Number 51 Year 2025

Journal of New Theory

ISSN: 2149-1402

Naim Çağman

Editor-in-Chief

www.dergipark.org.tr/en/pub/jnt

Journal of New Theory (abbreviated as J. New Theory or JNT) is an international, peer-reviewed, and open-access journal.

J. New Theory is a mathematical journal focusing on new mathematical theories or their applications to science.

JNT was founded on 18 November 2014, and its first issue was published on 27 January 2015. **Language:** As of 2023, JNT accepts contributions in **American English** only.

Frequency: 4 Issues Per Year

Publication Dates: March, June, September, and December

ISSN: 2149-1402

Editor-in-Chief: Naim Çağman

E-mail: journalofnewtheory@gmail.com

Publisher: Naim Çağman

APC: JNT incurs no article processing charges.

Review Process: Blind Peer Review

DOI Numbers: The published papers are assigned DOI numbers.

Journal Boards

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Journal of New Theory
dergipark.org.tr/tr/pub/jnt
Open Access

ISSN: 2149-1402 (51) (2025) 1-9 doi.org/10.53570/jnt.1647686

Semiconformal Curvature Tensor on (κ, μ)-Paracontact Metric Manifolds

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Article Info
Received: 26 Feb 2025
Accepted: 18 May 2025
Published: 30 Jun 2025
Research Article

Abstract— This paper investigates several properties of the semiconformal curvature tensor on a (κ, μ)-paracontact metric manifold. It first examines the results arising when such a manifold is both semiconformal and semisymmetric. Based on these findings, this study provides characterizations of the manifold. It then explores the derivative interactions between various curvature tensors and the semiconformal curvature tensor. According to the results, the present paper establishes the conditions under which a (κ, μ)-paracontact metric manifold reduces to a (κ, μ)-paracontact metric manifold.

Keywords — *Semiconformal curvature tensor, (κ, μ)-paracontact metric manifolds, semisymmetric*

Mathematics Subject Classification (2020) 53C15, 53D10

1. Introduction

In 1985, the first studies on paracontact structures and paracontact geometry were presented by Kaneyuki and Williams [1]. Later, paracontact geometry studies were continued by Zamkovoy [2] with his systematic studies on paracontact manifolds and submanifolds. The more general case of paracontact metric manifolds, (κ, μ)-paracontact metric manifolds and (κ, μ, ν)- paracontact metric manifolds continue to be studied by many authors [3–7].

Let M be a $(2n+1)$-dimensional smooth manifold, and let ϕ be a $(1,1)$-tensor field, ξ a characteristic vector field, and η a 1-form on M. The triple (ϕ, ξ, η) is called an almost paracontact structure on M if it satisfies the following conditions:

i. $\phi(\xi) = 0$, $\eta o \phi = 0$, and $\eta(\xi) = 1$

ii. $\phi^2 \varrho_1 = \varrho_1 - \eta(\varrho_1)\xi$ and $\varrho_1 \in \chi(M)$

iii. The tensor field ϕ induces an almost paracomplex structure on each fibre of $D = \ker(\eta)$, meaning that the eigendistributions $D^+\phi$ and $D^-\phi$ of ϕ, corresponding to the eigenvalues 1 and -1, respectively, have equal dimension n.

The (M, ϕ, ξ, η) quadruple together with the (ϕ, ξ, η) structure on M is called an almost paracontact manifold [2]. There exists a semi-Riemannian metric g on the almost paracontact metric manifold M such that

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$$g(\phi\varrho_1,\phi\varrho_2) = -g(\varrho_1,\varrho_2) + \eta(\varrho_1)\eta(\varrho_2), g(\varrho_1,\xi) = \eta(\varrho_1) \tag{1.1}$$

and

$$g(\phi\varrho_1,\varrho_2) + g(\varrho_1,\phi\varrho_2) = 0$$

Let (M^{2n+1},ϕ,ξ,η) be an almost paracontact manifold. Then, the semi-Riemannian metric g is called an almost paracontact metric on (M^{2n+1},ϕ,ξ,η), the structure (ϕ,ξ,η,g) is called an almost paracontact metric structure, and the quintet $(M^{2n+1},\phi,\xi,\eta,g)$ is called an almost paracontact metric manifold [2]. If

$$d\eta(\varrho_1,\varrho_2) = \frac{1}{2}(\varrho_1\eta(\varrho_2) - \varrho_2\eta(\varrho_1) - \eta([\varrho_1,\varrho_2]))$$

is defined, then

$$\Phi(\varrho_1,\varrho_2) = g(\varrho_1,\phi\varrho_2) = d\eta(\varrho_1,\varrho_2)$$

The (ϕ,ξ,η,g) quadruplet is called a paracontact metric structure, and the quintet (M,ϕ,ξ,η,g) is called a paracontact metric manifold [2]. If M is a $(2n+1)$-dimensional almost paracontact metric manifold and the vector field ξ in the structure (ϕ,ξ,η,g) is a Killing vector field concerning g. The paracontact structure on M is called a K-paracontact structure, and M is called a K-paracontact metric manifold [2]. On a paracontact metric manifold, h is a symmetric operator, and the following properties hold:

$$h\xi = 0, \quad h\phi = -\phi h, \quad \text{and} \quad \operatorname{Tr} h = \operatorname{Tr}\phi h = 0 \tag{1.2}$$

$$2h\varrho_1 = (L_\xi\phi)\varrho_1 = L_\xi\phi\varrho_1 - \phi L_\xi\varrho_1 = [\xi,\phi\varrho_1] - \phi[\xi,\varrho_1]$$

$$\nabla_{\varrho_1}\xi = -\phi\varrho_1 + \phi h\varrho_1$$

Here, L is the Lie derivative [2].

Motivated by the above studies, this work investigates some symmetry conditions of a (κ,μ)-paracontact metric manifold. This paper consists of four sections. The first section provides basic information about almost paracontact metric manifolds. The second section introduces basic definitions and properties of (κ,μ)-paracontact metric manifolds and semiconformal curvature tensors. The third section obtains the results when a (κ,μ)-paracontact metric manifold is semiconformally semisymmetric. The fourth section investigates the condition for a (κ,μ)-paracontact metric manifold to be semiconformally Ricci semisymmetric and characterizes the manifold according to the obtained results. The last section concludes the paper.

2. Preliminaries

This section presents some notions to be needed in the following sections.

Definition 2.1. [4] A paracontact metric manifold is said to be a (κ,μ)-paracontact manifold if the curvature tensor R satisfies the following conditions

$$R(\varrho_1,\varrho_2)\xi = \kappa(\eta(\varrho_2)\varrho_1 - \eta(\varrho_1)\varrho_2) + \mu(\eta(\varrho_2)h\varrho_1 - \eta(\varrho_1)h\varrho_2) \tag{2.1}$$

for all $\varrho_1,\varrho_2 \in \chi(M)$ and κ and μ are real constants.

Here, if $\mu = 0$, then the (κ,μ)-paracontact metric manifold is called $N(\kappa)$-paracontact metric manifold.

In a (κ,μ)-paracontact metric manifold $(M^{2n+1},\phi,\xi,\eta,g)$ such that $n>1$, the following relations hold:

$$h^2=(\kappa+1)\phi^2$$

$$(\nabla_{\varrho_1}\phi)\varrho_2=-g(\varrho_1-h\varrho_1,\varrho_2)\xi+\eta(\varrho_2)(\varrho_1-h\varrho_1)$$

for $\kappa\neq-1$.

$$S(\varrho_1,\varrho_2)=(2(1-n)+\nu\mu)g(\varrho_1,\varrho_2)+(2(n-1)+\mu)g(h\varrho_1,\varrho_2)+(2(n-1)+n(2\kappa-\mu))\eta(\varrho_1)\eta(\varrho_2)$$

$$S(\varrho_1,\xi)=2n\kappa\eta(\varrho_1) \tag{2.2}$$

$$Q\varrho_1=(2(1-n)+n\mu)\varrho_1+(2(n-1)+\mu)h\varrho_1+(2(n-1)+n(2\kappa-\mu))\eta(\varrho_1)\xi$$

$$\phi\xi=2n\kappa\xi$$

and

$$Q\phi-\phi Q=2(2(n-1)+\mu)h\phi$$

In 2017, Kim [8] defined a curvature tensor of $(1,3)-$type that remains invariant under conharmonic transformation, called semiconformal curvature tensor, and obtained some results. The semiconformal curvature tensor is a generalization of the conformal curvature tensor and the conharmonic curvature tensor. Recently, many conditions of the semiconformal curvature tensor have been studied by many researchers [8–20]. The semiconformal curvature tensor P of $(1,3)-$type on a Riemann manifold (M^{2n+1},g), $n>1$, is as follows:

$$P(\varrho_1,\varrho_2)\varrho_3=-(n-2)bC(\varrho_1,\varrho_2)\varrho_3+(a+(n-2)b)H(\varrho_1,\varrho_2)\varrho_3 \tag{2.3}$$

where a and b are constants and not simultaneously zero, $C(\varrho_1,\varrho_2)\varrho_3$ denotes the conformal curvature tensor of $(1,3)-$type, and $H(\varrho_1,\varrho_2)\varrho_3$ denotes the conharmonic curvature tensor of $(1,3)-$type. The conformal curvature tensor of $(1,3)-$type and the conharmonic curvature tensor of $(1,3)-$type are given as:

$$\begin{aligned}C(\varrho_1,\varrho_2)\varrho_3 =& R(\varrho_1,\varrho_2)\varrho_3-\tfrac{1}{n-2}(S(\varrho_2,\varrho_3)\varrho_1-S(\varrho_1,\varrho_3)\varrho_2+g(\varrho_2,\varrho_3)Q\varrho_1-g(\varrho_1,\varrho_3)Q\varrho_2)\\ &+\tfrac{r}{(n-1)(n-2)}(g(\varrho_2,\varrho_3)\varrho_1-g(\varrho_1,\varrho_3)\varrho_2)\end{aligned} \tag{2.4}$$

and

$$H(\varrho_1,\varrho_2)\varrho_3=R(\varrho_1,\varrho_2)\varrho_3-\frac{1}{n-2}(S(\varrho_2,\varrho_3)\varrho_1-S(\varrho_1,\varrho_3)\varrho_2+g(\varrho_2,\varrho_3)Q\varrho_1-g(\varrho_1,\varrho_3)Q\varrho_2) \tag{2.5}$$

where r is scalar curvature, R is Riemann curvature tensor of $(1,3)-$type of the manifold M^{2n+1}, and S is the Ricci tensor of the manifold, given by $g(Q\varrho_1,\varrho_2)=S(\varrho_1,\varrho_2)$, where Q is the Ricci operator. Here, if (2.5) and (2.4) are used in (2.3), then (2.3) is reduced to the following form:

$$P(\varrho_1,\varrho_2)\varrho_3=aH(\varrho_1,\varrho_2)\varrho_3-\frac{br}{n-1}(g(\varrho_2,\varrho_3)\varrho_1-g(\varrho_1,\varrho_3)\varrho_2) \tag{2.6}$$

Putting $Z=\xi$ and using (1.1), (2.1), and (2.2),

$$H(\varrho_1,\varrho_2)\xi=\left(\tfrac{-(\kappa+2)-n(\mu-2)}{2n-1}\right)(\eta(\varrho_2)\varrho_1-\eta(\varrho_1)\varrho_2)+\left(\tfrac{(2\mu-2)(n-1)}{2n-1}\right)(\eta(\varrho_2)h\varrho_1-\eta(\varrho_1)h\varrho_2) \tag{2.7}$$

Put $\varrho_2=\xi$ in (2.7),

$$H(\varrho_1,\varrho_2)\xi=\left(\frac{-(\kappa+2)-n(\mu-2)}{2n-1}\right)(\varrho_1-\eta(\varrho_1)\xi)+\left(\frac{(2\mu-2)(n-1)}{2n-1}\right)h\varrho_1 \tag{2.8}$$

Moreover, putting $\varrho_1 = \xi$ in (2.5),

$$H(\xi, \varrho_2)\varrho_3 = \left(\frac{2n - n\mu - 4}{2n-1}\right)(g(\varrho_2, \varrho_3)\xi - \eta(\varrho_3)\varrho_2) - \left(\frac{2(n-1)(\mu-1)}{2n-1}\right)(g(h\varrho_2, \varrho_3)\xi - \eta(\varrho_3)h\varrho_2) \quad (2.9)$$

Putting $\varrho_1 = \xi$ in (2.6) and using (2.9),

$$P(\xi, \varrho_2)\varrho_3 = \left(\frac{a[2n-n\mu-4]}{2n-1} - \frac{br}{2n}\right)(g(\varrho_2, \varrho_3)\xi - \eta(\varrho_3)\varrho_2) - \left(\frac{2a(n-1)(\mu-1)}{2n-1}\right)(g(h\varrho_2, \varrho_3)\xi - \eta(\varrho_3)h\varrho_2) \quad (2.10)$$

Similarly, choosing $\varrho_3 = \xi$ in (2.6) and (2.8),

$$P(\varrho_1, \varrho_2)\xi = \left(\frac{-a(\kappa+2) - an(\mu-2)}{2n-1} - \frac{br}{2n}\right)(\eta(\varrho_2)\varrho_1 - \eta(\varrho_1)\varrho_2) + \left(\frac{2a(n-1)(\mu-1)}{2n-1}\right)(\eta(\varrho_2)h\varrho_1 - \eta(\varrho_1)h\varrho_2) \quad (2.11)$$

3. Semiconformally Semisymmetric (κ, μ)-paracontact Metric Manifold

This section introduces a semiconformally semisymmetric (κ, μ)-paracontact metric manifold.

Definition 3.1. A (κ, μ)-paracontact metric manifold M^{2n+1} is said to be semiconformally semisymmetric if the semiconformal curvature tensor satisfies the condition

$$R(\varrho_1, \varrho_2)P = 0$$

for all vector fields ϱ_1 and ϱ_2 on M.

Theorem 3.2. Let M be a $(2n+1)-$dimensional (κ, μ)-paracontact metric manifold. Then, M is semiconformally semisymmetric if and only if at least one of the following statements is true:

i. M is a $(\kappa, 1)$-paracontact metric manifold.

ii. The semiconformal curvature tensor P of M reduces to the form

$$P(\varrho_1, \varrho_2)\varrho_3 = \frac{-br}{2n-1}(g(\varrho_2, \varrho_3)\varrho_1 - g(\varrho_1, \varrho_3)\varrho_2)$$

for all $\varrho_1, \varrho_2, \varrho_3 \in \chi(M)$.

iii. $(\kappa+1)(\kappa - \mu(\kappa+1))h^2 + (\mu - \kappa)h = 0$.

Proof. Assume that the $(2n+1)-$dimensional (κ, μ)-paracontact metric manifold M is semiconformally semisymmetric. Then, for all $\varrho_1, \varrho_2, \varrho_3, \varrho_4, \varrho_5 \in \chi(M)$,

$$\begin{aligned}(R(\varrho_1, \varrho_2)P)(\varrho_4, \varrho_5, \varrho_3) &= R(\varrho_1, \varrho_2)P(\varrho_4, \varrho_5)\varrho_3 - P(R(\varrho_1, \varrho_2)\varrho_4, \varrho_5)\varrho_3 - P(\varrho_4, R(\varrho_1, \varrho_2)\varrho_5)\varrho_3 \\ &\quad - P(\varrho_4, \varrho_5)R(\varrho_1, \varrho_2)\varrho_3 \\ &= 0\end{aligned} \quad (3.1)$$

Choosing $\varrho_3 = \xi$ in (3.1),

$$R(\varrho_1, \varrho_2)P(\varrho_4, \varrho_5)\xi - P(R(\varrho_1, \varrho_2)\varrho_4, \varrho_5)\xi - P(\varrho_4, R(\varrho_1, \varrho_2)\varrho_5)\xi - P(\varrho_4, \varrho_5)R(\varrho_1, \varrho_2)\xi = 0 \quad (3.2)$$

Using (2.1) and (2.11) in (3.2),

$$\begin{aligned}0 &= A\eta(\varrho_5)R(\varrho_1, \varrho_2)\varrho_4 - A\eta(\varrho_4)R(\varrho_1, \varrho_2)\varrho_5 + B\eta(\varrho_5)R(\varrho_1, \varrho_2)h\varrho_4 \\ &\quad - B\eta(\varrho_4)R(\varrho_1, \varrho_2)h\varrho_5 - A\eta(\varrho_5)R(\varrho_1, \varrho_2)\varrho_4 + A\eta(R(\varrho_1, \varrho_2)\varrho_4)\varrho_5 \\ &\quad - B\eta(\varrho_5)hR(\varrho_1, \varrho_2)\varrho_4 + B\eta(R(\varrho_1, \varrho_2)\varrho_4)h\varrho_5 - A\eta(R(\varrho_1, \varrho_2)\varrho_5)\varrho_4 \\ &\quad + A\eta(\varrho_4)R(\varrho_1, \varrho_2)\varrho_5 - B\eta(R(\varrho_1, \varrho_2)\varrho_5)h\varrho_4 + B\eta(\varrho_4)hR(\varrho_1, \varrho_2)\varrho_5 \\ &\quad - \kappa\eta(\varrho_2)P(\varrho_4, \varrho_5)\varrho_1 + \kappa\eta(\varrho_1)P(\varrho_4, \varrho_5)\varrho_2 - \mu\eta(\varrho_2)P(\varrho_4, \varrho_5)h\varrho_1 \\ &\quad + \mu\eta(\varrho_1)P(\varrho_4, \varrho_5)h\varrho_2\end{aligned}$$

where

$$A=\frac{-a(\kappa+2)-an(\mu-2)}{2n-1}-\frac{br}{2n}\quad\text{and}\quad B=\frac{2a(n-1)(\mu-1)}{2n-1}$$

Thus,

$$\begin{aligned}0=\ &B\eta(\varrho_5)R(\varrho_1,\varrho_2)h\varrho_4-B\eta(\varrho_4)R(\varrho_1,\varrho_2)h\varrho_5+A\eta(R(\varrho_1,\varrho_2)\varrho_4)\varrho_5\\&-B\eta(\varrho_5)hR(\varrho_1,\varrho_2)\varrho_4+B\eta(R(\varrho_1,\varrho_2)\varrho_4)h\varrho_5-A\eta(R(\varrho_1,\varrho_2)\varrho_5)\varrho_4\\&-B\eta(R(\varrho_1,\varrho_2)\varrho_5)h\varrho_4+B\eta(\varrho_4)hR(\varrho_1,\varrho_2)\varrho_5-\kappa\eta(\varrho_2)P(\varrho_4,\varrho_5)\varrho_1\\&+\kappa\eta(\varrho_1)P(\varrho_4,\varrho_5)\varrho_2-\mu\eta(\varrho_2)P(\varrho_4,\varrho_5)h\varrho_{11}+\mu\eta(\varrho_1)P(\varrho_4,\varrho_5)h\varrho_2\end{aligned}\tag{3.3}$$

Putting $Y=U=\xi$ in (3.3) and using (1.1), (1.2), (2.1), (2.10), and (2.11),

$$\begin{aligned}0=\ &B\kappa g(\varrho_1,\varrho_5)\xi-B\kappa\eta(\varrho_5)\varrho_1+Bg(h\varrho_1,\varrho_5)\xi-B\mu\eta(\varrho_5)h\varrho_1-A\eta(\varrho_1)\varrho_5+B\kappa\eta(\varrho_1)h\varrho_5\\&-B\eta(\varrho_1)h\varrho_5+A\kappa\eta(\varrho_1)\varrho_5-B\kappa g(h\varrho_5,\varrho_1)\xi+B\kappa\eta(\varrho_1)h\varrho_5+B\kappa\eta(\varrho_1)h\varrho_5-\mu Bg(h\varrho_5,h\varrho_1)\xi\end{aligned}\tag{3.4}$$

In (3.4), taking inner product with $\xi\in\chi(M)$,

$$B\left(\kappa\left(g(\varrho_1,\varrho_5)-\eta(\varrho_5)\eta(\varrho_1)-g(h\varrho_5,\varrho_1)\right)+\mu\left(g(h\varrho_1,\varrho_5)-g(h\varrho_5,h\varrho_1)\right)\right)=0\tag{3.5}$$

This equation is satisfied for the following three cases:

i. $B=\frac{2a(n-1)(\mu-1)}{2n-1}=0$ and $\kappa\left(g(\varrho_1,\varrho_5)-\eta(\varrho_5)\eta(\varrho_1)-g(h\varrho_5,\varrho_1)\right)+\mu\left(g(h\varrho_1,\varrho_5)-g(h\varrho_5,h\varrho_1)\right)\neq 0$

ii. $B=\frac{2a(n-1)(\mu-1)}{2n-1}\neq 0$ and $\kappa\left(g(\varrho_1,\varrho_5)-\eta(\varrho_5)\eta(\varrho_1)-g(h\varrho_5,\varrho_1)\right)+\mu\left(g(h\varrho_1,\varrho_5)-g(h\varrho_5,h\varrho_1)\right)=0$

iii. $B=\frac{2a(n-1)(\mu-1)}{2n-1}=0$ and $\kappa\left(g(\varrho_1,\varrho_5)-\eta(\varrho_5)\eta(\varrho_1)-g(h\varrho_5,\varrho_1)\right)+\mu\left(g(h\varrho_1,\varrho_5)-g(h\varrho_5,h\varrho_1)\right)=0$

Here, if $B=\frac{2a(n-1)(\mu-1)}{2n-1}=0$, then $\mu=1$ and M is reduced to a $(\kappa,1)$ manifold. If $a=0$, then from (2.6)

$$P(X\varrho_1,\varrho_2)\varrho_3=-\frac{br}{2n-1}(g(\varrho_2,\varrho_3)\varrho_1-g(\varrho_1,\varrho_3)\varrho_2)$$

Finally, from (3.5),

$$\kappa(g(\varrho_1,\varrho_5)-\eta(\varrho_5)\eta(\varrho_1)-g(h\varrho_5,\varrho_1))+\mu(g(h\varrho_1,\varrho_5)-g(h\varrho_5,h\varrho_1))=0$$

Since h is symmetric, then

$$\kappa(g(\varrho_1,\varrho_5)-\eta(\varrho_5)\eta(\varrho_1)-g(h\varrho_5,\varrho_1))+\mu(g(h\varrho_1,\varrho_5)-g(h^2\varrho_5,\varrho_1))=0\tag{3.6}$$

Using (1.1) in (3.6),

$$\kappa(g(\phi\varrho_1,\phi\varrho_5)-g(h\varrho_1,\varrho_5))+\mu(g(h\varrho_1,\varrho_5)-(\kappa+1)g(\phi^2\varrho_1,\varrho_5))=0$$

and thus

$$\kappa\left(-g(\phi^2\varrho_1,\varrho_5)-g(h\varrho_1,\varrho_5)\right)+\mu\left(g(h\varrho_1,\varrho_5)-(\kappa+1)g(\phi^2\varrho_1,\varrho_5)\right)=0\tag{3.7}$$

From (3.7),

$$g(\phi^2\varrho_1,\varrho_5)[-\kappa-\mu(\kappa+1)]+g(h\varrho_1,\varrho_5)(\mu-\kappa)=0$$

and thus

$$(\kappa+1)(-\kappa-\mu(\kappa+1))g(h\varrho_1,h\varrho_5)+(\mu-\kappa)g(h\varrho_1,\varrho_5)=0$$

Therefore,

$$(\kappa+1)(-\kappa-\mu(\kappa+1))h^2+(\mu-\kappa)h=0$$

□

Moreover, if the trace of (3.6) is considered,

$$2n\kappa - \mu trh^2 = 2n\kappa - (\kappa+1)\mu\phi^2 = 2n\kappa - 2n(\kappa+1)\mu = 0$$

It means that

$$\kappa - (\kappa+1)\mu = 0$$

If $\kappa+1=0$ then $\kappa=0$. However, this contradicts $\kappa+1=0$. So $\kappa+1$ can never vanish (zero). Hence,

$$\mu = \frac{\kappa}{\kappa+1}$$

Here, $\mu=0$ if and only if $\kappa=0$. In this case, $R(X,Y)\xi=0$. Using Zamkovoy classification, M is $H^{2n}\times\mathbb{R}$.

4. Semiconformally Ricci Semisymmetric (κ,μ)-Paracontact Metric Manifold

This section defines a semiconformally Ricci semisymmetric (κ,μ)-paracontact metric manifold.

Definition 4.1. A (κ,μ)-paracontact metric manifold M^{2n+1} is said to be semiconformally Ricci semisymmetric if the semiconformal curvature tensor satisfies the condition

$$P(\varrho_1,\varrho_2)S=0$$

for all vector fields ϱ_1 and ϱ_2 on M.

Definition 4.2. A (κ,μ)-paracontact metric manifold M^{2n+1} is said to be $\eta-$Einstein manifold if its Ricci tensor S satisfies the condition

$$S(\varrho_1,\varrho_2)=\alpha g(\varrho_1,\varrho_2)+\beta\eta(\varrho_1)\eta(\varrho_2)$$

for all vector fields ϱ_1 and ϱ_2 and some real constants α and β. For $\beta=0$, it reduces to an Einstein manifold.

Theorem 4.3. Let M be a $(2n+1)-$dimensional (κ,μ)-paracontact metric manifold. Then, M is semiconformally semisymmetric if and only if M is an Einstein manifold.

Proof. Suppose that M is a $(2n+1)-$dimensional (κ,μ)-paracontact metric manifold that is semiconformally semisymmetric. Then, for all $\varrho_1,\varrho_2,\varrho_3,\varrho_4\in\chi(M)$,

$$P(\varrho_1,\varrho_2)S(\varrho_3,\varrho_4)=0 \tag{4.1}$$

From (4.1),

$$S(P(\varrho_1,\varrho_2)\varrho_3,\varrho_4)+S(\varrho_3,P(\varrho_1,\varrho_2)\varrho_4)=0 \tag{4.2}$$

Choosing $\varrho_1=\varrho_3=\xi$ in (4.2),

$$S(P(\xi,\varrho_2)\xi,\varrho_4)+2n\kappa\eta(P(\xi,\varrho_2)\varrho_4)=0 \tag{4.3}$$

Using (2.10) and (2.11) in (4.3),

$$\begin{aligned}0=&\left(\tfrac{a(2n-n\mu-4)}{2n-1}-\tfrac{br}{2n}\right)\eta(\varrho_2)S(\xi,\varrho_4)-\left(\tfrac{a(2n-n\mu-4)}{2n-1}-\tfrac{br}{2n}\right)S(\varrho_2,\varrho_4)+\left(\tfrac{2a(n-1)(\mu-1)}{2n-1}\right)S(h\varrho_2,\varrho_4)\\&+\left(\tfrac{a(2n-n\mu-4)}{2n-1}-\tfrac{br}{2n}\right)2n\kappa g(\varrho_2,\varrho_4)-\left(\tfrac{a(2n-n\mu-4)}{2n-1}\tfrac{br}{2n}\right)2n\kappa\eta(\varrho_2)\eta(\varrho_4)-\left(\tfrac{2a(n-1)(\mu-1)}{2n-1}\right)2n\kappa g(h\varrho_2,\varrho_4)\end{aligned}$$

which implies

$$\begin{aligned}0 = &\left(\tfrac{a(2n-n\mu-4)}{2n-1} - \tfrac{br}{2n}\right) S(\varrho_2,\varrho_4) - \left(\tfrac{a(2n-n\mu-4)}{2n-1} - \tfrac{br}{2n}\right) 2n\kappa g(\varrho_2,\varrho_4) + \left(\tfrac{2a(n-1)(\mu-1)}{2n-1}\right) S(h\varrho_2,\varrho_4)\\ &- \left(\tfrac{2a(n-1)(\mu-1)}{2n-1}\right) 2n\kappa g(hY\varrho_2,\varrho_4)\end{aligned} \tag{4.4}$$

Choosing $Y = hY$ in (4.4),

$$\begin{aligned}0 = &\left(\tfrac{a(2n-n\mu-4)}{2n-1} - \tfrac{br}{2n}\right) S(h\varrho_2,\varrho_4) - \left(\tfrac{a(2n-n\mu-4)}{2n-1} - \tfrac{br}{2n}\right) 2n\kappa(h\varrho_2,\varrho_4)\\ &+ \left(\tfrac{2a(n-1)(\mu-1)}{2n-1}\right)(\kappa+1)S(\varrho_2,\varrho_4) - \left(\tfrac{2a(n-1)(\mu-1)}{2n-1}\right)(\kappa+1)(\kappa+1)2n\kappa g(\varrho_2,\varrho_4)\end{aligned} \tag{4.5}$$

From (4.4) and (4.5),

$$S(\varrho_2,\varrho_4) = 2n\kappa g(\varrho_2,\varrho_4)$$

This shows that a (κ,μ)-paracontact metric manifold with $(2n+1)$-dimensional semiconformally Ricci semisymmetric is an Einstein manifold. □

5. Conclusion

This paper obtained significant and manifold-characterizing results for the semiconformally semisymmetry case of a (κ,μ)-paracontact metric manifold. First, it can be observed that the (κ,μ)-paracontact metric manifold reduces to a more special case of the $(\kappa,1)-$ paracontact metric manifold. Another result of the theorem is that the semiconformal curvature tensor on the manifold is reduced to the following form:

$$P(\varrho_1,\varrho_2)\varrho_3 = -\frac{br}{2n}[g(\varrho_2,\varrho_3)\varrho_1 - g(\varrho_1,\varrho_3)\varrho_2]$$

When this case is considered, if

$$P(\varrho_1,\varrho_2)\varrho_3 = -\frac{br}{2n}[g(\varrho_2,\varrho_3)\varrho_1 - g(\varrho_1,\varrho_3)\varrho_2]$$

then the manifold is said to be reduced to the real space form with constant section curvature. However, there is no classification expressed in the literature regarding the condition obtained. It is an open problem whether a classification, such as semiconformal real space form, can be made by examining this situation. In addition, as another result of the theorem, the following relation is obtained:

$$(\kappa+1)(-\kappa-\mu(\kappa+1))h^2 + (\mu-\kappa)h = 0$$

A second open problem is to determine to which special case of a (κ,μ)-paracontact metric manifold this relation reduces.

Author Contributions

All the authors equally contributed to this work. They all read and approved the final version of the paper.

Conflicts of Interest

All the authors declare no conflict of interest.

Ethical Review and Approval

No approval from the Board of Ethics is required.

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Journal of New Theory
dergipark.org.tr/tr/pub/jnt
Open Access

ISSN: 2149-1402 (51) (2025) 10-25 doi.org/10.53570/jnt.1651994

Toward the Determination of Vietoris-like Polynomials

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Article Info
Received: 5 Mar 2025
Accepted: 12 May 2025
Published: 30 Jun 2025
Research Article

Abstract— This paper studies the relationship between polynomials and classical number sequences, focusing on their structural properties and mathematical significance. It explores a specific class of polynomials inspired by Vietoris' number sequences, referred to as Vietoris-like polynomials. The primary objective is to analyze their fundamental algebraic properties, recurrence relations, and special identities. The study employs algebraic methods to derive the recurrence relations and explicit formulas for these polynomials. Moreover, it establishes Catalan-like, Cassini-like, and d'Ocagne-like identities.

Keywords — *Polynomials, Vietoris' sequence, Vietoris-like polynomials*

Mathematics Subject Classification (2020) 11B83, 11K31

1. Introduction

Polynomial forms of number sequences, beginning with Fibonacci polynomials, hold an important place in various subfields of mathematics such as geometry and algebra [1–3]. Fibonacci and Lucas polynomials constitute significant recursive sequences with remarkable algebraic and combinatorial properties. These polynomials have been extensively studied for their theoretical importance and applicability in interdisciplinary fields such as coding theory, quantum computing, and symbolic computation. Their structural characteristics enable efficient formulations in both pure and applied mathematics. In particular, Fibonacci-type polynomials have considerable applications in number theory [4–7]. For any variable quantity x, the Fibonacci polynomial $F_n(x)$ is defined as

$$F_n(x) = xF_{n-1}(x) + F_{n-2}(x), \quad \text{for all } n \geq 2$$

with $F_0(x) = 0$ and $F_1(x) = 1$. With a similar idea, the Lucas polynomial $L_n(x)$ is defined as

$$L_n(x) = xL_{n-1}(x) + L_{n-2}(x), \quad \text{for all } n \geq 2$$

with $L_0(x) = 2$ and $L_1(x) = x$. For more details, see [8, 9].

In 1958, Vietoris used Appell polynomials in connection with positivity problems of trigonometric sums [10]. Positivity as an interdisciplinary subject was an active research field, and several works were conducted using Vietoris' results [11]. Later on, in [12], the authors studied Vietoris' number

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sequence $\{v_s\}_{s\geq 0}$ with s-th term formula

$$v_s = \frac{1}{2^s}\binom{s}{\lfloor \frac{s}{2} \rfloor} \tag{1.1}$$

where $\binom{s}{\lfloor \frac{s}{2} \rfloor}$ is the central binomial coefficient [13] and $\lfloor . \rfloor$ represents the floor function. This sequence is associated with the sequence A283208 in the Online Encyclopedia of Integer Sequences (OEIS) [14]. As can be observed from [13, 15–19], Vietoris' sequence is one of the members of rational sequences, and some terms are as follows:

$$1, \frac{1}{2}, \frac{1}{2}, \frac{3}{8}, \frac{3}{8}, \frac{5}{16}, \frac{5}{16}, \frac{35}{128}, \frac{35}{128}, \frac{63}{256}, \frac{63}{256}, \ldots$$

In addition, the sequence of Appell-Vietoris polynomials [20], namely $\{\mathbb{V}_n(x)\}_{n\geq 0}$, is defined. For this sequence,

$$\mathbb{V}_n(x) = \sum_{k=0}^{n} \mathbb{T}_k^n x^k = \sum_{k=0}^{n} \binom{n}{k} c_{n-k} x^k$$

where $\mathbb{T}_k^n$ and c_k are triangle, i.e., these numbers form a triangular array with $n+1$ rows, indexed from $k = 0$ to $k = n$, and $k-$th term of the Vietoris sequence, respectively. In [20], it can be seen that Vietoris' sequence via the sequence of Appell-Vietoris polynomials for $x = 0$.

In this paper, we investigate the following questions: Is it possible to determine a special type of Vietoris-like polynomials by considering the properties of Vietoris' numbers? If so, what relations, identities, and properties do they satisfy? What conditions must be imposed on Vietoris-like polynomials to obtain meaningful results? This paper aims to explore and provide answers to the questions posed.

The rest of this study is structured as follows: Section 2 introduces the fundamental concepts to be utilized throughout the paper. Section 3 defines special Vietoris-like polynomials, investigates some of their basic properties, and analyzes their recurrence relations, special equalities, and identities such as those of Catalan, Cassini, and d'Ocagne. Finally, Section 4 provides the conclusions.

2. Preliminaries

This section discusses the basic properties of Vietoris' number sequence $\{v_s\}_{s\geq 0}$ with the s-th element in (1.1), For more details, see [10–18]. Even members of $\{v_s\}_{s\geq 0}$ are as follows:

$$v_{2n} = \frac{1}{2^{2n}}\binom{2n}{n}, \quad n \geq 0$$

where $v_{2n} = v_{2n-1}$. The two-term recurrence relation for $\{v_{2n}\}_{n\geq 0}$ is as follows:

$$v_{2n+2} = \mathcal{L}(2n)v_{2n}, \quad n \geq 0 \tag{2.1}$$

where

$$\mathcal{L}(k) = \frac{k+1}{k+2}, \quad k \geq 0 \tag{2.2}$$

Thus, the expression for v_{2n} in terms of any v_{2k} is as follows:

$$v_{2n} = \prod_{l=1}^{n-k} \mathcal{L}(2n-2l)v_{2k}, \quad n > k$$

Similarly, v_{2n} in terms of v_0 is as follows:

$$v_{2n+2} = \prod_{i=0}^{n} \mathcal{L}(2i) v_0 = \frac{(2n+1)!!}{(2n+2)!!}$$

Here, the double factorial of a number is defined as the product of all positive integers up to this number that shares the same parity (odd or even) as itself. The three consecutive-term recurrence relation for $\{v_{2n}\}_{n\geq 0}$ is as follows [17]:

$$v_{2n+2} = \frac{1}{2} v_{2n+1} + \frac{1}{2}\mathcal{L}(2n) v_{2n}, \quad n \geq 0 \tag{2.3}$$

The characteristic equation for the recurrence relation in (2.3) is given by [17]:

$$t^2 - \frac{1}{2}t - \frac{1}{2}\mathcal{L}(2n) = 0$$

with roots

$$r_{2n}^{\dagger_1} = \frac{1}{4}\left(1 - \sqrt{1 + 8\mathcal{L}(2n)}\right) \quad \text{and} \quad r_{2n}^{\dagger_2} = \frac{1}{4}\left(1 + \sqrt{1 + 8\mathcal{L}(2n)}\right) \tag{2.4}$$

According to the roots in (2.4), Vietoris' number sequence provides the following Binet-like formula [17]:

$$v_{2n} = c_{2n}^{\dagger_1}\left(r_{2n}^{\dagger_1}\right)^{2n} + c_{2n}^{\dagger_2}\left(r_{2n}^{\dagger_2}\right)^{2n}$$

where

$$c_{2n}^{\dagger_1} = \frac{\left(r_{2n}^{\dagger_2}\right)^{2n} - v_2}{\left(r_{2n}^{\dagger_2}\right)^{2n} - \left(r_{2n}^{\dagger_1}\right)^{2n}} \prod_{k=1}^{n-1}\left(2r_{2k}^{\dagger_1} - 1\right) r_{2k}^{\dagger_1} \quad \text{and} \quad c_{2n}^{\dagger_2} = \frac{v_2 - \left(r_{2n}^{\dagger_1}\right)^{2n}}{\left(r_{2n}^{\dagger_2}\right)^{2n} - \left(r_{2n}^{\dagger_1}\right)^{2n}} \prod_{k=1}^{n-1}\left(2r_{2k}^{\dagger_2} - 1\right) r_{2k}^{\dagger_2}$$

By the roots in (2.4), the following holds: $r_0^{\dagger_2} = \frac{1+\sqrt{5}}{4}$ (half of the golden ratio), $r_{2n}^{\dagger_1} + r_{2n}^{\dagger_2} = \dfrac{1}{2}$, and $r_{2n}^{\dagger_1} r_{2n}^{\dagger_2} = -\frac{\mathcal{L}(2n)}{2}$ [17]. Using (2.1), (2.3) of order two for the even index is rewritten as [17]:

$$v_{2n+2} = \frac{1}{2}\mathcal{L}(2n) v_{2n} + \frac{1}{2}\mathcal{L}(2n)\mathcal{L}(2n-2) v_{2n-2}, \quad n \geq 1$$

The characteristic equation of this recurrence is as follows [17]:

$$t^2 - \frac{1}{2}\mathcal{L}(2n)t - \frac{1}{2}\mathcal{L}(2n)\mathcal{L}(2n-2) = 0$$

with roots

$$\mathbf{r}_{2n}^{\dagger_1} = \frac{\mathcal{L}(2n)}{4}\left(1 - \sqrt{1 + 8\frac{\mathcal{L}(2n-2)}{\mathcal{L}(2n)}}\right) \quad \text{and} \quad \mathbf{r}_{2n}^{\dagger_2} = \frac{\mathcal{L}(2n)}{4}\left(1 + \sqrt{1 + 8\frac{\mathcal{L}(2n-2)}{\mathcal{L}(2n)}}\right) \tag{2.5}$$

According to these roots, Vietoris' number sequence provides the Binet-like formula [17]:

$$v_{2n} = \mathbf{c}_{2n}^{\dagger_1}\left(\mathbf{r}_{2n}^{\dagger_1}\right)^{2n} + \mathbf{c}_{2n}^{\dagger_2}\left(\mathbf{r}_{2n}^{\dagger_2}\right)^{2n}$$

where

$$\mathbf{c}_{2n}^{\dagger_1} = \frac{(2n-1)!!\left(-\mathcal{L}(2n)\left(\mathbf{r}_{2n}^{\dagger_2}\right)^{2n} + \left(\mathbf{r}_{2n+2}^{\dagger_2}\right)^{2n+2}\right)}{2^n n!\left(\left(\mathbf{r}_{2n}^{\dagger_1}\right)^{2n}\left(\mathbf{r}_{2n+2}^{\dagger_2}\right)^{2n+2} - \left(\mathbf{r}_{2n}^{\dagger_2}\right)^{2n}\left(\mathbf{r}_{2n+2}^{\dagger_1}\right)^{2n+2}\right)}$$

and

$$\mathbf{c}_{2n}^{\dagger_2} = \frac{(2n-1)!!\left(\mathcal{L}(2n)\left(\mathbf{r}_{2n}^{\dagger_1}\right)^{2n} - \left(\mathbf{r}_{2n+2}^{\dagger_1}\right)^{2n+2}\right)}{2^n n!\left(\left(\mathbf{r}_{2n}^{\dagger_1}\right)^{2n}\left(\mathbf{r}_{2n+2}^{\dagger_2}\right)^{2n+2} - \left(\mathbf{r}_{2n}^{\dagger_2}\right)^{2n}\left(\mathbf{r}_{2n+2}^{\dagger_1}\right)^{2n+2}\right)}$$

By the roots in (2.5), the following hold: $\mathbf{r}_{2n}^{\dagger_1} + \mathbf{r}_{2n}^{\dagger_2} = \dfrac{\mathcal{L}(2n)}{2}$ and $\mathbf{r}_{2n}^{\dagger_1}\mathbf{r}_{2n}^{\dagger_2} = -\dfrac{\mathcal{L}(2n)\mathcal{L}(2n-2)}{2}$ [17]. Moreover, the generating function is given by [15]:

$$g(z) = \frac{\sqrt{1+z} - \sqrt{1-z}}{z\sqrt{1-z}} = \sum_{p=0}^{\infty} v_p z^p, \quad 0 < |z| < 1$$

3. Special Vietoris-like Polynomials

This section introduces special Vietoris-like polynomials and presents several of their properties.

Definition 3.1. For real variable x, the s-th element of Vietoris-like polynomial sequence $\{\mathcal{V}_s(x)\}_{s\geq 0}$ is defined by

$$\mathcal{V}_s(x) = \begin{cases} \mathcal{L}(s-1)\mathcal{V}_{s-1}(x), & \text{if } s \text{ is odd} \\ \frac{x+1}{2}\mathcal{L}(s-2)\mathcal{V}_{s-2}(x), & \text{if } s \text{ is even} \end{cases} \tag{3.1}$$

where $\mathcal{V}_0(x) = 1$.

The first few Vietoris-like polynomials are

$$1,\ \tfrac{1}{2},\ \tfrac{x+1}{4},\ \tfrac{3(x+1)}{16},\ \tfrac{3(x+1)^2}{32},\ \tfrac{5(x+1)^2}{64},\ \tfrac{5(x+1)^3}{128},\ \tfrac{35(x+1)^3}{1024},\ \tfrac{35(x+1)^4}{2048},\ \tfrac{63(x+1)^4}{4096},\ \tfrac{63(x+1)^5}{8192},\ \cdots \tag{3.2}$$

In particular, for $x = 1$, Vietoris-like polynomials are equal to Vietoris' sequence. For $x = -1, \mathcal{V}_s(x) = 0$ where $s \geq 2$. It can be observed the graphs of the first eleven elements of Vietoris-like polynomial sequence in Figure 1, for $-5 \leq x \leq 5$.

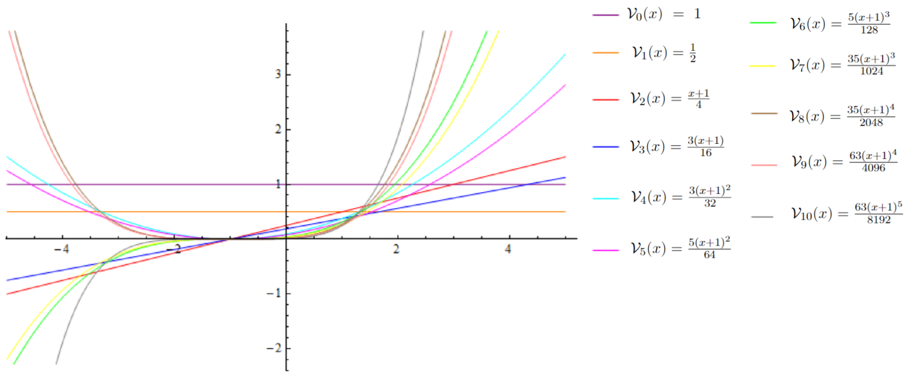

Figure 1. First eleven elements of Vietoris-like polynomial sequence

Corollary 3.2. Let $\{\mathcal{V}_s(x)\}_{s\geq 0}$ be Vietoris-like polynomial sequence. Then, two-term recurrence relation, for $\{\mathcal{V}_{2n}(x)\}_{n\geq 0}$ is obtained from (3.1), for $s = 2n+2$ as follows:

$$\mathcal{V}_{2n+2}(x) = \frac{x+1}{2}\mathcal{L}(2n)\mathcal{V}_{2n}(x) \tag{3.3}$$

Moreover, even members can be also written using (2.2) such that:

$$\mathcal{V}_{2n}(x) = \left(\frac{x+1}{2}\right)^n \frac{1}{2^{2n}}\binom{2n}{n}, \quad n \geq 0 \tag{3.4}$$

Corollary 3.3. Let $\{\mathcal{V}_s(x)\}_{s\geq 0}$ be Vietoris-like polynomial sequence. Considering (3.3) in terms of $\{\mathcal{V}_{2n}(x)\}_{n\geq 0}$,

$$\mathcal{V}_{2n+2}(x) = \left(\frac{x+1}{2}\right)^2 \mathcal{L}(2n)\mathcal{L}(2n-2)\mathcal{V}_{2n-2}(x), \quad n \geq 1 \tag{3.5}$$

Additionally, the term $\mathcal{V}_{2n}(x)$ in terms of any $\mathcal{V}_{2k}(x)$ is as follows:

$$\mathcal{V}_{2n}(x) = \prod_{l=1}^{n-k} \left(\frac{x+1}{2}\right)^{n-k} \mathcal{L}(2n-2l)\mathcal{V}_{2k}(x), \quad n > k \tag{3.6}$$

The following equality in terms of $\mathcal{V}_0(x)$ is obtained:

$$\mathcal{V}_{2n+2}(x) = \left(\frac{x+1}{2}\right)^{n+1} \prod_{i=0}^{n} \mathcal{L}(2i)\mathcal{V}_0(x) = \left(\frac{x+1}{2}\right)^{n+1} \frac{(2n+1)!!}{(2n+2)!!} \tag{3.7}$$

Proof. By putting $s = 2n+2$ and $s = 2n$ in (3.1), (3.3) and

$$\mathcal{V}_{2n}(x) = \frac{x+1}{2}\mathcal{L}(2n-2)\mathcal{V}_{2n-2}(x) \tag{3.8}$$

calculates, respectively. When (3.8) is substituted into (3.3), (3.5) is obtained. If this process continues, (3.6) is obtained. Moreover, for a particular value $k = 0$, (3.6) is transformed into (3.7). Here, it is clear that $\prod_{i=0}^{n} \mathcal{L}(2i) = \frac{(2n+1)!!}{(2n+2)!!}$ via (2.2) □

Corollary 3.4. The three consecutive-term recurrence relation for $\{\mathcal{V}_{2n}(x)\}_{n\geq 0}$ is as follows:

$$\mathcal{V}_{2n+2}(x) = \frac{x}{2}\mathcal{V}_{2n+1}(x) + \frac{\mathcal{L}(2n)}{2}\mathcal{V}_{2n}(x) \tag{3.9}$$

Proof. From (3.3), $\mathcal{V}_{2n+2}(x) = \frac{x+1}{2}\mathcal{L}(2n)\mathcal{V}_{2n}(x)$. Then, it follows $\mathcal{V}_{2n+2}(x) = \frac{x}{2}\mathcal{L}(2n)\mathcal{V}_{2n}(x) + \frac{1}{2}\mathcal{L}(2n)\mathcal{V}_{2n}(x)$. From (3.1), $\mathcal{L}(2n)\mathcal{V}_{2n}(x)$ for $\mathcal{V}_{2n+1}$. This ultimately leads to the three-consecutive-term recurrence relation (3.9). □

Corollary 3.5. Let $\{\mathcal{V}_s(x)\}_{s\geq 0}$ be Vietoris-like polynomial sequence. Then, two-term recurrence relation, for $\{\mathcal{V}_{2n+1}(x)\}_{n\geq 0}$ is obtained from (3.1), for $s = 2n+1$ as follows:

$$\mathcal{V}_{2n+1}(x) = \frac{x+1}{2}\mathcal{L}(2n)\mathcal{V}_{2n-1}(x) \tag{3.10}$$

Moreover, odd members can be also written using (2.2) such that:

$$\mathcal{V}_{2n-1}(x) = \left(\frac{x+1}{2}\right)^{n-1} \frac{1}{2^{2n}} \binom{2n}{n}, \quad n \geq 0$$

Corollary 3.6. Let $\{\mathcal{V}_s(x)\}_{s\geq 0}$ be Vietoris-like polynomial sequence. Considering (3.10) in terms of $\{\mathcal{V}_{2n+1}(x)\}_{n\geq 0}$,

$$\mathcal{V}_{2n+1}(x) = \left(\frac{x+1}{2}\right)^2 \mathcal{L}(2n)\mathcal{L}(2n-2)\mathcal{V}_{2n-3}(x), \quad n \geq 1 \tag{3.11}$$

Additionally, by using (3.10), the term $\mathcal{V}_{2n+1}(x)$ in terms of any $\mathcal{V}_{2k+1}(x)$ is as follows:

$$\mathcal{V}_{2n-1}(x) = \prod_{l=1}^{n-k} \left(\frac{x+1}{2}\right)^{n-k} \mathcal{L}(2n-2l)\mathcal{V}_{2k-1}(x), \quad n > k \tag{3.12}$$

the term $\mathcal{V}_{2n+1}(x)$ in terms of $\mathcal{V}_0(x)$ is obtained as follows:

$$\mathcal{V}_{2n+1}(x) = \left(\frac{x+1}{2}\right)^{n} \prod_{i=0}^{n} \mathcal{L}(2i)\mathcal{V}_0(x) = \left(\frac{x+1}{2}\right)^{n} \frac{(2n+1)!!}{(2n+2)!!} \tag{3.13}$$

Proof. By putting $s = 2n+1$ and $s = 2n-1$ in (3.1), (3.10) and

$$\mathcal{V}_{2n-1}(x) = \frac{x+1}{2}\mathcal{L}(2n-2)\mathcal{V}_{2n-3}(x), \tag{3.14}$$

is calculated, respectively. When (3.14) is substituted into (3.10), (3.11) is obtained. If this process continues, (3.12) is obtained. Moreover, for a particular value $k = 0$, (3.12) is transformed into (3.13). Here, it is clear that $\prod_{i=0}^{n} \mathcal{L}(2i) = \frac{(2n+1)!!}{(2n+2)!!}$ via (2.2) □

Theorem 3.7 (Binet-like Formula-Form 1)**.** Let $\{\mathcal{V}_{2n}(x)\}_{n\geq 0}$ be Vietoris-like polynomial sequence. Then, for $n > 1$, it provides Binet-like formula:

$$\mathcal{V}_{2n}(x) = C^{\dagger_1}_{2n}(x)(R^{\dagger_1}_{2n}(x))^{2n} + C^{\dagger_2}_{2n}(x)(R^{\dagger_2}_{2n}(x))^{2n} \tag{3.15}$$

where

$$R^{\dagger_1}_{2n}(x) = \frac{1}{4}\left(x - \sqrt{x^2 + 8\mathcal{L}(2n)}\right), \; R^{\dagger_2}_{2n}(x) = \frac{1}{4}\left(x + \sqrt{x^2 + 8\mathcal{L}(2n)}\right) \tag{3.16}$$

and

$$\begin{cases} C^{\dagger_1}_{2n}(x) &= \left(\frac{x+1}{2}\right)^{n-1} \dfrac{(R^{\dagger_2}_{2n}(x))^{2n} - \mathcal{V}_2(x)}{(R^{\dagger_2}_{2n}(x))^{2n} - (R^{\dagger_1}_{2n}(x))^{2n}} \prod\limits_{k=1}^{n-1} (2R^{\dagger_1}_{2k}(x) - x)R^{\dagger_1}_{2k}(x) \\ C^{\dagger_2}_{2n}(x) &= \left(\frac{x+1}{2}\right)^{n-1} \dfrac{\mathcal{V}_2(x) - (R^{\dagger_1}_{2n}(x))^{2n}}{(R^{\dagger_2}_{2n}(x))^{2n} - (R^{\dagger_1}_{2n}(x))^{2n}} \prod\limits_{k=1}^{n-1} (2R^{\dagger_2}_{2k}(x) - x)R^{\dagger_2}_{2k}(x) \end{cases} \tag{3.17}$$

Proof. Considering (3.9), characteristic equation for $\{\mathcal{V}_{2n}(x)\}_{n\geq 0}$ is written by

$$t^2 - \frac{1}{2}xt - \frac{1}{2}\mathcal{L}(2n) = 0 \tag{3.18}$$

Thus, its roots $R^{\dagger_1}_{2n}(x)$ and $R^{\dagger_2}_{2n}(x)$ are

$$R^{\dagger_1}_{2n}(x) = \frac{1}{4}\left(x - \sqrt{x^2 + 8\mathcal{L}(2n)}\right) \quad \text{and} \quad R^{\dagger_2}_{2n}(x) = \frac{1}{4}\left(x + \sqrt{x^2 + 8\mathcal{L}(2n)}\right) \tag{3.19}$$

By (3.17),

$$\begin{aligned} C^{\dagger_1}_{2n}(x)(R^{\dagger_1}_{2n}(x))^{2n} + C^{\dagger_2}_{2n}(x)(R^{\dagger_2}_{2n}(x))^{2n} &= \left(\frac{x+1}{2}\right)^{n-1} \frac{\left(\left(R^{\dagger_2}_{2n}(x)\right)^{2n} - \mathcal{V}_2(x)\right)\prod\limits_{k=1}^{n-1}(2R^{\dagger_1}_{2k}(x) - x)R^{\dagger_1}_{2k}(x)\left(R^{\dagger_1}_{2n}(x)\right)^{2n}}{\left(R^{\dagger_2}_{2n}(x)\right)^{2n} - \left(R^{\dagger_1}_{2n}(x)\right)^{2n}} \\ &+ \left(\frac{x+1}{2}\right)^{n-1} \frac{\left(\mathcal{V}_2(x) - \left(R^{\dagger_1}_{2n}(x)\right)^{2n}\right)\prod\limits_{k=1}^{n-1}(2R^{\dagger_2}_{2k}(x) - x)R^{\dagger_2}_{2k}(x)\left(R^{\dagger_2}_{2n}(x)\right)^{2n}}{\left(R^{\dagger_2}_{2n}(x)\right)^{2n} - \left(R^{\dagger_1}_{2n}(x)\right)^{2n}} \\ &= \left(\frac{x+1}{2}\right)^{n-1} \frac{\left(R^{\dagger_2}_{2n}(x)\right)^{2n}\left(R^{\dagger_1}_{2n}(x)\right)^{2n}\prod\limits_{k=1}^{n-1}(2R^{\dagger_1}_{2k}(x) - x)R^{\dagger_1}_{2k}(x) - \left(R^{\dagger_1}_{2n}(x)\right)^{2n}\left(R^{\dagger_2}_{2n}(x)\right)^{2n}\prod\limits_{k=1}^{n-1}(2R^{\dagger_2}_{2k}(x) - x)R^{\dagger_2}_{2k}(x)}{\left(R^{\dagger_2}_{2n}(x)\right)^{2n} - \left(R^{\dagger_1}_{2n}(x)\right)^{2n}} \\ &+ \left(\frac{x+1}{2}\right)^{n-1} \frac{-\mathcal{V}_2(x)\left(R^{\dagger_1}_{2n}(x)\right)^{2n}\prod\limits_{k=1}^{n-1}(2R^{\dagger_1}_{2k}(x) - x)R^{\dagger_1}_{2n}(2k) + \mathcal{V}_2(x)\left(R^{\dagger_2}_{2n}(x)\right)^{2n}\prod\limits_{k=1}^{n-1}(2R^{\dagger_2}_{2k}(x) - x)R^{\dagger_2}_{2k}(x)}{\left(R^{\dagger_2}_{2n}(x)\right)^{2n} - \left(R^{\dagger_1}_{2n}(x)\right)^{2n}} \\ &= \left(\frac{x+1}{2}\right)^{n-1} \frac{\left(R^{\dagger_2}_{2n}(x)\right)^{2n}\left(R^{\dagger_1}_{2n}(x)\right)^{2n}\prod\limits_{k=1}^{n-1}\left(2\left(R^{\dagger_1}_{2k}(x)\right)^2 - xR^{\dagger_1}_{2k}(x)\right) - \left(2\left(R^{\dagger_2}_{2k}(x)\right)^2 - xR^{\dagger_2}_{2k}(x)\right)}{\left(R^{\dagger_2}_{2n}(x)\right)^{2n} - \left(R^{\dagger_1}_{2n}(x)\right)^{2n}} \\ &+ \mathcal{V}_2(x)\left(\frac{x+1}{2}\right)^{n-1} \frac{-\left(R^{\dagger_1}_{2n}(x)\right)^{2n}\prod\limits_{k=1}^{n-1}\left(2\left(R^{\dagger_1}_{2k}(x)\right)^2 - xR^{\dagger_1}_{2k}(x)\right) + \left(R^{\dagger_2}_{2n}(x)\right)^{2n}\prod\limits_{k=1}^{n-1}\left(2\left(R^{\dagger_2}_{2k}(x)\right)^2 - xR^{\dagger_2}_{2k}(x)\right)}{\left(R^{\dagger_2}_{2n}(x)\right)^{2n} - \left(R^{\dagger_1}_{2n}(x)\right)^{2n}} \end{aligned}$$

Since $R^{\dagger_1}_{2n}(x)$ and $R^{\dagger_2}_{2n}(x)$ in (3.19) satisfies (3.18), then

$$2\left(R^{\dagger_1}_{2k}(x)\right)^2 - xR^{\dagger_1}_{2k}(x) = \mathcal{L}(2k) \quad \text{and} \quad 2\left(R^{\dagger_2}_{2k}(x)\right)^2 - xR^{\dagger_2}_{2k}(x) = \mathcal{L}(2k)$$

Then,

$$C_{2n}^{\dagger_1}(x)(R_{2n}^{\dagger_1}(x))^{2n} + C_{2n}^{\dagger_2}(x)(R_{2n}^{\dagger_2}(x))^{2n} = \mathcal{V}_2(x)\left(\frac{x+1}{2}\right)^{n-1} \frac{-\left(R_{2n}^{\dagger_1}(x)\right)^{2n}\prod_{k=1}^{n-1}\mathcal{L}(2k) + \left(R_{2n}^{\dagger_2}(x)\right)^{2n}\prod_{k=1}^{n-1}\mathcal{L}(2k)}{\left(R_{2n}^{\dagger_2}(x)\right)^{2n} - \left(R_{2n}^{\dagger_1}(x)\right)^{2n}}$$

$$= \mathcal{V}_2(x)\left(\frac{x+1}{2}\right)^{n-1}\prod_{k=1}^{n-1}\mathcal{L}(2k)$$

Furthermore, using (3.6), for $k=1$, the equality $\mathcal{V}_{2n}(x) = \prod_{l=1}^{n-1}\left(\frac{x+1}{2}\right)^{n-1}\mathcal{L}(2n-2l)\mathcal{V}_2(x)$ is obtained, and thus (3.15) is valid. □

Example 3.8. Calculate $\mathcal{V}_6(x)$ and $\mathcal{V}_8(x)$ using Binet-like Formula-Form 1 for $n=3$ and $n=4$, respectively. Through (3.16) and (3.17),

$$R_6^{\dagger_1}(x) = \frac{\left(x-\sqrt{x^2+7}\right)^6}{4096}$$

$$R_6^{\dagger_2}(x) = \frac{\left(x+\sqrt{x^2+7}\right)^6}{4096}$$

$$C_6^{\dagger_1}(x) = \frac{(x+1)^2\left(x+\sqrt{x^2+6}\right)\left(x+\sqrt{x^2+\frac{20}{3}}\right)\left(-x+\frac{1}{2}\left(x+\sqrt{x^2+6}\right)\right)\left(-x+\frac{1}{2}\left(x+\sqrt{x^2+\frac{20}{3}}\right)\right)\left(\frac{1}{4}(-1-x)+\frac{\left(x+\sqrt{x^2+7}\right)^6}{4096}\right)}{64\left(-\frac{\left(x-\sqrt{x^2+7}\right)^6}{4096}+\frac{\left(x+\sqrt{x^2+7}\right)^6}{4096}\right)}$$

and

$$C_6^{\dagger_2}(x) = \frac{(x+1)^2\left(x-\sqrt{x^2+6}\right)\left(x-\sqrt{x^2+\frac{20}{3}}\right)\left(-x+\frac{1}{2}\left(x-\sqrt{x^2+6}\right)\right)\left(-x+\frac{1}{2}\left(x-\sqrt{x^2+\frac{20}{3}}\right)\right)\left(\frac{1+x}{4}-\frac{\left(x-\sqrt{x^2+7}\right)^6}{4096}\right)}{64\left(-\frac{\left(x-\sqrt{x^2+7}\right)^6}{4096}+\frac{\left(x+\sqrt{x^2+7}\right)^6}{4096}\right)}$$

Then, $\mathcal{V}_6(x) = C_6^{\dagger_1}(x)(R_6^{\dagger_1}(x))^6 + C_6^{\dagger_2}(x)(R_6^{\dagger_2}(x))^6 = \frac{5(x+1)^3}{128}$. It can also be observed that $\mathcal{V}_6(x)$ via (3.2). Similarly, through (3.16) and (3.17),

$$R_8^{\dagger_1}(x) = \frac{\left(x-\sqrt{x^2+\frac{36}{5}}\right)^8}{65536}$$

$$R_8^{\dagger_2}(x) = \frac{\left(x+\sqrt{x^2+\frac{36}{5}}\right)^8}{65536}$$

$$C_8^{\dagger_1}(x) = \frac{(x+1)^3\left(-x+\sqrt{x^2+6}\right)\left(x+\sqrt{x^2+6}\right)\left(-\frac{x}{2}+\frac{1}{2}\sqrt{x^2+\frac{20}{3}}\right)\left(x+\sqrt{x^2+\frac{20}{3}}\right)\left(-x+\sqrt{x^2+7}\right)\left(x+\sqrt{x^2+7}\right)\left(\frac{1}{4}(-x-1)+\frac{\left(x+\sqrt{x^2+\frac{36}{5}}\right)^8}{65536}\right)}{2048\left(-\frac{\left(x-\sqrt{x^2+\frac{36}{5}}\right)^8}{65536}+\frac{\left(x+\sqrt{x^2+\frac{36}{5}}\right)^8}{65536}\right)}$$

and

$$C_8^{\dagger_2}(x) = \frac{(x+1)^3\left(-x-\sqrt{x^2+6}\right)\left(x-\sqrt{x^2+6}\right)\left(x-\sqrt{x^2+\frac{20}{3}}\right)\left(-\frac{x}{2}-\frac{1}{2}\sqrt{x^2+\frac{20}{3}}\right)\left(-x-\sqrt{x^2+7}\right)\left(x-\sqrt{x^2+7}\right)\left(\frac{1+x}{4}-\frac{\left(x-\sqrt{x^2+\frac{36}{5}}\right)^8}{65536}\right)}{2048\left(-\frac{\left(x-\sqrt{x^2+\frac{36}{5}}\right)^8}{65536}+\frac{\left(x+\sqrt{x^2+\frac{36}{5}}\right)^8}{65536}\right)}$$

Then, it follows that $\mathcal{V}_8(x) = C_8^{\dagger_1}(x)(R_8^{\dagger_1}(x))^8 + C_8^{\dagger_2}(x)(R_8^{\dagger_2}(x))^8 = \frac{35(x+1)^4}{2048}$. It can also be checked via (3.2).

Remark 3.9. The following hold for $R_{2n}^{\dagger_1}(x)$ and $R_{2n}^{\dagger_2}(x)$:

i. $R_0^{\dagger_2}(1) = \frac{1+\sqrt{5}}{4}$, which is half of the golden ratio *ii.* $R^{\dagger_1}_{2n}(x) + R^{\dagger_2}_{2n}(x) = \frac{x}{2}$

iii. $R^{\dagger_1}_{2n}(x) R^{\dagger_2}_{2n}(x) = -\frac{\mathcal{L}(2n)}{2}$

Theorem 3.10 (Binet-like Formula-Form 2)**.** Let $\{\mathcal{V}_{2n}(x)\}_{n\geq 0}$ be Vietoris-like polynomial sequence. Then, it provides Binet-like formula

$$\mathcal{V}_{2n}(x) = \mathcal{C}^{\dagger_1}_{2n}(x)(\mathcal{R}^{\dagger_1}_{2n}(x))^{2n} + \mathcal{C}^{\dagger_2}_{2n}(x)(\mathcal{R}^{\dagger_2}_{2n}(x))^{2n} \tag{3.20}$$

where

$$\begin{cases} \mathcal{R}^{\dagger_1}_{2n}(x) = \dfrac{\mathcal{L}(2n)}{4}\left(x - \sqrt{x^2 + \dfrac{4(x+1)\mathcal{L}(2n-2)}{\mathcal{L}(2n)}}\right) \\ \mathcal{R}^{\dagger_2}_{2n}(x) = \dfrac{\mathcal{L}(2n)}{4}\left(x + \sqrt{x^2 + \dfrac{4(x+1)\mathcal{L}(2n-2)}{\mathcal{L}(2n)}}\right) \end{cases} \tag{3.21}$$

and

$$\begin{cases} \mathcal{C}^{\dagger_1}_{2n}(x) = \left(\dfrac{x+1}{2}\right)^n \dfrac{(2n-1)!!\left(-\mathcal{L}(2n)(\mathcal{R}^{\dagger_2}_{2n}(x))^{2n} + (\mathcal{R}^{\dagger_2}_{2n+2}(x))^{2n+2}\right)}{2^n n!\left((\mathcal{R}^{\dagger_1}_{2n}(x))^{2n}(\mathcal{R}^{\dagger_2}_{2n+2}(x))^{2n+2} - (\mathcal{R}^{\dagger_2}_{2n}(x))^{2n}(\mathcal{R}^{\dagger_1}_{2n+2}(x))^{2n+2}\right)} \\ \mathcal{C}^{\dagger_2}_{2n}(x) = \left(\dfrac{x+1}{2}\right)^n \dfrac{(2n-1)!!\left(\mathcal{L}(2n)(\mathcal{R}^{\dagger_1}_{2n}(x))^{2n} - (\mathcal{R}^{\dagger_1}_{2n+2}(x))^{2n+2}\right)}{2^n n!\left((\mathcal{R}^{\dagger_1}_{2n}(x))^{2n}(\mathcal{R}^{\dagger_2}_{2n+2}(x))^{2n+2} - (\mathcal{R}^{\dagger_2}_{2n}(x))^{2n}(\mathcal{R}^{\dagger_1}_{2n+2}(x))^{2n+2}\right)} \end{cases} \tag{3.22}$$

Proof. Considering (3.5), the characteristic equation of Vietoris-like polynomials is as follows:

$$t^2 - \frac{x}{2}\mathcal{L}(2n)t - \frac{x+1}{2}\frac{\mathcal{L}(2n)\mathcal{L}(2n-2)}{2} = 0$$

Thus, its roots $R^{\dagger_1}_{2n}(x)$ and $\mathcal{R}^{\dagger_2}_{2n}(x)$ are as follows:

$$\mathcal{R}^{\dagger_1}_{2n}(x) = \frac{\mathcal{L}(2n)}{4}\left(x - \sqrt{x^2 + \frac{4(x+1)\mathcal{L}(2n-2)}{\mathcal{L}(2n)}}\right)$$

and

$$\mathcal{R}^{\dagger_2}_{2n}(x) = \frac{\mathcal{L}(2n)}{4}\left(x + \sqrt{x^2 + \frac{4(x+1)\mathcal{L}(2n-2)}{\mathcal{L}(2n)}}\right)$$

By using (3.22), calculate $\mathcal{C}^{\dagger_1}_{2n}(x)(\mathcal{R}^{\dagger_1}_{2n}(x))^{2n} + \mathcal{C}^{\dagger_2}_{2n}(x)(\mathcal{R}^{\dagger_2}_{2n}(x))^{2n}$ as:

$$\begin{aligned}
&= \left(\frac{x+1}{2}\right)^n \frac{(2n-1)!!\left(-\mathcal{L}(2n)(\mathcal{R}^{\dagger_2}_{2n}(x))^{2n} + (\mathcal{R}^{\dagger_2}_{2n+2}(x))^{2n+2}\right)(\mathcal{R}^{\dagger_1}_{2n}(x))^{2n}}{2^n n!\left((\mathcal{R}^{\dagger_1}_{2n}(x))^{2n}(\mathcal{R}^{\dagger_2}_{2n+2}(x))^{2n+2} - (\mathcal{R}^{\dagger_2}_{2n}(x))^{2n}(\mathcal{R}^{\dagger_1}_{2n+2}(x))^{2n+2}\right)} \\
&\quad + \left(\frac{x+1}{2}\right)^n \frac{(2n-1)!!\left(\mathcal{L}(2n)(\mathcal{R}^{\dagger_1}_{2n}(x))^{2n} - (\mathcal{R}^{\dagger_1}_{2n+2}(x))^{2n+2}\right)(\mathcal{R}^{\dagger_2}_{2n}(x))^{2n}}{2^n n!\left((\mathcal{R}^{\dagger_1}_{2n}(x))^{2n}(\mathcal{R}^{\dagger_2}_{2n+2}(x))^{2n+2} - (\mathcal{R}^{\dagger_2}_{2n}(x))^{2n}(\mathcal{R}^{\dagger_1}_{2n+2}(x))^{2n+2}\right)} \\
&= \left(\frac{x+1}{2}\right)^n \frac{(2n-1)!!\left((\mathcal{R}^{\dagger_1}_{2n}(x))^{2n}(\mathcal{R}^{\dagger_2}_{2n+2}(x))^{2n+2} - (\mathcal{R}^{\dagger_2}_{2n}(x))^{2n}(\mathcal{R}^{\dagger_1}_{2n+2}(x))^{2n+2}\right)}{2^n n!\left((\mathcal{R}^{\dagger_1}_{2n}(x))^{2n}(\mathcal{R}^{\dagger_2}_{2n+2}(x))^{2n+2} - (\mathcal{R}^{\dagger_2}_{2n}(x))^{2n}(\mathcal{R}^{\dagger_1}_{2n+2}(x))^{2n+2}\right)} \\
&= \left(\frac{x+1}{2}\right)^n \frac{(2n-1)!!}{2^n n!} \\
&= \left(\frac{x+1}{2}\right)^n \frac{(2n-1)!!}{(2n)!!}
\end{aligned}$$

Using (3.7), (3.20) is obtained. $\square$

Example 3.11. Calculate $\mathcal{V}_6(x)$ with Binet-like Formula-Form 2 for $n = 3$. Through (3.21) and (3.22),

$$\mathcal{R}_6^{\dagger_1}(x) = \frac{117649}{64}\left(x - \sqrt{x^2 + \frac{80\,(x+1)}{21}}\right)^6$$

$$\mathcal{R}_6^{\dagger_2}(x) = \frac{117649}{64}\left(x - \sqrt{x^2 + \frac{80\,(x+1)}{21}}\right)^6$$

$$\mathcal{C}_6^{\dagger_1} = \frac{1953125(x+1)^3\left(\frac{1679616\left(3x+\sqrt{9x^2+35x+35}\right)^8}{390625} - \frac{823543}{512}\left(x+\sqrt{x^2+\frac{80(x+1)}{21}}\right)^6\right)}{4608\left(\left(3x+\sqrt{9x^2+35x+35}\right)^8\left(-21x+\sqrt{21}\sqrt{21x^2+80x+80}\right)^6 - \left(-3x+\sqrt{9x^2+35x+35}\right)^8\left(21x+\sqrt{21}\sqrt{21x^2+80x+80}\right)^6\right)}$$

and

$$\mathcal{C}_6^{\dagger_2}(x) = \frac{1953125(x+1)^3\left(-\frac{1679616\left(-3x+\sqrt{9x^2+35x+35}\right)^8}{390625} + \frac{823543}{512}\left(x-\sqrt{x^2+\frac{80(x+1)}{21}}\right)^6\right)}{4608\left(\left(3x+\sqrt{9x^2+35x+35}\right)^8\left(-21x+\sqrt{21}\sqrt{21x^2+80x+80}\right)^6 - \left(-3x+\sqrt{9x^2+35x+35}\right)^8\left(21x+\sqrt{21}\sqrt{21x^2+80x+80}\right)^6\right)}$$

Then, $\mathcal{V}_6(x) = \mathcal{C}_6^{\dagger_1}(x)(\mathcal{R}_6^{\dagger_1}(x))^6 + \mathcal{C}_6^{\dagger_2}(x)(\mathcal{R}_6^{\dagger_2}(x))^6 = \frac{5(x+1)^3}{128}$. It can be checked via (3.2). It can also be observed that $\mathcal{R}(x)$ and $\mathcal{C}(x)$ values obtained in this example are different from $R(x)$ and $C(x)$ values found in Example 3.8.

Remark 3.12. The following hold for $\mathcal{R}_{2n}^{\dagger_1}(x)$ and $\mathcal{R}_{2n}^{\dagger_2}(x)$:

i. $\mathcal{R}_{2n}^{\dagger_1}(x) + \mathcal{R}_{2n}^{\dagger_2}(x) = \dfrac{\mathcal{L}(2n)x}{2}$

ii. $\mathcal{R}_{2n}^{\dagger_1}(x)\mathcal{R}_{2n}^{\dagger_2}(x) = -\dfrac{\mathcal{L}(2n)\mathcal{L}(2n-2)(x+1)}{4}$

Remark 3.13. By setting $x = 1$ in the previously obtained results, the concepts related to Vietoris' number sequence $\{v_s\}_{s\geq 0}$ can be observed.

It can be observed that Theorem 3.7 presents Binet-like formula based on the three consecutive-term recurrence relation (3.9). Theorem 3.10 adapts the recurrence relation (3.1) into (3.9) and also derives Binet-like formula again. This leads to two alternative expressions, referred to as Form 1 and Form 2, for the Binet-like formula.

3.1. Some Identities for Vietoris-like Polynomials

This subsection investigates several identities for Vietoris-like polynomial sequence $\{\mathcal{V}_s(x)\}_{s\geq 0}$.

Proposition 3.14. Let $\{\mathcal{V}_s(x)\}_{s\geq 0}$ be Vietoris-like polynomial sequence. Then, the following properties hold:

i. $\mathcal{V}_{2n}(x) + \mathcal{V}_{2n-1}(x) = \dfrac{x+3}{2}\mathcal{L}(2n-2)\mathcal{V}_{2n-2}(x)$

ii. $\mathcal{V}_{2n}(x) - \mathcal{V}_{2n-1}(x) = \dfrac{x-1}{2}\mathcal{L}(2n-2)\mathcal{V}_{2n-2}(x)$

iii. $\mathcal{V}_{2n+1}(x) + \mathcal{V}_{2n-1}(x) = \left(\frac{x+1}{2}\mathcal{L}(2n) + 1\right)\mathcal{L}(2n-2)\mathcal{V}_{2n-2}(x)$

iv. $\mathcal{V}_{2n+1}(x) - \mathcal{V}_{2n-1}(x) = \left(\frac{x+1}{2}\mathcal{L}(2n) - 1\right)\mathcal{L}(2n-2)\mathcal{V}_{2n-2}(x)$

v. $\mathcal{V}_{2n}(x) + \mathcal{V}_{2n+2}(x) = \left(\frac{x+1}{2}\mathcal{L}(2n) + 1\right)\mathcal{V}_{2n}(x) = \left(\frac{\mathcal{V}_{2n+2}(x)}{\mathcal{V}_{2n}(x)} + 1\right)\mathcal{V}_{2n}(x)$

vi. $\mathcal{V}_{2n}(x) - \mathcal{V}_{2n+2}(x) = \left(\frac{x+1}{2}\mathcal{L}(2n) - 1\right)\mathcal{V}_{2n}(x) = \left(\frac{\mathcal{V}_{2n+2}(x)}{\mathcal{V}_{2n}(x)} - 1\right)\mathcal{V}_{2n}(x)$

Proof. Let $\{\mathcal{V}_s(x)\}_{s\geq 0}$ be Vietoris-like polynomial sequence.

i. From (3.8) and (3.1), $\mathcal{V}_{2n}(x) = \frac{x+1}{2}\mathcal{L}(2n-2)\mathcal{V}_{2n-2}(x)$ and $\mathcal{V}_{2n-1}(x) = \mathcal{L}(2n-2)\mathcal{V}_{2n-2}(x)$, respectively. The proof is completed when these equations are added side by side.

ii. From (3.3) and (3.8), $\mathcal{V}_{2n}(x) + \mathcal{V}_{2n+2}(x) = \left(\frac{x+1}{2}\mathcal{L}(2n) + 1\right)\mathcal{V}_{2n}(x)$. Since $\frac{\mathcal{V}_{2n+2}(x)}{\mathcal{V}_{2n}(x)} = \frac{x+1}{2}\mathcal{L}(2n)$, the desired result is obtained.

The other proofs are similar. □

Proposition 3.15 (Catalan-like Identity)**.** Let $\{\mathcal{V}_s(x)\}_{s\geq 0}$ be Vietoris-like polynomial sequence. For $s > t$ and $\mathcal{K} = \mathcal{V}_{s+t}(x)\mathcal{V}_{s-t}(x) - (\mathcal{V}_s(x))^2$, the following relation is valid: For all $n \geq 1$ and $m > 1$,

$$\mathcal{K} = \begin{cases} \left(\prod_{l=1}^{t/2}\left(\frac{x+1}{2}\right)^{t/2}\mathcal{L}(s+t-2l) - \prod_{l=1}^{t/2}\left(\frac{x+1}{2}\right)^{t/2}\mathcal{L}(s-2l)\right)\mathcal{V}_s(x)\mathcal{V}_{s-t}(x), & s = 2n \text{ and } t = 2m \\ \left(\mathcal{L}(s-t-1)\prod_{l=1}^{(t+1)/2}\left(\frac{x+1}{2}\right)^{(t-1)/2}\mathcal{L}(s+t+1-2l) - \prod_{l=1}^{(t+1)/2}\left(\frac{x+1}{2}\right)^{(t+1)/2}\mathcal{L}(s-2l)\right)\mathcal{V}_{s-t-1}(x)\mathcal{V}_s(x), & s = 2n \text{ and } t = 2m-1 \\ \left(\prod_{l=1}^{t/2}\left(\frac{x+1}{2}\right)^{(t-4)/2}\mathcal{L}(s+t+1-2l) - \prod_{l=1}^{t/2}\left(\frac{x+1}{2}\right)^{(t-4)/2}\mathcal{L}(s+1-2l)\right)\mathcal{V}_{s+1}(x)\mathcal{V}_{s-t+1}(x), & s = 2n-1 \text{ and } t = 2m \\ \left(\prod_{l=1}^{\frac{t-1}{2}}\left(\frac{x+1}{2}\right)^{\frac{t-1}{2}}\mathcal{L}(s+t-2l) - \prod_{l=1}^{\frac{t+1}{2}}\left(\frac{x+1}{2}\right)^{\frac{t-3}{2}}\mathcal{L}(s+1-2l)\right)\mathcal{V}_{s+1}(x)\mathcal{V}_{s-t}(x), & s = 2n-1 \text{ and } t = 2m-1 \end{cases}$$

Proof. Consider (3.6). For $s = 2n$ and $t = 2m$,

$$\begin{aligned} \mathcal{V}_{s+t}(x)\mathcal{V}_{s-t}(x) - (\mathcal{V}_s(x))^2 =& \mathcal{V}_{2n+2m}(x)\mathcal{V}_{2n-2m}(x) - (\mathcal{V}_{2n}(x))^2 \\ =& \prod_{l=1}^{m}\left(\frac{x+1}{2}\right)^{m}\mathcal{L}(2n+2m-2l)\mathcal{V}_{2n}(x)\mathcal{V}_{2n-2m}(x) \\ & -\prod_{l=1}^{m}\left(\frac{x+1}{2}\right)^{m}\mathcal{L}(2n-2l)\mathcal{V}_{2n}(x)\mathcal{V}_{2n-2m}(x) \\ =& \left(\prod_{l=1}^{m}\left(\frac{x+1}{2}\right)^{m}\mathcal{L}(2n+2m-2l) - \prod_{l=1}^{m}\left(\frac{x+1}{2}\right)^{m}\mathcal{L}(2n-2l)\right)\mathcal{V}_{2n}(x)\mathcal{V}_{2n-2m}(x) \\ =& \left(\prod_{l=1}^{t/2}\left(\frac{x+1}{2}\right)^{t/2}\mathcal{L}(s+t-2l) - \prod_{l=1}^{t/2}\left(\frac{x+1}{2}\right)^{t/2}\mathcal{L}(s-2l)\right)\mathcal{V}_s(x)\mathcal{V}_{s-t}(x) \end{aligned}$$

For $s = 2n$ and $t = 2m-1$, using (3.1),

$$\begin{aligned} \mathcal{V}_{s+t}(x)\mathcal{V}_{s-t}(x) - (\mathcal{V}_s(x))^2 =& \mathcal{V}_{2n+2m-1}(x)\mathcal{V}_{2n-2m+1}(x) - (\mathcal{V}_{2n}(x))^2 \\ =& \frac{2\mathcal{V}_{2n+2m}(x)}{x+1}\frac{2\mathcal{V}_{2n-2m+2}(x)}{x+1} - (\mathcal{V}_{2n}(x))^2 \\ =& \prod_{l=1}^{m}\left(\frac{x+1}{2}\right)^{m-2}\mathcal{L}(2n+2m-2l)\mathcal{V}_{2n}(x)\mathcal{V}_{2n-2m+2}(x) \\ & - \mathcal{V}_{2n}(x)\prod_{l=1}^{m-1}\left(\frac{x+1}{2}\right)^{m-1}\mathcal{L}(2n-2l)\mathcal{V}_{2n-2m+2}(x) \\ =& \left(\prod_{l=1}^{(t+1)/2}\left(\frac{x+1}{2}\right)^{(t-3)/2}\mathcal{L}(s+t+1-2l) - \prod_{l=1}^{(t-1)/2}\left(\frac{x+1}{2}\right)^{(t-1)/2}\mathcal{L}(s-2l)\right)\mathcal{V}_{s-t+1}(x)\mathcal{V}_s(x) \end{aligned}$$

For $s = 2n - 1$ and $t = 2m$,

$$\begin{aligned}
\mathcal{V}_{s+t}(x)\mathcal{V}_{s-t}(x) - (\mathcal{V}_n(x))^2 =& \mathcal{V}_{2n+2m-1}(x)\mathcal{V}_{2n-2m-1}(x) - (\mathcal{V}_{2n-1}(x))^2 \\
=& \frac{2\mathcal{V}_{2n+2m}(x)}{x+1}\frac{2\mathcal{V}_{2n-2m}(x)}{x+1} - \left(\frac{2\mathcal{V}_{2n}(x)}{x+1}\right)^2 \\
=& \left(\prod_{l=1}^{m}\left(\frac{x+1}{2}\right)^{m-2}\mathcal{L}(2n+2m-2l) - \prod_{l=1}^{m}\left(\frac{x+1}{2}\right)^{m-2}\mathcal{L}(2n-2l)\right)\mathcal{V}_{2n}(x)\mathcal{V}_{2n-2m}(x) \\
=& \left(\prod_{l=1}^{t/2}\left(\frac{x+1}{2}\right)^{(t-4)/2}\mathcal{L}(s+t+1-2l) - \prod_{l=1}^{t/2}\left(\frac{x+1}{2}\right)^{(t-4)/2}\mathcal{L}(s+1-2l)\right)\mathcal{V}_{s+1}(x)\mathcal{V}_{s-t+1}(x)
\end{aligned}$$

For $s = 2n - 1$ and $t = 2m - 1$,

$$\begin{aligned}
\mathcal{V}_{s+t}(x)\mathcal{V}_{s-t}(x) - (\mathcal{V}_s(x))^2 &= \mathcal{V}_{2n+2m-2}(x)\mathcal{V}_{2n-2m}(x) - (\mathcal{V}_{2n-1}(x))^2 \\
&= \mathcal{V}_{2n+2m-2}(x)\mathcal{V}_{2n-2m}(x) - \left(\frac{2\mathcal{V}_{2n}(x)}{x+1}\right)^2 \\
&= \left(\prod_{l=1}^{m-1}\left(\frac{x+1}{2}\right)^{m-1}\mathcal{L}(2n+2m-2-2l)\prod_{l=1}^{m}\left(\frac{x+1}{2}\right)^{m-2}\mathcal{L}(2n-2l)\right)\mathcal{V}_{2n}(x)\mathcal{V}_{2n-2m}(x) \\
&= \left(\prod_{l=1}^{\frac{t-1}{2}}\left(\frac{x+1}{2}\right)^{\frac{t-1}{2}}\mathcal{L}(s+t-2l) - \prod_{l=1}^{\frac{t+1}{2}}\left(\frac{x+1}{2}\right)^{\frac{t-3}{2}}\mathcal{L}(s+1-2l)\right)\mathcal{V}_{s+1}(x)\mathcal{V}_{s-t}(x)
\end{aligned}$$

□

The above proposition is also valid for $s > t > 2$.

Example 3.16. Considering (3.2), we compute $\mathcal{V}_{10}(x)\mathcal{V}_2(x) - \mathcal{V}_6(x)^2 = \frac{13(x+1)^6}{32768}$, where $s = 6$ and $t = 4$. Besides, using the above formula, we obtain the same results

$$\begin{aligned}
\mathcal{V}_{10}(x)\mathcal{V}_2(x) - \mathcal{V}_6(x)^2 &= \left(\prod_{l=1}^{2}\left(\tfrac{x+1}{2}\right)^2\mathcal{L}(10-2l) - \prod_{l=1}^{2}\left(\tfrac{x+1}{2}\right)^2\mathcal{L}(6-2l)\right)\mathcal{V}_6(x)\mathcal{V}_2(x) \\
&= \frac{13(x+1)^6}{32768}
\end{aligned}$$

Similarly, for $s = 6$ and $t = 3$,

$$\begin{aligned}
\mathcal{V}_9(x)\mathcal{V}_3(x) - \mathcal{V}_6(x)^2 &= \left(\mathcal{L}(2)\prod_{l=1}^{2}\left(\tfrac{x+1}{2}\right)\mathcal{L}(10-2l) - \prod_{l=1}^{2}\left(\tfrac{x+1}{2}\right)^2\mathcal{L}(6-2l)\right)\mathcal{V}_6(x)\mathcal{V}_2(x) \\
&= -\frac{(x+1)^5(-89+100x)}{65536}
\end{aligned}$$

For $s = 9$ and $t = 4$,

$$\begin{aligned}
\mathcal{V}_{13}(x)\mathcal{V}_5(x) - \mathcal{V}_9(x)^2 &= \left(\prod_{l=1}^{2}\mathcal{L}(14-2l) - \prod_{l=1}^{2}\mathcal{L}(10-2l)\right)\mathcal{V}_{10}(x)\mathcal{V}_6(x) \\
&= \frac{321(1+x)^8}{16777216}
\end{aligned}$$

For $s = 5$ and $t = 3$,

$$\begin{aligned}
\mathcal{V}_8(x)\mathcal{V}_2(x) - \mathcal{V}_5(x)^2 &= \left(\tfrac{x+1}{2}\mathcal{L}(6) - \prod_{l=1}^{2}\mathcal{L}(6-2l)\right)\mathcal{V}_6(x)\mathcal{V}_2(x) \\
&= \frac{5(x+1)^4(-3+7x)}{8192}
\end{aligned}$$

Proposition 3.17. For $n \geq 1$,

$$\mathcal{V}_{s+2}(x)\mathcal{V}_{s-2}(x) - (\mathcal{V}_s(x))^2 = \begin{cases} \dfrac{x+1}{2}\dfrac{2}{s(s+2)}\mathcal{V}_s(x)\mathcal{V}_{s-2}(x), & s = 2n \\ \dfrac{x+1}{2}\dfrac{2}{(s+1)(s+3)}\mathcal{V}_s(x)\mathcal{V}_{s-2}(x), & s = 2n-1 \end{cases}$$

Moreover, for $x = 1$, considering (3.1), we obtain the following result as in [18]:

$$v_{s+2}v_{s-2} - (v_s)^2 = \begin{cases} \dfrac{2}{s(s+2)}v_s v_{s-2}, & s = 2n \\ \dfrac{2}{(s+1)(s+3)}v_{s+1}v_{s-1}, & s = 2n-1 \end{cases}$$

Proposition 3.18 (Cassini-like Identity)**.** Let $\{\mathcal{V}_s(x)\}_{s\geq 0}$ be Vietoris-like polynomial sequence. Then,

$$\mathcal{V}_{s+1}(x)\mathcal{V}_{s-1}(x) - (\mathcal{V}_s(x))^2 = \begin{cases} \frac{x+1}{2}\left(\mathcal{L}(s) - \frac{x+1}{2}\right)(\mathcal{V}_{s-1}(x))^2, & s = 2n \\ \mathcal{L}(s-1)\left(\frac{x+1}{2} - \mathcal{L}(s-1)\right)(\mathcal{V}_{s-1}(x))^2, & s = 2n-1 \end{cases}$$

where $n \geq 1$.

Proof. Consider (3.12). For $s = 2n$,

$$\begin{aligned} \mathcal{V}_{s+1}(x)\mathcal{V}_{s-1}(x) - (\mathcal{V}_s(x))^2 &= \mathcal{V}_{2n+1}(x)\mathcal{V}_{2n-1}(x) - (\mathcal{V}_{2n}(x))^2 \\ &= \frac{x+1}{2}\mathcal{L}(2n)\mathcal{V}_{2n-1}(x)\mathcal{V}_{2n-1}(x) - \left(\frac{x+1}{2}\mathcal{V}_{2n-1}\right)^2 \\ &= \frac{x+1}{2}\left(\mathcal{L}(2n) - \frac{x+1}{2}\right)(\mathcal{V}_{2n-1}(x))^2 \\ &= \frac{x+1}{2}\left(\mathcal{L}(s) - \frac{x+1}{2}\right)(\mathcal{V}_{s-1}(x))^2 \end{aligned}$$

For $s = 2n-1$,

$$\begin{aligned} \mathcal{V}_{s+1}(x)\mathcal{V}_{s-1}(x) - (\mathcal{V}_s(x))^2 &= \mathcal{V}_{2n}(x)\mathcal{V}_{2n-2}(x) - (\mathcal{V}_{2n-1}(x))^2 \\ &= \frac{x+1}{2}\mathcal{L}(2n-2)\mathcal{V}_{2n-2}(x)\mathcal{V}_{2n-2}(x) - \mathcal{L}^2(2n-2)(\mathcal{V}_{2n-2}(x))^2 \\ &= \mathcal{L}(2n-2)\left(\frac{x+1}{2} - \mathcal{L}(2n-2)\right)(\mathcal{V}_{2n-2}(x))^2 \\ &= \mathcal{L}(s-1)\left(\frac{x+1}{2} - \mathcal{L}(s-1)\right)(\mathcal{V}_{s-1}(x))^2 \end{aligned}$$

□

Proposition 3.19 (d'Ocagne-like Identity)**.** Let $\{\mathcal{V}_s(x)\}_{s\geq 0}$ be Vietoris-like polynomial sequence. Then,

$$\mathcal{V}_s(x)\mathcal{V}_{t+1}(x) - \mathcal{V}_{s+1}(x)\mathcal{V}_t(x) = \begin{cases} (\mathcal{L}(t) - \mathcal{L}(s))\,\mathcal{V}_s(x)\mathcal{V}_t(x), & s = 2n \text{ and } t = 2m \\ \left(1 - \frac{2\mathcal{L}(s)}{x+1}\right)\mathcal{V}_s(x)\mathcal{V}_{t+1}(x), & s = 2n \text{ and } t = 2m-1 \\ \left(\frac{2\mathcal{L}(t)}{x+1} - 1\right)\mathcal{V}_{s+1}(x)\mathcal{V}_t(x), & s = 2n-1 \text{ and } t = 2m \\ 0, & s = 2n-1 \text{ and } t = 2m-1 \end{cases} \tag{3.23}$$

where $n, m \geq 1$.

PROOF. Consider (3.3). For $s = 2n$ and $t = 2m$,

$$\begin{aligned}
\mathcal{V}_s(x)\mathcal{V}_{t+1}(x) - \mathcal{V}_{s+1}(x)\mathcal{V}_t(x) &= \mathcal{V}_{2n}(x)\mathcal{V}_{2m+1}(x) - \mathcal{V}_{2n+1}(x)\mathcal{V}_{2m}(x) \\
&= \mathcal{V}_{2n}(x)\mathcal{L}(2m)\mathcal{V}_{2m}(x) - \mathcal{L}(2n)\mathcal{V}_{2n}(x)\mathcal{V}_{2m}(x) \\
&= \left(\mathcal{L}(2m) - \mathcal{L}(2n)\right)\mathcal{V}_{2n}(x)\mathcal{V}_{2m}(x) \\
&= \left(\mathcal{L}(t) - \mathcal{L}(s)\right)\mathcal{V}_s(x)\mathcal{V}_t(x)
\end{aligned}$$

Additionally, by (2.2),

$$\mathcal{V}_s(x)\mathcal{V}_{t+1}(x) - \mathcal{V}_{s+1}(x)\mathcal{V}_t(x) = \left(\frac{t-s}{(s+2)(t+2)}\right)\mathcal{V}_s(x)\mathcal{V}_t(x)$$

Then, for $s = 2n$ and $t = 2m-1$,

$$\begin{aligned}
\mathcal{V}_s(x)\mathcal{V}_{t+1}(x) - \mathcal{V}_{s+1}(x)\mathcal{V}_t(x) &= \mathcal{V}_{2n}(x)\mathcal{V}_{2m}(x) - \mathcal{V}_{2n+1}(x)\mathcal{V}_{2m-1}(x) \\
&= \mathcal{V}_{2n}(x)\mathcal{V}_{2m}(x) - \mathcal{L}(2n)\mathcal{V}_{2n}(x)\frac{2\mathcal{V}_{2m}(x)}{x+1} \\
&= \left(1 - \frac{2\mathcal{L}(2n)}{x+1}\right)\mathcal{V}_{2n}(x)\mathcal{V}_{2m}(x) \\
&= \left(1 - \frac{2\mathcal{L}(s)}{x+1}\right)\mathcal{V}_s(x)\mathcal{V}_{t+1}(x)
\end{aligned}$$

For $s = 2n-1$ and $t = 2m$,

$$\begin{aligned}
\mathcal{V}_s(x)\mathcal{V}_{t+1}(x) - \mathcal{V}_{s+1}(x)\mathcal{V}_t(x) &= \mathcal{V}_{2n-1}(x)\mathcal{V}_{2m+1}(x) - \mathcal{V}_{2n}(x)\mathcal{V}_{2m}(x) \\
&= \frac{2\mathcal{V}_{2n}(x)}{x+1}\mathcal{L}(2m)\mathcal{V}_{2m}(x) - \mathcal{V}_{2n}(x)\mathcal{V}_{2m}(x) \\
&= \left(\frac{2\mathcal{L}(2m)}{x+1} - 1\right)\mathcal{V}_{2n}(x)\mathcal{V}_{2m}(x) \\
&= \left(\frac{2\mathcal{L}(t)}{x+1} - 1\right)\mathcal{V}_{s+1}(x)\mathcal{V}_t(x)
\end{aligned}$$

For $s = 2n-1$ and $t = 2m-1$,

$$\begin{aligned}
\mathcal{V}_s(x)\mathcal{V}_{t+1}(x) - \mathcal{V}_{s+1}(x)\mathcal{V}_t(x) &= \mathcal{V}_{2n-1}(x)\mathcal{V}_{2m}(x) - \mathcal{V}_{2n}(x)\mathcal{V}_{2m-1}(x) \\
&= \frac{2\mathcal{V}_{2n}(x)}{x+1}\mathcal{V}_{2m}(x) - \mathcal{V}_{2n}(x)\frac{2\mathcal{V}_{2m}(x)}{x+1} \\
&= 0
\end{aligned}$$

□

Remark 3.20. For $x = 1$, (3.23) becomes the following formula as in [18]:

$$v_s v_{t+1} - v_{s+1} v_t = \begin{cases} \dfrac{t-s}{(s+2)(t+2)} v_s v_t, & s = 2n \text{ and } t = 2m \\ \dfrac{1}{s+2} v_s v_{t+1}, & s = 2n \text{ and } t = 2m-1 \\ -\dfrac{1}{t+2} v_{s+1} v_t, & s = 2n-1 \text{ and } t = 2m \\ 0, & s = 2n-1 \text{ and } t = 2m-1 \end{cases}$$

Proposition 3.21. Let $\{\mathcal{V}_s(x)\}_{s\geq 0}$ be Vietoris-like polynomial sequence and $n \geq 2$. Then,

$$\mathcal{V}_{2n+2}(x) = \left(\frac{x+1}{2}\right)^2 \frac{2n-1}{2^2(n+1)}\mathcal{V}_{2n-2}(x) + \left(\frac{x+1}{4}\right)\left(\frac{1}{2}\mathcal{V}_{2n}(x) + \frac{x+1}{4}\mathcal{V}_{2n-1}(x)\right) \tag{3.24}$$

Proof. From (3.4) and the Pascal's identity $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$,

$$\begin{aligned}
\mathcal{V}_{2n+2}(x) &= \left(\frac{x+1}{2}\right)^{n+1} \frac{1}{2^{2n+2}}\binom{2n+2}{n+1} \\
&= \left(\frac{x+1}{2}\right)^{n+1} \frac{1}{2^{2n+2}}\left(\binom{2n+1}{n} + \binom{2n+1}{n+1}\right) \\
&= \left(\frac{x+1}{2}\right)^{n+1} \frac{1}{2^{2n+2}}\left(\binom{2n}{n-1} + \binom{2n}{n} + \binom{2n}{n} + \binom{2n}{n+1}\right) \\
&= \left(\frac{x+1}{2}\right)^{n+1} \left(\frac{1}{2^{2n+2}}\left(\binom{2n}{n-1} + \binom{2n}{n+1}\right) + \frac{1}{2^{2n+1}}\binom{2n}{n}\right)
\end{aligned}$$

Using $\binom{n}{k} = \binom{n}{n-k}$,

$$\begin{aligned}
\mathcal{V}_{2n+2}(x) &= \left(\frac{x+1}{2}\right)^{n+1} \left(\frac{1}{2^{2n+2}}\left(\binom{2n}{n-1} + \binom{2n}{2n-n-1}\right) + \frac{1}{2^{2n+1}}\binom{2n}{n}\right) \\
&= \left(\frac{x+1}{2}\right)^{n+1} \left(\frac{1}{2^{2n+1}}\binom{2n}{n-1} + \frac{1}{2^{2n+1}}\binom{2n}{n}\right)
\end{aligned}$$

Using $\binom{n}{k} = \frac{n}{n-k}\binom{n-1}{k}$ and (3.4),

$$\begin{aligned}
\mathcal{V}_{2n+2}(x) &= \left(\frac{x+1}{2}\right)^{n+1} \frac{1}{2^{2n+1}}\frac{2n}{(n+1)}\binom{2n-1}{n-1} + \left(\frac{x+1}{4}\right)\left(\frac{x+1}{2}\right)^{n}\frac{1}{2^{2n}}\binom{2n}{n} \\
&= \left(\frac{x+1}{2}\right)^{2} \frac{2n-1}{2^2(n+1)}\left(\frac{x+1}{2}\right)^{n-1}\frac{1}{2^{2n-2}}\binom{2n-2}{n-1} + \left(\frac{x+1}{4}\right)\mathcal{V}_{2n}(x) \\
&= \left(\frac{x+1}{2}\right)^{2} \frac{2n-1}{2^2(n+1)}\mathcal{V}_{2n-2}(x) + \left(\frac{x+1}{4}\right)\mathcal{V}_{2n}(x)
\end{aligned}$$

From (3.1) and (3.3),

$$\mathcal{V}_{2n+2}(x) = \left(\frac{x+1}{2}\right)^2 \frac{2n-1}{2^2(n+1)}\mathcal{V}_{2n-2}(x) + \left(\frac{x+1}{4}\right)\left(\frac{1}{2}\mathcal{V}_{2n}(x) + \frac{x+1}{4}\mathcal{V}_{2n-1}(x)\right)$$

□

Remark 3.22. For $x = 1$, (3.24) becomes the following equality as in [18]:

$$v_{2n+2} = \left(\frac{1}{4}v_{2n} + \frac{1}{4}v_{2n-1}\right) + \frac{2n-1}{4(n+1)}v_{2n-2}, \quad n \geq 2$$

4. Conclusion

Many researchers have studied number sequences and their properties, which play an essential role in mathematics. Hence, the polynomial forms of these number sequences for any variable quantity x have also become an area of significant interest. The Fibonacci polynomials were among the first polynomial forms considered. Since Fibonacci-type polynomials have significant applications in geometry and algebra, various researchers have extensively studied them in number theory. In this paper, we provided an affirmative answer to a question related to the existence of special Vietoris-like polynomials by using the properties of Vietoris' numbers. Hence, we derived special Vietoris-like polynomials and investigated their basic properties, recurrence relations, and special equalities. We also constructed an analogy with the studies [10–12, 15–18] using Vietoris-like polynomial approach and established some conditions for obtaining interesting results inspired by studies [2–9]. We determined Catalan-like, Cassini-like and d'Ocagne-like identities. We also presented their special cases corresponding to the existing identities in Vietoris' number sequence. We believe that the calculations of this work contribute to the broader understanding of polynomial structures and their connections with well-known number sequences and enable new studies. Specifically, the results of Vietoris-like polynomials and the properties of Vietoris' hybrid numbers (for more details on hybrid numbers, see [21]) of the form $\mathcal{VH}s = v_s + v_{s+1}i + \varepsilon v_{s+2} + hv_{s+3}$ where $i^2 = -1$, $\varepsilon^2 = 0$, $h^2 = 1$, and $ih = -hi = \varepsilon + i$ [22], Vietoris-like hybrid binomial sequence and its remarkable features represent key areas for future research.

Author Contributions

All the authors equally contributed to this work. They all read and approved the final version of the paper.

Conflicts of Interest

All the authors declare no conflict of interest.

Ethical Review and Approval

No approval from the Board of Ethics is required.

Acknowledgment

The second author is supported by the 2211-A Domestic Doctoral Fellowship by the Scientific and Technological Research Council of Türkiye (TÜBİTAK), Grant number: 1649B032103711.

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Journal of New Theory
dergipark.org.tr/tr/pub/jnt
Open Access

ISSN: 2149-1402 (51) (2025) 26-32 doi.org/10.53570/jnt.1663409

Combinatorial Results on Some Nilpotent Subsemigroups of a Semigroup of Order-Decreasing Full Transformations

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Article Info
Received: 22 Mar 2025
Accepted: 10 Jun 2025
Published: 30 Jun 2025
Research Article

Abstract— Let $\mathcal{D}_n$ be the semigroup of all order-decreasing full transformations on $X_n = \{1, 2, \dots, n\}$ under its natural order, and let $N(\mathcal{D}_n)$ be the subsemigroup of all nilpotent elements of $\mathcal{D}_n$, where $n \in \mathbb{Z}^+$, the set of all positive integers. In this paper, for $1 \leq r \leq n-1$, we determine the cardinality and rank of nilpotent subsemigroup $N(\mathcal{D}_{n,r}) = \{\alpha \in N(\mathcal{D}_n) : |\text{im}(\alpha)| \leq r\}$ of $N(\mathcal{D}_n)$. We then find the cardinalities of $\mathcal{D}_n^{2,2}$ and $N(\mathcal{D}_n)^{p,p}$. Furthermore, we present an alternative combinatorial approach to determine the cardinality and rank of $\mathcal{D}_n(\xi) = \{\alpha \in \mathcal{D}_n : \alpha^k = \xi, \text{ for some } k \in \mathbb{Z}^+\}$, for all idempotent $\xi \in \mathcal{D}_n$ within the scope of this study. Here, for all $\alpha \in \mathcal{D}_n$, $\text{im}^c(\alpha) = \{t \in \text{im}(\alpha) : |t\alpha^{-1}| \geq 2\}$. Besides, for all $2 \leq p \leq r \leq n$ and $\mathcal{C} \in \{N(\mathcal{D}_n), \mathcal{D}_n\}$, $\mathcal{C}^p = \left\{\alpha \in \mathcal{C} : t \in \text{im}^c(\alpha) \text{ and } |t\alpha^{-1}| = p\right\}$ and $\mathcal{C}^{p,r} = \left\{\alpha \in \mathcal{C}^p : \left|\bigcup_{t \in \text{im}^c(\alpha)} t\alpha^{-1}\right| = r\right\}$.

Keywords — *Order-decreasing, transformations, collapse, nilpotent, rank*

Mathematics Subject Classification (2020) 06F05, 20M20

1. Introduction

For an arbitrary set $X_n = \{1, 2, \dots, n\}$, ordered standard way, such that $n \in \mathbb{Z}^+$, the set of all positive integers, the notation $\mathcal{T}_n$ denote the full transformation semigroup on X_n, i.e., all mappings from X to X, under the operation of composition. We compose the functions from left to right. A transformation $\alpha \in \mathcal{T}_n$ is called order-preserving if $x \leq y$ implies $x\alpha \leq y\alpha$, for all $x, y \in X_n$, and decreasing if $x\alpha \leq x$ ($x\alpha \geq x$), for all $x \in X_n$. Let

$$\mathcal{D}_n = \{\alpha \in T_n : \forall x \in X_n,\ x\alpha \leq x\}$$

be the semigroup of all order-decreasing full transformations. For any transformation $\alpha \in \mathcal{T}_n$, the collapse, the image, and the fix of α are defined as follows, respectively:

$$c(\alpha) = \bigcup_{t \in \text{im}(\alpha)} \{t\alpha^{-1} : |t\alpha^{-1}| \geq 2\}$$

$$\text{im}(\alpha) = \{x\alpha : x \in X_n\}$$

and

$$\text{fix}(\alpha) = \{x \in X_n : x\alpha = x\}$$

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For any semigroup S, an element e in S is said to be an idempotent if $e^2 = e$. It is known that $\alpha \in E(\mathcal{T}_n)$ if and only if $\mathrm{fix}(\alpha) = \mathrm{im}(\alpha)$. Let S be a semigroup with zero element 0. An element a in S is said to be a nilpotent element if there exists a $k \in \mathbb{Z}^+$ such that $a^k = 0$. A subsemigroup $T \subseteq S$ is called nilpotent if there exists an $m \in \mathbb{Z}^+$ such that $T^m = \{0\}$ or T is a nilpotent as a semigroup with different 0 which means that there exists an idempotent $e \in T$ differs from 0 such that $T^k = \{e\}$, for some $k \in \mathbb{Z}^+$. It is proven in [1] that a finite semigroup S with 0 is nilpotent if and only if the unique idempotent of S is 0. The set of all nilpotent subsemigroups of S is partially ordered with respect to inclusions, and each maximal elements of this set are called maximal nilpotent subsemigroup of S. Throughout this study, let $E(S)$ and $N(S)$ denote the set of all idempotent and nilpotent elements of S, respectively. It should be noted that $N(S)$ may not be a subsemigroup of S.

A subset W of a semigroup S is a generating set of S if every element of S is expressible as a product of the elements of W. Further, $\langle W \rangle$ denotes the subsemigroup generated by a non-empty subset W of S. If S is finitely generated, then its rank and idempotent rank are defined as follows, respectively:

$$\mathrm{rank}(S) = \min\{|W| : W \subseteq S \text{ and } \langle W \rangle = S\}$$

and

$$\mathrm{idrank}(S) = \min\{|W| : W \subseteq E(S) \text{ and } \langle W \rangle = S\}$$

The problem of determining the cardinality and aforementioned ranks of a certain finite transformation semigroups is closely related to combinatorics, is classical, and has been extensively explored. The authors [2] considered the ideal (and hence subsemigroup) $\mathcal{K}_{n,r} = \{\alpha \in \mathcal{T}_n : \mathrm{im}(\alpha) \leq r\}$ and demonstrated that $\mathrm{rank}(\mathcal{T}_{n,r}) = \mathrm{idrank}(\mathcal{T}_{n,r}) = S(n,r)$. Afterward, Ruškuc [3] gave an alternative proof for the rank of $\mathcal{K}_{n,r}$. These results provided that to be applicable in other semigroups as well as $\mathcal{D}_n$. This was followed by a number of articles on $\mathcal{D}_n$ in terms of algebraic, combinatorial, and (idempotent) rank properties [4–9]. In particular,

$$\mathrm{rank}(\mathcal{D}_n) = \mathrm{idrank}(\mathcal{D}_n) = \frac{n(n-1)}{2}$$

$$\mathrm{rank}(N(\mathcal{D}_n)) = (n-2)\,!\,(n-2)$$

$$|\mathcal{D}_n| = n!$$

$$|N(\mathcal{D}_n)| = (n-1)!$$

and

$$|E(\mathcal{D}_n)| = \sum_{r=0}^{n} S(n,r)$$

For further research on transformations semigroups within the scope of this study, see [10–12], and for more information about semigroup theory, see [1, 13].

The rest of the paper is organized as follows: Section 2 provides some combinatorial results on a nilpotent subsemigroup of $\mathcal{D}_n$ and certain invariants related to the collapse. Section 3 demonstrates the efficacy of enumerative techniques by providing alternative proofs for the cardinality and rank of maximal nilpotent subsemigroups of $\mathcal{D}_n$. The last section concludes the paper.

2. Cardinality and Rank Properties

Given a finite semigroup S with zero 0, S is nilpotent if and only if $S^m = \{0\}$, for some positive integer m. Let S be a finite nilpotent semigroup with $|S| \geq 2$. In [15], it was shown that $S \setminus S^2$ is the minimum generating set of S, and thus

$$\operatorname{rank}(S) = |S| - |S^2|$$

Therefore, throughout this paper, we consider non-trivial nilpotent semigroups by determining their ranks.

For $1 \leq r \leq n-1$, let α and β be two elements in

$$N(\mathcal{D}_{n,r}) = \{\alpha \in N(\mathcal{D}_n) : |\operatorname{im}(\alpha)| \leq r\}$$

Given that $\operatorname{im}(\alpha\beta) \subseteq \operatorname{im}(\beta)$ implies $|\operatorname{im}(\alpha\beta)| \leq r$, then $N(\mathcal{D}_{n,r})$ is a nilpotent subsemigroup of both $\mathcal{D}_n$ and $N(\mathcal{D}_n)$ with the zero element 0_n. It can be observed that $N(\mathcal{D}_{n,1}) = \{0_n\}$ and $N(\mathcal{D}_{n,n-1}) = N(\mathcal{D}_n)$. We aim to discover a formula for the cardinality of $N(\mathcal{D}_{n,r})$, for $2 \leq r \leq n-1$, and then utilize this formula to determine the rank of $N(\mathcal{D}_{n,r})$.

Lemma 2.1. For $2 \leq r \leq n-1$,

$$|N(\mathcal{D}_{n,r})| = \sum_{k=1}^{r} \sum_{i=0}^{k-1} (-1)^i \binom{n}{i} (k-i)^{n-1}$$

Proof. For $\alpha \in \mathcal{D}_{n-1}$, let $\widehat{\alpha} : X_n \to X_n$ be defined by $1\widehat{\alpha} = 1\alpha = 1$ and $i\widehat{\alpha} = (i-1)\alpha$, for all $2 \leq i \leq n$. It can be observed that $\widehat{\alpha} \in N(\mathcal{D}_n)$. For all $\alpha \in \mathcal{D}_{n-1}$, since the function $\varphi : \mathcal{D}_{n-1} \to N(\mathcal{D}_n)$ defined by $(\alpha)\varphi = \widehat{\alpha}$ is a bijection, then $|\mathcal{D}_{n-1}| = |N(\mathcal{D}_n)|$. For $1 \leq k \leq r \leq n-1$, consider the sets

$$\mathcal{D}_n(k) = \{\alpha \in \mathcal{D}_n : |\operatorname{im}(\alpha)| = k\} \quad \text{and} \quad N(\mathcal{D}_n(k)) = \{\alpha \in N(\mathcal{D}_n) : |\operatorname{im}(\alpha)| = k\}$$

It is known from [6] that $|\mathcal{D}_n(k)| = \sum_{i=0}^{k-1} (-1)^i \binom{n+1}{i} (k-i)^n$. It can be observed from the aforementioned bijection that $\alpha \in \mathcal{D}_{n-1}(k)$ if and only if $\widehat{\alpha} \in N(\mathcal{D}_n(k))$. Hence,

$$|N(\mathcal{D}_n(k))| = |\mathcal{D}_{n-1}(k)| = \sum_{i=0}^{k-1} (-1)^i \binom{n}{i} (k-i)^{n-1}$$

Since $N(\mathcal{D}_{n,r})$ is the union of disjoint sets $N(\mathcal{D}_n(k))$, for $1 \leq k \leq r$, then

$$|N(\mathcal{D}_{n,r})| = \sum_{k=1}^{r} \sum_{i=0}^{k-1} (-1)^i \binom{n}{i} (k-i)^{n-1}$$

□

Afterward, we present one of the key results of this study.

Theorem 2.2. For $2 \leq r \leq n-2$,

$$\operatorname{rank}(N(\mathcal{D}_{n,r})) = \sum_{k=1}^{r} \sum_{i=0}^{k-1} (-1)^i \binom{n}{i} (k-i)^{n-1} - \sum_{k=1}^{r} \sum_{i=0}^{k-1} (-1)^i \binom{n-1}{i} (k-i)^{n-2}$$

Proof. Since $N(\mathcal{D}_{n,r})$ is a nilpotent semigroup, it follows that

$$\operatorname{rank}(N(\mathcal{D}_{n,r})) = |N(\mathcal{D}_{n,r}) \setminus N(\mathcal{D}_{n,r})^2| = |N(\mathcal{D}_{n,r})| - |N(\mathcal{D}_{n,r})^2|$$

The cardinality of $N(\mathcal{D}_{n,r})^2$ can be calculated by constructing the well-defined bijection $f : N(\mathcal{D}_{n-1,r}) \to N(\mathcal{D}_{n,r})^2$ in the following way:

$$\begin{pmatrix} 1 & 2 & 3 & 4 & \cdots & n-1 \\ 1 & 1 & c_2 & c_3 & \cdots & c_{n-1} \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & \cdots & n \\ 1 & 1 & 1 & c_2 & c_3 & \cdots & c_{n-1} \end{pmatrix}$$

where $1 \leq c_i \leq i$, for $2 \leq i \leq n-1$. As a result, it follows from Lemma 2.1 that

$$|N(\mathcal{D}_{n,r})^2| = |N(\mathcal{D}_{n-1,r})| = \sum_{k=1}^{r}\sum_{i=0}^{k-1}(-1)^i\binom{n-1}{i}(k-i)^{n-2}$$

and thus

$$\operatorname{rank}(N(\mathcal{D}_{n,r})) = \sum_{k=1}^{r-1}\sum_{i=0}^{k-1}(-1)^i\binom{n}{i}(k-i)^{n-1} - \sum_{k=1}^{r}\sum_{i=0}^{k-1}(-1)^i\binom{n-1}{i}(k-i)^{n-2}$$

□

Example 2.3. Let $N(\mathcal{D}_{6,4}) = \{\alpha \in N(\mathcal{D}_6) : |\mathrm{im}(\alpha)| \leq 4\}$. Then,

$$|N(\mathcal{D}_6)(1)| = 1$$

$$|N(\mathcal{D}_6)(2)| = \sum_{i=0}^{1}(-1)^i\binom{6}{i}(2-i)^5 = 19$$

$$N(\mathcal{D}_6)(3)| = \sum_{i=0}^{2}(-1)^i\binom{6}{i}(3-i)^5 = 66$$

$$|N(\mathcal{D}_6)(4)| = \sum_{i=0}^{3}(-1)^i\binom{6}{i}(4-i)^5 = 26$$

and

$$|N(\mathcal{D}_{6,4}| = \sum_{k=1}^{4}\sum_{i=0}^{k-1}(-1)^i\binom{6}{i}(k-i)^5 = 112$$

Furthermore,

$$|N(\mathcal{D}_{6,4})^2| = |N(\mathcal{D}_{5,4})| = |N(\mathcal{D}_5)| = \sum_{k=1}^{4}\sum_{i=0}^{k-1}(-1)^i\binom{5}{i}(k-i)^4 = 24$$

Therefore,

$$\operatorname{rank}(N(\mathcal{D}_{6,4})) = |N(\mathcal{D}_{6,4}| - |N(\mathcal{D}_{6,4})^2| = 112 - 24 = 88$$

Lemma 2.4. For $n \geq 2$,

$$|\mathcal{D}_n^{2,2}| = 2^{n-1} - 1$$

Proof. For a given $\alpha \in \mathcal{D}_n^{2,2}$, there exists an $i \in \mathrm{im}(\alpha)$ such that $|i\alpha^{-1}| = 2$ and $\min\{i\alpha^{-1}\} = i$. Thus,

$$\alpha = \begin{pmatrix} \{1\} & \{2\} & \cdots & \{i-1\} & \{i, i+1\} & \{i+2\} & \cdots & \{n\} \\ 1 & 2 & \cdots & i-1 & i & (i+2)\alpha & \cdots & n\alpha \end{pmatrix}$$

where $1 \leq i \leq n-1$. It can be observed that there is only one transformation in $\mathcal{D}_n^{2,2}$, for $i = n-1$. Hence, suppose that $i \neq n-1$. Since each block $\{j\}$ is a singleton for $i+2 \leq j \leq n$, we deduce that there are exactly two possibilities for $j\alpha$. Consequently, the number of transformations in $\mathcal{D}_n^{2,2}$ for a fixed i is given by 2^{n-i-1}. Since $1 \leq i \leq n-2$, then

$$\left|\mathcal{D}_n^{2,2}\right| = 1 + \sum_{i=1}^{n-2} 2^{n-i-1} = 2^{n-1} - 1$$

□

Lemma 2.5. For $n \geq 3$ and $2 \leq p \leq n-1$, let $\left|1\alpha^{-1}\right| = p$ and $X_n \setminus 1\alpha^{-1} = \{b_1, \ldots, b_{n-p}\}$. Then,

$$|N(\mathcal{D}_n)^{p,p}| = (b_1 - 2)(b_2 - 3)(b_3 - 3)\ldots(b_{n-k} - 3)$$

Proof. Given $\alpha \in N(\mathcal{D}_n)^{p,p}$ the conditions $1\alpha = 2\alpha = 1$ implies that $|1\alpha^{-1}| = p$. If $X_n \setminus 1\alpha^{-1} = \{b_1, \ldots, b_{n-p}\}$, then

$$\alpha = \begin{pmatrix} A & \{b_1\} & \{b_2\} & \cdots & \{b_{n-p}\} \\ 1 & b_1\alpha & b_2\alpha & \cdots & b_{n-p}\alpha \end{pmatrix}$$

Recall that the conditions $1\alpha = 2\alpha = 1$ implies that $b_j \geq 3$, for each $1 \leq j \leq n-p$. Since $\alpha \in N(\mathcal{D}_n)$ and $\{b_j\}$ is a singletion, we deduce that there are exactly $b_1 - 2$ possibilities, for $b_1\alpha$, and $b_i - 3$ possibilities, for $b_i\alpha$, where $2 \leq i \leq n-p$. Hence,

$$|N(\mathcal{D}_n)^{p,p}| = (b_1 - 2)(b_2 - 3)(b_3 - 3)\ldots(b_{n-k} - 3)$$

□

Example 2.6. Consider the sets

$$\mathcal{D}_4^{2,2} = \left\{ \begin{pmatrix} \{1,2\} & \{3\} & \{4\} \\ 1 & 2 & 3 \end{pmatrix}, \begin{pmatrix} \{1,2\} & \{3\} & \{4\} \\ 1 & 2 & 4 \end{pmatrix}, \begin{pmatrix} \{1,2\} & \{3\} & \{4\} \\ 1 & 3 & 2 \end{pmatrix}, \begin{pmatrix} \{1,2\} & \{3\} & \{4\} \\ 1 & 3 & 4 \end{pmatrix}, \right.$$
$$\left. \begin{pmatrix} \{1\} & \{2,3\} & \{4\} \\ 1 & 2 & 3 \end{pmatrix}, \begin{pmatrix} \{1\} & \{2,3\} & \{4\} \\ 1 & 2 & 4 \end{pmatrix}, \begin{pmatrix} \{1\} & \{2\} & \{3,4\} \\ 1 & 2 & 3 \end{pmatrix} \right\}$$

and

$$N(\mathcal{D}_5)^{3,3} = \left\{ \begin{pmatrix} \{1,2,3\} & \{4\} & \{5\} \\ 1 & 2 & 3 \end{pmatrix}, \begin{pmatrix} \{1,2,3\} & \{4\} & \{5\} \\ 1 & 2 & 4 \end{pmatrix}, \begin{pmatrix} \{1,2,3\} & \{4\} & \{5\} \\ 1 & 3 & 2 \end{pmatrix}, \begin{pmatrix} \{1,2,3\} & \{4\} & \{5\} \\ 1 & 3 & 4 \end{pmatrix} \right\}$$

Thus, $\left|\mathcal{D}_4^{2,2}\right| = 2^{4-1} - 1 = 7$ and $\left|N(\mathcal{D}_5)^{3,3}\right| = (4-2)(5-3) = 4$, where $1\alpha^{-1} = \{1,2,3\}$ and $X_n \setminus 1\alpha^{-1} = \{4,5\} = \{b_1, b_2\}$.

3. Maximal Nilpotent Subsemigroups of $\mathcal{D}_n$

This section demonstrates the efficacy of enumerative techniques by providing alternative proofs for the cardinality and rank of maximal nilpotent subsemigroups of $\mathcal{D}_n$. If $\alpha \in \mathcal{D}_n$, then

$$\alpha = \begin{pmatrix} A_1 & A_2 & \cdots & A_r \\ a_1 & a_2 & \cdots & a_r \end{pmatrix}$$

to indicate that $\mathrm{im}(\alpha) = \{1 = a_1, a_2, \ldots, a_r\}$ and $a_i\alpha^{-1} = A_i$, for each $1 \leq i \leq r$, and A_1, A_2, …, A_r also called blocks of α are all non-empty. It is known that every transformation $\alpha \in \mathcal{D}_n$ is idempotent if and only if $a_i = \min A_i$, for each $1 \leq i \leq r$. Every transformation $\alpha \in \mathcal{D}_n$ is nilpotent if and only if $\mathrm{fix}(\alpha) = \{1\}$. In this paper, we denote the zero and the identity elements of $\mathcal{D}_n$ by 0_n and 1_n, respectively.

For each $\xi \in E(\mathcal{D}_n)$, let $\mathcal{D}_n(\xi) = \{\alpha \in \mathcal{D}_n : \alpha^k = \xi, \text{ for some } k \in \mathbb{Z}^+\}$ be the maximum nilpotent subsemigroup of $\mathcal{D}_n$ with the zero element ξ. For any $\xi \in E(\mathcal{D}_n)$, the cardinality and rank of $\mathcal{D}_n(\xi)$ were computed in [14].

In this section, we demonstrate the efficacy of enumerative techniques by providing alternative proofs for the cardinality and rank of $\mathcal{D}_n(\xi)$. These proofs, derived through distinct block enumerations of ξ, highlight the potency of these techniques in tackling complex combinatorial challenges.

Theorem 3.1. For any $\xi \in E(\mathcal{D}_n)$, let $\mathrm{fix}(\xi) = \{1 = a_1 < a_2 < \cdots < a_r\}$ and A_i be blocks of ξ with $|A_i| = k_i$, for each $1 \leq i \leq r$. Then,

$$|\mathcal{D}_n(\xi)| = \prod_{i=1}^{r} (k_i - 1)!$$

PROOF. Let $A_i = \{a_i + s_{i_1}, a_i + s_{i_2}, \ldots, a_i + s_{i_{k_i}}\}$ with $s_{i_1} = 0$. For $\alpha \in \mathcal{D}_n(\xi)$, if $(a_i + s_{i_j})\alpha = a_i + s_{i_{m_j}}$, for $1 \leq j \leq k_i$, then $m_1 = 1$ and $m_j \in \{1, 2, \ldots, j-1\}$, for each $2 \leq j \leq k_i$. Consider

$$\alpha_i = \begin{pmatrix} 1 & 2 & \cdots & k_i \\ m_1 & m_2 & \cdots & m_{k_i} \end{pmatrix}$$

It can be observed that $\alpha_i \in N(\mathcal{D}_{k_i})$, for each $1 \leq i \leq r$. Therefore, the function

$$f : \mathcal{D}_n(\xi) \to N(\mathcal{D}_{k_1}) \times N(\mathcal{D}_{k_2}) \times \cdots \times N(\mathcal{D}_{k_r})$$

defined by $\alpha f = (\alpha_1, \alpha_2, \ldots, \alpha_r)$ is a well-defined bijection. Thus,

$$|\mathcal{D}_n(\xi)| = \prod_{i=1}^{r} (k_i - 1)!$$

□

Theorem 3.2. For any $\xi \in E(\mathcal{D}_n)$, let $\mathrm{fix}(\xi) = \{1 = a_1 < a_2 < \cdots < a_r\}$ and A_i be blocks of ξ with $|A_i| = k_i$, for each $1 \leq i \leq r$. Then,

$$\mathrm{rank}(\mathcal{D}_n(\xi)) = \prod_{i=1}^{r} (k_i - 1)! - \prod_{i=1}^{r} (k_i - 2)!$$

PROOF. Let $A_i = \{a_i + s_{i_1}, a_i + s_{i_2}, \ldots, a_i + s_{i_{k_i}}\}$ with $s_{i_1} = 0$. For $\alpha \in \mathcal{D}_n(\xi)^2$, let $(a_i + s_{ij})\alpha = a_i + s_{i_{m_j}}$, for $1 \leq j \leq k_i$. Then, $m_1 = m_2 = 1$ and $m_j \in \{1, \ldots, j-2\}$, for each $3 \leq j \leq k_i$. Consider

$$\alpha_i = \begin{cases} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, & k_i \in \{1, 2\} \\ \begin{pmatrix} 1 & 2 & \cdots & k_i - 1 \\ m_1 & m_2 & \cdots & m_{k_i - 1} \end{pmatrix}, & k_i \geq 3 \end{cases}$$

Assuming that $N(\mathcal{D}_{k_i-1}) = N(\mathcal{D}_1)$, for $k_i = 1$, then $\alpha_i \in N(\mathcal{D}_{k_i-1})$, for each $1 \leq i \leq r$. Consequently, the function

$$f : \mathcal{D}_n(\xi)^2 \to N(\mathcal{D}_{k_1-1}) \times N(\mathcal{D}_{k_2-1}) \times \cdots \times N(\mathcal{D}_{k_r-1})$$

defined by $\alpha f = (\alpha_1, \alpha_2, \ldots, \alpha_r)$ is a well-defined bijection. By taking into account the assumption that $N(\mathcal{D}_{k_i-1}) = N(\mathcal{D}_1)$ when $k_i = 1$, that is, $|N(\mathcal{D}_{k_i-1})| = (k_i - 2)! = 1$ for $k_i = 1$, the result follows from the fact that $\mathcal{D}_n(\xi) \setminus \mathcal{D}_n(\xi)^2$ is the minimum generating set of $\mathcal{D}_n(\xi)$. Therefore,

$$\mathrm{rank}(\mathcal{D}_n(\xi)) = \left|\mathcal{D}_n(\xi) \setminus \mathcal{D}_n(\xi)^2\right| = \left|\mathcal{D}_n(\xi)\right| - \left|\mathcal{D}_n(\xi)^2\right| = \prod_{i=1}^{r} (k_i - 1)! - \prod_{i=1}^{r} (k_i - 2)!$$

□

4. Conclusion

In this study, we determined the cardinality and rank of the nilpotent subsemigroup $N(\mathcal{D}_{n,r})$ of $N(\mathcal{D}_n)$ and, consequently, of $\mathcal{D}_n$. We also found the cardinalities of $\mathcal{D}_n^{2,2}$ and $N(\mathcal{D}_n)^{p,p}$. Furthermore, we provided an alternative combinatorial approach to determine the cardinality and rank of $\mathcal{D}_n(\xi)$ for each $\xi \in \mathcal{D}_n$. Future studies may delve deeper into the structural properties of nilpotent subsemigroups within other transformation semigroups, expanding the scope beyond $\mathcal{D}_n$. Moreover, exploring the interrelations between combinatorial approaches and other algebraic methods could yield new insights and simplify complex calculations.

Author Contributions

The author read and approved the final version of the paper.

Conflicts of Interest

The author declares no conflict of interest.

Ethical Review and Approval

No approval from the Board of Ethics is required.

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Journal of New Theory
dergipark.org.tr/tr/pub/jnt
Open Access

ISSN: 2149-1402 (51) (2025) 33-41 doi.org/10.53570/jnt.1671463

Principally 1-Absorbing Right Primary Ideals

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Article Info
Received: 7 Apr 2025
Accepted: 18 Jun 2025
Published: 30 Jun 2025
Research Article

Abstract— This paper first defines the 1-absorbing version of principally right primary ideals (P1ARP ideals), generalizing prime ideals, for noncommutative rings. It then investigates various properties of this ideal structure in different ring settings. It obtains some essential results in ring extensions, such as homomorphic images, product rings, local rings, and idealization. While this study enables the obtaining of original results due to structural differences between commutative and noncommutative rings, it shows that some properties valid in commutative rings are preserved. Finally, the paper concludes by discussing two open problems that could guide future studies.

Keywords — 1-*Absorbing primary ideals, noncommutative rings, prime ideals, pseudo-radicals*

Mathematics Subject Classification (2020) 16N99, 16L30

1. Introduction

Working with non-commutative rings is generally more challenging than working with commutative rings. Consequently, numerous results proven for commutative rings still remain unresolved in the setting of non-commutative rings. In this study, we aim to introduce a suitable version of 1-absorbing primary ideals–originally defined for commutative rings–for non-commutative rings. As a starting point, we conduct a literature review and investigate similar definitions and results in the context of commutative rings. This will serve to illustrate the background and motivation for our work.

The study of prime ideal structures and their various generalizations in commutative rings has been a significant area of research in ring theory, as these concepts contribute significantly to understanding the properties and classification of rings. One of these generalizations is the concept of 1-absorbing primary ideals, introduced by Badawi and Çelikel [1]. The authors proved that a ring containing a 1-absorbing primary ideal that is not primary must be a quasi-local ring. Additionally, they explored the relationship between these ideals and the connection between Noetherian domains and Dedekind domains, presenting several significant results. Subsequently, Nikandish et al. [2] investigated various properties of these ideals and introduced a more general version known as the weakly 1-absorbing primary ideals, integrating the characteristics of weakly prime ideals and 1-absorbing primary ideals. They studied the key properties of these ideals in the context of polynomial rings, principal ideal domains (PIDs), and idealization structures.

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The study of primary ideals, initially focused on commutative rings, has been expanded to noncommutative settings to investigate more extensive ideal structures. In particular, Birkenmeier et al. [3] introduced various generalizations of primary ideals to address the complexities of noncommutative rings. These generalizations establish a foundation for exploring ideal structures in noncommutative rings, enabling the extension of classical commutative algebra results to broader algebraic systems. Investigating these properties has contributed to significant advancements in understanding the decomposition of ideals in noncommutative rings and their algebraic representations [4–11].

In 2022, Groenewald [11] defined the concept of a weakly right primary ideal in noncommutative rings. Additionally, in 2021, Groenewald [10] defined p-2-absorbing right primary ideals, which generalize principally right primary ideals. The aforementioned generalizations of prime ideals are as in Figure 1.

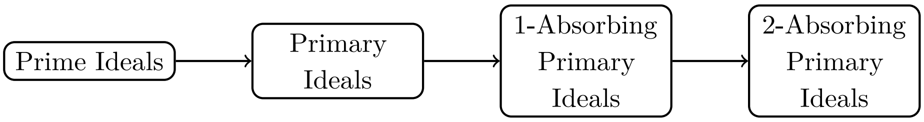

Figure 1. Relations between the aforementioned generalizations of prime ideals

Therefore, this study fills the gap between right principally primary ideals and principally 2-absorbing right primary ideals, which are defined in noncommutative rings, by introducing principally 1-absorbing right primary ideals. This new class of ideals plays a significant role in the characterization of local rings, as demonstrated in Theorem 3.12 and Proposition 3.14. The remainder of this study is organized as follows: Section 2 presents some basic notions to be needed in the following section. Section 3 introduces principally 1-absorbing right primary ideals and explores some of their basic properties. The last section discusses the need for further research.

2. Preliminaries

This section presents some basic definitions and properties to be used in the following section. Throughout this paper, $\mathcal{H}$ denotes a noncommutative ring unless otherwise specified.

Definition 2.1. [3] Let J be a proper ideal of $\mathcal{H}$. Then, the pseudo-radical of J is defined by

$$\sqrt{J} := \sum \left\{ K \lhd \mathcal{H} \mid K^n \subseteq J \text{ for some } n \in \mathbb{Z}^+ \right\}$$

It is clear that $\sqrt{J}$ is an ideal of $\mathcal{H}$. In this context, the term $\mathcal{P}^*(J)$ denotes the prime radical of J, defined as the intersection of all prime ideals of $\mathcal{H}$ containing J. In commutative rings, the definition of the radical of an ideal is the same as this concept. The radical of an ideal in noncommutative rings differs slightly from its definition in commutative rings. It is known that $\sqrt{J} \subseteq \mathcal{P}^*(J)$, and as established in Lemma 2.7 *iv*, $\sqrt{J}$ is strictly contained within $\mathcal{P}^*(J)$. Therefore, this study differs from generalizations related to primary ideals defined in commutative rings. Specifically, the collection of all prime ideals of $\mathcal{H}$, represented by $P(\mathcal{H})$, corresponds to $\mathcal{P}^*(0)$.

Definition 2.2. [1] A proper ideal A of a commutative ring $\mathcal{H}$ is said to be a 1-absorbing primary ideal if, for any nonunit elements $u, v, z \in \mathcal{H}$, the condition $uvz \in A$ implies that either $uv \in A$ or $z \in \sqrt{A}$.

Definition 2.3. [3] An ideal A of $\mathcal{H}$ is defined as a (principally) right primary ideal if, for any (principal) ideals K and L of $\mathcal{H}$ satisfying $KL \subseteq A$, it follows that either $K \subseteq A$ or $L^n \subseteq A$ for some $n \in \mathbb{Z}^+$.

Definition 2.4. [3] An ideal A of $\mathcal{H}$ is referred to as a (principally) semiprimary ideal if, for any (principal) ideals K and L of $\mathcal{H}$ such that $KL \subseteq A$, it holds that either $K^l \subseteq A$ or $L^n \subseteq A$ for some positive integers l and n.

Definition 2.5. [11] An ideal A of $\mathcal{H}$ is defined as a weakly (principally) right primary ideal if, any time K and L are (principal) ideals of $\mathcal{H}$ satisfying $\{0\} \neq KL \subseteq A$, then either $K \subseteq A$ or there is $n \in \mathbb{Z}^+$ such that $L^n \subseteq A$.

Definition 2.6. [10] An $A \lhd \mathcal{H}$ is defined as a p-right 2-absorbing primary ideal if, for any elements $u, v, k \in \mathcal{H}$, the condition $u\mathcal{H}v\mathcal{H}k \subseteq A$ implies that at least one of the following holds: $uv \in A$, $uk \in \sqrt{A}$ or $vk \in \sqrt{A}$.

Lemma 2.7. [3] The following properties hold for some ideals K, L, and J in $\mathcal{H}$:

i. If $K \subseteq L$, then $\sqrt{K} \subseteq \sqrt{L}$.

ii. If $K \subseteq \sqrt{J}$, then $K^n \subseteq J$ for some $n \in \mathbb{Z}^+$ under the condition that K is finitely generated or there exists an $m \in \mathbb{Z}^+$ such that $(\sqrt{J})^m \subseteq J$. In particular, if $\sqrt{J}$ is finitely generated, then there exists an $n \in \mathbb{Z}^+$ such that $(\sqrt{J})^n \subseteq J$.

iii. $\sqrt{KL} = \sqrt{K \cap L} = \sqrt{K} \cap \sqrt{L}$

iv. If $(\sqrt{J})^l \subseteq J$, for an $l \in \mathbb{Z}^+$, then $\sqrt{J} = \mathcal{P}^*(J) = \sqrt{\sqrt{J}}$.

Definition 2.8. [10] An ideal J of $\mathcal{H}$ is called a principally 2-absorbing right primary ideal of $\mathcal{H}$ if, for all $x, y, z \in \mathcal{H}$, whenever $x\mathcal{H}y\mathcal{H}z \subseteq J$, this means that $xy \in J$ or $xz \in \sqrt{J}$ or $yz \in \sqrt{J}$.

Note that the definition of semiprime in noncommutative rings is as follows:

Definition 2.9. An ideal L of $\mathcal{H}$ is said to be semiprime if, for any ideal J of $\mathcal{H}$, whenever a positive power of J, say J^k, is contained in L for some natural number k, then J itself must also be contained in L.

Lemma 2.10. [10] For a ring homomorphism $\varphi : H_1 \to H_2$ and ideals $A_1 \lhd H_1$ and $A_2 \lhd H_2$, the following hold:

i. $\varphi^{-1}(\sqrt{A_2}) = \sqrt{\varphi^{-1}(A_2)}$

ii. If $\mathrm{Ker}(\varphi) \subseteq A_1$, then $\varphi(\sqrt{A_1}) \subseteq \sqrt{\varphi(A_1)}$.

Lemma 2.11. [12] If, for each nonunit $x \in \mathcal{H}$ and each unit $y \in \mathcal{H}$, the sum $x + y$ is a unit, then $\mathcal{H}$ is a local ring.

Remark 2.12. [13] If $x \in \sqrt{J}$, then there exists $n \in \mathbb{Z}^+$ such that $(\langle x \rangle)^n \subseteq J$. Thus, for any ideals I, J of $\mathcal{H}$, if $I \nsubseteq \sqrt{J}$ then $I^n \nsubseteq J$, for each $n \in \mathbb{N}$.

Unless otherwise stated, throughout this paper, all rings are assumed to be noncommutative and possess a nonzero identity element, denoted by $\mathcal{H}$.

3. Main Results

This section defines principally 1-absorbing right primary ideals and investigates some of their basic properties.

Definition 3.1. A proper ideal J of a ring $\mathcal{H}$ is called a principally 1-absorbing right primary ideal if, for all nonunit elements $x, y, z \in \mathcal{H}$, the condition $x\mathcal{H}y\mathcal{H}z \subseteq J$ implies that either $xy \in J$ or $z \in \sqrt{J}$.

For the sake of convenience, we will use the abbreviations P1ARP to refer to "principally 1-absorbing right primary," and PRP refer to "principally right primary" for the remainder of our work.

We should note that a principally 2-absorbing right primary ideal is more general than a P1ARP ideal. Hence, every P1ARP ideal is a subclass of a principally 2-absorbing right primary ideal. To support this claim, we provide a counterexample.

Example 3.2. Let $\mathcal{H} = M_2(\mathbb{Z})$ and $J = M_2(\langle 12\rangle)$. From [10], J is a principally 2-absorbing right primary ideal. However, J is not a P1ARP ideal since, although

$$\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \mathcal{H} \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \mathcal{H} \begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix} \subseteq J$$

it can be observed that

$$\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \notin J \quad \text{and} \quad \begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix} \notin \sqrt{J}$$

Theorem 3.3. Let J be a P1ARP ideal of $\mathcal{H}$. If $x\mathcal{H}yK \subseteq J$, for all nonunits $x, y \in \mathcal{H}$ and proper ideals K of $\mathcal{H}$, then $xy \in J$ or $K \subseteq \sqrt{J}$.

Proof. Suppose that $x\mathcal{H}yK \subseteq J$, for nonunits $x, y \in \mathcal{H}$ and $K \lhd \mathcal{H}$ with $xy \notin J$ and $K \nsubseteq \sqrt{J}$. Then, there exists an $a \in K$ such that $a \notin \sqrt{J}$. Hence, $x\mathcal{H}y\mathcal{H}a \subseteq J$, while neither $xy \in J$ nor $a \in \sqrt{J}$, contradicting the fact that J is a P1ARP. □

Theorem 3.4. Let K be a proper ideal of $\mathcal{H}$. The following statements are equivalent:

i. K is a P1ARP ideal of $\mathcal{H}$.

ii. For some proper ideals J_1, J_2, J_3 of $\mathcal{H}$ with $J_1J_2J_3 \subseteq K$, $J_1J_2 \subseteq K$ or $J_3 \subseteq \sqrt{K}$.

Proof. Assume that K is a proper ideal of $\mathcal{H}$.

$i \Rightarrow ii$: Let K be a P1ARP ideal and $J_1J_2J_3 \subseteq K$ with $J_1J_2 \nsubseteq K$, for some proper ideals J_1, J_2, J_3 of $\mathcal{H}$. Then, there exist nonunits $x \in J_1$ and $y \in J_2$ such that $xy \notin K$. Since $x\mathcal{H}yJ_3 \nsubseteq K$ and $xy \notin K$, then $J_3 \subseteq \sqrt{K}$ by Theorem 3.3.

$ii \Rightarrow i$: Let x, y, z be some nonunits of $\mathcal{H}$ such that $x\mathcal{H}y\mathcal{H}z \subseteq K$ with $xy \notin K$. Assume that $J_1 = \mathcal{H}x\mathcal{H}$, $J_2 = \mathcal{H}y\mathcal{H}$, and $J_3 = \mathcal{H}z\mathcal{H}$. Then, $J_1J_2J_3 \subseteq K$ and $J_1J_2 \subseteq K$. Hence, $J_3 \subseteq \sqrt{K}$, and thus $z \in \sqrt{K}$. □

Proposition 3.5. Assume that $\mathcal{H}$ has a nonzero identity. For $K \lhd H$, if $\sqrt{K}$ is a PRP ideal, then K is a P1ARP1.

Proof. Suppose that $uRvRz \subseteq K$, for some nonunits $u, v, z \in \mathcal{H}$ and $z \notin \sqrt{K}$. Consider that $uvRz \subseteq uRvRz \subseteq K \subseteq \sqrt{K}$. Since $\sqrt{K}$ is a PRP ideal, then $uv \in K$. Therefore, K is a P1ARP. □

Theorem 3.6. If J is a semiprime ideal, then the following condition holds:

$$J \text{ is a prime ideal} \Leftrightarrow J \text{ is a P1ARP ideal}$$

Proof. Let J be a semiprime ideal.

($\Rightarrow$) The proof follows directly from the definitions.

($\Leftarrow$) Assume that $\langle x\rangle\langle y\rangle\langle z\rangle \subseteq J$ and $\langle x\rangle\langle y\rangle \nsubseteq J$, for nonunits $x, y, z \in \mathcal{H}$. Since $xy\mathcal{H}z \subseteq x\mathcal{H}y\mathcal{H}z \subseteq \langle x\rangle\langle y\rangle\langle z\rangle \subseteq J$ and J is a P1ARP ideal, then $\langle z\rangle^n \subseteq J$ by Remark 2.12, and since J is a semiprime ideal, then $\langle z\rangle \subseteq J$. Therefore, $xy \notin J$ and $z \in J$ and hence J is a prime ideal. □

Proposition 3.7. If the radical of J is a prime ideal in $\mathcal{H}$, then J is a P1ARP ideal.

Proof. Let $x\mathcal{H}y\mathcal{H}z \subseteq J$ with $xy \notin J$, for nonunits $x, y, z \in \mathcal{H}$. Then, $xy\mathcal{H}z \subseteq x\mathcal{H}y\mathcal{H}z \subseteq J \subseteq \sqrt{J}$ since $\sqrt{J}$ is prime and $xy \notin J$, i.e., $xy \notin \sqrt{J}$, then $z \in \sqrt{J}$. Hence, J is a P1ARP ideal. □

Theorem 3.8. Let A be a P1ARP ideal of $\mathcal{H}$. If $k \in \mathcal{H} \setminus A$ is a nonunit element, then $(A : \langle k\rangle) = \{t \in \mathcal{H} : \langle k\rangle t \subseteq A\}$ is a PRP ideal of $\mathcal{H}$.

Proof. Let A be a P1ARP ideal of $\mathcal{H}$ and $k \in \mathcal{H} \setminus A$ be a nonunit element. For some nonunits $u, v \in \mathcal{H}$, assume that $u\mathcal{H}v \subseteq (A : \langle k\rangle)$. If $u \notin (A : \langle k\rangle)$, then $\langle k\rangle u \nsubseteq A$. Since $\langle k\rangle uHv \subseteq A$ and $\langle k\rangle u \nsubseteq A$, then $kHuHv \subseteq A$ and $ku \notin A$. Therefore, $v \in \sqrt{A} \subseteq \sqrt{(A : \langle k\rangle)}$ since A is a P1ARP ideal. Hence, $(A : \langle k\rangle)$ is a PRP ideal of $\mathcal{H}$. □

3.1. Homomorphic Images

The subsection investigates the relations between ring homomorphisms and P1ARP ideals.

Theorem 3.9. The following conditions hold under the surjective ring homomorphism $\varphi : H_1 \to H_2$.

i. If A_2 is a P1ARP ideal of H_2, then $\varphi^{-1}(A_2)$ is a P1ARP ideal of H_1.

ii. If A_1 is a P1ARP ideal with $\mathrm{Ker}(\varphi) \subseteq A_1$, then $\varphi(A_1)$ is a P1ARP ideal of H_2.

Proof. Let $\varphi : H_1 \to H_2$ be a surjective ring homomorphism.

i. For some nonunits $u, v, z \in H_1$, suppose that $uH_1vH_1z \subseteq \varphi^{-1}(A_2)$. Then, $\varphi(uH_1vH_1z) \subseteq \varphi(u)H_2\varphi(v)H_2\varphi(z) \subseteq A_2$. Thus, $\varphi(u)\varphi(v) \in A_2$ or $\varphi(z) \in \sqrt{A_2}$, i.e., $uv \in \varphi^{-1}(A_2)$ or $z \in \varphi^{-1}(\sqrt{A_2}) = \sqrt{\varphi^{-1}(A_2)}$. Hence, $\varphi^{-1}(A_2)$ is a P1ARP ideal.

ii. For some nonunits $u, v, z \in H_2$, assume that $uH_2vH_2z \subseteq \varphi(A_1)$. Hence, there exist $\varphi(k) = u$, $\varphi(l) = v$, and $\varphi(m) = z$ such that $\varphi(kH_1lH_1m) = uH_2vH_2z \subseteq \varphi(A_1)$. Since $\mathrm{Ker}(\varphi) \subseteq A_1$, then $kH_1lH_1m \subseteq A_1$. Thus, $kl \in A_1$ or $m \in \sqrt{A_1}$. Therefore, $uv \in \varphi(A_1)$ or $z \in \varphi(\sqrt{A_1}) \subseteq \sqrt{\varphi(A_1)}$. Consequently, $\varphi(A_1)$ is a P1ARP ideal of H_2.

□

Corollary 3.10. Let A_1 and A_2 be proper ideals of $\mathcal{H}$ satisfying the condition $A_1 \subseteq A_2$. Then, A_1 is a P1ARP ideal if and only if A_2/A_1 is a P1ARP ideal of $\mathcal{H}/A_1$.

Proof. Take $\varphi : \mathcal{H} \to \mathcal{H}/A_1$ with $\varphi(x) = x + A_1$. Assume that A_2 is a P1ARP ideal of $\mathcal{H}$. By Theorem 3.9 *ii*, $\varphi(A_2) = A_2/A_1$ is a P1ARP since $\mathrm{Ker}(\varphi) = A_1 \subseteq A_2$. Conversely, assume that A_2/A_1 is a P1ARP ideal of $\mathcal{H}/A_1$. By Theorem 3.9 *i*, $\varphi^{-1}(A_2/A_1) = A_2$ is a P1ARP. □

3.2. Product Rings

The following identities are well known:

$$\sqrt{J_1 \times \mathcal{H}_2} = \sqrt{J_1} \times \mathcal{H}_2$$

and

$$\sqrt{\mathcal{H}_1 \times J_2} = \mathcal{H}_1 \times \sqrt{J_2}$$

Theorem 3.11. Let $\mathcal{H}$ be the product of two unital rings $\mathcal{H}_1$ and $\mathcal{H}_2$, i.e., $\mathcal{H} = \mathcal{H}_1 \times \mathcal{H}_2$. Then, the following properties are valid.

i. If J_1 is a P1ARP ideal of $\mathcal{H}_1$, then $J_1 \times \mathcal{H}_2$ is a P1ARP ideal of $\mathcal{H}$.

ii. If J_2 is a P1ARP ideal of $\mathcal{H}_2$, then $\mathcal{H}_1 \times J_2$ is a P1ARP ideal of $\mathcal{H}$.

Proof. Let $\mathcal{H}$ be the product of two unital rings $\mathcal{H}_1$ and $\mathcal{H}_2$, i.e., $\mathcal{H} = \mathcal{H}_1 \times \mathcal{H}_2$.

i. Assume that $(x, y)\mathcal{H}(z, d)\mathcal{H}(k, l) = x\mathcal{H}_1 z\mathcal{H}_1 k \times y\mathcal{H}_2 d\mathcal{H}_2 l \subseteq J_1 \times \mathcal{H}_2$ for nonunits $(x, y), (z, d), (k, l) \in \mathcal{H}$. Since J_1 is a P1ARP ideal, $xz \in J_1$ or $k \in \sqrt{J_1}$. Thus, $(x, y)(z, d) \in J_1 \times \mathcal{H}_2$ or $(k, l) \in \sqrt{J_1} \times \mathcal{H}_2 = \sqrt{J_1 \times \mathcal{H}_2}$. Therefore, $J_1 \times \mathcal{H}_2$ is a P1ARP ideal of $\mathcal{H}$.

ii. The proof is similar to the proof of *i.*

□

3.3. Results in Local Rings

This subsection presents the following useful results on local rings.

Theorem 3.12. If $\mathcal{H}$ has a P1ARP ideal that is not PRP, then $\mathcal{H}$ is local.

Proof. If J is a P1ARP ideal of $\mathcal{H}$ that is not PRP, then there exist nonunits $x, y \in \mathcal{H}$ such that $x\mathcal{H}y \subseteq J$, but $x \notin J$ and $\langle y\rangle^n \nsubseteq J$. Assume that k is a nonunit and l is a unit element of $\mathcal{H}$. Suppose that $k+l$ is a nonunit. Since J is a P1ARP ideal and $k\mathcal{H}x\mathcal{H}y \subseteq J$, then $kx \in J$. Thus, $(k+l)\mathcal{H}x\mathcal{H}y \subseteq J$ and hence $(k+l)x \in J$, i.e., $lx \in J$. But since l is a unit, then $x \in J$, which contradicts the assumption $x \notin J$. Therefore, $\mathcal{H}$ is local by Lemma 2.11. □

Theorem 3.13. Let $\mathcal{H}$ be a local ring with the unique maximal ideal M. Thus, the following are identical:

i. J is a P1ARP ideal.

ii. J is a PRP ideal or $M^2 \subseteq J \subseteq M$.

Proof. Assume that $\mathcal{H}$ is a local ring with the unique maximal ideal M.

$i \Rightarrow ii$: Suppose that J is a P1ARP ideal that is not prime. Then, $J \subsetneq M$, and there exist $x, y \in M \setminus J$ with $x\mathcal{H}y \subseteq J$. Let $k, l \in M$. Then, $(k\mathcal{H}l\mathcal{H})\mathcal{H}x\mathcal{H}y \subseteq J$. Consider that M is the unique maximal ideal of $\mathcal{H}$, $(k\mathcal{H}l) \subseteq M$, $x, y \in M$, and $y \notin J \subseteq \sqrt{J}$. Noting that J is a P1ARP ideal, then $(k\mathcal{H}l)x \subseteq J$. Moreover, since $k, l, x \in M$ and $x \notin J \subseteq \sqrt{J}$, then $kl \in J$.

$ii \Rightarrow i$: It can be observed that if J is a PRP ideal, then it is a P1ARP ideal. Assume that $M^2 \subseteq J \subseteq M$. Hence, J is a proper ideal. Suppose that $x\mathcal{H}y\mathcal{H}z \subseteq J$, for $x, y, z \in M$. Thus, $xy \in M^2 \subseteq J$. Therefore, J is a P1ARP ideal. □

Proposition 3.14. A is not a PRP ideal of $\mathcal{H}$ that is a P1ARP ideal if and only if $\mathcal{H}$ is a local ring whose maximal ideal M fulfills $M^2 \neq M$.

Proof. $(\Rightarrow)$ Assume that A is not a PRP ideal which is a P1ARP. Then, by Theorem 3.13, $\mathcal{H}$ is a local ring and its maximal ideal M satisfies $M^2 \subseteq A \subset M$ and thus $M^2 \neq M$.

$(\Leftarrow)$ Assume that M is a maximal ideal of a local ring $\mathcal{H}$ and $M^2 \neq M$. Thus, $M^2 \subset M$. Suppose that $x, y \in M \setminus M^2$. Thus, $\langle x\rangle\langle y\rangle \subseteq M^2$, but neither $\langle x\rangle \subseteq M^2$ nor $\langle y\rangle^n \subset \langle y\rangle \subseteq M^2$. Thus, M^2 is not a P1ARP ideal. If $x, y, z \in \mathcal{H}$ are nonunits with $x\mathcal{H}y\mathcal{H}z \subseteq M^2$, then M^2 is a P1ARP ideal which is not PRP since $xy \in M^2$. □

Proposition 3.15. Let K and L be P1ARP ideals of $\mathcal{H}$ which are not PRP. Then, $K + M$ and $K \cap L$ (or KL) are P1ARP ideals.

Proof. By Theorem 3.12, $\mathcal{H}$ is a local ring, and from Theorem 3.13, $M^2 \subseteq K \cap L$. Suppose that $uRvRk \subseteq K \cap L$, where $u, v, k \in H$ are nonunits and $k \notin \sqrt{K \cap L} = \sqrt{KL}$. Since $u, v \in M$, then $uv \in M^2 \subseteq K \cap L \subseteq KL$. Therefore, $K \cap L$ (or KL) is a P1ARP ideal. With a similar way, it can be observed that $K + L$ is a P1ARP ideal. □

Proposition 3.16. Let H_1 and H_2 be unital rings, and define $\mathcal{H}$ as their direct product, i.e., $\mathcal{H} = H_1 \times H_2$. If $A_1 \times A_2$ is a P1ARP ideal of $\mathcal{H}$, where A_1 and A_2 are ideals of H_1 and H_2, respectively, then A_1 and A_2 are P1ARP ideals of H_1 and H_2, respectively.

Proof. For some nonunits $u, v, z \in H_1$, assume that $uH_1vH_1z \subseteq A_1$. Then, for $a \in H_2$,

$$(uH_1vH_2z, aH_2aH_2a) \subseteq A_1 \times A_2$$

Since $A_1 \times A_2$ is a P1ARP ideal, then $(u, a)(v, a) \in A_1 \times A_2$ or $(z, a) \in \sqrt{A_1 \times A_2} = \sqrt{A_1} \times \sqrt{A_2}$. Hence, $uv \in A_1$ or $z \in \sqrt{A_1}$ and thus A_1 is a P1ARP ideal of H_1. In similar manner, it can be observed that A_2 is a P1ARP ideal of H_2. □

3.4. Idealization

This section explores certain properties of P1ARP ideals within the idealization of a ring. Recall that the structure $\mathcal{H} \boxplus M$ is referred to as the idealization, where $\mathcal{H}$ is a ring and M is an $\mathcal{H}$-$\mathcal{H}$-bimodule. The multiplication in this ring is defined as follows: Let $k, l \in H$ and $z, t \in M$, then $(k, z)(l, t) = (kl, kt + zl)$. In Remark 3.1 of [12], it is established that an element $(u, v) \in \mathcal{H} \boxplus M$ is a nonunit if and only if u is a nonunit in $\mathcal{H}$.

Theorem 3.17. Let $\mathcal{H}$ be a unital ring and M be a $\mathcal{H}$-$\mathcal{H}$-bimodule. For $K \lhd \mathcal{H}$, $K \boxplus M$ is a P1ARP ideal of $\mathcal{H} \boxplus M$ if and only if K is a P1ARP ideal of $\mathcal{H}$.

Proof. Suppose that $K \boxplus M$ is a P1ARP ideal of $\mathcal{H} \boxplus M$ and that $uRvRz \subseteq K$, where u, v, and z are nonunits in $\mathcal{H}$. Then, for nonunits $(u, 0), (v, 0), (z, 0) \in \mathcal{H} \boxplus M$,

$$(u, 0)(\mathcal{H} \boxplus M)(v, 0)(\mathcal{H} \boxplus M)(z, 0) \subseteq K \boxplus M$$

Since $K \boxplus M$ is a P1ARP ideal of $\mathcal{H} \boxplus M$, then $(u, 0)(v, 0) \in K \boxplus M$ or $(z, 0) \in \sqrt{K \boxplus M} = \sqrt{K} \boxplus M$. Hence, $uv \in K$ or $z \in \sqrt{K}$. Then, K is a P1ARP ideal of $\mathcal{H}$. □

Remark 3.18. For an $\mathcal{H}$-$\mathcal{H}$-bisubmodule N of M and $A \lhd \mathcal{H}$, $A \boxplus N$ is an ideal of $\mathcal{H} \boxplus M$ if and only if $AM + MA \subseteq N$.

Theorem 3.19. Let $\sqrt{A}$ be a PRP ideal and N be an $\mathcal{H}$-$\mathcal{H}$-bisubmodule with $AM + MA \subseteq N$. Then, the ideal $A \boxplus N$ is a P1ARP ideal of $\mathcal{H} \boxplus M$.

Proof. From [14], $\sqrt{A \boxplus N} = \sqrt{A} \boxplus N$ is a PRP ideal of $\mathcal{H} \boxplus M$ since $\sqrt{A}$ is a PRP ideal of $\mathcal{H}$. Thus, by Proposition 3.5, $A \boxplus N$ is a P1ARP ideal of $\mathcal{H} \boxplus M$. □

4. Conclusion

In this study, we have investigated the structural properties of P1ARP ideals, their behaviour under homomorphisms, and the conditions they satisfy in ring extensions such as idealization, leading to significant results. On the other hand, certain localized studies in commutative ring theory—such as the relationship between PIDs and Dedekind domains, as well as ring localizations defined via multiplicatively closed subsets, i.e., S, leading to $S^{-1}H$, remain open problems for the context of noncommutative rings. Since ring localizations defined through structures like Ore extensions differ significantly from those in commutative settings, it is natural to ask whether similar ideal-theoretic properties can be established in these cases.

As a result, we pose the following two open problems for further research:

i. What are the structural properties of P1ARP ideals in noncommutative domains? In such settings, can a meaningful relationship be established between prime ideals and PRP ideals?

ii. What structural properties do P1ARP ideals exhibit in ring localizations defined via Ore or Dorroh extensions? Under what conditions—analogous to those in commutative rings—can similar behaviors be observed in these noncommutative settings?

Author Contributions

The author read and approved the final version of the paper.

Conflicts of Interest

The author declares no conflict of interest.

Ethical Review and Approval

No approval from the Board of Ethics is required.

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Journal of New Theory
dergipark.org.tr/tr/pub/jnt
Open Access

ISSN: 2149-1402 (51) (2025) 42-51 doi.org/10.53570/jnt.1675521

A Novel Algorithm for Permanent Computation

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Article Info
Received: 15 Apr 2025
Accepted: 20 Jun 2025
Published: 30 Jun 2025
Research Article

Abstract— This study computes the permanent of a square matrix by reducing it to triangular form. To achieve the triangularization of a matrix, this paper employs additive row operations. Although applying an additive row operation does not alter the determinant, it does affect the permanent, thereby increasing the complexity of the computational process. This difficulty has discouraged previous attempts to compute the permanent via triangularization. This paper addresses this challenge and introduces a novel approach for computing the permanent of a square matrix.

Keywords — *Permanent, computational complexity, algorithm, triangularization*

Mathematics Subject Classification (2020) 15A15, 03D15

1. Introduction

In linear algebra, the determinant is a well-known and extensively studied function of a matrix, with numerous applications in fields as varied as mathematics, physics, and engineering. The permanent, a function analogous to the determinant but without the alternating sign characteristic, has also attracted broad interest, particularly for its applications in combinatorial enumeration problems and quantum computing. For example, the connections between the permanent and the quantum entanglement have been investigated in [1].

The permanent computation of a matrix according to the classical Binet's formula, also called the naive algorithm, is as follows: Let $A = [a_{ij}]$ be a square matrix of size n by n with elements in a field F. Then, the permanent of A is defined by

$$\operatorname{per}(A) = \sum_{\sigma \in S_n} \prod_{i=1}^{n} a_{i\sigma_i} \tag{1.1}$$

where S_n is the symmetric group which consists of all permutations of $\{1, 2, \ldots, n\}$, and σ is an element of this symmetric group where $\sigma = (\sigma_1, \sigma_2, ..., \sigma_n)$ [2].

Despite the permanent similarity to the determinant, this function differs significantly in its properties. For example, the Gaussian elimination method, which can be used for reducing a matrix, allows efficient evaluation of the determinant by row reduction. However, using this method to compute the permanent is far more complex. The elementary row operations affect these two functions differently

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because the permanent is unstable, as the determinant is under some matrix operations. For example, adding a non-zero scalar multiple of one row to another does not vary the determinant of a matrix but affects the permanent. While the determinant of a matrix with two identical rows is zero, the permanent of such a matrix does not necessarily have to be zero. Multiplying a row by a scalar requires multiplying the determinant by the same scalar. This is also valid for the permanent. In addition, interchanging two rows changes the sign of the determinant, but keeps the permanent unchanged. These differences complicate the computation of the permanent and offer possibilities for further research on computational complexity.

The problem of computing the permanent was proven to be #P-complete by Valiant [3]. It is therefore unlikely to have a polynomial-time solution. The algorithms for evaluating the permanent of a matrix have an exponential time complexity. Any method to evaluate the permanent is thus of fundamental interest to complexity theory [4]. Ryser [5] introduced an algorithm to calculate the permanent of an $n \times n$ matrix with complexity time $\mathcal{O}(n^2 2^{n-1})$. An improved version, known as the Ryser-NW algorithm, was later improved by Nijenhuis and Wilf [6], achieving a time complexity of $\mathcal{O}(n 2^{n-1})$. Another formula, also with the same time complexity, was presented by Glynn [7].

Recent literature has introduced a range of novel perspectives on the computation of matrix permanents. Küçük [8] provides a combinatorial interpretation of the rectangular permanent problem, linking its evaluation to the solution of a structured combinatorial problem. Baykasoglu [9] offered a visual explanation of permanent computation by using directed graphs and subgraph enumeration. Chabaud et al. [10] used quantum-inspired methods to give quick proofs for several key theorems related to the permanent, including the MacMahon Master Theorem. Masschelein [11] adapted Glynn's algorithm for rectangular matrices and incorporated it into a computational framework that selects the most suitable algorithm − Naive, Ryser's, or Glynn's − based on the matrix's characteristics. Another relevant contribution is SUperman, an open-source tool introduced in [12], which provides a practical framework for computing matrix permanents efficiently across different matrix types and application domains. One notable approach involves using the permanents of submatrices, as presented in [13]. This method leverages the permanents of smaller submatrices to construct the permanent of the original matrix. This approach reduces the overall computational complexity by systematically breaking down the original matrix into smaller components.

However, despite the advancements in the usage of submatrix permanents, challenges remain in achieving computational efficiency, particularly for large matrices. The present study addresses these difficulties by proposing an approach that utilizes the triangularization of a matrix to compute its permanent. The triangularization process involves additive row operations, which alter the permanent, although they do not affect the determinant, introducing additional complexity into the calculation. In the previous study [14], we dealt with this problem by examining the variation in the permanent. We established a theorem that quantifies the variation in the permanent when a scalar multiple of one row is added to another row in a square matrix.

On this basis, the current study introduces an algorithm for computing the permanent of a square matrix using a triangularization process. Traditional methods for computing determinants, such as Gaussian elimination, are known for their efficiency and stability. However, these methods are not directly applicable to the computation of permanents due to the instability mentioned earlier under additive row operations. The new approach herein formulates a systematic method to compute the permanent by transforming the matrix into a triangular form, considering the variations introduced by additive row operations.

Building on [14], we offer a novel perspective on permanent computation, providing a systematic and theoretically grounded method. This work does not claim superiority over existing algorithms regarding efficiency or computational complexity. Instead, it aims to expand availability for researchers, offering a new approach that could inspire further exploration in matrix permanents.

The rest of this paper is organized as follows: Section 2 presents the preliminaries and necessary notations. Section 3 introduces the obtained main results herein, including the new variation formula and a novel algorithm for permanent computation. Section 4 details the proposed algorithm for permanent computation via an illustrative example. Finally, Section 5 discusses the obtained results.

2. Preliminaries

In this section, we present the notations used in this study and explores the potential improvement and efficiency of the variation formula introduced in [14].

Let $A = [a_{i,j}]$ be an $n \times n$ matrix. Then, $\widetilde{A}_{r|t}$ denotes the submatrix obtained by deleting the r^{th} row and the t^{th} column of A. Similarly, $\widetilde{A}_{i,r|j,t}$ denotes the submatrix obtained by deleting the i^{th} and r^{th} rows and the j^{th} and t^{th} columns of A. Moreover, $\widetilde{A}_{i,r,m|j,t,z}$ denotes the submatrix obtained by deleting the i^{th}, r^{th}, and m^{th} rows, as well as the j^{th}, t^{th}, and z^{th} columns of A. Here, the submatrix $\widetilde{A}_{r|t}$ is of order $n-1$ by $n-1$, the submatrix $\widetilde{A}_{i,r|j,t}$ is of order $n-2$ by $n-2$, and the submatrix $\widetilde{A}_{i,r,m|j,t,z}$ is of order $n-3$ by $n-3$. To exemplify, consider the 4×4 matrix as follows:

$$A = \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{bmatrix}$$

Then,

$$\widetilde{A}_{1|4} = \begin{bmatrix} b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \\ d_1 & d_2 & d_3 \end{bmatrix}, \quad \widetilde{A}_{1,2|3,4} = \begin{bmatrix} c_1 & c_2 \\ d_1 & d_2 \end{bmatrix}, \quad \text{and} \quad \widetilde{A}_{1,2,3|1,2,3} = \begin{bmatrix} d_4 \end{bmatrix}$$

Throughout this study, let $[n] := \{1, 2, 3, \ldots, n\}$, for all $n \in \mathbb{Z}^+$, the set of all positive integers.

Theorem 2.1. [14] Let $A = [a_{i,j}]$ be a matrix of order $n \times n$, and let B be the matrix obtained by adding k times of the i^{th} row to the r^{th} row of matrix A. If

$$V := \operatorname{per}(B) - \operatorname{per}(A)$$

then

$$V = 2k \sum_{(j,t)\in\Omega} a_{i,j} a_{i,t} \operatorname{per}\left(\widetilde{A}_{i,r|j,t}\right) \tag{2.1}$$

where the summation extends over the set

$$\Omega = \{(j,t) \mid j < t \text{ and } j,t \in [n]\}$$

Consider a matrix $A = [a_{ij}]$ with order 5×5. Add k times of the 4^{th} row to the 5^{th} row of matrix A. Here, $i = 4$ and $r = 5$. Thus, the variation formula seen by (2.1) is in the form of

$$V = 2k \sum_{\Omega} a_{4,j} a_{4,t} \operatorname{per}\left(\widetilde{A}_{4,5|j,t}\right)$$

where

$$\Omega = \{(1,2), (1,3), (1,4), (1,5), (2,3), (2,4), (2,5), (3,4), (3,5), (4,5)\}$$

In that case, for example, if $j \geq 3$, then the following permanents occur:

$$\operatorname{per}\left(\widetilde{A}_{4,5|3,4}\right) = \operatorname{per}\begin{bmatrix} a_{11} & a_{12} & a_{15} \\ a_{21} & a_{22} & a_{25} \\ a_{31} & a_{32} & a_{35} \end{bmatrix}$$

$$\operatorname{per}\left(\widetilde{A}_{4,5|3,5}\right) = \operatorname{per}\begin{bmatrix} a_{11} & a_{12} & a_{14} \\ a_{21} & a_{22} & a_{24} \\ a_{31} & a_{32} & a_{34} \end{bmatrix}$$

and

$$\operatorname{per}\left(\widetilde{A}_{4,5|4,5}\right) = \operatorname{per}\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Two columns of the matrices $\widetilde{A}_{4,5|3,4}$, $\widetilde{A}_{4,5|3,5}$, and $\widetilde{A}_{4,5|4,5}$ are the same, but only one column is different. Upon these two same columns, the following identical submatrices appear:

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \quad \begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{bmatrix}$$

These identical submatrices can be determined by expanding the permanents of the matrices $\widetilde{A}_{4,5|3,4}$, $\widetilde{A}_{4,5|3,5}$, and $\widetilde{A}_{4,5|4,5}$. From this point of view, it seems that the variation formula given by (2.1) for a matrix $A = [a_{i,j}]$ of order 5×5 can be formalized in terms of the permanents of submatrices of order 2×2.

3. Main Results

In this section, first, we present an improved variation formula that replaces the formula provided by (2.1), achieved by using $n-3$ order permanents instead of $n-2$ order permanents. Then, we presents a new method and algorithm to compute the permanent of order n.

Theorem 3.1. Let $A = [a_{i,j}]$ be $n \times n$ matrix, and let B be the matrix obtained by adding k times of the i^{th} row to the r^{th} row of the matrix A. Let $\mathcal{E}$ denote the additive row operation applied to matrix A, and let $\mathbb{V}_{\mathcal{E}}$ denote the variation between the permanents of the matrices A and B, i.e.,

$$\mathcal{E}(A) = B$$

and

$$\mathbb{V}_{\mathcal{E}} := \operatorname{per}(B) - \operatorname{per}(A)$$

Then,

$$\mathbb{V}_{\mathcal{E}} = 2k \sum_{\Delta} (a_{i,p}a_{i,t}a_{m,z} + a_{i,p}a_{i,z}a_{m,t} + a_{i,t}a_{i,z}a_{m,p}) \operatorname{per}\left(\widetilde{A}_{i,r,m|p,t,z}\right) \tag{3.1}$$

where the summation extends over the set

$$\Delta = \{(p,t,z) \mid p < t < z \text{ and } p,t,z \in [n]\}$$

Moreover, m is chosen arbitrarily from the set $M = [n] \setminus \{i, r\}$.

Proof. In the variation formula given by (2.1), the Laplace expansion is applied to each of $n-2$ ordered permanents along the m^{th} rows. These expansion process produces $n-3$ ordered permanents, denoted as $\operatorname{per}\left(\widetilde{A}_{i,r,m|p,t,z}\right)$. The permanents of order $n-3$ are rearranged by parenthesizing the identical ones, and the formula given with (3.1) is obtained. $\square$

3.1. Computing the Permanent of a Square Matrix by Using the Proposed New Variation Formula

In this subsection, we establish the theoretical basis for the proposed new variation formula, focusing on the method used to compute the permanent of a square matrix. The method involves reducing a square matrix, for which we wish to compute the permanent, to upper triangular form. To achieve this, we will utilize only the elementary row operation, which adds a row, multiplied by a constant, to another row. Throughout this study, we will refer to the term "additive row operation" to denote the operation of adding $c \in \mathbb{R}$ times of the i^{th} row to the j^{th} row of a matrix, and we represent it by $cR_i + R_j$.

3.1.1. Description of the Method

For a matrix $A = [a_{i,j}]$ with $n \times n$, define the elementary row operation

$$\mathcal{E}_i^j := k(i,j)R_j + R_{i+j} \tag{3.2}$$

where

$$k(i,j) = -\frac{a_{i+j,j}}{a_{j,j}} \in \mathbb{R} \tag{3.3}$$

such that $j \in [n-1]$ and $i := i(j) \in [n-j]$. Let $B_{f(i,j)}$ represent the matrices obtained by successively applying elementary row operations defined by (3.2), and let

$$\mathcal{E}_i^j(B_{f(i,j)-1}) = B_{f(i,j)}$$

where

$$f(i,j) = n(j-1) + i - 1 - \frac{(j-2)(j+1)}{2}$$

and with the initial condition that $B_0 = A$. Here, the indices range over $1 \le j < n$ and $1 \le i < n - j$. By considering Theorem 3.1, the following equalities can be written for the matrices $B_{f(i,j)}$:

$$\operatorname{per}\left(B_{f(1,1)}\right) - \operatorname{per}(A) = \mathbb{V}_{\mathcal{E}_1^1} \tag{3.4}$$

$$\operatorname{per}\left(B_{f(2,1)}\right) - \operatorname{per}\left(B_{f(1,1)}\right) = \mathbb{V}_{\mathcal{E}_2^1} \tag{3.5}$$

$$\vdots$$

$$\operatorname{per}\left(B_{f(2,n-2)}\right) - \operatorname{per}\left(B_{f(1,n-2)}\right) = \mathbb{V}_{\mathcal{E}_2^{n-2}} \tag{3.6}$$

and

$$\operatorname{per}\left(B_{f(1,n-1)}\right) - \operatorname{per}\left(B_{f(2,n-2)}\right) = \mathbb{V}_{\mathcal{E}_1^{n-1}} \tag{3.7}$$

From (3.4)-(3.7),

$$\operatorname{per}\left(B_{f(1,n-1)}\right) - \operatorname{per}(A) = \sum_{\substack{j\in[n-1]\\ i\in[n-j]}} \mathbb{V}_{\mathcal{E}_i^j}$$

Since $B_{f(1,n-1)}$ is an upper triangular matrix, then

$$\operatorname{per}\left(B_{f(1,n-1)}\right) = \prod_{r=1}^{n} \alpha_r$$

where α_r represents the r^{th} diagonal element of the matrix $B_{f(1,n-1)}$. Thus,

$$\operatorname{per}(A) = \prod_{r=1}^{n} \alpha_r - \sum_{\substack{j\in[n-1]\\ i\in[n-j]}} \mathbb{V}_{\mathcal{E}_i^j}$$

In the proposed method, during the computation of the k value given by (3.3), it is assumed that $a_{jj} \neq 0$. If $a_{jj} = 0$, the procedure continues by interchanging the jth row with one of the rows below it. This case is explicitly incorporated into the algorithm presented in the following section. Moreover, as mentioned in the Introduction, interchanging two rows does not alter the value of the permanent.

3.2. Proposed Algorithm for Computing the Permanent

In this subsection, we outline the algorithm designed to implement the method described in the previous subsection for computing the permanent of a square matrix.

Algorithm 1 Computation of the Permanent via Triangularization

Require: A matrix A ▷ of order n by n

$\omega \leftarrow 1$ and $\beta \leftarrow 0$
for $j = 1$ to $n-1$ **do**
  for $i = 1$ to $n-j$ **do**
    if $a_{i+j,j} \neq 0$ **then**
      if $a_{j,j} = 0$ **then**
        $A_{j+i} \longleftrightarrow A_j$ ▷ the jth and $(j+i)$th rows of matrix A have interchanged.
      else
        $k(i,j) \leftarrow -(a_{i+j,j}/a_{j,j})$
        $\beta \leftarrow \beta + \mathbb{V}_{\mathcal{E}}$ ▷ the variation $\mathbb{V}_{\mathcal{E}}$ defined by (3.1)
        $A \leftarrow \mathcal{E}_i^j(A)$ ▷ $\mathcal{E}_i^j$ is defined by (3.2)
      end if
    end if
  end for
end for
for $i = 1$ to n **do**
  $\omega \leftarrow \omega * a_{i,i}$
end for
return $\omega - \beta$

4. Illustrative Example

Consider a 4×4 matrix A and use the diagram below to illustrate the row operations necessary to reduce it to the upper triangular form:

$$A \xrightarrow{\mathcal{E}_1^1} B_1 \xrightarrow{\mathcal{E}_2^1} B_2 \xrightarrow{\mathcal{E}_3^1} B_3 \xrightarrow{\mathcal{E}_1^2} B_4 \xrightarrow{\mathcal{E}_2^2} B_5 \xrightarrow{\mathcal{E}_1^3} B_6$$

More explicitly,

$$\mathcal{E}_1^1(A) = B_1, \quad \mathcal{E}_2^1(B_1) = B_2, \quad \mathcal{E}_3^1(B_2) = B_3, \quad \mathcal{E}_1^2(B_3) = B_4, \quad \mathcal{E}_2^2(B_4) = B_5, \quad \text{and} \quad \mathcal{E}_1^3(B_5) = B_6$$

Some of the matrices B_i that emerge in this process are as follows:

$$B_3 = \begin{bmatrix} \alpha_1 & * & * & * \\ 0 & \alpha_2 & * & * \\ 0 & * & * & * \\ 0 & * & * & * \end{bmatrix}, \quad B_5 = \begin{bmatrix} \alpha_1 & * & * & * \\ 0 & \alpha_2 & * & * \\ 0 & 0 & \alpha_3 & * \\ 0 & 0 & * & * \end{bmatrix}, \quad \text{and} \quad B_6 = \begin{bmatrix} \alpha_1 & * & * & * \\ 0 & \alpha_2 & * & * \\ 0 & 0 & \alpha_3 & * \\ 0 & 0 & 0 & \alpha_4 \end{bmatrix} \tag{4.1}$$

It is clear that $\text{per}(B_6) = \prod_{r=1}^{4} \alpha_r$. Moreover, according to Theorem 3.1, we have the following equations:

$$\mathbb{V}_{\mathcal{E}_1^3} = \text{per}\,(B_6) - \text{per}\,(B_5) \tag{4.2}$$

$$\mathbb{V}_{\mathcal{E}_2^2} = \text{per}\,(B_5) - \text{per}\,(B_4) \tag{4.3}$$

$$\mathbb{V}_{\mathcal{E}_1^2} = \text{per}\,(B_4) - \text{per}\,(B_3) \tag{4.4}$$

$$\mathbb{V}_{\mathcal{E}_3^1} = \text{per}\,(B_3) - \text{per}\,(B_2) \tag{4.5}$$

$$\mathbb{V}_{\mathcal{E}_2^1} = \text{per}\,(B_2) - \text{per}\,(B_1) \tag{4.6}$$

and

$$\mathbb{V}_{\mathcal{E}_1^1} = \text{per}\,(B_1) - \text{per}\,(A) \tag{4.7}$$

Finally, from (4.2)-(4.7),

$$\mathbb{V}_{\mathcal{E}_1^3} + \mathbb{V}_{\mathcal{E}_2^2} + \mathbb{V}_{\mathcal{E}_1^2} + \mathbb{V}_{\mathcal{E}_3^1} + \mathbb{V}_{\mathcal{E}_2^1} + \mathbb{V}_{\mathcal{E}_1^1} = \text{per}(B_6) - \text{per}(A)$$

where the calculation of each $\mathbb{V}_{\mathcal{E}_i^j}$ is based on the formula given by (3.1). To exemplify, the calculation of the variation $\mathbb{V}_{\mathcal{E}_1^3}$, necessary due to the transformation $\mathcal{E}_1^3(B_5) = B_6$, is as follows: For the matrix $B_5 = [b_{ij}]$, according to (3.2) and (3.3),

$$\mathcal{E}_1^3 = -\frac{b_{43}}{b_{33}} R_3 + R_4$$

where R_3 and R_4 denote the third and fourth rows of the matrix B_5, respectively. Moreover, according to Theorem 3.1,

$$\Delta = \{(1,2,3), (1,2,4), (1,3,4), (2,3,4)\}$$

for a 4×4 matrix. Since $i = 3$ and $r = 4$, $m = 1$ or $m = 2$. Consider $m = 1$. Thus,

$$\begin{aligned} \mathbb{V}_{\mathcal{E}_1^3} = -2\tfrac{b_{43}}{b_{33}} \Big(& (b_{31}b_{32}b_{13} + b_{31}b_{33}b_{12} + b_{32}b_{33}b_{11})\, \text{per}\left(\widetilde{A}_{3,4,1|1,2,3}\right) \\ & + (b_{31}b_{32}b_{14} + b_{31}b_{34}b_{12} + b_{32}b_{34}b_{11})\, \text{per}\left(\widetilde{A}_{3,4,1|1,2,4}\right) \\ & + (b_{31}b_{33}b_{14} + b_{31}b_{34}b_{13} + b_{33}b_{34}b_{11})\, \text{per}\left(\widetilde{A}_{3,4,1|1,3,4}\right) \\ & + (b_{32}b_{33}b_{14} + b_{32}b_{34}b_{13} + b_{33}b_{34}b_{12})\, \text{per}\left(\widetilde{A}_{3,4,1|2,3,4}\right)\Big) \end{aligned} \tag{4.8}$$

We note that $\widetilde{A}$'s in (4.8) are the submatrices of $B_5 = [b_{ij}]$. Since the matrix B_5 has the form seen by (4.1), then

$$\mathbb{V}_{\mathcal{E}_1^3} = -2b_{43}\,(b_{34}b_{11}b_{22} + b_{34}b_{12}b_{21})$$

5. Conclusion

The section begins with a performance analysis of the new variation formula in (3.1), in comparison with the earlier formula (2.1), originally proposed in [14]. Then, the computational complexity analysis of the proposed permanent algorithm is provided, along with a comparison to Ryser's classical formula.

Let the variation formula given by (2.1) be denoted $\mathbb{V}_1$, and the variation formula given by (3.1) be denoted $\mathbb{V}_2$. Then,

$$\mathcal{O}(\mathbb{V}_1) = \binom{n}{n-2}\left[\mathcal{O}\left(\text{per}\left(\widetilde{A}_{i,r|j,t}\right)\right)+2\right]$$

and

$$\mathcal{O}(\mathbb{V}_2) = \binom{n}{n-3}\left[\mathcal{O}\left(\text{per}\left(\widetilde{A}_{i,r,m|j,t,z}\right)\right)+9\right]$$

where $\mathcal{O}(\mathbb{V})$ denotes the number of arithmetic operations required for calculating the variation formula $\mathbb{V}$. The computation of the permanent of an $n \times n$ matrix $A = [a_{i,j}]$ by using the Binet formula requires $(n-1)n!$ multiplication operations and $n!-1$ addition operations. Therefore, the total number of arithmetic operations required to compute the permanent of the matrix A via (1.1) is $n(n!)-1$. Accordingly,

$$\mathcal{O}\left(\text{per}\left(\widetilde{A}_{n-2\times n-2}\right)\right) = (n-2)(n-2)!-1$$

for each $n-2$ by $n-2$ submatrix $\widetilde{A}_{i,r|j,t}$, and

$$\mathcal{O}\left(\text{per}\left(\widetilde{A}_{n-3\times n-3}\right)\right) = (n-3)(n-3)!-1$$

for each $n-3$ by $n-3$ submatrix $\widetilde{A}_{i,r,m|j,t,z}$. Taking these into account,

$$\mathcal{O}(\mathbb{V}_1) = \binom{n}{n-2}\left[(n-2)(n-2)!+1\right]$$

and

$$\mathcal{O}(\mathbb{V}_2) = \binom{n}{n-3}\left[(n-3)(n-3)!+8\right]$$

and therefore

$$\mathcal{O}(\mathbb{V}_1) - \mathcal{O}(\mathbb{V}_2) = n!\left(2n-3+\frac{19-8n}{(n-2)!}\right)$$

The result of $\mathcal{O}(\mathbb{V}_1) - \mathcal{O}(\mathbb{V}_2)$ is positive for $n \geq 5$. Therefore, the variation formula $\mathbb{V}_2$ yields results in a shorter time for square matrices of order at least 5×5. Moreover, since $\mathcal{O}(\mathbb{V}_2)$ and $\mathcal{O}(\mathbb{V}_1)$ are sequences with positive terms, they can be subjected to the limit comparison test:

$$\lim_{n\to\infty}\frac{\mathcal{O}(\mathbb{V}_2)}{\mathcal{O}(\mathbb{V}_1)} = \frac{1}{3}$$

Thus, it is clear that the variation $\mathbb{V}_2$ yields results in a shorter time. These results collectively confirm that $\mathbb{V}_2$ is the more efficient and preferable variation formula for square matrices of order at least 5×5.

An analysis of the computational complexity of the proposed algorithm yields the following results: Given an $n \times n$ matrix A, the algorithm requires the computation of permanents of $(n-3) \times (n-3)$ submatrices of A. These sub-permanents can be computed using any known method from the literature; such as R-NW algorithm, which has a computational complexity of $\mathcal{O}(n2^{n-1})$. Accordingly, the complexity of computing a permanent of size $n-3$ is $\mathcal{O}((n-3)2^{n-4})$.

In our proposed method, the triangularization of an $n \times n$ matrix A requires applying the variation formula $\frac{n(n-1)}{2}$ times. At each application of the variation formula, $\binom{n}{n-3}$ distinct submatrices of size $n-3$ are produced, and their permanents need to be computed. Here, assuming that the permanent of each submatrix is computed using Ryser's algorithm, the overall computational complexity of the proposed algorithm becomes

$$\frac{n^2(n-1)^2(n-2)}{12}\mathcal{O}\left((n-3)2^{n-4}\right)$$

Beyond providing a method for computing the permanent of a square matrix, the approach introduced in this study also contributes to the theoretical background of the permanent function. Furthermore, the algorithm developed here holds potential for enhancement through techniques such as parallel computation. By exploiting such improvements, more efficient algorithms could be realized, paving the way for future research, particularly in areas where the permanent plays a central role, including its emerging applications in quantum mechanics.

Author Contributions

All the authors equally contributed to this work. They all read and approved the final version of the paper.

Conflicts of Interest

All the authors declare no conflict of interest.

Ethical Review and Approval

No approval from the Board of Ethics is required.

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Journal of New Theory
dergipark.org.tr/tr/pub/jnt
Open Access

ISSN: 2149-1402 (51) (2025) 52-64 doi.org/10.53570/jnt.1686180

Soft Set Extensions via Isotonic Spaces: Theory and Application to Risk-Based Decision Making

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Article Info
Received: 28 Apr 2025
Accepted: 14 Jun 2025
Published: 30 Jun 2025
Research Article

Abstract— This paper investigates the theoretical structure and practical implications of isotonic extensions of soft sets. It utilizes isotonic operators—functions satisfying groundedness and order-preserving properties—to derive new soft sets that reflect observed attributes and potential latent associations within a system. This study presents foundational results on preserving key soft set structures under isotonic extension and examines how internal approximation relations evolve under such operators. The study provides an application to infectious disease risk modeling in a hospital environment as a practical demonstration. Here, isotonic extensions enable the identification of asymptomatic but exposed individuals, offering a novel mathematical approach to decision-making under uncertainty.

Keywords — *Soft sets, closure operators, isotonic spaces, decision making, medical diagnosis*

Mathematics Subject Classification (2020) 03E47, 54A05

1. Introduction

The necessity of coping with uncertainty and incomplete information is a fundamental challenge in modern mathematical modeling. In this context, the soft set theory, introduced by Molodtsov in 1999 [1], presents a significant innovation, particularly in modeling uncertainty through parametric representations. By providing a more flexible structure compared to classical logic frameworks, soft sets have found applications in numerous fields, such as multi-criteria decision making, information systems, medicine, engineering, and economics. However, existing approaches in the literature predominantly focus on observable, direct data, living out the modeling of indirect, implicit, or potentially risky relationships. This limitation leads to the inadequacy of decision models, especially in areas where indirect interactions are decisive, such as epidemiology, security analysis, and network theory.

The definition of a topology on a set extends beyond the traditional axioms for open sets, encompassing collections of closed sets, neighborhood systems, closure operators, and interior operators, among other constructs. For instance, Day [2] and Hausdorff [3] have developed topological concepts by leveraging the notions of convergence, closure, and neighborhoods. Kuratowski [4] has pioneered a distinct approach to constructing a topological structure on a non-empty set U through the definition of a closure operator $\mu : P(U) \to P(U)$, where $P(U)$ is the power set of U. Utilizing this framework, the closure operator satisfying the established axioms enables the definition of the topological space

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(U, μ) by identifying closed sets as those satisfying $\mu(X) = X$. Furthermore, Kuratowski has broadened the scope of topological spaces by relaxing the axiom $\mu(X \cup Y) \subseteq \mu(X) \cup \mu(Y)$, thereby defining closure spaces. Conversely, Čech's approach [5] to the definition of closure spaces omits the idempotence axiom $\mu(\mu(X)) = \mu(X)$. The terms "Kuratowski closure space" and "Čech closure space" are employed in the literature to mitigate terminological ambiguity. Additionally, Gnilka [6–8] and Hammer [9, 10] have preferred the term "extended topological space" over "closure space". These studies have investigated fundamental concepts, such as compactness, quasi-metrizability, symmetry, and continuity through the lens of closure operators. More recently, Stadler and Stadler [11] and Stadler et al. [12, 13] have unveiled a topological approach to chemical organizations, evolutionary theory, and combinatorial chemistry, elucidating the relationships between topological concepts, such as similarity, neighborhood, connectedness, and continuity within chemical and biological contexts. In these interdisciplinary studies, the authors have considered the foundational concepts of closure and isotonic spaces, defining an isotonic space as a closure space (U, μ) that satisfies only the axioms of groundedness, i.e., the condition $\mu(\emptyset) = \emptyset$, and isotonicity, i.e., the condition $X \subseteq Y \Rightarrow \mu(X) \subseteq \mu(Y)$, for all $X, Y \in P(U)$. Moreover, Habil and Elzenati [14, 15] have explored the notions of connectedness and lower and upper separation axioms in isotonic spaces.

Current research concerning soft sets primarily concentrates on fundamental set operations, equivalence structures, decision-making algorithms, and generalized operators. For instance, Maji et al. [16] have defined basic operations on soft sets; Ali et al. [17] extended these operations; and Molodtsov [18] provided a theoretical foundation for correct operation definitions in soft sets. Rapid advancements in soft set theory have led to the definition of a multitude of novel operations, such as multiplication and complementation on soft sets, along with their various modifications. The theoretical properties of these operations have been extensively studied in [19–28]. Alongside these, numerous studies concern variations of soft sets and their applications to decision-making problems [29–43]. Nevertheless, the vast majority of these studies are based on models where only existing information is processed. A framework for systematically including implicit relationships, chains of contact, or potential impacts into the model remains absent within the classical structure. This gap can lead to serious consequences, particularly in decision-making problems involving high uncertainty, such as the detection asymptomatic infections.

This study proposes a novel mathematical approach by extending soft sets through isotonic operators in this context. These operators consider not only the observed information but also the potential relationships arising from the structural nature of the system. Thus, elements that are not directly observable but are systemically at risk can be incorporated into the model. For instance, in a hospital, an asymptomatic individual, while not exhibiting direct symptoms, may carry a risk due to past contact with symptomatic individuals. The inability to integrate such indirect information into the classical soft set structure leads to deficient decision-making processes; the isotonic operator-extended soft sets aim to bridge this gap.

The core problem of this study is the inability of classical soft sets to systematically model indirect and potential information; the central hypothesis, on the other hand, posits that "soft sets extended with isotonic operators will be an effective tool in incorporate implicit relationships into decision systems by enhancing their sensitivity". The studies conducted in this direction in the literature are quite limited and mostly confined to specific examples. This study aims to reveal the structural properties of the isotonic extension at a theoretical level and demonstrate the model's functionality through a real-world application scenario.

Within this framework, the structure of the study is organized as follows: The second section presents the necessary preliminary information and conceptual foundation. The third section provides definitions of soft sets defined on isotonic spaces and the fundamental definitions for the extension of these structures. The fourth section examines the structural properties of the isotonic extension in detail and presents various theoretical results. The fifth section, on the other hand, conducts an exemplary application on infectious disease risk in a hospital setting, discussing the advantages of the isotonic extension compared to classical models. Finally, the conclusion section provides a general evaluation based on the findings obtained and offered by suggestions for future research.

2. Preliminaries

In this section, we lay the groundwork by introducing essential definitions and concepts from soft set theory and topology, fundamental to understanding the proposed methodology.

Definition 2.1. [1] Let U be a universe of discourse and E be the set of all parameters associated with the elements of U. The ordered pair (F, E) is called a soft set over U, where $F : E \to P(U)$ is a set-valued function.

Definition 2.2. [18] Let (F, E) be a soft set over U. Then, the family

$$APP(F, E) = \{F(p) \mid p \in E\}$$

is designated as a family of approximate descriptions, contingent upon the selection of E.

Definition 2.3. [18] Let (F, E_1) and (G, E_2) be two soft sets over a universe U.

i. (F, E_1) and (G, E_2) are termed equal soft sets, denoted by $(F, E_1) = (G, E_2)$, if $E_1 = E_2$ and $F = G$.

ii. (F, E_1) and (G, E_2) are termed equivalent soft sets, denoted by $(F, E_1) \cong (G, E_2)$, if $APP(F, E_1) = APP(G, E_2)$.

Note that this equivalence holds if and only if for every $p \in E_1$, there exists a $q \in E_2$ such that $F(p) = G(q)$, and for every $q \in E_2$, there exists a $p \in E_1$ such that $G(q) = F(p)$.

Definition 2.4. [18] Let U be a universe of discourse.

i. A unary operation Φ on soft sets is a mapping over U that associates a soft set (F, E_1) with another soft set (G, E_2), i.e., $\Phi(F, E_1) = (G, E_2)$. Moreover, Φ is deemed correct if $(F, E_1) \cong (G, E_2) \Rightarrow \Phi(F, E_1) \cong \Phi(G, E_2)$.

ii. A binary operation Θ on soft sets is a mapping that assigns to any two soft sets (F, E_1) and (G, E_2) over U, a novel soft set (H, E_3), i.e., $\Theta((F, E_1), (G, E_2)) = (H, E_3)$. Moreover, Θ is considered correct if $(F_1, E_1) \cong (F_2, E_2) \wedge (G_1, E_3) \cong (G_2, E_4) \Rightarrow \Theta((F_1, E_1), (G_1, E_3)) \cong \Theta((F_2, E_2), (G_2, E_4))$.

iii. A relationship Ω between two soft sets (F, E_1) and (G, E_2) is a mapping assigning the values 0 or 1 to $\Omega((F, E_1), (G, E_2))$. If $\Omega((F, E_1), (G, E_2)) = 1$, then it is denoted by $(F, E_1)\Omega(G, E_2)$. Moreover, Ω is correct if $(F_1, E_1) \cong (F_2, E_2) \wedge (G_1, E_3) \cong (G_2, E_4) \Rightarrow \Omega((F_1, E_1), (G_1, E_3)) = \Omega((F_2, E_2), (G_2, E_4))$.

Definition 2.5. [18] The complement of a soft set (F, E) is defined as a unary operation, denoted by $(F, E)^c = (F^c, E)$, where $F^c(p) = U \setminus F(p)$, for all $p \in E$.

Definition 2.6. [18] Let (F, E_1) and (G, E_2) be two soft sets over a universe U.

i. The intersection of (F, E_1) and (G, E_2) is a binary operation denoted by $(F, E_1)\widetilde{\cap}(G, E_2) = (H, E_1 \times E_2)$, where $H(p, q) = F(p) \cap G(q)$, for all $(p, q) \in E_1 \times E_2$.

ii. The union of (F, E_1) and (G, E_2) is a binary operation denoted by $(F, E_1)\widetilde{\cup}(G, E_2) = (K, E_1 \times E_2)$, where $K(p, q) = F(p) \cup G(q)$, for all $(p, q) \in E_1 \times E_2$.

Definition 2.7. [18] Let (F, E_1) and (G, E_2) be two soft sets over a universe U.

i. (F, E_1) is termed an internal approximation of (G, E_2), denoted by $(F, E_1)\widetilde{\subseteq}(G, E_2)$, if, for all $q \in E_2$ such that $G(q) \neq \emptyset$, there exists a $p \in E_1$ that satisfies $\emptyset \neq F(p) \subseteq G(q)$.

ii. (F, E_1) is termed an external approximation for (G, E_2), denoted by $(F, E_1)\widetilde{\supseteq}(G, E_2)$, if, for all $q \in E_2$ such that $G(q) \neq \emptyset$, there exists a $p \in E_1$ which satisfies $U \neq F(p) \supseteq G(q)$.

Definition 2.8. [18] Let (F, E) be a soft set over a universe U.

i. (F, E) is classified as a null soft set, denoted by $\widetilde{\emptyset}$, if and only if $APP(F, E) = \{\emptyset\}$.

ii. (F, E) is classified as an absolute soft set, denoted by $\widetilde{\mathbb{U}}$, if and only if $APP(F, E) = \{U\}$.

Definition 2.9. [4] Let $U \neq \emptyset$. Then, a function $\mu : P(U) \to P(U)$ is called a Kuratowski closure operator if it satisfies the following properties, for all $X, Y \in P(U)$:

(K0) $\mu(\emptyset) = \emptyset$ (groundedness)

(K1) $X \subseteq Y \Rightarrow \mu(X) \subseteq \mu(Y)$ (isotonicity)

(K2) $X \subseteq \mu(X)$ (expansiveness)

(K3) $\mu(X \cup Y) \subseteq \mu(X) \cup \mu(Y)$ (sub-additivity)

(K4) $\mu(\mu(X)) = \mu(X)$ (idempotence)

Definition 2.10. [4,5] Let $U \neq \emptyset$ and $\mu : P(U) \to P(U)$ be a function.

i. A topological space (U, μ) can be defined by a Kuratowski closure operator μ, where closed sets are the sets $X \subseteq U$ satisfying the condition $\mu(X) = X$.

ii. An ordered pair (U, μ) is called a closure space such that μ satisfies the conditions (K0)-(K3), where μ is called a closure operator.

Lemma 2.11. [5] Let (U, μ) be a closure space. Then, the following hold:

i. $\mu(X) \cup \mu(Y) \subseteq \mu(X \cup Y)$, for all $X, Y \in P(U)$

ii. $\mu(X \cap Y) \subseteq \mu(X) \cap \mu(Y)$, for all $X, Y \in P(U)$

Definition 2.12. [11–13] Let $U \neq \emptyset$ and $\mu : P(U) \to P(U)$ be a function. Then, the ordered pair (U, μ) is called an isotonic space if μ satisfies the conditions (K0) and (K1).

Example 2.13. Let $U = \{a, b\}$ and consider the function $\mu : P(U) \to P(U)$ defined by $\mu(\emptyset) = \emptyset$, $\mu(\{a\}) = \{b\}$, $\mu(\{b\}) = \{b\}$, and $\mu(U) = U$. It can be observed that μ is grounded. Furthermore, it is isotonic because $\mu(A) \subseteq \mu(B)$, for all $A \subseteq B$. Therefore, (U, μ) is an isotonic space. It must be noted that μ is not a Kuratowski closure operator and not a closure operator because $\{a\} \nsubseteq \mu(\{a\})$.

3. Soft Sets over Isotonic Spaces

In this section, we provide some results based on the relationship between soft sets and isotonic and closure spaces. Unless otherwise claimed, we consider the parameter set E for all soft sets.

Definition 3.1. Let (U, μ) be an isotonic space and (F, E) and (G, E) be soft sets over U. If $\mu \circ F = G$, i.e., $(\mu \circ F)(p) = G(p)$, for all $p \in E$, then (G, E) is called an isotonic extension of (F, E) and denoted by $(\mu F, E)$.

This definition allows for to extend a soft set by incorporating external information, such as indirect contact or inferred proximity, through the isotonic operator. It can be observed that Definition 3.1

yields the following commutative diagram:

$$\begin{tikzcd} E \arrow[r, "F"] \arrow[dr, dotted, "\mu F"'] & P(U) \arrow[d, "\mu"] \\ & P(U) \end{tikzcd}$$

Furthermore, since the operator μ defines a new soft set, it functions as a unary operation among soft sets. Consequently, the following proposition is derived.

Proposition 3.2. Let (U, μ) be an isotonic space and (F, E) and (G, E) be soft sets over U. If $(F, E) \cong (G, E)$, then $(\mu F, E) \cong (\mu G, E)$.

Proof. Let (U, μ) be an isotonic space, (F, E) and (G, E) be soft sets over U, and $(F, E) \cong (G, E)$. Then, there exist $p, q \in E$ such that $F(p) = G(q)$. Thus, $\mu(F(p)) = \mu(G(q))$. Hence, there exist $p, q \in E$ such that $(\mu \circ F)(p) = (\mu \circ G)(q)$. Therefore, $APP(F, E) = APP(G, E)$ and thus $(\mu F, E) \cong (\mu G, E)$. □

It should be noted that the converse of this proposition is not always true. For example, let $U = \{1, 2, 3\}$ and $E = \{p\}$ and define the soft sets (F, E) and (G, E) such that $F(p) = \{1\}$ and $G(p) = \{1, 2\}$. Consider the isotonic operator $\mu : P(U) \to P(U)$ given by $\mu(X) = \begin{cases} X \cup \{2\}, & X = \{1\} \\ X, & \text{otherwise} \end{cases}$. Since $\mu(\{1\}) = \{1, 2\}$ and $\mu(\{1, 2\}) = \{1, 2\}$, then $(\mu \circ F)(p) = \mu(F(p)) = \mu(\{1\}) = \{1, 2\}$ and $(\mu \circ G)(p) = \mu(G(p)) = \mu(\{1, 2\}) = \{1, 2\}$ and thus $(\mu F, E) \cong (\mu G, E)$ and $(F, E) \ncong (G, E)$.

Corollary 3.3. Let (U, μ) be an isotonic space. Then, the isotonic extension operation Φ on soft sets over U defined by $\Phi(F, E) = (\mu F, E)$ is correct.

Proposition 3.4. Let (U, μ) be an isotonic space. Then, the null soft set $\widetilde{\emptyset}$ is preserved the under isotonic extension in Corollary 3.3.

Proof. Let (U, μ) be an isotonic space and $\widetilde{\emptyset} = (F, E)$. Then, $APP\,(F, E) = \{\emptyset\}$. By the property of groundedness, $\mu(\emptyset) = \emptyset$. Thus, $APP(\mu F, E) = \{\emptyset\}$. Hence, $\Phi\left(\widetilde{\emptyset}\right) = \widetilde{\emptyset}$. □

This result implies that the isotonic operator preserves the structure of complete absence (null soft set), ensuring no unintended elements are added during extension.

Proposition 3.5. Let (U, μ) be an isotonic space. Then, the absolute soft set $\widetilde{\mathbb{U}}$ is preserved under isotonic extension if and only if $\mu(U) = U$.

Proof. Let (U, μ) be an isotonic space and $(F, E) = \widetilde{\mathbb{U}}$.

$(\Rightarrow)$: Suppose that the absolute soft set (F, E) is preserved under isotonic extension. Then, $APP(F, E) = \{U\} = \{\mu(U)\} = APP(\mu F, E)$. Thus, $\mu(U) = U$.

$(\Leftarrow)$: Suppose that $\mu(U) = U$. Then, $\mu(F(p)) = \mu(U) = U$, for all $p \in E$. Thus, $APP(\mu F, E) = \{U\}$. Hence, $\Phi(F, E) = (F, E)$. Therefore, the absolute soft set (F, E) is preserved under isotonic extension. □

3.1. Structural Properties of Isotonic Extensions

In this subsection, we provide structural properties of isotonic extensions. These properties collectively show that isotonic extension behaves structurally consistently, preserving logical relations among soft sets and enabling risk propagation mechanisms.

Proposition 3.6. Let (U,μ) be an isotonic space, (F,E) and (G,E) be two soft sets over U, and $(H,E\times E)=(F,E)\widetilde{\cup}(G,E)$. Then, the following hold:

i. $(\mu F,E)\widetilde{\cup}(\mu G,E)\widetilde{\subseteq}(\mu H,E\times E)$

ii. If μ is preserved under the union operation, then $(\mu H,E\times E)=(\mu F,E)\widetilde{\cup}(\mu G,E)$

The proof can be observed from Lemma 2.11 (*i*).

Proposition 3.7. Let (U,μ) be an isotonic space, (F,E) and (G,E) be two soft sets over U, and $(H,E\times E)=(F,E)\widetilde{\cap}(G,E)$. Then,

i. $(\mu H,E\times E)\widetilde{\subseteq}(\mu F,E)\widetilde{\cap}(\mu G,E)$

ii. If μ is preserved under the intersection operation, then $(\mu H,E\times E)=(\mu F,E)\widetilde{\cap}(\mu G,E)$

The proof can be observed from Lemma 2.11 (*ii*).

It is important to note that the complement of the isotonic extension of a soft set may not be equal to the isotonic extension of the complement of the soft set. Indeed, consider the soft set (F,E), where $U=\{1,2,3\}$, $E=\{p\}$, and $F(p)=\{1\}$. Let the isotonic operator $\mu:\mathcal{P}(U)\to\mathcal{P}(U)$ be defined by

$$\mu(X)=\begin{cases} X\cup\{2\}, & X=\{1\} \\ X, & \text{otherwise} \end{cases}$$

Since $F^c(p)=U\setminus F(p)=\{2,3\}$, then $(\mu\circ F^c)(p)=\mu(\{2,3\})=\{2,3\}$. Furthermore, $(\mu\circ F)(p)=\mu(\{1\})=\{1,2\}$ and $(\mu\circ F)^c(p)=U\setminus\{1,2\}=\{3\}$. Therefore, $(\mu\circ F^c)(p)\neq(\mu\circ F)^c(p)$.

Proposition 3.8. Let (U,μ) be an isotonic space. If $(F,E)\widetilde{\subseteq}(G,E)$, then $(\mu F,E)\widetilde{\subseteq}(\mu G,E)$.

Proof. Let (U,μ) be an isotonic space and $(F,E)\widetilde{\subseteq}(G,E)$. According to the definition of internal approximation, for all $q\in E$, there exists a $p\in E$ such that $\emptyset\neq F(p)\subseteq H(q)$. Since μ is grounded and isotonic, $\mu(F(p))\subseteq\mu(H(q))$. Thus, $(\mu F,E)\widetilde{\subseteq}(\mu H,E)$. □

Proposition 3.9. Let (U,μ) be an isotonic space. If the closure operator is extensive, then $(F,E)\widetilde{\subseteq}(\mu F,E)$.

Proof. Let (U,μ) be an isotonic space and μ be extensive. Then, for all $p\in E$ such that $(\mu F)(p)\neq\emptyset$, $\emptyset\neq F(p)\subseteq(\mu F)(p)$. Thus, (F,E) is an internal approximation of $(\mu F,E)$, i.e., $(F,E)\widetilde{\subseteq}(\mu F,E)$. □

Definition 3.10. Let (U,μ) be an isotonic space and (F,E) be a soft set over U. If $(\mu F,E)=(F,E)$, then (F,E) is called a μ-closed soft set over U.

Proposition 3.11. Let (U,μ) be an isotonic space and (F,E) be a soft set over U. Then, (F,E) is a μ-closed soft set over U if and only if for all $p\in E$, $F(p)$ is closed in the isotonic space (U,μ).

Proof. Let (U,μ) be an isotonic space and (F,E) be a soft set over U.

($\Rightarrow$): Suppose that (F,E) is a μ-closed soft set over U. Then, $(\mu F,E)=(F,E)$. Thus, $\mu(F(p))=F(p)$, for all $p\in E$. Hence, $F(p)$ is closed in the isotonic space (U,μ), for all $p\in E$.

($\Leftarrow$): Suppose that $F(p)$ is closed in the isotonic space (U,μ), for all $p\in E$. Then, $\mu(F(p))=F(p)$, for all $p\in E$. Hence, $(\mu F,E)=(F,E)$. Therefore, (F,E) is a μ-closed soft set over U. □

Proposition 3.12. Let (U,μ) be an isotonic space and (F,E) and (G,E) be two soft sets over U. If (F,E) and (G,E) are μ-closed soft sets over U and μ is extensive, then $(F,E)\widetilde{\cap}(G,E)$ is a μ-closed soft set over U.

Proof. Let (U, μ) be an isotonic space, (F, E) and (G, E) be two soft sets over U, (F, E) and (G, E) be μ-closed soft sets over U, μ be extensive, and $(F, E)\tilde{\cap}(G, E) = (H, E \times E)$. By Proposition 3.7 (i), $(\mu H, E \times E)\tilde{\subseteq}(\mu F, E)\tilde{\cap}(\mu G, E)$. Since (F, E) and (G, E) are μ-closed soft sets over U, $(\mu F, E) = (F, E)$ and $(\mu G, E) = (G, E)$, which implies $(\mu H, E \times E)\tilde{\subseteq}(H, E \times E)$. Moreover, since μ is extensive, $H(p, q) \subseteq \mu(H(p, q))$, for all $(p, q) \in E \times E$. Thus, $(\mu H, E \times E) = (H, E \times E)$. Hence, $(F, E)\tilde{\cap}(G, E)$ is a μ-closed soft set over U. □

Proposition 3.13. Let (U, μ) be an isotonic space and (F, E) and (G, E) be two soft sets over U. If (F, E) and (G, E) are μ-closed soft sets over U and μ is sub-additive, then $(F, E)\tilde{\cup}(G, E)$ is a μ-closed soft sets over U.

The proof can be observed from Proposition 3.6 and the property of sub-additivity.

4. An Application of Isotonic Extensions of Soft Sets to Medical Diagnosis

To demonstrate the practical utility of isotonic extensions of soft sets, we consider an example involving infectious disease surveillance in a hospital setting. The goal is to detect symptomatic patients and those with potential exposure risks. This section details a novel approach to infectious disease surveillance through soft set-based risk modeling.

Algorithm 1 Core Algorithm Implemented in the Application

Input

1. The universal set U is defined as the collection of all patients or individuals.

2. The parameter set E whose elements represent symptoms or risk indicators.

3. The initial soft set (F, E), where the observed individuals are considered for each parameter.

4. The exposure rule $\mathcal{R} = \{(C_i, a_i) \mid C_i \subseteq U, a_i \in U\}$.

Output

1. The isotonic extension $(\mu F, E)$.

2. Risk frequencies obtained by the function $risk : U \to \mathbb{N}$ defined by $risk(h) = |\{p \in E \mid h \in (\mu F)(p)\}|$.

3. Priority ranking descending by risk scores.

Steps

Step 1. Define the isotonic operator $\mu(X) = X \cup \{a_i \mid (C_i, a_i) \in \mathcal{R} \text{ and } X \cap C_i \neq \emptyset\}$.

Step 2. Obtain the isotonic extension $(\mu F, E)$.

Step 3. Calculete risk frequencies, for each individual.

Step 4. Apply the prioritization rule, i.e., rank individuals in descending order according to their risk frequencies. Determine arbitrary priority for equal risk scores and consider additional criteria.

Following a respiratory disease outbreak, a metropolitan hospital faces a subtle yet significant challenge: identifying not only patients who exhibit overt symptoms but also asymptomatically infected individuals who may be silently transmitting the disease within the hospital environment. Conventional diagnostic methodologies predominantly focus on symptomatic individuals; however, the transmission dynamics of infectious diseases often transcend such clinical presentations.

In this paper, we introduce an innovative soft set-based decision support model that extends beyond the analysis of observable symptoms by incorporating exposure-based information through the application of an isotonic operator. The central objective is to develop a mathematical framework, utilizing soft sets enriched with contact-tracing semantics, to effectively model latent infection risk within a hospital milieu.

Consider patients who present without symptoms yet have a documented history of sharing rooms or interacting with confirmed cases. Their seemingly benign status raises a critical question: Are they truly risk-free?

4.1. Soft Set Modeling of Symptom Data

We define the universal set of patients as follows:

$$U = \{h_1, h_2, h_3, h_4, h_5, h_6\}$$

Let the set of pertinent symptoms be represented by $E = \{p_1, p_2, p_3\}$, where p_1: Fever, p_2: Cough, and p_3: Sore throat. Consider the soft set (F, E) defined by

$$F(p_1) = \{h_1, h_2\}, \quad F(p_2) = \{h_1, h_3\}, \quad \text{and} \quad F(p_3) = \{h_5\}$$

Moreover, (F, E) can also be represented as shown in Table 1:

Table 1. A representation of (F, E)

Pertinent symptoms	**Patients exhibiting the symptom**
Fever (p_1)	$\{h_1, h_2\}$
Cough (p_2)	$\{h_1, h_3\}$
Sore throat (p_3)	$\{h_5\}$

4.2. Integration of Related Contact Data (Exposure (Semantics) Rules)

To enhance the model's granularity, we incorporate hospital contact data:

i. Patient h_4 shared a room with patients h_1 and h_2.

ii. Patient h_6 had close contact with patient h_5.

These documented connections serve as the basis for the subsequent application of the considered isotonic operator. This relationship is illustrated in Figure 1:

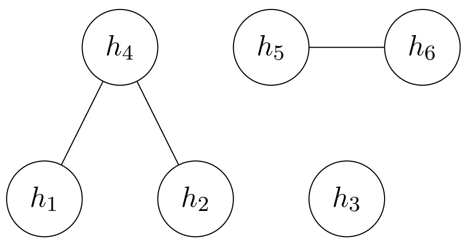

Figure 1. Contact network among patients

Thus, the exposure rule $\mathcal{R}$ is obtained as follows:

$$\mathcal{R} = \{(\{h_1, h_2\}, h_4), (\{h_5\}, h_6)\}$$

where $C_1 = \{h_1, h_2\}$ and $C_2 = \{h_5\}$; $a_1 = h_4$ and $a_2 = h_6$.

4.3. Isotonic Extension Operator: Capturing Exposure Risk

Consider the function $\mu : P(U) \to P(U)$ defined by

$$\mu(X) = X \cup \{a_i \mid (C_i, a_i) \in \mathcal{R} \text{ and } X \cap C_i \neq \emptyset\}$$

Then, μ is an isotonic operator on U. This operator is designed to capture the risk associated with indirect exposure and embodies a proactive infection control strategy by identifying individuals at elevated risk due to their proximity to confirmed cases. The isotonic extension effectively models the amplification of infection risk based on spatial and social proximity within the hospital environment. The isotonic extension of the soft set can be illustrated in Figure 2.

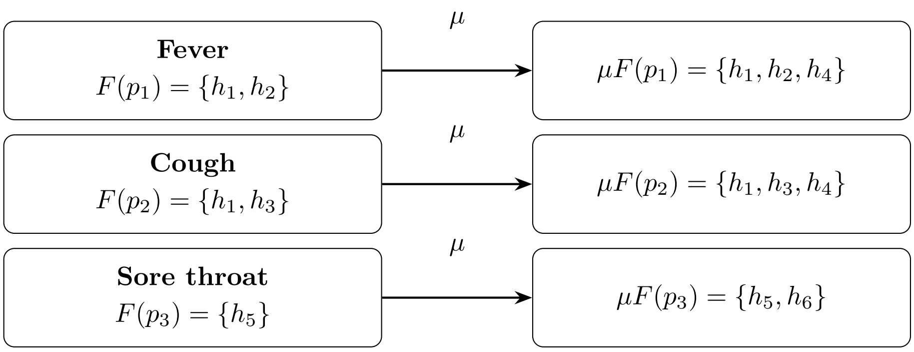

Figure 2. Diagram of the isotonic extension of (F, E)

4.4. Isotonic Extension of the Soft Set and Risk Frequencies

Applying the isotonic operator μ to each symptom-based patient set yields the following sets:

$$\mu F(p_1) = \{h_1, h_2, h_4\}, \quad \mu F(p_2) = \{h_1, h_3, h_4\}, \quad \text{and} \quad \mu F(p_3) = \{h_5, h_6\}$$

From these sets, we compute risk frequencies for each patient (see Table 2):

Table 2. Risk Frequencies of Patients

Patients	Frequencies
h_1	2
h_2	1
h_3	1
h_4	2
h_5	1
h_6	1

The visual representation illustrating the risk frequencies for each patient is provided in Figure 3:

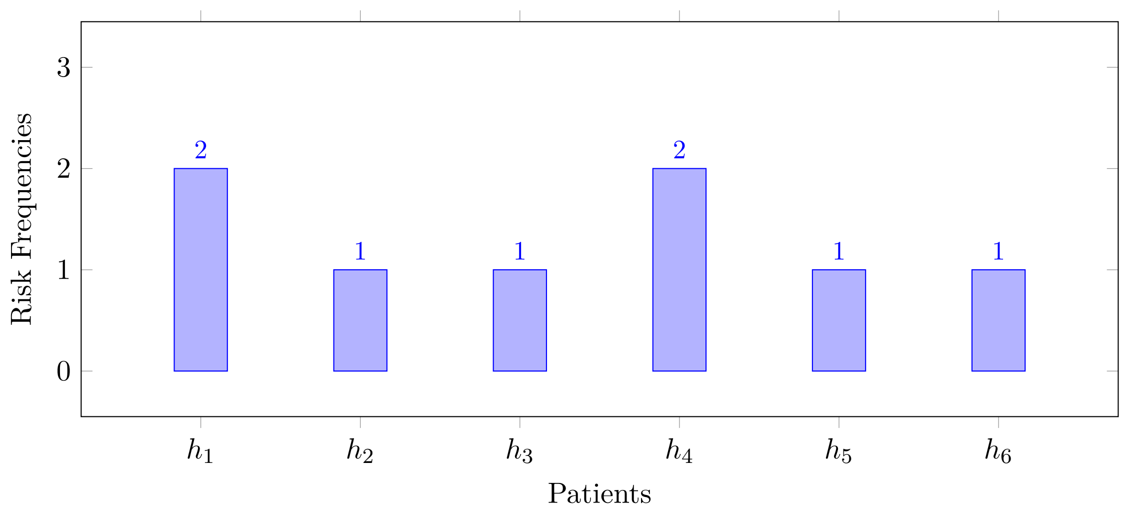

Figure 3. Risk frequencies for each patient computed from $(\mu F, E)$

4.5. Decision Outcome: Prioritization for Intervention

Employing a straightforward decision rule, prioritize the patient who appears in the highest number of extended symptom sets, the analysis reveals:

i. Patients h_1 and h_4 appear in two symptom sets of the isotonic extension.

ii. Patient h_1 is symptomatic.

iii. Patient h_4 is asymptomatic but identified as high-risk due to exposure.

Consequently, the proposed system recommends that both patients should be prioritized for isolation and further diagnostic testing. This simple yet powerful rule highlights how isotonic extension enhances the soft set model: it successfully identifies asymptomatic individuals (e.g., h_4) who, despite not presenting symptoms, pose a risk due to documented exposure. Traditional soft set models would fail to flag such individuals.

5. Conclusion

This paper presents a theoretical and applied framework for extending soft sets via isotonic operators. The proposed approach addresses a key limitation in classical soft set theory: the inability to represent indirect or latent information such as exposure risk. The study establishes a consistent and correct unary operation that preserves equivalence on soft sets, the null soft set, and the absolute soft set under certain conditions by introducing and formalizing the isotonic extension of a soft set. From a practical standpoint, the application to hospital-based infection surveillance demonstrates the real-world relevance of isotonic extensions of soft sets. The model identifies high-risk individuals not based solely on observed symptoms but also on indirect contact information, an essential advancement in decision-making under uncertainty. Future research may explore further generalizations using parameter-dependent isotonic operators or the integration of temporal dynamics, enabling real-time risk modeling in evolving systems.

Author Contributions

The author read and approved the final version of the paper.

Conflicts of Interest

The author declares no conflict of interest.

Ethical Review and Approval

No approval from the Board of Ethics is required.

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Journal of New Theory
dergipark.org.tr/tr/pub/jnt
Open Access

ISSN: 2149-1402 (51) (2025) 65-75 doi.org/10.53570/jnt.1696750

Sharp B-Maximal Function Estimates and Boundedness for Some Integral Operators to the Inequalities

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Article Info
Received: 10 May 2025
Accepted: 26 Jun 2025
Published: 30 Jun 2025
Research Article

Abstract— In this paper, we first establish the relation between B-maximal and sharp B-maximal functions generated by the generalized translation operator connected with the Laplace-Bessel differential operator. We then prove some sharp B-maximal function estimates and present an application using these sharp estimates to study singular integral operators. We finally obtain the boundedness of the Littlewood-Paley g-function related to the Laplace-Bessel differential operator on generalized B-Morrey spaces.

Keywords — *B-Morrey space, Laplace-Bessel differential operator, Littlewood-Paley g-function, sharp B-maximal function*

Mathematics Subject Classification (2020) 42B25, 42B35

1. Introduction

In 1938, Morrey [1] introduced the classical Morrey spaces, an extension of the classical Lebesgue spaces. In Morrey spaces, numerous researchers have studied the boundedness and compactness properties of maximal and singular integral operators. Due to the applications in the study of Morrey spaces, this space has aroused widespread interest and curiosity [2]. Thus far, many papers have focused on various Morrey spaces. They extended Morrey spaces to different settings. For example, Guliyev [3, 4], Sawano [5], and Nakai [6] introduced the generalized Morrey spaces. Moreover, they investigated the similar boundedness problems of maximal and singular integral operators in these spaces.

Additionally, weighted inequalities are crucial in Fourier analysis and have numerous applications in solving boundedness problems for certain integral operators. In particular, weight theory is critical in studying boundary value problems for the Laplace equation on Lipschitz regions. Muckenhoupt's characterization provides the foundation for defining the class A_p and developing weighted inequalities, ensuring that the Hardy–Littlewood maximal operator maps the weighted Lebesgue space $L^p(w) \equiv L^p(\mathbb{R}^n, w)$ onto itself.

The study of the Littlewood-Paley g-theory enjoys a natural motivation and great interest. Many works and topics have been studied. To study the dyadic decomposition of Fourier series, Littlewood and Paley [7–9] introduced the g-function of one dimension. The function g is basic in the Littlewood-Paley theory of Fourier series [10]. Littlewood and Paley proved that

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$$A_p\|f\|_p \leq \|g(f)\|_p \leq B_p\|f\|_p \tag{1.1}$$

where on the left side of the above inequality, it was assumed that $\int\limits_0^{2\pi} f(\theta)d\theta = 0$. Later, Stein [11] defined the following n-dimensional form of the Littlewood-Paley g-function and obtained the same norm inequality as (1.1),

$$g(f)(x) = \left(\int\limits_0^{\infty} t|\nabla u(x,t)|^2 dt\right)^{1/2}$$

where $u(x,t) = P_t * f(x)$ denotes the Poisson integral of f. Afterward, many mathematicians have studied Littlewood-Paley g-function of higher dimensions with more general kernels [12–16].

Over the past 30 years, considerable developments have been made to extend the classical Littlewood-Paley g-function to some different settings. Akbulut et al. [17] are interested in problems related to weighted inequalities for the g-Littlewood-Paley functions associated with the Laplace-Bessel differential operators. However, in [18], they establish some sharp maximal function estimates for certain Toeplitz-type operators (including the Littlewood-Paley operator) associated with some fractional integral operators with a general kernel. Moreover, Lerner [19] has established sharp weighted estimates for any convolution Calderón-Zygmund operator, for all $1 < p \leq 3/2$ and $3 \leq p < \infty$.

Highly inspired by [12–19], in this paper, we are interested in problems related to weighted inequalities for the Littlewood-Paley g-functions connected with the Laplace-Bessel differential operators Δ_ν. Moreover, we are motivated by the work of Akbulut et al. [20] in which there is a different setting of the Littlewood-Paley g-function has different settings. We obtained a similar Fefferman-Stein boundedness result for this operator on generalized B-Morrey spaces by utilizing B-sharp maximal functions related to the Laplace-Bessel operator.

The rest of the paper is organized as follows: Section 2 presents the basic notations needed throughout this paper. Section 3 concerns maximal functions related to the Laplace-Bessel differential operator and its properties. Section 4 defines the integral operator g. The last section indicates the boundedness of this integral operator on generalized B-Morrey spaces.

2. Preliminaries

This section presents the basic notations and concepts to be required. Throughout this paper, let Q denote a cube of $\mathbb{R}^n_+$, the upper half region of $\mathbb{R}^n$, n dimensional Euclidean space with sides parallel to the axes and $x = (x', x_n)$, $x' = (x_1, \ldots, x_{n-1}) \in \mathbb{R}^{n-1}$. Moreover, let $E(x,t) = \{y \in \mathbb{R}^n_+ \ ; \ |x-y| < t\}$ and $E(x,t)^c = \mathbb{R}^n_+ \backslash E(x,t)$. If E is a Lebesgue measurable set, then χ_E is the characteristic function of E, and the weighted Lebesgue measure of E denoted by $|E|_\nu$, where $|E|_\nu = \int\limits_E x_n^\nu dx$ such that $\nu > 0$. Besides, let $|E(0,r)|_\nu = w(n,\nu) r^{n+\nu}$, where

$$w(n,\nu) = \int\limits_{E(0,1)} x_n^\nu dx = \frac{\pi^{\frac{n-1}{2}}\Gamma\left(\frac{\nu+1}{2}\right)}{2\Gamma\left(\frac{n+\nu-2}{2}\right)}$$

The weight w is a nonnegative locally integrable function on $\mathbb{R}^n_+$ that takes values in $(0, +\infty)$ almost everywhere.

Let the class $A_{p,\nu}$ consist of those weights w for which

$$\left(\frac{1}{|E|_\nu}\int\limits_E w(x)x_n^\nu\,dx\right)^{1/p}\left(\frac{1}{|E|_\nu}\int\limits_E w(x)^{-p'/p}\,x_n^\nu dx\right)^{1/p'}\le C$$

Here, p' is the dual of p such that $\frac{1}{p}+\frac{1}{p'}=1$, for $1<p<\infty$. The class $A_{1,\nu}$ is defined replacing the above inequality by

$$\frac{1}{|E|_\nu}\int\limits_E w(x)\,x_n^\nu dx\le C\operatorname*{ess\,inf}_{x\in E} w(x)$$

for every ball $E\subseteq\mathbb{R}_+^n$. Further, let $A_\infty=\bigcup\limits_{1\le p<\infty}A_{p,\nu}$. It is well known from [21] that if $w\in A_{p,\nu}$ with $1\le p<\infty$ (or $w\in A_\infty$), then w satisfies the doubling condition; that is, for all ball E, there exists an absolute constant $C>0$ such that $w(2E)\le C\,w(E)$. Furthermore, if $w\in A_\infty$, then for all ball B and all measurable subset E of a ball B, there exists a number $\delta>0$ independent of E and B such that $\frac{w(E)}{w(B)}\le C\left(\frac{|E|}{|B|}\right)^\delta$ [21]. Given a weight function w on $\mathbb{R}_+^n$, the weighted Lebesgue space $L_{p,w,\nu}(\mathbb{R}_+^n)$, for $1\le p<\infty$, is defined as the set of all functions f for which

$$\|f\|_{L_{p,w,\nu}}:=\left(\int\limits_{\mathbb{R}_+^n}|f(x)|^p w(x)\,x_n^\nu dx\right)^{1/p}<\infty$$

In addition, let $WL_{p,w,\nu}(\mathbb{R}^n)$, $1\le p<\infty$, denote the weighted weak Lebesgue space consisting of all measurable functions f such that

$$\|f\|_{WL_{p,w,\nu}}:=\sup_{\lambda>0}\lambda\Big[w\big(\{x\in\mathbb{R}^n:|f(x)|>\lambda\}\big)\Big]^{1/p}<\infty$$

Additionally, the generalized translate operator T^y is defined by

$$T^y f(x)=c_\nu\int\limits_0^\pi\dots\int\limits_0^\pi f\left(x'-y',(x_n,y_n)_\theta\right)\,d\mu(\theta)$$

where $c_\nu=\dfrac{\pi^{-\frac{1}{2}}\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)}$, $(x_n,y_n)_\theta=\sqrt{x_n^2-2x_ny_n\cos\theta+y_n^2}$, and $d\mu\left(\theta\right)=\sin^{\nu-1}\theta\;\,d\theta$.

The operator T^y acts from $L_p(\mathbb{R}_+^n,d\mu)$ to $L_p(\mathbb{R}_+^n,d\mu)$ and satisfies the conditions $||T^yf||_p<||f||_p$, $T^y1=1$, and L_p-boundedness. We remark that the generalized translate operator T^y is closely related to the Laplace-Bessel differential operator Δ_ν defined by

$$\Delta_\nu=\sum_{i=1}^{n-1}\frac{\partial^2}{\partial x_i^2}+B\quad\text{ where }\quad B=\frac{\partial^2}{\partial x_n^2}+\frac{\nu}{x_n}\frac{\partial}{\partial x_n}\quad\text{ such that }\quad\nu>0$$

For $n=1$ and $n>1$, see [22–26]. The generalized translate operator T^y generates the corresponding B-convolution

$$(f\otimes g)(x)=\int\limits_{\mathbb{R}_+^n}f(y)T^yg(x)y_n^\nu dy$$

for which the following Young inequality holds:

$$\|f\otimes g\|_{L_{r,\nu}}\le\|f\|_{L_{p,\nu}}\|g\|_{L_{q,\nu}}$$

such that $1\le p,q\le r\le\infty$ and $\frac{1}{p}+\frac{1}{q}=\frac{1}{r}+1$.

Lemma 2.1. [27] For all $x \in \mathbb{R}_+^n$, the following equality holds:

$$\int_{E_t} T^y g(x) y_n^\nu dy = \int_{E((x,0),t)} g\left(z', \sqrt{z_n^2 + \overline{z}_n^2}\right) d\mu(z', \overline{z_n})$$

where $E((x,0),t) = \{(z, \overline{z_n}) \in \mathbb{R}^n \times (0,\infty) : |(x - z, \overline{z_n})| < t\}$.

Lemma 2.2. [27] For all $x \in \mathbb{R}_+^n$, the following equality holds:

$$\int_{\mathbb{R}_+^n} T^y g(x)\varphi(y) M_\nu \chi_{E_r}(y) y_n^\nu dy = \int_{\mathbb{R}^n \times (0,\infty)} g\left(z', \sqrt{z_n^2 + \overline{z}_n^2}\right) \varphi(z', \overline{z_n}) M_\nu \chi_{E((x,0),r)}(z', \overline{z_n}) d\mu(z', \overline{z_n})$$

Lemmas 2.1 and 2.2 can be obtained via the following substitutions: $z' = x'$, $z_n = y_n \cos\theta$, and $\overline{z_n} = y_n \sin\alpha$, where $0 \le \theta < \pi$, $y \in \mathbb{R}_+^n$, and $(z, \overline{z_n}) \in \mathbb{R}^n \times (0,\infty)$. Let $\mathcal{S}'_+ = \mathcal{S}'_+\left(\mathbb{R}_+^n\right)$ denote the topological dual of $\mathcal{S}_+$, the collection of all tempered distributions on $\mathbb{R}_+^n$.

Definition 2.3. [27,28] Let ω be a positive measurable weight function. Then, $\mathcal{M}_{p,\omega,\nu}(\mathbb{R}_+^n)$ denotes the generalized B-Morrey spaces as the set of all locally integrable functions f with finite quasi-norm

$$||f||_{\mathcal{M}_{p,\omega,\nu}} = \sup_{x \in \mathbb{R}_+^n, r>0} \left(\frac{r^{-\frac{n+\nu}{p}}}{\omega(r)} \int_{E(0,r)} T^y[|f(x)|]^p \, y_n^\nu dy \right)^{\frac{1}{p}} < \infty$$

Note that

i. if $\omega(r) = r^{-\frac{n+\nu}{p}}$, then $\mathcal{M}_{p,\omega,\nu}(\mathbb{R}_+^n) \equiv L_\nu^p(\mathbb{R}_+^n)$.

ii. if $\omega(r) = r^{\frac{\lambda - n - \nu}{p}}$ and $0 \le \lambda < n + \nu$, then $\mathcal{M}_{p,\omega,\nu}(\mathbb{R}_+^n) = \mathcal{M}_{p,\lambda,\nu}(\mathbb{R}_+^n)$.

3. B-Maximal Functions

This section includes a modified version of the sharp maximal operator, as introduced by Fefferman and Stein [29]. A variant of the sharp maximal function, namely the sharp B-maximal function $M_\nu^\# f$, associated with the Laplace–Bessel differential operator, was introduced in [30] as follows:

$$M_\nu^\# f(x) = \sup_{x \in Q} \inf_c \frac{1}{|Q|_\nu} \int_Q \left| T^y f(x) - c \right| y_n^\nu \, dy \approx \sup_{x \in Q} \frac{1}{|Q|_\nu} \int_Q |T^y f(x) - f_Q| y_n^\nu \, dy$$

Here, $f_Q = \frac{1}{|Q|_\nu} \int_Q T^y f(x) y_n^\nu \, dy$ denotes the average of f over E. Moreover, for $\delta > 0$,

$$M_{\nu,\delta}^\# f(x) = M_\nu^\# \left(|f|^\delta\right)(x)^{\frac{1}{\delta}}$$

which is useful for the sharp B-maximal operator below. We denote the Hardy-Littlewood maximal function, i.e., B-maximal function, by $M_\nu f$, defined as follows [30]:

$$M_\nu f(x) = \sup_{r>0} \frac{1}{|Q|_\nu} \int_Q T^y |f(x)| y_n^\nu \, dy$$

It is well known that the B-maximal function provides control over the mean value of a function concerning any radially decreasing function in $L_{1,\nu}$. Moreover, boundedness estimates for M_ν can be established in the framework of generalized B-Morrey spaces.

Theorem 3.1. [20] Let $1 \le p < \infty$ and ω be positive measurable weight function on $\mathbb{R}_+^n$ satisfying the condition

$$\int_t^\infty \omega(x,\tau)\frac{d\tau}{\tau} \leq C\omega(x,t) \tag{3.1}$$

where the constant C is independent of x and t. Then, for $p > 1$, the maximal operator M_ν is bounded on $\mathcal{M}_{p,\omega,\nu}(\mathbb{R}^n_+)$ and for $p = 1$, it is bounded on $\mathcal{M}_{1,\omega,\nu}(\mathbb{R}^n_+)$.

The proof can be obtained using a similar way to the one employed in the proof of Theorem 4.1 in [20].

The following inequalities, inspired by the work of Fefferman and Stein [29], will be used in the remainder of this section.

Lemma 3.2. [29] Let $1 \leq p < \infty$ and ω be an A_∞ weight. Then, there exists a constant C such that the following inequality holds for every function f for which the left-hand side is finite:

$$\int_{\mathbb{R}^n_+} |M_\nu f(x)|^p \omega(x)\ x_n^\nu\,dx \leq C \int_{\mathbb{R}^n_+} \left|M_\nu^\# f(x)\right|^p \omega(x)\ x_n^\nu\,dx$$

4. Littlewood-Paley g-Function

This section is devoted to defining and investigating the Littlewood-Paley g-function associated with the Laplace-Bessel differential operator in the generalized B-Morrey space $\mathcal{M}_{p,w,\nu}(\mathbb{R}^n_+)$.

Definition 4.1. [31] Let $f \in S(\mathbb{R}^n_+)$, the space of infinitely differentiable functions on $\mathbb{R}^n_+$ that decrease rapidly at infinity together with all their derivatives. For $t > 0$, the Poisson-type integral $u_t(f)$ is defined by

$$u(f)(x,t) = u_t(f)(x) := \int_{\mathbb{R}^n_+} p_t(y)\, T^y f(x)\, y_n^\nu dy$$

where $x \in \mathbb{R}^n_+$ and p_t denotes the Poisson-type kernel given by

$$p_t(x) = p(x,t) = c_\nu \frac{t}{(t^2+|x|^2)^{\frac{n+\nu+1}{2}}} \quad \text{where} \quad c_\nu = \frac{2^{n+\nu}\Gamma\left(\frac{n+\nu+1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)}$$

Recall that the Poisson-type integral $u_t(f)$, defined by $u_t(f) = (p_t \otimes f)(x)$, is a B-convolution type singular integral operator and satisfies the following properties:

Proposition 4.2. [20] Let $f \in S(\mathbb{R}^n_+)$ be a positive function and $p > 1$. Then,

i. $u_t(x) \leq C(t^2+|x|^2)^{\left(\frac{n-\nu+3}{2}\right)}$

ii. $\dfrac{\partial u}{\partial t}(x) \leq Ct^{n-\nu+4}$

iii. $\dfrac{\partial u}{\partial x_i}(x) \leq C(t^2+|x|^2)^{\left(\frac{n-\nu+4}{2}\right)}$, for all $1 \leq i \leq n$

iv. $(p_t * f)(x) \leq M_\nu f(x)$

where $M_\nu f$ is the B-maximal function.

Definition 4.3. [20] Let $f \in S(\mathbb{R}^n_+)$. Then, a g-function associated with the Laplace-Bessel differential operators is defined by

$$g(f)(x) = \left(\int_0^\infty |\nabla u_t(x)|^2\, t\ dt\right)^{1/2} \tag{4.1}$$

where $x \in \mathbb{R}^n_+$, u_t is the Poisson-type integral, and $|\nabla_t(x)|^2 = \left|\frac{\partial u}{\partial t}(x)\right|^2 + \sum\limits_{i=1}^{n} \left|\frac{\partial u}{\partial x_i}(x)\right|^2$.

Theorem 4.4. [20] Let $1 < p < 2$ and ω be positive measurable weight function on $\mathbb{R}^n_+ \times [0,\infty)$ satisfying (3.1). Then, there exists a positive constant $C_{p,\nu}$ such that for all $f \in \mathcal{M}_{p,\omega,\nu}(\mathbb{R}^n_+)$,

$$\|g(f)\|_{\mathcal{M}_{p,\omega,\nu}} \leq C_{p,\nu} \|f\|_{\mathcal{M}_{p,\omega,\nu}}$$

The proof follows from the proof of Theorem 6.1 in [20].

5. Main Results

This section primarily employs similar techniques to those in [32, 33] to derive the following lemmas, which play a crucial role in establishing the boundedness of the Littlewood-Paley g-function. Afterward, it provides the main result, Theorem 5.5. This section considers the Littlewood-Paley g-function with the same kind of kernel as in (4.1). This leads to the following definition:

Definition 5.1. Let $\varepsilon > 0$ and φ be a fixed function satisfying the following properties:

i. $\int\limits_{\mathbb{R}^n_+} \varphi(x) x_n^\nu dx = 0$

ii. $|T^y \varphi(x)| \leq T^y |\varphi(x)| \leq C(1 + |x|)^{-(n+\nu+1)}$

iii. If $2|y| < |x|$, then $|T^y \varphi(x) - \varphi(y)| \leq C|y|^\varepsilon (1 + |x|)^{-(n+\nu+1+\varepsilon)}$

Here, $C > 0$ is a constant independent of x. Thus, the Littlewood-Paley g-function is defined by

$$g(f)(x) = \left(\int\limits_0^\infty |\varphi_t \otimes f(x)|^2 \frac{dt}{t} \right)^{1/2}$$

where φ_t is the dilation of φ given by $\varphi_t(x) = t^{n+\nu} \varphi(\frac{x}{t})$.

Lemma 5.2. Let $1 < p < \infty$ and $0 < D < 2^{n+\nu}$. Then, for any smooth function f for which the left-hand side is finite,

$$||M_\nu f||_{\mathcal{M}_{p,\varphi,\nu}} \leq C ||M_\nu^\# f||_{\mathcal{M}_{p,\varphi,\nu}}$$

Proof. For any cube $Q = Q(x,r)$ in $\mathbb{R}^n_+$, $M_\nu(\chi_Q)(x) \in A_{1,\nu}$ by [34]. It must be noted that $M_\nu(\chi_Q)(x) \leq 1$. By Lemma 3.2, for $f \in \mathcal{M}_{p,\omega,\nu}(\mathbb{R}^n_+)$,

$$\begin{aligned}
\int\limits_Q T^y |M_\nu f(x)|^p(x)\, y_n^\nu\, dy &= \int\limits_{\mathbb{R}^n_+} T^y |M_\nu f(x)|^p(x)\ (\chi_Q)(x)\ y_n^\nu\, dy \\
&\leq \int\limits_{\mathbb{R}^n_+} T^y |M_\nu f(x)|^p(x) M_\nu(\chi_Q)(x)\ y_n^\nu\, dy \\
&= \int\limits_{\mathbb{R}^n_+} |M_\nu f(y)|^p\ T^y M_\nu(\chi_Q)(x)\ y_n^\nu\, dy \\
&\leq C \int\limits_{\mathbb{R}^n_+} |M_\nu^\# f(y)|^p T^y M_\nu(\chi_Q)(x)\ y_n^\nu\, dy \\
&\leq C \left[\int\limits_Q |M_\nu^\# f(y)|^p T^y M_\nu(\chi_Q)(x) y_n^\nu dy \right. \\
&\quad \left. + \sum_{k=0}^\infty \int\limits_{2^{k+1}Q \setminus 2^k Q} |M_\nu^\# f(y)|^p T^y M_\nu(\chi_Q)(x) y_n^\nu\, dy \right]
\end{aligned}$$

$$\leq C\left[\int\limits_{Q} T^y|M_\nu^{\#}f(x)|^p\, y_n^\nu\, dy + \sum_{k=0}^{\infty}\int\limits_{2^{k+1}Q\backslash 2^kQ} T^y|M_\nu^{\#}f(x)|^p\, \frac{|Q|_\nu}{|2^{k+1}Q|_\nu}\, y_n^\nu\, dy\right]$$

$$\leq C\left[\int\limits_{Q} T^y|M_\nu^{\#}f(x)|^p\, y_n^\nu\, dy + \sum_{k=0}^{\infty}\int\limits_{2^{k+1}Q} T^y|M_\nu^{\#}f(x)|^p\, 2^{-k(n+\nu)}\, y_n^\nu\, dy\right]$$

$$\leq C||T^y(M_\nu^{\#}(f))||^p_{\mathcal{M}_{p,\varphi,\nu}}\sum_{k=0}^{\infty}2^{-k(n+\nu)}\varphi(2^{k+1}r)$$

$$\leq C||T^y(M_\nu^{\#}(f))||^p_{\mathcal{M}_{p,\varphi,\nu}}\sum_{k=0}^{\infty}(2^{-(n+\nu)}D)^k\varphi(r)$$

$$\leq C||T^y(M_\nu^{\#})(f)||^p_{\mathcal{M}_{p,\varphi,\nu}}\varphi(r)$$

Thus,

$$\left(\frac{1}{\varphi(r)}\int\limits_{Q} T^y|M_\nu f(x)|^p x_n^\nu dx\right)^{1/p} \leq C\left(\frac{1}{\varphi(r)}\int\limits_{Q} T^y|M_\nu^{\#}f(x)|^p x_n^\nu dx\right)^{1/p}$$

Hence,

$$||M_\nu(f)||_{\mathcal{M}_{p,\varphi,\nu}} \leq C||M_\nu^{\#}(f)||_{\mathcal{M}_{p,\varphi,\nu}}$$

□

Theorem 5.3. [35] Let T be a convolution-type singular integral operator. Then, there exists a constant $C > 0$ for $\omega \in A_{1,\nu}$, for $0 < p < 1$, and for every ball Q such that

$$\int\limits_{Q} |Tf|^p\omega(x)x_n^\nu\, dx \leq C(n,p,[\omega])_{A_{1,\nu}}\omega(E)^{1-p}\left(\int\limits_{\mathbb{R}_+^n}|f(x)|\omega(x)x_n^\nu\, dx\right)^p$$

Lemma 5.4. Let $0 < \delta < 1$. Then, there exists a constant $C > 0$ only depending on δ such that

$$M_\nu^{\#}(g(f))(x) \leq CM_\nu(f)(x) \tag{5.1}$$

where

$$M_\nu^{\#}(g(f))(x) = \left(\sup_{x\in Q}\inf_{c\in\mathbb{R}}\frac{1}{|Q|_\nu}\int\limits_{Q}\Big|T^y|g(f)|^\delta - |c|^\delta\Big|y_n^\nu\, dy\right)^{\frac{1}{\delta}}$$

Proof. Let $x \in \mathbb{R}_+^n$ and Q be a cube containing x. To obtain (5.1), it suffices to show that

$$\left(\frac{1}{|Q|_\nu}\int\limits_{Q}\Big||T^y[g(f)]|^\delta - |c|^\delta\Big|\, y_n^\nu\, dy\right)^{\frac{1}{\delta}} \leq CM_\nu f(x)$$

for some constant c to be determined. Using the inequality $\Big||u|^\delta - |v|^\delta\Big| \leq |u-v|^\delta$ such that $0 < \delta < 1$, define

$$\Big(g(f)\Big)_Q = \frac{1}{|Q|_\nu}\int\limits_{Q} T^y\Big(g(f)\Big)(x)y_n^\nu dy$$

Denote f as $f = f_1 + f_2$, where $f_1 = \lambda_{2Q}$. We will show that $c = \Big(g(f_2)\Big)_Q$ satisfies the required inequality. By the linearity of the Littlewood-Paley operator $g(f)$,

$$\begin{aligned}
\mathcal{M}^{\#}_{\nu,\delta}([g(f)])(x) &:= \left(\sup_Q \mathcal{M}^{\#}_{\nu}\Big([g(f)]^{\delta}\Big)\right)^{\frac{1}{\delta}} \\
&= \left(\frac{1}{|Q|_\nu}\int_Q \left|\left|T^y\Big(g(f)\Big)(x)\right|^{\delta} - |c|^{\delta}\right| y_n^{\nu}\, dy\right)^{\frac{1}{\delta}} \\
&\le C\left[\left(\frac{1}{|Q|_\nu}\int_Q \left|g\Big(T^y f_1(x)\Big)\right|^{\delta} y_n^{\nu}\, dy\right)^{\frac{1}{\delta}} + \left(\frac{1}{|Q|_\nu}\int_Q \left|g\Big(T^y f_2(x)\Big) - c\right|^{\delta} y_n^{\nu}\, dy\right)^{\frac{1}{\delta}}\right] \\
&:= C(I_1 + I_2)
\end{aligned}$$

First, we show the estimate I_1. For $0 < \delta < 1$, applying Theorem 5.3 (Kolmogorov's estimate of Bessel type),

$$I_1 = \left(\frac{1}{|Q|_\nu}\int_Q \left|g\Big(T^y f(x)\Big)\right|^{\delta} y_n^{\nu}\, dy\right)^{\frac{1}{\delta}} \le \frac{1}{|Q|_\nu}|Q|_\nu^{1-\delta}\left[\left(\int_Q |T^y f_1(x)|\, y_n^{\nu}\, dy\right)^{\delta}\right]^{\frac{1}{\delta}} \le CM_\nu f(x)$$

For the estimate of I_2, if $|x-y| > 2r$, by the Jensen inequality and Fubini's theorem for integrals,

$$\begin{aligned}
I_2 &= \frac{1}{|Q|_\nu}\int_Q \left|T^y\left(g(f_2)(x)\right) - \Big(g(f_2)\Big)_Q\right| y_n^{\nu}\, dy \\
&= \frac{1}{|Q|_\nu}\int_Q \left|T^y\Big(gf_2(x)\Big) - \frac{1}{|Q|_\nu}\int_Q T^y\Big(g(f_2)(x)\Big) z_n^{\nu}\, dz\right| y_n^{\nu}\, dy \\
&\le \frac{1}{|Q|_\nu}\int_Q \left(\frac{1}{|Q|_\nu}\int_Q \left|g\Big(T^y f_2(x)\Big) - gf_2(y)\right| z_n^{\nu}\, dz\right) y_n^{\nu}\, dy \\
&\le \frac{1}{|Q|_\nu}\int_Q \frac{1}{|Q|_\nu}\left(\int_Q \left|\int_{\mathbb{R}^n_+}\varphi(\tau)T^{\tau}\Big(T^y f_2(x)\Big)\tau_n^{\nu}\, d\tau - \int_{\mathbb{R}^n_+}\varphi(\tau)T^{\tau} f_2(x)\tau_n^{\nu}\, d\tau\right| z_n^{\nu}\, dz\right) y_n^{\nu}\, dy \\
&\le \frac{1}{|Q|_\nu}\int_Q \frac{1}{|Q|_\nu}\left(\int_Q \left|\int_{\mathbb{R}^n_+}\varphi(\tau)T^{y}T^{\tau} f_2(x)\tau_n^{\nu}\, d\tau - \int_{\mathbb{R}^n_+}\varphi(\tau)T^{\tau} f_2(x)\tau_n^{\nu}\, d\tau\right| z_n^{\nu}\, dz\right) y_n^{\nu}\, dy \\
&\le \frac{1}{|Q|_\nu}\int_Q \frac{1}{|Q|_\nu}\left(\int_Q \left|\int_{\mathbb{R}^n_+}T^{y}\varphi(\tau)T^{\tau} f_2(x)\tau_n^{\nu}\, d\tau - \int_{\mathbb{R}^n_+}\varphi(\tau)T^{\tau} f_2(x)\tau_n^{\nu}\, d\tau\right| z_n^{\nu}\, dz\right) y_n^{\nu}\, dy
\end{aligned}$$

$$\leq \frac{1}{|Q|_\nu} \int\limits_Q \frac{1}{|Q|_\nu} \left(\left| \int\limits_Q \Big[T^y \varphi(\tau) - \varphi(\tau) \Big] T^\tau f_2(x) \tau_n^\nu \, d\tau \right| z_n^\nu \, dz \right) y_n^\nu \, dy$$

$$\leq \frac{1}{|Q|_\nu^2} \int\limits_Q z_n^\nu \, dz \int\limits_Q \left[\int\limits_Q \Big| T^y \varphi(\tau) - \varphi(\tau) \Big| y_n^\nu \, dy \right] T^\tau f_2(x) \tau_n^\nu \, d\tau$$

$$\leq C \frac{1}{|Q|_\tau} \int\limits_Q T^\tau |f_2(x)| \tau_n^\nu \, d\tau \leq C M_\nu f(x)$$

□

Theorem 5.5. Let $1 \leq p < \infty$ and $\omega \in A_{1,\nu}(\mathbb{R}_+^n)$. Then, there exists a positive constant $C > 0$ such that

$$||g(f)||_{\mathcal{M}_{p,\omega,\nu}} \leq C \|f\|_{\mathcal{M}_{p,\omega,\nu}}$$

The proof can be easily observed from Lemmas 5.2 and 5.4.

6. Conclusion

This paper presents a Fefferman-Stein type boundedness result for the Littlewood-Paley g-operator on generalized B-Morrey spaces by utilizing B-sharp maximal functions. The importance and fundamental difference of this paper lie in its use of different transformations (generalized transformations) of the obtained results. Future studies can extend this work to encompass multilinear analogues of the results presented here. B-Sharp maximal function estimates for multilinear singular integrals and their commutators can be constructed. In addition, their boundedness properties can be investigated on generalized B-Morrey spaces.

Author Contributions

All the authors equally contributed to this work. This paper is derived from the second author's undergraduate thesis supervised by the first author. They all read and approved the final version of the paper.

Conflicts of Interest

All the authors declare no conflict of interest.

Ethical Review and Approval

No approval from the Board of Ethics is required.

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Journal of New Theory
dergipark.org.tr/tr/pub/jnt
Open Access

ISSN: 2149-1402 (51) (2025) 76-91 doi.org/10.53570/jnt.1700338

A Multi-Species Keller-Segel Chemotaxis-Competition Model: Global Existence, Boundedness, and Mass Persistence

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Article Info
Received: 15 May 2025
Accepted: 18 Jun 2025
Published: 30 Jun 2025
Research Article

Abstract— This paper investigates the population dynamics of solutions to a parabolic-parabolic-elliptic type of multi-species Keller-Segel chemotaxis system under the Neumann boundary conditions in a smoothly bounded domain. It studies dynamical properties such as L^p-bounds, global existence, global boundedness, and combined mass persistence of solutions for the aforementioned system. Under certain specified parameter conditions, the paper shows that the system admits a unique global classical solution that remains uniformly bounded from above. Furthermore, it establishes that the entire population persists at all times; in other words, this study proves that any globally bounded classical solution maintains a positive lower mass bound.

Keywords — *Keller-Segel system, chemotaxis, global existence, global boundedness, mass persistence*

Mathematics Subject Classification (2020) 35K57, 92C17

1. Introduction

Chemotaxis is the process of directed movement of mobile organisms or cells in response to a chemical gradient. Keller and Segel [1, 2] first established a mathematical model to explain this phenomenon in the late 1970s. This phenomenon plays a key role in many biological processes, including population dynamics, immune cell migration, and tumor growth. In the aftermath of this period, many authors investigated various chemotaxis models from various perspectives, including local existence, uniqueness, finite time blow-up, global existence and boundedness, persistence, stability, and special solutions in various research publications, making significant contributions to the mentioned problems above. For further details, see [3–5].

Regarding these problems in more general frameworks, including two species with chemical signals, some variants of the model of (1.1) have also been researched in various ways. A comprehensive comparison exists between one-species and multi-species chemotaxis models, addressing their mathematical frameworks, biological implications, and essential dynamical characteristics, and they are applicable in symbiotic or competitive systems, predator-prey dynamics, and host-pathogen interactions. The main difference between one-species and two-species chemotaxis models lies in the number of interacting populations and their interaction with chemical signals. In simpler terms, in a one-species system,

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the cell population reacts to a single stimuli, while in two-species models, the system comprises two populations that interact with one another and respond to a single chemical. It is well-known that multi-species chemotaxis models offer a more accurate representation of biological scenarios. Nevertheless, these models, while more realistic, working on those ones mathematically is quite challenging. It is also natural to regard competition and cooperation in chemotaxis models. In this context, the paper presents a model that incorporates three interacting populations responding to a single chemical, which allow for to study new challenges and areas of research, such as coexistence and extinction, provided that a globally bounded classical solution exists. In this regard, this article first studies global existence, global boundedness, and persistence of solutions within the following model, thereby providing a way for an exploration of the model's long-term behaviors. However, we leave open the topics associated with the large time behaviors to investigate somewhere else.

This paper aims to investigate a more realistic scenario in a biological environment by illustrating the interactions among three different cell populations as they react to one chemical. This is far more realistic compared to the previous works, revealing numerous intriguing dynamic scenarios within such chemotaxis systems. In this respect, this research paper analyzes the dynamical characteristics of the population as described by the subsequent parabolic-parabolic-parabolic-elliptic chemotaxis growth model involving strong logistic kinetics:

$$\begin{cases} u_t = \Delta u - \chi_1 \nabla \cdot (u \nabla z) + u(h_1 - k_1 u) \\ v_t = \Delta v - \chi_2 \nabla \cdot (v \nabla z) + v(h_2 - k_2 v) \\ w_t = \Delta v - \chi_3 \nabla \cdot (w \nabla z) + w(h_3 - k_3 w) \\ 0 = \Delta z - az + bu + cv + dw \end{cases} \tag{1.1}$$

with the Neumann boundary conditions

$$\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = \frac{\partial w}{\partial \nu} = \frac{\partial z}{\partial \nu} = 0$$

and the initial data $u_0(x) := u(0, x; u_0)$, $v_0(x) := v(0, x; u_0)$, and $w_0(x) := w(0, x; u_0)$ satisfying

$$u_0, v_0, w_0 \in C^0(\bar{S}) \quad \text{and} \quad u_0, v_0, w_0 \geq 0 \tag{1.2}$$

where $S \subseteq \mathbb{R}^n$ with $n \geq 1$ is a smooth bounded domain; $a, b, c, d > 0$, and $\chi_i, h_i, k_i > 0$, for $i \in \{1, 2, 3\}$. Moreover, assume that

$$k_1 > (n-2) \left\{ \frac{b\chi_1}{n} + \frac{(c+d)\chi_1}{n+2} + \frac{2b(\chi_2 + \chi_3)}{n(n+2)} \right\} \tag{1.3}$$

$$k_2 > (n-2) \left\{ \frac{(b+d)\chi_2}{n+2} + \frac{c\chi_2}{n} + \frac{2c(\chi_1 + \chi_3)}{n(n+2)} \right\} \tag{1.4}$$

and

$$k_3 > (n-2) \left\{ \frac{(b+c)\chi_3}{n+2} + \frac{2d(\chi_1 + \chi_2)}{n(n+2)} + \frac{d\chi_3}{n} \right\} \tag{1.5}$$

From a biological standpoint, the system described by (1.1) illustrates the evolution of three competing mobile species, namely u, v, and w, which are affected by a single chemical substance z. In this context, the mobile cells u, v, and z are attracted by the chemical substance z. In the framework of (1.1), the unknown functions $u(x, t)$, $v(x, t)$, and $w(x, t)$ represent the density of cells, while $z(x, t)$ indicates the concentration of the chemical signal at time t and space $x \in \Omega$; the cross-diffusion terms $-\chi_1 \nabla \cdot (u \nabla z)$, $-\chi_2 \nabla \cdot (v \nabla z)$, and $-\chi_3 \nabla \cdot (w \nabla z)$ reflect the chemotactic movement, where $\chi_1, \chi_2, \chi_3 > 0$ are the chemotactic sensitivity coefficients. The parameters $h_1, h_2, h_3 > 0$ represent the intrinsic growth rates, while the parameters $k_1, k_2, k_3 > 0$ indicate the self-limitation effects of the species u, v and w, respectively. Additionally, the parameters $a > 0$ indicate the degradation rate of chemical substance w; the parameters $b, c, d > 0$ denote the production rate of the mobile cells u, v, and w.

In the competitive scenario, all three species strive to generate stimuli to attract their rivals to gain dominance. Multi-species chemotaxis models have a great biological importance in real-world scenarios, as they provide insights into the movement of different cell types or organisms in response to chemical signals, particularly when interacting among various species or cell types. From the biological perspective, the model in (1.1) describes the evolution of three competitive species subject to one chemical substance. It is essential to highlight that the system represented by (1.1) is under investigation for the first time. It is particularly noteworthy that the system incorporates three species and one stimulus with regular sensitivity, which gives us the opportunity to compare and discuss these three distinct cell populations at the same time. Hence, we explore the interactions among all the species and their mutual influences on the dynamic properties of the system in (1.1). Throughout this study, investigate the L^ρ-boundedness, global existence, global boundedness, and combined mass persistence of solutions to the system in (1.1).

Various versions of the system in (1.1), such as one-species or multi-species and one-multi type chemical substance models, have been analyzed in many research papers so far. First, assume that $v(x,t) = w(t,x) = 0$. Then, the following system is obtained:

$$\begin{cases} u_t = \Delta u - \chi \nabla \cdot (u \nabla z) + h_1 u - k_1 u^2 \\ 0 = \Delta z - az + bu \end{cases} \tag{1.6}$$

For the case $n \geq 2$ and $h_1 = k_1 = 0$, (1.6) has a finite-time blows-up in solutions of (1.1) under some restriction on the initial data, see [6–9]. For the case $a = b = 1$ and $h_1, k_1 > 0$, (1.6) has a bounded classical solution if $n < 2$ or $n \geq 3$ whenever $\chi_1 < \frac{k_1 n}{n-2}$ [10]. Moreover, the global existence and boundedness of this model was obtained at the critical point, which is $\chi_1 = \frac{k_1 n}{n-2}$ with $n \geq 3$ [11]. In addition, the mass persistence of solutions of (1.6) was first studied in [12], and it was shown that in any space dimensional setting, when S is a convex domain, all positive solutions to the model in (1.1) always persists as a whole, that is,

$$\int_S u \geq c_* > 0 \tag{1.7}$$

Recently, the convexity condition for the persistence of mass of solutions has been eliminated in [13] under the the following explicit conditions, which means (1.7) also holds for any domain $S \subseteq \mathbb{R}^n$ if

$$n \leq 2 \quad \text{or} \quad \chi \leq \frac{k_1}{b} \cdot \frac{n}{n-2} \ \text{ with } \ n \geq 3$$

For the other dynamical behaviors of solutions, including weak solutions, stability, and persistence, see [11, 14–26].

A selection of known results concerning similar models of (1.1) can be outlined as follows: Consider the subsequent two-species one chemoattractant Keller-Segel model

$$\begin{cases} u_t = d_1 \Delta u - \chi_1 \nabla \cdot (u \nabla z) + \mu_1 u(1 - u - a_1 v) \\ v_t = d_2 \Delta v - \chi_2 \nabla \cdot (u \nabla z) + \mu_2 v(1 - a_2 u - v) \\ 0 = d_3 \Delta z - \gamma z + \alpha u + \beta v \end{cases} \tag{1.8}$$

Tello and Winkler [27] established the global existence, boundedness, and stability of solutions of (1.8) under the conditions $d_3 = \alpha = \beta = 1$, $2(\chi_1 + \chi_2) + a_2 \mu_1 < \mu_2$, and $2(\chi_1 + \chi_2) + a_1 \mu_2 < \mu_1$.

The same results [28] were achieved under the relaxed conditions provided that $\frac{\chi_1}{\mu_1} < \min\left\{\frac{d_3}{2\alpha}, \frac{a_1 d_3}{\beta}\right\}$, $\frac{\chi_2}{\mu_2} < \min\left\{\frac{d_3}{2\beta}, \frac{a_2 d_3}{\alpha}\right\}$, and $a_1 a_2 d_3^2 < \left(d_3 - \frac{2\alpha\chi_1}{\mu_1}\right)\left(d_3 - \frac{2\beta\chi_2}{\mu_2}\right)$. The long-time behaviors of solutions to the system in (1.8) has been established in [29] provided that $a_1 > 1 > a_2$, $d_3 = \beta = 1$, $\frac{\chi_1}{\mu_1} \le a_1$, $\frac{\chi_2}{\mu_2} \le \frac{1}{2}$, and $\frac{\chi_1}{\mu_1} + \max\left\{\frac{\chi_2}{\mu_2}, \frac{a_2(\mu_2-\chi_2)}{\mu_2-2\chi_2}, \frac{(\alpha-a_2)\chi_2}{\mu_2-2\chi_2}\right\} < 1$. Afterward, in the general case, i.e., $a_1, a_2 > 0$, it was demonstrated in [30] that the system in (1.8) has a global bounded classical solutions if $n \le 2$ or $n \ge 3$ with $\frac{\chi_1}{\mu_1} < \frac{d_3 n}{n-2}\min\left\{\frac{1}{\alpha}, \frac{a_1}{\beta}\right\}$ and $\frac{\chi_2}{\mu_2} < \frac{d_3 n}{n-2}\min\left\{\frac{1}{\beta}, \frac{a_2}{\alpha}\right\}$. This result was improved in [31] when $\alpha = \beta = \gamma = 1$ if $\frac{\chi_1}{\mu_1} + \frac{\chi_2}{\mu_2} < d_3$. Finally, in [32], the most general case for the arbitrary parameters, the system in (1.8) admits a bounded solution under the much milder suitable conditions on the parameters. For the existence, boundedness, long-term behavior of solutions, such as asymptotic stability, persistence, competitive exclusion, and coexistence, for similar models of (1.8), see [33–41].

The remainder of this paper is structured as follows: Section 2 focuses on presenting key estimates and L^ρ-bounds and discussing the global existence and boundedness of solutions to (1.1). Section 3 analyzes the persistence of the mass of globally bounded solutions to (1.1). The last section discusses the need for further research.

2. Preliminaries

This section aims to introduce several fundamental lemmas. Initially, it discusses the local existence and uniqueness of the solution to (1.1).

Lemma 2.1. For all u_0 and v_0 satisfying (1.2), there exists a $T_{\max}(u_0, v_0, w_0) \in (0, \infty]$ such that the system described by (1.1) and (1.2) admit a classical solution on $(0, T_{\max})$ with initial conditions $u(0, x) = u_0(x)$, $v(0, x) = v_0(x)$, and $w(0, x) = w_0(x)$ satisfying

$$\lim_{t\to 0} \|u(t,\cdot) - u_0(\cdot)\|_{L^\infty(\bar{S})} = \lim_{t\to 0} \|v(t,\cdot) - v_0(\cdot)\|_{L^\infty(\bar{S})} = \lim_{t\to 0} \|w(t,\cdot) - w_0(\cdot)\|_{L^\infty(\bar{S})} = 0$$

where $u, v, w \in C((0, T_{\max}) \times \bar{S}) \cap C^{2,1}((0, T_{\max}) \times \bar{S}))$ and $z \in C^{2,0}((0, T_{\max}) \times \bar{S}))$. In addition, being $T_{\max}(u_0, v_0, w_0) < \infty$ also implies

$$\|u(t,\cdot) + v(t,\cdot) + w(t,\cdot)\|_{L^\infty(\bar{S})} = \infty \quad \text{as} \quad t \to T_{\max}$$

The proof can be obtained from the similar operations of Theorem 2.1 in [10].

In the subsequent discussion, we establish upper bounds for the solutions that serve as a foundation for proving the main results herein. Note that we prove the following lemmas in the interval $t \in (0, T_{\max})$, for all $0 < t < T_{\max}(u_0, v_0, w_0) \in (0, \infty]$.

Lemma 2.2. The following hold:

i. Let $|S|$ be the Lebesgue measure of S. Then,

$$\int_S u \le m_1 := \max\Big\{\frac{h_1}{k_1}|S|, \int_S u_0\Big\}$$

$$\int_S v \le m_2 := \max\Big\{\frac{h_2}{k_2}|S|, \int_S v_0\Big\}$$

and

$$\int_S w \le m_3 := \max\Big\{\frac{h_3}{k_3}|S|, \int_S v_0\Big\}$$

for all $t \in (0, T_{\max})$.

ii. Let $\xi > 1$. For all $\varepsilon > 0$, there are $C(\varepsilon, \xi, m_1) > 0, C(\varepsilon, \xi, m_2) > 0$, and $C(\varepsilon, \xi, m_3)$ such that

$$\int_S u^\xi \leq \varepsilon \int_S u^{\xi-2}|\nabla u|^2 + C(\varepsilon, \xi, m_1)$$

$$\int_S v^\xi \leq \varepsilon \int_S v^{\xi-2}|\nabla v|^2 + C(\varepsilon, \xi, m_2)$$

and

$$\int_S w^\xi \leq \varepsilon \int_S w^{\xi-2}|\nabla w|^2 + C(\varepsilon, \xi, m_3)$$

for all $t \in (0, T_{\max})$.

Proof. *i.* Integrating the first and second equalities in (1.1) and using Hölder inequality,

$$\frac{d}{dt}\int_S u = h_1 \int_S u - k_1 \int_S u^2 \leq h_1 \int_S u - \frac{k_1}{|S|}\Big(\int_S u\Big)^2$$

$$\frac{d}{dt}\int_S v = h_2 \int_S v - k_2 \int_S v^2 \leq h_2 \int_S v - \frac{k_2}{|S|}\Big(\int_S v\Big)^2$$

and

$$\frac{d}{dt}\int_S w = h_3 \int_S v - k_3 \int_S w^2 \leq h_3 \int_S w - \frac{k_3}{|S|}\Big(\int_S w\Big)^2$$

for all $t \in (0, T_{\max})$. Then, i follows from the ODE's comparison principle.

ii. By the Erhling type lemma, for $\varepsilon > 0$, $C(\varepsilon, \xi) > 0$ such that

$$\int_S u^\xi \leq \frac{4\varepsilon}{\xi^2}\int_S |\nabla u^{\frac{\xi}{2}}|^2 + C(\varepsilon, \xi)\Big(\int_S u\Big)^\xi \leq \varepsilon \int_S u^{\xi-2}|\nabla u|^2 + C(\varepsilon, \xi)(m_1)^\xi$$

$$\int_S v^\xi \leq \frac{4\varepsilon}{\xi^2}\int_S |\nabla v^{\frac{\xi}{2}}|^2 + C(\varepsilon, \xi)\Big(\int_S v\Big)^\xi \leq \varepsilon \int_S v^{\xi-2}|\nabla v|^2 + C(\varepsilon, \xi)(m_2)^\xi$$

and

$$\int_S w^\xi \leq \frac{4\varepsilon}{\xi^2}\int_S |\nabla w^{\frac{\xi}{2}}|^2 + C(\varepsilon, \xi)\Big(\int_S w\Big)^\xi \leq \varepsilon \int_S w^{\xi-2}|\nabla w|^2 + C(\varepsilon, \xi)(m_3)^\xi$$

for all $t \in (0, T_{\max})$. Then, ii follows. □

Lemma 2.3. Let $\xi > 0$. Then, for all $t \in (0, T_{\max})$,

$$\int_S u^{\xi-1}\nabla u \cdot \nabla z \leq \left[\frac{b}{\xi} + \frac{c}{\xi+1} + \frac{d}{\xi+1}\right]\int_S u^{\xi+1} + \frac{c}{\xi(\xi+1)}\int_S v^{\xi+1} + \frac{d}{\xi(\xi+1)}\int_S w^{\xi+1}$$

$$\int_S v^{\xi-1}\nabla v \cdot \nabla z \leq \frac{b}{\xi(\xi+1)}\int_S u^{\xi+1} + \left[\frac{b}{\xi+1} + \frac{c}{\xi} + \frac{d}{\xi+1}\right]\int_S v^{\xi+1} + \frac{d}{\xi(\xi+1)}\int_S w^{\xi+1}$$

and

$$\int_S w^{\xi-1}\nabla w \cdot \nabla z \leq \frac{b}{\xi(\xi+1)}\int_S u^{\xi+1} + \frac{c}{\xi(\xi+1)}\int_S v^{\xi+1} + \left[\frac{b}{\xi+1} + \frac{c}{\xi+1} + \frac{d}{\xi}\right]\int_S w^{\xi+1}$$

Proof. By multiplying the third equality in (1.1) by $u^{\xi-1}$ and integrating by parts over S,

$$\int_S u^{\xi-1}\cdot(\Delta z - az + bu + cv + dw) = 0$$

which gives by Young's inequality

$$\begin{aligned}\xi\int_S u^{\xi-1}\nabla u \cdot \nabla z + a\int_S zu^\xi &= b\int_S u^{\xi+1} + c\int_S vu^\xi + d\int_S wu^\xi \\ &\leq b\int_S u^{\xi+1} + \frac{c}{\xi+1}\int_S v^{\xi+1} + \frac{c\xi}{\xi+1}\int_S u^{\xi+1} + \frac{d}{\xi+1}\int_S w^{\xi+1} + \frac{d\xi}{\xi+1}\int_S u^{\xi+1}\end{aligned}$$

for all $t \in (0, T_{\max})$. Similarly,

$$\begin{aligned}\xi \int_S v^{\xi-1}\nabla v \cdot \nabla z + a\int_S zv^\xi &= b\int_S uv^\xi + c\int_S v^{\xi+1} + d\int_S wv^\xi \\ &\le \frac{b}{\xi+1}\int_S u^{\xi+1} + \frac{b\xi}{\xi+1}\int_S v^{\xi+1} + c\int_S v^{\xi+1} + \frac{d}{\xi+1}\int_S w^{\xi+1} + \frac{d\xi}{\xi+1}\int_S v^{\xi+1}\end{aligned}$$

and

$$\begin{aligned}\xi \int_S w^{\xi-1}\nabla w \cdot \nabla z + a\int_S zw^\xi &= b\int_S uw^\xi + c\int_S vw^\xi + d\int_S w^{\xi+1} \\ &\le \frac{b}{\xi+1}\int_S u^{\xi+1} + \frac{b\xi}{\xi+1}\int_S w^{\xi+1} + \frac{c}{\xi+1}\int_S v^{\xi+1} + \frac{c\xi}{\xi+1}\int_S w^{\xi+1} + d\int_S w^{\xi+1}\end{aligned}$$

for all $t \in (0, T_{\max})$. □

The subsequent lemma represents a significant estimate for the L^ρ-bounds of $u + v$.

Lemma 2.4. Assume that (1.3)-(1.5) holds. Then for all k_1, k_2 and k_3, there is a $\xi := \xi(k_1, k_2, k_3) > 1$ such that

$$\int_S u^\xi + \int_S v^\xi + \int_S w^\xi \le C$$

for all $t \in (0, T_{\max})$.

Proof. Multiplying the first equality in (1.1) by $u^{\xi-1}$ and integrating it over S,

$$\begin{aligned}\frac{1}{\xi}\cdot\frac{d}{dt}\int_S u^\xi &= \int_S u^{\xi-1}\Delta u - \chi_1\int_S u^{\xi-1}\nabla\cdot(u\nabla z) + h_1\int_S u^\xi - k_1\int_S u^{\xi+1} \\ &= -(\xi-1)\int_S u^{\xi-2}|\nabla u|^2 + (\xi-1)\chi_1\int_S u^{\xi-1}\nabla u\cdot\nabla z + h_1\int_S u^\xi - k_1\int_S u^{\xi+1}\end{aligned} \tag{2.1}$$

for all $t \in (0, T_{\max})$. Similarly,

$$\frac{1}{\xi}\cdot\frac{d}{dt}\int_S v^\xi = -(\xi-1)\int_S v^{\xi-2}|\nabla v|^2 + (\xi-1)\chi_2\int_S v^{\xi-1}\nabla v\cdot\nabla z + h_2\int_S v^\xi - k_2\int_S v^{\xi+1} \tag{2.2}$$

and

$$\frac{1}{\xi}\cdot\frac{d}{dt}\int_S w^\xi = -(\xi-1)\int_S w^{\xi-2}|\nabla w|^2 + (\xi-1)\chi_3\int_S w^{\xi-1}\nabla w\cdot\nabla z + h_3\int_S w^\xi - k_3\int_S w^{\xi+1} \tag{2.3}$$

for all $t \in (0, T_{\max})$. By adding (2.1)-(2.3),

$$\begin{aligned}\frac{1}{\xi}\cdot\frac{d}{dt}\left(\int_S u^\xi + \int_S v^\xi + \int_S w^\xi\right) &= -(\xi-1)\int_S u^{\xi-2}|\nabla u|^2 - (\xi-1)\int_S v^{\xi-2}|\nabla v|^2 - (\xi-1)\int_S w^{\xi-2}|\nabla w|^2 \\ &+ (\xi-1)\chi_1\int_S u^{\xi-1}\nabla u\cdot\nabla z + (\xi-1)\chi_2\int_S v^{\xi-1}\nabla v\cdot\nabla z + (\xi-1)\chi_3\int_S w^{\xi-1}\nabla w\cdot\nabla z \\ &+ h_1\int_S u^\xi + h_2\int_S v^\xi + h_3\int_S w^\xi - k_1\int_S u^{\xi+1} - k_2\int_S v^{\xi+1} - k_3\int_S w^{\xi+1}\end{aligned}$$

for all $t \in (0, T_{\max})$. By Lemma 2.2, there is a positive number $C > 0$ such that

$$h_1\int_S u^\xi \le (\xi-1)\int_S u^{\xi-2}|\nabla u|^2 + \frac{C}{3}$$

$$h_2\int_S v^\xi \le (\xi-1)\int_S v^{\xi-2}|\nabla v|^2 + \frac{C}{3}$$

and

$$h_3\int_S w^\xi \le (\xi-1)\int_S w^{\xi-2}|\nabla w|^2 + \frac{C}{3}$$

for all $t \in (0, T_{\max})$. Moreover, by Lemma 2.3,

$$(\xi-1)\chi_1 \int_S u^{\xi-1}\nabla u\cdot\nabla z \le (\xi-1)\chi_1\left[\frac{b}{\xi}+\frac{c}{\xi+1}+\frac{d}{\xi+1}\right]\int_S u^{\xi+1}+\frac{c(\xi-1)\chi_1}{\xi(\xi+1)}\int_S v^{\xi+1}+\frac{d(\xi-1)\chi_1}{\xi(\xi+1)}\int_S w^{\xi+1}$$

$$(\xi-1)\chi_2 \int_S v^{\xi-1}\nabla v\cdot\nabla z \le (\xi-1)\chi_2\left[\frac{b}{\xi+1}+\frac{c}{\xi}+\frac{d}{\xi+1}\right]\int_S v^{\xi+1}+\frac{b(\xi-1)\chi_2}{\xi(\xi+1)}\int_S u^{\xi+1}+\frac{d(\xi-1)\chi_2}{\xi(\xi+1)}\int_S w^{\xi+1}$$

and

$$(\xi-1)\chi_3 \int_S w^{\xi-1}\nabla w\cdot\nabla z \le \frac{b(\xi-1)\chi_3}{\xi(\xi+1)}\int_S u^{\xi+1}+\frac{c(\xi-1)\chi_3}{\xi(\xi+1)}\int_S v^{\xi+1}+(\xi-1)\chi_3\left[\frac{b}{\xi+1}+\frac{c}{\xi+1}+\frac{d}{\xi}\right]\int_S w^{\xi+1}$$

which yields

$$\begin{aligned}(\xi-1)\left[\chi_1\int_S u^{\xi-1}\nabla u\cdot\nabla z+\chi_2\int_S v^{\xi-1}\nabla v\cdot\nabla z+\chi_3\int_S w^{\xi-1}\nabla w\cdot\nabla z\right] &\le (\xi-1)\left[\frac{b\chi_1}{\xi}+\frac{(c+d)\chi_1}{\xi+1}+\frac{b(\chi_2+\chi_3)}{\xi(\xi+1)}\right]\int_S u^{\xi+1}\\ &+(\xi-1)\left[\frac{(b+d)\chi_2}{\xi+1}+\frac{c\chi_2}{\xi}+\frac{c(\chi_1+\chi_3)}{\xi(\xi+1)}\right]\int_S v^{\xi+1} \quad (2.4)\\ &+(\xi-1)\left[\frac{(b+c)\chi_3}{\xi+1}+\frac{d(\chi_1+\chi_2)}{\xi(\xi+1)}+\frac{d\chi_3}{\xi}\right]\int_S w^{\xi+1}\end{aligned}$$

for all $t \in (0, T_{\max})$. It then follows that

$$\begin{aligned}\frac{1}{\xi}\cdot\frac{d}{dt}\left(\int_S u^\xi+\int_S v^\xi+\int_S w^\xi\right) &\le \left\{(\xi-1)\left(\frac{b\chi_1}{\xi}+\frac{(c+d)\chi_1}{\xi+1}+\frac{b(\chi_2+\chi_3)}{\xi(\xi+1)}\right)-k_1\right\}\int_S u^{\xi+1}\\ &+\left\{(\xi-1)\left(\frac{(b+d)\chi_2}{\xi+1}+\frac{c\chi_2}{\xi}+\frac{c(\chi_1+\chi_3)}{\xi(\xi+1)}\right)-k_2\right\}\int_S v^{\xi+1}\\ &+\left\{(\xi-1)\left(\frac{(b+c)\chi_3}{\xi+1}+\frac{d(\chi_1+\chi_2)}{\xi(\xi+1)}+\frac{d\chi_3}{\xi}\right)-k_3\right\}\int_S v^{\xi+1}+C\end{aligned}$$

for all $t \in (0, T_{\max})$. Fix $\xi > 1$ sufficiently close to 1 such that $\xi := 1+\varepsilon$, for $\varepsilon \ll 1$. By (1.3)-(1.5), for all k_1, k_2, and k_3,

$$\varepsilon\cdot\left[\frac{b\chi_1}{1+\varepsilon}+\frac{(c+d)\chi_1}{2+\varepsilon}+\frac{b(\chi_2+\chi_3)}{(1+\varepsilon)(2+\varepsilon)}\right]-k_1<0$$

$$\varepsilon\cdot\left[\frac{(b+d)\chi_2}{2+\varepsilon}+\frac{c\chi_2}{1+\varepsilon}+\frac{c(\chi_1+\chi_3)}{(1+\varepsilon)(2+\varepsilon)}\right]-k_2<0$$

and

$$\varepsilon\cdot\left[\frac{(b+c)\chi_3}{2+\varepsilon}+\frac{d(\chi_1+\chi_2)}{(1+\varepsilon)(2+\varepsilon)}+\frac{d\chi_3}{1+\varepsilon}\right]-k_3<0$$

Then, by Young's inequality with some elementary arrangements, there is a $k^* > 0$ such that

$$\frac{1}{\xi}\cdot\frac{d}{dt}\left(\int_S u^\xi+\int_S v^\xi+\int_S w^\xi\right)\le -k^*\left(\int_S u^\xi+\int_S v^\xi+\int_S w^\xi\right)+C^*$$

for all $t \in (0, T_{\max})$. Let $y(t) := \int_S u^\xi+\int_S v^\xi+\int_S w^\xi$, which yields $y' \le -\xi k^* y+\xi C^*$. Then, the ODE's comparison principle yields

$$\int_S u^\xi+\int_S v^\xi+\int_S w^\xi \le C, \qquad \text{for all } t \in (0, T_{\max})$$

□

3. Main Results

This section provides the obtained primary results.

3.1. L^ρ-Bounds

This subsection establishes L^ρ-bounds of $u+v+w$.

Theorem 3.1 (L^ρ-boundedness)**.** Suppose that the initial functions u_0, v_0, and w_0 satisfy (1.2), and the assumptions in (1.3)-(1.5) are valid. Then, for any given $\frac{n}{2}<\rho<n$,

$$\int_S u^\rho+\int_S v^\rho+\int_S w^\rho \le C, \quad \text{for all } t\in(0,T_{\max})$$

PROOF. Fix $\frac{n}{2}<\xi<n$. Then, the main assumptions in (1.3)-(1.5) guarantee that the following hold:

$$(\xi-1)\left(\frac{b\chi_1}{\xi}+\frac{(c+d)\chi_1}{\xi+1}+\frac{b(\chi_2+\chi_3)}{\xi(\xi+1)}\right)-k_1<0$$

$$(\xi-1)\left(\frac{(b+d)\chi_2}{\xi+1}+\frac{c\chi_2}{\xi}+\frac{c(\chi_1+\chi_3)}{\xi(\xi+1)}\right)-k_2<0$$

and

$$(\xi-1)\left(\frac{(b+c)\chi_3}{\xi+1}+\frac{d(\chi_1+\chi_2)}{\xi(\xi+1)}+\frac{d\chi_3}{\xi}\right)-k_3<0$$

Hence, by Lemma 2.4,

$$\int_S u^\xi+\int_S v^\xi+\int_S w^\xi\le C_\xi$$

for all $t\in(0,T_{\max})$. Moreover, by the Gagliardo-Nirenberg embedding theorem and Young's inequality, for all $\varepsilon>0$,

$$\begin{aligned}\int_S u^{\rho+1}&=\|u^{\frac{\rho}{2}}\|_{L^{\frac{2(\rho+1)}{\rho}}(S)}^{\frac{2(\rho+1)}{\rho}}\\&\le C\|\nabla u^{\frac{\rho}{2}}\|_{L^2(S)}^{\frac{2(\rho+1)\theta}{\rho}}\|u^{\frac{\rho}{2}}\|_{L^{\frac{2\xi}{\rho}}(S)}^{\frac{2(\rho+1)(1-\theta)}{\rho}}+C\|u^{\frac{\rho}{2}}\|_{L^{\frac{2\xi}{\rho}}(S)}^{\frac{2(\rho+1)\theta}{\rho}}\\&\le C\left(\frac{\rho^2}{4}\int_S u^{\rho-2}|\nabla u|^2\right)^{\frac{(\rho+1)\theta}{\rho}}(C_\xi)^{\frac{(\rho+1)(1-\theta)}{\xi}}+C(C_\xi)^{\frac{(\rho+1)\theta}{\xi}}\\&\le\varepsilon\int_S u^{\rho-2}|\nabla u|^2+C(\rho,\xi,\varepsilon,\theta,C_\xi,|S|)\quad\text{for all } t\in(0,T_{\max})\end{aligned}\tag{3.1}$$

where

$$\theta=\frac{\frac{\rho}{2\xi}-\frac{\rho}{2(\rho+1)}}{\frac{1}{n}+\frac{\rho}{2\xi}-\frac{1}{2}}=\frac{\rho n}{\rho+1}\cdot\frac{\rho+1-\xi}{2\xi+n(p-\xi)}\in(0,1)\quad\text{and}\quad\frac{(\rho+1)\theta}{\rho}<1$$

due to the fact that (1.3) implies

$$\chi_1<\frac{n}{n-2}\cdot\frac{k_1}{b_1}\quad\text{for all }\frac{n-2}{2}<\frac{n}{2}<\rho<n$$

Similarly,

$$\int_S v^{\rho+1}\le\varepsilon\int_S v^{\rho-2}|\nabla v|^2+C(\rho,\xi,\varepsilon,\theta,C_\xi,|S|)\tag{3.2}$$

and

$$\int_S w^{\rho+1}\le\varepsilon\int_S w^{\rho-2}|\nabla w|^2+C(\rho,\xi,\varepsilon,\theta,C_\xi,|S|)\tag{3.3}$$

for all $t\in(0,T_{\max})$. Besides, multiplying the first equality in (1.1) by $u^{\rho-1}$ with $\rho>1$, the second equality in (1.1) by $v^{\rho-1}$ with $\rho>1$, and the third equality in (1.1) by $w^{\rho-1}$ with $\rho>1$, integrating them over S, and adding these equations,

$$\begin{aligned}\frac{1}{\rho}\frac{d}{dt}\left(\int_S u^\rho + \int_S v^\rho + \int_S w^\rho\right) &= -(\rho-1)\int_S u^{\rho-2}|\nabla u|^2 - (\rho-1)\int_S v^{\rho-2}|\nabla v|^2 - (\rho-1)\int_S w^{\rho-2}|\nabla w|^2 \\ &\quad + (\rho-1)\chi_1\int_S u^{\rho-1}\nabla u\cdot\nabla z + (\rho-1)\chi_2\int_S v^{\rho-1}\nabla v\cdot\nabla z + (\rho-1)\chi_3\int_S w^{\rho-1}\nabla w\cdot\nabla z \\ &\quad + h_1\int_S u^\rho + h_2\int_S v^\rho + h_3\int_S w^\rho - k_1\int_S u^{\rho+1} - k_2\int_S v^{\rho+1} - k_3\int_S w^{\rho+1}\end{aligned} \tag{3.4}$$

for all $t \in (0, T_{\max})$. In view of (2.4), (3.1), (3.2), and (3.3),

$$\begin{aligned}(\rho-1)\left[\chi_1\int_S u^{\rho-1}\nabla u\cdot\nabla z + \chi_2\int_S v^{\rho-1}\nabla v\cdot\nabla z + \chi_3\int_S w^{\rho-1}\nabla w\cdot\nabla z\right] &\le (\xi-1)\left[\frac{b\chi_1}{\xi} + \frac{(c+d)\chi_1}{\xi+1} + \frac{b(\chi_2+\chi_3)}{\xi(\xi+1)}\right]\int_S u^{\rho+1} \\ &\quad + (\xi-1)\left[\frac{(b+d)\chi_2}{\xi+1} + \frac{c\chi_2}{\xi} + \frac{c(\chi_1+\chi_3)}{\xi(\xi+1)}\right]\int_S v^{\rho+1} \\ &\quad + (\xi-1)\left[\frac{(b+c)\chi_3}{\xi+1} + \frac{d(\chi_1+\chi_2)}{\xi(\xi+1)} + \frac{d\chi_3}{\xi}\right]\int_S w^{\rho+1} \\ &\le (\rho-1)\int_S v^{\rho-2}|\nabla v|^2 + (\rho-1)\int_S v^{\rho-2}|\nabla v|^2 \\ &\quad + (\rho-1)\int_S w^{\rho-2}|\nabla w|^2 + C\end{aligned} \tag{3.5}$$

for $t \in (0, T_{\max})$. Moreover, by Young's inequality,

$$h_1\int_S u^\rho \le \frac{k_1}{2}\int_S u^{\rho+1} + C(h_1, k_1, |S|) \tag{3.6}$$

$$h_2\int_S v^\rho \le \frac{k_2}{2}\int_S v^{\rho+1} + C(h_2, k_2, |S|) \tag{3.7}$$

and

$$h_3\int_S w^\rho \le \frac{k_3}{2}\int_S w^{\rho+1} + C(h_3, k_3, |S|) \tag{3.8}$$

for all $t \in (0, T_{\max})$. Collecting (3.4)-(3.8),

$$\frac{1}{\rho}\frac{d}{dt}\left(\int_S u^\rho + \int_S v^\rho + \int_S w^\rho\right) \le -\min\left\{\frac{k_1}{2}, \frac{k_2}{2}, \frac{k_3}{2}\right\}\left(\int_S u^\rho + \int_S u^\rho + \int_S w^\rho\right) + C^*$$

which implies by the ODE's comparison principle that

$$\int_S u^\rho + \int_S v^\rho + \int_S w^\rho \le \max\left\{\int_S (u_0^\rho + v_0^\rho + w_0^\rho),\ \frac{4C^*}{\min\{k_1, k_2, k_3\}}\right\}$$

for all $t \in (0, T_{\max})$. The proof is thus over. □

3.2. Global Existence and Boundedness

This subsection presents the subsequent observation related to the global existence and boundedness of solutions to (1.1).

Theorem 3.2 (Global existence and boundedness)**.** Assume that the initial functions u_0, v_0, and w_0 satisfy (1.2), and (1.3) and (1.4) hold. Then, the solution (u, v, w, z) is global, i.e.,

$$T_{\max}(u_0, v_0, w_0) = \infty$$

Moreover, there is a $K_\infty > 0$ such that

$$\|u+v+w\|_{L^\infty(S)} \le K_\infty, \quad \text{for all } t>0$$

Proof. It is well known that if $\rho > \frac{n}{2}$, then L^ρ-boundedness of solutions in time implies the L^∞-boundedness in time of solutions. Thus, by Theorem 3.1 and similar operations in the proof of Theorem 2.5 in [27], $T_{\max}(u_0, v_0, w_0) = \infty$ and

$$\sup \|u(t,\cdot)+v(t,\cdot)+w(t,\cdot)\|_{L^\infty(S)} < \infty, \quad \text{for all } t>0$$

□

3.3. Combined Mass Persistence

This section analyzes the combined mass persistence of solutions to (1.1). It first present the following key estimate.

Lemma 3.3. Let β_0, β_1, and β_2 be positive, $\theta_1 > 1$, $\theta_2 > 1$, $t_0 \in \mathbb{R}$, and $y \in C^1([t_0,\infty))$ be nonnegative and satisfy the following inequality, for all $t>0$:

$$y'(t) \ge \beta_0 y(t) - \beta_1 y^{\theta_1}(t) - \beta_2 y^{\theta_2}(t)$$

Then,

$$y(t) \ge \min\left\{ y(t_0), \left(\frac{\beta_0}{2\beta_1}\right)^{\frac{1}{\theta_1-1}}, \left(\frac{\beta_0}{2\beta_2}\right)^{\frac{1}{\theta_2-1}} \right\}$$

The proof follows from the argument of Lemma 2.5 in [42]

Afterward, we provide an estimate from below for $u(t,x)+v(t,x)+w(t,x)$.

Lemma 3.4. Assume that $\delta \in (0,1)$. Then, there is a $\sigma > 0$ such that

$$\int_S (u^\delta(t,x)+v^\delta(t,x)+w^\delta(t,x))\,dx \ge \sigma, \quad \text{for all } t>0$$

Proof. Let $\delta\in(0,1)$. Then, multiplying the first equality in (1.1) by $u^{\delta-1}$ with , the second equality in (1.1) by $v^{\delta-1}$ with $\delta\in(0,1)$, and the third equality in (1.1) by $w^{\delta-1}$ with $\delta\in(0,1)$, integrating them over S, and adding these equations, for all $t>0$,

$$\begin{aligned}\frac{1}{\delta}\cdot\frac{d}{dt}\left(\int_S u^\delta + \int_S v^\delta + \int_S w^\delta\right) &= (1-\delta)\int_S u^{\delta-2}|\nabla u|^2 + (1-\delta)\int_S v^{\delta-2}|\nabla v|^2 + (1-\delta)\int_S w^{\delta-2}|\nabla w|^2 \\ &- (1-\delta)\chi_1\int_S u^{\delta-1}\nabla u\cdot\nabla z - (1-\delta)\chi_2\int_S v^{\delta-1}\nabla v\cdot\nabla z \\ &- (1-\delta)\chi_3\int_S w^{\delta-1}\nabla w\cdot\nabla z + h_1\int_S u^\delta + h_2\int_S v^\delta + h_3\int_S w^\delta \\ &- k_1\int_S u^{\delta+1} - k_2\int_S v^{\delta+1} - k_3\int_S w^{\delta+1}\end{aligned} \tag{3.9}$$

From Lemma 2.3, the third equality in (1.1), and integration by parts over S,

$$(1-\delta)\chi_1\int_S u^{\delta-1}\nabla u\cdot\nabla z \le (1-\delta)\chi_1\left(\frac{b}{\delta}+\frac{c+d}{\delta+1}\right)\int_S u^{\delta+1} + \frac{(1-\delta)c\chi_1}{\delta(\delta+1)}\int_S v^{\delta+1} + \frac{(1-\delta)d\chi_1}{\delta(\delta+1)}\int_S w^{\delta+1}$$

$$(1-\delta)\chi_2\int_S v^{\delta-1}\nabla v\cdot\nabla z \le \frac{(1-\delta)b\chi_2}{\delta(\delta+1)}\int_S u^{\delta+1} + (1-\delta)\chi_2\left(\frac{b+d}{\delta+1}+\frac{c}{\delta}\right)\int_S v^{\delta+1} + \frac{(1-\delta)b\chi_2}{\delta(\delta+1)}\int_S w^{\delta+1}$$

and

$$(1-\delta)\chi_3\int_S w^{\delta-1}\nabla w\cdot\nabla z \le \frac{b(\delta-1)\chi_3}{\delta(\delta+1)}\int_S u^{\delta+1} + \frac{c(\delta-1)\chi_3}{\delta(\delta+1)}\int_S v^{\delta+1} + (\delta-1)\chi_3\left(\frac{b}{\delta+1}+\frac{c}{\delta+1}+\frac{d}{\delta}\right)\int_S w^{\delta+1}$$

which entail for all $t > 0$ that

$$(1-\delta)\left[\chi_1\int_S u^{\delta-1}\nabla u\cdot\nabla z+\chi_2\int_S v^{\delta-1}\nabla v\cdot\nabla z+\chi_3\int_S w^{\delta-1}\nabla w\cdot\nabla z\right]\le C_1\int_S u^{\delta+1}+C_2\int_S v^{\delta+1}+C_3\int_S w^{\delta+1} \tag{3.10}$$

where

$$C_1=(1-\delta)\chi_1\left(\frac{b}{\delta}+\frac{c+d}{\delta+1}\right)+\frac{(1-\delta)b\chi_2}{\delta(\delta+1)}\frac{b(\delta-1)\chi_3}{\delta(\delta+1)}$$

$$C_2=\frac{(1-\delta)c\chi_1}{\delta(\delta+1)}+(1-\delta)\chi_2\left(\frac{b+d}{\delta+1}+\frac{c}{\delta}\right)+\frac{c(\delta-1)\chi_3}{\delta(\delta+1)}$$

and

$$C_3=\frac{(1-\delta)d\chi_1}{\delta(\delta+1)}+\frac{(1-\delta)b\chi_2}{\delta(\delta+1)}+(\delta-1)\chi_3\left(\frac{b}{\delta+1}+\frac{c}{\delta+1}+\frac{d}{\delta}\right)$$

Define $\zeta > 0$ such that

$$0<\frac{\xi(n-2\delta)}{n(\xi-\delta)}<\zeta<1<\xi$$

where $\xi > 1$ is as in Lemma 2.4. By Hölder's inequality, for all $t > t_0 > 0$,

$$\int_S u^{\delta+1}=\int_S u^{\zeta}\cdot u^{\delta+1-\zeta}\le\Big(\int_S u^{\xi}\Big)^{\frac{\zeta}{\xi}}\Big(\int_S u^{\frac{\xi(\delta+1-\zeta)}{\xi-\zeta}}\Big)^{\frac{\xi-\zeta}{\xi}}\le(C_\xi)^{\frac{\zeta}{\xi}}\Big(\int_S u^{\frac{\xi(\delta+1-\zeta)}{\xi-\zeta}}\Big)^{\frac{\xi-\zeta}{\xi}} \tag{3.11}$$

Employing the Gagliardo–Nirenberg Theorem and Young's inequality yields that

$$\begin{aligned}
\Big(\int_S u^{\frac{\xi(\delta+1-\zeta)}{\xi-\zeta}}\Big)^{\frac{\xi-\zeta}{\xi}} &= \|u^{\frac{\delta}{2}}\|^{\frac{2(\delta+1-\zeta)}{\delta}}_{L^{\frac{2q(\delta+1-\zeta)}{\delta(\xi-\zeta)}}(S)}\\
&\le C\|\nabla u^{\frac{\delta}{2}}\|^{\frac{2(\delta+1-\zeta)\theta}{\delta}}_{L^2(S)}\|u^{\frac{\delta}{2}}\|^{\frac{2(\delta+1-\zeta)(1-\theta)}{\delta}}_{L^2(S)}+C\|u^{\frac{\delta}{2}}\|^{\frac{2(\delta+1-\zeta)}{\delta}}_{L^2(S)}\\
&\le C\Big(\int_S u^{\delta-2}|\nabla u|^2\Big)^{\frac{(\delta+1-\zeta)\theta}{\delta}}\Big(\int_S u^{\delta}\Big)^{\frac{(\delta+1-\zeta)(1-\theta)}{\delta}}+C\Big(\int_S u^{\delta}\Big)^{\frac{\delta+1-\zeta}{r}}\\
&\le(1-\delta)C_1^{-1}(C_\xi)^{-\frac{\zeta}{\xi}}\int_S u^{\delta-2}|\nabla u|^2+\tilde{C}\Big(\int_S u^{\delta}\Big)^{\frac{(\delta+1-\zeta)(1-\theta)}{\delta-\theta(\delta+1-\zeta)}}+C\Big(\int_S u^{\delta}\Big)^{\frac{\delta+1-\zeta}{\delta}}
\end{aligned}$$

where

$$\theta=\frac{\frac{1}{2}-\frac{\delta(\xi-\zeta)}{2\xi(\delta+1-\zeta)}}{\frac{1}{n}+\frac{1}{2}-\frac{1}{2}}=\frac{n}{2\xi}\cdot\frac{\xi-\zeta\xi+\zeta\delta}{\delta+1-\zeta}\in(0,1)$$

$$\frac{(\delta+1-\zeta)\theta}{\delta}=\frac{n(\xi-\xi\zeta+\delta\zeta)}{2\xi\delta}\in(0,1)$$

$$\frac{(\delta+1-\zeta)(1-\theta)}{\delta-\theta(\delta+1-\zeta)}=1+\frac{1-\zeta}{\delta-\theta(\delta+1-\zeta)}>1$$

and

$$\frac{\delta+1-\zeta}{\delta}>1$$

It then follows that for all $t > t_0$,

$$C_1\int_S u^{r+1}\le(1-\delta)\int_S u^{\delta-2}|\nabla u|^2+\beta_1\Big(\int_S u^{\delta}\Big)^{\theta_1}+\beta_2\Big(\int_S u^{\delta}\Big)^{\theta_2} \tag{3.12}$$

where $\beta_1, \beta_2 > 0$ are certain positive constants and $\theta_1, \theta_2 > 1$. Similarly, for all $t > t_0$,

$$C_2\int_S v^{\delta+1}\le(1-\delta)\int_S v^{\delta-2}|\nabla v|^2+\beta_3\Big(\int_S v^{\delta}\Big)^{\theta_1}+\beta_4\Big(\int_S v^{\delta}\Big)^{\theta_2} \tag{3.13}$$

and

$$C_3\int_S w^{\delta+1}\le(1-\delta)\int_S w^{\delta-2}|\nabla w|^2+\beta_5\Big(\int_S w^{\delta}\Big)^{\theta_1}+\beta_6\Big(\int_S w^{\delta}\Big)^{\theta_2} \tag{3.14}$$

where $\beta_3, \beta_4, \beta_5, \beta_6 > 0$ are certain positive constants. Hence, from (3.10)-(3.14),

$$\begin{aligned}
(1-\delta)\left[\chi_1 \int_S u^{\delta-1}\nabla u\cdot\nabla z + \chi_2 \int_S v^{\delta-1}\nabla v\cdot\nabla z + \chi_3 \int_S w^{\delta-1}\nabla w\cdot\nabla z\right] &\le (1-\delta)\left[\int_S u^{\delta-2}|\nabla u|^2 + \int_S v^{\delta-2}|\nabla v|^2 + \int_S w^{\delta-2}|\nabla w|^2\right] \\
&\quad + \beta_1\Big(\int_S u^\delta\Big)^{\theta_1} + \beta_3\Big(\int_S v^\delta\Big)^{\theta_1} + \beta_5\Big(\int_S w^\delta\Big)^{\theta_1} \\
&\quad + \beta_2\Big(\int_S u^\delta\Big)^{\theta_2} + \beta_4\Big(\int_S v^\delta\Big)^{\theta_2} + \beta_6\Big(\int_S w^\delta\Big)^{\theta_2} \\
&\le (1-\delta)\left[\int_S u^{\delta-2}|\nabla u|^2 + \int_S v^{\delta-2}|\nabla v|^2 + \int_S w^{\delta-2}|\nabla w|^2\right] \\
&\quad + \beta_7\Big(\int_S u^\delta + \int_S v^\delta + \int_S w^\delta\Big)^{\theta_1} \\
&\quad + \beta_8\Big(\int_S u^\delta + \int_S v^\delta + \int_S w^\delta\Big)^{\theta_2}
\end{aligned}$$

for some $\beta_7, \beta_8 > 0$. Together with (3.9), this yields that

$$\begin{aligned}
\frac{1}{\delta}\cdot\frac{d}{dt}\left(\int_S u^\delta + \int_S v^\delta + \int_S w^\delta\right) &\ge \min\{h_1,h_2,h_3\}\left(\int_S u^\delta + \int_S v^\delta + \int_S w^\delta\right) - \beta_5\Big(\int_S u^\delta + \int_S v^\delta + \int_S w^\delta\Big)^{\theta_1} \\
&\quad - \beta_6\Big(\int_S u^\delta + \int_S v^\delta + \int_S w^\delta\Big)^{\theta_2}
\end{aligned}$$

for all $t > t_0$. Consequently, by Lemma 3.3, there is a $\sigma > 0$ such that for all $t > t_0$,

$$\int_S u^\delta + \int_S v^\delta + \int_S w^\delta \ge \sigma$$

□

Theorem 3.5 (Combined mass persistence)**.** Suppose that initial functions u_0, v_0, and w_0 satisfy (1.2), and the main assumptions in (1.3) and (1.4) are valid. Then, there is a $\sigma_* > 0$ such that

$$\int_S (u+v+w) \ge \sigma_*, \qquad \text{for all } t > 0$$

Proof. By Hölder inequality, for all $\delta \in (0,1)$ and for all $t > 0$,

$$\int_S (u+v+w) \ge |S|^{\frac{\delta-1}{\delta}}\left(\int_S (u+v+w)^\delta\right)^{\frac{1}{\delta}} \ge |S|^{\frac{\delta-1}{\delta}}\left\{\int_S u^\delta + \int_S v^\delta + \int_S w^\delta\right\}^{\frac{1}{\delta}}$$

Afterward, by Lemma 3.4, for all $\delta \in (0,1)$, there is a $\sigma > 0$ such that for all $t > 0$,

$$\int_S u^\delta + \int_S v^\delta + \int_S w^\delta \ge \sigma$$

Therefore, for all $t > 0$,

$$\int_S (u+v+w) \ge |S|^{\frac{\delta-1}{\delta}}\left\{\int_S u^\delta + \int_S v^\delta + \int_S w^\delta\right\}^{\frac{1}{\delta}} \ge |S|^{\frac{\delta-1}{\delta}}\sigma^{\frac{1}{\delta}}$$

□

4. Conclusion

In this section, we analyze the obtained findings, outline open problems related to the system in (1.1), and suggest potential directions for future research. To begin with, we remark that the system in (1.1) represents a mixed-type Keller-Segel chemotaxis model, incorporating three mobile species and a single chemical stimulus. This model combines standard sensitivities with competitive dynamics defined by weak logistic sources. Furthermore, it is significant to point out that this is the inaugural study documented related to the system in (1.1) in the related literature.

Afterward, we discuss the results obtained in Theorems 3.1, 3.2, and 3.5. To begin with, we note that compared to the Lotka-Volterra kinetics, which involves interactions among multiple species such as competition, predation, or mutualism, the current logistic source for any cell represented by u, v, w in the system described in (1.1) does not interact with other species. This situation presents both advantages and disadvantages for the system in (1.1). The benefit of the existing logistic source is its ability to prevent species extinction, refers to persistence. In contrast, the limitations are connected to the continuous evolution of the cell population over time, avoiding infinite growth or collapse within a finite period, which relates to global existence and boundedness, under more stronger assumptions regarding the parameters, particularly k_1, k_2, and k_3. The Lotka-Volterra kinetics offers more beneficial conditions for achieving outcomes associated with global existence and boundedness; however, it can also cause the extinction of one or two species as time progresses. While the current logistic kinetics requires more rigorous conditions on the parameters to secure the stated results on existence, it will ensure the strict positivity of species at any moment they exist. Therefore, the assumptions herein in establishing the main results presented in Theorem 3.5 compared to the previous works are considerably more stronger than those in earlier studies [12, 35] in terms of the persistency of species. On the other hand, these current results indicate the upper bounds for global existence, boundedness, and persistence in the (1.1) if the Lotka-Volterra kinetics is integrated into the system. We highlight that the global existence, boundedness and combined mass persistence of the current system has been established for the first time in this paper. Hence, future works associated with the system in (1.1) can focus on the following topics:

i. If (1.3) and (1.4) are not valid, then the global existence, boundedness, and mass persistence of the solution to the system in (1.1) is still open. The next phases of this research could involve an analysis of the asymptotic stability, co-existence, extinction, and bifurcation analysis of solutions, along with their numerical simulations.

ii. Another future works related to system in (1.1) may involve integrating Lotka-Volterra kinetics into the system to investigate its dynamics, followed by comparing the results from each model.

Author Contributions

All the authors equally contributed to this work. They all read and approved the final version of the paper.

Conflicts of Interest

All the authors declare no conflict of interest.

Ethical Review and Approval

No approval from the Board of Ethics is required.

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Journal of New Theory
dergipark.org.tr/tr/pub/jnt
Open Access

ISSN: 2149-1402 (51) (2025) 92-106 doi.org/10.53570/jnt.1702949

Exploring Soliton Solutions of the Nonlinear Time-Fractional Schrödinger Model via M-Truncated and Atangana-Baleanu Fractional Operators

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Article Info
Received: 20 May 2025
Accepted: 21 Jun 2025
Published: 30 Jun 2025
Research Article

Abstract— This study investigates the nonlinear time-fractional Schrödinger model by utilizing this prototype in fields like nonlinear optics, plasma physics, soliton theory, quantum field theory, and dark matter/neural network modeling. It analyzes the equation to reveal key insights into fundamental physical phenomena, advancing novel technological applications. The paper presents fractional derivatives using M-truncated and Atangana-Baleanu operators. The approach employs Bäcklund transformation and Wang's direct mapping method to derive soliton solutions, including exponential, sin-cos, sinh-cosh, rational, trigonometric, and hyperbolic forms. The present study constructs the energy balance method via the problem's Hamiltonian and variational principle, offering a promising approach. It complements analytical results with numerical simulations to enhance understanding of solution behavior. The study provides foundations for further exploration, ensuring practical, reliable solutions for complex nonlinear problems. The methods prove robust, efficient, and applicable to diverse nonlinear PDEs.

Keywords — *Bäcklund transformation-based approach, Wang's direct mapping method, energy balance method, soliton solutions*

Mathematics Subject Classification (2020) 35A24, 35C08

1. Introduction

Fractional calculus plays a key role by offering a more detailed and accurate mathematical framework for modeling, analyzing, and understanding complex systems and phenomena. It bridges the gap between traditional calculus and the intricate dynamics observed in real-world applications, thereby enabling advancements in technology, scientific research, and various applied fields. Considerable strides have been made in fractional calculus to overcome the limitations of classical differential operators. Consequently, researchers have developed new operators or modified existing ones. The non-singular operator introduced by Caputo and Fabrizio [1] was proposed to resolve the singularity issues present in traditional definitions. However, the Caputo–Fabrizio operator still exhibits non-local behavior, which can be problematic in contexts requiring localized modeling. To simultaneously overcome both singularity and non-locality challenges, Atangana and Baleanu [2] introduced a new fractional operator. This operator forms provides an effective alternative by reducing non-local effects while eliminating singularities.

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Fractional derivatives have a wide range of implications in various fields, such as physics, transport, fluid motion, elastic media, robotics, mechanics, electromagnetic theory, engineering, geophysics, signal processing, and biology. ecent advancements have shown that fractional derivatives can also be used to describe financial [3] and economic systems [4]. Given that many complex problems can be expressed as fractional differential equations (FDEs), the exploration of both analytical and numerical methods for solving such equations is of significant interest [5,6]. These models are valuable for studying various real-world nonlinear phenomena that arise across diverse disciplines, including engineering, physics, and applied mathematics.

Classical definitions of fractional derivatives often face limitations regarding physical interpretability and are highly sensitive to initial conditions. In contrast, the M-truncated and Atangana–Baleanu (AB) fractional derivatives employed in this study offer significant advantages for modeling complex memory effects and damping behaviors in physical systems. Notably, the Mittag–Leffler kernel associated with the AB derivative enhances its ability to capture long-term effects more accurately than classical approaches. On the other hand, the M-truncated derivative provides greater flexibility in analytical treatment, making it more suitable for various solution strategies. Consequently, both operators serve as effective modeling tools, capable of representing physical processes with both mathematical rigor and practical relevance.

The Schrödinger equation is one of the fundamental models in quantum theory. In the Schrödinger equation, Naber [7] included the time-fractional derivative. This equation is applied in a variety of physics fields, including particle physics, biological systems, nuclear physics, atomic physics, molecular chemistry, astrophysics, solid-state physics, quantum mechanics in nanostructures, condensed matter physics, and quantum information and computing [8].

Several analytical methods have been used to obtain exact soliton solutions. These include the modified generalized Riccati equation mapping method [9], the auxiliary equation method [10], the improved modified Sardar sub-equation method [11], the new version of the generalized exponential rational function method [12], the modified Kudryashov method [13], the sine-cose method [14], the modified extended Tanh expansion method [15], the extended simplest equation method [16], and Lie symmetry analysis and conservation laws [17]. In this study, a nonlinear time-fractional model with M-truncated and AB fractional operators are used. These investigations include both singular and non-singular operators. The Bäcklund transformation-based method, Wang's direct mapping method, and the energy balance method (EBM) are utilized to obtain the soliton solutions. Additionally, these methods are novel in the literature for the investigated model.

The remaining sections of this paper are organized as follows: Section 2 describes the model used in the study. Section 3 presents the preliminary definitions. Section 4 proposes the fractional model. Section 5 uses the homogeneous balancing approach to obtain the Bäcklund transformation and the ansatz function schemes to construct numerous exact solutions through symbolic computation. Section 6 investigates the various soliton solutions using Wang's direct mapping method. Section 7 implements the EBM. Section 8 provides comparisons. Section 9 presents the physical properties of the obtained results in graphical form. Finally, Section 10 provides the conclusions.

2. A Summary of the Model

Consider the nonlinear time-fractional Schrödinger equation [18]:

$$iV_t + a_1 V_{xx} + a_2 |V|^2 V = 0 \tag{2.1}$$

where $V = V(x,t)$ represents a complex-valued function depend on both the spatial variable x and the temporal variable t, where $t > 0$. The nonlinear Schrödinger equation is a fundamental equation in quantum mechanics, playing a pivotal role in understanding various phenomena. The model presented is of significant importance in contemporary scientific research and is regarded as a widely applicable indirect model. It arises from diverse applied mathematics and physics areas, including astrophysics, biophysics, plasma dynamics, quantum optics, fluid dynamics, chaos theory, nonlinear wave propagation, and quantum information theory [19]. The specific form considered here is a well-known variant of the classical Schrödinger equation, where the constant parameter a_2 determines the type of soliton solution: a positive a_2 leads to bright solitons, while a negative a_2 produces dark solitons [18]. Previous studies have investigated this model using different fractional derivatives. For example, Gurefe [20] applied Atangana's conformable fractional derivative and obtained five exact solutions via the generalized Kudryashov method. Ahmad et al. [21] used the unified method, and Asjad et al. [22] employed an extended direct algebraic technique to construct multiple exact solutions for the same model.

3. Preliminaries

This section presents the AB fractional operator and the M-truncated operator, along with a description of the operators used.

Definition 3.1. [2] Let $g : [a,b] \to \mathbb{R}$ be a continuous function and $0 < \varepsilon < 1$. Then, the AB-type fractional operator in the sense of the Riemann–Liouville operator is defined by

$$ {}_{0}^{ABR}\digamma_{a^+}^{\varepsilon}(g(t)) = \frac{AB(\varepsilon)}{(1-\varepsilon)} \frac{d}{dt} \int_a^t g(t) E_\varepsilon \left(\frac{-\varepsilon(t-t)^\varepsilon}{1-\varepsilon} \right) dt $$

where the normalization is represented by $AB(\varepsilon)$, and the Mittag-Leffer function is shown by E_ε. Consequently,

$$ {}_{0}^{ABR}\digamma_{a^+}^{\varepsilon}(g(t)) = \frac{AB(\varepsilon)}{(1-\varepsilon)} \sum_{j=0}^{\infty} \left(\frac{-\varepsilon}{1-\varepsilon} \right)^j \, {}^{RL}I_\alpha^{\varepsilon j} g(t) $$

Definition 3.2. [23] Let $h : \mathbb{R}^+ \to \mathbb{R}$ be a differentiable function, $0 < \varepsilon < 1$, $\gamma > 0$, and $t > 0$. Then, the generalized fractional operator is defined by

$$ {}_{\iota}\digamma_M^{\varepsilon,\gamma}(h(t)) = \lim_{\varepsilon \to 0} \frac{h(t +_\iota E_\gamma(\varepsilon t^{-\varepsilon})) - h(t)}{\varepsilon} $$

in which ${}_{\iota}E_\gamma(\cdot)$ is a single-parameter Mittag-Leffer function.

Definition 3.3. [24] Let $\gamma > 0$, $Z \in \mathbb{C}$, and $\iota \in \mathbb{N}$. Then, ${}_{\iota}E_\gamma$ is defined as follows:

$$ {}_{\iota}E_\gamma(Z) = \sum_{j=0}^{\iota} \frac{Z^j}{\Gamma(\gamma j + 1)} $$

Theorem 3.4. Let $h : \mathbb{R}^+ \to \mathbb{R}$ be a function that is continuous at $t = 0$, and differentiable at some $t_0 > 0$ both in the classical and fractional sense of order $\alpha \in (0,1)$, with a fixed parameter $\gamma > 0$. Then, the function h satisfies the required conditions for the application of the generalized fractional operator.

Theorem 3.5. [23] Let $c_1, c_2, c_3 \in \mathbb{R}$, $0 < \varepsilon \leq 1$, $\gamma > 0$, and $g, h : \mathbb{R}^+ \to \mathbb{R}$ be differentiable functions for $t > 0$. Then, the following properties hold for the fractional operator ${}_{\iota}\digamma_M^{\varepsilon,\gamma}$:

$$ {}_{\iota}\digamma_M^{\varepsilon,\gamma}(c_3) = 0 $$

$$ {}_{\iota}\digamma_M^{\varepsilon,\gamma}(k(g(t))) = k \, {}_{\iota}\digamma_M^{\varepsilon,\gamma}(g(t)) $$

$$
{}_{\iota}F_{M}^{\varepsilon,\gamma}(c_1(g(t))+c_2(h(t)))=c_1\ {}_{\iota}F_{M}^{\varepsilon,\gamma}(g(t))+c_2\ {}_{\iota}F_{M}^{\varepsilon,\gamma}(h(t))
$$

$$
{}_{\iota}F_{M}^{\varepsilon,\gamma}(h(t)*g(t))=(h(t))\ {}_{\iota}F_{M}^{\varepsilon,\gamma}(g(t))*(g(t))\ {}_{\iota}F_{M}^{\varepsilon,\gamma}(h(t))
$$

$$
{}_{\iota}F_{M}^{\varepsilon,\gamma}\left(\frac{h(t)}{g(t)}\right)=\frac{(g(t))\ {}_{\iota}F_{M}^{\varepsilon,\gamma}(h(t))-(h(t))\ {}_{\iota}F_{M}^{\varepsilon,\gamma}(g(t))}{(g(t))^2}
$$

$$
{}_{\iota}F_{M}^{\varepsilon,\gamma}(t^{\vartheta})=T^{\vartheta-\varepsilon},\quad \vartheta\in\mathbb{R}
$$

and

$$
{}_{\iota}F_{M}^{\varepsilon,\gamma}(g(h))(t)=g^{'}(h\ {}_{\iota}F_{M}^{\varepsilon,\gamma}(h))
$$

4. Fractional Representations of the Aforementioned Model

This section presents the fractional structure of the nonlinear partial differential equation (NLPDE) using two different fractional operators, including both singular and nonsingular kernels.

i. The R-L sense form of (2.1) has the fractional operator AB as

$$
\iota{}_{0}^{AB}F_{t}^{\varepsilon}V+a_1V_{xx}+a_2\left|V\right|^2V=0
$$

where the AB in the context of R-L operators with regard to t is represented by ${}_{0}^{AB}F_{t}^{\varepsilon}$.

ii. The modified M-truncated form of the fractional operator of (2.1) is

$$
\iota F_{M,t}^{\varepsilon}V+a_1V_{xx}+a_2\left|V\right|^2V=0
$$

where the modified M-truncated operators based on t is represented by $F_{M,t}^{\varepsilon}$.

4.1. Construction of the Operational Transformation Scheme

Consider the following wave transformation:

$$
V=V(x,t)=L(\xi)e^{i\Theta} \tag{4.1}
$$

The analysis of the fractional derivatives will be carried out with respect to ξ and Θ.

i. For the M-truncated operator, we produce ξ and Θ as follows:

$$
\xi=x-\frac{2a_4a_5}{\varepsilon}\left(t+\frac{1}{\Gamma(\varepsilon)}\right)^{\varepsilon}
$$

and

$$
\Theta=xa_3+\frac{a_4}{\varepsilon}\left(t+\frac{1}{\Gamma(\varepsilon)}\right)^{\varepsilon}+\theta \tag{4.2}
$$

ii. For the AB operator in RL sense, we produce ξ and Θ as follows:

$$
\xi=x-\frac{(1-\varepsilon)(2a_4a_5t^{-n\varepsilon})}{AB(\varepsilon)\sum\limits_{n=0}^{\infty}\left(\frac{-\varepsilon}{1-\varepsilon}\right)\Gamma(1-\varepsilon n)}
$$

and

$$
\Theta=a_3x+\frac{(1-\varepsilon)(a_4t^{-n\varepsilon})}{AB(\varepsilon)\sum\limits_{n=0}^{\infty}\left(\frac{-\varepsilon}{1-\varepsilon}\right)\Gamma(1-\varepsilon n)}+\theta \tag{4.3}
$$

5. Methodology

Consider the subsequent NLPDE:

$$P_1(V, V_x, V_t, V_{xx}, V_{xt}, ...) = 0 \tag{5.1}$$

in which (5.1) is satisfied by $V = V(x,t)$. By applying the transformation given in (4.1), where ξ is defined accordingly, the formulation is considered both for the M-truncated operator as in (4.2) and for the AB operator in the Riemann–Liouville sense as in (4.3). Consequently, (5.1) is transformed into an nonlinear ordinary differential equation (NODE) as follows:

$$P_2(L, L', L'', ...) = 0 \tag{5.2}$$

Applying (4.1) into (2.1), the imaginary and real components are obtained as follows, respectively:

$$a_5 = \frac{a_1 a_3}{a_4}$$

and

$$a_1 L'' - \left(a_4 + a_1 a_3^2\right) L + a_2 L^3 = 0 \tag{5.3}$$

5.1. Bäcklund Transformation and Implementing

Assume that (5.2) provides the subsequent solution form [25]:

$$L = A_1 \frac{\partial^{\varrho} \hbar}{\partial \xi^{\varrho}} + A_2 \tag{5.4}$$

To calculate the value of the equilibrium parameter ϱ, we utilize (5.3) to balance L'' and L^3. Thus, $\varrho = 1$.

5.1.1. Type-I: Exponential Function

Assume that

$$\hbar(\xi) = \ln\left(n_1 \exp(-\xi) + n_0 + n_2 \exp(\xi)\right) \tag{5.5}$$

where n_i such that $i \in \{0, 1, 2\}$ are real parameters. By putting (5.4) with (5.5) into (5.3) and taking the coefficients of $\exp(\xi)$ as zero, an algebraic equation system is produced. The solutions of this system are obtained as follows:

Set I.

$$n_0 = n_0, \quad n_1 = 0, \quad n_2 = n_2, \quad a_1 = -\frac{2a_4}{2a_3^2 + 1}, \quad A_1 = -\sqrt{\frac{4a_4}{(2a_2 a_3^2 + a_2)}}, \quad \text{and} \quad A_2 = \sqrt{\frac{a_4}{(2a_2 a_3^2 + a_2)}}$$

Combining (5.4), (5.5), and (4.1) with above results, the exponential function solution of (2.1) is obtained as follows:

$$V_{1,1,1}(x,t) = -\frac{\sqrt{\frac{a_4}{a_2\left(2a_3^2+1\right)}}\left(n_2 \exp(\xi) - n_0\right)}{n_0 + n_2 \exp(\xi)} e^{i(\Theta)} \tag{5.6}$$

Here, ξ and Θ are expressed in (4.2) for the M-truncated operator form and in (4.3) for the AB operator form in RL sense, respectively.

Set II.

$$n_0 = n_0, \quad n_1 = n_1, \quad n_2 = 0, \quad a_1 = -\frac{2a_4}{2a_3^2 + 1}, \quad A_1 = \sqrt{\frac{4a_4}{(2a_2 a_3^2 + a_2)}}, \quad \text{and} \quad A_2 = \sqrt{\frac{a_4}{(2a_2 a_3^2 + a_2)}}$$

Combining (5.4), (5.5), and (4.1) with above results, the exponential function solution of (2.1) is obtained as follows:

$$V_{1,1,2}(x,t) = -\frac{\sqrt{\frac{a_4}{a_2\left(2a_3^2+1\right)}}\left(n_1 \exp\left(-\xi\right) - n_0\right)}{n_0 + n_1 \exp\left(-\xi\right)} e^{i(\Theta)}$$

Here, ξ and Θ are expressed in (4.2) for the M-truncated operator form and in (4.3) for the AB operator form in RL sense, respectively.

Set III.

$$n_0 = 2\sqrt{n_1 n_2}, \quad n_1 = n_1, \quad n_2 = n_2, \quad a_1 = -\frac{2a_4}{2a_3^2+1}, \quad A_1 = \sqrt{\frac{a_4}{\left(2a_2a_3^2 + a_2\right)}}, \quad \text{and} \quad A_2 = 0$$

Combining (5.4), (5.5), and (4.1) with above results, the exponential function solution of (2.1) is obtained as follows:

$$V_{1,1,3}(x,t) = -\frac{\sqrt{\frac{a_4}{a_2\left(2a_3^2+1\right)}}\left(n_1 \exp\left(-\xi\right) - n_2 \exp\left(\xi\right)\right)}{n_1 \exp\left(-\xi\right) + 2\ \sqrt{n_1 n_2} + n_2 \exp\left(\xi\right)} e^{i(\Theta)}$$

Here, ξ and Θ are expressed in (4.2) for the M-truncated operator form and in (4.3) for the AB operator form in RL sense, respectively.

Set IV.

$$n_0 = 0, \quad n_1 = n_1, \quad n_2 = n_2, \quad a_1 = -\frac{a_4}{a_3^2+2}, \quad A_1 = \sqrt{\frac{4a_4}{\left(2a_2a_3^2 + a_2\right)}}, \quad \text{and} \quad A_2 = 0$$

Combining (5.4), (5.5), and (4.1) with above results, the exponential function solution of (2.1) is obtained as follows:

$$V_{1,1,4}(x,t) = -\frac{\sqrt{\frac{4a_4}{a_2\left(2a_3^2+1\right)}}\left(n_1 \exp\left(-\xi\right) - n_2 \exp\left(\xi\right)\right)}{n_1 \exp\left(-\xi\right) + n_2 \exp\left(\xi\right)} e^{i(\Theta)}$$

Here, ξ and Θ are expressed in (4.2) for the M-truncated operator form and in (4.3) for the AB operator form in RL sense, respectively.

5.1.2. Type-II: Sin-Cos Function

Assume that

$$\hbar(\xi) = \ln\left(n_0 \cos(\xi) + n_1 + n_2 \sin(\xi)\right) \tag{5.7}$$

where n_i such that $i \in \{0,1,2\}$ are real parameters. By putting (5.4) with (5.7) into (5.3) and taking the coefficients of $\cos(\xi)$ and $\sin(\xi)$ as zero, an algebraic equation system is produced. The solution of this system is reached as follows:

Set I.

$$n_0 = n_0, \quad n_1 = n_1, \quad n_2 = -\frac{a_2 n_0\left(2a_3^2 - 1\right)}{a_4}\sqrt{-\frac{a_4}{\left(2a_2a_3^2 - a_2\right)}}\sqrt{\frac{a_4}{\left(2a_2a_3^2 - a_2\right)}}, \quad a_1 = -\frac{2a_4}{2a_3^2 - 1}$$

$$A_1 = \sqrt{\frac{4a_4}{\left(2a_2a_3^2 - a_2\right)}}, \quad \text{and} \quad A_2 = \sqrt{-\frac{a_4}{\left(2a_2a_3^2 - a_2\right)}}$$

Combining (5.4), (5.7), and (4.1) with above results, the sin-cos function solution of (2.1) is obtained as follows:

$$V_{2,1,1}(x,t)=\frac{a_4\begin{pmatrix}\sin(\xi)\,n_0\sqrt{\frac{a_4}{a_2\left(2a_3^2-1\right)}}\\+\cos(\xi)\,n_0\sqrt{-\frac{a_4}{a_2\left(2a_3^2-1\right)}}\\-\sqrt{-\frac{a_4}{a_2\left(2a_3^2-1\right)}}n_1\end{pmatrix}}{\begin{pmatrix}\left(2\sin(\xi)\,a_2a_3^2n_0-\sin(\xi)\,a_2n_0\right)\sqrt{\frac{a_4}{a_2\left(2a_3^2-1\right)}}\sqrt{-\frac{a_4}{a_2\left(2a_3^2-1\right)}}\\-\cos(\xi)\,a_4n_0-a_4n_1\end{pmatrix}}e^{i(\Theta)} \tag{5.8}$$

Here, ξ and Θ are expressed in (4.2) for the M-truncated operator form and in (4.3) for the AB operator form in RL sense, respectively.

5.1.3. Type-III: Sinh–Cosh Function

Assume that

$$\hbar(\xi)=\ln\left(n_0\cosh(\xi)+n_1+n_2\sinh(\xi)\right) \tag{5.9}$$

where n_i such that $i\in\{0,1,2\}$ are real parameters. By putting (5.4) with (5.9) into (5.3) and taking the coefficients of $\cos(\xi)$ and $\sin(\xi)$ as zero, an algebraic equation system is produced. The solution of this system is attained as follows:

Set I.

$$n_0=n_0,\quad n_1=n_1,\quad n_2=-\frac{a_2n_0\left(2a_3^2+1\right)}{\left(2a_2a_3^2+a_2\right)},\quad a_1=-\frac{2a_4}{2a_3^2+1},$$

$$A_1=\sqrt{\frac{4a_4}{\left(2a_2a_3^2+a_2\right)}},\quad\text{and}\quad A_2=\sqrt{\frac{a_4}{\left(2a_2a_3^2+a_2\right)}}$$

Combining (5.4), (5.9), and (4.1) with above results, the sinh–cosh function solution of (2.1) is obtained as follows:

$$V_{3,1,1}(x,t)=-\frac{\sqrt{\frac{a_4}{a_2\left(2a_3^2+1\right)}}\left(\sinh(\xi)\,n_0-\cosh(\xi)\,n_0+n_1\right)}{\left(\sinh(\xi)\,n_0-\cosh(\xi)\,n_0-n_1\right)}e^{i(\Theta)}$$

Here, ξ and Θ are expressed in (4.2) for the M-truncated operator form and in (4.3) for the AB operator form in RL sense, respectively.

5.1.4. Type-IV: Rational Function

Assume that

$$\hbar(\xi)=\ln\left(n_0+n_1(\xi)\right) \tag{5.10}$$

where n_i such that $i\in\{0,1,2\}$ are real parameters. By putting (5.4) with (5.10) into (5.3) and taking the coefficients of $\cos(\xi)$ and $\sin(\xi)$ as zero, an algebraic equation system is produced. The solution of this system is gained as follows:

Set I.

$$n_0=n_0,\quad n_1=n_1,\quad n_2=n_2,\quad a_1=-\frac{a_4}{a_3^2},\quad A_1=\frac{\sqrt{\frac{2a_4}{a_2}}}{a_3},\quad\text{and}\quad A_2=0$$

Combining (5.4), (5.10), and (4.1) with above results, the rational function solution of (2.1) is obtained as follows:

$$V_{4,1,1}(x,t)=-\frac{\sqrt{\frac{a_4}{a_2\left(2a_3^2+1\right)}}\left(\sinh(\xi)\,n_0-\cosh(\xi)\,n_0+n_1\right)}{\left(\sinh(\xi)\,n_0-\cosh(\xi)\,n_0-n_1\right)}e^{i(\Theta)}$$

Here, ξ and Θ are expressed in (4.2) for the M-truncated operator form and in (4.3) for the AB operator form in RL sense, respectively.

6. Wang's Direct Mapping Approach

In this section, (5.3) is addressed using Wang's direct mapping approach. This approach gives the auxiliary function listed below [25]:

$$\left(\Lambda'(\xi)\right)^2 = \varphi\tau^2\Lambda^2(\xi) + \frac{\vartheta\tau^2}{\rho^2}\Lambda^4(\xi)$$

where

$$\Lambda(\xi) = \rho\sec h(\tau\xi), \quad \varphi = 1 \text{ and } \vartheta = -1$$

$$\Lambda(\xi) = \rho\csc h(\tau\xi), \quad \varphi = 1 \text{ and } \vartheta = 1$$

$$\Lambda(\xi) = \rho\sec(\tau\xi), \quad \varphi = -1 \text{ and } \vartheta = 1$$

and

$$\Lambda(\xi) = \rho\csc(\tau\xi), \quad \varphi = -1 \text{ and } \vartheta = 1 \tag{6.1}$$

If (5.3) is integrated with respect to ξ and the integral constant is set to zero, the following results are obtained:

$$\left(L'\right)^2 = -\frac{a_2}{2a_1}L^4 + \frac{(a_1a_3^2 + a_4)}{a_1}L^2 \tag{6.2}$$

Using (6.2) to map (6.1), the solution are derived as follows:

Set I.

$$\varphi = 1, \quad \vartheta = -1, \quad \tau^2 = \frac{a_1a_3^2 + a_4}{a_1}, \quad \frac{\tau^2}{\rho^2} = \frac{a_2}{2a_1}, \quad \tau = \frac{\sqrt{a_1\left(a_1a_3^2 + a_4\right)}}{a_1}, \quad \text{and} \quad \rho = \frac{\sqrt{2a_2\left(a_1a_3^2 + a_4\right)}}{a_1}$$

Applying the method's steps together with the above results, the solution to (2.1) is obtained as follows:

$$V_{2,2,1}(x,t) = \frac{\sqrt{2a_2\left(a_1a_3^2 + a_4\right)}\sec h\left(\frac{\sqrt{a_1\left(a_1a_3^2+a_4\right)}}{a_1}\xi\right)}{a_2}e^{i(\Theta)} \tag{6.3}$$

Set II.

$$\varphi = 1, \quad \vartheta = 1, \quad \tau^2 = \frac{a_1a_3^2 + a_4}{a_1}, \quad \frac{\tau^2}{\rho^2} = -\frac{a_2}{2a_1}, \quad \tau = \frac{\sqrt{a_1\left(a_1a_3^2 + a_4\right)}}{a_1}, \quad \text{and} \quad \rho = \frac{\sqrt{-2a_2\left(a_1a_3^2 + a_4\right)}}{a_1}$$

Applying the method's steps together with the above results, the solution to (2.1) is obtained as follows:

$$V_{2,2,2}(x,t) = \frac{\sqrt{-2a_2\left(a_1a_3^2 + a_4\right)}\csc h\left(\frac{\sqrt{a_1\left(a_1a_3^2+a_4\right)}}{a_1}\xi\right)}{a_2}e^{i(\Theta)}$$

Set III.

$$\varphi = 1, \quad \vartheta = -1, \quad \tau^2 = \frac{a_1a_3^2 + a_4}{a_1}, \quad \frac{\tau^2}{\rho^2} = \frac{a_2}{2a_1}, \quad \tau = \frac{\sqrt{a_1\left(a_1a_3^2 + a_4\right)}}{a_1}, \quad \text{and} \quad \rho = \frac{\sqrt{2a_2\left(a_1a_3^2 + a_4\right)}}{a_1}$$

Applying the method's steps together with the above results, the solution to (2.1) is obtained as follows:

$$V_{2,2,3}(x,t) = \frac{\sqrt{2a_2\left(a_1a_3^2 + a_4\right)}\sec\left(\frac{\sqrt{a_1\left(a_1a_3^2+a_4\right)}}{a_1}\xi\right)}{a_2}e^{i(\Theta)}$$

Set IV.

$$\varphi=1,\quad \vartheta=-1,\quad \tau^2=\frac{a_1a_3^2+a_4}{a_1},\quad \frac{\tau^2}{\rho^2}=\frac{a_2}{2a_1},\quad \tau=\frac{\sqrt{a_1\left(a_1a_3^2+a_4\right)}}{a_1},\quad \text{and}\quad \rho=\frac{\sqrt{2a_2\left(a_1a_3^2+a_4\right)}}{a_1}$$

Applying the method's steps together with the above results, the solution to (2.1) is obtained as follows:

$$V_{2,2,4}(x,t)=\frac{\sqrt{2a_2\left(a_1a_3^2+a_4\right)}\csc\left(\frac{\sqrt{a_1\left(a_1a_3^2+a_4\right)}}{a_1}\xi\right)}{a_2}e^{i(\Theta)}$$

7. EBM

To apply the EBM, we refer to (5.3), as presented in [26]:

$$L''-\frac{\left(a_4+a_1a_3^2\right)}{a_1}L+\frac{a_2}{a_1}L^3=0 \tag{7.1}$$

The subsequent is the expression for the relevant variational principle:

$$Y(L)=\int_0^{\frac{\pi}{4}}\left(\frac{1}{2}L'+\frac{\left(a_4+a_1a_3^2\right)}{2a_1}L^2-\frac{a_2}{4a_1}L^4\right)d\xi$$

This implies that

$$Y(L)\;=\;\int_0^{\frac{\pi}{4}}\left(\frac{1}{2}L'-\left[\frac{a_2}{4a_1}L^4-\frac{\left(a_4+a_1a_3^2\right)}{2a_1}L^2\right]\right)d\xi=\int_0^{\frac{\pi}{4}}(R-P)\,d\xi$$

Here, R and P represents kinetic energy and potential energy and are expressed as follows, respectively:

$$R=\frac{1}{2}L'\quad\text{and}\quad P=\frac{a_2}{4a_1}L^4-\frac{\left(a_4+a_1a_3^2\right)}{2a_1}L^2$$

Therefore, the Hamiltonian invariant is given by

$$H=R+P=\frac{1}{2}L'+\frac{a_2}{4a_1}L^4-\frac{\left(a_4+a_1a_3^2\right)}{2a_1}L^2$$

The solution to (7.1) can be written as follows:

$$L(\xi)=K\cos(\varphi\xi) \tag{7.2}$$

Based on the EBM, the Hamiltonian invariant must remain constant, i.e.,

$$H=P+R=\frac{1}{2}L'+\frac{a_2}{4a_1}L^4-\frac{\left(a_4+a_1a_3^2\right)}{2a_1}L^2=H_0 \tag{7.3}$$

For (7.2), the initial conditions are expressed as:

$$L'(0)=0\quad\text{and}\quad L(0)=K$$

By replacing these into (7.3),

$$L_0=\frac{a_2}{4a_1}K^4-\frac{\left(a_4+a_1a_3^2\right)}{2a_1}K^2 \tag{7.4}$$

Inserting the expressions from (7.4) and (7.2) into (7.3) leads to the following equality:

$$\frac{1}{2}\left(-\zeta K\cos(\zeta\xi)\right)^2+\frac{a_2}{4a_1}\left(K\cos(\zeta\xi)\right)^4-\frac{\left(a_4+a_1a_3^2\right)}{2a_1}\left(K\cos(\zeta\xi)\right)^2=\frac{a_2}{4a_1}K^4-\frac{\left(a_4+a_1a_3^2\right)}{2a_1}K^2 \tag{7.5}$$

After applying $\zeta\xi=\frac{\pi}{4}$ to (7.5),

$$\frac{1}{4}\zeta^2K^2+\frac{a_2}{16a_1}K^4-\frac{\left(a_4+a_1a_3^2\right)}{4a_1}K^2=\frac{a_2}{4a_1}K^4-\frac{\left(a_4+a_1a_3^2\right)}{2a_1}K^2$$

Consequently,

$$\zeta = \frac{\sqrt{a_1\left(3K^2a_2 - 4a_1a_3^2 - 4a_4\right)}}{2a_1}$$

Therefore, the following statement is the outcomes to (5.2):

$$L\left(x,t\right) = K\cos\left(\zeta\xi\right)$$

After entering L's value into (5.1),

$$V_{3,1,1}(x,t) = K\cos\left(\zeta\xi\right)e^{i(\Theta)} \tag{7.6}$$

8. Comparisons

In this paper, the Bäcklund transformation-based method, Wang's direct mapping methodology, and the EBM have been used to generate new soliton solutions for the mathematical and physical problems describing by nonlinear fractional model. Gurefe [20] analyzed the model using Atangana's conformable fractional derivative. He produced only five results using the generalized Kudryashov technique. Ahmad et al. [21] applied the unified technique. Asjad et al. [22] used the novel extended direct algebraic technique to obtain several solutions for the model. Notably, the current study has yielded new solutions with various physical properties, such as bright solitons and periodic-type wave structures. The soliton solutions have a wide range of critical real-world applications in fields such as nonlinear optics, phase evolution, chaos theory, condensed matter physics, astronomy, fluid mechanics, biology, and nonlinear quantum field theory. The obtained solutions provide valuable insights into the physical behaviors of the modal. Moreover, a visual depiction of the solutions, systematically derived through analytical methods, is included.

9. Physical Interpretation

This section provides a graphical interpretation of some of the solutions derived from the nonlinear fractional model. Various soliton solutions of the prototype were constructed using the Bäcklund transformation-based method, Wang's direct mapping approach, and EBM.

In Figures 1 and 2, the periodic waves behavior are investigated using M-truncated operators in subfigures a, b, and c and AB fractional operators in subfigures d, e, and f. In Figures 3 and 4, the dark-bright soliton behavior are investigated using M-truncated operators in subfigures a, b, and c and AB fractional operators in subfigures d, e, and f. In Figure 1, the 3d, 2d, and contour plots for the real part of $V_{1,1,1}(x,t)$ in (5.6) with fractional order $\varepsilon = 0.5$, by choosing the values $a_4 = 3$, $a_2 = 2$, $a_3 = 1$, $n_0 = 1.5$, $n_2 = 7$, $a_1 = 0.7$, $n = 0.5$, and $\theta = 0$. In Figure 2, the 3d, 2d, and contour plots for the real part of $V_{2,1,1}(x,t)$ in (5.8) with fractional order $\varepsilon = 0.5$, by choosing the values $a_4 = 1.7$, $a_2 = 0.9$, $a_3 = 0.2$, $n_0 = 1$, $n_1 = 2$, $a_1 = 0.4$, $n = 1$, and $\theta = 0$. In Figure 3, the 3d, 2d, and contour plots for the real part of $V_{2,2,1}(x,t)$ in (6.3) with fractional order $\varepsilon = 0.5$, by choosing the values $a_4 = 0.7$, $a_2 = 1.2$, $a_3 = 0.5$, $a_1 = 0.07$, $n = 0.5$, and $\theta = 0$. In Figure 4, the 3d, 2d, and contour plots for the real part of $V_{3,1,1}(x,t)$ in (7.6) with fractional order $\varepsilon = 0.5$, by choosing the values $a_4 = 0.02$, $a_2 = 4$, $a_3 = 0.5$, $a_1 = 1.9$, $K = 1$, $n = 0.5$, and $\theta = 0$.

The parameters in the diagrams have distinct physical meanings. Whereas a_2 describes the type and degree of nonlinearity in the system, a_1 is the dispersion coefficient that controls the wave spreading behavior throughout space. Whether a solution covers a bright or dark soliton depends on the sign of a_2. Whereas a_4 aids in the temporal phase evolution, a_3 influences the spatial phase modulation. The memory effect in the system is controlled by the parameter ε, which represents the order of the fractional derivative. Stronger nonlocal temporal effects are associated with lower levels of ε. Together, these parameters determine the outcome of soliton structures' propagation, shape, and stability properties.

In particular, the effect of the fractional order ε changes depending on whether the M-truncated or AB fractional operators are used. By changing the fractional order ε, the impact of previous states on the system's current behavior is directly changed. In the AB operator situation, the dispersive smoothing in the wave propagation is improved by decreasing ε because it increases the nonlocal interactions caused by the Mittag-Leffler kernel. Energy spreads throughout a greater area, making solutions smoother and broader. In contrast, because the influence range is bounded and the kernel is truncated for the M-truncated operator, the fractional order has a more localized mathematical effect. Sharp, high-amplitude solitons can appear for small ε because of increased nonlinearity and limited dispersion, but solutions shift to more classical waveforms as ε grows. This shows that the two operators are not affected by the same fractional order ε in the same way; the M-truncated derivative shows more sensitive and localized alterations, whilst the AB derivative reacts more slowly and universally.

By assigning specific values to the parameters, periodic waves and dark-bright soliton solutions were obtained from these results. Periodic solitons are wave structures that continuously repeat in a specific pattern and are often observed in media, such as optical fibers, water waves, or plasma. They enable the undistorted and stable propagation of energy-carrying waves through a medium. Dark-bright solitons combine structures in which the intensity decreases in one part of a wave packet dark soliton, and increases in the other part, a bright soliton. Such solutions have critical applications, particularly in optical systems and atomic physics, because combining two opposite interactions offers opportunities for innovative applications in areas such as energy and information transport, lasers, or quantum computing. These solutions are fundamental tools for understanding and controlling physical processes described by nonlinear equations. It is important to emphasize that the findings and solutions presented in this paper are novel and have not been previously documented.

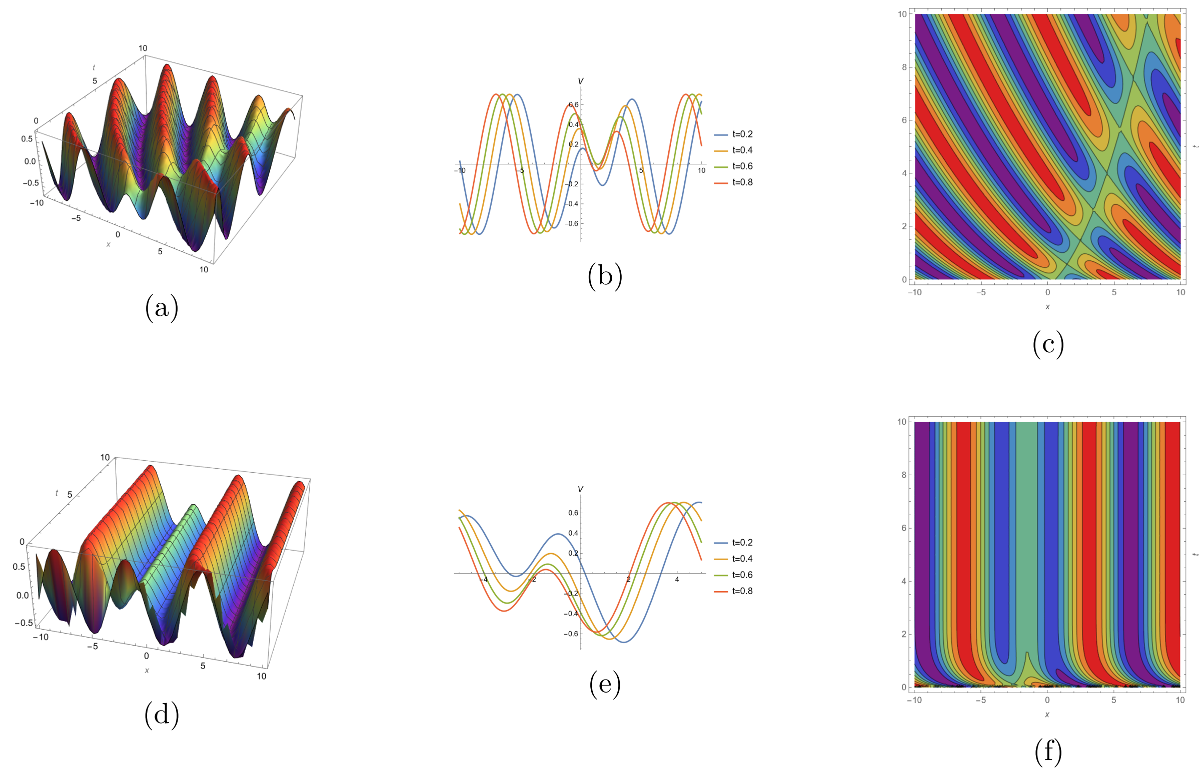


Figure 1. Effects of (a)-(c) the M-truncated and (d)-(f) AB fractional operators of $V_{1,1,1}(x,t)$

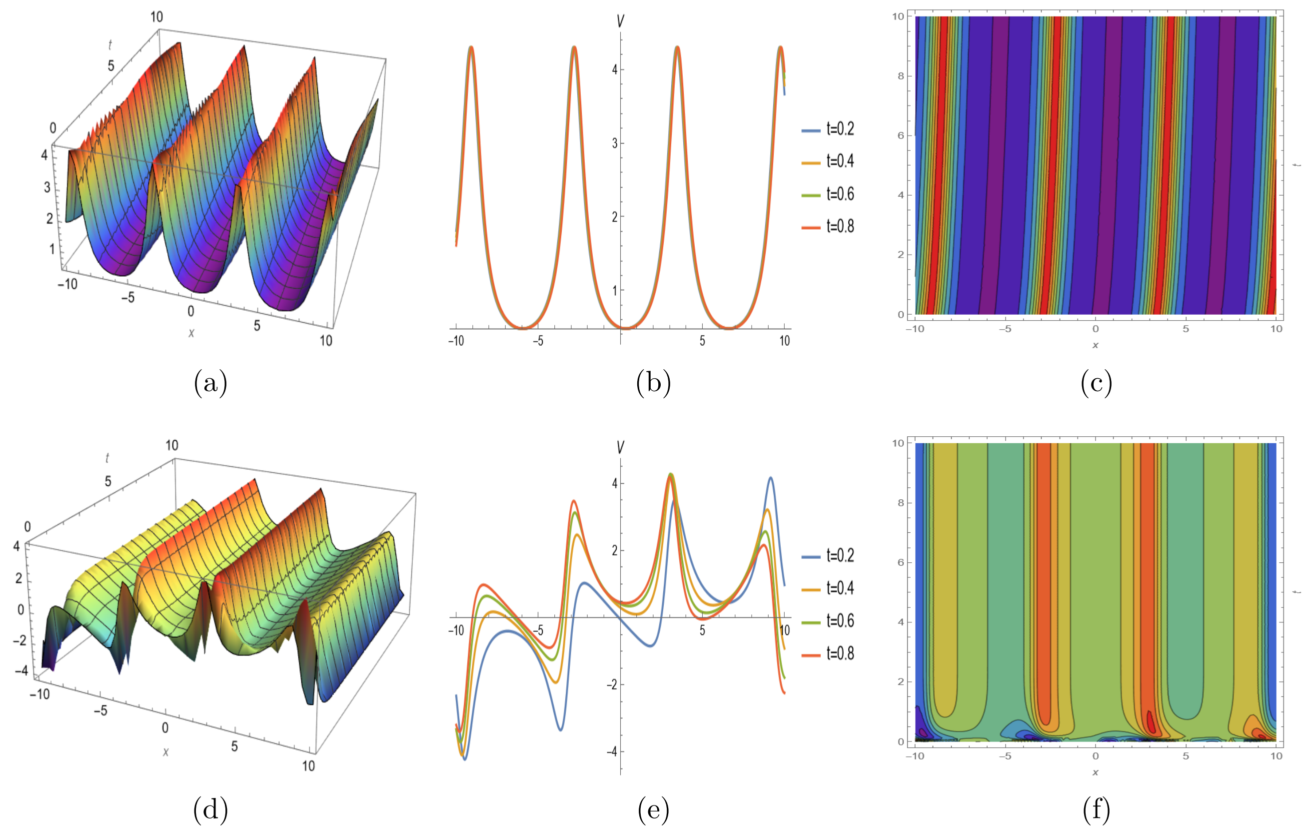


Figure 2. Effects of (a)-(c) the M-truncated and (d)-(f) AB fractional operators of $V_{2,1,1}(x,t)$

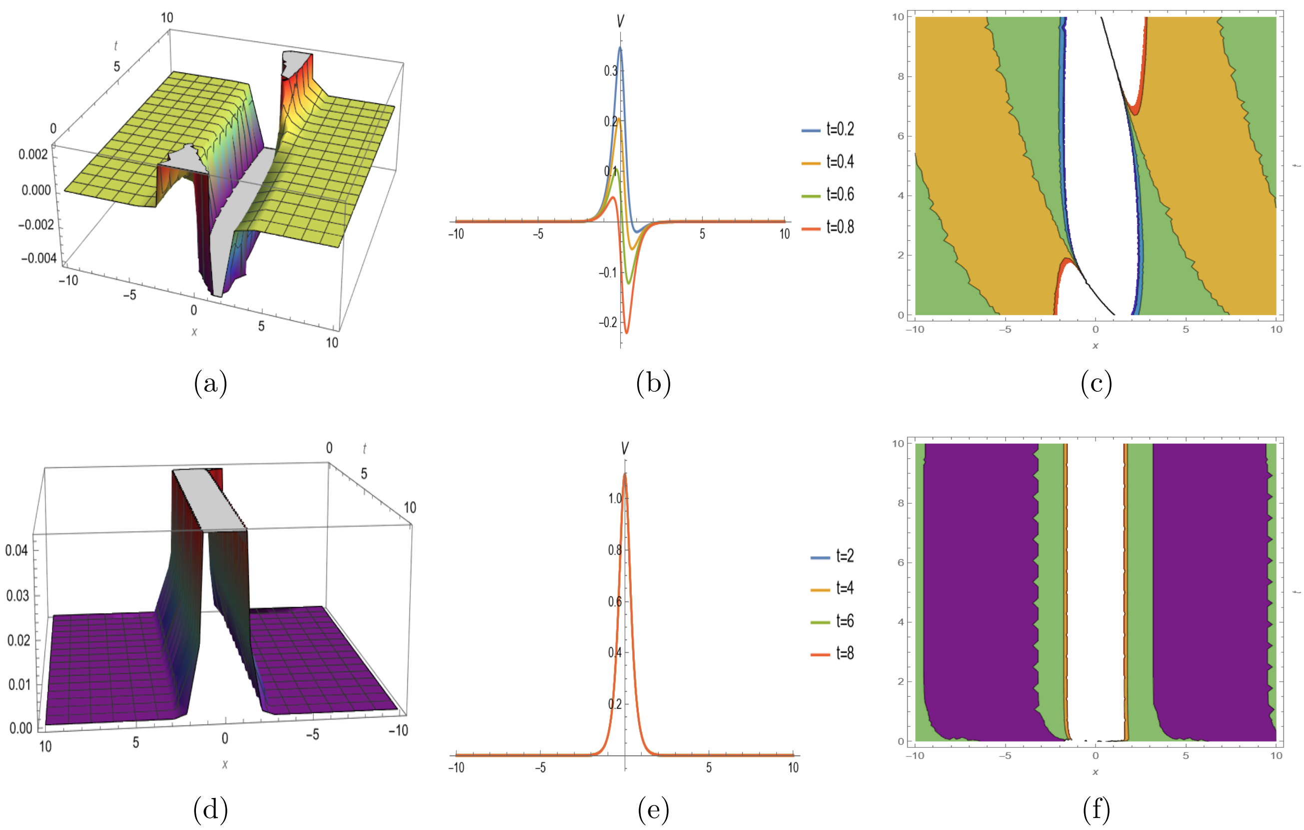
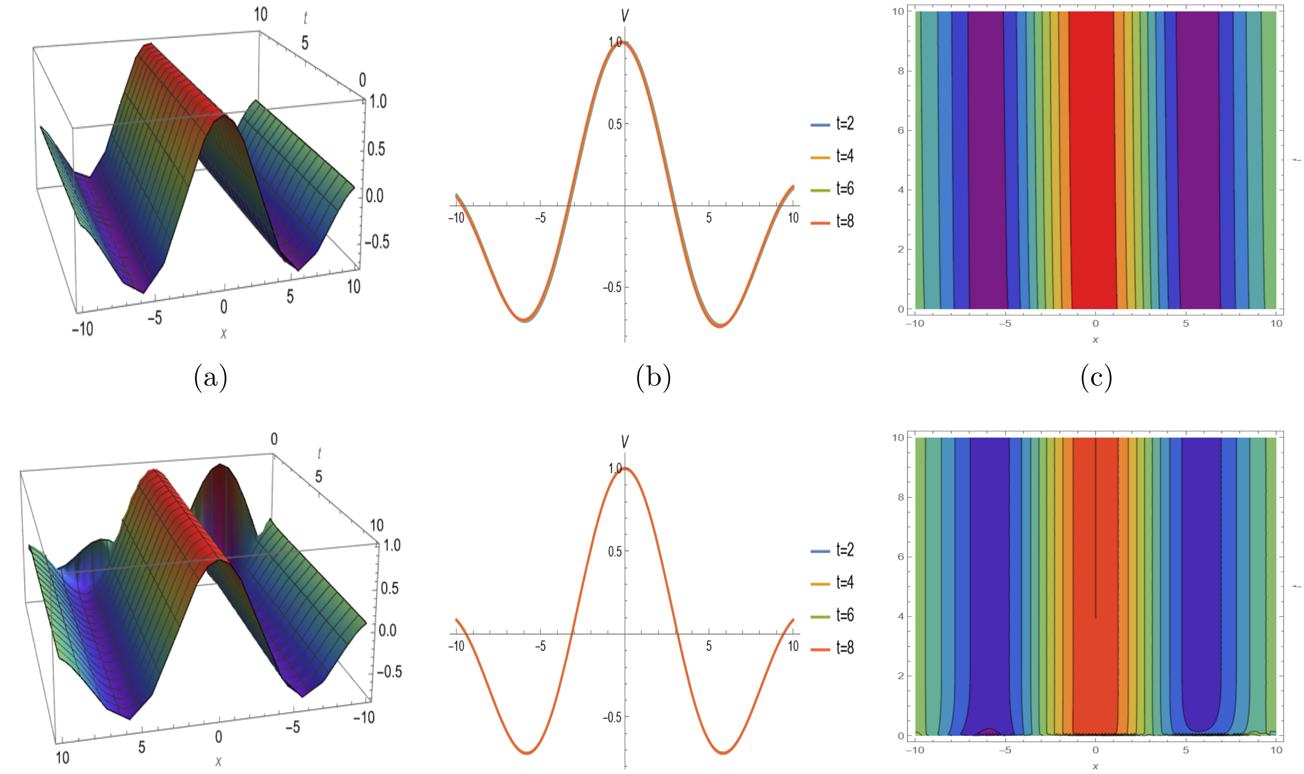


Figure 3. Effects of (a)-(c) the M-truncated and (d)-(f) AB fractional operators of $V_{2,2,1}(x,t)$



(a) (b) (c)

(d) (e) (f)

Figure 4. Effects of (a)-(c) the M-truncated and (d)-(f) AB fractional operators of $V_{3,1,1}(x,t)$

10. Conclusion

This study investigated the nonlinear time-fractional Schrödinger model's wave propagation, which arises in different fields of physics and mathematics. Fractional transformations for the wave variables ξ and θ using M-truncated and AB fractional operators were applied to the prototype and converted to a nonlinear ordinary differential equation. Two new approaches, the Bäcklund transformation-based method and Wang's direct mapping method, were used to find a wide range of optical soliton solutions. The solutions for a wide range of solitons were obtained by expressing them as exponential wave solutions, sin-cos wave solutions, sinh-cosh wave solutions, rational wave solutions, trigonometric functions, and hyperbolic function solutions. The EBM was also used, providing an efficient approach by deriving the Hamiltonian and applying the variational principle to the problem. In addition, 3d, 2d, and contour plots were used to illustrate the profiles of different soliton solutions. The findings obtained by the investigated techniques demonstrate that these techniques effectively examine nonlinear wave equations, enhancing the comprehension of their complicated structures and expanding the potential for theoretical investigation. The applied methods are adaptable and applicable to many different types of NLPDEs. Although the methods are pretty flexible, they can not be suitable for all NLPDEs, particularly those that do not satisfy the predetermined criteria or structures that these methods can handle. Future research could concentrate on analyzing these solutions' behavior under various circumstances and investigating their physical effects. This will help to improve science in physics and its broader applications while offering a greater knowledge of nonlinear wave processes.

Author Contributions

The author read and approved the final version of the paper.

Conflicts of Interest

The author declares no conflict of interest.

Ethical Review and Approval

No approval from the Board of Ethics is required.

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