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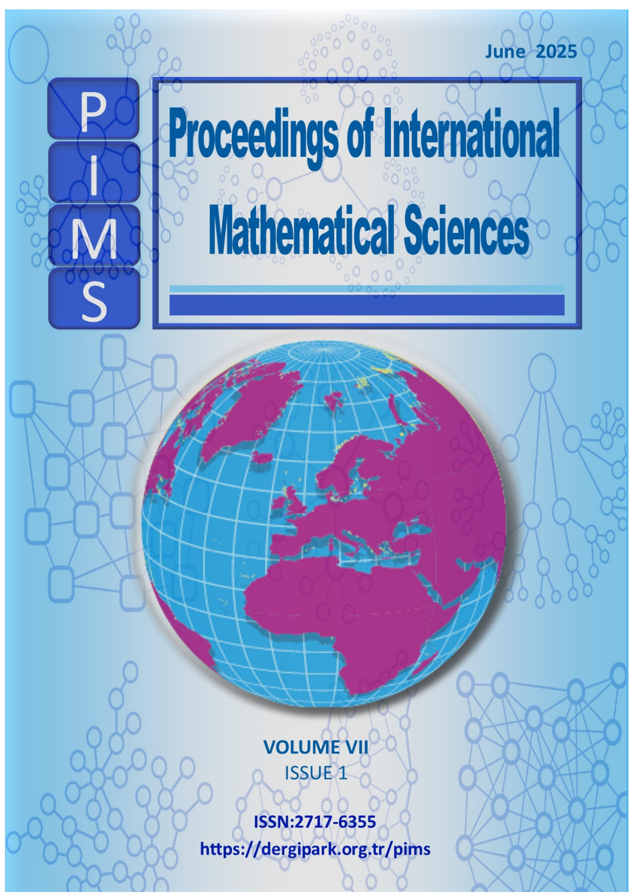

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TWO FIXED POINT RESULTS ON b-METRIC SPACE

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Abstract. In this paper, we present two fixed point theorems on b-metric spaces. First, we obtain a general result, which includes many fixed point theorems on b-metric space in the literature as a special case, by using implicit relation technique, Then, we define the concept of nearly Lipschitzian mapping in b-metric spaces, and we obtain a fixed point theorem for such mappings.

1. Introduction and preliminaries

The notion of metric, which has an important place in mathematics, has been generalized in several ways by weakening some of the metric axioms [2, 3, 10]. For example, by weakening the axiom that the necessary and sufficient condition for two points to be equal is that the distance between them is zero, the concepts of pseudometric have been introduced. Again by neglecting the symmetry axiom and triangular inequality axiom respectively the notion of quasimetric and the notion of semimetric have been defined. On the other hand, by weakening the triangle inequality, the concept of b-metric has been introduced, which is a concept that has been frequently studied recently.

Czerwik in [6] introduced the concept of b-metric space (in short b-ms).

Definition 1.1 ([6]). *Let B be a non-empty set and $s \geq 1$ be a fixed real number. A function $b : B \times B \to \mathbb{R}_+ = [0, \infty)$ is called a b-metric (in short b-m) on B if, for all $\omega, \varpi, \upsilon \in B$, the following conditions hold:*

(b1) $b(\omega, \varpi) = 0$ *if and only if* $\omega = \varpi$,
(b2) $b(\omega, \varpi) = b(\varpi, \omega)$,
(b3) $b(\omega, \upsilon) \leq s\,[b(\omega, \varpi) + b(\varpi, \upsilon)]$.

We call the triplet (B, b, s) a b-ms.

It is important to note that the collection of b-ms's is strictly broader than that of metric spaces, as a b-metric reduces to a metric when $s = 1$.

There are many examples of b-ms in the literature. Let us present one example here for the sake of completeness.

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Example 1.1 ([1]). *Let (B, ρ) be a metric space, $p > 1$ be a real number and let $b(\omega, \varpi) = (\rho(\omega, \varpi))^p$. Then the triplet $(B, b, 2^{p-1})$ is a b-ms. Indeed (b1) and (b2) are obviously hold. On the other hand, since the function $f(\omega) = \omega^p$ $(\omega \geq 0)$ is convex, then*

$$\left(\frac{\lambda + \mu}{2}\right)^p \leq \frac{1}{2}\left(\lambda^p + \mu^p\right),$$

and hence

$$(\lambda + \mu)^p \leq 2^{p-1}(\lambda^p + \mu^p)$$

holds for $\lambda, \mu \geq 0$. Thus, for each $\omega, \varpi, \upsilon \in B$ we obtain

$$b(\omega, \varpi) = (\rho(\omega, \varpi))^p \leq 2^{p-1}[b(\omega, \upsilon) + b(\upsilon, \varpi)].$$

So the condition (b3) of Definition 1.1 is hold and so b is a b -metric.

Definition 1.2 ([4]). *Let (B, b, s) be a b-ms. A sequence $\{\omega_n\}$ in B is defined as:*

(a) *convergent if there exists an element $\omega \in B$ such that $b(\omega_n, \omega) \to 0$ as $n \to \infty$. In this case, we write $\lim_{n\to\infty} \omega_n = \omega$.*
(b) *Cauchy if $b(\omega_n, \omega_m) \to 0$ as $n, m \to \infty$, meaning that for every $\varepsilon > 0$, there is a natural number n_0 such that for all $n, m \geq n_0$, $b(\omega_n, \omega_m) < \varepsilon$.*

The b-ms (B, b, s) is considered complete (in short cb-ms) if every Cauchy sequence in B converges.

Proposition 1.1 ([4], Remark 2.1). *Consider a b-ms (B, b, s). The following statements are true:*

(i) *A convergent sequence has at most one limit,*
(ii) *Every convergent sequence is also Cauchy,*
(iii) *In general, a b-metric does not exhibit continuity.*

Although the b-metric function is not continuous in general, it satisfies the following property.

Lemma 1.2 ([1]). *Let (B, b, s) be a b-ms, and assume that the sequences $\{\omega_n\}$ and $\{\varpi_n\}$ converge to ω and ϖ, respectively. Then, the following inequality holds:*

$$\frac{1}{s^2} b(\omega, \varpi) \leq \liminf_{n\to\infty} b(\omega_n, \varpi_n) \leq \limsup_{n\to\infty} b(\omega_n, \varpi_n) \leq s^2 b(\omega, \varpi).$$

In particular, if $\omega = \varpi$, we have $\lim_{n\to\infty} b(\omega_n, \varpi_n) = 0$.

Furthermore, for any $u \in B$, the following inequality holds:

$$\frac{1}{s} b(\omega, u) \leq \liminf_{n\to\infty} b(\omega_n, u) \leq \limsup_{n\to\infty} b(\omega_n, u) \leq s b(\omega, u).$$

The aim of this paper is to present some fixed point theorems in b-metric space inspired by metric fixed point theory. As it is known, there are basically three stages in the proof of metric fixed point theorems. The first is to construct an iteration sequence, which is usually a Picard iteration sequence, the second stage is to show that this sequence is a Cauchy sequence, which is the real difficulty in the proof. The third is to show that the point to which the sequence converges, which is guaranteed by the completeness of the space, is a fixed point. In showing that the sequence in question is a Cauchy sequence, sometimes direct proof, sometimes proof by contradiction method is used according to the hypotheses. For example, in a metric space (B, ρ) if there exists $\lambda \in [0, 1)$ such that

$$\rho(\omega_n, \omega_{n+1}) \leq \lambda \rho(\omega_{n-1}, \omega_n) \tag{1.1}$$

for all $n \in \mathbb{N}$, then $\{\omega_n\}$ is a Cauchy sequence. Indeed, by using (1.1) and the triangular inequality of the metric ρ, we have

$$\begin{aligned} \rho(\omega_n, \omega_m) &\leq \rho(\omega_n, \omega_{n+1}) + \rho(\omega_{n+1}, \omega_{n+2}) + \cdots + \rho(\omega_{m-1}, \omega_m) \\ &\leq \frac{\lambda^n}{1-\lambda}\rho(\omega_0, \omega_1), \end{aligned}$$

for $m > n$, which shows that $\{\omega_n\}$ is Cauchy. Similarly, by using the triangular inequality of the metric ρ, we have, for $m > n$

$$\begin{aligned} \rho(\omega_n, \omega_m) &\leq \rho(\omega_n, \omega_{n+1}) + \rho(\omega_{n+1}, \omega_{n+2}) + \cdots + \rho(\omega_{m-1}, \omega_m) \\ &\leq \sum_{k=n}^{\infty} \rho(\omega_k, \omega_{k+1}) \end{aligned}$$

and so the convergence of the series

$$\sum_{n=1}^{\infty} \rho(\omega_n, \omega_{n+1}) \tag{1.2}$$

is guaranteed that $\{\omega_n\}$ is a Cauchy sequence. Therefore, before presenting a fixed point theorem in a b-ms, it is important to investigate whether these situations are also valid in a b-ms.

As a result of the research carried out in this context, Suzuki [12] presented an example showing that the convergence of the series given in (1.2) in a b-ms is not sufficient for the sequence $\{\omega_n\}$ to be a Cauchy. However, the following lemmas helps us in this direction.

Lemma 1.3 ([7]). *Let $\{\omega_n\}$ be a sequence in a b-ms (B, b, s). The sequence $\{\omega_n\}$ is Cauchy if there exists a constant $\gamma > \log_2 s$ such that the series*

$$\sum_{n=1}^{\infty} n^{\gamma} b(\omega_n, \omega_{n+1})$$

is convergent.

Lemma 1.4 ([7]). *Let $\{\omega_n\}$ be a sequence in a b-ms (B, b, s). The sequence $\{\omega_n\}$ is Cauchy if there exists $\alpha > 1$ such that the series*

$$\sum_{n=1}^{\infty} \alpha^{n} b(\omega_n, \omega_{n+1})$$

is convergent.

On the other hand, it may be thought that the inequality (1.1) may not be sufficient for the sequence $\{\omega_n\}$ to be a Cauchy sequence in a b-metric space, and in fact it is clear that $\lambda < \frac{1}{s}$ will guarantee this. However, the following lemma shows that this is valid even if $\lambda < 1$.

Lemma 1.5 ([8, 12]). *Consider (B, b, s), a b-ms, and a sequence $\{\omega_n\}$ within B. Suppose there exists $\alpha \in [0, 1)$ such that*

$$b(\omega_n, \omega_{n+1}) \leq \alpha b(\omega_{n-1}, \omega_n),$$

for any $n \in \mathbb{N}$. Under this condition, the sequence $\{\omega_n\}$ is Cauchy.

Considering the above lemmas and similar ones, many fixed point theorems have been obtained in b-ms. Let us mention a few here.

Throughout the remainder of this paper, the term ”fixed point” will be abbreviated as ”fp” for brevity.

Theorem 1.6 ([9]). *Let (B, b, s) be a cb-ms, and let $\Upsilon : B \to B$ be a mapping such that:*

$$b(\Upsilon\omega, \Upsilon\varpi) \leq \lambda b(\omega, \varpi)$$

for all $\omega, \varpi \in B$, where $\lambda \in [0, 1)$. Then Υ admits a unique fp υ, and for any starting point $\omega_0 \in B$, the sequence $\{\Upsilon^n\omega_0\}$ converges to υ.

Theorem 1.7 ([9]). *Let (B, b, s) be a cb-ms, and let $\Upsilon : B \to B$ be a mapping such that:*

$$b(\Upsilon\omega, \Upsilon\varpi) \leq \alpha b(\omega, \varpi) + \beta b(\omega, \Upsilon\omega) + \gamma b(\varpi, \Upsilon\varpi)$$

for all $\omega, \varpi \in B$, where $\alpha, \beta, \gamma \in [0, 1)$ and $\alpha + \beta + \gamma < 1$. Then Υ admits a unique fp υ, and for any starting point $\omega_0 \in B$, the sequence $\{\Upsilon^n\omega_0\}$ converges to υ, provided that one of the following hold:

- *Υ is continuous,*
- *$s\beta < 1$*
- *$s\gamma < 1$.*

Theorem 1.8 ([5]). *Let (B, b, s) be a cb-ms, and let $\Upsilon : B \to B$ be a mapping such that:*

$$b(\Upsilon\omega, \Upsilon\varpi) \leq q \max\left\{b(\omega, \varpi), b(\omega, \Upsilon\omega), b(\varpi, \Upsilon\varpi), \frac{1}{2}[b(\omega, \Upsilon\varpi) + b(\varpi, \Upsilon\omega)]\right\}$$

for all $\omega, \varpi \in B$, where $q \in [0, \frac{1}{s})$. Then Υ admits a unique fp υ, and for any starting point $\omega_0 \in B$, the sequence $\{\Upsilon^n\omega_0\}$ converges to υ, provided that Υ is continuous.

2. Result via implicit relation

In this section, we will first consider an implicit relation with some properties and give a few examples. Then, we will present a fp theorem in the b-ms, taking this relation into account.

Let $\mathcal{F}$ be the set of all functions $F : \mathbb{R}_+^6 \to \mathbb{R}$ satisfying the following conditions:

(F1) $F(\xi_1, \cdots, \xi_6)$ is nonincreasing in variables ξ_5 and ξ_6.

(F2) there exists $h \in [0, 1)$ such that for $\eta > 0$

$$F(\eta, \mu, \mu, \eta, \eta + \mu, 0) \leq 0$$

or

$$F(\eta, \mu, \eta, \mu, 0, \eta + \mu) \leq 0$$

implies $\eta \leq h\mu$.

(F3) $F(\eta, \eta, 0, 0, \eta, \eta) > 0$ for $\eta > 0$.

Also, for $s \geq 1$, let $\mathcal{F}_s$ be the set of all continuous functions $F \in \mathcal{F}$ satisfying the following conditions:

(Fs1) $F(\xi_1, \cdots, \xi_6)$ is nondecreasing in variable ξ_1.

(Fs2) $F(\frac{\eta}{s}, 0, 0, \eta, \eta, 0) > 0$ or $F(\frac{\eta}{s}, 0, \eta, 0, 0, \eta) > 0$ for $\eta > 0$.

It is clear that $\mathcal{F}_s \subseteq \mathcal{F}$.

Example 2.1. *Let $F(\xi_1, \cdots, \xi_6) = \xi_1 - \alpha\xi_2$, where $\alpha \in [0, 1)$. Then $F \in \mathcal{F}_s$ for all $s \geq 1$.*

Example 2.2. *Let $F(\xi_1, \cdots, \xi_6) = \xi_1 - \alpha\xi_2 - \beta\xi_3 - \gamma\xi_4$, where $\alpha, \beta, \gamma \in [0, 1)$ and $\alpha + \beta + \gamma < 1$. Then $F \in \mathcal{F}$. Also if $s\beta < 1$ or $s\gamma < 1$ then $F \in \mathcal{F}_s$.*

Example 2.3. *Let $F(\xi_1, \cdots, \xi_6) = \xi_1 - q \max\left\{\xi_2, \xi_3, \xi_4, \frac{\xi_5+\xi_6}{2}\right\}$, where $q \in [0, 1)$. Then $F \in \mathcal{F}$. Also if $qs < 1$, then $F \in \mathcal{F}_s$.*

Example 2.4. *Let $F(\xi_1, \cdots, \xi_6) = \xi_1 - c\{\xi_2 + |\xi_3 - \xi_4|\}$, where $c \in [0, 1)$. Then $F \in \mathcal{F}$. Also if $cs < 1$, then $F \in \mathcal{F}_s$.*

Example 2.5. *Let $F(\xi_1, \cdots, \xi_6) = \xi_1 - \alpha \max\{\xi_2, \xi_3, \xi_4\} - \beta\frac{\xi_5+\xi_6}{2}$, where $\alpha, \beta \in [0, 1)$ and $\alpha + \beta < 1$. Then $F \in \mathcal{F}$. Also if $(2\alpha + \beta)s < 2$, then $F \in \mathcal{F}_s$.*

Theorem 2.1. *Let (B, b, s) be a cb-ms, $\Upsilon : B \to B$ be a mapping satisfying:*

$$F\left(b(\Upsilon\omega, \Upsilon\varpi), b(\omega, \varpi), b(\omega, \Upsilon\omega), b(\varpi, \Upsilon\varpi), \frac{b(\omega, \Upsilon\varpi)}{s}, \frac{b(\varpi, \Upsilon\omega)}{s}\right) \leq 0 \tag{2.1}$$

for all $\omega, \varpi \in B$, where $F \in \mathcal{F}$. Then Υ has a unique fp υ, and for every $\omega_0 \in B$, the sequence $\{\Upsilon^n\omega_0\}$ converges to υ provided that Υ is continuous.

Proof. Let $\omega_0 \in B$ be arbitrary. Define a sequence $\{\omega_n\}$ by $\omega_{n+1} = \Upsilon\omega_n$ for all $n \geq 0$. For the sake of brevity let $b_n = b(\omega_n, \omega_{n+1})$ for $n \geq 0$. If there exists $n_0 \in \mathbb{N}$ such that $b_{n_0} = 0$, then ω_{n_0} is a fp of Υ. In this case, the sequence $\{\omega_n\}$ will be an eventually constant sequence with the constant ω_{n_0} and will converge to ω_{n_0} . Now, let $b_n > 0$ for all $n \in \mathbb{N}$. Then from (2.1) we have,

$$F\left(\begin{array}{r} b(\Upsilon\omega_n, \Upsilon\omega_{n+1}), b(\omega_n, \omega_{n+1}), b(\omega_n, \Upsilon\omega_n), b(\omega_{n+1}, \Upsilon\omega_{n+1}), \\ \frac{1}{s}b(\omega_n, \Upsilon\omega_{n+1}), \frac{1}{s}b(\omega_{n+1}, \Upsilon\omega_n) \end{array}\right) \leq 0$$

for all $n \in \mathbb{N}$. Thus, we have

$$F\left(b(\omega_{n+1}, \omega_{n+2}), b(\omega_n, \omega_{n+1}), b(\omega_n, \omega_{n+1}), b(\omega_{n+1}, \omega_{n+2}), \frac{b(\omega_n, \omega_{n+2})}{s}, 0\right) \leq 0$$

for all $n \in \mathbb{N}$. Now, from the last inequality and (F1) we have

$$F(b_{n+1}, b_n, b_n, b_{n+1}, b_n + b_{n+1}, 0) \leq 0$$

and hence from (F2), there exists $h \in [0, 1)$ such that

$$b_{n+1} \leq hb_n$$

or equivalently

$$b(\omega_{n+1}, \omega_{n+2}) \leq hb(\omega_n, \omega_{n+1})$$

for all $n \in \mathbb{N}$. From Lemma 1.5, we have $\{\omega_n\}$ is a Cauchy sequence in the cb-ms (B, b, s). Hence there exists $\upsilon \in B$ such that $\lim \omega_n = \upsilon$, that is $\lim b(\omega_n, \upsilon) = 0$. Since Υ is continuous we have $\lim b(\omega_{n+1}, \Upsilon\upsilon) = \lim b(\Upsilon\omega_n, \Upsilon\upsilon) = 0$, and so by the uniqueness of the limit of $\{\omega_n\}$, we get $\upsilon = \Upsilon\upsilon$. Now, let w be also a fp of Υ with $b(\upsilon, w) > 0$, then from (2.1) we have

$$F\left(b(\Upsilon\upsilon, \Upsilon w), b(\upsilon, w), b(\upsilon, \Upsilon\upsilon), b(w, \Upsilon w), \frac{b(\upsilon, \Upsilon w)}{s}, \frac{b(w, \Upsilon\upsilon)}{s}\right) \leq 0$$

or equivalently we have

$$F\left(b(\upsilon, w), b(\upsilon, w), 0, 0, \frac{b(\upsilon, w)}{s}, \frac{b(w, \upsilon)}{s}\right) \leq 0.$$

Hence from (F1) we have

$$F(b(\upsilon, w), b(\upsilon, w), 0, 0, b(\upsilon, w), b(\upsilon, w)) \leq 0,$$

which is contradict to (F3). Hence, the fp of Υ is unique. □

In Theorem 2.1, by considering the class $\mathcal{F}_s$, we can remove the continuity condition of Υ.

Theorem 2.2. *Let (B, b, s) be a cb-ms, $\Upsilon : B \to B$ be a mapping satisfying (2.1) for all $\omega, \varpi \in B$, where $F \in \mathcal{F}_s$. Then Υ has a unique fp υ, and for every $\omega_0 \in B$, the sequence $\{\Upsilon^n \omega_0\}$ converges to υ.*

Proof. As in the proof of Theorem 2.1, we can show that the iteration sequence $\{\omega_n\}$ is Cauchy and hence converges to some $\upsilon \in B$. Now, let $b(\upsilon, \Upsilon\upsilon) > 0$. Then, from (2.1) we have

$$F\left(b(\Upsilon\omega_n, \Upsilon\upsilon), b(\omega_n, \upsilon), b(\omega_n, \Upsilon\omega_n), b(\upsilon, \Upsilon\upsilon), \frac{b(\omega_n, \Upsilon\upsilon)}{s}, \frac{b(\upsilon, \Upsilon\omega_n)}{s}\right) \leq 0$$

or equivalently we have

$$F\left(b(\omega_{n+1}, \Upsilon\upsilon), b(\omega_n, \upsilon), b(\omega_n, \omega_{n+1}), b(\upsilon, \Upsilon\upsilon), \frac{b(\omega_n, \Upsilon\upsilon)}{s}, \frac{b(\upsilon, \omega_{n+1})}{s}\right) \leq 0.$$

Hence, by taking the continuity of F into account, from Lemma 1.2, (F1) and (Fs1) we have

$$F\left(\frac{b(\upsilon, \Upsilon\upsilon)}{s}, 0, 0, b(\upsilon, \Upsilon\upsilon), b(\upsilon, \Upsilon\upsilon), 0\right) \leq 0.$$

which is contradict to (Fs2). Hence we have $b(\upsilon, \Upsilon\upsilon) = 0$. The uniqueness follows from (F3). □

If we consider special functions F in Theorem 2.1 or Theorem 2.2, we can obtain some fp theorems obtained in the literature, including Theorem 1.6, Theorem 1.7 and Theorem 1.8. Here we present two results obtained with Example 2.4 and Example 2.5.

Corollary 2.3. *Let (B, b, s) be a cb-ms, $\Upsilon : B \to B$ be a mapping satisfying:*

$$b(\Upsilon\omega, \Upsilon\varpi) \leq c\{b(\omega, \varpi) + |b(\omega, \Upsilon\omega) - b(\varpi, \Upsilon\varpi)|\}$$

for all $\omega, \varpi \in B$, where $c \in [0, 1)$. Then Υ has a unique fp υ, and for every $\omega_0 \in B$, the sequence $\{\Upsilon^n \omega_0\}$ converges to υ provided that one of the following hold:

- *Υ is continuous,*
- *$sc < 1$.*

Corollary 2.4. *Let (B, b, s) be a cb-ms, $\Upsilon : B \to B$ be a mapping satisfying:*

$$b(\Upsilon\omega, \Upsilon\varpi) \leq \alpha \max\{b(\omega, \varpi), b(\omega, \Upsilon\omega), b(\varpi, \Upsilon\varpi)\} + \beta\frac{b(\omega, \Upsilon\varpi) + b(\varpi, \Upsilon\omega)}{2s}$$

for all $\omega, \varpi \in B$, where $\alpha, \beta \in [0, 1)$ and $\alpha + \beta < 1$. Then Υ has a unique fp υ, and for every $\omega_0 \in B$, the sequence $\{\Upsilon^n \omega_0\}$ converges to υ provided that one of the following hold:

- *Υ is continuous,*
- *$(2\alpha + \beta)s < 2$.*

3. Results for nearly Lipschitzian mappings

In this section, we will present the b-ms version of the notion of a nearly Lipschitzian mappings defined by Sahu [11] in the metric space and give a fp theorem for such mappings. Recall that a self mapping Υ of a b-ms (B, b, s) is said to be Lipschitz mapping if there exists $\lambda \geq 0$ such that

$$b(\Upsilon\omega, \Upsilon\varpi) \leq \lambda b(\omega, \varpi)$$

for all $\omega, \varpi \in B$. It is clear that every Lipschitz mapping is continuous.

Definition 3.1. *Let (B, b, s) be a b-ms, $\Upsilon : B \to B$ be a mapping and $\{\alpha_n\}$ be a sequence in $\mathbb{R}_+$ such that $\alpha_n \to 0$. Then, Υ is said to be nearly Lipschitzian with respect to $\{\alpha_n\}$ if for each $n \in \mathbb{N}$, there exists a constant $\lambda_n \geq 0$ such that*

$$b(\Upsilon^n \omega, \Upsilon^n \varpi) \leq \lambda_n \{b(\omega, \varpi) + \alpha_n\} \tag{3.1}$$

for all $\omega, \varpi \in B$. The infimum of constants λ_n for which (3.1) holds is denoted by $\phi(\Upsilon^n)$ and called the nearly Lipschitz constant.

Example 3.1. *Let $B = [0, 2]$ with the b-metric $b(\omega, \varpi) = (\omega - \varpi)^2$ and $\Upsilon : B \to B$ a mapping defined by*

$$\Upsilon\omega = \begin{cases} \omega - \frac{1}{4} & \omega > 1 \\ \\ \frac{1}{2} \quad , & \omega \leq 1 \end{cases}.$$

Since Υ is discontinuous, then it is non-Lipschitz mapping. However, by the routine calculation we can see

$$b(\Upsilon^n \omega, \Upsilon^n \varpi) \leq \frac{n}{2^n} \{b(\omega, \varpi) + \alpha_n\},$$

where

$$\alpha_n = \begin{cases} 16 & n \leq 4 \\ \\ \frac{1}{n} \quad , & n > 4 \end{cases}.$$

Hence, Υ is nearly Lipschitzian mapping.

Theorem 3.1. *Let (B, b, s) be a cb-ms and $\Upsilon : B \to B$ be a continuous nearly Lipschitzian mapping with nearly Lipschitz constant $\phi(\Upsilon^n)$. If*

$$\limsup_{n\to\infty} \sqrt[n]{\phi(\Upsilon^n)} < 1, \tag{3.2}$$

then Υ has a unique fp υ, and for every $\omega_0 \in B$, the sequence $\{\Upsilon^n \omega_0\}$ converges to υ.

Proof. Let $\omega_0 \in B$ be arbitrary. Define a sequence $\{\omega_n\}$ by $\omega_{n+1} = \Upsilon\omega_n$ for all $n \geq 0$. For the sake of brevity let $b_n = b(\omega_n, \omega_{n+1})$ for $n \geq 0$. If there exists $n_0 \in \mathbb{N}$ such that $b_{n_0} = 0$, then ω_{n_0} is a fp of Υ. In this case, the sequence $\{\omega_n\}$ will be an eventually constant sequence with the constant ω_{n_0} and will converge to ω_{n_0}. Now, let $b_n > 0$ for all $n \in \mathbb{N}$. Hence

$$0 < b_n = b(\Upsilon^n \omega_0, \Upsilon^{n+1} \omega_0) \leq \phi(\Upsilon^n) \{b(\omega_0, \omega_1) + \alpha_n\} \tag{3.3}$$

for all $n \in \mathbb{N}$. Now, define $\limsup_{n\to\infty} \sqrt[n]{\phi(\Upsilon^n)} = \delta$, then from (3.2) and (3.3), we have $0 < \delta < 1$. Choose $\alpha \in (1, \frac{1}{\delta})$, then from (3.3) we have

$$\alpha^n b_n \leq \alpha^n \phi(\Upsilon^n) \{b(\omega_0, \omega_1) + M\}, \tag{3.4}$$

where $M = \sup \alpha_n$, for all $n \in \mathbb{N}$. Hence, from (3.4) we have

$$\sum_{n=1}^{\infty} \alpha^n b_n \leq \{b(\omega_0, \omega_1) + M\} \sum_{n=1}^{\infty} \alpha^n \phi(\Upsilon^n).$$

By the root test for the convergence of the series we have

$$\sum_{n=1}^{\infty} \alpha^n \phi(\Upsilon^n) < \infty$$

since

$$\limsup_{n\to\infty} \sqrt[n]{\alpha^n \phi(\Upsilon^n)} = \alpha\delta < 1.$$

Therefore, by the comparison test we have

$$\sum_{n=1}^{\infty} \alpha^n b_n = \sum_{n=1}^{\infty} \alpha^n b(\omega_n, \omega_{n+1}) < \infty,$$

and so from Lemma 1.4, $\{\omega_n\}$ is a Cauchy sequence in the cb-ms (B, b, s). Hence there exists $\upsilon \in B$ such that $\lim \omega_n = \upsilon$, that is $\lim b(\omega_n, \upsilon) = 0$. Since Υ is continuous we have $\lim b(\omega_{n+1}, \Upsilon\upsilon) = \lim b(\Upsilon\omega_n, \Upsilon\upsilon) = 0$, and so by the uniqueness of the limit of $\{\omega_n\}$, we get $\upsilon = \Upsilon\upsilon$. Now, let w be also a fp of Υ with $b(\upsilon, w) > 0$, then from we have

$$\begin{aligned}
\sum_{n=1}^{\infty} b(\upsilon, w) &= \sum_{n=1}^{\infty} b(\Upsilon^n \upsilon, \Upsilon^n w) \\
&\leq \sum_{n=1}^{\infty} \phi(\Upsilon^n) \{b(\upsilon, w) + \alpha_n\} \\
&\leq \{b(\upsilon, w) + M\} \sum_{n=1}^{\infty} \phi(\Upsilon^n) \\
&< \infty,
\end{aligned}$$

which is a contradiction, since $\sum\limits_{n=1}^{\infty} b(\upsilon, w) = \infty$. Hence, Υ has a unique fp. □

4. Conclusion

In conclusion, this paper presents significant advancements in the theory of fixed point theorems in the context of b-metric spaces. The general result established in the first part of the paper not only provides a broader framework for fixed point theorems in b-metric spaces but also encompasses many existing results from the literature as special cases. By employing the implicit relation technique, we have extended the applicability of previous fixed point theorems to a more general setting. Furthermore, the introduction of the concept of nearly Lipschitzian mappings in b-metric spaces has led to the derivation of a new fixed point theorem, broadening the scope of mappings for which fixed points can be guaranteed. These contributions offer valuable insights and open up potential avenues for further research in the field of metric and b-metric spaces, particularly in the study of fixed point theory and its applications.

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HIGHER-ORDER EQUATIONS WITH ROBIN BOUNDARY CONDITIONS IN THE UPPER HALF COMPLEX PLANE

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Abstract. This study explores the solvability and solution of a Robin problem for a higher-order differential equation in the upper half-plane. Using the framework of the higher-order Cauchy-Riemann operator, we extend classical techniques to handle complex boundary interactions.

Necessary conditions are established by analyzing the boundary operator's structure. To construct solutions, we apply an integral method that reduces the problem to a more manageable form, employing higher-order Cauchy-type transforms and kernel functions suited to the upper half-plane.

1. Introduction

Different boundary value issues are investigated in [1-5] for various domains with explicit solutions. Boundary value problems play a fundamental role in mathematical analysis and its applications, particularly in understanding physical phenomena modeled by partial differential equations. Among these, Robin boundary value problems represent a versatile class that interpolates between Dirichlet and Neumann conditions, offering a rich framework for both theoretical exploration and practical applications. While significant progress has been made in the study of classical boundary value problems in the complex plane, the investigation of higher-order equations in the upper half-plane, especially under Robin boundary conditions, remains an open and challenging area of research. The current work focuses on a Robin boundary value problem associated with a higher-order partial differential equation in the upper half-plane. These problems are of particular interest due to their intricate coupling of boundary and interior conditions, as well as their connection to advanced operators such as the higher-order Cauchy-Riemann operator. Such equations naturally arise in a variety of contexts, including fluid dynamics, elasticity, and electromagnetic theory, where higher-order derivatives and mixed boundary conditions are integral to the models. Our primary objective is to establish the conditions under which the Robin boundary value problem admits a solution and to provide an explicit integral representation of the solution. By employing tools from complex analysis, we develop a framework that reduces the complexity of the higher-order equation and reveals the interplay between

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the boundary conditions and the solution's analytic structure. The integral representation not only provides a practical method for constructing solutions but also offers deeper insights into the properties of the underlying operator. This study contributes to the broader theory of boundary value problems in complex analysis and lays the groundwork for further investigations into higher-order equations in more general domains. Additionally, the techniques and results presented here may find applications in mathematical physics and engineering, where the study of such equations is both theoretical and practical.

2. Preliminaries

Let $\mathbb{H} := \{z \in \mathbb{C} : Imz > 0\}$ denotes the upper half plane and $\mathbb{R}$ its boundary (the x-axis). Here $L_{p,2}(\mathbb{H};\mathbb{C})$ means the space of complex-valued functions f defined in $\mathbb{H}$. T is Pompeiu integral operator. The Cauchy–Riemann operator is a fundamental tool in complex analysis and is defined as

$$\frac{\partial}{\partial \bar{z}} := \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right),$$

where $z = x + iy$. A complex-valued function $\omega : \mathbb{H} \to \mathbb{C}$ is said to be analytic if and only if $\omega_{\bar{z}} = 0$, that is, if it lies in the kernel of the Cauchy–Riemann operator. The inhomogeneous Cauchy–Riemann equation $\omega_{\bar{z}} = f$, where f is a given function, arises naturally in the context of boundary value problems involving non-analytic functions and serves as a starting point for constructing solutions using integral representations.

In this study, we extend the classical theory by considering higher-order versions of the Cauchy–Riemann operator. These operators are crucial for analyzing more intricate boundary conditions, such as those appearing in Robin-type problems for higher-order partial differential equations.

In this study, we formulate the problem in the upper half-plane $\mathbb{H}$ due to its analytical convenience and compatibility with classical integral operators such as the Cauchy and Pompeiu transforms. The upper half-plane provides a natural setting where the boundary $\mathbb{R}$ allows explicit use of well-known kernels and boundary value techniques. Although the analysis is carried out in $\mathbb{H}$, the approach can be extended to the lower half-plane $\mathbb{L} := \{z \in \mathbb{C} : \operatorname{Im} z < 0\}$ by symmetry or by adapting the kernel functions accordingly.

Theorem 2.1. [3] *The Robin problem $\omega_{\bar{z}} = 0$ in $\mathbb{H}$, $\alpha\omega + \partial_\nu \omega = \gamma$ on $\mathbb{R}, \omega(i) = C_0$ for $\alpha \in O(\mathbb{H}) \cap C\left(\overline{\mathbb{H}};\mathbb{C}\right), \gamma \in L^2(\mathbb{R};\mathbb{C}) \cap C(\mathbb{R};\mathbb{C})$ and $C_0 \in \mathbb{C}$ is uniquely solvable if and only if*

$$\frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{\gamma(t)}{t-\bar{z}}\, dt = 0 \tag{2.1}$$

for $z \in \mathbb{H}$. The solution is given by

$$\omega(z) = C_0 e^{-\int_i^z i\alpha(\varsigma)d\varsigma} - \frac{i}{\pi}\int_{-\infty}^{\infty} \gamma(t) \int_i^z \frac{Im(s)}{|t-s|^2} e^{\int_s^z i\alpha(\varsigma)d\varsigma}\, ds\, dt. \tag{2.2}$$

Here $O(\mathbb{H})$ denotes the set of analytic functions in $\mathbb{H}$.

Proposition 2.2. [5] *The Robin problem for the inhomogeneous Cauchy–Riemann equation*

$$\omega_{\bar{z}} = f \text{ in } \mathbb{H},$$
$$\omega - \partial_y \omega = \gamma \text{ on } \mathbb{R},$$
$$\omega(0) = c + Tf(0)$$

for given $f \in L_{p,2}(\mathbb{H};\mathbb{C}) \cap C^1(\overline{\mathbb{H}};\mathbb{C})$, $p > 2$ *and* $\gamma \in L^2(\mathbb{R};\mathbb{C}) \cap C(\mathbb{R};\mathbb{C})$ *is uniquely solvable if and only if for* $z \in \mathbb{H}$

$$\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{i\gamma(t)+f(t)}{t-\overline{z}}\,dt + iTf(\overline{z}) + T\partial_\zeta f(\overline{z}) = 0. \tag{2.3}$$

The solution is given by

$$\omega(z) = ce^{-iz} + \frac{1}{2\pi i}\int_{-\infty}^{\infty}(\gamma(t)-2if(t))\,e^{i(t-z)}\int_{-t}^{z-t}\frac{e^{i\zeta}}{i\zeta}\,d\zeta\,dt + Tf(z) \tag{2.4}$$

3. Robin Boundary Value Problem For Higher Order Equation

This section explores the Robin Boundary value problem associated with higher order partial differential equations in the upper half complex plane. By generalizing the classical techniques, we aim to establish conditions fort he existence and uniqueness of the solutions. Additionally, explicit representations are derived, highlighting the interplay between boundary conditions and solution's analytical structure.

Theorem 3.1. *Let* $\gamma_i \in L_2(\mathbb{R};\mathbb{C}) \cap C(\mathbb{R};\mathbb{C})$, $p > 2$, $w \in L_{p,2}(\mathbb{H};\mathbb{C}) \cap C^n(\overline{\mathbb{H}};\mathbb{C}) \cap L_2(\mathbb{R};\mathbb{C})$, *and* $c_i \in \mathbb{C}$ *for* $i = 1,\dots,n$. *The Robin boundary value problem for the homogeneous Cauchy-Riemann equation is:*

$$\partial_{\overline{z}}^n w = 0 \text{ in } \mathbb{H} \tag{3.1}$$
$$\partial_{\overline{z}}^k w - \partial_y \partial_{\overline{z}}^k w = \gamma_{k+1} \text{ on } \mathbb{R}, \tag{3.2}$$
$$\partial_{\overline{z}}^{n-1} w + \partial_y \partial_{\overline{z}}^{n-1} w = \gamma_n \text{ on } \mathbb{R}, \tag{3.3}$$
$$\partial_{\overline{z}}^k w(0) = c_{k+1} + T\partial_{\overline{z}}^{k+1} w(0), \tag{3.4}$$
$$\partial_{\overline{z}}^{n-1} w(i) = c_n, \tag{3.5}$$

is uniquely solvable if and only if for $z \in \mathbb{H}$,

$$\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{i\gamma_{k+1}(t)+u_{k+1}(t)}{t-\overline{z}}\,dt + iTu_{k+1}(\overline{z}) + T\partial_\zeta u_{k+1}(\overline{z}) = 0 \quad \text{for } 0 \le k \le n-2, \tag{3.6}$$
$$\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\gamma_n(t)}{t-\overline{z}}\,dt = 0 \quad \text{for } k = n-1, \tag{3.7}$$

and the solution is given by

$$w(z) = c_1 e^{-iz} + Tu_1(z) + \frac{1}{2\pi i}\int_{-\infty}^{\infty}(\gamma_1(t)-2iu_1(t))\,e^{i(t-z)}\int_{-t}^{z-t}\frac{e^{i\zeta}}{i\zeta}\,d\zeta\,dt, \tag{3.8}$$

where the auxiliary functions $u_{k+1}(t)$ *and* $u_{k+1}(z)$ *are defined as:*

$$u_{k+1}(t) = c_{k+2}e^{-it} + \int_0^t (i\gamma_{k+2}(\varsigma) + u_{k+2}(\varsigma))\,e^{i(\varsigma-t)}\,d\varsigma, \tag{3.9}$$

$$u_{k+1}(z) = c_{k+2}e^{-iz} + Tu_{k+2}(z) + \frac{1}{2\pi i}\int_{-\infty}^{\infty}(\gamma_{k+2}(t)-2iu_{k+2}(t))\,e^{i(t-z)}\int_{-t}^{z-t}\frac{e^{i\zeta}}{i\zeta}\,d\zeta\,dt \tag{3.10}$$

for $0 \le k \le n-3$,
and

$$u_{n-1}(t) = c_n e^{it} - \int_0^t i\gamma_n(\varsigma)e^{-i(\varsigma-t)}\,d\varsigma, \tag{3.11}$$

$$u_{n-1}(z) = c_n e^{1+iz} - \frac{i}{\pi}\int_{-\infty}^{\infty}\gamma(t)\int_i^z e^{i(z-s)}\frac{Im(s)}{|t-s|^2}\,ds\,dt \tag{3.12}$$

for $k = n-1$.

Proof. For $n = 1$, we consider $w(z)$ as

$$w(z) = u_1(z): \tag{3.13}$$

$$w(z) = c_1 e^{1+iz} - \frac{i}{\pi}\int_{-\infty}^{\infty}\gamma(t)\int_i^z e^{i(z-s)}\frac{\mathrm{Im}(s)}{|t-s|^2}\,ds\,dt. \tag{3.14}$$

For $n > 1$, the problem reduces to the following sequence of equations:

(1) $w_{\bar z} = u_1$, $w - \partial_y w = \gamma_1$ on $\mathbb{R}$,

$$w(0) = c_1 + Tu_1(0);$$

(2) $u_{i_{\bar z}} = u_{i+1}$, $u_i - \partial_y u_i = \gamma_{i+1}$ on $\mathbb{R}$,

$$u_i(0) = c_{i+1} + Tu_{i+1}(0), \quad 1 \le i \le n-2;$$

(3) $(u_{n-1})_{\bar z} = 0$ in H, $u_{n-1} + \partial_y u_{n-1} = \gamma_n$ on $\mathbb{R}$,

$$u_{n-1}(i) = c_n.$$

The condition $u_i - \partial_y u_i = \gamma_{i+1}$ implies

$$u_i'(t) + iu_i(t) = i\gamma_{i+1}(t) + u_{i+1}(t) \quad \text{on } \mathbb{R}.$$

Solving this differential equation, we obtain

$$u_i(t) = c_{i+1}e^{it} + \int_0^t (i\gamma_{i+1}(\varsigma) + u_{i+1}(\varsigma))\, e^{i(\varsigma-t)}\, d\varsigma.$$

Combining the solutions and solvability conditions completes the proof. □

Theorem 3.2. *Let $\gamma_i \in L_2(\mathbb{R};\mathbb{C}) \cap C(\mathbb{R};\mathbb{C})$, $p > 2$, $w \in L_{p,2}(\mathbb{H};\mathbb{C}) \cap C^n(\overline{\mathbb{H}};\mathbb{C}) \cap L_2(\mathbb{R};\mathbb{C})$, and $c_i \in \mathbb{C}$ for $i = 1,\dots,n$. The Robin boundary value problem for the higher-order inhomogeneous Cauchy-Riemann equation is formulated as*

$$\partial_{\bar z}^n w = f(z) \quad \text{in } \mathbb{H}, \tag{3.15}$$

with the following boundary and point conditions:

$$\partial_{\bar z}^k w - \partial_y \partial_{\bar z}^k w = \gamma_{k+1} \text{ on } \mathbb{R},\ 0 \le k \le n-1, \tag{3.16}$$

$$\partial_{\bar z}^k w(0) = c_{k+1} + Tu_{k+1}(0), \quad 0 \le k \le n-2, \tag{3.17}$$

and

$$\partial_{\bar z}^{n-1} w(0) = c_n + Tf(0). \tag{3.18}$$

The problem is uniquely solvable if and only if the following solvability conditions hold for $z \in \mathbb{H}$

$$\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{i\gamma_k(t) + u_k(t)}{t - \bar z}\,dt + iTu_k(\bar z) + T\partial_\zeta u_k(\bar z) = 0, \quad \text{for } 1 \le k \le n-1, \tag{3.19}$$

$$\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{i\gamma_n(t) + f(t)}{t - \bar z}\,dt + iTf(\bar z) + T\partial_\zeta f(\bar z) = 0, \quad \text{for } k = n-1. \tag{3.20}$$

The solution $w(z)$ is given by

$$w(z) = c_1 e^{-iz} + Tu_1(z) + \frac{1}{2\pi i}\int_{-\infty}^{\infty}(\gamma_1(t) - 2iu_1(t))\, e^{i(t-z)}\int_{-t}^{z-t}\frac{e^{i\zeta}}{i\zeta}\,d\zeta\,dt, \tag{3.21}$$

where $u_k(z)$ and $u_k(t)$ are defined recursively:

$$u_k(z) = c_{k+1}e^{-iz} + Tu_{k+1}(z) + \frac{1}{2\pi i}\int_{-\infty}^{\infty}(\gamma_{k+1}(t) - 2iu_{k+1}(t))\, e^{i(t-z)}\int_{-t}^{z-t}\frac{e^{i\zeta}}{i\zeta}\,d\zeta\,dt, \tag{3.22}$$

$$u_k(t) = c_{k+1}e^{-it} + \int_0^t (i\gamma_{k+1}(\varsigma) + u_{k+1}(\varsigma))\, e^{i(\varsigma-t)}\, d\varsigma \tag{3.23}$$

for $1 \leq k \leq n-2$.

For $k = n-1$*:*

$$u_{n-1}(z) = c_n e^{-iz} + Tf(z) + \frac{1}{2\pi i}\int_{-\infty}^{\infty} (\gamma_1(t) - 2if(t))\, e^{i(t-z)} \int_{-t}^{z-t} \frac{e^{i\zeta}}{i\zeta}\, d\zeta\, dt, \qquad (3.24)$$

$$u_{n-1}(t) = c_n e^{-it} + \int_0^t (i\gamma_n(\varsigma) + f(\varsigma))\, e^{i(\varsigma - t)}\, d\varsigma. \qquad (3.25)$$

Proof. For $n = 1$, take $u_1(t) = f(t)$ and $u_1(z) = f(z)$. Substituting these into $w(z)$ verifies the result. Assume the theorem holds for n-1 and consider the case for n. The problem decomposes into two systems

$$\partial_{\bar{z}}^{n-1} w(z) = u(z) \text{ in } \mathbb{H}, \qquad (3.26)$$

$$\partial_{\bar{z}}^{k} w - \partial_y \partial_{\bar{z}}^{k} w = \gamma_{k+1} \quad \text{on } \mathbb{R},\ 0 \leq k \leq n-2, \qquad (3.27)$$

$$\partial_{\bar{z}}^{k} w(0) = c_{k+1} + Tu_{k+1}(0), \quad 0 \leq k \leq n-3, \qquad (3.28)$$

$$\partial_{\bar{z}}^{n-2} w(0) = c_{n-1} + Tf(0); \qquad (3.29)$$

and

$$u_{\bar{z}} = f(z) \text{ in } \mathbb{H},\ u - \partial_y u = \gamma_n \quad \text{on } \mathbb{R}, \qquad (3.30)$$

$$u(0) = c_n + Tf(0). \qquad (3.31)$$

Solving these systems recursively using the solvability conditions and integrating the solutions yields the desired result.

□

4. Conclusion

In this study, we analyzed the Robin boundary value problem for higher-order equations in the upper half complex plane. Through a systematic exploration of existence and uniqueness conditions, we provided explicit integral representations for solutions. Our findings demonstrate the feasibility of generalizing classical boundary value problem techniques to higher-order equations, with particular emphasis on the complex interaction between boundary conditions and the operator's structure.The results have potential applications in mathematical physics and engineering, especially in problems involving mixed boundary conditions and higher-order derivatives. Future work may extend these methods to more complex domains and investigate numerical approaches for practical implementations. The theoretical results obtained from the Robin boundary value problem for higher-order equations, particularly in the context of the Cauchy-Riemann equations, can be applied in various fields, including signal processing in communication systems.

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ON STAKHOV FUNCTIONS AND NEW HYPERBOLOID SURFACES

ABSTRACT. This paper presents an investigation into the generalization of hyperbolic Fibonacci sine and cosine functions, as well as Fibonacci spirals. Initially, we establish the main definitions and theoretically model them, listing several special cases. We then uncover fundamental results, including the De Moivre and Pythagorean formulas. Based on these new definitions, we introduce new classes of three-dimensional hyperboloid surfaces and compute their Gauss and mean curvatures. Notably, we demonstrate that these surfaces are geodesic.

1. INTRODUCTION

The usual Fibonacci numbers are defined by the following recurrence relation: for $n \geqslant 0$

$$F_{n+2} = F_{n+1} + F_n, \tag{1.1}$$

where $F_0 = 0$ and $F_1 = 1$. These numbers can also be produced by using the Binet's formula in the form of

$$F_n = \frac{\alpha^n - \beta^n}{\sqrt{5}}, \tag{1.2}$$

where α and β are the positive and negative roots of the equation $x^2 - x - 1 = 0$, respectively.

In the literature, many interesting properties and applications of the recurrence sequences have been studied by many authors; see for example, [1], [2], [3]. In 1993, the Ukrainian mathematicians Stakhov and Tkachenko put forth a new idea to describe hyperbolic geometry [4]. Inspired by the Binet's formula in Eq. (1.2), the authors introduced a new class of hyperbolic functions, which are called the Hyperbolic Fibonacci and Lucas functions. In [5], Stakhov provided detailed information with applications to the available literature. In [6], Stakhov and Rozin further developed the ideas of the hyperbolic Fibonacci and Lucas functions, and

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defined the symmetric hyperbolic Fibonacci and Lucas functions as follows:

$$sFs(x)=\frac{\alpha^{x}-\alpha^{-x}}{\sqrt{5}},\ cFs(x)=\frac{\alpha^{x}+\alpha^{-x}}{\sqrt{5}},\ sLs(x)=\alpha^{x}-\alpha^{-x},\ \text{and}\ cLs(x)=\alpha^{x}+\alpha^{-x},\tag{1.3}$$

where x is any real numbers. To be clear, for $k\in Z$, $sFs(2k)=F_{2k}$, $cLs(2k)=L_{2k}$, $cFs(2k+1)=F_{2k+1}$, and $sLs(2k+1)=L_{2k+1}$. In [7], Stakhov and Rozin defined the quasi-sine Fibonacci functions and Fibonacci spirals to eliminate the discrete case in Eq. (1.3) as follows:

$$FF(x)=\frac{\alpha^{x}-\cos(\pi x)\,\alpha^{-x}}{\sqrt{5}}\ \text{and}\ CFF(x)=\frac{\alpha^{x}-\cos(\pi x)\,\alpha^{-x}}{\sqrt{5}}+i\frac{\sin(\pi x)\,\alpha^{-x}}{\sqrt{5}},\tag{1.4}$$

where i is the complex unit. Note that in [8], Stakhov and Rozin presented a brief description of these hyperbolic phenomenons in the world.

According to these developments, in [9], Falcón and Plaza defined a new class of hyperbolic sine&cosine, quasi-sine, and spiral-like functions using the k-Fibonacci sequence as follows:

$$sF_kh(x)=\frac{\sigma^{x}-\sigma^{-x}}{\sigma+\sigma^{-1}},cF_kh(x)=\frac{\sigma^{x}+\sigma^{-x}}{\sigma+\sigma^{-1}},FF_kh(x)=\frac{\sigma^{x}-\cos(\pi x)\,\sigma^{-x}}{\sigma+\sigma^{-1}},\ \text{and}\ CFF_k(x)=\frac{\sigma^{x}-\cos(\pi x)\,\sigma^{-x}}{\sigma+\sigma^{-1}}+i\frac{\sin(\pi x)\,\sigma^{-x}}{\sigma+\sigma^{-1}},\tag{1.5}$$

where σ is the positive root of $\sigma^2=k\sigma+1$ and k is any positive real number. Motivated by the definitions of Stakhov and Rozin [6,7], and Falcón and Plaza [9], Daşdemir et al. gave a generalized version of the functions in Eqs. (1.3)-(1.4) as follows [10]:

$$\mathcal{H}_s(x)=\frac{A\alpha^{x}-B\alpha^{-x}}{\Delta},\ \mathcal{H}_c(x)=\frac{A\alpha^{x}+B\alpha^{-x}}{\Delta},\ \mathcal{H}(x)=\frac{A\alpha^{x}-\cos(\pi x)\,B\alpha^{-x}}{\Delta},\ \text{and}\ \mathcal{CH}(x)=\frac{A\alpha^{x}-\cos(\pi x)\,B\alpha^{-x}}{\Delta}+i\frac{\sin(\pi x)\,B\alpha^{-x}}{\Delta},\tag{1.6}$$

which are called the Horadam hyperbolic sine function, the Horadam hyperbolic cosine function, the quasi-sine Horadam function, and Horadam spiral, respectively. Here, α is the positive root of $\lambda^2=f(x)\lambda+1$, $\Delta=\sqrt{f^2(x)+4}$, $A=b(x)+a(x)\alpha^{-1}$, $B=b(x)-a(x)\alpha$, and $a(x)$ and $b(x)$ are any continue real-valued function.

As the above literature survey reveals, the functions in (1.3)-(1.5) only vary on the real variable x, while other parameters are constant. In Eq. (1.6), the parameters α and β depend on a continuous function of x. Consequently, it would be interesting to consider the mentioned functions in a more general form such that the roots of the algebraic equation depend on two real-valued functions. To be clear, this consideration is due to the generalized second-order sequence designated by Horadam [11]. This motivates us to revise the mentioned functions. For this purpose, presented herein is to generalize the definitions introduced by Stakhov and Tkachenko [4], Stakhov and Rozin [5], Falcón and Plaza [9], and Daşdemir et. al [10]. This is the main focus of the present paper, and a particular concern will be paid to some elementary results and geometrical considerations.

2. Main Results

In this section, we will present the outcomes of the paper, including some definitions, fundamental considerations, and elementary properties.

2.1. **Fundamental definitions.** Let $f(x)$ and $g(x)$ be an arbitrary non-zero continuous functions of real number x. Consider the second-order equation

$$\lambda^2 - f(x)\lambda - g(x) = 0. \tag{2.1}$$

Hence, Eq. (2.1) has the following distinct two roots

$$\lambda_1 = \alpha(x) = \frac{f(x) + \sqrt{f^2(x) + 4g(x)}}{2} \text{ and } \lambda_2 = \beta(x) = \frac{f(x) - \sqrt{f^2(x) + 4g(x)}}{2}.$$

To ensure that the solution is real, we assume that condition $f^2(x) + 4g(x) > 0$ is met. Here, we can write

$$\alpha(x) + \beta(x) = f(x), \ \alpha(x)\beta(x) = -g(x), \alpha(x) - \beta(x) = \Delta(x), \tag{2.2}$$

where $= \Delta(x) = \sqrt{f^2(x) + 4g(x)}$. In consequence, we obtained two distinct solutions. Therefore, their linear combination, i.e., $c_1\{\alpha(x)\}^x + c_2\{\beta(x)\}^x$, is also a solution of Eq. (2.1). Solving the system of equations for $x = 0$ and $x = 1$, we find

$$c_1 = \frac{b(x) - a(x)\beta(x)}{\alpha(x) - \beta(x)} \text{ and } c_2 = -\frac{b(x) - a(x)\alpha(x)}{\alpha(x) - \beta(x)}.$$

As a result, we can give the following definition.

Definition 2.1. *Let $a(x)$ and $b(x)$ be an arbitrary continuous function. Then, the Horadam functions are defined as*

$$\mathcal{H}(a, b, f, g, x) = \mathcal{H}(x) = \frac{\tilde{A}(x)[\alpha(x)]^x - \tilde{B}(x)[\beta(x)]^x}{\alpha(x) - \beta(x)}, \tag{2.3}$$

where $\tilde{A}(x) = b(x) + a(x)[\alpha(x)]^{-1}$ and $\tilde{B}(x) = b(x) - a(x)\alpha(x)$.

This is a similar form to the generalized second-order sequence given by Horadam [11]. We can, therefore, call Eq. (2.3) the *Horadam functions* due to Australian mathematician Alwyn Francis Horadam's great contributions to the available literature. Note that, for the sake of presentation simplicity, all the functions will be represented in the non-parentheses form.

Substituting $\alpha\beta = -g$ into Eq. (2.3), we can write

$$\mathcal{H}(x) = \frac{\tilde{A}\alpha^x - \tilde{B}\left(-g\alpha^{-1}\right)^x}{\Delta} = \frac{\tilde{A}\alpha^x - (-1)^x\tilde{B}g^x\alpha^{-x}}{\Delta},$$

Here, we run into the problem of what the real power of -1 will be. To address this issue, from the famous Euler's formula, we can write $e^{\mp i\pi} = \cos\pi \mp i\sin\pi = -1$, where e is Euler's constant and i is the imaginary unit. As a result, we can give the following definition.

Definition 2.2. *Let a and b be any continuous function. Then, the Stakhov spiral is defined as*

$$\mathcal{SR}(a, b, f, g, x) = \mathcal{SR}(x) = \frac{A\alpha^x - \cos(\pi x)B\alpha^{-x}}{\Delta} + i\frac{\sin(\pi x)B\alpha^{-x}}{\Delta}, \tag{2.4}$$

where $A = b + a\alpha^{-1}$ and $B = g^x(b - a\alpha)$.

Table 1. Special cases for the Stakhov spiral functions

Stakhov spiral functions	Symbols $\mathcal{SR}(x)$	a	b	f	g
Horadam spiral [10]	$\mathcal{H}(x)$				1
Fibonacci spiral [6]	$\mathrm{CFF}(x)$	0	1	1	1
Lucas spiral	$\mathrm{CLL}(x)$	2	1	1	1
k-Fibonacci spiral [9]	$\mathrm{CFF}_k(x)$	0	1	k	1
Pell spiral	$\mathrm{CPP}(x)$	0	1	2	1
Modified Pell spiral	$\mathrm{CRR}(x)$	1	1	2	1
Pell-Lucas spiral	$\mathrm{CQQ}(x)$	2	2	2	1
Jacobsthal spiral	$\mathrm{CJJ}(x)$	0	1	1	2
Jacobsthal-Lucas spiral	$\mathrm{CJJL}(x)$	2	1	1	2
Fermat spiral	$\mathrm{CFFR}(x)$	1	3	3	2

This looks like a three-dimensional spiral-like curve and is the most general form of Eq. (1.4) given by Stakhov and Rozin [7]. Table 1 indicates the special cases that can be obtained depending on the particular choice of a, b, f, and g. For integer values of x, the imaginary part of the function $\mathcal{SR}(x)$ vanishes. The reason for the name "Stakhov spiral function" is that the great Ukrainian mathematician Stakhov has attributed an indescribable contribution both to the subject of this paper and to the literature on Fibonacci numbers.

For the concrete examples, we consider the following cases:

$$\text{Case I}: a(x) = \sin x,\ b(x) = \cos x,\ f(x) = \ln\left(1+x^2\right),\ \text{and } g(x) = \cosh(x)$$

$$\text{Case II}: a(x) = \sqrt[3]{x},\ b(x) = x,\ f(x) = \operatorname{arcsinh}\left(1+x^2\right),\ \text{and } g(x) = e^{-x}$$

Fig. 1 displays the three-dimensional graphs of the Stakhov spirals for Case I (Fig. 1.a) and Case II (Fig. 1.b), respectively. As seen, the distributions are a spiral-like curve.

Under the assumption that the $Oy-$ and $Oz-$axes are real and imaginary directions, respectively, we can build up the following system of equations:

$$\begin{cases} y - \frac{A\alpha^x}{\Delta} = -\frac{\cos(\pi x)B\alpha^{-x}}{\Delta} \\ z = \frac{\sin(\pi x)B\alpha^{-x}}{\Delta} \end{cases}$$

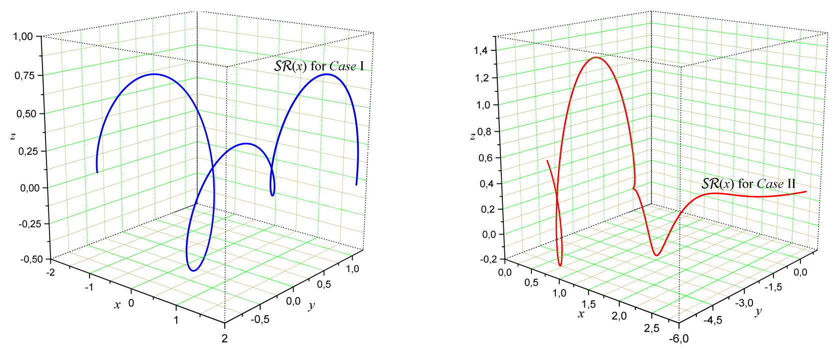
Thus, after some operations, we get the following equation:

$$\left(y - \frac{A\alpha^x}{\Delta}\right)^2 + z^2 = \left(\frac{B\alpha^{-x}}{\Delta}\right)^2 \tag{2.5}$$

or in the re-organized form

$$z^2 = \left(\frac{A\alpha^x + B\alpha^{-x}}{\Delta} - y\right)\left(y - \frac{A\alpha^x - B\alpha^{-x}}{\Delta}\right). \tag{2.6}$$

Note that Eq. (2.4) is a complex-valued function. However, we are usually not concerned with what is going on in the imaginary axis, as we generally work in real space. This idea coincides also with the approaches by the references [6], [9], and [10]. Thus, considering the real part of the Stakhov spiral functions, we can express the following definition.



(a) Case I **(b)** Case II

Figure 1. Distributions of the Stakhov spirals for Cases I and II

Definition 2.3. *For the functions a and b, the quasi-sine Stakhov function is defined by*

$$\mathcal{S}_q(a,b,f,g,x) = \mathcal{QS}(x) = \frac{A\alpha^x - \cos(\pi x)\, B\alpha^{-x}}{\Delta}. \tag{2.7}$$

Meanwhile, Eqs. (2.6) and (2.7) are whispering some new definitions. In Eq. (2.6), each function in parentheses has a hyperbolic structure. Besides, combining these functions utilizing the character of $\cos(\pi x)$ yields Eq. (2.7). Based on this idea, we can then discretize the function $\mathcal{S}_q(x)$ as follows.

Definition 2.4. *Let a and b be any continuous function. Then, the hyperbolic Stakhov sine and cosine functions are defined by*

$$\mathcal{S}_s(a,b,f,g,x) = \mathcal{S}_s(x) = \frac{A\alpha^x - B\alpha^{-x}}{\Delta} \tag{2.8}$$

and

$$\mathcal{S}_c(a,b,f,g,x) = \mathcal{S}_c(x) = \frac{A\alpha^x + B\alpha^{-x}}{\Delta}, \tag{2.9}$$

respectively.

From the last definitions, we can give the following additional definitions.

Definition 2.5. *Let a and b be continuous. Then, the hyperbolic Stakhov tangent and cotangent functions are defined by*

$$\mathcal{S}_t(a,b,f,g,x) = \frac{\mathcal{S}_s(x)}{\mathcal{S}_c(x)} = \frac{A\alpha^x - B\alpha^{-x}}{A\alpha^x + B\alpha^{-x}} = 1 - \frac{2B}{B + A\alpha^{2x}}, \tag{2.10}$$

and

$$\mathcal{S}_{ct}(a,b,f,g,x) = \frac{\mathcal{S}_c(x)}{\mathcal{S}_s(x)} = \frac{A\alpha^x + B\alpha^{-x}}{A\alpha^x - B\alpha^{-x}} = 1 - \frac{2B}{B - A\alpha^{2x}}, \tag{2.11}$$

respectively.

In working with the above-stated functions, it is useful to consider the following special cases:

- Generalized Stakhov-Fibonacci spirals

$$\mathcal{SR}(0,1,f,g,x) = \mathcal{S}_u(x) = \frac{\alpha^x - \cos(\pi x)\, g^x\alpha^{-x}}{\Delta} + i\frac{\sin(\pi x)\, g^x\alpha^{-x}}{\Delta} \tag{2.12}$$

- Generalized Stakhov-Lucas spirals

$$\mathcal{SR}\left(2,f,f,g,x\right)=\mathcal{S}_v\left(x\right)=\alpha^x+\cos\left(\pi x\right)g^x\alpha^{-x}-i\sin\left(\pi x\right)g^x\alpha^{-x} \tag{2.13}$$

- Generalized quasi-sine Fibonacci functions

$$\mathcal{S}_q\left(0,1,f,g,x\right)=\operatorname{Re}\left(\mathcal{S}_u\left(x\right)\right)=\mathcal{U}_q\left(x\right)=\frac{\alpha^x-\cos\left(\pi x\right)g^x\alpha^{-x}}{\Delta} \tag{2.14}$$

- Generalized quasi-sine Lucas functions

$$\mathcal{S}_q\left(2,f,f,g,x\right)=\operatorname{Re}\left(\mathcal{S}_v\left(x\right)\right)=\mathcal{V}_q\left(x\right)=\alpha^x+\cos\left(\pi x\right)g^x\alpha^{-x} \tag{2.15}$$

- Generalized hyperbolic Fibonacci sine and cosine functions

$$\mathcal{S}_s\left(0,1,f,g,x\right)=\mathcal{F}_s\left(x\right)=\frac{\alpha^x-g^x\alpha^{-x}}{\Delta},\ \mathcal{S}_c\left(0,1,f,g,x\right)=\mathcal{F}_c\left(x\right)=\frac{\alpha^x+g^x\alpha^{-x}}{\Delta} \tag{2.16}$$

- Generalized hyperbolic Lucas cosine and sine functions

$$\mathcal{S}_s\left(2,f,f,g,x\right)=\mathcal{L}_c\left(x\right)=\alpha^x+g^x\alpha^{-x},\ \mathcal{S}_c\left(2,f,f,g,x\right)=\mathcal{L}_s\left(x\right)=\alpha^x-g^x\alpha^{-x} \tag{2.17}$$

- Generalized hyperbolic Lucas tangent and cotangent functions

$$\mathcal{S}_t\left(0,1,f,g,x\right)=\mathcal{F}_t\left(x\right)=\frac{\alpha^x-g^x\alpha^{-x}}{\alpha^x+g^x\alpha^{-x}},\ \mathcal{S}_{ct}\left(0,1,f,g,x\right)=\mathcal{F}_{ct}\left(x\right)=\frac{\alpha^x+g^x\alpha^{-x}}{\alpha^x-g^x\alpha^{-x}} \tag{2.18}$$

2.2. **Some features.** In this section, some elementary formulas regarding the hyperbolic Stakhov functions will bw developed. We can thus start with the following results.

Theorem 2.1. *The following non-homogeneous recurrence relations hold for any real number x:*

$$\mathcal{S}_s\left(x+2\right)=f\mathcal{S}_c\left(x+1\right)+g\mathcal{S}_s\left(x\right) \tag{2.19}$$

and

$$\mathcal{S}_c\left(x+2\right)=f\mathcal{S}_s\left(x+1\right)+g\mathcal{S}_c\left(x\right). \tag{2.20}$$

Proof. Substituting Eqs. (2.10) and (2.11) into Eq. (2.19) yields

$$\begin{aligned} f\mathcal{S}_s\left(x+1\right)+g\mathcal{S}_c\left(x\right) &= f\frac{A\alpha^{(x+1)}-B\alpha^{-(x+1)}}{\Delta}+g\frac{A\alpha^x+B\alpha^{-x}}{\Delta} \\ &= \frac{A\alpha^x\left(f\alpha+g\right)+B\alpha^{-x}\left(1-\frac{f}{\alpha}\right)}{\Delta}. \end{aligned}$$

Considering Eqs. (2.1) and (2.2), we can write

$$\alpha^2=f\alpha+g \text{ and } 1-\frac{f}{\alpha}=\frac{g}{\alpha^2},$$

which completes the proof. □

Remark. *Theorem 2.1 indicates that symmetric exchange between the functions $\mathcal{S}_s\left(x\right)$ and $\mathcal{S}_c\left(x\right)$ is possible in all linear relations of the hyperbolic Stakhov functions.*

The recurrence relations in Eqs. (2.19) and (2.20) can be homogenized as follows. Subtracting the resultant equations after multiplying the values of these functions for the real numbers x and $x+2$ with appropriate factors, we obtain

$$\mathcal{S}_s(x+2) = \left(f^2+2g\right)\mathcal{S}_s(x) - g^2\mathcal{S}_s(x-2) \tag{2.21}$$

and

$$\mathcal{S}_c(x+2) = \left(f^2+2g\right)\mathcal{S}_c(x) - g^2\mathcal{S}_c(x-2). \tag{2.22}$$

Note that the new recurrence relations have a forth-order homogeneous form.

The next theorem presents the inverse hyperbolic functions.

Theorem 2.2. *The hyperbolic Stakhov sine and cosine functions have an inverse in the form of*

$$S_s{}^{-1}(x) = \log_\alpha(\tilde{x}) \ \ \text{and} \ S_c{}^{-1}(x) = \log_\alpha(\tilde{x}), \tag{2.23}$$

where $\tilde{x} = \frac{\Delta x + \sqrt{\Delta^2 x^2 + 4AB}}{2A}$.

Proof. From the definition of the hyperbolic Stakhov functions, we can write

$$x = \frac{A\alpha^y - B\alpha^{-y}}{\Delta} \Rightarrow \Delta x\alpha^y = A\alpha^{2y} - B \Rightarrow A(\alpha^y)^2 - \Delta x\alpha^y - B = 0,$$

which is second-order equation. Since $\alpha^y > 0$, there is a unique solution, namely $\alpha^y = \frac{\Delta x + \sqrt{\Delta^2 x^2 + 4AB}}{2A}$. As a result, the first equation is obtained. The latter can also be proved after a similar process. □

We give the Pythagorean formula for hyperbolic Stakhov functions.

Theorem 2.3 (Pythagorean formula)**.** *For any real number x, we have*

$$\left[\mathcal{S}_c(x)\right]^2 - \left[\mathcal{S}_s(x)\right]^2 = \frac{4AB}{\Delta^2}. \tag{2.24}$$

Proof. Subtracting the resultant equations after substituting Eqs. (2.8) and (2.9) into the left-hand side of Eq. (2.24) yields the claimed result. □

The next theorem presents a similar result to the famous De Moivre's formula.

Theorem 2.4 (De Moivre-type formula)**.** *Let x be any real number. Then the following identities hold for any positive integer n:*

$$\left[\mathcal{S}_c(x) + \mathcal{S}_s(x)\right]^n = \left(\frac{2A}{\Delta}\right)^{n-1}\left[\mathcal{S}_c(nx) + \mathcal{S}_s(nx)\right] \tag{2.25}$$

and

$$\left[\mathcal{S}_c(x) - \mathcal{S}_s(x)\right]^n = \left(\frac{2B}{\Delta}\right)^{n-1}\left[\mathcal{S}_c(nx) - \mathcal{S}_s(nx)\right]. \tag{2.26}$$

Proof. We use the induction method to show the validity of theorem. It is clear that Eq. (2.25) holds for $n=1$. Based on the assumption such that this equation is valid for any positive integer n, we can write

$$\begin{aligned}
\left[\mathcal{S}_c(x) + \mathcal{S}_s(x)\right]^{n+1} &= \left[\mathcal{S}_c(x) + \mathcal{S}_s(x)\right]^n\left[\mathcal{S}_c(x) + \mathcal{S}_s(x)\right] \\
&= \left(\frac{2A}{\Delta}\right)^{n-1}\left[\mathcal{S}_c(nx) + \mathcal{S}_s(nx)\right]\left[\frac{A\alpha^x + B\alpha^{-x}}{\Delta} - \frac{A\alpha^x - B\alpha^{-x}}{\Delta}\right] \\
&= \left(\frac{2A}{\Delta}\right)^n\left[\frac{A\alpha^{nx} + B\alpha^{-nx}}{\Delta} + \frac{A\alpha^{nx} - B\alpha^{-nx}}{\Delta}\right]\alpha^x
\end{aligned}$$

$$= \left(\frac{2A}{\Delta}\right)^n \left[\frac{2A\alpha^{(n+1)x} + B\alpha^{-(n+1)x} - B\alpha^{-(n+1)x}}{\Delta}\right]$$
$$= \left(\frac{2A}{\Delta}\right)^n \left[\mathcal{S}_c\left((n+1)\,x\right) + \mathcal{S}_s\left((n+1)\,x\right)\right],$$

which completes the proof process. Repeating the same procedure, the other can be demonstrated. □

Up to now, we only make our investigation for real values of x. So, what properties will the hyperbolic Stakhov functions have in the complex space? The answer is presented in the following.

Theorem 2.5. *For complex variable $z = x + iy$, we have*

$$\mathcal{S}_s\left(z\right) = \frac{1}{2AB}\left[u\cos\left(y\ln\alpha\right) + iv\sin\left(y\ln\alpha\right)\right] \tag{2.27}$$

and

$$\mathcal{S}_c\left(z\right) = \frac{1}{2AB}\left[v\cos\left(y\ln\alpha\right) + iu\sin\left(y\ln\alpha\right)\right], \tag{2.28}$$

where $u = (A+B)\,\mathcal{S}_s\left(x\right) - (A-B)\,\mathcal{S}_c\left(x\right)$ and $v = (A+B)\,\mathcal{S}_c\left(x\right) - (A-B)\,\mathcal{S}_s\left(x\right)$.

Proof. Considering

$$\mathcal{S}_s\left(z\right) = \frac{A\alpha^z - B\alpha^{-z}}{\Delta} \text{ and } \mathcal{S}_c\left(z\right) = \frac{A\alpha^z + B\alpha^{-z}}{\Delta},$$

we can write

$$\begin{aligned} A\alpha^z = A\alpha^{x+iy} = A\alpha^x\alpha^{iy} &= \frac{\Delta}{2A}\left(\mathcal{S}_c\left(x\right) + \mathcal{S}_s\left(x\right)\right)\alpha^{iy} \\ &= \frac{\Delta}{2A}\left(\mathcal{S}_s\left(x\right) + \mathcal{S}_c\left(x\right)\right)\left[\cos\left(y\ln\alpha\right) + i\sin\left(y\ln\alpha\right)\right] \end{aligned}$$

and

$$\begin{aligned} B\alpha^{-z} = B\alpha^{-x-iy} = B\alpha^{-x}\alpha^{-iy} &= \frac{\Delta}{2B}\left(\mathcal{S}_c\left(x\right) - \mathcal{S}_s\left(x\right)\right)\alpha^{-iy} \\ &= \frac{\Delta}{2B}\left(\mathcal{S}_c\left(x\right) - \mathcal{S}_s\left(x\right)\right)\left[\cos\left(y\ln\alpha\right) - i\sin\left(y\ln\alpha\right)\right]. \end{aligned}$$

Here, we used the well-known Euler's formula. As a result, combining the last two equations, the proof is completed. □

As an example, we give the following special case.

Example 2.1. *Consider $z = \frac{i\pi}{\ln\alpha}$. Let us compute $\mathcal{S}_s\left(z\right)$ and $\mathcal{S}_c\left(z\right)$. In this case, $x = 0$ and $y = \frac{\pi}{\ln\alpha}$. Inserting these values into Eqs. (2.32) and (2.35), we can readily obtain $\mathcal{S}_s\left(z\right) = 0$ and $\mathcal{S}_c\left(z\right) = -\frac{2}{\Delta}$.*

2.3. **Geometrical considerations.** In this section, some geometrical approaches will be developed. For this purpose, we first introduce the following equations:

$$x = \mp\frac{A\alpha^t - B\alpha^{-t}}{\Delta} \text{ and } y = \frac{A\alpha^t + B\alpha^{-t}}{\Delta}, \tag{2.29}$$

where the parameter t is the hyperbolic angle. From this, we can write

$$y^2 - x^2 = \frac{4AB}{\Delta^2}$$

or

$$\frac{y^2}{\left(\frac{2\sqrt{AB}}{\Delta}\right)^2} - \frac{x^2}{\left(\frac{2\sqrt{AB}}{\Delta}\right)^2} = 1, \tag{2.30}$$

which is a rectangular hyperbola equation, with center at the origin, and with a horizontal real axis. We may call Eq. (2.30) the Stakhov hyperbola. Note that the exchange of preferences in Eq. (2.29) yields to obtain the equation of the conjugate hyperbola.

According to Eq. (2.30), the foci, vertices, and co-vertices of the hyperbola lie in $\left(\mp\frac{2\sqrt{2AB}}{\Delta}, 0\right)$, $\left(\mp\frac{2\sqrt{AB}}{\Delta}, 0\right)$, and $\left(0, \mp\frac{2\sqrt{AB}}{\Delta}\right)$, respectively. In addition, the equations of asymptotes and directrices are $y = \mp x$ and $x = \mp\frac{\sqrt{2AB}}{\Delta}$. In particular, the Modified Pell hyperbola is unit. This means that since the Pseudo Euclidean plane is represented by a unit hyperbola that also describes Minkowski space-time, the Modified Pell hyperbola can be used.

On the other hand, rotating the Stakhov hyperbola completely around the vertical axis generates a hyperboloid of one sheet. In this case, we have the equation

$$\frac{x^2}{\left(\frac{2\sqrt{AB}}{\Delta}\right)^2} + \frac{y^2}{\left(\frac{2\sqrt{AB}}{\Delta}\right)^2} - \frac{z^2}{\left(\frac{2\sqrt{AB}}{\Delta}\right)^2} = 1 \tag{2.31}$$

and its parametric representation is

$$\begin{cases} x = \frac{2\sqrt{AB}}{\Delta}\mathcal{S}_c(t)\cos\theta \\ y = \frac{2\sqrt{AB}}{\Delta}\mathcal{S}_c(t)\sin\theta \\ z = \frac{2\sqrt{AB}}{\Delta}\mathcal{S}_s(t) \end{cases} \tag{2.32}$$

where θ is azimuth angle and $t \in [0, \infty)$. By the way, this may be called the hyperbolic Stakhov hyperboloid. Further, for $z \in [0, \infty)$, the projection of the hyperbolic Stakhov hyperboloid on xy-plane is a planar spiral that looks like an Archimedean spiral.

If rotation is made along the horizantal axis, an hyperboloid of two sheets occurs. So, we have the hyperboloid equation as follows:

$$\frac{x^2}{\left(\frac{2\sqrt{AB}}{\Delta}\right)^2} + \frac{y^2}{\left(\frac{2\sqrt{AB}}{\Delta}\right)^2} - \frac{z^2}{\left(\frac{2\sqrt{AB}}{\Delta}\right)^2} = -1 \tag{2.33}$$

and its parametric representation is

$$\begin{cases} x = \frac{2\sqrt{AB}}{\Delta}S_s(t)\cos\theta \\ y = \frac{2\sqrt{AB}}{\Delta}S_s(t)\sin\theta \\ z = \frac{2\sqrt{AB}}{\Delta}S_c(t) \end{cases} \tag{2.34}$$

Similarly, the last surface may be called the elliptic Stakhov hyperboloid. Since the denominators are equal, there is a hyperboloid of revolution in both cases. It should be noted that there are no umbilics on a hyperboloid of one sheet, but two on each sheet of the two-sheeted variety.

Fig. 2 displays special forms of the hyperbolic and elliptic Stakhov hyperboloids for cases where the hyperbolic and elliptic Fibonacci (Fig. 2. a & b), the hyperbolic

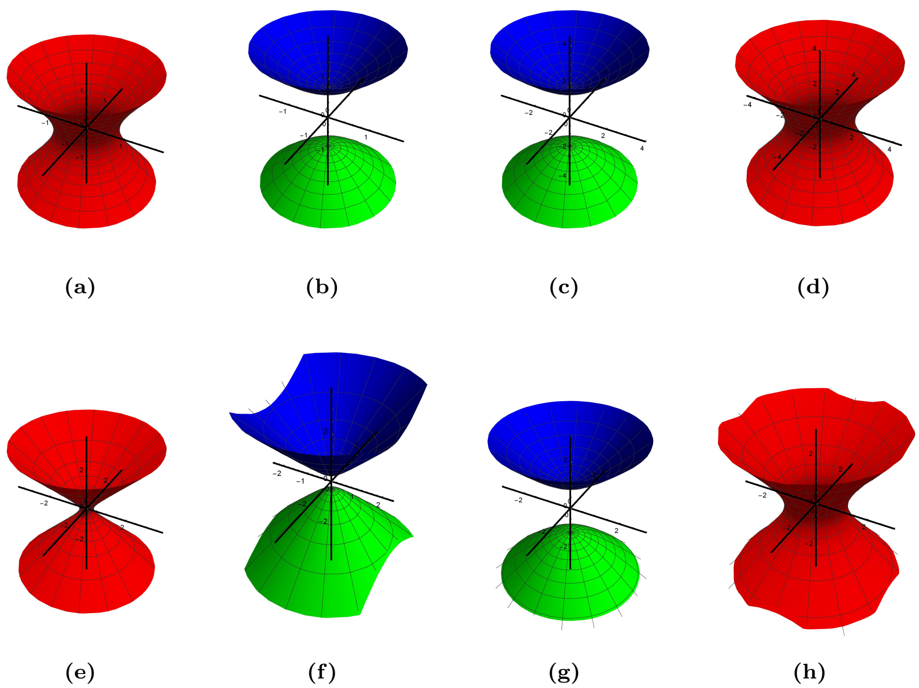


Figure 2. Some special surfaces: (a) Hyperbolic Fibonacci, (b) Elliptic Fibonacci, (c) Hyperbolic Lucas, (d) Elliptic Lucas, (e) Hyperbolic Pell, (f) Elliptic Pell, (g) Hyperbolic Modified Pell, and (h) Elliptic Modified Pell

and elliptic Lucas (Fig. 2. c & d), the hyperbolic and elliptic Pell (Fig. 2. e & f), and the hyperbolic and elliptic Modified Pell (Fig. 2. g & h) hyperboloids according to Table 1. Fig. 2 reveals that oscillating characters of the Fibonacci and Pell surface symmetrically exchange with the ones of Lucas and Modified Pell.

Theorem 2.6. *The Gaussian and mean curvatures of the hyperbolic Stakhov hyperboloid are given by*

$$K_h = -\frac{1}{\left[\{S_s(t)\}^2 + \{S_c(t)\}^2\right]^2} \; and \, H_h = \frac{\Delta[S_s(t)]^2}{2\sqrt{AB}\left[\{S_s(t)\}^2 + \{S_c(t)\}^2\right]^{\frac{3}{2}}}. \quad (2.35)$$

Proof. It is clear that the hyperbolic Stakhov hyperboloid is a regular surface with a differentiable field of unit normal vectors N. Let us compute the coefficients of the first and second fundamental forms of hyperboloid so that the Gaussian and mean curvatures can be obtained in terms of whose coefficients. To do this, we shall express dN as a matrix in terms of the natural basis X_u, X_v, where $X(t,\theta) = \frac{2\sqrt{AB}}{\Delta}(S_c(t)\cos\theta, S_c(t)\sin\theta, S_s(t))$. In this case, the coefficients of the first fundamental form are given by

$$E = \langle X_t, X_t \rangle = \left\{ \tfrac{2\sqrt{AB}\ln\alpha}{\Delta} \right\}^2 \left[\{S_s(t)\}^2 + \{S_c(t)\}^2 \right],$$
$$F = \langle X_t, X_\theta \rangle = 0, \text{ and } G = \langle X_\theta, X_\theta \rangle = \left\{ \tfrac{2\sqrt{AB}}{\Delta} \right\}^2 \{S_c(t)\}^2.$$

Besides, we can write down

$$N = \frac{X_t \times X_\theta}{\|X_t \times X_\theta\|} = \frac{-iS_c(t)\cos\theta - jS_c(t)\sin\theta + kS_s(t)}{\left\{ \{S_c(t)\}^2 + \{S_s(t)\}^2 \right\}^{\frac{1}{2}}},$$

and from this, the coefficients of the second fundamental form can be computed as

$$e = \langle N, X_{tt} \rangle = -\tfrac{8AB\sqrt{AB}(\ln\alpha)^2}{\Delta^3\left\{ \{S_c(t)\}^2 + \{S_s(t)\}^2 \right\}^{\frac{1}{2}}}, \ f = \langle N, X_{t\theta} \rangle = 0,$$
$$\text{and } g = \langle N, X_{\theta\theta} \rangle = \tfrac{2\sqrt{AB}\{S_c(t)\}^2}{\Delta\left\{ \{S_c(t)\}^2 + \{S_s(t)\}^2 \right\}^{\frac{1}{2}}}.$$

Considering

$$K = \frac{eg - f^2}{EG - F^2} \text{ and } H = \frac{1}{2}\frac{eG - 2fF + gE}{EG - F^2},$$

the results follows after some mathematical operations. □

Since $K < 0$, the principal curvatures κ_1 and κ_2 are of opposite sign at any point P. So the surface near P is a hyperboloid. We can call P a hyperbolic point of the surface.

Theorem 2.7. *The Gaussian and mean curvatures of the elliptic Stakhov hyperboloid are given by*

$$K_e = \frac{1}{\left[\{S_s(t)\}^2 + \{S_c(t)\}^2 \right]^2} \text{ and } H_e = \frac{\Delta [S_c(t)]^2}{2\sqrt{AB}\left[\{S_s(t)\}^2 + \{S_c(t)\}^2 \right]^{\frac{3}{2}}}. \quad (2.36)$$

Proof. Repeating the same procedure in the previous theorem, the proof can easily be done. □

Since $K > 0$, the sings of the principal curvatures κ_1 and κ_2 are the same. The normal curvature κ in any tangent direction t is equal to $\kappa = \kappa_1\cos\theta + \kappa_2\sin\theta$, where θ is the angle between t and the principal vector corresponding to κ_1. So the sign of κ is the same as that of κ_1 and κ_2. The surface is bending away from its tangent plane in all tangent directions at any point P. The quadratic approximation of the surface near P is the paraboloid $z^2 = \kappa_1 x^2 + \kappa_2 y^2$. In addition, the Gaussian curvatures K of two surfaces are invariant by local isometries.

Theorem 2.8. *Both hyperbolic and elliptic Stakhov hyperboloids are geodesic. To be clear, we have*

$$\kappa_h = 0 \text{ and } \kappa_e = 0. \quad (2.37)$$

Proof. We only present proof for the hyperbolic Stakhov hyperboloid here. Other can be proved similarly. Let χ be cut out of the hyperbolic Stakhov hyperboloid by the form $z = c$. In this case, $\frac{2\sqrt{AB}}{\Delta}\mathcal{S}_s(t)$ is constant and so t is also constant. Then a unit-speed parameterization of χ can be defined as

$$\chi(s) = \frac{2\sqrt{AB}}{\Delta}\left(S_c(\tilde{s})\cos t_0, S_c(\tilde{s})\sin t_0, S_s(\tilde{s}) \right), \ \tilde{s} = \frac{s}{\frac{2\sqrt{AB}}{\Delta}S_s(t_0)}.$$

Then we can compute the Frenet frame as follow:

$$\mathbf{t}(s)=\frac{\left(S_s\left(\frac{s}{\frac{2\sqrt{AB}}{\Delta}S_s(t_0)}\right)\cos t_0, S_s\left(\frac{s}{\frac{2\sqrt{AB}}{\Delta}S_s(t_0)}\right)\sin t_0, S_c\left(\frac{s}{\frac{2\sqrt{AB}}{\Delta}S_s(t_0)}\right)\right)}{\sqrt{\left\{S_c\left(\frac{s}{\frac{2\sqrt{AB}}{\Delta}S_s(t_0)}\right)\right\}^2+\left\{S_s\left(\frac{s}{\frac{2\sqrt{AB}}{\Delta}S_s(t_0)}\right)\right\}^2}}$$

$$\mathbf{n}(s)=\frac{\left(S_c\left(\frac{s}{\frac{2\sqrt{AB}}{\Delta}S_s(t_0)}\right)\cos t_0, S_c\left(\frac{s}{\frac{2\sqrt{AB}}{\Delta}S_s(t_0)}\right)\sin t_0, S_s\left(\frac{s}{\frac{2\sqrt{AB}}{\Delta}S_s(t_0)}\right)\right)}{\sqrt{\left\{S_c\left(\frac{s}{\frac{2\sqrt{AB}}{\Delta}S_s(t_0)}\right)\right\}^2+\left\{S_s\left(\frac{s}{\frac{2\sqrt{AB}}{\Delta}S_s(t_0)}\right)\right\}^2}}$$

$$\mathbf{b}(s)=\frac{\frac{4AB}{\Delta^2}\left(-\sin t_0, \cos t_0, 0\right)}{\left\{S_c\left(\frac{s}{\frac{2\sqrt{AB}}{\Delta}S_s(t_0)}\right)\right\}^2+\left\{S_s\left(\frac{s}{\frac{2\sqrt{AB}}{\Delta}S_s(t_0)}\right)\right\}^2}$$

From $\kappa_h=\langle\chi''(s),\mathbf{b}(s)\rangle$, the result follows. □

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NONDIFFERENTIABLE DESIRABILITY FUNCTIONS: DERIVATIVE FREE OPTIMIZATION WITH MATLAB/NOMAD

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Abstract. Nondifferentiable desirability functions are one of the most preferred multi-response optimization methods in nonlinear robust parameter design. Their nondifferentiability makes the optimization problem hard to solve and researchers and scientists look for new softwares and new desirability function structures to overcome this problem. In this study, we suggest a new implementation of derivative free mash adaptive direct search algorithm (MADS) with MATLAB/NOMAD to nondifferentiable desirability functions. For doing this, we need to model the optimization problem of desirability functions as a mixed-integer nonlinear optimization program (MINLP) by introducing a new binary variable to the model. This integer shows the side of the two-sided desirability function which is active. Hence, the model of our problem becomes nondifferentiable nonconvex MINLP. We show our implementation on three well-known optimization problem from the multi-response optimization literature. We finally conclude with an outlook and future research projects.

1. Introduction

Taguchi studied quality improvement through robust design which made the field of robust design widespread among industrial quality engineers [32, 33]. Robust design aims at designing a product or process to which the effect of noise factors is minimum. Robust design is important in terms of minimizing variance of a product or process performance while keeping the difference between mean and the target of the responses (output variable) as small as possible which improves the quality during the design phase of a product or process. If there are more than one responses, we must solve a multi-response optimization problem to take into account all the characteristics of a product or process and to obtain an optimal factor setting reflecting the optimal properties of a product or process, simultaneously.

In the literature, there are many methods developed for multi-response optimization problems. Since these methods stem from multi objective optimization, they are classified according to articulation of preference information of a decision maker: no articulation,

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prior articulation, interactive articularion and posterior articularion [18, 19, 20, 21, 22, 23, 24, 25, 27, 30]. Prior articulation methods are the most popular methods where pitfalls needs to be studied and overcome. These methods collect the decision makers preferences and articulates before the optimization algorithm run.

The so called loss functions which is a prior articulation method in multi-response optimization ignore mean and variance information of responses to handle the different scales of responses in computations. Another important prior articulation method used for solving multi-response optimization is desirability functions. This method uses mean and variance information and has been improved a lot in the last decades. Desirability functions method overcomes the different scales of responses by assigning a desirability function, which takes value between 0 and 1, to each response and then combine them to an overall desirability function to be optimized.

Desirability functions developed by Derringer and Suich has a drawback of containing nondifferentiable points [13, 14] occuring at the target points of two-sided desirability functions [1]. They are obtained by the composition of nonsmooth piecewise functions with signomial response functions (depend on factor variables (input vector)). Before, optimization of desirability function was solved by either direct search techniques or by smoothing nondifferentible desirability functions with polynomial approximations or by changing the formula of desirability functions. Therefore new advances in numerical optimization made us suggest using these new methods for the optimization of desirability functions. In [1], we developed nonconvex model of desirability functions [7] and we obtained continuous optimization relaxations and convex relaxations of this model. We extend this nonconvex model to MIP relaxations [34] in an upcoming paper [5]. These relaxations are solved by GAMS/CPLEX [12], GAMS/BARON [6] and GAMS/CONOPT [11] in [1] and [5].

In [2], we made a topological generalization of desirability functions used in practice to provide the robust optimization [35] of the nondifferentiable desirability functions and solved it with generalized semi-definite programming and disjunctive optimization by using GAMS/BARON. In [4], we analyzed the topological structure of generalized desirability functions to explain the mechanism behind these functions that enables researchers and scientists to develop new desirability functions with better structural properties.

In this paper, we solve a mixed-integer nonlinear optimization model for the desirability functions first given in [1, 10]. We apply derivative-free optimization techniques (mesh adaptive direct search) [10] to desirability functions that is mentioned in Table 1 of [1]. For the researchers and scientists, who do not use GAMS environment [15] and its solvers, this method is available under MATLAB and NOMAD solver [26, 29]. This method is easy to use and have proven superiority in the literature for nonconvex MINLP [7]. In this paper, we show implementation of MATLAB/NOMAD solver on nonsmooth nonconvex MINLP [28] formulation of desirability functions of Derringer and Suich for wire-bonding process optimization problem [9] which includes quantitative variables [8], for tire-tread compound problem [13] and for a chemical process problem [17].

In this study, after introducing notation of desirability functions in Section 2. We will give numerical examples' statements and results in Section 3. We will finish with a conclusion and outlook to the future which will be given in Section 4. We present response models in Appendix A and MATLAB/NOMAD implementations in appendix B.

1.1. **Derringer and Suich Type of DFs.** An average or expected value of a response can be written as $Y_{ji} = f_j(x_1, x_2, ..., x_n) + \epsilon_{ji}$ $(i = 1, 2, ..., n)$, $(j = 1, 2, ..., m)$ where Y_{ji} is measured through design of experiment. These average value Y_{ji}s are related to factor variables by the polynomial expressions f_j with expected value of $\epsilon_{ji} = 0$ and covariance matrix $\alpha^2 I$. Polynomial expressions f_j are approximated through polynomial functions. Here, expected value of responses are estimated by $\hat{Y}_j$ using regression by second degree polynomials for better fit. In this study, we will show estimators of expected value of responses by Y_j where $Y_j(\mathbf{x}) = z(\mathbf{X})\beta_j$ with β_j is the vector of regression coefficient estimates and $z(\mathbf{x})$ is the vector of regression variables, i.e,
$(1, x_1, x_2, x_3, x_1x_2, x_1x_3, x_2x_3, x_1^2, x_2^2, x_3^2, x_1x_2x_3, ...)$. Here, β_j is the mean of the unique least squares estimator $\hat{\beta} = (X'X)^{-1}X'Y$ since $X'X$ is always a nonsingular matrix.

Since there are more than one-response in a multi-response optimization problem, desirability functions converts these response values to desirability values and combine them by geometic mean to obtain a single objective function. This objective function is optimized to find the best trade-off between responses. There are two types of desirability functions: one sided (for smaller-the-better responses type and larger-the-better type responses) and two-sided (for nominal-is-the-best type responses) [1]. The desirability functions considered in this study are of Derringer and Suich's type [13]. They can be linear or nonlinear; usually piecewise smooth including a finite number of nondifferentiable points at their target value, where the maximum desirability occurs. The optimization of overall desirability functions becomes a complicated task when there are two-sided individual desirability functions in the problem. Below, we give the optimization problem of overall desirability function as a nonsmooth MINLP problem:

$$\begin{array}{l} \text{maximize} \quad D(\mathbf{y}, \mathbf{z}) \\ \text{subject to} \\ \quad x_i \in [-1, 1] \quad (i = 1, 2, \ldots, n), \\ \quad 0 \leq d_j(y_j, z_j) \leq 1, \\ \quad 0 \leq d_j(y_j) \leq 1, \\ \quad z_j \in \{0, 1\} \quad (j = 1, 2, \ldots, m) \end{array} \tag{1.1}$$

where $D(\mathbf{y}, \mathbf{z}) = (d_1(y_1, z_1)^{(w_1)} \cdot d_2(y_2, z_2)^{(w_2)} \cdot \ldots \cdot d_m(y_m, z_m)^{(w_m)})^{(\frac{1}{w_1+w_2+\ldots+w_m})}$. Here, $d_j(y_j, z_j) = z_j((y_j - l_j)/(t_j - l_j)) + (1 - z_j)((y_j - u_j)/(t_j - u_j))$ $(j = 1, 2, \ldots, m)$ for two-sided desirability functions, $d_j(y_j) = (y_j - l_j)/(t_j - l_j)$ for upper-the best one-sided desirability functions, $d_j(y_j) = (y_j - u_j)/(t_j - u_j)$ for lower-is-the- better one-sided desirability functions and $d_j(y_j) = (y_j - l_j)/(t_j - l_j)$ for upper-is-the-better one-sided desirability functions. Here, l_j,u_j,t_j corresponds to lower, upper and target of a response $y_j = Y_j(\mathbf{x})$ $(j = 1, 2, \ldots, m)$.

2. Examples and Results

In this Section, we solve three optimization problems with Derringer and Suich nondifferentiable desirability functions. We state the response models and necessary information in Appendix A. The problem given in Example 1 is solved by a modified desirability functions approach using Microsoft Excel GRG solver [9]. The problem given in Example 2 is solved by univariate direct search implemented under FORTRAN [13]. Lastly, Example 3 is solved by a hybrid genetic algorithm in combination with pattern search [17]. In this study, we obtained responses' models with better fits than those previously done by Design-Expert [1, 2] and solved the optimization problem of overall desirability by mesh adaptive direct search (MADS) implemented under MATLAB/NOMAD [26, 29].

2.1. **Numerical example: Wire Bonding Process Optimization.** The problem of wire bonding process optimization in semiconductor manufacturing has originally been presented in [9]. We use the 3 response models case given in [1]. In this problem, the overall desirability function $D^{\mathbf{Y}}(\mathbf{x},\mathbf{z}) = D(\mathbf{y},\mathbf{z}) = D(\mathbf{Y}(\mathbf{x}),\mathbf{z})$ with $y_j = Y_j(\mathbf{x})$ $(j = 1, 2, 3)$ is:

$$\begin{aligned}
&D^{\mathbf{Y}}(\mathbf{x},\mathbf{z}) = ((((z_1(174.9333 + 23.3750x_2 + 3.6250x_3 - 19.0000x_2x_3 - 185)/(190 - 185))+\\
&((1 - z_1)(174.9333 + 23.3750x_2 + 3.6250x_3 - 19.0000x_2x_3 - 195)/(190 - 195)))\cdot\\
&(z_2((154.8571 + 8.5000x_1 + 30.6250x_2 + 7.8750x_3 - 12.8571x_1^2 + 11.2500x_1x_2 - 185)/(190 - 185))+\\
&(1 - z_2)((154.8571 + 8.5000x_1 + 30.6250x_2 + 7.8750x_3 - 12.8571x_1^2 + 11.2500x_1x_2 - 195)/(190 - 195)))\cdot\\
&(z_3((140.2333 + 5.3437x_1 + 18.2500x_2 + 19.5938x_3 - 170)/(185 - 170))+\\
&(1 - z_3)((140.2333 + 5.3437x_1 + 18.2500x_2 + 19.5938x_3 - 195)/(185 - 195))))^(1/3))
\end{aligned} \tag{2.1}$$

where the decision variables are $\mathbf{x} = (x_1, x_2, x_3)$ and $\mathbf{z} = (z_1, z_2, z_3)$ since all the desirability functions of the problem are two-sided. We added the nonlinear constraints of individual desirability functions being 0 and 1 to the model on which MATLAB/NOMAD is implemented.

TABLE 1. Optimal solutions of the Wire Bonding Process Optimization problem with 3 responses with MATLAB/NOMAD [29].

Method	$(x_1^0, x_2^0, x_3^0, z_1^0, z_2^0, z_3^0)$	$(x_1^*, x_2^*, x_3^*, z_1^*, z_2^*, z_3^*)$
1	$(0, 0, 0, 0, 0, 0)$	$(-0.4854, 0.9945, 1, 1, 1, 1)$
2	$(0.0920, 1.0000, 0.8170, 1, 1, 1)$	$(0.9999, 0.8317, 0.5932, 1, 1, 1)$
3	$(1.0000, 0.8630, 0.5880, 1, 0, 1)$	$(1.0000, 0.8630, 0.5880, 1, 0, 1)$

	$(d_1(y_1^*), d_2(y_2^*), d_3(y_3^*))$	D^*
1	$(-0.4182, -0.8795, 0.3589)$	infeasible
2	$(0.4301, 0.9999, 0.1585)$	0.4085
3	$(0.5223, 0.7492, 0.1880)$	0.4190

2.1.1. *Results.* We run MATLAB/NOMAD [29] to solve this nonsmooth MINLP problem. We use three different initial points to find the global optimal. In Table 1, on the first line, we give $(0, 0, 0, 0, 0, 0)$as the initial point and MATLAB/NOMAD finds an infeasible solution. In the second line, we give $(0.0920, 1.0000, 0.8170, 1, 1, 1)$ as the initial point which is the local solution produced by GAMS/CONOPT in combination with MSG (see [1]) and MATLAB/NOMAD converges with a deep local solution which is very close to global optimal. When we give the global optimal (that we know from the literature [1]) as the initial point $(1.0000, 0.8630, 0.5880, 1, 0, 1)$, MATLAB/NOMAD finds the global optimal given in the third line. We present the MATLAB/NOMAD implementation of this problem in Appendix B.

2.2. **Numerical example: Tire Tread Compound Optimization.** The problem of tire tread compund optimization has originally been presented in [13]. We use the 4 response models given in [1] (see Appendix A). In this problem, the overall desirability function $D^{\mathbf{Y}}(\mathbf{x},\mathbf{z}) = D(\mathbf{y},\mathbf{z}) = D(\mathbf{Y}(\mathbf{x}),\mathbf{z})$ with $y_j = Y_j(\mathbf{x})$ $(j = 1, 2, 3, 4)$ is:

$$D^{\mathbf{Y}}(\mathbf{x},\mathbf{z}) = ((((139.1192 + 16.4936 * x_1 + 17.8808 * x_2 + 10.9065 * x_3 - 4.0096 * x_1 * x_1 - 3.4471 * x_2 * x_2 - 1.5721 * x_3 * x_3 + 5.1250 * x_1 * x_2 + 7.1250 * x_1 * x_3 + 7.8750 * x_2 * x_3 - 120)/(170 - 120))* ((1261.1331 + 268.1511 * x_1 + 246.5032 * x_2 + 139.4845 * x_3 - 83.5659 * x_1 * x_1 - 124.8155 * x_2 * x_2 + 199.1818 * x_3 * x_3 + 69.3750 * x_1 * x_2 + 94.1250 * x_1 * x_3 + 104.3750 * x_2 * x_3 - 1000)/(1300 - 1000))* (z_1 * ((400.3846 - 99.6664 * x_1 - 31.3964 * x_2 - 73.9190 * x_3 + 7.9327 * x_1 * x_1 + 17.3076 * x_2 * x_2 + +0.4328 * x_3 * x_3 + 8.7500 * x_1 * x_2 + 6.250 * x_1 * x_3 + 1.2500 * x_2 * x_3 - 400)/(500 - 400))+ (1 - z_1) * ((400.3846 - 99.6664 * x_1 - 31.3964 * x_2 - 73.9190 * x_3 + 7.9327 * x_1 * x_1 + 17.3076 * x_2 * x_2 + +0.4328 * x_3 * x_3 + 8.7500 * x_1 * x_2 + 6.250 * x_1 * x_3 + 1.2500 * x_2 * x_3 - 600)/(500 - 600)))* (z_2 * ((68.9096 - 1.4099 * x_1 + 4.3197 * x_2 + 1.6348 * x_3 + 1.5577 * x_1 * x_1 + 0.0577 * x_2 * x_2 - 0.3173 * x_3 * x_3 - 1.6250 * x_1 * x_2 + 0.1250 * x_1 * x_3 - 0.2500 * x_2 * x_3 - 60)/(67.5 - 60))+ (1 - z_2) * ((68.9096 - 1.4099 * x_1 + 4.3197 * x_2 + 1.6348 * x_3 + 1.5577 * x_1 * x_1 + 0.0577 * x_2 * x_2 - 0.3173 * x_3 * x_3 - 1.6250 * x_1 * x_2 + 0.1250 * x_1 * x_3 - 0.2500 * x_2 * x_3 - 75)/(67.5 - 75))))^(1/4)) \tag{2.2}$$

where the decision variables are $\mathbf{x} = (x_1, x_2, x_3)$ and $\mathbf{z} = (z_1, z_2)$ since there are two two-sided desirability function. We added the nonlinear constraints of individual desirability functions being 0 and 1 to the model on which MATLAB/NOMAD is implemented.

TABLE 2. Optimal solutions of the Wire Bonding Process Optimization problem with 3 responses with MATLAB/NOMAD [29].

Method	$(x_1^0, x_2^0, x_3^0, z_1^0, z_2^0)$	$(x_1^*, x_2^*, x_3^*, z_1^*, z_2^*)$
1	$(0, 0, 0, 0, 0)$	$(-0.0519, 0.1507, -0.8662, 1.0000, 0)$
2	$(0.0610, 0.0500, -0.8150, 1, 0)$	$(-0.0525, 0.1480, -0.8684, 1, 0)$
3	$(-0.0520, 0.1480, -0.8690, 1, 0)$	$(-0.0525, 0.1482, -0.8683, 1, 0)$

	$(d_1(y_1^*), d_2(y_2^*), d_3(y_3^*), d_4(y_4^*))$	D^*
1	$(0.1899, 1, 0.6564, 0.9285)$	0.5833
2	$(0.1886, 1, 0.6595, 0.9307)$	0.5833
3	$(0.1887, 1, 0.6593, 0.9305)$	0.5833

2.2.1. *Results.* We run MATLAB/NOMAD [29] to solve this nonsmooth MINLP problem. We use three different initial points to find the global optimal. In Table 2, on the first line, we give $(0, 0, 0, 0, 0, 0)$as the initial point (an arbitrary point) and MATLAB/NOMAD finds the global optimal. In the second line, we give $(0.0610, 0.0500, -0.8150, 1, 0)$ as the initial point which is the local solution produced by GAMS/CONOPT in combination with MSG (see [1]) and MATLAB/NOMAD converges to global optimal. When we give the global optimal (that we know from the literature [1]) as the initial point $(-0.0520, 0.1480, -0.8690, 1, 0)$, MATLAB/NOMAD finds the global optimal given in the third line. We present the MATLAB/NOMAD implementation of this problem in Appendix C. Hence, in all three cases, MATLAB/NOMAD finds the global optima for this problem.

2.3. **Numerical example : A Chemical Process Optimization.** The problem of a chemical process optimization has originally been presented in [17]. We use the 3 response models case given in [1] (see appendix A). In this problem, the overall desirability function $D^{\mathbf{Y}}(\mathbf{x}, \mathbf{z}) = D(\mathbf{y}, \mathbf{z}) = D(\mathbf{Y}(\mathbf{x}), \mathbf{z})$ with $y_j = Y_j(\mathbf{x})$ $(j = 1, 2, 3)$ is:

$$
\begin{aligned}
&D^{\mathbf{Y}}(\mathbf{x},\mathbf{z}) = 0.7*((79.940 + 0.995*x_1 + 0.515*x_2 - 0.1376*x_1*x_1 - 1.001*x_2*x_2 + \\
&0.250*x_1*x_2 - 78.5)/(80-78.5))* \\
&(0.2*z*((69.552 - 0.948*x_2 - 6.598*x_2*x_2 - 62)/(65-62)) + \\
&0.2*(1-z)*((69.552 - 0.948*x_2 - 6.598*x_2*x_2 - 68)/(65-68)))* \\
&0.1*((3386.2 + 205.10*x_1 + 177.4*x_2 - 3450)/(3100-3450))
\end{aligned} \tag{2.3}
$$

where the decision variables are $\mathbf{x} = (x_1, x_2)$ and $\mathbf{z} = (z)$ since there is only one two-sided desirability function. We added the nonlinear constraints of individual desirability functions being 0 and 1 to the model on which MATLAB/NOMAD is implemented.

2.3.1. *Results.* We run MATLAB/NOMAD [29] to solve this nonsmooth MINLP problem. We use two different initial points to find the global optimal. In Table 3, on the first line, we give $(0, 0, 0)$ as the initial point (an arbitrary point) and MATLAB/NOMAD converges. When we give the global optimal (that we know from the literature [2]) as the initial point $(0.1723, -0.8516, 0)$, MATLAB/NOMAD converges. We present the MATLAB/NOMAD implementation of this problem in Appendix D. Here, we note that although MATLAB/NOMAD converges, it gives an inferior solution than found in [17].

Table 3. Optimal solutions of the Chemical Process Optimization problem with 3 responses with MATLAB/NOMAD [29].

Method	(x_1^0, x_2^0, z^0)	(x_1^*, x_2^*, z^*)	$(d_1(y_1^*), d_2(y_2^*), d_3(y_3^*))$	D^*
1	$(0, 0, 0)$	$(-0.3774, -0.8865, 0)$	$(0.2189, 0.1862, 0.0410)$	0.0017
2	$(0.1723, -0.8516, 0)$	$(0.3774, -0.8865, 0)$	$(0.2189, 0.1862, 0.0410)$	0.0017

3. Conclusion and Future Outlook

In this work, we investigate the derivative free optimization [10] to find out advantages of them over the global optimization approaches on wire bonding process optimization problem, tire tread compund problem and a chemical process problem. Although, MATLAB/NOMAD is a nonconvex MINLP solver, it highly depends on initial point selection. On wire bonding process optimization problem, we tried three different initial points to find out if it gives the global optimal however, it did not produce the global optimal unless the global optimal is the initial point itself. On tire tread compound problem, MATLAB/NOMAD succeed to find the global optimal whatever the initial point is. On chemical process optimization problem, MATLAB/NOMAD converges, however the optimal value is inferior than the results reported in the literature.

Another important issue which we faced with in our implementation is related with bound selection of decision variables, which effects the convergence of MATLAB/NOMAD. Anyway, MATLAB/NOMAD is a useful software when we know the global optima. This is important for the researchers and scientists who do not have the global optimizers available.

In this study, our work describes a new approach to model nondifferentiable functions via integer variables using a new tool. The methodology is new in the sense that we try different initial points one of which is global optimal. The solution process can be improved further by studying selections for bounds and initial values. We tested our computational approaches on different examples from the literature including one-sided and two-sided desirability functions.

In the future, it is possible to implement the desirability function which includes more than one nondifferentiable points [9, 3] since we have already tested global optimization [1], convex optimization [1], semi-infinite programming [2], mixed integer linear programming [5] and derivative free optimization on desirability functions including one nondifferentiable point. It is also possible to apply our derivative free approach to signomial [31] cases of desirability functions. This study is also connected in a broder sense to optimal control.

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Appendix A. Responses of Wire bonding process optimization problem

$$
\begin{aligned}
Y_1(\mathbf{x}) &= 174.9333 + 23.3750x_2 + 3.6250x_3 - 19.0000x_2x_3, \\
Y_2(\mathbf{x}) &= 154.8571 + 8.5000x_1 + 30.6250x_2 + 7.8750x_3 - 12.8571x_1^2 + 11.2500x_1x_2, \\
Y_3(\mathbf{x}) &= 140.2333 + 5.3437x_1 + 18.2500x_2 + 19.5938x_3.
\end{aligned} \tag{A.1}
$$

Corresponding lower, target and upper values is given in 4

Table 4. Desirability Parameters of the responses for the Wire Bonding Problem [9].

	l_j	t_j	u_j	$d_j(l_j)$	$d_j(t_j)$	$d_j(u_j)$
y_1	185	190	195	0	1	0
y_2	185	190	195	0	1	0
y_3	170	185	195	0	1	0

Responses of Tire tread compound problem

$$
\begin{aligned}
Y_1(\mathbf{x}) = \quad & 139.1192 + 16.4936x_1 + 17.8808x_2 + 10.9065x_3 - 4.0096x_1x_1 - 3.4471x_2x_2 \\
& -1.5721x_3x_3 + 5.1250x_1x_2 + 7.1250x_1x_3 + 7.8750x_2x_3, \\
Y_2(\mathbf{x}) = \quad & 1261.1331 + 268.1511x_1 + 246.5032x_2 + 139.4845x_3 - 83.5659x_1x_1 - 124.8155x_2x_2 \\
& +199.1818x_3x_3 + 69.3750x_1x_2 + 94.1250x_1x_3 + 104.3750x_2x_3, \\
Y_3(\mathbf{x}) = \quad & 400.3846 - 99.6664x_1 - 31.3964x_2 - 73.9190x_3 + 7.9327x_1x_1 + 17.3076x_2x_2 \\
& +0.4328x_3x_3 + 8.7500x_1x_2 + 6.250x_1x_3 + 1.2500x_2x_3, \\
Y_4(\mathbf{x}) = \quad & 68.9096 - 1.4099x_1 + 4.3197x_2 + 1.6348x_3 + 1.5577x_1x_1 + 0.0577x_2x_2 \\
& -0.3173x_3x_3 - 1.6250x_1x_2 + 0.1250x_1x_3 - 0.2500x_2x_3.
\end{aligned} \tag{A.2}
$$

Corresponding lower, target and upper values is given in 5

TABLE 5. Desirability Parameters of the responses for the Tire tread compound problem [9].

	l_j	t_j	u_j	$d_j(l_j)$	$d_j(t_j)$	$d_j(u_j)$
y_1	120	–	170	0	1	0
y_2	1000	–	1300	0	1	0
y_3	400	500	600	0	1	0
y_4	60	67.5	75	0	1	0

Responses of a Chemical process

$$\begin{aligned} Y_1(\mathbf{x}),\ y = &\quad 79.940 + 0.995x_1 + 0.515x_2 - 1.376x_1x_1 \\ &-1.001x_2x_2 + 0.250x_1x_2, \\ Y_2(\mathbf{x}),\ y = &\quad 69.522 - 0.948x_2 - 6.598x_2x_2, \\ Y_3(\mathbf{x}),\ y = &\quad 3386.2 + 205.10x_1 + 177.4x_2. \end{aligned} \tag{A.3}$$

Corresponding lower, target and upper values is given in 6. The weights of the responses are $0.7, 0.2$ and 0.1, respectively.

TABLE 6. Desirability Parameters of the responses for the Tire tread compound problem [9].

	l_j	t_j	u_j	$d_j(l_j)$	$d_j(t_j)$	$d_j(u_j)$
y_1	78.5	–	80	0	1	0
y_2	62	65	68	0	1	0
y_3	3100	–	3450	0	1	0

APPENDIX B. MATLAB/NOMAD IMPLEMENTATION OF WIRE BONDING PROCESS OPTIMIZATION PROBLEM

```
clc
fun=@(x)-((((x(4)*(174.9333+23.3750*x(2)+3.6250*x(3)...
-19.0000*x(2)*x(3)-185)/(190-185))+...
((1-x(4))*(174.9333+23.3750*x(2)+3.6250*x(3)...
-19.0000*x(2)*x(3)-195)/(190-195)))*...
(x(5)*((154.8571+8.5000*x(1)+30.6250*x(2)+7.8750*x(3)...
-12.8571*x(1)^2+11.2500*x(1)*x(2)-185)/(190-185))+...
(1-x(5))*((154.8571+8.5000*x(1)+30.6250*x(2)+7.8750*x(3)...
-12.8571*x(1)^2+11.2500*x(1)*x(2)-195)/(190-195)))*...
(x(6)*((140.2333+ 5.3437*x(1)+18.2500*x(2)...
+19.5938*x(3)-170)/(185-170))+...
(1-x(6))*((140.2333+ 5.3437*x(1)+18.2500*x(2)...
+19.5938*x(3)-195)/(185-195))))^(1/3))

%x0 = [0.0920 1.0000 0.8170 1 1 1]';
x0 = [1.0000 0.8630 0.5880 1 0 1]';
```

```
%x0 = [0 0 0 0 0 0]';
lb = [-1;-1;-1;1;0;1];
ub = [1;1;1;1;0;1];

nlcon = @(x)[((x(4)*(174.9333+23.3750*x(2)+3.6250*x(3)...
-19.0000*x(2)*x(3)-185)/(190-185))+...
((1-x(4))*(174.9333+23.3750*x(2)+3.6250*x(3)...
-19.0000*x(2)*x(3)-195)/(190-195)))
(x(5)*((154.8571+8.5000*x(1)+30.6250*x(2)+7.8750*x(3)...
-12.8571*x(1)^2+11.2500*x(1)*x(2)-185)/(190-185))+...
(1-x(5))*((154.8571+8.5000*x(1)+30.6250*x(2)...
+7.8750*x(3)-12.8571*x(1)^2+11.2500*x(1)*x(2)-195)/(190-195)))...
(x(6)*((140.2333+ 5.3437*x(1)+18.2500*x(2)+19.5938*x(3)-170)/(185-170))+...
(1-x(6))*((140.2333+ 5.3437*x(1)+18.2500*x(2)+19.5938*x(3)-195)/(185-195)))];
cl=[0 0 0]';
cu=[1 1 1]';

xtype='CCCBBB';
opts=optiset('solver','nomad','display','iter')
Opt=opti('fun',fun,'bounds',lb,ub,'nl',nlcon,cl,cu,'xtype',xtype,'options',opts)
[x,fval,exitflag,info] = solve(Opt,x0)
```

Appendix C. MATLAB/NOMAD Implementation of Tire tread compound optimization problem

```
clc

fun=@(x) -((((139.1192+16.4936*x(1)+17.8808*x(2)+10.9065*x(3)...
-4.0096*x(1)*x(1)-3.4471*x(2)*x(2)-1.5721*x(3)*x(3)+5.1250*x(1)*x(2)...
+7.1250*x(1)*x(3)+7.8750*x(2)*x(3)-120)/(170-120))*...
((1261.1331+268.1511*x(1)+246.5032*x(2)+139.4845*x(3)...
-83.5659*x(1)*x(1)-124.8155*x(2)*x(2)+...
199.1818*x(3)*x(3)+69.3750*x(1)*x(2)+...
94.1250*x(1)*x(3)+104.3750*x(2)*x(3)-1000)/(1300-1000))*...
(x(4)*((400.3846-99.6664*x(1)-31.3964*x(2)-73.9190*x(3)...
+7.9327*x(1)*x(1)+17.3076*x(2)*x(2)+...
+0.4328*x(3)*x(3)+8.7500*x(1)*x(2)+6.250*x(1)*x(3)+...
1.2500*x(2)*x(3)-400)/(500-400))+...
(1-x(4))*((400.3846-99.6664*x(1)-31.3964*x(2)-...
73.9190*x(3)+7.9327*x(1)*x(1)+17.3076*x(2)*x(2)+...
+0.4328*x(3)*x(3)+8.7500*x(1)*x(2)+6.250*x(1)*x(3)+...
1.2500*x(2)*x(3)-600)/(500-600)))*...
(x(5)*((68.9096-1.4099*x(1)+4.3197*x(2)+1.6348*x(3)+...
1.5577*x(1)*x(1)+0.0577*x(2)*x(2)-...
0.3173*x(3)*x(3)-1.6250*x(1)*x(2)+0.1250*x(1)*x(3)-...
```

```
0.2500*x(2)*x(3)-60)/(67.5-60))+...
(1-x(5))*((68.9096-1.4099*x(1)+4.3197*x(2)+...
1.6348*x(3)+1.5577*x(1)*x(1)+0.0577*x(2)*x(2)-...
0.3173*x(3)*x(3)-1.6250*x(1)*x(2)+0.1250*x(1)*x(3)-...
0.2500*x(2)*x(3)-75)/(67.5-75))))^(1/4))

%x0 = [-0.0520 0.1480 -0.8690 1 0]';
%x0 = [0.0610 0.0500 -0.8150 1 0 ]';
x0 = [0 0 0 0 0 ]';
lb = [-1;-1;-1;0;0];
ub = [1;1;1;1;1];

nlcon = @(x)[((139.1192+16.4936*x(1)+17.8808*x(2)...
+10.9065*x(3)-4.0096*x(1)*x(1)-3.4471*x(2)*x(2)-...
1.5721*x(3)*x(3)+5.1250*x(1)*x(2)+7.1250*x(1)*x(3)...
+7.8750*x(2)*x(3)-120)/(170-120))

((1261.1331+268.1511*x(1)+246.5032*x(2)...
+139.4845*x(3)-83.5659*x(1)*x(1)-124.8155*x(2)*x(2)+...
199.1818*x(3)*x(3)+69.3750*x(1)*x(2)...
+94.1250*x(1)*x(3)+104.3750*x(2)*x(3)-1000)/(1300-1000))

(x(4)*((400.3846-99.6664*x(1)-31.3964*x(2)...
-73.9190*x(3)+7.9327*x(1)*x(1)+17.3076*x(2)*x(2)+...
+0.4328*x(3)*x(3)+8.7500*x(1)*x(2)...
+6.250*x(1)*x(3)+1.2500*x(2)*x(3)-400)/(500-400))+...
(1-x(4))*((400.3846-99.6664*x(1)...
-31.3964*x(2)-73.9190*x(3)+7.9327*x(1)*x(1)+...
17.3076*x(2)*x(2)+...
+0.4328*x(3)*x(3)+8.7500*x(1)*x(2)+6.250*x(1)*x(3)+...
1.2500*x(2)*x(3)-600)/(500-600)))

(x(5)*((68.9096-1.4099*x(1)+4.3197*x(2)+...
1.6348*x(3)+1.5577*x(1)*x(1)+0.0577*x(2)*x(2)-...
0.3173*x(3)*x(3)-1.6250*x(1)*x(2)+0.1250*x(1)*x(3)-...
0.2500*x(2)*x(3)-60)/(67.5-60))+...
(1-x(5))*((68.9096-1.4099*x(1)+4.3197*x(2)+1.6348*x(3)+...
1.5577*x(1)*x(1)+0.0577*x(2)*x(2)-...
0.3173*x(3)*x(3)-1.6250*x(1)*x(2)+0.1250*x(1)*x(3)-...
0.2500*x(2)*x(3)-75)/(67.5-75)))];

cl=[0 0 0 0]';
cu=[1 1 1 1]';
xtype='CCCBB';

opts=optiset('solver','nomad','display','iter')
Opt=opti('fun',fun,'bounds',lb,ub,'nl',nlcon,cl,cu,'xtype',xtype,'options',opts)
[x,fval,exitflag,info] = solve(Opt,x0)
```

Appendix D. MATLAB/NOMAD Implementation of a Chemical process optimization problem

```
clc

fun=@(x)(-( 0.7*((79.940+0.995*x(1)+0.515*x(2)-0.1376*x(1)*x(1)-1.001*x(2)*x(2)+...
0.250*x(1)*x(2)-78.5)/(80-78.5))*...
(0.2*x(3)*((69.552-0.948*x(2)-6.598*x(2)*x(2)-62)/(65-62))+...
0.2*(1-x(3))*((69.552-0.948*x(2)-6.598*x(2)*x(2)-68)/(65-68)))*...
0.1*((3386.2+205.10*x(1)+177.4*x(2)-3450)/(3100-3450))))

%x0 = [0.1723 -0.8516 0]';
x0 = [0 0 0]';
lb = [-1;-1;0];
ub = [1;1;0];

nlcon = @(x) [ 0.7*((79.940+0.995*x(1)+0.515*x(2)-0.1376*x(1)*x(1)-1.001*x(2)*x(2)+...
0.250*x(1)*x(2)-78.5)/(80-78.5))
(0.2*x(3)*((69.552-0.948*x(2)-6.598*x(2)*x(2)-62)/(65-62))+...
0.2*(1-x(3))*((69.552-0.948*x(2)-6.598*x(2)*x(2)-68)/(65-68)))
0.1*((3386.2+205.10*x(1)+177.4*x(2)-3450)/(3100-3450))];

cl = [0 0 0]';
cu=[1 1 1]';

 xtype='CCB';
 opts=optiset('solver','nomad','display','iter')
 Opt=opti('fun',fun,'bounds',lb,ub,'nl',nlcon,cl,cu,'xtype',xtype,'options',opts)
 [x,fval,exitflag,info] = solve(Opt,x0)
```

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