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Editorial

I am delighted to welcome you to the first issue of the Journal of New Theory (JNT). After receiving first manuscript at 12.11.2014, this issue is now complete with 9 papers. One of the papers is about computer sciences and the others are all about mathematics.

It is truly a collaborative effort and thanks go to many scientists who have helped start this journal. ISSN: 2149-1402 is obtained at 27.01.2015.

We would like to express our deepest thanks to all of the members of the editorial board and reviewers of the papers in this issue who are U. Orhan, A. Filiz, A. Fenercioğlu, A. Sarı, A. Yıldırım, A. S. Sezer, B. Mehmetoğlu, B. H. Çadırcı, C. Kaya, Ç. Çekiç, E. Altuntaş, E. Turgut, F. Karaaslan, F. Smarandache, G. Erdal, H. Aktaş, H. M. Doğan, H. Günal, H. Kızılaslan, H. Önen, H. Şimşek, İ. Zorlutuna, İ. Deli, İ. Gökçe, İ. Türkekul, İ. Parmaksız, J. Ye, M. Akar, M. Akdağ, M. Ali, M. Çavuş, M. Demirci, M. Sağlam, N. Yeşilayar, O. Muhtaroğlu, P. K. Maji, R. Yayar, S. Broumi, S. Karaman, S. Tarhan, S. Enginoğlu, S. Demiriz, S. Karataş, S. Öztürk, S. Eğri, Ş. Sözen, Y. Budak, Y. Karadağ, S. J. John, M. Ali and O. Ravi.

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Pleases, write any original idea. If it is true, it gives an opportunity to use. If it is incomplete, it gives an opportunity to complete. If it is incorrect, it gives an opportunity to correct.

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We hope you will enjoy this issue of JNT. We are looking forward to hearing your feedback and receiving your contributions.

Happy reading!

27 January 2015

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Original Article[**]

Connectedness on Intuitionistic Fuzzy Soft Topological Spaces

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Abstract – In this study, we introduce intuitionistic fuzzy soft connected sets in intuitionistic fuzzy soft topological spaces and some properties. Moreover, we extend the notion of C_i connectedness $(i = 1, 2, 3, 4)$ to intuitionistic fuzzy soft topological spaces.

Keywords – *Intuitionistic fuzzy soft set, intuitionistic fuzzy soft topological space, intuitionistic fuzzy soft connectedness.*

1 Introduction

Nowadays, several researchers investigate to model the uncertainties. They use different set theories for this, for example fuzzy set theory [1] and intuitionistic fuzzy set theory [2] are the most common. But, such theories have their own difficulties such as constructing membership function. Therefore, Molodtsov [6] proposed a new mathematical tool for uncertainties, called soft set theory. In this theory, it is not necessary which constructing membership function. Soft sets can apply several areas such as Riemann-integration, Perron integration, game theory, operations research, probability theory, etc.

Many researchers study on soft set theory, especially soft topological structures. For example, soft topology and related properties were studied in [8, 9, 10, 11, 12, 13, 14, 15, 16, 17]. Then, several paper were published about fuzzy soft topological spaces [18, 19, 20, 21, 22, 23]. Moreover, recently, some authors have studied over intuitionistic fuzzy soft topological spaces [26, 27, 28, 29].

In this article, we introduce the connectedness on intuitionisitic fuzzy soft topological spaces. Then, we are compare the *ifs* C_i themselves.

2 Preliminary

In this section, we will give basic definitions and theorems with *ifs*-sets, intuitionistic fuzzy soft topology and intuitionistic fuzzy soft continuous functions. Throughout this paper, $\mathcal{P}(X)$, E and $\mathcal{IF}(X)$ denote power set of X, set of parameter and set of all intuitionistic fuzzy sets over X, respectively.

[**]*Edited by* Naim Çağman *(Editors-in-Chief) and* Oktay Muhtaroğlu *(Area Editor).*
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Definition 2.1. [2] Let X be a nonempty set. An intuitionistic fuzzy set A is defined by

$$A = \Big\{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\Big\}$$

where $\mu_A : X \to [0,1]$ and $\nu_A : X \to [0,1]$ denote membership and nonmembership functions respectively. Therefore, $\mu_A(x)$ and $\nu_A(x)$ are membership and nonmembership degree of each element $x \in X$ to the intuitionistic fuzzy set A and $0 \leq \mu_A(x) + \nu_A(x) \leq 1$ for each $x \in X$.

Definition 2.2. [2] Let $\{A_i\}_{i\in I} \subseteq \mathcal{IF}(X)$, $A = \big\{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\big\}$ and $B = \big\{\langle x, \mu_B(x), \nu_B(x)\rangle : x \in X\big\}$ be two intuitionistic fuzzy sets on X. Then, some basic set operations of intuitionistic fuzzy sets are defined as follows.

i. $A \subseteq B \Leftrightarrow \mu_B(x) \geq \mu_A(x)$ and $\nu_B(x) \leq \nu_A(x)$ for all $x \in X$

ii. $A = B \Leftrightarrow A \subseteq B$ and $B \subseteq A$.

iii. $\bigcup_{i\in I} A_i = \Big\{\langle x, \bigvee_{i\in I} \mu_{A_i}(x), \bigwedge_{i\in I} \nu_{A_i}(x)\rangle : x \in X\Big\}$

iv. $\bigcap_{i\in I} A_i = \Big\{\langle x, \bigwedge_{i\in I} \mu_{A_i}(x), \bigvee_{i\in I} \nu_{A_i}(x)\rangle : x \in X\Big\}$

v. $\Box A = \Big\{\langle x, \mu_A(x), 1 - \mu_A(x)\rangle : x \in X\Big\}$

vi. $\Diamond A = \Big\{\langle x, 1 - \nu_A(x), \nu_A(x)\rangle : x \in X\Big\}$

vii. $A^c = \big\{\langle x, \nu_A(x), \mu_A(x)\rangle : x \in X\big\}$

viii. $\tilde{1} = \big\{\langle x, 1, 0\rangle : x \in X\big\}$ and $\tilde{0} = \big\{\langle x, 0, 1\rangle : x \in X\big\}$.

Theorem 2.3. [3] Let $A, B, C \in \mathcal{IF}(X)$. Then

i. $A \subseteq B$ and $B \subseteq C \Rightarrow\ A \subseteq C$

ii. $A \subseteq B \Rightarrow A \cup C \subseteq B \cup C$ and $A \cap C \subseteq B \cap C$

iii. $(A \cap B)^c = A^c \cup B^c$ and $(A \cup B)^c = A^c \cap B^c$

iv. $(A^c)^c = A$, $\tilde{1}^c = \tilde{0}$ and $\tilde{0}^c = \tilde{1}$

v. $A \subseteq B \Rightarrow B^c \subseteq A^c$

Definition 2.4. [6] A pair (F, A) is called a soft set over X, if F is a mapping defined by $F : A \to \mathcal{P}(X)$, where $A \subseteq E$.

Now, we will give a new soft set definition who was given by Çağman [7]. The definition is a new comment for the soft sets.

Definition 2.5. [7] A soft set F over X is a set valued function from E to $\mathcal{P}(X)$. It can be written a set of ordered pairs

$$F = \big\{(e, F(e)) : e \in E\big\}.$$

Note that if $F(e) = \emptyset$, then the element $(e, F(e))$ is not appeared in F. Set of all soft sets over X is denoted by $\mathbb{S}$.

According to Definition 2.5 we will redefine *ifs*-set and its set operations.

Definition 2.6. An intuitionistic fuzzy soft set (or namely *ifs*-set) f over X is a set valued function from E to $\mathcal{IF}(X)$. It can be written a set of ordered pairs

$$f = \Big\{\big(e, \{\langle x, \mu_{f(e)}(x), \nu_{f(e)}(x)\rangle : x \in X\}\big) : e \in E\Big\}$$

Note that if $f(e) = \tilde{0}$, then the element $(e, f(e))$ is not appeared in f. Set of all *ifs*-sets over X is denoted by $\mathbb{IFS}_X^E$.

Definition 2.7. Let $f, g, h \in \mathbb{IFS}_X^E$. Then some basic set operations of *ifs*-sets are defined as follows:

i. (Inclusion) $f \sqsubseteq g$ iff $f(e) \subseteq g(e)$ for all $e \in E$.

ii. (Equality) $f = g$ iff $f \sqsubseteq g$ and $g \sqsubseteq f$

iii. (Union) $h = f \sqcup g$ iff $h(e) = f(e) \cup g(e)$ for all $e \in E$.

iv. (Intersection) $h = f \sqcap g$ iff $h(e) = f(e) \cap g(e)$ for all $e \in E$.

v. (Complement) $h = f^{\tilde{c}}$ iff $h(e) = \big(f(e)\big)^{\tilde{c}}$ for all $e \in E$

vi. (Null *ifs*-set) f is called the null *ifs*-set and denoted by Φ, if $f(e) = \tilde{0}$ for all $e \in E$.

vii. (Universal *ifs*-set) f is called the universal *ifs*-set and denoted by $\tilde{X}$, if $f(e) = \tilde{1}$ for all $e \in E$.

Theorem 2.8. Let $\{f_i\}_{i\in\Lambda} \subseteq \mathbb{IFS}_X^E$ and $g \in \mathbb{IFS}_X^E$. Then

i. $g \sqcap \Big(\bigsqcup_{i\in\Lambda} f_i\Big) = \bigsqcup_{i\in\Lambda}(g \sqcap f_i)$

ii. $g \sqcup \Big(\bigsqcap_{i\in\Lambda} f_i\Big) = \bigsqcap_{i\in\Lambda}(g \sqcup f_i)$

iii. $\Big(\bigsqcup_{i\in\Lambda} f_i\Big)^{\tilde{c}} = \bigsqcap_{i\in\Lambda} f_i^{\tilde{c}}$

iv. $\Big(\bigsqcap_{i\in\Lambda} f_i\Big)^{\tilde{c}} = \bigsqcup_{i\in\Lambda} f_i^{\tilde{c}}$

v. $\Phi \sqsubseteq f \sqsubseteq \tilde{X}$, $\tilde{X}^{\tilde{c}} = \Phi$ and $\Phi^{\tilde{c}} = \tilde{X}$,

vi. $g \sqcup g^{\tilde{c}} = \tilde{X}$ and $\big(g^{\tilde{c}}\big)^{\tilde{c}} = g$.

Definition 2.9. [25, 29] Let $\mathbb{IFS}_X^E$ and $\mathbb{IFS}_Y^K$ be sets of all *ifs*-sets on X and Y, respectively. Let $\varphi : X \to Y$ and $\psi : E \to K$ be two mappings. Then a mapping $\varphi_\psi : \mathbb{IFS}_X^E \to \mathbb{IFS}_Y^K$ is defined as:

i. For $f \in \mathbb{IFS}_X^E$, the image of f under φ_ψ, denoted $\varphi_\psi(f)$, is an *ifs*-set in $\mathbb{IFS}_Y^K$ given by

$$\mu_{\varphi(f)}(k)(y) = \begin{cases} \sup_{e\in\psi^{-1}(k),\ x\in\varphi^{-1}(y)} \mu_{f(e)}(x), & \text{if } \varphi^{-1}(y) \neq \emptyset \\ 0, & \text{otherwise} \end{cases}$$

and

$$\nu_{\varphi(f)}(k)(y) = \begin{cases} \inf_{e\in\psi^{-1}(k),\ x\in\varphi^{-1}(y)} \nu_{f(e)}(x), & \text{if } \varphi^{-1}(y) \neq \emptyset \\ 1, & \text{otherwise} \end{cases}$$

ii. For $g \in \mathbb{IFS}_Y^K$, the inverse image of g under φ_ψ, denoted by $\varphi_\psi^{-1}(g)$ is an *ifs*-set in $\mathbb{IFS}_X^E$ given by

$$\mu_{\varphi^{-1}(g)}(e)(x) = \mu_{g(\psi(e))}(\varphi(x)) \text{ and } \nu_{\varphi^{-1}(g)}(e)(x) = \nu_{g(\psi(e))}(\varphi(x))$$

for all $e \in E$ and $x \in X$.

If φ and ψ are injective (surjective) then the *ifs*-mapping φ_ψ is said to be *ifs*-injective (*ifs*-surjective).

Theorem 2.10. [25] Let $\varphi_\psi : \mathbb{IFS}_X^E \to \mathbb{FS}_Y^K$ be a intuitionistic fuzzy soft mapping, $f \in \mathbb{IFS}_X^E$ and $\{f_i\}_{i\in\Lambda} \subseteq \mathbb{IFS}_X^E$. Then

i. If $f_1 \sqsubseteq f_2$, then $\varphi_\psi(f_1) \sqsubseteq \varphi_\psi(f_2)$

ii. $\varphi_\psi\Big(\bigsqcup_{i\in\Lambda} f_i\Big) = \bigsqcup_{i\in\Lambda} \varphi_\psi(f_i)$

iii. $\varphi_\psi\Big(\bigsqcap_{i\in\Lambda} f_i\Big) \sqsubseteq \bigsqcap_{i\in\Lambda} \varphi_\psi(f_i)$

iv. $\big(\varphi_\psi(f)\big)^{\tilde{c}} \sqsubseteq \varphi_\psi\big(f^{\tilde{c}}\big)$

v. If φ_ψ surjective, then $\varphi_\psi(\tilde{X}) = \tilde{Y}$

vi. $f \sqsubseteq \varphi_\psi^{-1}\big(\varphi_\psi(f)\big)$, the equality holds if φ_ψ is *ifs*-injective.

Theorem 2.11. [25] Let $\varphi_\psi : \mathbb{IFS}_X^E \to \mathbb{IFS}_Y^K$ be a intuitionistic fuzzy soft mapping, $g \in \mathbb{IFS}_Y^K$ and $\{g_j\}_{j\in J} \subseteq \mathbb{IFS}_Y^K$. Then

i. If $g_1 \sqsubseteq g_2$, then $\varphi_\psi^{-1}(g_1) \sqsubseteq \varphi_\psi^{-1}(g_2)$

ii. $\varphi_\psi^{-1}\Big(\bigsqcup_{i\in J} g_j\Big) = \bigsqcup_{j\in J} \varphi_\psi^{-1}(g_j)$

iii. $\varphi_\psi^{-1}\Big(\bigsqcap_{i\in J} g_j\Big) = \bigsqcap_{j\in J} \varphi_\psi^{-1}(g_j)$

iv. $\big(\varphi_\psi^{-1}(g)\big)^{\tilde{c}} = \varphi_\psi^{-1}\big(g^{\tilde{c}}\big)$

v. $\varphi_\psi^{-1}(\tilde{Y}) = \tilde{X}$ and $\varphi_\psi^{-1}(\Phi) = \Phi$

vi. $\varphi_\psi\big(\varphi_\psi^{-1}(g)\big) \sqsubseteq g$, the equality holds if φ_ψ is *ifs*-surjective.

Definition 2.12. [26] An *ifs*-topological space is a triplet (X, τ, E) where X is a nonempty set and τ a family of *ifs*-sets over X satisfying the following properties:

i. $\Phi, \tilde{X} \in \tau$,

ii. If $f, g \in \tau$, then $f \sqcap g \in \tau$,

iii. If $\{f_i\}_{i\in\Lambda} \subseteq \tau$, then $\bigsqcup_{i\in\Lambda} f_i \in \tau$.

Then, the family τ is called an *ifs*-topology on X. Every member of τ is called *ifs*-open. g is called *ifs*-closed in (X, τ, E) if $g^{\tilde{c}} \in \tau$.

If f is *ifs*-open and *ifs*-closed, then it is called *ifs*-clopen set. In case $f \neq \tilde{X}$ and $f \neq \Phi$, f is called *ifs*-proper set.

Example 2.13. $\tau^0 = \{\tilde{X}, \Phi\}$ and $\tau^1 = \mathbb{IFS}_X^E$ are *ifs*-topologies on X.

Definition 2.14. [26] Let (X, τ, E) be a *ifs*-topological space and $f \in \mathbb{IFS}_X^E$. Then, *ifs*-interior of f denoted by f° is the union of all *ifs*-open subsets of f. So, we can write the *ifs*-interior of f as

$$f^\circ = \bigsqcup_{\substack{g\sqsubseteq f\\ g\in\tau}} g.$$

Definition 2.15. [26] Let (X, τ, E) be a *ifs*-topological space and $f \in \mathbb{IFS}_X^E$. Then, *ifs*-closure of f denoted by $\overline{f}$ is the intersection of all *ifs*-closed supersets of f. So, we can write the *ifs*-closure of f as

$$\overline{f} = \bigsqcap_{\substack{f\sqsubseteq h\\ h^{\tilde{c}}\in\tau}} h.$$

It can be seen clearly that f° and $\overline{f}$ are the largest *ifs*-open set which contained in f and the smallest *ifs*-closed set which contains f over X, respectively.

Definition 2.16. Let (X, τ, E) be a *ifs*-topological space and $f \in \mathbb{IFS}_X^E$. If $f = \big(\overline{f}\big)^\circ$, then f is called *ifs*-regular open set. If If $f = \overline{f^\circ}$, then f is called *ifs*-regular closed set.

Theorem 2.17. [26] Let (X, τ, E) be a *ifs*-topological space and $f,\ g \in \mathbb{IFS}_X^E$. Then,

i. If $f \sqsubseteq g$, then $f^\circ \sqsubseteq g^\circ$ and $\overline{f} \sqsubseteq \overline{g}$

ii. f is a soft open set iff $f^\circ = f$

iii. f is a soft closed set iff $\overline{f} = f$

iv. $\big(\overline{f}\big)^{\tilde{c}} = \big(f^{\tilde{c}}\big)^\circ$ and $\overline{\big(f^{\tilde{c}}\big)} = \big(f^\circ\big)^{\tilde{c}}$

Definition 2.18. [29] Let (X, τ, E) and (Y, σ, K) be two *ifs*-topological spaces. An *ifs*-mapping $\varphi_\psi : (X, \tau, E) \to (Y, \sigma, K)$ is called an *ifs*-continuous mapping if $\varphi_\psi^{-1}(g) \in \tau$ for all $g \in \sigma$.

Example 2.19. [29] In Example 2.13, every *ifs*-mapping $\varphi_\psi : (X, \tau^1, E) \to (Y, \sigma, K)$ is an *ifs*-continuous mapping.

3 Intuitionistic Fuzzy Soft Connectedness

In this section, we will give definition of *ifs*-connected spaces and their some properties. Further, we will introduce *ifs* C_i-connectedness $(i = 1, 2, 3, 4)$ and *ifs*-super connectedness.

Definition 3.1. Let (X, τ, E) be a *ifs*-topological space and $f \in \mathbb{IFS}_X^E$. If there are two *ifs*-proper open sets g_1 and g_2 such that $f \sqsubseteq g_1 \sqcup g_2$ and $g_1 \sqcap g_2 = \Phi$, then the *ifs*-set f is called *ifs*-disconnected set. If there does not exist such two *ifs*-proper open sets, then the *ifs*-set f is called *ifs*-connected set. If we take $\tilde{X}$ instead of f, then the (X, τ, E) is called *ifs*-disconnected (connected) space.

Example 3.2. Let consider the *ifs*-topological spaces (X, τ^0, E) and (X, τ^1, E) in Example 2.13, (X, τ^0, E) is an *ifs*-connected topological space, but (X, τ^1, E) is an *ifs*-disconnected topological space.

Theorem 3.3. Let (X, τ, E) be a *ifs*-topological space. (X, τ, E) *ifs*-connected if and only if there does not exist a *ifs*-proper clopen set f in (X, τ, E).

Proof. $(\Rightarrow)$: Let (X, τ, E) be a *ifs*-connected space. Suppose that there exist a *ifs*-proper clopen set f in (X, τ, E) such that $f \sqcup f^{\tilde{c}} = \tilde{X}$ and $f \sqcap f^{\tilde{c}} = \Phi$. It is a contradiction.

$(\Leftarrow)$: It is clear.

Theorem 3.4. Let (X, τ, E) be a *ifs*-topological space and $\sigma \subseteq \tau$. Then, (X, σ, E) is a connected *ifs*-topological space.

Proof. It is clear.

Theorem 3.5. Let (X, τ, E) and (Y, σ, K) be two *ifs*-topological spaces, $f \in \mathbb{IFS}_X^E$ and $\varphi_\psi : (X, \tau, E) \to (Y, \sigma, K)$ be an *ifs*-continuous mapping. If f is an *ifs*-connected set, then $\varphi_\psi(f)$ is an *ifs*-connected set.

Proof. Assume that $\varphi_\psi(f)$ is an *ifs*-disconnected set. Therefore, there exist two *ifs*-proper open sets g and h such that $\varphi_\psi(f) \sqsubseteq g \sqcup h$ and $g \sqcap h = \Phi$. By Theorem 2.11, we have

$$f \sqsubseteq \varphi_\psi^{-1}\big(\varphi_\psi(f)\big) \sqsubseteq \varphi_\psi^{-1}(g) \sqcup \varphi_\psi^{-1}(h)$$

and

$$\varphi_\psi^{-1}(g) \sqcap \varphi_\psi^{-1}(h) = \varphi_\psi^{-1}(\Phi) = \Phi.$$

It is a contradiction and this complete the proof.

Theorem 3.6. Let (X, τ, E) and (Y, σ, K) be two *ifs*-topological spaces and $\varphi_\psi : (X, \tau, E) \to (Y, \sigma, K)$ be an *ifs*-continuous and *ifs*-surjective mapping. If (X, τ, E) is an *ifs*-connected space, then (Y, σ, K) is also an *ifs*-connected space.

Proof. Assume that (Y, σ, K) is an *ifs*-disconnected space. So, there exist two *ifs*-proper open sets g_1 and g_2 such that $g_1 \sqcup g_2 = \tilde{Y}$, $g_1 \sqcap g_2 = \Phi$. By Theorem 2.11 $\varphi_\psi^{-1}(g_1) \sqcup \varphi_\psi^{-1}(g_2) = \tilde{X}$ and $\varphi_\psi^{-1}(g_1) \sqcap \varphi_\psi^{-1}(g_2) = \Phi$. This contradiction completes the proof.

Definition 3.7. Let (X, τ, E) be an *ifs*-topological space. If there exist $f, g \in \mathbb{IFS}_X^E$ which are *ifs*-proper, such that $\overline{f} \sqcap g = \Phi$ and $f \sqcap \overline{g} = \Phi$ then the *ifs*-sets f and g are called *ifs*-separated sets.

Theorem 3.8. Let (X, τ, E) be a *ifs*-topological space, f and g be two *ifs*-open sets. If $f \sqcap g = \Phi$, then f and g are *ifs*-separated sets.

Proof. Let $f, g \in \tau$ and $f \sqcap g = \Phi$. Then, $f^{\tilde{c}} \sqcup g^{\tilde{c}} = \tilde{X}$. So, $f \sqsubseteq g^{\tilde{c}}$ and $g \sqsubseteq f^{\tilde{c}}$. $f^{\tilde{c}}$ and $g^{\tilde{c}}$ are *ifs*-closed sets. By 2.17, we have

$$\overline{f} \sqsubseteq \overline{g^{\tilde{c}}} = g^{\tilde{c}} \text{ and } \overline{g} \sqsubseteq \overline{f^{\tilde{c}}} = f^{\tilde{c}}.$$

Therefore, $\overline{f} \sqcap g = \Phi$ and $f \sqcap \overline{g} = \Phi$.

Theorem 3.9. Let (X, τ, E) be an *ifs*-topological space, f and g be two *ifs*-closed sets. If $f \sqcap g = \Phi$, then f and g are *ifs*-separated sets.

Proof. From Theorem 2.17, it is clear.

Theorem 3.10. An *ifs*-topological space (X, τ, E) is connected if and only if $\tilde{X}$ cannot be written as union of *ifs*-separated sets.

Proof. $(\Rightarrow)$: Assume that $\tilde{X}$ can be written as union of *ifs*-separated sets f and g. Thus, $\tilde{X} = f \sqcup g$, $\overline{f} \sqcap g = \Phi$ and $f \sqcap \overline{g} = \Phi$. So, we have $f \sqcap g = \Phi$, $f = g^{\tilde{c}}$ and $g = f^{\tilde{c}}$. Furthermore

$$\begin{aligned} \overline{f} &= \overline{f} \sqcap \tilde{X} \\ &= \overline{f} \sqcap (f \sqcup g) \\ &= (\overline{f} \sqcap f) \sqcup (\overline{f} \sqcap g) \\ &= f. \end{aligned}$$

Thus, f is an *ifs*-closed set. With similar way, it can be seen clearly that g is also an *ifs*-closed set. This is a contradiction because $f = g^{\tilde{c}}$ and $g = f^{\tilde{c}}$, f and g are *ifs*-open sets.

$(\Leftarrow)$: Assume that (X, τ, E) is not an *ifs*-connected space. Thus, there exist an *ifs*-proper clopen set f. But it contradicts by hypothesis.

Theorem 3.11. Let (X, τ, E) be an *ifs*-topological space and $f \in \mathbb{IFS}_X^E$ be an *ifs*-open connected set. If $f \sqsubseteq g \sqsubseteq \overline{f}$, then g is an *ifs*-connected set.

Proof. Suppose that g is an *ifs*-disconnected set. Then, there exist two *ifs*-open proper sets h_1 and h_2 such that

$$h_1 \sqcap h_2 = \Phi \text{ and } g \sqsubseteq h_1 \sqcup h_2.$$

So,

$$f = [f \sqcap h_1] \sqcup [f \sqcap h_2]$$

and

$$[f \sqcap h_1] \sqcap [f \sqcap h_2] = \Phi.$$

But it is a contradiction. Thus g is an *ifs*-connected set.

Remark 3.12. Let (X, τ, E) be an *ifs*-topological space and $f \in \mathbb{IFS}_X^E$ be an *ifs*-open set. If f is an *ifs*-connected set, then $\overline{f}$ is an *ifs*-connected set.

Definition 3.13. Let (X, τ, E) be an *ifs*-topological space. If there exist an *ifs*-regular open proper set f, then (X, τ, E) is called *ifs*-super disconnected.

Example 3.14. Let $X = \{x_1, x_2, x_3\}$ and $E = \{e_1, e_2\}$. Then, for

$$\begin{aligned} f &= \Big\{\big(e_1, \{\langle x_1, 0.4, 0.6\rangle, \langle x_2, 0.6, 0.3\rangle, \langle x_3, 0.2, 0.3\rangle\}\big), \\ &\quad \big(e_2, \{\langle x_1, 0.6, 0.4\rangle, \langle x_2, 0.3, 0.6\rangle, \langle x_3, 0.3, 0.2\rangle\}\big)\Big\} \\ g &= \Big\{\big(e_1, \{\langle x_1, 0.5, 0.2\rangle, \langle x_2, 0.3, 0.6\rangle, \langle x_3, 0.4, 0.3\rangle\}\big), \\ &\quad \big(e_2, \{\langle x_1, 0.2, 0.5\rangle, \langle x_2, 0.6, 0.3\rangle, \langle x_3, 0.3, 0.4\rangle\}\big)\Big\} \\ h &= \Big\{\big(e_1, \{\langle x_1, 0.5, 0.4\rangle, \langle x_2, 0.4, 0.5\rangle, \langle x_3, 0.2, 0.4\rangle\}\big), \\ &\quad \big(e_2, \{\langle x_1, 0.4, 0.5\rangle, \langle x_2, 0.5, 0.4\rangle, \langle x_3, 0.4, 0.2\rangle\}\big)\Big\} \end{aligned}$$

$\tau = \{\tilde{X}, \Phi, f, g, h\}$ is an *ifs*-topology on X and (X, τ, E) is an *ifs*-super connected space.

Theorem 3.15. The followings are equivalent.

i. (X, τ, E) is an *ifs*-super connected space

ii. For each f such that $f \neq \Phi$, $\overline{f} = \tilde{X}$

iii. For each f such that $f \neq \Phi$, $f^\circ = \Phi$

iv. There exist no *ifs*-open sets f and g such that $f \neq \Phi$, $g \neq \Phi$ and $f \sqsubseteq g^{\tilde{c}}$

v. There exist no *ifs*-open sets $f$and g such that $f \neq \Phi$, $g \neq \Phi$, $g = (\overline{f})^{\tilde{c}}$ and $f = (\overline{g})^{\tilde{c}}$

vi. There exist no *ifs*-closed sets $f$and g such that $f \neq \tilde{X}$, $g \neq \tilde{X}$, $g = (f^\circ)^{\tilde{c}}$ and $f = (g^\circ)^{\tilde{c}}$

Proof. ($i. \Rightarrow ii.$): Suppose that there exists an *ifs*-open f such that $f \neq \Phi$ and $\overline{f} \neq \tilde{X}$. If we take $g = (\overline{f})^\circ$, then g is an *ifs*-proper and regular open set. But it is a contradiction.

($ii. \Rightarrow iii.$): Let $f \neq \tilde{X}$ be an *ifs*-closed set. If we take $g = f^{\tilde{c}}$, then g is an *ifs*-open and $g \neq \Phi$. For $\overline{g} = \tilde{X}$, we have $(g^\circ)^{\tilde{c}} = \Phi$ and $(\overline{g})^\circ = \Phi$. So, $f^\circ = \Phi$.

($iii. \Rightarrow iv.$): Let f and g be *ifs*-open sets such that $f \neq \Phi$, $g \neq \Phi$ and $f \sqsubseteq g^{\tilde{c}}$. Thus, $g^{\tilde{c}}$ is an *ifs*-closed set and because of $g \neq \Phi$, $g^{\tilde{c}} \neq \tilde{X}$. So, we obtain $(g^{\tilde{c}})^\circ = \Phi$. But, with $f \sqsubseteq g^{\tilde{c}}$, we can write $\Phi \neq f = f^\circ \sqsubseteq (g^{\tilde{c}})^\circ = \Phi$. It is a contradiction

($iv. \Rightarrow i.$): Let f be an *ifs*-regular open proper. If we take $g = (\overline{f})^{\tilde{c}}$, we obtain $g \neq \Phi$. (Otherwise, $(\overline{f})^{\tilde{c}} = \Phi \Rightarrow \overline{f} = \tilde{X}$ and so, $f = (\overline{f})^\circ = \tilde{X}$. But it contradicts the fact $f \neq \tilde{X}$.)

($i. \Rightarrow v.$): Let f and g be *ifs*-open sets such that $f \neq \Phi$, $g \neq \Phi$, $g = (\overline{f})^{\tilde{c}}$ and $f = (\overline{g})^{\tilde{c}}$. Then we have $(\overline{f})^\circ = (g^{\tilde{c}})^\circ = (\overline{g})^{\tilde{c}} = f$ where $f \neq \Phi$ and $f \neq \tilde{X}$. (Otherwise, if $f = \tilde{X}$, then $\tilde{X} = (\overline{g})^{\tilde{c}}$ and thus $\Phi = \overline{g}$.) But it is a contradiction.

($v. \Rightarrow i.$): Let f be an *ifs*-open proper set such that $f = (\overline{f})^\circ$. If we take $g = (\overline{f})^{\tilde{c}}$, then we have $g \neq \Phi$, $g \in \tau$, $g = (\overline{f})^{\tilde{c}}$ and so

$$(\overline{g})^{\tilde{c}} = \left(\overline{(\overline{f})^{\tilde{c}}}\right)^{\tilde{c}} = \left(((\overline{f})^\circ)^{\tilde{c}}\right)^{\tilde{c}} = (\overline{f})^\circ = f$$

but it is a contradiction.

($v. \Rightarrow vi.$): Let f and g be *ifs*-closed sets such that $f \neq \tilde{X}$, $g \neq \tilde{X}$, $g = (f^\circ)^{\tilde{c}}$ and $f = (g^\circ)^{\tilde{c}}$. If we take $h_1 = f^{\tilde{c}}$ and $h_2 = g^{\tilde{c}}$, then h_1 and h_2 become *ifs*-open sets such that $h_1 \neq \Phi$ and $h_2 \neq \Phi$. Thus $(\overline{h_1})^{\tilde{c}} = (\overline{f^{\tilde{c}}})^{\tilde{c}} = \left((f^\circ)\right)^{\tilde{c}} = f^\circ = g^{\tilde{c}} = h_2$ and similarly $(\overline{h_2})^{\tilde{c}} = h_1$. But this is a contradiction, clearly.

($vi. \Rightarrow v.$): It can be proved similar way in ($v. \Rightarrow vi.$)

Now, we will introduce *ifs* C_i-connected spaces ($i = 1, 2, 3, 4$) by helping of fuzzy C_i-connectedness in intuitionistic fuzzy sets [4]. Definitions of *ifs* C_i-connected spaces can be seen as an extension of intuitionistic fuzzy connected space.

Definition 3.16. Let (X, τ, E) be a *ifs*-topological space and $f \in \mathbb{IFS}_X^E$. f is called

i. *ifs* C_1-connected iff does not exist two non null *ifs*-open sets g and h such that $f \sqsubseteq g \sqcup h$, $g \sqcap h \sqsubseteq f^{\tilde{c}}$, $f \sqcap g \neq \Phi$ and $f \sqcap h \neq \Phi$.

ii. *ifs* C_2-connected iff does not exist two non null *ifs*-open sets g and h such that $f \sqsubseteq g \sqcup h$, $f \sqcap g \sqcap h = \Phi$ $f \sqcap g \neq \Phi$ and $f \sqcap h \neq \Phi$.

iii. *ifs* C_3-connected iff does not exist two non null *ifs*-open sets g and h such that $f \sqsubseteq g \sqcup h$, $g \sqcap h \sqsubseteq f^{\tilde{c}}$, $g \not\sqsubseteq f^{\tilde{c}}$ and $h \not\sqsubseteq f^{\tilde{c}}$.

iv. *ifs* C_4-connected iff does not exist two non null *ifs*-open sets g and h such that $f \sqsubseteq g \sqcup h$, $f \sqcap g \sqcap h = \Phi$, $g \not\sqsubseteq f^{\tilde{c}}$ and $h \not\sqsubseteq f^{\tilde{c}}$.

From Definition 3.16, relations between *ifs* C_i-connectedness ($i = 1, 2, 3, 4$) can be described by the following diagram:

$$\begin{array}{ccc} \textit{ifs } C_1 \text{ connectedness} & \longrightarrow & \textit{ifs } C_2 \text{ connectedness} \\ \big\downarrow & & \big\downarrow \\ \textit{ifs } C_3 \text{ connectedness} & \longrightarrow & \textit{ifs } C_4 \text{ connectedness} \end{array}$$

In the following examples, we illustrate all reverse implications.

Example 3.17. Let $X = [0,1]$ and $E = \{a,b\}$. Moreover, define soft sets f, g and h as following:

$$\begin{aligned} f &= \Big\{\big(a, \{\langle x, \mu_{f(a)}(x), \nu_{f(a)}(x)\rangle : x \in X\}\big), \\ &\quad \big(b, \{\langle x, \mu_{f(b)}(x), \nu_{f(b)}(x)\rangle : x \in X\}\big)\Big\} \\ g &= \Big\{\big(a, \{\langle x, \mu_{g(a)}(x), \nu_{g(a)}(x)\rangle : x \in X\}\big), \\ &\quad \big(b, \{\langle x, \mu_{g(b)}(x), \nu_{g(b)}(x)\rangle : x \in X\}\big)\Big\} \\ h &= \Big\{\big(a, \{\langle x, \mu_{h(a)}(x), \nu_{h(a)}(x)\rangle : x \in X\}\big), \\ &\quad \big(b, \{\langle x, \mu_{h(b)}(x), \nu_{h(b)}(x)\rangle : x \in X\}\big)\Big\} \end{aligned}$$

where

$$\mu_{g(a)}(x) = \begin{cases} \frac{1}{3}, & \text{if } \frac{1}{3} < x \leq 1 \\ 1, & \text{if } 0 \leq x \leq \frac{1}{3} \end{cases} \text{ and } \mu_{g(b)}(x) = \begin{cases} 1, & \text{if } \frac{1}{3} < x \leq 1 \\ \frac{1}{3}, & \text{if } 0 \leq x \leq \frac{1}{3}, \end{cases}$$

$$\nu_{g(a)}(x) = \begin{cases} 1, & \text{if } \frac{1}{3} < x \leq 1 \\ \frac{1}{3}, & \text{if } 0 \leq x \leq \frac{1}{3}, \end{cases} \text{ and } \nu_{g(b)}(x) = \begin{cases} \frac{1}{3}, & \text{if } \frac{1}{3} < x \leq 1 \\ 1, & \text{if } 0 \leq x \leq \frac{1}{3} \end{cases}$$

$$\mu_{h(a)}(x) = \begin{cases} 1, & \text{if } \frac{1}{3} < x \leq 1 \\ \frac{1}{3}, & \text{if } 0 \leq x \leq \frac{1}{3} \end{cases} \text{ and } \mu_{h(b)}(x) = \begin{cases} \frac{1}{3}, & \text{if } \frac{1}{3} < x \leq 1 \\ 1, & \text{if } 0 \leq x \leq \frac{1}{3} \end{cases}$$

$$\nu_{h(a)}(x) = \begin{cases} \frac{1}{3}, & \text{if } \frac{1}{3} < x \leq 1 \\ 1, & \text{if } 0 \leq x \leq \frac{1}{3} \end{cases} \text{ and } \nu_{h(b)}(x) = \begin{cases} 1, & \text{if } \frac{1}{3} < x \leq 1 \\ \frac{1}{3}, & \text{if } 0 \leq x \leq \frac{1}{3} \end{cases}$$

$\mu_{f(a)}(x) = \mu_{f(b)}(x) = \nu_{f(a)}(x) = \nu_{f(b)}(x) = 3/4$ for all $x \in [0,1]$. $\tau = \{\Phi, \tilde{X}, g, h, g \sqcap h\}$ is a *ifs*-topology on X. It can be see clearly that f is *ifs* C_4-connected but *ifs* C_3-disconnected.

Example 3.18. Let $X = [0,1]$ and $E = \{a,b\}$. Moreover, define soft sets g, h and f as following:

$$\begin{aligned} g &= \Big\{\big(a, \{\langle x, \mu_{g(a)}(x), \nu_{g(a)}(x)\rangle : x \in X\}\big), \\ &\quad \big(b, \{\langle x, \mu_{g(b)}(x), \nu_{g(b)}(x)\rangle : x \in X\}\big)\Big\} \\ h &= \Big\{\big(a, \{\langle x, \mu_{h(a)}(x), \nu_{h(a)}(x)\rangle : x \in X\}\big), \\ &\quad \big(b, \{\langle x, \mu_{h(b)}(x), \nu_{h(b)}(x)\rangle : x \in X\}\big)\Big\} \\ f &= g \sqcup h \end{aligned}$$

where

$$\mu_{g(a)}(x) = \begin{cases} 0, & \text{if } \frac{1}{3} < x \leq 1 \\ \frac{1}{3}, & \text{if } 0 \leq x \leq \frac{1}{3} \end{cases} \text{ and } \mu_{g(b)}(x) = \begin{cases} \frac{1}{3}, & \text{if } \frac{1}{3} < x \leq 1 \\ 0, & \text{if } 0 \leq x \leq \frac{1}{3}, \end{cases}$$

$$\nu_{g(a)}(x) = \begin{cases} \frac{1}{3}, & \text{if } \frac{1}{3} < x \leq 1 \\ 0, & \text{if } 0 \leq x \leq \frac{1}{3}, \end{cases} \text{ and } \nu_{g(b)}(x) = \begin{cases} 0, & \text{if } \frac{1}{3} < x \leq 1 \\ \frac{1}{3}, & \text{if } 0 \leq x \leq \frac{1}{3} \end{cases}$$

$$\mu_{h(a)}(x) = \begin{cases} \frac{1}{3}, & \text{if } \frac{1}{3} < x \leq 1 \\ 0, & \text{if } 0 \leq x \leq \frac{1}{3}, \end{cases} \text{ and } \mu_{h(b)}(x) = \begin{cases} 0, & \text{if } \frac{1}{3} < x \leq 1 \\ \frac{1}{3}, & \text{if } 0 \leq x \leq \frac{1}{3} \end{cases}$$

$$\nu_{h(a)}(x) = \begin{cases} 0, & \text{if } \frac{1}{3} < x \leq 1 \\ \frac{1}{3}, & \text{if } 0 \leq x \leq \frac{1}{3} \end{cases} \text{ and } \nu_{h(b)}(x) = \begin{cases} \frac{1}{3}, & \text{if } \frac{1}{3} < x \leq 1 \\ 0, & \text{if } 0 \leq x \leq \frac{1}{3}, \end{cases}$$

$\tau = \{\Phi, \tilde{X}, g, h, g \sqcup h\}$ is a *ifs*-topology on X. It can be seen clearly that f is *ifs* C_4-connected but *ifs* C_2-disconnected.

Example 3.19. Let $X = [0,1]$ and $E = \{a, b\}$. Moreover, define soft sets f, g and h as following:

$$\begin{aligned} f &= \Big\{\big(a, \{\langle x, \mu_{f(a)}(x), \nu_{f(a)}(x)\rangle : x \in X\}\big), \\ &\quad \big(b, \{\langle x, \mu_{f(b)}(x), \nu_{f(b)}(x)\rangle : x \in X\}\big)\Big\} \\ g &= \Big\{\big(a, \{\langle x, \mu_{g(a)}(x), \nu_{g(a)}(x)\rangle : x \in X\}\big), \\ &\quad \big(b, \{\langle x, \mu_{g(b)}(x), \nu_{g(b)}(x)\rangle : x \in X\}\big)\Big\} \\ h &= \Big\{\big(a, \{\langle x, \mu_{h(a)}(x), \nu_{h(a)}(x)\rangle : x \in X\}\big), \\ &\quad \big(b, \{\langle x, \mu_{h(b)}(x), \nu_{h(b)}(x)\rangle : x \in X\}\big)\Big\} \end{aligned}$$

where

$$\mu_{g(a)}(x) = \begin{cases} \frac{1}{3}, & \text{if } \frac{1}{3} < x \leq 1 \\ \frac{2}{3}, & \text{if } 0 \leq x \leq \frac{1}{3} \end{cases} \quad \text{and} \quad \mu_{g(b)}(x) = \begin{cases} \frac{2}{3}, & \text{if } \frac{1}{3} < x \leq 1 \\ \frac{1}{3}, & \text{if } 0 \leq x \leq \frac{1}{3}, \end{cases}$$

$$\nu_{g(a)}(x) = \begin{cases} \frac{2}{3}, & \text{if } \frac{1}{3} < x \leq 1 \\ \frac{1}{3}, & \text{if } 0 \leq x \leq \frac{1}{3}, \end{cases} \quad \text{and} \quad \nu_{g(b)}(x) = \begin{cases} \frac{1}{3}, & \text{if } \frac{1}{3} < x \leq 1 \\ \frac{2}{3}, & \text{if } 0 \leq x \leq \frac{1}{3} \end{cases}$$

$$\mu_{h(a)}(x) = \begin{cases} \frac{2}{3}, & \text{if } \frac{1}{3} < x \leq 1 \\ \frac{1}{3}, & \text{if } 0 \leq x \leq \frac{1}{3}, \end{cases} \quad \text{and} \quad \mu_{h(b)}(x) = \begin{cases} \frac{1}{3}, & \text{if } \frac{1}{3} < x \leq 1 \\ \frac{2}{3}, & \text{if } 0 \leq x \leq \frac{1}{3} \end{cases}$$

$$\nu_{h(a)}(x) = \begin{cases} \frac{1}{3}, & \text{if } \frac{1}{3} < x \leq 1 \\ \frac{2}{3}, & \text{if } 0 \leq x \leq \frac{1}{3} \end{cases} \quad \text{and} \quad \nu_{h(b)}(x) = \begin{cases} \frac{2}{3}, & \text{if } \frac{1}{3} < x \leq 1 \\ \frac{1}{3}, & \text{if } 0 \leq x \leq \frac{1}{3}, \end{cases}$$

$\mu_{f(a)}(x) = \mu_{f(b)}(x) = \nu_{f(a)}(x) = \nu_{f(b)}(x) = 1/3$ for all $x \in [0,1]$. $\tau = \{\Phi, \tilde{X}, g, h, g \sqcap h, g \sqcup h\}$ is a *ifs*-topology on X. It can be seen clearly that f is *ifs* C_3-connected and *ifs* C_2-connected but *ifs* C_1-disconnected.

Example 3.20. Let $X = [0,1]$ and $E = \{a, b\}$. Moreover, define soft sets f, g and h as following:

$$\begin{aligned} f &= \Big\{\big(a, \{\langle x, \mu_{f(a)}(x), \nu_{f(a)}(x)\rangle : x \in X\}\big), \\ &\quad \big(b, \{\langle x, \mu_{f(b)}(x), \nu_{f(b)}(x)\rangle : x \in X\}\big)\Big\} \\ g &= \Big\{\big(a, \{\langle x, \mu_{g(a)}(x), \nu_{g(a)}(x)\rangle : x \in X\}\big), \\ &\quad \big(b, \{\langle x, \mu_{g(b)}(x), \nu_{g(b)}(x)\rangle : x \in X\}\big)\Big\} \\ h &= \Big\{\big(a, \{\langle x, \mu_{h(a)}(x), \nu_{h(a)}(x)\rangle : x \in X\}\big), \\ &\quad \big(b, \{\langle x, \mu_{h(b)}(x), \nu_{h(b)}(x)\rangle : x \in X\}\big)\Big\} \end{aligned}$$

where

$$\mu_{g(a)}(x) = \begin{cases} 0, & \text{if } \frac{2}{3} < x \leq 1 \\ \frac{2}{3}, & \text{if } 0 \leq x \leq \frac{2}{3} \end{cases} \quad \text{and} \quad \mu_{g(b)}(x) = \begin{cases} \frac{1}{3}, & \text{if } \frac{2}{3} < x \leq 1 \\ 0, & \text{if } 0 \leq x \leq \frac{2}{3}, \end{cases}$$

$$\nu_{g(a)}(x) = \begin{cases} \frac{1}{3}, & \text{if } \frac{2}{3} < x \leq 1 \\ 0, & \text{if } 0 \leq x \leq \frac{2}{3}, \end{cases} \quad \text{and} \quad \nu_{g(b)}(x) = \begin{cases} 0, & \text{if } \frac{2}{3} < x \leq 1 \\ \frac{2}{3}, & \text{if } 0 \leq x \leq \frac{2}{3} \end{cases}$$

$$\mu_{h(a)}(x) = \begin{cases} \frac{1}{3}, & \text{if } \frac{2}{3} < x \leq 1 \\ 0, & \text{if } 0 \leq x \leq \frac{2}{3}, \end{cases} \quad \text{and} \quad \mu_{h(b)}(x) = \begin{cases} 0, & \text{if } \frac{2}{3} < x \leq 1 \\ \frac{2}{3}, & \text{if } 0 \leq x \leq \frac{2}{3} \end{cases}$$

$$\nu_{h(a)}(x) = \begin{cases} 0, & \text{if } \frac{2}{3} < x \leq 1 \\ \frac{2}{3}, & \text{if } 0 \leq x \leq \frac{2}{3} \end{cases} \quad \text{and} \quad \nu_{h(b)}(x) = \begin{cases} \frac{1}{3}, & \text{if } \frac{2}{3} < x \leq 1 \\ 0, & \text{if } 0 \leq x \leq \frac{2}{3}, \end{cases}$$

$$\mu_{f(a)}(x) = \begin{cases} \frac{1}{3}, & \text{if } \frac{2}{3} < x \leq 1 \\ \frac{2}{3}, & \text{if } 0 \leq x \leq \frac{2}{3} \end{cases} \quad \text{and} \quad \mu_{f(b)}(x) = \begin{cases} \frac{2}{3}, & \text{if } \frac{2}{3} < x \leq 1 \\ \frac{1}{3}, & \text{if } 0 \leq x \leq \frac{2}{3} \end{cases}$$

$$\nu_{f(a)}(x) = \begin{cases} \frac{2}{3}, & \text{if } \frac{2}{3} < x \leq 1 \\ \frac{1}{3}, & \text{if } 0 \leq x \leq \frac{2}{3} \end{cases} \quad \text{and} \quad \nu_{f(b)}(x) = \begin{cases} \frac{1}{3}, & \text{if } \frac{2}{3} < x \leq 1 \\ \frac{2}{3}, & \text{if } 0 \leq x \leq \frac{2}{3} \end{cases}$$

$\tau = \{\Phi, \tilde{X}, g, h, g \sqcup h\}$ is a *ifs*-topology on X. It can be seen clearly that f is *ifs* C_3-connected but *ifs* C_2-disconnected and *ifs* C_1-disconnected.

Example 3.21. In the Example 3.19, if we take $\mu_{f(a)}(x) = \mu_{f(b)}(x) = \nu_{f(a)}(x) = \nu_{f(b)}(x) = \frac{2}{3}$ for all $x \in [0, 1]$, then f is *ifs* C_2-connected but *ifs* C_3-disconnected.

Theorem 3.22. Let $\varphi_\psi : (X, \tau, E) \to (Y, \sigma, K)$ be a *ifs*-surjective continuous mapping and $f \in \mathbb{IFS}_X^E$. If f is a *ifs* C_1-connected, then $\varphi_\psi(f)$ is *ifs* C_1-connected.

Proof. Suppose that $\varphi_\psi(f)$ is not *ifs* C_1-connected. Then, there exist two non null *ifs*-open sets g and h in (Y, σ, K) such that

$$\begin{aligned} \varphi_\psi(f) &\sqsubseteq g \sqcup h, \\ g \sqcap h &\sqsubseteq \big(\varphi_\psi(f)\big)^{\tilde{c}}, \\ \varphi_\psi(f) \sqcap g &\neq \Phi, \\ \varphi_\psi(f) \sqcap h &\neq \Phi. \end{aligned}$$

Thus, by Theorem 2.11 we have

$$\begin{aligned} f &\sqsubseteq \varphi_\psi^{-1}(g) \sqcup \varphi_\psi^{-1}(h), \\ \varphi_\psi^{-1}(g) \sqcap \varphi_\psi^{-1}(h) &\sqsubseteq f^{\tilde{c}} \\ \varphi_\psi^{-1}(g) \sqcap f &\neq \Phi, \\ \varphi_\psi^{-1}(h) \sqcap f &\neq \Phi. \end{aligned}$$

But this contradict by hypothesis. So, $\varphi_\psi(f)$ is an *ifs* C_1-connected.

Theorem 3.23. Let $\varphi_\psi : (X, \tau, E) \to (Y, \sigma, K)$ be a *ifs*-surjective continuous mapping and $f \in \mathbb{IFS}_X^E$. If f is a *ifs* C_2-connected, then $\varphi_\psi(f)$ is *ifs* C_2-connected.

Proof. it can be proved similar way to above theorem.

Theorem 3.24. Let $\varphi_\psi : (X, \tau) \to (Y, \sigma)$ be *ifs*-continuous surjective mapping and $f \in \mathbb{IFS}_X^E$. If f is a *ifs* C_3-connected, then $\varphi_\psi(f)$ is a *ifs* C_3-connected.

Proof. Assume that, $\varphi_\psi(f)$ is not *ifs* C_3-connected. Then, there exist two non null *ifs*-open sets g and h in (Y, σ, K) such that

$$\begin{aligned} \varphi_\psi(f) &\sqsubseteq g \sqcup h, \\ g \sqcap h &\sqsubseteq \big(\varphi_\psi(f)\big)^{\tilde{c}}, \\ g &\not\sqsubseteq \big(\varphi_\psi(f)\big)^{\tilde{c}}, \\ h &\not\sqsubseteq \big(\varphi_\psi(f)\big)^{\tilde{c}}. \end{aligned}$$

By Theorem 2.11,

$$f \sqsubseteq \varphi_\psi^{-1}\big(\varphi_\psi(f)\big) \sqsubseteq \varphi_\psi^{-1}\big(g \sqcup h\big) = \varphi_\psi^{-1}(g) \sqcup \varphi_\psi^{-1}(h)$$

and

$$\varphi_\psi^{-1}\big(g \sqcap h\big) = \varphi_\psi^{-1}(g) \sqcap \varphi_\psi^{-1}(h) \sqsubseteq f^{\tilde{c}}.$$

Since, $f \sqsubseteq \varphi_\psi^{-1}\big(\varphi_\psi(f)\big)$ implies $\big(\varphi_\psi^{-1}(\varphi_\psi(f))\big)^{\tilde{c}} \sqsubseteq f^{\tilde{c}}$ and φ_ψ is a *ifs*-continuous function, so $\varphi_\psi^{-1}(g), \varphi_\psi^{-1}(h) \in \tau$. Moreover, from $g \not\sqsubseteq \big(\varphi_\psi(f)\big)^{\tilde{c}}$ and $h \not\sqsubseteq \big(\varphi_\psi(f)\big)^{\tilde{c}}$, there exist $y_1, y_2 \in Y$ such that

$$g_e(y_1) \geq 1 - \varphi_\psi(f)(k)(y_1) \tag{1}$$

$$h_e(y_2) \geq 1 - \varphi_\psi(f)(k)(y_2) \quad (2)$$

We claim that $\varphi_\psi^{-1}(g) \not\sqsubseteq f^{\tilde{c}}$ and $\varphi_\psi^{-1}(h) \not\sqsubseteq f^{\tilde{c}}$. To prove the claim, we suppose $\varphi_\psi^{-1}(g) \sqsubseteq f^{\tilde{c}}$. Clearly, this claim contradicts by (1). Similarly, $\varphi_\psi^{-1}(h) \sqsubseteq f^{\tilde{c}}$ contradicts by (2). So, $\varphi_\psi(f)$ is *ifs* C_3-connected.

Theorem 3.25. Let $\varphi_\psi : (X, \tau) \to (Y, \sigma)$ be *ifs*-continuous surjevtive mapping and $f \in \mathbb{IFS}_X^E$. If f is a *ifs* C_4-connected, then $\varphi_\psi(f)$ is a SC_4 connected.

Proof. It can be proved similarly way in Theorem 3.24.

Theorem 3.26. Let (X, τ, E) be a *ifs*-topological space, f_1 and f_2 be two *ifs* C_1-connected *ifs*-sets such that $f_1 \sqcap f_2 \neq \Phi$. Then, $f_1 \sqcup f_2$ is *ifs* C_1-connected.

Proof. It is easy.

Remark 3.27. From Theorem 3.26, we can say easily that if f_1 and f_2 be two *ifs* C_2-connected *ifs*-sets such that $f_1 \sqcap f_2 \neq \Phi$, then $f_1 \sqcup f_2$ is *ifs* C_2-connected.

Theorem 3.28. Let (X, τ, E) be a *ifs*-topological space and $\{f_k\}_{k\in\Lambda} \subseteq \mathbb{IFS}_X^E$ be family of *ifs* C_1-connected *ifs*-sets such that $f_i \sqcap f_j \neq \Phi$ for $i, j \in \Lambda$ $(i \neq j)$. Then, $\bigsqcup_{k\in\Lambda} f_k$ is is a *ifs* C_1-connected *ifs*-set.

Proof. It can be proved by using Theorem 3.26.

4 Conclusion

In this paper we introduced *ifs*-connectedness which super *ifs* connectedness and *ifs* C_i $(i = 1, 2, 3, 4)$ connectedness and presented fundamentals properties. For future works, we consider to study on *ifs* C_M and C_5 connected sets in *ifs* topological spaces.

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A Novel Supervised Learning Based on Density

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Abstract – Because prototype based classifiers are both easy and reasonable methods, there have been many studies on similarity based supervised learning. In order to detect each class region, they should not only appropriately locate the prototypes, but also deal with overfitting and instability. In this study, by considering all these criteria, we develop a new classifier method based on the prototypes selected from dense patterns. While the method determines details of the prototypes, it evades overfitting according to relation of the correct classification accuracy and the number of prototypes. Because of its similarity in point of architecture, we compare it with learning vector quantization (LVQ) method by using some synthetic and benchmark datasets. This comparison shows that our method is better than the other, and it may cause new suggestions on classification and some real applications.

Keywords – *Supervised learning, prototype classifier, learning vector quantization (LVQ), overfitting.*

1. Introduction

The simplest supervised learning method classifies samples based on similarity according to their class labels. Because prototype based classifiers are both easy and reasonable methods, there have been many studies on similarity based supervised learning. Each prototype represents a group of patterns with the same class. There are different viewpoints to prototype term. In the some of the approaches, each pattern acts as a prototype, but the approaches using fewer prototypes are more widespread. Learning Vector Quantization (LVQ) which finds prototypes using cluster analyze [7] and self-generating neural tree [19] are well-known prototype classifiers. Some other prototype classifiers are hyper-spheres [13], hyper-ellipsoids [8] and hyper-rectangles [14] based methods.

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In some studies, special terms are used instead of prototypes. For instance Expectation Maximization (EM) and Gaussian Mixture (GM) models use components instead of prototypes in estimation of the class densities [6]. Minimum enclosing axis-parallel boxes in the method of Kudo et al. [9] and Takigawa et al. [15] represent also prototypes. In two algorithms offered by Takigawa et al. [16], convex balls are regarded as prototypes. Besides some researchers proposed some methods on logical analysis of box-based data [1, 4], ball-based combinatorial classifier [2, 3, 10, 12], and support vector machines (SVM) [18]. SVM is a classifier which selects the vectors on the margin of the classes. If the selected vectors are considered as the prototypes, SVM can be regarded as a prototype based classifier. In contrast to other prototype classifiers, SVM selects the weakest patterns as prototypes. Fayed et al. [5] suggest starting with one prototype for each class, assigns patterns into prototypes, and reduces prototypes.

The commonest handicaps of all prototype based methods are determining the number and starting positions of prototypes. In order to prevent these disadvantages, the most methods are run for different numbers of prototypes. In this paper, we offer a novel supervised learning algorithm called supervised learning based on the prototypes selected from dense patterns (SLDP) which no need running more times.

The paper is organized as follows: section 2 describes the structure of new method, section 3 presents some applications on some artificial and real datasets to verify the effectiveness of the proposed method, and finally, section 4 gives the conclusions.

2. Supervised Learning Based on the Prototypes Selected from Dense Patterns

A dataset usually includes many regions which have different densities based on the distances among the patterns. Dense regions within each class can be symbolized by the prototypes.

Inspired by gravity, we suppose that each pattern has potential energy, and this energy (or weight) can be computed by using neighborhood among patterns. If we accept that each pattern has unit mass [11], the potential weight can be easily calculated as follow.

$$w_j = \sum_{i=1}^{n} \frac{1}{\left\| x_i - x_j \right\|^2}, \quad \text{for } x_i, x_j \in C_k \text{ where } k \in \{1,2,...,c\} \qquad (1)$$

where n is the number of patterns, $\left\| x_i - x_j \right\|$ is Euclidean distance between patterns x_i and x_j, C_k is class k and c is the number of classes in the dataset.

The pattern with the maximum potential weight is selected as a prototype. The location (x_i) and the potential weight (w_i) of the selected pattern are assigned the location (X_i) of the prototype and its absolute weight (W_i) respectively. According to the second step of the study, the classification process is operated by the determined prototypes. The classification effect of each class (C_k) on a new pattern (Y) is calculated by

$$\varepsilon_k = \sum_j \frac{W_j}{\|X_j - Y\|^2}, \quad \text{for all } X_j \in C_k \text{ where } k = 1, 2, \ldots, c \tag{2}$$

where *j* values list is increased as new prototypes are discovered. Then the class of the new pattern (*Z*) can be estimated by

$$Z = \arg max(\varepsilon_k), \text{ for } k = 1, 2, \ldots, c \tag{3}$$

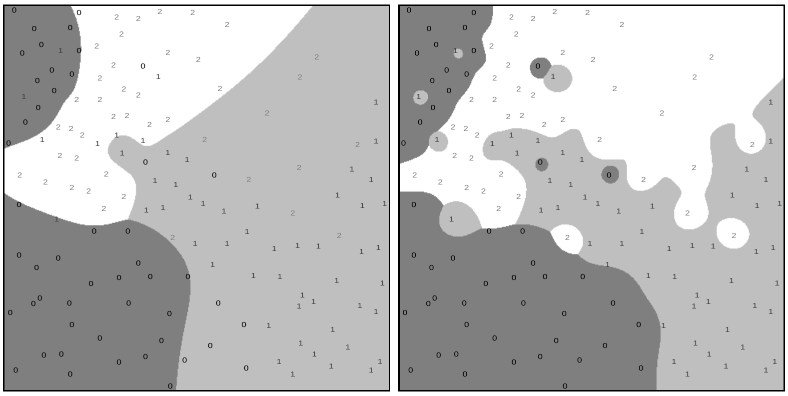
where *Z* is the estimated class of pattern *Y*. In training process, the pattern is identified as a misclassified pattern, if its desired class is not the same with *Z*. Then, the one with maximum potential weight among the misclassified patterns is selected as a new prototype. Training process stays in this loop until the gradient between number of prototypes and misclassification error get less than a predefined threshold value. Our learning algorithm consists of the following steps.

Step 1. Calculate w_j using Equation 1, and choose one prototype for each class.
Step 2. For each pattern in dataset, calculate $\varepsilon_{i=1,..,k}$ using Equation 2, and decide *Z* by Equation 3. If $Z \neq C_Y$, signify *Y* as a misclassified pattern.
Step 3. Compute the gradient between misclassification error and the number of prototypes. If it is less than a predefined threshold value, stop the algorithm.
Step 4. Set the misclassified pattern with the maximum weight as new prototype.
Step 5. Go to Step 2.

Unlike a common LVQ network, the locations and weights of the prototypes does not change in SLDP. In each iteration of the training process, only one prototype is discovered.

3. Numerical Results and Comparisons

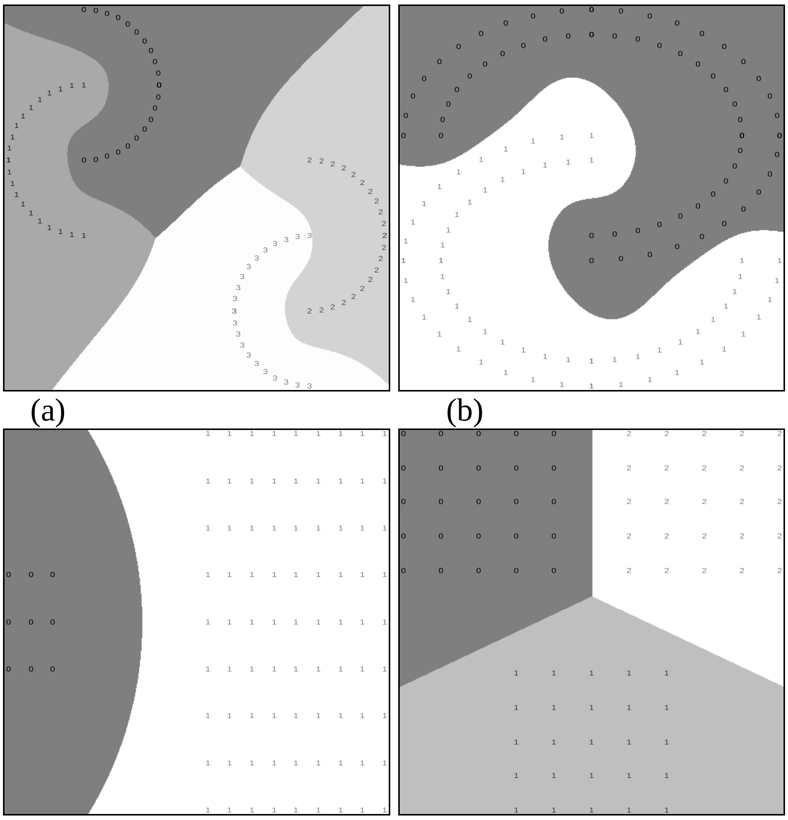
In the numerical experiments, we use some synthetic and real datasets. To show behaviors of the proposed method, five synthetic datasets are preferred as two dimensional, and to prove success of it, four real datasets are chosen from multidimensional benchmark datasets. The classification behaviors of the method are illustrated in Figure 1(a) and 1(b). In the first experiment, the method classifies an asymmetric dataset with 171 patterns. In Figure 1(a), the algorithm reaches success 81.38% with 12 prototypes by avoiding overfitting. As seen in Figure 1(b), if the algorithm does not consider overfitting, it can reach 100% success with 44 prototypes.



(a) (b)

Figure 1. The classification maps of SLDP for a dataset including outliers (a) by avoiding overfitting (b) by not avoiding overfitting.

Frequently, high success brings to mind overfitting. The success of the algorithm is 81.38% for 12 prototypes in Figure 1(a). Even though 32 new prototypes are discovered, the success increases only 18.62%. This increase values cannot be accepted as consistent.
We have also prepared four synthetic datasets which are discrete, complex, symmetric and asymmetric. Figure 2 shows the classification maps of the method for these four synthetic datasets.



(a) (b) (c) (d)

Figure 2. The classification maps of SLDP for synthetic datasets with (a) 4 symmetric classes (b) chain shaped 2 classes (c) 2 asymmetric classes (d) 3 symmetric classes

In order to compare our algorithm with some other methods, we applied it to some benchmark datasets listed in Table 1 [17].

Table 1. Summary of datasets.

Dataset	Instances	Attributes	The number of classes
Iris	150	4	3
Parkinson	195	22	2
Spect Heart	267	44	2
Statlog (Landsat)	6435	36	6

The numbers of prototypes found by SLDP for Iris, Parkinson, Spect Heart, and Statlog are 5, 6, 3, and 7 respectively. They are also used as the number of prototypes of LVQ for real datasets. For example, the new method finds 1, 2, and 2 prototypes for each class of Iris dataset. LVQ is started with the same distribution of prototypes for each class. Table 2 shows the correct classification values of LVQ and SLDP for real datasets.

Table 2. The correct classification values of LVQ and SLDP for real datasets.

Datasets	**LVQ** (%)	**SLDP** (%)
Iris	95,33	96,00
Parkinson	60,00	88,72
Spect Heart	53,18	81,27
Statlog (Landsat)	20,45	68,47

The novel method reaches the same result each time. But LVQ is run 500 times and 500 iterations for each dataset and the highest results are selected for this comparison. As seen in Table 2, the most successful method is SLDP.

There is no method which is able to reach high success for every dataset without overfitting. The relation between the misclassification error and the number of prototypes is very important in dealing with overfitting. In Figure 3, we can see the regions with overfitting for four real datasets.

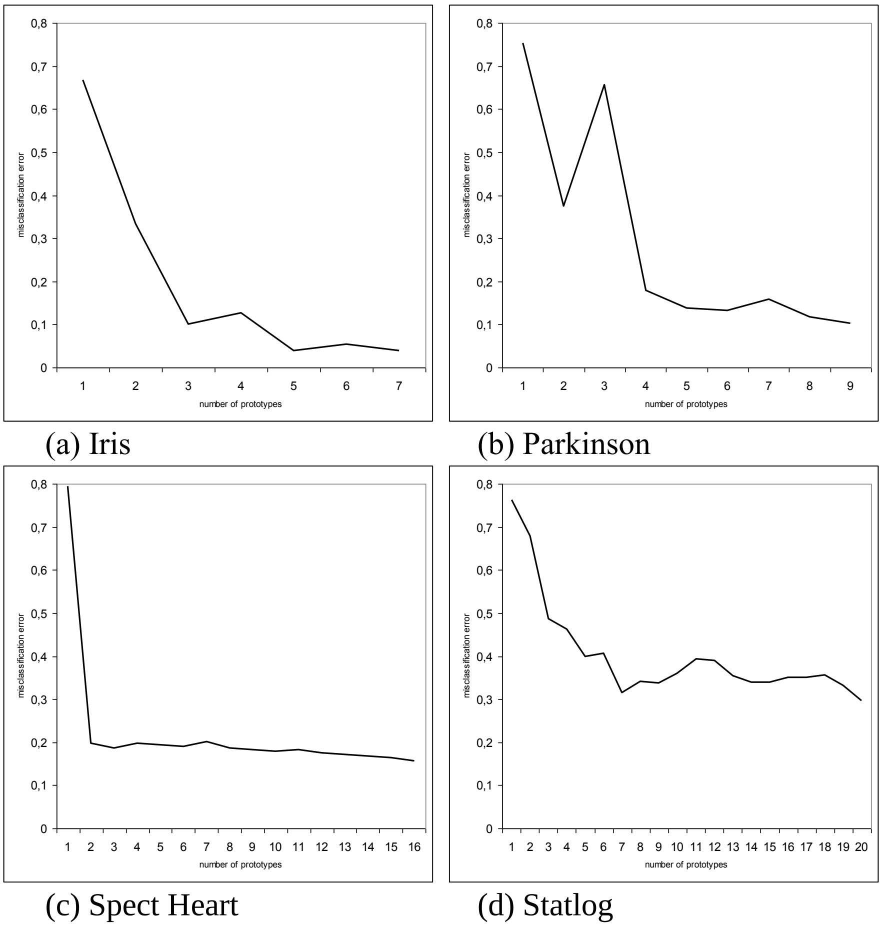


Figure 3. The relation between misclassification error and the number of prototypes for real datasets.

In Figure 3, if gradient of the relation is less than 0.01, overfitting starts. In this situation, the algorithm must be stopped. Otherwise, it will continue overfitting. Thus there is no learning after 5 prototypes for Iris, 6 prototypes for Parkinson, 3 prototypes for Spect Heart, and 7 prototypes for Statlog.

4. Conclusion

In this paper, we proposed a new method called supervised learning based on the prototypes selected from dense patterns (SLDP). While the method learns the number of prototypes and its locations by avoiding overfitting, it determines prototypes by using the potential weights of each pattern. Its other two advantages are that it does not depend on the sequence of patterns in a dataset and does not require any input parameters. The method also offers a new approach to control overfitting. Although it needs many more experiments, we hope that SLDP will be a source of inspiration for new methods.

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New Operations on Interval Neutrosophic Sets

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Abstract - An interval neutrosophic set is an instance of a neutrosophic set, which can be used in real scientific and engineering. In this paper, three new operations on interval neutrosophic sets based on the arithmetic mean, geometrical mean, and respectively harmonic mean are defined on interval neutrosophic set.

Keywords - *Neutrosophic Sets, Interval Valued Neutrosophic Sets.*

1. Introduction

In recent decades, several types of sets, such as fuzzy sets [1], interval-valued fuzzy sets [2], intuitionistic fuzzy sets [3, 4], interval-valued intuitionistic fuzzy sets [5], type 2 fuzzy sets [6, 7], type n fuzzy sets [6], and hesitant fuzzy sets [8], neutrosophic set theory [9], interval valued neutrosophic set [10] have been introduced and investigated widely. The concept of neutrosophic sets introduced by Smarandache [6, 9] is interesting and useful in modeling several real life problems.

The neutrosophic set theory (NS for short) which is a generalization of intuitionistic fuzzy set has three associated defining functions, namely the membership function, the non-membership function and the indeterminacy function which are completely independent. After the pioneering work of Smarandache [9], Wang et al.[10] introduced the notion of interval neutrosophic sets theory (INS for short) which is a special set of neutrosophic sets. This concept is characterized by a membership function, a non-membership function and indeterminacy function whose values are intervals rather than real number, INS is more powerful in dealing with vagueness and uncertainty than NS, also INS is regarded as a

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useful and practical tool for dealing with indeterminate and inconsistent information in real world .

The theories of both neutrosophic set (NS) and interval neutrosophic set (INS) have achieved great success in various areas such as medical diagnosis [11], database [12,13], topology [14], image processing [15,16,17], and decision making problem [18].

Recently, Ye [19] defined the similarity measures between INSs on the basis of the hamming and Euclidean distances, and a multicriteria decision–making method based on the similarity degree is proposed. Some set theoretic operations such as union, intersection and complement on interval neutrosophic sets have also been proposed by Wang et al. [10]. Later on, Broumi and Smarandache [20] also defined correlation coefficient of interval neutrosophic set. In 2013, Peide Liu [21] have presented some new operational laws for interval neutrosophic sets (INSs) and studied their properties and proposed some aggregation operators, include interval neutrosophic power generalized weighted aggregation (INPGWA) operator and interval neutrosophic power generalized ordered weighted aggregation (INPGOWA) operator and gave a decision making method based on these operators.

In this paper, our aim is to propose three new operations on interval neutrosophic sets (INSs) and study their properties.

Therefore, the rest of the paper is set out as follows: In Section 2, some basic definitions related to neutrosophic sets, and interval valued neutrosophic set are briefly discussed. In Section 3, three new operations on interval neutrosophic sets have been proposed and some properties of the proposed operations on interval neutrosophic sets are proved. In section 4 we concludes the paper.

2. Preliminaries

In this section, we mainly recall some notions related to neutrosophic sets, and interval neutrosophic sets relevant to the present work. See especially [9, 10, 21] for further details and background.

Definition 2.1 ([9]). Let U be an universe of discourse; then the neutrosophic set A is an object having the form $A = \{< x: T_A(x), I_A(x), F_A(x)>, x \in U\}$, where the functions T, I, F : $U \rightarrow]^-0,1^+[$ define respectively the degree of membership, the degree of indeterminacy, and the degree of non-membership of the element $x \in U$ to the set A with the condition:

$$^-0 \leq T_A(x)+ I_A(x)+ F_A(x)\leq 3^+. \quad (1)$$

From philosophical point of view, the neutrosophic set takes the value from real standard or non-standard subsets of $]^-0,1^+[$.So instead of $]^-0,1^+[$ we need to take the interval [0,1] for technical applications, because $]^-0,1^+$ [will be difficult to apply in the real applications such as in scientific and engineering problems.

Definition 2.2 [10]. Let X be a space of points (objects) with generic elements in X denoted by x. An interval neutrosophic set (for short INS) A in X is characterized by truthmembership function $T_A(x)$, indeteminacy-membership function $I_A(x)$ and falsity-membership function $F_A(x)$. For each point x in X, we have that

$$T_A(x), I_A(x), F_A(x) \in [0, 1]$$

For convenience, we can use x =([T^L , T^U], [I^L , I^U], [F^L , F^U]) to represent an element in INS.

Remark 1. An INS is clearly a NS.

Definition 2.3 [10]. Let A ={([T_A^L , T_A^U] , [I_A^L , I_A^U] , [F_A^L , F_A^U])}

i. An INS A is empty if $T_A^L = T_A^U = 0$, $I_A^L = I_A^U = 1$, $F_A^L = F_A^U = 1$, for all x in A.
ii. Let $\underline{0}$ = <0, 1 ,1> and $\underline{1}$ = <1, 0 ,0>

In the following, we introduce some basic concepts related to INSs.

Definition 2.4 [21] Let $\tilde{n}_1$ ={([T_1^L , T_1^U] , [I_1^L , I_1^U] , [F_1^L , F_1^U])} and $\tilde{n}_2$ = {([T_2^L , T_2^U] , [I_2^L , I_2^U] , [F_2^L , F_2^U])} be two INSs.

i. $\tilde{n}_1 \cup \tilde{n}_2$ =[max(T_1^L, T_2^L), max(T_1^U ,T_2^U)], [min (I_1^L, I_2^L),min(I_1^U, I_2^U)] , [min (F_1^L, F_2^L),min(F_1^U, F_2^U)] }
ii. $\tilde{n}_1 \cap \tilde{n}_2$ =[min(T_1^L, T_2^L), min(T_1^U ,T_2^U)], [max (I_1^L, I_2^L)max(I_1^U, I_2^U)] , [max (F_1^L, F_2^L),max(F_1^U, F_2^U)] }

Definition 2.5. Let $\tilde{n}_1$ ={([T_1^L , T_1^U] , [I_1^L , I_1^U] , [F_1^L , F_1^U])} and $\tilde{n}_2$ = {([T_2^L , T_2^U] , [I_2^L , I_2^U] , [F_2^L , F_2^U])} be two INSs, then the operational laws are defined as follows.

i. $\tilde{n}^c = [F^L, F^U], [1 - I^L, 1 - I^U], [T^L, T^U]$
ii. $\tilde{n}_1 \oplus \tilde{n}_2 = ([T_1^L + T_2^L - T_1^L T_2^L, T_1^U + T_2^U - T_1^U T_2^U], [I_1^L I_2^L, I_1^U I_2^U], [F_1^L F_2^L, F_1^U F_2^U])$
iii. $\tilde{n}_1 \otimes \tilde{h}_2 = [T_1^L T_2^L, T_1^U T_2^U)], [I_1^L + I_2^L - I_2^L I_2^L, I_1^U + I_2^U - I_1^U I_2^U], [F_1^L + F_2^L - F_2^L F_2^L, F_1^U + F_2^U - F_1^U F_2^U]$
iv. $\lambda \tilde{n} = \{([1 - (1 - T^L)^\lambda, 1 - (1 - T^U)^\lambda], [(I^L)^\lambda, (I^U)^\lambda], [(F^L)^\lambda, (F^U)^\lambda])\}$.

3. Three New Operations on INSs

Definition 3.1 Let $\tilde{n}_1$ and $\tilde{n}_2$ two interval neutrosophic set, we propose the following operations on INSs as follows:

$$\tilde{n}_1 \ @ \ \tilde{n}_2 = \{([\frac{T_1^L + T_2^L}{2}, \frac{T_1^U + T_2^U}{2}], [\frac{I_1^L + I_2^L}{2}, \frac{I_1^U + I_2^U}{2}], [\frac{F_1^L + F_2^L}{2}, \frac{F_1^U + F_2^U}{2}],$$

where

$< T_1, I_1, F_1 > \in \tilde{n}_1, < T_2, I_2, F_2 > \in \tilde{n}_2$ }

$\tilde{n}_1$ \$ $\tilde{n}_2$ = {([$\sqrt{T_1^L\, T_2^L}$, $\sqrt{T_1^U\, T_2^U}$], [$\sqrt{I_1^L\, I_2^L}$, $\sqrt{I_1^U\, I_2^U}$] , [$\sqrt{F_1^L\, F_2^L}$, $\sqrt{F_1^U\, F_2^U}$] }

where

$< T_1, I_1, F_1 > \in \tilde{n}_1$,$< T_2, I_2, F_2 > \in \tilde{n}_2$ }

$\tilde{n}_1$ # $\tilde{n}_2$ = {([$\frac{2\,T_1^L\,T_2^L}{T_1^L+T_2^L}$, $\frac{2\,T_1^U\,T_2^U}{T_1^U+T_2^U}$], [$\frac{2\,I_1^L\,I_2^L}{I_1^L+I_2^L}$, $\frac{2\,I_1^U\,I_2^U}{I_1^U+I_2^U}$] , [$\frac{2\,F_1^L\,F_2^L}{F_1^L+F_2^L}$, $\frac{2\,F_1^U\,F_2^U}{F_1^U+F_2^U}$]} ,

where

$< T_1, I_1, F_1 > \in \tilde{n}_1$,$< T_2, I_2, F_2 > \in \tilde{n}_2$ }

With

T_1= [T_1^L , T_1^U] , I_1= [I_1^L , I_1^U] , F_1= [F_1^L , F_1^U] and T_2= [T_2^L , T_2^U] , I_2= [I_2^L , I_2^U] , F_2= [F_2^L,F_2^U]

Obviously, for every two $\tilde{n}_1$ and $\tilde{n}_2$,($\tilde{n}_1$@$\tilde{n}_2$), ($\tilde{n}_1$\$$\tilde{n}_2$) and ($\tilde{n}_1$#$\tilde{n}_2$) are also INSs.

Example 3.2 Let $\tilde{n}_1$(x)= {([0.2, 0.3], [0.5, 0.6] , [0.2 , 0.4]),([0.5, 0.8], [0.1, 0.2], [0.6 ,0.1])} and $\tilde{n}_2$ (x)= {([0.4, 0.6],[0.3, 0.4], [0.3 , 0.5]), ([0.3, 0.5], [0.1, 0.2], [0.5 ,0.1]) be two interval neutrosophic sets. Then we have

($\tilde{n}_1$ @ $\tilde{n}_2$) = {([0.3, 0.45], [0.4, 0.5],[0.25 ,0.45]),(b, [0.4, 0.65], [0.1, 0.2], [0.55 , 0.1])}

($\tilde{n}_1$ \$ $\tilde{n}_2$) ={(a, [0.28, 0.42], [0.38, 0.48], [0.24 ,0.44]),(b, [0.38, 0.63], [0.1, 0.2], [0.55 ,0.1])}

($\tilde{n}_1$#$\tilde{n}_2$) = {(a, [0.26, 0.4], [0.37, 0.48] , [0.24 , 0.44]), (b, [0.37, 0.61], [0.1, 0.2], [0.54 , 0.1])}

With these operations, several results follow.

Theorem 3.4 For $\tilde{n}_1, \tilde{n}_2 \in$ INSs(X),

(i) $\tilde{n}_1$@ $\tilde{n}_2$=$\tilde{n}_2$ @$\tilde{n}_1$;
(ii) $\tilde{n}_1$\$ $\tilde{n}_2$=$\tilde{n}_2$ \$ $\tilde{n}_1$;
(iii) $\tilde{n}_1$# $\tilde{n}_2$=$\tilde{n}_2$# $\tilde{n}_1$;

Proof. These also follow from definitions.

Theorem 3.5 For $\tilde{n}_1, \tilde{n}_2 \in$ INSs(X), (${\tilde{n}_1}^c$@ ${\tilde{n}_2}^c$) ^{c} = $\tilde{n}_1$@$\tilde{n}_2$.

Proof. $\tilde{n}_1$ @ $\tilde{n}_2$ = {([$\frac{T_1^L+T_2^L}{2}$, $\frac{T_1^U+T_2^U}{2}$], [$\frac{I_1^L+I_2^L}{2}$, $\frac{I_1^U+I_2^U}{2}$] ,[$\frac{F_1^L+F_2^L}{2}$, $\frac{F_1^U+F_2^U}{2}$]

where

$< T_1, I_1, F_1 > \in \tilde{n}_1$,$< T_2, I_2, F_2 > \in \tilde{n}_2$ }

${\tilde{n}_1}^c$ ={([F_1^L , F_1^U] , [1- I_1^L , $1 - I_1^U$] , [T_1^L , T_1^U])}

$\tilde{n}_2{}^c$ = {([F_2^L , F_2^U] , [1- I_2^L , $1 - I_2^U$] , [T_2^L , T_2^U])}

$\tilde{n}_1{}^c @ \ \tilde{n}_2{}^c$ ={ [$\frac{F_1^L + F_2^L}{2}$, $\frac{F_1^U + F_2^U}{2}$], [$\frac{(1-I_1^L)+(1-I_2^L)}{2}$, $\frac{(1-I_1^U)+(1-I_2^U)}{2}$] ,[$\frac{T_1^L + T_2^L}{2}$, $\frac{T_1^U + T_2^U}{2}$] }

$$(\tilde{n}_1{}^c @ \ \tilde{n}_2{}^c)^c = \left([\frac{F_1^L + F_2^L}{2} , \frac{F_1^U + F_2^U}{2}], [\frac{(1-I_1^L)+(1-I_2^L)}{2} , \frac{(1-I_1^U)+(1-I_2^U)}{2}] , [\frac{T_1^L + T_2^L}{2} , \frac{T_1^U + T_2^U}{2}] \right)^c$$

$$= \left([\frac{T_1^L + T_2^L}{2} , \frac{T_1^U + T_2^U}{2}], [1 - \frac{(1-I_1^L)+(1-I_2^L)}{2} , 1 - \frac{(1-I_1^U)+(1-I_2^U)}{2}] , [\frac{F_1^L + F_2^L}{2} , \frac{F_1^U + F_2^U}{2}] \right)$$

$$= \left([\frac{T_1^L + T_2^L}{2} , \frac{T_1^U + T_2^U}{2}], [\frac{2-[2-(I_1^L + I_2^L)]}{2} , \frac{2-[2-(I_1^U + I_2^U)]}{2}] , [\frac{F_1^L + F_2^L}{2} , \frac{F_1^U + F_2^U}{2}] \right)$$

$$= \left([\frac{T_1^L + T_2^L}{2} , \frac{T_1^U + T_2^U}{2}], [\frac{(I_1^L + I_2^L)}{2} , \frac{(I_1^U + I_2^U)}{2}] , [\frac{F_1^L + F_2^L}{2} , \frac{F_1^U + F_2^U}{2}] \right)$$

Then $(\tilde{n}_1{}^c @ \ \tilde{n}_2{}^c)^c = \tilde{n}_1 @ \ \tilde{n}_2$

This proves the theorem.

Note 1: One can easily verify that

(i) $(\tilde{n}_1{}^c \$ \ \tilde{n}_2{}^c)^c \neq \tilde{n}_1 \$ \ \tilde{n}_2$
(ii) $(\tilde{n}_1{}^c \# \ \tilde{n}_2{}^c)^c \neq \tilde{n}_1 \# \tilde{n}_2$

Theorem 3.6 For $\tilde{n}_1, \tilde{n}_2$ and $\tilde{n}_3$ ∈ INSs(X), we have the following identities:

(i) $(\tilde{n}_1 \cup \tilde{n}_2) @ \tilde{n}_3 = (\tilde{n}_1 @ \tilde{n}_3) \cup (\tilde{n}_2 @ \tilde{n}_3)$;
(ii) $(\tilde{n}_1 \cap \tilde{n}_2) @ \tilde{n}_3 = (\tilde{n}_1 @ \tilde{n}_3) \cap (\tilde{n}_2 @ \tilde{n}_3)$
(iii) $(\tilde{n}_1 \cup \tilde{n}_2) \$ \tilde{n}_3 = (\tilde{n}_1 \$ \tilde{n}_3) \cup (\tilde{n}_2 \$ \tilde{n}_3)$;
(iv) $(\tilde{n}_1 \cap \tilde{n}_2) \$ \tilde{n}_3 = (\tilde{n}_1 \$ \tilde{n}_3) \cap (\tilde{n}_2 \$ \tilde{h}_3)$;
(v) $((\tilde{n}_1 \cup \tilde{n}_2)) \# \tilde{n}_3 = (\tilde{n}_1 \# \tilde{n}_3) \cup (\tilde{n}_2 \# \tilde{n}_3)$;
(vi) $(\tilde{n}_1 \cap \tilde{n}_2) \# \tilde{n}_3 = (\tilde{n}_1 \# \tilde{n}_3) \cap (\tilde{n}_2 \# \tilde{n}_3)$;
(vii) $(\tilde{n}_1 @ \tilde{n}_2) \oplus \tilde{n}_3 = (\tilde{n}_1 \oplus \tilde{n}_3) @ (\tilde{n}_2 \oplus \tilde{n}_3)$;
(viii) $(\tilde{n}_1 @ \tilde{n}_2) \otimes \tilde{n}_3 = (\tilde{n}_1 \otimes \tilde{n}_3) @ (\tilde{n}_2 \otimes \tilde{n}_3)$

Proof . We prove (i), (iii), (v), (vii) and (ix), results (ii), (iv), (vi), (viii) and (x) can be proved analogously

(i) Using definitions in 2.4, 2.5 and 3.1, we have

$\tilde{n}_1$ ={([T_1^L , T_1^U] , [I_1^L , I_1^U] , [F_1^L , F_1^U])}

$\tilde{n}_2$ = {([T_2^L , T_2^U] , [I_2^L , I_2^U] , [F_2^L , F_2^U])}

$(\tilde{n}_1 \cup \tilde{n}_2) @ \tilde{n}_3$ = {([max(T_1^L, T_2^L) ,max(T_1^U, T_2^U)] , [min (I_1^L, I_2^L) ,min(I_1^U,I_2^U)] ,

[min(F_1^L , F_2^L),min(F_1^U,F_2^U)])} @ {([T_3^L , T_3^U] , [I_3^L , I_3^U] , [F_3^L , F_3^U])}

$=\{[\frac{\max(T_1^L,T_2^L)+T_3^L}{2}, \frac{\max(T_1^U,T_2^U)+T_3^U}{2}], [\frac{\min(I_1^L,I_2^L)+I_3^L}{2}, \frac{\min(I_1^U,I_2^U)+I_3^U}{2}],$
$[\frac{\min(F_1^L,F_2^L)+F_3^L,}{2}, \frac{\min(F_1^U,F_2^U)+F_3^U}{2}]\}$

$=\{[\max(\frac{T_1^L+T_3^L}{2}, \frac{T_2^L+T_3^U}{2}), \max(\frac{T_1^U+T_3^U}{2}, \frac{T_2^U+T_3^U}{2})], [\min(\frac{I_1^L+I_3^L}{2}, \frac{I_2^L+I_3^L}{2}),$
$\min(\frac{I_1^U+I_3^U}{2}, \frac{I_2^U+I_3^U}{2})], [\min(\frac{F_1^L+F_3^L}{2}, \frac{F_2^L+F_3^L}{2}), \min(\frac{F_1^U+F_3^U}{2}, \frac{F_2^U+F_3^U}{2}]\}$

$= (\tilde{n}_1 @ \tilde{n}_3) \cup (\tilde{n}_2 @ \tilde{n}_3)$

This proves (i)

(iii) From definitions in 2.4, 2.5 and 3.1, we have

$(\tilde{n}_1 \cup \tilde{n}_2) \$ \tilde{n}_3 = \{([\max(T_1^L, T_2^L), \max(T_1^U, T_2^U)], [\min(I_1^L, I_2^L), \min(I_1^U, I_2^U)],$
$[\min(F_1^L, F_2^L), \min(F_1^U, F_2^U)])\} \$ \{([T_3^L, T_3^U], [I_3^L, I_3^U], [F_3^L, F_3^U])\}$

$= \{[\sqrt{max(T_1^L, T_2^L)\, T_3^L}, \sqrt{\max(T_1^U, T_2^U)\, T_3^U}], [\sqrt{\min(I_1^L, I_2^L)\, I_3^L},$
$\sqrt{\min(I_1^U, I_2^U)\, I_3^U}], [\sqrt{\min(F_1^L, F_2^L)\, F_3^L}, \sqrt{\min(F_1^U, F_2^U)\, F_3^U}]\}.$

$=\{[\max(\sqrt{T_1^L\, T_3^L}, \sqrt{T_2^L\, T_3^L}), \max(\sqrt{T_1^U\, T_3^U}, \sqrt{T_2^U\, T_3^U})], [\min(\sqrt{I_1^L\, I_3^L}, \sqrt{I_2^L\, I_3^L}),$
$\min(\sqrt{I_1^U\, I_3^U}, \sqrt{I_2^U\, I_3^U})], [\min(\sqrt{F_1^L F_3^L}, \sqrt{F_2^L\, F_3^L}), \min(\sqrt{F_1^U\, F_3^U}, \sqrt{F_2^U\, F_3^U})]\}$

$=(\tilde{n}_1 \$ \tilde{n}_3) \cup (\tilde{n}_2 \$ \tilde{n}_3)$

This proves (iii).

(v) Using definitions in 2.4, 2.5 and 3.1, we have

$((\tilde{n}_1 \cup \tilde{n}_2)) \# \tilde{n}_3 = \{([\max(T_1^L, T_2^L), \max(T_1^U, T_2^U)], [\min(I_1^L, I_2^L), \min(I_1^U, I_2^U)],$
$[\min(F_1^L, F_2^L), \min(F_1^U, F_2^U)])\} \# \{([T_3^L, T_3^U], [I_3^L, I_3^U], [F_3^L, F_3^U])\}$

$= \{[\frac{2\max(T_1^L,T_2^L)\, T_3^L}{\max(T_1^L,T_2^L)+T_3^L}, \frac{2\max(T_1^U,T_2^U)\, T_3^U}{\max(T_1^U,T_2^U)+T_3^U}], [\frac{2\min(I_1^L,I_2^L)\, I_3^L}{\min(I_1^L,I_2^L)+I_3^L}, \frac{2\min(I_1^U,I_2^U)\, I_3^U}{\min(I_1^U,I_2^U)+I_3^U}],$
$[\frac{2\min(F_1^L,F_2^L)\, F_3^L}{\min(F_1^L,F_2^L)+F_3^L}, \frac{2\min(F_1^U,F_2^U)\, F_3^U}{\min(F_1^U,F_2^U)+F_3^U}]\}$

$=\{[\max(\frac{2\, T_1^L\, T_3^L}{T_1^L+T_3^L}, \frac{2\, T_2^L\, T_3^L}{T_2^L+T_3^L}), \max(\frac{2\, T_1^U\, T_3^U}{T_1^U+T_3^U}, \frac{2\, T_2^U\, T_3^U}{T_2^U+T_3^U})],$
$[\min(\frac{2\, I_1^L I_3^L}{I_1^L+I_3^L}, \frac{2\, I_2^L I_3^L}{I_2^L+I_3^L}), \min(\frac{2\, I_1^U I_3^U}{I_1^U+I_3^U}, \frac{2\, I_2^U I_3^U}{I_2^U+I_3^U})], [\min(\frac{2\, F_1^L F_3^L}{F_1^L+F_3^L}, \frac{2\, F_2^L F_3^L}{F_2^L+F_3^L}),$
$\min(\frac{2\, F_1^U F_3^U}{F_1^U+F_3^U}, \frac{2\, F_2^U F_3^U}{F_2^U+F_3^U})]\}$

$=(\tilde{n}_1 \# \tilde{n}_3) \cup (\tilde{n}_2 \# \tilde{n}_3)$

This proves (v)

(vii) Using definitions in 2.4, 2.5 and 3.1, we have

$(\tilde{n}_1 @ \tilde{n}_2) \oplus \tilde{n}_3 = (\tilde{n}_1 \oplus \tilde{n}_3) @ (\tilde{n}_2 \oplus \tilde{n}_3)$;

$\tilde{n}_1 = \{([T_1^L, T_1^U], [I_1^L, I_1^U], [F_1^L, F_1^U])\}$

$\tilde{n}_2 = \{([T_2^L, T_2^U], [I_2^L, I_2^U], [F_2^L, F_2^U])\}$

$\tilde{n}_3 = \{([T_3^L, T_3^U], [I_3^L, I_3^U], [F_3^L, F_3^U])\}$

$= \{([\frac{T_1^L + T_2^L}{2}, \frac{T_1^U + T_2^U}{2}], [\frac{I_1^L + I_2^L}{2}, \frac{I_1^U + I_2^U}{2}], [\frac{F_1^L + F_2^L}{2}, \frac{F_1^U + F_2^U}{2}])\}$
$\oplus \{([T_3^L, T_3^U], [I_3^L, I_3^U], [F_3^L, F_3^U])\}$

$= \{ [\frac{T_1^L + T_2^L}{2} + T_3^L - \frac{T_1^L + T_2^L}{2} T_3^L, \frac{T_1^U + T_2^U}{2} + T_3^U - \frac{T_1^U + T_2^U}{2} T_3^U], [\frac{I_1^L + I_2^L}{2} I_3^L, \frac{I_1^U + I_2^U}{2} I_3^U],$
$[\frac{F_1^L + F_2^L}{2} F_3^L, \frac{F_1^U + F_2^U}{2} F_3^U] \}$

$= \{ [\frac{(T_1^L + T_3^L - T_1^L T_3^L) + (T_2^L + T_3^L - T_2^L T_3^L)}{2}, \frac{(T_1^U + T_3^U - T_1^U T_3^U) + (T_2^U + T_3^U - T_2^U T_3^U)}{2}], [\frac{I_1^L + I_2^L}{2} I_3^L, \frac{I_1^U + I_2^U}{2} I_3^U],$
$[\frac{F_1^L + F_2^L}{2} F_3^L, \frac{F_1^U + F_2^U}{2} F_3^U] \}$

$= (\tilde{n}_1 \oplus \tilde{n}_3) @ (\tilde{n}_2 \oplus \tilde{n}_3)$

This proves (vi)

Theorem 3.7. For $\tilde{n}_1$and $\tilde{n}_2 \in$ INSs(X), we have the following identities:

(i) $(\tilde{n}_1 \oplus \tilde{n}_2) \cap (\tilde{n}_1 \otimes \tilde{n}_2) = \tilde{n}_1 \otimes \tilde{n}_2$;
(ii) $(\tilde{n}_1 \oplus \tilde{n}_2) \cup (\tilde{n}_1 \otimes \tilde{n}_2) = \tilde{n}_1 \oplus \tilde{n}_2$;
(iii) $(\tilde{n}_1 \oplus \tilde{n}_2) \cap (\tilde{n}_1 @ \tilde{n}_2) = \tilde{n}_1 @ \tilde{n}_2$;
(iv) $(\tilde{n}_1 \oplus \tilde{n}_2) \cup (\tilde{n}_1 @ \tilde{n}_2) = \tilde{n}_1 \oplus \tilde{n}_2$;
(v) $(\tilde{n}_1 \otimes \tilde{n}_2) \cap (\tilde{n}_1 @ \tilde{n}_2) = \tilde{n}_1 \otimes \tilde{n}_2$;
(vi) $(\tilde{n}_1 \otimes \tilde{n}_2) \cup (\tilde{n}_1 @ \tilde{n}_2) = \tilde{n}_1 @ \tilde{n}_2$;
(vii) $(\tilde{n}_1 \oplus \tilde{n}_2) \cap (\tilde{n}_1 \$ \tilde{n}_2) = \tilde{n}_1 \$ \tilde{n}_2$;
(viii) $(\tilde{n}_1 \oplus \tilde{n}_2) \cup (\tilde{n}_1 \$ \tilde{n}_2) = \tilde{n}_1 \oplus \tilde{n}_2$;
(ix) $(\tilde{n}_1 \otimes \tilde{n}_2) \cap (\tilde{n}_1 \$ \tilde{n}_2) = \tilde{n}_1 \otimes \tilde{n}_2$;
(x) $(\tilde{n}_1 \otimes \tilde{n}_2) \cup (\tilde{n}_1 \$ \tilde{n}_2) = \tilde{n}_1 \$ \tilde{n}_2$;
(xi) $(\tilde{n}_1 \oplus \tilde{n}_2) \cap (\tilde{n}_1 \# \tilde{n}_2) = \tilde{n}_1 \# \tilde{n}_2$;
(xii) $(\tilde{n}_1 \oplus \tilde{n}_2) \cup (\tilde{n}_1 \# \tilde{n}_2) = \tilde{n}_1 \oplus \tilde{n}_2$;
(xiii) $(\tilde{n}_1 \otimes \tilde{n}_2) \cap (\tilde{n}_1 \# \tilde{n}_2) = \tilde{n}_1 \otimes \tilde{n}_2$;
(xiv) $(\tilde{n}_1 \otimes \tilde{n}_2) \cup (\tilde{n}_1 \# \tilde{n}_2) = \tilde{n}_1 \# \tilde{n}_2$

Proof .We prove (i), (iii), (v), (vii), (ix), (xi) and (xii), other results can be proved analogously.

(i) From definitions in 2.4, 2.5 and 3.1, we have

$(\tilde{n}_1 \oplus \tilde{n}_2) \cap (\tilde{n}_1 \otimes \tilde{n}_2)$

$\tilde{n}_1 = \{([T_1^L, T_1^U], [I_1^L, I_1^U], [F_1^L, F_1^U])\}$

$\tilde{n}_2 = \{([T_2^L, T_2^U], [I_2^L, I_2^U], [F_2^L, F_2^U])\}$

$=([T_1^L + T_2^L - T_1^L T_2^L, T_1^U + T_2^U - T_1^U T_2^U], [I_1^L I_2^L, I_1^U I_2^U], [F_1^L F_2^L, F_1^U F_2^U]) \cap \{[T_1^L T_2^L, T_1^U T_2^U], [I_1^L + I_2^L - I_2^L I_2^L, I_1^U + I_2^U - I_1^U I_2^U], [F_1^L + F_2^L - F_2^L F_2^L, F_1^U + F_2^U - F_1^U F_2^U]\}$

$=\{[\min(T_1^L + T_2^L - T_1^L T_2^L, T_1^L T_2^L), \min(T_1^U + T_2^U - T_1^U T_2^U, T_1^U T_2^U)],$
$[\max(I_1^L I_2^L, I_1^L + I_2^L - I_2^L I_2^L), \max(I_1^U I_2^U, I_1^U + I_2^U - I_1^U I_2^U)],$
$[\max(F_1^L F_2^L, F_1^L + F_2^L - F_2^L F_2^L), \max(F_1^U F_2^U, F_1^U + F_2^U - F_1^U F_2^U)]\}$

$=[T_1^L T_2^L, T_1^U T_2^U)], [I_1^L + I_2^L - I_2^L I_2^L, I_1^U + I_2^U - I_1^U I_2^U], [F_1^L + F_2^L - F_2^L F_2^L, F_1^U + F_2^U - F_1^U F_2^U]$

$=\tilde{n}_1 \otimes \tilde{n}_2$

This proves (i)

(iii) Using definitions in 2.4, 2.5 and 3.1, we have

$(\tilde{n}_1 \oplus \tilde{n}_2) \cap (\tilde{n}_1 @ \tilde{n}_2) = \tilde{n}_1 @ \tilde{n}_2;$

$=\{([T_1^L + T_2^L - T_1^L T_2^L, T_1^U + T_2^U - T_1^U T_2^U], [I_1^L I_2^L, I_1^U I_2^U], [F_1^L F_2^L, F_1^U F_2^U]) \cap ([\frac{T_1^L + T_2^L}{2}, \frac{T_1^U + T_2^U}{2}], [\frac{I_1^L + I_2^L}{2}, \frac{I_1^U + I_2^U}{2}], [\frac{F_1^L + F_2^L}{2}, \frac{F_1^U + F_2^U}{2}])$

$= \{[\min(T_1^L + T_2^L - T_1^L T_2^L, \frac{T_1^L + T_2^L}{2}), \min(T_1^U + T_2^U - T_1^U T_2^U, \frac{T_1^U + T_2^U}{2}])],$
$[\max(I_1^L I_2^L, \frac{I_1^L + I_2^L}{2}), \max(I_1^U I_2^U, \frac{I_1^U + I_2^U}{2})], [\max(F_1^L F_2^L, \frac{F_1^L + F_2^L}{2}), \max(F_1^U F_2^U, \frac{F_1^U + F_2^U}{2})]\}.$

$=\{[\frac{T_1^L + T_2^L}{2}, \frac{T_1^U + T_2^U}{2}], [\frac{I_1^L + I_2^L}{2}, \frac{I_1^U + I_2^U}{2}], [\frac{F_1^L + F_2^L}{2}, \frac{F_1^U + F_2^U}{2}]$

$= \tilde{n}_1 @ \tilde{n}_2$

This proves (iii).

(v) From definitions in 2.4, 2.5 and 3.1, we have

$(\tilde{n}_1 \otimes \tilde{n}_2) \cap (\tilde{n}_1 @ \tilde{n}_2) = \tilde{n}_1 \otimes \tilde{n}_2;$

$\{[T_1^L T_2^L, T_1^U T_2^U)], [I_1^L + I_2^L - I_2^L I_2^L, I_1^U + I_2^U - I_1^U I_2^U],$
$[F_1^L + F_2^L - F_2^L F_2^L, F_1^U + F_2^U - F_1^U F_2^U]\} \cap \{[\frac{T_1^L + T_2^L}{2}, \frac{T_1^U + T_2^U}{2}], [\frac{I_1^L + I_2^L}{2}, \frac{I_1^U + I_2^U}{2}],$
$[\frac{F_1^L + F_2^L}{2}, \frac{F_1^U + F_2^U}{2}]\}.$

$=\{[\min(T_1^L T_2^L, \frac{T_1^L + T_2^L}{2}), \min(T_1^U T_2^U, \frac{T_1^U + T_2^U}{2})], [\max(I_1^L + I_2^L - I_2^L I_2^L, \frac{I_1^L + I_2^L}{2}),$
$\max(I_1^U + I_2^U - I_1^U I_2^U, \frac{I_1^U + I_2^U}{2})], [\max(F_1^L + F_2^L - F_2^L F_2^L, \frac{F_1^L + F_2^L}{2}),$
$\max(F_1^U + F_2^U - F_2^U F_2^U, \frac{F_1^U + F_2^U}{2})]\}.$

$= \{[\ T_1^L\, T_2^L\ ,\ T_1^U\, T_2^U\ ], [I_1^L + I_2^L - I_2^L I_2^L\ ,\ I_1^U + I_2^U - I_1^U I_2^U],$
$[\ F_1^L + F_2^L - F_2^L F_2^L\ ,\ F_1^U + F_2^U - F_2^U F_2^U\ ]\ \}$

$= \tilde{n}_1 \otimes \tilde{n}_2$

This proves (v).

(vii) Using definitions in 2.4, 2.5 and 3.1, we have

$(\tilde{n}_1 \oplus \tilde{n}_2) \cap (\ \tilde{n}_1 \$\ \tilde{n}_2) = \tilde{n}_1\ \$\ \tilde{n}_2$

$= ([T_1^L + T_2^L - T_1^L T_2^L\ , T_1^U + T_2^U - T_1^U T_2^U], [\ I_1^L\ I_2^L\ , I_1^U\ I_2^U]\ , [F_1^L F_2^L\ , F_1^U F_2^U])$
$\cap \{(\ [\sqrt{T_1^L\ T_2^L}, \sqrt{T_1^U\ T_2^U}\ ], [\sqrt{I_1^L\ I_2^L}, \sqrt{I_1^U\ I_2^U}\ ]\ , [\sqrt{F_1^L\ F_2^L}, \sqrt{F_1^U\ F_2^U}\ ]\}$

$= \{\ [\ \min\ (T_1^L + T_2^L - T_1^L T_2^L\ , \sqrt{T_1^L\ T_2^L}\ )\ , \min\ (T_1^U + T_2^U - T_1^U T_2^U\ , \sqrt{T_1^U\ T_2^U}\ )],$
$[\max\ (\ I_1^L\ I_2^L, \sqrt{I_1^L\ I_2^L}\ )\ , \max\ (I_1^U\ I_2^U\ , \sqrt{I_1^U\ I_2^U}\ )\ ], [\max\ (\ F_1^L\ F_2^L, \sqrt{F_1^L\ F_2^L}\ )\ ,$
$\max\ (F_1^U\ F_2^U\ , \sqrt{F_1^U\ F_2^U}\ )\ ]\}$

$= \{[\sqrt{T_1^L\ T_2^L}\ , \sqrt{T_1^U\ T_2^U}\ ]\ , [\ \sqrt{I_1^L\ I_2^L}\ , \sqrt{I_1^U\ I_2^U}\ ]\ , [\ \sqrt{F_1^L\ F_2^L}\ , \sqrt{F_1^U\ F_2^U}\ ]\ \}$

$= \tilde{n}_1\ \$\ \tilde{n}_2$

This proves (vii)

(ix) From definitions in 2.4, 2.5 and 3.1, we have

$(\tilde{n}_1 \otimes \tilde{n}_2) \cap (\ \tilde{n}_1 \$\ \tilde{n}_2) = \tilde{n}_1 \otimes \tilde{n}_2;$

$= \{[T_1^L\ T_2^L\ ,\ T_1^U\ T_2^U)], [\ I_1^L + I_2^L - I_2^L I_2^L\ , I_1^U + I_2^U - I_1^U I_2^U], [F_1^L + F_2^L - F_2^L F_2^L\ , F_1^U + F_2^U - F_1^U F_2^U\ ]\}$
$\cap \{[\ \sqrt{T_1^L\ T_2^L}\ , \sqrt{T_1^U\ T_2^U}\ ]\ , [\ \sqrt{I_1^L\ I_2^L}\ , \sqrt{I_1^U\ I_2^U}\ ]\ , [\ \sqrt{F_1^L\ F_2^L}\ , \sqrt{F_1^U\ F_2^U}\ ]\}$

$= \{[\ \min\ (T_1^L\ T_2^L\ , \sqrt{T_1^L\ T_2^L})\ , \min\ (\ T_1^U\ T_2^U, \sqrt{T_1^U\ T_2^U}\ )], [\max\ (I_1^L + I_2^L - I_2^L I_2^L\ , \sqrt{I_1^L\ I_2^L})\ ,$
$\max\ (I_1^U + I_2^U - I_1^U I_2^U, \sqrt{I_1^U\ I_2^U})], [\max\ (F_1^L + F_2^L - F_2^L F_2^L\ , \sqrt{F_1^L\ F_2^L})\ ,$
$\max\ (F_1^U + F_2^U - F_1^U F_2^U, \sqrt{F_1^U\ F_2^U})]\}$

$= \{[\ T_1^L\ T_2^L\ , T_1^U\ T_2^U\ ]\ , [\ I_1^L + I_2^L - I_2^L I_2^L\ , I_1^U + I_2^U - I_1^U I_2^U\ ],$
$[\ F_1^L + F_2^L - F_2^L F_2^L\ , F_1^U + F_2^U - F_1^U F_2^U\ ]\}$

$= \tilde{n}_1 \otimes \tilde{n}_2$

This proves (ix)

(xiii) From definitions in 2.3, 2.5 and 3.1, we have

$(\tilde{n}_1 \otimes \tilde{n}_2) \cap (\ \tilde{n}_1 \# \tilde{n}_2) = \tilde{n}_1 \otimes \tilde{n}_2;$

$= \{[T_1^L\ T_2^L\ ,\ T_1^U\ T_2^U)], [\ I_1^L + I_2^L - I_2^L I_2^L\ , I_1^U + I_2^U - I_1^U I_2^U],$

$[F_1^L + F_2^L - F_2^L F_2^L \ , F_1^U + F_2^U - F_1^U F_2^U \ ]\}$ $\cap \{([\frac{2\,T_1^L\,T_2^L}{T_1^L + T_2^L}, \frac{2\,T_1^U\,T_2^U}{T_1^U + T_2^U}], [\frac{2\,I_1^L\,I_2^L}{I_1^L + I_2^L}, \frac{2\,I_1^U\,I_2^U}{I_1^U + I_2^U}]$,
$[\frac{2\,F_1^L\,F_2^L}{F_1^L + F_2^L}, \frac{2\,F_1^U\,F_2^U}{F_1^U + F_2^U}]\}$

={ [min ($T_1^L\,T_2^L\,,\frac{2\,T_1^L\,T_2^L}{T_1^L + T_2^L}$) , min ($T_1^U\,T_2^U\,,\frac{2\,T_1^U\,T_2^U}{T_1^U + T_2^U}$)] , [max ($I_1^L + I_2^L - I_2^L I_2^L\,,\frac{2\,I_1^L\,I_2^L}{I_1^L + I_2^L}$) ,
max($I_1^U + I_2^U - I_1^U I_2^U\,,\frac{2\,I_1^U\,I_2^U}{I_1^U + I_2^U}$], [max ($F_1^L + F_2^L - F_2^L F_2^L \ ,\frac{2\,F_1^L\,F_2^L}{F_1^L + F_2^L}$) ,
max($F_1^U + F_2^U - F_1^U F_2^U \ ,\frac{2\,F_1^U\,F_2^U}{F_1^U + F_2^U}$]}

={ $[T_1^L\,T_2^L, T_1^U\,T_2^U]$, [$I_1^L + I_2^L - I_2^L I_2^L\,, I_1^U + I_2^U - I_1^U I_2^U$] ,
[$F_1^L + F_2^L - F_2^L F_2^L \ , F_1^U + F_2^U - F_1^U F_2^U$]}

=$\tilde{n}_1 \otimes \tilde{n}_2$

This proves (xiii). This proves the theorem.

Theorem 3.8 . For $\tilde{n}_1$ and $\tilde{n}_2 \in$ INSs(X), then following relations are valid:

(i) $(\tilde{n}_1 \# \tilde{n}_2)\ \$\ (\tilde{n}_1 \# \tilde{n}_2) = \tilde{n}_1 \#\, \tilde{n}_2$;
(ii) $(\tilde{n}_1 \oplus \tilde{n}_2)\ \$\ (\tilde{n}_1 \oplus \quad \tilde{n}_2) = \tilde{n}_1 \oplus \tilde{n}_2$;
(iii) $(\tilde{n}_1 \otimes \tilde{n}_2)\ \$\ (\tilde{n}_1 \otimes \tilde{h}_2) = \tilde{n}_1 \otimes \tilde{n}_2$;
(iv) $(\tilde{n}_1 @ \tilde{n}_2)\ \$\ (\tilde{n}_1 @ \tilde{n}_2) = \tilde{n}_1 @\, \tilde{n}_2$;
(v) $(\tilde{n}_1 \# \tilde{n}_2)\ @\ (\tilde{n}_1 \# \tilde{n}_2) = \tilde{n}_1 \# \tilde{n}_2$;
(vi) $(\tilde{n}_1 \oplus \tilde{n}_2)\ @\ (\tilde{n}_1 \otimes \tilde{n}_2) = \tilde{n}_1 @ \tilde{n}_2$;
(vii) $(\tilde{n}_1 \cup \tilde{n}_2)\ @\ (\tilde{n}_1 \cap \tilde{n}_2) = \tilde{n}_1 @ \tilde{n}_2$;
(viii) $(\tilde{n}_1 \cup \tilde{n}_2)\ \$\ (\tilde{n}_1 \cap \tilde{n}_2) = \tilde{n}_1 \$\, \tilde{n}_2$;
(ix) $(\tilde{n}_1 \cup \tilde{n}_2)\ \#\ (\tilde{n}_1 \cap \tilde{n}_2) = \tilde{n}_1 \# \tilde{n}_2$;

Proof. The proofs of these results are the same as in the above proof

Theorem 3.9 For every two $\tilde{n}_1$ and $\tilde{n}_2 \in$ INSs(X), we have:

(i) $((\tilde{n}_1 \cup \tilde{n}_2) \oplus (\tilde{n}_1 \cap \tilde{n}_2))\ @\ ((\tilde{n}_1 \cup \tilde{n}_2) \otimes (\tilde{n}_1 \cap \tilde{n}_2)) = \tilde{n}_1 @ \tilde{n}_2$;
(ii) $((\tilde{n}_1 \cup \tilde{n}_2) \# (\tilde{n}_1 \cap \tilde{n}_2))\ \$\ ((\tilde{n}_1 \cup \tilde{n}_2)\ @\ (\tilde{n}_1 \cap \tilde{n}_2)) = \tilde{n}_1 \$ \tilde{n}_2$
(iii) $((\tilde{n}_1 \oplus \tilde{n}_2) \cup (\tilde{n}_1 \otimes \tilde{n}_2))\ @\ ((\tilde{n}_1 \oplus \tilde{n}_2) \cap (\tilde{n}_1 \otimes \tilde{n}_2)) = \tilde{n}_1 @ \tilde{n}_2$;
(iv) $((\tilde{n}_1 \oplus \tilde{n}_2) \cup (\tilde{n}_1 @ \tilde{n}_2))\ @\ ((\tilde{n}_1 \otimes \tilde{n}_2) \cap (\tilde{n}_1 @ \tilde{n}_2)) = \tilde{n}_1 @ \tilde{n}_2$;
(v) $((\tilde{n}_1 \oplus \tilde{n}_2) \cup (\tilde{n}_1 \# \tilde{n}_2))\ @\ ((\tilde{n}_1 \otimes \tilde{n}_2) \cap (\tilde{n}_1 \# \tilde{n}_2)) = \tilde{n}_1 @ \tilde{n}_2$;
(vi) $((\tilde{n}_1 \oplus \tilde{n}_2) \cup (\tilde{n}_1 \$ \tilde{n}_2))\ @\ ((\tilde{n}_1 \otimes \tilde{n}_2) \cap (\tilde{n}_1 \$ \tilde{n}_2)) = \tilde{n}_1 @ \tilde{n}_2$;
(vii) $((\tilde{n}_1 \oplus \tilde{n}_2) \cup (\tilde{n}_1 @ \tilde{n}_2))\ @\ ((\tilde{n}_1 \oplus \tilde{n}_2) \cap (\tilde{n}_1 \# \tilde{n}_2)) = \tilde{n}_1 \$ \tilde{n}_2$.

Proof . In the following, we prove (i) and (iii), other results can be proved analogously.

(i) From definitions in 2.4, 2.5 and 3.1, we have

$((\tilde{n}_1 \cup \tilde{n}_2) \oplus (\tilde{n}_1 \cap \tilde{n}_2))\ @\ ((\tilde{n}_1 \cup \tilde{n}_2) \otimes (\tilde{n}_1 \cap \tilde{n}_2))$

$\tilde{n}_1$ ={([T_1^L , T_1^U] , [I_1^L , I_1^U] , [F_1^L , F_1^U])}

$\tilde{n}_2$ = {([T_2^L , T_2^U] , [I_2^L , I_2^U] , [F_2^L , F_2^U])}

$\tilde{n}_3$= {([T_3^L , T_3^U] , [I_3^L , I_3^U] , [F_3^L , F_3^U])}

(($\tilde{n}_1 \cup \tilde{n}_2$) ⊕ ($\tilde{n}_1 \cap \tilde{n}_2$))

={ [max(T_1^L, T_2^L), max(T_1^U ,T_2^U)], [min (I_1^L, I_2^L),min(I_1^U, I_2^U)] ,
[min (F_1^L, F_2^L), min(F_1^U, F_2^U)] } ⊕ { [min(T_1^L, T_2^L), min(T_1^U ,T_2^U)],
[max (I_1^L, I_2^L), max(I_1^U, I_2^U)] , [max (F_1^L, F_2^L), max(F_1^U, F_2^U)] }.

={[max(T_1^L, T_2^L)+ min(T_1^L, T_2^L) - max(T_1^L, T_2^L) min(T_1^L, T_2^L), max(T_1^U ,T_2^U) +
min(T_1^U ,T_2^U) - max(T_1^U ,T_2^U min(T_1^U ,T_2^U)] ,[min(I_1^L, I_2^L) max (I_1^L, I_2^L),
min(I_1^U, I_2^U) max(I_1^U, I_2^U)], [min(F_1^L, F_2^L) max (F_1^L, F_2^L), min(F_1^U, F_2^U) max(F_1^U, F_2^U)]}.

($\tilde{n}_1 \cup \tilde{n}_2$) ⊗ ($\tilde{n}_1 \cap \tilde{n}_2$)

={ [max(T_1^L, T_2^L), max(T_1^U ,T_2^U)], [min (I_1^L, I_2^L),min(I_1^U, I_2^U)] , [min (F_1^L, F_2^L), min(F_1^U, F_2^U)
] } ⊗ { [min(T_1^L, T_2^L), min(T_1^U ,T_2^U)], [max (I_1^L, I_2^L), max(I_1^U, I_2^U)] ,
[max (F_1^L, F_2^L), max(F_1^U, F_2^U)] }.

= {[max(T_1^L, T_2^L) min(T_1^L, T_2^L) , max(T_1^U ,T_2^U) min(T_1^U ,T_2^U)] ,
[min (I_1^L, I_2^L) + max (I_1^L, I_2^L) - min (I_1^L, I_2^L) max (I_1^L, I_2^L), min (I_1^U, I_2^U) +
max (I_1^U, I_2^U) - min (I_1^U, I_2^U) max (I_1^U, I_2^U)] , [min (F_1^L, F_2^L) + max (F_1^L, F_2^L) - min (F_1^L, F_2^L)
max (F_1^L, F_2^L), min (F_1^U, F_2^U) + max (F_1^U, F_2^U) - min (F_1^U, FI_2^U) max (F_1^U, F_2^U)]}.

(($\tilde{n}_1 \cup \tilde{n}_2$) ⊕ ($\tilde{n}_1 \cap \tilde{n}_2$)) @ (($\tilde{n}_1 \cup \tilde{n}_2$) ⊗ ($\tilde{n}_1 \cap \tilde{n}_2$))

$$=\{[\frac{\max(T_1^L,T_2^L)+\min(T_1^L,T_2^L)-\max(T_1^L,T_2^L)\min(T_1^L,T_2^L)+\max(T_1^L,T_2^L)\min(T_1^L,T_2^L)}{2},$$
$$\frac{\max(T_1^U,T_2^U)+\min(T_1^U,T_2^U)-\max(T_1^U,T_2^U)\min(T_1^U,T_2^U)+\max(T_1^U,T_2^U)\min(T_1^U,T_2^U)}{2}],$$
$$[\frac{[\min(I_1^L,I_2^L)\max(I_1^L,I_2^L)+\min(I_1^L,I_2^L)+\max(I_1^L,I_2^L)-\min(I_1^L,I_2^L)\max(I_1^L,I_2^L))}{2},$$
$$\frac{\min(I_1^U,I_2^U)\max(I_1^U,I_2^U)+\min(I_1^U,I_2^U)+\max(I_1^U,I_2^U)-\min(I_1^U,I_2^U)\max(I_1^U,I_2^U)}{2}]$$

$$[\frac{[\min(F_1^L,F_2^L)\max(F_1^L,F_2^L),+\min(F_1^L,F_2^L)+\max(F_1^L,F_2^L)-\min(F_1^L,F_2^L)\max(F_1^L,F_2^L))}{2},$$
$$\frac{\min(F_1^U,F_2^U)\max(F_1^U,F_2^U)+\min(F_1^U,F_2^U)+\max(F_1^U,F_2^U)-\min(F_1^U,F_2^U)\max(F_1^U,F_2^U)}{2}]\}$$
$$=[\frac{\max(T_1^L,T_2^L)+\min(T_1^L,T_2^L)}{2},\frac{\max(T_1^U,T_2^U)+\min(T_1^U,T_2^U)}{2}],$$
$$[\frac{\min(I_1^L,I_2^L)+\max(I_1^L,I_2^L)}{2},\frac{\min(I_1^U,I_2^U)+\max(I_1^U,I_2^U)}{2}],[\frac{\min(F_1^L,F_2^L)+\max(F_1^L,F_2^L)}{2},$$
$$\frac{\min(I_1^U,I_2^U)+\max(I_1^U,I_2^U)}{2}]$$

$$=\{[\frac{T_1^L+T_2^L}{2},\frac{T_1^U+T_2^U}{2}],[\frac{I_1^L+I_2^L}{2},\frac{I_1^L+I_2^L}{2}],[\frac{F_1^L+F_2^L}{2},\frac{F_1^L+F_2^L}{2}]\}$$

= $\tilde{n}_1 @ \tilde{n}_2$

This proves (i).

(iii) From definitions in 2.4, 2.5 and 3.1, we have

$((\tilde{n}_1 \oplus \tilde{n}_2) \cup (\tilde{n}_1 \otimes \tilde{n}_2))$ @ $((\tilde{n}_1 \oplus \tilde{n}_2) \cap (\tilde{n}_1 \otimes \tilde{n}_2))$

= $\tilde{n}_1$@$\tilde{n}_2$;$(\tilde{n}_1 \oplus \tilde{n}_2) \cap (\tilde{n}_1 \otimes \tilde{n}_2)$

$=\{([T_1^L + T_2^L - T_1^L T_2^L, T_1^U + T_2^U - T_1^U T_2^U], [I_1^L I_2^L, I_1^U I_2^U], [F_1^L F_2^L, F_1^U F_2^U])\}$
$\cap \{[T_1^L T_2^L, T_1^U T_2^U)], [I_1^L + I_2^L - I_2^L I_2^L, I_1^U + I_2^U - I_1^U I_2^U],$
$[F_1^L + F_2^L - F_2^L F_2^L, F_1^U + F_2^U - F_1^U F_2^U]\}$

$=\{[\min(T_1^L + T_2^L - T_1^L T_2^L, T_1^L T_2^L), \min(T_1^U + T_2^U - T_1^U T_2^U, T_1^U T_2^U)],$
$[\max(I_1^L I_2^L, I_1^L + I_2^L - I_2^L I_2^L), \max(I_1^U I_2^U, I_1^U + I_2^U - I_1^U I_2^U)],$
$[\max(F_1^L F_2^L, F_1^L + F_2^L - F_2^L F_2^L), \max(F_1^U F_2^U, F_1^U + F_2^U - F_1^U F_2^U)]\}.$

$= \{[T_1^L T_2^L, T_1^U T_2^U], [I_1^L + I_2^L - I_2^L I_2^L, I_1^U + I_2^U - I_1^U I_2^U],$
$[F_1^L + F_2^L - F_2^L F_2^L, F_1^U + F_2^U - F_1^U F_2^U]\}$

$(\tilde{n}_1 \oplus \tilde{n}_2) \cup (\tilde{n}_1 \otimes \tilde{n}_2)$

$=\{([T_1^L + T_2^L - T_1^L T_2^L, T_1^U + T_2^U - T_1^U T_2^U], [I_1^L I_2^L, I_1^U I_2^U], [F_1^L F_2^L, F_1^U F_2^U])\}$
$\cup \{[T_1^L T_2^L, T_1^U T_2^U)], [I_1^L + I_2^L - I_2^L I_2^L, I_1^U + I_2^U - I_1^U I_2^U],$
$[F_1^L + F_2^L - F_2^L F_2^L, F_1^U + F_2^U - F_1^U F_2^U]$

$=\{[\max(T_1^L + T_2^L - T_1^L T_2^L, T_1^L T_2^L), \max(T_1^U + T_2^U - T_1^U T_2^U, T_1^U T_2^U)],$
$[\min(I_1^L I_2^L, I_1^L + I_2^L - I_2^L I_2^L), \min(I_1^U I_2^U, I_1^U + I_2^U - I_1^U I_2^U)],$
$[\min(F_1^L F_2^L, F_1^L + F_2^L - F_2^L F_2^L), \min(F_1^U F_2^U, F_1^U + F_2^U - F_1^U F_2^U)]\}$

$= \{[T_1^L + T_2^L - T_1^L T_2^L, T_1^U + T_2^U - T_1^U T_2^U], [I_1^L I_2^L, I_1^U I_2^U], [F_1^L F_2^L, F_1^U F_2^U]\}$

$((\tilde{n}_1 \oplus \tilde{n}_2) \cup (\tilde{n}_1 \otimes \tilde{n}_2))$ @ $((\tilde{n}_1 \oplus \tilde{n}_2) \cap (\tilde{n}_1 \otimes \tilde{n}_2))$
$=\{[\frac{T_1^L T_2^L + T_1^L + T_2^L - T_1^L T_2^L}{2}, \frac{T_1^U T_2^U + T_1^U + T_2^U - T_1^U T_2^U}{2}], [\frac{I_1^L + I_2^L - I_2^L I_2^L + I_1^L I_2^L}{2},$
$\frac{I_1^U + I_2^U - I_1^U I_2^U + I_1^U I_2^U}{2}], [\frac{F_1^L + F_2^L - F_2^L F_2^L + F_1^L F_2^L,}{2}, \frac{F_1^U + F_2^U - F_1^U F_2^U + F_1^U F_2^U}{2}]\}$

$=\{[\frac{T_1^L + T_2^L}{2}, \frac{T_1^U + T_2^U}{2}], [\frac{I_1^L + I_2^L}{2}, \frac{I_1^U + I_2^U}{2}], [\frac{F_1^L + F_2^L}{2}, \frac{F_1^U + F_2^U}{2}]$

Hence ,

$((\tilde{n}_1 \oplus \tilde{n}_2) \cup (\tilde{n}_1 \otimes \tilde{n}_2))$ @ $((\tilde{n}_1 \oplus \tilde{n}_2) \cap (\tilde{n}_1 \otimes \tilde{n}_2)) = \tilde{n}_1$@$\tilde{n}_2$

This proves (iii).

4. Conclusion

In this paper we have defined three new operations on interval neutrosophic sets based on the arithmetic mean, geometrical mean, and respectively harmonic mean, which involve different defining functions. Some related results have been proved and bring out the characteristics of the interval neutrosophic sets.

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Connectedness in Ditopological Texture Spaces Via Ideal

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Abstract – The main purpose of this paper, is to introduce the notion of ditopological texture spaces with ideal. We study the notions of $\star$-connected ditopological texture spaces with an ideal and $\star$-connected sets in ditopological texture spaces with ideal. Some new types of connectedness in $\star$-ditopological texture spaces namely, locally $\star$-connectedness, totally $\star$-disconnectedness, hyperconnectedness and $\star$-hyperconnectedness have investigated.

Keywords – *Texturing, Texture space, Bitopology, Ditopology, Connectedness, $\star$-connected sets, Ditopological texture spaces with ideal, $\star$-ditopological texture spaces, $\star$-separated sets, $\star$-connectedness, $\star$-component, Locally $\star$-connectedness, Totally $\star$-disconnectedness, $\star$-hyperconnected.*

1 Introduction

The notion of a texture space, under the name of fuzzy structure, was introduced by Brown in [2]. The motivation for the study of texture spaces is that they allow us to represent, for instance, classical fuzzy sets, L-fuzzy sets [14], intuitionistic fuzzy sets [1] and intuitionistic sets [9], as lattices of crisp subsets of some base set S. A detailed analysis of this relation between texture spaces and lattices of fuzzy sets of various kinds may be found in [5, 6, 9]. The concept of a ditopology on a texture space is introduced in [3] and corresponds in a natural way to a fuzzy topology. In general ditopological texture spaces may be regarded as natural generalizations of both topological spaces and bitopological spaces [16]. The notion of connectedness in ditopological texture space was introduced in [10], which is being extended by Tantawy et al. in [12]. In this paper, we introduce the notion of ditopological texture spaces with ideal. Also connectedness in ditopological texture spaces with an ideal is studied. We study the notions of $\star$-connectedness in ditopological texture spaces with ideal. Some new types of connectedness in $\star$-ditopological texture spaces namely, locally $\star$-connectedness, totally $\star$-disconnectedness, hyperconnectedness and $\star$-hyperconnectedness have investigated.

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2 Preliminary

The aim of this section is to collect the relevant definitions and results from texture space and ditopology which will be needed in the sequel.

Definition 2.1. [2]. Let X be a set. Then $L \subseteq P(X)$ is called texturing of X and X is said to be textured by L if L is separates the points of X, complete, completely distributive lattice with respect to inclusion, which contains X, ϕ, and for which arbitrary meet coincides with intersection and finite joins coincide with unions. The pair (X,L) is then known as a texture space.
In any texture space, the p-sets and q-sets for each $x \in X$ are the sets $p_x = \bigcap\{A \in L : x \in A\}$ and $q_x = \bigvee\{A \in L : x \notin A\}$.
A surjection $\sigma : L \to L$ is called a complementation if $\sigma^2(A) = A \; \forall A \in L$ and $A \subseteq B$ in L implies $\sigma(B) \subseteq \sigma(A)$. A texture with a complementation is said to be complemented.
We now recall the definition of a dichotomous topology (or ditopology for short) on a texture given in [2].

Definition 2.2. [3]. (L, τ, K) is called a ditopological texture space on X if

(1) $\tau \subseteq L$ satisfies

- **(a)** $X,\ \phi \in \tau$,
- **(b)** $G_1,\ G_2 \in \tau \Rightarrow G_1 \cap G_2 \in \tau$, and
- **(c)** $G_i \in \tau, i \in I \Rightarrow \bigvee_{i\in I} G_i \in \tau$, and

(2) $K \subseteq L$ satisfies

- **(a)** $X,\ \phi \in K$,
- **(b)** $F_1,\ F_2 \in K \Rightarrow F_1 \cup F_2 \in K$, and
- **(c)** $F_i \in K, i \in I \Rightarrow \bigwedge_{i\in I} F_i \in K$.

The elements of τ are called open and those of K are called closed. We refer to τ as the topology and to K as the cotopology of (τ,K).
In general there is no a priori relation between τ and K , but if σ is a complementation on (X, L), and τ, k are related by the relation $K = \sigma(\tau)$, then we call (τ, K) a complemented ditopology on (X, L, σ).
Finally, let $Z \subseteq X$. Then the closure of Z is the set $[Z] = \bigcap\{F \in K : Z \subseteq F\}$, the interior is $]Z[= \bigvee\{G \in \tau : G \subseteq Z\}$, the exterior is $ext(Z) = \bigvee\{G \in \tau : G \cap Z = \phi\}$ and Z is called dense in X if $[Z] = X$. Also, if $A \nsubseteq F \; \forall F \in K - \{X\}$, we say A is co-dense.

Example 2.1. [8].

(1) For any texture (X, L), a ditopology (τ, K) with $\tau = L$ is called discrete, and one with $K = L$ is called co-discrete.

(2) For any texture (X, L), a ditopology (τ, K) with $\tau = \{X, \phi\}$ is called indiscrete, and one with $K = \{X, \phi\}$ is called co-indiscrete.

(3) For any topology τ on X, (τ, τ'), $\tau' = \{X - G : G \in \tau\}$, is a complemented ditopology on the usual(crisp) set structure $(X, P(X), \sigma_X)$ of X, where $\sigma_X : P(X) \to P(X)$ defined by $\sigma_X(A) = A'$ where $A' = X - A \; \forall A \in P(X)$.

(4) For any bitopology (τ_1, τ_2) on X, (τ_1, τ_2') is a ditopology on$(X, P(X))$.

Definition 2.3. [10]. Let (X,L) be a texture space, $A \subseteq X$.Then, We define $\lambda(A)$ by $\lambda(A) = \bigvee_{x\in A} P_x$. Hence, $\lambda(A)$ is the smallest element of L containing A.

Definition 2.4. [10]. Let (X_i,L_i) be a texture spaces on X_i,$i = 1, 2$ and$f : X_1 \to X_2$ a mapping. We define the mapping $\tilde{f}^{-1} : L_2 \to L_1$ by $\tilde{f}^{-1}(l) = \lambda_1(f^{-1}(l) \; \forall l \in L_2$, where λ_1 is defined for L_1 as in definition 2.3.

Theorem 2.1. [10].The following are equivalent for a function $f : X_1 \to X_2$.

(1) $f^{-1}(l) \in L_1 \ \forall l \in L_2$.

(2) $\tilde{f}^{-1}(l) = f^{-1}(l) \ \forall l \in L_2$.

Definition 2.5. [10]. Let (X_i,L_i,K_i) be a ditopological texture space on X_i, $i = 1, 2$ and $f : X_1 \to X_2$ a mapping. We say that f is continuous if
(1) $\tilde{f}^{-1}(G) \in \tau_1 \ \forall G \in \tau_2$, and
(2) $\tilde{f}^{-1}(F) \in K_1 \ \forall F \in K_2$.

Definition 2.6. [10]. Let(X,L) be a texture space and $\phi \neq Z \subseteq X$. $\{A, B\} \subseteq P(X)$ is said to be a partition of Z if $A \cap Z \neq \phi$, $Z \nsubseteq B$ and $A \cap Z = B \cap Z$. Here we may interchange the roles of A and B. Indeed if $\{A, B\}$ is a partition of Z, then we also have $B \cap Z \neq \phi$ and $Z \nsubseteq A$.

Definition 2.7. [10]. Let (L, τ, K) be a ditopological texture space on X and $Z \subseteq X$. Z is said to be connected if there exists no partition $\{G, F\}$ with $G \in \tau$ and $F \in K$.

Theorem 2.2. [10]. Let (X, L, τ, K) be a ditopological texture space, then X is connected if and only if $\tau \cap K = \{X, \phi\}$.

Theorem 2.3. [10]. Let Z be a connected set in a ditopological texture space (X_1, L_1, τ_1, K_1) and f be a continuous function of X_1 in to a ditopological texture space (X_2, L_2, τ_2, K_2) satisfying one of the equivalent conditions of Theorem 2.1. Then $f(Z)$ is connected in X_2.

Definition 2.8. [12]. A set which is τ-open as well as K-closed is said to be clopen.

Theorem 2.4. [12]. Let (X, L, τ, K) be a ditopological texture space, then the following are equivalent:

(1) X is connected.

(2) X has no a partition $\{A, B\} \subseteq P(X)$ with $A \in \tau$ and $B \in K$.

(3) There is no proper subset A of X which is clopen.

(4) X can not be expressed as an union of two nonempty disjoint subsets A, B of X with $A \in \tau$ and $B \in K'$.

(5) X can not be expressed as an union of two nonempty disjoint subsets A, B of X with $A \in \tau'$ and $B \in K$.

(6) X can not be expressed as an union of two separated subsets A, B of X.

Definition 2.9. [12]. Let (X, L, τ, K) be a ditopological texture space and let $Z \subseteq X$ with $x \in Z$. Then the component of Z w.r.t x is the maximal of all connected subsets of Z containing the point x and denoted by $C(Z, x)$, i.e

$$C(Z, x) = \bigvee \{Y \subseteq Z : x \ \in Y, \ Y \ is \ connected\}.$$

Theorem 2.5. [12]. Every clopen connected subset of a ditopological texture space (X, L, τ, K) is a component of X.

Theorem 2.6. [12]. Let (Y, L_Y, τ_Y, K_Y) be a subspace of a ditopological texture space (X, L, τ, K) and $A \subseteq Y$. Then

(1) $Cl_Y(A) = [A] \cap Y$.

(2) $]A[\subseteq Int_Y(A)$.

(3) $ext_Y(A) = Y \cap ext(A)$.

Definition 2.10. [12]. A ditopological texture space (X, L, τ, K) is said to be locally connected at a point $x \in X$ if and only if every open subset of X containing x contains a connected open set containing x. X is said to be locally connected if and only if it is locally connected at each of its points.

Theorem 2.7. [12]. Every connected ditopological texture space is a locally connected.

Definition 2.11. [12]. A ditopological texture space (X, L, τ, K) is said to be totally disconnected if and only if $\forall x,\ y \in X$ s.t $x \neq y$ $\exists$ non empty disjoint clopen proper subsets $A,\ B$ of X s.t $x \in A$ and $y \in B$.

Definition 2.12. [12]. A ditopological texture space (X, L, τ, K) is said to be extremely disconnected if for every open set $G \subseteq X$ we have $[G]$ is open in X.

Theorem 2.8. [12]. Let (Y, L_Y, τ_Y, K_Y) be a subspace of a ditopological texture space (X, L, τ, K). Then

(1) Every τ_Y open set is τ open set if and only if $Y \in \tau$.

(2) Every K_Y closed set is K closed set if and only if $Y \in K$.

Definition 2.13. [17]. A topological space (X, τ) is said to be hyperconnected if for every pair of nonempty open sets of X has a nonempty intersection.

Definition 2.14. [15]. A nonempty collection I of subsets of a nonempty set X is said to be an ideal on X, if it satisfies the following two conditions:

(1) $A \in I$ and $B \subseteq A \Rightarrow B \in I$,

(2) $A \in I$ and $B \in I \Rightarrow A \cup B \in I$.

Given a topological space (X, τ) with an ideal I on X and if $P(X)$ is the set of all subsets of X, a set operator $()^* : P(X) \to P(X)$, called a local function of A with respect to τ and I, is denoted by $A^*(I, \tau)$ or $A^*(I)$ and defined as follows, for $A \subseteq X$, $A^*(I, \tau) = \{x \in X : O_x \cap A \notin I\ \forall\ O_x \in \tau\}$. A Kuratowski closure operator for the topology $\tau^*(I, \tau)$, called the $\star$-topology, finer than τ, is defined by $Cl^*(A) = A \cup A^*$ and $\tau^*(I, \tau)$ or $\tau^*(I)$ is defined by $\tau^*(I) = \{A \subseteq X : Cl^*(A') = A'\}$. Also, (X, τ, I) is called an ideal topological space or simply an ideal space.
For any ideal space (X, τ, I), the collection$\{G - V : G \in \tau, V \in I\}$ is a basis for τ^*.

Definition 2.15. [13]. Nonempty subsets $A,\ B$ of a topological space with an ideal I on X (X, τ, I) are said to be $\star$-separated sets if $Cl^*(A) \cap B = A \cap Cl(B) = \phi$.

Definition 2.16. [13]. A subset A of a topological space (X, τ, I) with an ideal I on X is said to be $\star_s$-connected if A is not the union of two $\star$-separated sets in (X, τ, I).

Definition 2.17. [13]. Let (X, τ, I) be a topological space with an ideal I on X and $x \in X$. The union of all $\star_s$-connected subsets of X containing x is called the $\star_s$-component of X containing x.

Definition 2.18. [11]. A subset A of an ideal topological space (X, τ, I) is said to be $\star$-dense if $Cl^*(A) = X$. An ideal topological space (X, τ, I) is said to be $\star$-hyperconnected if A is $\star$-dense for every nonempty open subset A of X has a nonempty intersection.

3 Ideal Ditopological Texture Spaces

In this section we introduce a ditopological texture space finer than the given ditopological texture space (X, L, τ, K) on the same set X by using the ideal notion. We extend the notion of connectedness to such spaces and study some of its basic properties. We denote by (X, L, τ, K, I) as a ditopological texture space with an ideal I on X.

Definition 3.1. Let (τ, K) be a ditopological space on any texture space with an ideal (X, L, I). Then

(1) define the local function $()^*_\tau : P(X) \to P(X)$ by $A^*(I, \tau) = \{x \in X : O_x \cap A \notin I\ \forall\ O_x \in \tau\}\ \forall\ A \in P(X)$. A Kuratowski closure operator $Cl^*_\tau(.)$ for the topology $\tau^*(I, \tau)$, called the $*$-topology, finer than τ, induced by $Cl^*_\tau(A) = A \cup A^*(I, \tau)$, where $\tau^* = \{G \subseteq X : Cl^*_\tau(G') = G'\}$.

(2) let $K' = \{X - F : F \in K\}$, which is a topology on X, so we again define a local function $()^*_{K'} : P(X) \to P(X)$, where $A^*(I, K')$ is the local function of A w.r.t $I,\ K'$. Also a Kuratowski closure operator $Cl^*_{K'}(.)$ for the topology $K'^*(I, K')$, called the $*$-topology, finer than K'. Hence, $K'^{*'} = \mathcal{K}^*$ is a family of closed subsets of X finer than K.

(3) let (X, L^*) be the smallest texture structure space containing L, τ^* and $\mathcal{K}^*$. Hence, $(\tau^*, \mathcal{K}^*)$ is called the $\star$-ditopology on (X, L^*), finer than (τ, K) on (X, L).

Finally, let $Z \subseteq X$. Then the $\star$-closure of Z is the set $[Z]^{\mathcal{K}^*} = \bigcap\{F \in \mathcal{K}^* : Z \subseteq F\}$, the $\star$-interior is $]Z[^{\tau^*} = \bigvee\{G \in \tau^* : G \subseteq Z\}\}$, the $\star$-exterior is $ext^{\tau^*}(Z) = \bigvee\{G \in \tau^* : G \cap Z = \phi\}$ and Z is called $\star$-dense in X if $[Z]^{\mathcal{K}^*} = X$. Also if $A \nsubseteq F \ \ \forall F \in \mathcal{K}^* - \{X\}$, we say A is $\star$-co-dense.

Examples 3.1. **(1)** If $I = \phi$, then $A^*(I, \tau) = [A]^\tau$ and $A^*(I, K') = [A]^K \ \forall A \in P(X)$. Hence, $Cl^*_\tau(A) = [A]^\tau$, $Cl^*_{K'}(A) = [A]^K$, $\tau^* = \tau$ and $\mathcal{K}^* = K$.

(2) If $I = P(X)$, then $A^*(I, \tau) = \phi$ and $A^*(I, K') = \phi \ \forall A \in P(X)$. Hence, $Cl^*_\tau(A) = A$, $Cl^*_{K'}(A) = A$, $\tau^* = P(X)$ and $\mathcal{K}^* = P(X)$.

(3) If $I \subseteq J$, then $A^*(I, \tau) \subseteq A^*(J, \tau)$ and $A^*(I, K') \subseteq A^*(J, K')$. Hence, the $\star$- ditopological texture space $(X, L^*, \tau^*(J), \mathcal{K}^*(J))$ is finer than the $\star$- ditopological texture space $(X, L^*, \tau^*(I), \mathcal{K}^*(I))$.

Remark 3.1. In the case of $\star$- ditopological texture space, we choose $L^* \supseteq L$. Indeed sometimes $\tau^*, \mathcal{K}^* \nsubseteq L$, as in the following examples.

Examples 3.2. **(1)** Let $X = \{a, b, c\}$, $L = \{X, \phi, \{b\}, \{c\}, \{b, c\}\}$, $\tau = \{X, \phi, \{b\}\}$, $K = \{X, \phi, \{c\}\}$ and $I = \{\phi, \{b\}\}$ be an ideal on X. Then $\tau^* = \{X, \phi, \{b\}, \{a, c\}\}$, $\mathcal{K}^* = \{X, \phi, \{b\}, \{c\}, \{b, c\}\}$ and $\tau^* \nsubseteq L$. Hence, $L^* = \{X, \phi, \{b\}, \{c\}, \{b, c\}, \{a, c\}\}$.

(2) Let $X = \{a, b, c\}$, $L = \{X, \phi, \{b\}, \{c\}, \{b, c\}, \{a, b\}\}$, $\tau = \{X, \phi, \{b\}, \{a, b\}$, $K = L$ and $I = \{\phi, \{a\}, \{b\}, \{a, b\}\}$ be an ideal on X. Then $\tau^* = P(X)$, $\mathcal{K}^* = P(X)$ and τ^*, $\mathcal{K}^* \nsubseteq L$. Hence, $L^* = P(X)$.

(3) Let $X = [0, 1]$, $L = \{[0, r] : r \in X\} \cup \{\phi\}$, $\tau = K = \{X, \phi\}$, $I = \{A \subseteq X : A \ is \ finite\}$. Then $\tau^* = \tau_\infty$ [15], where τ_∞ is the cofinite topology, $\tau_\infty \nsubseteq L$, also $\mathcal{K}^* = \tau'_\infty$ and $\mathcal{K}^* \nsubseteq L$. Hence, $L^* = P(X)$.

Theorem 3.1. Let (L, τ, K) be a ditopological texture space on X, I be an ideal on X and $(X, L^*, \tau^*, \mathcal{K}^*$ be the $\star$-ditopological texture space w.r.t I. Then

(1) $\beta(I, \tau) = \{G - V : G \in \tau, V \in I\}$ is a basis of τ^*.

(2) $\beta(K', I) = \{F - V : F \in K', V \in I\}$ is a basis of $\mathcal{K}^{*'}$.

Proof. **(1)** Since $X \in \tau, \phi \in I$, then $X - \phi \in \beta$, hence $X \in \beta$ and $\bigcup_{i \in I}(G_i - V_i) = X$. Also, let $G_1 - V_1, G_2 - V_2 \in \beta$, s.t $x \in (G_1 - V_1) \cap (G_2 - V_2)$, then $x \in (G_1 \cap G_2) - (V_1 \cup V_2) \in \beta$. Therefore, $x \in (G_1 \cap G_2) - (V_1 \cup V_2) \subseteq (G_1 - V_1) \cap (G_2 - V_2)$. Hence, β is a basis of τ^*.

(2) By a similar way.

□

Definition 3.2. Let (X, L, τ, K, I) be a ditopological texture space with an ideal, $Y \subseteq X$ s.t $Y \in L$ and let L_Y, τ_Y, K_Y, I_Y are the restriction of L, τ, K, I on I, then $(Y, L_Y, \tau_Y, K_Y, I_Y)$ is a ditopological texture subspace with an ideal I_Y on Y.

Theorem 3.2. Let (X, L, τ, K, I) be a ditopological texture space with an ideal I on X and $A \subseteq Y \subseteq X$. Then

(1) $A^*(\tau_Y, I_Y) = Y \cap A^*(\tau, I)$.

(2) $A^*(K'_Y, I_Y) = Y \cap A^*(K', I)$.

Proof. **(1)** $A^*(\tau_Y, I_Y) = \{y \in Y : O_y \cap A \notin I_Y \ \forall \ O_y \in \tau_Y\} = \{y \in Y : (Y \cap G) \cap A \notin I_Y \ \forall \ Y \cap G \in \tau_Y\} = \{y \in Y : G \cap A \notin I \ \forall \ G \in \tau\} = Y \cap \{y \in X : G \cap A \notin I \ \forall \ G \in \tau\} = Y \cap A^*(\tau, I)$.

(2) By a similar way.

□

Theorem 3.3. Let $(Y, L_Y, \tau_Y, K_Y, I_Y)$ be a ditopological texture subspace with an ideal I_Y on Y of a ditopological texture space (X, L, τ, K, I) and $A \subseteq Y \subseteq X$. Then

(1) $[A]^{\mathcal{K}^*Y} = Y \cap [A]^{\mathcal{K}^*}$, where $\mathcal{K}^*{}_Y$ is the family of $\mathcal{K}^*$-closed subsets of Y.

(2) $]A[^{\tau^*} \subseteq Int^{\tau^*_Y}(A)$, where τ^*_Y is the family of τ^*-open subsets of Y.

(3) $ext^{\tau^*_Y}(A) = Y \cap ext^{\tau^*}(A)$.

Proof. Immediate from Theorem 2.6, Definition 3.1 and Theorem 3.2. □

Remark 3.2. Note that, the equality in Theorem 3.3(2) holds for all subsets of Y if and only if Y is τ^*-open. Indeed, if $x \in Int^{\tau^*_Y}(A)$. Then $x \in \bigvee\{G \cap Y \in \tau^*_Y : G \cap Y \subseteq A\}$. Since $Y \in \tau^*$, then $x \in]A[^{\tau^*}$.

Theorem 3.4. Let $(Y, L_Y, \tau_Y, K_Y, I_Y)$ be a ditopological texture subspace with an ideal I_Y on Y of a ditopological texture space (X, L, τ, K, I) and consider the $\star$-ditopological texture space $(X, L^*, \tau^*, \mathcal{K}^*)$. Then

(1) Every τ^*_Y-open set is τ^*-open set if and only if $Y \in \tau^*$.

(2) Every $\mathcal{K}^*{}_Y$-closed set is $\mathcal{K}^*$-closed set if and only if $Y \in \mathcal{K}^*$.

Proof. **(i)** Suppose that every τ^*_Y-open set is τ^*-open set, then $Y \in \tau^*_Y \subseteq \tau^*$. Conversely, if $Y \in \tau^*$ and $A \subseteq Y$ is τ^*_Y-open, then $A = Y \cap G$ for some $G \in \tau^*$, but $Y \in \tau^*$, hence $A \in \tau^*$.

(2) By a similar way.

□

Corollary 3.1. Let $(Y, L_Y, \tau_Y, K_Y, I_Y)$ be a τ-open subspace of (X, L, τ, K, I), consider the $\star$-ditopological texture space $(X, L^*, \tau^*, \mathcal{K}^*)$ and $A \subseteq Y$. Then A is τ^*_Y-open set if and only if it is τ^*-open set.

Proof. Immediate from Theorem 3.4. □

Corollary 3.2. Let $(Y, L_Y, \tau_Y, K_Y, I_Y)$ be a K-closed subspace of (X, L, τ, K, I), consider the $\star$-ditopological texture space $(X, L^*, \tau^*, \mathcal{K}^*)$ and $A \subseteq Y$. Then A is $\mathcal{K}^*_Y$-closed set if and only if it is $\mathcal{K}^*$-closed set .

Proof. Immediate from Theorem 3.4. □

Definition 3.3. Let (X, L, τ, K, I) be a ditopological texture space with an ideal on X and $Z \subseteq X$. $\{A, B\} \subseteq P(X)$, where $(A, B) \in \tau^* \times K$ or $(A, B) \in \mathcal{K}^* \times \tau$, is said to be a $\star$-partition of Z if $A \cap Z \neq \phi$, $Z \nsubseteq B$ and $A \cap Z = B \cap Z$.

Definition 3.4. Let (X, L, τ, K, I) be a ditopological texture space with an ideal on X and $Z \subseteq X$. Z is said to be $\star$-connected if there exists no $\star$-partition of Z.

Theorem 3.5. Let (X, L, τ, K, I) be a ditopological texture space with an ideal on X and $(X, L^*, \tau^*, \mathcal{K}^*)$ be a $\star$-ditopological texture space, then the following are equivalent:

(1) X is $\star$-connected.

(2) There is no proper subset A of X with $A \in \tau^* \cap K$ and $A \in \mathcal{K}^* \cap \tau$.

(3) X can not be expressed as an union of two nonempty disjoint subsets A, B of X with $A \in \tau$ and $B \in \mathcal{K}^{*'}$ (resp. $A \in \tau^*$ and $B \in K'$).

(4) X can not be expressed as an union of two nonempty disjoint subsets A, B of X with $A \in \tau'$ and $B \in \mathcal{K}^*$ (resp. $A \in \tau^{*'}$ and $B \in K$).

Proof. Immediate by Theorem 2.4 and Definition 3.4. □

Theorem 3.6. Let (X, L, τ, K, I) be a $\star$-connected ditopological texture space with an ideal. Then (X, L, τ, K) is connected.

Proof. Immediate. □

Remark 3.3. The converse of Theorem 3.6 is not true in general, as in the following example.
Let $X = \{a, b, c\}$, $L = \{X, \phi, \{b\}, \{c\}, \{b, c\}, \{a, b\}\}$, $\tau = \{X, \phi, \{b\}, \{a, b\}\}$, $K = \{X, \phi, \{c\}\}$, and $I = \{\phi, \{a\}, \{b\}, \{a, b\}\}$ be an ideal on X, then $\tau^* = P(X)$, $\mathcal{K}^* = P(X)$ and $L^* = P(X)$. Then (X, L, τ, K) is connected but (X, L, τ, K, I) is $\star$-disconnected.

Theorem 3.7. Let (X, L, τ, K, I) be a $\star$-connected ditopological texture space with an ideal. Then (X, L, τ, K) is locally connected.

Proof. Immediate by Theorem 2.7 and Theorem 3.6. □

Theorem 3.8. X is $\star$-connected if for all pair of point $x,\ y \in X$ with $x \neq y$ there exists a $\star$-connected set $Z \subseteq X$ with $x,\ y \in Z$.

Proof. Suppose that X is $\star$-disconnected. Then by Theorem 3.5 there exists a proper subset A of X s.t $A \in \tau^* \cap K$ or $A \in \tau \cap \mathcal{K}^*$. If $A \in \tau^* \cap K$. Then we choose $x,\ y \in X$ with $x \in A$ and $y \notin A$. If there exists a $\star$-connected set Z with $x,\ y \in Z$, then $Z \nsubseteq A$ and $A \cap Z \neq \phi$. Hence, $\{A, A\}$ is a partition of Z, which is a contradiction with the $\star$-connectedness of Z. By a similar way if $A \in \tau \cap \mathcal{K}^*$. Hence, we get the proof. □

Corollary 3.3. Let Z be a $\star$-connected set in a ditopological texture space with an ideal $(X_1, L_1, \tau_1, K_1, I_1)$ and f be a continuous function of X_1 into a ditopological texture space with an ideal $(X_2, L_2, \tau_2, K_2, I_2)$ satisfying one of the equivalent conditions of Theorem 2.1. Then $f(Z)$ is $\star$-connected in X_2.

Proof. Suppose that $f(Z)$ is not $\star$-connected in X_2. Let $\{G, F\} \subseteq P(X)$ be a partition of $f(Z)$ with $G \in \tau_2^*$ and $F \in K_2$ or $G \in \tau_2$ and $F \in \mathcal{K}_2^*$. If $G \in \tau_2^*$ and $F \in K_2$. Then $f(Z) \cap G \neq \phi$, $f(Z) \nsubseteq F$ and $f(Z) \cap G = f(Z) \cap F$. Since $\tilde{f}^{-1}(G) = f^{-1}(G)$ and $\tilde{f}^{-1}(F) = f^{-1}(F)$, then $\{f^{-1}(G), f^{-1}(F)\}$ is a partition of Z where $f^{-1}(G) \in \tau_1^*$ and $f^{-1}(F) \in K_1$, which is a contradiction with the $\star$-connectedness of Z. By a similar way if $G \in \tau_2$ and $F \in \mathcal{K}_2^*$. Hence, we get the proof.

□

Definition 3.5. Let (X, L, τ, K, I) be a ditopological texture space with an ideal and let $Z \subseteq X$ with $x \in Z$.Then the $\star$-component of Z w.r.t x is the maximal of all $\star$-connected subsets of Z containing the point x and denoted by $C(Z, x)$, i.e

$$C(Z, x) = \bigvee\{Y \subseteq Z : \text{x} \in Y, Y is \star -connected\}.$$

Theorem 3.9. If Z is a $\star$-component and $\text{ext}^{\tau^*}(Z) = \phi$, then $Z = [Z]^*$.

Proof. We want to prove that $[Z]^* \subseteq Z$. So Let (X, L, τ, K, I) be a ditopological texture space with an ideal, Z be a $\star$-component subset of $(X, L^*, \tau^*, \mathcal{K}^*)$ and $x \notin Z$, but Z is a maximal $\star$-connected set, then $Z \cup \{x\}$ can not be $\star$-connected set. Take a partition $\{A, B\}$ of $Z \cup \{x\}$ s.t $(A, B) \in \tau^* \times K$ with $A \cap (Z \cup \{x\}) \neq \phi$, $Z \cup \{x\} \nsubseteq B$ and $(Z \cup \{x\}) \cap A = (Z \cup \{x\}) \cap B$. Since $Z \subseteq Z \cup \{x\}$, then $Z \cap A = Z \cap B$ and Z is $\star$-connected. Hence, either $Z \cap A = \phi$ or $Z \subseteq B$. Suppose $Z \cap A = \phi$. Since $x \notin Z$ and $A \cap (Z \cup \{x\}) \neq \phi$, then $x \in A$. Hence, $x \in ext^{\tau^*}(Z)$, which is a contradiction. If $Z \subseteq B$ and $Z \cup \{x\} \nsubseteq B$, then $x \notin B$ and $[Z]^* \subseteq B$. Hence, $x \notin [Z]^*$. By a similar way if we take a partition $\{A, B\}$ of $Z \cup \{x\}$ s.t $(A, B) \in \tau \times \mathcal{K}^*$. This completes the proof. □

Theorem 3.10. Let $\{Z_i : i \in I\}$ be a family of $\star$-connected subsets in L^* with $\bigcap_{i \in I} Z_i \neq \phi$, then $\bigvee_{i \in I} Z_i$ is also $\star$-connected.

Proof. Suppose that $Z = \bigvee_{i \in I} Z_i$ is $\star$-disconnected. Then we may choose a partition $\{A, B\}$ of Z s.t $(A, B) \in \tau^* \times K$ or $(A, B) \in \tau \times \mathcal{K}^*$ with $Z \cap A \neq \phi$, $Z \nsubseteq B$ and $A \cap Z = B \cap Z$. Since $Z_i \subseteq Z\ \forall i \in I$, then $A \cap Z_i = B \cap Z_i\ \forall i \in I$. But Z_i is $\star$-connected, then either $Z_i \cap B = \phi$ or $Z_i \subseteq A$. Now we choose $x \in \bigcap_{i \in I} Z_i$, then $x \in Z_i\ \forall i \in I$, so either $x \in A$ or $x \notin B$. Suppose $x \in A$, then $A \cap Z_i \neq \phi\ \forall i \in I$, $A \cap Z_i = B \cap Z_i$ and Z_i is $\star$-connected, then $Z_i \subseteq B\ \forall i \in I$. Hence, $Z = \bigvee_{i \in I} Z_i \subseteq B$, which is a contradiction. Now suppose $x \notin B$, since $x \in Z_i\ \forall i \in I$, then $Z_i \nsubseteq B$, $A \cap Z_i = B \cap Z_i\ \forall i \in I$ and Z_i is $\star$-connected. Hence, $A \cap Z_i = \phi$ and $A \cap Z = A \cap (\bigvee_{i \in I} Z_i) = \bigvee_{i \in I}(A \cap Z_i) = \bigvee_{i \in I}(\phi) = \phi$, which is a contradiction. □

Corollary 3.4. Let $\{Z_i : i \in I\}$ be a family of $\star$-connected subsets of a ditopological texture space with an ideal (X, L, τ, K, I) s.t one of the members of the family intersects every other members, then $Z = \bigvee_{i \in I} Z_i$ is $\star$-connected.

Proof. Let $Z_{i0} \in \{Z_i : i \in I\}$ s.t $Z_{i0} \cap Z_i \neq \phi \quad \forall i \in I$. Then $Z_{i0} \bigvee Z_i$ is $\star$-connected $\quad \forall i \in I$ by Theorem 3.10., hence the collection $\{Z_{i0} \vee Z_i : i \in I\}$ is a collection of a $\star$-connected subsets of X, which having a non-empty intersection. So $Z = \bigvee_{i \in I} Z_i$ is $\star$-connected by Theorem 3.10. □

Theorem 3.11. Let $Z \subseteq X$ be a $\star$-connected set, $Z \subseteq Y \subseteq [Z]^*$ and $ext^{\tau^*}(Z) \cap Y = \phi$, then Y is $\star$-connected.

Proof. Suppose that Y is $\star$-disconnected. Take a partition $\{A, B\}$ of Y with $(A, B) \in \tau^* \times K$. Then $Y \cap A \neq \phi$, $Y \nsubseteq B$ and $Y \cap A = Y \cap B$. Since $Z \subseteq Y$, then $Z \cap A = Z \cap B$ but Z is $\star$-connected, so either $Z \cap A = \phi$ or $Z \subseteq B$. Suppose $Z \cap A = \phi$, then $A \subseteq ext^*(Z)$ and $Y \cap A \subseteq Y \cap ext^*(Z)$. Hence, $A \cap Y = \phi$ is a contradiction. Now suppose $Z \subseteq B$, then $[Z]^* \subseteq B$, hence $Y \subseteq B$, which is a contradiction. □

Corollary 3.5. The $\mathcal{K}^*$-closure of $\star$-connected subset of a ditopological texture space (X, L, τ, K, I) with an ideal is $\star$-connected.

Proof. Immediate by Theorem 3.11. □

Corollary 3.6. Every $\star$-component of a ditopological texture space (X, L, τ, K, I) with an ideal is $\mathcal{K}^*$-closed set.

Proof. Immediate from Definition 3.5 and Corollary 3.5. □

4 $\star_s$-Connectedness in Ditopological Texture Spaces Modulo Ideal

Definition 4.1. Nonempty subsets A, B of a ditopological texture space with (X, L, τ, K, I) an ideal are said to be $\star$-separated sets if either $A \cap [B]^{\tau^*} = B \cap [A]^K = \phi$ or $A \cap [B]^{\mathcal{K}^*} = B \cap [A]^\tau = \phi$.

Theorem 4.1. Let (X, L, τ, K, I) be a ditopological texture space with an ideal and $(X, L^*, \tau^*, \mathcal{K}^*)$ be a $\star$-ditopological texture space, then the following are equivalent:

(1) X is $\star$-connected.

(2) There is no proper subset A of X with $A \in \tau^* \cap K$ and $A \in \mathcal{K}^* \cap \tau$.

(3) X can not be expressed as an union of two nonempty disjoint subsets A, B of X with $A \in \tau$ and $B \in \mathcal{K}^{*'}$ (resp. $A \in \tau^*$ and $B \in K'$).

(4) X can not be expressed as an union of two nonempty disjoint subsets A, B of X with $A \in \tau'$ and $B \in \mathcal{K}^*$ (resp. $A \in \tau^{*'}$ and $B \in K$).

(5) X can not be expressed as an union of two $\star$-separated sets.

Proof. Immediate from Theorem2.4 and Definition 4.1.

□

Theorem 4.2. Let (X, L, τ, K, I) be a ditopological texture space with an ideal I. If A and B are $\star$-separated sets of X s.t $A \cup B \in \tau \cap K$, then either A $(resp.B) \in \tau^* \cap K$ or A $(resp.B) \in \mathcal{K}^* \cap \tau$.

Proof. Suppose that A, B be a $\star$-separated sets s.t $A \cup B \in \tau \cap K$, then $[A \cup B]^{\tau^*} \in \tau^{*'}$. Since $[B]^{\tau^*} \in \tau^{*'}$, then $([B]^{\tau^*})' \in \tau^*$, it follows that $(A \cup B) \cap ([B]^{\tau^*})' \in \tau^*$. Then $A = (A \cap ([B]^{\tau^*})') \cup (B \cap ([B]^{\tau^*})' \in \tau^*$, hence $A \in \tau^*$. Since $A \cup B \in K$ and $[A]^K \in K$, then $(A \cup B) \cap [A]^K \in K$. Then, $A = (A \cap [A]^K) \cup (B \cap [A]^K) \in K$, it follows that $A \in K$. This means that, $A \in \tau^* \cap K$. The rest of the proof by s similar way.

□

Definition 4.2. A subset Z of a ditopological texture space with an ideal (X, L, τ, K, I) is called $\star_s$-connected if Z is not the union of two $\star$-separated sets in (X, L, τ, K, I).

Theorem 4.3. Let Y be a clopen subset of a ditopological texture space with an ideal (X, L, τ, K, I). Then Y is $\star_s$-connected if and only if it is $\star$-connected.

Proof. $\Rightarrow$**:** Suppose that Y is $\star$-disconnected, then either $\exists$ nonempty disjoint τ^*_Y-open and K_Y-open or $\mathcal{K}^*$-closed and τ-open subsets $A,\ B$ of Y s.t $Y = A \cup B$. Since $Y \in \tau \cap K$, by Theorem 2.8 and Theorem 3.4, A and B are τ^*-open and K-open or $\mathcal{K}^*$-closed and τ-closed subsets of X. Since A and B are disjoint, then either $B \cap [A]^{\tau^*} = A \cap [B]^K = \phi$ or $A \cap [B]^{\mathcal{K}^*} = B \cap [A]^{\tau} = \phi$. This implies that, $A,\ B$ are $\star$-separated sets in X s.t $Y = A \cup B$. Hence, Y is not $\star_s$-connected, which is a contradiction.

$\Leftarrow$**:** Suppose that Y is not $\star_s$-connected in X, then $\exists$ $\star$-separated sets $A,\ B$ s.t $Y = A \cup B$. By Theorem 4.2 $A \in \tau^* \cap K$. By Theorem 2.8 and Theorem 3.4, $A \in \tau^*_Y \cap K_Y$. Hence, Y is $\star$-disconnected by Theorem 4.1, which is a contradiction.

□

Theorem 4.4. Let Z be a $\star_s$-connected subset of a ditopological texture space with an ideal I on X (X, L, τ, K, I) and $A,\ B$ are $\star$-separated subsets of X with $Z \subseteq A \cup B$, then either $Z \subseteq A$ or $Z \subseteq B$.

Proof. Let $Z \subseteq A \cup B$ for some $\star$-separated subsets $A,\ B$ of X. Since $Z = (Z \cap A) \cup (Z \cap B)$, then $(Z \cap A) \cap ([Z \cap B]^{\tau^*}) \subseteq A \cap [B]^{\tau^*} = \phi$. By a similar way, we have $(Z \cap B) \cap ([Z \cap A]^K) = \phi$, $(Z \cap A) \cap ([Z \cap B]^{\mathcal{K}^*}) = \phi$ and $(Z \cap B) \cap ([Z \cap A]^{\tau}) = \phi$. Suppose that $Z \cap A$ and $Z \cap B$ are nonempty. Then Z is not $\star_s$-connected, which is a contradiction. Thus, either $Z \cap A = \phi$ or $Z \cap B = \phi$. This implies that, $Z \subseteq A$ or $Z \subseteq B$. □

Theorem 4.5. Let $\{Z_i : i \in J\}$ be a nonempty family of $\star_s$-connected subsets of a ditopological texture space with an ideal (X, L, τ, K, I) with $\bigcap_{i \in J} Z_i \neq \phi$, then $\bigvee_{i \in J} Z_i$ is also $\star_s$-connected.

Proof. Suppose that $Z = \bigvee_{i \in J} Z_i$ is not $\star_s$-connected. Then $Z = A \cup B$ for some two $\star$-separated subsets $A,\ B$ of X. Since $\bigcap_{i \in J} Z_i \neq \phi$, then $\exists\ x \in \bigcap_{i \in J} Z_i\ \forall i \in J$, so $x \in Z_i\ \forall i \in J$ and $x \in A$ or $x \in B$. Suppose that $x \in A$. Since $Z_i \subseteq A \cup B\ \forall i \in J$, then $Z_i \subseteq A$ or $Z_i \subseteq B\ \forall i \in J$ by Theorem 4.4. Since $A \cap B = \phi$, $Z_i \subseteq A; \forall i \in J$, then $Z \subseteq A$. This implies that, $B = \phi$, which is a contradiction. The rest of the proof is similar. □

Corollary 4.1. Let $\{Z_i : i \in J\}$ be a nonempty family of $\star_s$-connected subsets of a ditopological texture space with an ideal (X, L, τ, K, I) s.t one of the members of the family intersects every other members, then $Z = \bigvee_{i \in J} Z_i$ is $\star_s$-connected.

Proof. The proof is similar to Corollary 3.4. □

Definition 4.3. Let (X, L, τ, K, I) be a ditopological texture space with an ideal and let $Z \subseteq X$ with $x \in Z$. Then the $\star_s$-component of Z w.r.t x is the maximal of all $\star_s$-connected subsets of Z containing the point x.

Theorem 4.6. Every $\star_s$-component of a ditopological texture space with an ideal (X, L, τ, K, I) is a maximal $\star_s$-connected subset of X.

Proof. Immediate from Definition 4.3. □

Theorem 4.7. Let (X, L, τ, K, I) be a ditopological texture space with an ideal. Then:

(1) Each point in X is contained in exactly one component of X.

(2) Any two components w.r.t two different points of X are either disjoint or identical.

(3) Every τ^*-open and K-closed $\star_s$-connected subset of X is a $\star_s$-component of X.

Proof. Immediate from Theorem 2.5 and Theorem4.6. □

Corollary 4.2. The set of all distinct $\star_s$-components of a ditopological texture space with an ideal (X, L, τ, K, I) partition the set X.

Proof. Immediate from Theorem 4.7. □

5 Relation Between the $\star$-Connectedness

In this section we introduce some new types of a $\star$-connectedness in a ditopological texture spaces with an ideal (X, L, τ, K, I).

Definition 5.1. A ditopological texture space with an ideal (X, L, τ, K, I) is said to be locally $\star$-connected at a point $x \in X$ if and only if every τ^*-open and K-closed set and for every $\mathcal{K}^*$-closed and τ-open set containing x contains a $\star$-connected open set containing x and is said to be locally $\star$-connected if and only if it is locally $\star$-connected at each of its points.

Theorem 5.1. Every $\star$-connected space is a locally $\star$-connected space.

Proof. Suppose that (X, L, τ, K, I) be a $\star$-connected ditopological texture space with an ideal and $(X, L^*, \tau^*, \mathcal{K}^*)$ be a $\star$-ditopological texture space. Then $\tau^* \cap K = \{X, \phi\}$ and $\mathcal{K}^* \cap \tau = \phi$, hence $\forall x \in X \ \exists \ X \in \tau^*$ which is $\star$-connected set and $x \in X \subseteq X$. Then X is locally $\star$-connected. □

Theorem 5.2. Every $\star$-component of a locally $\star$-connected ditopological texture space with an ideal (X, L, τ, K, I) is a $\star$-open set.

Proof. Let (X, L, τ, K, I) be a locally $\star$-connected ditopological texture space with an ideal, $x \in X$ and C be a $\star$-component of X w.r.t x. Since (X, L, τ, K, I) is a locally $\star$-connected space. Therefore, every τ^*-open and K-closed set and every $\mathcal{K}^*$-closed and τ-open set containing x contains a $\star$-connected open set G containing x, but C is the largest $\star$-connected set containing x, then $x \in G \subseteq C$, i.e C is a τ^*-nbd of x. Then C is a τ^*-nbd of each of its points. This implies that, C is a $\star$-open set. □

Definition 5.2. A ditopological texture space with an ideal (X, L, τ, K, I) is said to be totally $\star$-disconnected if and only if $\forall x,\ y \in X$ s.t $x \neq y \ \exists$ a non empty disjoint τ^*-open and K-closed or $\mathcal{K}^*$-closed and τ-open subsets $A,\ B$ of X s.t $x \in A$ and $y \in B$.

Theorem 5.3. The $\star$-components of a totally $\star$-disconnected ditopological texture space with an ideal (X, L, τ, K, I) are the singleton subsets of X.

Proof. Suppose that Y be a subset of a totally $\star$-disconnected ditopological texture space with an ideal (X, L, τ, K, I), which containing more than one point of X. Let $y_1,\ y_2 \in Y \subseteq X$ s.t $y_1 \neq y_2$, since X is totally $\star$-disconnected, then $\exists$ a non empty disjoint τ^*-open and K-closed or $\mathcal{K}^*$-closed and τ-open proper subsets $A,\ B$ of X s.t $y_1 \in A$ and $y_2 \in B$. Clearly, $\{A, A\}$ is a partition of Y in both cases, then Y is $\star$-disconnected set, but the $\star$-components are $\star$-connected set, hence no subsets of X containing more than one point can be a $\star$-component of X. □

Definition 5.3. Let (X, L, τ, K) be a ditopological texture space. Then X is said to be hyperconnected if every pair of nonempty τ-open and K-open proper subsets $A,\ B$ of X respectively, has a nonempty intersection, i.e
(X, L, τ, K) is said to be hyperconnected if $\forall\ \ A \in \tau$ and $B \in K'$ we have $A \cap B \neq \phi$.

Theorem 5.4. Every hyperconnected ditopological texture space is connected.

Proof. Suppose that (X, L, τ, K) be a disconnected ditopological texture space, then there exists a proper subset A of X with $A \in \tau \cap K$. Then $A \in \tau$ and $A' \in K$ s.t $A \cap A' = \phi$, hence X is not hyperconnected, which is a contradiction. □

Remark 5.1. The converse of Theorem 5.4 is not true in general, for the following example,
let $X = \{a, b, c\}$, $L = \{X, \phi, \{a\}, \{c\}, \{a, c\}, \{a, b\}\}$, $\tau = \{X, \phi, \{a\}\}$, $K = \{X, \phi, \{b\}\}$. Then (X, L, τ, K) is connected but not hyperconnected.

Theorem 5.5. Every hyperconnected space is a locally connected space.

Proof. Immediate by Theorem 2.7 and Theorem 5.4. □

Definition 5.4. Let (X, L, τ, K, I) be a ditopological texture space with an ideal and $(X, L^*, \tau^*, \mathcal{K}^*)$ be a $\star$-ditopological texture space. Then X is said to be a $\star$-hyperconnected if A is $\star$-dense for ever nonempty τ-open subset A of X.

Theorem 5.6. Every $\star$-hyperconnected ditopological texture space is $\star$-connected.

Proof. Suppose that (X, L, τ, K, I) be a $\star$-disconnected ditopological texture space with an ideal. Then either $\exists\ A \in \tau$ and $B \in \mathcal{K}^{*'}$ or $A \in \tau^*$ and $B \in K'$ s.t $A \cap B = \phi$ and $X = A \cup B$. Then $B = \phi$, which is a contradiction.Then (X, L, τ, K, I) be a $\star$-connected. □

Theorem 5.7. Every $\star$-hyperconnected ditopological texture space is locally $\star$-connected.

Proof. Immediate by Theorem 5.1 and Theorem 5.6. □

Theorem 5.8. Let (X, L, τ, K, I) be a $\star$-hyperconnected ditopological texture space with an ideal, then (X, L, τ, K) is hyperconnected.

Proof. Immediate. □

Remark 5.2. The converse of Theorem 5.8 not true in general, for the following example, let $X = \{a, b, c\}$, $L = \{X, \phi, \{b\}, \{c\}, \{b, c\}\}$, $\tau = \{X, \phi, \{b\}\}$, $K = \{X, \phi, \{c\}\}$ and $I = \{\phi, \{b\}\}$ be an ideal on$X$, then $\tau^* = \{X, \phi, \{b\}, \{a, c\}\}$ and $\mathcal{K}^* = \{X, \phi, \{b\}, \{c\}, \{b, c\}\}$. Hence, $L^* = \{X, \phi, \{b\}, \{c\}, \{b, c\}, \{a, c\}\}$. Then (X, L, τ, K) is hyperconnected and (X, L, τ, K, I) is not $\star$-hyperconnected.

Theorem 5.9. . Let (X, L, τ, K, I) be a $\star$-hyperconnected ditopological texture space with an ideal, then (X, L, τ, K) is connected.

Proof. Immediate by Theorem 3.6 and Theorem 5.6. □

Theorem 5.10. Let (X, L, τ, K, I) be a $\star$-hyperconnected ditopological texture space with an ideal, then (X, L, τ, K) is locally connected.

Proof. Immediate by Theorem 2.7, Theorem 3.6 and Theorem 5.6. □

Theorem 5.11. The following implications hold for a ditopological texture space (X, L, τ, K, I) with an ideal.

(X, L, τ, K, I) is $\star$-hyperconnected	$\Rightarrow$	(X, L, τ, K) is hyperconnected
$\Downarrow$		$\Downarrow$
(X, L, τ, K, I) is $\star$-connected	$\Rightarrow$	(X, L, τ, K) is connected
$\Downarrow$		$\Downarrow$
(X, L, τ, K, I) is locally $\star$-connected		(X, L, τ, K) is locally connected

Proof. Immediate by Theorem 2.7, Theorem 3.6, Theorem 5.5, Theorem 5.6, Theorem 5.8 and Theorem 5.10. □

6 Conclusion

Topology is an important and major area of mathematics and it can give many relationships between other scientific areas and mathematical models. The notion of a texture space, under the name of fuzzy structure, was introduced by Brown in [2], as a means of representing a lattice of fuzzy sets as a lattice of crisp subsets of some base set. The notion of connectedness in ditopological texture spaces was initiated by Diker in [10], which is being extended in [12]. The main purpose of this paper, is to introduce the notion of ditopological texture spaces with ideal (X, L, τ, K, I), which is finer than the given ditopological texture space (X, L, τ, K) on the same set X. We study the notions of $\star$-connected ditopological texture spaces with an ideal and $\star$-connected sets in ditopological texture spaces with ideal. Moreover, we introduce new types of connectedness in $\star$-ditopological texture spaces namely, locally $\star$-connectedness, totally $\star$-disconnectedness, hyperconnectedness and $\star$-hyperconnectedness have investigated.

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On Soft Multi Continuous Functions

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Abstract – In this article, by using basic properties of soft multi topology and soft multi function, we define the notion of soft multi continuous function. We also introduce some basic definitions and theorems of the concept.

Keywords – *Soft multiset, Soft multi function, Soft multi topology, Soft multi continuous function.*

1 Introduction

Classical mathematical methods are not enough to solve the problems of daily life and also are not enough to meet the new requirements. Therefore, some theories such as Fuzzy set theory [23], Rough set theory [15], Soft set theory [12] and Multiset (or Bag) theory [22] have been developed to solve those problems.

Applications of these theories exists in many areas of mathematics. Shabir and Naz [17] defined the soft topological space and studied the concepts of soft open set, soft multi interior point, soft neighborhood of a point, soft separation axioms, and subspace of a soft topological space. Aygunoglu and Aygun [2] introduced the soft continuity of soft mapping, soft product topology and studied soft compactness and generalized Tychonoff theorem to the soft topological space. Min [11] gave some results on soft topological spaces. Zorlutuna et al. [24] also investigated soft interior point and soft neighborhood. There are some other studies on the structure of soft topological spaces ([3],[21]). Maji et al. [9] also initiated the more generalized concept of fuzzy soft sets which is a combination of fuzzy set and soft set. Tanay and Kandemir introduced topological structure of fuzzy soft set in [18] and gave a introductory theoretical base to carry further study on this concept. Following this study, some others ([20],[8],[1],[16]) studied on the concept of fuzzy soft topological spaces.

The concept of soft multisets which is combination of soft sets and multisets can be used to solve some real life problems. Also this concept can be used in many areas, such as data storage, computer science, information science, medicine, engineering, etc. The concept of soft multisets was introduced in [19]. Moreover, in [19],[13] and [14] soft multi topology and its some properties were given.

In this work we will recall soft multi function between two soft multiset. Then we will introduce soft multi continuous function on soft multi topological spaces and will give basic definitions and theorems of soft multi continuity.

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2 Preliminaries

2.1 Soft Sets, Multisets and Soft Multisets

In this section, we present the basic definitions of soft set, multiset and soft multiset which may be found in earlier studies [6, 12, 19].

Definition 2.1 (Soft set)**.** [12] Let U be an initial universe set and E be set of parameters. Let $P(U)$ denotes the power set of U and $A \subseteq U$. A pair (F, A) is called a soft set over U, where F is a mapping given by $F : A \to P(U)$.

Definition 2.2 (Multiset)**.** [6] An mset M drawn from the set X is represented by a function *Count* M or C_M defined as $C_M : X \to \mathbb{N}$. The word "multiset"is often shortened to "mset".

Let M be an mset from X with x appearing n times in M. It is denoted by $x \in^n M$. If $M = \{k_1/x_1, k_2/x_2, ..., k_n/x_n\}$, then, x_1 appearing k_1 times, x_2 appearing k_2 times and so on.

Definition 2.3. [6] Let M be an mset drawn from a set X. The support set of M denoted by M^* is a subset of X and $M^* = \{x \in X : C_M(x) > 0\}$. i.e., M^* is an ordinary set and it is also called root set.

The power set of an mset is the support set of the power mset and is denoted by $P^*(M)$.

Example 2.4. [5] Let $M = \{2/x, 3/y\}$ be an multiset. Then $M^* = \{x, y\}$ is the support set of M. The collection

$P(M) = \{3/\{2/x, 1/y\}, 3/\{2/x, 2/y\}, 6/\{1/x, 1/y\}, 6/\{1/x, 2/y\}, 2/\{1/x, 3/y\}, 1/\{2/x\}, 2/\{1/x\}, 1/\{3/y\}, 3/\{2/y\}, 3/\{1/y\}, M, \emptyset\}$

is the power multiset of M. The collection

$P^*(M) = \{\{2/x, 1/y\}, \{2/x, 2/y\}, \{1/x, 1/y\}, \{1/x, 2/y\}, \{1/x, 3/y\}, \{2/x\}, \{1/x\}, \{3/y\}, \{2/y\}, \{1/y\}, M, \emptyset\}$

is the support set of $P(M)$.

Definition 2.5 (Soft multiset)**.** [19] Let U be an multiset, E be set of parameters and $A \subseteq E$. Then a pair (F, A) is called a soft multiset where F is a mapping given by $F : A \to P^*(U)$. For $\forall e \in A$, multiset $F(e)$ represent by count function $C_{F(e)} : U^* \to \mathbb{N}$.

Example 2.6. [19] Let $U = \{1/x, 5/y, 3/z, 4/w\}$ and $E = \{p, q, r\}$. Define a mapping $F : E \to P^*(U)$ as follows:

$$F(p) = \{1/x, 2/y, 3/z\}, F(q) = \{4/w\} \text{ and } F(r) = \{3/y, 1/z, 2/w\}.$$

Then (F, A) is a soft multiset where for $\forall e \in A$, $F(e)$ multiset represent by count function $C_{F(e)} : U^* \to \mathbb{N}$, which are defined as follows:

$$\begin{array}{llll} C_{F(p)}(x) = 1, & C_{F(p)}(y) = 2, & C_{F(p)}(z) = 3, & C_{F(p)}(w) = 0, \\ C_{F(q)}(x) = 0, & C_{F(q)}(y) = 0, & C_{F(q)}(z) = 0, & C_{F(q)}(w) = 4, \\ C_{F(r)}(x) = 0, & C_{F(r)}(y) = 3, & C_{F(r)}(z) = 1, & C_{F(r)}(w) = 2. \end{array}$$

Then $(F, A) = \{F(p), F(q), F(r)\} = \{\{1/x, 2/y, 3/z\}, \{4/w\}, \{3/y, 1/z, 2/w\}\}$.

Definition 2.7. [19] For two soft multisets (F, A) and (G, B) over U, we say that (F, A) is a soft submultiset of (G, B) if

i. $A \subseteq B$

ii. $C_{F(e)}(x) \leq C_{G(e)}(x), \forall x \in U^*, \forall e \in A$

We write $(F, A)\tilde{\subset}(G, B)$.

In addition to (F, A) is a whole soft submultiset of (G, B) if $C_{F(e)}(x) = C_{G(e)}(x), \forall x \in U^*, \forall e \in A$.

Definition 2.8. [19] Let (F, A) and (G, B) be two soft multisets over U.

Equality $(F, A) = (G, B) \Leftrightarrow (F, A)\tilde{\subseteq}(G, B)$ and $(F, A)\tilde{\supseteq}(G, B)$.

Union $(H, C) = (F, A)\tilde{\cup}(G, B)$ where $C = A \cup B$ and $C_{H(e)}(x) = \max\{C_{F(e)}(x), C_{G(e)}(x)\}, \forall e \in A \cup B, \forall x \in U^*$.

Intersection $(H,C) = (F,A)\tilde{\cap}(G,B)$ where $C = A \cap B$ and $C_{H(e)}(x) = \min\{C_{F(e)}(x), C_{G(e)}(x)\}, \forall e \in A \cap B, \forall x \in U^*$.

Difference $(H,E) = (F,E) \backslash (G,E)$ where $C_{H(e)}(x) = \max\left\{C_{F(e)}(x) - C_{G(e)}(x), 0\right\}, \forall x \in U^*$.

Null A soft multiset (F,A) is said to be a NULL soft multiset denoted by Φ if for all $e \in A$, $F(e) = \emptyset$.

Complement The complement of a soft multiset (F,A) is denoted by $(F,A)^c$ and is defined by $(F,A)^c = (F^c, A)$ where $F^c : A \to P^*(U)$ is a mapping given by $F^c(e) = U \backslash F(e)$ for all $e \in A$ where $C_{F^c(e)}(x) = C_U(x) - C_{F(e)}(x), \forall x \in U^*$.

Definition 2.9. Let (F,E) be a soft set over U. The soft set (F,E) is called a soft multi point, denoted by x_e, if for the element $e \in E$, $F(e) = \{1/x\} = \{x\}$ and $F(\acute{e}) = \emptyset$, for all $\acute{e} \in E - \{e\}$.

Definition 2.10. Let (F,E) be a soft set over U and $e \in E$. $x_e \in (F,E)$ if for $C_{F(e)}(x) = n$, $n \geq 1$.

Example 2.11. Let $U = \{2/x, 1/y, 1/z, 3/w\}$, $E = \{e_1, e_2, e_3\}$ and $(F,A) = \{F(e_1), F(e_2)\} = \{\{1/x, 1/z, 2/w\}, \{1/y, 2/w\}\}$. Then $y_{e_2} \in (F,A)$ since $C_{F(e_2)}(y) = 1$. But $y_{e_1} \in (F,A)$ since $C_{F(e_1)}(y) = 0$.

Definition 2.12. [19] Let V be a non-empty submultiset of U, then $\tilde{V}$ denotes the soft multiset (V,E) over U for which $V(e) = V$, for all $e \in E$.

In particular, (U,E) will be denoted by $\tilde{U}$.

2.2 Soft Multi Function

In this section, we recall soft multi function which was given in [13].

Definition 2.13. [13] Let X be multiset and E be set of parameters. Then the collection of all soft multisets over X with parameters from E is called a soft multi class and is denoted as X_E.

Definition 2.14. [13] Let X_E and Y_K be two soft multi class. Let $\varphi : X^* \to Y^*$ and $\psi : E \to K$ be two function. Then the pair (φ, ψ) is called a soft multi function and denoted by $f = (\varphi, \psi) : X_E \to Y_K$ is defined as follows:
Let (F,E) be a soft multiset in X_E. Then the image of (F,E) under soft multi function f is soft multiset in Y_K defined by $f(F,E)$, where for $k \in \psi(E) \subseteq K$ and $y \in Y^*$,

$$C_{f(F,E)(k)}(y) = \begin{cases} \sup\limits_{e \in \psi^{-1}(k) \cap E, x \in \varphi^{-1}(y)} C_{F(e)}(x), & \text{if } \psi^{-1}(k) \neq \emptyset, \varphi^{-1}(y) \neq \emptyset; \\ 0, & \text{otherwise.} \end{cases}$$

Let (G,K) be a soft multiset in Y_K. Then the inverse image of (G,K) under soft multi function f is soft multiset in X_E defined by $f^{-1}(G,K)$, where for $e \in \psi^{-1}(K) \subseteq E$ and $x \in X^*$,

$$C_{f^{-1}(G,K)(e)}(x) = C_{G(\psi(e))}(\varphi(x)).$$

f is said to be injective (onto or surjective) if both $\varphi : X^* \to Y^*$ and $\psi : E \to K$ are injective (onto or surjective) mappings. If f is both injective as well as surjective, then f is said to be a soft multi bijective function.

Example 2.15. [13] Let $X = \{2/a, 3/b, 4/c, 5/d\}$, $Y = \{5/x, 4/y, 3/z, 2/w\}$, $E = \{e_1, e_2, e_3, e_4\}$, $K = \{k_1, k_2, k_3\}$ and X_E, Y_K, classes of soft multisets. Let $\varphi : X^* \to Y^*$ and $\psi : E \to K$ be two function defined as

$$\varphi(a) = z, \quad \varphi(b) = y, \quad \varphi(c) = y, \quad \varphi(d) = x,$$
$$\psi(e_1) = k_1, \quad \psi(e_2) = k_3, \quad \psi(e_3) = k_2, \quad \psi(e_4) = k_1.$$

Choose two soft multisets in X_E and Y_K, respectively, as

$$(F,A) = \{e_1 = \{1/a, 2/b, 1/d\}, e_3 = \{3/b, 2/c, 1/d\}, e_4 = \{2/a, 5/d\}\},$$
$$(G,B) = \{k_1 = \{4/x, 2/w\}, k_2 = \{1/x, 1/y, 2/z, 2/w\}\}$$

Then soft multiset image of (F,A) under $f : X_E \to Y_K$ is obtained as

$$C_{f(F,A)(k_1)}(x) = \begin{cases} \sup\limits_{e\in\psi^{-1}(k_1)\cap A, a\in\varphi^{-1}(x)} C_{F(e)}(a), & \text{if } \psi^{-1}(k_1) \neq \emptyset, \varphi^{-1}(x) \neq \emptyset; \\ 0, & \text{otherwise.} \end{cases}$$
$$= \begin{cases} \sup\limits_{e\in\{e_1,e_4\}, a\in\{d\}} C_{F(e)}(a), & \text{if } \psi^{-1}(k_1) \neq \emptyset, \varphi^{-1}(x) \neq \emptyset; \\ 0, & \text{otherwise.} \end{cases}$$
$$= \sup\left\{C_{F(e_1)}(d), C_{F(e_4)}(d)\right\}$$
$$= 5,$$
$$C_{f(F,A)(k_1)}(y) = \sup\left\{C_{F(e_1)}(b), C_{F(e_4)}(b), C_{F(e_1)}(c), C_{F(e_4)}(c)\right\} = 2,$$
$$C_{f(F,A)(k_1)}(z) = \sup\left\{C_{F(e_1)}(a), C_{F(e_4)}(a)\right\} = 2,$$
$$C_{f(F,A)(k_1)}(w) = 0 \text{ (since } \varphi^{-1}(w) = \emptyset\text{)},$$
$$C_{f(F,A)(k_2)}(x) = \begin{cases} \sup\limits_{e\in\psi^{-1}(k_2)\cap A, a\in\varphi^{-1}(x)} C_{F(e)}(a), & \text{if } \psi^{-1}(k_2) \neq \emptyset, \varphi^{-1}(x) \neq \emptyset; \\ 0, & \text{otherwise.} \end{cases}$$
$$= \begin{cases} \sup\limits_{e\in\{e_3\}, a\in\{d\}} C_{F(e)}(a), & \text{if } \psi^{-1}(k_2) \neq \emptyset, \varphi^{-1}(x) \neq \emptyset; \\ 0, & \text{otherwise.} \end{cases}$$
$$= \sup\left\{C_{F(e_3)}(d)\right\}$$
$$= 1,$$
$$C_{f(F,A)(k_2)}(y) = \sup\left\{C_{F(e_3)}(b), C_{F(e_3)}(c)\right\} = 3,$$
$$C_{f(F,A)(k_2)}(z) = \sup\left\{C_{F(e_3)}(a)\right\} = 0,$$
$$C_{f(F,A)(k_2)}(w) = 0 \text{ (since } \varphi^{-1}(w) = \emptyset\text{)}.$$

Consequently, we have
$$(f(F,A),B) = \{k_1 = \{5/x, 2/y, 2/z\}, k_2 = \{1/x, 3/y\}\}.$$

Soft multiset inverse image of (G,B) under $f : X_E \to Y_K$ is obtained as
$$C_{f^{-1}(G,B)(e_3)}(a) = C_{G(\psi(e_3))}(\varphi(a)) = C_{G(k_2)}(z) = 2,$$
$$C_{f^{-1}(G,B)(e_3)}(b) = C_{G(\psi(e_3))}(\varphi(b)) = C_{G(k_2)}(y) = 1,$$
$$C_{f^{-1}(G,B)(e_3)}(c) = C_{G(\psi(e_3))}(\varphi(c)) = C_{G(k_2)}(y) = 1,$$
$$C_{f^{-1}(G,B)(e_3)}(d) = C_{G(\psi(e_3))}(\varphi(d)) = C_{G(k_2)}(x) = 1,$$
$$C_{f^{-1}(G,B)(e_4)}(a) = C_{G(\psi(e_4))}(\varphi(a)) = C_{G(k_1)}(z) = 0,$$
$$C_{f^{-1}(G,B)(e_4)}(b) = C_{G(\psi(e_4))}(\varphi(b)) = C_{G(k_1)}(y) = 0,$$
$$C_{f^{-1}(G,B)(e_4)}(c) = C_{G(\psi(e_4))}(\varphi(c)) = C_{G(k_1)}(y) = 0,$$
$$C_{f^{-1}(G,B)(e_4)}(d) = C_{G(\psi(e_4))}(\varphi(d)) = C_{G(k_1)}(x) = 4.$$
Consequently, we have
$$(f^{-1}(G,B),D) = \{e_3 = \{2/a, 1/b, 1/c, 1/d\}, e_4 = \{4/d\}\}.$$

Theorem 2.16. [13] Let $f : X_E \to Y_K$ be a soft multi function, $(F,A),(F_i,A)$ soft multisets in X_E and $(G,B),(G_i,B)$ soft multisets in Y_K.

(1) $f(\Phi) = \Phi$, $f(\tilde{X})\tilde{\subseteq}\tilde{Y}$,

(2) $f^{-1}(\Phi) = \Phi$, $f^{-1}(\tilde{Y}) = \tilde{X}$,

(3) $f((F_1,A_1)\tilde{\cup}(F_2,A_2)) = f(F_1,A_1)\tilde{\cup}f(F_2,A_2)$.
In general, $f(\tilde{\cup}_{i\in I}(F_i,A_i)) = \tilde{\cup}_{i\in I}\ f(F_i,A_i)$,

(4) $f^{-1}((G_1,B)\tilde{\cup}(G_2,B)) = f^{-1}(G_1,B)\tilde{\cup}f^{-1}(G_2,B)$.
In general, $f^{-1}(\tilde{\cup}_{i\in I}(G_i,B)) = \tilde{\cup}_{i\in I}\ f^{-1}(G_i,B)$,

(5) $f((F_1,A)\tilde{\cap}(F_2,A))\tilde{\subseteq}f(F_1,A)\tilde{\cap}f(F_2,A)$.
In general, $f(\tilde{\cap}_{i\in I}(F_i,A))\tilde{\subseteq}\tilde{\cap}_{i\in I}\ f(F_i,A)$,

(6) $f^{-1}((G_1,B)\tilde{\cap}(G_2,B)) = f^{-1}(G_1,B)\tilde{\cap}f^{-1}(G_2,B)$.
In general, $f^{-1}(\tilde{\cap}_{i\in I}(G_i,B)) = \tilde{\cap}_{i\in I}\ f^{-1}(G_i,B)$,

(7) If $(F_1,A)\tilde{\subseteq}(F_2,A)$, then $f(F_1,A)\tilde{\subseteq}f(F_2,A)$,

(8) If $(G_1,B)\tilde{\subseteq}(G_2,B)$, then $f^{-1}(G_1,B)\tilde{\subseteq}f^{-1}(G_2,B)$.

2.3 Soft Multi Topology

In this section, we recall soft multi topology which was given in [19].

Definition 2.17. [19] Let $\tau \subseteq X_E$, then τ is said to be a soft multi topology on X if the following conditions hold.

i. Φ,$\tilde{X}$ belong to τ.

ii. The union of any number of soft multisets in τ belongs to τ.

iii. The intersection of any two soft multisets in τ belongs to τ.

τ is called a soft multi topology over X and the binary (X_E, τ) is called a soft multi topological space over X.

The members of τ are said to be soft multi open sets in X.

A soft multiset (F, E) over X is said to be a soft multi closed set in X, if its complement $(F, E)^c$ belongs to τ .

Example 2.18. [19] Let $X = \{2/x, 3/y, 4/z, 5/w\}$, $E = \{p, q\}$ and $\tau = \{\Phi, \tilde{X}, (F_1, E), (F_2, E), (F_3, E)\}$ where $(F_1, E), (F_2, E), (F_3, E)$ are soft multisets over X, defined as follows

$F_1 (p) = \{1/x, 2/y, 3/z\}$, $F_1 (q) = \{4/w\}$

$F_2 (p) = X$, $F_2 (q) = \{1/x, 3/y, 4/z, 5/w\}$

$F_3 (p) = \{2/x, 3/y, 3/z, 1/w\}$, $F_3 (q) = \{1/x, 4/w\}$.

Then τ defines a soft multi topology on X and hence (X_E, τ) is a soft multi topological space over X.

Remark 2.19. In the example above, without allowing the repetitions for the elements of X, we obtain soft topology on X. Thus the concept of soft multi topology is more general than that of soft topology.

Definition 2.20. [19] Let X be multiset, E be the set of parameters.

- let τ^1 be the collection of all soft multisets which can be defined over X. Then τ^1 is called the soft multi discrete topology on X and (X_E, τ^1) is said to be a soft multi discrete space over X.
- $\tau^0 = \{\Phi, \tilde{X}\}$ is called the soft multi indiscrete topology on X and (X_E, τ^0) is said to be a soft indiscrete space over X.

3 Soft Multi Continuous Functions

In this section, we define soft multi continuous functions and examine their properties.

Definition 3.1. Let (X_E, τ) and (Y_K, σ) be two soft multi topological spaces, $f : (X_E, \tau) \to (Y_K, \sigma)$ be a soft multi function. For each soft multi open neighbourhood (G, B) of $f(x)_k$, if there exists a soft multi open neighbourhood (F, A) of x_e such that $f((F, A))\tilde{\subset}(G, B)$ then f is said to be soft multi continuous function on x_e. If f is soft multi continuous function for all x_e, then f is called soft multi continuous function.

Theorem 3.2. Let (X_E, τ) and (Y_K, σ) be two soft multi topological spaces, $f : (X_E, \tau) \to (Y_K, \sigma)$ be a soft multi function. Then f is soft multi continuous function if and only if $f^{-1}((G, B))$ is a soft multi open set in X_E, for each soft multi open set (G, B) in Y_K.

Proof. $\Rightarrow$: Let (G, B) be a soft multi open set in Y_K and $x_e \in f^{-1}((G, B))$ be an arbitrary soft multi point. Then $f(x)_k = f(x_e) \in f(f^{-1}((G, B)))\tilde{\subseteq}(G, B)$. Since f is soft multi continuous function, there exists soft multi open set $x_e \in (F, A)$ such that $f((F, A))\tilde{\subset}(G, B)$. This implies that $x_e \in (F, A)\tilde{\subseteq} f^{-1}(f((F, A)))\tilde{\subseteq}(f^{-1}((G, B))$ is a soft multi open set in X_E. □

$\Leftarrow$:Let x_e be a soft multi point and $f(x)_k \in (G, B)$ be an arbitrary soft multi open neighbourhood. Then $(F, A) = f^{-1}((G, B))$ is a soft multi open set in X_E, $x_e \in (F, A)$ and $f((F, A))\tilde{\subset}(G, B)$.

Theorem 3.3. Let (X_E, τ) and (Y_K, σ) be two soft multi topological spaces, $f : (X_E, \tau) \to (Y_K, \sigma)$ be a soft multi function. Then f is soft multi continuous function if and only if $f^{-1}((G, B))$ is a soft multi closed set in X_E, for each soft multi closed set (G, B) in Y_K.

Proof. This proof is similar to Theorem 3.2. □

Example 3.4. Let (X_E, τ) and (Y_K, σ) be two soft multi topological spaces. If $\tau = \tau^1$, then every function $f : (X_E, \tau) \to (Y_K, \sigma)$ is soft multi continuous. Also if $\sigma = \tau^0$, then every function $f : (X_E, \tau) \to (Y_K, \sigma)$ is soft multi continuous.

Example 3.5. Let $X = \{6/a, 7/b, 8/c\}$, $Y = \{8/x, 9/y, 7/z\}$, $E = \{e_1, e_2, e_3, e_4\}$ and $K = \{k_1, k_2, k_3\}$. Let $\tau = \{\Phi, \tilde{X}, (F_1, A_1), (F_2, A_2)\}$ and $\sigma = \{\Phi, \tilde{Y}, (G_1, B_1), (G_2, B_2)\}$ where soft multisets defined as follows

$(F_1, A_1) = \{F_1(e_1), F_1(e_2)\} = \{\{1/a, 2/b, 2/c\}, \{1/a, 2/b, 2/c\}\}$,
$(F_2, A_2) = \{F_2(e_1), F_2(e_2), F_2(e_2)\} = \{\{6/a, 7/b, 7/c\}, \{6/a, 7/b, 7/c\}, \{6/a, 4/b, 4/c\}\}$,
$(G_1, B_1) = \{G_1(k_1), G_1(k_2)\} = \{\{3/x, 2/y, 3/z\}, \{5/x, 9/y, 6/z\}\}$,
$(G_2, B_2) = \{G_2(k_1), G_2(k_2), G_2(k_3)\} = \{\{3/x, 7/y, 6/z\}, \{8/x, 9/y, 7/z\}, \{2/x, 4/y, 6/z\}\}$.

Then (X_E, τ) and (Y_K, σ) are soft multi topological spaces over X and Y, respectively. Let $\varphi : X^* \to Y^*$ and $\psi : E \to K$ be two function defined as

$\varphi(a) = z, \quad \varphi(b) = y, \quad \varphi(c) = y,$
$\psi(e_1) = k_1, \quad \psi(e_2) = k_1, \quad \psi(e_3) = k_3, \quad \psi(e_4) = k_2.$

Since $f^{-1}(\Phi) = \Phi$, $f^{-1}(\tilde{Y}) = \tilde{X}$, $f^{-1}((G_1, B_1)) = (F_1, A_1)$ and $f^{-1}((G_2, B_2)) = (F_2, A_2)$, then function $f : (X_E, \tau) \to (Y_K, \sigma)$ is soft multi continuous.

Theorem 3.6. Let (X_E, τ) and (Y_K, σ) be two soft multi topological spaces, $f : (X_E, \tau) \to (Y_K, \sigma)$ be a soft multi function. Then the following statements are equivalent:

i. f is soft multi continuous function,

ii. For each soft multi set (F, A) in X_E, $f(\overline{(F, A)})\tilde{\subseteq}\overline{f((F, A))}$,

iii. For each soft multi set (G, B) in Y_K, $\overline{f^{-1}((G, B))}\tilde{\subseteq} f^{-1}(\overline{(G, B)})$.

iv. For each soft multi set (G, B) in Y_K, $f^{-1}((G, B)^\circ)\tilde{\subseteq} f^{-1}((G, B))^\circ$

Proof. **i. ⇒ ii.** Let f be soft multi continuous function and (F, A) be soft multi set in X_E. Since $f((F, A))\tilde{\subseteq}\overline{f((F, A))}$, then $(F, A)\tilde{\subseteq} f^{-1}(f((F, A)))\tilde{\subseteq} f^{-1}(\overline{f((F, A))})$. Using this statement and continuity of f , we have soft multi closed set $f^{-1}(\overline{f((F, A))})$ in Y_K and $\overline{f^{-1}(\overline{f((F, A))})} = f^{-1}(\overline{f((F, A))})$. Then $\overline{(F, A)}\tilde{\subseteq} f^{-1}(\overline{f((F, A))})$ and so $f(\overline{(F, A)})\tilde{\subseteq}\overline{f((F, A))}$.

ii. ⇒ iii. Let (G, B) be a soft multi set in Y_K and $f^{-1}((G, B)) = (F, A)$. By part ii., we have $f(\overline{(F, A)}) = f(\overline{f^{-1}((G, B))})\tilde{\subseteq}\overline{f(f^{-1}((G, B)))}\tilde{\subseteq}\overline{(G, B)}$. Then $\overline{f^{-1}((G, B))} = \overline{(F, A)}\tilde{\subseteq} f^{-1}(\overline{(F, A)})\tilde{\subseteq} f^{-1}(\overline{(G, B)})$.

iii. ⇒ iv. Let (G, B) be a soft multi set in Y_K. Using part iii., we have $\overline{f^{-1}((G, B)^c)}\tilde{\subseteq} f^{-1}(\overline{(G, B)^c})$. Since $(G, B)^\circ = (\overline{(G, B)^c})^c$, then $f^{-1}((G, B)^\circ) = f^{-1}((\overline{(G, B)^c})^c) = (f^{-1}(\overline{(G, B)^c}))^c\tilde{\subseteq}(\overline{f^{-1}((G, B)^c)})^c = f^{-1}((G, B))^\circ$.

iv. ⇒ i. Let (G, B) be a soft multi open set in Y_K . Since $f^{-1}((G, B)) = f^{-1}((G, B)^\circ)\tilde{\subseteq} f^{-1}((G, B))^\circ$, we have $f^{-1}((G, B))^\circ = f^{-1}((G, B))$. Then $f^{-1}((G, B))$ is soft multi open set in X_E and so f is soft multi continuous function.

□

Definition 3.7. Let (X_E, τ) and (Y_K, σ) be two soft multi topological spaces, $f : (X_E, \tau) \to (Y_K, \sigma)$ be a soft multi function.

– [13] Soft multi function f is called soft multi open if $f((F, A))$ is a soft multi open set in Y_K, for each soft multi open set (F, A) in X_E.

– Soft multi function f is called soft multi closed if $f((F,A))$ is a soft multi closed set in Y_K, for each soft multi closed set (F,A) in X_E.

Theorem 3.8. Let (X_E,τ) and (Y_K,σ) be two soft multi topological spaces, $f:(X_E,\tau)\to(Y_K,\sigma)$ be a soft multi function.

i. f is a soft multi open function if and only if for each soft multi set (F,A) in X_E, $f((F,A)^\circ)\tilde{\subseteq}(f((F,A)))^\circ$ is satisfied.

ii. f is a soft multi closed function if and only if for each soft multi set (F,A) in X_E, $\overline{f((F,A))}\tilde{\subseteq}f(\overline{(F,A)})$ is satisfied.

Proof. **i.** Let f be a soft multi open function and (F,A) be a soft multiset in X_E. Since $(F,A)^\circ\tilde{\subseteq}(F,A)$ and f is soft multi open function, then $f((F,A)^\circ)\tilde{\subseteq}f((F,A))$ and $f((F,A)^\circ)$ is a soft multi open set in Y_K. Thus $f((F,A)^\circ)\tilde{\subseteq}(f((F,A)))^\circ$.

Let (F,A) be any soft open multiset in X_E. Using $(F,A)=(F,A)^\circ$, we have $f((F,A)) = f((F,A)^\circ)\tilde{\subseteq}(f((F,A)))^\circ$. Then $f((F,A))=(f((F,A)))^\circ$ and so f is a soft multi open function.

ii. Let f be a soft multi closed function and (F,A) be a soft multiset in X_E. Since $(F,A)\tilde{\subseteq}\overline{(F,A)}$ and f is soft multi closed function, then $f((F,A))\tilde{\subseteq}f(\overline{(F,A)})$ and $f(\overline{(F,A)})$ is a soft multi closed set in Y_K. Thus $\overline{f((F,A))}\tilde{\subseteq}f(\overline{(F,A)})$.

Let (F,A) be any soft closed multiset in X_E. Using $(F,A)=\overline{(F,A)}$, we have $\overline{f((F,A))}\tilde{\subseteq}f(\overline{(F,A)}) = f((F,A))$. Then $f((F,A))=\overline{f((F,A))}$ and so f is a soft multi closed function. □

Theorem 3.9. Let (X_E,τ) and (Y_K,σ) be two soft multi topological spaces, $f:(X_E,\tau)\to(Y_K,\sigma)$ be a soft multi bijection function. f is a soft multi open function if and only if f is a soft multi closed function.

Proof. For any soft multiset (F,A) in X_E, $f((F,A)^c)=(f((F,A)))^c$.

Let (F,A) be any soft closed multiset in X_E. Then $(F,A)^c$ is soft open multiset in X_E. Since f is a soft multi open function and $f((F,A)^c)=(f((F,A)))^c$, then $(f((F,A)))^c$ is soft open multiset in Y_K. Thus $f((F,A))$ is soft closed multiset in Y_K and so f is a soft multi closed function.

Let (F,A) be any soft open multiset in X_E. Then $(F,A)^c$ is soft closed multiset in X_E. Since f is a soft multi closed function and $f((F,A)^c)=(f((F,A)))^c$, then $(f((F,A)))^c$ is soft closed multiset in Y_K. Thus $f((F,A))$ is soft open multiset in Y_K and so f is a soft multi open function. □

Theorem 3.10. Let (X_E,τ) and (Y_K,σ) be two soft multi topological spaces, $f:(X_E,\tau)\to(Y_K,\sigma)$ be a soft multi injective function. f is a soft multi continuous and a soft multi open function if and only if for each soft multi set (F,A) in X_E, $(f((F,A)))^\circ=f((F,A)^\circ)$ is satisfied.

Proof. It can be proved easily using Theorem 3.6 and Theorem 3.8. □

Theorem 3.11. Let (X_E,τ) and (Y_K,σ) be two soft multi topological spaces, $f:(X_E,\tau)\to(Y_K,\sigma)$ be a soft multi function. f is a soft multi continuous and a soft multi closed function if and only if for each soft multi set (F,A) in X_E, $\overline{f((F,A))}=f(\overline{(F,A)})$ is satisfied.

Proof. It can be proved easily using Theorem 3.6 and Theorem 3.8. □

Theorem 3.12. Let (X_E,τ) and (Y_K,σ) be two soft multi topological spaces, $f:(X_E,\tau)\to(Y_K,\sigma)$ be a soft multi bijection function. f is a soft multi continuous if and only if f is a soft multi open (closed) function.

Proof. Let (G,B) be a soft multi open (closed) set in Y_K. Since f is soft multi continuous, then $f^{-1}((G,B))$ is soft multi open (closed) set in X_E. Thus it clear that f^{-1} is soft multi open (closed) function.

Let (G,B) be a soft multi open (closed) set in Y_K. Since f^{-1} is soft multi open (closed) function, then $f^{-1}((G,B))$ is soft multi open (closed) set in X_E. Thus it clear that f^{-1} is soft multi continuous function. □

Definition 3.13. Let (X_E, τ) and (Y_K, σ) be two soft multi topological spaces, $f : (X_E, \tau) \rightarrow (Y_K, \sigma)$ be a soft multi function. If f is a soft multi bijection, soft multi continuous and f^{-1} is a soft multi continuous function, then f is said to be soft multi homeomorphism from X to Y . When a homeomorphism f exists between X and Y , we say that X is soft multi homeomorphic to Y.

Theorem 3.14. Let (X_E, τ) and (Y_K, σ) be two soft multi topological spaces, $f : (X_E, \tau) \rightarrow (Y_K, \sigma)$ be a soft multi function. Then the following statements are equivalent:

i. f is a soft multi homeomorphism,

ii. f is a soft multi continuous and soft multi open function,

iii. f is a soft multi continuous and soft multi closed function,

iv. For each soft multi set (F, A) in X_E, $\overline{f((F, A))} = f(\overline{(F, A)})$.

Proof. **i.**⇔ **ii.** İt is clear from Theorem 3.12.

ii.⇔ **iii.** İt is clear from Theorem 3.9.

iii.⇔ **iv.** İt is clear from Theorem 3.11.

□

4 Conclusion

In this work, we introduced soft multi continuous function , soft multi open function, soft multi closed function and soft multi homeomorphism. Also we gave some basic properties of these concepts.

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Upper and Lower Continuity of Fuzzy Soft Multifunctions

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Abstract – In this paper, we define the upper and lower inverse of a fuzzy soft multifunction and prove some basic identities. Then by using these ideas we introduced the concept of fuzzy soft continuity and obtain many interesting properties of upper and lower fuzzy soft continuous multifunctions.

Keywords – Fuzzy soft sets, Fuzzy soft multifunction, Fuzzy soft continuity.

1 Introduction

Engineering, physics, computer sciences, economics, social sciences, medical sciences and many other diverse fields deal with the uncertain data that may not be successfully modeled by the classical mathematics. There are some mathematical tools for dealing with uncertainties; two of them are fuzzy set theory, developed by Zadeh [24], and soft set theory, introduced by Molodtsov [19], that are related to this work. At present, works on the soft set theory and its applications are progressing rapidly. Maji et al [14] defined operations of soft sets to make a detailed theoretical study on the soft sets. By using these definitions, soft set theory has been applied in several directions, such as topology [5, 17, 22, 23, 25], various algebraic structures [2, 3, 7, 11], operations research [4, 9, 10] especially decision-making [6, 8, 13, 15, 20]. In recent times, researchers have contributed a lot towards fuzzification of soft set theory. Maji et al. [16] introduced the concept of fuzzy soft set and some properties regarding fuzzy soft union, intersection, complement of a fuzzy soft set, De Morgan Law etc. These results were further revised and improved by Ahmad and Kharal [1]. Tanay and Kandemir [23] introduced the definition of fuzzy soft topology over a subset of the initial universe set. Later, Roy and Samanta [21] gave the definition of fuzzy soft topology over the initial universe set. There are various types of functions which play an important role in the classical theory of set topology. A great deal of works on such functions has been extended to the setting of multifunctions. A multifunction is a set-valued function. The theory of multifunctions was first codified by Berge [26]. In the last three decades, the theory of multifunctions has advanced in a variety of ways and applications of this theory, can be found for example, in economic theory, noncooparative games, artificial intelligence, medicine,

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information sciences and decision theory. Papageorgiou [27], Allbrycht and Maltoka [28] , Beg [29], Heilpein [30] and Butnairu [31] have started the study of fuzzy multifunctions and obtained several fixed point theorems for fuzzy mappings.

In this paper our purpose is two fold. First, we define upper and lower inverse of a fuzzy soft multifunction and study their various properties. Next, we use these ideas to introduce upper fuzzy soft continuous multifunctions and lower soft continuous multifunctions. Moreover, we obtain some characterizations and several properties concerning such multifunctions.

2 Preliminary

Throughout this paper X denotes initial universe, E denotes the set of all possible parameters which are attributes, characteristic or properties of the objects in X, and the set of all subsets of X will be denoted by $P(X)$.

Definition 2.1. [24] A fuzzy set A of a non-empty set X is characterized by a membership function $\mu_A : X \to [0,1]$ whose value $\mu_A(x)$ represents the "grade of membership" of x in A for $x \in X$.

Let I^X denotes the family of all fuzzy sets on X. If $A, B \in I^X$, then some basic set operations for fuzzy sets are given by Zadeh as follows:
(1) $A \leq B \Leftrightarrow \mu_A(x) \leq \mu_B(x)$, for all $x \in X$.
(2) $A = B \Leftrightarrow \mu_A(x) = \mu_B(x)$, for all $x \in X$.
(3) $C = A \vee B \Leftrightarrow \mu_C(x) = \mu_A(x) \vee \mu_B(x)$, for all $x \in X$.
(4) $D = A \wedge B \Leftrightarrow \mu_D(x) = \mu_A(x) \wedge \mu_B(x)$, for all $x \in X$.
(5) $E = A^C \Leftrightarrow \mu_E(x) = 1 - \mu_A(x)$, for all $x \in X$.

A fuzzy point in X, whose value is α $(0 < \alpha \leq 1)$ at the support $x \in X$, is denoted by x_α [24]. A fuzzy point $x_\alpha \in A$, where A is a fuzzy set in X iff $\alpha \leq \mu_A(x)$ [24]. The class of all fuzzy points will be denoted by $S(X)$.

Definition 2.2. [18] For two fuzzy sets A and B in X, we write AqB to mean that A is quasi-coincident with B, i.e., there exists at least one point $x \in X$ such that $\mu_A(x) + \mu_B(x) > 1$. If A is not quasi-coincident with B, then we write $A\overline{q}B$.

Definition 2.3. [19] Let X be the initial universe set and E be the set of parameters. A pair (F, A) is called a soft set over X where F is a mapping given by $F : A \longrightarrow P(X)$ and $A \subseteq E$.

In the other words, the soft set is a parametrized family of subsets of the set X. Every set $F(e)$, for every $e \in A$, from this family may be considered as the set of e-elements of the soft set (F, A).

Definition 2.4. [16] Let $A \subseteq E$. A pair (f, A) is called a fuzzy soft set over X if $f : A \longrightarrow I^X$ is a function.

We will use $FS(X, E)$ instead of the family of all fuzzy soft sets over X.

Roy and Samanta [21] did some modifications in above definition analogously ideas made for soft sets.

Definition 2.5. [21] Let $A \subseteq E$. A fuzzy soft set f_A over universe X is mapping from the parameter set E to I^X, i.e., $f_A : E \longrightarrow I^X$, where $f_A(e) \neq 0_X$ if $e \in A \subset E$ and $f_A(e) = 0_X$ if $e \notin A$, where 0_X denotes empty fuzzy set on X.

Definition 2.6. [21] The fuzzy soft set $f_\emptyset \in FS(X, E)$ is called null fuzzy soft set, denoted by $\widetilde{0}_E$, if for all $e \in E$, $f_\emptyset(e) = 0_X$.

Definition 2.7. [21] Let $f_E \in FS(X, E)$. The fuzzy soft set f_E is called universal fuzzy soft set, denoted by $\widetilde{1}_E$, if for all $e \in E$, $f_E(e) = 1_X$ where $1_X(x) = 1$ for all $x \in X$.

Definition 2.8. [21] Let $f_A, g_B \in FS(X, E)$. f_A is called a fuzzy soft subset of g_B if $f_A(e) \leq g_B(e)$ for every $e \in E$ and we write $f_A \sqsubseteq g_B$.

Definition 2.9. [21] Let $f_A, g_B \in FS(X,E)$. f_A and g_B are said to be equal, denoted by $f_A = g_B$ if $f_A \sqsubseteq g_B$ and $g_B \sqsubseteq f_A$.

Definition 2.10. [21] Let $f_A, g_B \in FS(X,E)$. Then the union of f_A and g_B is also a fuzzy soft set h_C, defined by $h_C(e) = f_A(e) \vee g_B(e)$ for all $e \in E$, where $C = A \cup B$. Here we write $h_C = f_A \sqcup g_B$.

Definition 2.11. [21] Let $f_A, g_B \in FS(X,E)$. Then the intersection of f_A and g_B is also a fuzzy soft set h_C, defined by $h_C(e) = f_A(e) \wedge g_B(e)$ for all $e \in E$, where $C = A \cap B$. Here we write $h_C = f_A \sqcap g_B$.

Definition 2.12. [23] Let $f_A \in FS(X,E)$. The complement of f_A, denoted by f_A^c, is a fuzzy soft set defined by $f_A^c(e) = 1 - f_A(e)$ for every $e \in E$.

Let us call f_A^c to be fuzzy soft complement function of f_A. Clearly $(f_A^c)^c = f_A$, $(\widetilde{1}_E)^c = \widetilde{0}_E$ and $(\widetilde{0}_E)^c = \widetilde{1}_E$.

Definition 2.13. [12] Let $FS(X,E)$ and $FS(Y,K)$ be the families of all fuzzy soft sets over X and Y, respectively. Let $u : X \to Y$ and $p : E \to K$ be two functions. Then f_{up} is called a fuzzy soft mapping from X to Y and denoted by $f_{up} : FS(X,E) \to FS(Y,K)$.

(1) Let $f_A \in FS(X,E)$, then the image of f_A under the fuzzy soft mapping f_{up} is the fuzzy soft set over Y defined by $f_{up}(f_A)$, where

$$f_{up}(f_A)(k)(y) = \begin{cases} \vee_{x\in u^{-1}(y)}(\vee_{e\in p^{-1}(k)\cap A} f_A(e))(x) & \text{if } u^{-1}(y) \neq \varnothing,\ p^{-1}(k)\cap A \neq \varnothing; \\ 0_Y & \text{otherwise.} \end{cases}$$

(2) Let $g_B \in FS(Y,K)$, then the preimage of g_B under the fuzzy soft mapping f_{up} is the fuzzy soft set over X defined by $f_{up}^{-1}(g_B)$, where

$$f_{up}^{-1}(g_B)(e)(x) = \begin{cases} g_B(p(e))(u(x)) & \text{for } p(e) \in B; \\ 0_X & \text{otherwise.} \end{cases}$$

If u and p are injective then the fuzzy soft mapping f_{up} is said to be injective. If u and p are surjective then the fuzzy soft mapping f_{up} is said to be surjective. The fuzzy soft mapping f_{up} is called constant, if u and p are constant.

Theorem 2.14. [12] Let $f_A, \in FS(X,E)$, $\{f_{A_i}\}_{i\in J} \subset FS(X,E)$ and $g_B \in FS(Y,K)$, $\{g_{B_i}\}_{i\in J} \subset FS(Y,K)$, where J is an index set.

(1) If $(f_{A_1}) \sqsubseteq (f_{A_2})$, then $f_{up}(f_{A_1}) \sqsubseteq f_{up}(f_{A_2})$.
(2) If $(g_{B_1}) \sqsubseteq (g_{B_2})$, then $f_{up}^{-1}(g_{B_1}) \sqsubseteq f_{up}^{-1}(g_{B_2})$.
(3) $f_{up}(\sqcup_{i\in J}(f_{A_i})) = \sqcup_{i\in J} f_{up}(f_{A_i})$.
(4) $f_{up}(\sqcap_{i\in J}(f_{A_i})) \sqsubseteq \sqcap_{i\in J} f_{up}(f_{A_i})$.
(5) $f_{up}^{-1}(\sqcup_{i\in J}(g_{B_i})) = \sqcup_{i\in J} f_{up}^{-1}(g_{B_i})$.
(6) $f_{up}^{-1}(\sqcap_{i\in J}(g_{B_i})) = \sqcap_{i\in J} f_{up}^{-1}(g_{B_i})$.
(7) $f_{up}^{-1}(\widetilde{1}_K) = \widetilde{1}_E$ and $f_{up}^{-1}(\widetilde{0}_K) = \widetilde{0}_E$.
(8) $f_{up}(\widetilde{0}_E) = \widetilde{0}_K$ and $f_{up}(\widetilde{1}_E) \sqsubseteq \widetilde{1}_K$.

Theorem 2.15. [32] Let $f_A, \in FS(X,E)$, $\{f_{A_i}\}_{i\in J} \subset FS(X,E)$ and $g_B \in FS(Y,K)$, $\{g_{B_i}\}_{i\in J} \subset FS(Y,K)$, where J is an index set.

(1) $f_{up}(\sqcap_{i\in J}(f_{A_i})) = \sqcap_{i\in J} f_{up}(f_{A_i})$ if f_{up} is injective.
(2) $f_{up}(\widetilde{1}_E) = \widetilde{1}_K$ if f_{up} is surjective.
(3) $f_{up}^{-1}(f_A^c) = (f_{up}^{-1}(f_A))^c$.

2.1 Soft quasi-coincidence

Definition 2.16. [32] The fuzzy soft set $f_A \in FS(X,E)$ is called fuzzy soft point if $A = \{e\} \subseteq E$ and $f_A(e)$ is a fuzzy point in X, i.e., there exists $x \in X$ such that $f_A(e)(x) = \alpha$ $(0 < \alpha \leq 1)$ and $f_A(e)(y) = 0$ for all $y \in X - \{x\}$. We denote this fuzzy soft point $f_A = e_x^\alpha = \{(e, x_\alpha)\}$.

Definition 2.17. [32] Let e_x^α, $f_A \in FS(X,E)$. We say that $e_x^\alpha \widetilde{\in} f_A$ read as e_x^α belongs to the fuzzy soft set f_A if for the element $e \in A$, $\alpha \leq f_A(e)(x)$.

Proposition 2.18. [32] Every non null fuzzy soft set f_A can be expresssed as the union of all the fuzzy soft points which belong to f_A.

Definition 2.19. [32] Let $x_\alpha \in S(X)$ and $f_A \in FS(X,E)$. We say that $x_\alpha \in f_A$ read as x_α belongs to the fuzzy soft set f_A whenever $x_\alpha \in f_A(e)$, i.e., $\alpha \leq f_A(e)(x)$ for all $e \in A$.

Definition 2.20. [32] Let f_A, $g_B \in FS(X,E)$. f_A is said to be soft quasi-coincident with g_B, denoted by $f_A q g_B$, if there exist $e \in E$ and $x \in X$ such that $f_A(e)(x) + g_B(e)(x) > 1$.

If f_A is not soft quasi-coincident with g_B, then we write $f_A \overline{q} g_B$.

Definition 2.21. [32] Let $x_\alpha \in S(X)$ and $f_A \in FS(X,E)$. x_α is said to be soft quasi-coincident with f_A, denoted by $x_\alpha q f_A$, if and only if there exists an $e \in E$ such that $\alpha + f_A(e)(x) > 1$.

Proposition 2.22. [32] Let f_A, $g_B \in FS(X,E)$, then the followings are true.
(1) $f_A \sqsubseteq g_B \Leftrightarrow f_A \overline{q} g_B^c$.
(2) $f_A q g_B \Rightarrow f_A \sqcap g_B \neq \widetilde{0}_E$.
(3) $x_\alpha \overline{q} f_A \Leftrightarrow x_\alpha \in f_A^c$.
(4) $f_A \overline{q} f_A^c$.
(5) $f_A \sqsubseteq g_B \Rightarrow x_\alpha q f_A$ implies $x_\alpha q g_B$.
(6) $f_A q g_B \Leftrightarrow$there exists an $e_x^\alpha \widetilde{\in} f_A$ such that $e_x^\alpha q g_B$.
(7) $e_x^\alpha \overline{q} f_A \Leftrightarrow e_x^\alpha \widetilde{\in} f_A^c$.
(8) $f_A \sqsubseteq g_B \Leftrightarrow$ If $e_x^\alpha q f_A$, then $e_x^\alpha q g_B$ for all $e_x^\alpha \in FS(X,E)$.

Definition 2.23. (see [23] , [21]) A fuzzy soft topological space is a pair (X,τ) where X is a nonempty set and τ is a family of fuzzy soft sets over X satisfying the following properties:
(1) $\widetilde{0}_E, \widetilde{1}_E \in \tau$
(2) If f_A, $g_B \in \tau$, then $f_A \sqcap g_B \in \tau$
(3) If $f_{Ai} \in \tau$, $\forall i \in J$, then $\sqcup_{i\in J} f_{Ai} \in \tau$.
Then τ is called a topology of fuzzy soft sets on X. Every member of τ is called fuzzy soft open. g_B is called fuzzy soft closed in (X,τ) if $(g_B)^c \in \tau$.

Theorem 2.24. [32] Let (X,τ) be a fuzzy soft topological space and τ' denotes the collection of all fuzzy soft closed sets. Then
(1) $\widetilde{0}_E, \widetilde{1}_E \in \tau'$
(2) If f_A, $g_B \in \tau'$, then $f_A \sqcup g_B \in \tau'$
(3) If $f_{Ai} \in \tau'$, $\forall i \in J$, then $\sqcap_{i\in J} f_{Ai} \in \tau'$.

Definition 2.25. [23] Let (X,τ) be a fuzzy soft topological space and $f_A \in FS(X,E)$. The fuzzy soft closure of f_A denoted by $cl\,(f_A)$ is the intersection of all fuzzy soft closed supersets of f_A.

Clearly, $cl\,(f_A)$is the smallest fuzzy soft closed set over X which contains f_A.

Definition 2.26. [23] Let (X,τ) be a fuzzy soft topological space and $f_A \in FS(X,E)$. The fuzzy soft interior of f_A denoted by f_A° is the union of all fuzzy soft open subsets of f_A.

Clearly, f_A° is the largest fuzzy soft open set over X which contained in f_A.

Theorem 2.27. [32] Let (X,τ) be a fuzzy soft topological space and $f_A, g_B \in FS(X,E)$. Then,
(1) $(\overline{f_A})^c \sqsubseteq (f_A^c)^\circ$.
(2) $(f_A^\circ)^c \sqsubseteq \overline{(f_A^c)}$.

Definition 2.28. [32] A fuzzy soft set f_A in $FS(X,E)$ is called Q-neighborhood (briefly, Q-nbd) of g_B if and only if there exists a fuzzy soft open set h_C in τ such that $g_B q h_C \sqsubseteq f_A$.

Theorem 2.29. [32] Let e_x^α, $f_A \in FS(X,E)$. Then $e_x^\alpha \widetilde{\in} \overline{f_A}$ if and only if each Q-nbd of e_x^α is soft quasi-coincident with f_A.

Definition 2.30. [23] Let (X,τ) be a fuzzy soft topological space and β be a subfamily of τ. If every element of τ can be written as the arbitrary fuzzy soft union of some elements of β, then β is called a fuzzy soft basis for the fuzzy soft topology τ.

Proposition 2.31. [32] Let (X, τ) be a fuzzy soft topological space and β is subfamily of τ. β is a base for τ if and only if for each e_x^α in $FS(X, E)$ and for each fuzzy soft open Q-nbd f_A of e_x^α, there exists a $g_B \in \beta$ such that $e_x^\alpha q g_B \sqsubseteq f_A$.

Definition 2.32. [23] A fuzzy soft set g_B in a fuzzy soft topological space (X, τ) is called a fuzzy soft neighborhood (briefly: nbd) of the fuzzy soft set f_A if there exists a fuzzy soft open set h_C such that $f_A \sqsubseteq h_C \sqsubseteq g_B$.

Theorem 2.33. [32] g_B is fuzzy soft open if and only if for each fuzzy soft set f_A contained in g_B, g_B is a fuzzy soft neighborhood of f_A.

Definition 2.34. [32] Let (X, τ_1) and (Y, τ_2) be two fuzzy soft topological spaces. A fuzzy soft mapping $f_{up} : (X, \tau_1) \to (Y, \tau_2)$ is called fuzzy soft continuous if $f_{up}^{-1}(g_B) \in \tau_1$ for all $g_B \in \tau_2$.

Theorem 2.35. [32] Let (X, τ_1) and (Y, τ_2) be fuzzy soft topological spaces. For a function f_{up} : $FS(X, E) \longrightarrow FS(Y, K)$, the following statements are equivalent:
(a) f_{up} is fuzzy soft continuous;
(b) for each fuzzy soft set f_A in $FS(X, E)$, the inverse image of every nbd of $f_{up}(f_A)$ is a nbd of f_A;
(c) for each soft set f_A in $FS(X, E)$ and each nbd h_C of $f_{up}(f_A)$, there is a nbd g_B of f_A such that $f_{up}(g_B) \sqsubseteq h_C$.

Theorem 2.36. [32] A mapping $f_{up} : (X, E) \to (Y, K)$ is fuzzy soft continuous if and only if corresponding fuzzy soft open Q-nbd g_B of k_y^α in $FS(Y, K)$ there exists a fuzzy soft open Q-nbd f_A of e_x^α in $FS(X, E)$ such that $f_{up}(f_A) \sqsubseteq g_B$, where $f_{up}(e_x^\alpha) = k_y^\alpha$.

Theorem 2.37. [32] Let (X, τ_1) and (Y, τ_2) be two fuzzy soft topological spaces and $f_{up} : FS(X, E) \to FS(Y, K)$ be a fuzzy soft mapping. Then the followings are equivalent:
(1) f_{up} is continuous;
(2) $f_{up}^{-1}(h_C) \sqsubseteq (\ f_{up}^{-1}(h_C))^\circ$, $\forall h_C \in \tau_2$;
(3) $f_{up}(cl\,(f_A)) \sqsubseteq cl\,(f_{up}(f_A))$, $\forall f_A \in FS(X, E)$;
(4) $cl\left(f_{up}^{-1}(g_B)\right) \sqsubseteq f_{up}^{-1}(cl\,(g_B))$, $\forall g_B \in FS(Y, K)$;
(5) $f_{up}^{-1}(g_B^\circ) \sqsubseteq (f_{up}^{-1}(g_B))^\circ$, $\forall g_B \in FS(Y, K)$.

3 Continuity of Fuzzy Soft Multifunctions

Let Y be an initial universe set and E be the non-empty set of parameters.

Definition 3.1. A soft multifunction F from an ordinary topological space (X, τ) into a fuzzy soft topological space (Y, σ, E) assings to each x in X a soft set $F(x)$ over Y. A fuzzy soft multifunction will be denoted by $F : (X, \tau) \to (Y, \sigma, E)$. F is said to be onto if for each fuzzy soft set g_B over Y, there exists a point $x \in X$ such that $F(x) = g_B$.

Definition 3.2. For a fuzzy soft multifunction $F : (X, \tau) \to (Y, \sigma, E)$, the upper inverse $F^+(g_B)$ and the lower inverse $F^-(g_B)$ of a fuzzy soft set g_B over Y are defined as follows: $F^+(g_B) = \{x \in X : F(x) \tilde{\sqsubseteq} g_B\}$ and $F^-(g_B) = \{x \in X : F(x) \tilde{\sqcap} g_B \neq \widetilde{\Phi}\}$. Moreover, for a subset M of X, $F(M) = \tilde{\sqcup}\{F(x) : x \in X\}$.

Definition 3.3. [27] Let (X, τ) be an ordinary topological space and (Y, ϑ) be a fuzzy topological space. $F : (X, \tau) \to (Y, \vartheta)$ is called a fuzzy multifunction iff for every $x \in X$, $F(x)$ is a fuzzy set in Y.

Remark 3.4. Since every fuzzy set is a soft set, then every fuzzy multifunction is a soft multifunction.

Proposition 3.5. Let M be a subset of X. Then the follows are true for a fuzzy soft multifunction $F : (X, \tau) \to (Y, \sigma, E)$;
(a) $M \subset F^+(F\,(M))$. If F is onto $M = F^+(F\,(M))$.
(b) $M \subset F^-(F\,(M))$. If F is onto $M = F^-(F\,(M))$.

Proof. (a) Let $x \in M$. Then $F(x)\widetilde{\subseteq}F(M) = \widetilde{\sqcup}\{F(x) : x \in M\}$ and so $x \in F^+(F(M))$. Hence, $M \subset F^+(F(M))$.
(b) The proof is similar to (a). □

Proposition 3.6. Let g_B be a fuzzy soft set over Y. Then the followings are true for a fuzzy soft multifunction $F : (X, \tau) \to (Y, \sigma, E)$:
(a) $F^+((g_B)^c) = X - F^-(g_B)$
(b) $F^-((g_B)^c) = X - F^+(g_B)$.

Proof. (a) If $x \in X - F^-(g_B)$ then $x \notin F^-(g_B)$ which implies $F(x)\widetilde{\sqcap}(g_B) = \widetilde{\Phi}$ and therefore $F(x)\widetilde{\subseteq}(g_B)^c$. Thus $x \in F^+((g_B)^c)$ and $X - F^-(g_B)\widetilde{\subseteq}F^+((g_B)^c)$.

Conversely, if $x \in F^+((g_B)^c)$ then $F(x)\widetilde{\subseteq}(g_B)^c$ which implies $F(x)\widetilde{\sqcap}(g_B) = \widetilde{\Phi}$ and therefore $x \notin F^-(g_B)$. Thus $x \in X - F^-(g_B)$ and $F^+((g_B)^c)\widetilde{\subseteq}X - F^-(g_B)$.

(b) If $x \in X - F^+(g_B)$ then $x \notin F^+(g_B)$ which implies $F(x)\widetilde{\nsubseteq}(g_B)$ and therefore $F(x)\widetilde{\sqcap}((g_B)^c) \neq \widetilde{\Phi}$. Thus $x \in F^-((g_B)^c)$ and $X - F^+(g_B)\widetilde{\subseteq}F^-((g_B)^c)$.

Conversely, if $x \in F^-((g_B)^c)$ then $F(x)\widetilde{\sqcap}((g_B)^c) \neq \widetilde{\Phi}$ which implies $F(x) \widetilde{\nsubseteq} (g_B)$ and therefore $x \notin F^+(g_B)$. Thus $x \in X - F^+(g_B)$ and $F^-((g_B)^c)\widetilde{\subseteq}X - F^+(g_B)$. □

Proposition 3.7. Let (g_{Bi}) be fuzzy soft sets over Y for each $i \in I$. Then the follows are true for a fuzzy soft multifunction $F : (X, \tau) \to (Y, \sigma, E)$;
(a) $F^-(\underset{i\in I}{\widetilde{\sqcup}} g_{Bi}) = \underset{i\in I}{\sqcup}(F^-(g_{Bi}))$.
(b) $F^+(\underset{i\in I}{\widetilde{\sqcap}} g_{Bi}) = \underset{i\in I}{\sqcap}(F^-(g_{Bi}))$.

Proof. (a) For every $x \in F^-(\underset{i\in I}{\widetilde{\sqcup}} g_{Bi})$, $F(x)\widetilde{\sqcap}(\underset{i\in I}{\sqcup} g_{Bi}) \neq \widetilde{\Phi}$. There exists $i \in I$ such that $F(x)\widetilde{\sqcap}(g_{Bi}) \neq \widetilde{\Phi}$. For the same $i \in I$, $x \in F^-(g_{Bi})$. Therefore $x \in \underset{i\in I}{\sqcup}(F^-(g_{Bi})$. Thus $F^-(\underset{i\in I}{\widetilde{\sqcup}} g_{Bi}) \sqsubseteq \underset{i\in I}{\sqcup}(F^-(g_{Bi})$.
Conversely, for every $x \in \underset{i\in I}{\sqcup}(F^-(g_{Bi})$, there exists $i \in I$ such that $x \in F^-(g_{Bi})$. For the same $i \in I$, $F(x)\widetilde{\sqcap}(g_{Bi}) \neq \Phi$. Therefore, $F(x)\widetilde{\sqcap}(\underset{i\in I}{\widetilde{\sqcup}} g_{Bi}) \neq \phi$ and $x \in F^-(\underset{i\in I}{\widetilde{\sqcup}} g_{Bi})$. Thus $\underset{i\in I}{\sqcup}(F^-(g_{Bi}) \sqsubseteq F^-(\underset{i\in I}{\widetilde{\sqcup}} g_{Bi})$.
(b)The proof is similar of (a). □

Definition 3.8. Let (X, τ) be an ordinary topological space and (Y, σ, E) be a fuzzy soft topological space. Then a fuzzy soft multifunction $F : (X, \tau) \to (Y, \sigma, E)$ is said to be;

(a) upper fuzzy soft continuous (briefly: u.fuzzy soft c.) at a point $x \in X$ if for each fuzzy soft open g_B such that $F(x)\widetilde{\subseteq}(g_B)$, there exists an open neighborhood $P(x)$ of x such that $F(z)\widetilde{\subseteq}g_B$ for all $z \in P(x)$.

(b) lower fuzzy soft continuous (briefly: l. fuzzy soft c.) at a point $x \in X$ if for each fuzzy soft open g_B such that $F(x)\widetilde{\sqcap}g_B \neq \widetilde{\Phi}$, there exists an open neighborhood $P(x)$ of x such that $F(z)\widetilde{\sqcap}g_B \neq \widetilde{\Phi}$ for all $z \in P(x)$.

(c) upper(lower) fuzzy soft continuous if F has this property at every point of X.

Proposition 3.9. A fuzzy soft multifunction $F : (X, \tau) \to (Y, \sigma, E)$ is upper fuzzy soft continuous if and only if for all fuzzy soft open set g_B over Y, $F^+(g_B)$ is open in X.

Proof. First suppose that F is upper fuzzy soft continuous. Let g_B is be fuzzy soft open set over Y and $x \in F^+(g_B)$. Then from Definition 29, we know that there exists an open neighborhood $P(x)$ of x such that for all $z \in P(x)$, $F(z)\widetilde{\subseteq}F^+(g_B)$ which means that $F^+(g_B)$ is open as claimed. The direction is just the definition of upper fuzzy soft continuity of F. □

Proposition 3.10. $F : (X, \tau) \to (Y, \sigma, E)$ is lower fuzzy soft continuous multifunction if and only if for every fuzzy soft open set g_B over Y, $F^-(g_B)$ is open set in X.

Proof. First assume that F is lower fuzzy soft continuous. Let g_B fuzzy soft open over Y and $x \in F^-(g_B)$. Then there is an open neighborhood $P(x)$ of x such that $F(z)\widetilde{\sqcap}\, g_B \neq \widetilde{\Phi}$ for all $z \in P(x)$. So $P(x) \subseteq F^-(g_B)$ which implies that $F^-(g_B)$ is open in X. Now suppose that $F^-(g_B)$ is open. Let $x \in F^-(g_B)$.Then $F^-(g_B)$ is an open neighborhood of x and for all $z \in F^-(g_B)$ we have $F(z)\widetilde{\sqcap}\, g_B \neq \widetilde{\Phi}$. So, F is lower fuzzy soft continuous. □

Theorem 3.11. The followings are equivalent for a fuzzy soft multifunction $F : (X, \tau) \to (Y, \sigma, E)$;
(a) F is upper fuzzy soft continuous
(b) for each fuzzy soft closed set g_B over Y, $F^-(g_B)$ is closed in X.
(c) for each fuzzy soft set g_B over Y, $cl(F^-(g_B) \subseteq F^-(cl(g_B)$.
(d) for each fuzzy soft set g_B over Y, $F^+(Int(g_B)) \subseteq Int(F^+(g_B)$.

Proof. (a)$\Longrightarrow$(b) Let g_B be a closed fuzzy soft over Y. Then proposition1 implies $(g_B)^c$ is fuzzy soft open and $F^+((g_B)^c) = X - F^-(g_B)$, then since $F^-(g_B)$ is open and so $F^-(g_B)$ is closed.

(b)$\Longrightarrow$(c) Let g_B be any fuzzy soft set over Y. Then $cl(g_B)$ is fuzzy soft closed set. By (b) $F^-(cl(g_B))$ is closed in X. Hence, $cl(F^-(g_B) \subseteq F^-(cl(g_B))$ and since $F^-(g_B) \subseteq F^-(cl(g_B))$. Thus, $cl(F^-(g_B)) \subseteq F^-(cl(g_B))$.

(c)$\Longrightarrow$(d) Let g_B be any fuzzy soft set over Y. By (c), $cl(F^-((g_B)^c) \subseteq F^-(cl(g_B)^c)$, $X - F^-((int(g_B)^c) \subseteq int(X - F^-((int(g_B)^c))$, $X - X - F^+(int(g_B)) \subseteq intF^+(g_B)$.

(d)$\Longrightarrow$(a) Let g_B be any fuzzy soft set over Y. By (d), $F^+(int(g_B)) = F^+(g_B) \subseteq int(F^+(g_B)$ and so $F^+(g_B)$ is open in X. They by proposition (1), F is upper fuzzy soft continuous. □

Theorem 3.12. The following are equivalent for a fuzzy soft multifunction $F : (X, \tau) \to (Y, \sigma, E)$;
(a) F is lower fuzzy soft continuous.
(b) for each fuzzy soft closed set g_B over Y, $F^+(g_B)$ is closed in X.
(c) for each fuzzy soft set g_B over Y, $cl(F^+(g_B) \subseteq F^+(cl(g_B))$.
(d) for each fuzzy soft set g_B over Y, $F^-(int(g_B) \subseteq int(F^-(g_B)$.

Proof. It is similar the proof of Theorem 4. □

Definition 3.13. For a fuzzy soft multifunction $F : (X, \tau) \to (Y, \sigma, E)$, the graph fuzzy soft multifunction $G_F : X \to X \times Y$ is defined as follows: $G_F(x) = \{x\} \times F(x)$, for every $x \in X$.

Lemma 3.14. For a fuzzy soft multifunction $F : (X, \tau) \to (Y, \sigma, E)$, the followings are hold:
(a) $G_F^+(M \times h_B) = M \cap F^+(h_B)$
(b) $G_F^-(M \times h_B) = M \cap F^-(h_B)$

Proof. (a)Let M be any subset of X and let h_B be any fuzzy soft set over Y. Let $x \in G_F^+(M \times h_B)$. Then $G_F(x)\widetilde{\subseteq}(M \times h_B)$ that is $(\{x\} \times F(x))\widetilde{\subseteq}M \times h_B$. Therefore, we have $x \in M$ and $F(x)\widetilde{\subseteq}h_B$. Hence $x \in M \cap F^+(h_B)$.
Conversely, let $x \in M \cap F^+(h_B)$. Then $x \in M$ and $x \in F^+(h_B)$. Thus $x \in M$ and $F(x)\widetilde{\subseteq}h_B$ that is $G_F(x)\widetilde{\subseteq}(M \times h_B)$. Therefore $x \in G_F^+(M \times h_B)$.
(b)Let M be any subset of X and let h_B be any fuzzy soft set over Y. Let $x \in G_F^-(M \times h_B)$. Then $\widetilde{\Phi} \neq G_F(x)\widetilde{\sqcap}(M \times h_B) = (\{x\} \times F(x))\widetilde{\sqcap}(M \times h_B) = (\{x\} \cap M) \times (F(x)\widetilde{\sqcap}h_B)$. Therefore, we have $x \in M$ and $F(x)\widetilde{\sqcap}h_B \neq \widetilde{\widetilde{\Phi}}$. Hence $x \in M \cap F^-(h_B)$.
Conversely, let $x \in M \cap F^-(h_B)$. Then $x \in M$ and $x \in F^-(h_B)$. Thus $x \in M$ and $F(x)\widetilde{\sqcap}h_B \neq \widetilde{\widetilde{\Phi}}$ that is $G_F(x)\widetilde{\sqcap}(M \times h_B) \neq \widetilde{\widetilde{\Phi}}$. Therefore $x \in G_F^-(M \times h_B)$. □

Theorem 3.15. Let $F : (X, \tau) \to (Y, \sigma, E)$ be a fuzzy soft multifunction. If the graph fuzzy soft function of F is lower (upper) fuzzy soft continuous, then F is lower (upper) fuzzy soft continuous.

Proof. For a subset V of X and h_B a fuzzy soft set over Y, we take $(V \times h_B)(z, y) = \begin{cases} \Phi, & if\ z \notin V \\ h_B(y), & if\ z \in V \end{cases}$
Let $x \in X$ and let h_B be fuzzy soft open set such that $x \in F^-(h_B)$. Then we obtain that $x \in G_F^-(X \times h_B)$ and $X \times h_B$ is a fuzzy soft set over Y. Since fuzzy soft graph multifuntion G_F is lower fuzzy soft continuons, it follows that there exists an open set P containg x such that $P \subseteq G_F^-(X \times h_B)$. From here, we obtain that $P \subseteq F^-(h_B)$. Thus, F is lower fuzzy soft continuous.
The proof of the upper fuzzy soft continuity of F is similar to the above. □

Theorem 3.16. Let $F : (X, \tau) \to (Y, \sigma, E)$ be a fuzzy soft multifunction and M be an open set of X. Then the restriction $F|_M$ is upper fuzzy soft continuous if F is upper fuzzy soft continuous.

Proof. Let h_B be any fuzzy soft open set over Y such that $(F|_M)(x)\widetilde{\subseteq}h_B$. Since F is upper fuzzy soft continuous and $F(x) = (F_{|M})(x)\widetilde{\subseteq}h_B$, there exists open set $U \subseteq X$ containing x such that $F(z)\widetilde{\subseteq}h_B$ for all $z \in U$. Put $U_1 = U \cap M$ then we have U_1 is open set in M containing x and $F(U_1) = (F|_M)(U_1)\widetilde{\subseteq}h_B$. This shows that $F|_M$ is upper fuzzy soft continuous. □

Theorem 3.17. Let $F : (X, \tau) \to (Y, \sigma, E)$ be a fuzzy soft multifunction and M be an open set of X. Then F is lower fuzzy soft continuous if and only if the restriction $F|_M$ is lower fuzzy soft continuous.

Proof. Let h_B be any fuzzy soft open set over Y such that $(F|_M)(x)\widetilde{\sqcap}h_B \neq \widetilde{\Phi}$. Since $F(x) = (F|_M)(x)$, then $F(x)\widetilde{\sqcap}h_B \neq \widetilde{\Phi}$. Also since F is lower fuzzy soft continuous there exists an open set $U \subseteq X$ containing x such that $F(z)\widetilde{\sqcap}h_B \neq \widetilde{\Phi}$ for all $z \in U$. Put $U_1 = U \cap M$ then we have U_1 is open set in M containing x and $F(U_1)\widetilde{\sqcap}h_B \neq \widetilde{\Phi}$.Therefore $(F|_M)(U_1)\widetilde{\sqcap}h_B \neq \widetilde{\Phi}$.This shows that $F|_M$ is lower fuzzy soft continuous. □

Remark 3.18. Let $F : (X, \tau) \to (Y, \sigma, E)$ be a fuzzy soft multifunction and $\{M_i : i \in I\}$ be an open cover set of X. The followings are hold :

(a) F is lower fuzzy soft continuous if and only if the restriction $F|_{M_i}$ is lower fuzzy soft continuous for every $i \in I$.

(b) F is upper fuzzy soft continuous if and only if the restriction $F|_{M_i}$ is upper fuzzy soft continuous for every $i \in I$.

Definition 3.19. Let $F : (X, \tau) \to (Y, \sigma)$ be a multifunction and let $G : (Y, \sigma) \to (Z, \vartheta, E)$ be a fuzzy soft multifunction. Then the fuzzy soft multifunction $G \circ F : (X, \tau) \to (Z, \vartheta, E)$ is defined by $(G \circ F)(x) = G(F(x))$.

Proposition 3.20. Let $F : (X, \tau) \to (Y, \sigma)$ be a multifunction and let $G : (Y, \sigma) \to (Z, \vartheta, E)$ be a fuzzy soft multifunction. Then we have

(a) $(G \circ F)^+(h_B) = F^+(G^+(h_B))$

(b) $(G \circ F)^-(h_B) = F^-(G^-(h_B))$

Proof. Clear from the Definitions 17 and 20. □

Definition 3.21. [1] Let (X, τ) and (Y, σ) be two ordinary topological spaces. Then a multifunction $F : (X, \tau) \to (Y, \sigma)$ is said to be

(a) upper semi continuous if for each open V in Y, $F^+(V)$ is an open set in X.

(b) lower semi continuous if for each soft open V in Y, $F^-(V)$ is an open set in X.

Theorem 3.22. Let $F : (X, \tau) \to (Y, \sigma)$ be a multifunction and let $G : (Y, \sigma) \to (Z, \vartheta, E)$ be a fuzzy soft multifunction. If F is upper semi continuous and G is upper fuzzy soft continuous then $G \circ F$ is upper fuzzy soft continuous.

Proof. Let h_B be any fuzzy soft open subset of Z. Since G is upper fuzzy soft continuous then $G^+(h_B)$ is open in Y. Since F is upper semi continuous then $F^+(G^+(h_B)) = (G \circ F)^+(h_B)$ is open in X. Therefore $G \circ F$ is upper fuzzy soft continuous. □

Definition 3.23. A family Ψ of fuzzy soft sets is a cover of a soft set h_B if $h_B \widetilde{\subset}\widetilde{\bigcup} \{h_{B_i} : h_{B_i} \in \Psi, i \in I\}$. It is fuzzy soft open cover if each of Ψ is a fuzzy soft open set. A subcover of Ψ is a subfamily of Ψ which is also cover.

Definition 3.24. A fuzzy fuzzy soft topological space (Y, σ, E) is fuzzy soft compact if each fuzzy soft open cover of $\widetilde{Y}$ has a finite subcover.

Theorem 3.25. The image of a fuzzy soft compact set under upper fuzzy soft continuous multifunction is fuzzy soft compact.

Proof. Let $F : (X, \tau) \to (Y, \sigma, E)$ be an onto fuzzy soft multifunction and let $\Psi = \{h_{B_i} : i \in I\}$ be a cover of $\widetilde{Y}$ by fuzzy soft open sets. Then since F is upper fuzzy soft continuous, the family of all open sets of the form $F^+(h_{B_i})$, for $h_{B_i} \in \Psi$ is an open cover of X which has a finite subcover. However since F is surjective, then it is eaisly seen that $F(F^+(h_{B_i})) = h_{B_i}$ for any fuzzy soft set h_{B_i} over Y. There the family of image members of subcover is a finite subfamily of Ψ which covers $\breve{Y}$. Conseqventhy (Y, σ, E) is fuzzy soft compact. □

4 Conclusion

In the present work, we have continued to study the properties of fuzzy soft topological spaces. We introduce soft quasi-coincidence and have established several interesting properties. We hope that the findings in this paper will help researcher enhance and promote the further study on fuzzy soft topology to carry out a general framework for their applications in practical life.

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On $(1,2)^{\star}$-$g^{\#}$-Continuous Functions

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Abstract − The aim of this paper is to study and characterize $(1,2)^{\star}$-$g^{\#}$-continuous functions and $(1,2)^{\star}$-$g^{\#}$-irresolute functions formed with the help of $(1,2)^{\star}$-$g^{\#}$-closed sets.

Keywords − *Bitopological space, $(1,2)^{\star}$-$g^{\#}$-closed set, $(1,2)^{\star}$-$g^{\#}$-continuous function, $(1,2)^{\star}$-$g^{\#}$-irresolute function.*

1 Introduction

Several authors ([1, 4, 5, 19]) working in the field of general topology have shown more interest in studying the concepts of generalizations of continuous functions. A weak form of continuous functions called g-continuous functions were introduced by Balachandran et al [3]. Recently Sheik John [18] have introduced and studied another form of generalized continuous functions called ω-continuous functions.

In this paper, we first study $(1,2)^{\star}$-$g^{\#}$-continuous functions and investigate their relations with various generalized $(1,2)^{\star}$-continuous functions. We also discuss some properties of $(1,2)^{\star}$-$g^{\#}$-continuous functions. We also introduce $(1,2)^{\star}$-$g^{\#}$-irresolute functions and study some of its applications. Finally using $(1,2)^{\star}$-$g^{\#}$-continuous function we obtain a decomposition of $(1,2)^{\star}$-continuity.

2 Preliminary

Throughout this paper, X, Y and Z denote bitopological spaces (X, τ_1, τ_2), (Y, σ_1, σ_1) and (Z, η_1, η_2) respectively.

Definition 2.1. *Let A be a subset of a bitopological space X. Then A is called $\tau_{1,2}$-open [9] if $A = P \cup Q$, for some $P \in \tau_1$ and $Q \in \tau_2$. The complement of $\tau_{1,2}$-open set is called $\tau_{1,2}$-closed.*

The family of all $\tau_{1,2}$-open (resp. $\tau_{1,2}$-closed) sets of X is denoted by $(1,2)^{\star}$-O(X) (resp. $(1,2)^{\star}$-C(X)).

Definition 2.2. *Let A be a subset of a bitopological space X. Then*

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1. *the $\tau_{1,2}$-interior of A, denoted by $\tau_{1,2}$-int(A), is defined by $\cup$ { U : $U \subseteq A$ and U is $\tau_{1,2}$-open};*
2. *the $\tau_{1,2}$-closure of A, denoted by $\tau_{1,2}$-cl(A), is defined by $\cap$ { U : $A \subseteq U$ and U is $\tau_{1,2}$-closed}.*

Remark 2.3. *Notice that $\tau_{1,2}$-open subsets of X need not necessarily form a topology.*

Definition 2.4. *Let A be a subset of a bitopological space X is called*

1. *$(1,2)^\star$-semi-open set [9] if $A \subseteq \tau_{1,2}$-cl($\tau_{1,2}$-int(A)).*
2. *$(1,2)^\star$-preopen set [9] if $A \subseteq \tau_{1,2}$-int($\tau_{1,2}$-cl(A)).*
3. *$(1,2)^\star$-α-open set [9] if $A \subseteq \tau_{1,2}$-int($\tau_{1,2}$-cl($\tau_{1,2}$-int(A))).*
4. *$(1,2)^\star$-β-open set [12] if $A \subseteq \tau_{1,2}$-cl($\tau_{1,2}$-int($\tau_{1,2}$-cl(A))).*
5. *$(1,2)^\star$-regular open set [13] if $A = \tau_{1,2}$-int($\tau_{1,2}$-cl(A)).*

The complements of the above mentioned open sets are called their respective closed sets.

The $(1,2)^\star$-preclosure [11] (resp. $(1,2)^\star$-semi-closure [11], $(1,2)^\star$-α-closure [11], $(1,2)^\star$-β-closure [16]) of a subset A of X, denoted by $(1,2)^\star$-pcl(A) (resp. $(1,2)^\star$-scl(A), $(1,2)^\star$-αcl(A), $(1,2)^\star$-βcl(A)) is defined to be the intersection of all $(1,2)^\star$-preclosed (resp. $(1,2)^\star$-semi-closed, $(1,2)^\star$-α-closed, $(1,2)^\star$-β-closed) sets of X containing A. It is known that $(1,2)^\star$-pcl(A) (resp. $(1,2)^\star$-scl(A), $(1,2)^\star$-αcl(A), $(1,2)^\star$-βcl(A)) is a $(1,2)^\star$-preclosed (resp. $(1,2)^\star$-semi-closed, $(1,2)^\star$-α-closed, $(1,2)^\star$-β-closed) set. For any subset A of an arbitrarily chosen bitopological space, the $(1,2)^\star$-semi-interior [11] (resp. $(1,2)^\star$-α-interior [11], $(1,2)^\star$-preinterior [11]) of A, denoted by $(1,2)^\star$-sint(A) (resp. $(1,2)^\star$-αint(A), $(1,2)^\star$-pint(A)), is defined to be the union of all $(1,2)^\star$-semi-open (resp. $(1,2)^\star$-α-open, $(1,2)^\star$-preopen) sets of X contained in A.

Definition 2.5. *Let A be a subset of a bitopological space X is called*

1. *a $(1,2)^\star$-generalized closed (briefly, $(1,2)^\star$-g-closed) set [17] if $\tau_{1,2}$-cl(A) $\subseteq U$ whenever $A \subseteq U$ and U is $\tau_{1,2}$-open in X.*
   The complement of $(1,2)^\star$-g-closed set is called $(1,2)^\star$-g-open set.
2. *a $(1,2)^\star$-$g^\star$-closed set [17] if $\tau_{1,2}$-cl(A) $\subseteq U$ whenever $A \subseteq U$ and U is $(1,2)^\star$-g-open in X.*
   The complement of $(1,2)^\star$-$g^\star$-closed set is called $(1,2)^\star$-$g^\star$-open set.
3. *a $(1,2)^\star$-semi-generalized closed (briefly, $(1,2)^\star$-sg-closed) set [2] if $(1,2)^\star$-scl(A) $\subseteq U$ whenever $A \subseteq U$ and U is $(1,2)^\star$-semi-open in X.*
   The complement of $(1,2)^\star$-sg-closed set is called $(1,2)^\star$-sg-open set.
4. *a $(1,2)^\star$-generalized semi-closed (briefly, $(1,2)^\star$-gs-closed) set [2] if $(1,2)^\star$-scl(A) $\subseteq U$ whenever $A \subseteq U$ and U is $\tau_{1,2}$-open in X.*
   The complement of $(1,2)^\star$-gs-closed set is called $(1,2)^\star$-gs-open set.
5. *an $(1,2)^\star$-α-generalized closed (briefly, $(1,2)^\star$-αg-closed) set [6] if $(1,2)^\star$-αcl(A) $\subseteq U$ whenever $A \subseteq U$ and U is $\tau_{1,2}$-open in X.*
   The complement of $(1,2)^\star$-αg-closed set is called $(1,2)^\star$-αg-open set.
6. *a $(1,2)^\star$-generalized semi-preclosed (briefly, $(1,2)^\star$-gsp-closed) set [6] if $(1,2)^\star$-βcl(A) $\subseteq U$ whenever $A \subseteq U$ and U is $\tau_{1,2}$-open in X.*
   The complement of $(1,2)^\star$-gsp-closed set is called $(1,2)^\star$-gsp-open set.
7. *a $(1,2)^\star$-$g\alpha$-closed set [15] if $(1,2)^\star$-αcl(A) $\subseteq U$ whenever $A \subseteq U$ and U is $(1,2)^\star$-α-open in X.*
   The complement of $(1,2)^\star$-$g\alpha$-closed set is called $(1,2)^\star$-$g\alpha$-open set.

Remark 2.6. *The collection of all $(1,2)^\star$-$g^\star$-closed (resp. $(1,2)^\star$-g-closed, $(1,2)^\star$-gs-closed, $(1,2)^\star$-gsp-closed, $(1,2)^\star$-αg-closed, $(1,2)^\star$-sg-closed, $(1,2)^\star$-α-closed, $(1,2)^\star$-semi-closed) sets of X is denoted by $(1,2)^\star$-$G^\star C(X)$ (resp. $(1,2)^\star$-GC(X), $(1,2)^\star$-GSC(X), $(1,2)^\star$-GSPC(X), $(1,2)^\star$-$\alpha GC(X)$, $(1,2)^\star$-SGC(X), $(1,2)^\star$-$\alpha C(X)$, $(1,2)^\star$-SC(X)).*

The collection of all $(1,2)^\star$-$g^\star$-open (resp. $(1,2)^\star$-g-open, $(1,2)^\star$-gs-open, $(1,2)^\star$-gsp-open, $(1,2)^\star$-αg-open, $(1,2)^\star$-sg-open, $(1,2)^\star$-α-open, $(1,2)^\star$-semi-open) sets of X is denoted by $(1,2)^\star$-$G^\star O(X)$ (resp. $(1,2)^\star$-GO(X), $(1,2)^\star$-GSO(X), $(1,2)^\star$-GSPO(X), $(1,2)^\star$-$\alpha GO(X)$, $(1,2)^\star$-SGO(X), $(1,2)^\star$-$\alpha O(X)$, $(1,2)^\star$-SO(X)).

We denote the power set of X by P(X).

Definition 2.7. *[10] Let A be a subset of a bitopological space X. Then A is called*

1. *$(1,2)^\star$-$g^\#$-closed set if $\tau_{1,2}$-cl(A) $\subseteq$ U whenever A $\subseteq$ U and U is $(1,2)^\star$-αg-open in X.*
   The family of all $(1,2)^\star$-$g^\#$-closed sets in X is denoted by $(1,2)^\star$-$G^\# C(X)$.
2. *$(1,2)^\star$-$g_\alpha^\#$-closed set if $(1,2)^\star$-$\alpha cl(A) \subseteq U$ whenever A $\subseteq$ U and U is $(1,2)^\star$-αg-open in X.*
   The family of all $(1,2)^\star$-$g_\alpha^\#$-closed sets in X is denoted by $(1,2)^\star$-$G_\alpha^\# C(X)$.

Definition 2.8. *A function $f : X \to Y$ is called:*

1. *$(1,2)^\star$-$g^\star$-continuous [7] if $f^{-1}(V)$ is a $(1,2)^\star$-$g^\star$-closed set in X for every $\sigma_{1,2}$-closed set V of Y.*
2. *$(1,2)^\star$-g-continuous [7] if $f^{-1}(V)$ is a $(1,2)^\star$-g-closed set in X for every $\sigma_{1,2}$-closed set V of Y.*
3. *$(1,2)^\star$-αg-continuous [16] if $f^{-1}(V)$ is an $(1,2)^\star$-αg-closed set in X for every $\sigma_{1,2}$-closed set V of Y.*
4. *$(1,2)^\star$-gs-continuous [16] if $f^{-1}(V)$ is a $(1,2)^\star$-gs-closed set in X for every $\sigma_{1,2}$-closed set V of Y.*
5. *$(1,2)^\star$-gsp-continuous [16] if $f^{-1}(V)$ is a $(1,2)^\star$-gsp-closed set in X for every $\sigma_{1,2}$-closed set V of Y.*
6. *$(1,2)^\star$-sg-continuous [14] if $f^{-1}(V)$ is a $(1,2)^\star$-sg-closed set in X for every $\sigma_{1,2}$-closed set V of Y.*
7. *$(1,2)^\star$-semi-continuous [11] if $f^{-1}(V)$ is a $(1,2)^\star$-semi-open set in X for every $\sigma_{1,2}$-open set V of Y.*
8. *$(1,2)^\star$-α-continuous [11] if $f^{-1}(V)$ is an $(1,2)^\star$-α-closed set in X for every $\sigma_{1,2}$-closed set V of Y.*

Definition 2.9. *A function $f : X \to Y$ is called:*

1. *$(1,2)^\star$-αg-irresolute [16] if the inverse image of every $(1,2)^\star$-αg-closed (resp. $(1,2)^\star$-αg-open) set in Y is $(1,2)^\star$-αg-closed (resp. $(1,2)^\star$-αg-open) in X.*
2. *$(1,2)^\star$-gc-irresolute [7] if the inverse image of every $(1,2)^\star$-g-closed set in Y is $(1,2)^\star$-g-closed in X.*
3. *$(1,2)^\star$-sg-irresolute [16] if the inverse image of every $(1,2)^\star$-sg-closed (resp. $(1,2)^\star$-sg-open) set in Y is $(1,2)^\star$-sg-closed (resp. $(1,2)^\star$-sg-open) in X.*

Definition 2.10. *[16] A function $f : X \to Y$ is called pre-$(1,2)^\star$-αg-closed if f(U) is $(1,2)^\star$-αg-closed in Y, for each $(1,2)^\star$-αg-closed set U in X.*

Definition 2.11. *A bitopological space X is called:*

1. *$(1,2)^\star$-$T_{1/2}$-space [14] if every $(1,2)^\star$-g-closed set in it is $\tau_{1,2}$-closed.*
2. *$(1,2)^\star$-$T_{\star 1/2}$-space [12] if every $(1,2)^\star$-$^\star g$-closed set in it is $\tau_{1,2}$-closed.*
3. *$(1,2)^\star$-$^\star T_{1/2}$-space [12] if every $(1,2)^\star$-g-closed set in it is $(1,2)^\star$-$g^\star$-closed.*
4. *$(1,2)^\star$-T_b-space [12] if every $(1,2)^\star$-gs-closed set in it is $\tau_{1,2}$-closed.*

5. $(1,2)^\star$-$_\alpha T_b$-space [16] if every $(1,2)^\star$-αg-closed set in it is $\tau_{1,2}$-closed.
6. $(1,2)^\star$-T_d-space [16] if every $(1,2)^\star$-αg-closed set in it is $(1,2)^\star$-g-closed.
7. $(1,2)^\star$-α-space [11] if every $(1,2)^\star$-α-closed set in it is $\tau_{1,2}$-closed.
8. $(1,2)^\star$-$T_{\#_g}$-space [10] if every $(1,2)^\star$-$g^\#$-closed set in it is $\tau_{1,2}$-closed.

Theorem 2.12. *[10] A set A of X is $(1,2)^\star$-$g^\#$-open if and only if $F \subseteq \tau_{1,2}$-int(A) whenever F is $(1,2)^\star$-αg-closed and $F \subseteq A$.*

Theorem 2.13. *[10] For a space X, the following properties are equivalent:*

1. *X is a $(1,2)^\star$-$T_g^\#$-space.*
2. *Every singleton subset of X is either $(1,2)^\star$-αg-closed or $\tau_{1,2}$-open.*

3 $(1,2)^\star$-$g^\#$-Continuous Functions

We introduce the following definitions:

Definition 3.1. *A function $f : X \to Y$ is called:*

1. *$(1,2)^\star$-$g^\#$-continuous if the inverse image of every $\sigma_{1,2}$-closed set in Y is $(1,2)^\star$-$g^\#$-closed set in X.*
2. *$(1,2)^\star$-$g_\alpha^\#$-continuous if $f^{-1}(V)$ is an $(1,2)^\star$-$g_\alpha^\#$-closed set in X for every $\sigma_{1,2}$-closed set V of Y.*
3. *strongly $(1,2)^\star$-$g^\#$-continuous if the inverse image of every $(1,2)^\star$-$g^\#$-open set in Y is $\tau_{1,2}$-open in X.*

Example 3.2. *Let $X = \{a, b, c\}$, $\tau_1 = \{\phi, \{c\}, X\}$ and $\tau_2 = \{\phi, \{a, c\}, X\}$. Then the sets in $\{\phi, \{c\}, \{a, c\}, X\}$ are called $\tau_{1,2}$-open and the sets in $\{\phi, \{b\}, \{a, b\}, X\}$ are called $\tau_{1,2}$-closed. Let $Y = \{a, b, c\}$, $\sigma_1 = \{\phi, Y\}$ and $\sigma_2 = \{\phi, \{c\}, Y\}$. Then the sets in $\{\phi, \{c\}, Y\}$ are called $\sigma_{1,2}$-open and the sets in $\{\phi, \{a, b\}, Y\}$ are called $\sigma_{1,2}$-closed. We have $(1,2)^\star$-$G^\# C(X) = \{\phi, \{b\}, \{a, b\}, X\}$. Let $f : X \to Y$ be the identity function. Then f is $(1,2)^\star$-$g^\#$-continuous.*

Proposition 3.3. *Every $(1,2)^\star$-continuous function is $(1,2)^\star$-$g^\#$-continuous but not conversely.*

Example 3.4. *Let $X = \{a, b, c\}$, $\tau_1 = \{\phi, X\}$ and $\tau_2 = \{\phi, \{a, b\}, X\}$. Then the sets in $\{\phi, \{a, b\}, X\}$ are called $\tau_{1,2}$-open and the sets in $\{\phi, \{c\}, X\}$ are called $\tau_{1,2}$-closed. Let $Y = \{a, b, c\}$, $\sigma_1 = \{\phi, \{b\}, Y\}$ and $\sigma_2 = \{\phi, Y\}$. Then the sets in $\{\phi, \{b\}, Y\}$ are called $\sigma_{1,2}$-open and the sets in $\{\phi, \{a, c\}, Y\}$ are called $\sigma_{1,2}$-closed. We have $(1,2)^\star$-$G^\# C(X) = \{\phi, \{c\}, \{a, c\}, \{b, c\}, X\}$. Let $f : X \to Y$ be the identity function. Then f is $(1,2)^\star$-$g^\#$-continuous but not $(1,2)^\star$-continuous, since $f^{-1}(\{a, c\}) = \{a, c\}$ is not $\tau_{1,2}$-closed in X.*

Proposition 3.5. *Every $(1,2)^\star$-$g^\#$-continuous function is $(1,2)^\star$-$g_\alpha^\#$-continuous but not conversely.*

Example 3.6. *Let $X = \{a, b, c\}$, $\tau_1 = \{\phi, X\}$ and $\tau_2 = \{\phi, \{b\}, X\}$. Then the sets in $\{\phi, \{b\}, X\}$ are called $\tau_{1,2}$-open and the sets in $\{\phi, \{a, c\}, X\}$ are called $\tau_{1,2}$-closed. Let $Y = \{a, b, c\}$, $\sigma_1 = \{\phi, Y\}$ and $\sigma_2 = \{\phi, \{b, c\}, Y\}$. Then the sets in $\{\phi, \{b, c\}, Y\}$ are called $\sigma_{1,2}$-open and the sets in $\{\phi, \{a\}, Y\}$ are called $\sigma_{1,2}$-closed. We have $(1,2)^\star$-$G^\# C(X) = \{\phi, \{a, c\}, X\}$ and $(1,2)^\star$-$G_\alpha^\# C(X) = \{\phi, \{a\}, \{c\}, \{a, c\}, X\}$. Let $f : X \to Y$ be the identity function. Then f is $(1,2)^\star$-$g_\alpha^\#$-continuous but not $(1,2)^\star$-$g^\#$-continuous, since $f^{-1}(\{a\}) = \{a\}$ is not $(1,2)^\star$-$g^\#$-closed in X.*

Proposition 3.7. *Every $(1,2)^\star$-$g^\#$-continuous function is $(1,2)^\star$-$g^\star$-continuous but not conversely.*

Example 3.8. *Let $X = \{a, b, c\}$, $\tau_1 = \{\phi, \{c\}, X\}$ and $\tau_2 = \{\phi, \{a, c\}, X\}$. Then the sets in $\{\phi, \{c\}, \{a, c\}, X\}$ are called $\tau_{1,2}$-open and the sets in $\{\phi, \{b\}, \{a, b\}, X\}$ are called $\tau_{1,2}$-closed. Let $Y = \{a, b, c\}$, $\sigma_1 = \{\phi, Y\}$ and $\sigma_2 = \{\phi, \{a\}, Y\}$. Then the sets in $\{\phi, \{a\}, Y\}$ are called $\sigma_{1,2}$-open and the sets in $\{\phi, \{b, c\}, Y\}$ are called $\sigma_{1,2}$-closed. We have $(1,2)^\star$-$G^\# C(X) = \{\phi, \{b\}, \{a, b\}, X\}$ and $(1,2)^\star$-$G^\star C(X) = \{\phi, \{b\}, \{a, b\}, \{b, c\}, X\}$. Let $f : X \to Y$ be the identity function. Then f is $(1,2)^\star$-$g^\star$-continuous but not $(1,2)^\star$-$g^\#$-continuous, since $f^{-1}(\{b, c\}) = \{b, c\}$ is not $(1,2)^\star$-$g^\#$-closed in X.*

Proposition 3.9. *Every* $(1,2)^\star$-$g^\#$*-continuous function is* $(1,2)^\star$-*g-continuous but not conversely.*

Example 3.10. *Let* $X = \{a, b, c\}$, $\tau_1 = \{\phi, \{a\}, X\}$ *and* $\tau_2 = \{\phi, \{b, c\}, X\}$. *Then the sets in* $\{\phi, \{a\}, \{b, c\}, X\}$ *are called* $\tau_{1,2}$*-open and the sets in* $\{\phi, \{a\}, \{b, c\}, X\}$ *are called* $\tau_{1,2}$*-closed. Let* $Y = \{a, b, c\}$, $\sigma_1 = \{\phi, Y\}$ *and* $\sigma_2 = \{\phi, \{c\}, Y\}$. *Then the sets in* $\{\phi, \{c\}, Y\}$ *are called* $\sigma_{1,2}$*-open and the sets in* $\{\phi, \{a, b\}, Y\}$ *are called* $\sigma_{1,2}$*-closed. We have* $(1,2)^\star$-$G^\# C(X) = \{\phi, \{a\}, \{b, c\}, X\}$ *and* $(1,2)^\star$-$GC(X) = P(X)$. *Let* $f : X \to Y$ *be the identity function. Then f is* $(1,2)^\star$*-g-continuous but not* $(1,2)^\star$-$g^\#$*-continuous, since* $f^{-1}(\{a, b\}) = \{a, b\}$ *is not* $(1,2)^\star$-$g^\#$*-closed in X.*

Proposition 3.11. *Every* $(1,2)^\star$-$g^\#$*-continuous function is* $(1,2)^\star$-αg*-continuous but not conversely.*

Example 3.12. *Let* $X = \{a, b, c\}$, $\tau_1 = \{\phi, \{a\}, X\}$ *and* $\tau_2 = \{\phi, \{b, c\}, X\}$. *Then the sets in* $\{\phi, \{a\}, \{b, c\}, X\}$ *are called* $\tau_{1,2}$*-open and the sets in* $\{\phi, \{a\}, \{b, c\}, X\}$ *are called* $\tau_{1,2}$*-closed. Let* $Y = \{a, b, c\}$, $\sigma_1 = \{\phi, Y\}$ *and* $\sigma_2 = \{\phi, \{b\}, Y\}$. *Then the sets in* $\{\phi, \{b\}, Y\}$ *are called* $\sigma_{1,2}$*-open and the sets in* $\{\phi, \{a, c\}, Y\}$ *are called* $\sigma_{1,2}$*-closed. We have* $(1,2)^\star$-$G^\# C(X) = \{\phi, \{a\}, \{b, c\}, X\}$ *and* $(1,2)^\star$-$\alpha GC(X) = P(X)$. *Let* $f : X \to Y$ *be the identity function. Then f is* $(1,2)^\star$-αg*-continuous but not* $(1,2)^\star$-$g^\#$*-continuous, since* $f^{-1}(\{a, c\}) = \{a, c\}$ *is not* $(1,2)^\star$-$g^\#$*-closed in X.*

Proposition 3.13. *Every* $(1,2)^\star$-$g^\#$*-continuous function is* $(1,2)^\star$*-gs-continuous but not conversely.*

Example 3.14. *Let* $X = \{a, b, c\}$, $\tau_1 = \{\phi, X\}$ *and* $\tau_2 = \{\phi, \{a\}, X\}$. *Then the sets in* $\{\phi, \{a\}, X\}$ *are called* $\tau_{1,2}$*-open and the sets in* $\{\phi, \{b, c\}, X\}$ *are called* $\tau_{1,2}$*-closed. Let* $Y = \{a, b, c\}$, $\sigma_1 = \{\phi, Y\}$ *and* $\sigma_2 = \{\phi, \{a, b\}, Y\}$. *Then the sets in* $\{\phi, \{a, b\}, Y\}$ *are called* $\sigma_{1,2}$*-open and the sets in* $\{\phi, \{c\}, Y\}$ *are called* $\sigma_{1,2}$*-closed. We have* $(1,2)^\star$-$G^\# C(X) = \{\phi, \{b, c\}, X\}$ *and* $(1,2)^\star$-$GSC(X) = \{\phi, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, X\}$. *Let* $f : X \to Y$ *be the identity function. Then f is* $(1,2)^\star$*-gs-continuous but not* $(1,2)^\star$-$g^\#$*-continuous, since* $f^{-1}(\{c\}) = \{c\}$ *is not* $(1,2)^\star$-$g^\#$*-closed in X.*

Proposition 3.15. *Every* $(1,2)^\star$-$g^\#$*-continuous function is* $(1,2)^\star$*-gsp-continuous but not conversely.*

Example 3.16. *Let* $X = \{a, b, c\}$, $\tau_1 = \{\phi, X\}$ *and* $\tau_2 = \{\phi, \{b\}, X\}$. *Then the sets in* $\{\phi, \{b\}, X\}$ *are called* $\tau_{1,2}$*-open and the sets in* $\{\phi, \{a, c\}, X\}$ *are called* $\tau_{1,2}$*-closed. Let* $Y = \{a, b, c\}$, $\sigma_1 = \{\phi, Y\}$ *and* $\sigma_2 = \{\phi, \{a, b\}, Y\}$. *Then the sets in* $\{\phi, \{a, b\}, Y\}$ *are called* $\sigma_{1,2}$*-open and the sets in* $\{\phi, \{c\}, Y\}$ *are called* $\sigma_{1,2}$*-closed. We have* $(1,2)^\star$-$G^\# C(X) = \{\phi, \{a, c\}, X\}$ *and* $(1,2)^\star$-$GSPC(X) = \{\phi, \{a\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, X\}$. *Let* $f : X \to Y$ *be the identity function. Then f is* $(1,2)^\star$*-gsp-continuous but not* $(1,2)^\star$-$g^\#$*-continuous, since* $f^{-1}(\{c\}) = \{c\}$ *is not* $(1,2)^\star$-$g^\#$*-closed in X.*

Proposition 3.17. *Every* $(1,2)^\star$-$g^\#$*-continuous function is* $(1,2)^\star$*-sg-continuous but not conversely.*

Example 3.18. *Let* $X = \{a, b, c\}$, $\tau_1 = \{\phi, \{a\}, X\}$ *and* $\tau_2 = \{\phi, \{b, c\}, X\}$. *Then the sets in* $\{\phi, \{a\}, \{b, c\}, X\}$ *are called* $\tau_{1,2}$*-open and the sets in* $\{\phi, \{a\}, \{b, c\}, X\}$ *are called* $\tau_{1,2}$*-closed. Let* $Y = \{a, b, c\}$, $\sigma_1 = \{\phi, Y\}$ *and* $\sigma_2 = \{\phi, \{a, b\}, Y\}$. *Then the sets in* $\{\phi, \{a, b\}, Y\}$ *are called* $\sigma_{1,2}$*-open and the sets in* $\{\phi, \{c\}, Y\}$ *are called* $\sigma_{1,2}$*-closed. We have* $(1,2)^\star$-$G^\# C(X) = \{\phi, \{a\}, \{b, c\}, X\}$ *and* $(1,2)^\star$-$SGC(X) = P(X)$. *Let* $f : X \to Y$ *be the identity function. Then f is* $(1,2)^\star$*-sg-continuous but not* $(1,2)^\star$-$g^\#$*-continuous, since* $f^{-1}(\{c\}) = \{c\}$ *is not* $(1,2)^\star$-$g^\#$*-closed in X.*

Remark 3.19. *The following examples show that* $(1,2)^\star$-$g^\#$*-continuity is independent of* $(1,2)^\star$-α*-continuity and* $(1,2)^\star$*-semi-continuity.*

Example 3.20. *Let* $X = \{a, b, c\}$, $\tau_1 = \{\phi, X\}$ *and* $\tau_2 = \{\phi, \{a, b\}, X\}$. *Then the sets in* $\{\phi, \{a, b\}, X\}$ *are called* $\tau_{1,2}$*-open and the sets in* $\{\phi, \{c\}, X\}$ *are called* $\tau_{1,2}$*-closed. Let* $Y = \{a, b, c\}$, $\sigma_1 = \{\phi, Y\}$ *and* $\sigma_2 = \{\phi, \{a\}, Y\}$. *Then the sets in* $\{\phi, \{a\}, Y\}$ *are called* $\sigma_{1,2}$*-open and the sets in* $\{\phi, \{b, c\}, Y\}$ *are called* $\sigma_{1,2}$*-closed. We have* $(1,2)^\star$-$G^\# C(X) = \{\phi, \{c\}, \{a, c\}, \{b, c\}, X\}$ *and* $(1,2)^\star$-$\alpha C(X) = (1,2)^\star$-$SC(X) = \{\phi, \{c\}, X\}$. *Let* $f : X \to Y$ *be the identity function. Then f is* $(1,2)^\star$-$g^\#$*-continuous but it is neither* $(1,2)^\star$-α*-continuous nor* $(1,2)^\star$*-semi-continuous, since* $f^{-1}(\{b, c\}) = \{b, c\}$ *is neither* $(1,2)^\star$-α*-closed nor* $(1,2)^\star$*-semi-closed in X.*

Example 3.21. *In Example 3.14, we have* $(1,2)^\star$-$G^\# C(X) = \{\phi, \{b, c\}, X\}$ *and* $(1,2)^\star$-$\alpha C(X) = (1,2)^\star$-$SC(X) = \{\phi, \{b\}, \{c\}, \{b, c\}, X\}$. *Let* $f : X \to Y$ *be the identity function. Then f is both* $(1,2)^\star$-α*-continuous and* $(1,2)^\star$*-semi-continuous but it is not* $(1,2)^\star$-$g^\#$*-continuous, since* $f^{-1}(\{c\}) = \{c\}$ *is not* $(1,2)^\star$-$g^\#$*-closed in X.*

Proposition 3.22. *A function $f: X \to Y$ is $(1,2)^\star$-$g^\#$-continuous if and only if $f^{-1}(U)$ is $(1,2)^\star$-$g^\#$-open in X for every $\sigma_{1,2}$-open set U in Y.*

Proof. Let $f : X \to Y$ be $(1,2)^\star$-$g^\#$-continuous and U be an $\sigma_{1,2}$-open set in Y. Then U^c is $\sigma_{1,2}$-closed in Y and since f is $(1,2)^\star$-$g^\#$-continuous, $f^{-1}(U^c)$ is $(1,2)^\star$-$g^\#$-closed in X. But $f^{-1}(U^c) = (f^{-1}(U))^c$ and so $f^{-1}(U)$ is $(1,2)^\star$-$g^\#$-open in X.

Conversely, assume that $f^{-1}(U)$ is $(1,2)^\star$-$g^\#$-open in X for each $\sigma_{1,2}$-open set U in Y. Let F be a $\sigma_{1,2}$-closed set in Y. Then F^c is $\sigma_{1,2}$-open in Y and by assumption, $f^{-1}(F^c)$ is $(1,2)^\star$-$g^\#$-open in X. Since $f^{-1}(F^c) = (f^{-1}(F))^c$, we have $f^{-1}(F)$ is $(1,2)^\star$-$g^\#$-closed in X and so f is $(1,2)^\star$-$g^\#$-continuous.

Remark 3.23. *The composition of two $(1,2)^\star$-$g^\#$-continuous functions need not be a $(1,2)^\star$-$g^\#$-continuous function as is shown in the following example.*

Example 3.24. *Let $X = \{a, b, c\}$, $\tau_1 = \{\phi, \{a\}, \{a, c\}, X\}$ and $\tau_2 = \{\phi, \{a, b\}, X\}$. Then the sets in $\{\phi, \{a\}, \{a, b\}, \{a, c\}, X\}$ are called $\tau_{1,2}$-open and the sets in $\{\phi, \{b\}, \{c\}, \{b, c\}, X\}$ are called $\tau_{1,2}$-closed. Let $Y = \{a, b, c\}$, $\sigma_1 = \{\phi, Y\}$ and $\sigma_2 = \{\phi, \{a, b\}, Y\}$. Then the sets in $\{\phi, \{a, b\}, Y\}$ are called $\sigma_{1,2}$-open and the sets in $\{\phi, \{c\}, Y\}$ are called $\sigma_{1,2}$-closed. Let $Z = \{a, b, c\}$, $\eta_1 = \{\phi, Z\}$ and $\eta_2 = \{\phi, \{b\}, Z\}$. Then the sets in $\{\phi, \{b\}, Z\}$ are called $\eta_{1,2}$-open and the sets in $\{\phi, \{a, c\}, Z\}$ are called $\eta_{1,2}$-closed. Let $f : X \to Y$ and $g : Y \to Z$ be the identity functions. Then f and g are $(1,2)^\star$-$g^\#$-continuous but their $g\ o\ f : X \to Z$ is not $(1,2)^\star$-$g^\#$-continuous, since for the set $V = \{a, c\}$ is $\eta_{1,2}$-closed in Z, $(g\ o\ f)^{-1}(V) = f^{-1}(g^{-1}(V)) = f^{-1}(g^{-1}(\{a, c\})) = f^{-1}(\{a, c\}) = \{a, c\}$ is not $(1,2)^\star$-$g^\#$-closed in X.*

Proposition 3.25. *Let X and Z be bitopological spaces and Y be a $(1,2)^\star$-$T_g^\#$-space. Then the composition $g\ o\ f : X \to Z$ of the $(1,2)^\star$-$g^\#$-continuous functions $f : X \to Y$ and $g : Y \to Z$ is $(1,2)^\star$-$g^\#$-continuous.*

Proof. Let F be any $\eta_{1,2}$-closed set of Z. Then $g^{-1}(F)$ is $(1,2)^\star$-$g^\#$-closed in Y, since g is $(1,2)^\star$-$g^\#$-continuous. Since Y is a $(1,2)^\star$-$T_g^\#$-space, $g^{-1}(F)$ is $\sigma_{1,2}$-closed in Y. Since f is $(1,2)^\star$-$g^\#$-continuous, $f^{-1}(g^{-1}(F))$ is $(1,2)^\star$-$g^\#$-closed in X. But $f^{-1}(g^{-1}(F)) = (g\ o\ f)^{-1}(F)$ and so $g\ o$ f is $(1,2)^\star$-$g^\#$-continuous.

Proposition 3.26. *Let X and Z be bitopological spaces and Y be a $(1,2)^\star$-$T_{1/2}$-space (resp. $(1,2)^\star$-T_b-space, $(1,2)^\star$-${}_\alpha T_b$-space). Then the composition $g\ o\ f : X \to Z$ of the $(1,2)^\star$-$g^\#$-continuous function f $: X \to Y$ and the $(1,2)^\star$-g-continuous (resp. $(1,2)^\star$-gs-continuous, $(1,2)^\star$-αg-continuous) function g : $Y \to Z$ is $(1,2)^\star$-$g^\#$-continuous.*

Proof. Similar to Proposition 3.25.

Proposition 3.27. *If $f : X \to Y$ is $(1,2)^\star$-$g^\#$-continuous and $g : Y \to Z$ is $(1,2)^\star$-continuous, then their composition $g\ o\ f : X \to Z$ is $(1,2)^\star$-$g^\#$-continuous.*

Proof. Let F be any $\eta_{1,2}$-closed set in Z. Since $g : Y \to Z$ is $(1,2)^\star$-continuous, $g^{-1}(F)$ is $\sigma_{1,2}$-closed in Y. Since $f : X \to Y$ is $(1,2)^\star$-$g^\#$-continuous, $f^{-1}(g^{-1}(F)) = (g\ o\ f)^{-1}(F)$ is $(1,2)^\star$-$g^\#$-closed in X and so $g\ o$ f is $(1,2)^\star$-$g^\#$-continuous.

Proposition 3.28. *Let A be $(1,2)^\star$-$g^\#$-closed in X. If $f : X \to Y$ is $(1,2)^\star$-αg-irresolute and $(1,2)^\star$-closed, then f(A) is $(1,2)^\star$-$g^\#$-closed in Y.*

Proof. Let U be any $(1,2)^\star$-αg-open in Y such that $f(A) \subseteq U$. Then $A \subseteq f^{-1}(U)$ and by hypothesis, $\tau_{1,2}$-cl(A) $\subseteq f^{-1}(U)$. Thus $f(\tau_{1,2}\text{-cl}(A)) \subseteq U$ and $f(\tau_{1,2}\text{-cl}(A))$ is a $\sigma_{1,2}$-closed set. Now, $\sigma_{1,2}\text{-cl}(f(A)) \subseteq \sigma_{1,2}\text{-cl}(f(\tau_{1,2}\text{-cl}(A))) = f(\tau_{1,2}\text{-cl}(A)) \subseteq U$. i.e., $\sigma_{1,2}\text{-cl}(f(A)) \subseteq U$ and so f(A) is $(1,2)^\star$-$g^\#$-closed in Y.

Theorem 3.29. *Let $f : X \to Y$ be a pre-$(1,2)^\star$-αg-closed and $(1,2)^\star$-open bijection. If X is a $(1,2)^\star$-$T_{g^\#}$-space, then Y is also a $(1,2)^\star$-$T_{g^\#}$-space.*

Proof. Let $y \in Y$. Since f is bijective, $y = f(x)$ for some $x \in X$. Since X is a $(1,2)^\star$-$T_{g^\#}$-space, $\{x\}$ is $(1,2)^\star$-αg-closed or $\tau_{1,2}$-open by Theorem 2.13. If $\{x\}$ is $(1,2)^\star$-αg-closed then $\{y\} = f(\{x\})$ is $(1,2)^\star$-αg-closed, since f is pre-$(1,2)^\star$-αg-closed. Also $\{y\}$ is $\sigma_{1,2}$-open if $\{x\}$ is $\tau_{1,2}$-open since f is $(1,2)^\star$-open. Therefore by Theorem 2.13, Y is a $(1,2)^\star$-$T_{g^\#}$-space.

Theorem 3.30. *If $f : X \to Y$ is $(1,2)^{\star}$-$g^{\#}$-continuous and pre-$(1,2)^{\star}$-αg-closed and if A is an $(1,2)^{\star}$-$g^{\#}$-open (or $(1,2)^{\star}$-$g^{\#}$-closed) subset of Y, then $f^{-1}(A)$ is $(1,2)^{\star}$-$g^{\#}$-open (or $(1,2)^{\star}$-$g^{\#}$-closed) in X.*

Proof. Let A be an $(1,2)^{\star}$-$g^{\#}$-open set in Y and F be any $(1,2)^{\star}$-αg-closed set in X such that F $\subseteq$ $f^{-1}(A)$. Then f(F) $\subseteq$ A. By hypothesis, f(F) is $(1,2)^{\star}$-αg-closed and A is $(1,2)^{\star}$-$g^{\#}$-open in Y. Therefore, f(F) $\subseteq$ $\sigma_{1,2}$-int(A) by Theorem 2.12, and so F $\subseteq$ $f^{-1}(\sigma_{1,2}\text{-int(A)})$. Since f is $(1,2)^{\star}$-$g^{\#}$-continuous and $\sigma_{1,2}$-int(A) is $\sigma_{1,2}$-open in Y, $f^{-1}(\sigma_{1,2}\text{-int(A)})$ is $(1,2)^{\star}$-$g^{\#}$-open in X. Thus F $\subseteq$ $\tau_{1,2}$-int($f^{-1}(\sigma_{1,2}$-int(A))) $\subseteq$ $\tau_{1,2}$-int(f^{-1}(A)). i.e., F $\subseteq$ $\tau_{1,2}$-int(f^{-1}(A)) and by Theorem 2.12, f^{-1}(A) is $(1,2)^{\star}$-$g^{\#}$-open in X. By taking complements, we can show that if A is $(1,2)^{\star}$-$g^{\#}$-closed in Y, f^{-1}(A) is $(1,2)^{\star}$-$g^{\#}$-closed in X.

Corollary 3.31. *If $f : X \to Y$ is $(1,2)^{\star}$-continuous and pre-$(1,2)^{\star}$-αg-closed and if B is a $(1,2)^{\star}$-$g^{\#}$-closed (or $(1,2)^{\star}$-$g^{\#}$-open) subset of Y, then $f^{-1}(B)$ is $(1,2)^{\star}$-$g^{\#}$-closed (or $(1,2)^{\star}$-$g^{\#}$-open) in X.*

Proof. Follows from Proposition 3.3, and Theorem 3.30.

Corollary 3.32. *Let X, Y and Z be any three bitopological spaces. If $f : X \to Y$ is $(1,2)^{\star}$-$g^{\#}$-continuous and pre-$(1,2)^{\star}$-αg-closed and $g : Y \to Z$ is $(1,2)^{\star}$-$g^{\#}$-continuous, then their composition $g\ o\ f : X \to Z$ is $(1,2)^{\star}$-$g^{\#}$-continuous.*

Proof. Let F be any $\eta_{1,2}$-closed set in Z. Since g : Y $\to$ Z is $(1,2)^{\star}$-$g^{\#}$-continuous, g^{-1}(F) is $(1,2)^{\star}$-$g^{\#}$-closed in Y. Since f : X $\to$ Y is $(1,2)^{\star}$-$g^{\#}$-continuous and pre-$(1,2)^{\star}$-αg-closed, by Theorem 3.30, $f^{-1}(g^{-1}(F))$ = (g o f) $^{-1}$(F) is $(1,2)^{\star}$-$g^{\#}$-closed in X and so g o f is $(1,2)^{\star}$-$g^{\#}$-continuous.

4 $(1,2)^{\star}$-$g^{\#}$-Irresolute Functions

We introduce the following definition.

Definition 4.1. *A function $f : X \to Y$ is called an $(1,2)^{\star}$-$g^{\#}$-irresolute if the inverse image of every $(1,2)^{\star}$-$g^{\#}$-closed set in Y is $(1,2)^{\star}$-$g^{\#}$-closed in X.*

Remark 4.2. *The following examples show that the notions of $(1,2)^{\star}$-sg-irresolute functions and $(1,2)^{\star}$-$g^{\#}$-irresolute functions are independent.*

Example 4.3. *Let X = {a, b, c}, τ_1 = {ϕ, X} and τ_2 = {ϕ, {a, b}, X}. Then the sets in {ϕ, {a, b}, X} are called $\tau_{1,2}$-open and the sets in {ϕ, {c}, X} are called $\tau_{1,2}$-closed. Let Y = {a, b, c}, σ_1 = {ϕ, {a}, {a, b}, Y} and σ_2 = {ϕ, {b}, Y}. Then the sets in {ϕ, {a}, {b}, {a, b}, Y} are called $\sigma_{1,2}$-open and the sets in {ϕ, {c}, {a, c}, {b, c}, Y} are called $\sigma_{1,2}$-closed. We have $(1,2)^{\star}$-$G^{\#}C$(X) = {ϕ, {c}, {a, c}, {b, c}, X}, $(1,2)^{\star}$-SGC(X) = {ϕ, {c}, {a, c}, {b, c}, X}, $(1,2)^{\star}$-$G^{\#}C$(Y) = {ϕ, {c}, {a, c}, {b, c}, Y} and $(1,2)^{\star}$-SGC(Y) = {ϕ, {a}, {b}, {c}, {a, c}, {b, c}, Y}. Let $f : X \to Y$ be the identity function. Then f is $(1,2)^{\star}$-$g^{\#}$-irresolute but it is not $(1,2)^{\star}$-sg-irresolute, since $f^{-1}(\{b\})$ = {b} is not $(1,2)^{\star}$-sg-closed in X.*

Example 4.4. *Let X = {a, b, c}, τ_1 = {ϕ, {a}, {a, b}, X} and τ_2 = {ϕ, {b}, X}. Then the sets in {ϕ, {a}, {b}, {a, b}, X} are called $\tau_{1,2}$-open and the sets in {ϕ, {c}, {a, c}, {b, c}, X} are called $\tau_{1,2}$-closed. Let Y = {a, b, c}, σ_1 = {ϕ, {b}, Y} and σ_2 = {ϕ, {b, c}, Y}. Then the sets in {ϕ, {b}, {b, c}, Y} are called $\sigma_{1,2}$-open and the sets in {ϕ, {a}, {a, c}, Y} are called $\sigma_{1,2}$-closed. We have $(1,2)^{\star}$-$G^{\#}C$(X) = {ϕ, {c}, {a, c}, {b, c}, X} and $(1,2)^{\star}$-SGC(X) = {ϕ, {a}, {b}, {c}, {a, c}, {b, c}, X}, $(1,2)^{\star}$-$G^{\#}C$(Y) = {ϕ, {a}, {a, c}, Y} and $(1,2)^{\star}$-SGC(Y) = {ϕ, {a}, {c}, {a, c}, Y}. Let $f : X \to Y$ be the identity function. Then f is $(1,2)^{\star}$-sg-irresolute but it is not $(1,2)^{\star}$-$g^{\#}$-irresolute, since $f^{-1}(\{a\})$ = {a} is not $(1,2)^{\star}$-$g^{\#}$-closed in X.*

Proposition 4.5. *A function $f : X \to Y$ is $(1,2)^{\star}$-$g^{\#}$-irresolute if and only if the inverse of every $(1,2)^{\star}$-$g^{\#}$-open set in Y is $(1,2)^{\star}$-$g^{\#}$-open in X.*

Proof. Similar to Proposition 3.22.

Proposition 4.6. *If a function $f : X \to Y$ is $(1,2)^{\star}$-$g^{\#}$-irresolute then it is $(1,2)^{\star}$-$g^{\#}$-continuous but not conversely.*

Example 4.7. *Let $X = \{a,\ b,\ c\}$, $\tau_1 = \{\phi,\ X\}$ and $\tau_2 = \{\phi,\ \{b\},\ X\}$. Then the sets in $\{\phi,\ \{b\},\ X\}$ are called $\tau_{1,2}$-open and the sets in $\{\phi,\ \{a,\ c\},\ X\}$ are called $\tau_{1,2}$-closed. Let $Y = \{a,\ b,\ c\}$, $\sigma_1 = \{\phi,\ Y\}$ and $\sigma_2 = \{\phi,\ \{a,\ b\},\ Y\}$. Then the sets in $\{\phi,\ \{a,\ b\},\ Y\}$ are called $\sigma_{1,2}$-open and the sets in $\{\phi,\ \{c\},\ Y\}$ are called $\sigma_{1,2}$-closed. We have $(1,2)^\star$-$G^\# C(X) = \{\phi,\ \{a,\ c\},\ X\}$ and $(1,2)^\star$-$G^\# C(Y) = \{\phi,\ \{c\},\ \{a,\ c\},\ \{b,\ c\},\ Y\}$. Let $f : X \to Y$ be the identity function. Then f is $(1,2)^\star$-$g^\#$-continuous but it is not $(1,2)^\star$-$g^\#$-irresolute, since $f^{-1}(\{a\}) = \{a\}$ is not $(1,2)^\star$-$g^\#$-open in X.*

Proposition 4.8. *Let X be any bitopological space, Y be a $(1,2)^\star$-$T_{g^\#}$-space and $f : X \to Y$ be a function. Then the following are equivalent:*

1. *f is $(1,2)^\star$-$g^\#$-irresolute.*
2. *f is $(1,2)^\star$-$g^\#$-continuous.*

Proof. (1) $\Rightarrow$ (2) Follows from Proposition 4.6.

(2) $\Rightarrow$ (1) Let F be a $(1,2)^\star$-$g^\#$-closed set in Y. Since Y is a $(1,2)^\star$-$T_{g^\#}$-space, F is a $\sigma_{1,2}$-closed set in Y and by hypothesis, $f^{-1}(F)$ is $(1,2)^\star$-$g^\#$-closed in X. Therefore f is $(1,2)^\star$-$g^\#$-irresolute.

Definition 4.9. *A function $f : X \to Y$ is called pre-$(1,2)^\star$-αg-open if f(U) is $(1,2)^\star$-αg-open in Y, for each $(1,2)^\star$-αg-open set U in X.*

Proposition 4.10. *If $f : X \to Y$ is bijective pre-$(1,2)^\star$-αg-open and $(1,2)^\star$-$g^\#$-continuous then f is $(1,2)^\star$-$g^\#$-irresolute.*

Proof. Let A be $(1,2)^\star$-$g^\#$-closed set in Y. Let U be any $(1,2)^\star$-αg-open set in X such that $f^{-1}(A) \subseteq U$. Then $A \subseteq f(U)$. Since A is $(1,2)^\star$-$g^\#$-closed and f(U) is $(1,2)^\star$-αg-open in Y, $\sigma_{1,2}$-cl(A) $\subseteq$ f(U) holds and hence $f^{-1}(\sigma_{1,2}\text{-cl}(A)) \subseteq U$. Since f is $(1,2)^\star$-$g^\#$-continuous and $\sigma_{1,2}$-cl(A) is $\sigma_{1,2}$-closed in Y, $f^{-1}(\sigma_{1,2}\text{-cl}(A))$ is $(1,2)^\star$-$g^\#$-closed and hence $\tau_{1,2}\text{-cl}(f^{-1}(\sigma_{1,2}\text{-cl}(A))) \subseteq U$ and so $\tau_{1,2}\text{-cl}(f^{-1}(A)) \subseteq U$. Therefore, $f^{-1}(A)$ is $(1,2)^\star$-$g^\#$-closed in X and hence f is $(1,2)^\star$-$g^\#$-irresolute.

The following examples show that no assumption of Proposition 4.10 can be removed.

Example 4.11. *The identity function defined in Example 4.7 is $(1,2)^\star$-$g^\#$-continuous and bijective but not pre-$(1,2)^\star$-αg-open and so f is not $(1,2)^\star$-$g^\#$-irresolute.*

Example 4.12. *Let $X = \{a,\ b,\ c\}$, $\tau_1 = \{\phi,\ \{a\},\ \{a,\ b\},\ X\}$ and $\tau_2 = \{\phi,\ \{b\},\ X\}$. Then the sets in $\{\phi,\ \{a\},\ \{b\},\ \{a,\ b\},\ X\}$ are called $\tau_{1,2}$-open and the sets in $\{\phi,\ \{c\},\ \{a,\ c\},\ \{b,\ c\},\ X\}$ are called $\tau_{1,2}$-closed. Let $Y = \{a,\ b,\ c\}$, $\sigma_1 = \{\phi,\ \{a\},\ Y\}$ and $\sigma_2 = \{\phi,\ \{b,\ c\},\ Y\}$. Then the sets in $\{\phi,\ \{a\},\ \{b,\ c\},\ Y\}$ are called $\sigma_{1,2}$-open and the sets in $\{\phi,\ \{a\},\ \{b,\ c\},\ Y\}$ are called $\sigma_{1,2}$-closed. We have $(1,2)^\star$-$G^\# C(X) = \{\phi,\ \{c\},\ \{a,\ c\},\ \{b,\ c\},\ X\}$ and $(1,2)^\star$-$SGC(X) = \{\phi,\ \{a\},\ \{b\},\ \{c\},\ \{a,\ c\},\ \{b,\ c\},\ X\}$, $(1,2)^\star$-$G^\# C(Y) = \{\phi,\ \{a\},\ \{b,\ c\},\ Y\}$ and $(1,2)^\star$-$SGC(Y) = P(Y)$. Let $f : X \to Y$ be the identity function. Then f is bijective and pre-$(1,2)^\star$-αg-open but not $(1,2)^\star$-$g^\#$-continuous and so f is not $(1,2)^\star$-$g^\#$-irresolute, since $f^{-1}(\{a\}) = \{a\}$ is not $(1,2)^\star$-$g^\#$-closed in X.*

Proposition 4.13. *If $f : X \to Y$ is bijective $(1,2)^\star$-closed and $(1,2)^\star$-αg-irresolute then the inverse function $f^{-1} : Y \to X$ is $(1,2)^\star$-$g^\#$-irresolute.*

Proof. Let A be $(1,2)^\star$-$g^\#$-closed in X. Let $(f^{-1})^{-1}(A) = f(A) \subseteq U$ where U is $(1,2)^\star$-αg-open in Y. Then $A \subseteq f^{-1}(U)$ holds. Since $f^{-1}(U)$ is $(1,2)^\star$-αg-open in X and A is $(1,2)^\star$-$g^\#$-closed in X, $\tau_{1,2}$-cl(A) $\subseteq f^{-1}(U)$ and hence $f(\tau_{1,2}\text{-cl}(A)) \subseteq U$. Since f is $(1,2)^\star$-closed and $\tau_{1,2}$-cl(A) is $\tau_{1,2}$-closed in X, $f(\tau_{1,2}\text{-cl}(A))$ is $\sigma_{1,2}$-closed in Y and so $f(\tau_{1,2}\text{-cl}(A))$ is $(1,2)^\star$-$g^\#$-closed in Y. Therefore $\sigma_{1,2}\text{-cl}(f(\tau_{1,2}\text{-cl}(A))) \subseteq U$ and hence $\sigma_{1,2}\text{-cl}(f(A)) \subseteq U$. Thus f(A) is $(1,2)^\star$-$g^\#$-closed in Y and so f^{-1} is $(1,2)^\star$-$g^\#$-irresolute.

5 Applications

To obtain a decomposition of $(1,2)^\star$-continuity, we introduce the notion of $(1,2)^\star$-$\alpha glc^\#$-continuous function in bitopological spaces and prove that a function is $(1,2)^\star$-continuous if and only if it is both $(1,2)^\star$-$g^\#$-continuous and $(1,2)^\star$-$\alpha glc^\#$-continuous.

Definition 5.1. *A subset A of a bitopological space X is called $(1,2)^\star$-$\alpha glc^\star$-set if $A = M \cap N$, where M is $(1,2)^\star$-αg-open and N is $\tau_{1,2}$-closed in X.*

The family of all $(1,2)^{\star}$-$\alpha glc^{\star}$-sets in a space X is denoted by $(1,2)^{\star}$-$\alpha glc^{\star}$(X).

Example 5.2. *Let $X = \{a, b, c\}$, $\tau_1 = \{\phi, X\}$ and $\tau_2 = \{\phi, \{c\}, X\}$. Then the sets in $\{\phi, \{c\}, X\}$ are called $\tau_{1,2}$-open and the sets in $\{\phi, \{a, b\}, X\}$ are called $\tau_{1,2}$-closed. Then $\{a\}$ is $(1,2)^{\star}$-$\alpha glc^{\star}$-set in X.*

Remark 5.3. *Every $\tau_{1,2}$-closed set is $(1,2)^{\star}$-$\alpha glc^{\star}$-set but not conversely.*

Example 5.4. *Let $X = \{a, b, c\}$, $\tau_1 = \{\phi, X\}$ and $\tau_2 = \{\phi, \{a\}, X\}$. Then the sets in $\{\phi, \{a\}, X\}$ are called $\tau_{1,2}$-open and the sets in $\{\phi, \{b, c\}, X\}$ are called $\tau_{1,2}$-closed. Then $\{a, b\}$ is $(1,2)^{\star}$-$\alpha glc^{\star}$-set but not $\tau_{1,2}$-closed in X.*

Remark 5.5. *$(1,2)^{\star}$-$g^{\#}$-closed sets and $(1,2)^{\star}$-$\alpha glc^{\star}$-sets are independent of each other.*

Example 5.6. *Let $X = \{a, b, c\}$, $\tau_1 = \{\phi, X\}$ and $\tau_2 = \{\phi, \{a, c\}, X\}$. Then the sets in $\{\phi, \{a, c\}, X\}$ are called $\tau_{1,2}$-open and the sets in $\{\phi, \{b\}, X\}$ are called $\tau_{1,2}$-closed. Then $\{b, c\}$ is a $(1,2)^{\star}$-$g^{\#}$-closed set but not $(1,2)^{\star}$-$\alpha glc^{\star}$-set in X.*

Example 5.7. *Let $X = \{a, b, c\}$, $\tau_1 = \{\phi, X\}$ and $\tau_2 = \{\phi, \{b\}, X\}$. Then the sets in $\{\phi, \{b\}, X\}$ are called $\tau_{1,2}$-open and the sets in $\{\phi, \{a, c\}, X\}$ are called $\tau_{1,2}$-closed. Then $\{a, b\}$ is an $(1,2)^{\star}$-$\alpha glc^{\star}$-set but not $(1,2)^{\star}$-$g^{\#}$-closed set in X.*

Proposition 5.8. *Let X be a bitopological space. Then a subset A of X is $\tau_{1,2}$-closed if and only if it is both $(1,2)^{\star}$-$g^{\#}$-closed and $(1,2)^{\star}$-$\alpha glc^{\star}$-set.*

Proof. Necessity is trivial. To prove the sufficiency, assume that A is both $(1,2)^{\star}$-$g^{\#}$-closed and $(1,2)^{\star}$-$\alpha glc^{\star}$-set. Then A = M ∩ N, where M is $(1,2)^{\star}$-αg-open and N is $\tau_{1,2}$-closed in X. Therefore, A ⊆ M and A ⊆ N and so by hypothesis, $\tau_{1,2}$-cl(A) ⊆ M and $\tau_{1,2}$-cl(A) ⊆ N. Thus $\tau_{1,2}$-cl(A) ⊆ M ∩ N = A and hence $\tau_{1,2}$-cl(A) = A i.e., A is $\tau_{1,2}$-closed in X.

We introduce the following definition.

Definition 5.9. *A function $f : X \to Y$ is said to be $(1,2)^{\star}$-$\alpha glc^{\#}$-continuous if for each $\sigma_{1,2}$-closed set V of Y, $f^{-1}(V)$ is an $(1,2)^{\star}$-$\alpha glc^{\star}$-set in X.*

Example 5.10. *Let $X = \{a, b, c\}$, $\tau_1 = \{\phi, X\}$ and $\tau_2 = \{\phi, \{a\}, X\}$. Then the sets in $\{\phi, \{a\}, X\}$ are called $\tau_{1,2}$-open and the sets in $\{\phi, \{b, c\}, X\}$ are called $\tau_{1,2}$-closed. Let $Y = \{a, b, c\}$, $\sigma_1 = \{\phi, \{a\}, Y\}$ and $\sigma_2 = \{\phi, \{b, c\}, Y\}$. Then the sets in $\{\phi, \{a\}, \{b, c\}, Y\}$ are called $\sigma_{1,2}$-open and the sets in $\{\phi, \{a\}, \{b, c\}, Y\}$ are called $\sigma_{1,2}$-closed. Let $f : X \to Y$ be the identity function. Then f is $(1,2)^{\star}$-$\alpha glc^{\#}$-continuous function.*

Remark 5.11. *From the definitions it is clear that every $(1,2)^{\star}$-continuous function is $(1,2)^{\star}$-$\alpha glc^{\#}$-continuous but not conversely.*

Example 5.12. *Let $X = \{a, b, c\}$, $\tau_1 = \{\phi, X\}$ and $\tau_2 = \{\phi, \{b\}, X\}$. Then the sets in $\{\phi, \{b\}, X\}$ are called $\tau_{1,2}$-open and the sets in $\{\phi, \{a, c\}, X\}$ are called $\tau_{1,2}$-closed. Let $Y = \{a, b, c\}$, $\sigma_1 = \{\phi, \{b\}, Y\}$ and $\sigma_2 = \{\phi, \{a, c\}, Y\}$. Then the sets in $\{\phi, \{b\}, \{a, c\}, Y\}$ are called $\sigma_{1,2}$-open and the sets in $\{\phi, \{b\}, \{a, c\}, Y\}$ are called $\sigma_{1,2}$-closed. Let $f : X \to Y$ be the identity function. Then f is $(1,2)^{\star}$-$\alpha glc^{\#}$-continuous function but not $(1,2)^{\star}$-continuous. Since for the $\sigma_{1,2}$-closed set $\{b\}$ in Y, $f^{-1}(\{b\}) = \{b\}$, which is not $\tau_{1,2}$-closed in X.*

Remark 5.13. *$(1,2)^{\star}$-$g^{\#}$-continuity and $(1,2)^{\star}$-$\alpha glc^{\#}$-continuity are independent of each other.*

Example 5.14. *Let $X = \{a, b, c\}$, $\tau_1 = \{\phi, X\}$ and $\tau_2 = \{\phi, \{a, b\}, X\}$. Then the sets in $\{\phi, \{a, b\}, X\}$ are called $\tau_{1,2}$-open and the sets in $\{\phi, \{c\}, X\}$ are called $\tau_{1,2}$-closed. Let $Y = \{a, b, c\}$, $\sigma_1 = \{\phi, Y\}$ and $\sigma_2 = \{\phi, \{a\}, Y\}$. Then the sets in $\{\phi, \{a\}, Y\}$ are called $\sigma_{1,2}$-open and the sets in $\{\phi, \{b, c\}, Y\}$ are called $\sigma_{1,2}$-closed. Let $f : X \to Y$ be the identity function. Then f is $(1,2)^{\star}$-$g^{\#}$-continuous function but not $(1,2)^{\star}$-$\alpha glc^{\#}$-continuous.*

Example 5.15. *Let $X = \{a, b, c\}$, $\tau_1 = \{\phi, X\}$ and $\tau_2 = \{\phi, \{a\}, X\}$. Then the sets in $\{\phi, \{a\}, X\}$ are called $\tau_{1,2}$-open and the sets in $\{\phi, \{b, c\}, X\}$ are called $\tau_{1,2}$-closed. Let $Y = \{a, b, c\}$, $\sigma_1 = \{\phi, Y\}$ and $\sigma_2 = \{\phi, \{b, c\}, Y\}$. Then the sets in $\{\phi, \{b, c\}, Y\}$ are called $\sigma_{1,2}$-open and the sets in $\{\phi, \{a\}, Y\}$ are called $\sigma_{1,2}$-closed. Let $f : X \to Y$ be the identity function. Then f is $(1,2)^{\star}$-$\alpha glc^{\#}$-continuous function but not $(1,2)^{\star}$-$g^{\#}$-continuous.*

We have the following decomposition for $(1,2)^{\star}$-continuity.

Theorem 5.16. *A function $f : X \to Y$ is $(1,2)^{\star}$-continuous if and only if it is both $(1,2)^{\star}$-$g^{\#}$-continuous and $(1,2)^{\star}$-$\alpha glc^{\#}$-continuous.*

Proof. Assume that f is $(1,2)^{\star}$-continuous. Then by Proposition 3.3 and Remark 5.11, f is both $(1,2)^{\star}$-$g^{\#}$-continuous and $(1,2)^{\star}$-$\alpha glc^{\#}$-continuous.

Conversely, assume that f is both $(1,2)^{\star}$-$g^{\#}$-continuous and $(1,2)^{\star}$-$\alpha glc^{\#}$-continuous. Let V be a $\sigma_{1,2}$-closed subset of Y. Then $f^{-1}(V)$ is both $(1,2)^{\star}$-$g^{\#}$-closed set and $(1,2)^{\star}$-$\alpha glc^{\star}$-set. By Proposition 5.8, $f^{-1}(V)$ is a $\tau_{1,2}$-closed set in X and so f is $(1,2)^{\star}$-continuous.

6 Conclusion

The notions of the sets, functions and spaces in bitopological spaces are highly developed and used extensively in many practical and engineering problems, computational topology for geometric design, computer-aided geometric design, engineering design research and mathematical sciences. Also, topology plays a significant role in space time geometry and high-energy physics. Thus generalized continuity is one of the most important subjects on topological spaces. Hence we studied new types of generalizations of non-continuous functions, obtained some of their properties in bitopological spaces.

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Smarandache Soft Semigroups and their Properties

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Abstract – In this paper, the notions of smarandache soft semigroups (SS-semigroups) introduced for the first time. An SS-semigroup (F, A) is basically a parameterized collection of subsemigroups which has atlest a proper soft subgroup of (F, A). Some new type of SS-semigroup is also presented here such as smarandache weak commutative semigroup, smarandache weak cyclic semigroup, smarandache hyper subsemigroup etc. Some of their related properties and other notions have been discussed with sufficient amount of examples.

Keywords – Smarndache semigroup, soft set, soft semigroup, smarandache soft semigroup.

1 Introduction

Raul [26] introduced in 1998, the notions of Smarandache semigroup in the article "Smaradache Algebraic Structures". Smarandache semigroup is analogous to the smarandache group. F. Smarandache in [30] first introduced the theory of Smarandach algebraic structures in a paper "Special Algebraic Structures". The Smarandache semigroups exhibit characteristics and features of both groups and semigroups simultaneously. The Smarandache semigroups are a class of innovative and conceptually a new structure in nature. The concept of Smarandache algebraic structures almost exist in every algebraic structure such as Smarandache groupoid which are discussed in [17], Smarandache rings [20], Smarandache semirings, semifields, semivector spaces [18], Smarandache loops [19] etc. Kandassamy have written several books on Smarandache algebraic structures and their related theory.

Molodtsov [25] initiated the theory of soft sets in 1995. Soft set theory is a mathematical tool which is free from parameterization inadequacy, syndrome of fuzzy set theory, rough set theory, probability theory and so on. This theory has been applications in many fields

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such as smoothness of functions, game theory, operation research, Riemann integration, Perron integration, and probability etc. Recently soft set theory gain much attention of the researchers since its introduction. There are a lot of soft algebraic structures introduced in soft set theory successfully. H. Aktas and N. Cagmann [1] introduced soft groups, soft semigroups [15]. The work which is based on several operations of soft sets discussed in [12,13]. Some properties and related algebra may be found in [14]. Some other concepts and notions together with fuzzy set and rough set were studied in [23,24]. Some useful study about soft neutrosophic algebraic structures have been discussed in [3,4,5,6,7,8,9,10,11,28,29,31].

The organization of this paper is below. In first section, some basic concepts and notions about smarandache semigroups, soft sets, and soft semigroups are presented. In the next section Smarandache soft semigroup shortly SS-semigroups is introduced. In this section some related theory and characterization is also presented with illustrative examples. In the further section, Smarandache hyper soft semigroup is studied with some of their core properties.

2 Basic Concepts

In this section, fundamental concepts about Smarandache semigroups, soft sets, and soft semigroups is presented with some of their basic properties.

2.1 Smarandache Semigroups

Definition 2.1.1: A smarandache semigroup is define to be a semigroup S such that a proper subset of S is a group with respect to the same induced operation. A smarandache semigroup S is denoted by S-semigroup.

Definition 2.1.2: Let S be a smarandache semigroup. If every proper subset A in S which is a group is commutative, then S is said to be a smarandache commutative semigroup.
If S has atleast one proper subgroup, then S is called a weak smarandache commutative semigroup.

Definition 2.1.3: Let S be a smarandache semigroup. If every proper subset A of S is a cyclic group, then S is said to be a smarandache cyclic semigroup.
If S has atleast one proper cyclic subgroup, then S is called a weak smarandache commutative semigroup.

Definition 2.1.4: Let S be a smarandache semigroup. A proper subset A of S is called a samarandache subsemigroup if A itself is a smarandache semigroup under the operation of S.

Definition 2.1.5: Let S be a smarandache semigroup. If A be a proper subset of S which subsemigroup of S and A contains the largest group of S.Then A is called a smarandache hyper subsemigroup.

2.2 Soft Sets

Throughout this subsection U refers to an initial universe, E is a set of parameters, $P(U)$ is the power set of U, and $A, B \subset E$. Molodtsov defined the soft set in the following manner:

Definition 2.2.1: A pair (F, A) is called a soft set over U where F is a mapping given by $F : A \to P(U)$. In other words, a soft set over U is a parameterized family of subsets of the universe U. For $a \in A$, $F(\mathrm{a})$ may be considered as the set of a-elements of the soft set (F, A), or as the set of a-approximate elements of the soft set.

Definition 2.2.2: For two soft sets (F, A) and (H, B) over U, (F, A) is called a soft subset of (H, B) if

1. $A \subseteq B$ and
2. $F(a) \subseteq H(a)$, for all $x \in A$

This relationship is denoted by $(F, A) \subset (H, B)$. Similarly (F, A) is called a soft superset of (H, B) if (H, B) is a soft subset of (F, A) which is denoted by $(F, A) \supset (H, B)$.

Definition 2.2.3: Two soft sets (F, A) and (H, B) over U are called soft equal if (F, A) is a soft subset of (H, B) and (H, B) is a soft subset of (F, A).

Definition 2.2.4: Let (F, A) and (K, B) be two soft sets over a common universe U such that $A \cap B \neq \phi$. Then their restricted intersection is denoted by $(F, A) \cap_R (K, B) = (H, C)$ where (H, C) is defined as $H(c) = F(c) \cap \mathrm{K}(c)$ for all $c \in C = A \cap B$.

Definition 2.2.5: The extended intersection of two soft sets (F, A) and (K, B) over a common universe U is the soft set (H, C), where $C = A \cup B$, and for all $c \in C$, $H(c)$ is defined as

$$H(c) = \begin{cases} F(c) & \text{if c} \in A - B, \\ G(c) & \text{if c} \in B - A, \\ F(c) \cap G(c) & \text{if c} \in A \cap B. \end{cases}$$

We write $(F, A) \cap_\varepsilon (K, B) = (H, C)$.

Definition 2.2.6: The restricted union of two soft sets (F, A) and (K, B) over a common universe U is the soft set (H, C), where $C = A \cup B$, and for all $c \in C$, $H(c)$ is defined as $H(c) = F(c) \cup G(c)$ for all $c \in C$. We write it as
$(F, A) \cup_R (K, B) = (H, C)$.

Definition 2.2.7: The extended union of two soft sets (F, A) and (K, B) over a common universe U is the soft set (H, C), where $C = A \cup B$, and for all $c \in C$, $H(c)$ is defined as

$$H(c) = \begin{cases} F(c) & \text{if c} \in A - B, \\ G(c) & \text{if c} \in B - A, \\ F(c) \cup G(c) & \text{if c} \in A \cap B. \end{cases}$$

We write $(F,A) \cup_{\varepsilon} (K,B) = (H,C)$.

Definition 2.2.8: A soft set (F,A) over S is called a soft semigroup over S if $(F,A)\hat{\circ}(F,A) \subseteq (F,A)$.

It is easy to see that a soft set (F,A) over S is a soft semigroup if and only if $\phi \neq F(a)$ is a subsemigroup of S for all $a \in A$.

3 Smarandache Soft Semigroups

In this section we define smarandache soft semigroups and give some of their properties with sufficient amount of examples.

Definition 3.1: Let S be a semigroups and (F,A) be a soft semigroup over S. Then (F,A) is called a smarandache soft semigroup over U if a proper soft subset (G,B) of (F,A) is a soft group under the operation of S. We denote a smarandache soft semigroup by SS -semigroup.

A smarandache soft semigroup is a parameterized collection of smarandache subsemigroups of S.

Example 3.2: Let $\mathbb{Z}_{12} = \{0,1,2,3,...,11\}$ be the semigroup under multiplication modulo 12. Let $A = \{a_1, a_2, a_3, a_4, a_5, a_6\}$ be a set of parameters. Let (F,A) be a soft semigroup over $\mathbb{Z}_{12}$, where

$F(a_1) = \{1,3,5,9\}, \mathrm{F}(a_2) = \{1,4,7,8\}$,
$F(a_3) = \{1,5,7,11\}$, $F(a_4) = \{3,4,8,9\}$,
$F(a_5) = \{1,3,9,11\}$, $F(a_6) = \{1,4,5,8\}$.

Let $B = \{a_1, a_2, a_4, a_5\} \subset A$. Then (G,B) is a soft subgroup of (F,A) over U, where

$G(a_1) = \{1,5\}, G(a_2) = \{4,8\}$,
$G(a_4) = \{3,9\}, G(a_5) = \{1,11\}$.

Thus clearly (F,A) is a smarandache semigroup over $\mathbb{Z}_{12}$.

Proposition 3.3: If S is a smarandache semigroup, then (F,A) is also a smarandache soft semigroup over S.

Proposition 3.4: The extended union of two SS -semigroups (F, A) and (G, B) over S is a SS -semigroup over S .

Proposition 3.5: The extended intersection of two SS -semigroups (F, A) and (G, B) over S is a SS -semigroup over S .

Proposition 3.6: The restricted union of two SS -semigroups (F, A) and (G, B) over S is a SS -semigroup over S .

Proposition 3.7: The restricted intersection of two SS -semigroups (F, A) and (G, B) over S is a SS -semigroup over S .

Proposition 3.8: The AND operation of two SS -semigroups (F, A) and (G, B) over S is a SS -semigroup over S .

Proposition 3.9: The OR operation of two SS -semigroups (F, A) and (G, B) over S is a SS -semigroup over S .

Definition 3.10: Let (F, A) be a SS -semigroup over a semigroup S . Then (F, A) is called a commutative SS -semigroup if each proper soft subset (G, B) of (F, A) is a commutative group.

Definition 3.11: Let (F, A) be a SS -semigroup over a semigroup S . Then (F, A) is called a weakly commutative SS -semigroup if atleast one proper soft subset (G, B) in (F, A) is a commutative group.

Proposition 3.12: If S is a commutative S -semigroup, then (F, A) over S is also a commutative SS -semigroup.

Definition 3.13: Let (F, A) be a SS -semigroup over a semigroup S . Then (F, A) is called a cyclic SS -semigroup if each proper soft subset (G, B) of (F, A) is a cyclic subgroup.

Proposition 3.14: Let (F, A) and (H, B) be two strong soft groups over a semigroup S . Then

1. $(F, A) \cap_R (H, B)$ is a strong soft group over S .
2. $(F, A) \cap_E (H, B)$ is a strong soft group over S .

Definition 3.15: Let (F, A) be a SS -semigroup over a semigroup S . If there exist atleast one proper soft subset (G, B) of (F, A) which is a cyclic subgroup. Then (F, A) is termed as weakly cyclic SS -semigroup.

Proposition 3.16: If S is a cyclic S -semigroup, then (F, A) over S is also a cyclic SS -semigroup.

Proposition 3.17: If S is a cyclic S-semigroup, then (F,A) over S is a commutative SS-semigroup.

Definition 3.18: Let S be a semigroup and (F,A) be a SS-semigroup. A proper soft subset (G,B) of (F,A) is said to be a smarandache soft subsemigroup if (G,B) is itself a smarandache soft semigroup over S.

Definition 3.19: Let S be a semigroup and (F,A) be a soft set over S. Then S is called a parameterized smarandache semigroup if $F(a) \subset \mathrm{S}$ such that $F(a)$ is a group under the operation of S for all $a \in A$. In this case (F,A) is called a strong soft group.

A strong soft group is a parameterized collection of the subgroups of the semigroup S.

Proposition 3.20: Let S be a semigroup and (F,A) be a soft set over S. Then S is a parameterized smarandache semigroup if (F,A) is a soft group over S.

Proof: Suppose that (F,A) is a soft group over S. This implies that each $F(a)$ is a subgroup of the semigroup S for al $a \in A$ and thus S is a parameterized smarandache semigroup.

Example 3.21: Let $\mathbb{Z}_{12} = \{0,1,2,3,...,11\}$ be the semigroup under multiplication modulo 12. Let $A = \{a_1, a_2, a_3\}$ be a set of parameters. Let (F,A) be a soft semigroup over $\mathbb{Z}_{12}$, where

$F(a_1) = \{3,9\}, \mathrm{F}(a_2) = \{1,7\}$,
$F(a_3) = \{1,5\}$.

Then $\mathbb{Z}_{12}$ is a parameterized smarandache semigroup.

Proposition 3.22: Let (F,A) and (H,B) be two strong soft groups over a semigroup S. Then

1. $(F,A) \cap_R (H,B)$ is a strong soft group over S.
2. $(F,A) \cap_E (H,B)$ is a strong soft group over S.

Remark 3.23: Let (F,A) and (H,B) be two strong soft groups over a semigroup S. Then

1. $(F,A) \cup_R (H,B)$ need not be strong soft group over S.
2. $(F,A) \cup_E (H,B)$ need not be strong soft group over S.

For this, we take the following example.

Example 3.24: Let $\mathbb{Z}_{12} = \{0,1,2,3,...,11\}$ be the semigroup under multiplication modulo 12. Let $A = \{a_1, a_2, a_3, a_4, a_5\}$ be a set of parameters. Let (F, A) be a strong soft group over $\mathbb{Z}_{12}$, where

$F(a_1) = \{1,5\}, \mathrm{F}(a_2) = \{4,8\}$,
$F(a_3) = \{7,11\}$, $F(a_4) = \{3,9\}$,
$F(a_5) = \{1,11\}$.

Let $B = \{a_1, a_2, a_7\}$. Then (H, B) is another strong soft group over $\mathbb{Z}_{12}$, where

$\mathrm{H}(a_1) = \{3,9\}, \mathrm{H}(a_2) = \{1,5\}$,
$\mathrm{H}(a_7) = \{3,9\}$.

Then clearly $C = A \cap B = \{a_1, a_2\}$. Now $F(a_1) \cap H(a_1) = \{1,3,5,9\}$ and $F(a_2) \cap H(a_2) = \{1,4,5,8\}$ are not subgroups of $\mathbb{Z}_{12}$. Thus $(F, A) \cup_R (H, B)$ is not a strong soft group over $S = \mathbb{Z}_{12}$.

One can easily show 2 with the help of examples.

4 Smarandache Hyper Soft Subsemigroups

Definition 4.1: Let (F, A) be a SS-semigroup over S and (H, B) be a SS-subsemigroup of (F, A). Then (H, B) is called a smarandache hyper soft subsemigroup if (H, B) contains a proper soft subset (K, C) such that $K(c)$ is a smarandache hyper subsemigroup of S for all $c \in B$.

Theorem 4.2: Every smarandache hyper subsemigroup is a smarandache subsemigroup.
Proof: Its obvious.

Definition 4.3: Let (F, A) be a SS-semigroup. Then (F, A) is called simple SS-semigroup if (F, A) has no smarandach hyper subsemigroup.

Theorem 4.4: If S is a simple smarandache semigroup. Then (F, A) over S is also a simple SS-semigroup.

Proof. The proof is simple.

Theorem 4.5: If S is a smarandache semigroup of prime order p. Then (F, A) is a simple SS-semigroup over S.

Conclusion

In this paper Smaradache soft semigroups are introduced. Their related properties and results are explained with many illustrative examples. This theory opens a new way for researchers to define these type of soft algebraic strucutres in almost all areas of algebra in the future.

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Intuitionistic Fuzzy Soft Expert Sets and its Application in Decision Making

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Abstract - In this paper, we first introduced the concept of intuitionistic fuzzy soft expert sets (IFSESs for short) which combines intuitionistic fuzzy sets and soft expert sets. We also define its basic operations, namely complement, union, intersection, AND and OR, and study some of their properties. This concept is a generalization of fuzzy soft expert sets (FSESs). Finally, an approach for solving MCDM problems is explored by applying intuitionistic fuzzy soft expert sets, and an example is provided to illustrate the application of the proposed method.

Keywords - Intuitionistic fuzzy sets, soft expert sets, intuitionistic fuzzy soft expert sets, decision making.

1. Introduction

Intuitionistic fuzzy set (IFS in short) on a universe was introduced by Atanassov [7] in 1983, as a generalization of fuzzy set [13]. The conception of IFS can be viewed as an appropriate /alternative approach in case where available information is not sufficient to define the impreciseness by the conventional fuzzy set. In fuzzy sets the degree of acceptance is considered only but IFS is characterized by a membership function and a non-membership function so that the sum of both values is less than one. A detailed theoretical study may be found in [7].

Soft set theory was originally introduced by Molodtsov [3] as a general mathematical tool for dealing with uncertainties which traditional mathematical tools cannot handle and how soft set theory is free from the parameterization inadequacy syndrome of fuzzy set theory,

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rough set theory, and probability theory. A soft set is in fact a set-valued map which gives an approximation description of objects under consideration based on some parameters. After Molodtsov's work, Maji et al. [26] introduced the concept of fuzzy soft set, a more generalized concept, which is a combination of fuzzy set and soft set and studied its properties and also discussed their properties. Also, Maji et al. [27] devoted the concept of intuitionistic fuzzy soft sets by combining intuitionistic fuzzy sets with soft sets. Then, many interesting results of soft set theory have been studied on fuzzy soft sets [19, 20, 24, 25], on intuitionistic fuzzy soft set theory [21, 22, 23, 27], on possibility fuzzy soft set [31], on generalized fuzzy soft sets [5,29], on generalized intuitionistic fuzzy soft [12, 28], on possibility intuitionistic fuzzy soft set [14], on possibility vague soft set [8] and so on. All these research aim to solve most of our real life problems in medical sciences, engineering, management, environment and social science which involve data that are not crisp and precise. Moreover all the models created will deal only with one expert. To redefine this one expert opinion, Alkhazaleh and Salleh in 2011 [29] defined the concept of soft expert set in which the user can know the opinion of all the experts in one model and give an application of this concept in decision making problem. Also, they introduced the concept of the fuzzy soft expert set [30] as a combination between the soft experts set and the fuzzy set. After Alkhazaleh's work, many researchers have worked with the concept of soft expert sets [1, 2, 4, 6, 9, 10, 11, 15, 16, 18, 33].

Until now, there is no study on soft experts in intuitionistic fuzzy environment, so there is a need to develop a new mathematical tool called "intuitionistic fuzzy soft expert sets.

The paper is organized as follows. In Section 2, we first recall the necessary background on intuitionistic fuzzy sets, soft set, intuitionistic fuzzy soft sets, soft expert sets, fuzzy soft expert sets. Section 3 reviews various proposals for the definition of intuitionistic fuzzy soft expert sets and derive their respective properties. Section 4 presents basic operations on intuitionistic fuzzy soft expert sets. Section 5 presents an application of this concept in solving a decision making problem. Finally, we conclude the paper.

2. Preliminaries

In this section, we will briefly recall the basic concepts of intuitionistic fuzzy sets, soft set, soft expert sets and fuzzy soft expert sets.

Let U be an initial universe set of objects and E the set of parameters in relation to objects in U. Parameters are often attributes, characteristics or properties of objects. Let P (U) denote the power set of U and $A \subseteq E$.

2.1. Intuitionistic Fuzzy Set

Definition 2.1 [7]: Let U be an universe of discourse then the intuitionistic fuzzy set A is an object having the form $A = \{< x, \mu_A(x), \omega_A(x)>, x \in U\}$,where the functions $\mu_A(x)$, $\omega_A(x) : U \to [0,1]$ define respectively the degree of membership, and the degree of non-membership of the element $x \in X$ to the set A with the condition.

$$0 \leq \mu_A(x) + \omega_A(x) \leq 1.$$

For two IFS,

A_{IFS}= {<x, μ_A (x), ω_A(x)> | x ∈ X }

and

B_{IFS}= {<x, μ_B (x), ω_B (x)> | x ∈ X }

Then,

1. $A_{\mathrm{IFS}} \subseteq B_{\mathrm{IFS}}$ if and only if

$$\mu_A(x) \leq \mu_B(x),\ \omega_A(x) \geq \omega_B(x)$$

2. $A_{\mathrm{IFS}} = B_{\mathrm{IFS}}$ if and only if ,

$\mu_A(x) = \mu_B(x)$, $\omega_A(x) = \omega_B(x)$ for anyx ∈ X.

3. The complement of A_{IFS} is denoted by A^o_{IFS} and is defined by

A^o_{IFS}= {<x, ω_A (x), μ_A (x) > | x ∈ X }

4. A∩B = {<x, $\min\{\mu_A(x), \mu_B(x)\}$, $\max\{\omega_A(x), \omega_B(x)\}$> | x ∈ X }

5. A∪B = {<x, $\max\{\mu_A(x), \mu_B(x)\}$, $\min\{\omega_A(x), \omega_B(x)\}$> | x ∈ X }

As an illustration, let us consider the following example.

Example 2.2. Assume that the universe of discourse U={x_1,x_2,x_3,x_4}. It may be further assumed that the values of x_1, x_2, $x_3$and $x_4$are in [0, 1] Then, A is an intuitionistic fuzzy set (IFS) of U, such that,

A= {< x_1, 0.4, 0.6>, < x_2, 0.3, 0.7>, < x_3, 0.2,0.8>,<x_4, 0.2,0.8>}

2.2. Soft Set

Definition 2.3. [3] Let U be an initial universe set and E be a set of parameters. Let P(U) denote the power set of U. Consider a nonempty set A, A ⊂ E. A pair (K, A) is called a soft set over U, where K is a mapping given by K : A → P(U).

As an illustration, let us consider the following example.

Example 2.4 . Suppose that U is the set of houses under consideration, say U = {h_1, h_2, . . ., h_5}. Let E be the set of some attributes of such houses, say E = {e_1, e_2, . . ., e_8}, where e_1, e_2, . . ., e_8 stand for the attributes “beautiful”, “costly”, “in the green surroundings”, “moderate”, respectively.

In this case, to define a soft set means to point out expensive houses, beautiful houses, and so on. For example, the soft set (K, A) that describes the "attractiveness of the houses" in the opinion of a buyer, says Thomas, and may be defined like this:

A={e_1,e_2,e_3,e_4,e_5};

K(e_1) = {h_2, h_3, h_5}, K(e_2) = {h_2, h_4}, K(e_3) = {h_1}, K(e_4) = U, K(e_5) = {h_3, h_5}.

2.3. Intuitionistic Fuzzy Soft Sets

Definition 2.5 [27] Let U be an initial universe set and $A \subset E$ be a set of parameters. Let IFS(U) denotes the set of all intuitionistic fuzzy subsets of U. The collection (F, A) is termed to be the intuitionistic fuzzy soft set over U, where F is a mapping given by $F: A \to IFS(U)$.

Example 2.6 Let U be the set of houses under consideration and E is the set of parameters. Each parameter is a word or sentence involving intuitionistic fuzzy words. Consider E ={beautiful, wooden, costly, very costly, moderate, green surroundings, in good repair, in bad repair, cheap, expensive}. In this case, to define a intuitionistic fuzzy soft set means to point out beautiful houses, wooden houses, houses in the green surroundings and so on. Suppose that, there are five houses in the universe U given by $U = \{h_1, h_2, \ldots, h_5\}$ and the set of parameters

$A = \{e_1, e_2, e_3, e_4\}$,where e_1 stands for the parameter `beautiful', $e_2$ stands for the parameter `wooden', e_3 stands for the parameter `costly' and the parameter $e_4$ stands for `moderate'. Then the intuitionistic fuzzy set (F, A) is defined as follows:

$$(F, A) = \left\{ \begin{array}{l} \left(e_1 \left\{\frac{h_1}{(0.1,0.6)}, \frac{h_2}{(0.2,0.7)}, \frac{h_3}{(0.6,0.2)}, \frac{h_4}{(0.7,0.3)}, \frac{h_5}{(0.2,0.3)}\right\}\right) \\ \left(e_2 \left\{\frac{h_1}{(0.3,0.5)}, \frac{h_2}{(0.2,0.4)}, \frac{h_3}{(0.1,0.2)}, \frac{h_4}{(0.1,0.3)}, \frac{h_5}{(0.3,0.6)}\right\}\right) \\ \left(e_3 \left\{\frac{h_1}{(0.4,0.3)}, \frac{h_2}{(0.6,0.3)}, \frac{h_3}{(0.2,0.5)}, \frac{h_4}{(0.2,0.6)}, \frac{h_5}{(0.7,0.3)}\right\}\right) \\ \left(e_4 \left\{\frac{h_1}{(0.1,0.6)}, \frac{h_2}{(0.3, 0.6)}, \frac{h_3}{(0.6,0.4)}, \frac{h_4}{(0.4,0.2)}, \frac{h_5}{(0.5,0.3)}\right\}\right) \end{array} \right\}$$

2.5. Soft Expert Sets

Definition 2.7 [29] Let U be a universe set, E be a set of parameters and X be a set of experts (agents). Let O= {1=agree, 0=disagree} be a set of opinions. Let Z= E × X × O and A ⊆ Z

A pair (F, E) is called a soft expert set over U, where F is a mapping given by F : A → P(U) and P(U) denote the power set of U.

Definition 2.8 [29] An agree- soft expert set $(F, A)_1$ over U, is a soft expert subset of (F,A) defined as :

$$(F, A)_1 = \{F(\alpha) \mid \alpha \in E \times X \times \{1\}\}.$$

Definition 2.9[29] A disagree- soft expert set $(F, A)_0$ over U, is a soft expert subset of (F,A) defined as :

$$(F, A)_0 = \{F(\alpha) \mid \alpha \in E \times X \times \{0\}\}.$$

2.6. Fuzzy Soft Expert Sets

Definition 2.10 [30] A pair (F, A) is called a fuzzy soft expert set over U, where F is a mapping given by $F : A \rightarrow I^U$,and I^U denote the set of all fuzzy subsets of U.

3. Intuitionistic Fuzzy Soft Expert Sets

In this section, we generalize the fuzzy soft expert sets as introduced by Alkhazaleh and Salleh [30] to intuitionistic fuzzy soft expert sets and give the basic properties of this concept.

Let U be universal set of elements, E be a set of parameters, X be a set of experts (agents), O= {1=agree, 0=disagree} be a set of opinions. Let Z= E $\times$ X $\times$O and

Definition 3.1 Let U= $\{ u_1, u_2, u_3, \ldots, u_n \}$ be a universal set of elements, E=$\{ e_1, e_2, e_3, \ldots, e_m \}$ be a universal set of parameters, X=$\{ x_1, x_2, x_3, \ldots, x_i \}$ be a set of experts (agents) and O= {1=agree, 0=disagree} be a set of opinions. Let Z= { E $\times$ X $\times$ Q } and A $\subseteq$ Z. Then the pair (U, Z) is called a soft universe. Let F: Z $\rightarrow$ IF^U where IF^U denotes the collection of all intuitionistic fuzzy subsets of U. Suppose F: Z $\rightarrow$ IF^U be a function defined as:

$F(z) = F(z)(u_i)$, for all $u_i \in U$.

Then $F(z)$ is called an intuitionistic fuzzy soft expert set (IFSES in short) over the soft universe (U, Z).

For each $z_i \in Z$. $F(z) = F(z_i)(u_i)$ where $F(z_i)$ represents the degree of belongingness and non-belongingness of the elements of U in $F(z_i)$. Hence $F(z_i)$ can be written as:

$$F(z_i) = \{(\frac{u_i}{F(z_i)(u_i)}), \ldots, (\frac{u_i}{F(z_i)(u_i)})\}, \text{ for } i=1,2,3,\ldots,n$$

where $F(z_i)(u_i) = < \mu_{F(z_i)}(u_i), \omega_{F(z_i)}(u_i)>$ with $\mu_{F(z_i)}(u_i)$ and $\omega_{F(z_i)}(u_i)$ representing the membership function and non-membership function of each of the elements $u_i \in U$ respectively.

Sometimes we write F as (F, Z) . If $A \subseteq Z$. we can also have IFSES (F, A).

Example 3.2 Let U=$\{u_1,u_2, u_3\}$ be a set of elements, E=$\{e_1,e_2\}$ be a set of decision parameters, where e_i(i= 1, 2,3} denotes the parameters E =$\{e_1$= beautiful, e_2= cheap} and X= $\{x_1,x_2\}$ be a set of experts. Suppose that F :Z→ IF^U is function defined as follows:

$F(e_1,x_1, 1) = \{ (\frac{u_1}{<0.1,0.8>}), (\frac{u_2}{<0.1,0.6,>}), (\frac{u_3}{<0.4,0.5>}) \}$,
$F(e_2, x_1,1) = \{ (\frac{u_1}{<0.5,0.25>}), (\frac{u_2}{<0.25,0.6>}), (\frac{u_3}{<0.4,0.4>}) \}$,
$F(e_1, x_2,1) = \{ (\frac{u_1}{<0.2,0.7>}), (\frac{u_2}{<0.4,0.3>}), (\frac{u_3}{<0.6,0.2>}) \}$,
$F(e_2, x_2,1) = \{ (\frac{u_1}{<0.2,0.6>}), (\frac{u_2}{<0.3,0.2>}), (\frac{u_3}{<0.3,0.5>}) \}$,
$F(e_1, x_1,0) = \{ (\frac{u_1}{<0.2,0.4>}), (\frac{u_2}{<0.1,0.9>}), (\frac{u_3}{<0.2,0.5>}) \}$,
$F(e_2, x_1,0) = \{ (\frac{u_1}{<0.3,0.4>}), (\frac{u_2}{<0.2,0.7>}), (\frac{u_3}{<0.5,0.2>}) \}$,
$F(e_1, x_2,0) = \{ (\frac{u_1}{<0.3,0.4>}), (\frac{u_2}{<0.1,0.6>}), (\frac{u_3}{<0.6,0.3>}) \}$
$F(e_2, x_2,0) = \{ (\frac{u_1}{<0.4,0.4>}), (\frac{u_2}{<0.8,0.2>}), (\frac{u_3}{<0.2,0.4>})$

Then we can view the intuitionistic fuzzy soft expert set (F, Z) as consisting of the following collection of approximations:

$(F, Z) =\{ (e_1,x_1, 1) = \{ (\frac{u_1}{<0.1,0.8>}), (\frac{u_2}{<0.1,0.6,>}), (\frac{u_3}{<0.4,0.5>}) \}\}$,
$\{(e_2, x_1,1) = \{ (\frac{u_1}{<0.5,0.25>}), (\frac{u_2}{<0.25,0.6>}), (\frac{u_3}{<0.4,0.4>}) \}\}$,
$\{(e_1, x_2,1) = \{ (\frac{u_1}{<0.2,0.7>}), (\frac{u_2}{<0.4,0.3>}), (\frac{u_3}{<0.6,0.2>}) \}\}$,
$\{(e_2, x_2,1) = \{ (\frac{u_1}{<0.2,0.6>}), (\frac{u_2}{<0.3,0.2>}), (\frac{u_3}{<0.3,0.5>}) \}\}$,
$\{(e_1, x_1,0) = \{ (\frac{u_1}{<0.2,0.4>}), (\frac{u_2}{<0.1,0.9>}), (\frac{u_3}{<0.2,0.5>}) \}\}$,
$\{(e_2, x_1,0) = \{ (\frac{u_1}{<0.3,0.4>}), (\frac{u_2}{<0.2,0.7>}), (\frac{u_3}{<0.5,0.2>}) \}\}$,
$\{(e_1, x_2,0) = \{ (\frac{u_1}{<0.3,0.4>}), (\frac{u_2}{<0.1,0.6>}), (\frac{u_3}{<0.6,0.3>}) \}\}$,
$\{(e_2, x_2,0) = \{ (\frac{u_1}{<0.4,0.4>}), (\frac{u_2}{<0.8,0.2>}), (\frac{u_3}{<0.2,0.4>}) \}\}$.

Then (F, Z) is an intuitionistic fuzzy soft expert set over the soft universe (U, Z).

Definition 3.3. For two intuitionistic fuzzy soft expert sets (F,A) and (G,B) over a soft universe (U, Z). Then (F, A) is said to be an intuitionistic fuzzy soft expert subset of (G,B) if

i. $B \subseteq A$
ii. $F(\varepsilon)$ is an intuitionistic fuzzy subset of $G(\varepsilon)$, for all $\varepsilon \in A$

This relationship is denoted as $(F, A) \tilde{\subseteq} (G, B)$. In this case, (G, B) is called an intuitionistic fuzzy soft expert superset (IFSES superset) of (F, A) .

Definition 3.4. Two intuitionistic fuzzy soft expert sets (F, A) and (G, B) over soft universe (U, Z) are said to be equal if (F, A) is a intuitionistic fuzzy soft expert subset of (G, B) and (G, B) is an intuitionistic fuzzy soft expert subset of (F, A).

Definition 3.5. An IFSES (F, A) is said to be a null intuitionistic fuzzy soft expert sets denoted $(\tilde{\emptyset}, A)$ and defined as:

$$(\tilde{\emptyset}, A) = F(\alpha) \text{ where } \alpha \in Z.$$

Where F(α)= < 0, 1>, that is $\mu_{F(\alpha)}$=0 and $\omega_{F(\alpha)}$= 1 for all $\alpha \in Z$.

Definition 3.6. An IFSES (F, A) is said to be an absolute intuitionistic fuzzy soft expert sets denoted $(F, A)_{\text{abs}}$ and defined as:

$$(F, A)_{\text{abs}} = F(\alpha), \text{ where } \alpha \in Z.$$

Where F(α)= <1, 0>, that is $\mu_{F(\alpha)}$= 1 and $\omega_{F(\alpha)}$= 0 , for all $\alpha \in Z$.

Definition 3.7. Let (F, A) be an IFSES over a soft universe (U,Z). An agree- intuitionistic fuzzy soft expert set (agree- IFSES) over U, denoted as $(F, A)_1$ is an intuitionistic fuzzy soft expert subset of (F, A) which is defined as :

$$(F, A)_1 = \{F(\alpha) \mid \alpha \in E \times X \times \{1\}\}.$$

Definition 3.8. Let (F, A) be a IFSES over a soft universe (U, Z). A disagree- intuitionistic fuzzy soft expert set (disagree- IFSES) over U, denoted as $(F, A)_0$ is a intuitionistic fuzzy soft expert subset of (F, A) which is defined as :

$$(F, A)_0 = \{F(\alpha) \mid \alpha \in E \times X \times \{0\}\}.$$

Example 3.9 consider Example 3.2 .Then the agree- intuitionistic fuzzy soft soft expert set

$$\begin{aligned}(F, A)_1 = \{&((e_1, x_1, 1), \{(\tfrac{u_1}{<0.1,0.8>}), (\tfrac{u_2}{<0.1,0.6>}), (\tfrac{u_3}{<0.4,0.5>})\}),\\
&((e_2, x_1, 1), \{(\tfrac{u_1}{<0.5,0.25>}), (\tfrac{u_2}{<0.25,0.6>}), (\tfrac{u_3}{<0.4,0.4>})\}),\\
&((e_1, x_2, 1), \{(\tfrac{u_1}{<0.2,0.7>}), (\tfrac{u_2}{<0.4,0.3>}), (\tfrac{u_3}{<0.6,0.2>})\}),\\
&((e_2, x_2, 1), \{(\tfrac{u_1}{<0.2,0.6>}), (\tfrac{u_2}{<0.3,0.2>}), (\tfrac{u_3}{<0.3,0.5>})\})\}\end{aligned}$$

And the disagree-intuitionistic fuzzy soft expert set over U

$$(F, A)_0 = \{((e_1, x_1, 0), \{(\tfrac{u_1}{<0.2,0.4>}), (\tfrac{u_2}{<0.1,0.9>}), (\tfrac{u_3}{<0.2,0.5>})\}),$$

$$((e_2, x_1, 0), \{(\tfrac{u_1}{<0.3,0.4,0.6>}), (\tfrac{u_2}{<0.2,0.7>}), (\tfrac{u_3}{<0.5,0.2>})\}),$$

$((e_1, x_2, 0), \{ (\frac{u_1}{<0.3,0.4>}), (\frac{u_2}{<0.1,0.6>}), (\frac{u_3}{<0.6,0.3>}) \})$

$((e_2, x_2, 0), \{ (\frac{u_1}{<0.4,0.4>}), (\frac{u_2}{<0.8,0.2>}), (\frac{u_3}{<0.2,0.4>}) \})\}$

4. Basic Operations on Intuitionistic Fuzzy Soft Expert Sets

In this section, we introduce some basic operations on IFSES, namely the complement, AND, OR, union and intersection of IFSES, derive their properties, and give some examples.

Definition 4.1 Let (F, A) be an IFSES over a soft universe (U, Z). Then the complement of (F, A) denoted by $(F, A)^c$ is defined as:

$(F, A)^c = \tilde{c}$ (F(α)) for all $\alpha \in$ U.

where $\tilde{c}$ is an intuitionistic fuzzy complement .

Example 4.2 Consider the IFSES (F, Z) over a soft universe (U, Z) as given in Example 3.2. By using the intuitionistic fuzzy complement for F(α), we obtain $(F, Z)^c$ which is defined as:

$(F, Z)^c = \{ (e_1, x_1, 1) = \{ (\frac{u_1}{<0.8,0.1>}), (\frac{u_2}{<0.6,0.1,>}), (\frac{u_3}{<0.5,0.4>}) \}\},$

$\{(e_2, x_1, 1) = \{ (\frac{u_1}{<0.25,0.5>}), (\frac{u_2}{<0.6,0.25>}), (\frac{u_3}{<0.4,0.4>}) \}\},$

$\{(e_1, x_2, 1) = \{ (\frac{u_1}{<0.7,0.2>}), (\frac{u_2}{<0.3,0.4>}), (\frac{u_3}{<0.2,0.6>}) \}\},$

$\{(e_2, x_2, 1) = \{ (\frac{u_1}{<0.6,0.2>}), (\frac{u_2}{<0.2,0.3>}), (\frac{u_3}{<0.5,0.3>}) \}\},$

$\{(e_1, x_1, 0) = \{ (\frac{u_1}{<0.4,0.2>}), (\frac{u_2}{<0.9,0.1>}), (\frac{u_3}{<0.5,0.2>}) \}\},$

$\{(e_2, x_1, 0) = \{ (\frac{u_1}{<0.4,0.3>}), (\frac{u_2}{<0.7,0.2>}), (\frac{u_3}{<0.2,0.5>}) \}\},$

$\{(e_1, x_2, 0) = \{ (\frac{u_1}{<0.4,0.3>}), (\frac{u_2}{<0.6,0.1>}), (\frac{u_3}{<0.3,0.6>}) \}\},$

$\{(e_2, x_2, 0) = \{ (\frac{u_1}{<0.4,0.4>}), (\frac{u_2}{<0.2,0.8>}), (\frac{u_3}{<0.4,0.2>}) \}\}.$

Proposition 4.3 If (F, A) is an IFSES over a soft universe (U, Z), then,

$$((F, A)^c)^c = (F, A).$$

Proof. Suppose that is (F, A) an IFSES over a soft universe (U, Z) defined as (F, A) = F(e). Now let IFSES$(F, A)^c = (G, B)$. Then by Definition 4.1, (G, B) = G(e) such that G(e) $= \tilde{c}$ (F(e)), Thus it follows that:

$(G, B)^c = \tilde{c}$ (G(e)) =($\tilde{c}$ ($\tilde{c}$ (F(e))) = F(e)=(F, A).

Therefore

$((F,\mathrm{A})^c)^c = (G,\mathrm{B})^c = (\mathrm{F},\mathrm{A})$.Hence it is proven that $((F,\mathrm{A})^c)^c = (\mathrm{F},\mathrm{A})$.

Definition 4.4 Let (F,A) and (G,B) be any two IFSES s over a soft universe (U, Z). Then the union of (F,A) and (G,B), denoted by $(F,\mathrm{A}) \tilde{\cup} (G,\mathrm{B})$ is an IFSES defined as $(F,\mathrm{A}) \tilde{\cup} (G,\mathrm{B}) = (H,\mathrm{C})$, where C= A ∪ B and

$$\mathrm{H}(\alpha) = \mathrm{F}(\alpha) \tilde{\cup} \mathrm{G}(\alpha), \text{ for all } \alpha \in \mathrm{C}$$

where

$$\mathrm{H}(\alpha) = \begin{cases} \mathrm{F}(\alpha) & \alpha \in A - B \\ \mathrm{G}(\alpha) & \alpha \in A - B \\ s(F(\alpha), G(\alpha)) & \alpha \in A \cap B \end{cases}$$

Where s is a s- norm.

Proposition 4.5 Let (F,A), (G,B) and (H,C) be any three IFSES over a soft universe (U, Z).Then the following properties hold true.

(i) $(F,\mathrm{A}) \tilde{\cup} (G,\mathrm{B}) = (G,\mathrm{B}) \tilde{\cup} (F,\mathrm{A})$
(ii) $(F,\mathrm{A}) \tilde{\cup} ((G,\mathrm{B}) \tilde{\cup} (H,\mathrm{C})) = ((F,\mathrm{A}) \tilde{\cup} (G,\mathrm{B})) \tilde{\cup} (H,\mathrm{C})$
(iii) $(F,\mathrm{A}) \tilde{\cup} (F,\mathrm{A}) \subseteq (F,\mathrm{A})$
(iv) $(F,\mathrm{A}) \tilde{\cup} (\Phi, A) = (\Phi, A)$

Proof
(i) Let $(F,\mathrm{A}) \tilde{\cup} (G,\mathrm{B}) = (H,\mathrm{C})$. Then by definition 4.4, for all $\alpha \in \mathrm{C}$, we have $(H,\mathrm{C}) = \mathrm{H}(\alpha)$

Where

$\mathrm{H}(\alpha) = \mathrm{F}(\alpha) \tilde{\cup} \mathrm{G}(\alpha)$ However $\mathrm{H}(\alpha) = \mathrm{F}(\alpha) \tilde{\cup} \mathrm{G}(\alpha) = \mathrm{G}(\alpha) \tilde{\cup} \mathrm{F}(\alpha)$ since the union of these sets are commutative by definition 4.4. Therfore $(H,\mathrm{C}) = (G,\mathrm{B}) \tilde{\cup} (F,\mathrm{A})$. Thus the union of two IFSES are commutative i.e $(F,\mathrm{A}) \tilde{\cup} (G,\mathrm{B}) = (G,\mathrm{B}) \tilde{\cup} (F,\mathrm{A})$.

(ii) The proof is similar to proof of part(i) and is therefore omitted

(iii) The proof is straightforward and is therefore omitted.

(iv) The proof is straightforward and is therefore omitted.

Definition 4.6 Let (F,A) and (G,B) be any two IFSES over a soft universe (U, Z). Then the intersection of (F,A) and (G,B), denoted by $(F,\mathrm{A}) \tilde{\cap} (G,\mathrm{B})$ is an IFSES defined as $(F,\mathrm{A}) \tilde{\cap} (G,\mathrm{B}) = (H,\mathrm{C})$ where C= A ∪ B and

$$\mathrm{H}(\alpha) = \mathrm{F}(\alpha) \tilde{\cap} \mathrm{G}(\alpha), \text{ for all } \alpha \in \mathrm{C}$$

where

$$\mathrm{H}(\alpha) = \begin{cases} \mathrm{F}(\alpha) \alpha \in A - B \\ \mathrm{G}(\alpha) & \alpha \in A - B \\ \mathrm{t}(F(\alpha), G(\alpha)) & \alpha \in A \cap B \end{cases}$$

Where t is a t-norm

Proposition 4.7 If (F, A) , (G, B) and (H, C) are three IFSES over a soft universe (U, Z), then,

(i) $(F, \mathrm{A}) \tilde{\cap} (G, \mathrm{B}) = (G, \mathrm{B}) \tilde{\cap} (F, \mathrm{A})$
(ii) $(F, \mathrm{A}) \tilde{\cap} ((G, \mathrm{B}) \tilde{\cap} (H, \mathrm{C})) = ((F, \mathrm{A}) \tilde{\cap} (G, \mathrm{B})) \tilde{\cap} (H, \mathrm{C})$
(iii) $(\mathrm{F}, \mathrm{A}) \tilde{\cap} (\mathrm{F}, \mathrm{A}) \subseteq (\mathrm{F}, \mathrm{A})$
(iv) $(F, A) \tilde{\cap} (\Phi, A) = (\Phi, A)$

Proof

(i) The proof is similar to that of Propositio 4.5 (i) and is therefore omitted
(ii) The prof is similar to the prof of part (i) and is therefore omitted
(iii) The proof is straightforward and is therefore omitted.
(iv) The proof is straightforward and is therefore omitted.

Proposition 4.8. If (F, A), (G, B) and (H, C) are three IFSES over a soft universe (U, Z), then,

(i) $(F, \mathrm{A}) \tilde{\cup} ((G, \mathrm{B}) \cap (H, \mathrm{C})) = ((F, \mathrm{A}) \tilde{\cup} (G, \mathrm{B})) \tilde{\cap} ((F, \mathrm{A}) \tilde{\vee} (H, \mathrm{C}))$
(ii) $(F, \mathrm{A}) \tilde{\cap} ((G, \mathrm{B}) \tilde{\cup} (H, \mathrm{C})) = ((F, \mathrm{A}) \tilde{\cap} (G, \mathrm{B})) \tilde{\cup} ((F, \mathrm{A}) \tilde{\cap} (H, \mathrm{C}))$

Proof. The proof is straightforward by definitions 4.4 and 4.6 and is therefore omitted.

Proposition 4.9 If (F, A), (G, B) are two IFSES over a soft universe (U, Z), then,

i. $((F, \mathrm{A}) \tilde{\cup} (G, \mathrm{B}))^c = (F, \mathrm{A})^{\mathrm{c}} \tilde{\cap} (G, \mathrm{B})^{\mathrm{c}}$.
ii. $((F, \mathrm{A}) \tilde{\cap} (G, \mathrm{B}))^c = (F, \mathrm{A})^{\mathrm{c}} \tilde{\cup} (G, \mathrm{B})^{\mathrm{c}}$.

Proof.
(i) suppose that (F, A) and(G, B) be IFSES over a soft universe (U, Z) defined as:

$(F, \mathrm{A}) = \mathrm{F}(\alpha)$ for all $\alpha \in \mathrm{A} \subseteq \mathrm{Z}$ and $(G, \mathrm{B}) = \mathrm{G}(\alpha)$ for all $\alpha \in \mathrm{B} \subseteq \mathrm{Z}$. Now , due to the commutative and associative properties of IFSES, it follows that: by Definition 4.10 and 4.11, it follows that:

$$
\begin{aligned}
(F, \mathrm{A})^{\mathrm{c}} \tilde{\cap} (G, \mathrm{B})^{\mathrm{c}} &= (F(\alpha))^c \tilde{\cap} (G(\alpha))^c \\
&= (\tilde{c}\ (\mathrm{F}(\alpha))) \tilde{\cap} (\tilde{c}\ (\mathrm{G}(\alpha)) \\
&= (\ \tilde{c}\ (\mathrm{F}(\alpha) \tilde{\cap} \mathrm{G}(\alpha)) \\
&= ((F, \mathrm{A}) \tilde{\cup} (G, \mathrm{B}))^c.
\end{aligned}
$$

(ii) The proof is similar to the proof of part (i) and is therefore omitted.

Definition 4.10 Let (F, A) and (G, B) be any two IFSES over a soft universe (U, Z). Then " (F, A) AND (G, B) " denoted $(F, \mathrm{A}) \tilde{\wedge} (G, \mathrm{B})$ is a defined by:

$(F, \mathrm{A}) \tilde{\wedge} (G, \mathrm{B}) = (H, \mathrm{A} \times \mathrm{B})$

Where $(H, \text{A} \times \text{B})$ = H(α, β), such that H(α, β) = F(α) ∩ G(β) , for all $(\alpha, \beta) \in \text{A} \times \text{B}$. and ∩ represent the basic intersection.

Definition 4.11 Let (F, A) and (G, B) be any two IFSES over a soft universe (U, Z). Then " (F, A) OR (G, B) " denoted $(F, \text{A}) \tilde{\vee} (G, \text{B})$ is a defined by:

$(F, \text{A}) \tilde{\vee} (G, \text{B}) = (H, \text{A} \times \text{B})$

Where $(H, \text{A} \times \text{B})$ = H(α, β) such that H(α, β) = F(α) ∪ G(β), for all $(\alpha, \beta) \in \text{A} \times \text{B}$. and ∪ represent the basic union.

Proposition 4.12 If (F, A), (G, B) and (H, C) are three IFSES over a soft universe (U, Z), then,

i. $(F, \text{A}) \tilde{\wedge} ((G, \text{B}) \tilde{\wedge} (H, \text{C})) = ((F, \text{A}) \tilde{\wedge} (G, \text{B})) \tilde{\wedge} (H, \text{C})$
ii. $(F, \text{A}) \tilde{\vee} ((G, \text{B}) \tilde{\vee} (H, \text{C})) = ((F, \text{A}) \tilde{\vee} (G, \text{B})) \tilde{\vee} (H, \text{C})$
iii. $(F, \text{A}) \tilde{\vee} ((G, \text{B}) \tilde{\wedge} (H, \text{C})) = ((F, \text{A}) \tilde{\vee} (G, \text{B})) \tilde{\wedge} ((F, \text{A}) \tilde{\vee} (H, \text{C}))$
iv. $(F, \text{A}) \tilde{\wedge} ((G, \text{B}) \tilde{\vee} (H, \text{C})) = ((F, \text{A}) \tilde{\wedge} (G, \text{B})) \tilde{\vee} ((F, \text{A}) \tilde{\wedge} (H, \text{C}))$

Proof. The proofs are straightforward by Definitions 4.10 and 4.11 and are therefore omitted.

Note: The "AND" and "OR" operations are not commutative since generally A × B ≠ B ×A.

Proposition 4.13. If (F, A) and (G, B) are two IFSES over a soft universe (U, Z), then,

i. $((F, \text{A}) \tilde{\wedge} (G, \text{B}))^c = (F, \text{A})^c \tilde{\vee} (G, \text{B})^c$.
ii. $((F, \text{A}) \tilde{\vee} (G, \text{B}))^c = (F, \text{A})^c \tilde{\wedge} (G, \text{B})^c$.

Proof.
(i) suppose that (F, A) and (G, B) be IFSES over a soft universe (U, Z) defined as:

(F, A) = (F(α) for all $\alpha \in \text{A} \subseteq \text{Z}$ and (G, B) = G(β) for all $\beta \in \text{B} \subseteq \text{Z}$. Then by Definition 4.10 and 4.11, it follows that:

$$\begin{aligned}
((F, \text{A}) \tilde{\wedge} (G, \text{B}))^c &= ((F(\alpha) \tilde{\wedge} G(\beta))^c \\
&= (F(\alpha) \cap G(\beta))^c \\
&= (\tilde{c}\ (\text{F}(\alpha) \cap \text{G}(\beta)) \\
&= (\ \tilde{c}\ (\text{F}(\alpha)) \cup \tilde{c}\ (\text{G}(\beta))) \\
&= (F(\alpha))^c \tilde{\vee} (G(\beta))^c \\
&= (F, \text{A})^c \tilde{\vee} (G, \text{B})^c.
\end{aligned}$$

(ii) the proof is similar to that of part (i) and is therefore omitted.

5. Application of Intuitionistic Fuzzy Soft Expert Sets in a Decision Making Problem.

In this section, we introduce a generalized algorithm which will be applied to the IFSES model introduced in Section 3 and used to solve a hypothetical decision making problem.

Suppose that company Y is looking to hire a person to fill in the vacancy for a position in their company. Out of all the people who applied for the position, three candidates were shortlisted and these three candidates form the universe of elements, U= $\{u_1, u_2, u_3\}$ The hiring committee consists of the hiring manager, head of department and the HR director of the company and this committee is represented by the set {p, q, r }(a set of experts) while the set Q= {1=agree, 0=disagree } represents the set of opinions of the hiring committee members. The hiring committee considers a set of parameters, E={ e_1, e_2, e_3, e_4 } where the parameters e_i represent the characteristics or qualities that the candidates are assessed on, namely "relevant job experience", "excellent academic qualifications in the relevant field", "attitude and level of professionalism" and "technical knowledge" respectively. After interviewing all the three candidates and going through their certificates and other supporting documents, the hiring committee constructs the following IFSES.

$(F, Z) = \{ (e_1, p, 1) = \{ (\frac{u_1}{<0.2,0.4>}), (\frac{u_2}{<0.1,0.4>}), (\frac{u_3}{<0.1,0.7>}) \}\},$

$\{(e_2, p, 1) = \{ (\frac{u_1}{<0.3,0.2,>}), (\frac{u_2}{<0.25,0.2>}), (\frac{u_3}{<0.2,0.6>}) \}\},$

$\{(e_3, p, 1) = \{ (\frac{u_1}{<0.2,0.7>}), (\frac{u_2}{<0.4,0.3>}), (\frac{u_3}{<0.1,0.6>}) \}\},$

$\{ (e_4, p, 1) = \{ (\frac{u_1}{<0.2,0.6>}), (\frac{u_2}{<0.3,0.2>}), (\frac{u_3}{<0.3,0.1>}) \}\},$

$\{ (e_1, q, 1) = \{ (\frac{u_1}{<0.4,0.6>}), (\frac{u_2}{<0.2,0.3>}), (\frac{u_3}{<,0.3,0.2>}) \}\},$

$\{ (e_2, q, 1) = \{ (\frac{u_1}{<0.3,0.3>}), (\frac{u_2}{<0.9,0.1>}), (\frac{u_3}{<0.1,0.2>}) \}\},$

$\{ (e_3, q, 1) = \{ (\frac{u_1}{<0.1,0.4>}), (\frac{u_2}{<0.6,0.2>}), (\frac{u_3}{<0.2,0.4>}) \}\}.$

$\{ (e_4, q, 1) = \{ (\frac{u_1}{<0.5,0.3>}), (\frac{u_2}{<0.8,0.2>}), (\frac{u_3}{<0.3,0.4>}) \}\}.$

$\{(e_1, r, 1) = \{ (\frac{u_1}{<0.4,0.5>}), (\frac{u_2}{<0.6,0.4>}), (\frac{u_3}{<0.2,0.4>}) \}\}.$

$\{(e_2, r, 1) = \{ (\frac{u_1}{<0.3,0.7>}, (\frac{u_2}{<0.3,0.2>}), (\frac{u_3}{<0.2,0.2>}) \}\}.$

$\{(e_3, r, 1) = \{ (\frac{u_1}{<0.5,0.2>}), (\frac{u_2}{<0.1,0.6>}), (\frac{u_3}{<0.3,0.2>}) \}\}.$

$\{(e_1, p, 0) = \{ (\frac{u_1}{<0.1,0.4>}), (\frac{u_2}{<0.3,0.2>}), (\frac{u_3}{<0.2,0.4>}) \}\}.$

$\{(e_3, p, 0) = \{ (\frac{u_1}{<0.3,0.2>}), (\frac{u_2}{<0.2,0.4>}), (\frac{u_3}{<0.3,0.1>}) \}\}.$

$\{(e_4, p, 0) = \{ (\frac{u_1}{<0.3,0.2>}), (\frac{u_2}{<0.6,0.4,>}), (\frac{u_3}{<0.4,0.5>}) \}\}.$

$\{(e_1, q, 0) = \{ (\frac{u_1}{<0.2,0.4>}), (\frac{u_2}{<0.1,0.9>}), (\frac{u_3}{<0.1,0.2>}) \}\},$

$\{ (e_2, q, 0) = \{ (\frac{u_1}{<0.3,0.4>}), (\frac{u_2}{<0.2,0.7>}), (\frac{u_3}{<0.3,0.5>}) \}\},$

$\{(e_3, q, 0) = \{ (\frac{u_1}{<0.2,0.8>}), (\frac{u_2}{<0.1,0.2>}), (\frac{u_3}{<0.6,0.3>}) \}\},$

$\{(e_4, q, 0) = \{ (\frac{u_1}{<0.2,0.4>}), (\frac{u_2}{<0.6,0.2>}), (\frac{u_3}{<0.3,0.4>}) \}\}.$

$\{(e_1, r, 0) = \{ (\frac{u_1}{<0.3,0.4>}), (\frac{u_2}{<0.3,0.6>}), (\frac{u_3}{<0.25,0.2>}) \}\}.$

$\{(e_2, r, 0) = \{ (\frac{u_1}{<0.4,0.5>}), (\frac{u_2}{<0.4,0.2>}), (\frac{u_3}{<0.4,0.3>}) \}\}.$

$\{(e_3, r, 0) = \{ (\frac{u_1}{<0.3,0.2>}), (\frac{u_2}{<0.3,0.5>}), (\frac{u_3}{<0.5,0.1>}) \}\}.$

Next the IFSES (F , Z) is used together with a generalized algorithm to solve the decision making problem stated at the beginning of this section. The algorithm given below is employed by the hiring committee to determine the best or most suitable candidate to be hired for the position. This algorithm is a generalization of the algorithm introduced by Alkhazaleh and Salleh (see [30]) which is used in the context of the IFSES model that is introduced in this paper. The generalized algorithm is as follows:

Algorithm

1. Input the IFSES (F , Z).

2. Find the values of $\mu_{F(z_i)}(u_i)$- $\omega_{F(z_i)}(u_i)$ for each element $u_i \in$ U where $\mu_{F(z_i)}(u_i)$, and $\omega_{F(z_i)}(u_i)$ are the membership function and non-membership function of each of the elements $u_i \in$ U respectively.

3. Find the highest numerical grade for the agree- IFSES and disagree- IFSES.

4. Compute the score of each element $u_i \in$ U by taking the sum of the products of the numerical grade of each element for the agree- IFSES and disagree IFSES, denoted by A_i and D_i respectively.

5. Find the values of the score $r_i = A_i - D_i$ for each element $u_i \in$ U.

Table I. Values of $\mu_{F(z_i)}(u_i)$- $\omega_{F(z_i)}(u_i)$ for all $u_i \in$ U.

	u_1	u_2	u_3		u_1	u_2	u_3
$(e_1, p, 1)$	-0.2	-0.3	-0.6	$(e_3, p, 0)$	0.1	-0.2	0.2
$(e_2, p, 1)$	0.1	0.05	-0.4	$(e_4, p, 0)$	0.1	-0.2	- 0.1
$(e_3, p, 1)$	-0.5	0.1	-0.5	$(e_1, q, 0)$	-0.2	-0.8	-0.1
$(e_4, p, 1)$	-0.4	0.1	0.2	$(e_2, q, 0)$	-0.1	-0.5	-0.2
$(e_1, q, 1)$	-0.2	-0.1	0.1	$(e_3, q, 0)$	-0.6	-0.1	0.3
$(e_2, q, 1)$	0	0.8	-0.1	$(e_4, q, 0)$	-0.1	0.4	-0.1
$(e_3, q, 1)$	-0.3	0.4	-0.2	$(e_1, r, 0)$	-0.1	-0.3	0.05
$(e_4,q, 1)$	0.2	0.6	-0.1	$(e_2, r, 0)$	-0.1	0.2	0.1
$(e_1, r, 1)$	-0.1	0.2	-0.2	$(e_4, r, 0)$	0.1	-0.2	0.4
$(e_2, r, 1)$	-0.4	0.1	0				
$(e_3, r, 1)$	0.3	-0.5	0.1				
$(e_1, p, 0)$	-0.3	0.1	-0.2				

6. Determine the value of the highest score, s= $\max_{u_i}\{r_i\}$. Then the decision is to choose element as the optimal or best solution to the problem. If there is more than one element with the highest r_i score, then any one of those elements can be chosen as the optimal solution.

Then we can conclude that the optimal choice for the hiring committee is to hire candidate u_i to fill the vacant position

Table I gives the values of $\mu_{F(z_i)}(u_i)$- $\omega_{F(z_i)}(u_i)$ for each element $u_i \in U$ The notation a ,b gives the values of $\mu_{F(z_i)}(u_i)$- $\omega_{F(z_i)}(u_i)$.

In Table II and Table III, we gives the highest numerical grade for the elements in the agree- IFSES and disagree IFSES respectively.

Table II. Numerical Grade for Agree- IFSES

	u_i	Highest Numeric Grade
$(e_1, p, 1)$	u_1	-0.2
$(e_2, p, 1)$	u_1	0.1
$(e_3, p, 1)$	u_2	0.1
$(e_4, p, 1)$	u_3	0.2
$(e_1, q, 1)$	u_3	0.1
$(e_2, q, 1)$	u_2	0.8
$(e_3, q, 1)$	u_2	0.4
$(e_4,q, 1)$	u_2	0. 6
$(e_1, r, 1)$	u_2	0.2
$(e_2, r, 1)$	u_2	0.1
$(e_3, r, 1)$	u_1	0.3

Score (u_1) = -0.1 + 0.3 = 0.2
Score (u_2) = 0.1+0.80.4+0.6+0.2+0.1 = 2.2
Score (u_3) =0.2+0.1 = 0.3

Table III. Numerical Grade for Disagree-IFSES

	u_i	Highest Numeric Grade
$(e_1, p, 0)$	u_2	0.1
$(e_3, p, 0)$	u_3	0.2
$(e_4, p, 0)$	u_1	0.1
$(e_1, q, 0)$	u_3	-0.1
$(e_2, q, 0)$	u_1	-0.1
$(e_3, q, 0)$	u_3	0.3
$(e_4, q, 0)$	u_2	0.4
$(e_1, r, 0)$	u_3	0.05
$(e_2, r, 0)$	u_2	0.2
$(e_4, r, 0)$	u_3	0.4

Score (u_1) = 0.1-0.1 =0
Score (u_2) = 0.1+0.4+0.2 = 0.7
Score (u_3) = 0.2 -0.1 +0.3 + 0.05 +0.4 = 0.85

Let A_i and D_i represent the score of each numerical grade for the agree- IFSES and disagree- IFSES respectively. These values are given in Table IV.

Table IV. The score r_i = A_i - D_i

A_i	D_i	r_i
Score (u_1) = 0.2	Score (u_1) = 0	**0.2**
Score (u_2) = 2.2	Score (u_2) = 0.7	**1.45**
Score (u_3) = 0.3	Score (u_3) = 0.85	**-0.55**

Then s= $\max_{u_i} \{ r_i \} = r_2$, the hiring committee should hire candidate u_2 to fill in the vacant position

6. Conclusion

In this paper we have introduced the concept of intuitionistic fuzzy soft expert soft set and studied some of its properties. The complement, union, intersection, AND or OR operations have been defined on the intuitionistic fuzzy soft expert set. Finally, an application of this concept is given in solving a decision making problem. This new extension will provide a significant addition to existing theories for handling uncertainties, and lead to potential areas of further research and pertinent applications.

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