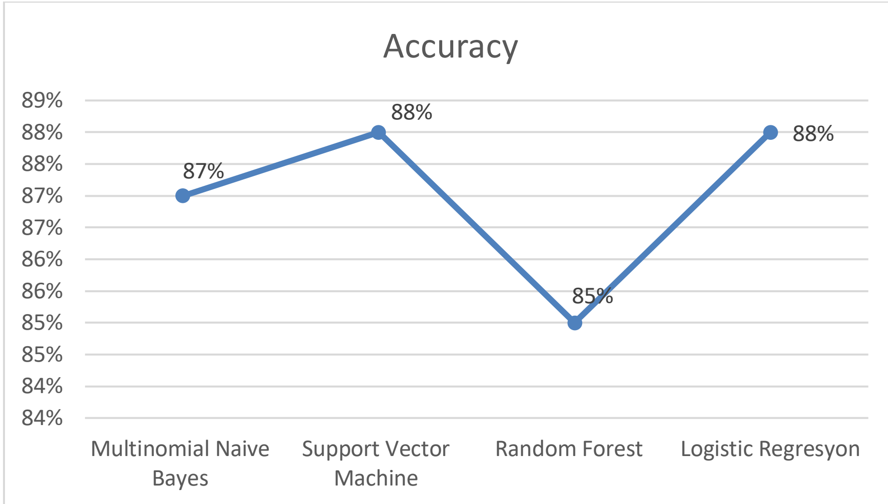

Figure 3. Accuracy rates

Figure 4 shows the precision values of the algorithms used in the study. It is calculated with the mathematical expression is shown in equation (2).

$$TP / (TP + FP) \tag{2}$$

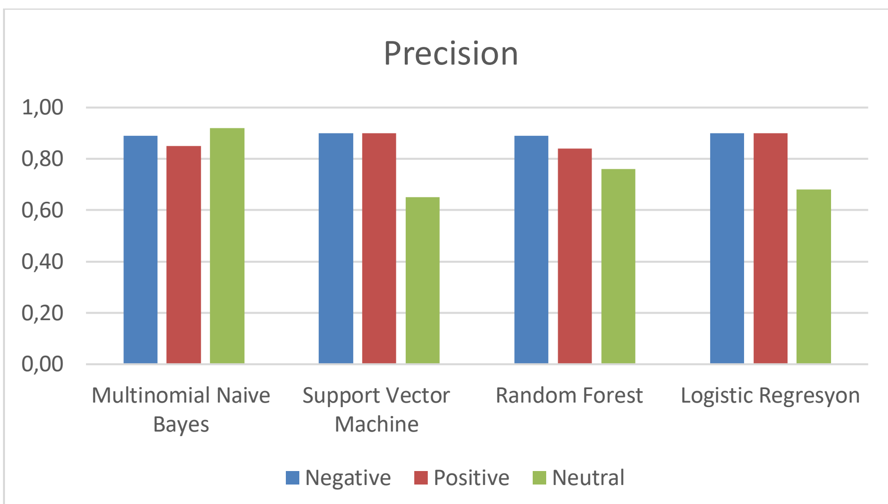

Figure 4. Precision values of the algorithms

Figure 5 shows the recall values of the algorithms used in the study. Mathematical expression of the recall is shown in equation (3).

$$TP/(TP+FN) \tag{3}$$

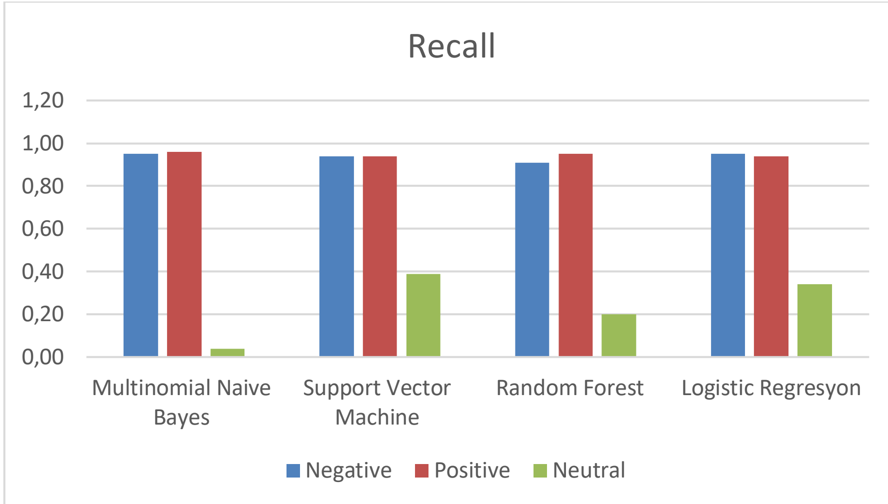

Figure 5. Recall Values of algorithms

The F1 score is the harmonic mean of Precision and Recall metrics. It is calculated with the formula is shown in equation (4) [9].

$$2*(Precision * Recall) / (Precision + Recall) \tag{4}$$

Figure 6 shows the F1 score values of the algorithms used in the study.

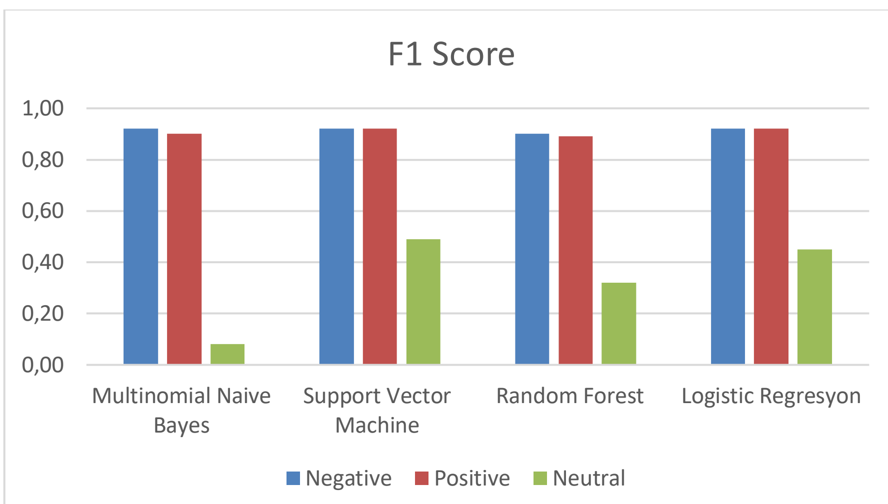

Figure 6. F1 scores of algorithms

After training the model, a randomly selected comment from the dataset was classified, and the automatically generated response corresponding to the predicted sentiment was displayed. The same process was applied to a comment received from the user and the result was printed on the screen.

Figure 7 shows the comments returned from the dataset, the comments received from the user and the automatic responses. “0” represents Negative, ‘1’ Positive, ‘2’ Neutral data.

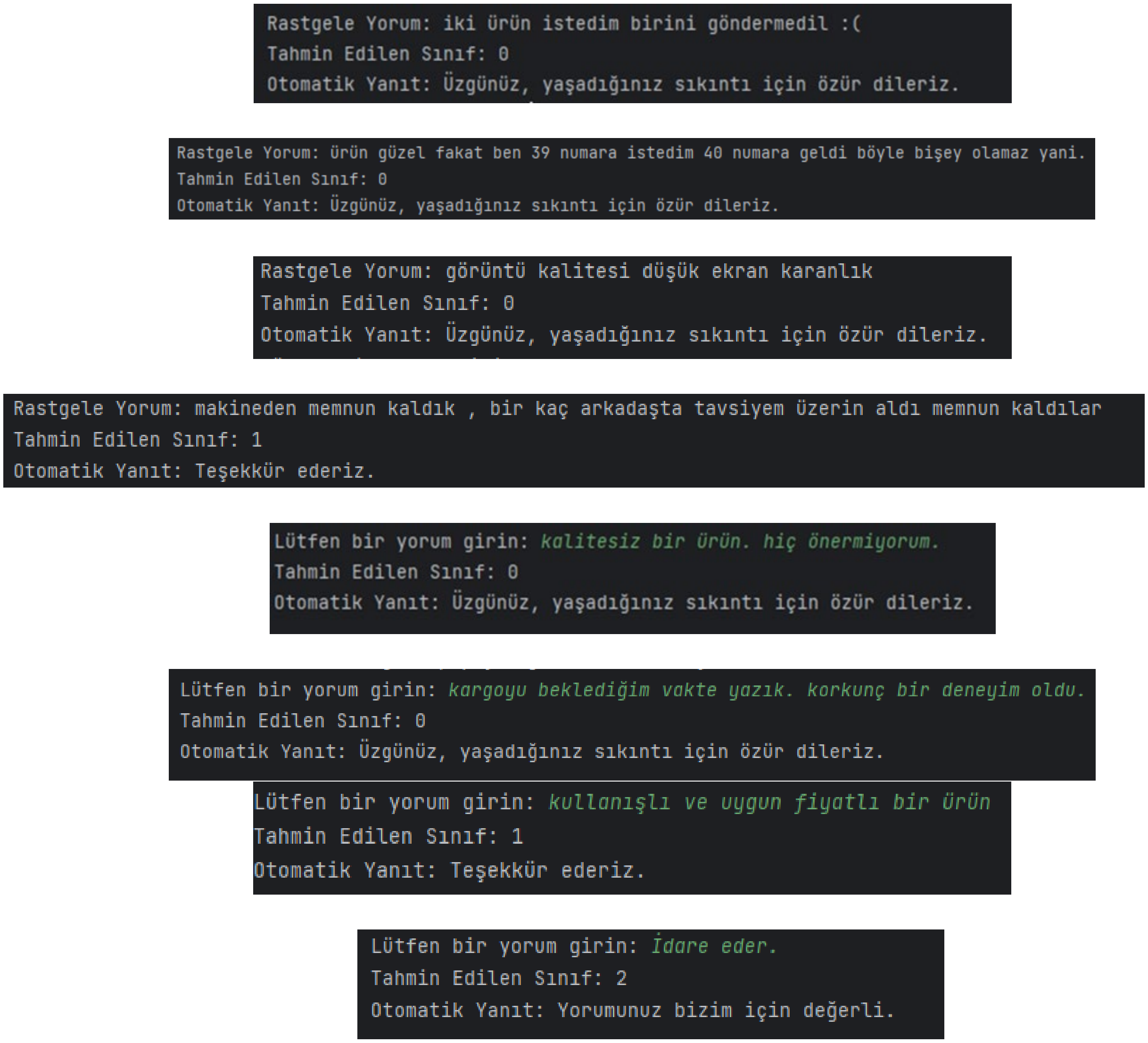

Figure 7. The comments results

CONCLUSION

Multinomial Naive Bayes, Support Vector Machine, Random Forest, Logistic Regression algorithms and the accuracy values are 87%, 88%, 85% and 88% respectively. For the neutral class, where sample sizes were insufficient, the Support Vector Machine algorithm produced the best results. The performance of the study can be improved by using larger and more diverse datasets, improving the data preprocessing steps and trying more complex algorithms. In this way, the study can be further developed and used as an industrial product. E-commerce sellers can enhance customer relationships and satisfaction by using such automatic analysis tools.

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LSTM-Based Deep Learning Model for Air Temperature Prediction

Research Article

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Abstract

With escalating temperatures at its core, climate warming triggers glaciers melting, rising sea levels, extreme weather phenomena, biodiversity loss, food chain disruptions, and heightened risks of natural disasters such as typhoons, tsunamis, landslides, and soil erosion. Air temperature serves as a pivotal indicator for assessing energy and hydrological balance, greenhouse effects, solar radiation levels, and air pollution. Consequently, temperature variation is marked by dynamic, uncertain, and nonlinear patterns. In this study, LSTM (Long Short-Term Memory) architecture, one of the deep learning methods, was applied to the 5-year daily average air temperature data of Izmir. With the LSTM, long-term temperature trends are determined by analyzing historical temperature data. This method is particularly effective for modeling complex and variable data, such as air temperature. In order to measure the effectiveness of the developed method, different machine learning algorithms were applied, and their performance values were compared. The R-squared (R^2) value, which indicates the correlation between actual and predicted values was found to be 0.963 for Linear Regression (LR), 0.948 for Random Forest (RF), 0.949 for Support Vector Machine (SVM), 0.949 for Convolutional Neural Network (CNN), 0.950 for Multilayer Perceptron (MLP), and 0.963 for LSTM. The high prediction accuracy of LSTM networks has shown that they can be successfully applied in temperature time series forecasting.

Keywords: Artificial Intelligent, Air Temperature, Deep learning, LSTM.

1. INTRODUCTION

Characterized by rising temperatures, climate warming leads to glaciers melting, rising sea levels, extreme weather events, loss of biodiversity, disruption of food chains, and an increased likelihood of natural disasters such as typhoons, tsunamis, landslides, and soil erosion. Air temperature is a critical indicator for evaluating energy balance, hydrological balance, greenhouse effects, total solar radiation estimation, and air pollution. Therefore, temperature change is dynamic, uncertain, and nonlinear.

Various methods are used to estimate air temperature. Artificial neural networks (ANN) and ANFIS are some of these methods. Hayati and Mohebi [1] developed artificial neural networks based on multilayer perceptron's in the temperature estimation of Kermanshah city in western Iran. They used meteorological data for the years 1996-2006 in the training and testing phase. As a result, it was determined that the network gave minimum errors, and the model was a suitable method for short-term temperature estimations. Dombayci and Gölcü [2] created an ANN model for the estimation of daily average temperature in Denizli. In the study, the data of the State Meteorological Directorate between the years 2003-2005 were used for training data and the data for the year 2006 was used for testing data. Different algorithms and neuron numbers were tried in the created model. As a result, they determined and compared the R2 and Root Mean Squared Error (RMSE) values of the network. It has been emphasized that the ANN approach is a reliable method in temperature estimation. In their study, Bilgili and Şahin [3] developed an ANN model for long-term monthly temperature and precipitation estimation according to measurement point data at any point in Turkiye. They used 59 station data from the State Meteorology Directorate between 1975-2006 for model training and 17 station data for model testing. The input parameters of the model they created are latitude, longitude, altitude and time; the output is long-term air temperature and precipitation. They compared the ANN model results with real values and stated that the errors in the model were at acceptable levels. In their study, İbrikçi and Soylu [4] estimated the air temperature of Adana according to meteorological parameters. The input parameters consist of wind, atmospheric pressure and relative humidity, while the temperature constitutes the output parameter. In this study, the efficiency of the model was compared with the mean square error (MSE) and the mean absolute percentage error (MAPE) statistical methods. Süzen and Kayaalp [5] conducted a temperature forecast study for Isparta province using LSTM, one of the deep learning algorithms. After the developed system was trained, the daily average temperatures of Isparta province for the next 4 years were estimated. According to their results, it was revealed that their developed model can be used successfully in predicting the future with deep learning algorithms. Singh and Mohapatra [6] developed an ARIMA model for very short-term wind speed prediction and investigated the shortcomings of ARIMA and WT-ARIMA models, which are recently popular techniques for short-term and very short-term prediction of wind speed and proposed a new Iterative WT-based ARIMA (RWT-ARIMA) model with improved accuracy for very short-term wind speed prediction. Qui et al. [7] developed a deep learning prediction methodology to investigate the river water temperature that changes in the thermal regime caused by climate and dam construction. Their results revealed that LSTM outperformed other prediction methods to estimate the average daily water temperature in rivers and accurately captured the daily average changes in the thermal regime. Fang et al. [8] in their study on multi-region indoor temperature prediction with LSTM-based sequencing proposed a LSTM-based sequence-to-sequence (seq2seq) model to perform multi-step forward prediction to make the most of different input variables. It was shown that the existing model is much more capable and reliable in very short-term predictions. A cross-serial learning strategy was adopted to provide multi-region indoor temperature prediction with a more generalized model. Uluocak and Bilgili [9] focus on developing innovative hybrid models to enhance the accuracy of one-day-ahead air temperature (AT) predictions. By integrating convolutional neural networks (CNN) with LSTM neural networks and gated recurrent units (GRU), two hybrid models. Their result indicated that the proposed hybrid models outperformed traditional prediction approaches, offering a more robust and reliable solution for short-term air temperature predictions under varying climatic conditions.

2. MATERIALS AND METHODS

LSTM is a type of Recurrent Neural Network (RNN) commonly used in sequential data such as time series, language processing, and speech recognition. LSTM was developed to overcome this limitation due to the vanishing gradient problem of RNNs. LSTM is quite effective in capturing long-term dependencies. It has cell state structures. Cell state contains mechanisms that ensure that information is preserved for a long time and unnecessary information is forgotten. The basic components of LSTM are the input gate, forgetting gate and output gate. The input gate decides how much new information will be included in the cell state. The forget gate determines which information will be removed from the cell state. The output gate controls which information from the cell state will be used as output. These gates are optimized during the learning process using the sigmoid and tanh activation functions.

One of the biggest advantages of LSTM is its ability to learn long-term dependencies in sequential data. For example, in language processing, the meaning of a word is influenced by the context of the surrounding sentence. By preserving this context, LSTM can achieve successful results in tasks such as language modelling and translation. At the same time, LSTM models are widely used in areas such as time series prediction, speech recognition, and analysis of biomedical signals. Its success in these areas has made it preferred especially in cases where complex and long-term relationships need to be analysed. LSTM is generally more complex than other neural networks and the training process can take longer. This is because it requires more computation to optimize the gate mechanisms at each step. However, this complexity is offset by the effective learning of long-term dependencies and usually pays off in terms of performance.

Today, to successfully combat climate change's increasing effects, effective air temperature modeling has become a priority. Accurately estimating temperature values is essential for application areas such as agriculture, tourism, industry, energy management, and traffic planning. Artificial intelligence-based models can predict future temperature trends using past temperature observation data. Deep learning models, particularly effective in time series analysis such as LSTM, can make temperature predictions with high accuracy. LSTM effectively learns complex dependencies and long-term relationships in time series data and can make more accurate and reliable predictions than traditional air temperature prediction models. General structure of LSTM model for air temperature prediction is given in Figure 1.

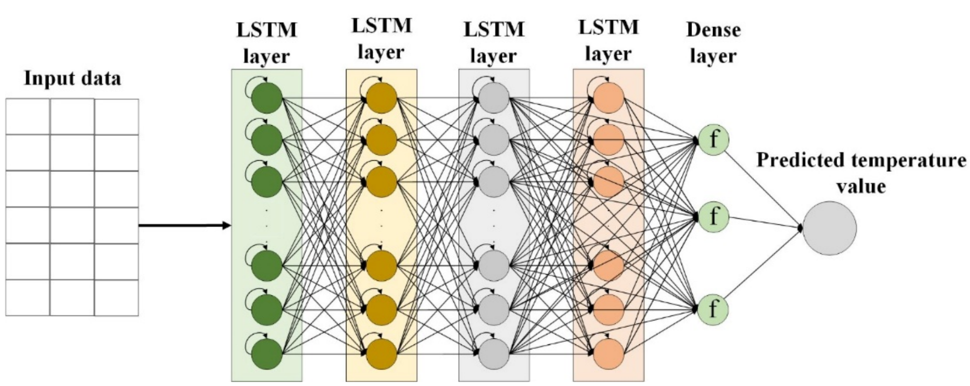

Figure 1. LSTM network structure for predicting air temperature

In this study, a comprehensive analysis of popular prediction models such as LR, RF, SVM, CNN, MLP, and LSTM for the prediction of daily mean temperature values of Izmir was presented.

2.1. Dataset

In this study, a comprehensive analysis of popular prediction models for the prediction of daily average temperature values was presented. In the study, daily average temperature values of Izmir between 01/01/2018 and 31/08/2022 were used. Figure 2 shows the change in daily average temperature values over time.

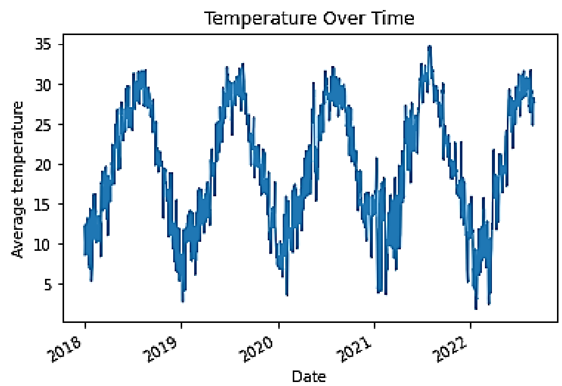

Figure 2. The change in daily average temperature values over time

The dataset comprises 1,704 entries of daily average temperature observations. The temperature ranged from a minimum of 1.80 °C to a maximum of 34.70 °C, with an average of 19.55 °C.

2.2. Prediction Models

LR is a machine learning model used to express a dependent variable with a linear model by one or more independent variables [10]. The main purpose is to understand the relationship between the dependent variable and the independent variables and to predict the value of the dependent variable using this relationship. LR is a basic statistical method used to model the linear relationship between dependent and independent variables. It is used in the analysis of continuous data such as air temperature forecasting. It is preferred due to its simplicity and fast operation, but its ability to model complex and nonlinear relationships is limited. Random Forest (RF) is an ensemble learning method composed of multiple decision trees. Each decision tree is trained on a randomly sampled subset of data, and these trees are created with random features [11]. This process ensures that the trees are independent of each other and can make different decisions. Each decision tree makes its prediction. In classification problems, each tree makes a class prediction, and the class with the most votes is the final prediction of the model. In regression problems, the final result is obtained by taking the average of the predictions of the trees [12]. RF is robust to data noise in air temperature forecasting. SVM is used to classify data as belonging to a particular class. SVM works to separate data with a line or hyperplane [13]. This line or hyperplane is chosen to separate the data as best as possible. SVM works by maximizing the distance between two classes on this line or hyperplane. SVM can be used to classify both linear and non-linear data. Linear data is data that a line can separate. Non-linear data cannot be separated by a line [14]. SVM uses different kernel functions to classify non-linear data as well. CNN, which is used in the form of 1D in the time series, effectively removes local patterns in the data [15]. In the time series context, trends and patterns in the data are removed at a particular time. Convolution layers make the property subtraction by processing the data in a rocking window of a specific size. Pooling layers reduce the data size to learn more general features [16]. Full-linked layers are used to learn the features of higher levels than those extracted by conviction, and pooling layers are used to learn and create final predictions. MLP is a type of artificial neural network and can be used in estimating continuous values such as air temperature. It can model non-linear relationships by passing input data through different layers. Backpropagation algorithm is used to reduce the error rate during training. The primary purpose of LSTM is to solve the problem of learning long-term addictions in Recurrent Neural Network (RNN) models. Although RNNs theoretically have the potential to learn long-term addictions, they may encounter problems with the disappearance of a gradient in practice [17]. LSTM can solve these problems with unique cell structures (forgetting, input, and output), keep the information in memory for long periods of time, and use it when necessary [18]. These cells determine what to remember and what to forget. If the incoming input is trivial, it is forgotten; if it is essential, it is transferred to the next stage. Decide which information will be kept or forgotten. Information from the previous hidden layer, along with the current input, passes through the Sigmoid activation function [19]. The closer the value is to 0, the more information is forgotten; the closer it is to 1, the more information is retained. This ability to capture complex and long-term dependencies makes LSTM ideal for time-dependent data, such as air temperature. By remembering past information, LSTM can make accurate predictions based on historical patterns in the data.

3. EXPERIMENTAL RESULTS

In the study, null and erroneous rows in the dataset were first checked. Since the dataset was a time series dataset, the dataset was converted to a supervised learning problem format using the sliding-window method. This method allows the observation data to be structured as input for a specified window size and the data in the following observation step as output. In this way, the sliding window method converts the time series data into a supervised learning format by structuring it as input-output samples. As a result of the experimental studies, the sliding-window size was determined as 3. 80% of the dataset was split for training the models and 20% for testing the models. Using the grid search method, 10% of the training data was used to determine the model hyper-parameters. Grid research involves generating and testing models across a predefined hyperparameter space to identify the optimal configuration. After determining the most successful hyper-parameters for each model, the models were created and tested with Mean Absolute Error (MAE), R-squared (R2), and RMSE. RMSE is calculated by taking the square root of the sum of the squares of the model's prediction errors. In this study, RMSE, MAE and R2 metrics were used to evaluate the models. RMSE, MAE, and R^2 are statistical metrics used to measure the predictive performance of a model. They are used to analyze the difference between the predicted values and the actual values, especially in regression problems. RMSE is the root square of the errors between the predicted values and the actual values. MAE represents the average of the absolute differences between predicted and actual values. R2 measures how well the model explains the relationship between the independent variables and the dependent variable. R2 expresses the explanatory power of the model. The closer the RMSE is to 0, the better the model performs. MAE is the mean of the model's absolute prediction errors, and like RMSE, the closer the MAE is to 0, the better the model performs. R2 indicates how well the model explains the variance of the data, and the closer the R^2 value is to 1, the better the model performs. Table 1 and Figure 3 show the experimental findings of the models.

Table 1. The experimental findings of the models

Model	RMSE	MAE	R2
LR	1.803	1.327	0.948
RF	1.793	1.311	0.948
SVM	1.774	1.301	0.949
CNN	1.769	1.291	0.949
MLP	1.738	1.263	0.950
LSTM	1.537	1.130	0.963

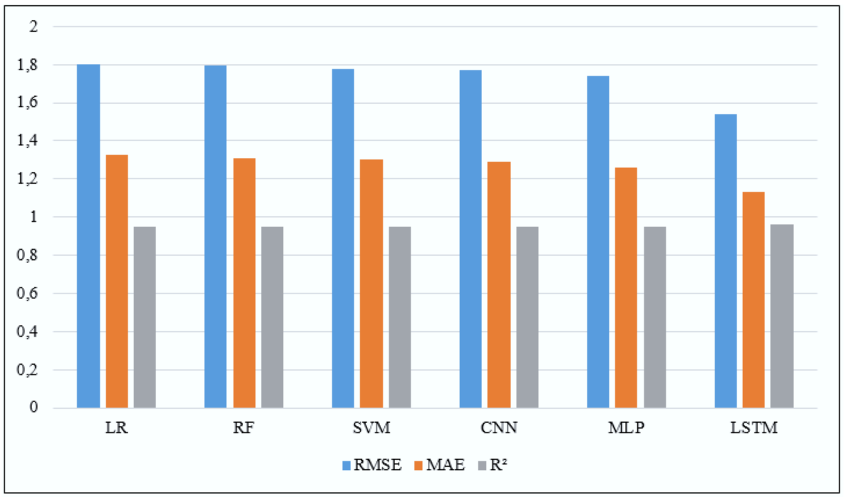

Figure 3. The experimental results of the compared models

As seen in Table 1 and Figure 2, the experimental findings showed that LSTM has better prediction performance than the compared models. LSTM has lower error values than the compared models regarding RMSE and MAE metrics. The R2 value is higher than that of the compared models, which is 0.963. LSTM is highly effective at learning long-term dependencies due to its memory cells. Depending on a specific time index, LSTM can learn long-term trends of past temperature observation values and perform better predictions in temperature data. The other compared models are limited in learning long-term dependencies.

4. CONCLUSION

Accurately predicting air temperature values is essential for planning and developing strategies for many sectors. Traditional weather prediction methods are often insufficient for capturing long-term climate trends. Artificial intelligence methods increase forecast accuracy by quickly processing large amounts of data and learning complex relationships in this data. For example, it is essential in the agricultural sector to plan harvest times and determine irrigation requirements, in the energy sector to predict energy consumption values according to temperature data, and in the transportation sector to plan traffic density.

In this study, an LSTM-based weather forecast model was developed using a dataset consisting of daily average temperature values of Izmir between 01/01/2018 and 31/08/2022. The developed model was compared with LR, SVM, RF, CNN, and MLP. Experiments showed that LSTM has a better forecast performance than the compared models with 0.963 R2. The study showed that deep learning models were more successful than machine learning models. Future research will explore hybrid deep learning models trained on more comprehensive datasets.

LR failed in data containing complex and non-linear relationships such as air temperature, because it can only model linear relationships. RF and SVM are more successful models than LR in non-linear relationships and variations in data, but they were not successful enough, especially because they did not take into account the past information in time series data. Since CNN cannot successfully model the dynamics of time series data, its performance is low compared to LSTM. Similar to CNN in MLP, it is limited in modeling the dynamic structure in time series data, but since the data is processed with multi-layer fully connected neural networks, it was more successful than CNN. Among the models evaluated, LSTM demonstrated the highest prediction accuracy. LSTM can learn long-term dependencies in time series data. Low RMSE and MAE, high R^2 value showed that LSTM is more successful than other models in the problem of air temperature prediction.

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An Inexpensive Tensile Test Machine

Research Article

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Abstract

Tensile testing is a fundamental method for understanding the mechanical properties of materials. In this study, a low-cost tensile testing setup was developed. The design aims to provide a practical and economical solution for educational institutions and small-scale research projects. The developed system is built from locally available components and is based on simple mechanical principles. Using force sensors and linear motion mechanisms, the tensile strength and elongation properties of the specimens were successfully measured. The experimental results showed that the system works with high precision and successfully measured important properties of the tested materials such as tensile strength. Compared to commercial devices, this setup provides a substantial cost advantage while maintaining acceptable levels of accuracy. As a result, this setup, which stands out as a low-cost and practical alternative, is considered to be especially suitable for use in educational laboratories and small-scale projects.

Keywords: Tensile Testing, Tensile Strength, Material Testing System, Mechanical Properties.

1. INTRODUCTION

Tensile testing is one of the most widely used methods for evaluating the mechanical properties of materials. It provides essential information such as tensile strength, elongation, and modulus of elasticity, which are critical for material selection and engineering design. Despite its importance, commercial tensile testing machines are often expensive, making them inaccessible to many educational institutions and small-scale research facilities[1].

The need for affordable and practical solutions has become increasingly evident, particularly for academic laboratories where cost constraints often limit the availability of advanced testing equipment. Low-cost testing setups Load cells designed with locally sourced components, present an opportunity to bridge this gap while maintaining acceptable levels of accuracy and reliability[2,3]. This study focuses on the development of an inexpensive tensile test machine tailored to meet the needs of educational and small-scale research applications. The proposed system combines simplicity in design with cost-efficiency, using readily available materials and straightforward mechanical principles. By integrating force sensors and a linear motion mechanism, the setup is capable of measuring the tensile properties of materials with a high degree of precision[4-6].

In the following sections, the design, fabrication, and performance evaluation of the developed tensile test machine are presented. The results demonstrate the viability of this system as a cost-effective alternative to commercial devices, particularly in environments with limited financial resources[7-9]. Load cells are widely used to measure forces in various applications, including weighing scales, where the strain of the load cell determines an object's weight. A load cell typically consists of a counterforce and force transducers, such as strain gauges, which convert force into an electrical signal. These transducers are commonly arranged in a Wheatstone bridge configuration for precision. In industrial applications such as checkweighers, load cells play a critical role, as these applications demand high-speed and accurate measurements [10–13]. This study focuses on developing a 3D force transducer and a real-time force measurement system tailored for agricultural machinery. The system is designed to monitor force variations at the three-point linkage of tractors, enabling real-time data transmission via USB communication. Key design considerations include real-time and graphical data visualization, robustness, linear output, dynamic calibration, and high accuracy. Figure 1 illustrates the developed transducers and the accompanying electronics. This system provides an efficient solution for monitoring force dynamics in agricultural operations [14-16,19].

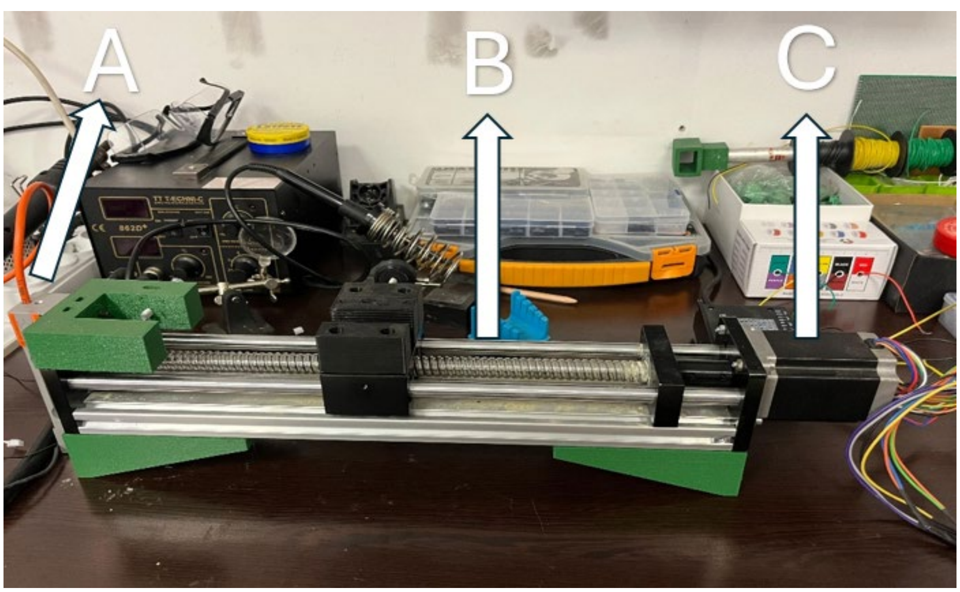

Figure 1: The Developed Tensile Test System. A: LoadCell, B: Linear Rail Guide, C: Motor.

2. MATERIAL AND METHOD

A. Mechanics

Due to their affordability and precision, load cells are extensively used in industrial instrumentation [17,18]. Strain gauge load cells, a common type for force or load measurement, offer accuracy levels ranging from 0.03% to 1%. In this study, strain gauge-based load cells are employed as force transducers. The mechanical structure of these transducers, designed specifically for detecting three-dimensional force variations, is depicted in Figure 2. These custom-designed 3D force transducers enable precise measurement of forces in multiple directions.

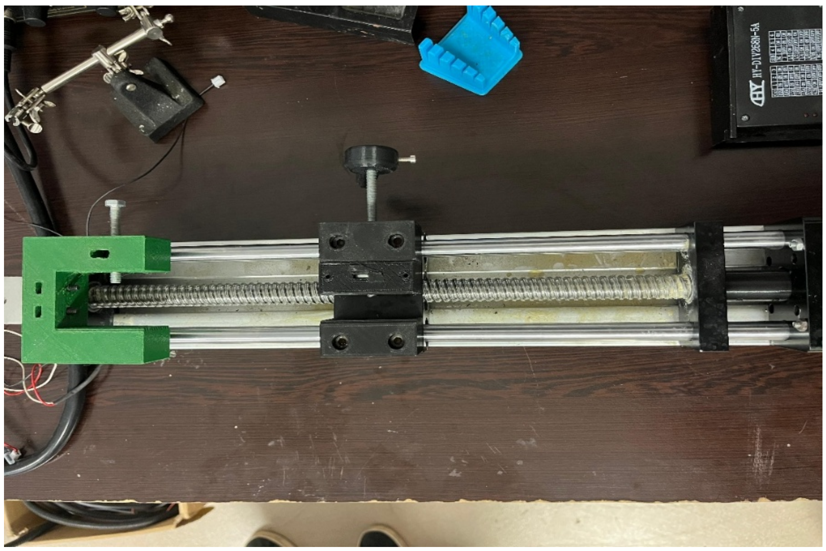

Figure 2: Tensile Mechanism.

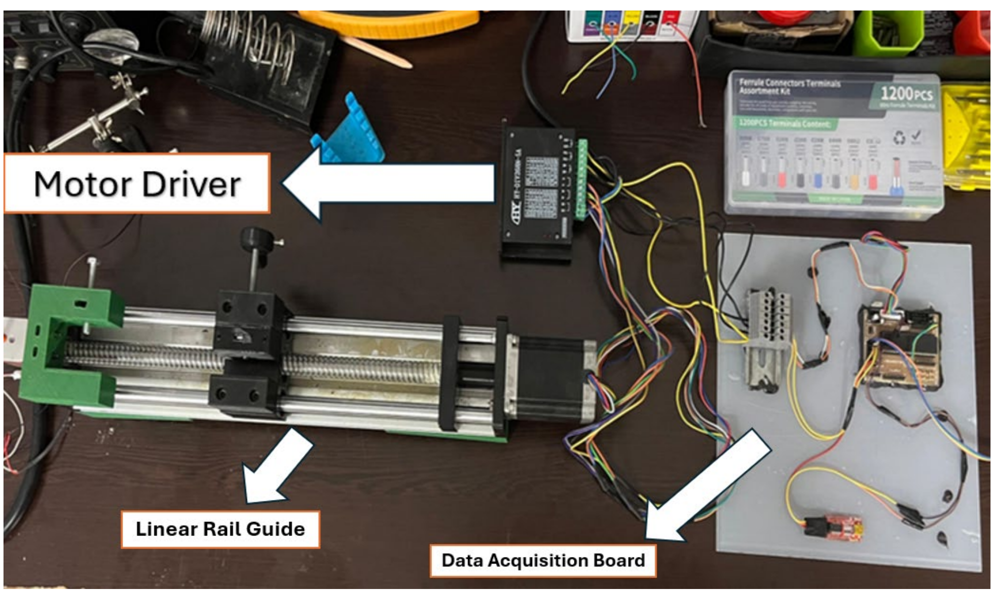

Figure 3. General Structure of the System.

The mechanical design criteria for the 3D force transducer used in the load cells are outlined in Figure 5.

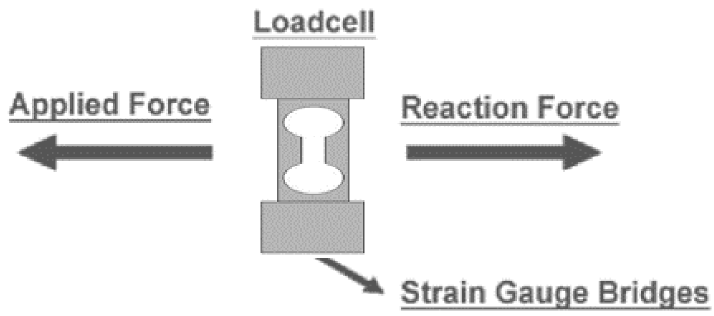

Figure 4: Force Equilibrium of the Load cells.

B. Electronics

A 3D force measurement system was designed to monitor forces acting on a three-point linkage using load cells. The system comprises three load cells, a microcontroller unit (MCU), an analog-to-digital converter (ADC), an USB interface, and a computer. Strain gauges were integrated into the load cells to detect force variations, and a Wheatstone bridge configuration was employed for signal conditioning. The analog voltage signals generated by the bridges were digitized using a 24-bit ADC. The processed digital data was transmitted to the computer via RF communication, where it was graphically displayed in real-time. The system's block diagram is illustrated in Figure 6.

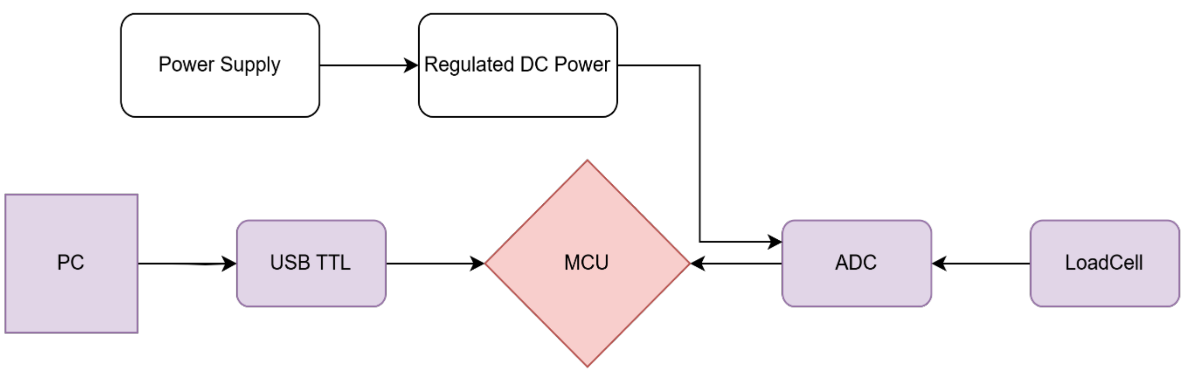

Figure 5. Block Diagram of the Measurement System.

In this system, regulated DC power supplies were used to provide power to the Wheatstone bridges on each load cell. Each load cell was independently powered using a dedicated power unit. The analog signals generated by the bridges were captured using 24-bit ADCs, which converted the signals into digital data. This digital data was then transmitted to the microcontroller through serial communication. The microcontroller sent the processed information via an USB interface, while the receiver unit forwarded the data to the computer through a USB connection.

To minimize noise in the load cells, a regulated DC power supply was used to power the Wheatstone bridges. Additionally, cable shielding was grounded to further reduce interference, as illustrated in Figure 7.

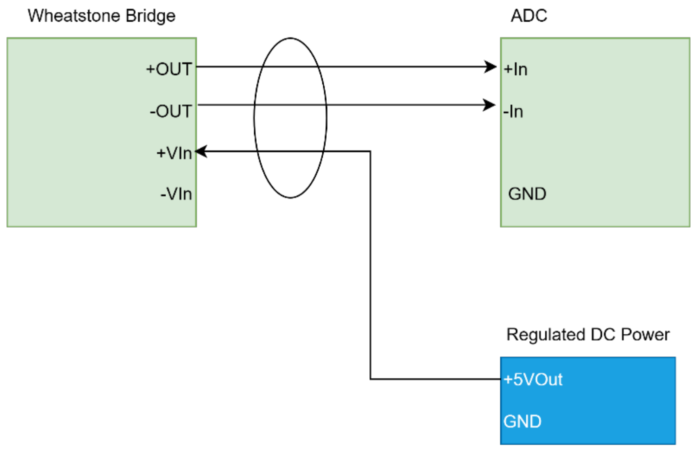

Figure 6. Shield Grounding for Noise Reduction.

The system incorporates full bridge strain gauges, as shown in Figure 8, on the load cells for thermal compensation. A 5V excitation voltage is used to power the bridges, and the strain gauges have a resistance of 120Ω.

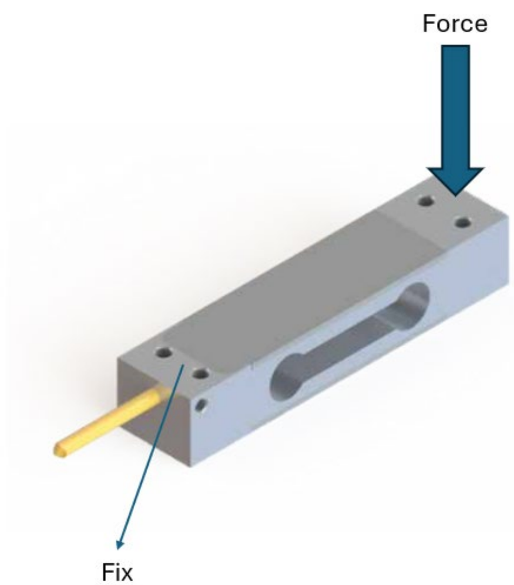

Figure 7. LoadCell Layout.

The electronic system processes force data from the load cells and transmits it to a computer. This measurement setup supports remote data collection using RF signals, with a robust USB interface ensuring reliable communication between field and remote units. A custom-designed interface developed using C++ software is utilized for real-time data analysis and visualization. The user interface allows interactive monitoring of factors such as offset, force variations, and graphical outputs for each transducer. A microcontroller (MCU) handles data acquisition, while an analog-to-digital converter (ADC) converts analog force signals into digital values. The USB interface is connected to the computer via a USB interface for seamless data transfer.

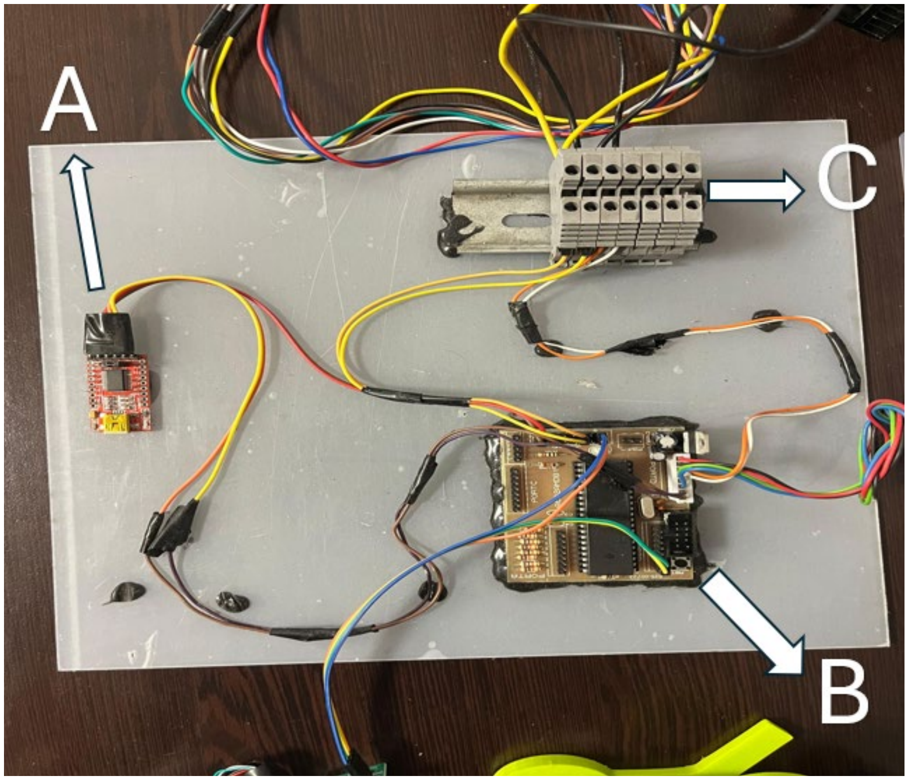

Figure 8: Electronical Parts of the System. A: USB Interface, B: MCU, C: MCU Motor Driver Connection Terminal.

3. RESULTS

Prior to the development of the electronic system and mechanical design, theoretical calculations were conducted to determine the upper and lower limits of the measurement system, ensuring the desired measurement range and step response. Based on these calculations, key characteristics of the mechanical transducer and electronic system were identified to guide the development process. During the design phase, theoretical evaluations were performed specifically for a full-bridge strain gauge load cell. The outcomes of these calculations are presented in Table 1, which illustrates the expected voltage output corresponding to both minimum and maximum load conditions.

PS Series Load cell model was used that manufactured by Pulse Electronic [20] for calibrating the designed 3D force transducers. Table 1 is based on the capacity and output signal of the load cell with a load capacity of 20 Kg. For basic calculations the maximum Capacity (Emax) 20 kg, output signal 2 mV/V, excitation voltage 10 V are used. The output voltage was calculated as follows:

$$V_{Out} = Output\,Voltage\ x\ Excitation\,Voltage\ x\ \frac{Force}{Maximum\ Capacity}$$

Table 1: Applied Force versus Tension & Compression.

Force (kg)	Output Voltage 10V (mV)	Output Voltage 12V (mV)	Amplified Voltage (mV)
0.00	0.00	0.00	0.00
2.86	2.86	3.43	438.86
5.71	5.71	6.86	877.71
8.57	8.57	10.29	1316.57
11.43	11.43	13.71	1755.43
14.29	14.29	17.14	2194.29
17.14	17.14	20.57	2633.14
20.00	20.00	24.00	3072.00

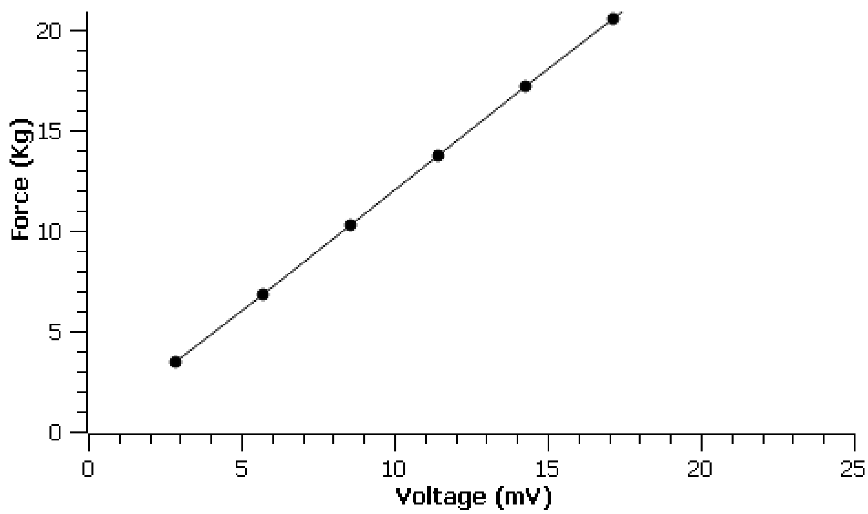

Figure 9. Force as a Function of Voltage. (12V).

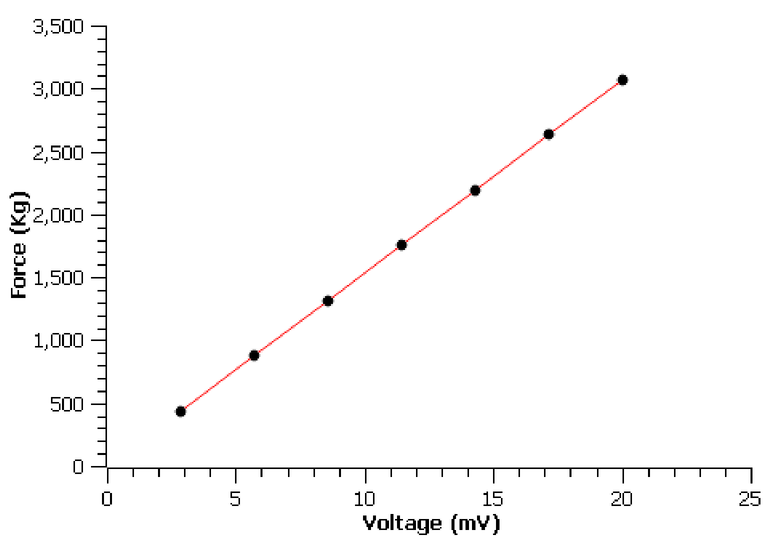

Figure 10. Force as a Function of Voltage (Amplified Voltage).

Table 1, Figure 9, and Figure 10 illustrate the linear relationship between load (force) and voltage in the context of a towing test rig. The load cell used in the towing test rig measures the force applied on it and converts this force into a low-voltage electrical signal. The table shows the output signals from the load cell under a 12V supply voltage and their amplification by an amplifier. For example, if the load cell has an output capability of 2 mV/V, it produces an output proportional to the applied load, and this signal is amplified to a readable level (e.g., 3072 mV) with an amplifier. Figure 9 visualizes how the output voltage varies with force under a 12V supply, while Figure 10 shows the amplified voltage. Both figures confirm the linear increase between voltage and load, demonstrating the accuracy of the calibration of the system and the linear performance of the load cell. With this data, the tensile test rig enables precise and reliable measurements when testing the tensile strength or elastic behavior of a material. This connection between voltage and force represents the basic operating principle of the setup used in materials testing.
According to the results obtained, when compared with the traditional methods in the literature, the following results were revealed in Table 2:

Table 2: Comparison of the Designed System with Traditional Systems.

Comparison Criteria	Traditional Tensile Testing Machines	Developed Low-Cost Tensile Testing Machine
Cost	High; expensive and designed for industrial and commercial use.	Low; built using locally available and cost-effective components.
Accuracy & Performance	High precision, wide measurement range, and compliant with industry standards.	Provides acceptable accuracy with a linear force-voltage relationship.
Application Area	Suitable for large laboratories, industrial research, and extensive material testing.	Ideal for educational institutions and small-scale research projects.
Technical Hardware	Equipped with high-resolution sensors, powerful servo motors, and specialized software.	Uses a microcontroller (MCU), ADC, and load cells in a simple design.
Data Collection & Processing	Automated reporting and advanced data analysis using specialized software.	Data transmission via RF and USB; real-time analysis using a C++-based interface.
Measurement Capacity	Capable of high-capacity and large-scale testing.	Limited capacity, suitable for small-scale tests.
User-Friendliness	Often requires expertise to operate, with complex user interfaces.	Simple and easy to use, making it suitable for educational purposes.
Calibration & Maintenance	Requires regular maintenance and calibration, which can be costly.	Requires minimal maintenance and offers

		simpler, cost-effective calibration.
Motion & Mechanism	Uses precision lead screws and powerful motors for high accuracy.	Operates with linear rail guides and low-cost motors.

Due to limited facilities, we were unable to conduct direct comparisons of test samples. However, we compared the results of our study with those reported in the literature by analyzing force and measurement outputs.

4. CONCLUSION

In this study, a low-cost and effective tensile testing machine was developed, providing accessible and reliable solutions for material testing. The tensile testing machine produces consistent stress-strain curves, accurately capturing material properties such as yield strength, tensile strength, and elongation. While it lacks certain advanced features found in commercial machines, its performance is comparable to higher-cost alternatives, making it an ideal solution for educational institutions and research settings with limited budgets. Additionally, the results demonstrated the system's ability to deliver dependable data, as evidenced by the linear force-voltage relationship observed in the calibration and testing phases. This relationship, visualized in the Voltage-Force graph, highlights the precise output of the system, with force measurements extending linearly up to 20 kg and amplified voltage outputs aligning with expected performance.

The force measurement system further enhances the setup, offering accurate three-dimensional force measurements and the ability to collect data remotely over distances up to 5 km. Its graphical interface, developed specifically for this project, enables real-time monitoring and analysis of field data. These features, combined with the affordability and reliability of the tensile testing machine, showcase that effective material testing and force measurement systems can be developed without the prohibitive costs typically associated with commercial alternatives. This study demonstrates that accessible and cost-effective tools can enhance educational and research opportunities in material science, bridging the gap between resource constraints and the need for precise, repeatable data.

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A Comprehensive Study on Restricted and Extended Intersection Operations of Soft Sets

Research Article

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Abstract

Soft set theory has gained prominence as a revolutionary approach for handling and modeling uncertainty since it was proposed by Molodtsov. The concept of soft set operations, which is the major notion for the theory, has served as the foundation for theoretical and practical advances in the theory, therefore deriving the algebraic properties of the soft set operations and studying the algebraic structure of soft sets associated with these operations have attracted the researchers' interest continuously. In the theory of soft set, many soft intersection operations have been defined up to now among which there are some differences, and some of which are no longer preferred for use as they are essentially not useful and functional. While the restricted intersection definition is widely accepted and used in literature, it remains incomplete, as it ignores certain cases where the parameter sets of soft sets may be disjoint, thus not all conditions in the theorems are considered in the related proofs, leading to inaccuracies or deficiencies in the studies where this operation is used or its properties are investigated. There is a critical lack of comprehensive research in the existing literature on the correctly defined restricted intersection operation, along with the extended intersection, including their proper properties and distributions and the correct algebraic structures associated with these soft set operations. In this study, we primarily intend to fill this crucial gap by first correcting the deficiencies in the presentation of the definition of restricted intersection and revising it. Moreover, in many papers related to this operation, several theorems were presented without their proofs, or there were some incorrect parts in the proofs. In this study, all the proofs based on the function-equality are regularly provided. Besides, the relationships between the concept of soft subset and restricted and extended intersection operations are presented for the first time with their detailed proofs. Furthermore, we obtain many new properties of these operations as analogies and counterparts of the intersection operation in classical set theory. Moreover, the operations' full properties and distributions over other soft set operations are thoroughly investigated to determine the correct algebraic structures the operations form individually and in combination with other soft set operations both in the set of soft sets over the universe and with a fixed parameter set. This study demonstrates that the restricted/extended intersection operations, when combined with other kinds of soft set operations, form several significant algebraic structures, such as monoid, bounded semi-lattice, semiring, hemiring, bounded distributive lattice, Bool algebra, De Morgan Algebra, Kleene Algebra, Stone algebra, and MV-algebra but with detailed explanations. Accordingly, this study offers the most comprehensive analysis of restricted and extended intersection operations to date. It corrects earlier theorems and proofs, thereby advancing the theory and addressing a significant gap in the literature. Furthermore, it serves as a guide for beginners and sheds light on future research directions in soft set theory.

Keywords: Soft sets, Soft set operations, Restricted intersection operation, Extended intersection operation.

1. INTRODUCTION

It is difficult to explain and precisely describe many events in our daily lives, including uncertainty. Modeling situations involving uncertainty using classical mathematics or Aristotelian reasoning is extremely challenging. Set theory is considered a fundamental tool in mathematics, as it forms the basis for nearly all mathematical disciplines. To address and overcome uncertainty, many scientists from various fields have conducted research and proposed new theories. Among them, fuzzy set theory, introduced by Zadeh (1965), is one of the most widely used methods for dealing with uncertainty. In Aristotelian logic, the truth value of a proposition is either 0 or 1, whereas in fuzzy logic, it can be any real number within the range [0,1]. However, despite its popularity, fuzzy set theory faces certain limitations. The construction of the membership function is very subjective, which can lead to varying outcomes for the same problem. These challenges created a need for a new theory to address both uncertainty and cases of certainty. As an alternative, Molodtsov (1999) established "Soft Set (SS) Theory" as a mathematical technique to deal with uncertainty. SS theory's lack of a membership function construction issue makes it more practical. This advantage has led to its rapid application in fields such as mathematics, engineering, medicine, social sciences, and daily life situations like information systems and decision-making problems. Additionally, Molodtsov (1999) effectively used SS theory in domains like game theory, operations research, continuous differentiable functions, probability, measurement theory, Riemann integration, and Perron integration.

The fundamental notions of the SS were initially presented by Molodtsov (1999) in his pioneer study, and they were further expanded upon by a theoretical study of Maji et al. (2003) which introduces intersection and union operation with AND and OR operations, as well as the concepts of soft subset, soft equality, soft complement, NULL SSs, and absolute SSs. In contrast to Maji et al. (2003), Pei and Miao (2005) introduced a new intersection operation and proposed a new soft subset concept in their study on soft-based information systems. For the intersection operation, Feng et al. (2008) proposed an alternative concept known as "bi-intersection" (double intersection). To better understand the evolution and variations in soft set operations, it is essential to review key contributions in literature. For instance, introducing new definitions such as restricted union, intersection, difference, and extended intersection of SSs, Ali et al. (2009) aimed to resolve certain limitations in earlier operations defined by Maji et al. (2003). These foundational modifications laid the groundwork for exploring algebraic properties. Building on this, the concept of "relative complement" was developed, and it was shown that De Morgan's laws hold in soft set theory under these refined operations. Subsequently, Qin and Hong (2010) introduced a new form of "soft equality" and explored the algebraic structures of SSs, applying absorption laws to investigate whether these structures form lattices. In their extensive study, Ali et al. (2011) showed that some operations in the fixed-parameter SSs form MV-algebras and BCK-algebras, and they also showed that the SS operations defined by Ali et al. (2009) form a variety of algebraic structures, including monoids, hemirings, and lattices in the collection of SSs over the universe as well as in the fixed-parameter SSs. Sezgin and Atagün (2011) introduced restricted symmetric difference for SSs and investigated its characteristics. They also further explored the fundamental properties of restricted and extended intersection and union operation defined by Maji et al. (2003) and Ali et al. (2009). Redefining the notion of an SS's complement, Singh and Onyeozili (2012a, 2012b, 2012c, 2012d) published research on SS operations, the distributive and absorption laws of SS operations. Sen (2014) showed that restricted and extended intersection and union operations constitute a Boolean algebra in the set of the SSs with a fixed parameter set. A new SS operation known as "extended difference" was added to the list of extended operations in SSs by Sezgin et al. (2019) Additionally, Sezgin et al. (2019) investigated the characteristics of the operation and its connections to other SS operations. By proposing and examining the operation of extended symmetric difference, Stojanovic (2021) addressed a gap in the literature about the extended operation in SS theory. Some papers, such as Neog and Sut, 2011; Fu, 2011; Ge and Yang, 2011; Zhu and Wen, 2013; Onyeozili and Gwary, 2014; Husain and Shivani, 2018 contain incorrect assertions that must be corrected.

As we see, "restricted" and "extended" SS operations are two primary categories under which the advancements in SS operations may be divided after a study of the research done up to this point. In contrast to the restricted and extended operation forms, the "soft binary piecewise difference operation" was an innovative SS operation that Eren and Çalışıcı (2019) described and investigated the characteristics of. A thorough investigation of the properties of the soft binary piecewise difference operation was explored by Sezgin and Çalışıcı (2024). Sezgin et al. (2023a) studied several novel binary set operations, motivated by Çağman (2021) work on conditional complements of sets. These binary set operations were transferred to SSs by Aybek (2024), who also defined novel restricted and extended SS operations, investigated their characteristics, and explored how they related to other SS operations. Additionally, Akbulut (2024), Demirci (2024), and Sarıalioğlu (2024) investigated a new type of SS operations known as "complementary extended SS operations". Yavuz (2024), Sezgin and Yavuz (2023a), and Sezgin and Yavuz (2024) defined and thoroughly examined a number of new soft binary piecewise operations, all of which were defined within the framework of the soft binary piecewise operation that was first presented in the study of Eren and Çalışıcı (2019). Besides, several authors (Sezgin et al., 2023b, Sezgin et al. 2023c; Sezgin and Dagtoros, 2023; Sezgin and Demirci, 2023; Sezgin and Yavuz, 2023b; Sezgin and Sarıalioğlu; 2024a; Sezgin and Sarıalioğlu; 2024b; Sezgin and Çağman, 2024; Sezgin and Şenyiğit, 2025)

An algebraic structure is made up of a set that has one or more binary operations defined on it along with those binary operations. Classifying algebraic structures and finding, showing, and deriving results from their common features are the goals of abstract algebra. It conducts this regardless of the sets and binary operations that make up these structures. This is the reason abstract algebra is the name given to this area of mathematics. Fundamentally, algebraic structures are involved in many branches of mathematics. Mathematicians have studied algebraic structures for millennia as they offer a universal and abstract approach to understanding and comprehending mathematical subjects. Understanding the properties of algebraic structures enables mathematicians to solve challenging problems, create new theories, and apply ideas to a variety of mathematical, scientific, and engineering domains. Furthermore, applications frequently provide special examples of algebraic structures, which help to clarify specific circumstances and make it easier to examine more general scenarios. When a particular set S is recognized as an illustration of a well-known algebraic structure, all of the well-known results regarding this algebraic structure also inherently hold for S. Abstraction is primarily motivated by this advantage. As a result, algebraic structures play a significant role in abstract algebra and mathematics.

One of the most well-known binary algebraic structures, which is a generalization of rings, is the notion of semirings which has been a subject of study and fascination for scholars from the past to the present. Vandiver (1934) introduced the concept of semirings. Several researchers have also studied semirings with additive inverses (Karvellas,1974; Goodearl, 1976; Petrich, 1973). While semirings are especially important in geometry, they are also important in pure mathematics and are critical for resolving problems in many practical mathematics and information science applications. Hemiring is a special class of semirings with commutative addition and a zero element. Additionally, there are several algebras related to logic. MV algebras are suited for multi-valued logic, while Boolean algebras are connected to traditional two-valued Aristotelean logic.

Just as the basic operations such as addition, subtraction, multiplication, and division in the set of integers and intersection, union, difference, complement, and symmetric difference in the set of sets are fundamental for the related theories, operations on SSs are equally vital in SS theory. SS operations serve as the theoretical basis for several soft computing and decision-making approaches. Furthermore, a thorough understanding of the algebraic structure of SSs may be obtained by looking at the algebraic structures formed by SSs and operations. This improves the comprehension of applications and makes it possible to see how SS algebra can be used in both classical and non-classical logic, which paves the way for a number of uses, such as the development of new SS-based cryptography techniques and decision-making processes. We refer to the study by Alcantud et al. (2024), where a comprehensive survey of SS theory, encompassing its foundational concepts, developments, and applications are presented. As regards the studies on soft algebraic structures for all of which SS operations have been the basis, we refer to Aktaş and Çağman, 2007; Jun, 2008; Jun and Park, 2008; Park et al., 2008; Feng et al., 2008; Sun et al., 2008; Acar et al., 2010; Zhan and Jun, 2010; Sezer et al., 2013, Sezer et al., 2014; Atagün and Sezgin, 2015; Sezer et al, 2015; Muştuoğlu et al., 2016; Mahmood et al., 2015; Sezer and Atagün, 2016; Tunçay and Sezgin, 2016; Sezer et al., 2017; Khan et al., 2017; Atagün and Sezgin, 2017; Sezgin et al., 2017; Atagün and Sezer, 2018; Ullah et al., 2018; Iftikhar and Mahmood; 2018; Gulistan et al., 2018; Sezgin, 2018; Atagün et al., 2019; Jana et al., 2019; Karaaslan, 2019; Özlü and Sezgin, 2020; Karaaslan et al., 2021; Sezgin et al., 2022, Atagün and Sezgin, 2022; Sezgin and Orbay, 2022, Riaz et al., 2023; Manikantan et al., 2023; Sezgin and İlgin, 2024; Sezgin and Onur, 2024; Sezgin et al., 2024).

In the theory of SS, many soft intersection operations have been defined up to now. There are some differences among them, and some definitions are no longer preferred for use as they are essentially not very useful. The intersection of SSs was first defined by Maji et al. (2003), however, it is problematic as it is obvious from the nature of the definition of SS that the condition put in the definition is not necessarily the case. This problematic nature of the definition was detailed by Ali et al. (2009) and Pei and Miao (2005). Pei and Miao (2005) defined a new intersection operation for SS, which they believed would be more functional, however in this definition it was not addressed what the result of the operation would be in the case where the parameter sets of the SSs are disjoint. Feng et al. (2008) defined an alternative intersection operation for SS, called the “bi-intersection” of SSs. This definition is problematic as well, as it is not addressed what the result of the operation would be in the case where the parameter sets of the SSs are disjoint. Ali et al. (2009) defined a new intersection operation for SSs called the “restricted intersection operation”. Unlike the definition by Feng et al. (2008), this definition starts with the condition that the parameter sets of the SSs whose intersection is calculated should be disjoint. Moreover, it did not address what the result of the operation would be in the case where the parameter sets of the SSs are disjoint as well. Ali et al. (2011) evaluated the case of the intersection of the parameter sets of two SSs being empty, which was not considered in the restricted soft intersection operation defined by Ali et al. (2009), and updated the definition by adding the note that if the parameter sets of the SSs whose restricted intersection is calculated are disjoint, then the result of the operation is the empty SS. This is the first study to provide information on the result of the restricted intersection operation when the intersection of the parameter sets is empty set. Although the most current and useful definition for the restricted intersection operation is the one provided by Ali et al. (2011), in this definition, the condition that the parameter sets of the SSs should not be disjoint to calculate their restricted intersection was included as a necessary condition; however this is not the case, because whether the intersection of the parameter sets of the two SSs is an empty set or not, the restricted intersection of these two SSs can be calculated in any case. The intersection of the parameter sets of these two SSs being non-empty is never a necessary condition for their restricted intersection to be calculated. In this sense, from a chronological perspective, although the idea of the restricted intersection operation in SSs was first proposed by Pei and Miao (2005), as in their definition, the case where the intersection of the parameter sets of the SSs is empty was not considered, and this was addressed for the first time by Ali et al. (2011), the study by Ali et al. (2011) is of great importance. Besides, inspired by the union definition of SSs by Maji et al. (2013), a similar type operation defined as the "extended intersection operation" of SSs was proposed by Ali et al. (2009), and its properties and distributive rules were studied by various authors (Ali et al. 2009; Ali et al. 2011; Qin and Hong 2010); Sezgin and Atagün, 2011; Singh and Onyeozili, 2012c).

As restricted and extended intersection operations are anyway existing concepts in the literature, the properties of them were already studied in many studies, thus it may seem that some properties included in this paper were already presented in the previous studies (Ali et al., 2009; Ali et al., 2011; Feng et al. 2008; Maji et al., 2003; Pei and Miao, 2005; Qin and Hong, 2010; Sezgin and Atagün, 2011). However, we find it beneficial to note that the properties of the operations together with their distribution rules were not handled in the mentioned papers by considering the important point that the parameter sets of the SSs may be disjoint. This is due to the incomplete definition of restricted intersection operation and thus, ignoring some of the cases in the theorems and proofs. From this perspective, there is a significant gap in the literature for providing a comprehensive study of restricted and extended intersection operations by taking into account these ignored cases. Moreover, as SS operations serve as both the theoretical and practical basis for the theory, and in many studies as regards the soft algebraic structures, these basic two SS operations are always used while exploring the properties of the soft structures, this critical gap needs to be filled immediately beginning with a precise exposition of the definition of restricted intersection. This study aims to encompass all the prior studies regarding these operations. Moreover, in the above-mentioned papers together with (Neog and Sut, 2011; Fu, 2011; Ge and Yang S, 2011; Zhu and Wen, 2013; Onyeozili and Gwary, 2014; Husain and Shivani, 2018) several theorems and propositions were presented without their proofs, or there were some incorrect or missing parts in the proofs due to the incomplete definition of restricted intersection; however, in this study, the proofs based on the function equality are regularly provided, and thus all the incorrect parts are corrected. Additionally, as the definition of subset by Pei and Miao (2005) is more functional and rational, and thus has a wide-spread usage than that of Maji et al. (2003), and since in the existing studies, (especially in the study of Sezgin and Atagün, 2011), the relationships between restricted and extended intersection operations and soft subset were handled with regard to the definition of subset proposed by Maji et al. (2003), in the literature there is a wide gap needs to be filled in this regard as well. In this study, the relationships in this regard which were not addressed in previous studies, are presented for the first time with detailed proofs and with their classical set counterparts as well. We do not only correct the problematics parts in the existing papers, but also we obtain many new properties of restricted and extended intersection operations together with their relationships with the SS operations defined by Aybek (2024) and Yavuz (2024). By looking at the distribution rules, the algebraic structures formed by these operations in the set of SSs with a fixed parameter set and in the set of sets over the universe are examined and presented thoroughly with their detailed proofs. Furthermore, when a distribution rule does not hold, unlike the studies by Ali et al. (2011) and Qin and Hong (2010), where the distribution rules are investigated to obtain the algebraic structures of SSs associated with the operations, we also explore and put forward the condition(s) under which the assertions hold. In this sense, we obtain many new algebraic structures related to SSs and restricted and extended intersection operations. Thus, this study presents a detailed and complete examination of all the properties of restricted intersection and extended intersection operation, which are the basic SS operations. As the intersection operation exists in classical set theory, all of the properties of the operations together with their counterparts in classical sets have been thoroughly investigated without omission. Additionally, the properties that were previously handled with incorrect/lengthy proofs or without proof in earlier studies by Ali et al. (2009), Ali et al. (2011), Sezgin and Atagün, 2011, Singh and Onyeozili (2012c) are handled again by presenting them in their correct forms. In order to find out whether the collection of SSs and restricted and extended intersection operations form lattice structures in the collection of SSs over the universe and in the collection of SS with a fixed parameter set, the so-called absorption laws are examined with detailed their proofs. Although the absorption laws were presented in previous works by Ali et al. (2011), Qin and Hong (2010), Singh and Onyeozili (2012c) presented the results only with a table without proofs, and since the proofs in other studies are element-based and relatively long proofs, they are presented in this study with their more rational proofs. Furthermore, in this study, the absorption laws for the SSs with a fixed parameter set are given in detail for the newly-defined operations by Yavuz (2024) and Aybek (2024) as well. Additionally, the distributive rules are presented collectively in a table. Finally, we systematically, in detail, and collectively present the unary and binary algebraic structures, and lattice structures formed by the restricted intersection and extended intersection together with other SS operations both in the collection of SS over the universe and in the collection of SSs with a fixed parameter set together with their detailed explanations and with the corrected ones.

The stream of the paper is as follows: In Section 2, we review the fundamental concepts regarding SSs and certain algebraic structures which are obtained to be associated with the SSs throughout the paper. In Section 3, first of all, we give the original definitions of intersection operations of SSs proposed up now together with the historical improvements of these operations in chronological order to indicate what deficiencies these definitions have and to contribute to the comprehensibility of the study. Then, the revised and updated definition of restricted intersection and all the properties of the restricted and extended intersection operations are presented and examined in detail and demonstrated with their complete proofs. When investigating the properties and distributive rules, the case where the intersection of the parameter sets of the SSs is empty is always considered in the assertions and the proofs. As intersection operation also exists in classical sets, special attention is given to show how the properties of intersection operation in classical sets reflect these operations to obtain their counterparts and analogies in SS theory. Thus, numerous new properties have also been added to those previously presented in this field. Besides, previously presented properties that were either unproven or had lengthy or erroneous proofs are presented with simplified proofs. Incorrect parts in previous studies, as regards these operations, are corrected with detailed explanations. Additionally, in order to see which algebraic structures these basic SS operations form, the distributions of restricted and extended intersection operations over other types of SS operations are examined in Section 3, and the absorption laws are investigated in detail in Section 4, and it is observed that these SS operations individually and together with other types of SSs form a wide variety of algebraic structures in the set of SSs over the universe and in the set of SSs with a fixed parameter set, such as monoid, bounded semi-lattice, semiring, hemiring, bounded distributive lattice, Bool algebra, De Morgan Algebra, Kleene Algebra, Stone algebra and MV-algebra, which are given collectively and with their detailed explanations in Section 4. We also, by providing a methodical study, correct some algebraic structures associated with the restricted and extended intersection operations obtained by Ali et al. (2009) presenting the corrected new ones. In the conclusion section, we highlight the significance of the study's results and their possible impact on both the SS theory, classical algebra, and real-world scenario. Taking all of these into account, this paper is the most comprehensive study in the existing literature of SSs as regards the restricted and extended intersection operations which encompass all the previous studies on this subject (Maji et al., 2003; Pei and Miao, 2005; Ali et al., 2009; Qin and Hong, 2010; Sezgin and Atagün, 2011; Ali et al., 2011; Singh and Onyeozili, 2012c, Sen, 2014) and (Neog and Sut, 2011; Fu, 2011; Ge and Yang, 2011; Zhu and Wen, 2013; Onyeozili and Gwary, 2014; Husain and Shivani, 2018) serving as a handbook for those who start to study SS theory and advancing the theory by closing the big gap in the literature in this regard, as such an inclusive study does currently not exist in the literature and it is quite necessary in terms of shedding light on the future studies and preventing possible errors in the theory.

2. PRELIMINARIES

In this section, several algebraic structures and several fundamental concepts in SS theory are provided. Soft set first proposed by Molodtsov (1999); however its definition was revised by Maji et al. (2003).

Definition 1. Let E be the parameter set, Ɲ⊆ E, U be the universal set, P(U) be the power set of U, A pair (Ꞙ,Ɲ) is called an SS over U, where Ꞙ is a function given by Ꞙ : Ɲ → P(U) (Maji et al., 2003).

While the SS (Ꞙ,Ɲ) is denoted as $F_{Ɲ}$ in some papers, we prefer the commonly-held representation "(Ꞙ,Ɲ)" in this study. Besides, the definition of SS, proposed by Maji et al. (1999), has been reorganized by Çağman and Enginoğlu (2010) however, we use the definition of Maji et al. (2003) to be faithful to the original definition of SS throughout this paper.

More than one SS can be defined with a subset Ɲ of the set of parameters E. In this case, these SSs are denoted as (Ꞙ,Ɲ) (ℭ, Ɲ), (Ƕ, Ɲ), etc. Also, more than one SS can be defined with different subsets Ɲ,Y, Ᵽ etc. of the set of parameters E. In this case, the SSs are denoted as (Ꞙ,Ɲ), (Ꞙ,Y), (Ꞙ,Ᵽ), etc. (Maji et al, 2003). The collection of all SSs over U is denoted by S_E(U), and $S_{Ɲ}$(U) indicates the collection of all SSs over U with a fixed parameter set Ɲ, where Ɲ is a subset of E.

The definitions of "NULL SS" and "absolute SS" were first introduced by Maji et al. (2003), where a NULL SS (Ꞙ,A) was represented by Φ, and an absolute SS (Ꞙ,A) by $\tilde{A}$. However, it was extensively shown by Ali et al. (2009), Sezgin and Atagün (2011), and Yang (2008) that these definitions and notations, unfortunately, pose certain mathematical problems. Specifically, these definitions create many problematic situations in theorems and propositions, as the parameter set of the SS need not be a fixed set changing from SS to SS.

To address these problematic situations, Ali et al. (2009) updated these definitions, introducing the definitions of "relative null SS with respect to parameter set Ɲ" and "relative whole SS with respect to parameter set Ɲ" which are all determined by the parameter set of the SS. Consequently, in several significant studies (Ali et al. (2009), Ali et al. (2011) Sezgin and Atagün (2011)), the mathematically correct versions of all problematic theorems and propositions related to NULL SS and absolute SS operations were provided. Throughout this paper, in order to avoid confusion, we use the definitions of Ali et al. (2011) for absolute SS, null SS, and whole SS.

A function whose domain is the empty set is known as the empty function. Since the empty function is also a function, it is evident that by taking the domain as ∅, an SS can be defined as F: ∅ ⟶ P(U). This type of SS is referred to as an empty SS and is represented by $\emptyset_{\emptyset}$. As stated by Ali et al. (2011), the only SS with an empty parameter set is $\emptyset_{\emptyset}$. In this study, unless otherwise stated, all the SSs over U are different from $\emptyset_{\emptyset}$.

Definition 2. Let (Ꞙ,Ɲ)∈ $S_E(U)$. If Ꞙ(ʂ)=∅ for all ʂ∈ Ɲ, then (Ꞙ,Ɲ) is referred to as a null SS with respect to Ɲ, signified by $\emptyset_{Ɲ}$ (Ali et al, 2009). An SS with an empty parameter set is denoted as $\emptyset_{\emptyset}$ and is called an empty SS (Ali et al., 2011).

Definition 3. Let (Ꞙ,Ɲ)∈ $S_E(U)$. If Ꞙ(ʂ)=U for all ʂ∈ Ɲ, then (Ꞙ,Ɲ) is referred to as a relative whole SS with respect to Ɲ, signified by $U_{Ɲ}$. The relative whole SS U_E with respect to the universe set of parameters E is called the absolute SS over U (Ali et al., 2009).

Soft subsets and soft equal relations, in the framework of SS theory, are core concepts as well. Maji et al. (2003) were the pioneers in using a very strict definition of soft subsets. The definition is as follows:

Definition 4. Let (Ꞙ,Ɲ), (ℭ,Ɏ)∈ $S_E(U)$. If

i. Ɲ⊆ Ɏ and

ii. For all w ∈ Ɲ, Ꞙ(w)= ℭ(w)

then (Ꞙ,Ɲ) is called a soft subset of (ℭ, Ɏ), denoted as (Ꞙ,Ɲ) $\widetilde{\subseteq}$ (ℭ, Ɏ).

If (ℭ, Ɏ) is a soft subset of (Ꞙ,Ɲ), then (Ꞙ,W) is a soft superset of (ℭ, Ɏ), denoted as (Ꞙ,Ɲ) $\widetilde{\supseteq}$ (ℭ, Ɏ). If (Ꞙ,Ɲ) $\widetilde{\subseteq}$ (ℭ, Ɏ) and (ℭ, Ɏ) $\widetilde{\subseteq}$ (Ꞙ,Ɲ), then (Ꞙ,Ɲ) and (ℭ, Ɏ) are said to be (soft) equal sets (Maji et al., 2003).

In the study by Sezgin and Atagün (2011), the properties of restricted intersection and restricted union, and extended intersection and extended union operations were examined according to the soft subset definition by Maji et al. (2003). However, since the soft subset definition in the study by Pei and Miao (2005) is more related to classical sets, and thus is more useful and functional, the soft subset definition by Pei and Miao (2005) is used throughout this study.

Definition 5. Let (Ꞙ,Ɲ), (ℭ, Ɏ)∈ $S_E(U)$. If Ɲ ⊆ Ɏ and Ꞙ(ʂ) ⊆ ℭ(ʂ), for all ʂ∈ Ɲ, then (Ꞙ,Ɲ) is a soft subset of (ℭ,Ɏ), indicated by (Ꞙ,Ɲ)$\widetilde{\subseteq}$(ℭ,Ɏ). If (ℭ,Ɏ) is a soft subset of (Ꞙ,Ɲ), then (Ꞙ,Ɲ) is a soft superset of (ℭ,Ɏ), indicated by (Ꞙ,Ɲ)$\widetilde{\supseteq}$(ℭ, Ɏ). If (Ꞙ,Ɲ)$\widetilde{\subseteq}$(ℭ,Ɏ) and (ℭ,Ɏ)$\widetilde{\subseteq}$(Ꞙ,W), then (Ꞙ,Ɲ) and (ℭ,Ɏ) are called soft equal sets (Pei and Mio, 2005).

In the literature, various and updated definitions of soft subset and soft equal set have been introduced. For these definitions and the relationships between them, we refer to the studies by Qin and Hong, 2010); Jun and Yang, 2011; Liu et al. 2012; Feng and Li, 2013; Abbas et al., 2014; Mujahid et al., 2017; Abbas et al., 2017; Al-Shami, 2019; Al-Shasi and El-Shafei, 2020; and Ali et al., 2022.

The concept of the complement of an SS was first introduced by Maji et al. (2003). In this definition, when the complement of an SS (Ꞙ,Ɲ) is calculated, the complement of Ɲ is also conducted, thus the parameter set of the SS changes. It was shown by Ali et al. (2009) that this causes problematic situations in important aspects such as De Morgan's laws. To overcome this confusion, Ali et al. (2009) introduced the concept of "relative complement" of an SS which is more rational. In this definition, when the complement of an SS is conducted, the parameter set remains unchanged, that is, it is preserved. This definition became preferred, as it is more functional than the complement defined by Maji et al. (2003). To avoid confusion, in the study by Ali et al. (2009), the complement defined by Maji et al. (2003) is called the "neg-complement", and the updated complement concept for SSs is called the "relative complement" (briefly soft complement). Below, the concept of (relative) complement introduced by Ali et al. (2011) is presented and it used throughout this study is provided.

Definition 6. Let (Ꞙ,Ɲ)∈ $S_E(U)$. The relative complement of (Ꞙ,Ɲ), indicated by $(Ꞙ,Ɲ)^r$ =($Ꞙ^r$,Ɲ), is defined as follows: $Ꞙ^r$(ʂ)=U-Ꞙ(ʂ), for all ʂ∈ Ɲ (Ali et al, 2011).

From here, $(\emptyset_{A})^{r}$=U_A, $(U_{A})^{r}$=$\emptyset_{A}$, $(\emptyset_{E})^{r}$= U_E, $(U_{E})^{r}$=$\emptyset_{E}$, $(\emptyset_{\emptyset})^{r}$=$\emptyset_{\emptyset}$ (Since the parameter set remains unchanged and $\emptyset_{\emptyset}$ is the only SS that has an empty parameter set), and $((Ғ, Ɲ)^{r})^{r}$=(Ғ, Ɲ).

Moreover, it is clear that $\emptyset_{A}$ $\tilde{\subseteq}$(Ғ,A) $\tilde{\subseteq}$ U_A $\tilde{\subseteq}$ U_E (Ali et al., 2011). Here we want to point out one issue: In Ali et al. (2011) it is stated that $\emptyset_{\emptyset}$ $\tilde{\subseteq}$ $\emptyset_{A}$ $\tilde{\subseteq}$(Ғ,A) $\tilde{\subseteq}$ U_A $\tilde{\subseteq}$ U_E; however it is unknown whether $\emptyset_{\emptyset}$ is a subset of $\emptyset_{A}$ or not due to the definition of empty SS.

Çağman (2021) introduced two new complements as the inclusive complement (Here, we denote by +) and the exclusive complement (Here, we denote by θ). For two sets Ɲ and ℑ, these binary operations are defined as Ɲ + ℑ = Ɲ'∪ℑ and Ɲ θℑ = Ɲ'∩ℑ'. Sezgin et al. (2023a) investigated the relationship between these two operations and also introduced new binary operations: For the sets Ɲ and ℑ, Ɲ *ℑ = Ɲ'∪ℑ', Ɲ γℑ = Ɲ'∩ℑ, Ɲ λℑ = Ɲ ∪ℑ'. Let the set operations be denoted by "⊛" (that is, ⊛ can be ∩, ∪,\, Δ, +,θ, *, λ,γ), then the following definitions are applied to all forms of SS operations:

Definition 7. Let (Ғ, Ɲ), (ℭ,Ɏ)∈ $S_E(U)$ such that Ɲ∩ Ɏ≠ ∅. The restricted ⊛ operation of (Ғ, Ɲ) and (ℭ,Ɏ) is the SS (Ԋ,Ꝑ), denoted by (Ғ,Ɲ) $⊛_R$ (ℭ, Ɏ)= (Ԋ,Ꝑ), where Ꝑ = Ɲ ∩Ɏ and for all ɿ∈Ꝑ, Ԋ(ɿ) = Ғ(ɿ)⊛ ℭ (ɿ). Here, if Ꝑ= Ɲ ∩ Ɏ = ∅, then (Ғ,Ɲ) $⊛_R$(ℭ, Ɏ)= $\emptyset_{\emptyset}$ (Ali et al., 2011; Pei and Mia, 2005; Sezgin and Atagün, 2011).

Definition 8. Let (Ғ,Ɲ), (ℭ,Ɏ)∈ $S_E(U)$. The extended ⊛ operation (Ғ,Ɲ) and (ℭ,Ɏ) is the SS (Ԋ,Ꝑ), denoted by (Ғ, Ɲ) $⊛_ε$(ℭ,Ɏ) = (Ԋ, Ꝑ), where Ꝑ= Ɲ∪Ɏ, and for all ɿ∈Ꝑ,

$$Ԋ(ɿ) = \begin{cases} Ғ(ɿ), & ɿ \in Ɲ - Ɏ \\ ℭ(ɿ), & ɿ \in Ɏ - Ɲ \\ Ғ(ɿ) ⊛ ℭ(ɿ), & ɿ \in Ɲ \cap Ɏ \end{cases}$$

(Maji et al., 2003; Ali et al., 2009; Ali et al., 2011; Sezgin et al, 2019; Stojanavic, 2021; Aybek, 2024).

Definition 9. Let (Ғ,Ɲ), (ℭ, Ɏ)∈ $S_E(U)$. The complementary extended ⊛ operation (Ғ,Ɲ) and (ℭ,Y) is the SS (Ԋ,Ꝑ), denoted by (Ғ,Ɲ) $\overset{*}{⊛_ε}$(ℭ, Ɏ) = (Ԋ, Ꝑ), where Ꝑ = Ɲ∪ Ɏ, and for all ɿ∈Ꝑ,

$$Ԋ(ɿ) = \begin{cases} Ғ'(ɿ), & ɿ \in Ɲ - Ɏ \\ ℭ'(ɿ), & ɿ \in Ɏ - Ɲ \\ Ғ(ɿ) ⊛ ℭ(ɿ), & ɿ \in Ɲ \cap Ɏ \end{cases}$$

(Akbulut, 2024; Demirci, 2024; Sarıalioğlu, 2024; Sezgin and Sarıalioğlu, 2024b)

Definition 10. Let (Ғ,Ɲ), (ℭ, Ɏ)∈ $S_E(U)$. The soft binary piecewise ⊛ operation of (Ғ,Ɲ) and (ℭ, Ɏ) is the SS (Ԋ, Ɲ), denoted by (Ғ,Ɲ)$\underset{⊛}{\sim}$(ℭ,Ɏ) = (Ԋ,Ɲ), where for all ɿ∈ Ɲ,

$$Ԋ(ɿ) = \begin{cases} Ғ(ɿ), & ɿ \in Ɲ - Ɏ \\ Ғ(ɿ) ⊛ ℭ(ɿ), & ɿ \in Ɲ \cap Ɏ \end{cases}$$

(Eren and Çalışıcı, 2019; Sezgin and Yavuz, 2023a; Sezgin and Çalışıcı, 2024; Yavuz, 2024)

Definition 11. Let (Ғ,Ɲ), (ℭ, Ɏ)∈ $S_E(U)$. The complementary soft binary piecewise ⊛ operation of (Ғ,Ɲ) and (ℭ, Ɏ) is the SS (Ԋ, Ɲ), denoted by (Ғ,Ɲ)$\underset{⊛}{\overset{*}{\sim}}$(ℭ,Ɏ) = (Ԋ,Ɲ), where for all ɿ∈ Ɲ,

$$Ԋ(ɿ) = \begin{cases} Ғ'(ɿ), & ɿ \in Ɲ - Ɏ \\ Ғ(ɿ) ⊛ ℭ(ɿ), & ɿ \in Ɲ \cap Ɏ \end{cases}$$

(Sezgin and Demirci, 2023; Sezgin et al. 2023a, 2023b; Sezgin and Yavuz, 2023b; Sezgin and Dagtoros, 2023; Sezgin and Çağman, 2024; Sezgin and Sarıalioğlu, 2024a)

Definition 12. An algebraic structure (Ś,⋆) is said to be idempotent if $s^2=s$ for all s∈ Ś, then. An idempotent semigroup is said to be a band, a commutative band is called a semi-lattice, and a semi-lattice with an identity is called a bounded semi-lattice (Clifford, 1954).

Definition 13. Let Ś be a non-empty set and "+" and "⋆" be two binary operations defined on Ś. If the algebraic structure (Ś, +, ⋆) satisfies the following properties, then it is called a semiring:

i. (Ś, +) is a semigroup.
ii. (Ś, ⋆) is a semigroup,
iii. For all ƀ, đ, z ∈ Ś, ƀ⋆(đ + z) = ƀ⋆ đ + ƀ⋆z and (ƀ + đ) ⋆z = ƀ⋆z + đ⋆z

If ƀ + đ = đ + ƀ for all ƀ,đ∈ Ś, then Ś is called an additive commutative semiring. If ƀ⋆ đ =đ⋆ ƀ for all ƀ,đ∈ Ś, then Ś is called a multiplicative commutative semiring. If there exists an element 1∈ Ś such that ƀ⋆1=1⋆ ƀ = ƀ for all ƀ∈ Ś (multiplicative identity), then Ś is called semiring with unity. If there exists 0∈ Ś such that for all ƀ ∈ Ś, 0⋆ ƀ =ƀ⋆0=0 and 0+ ƀ = ƀ +0= ƀ, then 0 is called the zero of Ś. A semiring with commutative addition and a zero element, is called a hemiring (Vandiver, 1934).

Definition 14. Let ζ be a non-empty set, and let "∨" and "∧" be two binary operations defined on ζ. If the algebraic structure. (ζ,∨,∧) satisfies the following properties, then it is called a lattice:

i. (ζ, ∨) is a semi-lattice
ii. (ζ, ∧) is a semi-lattice
iii. For all ꝗ,ȣ ∈ ζ , ꝗ ∨(ꝗ ∧ ȣ) =ꝗ ∧(ꝗ ∨ ȣ) (absorption law)

A lattice with an identity element according to both operations is called a bounded lattice. In a bounded lattice, the identity element of ζ with respect to the ∧ operation is usually denoted by 1, while the identity element with respect to the ∨ operation is denoted by 0. If the bounded lattice ζ has an element ꝗ′ such that ꝗ ∧ ꝗ′ = 0 and ꝗ∨ ꝗ′=1 for all ꝗ ∈ ζ, then ζ is called a complemented lattice. A lattice holding distribution law is called a distributive lattice. A lattice that is bounded, distributive, and at the same time complemented is called Boolean algebra. The lattice with De Morgan's law, i.e. (ꝗ ∨ ȣ)′ = ꝗ ′∧ ȣ ′ and (ꝗ ∧ ȣ)′ = ꝗ ′ ∨ ȣ ′ for all ꝗ, ȣ ∈ ζ is called De Morgan algebra. If De Morgan algebra satisfies the condition ꝗ ∧ ꝗ ′≤ ȣ∨ ȣ′ for all ꝗ, ȣ ∈ ζ, then it is called a Kleene algebra. For ꝗ ∈ ζ, $ꝗ^*$ is called the pseudo-complete of ꝗ if ꝗ ∧ $ꝗ^*$= 0 and ȣ ≤ $ꝗ^*$ whenever ꝗ ∧ȣ =0. The equality $ꝗ^*$∨ $ꝗ^{**}$=1 is called the Stone's identity. A pseudo-complemented distributive lattice satisfying the Stone's identity is called a Stone algebra (Ali et al., 2011)

Definition 15. Let ☡ be a non-empty set with binary operation "⊕" and a unary operation "*" defined on ☡. If 0 is a constant that fulfills the following axioms for each ȥ and y in ☡, then the structure (☡,⊕,∗ ,0) is called an MV-algebra:

i. (☡, ⊕, 0) commutative monoid.
ii. $(ȥ^*)^*$= ȥ
iii. 0^*⊕ ȥ =0^*
iv. $(ȥ^* ⊕ y)^* ⊕ y=(y^* ⊕ ȥ)^* ⊕ ȥ$

The concept of MV-algebras was introduced by Chang (1959) with the aim of providing an algebraic proof for Lukasiewicz logic, a many-valued logic introduced by Lukasiewicz in the 1920s. We refer to Pant et al. (2024) regarding the possible applications of network analysis and graph applications on SSs, and to Ali et al. (2015), Jan et al. (2020), Irfan Siddique et al. (2021), and Mahmood (2020) for bipolar soft sets, double framed soft sets and lattice ordered soft sets. For more about soft AG-groupoids, soft KU-algebras, and picture soft sets, see Khan et al. (2015), Gulistan and Shahzad (2014), Memiş (2022), and Naeem and Memiş (2023), respectively.

3. MORE ON RESTRICTED AND EXTENDED INTERSECTION OPERATIONS

In the theory of SS, which was proposed in 1999, many soft intersection operations have been defined. There are some differences among them, and some definitions are no longer preferred for use because they are essentially not very useful. We find it beneficial to start this section by recalling these points. From this perspective, we first aim to present the soft intersection operations existing in the literature in chronological order, and to indicate what deficiencies these definitions have, as we believe this will contribute to the comprehensibility of the study.

In 2003, the intersection of SSs was first defined by Maji et al. (2003) as follows: Let (Ғ,A) and (ℭ,B) be two SSs over U. The intersection operation of these SSs, denoted by (Ғ,A)$\tilde{\cap}$(ℭ,B), is defined as (Ғ,A)$\tilde{\cap}$(ℭ,B)=(Ꮋ, C), where C=A∩B and Ꮋ(α)=Ғ(α) or Ꮋ(α)=ℭ(α), for all α ∈C (as both are the same sets) (Maji et al, 2003). Although it was claimed in this definition that Ғ(α) = ℭ(α), for α ∈ C, it is clear from the definition and nature of the SS that such a situation is not necessarily the case. Therefore, the problematic nature of this definition has been detailed in the studies by Ali et al. (2009) and Pei and Miao (2005).

Pei and Miao (2005) defined a new soft intersection operation, which they believed would be more functional, as follows: Let (Ғ, A) and (ℭ, B) be two SSs over U. The restricted intersection of these SSs, denoted by (Ғ,A)∩(ℭ,B), is defined as (Ғ,A)∩(ℭ,B)=(Ꮋ,C), where C=A∩B and for all α ∈C, Ꮋ(α) = Ғ(α)∩ℭ(α) (Pei and Mia, 2005). In this definition, however, it is not addressed what the result of the operation (Ғ,A)∩(ℭ,B) would be in the case where the parameter sets of the SSs are disjoint, and the notation for the restricted soft intersection operation is chosen to be similar to the intersection operation in classical sets.

Feng et al. (2008) defined an intersection operation, called the bi-intersection operation, as follows: Let (Ғ, A) and (ℭ, B) be two SSs over U. The bi-intersection of these SSs, where C=A∩B and for all ϛ ∈ C, H: C → P(U) is defined as Ꮋ(ϛ)=Ғ(ϛ)∩ℭ(ϛ), and is denoted by (Ғ, A) $\tilde{\sqcap}$ (ℭ, B) = (Ꮋ, C). In this definition, as well, it is not addressed what the result of the operation (Ғ, A) $\tilde{\sqcap}$ (ℭ, B) would be in the case where A∩B=∅.

Ali et al. (2009) defined an SS operation called the restricted intersection operation as follows: Let (Ғ, A) and (ℭ, B) be two SSs over U such that A∩B≠ ∅. The restricted intersection operation of (Ғ, A) and (ℭ, B) denoted by (Ғ,A)⋒(ℭ,B), is defined as (Ғ,A)⋒(ℭ,B)=(Ꮋ,C), where C=A∩B, and for all α ∈C, Ꮋ(α)= Ғ(α)∩ℭ(α). Unlike the definition by Feng et al. (2008), this definition starts with the condition "Let (Ғ,A) and (ℭ,B) be two SSs such that A∩B ≠ ∅," treating this condition as a necessary condition. Moreover, it does not address what the result of the operation (Ғ,A)⋒(ℭ,B) would be in the case where A∩B = ∅.

Ali et al. (2011) evaluated the case of the intersection of the parameter sets of two SSs being empty, which was not considered in the restricted soft intersection operation defined by Ali et al. (2009), and updated the definition of the restricted intersection operation as follows: "Let (Ғ, A) and (ℭ, B) be two SSs over U such that A∩B=∅. The restricted intersection operation (Ғ,A) and (ℭ,B) is denoted by (Ғ,A)$\cap_R$(ℭ,B), and is defined as (Ғ,A) $\cap_R$(ℭ,B)=(Ꮋ,C), where C = A∩B, and for all α ∈ C, Ꮋ(α)= Ғ(α)∩ℭ(α). If A∩B=∅, then (Ғ,A) $\cap_R$(ℭ,B)= $\emptyset_\emptyset$. "This was the first study to provide information on the result of the restricted intersection operation when the intersection of the parameter sets is empty. Additionally, this study preferred to use the symbol “$\cap_R$” which is the most useful notation for the restricted intersection operation. The letter "R" under the intersection symbol is in harmony with and meaningful because it stands for "restricted" in English. Indeed, in subsequent studies on SS operations, this notation form was preferred for restricted SS operations. The most current and useful definition of the restricted intersection definition is the one provided by Ali et al. (2011). However, in this definition, for two SSs (Ғ,A) and (ℭ, B) over the same universe, the condition $A\cap B\neq \emptyset$ is again included as a necessary condition for the restricted intersection of these two SSs to be calculated by adding the expression “Let (Ғ,A) and (ℭ, B) be two SSs over U such that $A\cap B\neq \emptyset$.” However, even if $A\cap B= \emptyset$, the restricted intersection operation of these SSs is still defined, and in this case, (Ғ,A) $\cap_R$(ℭ,B)= $\emptyset_\emptyset$. That is, whether the intersection of the parameter sets of the two SSs to be intersected is an empty set or not, the restricted intersection of the two SSs can be calculated in any case. The parameter sets of the two SSs being not disjoint is never a necessary condition for their restricted intersection to be calculated.

In this sense, from a chronological perspective, although the idea of the restricted intersection operation in SSs was first proposed by Pei and Miao (2005), as in their definition, the case where the intersection of the parameter sets of the SSs to be intersected is empty was not considered and it is addressed firstly in the study of Ali et al. (2011), the updated definition for the restricted intersection provided in this section as Definition 3.1.1 will be given and used.

In this section, the algebraic properties of the restricted intersection and extended intersection operations of SSs, updated considering the above-mentioned points, are examined in comparison with the properties of the intersection operation in classical sets. We investigate the distribution rules so as to obtain the algebraic structures formed by these operations in the collection of SSs with a fixed parameter set, and in the collection of softs over the universe in the following section.

Here, we find it beneficial to note the following: Although the restricted intersection operation exists in the literature as the restricted and extended intersection operations are pre-defined operations, many properties included in this section have been given in previous works (Ali et al. 2009; Ali et al., 2011; Feng et al. 2008; Pei and Mia, 2005; Maji et al, 2003; Qin and Hong, 2010; Sezgin and Atagün, 2011), but without considering the important points detailed in this section. Moreover, in most studies, many of these properties are provided without proofs, or with insufficiently detailed proofs. In this study, all proofs are systematically provided based on function equality. Additionally, the relationships between Pei and Miao's (2005) definition of soft subsets and restricted/extended intersection operations, which were not addressed in previous studies, are presented for the first time in this study with detailed proofs and their classical set counterparts.

Moreover, since all properties of the fundamental SS operations, namely the restricted and extended intersection operations, are provided together with their proofs, and since this study takes care the case where the intersection of the parameter sets of the SSs may be empty set for each property-previously overlooked in other studies-this work is comprehensive of all previous works. From this perspective, we consider this study to be of significant importance, and hope it serves as a handbook for beginners in SS theory.

3.1. More on Restricted Intersection Operation

In this subsection, the updated and revised presentation of the definition of restricted intersection defined by Ali et al. (2011), its example, and all its properties are provided with their detailed proofs.

Definition 16 Let $(\text{Ꞙ}, \text{Ʝ})$ and $(\mathfrak{C}, \text{Ᵽ})$ be SSs over U. The restricted intersection of $(\text{Ꞙ}, \text{Ʝ})$ and $(\mathfrak{C}, \text{Ᵽ})$, denoted by $(\text{Ꞙ}, \text{Ʝ}) \cap_R (\mathfrak{C}, \text{Ᵽ})$, is defined as $(\text{Ꞙ}, \text{Ʝ}) \cap_R (\mathfrak{C}, \text{Ᵽ}) = (\text{Ƕ}, C)$, where C= Ʝ ∩Ᵽ. Here, if C= Ʝ ∩Ᵽ≠∅, then Ƕ(α)= Ꞙ(α) ∩$\mathfrak{C}$(α) for all α∈C, and if C= Ʝ ∩Ᵽ=∅, then (Ꞙ,Ʝ) $\cap_R$ ($\mathfrak{C}$,Ᵽ)=(Ƕ, C)= $\emptyset_\emptyset$.

Since the only SS with an empty parameter set is $\emptyset_\emptyset$, it is clear from the definition that if C=Ʝ∩Ᵽ=∅, then (Ꞙ,Ʝ) $\cap_R$ ($\mathfrak{C}$,Ᵽ)= $\emptyset_\emptyset$. Therefore, it can be seen that there is no requirement for Ʝ∩Ᵽ≠ ∅ for the restricted intersection operation to be defined, where (Ꞙ,Ʝ) and ($\mathfrak{C}$, Ᵽ) are two SSs over U.

The reason we have not cited Ali et al. (2009), Ali et al. (2011), Feng et al. (2008), and Pei and Miao (2005) in Definition 16 is due to the detailed explanations we provided in the introduction of Section 3, where we updated the definition by considering the case of the intersection of parameter sets being empty. In many studies (Ali et al., 2009; Ali et al., 2011; Sezgin and Atagün, 2011; Singh and Onyeozili, 2012c), properties related to the restricted intersection operation were stated overlooking this case, but throughout this study, it has been shown that this condition is not necessary.

The symbol "$\cap_R$" used to denote the restricted intersection operation aligns well with the English word "restricted," forming a meaningful whole. This notation form has been preferred for restricted SS operations in other studies on SS operations as well.

Example 1 Let E=$\{e_1,e_2,e_3,e_4\}$ be the parameter set Ʝ =$\{e_1, e_3\}$ and Ᵽ =$\{e_2, e_3, e_4\}$ be the subsets of E and U=$\{h_1,h_2,h_3,h_4,h_5\}$ be the initial universe set. Assume that (Ꞙ,Ʝ) and ($\mathfrak{C}$,Ᵽ) are the SSs over U defined as follows:

(Ꞙ,Ʝ)=$\{(e_1,\{h_2,h_5\}),(e_3,\{h_1,h_2,h_5\})\}$, ($\mathfrak{C}$,Ᵽ)=$\{(e_2,\{h_1,h_4,h_5\}), (e_3,\{h_2,h_3,h_4\}),(e_4,\{h_3,h_5\})\}$.

Let (Ꞙ,Ʝ)$\cap_R$($\mathfrak{C}$,Ᵽ)=(Ƕ,Ʝ∩Ᵽ), where for all α ∈ Ʝ ∩ Ᵽ = $\{e_3\}$,

Ƕ(e_3)=Ꞙ(e_3)∩$\mathfrak{C}$(e_3)=$\{h_1,h_2,h_5\}\cap\{h_2,h_3,h_4\}=\{h_2\}$.

Thus, (Ꞙ,Ʝ) $\cap_R$($\mathfrak{C}$,Ᵽ)=$\{(e_3,\{h_2\})\}$.

Proposition 1 The set $S_E(U)$ is closed under the operation $\cap_R$. That is, when (Ꞙ,Ʝ) and ($\mathfrak{C}$,Ᵽ) are two SSs over U, then so is (Ꞙ, Ʝ) $\cap_R$ ($\mathfrak{C}$, Ᵽ).

Proof: It is clear that $\cap_R$ is a binary operation in $S_E(U)$. That is,

$\cap_R$: $S_E(U)$x $S_E(U)\rightarrow S_E(U)$

((Ꞙ,Ʝ), ($\mathfrak{C}$,Ᵽ)) → (Ꞙ, Ʝ) $\cap_R$ ($\mathfrak{C}$, Ᵽ)=(Ƕ,Ʝ∩Ᵽ)

Hence, the set $S_E(U)$ is closed under the operation $\cap_R$. Similarly,

$\cap_R$: $S_{Ʝ}(U)$ x $S_{Ʝ}(U)\rightarrow S_{Ʝ}(U)$

((Ꞙ,Ʝ), ($\mathfrak{C}$,Ʝ)) → (Ꞙ, Ʝ) $\cap_R$ ($\mathfrak{C}$, Ʝ)=(K,Ʝ∩Ʝ)=(K,Ʝ)

That is, $\cap_R$ is also closed in $S_{Ʝ}(U)$, where Ʝ is a fixed subset of E.

Proposition 2 Let (Ꞙ,Ʝ), ($\mathfrak{C}$,Ᵽ), and (Ƕ,Ƹ) be SSs over U. Then, [(Ꞙ,Ʝ) $\cap_R$ ($\mathfrak{C}$,Ᵽ)] $\cap_R$ (Ƕ,Ƹ) = (Ꞙ,Ʝ) $\cap_R$ [($\mathfrak{C}$,Ᵽ) $\cap_R$(Ƕ,Ƹ)] (Pei and Miao, 2005).

Proof: Pei and Miao (2005) presented this property without its proof; however, we provide it in detail with its rigorous proof. Let (Ꞙ,Ʝ)$\cap_R$(ℭ,Ᵽ)=(S,Ʝ∩Ᵽ), where for all α∈Ʝ∩Ᵽ, S(α)=Ꞙ(α)∩ℭ(α). Let (S,Ʝ∩Ᵽ)$\cap_R$ (Ӈ, Ꝣ) =(R,(Ʝ∩Ᵽ)∩Ꝣ)), where for all α∈(Ʝ∩Ᵽ)∩Ꝣ, R(α)=S(α)∩Ӈ(α). Thus,

R(α)=[Ꞙ(α)∩ℭ(α)]∩Ӈ(α)

Let (ℭ,Ᵽ)$\cap_R$(Ӈ,Ꝣ)=(K,Ᵽ∩Ꝣ), where for all α∈Ᵽ∩Ꝣ, K(α)=ℭ(α)∩Ӈ(α). Let (Ꞙ,Ʝ)$\cap_R$ (K, Ᵽ ∩ Ꝣ) =(L,Ʝ∩(Ᵽ∩Ꝣ)), where for all α∈Ʝ∩(Ᵽ∩Ꝣ), L(α)=Ꞙ(α)∩K(α). Thus,

L(α)=Ꞙ(α)∩[ℭ(α)∩Ӈ(α)]

Here it is seen that (R,(Ʝ ∩ Ᵽ) ∩ Ꝣ) =(L, Ʝ∩(Ᵽ∩Ꝣ)). That is, $\cap_R$ is associative in the S_E(U).

Here, it is obvious that if Ʝ∩Ᵽ=∅ or Ᵽ∩Ꝣ=∅ or Ʝ∩Ꝣ=∅, then (R,(Ʝ ∩ Ᵽ) ∩ Ꝣ) =(L, Ʝ∩(Ᵽ∩Ꝣ))= $\emptyset_\emptyset$, thus $\cap_R$ is associative under these conditions as well.

Proposition 3 Let (Ꞙ,Ʝ), (ℭ,Ʝ), and (Ӈ,Ʝ) be SSs over U. Then, [(Ꞙ,Ʝ) $\cap_R$(ℭ,Ʝ)] $\cap_R$(Ӈ,Ʝ)=(Ꞙ,Ʝ) $\cap_R$[(ℭ,Ʝ) $\cap_R$(Ӈ,Ʝ)].

Proof: Let (Ꞙ,Ʝ)$\cap_R$(ℭ,Ʝ)=(K,Ʝ), where for all α∈Ʝ∩Ʝ=Ʝ, K(α)=Ꞙ(α) ∩ℭ(α). Let (K,Ʝ)$\cap_R$ (Ӈ, Ʝ) =(R,Ʝ), where for all α∈Ʝ∩Ʝ=Ʝ, R(α)=K(α)∩Ӈ(α). Thus,

R(α)=[Ꞙ(α)∩ℭ(α)]∩Ӈ(α)

Let (ℭ,Ʝ)$\cap_R$(Ӈ,Ʝ)=(L,Ʝ), where for all α∈Ʝ∩Ʝ, L(α)=ℭ(α)∩Ӈ(α). Let (Ꞙ,Ʝ) $\cap_R$(L,Ʝ)= (Ǹ,Ʝ), where for all α∈Ʝ∩Ʝ, Ǹ(α)= Ꞙ(α)∩L(α). Thus,

Ǹ(α)=Ꞙ(α) ∩ [ℭ(α)∩Ӈ(α)]

It is seen that (R,Ʝ)=(Ǹ,Ʝ). That is, $\cap_R$ is associative in $S_Ʝ$(U).

Proposition 4 Let (Ꞙ,Ʝ), and (ℭ,Ᵽ) be SSs over U. Then, (Ꞙ,Ʝ) $\cap_R$(ℭ,Ᵽ)=(ℭ,Ᵽ) $\cap_R$(Ꞙ,Ʝ) (Qin and Hong, 2010).

Proof: Qin and Hong (2010) presented this property without proof in their study; however, we provide it in detail with its rigorous proof. Let (Ꞙ,Ʝ) $\cap_R$(ℭ,Ᵽ)=(Ӈ,Ʝ∩Ᵽ), where for all α∈Ʝ∩Ᵽ, Ӈ(α)=Ꞙ(α)∩ℭ(α). Let (ℭ,Ᵽ)$\cap_R$(Ꞙ,Ʝ)=(S,Ᵽ∩Ʝ), where for all α∈Ᵽ∩Ʝ, S(α)=ℭ(α)∩Ꞙ(α). Thus,

(Ꞙ,Ʝ)$\cap_R$(ℭ,Ᵽ)=(ℭ,Ᵽ) $\cap_R$ (Ꞙ, Ʝ).

Hence, $\cap_R$ is commutative in S_E(U). It is obvious that (Ꞙ,Ʝ)$\cap_R$(ℭ,Ʝ)=(ℭ,Ʝ)$\cap_R$(Ꞙ,Ʝ). That is, $\cap_R$ is commutative in $S_Ʝ$(U) as well. Here, it is also obvious that if Ʝ∩Ᵽ=∅, then (Ӈ, ∩ Ᵽ) =(S, Ᵽ∩ Ʝ)= $\emptyset_\emptyset$, thus $\cap_R$ is commutative under this condition as well.

Proposition 5 Let (Ꞙ,Ʝ) be an SS over U. Then, (Ꞙ,Ʝ) $\cap_R$(Ꞙ,Ʝ)=(Ꞙ,Ʝ) (Pei and Miao, 2005).

Proof: Pei and Miao (2005) presented this property without proof in their study; however, we provide it in detail with its rigorous proof. Let (Ꞙ,Ʝ)$\cap_R$(Ꞙ,Ʝ)=(Ӈ,Ʝ∩Ʝ), where for all α∈Ʝ, Ӈ(α)=Ꞙ(α)∩Ꞙ(α)=Ꞙ(α). Thus. (Ӈ,Ʝ)= (Ꞙ, Ʝ). That is, $\cap_R$ is idempotent in S_E(U).

Proposition 6 Let (Ꞙ,Ʝ) be an SS over U. Then, (Ꞙ,Ʝ)$\cap_R$ $U_Ʝ$= $U_Ʝ$ $\cap_R$(Ꞙ,Ʝ)=(Ꞙ,Ʝ) (Ali et al. 2011).

Proof: Ali et al. (2011) presented this property with the relative whole SS with respect to Ʝ only on the right side, and without its proof; however, we provide the property with the relative whole SS with respect to Ʌ on both the right and left sides, along with its detailed proof. Let $U_{Ʌ}$=(K,Ʌ), where for all α∈Ʌ, K(α)=U. Let (Ғ,Ʌ) $\cap_R$ (K, Ʌ)=(Ӈ,Ʌ∩Ʌ), where for all α∈ Ʌ, Ӈ(α)= Ғ(α)∩ K(α)=Ғ(α)∩ U=Ғ(α). Thus, (Ӈ,Ʌ)= (Ғ,Ʌ) implying that (Ғ,Ʌ)$\cap_R$ U_T=(Ғ,Ʌ), and by Proposition 4, $U_{Ʌ}$ $\cap_R$(Ғ,Ʌ)=(Ғ,Ʌ) as well. That is, $U_{Ʌ}$ is the identity element of $\cap_R$ in $S_{Ʌ}$(U).

Here, it is obvious that there is no inverse element for the operation $\cap_R$ other than $U_{Ʌ}$ in $S_{Ʌ}$(U). Naturally, $U_{Ʌ}$ itself is the identity element for the operation $\cap_R$ in $S_{Ʌ}$(U).

Proposition 7 Let (Ғ,Ʌ) be an SS over U. Then, (Ғ,Ʌ) $\cap_R$ $\emptyset_{Ʌ}$=$\emptyset_{Ʌ}$ $\cap_R$(Ғ,Ʌ)=$\emptyset_{Ʌ}$ (Ali et al. 2011).

Proof: Ali et al. (2011) presented this property with the relative null SS with respect to Ʌ only on the right side, and without its proof; however, we provide the property with the relative null SS with respect to Ʌ on both the right and left sides, along with its detailed proof. Let $\emptyset_{Ʌ}$=(S,Ʌ), where for all α∈ Ʌ, S(α)= ∅. Let (Ғ,Ʌ) $\cap_R$(S,Ʌ)=(Ӈ,Ʌ∩Ʌ), where for all α∈Ʌ, Ӈ(α)=Ғ(α)∩S(α)=Ғ(α)∩ ∅=∅. Thus, (Ӈ,Ʌ)=$\emptyset_{Ʌ}$, implying that (Ғ,Ʌ) $\cap_R$ $\emptyset_{Ʌ}$=$\emptyset_{Ʌ}$, and by Proposition 4, $\emptyset_{Ʌ}$ $\cap_R$(Ғ,Ʌ)=$\emptyset_{Ʌ}$ as well. That is, $\emptyset_{Ʌ}$ is the absorbing element of $\cap_R$ in $S_{Ʌ}$(U).

Theorem 1 ($S_{Ʌ}$(U),$\cap_R$) is a bounded semi-lattice, whose identity is $U_{Ʌ}$ and the absorbing element is $\emptyset_{Ʌ}$.

Proof: By Proposition 1, Proposition 3, Proposition 4, Proposition 5, Proposition 6, and Proposition 7 (S_T(U),$\cap_R$) is a commutative, idempotent monoid whose identity is $U_{Ʌ}$ and absorbing element $\emptyset_{Ʌ}$, that is, a bounded semi-lattice.

Proposition 8 Let (Ғ,Ʌ) be an SS over U. Then, (Ғ,Ʌ)$\cap_R$ U_E=U_E $\cap_R$(Ғ,Ʌ)=(Ғ,Ʌ).

Proof: Let U_E=(K,E), where for all α∈E, K(α)=U. Let (Ғ, Ʌ) $\cap_R$ (K, E) = (Ӈ, Ʌ ∩ E), where for all α∈Ʌ∩E=Ʌ, Ӈ(α)=Ғ(α)∩K(α)=Ғ(α)∩ U=Ғ(α). Thus, (Ӈ,Ʌ)=(Ғ,Ʌ), implying that (Ғ,Ʌ)$\cap_R$ U_E=(Ғ,Ʌ), and by Proposition 4, U_E $\cap_R$(Ғ,Ʌ)=(Ғ,Ʌ) as well. That is, U_E is the identity element of $\cap_R$ in S_E(U).

From this, we can conclude that in S_E(U), no element has an inverse element for the operation $\cap_R$ other than U_E, which is the identity element. Naturally, the SS U_E itself is the identity element of the operation $\cap_R$ in S_E(U).

Proposition 9 Let (Ғ,Ʌ) be an SS over U. Then, (Ғ,Ʌ)$\cap_R$ $\emptyset_\emptyset$=$\emptyset_\emptyset$ $\cap_R$(Ғ,Ʌ)= $\emptyset_\emptyset$.

Proof: Let $\emptyset_\emptyset$ =(S,∅). (Ғ,Ʌ)$\cap_R$ (S,∅)=(Ӈ,Ʌ ∩ ∅)=(Ӈ, ∅), $\emptyset_\emptyset$ since the parameter set is the only SS that is the empty set, (Ӈ, ∅)=$\emptyset_\emptyset$, implying that (Ғ,Ʌ)$\cap_R$ $\emptyset_\emptyset$= $\emptyset_\emptyset$, and by Proposition 4, $\emptyset_\emptyset$ $\cap_R$(Ғ,Ʌ)= $\emptyset_\emptyset$ as well. That is, $\emptyset_\emptyset$ is the absorbing element of $\cap_R$ in S_E(U).

Theorem 2 (S_E(U),$\cap_R$) is a bounded semi-lattice, whose identity is U_E and the absorbing element is $\emptyset_\emptyset$.

Proof: By Proposition 1, Proposition 2, Proposition 4, Proposition 5, Proposition 8, and Proposition 9, (S_E(U),$\cap_R$) is a commutative, idempotent monoid whose identity is U_E, that is, a bounded semi-lattice.

Proposition 10 Let (Ꞙ,Ꭻ) be an SS over U. Then, (Ꞙ,Ꭻ)$\cap_R$ $\emptyset_E$=$\emptyset_E$ $\cap_R$ (Ꞙ,Ꭻ)= $\emptyset_{Ꭻ}$.

Proof: Let $\emptyset_E$=(S,E), where for all α∈E, S(α)=∅. Let (Ꞙ,Ꭻ)$\cap_R$(S,E)=(Ⱨ,Ꭻ∩E), where for all α∈Ꭻ∩E=Ꭻ, Ⱨ(α)=Ꞙ(α)∩S(α)=Ꞙ(α)∩ ∅=∅. Thus, (Ⱨ,Ꭻ)= $\emptyset_T$, implying that (Ꞙ,Ꭻ)$\cap_R$ $\emptyset_E$=$\emptyset_{Ꭻ}$, and by Proposition 4, $\emptyset_E$ $\cap_R$ (Ꞙ,Ꭻ)= $\emptyset_{Ꭻ}$ as well.

Proposition 11 Let (Ꞙ,Ꭻ) be an SS over U. Then, (Ꞙ,Ꭻ) $\cap_R$(Ꞙ,Ꭻ)^r=(Ꞙ,Ꭻ)^r $\cap_R$(Ꞙ,Ꭻ)= $\emptyset_{Ꭻ}$ (Sezgin and Atagün, 2011; Ali et al., 2011).

Proof: In the studies by Sezgin and Atagün (2011) and Ali et al. (2011), this property was presented with the relative complement of the SS (Ꞙ,Ꭻ) only on the right side, and without its proof. However, we provide the property with the relative complement of the SS (Ꞙ,Ꭻ) on both the right and left sides, along with its detailed proof. Let (Ꞙ,Ꭻ)^r=(Ⱨ,Ꭻ), where for all α∈ Ꭻ, Ⱨ(α)=Ꞙ'(α). Let (Ꞙ,Ꭻ)$\cap_R$ (Ⱨ, Ꭻ)=(L,Ꭻ∩Ꭻ), where for all α∈Ꭻ, L(α)=Ꞙ(α)∩Ⱨ(α)=Ꞙ(α)∩Ꞙ'(α)=∅. Thus, (L,Ꭻ)= $\emptyset_{Ꭻ}$, implying that (Ꞙ,Ꭻ)$\cap_R$(Ꞙ,Ꭻ)^r=$\emptyset_{Ꭻ}$, and by Proposition 4, (Ꞙ,Ꭻ)^r $\cap_R$(Ꞙ,Ꭻ)= $\emptyset_{Ꭻ}$ as well.

Proposition 12 Let (Ꞙ,Ꭻ) and (ℭ,Ᵽ) be SSs over U. Then, [(Ꞙ,Ꭻ) $\cap_R$(ℭ,Ᵽ)]^r=(Ꞙ,Ꭻ)^r $\cup_R$(ℭ,Ᵽ)^r (De Morgan Law) (Ali et al., 2009).

Proof: In the study by Ali et al. (2009), the De Morgan property was presented with the condition Ꭻ∩Ᵽ≠∅ and the proof was relatively lengthy. In this study, we state that the condition Ꭻ∩Ᵽ≠∅ is not a necessary condition for the proposition and provide a simpler proof. Let (Ꞙ,Ꭻ)$\cap_R$(ℭ,Ᵽ)=(Ⱨ,Ꭻ∩Ᵽ), where for all α∈Ꭻ∩Ᵽ, Ⱨ(α)=Ꞙ(α)∩ℭ(α). Let (Ⱨ,Ꭻ∩Ᵽ)^r=(K,Ꭻ∩Ᵽ), where for all α∈Ꭻ∩Ᵽ, K(α)=H'(α)=Ꞙ'(α)∪G'(α). Thus, (K,Ꭻ∩Ᵽ)=(Ꞙ,Ꭻ)^r $\cup_R$(ℭ,Ᵽ)^r. Here, if Ꭻ∩Ᵽ=∅, then the equality is again satisfied since the right and left sides will be $\emptyset_{\emptyset}$. So, Ꭻ∩Ᵽ≠∅ is not a necessary condition for this proposition.

Proposition 13 Let (Ꞙ,Ꭻ) and (ℭ,Ꭻ) be SSs over U. Then, (Ꞙ,Ꭻ) $\cap_R$ (ℭ, Ꭻ)= $U_{Ꭻ}$ ⇔ (Ꞙ, Ꭻ) = $U_{Ꭻ}$ and (ℭ, Ꭻ) = $U_{Ꭻ}$.

Proof: Let (Ꞙ, Ꭻ) $\cap_R$ (ℭ, Ꭻ) = (K,Ꭻ∩Ꭻ), where for all α∈Ꭻ, K(α)= Ꞙ(α)∩ℭ(α). Since (K,Ꭻ)= U_T, K(α)=U, for all α∈Ꭻ. Thus, K(α)=Ꞙ(α)∩ℭ(α)=U, for all α∈Ꭻ ⇔Ꞙ(α)= U and ℭ(α)= U, for all α ∈ Ꭻ ⇔ (Ꞙ, Ꭻ) = $U_{Ꭻ}$ and (ℭ,Ꭻ)= $U_{Ꭻ}$.

In the study by Sezgin and Atagün (2011), the properties related to restricted intersection and soft subsets were examined according to the definition of soft subsets given in the study by Maji et al. (2003). In this study, we examine the properties related to soft subsets according to the definition given in the study by Pei and Miao (2005), which is widely accepted. Therefore, the following properties related to soft subsets have not been included in previous studies.

Proposition 14 Let (Ꞙ,Ꭻ) and (ℭ,Ᵽ) be SSs over U. Then, $\emptyset_{Ꭻ∩Ᵽ}$ $\tilde{\subseteq}$(Ꞙ,Ꭻ) $\cap_R$(ℭ,Ᵽ). Moreover, (Ꞙ,Ꭻ) $\cap_R$(ℭ,Ᵽ) $\tilde{\subseteq}$ $U_{Ꭻ}$, and (Ꞙ,Ꭻ) $\cap_R$(ℭ,Ᵽ) $\tilde{\subseteq}$ $U_{Ᵽ}$.

Proof: The proof is evident from the fact that the empty set is a subset of every set and the universal set includes every set.

Proposition 15 Let (Ꞙ,Ɉ) and (ℭ,Ᵽ) be SSs over U. Then, (Ꞙ,Ɉ)$\cap_R$(ℭ,Ᵽ) ≦̃ (Ꞙ,Ɉ) and (Ꞙ,Ɉ)$\cap_R$(ℭ,Ᵽ) ≦̃(ℭ,Ᵽ), where Ɉ∩Ᵽ≠∅.

Proof: Let (Ꞙ,Ɉ)$\cap_R$(ℭ,Ᵽ)=(Ή,Ɉ∩Ᵽ), where for all α∈Ɉ∩Ᵽ, Ή(α)=Ꞙ(α) ∩ℭ(α). Thus, Ή(α)=Ꞙ(α) ∩ ℭ(α) ⊆Ꞙ(α), for all α∈Ɉ∩Ᵽ. Hence, (Ꞙ,Ɉ)$\cap_R$(ℭ,Ᵽ)≦̃(Ꞙ,Ɉ). Furthermore, since Ꞙ(α) ∩ ℭ(α) ⊆ℭ(α), (Ꞙ,Ɉ)$\cap_R$(ℭ,Ᵽ) ≦̃ (ℭ,Ᵽ) is obvious.

Proposition 16 Let (Ꞙ,Ɉ) and (ℭ,Ᵽ) be SSs over U. Then, (Ꞙ,Ɉ)≦̃ (ℭ,Ᵽ) if and only if (Ꞙ,Ɉ) $\cap_R$(ℭ,Ᵽ)=(Ꞙ,Ɉ)**.**

Proof: Let (Ꞙ,Ɉ)≦̃ (ℭ,Ᵽ) and (Ꞙ,Ɉ)$\cap_R$(ℭ,Ᵽ)=(K,Ɉ∩Ᵽ=Ɉ). Thus, Ɉ⊆Ᵽ and Ꞙ(α)⊆ ℭ(α), for all α∈Ɉ, and so K(α)=Ꞙ(α)∩ℭ(α)=Ꞙ(α), for all α∈Ɉ. Therefore, (K,Ɉ)= (Ꞙ,Ɉ) $\cap_R$(ℭ,Ᵽ) =(Ꞙ,Ɉ)**.** Conversely, let (Ꞙ,Ɉ)$\cap_R$(ℭ,Ᵽ) =(Ꞙ,Ɉ). Hence, Ɉ∩Ᵽ=Ɉ, implying that Ɉ⊆Ᵽ**.** Moreover, since Ꞙ(α)∩ℭ(α)=Ꞙ(α), for all α∈Ɉ, this implies that Ꞙ(α)⊆ℭ(α). Thereby, (Ꞙ,Ɉ) ≦̃ (ℭ,Ᵽ).

Proposition 17 Let (Ꞙ,Ɉ), (ℭ,Ᵽ), and (Ή,Ƨ) be SSs over U such that Ɉ∩Ƨ≠∅. If (Ꞙ,Ɉ) ≦̃ (ℭ,Ᵽ), then (Ꞙ,Ɉ) $\cap_R$ (Ή,Ƨ) ≦̃ (ℭ,Ᵽ) $\cap_R$ (Ή,Ƨ).

Proof: Let (Ꞙ,Ɉ)≦̃ (ℭ,Ᵽ). Then, Ɉ⊆Ᵽ, and Ꞙ(α)⊆ ℭ(α), for all α∈Ɉ. Let (Ꞙ,Ɉ)$\cap_R$(Ή,Ƨ)=(K,Ɉ∩Ƨ). Thus, K(α)=Ꞙ(α)∩Ή(α), for all α∈Ɉ∩Ƨ. Let (ℭ,Ᵽ)$\cap_R$(Ή,Ƨ)=(L,Ᵽ∩Ƨ). Hence, L(α)=ℭ(α)∩Ή(α), for all α∈Ᵽ∩Ƨ. Hence, Ɉ∩Ƨ ⊆ Ᵽ∩Ƨ, and K(α)=Ꞙ(α)∩Ή(α)⊆ℭ(α)∩Ή(α)=L(α), for all α∈Ɉ∩Ƨ. Thereby, (Ꞙ,Ɉ)$\cap_R$(Ή,Ƨ) ≦̃(ℭ,Ᵽ) $\cap_R$(Ή,Ƨ).

Here, if Ᵽ∩Ƨ=∅, this would require Ɉ∩Ƨ=∅ (as Ɉ∩Ƨ⊆Ᵽ∩Ƨ) making the proof evident once again.

Proposition 18 Let (Ꞙ,Ɉ), (ℭ,Ᵽ), and (Ή,Ƨ) be SSs over U such that Ɉ∩Ƨ≠∅. If (Ꞙ,Ɉ) $\cap_R$ (Ή,Ƨ) ≦̃ (ℭ,Ᵽ) $\cap_R$ (Ή,Ƨ), thn (Ꞙ,Ɉ) ≦̃ (ℭ,Ᵽ) needs not be true. That is, the converse of Proposition 17 is not true.

Proof: Let us give an example to show that the converse of Proposition 17 is not true. Let E={e_1,e_2,e_3,e_4,e_5} be the parameter set Ɉ={e_1,e_3}, Ᵽ={e_1,e_3,e_5}, Ƨ={e_1,e_3,e_5, e_6} be the subsets of E and U={h_1,h_2,h_3,h_4,h_5} be the universel set. Assume that (Ꞙ,Ɉ), (ℭ,Ᵽ), and (Ή,Ƨ) are the SSs over U defined as follows:

(Ꞙ,Ɉ)={(e_1,{h_2, h_5}),(e_3,{h_1,h_2,h_5})},

(ℭ,Ᵽ)={(e_1,{h_2}),(e_3,{h_1,h_2}),(e_5,{h_3})},

(Ή,Ƨ)={ e_1,∅),(e_3,∅),(e_5, ∅),(e_6,{h_1, h_5})}.

Let (Ꞙ,Ɉ)$\cap_R$(Ή,Ƨ)=(L,Ɉ∩Ƨ), where for all α∈Ɉ ∩ Ƨ = {e_1, e_3}, L(α)=Ꞙ(α) ∩Ή(α). Thus, L(e_1)=Ꞙ(e_1) ∩ Ή(e_1)=∅, L(e_3)=Ꞙ(e_3) ∩ Ή(e_3)=∅. Hence, (Ꞙ,Ɉ)$\cap_R$(Ⱨ,Ꝛ)={(e_1,∅),(e_3,∅)}.

Now let (ℭ,Ᵽ)$\cap_R$(Ⱨ,Ꝛ)=(K,Ᵽ∩Ꝛ), where for all α∈Ᵽ ∩ Ꝛ = {e_1, e_3, e_5}, K(α)=ℭ(α) ∩Ⱨ(α). Hence, K(e_1)=ℭ(e_1) ∩ Ⱨ(e_1)=∅, K(e_3)=ℭ(e_3) ∩ Ⱨ(e_3)=∅, and K(e_5)=ℭ(e_5) ∩ Ⱨ(e_5)=∅. Thus,

(ℭ,Ᵽ) $\cap_R$(Ⱨ,Ꝛ)={(e_1,∅), (e_3,∅),(e_5,∅)}.

Hence, (Ꞙ,Ɉ) $\cap_R$(Ⱨ,Ꝛ) $\widetilde{\subseteq}$(ℭ,Ᵽ) $\cap_R$(Ⱨ,Ꝛ), but (Ꞙ,Ɉ) isn't a subset of ℭ,Ᵽ).

Proposition 19 Let (Ꞙ,Ɉ), (ℭ,Ᵽ), (K,V) and (L,W) be SSs over U such that Ɉ∩V≠∅. If (Ꞙ,Ɉ) $\widetilde{\subseteq}$ (ℭ,Ᵽ) and (K,V) $\widetilde{\subseteq}$ (L,W), then (Ꞙ,Ɉ) $\cap_R$ (K,V) $\widetilde{\subseteq}$ (ℭ,Ᵽ) $\cap_R$ (L,W)

Proof: Let (Ꞙ,Ɉ) $\widetilde{\subseteq}$ (ℭ,Ᵽ) and (K,V) $\widetilde{\subseteq}$ (L,W). Hence, for all α∈Ɉ, Ꞙ(α)⊆ ℭ(α) and for all α∈V, K(α)⊆ L(α). Let (Ꞙ,Ɉ)$\cap_R$(K,V)=(M,Ɉ∩V). Thus, for all α∈Ɉ∩V, M(α)=Ꞙ(α)∩K(α). Let (ℭ,Ᵽ)$\cap_R$(L,W)=(Ǹ,Ᵽ∩W). Thus, for all α∈Ᵽ∩W, Ǹ(α)=ℭ(α)∩L(α). Hence, for all α∈ Ɉ∩V, M(α)=Ꞙ(α)∩K(α)⊆ℭ(α)∩L(α)=Ǹ(α). Hence, (Ꞙ,Ɉ)$\cap_R$(K,V)$\widetilde{\subseteq}$(ℭ,Ᵽ) $\cap_R$(L,W). Here, if Ᵽ∩W=∅, this would require Ɉ∩V=∅ (since Ɉ∩V⊆Ᵽ∩W) making the proof clear once again.

Proposition 20 Let (Ꞙ,Ɉ) and (ℭ,Ᵽ) be SSs over U. If (Ꞙ,Ɉ) $\widetilde{\subseteq}$ (ℭ,Ᵽ)^r, then (Ꞙ,Ɉ) $\cap_R$(ℭ,Ᵽ) = $\emptyset_{Ɉ}$. Moreover, (Ꞙ,Ɉ) $\widetilde{\subseteq}$ (ℭ,Ɉ)^r if and only if (Ꞙ,Ɉ) $\cap_R$(ℭ,Ɉ) = $\emptyset_{Ɉ}$.

Proof: Let (Ꞙ,Ɉ) $\widetilde{\subseteq}$ (ℭ,Ᵽ)^r. Hence, Ɉ ⊆ Ᵽ and Ꞙ(α) ⊆ ℭ'(α), for all α∈Ɉ. Let (Ꞙ,Ɉ) $\cap_R$(ℭ,Ᵽ) =(Ⱨ, Ɉ∩Ᵽ=Ɉ). Hence, Ⱨ(α)= Ꞙ(α)∩ℭ(α)=∅, for all α∈Ɉ. Thus, (Ⱨ,Ɉ) = $\emptyset_{Ɉ}$. Similarly, it can be shown that (Ꞙ,Ɉ) $\widetilde{\subseteq}$(ℭ,Ɉ)^r ⇔ (Ꞙ,Ɉ) $\cap_R$(ℭ,Ɉ)= $\emptyset_{Ɉ}$. In other words, for the converse of the theorem to be true, the parameter sets of the SSs must be the same. It was given without proof in Ali et al. (2011) that if (Ꞙ,Ɉ) $\cap_R$ (ℭ,Ɉ)= $\emptyset_{Ɉ}$, then (Ꞙ,Ɉ) $\widetilde{\subseteq}$(ℭ,Ɉ)^r . However, it is also evident that if (Ꞙ,Ɉ) $\widetilde{\subseteq}$(ℭ,Ɉ)^r, then (Ꞙ,Ɉ) $\cap_R$ (ℭ,Ɉ) = $\emptyset_{Ɉ}$.

3.1.1. The distributions of the restricted intersection operation over other SS operations:

In this subsection, the distributions of restricted intersection operation over other SS operations such as restricted SS operations, extended SS operations, and soft binary piecewise operations are examined in detail and many interesting results are obtained.

3.1.1.1. The distributions of the restricted intersection operation over other restricted SS operations:

Here, the distributions of the restricted intersection operation over other restricted operations have been examined. First, the left distributions, and then the right distributions were investigated. It is worth mentioning an important point here. Although Sezgin and Atagün (2011) showed that the restricted intersection operation distributes over the restricted union and restricted difference from both the right and the left, their proofs repeatedly emphasized that the intersections of the parameter sets of the SSs involved in the restricted operations should be non-empty. However, even if the intersections of the parameter sets of the SSs involved in the restricted operations are empty, these distributions still hold. Besides, in the study by Ali et al. (2011), only the left distributions were presented in a table and without proofs. Therefore, in this subsection, considering all these conditions, detailed proof is provided for the distributions.

a) LHS distributions of restricted intersection over other restricted SS operations:

Let (Ꞙ,Ɉ), (ℭ,Ᵽ), and (Ꜧ,Ƨ) be SSs over U. Then, we have the following distributions:

i) (Ꞙ,Ɉ)$\cap_R$[(ℭ,Ᵽ)$\cup_R$(Ꜧ,Ƨ)]=[(Ꞙ,Ɉ) $\cap_R$(ℭ,Ᵽ)] $\cup_R$ [(Ꞙ,Ɉ) $\cap_R$(Ꜧ,Ƨ)] (Sezgin and Atagün, 2011).

Proof: In their study, Sezgin and Atagün (2011) provided the proof of this property under the condition that the intersection of the parameter sets of the SSs involved in the restricted operations should be non-empty. However, this proof will specifically indicate that this property holds even if the intersection of the parameter sets of the SSs involved in the restricted operations is empty.

First, let's consider the left-hand side (LHS), and let (ℭ,Ᵽ)$\cup_R$(Ꜧ,Ƨ)=(R,Ᵽ∩Ƨ), where for all α∈Ᵽ∩Ƨ, R(α)=ℭ(α)∪Ꜧ(α). Let (Ꞙ,Ɉ) $\cap_R$(R,Ᵽ∩Ƨ)=(Ǹ,Ɉ∩(Ᵽ∩Ƨ)), where for all α∈Ɉ∩(Ᵽ∩Ƨ), Ǹ(α)=Ꞙ(α)∩R(α). Hence, for all α∈Ɉ∩Ᵽ∩Ƨ,

Ǹ(α)= Ꞙ(α)∩[(ℭ(α)∪Ꜧ(α)].

Now let's handle the right hand side (RHS). Let (Ꞙ,Ɉ)$\cap_R$ (ℭ,Ᵽ)=(V,Ɉ∩Ᵽ), where for all α∈Ɉ∩Ᵽ, V(α)=Ꞙ(α)∩ℭ(α), and let (Ꞙ,Ɉ)$\cap_R$ (Ꜧ,Ƨ)=(W,Ɉ∩Ƨ), where for all α∈Ɉ∩Ƨ, W(α)=Ꞙ(α)∩Ꜧ(α). Let (V,Ɉ∩Ᵽ) $\cup_R$(W,Ɉ∩Ƨ)=(S,(Ɉ∩Ᵽ)∩(Ɉ∩Ƨ)), where for all α∈Ɉ∩Ᵽ∩Ƨ, S(α)=V(α)∪W(α). Thereby,

S(α)= [Ꞙ(α)∩ℭ(α)]∪[Ꞙ(α)∩Ꜧ(α)]

Thus, it is seen that (Ǹ,Ɉ∩Ᵽ∩Ƨ)=(S,Ɉ∩Ᵽ∩Ƨ). Here, if Ᵽ∩Ƨ=∅ or Ɉ∩Ᵽ=∅ or Ɉ∩Ƨ=∅, then in every case, both the left side and the right side will be $\emptyset_\emptyset$. Thus, the equality holds in this case as well. Therefore, there is no need to impose the condition that these sets are non-empty.

ii) (Ꞙ,Ɉ) $\cap_R$[(ℭ,Ᵽ)$\cap_R$(Ꜧ,Ƨ)] = [(Ꞙ,Ɉ) $\cap_R$(ℭ,Ᵽ)] $\cap_R$ [(Ꞙ,Ɉ) $\cap_R$ (Ꜧ,Ƨ)].

iii) (Ꞙ,Ɉ) $\cap_R$ [(ℭ,Ᵽ)$\setminus_R$(Ꜧ,Ƨ)] = [(Ꞙ,Ɉ) $\cap_R$(ℭ,Ᵽ)] $\setminus_R$ [(Ꞙ,Ɉ) $\cap_R$ (Ꜧ,Ƨ)] (Sezgin and Atagün, 2011).

iv) (Ꞙ,Ɉ) $\cap_R$ [(ℭ,Ᵽ) γ_R (Ꜧ,Ƨ)] = [(Ꞙ,Ɉ) $\cap_R$(ℭ,Ᵽ)] γ_R [(Ꞙ,Ɉ) $\cap_R$ (Ꜧ,Ƨ)].

v) (Ꞙ,Ɉ) $\cap_R$ [(ℭ,Ᵽ) Δ_R(Ꜧ,Ƨ)]=[(Ꞙ,Ɉ) $\cap_R$(ℭ,Ᵽ)]Δ_R[(Ꞙ,Ɉ) $\cap_R$ (Ꜧ,Ƨ)] (Singh and Onyeozili, 2012c).
Although Singh and Onyeozili (2012c) provided this property, their proof contains numerous mathematical errors. Therefore, we are presenting the proof again in a structured and corrected manner. First, let's consider the LHS, and let (ℭ,Ᵽ)Δ_R(Ꜧ,Ƨ)=(R,Ᵽ∩Ƨ), where for all α∈Ᵽ∩Ƨ, R(α)=ℭ(α)ΔꜦ(α). Let (Ꞙ,Ɉ) $\cap_R$(R,Ᵽ∩Ƨ)=(Ǹ,Ɉ∩(Ᵽ∩Ƨ)), for all α∈Ɉ∩(Ᵽ∩Ƨ), Ǹ(α)= Ꞙ(α)∩R(α). Hence, for all α∈Ɉ∩Ᵽ∩Ƨ,

Ǹ(α)= Ꞙ(α)∩[(ℭ(α)ΔꜦ(α)].

Now let's handle the RHS. Let (Ꞙ,Ɉ)$\cap_R$ (ℭ,Ᵽ)=(V,Ɉ∩Ᵽ), where for all α∈Ɉ∩Ᵽ, V(α)=Ꞙ(α)∩ℭ(α). Now let (Ꞙ,Ɉ)$\cap_R$ (Ꜧ,Ƨ)=(W,Ɉ∩Ƨ), where for all α∈Ɉ∩Ƨ, W(α)=Ꞙ(α)∩Ꜧ(α). Let (V,Ɉ∩Ᵽ) Δ_R(W,Ɉ∩Ƨ)=(S,(Ɉ∩Ᵽ)∩(Ɉ∩Ƨ)). Hence, for all α∈Ɉ∩Ᵽ∩Ƨ, S(α)=V(α)ΔW(α),

S(α)= [Ꞙ(α)∩ℭ(α)]Δ[Ꞙ(α)∩Ꜧ(α)]

Thus, it is seen that (Ǹ,Ɉ∩Ᵽ∩Ƨ)=(S,Ɉ∩Ᵽ∩Ƨ). Here, if Ᵽ∩Ƨ=∅ or Ɉ∩Ᵽ=∅ or Ɉ∩Ƨ=∅, then in every case, both the left side and the right side will be $\emptyset_\emptyset$. Thus, the equality holds in this case as well. Therefore, there is no need to impose the condition that these sets are non-empty.

b) RHS distributions of restricted intersection over other restricted SS operations:
Let (Ꞙ,Ɉ), (ℭ,Ᵽ), and (Ꜧ,Ȝ) be SSs over U. Then, we have the following distributions:

i) [(₣,Ꭻ)∪$_R$(ℭ,Ᵽ)] ∩$_R$(Ꜧ,Ȝ)=[(₣,Ꭻ) ∩$_R$(Ꜧ,Ȝ)] ∪$_R$[(ℭ,Ᵽ) ∩$_R$ (Ꜧ,Ȝ)] (Sezgin and Atagün, 2011).

Proof: Sezgin and Atagün (2011) presented this property without proof in their study; however, we provide it with its detailed proof. First, let's handle the LHS. Let (₣,Ꭻ) ∪$_R$(ℭ,Ᵽ)=(R,Ꭻ∩Ᵽ), where for all α∈Ꭻ∩Ᵽ, R(α)=₣(α)∪ℭ(α). Let (R,Ꭻ∩Ᵽ) ∩$_R$(Ꜧ,Ƨ)=(Ǹ,(Ꭻ∩Ᵽ)∩Ƨ)), where for all α∈(Ꭻ∩Ᵽ)∩Ƨ, Ǹ(α)=R(α)∩Ꜧ(α). Hence,

Ǹ(α)= [₣(α)∪ℭ(α)]∩Ꜧ(α)

Now let's handle the RHS. Let (₣,Ꭻ)∩$_R$(Ꜧ,Ƨ)=(S,Ꭻ∩Ƨ), where for all α∈Ꭻ∩Ƨ, S(α)=₣(α)∩Ꜧ(α). Let (ℭ,Ᵽ) ∩$_R$(Ꜧ,Ƨ)=(K,Ᵽ∩Ƨ), where K(α)=ℭ(α)∩Ꜧ(α), for all α∈Ᵽ∩Ƨ. Let (S,Ꭻ∩Ƨ)∩$_R$(K,Ᵽ∩Ƨ)=(L,(Ꭻ∩Ᵽ∩Ƨ)), where for all α∈(Ꭻ∩Ƨ)∩(Ᵽ∩Ƨ) , L(α)=S(α)∩K(α). Hence,

L(α)= ([₣(α)∩Ꜧ(α)] ∪[ℭ(α)∩Ꜧ(α)]

Thus, it is seen that (Ǹ,Ꭻ∩Ᵽ∩Ƨ)=(L,Ꭻ∩Ᵽ∩Ƨ). Here, if Ᵽ∩Ƨ=∅ or Ꭻ∩Ᵽ=∅ or Ꭻ∩Ƨ=∅, =∅, then in every case, both the left side and the right side will be $\emptyset_\emptyset$. Thus, the equality holds in this case as well. Therefore, there is no need to impose the condition that these sets are non-empty.

ii) [(₣,Ꭻ) ∩$_R$ (ℭ,Ᵽ)] ∩$_R$ (Ꜧ,Ƨ)=[(₣,Ꭻ) ∩$_R$(Ꜧ,Ƨ)] ∩$_R$[(ℭ,Ᵽ) ∩$_R$(Ꜧ,Ƨ)].

iii) [(₣,Ꭻ) \\$_R$ (ℭ,Ᵽ)] ∩$_R$ (Ꜧ,Ƨ)= [(₣,Ꭻ)∩$_R$(Ꜧ,Ƨ)] \\$_R$ [(ℭ,Ᵽ)∩$_R$(Ꜧ,Ƨ)] (Sezgin and Atagün, 2011).

iv) [(₣,Ꭻ)γ_R(ℭ,Ᵽ)] ∩$_R$(Ꜧ,Ƨ)= [(₣,Ꭻ) ∩$_R$ (Ꜧ,Ƨ)] γ_R [(ℭ,Ᵽ) ∩$_R$ (Ꜧ,Ƨ)].

v) [(₣,Ꭻ) Δ_R (ℭ,Ᵽ)] ∩$_R$ (Ꜧ,Ƨ)= [(₣,Ꭻ)∩$_R$(Ꜧ,Ƨ)] Δ_R [(ℭ,Ᵽ)∩$_R$(Ꜧ,Ƨ)].

3.1.1.2. The distributions of the restricted intersection operation over extended SS operations:

Here, the distributions of the restricted intersection operation over extended operations have been examined. First, left distributions were investigated, followed by right distributions. It is important to note a significant point here. In some studies, although it has been stated that the restricted intersection operation distributes over extended union, intersection, and difference operations from the left side, no attention has been given to the right distributions, and all right distributions have been provided without proofs in the study by Ali et al. (2011). In other studies where proofs are provided, emphasis has been placed on the requirement that the intersection of the parameter sets of the SSs involved in the restricted operations must be non-empty. However, even if the intersection of the parameter sets of the SSs involved in the restricted operations is the empty set, these distributions are still valid. Therefore, detailed proofs are provided taking into account these considerations, especially in the distributions under this subsection.

a) LHS distributions of restricted intersection operation over extended SS operations:

Let (₣,Ꭻ), (ℭ,Ᵽ), and (Ꜧ,Ƨ) be SSs over U. Then, we have the following distributions:

i) (₣,Ꭻ) ∩$_R$[(ℭ,Ᵽ) ∪$_\varepsilon$ (Ꜧ,Ƨ)] = [(₣,Ꭻ) ∩$_R$(ℭ,Ᵽ)] ∪$_\varepsilon$ [(₣,Ꭻ) ∩$_R$ (Ꜧ,Ƨ)] (Pei and Miao, 2005)

Proof: Pei and Miao (2005) presented this property without proof in their study; however, we provide it with its detailed proof here. We also state and prove that the property holds even when the intersection of the parameter sets of the SSs involved in the restricted operations is empty.

First, let’s consider the LHS, and let (ℭ,Ᵽ) $\cup_\varepsilon$ (Ꮋ,Ꙅ)=(R,Ᵽ∪Ꙅ), where for all α∈Ᵽ∪Ꙅ,

R(α)= { ℭ(α) α∈Ᵽ\Ꙅ ; Ꮋ(α) α∈Ꙅ\Ᵽ ; ℭ(α)∪Ꮋ(α) α∈Ᵽ∩Ꙅ

Let (Ꞙ,Ꭻ) $\cap_R$ (R,Ᵽ∪Ꙅ) =(Ṅ,(Ꭻ∩(Ᵽ∪Ꙅ)), where for all α∈Ꭻ∩(Ᵽ∪Ꙅ), Ṅ(α)=Ꞙ(α)∩R(α). Thus,

Ṅ(α)= { Ꞙ(α)∩ℭ(α) α∈Ꭻ∩(Ᵽ\Ꙅ)=Ꭻ∩Ᵽ∩Ꙅ’ ; Ꞙ(α)∩Ꮋ(α) α∈Ꭻ∩(Ꙅ\Ᵽ)=Ꭻ∩Ᵽ’∩Ꙅ ; Ꞙ(α)∩[ℭ(α)∪Ꮋ(α)] α∈Ꭻ∩(Ᵽ∩Ꙅ)=Ꭻ∩Ᵽ∩Ꙅ

Now let’s handle the RHS. Let (Ꞙ,Ꭻ)$\cap_R$(ℭ,Ᵽ)=(K,Ꭻ∩Ᵽ), where for all α∈Ꭻ∩Ᵽ, K(α)=Ꞙ(α)∩ℭ(α). Let (Ꞙ,Ꭻ) $\cap_R$(Ꮋ,Ꙅ)=(S,Ꭻ∩Ꙅ), where for all α∈Ꭻ∩Ꙅ, S(α)=Ꞙ(α)∩Ꮋ(α). Let (K,Ꭻ∩Ᵽ) $\cup_\varepsilon$(S,Ꭻ∩Ꙅ)=(L,(Ꭻ∩Ᵽ)∪(Ꭻ∩Ꙅ)), where for all α∈(Ꭻ∩Ᵽ)∪(Ꭻ∩Ꙅ),

L(α)= { K(α) α∈(Ꭻ∩Ᵽ)\(Ꭻ∩Ꙅ)=Ꭻ∩(Ᵽ\Ꙅ) ; S(α) α∈(Ꭻ∩Ꙅ)\(Ꭻ∩Ᵽ)=Ꭻ∩(Ꙅ\Ᵽ) ; K(α)∪S(α) α∈(Ꭻ∩Ᵽ)∩(Ꭻ∩Ꙅ)=Ꭻ∩(Ᵽ∩Ꙅ)

Hence,

L(α)= { Ꞙ(α)∩ℭ(α) α∈Ꭻ∩Ᵽ∩Ꙅ’ ; Ꞙ(α)∩Ꮋ(α) α∈Ꭻ∩Ᵽ’∩Ꙅ ; [Ꞙ(α)∩ℭ(α)] ∪[Ꞙ(α)∩Ꮋ(α)] α∈Ꭻ∩Ᵽ∩Ꙅ

It is seen that (Ṅ,Ꭻ∩(Ᵽ∪Ꙅ))=(L,(Ꭻ∩Ᵽ)∪(Ꭻ∩Ꙅ)). Here, if Ꭻ∩Ᵽ=∅, then Ṅ(α)=L(α)=Ꞙ(α)∩Ꮋ(α), and if Ꭻ∩Ꙅ=∅, then Ṅ(α)=L(α)=Ꞙ(α)∩ℭ(α). That is, in these cases, the left-hand side and the right-hand side are still equal. Therefore, there is no need to impose the condition that these sets are non-empty.

ii) (Ꞙ,Ꭻ) $\cap_R$[(ℭ,Ᵽ) $\cap_\varepsilon$(Ꮋ,Ꙅ)]=[(Ꞙ,Ꭻ) $\cap_R$(ℭ,Ᵽ)] $\cap_\varepsilon$[(Ꞙ,Ꭻ) $\cap_R$ (Ꮋ,Ꙅ)] (Singh and Onyeozili, 2012c).

iii) (Ꞙ,Ꭻ) $\cap_R$[(ℭ,Ᵽ) $\setminus_\varepsilon$(Ꮋ,Ꙅ)]=[(Ꞙ,Ꭻ) $\cap_R$(ℭ,Ᵽ)]$\setminus_\varepsilon$[(Ꞙ,Ꭻ) $\cap_R$ (Ꮋ,Ꙅ)] (Sezgin et al., 2019).

iv) (Ꞙ,Ꭻ) $\cap_R$[(ℭ,Ᵽ) γ_ε(Ꮋ,Ꙅ)]=[(Ꞙ,Ꭻ) $\cap_R$(ℭ,Ᵽ)] γ_ε[(Ꞙ,Ꭻ) $\cap_R$ (Ꮋ,Ꙅ)].

v) (Ꞙ,Ꭻ) $\cap_R$[(ℭ,Ᵽ) Δ_ε(Ꮋ,Ꙅ)]=[(Ꞙ,Ꭻ) $\cap_R$(ℭ,Ᵽ)] Δ_ε[(Ꞙ,Ꭻ) $\cap_R$ (Ꮋ,Ꙅ)].

vi) (Ꞙ,Ꭻ) $\cap_R$[(ℭ,Ᵽ) $+_\varepsilon$ (Ꮋ,Ꙅ)] = [(Ꞙ,Ꭻ) $\cap_R$(ℭ,Ᵽ)] $+_\varepsilon$[(Ꞙ,Ꭻ) $\cap_R$ (Ꮋ,Ꙅ)], where Ꭻ∩Ᵽ∩Ꙅ=∅.

vii) (₣,Ɉ) $\cap_R$[(ℭ,Ᵽ) λ_ε (Ⱨ,Ƨ)] = [(₣,Ɉ) $\cap_R$(ℭ,Ᵽ)] λ_ε[(₣,Ɉ) $\cap_R$ (Ⱨ,Ƨ)], where Ɉ∩Ᵽ∩Ƨ=∅.

viii) (₣,Ɉ) $\cap_R$[(ℭ,Ᵽ) $*_\varepsilon$ (Ⱨ,Ƨ)] = [(₣,Ɉ) $\cap_R$(ℭ,Ᵽ)] $*_\varepsilon$[(₣,Ɉ) $\cap_R$ (Ⱨ,Ƨ)], where Ɉ∩Ᵽ∩Ƨ=∅.

ix) (₣,Ɉ) $\cap_R$[(ℭ,Ᵽ) θ_ε (Ⱨ,Ƨ)] = [(₣,Ɉ) $\cap_R$(ℭ,Ᵽ)] θ_ε[(₣,Ɉ) $\cap_R$ (Ⱨ,Ƨ)], where Ɉ∩Ᵽ∩Ƨ=∅.

b) RHS distributions of restricted intersection over extended SS operations:
Let (₣,Ɉ), (ℭ,Ᵽ), and (Ⱨ,Ƨ) be SSs over U. Then, we have the following distributions:

i)[(₣,Ɉ) $\cup_\varepsilon$ (ℭ,Ᵽ)] $\cap_R$ (Ⱨ,Ƨ) = [(₣,Ɉ) $\cap_R$ (Ⱨ,Ƨ)] $\cup_\varepsilon$ [(ℭ,Ᵽ) $\cap_R$ (Ⱨ,Ƨ)].

Proof: First, let's consider the LHS, and let (₣,Ɉ) $\cup_\varepsilon$(ℭ,Ᵽ)=(R,Ɉ∪Ᵽ), where for all α∈ Ɉ∪Ᵽ,

$$R(\alpha)=\begin{cases}\text{₣}(\alpha) & \alpha\in\text{Ɉ}\setminus\text{Ᵽ}\\ \text{ℭ}(\alpha) & \alpha\in\text{Ᵽ}\setminus\text{Ɉ}\\ \text{₣}(\alpha)\cup\text{ℭ}(\alpha) & \alpha\in\text{Ɉ}\cap\text{Ᵽ}\end{cases}$$

Let (R,Ɉ∪Ᵽ) $\cap_R$(Ⱨ,Ƨ) =(Ǹ,(Ɉ∪Ᵽ)∩Ƨ), where for all α∈(Ɉ∪Ᵽ)∩Ƨ, Ǹ(α)=R(α)∩Ⱨ(α). Thus,

$$\grave{N}(\alpha)=\begin{cases}\text{₣}(\alpha)\cap\text{Ⱨ}(\alpha) & \alpha\in(\text{Ɉ}\setminus\text{Ᵽ})\cap\text{Ƨ}=\text{Ɉ}\cap\text{Ᵽ}'\cap\text{Ƨ}\\ \text{ℭ}(\alpha)\cap\text{Ⱨ}(\alpha) & \alpha\in(\text{Ᵽ}\setminus\text{Ɉ})\cap\text{Ƨ}=\text{Ɉ}'\cap\text{Ᵽ}\cap\text{Ƨ}\\ [\text{₣}(\alpha)\cup\text{ℭ}(\alpha)]\cap\text{Ⱨ}(\alpha) & \alpha\in(\text{Ɉ}\cap\text{Ᵽ})\cap\text{Ƨ}=\text{Ɉ}\cap\text{Ᵽ}\cap\text{Ƨ}\end{cases}$$

Now let's handle the RHS. Let (₣,Ɉ)$\cap_R$(Ⱨ,Ƨ)=(K,Ɉ∩Ƨ), where for all α∈Ɉ∩Ƨ, K(α)=₣(α)∩Ⱨ(α). Let (ℭ,Ᵽ) $\cap_R$(Ⱨ,Ƨ)=(S,Ᵽ∩Ƨ), where for all α∈Ᵽ∩Ƨ, S(α)=ℭ(α)∩Ⱨ(α). Then, let (K,Ɉ∩Ƨ) $\cup_\varepsilon$(S,Ᵽ∩Ƨ)=(L,(Ɉ∩Ƨ)∪(Ᵽ∩Ƨ)), where for all α∈(Ɉ∩Ƨ)∪(Ᵽ∩Ƨ),

$$L(\alpha)=\begin{cases}K(\alpha) & \alpha\in(\text{Ɉ}\cap\text{Ƨ})\setminus(\text{Ᵽ}\cap\text{Ƨ})=(\text{Ɉ}\setminus\text{Ᵽ})\cap\text{Ƨ}\\ S(\alpha) & \alpha\in(\text{Ᵽ}\cap\text{Ƨ})\setminus(\text{Ɉ}\cap\text{Ƨ})=(\text{Ᵽ}\setminus\text{Ɉ})\cap\text{Ƨ}\\ K(\alpha)\cup S(\alpha) & \alpha\in(\text{Ɉ}\cap\text{Ƨ})\cap(\text{Ᵽ}\cap\text{Ƨ})=(\text{Ɉ}\cap\text{Ᵽ})\cap\text{Ƨ}\end{cases}$$

Hence,

$$L(\alpha)=\begin{cases}\text{₣}(\alpha)\cap\text{Ⱨ}(\alpha) & \alpha\in\text{Ɉ}\cap\text{Ᵽ}'\cap\text{Ƨ}\\ \text{ℭ}(\alpha)\cap\text{Ⱨ}(\alpha) & \alpha\in\text{Ɉ}'\cap\text{Ᵽ}\cap\text{Ƨ}\\ [\text{₣}(\alpha)\cap\text{Ⱨ}(\alpha)]\cup[\text{ℭ}(\alpha)\cap\text{Ⱨ}(\alpha)] & \alpha\in\text{Ɉ}\cap\text{Ᵽ}\cap\text{Ƨ}\end{cases}$$

It is seen that (Ǹ,(Ɉ∪Ᵽ)∩Ƨ)=(L,(Ɉ∩Ƨ)∪(Ᵽ∩Ƨ)). Here, if Ɉ∩Ᵽ=∅ and α∈Ɉ∩Ᵽ'∩Ƨ, then Ǹ(α)=L(α)=₣(α)∩Ⱨ(α); if Ɉ∩Ƨ=∅ and α∈Ɉ'∩Ᵽ∩Ƨ, then Ǹ(α)=L(α)=ℭ(α)∩Ⱨ(α). If Ᵽ∩Ƨ=∅, then

Ǹ(α)=L(α)=Ꞙ(α)∩Ⱨ(α). Since the right and left sides are equal in these cases, it is not necessary to impose the condition that these sets must be non-empty.

ii) $[(Ꞙ,Ꮭ) \cap_\varepsilon (ℭ,Ꝑ)] \cap_R (Ⱨ,Ꝣ)= [(Ꞙ,Ꮭ) \cap_R (Ⱨ,Ꝣ)] \cap_\varepsilon [(ℭ,Ꝑ) \cap_R (Ⱨ,Ꝣ)]$.

iii) $[(Ꞙ,Ꮭ) \setminus_\varepsilon (ℭ,Ꝑ)] \cap_R (Ⱨ,Ꝣ)=[(Ꞙ,Ꮭ) \cap_R (Ⱨ,Ꝣ)] \setminus_\varepsilon [(ℭ,Ꝑ) \cap_R (Ⱨ,Ꝣ)]$ (Sezgin et al, 2019).

iv) $[(Ꞙ,Ꮭ) \gamma_\varepsilon (ℭ,Ꝑ)] \cap_R (Ⱨ,Ꝣ)= [(Ꞙ,Ꮭ) \cap_R (Ⱨ,Ꝣ)] \gamma_\varepsilon [(ℭ,Ꝑ) \cap_R (Ⱨ,Ꝣ)]$.

v) $[(Ꞙ,Ꮭ) \Delta_\varepsilon (ℭ,Ꝑ)] \cap_R (Ⱨ,Ꝣ)= [(Ꞙ,Ꮭ) \cap_R (Ⱨ,Ꝣ)] \Delta_\varepsilon [(ℭ,Ꝑ) \cap_R (Ⱨ,Ꝣ)]$ (Sezgin and Çağman, 2025).

vi) $[(Ꞙ,Ꮭ) +_\varepsilon (ℭ,Ꝑ)] \cap_R (Ⱨ,Ꝣ)= [(Ꞙ,Ꮭ) \cap_R (Ⱨ,Ꝣ)] +_\varepsilon [(ℭ,Ꝑ) \cap_R (Ⱨ,Ꝣ)]$, where Ꮭ∩Ꝑ∩Ꝣ=∅.

vii) $[(Ꞙ,Ꮭ) \lambda_\varepsilon (ℭ,Ꝑ)] \cap_R (Ⱨ,Ꝣ)= [(Ꞙ,Ꮭ) \cap_R (Ⱨ,Ꝣ)] \lambda_\varepsilon [(ℭ,Ꝑ) \cap_R (Ⱨ,Ꝣ)]$, where Ꮭ∩Ꝑ∩Ꝣ=∅.

viii) $[(Ꞙ,Ꮭ) \theta_\varepsilon (ℭ,Ꝑ)] \cap_R (Ⱨ,Ꝣ)= [(Ꞙ,Ꮭ) \cap_R (Ⱨ,Ꝣ)] \theta_\varepsilon [(ℭ,Ꝑ) \cap_R (Ⱨ,Ꝣ)]$, where Ꮭ∩Ꝑ∩Ꝣ=∅.

ix) $[(Ꞙ,Ꮭ) *_\varepsilon (ℭ,Ꝑ)] \cap_R (Ⱨ,Ꝣ)= [(Ꞙ,Ꮭ) \cap_R (Ⱨ,Ꝣ)] *_\varepsilon [(ℭ,Ꝑ) \cap_R (Ⱨ,Ꝣ)]$, where Ꮭ∩Ꝑ∩Ꝣ=∅.

3.1.1.3. The distributions of the restricted intersection operation over soft binary piecewise operations:

In this subsection, the distributions of the restricted intersection operation to soft binary piecewise operations have been examined. First, left distributions were investigated, followed by right distributions. It is worth noting here that these distributions are satisfied even if the intersection of the parameter sets of the SSs involved in the restricted operations is the empty set.

a) LHS distributions of restricted intersection operation over soft binary piecewise operations:
Let (Ꞙ,Ꮭ), (ℭ,Ꝑ), and (Ⱨ,Ꝣ) be SSs over U. Then, we have the following distributions:

i) $(Ꞙ,Ꮭ) \cap_R [(ℭ,Ꝑ) \overset{\sim}{\cup} (Ⱨ,Ꝣ)] = [(Ꞙ,Ꮭ) \cap_R (ℭ,Ꝑ)] \overset{\sim}{\cup} [(Ꞙ,Ꮭ) \cap_R (Ⱨ,Ꝣ)]$.

Proof: First, let's consider the LHS, and let $(ℭ,Ꝑ) \overset{\sim}{\cup} (Ⱨ,Ꝣ)=(R,Ꝑ)$, where for all α∈Ꝑ,

$$R(\alpha)=\begin{cases} ℭ(\alpha) & \alpha\in Ꝑ\setminus Ꝣ \\ ℭ(\alpha)\cup Ⱨ(\alpha) & \alpha\in Ꝑ\cap Ꝣ \end{cases}$$

Let $(Ꞙ,Ꮭ) \cap_R (R,Ꝑ) =(Ǹ,Ꮭ\cap Ꝑ)$, where for all α∈Ꮭ∩Ꝑ, Ǹ(α)=Ꞙ(α)∩R(α). Hence,

$$Ṅ(\alpha)=\begin{cases} ɟ(\alpha)\cap ℭ(\alpha) & \alpha\in Ʝ\cap(Ꝑ\setminus Ƨ) \\ ɟ(\alpha)\cap [ℭ(\alpha)\cup Ή(\alpha)] & \alpha\in Ʝ\cap(Ꝑ\cap Ƨ) \end{cases}$$

for all $\alpha\in Ʝ\cap Ꝑ$. Now let's handle the RHS. Let $(ɟ,Ʝ)\cap_R(ℭ,Ꝑ)=(K,Ʝ\cap Ꝑ)$, where for all $\alpha\in Ʝ\cap Ꝑ$, $K(\alpha)=ɟ(\alpha)\cap ℭ(\alpha)$. Let $(ɟ,Ʝ)\cap_R(Ή,Ƨ)=(S,Ʝ\cap Ƨ)$, where for all $\alpha\in Ʝ\cap Ƨ$, $S(\alpha)=ɟ(\alpha)\cap Ή(\alpha)$. Assume that $(K,Ʝ\cap Ꝑ)\tilde{\cup}(S,Ʝ\cap Ƨ)=(L,Ʝ\cap Ꝑ)$, where for all $\alpha\in Ʝ\cap Ꝑ$,

$$L(\alpha)=\begin{cases} K(\alpha) & \alpha\in(Ʝ\cap Ꝑ)\setminus(Ʝ\cap Ƨ) \\ K(\alpha)\cup S(\alpha) & \alpha\in(Ʝ\cap Ꝑ)\cap(Ʝ\cap Ƨ) \end{cases}$$

Hence,

$$L(\alpha)=\begin{cases} ɟ(\alpha)\cap ℭ(\alpha) & \alpha\in(Ʝ\cap Ꝑ)\setminus(Ʝ\cap Ƨ)=Ʝ\cap(Ꝑ\setminus Ƨ) \\ [ɟ(\alpha)\cap ℭ(\alpha)]\cup[ɟ(\alpha)\cap Ή(\alpha)] & \alpha\in(Ʝ\cap Ꝑ)\cap(Ʝ\cap Ƨ)=Ʝ\cap(Ꝑ\cap Ƨ) \end{cases}$$

It is seen that $(Ṅ,Ʝ\cap Ꝑ)=(L,Ʝ\cap Ꝑ)$. Here, if $Ʝ\cap Ꝑ=\emptyset$, then $(Ṅ,Ʝ\cap Ꝑ)=(L,Ʝ\cap Ꝑ)=\emptyset_{\emptyset}$, and if $Ʝ\cap Ƨ=\emptyset$, then $Ṅ(\alpha)=L(\alpha)=ɟ(\alpha)\cap ℭ(\alpha)$. Since the right and left sides are equal in these cases, it is not necessary to impose the condition that these sets must be non-empty.

ii) $(ɟ,Ʝ)\cap_R[(ℭ,Ꝑ)\tilde{\cap}(Ή,Ƨ)]=[(ɟ,Ʝ)\cap_R(ℭ,Ꝑ)]\tilde{\cap}[(ɟ,Ʝ)\cap_R(Ή,Ƨ)]$.

iii) $(ɟ,Ʝ)\cap_R[(ℭ,Ꝑ)\tilde{\setminus}(Ή,Ƨ)]=[(ɟ,Ʝ)\cap_R(ℭ,Ꝑ)]\tilde{\setminus}[(ɟ,Ʝ)\cap_R(Ή,Ƨ)]$.

iv) $(ɟ,Ʝ)\cap_R[(ℭ,Ꝑ)\tilde{\gamma}(Ή,Ƨ)]=[(ɟ,Ʝ)\cap_R(ℭ,Ꝑ)]\tilde{\gamma}[(ɟ,Ʝ)\cap_R(Ή,Ƨ)]$.

v) $(ɟ,Ʝ)\cap_R[(ℭ,Ꝑ)\tilde{\Delta}(Ή,Ƨ)]=[(ɟ,Ʝ)\cap_R(ℭ,Ꝑ)]\tilde{\Delta}[(ɟ,Ʝ)\cap_R(Ή,Ƨ)]$.

vi) $(ɟ,Ʝ)\cap_R[(ℭ,Ꝑ)\tilde{+}(Ή,Ƨ)]=[(ɟ,Ʝ)\cap_R(ℭ,Ꝑ)]\tilde{+}[(ɟ,Ʝ)\cap_R(Ή,Ƨ)]$, where $Ʝ\cap Ꝑ\cap Ƨ=\emptyset$.

vii) $(ɟ,Ʝ)\cap_R[(ℭ,Ꝑ)\tilde{\lambda}(Ή,Ƨ)]=[(ɟ,Ʝ)\cap_R(ℭ,Ꝑ)]\tilde{\lambda}[(ɟ,Ʝ)\cap_R(Ή,Ƨ)]$, where $Ʝ\cap Ꝑ\cap Ƨ=\emptyset$.

viii) $(ɟ,Ʝ)\cap_R[(ℭ,Ꝑ)\tilde{\theta}(Ή,Ƨ)]=[(ɟ,Ʝ)\cap_R(ℭ,Ꝑ)]\tilde{\theta}[(ɟ,Ʝ)\cap_R(Ή,Ƨ)]$, where $Ʝ\cap Ꝑ\cap Ƨ=\emptyset$.

ix) $(ɟ,Ʝ)\cap_R[(ℭ,Ꝑ)\tilde{*}(Ή,Ƨ)]=[(ɟ,Ʝ)\cap_R(ℭ,Ꝑ)]\tilde{*}[(ɟ,Ʝ)\cap_R(Ή,Ƨ)]$, where $Ʝ\cap Ꝑ\cap Ƨ=\emptyset$.

b) RHS distributions of restricted intersection operation over soft binary piecewise operations:
Let (Ғ,Ɉ), (ℭ,₱), and (Ή,Ƨ) be SSs over U. Then, we have the following distributions:

i) $[(Ғ,Ɉ)\ \widetilde{\cup}\,(ℭ,₱)]\cap_R(Ή,Ƨ)=[(Ғ,Ɉ)\cap_R(Ή,Ƨ)]\ \widetilde{\cup}\ [(ℭ,₱)\cap_R(Ή,Ƨ)]$.

Proof: First, let's consider the LHS, and let $(Ғ,Ɉ)\ \widetilde{\cup}\ (ℭ,₱)=(R,Ɉ)$, where for all $\alpha\in Ɉ$,

$$R(\alpha)=\begin{cases} Ғ(\alpha) & \alpha\in Ɉ\backslash ₱ \\ Ғ(\alpha)\cup ℭ(\alpha) & \alpha\in Ɉ\cap ₱ \end{cases}$$

Let $(R,Ɉ)\cap_R(Ή,Ƨ)=(\dot{N},Ɉ\cap Ƨ)$, where for all $\alpha\in Ɉ\cap Ƨ$, $\dot{N}(\alpha)=R(\alpha)\cap Ή(\alpha)$. Thus,

$$\dot{N}(\alpha)=\begin{cases} Ғ(\alpha)\cap Ή(\alpha) & \alpha\in(Ɉ\backslash ₱)\cap Ƨ \\ [Ғ(\alpha)\cup ℭ(\alpha)]\cap Ή(\alpha) & \alpha\in(Ɉ\cap ₱)\cap Ƨ \end{cases}$$

Now let's handle the RHS. Let $(Ғ,Ɉ)\cap_R(Ή,Ƨ)=(K,Ɉ\cap Ƨ)$, where for all $\alpha\in Ɉ\cap Ƨ$, $K(\alpha)=Ғ(\alpha)\cap Ή(\alpha)$. Let $(ℭ,₱)\cap_R(Ή,Ƨ)=(S,₱\cap Ƨ)$, where for all $\alpha\in ₱\cap Ƨ$, $S(\alpha)=ℭ(\alpha)\cap Ή(\alpha)$. Let $(K,Ɉ\cap Ƨ)\ \widetilde{\cup}\,(S,₱\cap Ƨ)=(L,Ɉ\cap Ƨ)$, where for all $\alpha\in Ɉ\cap Ƨ$,

$$L(\alpha)=\begin{cases} K(\alpha) & \alpha\in(Ɉ\cap Ƨ)\backslash(₱\cap Ƨ) \\ K(\alpha)\cup S(\alpha) & \alpha\in(Ɉ\cap Ƨ)\cap(₱\cap Ƨ) \end{cases}$$

Thus,

$$L(\alpha)=\begin{cases} Ғ(\alpha)\cap Ή(\alpha) & \alpha\in(Ɉ\cap Ƨ)\backslash(₱\cap Ƨ)=(Ɉ\backslash ₱)\cap Ƨ \\ [Ғ(\alpha)\cap Ή(\alpha)]\cup[ℭ(\alpha)\cap Ή(\alpha)] & \alpha\in(Ɉ\cap Ƨ)\cap(₱\cap Ƨ)=(Ɉ\cap ₱)\cap Ƨ \end{cases}$$

It is seen that $(\dot{N},Ɉ\cap Ƨ)=(L,Ɉ\cap Ƨ)$. Here, if $Ɉ\cap Ƨ=\emptyset$, then $(\dot{N},Ɉ\cap Ƨ)=(L,Ɉ\cap Ƨ)=\emptyset_{\emptyset}$, and if $₱\cap Ƨ=\emptyset$, then $\dot{N}(\alpha)=L(\alpha)=Ғ(\alpha)\cap Ή(\alpha)$. Since the right and left sides are equal in these cases, it is not necessary to impose the condition that these sets must be non-empty.

ii) $[(Ғ,Ɉ)\ \widetilde{\cap}\,(ℭ,₱)]\cap_R(Ή,Ƨ)=[(Ғ,Ɉ)\cap_R(Ή,Ƨ)]\ \widetilde{\cap}\ [(ℭ,₱)\cap_R(Ή,Ƨ)]$.

iii) $[(Ғ,Ɉ)\ \widetilde{\backslash}\,(ℭ,₱)]\cap_R(Ή,Ƨ)=[(Ғ,Ɉ)\cap_R(Ή,Ƨ)]\ \widetilde{\backslash}\,[(ℭ,₱)\cap_R(Ή,Ƨ)]$.

iv) $[(\text{Ꞙ},\text{Ꞁ})\ \tilde{\gamma}\ (\mathfrak{C},\text{Ᵽ})]\cap_R(\text{Ꞁ},\text{Ƨ})= [(\text{Ꞙ},\text{Ꞁ})\cap_R(\text{Ꞁ},\text{Ƨ})]\ \tilde{\gamma}\ [(\mathfrak{C},\text{Ᵽ})\cap_R(\text{Ꞁ},\text{Ƨ})]$.

v) $[(\text{Ꞙ},\text{Ꞁ})\ \tilde{\Delta}\ (\mathfrak{C},\text{Ᵽ})]\cap_R(\text{Ꞁ},\text{Ƨ})= [(\text{Ꞙ},\text{Ꞁ})\cap_R(\text{Ꞁ},\text{Ƨ})]\ \tilde{\Delta}\ [(\mathfrak{C},\text{Ᵽ})\cap_R(\text{Ꞁ},\text{Ƨ})]$.

vi) $[(\text{Ꞙ},\text{Ꞁ})\ \tilde{+}\ (\mathfrak{C},\text{Ᵽ})]\cap_R(\text{Ꞁ},\text{Ƨ})= [(\text{Ꞙ},\text{Ꞁ})\cap_R(\text{Ꞁ},\text{Ƨ})]\ \tilde{+}\ [(\mathfrak{C},\text{Ᵽ})\cap_R(\text{Ꞁ},\text{Ƨ})]$, where $\text{Ꞁ}\cap\text{Ᵽ}\cap\text{Ƨ}=\emptyset$.

vii) $[(\text{Ꞙ},\text{Ꞁ})\ \tilde{\lambda}\ (\mathfrak{C},\text{Ᵽ})]\cap_R(\text{Ꞁ},\text{Ƨ})= [(\text{Ꞙ},\text{Ꞁ})\cap_R(\text{Ꞁ},\text{Ƨ})]\ \tilde{\lambda}\ [(\mathfrak{C},\text{Ᵽ})\cap_R(\text{Ꞁ},\text{Ƨ})]$, where $\text{Ꞁ}\cap\text{Ᵽ}\cap\text{Ƨ}=\emptyset$.

viii) $[(\text{Ꞙ},\text{Ꞁ})\ \tilde{\theta}\ (\mathfrak{C},\text{Ᵽ})]\cap_R(\text{Ꞁ},\text{Ƨ})= [(\text{Ꞙ},\text{Ꞁ})\cap_R(\text{Ꞁ},\text{Ƨ})]\ \tilde{\theta}\ [(\mathfrak{C},\text{Ᵽ})\cap_R(\text{Ꞁ},\text{Ƨ})]$, where $\text{Ꞁ}\cap\text{Ᵽ}\cap\text{Ƨ}=\emptyset$.

ix) $[(\text{Ꞙ},\text{Ꞁ})\ \tilde{*}\ (\mathfrak{C},\text{Ᵽ})]\cap_R(\text{Ꞁ},\text{Ƨ})= [(\text{Ꞙ},\text{Ꞁ})\cap_R(\text{Ꞁ},\text{Ƨ})]\ \tilde{*}\ [(\mathfrak{C},\text{Ᵽ})\cap_R(\text{Ꞁ},\text{Ƨ})]$, where $\text{Ꞁ}\cap\text{Ᵽ}\cap\text{Ƨ}=\emptyset$.

3.2. More on Extended Intersection Operation

To further clarify the conceptual expansion, this subsection is inspired by the extended union definition for SSs by Maji et al. (2003), a similar operation defined as the extended intersection operation of SSs by Ali et al. (2009) is examined in detail. Its properties similar to the intersection operation in classical sets, distributive rules, and relationships with other operations are thoroughly investigated. Since the extended intersection operation for SSs is not a new definition, some of its properties and its distributive rules have already been studied by various authors (Ali et al. (2009), Ali et al. (2011), Qin and Hong (2010), Sezgin and Atagün (2011)) However, in most studies, these properties have been presented without their proofs. From this perspective, we want to emphasize the importance of this study, as it includes all the properties of the extended intersection operation with their proofs, and provides detailed proofs of many new properties, especially those relating to their counterparts in classical set theory as regards intersection operation.

Definition 17 Let $(\text{Ꞙ},\text{Ꞁ})$ and $(\mathfrak{C},\text{Ᵽ})$ be SSs over U. The extended intersection $(\text{Ꞙ},\text{Ꞁ})$ and $(\mathfrak{C},\text{Ᵽ})$ is the SS (Ꞁ,C) denoted by $(\text{Ꞙ},\text{Ꞁ})\cap_\varepsilon(\mathfrak{C},\text{Ᵽ}) = (\text{Ꞁ},\text{C})$, where $\text{C}=\text{Ꞁ}\cup\text{Ᵽ}$ and for all $\alpha\in\text{C}$,

$$\text{Ꞁ}(\alpha)=\begin{cases}\text{Ꞙ}(\alpha) & \alpha\in\text{Ꞁ}\setminus\text{Ᵽ}\\ \mathfrak{C}(\alpha) & \alpha\in\text{Ᵽ}\setminus\text{Ꞁ}\\ \text{Ꞙ}(\alpha)\cap\mathfrak{C}(\alpha) & \alpha\in\text{Ꞁ}\cap\text{Ᵽ}\end{cases}$$

(Ali et al., 2009).

Here note that the letter "ε" written below the symbol " ∩" which represents the extended intersection operation, forms a meaningful and consistent whole with its English meaning "extended". In other studies on SS operations, extended SS operations are also represented in this form. From the definition, it is obvious that if $\mathcal{J}=\emptyset$, then $(Ғ,\mathcal{J})\cap_{\varepsilon}(\mathfrak{C},Ᵽ)=(\mathfrak{C},Ᵽ)$; if $Ᵽ=\emptyset$, then $(Ғ,\mathcal{J})\cap_{\varepsilon}(\mathfrak{C},Ᵽ)=(Ғ,\mathcal{J})$, and if $\mathcal{J}=Ᵽ=\emptyset$, then $(Ғ,\mathcal{J})\cap_{\varepsilon}(\mathfrak{C},Ᵽ)=\emptyset_{\emptyset}$.

Example 2 Let $E=\{e_1,e_2,e_3,e_4\}$ be the parameter set $\mathcal{J}=\{e_1,e_3\}$ and $Ᵽ=\{e_2,e_3,e_4\}$ be the subsets of E and $U=\{h_1,h_2,h_3,h_4,h_5\}$ be the initial universe set. Assume that $(Ғ,\mathcal{J})$ and $(\mathfrak{C},Ᵽ)$ are the SSs over U defined as follows: $(Ғ,\mathcal{J})=\{(e_1,\{h_2,h_5\}), (e_3,\{h_1,h_2,h_5\})\}$, $(\mathfrak{C},Ᵽ)=\{(e_2,\{h_1,h_4,h_5\}), (e_3,\{h_2,h_3,h_4\}),(e_4,\{h_3,h_5\})\}$. Let $(Ғ,\mathcal{J})\cap_{\varepsilon}(\mathfrak{C},Ᵽ)=(Ԋ,\mathcal{J}\cup Ᵽ)$, where for all $\alpha\in\mathcal{J}\cup Ᵽ$,

$$Ԋ(\alpha)=\begin{cases} Ғ(\alpha) & \alpha\in\mathcal{J}\setminus Ᵽ \\ \mathfrak{C}(\alpha) & \alpha\in Ᵽ\setminus\mathcal{J} \\ Ғ(\alpha)\cap\mathfrak{C}(\alpha) & \alpha\in\mathcal{J}\cap Ᵽ \end{cases}$$

Since $\mathcal{J}\cup Ᵽ=\{e_1,e_2,e_3,e_4\}$, $\mathcal{J}\setminus Ᵽ=\{e_1\}$, $Ᵽ\setminus\mathcal{J}=\{e_2,e_4\}$, and $\mathcal{J}\cap Ᵽ=\{e_3\}$, $Ԋ(e_1)=Ғ(e_1)=\{h_2,h_5\}$, $Ԋ(e_2)=\mathfrak{C}(e_2)=\{h_1,h_4,h_5\}$, $Ԋ(e_4)=\mathfrak{C}(e_4)=\{h_3,h_5\}$, $Ԋ(e_3)=Ғ(e_3)\cap\mathfrak{C}(e_3)=\{h_1,h_2,h_5\}\cap\{h_2,h_3,h_4\}=\{h_2\}$. Thus,

$$(Ғ,\mathcal{J})\cap_{\varepsilon}(\mathfrak{C},Ᵽ)=\{(e_1,\{h_2,h_5\}),(e_2,\{h_1,h_4,h_5\}),(e_3,\{h_2\}),(e_4,\{h_3,h_5\})\}.$$

Note 1 The restricted intersection and extended intersection operations in $S_{Ꞩ}(U)$ are coincident, where Ꞩ is a fixed subset of E. That is, $(Ғ,Ꞩ)\cap_{\varepsilon}(\mathfrak{C},Ꞩ)=(Ғ,Ꞩ)\cap_{R}(\mathfrak{C},Ꞩ)$.

Proposition 21 The set $S_E(U)$ is closed under the operation $\cap_{\varepsilon}$. That is, when $(Ғ,\mathcal{J})$ and $(\mathfrak{C},Ᵽ)$ are two SSs over U, then so is $(Ғ,\mathcal{J})\cap_{\varepsilon}(\mathfrak{C},Ᵽ)$.

Proof: It is clear that $\cap_{\varepsilon}$ is a binary operation in $S_E(U)$. That is,

$$\cap_{\varepsilon}: S_E(U)\times S_E(U)\rightarrow S_E(U)$$
$$((Ғ,\mathcal{J}),(\mathfrak{C},Ᵽ))\rightarrow(Ғ,\mathcal{J})\cap_{\varepsilon}(\mathfrak{C},Ᵽ)=(Ԋ,\mathcal{J}\cup Ᵽ)$$

Hence, the set $S_E(U)$ is closed under $\cap_{\varepsilon}$. Similarly,

$$\cap_{\varepsilon}: S_{\mathcal{J}}(U)\times S_{\mathcal{J}}(U)\rightarrow S_{\mathcal{J}}(U)$$
$$((Ғ,\mathcal{J}),(\mathfrak{C},\mathcal{J}))\rightarrow(Ғ,\mathcal{J})\cap_{\varepsilon}(\mathfrak{C},\mathcal{J})=(K,\mathcal{J}\cup\mathcal{J})=(K,\mathcal{J})$$

Let $(Ғ,\mathcal{J})$ and $(\mathfrak{C},\mathcal{J})$ be elements of the set $S_T(U)$, where $\mathcal{J}$ is a fixed subset of the set E. Then, $(Ғ,\mathcal{J})\cap_{\varepsilon}(\mathfrak{C},\mathcal{J})$ is an element of the set $S_{\mathcal{J}}(U)$. That is, the operation $\cap_{\varepsilon}$ is also closed in $S_T(U)$.

Proposition 22 Let $(Ғ,\mathcal{J})$, $(\mathfrak{C},Ᵽ)$ and $(Ԋ,Ꙅ)$ be SSs over U. Then, $[(Ғ,\mathcal{J})\cap_{\varepsilon}(\mathfrak{C},Ᵽ)]\cap_{\varepsilon}(Ԋ,Ꙅ)=(Ғ,\mathcal{J})\cap_{\varepsilon}[(\mathfrak{C},Ᵽ)\cap_{\varepsilon}(Ԋ,Ꙅ)]$ (Qin and Hong, 2010).

Proof: Qin and Hong (2010) presented this property without proof in their study; however, we provide it with its detailed proof. First, let's consider the LHS, and let $(Ғ,\mathcal{J})\cap_{\varepsilon}(\mathfrak{C},Ᵽ)=(S,\mathcal{J}\cup Ᵽ)$, where for all $\alpha\in\mathcal{J}\cup Ᵽ$,

$$S(\alpha)=\begin{cases} Ғ(\alpha) & \alpha\in\mathcal{J}\setminus Ᵽ \\ \mathfrak{C}(\alpha) & \alpha\in Ᵽ\setminus\mathcal{J} \\ Ғ(\alpha)\cap\mathfrak{C}(\alpha) & \alpha\in\mathcal{J}\cap Ᵽ \end{cases}$$

Let (S,ᎫUᵽ) $\cap_{\varepsilon}$ (Ɦ,Ƨ) =(Ṅ,(ᎫUᵽ)UƧ)), where for all α∈(ᎫUᵽ)UƧ,

Ṅ(α)=
- S(α) α∈(ᎫUᵽ)\Ƨ
- Ɦ(α) α∈Ƨ\(ᎫUᵽ)
- S(α)∩Ɦ(α) α∈(ᎫUᵽ)∩Ƨ

Thus,

Ṅ(α)=
- Ꞙ(α) α∈(Ꭻ\Ᵽ)\Ƨ=Ꭻ∩Ᵽ'∩Ƨ'
- ℭ(α) α∈(Ᵽ\Ꭻ)\Ƨ=Ꭻ'∩Ᵽ∩Ƨ'
- Ꞙ(α)∩ℭ(α) α∈(Ꭻ∩Ᵽ)\Ƨ=Ꭻ∩Ᵽ∩Ƨ'
- Ɦ(α) α∈Ƨ\(ᎫUᵽ)=Ꭻ'∩Ᵽ'∩Ƨ
- Ꞙ(α)∩H (α) α∈(Ꭻ\Ᵽ)∩Ƨ=Ꭻ∩Ᵽ'∩Ƨ
- ℭ(α)∩H (α) α∈(Ᵽ\Ꭻ)∩Ƨ=Ꭻ'∩Ᵽ∩Ƨ
- [Ꞙ(α)∩ℭ(α)]∩H (α) α∈(Ꭻ∩Ᵽ)∩Ƨ=Ꭻ∩Ᵽ∩Ƨ

Now let's handle the RHS, and let (ℭ,Ᵽ)$\cap_{\varepsilon}$(Ɦ,Ƨ)=(R,ᵽUƧ), where for all α∈ᵽUƧ,

R(α)=
- ℭ(α) α∈Ᵽ\Ƨ
- Ɦ(α) α∈Ƨ\Ᵽ
- ℭ(α)∩H (α) α∈Ᵽ∩Ƨ

Let (Ꞙ,Ꭻ)$\cap_{\varepsilon}$(R,ᵽUƧ) =(L,ᎫU(ᵽUƧ)), where for all α∈ᎫUᵽUƧ,

L(α)=
- Ꞙ(α) α∈Ꭻ\(ᵽUƧ)
- R(α) α∈(ᵽUƧ)\Ꭻ
- Ꞙ(α)∩R(α) α∈Ꭻ∩(ᵽUƧ)

Thus,

L(α)=
- Ꞙ(α) α∈Ꭻ\(ᵽUƧ)=Ꭻ∩Ᵽ'∩Ƨ'
- ℭ(α) α∈(Ᵽ\Ƨ)\Ꭻ=Ꭻ'∩Ᵽ∩Ƨ'
- Ɦ(α) α∈(Ƨ\Ᵽ)\Ꭻ=Ꭻ'∩Ᵽ'∩Ƨ
- ℭ(α)∩Ɦ(α) α∈(Ᵽ∩Ƨ)\Ꭻ=Ꭻ'∩Ᵽ∩Ƨ
- Ꞙ(α)∩ℭ(α) α∈Ꭻ∩(Ᵽ\Ƨ)=Ꭻ∩Ᵽ∩Ƨ'
- Ꞙ(α)∩H (α) α∈Ꭻ∩(Ƨ\Ᵽ)=Ꭻ∩Ᵽ'∩Ƨ
- Ꞙ(α)∩ [ℭ(α)∩Ɦ(α)] α∈Ꭻ∩(Ᵽ∩Ƨ)=Ꭻ∩Ᵽ∩Ƨ

It is seen that (Ṅ,(ᎫUᵽ)UƧ)=(L,ᎫU(ᵽUƧ)). That is, $\cap_{\varepsilon}$ is associative in S_E(U).

Proposition 23 Let (Ꞙ,Ɉ), (ℭ,Ɉ) and (Ꟶ,Ɉ) be SSs over U. Then, [(Ꞙ,Ɉ) $\cap_{\varepsilon}$(ℭ,Ɉ)] $\cap_{\varepsilon}$ (Ꟶ,Ɉ) = (Ꞙ,Ɉ) $\cap_{\varepsilon}$ [(ℭ,Ɉ) $\cap_{\varepsilon}$(Ꟶ,Ɉ)].

Proof: The proof follows from Note 1 and Proposition 3. That is, $\cap_{\varepsilon}$ is associative in $S_{Ɉ}$(U), where Ɉ is a fixed subset of E.

Proposition 24 Let (Ꞙ,Ɉ) and (ℭ,Ᵽ) be SSs over U. Then, (Ꞙ,Ɉ) $\cap_{\varepsilon}$(ℭ,Ᵽ)=(ℭ,Ᵽ) $\cap_{\varepsilon}$ (Ꞙ,Ɉ) (Qin and Hong, 2010)

Proof: Qin and Hong (2010) presented this property without proof in their study; however, we provide it with its detailed proof. Let (Ꞙ,Ɉ) $\cap_{\varepsilon}$ (ℭ,Ᵽ)=(Ꟶ,Ɉ∪Ᵽ), where for all α∈Ɉ∪Ᵽ,

$$Ꟶ(\alpha)=\begin{cases} Ꞙ(\alpha) & \alpha\in Ɉ\setminus Ᵽ \\ ℭ(\alpha) & \alpha\in Ᵽ\setminus Ɉ \\ Ꞙ(\alpha)\cap ℭ(\alpha) & \alpha\in Ɉ\cap Ᵽ \end{cases}$$

Let (ℭ,Ᵽ)$\cap_{\varepsilon}$(Ꞙ,Ɉ)=(S,Ᵽ∪Ɉ), where for all α∈Ᵽ∪Ɉ,

$$S(\alpha)=\begin{cases} ℭ(\alpha) & \alpha\in Ᵽ\setminus Ɉ \\ Ꞙ(\alpha) & \alpha\in Ɉ\setminus Ᵽ \\ ℭ(\alpha)\cap Ꞙ(\alpha) & \alpha\in Ᵽ\cap Ɉ \end{cases}$$

Thus, (Ꞙ,Ɉ) $\cap_{\varepsilon}$(ℭ,Ᵽ)=(ℭ,Ᵽ) $\cap_{\varepsilon}$(Ꞙ,Ɉ). Moreover, it is evident that (Ꞙ,Ɉ) $\cap_{\varepsilon}$(ℭ,Ɉ)=(ℭ,Ɉ)$\cap_{\varepsilon}$(Ꞙ,Ɉ). That is, $\cap_{\varepsilon}$ is commutative in both S_E(U) and S_T(U).

Proposition 25 Let (Ꞙ,Ɉ) be an SS over U. Then, (Ꞙ,Ɉ) $\cap_{\varepsilon}$(Ꞙ,Ɉ)=(Ꞙ,Ɉ) (Qin and Hong, 2010)

Proof: Qin and Hong (2010) presented this property without proof in their study; however, we provide it with its detailed proof. The proof is obtained from Note 1 and Proposition 5. That is, $\cap_{\varepsilon}$ is idempotent in S_E(U).

Proposition 26 Let (Ꞙ,Ɉ) be an SS over U. Then, (Ꞙ,Ɉ) $\cap_{\varepsilon}$ $U_{Ɉ}$=$U_{Ɉ}$ $\cap_{\varepsilon}$(Ꞙ,Ɉ)=(Ꞙ,Ɉ).

Proof: The proof is obtained from Note 1 and Proposition 6. That is, $U_{Ɉ}$ is the identity element of $\cap_{\varepsilon}$ in $S_{Ɉ}$(U).

Theorem 3 ($S_{Ɉ}$(U), $\cap_{\varepsilon}$) is a bounded semi-lattice, whose identity is $U_{Ɉ}$.

Proof: By Proposition 21, Proposition 23, Proposition 24, Proposition 25, and Proposition 26, ($S_{Ɉ}$(U),$\cap_{\varepsilon}$) is a commutative, idempotent monoid whose identity is $U_{Ɉ}$, that is, a bounded semi-lattice.

Proposition 27 Let (Ꞙ,Ꭻ) be an SS over U. Then, (Ꞙ,Ꭻ)$\cap_\varepsilon$ $\emptyset_\emptyset$=(Ꞙ,Ꭻ) (Ali et al., 2011).

Proof: Ali et al. (2011) presented this property without proof in their study; however, we provide it with its detailed proof. Let $\emptyset_\emptyset$=(S,∅) and (Ꞙ,Ꭻ) $\cap_\varepsilon$(S, ∅)=(Ҥ,Ꭻ∪∅), where for all α∈Ꭻ∪∅=Ꭻ,

$$\text{Ҥ}(\alpha)=\begin{cases} \text{Ꞙ}(\alpha) & \alpha\in \text{Ꭻ}\setminus\emptyset=\text{Ꭻ} \\ S(\alpha) & \alpha\in\emptyset\setminus \text{Ꭻ}=\emptyset \\ \text{Ꞙ}(\alpha)\cap S(\alpha) & \alpha\in \text{Ꭻ}\cap\emptyset=\emptyset \end{cases}$$

Thus, Ҥ(α)=Ꞙ(α), for all α∈Ꭻ, and (Ҥ,Ꭻ)=(Ꞙ,Ꭻ).

Proposition 28 Let (Ꞙ,Ꭻ) be an SS over U. Then, $\emptyset_\emptyset$ $\cap_\varepsilon$(Ꞙ,Ꭻ)=(Ꞙ,Ꭻ) (Ali et al., 2011).

Proof: Ali et al. (2011) presented this property without proof in their study; however, we provide it with its detailed proof. Let $\emptyset_\emptyset$ =(S,∅) and (S,∅) $\cap_\varepsilon$(Ꞙ,Ꭻ)=(Ҥ, ∅∪Ꭻ), where for all α∈∅∪Ꭻ= Ꭻ,

$$\text{Ҥ}(\alpha)=\begin{cases} S(\alpha) & \alpha\in\emptyset\setminus \text{Ꭻ}=\emptyset \\ \text{Ꞙ}(\alpha) & \alpha\in \text{Ꭻ}\setminus\emptyset=\text{Ꭻ} \\ S(\alpha)\cap \text{Ꞙ}(\alpha) & \alpha\in\emptyset\cap \text{Ꭻ}=\emptyset \end{cases}$$

Thus, Ҥ(α)=Ꞙ(α), for all α∈Ꭻ, and (Ҥ,Ꭻ)=(Ꞙ,Ꭻ).

By Proposition 27 and Proposition 28, the identity element of $\cap_\varepsilon$ is the SS $\emptyset_\emptyset$ in S_E(U).

In classical set theory, it is well-known that A∪B=∅ if and only if A=∅ and B=∅. By this fact, there does not exist (ℭ, K) ∈ S_E(U) such that (Ꞙ,Ꭻ)$\cap_\varepsilon$ (ℭ, K)= (ℭ,K) $\cap_\varepsilon$ (Ꞙ,Ꭻ)= $\emptyset_\emptyset$, as this requires Ꭻ ∪K=∅, and so Ꭻ =∅ and K=∅. Since $\emptyset_\emptyset$ is the only SS with an empty parameter in S_E(U), there is not any element in S_E(U), except the identity element $\emptyset_\emptyset$, which has an inverse with respect to $\cap_\varepsilon$. Of course, the inverse of $\emptyset_\emptyset$, which is the identity element, is itself, as usual.

Proposition 29 Let (Ꞙ,Ꭻ) be an SS over U. Then, $\emptyset_E$ $\cap_\varepsilon$(Ꞙ,Ꭻ)= (Ꞙ,Ꭻ) $\cap_\varepsilon$ $\emptyset_E$ =$\emptyset_E$.

Proof: Let $\emptyset_E$ =(S,E) ve (S,E) $\cap_\varepsilon$(Ꞙ,Ꭻ)=(Ҥ, E∪Ꭻ=E), where for all α∈E∪Ꭻ=E,

$$\text{Ҥ}(\alpha)=\begin{cases} S(\alpha), & \alpha\in E\setminus \text{Ꭻ}=\text{Ꭻ}' \\ \text{Ꞙ}(\alpha), & \alpha\in \text{Ꭻ}\setminus E=\emptyset \\ S(\alpha)\cap \text{Ꞙ}(\alpha), & \alpha\in E\cap \text{Ꭻ}=\text{Ꭻ} \end{cases}$$

Hence, for all α∈E∪Ꭻ=E,

$$\begin{cases} \emptyset, & \alpha\in E\setminus \text{Ꭻ}=\text{Ꭻ}' \end{cases}$$

Ⱨ(α)= Ꞙ(α) α∈Ɉ\E=∅
∅, α∈E∩Ɉ=Ɉ

Thus, Ⱨ(α)=∅, for all α∈ E, and (Ⱨ,Ɉ)= $\emptyset_E$. That is, $\emptyset_E$ is the absorbing element of $\cap_\varepsilon$ in $S_E(U)$.

Theorem 4 ($S_E(U)$, $\cap_\varepsilon$) is a bounded semi-lattice, whose identity is $\emptyset_\emptyset$ and the absorbing element is $\emptyset_E$.

Proof: By Proposition 21, Proposition 22, Proposition 24, Proposition 25, Proposition 27, Proposition 28, and Proposition 29, ($S_E(U)$,$\cap_\varepsilon$) is a commutative, idempotent monoid whose identity is $\emptyset_\emptyset$, that is, a bounded semi-lattice with the absorbing element is $\emptyset_E$.

Proposition 30 Let (Ꞙ,Ɉ) be an SS over U. Then, (Ꞙ,Ɉ) $\cap_\varepsilon$(Ꞙ,Ɉ)^r= (Ꞙ,Ɉ)^r$\cap_\varepsilon$ (Ꞙ,Ɉ)= $\emptyset_Ɉ$ (Sezgin and Atagün, 2011).

Proof: Sezgin and Atagün (2011) presented this property without proof; however, we give it here with proof. The proof is obtained from Note 1 and Proposition 11.

Ali et al. (2009) used the negative complement defined by Maji et al. (2003) for De Morgan's laws for extended intersection and extended union. On the other hand, Qin and Hong (2010) adopted the more commonly used relative complement defined by Ali et al. (2009) for the complement operation and provided De Morgan's laws accordingly. Their proofs were element-based in Qin and Hong (2010), while we present a simpler proof using function equality as follows:

Proposition 31 Let (Ꞙ,Ɉ) and (ℭ,Ᵽ) be SSs over U. Then, [(Ꞙ,Ɉ) $\cap_\varepsilon$(ℭ,Ᵽ)]^r=(Ꞙ,Ɉ)^r $\cup_\varepsilon$(ℭ,Ᵽ)^r (De Morgan Law) (Qin and Hong, 2010)

Proof: Let (Ꞙ,Ɉ)$\cap_\varepsilon$(ℭ,Ᵽ)=(Ⱨ,Ɉ∪Ᵽ), where for all α∈Ɉ∪Ᵽ,

Ⱨ(α)= { Ꞙ(α) α∈Ɉ\Ᵽ
ℭ(α) α∈Ᵽ\Ɉ
Ꞙ(α)∩ℭ(α) α∈Ɉ∩Ᵽ

Let (Ⱨ,Ɉ∪Ᵽ)^r =(K,Ɉ∪Ᵽ), where for all α∈Ɉ∪Ᵽ,

K(α)= { Ꞙ'(α) α∈Ɉ\Ᵽ
G'(α) α∈Ᵽ\Ɉ
Ꞙ'(α)∪G'(α) α∈Ɉ∩Ᵽ

Hence, (K,Ɉ∪Ᵽ)=(Ꞙ,Ɉ)^r $\cup_\varepsilon$(ℭ,Ᵽ)^r.

Proposition 32 Let (Ꞙ,Ɉ) and (ℭ,Ᵽ) be SSs over U. Then, (Ꞙ,Ɉ) $\cap_\varepsilon$ (ℭ,Ᵽ)= $U_{Ɉ\cup Ᵽ}$ if and only if (Ꞙ,Ɉ)= $U_Ɉ$ and (ℭ,Ᵽ) = $U_Ᵽ$.

Proof: Let $(Ғ, Ꭻ) \cap_{\varepsilon} (\mathfrak{C}, ℘) = (Ԋ, Ꭻ \cup ℘)$,

$$Ԋ(\alpha)=\begin{cases} Ғ(\alpha), & \alpha \in Ꭻ \setminus ℘ \\ \mathfrak{C}(\alpha), & \alpha \in ℘ \setminus Ꭻ \\ Ғ(\alpha) \cap \mathfrak{C}(\alpha), & \alpha \in Ꭻ \cap ℘ \end{cases}$$

Since $(Ԋ, Ꭻ \cup ℘) = U_{Ꭻ \cup ℘}$, $Ԋ(\alpha) = U$, for all $\alpha \in Ꭻ \cup ℘$. Hence, if $\alpha \in Ꭻ \setminus ℘$, then $Ғ(\alpha)=U$, if $\alpha \in ℘ \setminus Ꭻ$, then $\mathfrak{C}(\alpha)=U$, and if $\alpha \in Ꭻ \cap ℘$, then $Ғ(\alpha) \cap \mathfrak{C}(\alpha)=U$, implying that $Ғ(\alpha)=\mathfrak{C}(\alpha)=U$. Thus, $Ғ(\alpha)=U$, for all $\alpha \in Ꭻ$, and $\mathfrak{C}(\alpha)=U$, for all $\alpha \in ℘$. Hence, $(Ғ, Ꭻ) = U_{Ꭻ}$ and $(\mathfrak{C}, ℘) = U_{℘}$.

Conversely let $(Ғ, Ꭻ) = U_{Ꭻ}$ and $(\mathfrak{C}, ℘) = U_{℘}$. Then, $Ғ(\alpha)=U$, for all $\alpha \in Ꭻ$, and $\mathfrak{C}(\alpha)=U$, for all $\alpha \in ℘$. Then,

$$Ԋ(\alpha)=\begin{cases} U, & \alpha \in Ꭻ \setminus ℘ \\ U, & \alpha \in ℘ \setminus Ꭻ \\ U \cap U, & \alpha \in Ꭻ \cap ℘ \end{cases}$$

for all $\alpha \in Ꭻ \cup ℘$. Therefore, $(H, Ꭻ \cup ℘) = (Ғ, Ꭻ) \cap_{\varepsilon} (\mathfrak{C}, ℘) = U_{Ꭻ \cup ℘}$.

Proposition 33 Let $(Ғ, Ꭻ)$ and $(\mathfrak{C}, ℘)$ be SSs over U. Then, $\emptyset_{Ꭻ} \tilde{\subseteq} (Ғ, Ꭻ) \cap_{\varepsilon} (\mathfrak{C}, ℘)$, $\emptyset_{℘} \tilde{\subseteq} (Ғ, Ꭻ) \cap_{\varepsilon} (\mathfrak{C}, ℘)$. Also, $(Ғ, Ꭻ) \cap_{\varepsilon} (\mathfrak{C}, ℘) \tilde{\subseteq} U_{Ꭻ \cup ℘}$.

Proof: The proof is obvious since the empty set is a subset of every set and the universal set includes every set.

Proposition 34 Let $(Ғ, Ꭻ)$ and $(\mathfrak{C}, Ꭻ)$ be SSs over U. Then, $(Ғ, Ꭻ) \cap_{\varepsilon} (\mathfrak{C}, Ꭻ) \tilde{\subseteq} (Ғ, Ꭻ)$ and $(Ғ, Ꭻ) \cap_{\varepsilon} (\mathfrak{C}, Ꭻ) \tilde{\subseteq} (\mathfrak{C}, Ꭻ)$.

Proof: The proof is obtained from Note 1 and Proposition 15.

Proposition 35 Let $(Ғ, Ꭻ)$ and $(\mathfrak{C}, Ꭻ)$ be SSs over U. $(Ғ, Ꭻ) \tilde{\subseteq} (\mathfrak{C}, Ꭻ)$ if and only if $(Ғ, Ꭻ) \cap_{\varepsilon} (\mathfrak{C}, Ꭻ) = (Ғ, Ꭻ)$**.**

Proof: The proof is obtained from Note 1 and Proposition 16.

Proposition 36 Let $(Ғ, Ꭻ)$, $(\mathfrak{C}, Ꭻ)$, and (K, V) be SSs over U. If $(Ғ, Ꭻ) \tilde{\subseteq} (\mathfrak{C}, Ꭻ)$, then $(Ғ, Ꭻ) \cap_{\varepsilon} (K, V) \tilde{\subseteq} (\mathfrak{C}, Ꭻ) \cap_{\varepsilon} (K, V)$. However, the converse is not true.

Proof: Let $(Ғ, Ꭻ) \tilde{\subseteq} (\mathfrak{C}, Ꭻ)$. Hence, $Ғ(\alpha) \subseteq \mathfrak{C}(\alpha)$, for all $\alpha \in Ꭻ$. Let $(Ғ, Ꭻ) \cap_{\varepsilon} (K, V) = (Ԋ, Ꭻ \cup V)$, where for all $\alpha \in Ꭻ U V$,

$$\begin{cases} Ғ(\alpha) & \alpha \in Ꭻ \setminus V \end{cases}$$

$$Ħ(\alpha)= \begin{cases} K(\alpha) & \alpha\in V\setminus J \\ Ꞙ(\alpha)\cap K(\alpha) & \alpha\in J\cap V \end{cases}$$

Let $(\mathfrak{C},J)\cap_\varepsilon(K,V)=(S, J \cup V)$, where for all $\alpha\in J\cup V$,

$$S(\alpha)= \begin{cases} \mathfrak{C}(\alpha) & \alpha\in J\setminus V \\ K(\alpha) & \alpha\in V\setminus J \\ \mathfrak{C}(\alpha)\cap K(\alpha) & \alpha\in J\cap V \end{cases}$$

If $\alpha\in J\setminus V$, then $H(\alpha)=Ꞙ(\alpha)\subseteq Ꞙ(\alpha)=S(\alpha)$, if $\alpha\in V\setminus J$, then $H(\alpha)=K(\alpha)\subseteq K(\alpha)=S(\alpha)$, if $\alpha\in J\cap V$, then $H(\alpha)=Ꞙ(\alpha)\cap K(\alpha)\subseteq \mathfrak{C}(\alpha)\cap K(\alpha)= S(\alpha)$. Thus, $Ħ(\alpha)\subseteq S(\alpha)$, for all $\alpha\in J\cup V$, implying that $(Ꞙ,J)\cap_\varepsilon(K,V)\,\tilde{\subseteq}(\mathfrak{C},J)\cap_\varepsilon(K,V)$.

Let's give an example to show that the converse is not true. Let $E=\{e_1,e_2,e_3,e_4,e_5\}$ be the parameter set, $J=\{e_1,e_3\}$, $V=\{e_1,e_3,e_5\}$ be the subsets of E and, $U=\{h_1,h_2,h_3,h_4,h_5\}$ be the universel set. Assume that $(Ꞙ,J)$, $(\mathfrak{C},J)$ and (K,V) are the SSs over U defined as follows: $(Ꞙ,J)=\{(e_1,\{h_2,h_5\}),(e_3,\{h_1,h_2,h_5\})\}$,$(\mathfrak{C},J)=\{(e_1,\{h_2\}),(e_3,\{h_1,h_2\})\}$,$(K,V)=\{e_1,\emptyset),(e_3,\emptyset),(e_5,\{h_1,h_5\})\}$.

It is obvious that $(Ꞙ,J)\cap_\varepsilon K,V)=\{(e_1,\emptyset),(e_3,\emptyset),(e_5,\{h_1,h_5\})\}$ and $(\mathfrak{C},J)\cap_\varepsilon(K,V)=\{(e_1,\emptyset),(e_3,\emptyset),(e_5,\{h_1,h_5\})\}$. Hence, $(Ꞙ,J)\cap_\varepsilon(K,V)\tilde{\subseteq}(\mathfrak{C},J)\cap_\varepsilon(K,V)$; however, $(Ꞙ,J)$ is not soft subset of $(\mathfrak{C},J)$.

Proposition 37 Let $(Ꞙ,J)$, $(\mathfrak{C},J)$, (K,V), and (L,V) be SSs over U. If $(Ꞙ,J)\,\tilde{\subseteq}\,(\mathfrak{C},J)$ and $(K,V)\,\tilde{\subseteq}\,(L,V)$, then $(Ꞙ,J)\cap_\varepsilon(K,V)\,\tilde{\subseteq}\,(\mathfrak{C},J)\cap_\varepsilon(L,V)$.

Proof: Let $(Ꞙ,J)\,\tilde{\subseteq}\,(\mathfrak{C},J)$ and $(K,V)\,\tilde{\subseteq}\,(L,V)$. Hence, $Ꞙ(\alpha)\subseteq\mathfrak{C}(\alpha)$, for all $\alpha\in J$ and $K(\alpha)\subseteq L(\alpha)$, and for all $\alpha\in V$. Let $(Ꞙ,J)\cap_\varepsilon(K,V)=(Ħ, J\cup V)$, where for all $\alpha\in J\cup V$,

$$Ħ(\alpha)= \begin{cases} Ꞙ(\alpha) & \alpha\in J\setminus V \\ K(\alpha) & \alpha\in V\setminus J \\ Ꞙ(\alpha)\cap K(\alpha) & \alpha\in J\cap V \end{cases}$$

Let $(\mathfrak{C},J)\cap_\varepsilon(L,V)=(S, J\cup V)$, where for all $\alpha\in J\cup V$,

$$S(\alpha)= \begin{cases} \mathfrak{C}(\alpha) & \alpha\in J\setminus V \\ L(\alpha) & \alpha\in V\setminus J \\ \mathfrak{C}(\alpha)\cap L(\alpha) & \alpha\in J\cap V \end{cases}$$

If $\alpha\in J\setminus V$, then $H(\alpha)=Ꞙ(\alpha)\subseteq\mathfrak{C}(\alpha)=S(\alpha)$, if $\alpha\in V\setminus J$, then $H(\alpha)=K(\alpha)\subseteq L(\alpha)=S(\alpha)$, and if $\alpha\in\in J\cap V$, $H(\alpha)=Ꞙ(\alpha)\cap K(\alpha)\subseteq\mathfrak{C}(\alpha)\cap L(\alpha)= S(\alpha)$. Thus, $H(\alpha)\subseteq S(\alpha)$, for all $\alpha\in J\cup V$, implying that $(Ꞙ,J)\cap_\varepsilon(K,V)\,\tilde{\subseteq}(\mathfrak{C},J)\cap_\varepsilon(L,V)$.

Proposition 38 Let (Ꞙ,Ꞙ) and (ℭ,Ꞙ) be SSs over U. Then, $(Ꞙ,Ꞙ) \mathrel{\widetilde{\subseteq}} (ℭ,Ꞙ)^r$ if and only if $(Ꞙ,Ꞙ) \cap_\varepsilon (ℭ,Ꞙ) = \emptyset_{Ꞙ}$.

Proof: The proof is obtained from Note 1 and Proposition 20.

3.2.1. The distributions of the extended intersection operation over other SS operations:

In this subsection, the distributions of the extended intersection operation over restricted SS operations, extended operations, and soft binary partition operations have been examined.

3.2.1.1. The distributions of the extended intersection operation over restricted SS operations:

Here, the distributions of the extended intersection operation to restricted operations have been examined. First, distributions from the left, followed by distributions from the right, have been investigated. It is worth noting an important point here. In the study by Ali et al. (2011), the distributions of the extended intersection operation over the restricted union from the left were examined without proof, and demonstrated with an example that the extended intersection operation does not satisfy the property of distributions over the restricted intersection from the left. Singh and Onyeozili (2012c) showed that the extended intersection operation does not satisfy the property of distributions over the restricted difference from the left. Sezgin and Atagün (2011), although showed that the extended intersection operation distributes to the restricted union from both the right and the left, they overlooked some points in their proof. In this study, the distributive properties are presented with detailed proofs, considering the cases where the intersection of the parameter sets of the SSs involved in restricted operations is empty as well. Additionally, for those that do not satisfy the distributive property, the conditions under which they do satisfy the distributive property are also provided with detailed proofs.

a) LHS distributions of extended intersection over restricted SS operations:
Let (Ꞙ,Ꞙ), (ℭ,Ᵽ), and (Ƕ,Ƨ) be SSs over U. Then, we have the following distributions:

i) $(Ꞙ,Ꞙ) \cap_\varepsilon[(ℭ,Ᵽ) \cup_R (Ƕ,Ƨ)]=[(Ꞙ,Ꞙ) \cap_\varepsilon (ℭ,Ᵽ)]\cup_R[(Ꞙ,Ꞙ) \cap_\varepsilon(Ƕ,Ƨ)]$ (Sezgin and Atagün, 2011).

Proof: In the proof by Sezgin and Atagün (2011), a case in the parameter partitioning on the right-hand side was overlooked. Moreover, the proof emphasized that the intersection of the parameter sets of the SSs involved in restricted operations must be non-empty. However, even if the intersection of the parameter sets of the SSs involved in restricted operations is empty, this distributive property still holds. Therefore, in our proof, these cases are specifically considered and addressed by taking these situations into consideration.

First, let's consider the LHS, and let $(ℭ,Ᵽ) \cup_R(Ƕ,Ƨ)=(S,Ᵽ\cap Ƨ)$. Hence, for all $\alpha\in Ᵽ\cap Ƨ$, $S(\alpha)=ℭ(\alpha)\cup Ƕ(\alpha)$. Let $(Ꞙ,Ꞙ)\cap_\varepsilon(S,Ᵽ\cap Ƨ)=(Ṅ,Ꞙ\cup(Ᵽ\cap Ƨ))$, where for all $\alpha\in Ꞙ\cup(Ᵽ\cap Ƨ)$,

$$Ṅ(\alpha)=\begin{cases} Ꞙ(\alpha) & \alpha\in Ꞙ\setminus(Ᵽ\cap Ƨ) \\ S(\alpha) & \alpha\in(Ᵽ\cap Ƨ)\setminus Ꞙ \\ Ꞙ(\alpha)\cap S(\alpha) & \alpha\in Ꞙ\cap(Ᵽ\cap Ƨ) \end{cases}$$

Hence,

$$\text{Ǹ}(\alpha)=\begin{cases}\text{Ꞙ}(\alpha) & \alpha\in\text{Ɉ}\setminus(\text{Ᵽ}\cap\text{Ƨ})\\ \mathfrak{C}(\alpha)\cup\text{Ή}(\alpha) & \alpha\in(\text{Ᵽ}\cap\text{Ƨ})\setminus\text{Ɉ}=\text{Ɉ}'\cap\text{Ᵽ}\cap\text{Ƨ}\\ \text{Ꞙ}(\alpha)\cap[\mathfrak{C}(\alpha)\cup\text{Ή}(\alpha)] & \alpha\in\text{Ɉ}\cap(\text{Ᵽ}\cap\text{Ƨ})=\text{Ɉ}\cap\text{Ᵽ}\cap\text{Ƨ}\end{cases}$$

Now let's handle the RHS, and let $(\text{Ꞙ},\text{Ɉ})\cap_{\varepsilon}(\mathfrak{C},\text{Ᵽ})=(W,\text{Ɉ}\cup\text{Ᵽ})$, where for all $\alpha\in\text{Ɉ}\cup\text{Ᵽ}$,

$$W(\alpha)=\begin{cases}\text{Ꞙ}(\alpha) & \alpha\in\text{Ɉ}\setminus\text{Ᵽ}\\ \mathfrak{C}(\alpha) & \alpha\in\text{Ᵽ}\setminus\text{Ɉ}\\ \text{Ꞙ}(\alpha)\cap\mathfrak{C}(\alpha) & \alpha\in\text{Ɉ}\cap\text{Ᵽ}\end{cases}$$

Let $(\text{Ꞙ},\text{Ɉ})\cap_{\varepsilon}(\text{Ή},\text{Ƨ})=(K,\text{Ɉ}\cup\text{Ƨ})$, where for all $\alpha\in\text{Ɉ}\cup\text{Ƨ}$,

$$K(\alpha)=\begin{cases}\text{Ꞙ}(\alpha) & \alpha\in\text{Ɉ}\setminus\text{Ƨ}\\ \text{Ή}(\alpha) & \alpha\in\text{Ƨ}\setminus\text{Ɉ}\\ \text{Ꞙ}(\alpha)\cap\text{Ή}(\alpha) & \alpha\in\text{Ɉ}\cap\text{Ƨ}\end{cases}$$

Let $(W,\text{Ɉ}\cup\text{Ᵽ})\cup_{R}(K,\text{Ɉ}\cup\text{Ƨ})=(Y,(\text{Ɉ}\cup\text{Ᵽ})\cap(\text{Ɉ}\cup\text{Ƨ}))$, where for all $\alpha\in(\text{Ɉ}\cup\text{Ᵽ})\cap(\text{Ɉ}\cup\text{Ƨ})$, $Y(\alpha)=W(\alpha)\cup K(\alpha)$. Thereby,

$$Y(\alpha)=\begin{cases}\text{Ꞙ}(\alpha)\cup\text{Ꞙ}(\alpha) & \alpha\in(\text{Ɉ}\setminus\text{Ᵽ})\cap(\text{Ɉ}\setminus\text{Ƨ})=\text{Ɉ}\cap\text{Ᵽ}'\cap\text{Ƨ}'\\ \text{Ꞙ}(\alpha)\cup\text{Ή}(\alpha) & \alpha\in(\text{Ɉ}\setminus\text{Ᵽ})\cap(\text{Ƨ}\setminus\text{Ɉ})=\emptyset\\ \text{Ꞙ}(\alpha)\cup[\text{Ꞙ}(\alpha)\cap\text{Ή}(\alpha)] & \alpha\in(\text{Ɉ}\setminus\text{Ᵽ})\cap(\text{Ɉ}\cap\text{Ƨ})=\text{Ɉ}\cap\text{Ᵽ}'\cap\text{Ƨ}\\ \mathfrak{C}(\alpha)\cup\text{Ꞙ}(\alpha) & \alpha\in(\text{Ᵽ}\setminus\text{Ɉ})\cap(\text{Ɉ}\setminus\text{Ƨ})=\emptyset\\ \mathfrak{C}(\alpha)\cup\text{Ή}(\alpha) & \alpha\in(\text{Ᵽ}\setminus\text{Ɉ})\cap(\text{Ƨ}\setminus\text{Ɉ})=\text{Ɉ}'\cap\text{Ᵽ}\cap\text{Ƨ}\\ \mathfrak{C}(\alpha)\cup[\text{Ꞙ}(\alpha)\cap\text{Ή}(\alpha)] & \alpha\in(\text{Ᵽ}\setminus\text{Ɉ})\cap(\text{Ɉ}\cap\text{Ƨ})=\emptyset\\ [\text{Ꞙ}(\alpha)\cap\mathfrak{C}(\alpha)]\cup\text{Ꞙ}(\alpha) & \alpha\in(\text{Ɉ}\cap\text{Ᵽ})\cap(\text{Ɉ}\setminus\text{Ƨ})=\text{Ɉ}\cap\text{Ᵽ}\cap\text{Ƨ}'\\ [\text{Ꞙ}(\alpha)\cap\mathfrak{C}(\alpha)]\cup\text{Ή}(\alpha) & \alpha\in(\text{Ɉ}\cap\text{Ᵽ})\cap(\text{Ƨ}\setminus\text{Ɉ})=\emptyset\\ [\text{Ꞙ}(\alpha)\cap\mathfrak{C}(\alpha)]\cup[\text{Ꞙ}(\alpha)\cap\text{Ή}(\alpha)] & \alpha\in(\text{Ɉ}\cap\text{Ᵽ})\cap(\text{Ɉ}\cap\text{Ƨ})=\text{Ɉ}\cap\text{Ᵽ}\cap\text{Ƨ}\end{cases}$$

Thus,

$$Y(\alpha)=\begin{cases}\text{Ꞙ}(\alpha) & \alpha\in\text{Ɉ}\cap\text{Ᵽ}'\cap\text{Ƨ}'\\ \text{Ꞙ}(\alpha) & \alpha\in\text{Ɉ}\cap\text{Ᵽ}'\cap\text{Ƨ}\\ \mathfrak{C}(\alpha)\cup\text{Ή}(\alpha) & \alpha\in\text{Ɉ}'\cap\text{Ᵽ}\cap\text{Ƨ}\\ \text{Ꞙ}(\alpha) & \alpha\in\text{Ɉ}\cap\text{Ᵽ}\cap\text{Ƨ}'\\ \text{Ꞙ}(\alpha)\cap[\mathfrak{C}(\alpha)\cup\text{Ή}(\alpha)] & \alpha\in\text{Ɉ}\cap\text{Ᵽ}\cap\text{Ƨ}\end{cases}$$

Here, let's consider Ʝ\(Ᵽ∩Ƨ) in the function N. Since Ʝ\(Ᵽ∩Ƨ)=Ʝ∩(Ᵽ∩Ƨ)' and if an element is in the complement of (Ᵽ∩Ƨ), it is either in Ᵽ\Ƨ, in Ƨ\Ᵽ or in the complement of Ᵽ∪Ƨ, thus if α∈Ʝ\(Ᵽ∩Ƨ), then α∈Ʝ∩Ᵽ∩Ƨ' or α∈Ʝ∩Ᵽ'∩Ƨ or α∈Ʝ∩Ᵽ'∩Ƨ'. Therefore, Ǹ=Y.

Here, if Ᵽ∩Ƨ=∅, then Ǹ(α)=W(α)=Ꞙ(α), and thus equality is satisfied again. Similarly, when (Ʝ∪Ꝓ)∩(Ʝ∪Ƨ)=Ʝ∪(Ᵽ∩Ƨ)=∅, i.e. Ʝ=∅ and Ᵽ∩Ƨ=∅, then (Ǹ,Ʝ∪(Ᵽ∩Ƨ))=(Y,(Ʝ∪Ꝓ)∩(Ʝ∪Ƨ))=∅$_\emptyset$. Therefore, there is no need to require these sets to be different from the empty set.

ii) (Ꞙ,Ʝ)∩$_\varepsilon$ [(ℭ,Ꝓ) ∩$_R$(Ⱨ,Ƨ)]≠[(Ꞙ,Ʝ)∩$_\varepsilon$ (ℭ,Ꝓ)]∩$_R$[(Ꞙ,Ʝ) ∩$_\varepsilon$(Ⱨ,Ƨ)] (Ali et al., 2011), however
(Ꞙ,Ʝ) ∩$_\varepsilon$[(ℭ,Ꝓ) ∩$_R$ (Ⱨ,Ƨ)]=[(Ꞙ,Ʝ) ∩$_\varepsilon$ (ℭ,Ꝓ)]∩$_R$[(Ꞙ,Ʝ) ∩$_\varepsilon$(Ⱨ,Ƨ)], where Ʝ∩(ꝒΔ Ƨ)=∅.

Proof: Since the proof of the left distributive property of the extended intersection over the restricted intersection is very similar to (i), it is not repeated here. However, it is worth mentioning the following point. In classical sets, the intersection operation is left-distributive over the union operation. However, for extended intersection and restricted union operations, this situation does not hold, as shown by Ali et al. (2011) with a counter-example. In this study, we show that the distributivity can be achieved under the condition Ʝ∩(ꝒΔ Ƨ)=∅.

In classical sets, the intersection operation is left-distributive over both the difference and the symmetric difference operations. However, for extended intersection and restricted difference and restricted symmetric difference operations, this situation does not hold, as shown below:

iii) (Ꞙ,Ʝ)∩$_\varepsilon$[(ℭ,Ꝓ)\$_R$(Ⱨ,Ƨ)]≠[(Ꞙ,Ʝ)∩$_\varepsilon$ (ℭ,Ꝓ)]\$_R$[(Ꞙ,Ʝ)∩$_\varepsilon$(Ⱨ,Ƨ)] (Singh and Onyeozili, 2012c).

iv) (Ꞙ,Ʝ)∩$_\varepsilon$[(ℭ,Ꝓ)Δ$_R$(Ⱨ,Ƨ)]≠[(Ꞙ,Ʝ)∩$_\varepsilon$ (ℭ,Ꝓ)]Δ$_R$[(Ꞙ,Ʝ)∩$_\varepsilon$(Ⱨ,Ƨ)].

b) RHS distributions of extended intersection over restricted SS operations:
Let (Ꞙ,Ʝ), (ℭ,Ᵽ), and (Ⱨ,Ƨ) be SSs over U. Then, we have the following distributions:

i) [(Ꞙ,Ʝ) ∪$_R$(ℭ,Ᵽ)]∩$_\varepsilon$(Ⱨ,Ƨ)= [(Ꞙ,Ʝ)∩$_\varepsilon$(Ⱨ,Ƨ)] ∪$_R$ [(ℭ,Ᵽ) ∩$_\varepsilon$ (Ⱨ,Ƨ)] (Sezgin and Atagün, 2011).

Proof: Sezgin and Atagün (2011) presented this property without proof in their study; however, we provide it with its detailed proof. First, let's consider the LHS and let (Ꞙ,Ʝ) ∪$_R$(ℭ,Ᵽ)=(R,Ʝ∩Ᵽ), where for all α∈Ʝ∩Ᵽ, R(α)=Ꞙ(α)∪ℭ(α). Let (R,Ʝ∩Ᵽ) ∩$_\varepsilon$(Ⱨ,Ƨ)=(L,(Ʝ∩Ᵽ)∪Ƨ), where for all α∈(Ʝ∩Ᵽ)∪Ƨ,

L(α)=
- R(α)   α∈(Ʝ∩Ᵽ)\Ƨ
- Ⱨ(α)   α∈Ƨ\(Ʝ∩Ᵽ)
- R(α)∩Ⱨ(α)   α∈(Ʝ∩Ᵽ)∩Ƨ

Hence,

- Ꞙ(α)∪ℭ(α)   α∈(Ʝ∩Ᵽ)\Ƨ=Ʝ∩Ᵽ∩Ƨ'

L(α)= Ꟶ(α) α∈Ꝣ\(Ʝ∩Ᵽ)
[Ꞙ(α)∪ℭ(α)]∩Ꟶ(α) α∈(Ʝ∩Ᵽ)∩Ꝣ=Ʝ∩Ᵽ∩Ꝣ

Now let's handle the RHS, and let (Ꞙ,Ʝ) $\cap_{\varepsilon}$ (Ꟶ,Ꝣ)=(S,Ʝ∪Ꝣ), where for all α∈Ʝ∪Ꝣ,

S(α)=
- Ꞙ(α) α∈Ʝ\Ꝣ
- Ꟶ(α) α∈Ꝣ\Ʝ
- Ꞙ(α)∩Ꟶ(α) α∈Ʝ∩Ꝣ

Let (ℭ,Ᵽ) $\cap_{\varepsilon}$ (Ꟶ,Ꝣ)=(K,Ᵽ∪Ꝣ), where for all α∈Ᵽ∪Ꝣ,

K(α)=
- ℭ(α) α∈Ᵽ\Ꝣ
- Ꟶ(α) α∈Ꝣ\Ᵽ
- ℭ(α)∩Ꟶ(α) α∈Ᵽ∩Ꝣ

Let (S,Ʝ∪Ꝣ) $\cup_{R}$(K,Ᵽ∪Ꝣ)=(W,(Ʝ∪Ꝣ)∩(Ᵽ∪Ꝣ)), where for all α∈(Ʝ∪Ꝣ)∩(Ᵽ∪Ꝣ), W(α)=S(α)∪K(α). Thus,

W(α)=
- Ꞙ(α)∪ℭ(α) α∈(Ʝ\Ꝣ)∩(Ᵽ\Ꝣ)=Ʝ∩Ᵽ∩Ꝣ'
- Ꞙ(α)∪Ꟶ(α) α∈(Ʝ\Ꝣ)∩(Ꝣ\Ᵽ)=∅
- Ꞙ(α)∪[ℭ(α)∩Ꟶ(α)] α∈(Ʝ\Ꝣ)∩(Ᵽ∩Ꝣ)=∅
- Ꟶ(α)∪ℭ(α) α∈(Ꝣ\Ʝ)∩(Ᵽ\Ꝣ)= ∅
- Ꟶ(α)∪Ꟶ(α) α∈(Ꝣ\Ʝ)∩(Ꝣ\Ᵽ)=Ʝ'∩Ᵽ'∩Ꝣ
- Ꟶ(α)∪[ℭ(α)∩Ꟶ(α)] α∈(Ꝣ\Ʝ)∩(Ᵽ∩Ꝣ)=Ʝ'∩Ᵽ∩Ꝣ
- [Ꞙ(α)∩Ꟶ(α)]∪ℭ(α) α∈(Ʝ∩Ꝣ)∩(Ᵽ\Ꝣ)=∅
- [Ꞙ(α)∩Ꟶ(α)]∪Ꟶ(α) α∈(Ʝ∩Ꝣ)∩(Ꝣ\Ᵽ)=Ʝ∩Ᵽ'∩Ꝣ
- [Ꞙ(α)∩Ꟶ(α)]∪[ℭ(α)∩Ꟶ(α)] α∈(Ʝ∩Ꝣ)∩(Ᵽ∩Ꝣ)=Ʝ∩Ᵽ∩Ꝣ

Thus,

W(α)=
- Ꞙ(α)∪ℭ(α) α∈Ʝ∩Ᵽ∩Ꝣ'
- Ꟶ(α) α∈Ʝ'∩Ᵽ'∩Ꝣ
- Ꟶ(α) α∈Ʝ'∩Ᵽ∩Ꝣ
- Ꟶ(α) α∈Ʝ∩Ᵽ'∩Ꝣ
- [Ꞙ(α)∩Ꟶ(α)]∪[ℭ(α)∩Ꟶ(α)] α∈Ʝ∩Ᵽ∩Ꝣ

Here, let's consider Ꝣ\(Ʝ∩Ᵽ) in L. Since Ꝣ\(Ʝ∩Ᵽ)=Ꝣ∩(Ʝ∩Ᵽ)', if an element is in the complement of (Ʝ∩Ᵽ), it is either in Ʝ\Ᵽ , either in Ᵽ\Ʝ or in the complement of Ʝ∪ Z. Thus, if α∈Ꝣ\(Ʝ∩Ᵽ), then α∈Ꝣ∩Ʝ∩Ᵽ' or α∈ Ꝣ∩Ᵽ∩Ʝ' or α∈ Ꝣ∩Ʝ'∩Ᵽ'. Hence, L=W.

Here, if $Ꭻ∩Ꮴ=\emptyset$, then the equality will still be satisfied, since L(α)=W(α)=Ħ(α). Similarly, if (Ꭻ∪Ꞇ)∩(Ꮴ∪Ꞇ)=(Ꭻ∩Ꮴ)∪Ꞇ=∅, that is, Ꭻ∩Ꮴ=∅ and Ꞇ=∅, then (L,(Ꭻ∩Ꮴ)∪Ꞇ)=(W,(Ꭻ∪Ꞇ)∩(Ꮴ∪Ꞇ))= $\emptyset_{\emptyset}$. That is, in the theorem, there is no need to require these sets to be different from the empty.

ii) [(Ꞙ,Ꭻ) $\cap_R$(ℭ,Ꮴ)]$\cap_\varepsilon$(Ħ,Ꞇ)=[(Ꞙ,Ꭻ)$\cap_\varepsilon$(Ħ,Ꞇ)]$\cap_R$ [(ℭ,Ꮴ) $\cap_\varepsilon$ (Ħ,Ꞇ)], where (ᎫΔᏤ)∩Ꞇ=∅.

Proof: Since the proof of the right distributive property of the extended intersection over the restricted intersection is very similar to (i), it is not repeated here. However, it is worth mentioning the following point. In classical sets, the intersection operation is right-distributive over the intersection operation. However, for extended and restricted operations, this situation does not hold, but we state that the distributivity can be satisfied under the condition of (ᎫΔᏤ)∩Ꞇ=∅.

Similarly, in classical sets, the intersection operation is right-distributive over both the difference and the symmetric difference operations. However, for extended intersection and restricted difference and restricted symmetric difference operations, this situation does not hold, as given below:

iii) [(Ꞙ,Ꭻ) $\setminus_R$(ℭ,Ꮴ)]$\cap_\varepsilon$(Ħ, Ꞇ) ≠[(Ꞙ,Ꭻ)$\cap_\varepsilon$(Ħ,Ꞇ)]$\setminus_R$ [(ℭ,Ꮴ) $\cap_\varepsilon$ (Ħ,Ꞇ)].

iv) [(Ꞙ,Ꭻ) Δ_R(ℭ,Ꮴ)]$\cap_\varepsilon$(Ħ, Ꞇ) ≠[(Ꞙ,Ꭻ)$\cap_\varepsilon$(Ħ,Ꞇ)]Δ_R [(ℭ,Ꮴ) $\cap_\varepsilon$ (Ħ,Ꞇ)].

3.2.1.2. The distributions of the extended intersection operation over other extended SS operations:

Here, the distributive properties of the extended intersection operation over other extended operations are examined. First, left distributivity is considered, followed by right distributivity. It is important to note the following: In the study by Ali et al. (2011), the left distributive property of the extended intersection over the extended intersection was considered without proof, and it was shown with an example that the extended intersection does not satisfy the right distributive property over the extended union. In this study, these distributive properties are presented with detailed proofs. For those that do not satisfy the distributive property unconditionally, the conditions under which they do satisfy the distributive property are also proven.

a) LHS distributions of extended intersection over other extended SS operations:

Let (Ꞙ,Ꭻ), (ℭ,Ꮴ), and (Ħ,Ꞇ) be SSs over U. Then, we have the following distributions:

i) (Ꞙ,Ꭻ) $\cap_\varepsilon$[(ℭ,Ꮴ) $\cap_\varepsilon$(Ħ,Ꞇ)]=[(Ꞙ,Ꭻ) $\cap_\varepsilon$(ℭ,Ꮴ)]$\cap_\varepsilon$[(Ꞙ,Ꭻ)$\cap_\varepsilon$ (Ħ,Ꞇ)] (Ali et al., 2011).

Proof: Ali et al. (2011) presented this property without proof in their study; however, we provide it with its detailed proof. First, let's consider the LHS, and let (ℭ,Ꮴ) $\cap_\varepsilon$(Ħ,Ꞇ)=(R,Ꮴ∪Ꞇ), where for all α∈Ꮴ∪Ꞇ,

$$R(\alpha)=\begin{cases} ℭ(\alpha) & \alpha\in Ꮴ\setminus Ꞇ \\ Ħ(\alpha) & \alpha\in Ꞇ\setminus Ꮴ \\ ℭ(\alpha)\cap Ħ(\alpha) & \alpha\in Ꮴ\cap Ꞇ \end{cases}$$

Let (Ꞙ,Ʝ) $\cap_{\varepsilon}$(R,Ᵽ∪Ƨ) =(Ṅ,(Ʝ∪(Ᵽ∪Ƨ)), where for all α∈Ʝ∪(Ᵽ∪Ƨ),

$$\text{Ṅ}(\alpha)=\begin{cases} \text{Ꞙ}(\alpha) & \alpha\in \text{Ʝ}\setminus(\text{Ᵽ}\cup\text{Ƨ}) \\ R(\alpha) & \alpha\in(\text{Ᵽ}\cup\text{Ƨ})\setminus\text{Ʝ} \\ \text{Ꞙ}(\alpha)\cap R(\alpha) & \alpha\in\text{Ʝ}\cap(\text{Ᵽ}\cup\text{Ƨ}) \end{cases}$$

Hence,

$$\text{Ṅ}(\alpha)=\begin{cases} \text{Ꞙ}(\alpha) & \alpha\in\text{Ʝ}\setminus(\text{Ᵽ}\cup\text{Ƨ})=\text{Ʝ}\cap\text{Ᵽ}'\cap\text{Ƨ}' \\ \mathfrak{C}(\alpha) & \alpha\in(\text{Ᵽ}\setminus\text{Ƨ})\setminus\text{Ʝ}=\text{Ʝ}'\cap\text{Ᵽ}\cap\text{Ƨ}' \\ \text{Ⱨ}(\alpha) & \alpha\in(\text{Ƨ}\setminus\text{Ᵽ})\setminus\text{Ʝ}=\text{Ʝ}'\cap\text{Ᵽ}'\cap\text{Ƨ} \\ \mathfrak{C}(\alpha)\cap\text{Ⱨ}(\alpha) & \alpha\in(\text{Ᵽ}\cap\text{Ƨ})\setminus\text{Ʝ}=\text{Ʝ}'\cap\text{Ᵽ}\cap\text{Ƨ} \\ \text{Ꞙ}(\alpha)\cap\mathfrak{C}(\alpha) & \alpha\in\text{Ʝ}\cap(\text{Ᵽ}\setminus\text{Ƨ})=\text{Ʝ}\cap\text{Ᵽ}\cap\text{Ƨ}' \\ \text{Ꞙ}(\alpha)\cap\text{Ⱨ}(\alpha) & \alpha\in\text{Ʝ}\cap(\text{Ƨ}\setminus\text{Ᵽ})=\text{Ʝ}\cap\text{Ᵽ}'\cap\text{Ƨ} \\ \text{Ꞙ}(\alpha)\cap[\mathfrak{C}(\alpha)\cap\text{Ⱨ}(\alpha)] & \alpha\in\text{Ʝ}\cap(\text{Ᵽ}\cap\text{Ƨ})=\text{Ʝ}\cap\text{Ᵽ}\cap\text{Ƨ} \end{cases}$$

Now let's handle the RHS and let (Ꞙ,Ʝ) $\cap_{\varepsilon}$ ($\mathfrak{C}$,Ᵽ)=(K,Ʝ∪Ᵽ), where for all α∈Ʝ∪Ᵽ,

$$K(\alpha)=\begin{cases} \text{Ꞙ}(\alpha) & \alpha\in\text{Ʝ}\setminus\text{Ᵽ} \\ \mathfrak{C}(\alpha) & \alpha\in\text{Ᵽ}\setminus\text{Ʝ} \\ \text{Ꞙ}(\alpha)\cap\mathfrak{C}(\alpha) & \alpha\in\text{Ʝ}\cap\text{Ᵽ} \end{cases}$$

Let (Ꞙ,Ʝ) $\cap_{\varepsilon}$(Ⱨ,Ƨ)=(S,Ʝ∪Ƨ), where for all α∈Ʝ∪Ƨ,

$$S(\alpha)=\begin{cases} \text{Ꞙ}(\alpha) & \alpha\in\text{Ʝ}\setminus\text{Ƨ} \\ \text{Ⱨ}(\alpha) & \alpha\in\text{Ƨ}\setminus\text{Ʝ} \\ \text{Ꞙ}(\alpha)\cap\text{Ⱨ}(\alpha) & \alpha\in\text{Ʝ}\cap\text{Ƨ} \end{cases}$$

Let (K,Ʝ∪Ᵽ) $\cap_{\varepsilon}$(S,Ʝ∪Ƨ)=(L,(Ʝ∪Ᵽ)∪(Ʝ∪Ƨ)), where for all α∈(Ʝ∪Ᵽ)∪(Ʝ∪Ƨ),

$$L(\alpha)=\begin{cases} K(\alpha) & \alpha\in(\text{Ʝ}\cup\text{Ᵽ})\setminus(\text{Ʝ}\cup\text{Ƨ}) \\ S(\alpha) & \alpha\in(\text{Ʝ}\cup\text{Ƨ})\setminus(\text{Ʝ}\cup\text{Ᵽ}) \\ K(\alpha)\cap S(\alpha) & \alpha\in(\text{Ʝ}\cup\text{Ᵽ})\cap(\text{Ʝ}\cup\text{Ƨ}) \end{cases}$$

Hence,

L(α)=

Ꞙ(α)	α∈(Ʝ\Ᵽ)\(Ʝ∪Ƨ)=∅
ℭ(α)	α∈(Ᵽ\Ʝ)\(Ʝ∪Ƨ)=Ʝ’∩Ᵽ∩Ƨ’
Ꞙ(α)∩ℭ(α)	α∈(Ʝ∩Ᵽ)\(Ʝ∪Ƨ)=∅
Ꞙ(α)	α∈(Ʝ\Ƨ)\(Ʝ∪Ᵽ)=∅
Ɦ(α)	α∈(Ƨ\Ʝ)\(Ʝ∪Ᵽ)=Ʝ’∩Ᵽ’∩Ƨ
Ꞙ(α)∩Ɦ(α)	α∈(Ʝ∩Ƨ)\(Ʝ∪Ᵽ)=∅
Ꞙ(α)∩Ꞙ(α)	α∈(Ʝ\Ᵽ)∩(Ʝ\Ƨ)=Ʝ∩Ᵽ’∩Ƨ’
Ꞙ(α)∩Ɦ(α)	α∈(Ʝ\Ᵽ)∩(Ƨ\Ʝ)=∅
Ꞙ(α)∩ [Ꞙ(α)∩Ɦ(α)]	α∈(Ʝ\Ᵽ)∩(Ʝ∩Ƨ)=Ʝ∩Ᵽ’∩Ƨ
ℭ(α)∩Ꞙ(α)	α∈(Ᵽ\Ʝ)∩(Ʝ\Ƨ)=∅
ℭ(α)∩Ɦ(α)	α∈(Ᵽ\Ʝ)∩(Ƨ\Ʝ)= Ʝ’∩Ᵽ∩Ƨ
ℭ(α)∩ [Ꞙ(α)∩Ɦ(α)]	α∈(Ᵽ\Ʝ)∩(Ʝ∩Ƨ)=∅
[Ꞙ(α)∩ℭ(α)]∩Ꞙ(α)	α∈(Ʝ∩Ᵽ)∩(Ʝ\Ƨ)=Ʝ∩Ᵽ∩Ƨ’
[Ꞙ(α)∩ℭ(α)]∩Ɦ(α)	α∈(Ʝ∩Ᵽ)∩(Ƨ\Ʝ)=∅
[Ꞙ(α)∩ℭ(α)] ∩ [Ꞙ(α)∩Ɦ(α)]	α∈(Ʝ∩Ᵽ) ∩(Ʝ∩Ƨ)=Ʝ∩Ᵽ∩Ƨ

Hence,

L(α)=

ℭ(α)	α∈Ʝ’∩Ᵽ∩Ƨ’
Ɦ(α)	α∈Ʝ’∩Ᵽ’∩Ƨ
Ꞙ(α)	α∈Ʝ∩Ᵽ’∩Ƨ’
Ꞙ(α)∩Ɦ(α)	α∈Ʝ∩Ᵽ’∩Ƨ
ℭ(α)∩Ɦ(α)	α∈Ʝ’∩Ᵽ∩Ƨ
Ꞙ(α)∩ℭ(α)	α∈Ʝ∩Ᵽ∩Ƨ’
Ꞙ(α)∩ [ℭ(α) ∩ H (α)]	α∈Ʝ∩Ᵽ∩Ƨ

It is seen that Ǹ=L.

ii) (Ꞙ,Ʝ) $\cap_{\varepsilon}$[(ℭ,Ᵽ) $\cup_{\varepsilon}$(Ɦ,Ƨ)]≠[(Ꞙ,Ʝ) $\cap_{\varepsilon}$(ℭ,Ᵽ)] $\cup_{\varepsilon}$[(Ꞙ,Ʝ) $\cap_{\varepsilon}$ (Ɦ,Ƨ)] (Ali et al., 2011), and (Ꞙ,Ʝ) $\cap_{\varepsilon}$[(ℭ,Ᵽ) $\cup_{\varepsilon}$ (Ɦ,Ƨ)] = [(Ꞙ,Ʝ) $\cap_{\varepsilon}$(ℭ,Ᵽ)] $\cup_{\varepsilon}$[(Ꞙ,Ʝ) $\cap_{\varepsilon}$ (Ɦ,Ƨ)], where Ʝ∩(ᵽΔ Ƨ)=∅.

Proof: Since the proof of the left distributive property of the extended intersection over the extended union is very similar to (i), it is not repeated here. However, it is worth mentioning the following point. In classical sets, the intersection operation is left-distributive over the union operation. However, for extended intersection and extended union operations, this situation does not hold, as shown by Ali et al. (2011) with an example. In this study, we show that distributivity can be satisfied under the condition Ʝ∩(ZΔM)=∅.

iii) If Ʝ∩Ᵽ'∩Ƨ'=Ʝ∩(ᵽΔ Ƨ)=∅, then (Ꞙ,Ʝ)$\cap_{\varepsilon}$[(ℭ,Ᵽ) $\backslash_{\varepsilon}$(Ή,Ƨ)]=[(Ꞙ,Ʝ)$\cap_{\varepsilon}$(ℭ,Ᵽ)] $\backslash_{\varepsilon}$[(Ꞙ,Ʝ) $\cap_{\varepsilon}$ (Ή,Ƨ)].

Proof: Since the proof of the left distributive property of the extended intersection over the extended difference is very similar to (i), it is not repeated here. However, it is worth mentioning the following point. In classical sets, the intersection operation is left-distributive over the difference operation. However, for extended intersection and extended difference operations, this situation does not hold. We state that distributivity can be satisfied under the condition Ʝ∩Ᵽ'∩Ƨ'=Ʝ∩(ᵽΔ Ƨ)=∅.

iv) If Ʝ∩Ᵽ'∩Ƨ'=Ʝ∩(ᵽΔ Ƨ)=∅, then (Ꞙ,Ʝ)$\cap_{\varepsilon}$[(ℭ,Ᵽ) Δ_{ε}(Ή,Ƨ)]=[(Ꞙ,Ʝ)$\cap_{\varepsilon}$(ℭ,Ᵽ)] Δ_{ε}[(Ꞙ,Ʝ) $\cap_{\varepsilon}$ (Ή,Ƨ)].

Proof: Since the proof of the left distributive property of the extended intersection over the extended symmetric difference is very similar to (i), it is not repeated here. However, it is worth mentioning the following point. In classical sets, the intersection operation is left-distributive over the symmetric difference operation. However, for extended intersection and extended symmetric difference operations, this situation does not hold. We state that distributivity can be satisfied under the condition Ʝ∩Ᵽ'∩Ƨ'=Ʝ∩(ᵽΔ Ƨ)=∅.

v) If Ʝ∩Ᵽ'∩Ƨ'=Ʝ∩(ZΔM)=∅, then (Ꞙ,Ʝ)$\cap_{\varepsilon}$[(ℭ,Ᵽ) γ_{ε}(Ή,Ƨ)]=[(Ꞙ,Ʝ)$\cap_{\varepsilon}$(ℭ,Ᵽ)] γ_{ε}[(Ꞙ,Ʝ) $\cap_{\varepsilon}$ (Ή,Ƨ)].

b) RHS distributions of extended intersection operation over other extended SS operations:
Let (Ꞙ,Ʝ) and (ℭ,Ᵽ) be SSs over U. Then, we have the following distributions:

i)[(Ꞙ,Ʝ) $\cap_{\varepsilon}$ (ℭ,Ᵽ)] $\cap_{\varepsilon}$(Ή,Ƨ)= [(Ꞙ,Ʝ) $\cap_{\varepsilon}$(Ή,Ƨ)] $\cap_{\varepsilon}$[(ℭ,Ᵽ) $\cap_{\varepsilon}$ (Ή,Ƨ)].

Proof: First, let's consider the LHS, and let (Ꞙ,Ʝ) $\cap_{\varepsilon}$ (ℭ,Ᵽ)=(R,Ʝ∪Ꝑ), where for all α∈Ʝ∪Ꝑ,

R(α)= { Ꞙ(α) α∈Ʝ\Ᵽ ; ℭ(α) α∈Ᵽ\Ʝ ; Ꞙ(α)∩ℭ(α) α∈Ʝ∩Ᵽ

Let (R,Ʝ∪Ꝑ)$\cap_{\varepsilon}$(Ή,Ƨ) =(Ṅ,(Ʝ∪Ꝑ)∪Ƨ), where for all α∈(Ʝ∪Ꝑ)∪Ƨ,

Ṅ(α)= { R(α) α∈(Ʝ∪Ꝑ)\Ƨ ; Ή(α) α∈Ƨ\(Ʝ∪Ꝑ) ; R(α)∩Ή(α) α∈(Ʝ∪Ꝑ)∩Ƨ

Hence,

Ṅ(α)= { Ꞙ(α) α∈(Ʝ\Ᵽ)\Ƨ=Ʝ∩Ᵽ'∩Ƨ' ; ℭ(α) α∈(Ᵽ\Ʝ)\Ƨ=Ʝ'∩Ᵽ∩Ƨ' ; Ꞙ(α)∩ℭ(α) α∈(Ʝ∩Ᵽ)\Ƨ=Ʝ∩Ᵽ∩Ƨ' ; Ή(α) α∈Ƨ\(Ʝ∪Ꝑ)=Ʝ'∩Ᵽ'∩Ƨ

Ꞙ(α)∩Ⱨ(α)   α∈(Ʝ\Ᵽ)∩Ƨ=Ʝ∩Ᵽ'∩Ƨ
ℭ(α)∩Ⱨ(α)   α∈(Ᵽ\Ʝ)∩Ƨ =Ʝ'∩Ᵽ∩Ƨ
[Ꞙ(α)∩ℭ(α)]∩Ⱨ(α)   α∈(Ʝ∩Ᵽ)∩Ƨ=Ʝ∩Ᵽ∩Ƨ

Now let's handle the RHS, and let (Ꞙ,Ʝ) ∩$_{\varepsilon}$(Ⱨ,Ƨ)=(K,Ʝ∪Ƨ), where for all α∈Ʝ∪Ƨ,

K(α)= { Ꞙ(α)   α∈Ʝ\Ƨ
Ⱨ(α)   α∈Ƨ\Ʝ
Ꞙ(α)∩Ⱨ(α)   α∈Ʝ∩Ƨ

Let (ℭ,Ᵽ) ∩$_{\varepsilon}$(Ⱨ,Ƨ)=(S,Ᵽ∪Ƨ), where for all α∈Ᵽ∪Ƨ,

S(α)= { ℭ(α)   α∈Ᵽ\Ƨ
Ⱨ(α)   α∈Ƨ\Ᵽ
ℭ(α)∩Ⱨ(α)   α∈Ᵽ∩Ƨ

Let (K,Ʝ∪Ƨ)∩$_{\varepsilon}$(S,Ᵽ∪Ƨ)=(L,(Ʝ∪Ƨ)∪(Ᵽ∪Ƨ)), where for all α∈(Ʝ∪Ƨ)∪(Ᵽ∪Ƨ),

L(α)= { K(α)   α∈(Ʝ∪Ƨ)\(Ᵽ∪Ƨ)
S(α)   α∈(Ᵽ∪Ƨ)\(Ʝ∪Ƨ)
K(α)∩S(α)   α∈(Ʝ∪Ƨ)∩(Ᵽ∪Ƨ)

Hence,

L(α)= {
Ꞙ(α)   α∈(Ʝ\Ƨ)\(Ᵽ∪Ƨ)=Ʝ∩Ᵽ'∩Ƨ'
Ⱨ(α)   α∈(Ƨ\Ʝ)\(Ᵽ∪Ƨ)=∅
Ꞙ(α)∩Ⱨ(α)   α∈(Ʝ∩Ƨ)\(Ᵽ∪Ƨ)=∅
ℭ(α)   α∈(Ᵽ\Ƨ)\(Ʝ∪Ƨ)=Ʝ'∩Ᵽ∩Ƨ'
Ⱨ(α)   α∈(Ƨ\Ᵽ)\(Ʝ∪Ƨ)=∅
ℭ(α)∩Ⱨ(α)   α∈(Ᵽ∩Ƨ)\(Ʝ∪Ƨ)=∅
Ꞙ(α)∩ℭ(α)   α∈(Ʝ\Ƨ)∩(Ᵽ\Ƨ)=Ʝ∩Ᵽ∩Ƨ'
Ꞙ(α)∩Ⱨ(α)   α∈(Ʝ\Ƨ)∩(Ƨ\Ᵽ)=∅
Ꞙ(α)∩[ℭ(α)∩Ⱨ(α)]   α∈(Ʝ\Ƨ)∩(Ᵽ∩Ƨ)=∅
Ⱨ(α)∩ℭ(α)   α∈(Ƨ\Ʝ)∩(Ᵽ\Ƨ)=∅
Ⱨ(α)∩Ⱨ(α)   α∈(Ƨ\Ʝ)∩(Ƨ\Ᵽ)= Ʝ'∩Ᵽ'∩Ƨ
Ⱨ(α)∩[ℭ(α)∩Ⱨ(α)]   α∈(Ƨ\Ʝ)∩(Ᵽ∩Ƨ)=Ʝ'∩Ᵽ∩Ƨ
[Ꞙ(α)∩Ⱨ(α)]∩ℭ(α)   α∈(Ʝ∩Ƨ)∩(Ᵽ\Ƨ)=∅
[Ꞙ(α)∩Ⱨ(α)]∩Ⱨ(α)   α∈(Ʝ∩Ƨ)∩(Ƨ\Ᵽ)=Ʝ∩Ᵽ'∩Ƨ
[Ꞙ(α)∩Ⱨ(α)]∩[ℭ(α)∩Ⱨ(α)]   α∈(Ʝ∩Ƨ)∩(Ᵽ∩Ƨ)=Ʝ∩Ᵽ∩Ƨ

Thus,

L(α)= ⎧
- Ꞙ(α) α∈Ɉ∩Ᵽ'∩Ƨ'
- ℭ(α) α∈Ɉ'∩Ᵽ∩Ƨ'
- Ꞙ(α)∩ℭ(α) α∈Ɉ∩Ᵽ∩Ƨ'
- Ħ(α) α∈Ɉ'∩Ᵽ'∩Ƨ
- Ħ(α)∩ℭ(α) α∈Ɉ'∩Ᵽ∩Ƨ
- Ꞙ(α)∩Ħ(α) α∈Ɉ∩Ᵽ'∩Ƨ
- [Ꞙ(α)∩Ħ(α)]∩[ℭ(α)∩Ħ(α)] α∈Ɉ∩Ᵽ∩Ƨ

It is seen that Ǹ=L.

ii) [(Ꞙ,Ɉ) $\cup_\varepsilon$ (ℭ,Ᵽ)] $\cap_\varepsilon$(Ħ,Ƨ)= [(Ꞙ,Ɉ) $\cap_\varepsilon$(Ħ,Ƨ)] $\cup_\varepsilon$ [(ℭ,Ᵽ) $\cap_\varepsilon$ (Ħ,Ƨ)], where (Ɉ△Ᵽ)∩Ƨ=∅.

Proof: Since the proof of the right distributive property of the extended intersection over the extended union is very similar to (i), it is not repeated here. However, it is worth mentioning the following point. In classical sets, the intersection operation is right-distributive over the union operation. However, for extended intersection and extended union operations, this situation does not hold. We state that distributivity can be satisfied under the condition (Ɉ△Ᵽ)∩Ƨ=∅.

iii) [(Ꞙ,Ɉ) $\setminus_\varepsilon$(ℭ,Ᵽ)] $\cap_\varepsilon$(Ħ,Ƨ)=[(Ꞙ,Ɉ) $\cap_\varepsilon$(Ħ,Ƨ)]$\setminus_\varepsilon$ [(ℭ,Ᵽ) $\cap_\varepsilon$ (Ħ,Ƨ)], where (Ɉ△Ᵽ)∩Ƨ=Ɉ∩Ᵽ'∩Ƨ'=∅.

Proof: Since the proof of the right distributive property of the extended intersection over the extended difference is very similar to (i), it is not repeated here. However, it is worth mentioning the following point. In classical sets, the intersection operation is right-distributive over the difference operation. However, for extended intersection and extended difference operations, this situation does not hold. We state that distributivity can be satisfied under the condition (Ɉ△Ᵽ)∩Ƨ=Ɉ∩Ᵽ'∩Ƨ'=∅.

iv) [(Ꞙ,Ɉ) Δ_ε(ℭ,Ᵽ)] $\cap_\varepsilon$(Ħ,Ƨ)=[(Ꞙ,Ɉ) $\cap_\varepsilon$(Ħ,Ƨ)]Δ_ε [(ℭ,Ᵽ) $\cap_\varepsilon$ (Ħ,Ƨ)], where (Ɉ△Ᵽ)∩Ƨ=Ɉ∩Ᵽ'∩Ƨ'=∅.

Proof: Since the proof of the right distributive property of the extended intersection over the extended symmetric difference is very similar to (i), it is not repeated here. However, it is worth mentioning the following point. In classical sets, the intersection operation is right-distributive over the symmetric difference operation. However, for extended intersection and extended symmetric difference operations, this situation does not hold. We state that distributivity can be satisfied under the condition (Ɉ△Ᵽ)∩Ƨ=Ɉ∩Ᵽ'∩Ƨ'=∅.

v) [(Ꞙ,Ɉ) γ_ε(ℭ,Ᵽ)] $\cap_\varepsilon$(Ħ,Ƨ)=[(Ꞙ,Ɉ) $\cap_\varepsilon$(Ħ,Ƨ)]γ_ε [(ℭ,Ᵽ) $\cap_\varepsilon$ (Ħ,Ƨ)], where (Ɉ△Ᵽ)∩Ƨ=Ɉ∩Ᵽ'∩Ƨ'=∅.

3.2.1.3. The distributions of the extended intersection operation over soft binary piecewise operations:

Here, the distributions of the extended intersection operation to soft binary operations are investigated. First, distributions from the left side, followed by distributions from the right side are examined.

a) LHS distributions of extended intersection operation over soft binary piecewise operations:

Let (Ꞙ,Ꭻ), (ℭ,Ᵽ) and (Ͱ,Ƨ) be SSs over U. Then, we have the following distributions:

i) (Ꞙ,Ꭻ) $\cap_{\varepsilon}$[(ℭ,Ᵽ) $\tilde{\cup}$ (Ͱ,Ƨ)] = [(Ꞙ,Ꭻ)$\cap_{\varepsilon}$(ℭ,Ᵽ)] $\tilde{\cup}$[(Ꞙ,Ꭻ) $\cap_{\varepsilon}$ (Ͱ,Ƨ), where Ꭻ∩Ᵽ∩Ƨ'=∅.

Proof: First, let's consider the LHS, and let (ℭ,Ᵽ) $\tilde{\cup}$ (Ͱ,Ƨ)=(R,Ᵽ), where for all α∈Ᵽ,

$$R(\alpha)=\begin{cases} ℭ(\alpha) & \alpha\in Ᵽ\setminus Ƨ \\ ℭ(\alpha)\cup Ͱ(\alpha) & \alpha\in Ᵽ\cap Ƨ \end{cases}$$

Let (Ꞙ,Ꭻ) $\cap_{\varepsilon}$ (R,Ᵽ) =(Ǹ,Ꭻ∪Ᵽ), where for all α∈Ꭻ∪Ᵽ,

$$Ǹ(\alpha)=\begin{cases} Ꞙ(\alpha) & \alpha\in Ꭻ\setminus Ᵽ \\ R(\alpha) & \alpha\in Ᵽ\setminus Ꭻ \\ Ꞙ(\alpha)\cap R(\alpha) & \alpha\in Ꭻ\cap Ᵽ \end{cases}$$

Hence,

$$Ǹ(\alpha)=\begin{cases} Ꞙ(\alpha) & \alpha\in Ꭻ\setminus Ᵽ \\ ℭ(\alpha) & \alpha\in(Ᵽ\setminus Ƨ)\setminus Ꭻ=Ꭻ'\cap Ᵽ\cap Ƨ' \\ ℭ(\alpha)\cup Ͱ(\alpha) & \alpha\in(Ᵽ\cap Ƨ)\setminus Ꭻ=Ꭻ'\cap Ᵽ\cap Ƨ \\ Ꞙ(\alpha)\cap ℭ(\alpha) & \alpha\in Ꭻ\cap(Ᵽ\setminus Ƨ)=Ꭻ\cap Ᵽ\cap Ƨ' \\ Ꞙ(\alpha)\cap[ℭ(\alpha)\cup Ͱ(\alpha)] & \alpha\in Ꭻ\cap(Ᵽ\cap Ƨ)=Ꭻ\cap Ᵽ\cap Ƨ \end{cases}$$

Now let's handle the RHS, and let (Ꞙ,Ꭻ) $\cap_{\varepsilon}$(ℭ,Ᵽ)=(K,Ꭻ∪Ᵽ), where for all α∈Ꭻ∪Ᵽ,

$$K(\alpha)=\begin{cases} Ꞙ(\alpha) & \alpha\in Ꭻ\setminus Ᵽ \\ ℭ(\alpha) & \alpha\in Ᵽ\setminus Ꭻ \\ Ꞙ(\alpha)\cap ℭ(\alpha) & \alpha\in Ꭻ\cap Ᵽ \end{cases}$$

Let (Ꞙ,Ꭻ) $\cap_{\varepsilon}$ (Ͱ,Ƨ)=(S,Ꭻ∪Ƨ), where for all α∈Ꭻ∪Ƨ,

$$S(\alpha)=\begin{cases} Ꞙ(\alpha) & \alpha\in Ꭻ\setminus Ƨ \\ Ͱ(\alpha) & \alpha\in Ƨ\setminus Ꭻ \end{cases}$$

Ꞙ(α)∩Ƕ(α)   α∈Ɉ∩Ƨ

Let (K,Ɉ∪Ᵽ) $\tilde{\cup}$(S,Ɉ∪Ƨ)=(L,Ɉ∪Ᵽ), where for all α∈Ɉ∪Ᵽ

L(α)= { K(α)   α∈(Ɉ∪Ᵽ)\(Ɉ∪Ƨ)
K(α)∪S(α)   α∈(Ɉ∪Ᵽ)∩(Ɉ∪Ƨ)

Hence,

L(α)= {
Ꞙ(α)   α∈(Ɉ\Ꝑ)\(Ɉ∪Ƨ)=∅
ℭ(α)   α∈(Ꝑ\Ɉ)\(Ɉ∪Ƨ)=Ɉ'∩Ꝑ∩Ƨ'
Ꞙ(α)∩ℭ(α)   α∈(Ɉ∩Ꝑ)\(Ɉ∪Ƨ)=∅
Ꞙ(α)∪Ꞙ(α)   α∈(Ɉ\Ꝑ)∩(Ɉ\Ƨ)=Ɉ∩Ꝑ'∩Ƨ'
Ꞙ(α)∪Ƕ(α)   α∈(Ɉ\Ꝑ)∩(Ƨ\Ɉ)=∅
Ꞙ(α)∪ [Ꞙ(α)∩Ƕ(α)]   α∈(Ɉ\Ꝑ)∩(Ɉ∩Ƨ)=Ɉ∩Ꝑ'∩Ƨ
ℭ(α)∪Ꞙ(α)   α∈(Ꝑ\Ɉ)∩(Ɉ\Ƨ)=∅
ℭ(α)∪Ƕ(α)   α∈(Ꝑ\Ɉ)∩(Ƨ\Ɉ)= Ɉ'∩Ꝑ∩Ƨ
ℭ(α)∪[Ꞙ(α)∩Ƕ(α)]   α∈(Ꝑ\Ɉ)∩(Ɉ∩Ƨ)=∅
[Ꞙ(α)∩ℭ(α)]∪Ꞙ(α)   α∈(Ɉ∩Ꝑ)∩(Ɉ\Ƨ)=Ɉ∩Ꝑ∩Ƨ'
[Ꞙ(α)∩ℭ(α)]∪Ƕ(α)   α∈(Ɉ∩Ꝑ)∩(Ƨ\Ɉ)=∅
[Ꞙ(α)∩ℭ(α)]∪[Ꞙ(α)∩Ƕ(α)]   α∈(Ɉ∩Ꝑ) ∩(Ɉ∩Ƨ)=Ɉ∩Ꝑ∩Ƨ

Hence,

L(α)= {
ℭ(α)   α∈Ɉ'∩Ꝑ∩Ƨ'
Ꞙ(α)   α∈Ɉ∩Ꝑ'∩Ƨ'
Ꞙ(α)   α∈Ɉ∩Ꝑ'∩Ƨ
ℭ(α)∪Ƕ(α)   α∈Ɉ'∩Ꝑ∩Ƨ
Ꞙ(α)   α∈Ɉ∩Ꝑ∩Ƨ'
Ꞙ(α)∩[ℭ(α)∪Ƕ(α]   α∈Ɉ∩Ꝑ∩Ƨ

Here, if we consider Ɉ\Ꝑ in the function N, since Ɉ\Ꝑ=Ɉ∩Ꝑ', if an element is in the complement of Z, it is either in Ƨ\Ꝑ or in the complement of Ƨ∪Ᵽ. Hence, if α∈Ɉ\Ꝑ, then α∈Ɉ∩Ƨ∩Ꝑ' or α∈Ɉ∩Ƨ'∩Ꝑ'. Thus, Ǹ=L is satisfied with the condition Ɉ∩Ꝑ∩Ƨ'=∅.

ii)(Ꞙ,Ɉ) $\cap_\varepsilon$ [(ℭ,Ᵽ) $\tilde{\cap}$ (H,Ƨ)] = [(Ꞙ,Ɉ) $\cap_\varepsilon$ (ℭ,Ᵽ)] $\tilde{\cap}$ [(Ꞙ,Ɉ) $\cap_\varepsilon$ (H,Ƨ)], where Ɉ ∩ Ᵽ' ∩ Ƨ = ∅.

iii) (Ꞙ,Ɉ) $\cap_\varepsilon$[(ℭ,Ꝑ) $\tilde{\setminus}$(Ƕ,Ƨ)]=[(Ꞙ,Ɉ)$\cap_\varepsilon$(ℭ,Ꝑ)] $\tilde{\setminus}$[(Ꞙ,Ɉ) $\cap_\varepsilon$ (Ƕ,Ƨ)], where Ɉ∩Ꝑ'∩Ƨ'=Ɉ∩(ꝐΔ Ƨ)=∅.

iv) (Ꞙ,Ɉ) $\cap_\varepsilon$[(ℭ,Ꝑ) $\tilde{\gamma}$ (Ḣ,Ƨ)] = [(Ꞙ,Ɉ)$\cap_\varepsilon$(ℭ,Ꝑ)] $\tilde{\gamma}$[(Ꞙ,Ɉ) $\cap_\varepsilon$ (Ḣ,Ƨ)], where Ɉ∩Ꝑ’∩Ƨ’=Ɉ∩(ꝐΔ Ƨ)=∅.

v) (Ꞙ,Ɉ) $\cap_\varepsilon$[(ℭ,Ꝑ) $\tilde{\Delta}$ (Ḣ,Ƨ)] = [(Ꞙ,Ɉ)$\cap_\varepsilon$(ℭ,Ꝑ)] $\tilde{\Delta}$[(Ꞙ,Ɉ) $\cap_\varepsilon$ (Ḣ,Ƨ)], where Ɉ∩Ꝑ’∩Ƨ’=Ɉ∩(ꝐΔ Ƨ)=∅.

b) RHS distributions of extended intersection operation over soft binary piecewise operations:
Let (Ꞙ,Ɉ), (ℭ,Ꝑ) and (Ḣ,Ƨ) be SSs over U. Then, we have the following distributions:

i) [(Ꞙ,Ɉ) $\tilde{\cup}$ (ℭ,Ꝑ)] $\cap_\varepsilon$ (Ḣ,Ƨ)=[(Ꞙ,Ɉ)$\cap_\varepsilon$(Ḣ,Ƨ)] $\tilde{\cup}$ [(ℭ,Ꝑ) $\cap_\varepsilon$ (Ḣ,Ƨ)], where Ɉ∩Ꝑ’∩Ƨ=∅.

Proof: First, let’s consider the LHS, and let (Ꞙ,Ɉ) $\tilde{\cup}$ (ℭ,Ꝑ)=(R,Ɉ), where for all α∈Ɉ,

$$R(\alpha)=\begin{cases} Ꞙ(\alpha) & \alpha\in Ɉ\setminus Ꝑ \\ Ꞙ(\alpha)\cup ℭ(\alpha) & \alpha\in Ɉ\cap Ꝑ \end{cases}$$

Let (R,Ɉ) $\cap_\varepsilon$ (Ḣ,Ƨ) =(Ṅ,Ɉ∪Ƨ), where for all α∈Ɉ∪Ƨ,

$$Ṅ(\alpha)=\begin{cases} R(\alpha) & \alpha\in Ɉ\setminus Ƨ \\ Ḣ(\alpha) & \alpha\in Ƨ\setminus Ɉ \\ R(\alpha)\cap Ḣ(\alpha) & \alpha\in Ɉ\cap Ƨ \end{cases}$$

Hence,

$$Ṅ(\alpha)=\begin{cases} Ꞙ(\alpha) & \alpha\in(Ɉ\setminus Ꝑ)\setminus Ƨ=Ɉ\cap Ꝑ'\cap Ƨ' \\ Ꞙ(\alpha)\cup ℭ(\alpha) & \alpha\in(Ɉ\cap Ꝑ)\setminus Ƨ=Ɉ\cap Ꝑ\cap Ƨ' \\ Ḣ(\alpha) & \alpha\in Ƨ\setminus Ɉ \\ Ꞙ(\alpha)\cap Ḣ(\alpha) & \alpha\in(Ɉ\setminus Ꝑ)\cap Ƨ=Ɉ\cap Ꝑ'\cap Ƨ \\ [Ꞙ(\alpha)\cup ℭ(\alpha)]\cap Ḣ(\alpha) & \alpha\in(Ɉ\cap Ꝑ)\cap Ƨ=Ɉ\cap Ꝑ\cap Ƨ \end{cases}$$

Now let’s handle the RHS, and let (Ꞙ,Ɉ) $\cap_\varepsilon$ (Ḣ,Ƨ)=(K,Ɉ∪Ƨ), where for all α∈Ɉ∪Ƨ,

$$K(\alpha)=\begin{cases} Ꞙ(\alpha) & \alpha\in Ɉ\setminus Ƨ \\ Ḣ(\alpha) & \alpha\in Ƨ\setminus Ɉ \\ Ꞙ(\alpha)\cap Ḣ(\alpha) & \alpha\in Ɉ\cap Ƨ \end{cases}$$

Let (ℭ,Ꝑ) $\cap_\varepsilon$ (Ḣ,Ƨ)=(S, Ꝑ∪Ƨ), where for all α∈Ꝑ∪Ƨ,

$$\begin{cases} ℭ(\alpha) & \alpha\in Ꝑ\setminus Ƨ \\ & \\ & \end{cases}$$

S(α)= Ꜧ(α) α∈Ꝣ\Ᵽ
ℭ(α)∩Ꜧ(α) α∈Ᵽ∩Ꝣ

Let (K,Ɉ∪Ꝣ) $\tilde{\cup}$ (S,Ᵽ∪Ꝣ)=(L,(Ɉ∪Ꝣ), where for all α∈(Ɉ∪Ꝣ),

L(α)= { K(α) α∈(Ɉ∪Ꝣ)\(Ᵽ∪Ꝣ)
K(α)∪S(α) α∈(Ɉ∪Ꝣ)∩(Ᵽ∪Ꝣ)

Hence,

L(α)= { Ꞙ(α) α∈(Ɉ\Ꝣ)\(Ᵽ∪Ꝣ)=Ɉ∩Ᵽ'∩Ꝣ'
Ꜧ(α) α∈(Ꝣ\Ɉ)\(Ᵽ∪Ꝣ)=∅
Ꞙ(α)∩Ꜧ(α) α∈(Ɉ∩Ꝣ)\(Ᵽ∪Ꝣ)=∅
Ꞙ(α)∪ℭ(α) α∈(Ɉ\Ꝣ)∩(Ᵽ\Ꝣ)=Ɉ∩Ᵽ∩Ꝣ'
Ꞙ(α)∪Ꜧ(α) α∈(Ɉ\Ꝣ)∩(Ꝣ\Ᵽ)=∅
Ꞙ(α)∪[ℭ(α)∩Ꜧ(α)] α∈(Ɉ\Ꝣ)∩(Ᵽ∩Ꝣ)=∅
Ꜧ(α)∪ℭ(α) α∈(Ꝣ\Ɉ)∩(Ᵽ\Ꝣ)=∅
Ꜧ(α)∪Ꜧ(α) α∈(Ꝣ\Ɉ)∩(Ꝣ\Ᵽ)= Ɉ'∩Ᵽ'∩Ꝣ
Ꜧ(α)∪[ℭ(α)∩Ꜧ(α)] α∈(Ꝣ\Ɉ)∩(Ᵽ∩Ꝣ)=Ɉ'∩Ᵽ∩Ꝣ
[Ꞙ(α)∩Ꜧ(α)]∪ℭ(α) α∈(Ɉ∩Ꝣ)∩(Ᵽ\Ꝣ)=∅
[Ꞙ(α)∩Ꜧ(α)]∪Ꜧ(α) α∈(Ɉ∩Ꝣ)∩(Ꝣ\Ᵽ)=Ɉ∩Ᵽ'∩Ꝣ
[Ꞙ(α)∩Ꜧ(α)]∪[ℭ(α)∩Ꜧ(α)] α∈(Ɉ∩Ꝣ)∩(Ᵽ∩Ꝣ)=Ɉ∩Ᵽ∩Ꝣ

Hence,

L(α)= { Ꞙ(α) α∈Ɉ∩Ᵽ'∩Ꝣ'
Ꞙ(α)∪ℭ(α) α∈Ɉ∩Ᵽ∩Ꝣ'
Ꜧ(α) α∈Ɉ'∩Ᵽ'∩Ꝣ
Ꜧ(α) α∈Ɉ'∩Ᵽ∩Ꝣ
Ꜧ(α) α∈Ɉ∩Ᵽ'∩Ꝣ
[Ꞙ(α)∪ℭ(α)]∩Ꜧ(α) α∈Ɉ∩Ᵽ∩Ꝣ

Here, if we consider Ꝣ\Ɉ in the function N, since Ꝣ\Ɉ=Ꝣ∩Ɉ', if an element is in the complement of Ɉ, it is either in Ᵽ\Ɉ or in the complement of Ᵽ∪Ɉ. Hence, if α∈Ꝣ\Ɉ, then α∈ Ꝣ∩Ᵽ∩Ɉ' or α∈Ꝣ∩Ᵽ'∩Ɉ'. Thus, Ǹ=L is satisfied with the condition Ɉ∩Ᵽ'∩Ꝣ=∅.

ii)[(Ꞙ, Ɉ) $\tilde{\cap}$ (ℭ, Ᵽ)] $\cap_\varepsilon$ (H, Ꝣ) = [(F, Ɉ) $\cap_\varepsilon$ (H, Ꝣ)] $\tilde{\cap}$ [(, Ᵽ) $\cap_\varepsilon$ (H, Ꝣ)], where Ɉ' ∩ Ᵽ ∩ Ꝣ = ∅.

iii) [(Ꞙ,Ꭻ)$\tilde{\setminus}$ (ℭ,Ᵽ)] $\cap_\varepsilon$ (Ꟶ,Ƨ)=[(Ꞙ,Ꭻ)$\cap_\varepsilon$(Ꟶ,Ƨ)] $\tilde{\setminus}$ [(ℭ,Ᵽ) $\cap_\varepsilon$ (Ꟶ,Ƨ)], where Ꭻ’∩Ᵽ’∩Ƨ=(ꬺΔᵽ)∩Ƨ=∅.

iv) [(Ꞙ,Ꭻ)$\tilde{\gamma}$ (ℭ,Ᵽ)] $\cap_\varepsilon$ (Ꟶ,Ƨ)=[(Ꞙ,Ꭻ)$\cap_\varepsilon$(Ꟶ,Ƨ)] $\tilde{\gamma}$ [(ℭ,Ᵽ) $\cap_\varepsilon$ (Ꟶ,Ƨ)], where Ꭻ’∩Ᵽ’∩Ƨ=(ꬺΔᵽ)∩Ƨ=∅.

v) [(Ꞙ,Ꭻ)$\tilde{\Delta}$ (ℭ,Ᵽ)] $\cap_\varepsilon$ (Ꟶ,Ƨ)=[(Ꞙ,Ꭻ)$\cap_\varepsilon$(Ꟶ,Ƨ)] $\tilde{\Delta}$ [(ℭ,Ᵽ) $\cap_\varepsilon$ (Ꟶ,Ƨ)], where Ꭻ’∩Ᵽ’∩Ƨ=(ꬺΔᵽ)∩Ƨ=∅.

4. ABSORPTION LAWS FOR SOFT SETS AND ALGEBRAIC STRUCTURES OF SOFT SETS

In this section, in order to find out whether the collection of SSs and restricted and extended intersection operation form lattice structures in $S_E(U)$ and $S_Ꭻ(U)$, firstly the so-called absorption laws are examined with detailed proofs. Although the laws of absorption in $S_E(U)$ have been presented in previous works (Ali et al., 2009; Ali et al., 2011; Qin and Hong, 2010; Singh and Onyeozili, 2012c) presented the results only with a table without proofs, and since the proofs in other studies are element-based and relatively long proofs, they are presented here with simpler proofs. In addition, in this study, the absorption laws in $S_A(U)$ for the newly-defined operations by Aybek (2024) and Yavuz (2024) are given in detail as well. Additionally, the distributive rules obtained from Section 3.1.1 and Section 3.2.1 in $S_E(U)$ and $S_Ꭻ(U)$ are presented collectively in a table. Finally, we systematically, in detail, and collectively present the unary and binary algebraic structures formed by the restricted intersection and extended intersection together with other types of SS operations in $S_E(U)$ and $S_Ꭻ(U)$. We believe that this comprehensive study will fill a gap in the literature, as such an inclusive study is currently absent.

4.1. Absorption laws for SSs

4.1.1. Absorption laws in $S_E(U)$:

Let (Ꞙ,Ꭻ) and (ℭ,Ᵽ) be SSs over U. Then,

i) (Ꞙ,Ꭻ)$\cap_R$ [(Ꞙ,Ꭻ) $\cup_\varepsilon$ (ℭ,Ᵽ)]= (Ꞙ,Ꭻ) and (Ꞙ,Ꭻ) $\cup_\varepsilon$ [(Ꞙ,Ꭻ) $\cap_R$ (ℭ,Ᵽ)] = (Ꞙ,Ꭻ) (Qin and Hong, 2010; Singh and Onyeozili, 2012c).

Proof: Here, these absorption laws are proved with a simpler proof than the proofs given in Qin and Hong (2010) and Singh and Onyeozili (2012c) First, let’s handle the LHS, and let (Ꞙ,Ꭻ) $\cup_\varepsilon$(ℭ, Ᵽ)=(Q,Ꭻ∪Ᵽ), where for all α∈ Ꭻ∪Ᵽ,

Q(α)= { Ꞙ(α) α∈Ꭻ\Ᵽ ; ℭ(α) α∈Ᵽ\Ꭻ ; Ꞙ(α)∪ℭ(α) α∈Ꭻ∩Ᵽ

Let (Ꞙ,Ꭻ) $\cap_R$(Q,Ꭻ∪Ᵽ)=(M,Ꭻ∩(Ꭻ ∪ Ᵽ))=(M,Ꭻ), where for all α∈Ꭻ, M(α)=Ꞙ(α)∩Q(α). Hence,

M(α)= { Ꞙ(α)∩Ꞙ(α) α∈Ꭻ∩(Ꭻ\Ᵽ) =Ꭻ\Ᵽ ; Ꞙ(α)∩ℭ(α) α∈Ꭻ∩(Ᵽ\Ꭻ)=∅

Ꞙ(α)∩[Ꞙ(α)∪ℭ(α)] α∈Ɉ∩(Ɉ∩Ᵽ)=Ɉ∩Ᵽ

Thus,

$$M(\alpha)=\begin{cases} Ꞙ(\alpha) & \alpha\in Ɉ\setminus Ᵽ \\ Ꞙ(\alpha) & \alpha\in Ɉ\cap Ᵽ \end{cases}$$

Hence, (Ꞙ,Ɉ)$\cap_R$[(Ꞙ,Ɉ) $\cup_\varepsilon$ (ℭ,Ᵽ)] = (Ꞙ,Ɉ).

Now, show that (Ꞙ,Ɉ) $\cup_\varepsilon$ [(Ꞙ,Ɉ) $\cap_R$ (ℭ,Ᵽ)]) = (Ꞙ,Ɉ). Let (Ꞙ,Ɉ) $\cap_R$ (ℭ,Ᵽ)=(L,Ɉ∩Ᵽ), where for all α∈Ɉ∩Ᵽ, L(α)=Ꞙ(α) ∩ℭ(α). Let (Ꞙ,Ɉ) $\cup_\varepsilon$(L,Ɉ∩Ᵽ)=(W, Ɉ ∪(Ɉ∩Ᵽ))=(W,Ɉ), where for all α∈ Ɉ,

$$W(\alpha)=\begin{cases} Ꞙ(\alpha) & \alpha\in Ɉ\setminus(Ɉ\cap Ᵽ)=Ɉ\setminus Ᵽ \\ L(\alpha) & \alpha\in(Ɉ\cap Ᵽ)\setminus Ɉ=\emptyset \\ Ꞙ(\alpha)\cup L(\alpha) & \alpha\in Ɉ\cap(Ɉ\cap Ᵽ)=Ɉ\cap Ᵽ \end{cases}$$

Thus, for all α∈ Ɉ,

$$W(\alpha)=\begin{cases} Ꞙ(\alpha) & \alpha\in Ɉ\setminus Ᵽ \\ [Ꞙ(\alpha)\cup[Ꞙ(\alpha)\cap ℭ(\alpha)] & \alpha\in Ɉ\cap Ᵽ \end{cases}$$

Hence, for all α∈Ɉ

$$W(\alpha)=\begin{cases} Ꞙ(\alpha) & \alpha\in Ɉ\setminus Ᵽ \\ Ꞙ(\alpha) & \alpha\in Ɉ\cap Ᵽ \end{cases}$$

That is, (Ꞙ,Ɉ) $\cup_\varepsilon$ [(Ꞙ,Ɉ) $\cap_R$ (ℭ,Ᵽ)] = (Ꞙ,Ɉ). Thus, the absorption law is valid for the operations $\cup_\varepsilon$ and $\cap_R$ in S_E(U) as well. Here, even if Ɉ∩Ᵽ=∅, the equality still holds in every case because W(α)=Ꞙ(α) for all α∈Ɉ.

ii) (Ꞙ,Ɉ) $\cup_R$ [(Ꞙ,Ɉ) $\cap_\varepsilon$ (ℭ,Ᵽ)] = (Ꞙ,Ɉ) and (Ꞙ,Ɉ)$\cap_\varepsilon$ [(Ꞙ,Ɉ) $\cup_R$ (ℭ,Ᵽ)]= (Ꞙ,Ɉ) (Qin and Hong, 2010; Singh and Onyeozili, 2012c).

Remark 1 Absorption laws do not hold for the following cases. Here note that these cases and their proofs were given in Singh and Onyeozili (2012c); however, we present here once again, since there are some mathematical typos in the proofs of Singh and Onyeozili (2012c).

i) (Ꞙ,Ɉ)$\cap_R$ [(Ꞙ,Ɉ) $\cup_R$ (ℭ,Ᵽ)] $\widetilde{\subseteq}$ (Ꞙ,Ɉ) and (Ꞙ,Ɉ) $\cup_R$ [(Ꞙ,Ɉ) $\cap_R$ (ℭ,Ᵽ)] $\widetilde{\subseteq}$ (Ꞙ,Ɉ) (Singh and Onyeozili, 2012c).

Proof: First, let's consider the LHS, and let (Ꞙ,Ɉ) $\cup_R$(ℭ,Ᵽ)=(Q,Ɉ∩Ᵽ), where for all α∈Ɉ∩Ᵽ, Q(α)=Ꞙ(α)∪ℭ(α). Let (Ꞙ,Ɉ) $\cap_R$(Q,Ɉ∩Ᵽ)=(M,Ɉ∩ (Ɉ ∩ Ᵽ))=(M,Ɉ∩Ᵽ), where for all α∈Ɉ∩Ᵽ, M(α)=Ꞙ(α)∩Q(α). Hence, for all α∈Ɉ∩Ᵽ, M(α)=Ꞙ(α)∩[Ꞙ(α)∪ℭ(α)]=Ꞙ(α). Thus, (M,Ɉ∩Ᵽ) $\widetilde{\subseteq}$ (Ꞙ,Ɉ).That is, (Ꞙ,Ɉ)$\cap_R$ [(Ꞙ,Ɉ) $\cup_R$ (ℭ,Ᵽ) $\widetilde{\subseteq}$ (Ꞙ,Ɉ). Here, note that (Ꞙ,Ɉ)$\cap_R$ [(Ꞙ,Ɉ) $\cup_R$ (ℭ,Ᵽ) can not be soft equal to (Ꞙ,Ɉ), as they have different parameter sets. Similarly, one can show that (Ꞙ,Ɉ) $\cup_R$ [(Ꞙ,Ɉ) $\cap_R$ (ℭ,Ᵽ)]) $\widetilde{\subseteq}$ (Ꞙ,Ɉ).

Hence, the absorption law does not hold for the SS operations $\cup_R$ and $\cap_R$ in $S_E(U)$.

ii) (Ꞙ,Ɉ)$\widetilde{\subseteq}$ (Ꞙ,Ɉ) $\cap_\varepsilon$ [(Ꞙ,Ɉ) $\cup_\varepsilon$ (ℭ,Ᵽ)] and (Ꞙ,Ɉ) $\widetilde{\subseteq}$ (Ꞙ,Ɉ) $\cup_\varepsilon$ [(Ꞙ,Ɉ) $\cap_\varepsilon$ (ℭ,Ᵽ)] (Singh and Onyeozili, 2012c).

Proof: Let us show that (Ꞙ,Ɉ)$\widetilde{\subseteq}$ (Ꞙ,Ɉ) $\cap_\varepsilon$ [(Ꞙ,Ɉ) $\cup_\varepsilon$ (ℭ,Ᵽ)]. Let (Ꞙ,Ɉ) $\cup_\varepsilon$(ℭ,Ᵽ)=(Q,Ɉ∪Ᵽ), where for all α∈ Ɉ∪Ᵽ,

$$Q(\alpha)=\begin{cases} Ꞙ(\alpha), & \alpha\in Ɉ\backslash Ᵽ \\ ℭ(\alpha), & \alpha\in Ᵽ\backslash Ɉ \\ Ꞙ(\alpha)\cup ℭ(\alpha), & \alpha\in Ɉ\cap Ᵽ \end{cases}$$

Let (Ꞙ,Ɉ) $\cap_\varepsilon$(Q,Ɉ∪Ᵽ)=(M,Ɉ∪(Ɉ ∪ Ᵽ))=(M,Ɉ∪Ᵽ), where for all α∈Ɉ∪Ᵽ,

$$M(\alpha)=\begin{cases} Ꞙ(\alpha), & \alpha\in Ɉ\backslash(Ɉ\cup Ᵽ)=\emptyset \\ Q(\alpha), & \alpha\in(Ɉ\cup Ᵽ)\backslash Ɉ = Ᵽ \\ Ꞙ(\alpha)\cap Q(\alpha), & \alpha\in Ɉ\cap(Ɉ\cup Ᵽ)= Ɉ \end{cases}$$

Thus,

$$M(\alpha)=\begin{cases} Ꞙ(\alpha), & \alpha\in(Ɉ\backslash Ᵽ)\backslash Ɉ=\emptyset \\ ℭ(\alpha), & \alpha\in(Ᵽ\backslash Ɉ)\backslash Ɉ= Ᵽ\backslash Ɉ \\ Ꞙ(\alpha)\cup ℭ(\alpha), & \alpha\in(Ɉ\cap Ᵽ)\backslash Ɉ=\emptyset \\ Ꞙ(\alpha)\cap Ꞙ(\alpha), & \alpha\in Ɉ\cap(Ɉ\backslash Ᵽ)=Ɉ\backslash Ᵽ \\ Ꞙ(\alpha)\cap ℭ(\alpha), & \alpha\in Ɉ\cap(Ᵽ\backslash Ɉ)=\emptyset \\ Ꞙ(\alpha)\cap[Ꞙ(\alpha)\cup ℭ(\alpha)], & \alpha\in Ɉ\cap(Ɉ\cap Ᵽ)= Ɉ\cap Ᵽ \end{cases}$$

Thereby, for all α∈Ɉ∪Ᵽ,

$$M(\alpha)=\begin{cases}\mathfrak{C}(\alpha), & \alpha\in \text{Ꝑ}\setminus\text{Ꞁ}\\ \text{Ꞙ}(\alpha), & \alpha\in\text{Ꞁ}\setminus\text{Ꝑ}\\ \text{Ꞙ}(\alpha), & \alpha\in\text{Ꞁ}\cap\text{Ꝑ}\end{cases}$$

Thus, (Ꞙ,Ꞁ)$\cap_\varepsilon$ [(Ꞙ,Ꞁ) $\cup_\varepsilon$ ($\mathfrak{C}$,Ꝑ)] ≠ (Ꞙ,Ꞁ). Since Ꞙ(α)⊆M(α), for all α∈Ꞁ, it is evident that (Ꞙ,Ꞁ)$\widetilde{\subseteq}$ (Ꞙ,Ꞁ)$\cap_\varepsilon$ [(Ꞙ,Ꞁ) $\cup_\varepsilon$ ($\mathfrak{C}$,Ꝑ)]. Similarly, one can show that (Ꞙ,Ꞁ)$\widetilde{\subseteq}$ (Ꞙ,Ꞁ) $\cup_\varepsilon$ [(Ꞙ,Ꞁ) $\cap_\varepsilon$ ($\mathfrak{C}$,Ꝑ)]. Thereby, the absorption law does not hold for the SS operations $\cup_\varepsilon$ and $\cap_\varepsilon$ in $S_E(U)$.

When the absorption laws in Subsection 4.1.1 are considered, the following absorption laws exist in $S_E(U)$. In the table below, 1 indicates that the absorption law is satisfied, while 0 indicates that it is not.

	$\cap_R$	$\cup_R$	$\cap_\varepsilon$	$\cup_\varepsilon$
$\cap_R$	0	0	0	1
$\cup_R$	0	0	1	0
$\cap_\varepsilon$	0	1	0	0
$\cup_\varepsilon$	1	0	0	0

Table 1 Absorption Laws in $S_E(U)$ (Ali et al., 2011)

In the study by Ali et al. (2011), this table was provided without proving any of the absorption laws. In our study, before presenting the table, we have detailed the properties with thorough proofs.

4.1.2. Absorption laws in $S_{\text{Ꞁ}}(U)$:

Let (Ꞙ,Ꞁ), ($\mathfrak{C}$,Ꞁ) be soft sets over U. Then,

i)The following absorption laws are valid for $\cap_R$ in $S_{\text{Ꞁ}}(U)$:

- (Ꞙ,Ꞁ)$\cap_R$ [(Ꞙ,Ꞁ) $\cup_R$ ($\mathfrak{C}$,Ꞁ)]= (Ꞙ,Ꞁ) and (Ꞙ,Ꞁ) $\cup_R$ [(Ꞙ,Ꞁ) $\cap_R$ ($\mathfrak{C}$,Ꞁ)] = (Ꞙ,Ꞁ) .
- (Ꞙ,Ꞁ)$\cap_R$ [(Ꞙ,Ꞁ) $\cup_\varepsilon$ ($\mathfrak{C}$,Ꞁ)]= (Ꞙ,Ꞁ) and (Ꞙ,Ꞁ) $\cup_\varepsilon$ [(Ꞙ,Ꞁ) $\cap_R$ ($\mathfrak{C}$,Ꞁ)] = (Ꞙ,Ꞁ) .
- (Ꞙ,Ꞁ)$\cap_R$ [(Ꞙ,Ꞁ) $\overset{*}{\cup}_\varepsilon$ ($\mathfrak{C}$,Ꞁ)]= (Ꞙ,Ꞁ) and (Ꞙ,Ꞁ) $\overset{*}{\cup}_\varepsilon$ [(Ꞙ,Ꞁ) $\cap_R$ ($\mathfrak{C}$,Ꞁ)] = (Ꞙ,Ꞁ).
- (Ꞙ,Ꞁ)$\cap_R$ [(Ꞙ,Ꞁ)$\underset{\cup}{\sim}$($\mathfrak{C}$,Ꞁ)]= (Ꞙ,Ꞁ) and (Ꞙ,Ꞁ)$\underset{\cup}{\sim}$[(Ꞙ,Ꞁ) $\cap_R$ ($\mathfrak{C}$,Ꞁ)] = (Ꞙ,Ꞁ).

Proof: Since the operations of restricted union, extended union, complementary extended union, soft binary piecewise union are coincident, and these operations are commutative in $S_{\text{Ꞁ}}(U)$, the proof follows the Subsection of 4.1.1.

ii) The following absorption laws are valid for $\cap_{\varepsilon}$ in $S_{Л}(U)$:

- (Ꞙ,Л)$\cap_{\varepsilon}$ [(Ꞙ,Л) $\cup_{R}$ (ℭ,Л)]= (Ꞙ,Л) and (Ꞙ,Л) $\cup_{R}$ [(Ꞙ,Л) $\cap_{\varepsilon}$ (ℭ,Л)] = (Ꞙ,Л) .
- (Ꞙ,Л)$\cap_{\varepsilon}$ [(Ꞙ,Л) $\cup_{\varepsilon}$ (ℭ,Л)]= (Ꞙ,Л) and (Ꞙ,Л) $\cup_{\varepsilon}$ [(Ꞙ,Л) $\cap_{\varepsilon}$ (ℭ,Л)] = (Ꞙ,Л) .
- (Ꞙ,Л)$\cap_{\varepsilon}$ [Л $\overset{*}{\cup}_{\varepsilon}$ (ℭ,Л)]= (Ꞙ,Л) and (Ꞙ,Л) $\overset{*}{\cup}_{\varepsilon}$ [(Ꞙ,Л) $\cap_{\varepsilon}$ (ℭ,Л)] = (Ꞙ,Л).
- (Ꞙ,Л)$\cap_{\varepsilon}$ [(Ꞙ,Л)$\tilde{\cup}$(ℭ,Л)]= (Ꞙ,Л) and (Ꞙ,Л)$\tilde{\cup}$[(Ꞙ,Л) $\cap_{\varepsilon}$ (ℭ,Л)] = (Ꞙ,Л).

iii) The following absorption laws are valid for $\cup_{R}$ in $S_{Л}(U)$:

- (Ꞙ,Л)$\cup_{R}$ [(Ꞙ,Л) $\cap_{R}$ (ℭ,Л)]= (Ꞙ,Л) and (Ꞙ,Л) $\cap_{R}$ [(Ꞙ,Л) $\cup_{R}$ (ℭ,Л)] = (Ꞙ,Л) .
- (Ꞙ,Л)$\cup_{R}$ [(Ꞙ,Л) $\cap_{\varepsilon}$ (ℭ,Л)]= (Ꞙ,Л) and (Ꞙ,Л) $\cap_{\varepsilon}$ [(Ꞙ,Л) $\cup_{R}$ (ℭ,Л)] = (Ꞙ,Л) .
- (Ꞙ,Л)$\cup_{R}$ [(Ꞙ,Л) $\overset{*}{\cap}_{\varepsilon}$ (ℭ,Л)]= (Ꞙ,Л) and (Ꞙ,Л) $\overset{*}{\cap}_{\varepsilon}$ [(Ꞙ,Л) $\cup_{R}$ (ℭ,Л)] = (Ꞙ,Л).
- (Ꞙ,Л)$\cup_{R}$ [(Ꞙ,Л)$\tilde{\cap}$(ℭ,Л)]= (Ꞙ,Л) and (Ꞙ,Л)$\tilde{\cap}$[(Ꞙ,Л) $\cup_{R}$ (ℭ,Л)] = (Ꞙ,Л).

iv) The following absorption laws are valid for $\cup_{\varepsilon}$ in $S_{Л}(U)$:

- (Ꞙ,Л)$\cup_{\varepsilon}$ [(Ꞙ,Л) $\cap_{R}$ (ℭ,Л)]= (Ꞙ,Л) and (Ꞙ,Л) $\cap_{R}$ [(Ꞙ,Л) $\cup_{\varepsilon}$ (ℭ,Л)] = (Ꞙ,Л) .
- (Ꞙ,Л)$\cup_{\varepsilon}$ [(Ꞙ,Л) $\cap_{\varepsilon}$ (ℭ,Л)]= (Ꞙ,Л) and (Ꞙ,Л) $\cap_{\varepsilon}$ [(Ꞙ,Л) $\cup_{\varepsilon}$ (ℭ,Л)] = (Ꞙ,Л) .
- (Ꞙ,Л)$\cup_{\varepsilon}$ [(Ꞙ,Л) $\overset{*}{\cap}_{\varepsilon}$ (ℭ,Л)]= (Ꞙ,Л) and (Ꞙ,Л) $\overset{*}{\cap}_{\varepsilon}$ [(Ꞙ,Л) $\cup_{\varepsilon}$ (ℭ,Л)] = (Ꞙ,Л).
- (Ꞙ,Л)$\cup_{\varepsilon}$ [(Ꞙ,Л)$\tilde{\cap}$(ℭ,Л)]= (Ꞙ,Л) and (Ꞙ,Л)$\tilde{\cap}$[(Ꞙ,Л) $\cup_{\varepsilon}$ (ℭ,Л)] = (Ꞙ,Л).

When the absorption laws in Subsection 4.1.2. are considered, the following absorption laws exist in $S_{Л}(U)$. In the table below, 1 indicates that the absorption law is satisfied, while 0 indicates that it is not.

	$\cap_R$	$\cup_R$	$\cap_{\varepsilon}$	$\cup_{\varepsilon}$	$\tilde{\cap}$	$\tilde{\cup}$	$\overset{*}{\cap}_{\varepsilon}$	$\overset{*}{\cup}_{\varepsilon}$
$\cap_R$	0	1	0	1	0	1	0	1
$\cup_R$	1	0	1	0	1	0	1	0
$\cap_{\varepsilon}$	1	0	0	1	0	1	0	1
$\cup_{\varepsilon}$	1	0	1	0	1	0	1	0
$\tilde{\cap}$	0	1	0	1	0	1	0	1

$\tilde{\cup}$	1	0	1	0	1	0	1	0
$\overset{*}{\cap}_{\varepsilon}$	0	1	0	1	0	1	0	1
$\overset{*}{\cup}_{\varepsilon}$	1	0	1	0	1	0	1	0

Table 2 Absorption Laws in $S_{JJ}(U)$

In addition, this table includes the latest SS operations introduced in the literature in 2023 and 2024, such as complementary extended SS operations and soft binary piecewise operations,

4.2. Algebraic Structures of SSs Formed by Restricted and Extended Intersection SS Operations

In this subsection, it is examined in detail which algebraic structures are formed by the restricted and extended intersection SS operations together with other SS operations in $S_E(U)$ and $S_{JJ}(U)$, respectively. First of all, algebraic structures with one binary operation (restricted intersection and extended intersection), and then algebraic structures with two binary operations (respectively, one of them is restricted intersection SS operation and the other is other SS operations, then one of them is extended intersection SS operation and the other is other SS operations) are explored. In line with this aim, by considering all distributions in Section 3.1 and 3.2, the tables for the distributive laws in $S_E(U)$ and $S_{JJ}(U)$ are provided.

For the algebraic structures with one binary operation, all the properties such as the identity element, if any, the inverse element, the absorbing element, idempotent, and the commutative property of the algebraic structures are presented in detail. For the algebraic structures with two binary operations, the properties of the algebraic structures, such as the identity element (if any), commutative and idempotent properties for the first and second operations, and the zero element (if any), are also presented in detail without omission. Additionally, for the structures that form a lattice, it is specified whether the lattice is bounded or not. If it is bounded, the lower and upper bounds are given, as well as whether it is distributive, and if it satisfies the De Morgan properties or not. In this regard, we emphasize the importance of our study, as it is comprehensive, covering the works of Ali et al. (2011), Qin and Hong (2010), and Sen (2014), and serves as a handbook for those newly interested in SSs.

Now, first by considering all distributions in Section 3.1 and 3.2, we present the table for distributive laws in $S_E(U)$ and $S_{JJ}(U)$, respectively. In these tables, '1' indicates that the distributive law holds; '0' indicates that it does not. It is important to note the following: places marked with '1' indicate full distributivity, meaning both right and left distributivity are satisfied; places marked with '1*' indicate only right distributivity is satisfied; places marked with '0' indicate that neither right nor left distributivity is satisfied.

	$\cap_R$	$\cup_R$	$\backslash_R$	Δ_R	γ_R	$\cap_\varepsilon$	$\cup_\varepsilon$	$\backslash_\varepsilon$	Δ_ε	γ_ε	$\tilde{\cap}$	$\tilde{\cup}$	$\tilde{\backslash}$	$\tilde{\Delta}$	$\tilde{\gamma}$
$\cap_R$	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
$\cap_\varepsilon$	0	1	0	0	0	1	0	0	0	0	0	0	0	0	0

Table 3 Distributive laws in $S_E(U)$ for restricted and extended intersection operations

(*: Just right distributions)

Here note that in the study by Ali et al. (2011), they provided this table without proving any of the distributive laws, demonstrating only those that do not hold with examples. In our study, before presenting the table, we provided detailed proofs in Section 3.1 and Section 3.2. Additionally, we have included the soft binary piecewise operations, which are newly introduced in the literature in 2023 and 2024, in the first row of this table.

	$\cap_R$	$\cup_R$	$\backslash_R$	Δ_R	γ_R	$\cap_\varepsilon$	$\cup_\varepsilon$	$\backslash_\varepsilon$	Δ_ε	γ_ε	$\tilde{\cap}$	$\tilde{\cup}$	$\tilde{\backslash}$	$\tilde{\Delta}$	$\tilde{\gamma}$
$\cap_R$	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
$\cap_\varepsilon$	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1

Table 3 Distributive laws in $S_{\mathbb{J}}(U)$ for restricted and extended intersection operations

(*: Just right distributions)

4.2.1. Algebraic structures with one binary operation in set $S_E(U)$ and $S_{\mathbb{A}}(U)$ formed by for restricted and extended intersection operations

In this subsection, algebraic structures with one binary operation, specifically the binary operation is restricted intersection operation and extended intersection SS operation, respectively are examined in $S_E(U)$ and $S_{\mathbb{A}}(U)$, respectively.

4.2.1.1. Algebraic structures with one binary operation in $S_E(U)$ formed by restricted and extended intersection operations

1) $(S_E(U),\cap_R)$ is a commutative idempotent monoid with the identity U_E, namely, a bounded semi-lattice with the absorbing element $\emptyset_\emptyset$.

2) $(S_E(U), \cap_\varepsilon)$ is a commutative idempotent monoid with the identity $\emptyset_\emptyset$, namely, a bounded semi-lattice with the absorbing element $\emptyset_E$.

4.2.1.2. Algebraic structures with one binary operation in $S_{\mathbb{A}}(U)$ formed by restricted and extended intersection operations

1) $(S_{\mathbb{A}}(U), \cap_R)$ and $(S_{\mathbb{A}}(U), \cap_\varepsilon)$ are commutative idempotent monoids with the identity element $U_{\mathbb{A}}$, namely, a bounded semi-lattice with the absorbing element $\emptyset_{\mathbb{A}}$.

4.2.2. Algebraic structures with two binary operations in $S_E(U)$ and $S_{\mathbb{A}}(U)$ formed by restricted and extended intersection operations

In this subsection, algebraic structures with two binary operations, the second binary operation of which is restricted intersection operation and extended intersection operation, respectively are examined in $S_E(U)$

and $S_{Ⱥ}(U)$, respectively. Additionally, four mathematically incorrect algebraic structures in the study by Ali et al. [9] are corrected.

4.2.2.1. Algebraic structures with two binary operations in $S_E(U)$ formed by restricted and extended intersection operations

i) Algebraic structures in $S_E(U)$ with two binary operations, the second binary operation of which is the restricted intersection operation:

Let (Ꞙ,Ⱥ), (Ȼ,Ᵽ) and (Ⱨ,Ƨ) be SSs over U. Then,

1) $(S_E(U),\cap_R,\cap_R)$ is an additively and multiplicatively commutative and idempotent semiring with the identity element U_E and without zero.

2) $(S_E(U),\cup_R,\cap_R)$ is an additively and multiplicatively commutative and idempotent semiring with the identity element U_E and without zero.

Here, we also want to correct an error made in a previous study by Ali et al. (2011). It was stated that $(S_E(U),\cup_R,\cap_R)$ is a hemiring with the identity U_E. However, since (Ꞙ,Ⱥ)$\cup_R \emptyset_E=\emptyset_E \cup_R$(Ꞙ,Ⱥ)=(Ꞙ,Ⱥ) and (Ꞙ,Ⱥ)$\cap_R \emptyset_E = \emptyset_E \cap_R$(Ꞙ,Ⱥ)$\neq \emptyset_E$ (since (Ꞙ,Ⱥ)$\cap_R \emptyset_E = \emptyset_E \cap_R$(Ꞙ,Ⱥ)$= \emptyset_{Ⱥ}$), $(S_E(U),\cup_R,\cap_R)$ cannot be a hemiring.

3) $(S_E(U),\Delta_R,\cap_R)$ is an additively and multiplicatively commutative, multiplicatively idempotent semiring with the identity element U_E and without zero.

4) $(S_E(U),\cap_\varepsilon,\cap_R)$ is an additively and multiplicatively commutative and idempotent semiring with the identity element U_E.

Moreover, since (Ꞙ,Ⱥ)$\cap_\varepsilon \emptyset_\emptyset=\emptyset_\emptyset \cap_\varepsilon$(Ꞙ,Ⱥ)=(Ꞙ,Ⱥ) and (Ꞙ,Ⱥ)$\cap_R \emptyset_\emptyset=\emptyset_\emptyset \cap_R$(Ꞙ,Ⱥ)$=\emptyset_\emptyset$, $\emptyset_\emptyset$ is the zero of $(S_E(U),\cap_\varepsilon,\cap_R)$, and thus $(S_E(U),\cap_\varepsilon,\cap_R)$ is a hemiring.

5) $(S_E(U),\cup_\varepsilon,\cap_R)$is an additively and multiplicatively commutative and idempotent semiring with the identity element U_E.

Moreover, since (Ꞙ,Ⱥ) $\cup_\varepsilon \emptyset_\emptyset=\emptyset_\emptyset \cup_\varepsilon$(Ꞙ,Ⱥ)=(Ꞙ,Ⱥ) and (Ꞙ,Ⱥ)$\cap_R \emptyset_\emptyset=\emptyset_\emptyset \cap_R$(Ꞙ,Ⱥ)$=\emptyset_\emptyset$, $\emptyset_\emptyset$ is the zero of $(S_E(U),\cup_\varepsilon,\cap_R)$, and thus $(S_E(U),\cup_\varepsilon,\cap_R)$ is a hemiring. (Ali et al., 2011)

6) $(S_E(U),\Delta_\varepsilon,\cap_R)$ is an additively and multiplicatively commutative and multiplicatively idempotent semiring with the identity element U_E.

Moreover, since (Ꞙ,Ⱥ)$\Delta_\varepsilon\emptyset_\emptyset=\emptyset_\emptyset\Delta_\varepsilon$(Ꞙ,Ⱥ)=(Ꞙ,Ⱥ) and (Ꞙ,Ⱥ)$\cap_R \emptyset_\emptyset=\emptyset_\emptyset \cap_R$(Ꞙ,Ⱥ)$=\emptyset_\emptyset$, $\emptyset_\emptyset$ is the zero of $(S_E(U),\Delta_\varepsilon,\cap_R)$, and thus $(S_E(U),\Delta_\varepsilon,\cap_R)$ is a hemiring. (Sezgin and Çağman, 2025)

7) $(S_E(U),\backslash_\varepsilon,\cap_R)$ is a multiplicatively commutative and multiplicatively idempotent semiring with the identity element U_E, where Ⱥ∩Ᵽ∩Ꜫ=∅.

Although (Ꞙ,Ⱥ)$\backslash_\varepsilon\emptyset_\emptyset=\emptyset_\emptyset\backslash_\varepsilon$(Ꞙ,Ⱥ)=(Ꞙ,Ⱥ) and (Ꞙ,Ⱥ)$\cap_R\emptyset_\emptyset=\emptyset_\emptyset\cap_R$(Ꞙ,Ⱥ)=$\emptyset_\emptyset$, namely, $\emptyset_\emptyset$ is the zero of $(S_E(U),\backslash_\varepsilon,\cap_R)$, $(S_E(U),\backslash_\varepsilon,\cap_R)$ cannot be a hemiring, since it is not additively commutative, but it is semiring with zero.

8) $(S_E(U),\gamma_\varepsilon,\cap_R)$ is a multiplicatively commutative and multiplicatively idempotent semiring with the identity element U_E, where Ⱥ∩Ᵽ∩Ꜫ=∅.

Although (Ꞙ,Ⱥ)$\gamma_\varepsilon\emptyset_\emptyset=\emptyset_\emptyset\gamma_\varepsilon$(Ꞙ,Ⱥ)=(Ꞙ,Ⱥ) and (Ꞙ,Ⱥ)$\cap_R\emptyset_\emptyset=\emptyset_\emptyset\cap_R$(Ꞙ,Ⱥ)=$\emptyset_\emptyset$, namely, $\emptyset_\emptyset$ is the zero of $(S_E(U),\gamma_\varepsilon,\cap_R)$, $(S_E(U),\backslash_\varepsilon,\cap_R)$ cannot be a hemiring, since it is not additively commutative, but it is semiring with zero.

9) $(S_E(U),+_\varepsilon,\cap_R)$ is a multiplicatively commutative and multiplicatively idempotent semiring with the identity element U_E, where Ⱥ∩Ᵽ∩Ꜫ=∅.

Although (Ꞙ,Ⱥ)$+_\varepsilon\emptyset_\emptyset=\emptyset_\emptyset+_\varepsilon$(Ꞙ,Ⱥ)=(Ꞙ,Ⱥ) and (Ꞙ,Ⱥ)$\cap_R\emptyset_\emptyset=\emptyset_\emptyset\cap_R$(Ꞙ,Ⱥ)=$\emptyset_\emptyset$, namely, $\emptyset_\emptyset$ is the zero of $(S_E(U),+_\varepsilon,\cap_R)$, $(S_E(U),+_\varepsilon,\cap_R)$ cannot be a hemiring, since it is not additively commutative, but it is semiring with zero.

10) $(S_E(U),\lambda_\varepsilon,\cap_R)$ is a multiplicatively commutative and multiplicatively idempotent semiring with the identity element U_E, where Ⱥ ∩Ᵽ∩Ꜫ=∅.

Although (Ꞙ,Ⱥ)$\lambda_\varepsilon\emptyset_\emptyset=\emptyset_\emptyset\lambda_\varepsilon$(Ꞙ,Ⱥ)=(Ꞙ,Ⱥ) and (Ꞙ,Ⱥ)$\cap_R\emptyset_\emptyset=\emptyset_\emptyset\cap_R$(Ꞙ,Ⱥ)=$\emptyset_\emptyset$, namely, $\emptyset_\emptyset$ is the zero of $(S_E(U),\lambda_\varepsilon,\cap_R)$, $(S_E(U),\lambda_\varepsilon,\cap_R)$ cannot be a hemiring, since it is not additively commutative, but it is semiring with zero.

11) $(S_E(U),\theta_\varepsilon,\cap_R)$ is an additively and multiplicatively commutative and multiplicatively idempotent semiring with the identity element U_E, where Ⱥ∩Ᵽ∩Ꜫ=∅.

Moreover, since (Ꞙ,Ⱥ)$\theta_\varepsilon\emptyset_\emptyset=\emptyset_\emptyset\theta_\varepsilon$(Ꞙ,Ⱥ)=(Ꞙ,Ⱥ) and (Ꞙ,Ⱥ)$\cap_R\emptyset_\emptyset=\emptyset_\emptyset\cap_R$(Ꞙ,Ⱥ)=$\emptyset_\emptyset$, namely, $\emptyset_\emptyset$ is the zero of $(S_E(U),\theta_\varepsilon,\cap_R)$, $(S_E(U),\theta_\varepsilon,\cap_R)$ is a hemiring, where Ⱥ∩Ᵽ∩Ꜫ=∅.

12) $(S_E(U),*_\varepsilon,\cap_R)$ is an additively and multiplicatively commutative and multiplicatively idempotent a semiring with the identity element U_E, where Ⱥ∩Ᵽ∩Ꜫ=∅.

Moreover, since (Ꞙ,Ⱥ)$*_\varepsilon\emptyset_\emptyset=\emptyset_\emptyset*_\varepsilon$(Ꞙ,Ⱥ)=(Ꞙ,Ⱥ) and (Ꞙ,Ⱥ)$\cap_R\emptyset_\emptyset=\emptyset_\emptyset\cap_R$(Ꞙ,Ⱥ)=$\emptyset_\emptyset$, namely, $\emptyset_\emptyset$ is the zero of $(S_E(U),*_\varepsilon,\cap_R)$, $(S_E(U),*_\varepsilon,\cap_R)$ is a hemiring, where Ⱥ∩Ᵽ∩Ꜫ=∅.

13) $(S_E(U), \tilde{\cap}, \cap_R)$ is an additively and multiplicatively idempotent, multiplicatively commutative semiring with the identity element U_E, where Ꭿ∩Ᵽ'∩Ƨ=∅.

14) $(S_E(U), \tilde{\cup}, \cap_R)$ is an additively and multiplicatively idempotent, multiplicatively commutative semiring with the identity element U_E, where Ꭿ∩Ᵽ'∩Ƨ=∅.

15) $(S_E(U), \tilde{\Delta}, \cap_R)$ is a multiplicatively commutative and idempotent semiring with the identity element U_E, where Ꭿ∩Ᵽ'∩Ƨ=∅ (Sezgin and Yavuz, 2023b).

16) $(S_E(U), \cup_\varepsilon, \cap_R)$ is a bounded distributive lattice with the lower bound $\emptyset_\emptyset$ and the upper bound U_E.

In fact, $(S_E(U), \cup_\varepsilon)$ and $(S_E(U), \cap_R)$ are commutative idempotent monoids with the identity element $\emptyset_\emptyset$ and U_E, respectively (Ali et al., 2011) and restricted intersection distributes over extended intersection from both left and right sides in $S_E(U)$. Thus, $(S_E(U), \cup_\varepsilon, \cap_R)$ is a bounded distributive lattice with the lower bound $\emptyset_\emptyset$ and the upper bound U_E. Since (Ғ,Ꭿ)$\cup_\varepsilon$(Ғ,Ꭿ)$^r \neq U_E$ and (Ғ,Ꭿ)$\cap_R$(Ғ,Ꭿ)$^r \neq \emptyset_\emptyset$, the algebraic structure $(S_E(U), \cup_\varepsilon, \cap_R)$ is not complemented, thus it is not a Boolean algebra. (Ali et al., 2011; Qin and Hong, 2010)

<u>ii) Algebraic structures in $S_E(U)$ with two binary operations, the second binary operation of which is the extended intersection operation</u>

Let (Ғ,Ꭿ), (Ȼ,Ᵽ) and (Ħ,Ƨ) be SSs over U. Then,

1) $(S_E(U), \cup_R, \cap_\varepsilon)$ is an additively and multiplicatively commutative and idempotent semiring with the identity element $\emptyset_\emptyset$.

Moreover, since (Ғ,Ꭿ)$\cup_R \emptyset_E = \emptyset_E \cup_R$(Ғ,Ꭿ)=(Ғ,Ꭿ) and (Ғ,Ꭿ)$\cap_\varepsilon \emptyset_E = \emptyset_E \cap_\varepsilon$(Ғ,Ꭿ)=$\emptyset_E$, the algebraic structure **(**$S_E(U), \cup_R, \cap_\varepsilon$) is a hemiring (Ali et al., 2011).

2) $(S_E(U), \cap_R, \cap_\varepsilon)$ is an additively and multiplicatively commutative and idempotent semiring with the identity element $\emptyset_\emptyset$ where (ᎯΔᎵ)∩Ƨ=∅ and Ꭿ ∩(ᎵΔ Ƨ)=∅.

Here, we also want to correct an error made in a previous study by Ali et al. (2011). It was stated that $(S_E(U), \cap_R, \cap_\varepsilon)$ is a hemiring with the identity $\emptyset_\emptyset$. However, since (Ғ,Ꭿ)$\cap_R U_E = U_E \cap_R$(Ғ,Ꭿ)=(Ғ,Ꭿ) and (Ғ,Ꭿ)$\cap_\varepsilon U_E = U_E \cap_\varepsilon$(Ғ,Ꭿ)$\neq U_E$, $(S_E(U), \cap_R, \cap_\varepsilon)$ cannot be a hemiring. Moreover, since extended intersection distributes over restricted intersection from LHS and RHS, respectively where (ᎯΔᎵ)∩Ƨ=∅ and Ꭿ ∩(ᎵΔ Ƨ)=∅. $(S_E(U), \cap_R, \cap_\varepsilon)$ cannot be a hemiring.

3) $(S_E(U), \cap_\varepsilon, \cap_\varepsilon)$ is an additively and multiplicatively commutative and idempotent semiring with the identity element $\emptyset_\emptyset$ and without zero.

4) $(S_E(U),\tilde{\cup},\cap_\varepsilon)$ is a multiplicatively commutative, additively and multiplicatively idempotent semiring with the identity element $\emptyset_\emptyset$ and without zero, where $Ⱥ \cap (ⱣΔ Ƨ)=\emptyset$.

5) $(S_E(U),\tilde{\cap},\cap_\varepsilon)$ is a multiplicatively commutative, additively and multiplicatively idempotent semiring with the identity element $\emptyset_\emptyset$ and without zero, where $(ȺΔⱣ)\cap Ƨ=\emptyset$.

6) $(S_E(U),\cup_R,\cap_\varepsilon)$ is a bounded distributive lattice with the lower bound $\emptyset_E$ and the upper bound $\emptyset_\emptyset$. Since $(Ꞙ,Ⱥ)\cup_R(Ꞙ,Ⱥ)^r\neq \emptyset_\emptyset$ and $(Ꞙ,Ⱥ)\cap_\varepsilon(Ꞙ,Ⱥ)^r\neq U_E$, the algebraic structure $(S_E(U),\cup_R,\cap_\varepsilon)$ is non-complemented, bounded and distributive lattice; thus, it is not a Boolean algebra. (Ali et al., 2011; Qin and Hong, 2010)

<u>4.2.2.2. Algebraic structures with two binary operations in $S_{Ⱥ}(U)$:</u>

<u>i) Algebraic structures in $S_{Ⱥ}(U)$ with two binary operations, the second binary operation of which is the restricted intersection operation:</u>

1) $(S_{Ⱥ}(U),\cap_R,\cap_R)$, $(S_{Ⱥ}(U),\cap_\varepsilon,\cap_R)$, $(S_{Ⱥ}(U),\tilde{\cap},\cap_R)$ are additively and multiplicatively commutative and idempotent semirings with the identity element $U_{Ⱥ}$ and without zero.

2) $(S_{Ⱥ}(U),\cup_R,\cap_R)$ are additively and multiplicatively commutative and idempotent semirings with the identity element $U_{Ⱥ}$.

Moreover, since $(Ꞙ,Ⱥ)\cup_R \emptyset_{Ⱥ}=\emptyset_{Ⱥ} \cup_R(Ꞙ,Ⱥ)=(Ꞙ,Ⱥ)$ and $(Ꞙ,Ⱥ)\cap_R \emptyset_{Ⱥ}=\emptyset_{Ⱥ} \cap_R(Ꞙ,Ⱥ)=\emptyset_{Ⱥ}$, namely $\emptyset_{Ⱥ}$ is the zero of $(S_{Ⱥ}(U),\cup_R,\cap_R)$, $(S_{Ⱥ}(U),\cup_R,\cap_R)$ is a hemiring (Ali et al., 2011).

3) $(S_{Ⱥ}(U),\Delta_R,\cap_R)$ is an additively and multiplicatively commutative and multiplicatively idempotent semiring with the identity element $U_{Ⱥ}$.

Moreover, since $(Ꞙ,Ⱥ)\Delta_R\emptyset_{Ⱥ}=\emptyset_{Ⱥ}\Delta_R(Ꞙ,Ⱥ)=(Ꞙ,Ⱥ)$ and $(Ꞙ,Ⱥ)\cap_R \emptyset_{Ⱥ}=\emptyset_{Ⱥ} \cap_R(Ꞙ,Ⱥ)=\emptyset_{Ⱥ}$, namely $\emptyset_{Ⱥ}$ is the zero of $(S_{Ⱥ}(U),\Delta_R,\cap_R)$, $(S_{Ⱥ}(U),\Delta_R,\cap_R)$ is a hemiring.

Additionally, $(S_{Ⱥ}(U),\Delta_R,\cap_R)$ is a ring with the identity element, and since $(Ꞙ,Ⱥ)^2=(Ꞙ,Ⱥ)\cap_R(Ꞙ,Ⱥ)$, $(S_{Ⱥ}(U),\Delta_R,\cap_R)$ is a Boolean Ring. The fact that $(Ꞙ,Ⱥ)\Delta_R(Ꞙ,Ⱥ)=\emptyset_{Ⱥ}$ and $(Ꞙ,Ⱥ)\cap_R(ℭ,Ⱥ)=(ℭ,Ⱥ)\cap_R(Ꞙ,Ⱥ)$ is a natural consequence of the algebraic structure $(S_{Ⱥ}(U),\Delta_R,\cap_R)$ being a Bool ring (Eren and Çalışıcı, 2019).

4) $(S_{Ⱥ}(U), \cup_\varepsilon,\cap_R)$ is an additively and multiplicatively commutative and idempotent semiring with the identity element $U_{Ⱥ}$,

Moreover, since $(Ꞙ,Ⱥ)\cup_\varepsilon \emptyset_{Ⱥ}=\emptyset_{Ⱥ} \cup_\varepsilon(Ꞙ,Ⱥ)=(Ꞙ,Ⱥ)$ and $(Ꞙ,Ⱥ)\cap_R \emptyset_{Ⱥ}=\emptyset_{Ⱥ} \cap_R(Ꞙ,Ⱥ)= \emptyset_{Ⱥ}$, namely $\emptyset_{Ⱥ}$ is the zero of $(S_{Ⱥ}(U),\cup_\varepsilon,\cap_R)$, $(S_{Ⱥ}(U),\cup_\varepsilon,\cap_R)$ is a hemiring.

5) $(S_{Ⱥ}(U),\Delta_{\varepsilon},\cap_R)$ is an additively and multiplicatively commutative, multiplicatively idempotent semiring with the identity element $U_{Ⱥ}$,

Moreover, since $(Ꞙ,Ⱥ)\Delta_{\varepsilon}\emptyset_{Ⱥ}=\emptyset_{Ⱥ}\Delta_{\varepsilon}(Ꞙ,Ⱥ)=(Ꞙ,Ⱥ)$ and $(Ꞙ,Ⱥ)\cap_R \emptyset_{Ⱥ}=\emptyset_{Ⱥ} \cap_R(Ꞙ,Ⱥ)=\emptyset_{Ⱥ}$, namely $\emptyset_{Ⱥ}$ is the zero of $(S_{Ⱥ}(U),\Delta_{\varepsilon},\cap_R)$, $(S_{Ⱥ}(U),\Delta_{\varepsilon},\cap_R)$ is a hemiring (Sezgin and Çağman, 2025).

6) $(S_{Ⱥ}(U),\tilde{\cup},\cap_R)$ is an additively and multiplicatively commutative and idempotent is a semiring with the identity element $U_{Ⱥ}$.

Moreover, since $(Ꞙ,Ⱥ)\tilde{\cup}\emptyset_{Ⱥ}=\emptyset_{Ⱥ}\tilde{\cup}(Ꞙ,Ⱥ)=(Ꞙ,Ⱥ)$ and $(Ꞙ,Ⱥ)\cap_R \emptyset_{Ⱥ}=\emptyset_{Ⱥ} \cap_R(Ꞙ,Ⱥ)=\emptyset_{Ⱥ}$, namely $\emptyset_{Ⱥ}$ is the zero of $(S_{Ⱥ}(U),\tilde{\cup},\cap_R)$, $(S_{Ⱥ}(U),\tilde{\cup},\cap_R)$ is a hemiring.

7) $(S_{Ⱥ}(U),\tilde{\Delta},\cap_R)$ is an additively and multiplicatively commutative, multiplicatively idempotent semiring with the identity element $U_{Ⱥ}$.

Moreover, since $(Ꞙ,Ⱥ)\tilde{\Delta}\emptyset_{Ⱥ}=\emptyset_{Ⱥ}\tilde{\Delta}(Ꞙ,Ⱥ)=(Ꞙ,Ⱥ)$ and $(Ꞙ,Ⱥ)\cap_R \emptyset_{Ⱥ}=\emptyset_{Ⱥ} \cap_R(Ꞙ,Ⱥ)=\emptyset_{Ⱥ}$, namely $\emptyset_{Ⱥ}$ is the zero of $(S_{Ⱥ}(U),\tilde{\Delta},\cap_R)$, $(S_{Ⱥ}(U),\tilde{\Delta},\cap_R)$ is a hemiring (Sezgin and Yavuz, 2023b).

8) $(S_{Ⱥ}(U),\cup_{\varepsilon},\cap_R)$ is a complemented, bounded, distributive lattice with the lower bound $\emptyset_{Ⱥ}$ and the upper bound $U_{Ⱥ}$.

In fact, it was presented that $(S_{Ⱥ}(U),\cup_{\varepsilon})$ and $(S_{Ⱥ}(U),\cap_R)$ are commutative idempotent monoids with the identity element $\emptyset_{Ⱥ}$ and $U_{Ⱥ}$, respectively, $\cup_{\varepsilon}$ ve $\cap_R$ hold distributive laws in $S_{Ⱥ}(U)$, and restricted intersection distributes over extended union from both left and right sides in $S_{Ⱥ}(U)$.

Furthermore, since $(Ꞙ,Ⱥ)\cup_{\varepsilon}(Ꞙ,Ⱥ)^r= U_{Ⱥ}$ and $(Ꞙ,Ⱥ)\cap_R(Ꞙ,Ⱥ)^r= \emptyset_{Ⱥ}$, $(S_{Ⱥ}(U),\cup_{\varepsilon},\cap_R,{}^r)$ is a complemented, bounded and distributive lattice; thus, it is a Boolean algebra. Moreover, since it satisfies the De Morgan law, that is, $[(Ꞙ,Ⱥ)\cap_R(\mathfrak{C},Ⱥ)]^r=(Ꞙ,Ⱥ)^r\cup_{\varepsilon}(\mathfrak{C},Ⱥ)^r$ and $[(Ꞙ,Ⱥ)\cup_{\varepsilon}G,Ⱥ)]^r=(Ꞙ,Ⱥ)^r\cap_R (\mathfrak{C},Ⱥ)^r$. Thus, $(S_{Ⱥ}(U),\cup_{\varepsilon},\cap_R,{}^c)$ is a De Morgan Algebra.

Additionally, $(Ꞙ,Ⱥ)\cap_R (Ꞙ,Ⱥ)^r=\emptyset_{Ⱥ} \tilde{\subseteq}(\mathfrak{C},Ⱥ)\cup_{\varepsilon}(\mathfrak{C},Ⱥ)^r=U_{Ⱥ}$ for all $(Ꞙ,Ⱥ),(\mathfrak{C},Ⱥ)\in S_A(U)$, thus $(S_{Ⱥ}(U),\cup_{\varepsilon},\cap_R,{}^r)$ is a Kleene Algebra.

Additionally, it is known that $(Ꞙ,Ⱥ) \cap_R (Ꞙ,Ⱥ)^r= \emptyset_{Ⱥ}$ and if $(Ꞙ,Ⱥ)\cap_R(\mathfrak{C},Ⱥ)= \emptyset_{Ⱥ}$, then $(\mathfrak{C},Ⱥ) \tilde{\subseteq} (Ꞙ,Ⱥ)^r$. This shows that $(Ꞙ,Ⱥ)^r$ is the pseudo-complement of $(Ꞙ,Ⱥ)$. Furthermore, since $(Ꞙ,Ⱥ)^r\cup_{\varepsilon} ((Ꞙ,Ⱥ)^r)^r=U_{Ⱥ}$, $(S_{Ⱥ}(U),\cup_{\varepsilon},\cap_R,{}^r)$ satisfies Stone's unit property and thus, the algebraic structure $(S_{Ⱥ}(U),\cup_{\varepsilon},\cap_R,{}^r)$ is a Stone Algebra (Ali et al., 2011).

9) $(S_{Ⱥ}(U),\cup_R,\cap_R,{}^r)$ and $(S_{Ⱥ}(U),\tilde{\cup},\cap_R,{}^r)$ is a complemented, bounded, distributive lattice with the lower bound $\emptyset_{Ⱥ}$ and the upper bound $U_{Ⱥ}$, is therefore a Boolean algebra, De Morgan algebra, besides, Kleene algebra and Stone algebra.

Additionally $(S_{Ⱥ}(U),{}^r,\cap_R)$ is an MV-algebra with the constant $U_{Ⱥ}$ (Ali et al., 2011).

To show that $(S_{Ⱥ}(U),{}^r,\cap_R,U_{Ⱥ})$ is an MV-algebra, we need to show that it satisfies the MV-algebra conditions.

- (MV1) $(S_{Ⱥ}(U),\cap_R)$ is commutative monoid with the identity element $U_{Ⱥ}$.
- (MV2) $((Ꞙ,Ⱥ)^r)^r=(Ꞙ,Ⱥ)$.
- (MV3) $(U_{Ⱥ})^r\cap_R(Ꞙ,Ⱥ)=\emptyset_{Ⱥ}\cap_R(Ꞙ,Ⱥ)=\emptyset_{Ⱥ}=(U_{Ⱥ})^r$.
- (MV4) $[(Ꞙ,Ⱥ)^r\cap_R(ℭ,Ⱥ)]^r\cap_R(ℭ,Ⱥ)=((ℭ,Ⱥ)^r\cap_R(Ꞙ,Ⱥ)^r)^r\cap_R(Ꞙ,Ⱥ)$. Indeed,

$$
\begin{aligned}
[(Ꞙ,Ⱥ)^r\cap_R(ℭ,Ⱥ)]^r\cap_R(ℭ,Ⱥ) &=[((Ꞙ,Ⱥ)^r)^r\cup_R(ℭ,Ⱥ)^r]\cap_R(ℭ,Ⱥ)\\
&=[(Ꞙ,Ⱥ)\cup_R(ℭ,Ⱥ)^r]\cap_R(ℭ,Ⱥ)\\
&=[(Ꞙ,Ⱥ)\cap_R(ℭ,Ⱥ)]\cup_R[(ℭ,Ⱥ)^r\cap_R G,Ⱥ)]\\
&=[(Ꞙ,Ⱥ)\cap_R(ℭ,Ⱥ)]\cup_R[(Ꞙ,Ⱥ)\cap_R(Ꞙ,Ⱥ)^r]\\
&=(Ꞙ,Ⱥ)\cap_R[(ℭ,Ⱥ)\cup_R(Ꞙ,Ⱥ)^r]\\
&=(Ꞙ,Ⱥ)\cap_R[(ℭ,Ⱥ)^r\cap_R(Ꞙ,Ⱥ)]^r\\
&=[(ℭ,Ⱥ)^r\cap_R(Ꞙ,Ⱥ)]^r\cap_R(Ꞙ,Ⱥ)
\end{aligned}
$$

Thus, $(S_{Ⱥ}(U),{}^r,\cap_R)$ is an MV-algebra with the constant $U_{Ⱥ}$.

<u>ii) Algebraic structures in $S_{Ⱥ}(U)$ with two binary operations, the second binary operation of which is the extended intersection operation:</u>

1) $(S_{Ⱥ}(U),\cap_\varepsilon,\cap_\varepsilon)$, $(S_{Ⱥ}(U),\tilde{\cap},\cap_\varepsilon)$, $(S_{Ⱥ}(U),\cap_R,\cap_\varepsilon)$ are additively and multiplicatively commutative and idempotent semirings with the identity element $U_{Ⱥ}$ and without zero.

2) $(S_{Ⱥ}(U),\cup_R,\cap_\varepsilon)$ is an additively and multiplicatively commutative and idempotent semiring with the identity element U_A.

Moreover, since $(Ꞙ,Ⱥ)\cup_R\emptyset_{Ⱥ}=\emptyset_{Ⱥ}\cup_R(Ꞙ,Ⱥ)=(Ꞙ,Ⱥ)$ and $(Ꞙ,Ⱥ)\cap_\varepsilon\emptyset_{Ⱥ}=\emptyset_{Ⱥ}\cap_\varepsilon(Ꞙ,Ⱥ)=\emptyset_{Ⱥ}$, namely $\emptyset_{Ⱥ}$ is the zero of $(S_{Ⱥ}(U),\cup_R,\cap_\varepsilon)$, $(S_{Ⱥ}(U),\cup_R,\cap_\varepsilon)$ is a hemiring.

3) $(S_{Ⱥ}(U),\cup_\varepsilon,\cap_\varepsilon)$ is an additively and multiplicatively commutative and idempotent semiring with the identity element $U_{Ⱥ}$.

Moreover, since $(Ꞙ,Ⱥ)\cup_\varepsilon\emptyset_{Ⱥ}=\emptyset_{Ⱥ}\cup_\varepsilon(Ꞙ,Ⱥ)=(Ꞙ,Ⱥ)$ and $(Ꞙ,Ⱥ)\cap_\varepsilon\emptyset_{Ⱥ}=\emptyset_{Ⱥ}\cap_\varepsilon(Ꞙ,Ⱥ)=\emptyset_{Ⱥ}$, namely $\emptyset_{Ⱥ}$ is the zero of $(S_{Ⱥ}(U),\cup_\varepsilon,\cap_\varepsilon)$, $(S_{Ⱥ}(U),\cup_\varepsilon,\cap_\varepsilon)$ is a hemiring.

4) $(S_{A}(U),\tilde{\cup},\cap_{\varepsilon})$ is an additively and multiplicatively commutative and idempotent semiring with the identity element U_{A}.

Moreover, since $(F,A)\tilde{\cup}\emptyset_{A}=\emptyset_{A}\tilde{\cup}(F,A)=(F,A)$ and $(F,A)\cap_{\varepsilon}\emptyset_{A}=\emptyset_{A}\cap_{\varepsilon}(F,A)=\emptyset_{A}$, namely $\emptyset_{A}$ is the zero of $(S_{A}(U),\tilde{\cup},\cap_{\varepsilon})$, $(S_{A}(U),\tilde{\cup},\cap_{\varepsilon})$ is a hemiring.

5) $(S_{A}(U),\cup_{\varepsilon},\cap_{\varepsilon})$ is a complemented, bounded, distributive lattice with the lower bound $\emptyset_{A}$ and the upper bound U_{A}.

In fact, it was presented that $(S_{A}(U),\cup_{\varepsilon})$ and $(S_{A}(U),\cap_{\varepsilon})$ are commutative idempotent monoids with the identity element $\emptyset_{A}$ and U_{A}, respectively, $\cup_{\varepsilon}$ ve $\cap_{\varepsilon}$ hold distributive laws in $S_{A}(U)$, and extended intersection distributes over extended union from both left and right sides in $S_{A}(U)$.

Furthermore, since $(F,A)\cup_{\varepsilon}(F,A)^{r}=U_{A}$ and $(F,A)\cap_{\varepsilon}(F,A)^{r}=\emptyset_{A}$, $(S_{A}(U),\cup_{\varepsilon},\cap_{\varepsilon},^{r})$ is a complemented, bounded and distributive lattice; thus a Boolean algebra.

Moreover, since De Morgan law, that is $[(F,A)\cap_{\varepsilon}(\mathfrak{C},A)]^{r}=(F,A)^{r}\cup_{\varepsilon}(\mathfrak{C},A)^{r}$ and $[(F,A)\cup_{\varepsilon}G,A)]^{r}=(F,A)^{r}\cap_{\varepsilon}(\mathfrak{C},A)^{r}$ is satisfied, $(S_{A}(U),\cup_{\varepsilon},\cap_{\varepsilon},^{c})$ is a De Morgan Algebra.

Additionally, $(F,A)\cap_{\varepsilon}(F,A)^{r}=\emptyset_{A}\,\tilde{\subseteq}(\mathfrak{C},A)\cup_{\varepsilon}(\mathfrak{C},A)^{r}=U_{A}$, for all $(F,A),(\mathfrak{C},A)\in S_{A}(U)$, thus $(S_{A}(U),\cup_{\varepsilon},\cap_{\varepsilon},^{r})$ is a Kleene Algebra.

Additionally, it is known that $(F,A)\cap_{\varepsilon}(F,A)^{r}=\emptyset_{A}$ and if $(F,A)\cap_{\varepsilon}(\mathfrak{C},A)=\emptyset_{A}$, then $(\mathfrak{C},A)\,\tilde{\subseteq}(F,A)^{r}$. This shows that $(F,A)^{r}$ is the pseudo-complement of (F,A). Furthermore, since $(F,A)^{r}\cup_{\varepsilon}((F,A)^{r})^{r}=A$, $(S_{A}(U),\cup_{\varepsilon},\cap_{\varepsilon},^{r})$ satisfies Stone's unit property and thus, $(S_{A}(U),\cup_{\varepsilon},\cap_{\varepsilon},^{r})$ is a Stone Algebra.

6) $(S_{A}(U),\cup_{R},\cap_{\varepsilon},^{r})$ and $(S_{A}(U),\tilde{\cup},\cap_{\varepsilon},^{r})$ are complemented, bounded, distributive lattice with the lower bound $\emptyset_{A}$ and the upper bound U_{A}, is therefore, a Boolean algebra, De Morgan algebra, Kleene algebra, and Stone algebra. Additionally $(S_{A}(U),^{r},\cap_{\varepsilon})$ is an MV-algebra the constant element U_{A}.

To show that $(S_{A}(U),^{r},\cap_{\varepsilon},U_{A})$ is an MV-algebra, we neeed show that it satisfies the MV-algebra conditions.

- (MV1) $(S_{A}(U),\cap_{\varepsilon},U_{A})$ commutative monoid with U_{A}.
- (MV2) $((F,A)^{r})^{r}=(F,A)$.
- (MV3) $(U_{A})^{r}\cap_{\varepsilon}(F,A)=\emptyset_{A}\cap_{\varepsilon}(F,A)=\emptyset_{A}=(U_{A})^{r}$.
- (MV4) $[(F,A)^{r}\cap_{\varepsilon}(\mathfrak{C},A)]^{r}\cap_{\varepsilon}(\mathfrak{C},A)=(((\mathfrak{C},A)^{r}\cap_{\varepsilon}(F,A)^{r})^{r}\cap_{\varepsilon}(F,A)$. Indeed,

$$\begin{aligned}[(F,A)^{r}\cap_{\varepsilon}(\mathfrak{C},A)]^{r}\cap_{\varepsilon}(\mathfrak{C},A) &=[((F,A)^{r})^{r}\cup_{\varepsilon}(\mathfrak{C},A)^{r}]\cap_{\varepsilon}(\mathfrak{C},A)\\ &=[(F,A)\cup_{\varepsilon}(\mathfrak{C},A)^{r}]\cap_{\varepsilon}(\mathfrak{C},A)\\ &=[(F,A)\cap_{\varepsilon}(\mathfrak{C},A)]\cup_{\varepsilon}[(\mathfrak{C},A)^{r}\cap_{\varepsilon}(\mathfrak{C},A)]\end{aligned}$$

$=[(Ғ,A)\cap_{\varepsilon}(\mathfrak{C},A)]\cup_{\varepsilon}[(Ғ,A)\cap_{\varepsilon}(Ғ,A)^{r}]$

$= (Ғ,A)\cap_{\varepsilon}[(\mathfrak{C},A)\cup_{\varepsilon}(Ғ,A)^{r}]$

$= (Ғ,A)\cap_{\varepsilon}[(\mathfrak{C},A)^{r}\cap_{\varepsilon} (Ғ,A)]^{r}$

$= [(\mathfrak{C},A)^{r}\cap_{\varepsilon}(Ғ,A)]^{r} \cap_{\varepsilon} (F,A)$

Thus, $(S_{A}(U),{}^{r},\cap_{\varepsilon})$ is an MV-algebra with the constant U_{A}.

5. CONCLUSION

This work presents a thorough examination of all of the characteristics of restricted intersection and extended intersection operations, which are key concepts in SS theory. First of all, the intersection operations are viewed historically, demonstrating the incompleteness of the restricted intersection definition by Ali et al., 2009 and Ali et al., 2011. As the definition has rough edges, the claims in all papers examining the characteristics of the concept and applying it suffer from some problematic circumstances, as it is neglected that the parameter sets of the SSs contained in restricted intersection may also be disjoint. Following the inadequate definition of restricted intersection, some theorems and assertions in previous research on restricted and extended intersection operations were presented without proofs, or the proofs were wrong or missing sections. First and foremost, the presentation of the concept of restricted union is renewed in this study in a new manner that eliminates any incorrectness. This study typically gives proofs based on function equality and corrects any faulty parts in these studies. When evaluating the properties and distributive rules of restricted and extended intersection operations, the case in which the intersection of the parameter sets of the SSs is empty is always considered in the statements and proofs. Moreover, the relationships between restricted and extended intersection operations and the soft subset proposed by Pei and Miao (2005) are also examined in relation to their classical set counterparts. We also add many more properties to the properties that were previously supplied in this topic. In the set of SSs with a fixed parameter set and in the set of sets over the universe, the distribution rules and absorption laws are thoroughly investigated, and the algebraic structures formed by these operations individually and in combination with other SS operations are thoroughly examined with their detailed proofs by also correcting the incorrect parts in the literature in this regard. Boolean algebra, De Morgan algebra, MV-algebra, Kleene algebra, Stone algebra, semiring, hemiring, bounded distributive lattice, monoid, and bounded semi-lattice are some examples of these algebraic structures associated with restricted and extended intersection operations. Furthermore, if a distribution rule does not hold, we specify the condition(s) under which the assertions do. According to these perspectives, this paper represents the most comprehensive analysis of SSs in the literature that is currently available in terms of restricted and extended intersection operations, taking into account all of the earlier research on the topic such as Ali et al., 2009; Ali et al., 2011; Maji et al., 2003; Pei and Miao, 2005; Qin and Hong, 2010; Sen, 2014; Sezgin and Atagün, 2011; Singh and Onyeozili, 2012c as well as Neog and Sut, 2011; Fu, 2011; Ge and Yang S, 2011; Zhu and Wen, 2013; Onyeozili and Gwary, 2014; Husain and Shivani, 2018)), as there isn't any literature available at the moment with such a thorough analysis. As SS operations serve as the theoretical foundation for several approaches to soft computing, which open the door to a variety of applications, such as the development of new SS-based cryptography techniques and decision-making processes and the studies on soft algebraic structures have been the basis for understanding the applications of SS algebra in both classical and non-classical logic, this paper fills a significant gap for the past and future literature by advancing both the theoretical and practical aspects of SS theory. Future research can be employed from the perspective of this study to address other basic SS operations, such as restricted and extended union, difference, and symmetric difference operations.

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