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Editorial

I am delighted to welcome you to the second issue of the Journal of New Theory (JNT) is completed with 11 articles.

JNT publishes original research articles, reports, reviews and commentaries that are based on a theory of mathematics. However, the topics are not limited to only mathematics, but also include statistics, computer science, physics, engineering, chemistry, biology, economics or social sciences that use a theory of mathematics.

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We would like to express our deepest thanks to all of the members of the editorial board and reviewers of the papers in this issue who are U. Orhan, A. Filiz, A. Fenercioğlu, A. Sarı, A. Yıldırım, A. S. Sezer, B. Mehmetoğlu, B. H. Çadırcı, C. Kaya, Ç. Çekiç, E. Altuntaş, E. Turgut, F. Karaaslan, F. Smarandache, G. Erdal, H. Aktaş, H. M. Doğan, H. Günal, H. Kızılaslan, H. Önen, H. Şimşek, İ. Zorlutuna, İ. Deli, İ. Gökçe, İ. Türkekul, İ. Parmaksız, J. Ye, M. Akar, M. Akdağ, M. Ali, M. Çavuş, M. Demirci, M. Sağlam, N. Yeşilayar, O. Muhtaroğlu, P. K. Maji, R. Yayar, S. Broumi, S. Karaman, S. Tarhan, S. Enginoğlu, S. Demiriz, S. Karataş, S. Öztürk, S. Eğri, Ş. Sözen, Y. Budak, Y. Karadağ, P. G. Patil, S. Hussain, A. O. Akdemir, N. Çağman, E. E. Kara, M. Suresh, M. Öztürk, S. Halder, T.Som and I. Simşek.

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We hope you will enjoy this issue of JNT. We are looking forward to hearing your feedback and receiving your contributions.

Happy reading!

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Original Article[**]

NEW CONTINUOUS AND OPEN FUNCTIONS IN TOPOLOGICAL SPACES

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Abstract – The aim of this paper is to introduce new continuous and open functions called somewhat θgs-continuous and somewhat θgs-open functions using θgs-open sets.Its various characterisations and properties are established.

Keywords – *θgs-open set, Somewhat θgs-continuous, Somewhat θgs-irresolute , Somewhat θgs-open functoin.*

1 Introduction

Levine [6] introduced the notion of generalized closed set. This notion has been studied extensively in recent years by many toplogists. The investigation of generalized closed sets had led to several new and interesting concepts. Recently in [7] the notion of of θ-generalized semi closed (briefly,θgs-closed)set was introduced by G.B.Navalagi et al. Gentry and Hoyle[4] introduced and studied the conepts of somewhat continuous and somewhat open functions.In [11] the notion of somewhat $\omega\alpha$-continous and somewhat $\omega\alpha$-open functions are introduced.
In this paper, we will continue the study of related functions with θgs-closed and θgs-open sets. We introduce and charcterize the concept of somewhat θgs-continuous and somewhat θgs-irresolute and somewhat θgs-open functions.

2 Preliminary

Throughout this paper (X, τ), (Y, σ)(or simply X, Y)denote topological spaces on which no separation axioms are assumed unless explicitly stated. For a subset A of a space X the closure and interior of A with respect to τ are denoted by $Cl(A)$ and $Int(A)$ respectively.

[**]*Edited by* Oktay Muhtaroğlu *(Area Editor) and* Naim Çağman *(Editor-in-Chief).*
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Definition 2.1. A subset A of a space X is called
(1) a semi-open set [5] if $A \subset Cl(Int(A))$.
(2) a semi-closed set [2] if $Int(Cl(Int(A))) \subset A$.

Definition 2.2. [3] A point $x \in X$ is called a semi-θ-cluster point of A if $sCl(U) \cap A \neq \phi$, for each semi-open set U containing x. The set of all semi-θ-cluster point of A is called semi-θ-closure of A and is denoted by $sCl_\theta(A)$. A subset A is called semi-θ-closed set if $sCl_\theta(A) = A$. The complement of semi-θ-closed set is semi-θ-open set.

Definition 2.3. [7] A subset A of X is θgeneralized semi-closed(briefly, θgs-closed)set if $sCl_\theta(A) \subset U$ whenever $A \subset U$ and U is open in X. The complement of θgs-closed set is θgeneralized-semi open (briefly,θgs-open).The family of all θgs-closed sets of X is denoted by θGSC(X,τ) and θgs-open sets by θGSO(X,τ).

Definition 2.4. A function $f: (X,\tau) \to (Y,\sigma)$ is called:
(i) θ-generalized semi-irresolute (briefly,θgs-irresolute)[8] if $f^{-1}(F)$ is θgs-closed set in X for every θgs-closed set F of Y
(ii) θ-generalized semi-continuous (briefly,θgs-continuous)[8] if $f^{-1}(F)$ is θgs-closed set in X for every closed set F of Y.
(iii) somewhat continuous [4] if for $U \in \sigma$ and $f^{-1}(U) \neq \phi$ there exists an open set V in X such that $V \neq \phi$ and $V \subseteq f^{-1}(U)$.

Definition 2.5. [9] *A function $f: X \to Y$ is said to be θgs-open (resp., θgs-closed) if $f(V)$ is θgs-open (resp., θgs-closed) in Y for every open set (resp., closed) V in X.*

3 Somewhat θgs-Continuous Functions

Definition 3.1. A function $f : (X,\tau) \to (Y,\sigma)$ is said to be somewhat θgs-continuous function if for every $U \in \sigma$ and $f^{-1}(U) \neq \phi$ there exists a θgs-open set V in X such that and $V \neq \phi$ and $V \subseteq f^{-1}(U)$.

Theorem 3.2. Every somewhat continuous function is somewhat θgs-continuous funtion.

Proof: Let $f : X \to Y$ is somewhat θgs-continuous function. Let U be any open set in Y such that $f^{-1}(U) \neq \phi$. Since f is somewhat continuous function, there exists an open set V in X such that $V \neq \phi$ and $V \subseteq f^{-1}(U)$. Since every open set is θgs-open set ,there exists θgs-open set V such that $V \neq \phi$ and $V \subseteq f^{-1}(U)$, which implies f is somewhat θgs-continuous function.

Remark 3.3. . Converse of the above theorem need not be true in general which follows from the following example.

Example 3.4. . Let X = Y = $\{a,b,c\}$, $\tau = \{X,\phi,\{a\},\{b\},\{a,b\}\}$, $\sigma = \{X,\phi,\{a\},\{b,c\}\}$. We have θGSO(X) = $\{X,\phi,\{a\},\{b,c\}\}$. Then the identity function is somewhat θgs-continuous function but not somewhat continuous function.

Theorem 3.5. . If $f : (X,\tau) \to (Y,\sigma)$ and $g : (Y,\sigma) \to (Z,\eta)$ be any two functions. If f is somewhat θgs-continuous function and g is contonuous function, then $g \circ f$ is somewhat θgs-continuous funtion.

Proof: Let U be any open set in Z. Suppose that $g^{-1}(U) \neq \phi$. Since $U \in \eta$ and g is continuous function, $g^{-1}(U) \in \eta$. Suppose that $f^{-1}(g^{-1}(U)) \neq \phi$. By hypothesis f is somewhat θgs-continuous function, there exists a θgs-open set in V in X such that $V \neq \phi$ and $V \subseteq f^{-1}(g^{-1}(U)) = (g \circ f)^{-1}(U)$, which implies that $V \subseteq (g \circ f)^{-1}(U)$. Therefore $g \circ f$ is somewhat θgs-continuous funtion.

Definition 3.6. . A subset M of a topological space X is said to be θgs-dense in X if there is no proper θgs-closed set F in X such that $M \subset F \subset X$.

Theorem 3.7. . If $f : (X, \tau) \rightarrow (Y, \sigma)$ be a function. Then the following are equivalent;
(i) f is somewhat θgs-continuous function.
(ii) If F is a closed subset of Y such that $f^{-1}(F) \neq X$, then there is proper θgs-closed subset D of X such that $f^{-1}(F) \subset D$.
(iii) If M is a θgs-dense subset of X, then f(M) is a dense subset of Y.

Proof: (i)⇒(ii):Let F be a closed subset of Y such that $f^{-1}(F) \neq X$. Then $f^{-1}(Y - F) = X - f^{-1}(F) \neq \phi$. By hypothesis (i) there exists a θgs-open set V in X such that $V \neq \phi$ and $V \subset f^{-1}(Y - F) = X - f^{-1}(F)$. This imples $f^{-1}(F) \subset X - V$ and X-V = D is a θgs-closed set in X.Therefore(ii) holds.

(ii)⇒ (i): Let U be an open set in Y and $f^{-1}(U) \neq \phi$. Then $f^{-1}(Y - U) = X - f^{-1}(U) = \phi$. By hypothesis, there exists a proper θgs-closed set D such that $f^{-1}(Y - U) \subset D$. This implies that $X - D \subset f^{-1}(U)$ and X-D is θgs-open and $X - D \neq \phi$.

(ii)⇒(iii): Let M be any θgs-dense set in X. Suppose f(M) is not dense subset of Y, then there exists a proper θgs-closed set D such that $M \subset f^{-1}(F) \subset D \subset X$. This contradicts the fact that M is a θgs-dense set in X. Therefore (iii) holds.

(iii)⇒ (ii): Suppose (iii) is not true. Then there exists a closed set F in Y such that $f^{-1}(F) \neq X$. But there is no proper θgs-closed set that $f^{-1}(F) \subset D$. This means that $f^{-1}(F)$ is θgs-dense in X. But from hypothesis $f(f^{-1}(F)) = F$ must be dense in Y, which is contradiction to the choice of F. Hence (ii) hold.

Theorem 3.8. .If $f : (X, \tau) \rightarrow (Y, \sigma)$ be a function and $X = A \cup B$, A and B are open subsets of X such that (f/A) and (f/B) are somewhat θgs-continuous functions then f is somewhat θgs-continuous function.

Proof: Let U be an open set in Y such that $f^{-1}(U) \neq \phi$. Then $(f/A)^{-1}(U) \neq \phi$ of $(f/B)^{-1}(U) \neq \phi$ or both $(f/A)^{-1}(U) \neq \phi$ and $(f/B)^{-1}(U) \neq \phi$.

Case(i): Suppose $(f/A)^{-1}(U) \neq \phi$. Since (f/A) is somewhat θgs-continuous, then there exists a θgs-open set V in A such that $V \neq \phi$ and $V \subset (f/A)^{-1}(U) \subseteq f^{-1}(U)$. Since V is θgs-open set in A and A is open in X,V is θgs-open in X. Hence f is somewhat θgs-continuous function.

Case(ii): Suppose $(f/B)^{-1}(U) \neq \phi$. Since (f/B) is somewhat θgs-continuous function, then there exists a θgs-open set V in B such that $V \neq \phi$ and $V \subset (f/B)^{-1}(U) \subset f^{-1}(U)$. Since V is θgs-open in B and B is open in X, V is θgs-open in X. Hence f is somewhat θgs-continuous function.

Case(iii): Suppose $(f/A)^{-1}(U) \neq \phi$ and $(f/B)^{-1}(U) \neq \phi$. Follows from case(i) and case(ii).

Definition 3.9. A topological space X is said to be θgs-separable if there exists a countable subset B of X which is θgs-dense in X.

Theorem 3.10. . If f is somewhat θgs-continuous function from X onto Y and if X is θgs-separable, then Y is separable.

Proof: Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be somewhat θgs-continuous function such that X is θgs-separable. Then by definition there exists a countable subset B of X which is θgs-dense in X. Then by Theorem 3.7, f(B) is dense in Y. Since B is countable f(B) is also countable which is dense in Y, ehich implies that Y is separable.

4 Somewhat θgs-irresolute Function

Definition 4.1. A function $f : (X, \tau) \rightarrow (Y, \sigma)$ is said to be somewhat θgs-irresolute function if for $U \in \theta GSO(\sigma)$ and $f^{-1}(U) \neq \phi$ there exists a non-empty θgs-open set V in X such that $V \subset f^{-1}(U)$.

Theorem 4.2. . If f is somewhat θgs-irresolute function and g is θgs-irresolute function, then $g \circ f$ is somewhat θgs-irresolute function.

Proof: Let $U \in \theta GSO(\eta)$. Suppose that $g^{-1}(U) \neq \phi$. Since $U \in \theta GSO(\eta)$ and g is somewhat θgs-irresolute function,there exists a θgs-open set V in X such that $V \neq \phi$ and $V \subseteq f^{-1}(g^{-1}(U))$. But $f^{-1}(g^{-1}(U)) = (g \circ f)^{-1}(U)$, which implies that $V \subseteq (g \circ f)^{-1}(U)$. Therefore $g \circ f$ is somewhat θgs-irresolute function.

Theorem 4.3. . If $f : (X, \tau) \to (Y, \sigma)$ be a function. Then the following are equivalent;
(i) f is somewhat θgs-irresolute function.
(ii) If F is a closed subset of Y such that $f^{-1}(F) \neq X$, then there is proper θgs-closed subset D of X such that $f^{-1}(F) \subset D$.
(iii) If M is a θgs-dense subset of X, then f(M) is a dense subset of Y.

Proof: Obvious.

Theorem 4.4. .If $f : (X, \tau) \to (Y, \sigma)$ be a function and $X = A \cup B$, A and B are open subsets of X such that (f/A) and (f/B) are somewhat θgs-irresolute function then f is somewhat θgs-irresolute function.

Proof: Obvious.

Definition 4.5. .[1] If X is a set and τ and σ are topologies for X, then τ is equivalent to σ provided if $U \in \tau$ and $U \neq \phi$, then there is a open set V in (X, σ) such that $V \neq \phi$ and $V \subset U$ and if $U \in \sigma$ and $U \neq \phi$ then there is an open set V in (X, τ) such that $V \neq \phi$ and $V \subset U$.

Definition 4.6. If X is a set and τ and σ are topologies for X, then τ is said to be θgs-equivalent to σ provided if $U \in \tau$ and $U \neq \phi$, then there is a θgs-open set V in (X, σ) such that $V \neq \phi$ and $V \subset U$ and if $U \in \sigma$ and $U \neq \phi$ then there is θgs-open set V in (X, τ) such that $V \neq \phi$ and $V \subset U$.

Theorem 4.7. . Let $f : (X, \tau) \to (Y, \sigma)$ be a somewhat continuous function and let τ^* be a topology for X, which is θgs-equivalent to τ then the function $f : (X, \tau^*) \to (Y, \sigma)$ is somewhat θgs-continuous function.

Proof: Let U be any open set in (Y, σ) such that $f^{-1}(U) \neq \phi$. Since by hypothesis $f : (X, \tau) \to (Y, \sigma)$ is somewhat continuous function by definition there exists an open set in O in (X, τ) such that $O \neq \phi$ and $O \subseteq f^{-1}(U)$. Since O is an open set in (X, τ) such that $O \neq \phi$ and since by hypothesis τ is θgs-equivalent to τ^* by definition there exists a θgs-open set V in (X, τ^*) such that $V \neq \phi$ and $V \subset O \subset f^{-1}(U)$. Hence $O \subset f^{-1}(U)$. Thus for any open set U in (Y, σ) such that $f^{-1}(U) \neq \phi$ there exists a θgs-open set V in (X, τ^*) such that $V \subset f^{-1}(U)$. So $f : (X, \tau^*) \to (Y, \sigma)$ is somewhat θgs-continuous function.

Theorem 4.8. . Let $f : (X, \tau) \to (Y, \sigma)$ be a somewhat θgs-continuous function and let σ^* be a topology for Y, which is equivalent to σ. Then the function $f : (X, \tau) \to (Y, \sigma^*)$ is somewhat θgs-continuous function.

Proof: Let U be any open set in (Y, σ^*) such that $f^{-1}(U) \neq \phi$ which implies $U \neq \phi$. Since σ and σ^* are equivalent, then there exists an open set W in (Y, σ) such that $W \neq \phi$ and $W \subset U$. Now, W is open set such that $W \neq \phi$, which implies $f^{-1}(W) \neq \phi$. Now by hypothesis $f : (X, \tau) \to (Y, \sigma)$ is somewhat θgs-continuous function.Therefore there exists a θgs-open set in V in (X, τ) such that $V \subseteq f^{-1}(W)$. Now $W \subset U$ implies $f^{-1}(W) \subset f^{-1}(U)$. This implies $V \subset f^{-1}(W) \subset f^{-1}(U)$. So, we have $V \subset f^{-1}(U)$, which implies that $f : (X, \tau) \to (Y, \sigma^*)$ is somewhat θgs-continuous function.

Theorem 4.9. . Let $f : (X, \tau) \to (Y, \sigma)$ be a somewhat θgs-irresolute surjection and let τ^* be a topology for X, which is θgs-equivalent to τ then the function $f : (X, \tau^*) \to (Y, \sigma)$ is somewhat θgs-irresolute function.

Proof: Let U be any open set in (Y,σ) such that $f^{-1}(U) \neq \phi$. Since by hypothesis $f:(X,\tau) \to (Y,\sigma)$ is somewhat θgs-irresolute, by definition there exists a θgs-open set in O in (X,τ) such that $O \neq \phi$ and $O \subseteq f^{-1}(U)$. Since O is a θgs-open set in (X,τ) such that $O \neq \phi$ and since by hypothesis τ is θgs-equivalent to τ^* by definition there exists a θgs-open set V in (X,τ^*) such that $V \neq \phi$ and $V \subset O \subset f^{-1}(U)$. Hence $O \subset f^{-1}(U)$. Thus for any open set U in (Y,σ) such that $f^{-1}(U) \neq \phi$ there exists a θgs-open set V in (X,τ^*) such that $V \subset f^{-1}(U)$. So $f:(X,\tau^*) \to (Y,\sigma)$ is somewhat θgs-irresolute function.

Theorem 4.10. . Let $f:(X,\tau) \to (Y,\sigma)$ be a somewhat θgs-irresolute surjection function and let σ^* be a topology for Y, which is equivalent to σ. Then the function $f:(X,\tau) \to (Y,\sigma^*)$ is somewhat θgs-continuous function.

Proof: Let U be any open set in (Y,σ^*) such that $f^{-1}(U) \neq \phi$ which implies $U \neq \phi$. Since σ and σ^* are equivalent, then there exists an open set W in (Y,σ) such that $W \neq \phi$ and $W \subset U$. Now, W is open set such that $W \neq \phi$, which implies $f^{-1}(W) \neq \phi$. Now by hypothesis $f:(X,\tau) \to (Y,\sigma)$ is somewhat θgs-irresolute function. Therefore there exists a θgs-open set in V in X such that $V \subseteq f^{-1}(W)$. Now $W \subset U$ implies $f^{-1}(W) \subset f^{-1}(U)$. This implies $V \subset f^{-1}(W) \subset f^{-1}(U)$. So, we have $V \subset f^{-1}(U)$, which implies that $f:(X,\tau) \to (Y,\sigma^*)$ is somewhat θgs-irresolute function.

5 Somewhat θgs-Open Functions

Definition 5.1. A function $f:(X,\tau) \to (Y,\sigma)$ is said to be somewhat θgs-open function provided that for every $U \in \tau$ and $U \neq \phi$ there exists a θgs-open set V in Y such that and $V \neq \phi$ and $V \subseteq f(U)$.

Theorem 5.2. Every somewhat open function is somewhat θgs-open function.

Proof: Let $f:(X,\tau) \to (Y,\sigma)$ is said to be somewhat open function. Let $U \in \tau$ and $U \neq \phi$. Since f is somewhat open function, there exists an open set V in Y such that $V \neq \phi$ and $V \subseteq f(U)$. But every open is θgs-open. So there exists a θgs-open set V in Y such that $V \neq \phi$. Thus f is somewhat θgs-open function.

Remark 5.3. .Converse of the above theorem need not be true in general, which follows from the following example.

Example 5.4. .Let X = Y = $\{a,b,c\}$, $\tau = \{X,\phi,\{b\},\{c\},\{b,c\}\}$, $\sigma = \{Y,\phi,\{b\},\{c\},\{a\},\{a,c\}\}$. We have θGSO(Y) = $\{Y,\phi,\{b\},\{c\},\{a,b\},\{b,c\}\}$. Then the identity function is somewhat θgs-open function but not somewhat open function.

Theorem 5.5. . If $f:(X,\tau) \to (Y,\sigma)$ is an open function and $g:(Y,\sigma) \to (Z,\eta)$ somewhat θgs-open function then $g \circ f$ is somewhat θgs-open funtion.

Proof: Let $U \in \tau$. Suppose $U \neq \phi$. Since f is an open function, f(U) is open and $f(U) \neq \phi$. Thus $f(U) \in \sigma$ and $f(U) \neq \phi$. Since g is somewhat θgs-open function and $f(U) \in \sigma$ such that $f(U) \neq \phi$ there exists a θgs-open set in $V \in \eta$, $V \subset g(f(U))$, which implies $g \circ f$ is somewhat θgs-open funtion.

Theorem 5.6. . If $f:(X,\tau) \to (Y,\sigma)$ be a bejective function. Then the following are equivalent;
(i) f is somewhat θgs-open function.
(ii) If F is a closed subset of Y such that $f(F) \neq Y$, then there exists a θgs-closed subset D of Y such that $D \neq \phi$ and $f(F) \subset D$.

Proof: (i)⇒(ii):Let F be a closed subset of Y such that $f(F) \neq Y$. From (i), there exists a θgs-open set V in X such that $V \neq \phi$ such that $V \subset f(X-F)$. Put D = Y-V. Clearly D is a θgs-closed set in Y and we claim that $D \neq \phi$. If $D = Y$, then $V = \phi$ which is a contradiction. Since $V \subset f(X-F)$, $D = Y - V \subset Y - [f(X-F)] = f(F)$.

(ii)⇒ (i): Let U be any non empty open set in X.Put $F = X - U$. Then F is a closed subset of X and $f(X-U) = f(F) = Y - f(U)$ whicg implies $f(F) \neq \phi$. Therefore by (ii), there is a θgs-closed subset D of Y such that $f(U) \subset D$. Put V = X-D, clearly V is θgs-open set and $V \neq \phi$.Further $V = X - D \subset Y - f(F) = Y - [Y - f(F)] = f(U)$.

Theorem 5.7. . If $f : (X, \tau) \to (Y, \sigma)$ be somewhat θgs-open function and A be any open subset of X. Then $f/A : (A, \tau/A) \to (Y, \sigma)$ is also somewhat θgs-open funtion.

Proof: Let $U \in \tau/A$ such that $U \neq \phi$. Since U is open in A and A is open in (X, τ), U is open in (X, τ) and since by hypothesis f is somewhat θgs-open function, then there exists a θgs-open set in V in Y, such that $V \subset f(U)$. Thus, for any open set U in $(A, \tau/A)$ with $U \neq \phi$, there exists a θgs-open set V in Y such that $V \subset f(U)$ which implies f/A is somewhat θgs-open funtion.

Theorem 5.8. . Let $f : (X, \tau) \to (Y, \sigma)$ be a function such that f/A and f/B are somewhat θgs-open, then f is somewhat θgs-open funtion, where $X = A \cup B$, A and B are open subsets of X.

Proof: Let U be an open set in X such that $U \neq \phi$. Since $X = A \cup B$, either $A \cap U \neq \phi$ or $B \cap U \neq \phi$ or both $A \cap U \neq \phi$ and $B \cap U \neq \phi$. Since U is opn in X, U is open in both Then $A, \tau/A$ and $B, \tau/B$.

Case(i): Suppose $U \cap A \neq \phi$ where $U \cap A$ is open in $A, \tau/A)$. Since by hypothesis f/A is somewhat θgs-open function, then there exists a θgs-open set V in Y such that $V \subset f(U \cap A) \subset f(U)$,which implies f is somewhat θgs-open function.

Case(ii): Suppose $U \cap B \neq \phi$ where $U \cap B$ is open in $B, \tau/B)$. Since by hypothesis f/B is somewhat θgs-open function, then there exists a θgs-open set V in Y such that $V \subset f(U \cap B) \subset f(U)$,which implies f is somewhat θgs-open function.

Case(iii): Suppose that $U \cap A \neq \phi$ and $U \cap B \neq \phi$. Then obviously f is somewhat θgs-open function from case(i) and case(ii). Thus f is somewhat θgs-open function.

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CHARACTERIZATION OF b-OPEN SOFT SETS IN SOFT TOPOLOGICAL SPACES

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Abstract – In this paper, a new class of open soft sets in a soft topological space, called b-open soft sets, is introduced and studied. Moreover, the relations this class and these different types of subsets of soft topological spaces, which introduced in [9], is studied. In particular, this class is contained in the class of β-open soft sets and contains the classes of open soft sets, pre open soft sets, semi open soft sets and α-open soft sets. Also, the authors introduce the concept of b-continuous soft functions and study some of their properties in detail. As a consequence the relations of some soft continuities are shown in a diagram.

Keywords – *Soft set, Soft topological space, Pre-open soft set, α-open soft set, Semi-open soft set, β-open soft set, b-open soft sets.*

1 Introduction

The concept of soft sets was first introduced by Molodtsov [22] in 1999 as a general mathematical tool for dealing with uncertain objects. In [22, 23], Molodtsov successfully applied the soft theory in several directions, such as smoothness of functions,game theory, operations research,Riemann integration, Perron integration, probability, theory of measurement, and so on.
After presentation of the operations of soft sets [20], the properties and applications of soft set theory have been studied increasingly [4, 17, 23, 25]. In recent years, many interesting applications of soft set theory have been expanded by embedding the ideas of fuzzy sets [2, 3, 5, 18, 19, 20, 21, 23, 24, 28]. To develop soft set theory, the operations of the soft sets are redefined and a uni-int decision making method was constructed by using these new operations [6].
Recently, in 2011, Shabir and Naz [26] initiated the study of soft topological spaces. They defined soft topology on the collection τ of soft sets over X. Consequently, they defined basic notions of soft topological spaces such as open and closed soft sets, soft subspace, soft closure, soft nbd of a point, soft separation axioms, soft regular spaces and soft normal spaces and established their several properties. In [9], Kandil et. al. introduced some soft operations such as semi open soft, pre open soft, α-open soft and β-open soft and investigated their properties in detail. Studies on the soft topological spaces have been accelerated [7, 8, 10, 11, 12, 13, 14, 15, 16, 27].

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The main purpose of this paper, is to introduce a new class of open soft sets in a soft topological space, called b-open soft sets, to soft topological spaces. Also, the relations this class and these different types of subsets of soft topological spaces is studied. Moreover, the authors introduced the concept of b-continuous soft functions and study some of their properties in detail.

2 Preliminary

In this section, we present the basic definitions and results of soft set theory which will be needed in the sequel.

Definition 2.1. [22] Let X be an initial universe and E be a set of parameters. Let $P(X)$ denote the power set of X and A be a non-empty subset of E. A pair (F, A) denoted by F_A is called a soft set over X , where F is a mapping given by $F : A \to P(X)$. In other words, a soft set over X is a parametrized family of subsets of the universe X. For a particular $e \in A$, $F(e)$ may be considered the set of e-approximate elements of the soft set (F, A) and if $e \notin A$, then $F(e) = \phi$ i.e
$F_A = \{F(e) : e \in A \subseteq E,\ F : A \to P(X)\}$. The family of all these soft sets over X denoted by $SS(X)_A$.

Definition 2.2. [20] Let $F_A,\ G_B \in SS(X)_E$. Then F_A is soft subset of G_B, denoted by $F_A \tilde{\subseteq} G_B$, if

(1) $A \subseteq B$, and

(2) $F(e) \subseteq G(e), \forall e \in A$.

In this case, F_A is said to be a soft subset of G_B and G_B is said to be a soft superset of F_A, $G_B \tilde{\supseteq} F_A$.

Definition 2.3. [20] Two soft subset F_A and G_B over a common universe set X are said to be soft equal if F_A is a soft subset of G_B and G_B is a soft subset of F_A.

Definition 2.4. [4] The complement of a soft set (F, A), denoted by $(F, A)'$, is defined by $(F, A)' = (F', A)$, $F' : A \to P(X)$ is a mapping given by $F'(e) = X - F(e),\ \forall\, e \in A$ and F' is called the soft complement function of F.
Clearly $(F')'$ is the same as F and $((F, A)')' = (F, A)$.

Definition 2.5. [26] The difference of two soft sets (F, E) and (G, E) over the common universe X, denoted by $(F, E) - (G, E)$ is the soft set (H, E) where for all $e \in E$, $H(e) = F(e) - G(e)$.

Definition 2.6. [26] Let (F, E) be a soft set over X and $x \in X$. We say that $x \in (F, E)$ read as x belongs to the soft set (F, E) whenever $x \in F(e)$ for all $e \in E$.

Definition 2.7. [26] Let $x \in X$. Then the soft set (x, E) over X, where $x_E(e) = \{x\}\ \forall e \in E$, called the singleton soft point and denoted by x_E .

Definition 2.8. [20] A soft set (F, A) over X is said to be a NULL soft set denoted by $\tilde{\phi}$ or ϕ_A if for all $e \in A,\ F(e) = \phi$ (null set).

Definition 2.9. [20] A soft set (F, A) over X is said to be an absolute soft set denoted by $\tilde{A}$ or X_A if for all $e \in A,\ F(e) = X$. Clearly we have $X'_A = \phi_A$ and $\phi'_A = X_A$.

Definition 2.10. [20] The union of two soft sets (F, A) and (G, B) over the common universe X is the soft set (H, C), where $C = A \cup B$ and for all $e \in C$,

$$H(e) = \begin{cases} F(e),\ e \in A - B, \\ G(e),\ e \in B - A, \\ F(e) \cup G(e),\ e \in A \cap B \end{cases} .$$

Definition 2.11. [20] The intersection of two soft sets (F, A) and (G, B) over the common universe X is the soft set (H, C), where $C = A \cap B$ and for all $e \in C$, $H(e) = F(e) \cap G(e)$. Note that, in order to efficiently discuss, we consider only soft sets (F, E) over a universe X in which all the parameter set E are same. We denote the family of these soft sets by $SS(X)_E$.

Definition 2.12. [29] Let I be an arbitrary indexed set and $L = \{(F_i, E), i \in I\}$ be a subfamily of $SS(X)_E$.

(1) The union of L is the soft set (H, E), where $H(e) = \bigcup_{i\in I} F_i(e)$ for each $e \in E$. We write $\tilde{\bigcup}_{i\in I}(F_i, E) = (H, E)$.

(2) The intersection of L is the soft set (M, E), where $M(e) = \bigcap_{i\in I} F_i(e)$ for each $e \in E$. We write $\tilde{\bigcap}_{i\in I}(F_i, E) = (M, E)$.

Definition 2.13. [26] Let τ be a collection of soft sets over a universe X with a fixed set of parameters E, then $\tau \subseteq SS(X)_E$ is called a soft topology on X if

(1) $\tilde{X}, \tilde{\phi} \in \tau$, where $\tilde{\phi}(e) = \phi$ and $\tilde{X}(e) = X$, $\forall e \in E$,

(2) the union of any number of soft sets in τ belongs to τ,

(3) the intersection of any two soft sets in τ belongs to τ.

The triplet (X, τ, E) is called a soft topological space over X.

Definition 2.14. [29] The soft set $(F, E) \in SS(X)_E$ is called a soft point in X_E if there exist $x \in X$ and $e \in E$ such that $F(e) = \{x\}$ and $F(e') = \phi$ for each $e' \in E - \{e\}$, and the soft point (F, E) is denoted by x_e.

Definition 2.15. [29] The soft point x_e is said to be belonging to the soft set (G, A), denoted by $x_e \tilde{\in} (G, A)$, if for the element $e \in A$, $F(e) \subseteq G(e)$.

Definition 2.16. [1] Let $SS(X)_A$ and $SS(Y)_B$ be families of soft sets, $u : X \to Y$ and $p : A \to B$ be mappings. Let $f_{pu} : SS(X)_A \to SS(Y)_B$ be a mapping. Then;

(1) If $(F, A) \in SS(X)_A$. Then the image of (F, A) under f_{pu}, written as $f_{pu}(F, A) = (f_{pu}(F), p(A))$, is a soft set in $SS(Y)_B$ such that

$$f_{pu}(F)(b) = \begin{cases} \cup_{a\in p^{-1}(b)\cap A} \ u(F(a)), & p^{-1}(b) \cap A \neq \phi, \\ \phi, & otherwise. \end{cases}$$

for all $b \in B$.

(2) If $(G, B) \in SS(Y)_B$. Then the inverse image of (G, B) under f_{pu}, written as $f^{-1}_{pu}(G, B) = (f^{-1}_{pu}(G), p^{-1}(B))$, is a soft set in $SS(X)_A$ such that

$$f^{-1}_{pu}(G)(a) = \begin{cases} u^{-1}(G(p(a))), & p(a) \in B, \\ \phi, & otherwise. \end{cases}$$

for all $a \in A$.

The soft function f_{pu} is called surjective if p and u are surjective, also is said to be injective if p and u are injective.

Definition 2.17 ([29])**.** Let (X, τ, A) and (Y, τ^*, B) be soft topological spaces and $f_{pu} : SS(X)_A \to SS(Y)_B$ be a function. Then

(1) The function f_{pu} is called continuous soft (soft-cts) if $f^{-1}_{pu}(G, B) \in \tau \ \forall \ (G, B) \in \tau^*$.

(2) The function f_{pu} is called open soft if $f_{pu}(G, A) \in \tau^* \forall \ (G, A) \in \tau$.

Definition 2.18. [9] Let (X, τ, E) be a soft topological space and $(F, E) \in SS(X)_E$. Then (F, E) is said to be,

(1) Pre open soft set if $(F, E) \tilde{\subseteq} int(cl(F, E))$,

(2) Semi open soft set if $(F, E) \tilde{\subseteq} cl(int(F, E))$,

(3) α-open soft set if $(F, E) \tilde{\subseteq} int(cl(int(F, E)))$,

(4) β-open soft set if $(F, E) \tilde{\subseteq} cl(int(cl(F, E)))$.

The set of all pre open (resp. semi open, α-open, β-open) soft sets is denoted by $POS(X)$ (resp. $SOS(X)$, $\alpha OS(X)$, $\beta OS(X)$) and the set of all pre closed (resp. semi closed, α-closed, β-closed) soft sets is denoted by $PCS(X)$ (resp. $SCS(X)$, $\alpha CS(X)$, $\beta CS(X)$).

Definition 2.19. [9] Let (X, τ, E) be a soft topological space, $(F, E) \in SS(X)_E$. Then, the pre soft interior (resp. semi soft interior, α-soft interior, β-soft interior) of (F, E) is denoted by $PSint(F, E)$ (resp. $SSint(F, E)$, $\alpha Sint(F, E)$, $\beta Sint(F, E)$), which is the soft union of all pre open (resp. semi open, α-open, β-open) soft sets contained in (F, E).

Definition 2.20. [9] Let (X, τ, E) be a soft topological space, $(F, E) \in SS(X)_E$. Then, the pre soft closure (resp. semi soft closure, α-soft closure, β-soft closure) of (F, E) is denoted by $PScl(F, E)$ (resp. $SScl(F, E)$, $\alpha Scl(F, E)$, $\beta Scl(F, E)$), which is the soft intersection of all pre closed (resp. semi closed, α-closed, β-closed) soft sets containing (F, E).

Theorem 2.1. [9] Let (X, τ, E) be a soft topological space and $(F, E) \in SS(X)_E$. Then

(1) $(F, E) \in SOS(X)$ if and only if $cl(F, E) = cl(int(F, E))$.

(2) If $(G, E) \in OS(X)$. Then, $(G, E)\tilde{\cap} cl(F, E)\tilde{\subseteq} cl((F, E)\tilde{\cap}(F, E))$.

(3) If $(H, E) \in CS(X)$. Then, $int[(G, E)\tilde{\cup}(H, E)]\tilde{\subseteq} int(G, E)\tilde{\cup}(H, E)$.

Definition 2.21. [9] Let (X, τ_1, A) and (Y, τ_2, B) be soft topological spaces. Let $u : X \to Y$ and $p : A \to B$ be a mappings. Let $f_{pu} : SS(X)_A \to SS(Y)_B$ be a function. Then, the function f_{pu} is called,

(1) Pre-continuous soft (Pre-cts soft) if $f_{pu}^{-1}(G, B) \in POS(X)$
$\forall\ (G, B) \in \tau_2$.

(2) α-continuous soft (α-cts soft) if $f_{pu}^{-1}(G, B) \in \alpha OS(X)$
$\forall\ (G, B) \in \tau_2$.

(3) Semi-continuous soft (semi-cts soft) if $f_{pu}^{-1}(G, B) \in SOS(X)$
$\forall\ (G, B) \in \tau_2$.

(4) β-continuous soft (β-cts soft) if $f_{pu}^{-1}(G, B) \in \beta OS(X)$
$\forall\ (G, B) \in \tau_2$.

3 b-open soft sets in soft topological spaces

Definition 3.1. Let (X, τ, E) be a soft topological space and $(F, E) \in SS(X)_E$. Then (F, E) is called a b-open soft set if $(F, E)\tilde{\subseteq} cl(int(F, E))\tilde{\cup}$
$int(cl(F, E))$ and its complement is said to be b-closed soft set. The set of all b-open soft sets is denoted by $BOS(X, \tau, E)$, or $BOS(X)$ and the set of all b-closed soft sets is denoted by $BCS(X, \tau, E)$, or $BCS(X)$.

Theorem 3.1. Let (X, τ, E) be a soft topological space. Then

(1) Arbitrary soft union of b-open soft sets is b-open soft.

(2) Arbitrary soft intersection of b-closed soft sets is b-closed soft.

Proof.

(1) Let $\{F_{jE} : j \in J\} \subseteq BOS(X)$. Then, $\forall\ j \in J$, $F_{jE}\tilde{\subseteq}[int(cl(F_{jE}))]\tilde{\cup}$
$[cl(int(F_{jE}))]$. It follows that, $\tilde{\bigcup}_j F_{jE}\tilde{\subseteq}\tilde{\bigcup}_j[[int(cl(F_{jE}))]\tilde{\cup}[cl(int(F_{jE}))]]$

$$= \tilde{\bigcup}_j[int(cl(F_{jE}))]\tilde{\cup}[\tilde{\bigcup}_j[cl(int(F_{jE}))]]$$
$$\tilde{\subseteq} int(\tilde{\bigcup}_j cl(F_{jE}))\tilde{\cup}[cl(int[\tilde{\bigcup}_j(F_{jE})])]$$
$$= [int(cl(\tilde{\bigcup}_j F_{jE}))]\tilde{\cup}[[cl(int[\tilde{\bigcup}_j(F_{jE})])]].$$

Hence, $\tilde{\bigcup}_j F_{jE} \in BOS(X)\ \forall\ j \in J$.

(2) By a similar way.

Remark 3.1. A finite soft intersection of b-open soft sets need not to be b-open soft, as shown in the following example. Therefore, the family of all b-open soft sets may be fail to be soft topology.1

Example 3.1. Suppose that there are three computers in the universe X given by $X = \{h_1, h_2, h_3\}$. Let $E = \{e_1, e_2\}$ be the set of decision parameters which are stands for "expensive" and "beautiful" respectively.
Let (F_1, E) be a soft set over the common universe X, which describe the composition of the computers, where
$F(e_1) = \{h_1, h_3\}, \quad F(e_2) = \{h_2, h_3\}$.
Then $\tau = \{\tilde{X}, \tilde{\phi}, (F, E)\}$ defines a soft topology on X. Hence, the sets (G, E) and (H, E) which defined as follows:
$G(e_1) = \{h_1, h_2\}, \quad G(e_2) = \{h_1\}$,
$H(e_1) = \{h_2, h_3\}, \quad H(e_2) = \{h_1\}$,
are b-open soft sets of (X, τ, E), but their soft intersection $(G, E)\tilde{\cap}(H, E) = (M, E)$ where $M(e_1) = \{h_2\}, \quad M(e_2) = \{h_1\}$ is not b-open soft set.

Remark 3.2. Note that the family of all b-open soft sets on a soft topological space (X, τ, E) forms a supra soft topology, i.e τ contains $\tilde{X}, \tilde{\phi}$ and closed under arbitrary soft union.

Definition 3.2. Let (X, τ, E) be a soft topological space, $(F, E) \in SS(X)_E$ and $x_e \in SS(X)_E$. Then

(1) x_e is called a b-interior soft point of (F, E) if $\exists$ $(G, E) \in BOS(X)$ such that $x_e \in (G, E)\tilde{\subseteq}(F, E)$, the set of all b-interior soft points of (F, E) is called the b-soft interior of (F, E) and is denoted by $bSint(F, E))$ consequently, $bSint(F, E)) = \tilde{\bigcup}\{(G, E) : (G, E)\tilde{\subseteq}(F, E),\ (G, E) \in BOS(X)\}$.

(2) x_e is called a b-closure soft point of (F, E) if $(F, E)\tilde{\cap}(H, E) \neq \tilde{\phi}$ $\forall$ $(H, E) \in BOS(X)$. The set of all b-closure soft points of (F, E) is called b-soft closure of (F, E) and is denoted by $bScl(F, E))$ consequently, $bScl(F, E)) = \tilde{\bigcap}\{(H, E) : (H, E) \in BCS(X),\ (F, E)\tilde{\subseteq}(H, E)\}$.

Theorem 3.2. Let (X, τ, E) be a soft topological space. Then, the following properties are satisfied for the b-soft interior operators, denoted by $(bSint)$.

(1) $bSint(\tilde{X}) = \tilde{X}$ and $bSint(\tilde{\phi}) = \tilde{\phi}$.

(2) $bSint(F, E)\tilde{\subseteq}(F, E)$.

(3) $bSint(F, E)$ is the largest b-open soft set contained in (F, E).

(4) if $(F, E)\tilde{\subseteq}(G, E)$, then $bSint(F, E)\tilde{\subseteq}bSint(G, E)$.

(5) $bSint(bSint(F, E)) = bSint(F, E)$.

(6) $bSint(F, E)\tilde{\cup}bSint(G, E)\tilde{\subseteq}bSint[(F, E)\tilde{\cup}(G, E)]$.

(7) $bSint[(F, E)\tilde{\cap}(G, E)]\tilde{\subseteq}bSint(F, E)\tilde{\cap}bSint(G, E)$.

Proof. Immediate.

Theorem 3.3. Let (X, τ, E) be a soft topological space. Then, the following properties are satisfied for the b-soft closure operators, denoted by $(bScl)$.

(1) $bScl(\tilde{X}) = \tilde{X}$ and $bScl(\tilde{\phi}) = \tilde{\phi}$.

(2) $(F, E)\tilde{\subseteq}bScl(F, E)$.

(3) $bScl(F, E)$ is the smallest b-closed soft set contains (F, E).

(4) if $(F, E)\tilde{\subseteq}(G, E)$, then $bScl(F, E)\tilde{\subseteq}bScl(G, E)$.

(5) $bScl(bScl(F, E)) = bScl(F, E)$.

(6) $bScl(F, E)\tilde{\cup}bScl(G, E)\tilde{\subseteq}bScl[(F, E)\tilde{\cup}(G, E)]$.

(7) $bScl[(F, E)\tilde{\cap}(G, E)]\tilde{\subseteq}bScl(F, E)\tilde{\cap}bScl(G, E)$.

Proof. Immediate.

Theorem 3.4. Let (X, τ, E) be a soft topological space. Then, the following properties are satisfied:

(1) $PScl(F,E) = (F,E)\tilde{\cup}cl(int(F,E))$.

(2) $PSint(F,E) = (F,E)\tilde{\cap}int(cl(F,E))$.

(3) $\alpha Scl(F,E) = (F,E)\tilde{\cup}cl(int(cl(F,E)))$.

(4) $\alpha Sint(F,E) = (F,E)\tilde{\cap}int(cl(int(F,E)))$.

(5) $SScl(F,E) = (F,E)\tilde{\cup}int(cl(F,E))$.

(6) $SSint(F,E) = (F,E)\tilde{\cap}cl(int(F,E))$.

(7) $\beta Scl(F,E) = (F,E)\tilde{\cup}int(cl(int(F,E)))$.

(8) $\beta Sint(F,E) = (F,E)\tilde{\cap}cl(int(cl(F,E)))$.

Proof. We shall prove only the first statement, the other cases are similar. Since $cl(int[(F,E)\tilde{\cup}cl(int(F,E))])\tilde{\subseteq} cl[int(F,E)\tilde{\cup}cl(int(F,E))] = cl(int(F,E))\tilde{\cup}cl(int(F,E)) = cl(int(F,E))\tilde{\subseteq}(F,E)\tilde{\cup}cl(int(F,E))$ from Theorem 2.1 (3). This means that, $(F,E)\tilde{\cup}cl(int(F,E))$ is a pre closed soft set containing (F,E). So, $PScl(F,E)\tilde{\subseteq}(F,E)\tilde{\cup}cl(int(F,E))$. On the other hand, $PScl(F,E)$ is pre closed soft. So, we have $cl(int(F,E))\tilde{\subseteq} cl(int(PScl(F,E)))\tilde{\subseteq}PScl(F,E)$. Hence, $(F,E)\tilde{\cup}cl(int(F,E))\tilde{\subseteq}PScl(F,E)$. Therefore, $PScl(F,E) = (F,E)\tilde{\cup}cl(int(F,E))$. The rest of the proof by a similar way.

Theorem 3.5. Let (X, τ, E) be a soft topological space. Then, the following properties are satisfied:

(1) $PScl(PSint(F,E)) = PSint(F,E)\tilde{\cup}cl(int(F,E))$.

(2) $SScl(Sint(F,E)) = SSint(F,E)\tilde{\cup}cl(int(cl(F,E)))$.

Proof.

(1) Since $cl(int[PSint(F,E)\tilde{\cup}cl(int(F,E))])\tilde{\subseteq}cl[int(PSint(F,E))\tilde{\cup}cl(int(F,E))] = cl(int(PSint(F,E)))\tilde{\cup}cl(int(F,E)) = cl(int(F,E))\tilde{\subseteq}PSint(F,E)\tilde{\cup}cl(int(F,E))$ from Theorem 2.1 (3). This means that, $PSint(F,E)\tilde{\cup}cl(int(F,E))$ is a pre closed soft set containing $PSint(F,E)$. So, $PScl(PSint(F,E))\tilde{\subseteq}PSint(F,E)\tilde{\cup}cl(int(F,E))$. On the other hand, $PScl(PSint(F,E))$ is the largest pre closed soft set containing $PSint(F,E)$. Hence, $PSint(F,E)\tilde{\cup}cl(int(F,E))\tilde{\subseteq}PScl(PSint(F,E))$. Therefore, $PScl(PSint(F,E)) = PSint(F,E)\tilde{\cup}cl(int(F,E))$.

(2) By a similar way.

Theorem 3.6. Let (X, τ, E) be a soft topological space. Then, the following are equivalent:

(1) (F,E) is a b-open soft set.

(2) $(F,E) = PSint(F,E)\tilde{\cup}SSint(F,E)$.

(3) $(F,E)\tilde{\subseteq}PScl(PSint(F,E))$.

Proof.

(1) $\Rightarrow$ (2) Let (F,E) be a b-open soft set. Then, $(F,E)\tilde{\subseteq}cl(int(F,E))\tilde{\cup}int(cl(F,E))$. By Theorem 3.4, $PSint(F,E)\tilde{\cup}SSint(F,E) = [(F,E)\tilde{\cap}int(cl(F,E))]\tilde{\cup}[(F,E)\tilde{\cap}cl(int(F,E))] = (F,E)\tilde{\cap}[int(cl(F,E))\tilde{\cup}cl(int(F,E))] = (F,E)$.

(2) $\Rightarrow$ (3) $(F,E) = PSint(F,E)\tilde{\cup}SSint(F,E) = PSint(F,E)\tilde{\cup}[(F,E)\tilde{\cap}cl(int(F,E))]\tilde{\subseteq}PSint(F,E)\tilde{\cup}cl(int(F,E)) = PScl(PSint(F,E))$, from Theorem 3.4 (6) and Theorem 3.5 (1).

(3) $\Rightarrow$ (1) $(F,E)\tilde{\subseteq}PScl(PSint(F,E)) = PSint(F,E)\tilde{\cup}cl(int(F,E))\tilde{\subseteq}int(cl(F,E))\tilde{\cup}cl(int(F,E))$, from Theorem 3.4 (1) and Theorem 3.5 (1).

4 Relations between b-open soft sets and some types of open soft sets of soft topological spaces

In this section, we introduce the relations between b-open soft sets and some special subsets of a soft topological space (X, τ, E) mentioned in [9].

Theorem 4.1. In a soft topological space (X, τ, E), the following statements hold,

(1) Every open (resp. closed) soft set is b-open (resp. b-closed) soft.

(2) Every pre open (resp. pre closed) soft set is b-open (resp. b-closed) soft.

(3) Every semi open (resp. semi closed) soft set is b-open (resp. b-closed) soft.

(4) Every b-open (resp. b-closed) soft set is β-open (resp. β-closed) soft.

Proof. We prove the assertion in the case of b-open soft set in (4), the other case is clear. Let $(F, E) \in BOS(X)$. Then, $(F, E)\tilde{\subseteq}int(cl(F, E))\tilde{\cup}cl(int(F, E))$

$$\begin{aligned}&\tilde{\subseteq}cl(int(cl(F, E)))\tilde{\cup}cl(int(F, E))\\&= cl[int(cl(F, E))\tilde{\cup}int(F, E)]\\&\tilde{\subseteq}cl(int[cl(F, E)\tilde{\cup}int(F, E)])\\&= cl(int(cl[(F, E)])).\end{aligned}$$

Therefore, $(F, E) \in \beta OS(X)$.

Remark 4.1. It is obvious that $POS(X) \cup SOS(X) \subseteq BOS(X) \subseteq \beta OS(X)$. The following examples shall show that theses implications can not be reversed and the converse of Theorem 4.1 is not true in general.

Examples 4.1. **(1)** In Example 3.1, the soft set (G, E) is b-open soft set, but it is not open soft.

(2) Suppose that there are four alternatives in the universe of dresses $X = \{h_1, h_2, h_3, h_4\}$ and consider $E = \{e_1(cotton), e_2(woollen)\}$ be the set of parameters showing the material of the dresses.Let $(F_1, E), (F_2, E), (F_3, E), (F_4, E)$ be four soft sets over the common universe X which describe the goodness of the dresses, where
$F_1(e_1) = \{h_1\}, \quad F_1(e_2) = \{h_2\}$,
$F_2(e_1) = \{h_2\}, \quad F_2(e_2) = \{h_1\}$,
$F_3(e_1) = \{h_1, h_2\}, \quad F_3(e_2) = \{h_1, h_2\}$,
$F_4(e_1) = \{h_1, h_2, h_4\}, \quad F_4(e_2) = \{h_1, h_2, h_4\}$.
Then $\tau = \{\tilde{X}, \tilde{\phi}, (F_1, E), (F_2, E), (F_3, E), (F_4, E)\}$ defines a soft topology on X. Hence, the soft set (G, E) which defined by;
$G(e_1) = \{h_1, h_4\}, \quad G(e_2) = \{h_2, h_4\}$ is b-open soft set, but it is not pre open soft.

(3) In Example 3.1, the soft set (H, E) is b-open soft set, but it is not semi open soft.

(4) Suppose that there are four alternatives in the universe of houses $X = \{h_1, h_2, h_3, h_4\}$ and consider $E = \{e_1, e_2\}$ be two parameter "quality of houses" and "wooden" to be the linguistic variable. Let $(F_1, E), (F_2, E), (F_3, E)$ be three soft sets over the common universe X which describe the goodness of the houses, where
$F_1(e_1) = \{h_4\}, \quad F_1(e_2) = \{h_1, h_2\}$,
$F_2(e_1) = \{h_1, h_2\}, \quad F_2(e_2) = \{h_4\}$,
$F_3(e_1) = \{h_1, h_2, h_4\}, \quad F_3(e_2) = \{h_1, h_2, h_4\}$.
Then $\tau = \{\tilde{X}, \tilde{\phi}, (F_1, E), (F_2, E), (F_3, E)\}$ defines a soft topology on X. Hence, the soft set (G, E) which defined by;
$G(e_1) = \{h_1\}, \quad G(e_2) = \{h_3\}$ is β-open soft set of (X, τ, E), but it is not b-open soft.

Corollary 4.1. For a soft topological space (X, τ, E) we have:

$$\begin{array}{ccccccc} OS(X) & & \longrightarrow & & POS(X) & & \\ \downarrow & & & & \downarrow & & \\ \alpha OS(X) & \longrightarrow & SOS(X) & \longrightarrow & BOS(X) & \longrightarrow & \beta OS(X) \end{array}$$

Proof. It is follows from Theorem 4.1 and [[9], Remark 4.2].

Theorem 4.2. Let (X, τ, E) be a soft topological space and $(F, E) \in SS(X)_E$. Then the following hold:

(1) $bSint(F^c, E) = \tilde{X} - bScl(F, E)$.

(2) $bScl(F^c, E) = \tilde{X} - bSint(F, E))$.

Proof.

(1) $\tilde{X} - bScl(F, E) = [\tilde{\bigcap}\{(G, E) : (F, E)\tilde{\subseteq}(G, E),\ (G, E) \in BCS(X)\}]^c = \tilde{\bigcup}\{(G^c, E) : (G^c, E)\tilde{\subseteq}(F^c, E),\ (G^c, E) \in BOS(X)\} = bSint(F^c, E)$.

(2) $\tilde{X} - bSint(F, E) = [\tilde{\bigcup}\{(G, E) : (G, E)\tilde{\subseteq}(F, E),\ (G, E) \in BOS(X)\}]^c = \tilde{\bigcap}\{(G^c, E) : (F^c, E)\tilde{\subseteq}(G^c, E),\ (G^c, E) \in BCS(X)\} = bScl(F^c, E)$.

Theorem 4.3. Let (X, τ, E) be a soft topological space and $(G, E) \in BOS(X)$.

(1) If $(F, E) \in OS(X)$. Then, $F_E \tilde{\cap} G_E \in BOS(X)$.

(2) If $(F, E) \in \alpha OS(X)$. Then, $F_E \tilde{\cap} G_E \in BOS(X)$.

Proof.

(1) Let $(F, E) \in OS(X)$ and $(G, E) \in BOS(X)$. Then,
$(F, E)\tilde{\cap}(G, E)\tilde{\subseteq} int(F, E)\tilde{\cap}[cl(int(G, E))\tilde{\cup} int(cl(G, E))]$
$= [int(F, E)\tilde{\cap} cl(int(G, E))]\tilde{\cup}[int(F, E)\tilde{\cap} int(cl(G, E))]$
$\tilde{\subseteq} cl[int(F, E)\tilde{\cap} int(G, E)]\tilde{\cup} int[int(F, E)\tilde{\cap} cl(G, E)]$
$\tilde{\subseteq} cl(int[(F, E)\tilde{\cap}(G, E)])\tilde{\cup}\ int(cl[(F, E)\tilde{\cap}(G, E)])$.
from Theorem 2.1 (2). Therefore, $F_E \tilde{\cap} G_E \in BOS(X)$.

(2) Let $(F, E) \in \alpha OS(X)$ and $(G, E) \in BOS(X)$. Then,
$(F, E)\tilde{\cap}(G, E)\tilde{\subseteq} int(cl(int(F, E)))\tilde{\cap}[cl(int(G, E))\tilde{\cup} int(cl(G, E))]$
$= [int(cl(int(F, E)))\tilde{\cap} cl(int(G, E))]\tilde{\cup}[int(cl(int(F, E)))\tilde{\cap} int(cl(G, E))]$
$\tilde{\subseteq} cl[int(cl(int(F, E)))\tilde{\cap} int(G, E)]\tilde{\cup} int[cl(int(F, E))\tilde{\cap} int(cl(G, E))]$
$\tilde{\subseteq} cl(int[cl(int(F, E))\tilde{\cap} int(G, E)])\tilde{\cup} int(cl[int(F, E)\tilde{\cap} int(cl(G, E))])$
$\tilde{\subseteq} cl(int(cl[int(F, E)\tilde{\cap} int(G, E)]))\tilde{\cup} int(cl(int[int(F, E)\tilde{\cap} cl(G, E)]))$
$\tilde{\subseteq} cl(int(cl(int[(F, E)\tilde{\cap}(G, E)])))\tilde{\cup} int(cl(int(cl[(F, E)\tilde{\cap}(G, E)])))$
$\tilde{\subseteq} cl(int[(F, E)\tilde{\cap}(G, E)])\tilde{\cup} int(cl[(F, E)\tilde{\cap}(G, E)])$.
from Theorem 2.1 (2). Therefore, $F_E \tilde{\cap} G_E \in BOS(X)$.

Proposition 4.1. Let (X, τ, E) be a soft topological space and $(F, E) \in BOS(X)$.

(1) If $int(F, E) = \tilde{\phi}$, then (F, E) is a pre open soft set.

(2) If $cl(F, E) = \tilde{\phi}$, then (F, E) is a semi open soft set.

Proof. Obvious.

Proposition 4.2. Let (X, τ, E) be a soft topological space and $(F, E) \in SS(X)_E$. Then $(F, E) \in BCS(X)$ if and only if $cl(int(F, E))\tilde{\cap} int(cl(F, E))\tilde{\subseteq}(F, E)$.

Proof. Obvious.

Theorem 4.4. Let (X, τ, E) be a soft topological space and $(F, E) \in SS(X)_E$. Then, the following properties are satisfied:

(1) $bScl(F, E) = Scl(F, E) \tilde{\cap} PScl(F, E)$.

(2) $bSint(F, E) = Sint(F, E) \tilde{\cup} PSint(F, E)$.

Proof.

(1) Since $bScl(F, E)$ is a b-closed soft set. Then, $cl(int(bScl(F, E))) \tilde{\cap} int(cl(bScl(F, E))) \tilde{\subseteq} bScl(F, E)$. It follows that, $cl(int(F, E)) \tilde{\cap} int(cl(F, E)) \tilde{\subseteq} bScl(F, E)$. So, $(F, E) \tilde{\cup} [cl(int(F, E)) \tilde{\cap} int(cl(F, E))]$ $\tilde{\subseteq} (F, E) \tilde{\cup} bScl(F, E) = bScl(F, E)$. Hence, $[(F, E) \tilde{\cup} cl(int(F, E))] \tilde{\cap} [(F, E) \tilde{\cup} int(cl(F, E))]$ $= Scl(F, E) \tilde{\cap} PScl(F, E)$, from Theorem 3.4. This means that, $Scl(F, E) \tilde{\cap} PScl(F, E) \tilde{\subseteq} bScl(F, E)$. The reverse inclusion is obvious from Remark 4.1.

(2) By a similar way.

5 b-soft continuity

Definition 5.1. Let (X, τ_1, A) and (Y, τ_2, B) be soft topological spaces. Let $u : X \to Y$ and $p : A \to B$ be mappings. Let $f_{pu} : SS(X)_A \to SS(Y)_B$ be a function. Then, the function f_{pu} is called a b-continuous soft (b-cts soft) if $f_{pu}^{-1}(G, B) \in BOS(X) \forall\, (G, B) \in \tau_2$.

Theorem 5.1. Let (X, τ, A) and (Y, τ^*, B) be soft topological spaces. Let $u : X \to Y$ and $p : A \to B$ be mappings. Let $f_{pu} : SS(X)_A \to SS(Y)_B$ be a function.Then, the following are equivalent:

(1) f_{pu} is b-continuous soft function.

(2) $f_{pu}^{-1}(H, B) \in BCS(X)\ \forall\, (H, B) \in CS(Y)$.

(3) $f_{pu}(bScl(G, A) \subseteq cl_{\tau^*}(f_{pu}(G, A))\ \forall\, (G, A) \in SS(X)_A$.

(4) $bScl(f_{pu}^{-1}(H, B)) \subseteq f_{pu}^{-1}(cl_{\tau^*}(H, B))\ \forall\, (H, B) \in SS(Y)_B$.

(5) $f_{pu}^{-1}(int_{\tau^*}(H, B)) \subseteq bSint(f_{pu}^{-1}(H, B))\ \forall\, (H, B) \in SS(Y)_B$.

Proof.

(1) $\Rightarrow$ (2) Let (H, B) be a closed soft set over Y. Then, $(H, B)' \in OS(Y)$ and $f_{pu}^{-1}(H, B)' \in BOS(X)$ from (1). Since $f_{pu}^{-1}(H, B)' = (f_{pu}^{-1}(H, B))'$ from [[29], Theorem 3.14]. Thus, $f_{pu}^{-1}(H, B) \in BCS(X)$.

(2) $\Rightarrow$ (3) Let $(G, A) \in SS(X)_A$. Since $(G, A) \tilde{\subseteq} f_{pu}^{-1}(f_{pu}(G, A)) \tilde{\subseteq} f_{pu}^{-1}(cl_{\tau^*}(f_{pu}(G, A))) \in BCS(X)$ from (2) and [[29], Theorem 3.14]. Then $(G, A) \tilde{\subseteq} bScl(G, A) \tilde{\subseteq} f_{pu}^{-1}(cl_{\tau^*}(f_{pu}(G, A)))$. Hence, $f_{pu}(bScl(G, A)) \tilde{\subseteq} f_{pu}(f_{pu}^{-1}(cl_{\tau^*}(f_{pu}(G, A)))) \tilde{\subseteq} cl_{\tau^*}(f_{pu}(G, A)))$ from [[29], Theorem 3.14]. Thus, $f_{pu}(bScl(G, A)) \tilde{\subseteq} cl_{\tau^*}(f_{pu}(G, A))$.

(3) $\Rightarrow$ (4) Let $(H, B) \in SS(Y)_B$ and $(G, A) = f_{pu}^{-1}(H, B)$. Then $f_{pu}(bScl f_{pu}^{-1}(H, B)) \tilde{\subseteq} cl_{\tau^*}(f_{pu}(f_{pu}^{-1}(H, B)))$ From (3). Hence, $bScl(f_{pu}^{-1}\ (H, B)) \tilde{\subseteq} f_{pu}^{-1}(f_{pu}(bScl(f_{pu}^{-1}(H, B)))) \tilde{\subseteq} f_{pu}^{-1}(cl_{\tau^*}(f_{pu}(f_{pu}^{-1}(H, B)))) \tilde{\subseteq} f_{pu}^{-1}(cl_{\tau^*}(H, B))$ from [[29], Theorem 3.14]. Thus, $bScl(f_{pu}^{-1}(H, B)) \tilde{\subseteq} f_{pu}^{-1}(cl_{\tau^*}(H, B))$.

(4) $\Rightarrow$ (2) Let (H, B) be a closed soft set over Y. Then $bScl(f_{pu}^{-1}(H, B)) \tilde{\subseteq} f_{pu}^{-1}(cl_{\tau^*}(H, B)) = f_{pu}^{-1}(H, B)\ \forall\, (H, B) \in SS(Y)_B$ from (4). But clearly, $f_{pu}^{-1}(H, B) \tilde{\subseteq} bScl(f_{pu}^{-1}(H, B))$. This means that, $f_{pu}^{-1}(H, B) = bScl(f_{pu}^{-1}(H, B))$, and consequently $f_{pu}^{-1}(H, B) \in BCS(X)$.

(1) $\Rightarrow$ (5) Let $(H, B) \in SS(Y)_B$. Then, $f_{pu}^{-1}(int_{\tau^*}(H, B)) \in BOS(X)$ from (1). Hence, $f_{pu}^{-1}(int_{\tau^*}(H, B)) = bSint(f_{pu}^{-1} int_{\tau^*}(H, B)) \tilde{\subseteq} bSint(f_{pu}^{-1}(H, B))$. Thus, $f_{pu}^{-1}(int_{\tau^*}(H, B)) \tilde{\subseteq} bSint(f_{pu}^{-1}(H, B))$.

(5) $\Rightarrow$ (1) Let (H, B) be an open soft set over Y. Then $int_{\tau^*}(H, B) = (H, B)$ and $f_{pu}^{-1}(int_{\tau^*}(H, B)) = f_{pu}^{-1}((H, B)) \tilde{\subseteq} bSint(f_{pu}^{-1}(H, B))$ from (5). But, we have $bSint(f_{pu}^{-1}(H, B)) \tilde{\subseteq} f_{pu}^{-1}(H, B)$. This means that, $bSint(f_{pu}^{-1}(H, B)) = f_{pu}^{-1}(H, B) \in BOS(X)$. Thus, f_{pu} is continuous soft function.

Theorem 5.2. Let (X, τ, A), (Y, τ^*, B) be soft topological spaces and $f_{pu} : SS(X)_A \to SS(Y)_B$ be a function. Then

(1) Every continuous soft function is b-continuous soft function.

(2) Every pre-continuous soft function is b-continuous soft function.

(3) Every semi-continuous soft function is b-continuous soft function.

(4) Every b-continuous soft function is β-continuous soft function.

Proof. Immediate from Theorem 4.1.

On accounting of Theorem 5.2 and [[9], Corollary 5.1], we have the following corollary.

Corollary 5.1. For a soft topological space (X, τ, E) we have the following implications.

$$cts \quad soft \quad \longrightarrow \quad Pre - cts\ soft$$

$$\swarrow \qquad\qquad \downarrow$$

$$\alpha - cts\ soft \quad \longrightarrow semi - cts\ soft \longrightarrow b - cts\ soft \longrightarrow \beta - cts\ soft$$

6 Conclusion

Topology is an important and major area of mathematics and it can give many relationships between other scientific areas and mathematical models. Recently, many scientists have studied and improved the soft set theory, which is initiated by Molodtsov [22] and easily applied to many problems having uncertainties from social life. In this paper, a new class of open soft sets in a soft topological space, called b-open soft sets, is introduced and studied. Moreover, the relations this class and these different types of subsets of soft topological spaces ,which introduced in [9], is studied. In particular, this class is contained in the class of β-open soft sets and contains the classes of open soft sets, pre open soft sets, semi open soft sets and α-open soft sets. Also, the authors introduce the concept of b-continuous soft functions and study some of their properties in detail. As a consequence the relations of some soft continuities are shown in a diagram. In the next study, we extend the notion of b-open soft sets to supra soft topological spaces and and other topological properties. Also, we will use some topological tools in soft set application, like rough sets.

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EXTRACTION OF DYESTUFF FROM ONION (*Allium cepa* L.) AND INVESTIGATION OF DYEING PROPERTIES OF COTTON AND WOOL FABRICS USING (urea+ammonia+calcium oxalate) MIXTURE

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Abstract - The dyestuff from onion (*Allium cepa* L.) was extracted using Soxhlet apparatus with distilled water. Wool and cotton fabrics were pretreated with (urea+ammonia+calcium oxalate) mixtures, artificial animal urine system (AAUS) before dyeing. The solutions 0.1 M of $CuSO_4$, $FeSO_4$ and $AlK(SO_4)_2.12H_2O$ were used as mordant agents. Pre-mordanting, together mordanting and last mordanting methods were applied at pH =4 and pH= 7 for dyeing of fabrics. According to the fastness results, the best dyeing method was determined as together mordanting method at pH=4 for wool and last mordanting method at pH= 7 for cotton fabric. The results also reveal that the onion containing Quercetin dyestuff shall probably be an important raw material for dyeing process of natural textile fibers.

Key words - *Wool, Cotton, Oxalate, Dyeing, Fastness*

1 Introduction

Natural dyes have high importance in producing hand made carpets, kilim and similar industrial dyeing applications before of their advantage of high colour fastness, cheapness, long term colour stability and authentic properties. Nowadays, the natural dyes are being produced in Asian countries such as Turkey, Iran, India, Azerbaijani, and natural dye products are being used most countries of the world [1].

There are many industrial plants which contain natural dyes such as onion(*Allium cepa* L.) which has odoriferous, and is used as spices plant, commonly. Onion has major flavone molecule which can be used as dyestuff of 3,5,7-tri hydroxy-2-(3',4' dihydroxy chromen-2-on called as Quercetin. [2] (Figure 1). The molecule structure of Quercetin play important role on dyeing process of natural fabric.

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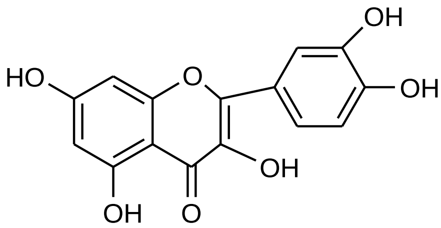

Figure 1. Chemical structure of quercetin

The acquired dyeing and fastness properties of woolen and cotton fabrics are very important characteristic in terms of user. The interaction of mordant compounds with wool and cotton fibers effects the affinity to fibers of dyestuffs. Improving the dyeing and fastness properties of textile fibers constitute the main subject of various studies [4,5,6]. In another different and last study, Onal-1 mordant mixtures in alkalin medium had been applied to wool fiber, feathered leather and cotton as a pretreatment process using *Rubai tinctorum* L. and *Hyperium scabrium* L. [7,8].

This study evaluates the average of dyeing properties of wool and cotton fabrics using Onion (*Allium cepa* L.*)* and the effect of (urea+ammonia +calcium oxalate) mixtures for each fabrics.

2. Experimental

Preparation of mordant solutions and dye-bath

Wool and cotton samples were treated with artificial animal urine system (AAUS). The stem and leaves of Onion (*A.cepa*) were supplied Plant Research Laboratory, Gaziosmanpasa University, in June, 2010. It was dried in shade, cleaned and powdered by grinder before the experiments. Extraction *of A. cepa* was performed by soxhlet apparatus with distilled water. 1 L of distilled water was used (for 100 g plant material) then the dyestuff was transferred to the aqueous media.

Reagents and equipments

All chemicals used in this work, were purchased from Merck. Distilled water was used for all steps. $FeSO_4.7H_2O$, $AlK(SO_4)_2.12H_2O$ and $CuSO_4.5H_2O$ were purchased from Merck. Extraction was performed by using soxhlet apparatus. Colour codes were determined by Pantone Colour Guide. The wash-, crock- (wet, dry) and light fastness of all dyed samples were established according to ISO 105-C06 and to CIS, respectively, and fastnesses were determined by Atlas Weather-ometer, a Launder-ometer and a 255 model crock-meter, respectively [9].

Dyeing procedures

Dyeing procedures of wool and cotton samples were firstly treated with artificial animal urine system (AAUS). The undyed materials were kept into AAUS included NH_3 (3%, v/v), CaC_2O_4 (3%, m/v) and urea (3%, m/v) for 24 h, at room temperature before dyeing procedures. At the end of the time, the samples rinsed with distilled water and dyed according to the dyeing methods that mentioned below.

Pre-mordanting method

The undyed material (1 g) which was treated with willow solution and AAUS for 24 h at room temperature, separately, was heated in 0.1 M mordant solution (100 mL) for 1 h at 90ºC. After cooling of sample, it was rinsed with distilled water and put into dye-bath solution (100 mL). It was heated at 90ºC for 1 h, at the end of the period, the dyed material removed, rinsed with distilled water and dried.

Together-mordanting method

Both mordant (in solid state which equivalent to 0.1 M mordant solution) and dyestuff solution poured into a flask and the sample placed in this mixture. The complication was heated at 90ºC for 1 h. After cooling, it was rinsed and dried.

Last-mordanting method

On the contrary to pre-mordanting method, the undyed material (1 g) was first treated with dyestuff solution for 1 h at 90ºC. After cooling the sample, it was rinsed with distilled water and put into 0.1 M mordant solution (100 mL) and heated for 1 h at 90ºC. Finally, the dyed material was rinsed with distilled water and dried.

3. Results and Discussion

Proposed dyeing mechanism

As the hydroxy (-OH) and carbonyl (C=O) groups forms coordine covalent bonds with mordant cation, such as Cu^{2+} (Figure 2, Figure 3 and Figure 4).

The dyeing mechanisms of wool with Salvigenin by pre-mordanting (1), together-mordanting (2) and last-mordanting (3) methods can be considered as follows [10] :

(1) Wool..........................Mordant (Me^{n+}).......................Dyestuff

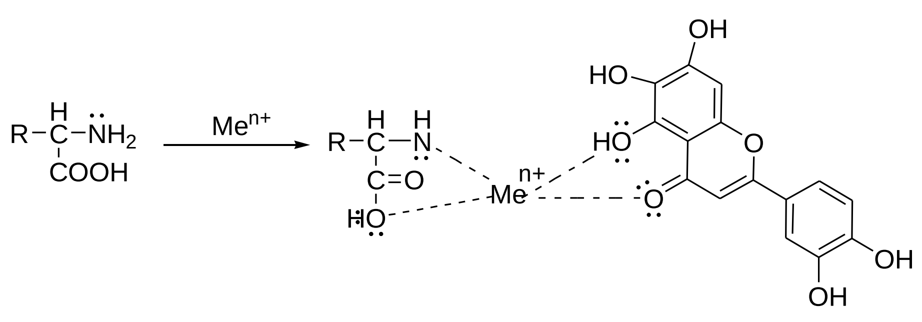

Figure 2. Proposed mordant-dye complex according to pre-mordanting method in dyeing of wool fibers

(2) Dyestuff......................Mordant (Me^{n+}).....................Dyestuff

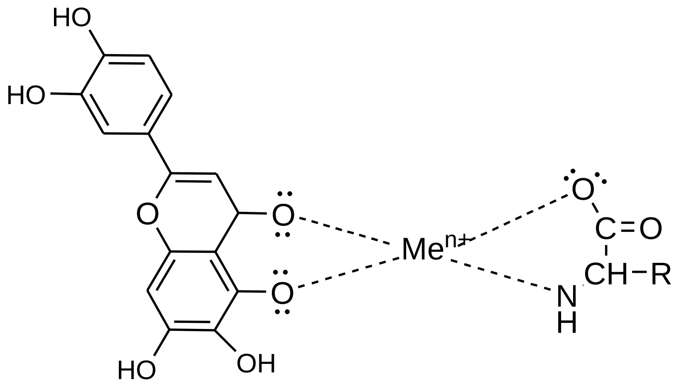

Figure 3. Proposed mordant-dye complex according to together-mordanting method in dyeing of wool fibers

(3) Wool............................Dyestuff (Me^{n+})..............................Mordant

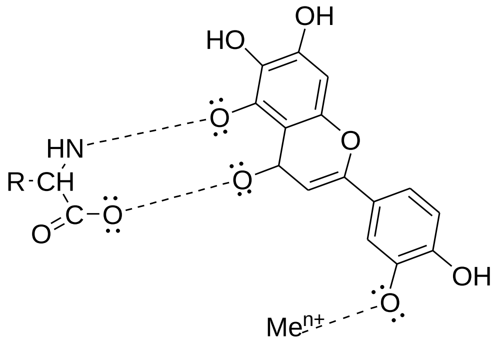

Figure 4. Proposed mordant-dye complex according to last-mordanting method in dyeing of wool fibers

Because of cotton has cellulosic structure, coordine covalent bonding occurs between CH_2O- groups of cellulose and metal cation. The suggested mechanism is given below (Figure 5)

The variation of average fastness for wool with respect to the mordant agent at Fig.6 and the variation of average fastness for cotton with respect to the mordant agent at fig.7.

As seen from the curves in Fig. 6 the average fastness for wool samples decreases in the order of Fe(II)>Cu(II)>Al(III). Best values for wool samples obtained by using Pre-mordanting method with Fe(II) and Al(III) mordants.

It can be clearly observed from the Fig.7, there is no considerable difference between together- and last-mordanting method with the use of Fe(II) and Al(III) mordants in dyeing of cotton fibers.

In general, from the Figures 6 and 7, the most effective mordant agent is Fe(II) and the most effective dyeing procedures are together- and last-mordanting method. This situation can be explained by the high stability of Fe(II) complex. Based on the results, it can be noted that treatment of natural fibers with AAUS assists to strenght the coordinate covalent bonding of Fe(II) salt to natural fiber.

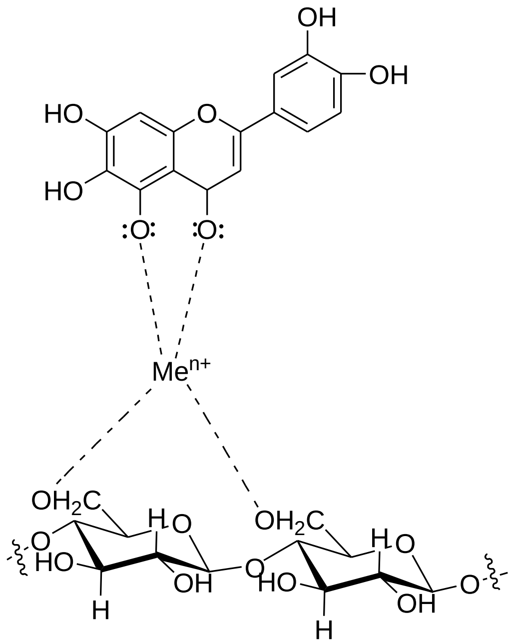

Figure 5: Proposed mordant-dye complex according to together-mordanting method in dyeing of cotton

When evaluated the dyed wool samples, green , brown and its tones were obtained in the presence of pre- and together-mordanting methods by $CuSO_4$ and $FeSO_4$ salts, and yellow tones were obtained by $AlK(SO_4)_2.12H_2O$ for three mordanting methods.

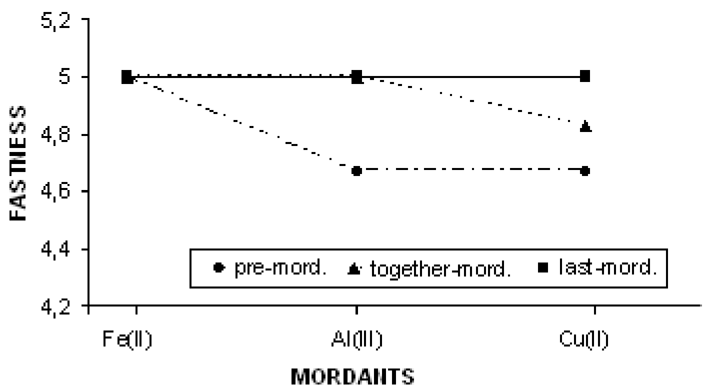

Figure 6. The variation of average fastness for wool with respect to the mordant agent

In dyeing of cotton samples, gray, light gray and cream tones were occured. According to the experimental results, however, the colours fastness of dyed cotton and wool samples have good degrees.

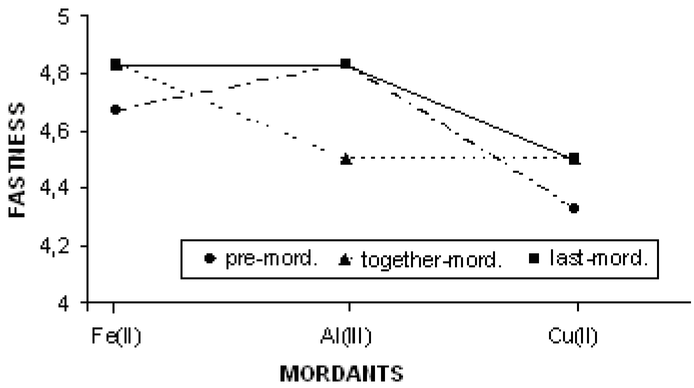

Figure 7. The variation of average fastness for cotton with respect to the mordant agent

The effect of AAUS was explained by Onal in 1996. Shortly, the components of AAUS (ammonia+ urea+ oxalate) have a great importance on the fastness of dyed fibers [10]. Here, ammonia helps the expanding of fiber misels so it facilitates the penetration of dye to the fiber. Urea serves as a pH regulator, and as last, oxalate plays an important role during the formation of complex structure which occurs between dye and natural fiber. It makes this complex very stable, and so the fastness values of the dyed samples increase in the presence of AAUS.

All the fastness values and colour codes are presented in Table I, Table II, for wool and cotton samples, respectively.

Table 1. Fastness values and color codes of dyed wool fabric (average values)

Mordant	**Dyeing Method**	**Wash-Fastness**	**Crock Fastness**		
			Wet	**Dry**	**Light fastness**
$FeSO_4.7H_2O$	Pre-mordanting	4	5	5	6
	Together-mordanting	4-5	5	5	6-7
	Last-mordanting	4-5	5	5	6
$CuSO_4.5H_2O$	Pre-mordanting	4	5	5	6
	Together-mordanting	3	5	5	7
	Last-mordanting	4-5	4-5	4-5	6
$AlK(SO_4)_2.12H_2O$	Pre-mordanting	3	5	5	6-7
	Together-mordanting	3-4	5	5	7
	Last-mordanting	4-5	4-5	4-5	6-7

It can be clearly seen that wet and dry fastness values are very good for dyed wool and cotton fibers .

Table 2. Fastness values and colour codes of dyed cotton fabric (average values)

Mordant	Dyeing Method	Wash-Fastness	Crock Fastness		
			Wet	Dry	Light fastness
$FeSO_4.7H_2O$	Pre-mordanting	5	5	5	5-6
	Together-mordanting	3-4	5	3-4	5-6
	Last-mordanting	5	5	5	6-7
$CuSO_4.5H_2O$	Pre-mordanting	4-5	5	4-5	6
	Together-mordanting	4-5	5	4	5-6
	Last-mordanting	5	5	5	6-7
$AlK(SO_4)_2.12H_2O$	Pre-mordanting	4	5	5	5
	Together-mordanting	3-4	4	4	5-6
	Last-mordanting	5	5	5	6-7

Consequently, the best dyeing conditions of wool materials are obtained with Fe(II) and Cu(II) mordants using pre- and together mordanting method. Generaly green and brown colour tones were obtained for wool samples. On the contrary to wool, the highest fastness values obtained for cotton fibers with Fe(II) according to all mordanting methods. The colours of cotton fibers are gray, yellow and cream tones. In addition, AAUS contributes the brightness of natural fibers dyed samples.

A. cepa may be evaluated as an important natural dyestuff source. However, AAUS which called as Onal-1 mordant system, may be used as pre- mordanting mixtures for cellulosic and protein fibers to increase the fastness and brightness of the textile products.

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HERMITE-HADAMARD TYPE INEQUALITIES FOR *LOG*−CONVEX STOCHASTIC PROCESSES

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Abstract – The main aim of the present note is to introduce log-convex stochastic processes and to contact correlation between convex stochastic processes and log-convex stochastic processes. We also prove some Hadamard-type inequalities for log-convex stochastic processes with the help of the special means.

Keywords – *Hermite Hadamard Inequality, log-convex functions, convex stochastic process, log-convex stochastic process*

1 Introduction

In 1980, Nikodem [13] introduced the convex stochastic processes in his article. Later in 1995, Skowronski [9] presented some further results on convex stochastic processes. Moreover, in 2011, Kotrys [7] derived some Hermite-Hadamard type inequalities for convex stochastic processes. In 2014, Maden *et.al.* [24] introduced the convex stochastic processes in the first sense and proved Hermite-Hadamard type inequalities to these processes. Also in 2014, Set *et.al.* [25] presented the convex stochastic processes in the second sense and they investigated Hermite-Hadamard type inequalities for these processes. Moreover, in recent papers [22, 23], strongly λ-GA-convex stochastic processes and preinvex stochastic processes has been introduced.

A function $f : I \to \mathbb{R}$, where $I \subseteq \mathbb{R}$ is an interval, is said to be a convex function on I if the inequality

$$f(tx + (1-t)y) \le tf(x) + (1-t)f(y) \tag{1}$$

holds for all $x, y \in I$ and $t \in [0, 1]$. If the reversed inequality in (1) holds, then f is concave. For some recent results related to this classic result, see the books [2, 4, 5, 6] and the papers [14, 15, 16, 17, 18, 19, 20, 21] where further references are given.

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Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function defined on the interval I of real numbers and $a < b$. The following double inequality

$$f(\frac{a+b}{2}) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2} \tag{2}$$

is well known in the literature as Hadamard's inequality. Both inequalities hold in the reversed direction if f is concave.

Recently, log-convex functions have gained much interest in mathematics and its sub-areas such as optimization theory. Let $f : I \to \mathbb{R}$ be a function where I is an interval of real numbers. f is said to be convex on I if the following inequality holds for all $x, y \in I$ and $\lambda \in [0, 1]$.

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y) \tag{3}$$

A function $f : I \to [0, \infty)$ is said to be *log*-convex (or multiplicatively convex) if $\log(f)$ is convex or namely the following inequality

$$f(\lambda x + (1-\lambda)y) \leq [f(x)]^{\lambda}[f(y)]^{(1-\lambda)} \tag{4}$$

holds for all $x, y \in I$ and $\lambda \in [0, 1]$. Moreover, any *log*-convex function is a convex function since the inequality

$$[f(x)]^{\lambda}[f(y)]^{(1-\lambda)} \leq \lambda f(x) + (1-\lambda)f(y) \tag{5}$$

holds for all $x, y \in I$ and $\lambda \in [0, 1]$. [1, p.7]

Let $f : I \subseteq \mathbb{R} \to [0, \infty)$ be a log-convex function defined on the interval I of real numbers and $a < b$. The following double inequality

$$f(\frac{a+b}{2}) \leq \exp\left[\frac{1}{b-a}\int_a^b \ln f(x)dx\right] \leq \sqrt{f(a)f(b)} \tag{6}$$

is well known in the literature as Hermite-Hadamard inequality for log-convex functions. Both inequalities hold in the reversed direction if f is concave.[18]

Furtermore, in [16], Dragomir and Mond proved that the following inequalities of Hermite–Hadamard type hold for log-convex functions:

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\leq \exp\left[\frac{1}{b-a}\int_a^b \ln f(t)dt\right] \\ &\leq \frac{1}{b-a}\int_a^b G\left(f\left(t\right) + fa+b-t\right)dt \\ &\leq \frac{1}{b-a}\int_a^b f\left(t\right)dt \\ &\leq L\left(f\left(a\right), f(b)\right) \end{aligned} \tag{7}$$

More information about *log*-convex functions and their properties can be found in [1, 10, 11, 12].

In this paper we propose the generalization of convexity of this kind for stochastic processes.

Let $(\Omega, \mathcal{F}, P)$ be an arbitrary probability space. A function $X : \Omega \to \mathbb{R}$ is called a random variable if it is $\mathcal{F} - measurable$. Let $(\Omega, \mathcal{F}, P)$ be an arbitrary probability space and let $T \subset \mathbb{R}$ be time. A collection of random variables $X(t, w)$, $t \in T$ with values in $\mathbb{R}$ is called a *stochastic process*. If $X(t, w)$ takes values in $S = \mathbb{R}^d$, it is called a $vector - valued$ stochastic process. If the time T can be a discrete subset of $\mathbb{R}$, then $X(t, w)$ is called a discrete time stochastic process. If time is an interval, $\mathbb{R}^+$ or $\mathbb{R}$, it is called a *stochastic process with continuous time*. For any fixed $\omega \in \Omega$, one can regard $X(t, w)$ as a function of t. It is called a *sample function* of the stochastic process. In the case of a $vector - valued$ process, it is a *sample path*, a curve in $\mathbb{R}^d$. Throughout the paper, we restrict our attention stochastic processes with continuous time, i.e. , index set $T = [0, \infty)$.

Definition 1.1. Let (Ω, A, P) be a probability space and $T \subset \mathbb{R}$ be an interval. We say that a stochastic process $X : T \times \Omega \to \mathbb{R}$ is

i. convex if

$$X(\lambda u + (1-\lambda)v, \cdot) \leq \lambda X(u, \cdot) + (1-\lambda) X(v, \cdot)$$

for all $u, v \in T$ *and* $\lambda \in [0,1]$. . This class of stochastic process are denoted by C.

ii. $\lambda-$*convex (where* λ *is a fixed number in* $(0,1)$ *if*

$$X(\lambda u + (1-\lambda)v, \cdot) \leq \lambda X(u, \cdot) + (1-\lambda) X(u, \cdot)$$

for all $u, v \in T$ *and* $\lambda \in (0,1)$. This class of stochastic process is denoted by C_λ.

iii. Wright-convex if

$$X(\lambda u + (1-\lambda)v, \cdot) + X((1-\lambda)u + \lambda v, \cdot) \leq X(u, \cdot) + X(v, \cdot)$$

for all $u, v \in T$ *and* $\lambda \in [0,1]$. This class of stochastic process is denoted by W.

iv. Jensen-convex if

$$X\left(\frac{u+v}{2}, \cdot\right) \leq \frac{X(u, \cdot) + X(v, \cdot)}{2}$$

[7, 8, 9, 13]

Clearly, $C \subseteq C_\lambda \subset W$ and $C_{\frac{1}{2}} \subseteq C_\lambda$, for all $\lambda \in (0,1)$. [9]

Definition 1.2. Let (Ω, A, P) be a probability space and $T \subset \mathbb{R}$ be an interval. We say that the stochastic process $X : T \times \Omega \to \mathbb{R}$ is called

i. continuous in probability in interval T *if for all* $t_0 \in T$

$$P - \lim_{t \to t_0} X(t, \cdot) = X(t_0, \cdot)$$

where $P - \lim$ *denotes the limit in probability;*

ii. mean-square continuous *in the interval* T *if for all* $t_0 \in T$

$$P - \lim_{t \to t_0} E[X(t, \cdot) - X(t_0, \cdot)] = 0$$

where $E[X(t, \cdot)]$ *denotes the expectation value of the random variable* $X(t, \cdot)$*;*

iii. increasing (decreasing) if for all $u, v \in T$ *such that* $t < s$*,*

$$X(u, \cdot) \leq X(v, \cdot), (X(u, \cdot) \geq X(v, \cdot))$$

iv. monotonic if it is increasing or decreasing;

v. differantiable at a point $t \in T$ *if there is a random variable* $X'(t, \cdot) : T \times \Omega \to \mathbb{R}$

$$X'(t, \cdot) = P - \lim_{t \to t_0} \frac{X(t, \cdot) - X(t_0, \cdot)}{t - t_0}$$

We say that a stochastic process $X : T \times \Omega \to \mathbb{R}$ is continuous (differantiable) if it is continuous (differantiable) at every point of the interval T. [7, 8, 9, 13]

Definition 1.3. Let (Ω, A, P) be a probability space and $T \subset \mathbb{R}$ be an interval with $E[X(t)^2] < \infty$ for all $t \in T$. Let $[a, b] \subset T$, $a = t_0 < t_1 < ... < t_n = b$ be a partition of $[a, b]$ and $\Theta_k \in [t_{k-1}, t_k]$ for $k = 1, ..., n$. A random variable $Y : \Omega \to \mathbb{R}$ is called mean-square integral of the process $X(t, \cdot)$ on $[a, b]$ if the following identity holds:

$$\lim_{n\to\infty} E[(X(\Theta_k (t_k - t_{k-1}) - Y)^2] = 0.$$

Then we can write

$$\int_a^b X(t,\cdot)dt = Y(\cdot) \ (a.e.).$$

Also, mean square integral operator is increasing, that is,

$$\int_a^b X(t,\cdot)dt \le \int_a^b Z(t,\cdot)dt \ (a.e.),$$

where $X(t,\cdot) \le Z(t,\cdot)$ *(a.e.)* in $[a,b]$ [3].

In throughout the paper, we will consider the stochastic processes that is *with continuous time* and *mean-square continuous.*

Now, we give the well-known Hermite-Hadamard integral inequality for convex stochastic processes:

If $X : T \times \Omega \to \mathbb{R}$ is Jensen-convex and mean-square continuous in the interval $T \times \Omega$, then for any $u, v \in T$, we have [7]

$$X\left(\frac{u+v}{2},\cdot\right) \le \frac{1}{v-u}\int_u^v X(t,\cdot)dt \le \frac{X(u,\cdot)+X(v,\cdot)}{2} \tag{8}$$

The main subject of this paper is to extend some well-known results concerning log-convex functions to log-convex stochastic processes. Also, we investigate the relationship between log-convex stochastic processes and convex stochastic processes. Moreover, we propose well-known Hermite-Hadamard type inequalities for $\log -$convex stochastic processes by the help of aritmetic and geometric means.

2 Hermite-Hadamard Inequality For $\log -$Convex Stochastic Process

Definition 2.1. Let (Ω, A, P) be a probability space and $T \subset \mathbb{R}$ be an interval. We say that a stochastic process $X : T \times \Omega \to [0,\infty)$ is *log−convex if*

$$X(\lambda s + (1-\lambda)t,\cdot) \le [X(s,\cdot)]^{\lambda} [X(t,\cdot)]^{1-\lambda} \tag{9}$$

for all $s,t \in T$ and $\lambda \in [0,1]$.

This class of stochastic process is denoted by C_l.

Proposition 2.2. If $X : T \times \Omega \to [0,\infty)$ is a log-convex stochastic process, then X is convex stochastic process. That is, $C_l \subseteq C$ for all $\lambda \in [0,1]$.

Proof. The proof is obvious from (9) and the arithmetic-geometric mean inequality which is known as the inequality

$$[X(s,\cdot)]^{\lambda} [X(t,\cdot)]^{1-\lambda} \le \lambda X(s,\cdot) + (1-\lambda) X(t,\cdot) \tag{10}$$

f*or all* $s,t \in T$and $\lambda \in [0,1]$. □

Proposition 2.3. Let $f : T \to [0,\infty)$ and $X : T \times \Omega \to [0,\infty)$ be a function and a stochastic process, respectively. If f and X are convex and f is increasing, then $f \circ X$ is convex.

Proof. Since f and X are convex and f is increasing

$$\begin{aligned}(f \circ X)(\lambda s + (1-\lambda)t, \cdot) &= f(X(\lambda s + (1-\lambda)t, \cdot)) \\ &\leq f(\lambda X(s, \cdot) + (1-\lambda) X(t, \cdot)) \\ &\leq \lambda f(X(s, \cdot)) + (1-\lambda) f(X(t, \cdot)) \\ &= \lambda (f \circ X)(s, \cdot) + (1-\lambda)(f \circ X)(X(t, \cdot))\end{aligned}$$

for all $s, t \in T$ and $\lambda \in [0,1]$.

Let us recall the Hermite-Hadamard inequality

$$X\left(\frac{u+v}{2}, \cdot\right) \leq \frac{1}{v-u}\int_u^v X(t, \cdot)\, dt \leq \frac{X(u, \cdot) + X(v, \cdot)}{2}$$

where $X : T \times \Omega \to \mathbb{R}$ is a convex stochastic process on the interval $T \times \Omega$, $u, v \in T$ and $u < v$. □

Note that if we apply the above inequality for the $log-$convex stochastic process $X : T\times\Omega \to (0, \infty)$, we have that

$$\ln\left[X\left(\frac{u+v}{2}, \cdot\right)\right] \leq \frac{1}{v-u}\int_u^v \ln[X(t, \cdot)]\, dt \leq \frac{\ln[X(u, \cdot)] + \ln[X(v, \cdot)]}{2} \tag{11}$$

from which we get

$$X\left(\frac{u+v}{2}, \cdot\right) \leq \exp\left[\frac{1}{v-u}\int_u^v \ln[X(t, \cdot)]\, dt\right] \leq \sqrt{X(u, \cdot) X(v, \cdot)} \tag{12}$$

which is an inequality of Hadamard's type for log-convex stochastic process.

Let us denote by $A(u, v)$ the arithmetic mean of the nonnegative real numbers, and by $G(u, v)$ the geometric mean of the same numbers.

Note that, by the use of these notations, Hadamard's inequality (8) can be written in the form:

$$X(A(u, v), \cdot) \leq \frac{1}{v-u}\int_u^v A(X(t, \cdot) + X(u+v-t, \cdot))dt \leq A(X(u, \cdot) + X(v, \cdot))$$

It is easy to see this as

$$\int_u^v X(t, \cdot)\, dt = \int_u^v X(u+v-t, \cdot)\, dt$$

We now prove a similar result for log-convex stochastic process and geometric means.

Theorem 2.4. Let $X : T \times \Omega \to [0, \infty)$ be a log-convex stochastic process on $T \times \Omega$ and $u, v \in T$ with $u < v$. Then one has the inequality:

$$X(A(u, v), \cdot) \leq \frac{1}{v-u}\int_u^v G(X(t, \cdot), X(u+v-t, \cdot))dt \leq G(X(u, \cdot), X(v, \cdot)) \tag{13}$$

Proof. Since X is log-convex, we have that

$$X(\lambda s + (1-\lambda)t, \cdot) \leq [X(s, \cdot)]^{\lambda}[X(t, \cdot)]^{1-\lambda}$$

for all $\lambda \in [0,1]$ and

$$X((1-\lambda)s + \lambda t, \cdot) \leq [X(s, \cdot)]^{1-\lambda}[X(t, \cdot)]^{\lambda}$$

for all $\lambda \in [0,1]$.

If we multiply the above inequalities and take square roots, we obtain

$$G(X(\lambda s+(1-\lambda)t,\cdot),X((1-\lambda)s+\lambda t,\cdot))\leq G(X(u,\cdot),X(v,\cdot))$$

Integrating this inequality on $[0,1]$ over λ, we get

$$\int_0^1 G(X(\lambda s+(1-\lambda)t,\cdot),X((1-\lambda)s+\lambda t,\cdot))d\lambda\leq G(X(u,\cdot),X(v,\cdot))$$

If we change the variable $t:=\lambda u+(1-\lambda)v$, $\lambda\in[0,1]$, we obtain

$$\begin{aligned}&\int_0^1 G(X(\lambda s+(1-\lambda)t,\cdot),X((1-\lambda)s+\lambda t,\cdot))d\lambda\\ =\ &\frac{1}{v-u}\int_u^v G(X(t,\cdot),X(u+v-t,\cdot))dt\end{aligned}$$

and the second inequality in (13) is proved.

Now, by (9) , for $\lambda=\frac{1}{2}$, we have that

$$X\left(\frac{s+t}{2},\cdot\right)\leq G(X(s,\cdot),X(t,\cdot))$$

for all $u,v\in T$.

If we choose $s:=\lambda u+(1-\lambda)v$, $t:=(1-\lambda)u+\lambda v$, we get the inequality

$$X\left(\frac{u+v}{2},\cdot\right)\leq G(X(\lambda u+(1-\lambda)v,\cdot),X((1-\lambda)u+\lambda v,\cdot)) \tag{14}$$

for all $\lambda\in[0,1]$. Integrating this inequality on $[0,1]$ over λ, the first inequality in (13) is proved. □

Corollary 2.5. With the above assumptions, $u\geq 0$ and X nondecreasing on $T\times\Omega$, we have the inequality:

$$X(G(u,v),\cdot)\leq\frac{1}{v-u}\int_u^v G(X(t,\cdot),X(u+v-t,\cdot))dt\leq G(X(u,\cdot),X(v,\cdot)) \tag{15}$$

The following result offers another inequality of Hadamard type for convex stochastic process.

Corollary 2.6. Let $X:T\times\Omega\to[0,\infty)$ be a convex stochastic process on $T\times\Omega$ and $u,v\in T$ with $u<v$. Then one has the inequalities:

$$\begin{aligned}&X\left(\frac{u+v}{2},\cdot\right)\\ \leq\ &\ln\left[\frac{1}{b-a}\int_u^v\exp\left[X(t,\cdot)+X(u+v-t,\cdot)\right]dt\right]\\ \leq\ &\frac{X(u,\cdot)+X(v,\cdot)}{2}\end{aligned} \tag{16}$$

Proof. Define the mapping $g:T\to(0,\infty)$, $g(t)=\exp(X(t,\cdot))$, which is clearly log-convex on I.

Now, if we apply Theorem 2.4, we obtain

$$\exp X(\frac{u+v}{2},\cdot)\leq\frac{1}{b-a}\int_u^v\sqrt{\exp X(t,\cdot)X(u+v-t,\cdot)}dt\leq\sqrt{\exp X(u,\cdot)X(v,\cdot)},$$

which implies (16). □

The following theorem for log-convex stochastic process also holds.

Theorem 2.7. Let $X : T \times \Omega \to (0, \infty)$ be a log-convex stochastic process on $T \times \Omega$ and $u, v \in T$ with $u < v$. Then, one has the inequalities:

$$\begin{aligned} X\left(\frac{u+v}{2}, \cdot\right) &\leq \exp\left[\frac{1}{v-u}\int_u^v \ln\left[X(t,\cdot)\right] dt\right] \\ &\leq \frac{1}{v-u}\int_u^v G\left(X(t,\cdot), X(u+v-t,\cdot)\right) dt \\ &\leq \frac{1}{v-u}\int_u^v X(t,\cdot)\, dt \\ &\leq L\left(X(u,\cdot), X(v,\cdot)\right), \end{aligned} \tag{17}$$

where $L(p,q) := \frac{p-q}{\ln p - \ln q}$ if $p \neq q$ and $L(p,p) := p$.

Proof. The first inequality in (17) was proved before. We now have that

$$G\left(X(t,\cdot) + X(u+v-t,\cdot)\right) = \exp\left[\ln G\left(X(t,\cdot) + X(u+v-t,\cdot)\right)\right]$$

for all $t \in [u, v]$.

Integrating this equality on $[u, v]$ and using the well-known Jensen's integral inequality for the convex mapping $\exp(\cdot)$, we have that

$$\begin{aligned} &\frac{1}{v-u}\int_u^v G\left(X(t,\cdot) + X(u+v-t,\cdot)\right) dt \\ = &\frac{1}{v-u}\int_u^v \exp\left[\ln\left(G\left(X(t,\cdot) + X(u+v-t,\cdot)\right)\right)\right] dt \\ \geq &\exp\left[\frac{1}{v-u}\int_u^v \ln\left(G\left(X(t,\cdot) + X(u+v-t,\cdot)\right)\right) dt\right] \\ = &\exp\left[\frac{1}{v-u}\int_u^v \frac{\ln X(t,\cdot) + \ln X(u+v-t,\cdot)}{2} dt\right] \\ = &\exp\left[\frac{1}{v-u}\int_u^v \ln X(t,\cdot)\, dt\right]. \end{aligned} \tag{18}$$

It is clear that

$$\int_u^v \ln X(t,\cdot)\, dt = \int_u^v \ln X(u+v-t,\cdot)\, dt.$$

By the aritmetic mean -geometric mean inequality we have that

$$G\left(X(t,\cdot), X(u+v-t,\cdot)\right) \leq \frac{X(t,\cdot) + X(u+v-t,\cdot)}{2}, t \in [u,v]$$

from which, by integration, we get

$$\frac{1}{v-u}\int_u^v G\left(X(t,\cdot), X(u+v-t,\cdot)\right) dt \leq \frac{1}{v-u}\int_u^v X(t,\cdot)\, dt$$

and the third inequality in (18) is proved.

To prove the last inequality, we observe, by the log-convexity of X, that

$$X(\lambda u + (1-\lambda) v, \cdot) \leq \left[X(u,\cdot)\right]^{\lambda} \left[X(v,\cdot)\right]^{1-\lambda} \tag{19}$$

for all $u, v \in T$.

Integrating (19) over λ in $[0, 1]$, we have

$$\int_0^1 X(\lambda u + (1-\lambda) v, \cdot)\, d\lambda \le \int_0^1 [X(u,\cdot)]^{\lambda} [X(v,\cdot)]^{1-\lambda}\, d\lambda.$$

As

$$\int_0^1 X(\lambda u + (1-\lambda) v, \cdot)\, d\lambda = \frac{1}{v-u}\int_u^v X(t,\cdot)\, dt$$

and

$$\int_0^1 [X(u,\cdot)]^{\lambda} [X(v,\cdot)]^{1-\lambda}\, d\lambda = L[X(u,\cdot), X(v,\cdot)],$$

the theorem is proved. □

Corollary 2.8. Let $X : T \times \Omega \to \mathbb{R}$ be a convex stochastic process on $T \times \Omega$ and $u, v \in T$ with $u < v$. Then one has the inequalities:

$$\begin{aligned} \exp\left[X\left(\frac{u+v}{2},\cdot\right)\right] &\le \exp\left[\frac{1}{v-u}\int_u^v X(t,\cdot)\, dt\right] && (20) \\ &\le \frac{1}{v-u}\int_u^v \exp\left[\frac{X(t,\cdot) + X(u+v-t,\cdot)}{2}\right] dt \\ &\le \frac{1}{v-u}\int_u^v \exp[X(t,\cdot)]\, dt \\ &\le E(X(u,\cdot), X(v,\cdot)), \end{aligned}$$

where E is the exponential mean, i.e.,

$$E(p,q) := \frac{\exp p - \exp q}{p-q} \text{ for } p \neq q \text{ and } E(p,p) = p.$$

Remark 2.9. Note that the inequality

$$\begin{aligned} &\exp\left(\frac{1}{v-u}\int_u^v \ln[X(t,\cdot)]\, dt\right) && (21) \\ \le\ & \frac{1}{v-u}\int_u^v G(X(t,\cdot), X(u+v-t,\cdot))\, dt && (22) \\ \le\ & \frac{1}{v-u}\int_u^v X(t,\cdot)\, dt \end{aligned}$$

holds for every strictly positive and integrable stochastic process $X : I \times \Omega \to \mathbb{R}$ and the inequality

$$\begin{aligned} &\exp\left[\frac{1}{v-u}\int_u^v \ln X(t,\cdot)\, dt\right] && (23) \\ \le\ & \frac{1}{v-u}\int_u^v \exp\left(\frac{X(t,\cdot) + X(u+v-t,\cdot)}{2}\right) dt && (24) \\ \le\ & \frac{1}{v-u}\int_u^v \exp X(t,\cdot)\, dt \end{aligned}$$

holds for every $X : T \times \Omega \to \mathbb{R}$ an integrable stochastic on $[u, v]$.

Taking into account that the above two inequalities hold, we can assert that for every $X : T \times \Omega \to (0, \infty)$ an integrable stochastic process on $[u, v]$ we have the inequalities:

$$\begin{aligned}
& \exp\left(\frac{1}{v-u}\int_u^v \ln X(t,\cdot)\,dt\right) \qquad (25)\\
\leq\ & \frac{1}{v-u}\int_u^v G\left(X(t,\cdot), X(u+v-t,\cdot)\right)dt\\
\leq\ & \frac{1}{v-u}\int_u^v X(t,\cdot)\,dt\\
\leq\ & \ln\left[\frac{1}{v-u}\int_u^v \exp A\left(X(t,\cdot), X(u+v-t,\cdot)\right)\right]dt\\
\leq\ & \ln\left[\frac{1}{v-u}\int_u^v \exp X(t,\cdot)\,dt\right],
\end{aligned}$$

which is of interest in itself.

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NUMERICAL METHODS FOR DISCONTINUOUS STURM-LIOUVILLE PROBLEMS

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Abstract − This study is devoted to determining the eigenvalues and eigenfunctions of a discontinuous Sturm-Liouville Problem. By modifying the finite difference method, we have developed a numerical approximation to the eigenvalues and eigenfunctions.

Keywords − *Sturm-Liouville, Discontinuous, Numerical Solution, Eigenvalues, Eigenfunctions, Transmission Conditions*

1 Introduction

Many physical systems are connected to a Sturm-Liouville problem. The computation of eigenvalues of Sturm-Liouville problems is therefore important to many problems in Mathematical Pyhsics. Sturm-Liouville systems arise from vibration problems in continuous media with non-uniform properties, such as the propagation of sonar in water and the seismic waves in the Earth [4]. In the classical sense, the Sturm-Liouville problem is replaced by a first order differential equation that is solved using Shooting methods. Added in this class are the Prüfer phase methods ([5] and [8]). In the recent years, pursued with considerable success by a number of researchers including Pruess ([9], [10]) and Paine and de Hoog [7], a simpler problem is constructed by replacing the coefficients in the Sturm-Liouville problem by piecewise constants. Anderssen and de Hoog [6] extend the results to the Liouville normal form with general boundary conditions. Moreover, some important results in this field have also been obtained for discontinuous Sturm-Liouville systems. It should be mentioned that O. Sh. Mukhtarov and his colleagues [1, 2, 3] have constructed boundary value problems with discontinuities where an eigenparameter appears not only in the differential equation, but also in the boundary and transmission conditions.

The main goal of this study is to extend the finite difference method to a particular discontinuous Sturm-Liouville problem.

Specifically, we shall compute approximations to the eigenvalues and eigenfunctions of a problem with the following transmission and boundary conditions:

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$$\begin{aligned} y^{''} + \lambda y &= 0 \quad x\epsilon[0,c) \cup (c,\pi] && (1)\\ y(0) &= y(\pi) = 0 && (2)\\ \gamma_1 y(c_+) &= \gamma_2 y(c_-) && (3) \end{aligned}$$

1.1 Finite Difference Method

Under certain conditions, we can use Taylor's formulae:

$$y(x+h) = y(x) + y'(x)h + \frac{y''(x)}{2!}h^2 + \cdots \quad (4)$$

and

$$y(x-h) = y(x) - y'(x)h + \frac{y''(x)}{2!}h^2 + \cdots . \quad (5)$$

Combining the two, we get

$$y(x+h) + y(x-h) \approx 2y(x) + y''(x)h^2, \quad (6)$$

discarding all higher-order terms. Thus, to approximate $y''(x)$, we can use the difference equation

$$y''(x) \approx \frac{y(x-h) - 2y(x) + y(x+h)}{h^2}. \quad (7)$$

Substituting this approximation into the Sturm-Liouville equation (1.1), we get

$$-\frac{y(x-h) - 2y(x) + y(x+h)}{h^2} \approx \lambda y(x). \quad (8)$$

We now partition the interval $[0, \pi]$ into $0 = x_0 < x_1 \ldots < x_N < x_{N+1} = \pi$ nodes, where $x_j = jh$ and $h = \frac{\pi}{N+1}$. We seek a solution to the Sturm-Liouville problem on these nodes. Naturally, we let

$$y(x_0) = y(0) = 0$$

and

$$y(x_{N+1}) = y(\pi) = 0$$

However, the solution is unknown at the interior points x_j, $j = 1, 2, \ldots, N$.

By evaluating the difference equation 8 at x_j, we obtain

$$-h^{-2}\big[y(x_j - h) - 2y(x_j) + y(x_j + h)\big] \approx \lambda y(x_j).$$

Because h is the increment between consecutive points on our partition, $x_j - h = x_{j-1}$ and $x_j + h = x_{j+1}$. Hence,

$$-h^{-2}\big[y(x_{j-1}) - 2y(x_j) + y(x_{j+1})\big] \approx \lambda y(x_j).$$

We now replace $y(x_j)$ with y_j and iterate the equation

$$-h^{-2}\big[y_{j-1} - 2y_j + y_{j+1}\big] = \lambda y_j \quad (9)$$

in order to find an approximate solution to the Sturm-Liouville problem.

It must be noted that $y(x_j)$ represents the *exact* value of the Sturm-Liouville solution evaluated at x_j, while the variable y_j represents the approximation to $y(x_j)$ at x_j.

For $j = 1$, equation 9 becomes

$$-h^{-2}\big[y_0 - 2y_1 + y_2\big] = \lambda y_1. \quad (10)$$

Similarly, for $j = 2$, 3, and 4, we get

$$\begin{aligned} -h^{-2}\big[y_1 - 2y_2 + y_3\big] &= \lambda y_2,\\ -h^{-2}\big[y_2 - 2y_3 + y_4\big] &= \lambda y_3,\\ -h^{-2}\big[y_3 - 2y_4 + y_5\big] &= \lambda y_4. \end{aligned}$$

Finally, for $j = N$, we get

$$-h^{-2}\big[y_{N-1} - 2y_N + y_{N+1}\big] = \lambda y_N. \tag{11}$$

However, the boundary conditions require

$$\begin{aligned} y_0 &= y(x_0) = y(0) = 0, \\ y_{N+1} &= y(x_{N+1}) = y(\pi) = 0, \end{aligned}$$

so equation 10 becomes $-h^{-2}\big[-2y_1 + y_2\big] = \lambda y_1$. Similarly, equation 11 becomes $-h^{-2}\big[y_{N-1} - 2y_N\big] = \lambda y_N$. The full set is a system of N equations in N unknowns.

$$\begin{array}{rcl} 2h^{-2}y_1 - h^{-2}y_2 & = & \lambda y_1, \\ -h^{-2}y_1 + 2h^{-2}y_2 - h^{-2}y_3 & = & \lambda y_2, \\ -h^{-2}y_2 + 2h^{-2}y_3 - h^{-2}y_4 & = & \lambda y_3, \\ -h^{-2}y_3 + 2h^{-2}y_4 - h^{-2}y_5 & = & \lambda y_4, \\ & \vdots & \\ -h^{-2}y_{N-1} + 2h^{-2}y_N & = & \lambda y_N. \end{array} \tag{12}$$

This system can be written in matrix form as

$$\begin{bmatrix} 2h^{-2} & -h^{-2} & 0 & 0 & \cdots & 0 \\ -h^{-2} & 2h^{-2} & -h^{-2} & 0 & \cdots & 0 \\ 0 & -h^{-2} & 2h^{-2} & -h^{-2} & \cdots & 0 \\ 0 & 0 & -h^{-2} & 2h^{-2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 2h^{-2} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ \vdots \\ y_N \end{bmatrix} = \lambda \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ \vdots \\ y_N \end{bmatrix}. \tag{13}$$

By denoting

$$M = \begin{bmatrix} 2h^{-2} & -h^{-2} & 0 & 0 & \cdots & 0 \\ -h^{-2} & 2h^{-2} & -h^{-2} & 0 & \cdots & 0 \\ 0 & -h^{-2} & 2h^{-2} & -h^{-2} & \cdots & 0 \\ 0 & 0 & -h^{-2} & 2h^{-2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 2h^{-2} \end{bmatrix} \quad \text{and} \quad \vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ \vdots \\ y_N \end{bmatrix}$$

the system (13) can be written in the form

$$M\vec{y} = \lambda\vec{y}.$$

1.2 Neumann Conditions

Let us consider the same Sturm-Liouville equation, but this time attach a Neumann condition at the right endpoint.

$$-y''(x) = \lambda y(x), \qquad y(0) = 0, \qquad y'(\pi) = 0 \tag{14}$$

Since we still have $-y''(x) = \lambda y(x)$, we can re-use equation (9), repeated here for convenience.

$$-h^{-2}\big[y_{j-1} - 2y_j + y_{j+1}\big] = \lambda y_j$$

Because of the Neumann condition $y'(\pi) = 0$, we do not know the value of the solution at the right endpoint (i.e., we do not know $y(\pi)$). Therefore, if we partition the interval as above, we will need to compute $y_1, y_2, \ldots, y_{N+1}$ instead of $y_1, y_2, \ldots, y_N$. Solving this system requires the addition of an extra equation for the variable y_{N+1}. The resulting system looks like this:

$$\begin{aligned}
-h^{-2}\big[y_0 - 2y_1 + y_2\big] &= \lambda y_1,\\
-h^{-2}\big[y_1 - 2y_2 + y_3\big] &= \lambda y_2,\\
-h^{-2}\big[y_2 - 2y_3 + y_4\big] &= \lambda y_3,\\
-h^{-2}\big[y_3 - 2y_4 + y_5\big] &= \lambda y_4,\\
-h^{-2}\big[y_4 - 2y_5 + y_6\big] &= \lambda y_5,\\
&\vdots\\
-h^{-2}\big[y_{N-2} - 2y_{N+1} + y_{N+2}\big] &= \lambda y_{N+1}.
\end{aligned} \tag{15}$$

However, this gives us $N+1$ equations in $N+3$ unknowns. Because of the Dirichlet condition at the left endpoint, we know that

$$y_0 = y(x_0) = y(0) = 0.$$

This condition eliminates the unknown y_0. However, we will need to obtain an estimate for y_{N+2}.

We can use a forward difference equation to approximate the first derivative:

$$y'(x) \approx \frac{y(x+h) - y(x)}{h}.$$

Using h as the step size in our partition of $[0,\pi]$, we have

$$y'(x_j) \approx \frac{y(x_{j+1}) - y(x_j)}{h}.$$

Using y_j as an approximation of $y(x_j)$, we can write

$$y'_j = \frac{y_{j+1} - y_j}{h}$$

or equivalently

$$y_{j+1} = y_j + hy'_j.$$

For $j = N+1$, this gives us the equation $y_{N+2} = y_{N+1} + hy'_{N+1}$. However, the Neumann condition on the right endpoint of $[0,\pi]$ is

$$y'_{N+1} = y'(x_{N+1}) = y'(\pi) = 0.$$

Thus, we get $y_{N+2} = y_{N+1}$.

Substituting $y_0 = 0$ and $y_{N+2} = y_{N+1}$ into the first and last equations respectively of system (15) gives us $N+1$ equations in $N+1$ unknowns:

$$\begin{aligned}
-h^{-2}\big[-2y_1 + y_2\big] &= \lambda y_1,\\
-h^{-2}\big[y_1 - 2y_2 + y_3\big] &= \lambda y_2,\\
-h^{-2}\big[y_2 - 2y_3 + y_4\big] &= \lambda y_3,\\
-h^{-2}\big[y_3 - 2y_4 + y_5\big] &= \lambda y_4,\\
-h^{-2}\big[y_4 - 2y_5 + y_6\big] &= \lambda y_5,\\
&\vdots\\
-h^{-2}\big[y_N - y_{N+1}\big] &= \lambda y_{N+1}.
\end{aligned}$$

This system can be written in matrix form.

$$\begin{bmatrix} 2h^{-2} & -h^{-2} & 0 & \cdots & 0 \\ -h^{-2} & 2h^{-2} & h^{-2} & \cdots & 0 \\ 0 & -h^{-2} & 2h^{-2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & h^{-2} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_{N+1} \end{bmatrix} = \lambda \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_{N+1} \end{bmatrix}. \tag{16}$$

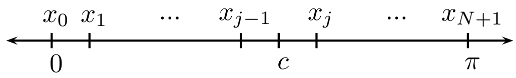

Figure 1: Partitioning the solution interval $[0;\pi]$

By denoting

$$N=\begin{bmatrix} 2h^{-2} & -h^{-2} & 0 & \cdots & 0 \\ -h^{-2} & 2h^{-2} & -h^{-2} & \cdots & 0 \\ 0 & -h^{-2} & 2h^{-2} & \cdots & 0 \\ 0 & 0 & -h^{-2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & h^{-2} \end{bmatrix} \quad \text{and} \quad \vec{y}=\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_{N+1} \end{bmatrix}$$

the system (16) reduced to the form

$$N\vec{y}=\lambda\vec{y}.$$

2 A Numerical Method for Solving Discontinuous Sturm-Liouville Problems

We now consider the Sturm-Liouville equation described in (1)- (3), which holds over the finite interval $[0,\pi]$ except at one inner point $c\in[0,\pi]$.

First, we partition the interval $[0,\pi]$ such that $x_{j-1}<c<x_j$ as in Figure 1. We then use equation (9) to find an approximate solution at those points to the left of c: $y_0,y_1,\ldots,y_{j-1}$ at $x_0,x_1,\ldots,x_{j-1}$ by . Once y_{j-1} is known, the transmission condition (3) gives us a value for y_j:

$$y_j=\frac{\gamma_2}{\gamma_1}y_{j-1}$$

at x_j, on the right-hand side of c. Finally, with y_j known, the values $y_{j+1},\ldots,y_n$ at $x_{j+1},\ldots,x_n$ can be calculated from equation (9).

To generalize this solution, consider the new variable Y_j:

$$Y_j=\begin{cases} y_j, & j\neq j^\star \\ \frac{\gamma_2}{\gamma_1}y_{j-1}, & j=j^\star \end{cases} \tag{17}$$

where

$$j^\star=min\{k|x_k>c\}.$$

System (12) then becomes

$$\begin{aligned} -h^{-2}\big[Y_0-2Y_1+Y_2\big]&=\lambda Y_1\\ -h^{-2}\big[Y_1-2Y_2+\frac{\gamma_1}{\gamma_2}Y_3\big]&=\lambda Y_2\\ -h^{-2}\big[\frac{\gamma_2}{\gamma_1}Y_2-2Y_3+\frac{\gamma_2}{\gamma_1}Y_4\big]&=\lambda Y_3\\ -h^{-2}\big[\frac{\gamma_1}{\gamma_2}Y_3-2Y_4+Y_5\big]&=\lambda Y_4\\ -h^{-2}\big[Y_4-2Y_5+Y_6\big]&=\lambda Y_5\\ &\vdots\\ -h^{-2}\big[Y_{N-1}-2Y_N+Y_{N+1}\big]&=\lambda Y_N. \end{aligned}$$

for $j = 1, \ldots, N$. The matrix form of this system is written as follows:

$$h^{-2}\begin{bmatrix} 2 & -1 & 0 & 0 & 0 & \cdots & 0 \\ -1 & 2 & -1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 2 & -\frac{\gamma_1}{\gamma_2} & 0 & \cdots & 0 \\ 0 & 0 & -\frac{\gamma_2}{\gamma_1} & 2 & -\frac{\gamma_2}{\gamma_1} & \cdots & 0 \\ 0 & 0 & 0 & -\frac{\gamma_1}{\gamma_2} & 2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 2 \end{bmatrix}\begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \\ Y_4 \\ Y_5 \\ \vdots \\ Y_N \end{bmatrix} = \lambda \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \\ Y_4 \\ Y_5 \\ \vdots \\ Y_N \end{bmatrix}. \tag{18}$$

3 Numerical Illustration

Example 3.1. For $N = 30$, $\frac{\gamma_1}{\gamma_2} = \frac{1}{2}$ and $x_2 < c < x_3$, we consider the discontinuous Sturm-Liouville problem:

$$-y''(x) = \lambda y(x) \quad x\epsilon[0, c) \cup (c, \pi] \tag{19}$$

$$y(0) = y(\pi) = 0$$

$$\gamma_2 y(c_+) = \gamma_1 y(c_-)$$

By applying the transformation (17) we obtain the following system of linear equations:

$$\begin{aligned} -h^{-2}\big[Y_0 - 2Y_1 + Y_2\big] &= \lambda Y_1 \\ -h^{-2}\big[Y_1 - 2Y_2 + \frac{\gamma_1}{\gamma_2}Y_3\big] &= \lambda Y_2 \\ -h^{-2}\big[\frac{\gamma_2}{\gamma_1}Y_2 - 2Y_3 + \frac{\gamma_2}{\gamma_1}Y_4\big] &= \lambda Y_3 \\ -h^{-2}\big[\frac{\gamma_1}{\gamma_2}Y_3 - 2Y_4 + Y_5\big] &= \lambda Y_4 \\ -h^{-2}\big[Y_4 - 2Y_5 + Y_6\big] &= \lambda Y_5 \\ &\vdots \\ -h^{-2}\big[Y_{28} - 2Y_{29} + Y_{31}\big] &= \lambda Y_{30} \end{aligned}$$

This system can be written as the following matrix equation:

$$h^{-2}\begin{bmatrix} 2 & -1 & 0 & 0 & 0 & \cdots & 0 \\ -1 & 2 & -1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 2 & -\frac{\gamma_1}{\gamma_2} & 0 & \cdots & 0 \\ 0 & 0 & -\frac{\gamma_2}{\gamma_1} & 2 & -\frac{\gamma_2}{\gamma_1} & \cdots & 0 \\ 0 & 0 & 0 & -\frac{\gamma_1}{\gamma_2} & 2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 2 \end{bmatrix}\begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \\ Y_4 \\ Y_5 \\ \vdots \\ Y_{30} \end{bmatrix} = \lambda \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \\ Y_4 \\ Y_5 \\ \vdots \\ Y_{30} \end{bmatrix}. \tag{20}$$

For this system, the following MatLAB commands generate approximations to the eigenvalue and the eigenfunctions, as shown in Figure 2.
v =

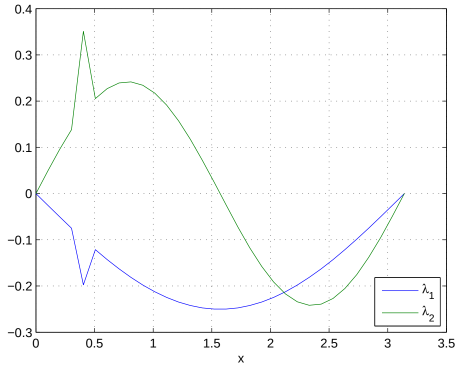

Figure 2: The first two eigenfunctions

Columns 1	through 5					Columns 29	through 30
-0.0253	0.0487	0.0703	-0.0917	0.1146	...	-0.2098	-0.2036
-0.0504	0.0954	0.1341	-0.1685	0.2005	...	0.1848	0.2154
-0.0749	0.1382	0.1857	-0.2181	0.2359	...	0.0470	-0.0243
-0.1974	0.3508	0.4404	-0.4645	0.4242	...	-0.4525	-0.3794
-0.1215	0.2054	0.2345	-0.2088	0.1350	...	0.1522	0.2250
-0.1430	0.2269	0.2273	-0.1515	0.0239	...	0.0922	-0.0483
-0.1630	0.2392	0.1993	-0.0696	-0.0932	...	-0.2334	-0.1739
-0.1814	0.2417	0.1529	0.0235	-0.1868	...	0.1134	0.2322
-0.1979	0.2343	0.0926	0.1129	-0.2335	...	0.1335	-0.0718
-0.2124	0.2173	0.0238	0.1839	-0.2215	...	-0.2310	-0.1563
-0.2247	0.1914	-0.0473	0.2251	-0.1539	...	0.0700	0.2371
-0.2347	0.1576	-0.1140	0.2299	-0.0476	...	0.1694	-0.0946
-0.2423	0.1174	-0.1702	0.1973	6 0.0707	...	-0.2191	-0.1370
-0.2474	0.0724	-0.2108	0.1328	0.1712	...	0.0236	0.2396
-0.2499	0.0245	-0.2321	0.0468	0.2287	...	0.1983	-0.1164
-0.2499	-0.0245	-0.2321	-0.0468	0.2287	...	-0.1983	-0.1164
-0.2474	-0.0724	-0.2108	-0.1328	0.1712	...	-0.0236	0.2396
-0.2423	-0.1174	-0.1702	-0.1973	0.0707	...	0.2191	-0.1370
-0.2347	-0.1576	-0.1140	-0.2299	-0.0476	...	-0.1694	-0.0946
-0.2247	-0.1914	-0.0473	-0.2251	-0.1539	...	-0.0700	0.2371
-0.2124	-0.2173	0.0238	-0.1839	-0.2215	...	0.2310	-0.1563
-0.1979	-0.2343	0.0926	-0.1129	-0.2335	...	-0.1335	-0.0718
-0.1814	-0.2417	0.1529	-0.0235	-0.1868	...	-0.1134	0.2322
-0.1630	-0.2392	0.1993	0.0696	-0.0932	...	0.2334	-0.1739
-0.1430	-0.2269	0.2273	0.1515	0.0239	...	-0.0922	-0.0483
-0.1215	-0.2054	0.2345	0.2088	0.1350	...	-0.1522	0.2250
-0.0987	-0.1754	0.2202	0.2322	0.2121	...	0.2262	-0.1897
-0.0749	-0.1382	0.1857	0.2181	0.2359	...	-0.0470	-0.0243
-0.0504	-0.0954	0.1341	0.1685	0.2005	...	-0.1848	0.2154
-0.0253	-0.0487	0.0703	0.0917	0.1146	...	0.2098	-0.2036

The eigenvalues are

n	1	2	3	.....	29	30
λ_n	0.9991	3.9863	8.9309	....	385.4923	388.4795

The first two eigenfunctions are shown in Figure 2.

Example 3.2. For $N = 30, \frac{\gamma_1}{\gamma_2} = \frac{1}{2}$ and $x_2 < c < x_3$, we consider the same discontinuous Sturm-Liouville problem with a Neumann boundary condition:

$$-y''(x) = \lambda y(x) \quad x \in [0, c) \cup (c, \pi] \tag{21}$$

$$y(0) = y^{'}(\pi) = 0$$
$$\gamma_2 y(c_+) = \gamma_1 y(c_-)$$

By applying the transformation (17) we obtain the following system of linear equations:

$$\begin{aligned}
-h^{-2}\big[-2Y_1 + Y_2\big] &= \lambda Y_1 \\
-h^{-2}\big[Y_1 - 2Y_2 + \frac{\gamma_1}{\gamma_2}Y_3\big] &= \lambda Y_2 \\
-h^{-2}\big[\frac{\gamma_2}{\gamma_1}Y_2 - 2Y_3 + \frac{\gamma_2}{\gamma_1}Y_4\big] &= \lambda Y_3 \\
-h^{-2}\big[\frac{\gamma_1}{\gamma_2}Y_3 - 2Y_4 + Y_5\big] &= \lambda Y_4 \\
-h^{-2}\big[Y_4 - 2Y_5 + Y_6\big] &= \lambda Y_5 \\
&\vdots \\
-h^{-2}\big[Y_{30} + Y_{31}\big] &= \lambda Y_{31}
\end{aligned}$$

The matrix form is

$$h^{-2}\begin{bmatrix} 2 & -1 & 0 & 0 & 0 & 0 & \cdots & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 2 & -\frac{\gamma_1}{\gamma_2} & 0 & 0 & \cdots & 0 \\ 0 & 0 & -\frac{\gamma_2}{\gamma_1} & 2 & -\frac{\gamma_2}{\gamma_1} & 0 & \cdots & 0 \\ 0 & 0 & 0 & -\frac{\gamma_1}{\gamma_2} & 2 & -1 & \cdots & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 1 \end{bmatrix}\begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \\ Y_4 \\ Y_5 \\ Y_6 \\ \vdots \\ Y_{31} \end{bmatrix} = \lambda\begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \\ Y_4 \\ Y_5 \\ Y_6 \\ \vdots \\ Y_{31} \end{bmatrix}. \quad (22)$$

The following commands were used to generate the eigenvalues and eigenfunctions.

v =

Columns 1	through 5					Columns 29	through 30
0.0257	-0.0483	-0.0673	0.0844	0.1018	...	0.0468	0.0248
0.0511	-0.0945	-0.1282	0.1548	0.1773	...	-0.0917	-0.0494
0.2281	-0.4102	-0.5310	0.5979	0.6209	...	0.3989	0.2205
0.1001	-0.1732	-0.2091	0.2105	0.1831	...	-0.1689	-0.0968
0.1232	-0.2023	-0.2213	0.1866	0.1120	...	0.1979	0.1192
0.1449	-0.2229	-0.2127	0.1316	0.0119	...	-0.2190	-0.1404
0.1651	-0.2341	-0.1839	0.0545	-0.0913	...	0.2313	0.1601
0.1835	-0.2353	-0.1377	-0.0316	-0.1709	...	-0.2342	-0.1782
0.2000	-0.2267	-0.0785	-0.1125	-0.2063	...	0.2278	0.1946
0.2144	-0.2084	-0.0119	-0.1745	-0.1884	...	-0.2121	-0.2089
0.2265	-0.1814	0.0558	-0.2075	-0.1218	...	0.1879	0.2212
0.2362	-0.1466	0.1183	-0.2058	-0.0237	...	-0.1561	-0.2312
0.2434	-0.1057	0.1696	-0.1697	0.0805	...	0.1180	0.2389
0.2480	-0.0603	0.2048	-0.1053	0.1639	...	-0.0752	-0.2442
0.2500	-0.0124	0.2207	-0.0234	0.2049	...	0.0294	0.2470
0.2493	0.0361	0.2157	0.0625	0.1930	...	0.0294	0.2470
0.2460	0.0830	0.1903	0.1379	0.1312	...	-0.0640	0.2451
0.2401	0.1264	0.1469	0.1904	0.0355	...	0.1077	-0.2404
0.2316	0.1645	0.0896	0.2110	-0.0694	...	-0.1472	0.2333
0.2207	0.1956	0.0238	0.1964	-0.1564	...	0.1806	-0.2239
0.2075	0.2185	-0.0442	0.1490	-0.2029	...	-0.2068	0.2122
0.1920	0.2321	-0.1081	0.0768	-0.1970	...	0.2247	-0.1984
0.1746	0.2360	-0.1617	-0.0083	-0.1402	...	-0.2335	0.1825
0.1552	0.2298	-0.2000	-0.0920	-0.0471	...	0.2329	-0.1648
0.1342	0.2140	-0.2194	-0.1603	0.0581	...	-0.2229	0.1454
0.1118	0.1890	-0.2181	-0.2019	0.1483	...	0.2040	-0.1246
0.0883	0.1562	-0.1961	-0.2097	0.2002	...	-0.1768	0.1025
0.0637	0.1167	-0.1556	-0.1826	0.2003	...	0.1425	-0.0794
0.0385	0.0722	-0.1003	-0.1249	0.1487	...	-0.1024	0.0555
0.0129	0.0248	-0.0356	-0.0464	0.0586	...	0.0583	-0.0310
-0.0128	-0.0238	0.0325	0.0398	-0.0466	...	-0.0117	0.0062

The eigenvalues are

n	1	2	3	.....	29	30
λ_n	1.032	4.1148	9.2101	....	385.5555	388.4954

The first two eigenfunctions are shown in Figure 3.

4 Conclusion

We have described a numerical method for approximating the eigenvalues and eigenfunctions of a discontinuous Sturm-Liouville problem. Since the present method gives real and positive eigenvalues at all nodes, if higher accuracy is required we can simply increase the number of nodes N.

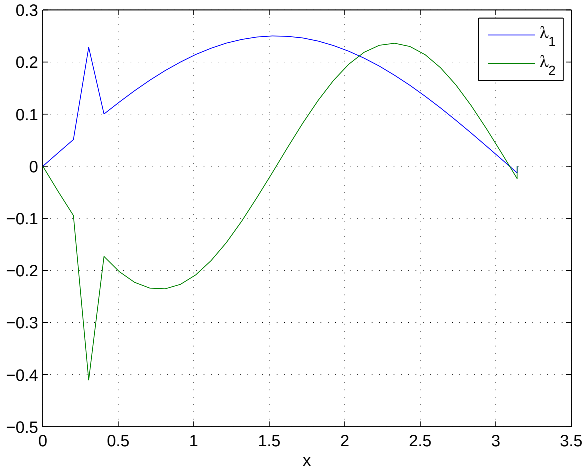

Figure 3: The first two eigenfunctions with a Neumann boundary condition

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RECENT RESULTS ON THE DOMAIN OF THE SOME LIMITATION METHODS IN THE SEQUENCE SPACES f_0 AND f

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Abstract − In the present paper, we summarize the literature on the sequence spaces almost $A-$null and almost $A-$convergent derived by using the domain of the certain $A-$ limitation matrix. Moreover, we introduce the spaces $\bar{f}(r,s,t)$ and $\bar{f}_0(r,s,t)$ and examine some properties of this spaces.

Keywords − *Almost convergence, Matrix domain, Generalized means, Matrix transformations.*

1 Introduction

By a *sequence space*, we mean any vector subspace of ω, the space of all real or complex valued sequences $x = (x_k)$. The well-known sequence spaces that we shall use throughout this paper are as following:

ℓ_∞: the space of all bounded sequences,
c: the space of all convergent sequences,
c_0: the space of all null sequences,
bs: the space of all sequences which forms bounded series,
cs: the space of all sequences which forms convergent series,
ℓ_1: the space of all sequences which forms absolutely convergent series,
ℓ_p: the space of all sequences which forms p-absolutely convergent series,
where $1 < p < \infty$.

Let λ, μ be two sequence spaces and $A = (a_{nk})$ be an infinite matrix of real or complex numbers a_{nk}, where $n, k \in \mathbb{N}$. Then, we say that A defines a matrix mapping from λ into μ, and we denote it by writing $A : \lambda \to \mu$, if for every sequence $x = (x_k) \in \lambda$ the sequence $Ax = \{(Ax)_n\}$, the A-transform of x, is in μ; where

$$(Ax)_n = \sum_k a_{nk}x_k, \quad (n \in \mathbb{N}). \tag{1}$$

For simplicity in notation, here and in what follows, the summation without limits runs from 0 to ∞. By $(\lambda : \mu)$, we denote the class of all matrices A such that $A : \lambda \to \mu$. Thus, $A \in (\lambda : \mu)$ if and only if the series on the right side of (1) converges for each $n \in \mathbb{N}$ and every $x \in \lambda$, and we have

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$Ax = \{(Ax)_n\}_{n\in\mathbb{N}} \in \mu$ for all $x \in \lambda$. A sequence x is said to be A-summable to α if Ax converges to α which is called as the A-limit of x.

If a normed sequence space λ contains a sequence (b_n) with the property that for every $x \in \lambda$ there is a unique sequence of scalars (α_n) such that

$$\lim_{n\to\infty} \|x - (\alpha_0 b_0 + \alpha_1 b_1 + ... + \alpha_n b_n)\| = 0,$$

then (b_n) is called a *Schauder basis* (or briefly *basis*) for λ. The series $\sum \alpha_k b_k$ which has the sum x is then called the expansion of x with respect to (b_n), and written as $x = \sum \alpha_k b_k$.

The $\beta-$dual of a subset X of ω is defined by

$$X^\beta = \{a = (a_k) \in \omega : ax = (a_k x_k) \in cs \text{ for all } x = (x_k) \in X\}.$$

The shift operator P is defined on ω by $(Px)_n = x_{n+1}$ for all $n \in \mathbb{N}$. A Banach limit L is defined on ℓ_∞, as a non-negative linear functional, such that $L(Px) = L(x)$ and $L(e) = 1$. A sequence $x = (x_k) \in \ell_\infty$ is said to be *almost convergent* to the generalized limit α if all Banach limits of x are α [1], and denoted by $f - \lim x_k = \alpha$. Let P^j be the composition of P with itself j times and define $t_{mn}(x)$ for a sequence $x = (x_k)$ by

$$t_{mn}(x) = \frac{1}{m+1}\sum_{j=0}^{m}(P^j x)_n \text{ for all } m, n \in \mathbb{N}.$$

Lorentz [1] proved that $f - \lim x_k = \alpha$ if and only if $\lim_{m\to\infty} t_{mn}(x) = \alpha$, uniformly in n. It is well-known that a convergent sequence is almost convergent such that its ordinary and generalized limits are equal. By f and f_0, we denote the space of all almost convergent sequences and almost convergent to zero sequences, respectively, i.e.,

$$f = \left\{x = (x_k) \in \omega : \exists \alpha \in \mathbb{C} \ni \lim_{m\to\infty}\sum_{k=0}^{m}\frac{x_{n+k}}{m+1} = \alpha \text{ uniformly in } n\right\}$$

and

$$f_0 = \left\{x = (x_k) \in \omega : \lim_{m\to\infty}\sum_{k=0}^{m}\frac{x_{n+k}}{m+1} = 0 \text{ uniformly in } n\right\}.$$

A matrix $A = (a_{nk})$ is called a *triangle* if $a_{nk} = 0$ for $k > n$ and $a_{nn} \neq 0$ for all $n \in \mathbb{N}$. It is trivial that $A(Bx) = (AB)x$ holds for triangle matrices A, B and a sequence x. Further, a triangle matrix U uniquely has an inverse $U^{-1} = V$ that is also a triangle matrix. Then, $x = U(Vx) = V(Ux)$ holds for all $x \in \omega$. We write by $\mathcal{U}$ and $\mathcal{U}_0$ for the sets of all sequences with non-zero terms and non-zero first terms, respectively. For $u \in \mathcal{U}$, let $1/u = (1/u_n)$.

Let us give the definition of some triangle limitation matrices which are needed in the text. Let $q = (q_k)$ be a sequence of positive reals and write

$$Q_n = \sum_{k=0}^{n} q_k, \quad (n \in \mathbb{N}).$$

Then the Cesàro mean of order one, Riesz mean with respect to the sequence $q = (q_k)$ and A^r- mean with $0 < r < 1$ are respectively defined by the matrices $C_1 = (c_{nk})$, $R^q = (r^q_{nk})$ and $A^r = (a^r_{nk})$; where

$$c_{nk} = \begin{cases} \frac{1}{n+1}, & (0 \le k \le n), \\ 0, & (k > n), \end{cases} \qquad r^q_{nk} = \begin{cases} \frac{q_k}{Q_n}, & (0 \le k \le n), \\ 0, & (k > n), \end{cases}$$

and

$$a^r_{nk} = \begin{cases} \frac{1+r^k}{1+n}, & (0 \le k \le n), \\ 0, & (k > n), \end{cases}$$

for all $k, n \in \mathbb{N}$. Additionally, the Euler mean of order r and the weighted mean matrix and the double band matrix are respectively defined by the matrices $E^r = (e^r_{nk})$, $G(u,v) = (g_{nk})$ and $B(r,s) = \{b_{nk}(r,s)\}$; where

$$e^r_{nk} = \begin{cases} \binom{n}{k}(1-r)^{n-k} r^k, & (0 \le k \le n) \\ 0, & (k > n) \end{cases} \quad \text{and} \quad g_{nk} = \begin{cases} u_n v_k, & (0 \le k \le n), \\ 0, & (k > n), \end{cases}$$

and

$$b_{nk}(r,s) = \begin{cases} r, & (k = n), \\ s, & (k = n-1), \\ 0, & \text{otherwise} \end{cases}$$

for all $k, n \in \mathbb{N}$ and $u, v \in \mathcal{U}$ and $r, s \in \mathbb{R} \backslash \{0\}$.

For a sequence space λ, the *matrix domain* λ_A of an infinite matrix A is defined by

$$\lambda_A = \{x = (x_k) \in \omega : Ax \in \lambda\}, \tag{2}$$

which is a sequence space. Although in the most cases the new sequence space λ_A generated in the limitation matrix A from a sequence space λ is the expansion or the contraction of the original space λ, it may be observed in some cases that those spaces overlap. Indeed, one can deduce that the inclusions $\lambda_S \subset \lambda$ strictly holds for $\lambda \in \{\ell_\infty, c, c_0\}$. As this, one can deduce that the inclusions $\ell_p \subset bv_p$ and $\lambda \subset \lambda_{\Delta^1}$ also strictly hold for $\lambda \in \{c, c_0\}$, where $1 \le p \le \infty$ and the space $(\ell_p)_{\Delta^{(1)}} = bv_p$ has been studied by Başar and Altay [2], (see also Çolak and Et and Malkowsky [3]). However, if we define $\lambda = c_0 \oplus z$ with $z = ((-1)^k)$, that is, $x \in \lambda$ if and only if $x = s + \alpha z$ for some $s \in c_0$ and some $\alpha \in \mathbb{C}$, and consider the matrix A with the rows A_n defined by $A_n = (-1)^n e^{(n)}$ for all $n \in \mathbb{N}$, we have $Ae = z \in \lambda$ but $Az = e \notin \lambda$ which lead us to the consequences that $z \in \lambda \setminus \lambda_A$ and $e \in \lambda_A \setminus \lambda$, where $e^{(n)}$ denotes the sequence whose only non-zero term is a 1 in n^{th} place for each $n \in \mathbb{N}$ and $e = (1,1,1,...)$. That is to say that the sequence spaces λ_A and λ overlap but neither contains the other. The approach constructing a new sequence space by means of the matrix domain of a particular limitation method has been recently employed by Wang [4], Ng and Lee [5], Aydın and Başar [6], Altay and Başar [7], and Altay et all. [8]. They introduced the sequence spaces $(\ell_\infty)_{N_q}$ and c_{N_q} in [4], $(\ell_p)_{C_1} = X_p$ in [5], $(c_0)_{A^r} = a^r_0$ and $c_{A^r} = a^r_c$ in [6], $(c_0)_{E^r} = e^r_0$ and $c_{E^r} = e^r_c$ in [7], $(\ell_p)_{E^r} = e^r_p$ and $(\ell_\infty)_{E^r} = e^r_\infty$ in [8]; where $1 \le p < \infty$.

In this study, we summarize some knowledge in the existing literature on the almost $A-$null and almost $A-$convergent sequence spaces derived by using the domain $A-$limitation matrix. Additionally, we introduce the new sequence spaces $\bar{f}_0(r,s,t)$ and $\bar{f}(r,s,t)$ and examine some properties of these sequence spaces.

2 Domain of the $A-$limitation matrix in the sequence spaces f_0 and f

In this section, we shortly give the knowledge on the sequence spaces derived by the $A-$limitation matrix from well-known almost convergent and almost null sequence spaces. For the concerning literature about the domain μ_A of an infinite limitation matrix A in a sequence space μ, Table 1 may be useful.

μ	A	μ_A	refer to
f_0, f	$B(r,s)$	$\widehat{f}, \widehat{f_0}$	[9]
f_0, f	C_1	$\widetilde{f}, \widetilde{f_0}$	[14]
f_0, f	R^q	$f_{R^q}, \{f_0\}_{R^q}$	[15]
f_0, f	A^r	$a^r_f, a^r_{f_0}$	[16]
f_0, f	$G(u,v)$	$f_0(G), f(G)$	[17]
f_0, f	E^r	$f(E), f_0(E)$	[18]
f_0, f	$B(r,s,t)$	$f(B), f_0(B)$	[19]
f_0, f	A_λ	$A_\lambda(f_0), A_\lambda(f)$	[20]

Table 1: The domains of the certain $A-$limitation matrix in the sequence spaces f_0 and f

The matrix domain of a certain limitation method on the sequence spaces f_0 and f firstly were studied by Başar and Kirişçi [9].

Başar and Kirişçi introduced the sequence spaces $\widehat{f_0}$ and $\widehat{f}$ in [9] as follows:

$$\widehat{f_0}: = \left\{x=(x_k)\in\omega : \lim_{m\to\infty}\sum_{j=0}^{m}\frac{sx_{k-1+j}+rx_{k+j}}{m+1}=0 \quad \text{uniformly in} \quad k\right\},$$
$$\widehat{f}: = \left\{x=(x_k)\in\omega : \exists\alpha\in\mathbb{C}\ni \lim_{m\to\infty}\sum_{j=0}^{m}\frac{sx_{k-1+j}+rx_{k+j}}{m+1}=\alpha \quad \text{uniformly in} \quad k\right\}.$$

It is trivial that the sequence spaces $\widehat{f_0}$ and $\widehat{f}$ are the domain of the matrix $B(r,s)$ in the spaces f_0 and f, respectively. Thus, with the notation of (2) we can redefine the spaces $\widehat{f_0}$ and $\widehat{f}$ by

$$\widehat{f_0} := \{f_0\}_{B(r,s)} \qquad \text{and} \qquad \widehat{f} := \{f\}_{B(r,s)}.$$

Define the sequence $y=(y_k)$ by the $B(r,s)-$transform of a sequence $x=(x_k)$, i.e.,

$$y_k := sx_{k-1}+rx_k \quad \text{for all} \quad k\in\mathbb{N}. \tag{3}$$

Since the matrix $B(r,s)$ is triangle, one can easily observe that $x=(x_k)\in\widehat{X}$ if and only if $y=(y_k)\in X$, where the sequences $x=(x_k)$ and $y=(y_k)$ are connected with the relation (3), and X denotes any of the sequence spaces f_0 and f. Therefore, one can easily see that the linear operator $T:\widehat{X}\to X$, $Tx=y=B(r,s)x$ which maps every sequence x in $\widehat{X}$ to the associated sequence y in X, is bijective and norm preserving, where $\|x\|_{\widehat{X}}=\|B(r,s)x\|_X$. This gives the fact that $\widehat{X}$ and X are norm isomorphic.

Başar and Kirişçi [9] proved that the sequence space f is a $BK-$space with the norm $\|.\|_\infty$ and non-separable closed subspace of ℓ_∞. So, the sequence space f has no Schauder basis. Jarrah and Malkowsky [12] showed that the matrix domain λ_A of a normed sequence space λ has a basis whenever $A=(a_{nk})$ is triangle. Then; our corollary concerning the space $\widehat{f_0}$ and $\widehat{f}$ is about their Schauder basis:

Corollary 2.1. [9, Corollary 4.2] The space $\widehat{f}$ has no Schauder basis.

The gamma- and beta-duals of the spaces $\widehat{f_0}$ and $\widehat{f}$ are determined. Also, some matrix transformations on these sequence spaces are characterized.

Quite recently, E. E. Kara and K. Elmaağaç [21] introduced the sequence space $\hat{c}^u$ as follows:

$$\hat{c}^u = \left\{x=(x_k)\in\omega : \exists\alpha\in\mathbb{C}\ni \lim_{m\to\infty}\sum_{j=0}^{m}\frac{u_{k+j}x_{k+j}+u_{k-1+j}x_{k-1+j}}{m+1}=\alpha \quad \text{uniformly in} \quad k\right\}.$$

It is trivial that the sequence space $\hat{c}^u$ is the domain of the matrix $A^u=(a^u_{nk})$ in the space f, where the matrix $A^u=(a^u_{nk})$ is defined by

$$a^u_{nk} = \begin{cases} (-1)^{n-k}u_k, & n-1\le k\le n, \\ 0, & 0\le k<n-1 \text{ or } k>n, \end{cases}$$

for all $k,n\in\mathbb{N}$. Also, they show that $\hat{c}^u$ is linearly isomorphic to the space $\hat{c}$. Further, they compute the $\beta-$dual of the space $\hat{c}^u$ and characterize the classes of infinite matrices related to sequence space $\hat{c}^u$.

3 Spaces of $\bar{A}(r,s,t)-$almost null and $\bar{A}(r,s,t)-$almost convergent sequences

In this section, we study some properties of the spaces of the $\bar{A}(r,s,t)-$almost null and $\bar{A}(r,s,t)-$almost convergent sequences.

For any sequences $s, t \in \omega$, the convolution $s * t$ is a sequence defined by

$$(s * t)_n = \sum_{k=0}^{n} s_{n-k} t_k; \quad (n \in \mathbb{N}).$$

Throughout this section, let $r, t \in \mathcal{U}$ and $s \in \mathcal{U}_0$. For any sequence $x = (x_n) \in \omega$, we define the sequence $\bar{x} = (\bar{x}_n)$ of generalized means of x by

$$\overline{x}_n = \frac{1}{r_n} \sum_{k=0}^{n} s_{n-k} t_k x_k; \quad (n \in \mathbb{N}), \tag{4}$$

that is $\overline{x}_n = (s * tx)_n / r_n$ for all $n \in \mathbb{N}$. Further, we define the infinite matrix $\bar{A}(r, s, t)$ of generalized means by

$$\{\bar{A}(r,s,t)\}_{nk} = \begin{cases} \dfrac{s_{n-k} t_k}{r_n}, & 0 \leq k \leq n, \\ 0, & k > n \end{cases} \tag{5}$$

for all $n, k \in \mathbb{N}$. Then, it follows by (4) that $\bar{x}$ is the $\bar{A}(r, s, t)-$transform of x, that is $\bar{x} = (\bar{A}(r, s, t)x)$ for all $x \in \omega$.

It is obvious by (5) that $\bar{A}(r, s, t)$ is a triangle. Moreover, it can easily be seen that $\bar{A}(r, s, t)$ is regular if and only if $s_{n-i} = o(r_n)$ for each $i \in \mathbb{N}$, $\sum_{k=0}^{n} |s_{n-k} t_k| = O(|r_n|)$ and $(s*t)_n / r_n \to 1$ $(n \to \infty)$.

The above definition of the matrix $\bar{A}(r, s, t)$ of generalized means given by (5) includes the following special cases:

(1) If $r_n = (s*t)_n \neq 0$ for all n, then $\bar{A}(r, s, t)$ reduces to the matrix (N, s, t) of generalized Nörlund means [22, 23]. In particular, if $t = e$ then $\bar{A}(r, s, t)$ reduces to the familiar matrix of Nörlund means [30, 4].

(2) If $\alpha > 0$, $r_k = \frac{\Gamma(\alpha+k+1)}{k!\Gamma(\alpha+1)}$, $s_k = \frac{\Gamma(\alpha+k)}{k!\Gamma(\alpha)}$ and $t_k = 1$ for all k, then $\bar{A}(r, s, t)$ reduces to the matrix (C, α) of Cesàro means of order α [24, 25]. In particular, if $\alpha = 1$ then $\bar{A}(r, s, t)$ reduces to the famous matrix $(C, 1)$ of arithmetic means [5, 26].

(3) If $0 < \alpha < 1$, $r_k = \frac{1}{k!}$, $s_k = \frac{(1-\alpha)^k}{k!}$ and $t_k = \frac{\alpha^k}{k!}$ for all k, then $\bar{A}(r, s, t)$ reduces to the matrix (E, α) of Euler means of order α [7, 10, 8].

(4) If $t_n > 0$ and $r_n = \sum_{k=0}^{n} t_k$ for all n, then $\bar{A}(r, s, t)$ reduces to the matrix $(\overline{N}, t)$ of weighted means [12, 27].

(5) If $0 < \alpha < 1$, $r_k = k + 1$, $s_k = 1$ and $t_k = 1 + \alpha^k$ for all k, then $\bar{A}(r, s, t)$ reduces to the matrix A^α studied by Aydın and Başar [6, 28].

(6) If $s = e^{(0)}$ and $t = e$, then $\bar{A}(r, s, t)$ reduces to the diagonal matrix $D_{1/r}$ studied by de Malafosse [29].

Now, since $\bar{A}(r, s, t)$ is a triangle, it has a unique inverse which is also a triangle. More precisely, by making a slight generalization of a work done in [30], we put $D_0^{(s)} = 1/s_0$ and

$$D_n^{(s)} = \frac{1}{s_0^{n+1}} \begin{vmatrix} s_1 & s_0 & 0 & 0 & \cdots & 0 \\ s_2 & s_1 & s_0 & 0 & \cdots & 0 \\ s_3 & s_2 & s_1 & s_0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ s_{n-1} & s_{n-2} & s_{n-3} & s_{n-4} & \cdots & s_0 \\ s_n & s_{n-1} & s_{n-2} & s_{n-3} & \cdots & s_1 \end{vmatrix}; \qquad (n = 1, 2, ...).$$

Then the inverse of $\bar{A}(r, s, t)$ is the triangle $\bar{B} = (\bar{b}_{nk})_{n,k=0}^{\infty}$ defined by

$$\bar{b}_{nk} = \begin{cases} (-1)^{n-k} D_{n-k}^{(s)} r_k \dfrac{1}{t_n}, & (0 \leq k \leq n), \\ 0, & (k > n), \end{cases}$$

for all $n, k \in \mathbb{N}$. For an arbitrary subset X of ω, the set $X(r, s, t)$ has recently been introduced in [31] as the matrix domain of the triangle $\bar{A}(r, s, t)$ in X.

We introduce the sequence spaces $\bar{f}(r, s, t)$ and $\bar{f}_0(r, s, t)$ as the sets of all sequences whose $\bar{A}(r, s, t)-$transforms are in the spaces f_0 and f, that is

$$\bar{f}_0(r,s,t) = \left\{x=(x_k)\in\omega : \lim_{m\to\infty}\frac{1}{m+1}\sum_{j=0}^{m}\sum_{k=0}^{n+j}\frac{s_{n+j-k}t_kx_k}{r_n}=0 \text{ uniformly in } n\right\},$$
$$\bar{f}(r,s,t) = \left\{x=(x_k)\in\omega : \exists l\in\mathbb{C}\ni \lim_{m\to\infty}\frac{1}{m+1}\sum_{j=0}^{m}\sum_{k=0}^{n+j}\frac{s_{n+j-k}t_kx_k}{r_n}=l \text{ uniformly in } n\right\}.$$

With the notation of (2), we can redefine the spaces $\bar{f}(r,s,t)$ and $\bar{f}_0(r,s,t)$ as follows:

$$\bar{f}(r,s,t)=\{f\}_{\bar{A}(r,s,t)} \qquad \text{and} \qquad \bar{f}_0(r,s,t)=\{f_0\}_{\bar{A}(r,s,t)}.$$

It is worth mentioning that the general forms of the well-known matrices of Nörlund, Cesàro, Euler and weighted means can be obtained as special cases of the matrix $\bar{A}(r,s,t)$ of generalized means. Therefore, all of the sequence spaces in Tablo 1 can be obtained by special choice from the sequence spaces $\bar{f}(r,s,t)$ and $\bar{f}_0(r,s,t)$ which are defined by using matrix domain of the matrix $\bar{A}(r,s,t)$.

Theorem 3.1. The sequence spaces $\bar{f}(r,s,t)$ and $\bar{f}_0(r,s,t)$ are $BK-$spaces with the same norm given by

$$\|x\|_{\bar{f}(r,s,t)}=\|\bar{A}(r,s,t)x\|_f=\sup_{m,n\in\mathbb{N}}|t_{mn}(\bar{A}(r,s,t)x)|, \tag{6}$$

where

$$\begin{aligned} t_{mn}(\bar{A}(r,s,t)x) &= \frac{1}{m+1}\sum_{j=0}^{n}(\bar{A}(r,s,t)x)_{n+j} \\ &= \frac{1}{m+1}\sum_{j=0}^{m}\sum_{k=0}^{n+j}\frac{s_{n+j-k}t_kx_k}{r_n} \end{aligned}$$

for all $m,n\in\mathbb{N}$.

Proof. f_0 and f endowed with the norm $\|.\|_\infty$ are $BK-$spaces [24, Example 7.3.2 (b)] and $\bar{A}(r,s,t)$ is a triangle matrix, Theorem 4.3.2 of Wilansky [32, p.61] gives the fact that $\bar{f}(r,s,t)$ and $\bar{f}_0(r,s,t)$ are $BK-$spaces with the norm $\|.\|_{\bar{f}(r,s,t)}$. □

Remark 3.2. It can easily be seen that the absolute property does not hold on the spaces $\bar{f}(r,s,t)$ and $\bar{f}_0(r,s,t)$, that is $\|x\|_{\bar{f}(r,s,t)}\neq\||x|\|_{\bar{f}(r,s,t)}$ for at least one sequence x in each of these spaces, where $|x|=(|x_k|)$. Thus, the spaces $\bar{f}(r,s,t)$ and $\bar{f}_0(r,s,t)$ are $BK-$spaces of non-absolute type.

Theorem 3.3. The sequence spaces $\bar{f}(r,s,t)$ and $\bar{f}_0(r,s,t)$ are norm isomorphic to the spaces f and f_0, respectively.

Proof. Since the fact $\bar{f}_0(r,s,t)\cong f_0$ can be similarly proved, we consider only the case $\bar{f}(r,s,t)\cong f$. To prove this, we should show the existence of a linear bijection between the spaces $\bar{f}(r,s,t)$ and f which preserves the norm. Consider the transformation T defined, with the notation of (4), from $\bar{f}(r,s,t)$ to f by $x\mapsto\bar{x}=Tx=\bar{A}(r,s,t)x$. The linearity of T is clear. Further, it is trivial that $x=\theta$ whenever $Tx=\theta$ and hence T is injective.

Let us take any $\bar{x}=(\bar{x}_k)\in f$ and define the sequence $x=(x_n)$ by

$$x_n=\frac{1}{t_n}\sum_{k=0}^{n}(-1)^{n-k}D^{(s)}_{n-k}r_k\bar{x}_k;\quad \text{for all } n\in\mathbb{N}. \tag{7}$$

Then, it is immediate that

$$\begin{aligned} \sum_{k=0}^{n+j}\frac{s_{n+j-k}t_kx_k}{r_n} &= \sum_{k=0}^{n+j}\frac{s_{n+j-k}t_k}{r_n}\frac{1}{t_k}\sum_{i=0}^{k}(-1)^{k-i}D^{(s)}_{k-i}r_i\bar{x}_i \\ &= \bar{x}_{n+j} \end{aligned}$$

which gives by a short calculation that

$$\frac{1}{m+1}\sum_{j=0}^{m}\sum_{k=0}^{n+j}\frac{s_{n+j-k}t_k x_k}{r_n} = \frac{1}{m+1}\sum_{j=0}^{m}\bar{x}_{n+j}.$$

Therefore, we have

$$\lim_{m\to\infty}\frac{1}{m+1}\sum_{j=0}^{m}\{\bar{A}(r,s,t)x\}_{n+j} = \lim_{m\to\infty}\frac{1}{m+1}\sum_{j=0}^{m}\bar{x}_{n+j} = l \quad \text{uniformly in } n.$$

This means that $x \in \bar{f}(r,s,t)$ and hence T is surjective. Thus, one can easily see from (6) that T is a norm preserving transformation. This completes the proof. □

Remark 3.4. It is known from Corollary of Başar and Kirişçi [9] that the Banach space f has no Schauder basis. It is also known from Theorem 2.3 of Jarrah and Malkowsky [12] that the domain λ_A of a matrix A in a normed sequence space λ has a basis if and only if λ has a basis whenever $A=(a_{nk})$ is a triangle. Combining these two facts one can immediately conclude that both the space $\bar{f}(r,s,t)$ and the space $\bar{f}_0(r,s,t)$ have no Schauder basis.

Now, we give the beta- and gamma-duals of the sequence spaces $\bar{f}(r,s,t)$ and $\bar{f}_0(r,s,t)$. For this, we need the following lemmas:

Lemma 3.5. [11] $A=(a_{nk}) \in (f:\ell_\infty)$ if and only if

$$\sup_{n\in\mathbb{N}}\sum_k |a_{nk}| < \infty. \tag{8}$$

Lemma 3.6. [11] $A=(a_{nk}) \in (f:c)$ if and only if (8) holds, and there are $\alpha_k, \alpha \in \mathbb{C}$ such that

$$\lim_{n\to\infty} a_{nk} = \alpha_k \text{ for each } k \in \mathbb{N}, \tag{9}$$

$$\lim_{n\to\infty}\sum_k a_{nk} = \alpha, \tag{10}$$

$$\lim_{n\to\infty}\sum_k \left|\Delta(a_{nk}-\alpha_k)\right| = 0. \tag{11}$$

Theorem 3.7. Define the sets $F_1(r,s,t), F_2(r,s,t), F_3(r,s,t), F_4(r,s,t), F_5(r,s,t)$ as follows:

$$\begin{aligned}
F_1(r,s,t) &= \left\{a=(a_k)\in\omega : \sup_{n\in\mathbb{N}}\sum_{k=0}^{n}\left|\sum_{j=k}^{n}\frac{1}{t_j}(-1)^{j-k}D^{(s)}_{j-k}r_k a_j\right| < \infty\right\},\\
F_2(r,s,t) &= \left\{a=(a_k)\in\omega : \lim_{n\to\infty}\sum_{j=k}^{n}\frac{1}{t_j}(-1)^{j-k}D^{(s)}_{j-k}r_k a_j \quad \text{exists}\right\},\\
F_3(r,s,t) &= \left\{a=(a_k)\in\omega : \lim_{n\to\infty}\sum_{k=0}^{n}\left[\sum_{j=k}^{n}\frac{1}{t_j}(-1)^{j-k}D^{(s)}_{j-k}r_k a_j\right] - \text{exists}\right\},\\
F_4(r,s,t) &= \left\{a=(a_k)\in\omega : \lim_{n\to\infty}\sum_{k=0}^{n}\left|\sum_{j=k}^{n}\frac{1}{t_j}(-1)^{j-k}D^{(s)}_{j-k}r_k a_j\right| = 0\right\},\\
F_5(r,s,t) &= \left\{a=(a_k)\in\omega : \lim_{n\to\infty}\sum_{k=n+1}^{\infty}\left|\sum_{j=n+1}^{\infty}(\Delta\bar{a}_{jk}-\alpha_k)\right| = 0\right\}.
\end{aligned}$$

Then, the $\beta-$dual of the sequence space $\bar{f}(r,s,t)$ is

$$\bigcap_{i=1}^{5} F_i(r,s,t).$$

Proof. Let $a = (a_k) \in \omega$ and consider the equality

$$\begin{aligned}\sum_{k=0}^{n} a_k x_k &= \sum_{k=0}^{n}\left[\frac{1}{t_k}\sum_{j=0}^{k}(-1)^{k-j}D_{k-j}^{(s)}r_j\bar{x}_j\right]a_k \\ &= \sum_{k=0}^{n}\left[\sum_{j=k}^{n}\frac{1}{t_j}(-1)^{j-k}D_{j-k}^{(s)}r_k a_j\right]\bar{x}_k = \{\bar{F}(r,s,t)\bar{x}\}_n,\end{aligned} \tag{12}$$

where $\bar{F}(r,s,t) = \{\bar{f}_{nk}(r,s,t)\}$ is defined by

$$\bar{f}_{nk}(r,s,t) = \begin{cases}\sum_{j=k}^{n}\frac{1}{t_j}(-1)^{j-k}D_{j-k}^{(s)}r_k a_j & (0 \le k \le n),\\ 0 & (k > n)\end{cases} \tag{13}$$

for all $n, k \in \mathbb{N}$. Thus, we deduce from Lemma 3.6 with (12) that $ax = (a_k x_k) \in cs$ whenever $x = (x_k) \in \bar{f}(r,s,t)$ if and only if $\bar{F}(r,s,t)\bar{x} \in c$ whenever $\bar{x} = (\bar{x}_k) \in f$, where $\bar{F}(r,s,t) = \{\bar{f}_{nk}(r,s,t)\}$ is defined by (13). Therefore, we derive from (8), (9), (10) and (11) that

$$\sup_{n\in\mathbb{N}}\sum_k\left|\sum_{j=k}^{n}\frac{1}{t_j}(-1)^{j-k}D_{j-k}^{(s)}r_k a_j\right| < \infty,$$

$$\lim_{n\to\infty}\sum_{j=k}^{n}\frac{1}{t_j}(-1)^{j-k}D_{j-k}^{(s)}r_k a_j = \alpha_k \quad \text{for each fixed } k \in \mathbb{N},$$

$$\lim_{n\to\infty}\sum_k\sum_{j=k}^{n}\frac{1}{t_j}(-1)^{j-k}D_{j-k}^{(s)}r_k a_j = \alpha,$$

$$\lim_{n\to\infty}\sum_k\left|\Delta\left[\sum_{j=k}^{n}\frac{1}{t_j}(-1)^{j-k}D_{j-k}^{(s)}r_k a_j\right]\right| = 0$$

which shows that

$$\{\bar{f}(r,s,t)\}^{\beta} = \bigcap_{i=1}^{5} F_i(r,s,t).$$

□

Theorem 3.8. The $\gamma-$dual of the sequence spaces $\bar{f}(r,s,t)$ and $\bar{f}_0(r,s,t)$ is the set $F_1(r,s,t)$.

Proof. This is similar to the proof of Theorem 3.7 with Lemma 3.5 instead of Lemma 3.6. So, we omit the detail. □

4 Matrix Transformations Related to The Sequence Space $\bar{f}(r,s,t)$

In the present section, we characterize the matrix transformations from $\bar{f}(r,s,t)$ into any given sequence space μ.

Since $\bar{f}(r,s,t) \cong f$, it is trivial that the equivalence "$x \in \bar{f}(r,s,t)$ if and only if $\bar{x} \in f$" holds.

Theorem 4.1. Suppose that the entries of the infinite matrices $A = (a_{nk})$ and $E = (e_{nk})$ are connected with the relation

$$e_{nk} = \sum_{j=k}^{\infty}\frac{1}{t_j}(-1)^{j-k}D_{j-k}^{(s)}r_k a_{nj} \tag{14}$$

for all $n, k \in \mathbb{N}$ and μ is any given sequence space. Then $A \in (\bar{f}(r,s,t) : \mu)$ if and only if $\{a_{nk}\}_{k\in\mathbb{N}} \in \{\bar{f}(r,s,t)\}^{\beta}$ for all $n \in \mathbb{N}$ and $E \in (f : \mu)$.

Proof. Let μ be any given sequence space. Suppose that (14) holds between $A = (a_{nk})$ and $E = (e_{nk})$, and take into account that the spaces $\bar{f}(r,s,t)$ and f are linearly isomorphic.

Let $A \in (\bar{f}(r,s,t) : \mu)$ and take any $\bar{x} = (\bar{x}_k) \in f$. Then $E\bar{A}(r,s,t)$ exists and $\{a_{nk}\}_{k\in\mathbb{N}} \in \cap_{i=1}^{5} F_i(r,s,t)$ which yields that $\{e_{nk}\}_{k\in\mathbb{N}} \in \ell_1$ for each $n \in \mathbb{N}$. Hence, $E\bar{x}$ exists and thus

$$\sum_k e_{nk}\bar{x}_k = \sum_k a_{nk}x_k$$

for all $n \in \mathbb{N}$. We have that $E\bar{x} = Ax$ which leads us to the consequence $E \in (f : \mu)$.

Conversely, let $\{a_{nk}\}_{k\in\mathbb{N}} \in \{\bar{f}(r,s,t)\}^{\beta}$ for each $n \in \mathbb{N}$ and $E \in (f : \mu)$ hold, and take any $x = (x_k) \in \bar{f}(r,s,t)$. Then, Ax exists. Therefore, we obtain from the equality

$$\sum_{k=0}^{m} a_{nk}x_k = \sum_{k=0}^{m}\left[\sum_{j=k}^{m}\frac{1}{t_j}(-1)^{j-k}D_{j-k}^{(s)}r_k a_{nj}\right]\bar{x}_k$$

for all $n \in \mathbb{N}$, as $m \to \infty$ that $E\bar{x} = Ax$ and this shows that $A \in (\bar{f}(r,s,t) : \mu)$. This step completes the proof. □

By changing the roles of the spaces $\bar{f}(r,s,t)$ and μ in Theorem (4.1), we have:

Theorem 4.2. Suppose that the elements of the infinite matrices $A = (a_{nk})$ and $C = (c_{nk})$ are connected with the relation

$$c_{nk} = \frac{1}{r_n}\sum_{j=0}^{n} s_{n-j}t_j a_{jk} \quad \text{for all } n,k \in \mathbb{N}.$$

Let μ be any given sequence space. Then, $A = (a_{nk}) \in (\mu : \bar{f}(r,s,t))$ if and only if $C \in (\mu : f)$.

Proof. Let $z = (z_k) \in \mu$ and consider the following equality

$$\sum_{k=0}^{m} c_{nk}z_k = \frac{1}{r_n}\sum_{j=0}^{n} s_{n-j}t_j\left(\sum_{k=0}^{m} a_{jk}z_k\right) \quad \text{for all } m,n \in \mathbb{N},$$

which yields as $m \to \infty$ that $(Cz)_n = \{\bar{A}(r,s,t)(Az)\}_n$ for all $n \in \mathbb{N}$. Therefore, one can observe from here that $Az \in \bar{f}(r,s,t)$ whenever $z \in \mu$ if and only if $Cz \in f$ whenever $z \in \mu$. This completes the proof. □

Of course, Theorems 4.1 and 4.2 have several consequences depending on the choice of the sequence space μ. Define $a(n,k)$, $a(n,k,m)$ and Δa_{nk} for all $k,m,n \in \mathbb{N}$ as follows;

$$a(n,k) = \sum_{j=0}^{n} a_{jk}, \quad a(n,k,m) = \frac{1}{m+1}\sum_{j=0}^{m} a_{n+j,k} \text{ and } \Delta a_{nk} = a_{nk} - a_{n,k+1}.$$

Prior to giving some results as an application of this idea, we give the following basic lemma, which is the collection of the characterizations of matrix transformations related to almost convergence:

Lemma 4.3. Let $A = (a_{nk})$ be an infinite matrix. Then, the following statements hold:

(i) [33, J. P. Duran]$A = (a_{nk}) \in (\ell_\infty : f)$ if and only if (8) holds and

$$\exists \alpha_k \in \mathbb{C} \ni f - \lim a_{nk} = \alpha_k \quad \text{for all } k \in \mathbb{N}, \tag{15}$$

$$\exists \alpha_k \in \mathbb{C} \ni \lim_{m\to\infty}\sum_k |a(n,k,m) - \alpha_k| = 0 \quad \text{uniformly in } n \tag{16}$$

also hold .

(ii) [33, J. P. Duran]$A = (a_{nk}) \in (f : f)$ if and only if (8) and (15) hold, and

$$\exists \alpha \in \mathbb{C} \ni f - \lim \sum_k a_{nk} = \alpha, \tag{17}$$

$$\exists\alpha_k \in \mathbb{C} \ni \lim_{m\to\infty}\sum_k \left|\Delta\big[a(n,k,m)-\alpha_k\big]\right| = 0 \quad \text{uniformly in } n \tag{18}$$

also hold .

(iii) [34, J. P. King]$A = (a_{nk}) \in (f : f)$ if and only if (8), (15) and (17) hold .

(iv) [35, Başar and Çolak]$A = (a_{nk}) \in (cs : f)$ if and only if (15) holds, and

$$\sup_{n\in\mathbb{N}}\sum_k |\Delta a_{nk}| < \infty \tag{19}$$

also holds .

(v) [36, Başar and Solak]$A = (a_{nk}) \in (bs : f)$ if and only if (15) and (19) hold, and

$$\lim_{k\to\infty} a_{nk} = 0 \quad \text{for all } n \in \mathbb{N}, \tag{20}$$

$$\exists\alpha_k \in \mathbb{C} \ni \lim_{q\to\infty}\sum_k \frac{1}{q+1}\left|\sum_{i=0}^{q}\Delta\big[a(n+i,k)-\alpha_k\big]\right| = 0 \quad \text{uniformly in } n \tag{21}$$

also hold .

(vi) [37, Başar]$A = (a_{nk}) \in (f : cs)$ if and only if the following conditions hold:

$$\sup_{n\in\mathbb{N}}\sum_k |a(n,k)| < \infty, \tag{22}$$

$$\exists\alpha_k \in \mathbb{C} \ni \sum_n a_{nk} = \alpha_k \quad \text{for all } k \in \mathbb{N}, \tag{23}$$

$$\exists\alpha \in \mathbb{C} \ni \sum_n\sum_k a_{nk} = \alpha, \tag{24}$$

$$\exists\alpha_k \in \mathbb{C} \ni \lim_{n\to\infty}\sum_k \left|\Delta\big[a(n,k)-\alpha_k\big]\right| = 0. \tag{25}$$

Now, we can give the following two corollaries as a direct consequence of Theorems 4.1 and 4.2 and Lemma 4.3:

Corollary 4.4. The following statements hold:

(i) $A = (a_{nk}) \in (\bar{f}(r,s,t) : \ell_\infty)$ if and only if $\{a_{nk}\}_{k\in\mathbb{N}} \in \{\bar{f}(r,s,t)\}^\beta$ and (8) holds with e_{nk} instead of a_{nk}.

(ii) $A = (a_{nk}) \in (\bar{f}(r,s,t) : c)$ if and only if $\{a_{nk}\}_{k\in\mathbb{N}} \in \{\bar{f}(r,s,t)\}^\beta$ and (8), (9), (10) and (11) hold with e_{nk} instead of a_{nk}.

(iii) $A = (a_{nk}) \in (\bar{f}(r,s,t) : bs)$ if and only if $\{a_{nk}\}_{k\in\mathbb{N}} \in \{\bar{f}(r,s,t)\}^\beta$ and (22) holds with e_{nk} instead of a_{nk}.

(iv) $A = (a_{nk}) \in (\bar{f}(r,s,t) : cs)$ if and only if $\{a_{nk}\}_{k\in\mathbb{N}} \in \{\bar{f}(r,s,t)\}^\beta$ and (22), (23), (24) and (25) hold with e_{nk} instead of a_{nk}.

(v) $A = (a_{nk}) \in (\bar{f}(r,s,t) : f)$ if and only if $\{a_{nk}\}_{k\in\mathbb{N}} \in \{\bar{f}(r,s,t)\}^\beta$ and (8), (15), (17) and (18) hold with e_{nk} instead of a_{nk}.

Corollary 4.5. The following statements hold:

(i) $A = (a_{nk}) \in (\ell_\infty : \bar{f}(r,s,t))$ if and only if (8), (15) and (17) hold with e_{nk} instead of a_{nk}.

(ii) $A = (a_{nk}) \in (f : \bar{f}(r,s,t))$ if and only if (8), (15), (17) and (18) hold with e_{nk} instead of a_{nk}.

(iii) $A = (a_{nk}) \in (c : \bar{f}(r,s,t))$ if and only if (8), (15) and (17) hold with e_{nk} instead of a_{nk}.

(iv) $A = (a_{nk}) \in (bs : \bar{f}(r,s,t))$ if and only if (15), (19), (20) and (21) hold with e_{nk} instead of a_{nk}.

(v) $A = (a_{nk}) \in (cs : \bar{f}(r,s,t))$ if and only if (15) and (19) hold with e_{nk} instead of a_{nk}.

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ON $I_{\pi g\alpha^\star}$-CLOSED SETS IN IDEAL TOPOLOGICAL SPACES

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Abstract – In this paper, a new class of sets called $I_{\pi g\alpha^\star}$-closed sets is introduced and its properties are studied in ideal topological space. Moreover $I_{\pi g\alpha^\star}$-continuity and the notion of quasi-$\alpha^\star$-I-normal spaces are introduced.

Keywords – *π-open set, $I_{\pi g\alpha^\star}$-closed set, $I_{\pi g\alpha^\star}$-continuity, quasi-$\alpha^\star$-I-normal space.*

1 Introduction and Preliminaries

An ideal topological space is a topological space (X, τ) with an ideal I on X, and is denoted by (X, τ, I). A$^*(I)$ = {x $\in$ X | U $\cap$ A $\notin I$ for each open neighborhood U of x} is called the local function of A with respect to I and τ [9]. When there is no chance for confusion A$^*(I)$ is denoted by A*. For every ideal topological space (X, τ, I), there exists a topology τ^* finer than τ, generated by the base $\beta(I, \tau)$ = {U$\backslash I$ | U$\in \tau$ and $I \in I$}. In general $\beta(I, \tau)$ is not always a topology [8]. Observe additionally that cl*(A) = A* $\cup$ A [14] defines a Kuratowski closure operator for τ^*. int*(A) will denote the interior of A in (X, $\tau^\star$).

In this paper, we define and study a new notion $I_{\pi g\alpha^\star}$-closed set by using the notion of $\alpha^\star_I$-open set. Some new notions depending on $I_{\pi g\alpha^\star}$-closed sets such as $I_{\pi g\alpha^\star}$-open sets, $I_{\pi g\alpha^\star}$-continuity and $I_{\pi g\alpha^\star}$-irresoluteness are also introduced and a decomposition of $\alpha^\star$-I-continuity is given. Also by using $I_{\pi g\alpha^\star}$-closed sets characterizations of quasi-$\alpha^\star$-I-normal spaces are obtained. Several preservation theorems for quasi-$\alpha^\star$-I-normal spaces are given.

Throughout this paper, space (X, τ) (or simply X) always means topological space on which no separation axioms are assumed unless explicitly stated. Let A be a subset of a space X. The closure of A and the interior of A are denoted by cl(A) and int(A), respectively.

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A subset A of a topological space (X, τ) is said to be regular open [13](resp. regular closed [13]) if A = int(cl(A)) (resp. A = cl(int(A))).

The finite union of regular open sets is said to be π-open [16] in (X, τ). The complement of a π-open set is π-closed [16].

A subset A of a topological space (X, τ) is said to be α-open [10] if A $\subseteq$ int(cl(int(A))) and the complement of an α-open set is called α-closed [10].

The intersection of all α-closed sets containing A is called the α-closure [10] of A and is denoted by αcl(A).

Note that αcl(A) = A $\cup$ cl(int(cl(A))).

A subset A of a space (X, τ) is said to be πg-closed [2] (resp. $\pi g\alpha$-closed [1]) if cl(A) $\subseteq$ U (resp. αcl(A) $\subseteq$ U) whenever A $\subseteq$ U and U is π-open in X.

A function f : (X, τ) $\rightarrow$ (Y, σ) is said to be m-π-closed [4] if f(V) is π-closed in (Y, σ) for every π-closed in (X, τ).

A function f : (X, τ) $\rightarrow$ (Y, σ) is said to be πg-continuous [2] (resp. $\pi g\alpha$-continuous [1]) if f^{-1}(V) is πg-closed (resp. $\pi g\alpha$-closed) in (X, τ) for every closed set V of (Y, σ).

A space (X, τ) is said to be quasi-normal [16] if for every pair of disjoint π-closed subsets A, B of X, there exist disjoint open sets U, V of X such that A $\subseteq$ U and B $\subseteq$ V.

An ideal I is said to be codense [3] if $\tau \cap I = \emptyset$.

A subset A of an ideal topological space X is said to be $\star$-dense-in-itself [7](resp. $\alpha^\star$-I-open or $\alpha_I^\star$-open [15], t-I-set [6], α-I-open [6]) if A $\subseteq$ $A^\star$ (resp. A $\subseteq$ int*(cl(int*(A))), int(A) = int(cl*(A)), A $\subseteq$ int(cl*(int(A)))).

The complement of $\alpha_I^\star$-open is $\alpha_I^\star$-closed.

A subset A of an ideal topological space X is said to be $I_{\pi g}$-closed [11] if $A^\star \subseteq$ U whenever A $\subseteq$ U and U is π-open in X.

A function f : (X, τ, I) $\rightarrow$ (Y, σ) is said to be $I_{\pi g}$-continuous [11] if f^{-1}(V) is $I_{\pi g}$-closed in (X, τ, I) for every closed set V of (Y, σ).

Lemma 1.1. *[12] Let (X, τ, I) be an ideal topological space and A $\subseteq$ X. If A $\subseteq$ $A^\star$, then $A^\star$ = cl($A^\star$) = cl(A) = cl*(A).*

Theorem 1.2. *[11] Every πg-closed set is $I_{\pi g}$-closed but not conversely.*

Theorem 1.3. *[11] For a function f : (X, τ, I) $\rightarrow$ (Y, σ), the following holds:*
Every πg-continuous function is $I_{\pi g}$-continuous but not conversely.

Theorem 1.4. *[1] Every πg-closed set is $\pi g\alpha$-closed but not conversely.*

Proposition 1.5. *[6] Every α-I-open set is α-open but not conversely.*

2 $I_{\pi g\alpha^\star}$-closed Sets

Theorem 2.1. *For a function f : (X, τ) $\rightarrow$ (Y, σ), the following holds:*
Every πg-continuous function is $\pi g\alpha$-continuous but not conversely.

Example 2.2. *Let X = {a, b, c, d}, τ ={X, $\emptyset$, {a}, {b}, {a, b}, {b, c}, {a, b, c}}, Y = {x, y, z} and σ = { Y, $\emptyset$, {y}, {y, z}}. Define a function f : (X, τ) $\rightarrow$ (Y, σ) as follows f(a) = f(b) = y, f(c)= x and f(d) = z. Then f is $\pi g\alpha$-continuous function but it is not an πg-continuous.*

Definition 2.3. *Let (X, τ, I) be an ideal topological space and let A be a subset of X. The union of all $\alpha_I^\star$-open sets contained in A is called the $\alpha_I^\star$-interior of A and is denoted by $\alpha_I^\star$int(A).*

Definition 2.4. *Let (X, τ, I) be an ideal topological space and let A be a subset of X. The intersection of all $\alpha_I^\star$-closed sets containing A is called the $\alpha_I^\star$-closure of A and is denoted by $\alpha_I^\star$cl(A).*

Lemma 2.5. *Let (X, τ, I) be an ideal topological space. For a subset A of X, the followings hold:*

1. *$\alpha_I^\star$cl(A) = A $\cup$ cl*(int(cl*(A))),*
2. *$\alpha_I^\star$int(A) = A $\cap$ int*(cl(int*(A))).*

Definition 2.6. *A subset A of an ideal topological space (X, τ, I) is called $I_{\pi g\alpha^\star}$-closed if $\alpha_I^\star cl(A) \subseteq$ U whenever $A \subseteq U$ and U is π-open in X.*

The complement of $I_{\pi g\alpha^\star}$-closed set is said to be $I_{\pi g\alpha^\star}$-open.

Proposition 2.7. *Every α-open set is $\alpha_I^\star$-open but not conversely.*

Proof. Let A be α-open set. Then A $\subseteq$ int(cl(int(A))) which implies A $\subseteq$ int*(cl(int*(A))). Hence A is $\alpha_I^\star$-open set.

Example 2.8. *Let X and τ be as in Example 2.2 and $I = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$. Then $\{a, c\}$ is $\alpha_I^\star$-open set but not an α-open set.*

Theorem 2.9. *Every $\star$-dense-in-itself and $I_{\pi g\alpha^\star}$-closed set is a $\pi g\alpha$-closed set.*

Proof. Let A $\subseteq$ U, and U is π-open in X. Since A is $I_{\pi g\alpha^\star}$-closed, $\alpha_I^\star$cl(A) $\subseteq$ U. By Lemmas 1.1 and 2.5, $\alpha_I^\star$cl(A) = A $\cup$ cl*(int(cl*(A))) = A $\cup$ cl(int(cl(A))) = αcl(A). Then, αcl(A) $\subseteq$ U. So A is $\pi g\alpha$-closed.

Theorem 2.10. *Every π-open and $I_{\pi g\alpha^\star}$-closed set is t-I-set.*

Proof. $\alpha_I^\star$cl(A) $\subseteq$ A, since A is π-open and $I_{\pi g\alpha^\star}$-closed. We have cl*(int(cl*(A))) $\subseteq$ A and int(cl*(A)) $\subseteq$ cl*(int(cl*(A))) $\subseteq$ A. It implies int(cl*(A)) $\subseteq$ int(A). Always int(A) $\subseteq$ int(cl*(A)). Therefore int(A) = int(cl*(A)), which shows that A is t-I-set.

Theorem 2.11. *Let A be $I_{\pi g\alpha^\star}$-closed in (X, τ, I). Then $\alpha_I^\star cl(A) \setminus A$ does not contain any non-empty π-closed set.*

Proof. Let F be a π-closed set such that F $\subseteq$ $\alpha_I^\star$cl(A) $\setminus$ A. Then F $\subseteq$ X $\setminus$ A implies A $\subseteq$ X $\setminus$ F. Therefore $\alpha_I^\star$cl(A) $\subseteq$ X $\setminus$ F. That is F $\subseteq$ X $\setminus$ $\alpha_I^\star$cl(A). Hence F $\subseteq$ $\alpha_I^\star$cl(A) $\cap$ (X $\setminus$ $\alpha_I^\star$cl(A))= $\emptyset$. This shows F = $\emptyset$.

Theorem 2.12. *If A is $I_{\pi g\alpha^\star}$-closed and $A \subseteq B \subseteq \alpha_I^\star cl(A)$, then B is $I_{\pi g\alpha^\star}$-closed.*

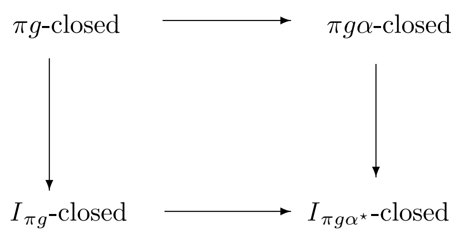
Proof. Let A be $I_{\pi g\alpha^\star}$-closed and B $\subseteq$ U, where U is π-open. Then A $\subseteq$ B implies A $\subseteq$ U. Since A is $I_{\pi g\alpha^\star}$-closed, $\alpha_I^\star$cl(A) $\subseteq$ U. B $\subseteq$ $\alpha_I^\star$cl(A) implies $\alpha_I^\star$cl(B) $\subseteq$ $\alpha_I^\star$cl(A). Therefore $\alpha_I^\star$cl(B) $\subseteq$ U and hence B is $I_{\pi g\alpha^\star}$-closed.

Proposition 2.13. *Let (X, τ, I) be an ideal topological space and $A \subseteq X$. Then the following properties hold:*

1. *If A is $\pi g\alpha$-closed, then A is $I_{\pi g\alpha^\star}$-closed,*
2. *If A is $I_{\pi g}$-closed, then A is $I_{\pi g\alpha^\star}$-closed.*

Proof. The proof is obvious.

Remark 2.14. *From Theorem 1.2, Theorem 1.4 and Proposition 2.13, we have the following diagram.*



where none of these implications is reversible as shown in the following examples.

Example 2.15. *(1) Let X and τ be as in Example 2.2. Then $\{c\}$ is $\pi g\alpha$-closed set but not an πg-closed.*
(2) In Example 2.8, $\{a\}$ is $I_{\pi g\alpha^\star}$-closed set but not $\pi g\alpha$-closed.
(3) In Example 2.8, $\{c\}$ is $I_{\pi g\alpha^\star}$-closed set but not $I_{\pi g}$-closed.

Remark 2.16. *The union of two $I_{\pi g\alpha^\star}$-closed sets need not be $I_{\pi g\alpha^\star}$-closed.*

Example 2.17. *In Example 2.8, $\{b\}$ and $\{c\}$ are $I_{\pi g\alpha^\star}$-closed sets but their union $\{b,\ c\}$ is not $I_{\pi g\alpha^\star}$-closed.*

Remark 2.18. *The intersection of two $I_{\pi g\alpha^\star}$-closed sets need not be $I_{\pi g\alpha^\star}$-closed.*

Example 2.19. *Let $X = \{a,\ b,\ c,\ d\}$, $\tau = \{X,\ \emptyset,\ \{a\},\ \{b\},\ \{a,\ b\},\ \{a,\ b,\ c\},\ \{a,\ b,\ d\}\}$ and $I = \{\emptyset,\ \{a\}\}$. Then $A = \{a,\ b,\ c\}$ and $B = \{a,\ b,\ d\}$ are $I_{\pi g\alpha^\star}$-closed sets but $A \cap B = \{a,\ b\}$ is not $I_{\pi g\alpha^\star}$-closed set.*

Definition 2.20. *[5] An ideal topological space (X, τ, I) is said to be $\star$-extremally disconnected if the $\star$-closure of every open subset of X is open.*

Theorem 2.21. *[5] For an ideal topological space (X, τ, I), the following properties are equivalent:*

1. *X is $\star$-extremally disconnected,*
2. *$cl^\star(int(V)) \subseteq int(cl^\star(V))$ for every subset V of X.*

Theorem 2.22. *Let (X, τ, I) be a $\star$-extremally disconnected ideal topological space. Then every subset of X is $I_{\pi g\alpha^\star}$-closed if and only if every π-open set is t-I-set.*

Proof. Necessity: It is obvious from Theorem 2.10.
Sufficiency: Suppose that every π-open set is t-I-set. Let A be a subset of X and U be π-open such that A $\subseteq$ U. By hypothesis $\mathrm{cl}^\star(\mathrm{int}(\mathrm{cl}^\star(\mathrm{A}))) \subseteq \mathrm{int}(\mathrm{cl}^\star(\mathrm{A})) \subseteq \mathrm{int}(\mathrm{cl}^\star(\mathrm{U})) = \mathrm{int}(\mathrm{U}) \subseteq \mathrm{U}$. Then $\alpha_I^\star\mathrm{cl}(\mathrm{A}) \subseteq \mathrm{U}$. So A is $I_{\pi g\alpha^\star}$-closed.

Theorem 2.23. *Let (X, τ, I) be an ideal topological space. $A \subseteq X$ is $I_{\pi g\alpha^\star}$-open if and only if $F \subseteq \alpha_I^\star int(A)$ whenever F is π-closed and $F \subseteq A$.*

Proof. Necessity: Let A be $I_{\pi g\alpha^\star}$-open and F be π-closed such that F $\subseteq$ A. Then X\A $\subseteq$ X\F where X\F is π-open. $I_{\pi g\alpha}$-closedness of X\A implies $\alpha_I^\star\mathrm{cl}(\mathrm{X}\backslash\mathrm{A}) \subseteq \mathrm{X}\backslash\mathrm{F}$. Then $\mathrm{F} \subseteq \alpha_I^\star\mathrm{int}(\mathrm{A})$.
Sufficiency: Suppose F is π-closed and F $\subseteq$ A implies $\mathrm{F} \subseteq \alpha_I^\star\mathrm{int}(\mathrm{A})$. Let X\A $\subseteq$ U where U is π-open. Then X\U $\subseteq$ A where X\U is π-closed. By hypothesis $\mathrm{X}\backslash\mathrm{U} \subseteq \alpha_I^\star\mathrm{int}(\mathrm{A})$. That is $\alpha_I^\star\mathrm{cl}(\mathrm{X}\backslash\mathrm{A}) \subseteq \mathrm{U}$. So, A is $I_{\pi g\alpha^\star}$-open.

Definition 2.24. *A subset A of an ideal topological space (X, τ, I) is called N_I-set if $A = U \cup V$ where U is π-closed and V is $\alpha_I^\star$-open.*

Proposition 2.25. *Every π-closed set is N_I-set but not conversely.*

Example 2.26. *In Example 2.19, $\{a\}$ is N_I-set but not π-closed set.*

Proposition 2.27. *Every $\alpha_I^\star$-open set is N_I-set but not conversely.*

Example 2.28. *In Example 2.19, $\{a,\ c,\ d\}$ is N_I-set but not $\alpha_I^\star$-open set.*

Proposition 2.29. *Every $\alpha_I^\star$-open set is $I_{\pi g\alpha^\star}$-open but not conversely.*

Proof. Let A be $\alpha_I^\star$-open set. Then A $\subseteq$ int*(cl(int*(V))). Assume that F is π-closed and F $\subseteq$ A. Then F $\subseteq$ int*(cl(int*(V))) which implies F $\subseteq$ A $\cap$ int*(cl(int*(V))) = $\alpha_I^\star$int(A) by Lemma 2.5. Hence, by Theorem 2.23, A is $I_{\pi g\alpha^\star}$-open.

Example 2.30. *In Example 2.19, $\{a,\ d\}$ is $I_{\pi g\alpha^\star}$-open set but not $\alpha_I^\star$-open set.*

Theorem 2.31. *For a subset A of (X, τ, I) the following conditions are equivalent:*

1. *A is $\alpha_I^\star$-open,*
2. *A is $I_{\pi g\alpha^\star}$-open and a N_I-set.*

Proof. (1) $\Rightarrow$ (2) It is obvious.
(2) $\Rightarrow$ (1) Let A be $I_{\pi g\alpha^\star}$-open and a N_I-set. Then there exist a π-closed set U and $\alpha_I^\star$-open set V such that A = U $\cup$ V. Since U $\subseteq$ A and A is $I_{\pi g\alpha^\star}$-open, by Theorem 2.23, U $\subseteq$ $\alpha_I^\star$int(A) and U $\subseteq$ int*(cl(int*(A))). Also, V $\subseteq$ int*(cl(int*(V))) $\subseteq$ int*(cl(int*(A))). Then A $\subseteq$ int*(cl(int*(A))). So A is $\alpha_I^\star$-open.

The following examples show that concepts of $I_{\pi g\alpha^\star}$-open set and N_I-set are independent.

Example 2.32. *Let (X, τ, I) be the same ideal topological space as in Example 2.19. Then $\{c,\ d\}$ is N_I-set but not $I_{\pi g\alpha^\star}$-open set.*

Example 2.33. *Let (X, τ, I) be the same ideal topological space as in Example 2.19. Then $\{a,\ c\}$ is $I_{\pi g\alpha^\star}$-open set but not a N_I-set.*

3 $I_{\pi g\alpha^\star}$-continuity and $I_{\pi g\alpha^\star}$-irresoluteness

Definition 3.1. *A function f : (X, τ, I) $\rightarrow$ (Y, σ) is said to be $I_{\pi g\alpha^\star}$-continuous (resp. $\alpha^\star$-I-continuous) if $f^{-1}(V)$ is $I_{\pi g\alpha^\star}$-closed (resp. $\alpha_I^\star$-closed) in X for every closed set V of Y.*

Definition 3.2. *A function f : (X, τ, I) $\rightarrow$ (Y, σ, J) is said to be $I_{\pi g\alpha^\star}$-irresolute if $f^{-1}(V)$ is $I_{\pi g\alpha^\star}$-closed in X for every $J_{\pi g\alpha^\star}$-closed set V of Y.*

Definition 3.3. *A function f : (X, τ, I) $\rightarrow$ (Y, σ) is said to be N_I-continuous if $f^{-1}(V)$ is N_I-set in (X, τ, I) for every closed set V of (Y, σ.*

Theorem 3.4. *A function f : (X, τ, I) $\rightarrow$ (Y, σ) is $\alpha^\star$-I-continuous if and only if it is N_I-continuous and $I_{\pi g\alpha^\star}$-continuous.*

Proof. This is an immediate consequence of Theorem 2.31.

The composition of two $I_{\pi g\alpha^\star}$-continuous functions need not be $I_{\pi g\alpha^\star}$-continuous. Consider the following Example:

Example 3.5. *Let X = $\{a,\ b,\ c,\ d\}$, τ = $\{X,\ \emptyset,\ \{b\},\ \{d\},\ \{b,\ d\},\ \{b,\ c,\ d\}\}$ and I = $\{\emptyset,\ \{c\},\ \{d\},\ \{c,\ d\}\}$. Let Y = $\{x,\ y,\ z\}$, σ = $\{Y,\ \emptyset,\ \{y,\ z\}\}$, J = $\{\emptyset,\ \{x\}\}$, Z = $\{1,\ 2\}$ and η = $\{Z,\ \emptyset,\ \{1\}\}$. Define f : (X, τ, I) $\rightarrow$ (Y, σ, J) by f(a)= f(c)= x, f(b)= y and f(d)= z and g : (Y, σ, J) $\rightarrow$ (Z, η) by g(x)= 1 and g(y)= g(z)= 2. Then f and g are $I_{\pi g\alpha^\star}$-continuous. $\{2\}$ is closed in (Z, η), $(g \circ f)^{-1}(\{2\}) = f^{-1}(g^{-1}(\{2\})) = f^{-1}(\{y,\ z\}) = \{b,\ d\}$ which is not $I_{\pi g\alpha^\star}$-closed in (X, τ, I). Hence $g \circ f$ is not $I_{\pi g\alpha^\star}$-continuous.*

Theorem 3.6. *Let f : (X, τ, I) $\rightarrow$ (Y, σ, J) and g : (Y, σ, J) $\rightarrow$ (Z, η, K) be any two functions. Then*

1. *$g \circ f$ is $I_{\pi g\alpha^\star}$-continuous, if g is continuous and f is $I_{\pi g\alpha^\star}$-continuous,*
2. *$g \circ f$ is $I_{\pi g\alpha^\star}$-continuous, if g is $J_{\pi g\alpha^\star}$-continuous and f is $I_{\pi g\alpha^\star}$-irresolute,*
3. *$g \circ f$ is $I_{\pi g\alpha^\star}$-irresolute, if g is $J_{\pi g\alpha^\star}$-irresolute and f is $I_{\pi g\alpha^\star}$-irresolute.*

Proof. (1) Let V be closed in Z. Then $g^{-1}(V)$ is closed in Y, since g is continuous. $I_{\pi g\alpha^\star}$-continuity of f implies that $f^{-1}(g^{-1}(V))$ is $I_{\pi g\alpha^\star}$-closed in X. Hence $g \circ f$ is $I_{\pi g\alpha^\star}$-continuous.
(2) Let V be closed in Z. Since g is $J_{\pi g\alpha^\star}$-continuous, $g^{-1}(V)$ is $J_{\pi g\alpha^\star}$-closed in Y. As f is $I_{\pi g\alpha^\star}$-irresolute, $f^{-1}(g^{-1}(V))$ is $I_{\pi g\alpha^\star}$-closed in X. Hence $g \circ f$ is $I_{\pi g\alpha^\star}$-continuous. (3) Let V be $K_{\pi g\alpha^\star}$-closed in Z. Then $g^{-1}(V)$ is $J_{\pi g\alpha^\star}$-closed in Y, since g is $J_{\pi g\alpha^\star}$-irresolute. Because f is $I_{\pi g\alpha^\star}$-irresolute, $f^{-1}(g^{-1}(V))$ is $I_{\pi g\alpha^\star}$-closed in X. Hence $g \circ f$ is $I_{\pi g\alpha^\star}$-irresolute.

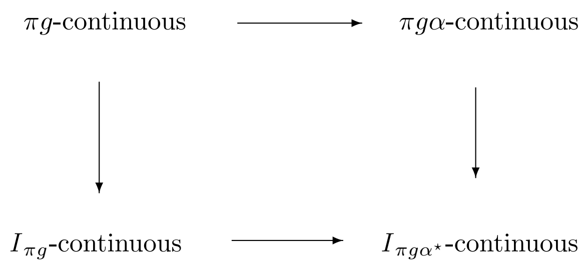
Remark 3.7. *The following Examples show that:*

1. *every $I_{\pi g\alpha^\star}$-continuous function is not $\pi g\alpha$-continuous,*
2. *every $I_{\pi g\alpha^\star}$-continuous function is not $I_{\pi g}$-continuous.*

Example 3.8. *Let (X, τ, I) be the same ideal topological space as in Example 2.8. Let $Y = \{x, y, z\}$ and $\sigma = \{Y, \emptyset, \{y, z\}\}$. Define a function $f : (X, \tau, I) \to (Y, \sigma)$ as follows: $f(a) = x$, $f(b) = f(c) = y$ and $f(d) = z$. Then f is $I_{\pi g\alpha^\star}$-continuous function but it is not $\pi g\alpha$-continuous.*

Example 3.9. *Let (X, τ, I) be the same ideal topological space as in Example 2.8. Let $Y = \{x, y, z\}$ and $\sigma = \{Y, \emptyset, \{y, z\}\}$. Define a function $f : (X, \tau, I) \to (Y, \sigma)$ as follows: $f(a) = f(b) = z$, $f(c) = x$ and $f(d) = y$. Then f is $I_{\pi g\alpha^\star}$-continuous function but it is not $I_{\pi g}$-continuous.*

Theorem 3.10. *For a function $f : (X, \tau, I) \to (Y, \sigma)$, the following properties hold:*



Proof. The proof is obvious by Remark 2.14.

4 Quasi-$\alpha^\star$-I-normal Spaces

Definition 4.1. *A space (X, τ) is said to be quasi-α-normal if for every pair of disjoint π-closed subsets A, B of X, there exist disjoint α-open sets U, V of X such that $A \subseteq U$ and $B \subseteq V$.*

Definition 4.2. *An ideal topological space (X, τ, I) is said to be quasi-$\alpha^\star$-I-normal if for every pair of disjoint π-closed subsets A, B of X, there exist disjoint $\alpha_I^\star$-open sets U, V of X such that $A \subseteq U$ and $B \subseteq V$.*

Proposition 4.3. *If X is a quasi-α-normal space, then X is quasi-$\alpha^\star$-I-normal.*

Proof. It is obtained from Proposition 2.7.

Theorem 4.4. *The following properties are equivalent for a space X:*

1. *X is quasi-$\alpha^\star$-I-normal,*
2. *for any disjoint π-closed sets A and B, there exist disjoint $I_{\pi g\alpha^\star}$-open sets U, V of X such that $A \subseteq U$ and $B \subseteq V$,*
3. *for any π-closed set A and any π-open set B containing A, there exists an $I_{\pi g\alpha^\star}$-open set U such that $A \subseteq U \subseteq \alpha_I^\star cl(U) \subseteq B$.*

Proof. (1) ⇒ (2) The proof is obvious.
(2) ⇒ (3) Let A be any π-closed set of X and B any π-open set of X such that A ⊆ B. Then A and X\B are disjoint π-closed subsets of X. Therefore, there exist disjoint $I_{\pi g\alpha^\star}$-open sets U and V such that A ⊆ U and X\B ⊆ V. By the definition of $I_{\pi g\alpha^\star}$-open set, We have that X\B ⊆ $\alpha_I^\star$int(V) and U ∩ $\alpha_I^\star$int(V) = ∅. Therefore, we obtain $\alpha_I^\star$cl(U) ⊆ $\alpha_I^\star$cl(X\V) and hence A ⊆ U ⊂ $\alpha_I^\star$cl(U) ⊆ B.
(3) ⇒ (1) Let A and B be any disjoint π-closed sets of X. Then A ⊆ X\B and X\B is π-open and hence there exists an $I_{\pi g\alpha^\star}$-open set G of X such that A ⊆ G ⊆ $\alpha_I^\star$cl(G) ⊆ X\B. Put U = $\alpha_I^\star$int(G) and V = X\ $\alpha_I^\star$cl(G). Then U and V are disjoint $\alpha_I^\star$-open sets of X such that A ⊆ U and B ⊆ V. Therefore, X is quasi-$\alpha^\star$-I-normal.

Theorem 4.5. *Let $f : X \to Y$ be an $I_{\pi g\alpha^\star}$-continuous m-π-closed injection. If Y is quasi-normal, then X is quasi-$\alpha^\star$-I-normal.*

Proof. Let A and B be disjoint π-closed sets of Y. Since f is m-π-closed injection, f(A) and f(B) are disjoint π-closed sets of Y. By the quasi-normality of X, there exist disjoint open sets U and V such that f(A) ⊆ U and f(B) ⊆ V. Since f is $I_{\pi g\alpha^\star}$-continuous, then f^{-1}(U) and f^{-1}(V) are disjoint $I_{\pi g\alpha^\star}$-open sets such that A ⊆ f^{-1}(U) and B ⊆ f^{-1}(V). Therefore X is quasi-$\alpha^\star$-I-normal by Theorem 4.4.

Theorem 4.6. *Let $f : X \to Y$ be an $I_{\pi g\alpha^\star}$-irresolute m-π-closed injection. If Y is quasi-$\alpha^\star$-I-normal, then X is quasi-$\alpha^\star$-I-normal.*

Proof. Let A and B be disjoint π-closed sets of Y. Since f is m-π-closed injection, f(A) and f(B) are disjoint π-closed sets of Y. By quasi-$\alpha^\star$-I-normality of Y, there exist disjoint $I_{\pi g\alpha^\star}$-open sets U and V such that f(A) ⊆ U and f(B) ⊆ V. Since f is $I_{\pi g\alpha^\star}$-irresolute, then f^{-1}(U) and f^{-1}(V) are disjoint $I_{\pi g\alpha^\star}$-open sets such that A ⊆ f^{-1}(U) and B ⊆ f^{-1}(V). Therefore X is quasi-$\alpha^\star$-I-normal.

Theorem 4.7. *Let (X, τ, I) be an ideal topological space where I is codense. Then X is quasi-$\alpha^\star$-I-normal if and only if it is quasi-α-normal.*

5 Conclusion

Topology is an area of Mathematics concerned with the properties of space that are preserved under continuous deformations including stretching and bending, but not tearing. By the middle of the 20th century, topology had become a major branch of Mathematics.

Topology as a branch of Mathematics can be formally defined as the study of qualitative properties of certain objects that are invariant under a certain kind of transformation especially those properties that are invariant under a certain kind of equivalence and it is the study of those properties of geometric configurations which remain invariant when these configurations are subjected to one-to-one bicontinuous transformations or homeomorphisms. Topology operates with more general concepts than analysis. Differential properties of a given transformation are nonessential for topology but bicontinuity is essential. As a consequence, topology is often suitable for the solution of problems to which analysis cannot give the answer.

Though the concept of topology has been identified as a difficult territory in Mathematics, we have taken it up as a challenge and cherishingly worked out this research study. Ideal Topology is a generalization of topology in classical mathematics, but it also has its own unique characteristics. It can also further up the understanding of basic structure of classical mathematics and offers new methods and results in obtaining significant results of classical mathematics. Moreover it also has applications in some important fields of Science and Technology.

A new class of sets called $I_{\pi g\alpha^\star}$-closed sets is introduced and its properties are studied in ideal topological space. Moreover $I_{\pi g\alpha^\star}$-continuity and the notion of quasi-$\alpha^\star$-I-normal spaces are introduced.

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ROUGH LATTICE OVER BOOLEAN ALGEBRA

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Abstract − Rough Set Theory (RST) is a mathematical formalism for representing uncertainty that can be considered as an extension of the classical set theory. It has been used in many different research areas, including those related to inductive machine learning and reduction of knowledge in knowledge-based systems. Rough partial order relation and rough lattice are two important concepts to introduce here based on RST. This paper provides some properties of rough relations, rough lattice, rough boolean lattice and established their validity. Some results are established to illustrate the paper.

Keywords − *Rough Set, Rough Approximation, Boolean Algebra, Rough Relation, Rough Partial Order, Boolean Lattice.*

1 Introduction

Rough set plays an important role for handling situations which are not crisp and deterministic but associated with impreciseness in the form of indiscernibility between the objects of a set. So, in case of dealing with some types of knowledge representation problems, rough algebraic structures are useful. The concepts of Lattices and Boolean algebra [1] are of cardinal importance in the theory and design of computers and of circuitry in general, besides having numerous other applications in mathematical logic, probability theory and other fields of engineering and mathematics. Lattice is an algebraic structure is of considerable importance, in view of its application in fields of mathematics and computer science. The notions of rough partial order relation and rough lattice are based on RST are needed in many applications, where experimental data are processes, in particular as a theoretical basis for rough relation. In [2] Jouni Järvinen has proposed several direction of lattice theory for rough set. We have also proposed lattice theory for rough set in different direction ([8],[9],[10],[11],[12]). This paper presents the main concepts related to rough partial order relations, some of its properties and related rough boolean algebra which are different but quite related with some special cases of Järvinen's work.

The remainder of this article is organized as follows. Section 2 gives account of previous work. Our new and exciting results are described in Section 3 and Section 4. Finally, Section 5 gives the conclusions.

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2 Definitions and Notations

2.1 Rough Set

Let U be a universe of discourse and E be an equivalence relation over U, called the indiscernibility relation. By U/E, we denote the family of all equivalence classes induced by E on U. These classes are referred to as categories or concepts of E and the equivalence class of an element $x \in U$, is denoted by x/E or $[x]_E$. The basic concept of rough set theory is the notion of an approximation space, which is an ordered pair $A = (U, E)$. For $x, y \in U$, if xEy then x and y are said to be indistinguishable in A. The elements of U/E are called elementary sets in A. It is assumed that the empty set is also elementary set for every approximation space. A definable set in A is any finite union of elementary sets in A.

2.2 Rough Approximations

Theory of rough set was introduced by Z. Pawlak [4], assumed that set is chosen from a universe U, but that elements of U can be specified only upto an indiscernibility equivalence relation E on U. If a subset $X \subseteq U$ contains an element indiscernible from some elements not in X, then X is rough. Also a rough set X is described by two approximations. Basically, in rough set theory, it is assumed that our knowledge is restricted by an indiscernibility relation. An indiscernibility relation is an equivalence relation E such that two elements of an universe of discourse U are E-equivalent if we cannot distinguish these two elements by their properties known by us. By the means of an indiscernibility relation E, we can partition the elements of U into three disjoint classes respect to any set $X \subseteq U$, defined as follows:

- The elements which are certainly in X. These are elements $x \in U$ whose E-class x/E is included in X.
- The elements which certainly are not in X. These are elements $x \in U$ such that their E-class x/E is included in X^{co}, which is the complement of X
- The elements which are possibly belongs to X. These are elements whose E-class intersects with both X and X^{co}. In other words, x/E is not included in X nor in X^{co}.

From this observation, we defined lower approximation set $X\downarrow$ of X to be the set of those elements $x \in U$ whose E-class is included in X, i.e, $X\downarrow = \{x \in U : x/E \subseteq X\}$ and for the upper approximation set $X\uparrow$ of X consists of elements $x \in U$ whose E-class intersect with X, i.e, $X\uparrow = \{x \in U : x/E \cap X \neq \emptyset\}$. The difference between $X\downarrow$ and $X\uparrow$ treated as the actual area of uncertainty.

3 Rough Relation

The notion of rough relation was introduced and their properties were studied by Pawlak ([6], [7]). Stepaniuk ([13], [14]) have established some more properties of rough relations and their applications.

Definition 3.1. Let $A_1 = (U_1, E_1)$ and $A_2 = (U_2, E_2)$ be two approximation spaces. The product of A_1 by A_2 is the approximation space denoted by $A = (U, S)$, where $U = U_1 \times U_2$ and the indiscernibility relation $S \subseteq (U \times U)^2$ is defined by $(x_1, y_1), (x_2, y_2) \in S \Leftrightarrow (x_1, x_2) \in E_1$ and $(y_1, y_2) \in E_2$, $(x_1, x_2) \in U_1$ and $(y_1, y_2) \in U_2$. It can be easily seen that S is an equivalence relation on $U \times U$. The elements (x_1, y_1) and (x_2, y_2) are indiscernible in S if and only if the elements x_1 and x_2 are indiscernible in E_1 and so are the elements y_1 and y_2 in E_2.

Definition 3.2. Let $(U_1 \times U_2, E)$ be an approximation space, where U_1 and U_2 are nonempty sets and $R \subseteq (U_1 \times U_2)^2$ be an equivalence relation. For any relation $S \subseteq U_1 \times U_2$, we define two relations $L(S)$ and $U(S)$ called lower and upper approximations of S respectively given by, $L(S) = \{(x_1, x_2) \in U_1 \times U_2 : [(x_1, x_2)]_E \subseteq S\}, U(S) = \{(x_1, x_2) \in U_1 \times U_2 : [(x_1, x_2)]_E \cap S \neq \emptyset\}$, where $[(x_1, x_2)]_E$ denotes the equivalence class of relation E containing the pair (x_1, x_2). Rough relation of S is defined as the pair $(L(S), U(S))$.

Definition 3.3. If V and W are relations in A, then $W * V$ is a relation such that $(a, b) \in V \ and \ (b, c) \in W \ for \ some \ b \in A\}$.

Proposition 3.4. If V, W, V_1, W_1 are relations in $A, V_1 \subseteq V$ and $W_1 \subseteq W$, then $W_1 * V_1 \subseteq W * V$

Proof: Let $(a, c) \in W_1 * V_1 \Rightarrow \exists b \in A$ such that $(a, b) \in V_1$ and $(b, c) \in W_1$. Then $(a, b) \in V$ and $(b, c) \in W$ so that $(a, c) \in W * V$

Proposition 3.5. Let $A = (U, R)$ be an approximation space and $B = (U^2, S)$ the approximation product space of $A \times A$. Then:

- $[(x, y)]_S = [x]_E \times [y]_E$, and
- $[(y, z)]_S * [(x, y)]_S = [(x, z)]_S$

Proof: The first result is trivially follows from the definition of the relation S.
For the second result let $(a, c) \in [(y, z)]_S * [(x, y)]_S$. Then there exist $a, b \in U$ such that $(a, b) \in [(x, y)]_S$ and $(b, c) \in [(y, z)]_S$. It follows that $(a, b)S(x, y)$ and $(b, c)S(y, z)$. Hence aEx, bEy and cEz hold. Consequently, $(a, c) \in [(x, z)]_S$
On the other hand, let $(a, c) \in [(x, z)]_S$. This gives $(a, c)S(x, z)$. We thus get aEx and cEz. This clearly implies $(a, y)S(x, y)$ and $(y, c)S(y, z)$. Hence $(a, y) \in [(x, y)]_S$ and $(y, c) \in [(y, z)]_S$, and therefore $(a, c) \in [(y, z)]_S * [(x, y)]_S$

Definition 3.6. Let $S = (U, E)$ be an approximation space and E_g be its generated relation of E, we say that $S_g = (U \times U, E_g)$ is general approximation space of S.

Definition 3.7. [6] We consider a non-null subset M of U and a relation T on M. The rough relation $E_g(T)(M \to M)$ is said to be Reflexive: if and only if $\forall \ m \in M, (m, m) \in E_g \uparrow (T)$. Symmetric: if and only if $\forall \ m_1, m_2 \in M, (m_1, m_2) \in E_g \uparrow (T) \Rightarrow (m_2, m_1)E_g \uparrow (T)$. Transitive: if and only if $\forall \ m_1, m_2, m_3 \in M, (m_1, m_2)$ and $(m_2, m_3)E_g \uparrow (T) \Rightarrow (m_1, m_3)E_g \uparrow (T)$. Antisymmetric: if and only if $\forall m_1, m_2 \in M, (m_1, m_2), (m_2, m_1) \in E_g \uparrow (T) \Rightarrow [m_1]_E = [m_2]_E$. We only consider the upper approximation as lower approximation is always subset of upper approximation.

Definition 3.8. A relation T is said to be a rough partially ordering if $E_g(T)$ is reflexive, symmetric and transitive.

3.1 Rough Membership Function

Rough sets can also be defined by the rough membership function instead of approximation [5]. We define the membership function of X with respect to E as $\mu_X^E : X \to [0, 1]$, such that $\mu_X^E = \frac{|x/E \cap X|}{|x/E|}$, where $||$ represents cardinality function on a set. The rough membership function can also be interpreted as the conditional probability, and can be interpreted as a degree of certainty to which x belongs to X. The rough membership function can be used to define the lower approximation, the upper approximation and the boundary region of a set, as follows: $E \downarrow (X) = \{x \in U : \mu_X^E(X) = 1\}, E \uparrow (X) = \{x \in U : \mu_X^E(X) > 0\}$ and
$BN_E(X) = \{x \in U : 0 < \mu_X^E(X) < 1\}$
$\mu_{A \cup B}^E(X) \geq max(\mu_A^E(X), \mu_B^E(X))$ for any $x \in U$.
$\mu_{A \cap B}^E(X) \leq min(\mu_A^E(X), \mu_B^E(X))$ for any $x \in U$.

Definition 3.9. For any rough partial ordering T on a non-null subset M of U, the dominating class of an element x in M is denoted by $T_{\geq[x]}$and is defined for every y in M as $T_{\geq[x]}(y) = r_T(x, y)$, where $r_T(x, y) = \frac{|[(x,y)]_{E_g} \cap T|}{|[(x,y)]_{E_g}|}$. For any rough partial ordering T on a non-null subset M of U, the dominating class of an element x in M is denoted by $T_{\leq[x]}$and is defined for every y in M as $T_{\leq[x]}(y) = r_T(y, x)$.

Definition 3.10. For any rough partial ordering T on a non-null subset M of U, the rough upper bound of M is the rough set denoted by $U(T, M)$ and is defined by $U(T, M) = \bigcap_{x \in M} T_{\geq[x]}$ Here, the operator "intersection" associates the minimum of the membership values in the constituents for each element in M.

Definition 3.11. For any rough partial ordering T on a non-null subset M of U, the rough greatest lower bound of M is a unique element x in $L(T, M)$ such that $L(T, M)(x) > 0$ and $r_T(y, x) > 0$ for all elements in the support of $L(T, M)$. The uniqueness of x is up to it equivalence class with respect to E

Definition 3.12. A crisp subset M of U with a rough partial ordering T is said to be a rough lattice if and only if for any subset $\{x, y\}$ in M, the least upper bound (l.u.b) and the greatest lower bound (g.l.b) exist in M. We denote the l.u.b. of $\{x, y\}$ by $x \vee y$ and the g.l.b of $\{x, y\}$ by $x \wedge y$. We say that (M, T) is a rough lattice on (U, E) and denoted it by $\mathbf{L}$.

Example-1: Let $A = (U, E)$ be an approximation space, where $U = \{a, b, c, d, e, f, g\}$ and $U/E = \{\{a, b\}, \{c, d\}, \{e, f\}, \{g\}\}$ as shown here $B = (U \times U, S) = \{(a, g), (b, g), (g, a), (g, b), (c, g), (d, g), (e, e), (f, f), (e, f), (f, e), (a, c), (a, d), (b, d), (b, c), (d, a), (d, b), (c, a), (c, b), (a, e), (b, e), (a, f), (b, f), (e, a), (e, b), (f, a), (f, b), (c, e), (d, e), (c, f), (d, f), (e, c), (e, d), (f, c), (f, d), (c, d), (d, c), (d, d), (c, c), (b, b), (a, b), (b, a), (a, a), (g, c), (g, d), (e, g), (f, g), (g, e), (g, f), (g, g)\}$.

1. Let us consider two non empty subsets $U_1 = \{a, b, c\}$ and $U_2 = \{f, g\}$ of U. We take a subset T of $U_1 \times U_2$ as $T = \{(a, g), (b, g), (c, f), (c, g)\}$. The $E_g \downarrow (T) = \{(a, g), (b, g)\}$ and $E_g \uparrow (T) = \{(a, g), (b, g), (c, e), (d, e), (c, f), (d, f), (c, g), (d, g)\}$. $r_T(a, f) = 0$ and $r_T(c, f) = \frac{1}{4}$.
2. Let us take $T = \{(a, b), (c, d), (e, f), (g, g)\}$. Then $E_g \uparrow (T) = \{(e, e), (e, f), (f, f), (f, e), (c, d), (d, c), (d, d), (c, c), (b, b), g(a, b), (b, a), (a, a), (g, g)\}$. It is easy to see that $R_g(T)$ is a rough equivalence relation.
3. Let $T = \{(a, g), (a, c), (c, e), (g, e), (g, g)\}$. Then $E_g \uparrow (T) = \{(a, g), (b, g), (a, c), (a, d), (b, d), (b, c), (c, e), (d, e), (c, f), g(d, f), (g, e), (g, f), (g, g)\}$. So, $E_g(T)$ is antisymmetric.
4. Let $T = \{(a, g), (e, f), (c, d), (a, b), (g, g)\}$. Then $E_g \uparrow (T) = \{(a, g), (b, g), (e, e), (e, f), (f, f), (f, e), (c, d), (d, c), (d, d), (c, c), g(b, b), (a, b), (b, a), (a, a), (g, g)\}$ So, $E_g(T)$ is clearly reflexive.

$E_g(T)$ is antisymmetric as $(e, f), (f, e) \in R_g \uparrow (T)$ and $[e]_E = [f]_E; (c, d), (d, c) \in R_g \uparrow (T)$ and $[c]_E = [d]_E; (a, b), (b, a) \in R_g \uparrow (T)$ and $[a]_E = [b]_E$. It is also clearly rough transitive. So, $E_g(T)$ is a rough partially ordered relation. if the universe is partitioned into at least three non singleton equivalence classes which will give ultimately " rough boolean lattice".

4 Rough Boolean Lattice

Let R be a reflexive relation on U and $X \subseteq U$. The set $R(X) = \{y \in U : xRy, \text{ for some } x \in X\}$ is the R-neighborhood of X. If $X = \{a\}$, then we write $R(a)$ instead of $R(\{a\})$. The approximations are defined as $X_R = \{x \in U : R(x) \subseteq X\}$ and $X^R = \{x \in U : R(x) \cap X \neq \emptyset\}$. A set $X \subseteq U$ is called R-closed if $R(X) = X$, and an element $x \in U$ is R-closed, if its singleton set $\{x\}$ is R-closed. The set of R-closed points is denoted by S. Let us assume that $(U; E)$ is an indiscernibility space. The set of lower approximations $B_E(U) = \{X_E : X \subseteq U\}$ and the set of upper approximations $B^E(U) = \{X^E : X \subseteq U\}$ coincide, so we denote this set simply by $B_E(U)$. The set $B_E(U)$ is a complete Boolean sublattice of $(P(U), \subseteq)$, where $P(U)$ denotes the set of all subsets of U. This means

that $B_E(U)$ forms a complete field of sets. Complete fields of sets are in one-to-one correspondence with equivalence relations, meaning that for each complete field of sets F on U, we can define an equivalence E such that $B_E(U) = F$. Note that S and all its subsets belong to $B_E(U)$, meaning that $P(S)$ is a complete sublattice of $B_E(U)$, and therefore in this sense S can be viewed to consist of completely defined objects. Each object in S can be separated from other points of U by the information provided by the indiscernibility relation E, meaning that for any $x \in S$ and $X \subseteq U, x \in X_E$ if and only if $x \in X^E$. The rough set of X is the equivalence class of all $Y \subseteq U$ such that $Y_E = X_E$ and $Y^E = X^E$. Since each rough set is uniquely determined by the approximation pair, one can represent the rough set of X as (X_E, X^E) . This is known as increasing representation [3]. This representations induce the sets $IR_E(U) = \{(X_E, X^E) : X \subseteq U\}$. The set $IR_E(U)$ can be ordered point wise $(X_E, X^E) \leq (Y_E, Y^E) \Leftrightarrow X_E \subseteq Y_E$. Therefore, $IR_E(U)$ can form completely distributive lattice. As shown in [8], $IR_E(U)$ is a complete sublattice of $P(U) \times P(U)$ ordered by the point wise set-inclusion relation, meaning that $IR_E(U)$ is an algebraic completely distributive lattice such that $\bigwedge\{(X_E, X^E) : X \subseteq H\} = (\bigcap_{X\in H} X_E, \bigcap_{X\in H} X^E)$ and $\bigvee\{(X_E, X^E) : X \subseteq H\} = (\bigcup_{X\in H} X_E, \bigcup_{X\in H} X^E)$ for all $H \subseteq IR_E(U)$.
Now we consider the rough lattice and rough boolean algebra which are parallel to fuzzy lattice and fuzzy boolean algebra [15].

Definition 4.1. A complemented distributive rough lattice (M, T) is known as rough Boolean algebra. Every complemented rough lattice need to be bounded. So every rough Boolean algebra is necessarily bounded rough lattice with bounds 0 and 1. Also every element a in M has an unique complement denoted by a^{co}

Lemma 4.2. Let **L** be a rough lattice on an approximation space (U, E) then for any two elements $a, b \in M$, and $b > a$ then $r_T(a, b) > 0 \Leftrightarrow a \wedge b = a \Leftrightarrow a \vee b = b$

Theorem 4.3. Let **L** be a rough lattice on the approximation space (U, E). Then for all $a, b, c \in M$, $a \wedge a = a$ and $a \vee a = a, a \wedge b = b \wedge a$ and $a \vee b = b \vee a, (a \wedge b) \wedge c = a \wedge (b \wedge c)$ and $(a \vee b) \vee c = a \vee (b \vee c), a \wedge (a \vee b) = a$ and $a \vee (a \wedge b) = a$

Theorem 4.4. Let **L** be a rough lattice on the approximation space (U, E). Then for all $a, b, c \in M, r_T((a \wedge b) \vee (a \wedge c), a \wedge (b \vee c)) > 0$. $r_T((a \vee (b \wedge c), (a \vee b) \wedge (a \vee c)) > 0$

Definition 4.5. A rough lattice **L** on the approximation space (U, E) is said to be complete if every subset of M has a l.u.b and a g.l.b. in (U, E)

Definition 4.6. A rough lattice **L** on the approximation space (U, E) is said to be bounded if $\exists$ two elements $0, 1 \in M$ such that $r_T(0, x) > 0$ and $r_T(x, 1) > 0$ for all $x \in M$

Definition 4.7. A rough lattice **L** on the approximation space (U, E) is said to be distributive if and only if for all $a, b, c \in M$, $P_1 : a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$. $P_2 : a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$. In this connection we can show that the statement P_1 , P_2 are equivalent

Theorem 4.8. In a rough lattice **L** on the approximation space (U, E) the cancellation laws hold, that is $a \vee b = a \vee c \Rightarrow b = c$ and $a \wedge b = a \wedge c \Rightarrow b = c$

Theorem 4.9. In a rough distributive lattice **L** on the approximation space (U, E), the De Morgan's laws hold true. That is, $(a \vee b)^{co} = a^{co} \wedge b^{co}$ and $(a \wedge b)^{co} = a^{co} \vee b^{co}$ for all $a, b \in \mathbf{L}$, where x^{co} stands for the complement of x

Definition 4.10. A rough chain is a partially ordered rough set (M, T) on the approximation space (U, E) in which for two elements $a, b \in L$, either $r_T(a, b) > 0$ *or* $r_T(b, a) > 0$

Definition 4.11. A rough lattice **L** on the approximation space (U, E) is said to be modular if $a \vee (b \wedge c) = (a \vee b) \wedge c$, whenever $r_T(a, c) > 0$ for all $a, b, c \in \mathbf{L}$

Lemma 4.12. Every rough chain is a distributive rough lattice and every distributive rough lattice is modular.

Lemma 4.13. In a complemented distributive rough lattice **L** on the approximation space $(U, E), a, b \in L, r_T(a, b) > 0 \Leftrightarrow a \wedge b^{co} = 0 \Leftrightarrow a^{co} \vee b = 1 \Leftrightarrow r_T(b^{co}, a^{co}) > 0$

5 Conclusion

In this paper, we have presented rough lattice through a rough partial ordering relation defined on a crisp set. We have introduced some important definitions, properties and lemmas of rough lattice, rough ordering relation based on rough approximation spaces, giving interesting example. The roughness of Boolean lattice is also studied, which is an interesting topic, we will extend it further in the future.

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SOME PERTURBED TRAPEZOID INEQUALITIES FOR m- AND (α, m)-CONVEX FUNCTIONS AND APPLICATIONS

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Abstract – In this paper, the Authors establish some new inequalities related to perturbed trapezoid inequality for the classes of functions whose second derivatives of absolute values are m and (α, m)-convex. After, applications to special means have also been presented.

Keywords – *Hermite-Hadamard inequalities, m- and (α, m)-convex functions, perturbed trapezoid inequality, means.*

1 Introduction

Definition 1.1. [11] A function $f : I \to \mathbb{R}$ is said to be convex on I if inequality

$$f(tu + (1-t)v) \le tf(u) + (1-t)f(v) \tag{1}$$

holds for all $u, v \in I$ and $t \in [0, 1]$. We say that f is concave if $(-f)$ is convex.

Geometrically, this means that if P, Q and R are three distinct points on the graph of f with Q between P and R, then Q is on or below the chord PR.

In [14], G. Toader defined m-convexity: another intermediate between the usual convexity and starshaped convexity.

Definition 1.2. [14] A function $f : [0, b] \to \mathbb{R}$ is said to be m-convex, where $m \in [0, 1]$, if we have

$$f(tx + m(1-t)y) \le tf(x) + m(1-t)f(y) \tag{2}$$

for all $x, y \in [0, b]$ and $t \in [0, 1]$ .We say that f is m-concave if $-f$ is m-convex. Denote by $K_m(b)$ the class of all m-convex functions on $[0, b]$ for which $f(0) \le 0$.

Remark 1.3. For $m = 1$ in (2), we recapture the concept of convex functions defined on $[0, b]$ and, for $m = 0$, the concept of star-shaped functions defined on $[0, b]$ is obtained.

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Definition 1.4. [1] The function $f:[0,b]\to\mathbb{R}$ is said to be (α,m)-convex, where $(\alpha,m)\in[0,1]^2$; if for every $u,v\in[0,b]$ and $t\in[0,1]$, we have

$$f\left(tu+(1-t)v\right)\leq t^{\alpha}f\left(u\right)+m\left(1-t^{\alpha}\right)f\left(v\right). \tag{3}$$

Remark 1.5. Note that for $(\alpha,m)\in\{(0,0),(\alpha,0),(1,0),(1,m),(1,1),(\alpha,1)\}$ one obtains the following classes of functions: increasing, α-starshaped, starshaped, m-convex, convex and α-convex.

Theorem 1.6. (The Hermite-Hadamard inequality) Let $f:I\subseteq\mathbb{R}\to\mathbb{R}$ be a convex function and $u,v\in I$ with $u<v$. The following double inequality:

$$f\left(\frac{u+v}{2}\right)\leq\frac{1}{v-u}\int_u^v f\left(x\right)dx\leq\frac{f\left(u\right)+f\left(v\right)}{2} \tag{4}$$

is known in the literature as Hadamard's inequality (or Hermite-Hadamard inequality) for convex functions. If f is a positive concave function, then the inequality is reversed.

In the literature [2]-[7] on numerical integration, the following estimation is well known as the trapezoid inequality:

$$\left|\int_u^v f\left(x\right)dx-\frac{1}{2}\left(v-u\right)\left(f\left(u\right)+f\left(v\right)\right)\right|\leq\frac{1}{12}M_2\left(v-u\right)^3, \tag{5}$$

where $f:[u,v]\to\mathbb{R}$ is supposed to be twice differentiable on the interval (u,v), with the second derivative bounded on (u,v) by $M_2=\sup_{x\in(u,v)}\left|f''\left(x\right)\right|<+\infty$.

For the perturbed trapezoid inequality, Dragomir et al. [4] obtained the following inequality by an application of the Grüss inequality:

$$\begin{aligned}&\left|\int_u^v f\left(x\right)dx-\frac{1}{2}\left(v-u\right)\left(f\left(u\right)+f\left(v\right)\right)+\frac{1}{12}\left(v-u\right)^2\left(f'\left(v\right)-f'\left(u\right)\right)\right|\\ \leq\;&\frac{1}{32}\left(\Gamma_2-\gamma_2\right)\left(v-u\right)^3,\end{aligned} \tag{6}$$

where f is supposed to be twice differentiable on the interval (u,v), with the second derivative bounded on (u,v) by $\Gamma_2=\sup_{x\in(u,v)}f''\left(x\right)<+\infty$ and $\gamma_2=\inf_{x\in(u,v)}f''\left(x\right)>-\infty$.

For recent results and generalizations concerning Hadamard's inequality, concepts of convexity, m-, (α,m)-convexity and trapezoid inequality see [1]-[19] and the references therein.

Throughout this paper we will use the following notations and conventions. Let $J=[0,\infty)\subset\mathbb{R}=(-\infty,+\infty)$, and $u,v\in J$ with $0<u<v$ and $f'\in L[u,v]$ and

$$A\left(u,v\right)=\frac{u+v}{2},\ G\left(u,v\right)=\sqrt{uv},\ I\left(u,v\right)=\frac{1}{e}\left(\frac{v^v}{u^u}\right)^{\frac{1}{v-u}}\ \text{(for } u\neq v\text{)},$$

be the arithmetic mean, geometric mean, identric mean, for $u,v>0$ respectively.

The aim of this paper is to establish some results connected with the perturbed trapezoid inequality for m and (α,m)-convex functions as well as to apply them for some elementary inequalities for real numbers and in numerical integration.

2 The New Results for m- and (α,m)-convex Functions

To prove perturbed trapezoid inequalities for m-convex and (α,m)-convex functions, we use following Lemma which was used by Tunç et al. (see [16])

Lemma 2.1. [16] Let $f : I^\circ \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, $a, b \in I^\circ$ with $a < b$. If $f'' \in L[a, b]$, then the following equality holds:

$$\begin{aligned} &\int_a^b f(x)\,dx - \frac{1}{2}(b-a)(f(a)+f(b)) + \frac{5}{4}(b-a)^2 (f'(b)-f'(a)) \\ = &\frac{(b-a)^3}{4}\int_0^1 (t+1)^2 [f''(ta+(1-t)b) + f''(tb+(1-t)a)]\,dt \end{aligned} \tag{7}$$

Theorem 2.2. [16] Let $f : I^\circ \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, $a, b \in I^\circ$ with $a < b$. If $|f''|$ is convex on $[a, b]$, then the following inequality holds:

$$\begin{aligned} &\left|\int_a^b f(x)\,dx - \frac{1}{2}(b-a)(f(a)+f(b)) + \frac{5}{4}(b-a)^2 (f'(b)-f'(a))\right| \\ \leq &\frac{7}{12}(b-a)^3 (|f''(a)| + |f''(b)|). \end{aligned} \tag{8}$$

Theorem 2.3. Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, $a, b \in I^\circ$ with $a < b$ and $m \in [0, 1]$. If $|f''|$ is m-convex on I, then the following inequality holds:

$$\begin{aligned} &\left|\int_a^b f(x)\,dx - \frac{1}{2}(b-a)(f(a)+f(b)) + \frac{5}{4}(b-a)^2 (f'(b)-f'(a))\right| \\ &\frac{(b-a)^3}{4}\left\{\frac{17[|f''(a)|+|f''(b)|]}{12} + m\frac{11\left[\left|f''\left(\frac{a}{m}\right)\right| + \left|f''\left(\frac{b}{m}\right)\right|\right]}{12}\right\}. \end{aligned} \tag{9}$$

Proof. Using Lemma 2.1 and Definition 1.2, it follows that

$$\begin{aligned} &\left|\int_a^b f(x)\,dx - \frac{1}{2}(b-a)(f(a)+f(b)) + \frac{5}{4}(b-a)^2 (f'(b)-f'(a))\right| \\ \leq &\frac{(b-a)^3}{4}\int_0^1 (t+1)^2 (|f''(ta+(1-t)b)| + |f''(tb+(1-t)a)|)\,dt \\ \leq &\frac{(b-a)^3}{4}\int_0^1 (t+1)^2 \left\{t|f''(a)| + m(1-t)\left|f''\left(\frac{b}{m}\right)\right|\right. \\ &\left. +t|f''(b)| + m(1-t)\left|f''\left(\frac{a}{m}\right)\right| dt\right\} \\ \leq &\frac{(b-a)^3}{4}\left\{\left([|f''(a)|+|f''(b)|]\int_0^1 t(t+1)^2\,dt\right)\right. \\ &\left. +m\left[\left|f''\left(\frac{a}{m}\right)\right| + \left|f''\left(\frac{b}{m}\right)\right|\right]\int_0^1 (t+1)^2(1-t)\,dt\right\} \\ \leq &\frac{(b-a)^3}{4}\left\{\frac{17[|f''(a)|+|f''(b)|]}{12} + m\frac{11\left[\left|f''\left(\frac{a}{m}\right)\right| + \left|f''\left(\frac{b}{m}\right)\right|\right]}{12}\right\}. \end{aligned}$$

□

Theorem 2.4. Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, $a, b \in I^\circ$ with $a < b$ and $(\alpha, m) \in [0, 1]^2$. If $|f''|$ is (α, m)-convex on I, then the following inequality holds:

$$\begin{aligned} &\left|\int_a^b f(x)\,dx - \frac{1}{2}(b-a)(f(a)+f(b)) + \frac{5}{4}(b-a)^2 (f'(b)-f'(a))\right| \\ \leq &\frac{(b-a)^3}{4}\left\{\frac{4\alpha^2+16\alpha+14}{\alpha^3+6\alpha^2+11\alpha+6}[|f''(a)|+|f''(b)|]\right. \\ &\left. +\left(\frac{7}{3} - \frac{4\alpha^2+16\alpha+14}{\alpha^3+6\alpha^2+11\alpha+6}\right)m\left[\left|f''\left(\frac{a}{m}\right)\right| + \left|f''\left(\frac{b}{m}\right)\right|\right]\right\}. \end{aligned} \tag{10}$$

Proof. Using Lemma 2.1 and Definition 1.4, it follows that

$$\begin{aligned}
&\left|\int_a^b f(x)\,dx-\frac{1}{2}(b-a)(f(a)+f(b))+\frac{5}{4}(b-a)^2(f'(b)-f'(a))\right| \\
\le\;& \frac{(b-a)^3}{4}\int_0^1 (t+1)^2\left(|f''(ta+(1-t)b)|+|f''(tb+(1-t)a)|\right)dt \\
\le\;& \frac{(b-a)^3}{4}\int_0^1 (t+1)^2\left\{t^\alpha|f''(a)|+m(1-t^\alpha)\left|f''\left(\frac{b}{m}\right)\right|\right. \\
&\left.+t^\alpha|f''(b)|+m(1-t^\alpha)\left|f''\left(\frac{a}{m}\right)\right|dt\right\} \\
\le\;& \frac{(b-a)^3}{4}\left\{\left([|f''(a)|+|f''(b)|]\int_0^1 t^\alpha(t+1)^2\,dt\right)\right. \\
&\left.+m\left[\left|f''\left(\frac{a}{m}\right)\right|+\left|f''\left(\frac{b}{m}\right)\right|\right]\int_0^1 (t+1)^2(1-t^\alpha)\,dt\right\} \\
\le\;& \frac{(b-a)^3}{4}\left\{\frac{4\alpha^2+16\alpha+14}{\alpha^3+6\alpha^2+11\alpha+6}[|f''(a)|+|f''(b)|]\right. \\
&\left.+m\left(\frac{7}{3}-\frac{4\alpha^2+16\alpha+14}{\alpha^3+6\alpha^2+11\alpha+6}\right)\left[\left|f''\left(\frac{a}{m}\right)\right|+\left|f''\left(\frac{b}{m}\right)\right|\right]\right\}.
\end{aligned}$$

□

Remark 2.5. i) In inequality (10), if we choose $\alpha = 1$, inequality (10) reduces to inequality (9).
ii) In inequality (10), if we take $\alpha = 1$, $m = 1$, inequality (10) reduces to inequality (8).

Theorem 2.6. [16] Let $f : I \subseteq R \to R$ be a differentiable mapping on I°, $a, b \in I^\circ$ with $a < b$, and let $p > 1$ with $1/p + 1/q = 1$. If the mapping $|f''|^q$ is convex on $[a, b]$ then the following inequality holds:

$$\begin{aligned}
&\left|\int_a^b f(x)\,dx-\frac{1}{2}(b-a)(f(a)+f(b))+\frac{5}{4}(b-a)^2(f'(b)-f'(a))\right| \qquad (11)\\
\le\;& \frac{(b-a)^3}{2}\left(\frac{2^{2p+1}-1}{2p+1}\right)^{\frac{1}{p}}\left(\frac{|f''(a)|^q+|f''(b)|^q}{2}\right)^{\frac{1}{q}}.
\end{aligned}$$

Theorem 2.7. Let $f : I \subseteq R \to \mathbb{R}$ be a differentiable mapping on I°, $a, b \in I^\circ$ with $a < b$ and $m \in [0, 1]$, and let $p > 1$ with $1/p + 1/q = 1$. If the mapping $|f''|^q$ is m-convex on I, then the following inequality holds:

$$\begin{aligned}
&\left|\int_a^b f(x)\,dx-\frac{1}{2}(b-a)(f(a)+f(b))+\frac{5}{4}(b-a)^2(f'(b)-f'(a))\right| \qquad (12)\\
\le\;& \frac{(b-a)^3}{4}\left(\frac{2^{2p+1}-1}{2p+1}\right)^{\frac{1}{p}} \\
&\times\left\{\left[\frac{|f''(a)|^q+m\left|f''\left(\frac{b}{m}\right)\right|^q}{2}\right]^{\frac{1}{q}}+\left[\frac{|f''(b)|^q+m\left|f''\left(\frac{a}{m}\right)\right|^q}{2}\right]^{\frac{1}{q}}\right\}.
\end{aligned}$$

Proof. Using Lemma 2.1, Definition 1.2 and Hölder's integral inequality, we get

$$\begin{aligned}
&\left|\int_a^b f(x)\,dx-\frac{1}{2}(b-a)(f(a)+f(b))+\frac{5}{4}(b-a)^2(f'(b)-f'(a))\right|\\
\leq\ & \frac{(b-a)^3}{4}\left[\int_0^1 |t+1|^2\,|f''(ta+(1-t)b)|\,dt\right.\\
&\left.+\int_0^1 |t+1|^2\,|f''(tb+(1-t)a)|\,dt\right]\\
\leq\ & \frac{(b-a)^3}{4}\left[\left(\int_0^1 |t+1|^{2p}\,dt\right)^{\frac{1}{p}}\left(\int_0^1 |f''(ta+(1-t)b)|^q\,dt\right)^{\frac{1}{q}}\right.\\
&\left.+\left(\int_0^1 |t+1|^{2p}\,dt\right)^{\frac{1}{p}}\left(\int_0^1 |f''(tb+(1-t)a)|^q\,dt\right)^{\frac{1}{q}}\right]\\
\leq\ & \frac{(b-a)^3}{4}\left(\frac{2^{2p+1}-1}{2p+1}\right)^{\frac{1}{p}}\left[\left(\int_0^1\left(t\,|f''(a)|^q+m(1-t)\left|f''\left(\frac{b}{m}\right)\right|^q\right)dt\right)^{\frac{1}{q}}\right.\\
&\left.+\left(\int_0^1\left(t\,|f''(b)|^q+m(1-t)\left|f''\left(\frac{a}{m}\right)\right|^q\right)dt\right)^{\frac{1}{q}}\right]\\
\leq\ & \frac{(b-a)^3}{4}\left(\frac{2^{2p+1}-1}{2p+1}\right)^{\frac{1}{p}}\\
&\times\left\{\left[\frac{|f''(a)|^q+m\left|f''\left(\frac{b}{m}\right)\right|^q}{2}\right]^{\frac{1}{q}}+\left[\frac{|f''(b)|^q+m\left|f''\left(\frac{a}{m}\right)\right|^q}{2}\right]^{\frac{1}{q}}\right\}.
\end{aligned}$$

□

Theorem 2.8. Let $f : I \subseteq R \to \mathbb{R}$ be a differentiable mapping on I°, $a, b \in I$ with $a < b$, $(\alpha, m) \in [0,1]^2$, and let $p > 1$ with $1/p + 1/q = 1$. If the mapping $|f''|^q$ is (α, m)-convex on I, then the following inequality holds:

$$\begin{aligned}
&\left|\int_a^b f(x)\,dx-\frac{1}{2}(b-a)(f(a)+f(b))+\frac{5}{4}(b-a)^2(f'(b)-f'(a))\right| \qquad (13)\\
\leq\ & \frac{(b-a)^3}{4}\left(\frac{2^{2p+1}-1}{2p+1}\right)^{\frac{1}{p}}\\
&\times\left\{\left[\frac{|f''(a)|^q}{\alpha+1}+\frac{m\alpha\left|f''\left(\frac{b}{m}\right)\right|^q}{\alpha+1}\right]^{\frac{1}{q}}+\left[\frac{|f''(b)|^q}{\alpha+1}+\frac{m\alpha\left|f''\left(\frac{a}{m}\right)\right|^q}{\alpha+1}\right]^{\frac{1}{q}}\right\}.
\end{aligned}$$

Proof. Using Lemma 2.1, Definition 1.4 and Hölder's integral inequality, we get

$$\begin{aligned}
&\left|\int_a^b f(x)\,dx-\frac{1}{2}(b-a)(f(a)+f(b))+\frac{5}{4}(b-a)^2(f'(b)-f'(a))\right| \\
\leq\; &\frac{(b-a)^3}{4}\left[\int_0^1 |t+1|^2\left|f''(ta+(1-t)b)\right|dt\right. \\
&\left.+\int_0^1 |t+1|^2\left|f''(tb+(1-t)a)\right|dt\right] \\
\leq\; &\frac{(b-a)^3}{4}\left[\left(\int_0^1 |t+1|^{2p}\,dt\right)^{\frac{1}{p}}\left(\int_0^1 \left|f''(ta+(1-t)b)\right|^q dt\right)^{\frac{1}{q}}\right. \\
&\left.+\left(\int_0^1 |t+1|^{2p}\,dt\right)^{\frac{1}{p}}\left(\int_0^1 \left|f''(tb+(1-t)a)\right|^q dt\right)^{\frac{1}{q}}\right] \\
\leq\; &\frac{(b-a)^3}{4}\left(\frac{2^{2p+1}-1}{2p+1}\right)^{\frac{1}{p}}\left[\left(\int_0^1\left(t^\alpha\left|f''(a)\right|^q+m(1-t^\alpha)\left|f''\left(\frac{b}{m}\right)\right|^q\right)dt\right)^{\frac{1}{q}}\right. \\
&\left.+\left(\int_0^1\left(t^\alpha\left|f''(b)\right|^q+m(1-t^\alpha)\left|f''\left(\frac{a}{m}\right)\right|^q\right)dt\right)^{\frac{1}{q}}\right] \\
\leq\; &\frac{(b-a)^3}{4}\left(\frac{2^{2p+1}-1}{2p+1}\right)^{\frac{1}{p}} \\
&\times\left\{\left[\frac{\left|f''(a)\right|^q}{\alpha+1}+\frac{m\alpha\left|f''\left(\frac{b}{m}\right)\right|^q}{\alpha+1}\right]^{\frac{1}{q}}+\left[\frac{\left|f''(b)\right|^q}{\alpha+1}+\frac{m\alpha\left|f''\left(\frac{a}{m}\right)\right|^q}{\alpha+1}\right]^{\frac{1}{q}}\right\}.
\end{aligned}$$

□

Remark 2.9. i) In (13), if we choose $\alpha=1$, we have the inequality in (12).
ii) In Theorem 2.8, if we choose $\alpha=m=1$, we obtain the inequality in (11).

Corollary 2.10. i) Under the assumptions of Theorem 2.7, if we choose $p=m=1$, we obtain the inequality;

$$\begin{aligned}
&\left|\int_a^b f(x)\,dx-\frac{1}{2}(b-a)(f(a)+f(b))+\frac{5}{4}(b-a)^2(f'(b)-f'(a))\right| \\
\leq\; &\frac{7(b-a)^3}{6}\left[\frac{\left|f''(a)\right|^q+\left|f''(b)\right|^q}{2}\right]^{\frac{1}{q}}.
\end{aligned}$$

ii) Under the assumptions of Theorem 2.8, if we choose $p=m=1$, we obtain the inequality;

$$\begin{aligned}
&\left|\int_a^b f(x)\,dx-\frac{1}{2}(b-a)(f(a)+f(b))+\frac{5}{4}(b-a)^2(f'(b)-f'(a))\right| \\
\leq\; &\frac{7(b-a)^3}{6}\left\{\left[\frac{\left|f''(a)\right|^q}{\alpha+1}+\frac{\alpha\left|f''(b)\right|^q}{\alpha+1}\right]^{\frac{1}{q}}+\left[\frac{\left|f''(b)\right|^q}{\alpha+1}+\frac{\alpha\left|f''(a)\right|^q}{\alpha+1}\right]^{\frac{1}{q}}\right\}.
\end{aligned}$$

Theorem 2.11. [16] Let $f: I\subseteq\mathbb{R}\to\mathbb{R}$ be a differentiable mapping on I°, $a,b\in I^\circ$ with $a<b$, and

let $p > 1$with $1/p+1/q=1$. If the mapping $|f''|^p$ convex on $[a,b]$, then the following inequality holds:

$$\begin{aligned} &\left|\int_a^b f(x)\,dx-\frac{1}{2}(b-a)(f(a)+f(b))+\frac{5}{4}(b-a)^2(f'(b)-f'(a))\right| \\ \leq\ & \frac{(b-a)^3}{4}\left(\frac{7}{3}\right)^{1-\frac{1}{p}}\left\{\left(\frac{17|f''(a)|^p+11|f''(b)|^p}{12}\right)^{\frac{1}{p}}\right. \\ &\left.+\left(\frac{17|f''(b)|^p+11|f''(a)|^p}{12}\right)^{\frac{1}{p}}\right\}. \end{aligned} \tag{14}$$

Theorem 2.12. Let $f: I\subseteq\mathbb{R}\to\mathbb{R}$ be a differentiable mapping on I°, $a,b\in I^\circ$ with $a<b$, and $m\in[0,1]$, and let $p>1$ with $1/p+1/q=1$. If the mapping $|f''|^q$ is m-convex on I, then the following inequality holds:

$$\begin{aligned} &\left|\int_a^b f(x)\,dx-\frac{1}{2}(b-a)(f(a)+f(b))+\frac{5}{4}(b-a)^2(f'(b)-f'(a))\right| \\ \leq\ & \frac{(b-a)^3}{4}\left(\frac{7}{3}\right)^{1-\frac{1}{p}}\left\{\left(\frac{17|f''(a)|^p+m11\left|f''\left(\frac{b}{m}\right)\right|^p}{12}\right)^{\frac{1}{p}}\right. \\ &\left.+\left(\frac{17|f''(b)|^p+m11\left|f''\left(\frac{a}{m}\right)\right|^p}{12}\right)^{\frac{1}{p}}\right\}. \end{aligned} \tag{15}$$

Proof. Using Lemma 2.1, Definition 1.2 and power mean integral inequality, we establish

$$\begin{aligned} &\left|\int_a^b f(x)\,dx-\frac{1}{2}(b-a)(f(a)+f(b))+\frac{5}{4}(b-a)^2(f'(b)-f'(a))\right| \\ \leq\ & \frac{(b-a)^3}{4}\int_0^1|t+1|^2\,|f''(ta+(1-t)b)+f''(tb+(1-t)a)|\,dt \\ \leq\ & \frac{(b-a)^3}{4}\left(\int_0^1|t+1|^2\,dt\right)^{1-\frac{1}{p}} \\ &\left\{\left(\int_0^1(t+1)^2\left(t|f''(a)|^p+m(1-t)\left|f''\left(\frac{b}{m}\right)\right|^p\right)dt\right)^{\frac{1}{p}}\right. \\ &\left.+\left(\int_0^1(t+1)^2\left(t|f''(b)|^p+m(1-t)\left|f''\left(\frac{a}{m}\right)\right|^p\right)dt\right)^{\frac{1}{p}}\right\} \\ \leq\ & \frac{(b-a)^3}{4}\left(\frac{7}{3}\right)^{1-\frac{1}{p}} \\ &\times\left\{\left(\frac{17|f''(a)|^p+m11\left|f''\left(\frac{b}{m}\right)\right|^p}{12}\right)^{\frac{1}{p}}+\left(\frac{17|f''(b)|^p+m11\left|f''\left(\frac{a}{m}\right)\right|^p}{12}\right)^{\frac{1}{p}}\right\}. \end{aligned}$$

□

Theorem 2.13. Let $f: I\subseteq\mathbb{R}\to\mathbb{R}$ be a differentiable mapping on I°, $a,b\in I^\circ$ with $a<b$, and $(\alpha,m)\in[0,1]^2$, and let $p>1$ with $1/p+1/q=1$. If the mapping $|f''|^q$ is (α,m)-convex on I then the following inequality holds:

$$\begin{aligned}
&\left|\int_a^b f(x)\,dx-\frac{1}{2}(b-a)(f(a)+f(b))+\frac{5}{4}(b-a)^2(f'(b)-f'(a))\right| \\
\leq\ & \frac{(b-a)^3}{4}\left(\frac{7}{3}\right)^{1-\frac{1}{p}} \\
&\times\left\{\left[\frac{4\alpha^2+16\alpha+14}{\alpha^3+6\alpha^2+11\alpha+6}|f''(a)|^p+\frac{m\alpha(7\alpha^2+30\alpha+29)}{3(\alpha+1)(\alpha+2)(\alpha+3)}\left|f''\left(\frac{b}{m}\right)\right|^p\right]^{\frac{1}{p}}\right. \\
&\left.+\left[\frac{4\alpha^2+16\alpha+14}{\alpha^3+6\alpha^2+11\alpha+6}|f''(b)|^p+\frac{m\alpha(7\alpha^2+30\alpha+29)}{3(\alpha+1)(\alpha+2)(\alpha+3)}\left|f''\left(\frac{a}{m}\right)\right|^p\right]^{\frac{1}{p}}\right\}.
\end{aligned} \tag{16}$$

Proof. Using Lemma 2.1, Definition 1.4 and power mean integral inequality, we obtain

$$\begin{aligned}
&\left|\int_a^b f(x)\,dx-\frac{1}{2}(b-a)(f(a)+f(b))+\frac{5}{4}(b-a)^2(f'(b)-f'(a))\right| \\
\leq\ & \frac{(b-a)^3}{4}\int_0^1 |t+1|^2\,|f''(ta+(1-t)b)+f''(tb+(1-t)a)|\,dt \\
\leq\ & \frac{(b-a)^3}{4}\left(\int_0^1 |t+1|^2\,dt\right)^{1-\frac{1}{p}} \\
&\left\{\left(\int_0^1 (t+1)^2\left(t^\alpha|f''(a)|^p+m(1-t^\alpha)\left|f''\left(\frac{b}{m}\right)\right|^p\right)dt\right)^{\frac{1}{p}}\right. \\
&\left.+\left(\int_0^1 (t+1)^2\left(t^\alpha|f''(b)|^p+m(1-t^\alpha)\left|f''\left(\frac{a}{m}\right)\right|^p\right)dt\right)^{\frac{1}{p}}\right\} \\
\leq\ & \frac{(b-a)^3}{4}\left(\frac{7}{3}\right)^{1-\frac{1}{p}} \\
&\times\left\{\left[\frac{4\alpha^2+16\alpha+14}{\alpha^3+6\alpha^2+11\alpha+6}|f''(a)|^p+\frac{m\alpha(7\alpha^2+30\alpha+29)}{3(\alpha+1)(\alpha+2)(\alpha+3)}\left|f''\left(\frac{b}{m}\right)\right|^p\right]^{\frac{1}{p}}\right. \\
&\left.+\left[\frac{4\alpha^2+16\alpha+14}{\alpha^3+6\alpha^2+11\alpha+6}|f''(b)|^p+\frac{m\alpha(7\alpha^2+30\alpha+29)}{3(\alpha+1)(\alpha+2)(\alpha+3)}\left|f''\left(\frac{a}{m}\right)\right|^p\right]^{\frac{1}{p}}\right\}.
\end{aligned}$$

□

Remark 2.14. i) In (16), if we choose $\alpha = 1$, we have the inequality in (15).
ii) In (16), if we choose $\alpha = m = 1$, we obtain the inequality in (14).

Corollary 2.15. i) Under the assumptions of Theorem 2.12, if we choose $p = m = 1$, we obtain the inequality;

$$\begin{aligned}
&\left|\int_a^b f(x)\,dx-\frac{1}{2}(b-a)(f(a)+f(b))+\frac{5}{4}(b-a)^2(f'(b)-f'(a))\right| \\
\leq\ & \frac{(b-a)^3}{2}\left(\frac{17|f''(a)|+11|f''(b)|}{12}\right).
\end{aligned}$$

ii) Under the assumptions of Theorem 2.13, if we choose $p = m = 1$, we obtain the inequality;

$$\left| \int_a^b f(x)\,dx - \frac{1}{2}(b-a)(f(a)+f(b)) + \frac{5}{4}(b-a)^2 (f'(b) - f'(a)) \right|$$
$$\leq \frac{(b-a)^3}{4} \frac{7\alpha^3 + 34\alpha^2 + 45\alpha + 14}{3(\alpha^3 + 6\alpha^2 + 11\alpha + 6)} \left[|f''(a)| + |f''(b)| \right].$$

3 Applications to Special Means

Now we shall use the results of Section 2 to prove the following new inequalities connecting the above means for arbitrary real numbers.

Proposition 3.1. Let $a, b \in (0, x)$ and $x > 0$, $m \in [0,1]$ with $a < b$. Then, the following inequality holds:

$$\left| -\ln I(a,b) + A(\ln a, \ln b) + \frac{5}{4} \frac{(b-a)^2}{G^2(a,b)} \right|$$
$$\leq \frac{(b-a)^2}{2} \frac{(17 + 11m^3)}{12} \frac{A(a^2, b^2)}{G^4(a,b)}.$$

Proof. The proof is immediate from Theorem 2.3 applied for $f(x) = -\ln x$, $x \in R$. □

Proposition 3.2. Let $(0, x)$, $a, b \in (0, x)$ and $x > 0$, $(\alpha, m) \in [0,1]^2$ with $a < b$. Then, the following inequality holds:

$$\left| -\ln I(a,b) + A(\ln a, \ln b) + \frac{5}{4} \frac{(b-a)^2}{G^2(a,b)} \right|$$
$$\leq \frac{(b-a)^2}{2} \left(\frac{4\alpha^2 + 16\alpha + 14}{\alpha^3 + 6\alpha^2 + 11\alpha + 6} (1 - m^3) + \frac{7m^3}{3} \right) \frac{A(a^2, b^2)}{G^4(a,b)}.$$

Proof. The proof is immediate from Theorem 2.4 applied for $f(x) = -\ln x$, $x \in R$. □

Proposition 3.3. Let $(0, x)$, $a, b \in (0, x)$ and $x > 0$, $m \in [0,1]$, $p > 1$ with $a < b$. Then, the following inequality holds:

$$\left| -\ln I(a,b) + A(\ln a, \ln b) + \frac{5}{4} \frac{(b-a)^2}{G^2(a,b)} \right|$$
$$\leq \frac{(b-a)^2}{2^{2+\frac{1}{q}} G^4(a,b)} \left(\frac{2^{2p+1} - 1}{2p+1} \right)^{1/p} \left\{ \left[b^{2q} + a^{2q} m^{1+q} \right]^{\frac{1}{q}} + \left[a^{2q} + b^{2q} m^{1+q} \right]^{\frac{1}{q}} \right\}.$$

Proof. The proof is immediate from Theorem 2.7 applied for $f(x) = -\ln x$, $x \in R$. □

Proposition 3.4. Let $(0, x)$, $a, b \in (0, x)$ and $x > 0$, $(\alpha, m) \in [0,1]^2$, $p > 1$ with $a < b$. Then, the following inequality holds:

$$\left| -\ln I(a,b) + A(\ln a, \ln b) + \frac{5}{4} \frac{(b-a)^2}{G^2(a,b)} \right|$$
$$\leq \frac{(b-a)^2}{4G^4(a,b)} \left(\frac{2^{2p+1} - 1}{2p+1} \right)^{1/p} \frac{1}{(\alpha+1)^{\frac{1}{q}}} \left\{ \left[b^{2q} + a^{2q} m^{1+q} \alpha \right]^{\frac{1}{q}} \right.$$
$$\left. + \left[a^{2q} + b^{2q} m^{1+q} \alpha \right]^{\frac{1}{q}} \right\}.$$

Proof. The proof is immediate from Theorem 2.8 applied for $f(x) = -\ln x$, $x \in R$.. □

Proposition 3.5. Let $(0,x)$, $a,b\in(0,x)$ and $x>0$, $m\in[0,1]$, $p>1$ with $a<b$. Then, the following inequality holds:

$$\begin{aligned}&\left|-\ln I(a,b)+A(\ln a,\ln b)+\frac{5}{4}\frac{(b-a)^2}{G^2(a,b)}\right|\\ \leq\;&\frac{7(b-a)^2}{12G^4(a,b)}\left(\frac{3}{84}\right)^{\frac{1}{p}}\left\{\left(17b^{2p}+11m^{p+1}a^{2p}\right)^{1/p}\right.\\&\left.+\left(17a^{2p}+11m^{p+1}b^{2p}\right)^{1/p}\right\}\end{aligned}$$

Proof. The proof is immediate from Theorem 2.12 applied for $f(x)=-\ln x$, $x\in\mathbb{R}$. □

Proposition 3.6. Let $(0,x)$, $a,b\in(0,x)$ and $x>0$, $(\alpha,m)\in[0,1]^2$, $p>1$ with $a<b$. Then, the following inequality holds:

$$\begin{aligned}&\left|-\ln I(a,b)+A(\ln a,\ln b)+\frac{5}{4}\frac{(b-a)^2}{G^2(a,b)}\right|\\ \leq\;&\frac{(b-a)^2}{4G^4(a,b)}\left(\frac{7}{3}\right)^{1-\frac{1}{p}}\\&\times\left\{\left(\frac{(4\alpha^2+16\alpha+14)\,b^{2p}}{(\alpha^3+6\alpha^2+11\alpha+6)}+\frac{m^{p+1}a^{2p}\left(7\alpha^3+30\alpha^2+29\alpha\right)}{3(\alpha^3+6\alpha^2+11\alpha+6)}\right)^{\frac{1}{p}}\right.\\&\left.+\left(\frac{(4\alpha^2+16\alpha+14)\,a^{2p}}{(\alpha^3+6\alpha^2+11\alpha+6)}+\frac{m^{p+1}b^{2p}\left(7\alpha^3+30\alpha^2+29\alpha\right)}{3(\alpha^3+6\alpha^2+11\alpha+6)}\right)^{\frac{1}{p}}\right\}.\end{aligned}$$

Proof. The proof is immediate from Theorem 2.13 applied for $f(x)=-\ln x$, $x\in\mathbb{R}$. □

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ON ABELIAN FUZZY MULTI GROUPS AND ORDERS OF FUZZY MULTI GROUPS

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Abstract – As a continuation of the study of various algebraic structures of Fuzzy Multisets, in this paper the concept of Abelian fuzzy multigroups, left and right cosets of fuzzy multi groups and fuzzy multi order of an element of a group are introduced and its various properties are discussed. In the last section some of the homomorphic properties between two Fuzzy multigroups are discussed.

Keywords – *fuzzy multi group, abelian fuzzy multi group, left and right cosets of fuzzy multi groups, fuzzy multi orders, homomorphism.*

1 Introduction

Modern set theory formulated by George Cantor is fundamental for the whole Mathematics. But to represent imprecise, vague data classical set theory is insufficient. So many non classical sets were put forward to overcome this problem. Some of them are fuzzy sets, soft sets, rough sets, multisets etc. To make these non classical sets even more powerful combinations of them were also introduced in time. One of them is Fuzzy Multisets.Fuzzy Multisetsis a powerful tool for modelling quantitative and qualitative properties of objects simultaneously.

Many fields of modern mathematics have been emerged by violating a basic principle of a given theory only because useful structures could be defined this way. Set is a well-defined collection of distinct objects, that is, the elements of a set are pair wise different. If we relax this restriction and allow repeated occurrences of any element, then we can get a mathematical structure that is known as Multisets or Bags. For example, the prime

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factorization of an integer n>0 is a Multiset whose elements are primes. The number 120 has the prime factorization $120 = 2^3 3^1 5^1$ which gives the Multiset {2, 2, 2, 3, 5}. A complete account of the development of multiset theory can be seen in [1,2, 9, 10,11,12,13]. As a generalization of multiset, Yager [6] introduced the concept of Fuzzy Multiset (FMS). An element of a Fuzzy Multiset can occur more than once with possibly the sameor different membership values.

2 Preliminaries

Definition 2.1.[11] Let X be a set. A multiset (mset) M drawn from X is represented by a function Count M or C_M defined as $C_M : X \to \{0,1, 2, 3,\ldots\}$.For each $x \in X$, $C_M(x)$ is the characteristic value of x in M. Here $C_M(x)$ denotes the number of occurrences of x in M.

Definition 2.2.[10] Let X be a group. A multi set G over X is a multi group over X if the count of G satisfies the following two conditions

1. $C_G(xy) \geq C_G(x) \wedge C_G(y) \quad \forall x,y \in X$;
2. $C_G(x^{-1}) \geq C_G(x) \ \forall x \in X$

Definition 2.3.[12] If $X$is a collection of objects, then a fuzzy set A in X is a set of ordered pairs: $A = \{(x,\mu_A(x)) : x \in X, \mu_A : X \to [0,1]\}$ where $\mu_A$is called the membership function of A, and is defined from X into [0, 1].

Definition 2.4.[2] Let G be a group and $\mu \in FP(G)$ (fuzzy power set of G), then μ is called fuzzy subgroup of G if

1. $\mu(xy) \geq \mu(x) \wedge \mu(y) \ \forall x, y \in G$ and
2. $\mu(x^{-1}) \geq \mu(x) \forall x \in G$

Definition 2.5.[9] Let X be a nonempty set. A Fuzzy Multiset (FMS) A drawn from X is characterized by a function, ‘count membership’ of A denoted by CM_A such that $CM_A : X \to Q$ where Q is the set of all crisp multisets drawn from the unit interval [0,1].

Then for any $x \in X$, the value $CM_A(x)$ is a crisp multiset drawn from [0,1]. For each $x \in X$, the membership sequence is defined as the decreasingly ordered sequence of elements in $CM_A(x)$. It is denoted by$\{ \mu_A^1(x), \mu_A^2(x), \mu_A^3(x), \ldots, \mu_A^p(x) \}$; $\mu_A^1(x) \geq \mu_A^2(x) \geq \mu_A^3(x) \geq \ldots \geq \mu_A^p(x)$.

When every $x \in X$ is mapped to a finite multiset of Q under the count membership function CM_A, then A is called a finite fuzzy multiset of X.The collection of all finite multisets of X is denoted by $FM(X)$. Throughout this paper fuzzy multisets are taken from $FM(X)$.

Definition 2.6.[7] Let $A \in FM(X)$ and $x \in A$. Then $L(x;\ A) = Max\{ j : \mu_A^j(x) \neq 0 \}$
When we define an operation between two fuzzy multisets, the length of their membership sequences should be set to equal. So if $A$and B are FMS at consideration, take $L(x;\ A,B) = Max\ \{L(x;A), L(x;B)\}$. When no ambiguity arises we denote the length of membership by $L(x)$.

Basic relations and operations, assuming that A and B are two fuzzy multisets of X is taken from [7] and is given below.

a) Inclusion
$A \subseteq B \Leftrightarrow \mu_A^j(x) \leq \mu_B^j(x),\ j = 1,2, \ldots\ldots, L(x)\ \forall x \in X$.

b) Equality
$A = B \Leftrightarrow \mu_A^j(x) = \mu_B^j(x),\ j = 1,2, \ldots\ldots, L(x)\ \forall x \in X$.

c) Union
$\mu_{A\cup B}^j(x) = \mu_A^j(x) \vee \mu_B^j(x),\ j = 1,2, \ldots\ldots, L(x)$ where $\vee$ is the maximum operation.

d) Intersection
$\mu_{A\cap B}^j(x) = \mu_A^j(x) \wedge \mu_B^j(x),\ j = 1,2, \ldots\ldots, L(x)$where $\wedge$ is the minimum operation.

By $CM_A(x) \geq CM_A(y)$ it is taken that $\mu_A^i(x) \geq \mu_A^i(y)$ $\forall i = 1, \ldots, \text{Max}\{L(x), L(y)\}$. And $CM_A(x) \wedge CM_A(y)$ means that $\{\mu_A^i(x) \wedge \mu_A^i(y)\}$ $\forall i = 1, \ldots, \text{Max}\{L(x), L(y)\}$. And by $CM_A(x) \vee CM_A(y)$ we mean $\{\mu_A^i(x) \vee \mu_A^i(y)\}$ $\forall i = 1, \ldots, \text{Max}\{L(x), L(y)\}$.

Definition 2.7.[8] Let $A \in FM(X)$. Then A^{-1} is defined as $CM_{A^{-1}}(x) = CM_A(x^{-1})$.

Definition 2.8.[8] Let $A, B \in FM(X)$. Then define $A\ o\ B$ as

$$CM_{AoB}(x) = \vee\{CM_A(y) \wedge CM_B(z)\ ;\ y, z \in X \text{ and } yz = x\}. \text{ Also}$$
$$CM_{AoB}(x) = \vee_{y\in X}\{CM_A(y) \wedge CM_B(y^{-1}x)\} \forall x \in X$$
$$= \vee_{y\in X}\{CM_A(xy^{-1}) \wedge CM_B(y)\} \forall x \in X.$$

Definition 2.9.[8] Let Xbe a group. A fuzzy multiset G over X is a fuzzy multi group (FMG) over X if the count (count membership) of G satisfies the following two conditions.

1. $CM_G(xy) \geq CM_G(x) \wedge CM_G(y)\ \ \forall x, y \in X$.
2. $CM_G(x^{-1}) \geq CM_G(x) \forall x \in X$.

Definition 2.10.[8] Let $A \in FM(X)$. Then
$A[\alpha, n] = \{x \in X: \mu_A^j(x) \geq \alpha\ ; L(x) \geq j \geq n \text{ and } j, n \in \mathbb{N}\}$. This is called n-α level set of A.

Definition 2.11.[8] Let $A \in FM(X)$. Then define $A^* = \{x \in X: CM_A(x) = CM_A(e)\}$.

Proposition 2.12.[8] Let $A \in FMG\ (X)$. Then

a) $CM_A(e) \geq CM_A(x)\ \ \forall x \in X$.
b) $CM_A(x^n) \geq CM_A(x)\ \ \forall x \in X$.
c) $A^{-1} \supseteq A$.

Proposition 2.13.[8] Let $A \in FM(X)$. Then $A \in FMG(X)$ iff $CM_A(xy^{-1}) \geq CM_A(x) \wedge CM_A(y)$ $\forall x, y \in X$.

Proposition 2.14.[8] If $A \in FMG(X)$, and H is a subgroup of X, then $A|_H$ (i.e. A restricted to H)$\in FMG(H)$ and is a fuzzy multi subgroup of A .

Proposition 2.15.[8] *Let* $A \in FMG(X)$*. Then A[α, n] are subgroups of X.*

Proposition 2.16.[8] *Let* $A \in FMG(X)$*. Then A* is a subgroup of X.*

Some of the basic properties of groups are given below.

Definition 2.17.[14]Let (G,*),(G',o) be two groups. A mapping $\emptyset : G \to G'$ is called a homomorphism if $\emptyset\,(a * b) = \emptyset(a) o \emptyset(b), a, b \in G.$

Definition 2.18.[14] Let $\emptyset : G \to G'$ be a homomorphism. Then the kennel of $\emptyset$ is the set of all those elements of G which are mapped to the identity element of G'. That is
$\text{Ker}\ \emptyset = K_{\emptyset} = \{x \in G : \emptyset\,(x) = e'\}$ where e'is the identity element of G'.

Proposition 2.19.[14] Let $\emptyset : G \to G'$ be a homomorphism. Then

$$\emptyset(e) = e', \emptyset(x^{-1}) = [\emptyset(x)]^{-1}$$

Proposition 2.20. [14] Let $\emptyset: G \to G$' with kernel K. Then K is a normal subgroup of G.

Definition 2.21.[14] A one-one homomorphism from G onto G' is called an isomorphism.

Definition 2.22.[14] Two groups G, and G* are said tobe isomorphic if there is an isomorphism of G onto G*.

Note :- If G and G* are isomorphic then both groups will have the same properties.

Definition 2.23.[14] An isomorphism of a group G to itself is called an Automorphism.

3. Abelian Fuzzy Multi Group

Proposition 3.1. Let $A \in FMG(x)$. Then the following assertions are equivalent.

a) $CM_A(xy) = CM_A(yx), \quad x, y \in X$
b) $CM_A(xyx^{-1}) = CM_A(y), \ x, y \in X$
c) $CM_A(xyx^{-1}) \geq CM_A(y), \ x, y \in X$
d) $CM_A(xyx^{-1}) \leq CM_A(y), \ x, y \in X$

Proof. (a) ⟹ (b) Let $x, y \in X$. Then $CM_A(xyx^{-1}) = CM_A(x^{-1}xy) = CM_A(y)$

(b) ⟹ (c)Straight forward

(c) ⟹ (d)$CM_A(xyx^{-1}) \leq CM_A[x^{-1}(xyx^{-1})(x^{-1})^{-1}] = CM_A(y)$

(d) ⟹ (a)Let $x, y \in X$
Then $CM_A(xy) = CM_A[x(yx)x^{-1}] \leq CM_A(yx) = CM_A[y(xy)y^{-1}] \leq CM_A(xy)$

Hence $CM_A(xy) = CM_A(xy)$.Thus the above assertions are equivalent.

Definition 3.2. G ∈*FMG*(*X*)is called an *Abelian fuzzy multi groupover X*, if $CM_G(xy) = CM_G(yx) \forall x, y \in X$. Let *AFMG*(*X*) denote the set of all abelian fuzzy multi groups over *X*.

Example 3.3. Let *X* be an abelian group and *G* be a FMG of *X*. Then *G* is an abelian FMG over *X*.

Proposition 3.4. Let *A*∈*AFMG*(*X*). Then A^*, $A[\alpha,n]$; $n \in N$, $\alpha \in [0,1]$ are normal subgroups of *X*.

Proof. By Propositions 2.15 and 2.16 A^* and $A[\alpha,n]$ are subgroups of *X*.

1. Let $x \in X$ and $y \in A^*$. So $CM_A(y) = CM_A(e)$. Since $A \in AFMG(X)$
$CM_A(xy) = CM_A(yx) \forall x, y \in X$. So $CM_A(xyx^{-1}) = CM_A(y) = CM_A(e)$ by (3.1.)
So $x^{-1} \in A^*$. Hence the proof by the definition of normal subgroup.

2. Let $x \in X$ and $y \in A[\alpha, n]$. Since $A \in AFMG(X)$, $CM_A(xy) = CM_A(yx) \forall x, y \in X$. So $CM_A(xyx^{-1}) = CM_A(y)$ by(3.1.) So $xyx^{-1} \in A[\alpha, n]$.Hence the proof by the definition of normal subgroup.

Proposition 3.5. Let *A*∈*AFMG*(*X*). Then A^j; $j \in N$, isnormal subgroup of *X* iff $\mu_A^{j+1}(xy^{-1}) = 0 \ \forall x, y \in A^j$.

Proof. In [8] it is proved thatA^j is a subgroup of *X*iff $\mu_A^{j+1}(xy^{-1}) = 0 \ \forall x, y \in A^j$. Let $x \in X$ and $y \in A^j$. So $\mu_A^j(y) > 0$ and$\mu_A^{j+1}(y) = 0$. Since *A*∈*AFMG*(*X*) , $CM_A(xy) = CM_A(yx) \ \forall x, y \in X$. So $CM_A(xyx^{-1}) = CM_A(y)$ by (3.1.). Then $\mu_A^j(xyx^{-1}) > 0 \ and \ \mu_A^{j+1}(xyx^{-1}) = 0$. So $xyx^{-1} \in A^j$. Hence the proof by the definition of normal subgroup.

Corollary 3.6. Let *A*∈*AFMG*(*X*). Then A^j ; $j \in N$, isnormal subgroup of *X* iff $A^j$is a subgroup of *X*.

Definition 3.7. Let *X* be a group and*A*∈*FM*(*X*).Then$[CM_A(e)]_x$ ∈*FM*(*X*) with only one element *x* and $CM_{[CM_A(e)]_x}(x) = CM_A(e)$.

Definition 3.8. Let *X*be a group *H*∈*FMG*(*X*) and *x*∈*X*. Also let *e* be the identity element of *X*. Then

a) the FMS $[CM_H(e)]_x \ o \ H$ is called a *left fuzzy multi coset* (LFMC) of *H* in *X* and is denoted by *xH*, where

$$\begin{aligned} CM_{xH}(z) &= \vee \{ CM_{[CM_H(e)]_x}(u) \wedge CM_H(v) \, ; uv = z \} \\ &= CM_{[CM_H(e)]_x}(x) \wedge CM_H(x^{-1}z) \\ &= CM_H(e) \wedge CM_H(x^{-1}z) \quad \text{by (3.9.)} \\ &= CM_H(x^{-1}z) \quad \text{by (2.10)} \end{aligned}$$

b) the FMS $H \, o[CM_H(e)]_x$is called a *right fuzzy multi coset* (RFMC) of *H* in *X* and is denoted by *Hx*, where

$$CM_{Hx}(z) = \vee \{ CM_H(u) \wedge CM_{[CM_H(e)]_x}(v) \ ; uv = z \}$$
$$= CM_H(zx^{-1}) \wedge CM_{[CM_H(e)]_x}(x)$$
$$= CM_H(zx^{-1}) \wedge CM_H(e)$$
$$= CM_H(zx^{-1})$$

Remark 3.9. If $H \in AFMG(X)$, then $xH = Hx$, $\forall x \in X$.

Proof. Let $H \in AFMG(X)$

$$CM_{Hx}(z) = CM_H(zx^{-1})$$
$$= CM_H(x^{-1}z)$$
$$= CM_{xH}(z)$$

Proposition 3.10. Let $H \in FMG(X)$. Then $\forall x, y \in X$,

a) $xH = yH \Leftrightarrow xH^* = yH^*$
b) $Hx = Hy \Leftrightarrow H^*x = H^*y$

Proof. a) Let $xH = yH$. Then $CM_{xH}(z) = CM_{yH}(z)$ and hence
$CM_H(x^{-1}z) = CM_H(y^{-1}z)$ $\forall z \in X$.
Now since z is arbitrary, put $z = y$, we get $CM_H(x^{-1}y) = CM_H(y^{-1}y) = CM_H(e)$
Thus $x^{-1}y \in H^*$ and hence $xH^* = yH^*$
Conversely, let $xH^* = yH^*$. Thus $x^{-1}y$, $y^{-1}x \in H^*$. …….(1)
Now $CM_H(x^{-1}z) = CM_H([x^{-1}y][y^{-1}z])$ by associativity of group
$\geq CM_H(x^{-1}y) \wedge CM_H(y^{-1}z)$ by (2.9)
$= CM_H(e) \wedge CM_H(y^{-1}z)$ by (1)
$= CM_H(y^{-1}z)$ $\forall z \in X$
Similarly $CM_H(y^{-1}z) \geq CM_H(x^{-1}z)$ $\forall z \in X$.
So $CM_{xH}(z) = CM_{yH}(z)$ $\forall z \in X$. Hence the proof.

b) Proof is similar to part (a).

Proposition 3.11. Let $H \in AFMG(X)$. If $xH = yH$, then $CM_H(x) = CM_H(y)$ $\forall x, y \in X$.

Proof. Let $xH = yH$. Then $CM_{xH}(z) = CM_{yH}(z)$ and hence $CM_H(x^{-1}z) = CM_H(y^{-1}z)$ $\forall z \in X$.
Now since z is arbitrary, put $z = y$, we get $CM_H(x^{-1}y) = CM_H(y^{-1}y) = CM_H(e)$.
Thus $x^{-1}y \in H^*$. Similarly $y^{-1}x \in H^*$. ………….(1)
Since $H \in AFMG(X)$, it follows that $CM_H(x) = CM_H(y^{-1}xy)$ by (3.1.)
$\geq CM_H(y^{-1}x) \wedge CM_H(y)$ by (2.9.)
$= CM_H(e) \wedge CM_H(y)$ by (1) and (2.10)
$= CM_H(y)$
Similarly $CM_H(y) \geq CM_H(x)$ and hence the proof.

Definition 3.12. Let $A \in FMG(X)$. Then
$[A] = \{x : x \in X \text{ and } CM_A(xy) = CM_A(yx) \ \forall \ y \in X\}$ is called the *normalizer of A in X*.

Proposition 3.13. Let $A \in FMG(X)$. Then $[A]$ is a subgroup of X and $A|_{[A]} \in AFMG([A])$.

Proof. Clearly $e \in [A]$. Let $x, y \in [A]$. Then $\forall\, z \in X$,

$$
\begin{aligned}
CM_A([xy^{-1}]z) &= CM_A(x[y^{-1}z]) \\
&= CM_A([y^{-1}z]x) \quad \text{by } x \in [A] \text{ and } y^{-1}z \in X. \\
&\geq CM_A([x^{-1}z^{-1}]y) \quad \text{by (2.9)} \\
&= CM_A(y[x^{-1}z^{-1}]) \quad \text{by } y \in [A] \text{ and } x^{-1}z^{-1} \in X. \\
&\geq CM_A(z\,[xy^{-1}]) \text{by (2.9)} \ \ldots\ldots\ldots\ldots\ldots\ldots\ (1)
\end{aligned}
$$

$$
\begin{aligned}
CM_A(z\,[xy^{-1}]) &\geq CM_A(y[x^{-1}z^{-1}]) \\
&= CM_A([x^{-1}z^{-1}]y) \\
&\geq CM_A([y^{-1}z]x) \\
&\geq CM_A([xy^{-1}]z) \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)
\end{aligned}
$$

From (1) & (2) $xy^{-1} \in [A]$.So $[A]$ is a subgroup of X. By (2.14) it is proved$A|_{[A]} \in FMG([A])$. And by the definition of AFMG the proof is complete.

Proposition 3.14. Let A, $B \in FMG(X)$ and $A \subseteq B$. Then the following assertions are equivalent.

a) $CM_A(xyx^{-1}) \geq CM_A(y) \wedge CM_B(x) \quad \forall \ x, y \in X$
b) $CM_A(yx) \geq CM_A(xy) \wedge CM_B(y) \quad \forall \ x, y \in X$
c) $[CM_A(e)]_x \ o\ A \supseteq (Ao[CM_A(e)]_x) \cap B$

Proof. (a) ⇒ (b)

Since $A \subseteq B CM_A(yx) = CM_A(yxyy^{-1}) \geq CM_A(xy) \wedge CM_B(y)$ by(a)

1. (b) ⇒ (c)

$$
\begin{aligned}
\forall\ z \in X, CM_{([CM_A(e)]_x\ o\ A)}(z) &= \vee_{x\in X}[CM_{[CM_A(e)]_x}(x) \wedge CM_A(x^{-1}z)] \\
&\geq CM_A(e) \wedge CM_A(x^{-1}z) \\
&= CM_A(x^{-1}z) \\
&\geq CM_A(z^{-1}x) \text{ by(2.9.)} \\
&\geq CM_A(xz^{-1}) \wedge CM_B(z^{-1}) \text{ by (b)} \\
&\geq CM_A(zx^{-1}) \wedge CM_B(z^{-1}) \text{ by (2.9.)} \\
&= CM_{Ax}(z) \wedge CM_B(z) \text{ by(3.8 and 2.9)} \\
&= \big(CM_{Ax}(z) \cap CM_B(z)\big)(z)
\end{aligned}
$$

2. (b) ⇒ (a) $\forall \ x, y \in X$

$$
\begin{aligned}
CM_A(x[yx^{-1}]) &\geq CM_A([yx^{-1}]x) \wedge CM_B(x) \\
&= CM_A(y) \wedge CM_B(x)
\end{aligned}
$$

3. (c) ⇒(b) $\forall \ x, y \in X$

$$
\begin{aligned}
CM_A(yx) &= CM_A(x^{-1}y^{-1}) \text{ by(2.9)} \\
&= CM_{xA}(y^{-1}) \\
&\geq CM_{(Ax)\cap B}(y^{-1}) \text{ by(c)} \\
&= CM_A(y^{-1}x^{-1}) \wedge CM_B(y^{-1}) \\
&\geq CM_A(xy) \wedge CM_B(y) \quad \text{Hence the proof by (2.9.) .}
\end{aligned}
$$

4 Fuzzy Multi Order

4.1 Fuzzy Multi Order of an Element of a Group

Throughout the rest of the paper we consider X as a group with finite order. And $A \in FMG(X)$. Also $x, y \in X$.

Definition 4.1.1. Let A be a FMG of X and $x \in X$. The least positive integer n such that $CM_A(x^n) = CM_A(e)$ is known as fuzzy multi order of x w.r.t. A and is denoted by ($O(x)$;A). If no such n exists, x is said to be of infinite order w.r.t. A.

Example 4.1.2. (Z_4, $+_4$) is a group. Let A = {(.6, .4, .3, .1)/2, (.9, .8, .7, .5, .1, .1)/0} is a fuzzy multi group. $CM_A(2^2) = CM_A(e)$. So (O(2);A) = 2.

Also $O(x) = O(y)$ does not imply ($O(x)$;A) = ($O(y)$;A). It is illustrated below. Consider the Klein four cycle X = {e, a, b, c}. Then A = {(.6, .4, .3, .1)/a, (.9, .8, .7, .5, .1, .1)/b, (.9, .8, .7, .5, .1, .1)/c, (.9, .8, .7, .5, .1, .1)/e}. Here O(a) = O(b) = O(c). But (O(a);A) ≠ (O(b); A) = (O(c);A).

Proposition 4.1.3. Let $A \in FMG(X)$. If $CM_A(x^m) = CM_A(e)$, for some positive integer m, then ($O(x)$;A)|m.

Proof. Let ($O(x)$; A) = n. Given $CM_A(x^m) = CM_A(e)$. Hence n≤m.
By division algorithm ∃integers s , t such that $m = ns+t$; $0 \leq t < n$. The

$$
\begin{aligned}
CM_A(x^t) &= CM_A(x^{m-ns}) \\
&= CM_A(x^m(x^n)^{-s}) \\
&\geq CM_A(x^m) \wedge CM_A(x^n)^{-s} \text{ by (2.9)} \\
&= CM_A(e) \wedge CM_A(x^{ns})^{-1} \\
&= CM_A(x^{ns})^{-1} \text{ by 2.12(a)} \\
&\geq CM_A(x^{ns}) \text{ by 2.9} \\
&= CM_A(x^n)^s) \\
&\geq CM_A(x^n) \text{ by 2.12(b)} \\
&= CM_A(e)
\end{aligned}
$$

So $CM_A(x^t) = CM_A(e)$. Hence t=0 by the minimality of n. i.e. m= ns. Hence the proof.

Proposition 4.1.4. Let $A \in FMG(X)$. Then$\forall\, x \in X$, ($O(x)$;A)|$O(x)$.

Proof. $O(x) = m$, ($O(x)$;A) =n
$CM_A(x^m) = CM_A(e)$
So ($O(x)$;A) = $n \leq O(x) = m$ (Since n is the least)
Let $n \nmid m$ and let $m = np + q$; $0 < q < n$. Then

$$
\begin{gathered}
x^m = x^{np+q} \\
e = x^{np}x^q \\
x^{-q} = x^{np} \\
CM_A(x^{-q}) = CM_A(x^{np})
\end{gathered}
$$

Thus $CM_A(x^{np}) \geq CM_A(x^q)$ by 2.9
$CM_A(x^q) = CM_A((x^n)^p) = CM_A(e)$ by (2.9.)
i.e. $\exists 0 < q < n$; $CM_A(x^q) = CM_A(e)$. This is a contradiction to (*O*(*x*); *A*) =*n*.
Hence the proof.

Proposition 4.1.5. Let $A \in FMG(X)$. Let *x*, $y \in X$ such that ((*O*(*x*) ;*A*), (*O*(*y*); *A*)) = 1 and *xy* = *yx*. Then if $CM_A(xy) = CM_A$(e), then $CM_A(x) = CM_A(y) = CM_A(e)$.

Proof. Let (*O*(*x*); *A*) = *n*, (*O*(*y*); *A*) = *m*. …….. (1)

$CM_A(e) = CM_A(xy)$ (given)
$\leq CM_A((xy)^m)$by 2.12(b)
$= CM_A(x^m y^m)$ …………… (2)

Hence $CM_A(e) = CM_A(x^m y^m)$ by (2.10.) Now

$$\begin{aligned} CM_A(x^m) &= CM_A(x^m y^m y^{-m}) \\ &\geq CM_A(x^m y^m) \wedge CM_A((y^m)^{-1}) \quad \text{by (2.9.)} \\ &\geq CM_A(e) \wedge CM_A(e) \quad \text{by (1) and (2) and 2.9} \\ &= CM_A(e). \end{aligned}$$

Thus $CM_A(x^m) = CM_A(e)$. Then n|m. (by 4.1.4). But (*n*, *m*) =1 (given). Thus *n* = 1.
i.e. $CM_A(x) = CM_A(x^n) = CM_A(e)$. Similarly $CM_A(y) = CM_A(e)$.

Corollary 4.1.6. Let $A \in FMG(X)$. Let *x*, $y \in X$ such that (*O*(*x*), *O*(*y*)) = 1and *xy* = *yx*. Then if $CM_A(xy) = CM_A$(e), then $CM_A(x) = CM_A(y) = CM_A(e)$.

Proof. (*O*(*x*), *O*(*y*)) = 1
(*O*(*x*) ;*A*)\ *O*(*x*). by (4.1.5.). Then
((*O*(*x*) ;*A*), (*O*(*y*); *A*)) = 1 Then the proof by (4.1.5)

Theorem 4.1.7. Let A $\in FMG(X)$. Let (*O*(*x*); *A*) = *n* ; $x \in X$. If *m* is an integer with (*m*,*n*) = *d*, then $(O(x^m); A) = n/d$.

Proof. Let $(O(x^m); A) = t$. Now

$$\begin{aligned} CM_A\left((x^m)^{n/d}\right) &= CM_A(x^{nk})\,; \quad m/d = k \in \mathbb{Z}^+ \\ &\geq CM_A(x^n) \quad \text{by (2.12)} \\ &= CM_A(e) \end{aligned}$$

i.e. $CM_A\left((x^m)^{n/d}\right) = CM_A(e)$. Thus *t*|(*n*/*d*) by(4.1.3) …………(1)
Now, since (*m*, *n*) = *d*, $\exists i, j \in \mathbb{Z}$such that $ni + mj = d$.So

$$\begin{aligned} CM_A(x^{td}) &= CM_A\left(x^{t(ni+mj)}\right) \\ &\geq CM_A\left((x^n)^{ti}\right) \wedge CM_A(((x^m)^t)^j) \\ &\geq CM_A(x^n) \wedge CM_A((x^m)^t) \quad \text{by (2. 12)} \\ &= CM_A(e) \end{aligned}$$

$$CM_A(x^{td}) = CM_A(e)$$

So $n|(td)$ by(4.1.3) i.e. $(n/d)|(td/d)$.⟹ $(n/d)|t$. ……..(2)
$t = (n/d)$ by (1) and (2). Hence the proof

Proposition 4.1.8. Let $A \in FMG(X)$. Let $(O(x); A) = n$; $x \in X$. If m is an integer with $(m, n) = 1$, then $CM_A(x^m) = CM_A(x)$.

Proof. Since $(m, n) = 1$, $\exists i, j \in \mathbb{Z}$ such that $ni + mj = 1$. We then have

$$\begin{aligned} CM_A(x) &= CM_A(x^{ni+mj}) \\ &\geq CM_A((x^n)^i) \wedge CM_A((x^m)^j) \quad \text{by (2.9)} \\ &\geq CM_A(x^n) \wedge CM_A(x^m) \\ &\geq CM_A(e) \wedge CM_A(x^m) \end{aligned}$$

$CM_A(x) \geq CM_A(x^m)$ So
$CM_A(x^m) = CM_A(x)$. by(2.12(b))

Theorem 4.1.9. Let $A \in FMG(X)$. Let $(O(x); A) = n$; $x \in X$. If $i \equiv j (mod n)$;$i, j \in \mathbb{Z}$. Then $(O(x^i); A) = (O(x^j); A)$.

Proof. $(O(x^i); A) = t$, $(O(x^j); A) = s$. Also $i = j + nk$; $k \in \mathbb{Z}$. So

$$\begin{aligned} CM_A((x^i)^s) &= CM_A((x^{j+nk})^s) \\ &\geq CM_A((x^j)^s) \wedge CM_A((x^n)^{ks}) \\ &\geq CM_A(e) \wedge CM_A(x^n) \\ &= CM_A(e). \end{aligned}$$

Then $CM_A((x^i)^s) = CM_A(e)$. by (2.12)
So $t|s$. Similarly by $CM_A((x^j)^t) = CM_A(e)$ we get $s|t$. Thus $t = s$.

Proposition 4.1.10. Let $A \in FMG(X)$. Let $x, y \in X$ such that $((O(x);A), (O(y); A)) = 1$ and $xy = yx$. Then $(O(xy);A) = [(O(x); A)][(O(y);A)]$.

Proof. Let $(O(xy);A) = n$, $(O(x); A) = s$, $(O(y);A) = t$. Then $(t,s) = 1$ (given)

$$\begin{aligned} CM_A((xy)^{st}) &\geq CM_A(x^{st}) \wedge CM_A(y^{st}) \\ &\geq CM_A(x^s) \wedge CM_A(y^t) \\ &= CM_A(e) \wedge CM_A(e) \\ &= CM_A(e). \end{aligned}$$

So $n|st$ by (4.1.3) ………………(1)
Now $CM_A(e) = CM_A((xy)^n) = CM_A(x^n y^n)$.
Since $n|st$ and $(t,s) = 1$, $n|s$ or $n|t$. Assume $n|t$, then $(n,s) = 1$.
So $(O(x^n); A) = s/(n,s) = s$. by (4.1.7)
Also by the same $(O(y^n); A) = t/(n, t)$. ……………..(a)
Since $(s,t) = 1$, we have $(s, (t/(n,t))) = 1$.
Thus $((O(x^n); A), (O(y^n); A)) = 1$. by (a)
Also $CM_A((xy)^n) = CM_A(x^n y^n) = CM_A(e)$. *Since* $(O(xy);A) = n$.

Also $CM_A(x^n) = CM_A(y^n) = CM_A(e)$
So $s|n$ and $t|n$ by(4.1.3).
Now since $(s, t) = 1$, $(st)|n$. ……………..(2) Then from (1) and (2) $n = st$.

Proposition4.1.11. Let $A \in FMG(X)$. Let $z \in X$. $(O(z); A) = mn$ with $(m, n) = 1$, then $\exists x, y \in X$ such that $xy = yx$, $(O(x); A) = m$, and $(O(y); A) = n$.

Proof. $(m,n) = 1 \Longrightarrow \exists s, t \in \mathbb{Z}$ such that $ms + nt = 1$. ……………..(1)
So $(m,t) = (n,s) = 1$. Let $x = z^{nt}$, $y = z^{ms}$. Then $xy = z^{nt} z^{ms} = z^{ms} z^{nt} = yx = z^{nt+ms} = z$ by (1)
Given $(O(z); A) = mn$. So by (4.1.7)
$(O(x); A) = (O(z^{nt}); A) = mn/(mn, nt) = m/(m,t) = m$. (since $(m, t) = 1$)
Similarly $(O(y); A) = n$. This proves the existence of x and y.

4.2 Fuzzy Multi Order in Cyclic Groups

In this section we consider X as a cyclic group with finite order. And $A \in FMG(X)$.

Lemma4.2.1. Let $A \in FMG(X)$. And let a, b be two generators of X. Then

$$(O(a); A) = (O(b); A).$$

Proof. Let $|X| = n$. $O(a) = O(b) = n$. Now $b = a^p$; $p \in \mathbb{N}$. So $(p,n) = 1$. Let $(O(a);\ A) = m$. Then $m|n$ by (4.1.4). Then
$(O(b);\ A) = (O(a^p);\ A) = m/(p, m) = m = (O(a);\ A)$. by (4.1.7)
and since $(p,n) = 1$. So $(O(a);A) = (O(b);A)$.

Theorem 4.2.2. Let $A \in FMG(X)$, with $|X| = n$. Then the following assertions hold $\forall\ x, y \in X$.

a) If $O(x) | O(y)$, then $(O(x);A) | (O(y);A)$.
b) If $O(x) = O(y)$, then $(O(x);A) = (O(y);A)$.
c) If $O(x) > O(y)$, then $(O(x);A) \geq (O(y);A)$.

Proof. Let $X = (a)$, $x = a^s$, $y = a^t$ and $(O(a);A) = m$. Hence $O(a) = n$. Now by (4.2.1.) m is independent of a particular choice of a generator a of X. Thus $O(x) = n/(s,n)$, By the property of a cyclic group $O(y) = n/(t,n)$ ………….(1)

$(O(x);A) = (O(a^s);A) = m/(s,m)$ by(4.1.7) Similarly $(O(y);A) = (O(a^t);A) = m/(t,m)$
By (4.1.4) $m|n$. …………..(2)

a) If $O(x) | O(y)$, then by (1) $\{n/(s,n)\} | \{n/(t,n)\} = (t,n)|(s,n)$.
Now by (2) $m|n \Rightarrow n = km$; $k \in Z$. So $(t, mk)|(s, mk)$. i.e. $(t,m)|(s,m)$.
Hence $m/(s,m) | m/(t,m)$. Hence the proof.

b) Result follows from (a)

c) $O(x) > O(y)$, then $n/(s,n) > n/(t,n)$ So $(s,n) < (t,n)$. So $(s,m) \leq (t,m)$ by $m|n$.

5 Homomorphism between Fuzzy Multigroups

Proposition 5.1 Let x, y be two groups and f: x→y be a homomorphism. If A∈ FMG$(x) then\ f(A)$ ∈ FMG(y)

Proof. Let U,V,∈Y

Case I:
Let u,$v \in f(x)$. Then
$CM_{f(A)}(u) \wedge CM_{f(A)}(v) = 0 \wedge 0 \leq CM_{f(A)}(uv)$
$CM_{f(A)}(u^{-1}) \geq 0 = CM_{f(A)}(u)$

Case II:
$u \notin f(x)$, (&$v \in f(x)$. similarly vice versa).
$CM_{f(A)}(u) \wedge CM_{f(A)}(v) = 0 \wedge CM_{f(A)}(v) = 0 \leq CM_{f(A)}(uv)$

Case III:
Let $u, v \in f(x)$ Then there exist $x, y \in$ X such that $f(x) = u, f(y) = v$
Now $CM_{f(A)}(u) = \vee \{CM_{(A)}(w): w \in X; f(w) = uv\}$ ⟶(1)
$\geq \{CM_{(A)}(xy): x, y \in X, f(x) = u, f(y) = v\}$
(Since $xy \in$ (1) by the definition of homomorphism. i.e,
$f(xy) = f(x)f(y) = uv) \geq \left\{CM_{(A)^{(U)}} \wedge CM_{(A)^{(y)}}: x, y \in X, f(x) = u, f(y) = v\right\}$
$since\ A \in FMG(X) =$ $[\vee \{CM_A(x): x \in X, f(x) = u\}] \wedge [\vee \{CM_A(y): y \in X: f(y) = v\}]$
$= CM_{f(A)}(u) \wedge CM_{f(A)}(v)$ ⟶(2)
Also
$CM_{f(A)}(u^{-1}) = \vee \{CM_A(z^{-1}) : z^{-1} \in X, f(z^{-1}) = u\}$
$\geq \vee \{CM_A(z): z \in X, f(z^{-1}) = u^{-1}\}$ $(A \in FMG(X))$
$= \{CM_A(z): z \in X; f(z) = u\}$

(f(z^{-1}) = u^{-1}⟹(f(z^{-1}))$^{-1}$ =(u)-1 ⟹ (f(z^{-1}))$^{-1}$ =u⟹ f(z)=u, property of h-ism)
= $CM_{f(A)}(u)$ → (3)

From (2) and (3) $f(A) \in FMG(y)$

Proposition 5.2 Let x,y, be two groups and f: x →y be a homomorphism. If

$$B \in FMG(y) than\ f^{-1} \in FMG(x).$$

Proof. Let x,y ∈ X

Case I, Case II, is similar to proposition 5.1.

Case III
Let x,y ∈ $f^{-1}(B)$. Then there exist u,v,∈ y *such that* $f^{-1}(u)$ = x and $f^{-1}(v) = y$.
Now $CM_{f^{-1}(B)}(xy)$ = $CM_B f(xy)$ (By definition of inverse)
$= CM_B(f)f(y)$ (Definition of homomorphism)
$\geq CM_B f(x) \wedge CM_B(f(y)$ (Since $B \in FMG(y)$

$= CM_{f^{-1}(B)}(x) \wedge CM_{f^{-1}(B)}(y)$

Now $CM_{f^{-1}(B)}(x^{-1}) = CM_B(f(x^{-1}))$ (By definition of inverse)

$= CM_B(f(x))^{-1}$ (By definition of homomorphism)

$\geq CM_B(f(x))$ (Since $B \in FMG(y)$)

$= CM_{f^{-1}(B)}(x)$

Proposition 5.3 Let $H \in AFMG(X)$ and Y be a group. Suppose that $f: X \to Y$ be an onto homomorphism. Then $f(H) \in AFMG(Y)$.

*Proof.*By proposition 5.1, $f(H) \in FMG(Y)$. Now let $y, z \in Y$. Since f is onto, there exist $u \in X$. Such that $f(u) = z$. Thus

$$CM_{f(H)}(zyz^{-1}) = \vee \{CM_H(w): w \in X, f(w) = zyz^{-1}\}$$
$$= \vee \{CM_H(u^{-1}wu): w \in X, f(u^{-1}wu) = y\}$$

$CM_H(w) = CM_H(u^{-1}wu)$ *since* $H \in AFMG(X)$byProposition 3.1

Now $f(w) = zyz^{-1} = f(u)y(f(u))^{-1}$
$\Rightarrow (f(u))^{-1} f(w)f(u) = y \Rightarrow f(u^{-1})f(w)f(u) = y$
$\Rightarrow \vee \{CM_A(v): \vee \in X, f(\vee) = y\} = CM_{f(H)}(y)$

Hence by proposition 3.1, $f(H) \in AFMG(y)$

Proposition 5.4 Let $H \in AFMG(y)$ and X be a group. Suppose that $f: x \to y$ be an into homomorphism. Then $f^{-1}(H) \in AFMG(X)$.

Proof. By proposition 3.1, $f^{-1}(H) \in AFMG(X)$. Let $x, z \in X$. Then

$$CM_{f^{-1}(H)}(xz) = CM_H[f(xz)] = CM_H[f(x)f(z)] = CM_H[f(z)f(x)]$$

Since f is a homomorphism and $H \in AFMG(X) = CM_H[f(zx)] = CM_{f^{-1}(H)}(zx)$
Hence $f^{-1}(H) \in AFMG(X)$.

6 Conclusions

In this paperwe introduced the concept of Abelian fuzzy multi groups and find out some of the normal subgroups of X. Also left and right cosets of fuzzy multi groups and fuzzy multi order of an element of groups are introduced and its various properties are discussed. And it became evident that Fuzzy multi order of an element of a group has some properties similar to that of order of an element in a group. And finally we discussed some of the homomorphic properties of Fuzzy multigroups.

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ERROR CORRECTING SOFT CODES FOR ODD NUMBERS WHICH ARE EQUAL OR LESS THEN (n/2-1)

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Abstract – The main purpose of this study is to determine the new encoding and decoding method. The encoding and decoding are an important tool for Coding Theory. İn this paper, we define soft codes by using definition soft sets. Also, we explain some algebraic properties of soft codes.

Keywords – *Soft Sets , Coding Theory, Soft Codes.*

1 Introduction

Soft set theory [1] was firstly introduced by Molodtsov in 1999 as general mathematical tool for dealing with uncertain, fuzzy, not clearly defined objects. He has shown several applications of this theory in solving many practical problems in economics, engineering, social science, medical science, etc. The soft set theory has been applied to many different fields with great success. Maji *et al.* [2] worked on theoretical study of soft sets in detail and [3] presented an application of soft set in the decision making problem using the reduction of rough sets [4]. Chen *et al.*[5] proposed parameterization reduction of soft sets, and then Kong *et al.* [6] presented the normal parameterization reduction of soft sets. We can say that The soft set has the similar applications with fuzzy sets and rough sets. H. Aktas and N. Cagman [7] has shown that every fuzzy set and every rough set can be considered as a soft set. In that sense we can say that this theory is much more general than its predecessors.

With the increasing importance of digital communications and data storage, there is a in the area of coding theory and channel modelling to design codes need for research for channels that are power limited or bandwidth limited. The purpose of a communication system is, in

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the broadest sense, the transmission of information from one point in space and time to another. We shall briefly explore the basic ideas of what information is and how it can be measured, and how these ideas relate to band width, capacity, signal-to-noise ratio, bit error rate and so on.

İn this paper, the coding theory which based digital communication is studied over soft set. Also this structure is used for single error- correcting. These codes have different applications from the other codes.

Through our study of error-control codes, we will model our data as strings of discrete symbols, often binary symbols $\{0,1\}$. When working with binary symbols, addition is done modulo 2. For example, $1+1 = 0 \ (mod\ 2)$. We will study channels that are corrected by additive white Gaussian noise, which we can model as a string of discrete symbols that get added symbol-wise to the code word. For example, if we wish to send the code word $c = 11111$, noise may corrupt the codeword so that the $r = 01101$ is received. In this case, we would say that the error vector is $e = 10010$, since the codeword was corrupted in the first and fourth positions. Notice that $c+e = r$, where the addition is done component-wise and modulo 2. The steps of encoding and decoding that concern us are as follows:

$$m \rightarrow Encode \rightarrow c \rightarrow Noise \rightarrow c + e = r \rightarrow Decode \rightarrow \tilde{m}$$

where m is the message, c is the code word, e is the error vector due to noise, r is the received word or vector, and $\tilde{m}$ is the decoded word or vector. The hope is that $\tilde{m} = m$.

2 Preliminaries and Notation

In this section, we present the basic definitions of soft set theory [8] and coding theory [9]. We consider a binary channel which can transmit either of two symbols 0 or 1. However, due to presence of noise a transmitted zero may sometimes be received as 1, and transmitted 1 may sometimes be received as 0. When this happens we say that there is an error in transmitting the symbol. The symbols successively presented to the channel for transmission constitute the input and the she symbols received constitute the output. Error control coding is a method to detect and possibly correct errors by introducing redundancy to the stream of bits to be sent to the channel. The Channel Encoder will add bits to the message bits to be transmitted systematically. After passing through the channel, the Channel decoder will detect and correct the errors. These definition sand more detailed explanations related to the soft sets and coding theory can be found in [10,11,12] and [13] respectively.

Throughout this work, U denotes to an set of vectors, E denotes the set of code words's weight, $A \subseteq E$ and n is the code's length, $P(U)$ is the power set of U, and $A \subseteq E$. Also, 1_n and 0_n denote that every position equals to 1 and 0, respectively.

Definition 2.1. [3] A pair (F, A) is called a soft set over U where F is a mapping given by

$$F: A \to P(U)$$

In other words, a soft set over U is parametriterized family of subsets of the universe U. For $\varepsilon \in A, F(\varepsilon)$ may be considered as the set of ε- elements of the soft set (F, A).

Definition 2.2. [3] For two soft sets (F, A) and (G, B)over U, (F, A) is called a soft subset of (G, B) if

(1) $A \subset B$ and

(2) $\forall \varepsilon \in A$ F(ε) and $G(\varepsilon)$ are identical approximations

This relationship is denoted by $(F, A) \tilde{\subset} (G, B)$.

Similarly, (F, A) is called a soft superset of (G, B), if (G, B) is a soft subset of (F, A). This relationship is denoted by $(F, A) \tilde{\supset} (G, B)$.

Definition 2.3. [3] Two soft sets $(F, A)$and (G, B) over U are called soft equal if (F, A) is a soft subset of (G, B) and (G, B) is a soft subset of (F, A).

Definition 2.4. [7] The intersection of two soft sets (F, A) and (G, B) over U is the soft set (H, C), where $C = A \cap B$ and $\forall \varepsilon \in C$, $H(\varepsilon) = F(\varepsilon)$ or $G(\varepsilon)$ (as both are same set). This is denoted by $(F, A) \tilde{\cap} (G, B) = (H, C)$.

Definition 2.5. [7] If (F, A) and (G, B) are two soft sets, then (F, A) and (G, B) is denoted $(F, A) \wedge (G, B)$. $(F, A) \wedge (G, B)$ is defined as $(H, A \times B)$, where $H(\alpha, \beta) = F(\alpha) \cap G(\beta)$ $, \forall (\alpha, \beta) \in A \times B$.

Definition 2.6. [7] The union of two soft sets (F, A) and (G, B) over U is the soft set (H, C), where $C = A \cup B$ and ∀ ε∈ C

$$H(e) = \begin{cases} F(\varepsilon) & if \quad \varepsilon \in A - B \\ G(\varepsilon) & if \ \varepsilon \in B - A \\ F(\varepsilon) U \ G(\varepsilon) & if \quad \varepsilon \in A \cap B \end{cases}$$

This relationship is denoted by $(F, A) \tilde{U} (G, B) = (H, C)$.

Definition 2.7.[14] The minimum distance of a code C is the minimum distance between any two code words in C. We can indicate as follows.

$$d(C) = min\{d(x, y) : x \neq y, x, \ y \in C\}.$$

Definition 2.8.[15] Weight $w(c)$ of a code word c is the number of nonzero components in the code words.

3 Soft Codes

Definition 3.1. Let U denotes set of vectors, $P(U)$ be the power set of U, E be the set of code words's weight, $A \subseteq E$ and n is the code's length. A soft set $(f_A, (E,n))$ on the universe U which defined by the set of ordered triads is called soft code.

$$(f_A, (E,n)) = \{(e, f_A(e), n): e \in E, f_A(e) \in P(U), n \in N\}$$

where $f_A: E \to P(U)$.

Example 3.2. Let $(F_A, (E,n))$ be a soft code over
$U = \{0, 1, 00,10,01,11,000,100,010,001,110,101,011,111 \dots\}$. We define $P(U)$ as the following for $n = 3$, $A = \{1,2\}$, $E = \{0,1,2,3\}$
$P(U) = \{000,100,010,001,110,101,011,111\}$. So that, we denote to soft code as follows
$(f_A, (E,n)) = \{(1, \{100,010,001\}, 3), (2, \{110,101,011\}, 3)\}$
$= \{\{(1, \{100\}, 3), (1, \{010\}, 3), (1, \{001\}, 3)\}, \{(2, \{110\}, 3), (2, \{101\}, 3), (2, \{101\}, 3)\}\}$

Definition 3.3. For a soft code $(F_A, (E,n))$ over U,

(a) $(f_A, (E,n))$ is said to be a zero soft code, if $A = \{0\}$. It is denoted $(e, f_A(e), n) = 0_n$.

(b) $(f_A, (E,n))$ is said to be a universal soft code, if $A = \{1\}$. It is denoted $(e, f_A(e), n) = 1_n$.

Definition 3.4. For three soft codes $(F_A, (E,n))$, $(G_B, (E,n))$, $(H_A, (E,n))$ over U,

(a) We define soft sub code as follows. $(G_B, (E,n))$ is soft sub code of $(F_A, (E,n))$, if $B \subseteq A$. It is denoted by $(G_B, (E,n)) \subseteq (F_A, (E,n))$.

(b) We define soft equal code as follows. $(F_A, (E,n))$ *and* $(G_B, (E,n))$ are equal soft codes, if $A = B$. It is denoted by $(G_B, (E,n)) = (F_A, (E,n))$.

Definition 3.5. Let $(F_A, (E,n))$, $(G_B, (E,n))$ and $(H_C, (E,n)) \in (P(U), (E,n))$;

(a) We define soft union code as follows. Union of $(F_A, (E,n))$ and $(G_B, (E,n))$ over U is soft code $(H_C, (E,n))$ where $C = A \cup B$, denoted by $(F_A, (E,n)) \cup (G_B, (E,n))$.

(b) We define soft intersection code as follows. İntersection of $(F_A, (E,n)$ and $(G_B, (E,n))$over U is soft code $(H_C, (E,n))$ where $C = A \cap B$, denoted by $(F_A, (E,n)) \cap (G_B, (E,n))$.

(c) Complement of $(F_A, (E,n))$over U, denoted by $(F_A, (E,n))^{\circ}$, $(G_B, (E,n)) = 1_n$ - $(F_A, (E,n))$.

(d) $(F_A, (E,n))$ and $(G_B, (E,n))$ are disjoint if $(F_A, (E,n)) \cap (G_B, (E,n)) = \emptyset$.

Proposition 3.6. Let $(F_A, (E,n)) \in (P(U), (E,n))$. Then

(a) $((F_A,(E,n))^{\circ})^{\circ} = (F_A,(E,n))$,
(b) $0_n{}^{\circ}=1_n$.

Proof. İt is clear from Definition 3.5.

Proposition 3.7. Let $(F_A,(E,n)), (G_B,(E,n), (H_C,(E,n)) \in (P(U),(E,n))$. Then

(a) $(F_A,(E,n)) \cup (F_A,(E,n)) = (F_A,(E,n))$,
(b) $(F_A,(E,n)) \cap (F_A,(E,n))^{\circ} = \emptyset$,
(c) $(F_A,(E,n)) \cup (G_B,(E,n) = (G_B,(E,n) \cup (F_A,(E,n))$,
(d) $((F_A,(E,n)) \cup (G_B,(E,n)) \cup (H_C,(E,n)) = (F_A,(E,n)) \cup ((G_B,(E,n)) \cup (H_C,(E,n))$.

Proof. İt is straightforward.

Proposition 3.4. Let$(F_A,(E,n)), (G_B,(E,n), (H_C,(E,n)) \in (P(U),(E,n))$. Then

(a) $(F_A,(E,n)) \cap (F_A,(E,n)) = (F_A,(E,n))$,
(b) $(F_A,(E,n)) \cap (G_B,(E,n) = (G_B,(E,n) \cap (F_A,(E,n))$,
(c)$((F_A,(E,n)) \cap (G_B,(E,n)) \cap (H_C,(E,n)) = (F_A,(E,n)) \cap ((G_B,(E,n)) \cap (H_C,(E,n))$

Proof. It is proved by using Definition 3.5.

Proposition 3.5. $(F_A,(E,n)), (G_B,(E,n) \in (P(U),(E,n))$. Then De Morgan's laws are valid

(a) $((F_A,(E,n)) \cap (G_B,(E,n))^{\circ} = ((F_A,(E,n)))^{\circ} \cap ((G_B,(E,n)))^{\circ}$,
(b) $(((F_A,(E,n)) \cup (G_B,(E,n)))^{\circ} = ((F_A,(E,n)))^{\circ} \cup ((G_B,(E,n)))^{\circ}$.

Proof.
(a) $$(((F_A,(E,n)) \cap (G_B,(E,n)))^{\circ} = 1_n - (((F_A,(E,n)) \cap (G_B,(E,n)\,)\,)$$
$$= (1_n - (((F_A,(E,n)) \cap (1_n - (G_B,(E,n)))$$
$$= (F_A,(E,n)))^{\circ} \cap (G_B,(E,n)))^{\circ}$$

(b) İt can be proved similarity.

Proposition 3.6. Let $(F_A,(E,n)), (G_B,(E,n), (H_C,(E,n)) \in (P(U),(E,n))$. Then

(a) $(((F_A,(E,n)) \cup (G_B,(E,n))) \cap (H_C,(E,n)) = (((F_A,(E,n) \cup (H_C,(E,n))) \cap ((G_B,(E,n)) \cup (H_C,(E,n)))$,

(b) $(((F_A,(E,n)) \cap (G_B,(E,n))) \cup (H_C,(E,n)) = ((F_A,(E,n) \cap (H_C,(E,n))) \cup ((G_B,(E,n))) \cap (H_C,(E,n)))$.

Proof. It is clear from Definition 3.1. and Definition 3.5.

3.1. Products of Soft Codes

In this part, we define three new definitions for soft encoding and decoding.

Definition 3.7. Let $(F_A, (E,n))$,$(G_B,(E,n)) \in (P(U),(E,n))$. We define vectorel multiplication as following.

$(F_A,(E,n)) = \{(e,\{a\},n): e \in E, \{a\} \in P(U), n \in N\}$
$(G_B,(E,n)) = \{(f,\{d\},n): f \in E, \{d\} \in P(U), n \in N\}$

Let's accept $a = (a_1 a_2 \dots a_j)$, $d = (d_1 d_2 \dots d_k)$, define vectorel multiplication as following. Also we show symbol of vectorel multiplication with "Λ".

$(F_A,(E,n)) \Lambda (G_B,(E,n)) = \{(max\ \{a_1,d_1\} max\ \{a_1,d_2\} \dots\ max\ \{a_1,d_k\}), (max\ \{a_2,d_1\}$ $max\{a_2,d_2\} \dots \{max\{a_2,d_k\}) \dots (max\ \{a_j,d_1\}, max\ \{a_j,d_2\} \dots max\ \{a_j,d_k\})$.

This multiplication is called as vectorel multiplication. Also this multiplication will create a basic for soft encoding and decoding. The soft encoding that set of a message which is showed by M is encoded by a soft code indicated by Ě. Also we make by using inverse operation decoding.

Definition 3.8. Let C be a soft code. The soft code has multiple of vectors. Each one of the vector has k information digits showed as follows.

$$(a_0, a_1, a_2, \dots, a_{k-1})$$

The each one of the soft code's elements is encoded by using Definition3.7. There are two multipliers of this product are called as message set and encoding set. The message set and encoding set is indicated M and Ě, respectively. $1_n$is not used for soft encoding and decoding.

Example 3.9. Let define the message set and the encoding set which are indicated $M$and Ě, respectively.

$M = (f_A,(E,4)) = \{(2,\{1100,1010,1001,0110,0101,0011\},4)\}$
Ě $= (f_A,(E,3)) = (0,\{000\},3)$

If we multiply sets of two codes,

C
$= \{(6,\{111111000000, 111000111000, 000111000111, 000111111000,000000111111\},12)\}$
.

Definition 3.10. The inverse operation of vectorel multiplication provides to find k information digit. This method is called soft decoding.

Example 3.11. Let's think Example 3.9. and try to solve the message which is called M. In this statement, we must note the following, while we multiply one digit with other digit the result code word consists from the large digits. If the digits equal one another, we write the common digit.
$m\Lambda$Ě $= \{(6,\{111111000000\},12)\}$

$\{xyzt\}\Lambda\{000\} = \{111111000000\}$
$x\,\Lambda\{000\} = 111 \rightarrow x = 1$
$y\,\Lambda\{000\} = 111 \rightarrow y = 1$
$z\,\Lambda\{111\} = 000 \rightarrow z = 0$
$t\,\Lambda\{111\} = 000 \rightarrow t = 0$

other elements of the message are found with similarity method.

4 $(\frac{n}{2} - 1)$ Error Correcting Soft Codes

Firstly, we proof a theorem for error correct. This theorem will generate a structure to correct.

Theorem 3.12. Distance of all of the codes which have same length and weight are always 2.

Proof.
Let x and y be same length and weight. We will examine two statements which $d(x,y)$ is even and odd.

(1) Let $d(x,y)$ be odd. In this statement, $d(x,y) = 2n + 1$ but this means $w(x) \neq w(y)$. This statement is contradiction with our acceptation.

(2) Let $d(x,y)$ be even. In this statement, let be $d(x,y) = 2n$. This sort codes are cyclic but not linear so ıf 10… is an element in code, 01… is an element in code from cyclic definition so distance is always 2.

Theorem 3.13. This collection can be 0_n or 1_n if all of the codes which have same length and weight are collected.

Proof. It is necessary to calculate the state of being one of each digit for this proof, examining all of the code words in code. Let's imagine a code which is w weight and n length.

a) We calculate the first position is 1 which are number of the code words that

$$\frac{(n-1)!}{(n-1-w+1)!(w-1)!}$$

b) Now, we calculate the second position is 1 which are number of the code words. There are two statements that $01\ldots$, $11\ldots$.

1) $01\ldots$

$$\downarrow$$

$$\frac{(n-2)!}{(n-2-w+1)!(w-1)!}$$

2) 11…

$\downarrow$

$$\frac{(n-2)!}{(n-2-w+2)!(w-2)!}$$

If we collect two statements, it will be like first statement.

$$\frac{(n-2)!}{(n-2-w+1)!(w-1)!}+\frac{(n-2)!}{(n-2-w+2)!(w-2)!}=\frac{(n-1)!}{(n-1-w+1)!(w-1)!}$$

…………………

n) …1

$\downarrow$

$$\frac{(n-1)!}{(n-1-w+1)!(w-1)!}$$

For end digit, we invent the same result.

Example 3.14. Let C be as following.

$$C=\{110,011,101\}.$$

In this statement, as can be seen in the code, the number of code words in which the first position 1 is 2. Number of Second and third positions respectively are repetition 2.

Theorem 3.15. The collection of elements of soft codes is 0_n or 1_n.

Proof. Let's choose two sets which are named with sets of message and encoding and show with M and E. We define as follows these sets, accept these sets have two elements. $M=\{x_1 \dots x_n, y_1 \dots y_n\}$ and $E=\{a_1 \dots a_n, b_1 \dots b_n\}$. Let's multiply by using definition 3.7..

$M\Lambda E=$
$\{(x_1 \dots x_n)\Lambda(a_1 \dots a_n),(x_1 \dots x_n)\Lambda(b_1 \dots b_n),(y_1 \dots y_n)\Lambda(a_1 \dots a_n),(y_1 \dots y_n)\Lambda(b_1 \dots b_n)\}$

If we collect these elements,

$=$
$\{(x_1 \dots x_n)\Lambda(a_1 \dots a_n)+(x_1 \dots x_n)\Lambda(b_1 \dots b_n)+(y_1 \dots y_n)\Lambda(a_1 \dots a_n)+$
$(y_1 \dots y_n)\Lambda(b_1 \dots b_n)\}$

$=\{\big((x_1 \dots x_n)+(y_1 \dots y_n)\big)\Lambda(a_1 \dots a_n)+\big((x_1 \dots x_n)+(y_1 \dots y_n)\big)\Lambda(b_1 \dots b_n)\}.$

We know that collection of soft codes can be 1_n or 0_n from Teorem 3.13. . Such as, $\big((x_1 \dots x_n)+(y_1 \dots y_n)\big)=1_n$ or 0_n. Let's accept this collection is 1_n. In this statement, $\{1_n\Lambda(a_1 \dots a_n)+1_n\Lambda(b_1 \dots b_n)\}=0_n$, ıf n is even. ıf n is odd, we can create as follows

$\{0_n\Lambda(a_1 \ldots a_n) + 0_n\Lambda(b_1 \ldots b_n)\} = (a_1 \ldots a_n) + (b_1 \ldots b_n)$,. Since E is a soft code, this result is either 1_n or 0_n.

Example 3.16. Let's think multiplication of codes $\{100{,}010{,}001\}\Lambda\{110{,}011{,}101\}$.

$100\Lambda\{110{,}101{,}011\} = \{111110110{,}111101101{,}111011011\}$ the code words' s collection is {111 000 000}= $1_3 0_3 0_3$

$010\Lambda\{110{,}101{,}011\} = \{110111110, 101111101, 011111011\}$ the total of these code words is {000 111 000}= $0_3 1_3 0_3$

$001\Lambda\{110{,}101{,}011\} = \{110110111, 101101111, 011011111\}$ the sum of these code words is {000 000 111}= $0_3 0_3 1_3$

If we write these sums in a set, it would be as follows

$$\{1_3 0_3 0_3,\ 0_3 1_3 0_3,\ 0_3 0_3 1_3\}$$

According to above theorem, the sum of the code words is either 0_9 or 1_9.

Error Correcting Soft Codes 4.1. To correct the error in the soft code the following algorithm is used.

Algorithm: These steps are followed for single error correcting soft codes.

(1) Elements of code are collected.
(2) If this collection has a mistake; this collection will be different from 0_n or 1_n.
(3) We analyses minimum distance of this collection by comparing with 0_n and 1_n.
(4) We know that minimum distance of this collection is close to 0_n or 1_n .
(5) The elements of code are compared with 0_n or 1_n.
(6) Such as we find an element which has a different distance, because this element is incorrect.
(7) All of the elements are collected but error element is not collected.
(8) This collection is collected with 0_n or 1_n, such as we find to correct element.

Example 3.17. Let's think a soft code as follows. Also we generate a mistake code word $C = \{111110110, 111011011, 111101101, 110111110, \mathbf{010001011}, 101111101, 110110111, 011011111, 101101111\}$

(1) If we collect elements of code, it is $\{110001111\}$

(2) $d(110001111, 111111111) = 3,\ \ d(110001111{,}000000000) = 6$

(3) this collection has to be 111111111

(4) Let find by using definition 2.7., $d(111111111, 111110110) = 2$
$d(111111111, 011111011) = 2$

……………………………….

$d(111111111, \mathbf{010001011}) = 4$ (Incorrect code word)

(5) We collect to code words but the incorrect code word is not collected. This collection is 100000100

(6) 111111111+100000100 = 011111011, it is a correct code word.

5 Conclusions

In this essay, we define a new method for low complication encoding and decoding for nonlinear binary product codes has been recommended. This technique provides an important error-correcting algorithm by using soft sets. Thus, we divide according to the weights of the linear code sets and these sets create elements of soft set. Also, a low complexity decoding algorithm was proposed for the developed nonlinear binary product codes. Finally, we provided an example illustrating the successfully application of this method.

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