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Editorial

I am delighted to welcome you to the third issue of the Journal of New Theory (JNT) is completed with eight articles.

JNT publishes original research articles, reports, reviews and commentaries that are based on a theory of mathematics. However, the topics are not limited to only mathematics, but also include statistics, computer science, physics, engineering, chemistry, biology, economics or social sciences that use a theory of mathematics.

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Papers in JNT are published free of charge.

We would like to express our deepest thanks to all of the members of the editorial board and reviewers of the papers in this issue who are U. Orhan, A. Filiz, A. Fenercioğlu, A. Sarı, A. Yıldırım, A. S. Sezer, B. Mehmetoğlu, B. H. Çadırcı, C. Kaya, Ç. Çekiç, E. Altuntaş, E. Turgut, F. Karaaslan, F. Smarandache, G. Erdal, H. Aktaş, H. M. Doğan, H. Günal, H. Kızılaslan, H. Önen, H. Şimşek, İ. Zorlutuna, İ. Deli, İ. Gökçe, İ. Türkekul, İ. Parmaksız, J. Ye, M. Akar, M. Akdağ, M. Ali, M. Çavuş, M. Demirci, M. Sağlam, N. Yeşilayar, O. Muhtaroğlu, P. K. Maji, R. Yayar, S. Broumi, S. Karaman, S. Tarhan, S. Enginoğlu, S. Demiriz, S. Karataş, S. Öztürk, S. Eğri, Ş. Sözen, Y. Budak, Y. Karadağ, A. R. Roy, A. B. Saeid, N. Çağman, A. K. Saad, R. Şahin, I. Simşek and S. K. Majumder.

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We hope you will enjoy this issue of JNT. We are looking forward to hearing your feedback and receiving your contributions.

Happy reading!

07 April 2015

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A FEW REMARKS ON FUZZY SOFT CONTINUOUS FUNCTIONS IN FUZZY SOFT TOPOLOGICAL SPACES

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Abstract – The aim of this paper is to appraise a few properties of fuzzy soft continuous functions and to define fuzzy soft compact set. Fuzzy soft continuous image of a fuzzy soft compact set is taken into account to be dealt with in this paper.

Keywords – *Soft sets, soft topology, soft bases, fuzzy soft functions, fuzzy soft compact sets.*

1 Introduction

Sometimes it is very complicated to come across precise solution of real life problems by using the classical mathematics. So in the present era such type problems are being solved approximately by using the concept of fuzzy set theory, soft set theory etc.

Fuzzy set theory, formulated by Zadeh [16] in 1965, is accepted as a potential mechanism for giving standardized technique to deal with uncertainties where classical theories fail to act upto. Thereafter a lot of works [5, 6, 15] have been done in this area during the last four decade. At that time a few uncertain problems were come out from engineering and computer sciences, which were not being solved by using the concepts of fuzzy set theory. To solve such problems, D. Molodtsov [11] formulated a new concept, viz. soft set theory, in 1999. Then many works [2, 3, 4, 8, 14] have been done in this field.

[**]*Edited by* Serkan Karataş *(Area Editor) and* Naim Çağman *(Editor-in-Chief).*
[*]*Corresponding Author.*

Right now, different hybrid concepts are coming out, as a result the above said two concepts are combined together [9, 10] and it is termed as fuzzy soft set. Thereafter different notions have been generalized on fuzzy soft set and topology is one of them which has been generalized on it [1, 12, 13]. In fact, many researchers have engaged themselves to deal with the fuzzy soft function and to find its application on fuzzy soft topological spaces.

Fuzzy soft function was first originated by Kharal and Ahmad [7] in 2009. Then this definition has been amended by Atmaca and Zorlutuna [1] in 2013. For the simplicity, we have taken a small modification of this definition. Then we have established some theorems related to neighbourhood properties. In section 3, necessary and sufficient conditions have been established for a fuzzy soft function to be continuous in fuzzy soft topological spaces. In this section, also the notions of open mappings and closed mappings are being generalized for a fuzzy soft mapping and then a few properties of such mappings were established. In section 4, we have defined the concept of cover for a fuzzy soft set and compact fuzzy soft set, which are substantiated by considering three examples. The first example shows the existence of a cover of a fuzzy soft set. The second one is considered for the existence of an open cover of fuzzy soft set and the third one shows that every open cover may or may not have a finite subcover. Lastly, a theorem related to the continuous function and fuzzy soft compact set is taken into account to be dealt with in this paper.

2 Preliminary

This section contains some basic definitions and theorem which will be needed in the sequel. Throughout this paper, E is considered as the set of parameters and U being the course of universe.

Definition 2.1. [12] Let $A \subseteq E$. A fuzzy soft set over (U, E) is a mapping $F_A : E \to I^U$. For an element $e \in E$, we denote the image of e by $\mu^e_{F_A}$, where $\mu^e_{F_A} = \overline{0}$ if $e \in E \setminus A$ and $\mu^e_{F_A} \neq \overline{0}$ if $e \in A$.
The set of all fuzzy soft set over (U, E) is denoted by $FS(U, E)$.

Note 2.2. [12] If A is a null set, then F_A is said to be a null fuzzy soft set and it is denoted by Φ.

Definition 2.3. [12] A fuzzy soft set F_E is called absolute fuzzy soft set if $F_E(e) = \overline{1}$ for all $e \in E$. This fuzzy soft set is denoted by $\widetilde{E}$.

Definition 2.4. [12] Let F_A, $G_B \in FS(U, E)$. Then F_A is said to be fuzzy soft subset of G_B, denoted by $F_A \sqsubseteq G_B$ if $F_A(e) \subseteq G_B(e)$ for all $e \in A$.

Definition 2.5. [12] Let F_A, $G_B \in FS(U, E)$. Then the union of F_A and G_B is also a fuzzy soft set $F_A \sqcup G_B$, defined by $(F_A \sqcup G_B)(e) = F_A(e) \cup G_B(e)$ for all $e \in A \cup B$.
Following the arbitrary union of fuzzy subsets and the union of two fuzzy soft sets, the definition of arbitrary union of fuzzy soft sets can be described in the similarly fashion.

Definition 2.6. [12] Let F_A, $G_B \in FS(U, E)$. Then the intersection of F_A and G_B is also a fuzzy soft set $F_A \sqcap G_B$, defined by $(F_A \sqcap G_B)(e) = F_A(e) \cap G_B(e)$ for all $e \in A \cap B$.

Definition 2.7. [12] Let τ be a collection of some fuzzy soft sets over (U, E). Then τ is said to be a fuzzy soft topology on (U, E) if τ satisfies the following properties
1. $\Phi, \widetilde{E} \in \tau$.
2. If F_A, $G_B \in \tau$ then $F_A \sqcap G_B \in \tau$.
3. If $F_{A_\alpha} \in \tau$ for all $\alpha \in \Lambda$, an index set, then $\sqcup_{\alpha \in \Lambda} F_{A_\alpha} \in \tau$.

Definition 2.8. [12] If τ is a fuzzy soft topology on (U, E), the triple $(U,\, E,\, \tau)$ is said to be a fuzzy soft topological space. Also each member of τ is called a fuzzy soft open set in $(U,\, E,\, \tau)$.

Definition 2.9. [12] A subfamily β of τ is called a fuzzy soft open base or simply a base of fuzzy soft topological space $(U,\, E,\, \tau)$ if the following conditions hold:
1. $\Phi \in \beta$.
2. $\sqcup\beta = \widetilde{E}$ i.e. for each $e \in E$ and $x \in U$, there exists $F_A \in \beta$ such that $\mu^e_{F_A}(x) = 1$.
3. If $F_A,\, G_B \in \beta$ then for each $e \in E$ and $x \in U$, there exists $H_C \in \beta$ such that $H_C \sqsubseteq F_A \sqcap G_B$ and $\mu^e_{H_C}(x) = \min\{\mu^e_{F_A}(x),\, \mu^e_{G_B}(x)\}$, where $C \subseteq A \cap B$.

Theorem 2.10. [12] Let β be a fuzzy soft base for a fuzzy soft topology τ_β on (U, E). Then $F_A \in \tau_\beta$ if and only if $F_A = \sqcup_{\alpha\in\Lambda} B_{A_\alpha}$ where $B_{A_\alpha} \in \beta$ for each $\alpha \in \Lambda$, Λ an index set.

Definition 2.11. [13] A fuzzy soft point F_e over (U, E) is a special fuzzy soft set, defined by

$$F_e(a) = \begin{cases} \mu_{F_e} & if \quad a = e, \quad where \quad \mu_{F_e} \neq \overline{0} \\ \overline{0} & if \quad a \neq e \end{cases}$$

Definition 2.12. [13] Let F_A be a fuzzy soft set over (U, E) and G_e be a fuzzy soft point over (U, E). Then we say that $G_e \in F_A$ if and only if $\mu_{G_e} \subseteq \mu^e_{F_A} = F_A(e)$ i.e., $\mu_{G_e}(x) \leq \mu^e_{F_A}(x)$ for all $x \in U$.

Definition 2.13. [13] A fuzzy soft set F_A is said to be a neighborhood of a fuzzy soft point G_e if there exists $H_B \in \tau$ such that $G_e \in H_B \sqsubseteq F_A$.
Then clearly, every open fuzzy soft set is a neighborhood of each of its points.

Definition 2.14. [13] The union of all fuzzy soft open subsets of F_A over (U, E) is called the interior of F_A and is denoted by $intF_A$.

2.1 Some Applications of Fuzzy Soft Functions

In this section we introduce some basic definitions and theorems of fuzzy soft functions.

In 2009, A Kharal and B. Ahmad first defined the fuzzy soft functions in their paper [7]. Then in 2013, S. Atmaca and I. Zorlutuna [1] have modified this definition. Here we also present this definition with a small modification as follows.

Definition 2.15. Let $FS(U, E)$ and $FS(V, E')$ be two collections of fuzzy soft sets over (U, E) and (V, E') respectively. Let $f : U \to V$ and $g : E \to E'$. Then a function $S : FS(U, E) \to FS(V, E')$ is defined by
$S(F_A)(\alpha) = \mu^\alpha_{S(F_A)}$ for all $F_A \in FS(U, E)$ and $\alpha \in E'$
where

$$\mu^\alpha_{S(F_A)}(y) = \begin{cases} \max\limits_{x\in f^{-1}(y),\, e\in g^{-1}(\alpha)} \mu^e_{F_A}(x) & \text{if } f^{-1}(y) \neq \phi \quad \text{and} \quad g^{-1}(\alpha) \neq \phi \\ 0 & \text{otherwise,} \end{cases}$$

for $y \in V$.
and $S^{-1}(H_B)(e) = \mu^e_{S^{-1}(H_B)}$ for all $H_B \in FS(V, E')$ and $e \in E$
where $\mu^e_{S^{-1}(H_B)}(x) = \mu^{g(e)}_{H_B}(f(x))$, for $x \in U$.

Definition 2.16. Let (U, E, τ) and (V, E', τ') be two fuzzy soft topological spaces and $S : (U, E, \tau) \to (V, E', \tau')$ be a fuzzy soft function. Then S is said to be
1. an open mapping if and only if $S(F_A) \in \tau'$ for all $F_A \in \tau$.
2. a closed mapping if and only if $S(F_A)$ is closed in (V, E', τ') for every fuzzy soft closed set F_A in (U, E, τ).
3. a continuous mapping [1] if and only if $S^{-1}(H_B) \in \tau$ for all $H_B \in \tau'$.

Theorem 2.17. Let $S : (U, E, \tau) \to (V, E', \tau')$ be a closed map. Then for any fuzzy soft set $H_B \in FS(V, E')$ and any fuzzy soft open set F_A containing $S^{-1}(H_B)$, there exists an open set I_C containing H_B such that $S^{-1}(I_C) \sqsubseteq F_A$.

Proof. Let $I_C = \widetilde{E}' \setminus S(\widetilde{E} \setminus F_A) = \widetilde{E}' \setminus S(F_A^c)$.
At first we show that $H_B \sqsubseteq I_C$.
Let $\alpha \in E'$ and $y \in V$. Now $I_C(\alpha) = \overline{1} - \mu^{\alpha}_{S(F_A^c)}$.
If $f^{-1}(y) = \phi$ or $g^{-1}(\alpha) = \phi$, then $\mu^{\alpha}_{S(F_A^c)}(y) = 0$. That is, $\mu^{\alpha}_{I_C}(y) = 1$. So $\mu^{\alpha}_{H_B}(y) \subseteq \mu^{\alpha}_{I_C}(y)$.
If not, then let $f(x) = y$ and $g(e) = \alpha$. Since $S^{-1}(H_B) \sqsubseteq F_A$, $\mu^{e}_{S^{-1}(H_B)}(x) = \mu^{g(e)}_{H_B}(f(x)) = \mu^{\alpha}_{H_B}(y) \leq \mu^{e}_{F_A}(x)$.

$$
\begin{aligned}
\text{Now } \mu^{\alpha}_{I_C}(y) &= 1 - \mu^{\alpha}_{S(F_A^c)}(y) \\
&= 1 - \max\{\mu^{e}_{F_A^c}(x) : x \in f^{-1}(y) \text{ and } e \in g^{-1}(\alpha)\} \\
&= \min\{1 - \mu^{e}_{F_A^c}(x) : x \in f^{-1}(y) \text{ and } e \in g^{-1}(\alpha)\} \\
&= \min\{\mu^{e}_{F_A}(x) : x \in f^{-1}(y) \text{ and } e \in g^{-1}(\alpha)\} \\
&\geq \mu^{\alpha}_{H_B}(y).
\end{aligned}
$$

Therefore $\mu^{\alpha}_{I_C}(y) \geq \mu^{\alpha}_{H_B}(y)$ for all $\alpha \in E'$ and $y \in V$. So, $H_B \sqsubseteq I_C$.
Again since S is closed and F_A is a fuzzy soft open set in (U, E, τ), I_C is open set in (V, E', τ') containing H_B.
Now $S^{-1}(I_C) = S^{-1}(\widetilde{E'} \setminus S(F_A^c))$. Let $e \in E$ and $x \in U$.

$$
\begin{aligned}
\text{Therefore } \mu^{e}_{S^{-1}(I_C)}(x) &= \mu^{e}_{S^{-1}(\widetilde{E'} \setminus S(F_A^c))}(x) \\
&= \mu^{g(e)}_{\widetilde{E'} \setminus S(F_A^c)}(f(x)) \\
&= 1 - \mu^{g(e)}_{S(F_A^c)}(f(x)) \\
&= 1 - \max\{\mu^{e_1}_{F_A^c}(x_1) : e_1 \in g^{-1}(g(e)) \text{ and } x_1 \in f^{-1}(f(x))\} \\
&= \min\{1 - \mu^{e_1}_{F_A^c}(x_1) : e_1 \in g^{-1}(g(e)) \text{ and } x_1 \in f^{-1}(f(x))\} \\
&= \min\{\mu^{e_1}_{F_A}(x_1) : e_1 \in g^{-1}(g(e)) \text{ and } x_1 \in f^{-1}(f(x))\} \\
&\leq \mu^{e}_{F_A}(x).
\end{aligned}
$$

Therefore $\mu^{e}_{S^{-1}(I_C)}(x) \leq \mu^{e}_{F_A}(x)$ for all $e \in E$ and $x \in U$.
Hence $S^{-1}(I_C) \sqsubseteq F_A$.

Theorem 2.18. Let $S : (U, E, \tau) \to (V, E', \tau')$ be an open map. Then for any fuzzy soft set $H_B \in FS(V, E')$ and any fuzzy soft closed set F_A containing $S^{-1}(H_B)$, there exists a closed set I_C containing H_B such that $S^{-1}(I_C) \sqsubseteq F_A$.

Proof. Same as previous theorem.

Theorem 2.19. The following four properties of a map $S : (U, E, \tau) \to (V, E', \tau')$ are equivalent
1. S is an open map.
2. $S(intF_A) \sqsubseteq int(S(F_A))$ for all $F_A \in FS(U, E)$.
3. S sends each member of a fuzzy soft open base for τ to a fuzzy soft open set in (V, E', τ').
4. For each fuzzy soft point F_e over (U, E) and each neighborhood G_A of F_e, there exists a neighborhood H_B of $S(F_e)$ in (V, E') such that $H_B \sqsubseteq S(G_A)$

Proof. (1) $\Rightarrow$ (2)
Since $intF_A \sqsubseteq F_A$, $S(intF_A) \sqsubseteq S(F_A)$.
By hypothesis, $S(intF_A)$ is open in (V, E', τ'). Again $int(S(F_A))$ is the union of all fuzzy soft open subsets of $S(F_A)$ over (V, E', τ'). Hence $S(intF_A) \sqsubseteq int(S(F_A))$.

(2) $\Rightarrow$ (3)
Let F_A be a member of fuzzy soft open base for (U, E, τ).
Then $S(F_A) = S(intF_A) \sqsubseteq int(S(F_A)) \sqsubseteq S(F_A)$. That is, $S(F_A) = int(S(F_A))$. Therefore $S(F_A)$ is an open set in (V, E', τ').

(3) $\Rightarrow$ (4)
Let F_e be a fuzzy soft point and G_A be a neighborhood of F_e. Then there exists a member I_C of fuzzy soft base for τ such that $F_e \in I_C \sqsubseteq G_A$. So, $S(F_e) \in S(I_C) \sqsubseteq S(G_A)$. By (3), $S(I_C)$ is a open fuzzy soft set in (V, E', τ'). So, there exists a neighborhood $S(I_C)$ of $S(F_e)$ such that $S(I_C) \sqsubseteq S(G_A)$.

(4) $\Rightarrow$ (1)
Let G_A be a fuzzy soft open set in (U, E, τ). Then by hypothesis, for each point $F_e \in G_A$, there exists a neighborhood H_B of $S(F_e)$ such that $S(F_e) \in H_B \sqsubseteq S(G_A)$. That is, there exists a fuzzy soft open set I_{F_e} containing $S(F_e)$ such that $I_{F_e} \sqsubseteq S(G_A)$. So, $S(G_A) = \sqcup\{I_{F_e} : F_e \in G_A\}$. Which shows that $S(G_A)$ is open fuzzy soft set. This completes the proof.

Theorem 2.20. A mapping $S : (U, E, \tau) \to (V, E', \tau')$ is closed if and only if $\overline{S(F_A)} \sqsubseteq S(\overline{F_A})$ for every $F_A \in FS(U, E)$.

Proof. Obvious.

Proposition 2.21. [7] Let $S : (U, E) \to (V, E')$ be a fuzzy soft function. Then $S^{-1}(\sqcup_{\alpha\in\Lambda} H_{B_\alpha}) = \sqcup_{\alpha\in\Lambda} S^{-1}(H_{B_\alpha})$ where $H_{B_\alpha} \in FS(V, E')$ for all $\alpha \in \Lambda$.

Proof.
$$\begin{aligned}\sqcup_{\alpha\in\Lambda} S^{-1}(H_{B_\alpha})(e) &= \cup\{\mu^e_{S^{-1}(H_{B_\alpha})} : \alpha \in \Lambda\}\\ &= \cup\{\mu^{g(e)}_{H_{B_\alpha}} : \alpha \in \Lambda\}\\ &= \mu^{g(e)}_{\sqcup_{\alpha\in\Lambda} H_{B_\alpha}}\\ &= \mu^e_{S^{-1}(\sqcup_{\alpha\in\Lambda} H_{B_\alpha})}\\ &= S^{-1}(\sqcup_{\alpha\in\Lambda} H_{B_\alpha})(e) \text{ for all } e \in E.\end{aligned}$$
Therefore $\sqcup_{\alpha\in\Lambda} S^{-1}(H_{B_\alpha}) = S^{-1}(\sqcup_{\alpha\in\Lambda} H_{B_\alpha})$.

Theorem 2.22. Let (U, E, τ) and (V, E', τ') be two fuzzy soft topological spaces and β be a basis for the topology τ'. Then $S : (U, E, \tau) \to (V, E', \tau')$ is continuous if and only if $S^{-1}(F_B)$ is open in (U, E, τ) for every $F_B \in \beta$.

Proof. If S is continuous, then the theorem is obvious.
Suppose $S^{-1}(F_B)$ is open in (U, E, τ) for every $F_B \in \beta$. We now show that S is continuous. Let G_A be an open fuzzy soft set in (V, E', τ'). Then by theorem 2.10 G_A is a union of some member of β, say $G_A = \sqcup G_{A_\alpha}$ where $G_{A_\alpha} \in \beta$ and union is taken over $\alpha \in \Lambda$.
Thus $S^{-1}(G_A) = S^{-1}(\sqcup G_{A_\alpha}) = \sqcup S^{-1}(G_{A_\alpha})$ by proposition 2.21.
By assumption, each $S^{-1}(G_{A_\alpha})$ is open in (U, E, τ). Therefore $S^{-1}(G_A)$ is open fuzzy soft set. Hence S is continuous.

2.2 Fuzzy Soft Compact Sets

Definition 2.23. Let F_A be a fuzzy soft set. A collection $\mathcal{C}$ of fuzzy soft sets $\{F_{A_\alpha} : \alpha \in \Lambda\}$, Λ being the index set, is said to be a cover of F_A if $F_A \sqsubseteq \sqcup\{F_{A_\alpha} : \alpha \in \Lambda\}$.
If $\mathcal{C}$ be a collection of fuzzy soft open sets, then $\mathcal{C}$ is said to be an open cover of F_A.

Example 2.24. Let $A = \{e_1, e_2, e_3\}$ and $U = \{x, y\}$. Also let $A_1 = \{e_1, e_2\}$, $A_2 = \{e_2, e_3\}$, $A_3 = \{e_1, e_3\}$
$\mu^{e_1}_{F_{A_1}} = \{.2, .3\}$, $\mu^{e_2}_{F_{A_1}} = \{.1, .3\}$
$\mu^{e_2}_{F_{A_2}} = \{.2, .5\}$, $\mu^{e_3}_{F_{A_2}} = \{.5, .7\}$
$\mu^{e_1}_{F_{A_3}} = \{.1, .4\}$, $\mu^{e_3}_{F_{A_3}} = \{.7, .7\}$
Let F_A be a fuzzy soft set, where $\mu^{e_1}_{F_A} = \{.2, .35\}$, $\mu^{e_2}_{F_A} = \{.1, .4\}$, $\mu^{e_3}_{F_A} = \{.6, .65\}$.
Then obviously, $F_A \sqsubseteq F_{A_1} \sqcup F_{A_2} \sqcup F_{A_3}$. Therefore $\{F_{A_1}, F_{A_2}, F_{A_3}\}$ is a cover of F_A.

Example 2.25. Here we recall the example 3.8 of our paper[13] as
Let $E = \{e_1,\ e_2,\ e_3\}$, $U = \{a,\ b,\ c\}$ and $A,\ B,\ C$ be the subsets of E, where $A = \{e_1,\ e_2\}$, $B = \{e_2,\ e_3\}$ and $C = \{e_1,\ e_3\}$ and also let $\tau = \{\phi,\ \widetilde{E},\ F_A,\ G_B,\ H_{e_2},\ I_E,\ J_B,\ K_{e_2}\}$ be a fuzzy soft topology over (U, E) where $F_A,\ G_B,\ H_{e_2},\ I_E,\ J_B,\ K_{e_2}$ are fuzzy soft set over (U, E), defined as follows
$\mu^{e_1}_{F_A} = \{.5,\ .75,\ .4\}$, $\mu^{e_2}_{F_A} = \{.3,\ .8,\ .7\}$,
$\mu^{e_2}_{G_B} = \{.4,\ .6,\ .3\}$, $\mu^{e_3}_{G_B} = \{.2,\ .4,\ .45\}$,
$\mu_{H_{e_2}} = \{.3,\ .6,\ .3\}$,
$\mu^{e_1}_{I_E} = \{.5,\ .75,\ .4\}$, $\mu^{e_2}_{I_E} = \{.4,\ .8,\ .7\}$, $\mu^{e_3}_{I_E} = \{.2,\ .4,\ .45\}$,
$\mu^{e_2}_{J_B} = \{.4,\ .8,\ .7\}$, $\mu^{e_3}_{J_B} = \{.2,\ .4,\ .45\}$,
$\mu_{K_{e_2}} = \{.3,\ .8,\ .7\}$.
Now consider a fuzzy soft set L_B as follows
$\mu^{e_2}_{L_B} = \{.37, .7, .3\}$, $\mu^{e_3}_{L_B} = \{.1, .4, .4\}$
Then $L_B \sqsubseteq F_A \sqcup G_B$. Therefore $\{F_A, G_B\}$ is an open cover of L_B.

Example 2.26. Let $E = \{e_1, e_2, e_3, \cdots\}$, $U = \{x_1, x_2, x_3, \cdots, x_p\}$ where p is any positive integer and for any $e_j \in E$, F_{e_j} be a fuzzy soft point over $(U,\ E)$.
For any subset A of E, we define a fuzzy soft set F_A as follows

$$\mu^{e_j}_{F_A}(x_i) = \begin{cases} \mu_{F_{e_j}}(x_i) & \text{if } e_j \in A \\ 0 & \text{otherwise.} \end{cases}$$

where $i = 1, 2, 3, \cdots, p$ and $j = 1, 2, 3, \cdots$.
Let $\tau = \{\phi,\ \widetilde{E},\ F_{e_1}, F_{e_2}, \cdots, F_{\{e_1,e_2\}}, F_{\{e_2,e_3\}}, \cdots, F_{\{e_1,e_2,e_3\}}, F_{\{e_2,e_3,e_4\}}, \cdots, F_{\{e_1,e_2,e_3,\cdots,e_m\}}, F_{\{e_2,e_3,e_4,\cdots,e_{m+1}\}}, F_{\{e_3,e_4,e_5,\cdots,e_{m+2}\}}, \cdots\}$.

Then clearly, τ is a fuzzy soft topology on (U, E). Now let us consider a fuzzy soft set G_A as follows

$$\mu_{G_A}^{e_j}(x_i) = \begin{cases} \frac{\mu_{F_{e_j}}(x_i)}{2} & \text{if } e_j \in A \\ 0 & \text{otherwise.} \end{cases}$$

where $i = 1, 2, \cdots, p$ and $j = 1, 2, \cdots$ and A is an infinite subset of E.
Then clearly, $G_A \sqsubseteq F_{e_1} \sqcup F_{e_2} \sqcup \cdots$
Therefore $\{F_{e_1}, F_{e_2}, \cdots\}$ is an open cover of G_A. But it has no finite subcover.
Again if we consider a fuzzy soft set H_B where B is a finite subset of E. Then obviously, every open cover of H_B has a finite subcover.

Definition 2.27. A fuzzy soft set F_A of a fuzzy soft topological space (U, E, τ) is said to be a compact fuzzy soft set if every open cover of F_A has a finite subcover.

Result 2.28. If $F_A \sqsubseteq S^{-1}(H_B)$, then $S(F_A) \sqsubseteq H_B$.

Proof. Let $F_A \sqsubseteq S^{-1}(H_B)$. Then $\mu_{F_A}^e \subseteq \mu_{S^{-1}(H_B)}^e$ for all $e \in E$.
That is, $\mu_{F_A}^e(x) \leq \mu_{S^{-1}(H_B)}^e(x) = \mu_{H_B}^{g(e)}(f(x))$ for all $e \in E$ and $x \in U$.
Now let $\alpha \in E'$ and $y \in V$.
Then $\mu_{S(F_A)}^{\alpha}(y) = \max\{\mu_{F_A}^e(x) : x \in f^{-1}(y) \text{ and } e \in g^{-1}(\alpha)\}$
$\leq \max\{\mu_{S^{-1}(H_B)}^e(x) : x \in f^{-1}(y) \text{ and } e \in g^{-1}(\alpha)\}$
$= \max\{\mu_{H_B}^{g(e)}(f(x)) : x \in f^{-1}(y) \text{ and } e \in g^{-1}(\alpha)\}$
$= \mu_{H_B}^{\alpha}(y)$
Therefore $\mu_{S(F_A)}^{\alpha}(y) \leq \mu_{H_B}^{\alpha}(y)$ for all $\alpha \in E'$ and $y \in V$. Hence $S(F_A) \sqsubseteq H_B$.

Theorem 2.29. Let $S : (U, E, \tau) \to (V, E', \tau')$ be a continuous mapping. Suppose F_A is a compact subset of $\widetilde{E}$. Then $S(F_A)$ is also compact.

Proof. Let $\mathcal{V}$ be an open cover of $S(F_A)$ and $\mathcal{U} = \{S^{-1}(G_B) : G_B \in \mathcal{V}\}$. We now show that $\mathcal{U}$ is a cover of F_A. Let $e \in A$ and H_e be any point of F_A. Then $S(H_e) \in S(F_A)$. Then there exists a subset $\{G_{B_\alpha} : \alpha \in \Lambda\}$ of $\mathcal{V}$ such that $S(H_e) \in \sqcup\{G_{B_\alpha} : \alpha \in \Lambda\}$. Therefore $H_e \in S^{-1}(\sqcup_{\alpha\in\Lambda} G_{B_\alpha}) = \sqcup_{\alpha\in\Lambda} S^{-1}(G_{B_\alpha})$ which shows that $\mathcal{U}$ is a cover of F_A. Again since S is continuous, each member of $\mathcal{U}$ is open. So, $\mathcal{U}$ is an open cover of F_A. Since F_A is compact, there exist finitely many members of $\mathcal{U}$, say $S^{-1}(G_{B_1})$, $S^{-1}(G_{B_2})$, $\cdots$, $S^{-1}(G_{B_n})$ where $G_{B_1}, G_{B_2}, \cdots, G_{B_n} \in \mathcal{V}$ such that
$F_A \sqsubseteq S^{-1}(G_{B_1}) \sqcup S^{-1}(G_{B_2}) \sqcup \cdots \sqcup S^{-1}(G_{B_n})$
$= S^{-1}(G_{B_1} \sqcup G_{B_2} \sqcup \cdots \sqcup G_{B_n})$ by proposition 2.21.
Therefore $S(F_A) \sqsubseteq G_{B_1} \sqcup G_{B_2} \sqcup \cdots \sqcup G_{B_n}$ by result 2.28.
So, $\{G_{B_1}, G_{B_2}, \cdots, G_{B_n}\}$ is a cover of $S(F_A)$. Hence $S(F_A)$ is compact.

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SMARANDACHE SOFT GROUPOIDS

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Abstract – In this paper, Smarandache soft groupoids shortly (SS-groupoids) are introduced as a generalization of Smarandache Soft semigroups (SS-semigroups) . A Smarandache Soft groupoid is an approximated collection of Smarandache subgroupoids of a groupoid . Further, we introduced parameterized Smarandache groupoid and strong soft semigroup over a groupoid Smarandache soft ideals are presented in this paper. We also discussed some of their core and fundamental properties and other notions with sufficient amount of examples. At the end, we introduced Smarandache soft groupoid homomorphism.

Keywords* - *Smarndache groupoid, soft set, soft groupoid, Smarandache soft groupoid.

1. Introduction

In 1998, Raul [27] introduced the notions of Smarandache semigroups in the article "Smaradache Algebraic Structures". Smarandache semigroups are analogous to the notions Smarandache groups. Smarandache in [33] first studied the theory of Smarandache algebraic structures in "Special Algebraic Structures". The Smarandache groupoid [18] was introduced by Kandassamy which exhibits the characteristics and features of both semigroups and groupoids simultaneously. The Smarandache groupoids are a class of compleltely and conceptually a new study of associative and non-associative strucutres in nature. Smarandache algebraic structures almost show their existance in all algebraic structure in some sense such as Smarandache semigroups [17], Smarandache rings [21], Smarandache semirings [19], Smarandache semifields [19], Smarandache semivector spaces [19], Smarandache loops [20] etc.

Molodtsov [26] introduced the innovative and novel concept of soft sets in 1995. Soft set theory is a kind of mathematical framework that is free from the inadequacy of parameterization, syndrome of fuzzy sets, rough sets, probability etc. Soft set theory has found their applications in several areas such as smoothness of functions, game theory, operation research, Riemann integration, Perron integration, and probability theory etc.

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Recently soft set gain much potential in the research since its beggining. Severalmalgebraic structures and their properties studied in the context of soft sets. Aktas and Cagmann [1] introduced soft groups. Other soft algebraic structures are soft semigroups [16], soft semirings, soft rings etc. Some more work on soft sets can be seen in [13,14]. Further, some other properties and related algebra may be found in [15]. Some other related concepts and notions together with fuzzy sets and rough sets were discussed in [24,25]. Some useful study on soft neutrosophic algebraic structures can be seen in [3,4,5,6,7,8,9,10,12,29,30,32]. Recently, Mumtaz studied Smarandache soft semigroups in [11].

Rest of the paper is organized as follows. In first section 2, we studied some basic concepts and notions of Smarandache groupoids, soft sets, and soft groupoid. In the next section 3, the notions of Smarandache soft groupoids, shortly SS-groupoids are introduced. In this section some related properties and characterization are also discussed with illustrative examples. In the further section 4, Smarandache soft ideals and Smarandache soft groupoid homomorphism is studied with some of their basic properties. Conclusion is given in section 5.

2. Literature Review

In this section, we presented the fundamental concepts of Smarandache groupoids, soft sets, soft groupoids, soft semigroups and Smarandache soft semigroups with some of their basic properties.

2.1 Smarandache Groupoids

Definition 2.1.1[18]: A Smarandache groupoid is define to be a groupoid G which has a proper subset S such that S is a semigroup with respect to the same induced operation of G. A Smarandache groupoid G is denoted by SG.

Definition 2.1.2 [18]: Let G be a Smarandache groupoid. If at least one proper subset A in G which is a semigroup is commutative, then G is said to be a Smarandache commutative groupoid.

Definition 2.1.4 [18]: Let G be a Smarandache groupoid. A proper subset A of G is called a Smarandache subgroupoid if A itself is a Smarandach groupoid under the binary operation of G.

Definition 2.1.5 [18]: A Smarandache left ideal A of the Smarandache groupoid G satisfies the following conditions.

1. A is a Smarandache subgroupoid
2. $x \in G$ and $a \in A$, then $xa \in A$.

2.2 Soft Sets

Throughout this subsection U refers to an initial universe, E is a set of parameters, $P(U)$ is the power set of U, and $A, B \subset E$. Molodtsov defined the soft set in the following manner:

Definition 2.2.1 [24,26]: A pair (F, A) is called a soft set over U where F is a mapping given by $F : A \rightarrow P(U)$.

In other words, a soft set over U is a parameterized family of subsets of the universe U. For $a \in A$, $F(a)$ may be considered as the set of a-elements of the soft set (F, A), or as the set of a-approximate elements of the soft set.

Definition 2.2.2 [24]: For two soft sets (F, A) and (H, B) over U, (F, A) is called a soft subset of (H, B) if

1. $A \subseteq B$ and
2. $F(a) \subseteq H(a)$, for all $x \in A$.

This relationship is denoted by $(F, A) \subset (H, B)$. Similarly (F, A) is called a soft superset of (H, B) if (H, B) is a soft subset of (F, A) which is denoted by $(F, A) \supset (H, B)$.

Definition 2.2.3 [24]: Two soft sets (F, A) and (H, B) over U are called soft equal if (F, A) is a soft subset of (H, B) and (H, B) is a soft subset of (F, A).

Definition 2.2.4 [24]: Let (F, A) and (K, B) be two soft sets over a common universe U such that $A \cap B \neq \phi$. Then their restricted intersection is denoted by $(F, A) \cap_R (K, B) = (H, C)$ where (H, C) is defined as $H(c) = F(c) \cap \mathrm{K}(c)$ for all $c \in C = A \cap B$.

Definition 2.2.5 [24]: The extended intersection of two soft sets (F, A) and (K, B) over a common universe U is the soft set (H, C), where $C = A \cup B$, and for all $c \in C$, $H(c)$ is defined as

$$H(c) = \begin{cases} F(c) & if \quad c \in A - B, \\ \mathrm{K}(c) & if \quad c \in B - A, \\ F(c) \cap \mathrm{K}(c) & if \quad c \in A \cap B. \end{cases}$$

We write $(F, A) \cap_\varepsilon (K, B) = (H, C)$.

Definition 2.2.6 [24]: The restricted union of two soft sets (F, A) and (K, B) over a common universe U is the soft set (H, C), where $C = A \cup B$, and for all $c \in C$, $H(c)$ is defined as $H(c) = F(c) \cup G(c)$ for all $c \in C$. We write it as $(F, A) \cup_R (K, B) = (H, C)$.

Definition 2.2.7 [24]: The extended union of two soft sets (F,A) and (K,B) over a common universe U is the soft set (H,C), where $C = A \cup B$, and for all $c \in C$, $H(c)$ is defined as

$$H(c) = \begin{cases} F(c) & if \quad c \in A - B, \\ \mathrm{K}(c) & if \quad c \in B - A, \\ F(c) \cup \mathrm{K}(c) & if \quad c \in A \cap B. \end{cases}$$

We write $(F,A) \cup_{\varepsilon} (K,B) = (H,C)$.

Definition 2.2.8: A soft set (F,A) over G is called a soft groupoid over G if and only if $\phi \neq F(a)$ is a subgroupoid of G for all $a \in A$.

Definition 2.2.9 [16]: Let S be a semigroup. A soft set (F,A) over S is called a soft semigroup over S if $(F,A)\hat{\circ}(F,A) \subseteq (F,A)$.

It is easy to see that a soft set (F,A) over S is a soft semigroup if and only if $\phi \neq F(a)$ is a subsemigroup of S for all $a \in A$.

Definition 2.2.10 [11]: Let S be a semigroups and (F,A) be a soft semigroup over S. Then (F,A) is called a smarandache soft semigroup over U if a proper soft subset (G,B) of (F,A) is a soft group under the operation of S. We denote a smarandache soft semigroup by SS-semigroup.

A smarandache soft semigroup is a parameterized collection of smarandache subsemigroups of S.

3. Smarandache Soft Groupoids

Definition 3.1: Let G be a groupoid and (F,A) be a soft groupoid over G. Then (F,A) is said to be a smarandache soft groupoid over G if a proper soft subset (K,B) of (F,A) is a soft semigroup with respect to the operation of G. We denote a smarandache soft groupoid by SS-groupoid.

In other words a smarandache soft groupoid is a parameterized collection of smarandache subgroupoids of G which has a parameterized family of subsemigroups of G.
We now give an examples to illustrate the notion of SS-groupoids.

Example 3.2: Let $G = \{0,1,2,3,4,5\}$ be a groupoid under the binary operation $*$ modulo 6 with the following table. We take G in [18].

*	0	1	2	3	4	5
0	0	3	0	3	0	3
1	1	4	1	4	1	4
2	2	5	2	5	2	5
3	3	0	3	0	3	0
4	4	1	4	1	4	1
5	5	2	5	2	5	2

Let $A=\{a_1,a_2,a_3,a_4\}$ be a set of parameters. Let (F,A) be a soft groupoid over G, where

$$F(a_1)=\{0,1,3,4\}, \mathrm{F}(a_2)=\{0,2,3,5\},\ F(a_3)=\{1,2,4,5\},\ F(a_4)=\{0,1,2,3,4,5\}.$$

Let $B=\{a_1,a_2,a_4\}\subset A$. Then (K,B) is a soft subsemigroup of (F,A) over G, where

$$\mathrm{K}(a_1)=\{0,3\}, \mathrm{K}(a_2)=\{2,5\},\ \mathrm{K}(a_4)=\{1,4\}.$$

This shows clearly that (F,A) is a smarandache soft groupoid over G.

Proposition 3.3: If G is a smarandache groupoid, then (F,A) over G is also a smarandache soft groupoid.

Proof: It is trivial.

Proposition 3.4: The extended union of two SS-groupoids (F,A) and (K,B) over G is a SS-groupoid over G.

Proposition 3.5: The extended intersection of two SS-groupoids (F,A) and (K,B) over G is again a SS-groupoid.

Proposition 3.6: The restricted union of two SS-groupoids (F,A) and (K,B) over G is a SS-groupoid over G.

Proposition 3.7: The restricted intersection of two SS-groupoids (F,A) and (K,B) over G is a SS-groupoid over G.

Proposition 3.8: The AND operation of two SS-groupoids (F,A) and (K,B) over G is a SS-groupoid over G.

Proposition 3.9: The OR operation of two SS-groupoids (F,A) and (K,B) over G is a SS-groupoid over G.

Definition 3.10: Let (F,A) be a SS-groupoid over a groupoid G. Then (F,A) is called a commutative SS-groupoid if atleast one proper soft subset (K,B) of (F,A) is a commutative semigroup.

Example 3.11: In Example 3.2, (F,A) is a commutative SS-groupoid over G.

Proposition 3.12: If G is a commutative S-groupoid, then (F,A) over G is also a commutative SS-groupoid.

Definition 3.13: Let (F,A) be a SS-groupoid over a groupoid G. Then (F,A) is called a cyclic SS-groupoid if each proper soft subset (K,B) of (F,A) is a cyclic subsemigroup.

Proposition 3.14: If G is a cyclic S-groupoid, then (F,A) over G is also a cyclic SS-groupoid.

Proposition 3.15: If G is a cyclic S-groupoid, then (F,A) over G is a commutative SS-groupoid.

Definition 3.16: Let G be a groupoid and (F,A) be a SS-groupoid. A proper soft subset (K,B) of (F,A) is said to be a Smarandache soft subgroupoid if (K,B) is itself a Smarandache soft groupoid over G.

Example 3.17: Let $G=\{0,1,2,3,4,5\}$ be a groupoid under the binary operation $*$ modulo 6 with the following table. Again G is taken from [18].

*	0	1	2	3	4	5
0	0	5	4	3	2	1
1	4	3	2	1	0	5
2	2	1	0	5	4	3
3	0	5	4	3	2	1
4	4	3	2	1	0	5
5	2	1	0	5	4	3

Let $A=\{a_1,a_2,a_3,a_4\}$ be a set of parameters. Let (F,A) be a soft groupoid over G, where

$$F(a_1)=\{0,3\}, \mathrm{F}(a_2)=\{0,2,4\}, F(a_3)=\{1,3,5\},\ F(a_4)=\{0,1,2,3,4,5\}.$$

Let $B=\{a_2,a_3,a_4\}\subset A$. Then (K,B) is a soft subgroupoid of (F,A) over G, where

$$\mathrm{K}(a_2)=\{0,2,4\}, \mathrm{K}(a_3)=\{1,3,5\},\ \mathrm{K}(a_4)=\{1,3,5\}.$$

Let $C=\{a_3,a_4\}\subset B$. Then (H,C) is aSsmarndache soft subsemigroup of (K,B) over G, where

$$\mathrm{K}(a_3)=\{1,3,5\}, \mathrm{K}(a_4)=\{1,3,5\}.$$

Thus clearly (K,B) is a Smarandache soft subgroupoid of (F,A) over G.

Remark: Every soft subgroupoid of a Smarandache soft groupoid need not be a Smarandache soft subgroupoid in general.

One can easily verify it by the help of examples.

Theorem: If a soft groupoid (F,A) contain a Smarandcahe soft subgroupoid, then (F,A) is SS-groupoid.

Proof: Let (F,A) be a soft groupoid and $(\mathrm{K},\mathrm{B})\subset(F,A)$ is a Smarandache soft subgroupoid. Therefore, (K,B) has a proper soft subgroupoid (H,C) which is a soft semigroup and this implies $(\mathrm{H},\mathrm{C})\subset(F,A)$ which completes the proof.

Definition 3.18: Let G be a groupoid and (F,A) be a soft set over G. Then G is called a parameterized Smarandache groupoid if $F(a)$ is a semigroup under the operation of G for all $a\in A$.

In this case (F,A) is termed a strong soft semigroup.

A strong soft semigroup (F,A) is a parameterized collection of the subsemigroups of the groupoid G.

Example 3.19: Let $G=\{0,1,2,3,4,5\}$ be a groupoid under the binary operation $*$ modulo 6 with the following table. We take G in [18].

*	0	1	2	3	4	5
0	0	3	0	3	0	3
1	1	4	1	4	1	4
2	2	5	2	5	2	5
3	3	0	3	0	3	0
4	4	1	4	1	4	1
5	5	2	5	2	5	2

Let $A=\{a_1,a_2,a_3\}$ be a set of parameters. Let (F,A) be a soft groupoid over G, where

$$F(a_1)=\{0,3\},\mathrm{F}(a_2)=\{2,5\},F(a_3)=\{1,4\}.$$

This clearly shows that (F,A) is a strong soft semigroup over G, and G is a parameterized groupoid.

Proposition 3.20: Let (F,A) and (H,B) be two strong soft semigroups over a groupoid G. Then

1. $(F,A)\bigcap_R(H,B)$ is a strong soft semigroup over G.
2. $(F,A)\bigcap_E(H,B)$ is a strong soft semigroup over G.
3. $(F,A)\bigcup_E(H,B)$ is a strong soft semigroup over G.
4. $(F,A)\bigcup_R(H,B)$ is a strong soft semigroup over G.

Proof: The proof of these are straightforward.

Proposition 3.21: Let G be a groupoid and (F,A) be a soft set over G. Then G is a parameterized Smarandache groupoid if (F,A) is a soft semigroup over G.

Proof: Suppose that (F,A) is a soft semigroup over G. This implies that each $F(a)$ is a subsemigroup of the groupoid G for al $a \in A$, and thus G is a parameterized smarandache groupoid.

Example 3.22: Let $Z_{12} = \{0,1,2,3,...,11\}$ be the groupoid with respect to multiplication modulo 12 and let $A = \{a_1, a_2, a_3\}$ be a set of parameters. Let (F,A) be a soft semigroup over Z_{12}, where

$$F(a_1) = \{3,9\}, \mathrm{F}(a_2) = \{1,7\},\ F(a_3) = \{1,5\}.$$

Then Z_{12} is a parameterized Smarandache groupoid.

4. Smarandache Soft Ideal over a Groupoid, Smarandache Soft Ideal of a Smarandache Soft Groupoid and Smarandache Soft Homomorphism

Definition 4.1: Let (F,A) be a SS-groupoid over G. Then (F,A) is called a Smarandache soft ideal over G if and only if $F(a)$ is a Smarandache ideal of G for all a in A.

Definition 4.2: Let (F,A) be a SS-groupoid and (K,B) be a soft subset of (F,A). Then (K,B) is called a Smarandache soft left ideal if the following conditions are hold.

1. (K,B) is a Smarandache soft subgroupoid of (F,A), and
2. $x \in G$ and $k \in K(b)$ implies $xk \in K(b)$ for all $b \in B$.

Similarly we can define a Smarandachesoft right ideal. A Smarandache soft ideal is one which is both Smarandache soft left and right ideal.

Theorem 4.3: Let (F,A) be a SS-groupoid over G. If (K,B) is a Smarandache soft ideal of (F,A), then (K,B) is a soft ideal of (F,A) over G.

Proposition 4.4: Let (F,A) and (H,B) be two Smarandache soft ideals over a groupoid G. Then

1. $(F,A) \cap_R (H,B)$ is a Smarandache soft ideal over G.
2. $(F,A) \cap_E (H,B)$ is a Smarandache soft ideal over G.
3. $(F,A) \cup_E (H,B)$ is a Smarandache soft ideal over G.
4. $(F,A) \cup_R (H,B)$ is a Smarandache soft ideal over G.

Proof: The proof of these are straightforward.

Definition 4.5: Let (F,A) and (K,B) be two SS-groupoid over G. Then (K,B) is called a Smarandache soft seminormal groupoid if

1. $B \subset A$, and
2. $K(a)$ is a Smaradache seminormal subgroupoid $F(a)$ for all $a \in A$.

Definition 4.6: Let (F,A) and (K,B) be two SS-groupoid over $(G,*)$ and $(G,\circ)$ respectively. A map $\phi:(F,A)\to(\mathrm{K},\mathrm{B})$ is said to be a Smaradache soft groupoid homomorphism if $\phi:(F',A')\to(\mathrm{K}',B')$ is a soft semigroup homomorphism where $(F',A')\subset(F,A)$ and $(\mathrm{K}',B')\subset(\mathrm{K},B)$ are soft semigroups respectively.

A Smarandache soft groupoid homomorphism is called Smarandache soft groupoid isomorphism if ϕ is a soft semigroup isomorphism.

5. Conclusion

In this paper Smaradache soft groupoids are introduced. Their related properties and results are discussed with illustrative examples to grasp the idea of these new notions. The theory of Smarandache soft groupoids open a new way for researchers to define these type of soft algebraic structures in other areas of soft algebraic structures in the future.

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CLUSTERING ALGORITHMS FOR WIRELESS SENSOR NETWORKS

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Abstract – A Wireless Sensor Network (WSN) Contains a large number of Small sensors can be an effective instrument for data collecting in diverse kinds of environments. The data gathering by each sensor is related to the sink, which unto the data to the end users. Clustering is presented to WSNs since it has proven to be an efficient method to prepare better data aggregation and scalability for great WSNs. Clustering also maintains the limited energy resources of the sensors. In This paper, we review the existing clustering algorithms in WSNs and compare them by different criteria; in addition, we describe the advantages and disadvantages of each algorithm.

Keywords – *Wireless Sensor Networks (WSNs), clustering, network lifetime, energy efficient.*

1 Introductıon

Wireless sensor networks contain a large number of low power multi functioning sensor nodes, operating in without care environment, with limited calculation and sensing abilities. Recent developments in low power wireless complex micro sensor technologies have made these sensor nodes available in large numbers, with a low cost, to be employed in a broad range of applications in martial and national security, environmental monitoring, and many other fields [1], [2].

Whereas wireless sensor nodes are power limited devices, long distance transfer should be maintained to minimum in order to prolong the network lifetime [3], [4], [5]. Therefore, straight connections among nodes and the sink are not persuasion. An efficient method for better efficiency is to order the network into some clusters, with each cluster selecting one

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node as its cluster head [6], [5]. Clustering algorithm in Wireless Sensor Network is known as a reliable method of self-organization [7].
The remainder of this paper is organized as follows. In section 2, we review clustering in WSNs and challenges in WSNs, also expressed some clustering parameters and process of clustering. In Section 3, we present most common classifications for WSNs clustering. In Section 4, we investigate several effective and popular clustering protocols for WSNs. Then, comparison discussed clustering protocols in Section 5. Finally, in Section 6 concludes the paper.

2 Clustering in WSNs and Challenges

Clustering can significantly effect on the overall scalability of the system, the lifetime and energy efficiency [8].

Clustering algorithms are used to reduce the number of nodes that transfer data to the sink or base station (BS). These algorithms, order the nodes based in the WSN into groups or clusters. One node in each cluster is identified as the leader of the cluster or the cluster head (CH). The nodes that are in a cluster though are not cluster head, become member nodes of that cluster. The member nodes will transmit their data to their CH, which is typically inside only a short distance, so consumed less energy [9].

Clustering in WSNs Is encountered some challenges, including confidence connectivity, electing the optimal frequency of CH circulation, calculation the optimal cluster sizes, and clustering the network in the attendance of a node duty cycle [11].

2.1 Clustering Parameters

Several important parameters with regard to the whole clustering method in WSNs are [10], [14]:

2.1.1 Number of Clusters (Cluster Count)

In most clustering algorithms the CH selection and cluster forming process, result in various numbers of clusters. However, in several spread method, the set of CHs are predefined and so the number of clusters are predetermined.

2.1.2 Intracluster Correlation

In certain basic clustering trends the relationship between a sensor and its CH is directly intended (one-hop communication). Whereas nowadays in most cases, multi-hop intracluster relationship is needed.

2.1.3 Mobility of Nodes and CH

If we suppose that the CHs and sensor nodes and are fixed; naturally encounter balanced and stable clusters with facilitated intracluster and intercluster network management. Against, if the nodes or the CHs themselves are supposed to be mobile, the cluster membership for each node should dynamically alteration, clusters that have recruited at any moment, and probably require permanent maintenance.

2.1.4 Nodes Types and Roles

In some provided network (i.e., heterogeneous environments) the CHs are supposed to be equipped with more calculation and communication resources than others. However, in many models of the usual network (i.e., homogeneous environments) all nodes have the same abilities and only a subset of the deployed nodes are selected as CHs.

2.1.5 Cluster-Head Selection

The leader nodes of the clusters (CHs) in some presented algorithms (mainly for heterogeneous environments) can be predetermined. However, in most cases (i.e., in homogeneous environments), the CHs are selected from the distributed set of nodes, in a probabilistic method or random methods or other specific criteria (Such as residual energy and connectivity, etc.).

2.2 Clustering Process

There are two basic stages in the clustering, which are CH selection and cluster establishment. But the evaluation degree of algorithms may differ in various steps. According to algorithm steps of all process of clustering algorithms, clustering routing protocols in WSNs can be classified into cluster formation based and data transition based ones [12], [13], [14].

3 Classification of Clustering Algorithms

There have been several different methods to classify the algorithms used for WSNs clustering, [10]. Four of the most common classifications are shown in Figure 1.

3.1 Clustering Algorithms Homogeneous or Heterogeneous Networks

This classification according to the characteristics and performance of sensor nodes in a cluster. In heterogeneous sensor networks, all nodes have the same specifications, hardware and processing capabilities. In these networks, which are common in nowadays applications, each node can be a CH. In addition to these networks, the CH role can be replaced between the nodes periodically (for the creation of better and more integrated load balancing energy). Against in *heterogeneous* sensor networks, generally, there are two types of sensor, the first type sensors with more processing abilities and complex hardware.

These sensors predetermined as a CH node. The other type conventional sensors, with lower abilities, which in fact, used to sense the environment properties.

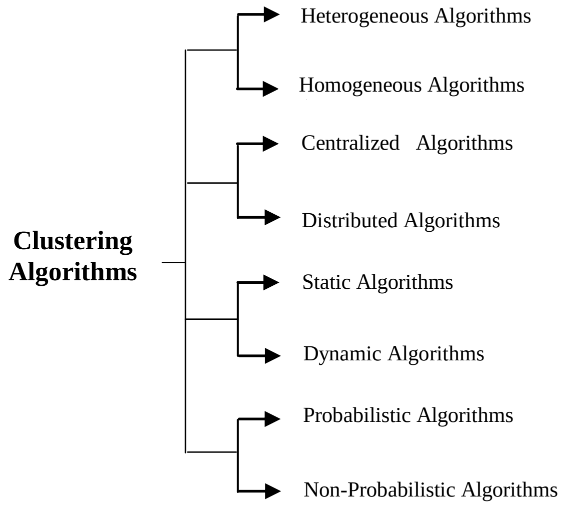

Figure 1. Common classification algorithms

3.2 Centralized or Distributed Clustering Algorithms

These clustering algorithms imply on the method utilized for shaping clusters. A distributed CH selection and shaping process are the most suitable way to obtain enhanced flexibility and faster convergence times independent of the number of nodes of the WSN. This approach is the most efficient method, particularly for large networks. Also, there are a few approaches using centralized or hybrid method, which one or more coordinate nodes or base stations (sink), responsible for the break up into detached and control of all network cluster members. These networks are not appropriate for overall objective large-scale practical application WSNs. They may be appropriate just for specific targets bounded-scale applications in which high-quality connectivity and network separation is needed.

3.3 Static and Dynamic Clustering Algorithms

Other conventional classification is static or dynamic clustering. Process shaping clusters is dynamic (otherwise as static) when it contains regular (periodic or event-oriented) CH re-election or includes cluster reorganization routine, these procedures may be effective in order to respond to changes in network topology and only accurately the cluster topology or proper movement with the purpose CH role between the nodes to obtain in energy saving. Dynamic cluster architectures make a better use of the sensors in a WSN and naturally result in improved energy consumption management and network lifetime.

3.4 Probabilistic and Non-Probabilistic Clustering Algorithms

This classification based on cluster shape parameters can be used to select the CH. These clustering algorithms are divided into two categories: probabilistic (random or hybrid) and non-probability (deterministic). Most clustering algorithms are known, they can be divided into two main categories. In the probabilistic clustering algorithm for determining the initial CH, a probability assigned to each node. Probabilistic clustering algorithms, beyond the more energy-efficient, often running faster at the convergence and reduce the volume of messages they exchange. In the non-probabilistic clustering algorithms basically criteria (deterministic) more specific for CH selection and cluster formation, take into consideration. These criteria are essentially based on the proximity of adjacent nodes like (connectivity, degree, etc.) and information received from other closely located nodes.

4 Clustering Protocols

In this section, we analyses and classify several effective and popular clustering protocols for WSNs.

4.1 Low-Energy Adaptive Clustering Hierarchy (LEACH)

This protocol is one of the most popular protocols for WSNs. LEACH [15], [13] shapes clusters by distributed protocol. That nodes make independent decisions without any centralized control. Each round of the protocol is divided into two stages. The first stage, which is the startup phase, the phase is formed clusters. The second stage is related to the normal operation of the network is called the steady-state phase. The protocol prepares a balancing of energy utilization by random rotation of CHs. It shapes clusters based on the received signal power, and usage the CH nodes as routers to the base station.

4.2 Low-Energy Adaptive Clustering Hierarchy Centralized (LEACH-C)

LEACH-C [23] is a centralized prescription of LEACH, this mean that the responsibility of the cluster creation is transmitted to the base station. Each node in beginning obligated to make a direct connection to the base station in order that an overall view of the network is shaped. Therefore an improved cluster formed method is done and a little better overall performance of the network is obtained.

4.3 Two-Level Low-Energy Adaptive Clustering Hierarchy (TL-LEACH)

TL-LEACH [16], [17] is an offered extension to the LEACH protocol. It uses two levels (two stage) of cluster heads (primary and secondary). In this protocol, for every cluster, the primary cluster heads can communicate with a secondary cluster head and also communicates with other sub cluster. Associated data from source node to sink in two stages:

- Secondary nodes gather data from the cluster nodes will perform. Also on this level is data-fusion operations performed.
- Primary node collects data from secondary nodes are responsible. Data-fusion can also be performed at this level.

The two-level structure of TL-LEACH decrease the amount of nodes that require to transfer to the base station, effectively decreases the total energy usage.

4.4 Hybrid Energy-Efficient Distributed Clustering (HEED)

HEED [18] is another popular and improved energy-efficient protocol. HEED is a multi-hop clustering protocol for WSNs, which periodically selects CHs based on a hybrid of the node remaining energy and a secondary parameter, such as node adjacency to its neighbors or node degree. The main objectives of HEED are:

- Distribute energy consumption to prolong the network lifetime.
- Minimize energy during the cluster head selection phase.
- Minimize the control overhead of the network.

4.5 Energy Efficient Hierarchical Clustering (EEHC)

EEHC [19], [16] a distributed, randomized clustering protocol for WSNs with the purpose of maximizing the network lifetime. CHs gathered the nodes readings in their individual clusters and send an aggregated report to the base station. EEHC protocol is based on the assumption that the communication is free of errors. This Method is according to two steps; In the first stage called single-level clustering, each node introduces itself as a CH with probability p to the neighboring nodes within its communication range. Each node that receives this information becomes part of the cluster head near it. Then processed in the second stage clustering protocol is developed to make the creation of multi-level cluster hierarchy level h.

This protocol guarantees the connectivity between CHs and base station. This protocol ensures the wasted energy of CHs that are far from the base station is reduced, because the cluster heads to the base station can transmit.

4.6 Partition Low-Energy Adaptive Clustering Hierarchy (PEACH)

P-LEACH [20] is a LEACH protocol based on partition, it is the first partitioning the network into sectors and then a CH in each sector, using the centralized calculations is selected. A node that has the highest energy is selected as CH node. P-LEACH protocol is performed in two steps. In the first stage, the optimal number of CH by the sink node is calculated and is based on the partitioning of the network into sectors. If k is the optimal number of cluster heads, the network is divided into k sectors that each have approximately the same number of nodes. In the next stage nodes with the highest energy as CH in each division will be chosen by the sink node. Then data selected CH by a sink node, will be sent to the entire network.

4.7 DSBCA

A Balanced Clustering Protocol with Distributed Self-Organization for Wireless Sensor Networks (DSBCA) [21]. The basic idea DSBCA, the connection density and distance from the base station to calculate k (cluster radius). DSBCA has three phases: cluster head selecting phase, clusters building phase and cycle phase. DSBCA objective is to balance the energy consumption of clusters as well as the connection density and location nodes, to create a balanced cluster.

4.8 Energy Efficient Clustering Scheme (EECS)

EECS [22], [12] is a clustering protocol, which better appropriate the periodical data collecting applications. EECS schema is like LEACH, where the network is divided into some clusters and single-hop connection among the CH and the base station is done. In EECS, CH candidates compete for the ability to raise to CH for a given round. This competition included candidates broadcasting their remaining energy to neighboring candidates. If a given node does not find a node with more remaining energy, it becomes a CH. Different from LEACH for cluster shaping, EECS deployments LEACH by the dynamic sizing of clusters according to cluster distance from the base station.

4.9 Energy Efficient Heterogeneous Clustered Scheme

EEHC [24] is an energy efficient clustered scheme for WSNs based on weighted election probabilities of each node to become the cluster head. It elects the cluster head in a distributed fashion in hierarchal WSN. This protocol is based on LEACH. This protocol works on the election processes of the cluster head in the presence of heterogeneity of nodes.

4.10 Fast Local Clustering Service

FLOC [25] is a distributed clustering protocol that produces approximately equal sized clusters with minimum overlap. A node can relationship reliably with the nodes that are in the inner band (i-band) range and unreliably with the nodes in its outer band (o-band) range. Therefore the i-band nodes suffer very little interference communicating with the CH, thereby it is a reliable communication. Messages from o-band nodes are unreliable during communication and hence it has the maximum probability of getting lost during communication. FLOC is fast and scalable, hence it achieves clustering in O(1) time, regardless of the size of the network. It also exhibits self-healing capabilities since o-band nodes can switch to i-band node in another cluster. It also achieves re-clustering within constant time and in a local manner. It also achieves locality, in that each node is only affected by the nodes within two units. These features stimulate FLOC protocol to be suitable for large scale WSNs.

5 Comparison of Discussed Clustering Protocols

Table 1 summaries the investigated clustering protocols in this paper.

6 Conclusion

Wireless sensor networks for military and civilian applications and deployment in remote and inaccessible areas, many researchers in the past few years has attracted. Due to lower power consumption in these networks, for reducing power consumption in these networks, clustering network seems inevitable.

Clustering protocols divide the network into several clusters, which each perform their work independently and each cluster has a head. This paper reviews some clustering methods have been investigated. It can be concluded that, different protocols based on power consumption, additional Information exchange rate, method clustering and determine the cluster head, is divided into several different groups.

Table 1. Comparison of the clustering protocols

Protocols / Criteria	Methodology	CH Selection	Cluster count	Node Mobility	Multi-Level	Residual Energy CH	Cluster Size	Distance from CH to BS	Hops
LEACH [15]	Distributed	Prob/random	Variable	Limited	No	No	No	No	1
LEACHC [23]	Centralized	Prob/random	Variable	Limited	Yes	Yes	Yes	Yes	1
TLLEACH [17]	Distributed	Prob/random	Variable	Limited	No	No	No	No	2
HEED [18]	Distributed	Prob/energy	Variable	Limited	No	Yes	Yes	No	1
EEHC [19]	Distributed	Prob/random	Variable	No	Yes	No	No	Yes	K
PLEACH [20]	Centralized	Prob/energy	Variable	Limited	No	Yes	No	No	1
DSBCA [21]	Distributed	Prob/ Connectivity	Variable	No	Yes	No	Yes	Yes	1
EECS [22]	Distributed	Prob/energy	Constant	No	No	Yes	Yes	Yes	1
EEHC [24]	Distributed	Prob/ Density	Variable	No	No	Yes	No	No	K
FLOC [25]	Distributed	Random	Variable	Possible	No	No	No	No	1

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Original Article[**]

A GROUP DECISION MAKING METHOD BASED ON TOPSIS UNDER FUZZY SOFT ENVIRONMENT

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Abstract – In this paper, we first briefly present conventional TOPSIS method developed by Hwang and Yoon[10, 19] as a multi-criteria decision making technique. We then give a group decision making method based on TOPSIS method under fuzzy soft environment, and finally give an application of proposed method to show operation and effectiveness of method.

Keywords – *Soft sets, Fuzzy set, Fuzzy soft sets, TOPSIS, Multi-criteria decision making.*

1 Introduction

Decision making is one of important processes that human being encounters many areas of the real world such as business, service, management, military, etc. But in real life, necessary informations for decision making may not be certain always. First step of the decision making process is to model such information involving uncertainty. Hence, in 1965, fuzzy set theory was suggested to model fuzzy data as mathematically by Zadeh [20]. However, in this theory, determining of membership function is rather difficult sometimes. Therefore, in 1999, Molodtsov [14] proposed a completely new approach for modeling uncertainty, free from this difficulty. Then Maji et al. [12] gave some operations of soft sets and their properties. To make some modifications to the operations of soft sets some researchers such

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as Ali et al. [1], Çağman and Enginoğlu [6], Zhu and Wen [22], Çağman [8] gave their contributions. Concept of fuzzy soft set and fuzzy soft set operations were first defined by Maji et al. [11] as a generalization of Molodtsov's soft set, in 2001. Also Roy and Maji [17] presented an application of fuzzy soft sets in a decision-making problem. Majumdar and Samanta[13] defined generalized fuzzy soft sets and studied on their properties. Zhou et al. [21] proposed and studied generalised interval-valued fuzzy soft sets.
Technique for Order Preference by Similarity to Ideal Solution (TOPSIS) being one of classical multi attributive decision making (MADM) methods such as PROMETHEE [2], VIKOR [15], ELECTRE [16], developed by Hwang and Yoon [10]. Chen et al. [4] extended the TOPSIS method for solving multi-criteria decision making (MCDM) problems in fuzzy environment. Boran et al.[3] developed TOPSIS method for MCDM problems based on intuitionistic fuzzy sets. Chi and Liu [5] extended TOPSIS to Interval neutrosophic sets (INSs), and with respect to the multiple attribute decision making problems in which the attribute weights were unknown and the attribute values take the form of INSs. Eraslan [9] gave a decision making method by using TOPSIS on soft set theory.

In this paper, we extend TOPSIS method to deal with group decision making problems in fuzzy soft environment. Then, we give an illustrative example to show the effectiveness of the suggested method.

The study is organized as follows: Section 2 introduces the basic definitions of soft set and fuzzy soft set with their basic operations. The main procedure for the conventional TOPSIS is described in a series of steps in Section 3. In Section 4, a group decision making method is developed by using TOPSIS on fuzzy soft set theory. Afterwards, an application of method is given to illustrate effectiveness of the method.

2 Preliminary

In this section, we summarize the preliminary definitions which are fuzzy set [20], soft set [14, 8], fuzzy soft set and their results that are required in this paper.

2.1 Fuzzy Sets

Definition 2.1. [20] Let U be a initial universe. A fuzzy set μ over U is defined by a membership

$$\mu : U \to [0, 1]$$

For $u \in U$; the membership value $\mu(u)$ essentially specifies the degree to which $u \in U$ belongs to the fuzzy set μ. Thus, a fuzzy set μ over U can be represented as follows,

$$\mu = \{(\mu(u)/u) : u \in U, \quad \mu(u) \in [0, 1]\}$$

Note that the set of all the fuzzy sets over U will be denoted by $F(U)$.

Example 2.2. Assume that $U = \{u_1, u_2, u_3, u_4, u_5\}$ is a universal set. Let be a fuzzy set μ over U can be represented as follows,

$$\mu = \{0.2/u_1, 0.5/u_2, 0.7/u_3, 0.9/u_4, 1.0/u_5\}$$

2.2 Soft Sets

Definition 2.3. [14] Consider a nonempty set A such that $A \subseteq E$. A pair (f, A) is called a soft set over U, where f is a mapping given by

$$f : A \to \mathcal{P}(U)$$

.

In this paper, we will benefit following definition defined by Çağman [8] for basic set operations on soft sets.

Definition 2.4. [8] A soft set f over U is a set valued function from E to $\mathcal{P}(U)$. It can be written a set of ordered pairs

$$f = \big\{(e, f(e)) : e \in E\big\}.$$

Note that if $f(e) = \emptyset$, then the element $(e, f(e))$ won't be appeared in soft set f. Set of all soft sets over U will be denoted by $\mathcal{S}(U)$.

Example 2.5. Let $U = \{u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8\}$ be the universe containing eight houses and $E = \{e_1, e_2, e_3, e_4, e_5, e_6\}$ be the set of parameters. Here, e_i $(i = 1, 2, 3, 4, 5, 6)$ stand for the parameters "*modern*", "*with parking*", "*expensive*", "*cheap*", "*large*" and "*near to city*" respectively. Then, following soft sets are described by Mr. A and Mr. B who want to buy a house, respectively

$$\begin{aligned} f &= \big\{(e_1, \{u_1, u_3, u_4\}), (e_2, \{u_1, u_4, u_7, u_8\}), (e_3, \{u_1, u_2, u_3, u_8\})\big\} \\ g &= \big\{(e_2\{u_1, u_3, u_6\}), (e_3, U), (e_5, \{u_2, u_4, u_5, u_6\})\big\}. \end{aligned}$$

Definition 2.6. [8] Let $f, g \in \mathcal{S}(U)$. Then,

1. If $f(e) = \emptyset$ for all $e \in E$, f is said to be a empty soft set, denoted by Φ.
2. If $f(e) = U$ for all $e \in E$, F is said to be universal soft set, denoted by $\hat{U}$.
3. f is soft subset of g, denoted by $f \tilde{\subseteq} g$, if $f(e) \subseteq g(e)$ for all $e \in E$.
4. $f = g$, if $f \tilde{\subseteq} g$ and $g \tilde{\subseteq} f$.
5. Soft union of f and g, denoted by $f \tilde{\cup} g$, is a soft set over U and defined by $f \tilde{\cup} g : E \to \mathcal{P}(U)$ such that $(f \tilde{\cup} g)(e) = f(e) \cup g(e)$ for all $e \in E$.
6. Soft intersection of f and g, denoted by $f \tilde{\cap} g$, is a soft set over U and defined by $f \tilde{\cap} g : E \to \mathcal{P}(U)$ such that $(f \tilde{\cap} g)(e) = f(e) \cap g(e)$ for all $e \in E$.
7. Soft complement of f is denoted by $f^{\tilde{c}}$ and defined by $f^{\tilde{c}} : E \to \mathcal{P}(U)$ such that $f^{\tilde{c}}(e) = U \setminus f(e)$ for all $e \in E$.

2.3 Fuzzy soft sets

Definition 2.7. [11] Let U be an initial universe set, X be a set of all parameters, μ be a fuzzy set over U for every $x \in X$ and $F(U)$ denote the set of all fuzzy sets in U. Then, a fuzzy soft set γ over U is defined by a function γ representing a mapping

$$\gamma : X \to F(U) \quad such \ \ that \ \ \gamma(x) = \emptyset \ \ if \ \ x \notin X$$

Here, for every $x \in X$, $\gamma(x)$ is a fuzzy set over U and it is called fuzzy value set of parameter x-element of the fs-set. Thus, an fs-set γ over U can be represented by the set of ordered pairs

$$\gamma = \{(x, \gamma(x)) : x \in X, \quad \gamma(x) \in F(U)\}$$

Note that from now on the sets of all fs-sets over U will be denoted by $FS(U)$.

Example 2.8. Assume that $U = \{u_1, u_2, u_3, u_4, u_5\}$ is a universal set and $X = \{x_1, x_2, x_3\}$ is a set of all parameters.
If $\gamma(x_1) = \{0.5/u_2, 0.9/u_4\}$, $\gamma(x_2) = U$, $\gamma(x_3) = \emptyset$, then the fs-set γ is written by

$$\gamma = \{(x_1, \{0.5/u_2, 0.9/u_4\}), (x_2, U)\}$$

3 TOPSIS Method

TOPSIS method is a practical and useful technique for ranking and selection of a number of externally determined alternatives through distance measures. The operations within the TOPSIS process include: decision matrix normalization, distance measures, and aggregation operators [18]. For more detail of TOPSIS, we refer to the earlier studies [10, 19]. The TOPSIS process is carried out as follows.

Throughout this paper, $I_n = \{1, 2, ..., n\}$ for all $n \in \mathbb{N}$.

Step 1. Constructing of decision matrix D.

$$D = \begin{matrix} \\ A_1 \\ A_2 \\ \vdots \\ A_i \\ \vdots \\ A_m \end{matrix} \begin{matrix} \begin{matrix} c_1 & c_2 & \cdots & c_n \end{matrix} \\ \begin{pmatrix} d_{11} & d_{12} & \cdots & d_{1n} \\ d_{21} & d_{22} & \cdots & d_{2n} \\ \vdots & \vdots & & \vdots \\ d_{i1} & d_{i2} & \cdots & d_{in} \\ \vdots & \vdots & & \vdots \\ d_{m1} & d_{m2} & \cdots & d_{mn} \end{pmatrix} \end{matrix} = [d_{ij}]_{m \times n} \tag{1}$$

here A_i $(i \in I_m)$ and c_j $(j \in I_n)$ denote alternatives and criteria, respectively.

Step 2. Creating of standard (normalized) decision matrix R.

$$r_{ij} = \frac{d_{ij}}{\sqrt{\sum_{k=1}^{m} d_{kj}^2}} \quad , \quad \forall d_{ij} \neq 0 \ \ and \ \ \forall i \in I_m \quad , \quad \forall j \in I_n \tag{2}$$

$$R = \begin{pmatrix} r_{11} & r_{12} & \cdots & r_{1n} \\ r_{21} & r_{22} & \cdots & r_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ r_{m1} & r_{m2} & \cdots & r_{mn} \end{pmatrix} = [r_{ij}]_{m \times n}$$

Step 3. Creating the weighted normalized decision matrix V.
$V = [v_{ij}]_{m\times n} = [w_j r_{ij}]_{m\times n}, i \in I_m$, where $w_j = W_j / \sum_{j=1}^{n} W_j, j = 1, 2, ..., n$ so that $\sum_{j=1}^{n} w_j = 1$, and W_j is the original weight given to the criteria $c_j, j \in I_n$.

$$V = \begin{pmatrix} v_{11} & v_{12} & \cdots & v_{1n} \\ v_{21} & v_{22} & \cdots & v_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ v_{m1} & v_{m2} & \cdots & v_{mn} \end{pmatrix} = [v_{ij}]_{m\times n}$$

Step 4. Determining of positive ideal solution (PIS), A^+ and negative ideal solution (NIS), A^- .

$$A^+ = \{v_1^+, \cdots, v_j^+, \cdots, v_n^+\} = \{(\underbrace{max}_{i} v_{ij} | j \in J_1) \ , \ (\underbrace{min}_{i} v_{ij} | j \in J_2) \ , \ i \in I_m\} \tag{3}$$

$$A^- = \{v_1^-, \cdots, v_j^-, \cdots, v_n^-\} = \{(\underbrace{min}_{i} v_{ij} | j \in J_1) \ , \ (\underbrace{max}_{i} v_{ij} | j \in J_2) \ , \ i \in I_m\} \tag{4}$$

where J_1 and J_2 are associated with the benefit and cost attribute sets, respectively.

Step 5. Calculating of separation measurements of positive ideal (S_i^+) and the negative ideal (S_i^-) solutions.

$$S_i^+ = \sqrt{\sum_{j=1}^{n} (v_{ij} - v_j^+)^2} \quad , \quad \forall i \in I_m \tag{5}$$

and

$$S_i^- = \sqrt{\sum_{j=1}^{n} (v_{ij} - v_j^-)^2} \quad , \quad \forall i \in I_m \tag{6}$$

Step 6. Calculating of relative closeness of alternatives to the ideal solution

$$C_i^+ = \frac{S_i^-}{(S_i^- + S_i^+)} \quad , \quad 0 \le C_i^+ \le 1 \quad , \quad \forall i \in I_m \tag{7}$$

Step 7. Ranking the preference order.

4 TOPSIS Method for group decision making with fuzzy soft information

In this section, we propose a new method by extending TOPSIS method to fuzzy soft environment. The main procedure of this method is presented with the following steps:

Linguistic Terms	FVs
Very Good / Very Important (VG/VI)	0.95
Good / Important(G /I)	0.85
Fair / Medium(F/M)	0.50
Bad / Unimportant (B / UI)	0.35
Very Bad / Very Unimportant (VB/VUI)	0.10

Table 1: Linguistic terms for evaluation of parameters.

Step 1. *Defining of problem.*

Let us assume that $DM = \{D_p,\ p \in I_n\}$ is set of decision makers, $U = \{u_i,\ i \in I_m\}$ denotes set of alternatives and $X = \{x_j,\ j \in I_n\}$ is a set of all parameters (criterion). Then, a fuzzy soft set γ over U is a function defined by

$$\gamma : X \to F(U)$$

Step *2.* *Constructing of weighed fuzzy parameter matrix D with choosing linguistic rating from Table 1.*

$$D = \begin{matrix} \\ D_1 \\ D_2 \\ \vdots \\ D_i \\ \vdots \\ D_m \end{matrix} \begin{matrix} \begin{matrix} x_1 & x_2 & \cdots & x_n \end{matrix} \\ \begin{pmatrix} d_{11} & d_{12} & \cdots & d_{1n} \\ d_{21} & d_{22} & \cdots & d_{2n} \\ \vdots & \vdots & & \vdots \\ d_{i1} & d_{i2} & \cdots & d_{in} \\ \vdots & \vdots & & \vdots \\ d_{m1} & d_{m2} & \cdots & d_{mn} \end{pmatrix} \end{matrix} = [d_{ij}]_{m\times n} \tag{8}$$

where d_{ij} is linguistic rating assigned by decision maker D_i the parameter x_j.

Step 3. *Constructing of weighted normalized fuzzy parameter matrix R and forming weighed vector* $W = (W_1, W_2, ..., W_n)$.

The weighted normalized elements of weighted normalized fuzzy parameter matrix R are calculated by using Eq (2) and weighed vector $W = (W_1, W_2, ..., W_n)$ is formed with aid of the formula

$$W_j = \frac{w_j}{\sum_{k=1}^{m} w_k}, \quad w_j = \frac{1}{m}\sum_{i=1}^{m} r_{ij} \tag{9}$$

$$R = \begin{pmatrix} r_{11} & r_{12} & \cdots & r_{1n} \\ r_{21} & r_{22} & \cdots & r_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ r_{m1} & r_{m2} & \cdots & r_{mn} \end{pmatrix} = [r_{ij}]_{m\times n}$$

Step 4. *Constructing fuzzy decision matrices D_k for each decision makers and building of fuzzy average decision matrix V*

Fuzzy decision matrices D_k are constructed similar way to classical TOPSIS Method (Step 2) and fuzzy average decision matrix V is constructed by using Eq (10).

$$D_k = \begin{array}{c} \\ u_1 \\ u_2 \\ \vdots \\ u_i \\ \vdots \\ u_m \end{array} \begin{array}{c} \begin{array}{cccc} x_1 & x_2 & \cdots & x_n \end{array} \\ \begin{pmatrix} d_{11} & d_{12} & \cdots & d_{1n} \\ d_{21} & d_{22} & \cdots & d_{2n} \\ \vdots & \vdots & & \vdots \\ d_{i1} & d_{i2} & \cdots & d_{in} \\ \vdots & \vdots & & \vdots \\ d_{m1} & d_{m2} & \cdots & d_{mn} \end{pmatrix} \end{array} = [d_{ij}^k]_{m \times n}$$

where $d_{ij}^k = \gamma_{X_k}(x_j)(u_i)$.

$$V = \frac{1}{n}\big(D_1 \oplus D_2 \oplus ... \oplus D_n\big) = [v_{ij}]_{m \times n} \quad (10)$$

where $\oplus$ indicates sum of matrices

Step 5. *Constructing of weighed fuzzy decision matrix $\mathcal{V}$.*

$$\mathcal{V} = \begin{pmatrix} \hat{v}_{11} & \hat{v}_{12} & \cdots & \hat{v}_{1n} \\ \hat{v}_{21} & \hat{v}_{22} & \cdots & \hat{v}_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ \hat{v}_{m1} & \hat{v}_{m2} & \cdots & \hat{v}_{mn} \end{pmatrix}$$

where

$$\hat{v}_{ij} = W_j \cdot v_{ij} \quad (11)$$

Step 6. *Finding of fuzzy valued positive ideal solution (FV-PIS) and fuzzy valued negative-ideal solution (FV-NIS).*

In the classic TOPSIS method, criteria are evaluated from aspect of benefit and cost. Assume that J_1 be a set of benefit criteria and J_2 be a set of cost criteria. Based on fuzzy set theory and principle of TOPSIS method, FV-PIS and FV-NIS can be found as follow respectively;

$$FV-PIS = \{\hat{v}_1^+, \hat{v}_2^+, ..., \hat{v}_j^+, ..., \hat{v}_n^+\} = \{(\underbrace{max}_{i}\,\hat{v}_{ij} | j \in J_1)\ ,\ (\underbrace{min}_{i}\,\hat{v}_{ij} | j \in J_2)\ ,\ i \in I_m\} \quad (12)$$

$$FV-NIS = \{\hat{v}_1^-, \hat{v}_2^-, ..., \hat{v}_j^-, ..., \hat{v}_n^-\} = \{(\underbrace{min}_{i}\,\hat{v}_{ij} | j \in J_1)\ ,\ (\underbrace{max}_{i}\,\hat{v}_{ij} | j \in J_2)\ ,\ i \in I_m\} \quad (13)$$

Step 7. *Calculating of the separation measurement for each parameter.*

Separation measurements (S_i^+) and (S_i^-) are found by using Eq(5) and Eq(6) .

Step 8. *Calculating of the relative closeness of alternative to the ideal solution.*

Relative closeness of alternatives to the ideal solution are calculating by using Eq(7)

$$C_i^+ = \frac{S_i^-}{(S_i^- + S_i^+)} \quad , \quad 0 \leq C_i^+ \leq 1 \quad , \quad \forall i \in I_m$$

Step 9. Ranking the preference order.

5 Application

In this section, we have presented an application for a group decision making method by using TOPSIS on fuzzy soft set theory. Now, by using the algorithm of this new group decision making method we can solve the following example (problem) step by step as follows:

Step 1. Defining the problem.

Assume that a real estate agent has a set of different types of houses $U = \{u_1, u_2, u_3\}$ which may be characterized by a set of all parameters $X = \{x_1, x_2, x_3\}$. For $j = 1, 2, 3$ the parameters x_j stand for "cheap", "modern", "large", respectively. Then we can give the following examples.

Suppose that three decision-makers come to the real estate agent to buy a house. Firstly, each decision-maker has to consider their own set of parameters. Then, they can construct their fuzzy soft sets. Next, by using TOPSIS on fuzzy soft set theory decision making method we select a house on the basis for the sets of decision-makers parameters.

Assume that decision-makers D_1, D_2 and D_3 construct fuzzy soft sets, respectively as follows;

$$\gamma_X^{(1)} = \{(x_1, \{0.5/u_1, 0.2/u_2, 0.5/u_3\}), (x_2, \{0.2/u_1, 0.6/u_2, 0.1/u_3\}), (x_3, \{0.3/u_1, 0.7/u_2, 0.2/u_3\})\}$$

$$\gamma_X^{(2)} = \{(x_1, \{0.1/u_1, 0.6/u_2, 0.8/u_3\}), (x_2, \{0.4/u_1, 0.9/u_2, 0.2/u_3\}), (x_3, \{0.2/u_1, 0.3/u_2, 0.7/u_3\})\}$$

$$\gamma_X^{(3)} = \{(x_1, \{0.3/u_1, 0.2/u_2, 0.7/u_3\}), (x_2, \{0.1/u_1, 0.5/u_2, 0.6/u_3\}), (x_3, \{0.6/u_1, 0.1/u_2, 0.1/u_3\})\}$$

Step 2. Weighed fuzzy parameter matrix D is as follow

$$D = \begin{array}{c} \\ D_1 \\ D_2 \\ D_3 \end{array} \begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \begin{pmatrix} 0.95 & 0.35 & 0.10 \\ 0.50 & 0.10 & 0.10 \\ 0.10 & 0.50 & 0.85 \end{pmatrix} \end{array} = [d_{ij}]_{m \times n}$$

Step 3. Weighted normalized fuzzy parameter matrix can be obtained as follow

$$R = \begin{pmatrix} 0,88 & 0,57 & 0,12 \\ 0,46 & 0,16 & 0,12 \\ 0,09 & 0,81 & 0,99 \end{pmatrix}.$$

And weighed vector W can be obtained using by Eq (9), as follow

$$W = (0.34, 0.37, 0.29)$$

Step 4. Fuzzy decision matrices can be constructed by decision makers as follows;

$$D_1 = \begin{pmatrix} 0,50 & 0,20 & 0,30 \\ 0,20 & 0,60 & 0,70 \\ 0,50 & 0,10 & 0,20 \end{pmatrix} \quad D_2 = \begin{pmatrix} 0,10 & 0,40 & 0,20 \\ 0,60 & 0,90 & 0,30 \\ 0,80 & 0,20 & 7,00 \end{pmatrix} \quad D_3 = \begin{pmatrix} 0,30 & 0,10 & 0,60 \\ 0,20 & 0,50 & 0,10 \\ 0,70 & 0,60 & 0,10 \end{pmatrix},$$

and from Eq (10) fuzzy average decision matrix is

$$V = \begin{pmatrix} 0,30 & 0,23 & 0,37 \\ 0,33 & 0,67 & 0,37 \\ 0,67 & 0,30 & 2,43 \end{pmatrix}.$$

Step 5. Weighed fuzzy decision matrix $\mathcal{V}$ is constructed with aid of Eq (11) as follow,

$$\mathcal{V} = \begin{pmatrix} 0,10 & 0,09 & 0,11 \\ 0,11 & 0,24 & 0,11 \\ 0,23 & 0,11 & 0,71 \end{pmatrix}.$$

Step 6. Positive ideal solution (FV-PIS) and fuzzy valued negative-ideal solution (FV-NIS) can be obtained using the Eq (12) and (13) as follow

$$A^+ = FV - PIS = \{\hat{v}_1^+ = 0.23,\ \hat{v}_2^+ = 0.24,\ \hat{v}_3^+ = 0.71\}$$

$$A^- = FV - NIS = \{\hat{v}_1^- = 0.10,\ \hat{v}_2^- = 0.09,\ \hat{v}_3^- = 0.11\}$$

Step 7. From Eq(5) and Eq(6), S_i^+ and S_i^-, for $i \in \{1, 2, 3\}$, we have

$$\begin{array}{rclrcl} S_1^+ & = & 0.63, & S_1^- & = & 0.00 \\ S_2^+ & = & 0.61, & S_2^- & = & 0.15 \\ S_3^+ & = & 0.13, & S_3^- & = & 0.61 \end{array}$$

Step 8. Relative closeness of alternatives to the ideal solution as follows

$$C_1^+ = 0.00$$
$$C_2^+ = 0.20$$
$$C_3^+ = 0.82$$

Step 9. Ranking the preference order is $u_1 < u_2 < u_3$.

6 Conclusion

In this paper, we have presented a group decision making method by using TOPSIS under fuzzy soft environment. Finally, we provided an example that demonstrated that this method can be successfully worked. It can be applied to decision making problems of many fields that contain uncertainty. However, the approach should be more comprehensive in the future to solve the related problems and a large number of examples could be recommended for test in future studies.

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TYPES OF FUZZY PAIRWISE S-COMPACTNESS MODULO SMOOTH IDEALS

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Abstract – In this paper, we introduce several types of fuzzy pairwise compactness and fuzzy pairwise compactness modulo a smooth ideal in smooth bitopological spaces by using the family of r-(τ_i, τ_j)-fuzzy semi-open sets as cover. Several characterizations and some properties of these spaces are discussed. Preservation of fuzzy pairwise compactness modulo a smooth ideal by some types of mappings is also investigated.

Keywords – *Fuzzy pairwise compactness, fuzzy pairwise S-compactness, smooth bitopological spaces.*

1 Introduction

Šostak [27] introduced the fundamental concept of a 'fuzzy topological structure' as an extension of both crisp topology and Chang's fuzzy topology [4], indicating that not only the objects were fuzzified, but also the axiomatics. Subsequently, Badard [3] introduced the concept of 'smooth topological space'. Chattopadhyay et al. [5] and Chattopadhyay and Samanta [6] re-introduced the same concept, calling it 'gradation of openess'. Ramadan [19] and his colleagues introduced a similar definition, namely a smooth topological space for lattice $L = [0, 1]$. Following Ramadan, several authors have re-introduced and further studied smooth topological space (cf. [5, 6, 9, 28]). Thus, the terms 'fuzzy topology', in Šostak's sense, 'gradation of openness' and 'smooth topology' are essentially referring to the same concept. In our paper, we adopt the term smooth topology. Further to this, Lee et al. [16] introduced the concept of smooth bitopological space as a generalization of smooth topological space and Kandil's fuzzy bitopological space [10].

The concept of fuzzy semi-open sets and fuzzy semi-continuous mapping in fuzzy topological spaces was studied by Azad [2]. Kumar [13] generalized the concepts of fuzzy semi-open sets, fuzzy semi-continuous mappings into fuzzy bitopological spaces. Kim et al. [12] as well as Lee and Lee

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[15], introduced the notion of fuzzy r-semi-open sets and fuzzy r-semi-continuous maps in smooth topological space which are generalizations of fuzzy semi-open sets and fuzzy semi-continuous maps in Chang's fuzzy topology. Ramadan and Abbas [21] introduced the notion of r-fuzzy semi-open sets in smooth bitopological spaces. El-sheikh [7] characterized the notion of r-fuzzy semi-open sets [21] and generalized the notions that introduced in smooth bitopological space [13], [20], [21]. Recently [29], we introduced the concept of r-τ_{12}-fuzzy semi-open sets in smooth supra topological space (X, τ_{12}) which were induced from smooth bitopological space (X, τ_1, τ_2). We have also shown that the present notion of fuzzy semi-open sets and the notion of r-(τ_i, τ_j)-fuzzy semi-open sets that introduced in [21] are independent.

Ideals are an important notions which was introduced into general topology by Kuratowski [14], where a nonempty family I of $P(X)$ is called an ideal if: (1) $A \in I$ and $B \subseteq A$ gives $B \in I$ (heredity) and (2) $A, B \in I$ gives $A \cup B \in I$ (finite additivity). Sarkar [26] introduced and studied the notion of ideal in Chang's sense. Ramadan et al. [22] introduced the notion of a smooth ideal in smooth topology.

The concept of compactness modulo an ideal was first introduced by Newcomb [18] and Rančin [23] and was studied by Hamlett and Janković [8]. Abd El-Monsef et al. [1] studied the relations between ideals and some types of weak compactness. Salama [25] defined and studied some other types of fuzzy compactness with respect to fuzzy ideals in Chang's fuzzy topologies. Saber and Abdel-Sattar [24] investigated some properties of smooth ideals and used these to introduce and study the concept of r-fuzzy ideal-compact, r-fuzzy quasi H-closed, and r-fuzzy compact modulo a smooth ideal in smooth topological spaces.

In the present paper we use the concept of r-(τ_i, τ_j)-fuzzy semi-open sets and a smooth ideal to introduce new types of compactness in smooth bitopological spaces, namely r-(τ_i, τ_j)-$FS\mathcal{I}$-compactness, r-(τ_i, τ_j)-$FS\mathcal{I}$-Lindelöfness and r-(τ_i, τ_j)-$F\mathcal{I}$-S-closedness that generalize r-(τ_i, τ_j)-FS-compactness, r-(τ_i, τ_j)-FS-Lindelöfness and r-(τ_i, τ_j)-FS-closedness respectively. We give the relation between these types of compactness and those introduced by Saber and Abdel-Sattar [24]. Also, we study some of the properties and characterizations. Moreover, the behavior of these types of compactness under some types of mappings is also investigated.

2 Preliminary

In this paper, X is a non-empty set, $I = [0, 1]$ and $I_0 = (0, 1]$. A fuzzy set μ of X is a mapping with $\mu : X \longrightarrow I$, and I^X the family of all fuzzy sets of X. For any $\mu_1, \mu_2 \in I^X$, $(\mu_1 \wedge \mu_2)(x) = \min\{\mu_1(x), \mu_2(x) : x \in X\}$, $(\mu_1 \vee \mu_2)(x) = \max\{\mu_1(x), \mu_2(x) : x \in X\}$ and $(\mu_1 - \mu_2)(x) = \min\{\mu_1(x), \bar{1} - \mu_2(x) : x \in X\}$. For a fuzzy set λ of X, $supp(\lambda) = \{x \in X | \ \lambda(x) > 0\}$. For $\lambda \in I^X$, $\bar{1} - \lambda$ denotes the complement of λ. For $\alpha \in I$, $\bar{\alpha}(x) = \alpha \ \forall x \in X$. By $\bar{0}$ and $\bar{1}$, we denote constant maps on X with values 0 and 1, respectively. For $\mu, \lambda \in I^X$, μ is called quasi-coincident with λ, denoted by $\mu \ q \ \lambda$, if $\mu(x) + \lambda(x) > 1$ for some $x \in X$. Otherwise we write $\mu \ \bar{q} \ \lambda$. For any λ_1 and $\lambda_2 \in I^X$, $\lambda_1 \leq \lambda_2 \Longleftrightarrow \lambda_1 \ \bar{q} \ \bar{1} - \lambda_2$. FP stands for fuzzy pairwise. The indices are $i, j \in \{1, 2\}$ and $i \neq j$. All other notations are standard notations of fuzzy set theory.

Definition 2.1. [3, 5, 19, 27] A smooth topology on X is a mapping $\tau : I^X \rightarrow I$ which satisfies the following properties:

(1) $\tau(\bar{0}) = \tau(\bar{1}) = 1$,

(2) $\tau(\mu_1 \wedge \mu_2) \geq \tau(\mu_1) \wedge \tau(\mu_2), \ \forall \ \mu_1, \mu_2 \in I^X$,

(3) $\tau(\bigvee_{i \in J} \mu_i) \geq \bigwedge_{i \in J} \tau(\mu_i)$, for any $\{\mu_i : i \in J\} \subseteq I^X$.

The pair (X, τ) is called a smooth topological space. The value of $\tau(\mu)$ is interpreted as the degree of openness of fuzzy set μ. For $r \in I_0$, μ is an r-open fuzzy set of X if $\tau(\mu) \geq r$, and μ is an r-closed fuzzy set of X if $\tau(\bar{1} - \mu) \geq r$. Note, Šostak [27] used the term 'fuzzy topology' and Chattopadhyay [5], the term 'gradation of openness' for a smooth topology τ.

Definition 2.2. [16] A triple (X, τ_1, τ_2) consisting of the set X endowed with smooth topologies τ_1 and τ_2 on X is called a smooth bitopological space (smooth bts). For $\lambda \in I^X$ and $r \in I_0$, r-τ_i-open (resp. closed) fuzzy set denotes the r-open (resp. closed) fuzzy set in (X, τ_i), for $i = 1, 2$.

Theorem 2.3. [6, 11] Let (X, τ_1, τ_2) be a smooth bts. For $\lambda \in I^X$ and $r \in I_0$, a τ_i-fuzzy closure of λ is a mapping $C_{\tau_i} : I^X \times I_0 \longrightarrow I^X$ defined as

$$C_{\tau_i}(\lambda, r) = \bigwedge\{\mu \in I^X | \ \mu \geq \lambda, \ \tau_i(\bar{1} - \mu) \geq r\}$$

And, a τ_i-fuzzy interior of λ is a mapping $I_{\tau_i} : I^X \times I_0 \longrightarrow I^X$ defined as

$$I_{\tau_i}(\lambda, r) = \bigvee\{\mu \in I^X | \ \mu \leq \lambda, \ \tau_i(\mu) \geq r\}.$$

Then:
(1) C_{τ_i} (resp. I_{τ_i}) is a fuzzy closure (resp. interior) operator.
(2) $\tau_{C_{\tau_i}} = \tau_{I_{\tau_i}} = \tau_i$.
(3) $I_{\tau_i}(\bar{1} - \lambda, r) = \bar{1} - C_{\tau_i}(\lambda, r), \ \forall \ r \in I_0, \lambda \in I^X$.

Definition 2.4. [24] A mapping $\mathcal{I} : I^X \longrightarrow I$ is called a smooth ideal on X if it satisfies the following conditions:
(S1) $\mathcal{I}(\bar{1}) = 0, \mathcal{I}(\bar{0}) = 1$,
(S2) $\mathcal{I}(\lambda \vee \mu) \geq \mathcal{I}(\lambda) \wedge \mathcal{I}(\mu)$, for $\lambda, \mu \in I^X$,
(S3) If $\lambda \leq \mu$, then $\mathcal{I}(\mu) \leq \mathcal{I}(\lambda)$, for $\lambda, \mu \in I^X$.

If $\mathcal{I}$ and $\mathcal{J}$ are smooth ideals on X, we say $\mathcal{I}$ is finer than $\mathcal{J}$ (or $\mathcal{J}$ is coarser than $\mathcal{I}$), denoted by $\mathcal{J} \leq \mathcal{I}$, if and only if $\mathcal{J}(\lambda) \leq \mathcal{I}(\lambda)$ for all $\lambda \in I^X$.

For each smooth ideal $\mathcal{I}$ on X and $\alpha \in I_0$, $\mathcal{I}_\alpha = \{\nu \in I^X \mid \mathcal{I}(\nu) \geq \alpha\}$ is a fuzzy ideal on X in the sense of Sarkar [26]. By a fuzzy ideal we mean a non-empty collection of fuzzy sets $\mathfrak{I}$ of a set X satisfying the following conditions:
(i) If $\mu \in \mathfrak{I}$ and $\nu \leq \mu$, then $\nu \in \mathfrak{I}$ [heredity],
(ii) If $\mu \in \mathfrak{I}$ and $\nu \in \mathfrak{I}$, then $\mu \vee \nu \in \mathfrak{I}$ [finite additivity].
The simplest smooth ideal on X is $\mathcal{I}^0 : I^X \longrightarrow I$ defined by $\mathcal{I}^0(\lambda) = 1$, if $\lambda = \bar{0}$ and 0 otherwise.
We denote the smooth bts (X, τ_1, τ_2) with a smooth ideal $\mathcal{I}$ by the quadruple $(X, \tau_1, \tau_2, \mathcal{I})$ and call it a smooth ideal bitopological space (smooth ideal bts).

Definition 2.5. [21] Let (X, τ_1, τ_2) be a smooth bts for $\lambda \in I^X$ and $r \in I_0$. Then:

(1) λ is an r-(τ_i, τ_j)-fuzzy semi-open set (r-(τ_i, τ_j)-fso), if there exists $\mu \in I^X$ with $\tau_i(\mu) \geq r$ such that $\mu \leq \lambda \leq C_{\tau_j}(\mu, r)$.

(2) λ is an r-(τ_i, τ_j)-fuzzy semi-closed set (r-(τ_i, τ_j)-fsc), if there exists $\mu \in I^X$ with $\tau_i(\bar{1} - \mu) \geq r$ such that $I_{\tau_j}(\mu, r) \leq \lambda \leq \mu$.

(3) The $r(i, j)$-fuzzy semi-interior of λ is denoted by $SI_{ij}(\lambda, r)$ and defined as

$$SI_{ij}(\lambda, r) = \bigvee\{\nu \in I^X | \ \nu \leq \lambda, \nu \ is \ r\text{-}(\tau_i, \tau_j)\text{-}fso\}.$$

(4) The $r(i, j)$-fuzzy semi-closure of λ is denoted by $SC_{ij}(\lambda, r)$ and defined as

$$SC_{ij}(\lambda, r) = \bigwedge\{\nu \in I^X | \ \nu \geq \lambda, \nu \ is \ r\text{-}(\tau_i, \tau_j)\text{-}fsc\}.$$

(5) λ is an r-(τ_i, τ_j)-fuzzy preopen set (r-(τ_i, τ_j)-fpo) if $\lambda \leq I_{\tau_i}(C_{\tau_j}(\lambda, r), r)$.

(6) λ is an r-(τ_i, τ_j)-fuzzy preclosed set (r-(τ_i, τ_j)-fpc) if $C_{\tau_i}(I_{\tau_j}(\lambda, r), r) \leq \lambda$.

Theorem 2.6. [7] Let (X, τ_1, τ_2) be a smooth bts for $\lambda \in I^X$ and $r \in I_0$. Then:

(1) λ is an r-(τ_i, τ_j)-fso iff $\lambda = SI_{ij}(\lambda, r)$.

(2) λ is an r-(τ_i, τ_j)-fsc iff $\lambda = SC_{ij}(\lambda, r)$.

(3) λ is an r-(τ_i, τ_j)-fso iff $\bar{1} - \lambda$ is an r-(τ_i, τ_j)-fsc.

(4) λ is an r-(τ_i, τ_j)-fso iff $\lambda \leq C_{\tau_j}(I_{\tau_i}(\lambda, r), r)$.

(5) $SC_{ij}(\bar{0}, r) = \bar{0}$.

(6) $\bar{1} - SC_{ij}(\lambda, r) = SI_{ij}(\bar{1} - \lambda, r)$.

Definition 2.7. [7] Let (X, τ_1, τ_2) and (Y, τ_1^*, τ_2^*) be smooth bts's. Let $f : X \longrightarrow Y$ be a mapping. Then f is called:

(1) FP-irresolute (resp. FP-semi-continuous [21]) iff $f^{-1}(\mu)$ is an r-(τ_i, τ_j)-fso set in X for each r-(τ_i^*, τ_j^*)-fso set μ in Y (resp. $\mu \in I^Y$, $\tau_i^*(\mu) \geq r$).

(2) FP-irresolute open iff $f(\mu)$ is an r-(τ_i^*, τ_j^*)-fso set in Y for each r-(τ_i, τ_j)-fso set μ in X.

Theorem 2.8. [7] Let (X, τ_1, τ_2) and (Y, τ_1^*, τ_2^*) be smooth bts's. Let $f : X \longrightarrow Y$ be a mapping. Then the following statements are equivalent:

(1) f is a FP-irresolute.

(2) For each r-(τ_i^*, τ_j^*)-fsc set $\mu \in I^Y$, $f^{-1}(\mu)$ is an r-(τ_i, τ_j)-fsc set in X.

(3) $SC_{ij}(f^{-1}(\mu), r) \leq f^{-1}(SC_{ij}(\mu, r))$, $\mu \in I^Y$.

Lemma 2.9. [17] Let $f : X \longrightarrow Y$ be a mapping and let λ and μ be fuzzy sets in X and Y, respectively. Then the following properties hold:

(1) $\lambda \leq f^{-1}(f(\lambda))$ and equality holds if f is injective.

(2) $f(f^{-1}(\mu)) \leq \mu$ and equality holds if f is surjective.

(3) For any fuzzy point x_t in X, $f(x_t)$ is a fuzzy point in Y and $f(x_t) = (f(x))_t$.

(4) If $f(\lambda) \leq \mu$, then $\lambda \leq f^{-1}(\mu)$.

Definition 2.10. [24] Let $(X, \tau, \mathcal{I})$ be a smooth ideal topological space and $r \in I_0$. Then X is called:

(1) An r-$F\mathcal{I}$-compact (resp. r-fuzzy ideal quasi H-closed (r-$F\mathcal{I}QHC$)) iff for every family $\{\lambda_i \in I^X | \ \tau(\lambda_i) \geq r, i \in J\}$ such that when $\bigvee_{i \in J} \lambda_i = \bar{1}$, there exists a finite set $J_0 \subset J$, such that $\mathcal{I}(\bar{1} - \bigvee_{i \in J_0} \lambda_i) \geq r$ (resp. $\mathcal{I}(\bar{1} - \bigvee_{i \in J_0} C_\tau(\lambda_i, r)) \geq r$).

(2) An r-fuzzy compact modulo fuzzy ideal space (r-fuzzy $C(\mathcal{I})$-compact) if for every $\beta \in I^X$, $\tau(\bar{1} - \beta) \geq r$ and each family $\{\lambda_i \in I^X | \ \tau(\lambda_i) \geq r, i \in J\}$ such that $\beta \leq \bigvee_{i \in J} \lambda_i$, there exists a finite set $J_0 \subset J$, such that $\mathcal{I}(\beta \wedge [\bar{1} - \bigvee_{i \in J_0} C_\tau(\lambda_i, r)]) \geq r$.

3 $FPS\mathcal{I}$-compact and $FPS\mathcal{I}$-Lindelöf Spaces

In this section we introduce the notion of FPS-compact (resp. Lindelöf) space in smooth bts (X, τ_1, τ_2) by using the family of r-(τ_i, τ_j)-fso sets as cover. Then, we generalize the same notions via smooth ideal $\mathcal{I}$ on X to obtain $FPS\mathcal{I}$-compact (resp. Lindelöf) space. We also give the relations between them and study some of their basic properties.

Definition 3.1. Let $(X, \tau_1, \tau_2, \mathcal{I})$ be a smooth ideal bts and $r \in I_0$. A fuzzy set $\rho \in I^X$ is called:

(1) An r-(τ_i, τ_j)-FS-compact if for every family $\{\mu_\alpha \in I^X | \ \mu_\alpha \ is \ r\text{-}(\tau_i, \tau_j)\text{-}fso \ set, \alpha \in J\}$, such that $\rho \leq \bigvee_{\alpha \in J} \mu_\alpha$, there exists a finite set $J_0 \subset J$, such that $\rho \leq \bigvee_{\alpha \in J_0} \mu_\alpha$. The space (X, τ_1, τ_2) is an r-(τ_i, τ_j)-FS-compact if X is an r-(τ_i, τ_j)-FS-compact as a fuzzy subset.

(2) An r-(τ_i, τ_j)-$FS\mathcal{I}$-compact if for every family $\{\mu_\alpha \in I^X | \ \mu_\alpha \ is \ r\text{-}(\tau_i, \tau_j)\text{-}fso \ set, \alpha \in J\}$, such that $\rho \leq \bigvee_{\alpha \in J} \mu_\alpha$, there exists a finite set $J_0 \subset J$, such that $\mathcal{I}(\rho \wedge [\bar{1} - \bigvee_{\alpha \in J_0} \mu_\alpha]) \geq r$. The space $(X, \tau_1, \tau_2, \mathcal{I})$ is an r-(τ_i, τ_j)-$FS\mathcal{I}$-compact if X is an r-(τ_i, τ_j)-$FS\mathcal{I}$-compact as a fuzzy subset.

(3) An r-(τ_i, τ_j)-FS-Lindelöf if for every family $\{\mu_\alpha \in I^X | \ \mu_\alpha \ is \ r\text{-}(\tau_i, \tau_j)\text{-}fso \ set, \alpha \in J\}$, such that $\rho \leq \bigvee_{\alpha \in J} \mu_\alpha$, there exists a countable set $J_0 \subset J$, such that $\rho \leq \bigvee_{\alpha \in J_0} \mu_\alpha$. The space (X, τ_1, τ_2) is an r-(τ_i, τ_j)-FS-Lindelöf if X is an r-(τ_i, τ_j)-FS-Lindelöf as a fuzzy subset.

(4) An r-(τ_i,τ_j)-$FS\mathcal{I}$-Lindelöf if for every family $\{\mu_\alpha \in I^X \mid \mu_\alpha\ is\ r\text{-}(\tau_i,\tau_j)\text{-}fso\ set, \alpha \in J\}$, such that $\rho \leq \bigvee_{\alpha\in J} \mu_\alpha$, there exists a countable set $J_0 \subset J$, such that $\mathcal{I}(\rho \wedge [\bar{1} - \bigvee_{\alpha\in J_0} \mu_\alpha]) \geq r$. The space $(X,\tau_1,\tau_2,\mathcal{I})$ is an r-(τ_i,τ_j)-$FS\mathcal{I}$-Lindelöf if X is an r-(τ_i,τ_j)-$FS\mathcal{I}$-Lindelöf as a fuzzy subset.

Definition 3.2. Let $(X,\tau_1,\tau_2,\mathcal{I})$ be a smooth ideal bts and $r \in I_0$. Then X is called:

(1) FPS-compact (resp. $FPS\mathcal{I}$-compact) if X is an r-(τ_i,τ_j)-FS-compact (resp. r-(τ_i,τ_j)-$FS\mathcal{I}$-compact) for each $r \in I_0$.

(2) FPS-Lindelöf (resp. $FPS\mathcal{I}$-Lindelöf) if X is an r-(τ_i,τ_j)-FS-Lindelöf (resp. r-(τ_i,τ_j)-$FS\mathcal{I}$-Lindelöf) for each $r \in I_0$.

From Definition 3.1 we have the following remark.

Remark 3.3. Let $(X,\tau_1,\tau_2,\mathcal{I})$ be a smooth ideal bts and $r \in I_0$. Then the following statements are true:

(1) If X is an r-(τ_i,τ_j)-$FS\mathcal{I}$-compact, then X is an r-τ_i-$F\mathcal{I}$-compact.

(2) If X is an r-(τ_i,τ_j)-FS-compact (resp. Lindelöf), then X is an r-(τ_i,τ_j)-$FS\mathcal{I}$-compact (resp. Lindelöf).

(3) If X is an r-(τ_i,τ_j)-FS-compact (resp. r-(τ_i,τ_j)-$FS\mathcal{I}$-compact), then X is an r-(τ_i,τ_j)-FS-Lindelöf (resp. r-(τ_i,τ_j)-$FS\mathcal{I}$-Lindelöf).

(4) If $\mathcal{I} = \mathcal{I}^0$, then r-(τ_i,τ_j)-FS-compact (resp. r-(τ_i,τ_j)-FS-Lindelöf) and r-(τ_i,τ_j)-$FS\mathcal{I}$-compact (resp. r-(τ_i,τ_j)-$FS\mathcal{I}$-Lindelöf) are equivalent.

It follows from the Definition 3.1, Remark 3.3 and the fact that every r-τ_i-open fuzzy set in X is an r-(τ_i,τ_j)-fso set that

$$\begin{array}{ccccc} r\text{-}(\tau_i,\tau_j)\text{-}FS\text{-}compact & \Longrightarrow & r\text{-}(\tau_i,\tau_j)\text{-}FS\mathcal{I}\text{-}compact & \Longrightarrow & r\text{-}\tau_i\text{-}F\mathcal{I}\text{-}compact \\ \Downarrow & & \Downarrow & & \\ r\text{-}(\tau_i,\tau_j)\text{-}FS\text{-}Lindel\ddot{o}f & & r\text{-}(\tau_i,\tau_j)\text{-}FS\mathcal{I}\text{-}Lindel\ddot{o}f & & \end{array}$$

Example 3.4. Let $X = \mathbb{N}$, where $\mathbb{N}$ is the set of natural numbers. Define fuzzy set $\lambda_n \in I^X$ as follows:

$$\lambda_n = \chi_{\{n\}}, \text{where } \chi_{\{n\}} \text{ is the characteristic function of } \{\text{n}\}, n \in \mathbb{N}.$$

Define smooth topologies $\tau_1 : I^X \longrightarrow I$ and $\tau_2 : I^X \longrightarrow I$ as follows:

$$\tau_1(\lambda) = \tau_2(\lambda) = \begin{cases} 1 & if\ \lambda = \bar{0}, \bar{1}, \\ \frac{1}{2} & if\ \lambda = \lambda_n,\ n \in \mathbb{N}, \\ \frac{3}{4} & otherwise. \end{cases}$$

Define smooth ideal $\mathcal{I} : I^X \longrightarrow I$ as follows:

$$\mathcal{I}(\lambda) = \begin{cases} 1 & if\ \lambda = \bar{0}, \\ 0 & if\ \lambda = \bar{1}, \\ \frac{3}{4} & otherwise. \end{cases}$$

Then clearly $(X,\tau_1,\tau_2,\mathcal{I})$ is a smooth ideal bts. Note that X is not a $\frac{1}{2}$-(τ_1,τ_2)-FS-compact since there exists a family

$$\{\lambda_n \in I^X \mid \lambda_n = \chi_{\{n\}}\ is\ \frac{1}{2}\text{-}(\tau_1,\tau_2)\text{-}fso\ set,\ n \in \mathbb{N}\} \text{ with } \bigvee_{n\in\mathbb{N}} \lambda_n = \bar{1}$$

such that there in no finite set $J_0 \subset \mathbb{N}$ with $\bigvee_{n\in J_0} \lambda_n = \bar{1}$.

But X is a $\frac{1}{2}$-(τ_1,τ_2)-$FS\mathcal{I}$-compact, since for any finite set $J_0 \subset \mathbb{N}$, $\mathcal{I}(\bar{1} - \bigvee_{n\in J_0} \lambda_n) = \frac{3}{4} \geq \frac{1}{2}$.

The following example shows that the finite spaces need not to be a r-(τ_i, τ_j)-FS-compact.

Example 3.5. Let $X = \{a, b, c\}$ and $\lambda_1, \lambda_2, \lambda_3$ and λ_4 be fuzzy sets of X defined as

$$\lambda_1 = a_s \vee b_{0.5} \vee c_s,\ \ \lambda_2 = a_{0.3} \vee b_s \vee c_s,\ \ s \in [0.9, 1),$$

$$\lambda_3 = a_k \vee b_{0.5} \vee c_k,\ \ \lambda_4 = a_{0.3} \vee b_k \vee c_k,\ \ k \in (0, 0.1].$$

Define smooth topologies $\tau_1 : I^X \longrightarrow I$ and $\tau_2 : I^X \longrightarrow I$ by

$$\tau_1(\lambda) = \begin{cases} 1 & if\ \lambda = \bar{0}, \bar{1}, \\ \frac{1}{3} & if\ \lambda = \lambda_1, \\ \frac{1}{3} & if\ \lambda = \lambda_2, \\ \frac{1}{3} & if\ \lambda = \lambda_1 \wedge \lambda_2, \\ \frac{1}{3} & if\ \lambda = \lambda_1 \vee \lambda_2, \\ 0 & otherwise, \end{cases} \quad and \quad \tau_2(\lambda) = \begin{cases} 1 & if\ \lambda = \bar{0}, \bar{1}, \\ \frac{1}{2} & if\ \lambda = \lambda_3, \\ \frac{1}{2} & if\ \lambda = \lambda_4, \\ \frac{1}{2} & if\ \lambda = \lambda_3 \wedge \lambda_4, \\ \frac{1}{2} & if\ \lambda = \lambda_3 \vee \lambda_4, \\ 0 & otherwise. \end{cases}$$

Then clearly (X, τ_1, τ_2) is a smooth bts. Note that X is finite set but it is not a $\frac{1}{3}$-(τ_1, τ_2)-FS-compact since there exists a family

$$\{a_s \vee b_s \vee c_s \in I^X \mid a_s \vee b_s \vee c_s\ is\ \frac{1}{3}\text{-}(\tau_1, \tau_2)\text{-}fso\ sets,\ s \in [0.9, 1)\} \text{ with } \bigvee_{s\in[0.9,1)} a_s \vee b_s \vee c_s = \bar{1}.$$

But there is no finite subset $J_0 \subset [0.9, 1)$ such that $\bigvee_{s\in J_0} a_s \vee b_s \vee c_s = \bar{1}$.

The following example corresponds to the concept of the ideal of finite (resp. countable) subsets of X in the ordinary sense.

Example 3.6. Define $\mathcal{I}_f, \mathcal{I}_c : I^X \longrightarrow I$ be two smooth ideals on X as follows:

$$\mathcal{I}_f(\mu) = \begin{cases} 1 & if \text{ supp } (\mu) = \text{finite subset of } X, \\ 0 & otherwise; \end{cases}$$

and

$$\mathcal{I}_c(\mu) = \begin{cases} 1 & if \text{ supp } (\mu) = \text{countable subset of } X, \\ 0 & otherwise. \end{cases}$$

That is mean:

(1) $\mathcal{I}_f$ is a smooth ideal on X such that for every $r \in I_0$, $(\mathcal{I}_f)_r = \{\mu \in I^X \mid \mathcal{I}_f(\mu) \geq r$ and $supp(\mu)$ is a finite subset of $X\}$ is a fuzzy ideal in Sarkar's sense.

(2) $\mathcal{I}_c$ is a smooth ideal on X such that for every $r \in I_0$, $(\mathcal{I}_c)_r = \{\mu \in I^X \mid \mathcal{I}_c(\mu) \geq r$ and $supp(\mu)$ is a countable subset of $X\}$ is a fuzzy ideal in Sarkar's sense.

Theorem 3.7. A smooth ideal bts $(X, \tau_1, \tau_2, \mathcal{I}_f)$ is an r-(τ_i, τ_j)-$FS\mathcal{I}_f$-compact iff (X, τ_1, τ_2) is an r-(τ_i, τ_j)-FS-compact.

Proof. Let $(X, \tau_1, \tau_2, \mathcal{I}_f)$ be an r-(τ_i, τ_j)-$FS\mathcal{I}_f$-compact and let $\{\mu_\alpha \in I^X \mid \mu_\alpha\ is\ r\text{-}(\tau_i, \tau_j)\text{-}fso\ set,$ $\alpha \in J\}$ be any family such that $\bigvee_{\alpha\in J} \mu_\alpha = \bar{1}$. Suppose that for any finite set $J_0 \subset J$ we have $\bigvee_{\alpha\in J_0} \mu_\alpha \neq \bar{1}$. This implies $\bar{1} - \bigvee_{\alpha\in J_0} \mu_\alpha \neq \bar{0}$. Therefore, $\mathcal{I}_f(\bar{1} - \bigvee_{\alpha\in J_0} \mu_\alpha) \neq 1$. Thus, from definition of $\mathcal{I}_f$, we have $\mathcal{I}_f(\bar{1} - \bigvee_{\alpha\in J_0} \mu_\alpha) = 0$ meaning that for any finite set $J_0 \subset J$, $\mathcal{I}_f(\bar{1} - \bigvee_{\alpha\in J_0} \mu_\alpha) < r$ which contradicts the hypothesis. Hence, (X, τ_1, τ_2) is an r-(τ_i, τ_j)-FS-compact.

Conversely, let $\{\mu_\alpha \in I^X \mid \mu_\alpha\ is\ r\text{-}(\tau_i, \tau_j)\text{-}fso\ set, \alpha \in J\}$ be any family such that $\bigvee_{\alpha\in J} \mu_\alpha = \bar{1}$. According to r-(τ_i, τ_j)-FS-compactness of X, there exists a finite set $J_0 \subset J$ such that $\bigvee_{\alpha\in J_0} \mu_\alpha = \bar{1}$. Since $\bar{1} - \bigvee_{\alpha\in J_0} \mu_\alpha = \bar{0}$ and $supp(\bar{0}) = \emptyset$, there is a finite subset of X. Then, $\mathcal{I}_f(\bar{1} - \bigvee_{\alpha\in J_0} \mu_\alpha) \geq r$. Hence, $(X, \tau_1, \tau_2, \mathcal{I}_f)$ is an r-(τ_i, τ_j)-$FS\mathcal{I}_f$-compact. □

Theorem 3.8. A smooth ideal bts $(X, \tau_1, \tau_2, \mathcal{I}_c)$ is an r-(τ_i, τ_j)-$FS\mathcal{I}_c$-compact iff (X, τ_1, τ_2) is an r-(τ_i, τ_j)-FS-Lindelöf.

Proof. Let $(X, \tau_1, \tau_2, \mathcal{I}_c)$ be an r-(τ_i, τ_j)-$FS\mathcal{I}_c$-compact and let $\{\mu_\alpha \in I^X | \ \mu_\alpha \ is \ r\text{-}(\tau_i, \tau_j)\text{-}fso \ set, \alpha \in J\}$ be any family such that $\bigvee_{\alpha \in J} \mu_\alpha = \bar{1}$. By r-(τ_i, τ_j)-$FS\mathcal{I}_c$-compactness of X, there exists a finite set $J_0 \subset J$ such that $\mathcal{I}_c(\bar{1} - \bigvee_{\alpha \in J_0} \mu_\alpha) \geq r$, meaning that $\bar{1} - \bigvee_{\alpha \in J_0} \mu_\alpha$ has a countable support. Therefore, there exists a countable set $J_c \subset J$, such that $\bar{1} - \bigvee_{\alpha \in J_0} \mu_\alpha \leq \bigvee_{\alpha \in J_c} \mu_\alpha$. Thus, $\bar{1} = \bigvee_{\alpha \in J_0} \mu_\alpha \vee \bigvee_{\alpha \in J_c} \mu_\alpha = \bigvee_{\alpha \in J_0 \cup J_c} \mu_\alpha$. Since $J_0 \cup J_c$ is countable subset of J, then (X, τ_1, τ_2) is an r-(τ_i, τ_j)-FS-Lindelöf.

Conversely, Theorem 3.7 is a similar proof. □

Corollary 3.9. If $(X, \tau_1, \tau_2, \mathcal{I}_c)$ is an r-(τ_i, τ_j)-$FS\mathcal{I}_c$-compact, then $(X, \tau_1, \tau_2, \mathcal{I}_c)$ is an r-(τ_i, τ_j)-$FS\mathcal{I}_c$-Lindelöf.

Corollary 3.10. If $(X, \tau_1, \tau_2, \mathcal{I}_c)$ is an r-(τ_i, τ_j)-$FS\mathcal{I}_c$-compact, then (X, τ_1, τ_2) is an r-τ_i-F-Lindelöf.

Theorem 3.11. Let $(X, \tau_1, \tau_2, \mathcal{I})$ be a smooth ideal bts. If $\lambda_1, \lambda_2 \in I^X$ are r-(τ_i, τ_j)-$FS\mathcal{I}$-compact fuzzy subsets, then $\lambda_1 \vee \lambda_2$ is an r-(τ_i, τ_j)-$FS\mathcal{I}$-compact.

Proof. Let $\{\mu_\alpha \in I^X | \ \mu_\alpha \ is \ r\text{-}(\tau_i, \tau_j)\text{-}fso \ set, \alpha \in J\}$ be a family such that $\lambda_1 \vee \lambda_2 \leq \bigvee_{\alpha \in J} \mu_\alpha$. Then, $\lambda_1 \leq \bigvee_{\alpha \in J} \mu_\alpha$ and $\lambda_2 \leq \bigvee_{\alpha \in J} \mu_\alpha$. Since λ_1 and λ_2 are r-(τ_i, τ_j)-$FS\mathcal{I}$-compact, there exists a finite set $J_1 \subset J$ and $J_2 \subset J$ such that $\mathcal{I}(\lambda_1 \wedge (\bar{1} - \bigvee_{\alpha \in J_1} \mu_\alpha)) \geq r$ and $\mathcal{I}(\lambda_2 \wedge (\bar{1} - \bigvee_{\alpha \in J_2} \mu_\alpha)) \geq r$. Therefore $\mathcal{I}(\lambda_1 \vee \lambda_2 \wedge (\bar{1} - \bigvee_{\alpha \in J_1 \cup J_2} \mu_\alpha)) \geq r$. Thus, $\lambda_1 \vee \lambda_2$ is an r-(τ_i, τ_j)-$FS\mathcal{I}$-compact. □

Theorem 3.12. Let $(X, \tau_1, \tau_2, \mathcal{I})$ be a smooth ideal bts and $r \in I_0$. If for each family $\{\mu_\alpha \in I^X | \ \mu_\alpha \ is \ r\text{-}(\tau_i, \tau_j)\text{-}fsc \ set, \ \alpha \in J\}$ with $\mathcal{I}(\bigwedge_{\alpha \in J} \mu_\alpha) \geq r$ there exists a finite set $J_0 \subset J$ such that $\mathcal{I}(\bigwedge_{\alpha \in J_0} \mu_\alpha) \geq r$, then X is an r-(τ_i, τ_j)-$FS\mathcal{I}$-compact.

Proof. Let $\{\mu_\alpha \in I^X | \ \mu_\alpha \ is \ r\text{-}(\tau_i, \tau_j)\text{-}fso \ set, \alpha \in J\}$ with $\bigvee_{\alpha \in J} \mu_\alpha = \bar{1}$ implies that $\mathcal{I}(\bigwedge_{\alpha \in J} \bar{1} - \mu_\alpha) \geq r$. The hypothesis suggests that there exists a finite set $J_0 \subset J$ such that $\mathcal{I}(\bigwedge_{\alpha \in J_0} \bar{1} - \mu_\alpha) \geq r$. Thus, $\mathcal{I}(\bar{1} - \bigvee_{\alpha \in J_0} \mu_\alpha) \geq r$. Hence, X is an r-(τ_i, τ_j)-$FS\mathcal{I}$-compact. □

Theorem 3.13. Let $(X, \tau_1, \tau_2, \mathcal{I})$ be an r-(τ_i, τ_j)-$FS\mathcal{I}$-compact and $r \in I_0$. If $\mathcal{J}$ is a smooth ideal on X such that $\mathcal{I} \leq \mathcal{J}$, then X is an r-(τ_i, τ_j)-$FS\mathcal{J}$-compact.

Proof. Let $\{\mu_\alpha \in I^X | \ \mu_\alpha \ is \ r\text{-}(\tau_i, \tau_j)\text{-}fso \ set, \alpha \in J\}$ with $\bigvee_{\alpha \in J} \mu_\alpha = \bar{1}$. Since X is an r-(τ_i, τ_j)-$FS\mathcal{I}$-compact, there exists a finite set $J_0 \subset J$ such that $\mathcal{I}(\bar{1} - \bigvee_{\alpha \in J_0} \mu_\alpha) \geq r$. Since $\mathcal{I} \leq \mathcal{J}$, then $\mathcal{J}(\bar{1} - \bigvee_{\alpha \in J_0} \mu_\alpha) \geq r$. Thus, X is an r-(τ_i, τ_j)-$FS\mathcal{J}$-compact. □

Theorem 3.14. Let $(X, \tau_1, \tau_2, \mathcal{I})$ be a smooth ideal bts and $r \in I_0$. Then the following statements are equivalent:

(1) $(X, \tau_1, \tau_2, \mathcal{I})$ is an r-(τ_i, τ_j)-$FS\mathcal{I}$-compact.

(2) For any collection $\{\mu_\alpha \in I^X | \ \mu_\alpha \ is \ r\text{-}(\tau_i, \tau_j)\text{-}fsc \ set, \alpha \in J\}$ with $\bigwedge_{\alpha \in J} \mu_\alpha = \bar{0}$, there exists a finite set $J_0 \subset J$ with $\mathcal{I}(\bigwedge_{\alpha \in J_0} \mu_\alpha) \geq r$.

Proof. (1) $\Longrightarrow$ (2) Let $\{\mu_\alpha \in I^X | \ \mu_\alpha \ is \ r\text{-}(\tau_i, \tau_j)\text{-}fsc \ set, \alpha \in J\}$ with $\bigwedge_{\alpha\in J} \mu_\alpha = \bar{0}$. This implies, $\bigvee_{\alpha\in J} (\bar{1} - \mu_\alpha) = \bar{1}$. Since $\{\bar{1} - \mu_\alpha, \alpha \in J\}$ is a family of r-(τ_i, τ_j)-fso sets and by r-(τ_i, τ_j)-$FS\mathcal{I}$-compactness of X, there exists a finite set $J_0 \subset J$ such that $\mathcal{I}(\bar{1} - \bigvee_{\alpha\in J_0} (\bar{1} - \mu_\alpha)) \geq r$ implies that $\mathcal{I}(\bigwedge_{\alpha\in J_0} \mu_\alpha) \geq r$.

(2) $\Longrightarrow$ (1) Let $\{\mu_\alpha \in I^X | \ \mu_\alpha \ is \ r\text{-}(\tau_i, \tau_j)\text{-}fso \ set, \alpha \in J\}$ be a family with $\bigvee_{\alpha\in J} \mu_\alpha = \bar{1}$. Then, $\bigwedge_{\alpha\in J} (\bar{1} - \mu_\alpha) = \bar{0}$. By (2), there exists a finite set $J_0 \subset J$ such that $\mathcal{I}(\bigwedge_{\alpha\in J_0} (\bar{1} - \mu_\alpha)) \geq r$. This implies that $\mathcal{I}(\bar{1} - \bigvee_{\alpha\in J_0} \mu_\alpha) \geq r$. Therefore, $(X, \tau_1, \tau_2, \mathcal{I})$ is an r-(τ_i, τ_j)-$FS\mathcal{I}$-compact. □

Definition 3.15. [24] A family $\{\mu_\alpha \in I^X | \ \alpha \in J\}$ has the finite intersection property ($\mathcal{I}$-FIP) iff $\mathcal{I}(\bigwedge_{\alpha\in J_0} \mu_\alpha) \geq r$ for any no finite subfamily $J_0 \subset J$.

Theorem 3.16. A smooth ideal bts $(X, \tau_1, \tau_2, \mathcal{I})$ is an r-(τ_i, τ_j)-$FS\mathcal{I}$-compact iff every collection $\{\mu_\alpha \in I^X | \ \mu_\alpha \ is \ r\text{-}(\tau_i, \tau_j)\text{-}fsc \ set, \alpha \in J\}$ having the $\mathcal{I}$-FIP has a non-empty intersection.

Proof. Suppose X is an r-(τ_i, τ_j)-$FS\mathcal{I}$-compact and let $\{\mu_\alpha \in I^X | \ \mu_\alpha \ is \ r\text{-}(\tau_i, \tau_j)\text{-}fsc \ set, \alpha \in J\}$ having the $\mathcal{I}$-FIP. Suppose $\bigwedge_{\alpha\in J} \mu_\alpha = \bar{0}$. Then $\bigvee_{\alpha\in J} (\bar{1} - \mu_\alpha) = \bar{1}$. Since $\bar{1} - \mu_\alpha$ is an r-(τ_i, τ_j)-fso set for each $\alpha \in J$. By r-(τ_i, τ_j)-$FS\mathcal{I}$-compactness of X, there exists a finite set $J_0 \subset J$ such that $\mathcal{I}(\bar{1} - \bigvee_{\alpha\in J_0} (\bar{1} - \mu_\alpha)) \geq r$, implies that $\mathcal{I}(\bigwedge_{\alpha\in J_0} \mu_\alpha) \geq r$ which is a contradiction. Hence, $\bigwedge_{\alpha\in J} \mu_\alpha \neq \bar{0}$.

Conversely, let $\{\mu_\alpha \in I^X | \ \mu_\alpha \ is \ r\text{-}(\tau_i, \tau_j)\text{-}fso \ set, \alpha \in J\}$ be a family with $\bigvee_{\alpha\in J} \mu_\alpha = \bar{1}$. Suppose for every finite set $J_0 \subset J$ we have $\mathcal{I}(\bar{1} - \bigvee_{\alpha\in J_0} \mu_\alpha) < r$. That is mean $\mathcal{I}(\bigwedge_{\alpha\in J_0} (\bar{1} - \mu_\alpha)) \geq r$ for every no finite $J_0 \subset J$, from hypothesis of $\mathcal{I}$-FIP we have, $\bigwedge_{\alpha\in J} (\bar{1} - \mu_\alpha) \neq \bar{0}$. This yields $\bigvee_{\alpha\in J} \mu_\alpha \neq \bar{1}$ which is a contradiction. Hence, $(X, \tau_1, \tau_2, \mathcal{I})$ is an r-(τ_i, τ_j)-$FS\mathcal{I}$-compact. □

Definition 3.17. Let (X, τ_1, τ_2) be a smooth bts and $r \in I_0$. A fuzzy set λ of X is called:

(1) an r-(τ_i, τ_j)-fuzzy regular(semi)open (r-(τ_i, τ_j)-fro (resp. r-(τ_i, τ_j)-frso)) if $\lambda = I_{\tau_i}(C_{\tau_j}(\lambda, r), r)$ (resp. $\lambda = SI_{ij}(C_{\tau_j}(\lambda, r), r)$).

(2) an r-(τ_i, τ_j)-fuzzy regular(semi)closed (r(i,j)-frc (resp. r-(τ_i, τ_j)-frsc)) if $\lambda = C_{\tau_i} \ (I_{\tau_j}(\lambda, r), r)$ (resp. $\lambda = SC_{ij}(I_{\tau_j}(\lambda, r), r)$).

Theorem 3.18. If $(X, \tau_1, \tau_2, \mathcal{I})$ is an r-(τ_i, τ_j)-$FS\mathcal{I}$-compact, then for every family $\{\mu_\alpha \in I^X | \ \mu_\alpha$ *is* $r\text{-}(\tau_j, \tau_i)\text{-}frc \ set, \alpha \in J\}$ such that $\bigvee_{\alpha\in J} \mu_\alpha = \bar{1}$, there exists a finite set $J_0 \subset J$ such that $\mathcal{I}(\bar{1} - \bigvee_{\alpha\in J_0} \mu_\alpha) \geq r$.

Proof. The proof derives from the fact that every r-(τ_j, τ_i)-frc set is an r-(τ_i, τ_j)-fso set. □

Theorem 3.19. If $(X, \tau_1, \tau_2, \mathcal{I})$ is an r-(τ_i, τ_j)-$FS\mathcal{I}$-compact, then for every family $\{\mu_\alpha \in I^X | \ \mu_\alpha$ *is* $r\text{-}(\tau_i, \tau_j)\text{-}frso \ set, \alpha \in J\}$ such that $\bigvee_{\alpha\in J} \mu_\alpha = \bar{1}$, there exists a finite set $J_0 \subset J$ such that $\mathcal{I}(\bar{1} - \bigvee_{\alpha\in J_0} \mu_\alpha) \geq r$.

Proof. Let $\{\mu_\alpha \in I^X | \ \mu_\alpha \ is \ an \ \ r\text{-}(\tau_i, \tau_j)\text{-}frso \ set, \alpha \in J\}$ such that $\bigvee_{\alpha\in J} \mu_\alpha = \bar{1}$. Then, from Definition 3.17(1), we have $\bigvee_{\alpha\in J} SI_{ij} \ (C_{\tau_j}(\mu_\alpha, r), r) = \bar{1}$. As $\{SI_{ij}(C_{\tau_j}(\mu_\alpha, r), r) \in I^X, \alpha \in J\}$ is a family of r-(τ_i, τ_j)-fso sets, and by r-(τ_i, τ_j)-$FS\mathcal{I}$-compactness of X there exists a finite set $J_0 \subset J$, such that $\mathcal{I}(\bar{1} - \bigvee_{\alpha\in J_0} SI_{ij}(C_{\tau_j}(\mu_\alpha, r), r)) \geq r$, then this $\mathcal{I}(\bar{1} - \bigvee_{\alpha\in J_0} \mu_\alpha) \geq r$. □

Theorem 3.20. If $(X, \tau_1, \tau_2, \mathcal{I})$ is an r-(τ_i, τ_j)-$FS\mathcal{I}$-compact, then for every family $\{\mu_\alpha \in I^X | \ \mu_\alpha$ *is* r-(τ_i, τ_j)-fpo *set*, $\alpha \in J\}$ such that $\bigvee_{\alpha \in J} \mu_\alpha = \bar{1}$, there exists a finite set $J_0 \subset J$ such that $\mathcal{I}(\bar{1} - \bigvee_{\alpha \in J_0} C_{\tau_j}(\mu_\alpha, r)) \geq r$.

Proof. Let $\{\mu_\alpha \in I^X | \ \mu_\alpha$ *is* r-(τ_i, τ_j)-fpo *set*, $\alpha \in J\}$ be any family such that $\bigvee_{\alpha \in J} \mu_\alpha = \bar{1}$. Since $\mu_\alpha \leq C_{\tau_j}(\mu_\alpha, r) \leq C_{\tau_j}(I_{\tau_i}(C_{\tau_j}(\mu_\alpha, r), r), r)$, then $\bar{1} = \bigvee_{\alpha \in J} C_{\tau_j}(\mu_\alpha, r)$ such that $\{C_{\tau_j}(\mu_\alpha, r) \in I^X, \alpha \in J\}$ is a family of r-(τ_i, τ_j)-fso set of X. By r-(τ_i, τ_j)-$FS\mathcal{I}$-compactness of X, there exists a finite $J_0 \subset J$ such that $\mathcal{I}(\bar{1} - \bigvee_{\alpha \in J_0} C_{\tau_j}(\mu_\alpha, r)) \geq r$. □

Corollary 3.21. If $(X, \tau_1, \tau_2, \mathcal{I})$ is an r-(τ_i, τ_j)-$FS\mathcal{I}$-compact, then for every family $\{\mu_\alpha \in I^X | \ \mu_\alpha$ *is* r-(τ_i, τ_j)-fpc *set*, $\alpha \in J\}$ such that $\bigwedge_{\alpha \in J} \mu_\alpha = \bar{0}$, there exists a finite set $J_0 \subset J$ such that $\mathcal{I}(\bigwedge_{\alpha \in J_0} I_{\tau_j}((\mu_\alpha, r)) \geq r$.

4 $FPSC(\mathcal{I})$-compact and $FP\mathcal{I}$-S-closed Spaces

In this section we introduce the notions of $FPSC$-compact, $FPSC(\mathcal{I})$-compact, FP-S-closed and $FP\mathcal{I}$-S-closed in a smooth bts (X, τ_1, τ_2) and study some of their basic properties. We give the relations between them. Furthermore, we show that $FPSC(\mathcal{I})$-compactness is not a generalization of $FPSC$-compactness.

Definition 4.1. Let $(X, \tau_1, \tau_2, \mathcal{I})$ be a smooth ideal bts and $r \in I_0$. Then, X is called:

(1) An r-(τ_i, τ_j)-FSC-compact, if for every r-(τ_i, τ_j)-fsc set ρ of X and every family $\{\mu_\alpha \in I^X | \ \mu_\alpha$ *is* r-(τ_i, τ_j)-fso *set*, $\alpha \in J\}$ with $\rho \leq \bigvee_{\alpha \in J} \mu_\alpha$, there exists a finite set $J_0 \subset J$ such that $\rho \leq \bigvee_{\alpha \in J_0} SC_{ij}(\mu_\alpha, r)$.

(2) An r-(τ_i, τ_j)-$FSC(\mathcal{I})$-compact, if for every r-(τ_i, τ_j)-fsc set ρ of X and every family $\{\mu_\alpha \in I^X | \ \mu_\alpha$ *is* r-(τ_i, τ_j)-fso *set*, $\alpha \in J\}$ with $\rho \leq \bigvee_{\alpha \in J} \mu_\alpha$, there exists a finite set $J_0 \subset J$ such that $\mathcal{I}(\rho \wedge [\bar{1} - \bigvee_{\alpha \in J_0} SC_{ij}(\mu_\alpha, r)]) \geq r$.

(3) An r-(τ_i, τ_j)-FS-closed, if for every family $\{\mu_\alpha \in I^X | \ \mu_\alpha$ *is* r-(τ_i, τ_j)-fso *set*, $\alpha \in J\}$ such that $\bigvee_{\alpha \in J} \mu_\alpha = \bar{1}$, there exists a finite set $J_0 \subset J$ such that $\bigvee_{\alpha \in J_0} SC_{ij}(\mu_\alpha, r) = \bar{1}$.

(4) An r-(τ_i, τ_j)-$F\mathcal{I}$-S-closed, if for every family $\{\mu_\alpha \in I^X | \ \mu_\alpha$ *is* r-(τ_i, τ_j)-fso *set*, $\alpha \in J\}$ such that $\bigvee_{\alpha \in J} \mu_\alpha = \bar{1}$, there exists a finite set $J_0 \subset J$ such that $\mathcal{I}(\bar{1} - \bigvee_{\alpha \in J_0} SC_{ij}(\mu_\alpha, r)) \geq r$.

Definition 4.2. Let $(X, \tau_1, \tau_2, \mathcal{I})$ be a smooth ideal bts and $r \in I_0$. Then X is called:

(1) $FPSC$-compact (resp. $FPSC(\mathcal{I})$-compact), if X is an r-(τ_i, τ_j)-FSC-compact (resp. r-(τ_i, τ_j)-$FSC(\mathcal{I})$-compact) for each $r \in I_0$.

(2) FP-S-closed (resp. $FP\mathcal{I}$-S-closed), if X is an r-(τ_i, τ_j)-FS-closed (resp. r-(τ_i, τ_j)-$F\mathcal{I}$-S-closed) for each $r \in I_0$.

From Definition 4.1 we have the following remark.

Remark 4.3. Let $(X, \tau_1, \tau_2, \mathcal{I})$ be a smooth ideal bts and $r \in I_0$. Then the following statements are true:

(1) If X is an r-(τ_i, τ_j)-$FSC(\mathcal{I})$-compact, then X is an r-τ_i-$FC(\mathcal{I})$-compact.

(2) If X is an r-(τ_i, τ_j)-$F\mathcal{I}$-S-closed, then X is an r-τ_i-$F\mathcal{I}QHC$.

(3) If X is an r-(τ_i, τ_j)-FS-closed, then X is an r-(τ_i, τ_j)-$F\mathcal{I}$-S-closed.

(4) If X is an r-(τ_i, τ_j)-FSC-compact (resp. r-(τ_i, τ_j)-$FSC(\mathcal{I})$-compact), then X is an r-(τ_i, τ_j)-FS-closed (resp. r-(τ_i, τ_j)-$F\mathcal{I}$-S-closed).

(5) If $\mathcal{I} = \mathcal{I}^0$, then r-(τ_i, τ_j)-FS-closed and r-(τ_i, τ_j)-$F\mathcal{I}$-S-closed are equivalent.

It follows from the Definition 4.1, Remark 4.3 and the fact that every r-τ_i-closed fuzzy set in X is an r-(τ_i, τ_j)-fsc set that.

$$\begin{array}{ccccc} r\text{-}(\tau_i,\tau_j)\text{-}FSC\text{-}compact & & r\text{-}(\tau_i,\tau_j)\text{-}FSC(\mathcal{I})\text{-}compact & \Longrightarrow & r\text{-}\tau_i\text{-}FC(\mathcal{I})\text{-}compact \\ \Downarrow & & \Downarrow & & \Downarrow \\ r\text{-}(\tau_i,\tau_j)\text{-}FS\text{-}closed & \Longrightarrow & r\text{-}(\tau_i,\tau_j)\text{-}F\mathcal{I}\text{-}S\text{-}closed & \Longrightarrow & r\text{-}\tau_i\text{-}F\mathcal{I}QHC \end{array}$$

Remark 4.4. The notion of r-(τ_i, τ_j)-$FSC(\mathcal{I})$-compactness of X is not a generalization of r-(τ_i, τ_j)-FSC-compactness. We show this in the next example.

Example 4.5. Let $X = \{a, b, c\}$. Define fuzzy sets λ_1, λ_2 and $\lambda_3 \in I^X$ as follows:

$$\lambda_1 = a_{0.5} \vee b_{0.1} \vee c_{0.5}, \quad \lambda_2 = a_{0.5} \vee b_{0.9} \vee c_{0.5}, \quad \lambda_3 = a_{0.6} \vee b_{0.2}.$$

Define smooth topologies $\tau_1 : I^X \longrightarrow I$ and $\tau_2 : I^X \longrightarrow I$ as follows:

$$\tau_1(\lambda) = \begin{cases} 1 & if\ \lambda = \bar{0}, \bar{1}, \\ \frac{1}{2} & if\ \lambda = \lambda_1,\ \lambda_2, \\ 0 & otherwise; \end{cases} \quad and \quad \tau_2(\lambda) = \begin{cases} 1 & if\ \lambda = \bar{0}, \bar{1}, \\ \frac{1}{2} & if\ \lambda = \lambda_1,\ \lambda_2,\ \lambda_3, \\ \frac{3}{4} & if\ \lambda = \lambda_1 \vee \lambda_3,\ \lambda_1 \wedge \lambda_3, \\ \frac{3}{4} & if\ \lambda = \lambda_2 \vee \lambda_3,\ \lambda_2 \wedge \lambda_3, \\ 0 & otherwise. \end{cases}$$

Define smooth ideal $\mathcal{I} : I^X \longrightarrow I$ by

$$\mathcal{I}(\lambda) = \begin{cases} 1 & if\ \lambda = \bar{0}, \\ 0 & if\ \lambda = \bar{1}, \\ \frac{1}{4} & otherwise. \end{cases}$$

Then $(X, \tau_1, \tau_2, \mathcal{I})$ is a smooth ideal bts. For $r = \frac{1}{2}$, $\{\bar{0}, \bar{1}, \lambda_1, \lambda_2\}$ is the family of all $\frac{1}{2}$-(τ_1, τ_2)-fso sets in X, and for any $\frac{1}{2}$-(τ_1, τ_2)-fsc set in X it easy to verify that (X, τ_1, τ_2) is a $\frac{1}{2}$-(τ_1, τ_2)-FSC-compact space. But $(X, \tau_1, \tau_2, \mathcal{I})$ is not a $\frac{1}{2}$-(τ_1, τ_2)-$FSC(\mathcal{I})$-compact space, as $\rho = a_{0.5} \vee b_{0.1} \vee c_{0.5}$ is a $\frac{1}{2}$-(τ_1, τ_2)-fsc set in X. However, for any J is $\frac{1}{2}$-(τ_1, τ_2)-fso set which cover ρ and for any finite subset J_0 of J we have,

$$\mathcal{I}(\rho \wedge [\bar{1} - \bigvee_{\alpha \in J_0} SC_{12}(\lambda_\alpha, \frac{1}{2})]) = \mathcal{I}(\rho) = \frac{1}{4} < \frac{1}{2}.$$

Theorem 4.6. Let $(X, \tau_1, \tau_2, \mathcal{I})$ be a smooth ideal bts and $r \in I_0$. Then the following statements are equivalent:

(1) $(X, \tau_1, \tau_2, \mathcal{I})$ is an r-(τ_i, τ_j)-$FSC(\mathcal{I})$-compact.

(2) For any collection $\{\mu_\alpha \in I^X |\ \mu_\alpha\ is\ r\text{-}(\tau_i, \tau_j)\text{-}fsc\ set, \alpha \in J\}$ and every r-(τ_i, τ_j)-fsc set ρ in X with $\rho\ \bar{q} \bigwedge_{\alpha \in J} \mu_\alpha$, there exists a finite set $J_0 \subset J$ such that $\mathcal{I}(\rho \wedge \bigwedge_{\alpha \in J_0} SI_{ij}(\mu_\alpha, r)) \geq r$.

(3) $\rho\ q \bigwedge_{\alpha \in J} \mu_\alpha$ holds for every collection $\{\mu_\alpha \in I^X |\ \mu_\alpha\ is\ r\text{-}(\tau_i, \tau_j)\text{-}fsc\ set, \alpha \in J\}$ and every r-(τ_i, τ_j)-fsc set ρ in X with $\{\rho \wedge SI_{ij}(\mu_\alpha, r), \alpha \in J\}$ has the $\mathcal{I}$-FIP.

Proof. (1) $\Longrightarrow$ (2) Let ρ be an r-(τ_i,τ_j)-fsc set in X and $\{\mu_\alpha \in I^X|\ \mu_\alpha\ is\ r\text{-}(\tau_i,\tau_j)\text{-}fsc\ set, \alpha\in J\}$ be any family with $\rho\ \bar{q} \bigwedge_{\alpha\in J}\mu_\alpha$. Then $\rho \leq \bigvee_{\alpha\in J}\bar{1}-\mu_\alpha$. For each $\alpha\in J$, $\bar{1}-\mu_\alpha$ is an r-(τ_i,τ_j)-fso set. Since X is an r-(τ_i,τ_j)-$FSC(\mathcal{I})$-compact, there exists a finite set $J_0\subset J$ such that $\mathcal{I}(\rho\wedge[\bar{1}-\bigvee_{\alpha\in J_0}SC_{ij}(\bar{1}-\mu_\alpha,r)])\geq r$. Since

$$\rho\wedge[\bar{1}-\bigvee_{\alpha\in J_0}SC_{ij}(\bar{1}-\mu_\alpha,r)]=\rho\wedge\bigwedge_{\alpha\in J_0}SI_{ij}(\mu_\alpha,r).\ \text{Thus},\ \mathcal{I}(\rho\wedge\bigwedge_{\alpha\in J_0}SI_{ij}(\mu_\alpha,r))\geq r.$$

(2) $\Longrightarrow$ (3) This is trivial.

(3) $\Longrightarrow$ (1) Let ρ be an r-(τ_i,τ_j)-fsc set in X and $\{\mu_\alpha \in I^X|\ \mu_\alpha\ is\ r\text{-}(\tau_i,\tau_j)\text{-}fso\ set, \alpha\in J\}$ where $\rho\leq\bigvee_{\alpha\in J}\mu_\alpha$. Suppose there is no finite subfamily $J_0\subset J$, then $\mathcal{I}(\rho\wedge[\bar{1}-\bigvee_{\alpha\in J_0}SC_{ij}(\mu_\alpha,r)])\geq r$. Since

$$\rho\wedge[\bar{1}-\bigvee_{\alpha\in J_0}SC_{ij}(\mu_\alpha,r)]=\rho\wedge\bigwedge_{\alpha\in J_0}(\bar{1}-SC_{ij}(\mu_\alpha,r))=\bigwedge_{\alpha\in J_0}\{\rho\wedge SI_{ij}(\bar{1}-\mu_\alpha,r)\}$$

the family $\{\rho\wedge SI_{ij}(\bar{1}-\mu_\alpha,r),\alpha\in J\}$ has $\mathcal{I}$-FIP. By (3), $\rho\ q \bigwedge_{\alpha\in J}(\bar{1}-\mu_\alpha)$ implies that $\bigvee_{\alpha\in J}\mu_\alpha\leq\rho$. This is a contradiction. □

Theorem 4.7. If $(X,\tau_1,\tau_2,\mathcal{I}_f)$ is an r-(τ_i,τ_j)-$FS\mathcal{I}_f$-compact, then (X,τ_1,τ_2) is an r-(τ_i,τ_j)-FS-closed.

Proof. Let $\{\mu_\alpha \in I^X|\ \mu_\alpha\ is\ r\text{-}(\tau_i,\tau_j)\text{-}fso\ set, \alpha\in J\}$ be any family such that $\bigvee_{\alpha\in J}\mu_\alpha=\bar{1}$. Since X is an r-(τ_i,τ_j)-$FS\mathcal{I}_f$-compact, there exists a finite set $J_0\subset J$ such that $\mathcal{I}_f(\bar{1}-\bigvee_{\alpha\in J_0}\mu_\alpha)\geq r$. This means $\bar{1}-\bigvee_{\alpha\in J_0}\mu_\alpha$ has a finite support, implying there exists a finite set $J_k\subset J$ such that $\bar{1}-\bigvee_{\alpha\in J_0}\mu_\alpha\leq\bigvee_{\alpha\in J_k}\mu_\alpha$. Therefore, $\bar{1}=\bigvee_{\alpha\in J_0\cup J_k}\mu_\alpha$. Since for any $\alpha\in J$, $\mu_\alpha\leq SC_{ij}(\mu_\alpha,r)$. Then, $\bar{1}=\bigvee_{\alpha\in J_0\cup J_k}SC_{ij}(\mu_\alpha,r)$. Hence, X is an r-(τ_i,τ_j)-FS-closed. □

Definition 4.8. A smooth topological space (X,τ) is called an r-$FQHC$ if for every family $\{\mu_\alpha\in I^X|\ \tau(\mu_\alpha)\geq r,\alpha\in J\}$ with $\bigvee_{\alpha\in J}\mu_\alpha=\bar{1}$, there exists a finite set $J_0\subset J$ such that $\bigvee_{\alpha\in J_0}C_\tau(\mu_\alpha,r)=\bar{1}$.

Theorem 4.9. If $(X,\tau_1,\tau_2,\mathcal{I}_f)$ is an r-(τ_i,τ_j)-$FS\mathcal{I}_f$-compact, then (X,τ_1,τ_2) is an r-τ_i-$FQHC$.

Proof. The proof is similar to the proof of Theorem 4.7. □

Theorem 4.10. Let $(X,\tau_1,\tau_2,\mathcal{I})$ be a smooth ideal bts and $r\in I_0$. Then the following statements are equivalent:

(1) $(X,\tau_1,\tau_2,\mathcal{I})$ is an r-(τ_i,τ_j)-$F\mathcal{I}$-S-closed.

(2) For any collection $\{\mu_\alpha \in I^X|\ \mu_\alpha\ is\ r\text{-}(\tau_i,\tau_j)\text{-}fsc\ set, \alpha\in J_0\}$ with $\bigwedge_{\alpha\in J}\mu_\alpha=\bar{0}$, there exists a finite set $J_0\subset J$ such that $\mathcal{I}(\bigwedge_{\alpha\in J_0}SI_{ij}(\mu_\alpha,r))\geq r$.

(3) $\bigwedge_{\alpha\in J}\mu_\alpha\neq\bar{0}$, holds for any collection $\{\mu_\alpha \in I^X|\ \mu_\alpha\ is\ r\text{-}(\tau_i,\tau_j)\text{-}fsc\ set, \alpha\in J_0\}$ such that $\{SI_{ij}(\mu_\alpha,r),\alpha\in J\}$ has $\mathcal{I}$-FIP.

Proof. (1) $\Longleftrightarrow$ (2) Similar to the proof of Theorem 3.14.

(1) $\Longrightarrow$ (3) Let $\{\mu_\alpha \in I^X|\ \mu_\alpha\ is\ r\text{-}(\tau_i,\tau_j)\text{-}fsc\ set, \alpha\in J_0\}$ be any family such that $\{SI_{ij}(\mu_\alpha,r),\alpha\in J\}$ has the $\mathcal{I}$-FIP. If $\bigwedge_{\alpha\in J}\mu_\alpha=\bar{0}$, then $\bigvee_{\alpha\in J}\bar{1}-\mu_\alpha=\bar{1}$. Since $(X,\tau_1,\tau_2,\mathcal{I})$ is an r-(τ_i,τ_j)-$F\mathcal{I}$-S-closed, there exists a finite set $J_0\subset J$ such that

$$\mathcal{I}(\bar{1}-\bigvee_{\alpha\in J_0}SC_{ij}(\bar{1}-\mu_\alpha,r))\geq r.$$

Since

$$\bar{1} - \bigvee_{\alpha \in J_0} SC_{ij}(\bar{1} - \mu_\alpha, r) = \bigwedge_{\alpha \in J_0} SI_{ij}(\mu_\alpha, r). \text{ Then, } \mathcal{I}(\bigwedge_{\alpha \in J_0} SI_{ij}(\mu_\alpha, r)) \geq r.$$

This is a contradiction.

(3) $\Longrightarrow$ (1) Let $\{\mu_\alpha \in I^X | \ \mu_\alpha \ is \ r\text{-}(\tau_i, \tau_j)\text{-}fso \ set, \alpha \in J_0\}$ be any family such that $\bigvee_{\alpha \in J} \mu_\alpha = \bar{1}$. Suppose there is no finite set $J_0 \subset J$ satisfying $\mathcal{I}(\bar{1} - \bigvee_{\alpha \in J_0} SC_{ij}(\mu_\alpha, r)) \geq r$. Since $\bar{1} - \bigvee_{\alpha \in J_0} SC_{ij}(\mu_\alpha, r) = \bigwedge_{\alpha \in J_0} SI_{ij}(\bar{1} - \mu_\alpha, r)$, then the family $\{SI_{ij}(\bar{1} - \mu_\alpha, r), \alpha \in J\}$ has $\mathcal{I}$-FIP. By (3), we have $\bigwedge_{\alpha \in J} \bar{1} - \mu_\alpha \neq \bar{0}$. Then $\bigvee_{\alpha \in J} \mu_\alpha \neq \bar{1}$. This is a contradiction.

□

Definition 4.11. A smooth bts (X, τ_1, τ_2) is called an r-(τ_i, τ_j)-fuzzy semiregular space iff for each r-(τ_i, τ_j)-fso set λ in X and $r \in I_0$, $\lambda = \bigvee\{\nu \in I^X | \ \nu \ is \ r\text{-}(\tau_i, \tau_j)\text{-}fso \ set, SC_{ij}(\nu, r) = \lambda\}$.

Theorem 4.12. If $(X, \tau_1, \tau_2, \mathcal{I})$ is an r-(τ_i, τ_j)-$F\mathcal{I}$-S-closed and r-(τ_i, τ_j)-fuzzy semiregular space, then $(X, \tau_1, \tau_2, \mathcal{I})$ is an r-(τ_i, τ_j)-$FS\mathcal{I}$-compact.

Proof. Let $\{\mu_\alpha \in I^X | \ \mu_\alpha \ is \ r\text{-}(\tau_i, \tau_j)\text{-}fso \ set, \alpha \in J\}$ be any family with $\bigvee_{\alpha \in J} \mu_\alpha = \bar{1}$. By r-(τ_i, τ_j)-fuzzy semiregular of X, for each $\alpha \in J$, $\mu_\alpha = \bigvee_{\alpha_k \in K_\alpha} \{\lambda_{\alpha_k} | \ \lambda_{\alpha_k} \ is \ r\text{-}(\tau_i, \tau_j)\text{-}fso, SC_{ij}(\lambda_{\alpha_k}, r) = \mu_\alpha\}$. Hence, $\bigvee_{\alpha \in J} \mu_\alpha = \bigvee_{\alpha \in J} (\bigvee_{\alpha_k \in K_\alpha} \lambda_{\alpha_k}) = \bar{1}$. Since X is an r-(τ_i, τ_j)-$F\mathcal{I}$-S-closed, there exist finite sets $J_0 \subset J$ and $K_{J_0} \subset K_\alpha$, such that

$$\mathcal{I}(\bar{1} - \bigvee_{\alpha \in J_0} (\bigvee_{\alpha_k \in K_{J_0}} SC_{ij}(\lambda_{\alpha_k}, r))) \geq r \quad \text{for each} \quad \alpha \in J_0.$$

Since,

$$\bigvee_{\alpha_k \in K_{J_0}} SC_{ij}(\lambda_{\alpha_k}, r) \leq \mu_\alpha.$$

This implies that,

$$\bigvee_{\alpha \in J_0} (\bigvee_{\alpha_k \in K_{J_0}} SC_{ij}(\lambda_{\alpha_k}, r)) \leq \bigvee_{\alpha \in J_0} \mu_\alpha \quad \text{which also implies } \bar{1} - \bigvee_{\alpha \in J_0} \mu_\alpha \leq \bar{1} - \bigvee_{\alpha \in J_0} (\bigvee_{\alpha_k \in K_{J_0}} SC_{ij}(\lambda_{\alpha_k}, r)).$$

Therefore,

$$\mathcal{I}(\bar{1} - \bigvee_{\alpha \in J_0} \mu_\alpha) \geq \mathcal{I}(\bar{1} - \bigvee_{\alpha \in J_0} (\bigvee_{\alpha_k \in K_{J_0}} SC_{ij}(\lambda_{\alpha_k}, r))).$$

Thus, $(X, \tau_1, \tau_2, \mathcal{I})$ is an r-(τ_i, τ_j)-$FS\mathcal{I}$-compact. □

5 FPS-compactness Modulo a Smooth Ideal and Mappings

In this section we show the types of FPS-compactness via a smooth ideal that is introduced in Section 3 and 4 which are preserved under some types of mappings. Throughout this section let $(X, \tau_1, \tau_2, \mathcal{I})$ and $(Y, \tau_1^*, \tau_2^*, \mathcal{J})$ be two smooth ideal bts's.

Theorem 5.1. Let $f : (X, \tau_1, \tau_2, \mathcal{I}) \longrightarrow (Y, \tau_1^*, \tau_2^*, \mathcal{J})$ be a surjective, FP-irresolute mapping. If $(X, \tau_1, \tau_2, \mathcal{I})$ is $FPS\mathcal{I}$-compact and $\mathcal{I}(\rho) \leq \mathcal{J}(f(\rho))$ for each $\rho \in I^X$, then $(Y, \tau_1^*, \tau_2^*, \mathcal{J})$ is $FPS\mathcal{J}$-compact.

Proof. Let $\{\mu_\alpha \in I^Y |\ \mu_\alpha\ is\ r\text{-}(\tau_i^*, \tau_j^*)\text{-}fso\ set, \alpha \in J\}$ be any family such that $\bigvee_{\alpha\in J} \mu_\alpha = \bar{1}$. Then $\bigvee_{\alpha\in J} f^{-1}(\mu_\alpha) = \bar{1}$. Since f is FP-irresolute, then for each $\alpha \in J$, $f^{-1}(\mu_\alpha)$ is an r-(τ_i, τ_j)-fso set. By $FPS\mathcal{I}$-compactness of X, there exists a finite set $J_0 \subset J$ with $\mathcal{I}(\bar{1} - \bigvee_{\alpha\in J_0} f^{-1}(\mu_\alpha)) \geq r$. Since $\mathcal{I}(\rho) \leq \mathcal{J}(f(\rho))$, then for each $\alpha \in J_0$ we have $\mathcal{J}(f(\bar{1} - \bigvee_{\alpha\in J_0} f^{-1}(\mu_\alpha))) \geq r$. From the surjective of f, we have $f(\bar{1} - \bigvee_{\alpha\in J_0} f^{-1}(\mu_\alpha)) = \bar{1} - \bigvee_{\alpha\in J_0} \mu_\alpha$. Thus $\mathcal{J}(\bar{1} - \bigvee_{\alpha\in J_0} \mu_\alpha) \geq r$. Hence, $(Y, \tau_1^*, \tau_2^*, \mathcal{J})$ is an $FPS\mathcal{J}$-compact. □

Theorem 5.2. Let $f : (X, \tau_1, \tau_2, \mathcal{I}) \longrightarrow (Y, \tau_1^*, \tau_2^*, \mathcal{J})$ be a surjective, FP-irresolute mapping. If $(X, \tau_1, \tau_2, \mathcal{I})$ is $FPSC(\mathcal{I})$-compact and $\mathcal{I}(\rho) \leq \mathcal{J}(f(\rho))$ for each $\rho \in I^X$, then $(Y, \tau_1^*, \tau_2^*, \mathcal{J})$ is $FPSC(\mathcal{J})$-compact.

Proof. Let ρ be an r-(τ_i^*, τ_j^*)-fsc set in Y, and let $\{\mu_\alpha \in I^Y |\ \mu_\alpha\ is\ r\text{-}(\tau_i^*, \tau_j^*)\text{-}fso\ set, \alpha \in J\}$ with $\rho \leq \bigvee_{\alpha\in J} \mu_\alpha$. Then, $f^{-1}(\rho) \leq \bigvee_{\alpha\in J} f^{-1}(\mu_\alpha)$. Since f is FP-irresolute, for each $\alpha \in J$, $f^{-1}(\mu_\alpha)$ is an r-(τ_i, τ_j)-fso set of X and $f^{-1}(\rho)$ is an r-(τ_i, τ_j)-fso set in X. By $FPSC(\mathcal{I})$-compactness in X, there exists a finite set $J_0 \subset J$, such that $\mathcal{I}(f^{-1}(\rho) \wedge [\bar{1} - \bigvee_{\alpha\in J_0} SC_{ij}(f^{-1}(\mu_\alpha), r)]) \geq r$. Since f is FP-irresolute mapping, then from Theorem 2.8(3), we have $SC_{ij}(f^{-1}(\lambda), r) \leq f^{-1}(SC_{ij}(\lambda, r))$ for every $\lambda \in I^Y$. Hence

$$f^{-1}(\rho) \wedge [\bar{1} - \bigvee_{\alpha\in J_0} SC_{ij}(f^{-1}(\mu_\alpha), r)] \geq f^{-1}(\rho) \wedge [\bar{1} - \bigvee_{\alpha\in J_0} f^{-1}(SC_{ij}(\mu_\alpha, r))].$$

Thus, $\mathcal{I}(f^{-1}(\rho) \wedge [\bar{1} - \bigvee_{\alpha\in J_0} f^{-1}(SC_{ij}(\mu_\alpha, r))]) \geq r$. Since $\mathcal{I}(\rho) \leq \mathcal{J}(f(\rho))$. Then, for each $\alpha \in J_0$, $\mathcal{J}(f(f^{-1}(\rho) \wedge [\bar{1} - \bigvee_{\alpha\in J_0} f^{-1}(SC_{ij}(\mu_\alpha, r))])) \geq r$. From the surjective of f,

$$f(f^{-1}(\rho) \wedge [\bar{1} - \bigvee_{\alpha\in J_0} f^{-1}(SC_{ij}(\mu_\alpha, r))]) = f(f^{-1}(\rho \wedge (\bar{1} - \bigvee_{\alpha\in J_0} SC_{ij}(\mu_\alpha, r)))) = \rho \wedge (\bar{1} - \bigvee_{\alpha\in J_0} SC_{ij}(\mu_\alpha, r)).$$

Thus, $\mathcal{J}(\rho \wedge (\bar{1} - \bigvee_{\alpha\in J_0} SC_{ij}(\mu_\alpha, r))) \geq r$. Hence Y is an $FPSC(\mathcal{J})$-compact. □

Theorem 5.3. Let $f : (X, \tau_1, \tau_2, \mathcal{I}) \longrightarrow (Y, \tau_1^*, \tau_2^*, \mathcal{J})$ be a surjective, FP-irresolute mapping. If $(X, \tau_1, \tau_2, \mathcal{I})$ is $FP\mathcal{I}$-S-closed and $\mathcal{I}(\rho) \leq \mathcal{J}(f(\rho))$ for each $\rho \in I^X$, then $(Y, \tau_1^*, \tau_2^*, \mathcal{J})$ is $FP\mathcal{J}$-S-closed.

Proof. Similar to proof of Theorem 5.2. □

In order to complete our study of the properties of FPS-compactness via a smooth ideal under mappings, we need now to introduce the notion of FP-weakly semi-continuous mapping.

Definition 5.4. Let $f : (X, \tau_1, \tau_2) \longrightarrow (Y, \tau_1^*, \tau_2^*)$ be a mapping. Then f is called a FP-weakly semi-continuous iff $f^{-1}(\mu) \leq SI_{ij}(f^{-1}(SC_{ij}(\mu, r)), r)$, $\mu \in I^Y$.

Theorem 5.5. Let $f : (X, \tau_1, \tau_2, \mathcal{I}) \longrightarrow (Y, \tau_1^*, \tau_2^*, \mathcal{J})$ be a surjective, FP-weakly semi-continuous mapping. If $(X, \tau_1, \tau_2, \mathcal{I})$ is $FPS\mathcal{I}$-compact and $\mathcal{I}(\rho) \leq \mathcal{J}(f(\rho))$ for each $\rho \in I^X$, then $(Y, \tau_1^*, \tau_2^*, \mathcal{J})$ is $FP\mathcal{J}$-S-closed.

Proof. Let $\{\mu_\alpha \in I^Y |\ \mu_\alpha\ is\ r\text{-}(\tau_i^*, \tau_j^*)\text{-}fso\ set, \alpha \in J\}$ be any family such that $\bigvee_{\alpha\in J} \mu_\alpha = \bar{1}$. Then $\bigvee_{\alpha\in J} f^{-1}(\mu_\alpha) = \bar{1}$. Since f is a FP-weakly semi-continuous, then for each $\alpha \in J$, $f^{-1}(\mu_\alpha) \leq SI_{ij}(f^{-1}(SC_{ij}(\mu_\alpha, r)), r)$. Hence, $\bigvee_{\alpha\in J} SI_{ij}(f^{-1}(SC_{ij}(\mu_\alpha, r)), r) = \bar{1}$. Since $\{SI_{ij}(f^{-1}(SC_{ij}(\mu_\alpha, r)), r) \in I^X, \alpha \in J\}$ is a family of r-(τ_i, τ_j)-fso sets and by $FPS\mathcal{I}$-compactness of X, there exists a finite set $J_0 \subset J$ such that $\mathcal{I}(\bar{1} - \bigvee_{\alpha\in J_0} SI_{ij}(f^{-1}(SC_{ij}(\mu_\alpha, r)), r)) \geq r$. Since $\mathcal{I}(\rho) \leq \mathcal{J}(f(\rho))$, $J(f(\bar{1} - \bigvee_{\alpha\in J_0} SI_{ij}(f^{-1}(SC_{ij}(\mu_\alpha, r)), r))) \geq r$ for each $\alpha \in J_0$. From the surjective of f,

$$f(\bar{1} - \bigvee_{\alpha\in J_0} SI_{ij}(f^{-1}(SC_{ij}(\mu_\alpha, r)), r)) \geq f(\bar{1} - \bigvee_{\alpha\in J_0} f^{-1}(SC_{ij}(\mu_\alpha, r))) = \bar{1} - \bigvee_{\alpha\in J_0} SC_{ij}(\mu_\alpha, r).$$

Thus, $\mathcal{J}(\bar{1} - \bigvee_{\alpha\in J_0} SC_{ij}(\mu_\alpha, r)) \geq r$. Hence, Y is $FP\mathcal{J}$-S-closed. □

Theorem 5.6. The image of $FPS\mathcal{I}$-compact set under surjective, FP-weakly semi-continuous mapping such that $\mathcal{I}(\rho) \leq \mathcal{J}(f(\rho))$ for each $\rho \in I^X$, is $FP\mathcal{J}$-S-closed.

Proof. Similar to proof of Theorem 5.5. □

The following theorem shows that the image of a smooth ideal is a smooth ideal.

Theorem 5.7. Let $f : (X, \tau) \longrightarrow (Y, \delta)$ be a mapping from a smooth topological space (X, τ) into a smooth topological space (Y, δ). If $\mathcal{I}$ is a smooth ideal on X, then $f(\mathcal{I})$ is a smooth ideal on Y defined as follows:

$$f(\mathcal{I})(\mu) = \begin{cases} \bigvee_{f(\lambda)=\mu} \mathcal{I}(\lambda) & if\ f^{-1}(\mu) \neq \bar{0} \\ 0 & if\ f^{-1}(\mu) = \bar{0} \end{cases}$$

Proof. Direct. □

Lemma 5.8. Let $f : (X, \tau) \longrightarrow (Y, \delta)$ be a mapping from a smooth topological space (X, τ) into a smooth topological space (Y, δ) and $\mathcal{I}$ be a smooth ideal on X. If $\mathcal{I}(\nu) \geq r$, then $f(\mathcal{I})(f(\nu)) \geq r$, $\nu \in I^X$.

Proof. Obvious. □

Theorem 5.9. Let $f : (X, \tau_1, \tau_2, \mathcal{I}) \longrightarrow (Y, \tau_1^*, \tau_2^*, f(\mathcal{I}))$ be a surjective, FP-irresolute mapping. If $(X, \tau_1, \tau_2, \mathcal{I})$ is $FPS\mathcal{I}$-compact, then $(Y, \tau_1^*, \tau_2^*, f(\mathcal{I}))$ is $FPSf(\mathcal{I})$-compact.

Proof. Let $\{\mu_\alpha \in I^Y \mid \mu_\alpha\ is\ r\text{-}(\tau_i^*, \tau_j^*)\text{-}fso\ set, \alpha \in J\}$ be any family such that $\bigvee_{\alpha\in J} \mu_\alpha = \bar{1}$. Since f is FP-irresolute, then $\{f^{-1}(\mu_\alpha) \in I^X \mid f^{-1}(\mu_\alpha)\ is\ r\text{-}(\tau_i, \tau_j)\text{-}fso\ set, \alpha \in J\}$ with $\bigvee_{\alpha\in J} f^{-1}(\mu_\alpha) = \bar{1}$. By hypothesis, there exists a finite set $J_0 \subset J$, such that $\mathcal{I}(\bar{1} - \bigvee_{\alpha\in J_0} f^{-1}(\mu_\alpha)) \geq r$. By Lemma 5.8 we have, $f(\mathcal{I})(f(\bar{1} - \bigvee_{\alpha\in J_0} f^{-1}(\mu_\alpha))) \geq r$. From the surjective of f, $f(\bar{1} - \bigvee_{\alpha\in J_0} f^{-1}(\mu_\alpha)) = \bar{1} - \bigvee_{\alpha\in J_0} \mu_\alpha$. Then, $f(\mathcal{I})(\bar{1} - \bigvee_{\alpha\in J_0} \mu_\alpha) \geq r$. Hence, $(Y, \tau_1^*, \tau_2^*, f(\mathcal{I}))$ is an $FPSf(\mathcal{I})$-compact. □

The following theorem shows that the inverse image of a smooth ideal is a smooth ideal.

Theorem 5.10. Let $f : (X, \tau) \longrightarrow (Y, \delta)$ be a mapping from a smooth topological space (X, τ) into a smooth topological space (Y, δ). If $\mathcal{J}$ is a smooth ideal on Y, then $f^{-1}(\mathcal{J})$ is a smooth ideal on X defined as follows:

$$f^{-1}(\mathcal{J})(\lambda) = \begin{cases} 0 & if\ \lambda = \bar{1} \\ \mathcal{J}(f(\lambda)) & \text{for all } \lambda \in I^X \end{cases}$$

Proof. Direct. □

Lemma 5.11. Let $f : (X, \tau) \longrightarrow (Y, \delta)$ be a surjective mapping from a smooth topological space (X, τ) into a smooth topological space (Y, δ) and let $\mathcal{J}$ be a smooth ideal on Y. If $\mathcal{J}(\nu) \geq r$, then $f^{-1}(\mathcal{J})(f^{-1}(\nu)) \geq r$.

Proof. Obvious. □

Theorem 5.12. Let $f : (X, \tau_1, \tau_2) \longrightarrow (Y, \tau_1^*, \tau_2^*, \mathcal{J})$ be a bijective, FP-irresolute open mapping. If $(Y, \tau_1^*, \tau_2^*, \mathcal{J})$ is $FPS\mathcal{J}$-compact, then $(X, \tau_1, \tau_2, f^{-1}(\mathcal{J}))$ is $FPSf^{-1}(\mathcal{J})$-compact.

Proof. Let $\{\mu_\alpha \in I^X \mid \mu_\alpha\ is\ r\text{-}(\tau_i, \tau_j)\text{-}fso\ set, \alpha \in J\}$ be any family with $\bigvee_{\alpha\in J} \mu_\alpha = \bar{1}$. From the surjective and FP-irresolute open of f we have $\bigvee_{\alpha\in J} f(\mu_\alpha) = \bar{1}$, such that for each $\alpha \in J$, $f(\mu_\alpha)$ is an r-(τ_i^*, τ_j^*)-fso set in Y. By $FPS\mathcal{J}$-compactness of Y, there exists the finite set $J_0 \subset J$ such that $\mathcal{J}(\bar{1} - \bigvee_{\alpha\in J_0} f(\mu_\alpha)) \geq r$. From Lemma 5.11, $f^{-1}(\mathcal{J})(f^{-1}(\bar{1} - \bigvee_{\alpha\in J_0} f(\mu_\alpha))) \geq r$. Since f is an injective, we have:

$$f^{-1}(\mathcal{J})(f^{-1}(\bar{1} - \bigvee_{\alpha\in J_0} f(\mu_\alpha))) = \mathcal{J}(f(f^{-1}(\bar{1} - \bigvee_{\alpha\in J_0} f(\mu_\alpha)))) = \mathcal{J}(\bar{1} - \bigvee_{\alpha\in J_0} f(\mu_\alpha)) = \mathcal{J}(f(\bar{1} - \bigvee_{\alpha\in J_0} \mu_\alpha)).$$

Then $f^{-1}(\mathcal{J})(\bar{1} - \bigvee_{\alpha\in J_0} \mu_\alpha)) \geq r$. Hence, $(X, \tau_1, \tau_2, f^{-1}(\mathcal{J}))$ is $FPSf^{-1}(\mathcal{J})$-compact. □

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SINGLE VALUED NEUTROSOPHIC SOFT EXPERT SETS AND THEIR APPLICATION IN DECISION MAKING

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Abstract - In this paper, we first introduce the concept of single valued neutrosophic soft expert sets (SVNSESs for short) which combines single valued neutrosophic sets and soft expert sets.We also defineits basic operations, namely complement, union, intersection, AND and OR, and study some related properties supporting proofs.This concept is a generalization of fuzzy soft expert sets (FSESs) and intuitionistic fuzzy soft expert sets (IFSESs). Finally, an approach for solving MCDM problems is explored by applying the single valued neutrosophic soft expert sets, and an example is provided to illustrate the application of the proposed method.

Keywords - Single valued neutrosophic sets, Soft expert sets, Single valued neutrosophic soft expert sets, Decision making

1. Introduction.

Neutrosophy has been introduced by Smarandache [12, 13, 14] as a new branch of philosophy and generalization of fuzzy logic, intuitionistic fuzzy logic, paraconsistent logic. Fuzzy sets [38] and intuitionistic fuzzy sets [32] are defined by membership functions while intuitionistic fuzzy sets are characterized by membership and non-membership functions, respectively. In some real life problems for proper description of an object in uncertain and ambiguous environment, we need to handle the indeterminate and incomplete information. But fuzzy sets and intuitionistic fuzzy sets don't handle the indeterminate and inconsistent information. Thus neutrosophic set (NS in short) is defined by Samarandache [13], as a new mathematical tool for dealing with problems involving incomplete, indeterminacy, inconsistent knowledge. In NS, the indeterminacy is quantified

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explicitly and truth-membership, indeterminacy membership, and false-membership are completely independent. From scientific or engineering point of view, the neutrosophic set and set- theoretic view, operators need to be specified. Otherwise, it will be difficult to apply in the real applications. Therefore, H. Wang et al [15] defined a single valued neutrosophic set (SVNS) and then provided the set theoretic operations and various properties of single valued neutrosophic sets. The works on single valued neutrosophic set (SVNS) and their hybrid structure in theories and application have been progressing rapidly [3, 4, 5, 6, 7, 8, 9, 11, 23, 24, 25, 26, 27, 28, 29, 30, 31, 39, 58, 66, 67, 68, 71, 75, 78, 79, 80, 81, 83,85].

In 1999, Molodtsov [10] initiated the theory of soft set theory as a general mathematical tool for dealing with uncertainty and vagueness and the soft set theory is free from the parameterization inadequacy syndrome of fuzzy set theory, rough set theory, probability theory. In fact, a soft set is a set-valued map which gives an approximation description of objects under consideration based on some parameters. Later Maji et al.[54] defined several operations on soft set. Many authors [33, 37, 40, 43, 45, 46, 47, 48, 49, 51, 52, 53, 56, 61] have combined soft sets with other sets to generate hybrid structures like fuzzy soft sets, generalized fuzzy soft sets, rough soft sets, intuitionstic fuzzy soft, intuitionistic fuzzy soft set theory, possibility fuzzy soft set, generalized intuitionistic fuzzy soft, generalized neutrosophic soft set, possibility vague soft set and so on. All these research aim to solvemost of our real life problems in medical sciences, engineering, management, environment and social science which involve data that are not crisp and precise. But most of these models deals with only one opinion (or) with only one expert. This causes a problem with the user when questioners are used for the data collection.Alkhazaleh and Salleh in 2011 [61] defined the concept of soft expert set and created a model in which the user can know the opinion of the experts in the model without any operations and presented an application of this concept in decision making problem. Also, they introduced the concept of the fuzzy soft expert set [60] as a combination between the soft expert set and the fuzzy set. Later on, many researchers have worked with the concept of soft expert sets [1,2, 15, 16, 20, 34, 35, 42, 44, 54, 55, 82, 84]. But most of these concepts cannot dealing with indeterminate and inconsistent information.

Based on [13], Maji [55] introduced the concept of neutrosophic soft set a more generalized concept, which is a combination of neutrosophic set and soft set and studied its properties. New operators on neutrosophic soft set presented by Şahin and Küçük [58]. Based on Çağman [46], Karaaslan [85] redefined neutrosophic soft sets and their operations. Various kinds of extended neutrosophic soft sets such as intuitionistic neutrosophic soft set [63, 65, 74], generalized neutrosophic soft set [57, 64], interval valued neutrosophic soft set [21], neutrosophic parameterized fuzzy soft set [70], Generalized interval valued neutrosophic soft sets [73], neutrosophic soft relation [18, 19], neutrosophic soft multiset theory [22] and cyclic fuzzy neutrosophic soft group [59] were presented. The combination of neutrosophic soft sets and rough set [72, 76, 75] is another interesting topic. Until now, there is no study on soft experts in neutrosophic environment, so there is a need to develop a new mathematical tool called “ single valued neutrosophic soft expert sets” .

The remaining part of this paper is organized as follows. In Section 2, we first recall the necessary background on neutrosophic sets, single valued neutrosophic sets, soft set, neutrosophic soft sets, soft expert sets, fuzzy soft expert sets and intutionistic fuzzy soft expert sets. Section 3 reviews various proposals for the definition of single valued neutrosophic soft expert sets and derive their respective properties. Section 4 presents basic operations on single valued neutrosophic soft expert sets. Section 5 presents an application of this concept in solving a decision making problem. Finally, we conclude the paper.

2. Preliminaries

In this section, we will briefly recall the basic concepts of neutrosophic sets, single valued neutrosophic sets, soft set, neutrosophic soft sets, soft expert sets, fuzzy soft expert sets, and intutionistic fuzzy soft expert sets.

Let U be an initial universe set of objects and E the set of parameters in relation to objects in U. Parameters are often attributes, characteristics or properties of objects. Let P (U) denote the power set of U and A ⊆ E.

2.1. Neutrosophic Set

Definition 2.1 [13]: Let U be an universe of discourse then the neutrosophic set A is an object having the form A = {< x: $\mu_A(x)$, $\nu_A(x)$, $\omega_A(x)$>,x ∈ U},where the functions

$$\mu_A(x),\ \nu_A(x),\omega_A(x) : U\rightarrow]^-0,1^+[$$

define respectively the degree of membership , the degree of indeterminacy, and the degree of non-membership of the element x ∈ U to the set A with the condition.

$$^-0 \le \mu_A(x)+ \nu_A(x)+ \omega_A(x)) \le 3^+$$

From philosophical point of view, the neutrosophic set takes the value from real standard or non-standard subsets of $]^-0,1^+[$. So instead of $]^-0,1^+[$ we need to take the interval [0,1] for technical applications, because $]^-0,1^+[$ will be difficult to apply in the real applications such as in scientific and engineering problems.

For two NS,

$$A_{NS}= \{<x, \mu_A(x), \nu_A(x), \omega_A(x)> \mid x \in U\}$$

and

$$B_{NS}= \{<x, \mu_B(x), \nu_B(x), \omega_B(x)> \mid x \in U\}$$

Then,

1. $A_{NS} \subseteq B_{NS}$ if and only if

$$\mu_A(x) \le \mu_B(x), \nu_A(x) \ge \nu_B(x), \omega_A(x) \ge \omega_B(x)$$

2. $A_{NS} = B_{NS}$ if and only if ,

$$\mu_A(x) = \mu_B(x), \nu_A(x) = \nu_B(x), \omega_A(x) = \omega_B(x) \text{ for any } x \in U.$$

3. The complement of A_{NS} is denoted by A_{NS}^{o} and is defined by

$$A_{NS}^{o} = \{<x, \omega_A(x), 1 - \nu_A(x), \mu_A(x) \mid x \in U\}$$

4. $A \cap B = \{<x, \min\{\mu_A(x), \mu_B(x)\}, \max\{\nu_A(x), \nu_B(x)\}, \max\{\omega_A(x), \omega_B(x)\}>: x \in U\}$

5. $A \cup B = \{<x, \max\{\mu_A(x), \mu_B(x)\}, \min\{\nu_A(x), \nu_B(x)\}, \min\{\omega_A(x), \omega_B(x)\}>: x \in U\}$

As an illustration, let us consider the following example.

Example 2.2. Assume that the universe of discourse $U=\{x_1,x_2,x_3,x_4\}$. It may be further assumed that the values of x_1, x_2, x_3 and x_4 are in [0, 1] Then, A is a neutrosophic set (NS) of U, such that,

A= {< x_1,0.4,0.6,0.5 >, < x_2,0.3,0.4, 0.7>, < x_3,0.4,0.4,0.6] >,< x_4,0.5,0.4,0.8 >}

2.2. Soft Sets

Definition 2.3. [10] Let U be an initial universe set and E be a set of parameters. Let P(U) denote the power set of U. Consider a nonempty set A, A ⊂ E. A pair (K, A) is called a soft set over U, where K is a mapping given by K : A → P(U).

As an illustration, let us consider the following example.

Example 2.4. Suppose that U is the set of houses under consideration, say U = {h_1, h_2, . . ., h_5}. Let E be the set of some attributes of such houses, say E = {e_1, e_2, . . ., e_8}, where e_1, e_2, . . ., e_8 stand for the attributes "beautiful", "costly", "in the green surroundings'", "moderate", respectively.

In this case, to define a soft set means to point out expensive houses, beautiful houses, and so on. For example, the soft set (K, A) that describes the "attractiveness of the houses" in the opinion of a buyer, say Thomas, may be defined like this:

$$A=\{e_1,e_2,e_3,e_4,e_5\};$$

$$K(e_1) = \{h_2, h_3, h_5\}, K(e_2) = \{h_2, h_4\}, K(e_3) = \{h_1\}, K(e_4) = U, K(e_5) = \{h_3, h_5\}.$$

2.3. Neutrosophic Soft Sets

Definition 2.5 [55, 85] Let U be an initial universe set and $A \subset E$ be a set of parameters. Let NS(U) denotes the set of all neutrosophic subsets of U. The collection (F, A) is termed to be the neutrosophic soft set over U, where F is a mapping given by $F: A \rightarrow NS(U)$.

Example 2.6 Let U be the set of houses under consideration and E is the set of parameters. Each parameter is a neutrosophic word or sentence involving neutrosophic words. Consider E ={beautiful, wooden, costly, very costly, moderate, green surroundings, in good repair, in bad repair, cheap, expensive}. In this case, to define a neutrosophic soft set means to point out beautiful houses, wooden houses, in the green surroundings houses and so on. Suppose that, there are five houses in the universe Ugiven by $U = \{h_1, h_2, \dots, h_5\}$ and the set of parameters

$A = \{e_1, e_2, e_3, e_4\}$,where e_1 stands for the parameter `beautiful', $e_2$ stands for the parameter `wooden', e_3 stands for the parameter `costly' and the parameter $e_4$stands for `moderate'. Then the neutrosophic set (F, A) is defined as follows:

$$(F,A) = \left\{ \begin{pmatrix} \left(e_1 \left\{\frac{h_1}{(0.5,0.6,0.3)}, \frac{h_2}{(0.4,0.7,0.6)}, \frac{h_3}{(0.6,0.2,0.3)}, \frac{h_4}{(0.7,0.3,0.2)}, \frac{h_5}{(0.8,0.2,0.3)}\right\}\right) \\ \left(e_2 \left\{\frac{h_1}{(0.6,0.3,0.5)}, \frac{h_2}{(0.7,0.4,0.3)}, \frac{h_3}{(0.8,0.1,0.2)}, \frac{h_4}{(0.7,0.1,0.3)}, \frac{h_5}{(0.8,0.3,0.6)}\right\}\right) \\ \left(e_3 \left\{\frac{h_1}{(0.7,0.4,0.3)}, \frac{h_2}{(0.6,0.7,0.2)}, \frac{h_3}{(0.7,0.2,0.5)}, \frac{h_4}{(0.5,0.2,0.6)}, \frac{h_5}{(0.7,0.3,0.4)}\right\}\right) \\ \left(e_4 \left\{\frac{h_1}{(0.8,0.6,0.4)}, \frac{h_2}{(0.7,0.9,0.6)}, \frac{h_3}{(0.7,0.6,0.4)}, \frac{h_4}{(0.7,0.8,0.6)}, \frac{h_5}{(0.9,0.5,0.7)}\right\}\right) \end{pmatrix} \right\}$$

2.4. Soft Expert Sets

Definition 2.7[61] Let U be a universe set, E be a set of parameters and X be a set of experts (agents). Let O= {1=agree, 0=disagree} be a set of opinions. Let Z= E × X × O and A ⊆ Z . A pair (F, A) is called a soft expert set over U, where F is a mapping given by

F: A→ P(U) and P(U) denote the power set of U.

Definition 2.8 [61] An agree- soft expert set $(F, A)_1$ over U, is a soft expert subset of (F, A) defined as :

$$(F, A)_1 = \{F(\alpha): \alpha \in E \times X \times \{1\}\}.$$

Definition 2.9 [61] A disagree- soft expert set $(F, A)_0$ over U, is a soft expert subset of (F, A) defined as :

$$(F, A)_0 = \{F(\alpha): \alpha \in E \times X \times \{0\}\}.$$

2.5. Fuzzy Soft Expert Sets

Definition 2.10 [42] Let O= {1=agree, 0=disagree} be a set of opinions. Let Z= E × X × O and A ⊆ Z .A pair (F, A) is called a fuzzy soft expert set over U, where F is a mapping given by F : A→ I^U,and I^Udenote the set of all fuzzy subsets of U.

2.6. Intuitiontistic Fuzzy Soft Expert Sets

Definition 2.11 [82] Let U= { $u_1, u_2, u_3, \ldots, u_n$} be a universal set of elements, E={ $e_1, e_2, e_3, \ldots, e_m$} be a universal set of parameters, X={ $x_1, x_2, x_3, \ldots, x_i$ } be a set of experts (agents) and O= {1=agree, 0=disagree} be a set of opinions. Let Z={ E × X ×O } and A ⊆ Z. Then the pair (U, Z) is called a soft universe. Let F: Z → $(I \times I)^U$ where $(I \times I)^U$ denotes the collection of all intuitionistic fuzzy subsets of U. Suppose F: Z → $(I \times I)^U$ a function defined as:

$$F(z) = F(z)(u_i), \text{ for all } u_i \in U.$$

Then $F(z)$ is called an intuitionistic fuzzy soft expert set (IFSES in short) over the soft universe (U, Z)

For each $z_i \in Z$. $F(z)$ = F(z_i)(u_i) where F(z_i) represents the degree of belongingnessand non-belongingness of the elements of U in F(z_i). Hence $F(z_i)$ can be written as:

$$F(z_i) = \{ \left(\frac{u_i}{F(z_i)(u_i)} \right), \ldots, \left(\frac{u_i}{F(z_i)(u_i)} \right) \} \text{ ,for i=1,2,3,…n}$$

where F(z_i)(u_i) = < $\mu_{F(z_i)}(u_i), \omega_{F(z_i)}(u_i)$> with $\mu_{F(z_i)}(u_i)$and $\omega_{F(z_i)}(u_i)$ representing the membership function and non-membership function of each of the elements $u_i \in U$ respectively.

Sometimes we write F as (F, Z) . If A ⊆ Z. we can also have IFSES (F, A).

3. Single Valued Neutrosophic Soft Expert Sets.

In this section, we generalize the fuzzy soft expert sets as introduced by Alhhazaleh and Salleh [60] and intuitionistic fuzzy soft expert sets as introduced by S. Broumi [83] to the single valued neutrosophic soft expert sets and give the basic properties of this concept.

Let U be universal set of elements, E be a set of parameters, X be a set of experts (agents), O= { 1=agree, 0=disagree} be a set of opinions. Let Z= E × X ×O and

Definition 3.1 Let U={ $u_1, u_2, u_3, \ldots, u_n$ } be a universal set of elements, E={ e_1, e_2, $e_3, \ldots, e_m$ } be a universal set of parameters, X={ $x_1, x_2, x_3, \ldots, x_i$ } be a set of experts (agents) and O= {1=agree, 0=disagree} be a set of opinions. Let Z= { E × X × O } and A ⊆ Z. Then the pair (U, Z) is called a soft universe. Let F: Z → SVN^U, where SVN^U denotes the collection of all single valued neutrosophic subsets of U.

Suppose F :Z→ SVN^U be a function defined as:

$$F(z) = \text{F(z)}(u_i) \text{ for all } u_i \in \text{U}.$$

Then $F(z)$ is called a single valued neutrosophic soft expert value (SVNSEV in short) over the soft universe (U, Z)

For each $z_i \in$ Z. $F(z)$ = F(z_i)(u_i), where F(z_i) represents the degree of belongingness, degree of indeterminacy and non-belongingness of the elements of U in F(z_i). Hence $F(z_i)$ can be written as:

$$F(z_i)\{(\frac{u_1}{F(z_1)(u_1)}),\ldots,(\frac{u_n}{F(z_n)(u_n)}),\}\text{ ,for i=1,2,3,…n}$$

where F(z_i)(u_i) =< $\mu_{F(z_i)}(u_i)$, $\nu_{F(z_i)}(u_i)$, $\omega_{F(z_i)}(u_i)$> with $\mu_{F(z_i)}(u_i)$,$\nu_{F(z_i)}(u_i)$ and $\omega_{F(z_i)}(u_i)$ representing the membership function, indeterminacy function and non-membership function of each of the elements $u_i \in$U respectively.

Sometimes we write F as (F, Z) . If A ⊆ Z. we can also have SVNSES (F, A).

Example 3.2 Let U={u_1,u_2, u_3} be a set of elements, E={e_1,e_2} be a set of decision parameters, where e_i(i= 1, 2,3} denotes the parameters E ={e_1= beautiful, e_2= cheap} and X= {x_1, x_2} be a set of experts. Suppose that F: Z→ SVN^U is function defined as follows:

$$F(e_1,x_1,1) = \{(\frac{u_1}{<0.1,0.8,0.3>}),(\frac{u_2}{<0.1,0.6,0.4>}),(\frac{u_3}{<0.4,0.7,0.2>})\},$$

$$F(e_2,x_1,1) = \{(\frac{u_1}{<0.7,0.5,0.25>}),(\frac{u_2}{<0.25,0.6,0.4>}),(\frac{u_3}{<0.4,0.4,0.6>})\},$$

$$F(e_1,x_2,1) = \{(\frac{u_1}{<0.3,0.2,0.7>}),(\frac{u_2}{<0.4,0.3,0.3>}),(\frac{u_3}{<0.1,0.6,0.2>})\},$$

$$F(e_2,x_2,1) = \{(\frac{u_1}{<0.2,0.2,0.6>}),(\frac{u_2}{<0.7,0.3,0.2>}),(\frac{u_3}{<0.3,0.1,0.5>})\},$$

$$F(e_1,x_1,0) = \{(\frac{u_1}{<0.2,0.4,0.5>}),(\frac{u_2}{<0.1,0.9,0.1>}),(\frac{u_3}{<0.1,0.2,0.5>})\},$$

$$F(e_2,x_1,0) = \{(\frac{u_1}{<0.3,0.4,0.6>}),(\frac{u_2}{<0.2,0.7,0.6>}),(\frac{u_3}{<0.1,0.5,0.2>})\},$$

$$F(e_1,x_2,0) = \{(\frac{u_1}{<0.2,0.8,0.4>}),(\frac{u_2}{<0.1,0.6,0.5>}),(\frac{u_3}{<0.7,0.6,0.3>})\}$$

$F(e_2, x_2, 0) = \{(\frac{u_1}{<0.4,0.4,0.7>}), (\frac{u_2}{<0.3,0.8,0.2>}), (\frac{u_3}{<0.6,0.2,0.4>})$

Then we can view the single valued neutrosophic soft expert set (F, Z) as consisting of the following collection of approximations:

$(F, \text{Z})=\{ (e_1, x_1, 1) = \{(\frac{u_1}{<0.1,0.8,0.3>}), (\frac{u_2}{<0.1,0.6,0.4>}), (\frac{u_3}{<0.4,0.7,0.2>})\}\},$

$\{(e_2, x_1, 1) = \{(\frac{u_1}{<0.7,0.5,0.25>}), (\frac{u_2}{<0.25,0.6,0.4>}), (\frac{u_3}{<0.4,0.4,0.6>})\}\},$

$\{(e_1, x_2, 1) = \{(\frac{u_1}{<0.3,0.2,0.7>}), (\frac{u_2}{<0.4,0.3,0.3>}), (\frac{u_3}{<0.1,0.6,0.2>})\}\},$

$\{(e_2, x_2, 1) = \{(\frac{u_1}{<0.2,0.2,0.6>}), (\frac{u_2}{<0.7,0.3,0.2>}), (\frac{u_3}{<0.3,0.1,0.5>})\}\},$

$\{(e_1, x_1, 0) = \{(\frac{u_1}{<0.2,0.4,0.5>}), (\frac{u_2}{<0.1,0.9,0.1>}), (\frac{u_3}{<0.1,0.2,0.5>})\}\},$

$\{(e_2, x_1, 0) = \{(\frac{u_1}{<0.3,0.4,0.6>}), (\frac{u_2}{<0.2,0.7,0.6>}), (\frac{u_3}{<0.1,0.5,0.2>})\}\},$

$\{(e_1, x_2, 0) = \{(\frac{u_1}{<0.2,0.8,0.4>}), (\frac{u_2}{<0.1,0.6,0.5>}), (\frac{u_3}{<0.7,0.6,0.3>})\}\},$

$\{(e_2, x_2, 0) = \{(\frac{u_1}{<0.4,0.4,0.7>}), (\frac{u_2}{<0.3,0.8,0.2>}), (\frac{u_3}{<0.6,0.2,0.4>})\}\}.$

Then (F, Z) is a single valued neutrosophic soft expert set over the soft universe (U, Z).

Definition 3.3. For two single valued neutrosophic soft expert sets (F, A) and (G, B) over a soft universe (U, Z). Then (F, A) is said to be a single valued neutrosophic soft expert subset of (G, B) if

i. B⊆A
ii. $F(\varepsilon)$ is a single valued neutrosophic subset of $G(\varepsilon)$, for all $\varepsilon \in \text{A}$.

This relationship is denoted as $(F, \text{A}) \tilde{\subseteq} (G, \text{B})$. In this case, (G, B) is called a single valued neutrosophic soft expert superset (SVNSE superset) of (F, A) .

Definition 3.4. Two single valued neutrosophic soft expert sets (F, A) and (G, B) over soft universe (U, Z) are said to be equal if (F, A) is a single valued neutrosophic soft expert subset of (G, B) and (G, B) is a single valued neutrosophic soft expert subset of (F, A).

Definition 3.5. A SVNSES (F, A) is said to be a null single valued neutrosophic soft expert set denoted $(\tilde{\emptyset}, \text{A})$ and defined as :

$$(\tilde{\emptyset}, \text{A}) = \text{F}(\alpha) \text{ where } \alpha \in \text{Z}.$$

Where F(α)= <0, 0, 1>, that is $\mu_{F(\alpha)}$=0, $\nu_{F(\alpha)}$= 0 and $\omega_{F(\alpha)}$= 1 for all $\alpha \in$ Z.

Definition 3.6. A SVNSES (F, A) is said to be an absolute single valued neutrosophic soft expert set denoted $(F, \mathrm{A})_{\mathrm{abs}}$ and defined as :

$$(F, \mathrm{A})_{\mathrm{abs}} = \mathrm{F}(\alpha), \text{ where } \alpha \in \mathrm{Z}.$$

Where F(α)= <1, 0, 0>, that is $\mu_{F(\alpha)}$= 1, $\nu_{F(\alpha)}$= 0 and $\omega_{F(\alpha)}$= 0 ,for all $\alpha \in$ Z.

Definition 3.7. Let (F, A) be a SVNSES over a soft universe (U, Z). An agree-single valued neutrosophic soft expert set (agree-SVNSES) over U, denoted as $(F, \mathrm{A})_1$ is a single valued neutrosophic soft expert subset of (F,A) which is defined as :

$$(F, \mathrm{A})_1 = \{\mathrm{F}(\alpha) : \alpha \in \mathrm{E} \times \mathrm{X} \times \{1\}\}.$$

Definition 3.8. Let (F, A) be a SVNSES over a soft universe (U, Z). A disagree-single valued neutrosophic soft expert set (disagree-SVNSES) over U, denoted as $(F, \mathrm{A})_0$ is asingle valued neutrosophic soft expert subset of (F, A) which is defined as:

$$(F, \mathrm{A})_0 = \{\mathrm{F}(\alpha) : \alpha \in \mathrm{E} \times \mathrm{X} \times \{0\}\}.$$

Example 3.9 Consider example 3.2.Then the Agree-single valued neutrosophic soft expert set $(F, \mathrm{A})_1$

$$(F, \mathrm{A})_1 = \{((e_1, x_1, 1), \{(\frac{u_1}{<0.1,0.8,0.3>}), (\frac{u_2}{<0.1,0.6,0.4>}), (\frac{u_3}{<0.4,0.7,0.2>})\}),$$
$$((e_2, x_1, 1), \{(\frac{u_1}{<0.7,0.5,0.25>}), (\frac{u_2}{<0.25,0.6,0.4>}), (\frac{u_3}{<0.4,0.4,0.6>})\}),$$
$$((e_1, x_2, 1), \{(\frac{u_1}{<0.3,0.2,0.7>}), (\frac{u_2}{<0.4,0.3,0.3>}), (\frac{u_3}{<0.1,0.6,0.2>})\}),$$
$$((e_2, x_2, 1), \{(\frac{u_1}{<0.2,0.2,0.6>}), (\frac{u_2}{<0.7,0.3,0.2>}), (\frac{u_3}{<0.3,0.1,0.5>})\})\}$$

And the disagree-single valued neutrosophic soft expert set over U

$$(F, \mathrm{A})_0 = \{((e_2, x_2, 1), \{(\frac{u_1}{<0.2,0.2,0.6>}), (\frac{u_2}{<0.7,0.3,0.2>}), (\frac{u_3}{<0.3,0.1,0.5>})\}),$$
$$((e_1, x_1, 0), \{(\frac{u_1}{<0.2,0.4,0.5>}), (\frac{u_2}{<0.1,0.9,0.1>}), (\frac{u_3}{<0.1,0.2,0.5>})\}),$$
$$((e_2, x_1, 0), \{(\frac{u_1}{<0.3,0.4,0.6>}), (\frac{u_2}{<0.2,0.7,0.6>}), (\frac{u_3}{<0.1,0.5,0.2>})\}),$$
$$((e_1, x_2, 0), \{(\frac{u_1}{<0.2,0.8,0.4>}), (\frac{u_2}{<0.1,0.6,0.5>}), (\frac{u_3}{<0.7,0.6,0.3>})\})$$

$((e_2, x_2, 0), \{(\frac{u_1}{<0.4,0.4,0.7>}), (\frac{u_2}{<0.3,0.8,0.2>}), (\frac{u_3}{<0.6,0.2,0.4>})\})\}$

4. Basic Operations on Single Valued Neutrosophic Soft Expert Sets

In this section, we introduce some basic operations on SVNSES, namely the complement, AND, OR, union and intersection of SVNSES, derive their properties, and give some examples.

Definition 4.1 Let (F, A) be a SVNSES over a soft universe (U, Z). Then the complement of (F, A) denoted by $(F, A)^c$ is defined as:

$$(F, A)^c = \tilde{c}\ (\mathrm{F}(\alpha)) \text{for all } \alpha \in \mathrm{U}.$$

where $\tilde{c}$ is single valued neutrosophic complement .

Example 4.2 Consider the SVNSES (F, Z) over a soft universe (U, Z) as given in Example 3.2. By using the single valued neutrosophic complement for F(α), we obtain $(F, Z)^c$ which is defined as:

$(F, Z)^c = \{ (e_1, x_1, 1) = \{(\frac{u_1}{<0.3,0.8,0.1>}), (\frac{u_2}{<0.4,0.6,0.1>}), (\frac{u_3}{<0.2,0.7,0.4>})\}\},$

$\{(e_2, x_1, 1) = \{(\frac{u_1}{<0.25,0.5,0.7>}), (\frac{u_2}{<0.4,0.6,0.25>}), (\frac{u_3}{<0.6,0.4,0.4>})\}\},$

$\{(e_1, x_2, 1) = \{(\frac{u_1}{<0.7,0.2,0.3>}), (\frac{u_2}{<0.3,0.3,0.4>}), (\frac{u_3}{<0.2,0.6,0.1>})\}\},$

$\{(e_2, x_2, 1) = \{(\frac{u_1}{<0.6,0.2,0.2>}), (\frac{u_2}{<0.2,0.3,0.7>}), (\frac{u_3}{<0.5,0.1,0.3>})\}\},$

$\{(e_1, x_1, 0) = \{(\frac{u_1}{<0.5,0.4,0.2>}), (\frac{u_2}{<0.1,0.9,0.1>}), (\frac{u_3}{<0.5,0.2,0.1>})\}\},$

$\{(e_2, x_1, 0) = \{(\frac{u_1}{<0.6,0.4,0.3>}), (\frac{u_2}{<0.6,0.7,0.2>}), (\frac{u_3}{<0.2,0.5,0.1>})\}\},$

$\{(e_1, x_2, 0) = \{(\frac{u_1}{<0.4,0.8,0.2>}), (\frac{u_2}{<0.5,0.6,0.1>}), (\frac{u_3}{<0.3,0.6,0.7>})\}\},$

$\{(e_2, x_2, 0) = \{(\frac{u_1}{<0.7,0.4,0.4>}), (\frac{u_2}{<0.2,0.8,0.3>}), (\frac{u_3}{<0.4,0.2,0.6>})\}\}.$

Proposition 4.3 If (F, A) is a SVNSES over a soft universe (U, Z). Then,

$$((F, \mathrm{A})^{\mathrm{c}})^{\mathrm{c}} = (\mathrm{F}, \mathrm{A}).$$

Proof. Suppose that is (F, A) is a SVNSES over a soft universe (U, Z) defined as (F, A)= F(e). Now let SVNSES $(\mathrm{F}, \mathrm{A})^{\mathrm{c}} = (G, \mathrm{B})$. Then by Definition 4.1, (G, B) = G(e) such that G(e) $= \tilde{c}$ (F(e)), Thus it follows that:

$$(G, \mathrm{B})^{\mathrm{c}} = \tilde{c}\,(\mathrm{G}(e)) = (\tilde{c}\,(\tilde{c}\,(\mathrm{F}(e))) = \mathrm{F(e)} = (\mathrm{F}, \mathrm{A}).$$

Therefore

$((F, \mathrm{A})^{\mathrm{c}})^{\mathrm{c}} = (G, \mathrm{B})^{\mathrm{c}} = (\mathrm{F}, \mathrm{A})$. Hence it is proven that $((F, \mathrm{A})^{\mathrm{c}})^{\mathrm{c}} = (\mathrm{F}, \mathrm{A})$.

Defintion 4.4 Let (F, A) and (G, B) be any two SVNSESs over a soft universe (U, Z). Then the union of (F, A) and (G, B), denoted by $(F, \mathrm{A}) \,\tilde{\cup}\, (G, \mathrm{B})$ is a SVNSES defined as $(F, \mathrm{A}) \,\tilde{\cup}\, (G, \mathrm{B}) = (H, \mathrm{C})$, where C= A ∪ B and

$$\mathrm{H}(\alpha) = \mathrm{F}(\alpha) \cup \mathrm{G}(\alpha), \text{ for all } \alpha \in \mathrm{C}$$

where

$$\mathrm{H}(\alpha) = \begin{cases} \mathrm{F}(\alpha), & \alpha \in A - B \\ \mathrm{G}(\alpha), & \alpha \in B - A \\ \cup\big(F(\alpha), G(\alpha)\big), & \alpha \in A \cap B, \end{cases}$$

Where $\cup\big(F(\alpha), G(\alpha)\big) = \{<u, \max\{\mu_{\mathrm{F}}(\alpha), \mu_G(\alpha)\}, \min\{\nu_{\mathrm{F}}(\alpha), \nu_G(\alpha)\}, \min\{\omega_{\mathrm{F}}(\alpha), \omega_G(\alpha)\}>: u \in U\}$

Proposition 4.5 Let (F, A), (G, B) and (H, C) be any three SVNSES over a soft universe (U, Z).Then the following properties hold true.

(i) $(F, \mathrm{A}) \,\tilde{\cup}\, (G, \mathrm{B}) = (G, \mathrm{B}) \,\tilde{\cup}\, (F, \mathrm{A})$

(ii) $(F, \mathrm{A}) \,\tilde{\cup}\, ((G, \mathrm{B}) \,\tilde{\cup}\, (H, \mathrm{C})) = ((F, \mathrm{A}) \,\tilde{\cup}\, (G, \mathrm{B})) \,\tilde{\cup}\, (H, \mathrm{C})$

(iii) $(F, \mathrm{A}) \,\tilde{\cup}\, (F, \mathrm{A}) \subseteq (F, \mathrm{A})$

(iv) $(F, \mathrm{A}) \,\tilde{\cup}\, (\Phi, A) = (F, \mathrm{A})$

Proof. (i) Let $(F, \mathrm{A}) \,\tilde{\cup}\, (G, \mathrm{B}) = (H, \mathrm{C})$. Then by definition 4.4,for all $\alpha \in$C, we have (H, C) =H(α). Where H(α) = F(α) ∪ G(α) However

$$\mathrm{H}(\alpha) = \mathrm{F}(\alpha) \cup \mathrm{G}(\alpha) = \mathrm{G}(\alpha) \cup \mathrm{F}(\alpha)$$

since the union of these sets are commutative by definition 4.4. Therfore

$$(H, \mathrm{C}) = (G, \mathrm{B}) \,\tilde{\cup}\, (F, \mathrm{A}).$$

Thus the union of two SVNSES are commutative i.e $(F, \mathrm{A}) \,\tilde{\cup}\, (G, \mathrm{B}) = (G, \mathrm{B}) \,\tilde{\cup}\, (F, \mathrm{A})$.

(ii) The proof is similar to proof of part(i) and is therefore omitted

(iii) The proof is straightforward and is therefore omitted.

(iv) The proof is straightforward and is therefore omitted.

Definition 4.6 Let (F, A) and (G, B) be any two SVNSES over a soft universe (U, Z). Then the intersection of $(F, \text{A})$and (G, B), denoted by $(F, \text{A}) \tilde{\cap} (G, \text{B})$is SVNSES defined as $(F, \text{A}) \tilde{\cap} (G, \text{B}) = (H, \text{C})$ where C= A ∪ B and H(α) = F(α) ∩ G(α), for all $\alpha \in$ C. Where

$$\text{H}(\alpha) = \begin{cases} \text{F}(\alpha), & \alpha \in A - B \\ \text{G}(\alpha), & \alpha \in B - A \\ \cap(F(\alpha), G(\alpha)), & \alpha \in A \cap B \end{cases}$$

Where $\cap(F(\alpha), G(\alpha))$ ={<u, min $\{\mu_F(\alpha), \mu_G(\alpha)\}$, max $\{\nu_F(\alpha), \nu_G(\alpha)\}$, max $\{\omega_F(\alpha), \omega_G(\alpha)\}$>:u ∈ U }

Proposition 4.7 If Let (F, A), $(G, \text{B})$and (H, C) are three SVNSES over a soft universe (U, Z). Then,

(i) $(F, \text{A}) \tilde{\cap} (G, \text{B}) = (G, \text{B}) \tilde{\cap} (F, \text{A})$
(ii) $(F, \text{A}) \tilde{\cap} ((G, \text{B}) \tilde{\cap} (H, \text{C})) = ((F, \text{A}) \tilde{\cap} (G, \text{B})) \tilde{\cap} (H, \text{C})$
(iii) $(\text{F}, \text{A}) \tilde{\cap} (\text{F}, \text{A}) \subseteq (\text{F}, \text{A})$
(iv) $(F, A) \tilde{\cap} (\Phi, A) = (\Phi, A)$

Proof.

(i) The proof is similar to that of Proposition 4.5 (i) and is therefore omitted

(ii) The prof is similar to the prof of part (i) and is therefore omitted

(iii) The proof is straightforward and is therefore omitted.

(iv) The proof is straightforward and is therefore omitted.

Proposition 4.8 If Let (F, A), (G, B) and (H, C) are three SVNSES over a soft universe (U, Z). Then,

(i) $(F, \text{A}) \tilde{\cup} ((G, \text{B}) \tilde{\cap} (H, \text{C})) = ((F, \text{A}) \tilde{\cup} (G, \text{B})) \tilde{\cap} ((F, \text{A}) \tilde{\cup} ((H, \text{C}))$
(ii) $(F, \text{A}) \tilde{\cap} ((G, \text{B}) \tilde{\cup} (H, \text{C})) = ((F, \text{A}) \tilde{\cap} (G, \text{B})) \tilde{\cup} ((F, \text{A}) \tilde{\cap} (H, \text{C}))$

Proof. The proof is straightforward by definitions 4.4 and 4.6 and is therefore omitted.

Proposition 4.9 If (F, A), (G, B) are two SVNSES over a soft universe (U, Z). Then,

i. $((F, \text{A}) \tilde{\cup} (G, \text{B}))^c = (F, \text{A})^c \tilde{\cap} (G, \text{B})^c$.
ii. $((F, \text{A}) \tilde{\cap} (G, \text{B}))^c = (F, \text{A})^c \tilde{\cup} (G, \text{B})^c$.

Proof. (i) Suppose that (F, A) and(G, B) be SVNSES over a soft universe (U, Z) defined as: (F, A)= F(α) for all $\alpha \in \text{A} \subseteq$ Z and (G, B) = G(α) for all $\alpha \in \text{B} \subseteq$ Z. Now , due to the commutative and associative properties of SVNSES, it follows that by definition 4.10 and 4.11, it follows that:

$$(F, \mathrm{A})^c \,\tilde{\cap}\, (G, \mathrm{B})^c = (F(\alpha))^c \,\tilde{\cap}\, (G(\alpha))^c$$
$$= (\tilde{c}\ (\mathrm{F}(\alpha))) \,\tilde{\cap}\, (\tilde{c}\ (\mathrm{G}(\alpha))$$
$$= (\tilde{c}\ (\mathrm{F}(\alpha) \,\tilde{\cap}\, \mathrm{G}(\alpha))$$
$$= ((F, \mathrm{A}) \,\tilde{\cup}\, (G, \mathrm{B}))^c.$$

(ii) The proof is similar to the proof of part (i) and is therefore omitted

Definition 4.10 Let (F, A) and (G, B) be any two SVNSES over a soft universe (U, Z). Then "(F, A) AND (G, B) " denoted $(F, \mathrm{A}) \,\tilde{\wedge}\, (G, \mathrm{B})$ is a defined by:

$$(F, \mathrm{A}) \,\tilde{\wedge}\, (G, \mathrm{B}) = (H, \mathrm{A} \times \mathrm{B})$$

Where $(H, \mathrm{A} \times \mathrm{B})$ = H(α, β), such that H(α, β) = F(α) $\cap$ G(β), for all $(\alpha, \beta) \in \mathrm{A} \times \mathrm{B}$. and $\cap$ represent the basic intersection.

Definition 4.11 Let (F, A) and (G, B) be any twoSVNSES over a soft universe (U, Z). Then "(F, A) OR (G, B) " denoted $(F, \mathrm{A}) \,\tilde{\vee}\, (G, \mathrm{B})$ is a defined by:

$$(F, \mathrm{A}) \,\tilde{\vee}\, (G, \mathrm{B}) = (H, \mathrm{A} \times \mathrm{B})$$

Where $(H, \mathrm{A} \times \mathrm{B})$ = H(α, β) such that H(α, β) = F(α) $\cup$ G(β), for all $(\alpha, \beta) \in \mathrm{A} \times \mathrm{B}$. and $\cup$ represent the basic union.

Proposition 4.12 If(F, A), (G, B) and (H, C) are three SVNSES over a soft universe (U, Z). Then,

i. $(F, \mathrm{A}) \tilde{\wedge} ((G, \mathrm{B}) \tilde{\wedge} (H, \mathrm{C})) = ((F, \mathrm{A}) \tilde{\wedge} (G, \mathrm{B})) \tilde{\wedge} (H, \mathrm{C})$
ii. $(F, \mathrm{A}) \tilde{\vee} ((G, \mathrm{B}) \tilde{\vee} (H, \mathrm{C})) = ((F, \mathrm{A}) \tilde{\vee} (G, \mathrm{B})) \tilde{\vee} (H, \mathrm{C})$
iii. $(F, \mathrm{A}) \tilde{\vee} ((G, \mathrm{B}) \tilde{\wedge} (H, \mathrm{C})) = ((F, \mathrm{A}) \tilde{\vee} (G, \mathrm{B})) \tilde{\wedge} ((F, \mathrm{A}) \tilde{\vee} (H, \mathrm{C}))$
iv. $(F, \mathrm{A}) \tilde{\wedge} ((G, \mathrm{B}) \tilde{\vee} (H, \mathrm{C})) = ((F, \mathrm{A}) \tilde{\wedge} (G, \mathrm{B})) \tilde{\vee} ((F, \mathrm{A}) \tilde{\wedge} (H, \mathrm{C}))$

Proof. The proofs are straightforward by definitions 4.10 and 4.11 and is therefore omitted.

Note: The "AND" and "OR" operations are not commutative since generally A $\times$B$\neq$B$\times$A.

Proposition 4.13 If (F, A) and (G, B) are two SVNSES over a soft universe (U, Z).Then,

i. $((F, \mathrm{A}) \,\tilde{\wedge}\, (G, \mathrm{B}))^c = (F, \mathrm{A})^c \,\tilde{\vee}\, (G, \mathrm{B})^c$.
ii. $((F, \mathrm{A}) \,\tilde{\vee}\, (G, \mathrm{B}))^c = (F, \mathrm{A})^c \,\tilde{\wedge}\, (G, \mathrm{B})^c$.

Proof. (i) suppose that (F, A) and (G, B) be SVNSES over a soft universe (U, Z) defined as:

(F, A) = F(α) for all $\alpha \in \mathrm{A} \subseteq \mathrm{Z}$ and (G, B)= G(β) for all $\beta \in \mathrm{B} \subseteq$ Z. Then by Definition 4.10 and 4.11, it follows that:

$$((F, \mathrm{A}) \,\tilde{\wedge}\, (G, \mathrm{B}))^c = ((F(\alpha) \,\tilde{\wedge}\, G(\beta))^c$$
$$= (F(\alpha) \cap G(\beta))^c$$

$= (\tilde{c}(\mathrm{F}(\alpha) \cap \mathrm{G}(\beta))$
$= (\tilde{c}(\mathrm{F}(\alpha)) \cup \tilde{c}(\mathrm{G}(\beta)))$
$= (F(\alpha))^c \,\tilde{\vee}\, (G(\beta))^c$
$= (F, \mathrm{A})^c \,\tilde{\vee}\, (G, \mathrm{B})^c.$

(ii) The proof is similar to that of part (i) and is therefore omitted.

5. Application of Single Valued Neutrosophic Soft Expert Sets in a Decision Making Problem.

In this section, we introduce a generalized algorithm which will be applied to the SVNSES model introduced in Section 3 and used to solve a hypothetical decision making problem.

Suppose that company Y is looking to hire a person to fill in the vacancy for a position in their company. Out of all the people who applied for the position, three candidates were shortlisted and these three candidates form the universe of elements, U= $\{u_1, u_2, u_3\}$ The hiring committee consists of the hiring manager, head of department and the HR director of the company and this committee is represented by the set {p,q,r }(a set of experts) while the set O= {1=agree, 0=disagree } represents the set of opinions of the hiring committee members. The hiring committee considers a set of parameters, E={ e_1,e_3,e_3,e_4} where the parameters e_i represent the characteristics or qualities that the candidates are assessed on, namely "relevant job experience", "excellent academic qualifications in the relevant field", "attitude and level of professionalism" and "technical knowledge" respectively. After interviewing all the three candidates and going through their certificates and other supporting documents, the hiring committee constructs the following SVNSES.

$$(F, \mathrm{Z}) = \{ (e_1, p, 1) = \{ (\frac{u_1}{<0.2,0.8,0.4>}), (\frac{u_2}{<0.3,0.2,0.4>}), (\frac{u_3}{<0.4,0.7,0.2>}) \}\},$$

$$\{(e_2, p, 1) = \{ (\frac{u_1}{<0.3,0.2,0.23>}), (\frac{u_2}{<0.25,0.2,0.3>}), (\frac{u_3}{<0.3,0.5,0.6>}) \}\},$$

$$\{(e_3, p, 1) = \{ (\frac{u_1}{<0.3,0.2,0.7>}), (\frac{u_2}{<0.4,0.3,0.3>}), (\frac{u_3}{<0.1,0.6,0.2>}) \}\},$$

$$\{(e_4, p, 1) = \{ (\frac{u_1}{<0.2,0.2,0.6>}), (\frac{u_2}{<0.7,0.3,0.2>}), (\frac{u_3}{<0.3,0.1,0.5>}) \}\},$$

$$\{ (e_1, q, 1) = \{ (\frac{u_1}{<0.4,0.6,0.3>}), (\frac{u_2}{<0.1,0.3,0.7>}), (\frac{u_3}{<0.6,0.3,0.7>}) \}\},$$

$$\{(e_2, q, 1) = \{ (\frac{u_1}{<0.3,0.3,0.5>}), (\frac{u_2}{<0.6,0.9,0.1>}), (\frac{u_3}{<0.1,0.2,0.7>}) \}\},$$

$$\{ (e_3, q, 1) = \{ (\frac{u_1}{<0.1,0.4,0.7>}), (\frac{u_2}{<0.4,0.6,0.2>}), (\frac{u_3}{<0.6,0.2,0.4>}) \}\}.$$

$$\{ (e_4, q, 1) = \{ (\frac{u_1}{<0.6,0.5,0.3>}), (\frac{u_2}{<0.7,0.8,0.2>}), (\frac{u_3}{<0.3,0.4,0.6>}) \}\}.$$

$\{(e_1,r,1) = \{(\frac{u_1}{<0.4,0.5,0.7>}),(\frac{u_2}{<0.3,0.8,0.4>}),(\frac{u_3}{<0.6,0.2,0.4>})\}\}.$

$\{(e_2,r,1) = \{(\frac{u_1}{<0.3,0.7,0.1>},(\frac{u_2}{<0.7,0.3,0.2>}),(\frac{u_3}{<0.8,0.2,0.2>})\}\}.$

$\{(e_3,r,1) = \{(\frac{u_1}{<0.6,0.5,0.2>}),(\frac{u_2}{<0.5,0.1,0.6>}),(\frac{u_3}{<0.3,0.2,0.1>})\}\}.$

$\{(e_1,p,0) = \{(\frac{u_1}{<0.1,0.4,0.3>}),(\frac{u_2}{<0.3,0.8,0.2>}),(\frac{u_3}{<0.6,0.2,0.4>})\}\}.$

$\{(e_3,p,0) = \{(\frac{u_1}{<0.6,0.3,0.2>}),(\frac{u_2}{<0.2,0.7,0.4>}),(\frac{u_3}{<0.3,0.1,0.6>})\}\}.$

$\{(e_4,p,0) = \{(\frac{u_1}{<0.3,0.2,0.5>}),(\frac{u_2}{<0.6,0.4,0.5>}),(\frac{u_3}{<0.5,0.4,0.3>})\}\}.$

$\{(e_1,q,0) = \{(\frac{u_1}{<0.2,0.4,0.7>}),(\frac{u_2}{<0.1,0.9,0.2>}),(\frac{u_3}{<0.1,0.2,0.5>})\}\},$

$\{(e_2,q,0) = \{(\frac{u_1}{<0.3,0.4,0.6>}),(\frac{u_2}{<0.2,0.7,0.6>}),(\frac{u_3}{<0.4,0.5,0.3>})\}\},$

$\{(e_3,q,0) = \{(\frac{u_1}{<0.2,0.8,0.4>}),(\frac{u_2}{<0.1,0.2,0.5>}),(\frac{u_3}{<0.7,0.6,0.3>})\}\},$

$\{(e_4,q,0) = \{(\frac{u_1}{<0.9,0.4,0.7>}),(\frac{u_2}{<0.5,0.6,0.2>}),(\frac{u_3}{<0.6,0.3,0.4>})\}\}.$

$\{(e_1,r,0) = \{(\frac{u_1}{<0.3,0.4,0.5>}),(\frac{u_2}{<0.3,0.6,0.2>}),(\frac{u_3}{<0.25,0.2,0.4>})\}\}.$

$\{(e_2,r,0) = \{(\frac{u_1}{<0.4,0.6,0.7>}),(\frac{u_2}{<0.6,0.4,0.2>}),(\frac{u_3}{<0.6,0.4,0.3>})\}\}.$

$\{(e_3,r,0) = \{(\frac{u_1}{<0.4,0.3,0.2>}),(\frac{u_2}{<0.3,0.5,0.7>}),(\frac{u_3}{<0.7,0.5,0.6>})\}\}.$

Next the SVNSES (F, Z) is used together with a generalized algorithm to solve the decision making problem stated at the beginning of this section. The algorithm given below is employed by the hiring committee to determine the best or most suitable candidate to be hired for the position. This algorithm is a generalization of the algorithm introduced by Alkhazaleh and Salleh [37] which is used in the context of the SVNSES model that is introduced in this paper. The generalized algorithm is as follows:

Algorithm

1. Input the SVNSES (F, Z)

2. Find the values of $\mu_{F(Z_i)}(u_i)$ - $\nu_{F(Z_i)}(u_i)$ -$\omega_{F(Z_i)}(u_i)$ for each element $u_i \in$ U where $\mu_{F(Z_i)}(u_i)$, $\nu_{F(Z_i)}(u_i)$ and $\omega_{F(Z_i)}(u_i)$ are the membership function, indeterminacy function and non-membership function of each of the elements $u_i \in$ U respectively.

3. Find the highest numerical grade for the agree-SVNSES and disagree-SVNSES.

4. Compute the score of each element $u_i \in$ U by taking the sum of the products of the numerical grade of each element for the agree-SVNSES and disagree SVNSES, denoted by A_i and D_i respectively.

5. Find the values of the score r_i= A_i-D_i for each element $u_i \in$ U.

6. Determine the value of the highest score, s = $\max_{u_i}$ $\{r_i\}$. Then the decision is to choose element as the optimal or best solution to the problem. If there are more than one element with the highest r_i. score, then any one of those elements can be chosen as the optimal solution.

Then we can conclude that the optimal choice for the hiring committee is to hire candidate u_i to fill the vacant position

Table I gives the values of $\mu_{F(Z_i)}(u_i)$ - $\nu_{F(Z_i)}(u_i)$ -$\omega_{F(Z_i)}(u_i)$ for each element $u_i \in$ U.

The notation a ,b gives the values of $\mu_{F(Z_i)}(u_i)$ - $\nu_{F(Z_i)}(u_i)$ -$\omega_{F(Z_i)}(u_i)$.

Table I. Values of $\mu_{F(Z_i)}(u_i)$ - $\nu_{F(Z_i)}(u_i)$ -$\omega_{F(Z_i)}(u_i)$ for all $u_i \in$ U.

	u_1	u_2	u_3		u_1	u_2	u_3
$(e_1, p, 1)$	-1	-0.3	-0.5	$(e_3, p, 0)$	0.1	-0.9	-0.4
$(e_2, p, 1)$	-0.13	-0.25	-0.8	$(e_4, p, 0)$	-0.4	-0.3	-0.2
$(e_3, p, 1)$	-0.6	-0.2	-0.7	$(e_1, q, 0)$	-0.9	-1	-0.6
$(e_4, p, 1)$	-0.6	0.2	-0.3	$(e_2, q, 0)$	-0.7	-1.1	-0.4
$(e_1, q, 1)$	-0.5	-0.9	-0.4	$(e_3, q, 0)$	-1	-0.6	-0.2
$(e_2, q, 1)$	-0.5	-0.4	-0.5	$(e_4, q, 0)$	-0.2	-0.3	-0.1
$(e_3, q, 1)$	-1	-0.4	0	$(e_1, r, 0)$	-0.6	-0.5	0.35
$(e_4,q, 1)$	-0.2	-0.3	-0.5	$(e_2, r, 0)$	-0.9	0	-0.1
$(e_1, r, 1)$	-0.8	-0.9	0	$(e_4, r, 0)$	-0.1	-0.9	-0.4
$(e_2, r, 1)$	-0.5	0.2	0.4				
$(e_3, r, 1)$	-0.1	-0.2	0				
$(e_1, p, 0)$	-0.6	-0.7	0				

In Table II and Table III, we gives the highest numerical grade for the elements in the agree-SVNSES and disagree SVNSES respectively.

Table II. Numerical Grade for Agree-SVNSES

	u_i	Highest Numeric Grade
$(e_1, p, 1)$	u_2	-0.3
$(e_2, p, 1)$	u_1	-0.13
$(e_3, p, 1)$	u_2	-0.2
$(e_4, p, 1)$	u_2	0.2
$(e_1, q, 1)$	u_3	-0.4
$(e_2, q, 1)$	u_2	-0.4
$(e_3, q, 1)$	u_3	0
$(e_4,q, 1)$	u_1	-0. 2
$(e_1, r, 1)$	u_3	0
$(e_2, r, 1)$	u_3	0.4
$(e_3, r, 1)$	u_3	0

Score (u_1) = -0.13 + -0.2 = -0.23

Score (u_2) = -0.3 + -0.2 + -0.2 + -0.4 = -0.11

Score (u_3) =-0.4 +0+ 0 +0.4 + 0

Table III. Numerical Grade for Disagree-SVNSE

	u_i	Highest Numeric Grade
$(e_1,p, 0)$	u_3	0
$(e_3,p, 0)$	u_1	0.1
$(e_4,p, 0)$	u_3	-0.2
$(e_1,q, 0)$	u_3	-0.6
$(e_2,q, 0)$	u_3	-0.4
$(e_3,q, 0)$	u_3	-0.2
$(e_4,q, 0)$	u_3	-0.1
$(e_1,r, 0)$	u_3	-0.35
$(e_2,r, 0)$	u_2	0
$(e_4,r, 0)$	u_1	-0.1

Score (u_1) = 0.1 + -0.1 = 0
Score (u_2) = 0
Score (u_3) = 0 -0.2 + -0.6 + -0.4 + -0.2 +-0.1 + -0.35 = - 1.85

Let A_i and D_i represent the score of each numerical grade for the agree-SVNSES and disagree-SVNSES respectively. These values are given in Table IV.

Table IV. The score $r_i = A_i - D_i$

A_i	D_i	r_i
Score (u_1) = - 0.23	Score (u_1) = 0	**-0.23**
Score (u_2) = -0.11	Score (u_2) = 0	**-0.11**
Score (u_3) = 0	Score (u_3) = -1.85	**1.85**

Then s $= \max_{u_i} \{r_i\} = r_3$, the hiring committee should hire candidate u_3 to fill in the vacant position.

6. Conclusion

In this paper we have introduced the concept of single valued neutrosophic soft expert soft set and studied some related properties with supporting proofs. The complement, union, intersection, AND or OR operations have been defined on the single valued neutrosophic soft expert set. Finally, an application of this concept is given in solving a decision making problem. This new extension will provide a significant addition to existing theories for handling indeterminacy, and lead to potential areas of further research and pertinent applications.

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Q-FUZZY IDEAL OF ORDERED Γ-SEMIRING

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Abstract − The notion of Q-fuzzy ideal in ordered Γ-semiring is introduced and studied along with some operations. Among all other results, it is shown that the set of all Q-fuzzy ideals of a Γ-semiring forms a complete lattice. They also form a zerosumfree Γ-semiring under the operations of sum and composition of Q-fuzzy ideals.

Keywords − *Γ-semiring, Q-fuzzy, intersection, lattice, normal.*

1 Introduction

The fundamental concept of fuzzy set, introduced by Zadeh [10], provides a natural frame-work for generalizing several basic notions of algebra . Jun and Lee [5] applied the concept of fuzzy sets to the theory of Γ-rings. The notion of Γ-semiring was introduced by Rao [9] as a generalization of Γ-ring as well as of semiring [3].
Majumder [8] introduced and studied the concept of Q-fuzzifcation of ideals of Γ-semigroups. Akram et al [1], Lekkoksung [6, 7] extended this concept in case of Γ-semigroup and ordered semigroups [4] and investigated some important properties.
Main object of the present paper is to define ordered Γ-semiring and study its ideals using the concept of Q-fuzzification.

2 Preliminary

Definition 2.1. A semiring is a system consisting of a non-empty set S on which operations addition and multiplication (denoted in the usual manner) have been defined such that $(S, +)$ is a semigroup, $(S, \cdot)$ is a semigroup and multiplication distributes over addition from either side.
A zero element of a semiring S is an element 0 such that $0 \cdot x = x \cdot 0 = 0$ and $0 + x = x + 0 = x$ for all $x \in S$. A semiring S is zerosumfree if and only if $s + s' = 0$ implies that $s = s' = 0$.

Definition 2.2. Let S and Γ be two additive commutative semigroups with zero. Then S is called a Γ-semiring if there exists a mapping $S \times \Gamma \times S \to S$ ($(a, \alpha, b) \mapsto a\alpha b$) satisfying the following conditions:

(i) $(a + b)\alpha c = a\alpha c + b\alpha c$,

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(ii) $a\alpha(b+c) = a\alpha b + a\alpha c$,

(iii) $a(\alpha+\beta)b = a\alpha b + a\beta b$,

(iv) $a\alpha(b\beta c) = (a\alpha b)\beta c$,

(v) $0_S\alpha a = 0_S = a\alpha 0_S$,

(vi) $a0_\Gamma b = 0_S = b0_\Gamma a$

for all $a, b, c \in S$ and for all $\alpha, \beta \in \Gamma$.
For simplification we write 0 instead of 0_S and 0_Γ.

Example 2.3. Let S be the set of all $m \times n$ matrices over $\mathbf{Z_0^-}$ (the set of all non-positive integers) and Γ be the set of all $n \times m$ matrices over $\mathbf{Z_0^-}$, then S forms a Γ-semiring with usual addition and multiplication of matrices.

Definition 2.4. A left ideal I of Γ-semiring S is a nonempty subset of S satisfying the following conditions:

(i) If $a, b \in I$ then $a + b \in I$,

(ii) If $a \in I$, $s \in S$ and $\gamma \in \Gamma$ then $s\gamma a \in I$.

A right ideal of S is defined in an analogous manner and an ideal of S is a nonempty subset which is both a left ideal and a right ideal of S.

Definition 2.5. An ordered Γ-semiring is a Γ-semiring S equipped with a partial order $\leq$ such that the operation is monotonic and constant 0 is the least element of S.

Definition 2.6. A left (resp. right) ideal I of S is called a left (resp. right) ordered ideal, if for any $a \in S$, $b \in I$, $a \leq b$ implies $a \in I$ (i.e., $(I] \subseteq I$). I is called an ordered ideal of S if it is both a left and a right ordered ideal of S.

Now we recall the example of ordered ideal from [2].

Example 2.7. Let $S = ([0,1], \vee, \cdot, 0)$ where [0,1] is the unit interval, $a \vee b = \max\{a, b\}$ and $a \cdot b = (a + b - 1) \vee 0$ for $a, b \in [0, 1]$. Then it is easy to verify that S equipped with the usual ordering $\leq$ is an ordered semiring and $I = [0, \frac{1}{2}]$ is an ordered ideal of S.

Definition 2.8. A fuzzy subset f of a non-empty set S is defined as a mapping from S to [0,1].

Definition 2.9. A function μ from $S \times Q$ to the real closed interval [0, 1] is called Q-fuzzy subset of S, where Q is a non-empty set.

Definition 2.10. Let μ be a Q-fuzzy subset of a set S and $t \in$[0,1]. The set

$$\mu_t = \{(x, q) \in S \times Q | \mu(x, q) \geq t\}$$

is called the level subset of μ. Clearly, $\mu_t \subseteq \mu_s$, whenever $t \geq s$.

Definition 2.11. The characteristic function $\chi_{A\times Q}$ of $A \times Q$, is the mapping of $S \times Q$ to $[0, 1]$ defined by

$$\begin{aligned} \chi_{A\times Q}(x, q) &= 1 \text{ if } (x, q) \in A \times Q \\ &= 0, \text{ if } (x, q) \notin A \times Q \end{aligned}$$

Definition 2.12. The union and intersection of two Q-fuzzy subsets μ and σ of a set S, denoted by $\mu \cup \sigma$ and $\mu \cap \sigma$ respectively, are defined as

$$(\mu \cup \sigma)(x, q) = \max\{\mu(x, q), \sigma(x, q)\} \text{ for all } x \in S,\ q \in Q$$

$$(\mu \cap \sigma)(x, q) = \min\{\mu(x, q), \sigma(x, q)\} \text{ for all } x \in S,\ q \in Q.$$

3 Main Results

Throughout this paper unless otherwise mentioned S denotes the ordered Γ-semiring.

Definition 3.1. Let μ and ν be two Q-fuzzy subsets of an ordered Γ-semiring S and $x,\ y,\ z \in S$, $\gamma \in \Gamma$, $q \in Q$. We define composition and sum of μ and ν as follows:

$$\begin{aligned} \mu \circ_1 \nu(x,q) &= \sup_{x \le y\gamma z}\{\min\{\mu(y,q), \nu(z,q))\}\} \\ &= 0, \text{ if } x \text{ cannot be expressed as } x \le y\gamma z \end{aligned}$$

and

$$\begin{aligned} \mu +_1 \nu(x,q) &= \sup_{x \le y+z}\{\min\{\mu(y,q), \nu(z,q))\}\} \\ &= 0, \text{ if } x \text{ cannot be expressed as } x \le y+z. \end{aligned}$$

Proposition 3.2. For any Q-fuzzy subset μ of an ordered Γ-semiring S, $(\chi_{S\times Q} o_1 \mu)(x,q) \ge (\chi_{S\times Q} o_1 \mu)(y,q)$ (resp. $(\chi_{S\times Q} +_1 \mu)(x,q) \ge (\chi_{S\times Q} +_1 \mu)(y,q)$) $\forall x,\ y \in S$, $q \in Q$ with $x \le y$.

Proof. Let μ be a Q-fuzzy subset of an ordered Γ-semiring S and $x,\ y \in S$ with $x \le y$. If y cannot be expressed as $y \le y_1\gamma y_2$ for $y_1,\ y_2 \in S$ and $\gamma \in \Gamma$ then the proof is trivial so we omit it.
Let y have such an expression. Then

$$(\chi_{S\times Q} o_1 \mu)(y,q) = \sup_{y \le y_1\gamma y_2}\{\min\{\chi_{S\times Q}(y_1,q), \mu(y_2,q)\}\} = \sup_{y \le y_1\gamma y_2}\{\mu(y_2,q)\}.$$

Since $x \le y \le y_1\gamma y_2$, we have

$$\begin{aligned} (\chi_{S\times Q} o_1 \mu)(x,q) &= \sup_{x \le x_1\gamma x_2}\{\min\{\chi_{S\times Q}(x_1,q), \mu(x_2,q)\}\} \\ &\ge \sup_{x \le y_1\gamma y_2}\{\min\{\chi_{S\times Q}(y_1,q), \mu(y_2,q)\}\} \\ &= \sup_{y \le y_1\gamma y_2}\{\mu(y_2,q)\} = (\chi_{S\times Q} o_1 \mu)(y,q). \end{aligned}$$

Similarly for $x \le y$, we can prove that $(\chi_{S\times Q} +_1 \mu)(x,q) \ge (\chi_{S\times Q} +_1 \mu)(y,q)$. □

Definition 3.3. Let μ be a non empty Q-fuzzy subset of an ordered Γ-semiring S (i.e., $\mu(x) \ne 0$ for some $x \in S$). Then μ is called a Q-fuzzy left ideal [resp. Q-fuzzy right ideal] of S if

(i) $\mu(x+y,q) \ge \min\{\mu(x,q), \mu(y,q)\}$,

(ii) $\mu(x\gamma y,q) \ge \mu(y,q)$ [resp. $\mu(x\gamma y,q) \ge \mu(x,q)$] and

(iii) $x \le y$ implies $\mu(x,q) \ge \mu(y,q)$.

for all $x, y \in S$, $\gamma \in \Gamma$ and $q \in Q$.
By a Q-fuzzy ideal we mean, it is both a Q-fuzzy left ideal as well as a Q-fuzzy right ideal.

Theorem 3.4. A Q-fuzzy subset μ of S is a Q-fuzzy ordered ideal if and only if its level subset μ_t, $t \in [0,1]$ is an ordered ideal of $S \times Q$.

Proof. We only prove the theorem only for left ordered ideal. For right ordered ideal it follows similarly.

Let μ be a Q-fuzzy left ordered ideal of S. Suppose $a \in S$ and $b \in \mu_t$ with $a \le b$. As μ is a Q-fuzzy left ordered ideal of S, $\mu(a,q) \ge \mu(b,q) \ge t$ so that $a \in \mu_t$ i.e., μ_t is a left ordered ideal of $S \times Q$.

Conversely, if μ_t is a left ordered ideal of $S \times Q$, then μ is a Q-fuzzy ideal of S. Now suppose $x, y \in S$ with $x \le y$. We have to show that $\mu(x,q) \ge \mu(y,q)$. Let $\mu(x,q) < \mu(y,q)$. Then there exists $t_1 \in [0,1]$ such that $\mu(x,q) < t_1 < \mu(y,q)$. Then $(y,q) \in \mu_{t_1}$ but $(x,q) \notin \mu_{t_1}$ which is a contradiction to the fact that μ_t is a left ordered ideal of $S \times Q$.

□

Definition 3.5. Let μ be a Q-fuzzy subset of an ordered Γ-semiring S and $a \in S$. We denote I_a the subset of $S \times Q$ defined as follows:

$$I_a = \{(b,q) \in S \times Q | \mu(b,q) \ge \mu(a,q)\}.$$

Proposition 3.6. Let S be an ordered Γ-semiring and μ be a Q-fuzzy right (resp. left) ideal of S. Then I_a is a right (resp. left) ideal of $S \times Q$ for every $a \in S$.

Proof. Let μ be a Q-fuzzy right ideal of S and $a \in S,\ q \in Q$. Then $I_a \neq \phi$ because $(a, q) \in I_a$ for every $(a, q) \in S \times Q$. Let $(b, q),\ (c, q) \in I_a$ and $x \in S$. Since $(b, q),\ (c, q) \in I_a,\ \mu(b, q) \geq \mu(a, q)$ and $\mu(c, q) \geq \mu(a, q)$. Now

$$\begin{aligned} \mu(b + c, q) &\geq \min\{\mu(b, q), \mu(c, q)\}[\because \mu \text{ is a } Q\text{-fuzzy right ideal}] \\ &\geq \mu(a, q). \end{aligned}$$

which implies $(b + c, q) \in I_a$.

Also $\mu(b\gamma x, q) \geq \mu(b, q) \geq \mu(a, q)$ i.e. $(b\gamma x, q) \in I_a$.

Let $(b, q) \in I_a$ and $S \ni x \leq b$. Then $\mu(x, q) \geq \mu(b, q) \geq \mu(a, q) \Rightarrow (x, q) \in I_a$.

Thus I_a is a right ideal of $S \times Q$.

Similarly, we can prove the result for left ideal also. □

Proposition 3.7. Intersection of a non-empty collection of Q-fuzzy right (resp. left) ideals is also a Q-fuzzy right (resp. left) ideal of S.

Proof. Let $\{\mu_i : i \in I\}$ be a non-empty family of Q-fuzzy right ideals of S and $x, y \in S,\ \gamma \in \Gamma,\ q \in Q$. Then

$$\begin{aligned} \bigcap_{i \in I} \mu_i(x + y, q) &= \inf_{i \in I}\{\mu_i(x + y, q)\} \geq \inf_{i \in I}\{\min\{\mu_i(x, q), \mu_i(y, q)\}\} \\ &= \min\{\inf_{i \in I} \mu_i(x, q), \inf_{i \in I} \mu_i(y, q)\} = \min\{\bigcap_{i \in I} \mu_i(x, q), \bigcap_{i \in I} \mu_i(y, q)\}. \end{aligned}$$

Again

$$\bigcap_{i \in I} \mu_i(x\gamma y, q) = \inf_{i \in I}\{\mu_i(x\gamma y, q)\} \geq \inf_{i \in I}\{\mu_i(x, q)\} = \bigcap_{i \in I} \mu_i(x, q).$$

Suppose $x \leq y$. Then $\mu_i(x, q) \geq \mu_i(y, q)$ for all $i \in I$ which implies $\bigcap_{i \in I} \mu_i(x, q) \geq \bigcap_{i \in I} \mu_i(y, q)$.

Hence $\bigcap_{i \in I} \mu_i$ is a Q-fuzzy right ideal of S.

Similarly, we can prove the result for Q-fuzzy left ideal also. □

Proposition 3.8. Let $\{\mu_i : i \in I\}$ be a family of Q-fuzzy ideals of S such that $\mu_i \subseteq \mu_j$ or $\mu_j \subseteq \mu_i$ for $i, j \in I$. Then $\underset{i \in I}{\cup} \mu_i$ is a Q-fuzzy ideal of S.

Proof. The proof follows by routine verification. □

Definition 3.9. Let f be a function from a set X to a set Y; μ be a Q-fuzzy subset of X and σ be a Q-fuzzy subset of Y.

Then image of μ under f, denoted by $f(\mu)$, is a Q-fuzzy subset of Y defined by

$$f(\mu)(y, q) = \begin{cases} \sup\limits_{x \in f^{-1}(y)} \mu(x, q) & \text{if } f^{-1}(y) \neq \phi \\ 0 & \text{otherwise} \end{cases}$$

The pre-image of σ under f, symbolized by $f^{-1}(\sigma)$, is a Q-fuzzy subset of X defined by

$$f^{-1}(\sigma)(x, q) = \sigma(f(x), q)\ \forall x \in X.$$

Proposition 3.10. Let $f : R \to S$ be a morphism of ordered Γ-semirings i.e. Γ-semiring homomorphism satisfying additional condition $a \leq b \Rightarrow f(a) \leq f(b)$. Then if ϕ is a Q-fuzzy left ideal of S, then $f^{-1}(\phi)$ is also a Q-fuzzy left ideal of R.

Proof. Let $f : R \to S$ be a morphism of ordered Γ-semirings and ϕ is a Q-fuzzy left ideal of S and $q \in Q$, $\gamma \in \Gamma$.

Now $f^{-1}(\phi)(0_R, q) = \phi(0_S, q) \geq \phi(x^{'}, q) \neq 0$ for some $x^{'} \in S$.

Therefore $f^{-1}(\phi)$ is non-empty.

Now, for any $r, s \in R$

$$\begin{aligned} f^{-1}(\phi)(r+s,q) &= \phi(f(r+s),q) = \phi(f(r)+f(s),q) \\ &\geq \min\{\phi(f(r),q), \phi(f(s),q)\} \\ &= \min\{(f^{-1}(\phi))(r,q), (f^{-1}(\phi))(s,q)\}. \end{aligned}$$

Again

$$(f^{-1}(\phi))(r\gamma s, q) = \phi(f(r\gamma s), q) = \phi(f(r)\gamma f(s), q) \geq \phi(f(s), q) = (f^{-1}(\phi))(s, q).$$

Also if $r \leq s$, then $f(r) \leq f(s)$. Then

$$(f^{-1}(\phi))(r, q) = \phi(f(r), q) \geq \phi(f(s), q) = (f^{-1}(\phi))(s, q).$$

Thus $f^{-1}(\phi)$ is a Q-fuzzy left ideal of R. □

Definition 3.11. Let μ and ν be Q-fuzzy subsets of X. The cartesian product of μ and ν is defined by $(\mu \times \nu)((x, y), q) = \min\{\mu(x, q), \nu(y, q)\}$ for all $x, y \in X$ and $q \in Q$.

Theorem 3.12. Let μ and ν be fuzzy left ideals of an ordered Γ-semiring S. Then $\mu \times \nu$ is a Q-fuzzy left ideal of $S \times S$.

Proof. Let $(x_1, x_2), (y_1, y_2) \in S \times S$, $\gamma \in \Gamma$ and $q \in Q$. Then

$$\begin{aligned} &(\mu \times \nu)((x_1, x_2) + (y_1, y_2), q) \\ &= (\mu \times \nu)((x_1 + y_1, x_2 + y_2), q) \\ &= \min\{\mu(x_1 + y_1, q), \nu(x_2 + y_2, q)\} \\ &\geq \min\{\min\{\mu(x_1, q), \mu(y_1, q)\}, \min\{\nu(x_2, q), \nu(y_2, q)\}\} \\ &= \min\{\min\{\mu(x_1, q), \nu(x_2, q)\}, \min\{\mu(y_1, q), \nu(y_2, q)\}\} \\ &= \min\{(\mu \times \nu)((x_1, x_2), q), (\mu \times \nu)((y_1, y_2), q)\} \end{aligned}$$

and

$$\begin{aligned} (\mu \times \nu)((x_1, x_2)\gamma(y_1, y_2), q) &= (\mu \times \nu)((x_1\gamma y_1, x_2\gamma y_2), q) \\ &= \min\{\mu(x_1\gamma y_1, q), \nu(x_2\gamma y_2, q)\} \\ &\geq \min\{\mu(y_1, q), \nu(y_2, q)\} \\ &= (\mu \times \nu)((y_1, y_2), q). \end{aligned}$$

Also if $(x_1, x_2) \leq (y_1, y_2)$, then

$$(\mu \times \nu)((x_1, x_2), q) = \min\{\mu(x_1, q), \nu(x_2, q)\} \geq \min\{\mu(y_1, q), \nu(y_2, q)\}.$$

Therefore $\mu \times \nu$ is a Q-fuzzy left ideal of $S \times S$. □

Proposition 3.13. For any three Q-fuzzy subset $\mu_1,\ \mu_2,\ \mu_3$ of an ordered Γ-semiring S, $\mu_1 o_1 (\mu_2 +_1 \mu_3) = (\mu_1 o_1 \mu_2) +_1 (\mu_1 o_1 \mu_3)$.

Proof. Let $\mu_1,\ \mu_2,\ \mu_3$ be any three fuzzy subset of an ordered Γ-semiring S and $x \in S$, $\gamma \in \Gamma$, $q \in Q$. Then

$$\begin{aligned} &(\mu_1 o_1 (\mu_2 +_1 \mu_3))(x, q) \\ &= \sup_{x \leq y\gamma z} \{\min\{\mu_1(y, q), (\mu_2 +_1 \mu_3)(z, q)\}\} \\ &= \sup_{x \leq y\gamma z} \{\min\{\mu_1(y, q), \sup_{z \leq a+b} \{\min\{\mu_2(a, q), \mu_3(b, q)\}\}\}\} \\ &= \sup\{\min\{\sup\{\min\{\mu_1(y, q), \mu_2(a, q)\}\}, \sup\{\min\{\mu_1(y, q), \mu_3(b, q)\}\}\}\} \\ &\leq \sup_{x \leq y\gamma a + y\gamma b}\{\min\{(\mu_1 o_1 \mu_2)(y\gamma a, q), (\mu_1 o_1 \mu_3)(y\gamma b, q)\}\} \\ &\leq ((\mu_1 o_1 \mu_2) +_1 (\mu_1 o_1 \mu_3))(x, q). \end{aligned}$$

Also

$$\begin{aligned}
&((\mu_1 o_1 \mu_2) +_1 (\mu_1 o_1 \mu_3))(x,q) \\
&= \sup_{x \leq x_1+x_2}\{\min\{(\mu_1 o_1 \mu_2)(x_1,q), (\mu_1 o_1 \mu_3)(x_2,q)\}\} \\
&= \sup_{x \leq x_1+x_2}\{\min\{\sup_{x_1 \leq c_1\gamma_1 d_1}\{\min\{\mu_1(c_1,q), \mu_2(d_1,q)\}\}, \sup_{x_2 \leq c_2\gamma_2 d_2}\{\min\{\mu_1(c_2,q), \mu_3(d_2,q)\}\}\}\} \\
&\leq \sup_{x \leq x_1+x_2 \leq c_1\gamma_1 d_1 + c_2\gamma_2 d_2 < (c_1+c_2)\gamma(d_1+d_2)}\{\min\{\mu_1(c_1+c_2,q), \sup\{\min\{\mu_2(d_1,q), \mu_3(d_2,q)\}\}\}\} \\
&\leq \sup_{x \leq c\gamma d}\{\min\{\mu_1(c,q), (\mu_2 +_1 \mu_3)(d,q)\}\} \\
&= (\mu_1 o_1 (\mu_2 +_1 \mu_3))(x,q).
\end{aligned}$$

Therefore $\mu_1 o_1 (\mu_2 +_1 \mu_3) = (\mu_1 o_1 \mu_2) +_1 (\mu_1 o_1 \mu_3)$. □

Theorem 3.14. If $\mu_1,\ \mu_2$ be any two Q-fuzzy ideals of an ordered Γ-semiring S, then $\mu_1 +_1 \mu_2$ is also so.

Proof. Assume that $\mu_1,\ \mu_2$ are any two fuzzy ideals of an ordered Γ-semiring S and $x,\ y \in S$, $\gamma \in \Gamma$, $q \in Q$. Then

$$\begin{aligned}
(\mu_1 +_1 \mu_2)(x+y,q) &= \sup_{x+y \leq c+d}\{\min\{\mu_1(c,q), \mu_2(d,q)\}\} \\
&\geq \sup_{x+y \leq (a_1+b_1)+(a_2+b_2)=(a_1+a_2)+(b_1+b_2)}\{\min\{\mu_1(a_1+a_2,q), \mu_2(b_1+b_2,q)\}\} \\
&\geq \sup\{\min\{\mu_1(a_1,q), \mu_1(a_2,q), \mu_2(b_1,q), \mu_2(b_2,q)\}\} \\
&= \min\{\sup_{x \leq a_1+b_1}\{\min\{\mu_1(a_1,q), \mu_2(b_1,q)\}\}, \sup_{y \leq a_2+b_2}\{\min\{\mu_1(a_2,q), \mu_2(b_2,q)\}\}\} \\
&= \min\{(\mu_1 +_1 \mu_2)(x,q), (\mu_1 +_1 \mu_2)(y,q)\}.
\end{aligned}$$

Now assume $\mu_1,\ \mu_2$ are as Q-fuzzy right ideals and we have

$$\begin{aligned}
(\mu_1 +_1 \mu_2)(x\gamma y,q) &= \sup_{x\gamma y \leq c+d}\{\min\{\mu_1(c,q), \mu_2(d,q)\}\} \\
&\geq \sup_{x\gamma y \leq (x_1+x_2)\gamma y}\{\min\{\mu_1(x_1\gamma y,q), \mu_2(x_2\gamma y,q)\}\} \\
&\geq \sup_{x \leq x_1+x_2}\{\min\{\mu_1(x_1,q), \mu_2(x_2,q)\}\} \\
&= (\mu_1 +_1 \mu_2)(x,q).
\end{aligned}$$

Similarly assuming $\mu_1,\ \mu_2$ are as fuzzy left ideal, we can show that

$$(\mu_1 +_1 \mu_2)(x\gamma y,q) \geq (\mu_1 +_1 \mu_2)(y,q).$$

Now suppose $x \leq y$. Then $\mu_1(x) \geq \mu_1(y)$ and $\mu_2(x) \geq \mu_2(y)$.

$$\begin{aligned}
(\mu_1 +_1 \mu_2)(x,q) &= \sup_{x \leq x_1+x_2}\{\min\{\mu_1(x_1,q), \mu_2(x_2,q)\}\} \\
&\geq \sup_{x \leq y \leq y_1+y_2}\{\min\{\mu_1(y_1,q), \mu_2(y_2,q)\}\} \\
&= \sup_{y \leq y_1+y_2}\{\min\{\mu_1(y_1,q), \mu_2(y_2,q)\}\} \\
&= (\mu_1 +_1 \mu_2)(y,q).
\end{aligned}$$

Hence $\mu_1 +_1 \mu_2$ is a Q-fuzzy ideal of S. □

Theorem 3.15. If $\mu_1,\ \mu_2$ be any two Q-fuzzy ideals an ordered Γ-semiring S, then $\mu_1 o_1 \mu_2$ is also so.

Proof. Let $\mu_1,\ \mu_2$ be any two Q-fuzzy ideals of an ordered Γ-semiring S and $x,\ y \in S$, $\gamma \in \Gamma$, $q \in Q$. Then

$$\begin{aligned}
(\mu_1 o_1 \mu_2)(x+y,q) &= \sup_{x+y \leq c\gamma d}\{\min\{\mu_1(c,q), \mu_2(d,q)\}\} \\
&\geq \sup_{x+y \leq c_1\gamma_1 d_1 + c_2\gamma_2 d_2 < (c_1+c_2)\gamma(d_1+d_2)}\{\min\{\mu_1(c_1+c_2,q), \mu_2(d_1+d_2,q)\}\} \\
&\geq \sup\{\min\{\mu_1(c_1,q), \mu_1(c_2,q), \mu_2(d_1,q), \mu_2(d_2,q)\}\} \\
&\geq \min\{\sup_{x \leq c_1\gamma_1 d_1}\{\min\{\mu_1(c_1,q), \mu_2(d_1,q)\}\}, \sup_{y \leq c_2\gamma_2 d_2}\{\min\{\mu_1(c_2,q), \mu_2(d_2,q)\}\}\} \\
&= \min\{(\mu_1 o_1 \mu_2)(x), (\mu_1 o_1 \mu_2)(y)\}.
\end{aligned}$$

Now assume $\mu_1,\ \mu_2$ are as Q-fuzzy right ideals and we have

$$\begin{aligned}(\mu_1 o_1 \mu_2)(x\gamma y) &= \sup_{x\gamma y\leq c\gamma d}\{\min\{\mu_1(c,q),\mu_2(d,q)\}\}\\ &\geq \sup_{x\gamma y\leq (x_1\gamma_1 x_2)\gamma_2 y}\{\min\ \{\mu_1(x_1,q),\mu_2(x_2\gamma_2 y,q)\}\}\\ &\geq \sup_{x\leq x_1\gamma_1 x_2}\{\min\{\mu_1(x_1,q),\mu_2(x_2,q)\}\}\\ &= (\mu_1 o_1 \mu_2)(x,q).\end{aligned}$$

Similarly assuming $\mu_1,\ \mu_2$ are as Q-fuzzy left ideal, we can show that $(\mu_1 o_1 \mu_2)(x\gamma y,q) \geq (\mu_1 o_1 \mu_2)(y,q)$.

Now suppose $x \leq y$. Then $\mu_1(x) \geq \mu_1(y)$ and $\mu_2(x) \geq \mu_2(y)$.

$$\begin{aligned}(\mu_1 o_1 \mu_2)(x,q) &= \sup_{x\leq x_1\gamma x_2}\{\min\{\mu_1(x_1,q),\mu_2(x_2,q)\}\}\\ &\geq \sup_{x\leq y\leq y_1\gamma y_2}\{\min\ \{\mu_1(y_1,q),\mu_2(y_2,q)\}\}\\ &= \sup_{y\leq y_1\gamma y_2}\{\min\{\mu_1(y_1,q),\mu_2(y_2,q)\}\}\\ &= (\mu_1 o_1 \mu_2)(y,q).\end{aligned}$$

Hence $\mu_1 o_1 \mu_2$ is a Q-fuzzy ideal of S. □

Theorem 3.16. The set of all Q-fuzzy left ideals of S form a complete lattice.

Proof. Suppose the set of all Q-fuzzy left ideals of S is denoted by $FLI(S)$.
Now, for $\mu_1, \mu_2 \in FLI(S)$, define a relation $\leq$ such that $\mu_1 \leq \mu_2$ if and only if $\mu_1(x,q) \leq \mu_2(x,q)$ for all $x \in S,\ q \in Q$. Then $FLI(S)$ is a poset with respect to the relation $\leq$.
Now, for every pair of elements say μ_1, μ_2 of $FLI(S)$, we see that $\mu_1 + \mu_2$ is the least upper bound and $\mu_1 \cap \mu_2$ is the greatest lower bound of μ_1 and μ_2. Thus $FLI(S)$ is a lattice.
Suppose ψ is a fuzzy subset of S such that $\psi(x,q) = 1$ for all $x \in S,\ q \in Q$. Then $\psi \in FLI(S)$ and for all $\mu \in FLI(S)$, $\mu(x,q) \leq \psi(x,q)$ for all $x \in S,\ q \in Q$. So, ψ is the greatest element of $FLI(S)$.
Let $\{\mu_i : i \in I\}$ be a non-empty family of Q-fuzzy left ideals of S. Then $\bigcap_{i\in I}\mu_i \in FLI(S)$. Also it is the greatest lower bound of $\{\mu_i : i \in I\}$.
Hence $FLI(S)$ is a complete lattice. □

Definition 3.17. Let μ be a Q-fuzzy subset of X and $\alpha \in [0,1\text{-}\sup\{\mu(x,q) : x \in X,\ q \in Q\}]$, $\beta \in [0,1]$. The mappings $\mu^T_\alpha : X \to [0,1]$, $\mu^M_\beta : X \to [0,1]$ and $\mu^{MT}_{\beta,\alpha} : X \to [0,1]$ are called a Q-fuzzy translation, Q-fuzzy multiplication and Q-fuzzy magnified translation of μ respectively if $\mu^T_\alpha(x,q) = \mu(x,q) + \alpha$, $\mu^M_\beta(x,q) = \beta.\mu(x,q)$ and $\mu^{MT}_{\beta,\alpha}(x,q) = \beta.\mu(x,q) + \alpha$ for all $x \in X,\ q \in Q$.

Theorem 3.18. Let μ be a Q-fuzzy subset of S and $\alpha \in [0,1\text{-}\sup\{\mu(x) : x \in X\}]$, $\beta \in (0,1]$. Then μ is a Q-fuzzy left ideal of S if and only if $\mu^{MT}_{\beta,\alpha}$, the Q-fuzzy magnified translation of μ, is also a Q-fuzzy left ideal of S.

Proof. Suppose μ is a Q-fuzzy left ideal of S. Let $x, y \in S,\ \gamma \in \Gamma,\ q \in Q$. Then

$$\begin{aligned}\mu^{MT}_{\beta,\alpha}(x+y,q) &= \beta.\mu(x+y,q) + \alpha\\ &\geq \beta.\min\{\mu(x,q),\mu(y,q)\} + \alpha\\ &= \min\{\beta.\mu(x,q),\beta.\mu(y,q)\} + \alpha\\ &= \min\{\beta.\mu(x,q)+\alpha,\beta.\mu(y,q)+\alpha\}\\ &= \min\{\mu^{MT}_{\beta,\alpha}(x,q),\mu^{MT}_{\beta,\alpha}(y,q)\}\end{aligned}$$

and

$$\mu^{MT}_{\beta,\alpha}(x\gamma y) = \beta.\mu(x\gamma y,q) + \alpha \geq \beta.\mu(y,q) + \alpha = \mu^{MT}_{\beta,\alpha}(y,q).$$

Therefore $\mu^{MT}_{\beta,\alpha}$ is a Q-fuzzy left ideal of S.
Conversely, suppose $\mu^{MT}_{\beta,\alpha}$ is a Q-fuzzy left ideal of S. Then for $x, y \in S,\ \gamma \in \Gamma,\ q \in Q$,

$$\begin{aligned}&\mu^{MT}_{\beta,\alpha}(x+y,q) \geq \min\{\mu^{MT}_{\beta,\alpha}(x,q),\mu^{MT}_{\beta,\alpha}(y,q)\}\\ &\Rightarrow \beta.\mu(x+y,q) + \alpha \geq \min\{\beta.\mu(x,q)+\alpha,\beta.\mu(y,q)+\alpha\}\\ &\Rightarrow \beta.\mu(x+y,q) + \alpha \geq \min\{\beta.\mu(x,q),\beta.\mu(y,q)\} + \alpha\\ &\Rightarrow \beta.\mu(x+y,q) \geq \beta.\min\{\mu(x,q),\mu(y,q)\}\\ &\Rightarrow \mu(x+y,q) \geq \min\{\mu(x,q),\mu(y,q)\}\end{aligned}$$

and

$$\begin{aligned}&\mu^{MT}_{\beta,\alpha}(x\gamma y,q)\geq\mu^{MT}_{\beta,\alpha}(y,q)\\&\Rightarrow\beta.\mu(x\gamma y,q)+\alpha\geq\beta.\mu(y,q)+\alpha\\&\Rightarrow\mu(x\gamma y,q)\geq\mu(y,q).\end{aligned}$$

Hence μ is a Q-fuzzy left ideal of S. □

Corollary 3.19. Let μ be a Q-fuzzy subset of S and $\alpha\in[0,1\text{-}\sup\{\mu(x):x\in X,\ q\in Q\}]$, $\beta\in(0,1]$. Then the following are equivalent

(i) μ is a Q-fuzzy left ideal of S

(ii) μ^T_α, the Q-fuzzy translation of μ, is a Q-fuzzy left ideal of S

(iii) μ^M_β, the Q-fuzzy multiplication of μ, is a Q-fuzzy left ideal of S.

Definition 3.20. A Q-fuzzy left ideal μ of an ordered Γ-semiring S, is said to be normal Q-fuzzy left ideal if there exists $x\in S,\ q\in Q$,such that $\mu(x,q)=1$.

Proposition 3.21. Given a Q-fuzzy left ideal μ of an ordered Γ-semiring S, let μ_+ be a Q-fuzzy set in S obtained by $\mu_+(x,q)=\mu(x,q)+1-\mu(0,q)$ for all $x\in S,\ q\in Q$. Then μ_+ is a normal Q-fuzzy left ideal of S, which contains μ.

Proof. For all $x,y\in S\ \gamma\in\Gamma,\ q\in Q$, we have $\mu_+(0,q)=\mu(0,q)+1-\mu(0,q)=1$. Now,

$$\begin{aligned}\mu_+(x+y,q)&=\mu(x+y,q)+1-\mu(0,q)\\&\geq\min\{\mu(x,q),\mu(y,q)\}+1-\mu(0,q)\\&=\min\{\{\mu(x,q)+1-\mu(0,q)\},\{\mu(y,q)+1-\mu(0,q)\}\}\\&=\min\{\mu_+(x),\mu_+(y)\}\end{aligned}$$

and

$$\mu_+(x\gamma y,q)=\mu(x\gamma y,q)+1-\mu(0,q)\geq\mu(y,q)+1-\mu(0,q)=\mu_+(y,q).$$

Suppose $x\leq y$. Then

$$\mu(x,q)\geq\mu(y,q)\Rightarrow\mu(x,q)+1-\mu(0,q)\geq\mu(y,q)+1-\mu(0,q)\Rightarrow\mu_+(x)\geq\mu_+(y).$$

Therefore, μ_+ is a normal Q-fuzzy left ideal of S and from definition of μ_+, $\mu\subseteq\mu_+$. □

Let $\mathcal{NQ}(S)$ denote the set of all normal Q-fuzzy left ideals of S. Then $\mathcal{NQ}(S)$ is a poset under inclusion.

Theorem 3.22. Let $\mu\in\mathcal{NQ}(S)$ be non-constant such that it is a maximal element of $(\mathcal{NQ}(S),\subseteq)$. Then μ takes only two values 0 and 1.

Proof. Since μ is normal, we have $\mu(0,q)=1$. Let $x_0(\neq0)\in S$ with $\mu(x_0,q)\neq1$. We claim that $\mu(x_0,q)=0$. If not, then $0<\mu(x_0)<1$. Define on S a Q-fuzzy set ν by $\nu(x,q)=\frac{1}{2}[\mu(x,q)+\mu(x_0,q)]$ for all $x\in S,q\in Q$. Then ν is well-defined and for all $x,y\in S$, $\gamma\in\Gamma$ and $q\in Q$ we have

$$\begin{aligned}\nu(x+y,q)&=\tfrac{1}{2}[\mu(x+y,q)+\mu(x_0,q)]\\&\geq\tfrac{1}{2}[\min[\mu(x,q),\mu(y,q)]+\mu(x_0,q)]\\&=\min[\tfrac{1}{2}[\mu(x,q)+\mu(x_0,q)],\tfrac{1}{2}[\mu(y,q)+\mu(x_0,q)]]\\&=\min[\nu(x,q),\nu(y,q)]\end{aligned}$$

and

$$\nu(x\gamma y,q)=\frac{1}{2}[\mu(x\gamma y,q)+\mu(x_0,q)]\geq\frac{1}{2}[\mu(y,q)+\mu(x_0,q)]=\nu(y,q).$$

Hence ν is a Q-fuzzy left ideal of S. Hence by Proposition 3.21, ν_+ is a normal Q-fuzzy left ideal of S. Now,

$$\nu_+(x,q)=\nu(x,q)+1-\nu(0,q)=\tfrac{1}{2}[\mu(x,q)+\mu(x_0,q)]+1-\tfrac{1}{2}[\mu(0,q)+\mu(x_0,q)]=\tfrac{1}{2}[\mu(x)+1]......(1)$$

In particular, $\nu_+(0,q)=\frac{1}{2}[\mu(0,q)+1]=1$ and $\nu_+(x_0,q)=\frac{1}{2}[\mu(x_0,q)+1]......(2)$.
From (1) we see that ν_+ is non-constant as μ is non-constant. From (2) we see that $\mu(x_0,q)<\nu_+(x_0,q)$. This violates the maximality of μ and so we get a contradiction. This completes the proof. □

Theorem 3.23. Let S be an ordered Γ-semiring. Then set of all Q-fuzzy ideals of S (in short $FI(S)$) is zerosumfree Γ-semiring with infinite element $\mathbf{1}$ under the operations of sum and composition of Q-fuzzy ideals of S.

Proof. Clearly $\phi \in FI(S)$. Suppose μ_1, μ_2, μ_3 to be three Q-fuzzy ideals of S. Then
(i) $\mu_1 +_1 \mu_2 \in FI(S)$,
(ii) $\mu_1 o_1 \mu_2 \in FI(S)$,
(iii) $\mu_1 +_1 \mu_2 = \mu_2 +_1 \mu_1$,
(iv) $\phi +_1 \mu_1 = \mu_1$,
(v) $\mu_1 +_1 (\mu_2 +_1 \mu_3) = (\mu_1 +_1 \mu_2) +_1 \mu_3$,
(vi) $\mu_1 o_1 (\mu_2 o_1 \mu_3) = (\mu_1 o_1 \mu_2) o_1 \mu_3$,
(vii) $\mu_1 o_1 (\mu_2 +_1 \mu_3) = (\mu_1 o_1 \mu_2) +_1 (\mu_1 o_1 \mu_3)$,
(viii) $(\mu_2 +_1 \mu_3) o_1 \mu_1 = (\mu_2 o_1 \mu_1) +_1 (\mu_3 o_1 \mu_1)$.
Also $\phi +_1 \mu_1 = \mu_1 +_1 \phi = \mu_1$.

Thus $FI(S)$ is a Γ-semiring under the operations of sum and composition of Q-fuzzy ideals of S.

Now $\mathbf{1} \subseteq \mathbf{1} +_1 \mu_1$ for $\mu_1 \in FI(S)$. Also

$(\mathbf{1} +_1 \mu)(x, q) = \sup\limits_{x \leq y+z} \{\min\{\mathbf{1}(y, q), \mu(z, q)\} : y, z \in S,\ q \in Q\} \leq 1 = \mathbf{1}(x, q)$ for all $x \in S,\ q \in Q$.

Therefore $\mathbf{1} +_1 \mu_1 \subseteq \mathbf{1}$ and hence $\mathbf{1} +_1 \mu_1 = \mathbf{1}$ for all $\mu_1 \in FI(S)$.

Thus $\mathbf{1}$ is an infinite element of $FI(S)$.

Next let $\mu_1 +_1 \mu_2 = \phi$ for $\mu_1, \mu_2 \in FI(S)$.

Then $\mu_1 \subseteq \mu_1 +_1 \mu_2 = \phi \subseteq \mu_1$ and so $\mu_1 = \phi$.

Similarly, it can be shown that $\mu_2 = \phi$.

Hence the Γ-semiring $FI(S)$ is zerosumfree. □

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RELATIONS ON FP-SOFT SETS APPLIED TO DECISION MAKING PROBLEMS

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Abstract – In this work, we first define relations on the fuzzy parametrized soft sets and study their properties. We also give a decision making method based on these relations. In approximate reasoning, relations on the fuzzy parametrized soft sets have shown to be of a primordial importance. Finally, the method is successfully applied to a problems that contain uncertainties.

Keywords – *Soft sets, fuzzy sets, FP-soft sets, relations on FP-soft sets, decision making.*

1 Introduction

In 1999, the concept of soft sets was introduced by Molodtsov [25] to deal with problems that contain uncertainties. After Molodtsov, the operations of soft sets are given in [4, 23, 28] and studied their properties. Since then, based on these operations, soft set theory has developed in many directions and applied to wide variety of fields. For instance; on the theory of soft sets [2, 4, 5, 9, 20, 23, 24, 28], on the soft decision making [16, 17, 18, 21, 22, 27], on the fuzzy soft sets [7, 10, 11] and soft rough sets [16] are some of the selected works. Some authors have also studied the algebraic properties of soft sets, such as [1, 3, 6, 19, 26, 29, 30].

The fuzzy parametrized soft sets (FP-soft sets), firstly studied by Çağman *et al.* [8], is a fuzzy parameterized soft sets. Then, FP-soft sets theory and its applications studied in detail, for example [12, 13, 14]. In this paper, after given most of the fundamental definitions of the operations of fuzzy sets, soft sets and FP-soft sets in next section, we define relations on FP-soft sets and we also give their properties in Section 3. In Section 4, we define symmetric, transitive and reflexive relations on the FP-soft sets. In Section 5, we construct a decision making method based on the FP-soft sets. We also give an application which shows that this methods successfully works. In the final section, some concluding comments are presented.

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2 Preliminary

In this section, we give the basic definitions and results of soft set theory [25] and fuzzy set theory [31] that are useful for subsequent discussions.

Definition 2.1. *[31] Let U be the universe. A fuzzy set X over U is a set defined by a membership function μ_X representing a mapping*

$$\mu_X : U \to [0, 1].$$

The value $\mu_X(x)$ for the fuzzy set X is called the membership value or the grade of membership of $x \in U$. The membership value represents the degree of x belonging to the fuzzy set X. Then a fuzzy set X on U can be represented as follows,

$$X = \{(\mu_X(x)/x) : x \in U, \mu_X(x) \in [0, 1]\}.$$

Note that the set of all fuzzy sets on U will be denoted by $F(U)$.

Definition 2.2. *[15] t-norms are associative, monotonic and commutative two valued functions t that map from $[0, 1] \times [0, 1]$ into $[0, 1]$. These properties are formulated with the following conditions:*

1. $t(0, 0) = 0$ *and* $t(\mu_{X_1}(x), 1) = t(1, \mu_{X_1}(x)) = \mu_{X_1}(x),\ x \in E$
2. *If* $\mu_{X_1}(x) \leq \mu_{X_3}(x)$ *and* $\mu_{X_2}(x) \leq \mu_{X_4}(x)$, *then* $t(\mu_{X_1}(x), \mu_{X_2}(x)) \leq t(\mu_{X_3}x), \mu_{X_4}(x))$
3. $t(\mu_{X_1}(x), \mu_{X_2}(x)) = t(\mu_{X_2}(x), \mu_{X_1}(x))$
4. $t(\mu_{X_1}(x), t(\mu_{X_2}(x), \mu_{X_3}(x))) = t(t(\mu_{X_1}(x), \mu_{X_2})(x), \mu_{X_3}(x))$

Definition 2.3. *[15] t-conorms or s-norm are associative, monotonic and commutative two placed functions s which map from $[0, 1] \times [0, 1]$ into $[0, 1]$. These properties are formulated with the following conditions:*

1. $s(1, 1) = 1$ *and* $s(\mu_{X_1}(x), 0) = s(0, \mu_{X_1}(x)) = \mu_{X_1}(x), x \in E$
2. *if* $\mu_{X_1}(x) \leq \mu_{X_3}(x)$ *and* $\mu_{X_2}(x) \leq \mu_{X_4}(x)$, *then* $s(\mu_{X_1}(x), \mu_{X_2}(x)) \leq s(\mu_{X_3}(x), \mu_{X_4}(x))$
3. $s(\mu_{X_1}(x), \mu_{X_2}(x)) = s(\mu_{X_2}(x), \mu_{X_1}(x))$
4. $s(\mu_{X_1}(x), s(\mu_{X_2}(x), \mu_{X_3}(x))) = s(s(\mu_{X_1}(x), \mu_{X_2})(x), \mu_{X_3}(x))$

t-norm and t-conorm are related in a sense of lojical duality. Typical dual pairs of non parametrized t-norm and t-conorm are complied below:

1. Drastic product:

$$t_w(\mu_{X_1}(x), \mu_{X_2}(x)) = \begin{cases} min\{\mu_{X_1}(x), \mu_{X_2}(x)\}, & max\{\mu_{X_1}(x)\mu_{X_2}(x)\} = 1 \\ 0, & otherwise \end{cases}$$

2. Drastic sum:

$$s_w(\mu_{X_1}(x), \mu_{X_2}(x)) = \begin{cases} max\{\mu_{X_1}(x), \mu_{X_2}(x)\}, & min\{\mu_{X_1}(x)\mu_{X_2}(x)\} = 0 \\ 1, & otherwise \end{cases}$$

3. Bounded product:

$$t_1(\mu_{X_1}(x), \mu_{X_2}(x)) = max\{0, \mu_{X_1}(x) + \mu_{X_2}(x) - 1\}$$

4. Bounded sum:

$$s_1(\mu_{X_1}(x), \mu_{X_2}(x)) = min\{1, \mu_{X_1}(x) + \mu_{X_2}(x)\}$$

5. Einstein product:

$$t_{1.5}(\mu_{X_1}(x), \mu_{X_2}(x)) = \frac{\mu_{X_1}(x).\mu_{X_2}(x)}{2 - [\mu_{X_1}(x) + \mu_{X_2}(x) - \mu_{X_1}(x).\mu_{X_2}(x)]}$$

6. Einstein sum:

$$s_{1.5}(\mu_{X_1}(x), \mu_{X_2}(x)) = \frac{\mu_{X_1}(x) + \mu_{X_2}(x)}{1 + \mu_{X_1}(x).\mu_{X_2}(x)}$$

7. Algebraic product:

$$t_2(\mu_{X_1}(x), \mu_{X_2}(x)) = \mu_{X_1}(x).\mu_{X_2}(x)$$

8. Algebraic sum:

$$s_2(\mu_{X_1}(x), \mu_{X_2}(x)) = \mu_{X_1}(x) + \mu_{X_2}(x) - \mu_{X_1}(x).\mu_{X_2}(x)$$

9. Hamacher product:

$$t_{2.5}(\mu_{X_1}(x), \mu_{X_2}(x)) = \frac{\mu_{X_1}(x).\mu_{X_2}(x)}{\mu_{X_1}(x) + \mu_{X_2}(x) - \mu_{X_1}(x).\mu_{X_2}(x)}$$

10. Hamacher sum:

$$s_{2.5}(\mu_{X_1}(x), \mu_{X_2}(x)) = \frac{\mu_{X_1}(x) + \mu_{X_2}(x) - 2.\mu_{X_1}(x).\mu_{X_2}(x)}{1 - \mu_{X_1}(x).\mu_{X_2}(x)}$$

11. Minumum:

$$t_3(\mu_{X_1}(x), \mu_{X_2}(x)) = min\{\mu_{X_1}(x), \mu_{X_2}(x)\}$$

12. Maximum:

$$s_3(\mu_{X_1}(x), \mu_{X_2}(x)) = max\{\mu_{X_1}(x), \mu_{X_2}(x)\}$$

Definition 2.4. *[25]. Let U be an initial universe set and let E be a set of parameters. Then, a pair (F, E) is called a soft set over U if and only if F is a mapping or E into the set of aft subsets of the set U.*

In other words, the soft set is a parametrized family of subsets of the set U. Every set $F(\varepsilon)$, $\varepsilon \in E$, from this family may be considered as the set of ε-elements of the soft set (F, E), or as the set of ε-approximate elements of the soft set.

It is worth noting that the sets $F(\varepsilon)$ may be arbitrary. Some of them may be empty, some may have nonempty intersection.

In this definition, E is a set of parameters that are describe the elements of the universe U. To apply the soft set in decision making subset $A, B, C, ...$ of the parameters set E are needed. Therefore, Çağman and Enginoğlu [4] modified the definition of soft set as follows.

Definition 2.5. *[4] Let U be a universe, E be a set of parameters that are describe the elements of U, and $A \subseteq E$. Then, a soft set F_A over U is a set defined by a set valued function f_A representing a mapping*

$$f_A : E \to P(U) \text{ such that } f_A(x) = \emptyset \text{ if } x \in E - A \tag{1}$$

where f_A is called approximate function of the soft set F_A. In other words, the soft set is a parametrized family of subsets of the set U, and therefore it can be written a set of ordered pairs

$$F_A = \{(x, f_A(x)) : x \in E, f_A(x) = \emptyset \text{ if } x \in E - A\}$$

The subscript A in the f_A indicates that f_A is the approximate function of F_A. The value $f_A(x)$ is a set called *x-element* of the soft set for every $x \in E$.

Definition 2.6. *[8] Let F_X be a soft set over U with its approximate function f_X and X be a fuzzy set over E with its membership function μ_X. Then, a $FP-$soft sets Γ_X, is a fuzzy parameterized soft set over U, is defined by the set of ordered pairs*

$$\Gamma_X = \{(\mu_X(x)/x, f_X(x)) : x \in E\}$$

where $f_X : E \to P(U)$ such that $f_X(x) = \emptyset$ if $\mu_X(x) = 0$ is called approximate function and $\mu_X : E \to [0, 1]$ is called membership function of $FP-$soft set Γ_X. The value $\mu_X(x)$ is the degree of importance of the parameter x and depends on the decision-maker's requirements.

Note that the sets of all $FP-$soft sets over U will be denoted by $FPS(U)$.

3 Relations on the FP-Soft Sets

In this section, after given the cartesian products of two FP-soft sets, we define a relations on FP-soft sets and study their desired properties.

Definition 3.1. *Let $\Gamma_X, \Gamma_Y \in FPS(U)$. Then, a cartesian product of Γ_X and Γ_Y, denoted by $\Gamma_X \widehat{\times} \Gamma_Y$, is defined as*

$$\Gamma_X \widehat{\times} \Gamma_Y = \Big\{ \big(\mu_{X \widehat{\times} Y}(x,y)/(x,y), f_{X \widehat{\times} Y}(x,y)\big) : (x,y) \in E \times E \Big\}$$

where

$$f_{X \widehat{\times} Y}(x,y) = f_X(x) \cap f_Y(y)$$

and

$$\mu_{X \widehat{\times} Y}(x,y) = min\{\mu_X(x), \mu_Y(y)\}$$

Here $\mu_{X \widehat{\times} Y}(x,y)$ is a t-norm.

Example 3.2. *Let $U = \{u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8, u_9, u_{10}, u_{11}, u_{12}, u_{13}, u_{14}, u_{15}\}$, $E = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8\}$, and $X = \{0.5/x_1, 0.7/x_2, 0.3/x_3, 0.9/x_4, 0.6/x_5\}$ and $Y = \{0.9/x_3, 0.1/x_6, 0.7/x_7, 0.3/x_8\}$ be two fuzzy subsets of E. Suppose that*

$$\begin{aligned} \Gamma_X = & \Big\{(0.5/x_1, \{u_1, u_3, u_4, u_6, u_7, u_8, u_{11}, u_{12}, u_{13}, u_{15}\}), (0.7/x_2, \{u_3, u_7, u_8, u_{14}, \\ & u_{15}\}), (0.3/x_3, \{u_1, u_2, u_4, u_5, u_6, u_9, u_{10}, u_{12}, u_{13}\}), (0.9/x_4, \{u_2, u_4, u_6, u_8, \\ & u_{12}, u_{13}\}), (0.6/x_5, \{u_3, u_4, u_6, u_7, u_9, u_{13}, u_{15}\})\Big\} \\ \Gamma_Y = & \Big\{(0.9/x_3, \{u_1, u_5, u_6, u_9, u_{10}, u_{13}\}), (0.1/x_6, \{u_3, u_5, u_7, u_8, u_9, u_{11}, u_{15}\}), \\ & (0.7/x_7, \{u_2, u_5, u_9, u_{10}, u_{11}, u_{14}\}), (0.3/x_8, \{u_2, u_5, u_8, u_{10}, u_{12}, u_{14}\})\Big\} \end{aligned}$$

Then, the cartesian product of Γ_X and Γ_Y is obtained as follows

$$\begin{aligned} \Gamma_X \widehat{\times} \Gamma_Y = & \Big\{(0.5/(x_1, x_3), \{u_1, u_6, u_{13}\}), (0.1/(x_1, x_6), \{u_3, u_7, u_8, u_{11}, u_{15}\}), \\ & (0.5/(x_1, x_7), \{u_{11}\}), (0.3/(x_1, x_8), \{u_8, u_{12}\}), (0.7/(x_2, x_3), \emptyset), \\ & (0.1/(x_2, x_6), \{u_3, u_7, u_8\}), (0.7(x_2, x_7), \{u_{14}\}), (0.3/(x_2, x_8), \\ & \{u_8, u_{14}\}), (0.3/(x_3, x_3), \{u_1, u_5, u_6, u_9, u_{10}, u_{13}\}), (0.1/(x_3, x_6), \\ & \{u_5, u_9\}), (0.3/(x_3, x_7), \{u_2, u_5, u_9, u_{10}\}), (0.3/(x_3, x_8), \\ & \{u_2, u_5, u_{10}, u_{12}\}), (0.9/(x_4, x_3), \{u_6\}), (0.1/(x_4, x_6), \emptyset), \\ & (0.7/(x_4, x_7), \{u_2, u_6\}), (0.3/(x_4, x_8), \{u_2, u_8, u_{12}\}), \\ & (0.6/(x_5, x_3), \{u_6, u_9, u_{13}\}), (0.1/(x_5, x_6), \{u_3, u_7, u_9, u_{11}, u_{15}\}), \\ & (0.6/(x_5, x_7), \{u_2\}), (0.3/(x_5, x_8), \emptyset)\Big\} \end{aligned}$$

Definition 3.3. *Let $\Gamma_X, \Gamma_Y \in FPS(U)$. Then, an FP-soft relation from Γ_X to Γ_Y, denoted by R_F, is an FP-soft subset of $\Gamma_X \widehat{\times} \Gamma_Y$. Any FP-soft subset of $\Gamma_X \times \Gamma_Y$ is called a FP-relation on Γ_X.*

Note that if $\alpha = (\mu_X(x), f_X(x)) \in \Gamma_X$ and $\beta = (\mu_Y(y), f_Y(y)) \in \Gamma_Y$, then

$$\alpha R_F \beta \Leftrightarrow \big(\mu_{X \widehat{\times} Y}(x,y)/(x,y), f_{X \widehat{\times} Y}(x,y)\big) \in R_F$$

Example 3.4. *Let us consider the Example 3.2. Then, we define an FP-soft relation R_F, from Γ_Y to Γ_X, as follows*

$$\alpha R_F \beta \Leftrightarrow \mu_{X \widehat{\times} Y}(x_i, x_j)/(x_i, x_j)) \geq 0.3 \quad (1 \leq i, j \leq 3)$$

Then

$$\begin{aligned} R_F = & \Big\{(0.5/(x_1, x_3), \{u_1, u_6, u_{13}\}), ((0.5/(x_1, x_7), \{u_{11}\}), (0.3/(x_1, x_8), \{u_8, \\ & u_{12}\}), (0.7(x_2, x_7), \{u_{14}\}), (0.3/(x_2, x_8), \{u_8, u_{14}\}), (0.3/(x_3, x_3), \{u_1, \\ & u_5, u_6, u_9, u_{10}, u_{13}\}), (0.3/(x_3, x_7), \{u_2, u_5, u_9, u_{10}\}), (0.3/(x_3, x_8), \{u_2, \\ & u_5, u_{10}, u_{12}\}), (0.9/(x_4, x_3), \{u_6\}), (0.7/(x_4, x_7), \{u_2, u_6\}), (0.3/(x_4, x_8), \\ & \{u_2, u_8, u_{12}\}), (0.6/(x_5, x_3), \{u_6, u_9, u_{13}\}), (0.6/(x_5, x_7), \{u_2\})\Big\} \end{aligned}$$

Definition 3.5. *Let $\Gamma_X, \Gamma_Y \in FPS(U)$ and R_F be an FP-soft relation from Γ_X to Γ_Y. Then domain and range of R_F respectively is defined as*

$$\begin{array}{rcl} D(R_F) & = & \{\alpha \in F_A : \alpha R_F \beta\} \\ R(R_F) & = & \{\beta \in F_B : \alpha R_F \beta\}. \end{array}$$

Example 3.6. *Let us consider the Example 3.4.*

$$D(R_F) = \Big\{(0.5/x_1, \{u_1, u_3, u_4, u_6, u_7, u_8, u_{11}, u_{12}, u_{13}, u_{15}\}), (0.7/x_2, \{u_3, u_7, u_8, u_{14}, u_{15}\}), (0.3/x_3, \{u_1, u_2, u_4, u_5, u_6, u_9, u_{10}, u_{12}, u_{13}\}), (0.9/x_4, \{u_2, u_4, u_6, u_8, u_{12}, u_{13}\}), (0.6/x_5, \{u_3, u_4, u_6, u_7, u_9, u_{13}, u_{15}\})\Big\}$$

$$R(R_F) = \Big\{(0.9/x_3, \{u_1, u_5, u_6, u_9, u_{10}, u_{13}\}), (0.7/x_7, \{u_2, u_5, u_9, u_{10}, u_{11}, u_{14}\}), (0.3/x_8, \{u_2, u_5, u_8, u_{10}, u_{12}, u_{14}\})\Big\}$$

Definition 3.7. *Let R_F be an FP-soft relation from Γ_X to Γ_Y. Then R_F^{-1} is from Γ_Y to Γ_X is defined as*

$$\alpha R_F^{-1} \beta = \beta R_F \alpha$$

Example 3.8. *Let us consider the Example 3.4. Then, ${R_F}^{-1}$ is from Γ_Y to Γ_X is obtained by*

$${R_F}^{-1} = \Big\{(0.5/(x_3, x_1), \{u_1, u_6, u_{13}\}), ((0.5/(x_7, x_1), \{u_{11}\}), (0.3/(x_8, x_1), \{u_8, u_{12}\}), (0.7(x_7, x_2), \{u_{14}\}), (0.3/(x_8, x_2), \{u_8, u_{14}\}), (0.3/(x_3, x_3), \{u_1, u_5, u_6, u_9, u_{10}, u_{13}\}), (0.3/(x_7, x_3), \{u_2, u_5, u_9, u_{10}\}), (0.3/(x_8, x_3), \{u_2, u_5, u_{10}, u_{12}\}), (0.9/(x_3, x_4), \{u_6\}), (0.7/(x_7, x_4), \{u_2, u_6\}), (0.3/(x_8, x_4), \{u_2, u_8, u_{12}\}), (0.6/(x_3, x_5), \{u_6, u_9, u_{13}\}), (0.6/(x_7, x_5), \{u_2\})\Big\}$$

Proposition 3.9. *Let R_{F_1} and R_{F_2} be two FP-soft relations. Then*

1. $(R_{F_1}^{-1})^{-1} = R_{F_1}$
2. $R_{F_1} \subseteq R_{F_2} \Rightarrow R_{F_1}^{-1} \subseteq R_{F_2}^{-1}$

Proof:

1. $\alpha(R_{F_1}^{-1})^{-1}\beta = \beta R_{F_1}^{-1} \alpha = \alpha R_{F_1} \beta$
2. $\alpha R_{F_1} \beta \subseteq \alpha R_{F_2} \beta \Rightarrow \beta R_{F_1}^{-1} \alpha \subseteq \beta R_{F_2}^{-1} \alpha \Rightarrow R_{F_1}^{-1} \subseteq R_{F_2}^{-1}$

Definition 3.10. *If R_{F_1} is a fuzzy parametrized soft relation from Γ_X to Γ_Y and R_{F_2} is a fuzzy parametrized soft relation from Γ_Y to Γ_Z, then a composition of two FP-soft relations R_{F_1} and R_{F_2} is defined by*

$$\alpha(R_{F_1} \circ R_{F_2})\gamma = (\alpha R_{F_1} \beta) \wedge (\beta R_{F_2} \gamma)$$

Proposition 3.11. *Let R_{F_1} and R_{F_2} be two FP-soft relation from Γ_X to Γ_Y. Then, $(R_{F_1} \circ R_{F_2})^{-1} = R_{F_2}^{-1} \circ R_{F_1}^{-1}$*

Proof:

$$\begin{aligned} \alpha(R_{F_1} \circ R_{F_2})^{-1})\gamma &= \gamma(R_{F_1} \circ R_{F_2})\alpha \\ &= (\gamma R_{F_1} \beta) \wedge (\beta R_{F_2} \alpha) \\ &= (\beta R_{F_2} \alpha) \wedge (\gamma R_{F_1} \beta) \\ &= (\alpha R_{F_2}^{-1} \beta) \wedge (\beta R_{F_1}^{-1} \gamma) \\ &= \alpha(R_{F_2}^{-1} \circ R_{F_1}^{-1})\gamma \end{aligned}$$

Therefore we obtain

$(R_{F_1} \circ R_{F_2})^{-1} = R_{F_2}^{-1} \circ R_{F_1}^{-1}$

Definition 3.12. *An FP-soft relation R_F on Γ_X is said to be an FP-soft symmetric relation if* $\alpha R_{F_1}\beta \Rightarrow \beta R_{F_1}\alpha, \forall\alpha, \beta \in \Gamma_X$.

Definition 3.13. *An FP-soft relation R_F on Γ_X is said to be an FP-soft transitive relation if $R_F \circ R_F \subseteq R_F$, that is, $\alpha R_F\beta$ and $\beta R_F\gamma \Rightarrow \alpha R_F\gamma, \forall\alpha, \beta, \gamma \in \Gamma_X$.*

Definition 3.14. *An FP-soft relation R_F on Γ_X is said to be an FP-soft reflexive relation if $\alpha R_F\alpha, \forall\alpha \in \Gamma_X$.*

Definition 3.15. *An FP-soft relation R_F on Γ_X is said to be an FP-soft equivalence relation if it is symmetric, transitive and reflexive.*

Example 3.16. *Let $U = \{u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8\}$, $E = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8\}$ and $X = \{0.5/x_1, 0.7/x_2, 0.3/x_3\}$ be a fuzzy subsets over E. Suppose that*

$$\Gamma_X = \Big\{(0.5/x_1, \{u_1, u_3, u_4, u_6, u_7, u_8\}), (0.7/x_2, \{u_3, u_7, u_8\}), (0.3/x_3, \{u_1, u_2, u_4, u_5, u_6, u_9\})\Big\}$$

Then, a cartesian product on Γ_X is obtained as follows

$$\begin{aligned}\Gamma_X \hat{\times} \Gamma_X = \Big\{&(0.5/(x_1, x_1), \{u_1, u_3, u_4, u_6, u_7, u_8\}),\\ &(0.5/(x_1, x_2), \{u_3, u_7, u_8\}), (0.3/(x_1, x_3), \{u_1, u_4, u_6\}),\\ &(0.5/(x_2, x_1), \{u_3, u_7, u_8\}), (0.7/(x_2, x_2), \{u_3, u_7, u_8\}),\\ &(0.3/(x_3, x_1), \{u_1, u_4, u_6\}), (0.3/(x_3, x_3), \{u_1, u_2, u_4, u_5, u_6, u_9\})\Big\}\end{aligned}$$

Then, we get a fuzzy parametrized soft relation R_F on F_X as follows

$$\alpha R_F\beta \Leftrightarrow \mu_{X\hat{\times}Y}(x_i, x_j)/(x_i, x_j)) \geq 0.3 \quad (1 \leq i, j \leq 3)$$

Then

$$\begin{aligned}R_F = \Big\{&(0.5/(x_1, x_1), \{u_1, u_3, u_4, u_6, u_7, u_8\}), (0.5/(x_1, x_2), \{u_3, u_7, u_8\}),\\ &(0.3/(x_1, x_3), \{u_1, u_4, u_6\}), (0.5/(x_2, x_1), \{u_3, u_7, u_8\}),\\ &(0.7/(x_2, x_2), \{u_3, u_7, u_8\}), (0.3/(x_3, x_1), \{u_1, u_4, u_6\}),\\ &(0.3/(x_3, x_3), \{u_1, u_2, u_4, u_5, u_6, u_9\})\Big\}\end{aligned}$$

R_F on Γ_X is an FP-soft equivalence relation because it is symmetric, transitive and reflexive.

Proposition 3.17. *If R_F is symmetric if and only if R_F^{-1} is so.*

Proof: If R_F is symmetric, then $\alpha R_F^{-1}\beta = \beta R_F\alpha = \alpha R_F\beta = \beta R_F^{-1}\alpha$. So, R_F^{-1} is symmetric.
Conversely, if R_F^{-1} is symmetric, then $\alpha R_F\beta = \alpha(R_F^{-1})^{-1}\beta = \beta(R_F^{-1})\alpha = \alpha(R_F^{-1})\beta = \beta R_F\alpha$ So, R_F is symmetric.

Proposition 3.18. *R_F is symmetric if and only if $R_F^{-1} = R_F$*

Proof: If R_F is symmetric, then $\alpha R_F^{-1}\beta = \beta R_F\alpha = \alpha R_F\beta$. So, $R_F^{-1} = R_F$.
Conversely, if $R_F^{-1} = R_F$, then $\alpha R_F\beta = \alpha R_F^{-1}\beta = \beta R_F\alpha$. So, R_F is symmetric.

Proposition 3.19. *If R_{F_1} and R_{F_2} are symmetric relations on Γ_X, then $R_{F_1} \circ R_{F_2}$ is symmetric on Γ_X if and only if $R_{F_1} \circ R_{F_2} = R_{F_2} \circ R_{F_1}$*

Proof: If R_{F_1} and R_{F_2} are symmetric, then it implies $R_{F_1}^{-1} = R_{F_1}$ and $R_{F_2}^{-1} = R_{F_2}$. We have $(R_{F_1} \circ R_{F_2})^{-1} = R_{F_2}^{-1} \circ R_{F_1}^{-1}$. then $R_{F_1} \circ R_{F_2}$ is symmetric. It implies $R_{F_1} \circ R_{F_2} = (R_{F_1} \circ R_{F_2})^{-1} = R_{F_2}^{-1} \circ R_{F_1}^{-1} = R_{F_2} \circ R_{F_1}$.
Conversely, $(R_{F_1} \circ R_{F_2})^{-1} = R_{F_2}^{-1} \circ R_{F_1}^{-1} = R_{F_2} \circ R_{F_1} = R_{F_1} \circ R_{F_2}$. So, $R_{F_1} \circ R_{F_2}$ is symmetric.

Corollary 3.20. *If R_F is symmetric, then R_F^n is symmetric for all positive integer n, where*

$$R_F^n = \underbrace{R_F \circ R_F \circ ... \circ R_F}_{n\ times}$$

Proposition 3.21. *If R_F is transitive, then R_F^{-1} is also transitive.*

Proof:

$$\begin{aligned} \alpha R_F^{-1}\beta &= \beta R_F \alpha \supseteq \beta(R_F \circ R_F)\alpha \\ &= (\beta R_F \gamma) \wedge (\gamma R_F \alpha) \\ &= (\gamma R_F \alpha) \wedge (\beta R_F \gamma) \\ &= (\alpha R_F^{-1}\gamma) \wedge (\gamma R_F^{-1}\beta) \\ &= \alpha(R_F^{-1} \circ R_F^{-1})\beta \end{aligned}$$

So, $R_F^{-1} \circ R_F^{-1} \subseteq R_F^{-1}$. The proof is completed.

Proposition 3.22. *If R_F is transitive then $R_F \circ R_F$ is so.*

Proof:

$$\begin{aligned} \alpha(R_F \circ R_F)\beta &= (\alpha R_F \gamma) \wedge (\gamma \text{-} R_F \beta) \\ &= \alpha(R_F \circ R_F)\gamma \wedge \gamma(R_F \circ R_F)\beta \\ &= \alpha(R_F \circ R_F \circ R_F \circ R_F)\beta \end{aligned}$$

So, $\alpha(R_F \circ R_F \circ R_F \circ R_F)\beta \subseteq \alpha(R_F \circ R_F)\beta$. The proof is completed.

Proposition 3.23. *If R_F is reflexive then R_F^{-1} is so.*

Proof: $\alpha R_F^{-1}\beta = \beta R_F \alpha \subseteq \alpha R_F \alpha = \alpha R_F^{-1}\alpha$ and $\beta R_F^{-1}\alpha = \alpha R_F \beta \subseteq \alpha R_F \alpha = \alpha R_F^{-1}\alpha$. The proof is completed.

Proposition 3.24. *If R_F is symmetric and transitive, then R_F is reflexive.*

Proof: Proof can be made easily by using Definition 4.1, Definition 4.2 and Definition 4.3.

Definition 3.25. *Let $\Gamma_X \in FPS(U)$, R_F be an FP-soft equivalence relation on Γ_X and $\alpha \in R_F$. Then, an equivalence class of α, denoted by $[\alpha]_{R_F}$, is defined as*

$$[\alpha]_{R_F} = \{\beta : \alpha R_F \beta\}.$$

Example 3.26. *Let us consider the Example 3.16. Then an equivalence class of $(x_1, \{u_1, u_3, u_4, u_6, u_7, u_8\})$ will be as follows.*

$$[(0.5/x_1, \{u_1, u_3, u_4, u_6, u_7, u_8\})]_{R_F} = \Big\{(0.5/x_1, \{u_1, u_3, u_4, u_6, u_7, u_8\}), (0.7/x_2, \{u_3, u_7, u_8\}), (0.3/x_3, \{u_1, u_2, u_4, u_5, u_6, u_9\})\Big\}$$

4 Decision Making Method

In this section, we construct a soft fuzzification operator and a decision making method on FP-soft relations.

Definition 4.1. *Let $\Gamma_X \in FPS(U)$ and R_F be a FP-soft relation on Γ_X. Then fuzzification operator, denoted by s_{R_F}, is defined by*

$$s_{R_F} : R_F \to F(U), \quad s_{R_F}(X \times X, U) = \{\mu_{R_F}(u)/u : u \in U\}$$

where

$$\mu_{R_F}(u) = \frac{1}{|X \times X|} \sum_j \sum_i \mu_{R_F}(x_i, x_j)\chi(u)$$

and where

$$\chi(u) = \begin{cases} 1, & u \in f_{R_F}(x_i, x_j) \\ 0, & u \notin f_{R_F}(x_i, x_j) \end{cases}$$

Note that $|X \times X|$ is the cardinality of $X \times X$.

Now; we can construct a decision making method on FP-soft relation by the following algorithm;

1. construct a feasible fuzzy subset X over E,
2. construct a FP-soft set Γ_X over U,
3. construct a FP-soft relation R_F over Γ_X according to the requests,
4. calculate the fuzzification operator s_{R_F} over R_F,
5. select the objects, from s_{R_F}, which have the largest membership value.

Example 4.2. *A customer, Mr. X, comes to the auto gallery agent to buy a car which is over middle class. Assume that an auto gallery agent has a set of different types of car* $U = \{u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8\}$, *which may be characterized by a set of parameters* $E = \{x_1, x_2, x_3, x_4\}$. *For* $i = 1, 2, 3, 4$ *the parameters* x_i *stand for "safety", "cheap", "modern" and "large", respectively. If Mr. X has to consider own set of parameters, then we select a car on the basis of the set of customer parameters by using the algorithm as follows.*

1. *Mr X constructs a fuzzy sets* X *over* E,
   $X = \{0.5/x_1, 0.7/x_2, 0.3/x_3\}$
2. *Mr X constructs a FP-soft set* Γ_X *over* U,
   $$\Gamma_X = \{(0.5/x_1, \{u_1, u_3, u_4, u_6, u_7, u_8\}), (0.7/x_2, \{u_3, u_7, u_8\}), (0.3/x_3, \{u_1, u_2, u_4, u_5, u_6, u_9\}$$
3. *the fuzzy parametrized soft relation* R_F *over* Γ_X *is calculated according to the Mr X's requests (The car must be a over middle class, it means the membership degrees are over 0.5),*
   $$R_F = \Big\{(0.5/(x_1, x_1), \{u_1, u_3, u_4, u_6, u_7, u_8\}), (0.5/(x_1, x_2), \{u_3, u_7, u_8\}), (0.5/(x_2, x_1), \{u_3, u_7, u_8\}), (0.7/(x_2, x_2), \{u_3, u_7, u_8\})\Big\}$$
4. *the soft fuzzification operator* s_{R_F} *over* R_F *is calculated as follows*
   $$s_{R_F} = \Big\{(0.055/u_1, 0.0/u_2, 0.244/u_3, 0.055/u_4, 0.0/u_5, 0.055/u_6, 0.244/u_7, 0.244/u_8\}\Big\}$$
5. *now, select the optimum alternative objects* u_3, u_7 *and* u_8 *which have the biggest membership degree 0.244 among the others.*

5 Conclusion

We first gave most of the fundamental definitions of the operations of fuzzy sets, soft sets and FP-soft sets are presented. We then defined relations on FP-soft sets and studied some of their properties. We also defined symmetric, transitive and reflexive relations on the FP-soft sets. Finally, we construct a decision making method and gave an application which shows that this method successfully works. We have used a t-norm, which is minimum operator, the above relation. However, application areas the relations can be expanded using the above other norms in the future.

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