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RESEARCH ARTICLE

Hybrid continuous multi-step method for second order problems in ordinary differential equations

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ABSTRACT

This study presents the development and analysis of a class of hybrid continuous methods designed for solving second-order initial value problems in ordinary differential equations. The formulation of the method is based on the application of a class of orthogonal and Chebyshev polynomials, which serve as a basis for the numerical approximation. The constructed scheme is subjected to a rigorous stability and convergence analysis, demonstrating its reliability and suitability for the class of problems under consideration. To evaluate the method's effectiveness, numerical experiments were conducted on selected benchmark problems from the literature. The results highlight the efficiency and accuracy of the proposed approach, showing improved numerical performance compared to existing methods. The hybrid continuous formulation ensures better approximation properties while maintaining computational efficiency. The stability properties confirm that the method remains robust across a range of problem scenarios, making it a viable tool for solving second-order differential equations. The study contributes to the ongoing advancement of numerical techniques for differential equations, particularly by leveraging hybrid continuous methods with polynomial-based approximations. The promising results from numerical experiments further establish the potential of this approach for broader applications in computational mathematics and applied sciences.

Mathematics Subject Classification (2020): 65L05

Keywords: Block Method, Hybrid method, Initial Value Problems, One-fourth Step Order, Interpolation and Collocation

1. INTRODUCTION

Ordinary differential equations (ODEs) play a fundamental role in modeling dynamic systems across various scientific and engineering disciplines. In particular, second-order ODEs arise in many real-world applications such as mechanical systems, electrical circuits, and celestial mechanics, where acceleration, forces, or other second derivatives are central to the system's behavior. Traditional numerical methods, such as Runge-Kutta and finite difference techniques, are often used to solve these problems. However, they can encounter limitations in efficiency and accuracy when applied to stiff or highly oscillatory systems. To address these related challenges, hybrid numerical methods have been developed, combining the strengths of different schemes to improve convergence and stability. One such approach is the hybrid continuous multi-step method, which leverages the advantages of both continuous and discrete numerical techniques. By incorporating multiple evaluation points within each step, this method offers enhanced accuracy for second-order ODEs while maintaining computational efficiency. Several numerical methods have been proposed by various authors to address the challenges of solving second-order ODEs efficiently and accurately. These methods often focus on improving convergence, stability, and computational cost through hybrid and multistep techniques. For instance, Odejide et al. (2012) developed a continuous five-step block method using a multistep collocation approach, which produced a class of eight discrete schemes. Their method was designed to provide a balance between computational efficiency and accuracy, particularly for stiff ODEs. By employing multiple collocation points, they were able to increase the accuracy without significantly increasing the computational load. Their work demonstrated the potential of block methods for solving complex ODE systems in a stable and efficient manner. Ibiyola et al. (2011) focused on the formation of hybrid block methods with higher step sizes, utilizing the continuous multistep collocation technique. His approach aimed at improving both the accuracy and stability of numerical solutions for ODEs. By extending the step size while maintaining stability, their method allowed for more efficient solutions to initial value problems (IVPs) in ODEs, especially when dealing with large systems or when high precision was

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required. Adeniyi et al. (2015) work led to the development of higher step hybrid block methods specifically tailored for solving IVPs in ODEs. Their method sought to refine the accuracy of existing block methods by integrating additional evaluation points and employing higher-order techniques, thereby enhancing both accuracy and computational efficiency. More recently, Taiwo et al. (2023) proposed a one-fifth hybrid block approach designed for second-order ODEs. Their method demonstrated significant improvements in accuracy and computational performance, particularly for systems characterized by high oscillatory behavior or stiffness. The use of hybrid block methods in this context highlighted the ongoing refinement and adaptation of multistep methods for complex differential equations. These works collectively emphasize the importance of hybrid and continuous multistep methods in solving ODEs, particularly when dealing with higher-order systems or stiff problems. Building on these foundational studies, our work introduces a new class of hybrid methods using the combination of two orthogonal polynomials, as the basis functions. By incorporating these polynomials into a higher step-size framework, we aim to further improve both the stability and accuracy of numerical solutions for second-order ODEs. The use of Chebyshev polynomials, known for their optimal properties in approximation theory, enhances the precision of our method while maintaining computational efficiency. Anake (2011) thesis offered foundational framework for the use of continuous implicit hybrid methods in numerical analysis, offering a robust alternative for solving a broad class of second-order ODEs with significant implications in various scientific and engineering applications.

Recently, Emmanuel et al. (2024) introduce multi-derivative hybrid block methods for singular initial value problems, demonstrating their applicability to complex systems. A related study by the same authors Emmanuel et al. (2024) explores the use of feed-forward neural networks to improve the computational efficiency of hybrid block derivative methods for second-order ODE systems. These works underscore the growing integration of machine learning techniques in numerical analysis. Ogunniran et al. (2023) generalize a class of uniformly optimized k-step hybrid block methods for solving two-point boundary value problems, extending the versatility of these approaches. Further expanding on rational approximations, Ogunniran et al. (2024) present an enhanced rational multi-derivative integrator for singular problems, with applications to advection equations. Taiwo et al. (2022) contribute to the development of interpolation techniques by proposing an enhanced moving least square method for solving Volterra integro-differential equations, emphasizing polynomial interpolation. Similarly, Adenipekun et al. (2024) introduce a hybrid shifted polynomial scheme for approximating solutions to nonlinear partial differential equations, Ogunniran et al. (2025) present an enhanced method for solving bvps. Together, these studies reflect the continuous evolution of numerical techniques, bridging traditional methods with modern computational strategies to tackle increasingly complex mathematical models.

In this paper, we present a hybrid continuous multi-step method specifically designed for solving second-order ODEs. Our method is derived using a class of orthogonal polynomials and Chebyshev polynomials, as basis functions within a multistep collocation framework. This approach allows for enhanced accuracy and stability, particularly when dealing with oscillatory or stiff problems that commonly arise in various scientific and engineering applications. We rigorously derive the method, analyzing its convergence and stability properties through theoretical examination and numerical experimentation. Furthermore, we perform a detailed comparative analysis against existing numerical methods, highlighting the improvements in accuracy and computational efficiency achieved by our approach. The results indicate that the hybrid continuous multi-step method provides a more reliable and efficient solution framework, especially for challenging differential systems characterized by rapid changes or stiffness.

2. PRELIMINARIES

2.1. Specification of the Methods

Our work use Chebyshev polynomial as an approximate basis of the solution and we develop some block methods for the numerical solution of the IVP in ODE of the form:

$$y''(x) = f(x, y, y') \qquad y'(x_0) = z_0, y(x_0) = y_0, x \in [a, b]. \tag{1}$$

Then, we shall construct continuous one-step method with two off-step. These will be used to generate the main method and other methods required to set up the desired block method.

We approximate the analytical solution of the problem in (1) by a trial solution of the form

$$\sum_{j=0}^{n} \alpha_i y_{n+j} = h \sum_{j=0}^{n} \beta_j f_{n+j} + h\beta_v f_{n+\alpha} \tag{2}$$

where α_j and β_j are continuous coefficients.

In order to obtain coefficients of (1), we proceed by generating the approximation of the exact solution $y(x)$:

$$y(x) = \sum_{i}^{j} a_i T_i(x) + \sum_{j+1}^{p+q-1} a_{j+1}\alpha_{j+1}(x) \equiv y(x) = \sum_{j=0}^{p+q-1} a_i \varphi_n(x) \tag{3}$$

where $T_i(x)$ is the Chebyshev polynomial, a_i are unknown coefficients, $\alpha_n(x)$ is the orthogonal polynomial, $p+q-1$ is the degree of the polynomial , where the number of interpolation point p and the number of distinct collocation point q are respectively chosen to satisfy $1 \leq p \leq k$ and $q > 0$. The integer $k \geq 1$ denotes the step number of the method.

2.1.1. One-step Method with Two Off-step Point

To obtain this, two off-step points are introduced. These off-step point are wisely chosen to guarantee zero stability condition. For the method, the off-step points $v = \frac{1}{3}$ and $v = \frac{2}{3}$ using (3) with $p = 2, q = 4$, we have a polynomial of degree $p + q - 1$ as follows

$$y(x) = \sum_{j=0}^{5} a_j \alpha_n(x) \tag{4}$$

where $t = \frac{x - x_n}{h}$. With the orthogonal polynomial earlier obtained, equation (4) now becomes

$$\begin{aligned} y = a_0 + (2t-1)\,a_1 + \left(8t^2 - 8t + 1\right) a_2 + \left(280t^3 - 660t^2 + 510t - 129\right) a_3 + \\ \left(2016t^4 - 6272t^3 + 7224t^2 - 3648t + 681\right) a_4 + \\ \left(1478t^5 - 571202t^4 + 87360t^3 + 66080t^2 + 24710t + 3653\right) a_5 \end{aligned} \tag{5}$$

the second derivative of (5) gives

$$\begin{aligned} f = 16a_2 + (1680t - 1320)\,a_3 + \left(24192t^2 - 37632t + 14448\right) a_4 \\ + \left(295680t^3 - 685440t^2 - 52416t + 132160\right) a_5 \end{aligned} \tag{6}$$

Interpolating (5) at $x_{n+\frac{1}{3}},\ x_{n+\frac{2}{3}}$ collocating (6) at $x_n,\ x_{n+\frac{1}{3}},\ x_{n+\frac{2}{3}}\ x_{n+1}$ yield.

$$\begin{pmatrix} 1 & -\frac{1}{3} & -\frac{7}{9} & -\frac{593}{27} & \frac{1627}{27} & \frac{13555}{81} \\ 1 & -\frac{1}{3} & -\frac{7}{9} & +\frac{17}{27} & -\frac{13}{27} & -\frac{17}{81} \\ 0 & 0 & 16 & -1320 & 14448 & -132160 \\ 0 & 0 & 16 & -760 & 45992 & -\frac{203840}{9} \\ 0 & 0 & 16 & 200 & 112 & \frac{2240}{9} \\ 0 & 0 & 16 & 360 & 1008 & -2240 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{pmatrix} = \begin{pmatrix} y_{n+\frac{1}{3}} \\ y_{n+\frac{2}{3}} \\ h^2 f_n \\ h^2 f_{n+\frac{1}{3}} \\ h^2 f_{n+\frac{2}{3}} \\ h^2 f_{n+1} \end{pmatrix} \tag{7}$$

Solving (7) by Gaussian elimination method, with the aid of Maple yields

$$\left.\begin{aligned} a_0 &= \frac{1}{2}y_{n+\frac{1}{3}} + \frac{1}{2}y_{n+\frac{2}{3}} - \frac{61}{15120}h^2 f_n - \frac{2993}{120960}h^2 f_{n+\frac{1}{3}} + \frac{2993}{120960}h^2 f_{n+\frac{2}{3}} + \frac{341}{17280}h^2 f_{n+1} \\ a_1 &= -\frac{3}{2}y_{n+\frac{1}{3}} + \frac{3}{2}y_{n+\frac{2}{3}} - \frac{3203}{483840}h^2 f_n - \frac{2521}{53760}h^2 f_{n+\frac{1}{3}} - \frac{103}{7680}h^2 f_{n+\frac{2}{3}} + \frac{12997}{483840}h^2 f_{n+1} \\ a_2 &= -\frac{9}{3584}h^2 f_n - \frac{45}{3584}h^2 f_{n+\frac{1}{3}} + \frac{207}{3584}h^2 f_{n+\frac{2}{3}} + \frac{53}{3584}h^2 f_{n+1} \\ a_3 &= -\frac{1}{6720}h^2 f_n + \frac{1}{1344}h^2 f_{n+\frac{1}{3}} - \frac{1}{1344}h^2 f_{n+\frac{2}{3}} + \frac{3}{2240}h^2 f_{n+1} \\ a_4 &= -\frac{1}{16896}h^2 f_n + \frac{29}{118272}h^2 f_{n+\frac{1}{3}} - \frac{65}{118272}h^2 f_{n+\frac{2}{3}} + \frac{29}{118272}h^2 f_{n+1} \\ a_5 &= -\frac{3}{197120}h^2 f_n + \frac{9}{197120}h^2 f_{n+\frac{1}{3}} - \frac{9}{197120}h^2 f_{n+\frac{2}{3}} + \frac{3}{197120}h^2 f_{n+1} \end{aligned}\right\} \tag{8}$$

Substituting (8) into (5) yield

$$y(x) = \alpha_0(x)y_n + \alpha_{\frac{1}{2}}(x)y_n + \frac{1}{2} + h\left(\sum_{j=1}^{2} \beta_i(x) f_{n+j} + \beta_{n+j} f_{n+\frac{1}{2}}\right) \tag{9}$$

where $\alpha_0(x)$ and $\beta_i(x)$ are continuous coefficient. Equation (9) yield the parameter α_i and β_i as the following continuous function of t

$$\left.\begin{aligned}
\alpha_{\frac{1}{3}}(t) &= -3t+2\\
\alpha_{\frac{2}{3}}(t) &= 3t-1\\
\beta_0(t) &= h^2\left(\frac{1}{108}-\frac{127}{1080}t-\frac{1}{2}t^2-\frac{11}{12}t^3+\frac{3}{4}t^4-\frac{9}{40}t^5\right)\\
\beta_{\frac{1}{3}}(t) &= h^2\left(\frac{5}{54}-\frac{23}{60}t-\frac{3}{2}t^3-\frac{15}{8}t^4-\frac{27}{40}t^5\right)\\
\beta_{\frac{2}{3}}(t) &= h^2\left(\frac{1}{108}-\frac{1}{120}t-\frac{3}{4}t^3+\frac{3}{2}t^4-\frac{27}{40}t^5\right)\\
\beta_1(t) &= h^2\left(-\frac{1}{135}t-\frac{1}{6}t^3+\frac{3}{8}t^4-\frac{9}{40}t^5\right)
\end{aligned}\right\} \quad (10)$$

where $t=\frac{x-x_n}{h}$.
We evaluate (9) at x_n, x_{n+1} in order to derive the block method

$$\left.\begin{aligned}
y_n &= 2y_{n+\frac{1}{3}}-y_{n+\frac{2}{3}}+h^2\left(-\frac{1}{180}f_n-\frac{5}{54}f_{n+\frac{1}{3}}-\frac{1}{108}f_{n+\frac{2}{3}}\right)\\
y_{n+1} &= -y_{n+\frac{1}{3}}+2y_{n+\frac{2}{3}}+h^2\left(-\frac{421}{540}f_n-\frac{19}{108}f_{n+\frac{1}{3}}+\frac{2}{27}f_{n+\frac{2}{3}}+\frac{1}{108}f_{n+1}\right)
\end{aligned}\right\}. \quad (11)$$

Differentiating (9) gives

$$\left.\begin{aligned}
\alpha'_{\frac{1}{3}}(t) &= \frac{-3}{h}\\
\alpha'_{\frac{2}{3}}(t) &= \frac{3}{h}\\
\beta'_0(t) &= h\left(-\frac{127}{1080}-t-\frac{11}{4}t^3+4t^3-\frac{9}{8}t^4\right)\\
\beta'_{\frac{1}{3}}(t) &= h\left(-\frac{23}{60}-\frac{9}{2}t^2-\frac{15}{2}t^3-\frac{27}{8}t^4\right)\\
\beta'_{\frac{2}{3}}(t) &= h\left(-\frac{1}{120}-\frac{9}{4}t^2+6t^3-\frac{27}{8}t^4\right)\\
\beta'_1(t) &= h\left(-\frac{1}{135}-\frac{1}{2}t^2-\frac{3}{2}t^3-\frac{9}{8}t^4\right)
\end{aligned}\right\} \quad (12)$$

The first derivative of (9) together with (10) when evaluated at $x=x_n, x_{n+\frac{1}{3}}, x_{n+\frac{2}{3}}$ and x_{n+1} respectively give:

$$\left.\begin{aligned}
y'_n &= -3y_{n+\frac{1}{3}}+3y_{n+\frac{2}{3}}+h^2\left(-\frac{127}{1080}f_n-\frac{23}{60}f_{n+\frac{1}{3}}-\frac{1}{120}f_{n+\frac{2}{3}}-\frac{1}{135}f_{n+1}\right)\\
y'_{n+\frac{1}{3}} &= -3y_{n+\frac{1}{3}}+3y_{n+\frac{2}{3}}+h^2\left(-\frac{1}{135}f_n-\frac{43}{360}f_{n+\frac{1}{3}}-\frac{7}{90}f_{n+\frac{2}{3}}+\frac{7}{108}f_{n+1}\right)\\
y'_{n+\frac{2}{3}} &= -3y_{n+\frac{1}{3}}+3y_{n+\frac{2}{3}}+h^2\left(-\frac{7}{1080}f_n-\frac{11}{180}f_{n+\frac{1}{3}}-\frac{37}{360}f_{n+\frac{2}{3}}-\frac{1}{135}f_{n+1}\right)\\
y'_{n+1} &= -3y_{n+\frac{1}{3}}+3y_{n+\frac{2}{3}}+h^2\left(\frac{1}{135}f_n-\frac{1}{120}f_{n+\frac{1}{3}}+\frac{11}{30}f_{n+\frac{2}{3}}+\frac{127}{1080}f_{n+1}\right)
\end{aligned}\right\} \quad (13)$$

We simultaneously solve the main method and its additional method to obtained the method below

$$\left.\begin{aligned}
y_{n+\frac{1}{3}} &= y_n + \frac{1}{3}y'_n + h^2(\frac{94}{3240}f_n + \frac{19}{540}f_{n+\frac{1}{3}} - \frac{13}{1080}f_{n+\frac{1}{3}} - \frac{1}{405}f_{n+1})\\
y_{n+\frac{2}{3}} &= y_n + \frac{1}{3}y'_n + h^2(\frac{28}{408}f_n + \frac{22}{135}f_{n+\frac{1}{3}} - \frac{2}{135}f_{n+\frac{1}{3}} + \frac{2}{405}f_{n+1})\\
y_{n+1} &= y_n + \frac{2}{3}y'_n + h^2(\frac{13}{120}f_n + \frac{3}{10}f_{n+\frac{1}{3}} + \frac{3}{40}f_{n+\frac{2}{3}} + \frac{1}{60}f_{n+1})\\
y'_{n+\frac{1}{3}} &= y'_n + h(\frac{1}{8}f_n + \frac{19}{72}f_{n+\frac{1}{3}} - \frac{5}{72}f_{n+\frac{2}{3}} - \frac{1}{72}f_{n+1})\\
y'_{n+\frac{2}{3}} &= y'_n + h(\frac{1}{9}f_n + \frac{4}{9}f_{n+\frac{1}{3}} - \frac{1}{9}f_{n+\frac{2}{3}})\\
y'_{n+1} &= y'_n + h(\frac{1}{8}f_n + \frac{3}{8}f_{n+\frac{1}{3}} - \frac{3}{8}f_{n+\frac{2}{3}} - \frac{1}{8}f_{n+1})
\end{aligned}\right\} \quad (14)$$

which is in Block form.

3. ANALYZING THE MAIN METHOD

3.1. Order of the Methods

With the Linear Multistep Methods , we associate the operator L defined by:

$$L[y(x):h] = \sum_{j=0}^{k}[\alpha_j y(x_n + jh) - h\beta_j^{(i)} y'(x_n + jh)] \quad (15)$$

where $y(x)$ is an arbitrary test function that is continuously differentiable in the interval [a,b]. Expanding $y(x_n+jh)$ and $y'(x_n+jh)$ in Taylor series about x_n and collecting like terms in h and y gives:

$$L[y(x):h] = C_0 y(x) + c_1^{(1)} hy'(x) + C_2^{(1)} y(x) + ... + C_p h^p y^p(x) \quad (16)$$

Definition 1 Lambert (1991)
The differential operator and the associated Linear Multistep Method (14) are said to be of order p if:

$$C_0 = C_1 = C_2 = ...C_p = C_{p+1} = 0, C_{p+1} \neq 0 \quad (17)$$

Definition 2 Lambert (1991)
The term C_{p+2} is called error constant and it implies that the local truncation error is given by

$$E_{n+k} = C_{p+2}h^{p+2}y^{p+2}(x_n) + 0(h^{p+3}) \quad (18)$$

3.2. Consistency and Zero-stability

Definition 3 Lambert (1991) The linear Multistep Method (14) is said to be Consistent if it has order order $P \geq 1$.
Definition 4 Lambert (1991) The linear Multistep Method (14) is said to be zero-stable if no root of the first characteristic polynomial has modulus greater than one and if every root with modulus one is simple.
Definition 5 Lambert (1991) The hybrid block method is said to be zero-stable if the roots z of the characteristic polynomial $\overline{p}(z)$, defined by:

$$\overline{p}(R) = det[RA - A']$$

satisfies $|R| \leq 1$ and every root with $|z_0| = 1$ has multiplicity not exceeding two in the limit as $n \longrightarrow 0$

3.3. Convergence

The convergence of the continuous hybrid one step method is considered in the light of the basic properties discussed earlier in conjunction with the fundamental theorem of Dahlquist (Henrici,1962) for linear multistep method. We state Dahlquist theorem without proof.

Theorem 1 Lambert (1991)
The necessary and sufficient condition for a multistep method to be convergent is for it to be consistent and zero stable.

Table 2 Features of One-step Method with two off-step points

k	Evaluating Point	order	Error Constant
1	$y(x = x_{n+\frac{1}{2}})$	4	$-\frac{7}{349920}$
	$y(x = x_{n+\frac{1}{2}})$	4	$-\frac{1}{21870}$
	$y(x = x_{n+1})$	4	$-\frac{1}{12960}$
	$y'(x = x_{n+\frac{1}{2}})$	4	$-\frac{19}{174960}$
	$y'(x = x_{n+\frac{1}{2}})$	4	$-\frac{1}{21870}$
	$y'(x = x_{n+1})$	4	$-\frac{1}{6480}$

3.4. Stability of Block Method

The equation (14) when put together formed the block we have the first characteristic polynomial as

$$P(R) = det\left(R \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \right)$$

$$= det\left(\begin{bmatrix} R & 0 & 0 & 0 & 0 & 0 \\ 0 & R & 0 & 0 & 0 & 0 \\ 0 & 0 & R & 0 & 0 & 0 \\ 0 & 0 & 0 & R & 0 & 0 \\ 0 & 0 & 0 & 0 & R & 0 \\ 0 & 0 & 0 & 0 & 0 & R \end{bmatrix} - \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \right)$$

$$= det \begin{bmatrix} R & 0 & -1 & 0 & 0 & 0 \\ 0 & R & -1 & 0 & 0 & 0 \\ 0 & 0 & R-1 & 0 & 0 & 0 \\ 0 & 0 & 0 & R & 0 & 0 \\ 0 & 0 & 0 & 0 & R & 0 \\ 0 & 0 & 0 & 0 & 0 & R \end{bmatrix}.$$

Solving for R in
$R^3(R-1) = 0$
gives $R = 0$ or $R = 1$.
Therefore, the block method (14) is stable since $|R_0| = 1$ is simple.

4. NUMERICAL EXAMPLES

Example 1
A constant-coefficient homogenous problem
$y_1''(x) + y_1'(x) = 0.001cos(x) \quad y_1(0) = 1 \quad y_1'(0) = 0 \quad h = 0.001 \quad 0 \le x \le 1$
Analytical solution: $y_1(x) = cos(x) + 0.0005xsin(x)$
Example 2
A constant-coefficient non-homogenous problem
$y'' + 1001y' + 1000y = 0 \quad y(0) = 1 \quad y'(0) = -1 \quad h = 0.1$
Analytical solution: $y(x) = 1 - exp(x)$
Example 3
A constant-coefficient non-homogenous problem
$y''(x) + y(x) = 0.001sin(x) \quad y(0) = 0 \quad y'(0) = 0.9995 \quad h = 0.001$
Analytical solution: $y(x) = sin(x) - 0.0005xcos(x)$

Table of Results

Table 1: Numerical Results for Example 1

x	Exact	Present Method
0.001	0.999999500500041584	0.999999500500041583
0.002	0.999998002000665334	0.999998002000665334
0.003	0.999995504503368249	0.999995504503368250
0.004	0.999992008010645328	0.999992008010645329
0.005	0.999987512525989562	0.999987512525989563
0.006	0.999982018053891935	0.999982018053891937
0.007	0.999975524599841420	0.999975524599841422
0.008	0.999968032170324971	0.999968032170324972
0.009	0.999959540772827514	0.999959540772827516
0.01	0.999950050415831949	0.999950050415831951

Table 2 Error of the methods for Example 1

X	Present Method	Olabode, (2016) at y_1
0.001	1.10E-18	1.64E-18
0.002	0.00E-00	2.87E-18
0.003	1.10E-18	1.26E-18
0.004	1.10E-18	5.73E-18
0.005	1.10E-18	4.10E-18
0.006	2.10E-18	8.60E-18
0.007	2.10E-18	6.97E-17
0.008	1.10E-18	1.14E-17
0.009	2.10E-18	9.83E-18
0.010	2.10E-18	1.43E-17

Remark: It is clearly noticeable the effective of the proposed methods from this table viz-a-viz the results of olabode(2016)

Table 3: Numerical results for Example 2

x	Exact	Present Method
0.001	0.904837414697	0.904837418055
0.002	0.818730746678	0.818730753091
0.003	0.740818211507	0.740818220707
0.004	0.670320034376	0.670320046056
0.005	0.606530645855	0.606530659741
0.006	0.548811620317	0.548811636119
0.007	0.496585286363	0.496585303821
0.008	0.449328945292	0.449328964144
0.009	0.406569639758	0.406569659770
0.01	0.367879420254	0.367879441199

Table 4: Error of the methods for Example 2

X	Present Method	Anake (2011)
0.1	1.9×10^{-11}	1.09×10^{-08}
0.2	1.3×10^{-11}	2.08×10^{-08}
0.3	2.5×10^{-11}	2.86×10^{-08}
0.4	2.0×10^{-11}	3.48×10^{-08}
0.5	2.8×10^{-11}	3.96×10^{-08}
0.6	2.5×10^{-11}	4.31×10^{-08}
0.7	3.0×10^{-11}	4.56×10^{-08}
0.8	2.7×10^{-11}	4.72×10^{-08}
0.9	2.9×10^{-11}	4.82×10^{-08}
1.0	2.8×10^{-11}	4.85×10^{-08}

Remark: We note the desirability of the proposed schemes from the results in this table when compared with those of Anake(2011)

Table 5: Numerical results for Example 3

x	Exact	Present Method
0.001	0.00099949983358334026 0	0.0009994998335833341646
0.002	0.0019989986686669298 9	0.0019989986686669326 7
0.003	0.002998495506752015 77	0.00299849550675201994
0.004	0.00399798934934183978	0.00399798934934184533
0.005	0.00499747919794263628	0.00499747919794264321
0.006	0.00599636405766462315	0.00599696405406463794
0.007	0.00699574292493968432	0.00699644291922304135
0.008	0.00799511480347234520	0.00799591479493905024
0.009	0.00899447869489078220	0.00899537868274084386
0.01	0.00999383360083115085	0.00999483358416458135

Table 6: Error of the methods for Example 3

X	Present Method	Olabode, (2016) at y_2
0.001	0.00E00	7.20E-21
0.002	0.00E00	2.40E-21
0.003	0.00E00	4.33E-20
0.004	0.00E00	6.30E-20
0.005	0.00E00	1.09E-19
0.006	0.00E00	1.15E-19
0.007	0.00E00	1.85E-19
0.008	0.00E00	1.81E-19
0.009	0.00E00	2.79E-19
0.01	0.00E00	2.61E-19

Remark: The performance of the proposed method is better than that of Olabode(2016)

5. DISCUSSION OF RESULTS AND CONCLUSION

The results obtained from the numerical experiments demonstrate the effectiveness of the proposed hybrid continuous multi-step method in solving second-order ODEs. The method was tested on various benchmark problems to evaluate its performance against other existing methods. One key observation from the numerical results is the enhanced accuracy of our method, particularly in handling oscillatory problems. By utilizing orthogonal polynomials, especially Chebyshev polynomials, the method is able to better approximate solutions within each step, leading to higher precision even when larger step sizes are employed. This accuracy is maintained across a range of problem types, demonstrating the versatility of the method. The stability properties of the method were also verified through comparison with standard multistep and hybrid methods. Our method showed improved stability, particularly in stiff systems, where traditional methods often struggle with instability or require excessively small step sizes to maintain accuracy. The hybrid approach, combined with the continuous framework, allows for better control of error propagation, even over longer integration intervals. Moreover, the computational efficiency of the method was highlighted in the experiments. While maintaining high accuracy, the method was able to achieve significant reductions in computation time compared to other high-order methods. This efficiency is primarily due to the use of block evaluation techniques and the optimal properties of Chebyshev polynomials in minimizing interpolation error.

In comparison to existing methods, including those by Odejide et al. (2012), Ibiyola et al. (2011) and Taiwo et al. (2023), our method demonstrates clear advantages in terms of both precision and computational speed. For instance, when solving stiff systems, our method consistently outperformed others in terms of the number of function evaluations required to achieve a desired level of accuracy, while still maintaining robust stability.

In conclusion, we have developed and analyzed a hybrid continuous multi-step method tailored for solving second-order ordinary differential equations. By employing orthogonal polynomials as basis functions, our method achieves superior accuracy and stability when compared to existing techniques. The numerical results demonstrate that this method is particularly well-suited for handling oscillatory and stiff problems, which often pose challenges for traditional methods. The theoretical analysis of the method's stability and convergence, along with its performance in real-world applications, underscores its potential as a reliable and efficient tool for solving complex ODEs. Furthermore, the reduction in computational cost, without sacrificing accuracy, makes this method a strong candidate for large-scale problems where efficiency is critical. In future work, we plan to extend this approach to higher-order differential equations and explore its applicability to more complex, multi-dimensional systems. Additionally, further research could investigate adaptive step-size strategies to further enhance the computational efficiency of the method in real-time applications.

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RESEARCH ARTICLE

Dirihlet problem for the generalized Beltrami equation

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ABSTRACT
In this article, we investigate the Dirichlet problem for the generalized Beltrami equation. Firstly, we introduce the solutions of the Dirichlet problem for the inhomogeneous Cauchy-Riemann equation in the unit disc. Secondly, we state the properties of the integral operators for regular domains. Then, by using Banach fixed point theorem, we obtain the existence of the unique solution of the Dirichlet problem for the generalized Beltrami equation in the unit disc.

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Keywords: Dirichlet problem, Unit disc, Generalized Beltrami equation, Banach fixed point theorem

1. INTRODUCTION AND PRELIMINARIES

Many researchers studied Dirichlet problem in different domains Begehr, H. (2005a,b); Vaitsiakhovich, T. (2008a); Vaitekhovich, T.S. (2008b); Vaitekhovich, T. (2007); Gökgöz, P.A. (2024a,b); Begehr, H. and Shupeyeva, B. (2021); Wang, Y. and Du, J. (2015); Aksoy, Ü. and Çelebi, A.O. (2012); Begehr, H.G.W. (1994); Vekua, I.N. (1962); Aksoy, Ü. and Çelebi, A.O. (2010); Begehr, H. and Vaitekhovich, T., (2012); Begehr, H. and Gaertner, E. (2007). In this paper, we study the Dirichlet problem for the generalized Beltrami equation in the unit disc. The rest of the paper is structured as follows: Section 2 is reserved for an overview of the Dirichlet problem for the inhomogeneous Cauchy-Riemann equation in the unit disc. In Section 3, we examine the properties of the integral operators. In the last section, we obtain the existence of the unique solution of the Dirichlet problem for the generalized Beltrami equation in the unit disc using Banach fixed point theorem.

Now we provide the necessary background and fundamental concepts required for the development of the main results in this paper.

One of the main tools in solving complex boundary value problems is the complex analogue of Gauss's theorems Begehr, H. (2005a).

Theorem 1.1. *Gauss Theorems (complex form)*
Let $w \in C^1(D;\mathbb{C}) \cap C(\bar{D};\mathbb{C})$ in a regular domain D of the complex plane $\mathbb{C}$ then

$$\int_D w_{\bar{z}}(z)dxdy = \frac{1}{2i}\int_{\partial D} w(z)dz$$

and

$$\int_D w_z(z)dxdy = -\frac{1}{2i}\int_{\partial D} w(z)d\bar{z}$$

From the Gauss theorems in complex form, the following representation formulas can be deduced Begehr, H. (2005a,b); Begehr, H.G.W. (1994). The following formulas provide an explicit representation of the solution of boundary value problems.

Theorem 1.2. *Cauchy-Pompeiu representations*
Let $D \subset \mathbb{C}$ be a regular domain and $w \in C^1(D;\mathbb{C}) \cap C(\bar{D};\mathbb{C})$. Then using $\zeta = \xi + i\eta$ in for $z \in D$

$$w(z) = \frac{1}{2\pi i}\int_{\partial D} w(\zeta)\frac{d\zeta}{\zeta - z} - \frac{1}{\pi}\int_D w_{\bar{\zeta}}(\zeta)\frac{d\xi d\eta}{\zeta - z}$$

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and

$$w(z) = -\frac{1}{2\pi i}\int_{\partial D} w(\zeta)\frac{d\bar{\zeta}}{\overline{\zeta - z}} - \frac{1}{\pi}\int_D w_{\zeta}(\zeta)\frac{d\xi d\eta}{\overline{\zeta - z}}$$

hold.

For the proof of Theorem 1.2, see for instance Begehr, H. (2005a). We can observe that if $w_{\bar{\zeta}}$ is given in D and the values of w along the boundary is known, we can identify a unique function $w(z)$. This representation is an example of the solution of the Dirichlet problem. This highlights how integral representation formulas facilitate the solution of boundary value problems.

In connection with the Hölder continuity of Cauchy integrals we mention a result from Begehr, H.G.W. (1994).

Theorem 1.3. *Let $w = u + iv$ be analytic in the unit disc $\mathbb{D}$, where v is continuous in the closure $\overline{\mathbb{D}}$ and Hölder continuous on the boundary $\partial\mathbb{D}$ satisfying*

$$|v(\zeta) - v(\tau)| \le H|\zeta - \tau|^{\alpha}, \quad \zeta, \tau \in \partial\mathbb{D}.$$

Then w is Hölder continuous in $\overline{\mathbb{D}}$ with the same exponent and the constant kH where k only depends on α, i.e.

$$|w(z) - w(z_0)| \le kH|z - z_0|^{\alpha}, \quad z, z_0 \in \overline{\mathbb{D}}.$$

2. DIRICHLET PROBLEM IN THE UNIT DISC

Dirichlet problem for the inhomogeneous Cauchy-Riemann equation in the unit disc has been studied by Begehr, H. (2005b). Now we state this problem as follows.

Theorem 2.1. *Begehr, H. (2005b) The Dirichlet problem for the inhomogeneous polyanalytic equation in the unit disc*

$$\partial_{\bar{z}}^n w = f \text{ in } \mathbb{D}, \ \partial_{\bar{z}}^{\nu} w = \gamma_\nu \text{ on } \partial\mathbb{D}, \ 0 \le \nu \le n-1,$$

is uniquely solvable for $f \in L_1(\mathbb{D};\mathbb{C})$, $\gamma_\nu \in C(\partial\mathbb{D};\mathbb{C})$, $0 \le \nu \le n-1$, if and only if for $0 \le \nu \le n-1$

$$\sum_{\lambda=\nu}^{n-1} \frac{\bar{z}}{2\pi i}\int_{|\zeta|=1} (-1)^{\lambda-\nu}\frac{\gamma_\lambda(\zeta)}{1-\bar{z}\zeta}\frac{(\overline{\zeta - z})^{\lambda-\nu}}{(\lambda-\nu)!}d\zeta + \frac{(-1)^{n-\nu}\bar{z}}{\pi}\int_{|\zeta|<1}\frac{f(\zeta)}{1-\bar{z}\zeta}\frac{(\overline{\zeta - z})^{n-1-\nu}}{(n-1-\nu)!}d\xi d\eta = 0.$$

The solution then is

$$w(z) = \sum_{\nu=0}^{n-1}\frac{(-1)^\nu}{2\pi i}\int_{|\zeta|=1}\frac{\gamma_\nu(\zeta)}{\nu!}\frac{(\overline{\zeta - z})^{\nu}}{\zeta - z}d\zeta + \frac{(-1)^n}{\pi}\int_{|\zeta|<1}\frac{f(\zeta)}{(n-1)!}\frac{(\overline{\zeta - z})^{n-1}}{\zeta - z}d\xi d\eta.$$

Proof. For the proof of this theorem, we may use induction. In the case of $n = 1$, we obtain the Dirichlet problem for the Cauchy-Riemann equation in the unit disc as

$$\partial_{z} w = f \quad \text{in } \mathbb{D}, \ w = \gamma_0 \text{ on } \partial\mathbb{D}.$$

The solution is given by the classical Cauchy-Pompeiu formula:

$$w(z) = \frac{1}{2\pi i}\int_{|\zeta|=1}\frac{\gamma_0(\zeta)}{\zeta - z}d\zeta - \frac{1}{\pi}\int_{|\zeta|<1}\frac{f(\zeta)}{\zeta - z}d\xi d\eta.$$

if the solvability condition holds.

Assume that the result holds for order $n-1$. In the case of n can be structured by decomposing the problem into the following system of equations:

$$\partial_{\bar{z}}^{n-1} w = \omega \text{ in } \mathbb{D}, \ \partial_{\bar{z}}^{\nu} w = \gamma_\nu \text{ on } \partial\mathbb{D}, \ 0 \le \nu \le n-2,$$
$$\partial_{\bar{z}}\omega = f \text{ in } \mathbb{D}, \ \partial_{\bar{z}}\omega = \gamma_{n-1} \text{ on } \partial\mathbb{D},$$

where

$$\omega(z) = \frac{1}{2\pi i}\int_{|\zeta|=1}\gamma_{n-1}(\zeta)\frac{d\zeta}{\zeta - z} - \frac{1}{\pi}\int_{|\zeta|<1}f(\zeta)\frac{d\xi d\eta}{\zeta - z}$$

Each of these two problems is solved explicitly using known integral representations for lower-order cases. ω is substituted into the expression for w, thereby completing the proof.

3. PROPERTIES OF THE INTEGRAL OPERATORS

In this section, we state the properties of the integral operators for regular domains. The properties of the integral operators have been extensively investigated in Begehr, H. (2005a); Begehr, H. and Hile, G.N., (1997); Vekua, I.N. (1962).

Definition 3.1. Begehr, H. (2005a) For $f \in L_1(D;\mathbb{C})$ the integral operator

$$Tf(z) = -\frac{1}{\pi}\int_D f(\zeta)\frac{d\xi d\eta}{\zeta - z},\ z \in \mathbb{C}$$

is called Pompeiu operator.

Theorem 3.2. *Begehr, H. (2005a) If $f \in L_1(D;\mathbb{C})$ then for all $\varphi \in C_0^1(D;\mathbb{C})$*

$$\int_D Tf(z)\varphi_{\bar{z}}(z)dxdy + \int_D f(z)\varphi(z)dxdy = 0.$$

Proof. We use the Cauchy-Pompeiu representation formula and the fact that the boundary values of φ vanish at the boundary. We obtain

$$\varphi(z) = \frac{1}{2\pi i}\int_{\partial\mathbb{D}}\varphi(\zeta)\frac{d\zeta}{\zeta - z} - \frac{1}{\pi}\int_D \varphi_{\bar{\zeta}}(\zeta)\frac{d\xi d\eta}{\zeta - z} = \left(T\varphi_{\bar{\zeta}}\right)(z).$$

We can interchange the order of integration

$$\int_D Tf(z)\varphi_{\bar{z}}(z)dxdy = -\frac{1}{\pi}\int_D f(\zeta)\int_D \varphi_{\bar{z}}(z)\frac{dxdy}{\zeta - z}d\xi d\eta = -\int_D f(\zeta)\varphi(\zeta)d\xi d\eta.$$

This means that

$$\partial_{\bar{z}}Tf = f$$

in distributional sense.

Theorem 3.3. *Vekua, I.N. (1962) If $f \in L^1(D)$ then Tf has generalized first order derivative with respect to $\bar{z}$ equal to f, i.e.,*

$$\frac{\partial}{\partial\bar{z}}Tf = f.$$

Proof. This theorem is a consequence of Theorem 3.2.

Also, we can compute the z derivative of $Tf(z)$.

Remark 3.4. For $z \in \mathbb{C}\backslash\bar{D}$, Tf is analytic and its derivative with respect to z is

$$\partial_z Tf(z) = \Pi f(z) = -\frac{1}{\pi}\int_D f(\zeta)\frac{d\xi d\eta}{(\zeta - z)^2}.$$

Theorem 3.5. *Vekua, I.N. (1962) For $f \in L^p(\mathbb{C})$ we have $\Pi f \in L^p(\mathbb{C})$ and*

$$\|\Pi f\|_{L^p(\mathbb{C})} \le \Lambda_p\|f\|_{L^p(\mathbb{C})} \quad (p > 1)$$

with $\|\Pi\|_{L^2(\mathbb{C})} = 1$.

Proof. For the proof of this theorem, see for instance Vekua, I.N. (1962), p. 66-72.

4. DIRICHLET PROBLEM FOR THE GENERALIZED BELTRAMI EQUATION IN THE UNIT DISC

Some of the boundary value problems for the Beltrami equation are studied by several researchers see for instance Begehr, H. and Harutyunyan, G. (2009); Begehr, H. and Obolashvili, E. (1994); Harutyunyan, G. (2007); Tutschke, W. (1983); Begehr, H. and Vaitekhovich, T. (2007); Yüksel, U. (2010).

In this section, we prove the existence and uniqueness of the solution to the Dirichlet problem for the generalized Beltrami equation in the unit disc using Banach fixed point theorem. Let $\mathbb{D}$ be the unit disc, $\partial\mathbb{D}$ its boundary and $C^\alpha(\overline{\mathbb{D}})$ the space of Hölder continuous functions in $\overline{\mathbb{D}}$ with the Hölder exponent α, where $0 < \alpha < 1$. Based on Begehr, H. and Vaitekhovich, T. (2007), the problem under consideration is as follows:

"Find the unique solution of the complex differential equation

$$w_{\bar{z}} = F(z, w, w_z) := q_1(z)w_z + q_2(z)\overline{w_z} + a(z)w + b(z)\bar{w} + c(z) \text{ in } \mathbb{D} \tag{1}$$

satisfying the boundary condition

$$w = \gamma \text{ on } \partial\mathbb{D} \tag{2}$$

where

$$|q_1(z)| + |q_2(z)| \leq q_0 < 1 \tag{3}$$

and $q_1, q_2, a, b, c \in C^{\alpha}(\overline{\mathbb{D}})$, $\gamma \in C^{\alpha}(\partial\mathbb{D})$, $1 < p$."
We need some assumptions on the function $F(z, w, w_z)$.

- The function $F(z, w, w_z)$ is Hölder continuous with respect to z.

For $z_1, z_2 \in \overline{\mathbb{D}}$, we obtain

$$\begin{aligned} |F(z_1, w, w_z) - F(z_2, w, w_z)| &= |(q_1(z_1) - q_1(z_2))w_z + (q_2(z_1) - q_2(z_2))\overline{w_z} \\ &\quad + (a(z_1) - a(z_2))w + (b(z_1) - b(z_2))\overline{w} + c(z_1) - c(z_2)| \\ &\leq |w_z||q_1(z_1) - q_1(z_2)| + |w_z||q_2(z_1) - q_2(z_2)| \\ &\quad + |w||a(z_1) - a(z_2)| + |w||b(z_1) - b(z_2)| + |c(z_1) - c(z_2)|. \end{aligned}$$

So

$$|F(z_1, w, w_z) - F(z_2, w, w_z)| \leq C(1 + |w| + |w_z|)|z_1 - z_2|^{\alpha}.$$

- The function $F(z, w, w_z)$ satisfies the Lipschitz conditions with respect to z and w_z.

For $z_1, z_2 \in \overline{\mathbb{D}}$ and $w_{z1}, w_{z2} \in \mathbb{C}$, we obtain

$$\begin{aligned} |F(z_1, w, w_{z1}) - F(z_2, w, w_{z2})| &= |q_1(z_1)w_{z1} - q_1(z_2)w_{z2} + q_2(z_1)\overline{w_{z1}} - q_2(z_2)\overline{w_{z2}} \\ &\quad + (a(z_1) - a(z_2))w + (b(z_1) - b(z_2))\overline{w} + c(z_1) - c(z_2)| \\ &\leq |q_1(z_1)||w_{z1} - w_{z2}| + |q_1(z_1) - q_1(z_2)||w_{z2}| \\ &\quad + |q_2(z_1)||w_{z1} - w_{z2}| + |q_2(z_1) - q_2(z_2)||w_{z2}| \\ &\quad + |a(z_1) - a(z_2)||w| + |b(z_1) - b(z_2)||w| + |c(z_1) - c(z_2)| \end{aligned}$$

We know that all coefficients of (1) are in $C^{\alpha}(\overline{\mathbb{D}})$. Also we have the condition (3). Using these, the result follows.

$$|F(z_1, w, w_{z1}) - F(z_2, w, w_{z2})| \leq q_0|w_{z1} - w_{z2}| + C(1 + |w| + |w_{z2}|)|z_1 - z_2|^{\alpha}.$$

Our aim is to transform the Dirichlet problem for the generalized Beltrami equation (1) with (2) to a fixed point problem.
We prove the following theorem.

Theorem 4.1. *A function $w \in C^{1,\alpha}(\bar{\Omega})$ is a solution to the Dirichlet problem (1) with the boundary condition (2) if and only if w can be written as the integral equation*

$$w(z) = \varphi(z) + T[F(z, w, w_z)](z), \tag{4}$$

where $\varphi \in C^{\alpha}(\bar{\Omega})$ is holomorphic in the domain $\mathbb{D}$ satisfying the Dirichlet boundary condition

$$\varphi(z) = \gamma(z) - T[F(z, w, w_z)] \text{ on } \partial\mathbb{D}. \tag{5}$$

Proof. *We assume that $w \in C^{1,\alpha}(\bar{\Omega})$ is a solution to the Dirichlet problem (1) with the boundary condition (2). We define a function φ by*

$$\varphi(z) = w(z) - T[F(z, w, w_z)](z).$$

Differentiating φ with respect to $\bar{z}$, we obtain

$$\varphi_{\bar{z}} = w_{\bar{z}} - [F(z, w, w_z)] = 0.$$

That is, φ is a holomorphic function in $\mathbb{D}$. The Dirichlet condition (2) becomes

$$\varphi(z) = \gamma(z) - T[F(z, w, w_z)] \text{ on } \partial\mathbb{D}.$$

for the holomorphic function φ. *We have* $\gamma \in C^{\alpha}(\partial\mathbb{D})$. *Further,* $F(z, w, w_z) \in C^{\alpha}(\bar{\mathbb{D}})$ *and therefore* $T_{\Omega}[F(z, w, w_z)] \in C^{\alpha}(\bar{\mathbb{D}})$. *Then, by Theorem 1.3,* $\varphi(z) \in C^{\alpha}(\bar{\Omega})$ *and therefore we have provided that* w *can be written as the integral equation* (4) *and Dirichlet boundary condition* (5) *is satisfied.*

Conversely, suppose that w *can be written as the integral equation* (4), *where* φ *is a holomorphic function satisfying* (5). *Differentiating* (4) *with respect to* $\bar{z}$, *we obtain*

$$w_{\bar{z}} = F(z, w, w_z).$$

This shows that $w \in C^{1,\alpha}(\bar{\mathbb{D}})$ *is a solution of the Dirichlet problem* (1) *with* (2).

We use the representation

$$w(z) = \varphi(z) + T[F(z, w, w_z)](z) \tag{6}$$

where $\varphi(z)$ is an analytic function in $\mathbb{D}$. Differentiating (6) with respect to z and using the properties of the Pompeiu operators, we obtain

$$w_z(z) = \varphi' + \Pi[F(z, w, w_z)](z). \tag{7}$$

The boundary condition becomes

$$\varphi(z) = \gamma(z) - T[F(z, w, w_z)] \text{ on } \partial\mathbb{D}. \tag{8}$$

The equations (6) and (7) form the following system of integro-differential equations.

$$\begin{aligned} w(z) &= \varphi(z) + T[F(z, w, w_z)](z) \\ w_z(z) &= \varphi'(z) + \Pi[F(z, w, w_z)](z) \end{aligned}$$

For the simplicity, we denote w_z by w^*. Let w and w^* be any functions in $C^{\alpha}(\bar{\mathbb{D}})$.

We define an operator Q by

$$Q : (w, w^*) \to (W, W^*)$$

where

$$\begin{aligned} W &= \varphi_{(w,w^*)} + T[F(\cdot, w, w^*)] \\ W^* &= \varphi'_{(w,w^*)} + \Pi[F(\cdot, w, w^*)], \end{aligned}$$

$\varphi_{(w,w^*)}$ is holomorphic in $\mathbb{D}$ and satisfies the Dirichlet boundary condition

$$\varphi_{(w,w^*)} = \gamma(z) - T[F(z, w, w^*)] \text{ on } \partial\mathbb{D}.$$

The fixed point of the operator Q provides the solution of the (1) satisfying Dirichlet boundary condition (2). We will obtain the conditions on the coefficients of F under which the operator Q has a fixed point.

We introduce function space

$$S = \left\{(w, w^*) : w, w^* \in C^{\alpha}(\bar{\mathbb{D}})\right\}$$

equipped with the norm

$$\|(w, w^*)\|_{*,\alpha} = \max\left(\|w\|_{\alpha}, \|w^*\|_{\alpha}\right),$$

where $\|\cdot\|_{\alpha}$ denotes the Hölder norm in $C^{\alpha}(\bar{\Omega})$. Since $C^{\alpha}(\bar{\Omega})$ is a Banach space, S is a Banach space.

We pick (w_1, w_1^*) and (w_2, w_2^*) from S. Now let $Q(w_1, w_1^*) = (W_1, W_1^*)$ and $Q(w_2, w_2^*) = (W_2, W_2^*)$. Then we have

$$\begin{aligned} W_1 &= \varphi_{(w_1,w_1^*)} + T\left[F\left(\cdot, w_1, w_1^*\right)\right] \\ W_1^* &= \varphi'_{(w_1,w_1^*)} + \Pi\left[F\left(\cdot, w_1, w_1^*\right)\right] \end{aligned}$$

and

$$\begin{aligned} W_2 &= \varphi_{(w_2,w_2^*)} + T\left[F\left(\cdot, w_2, w_2^*\right)\right] \\ W_2^* &= \varphi'_{(w_2,w_2^*)} + \Pi\left[F\left(\cdot, w_2, w_2^*\right)\right]. \end{aligned}$$

where

$$\varphi_{(w_1,w_1{}^*)} = \gamma(z) - T[F(z, w_1, w_1{}^*)] \text{ on } \partial\mathbb{D}.$$

and

$$\varphi_{(w_2,w_2^*)} = \gamma(z) - T[F(z,w_2,w_2{}^*)] \ on \ \partial\mathbb{D}$$

respectively.

We obtain

$$\begin{aligned}\|W_1 - W_2\|_\alpha \le & \left\|\varphi_{(w_1,w_1^*)} - \varphi_{(w_2,w_2^*)}\right\|_\alpha + \|T\left[F\left(\cdot, w_1, w_1^*\right)\right] \\ & - T\left[F\left(\cdot, w_2, w_2^*\right)\right]\|_\alpha\end{aligned}$$

$$\begin{aligned}\left\|W_1^* - W_2^*\right\|_\alpha \le & \left\|\varphi'_{(w_1,w_1^*)} - \varphi'_{(w_2,w_2^*)}\right\|_\alpha + \|\Pi\left[F\left(\cdot, w_1, w_1^*\right)\right] \\ & - \Pi\left[F\left(\cdot, w_2, w_2^*\right)\right]\|_\alpha.\end{aligned}$$

For these computations, we set

$$C_1 := \|T\|_\alpha \left(C_{11}\left\|w_1^* - w_2^*\right\|_\alpha + C_{12}\|w_1 - w_2\|_\alpha\right)$$

$$C_2 := \|\Pi\|_\alpha \left(C_{11}\left\|w_1^* - w_2^*\right\|_\alpha + C_{12}\|w_1 - w_2\|_\alpha\right)$$

where $C_{11} := \|q_1\|_\alpha + \|q_2\|_\alpha$ and $C_{12} = \|a\|_\alpha + \|b\|_\alpha$.

Now, we use the fact that the operators T and Π are bounded in $C^\alpha(\bar{\Omega})$.

$$\left\|T\left[F\left(\cdot, w_1, w_1^*\right)\right] - T\left[F\left(\cdot, w_2, w_2^*\right)\right]\right\|_\alpha \le \|T\|_\alpha \|F\left(\cdot, w_1, w_1^*\right) - F\left(\cdot, w_2, w_2^*\right)\|_\alpha \le C_1.$$

$$\|\Pi\left[F\left(\cdot, w_1, w_1^*\right)\right] - \Pi\left[F\left(\cdot, w_2, w_2^*\right)\right]\|_\alpha \le \|\Pi\|_\alpha \| F\left(\cdot, w_1, w_1^*\right) - F\left(\cdot, w_2, w_2^*\right)\|_\alpha \le C_2$$

and using Theorem 1.3, we have

$$\left\|\varphi_{(w_1,w_1^*)} - \varphi_{(w_2,w_2^*)}\right\|_\alpha \le C_1 k$$

where k depends only on α. Similarly,

$$\left\|\varphi'_{(w_1,w_1^*)} - \varphi'_{(w_2,w_2^*)}\right\|_\alpha \le C_2 \hat{k}$$

where $\hat{k}$ depends only on α.

Now we consider the distance $d\left(Q\left(w_1, w_1^*\right), Q\left(w_2, w_2^*\right)\right)$.

$$\begin{aligned}d\left(Q\left(w_1, w_1^*\right), Q\left(w_2, w_2^*\right)\right) &= \left\|\left(W_1 - W_2\right), \left(W_1^* - W_2^*\right)\right\| \\ &= \max\left(\|W_1 - W_2\|_\alpha, \left\|W_1^* - W_2^*\right\|_\alpha\right) \\ &\le \max\left(\left\|\varphi_{(w_1,w_1^*)} - \varphi_{(w_2,w_2^*)}\right\|_\alpha + \right. \\ &\quad \left\|T\left[F\left(\cdot, w_1, w_1^*\right)\right] - T\left[F\left(\cdot, w_2, w_2^*\right)\right]\right\|_\alpha, \|\varphi'_{(w_1,w_1^*)} - \\ &\quad \left.\varphi'_{(w_2,w_2^*)}\|_\alpha + \|\Pi\left[F\left(\cdot, \omega_1, w_1^*\right)\right] - \Pi\left[F\left(\cdot, w_2, w_2^*\right)\right]\right\|_\alpha \\ &\le \max\left(C_1 k, C_2 \hat{k}\right) \\ &\le \left(C_1 + C_2\right) d\left(\left(w_1, w_1^*\right), \left(w_2, w_2^*\right)\right) \max\left(k\|T\|_\alpha\right. \\ &\quad \left., \hat{k}\|\Pi\|_\alpha\right).\end{aligned}$$

The operator Q is contractive if

$$\|q_1\|_\alpha + \|q_2\|_\alpha + \|a\|_\alpha + \|b\|_\alpha < \frac{1}{\max\left(k\|T\|_\alpha + \hat{k}\|\Pi\|_\alpha\right)}$$

We obtain the existence of the unique solution (w, w^*) by using Banach fixed point theorem.

Remark 4.2. The methodology considered in this work suggests the potential for analogous studies regarding the solvability of the generalized Beltrami equation under alternative boundary conditions.

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RESEARCH ARTICLE

Curvature Relations for Lagrangian Submersions From Globally Conformal Kaehler Manifolds

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ABSTRACT
In this paper, we derive curvature identities for Lagrangian submersions from globally conformal Kaehler manifolds onto Riemannian manifolds. Then, we give a relation between the horizontal lift of the curvature tensor of the base manifold and the curvature tensor of a fiber. We examine the necessary and sufficient conditions for the total manifolds of Lagrangian submersions to be Einstein. We also obtain Ricci, scalar, sectional, holomorphic bisectional and holomorphic sectional curvatures for these submersions. Finally, we give some inequalities involving the scalar and Ricci curvatures, and we also provide Chen-Ricci inequality for Lagrangian submersions from globally conformal Kaehler space forms.

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Keywords: Riemannian submersion, Lagrangian submersion, globally conformal Kaehler manifold, Chen-Ricci inequality.

1. INTRODUCTION

Curvature invariants play the most important role in Riemannian geometry. They determine the intrinsic and extrinsic properties of Riemannian manifolds. Chen (1993) established a relationship between the intrinsic and extrinsic invariants. He also obtained an inequality between Ricci curvature and the squared mean curvature of a submanifold of a real space form (Chen (1999)). After then, he obtained a generalization of this inequality which is known as Chen-Ricci inequality (Chen (2005)).

On the other hand, the notion of Riemannian submersion is a generalization of an isometry between two Riemannian manifolds which was introduced by O'Neill (1966) and Gray (1967), independently. This notion was extended to almost complex and almost contact manifolds (Watson (1976), Chinea (1985)). After that, Riemannian submersions are studied widely in various kinds of structures for both almost complex and almost contact manifolds such as almost Hermitian (Şahin (2017)), almost contact (Taştan (2017)), cosymplectic (Taştan and Gerdan Aydın (2019)) and Sasakian (Taştan and Gerdan (2016)). These structures have also examined in different types of Riemannian submersions such as anti-invariant submersions (Şahin (2010)), Lagrangian submersions (Taştan (2014)) etc. Riemannian submersions have also been studied in globally conformal Kaehler manifolds which are a special class of Kaehler manifolds. The globally and locally conformal Kaehler manifolds were studied widely by Vaisman (1980). Then, locally conformal Kaehler submersions were introduced by Marrero and Rocha (1994) and studied by many researchers (Çimen et al. (2024), Pirinççi et al. (2023)).

In this paper, we study the curvature relations for Lagrangian submersions which are defined from globally conformal Kaehler manifolds onto Riemannian manifolds. First, we obtain curvature identities for Lagrangian submersions whose total manifolds are globally conformal Kaehler manifolds. Then, we give a relation between the horizontal lift of the curvature tensor of the base manifold and the curvature tensor of a fiber. We obtain Ricci curvatures and scalar curvatures for these submersions. Then, we give the necessary and sufficient conditions for the total manifolds of such submersions to be Einstein. We also obtain sectional, holomorphic bisectional and holomorphic sectional curvatures. Finally, we derive some inequalities involving the scalar curvature and Ricci curvature of Lagrangian submersions from globally conformal Kaehler space forms and give Chen-Ricci inequality for such submersions as well.

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2. GLOBALLY CONFORMAL KAEHLER MANIFOLDS

A Hermitian manifold (M^{2n}, J, g) with an almost complex structure J and a Hermitian metric g is called a *locally conformal Kaehler (l.c.K.) manifold*, if there exists an open cover $\{O_i\}_{i\in I}$ of M^{2n} with a family $\{\sigma_i\}_{i\in I}$ of smooth functions $\sigma_i : O_i \to \mathbb{R}$ such that

$$g_i = e^{-\sigma_i} g|_{O_i}$$

are Kaehler metrics for every $i \in I$ (Dragomir and Ornea (1998)). If $\tilde{g} = e^{-\sigma} g$ is Kaehlerian for a smooth function $\sigma : M^{2n} \to \mathbb{R}$, then (M^{2n}, J, g) is called a *globally conformal Kaehler (g.c.K.) manifold*. Dragomir and Ornea (1998) gave the following theorem for locally conformal Kaehler manifolds.

Theorem 2.1. *Let Φ be a 2-form defined by $\Phi(X, Y) = g(X, JY)$ on a Hermitian manifold (M^{2n}, J, g), where X, Y are vector fields on M^{2n}. Then (M^{2n}, J, g) is a locally conformal Kaehler manifold if and only if there exists a closed 1- form ω defined on M^{2n} globally such that $d\Phi = \omega \wedge \Phi$.*

If ω is exact, then (M^{2n}, J, g) is a g.c.K. manifold. In the case $\omega \equiv 0$, a g.c.K. manifold reduces a Kaehler manifold. The 1-form ω is called Lee form of (M^{2n}, J, g) and a g.c.K. manifold (M^{2n}, J, g) with Lee form ω is denoted by (M^{2n}, J, g, ω).

For the Riemannian connections ∇ of (M^{2n}, J, g, ω) and $\tilde{\nabla}$ of Kaehler metric $\tilde{g}$, we have

$$\tilde{\nabla}_X Y = \nabla_X Y - \frac{1}{2}\Big\{\omega(X)Y + \omega(Y)X - g(X, Y)B\Big\}, \tag{1}$$

where X, Y are vector fields on M^{2n} and B is the g- dual vector field of ω which is called *Lee vector field* of (M^{2n}, J, g, ω). $\tilde{\nabla}$ is a torsion-free connection and also satisfies $\tilde{\nabla} J = 0$. Hence, using (1), we have

$$(\nabla_X J)Y = \frac{1}{2}\Big\{\omega(JY)X - \omega(Y)JX - \Phi(X, Y)B + g(X, Y)JB\Big\}.$$

Now, from (1), Vaisman (1980) gave curvature identity between Riemannian curvature tensors of ∇ and $\tilde{\nabla}$ as follows:

$$\begin{aligned}\tilde{R}(X, Y)Z = \; & R(X, Y)Z \\ & - \frac{1}{2}\Big\{L(X, Z)Y - L(Y, Z)X - g(Y, Z)\Big[\nabla_X B + \frac{1}{2}\omega(X)B\Big] \\ & + g(X, Z)\Big[\nabla_Y B + \frac{1}{2}\omega(Y)B\Big]\Big\} \\ & - \frac{||\omega||^2}{4}\Big\{g(Y, Z)X - g(X, Z)Y\Big\},\end{aligned} \tag{2}$$

where $||\omega||^2 = g(B, B)$,

$$L(X, Y) = (\nabla_X \omega)(Y) + \frac{1}{2}\omega(X)\omega(Y) = g(\nabla_X B, Y) + \frac{1}{2}\omega(X)\omega(Y),$$

and X, Y, Z are vector fields on M^{2n}.

We note that L is a symmetric (0,2)-tensor on M^{2n}. From (2), he also obtained the well-known formula:

$$\begin{aligned}e^{\sigma}\tilde{R}(X, Y, Z, W) = \; & R(X, Y, Z, W) \\ & - \frac{1}{2}\Big\{L(X, Z)g(Y, W) - L(Y, Z)g(X, W) \\ & - L(X, W)g(Y, Z) + L(Y, W)g(X, Z)\Big\} \\ & - \frac{||\omega||^2}{4}\Big\{g(Y, Z)g(X, W) - g(X, Z)g(Y, W)\Big\},\end{aligned} \tag{3}$$

where $\tilde{R}$ denotes the Riemannian curvature tensor of the Kaehler metric $\tilde{g}$. Now, since $\tilde{g}$ is a Kaehler metric, the Riemannian curvature tensor $\tilde{R}$ satisfies $\tilde{R}(JX, JY, JZ, JW) = \tilde{R}(X, Y, Z, W)$. If we use the last equation in (3), then we get the following result as in the l.c.K submersion case in Pirinççi et al. (2023).

Theorem 2.2. *Let (M, J, g, ω) be a g.c.K. manifold. Then we have*

$$
\begin{aligned}
R(X,Y,Z,W) = {} & R(JX,JY,JZ,JW) \\
& + \frac{1}{2}\Big\{\delta(X,Z)g(Y,W) - \delta(Y,Z)g(X,W) \\
& - \delta(X,W)g(Y,Z) + \delta(Y,W)g(X,Z)\Big\},
\end{aligned} \tag{4}
$$

where

$$\delta(X,Y) = L(X,Y) - L(JX,JY),$$

and X, Y, Z, W are vector fields on M^{2n}.

3. LAGRANGIAN SUBMERSIONS

In this section, we will give the notion of Riemannian submersion and its special type Lagrangian submersion. We deduce the curvature relations for Lagrangian submersions.

Let (M_1^n, g_1) and (M_2^m, g_2) be Riemannian manifolds with dimensions n and m, respectively. O'Neill (1966) called a mapping ψ of (M_1^n, g_1) onto (M_2^m, g_2) that satisfies the following two conditions a *Riemannian submersion*:

(i) The rank of ψ is maximal;

which means that the derivative map ψ_* is surjective. Hence for each $y \in M_2^m$, $\psi^{-1}(y)$ is an $(n-m)$-dimensional closed submanifold of M_1^n. A submanifold $\psi^{-1}(y)$ is called a *fiber*. The vector fields on M_1^n which are tangent to a fiber is called *vertical*, and the vector fields on M_1^n orthogonal to a fiber is called *horizontal*. Vertical and horizontal distributions of the tangent space of M_1^n are denoted by $ker\psi_*$ and $(ker\psi_*)^\perp$, respectively. A horizontal vector field X on M_1^n is called *basic* if $\psi_*(X) = X_*$, for a vector field X_* on M_2^m.

(ii) ψ_* is a linear isometry on $(ker\psi_*)^\perp$.

Let E^v and E^h be the vertical and horizontal part of a vector field on M_1^n, respectively. Then, the covariant derivatives of vertical and horizontal vector fields are defined by O'Neill (1966) as follows:

$$\mathcal{T}_E F = (\nabla_{E^v} F^v)^h + (\nabla_{E^v} F^h)^v, \tag{5}$$

$$\mathcal{A}_E F = (\nabla_{E^h} F^v)^h + (\nabla_{E^h} F^h)^v, \tag{6}$$

where E and F are vector fields on M_1^n and ∇ is the Riemannian connection of g_1. The tensors $\mathcal{T}$ and $\mathcal{A}$ defined above are called *O'Neill's tensors*. $\mathcal{T}_E$ and $\mathcal{A}_E$ are skew-symmetric operators and each one reverses the vertical distribution to the horizontal distribution, and vice versa.

Lemma 3.1. *Let $\psi : (M_1^n, g_1) \to (M_2^m, g_2)$ be a Riemannian submersion, and X, Y be basic vector fields on M_1^n. Then,*

(i) $g_1(X,Y) = g_2(X_*, Y_*) \circ \psi$,

(ii) $\psi_*([X,Y]^h) = [X_*, Y_*]$,

(iii) $\psi_*((\nabla_X Y)^h) = \nabla^*_{X_*} Y_*$.

Using (5), (6) and Lemma 3.1 we obtain the following equations:

$$\mathcal{T}_U V = \mathcal{T}_V U, \tag{7}$$

$$\mathcal{A}_X Y = -\mathcal{A}_Y X = \frac{1}{2}[X,Y]^v, \tag{8}$$

$$\nabla_U V = \mathcal{T}_U V + (\nabla_U V)^v, \tag{9}$$

$$\nabla_U X = (\nabla_U X)^h + \mathcal{T}_U X, \tag{10}$$

$$\nabla_X U = \mathcal{A}_X U + (\nabla_X U)^v, \tag{11}$$

$$\nabla_X Y = (\nabla_X Y)^h + \mathcal{A}_X Y, \tag{12}$$

$$(\nabla_U X)^h = \mathcal{A}_X U, \qquad \textit{for a basic vector field } X, \tag{13}$$

where $U, V \in ker\psi_*$ and $X, Y \in (ker\psi_*)^\perp$. $\mathcal{T}$ is the second fundamental form of all the fibers. We say that the fibers are totally geodesic when $\mathcal{T} = 0$. If $\{U_1, ..., U_n\}$ is an ortonormal frame of $ker\psi_*$, then $H = \frac{1}{n}\sum_{i=1}^n \mathcal{T}_{U_i} U_i$ is called the *mean curvature vector field* of the fibers. For more information about Riemannian submersions, we refer to O'Neill (1966) and Falcitelli et al. (2004).

Now, using Lemma 3.1 and the equations (7)~(13) we have the following curvature relations for every $U, V, W, W' \in ker\psi_*$

and $X, Y, Z, Z' \in (ker\psi_*)^{\perp}$:

$$R_1(U,V,W,W') = \hat{R}(U,V,W,W') + g_1(\mathcal{T}_U W', \mathcal{T}_V W) - g_1(\mathcal{T}_U W, \mathcal{T}_V W'), \tag{14}$$

$$R_1(U,V,W,X) = g_1((\nabla_V \mathcal{T})_U W, X) - g_1((\nabla_U \mathcal{T})_V W, X), \tag{15}$$

$$R_1(X,Y,Z,Z') = R^*(X,Y,Z,Z') - 2g_1(\mathcal{A}_X Y, \mathcal{A}_Z Z') - g_1(\mathcal{A}_X Z, \mathcal{A}_Y Z') + g_1(\mathcal{A}_X Z', \mathcal{A}_Y Z), \tag{16}$$

$$R_1(X,Y,Z,U) = g_1((\nabla_Z \mathcal{A})_X Y, U) + g_1(\mathcal{A}_X Y, \mathcal{T}_U Z) + g_1(\mathcal{A}_X Z, \mathcal{T}_U Y) - g_1(\mathcal{A}_Y Z, \mathcal{T}_U X), \tag{17}$$

$$R_1(X,Y,U,V) = g_1((\nabla_U \mathcal{A})_X Y, V) - g_1((\nabla_V \mathcal{A})_X Y, U) + g_1(\mathcal{A}_X U, \mathcal{A}_Y V) - g_1(\mathcal{A}_X V, \mathcal{A}_Y U) - g_1(\mathcal{T}_U X, \mathcal{T}_V Y) + g_1(\mathcal{T}_V X, \mathcal{T}_U Y), \tag{18}$$

$$R_1(X,U,Y,V) = g_1((\nabla_X \mathcal{T})_U V, Y) + g_1((\nabla_U \mathcal{A})_X Y, V) - g_1(\mathcal{T}_U X, \mathcal{T}_V Y) + g_1(\mathcal{A}_X U, \mathcal{A}_Y V), \tag{19}$$

where R_1 and R_2 are Riemannian curvature tensors of M_1^n and M_2^m, respectively, R^* is the horizontal lift of the curvature tensor of R_2, i.e., $R^*(X,Y,Z,Z') = g_1(R^*(Z,Z')Y,X) = R_2(\psi_* X, \psi_* Y, \psi_* Z, \psi_* Z') \circ \psi$ and $\hat{R}$ is the curvature tensor of $\psi^{-1}(y)$ (see O'Neill (1966)).

We note that $(\nabla_E \mathcal{A})_F$ and $(\nabla_E \mathcal{T})_F$ are skew-symmetric and linear operators defined by

$$(\nabla_E \mathcal{A})_F G = \nabla_E(\mathcal{A}_F G) - \mathcal{A}_{(\nabla_E F)} G - \mathcal{A}_F(\nabla_E G),$$
$$(\nabla_E \mathcal{T})_F G = \nabla_E(\mathcal{T}_F G) - \mathcal{T}_{(\nabla_E F)} G - \mathcal{T}_F(\nabla_E G),$$

respectively, where E, F and G are vector fields on M_1^n. Moreover, $g_1((\nabla_E \mathcal{A})_X Y, U)$ is alternate in X and Y, $g_1((\nabla_E \mathcal{T})_U V, X)$ is symmetric in U and V, where $U, V \in ker\psi_*$ and $X, Y \in (ker\psi_*)^{\perp}$. Furthermore, (14) and (15) are the corresponding Gauss and Codazzi equations, (16) and (17) are their dual equations.

Definition 3.2. Let ψ be a Riemannian submersion from a Hermitian manifold (M_1^{2n}, J, g_1) onto a Riemannian manifold (M_2^m, g_2). ψ is called an *anti-invariant Riemannian submersion*, if its vertical distribution is anti-invariant with respect to J, i.e. $J(ker\psi_*) \subseteq (ker\psi_*)^{\perp}$. Especially, ψ is called a *Lagrangian submersion* when $J(ker\psi_*) = (ker\psi_*)^{\perp}$. In this case J reverses the vertical (horizontal) distributions to the horizontal (vertical) distributions, and $m = n$.

4. CURVATURE IDENTITIES FOR LAGRANGIAN SUBMERSIONS

In this section, we obtain curvature relations using the following result due to Pirinççi (2025). From now on, $(M_1^{2n}, J, g_1, \omega)$ represents a g.c.K. manifold and (M_2^n, g_2) represents a Riemannian manifold.

Lemma 4.1. *Let $\psi : (M_1^{2n}, J, g_1, \omega) \to (M_2^n, g_2)$ be a Lagrangian submersion. Then we have*

$$\mathcal{T}_U JV = J\mathcal{T}_U V + \frac{1}{2}\{\omega(JV)U + g_1(U,V)JB^h\},$$

$$\mathcal{T}_U JX = J\mathcal{T}_U X - \frac{1}{2}\{\omega(X)JU - g_1(JU,X)B^h\},$$

$$\mathcal{A}_X JU = J\mathcal{A}_X U - \frac{1}{2}\{\omega(U)JX + g_1(X,JU)B^v\},$$

$$\mathcal{A}_X JY = J\mathcal{A}_X Y + \frac{1}{2}\{\omega(JY)X + g_1(X,Y)JB^v\},$$

where $U, V \in ker\psi_*$ and $X, Y \in (ker\psi_*^{\perp})$.

Pirinççi (2025) showed that for a Lagrangian submersion ψ from a l.c.K. manifold onto a Riemannian manifold, if JU is a basic vector field for any $U \in ker\psi_*$, then the Lee vector field B cannot be vertical. Therefore, we will examine the curvature relations in the special case where the Lee vector field B is horizontal. In this case he showed that the horizontal distribution is integrable and totally geodesic, i.e., $\mathcal{A} \equiv 0$. Then we get the following result from (13)~(18):

Corollary 4.2. Let $\psi : (M_1^{2n}, J, g_1, \omega) \to (M_2^n, g_2)$ be a Lagrangian submersion and the Lee vector field B be horizontal. Then, we have the following curvature relations for every $U, V, W, W' \in ker\psi_*$ and $X, Y, Z, Z' \in (ker\psi_*)^\perp$:

$$R_1(U,V,W,W') = \hat{R}(U,V,W,W') + g_1(\mathcal{T}_U W', \mathcal{T}_V W) - g_1(\mathcal{T}_U W, \mathcal{T}_V W'), \tag{20}$$

$$R_1(U,V,W,X) = g_1((\nabla_V \mathcal{T})_U W, X) - g_1((\nabla_U \mathcal{T})_V W, X), \tag{21}$$

$$R_1(X,Y,Z,Z') = R^*(X,Y,Z,Z'), \tag{22}$$

$$R_1(X,Y,Z,U) = 0, \tag{23}$$

$$R_1(X,Y,U,V) = g_1(\mathcal{T}_V X, \mathcal{T}_U Y) - g_1(\mathcal{T}_U X, \mathcal{T}_V Y), \tag{24}$$

$$R_1(X,U,Y,V) = g_1((\nabla_X \mathcal{T})_U V, Y) - g_1(\mathcal{T}_U X, \mathcal{T}_V Y). \tag{25}$$

Now, if we use $\mathcal{A} \equiv 0$, (20) and (22) in (4), then we get the following relation between the horizontal lift of the curvature tensor of R_2 and the curvature tensor of a fiber.

Theorem 4.3. *Let $\psi : (M_1^{2n}, J, g_1, \omega) \to (M_2^n, g_2)$ be a Lagrangian submersion and the Lee vector field B be horizontal. Then, we have the following curvature relation for every $U, V, W, W' \in ker\psi_*$:*

$$\begin{aligned}\hat{R}(U,V,W,W') = {} & R^*(JU,JV,JW,JW') - g_1(\mathcal{T}_U W', \mathcal{T}_V W) + g_1(\mathcal{T}_U W, \mathcal{T}_V W') \\ & + \frac{1}{2}\Big\{\delta(U,W)g_1(V,W') - \delta(V,W)g_1(U,W') \\ & - \delta(U,W')g_1(V,W) + \delta(V,W')g_1(U,W)\Big\}.\end{aligned} \tag{26}$$

In a similar way, if we use (21) and (23) in (4), then we have

$$g_1((\nabla_V \mathcal{T})_U W, X) - g_1((\nabla_U \mathcal{T})_V W, X) = \frac{1}{2}\Big\{\delta(V,X)g_1(U,W) - \delta(U,X)g_1(V,W)\Big\}.$$

Using the skew-symmetry property of the operator $(\nabla_V \mathcal{T})_U$, we get

$$(\nabla_U \mathcal{T})_V X - (\nabla_V \mathcal{T})_U X = \frac{1}{2}\Big\{\delta(V,X)U - \delta(U,X)V\Big\}. \tag{27}$$

We will examine the conditions for M_1^{2n} to be an Einstein manifold. To obtain these conditions we will first find the Ricci and scalar curvatures of M_1^{2n} using the following notation:

$$\mathcal{T}_{ij}^k = g_1(\mathcal{T}_{U_i} U_j, JU_k), \tag{28}$$

$$||\mathcal{T}||^2 = \sum_{i,j=1}^n g_1(\mathcal{T}_{U_i} U_j, \mathcal{T}_{U_i} U_j) = \sum_{k=1}^n \sum_{i,j=1}^n (\mathcal{T}_{ij}^k)^2, \tag{29}$$

$$\delta(\mathcal{T}) = \sum_{i,j=1}^n g_1((\nabla_{JU_i} \mathcal{T})_{U_j} U_j, JU_i), \tag{30}$$

where $\{U_1, ..., U_n\}$ is an orthonormal frame of $ker\psi_*$.

Lemma 4.4. *Let $\psi : (M_1^{2n}, J, g_1, \omega) \to (M_2^n, g_2)$ be a Lagrangian submersion and the Lee vector field B be horizontal. Then the Ricci tensor Ric_1 and the scalar curvature ρ_1 of M_1^{2n} satisfy the following relations:*

$$\begin{aligned}Ric_1(U,V) = {} & \hat{Ric}(U,V) - \frac{1}{2}\omega(JU)\omega(JV) + g_1(U,V)||\omega||^2 \\ & - n g_1(H, \mathcal{T}_U V) + \sum_{i=1}^n g_1((\nabla_{JU_i} \mathcal{T})_U V, JU_i),\end{aligned} \tag{31}$$

$$Ric_1(U,X) = \frac{n-1}{2}(\nabla_U \omega)X, \tag{32}$$

$$Ric_1(X,Y) = Ric^*(X,Y) + \sum_{i=1}^n \Big\{g_1((\nabla_X \mathcal{T})_{U_i} U_i, Y) - g_1(\mathcal{T}_{U_i} X, \mathcal{T}_{U_i} Y)\Big\}, \tag{33}$$

$$\rho_1 = \hat{\rho} + \rho^* + (2n-1)||\omega||^2 - n^2||H||^2 - ||\mathcal{T}||^2 + 2\delta(\mathcal{T}), \tag{34}$$

where $U, V \in ker\psi_*$, $X, Y \in (ker\psi_*)^\perp$, $\{U_1, ..., U_n\}$ is an orthonormal frame of $ker\psi_*$, $\hat{Ric}$ is Ricci tensor of any fiber, Ric^* is the horizontal lift of Ricci tensor of M_2^n, $\hat{\rho}$ is scalar curvature of any fiber and ρ^* is the lift of scalar curvature of M_2^n.

Proof. Let $\{U_1, ..., U_n\}$ be an ortonormal frame of $ker\psi_*$. From the definition of Ricci tensor, (20) and (25) we have

$$\begin{aligned} Ric_1(U,V) &= \sum_{i=1}^{n} R_1(U_i,U,U_i,V) + \sum_{i=1}^{n} R_1(JU_i,U,JU_i,V) \\ &= \sum_{i=1}^{n} \Big\{\hat{R}(U_i,U,U_i,V) + g_1(\mathcal{T}_{U_i}V,\mathcal{T}_U U_i) - g_1(\mathcal{T}_{U_i}U_i,\mathcal{T}_U V)\Big\} \\ &\quad + \sum_{i=1}^{n} \Big\{g_1((\nabla_{JU_i}\mathcal{T})_U V, JU_i) - g_1(\mathcal{T}_U JU_i, \mathcal{T}_V JU_i)\Big\} \\ &= \hat{Ric}(U,V) - ng_1(H,\mathcal{T}_U V) \\ &\quad + \sum_{i=1}^{n} \Big\{g_1(\mathcal{T}_U U_i,\mathcal{T}_V U_i) + g_1((\nabla_{JU_i}\mathcal{T})_U V, JU_i) - g_1(\mathcal{T}_U JU_i,\mathcal{T}_V JU_i)\Big\}. \end{aligned} \tag{35}$$

Now, from Lemma 4.1 we have

$$\begin{aligned} \sum_{i=1}^{n} g_1(\mathcal{T}_U JU_i,\mathcal{T}_V JU_i) &= \sum_{i=1}^{n} g_1(\mathcal{T}_U U_i,\mathcal{T}_V U_i) + \frac{1}{2}\sum_{i=1}^{n} \Big\{\omega(JU_i)g_1(J\mathcal{T}_U U_i,V) \\ &\quad + g_1(V,U_i)g_1(\mathcal{T}_U U_i,B^h) + \omega(JU_i)g_1(U,J\mathcal{T}_V U_i) \\ &\quad + g_1(U,U_i)g_1(\mathcal{T}_V U_i,B^h)\Big\} \\ &= \sum_{i=1}^{n} g_1(\mathcal{T}_U U_i,\mathcal{T}_V U_i) - \frac{1}{2}\sum_{i=1}^{n} \Big\{g_1(U_i,JB)g_1(U_i,\mathcal{T}_U JV) \\ &\quad + g_1(V,U_i)g_1(U_i,\mathcal{T}_U B) + g_1(U_i,JB)g_1(\mathcal{T}_V JU,U_i) \\ &\quad + g_1(U,U_i)g_1(U_i,\mathcal{T}_V B)\Big\} \\ &= \sum_{i=1}^{n} g_1(\mathcal{T}_U U_i,\mathcal{T}_V U_i) \\ &\quad + \frac{1}{2}\Big\{\omega(J\mathcal{T}_U JV) + 2\omega(\mathcal{T}_U V) + \omega(J\mathcal{T}_V JU)\Big\}. \end{aligned} \tag{36}$$

Moreover, from Lemma 4.1, since

$$\begin{aligned} \omega(J\mathcal{T}_U JV) &= -\,g_1(\mathcal{T}_U JV, JB) \\ &= -\,g_1(J\mathcal{T}_U V + \frac{1}{2}\{\omega(JV)U + g_1(U,V)JB^h\}, JB) \\ &= -\,\omega(\mathcal{T}_U V) + \frac{1}{2}\omega(JV)\omega(JU) - g_1(U,V)||\omega||^2 \\ &= \omega(J\mathcal{T}_V JU), \end{aligned}$$

equation (36) becomes

$$\sum_{i=1}^{n} g_1(\mathcal{T}_U JU_i,\mathcal{T}_V JU_i) = \sum_{i=1}^{n} g_1(\mathcal{T}_U U_i,\mathcal{T}_V U_i) + \frac{1}{2}\omega(JU)\omega(JV) - g_1(U,V)||\omega||^2. \tag{37}$$

Using (37) in (35), we obtain (31).
Similarly, using (21), (22), (23), (25) and (27) we obtain (32) and (33). Now, from the definition of scalar curvature, (31) and (33)

we have

$$\begin{aligned}\rho_1 &= \sum_{j=1}^{n} Ric_1(U_j,U_j) + \sum_{j=1}^{n} Ric_1(JU_j,JU_j)\\ &= \sum_{j=1}^{n}\Big\{\hat{Ric}(U_j,U_j) - \frac{1}{2}\omega(JU_j)\omega(JU_j) + g_1(U_j,U_j)\|\omega\|^2\\ &\quad + \sum_{i=1}^{n}\Big[g_1((\nabla_{JU_i}\mathcal{T})_{U_j}U_j, JU_i) - g_1(\mathcal{T}_{U_i}U_i,\mathcal{T}_{U_j}U_j)\Big]\Big\}\\ &\quad + \sum_{j=1}^{n}\Big\{Ric^*(JU_j,JU_j) + \sum_{i=1}^{n}\Big[g_1((\nabla_{JU_j}\mathcal{T})_{U_i}U_i, JU_j) - g_1(\mathcal{T}_{U_i}JU_j,\mathcal{T}_{U_i}JU_j)\Big]\Big\}\\ &= \hat{\rho} + \rho^* + \frac{2n-1}{2}\|\omega\|^2 - n^2\|H\|^2\\ &\quad + 2\sum_{i,j=1}^{n} g_1((\nabla_{JU_i}\mathcal{T})_{U_j}U_j, JU_i) - \sum_{i,j=1}^{n} g_1(\mathcal{T}_{U_i}JU_j,\mathcal{T}_{U_i}JU_j)\end{aligned}$$

Finally, if we use (29), (30) and (37) in the last equation, we get (34).

Theorem 4.5. *Let $\psi : (M_1^{2n}, J, g_1, \omega) \to (M_2^n, g_2)$ be a Lagrangian submersion and the Lee vector field B be horizontal. Then, $(M_1^{2n}, J, g_1, \omega)$ is an Einstein manifold if and only if the following relations hold:*

$$\begin{aligned}\hat{Ric}(U,V) &= (\frac{\rho_1}{2n} - \|\omega\|^2)g_1(U,V) + ng_1(H,\mathcal{T}_UV) + \frac{1}{2}\omega(JU)\omega(JV)\\ &\quad - \sum_{i=1}^{n} g_1((\nabla_{JU_i}\mathcal{T})_UV, JU_i),\\ Ric^*(X,Y) &= \frac{\rho_1}{2n}g_1(X,Y) - \sum_{i=1}^{n}\Big\{g_1((\nabla_X\mathcal{T})_{U_i}U_i, Y) - g_1(\mathcal{T}_{U_i}X,\mathcal{T}_{U_i}Y)\Big\},\end{aligned}$$

and

$$(\nabla_U\omega)X = 0,$$

where $U,V \in ker\psi_*$, $X,Y \in (ker\psi_*)^\perp$ and $\{U_1,...,U_n\}$ is an orthonormal frame of $ker\psi_*$.

Proof. $(M_1^{2n}, J, g_1, \omega)$ is an Einstein manifold if and only if $Ric_1 = \frac{\rho_1}{2n}g_1$. Using this in (31), (32) and (33), we get the results.

Theorem 4.6. *Let $\psi : (M_1^{2n}, J, g_1, \omega) \to (M_2^n, g_2)$ be a Lagrangian submersion and the Lee vector field B be horizontal. If $U,V \in ker\psi_*$ and $X,Y \in (ker\psi_*)^\perp$, then the sectional curvature K_1 is given by*

$$\begin{aligned}K_1(U,V) &= \hat{K}(U,V) + \frac{\|\mathcal{T}_UV\|^2 - g_1(\mathcal{T}_UU,\mathcal{T}_VV)}{\|U\wedge V\|^2},\\ K_1(X,Y) &= K^*(X,Y),\\ K_1(X,U) &= \frac{g_1((\nabla_X\mathcal{T})_UU,X) - \|\mathcal{T}_UX\|^2}{\|X\|^2\|U\|^2},\end{aligned}$$

where $\|U\wedge V\|^2 = \|U\|^2\|V\|^2 - (g_1(U,V))^2$.

Proof. If we use the definition of the sectional curvature $K_1(E,F) = \frac{R_1(E,F,E,F)}{\|E\wedge F\|^2}$ in (20), then we have

$$\begin{aligned}K_1(U,V) &= \frac{R_1(U,V,U,V)}{\|U\wedge V\|^2}\\ &= \frac{1}{\|U\wedge V\|^2}\Big\{\hat{R}(U,V,U,V) + g_1(\mathcal{T}_UV,\mathcal{T}_VU) - g_1(\mathcal{T}_UU,\mathcal{T}_VV)\Big\}\\ &= \hat{K}(U,V) + \frac{\|\mathcal{T}_UV\|^2 - g_1(\mathcal{T}_UU,\mathcal{T}_VV)}{\|U\wedge V\|^2}.\end{aligned}$$

Similarly, using (22) and (25), we obtain the other two equations.

The *holomorphic bisectional curvature* and the *holomorphic sectional curvature* of an almost Hermitian manifold (M^{2n}, J, g) are defined for any nonzero vector fields E, F on M^{2n} as

$$B(E,F) = \frac{R(E,JE,F,JF)}{||E||^2||F||^2},$$

and

$$H(E) = B(E,E),$$

respectively. Hence, we have the following results.

Theorem 4.7. *Let $\psi : (M_1^{2n}, J, g_1, \omega) \to (M_2^n, g_2)$ be a Lagrangian submersion and the Lee vector field B be horizontal. If $U, V \in ker\psi_*$ and $X, Y \in (ker\psi_*)^\perp$, then the holomorphic bisectional curvature B_1 is given by*

$$B_1(U,V) = \frac{g_1((\nabla_{JU}\mathcal{T})_U V, JV) - g_1(\mathcal{T}_U JU, \mathcal{T}_V JV)}{||U||^2||V||^2},$$
$$B_1(X,Y) = \frac{g_1((\nabla_X\mathcal{T})_{JX} JY, Y) - g_1(\mathcal{T}_{JX} X, \mathcal{T}_{JY} Y)}{||X||^2||Y||^2},$$
$$B_1(X,U) = \frac{g_1(\mathcal{T}_{JX} X, \mathcal{T}_U JU) - g_1((\nabla_X\mathcal{T})_{JX} U, JU)}{||X||^2||U||^2}.$$

Proof. Using the definition of the holomorphic bisectional curvature in (25), we obtain the results immediately.

Theorem 4.8. *Let $\psi : (M_1^{2n}, J, g_1, \omega) \to (M_2^n, g_2)$ be a Lagrangian submersion and the Lee vector field B be horizontal. If $U \in ker\psi_*$ and $X \in (ker\psi_*)^\perp$, then the holomorphic sectional curvature H_1 is given by*

$$H_1(U) = \frac{g_1((\nabla_{JU}\mathcal{T})_U U, JU) - ||\mathcal{T}_U JU||^2}{||U||^4},$$
$$H_1(X) = \frac{g_1((\nabla_X\mathcal{T})_{JX} JX, X) - ||\mathcal{T}_{JX} X||^2}{||X||^4}.$$

Proof. Using the definition of the holomorphic bisectional curvature in Theorem 4.7, we obtain the above equations.

5. CHEN-RICCI INEQUALITY

A Kaehler manifold (M^{2n}, J, g) with constant holomorphic sectional curvature c is called a *complex space form* and denoted by $(M^{2n}(c), J, g)$. The curvature tensor R of $(M^{2n}(c), J, g)$ satisfies

$$\begin{aligned} R(X,Y,Z,W) = \frac{c}{4}\Big\{ & g(X,W)g(Y,Z) - g(X,Z)g(Y,W) + g(JX,W)g(JY,Z) \\ & - g(JX,Z)g(JY,W) - 2g(JX,Y)g(JZ,W)\Big\} \end{aligned} \tag{38}$$

for every vector fields X, Y, Z, W on $M^{2n}(c)$.

A g.c.K. manifold $(M_1^{2n}, J, g_1, \omega)$ with constant holomorphic sectional curvature c is called a *globally conformal complex space form* and denoted by $(M_1^{2n}(c), J, g_1, \omega)$. Using (3) and (38), we get

$$\begin{aligned} R_1(X,Y,Z,W) = & e^{-\sigma}\frac{c}{4}\Big\{ g_1(X,W)g_1(Y,Z) - g_1(X,Z)g_1(Y,W) + g_1(JX,W)g_1(JY,Z) \\ & - g_1(JX,Z)g_1(JY,W) - 2g_1(JX,Y)g_1(JZ,W)\Big\} \\ & + \frac{1}{2}\Big\{L(X,Z)g_1(Y,W) - L(Y,Z)g_1(X,W) \\ & - L(X,W)g_1(Y,Z) + L(Y,W)g_1(X,Z)\Big\} \\ & + \frac{||\omega||^2}{4}\Big\{g_1(Y,Z)g_1(X,W) - g_1(X,Z)g_1(Y,W)\Big\} \end{aligned} \tag{39}$$

for every vector fields X, Y, Z, W on $M_1^{2n}(c)$.

Let $\psi : (M_1^{2n}(c), J, g_1, \omega) \to (M_2^n, g_2)$ be a Lagrangian submersion and the Lee vector field B be horizontal. Now, using (20) and (22) in (39), we have

$$\hat{R}(U,V,W,W') = \left(\frac{ce^{-\sigma} + ||\omega||^2}{4}\right)\Big\{g_1(U,W')g_1(V,W) - g_1(U,W)g_1(V,W')\Big\}$$
$$+ \frac{1}{2}\Big\{L(U,W)g_1(V,W') - L(V,W)g_1(U,W') \quad (40)$$
$$- L(U,W')g_1(V,W) + L(V,W')g_1(U,W)\Big\}$$
$$+ g_1(\mathcal{T}_U W, \mathcal{T}_V W') - g_1(\mathcal{T}_U W', \mathcal{T}_V W)$$

and

$$R^*(X,Y,Z,Z') = \left(\frac{ce^{-\sigma} + ||\omega||^2}{4}\right)\Big\{g_1(X,Z')g_1(Y,Z) - g_1(X,Z)g_1(Y,Z')\Big\}$$
$$+ \frac{1}{2}\Big\{L(X,Z)g_1(Y,Z') - L(Y,Z)g_1(X,Z')$$
$$- L(X,Z')g_1(Y,Z) + L(Y,Z')g_1(X,Z)\Big\},$$

for every $U,V,W,W' \in ker\psi_*$ and $X,Y,Z,Z' \in (ker\psi_*)^{\perp}$.

We will use the following remark in the examination of the curvature relations.

Remark 5.1. Pirinççi (2025) showed that the vertical distribution of a Lagrangian submersion from a l.c.K. manifold onto a Riemannian manifold cannot be totally geodesic, i.e., $\mathcal{T} \neq 0$.

Proposition 5.2. *Let $\psi : (M_1^{2n}(c), J, g_1, \omega) \rightarrow (M_2^n, g_2)$ be a Lagrangian submersion and the Lee vector field B be horizontal. If $\{U_1, ..., U_n\}$ is an orthonormal frame of $ker\psi_*$, then we have*

$$\hat{Ric}(U_1) < (1-n)\left(\frac{ce^{-\sigma} + ||\omega||^2}{4}\right) + ng_1(\mathcal{T}_{U_1}U_1, H) \quad (41)$$
$$- \frac{1}{2}\Big\{(n-2)\omega(\mathcal{T}_{U_1}U_1) + n\omega(H)\Big\},$$

where

$$\hat{Ric}(U_1) = \sum_{i=1}^{n} \hat{R}(U_1, U_i, U_1, U_i). \quad (42)$$

Proof. We note that if the Lee vector field B is horizontal, then $L(U,V) = -\omega(\mathcal{T}_U V)$. So for every $U,V,W,W' \in ker\psi_*$, (40) becomes

$$\hat{R}(U,V,W,W') = \left(\frac{ce^{-\sigma} + ||\omega||^2}{4}\right)\Big\{g_1(U,W')g_1(V,W) - g_1(U,W)g_1(V,W')\Big\}$$
$$- \frac{1}{2}\Big\{\omega(\mathcal{T}_U W)g_1(V,W') - \omega(\mathcal{T}_V W)g_1(U,W') \quad (43)$$
$$- \omega(\mathcal{T}_U W')g_1(V,W) + \omega(\mathcal{T}_V W')g_1(U,W)\Big\}$$
$$+ g_1(\mathcal{T}_U W, \mathcal{T}_V W') - g_1(\mathcal{T}_U W', \mathcal{T}_V W).$$

Using (43) in (42), we have

$$\hat{Ric}(U_1) = \sum_{i=1}^{n}\left(\frac{ce^{-\sigma} + ||\omega||^2}{4}\right)\Big\{g_1(U_1,U_i)g_1(U_i,U_1) - g_1(U_1,U_1)g_1(U_i,U_i)\Big\}$$
$$- \frac{1}{2}\sum_{i=1}^{n}\Big\{\omega(\mathcal{T}_{U_1}U_1)g_1(U_i,U_i) - \omega(\mathcal{T}_{U_i}U_1)g_1(U_1,U_i)$$
$$- \omega(\mathcal{T}_{U_1}U_i)g_1(U_i,U_1) + \omega(\mathcal{T}_{U_i}U_i)g_1(U_1,U_1)\Big\}$$
$$+ \sum_{i=1}^{n}\Big\{g_1(\mathcal{T}_{U_1}U_1, \mathcal{T}_{U_i}U_i) - g_1(\mathcal{T}_{U_1}U_i, \mathcal{T}_{U_i}U_1)\Big\}$$
$$= (1-n)\left(\frac{ce^{-\sigma} + ||\omega||^2}{4}\right) + ng_1(\mathcal{T}_{U_1}U_1, H) - ||\mathcal{T}_{U_i}U_1||^2$$
$$- \frac{1}{2}\Big\{(n-2)\omega(\mathcal{T}_{U_1}U_1) + n\omega(H)\Big\}.$$

Hence, (41) comes from Remark 5.1.

Proposition 5.3. *Let* $\psi : (M_1^{2n}(c), J, g_1, \omega) \to (M_2^n, g_2)$ *be a Lagrangian submersion and the Lee vector field B be horizontal. If* $\{U_1, ..., U_n\}$ *is an orthonormal frame of* $ker\psi_*$*, then the scalar curvature of the vertical distribution holds*

$$\hat{\rho} < n(1-n)\Big(\frac{ce^{-\sigma} + ||\omega||^2}{4} + \omega(H)\Big) + n^2||H||^2, \tag{44}$$

where

$$\hat{\rho} = \sum_{i,j=1}^{n} \hat{R}(U_i, U_j, U_i, U_j). \tag{45}$$

Proof. If we use (43) in (45), then we have

$$\begin{aligned}
\hat{\rho} = & \sum_{i,j=1}^{n} \Big(\frac{ce^{-\sigma} + ||\omega||^2}{4}\Big)\Big\{g_1(U_i, U_j)g_1(U_j, U_i) - g_1(U_i, U_i)g_1(U_j, U_j)\Big\} \\
& - \frac{1}{2}\sum_{i,j=1}^{n} \Big\{\omega(\mathcal{T}_{U_i}U_i)g_1(U_j, U_j) - \omega(\mathcal{T}_{U_j}U_i)g_1(U_i, U_j) \\
& - \omega(\mathcal{T}_{U_i}U_j)g_1(U_j, U_i) + \omega(\mathcal{T}_{U_j}U_j)g_1(U_i, U_i)\Big\} \\
& + \sum_{i,j=1}^{n} \Big\{g_1(\mathcal{T}_{U_i}U_i, \mathcal{T}_{U_j}U_j) - g_1(\mathcal{T}_{U_i}U_j, \mathcal{T}_{U_j}U_i)\Big\} \\
= & \, n(1-n)\Big(\frac{ce^{-\sigma} + ||\omega||^2}{4}\Big) - ||\mathcal{T}||^2 + n^2||H||^2 + n(1-n)\omega(H).
\end{aligned}$$

The result is obtained by using Remark 5.1.

Proposition 5.4. *Let* $\psi : (M_1^{2n}(c), J, g_1, \omega) \to (M_2^n, g_2)$ *be a Lagrangian submersion and the Lee vector field B be horizontal. If* $\{U_1, ..., U_n\}$ *is an orthonormal frame of* $ker\psi_*$*, then the scalar curvature of the horizontal distribution holds*

$$\rho^* < n(1-n)\Big(\frac{ce^{-\sigma} + ||\omega||^2}{4} - \omega(H)\Big) + ||\mathcal{T}||^2 + (n-1)\mathrm{Trace}(L), \tag{46}$$

where

$$\rho^* = \sum_{i,j=1}^{n} R^*(JU_i, JU_j, JU_i, JU_j). \tag{47}$$

Proof. From (26) we have

$$\begin{aligned}\sum_{i,j=1}^{n}\hat{R}(U_i,U_j,U_i,U_j) &= \sum_{i,j=1}^{n}R^*(JU_i,JU_j,JU_i,JU_j)\\
&\quad-\sum_{i,j=1}^{n}g_1(\mathcal{T}_{U_i}U_j,\mathcal{T}_{U_j}U_i)+\sum_{i,j=1}^{n}g_1(\mathcal{T}_{U_i}U_i,\mathcal{T}_{U_j}U_j)\\
&\quad+\frac{1}{2}\sum_{i,j=1}^{n}\Big\{\delta(U_i,U_i)g_1(U_j,U_j)-\delta(U_j,U_i)g_1(U_i,U_j)\\
&\qquad-\delta(U_i,U_j)g_1(U_j,U_i)+\delta(U_j,U_j)g_1(U_i,U_i)\Big\}\\
&=\sum_{i,j=1}^{n}R^*(JU_i,JU_j,JU_i,JU_j)-||\mathcal{T}||^2+n^2||H||^2\\
&\quad+(n-1)\sum_{i=1}^{n}\delta(U_i,U_i)\\
&=\sum_{i,j=1}^{n}R^*(JU_i,JU_j,JU_i,JU_j)-||\mathcal{T}||^2+n^2||H||^2\\
&\quad+(n-1)\sum_{i=1}^{n}\Big\{L(U_i,U_i)-L(JU_i,JU_i)\Big\}\\
&=\sum_{i,j=1}^{n}R^*(JU_i,JU_j,JU_i,JU_j)-||\mathcal{T}||^2+n^2||H||^2\\
&\quad-n(n-1)\omega(H)-(n-1)\sum_{i=1}^{n}L(JU_i,JU_i).\end{aligned}$$

If we use (45), (47) and

$$\mathrm{Trace}(L)=\sum_{i=1}^{n}\Big\{L(U_i,U_i)+L(JU_i,JU_i)\Big\}=-n\omega(H)+\sum_{i=1}^{n}L(JU_i,JU_i),$$

in the last equation, then we have

$$\hat{\rho}=\rho^*-||\mathcal{T}||^2+n^2||H||^2-2n(n-1)\omega(H)-(n-1)\mathrm{Trace}(L). \tag{48}$$

Finally, using (44) in (48), we get (46).

Now, we give the Chen-Ricci inequality for a Lagrangian submersion from a g.c.K manifold onto a Riemannian manifold by using the following equation which was introduced by Gülbahar et al. (2017):

$$\begin{aligned}||\mathcal{T}||^2 &= \frac{n^2}{2}||H||^2+\frac{1}{2}\sum_{k=1}^{n}(\mathcal{T}_{11}^k-\mathcal{T}_{22}^k-\ldots-\mathcal{T}_{nn}^k)^2+2\sum_{k=1}^{n}\sum_{j=2}^{n}(\mathcal{T}_{1j}^k)^2 \\
&\quad-2\sum_{k=1}^{n}\sum_{2\le i<j}^{n}\Big\{\mathcal{T}_{ii}^k\mathcal{T}_{jj}^k-(\mathcal{T}_{ij}^k)^2\Big\}.\end{aligned} \tag{49}$$

Theorem 5.5. *Let* $\psi:(M_1^{2n}(c),J,g_1,\omega)\to(M_2^n,g_2)$ *be a Lagrangian submersion and the Lee vector field B be horizontal. If* $\{U_1,\ldots,U_n\}$ *is an orthonormal frame of* $ker\psi_*$*, then we have*

$$\begin{aligned}\hat{Ric}(U_1)+Ric^*(JU_1)+\delta(\mathcal{T})+\frac{(n^2+5n-2)ce^{-\sigma}}{4}+\frac{n^2+6n-4}{4}||\omega||^2\\
<||\mathcal{T}||^2+\frac{n^2}{4}||H||^2+\frac{n+1}{2}\mathrm{Trace}(L)\\
+\frac{n-2}{2}\Big\{L(U_1,U_1)+L(JU_1,JU_1)\Big\},\end{aligned}$$

where

$$Ric^*(JU_1)=\sum_{i=1}^{n}Ric^*(JU_1,JU_i,JU_1,JU_i).$$

Proof. Using (39), we have

$$\sum_{i,j=1}^{n} R_1(U_i, JU_j, U_i, JU_j) = \sum_{i,j=1}^{n} \Big[e^{-\sigma}\frac{c}{4}\Big\{ g_1(U_i, JU_j)g_1(JU_j, U_i) - g_1(U_i, U_i)g_1(JU_j, JU_j) - g_1(JU_i, JU_j)g_1(U_j, U_i) + g_1(JU_i, U_i)g_1(U_j, JU_j) - 2g_1(JU_i, JU_j)g_1(JU_i, JU_j)\Big\} + \frac{1}{2}\Big\{ L(U_i, U_i)g_1(JU_j, JU_j) - L(JU_j, U_i)g_1(U_i, JU_j) - L(U_i, JU_j)g_1(JU_j, U_i) + L(JU_j, JU_j)g_1(U_i, U_i)\Big\} + \frac{||\omega||^2}{4}\Big\{ g_1(JU_j, U_i)g_1(U_i, JU_j) - g_1(U_i, U_i)g_1(JU_j, JU_j)\Big\}\Big]$$
$$= \frac{n}{2}\mathrm{Trace}(L) - \frac{n(n+3)ce^{-\sigma}}{4} - \frac{n^2}{4}||\omega||^2.$$

If we substitute the above equation in the definition of the scalar curvature,

$$\rho_1 = \sum_{j=1}^{n} Ric_1(U_j, U_j) + \sum_{j=1}^{n} Ric_1(JU_j, JU_j) = \sum_{i,j=1}^{n} R_1(U_i, U_j, U_i, U_j) + 2\sum_{i,j=1}^{n} R_1(U_i, JU_j, U_i, JU_j) + \sum_{i,j=1}^{n} R_1(JU_i, JU_j, JU_i, JU_j), \tag{50}$$

then we have

$$\rho_1 = 2\sum_{1\le i<j}^{n} R_1(U_i, U_j, U_i, U_j) + 2\sum_{1\le i<j}^{n} R_1(JU_i, JU_j, JU_i, JU_j) + n\mathrm{Trace}(L) - \frac{n(n+3)ce^{-\sigma}}{2} - \frac{n^2}{2}||\omega||^2. \tag{51}$$

On the other hand if we use (20), (25), (30) and (37) , then we have

$$\sum_{i,j=1}^{n} R_1(U_i, U_j, U_i, U_j) = \sum_{i,j=1}^{n} \Big\{ \hat{R}(U_i, U_j, U_i, U_j) + g_1(\mathcal{T}_{U_i}U_j, \mathcal{T}_{U_j}U_i) - g_1(\mathcal{T}_{U_i}U_i, \mathcal{T}_{U_j}U_j)\Big\}$$
$$= 2\sum_{1\le i<j}^{n} \hat{R}(U_i, U_j, U_i, U_j) + ||\mathcal{T}||^2 - n^2||H||^2,$$

and

$$\sum_{i,j=1}^{n} R_1(U_i, JU_j, U_i, JU_j) = \sum_{i,j=1}^{n} R_1(JU_j, U_i, JU_j, U_i)$$
$$= \sum_{i,j=1}^{n} \Big\{ g_1((\nabla_{JU_j}\mathcal{T})_{U_i}U_i, JU_j) - g_1(\mathcal{T}_{U_i}JU_j, \mathcal{T}_{U_i}JU_j)\Big\}$$
$$= \delta(\mathcal{T}) - \sum_{i=1}^{n}\Big\{ \sum_{j=1}^{n} g_1(\mathcal{T}_{U_i}U_j, \mathcal{T}_{U_i}U_j) + \frac{1}{2}\omega(JU_i)\omega(JU_i) - g_1(U_i, U_i)||\omega||^2\Big\}$$
$$= \delta(\mathcal{T}) - ||\mathcal{T}||^2 + \Big(\frac{2n-1}{2}\Big)||\omega||^2.$$

If we write the last two equations in (50) and use (22), then we get

$$\rho_1 = 2\sum_{1\le i<j}^{n} \hat{R}(U_i, U_j, U_i, U_j) + 2\sum_{1\le i<j}^{n} R^*(JU_i, JU_j, JU_i, JU_j) + 2\delta(\mathcal{T}) + (2n-1)||\omega||^2 - n^2||H||^2 - ||\mathcal{T}||^2. \tag{52}$$

Making use of (51) and (52), we have

$$\begin{aligned}\sum_{1\le i<j}^{n} R_1(U_i,U_j,U_i,U_j) + \sum_{1\le i<j}^{n} R_1(JU_i,JU_j,JU_i,JU_j)&\\ +\frac{n}{2}\mathrm{Trace}(L) - \frac{n(n+3)ce^{-\sigma}}{4} - \frac{n^2+4n-2}{4}||\omega||^2&\\ = \sum_{1\le i<j}^{n} \hat{R}(U_i,U_j,U_i,U_j) + \sum_{1\le i<j}^{n} R^*(JU_i,JU_j,JU_i,JU_j)&\\ +\delta(\mathcal{T}) - \frac{n^2}{2}||H||^2 - \frac{1}{2}||\mathcal{T}||^2.&\end{aligned} \tag{53}$$

Now, using (28) and (29) in (20) we obtain

$$\sum_{2\le i<j}^{n} R_1(U_i,U_j,U_i,U_j) = \sum_{2\le i<j}^{n} \hat{R}(U_i,U_j,U_i,U_j) - \sum_{k=1}^{n}\sum_{2\le i<j}^{n}\Big\{\mathcal{T}_{ii}^k\mathcal{T}_{jj}^k - (\mathcal{T}_{ij}^k)^2\Big\}.$$

Substituting the last equation in (53), we get

$$\begin{aligned}\sum_{j=2}^{n} R_1(U_1,U_j,U_1,U_j)+\sum_{j=2}^{n} R_1(JU_1,JU_j,JU_1,JU_j) - \sum_{k=1}^{n}\sum_{2\le i<j}^{n}\Big\{\mathcal{T}_{ii}^k\mathcal{T}_{jj}^k - (\mathcal{T}_{ij}^k)^2\Big\}&\\ +\frac{n}{2}\mathrm{Trace}(L) - \frac{n(n+3)ce^{-\sigma}}{4} - \frac{n^2+4n-2}{4}||\omega||^2&\\ = \hat{Ric}(U_1) + Ric^*(JU_1) + \delta(\mathcal{T}) - \frac{n^2}{2}||H||^2 - \frac{1}{2}||\mathcal{T}||^2.&\end{aligned}$$

If we use (39) for the first and the second terms on the left hand side of the last equation, namely

$$\sum_{j=2}^{n} R_1(U_1,U_j,U_1,U_j) = (1-n)\Big(\frac{ce^{-\sigma}+||\omega||^2}{4}\Big) + \frac{1}{2}\Big\{(n-2)L(U_1,U_1) - n\omega(H)\Big\},$$

and

$$\begin{aligned}\sum_{j=2}^{n} R_1(JU_1,JU_j,JU_1,JU_j) = (1-n)\Big(\frac{ce^{-\sigma}+||\omega||^2}{4}\Big)&\\ +\frac{1}{2}\Big\{(n-2)L(JU_1,JU_1) + \mathrm{Trace}(L) + n\omega(H)\Big\},&\end{aligned}$$

then we have

$$\begin{aligned}\frac{n+1}{2}\mathrm{Trace}(L) + \frac{n-2}{2}\Big\{L(U_1,U_1)+L(JU_1,JU_1)\Big\} - \frac{(n^2+5n-2)ce^{-\sigma}}{4}&\\ -\frac{n^2+6n-4}{4}||\omega||^2 - \sum_{k=1}^{n}\sum_{2\le i<j}^{n}\Big\{\mathcal{T}_{ii}^k\mathcal{T}_{jj}^k - (\mathcal{T}_{ij}^k)^2\Big\}&\\ = \hat{Ric}(U_1) + Ric^*(JU_1) + \delta(\mathcal{T}) - \frac{1}{2}||\mathcal{T}||^2 - \frac{n^2}{2}||H||^2.&\end{aligned}$$

Now, using (49) in the last equation, we have

$$\begin{aligned}\hat{Ric}(U_1) + Ric^*(JU_1) + \delta(\mathcal{T}) - ||\mathcal{T}||^2 - \frac{n^2}{4}||H||^2&\\ = \frac{n+1}{2}\mathrm{Trace}(L) + \frac{n-2}{2}\Big\{L(U_1,U_1)+L(JU_1,JU_1)\Big\}&\\ -\frac{(n^2+5n-2)ce^{-\sigma}}{4} - \frac{n^2+6n-4}{4}||\omega||^2&\\ -\frac{1}{4}\sum_{k=1}^{n}(\mathcal{T}_{11}^k - \mathcal{T}_{22}^k - \dots - \mathcal{T}_{nn}^k)^2 - \sum_{k=1}^{n}\sum_{j=2}^{n}(\mathcal{T}_{1j}^k)^2.&\end{aligned}$$

The result comes from Remark 5.1.

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RESEARCH ARTICLE

Some remarks on the moment curves in $\mathbb{R}^4$ and $\mathbb{R}^3$

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ABSTRACT
The moment curves and their normalizations are key tools in obtaining the famous Kac formula from the theory of random polynomials. We study here the normalized moment curves $\Gamma_n \in S^n$ in the low dimensions n where S^n is the Euclidean n-dimensional unit sphere; more precisely we consider $n = 3$ and $n = 2$. First, we compute the image of the normalized moment curve Γ_3 under the well-known Hopf fibre map and show that this remarkable map reduces the length of Γ_3. Second, we analyze the curve Γ_2 using the theory of spherical Legendre curves. An image of Γ_2 is included.

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1. INTRODUCTION

The setting of this paper is provided by the space $\mathbb{R}^{n+1}$, $n \in \mathbb{N}^* = \{1, 2, ...\}$, which is an Euclidean vector space with respect to the canonical inner product:

$$\begin{cases} \langle u, v\rangle := u^1v^1 + ... + u^{n+1}v^{n+1}, u = (u^1, ..., u^{n+1}), v = (v^1, ..., v^{n+1}) \in \mathbb{R}^{n+1}, \\ 0 \leq \|u\|^2 := \langle u, u\rangle. \end{cases} \tag{1.1}$$

A special curve, called *moment curve*, is defined in (Edelman et al. 1995, p. 5-6) as:

$$\gamma_n : \mathbb{R} \to \mathbb{R}^{n+1}, \quad \gamma_n(t) := (1, t, ..., t^n). \tag{1.2}$$

It is a regular curve since the norm of its derivative is strictly positive: $\|\gamma_n'(t)\| > 0$ for all $t \in \mathbb{R}$. Also, *the normalized moment curve* is:

$$\Gamma_n : \mathbb{R} \to \mathbb{S}^n, \quad \Gamma_n(t) := \frac{\gamma_n(t)}{\|\gamma_n(t)\|}. \tag{1.3}$$

where S^n is the unit sphere of $\mathbb{E}^{n+1} := (\mathbb{R}^{n+1}, \langle\cdot,\cdot\rangle)$. A main result of Edelman et al. (1995) is that the expected number of real zeros E_n of a random polynomial of degree n if the coefficients are independent and distributed normally is given by:

$$E_n = \frac{L(\Gamma_n)}{\pi} \tag{1.4}$$

where $L(\Gamma_n)$ is the Euclidean length of Γ_n. The first few values of E_n are: 1, 1.29702, 1.49276, 1.64049. Also, it is easy to express the image of the curve Γ_n through the stereographic projection φ_N from the North pole $N(0, ..., 0, 1) \in S^n$:

$$\varphi_N(\Gamma_n(t)) = \frac{\gamma_{n-1}(t)}{\sqrt{1 + t^2 + ... + t^{2n}} - t^n} \in \mathbb{R}^n. \tag{1.5}$$

The present work concerns with the normalized moment curve Γ_n for the low values $n = 3$ and $n = 2$. More precisely, when $n = 3$ we use the well-known Hopf bundle and as result we obtain a lower length. The second case is treated in the framework of spherical Legendre curves since Γ_2 appears as a frontal curve in this theory. For both values of n we study the image of the normalized moment curve through the Veronese map. Other spherical curves in the case $n = 2$, as the Clelia curve and the spherical nephroid, are studied in Crasmareanu (2024).

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2. THE MOMENT CURVE IN $\mathbb{R}^4$ AND THE HOPF MAP

The setting of this section is provided by $n = 3$ since the sphere $S^3 = SU(2)$ is the total space of the famous Hopf bundle $H : S^3 \subset \mathbb{C}^2 \to S^2\left(\frac{1}{2}\right) \subset \mathbb{R} \times \mathbb{C}$:

$$H(z, w) = \left(\frac{1}{2}(|z|^2 - |w|^2), z\bar{w}\right). \tag{2.1}$$

We express the curve Γ_3 as:

$$\Gamma_3(t) = (z(t), w(t)), \quad z(t) = \frac{1 + it}{\sqrt{(1 + t^2)(1 + t^4)}}, \quad w(t) = t^2 z(t), \quad |z(t)|^2 = \frac{1}{1 + t^4} \tag{2.2}$$

and then a straightforward computation gives:

$$H(\Gamma_3(t)) = \frac{1}{2}\left(\frac{1 - t^4}{1 + t^4}, \frac{2t^2}{1 + t^4}, 0\right) \in S^2\left(\frac{1}{2}\right) \subset \mathbb{R} \times \mathbb{R} \times \{0\}. \tag{2.3}$$

Returning to the expectation numbers it is very easy to see that $E_1 = 1$. Indeed, for $\Gamma_1(t) = \frac{1}{\sqrt{1+t^2}}(1, t) \in S^1$ we consider the change of parameter $t = \tan\varphi$. It follows:

$$\Gamma_1(\varphi) = (\cos\varphi, \sin\varphi) \in S^1$$

and the condition $\cos\varphi = \frac{1}{\sqrt{1+t^2}} > 0$ yields the domain $\varphi \in \left(0, \frac{\pi}{2}\right) \cup \left(\frac{3\pi}{2}, 2\pi\right)$. Then we have the length $L(\gamma_1) = \frac{\pi}{2} + \frac{\pi}{2} = \pi$ and it results $E_1 = 1$.

For the case $n = 3$ we perform the change of parameter $t^2 = \tan\varphi \geq 0$ and then:

$$H(\Gamma_3(\varphi)) = \frac{1}{2}(\cos 2\varphi, \sin 2\varphi, 0) \in S^2\left(\frac{1}{2}\right) \tag{2.4}$$

but now $\varphi \in \left(0, \frac{\pi}{2}\right) \cup \left(\pi, \frac{3\pi}{2}\right)$. It follows that $2\varphi \in (0, \pi) \cup (2\pi, 3\pi)$ which implies that:

$$L(H \circ \Gamma_3) = \frac{1}{2}(\pi + \pi) = \pi < L(\Gamma_3) \simeq 1.49\pi. \tag{2.5}$$

Hence, the first conclusion of this section is that the Hopf map reduces the length of the normalized moment curve Γ_3.

Expressing $z = z_1 = q_1 + ip_1$, $w = z_2 = q_2 + ip_2$ the standard *contact form of* S^3 is the restriction of the 1-form:

$$\lambda_0 := \frac{1}{2}(p_1 dq_1 - q_1 dp_1 + p_2 dq_2 - q_2 dp_2) \tag{2.6}$$

to S^3. The tangent vector field of the normalized moment curve is:

$$\Gamma_3'(t) = \frac{1}{(1 + t^2 + t^4)^{\frac{3}{2}}}(-t - t^3, t, 2t + t^3 + t^5, 3t^2 + 2t^4 + 2t^6) \tag{2.7}$$

and then:

$$\lambda_0(\Gamma_3'(t)) = \frac{-t(1 + t + 2t^3 + t^5 + t^7)}{(1 + t^2 + t^4)^2}. \tag{2.8}$$

As element in the Lie group $SU(2)$ the normalized moment curve Γ_3 is:

$$\Gamma_3(t) = \frac{1}{\sqrt{(1 + t^2)(1 + t^4)}}\begin{pmatrix} t^2(t + i) & t + i \\ -(t - i) & t^2(t - i) \end{pmatrix} \in SU(2), \quad Tr\Gamma_3(t) = 2Im(w(t)). \tag{2.9}$$

Secondly, we recall the complex Veronese map $V : S^3 \subset \mathbb{C}^2 \to S^5 \subset \mathbb{C}^3$:

$$V(z = x + iy, w = u + iv) := (z^2, \sqrt{2}zw, w^2) =$$

$$= (x^2 - y^2, 2xy, \sqrt{2}(xu - yv), \sqrt{2}(xv + yu), u^2 - v^2, 2uv). \tag{2.10}$$

We obtain the new spherical curve:

$$V \circ \Gamma_3(t) = \frac{1}{(1 + t^2)(1 + t^4)}(1 - t^2, 2t, \sqrt{2}t^2(1 - t), \sqrt{2}t^2(1 + t), t^4(1 - t^2), 2t^5) \in S^5. \tag{2.11}$$

3. THE MOMENT CURVE IN $\mathbb{R}^3$ AND THE SPHERICAL LEGENDRE CURVES

Now we consider $n = 2$ and recall that the unit spherical bundle is a compact 3-dimensional contact metric manifold given by:

$$T_1S^2 := \{(u, v) \in S^2 \times S^2; \langle u, v\rangle = 0\} \tag{3.1}$$

see for example Crasmareanu (2016). There is a natural action of $O(3)$ on T_1S^2:

$$(A, (u, v)) \in O(3) \times T_1S^2 \to (Au, Av) \in T_1S^2, \quad \langle Au, Av\rangle = \langle u, v\rangle = 0. \tag{3.2}$$

For example, the complex Veronese map applied on a pair $(z, w) \in S^3$ with $|z| = |w| = \frac{1}{\sqrt{2}}$ gives the symmetric orthogonal matrix:

$$A = \frac{1}{2}\begin{pmatrix} 1 & \sqrt{2} & 1 \\ \sqrt{2} & 0 & -\sqrt{2} \\ 1 & -\sqrt{2} & 1 \end{pmatrix} \in Sym(3) \cap O^-(3),$$

where $O^-(3) := \{\Gamma \in O(3); \det \Gamma = -1\}$ and $Sym(3)$ is the linear subspace of $M_3(\mathbb{R})$ consisting in symmetric matrices. Considering this matrix as representing a conic in the plane $\mathbb{E}^2$ (see the formalism of Crasmareanu (2021)) it is a hyperbola.

The general notion of Legendre curves associated to a contact form is well-known, but we will work directly in our framework using the approach of Takahashi (2016) (see also CrasmareanuChapter (2024)):

Definition 3.1 The smooth map $LC := (\gamma, \nu) : I \subseteq \mathbb{R} \to T_1S^2$, $t \in I \to (\gamma(t), \nu(t))$ is *a spherical Legendre curve* if $\langle \gamma'(t), \nu(t)\rangle = 0$ for all t in the open interval I. The map γ is called *the frontal* and ν is *the dual of* γ.

Since $\mathbb{R}^3$ is endowed also with the cross product $\times$ we define $\mu = \gamma \times \nu$ and hence the triple $\mathcal{F} := \{\gamma, \nu, \mu\}^t$ is an positive oriented *moving frame* along the frontal γ; here t means the transposition, so $\mathcal{F}$ is a column matrix. Its moving equation is provided by the Proposition 2.2. of **?**:

$$\frac{d}{dt}\mathcal{F}(t) = \begin{pmatrix} 0 & 0 & k_1(t) \\ 0 & 0 & k_2(t) \\ -k_1(t) & -k_2(t) & 0 \end{pmatrix}\mathcal{F}(t), \quad \begin{pmatrix} 0 & 0 & k_1(t) \\ 0 & 0 & k_2(t) \\ -k_1(t) & -k_2(t) & 0 \end{pmatrix} \in o(3). \tag{3.3}$$

The pair of smooth functions (k_1, k_2) is called *the curvature* of the spherical Legendre curve $LC = (\gamma, \nu)$. Sometimes, it is more useful to denote a given LC with all its elements as $LC = (\gamma, \nu; \mu)$.

Let us consider now the Example 2.8. of Takahashi (2016); equivalently the Example 2.6 of CrasmareanuChapter (2024). Starting with the natural numbers (k, m, n) satisfying $m = k + n$ the LC is defined as:

$$\gamma(t) = \frac{1}{\sqrt{1 + t^{2n} + t^{2m}}}(1, t^n, t^m), \; \nu(t) = \frac{1}{\sqrt{n^2 + m^2t^{2k} + k^2t^{2m}}}(kt^m, -mt^k, n), \; t \in \mathbb{R} \tag{3.4}$$

with the associated curvature pair:

$$k_1(t) = -\frac{t^{n-1}\sqrt{n^2 + m^2t^{2k} + k^2t^{2m}}}{1 + t^{2n} + t^{2m}}, \quad k_2(t) = \frac{kmnt^{k-1}\sqrt{1 + t^{2n} + t^{2m}}}{n^2 + m^2t^{2k} + k^2t^{2m}}. \tag{3.5}$$

In fact, we have:

$$\nu = \frac{\gamma \times \gamma'}{\|\gamma \times \gamma'\|} = \frac{\gamma \times \gamma'}{\|\gamma'\|} = \gamma \times T \tag{3.6}$$

where $\{T, N, B\}$ is the Frenet frame of the γ as bi-regular space curve. We point out that the general normalized moment curve Γ_n is exactly the tangent vector field T of the curve:

$$t \in \mathbb{R} \to (\int \gamma_n)(t) = \left(t, \frac{t^2}{2}, \ldots, \frac{t^{n+1}}{n+1}\right).$$

Returning to our normalized moment curve Γ_2 it results that it corresponds exactly to the curve γ for $k = n = 1 < m = 2$; therefore ν can be called *the Legendre dual* of the normalized moment curve Γ_3. We have:

$$\Gamma_2'(t) = \frac{1}{(1 + t^2 + t^4)^{\frac{3}{2}}}(-(t + 2t^3), 1 - t^4, 2t + t^3), \quad \nu(t) = \frac{1}{\sqrt{1 + 4t^2 + t^4}}(t^2, -2t, 1) \tag{3.7}$$

and the curvature pair:

$$k_1(t) = -\frac{\sqrt{1 + 4t^2 + t^4}}{1 + t^2 + t^4} < 0, \quad k_2(t) = \frac{2\sqrt{1 + t^2 + t^4}}{1 + 4t^2 + t^4} > 0. \tag{3.8}$$

The length of the curve Γ_2 is approximately $4.07472 < 2\pi$.

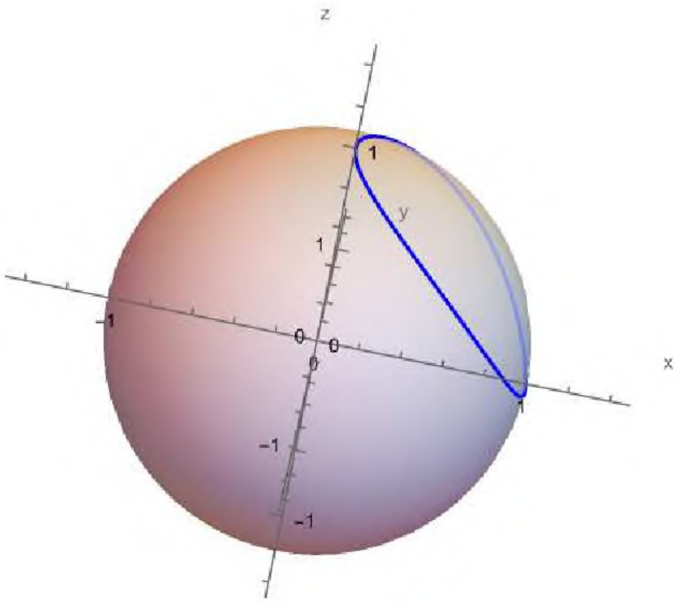

Figure 1. The curve Γ_2.

Recall the standard parametrization of S^2 as regular surface:

$$S^2 : \bar{r}(u,v) = (\cos u \cos v, \cos u \sin v, \sin u), \quad u \in (0, 2\pi), \quad v \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right). \tag{3.9}$$

The normalized moment curve $\Gamma_2 \in S^2$ is given then by $u = u(t)$, $v = v(t)$ with:

$$\begin{cases} \cos u(t) = \frac{\sqrt{1+t^2}}{\sqrt{1+t^2+t^4}} > 0, \quad \sin u(t) = \frac{t^2}{\sqrt{1+t^2+t^4}} \geq 0, \\ \cos v(t) = \frac{1}{\sqrt{1+t^2}} > 0, \quad \sin v(t) = \frac{t}{\sqrt{1+t^2}}. \end{cases} \tag{3.10}$$

With the change of parameter $t = \tan\varphi$ we have also:

$$\begin{cases} \Gamma_2(\varphi) = \frac{1}{\sqrt{\cos^2\varphi + \sin^4\varphi}}(\cos^2\varphi, \sin\varphi\cos\varphi, \sin^2\varphi), \\ \nu(\varphi) = \frac{1}{\sqrt{1+2\cos^2\varphi\sin^2\varphi}}(\sin^2\varphi, -\sin 2\varphi, \cos^2\varphi). \end{cases} \tag{3.11}$$

We point out that Γ_2 is not a Viviani curve being the intersection of the sphere S^2 not with a cylinder but with the elliptic cone $EC : xz = y^2$.

Moreover, the above spherical curves can be studied through the Veronese map:

$$V : S^2 \subset \mathbb{R}^3 \to S^4 \subset \mathbb{R}^5, V(u,v,w) := \left(\sqrt{3}vw, \sqrt{3}wu, \sqrt{3}uv, \frac{\sqrt{3}}{2}(u^2 - v^2), w^2 - \frac{u^2+v^2}{2}\right). \tag{3.12}$$

Then:

$$\begin{cases} V \circ \Gamma_2(t) = \frac{1}{1+t^2+t^4}\left(\sqrt{3}t^3, \sqrt{3}t^2, \sqrt{3}t, \frac{\sqrt{3}}{2}(1-t^2), t^4 - \frac{1+t^2}{2}\right), \\ V \circ \nu(t) = \frac{1}{1+2t^2+t^4}\left(-2\sqrt{3}t, \sqrt{3}t^2, -2\sqrt{3}t^3, \frac{\sqrt{3}t^2}{2}(t^2-4), 1 - \frac{t^4+4t^2}{2}\right). \end{cases} \tag{3.13}$$

An important remark is that the new curves, $V \circ \Gamma_2$ and $V \circ \nu$, are not orthogonal in $\mathbb{E}^6$.

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RESEARCH ARTICLE

An overview of complex boundary value problems

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ABSTRACT
In this overview we have pointed out some boundary value problems in a subset of complex plane. We start with Cauchy-Riemann operator and their conjugates. Then we introduce the Cauchy- Riemann operator and its conjugate. Firstly, we introduce the polyanalytic Pompeiu integral representation Tf and its conjugate which vanish at infinity. The basic polyanalytic Schwarz and polyanalytic Dirichlet problems are introduced. The later part is devoted to polyanalytic problems and discussions on polyharmonic problems. We have also summarized polyanalytic Neumann problem in the unit disk for $\partial_{\bar{z}} w = f$. In this case, we may have three types of boundary value problems. Those are polyharmonic Dirichlet problem, polyharmonic Neumann problem and polyharmonic Riquier (Navier) problem. In this later part we have given the iterated polyharmonic Green function.

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1. INTRODUCTION

Our aim is to discuss some boundary value problems related to Dirichlet problems in $\mathbb{C}$. This type of problems have started with Riemann and later on modified by Hilbert. The problems have been investigated over many different domains in $\mathbb{C}$, I.N. Vekua (1962); H. Begehr (2025); Aksoy at al. (2025).

1.1. Model representations

For a complex partial differential equation the variables are z and $\bar{z}$ and the operators $\partial_{\bar{z}}$ and ∂_z are known as Cauchy-Riemann operator and its conjugate. One of the main theorems in complex form is given in the following result.

Theorem 1.1. *(Gauss-Ostrogradskii theorem) In a regular domain $D \subset \mathbb{C}$, any function $w \in C(\overline{D};\mathbb{C}) \cap C^1(D;\mathbb{C})$ satisfies the following relations*

$$\int_D w_{\bar{z}}(z)\, dxdy = \frac{1}{2i}\int_{\partial D} w(z)\, dz \tag{1}$$

$$\int_D w_z(z)\, dxdy = -\frac{1}{2i}\int_{\partial D} w(z)\, d\bar{z}\,. \tag{2}$$

Now let us recall the Theorem 1.12 in I.N. Vekua (1962).

Theorem 1.2. *Let D be a bounded domain. If $f \in L_1(D)$ then the integrals*

$$Tf = -\frac{1}{\pi}\int_D \frac{f(\zeta)d\xi d\eta}{\zeta - z}$$

$$\bar{T}f = -\frac{1}{\pi}\int_D \frac{f(\zeta)d\xi d\eta}{\bar{\zeta} - \bar{z}}$$

exist for all points z outside $\overline{D}$, Tf and $\overline{T}f$ are holomorphic outside $\overline{D}$ with respect to z and $\bar{z}$, respectively, and vanish at infinity.

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Tf is called Pompeiu integral for Cauchy-Riemann operator and it satisfies $\partial_{\bar{z}}Tf = f$. This is a model equation for general first order complex differential equation.

2. POLYANALYTIC SCHWARZ AND DIRICHLET PROBLEMS

The basic representation formula for differential operator $\partial_{\bar{z}}$ is given below.

Theorem 2.1. *(Cauchy-Pompeiu integral representation)*
Any function $w \in C(\bar{D};\mathbb{C}) \cap C^1(D;\mathbb{C})$ can be represented in the form

$$w(z) = \frac{1}{2\pi i}\int\limits_{\partial\mathbb{D}} w(\zeta)\frac{d\zeta}{\zeta - z} - \frac{1}{\pi}\int\limits_{\mathbb{D}} w_{\bar{\zeta}}(\zeta)\frac{d\xi d\eta}{\zeta - z}, \quad \zeta = \xi + i\eta \tag{3}$$

for $z \notin \overline{D}$ the left-hand side must be replaced by 0.

Decomposing $\partial_{\bar{z}}^m w = f$ into a system of m Cauchy-Riemann equations we end up with the polyanalytic Cauchy-Pompeiu formula.

Theorem 2.2. *(Polyanalytic Cauchy-Pompeiu Integral formula)*
Any function $w \in C^{m-1}(\bar{D};\mathbb{C}) \cap C^m(D;\mathbb{C})$ can be represented in the form

$$\begin{aligned} w(z) = &\sum_{\mu=0}^{m-1}\frac{1}{2\pi i}\int_{\partial D}\frac{(-1)^\mu(\overline{\zeta - z})^\mu}{\mu!(\zeta - z)}\partial_{\bar{\zeta}}^\mu w(\zeta)d\xi \\ &-\frac{1}{\pi}\int_D \frac{(-1)^{m-1}(\overline{\zeta - z})^{m-1}}{(m-1)!(\zeta - z)}\partial_{\bar{\zeta}}^m w(\zeta)d\xi dz, \quad z \in D. \end{aligned}$$

For $z \notin \bar{D}$ the left-hand side must be replaced by 0 .

2.1. Polyanalytic Schwarz problem

We start with the unit disc $\mathbb{D} = \{|z| < 1\}$. Then we have the following lemma for Schwarz problem.

Lemma 2.3. *The Schwarz problem for the Cauchy-Riemann operator*

$$\partial_{\bar{z}}w = f \quad \text{in} \quad \mathbb{D}, \quad \text{Re}w = \gamma \quad \text{on} \quad \partial\mathbb{D}, \quad \text{Im}w(0) = c,$$

$$f \in L_p(\mathbb{D};\mathbb{C}), 2 < p, \quad \gamma \in C(\partial\mathbb{D};\mathbb{R}), \quad c \in \mathbb{R},$$

is uniquely solvable by

$$\begin{aligned} w(z) \quad = \quad & ic + \frac{1}{2\pi i}\int_{\partial\mathbb{D}}\gamma(\zeta)\frac{\zeta + z}{\zeta - z}\frac{d\zeta}{\zeta} \\ & -\frac{1}{2\pi}\int_{\mathbb{D}}\Big[\frac{f(\zeta)}{\zeta}\frac{\zeta + z}{\zeta - z} + \frac{\overline{f(\zeta)}}{\overline{\zeta}}\frac{1 + z\overline{\zeta}}{1 - z\overline{\zeta}}\Big]d\xi d\eta. \end{aligned}$$

Then we can give the Cauchy-Schwarz-Pompeiu representation for polyanalytic operators in $\mathbb{D}$.

Theorem 2.4. *(Cauchy-Schwarz-Pompeiu representation for the polyanalytic operator)*

Any $w \in C^{m-1}(\overline{D};\mathbb{C}) \cap C^m(D,\mathbb{C}),\ m \in \mathbb{N}$, *is representable as*

$$\begin{aligned} w(z) \quad = \quad & \sum_{\mu=0}^{m-1} \Big\{ \frac{i\mathrm{Im}\partial_{\overline{z}}^{\mu} w(z_0)}{\mu!} (z - z_0 + \overline{z - z_0})^{\mu} \\ & + \frac{(-1)^{\mu}}{2\pi i \mu!} \int_{\partial D} \mathrm{Re}\partial_{\overline{\zeta}}^{\mu} w(\zeta)(\zeta - z + \overline{\zeta - z})^{\mu} \\ & \times \Big[\frac{\zeta - z_0}{(\zeta - z)(\zeta - z_0)} d\zeta + \Big(h_{1\overline{\zeta}}(z,\zeta) - \frac{1}{\overline{\overline{\zeta - z_0}}} \Big) d\overline{\zeta} \Big] \Big\} \\ & + \frac{(-1)^m}{2\pi(m-1)!} \int_D \Big\{ \partial_{\overline{\zeta}}^{m} w(\zeta) \Big[\frac{\zeta + z - 2z_0}{(\zeta - z)(\zeta - z_0)} - h_{1\zeta}(z_0,\zeta) \Big] \\ & - \overline{\partial_{\overline{\zeta}}^{m} w(\zeta)} \Big[2h_{1\overline{\zeta}}(z,\zeta) - h_{1\overline{\zeta}}(z_0,\zeta) - \frac{1}{\overline{\overline{\zeta - z_0}}} \Big] \Big\} \\ & \times (\zeta - z + \overline{\zeta - z})^{m-1} d\xi d\eta. \end{aligned}$$

It is easy to observe that this formula is the solution of the polyanalytic problem

$$\partial_{\overline{z}}^{m} w = f \text{ in } \mathbb{D},\ f \in L_p(\mathbb{D};\mathbb{C}),\ 2 < p,$$
$$\mathrm{Re}\partial_{\overline{z}}^{\mu} w = \gamma_{\mu} \text{ on } \partial\mathbb{D},\ \gamma_{\mu} \in C(\partial\mathbb{D};\mathbb{R}),\ 0 \le \mu \le m-1,$$
$$\mathrm{Im}\partial_{\overline{z}}^{\mu} w(0) = c_{\mu},\ c_{\mu} \in \mathbb{R},\ 0 \le \mu \le m-1.$$

This solution may also be represented in terms of Green function $G(z,\zeta)$. We should note that a domain D whose Green function $G_1(z,\zeta)$ is such that

$$h_1(z,\zeta) = \log|\zeta - z|^2 + G_1(z,\zeta)$$

that satisfies for $\zeta \in \partial D, z \in D$,

$$\mathrm{Re}\Big[\frac{d\zeta}{\zeta - z} - h_{1\overline{\zeta}}(z,\zeta) d\overline{\zeta} \Big] = 0.$$

2.2. Polyanalytic Dirichlet problem

Firstly let us state the solution of the inhomogeneous polyanalytic equation

$$\partial_{\overline{z}}^{m} w = f \text{ in } D,\ f \in L_p(D;\mathbb{C}),\quad 2 < p,$$

satisfying the conditions

$$\partial_{\overline{z}}^{\mu} w = \gamma_{\mu} \text{ on } \partial D,\quad \gamma_{\mu} \in C(\partial D;\mathbb{R}),\quad 0 \le \mu \le m-1.$$

Now let us start the case $m = 1$.

Lemma 2.5. *For* $f \in L_p(D;\mathbb{C}), 2 < p$, *and* $\gamma \in C(\partial D;\mathbb{C})$, *the Dirichlet problem*

$$w_{\overline{z}} = f \text{ in } D, w = \gamma \text{ on } \partial D,$$

is uniquely solvable and the solution is given by the formula

$$w(z) = \frac{1}{2\pi i} \int_{\partial D} \gamma(\zeta) \frac{d\zeta}{\zeta - z} - \frac{1}{\pi} \int_D f(\zeta) \frac{d\xi d\eta}{\zeta - z}$$

if and only if the following condition holds:

$$\frac{1}{2\pi i} \int_{\partial D} \gamma(\zeta) h_{1\zeta}(z,\zeta) d\zeta = \frac{1}{\pi} \int_D f(\zeta) h_{1\zeta}(z,\zeta) d\xi d\eta\ .$$

Theorem 2.6. *The polyanalytic Dirichlet problem where* $f \in L_p(D;\mathbb{C}), 2 < p, \gamma_{\mu} \in C(\partial D;\mathbb{C}), 0 \le \mu \le m-1$, *has the solution*

$$\begin{aligned} w(z) = & \sum_{\mu=0}^{m-1} \frac{1}{2\pi i} \int_{\partial D} \frac{(-1)^{\mu} (\overline{\zeta - z})^{\mu}}{\mu!(\zeta - z)} \gamma_{\mu}(\zeta) d\zeta \\ & - \frac{1}{\pi} \int_D \frac{(-1)^{m-1} (\overline{\zeta - z})^{m-1}}{(m-1)!(\zeta - z)} f(\zeta) d\xi d\eta \end{aligned}$$

if and only if

$$\frac{1}{2\pi i}\int_{\partial D}\gamma_\mu(\zeta)\partial_\zeta h_1(z,\zeta)d\zeta + \sum_{\nu=\mu+1}^{m-1}\frac{1}{2\pi i}\int_{\partial D}\gamma_\nu(\zeta_{\nu-\mu+1})\Big(\frac{1}{\pi}\int_D\Big)^{\nu-\mu}\partial_{\zeta_1}h_1(z,\zeta_1) \times\prod_{\lambda=1}^{\nu-\mu}\frac{d\zeta_\lambda d\eta_\lambda}{\zeta_\lambda-\zeta_{\lambda+1}}d\zeta_{\nu-\mu+1} = \frac{1}{\pi}\int_D f(\zeta_{m-\mu})\Big(\frac{1}{\pi}\int_D\Big)^{m-\mu-1}\partial_{\zeta_1}h_1(z,\zeta_1) \times\prod_{\lambda=1}^{m-\mu-1}\frac{d\xi_\lambda d\eta_\lambda}{\zeta_\lambda-\zeta_{\lambda+1}}d\xi_{m-\mu}d\eta_{m-\mu}\ \text{for } 0\le\mu\le m-1$$

holds.

3. POLYANALYTIC NEUMANN-N PROBLEM IN THE UNIT DISK

Neumann boundary value problem demands to find functions with prescribed behaviour of its normal derivative on the boundary. The problem is not a well-posed problem. But it is solvable under solvability conditions. For $n = 1$ the statement of the problem may be reduced to Dirichlet problem. For $1 < n$ then we employ an iteration for Cauchy-Riemann equation.

Theorem 3.1. *The iterated Neumann-n problem for the polyanalytic operator in the unit disk $\mathbb{D}$ The iterated Neumann-n problem for the polyanalytic operator in the unit disk $\mathbb{D}$*

$$\partial_{\bar z}^n w = f \text{ in } \mathbb{D}, \partial_\nu\partial_{\bar z}^\mu w = \gamma_\mu \text{ on } \partial\mathbb{D}, \partial_{\bar z}^\mu w(0) = c_\mu, 0\le\mu\le n-1$$

for $f\in C^\alpha(\overline{\mathbb{D}};\mathbb{C}), 0<\alpha<1, \gamma_\mu\in C(\partial\mathbb{D};\mathbb{C}), c_\mu\in\mathbb{C}$, is uniquely solvable if and only if for any $\mu, 0\le\mu\le n-1$,

$$\sum_{\varrho=\mu}^{n-1}\frac{1}{2\pi i}\int_{\partial\mathbb{D}}\gamma_\varrho(\zeta)\frac{(-1)^{\varrho-\mu}}{(\varrho-\mu)!}\frac{\overline{(\zeta-z)}^{\varrho-\mu}}{1-\bar z\zeta}\frac{d\zeta}{\zeta} + \frac{1}{2\pi i}\int_{\partial\mathbb{D}}f(\zeta)\frac{(-1)^{n-1-\mu}}{(n-1-\mu)!}\frac{\overline{(\zeta-z)}^{n-1-\mu}}{1-\bar z\zeta}d\bar\zeta + \frac{\bar z}{\pi}\int_{\mathbb{D}}f(\zeta)\frac{(-1)^{n-1-\mu}}{(n-1-\mu)!}\frac{\overline{(\zeta-z)}^{n-1-\mu}}{(1-\bar z\zeta)^2}d\xi\, d\eta = 0 \tag{1}$$

are satisfied. The solution then is

$$w(z)=\sum_{\mu=0}^{n-1}\Big[\frac{c_\mu}{\mu!}\bar z^\mu - \frac{1}{2\pi i}\int_{\partial\mathbb{D}}\gamma_\mu(\zeta)\frac{(-1)^\mu}{\mu!}\overline{(\zeta-z)}^\mu\log(1-z\bar\zeta)\frac{d\zeta}{\zeta}\Big] - \frac{1}{2\pi i}\int_{\partial\mathbb{D}}f(\zeta)\frac{(-1)^{n-1}}{(n-1)!}\overline{(\zeta-z)}^{n-1}\log(1-z\bar\zeta)d\bar\zeta - \frac{z}{\pi}\int_{\mathbb{D}}f(\zeta)\frac{(-1)^{n-1}}{(n-1)!}\frac{\overline{(\zeta-z)}^{n-1}}{\zeta(\zeta-z)}d\xi\, d\eta. \tag{2}$$

4. POLYHARMONIC PROBLEMS

The poly-Poisson equation of order n

$$(\partial_z\partial_{\bar z})^n w = f \text{ in } D.$$

will assume different names depending on its boundary conditions

(i) If the boundary condition is

$$\partial_\nu^\mu w = \gamma_\mu, 0\le\mu\le n-1, \text{ on } \partial D,$$

we get the the classical polyharmonic Dirichlet problem.

(ii) If

$$\partial_\nu^\mu w = \gamma_\mu, 1 \le \mu \le n, \text{ on } \partial D,$$

we get the the polyharmonic Neumann problem.

(iii) If

$$(\partial_z \partial_{\overline{z}})^\mu w = \gamma_\mu, 0 \le \mu \le n-1, \text{ on } \partial D,$$

we get the the polyharmonic Riquier (Navier) problem.

We may also present the following boundary value problems for the polyharmonic n-Poisson equation:

Problem I

$$\partial_\nu (\partial_z \partial_{\overline{z}})^\mu w = \gamma_\mu, 0 \le \mu \le n-1, \text{ on } \partial D,$$

Problem II

$$(\partial_z \partial_{\overline{z}})^\mu w = \gamma_{0\mu}, 0 \le 2\mu \le n-1,$$

$$\partial_\nu (\partial_z \partial_{\overline{z}})^\mu w = \gamma_{1\mu}, 0 \le 2\mu \le n-2, \text{ on } \partial D.$$

4.1. Iterated polyharmonic Green functions

We rewrite the n-Poisson equation with the Riquier conditions which we can decompose it into Dirichlet problems for the Poisson equation

$$\partial_z \partial_{\overline{z}} w_\mu = w_{\mu+1} \text{ in } D, \quad w_\mu = \gamma_\mu \text{ on } \partial D, \quad 0 \le \mu \le n-1.$$

Naturally we assume $w_0 = w$ and $w_n = f$. Using iteration technique we start with solutions

$$w_\mu(z) = -\frac{1}{4\pi} \int_{\partial D} \gamma_\mu(\zeta) \partial_{\nu_\zeta} G_1(z,\zeta) ds_\zeta - \frac{1}{\pi} \int_D w_{\mu+1}(\zeta) G_1(z,\zeta) d\xi d\eta,$$

and find

$$w(z) = -\frac{1}{4\pi} \sum_{\mu=0}^{n-1} \int_{\partial D} \gamma_\mu(\zeta) \partial_{\nu_\zeta} G_{\mu+1}(z,\zeta) ds_\zeta - \frac{1}{\pi} \int_D f(\zeta) G_n(z,\zeta) d\xi d\eta$$

where

$$G_\mu(z,\zeta) = -\frac{1}{\pi} \int_D G_1(z,\widetilde{\zeta}) G_{\mu-1}(\widetilde{\zeta},\zeta) d\widetilde{\xi} d\widetilde{\eta}, \quad 2 \le \mu \le n.$$

Thus we have obtained the following theorem.

Theorem 4.1. *If $f \in L_p(D;\mathbb{C})$, $2 < p$, $\gamma_\mu \in C^{0,\alpha}(\partial D;\mathbb{C})$, $0 < \alpha < 1$, then the Riquier boundary value problem*

$$(\partial_z \partial_{\overline{z}})^n w = f \text{ in } D,$$
$$(\partial_z \partial_{\overline{z}})^\mu w = \gamma_\mu \text{ on } \partial D, \quad 0 \le \mu \le n-1,$$

is uniquely solvable and its solution has the form

$$w(z) = -\frac{1}{4\pi} \int_{\partial D} \sum_{\mu=0}^{n-1} \partial_{\nu_z} G_{\mu+1}(z,\zeta) \gamma_\mu(\zeta) ds_\zeta - \frac{1}{\pi} \int_D G_n(z,\zeta) f(\zeta) d\xi d\eta.$$

The iterated polyharmonic Green function $G_n(z,\zeta)$ has the following properties

- $G_n(\cdot,\zeta)$ is polyharmonic of order n in $D \backslash \{\zeta\}$,
- $G_n(z,\zeta) + \dfrac{|\zeta - z|^{2(n-1)}}{(n-1)!^2} \log|\zeta - z|^2$ is polyharmonic of order n for $z \in D, \zeta \in D$,
- $(\partial_z \partial_{\overline{z}})^\mu G_n(z,\zeta) = 0$ for $0 \le \mu \le n-1$ and $z \in \partial D, \zeta \in D$.
- $G_n(z,\zeta) = G_n(\zeta,z)$ for $z,\zeta \in D, z \ne \zeta$,
- $(\partial_z \partial_{\overline{z}}) G_n(z,\zeta) = G_{n-1}(z,\zeta)$ in D
- For any $\zeta \in D$, $G_n(z,\zeta) = 0$ on ∂D.

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