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NEW SUPRA TOPOLOGIES FROM OLD VIA IDEALS

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Abstract – In this paper, we define a supra topology obtained as an associated structure on a supra topological space (X, τ) induced by an ideal on X. Such a supra topology is studied in certain detail as to some of it is basic properties.

Keywords – *Ideals, Local function, Supra topology, Supra topological space.*

1 Introduction

The concept of ideal in topological space was first introduced by Kuratowski [4] and Vaidyanathswamy[9]. They also have defined local function in ideal topological space. Further Hamlett and Jankovic [2] studied the properties of ideal topological spaces and they have introduced another operator called Ψ- operator. They have also obtained a new topology from original ideal topological space. Using the local function, they defined a Kuratowski Closure operator in new topological space. Further, they showed that interior operator of the new topological space can be obtained by Ψ - operator. In [7], the authors introduced two operators $()^{*s}$ and ψ_τ in supra topological space. Mashhour et al[6] introduced the notion of supra topological space. El-Sheikh [1] studied the properties of supra topological space and he introduced the notion of supra closure operator which is generated a supra topological space. In this paper, we introduced a new supra topology from old via ideal. Further we have discussed the properties of this supra topology.

2 Preliminary

Definition 2.1. *[6] Let X be a nonempty set. A class τ of subsets of X is said to be a supra topology on X if it satisfies the following axioms:-*

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1. $X, \emptyset \in \tau$.
2. *The arbitrary union of members of* τ *is in* τ.

The members of τ are then called supra-open sets(s-open, for short). The pair (X, τ) is called a supra topological space. A subset A of a topological space (X, τ) is called a supra-closed set(s-closed, for short) if its complement A^c is an s-open set. The family of all s-closed sets is denoted by $\tau^c = \{F : F^c \in \tau\}$.

Definition 2.2. *[6] Let* (X, τ) *be a supra topological space and* $A \subseteq X$. *Then*

1. $Scl_\tau(A) = \cap\{F \in \tau^c : A \subseteq F\}$ *is called the supra-closure of* $A \in P(X)$.
2. $Sint_\tau(A) = \cup\{M \in \tau : M \subseteq A\}$ *is called the supra-interior of* $A \in P(X)$.

Definition 2.3. *[6] Let* (X, τ) *be a supra topological space and* $x \in X$ *be an arbitrary point. A set* $M \subseteq X$ *is called a supra-neighborhood (s-nbd, for short)of* x *if* $x \in M \in \tau$. *The family of all s-neighborhood of* x *is denoted by* $\tau(x) = \{M \subseteq X : x \in M \in \tau\}$. *We write* M_x *stands for the s-nbd of* x.

Theorem 2.1. *[6] Let* (X, τ) *be a supra topological space and* $A \subseteq X$. *Then*

(a) $x \in Scl_\tau(A) \Leftrightarrow M_x \cap A \neq \phi \; \forall M_x \in \tau(x)$.

(b) $[Sint_\tau(A^c)]^c = Scl_\tau(A)$.

Definition 2.4. *[6] Let* τ_1 *and* τ_2 *be two supra topologies on a set* X *such that* $\tau_1 \subseteq \tau_2$. *Then we say that* τ_2 *is stronger (finer) than* τ_1 *or* τ_1 *is weaker (coarser) than* τ_2.

Definition 2.5. *[6] Let* (X, τ) *be a supra topological space and* $\beta \subseteq \tau$. *Then* β *is called a base for the supra topology* τ *(s-base, for short) if every s-open set* $M \in \tau$ *is a union of members of* β.
Equivalently, β *is a supra-base for* τ *if for any point* p *belonging to a s-open set* M, *there exists* $B \in \beta$ *with* $p \in B \subseteq M$.

Definition 2.6. *[6] A mapping* $c : P(X) \to P(X)$ *is said to be a supra closure operator if it satisfies the following axioms:*

1. $c(\phi) = \phi$,
2. $A \subseteq c(A) \;\; \forall \; A \in P(X)$,
3. $c(A) \cup c(B) \subseteq c(A \cup B) \;\; \forall \; A, B \in P(X)$.
4. $c(c(A)) = c(A) \; \forall \; A \in P(X)$. *"idempotent condition",*

Theorem 2.2. *[6] Let* X *be a nonempty set and let the mapping* $c : P(X) \to P(X)$ *be a supra closure operator. Then*
the collection

$$\tau = \{G \subseteq X : c(G^c) = G^c\}$$

is a supra topology on X *induced by the supra closure operator* c.

Definition 2.7. *[7] Let* (X, τ) *be a supra topological space with an ideal* $\mathcal{I}$ *on* X. *Then*

$$A^{*^s}(\mathcal{I}) = \{x \in X : M_x \cap A \notin \mathcal{I} \;\; \forall M_x \in \tau(x)\}, \forall A \in P(X)$$

is called the supra-local function(s-local function, for short) of A *with respect to* $\mathcal{I}$ *and* τ*(here and henceforth also,* A^{*^s} *stands for* $A^{*^s}(\mathcal{I})$*).*

Theorem 2.3. *[7] Let* (X, τ) *be a supra topological space with ideals* $\mathcal{I}$ *and* $\mathcal{J}$ *on* X *and let* A *and* B *be two subsets of* X. *Then*

1. $\phi^{*^s} = \phi$.

2. $A \subseteq B \Rightarrow A^{*^s} \subseteq B^{*^s}$,
3. $\mathcal{I} \subseteq \mathcal{J} \Rightarrow A^{*^s}(\mathcal{J}) \subseteq A^{*^s}(\mathcal{I})$,
4. $A^{*^s} = Scl_\tau(A^{*^s}) \subseteq Scl_\tau(A)$,
5. $(A^{*^s})^{*^s} \subseteq A^{*^s}$,
6. $A^{*^s} \cup B^{*^s} \subseteq (A \cup B)^{*^s}$,
7. $(A \cap B)^{*^s} \subseteq A^{*^s} \cap B^{*^s}$
8. $M \in \tau \Rightarrow M \cap A^{*^s} = M \cap (M \cap A)^{*^s} \subseteq (M \cap A)^{*^s}$,
9. $H \in \mathcal{I} \Rightarrow (A \cup H)^{*^s} = A^{*^s} = (A \backslash H)^{*^s}$.

3 New Supra Topologies From Old via Ideals

In this section, we generate a supra topology obtained as an associated structure on a supra topological space (X, τ), induced by an ideal on X. Such a supra topology is studied in certain details as to some of its basic properties.

Lemma 3.1. *Let (X, τ) be a supra topological space, $A \subseteq X$ and $\mathcal{I}$ be an ideal on X. Then $M \in \tau$, $M \cap A \in \mathcal{I} \Rightarrow M \cap A^{*^s} = \phi$.*

Proof. *Let $x \in M \cap A^{*^s}$.Then $x \in M$, $x \in A^{*^s} \Rightarrow M_x \cap A \notin \mathcal{I}\ \forall M_x \in \tau(x)$. Since $x \in M \in \tau$, then $M \cap A \notin \mathcal{I}$.* ■

Lemma 3.2. *Let (X, τ) be a supra topological space and $\mathcal{I}$ be an ideal on X. Then $(A \cup A^{*^s})^{*^s} \subseteq A^{*^s}\ \forall A \in P(X)$.*

Proof. *Let $x \notin A^{*^s}$. Then there exists $M_x \in \tau(x)$ such that $M_x \cap A \in \mathcal{I} \Rightarrow M_x \cap A^{*^s} = \phi$ (By Lemma 3.1). Hence, $M_x \cap (A \cup A^{*^s}) = (M_x \cap A) \cup (M_x \cap A^{*^s}) = M_x \cap A \in \mathcal{I}$. Therefore, $x \notin (A \cup A^{*^s})^{*^s}$. Hence, $(A \cup A^{*^s})^{*^s} \subseteq A^{*^s}$.* ■

Theorem 3.1. *Let (X, τ) be a supra topological space and $\mathcal{I}$ be an ideal on X. Then the operator*

$$cl_{\mathcal{I}}^{*^s} : P(X) \rightarrow P(X)$$

defined by

$$cl_{\mathcal{I}}^{*^s}(A) = A \cup A^{*^s}\ \forall A \in P(X),$$

is a supra closure operator and hence it generates a supra topology

$$\tau^*(\mathcal{I}) = \{A \in P(X) : cl_{\mathcal{I}}^{*^s}(A^c) = A^c\}$$

which is finer than τ.
When there is no ambiguity we will write cl^{*^s} for $cl_{\mathcal{I}}^{*^s}$ and τ^* for $\tau^*(\mathcal{I})$.

Proof. *(i) By Theorem 2.3, $\phi^{*^s} = \phi$, we have $cl^{*^s}(\phi) = \phi$*

(ii) Clear that, $A \subseteq cl^{*^s}(A)\ \forall A \in P(X)$.

(iii) Let $A, B \in P(X)$. Then, $cl^{*^s}(A) \cup cl^{*^s}(B) = (A \cup A^{*^s}) \cup (B \cup B^{*^s}) = (A \cup B) \cup (A^{*^s} \cup B^{*^s}) \subseteq (A \cup B) \cup (A \cup B)^{*^s} = cl^{*^s}(A \cup B)$ (by using Theorem 2.3). Hence, $cl^{*^s}(A) \cup cl^{*^s}(B) \subseteq cl^{*^s}(A \cup B)$.

(iv) Let $A \in P(X)$. Since, by (ii), $A \subseteq cl^{*^s}(A)$, then $cl^{*^s}(A) \subseteq cl^{*^s}(cl^{*^s}(A))$. On the other hand, $cl^{*^s}(cl^{*^s}(A)) = cl^{*^s}(A \cup A^{*^s}) = (A \cup A^{*^s}) \cup (A \cup A^{*^s})^{*^s} \subseteq A \cup A^{*^s} \cup A^{*^s} = cl^{*^s}(A)$(by Lemma 3.2), it follows that $cl^{*^s}(cl^{*^s}(A)) \subseteq cl^{*^s}(A)$. Hence $cl^{*^s}(cl^{*^s}(A)) = cl^{*^s}(A)$. Consequently, cl^{*^s} is a supra closure operator. Also, it is easy to show that the collection $\tau^*(\mathcal{I}) = \{A \in P(X) : cl^{*^s}(A^c) = A^c\}$ is a supra topology on X which is called the supra topology induced by the supra closure operator.
Next, from Theorem 2.3(4) we have $A^{*^s} \subseteq Scl_\tau(A) \Rightarrow A \cup A^{*^s} \subseteq A \cup Scl_\tau(A) = Scl_\tau(A) \Rightarrow cl^{*^s}(A) \subseteq Scl_\tau(A)$. Hence $\tau \subseteq \tau^*(\mathcal{I})$. ■

Example 3.1. *Let (X,τ) be a supra topological space. If $\mathcal{I}=\{\phi\}$, then $\tau=\tau^*(\mathcal{I})$. In fact, if $x\in Scl(A)$, then, (by Theorem 2.1(a)), $M_x\cap A\neq\phi\ \forall M_x\in\tau(x)\Rightarrow M_x\cap A\notin\{\phi\}=\mathcal{I}\ \forall M_x\in\tau(x)\Rightarrow x\in A^{*^s}\Rightarrow x\in A\cup A^{*^s}=cl^{*^s}(A)$. Hence $Scl(A)\subseteq cl^{*^s}(A)$, but, by Theorem 3.1 , $cl^{*^s}(A)\subseteq Scl_\tau(A)$. Hence $cl^{*^s}(A)=Scl_\tau(A)\ \forall A\in P(X)$. Consequently, $\tau=\tau^*(\mathcal{I})=\tau^*(\{\phi\})$.* ■

Theorem 3.2. *Let (X,τ) be a supra topological space and let $\mathcal{I}_1$, $\mathcal{I}_2$ be two ideals on X. Then If $\mathcal{I}_1\subseteq\mathcal{I}_2$, then $\tau^*(\mathcal{I}_1)\subseteq\tau^*(\mathcal{I}_2)$.*

Proof. *Let $M\in\tau^*(\mathcal{I}_1)$. Then $cl^{*^s}_{\mathcal{I}_1}(M^c)=M^c\Rightarrow M^c=M^c\cup M^{c*^s}(\mathcal{I}_1)\Rightarrow M^{c*^s}(\mathcal{I}_1)\subseteq M^c\Rightarrow M^{c*^s}(\mathcal{I}_2)\subseteq M^c$ (by Theorem 2.3) implies $M^c=M^c\cup M^{c*^s}(\mathcal{I}_2)\Rightarrow cl^{*^s}_{\mathcal{I}_2}(M^c)=M^c\Rightarrow M\in\tau^*(\mathcal{I}_2)$.* ■

Theorem 3.3. *Let (X,τ) be a supra topological space and let $\mathcal{I}$ be an ideal on X. Then*

(1) $H\in\mathcal{I}\Rightarrow H^c\in\tau^*(\mathcal{I})$.

(2) $A^{*^s}=cl^{*^s}(A^{*^s})\ \forall A\in P(X)$, *i.e.* A^{*^s} *is a* $\tau^*(\mathcal{I})$*-closed* $\forall A\in P(X)$.

Proof. (1) *In Theorem 2.3(9), put $A=\phi\Rightarrow H^{*^s}=\phi\ \forall H\in\mathcal{I}$. Hence $cl^{*^s}(H)=H\cup\phi=H\Rightarrow H^c\in\tau^*(\mathcal{I})$ i.e. H is a $\tau^*(\mathcal{I})$-closed $\forall H\in\mathcal{I}$.*
(2) *From Theorem 2.3(5), we have $(A^{*^s})^{*^s}\subseteq A^{*^s}\Rightarrow A^{*^s}=A^{*^s}\cup(A^{*^s})^{*^s}=cl^{*^s}(A^{*^s})$. Hence A^{*^s} is a $\tau^*(\mathcal{I})$-closed.* ■

Lemma 3.3. *Let (X,τ) be a supra topological space and $\mathcal{I}$ be an ideal on X. Then F is a τ^*-closed if and only if $F^{*^s}\subseteq F$.*

Proof. *Straightforward.* ■

Theorem 3.4. *Let (X,τ_1) and (X,τ_2) be two supra topological spaces and $\mathcal{I}$ be an ideal on X. Then*

$$\tau_1\subseteq\tau_2\Rightarrow A^{*^s}(\mathcal{I},\tau_2)\subseteq A^{*^s}(\mathcal{I},\tau_1).$$

Proof. *Let $x\in A^{*^s}(\mathcal{I},\tau_2)$, then $M_x\cap A\notin\mathcal{I}\ \forall M_x\in\tau_2(x)\Rightarrow M_x\cap A\notin\mathcal{I}\ \forall M_x\in\tau_1(x)\Rightarrow x\in A^{*^s}(\mathcal{I},\tau_1)$. Hence, $A^{*^s}(\mathcal{I},\tau_2)\subseteq A^{*^s}(\mathcal{I},\tau_1)$.* ■

Corollary 3.1. *Let (X,τ_1) and (X,τ_2) be two supra topological spaces and $\mathcal{I}$ be an ideal on X. Then*

$$\tau_1\subseteq\tau_2\Rightarrow\tau_1^*(\mathcal{I})\subseteq\tau_2^*(\mathcal{I}).$$

Proof. *It follows from Theorem 3.4.* ■

Theorem 3.5. *Let (X,τ) be a supra topological space and $\mathcal{I}$ be an ideal on X. Then the collection*

$$\beta(\mathcal{I},\tau)=\{M-H:M\in\tau,H\in\mathcal{I}\}$$

is a base for the supra topology $\tau^*(\mathcal{I})$.

Proof. *Let $M\in\tau^*$ and $x\in M$. Then M^c is a τ^*-closed so that $cl^{*^s}(M^c)=M^c$, and hence $M^{c*^s}\subseteq M^c$(by Lemma 3.3). Then $x\notin M^{c*^s}$ and so there exists $V\in\tau(x)$ such that $V\cap M^c\in\mathcal{I}$. Putting $H=V\cap M^c$, then $x\notin H$ and $H\in\mathcal{I}$. Thus $x\in V\backslash H=V\cap H^c=V\cap(V\cap M^c)^c=V\cap(V^c\cup M)=V\cap M\subseteq M$. Hence, $x\in V\backslash H\subseteq M$, where $V\backslash H\in\beta(\mathcal{I},\tau)$. Hence M is the union of sets in $\beta(\mathcal{I},\tau)$.*
Note that, τ^* is a supra topology, so it is not closed under finite intersection, thus, we need only to prove that $M\in\tau^*$ is a union of sets in $\beta(\mathcal{I},\tau)$ as done above. ■

Theorem 3.6. *For any ideal on a supra topological space (X,τ), we have*

$$\tau\subseteq\beta(\mathcal{I},\tau)\subseteq\tau^*.$$

Proof. *Let* $M \in \tau$. *Then* $M = M\backslash\phi \in \beta(\mathcal{I},\tau)$. *Hence* $\tau \subseteq \beta(\mathcal{I},\tau)$. *Now, let* $G \in \beta(\mathcal{I},\tau)$, *then there exists* $M \in \tau$ *and* $H \in \mathcal{I}$ *such that* $G = M\backslash H$. *Then,* $cl^{*^s}(G^c) = cl^{*^s}(M\backslash H)^c = (M\backslash H)^c \cup ((M\backslash H)^c)^{*^s} = (M^c \cup H) \cup (M^c \cup H)^{*^s}$. *But,* $H \in \mathcal{I}$, *then, by Theorem 2.3(9),* $(M^c \cup H)^{*^s} = M^{c*^s}$ *and so,* $cl^{*^s}(M\backslash H)^c = M^c \cup H \cup M^{c*^s} \subseteq M^c \cup H$ *(by Lemma 3.3). Hence* $cl^{*^s}(M\backslash H)^c \subseteq M^c \cup H = (M\backslash H)^c$, *but* $(M\backslash H)^c \subseteq cl^{*^s}(M\backslash H)^c$. *Hence* $cl^{*^s}(M\backslash H)^c = (M\backslash H)^c$. *Therefore,* $G = M\backslash H \in \tau^*$. *Hence* $\beta(\mathcal{I},\tau) \subseteq \tau^*$. *Consequently,* $\tau \subseteq \beta(\mathcal{I},\tau) \subseteq \tau^*$. ■

Corollary 3.2. *Let* (X,τ) *be a supra topological space and* $\mathcal{I}$ *be an ideal on* X. *Then If* $\mathcal{I} = \{\phi\}$, *then* $\tau = \beta(\mathcal{I},\tau) = \tau^*$.

Proof. *It follows from Example 3.1 and Theorem 3.6 .* ■

Theorem 3.7. *Let* (X,τ) *be a supra topological space and* $\mathcal{I}$ *be an ideal on* X. *Then,* $\tau^{**} = \tau^*$.

Proof. *From Theorem 3.1, we have* $\tau^* \subseteq \tau^{**}$. *Now, let* $N \in \tau^{**}$, *then* N *can be written as* $N = \cup_{\alpha\in\Lambda}(M^*_\alpha \cap H^c_\alpha)$ *such that* $M^*_\alpha \in \tau^*$ *and* $H_\alpha \in \mathcal{I}$ $\forall\alpha \in \Lambda$. *But,* $M^*_\alpha = \cup_{j\in J}(M_{\alpha_j} \cap H^c_{\alpha_j})$ *where* $M_{\alpha_j} \in \tau$ *and* $H_{\alpha_j} \in \mathcal{I}$, *then*
$N = \cup_{\alpha\in\Lambda}(M^*_\alpha \cap H^c_\alpha)$
$= \cup_{\alpha\in\Lambda}[\cup_{j\in J}(M_{\alpha_j} \cap H^c_{\alpha_j}) \cap H^c_\alpha]$
$= \cup_{\alpha\in\Lambda}[\cup_{j\in J}(M_{\alpha_j} \cap (H^c_{\alpha_j} \cap H^c_\alpha))]$
$= \cup_{\alpha\in\Lambda}[\cup_{j\in J}(M_{\alpha_j} \cap (H_{\alpha_j} \cup H_\alpha)^c)]$
putting $S_{\alpha_j} = H_{\alpha_j} \cup H_\alpha$, *then*

$$N = \cup_{\alpha\in\Lambda}[\cup_{j\in J}(M_{\alpha_j} \cap S^c_{\alpha_j})].$$

Since $H_{\alpha_j}, H_\alpha (= H_{\alpha_j} \cup H_\alpha) \in \mathcal{I}$, *then* $S_{\alpha_j} \in \mathcal{I}$, *also* $\cup_{j\in J} M_{\alpha_j} \in \tau$, *it follows that* $\cup_{j\in J} M_{\alpha_j} \cap S^c_{\alpha_j} \in \beta(\mathcal{I},\tau)$. *Consequently,* $N \in \tau^*$. ■

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POSSIBILITY SINGLE VALUED NEUTROSOPHIC SOFT EXPERT SETS AND ITS APPLICATION IN DECISION MAKING

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Abstract - In this paper, we first introduced the concept of possibility single valued neutrosophic soft expert sets (PSVNSESs for short) which is a generalization of single valued neutrosophic soft expert sets (SVNSESs for short), possibility fuzzy soft expert sets (PFSESs) and possibility intuitionistic fuzzy soft expert sets (PIFSESs). We also define its basic operations, namely complement, union, intersection, AND and OR, and study some of their properties. Finally, an approach for solving MCDM problems is explored by applying the possibility single valued neutrosophic soft expert sets, and an example is provided to illustrate the application of the proposed method

***Keywords** - Single valued neutrosophic sets, soft expert sets, possibility single valued neutrosophic soft expert sets, decision making.*

1. Introduction

In 1999, F. Smarandache [12,13,14] proposed the concept of neutrosophic set (NS for short) by adding an independent indeterminacy-membership function. The concept of neutrosophic set is a generalization of classic set, fuzzy set [40], intuitionistic fuzzy set [34] and so on. In NS, the indeterminacy is quantified explicitly and truth-membership, indeterminacy membership, and false-membership are completely independent. From scientific or engineering point of view, the neutrosophic set and set- theoretic view, operators need to be specified. Otherwise, it will be difficult to apply in the real applications. Therefore, H. Wang et al [17] defined a single valued neutrosophic set (SVNS) and then provided the set theoretic operations and various properties of single valued neutrosophic sets. The works on single valued neutrosophic set (SVNS) and their

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hybrid structure in theories and application have been progressing rapidly (e.g, [3, 4, 5, 6, 7, 8, 9, 11, 25, 26, 27, 28, 29, 30, 31, 32, 33, 41, 60, 68, 69, 70, 73, 77, 80, 81, 82, 83, 86].

In the year 1999, Molodtsov a Russian researcher [10] firstly gave the soft set theory as a general mathematical tool for dealing with uncertainty and vagueness and how soft set theory is free from the parameterization inadequacy syndrome of fuzzy set theory, rough set theory, probability theory. A soft set is in fact a set-valued map which gives an approximation description of objects under consideration based on some parameters. Then, many interesting results of soft set theory have been studied on fuzzy soft sets [45, 47, 48, 53, 54], on intuitionistic fuzzy soft set theory [49, 50, 51, 55], on possibility fuzzy soft set [45, 63], on generalized fuzzy soft sets [58], on generalized intuitionistic fuzzy soft [39], on possibility intuitionistic fuzzy soft set [42], on possibility vague soft set [35] and so on. All these research aim to solve most of our real life problems in medical sciences, engineering, management, environment and social science which involve data that are not crisp and precise. Moreover all the models created will deal only with one expert .To redefine this one expert opinion, Alkhazaleh and Salleh in 2011 [63] defined the concept of soft expert set in which the user can know the opinion of all the experts in one model and give an application of this concept in decision making problem. Also, they introduced the concept of the fuzzy soft expert set [62] as a combination between the soft experts set and the fuzzy set. Therfore, Broumi and Smarandache [85] presented the concept of iintuitionstic fuzzy soft expert set, a more general concept, which combines intuitionstic fuzzy set and soft expert set and studied its application in decision making. Later on, many researchers have worked with the concept of soft expert sets and their hybrid structures [1, 2, 15, 16, 22, 36, 37, 44, 46]. But most of these concepts cannot deal with indeterminate and inconsistent information.

Combining neutrosophic set models with other mathematical models has attracted the attention of many researchers. Maji et al. presented the concept of neutrosophic soft set [57] which is based on a combination of the neutrosophic set and soft set models. Works on neutrosophic soft set theory are progressing rapidly. Based on [57], Maji [56] introduce the concept of weighted neutrosophic soft sets which is hybridization of soft sets and weighted parameter of neutrosophic soft sets. Also, Based on Çağman [48], Karaaslan [87] redefined neutrosophic soft sets and their operations. Various kinds of extended neutrosophic soft sets such as intuitionistic neutrosophic soft set [65, 67, 76], generalized neutrosophic soft set [59, 66], interval valued neutrosophic soft set [23], neutrosophic parameterized fuzzy soft set [72], Generalized interval valued neutrosophic soft sets [75], neutrosophic soft relation [20, 21], neutrosophic soft multiset theory [24] and cyclic fuzzy neutrosophic soft group [61] were presented. The combination of neutrosophic soft sets and rough set [74, 78, 79] is another interesting topic. In this paper, our objective is to generalize the concept of single valued neutrosophic soft expert sett. In our generalization of single valued neutrosophic soft expert set , a possibility of each element in the universe is attached with the parameterization of single valued neutrosophic sets while defining a single valued neutrosophic soft expert set The new model developed is called possibility single valued neutrosophic soft expert set (PSVNSES).

The paper is structured as follows. In Section 2, we first recall the necessary background on neutrosophic sets, single valued neutrosophic sets, soft set single valued neutrosophic soft sets, possibility single valued neutrosophic soft sets, single valued neutrosophic soft expert sets, soft expert sets, fuzzy soft expert sets, possibility fuzzy soft expert sets and possibility intutionistic fuzzy soft expert sets. Section 3 reviews various proposals for the definition of possibility single valued neutrosophic soft expert sets and derive their respective properties. Section 4 presents basic operations on possibility single valued neutrosophic soft expert sets. Section 5 presents an application of this concept in solving a decision making problem. Finally, we conclude the paper.

2. Preliminaries

In this section, we will briefly recall the basic concepts of neutrosophic sets, single valued neutrosophic sets, soft set single valued neutrosophic soft sets, possibility single valued neutrosophic soft sets, soft expert sets, fuzzy soft expert sets, possibility fuzzy soft expert sets and possibility intutionistic fuzzy soft expert sets
Let U be an initial universe set of objects and E the set of parameters in relation to objects in U . Parameters are often attributes, characteristics or properties of objects. Let P (U) denote the power set of U and $A \subseteq E$.

2.1 Neutrosophic Set

Definition 2.1 [13] Let U be an universe of discourse then the neutrosophic set A is an object having the form $A = \{< x: \mu_A(x), \nu_A(x), \omega_A(x)>, x \in U\}$, where the functions $\mu_A(x), \nu_A(x), \omega_A(x) : U \to]^-0,1^+[$ define respectively the degree of membership , the degree of indeterminacy, and the degree of non-membership of the element $x \in X$ to the set A with the condition.

$$^-0 \leq sup\ \mu_A(x) + sup\nu_A(x) + sup\omega_A(x)) \leq 3^+.$$

From philosophical point of view, the neutrosophic set takes the value from real standard or non-standard subsets of $]^-0,1^+[$. So instead of $]^-0,1^+[$ we need to take the interval [0,1] for technical applications, because $]^-0,1^+[$ will be difficult to apply in the real applications such as in scientific and engineering problems.

For two NS,

$A_{NS} = \{<x, \mu_A(x), \nu_A(x), \omega_A(x) > | x \in X\}$

and

$B_{NS} = \{<x, \mu_B(x), \nu_B(x), \omega_B(x) > | x \in X\}$

Then,

1. $A_{NS} \subseteq B_{NS}$ if and only if

$$\mu_A(x) \leq \mu_B(x), \nu_A(x) \geq \nu_B(x), \omega_A(x) \geq \omega_B(x).$$

2. $A_{NS} = B_{NS}$ if and only if ,

$\mu_A(x) = \mu_B(x), \nu_A(x) = \nu_B(x), \omega_A(x) = \omega_B(x)$ for any $x \in X$.

3. The complement of A_{NS} is denoted by A^{o}_{NS} and is defined by

A^{o}_{NS}= {<x, $\omega_{\mathrm{A}}(\mathrm{x}), 1 - \nu_{\mathrm{A}}(\mathrm{x})$, $\mu_{\mathrm{A}}(\mathrm{x})$ | x ∈ X }

4. A∩B = {<x, min{$\mu_{\mathrm{A}}(\mathrm{x}), \mu_{B}(\mathrm{x})$}, max{$\nu_{\mathrm{A}}(\mathrm{x}), \nu_{B}(\mathrm{x})$}, max{$\omega_{\mathrm{A}}(\mathrm{x}), \omega_{B}(\mathrm{x})$} >:x ∈ X }

5. A∪B = {<x, max{$\mu_{\mathrm{A}}(\mathrm{x}), \mu_{B}(\mathrm{x})$},min{$\nu_{\mathrm{A}}(\mathrm{x}), \nu_{B}(\mathrm{x})$}, min{$\omega_{\mathrm{A}}(\mathrm{x}), \omega_{B}(\mathrm{x})$} >:x ∈ X }

As an illustration, let us consider the following example.

Example 2.2. Assume that the universe of discourse U={x_1,x_2,x_3,x_4}. It may be further assumed that the values of x_1, x_2, $x_3$and x_4 are in [0, 1] Then, A is a neutrosophic set (NS) of U, such that,

A= {< x_1,0.4,0.6,0.5 >,< x_2, 0.3,0.4,0.7>,< x_3,0.4,0.4, 0.6] >,< x_4,0.5,0.4,0.8 >}

2.2 Soft Set

Definition 2.3. [10] Let U be an initial universe set and E be a set of parameters. Let P(U) denote the power set of U. Consider a nonempty set A, A ⊂ E. A pair (K, A) is called a soft set over U, where K is a mapping given by K : A → P(U).

As an illustration, let us consider the following example.

Example 2.4. Suppose that U is the set of houses under consideration, say U= {h_1,h_2,...,h_5}. Let E be the set of some attributes of such houses, say E={e_1,e_2, ... , e_8}, where e_1, e_2, ..., e_8 stand for the attributes "beautiful", "costly", "in the green surroundings'", "moderate", respectively.

In this case, to define a soft set means to point out expensive houses, beautiful houses, and so on. For example, the soft set (K, A) that describes the "attractiveness of the houses" in the opinion of a buyer, say Thomas, may be defined like this:

A={e_1,e_2,e_3,e_4,e_5};

K(e_1) = {h_2, h_3, h_5}, K(e_2) = {h_2, h_4}, K(e_3) = {h_1}, K(e_4) = U, K(e_5) = {h_3, h_5}.

2.3 Neutrosophic Soft Sets

Definition 2.5 [57,87] Let U be an initial universe set and $A \subset E$ be a set of parameters. Let NS(U) denotes the set of all neutrosophic subsets of U. The collection (F, A) is termed to be the neutrosophic soft set over U, where F is a mapping given by $F: A \rightarrow NS(U)$.

Example 2.6 [16] Let U be the set of houses under consideration and E is the set of parameters. Each parameter is a neutrosophic word or sentence involving neutrosophic words. Consider E ={beautiful, wooden, costly, very costly, moderate, green surroundings, in good repair, in bad repair, cheap, expensive}. In this case, to define a neutrosophic soft set means to point out beautiful houses, wooden houses, houses in the green surroundings and so on. Suppose that, there are five houses in the universe U given by$U = \{h_1, h_2, \dots, h_5\}$ and the set of parameters

$A = \{e_1, e_2, e_3, e_4\}$,where e_1 stands for the parameter `beautiful', $e_2$ stands for the parameter `wooden', e_3 stands for the parameter `costly' and the parameter $e_4$stands for `moderate'. Then the neutrosophic set (F, A) is defined as follows:

$$(F,A) = \left\{ \begin{matrix} \left(e_1 \left\{\frac{h_1}{(0.5,0.6,0.3)}, \frac{h_2}{(0.4,0.7,0.6)}, \frac{h_3}{(0.6,0.2,0.3)}, \frac{h_4}{(0.7,0.3,0.2)}, \frac{h_5}{(0.8,0.2,0.3)}\right\}\right) \\ \left(e_2 \left\{\frac{h_1}{(0.6,0.3,0.5)}, \frac{h_2}{(0.7,0.4,0.3)}, \frac{h_3}{(0.8,0.1,0.2)}, \frac{h_4}{(0.7,0.1,0.3)}, \frac{h_5}{(0.8,0.3,0.6)}\right\}\right) \\ \left(e_3 \left\{\frac{h_1}{(0.7,0.4,0.3)}, \frac{h_2}{(0.6,0.7,0.2)}, \frac{h_3}{(0.7,0.2,0.5)}, \frac{h_4}{(0.5,0.2,0.6)}, \frac{h_5}{(0.7,0.3,0.4)}\right\}\right) \\ \left(e_4 \left\{\frac{h_1}{(0.8,0.6,0.4)}, \frac{h_2}{(0.7,0.9,0.6)}, \frac{h_3}{(0.7,0.6,0.4)}, \frac{h_4}{(0.7,0.8,0.6)}, \frac{h_5}{(0.9,0.5,0.7)}\right\}\right) \end{matrix} \right\}$$

2.4 Possibility Single Valued Neutrosophic Soft Sets

Definition 2.7 [59] Let U= $\{u_1, u_2, u_3, \dots, u_n\}$ be a universal set of elements, E=$\{e_1, e_2, e_3, \dots, e_m\}$ be a universal set of parameters. The pair (U, E) will be called a soft universe. Let F: E → $(I \times I \times I)^U \times I^U$ where $(I \times I \times I)^U$ is the collection of all single valued neutrosophic subset of U and I^U is the collection of all fuzzy subset of U. Let p be a fuzzy subset of E, that is p: E → I^U

And let F_p: E → $(I \times I \times I)^U \times I^U$ be a function defined as follows:

$F_p(e)$= (F(e)(x), p(e)(x)), where F(e)(x)=(μ(x), $\nu(x)$,$\omega(x)$) for x∈ U.

Then F_p is called a possibility single valued neutrosophic soft set(PSVNSS) over the soft universe (U, E).

2.5 Soft Expert Sets

Definition 2.8 [63] Let U be a universe set, E be a set of parameters and X be a set of experts (agents). Let O= {1=agree, 0=disagree} be a set of opinions. Let Z= E × X × O and A ⊆ Z

A pair (F, E) is called a soft expert set over U, where F is a mapping given by F : A → P(U) and P(U) denote the power set of U.

Definition 2.9 [63] An agree- soft expert set $(F,\mathrm{A})_1$over U, is a soft expert subset of (F,A) defined as :

$$(F,\mathrm{A})_1 = \{F(\alpha): \alpha \in E \times X \times \{1\}\}.$$

Definition 2.10 [63] A disagree- soft expert set $(F, A)_0$ over U, is a soft expert subset of (F,A) defined as :

$$(F, A)_0 = \{F(\alpha) : \alpha \in E \times X \times \{0\}\}.$$

2.6 Fuzzy Soft Expert Sets

Definition 2.11 [62] A pair (F, A) is called a fuzzy soft expert set over U, where F is a mapping given by F : A→ I^U,and I^U denote the set of all fuzzy subsets of U.

2.7.Possibility Fuzzy Soft Expert Sets

Definition 2.12. [44] Let U={ $u_1, u_2, u_3, \ldots, u_n$ } be a universal set of elements, E={ $e_1, e_2, e_3, \ldots, e_m$ } be a universal set of parameters, X={ $x_1, x_2, x_3, \ldots, x_i$ } be a set of experts (agents) and O = { 1=agree, 0=disagree} be a set of opinions. Let Z= { E × X × Q } and A ⊆ Z. The pair (U, E) will be called a soft universe. Let F: E $\to I^U$ and μ be fuzzy subset o f E, i.e, μ :E $\to I^U$ where I^U is the collection of all fuzzy subsets of U. Let F_μ :E $\to I^U \times I^U$ be a function defined as follows:

$$F_\mu(e) = (\, F(e)(x), \mu(e)(x)), \text{ for all } x \in U.$$

Then F_μ is called a possibility fuzzy soft expert set (PFSES in short) over the soft universe (U, E)

For each parameter $e_i \in E$. $F_\mu(e_i) = (F(e_i)(x), \mu(e_i)(x))$ indicates not only the degree of belongingness of the elements of U in F(e_i), but also the degree of possibility of belongingness of the elements of U in F(e_i), which is represented by $\mu(e_i)$. So we can write $F_\mu(e_i)$ as follows:

$$F_\mu(e_i)\,\{(\frac{x_i}{F(e_i)(x_i)}), \mu(e_i)(x_i)\} \text{ ,for i=1,2,3,..,n}$$

Sometimes we write F_μ as (F_μ, E) . If A ⊆ E. we can also have PFSES (F_μ, A).

2.8 Possibility Intuitionstic Fuzzy Soft expert sets

Definition 2.13 [16] Let U= { $u_1, u_2, u_3, \ldots, u_n$ } be a universal set of elements, E={ $e_1, e_2, e_3, \ldots, e_m$ } be a universal set of parameters, X={ $x_1, x_2, x_3, \ldots, x_i$ } be a set of experts (agents) and O= {1=agree, 0=disagree} be a set of opinions. Let Z= { E × X × Q } and A ⊆ Z. Then the pair (U, Z) is called a soft universe. Let F: Z $\to I^U$ and λ be fuzzy subset o f Z defined as λ :Z $\to F^U$ where I^U

denotes the collection of all intuitionistic fuzzy subsets of U. Suppose F_λ :Z $\rightarrow$ I^U x F^U be a function defined as:

$$F_p(z) = (\text{F(z)}(u_i), \lambda \text{ (z)}(u_i)), \text{ for all } u_\text{i} \in \text{U}.$$

Then $F_\lambda(z)$ is called a possibility intuitionistic fuzzy soft expert set (PIFSES in short) over the soft universe (U, Z)

For each $z_\text{i} \in$Z. $F_\lambda(z)$= (F(z_i)(u_i), λ(z_i)(u_i)) where F(z_i) represents the degree of belongingness and non-belongingness of the elements of U in F(z_i) and λ(z_i) represents the degree of possibility of such belongingness. Hence $F_\lambda(z_i)$ can be written as:

$$F_\lambda(z_i)\{(\frac{u_\text{i}}{F(z_\text{i})(u_\text{i})}), \lambda(z_\text{i})(u_i)\} \text{ ,for i=1,2,3,…n}$$

where F(z_i)(u_i) = < $\mu_{\text{F}(z_\text{i})}(u_i)$, $\omega_{\text{F}(z_\text{i})}(u_i)$ > with $\mu_{\text{F}(z_\text{i})}(u_i)$ and $\omega_{\text{F}(z_\text{i})}(u_i)$ representing the membership function and non-membership function of each of the elements $u_\text{i} \in$U respectively.

Sometimes we write F_λ as (F_λ, Z) . If A ⊆ Z. we can also have PIFSES (F_λ, A).

2.9 Single Valued Neutrosophic Soft Expert Sets

Definition 2.14 [84] Let U= = { u_1, u_2, u_3,…, u_n} be a universal set of elements, E={ e_1, e_2, e_3,…, e_m} be a universal set of parameters, X={ x_1, x_2, x_3,…, x_i} be a set of experts (agents) and O= {1=agree, 0=disagree} be a set of opinions. Let Z= { E × X × Q } and A ⊆ Z. Then the pair (U, Z) is called a soft universe. Let F: Z $\rightarrow$ SVN^U, where SVN^U denotes the collection of all single valued neutrosophic subsets of U. Suppose F :Z $\rightarrow$ SVN^U be a function defined as:

$$F(z) = \text{F(z)}(u_i) \text{ for all } u_\text{i} \in \text{U}.$$

Then $F(z)$ is called a single valued neutrosophic soft expert set (SVNSES in short) over the soft universe (U, Z)

For each z_i ∈ Z. $F(z)$ = F(z_i)(u_i), where F(z_i) represents the degree of belongingness, degree of indeterminacy and non-belongingness of the elements of U in F(z_i).Hence $F(z_i)$ can be written as:

$$F(z_i)\{(\frac{u_1}{F(z_1)(u_1)}),\ldots,(\frac{u_\text{n}}{F(z_\text{n})(u_\text{n})}), \} \text{ ,for i=1,2,3,…n}$$

where $F(z_i)(u_i) = < \mu_{F(z_i)}(u_i), \nu_{F(z_i)}(u_i), \omega_{F(z_i)}(u_i) >$ with $\mu_{F(z_i)}(u_i)$, $\nu_{F(z_i)}(u_i)$ and $\omega_{F(z_i)}(u_i)$ representing the membership function, indeterminacy function and non-membership function of each of the elements $u_i \in U$ respectively.

Sometimes we write F as (F, Z) . If $A \subseteq Z$. we can also have SVNSES (F, A).

3. Possibility Single Valued Neutrosophic Soft Expert Sets

In this section, we generalize the possibility fuzzy soft expert sets as introduced by Alhhazaleh and Salleh [62] and possibility intuitionistic fuzzy soft expert sets as introduced by G. Selvachandran [16] to possibility single valued neutrosophic soft expert sets and give the basic properties of this concept.

Let U be universal set of elements, E be a set of parameters, X be a set of experts (agents), O= { 1=agree, 0=disagree } be a set of opinions. Let Z= $E \times X \times O$ and

Definition 3.1 Let U= = $\{u_1, u_2, u_3, \ldots, u_n\}$ be a universal set of elements, E=$\{e_1, e_2, e_3, \ldots, e_m\}$ be a universal set of parameters, X=$\{x_1, x_2, x_3, \ldots, x_i\}$ be a set of experts (agents) and O= {1=agree, 0=disagree} be a set of opinions. Let Z= $\{ E \times X \times Q \}$ and $A \subseteq Z$. Then the pair (U, Z) is called a soft universe. Let F: Z $\to SVN^U$ and p be fuzzy subset o f Z defined as p :Z $\to F^U$ where SVN^U denotes the collection of all single valued neutrosophic subsets of U. Suppose F_p :Z$\to SVN^U$ x F^U be a function defined as:

$$F_p(z) = (F(z)(u_i), p(z)(u_i)), \text{ for all } u_i \in U.$$

Then $F_p(z)$ is called a possibility single valued neutrosophic soft expert set (PSVNSES in short) over the soft universe (U, Z)

For each $z_i \in Z$. $F_p(z) = (F(z_i)(u_i), p(z_i)(u_i))$ where $F(z_i)$ represents the degree of belongingness, degree of indeterminacy and non-belongingness of the elements of U in $F(z_i)$ and $p(z_i)$ represents the degree of possibility of such belongingness. Hence $F_p(z_i)$ can be written as:

$$F_p(z_i)\{(\frac{u_i}{F(e_i)(u_i)}), p(z_i)(u_i)\}, \text{ for } i=1,2,3,\ldots$$

where $F(z_i)(u_i) = < \mu_{F(z_i)}(u_i), \nu_{F(z_i)}(u_i), \omega_{F(z_i)}(u_i) >$ with $\mu_{F(z_i)}(u_i)$, $\nu_{F(z_i)}(u_i)$ and $\omega_{F(z_i)}(u_i)$ representing the membership function, indeterminacy function and non-membership function of each of the elements $u_i \in U$ respectively.

Sometimes we write F_p as (F_p, Z) . If $A \subseteq Z$. we can also have PSVNSES (F_p, A).

Example 3.2 Let U=$\{u_1, u_2, u_3\}$ be a set of elements, E=$\{e_1, e_2\}$ be a set of decision parameters, where e_i(i= 1, 2,3} denotes the parameters E =$\{e_1$= beautiful, e_2= cheap} and X= $\{x_1, x_2\}$ be a set of experts. Suppose that F_p:Z→ $SVN^U \times F^U$ is function defined as follows:

$$F_p(e_1, x_1, 1) = \{ (\frac{u_1}{<0.1,0.8,0.3>},0.3), (\frac{u_2}{<0.1,0.6,0.4>},0.4), (\frac{u_3}{<0.4,0.7,0.2>},0.5) \},$$

$$F_p(e_2, x_1, 1) = \{ (\frac{u_1}{<0.7,0.5,0.25>},0.6), (\frac{u_2}{<0.25,0.6,0.4>},0.8), (\frac{u_3}{<0.4,0.4,0.6>},0.7) \},$$

$$F_p(e_1, x_2, 1) = \{ (\frac{u_1}{<0.3,0.2,0.7>},0.3), (\frac{u_2}{<0.4,0.3,0.3>},0.4), (\frac{u_3}{<0.1,0.6,0.2>},0.6) \},$$

$$F_p(e_2, x_2, 1) = \{ (\frac{u_1}{<0.2,0.2,0.6>},0.5), (\frac{u_2}{<0.7,0.3,0.2>},0.8), (\frac{u_3}{<0.3,0.1,0.5>},0.1) \},$$

$$F_p(e_1, x_1, 0) = \{ (\frac{u_1}{<0.2,0.4,0.5>},0.2), (\frac{u_2}{<0.1,0.9,0.1>},0.7), (\frac{u_3}{<0.1,0.2,0.5>},0.1) \},$$

$$F_p(e_2, x_1, 0) = \{ (\frac{u_1}{<0.3,0.4,0.6>},0.4), (\frac{u_2}{<0.2,0.7,0.6>},0.6), (\frac{u_3}{<0.1,0.5,0.2>},0.1) \},$$

$$F_p(e_1, x_2, 0) = \{ (\frac{u_1}{<0.2,0.8,0.4>},0.2), (\frac{u_2}{<0.1,0.6,0.5>},0.5), (\frac{u_3}{<0.7,0.6,0.3>},0.8) \}$$

$$F_p(e_2, x_2, 0) = \{ (\frac{u_1}{<0.4,0.4,0.7>},0.2), (\frac{u_2}{<0.3,0.8,0.2>},0.6), (\frac{u_3}{<0.6,0.2,0.4>},0.5)$$

Then we can view the possibility single valued neutrosophic soft expert set (F_p, Z) as consisting of the following collection of approximations:

(F_p, Z)=

$$\{ (e_1, x_1, 1) = \{ (\frac{u_1}{<0.1,0.8,0.3>},0.3), (\frac{u_2}{<0.1,0.6,0.4>},0.4), (\frac{u_3}{<0.4,0.7,0.2>},0.5) \}\},$$

$$\{ (e_2, x_1, 1) = \{ (\frac{u_1}{<0.7,0.5,0.25>},0.6), (\frac{u_2}{<0.25,0.6,0.4>},0.8), (\frac{u_3}{<0.4,0.4,0.6>},0.7) \}\},$$

$$\{ (e_1, x_2, 1) = \{ (\frac{u_1}{<0.3,0.2,0.7>},0.3), (\frac{u_2}{<0.4,0.3,0.3>},0.4), (\frac{u_3}{<0.1,0.6,0.2>},0.6) \}\},$$

$$\{ (e_2, x_2, 1) = \{ (\frac{u_1}{<0.2,0.2,0.6>},0.5), (\frac{u_2}{<0.7,0.3,0.2>},0.8), (\frac{u_3}{<0.3,0.1,0.5>},0.1) \}\},$$

$$\{ (e_1, x_1, 0) = \{ (\frac{u_1}{<0.2,0.4,0.5>}, 0.2), (\frac{u_2}{<0.1,0.9,0.1>}, 0.7), (\frac{u_3}{<0.1,0.2,0.5>}, 0.1) \}\},$$

$$\{ (e_2, x_1, 0) = \{ (\frac{u_1}{<0.3,0.4,0.6>}, 0.4), (\frac{u_2}{<0.2,0.7,0.6>}, 0.6), (\frac{u_3}{<0.1,0.5,0.2>}, 0.1) \}\},$$

$$\{ (e_1, x_2, 0) = \{ (\frac{u_1}{<0.2,0.8,0.4>}, 0.2), (\frac{u_2}{<0.1,0.6,0.5>}, 0.5), (\frac{u_3}{<0.7,0.6,0.3>}, 0.8) \}\},$$

$$\{ (e_2, x_2, 0) = \{ (\frac{u_1}{<0.4,0.4,0.7>}, 0.2), (\frac{u_2}{<0.3,0.8,0.2>}, 0.6), (\frac{u_3}{<0.6,0.2,0.4>}, 0.5) \}\}.$$

Then (F_p, Z) is a possibility single valued neutrosophic soft expert set over the soft universe (U, Z).

Definition 3.3. Let (F_p,A) and (G_q,B) be a PSVNSESs over a soft universe (U,Z). Then (F_p,A) is said to be a possibility single valued neutrosophic soft expert subset of (G_q,B) if A ⊆ B and for all $\varepsilon \in$ A, the following conditions are satisfied:

(i) $p(\varepsilon)$ is fuzzy subset of $q(\varepsilon)$
(ii) $F(\varepsilon)$ is a single valued neutrosophic subset of $G(\varepsilon)$.

This relationship is denoted as (F_p,A) ⊆ (G_q,B). In this case, (G_q,B) is called a possibility single valued neutrosophic soft expert superset (PSVNSE superset) of (F_p,A).

Definition 3.4. Let (F_p,A) and (G_q,B) be a PSVNSESs over a soft universe (U,Z). Then (F_p,A) and (G_q,B) are said to be equal if for all $\varepsilon \in$ E, the following conditions are satisfied:

(i) $p(\varepsilon)$ is equal $q(\varepsilon)$
(ii) $F(\varepsilon)$ is equal $G(\varepsilon)$

In other words, (F_p,A) = (G_q,B) if (F_p,A) is a PSVNSE subset of (G_q,B) and (G_q,B) is a PSVNSE subset of (F_p, A).

Definition 3.5. A PSVNSES (F_p,A) is said to be a null possibility single valued neutrosophic soft expert set denoted ($\widetilde{\emptyset}_p$, A) and defined as :

($\widetilde{\emptyset}_p$, A) = (F(α), p(α)), where $\alpha \in$ Z.

Where F(α)= <0, 0, 1>, that is $\mu_{F(\alpha)}$=0, $\nu_{F(\alpha)}$= 0 and $\omega_{F(\alpha)}$= 1 and p(α)= 0 for all $\alpha \in$ Z

Definition 3.6. A PSVNSES (F_p,A) is said to be an absolute possibility single valued neutrosophic soft expert set denoted (F_p, A) $_{\text{abs}}$ and defined as :

(F_p, A) $_{\text{abs}}$ = (F(α), p(α)), where $\alpha \in$ Z.

Where F(α)= <1, 0, 0>, that is $\mu_{F(\alpha)}$= 1, $\nu_{F(\alpha)}$= 0 and $\omega_{F(\alpha)}$= 0 and p(α)= 1 for all $\alpha \in$ Z

Definition 3.7. Let (F_p,A) be a PSVNSES over a soft universe (U,Z). An agree-possibility single valued neutrosophic soft expert set (agree- PSVNSES) over U, denoted as $(F_p, \mathrm{A})_1$ is a possibility single valued neutrosophic soft expert subset of (F_p,A) which is defined as :

$$(F_p, \mathrm{A})_1 = (\mathrm{F}(\alpha), \mathrm{p}(\alpha)), \text{ where } \alpha \in \mathrm{E} \times \mathrm{X} \times \{1\}$$

Definition 3.8. Let (F_p,A) be a PSVNSES over a soft universe (U,Z). A disagree-possibility single valued neutrosophic soft expert set (disagree- PSVNSES) over U, denoted as $(F_p, \mathrm{A})_0$ is a possibility single valued neutrosophic soft expert subset of (F_p,A) which is defined as :

$$(F_p, \mathrm{A})_0 = (\mathrm{F}(\alpha), \mathrm{p}(\alpha)), \text{ where } \alpha \in \mathrm{E} \times \mathrm{X} \times \{0\}$$

4. Basic Operations on Possibility Single Valued Neutrosophic Soft Expert Sets.

In this section, we introduce some basic operations on PSVNSES, namely the complement, AND, OR, union and intersection of PSVNSES, derive their properties, and give some examples.

Definition 4.1 Let (F_p, A) be a PSVNSES over a soft universe (U, Z). Then the complement of (F_p, A) denoted by $(F_p, A)^c$ is defined as:

$$(F_p, A)^c = (\tilde{c}(\mathrm{F}(\alpha)), \mathrm{c}(\mathrm{p}(\alpha))), \text{ for all } \alpha \in \mathrm{U}.$$

where $\tilde{c}$ is single valued neutrosophic complement and c is a fuzzy complement.

Example 4.2 Consider the PSVNSES (F_p, Z) over a soft universe (U, Z) as given in Example 3.2. By using the basic fuzzy complement for p(α) and the single valued neutrosophic complement for F(α), we obtain $(F_p, Z)^c$ which is defined as:

$(F_p, Z)^c =$

$$\{ (e_1, x_1, 1)=\{ (\frac{u_1}{<0.3,0.8,0.1>},0.7), (\frac{u_2}{<0.4,0.6,0.1>},0.6), (\frac{u_3}{<0.2,0.7,0.4>},0.5) \}\},$$

$$\{ (e_2, x_1, 1) = \{ (\frac{u_1}{<0.25,0.5,0.7>},0.4), (\frac{u_2}{<0.4,0.6,0.25>},0.2), (\frac{u_3}{<0.6,0.4,0.4>},0.3) \}\},$$

$$\{ (e_1, x_2, 1) = \{ (\frac{u_1}{<0.7,0.2,0.3>},0.7), (\frac{u_2}{<0.3,0.3,0.4>},0.6), (\frac{u_3}{<0.2,0.6,0.1>},0.4) \}\},$$

$$\{ (e_2, x_2, 1) = \{ (\frac{u_1}{<0.6,0.2,0.2>}, 0.5), (\frac{u_2}{<0.2,0.3,0.7>}, 0.2), (\frac{u_3}{<0.5,0.1,0.3>}, 0.9) \}\},$$

$$\{ (e_1, x_1, 0) = \{ (\frac{u_1}{<0.5,0.4,0.2>}, 0.8), (\frac{u_2}{<0.1,0.9,0.1>}, 0.3), (\frac{u_3}{<0.5,0.2,0.1>}, 0.9) \}\},$$

$$\{ (e_2, x_1, 0) = \{ (\frac{u_1}{<0.6,0.4,0.3>}, 0.6), (\frac{u_2}{<0.6,0.7,0.2>}, 0.4), (\frac{u_3}{<0.2,0.5,0.1>}, 0.9) \}\},$$

$$\{ (e_1, x_2, 0) = \{ (\frac{u_1}{<0.4,0.8,0.2>}, 0.8), (\frac{u_2}{<0.5,0.6,0.1>}, 0.5), (\frac{u_3}{<0.3,0.6,0.7>}, 0.2) \}\},$$

$$\{ (e_2, x_2, 0) = \{ (\frac{u_1}{<0.7,0.4,0.4>}, 0.8), (\frac{u_2}{<0.2,0.8,0.3>}, 0.4), (\frac{u_3}{<0.4,0.2,0.6>}, 0.5) \}\}.$$

Proposition 4.3 If (F_p, A) is a PSVNSES over a soft universe (U,Z), Then,

$$((F_p, A)^c)^c = (F_p, A).$$

Proof. Suppose that is (F_p, A) is a PSVNSES over a soft universe (U, Z) defined as (F_p, A) = (F(e), p(e)). Now let PSVNSES $(F_p, A)^c = (G_q, B)$. Then by Definition 4.1, (G_q, B) = (G(e), q(e)) such that G(e) $=\tilde{c}$ (F(e)), and q(e) =c(p(e)).Thus it follows that:

$$(G_q, B)^c = (\tilde{c}\,(G(e)), c(q(e))) = (\tilde{c}\,(\tilde{c}\,(F(e))), c(c(q(e)))) = (F(e), p(e)) = (F_p, A).$$

Therefore

$((F_p, A)^c)^c = (G_q, B)^c = (F_p, A)$.Hence it is proven that $((F_p, A)^c)^c = (F_p, A)$.

Definition 4.4 Let (F_p, A) and (G_q, B) be any two PSVNSESs over a soft universe (U, Z). Then the union of (F_p, A) and (G_q, B), denoted by $(F_p, A)\ \tilde{\cup}\ (G_q, B)$ is a PSVNSES defined as $(F_p, A)\ \tilde{\cup}\ (G_q, B) = (H_r, C)$, where C= A ∪ B and

$$r(\alpha) = \max (p(\alpha), q(\alpha)), \text{ for all } \alpha \in C.$$

and

$$H(\alpha) = F(\alpha) \tilde{\cup}\, G(\alpha), \text{ for all } \alpha \in C$$

where

$$H(\alpha) = \begin{cases} F(\alpha) & \alpha \in A - B \\ G(\alpha) & \alpha \in A - B \\ s_N(F(\alpha), G(\alpha)) & \alpha \in A \cap B \end{cases}$$

where s_N is a neutrosophic co- norm.

Proposition 4.5 Let (F_p, A), (G_q, B) and (H_r, C) be any three PSVNSES over a soft universe (U, Z).Then the following properties hold true.

(i) $(F_p, \text{A})\ \tilde{\cup}\ (G_q, \text{B}) = (G_q, \text{B})\ \tilde{\cup}\ (F_p, \text{A})$

(ii) $(F_p, \text{A})\ \tilde{\cup}\ ((G_q, \text{A})\ \tilde{\cup}\ (H_r, \text{C})) = ((F_p, \text{A})\ \tilde{\cup}\ (G_q, \text{B}))\ \tilde{\cup}\ (H_r, \text{C})$

(iii) $(F_p, \text{A})\ \tilde{\cup}\ (F_p, \text{A}) \subseteq (F_p, \text{A})$

(iv) $(F_p, \text{A})\ \tilde{\cup}\ (\Phi_p, A) = (\Phi_p, A)$

Proof

(i) Let $(F_p, \text{A})\ \tilde{\cup}\ (G_q, \text{B}) = (H_r, \text{C})$. Then by definition 4.4, for all $\alpha \in \text{C}$, we have (H_r, C) =(H(α), r(α))

Where

H(α) = F(α) $\tilde{\cup}$ G(α) and r(α) = max (p(α),q(α)). However H(α) = F(α) $\tilde{\cup}$ G(α)= G(α) $\tilde{\cup}$ F(α) since the union of these sets are commutative by definition 4.4. Also, r(α) = max (p(α), q(α)) = max (q(α), p(α)). Therfore $(H_r, \text{C}) = (G_q, \text{B})\ \tilde{\cup}\ (F_p, \text{A})$. Thus the union of two PSVNSES are commutative i.e $(F_p, \text{A})\ \tilde{\cup}\ (G_q, \text{B}) = (G_q, \text{B})\ \tilde{\cup}\ (F_p, \text{A})$.

(ii) The proof is similar to proof of part(i) and is therefore omitted

(iii) The proof is straightforward and is therefore omitted.

(iv) The proof is straightforward and is therefore omitted.

Definition 4.6 Let (F_p, A) and (G_q, B) be any two PSVNSES over a soft universe (U, Z). Then the intersection of (F_p, A) and (G_q, B), denoted by $(F_p, \text{A})\ \tilde{\cap}\ (G_q, \text{B})$ is PSVNSES defined as $(F_p, \text{A})\ \tilde{\cap}\ (G_q, \text{B}) = (H_r, \text{C})$ where C= A ∪ B and

$$\text{r}(\alpha) = \min(\text{p}(\alpha), \text{q}(\alpha)), \text{ for all } \alpha \in \text{C},$$

and

$$\text{H}(\alpha) = \text{F}(\alpha)\ \tilde{\cap}\ \text{G}(\alpha), \text{ for all } \alpha \in \text{C}$$

where

$$\text{H}(\alpha) = \begin{cases} \text{F}(\alpha) & \alpha \in A - B \\ \text{G}(\alpha) & \alpha \in A - B \\ \text{t}_\text{n}\big(F(\alpha), G(\alpha)\big) & \alpha \in A \cap B \end{cases}$$

where t_n is neutrosophic t-norm

Proposition 4.7 If (F_p, A), (G_q, B) and (H_r, C) are three PSVNSES over a soft universe (U, Z).Then,

(i) $(F_p, \text{A})\ \tilde{\cap}\ (G_q, \text{B}) = (G_q, \text{B})\ \tilde{\cap}\ (F_p, \text{A})$

(ii) $(F_p, \text{A})\ \tilde{\cap}\ ((G_q, \text{B})\ \tilde{\cap}\ (H_r, \text{C})) = ((F_p, \text{A})\ \tilde{\cap}\ (G_q, \text{B}))\ \tilde{\cap}\ (H_r, \text{C})$

(iii) $(F_p, A) \tilde{\cap} (F_p, A) \subseteq (F_p, A)$

(iv) $(F_p, A) \tilde{\cap} (\Phi_p, A) = (\Phi_p, A)$

Proof

(i) The proof is similar to that of Propositio 4.5 (i) and is therefore omitted

(ii) The prof is similar to the prof of part (i) and is therefore omitted

(iii) The proof is straightforward and is therefore omitted.

(iv) The proof is straightforward and is therefore omitted.

Proposition 4.8 If (F_p, A), (G_q, B) and (H_r, C) are three PSVNSES over a soft universe (U, Z). Then,

(i) $(F_p, A) \tilde{\cup} ((G_q, B) \cap (H_r, C)) = ((F_p, A) \tilde{\cup} (G_q, B)) \tilde{\cap} ((F_p, A) \tilde{\vee} (H_r, C))$

(ii) $(F_p, A) \tilde{\cap} ((G_q, B) \tilde{\cup} (H_r, C)) = ((F_p, A) \tilde{\cap} (G_q, B)) \tilde{\cup} ((F_p, A) \tilde{\cap} (H_r, C))$

Proof. The proof is straightforward by definitions 4.4 and 4.6 and is therefore omitted.

Proposition 4.9 If (F_p, A), (G_q, B) are two PSVNSES over a soft universe (U, Z). Then,

(i) $((F_p, A) \tilde{\cup} (G_q, B))^c = (F_p, A)^c \tilde{\cap} (G_q, B)^c$.

(ii) $((F_p, A) \tilde{\cap} (G_q, B))^c = (F_p, A)^c \tilde{\cup} (G_q, B)^c$.

Proof.

(i) Suppose that (F_p, A) and(G_q, B) be PSVNSES over a soft universe (U, Z) defined as:

$(F_p, A) = (F(\alpha), p(\alpha))$, for all $\alpha \in A \subseteq Z$ and $(G_q, B) = (G(\alpha), q(\alpha))$, for all $\alpha \in B \subseteq Z$. Now , due to the commutative and associative properties of PSVNSES, it follows that: by Definition 4.10 and 4.11, it follows that:

$$
\begin{aligned}
(F_p, A)^c \tilde{\cap} (G_q, B)^c &= (F(\alpha), p(\alpha))^c \tilde{\cap} (G(\alpha), q(\alpha))^c \\
&= (\tilde{c} (F(\alpha)) , c(p(\alpha))) \tilde{\cap} (\tilde{c} (G(\alpha)) , c(q(\alpha))) \\
&= (\tilde{c} (F(\alpha)) \tilde{\cap} \tilde{c} (G(\alpha))), \min(c(p(\alpha)), c(q(\alpha)))) \\
&= (\tilde{c} (F(\alpha) \tilde{\cap} G(\alpha)), c(\max(p(\alpha), q(\alpha))) \\
&= ((F_p, A) \tilde{\cup} (G_q, B))^c .
\end{aligned}
$$

(ii) The proof is similar to the proof of part (i) and is therefore omitted

Definition 4.10 Let (F_p, A) and (G_q, B) be any two PSVNSES over a soft universe (U, Z). Then " (F_p, A) AND (G_q, B) " denoted $(F_p, A) \tilde{\wedge} (G_q, B)$ is a defined by:

$$(F_p, A) \tilde{\wedge} (G_q, B) = (H_r, A \times B)$$

Where $(H_r, A \times B) = (H(\alpha, \beta), r(\alpha, \beta))$, such that $H(\alpha, \beta) = F(\alpha) \cap G(\beta)$ and $r(\alpha, \beta) = \min (p(\alpha),$

q(β)) for all $(\alpha, \beta) \in$ A × B. and ∩ represent the basic intersection.

Definition 4.11 Let (F_p, A) and (G_q, B) be any two PSVNSES over a soft universe (U, Z). Then " (F_p, A) OR (G_q, B) " denoted $(F_p, \text{A})\ \tilde{\vee}\ (G_q, \text{B})$ is a defined by:

$$(F_p, \text{A})\ \tilde{\vee}\ (G_q, \text{B}) = (H_r, \text{A} \times \text{B})$$

Where $(H_r, \text{A} \times \text{B})$ = (H(α, β), r(α, β)), such that H(α, β) = F(α) ∪ G(β) and r(α, β) = max (p(α),

q(β)) for all $(\alpha, \beta) \in$ A × B. and ∪ represent the basic union.

Proposition 4.12 If (F_p, A), (G_q, B) and (H_r, C) are three PSVNSES over a soft universe (U, Z).Then,

i. $(F_p, \text{A})\ \tilde{\wedge}\ ((G_q, \text{B})\ \tilde{\wedge}\ (H_r, \text{C})) = ((F_p, \text{A})\ \tilde{\wedge}\ (G_q, \text{B}))\ \tilde{\wedge}\ (H_r, \text{C})$

ii. $(F_p, \text{A})\ \tilde{\vee}\ ((G_q, \text{B})\ \tilde{\vee}\ (H_r, \text{C})) = ((F_p, \text{A})\ \tilde{\vee}\ (G_q, \text{B}))\ \tilde{\vee}\ (H_r, \text{C})$

iii. $(F_p, \text{A})\ \tilde{\vee}\ ((G_q, \text{B})\ \tilde{\wedge}\ (H_r, \text{C})) = ((F_p, \text{A})\ \tilde{\vee}\ (G_q, \text{B}))\ \tilde{\wedge}\ ((F_p, \text{A})\ \tilde{\vee}\ (H_r, \text{C}))$

iv. $(F_p, \text{A})\ \tilde{\wedge}\ ((G_q, \text{B})\ \tilde{\vee}\ (H_r, \text{C})) = ((F_p, \text{A})\ \tilde{\wedge}\ (G_q, \text{B}))\ \tilde{\vee}\ ((F_p, \text{A})\ \tilde{\wedge}\ (H_r, \text{C}))$

Proof. The proofs are straightforward by Definitions 4.10 and 4.11 and is therefore omitted.

Note: the " AND" and "OR" operations are not commutative since generally A × B ≠ B ×A.

Proposition 4.13 If (F_p, A) and (G_q, B) are two PSVNSES over a soft universe (U, Z). Then,

i. $((F_p, \text{A})\ \tilde{\wedge}\ (G_q, \text{B}))^c = (F_p, \text{A})^c\ \tilde{\vee}\ (G_q, \text{B})^c$.

ii. $((F_p, \text{A})\ \tilde{\vee}\ (G_q, \text{B}))^c = (F_p, \text{A})^c\ \tilde{\wedge}\ (G_q, \text{B})^c$.

Proof.

(i) Suppose that (F_p, A) and (G_q, B) be PSVNSES over a soft universe (U, Z) defined as:

(F_p, A) = (F(α), p(α)), for all $\alpha \in \text{A} \subseteq$ Z and (G_q, B) = (G(β), q(β)), for all $\beta \in \text{B} \subseteq$ Z. Then by Definition 4.10 and 4.11, it follows that:

$$\begin{aligned}
((F_p, \text{A})\ \tilde{\wedge}\ (G_q, \text{B}))^c &= ((F(\alpha), p(\alpha))\ \tilde{\wedge}\ (G(\beta), q(\beta)))^c \\
&= (F(\alpha)\ \cap\ G(\beta), \min(p(\alpha), q(\beta)))^c \\
&= (\ \tilde{c}\ (\text{F}(\alpha) \cap \text{G}(\beta)), \text{c}(\min(\ \text{p}(\alpha), \text{q}(\beta))) \\
&= (\ \tilde{c}\ (\text{F}(\alpha)) \cup \tilde{c}\ (\text{G}(\beta))), \max(\text{c}(\ \text{p}(\alpha)), \text{c}(\text{q}(\beta)))) \\
&= (F(\alpha), p(\alpha))^c\ \tilde{\vee}\ (G(\beta), q(\beta))^c \\
&= (F_p, \text{A})^c\ \tilde{\vee}\ (G_q, \text{B})^c.
\end{aligned}$$

(ii) the proof is similar to that of part (i) and is therefore omitted.

5. Application of Possibility Neutrosophic Soft Expert Sets in a Decision Making Problem.

In this section, we introduce a generalized algorithm which will be applied to the PNSES model introduced in Section 3 and used to solve a hypothetical decision making problem. The following example is adapted from [17] with minor changes.

Suppose that company Y is looking to hire a person to fill in the vacancy for a position in their company. Out of all the people who applied for the position, three candidates were shortlisted and these three candidates form the universe of elements, U= $\{u_1, u_2, u_3\}$ The hiring committee consists of the hiring manager, head of department and the HR director of the company and this committee is represented by the set {p, q, r }(a set of experts) while the set Q= {1=agree, 0=disagree } represents the set of opinions of the hiring committee members. The hiring committee considers a set of parameters, E={ e_1, e_2, e_3, e_4 } where the parameters e_i represent the characteristics or qualities that the candidates are assessed on, namely "relevant job experience", "excellent academic qualifications in the relevant field", "attitude and level of professionalism" and "technical knowledge" respectively. After interviewing all the three candidates and going through their certificates and other supporting documents, the hiring committee constructs the following PSVNSES.

(F_p, Z) =

$$\{ (e_1, p, 1) = \{ (\frac{u_1}{<0.2,0.8,0.4>},0.2), (\frac{u_2}{<0.3,0.2,0.4>},0.1), (\frac{u_3}{<0.4,0.7,0.2>},0.4) \}\},$$

$$\{ (e_2, p, 1) = \{ (\frac{u_1}{<0.3,0.2,0.23>},0.5), (\frac{u_2}{<0.25,0.2,0.3>},0.6), (\frac{u_3}{<0.3,0.5,0.6>},0.2) \}\},$$

$$\{ (e_3, p, 1) = \{ (\frac{u_1}{<0.3,0.2,0.7>},0.3), (\frac{u_2}{<0.4,0.3,0.3>},0.4), (\frac{u_3}{<0.1,0.6,0.2>},0.6) \}\},$$

$$\{ (e_4, p, 1) = \{ (\frac{u_1}{<0.2,0.2,0.6>},0.5), (\frac{u_2}{<0.7,0.3,0.2>},0.8), (\frac{u_3}{<0.3,0.1,0.5>},0.1) \}\},$$

$$\{ (e_1, q, 1) = \{ (\frac{u_1}{<0.4,0.6,0.3>},0.55), (\frac{u_2}{<0.1,0.3,0.7>},0.6), (\frac{u_3}{<0.6,0.3,0.7>},0.9) \}\},$$

$$\{ (e_2, q, 1) = \{ (\frac{u_1}{<0.3,0.3,0.5>},0.2), (\frac{u_2}{<0.6,0.9,0.1>},0.7), (\frac{u_3}{<0.1,0.2,0.7>},0.1) \}\},$$

$$\{ (e_3, q, 1) = \{ (\frac{u_1}{<0.1,0.4,0.7>},0.2), (\frac{u_2}{<0.4,0.6,0.2>},0.8), (\frac{u_3}{<0.6,0.2,0.4>},0.5) \}\}.$$

$$\{ (e_4, q, 1) = \{ (\frac{u_1}{<0.6,0.5,0.3>},0.1), (\frac{u_2}{<0.7,0.8,0.2>},0.6), (\frac{u_3}{<0.3,0.4,0.6>},0.7) \}\}.$$

$\{ (e_1, r, 1) = \{ (\frac{u_1}{<0.4,0.5,0.7>},0.2), (\frac{u_2}{<0.3,0.8,0.4>},0.6), (\frac{u_3}{<0.6,0.2,0.4>},0.5) \}\}.$

$\{ (e_2, r, 1) = \{ (\frac{u_1}{<0.3,0.7,0.1>},0.8, (\frac{u_2}{<0.7,0.3,0.2>},0.4), (\frac{u_3}{<0.8,0.2,0.2>},0.6) \}\}.$

$\{ (e_3, r, 1) = \{ (\frac{u_1}{<0.6,0.5,0.2>},0.2), (\frac{u_2}{<0.5,0.1,0.6>},0.9), (\frac{u_3}{<0.3,0.2,0.1>},0.1) \}\}.$

$\{ (e_1, p, 0) = \{ (\frac{u_1}{<0.1,0.4,0.3>},0.2), (\frac{u_2}{<0.3,0.8,0.2>},0.6), (\frac{u_3}{<0.6,0.2,0.4>},0.5) \}\}.$

$\{ (e_3, p, 0) = \{ (\frac{u_1}{<0.6,0.3,0.2>},0.4), (\frac{u_2}{<0.2,0.7,0.4>},0.9), (\frac{u_3}{<0.3,0.1,0.6>},0.7) \}\}.$

$\{ (e_4, p, 0) = \{ (\frac{u_1}{<0.3,0.2,0.5>},0.6), (\frac{u_2}{<0.6,0.4,0.5>},0.2), (\frac{u_3}{<0.5,0.4,0.3>},0.3) \}\}.$

$\{ (e_1, q, 0) = \{ (\frac{u_1}{<0.2,0.4,0.7>},0.3), (\frac{u_2}{<0.1,0.9,0.2>},0.7), (\frac{u_3}{<0.1,0.2,0.5>},0.1) \}\},$

$\{ (e_2, q, 0) = \{ (\frac{u_1}{<0.3,0.4,0.6>},0.4), (\frac{u_2}{<0.2,0.7,0.6>},0.3), (\frac{u_3}{<0.4,0.5,0.3>},0.4) \}\},$

$\{ (e_3, q, 0) = \{ (\frac{u_1}{<0.2,0.8,0.4>},0.2), (\frac{u_2}{<0.1,0.2,0.5>},0.6), (\frac{u_3}{<0.7,0.6,0.3>},0.8) \}\},$

$\{ (e_4, q, 0) = \{ (\frac{u_1}{<0.9,0.4,0.7>},0.68), (\frac{u_2}{<0.5,0.6,0.2>},0.5), (\frac{u_3}{<0.6,0.3,0.4>},0.55) \}\}.$

$\{ (e_1, r, 0) = \{ (\frac{u_1}{<0.3,0.4,0.5>},0.5), (\frac{u_2}{<0.3,0.6,0.2>},0.1), (\frac{u_3}{<0.25,0.2,0.4>},0.9) \}\}.$

$\{ (e_2, r, 0) = \{ (\frac{u_1}{<0.4,0.6,0.7>},0.3), (\frac{u_2}{<0.6,0.4,0.2>},1), (\frac{u_3}{<0.6,0.4,0.3>},0.25) \}\}.$

$\{ (e_3, r, 0) = \{ (\frac{u_1}{<0.4,0.3,0.2>},0.9), (\frac{u_2}{<0.3,0.5,0.7>},0.8), (\frac{u_3}{<0.7,0.5,0.6>},0.5) \}\}.$

Next the PSVNSES (F_p, Z) is used together with a generalized algorithm to solve the decision making problem stated at the beginning of this section. The algorithm given below is employed by the hiring committee to determine the best or most suitable candidate to be hired for the position. This algorithm is a generalization of the algorithm introduced by Alkhazaleh and Salleh (see [3]) which is used in the context of the PSVNSES model that is introduced in this paper. The generalized algorithm is as follows:

Algorithm

1. Input the PSVNSES (F_p , Z)

2. Find the values of $\mu_{F_p(z_i)}(u_i) - \nu_{F_p(z_i)}(u_i) - \omega_{F_p(z_i)}(u_i)$ for each element $u_i \in U$ where $\mu_{F_p(z_i)}(u_i)$, $\nu_{F_p(z_i)}(u_i)$ and $\omega_{F_p(z_i)}(u_i)$ are the membership function, indeterminacy function and non-membership function of each of the elements $u_i \in U$ respectively.

3. Find the highest numerical grade for the agree-PSVNSES and disagree-PSVNSES.

4. Compute the score of each element $u_i \in U$ by taking the sum of the products of the numerical grade of each element with the corresponding degree of possibility μ_i for the agree-PNSES and disagree PSVNSES, denoted by A_i and D_i respectively.

5. Find the values of the score $r_i = A_i - D_i$ for each element $u_i \in U$.

6. Determine the value of the highest score, s= $\max_{u_i} \{ r_i \}$. Then the decision is to choose element as the optimal or best solution to the problem. If there are more than one element with the highest r_i score, then any one of those elements can be chosen as the optimal solution.

Then we can conclude that the optimal choice for the hiring committee is to hire candidate u_i to fill the vacant position

Table I gives the values of $\mu_{F_p(z_i)}(u_i) - \nu_{F_p(z_i)}(u_i) - \omega_{F_p(z_i)}(u_i)$ for each element $u_i \in U$ The notation a ,b gives the values of $\mu_{F_p(z_i)}(u_i) - \nu_{F_p(z_i)}(u_i) - \omega_{F_p(z_i)}(u_i)$ and the degree of possibility of the element $\mu_i \in U$ respectively.

Table I. Values of $\mu_{F_p(z_i)}(u_i) - \nu_{F_p(z_i)}(u_i) - \omega_{F_p(z_i)}(u_i)$ for all $u_i \in U$

	u_1	u_2	u_3		u_1	u_2	u_3
$(e_1, p, 1)$	-1, 0.2	-0.3, 0.1	-0.5, 0.4	$(e_3, p, 0)$	0.1, 0.4	-0.9, 0.9	-0.4, 0.7
$(e_2, p, 1)$	-0.13, 0.5	-0.25, 0.6	-0.8, 0.2	$(e_4, p, 0)$	-0.4, 0.6	-0.3, 0.2	-0.2, 0.3
$(e_3, p, 1)$	-0.6, 0.3	-0.2, 0.4	-0.7, 0.6	$(e_1, q, 0)$	-0.9, 0.3	-1, 0.7	-0.6, 0.1
$(e_4, p, 1)$	-0.6, 0.5	0.2, 0.8	-0.3, 0.1	$(e_2, q, 0)$	-0.7, 0.4	-1.1, 0.3	-0.4, 0.4
$(e_1, q, 1)$	-0.5, 0.55	-0.9, 0.6	-0.4, 0.9	$(e_3, q, 0)$	-1, 0.2	-0.6, 0.6	-0.2, 0.8
$(e_2, q, 1)$	-0.5, 0.2	-0.4, 0.7	-0.5, 0.1	$(e_4, q, 0)$	-0.2, 0.68	-0.3,0.5	-0.1, 0.55
$(e_3, q, 1)$	-1, 0.2	-0.4, 0.8	0, 0.5	$(e_1, r, 0)$	-0.6, 0.5	-0.5, 0.1	0.35, 0.9
$(e_4,q, 1)$	-0.2, 0.1	-0.3, 0.6	-0.5,0.7	$(e_2, r, 0)$	-0.9, 0.3	0, 1	-0.1, 0.25
$(e_1, r, 1)$	-0.8, 0.2	-0.9, 0.6	0, 0.5	$(e_4, r, 0)$	-0.1, 0.9	-0.9,0.8	-0.4, 0.5
$(e_2, r, 1)$	-0.5, 0.8	0.2, 0.4	0.4, 0.6				
$(e_3, r, 1)$	-0.1, 0.2	-0.2, 0.9	0, 0.1				
$(e_1, p, 0)$	-0.6, 0.2	-0.7, 0.6	0, 0.5				

In Table II and Table III, we gives the highest numerical grade for the elements in the agree-PSVNSES and disagree PSVNSES respectively.

Table II. Numerical Grade for Agree-PSVNSES

	u_i	Highest Numeric Grade	Degree of possibility, μ_i
$(e_1, p, 1)$	u_2	-0.3	0.1
$(e_2, p, 1)$	u_1	-0.13	0.5
$(e_3, p, 1)$	u_2	-0.2	0.4
$(e_4, p, 1)$	u_2	0.2	0.8
$(e_1, q, 1)$	u_3	-0.4	0.9
$(e_2, q, 1)$	u_2	-0.4	0.7
$(e_3, q, 1)$	u_3	0	0.5
$(e_4,q, 1)$	u_1	-0. 2	0.1
$(e_1, r, 1)$	u_3	0	0.5
$(e_2, r, 1)$	u_3	0.4	0.6
$(e_3, r, 1)$	u_3	0	0.1

Score (u_1) = (-0.13 × 0.15) +(-0.2 × 0.1)
= - 0.0395

Score (u_2) = (-0.3 × 0.1) +(-0.2 × 0.4) +(-0.2 × 0.8) +(-0.4 × 0.7)
= - 0.55

Score (u_3) = (-0.4 × 0.9) +(0 × 0.5) +(0 × 0.5) +(0.4 × 0.6) +(0 × 0.1)
= - 0.12

Table III. Numerical Grade for Disagree-PSVNSES

	u_i	Highest Numeric Grade	Degree opossibility, μ_i
$(e_1, p, 0)$	u_3	0	0.5
$(e_3, p, 0)$	u_1	0.1	0.4
$(e_4, p, 0)$	u_3	-0.2	0.3
$(e_1, q, 0)$	u_3	-0.6	0.1
$(e_2, q, 0)$	u_3	-0.4	0.4
$(e_3, q, 0)$	u_3	-0.2	0.8
$(e_4, q, 0)$	u_3	-0.1	0.55
$(e_1, r, 0)$	u_3	-0.35	0.9
$(e_2, r, 0)$	u_2	0	1
$(e_4, r, 0)$	u_1	-0.1	0.9

Score (u_1) = (0.1 × 0.4) +(-0.1 × 0.9)
= - 0.05

Score (u_2) = (0 × 1)
= 0

Score (u_3) = (0 × 0.5) + (-0.2 × 0.3) +(-0.6 × 0.1) +(-0.4 × 0.4) +(-0.2 × 0.8) +(-0.1 × 0.55) +(-0.35 × 0.9) = - 0.81

Let A_i and D_i represent the score of each numerical grade for the agree-PSVNSES and disagree-PSVNSES respectively. These values are given in Table IV.

Table IV The score r_i = A_i - D_i

A_i	D_i	r_i
Score (u_1) = - 0.0395	Score (u_1) = -0.05	**0.0105**
Score (u_2) = -0.55	Score (u_2) = 0	**-0.55**
Score (u_3) = -0.12	Score (u_3) = -0.81	**0.69**

Then s= $\max_{u_i} \{ r_i \} = r_3$, the hiring committee should hire candidate u_3 to fill in the vacant position

6. Conclusion

In this paper we have introduced the concept of possibility single valued neutrosophic soft expert soft set and studied some of its properties. The complement, union, intersection, And or OR operations have been defined on the possibility single valued neutrosophic soft expert set. Finally, an application of this concept is given in solving a decision making problem. This new extension will provide a significant addition to existing theories for handling indeterminacy, and lead to potential areas of further research and pertinent applications.

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FURTHER DECOMPOSITIONS OF $*$-CONTINUITY^I

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Abstract – In this paper, we introduce the notions of $^*g$-$\mathcal{I}$-LC*-sets, $\mathcal{I}^*_{*g}$-closed sets and $\mathcal{I}$-$^*g_t$-sets. Also we define the notions of $^*g$-$\mathcal{I}$-LC*-continuous maps, $\mathcal{I}^*_{*g}$-continuous maps, $\mathcal{I}$-$^*g_t$-continuous maps and obtain decompositions of $*$-continuity.

Keywords – *G-$\mathcal{I}$-LC*-set, $^*g$-$\mathcal{I}$-LC*-set, $\mathcal{I}^*_g$-closed set, $\mathcal{I}^*_{*g}$-closed set, $\mathcal{I}$-$^*g_t$-set.*

1 Introduction and Preliminaries

The concept of ideals in topological spaces is treated in the classic text by Kuratowski [13] and Vaidyanathaswamy [23]. The notion of $\mathcal{I}$-open sets in topological spaces was introduced by Jankovic and Hamlett [11]. Dontchev et al. [3] introduced and studied the notion of $\mathcal{I}_g$-closed sets. An ideal $\mathcal{I}$ on a topological space (X, τ) is a non-empty collection of subsets of X satisfying the following properties:

1. A $\in \mathcal{I}$ and B $\subseteq$ A imply B $\in \mathcal{I}$ (heredity);
2. A $\in \mathcal{I}$ and B $\in \mathcal{I}$ imply A $\cup$ B $\in \mathcal{I}$ (finite additivity).

A topological space (X, τ) with an ideal $\mathcal{I}$ on X is called an ideal topological space and is denoted by (X, τ, $\mathcal{I}$). For a subset A $\subseteq$ X, $A^*(\mathcal{I}) = \{x \in X : U \cap A \notin \mathcal{I}$ for every U $\in \tau$(x)$\}$, is called the local function [13] of A with respect to $\mathcal{I}$ and τ. We simply write A^* in case there is no chance for confusion. A Kuratowski closure operator cl*(.) for a topology $\tau^*(\mathcal{I})$ called the $*$-topology finer than τ is defined by cl*(A) = A $\cup$ A* [23]. Let (X, τ) denote a topological space on which no separation axioms are assumed unless explicity stated. In a topological space (X, τ), the closure and the interior of any subset A of X will be denoted by cl(A) and int(A), respectively. A subset A of a topological space (X, τ) is said to be semi-open [15] if A $\subseteq$ cl(int(A)). A subset A of a topological space (X, τ) is said to be g-closed [14] (resp. ω-closed [21]) if cl(A) $\subseteq$ U whenever A $\subseteq$ U and U is open (resp. semi-open) in X. The complement of g-closed (resp. ω-closed) set is said to be g-open (resp. ω-open).

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A subset A of a topological space (X, τ) is said to be *g-closed [9] (resp. g*-closed [24]) if cl(A) $\subseteq$ U whenever A $\subseteq$ U and U is ω-open (resp. g-open) in X. The complement of *g-closed (resp. g*-closed) set is said to be *g-open (resp. g*-open). The intersection of all *g-closed sets of X containing a subset A of X is denoted by *gcl(A). Notice that the intersection of two *g-open sets is again a *g-open. A subset A of an ideal topological space (X, τ, $\mathcal{I}$) is called $*$-closed [11] (resp. $*$-perfect [6]) if $A^* \subseteq A$ (resp. $A = A^*$).

Definition 1.1. *A subset A of a topological space (X, τ) is called*

1. *locally closed set [4] (briefly LC-set) if A = $U \cap V$, where U is open and V is closed.*
2. *$^*g$-LC^*-set [16] if A = $U \cap V$, where U is $^*g$-open and V is closed.*
3. *t-set [22] if int(cl(A)) = int(A).*
4. *$^*g_t$-set [16] if A = $C \cap D$, where C is $^*g$-open and D is a t-set.*

Definition 1.2. *A subset A of an ideal topological space (X, τ, $\mathcal{I}$) is called*

1. *t-$\mathcal{I}$-set [5] if int(cl*(A)) = int(A).*
2. *α^*-$\mathcal{I}$-set [5] if int(cl*(int(A))) = int(A).*
3. *$\mathcal{I}$-LC set [2] if A = $C \cap D$, where $C \in \tau$ and D is $*$-perfect.*
4. *weakly-$\mathcal{I}$-LC set [12] if A = $C \cap D$, where $C \in \tau$ and D is $*$-closed.*
5. *$C_{\mathcal{I}}$-set [5] if A = $C \cap D$, where $C \in \tau$ and D is an α^*-$\mathcal{I}$-set.*
6. *G-$\mathcal{I}$-LC^*-set [8] if A = $C \cap D$, where C is g-open and D is $*$-closed.*

Notice that the intersection of two t-$\mathcal{I}$-sets is again a t-$\mathcal{I}$-set.

Definition 1.3. *[8] A subset A of an ideal topological space (X, τ, $\mathcal{I}$) is said to be $\mathcal{I}_g^*$-closed if $A^* \subseteq$ U whenever A $\subseteq$ U and U is g-open in X.*

Definition 1.4. *[7] A function f : (X, τ, $\mathcal{I}$) $\to$ (Y, σ) is said to be $*$-continuous if $f^{-1}(V)$ is $*$-closed in (X, τ, $\mathcal{I}$) for every closed set A in (Y, σ).*

Definition 1.5. *A function f : (X, τ) $\to$ (Y, σ) is said to be $^*g$-LC^*-continuous [16] (resp. g^*-continuous [24], $^*g_t$-continuous [16]) if $f^{-1}(A)$ is $^*g$-LC^*-set (resp. g^*-closed, $^*g_t$-set) in (X, τ) for every closed set A of (Y, σ).*

Definition 1.6. *A function f : (X, τ, $\mathcal{I}$) $\to$ (Y, σ) is said to be $\mathcal{I}_g^*$-continuous [8] (resp. G-$\mathcal{I}$-LC^*-continuous [8], weakly-$\mathcal{I}$-LC continuous [7]) if $f^{-1}(V)$ is $\mathcal{I}_g^*$-closed (resp. G-$\mathcal{I}$-LC^*-set, weakly-$\mathcal{I}$-LC set) in (X, τ, $\mathcal{I}$) for every closed set V in (Y, σ).*

For a subset A of an ideal topological space (X, τ, $\mathcal{I}$), if A is $*$-closed, then by [17] A is weakly-$\mathcal{I}$-LC. Also by Definition 1.3 it follows that if A is $*$-closed, then A is I_g^*-closed.

Lemma 1.7. *[11] Let (X, τ, $\mathcal{I}$) be an ideal topological space and A, B subsets of X. Then the following properties hold:*

1. *If A $\subseteq$ B then $A^* \subseteq B^*$;*
2. *$A^* = cl(A^*) \subseteq cl(A)$;*
3. *$(A^*)^* \subseteq A^*$;*
4. *$(A \cup B)^* = A^* \cup B^*$.*

Proposition 1.8. *[5] Let (X, τ, $\mathcal{I}$) be an ideal topological space and A a subset of X. Then the following hold:*

1. *If A is a t-$\mathcal{I}$-set, then A is an α^*-$\mathcal{I}$-set.*
2. *If A is an α^*-$\mathcal{I}$-set, then A is a $C_{\mathcal{I}}$-set.*

Remark 1.9. *[1] The following hold in an ideal topological space (X, τ, $\mathcal{I}$).*

$*$-*perfect* $\longrightarrow$ $*$-*closed* $\longrightarrow$ *t*-$\mathcal{I}$-*set* $\longrightarrow$ α^*-$\mathcal{I}$-*set*

Remark 1.10. *[19] The following hold in a topological space (X, τ).*

$$\begin{array}{ccc} closed & \longrightarrow & \omega\text{-}closed \\ \downarrow & & \downarrow \\ {}^*g\text{-}closed & \longrightarrow & g\text{-}closed \end{array}$$

*Notice that ω-closed sets and *g-closed sets are independent of each other.*

2 *g-$\mathcal{I}$-LC*-sets

Definition 2.1. *A subset A of an ideal topological space (X, τ, $\mathcal{I}$) is said to be an *g-$\mathcal{I}$-LC*-set if A = C $\cap$ D, where C is *g-open and D is $*$-closed.*

Proposition 2.2. *Let (X, τ, $\mathcal{I}$) be an ideal topological space and A $\subseteq$ X. Then the following hold:*

1. *If A is *g-open, then A is an *g-$\mathcal{I}$-LC*-set;*
2. *If A is $*$-closed, then A is an *g-$\mathcal{I}$-LC*-set;*
3. *If A is weakly-$\mathcal{I}$-LC set, then A is an *g-$\mathcal{I}$-LC*-set;*
4. *If A is an *g-$\mathcal{I}$-LC*-set, then A is an G-$\mathcal{I}$-LC*-set.*

The converses of Proposition 2.2 need not be true as seen from the following Examples.

Example 2.3. *Let X = {a, b, c}, τ = {$\emptyset$, X, {c}} and $\mathcal{I}$ = {$\emptyset$, {a}}. Then *g-$\mathcal{I}$-LC*-sets are P(X) and *g-open sets are $\emptyset$, X, {a}, {b}, {c}, {a, c}, {b, c}. It is clear that {a, b} is *g-$\mathcal{I}$-LC*-set but it is not *g-open.*

Example 2.4. *In Example 2.3, *g-$\mathcal{I}$-LC*-sets are P(X) and $*$-closed sets are $\emptyset$, X, {a}, {a, b}. It is clear that {a, c} is *g-$\mathcal{I}$-LC*-set but it is not $*$-closed.*

Example 2.5. *In Example 2.3, *g-$\mathcal{I}$-LC*-sets are P(X) and weakly-$\mathcal{I}$-LC sets are $\emptyset$, X, {a}, {c}, {a, b}. It is clear that {b, c} is *g-$\mathcal{I}$-LC*-set but it is not weakly-$\mathcal{I}$-LC set.*

Example 2.6. *Let X = {a, b, c}, τ = {$\emptyset$, X, {a}, {b, c}} and $\mathcal{I}$ = {$\emptyset$, {c}}. Then G-$\mathcal{I}$-LC*-sets are P(X) and *g-$\mathcal{I}$-LC*-sets are $\emptyset$, X, {a}, {c}, {a, c}, {b, c}. It is clear that {a, b} is G-$\mathcal{I}$-LC*-set but it is not *g-$\mathcal{I}$-LC*-set.*

Theorem 2.7. *Let (X, τ, $\mathcal{I}$) be an ideal topological space and A be an *g-$\mathcal{I}$-LC*-subset of X. Then the following hold:*

1. *If B is a $*$-closed set, then A $\cap$ B is an *g-$\mathcal{I}$-LC*-set;*
2. *If B is an *g-open set, then A $\cap$ B is an *g-$\mathcal{I}$-LC*-set;*
3. *If B is an *g-$\mathcal{I}$-LC*-set, then A $\cap$ B is an *g-$\mathcal{I}$-LC*-set.*

Proof. (1) Let B be $*$-closed and A is *g-$\mathcal{I}$-LC*-set, then A $\cap$ B = (C $\cap$ D) $\cap$ B = C $\cap$ (D $\cap$ B), where D $\cap$ B is $*$-closed. Hence A $\cap$ B is an *g-$\mathcal{I}$-LC*-set.

(2) Let B be *g-open and A is *g-$\mathcal{I}$-LC*-set, then A $\cap$ B = (C $\cap$ D) $\cap$ B = (C $\cap$ B) $\cap$ D, where C $\cap$ B is *g-open. Hence A $\cap$ B is an *g-$\mathcal{I}$-LC*-set.

(3) Let A and B be *g-$\mathcal{I}$-LC*-sets, then A $\cap$ B = (C $\cap$ D) $\cap$ (U $\cap$ V) = (C $\cap$ U) $\cap$ (D $\cap$ V), where C $\cap$ U is *g-open and D $\cap$ V is $*$-closed. Hence A $\cap$ B is an *g-$\mathcal{I}$-LC*-set.

Remark 2.8. *The union of any two *g-$\mathcal{I}$-LC*-sets need not be an *g-$\mathcal{I}$-LC*-set.*

Example 2.9. *Let X = {a, b, c}, τ = {$\emptyset$, X, {a, b}} and $\mathcal{I}$ = {$\emptyset$, {a}}. Then *g-$\mathcal{I}$-LC*-sets are $\emptyset$, X, {a}, {b}, {c}, {a, b}, {a, c}. It is clear that A = {b} and B = {c} are *g-$\mathcal{I}$-LC*-sets, but their union A$\cup$B = {b, c} is not *g-$\mathcal{I}$-LC*-set.*

Definition 2.10. *[20] A subset A of an ideal topological space (X, τ, $\mathcal{I}$) is said to be $\mathcal{I}^*_{*g}$-closed if A^* $\subseteq$ U whenever A $\subseteq$ U and U is $^*g$-open in X. The complement of $\mathcal{I}^*_{*g}$-closed set is called $\mathcal{I}^*_{*g}$-open.*

Theorem 2.11. *[20] If (X, τ, $\mathcal{I}$) is any ideal topological space and A $\subseteq$ X, then the following are equivalent.*

1. *A is $\mathcal{I}^*_{*g}$-closed,*
2. *cl*(A) $\subseteq$ U whenever A $\subseteq$ U and U is *g-open in X,*
3. *For all $x \in$ cl*(A), *gcl($\{x\}$)$\cap$A $\neq \emptyset$,*
4. *cl*(A) $-$ A contains no nonempty *g-closed set,*
5. *A* $-$ A contains no nonempty *g-closed set.*

Proposition 2.12. *Let (X, τ, $\mathcal{I}$) be an ideal topological space and A be a subset of X. If A is $\mathcal{I}^*_g$-closed, then A is $\mathcal{I}^*_{*g}$-closed.*

The converse of Proposition 2.12 need not be true as seen from the following Example.

Example 2.13. *In Example 2.6, $\mathcal{I}^*_{*g}$-closed sets are P(X) and $\mathcal{I}^*_g$-closed sets are $\emptyset$, X, $\{a\}$, $\{c\}$, $\{a, c\}$, $\{b, c\}$. It is clear that $\{a, b\}$ is $\mathcal{I}^*_{*g}$-closed set but it is not $\mathcal{I}^*_g$-closed.*

Definition 2.14. *A subset A of an ideal topological space (X, τ, $\mathcal{I}$) is said to be an $\mathcal{I}$-$^*g_t$-set if A = C $\cap$ D, where C is $^*g$-open and D is a t-$\mathcal{I}$-set.*

Proposition 2.15. *Let (X, τ, $\mathcal{I}$) be an ideal topological space and A be a subset of X. If A is an $^*g$-$\mathcal{I}$-LC*-set, then A is an $\mathcal{I}$-$^*g_t$-set.*

The converse of Proposition 2.15 need not be true as seen from the following Example.

Example 2.16. *Let X = $\{a, b, c, d\}$, τ = $\{\emptyset, X, \{a\}, \{a, b, c\}\}$ and $\mathcal{I}$ = $\{\emptyset, \{a\}\}$. Then $\mathcal{I}$-$^*g_t$-sets are $\emptyset$, X, $\{a\}$, $\{b\}$, $\{c\}$, $\{d\}$, $\{a, b\}$, $\{a, c\}$, $\{a, d\}$, $\{b, c\}$, $\{b, d\}$, $\{c, d\}$, $\{a, b, c\}$, $\{b, c, d\}$ and *g-$\mathcal{I}$-LC*-sets are $\emptyset$, X, $\{a\}$, $\{b\}$, $\{c\}$, $\{d\}$, $\{a, b\}$, $\{a, c\}$, $\{a, d\}$, $\{b, c\}$, $\{a, b, c\}$, $\{b, c, d\}$. It is clear that $\{c, d\}$ is $\mathcal{I}$-$^*g_t$-set but it is not *g-$\mathcal{I}$-LC*-set.*

Proposition 2.17. *Let (X, τ, $\mathcal{I}$) be an ideal topological space and A be a subset of X. If A is an $\mathcal{I}$-$^*g_t$-set, then A is a $C_{\mathcal{I}}$-set.*

The converse of Proposition 2.17 need not be true as seen from the following Example.

Example 2.18. *Let X = $\{a, b, c, d\}$, τ = $\{\emptyset, X, \{d\}, \{a, c\}, \{a, c, d\}\}$ and $\mathcal{I}$ = $\{\emptyset, \{a\}, \{d\}, \{a, d\}\}$. Then $C_{\mathcal{I}}$-sets are P(X) and $\mathcal{I}$-$^*g_t$-sets are $\emptyset$, X, $\{a\}$, $\{b\}$, $\{c\}$, $\{d\}$, $\{a, b\}$, $\{a, c\}$, $\{a, d\}$, $\{b, d\}$, $\{c, d\}$, $\{a, b, c\}$, $\{a, b, d\}$, $\{a, c, d\}$. It is clear that $\{b, c, d\}$ is $C_{\mathcal{I}}$-set but it is not $\mathcal{I}$-$^*g_t$-set.*

Remark 2.19. *From the above discussion, we have the following implications:*

$$\begin{array}{ccccccc} & & G\text{-}\mathcal{I}\text{-}LC^*\text{-set} & & & & \\ & & \uparrow & & & & \\ *\text{-closed} & \longrightarrow & {}^*g\text{-}\mathcal{I}\text{-}LC^*\text{-set} & \longrightarrow & \mathcal{I}\text{-}{}^*g_t\text{-set} & \longrightarrow & C_{\mathcal{I}}\text{-set} \\ & & \uparrow & & & & \\ \mathcal{I}\text{-}LC\text{-set} & \longrightarrow & \text{weakly-}\mathcal{I}\text{-}LC\text{ set} & & & & \end{array}$$

Theorem 2.20. *Let (X, τ, $\mathcal{I}$) be an ideal topological space and A be an $\mathcal{I}$-$^*g_t$-subset of X. Then the following hold:*

1. *If B is a t-$\mathcal{I}$-set, then A $\cap$ B is an $\mathcal{I}$-$^*g_t$-set;*
2. *If B is an $^*g$-open set, then A $\cap$ B is an $\mathcal{I}$-$^*g_t$-set;*
3. *If B is an $\mathcal{I}$-$^*g_t$-set, then A $\cap$ B is an $\mathcal{I}$-$^*g_t$-set.*

Remark 2.21. *The union of any two $\mathcal{I}$-$^*g_t$-sets need not be an $\mathcal{I}$-$^*g_t$-set.*

Example 2.22. *Let $X = \{a, b, c\}$, $\tau = \{\emptyset, X, \{c\}, \{b, c\}\}$ and $\mathcal{I} = \{\emptyset, \{a\}\}$. Then $\mathcal{I}$-$^*g_t$-sets are $\emptyset$, X, $\{a\}$, $\{b\}$, $\{c\}$, $\{a, b\}$, $\{b, c\}$. It is clear that $A = \{a\}$ and $B = \{c\}$ are $\mathcal{I}$-$^*g_t$-sets but their union $A \cup B = \{a, c\}$ is not $\mathcal{I}$-$^*g_t$-set.*

Theorem 2.23. *The following are equivalent for a subset A of an ideal topological space $(X, \tau, \mathcal{I})$:*

1. *A is $*$-closed;*
2. *A is a weakly-$\mathcal{I}$-LC set and an $\mathcal{I}_g^*$-closed set [10];*
3. *A is an $^*g$-$\mathcal{I}$-LC*-set and an $\mathcal{I}_g^*$-closed set;*
4. *A is an $^*g$-$\mathcal{I}$-LC*-set and an $\mathcal{I}_{*g}^*$-closed set;*
5. *A is an $\mathcal{I}$-$^*g_t$-set and an $\mathcal{I}_{*g}^*$-closed set.*

Proof. (1) $\Rightarrow$ (2): This is obvious.
(2) $\Rightarrow$ (3): Follows from Proposition 2.2.
(3) $\Rightarrow$ (4): Follows from Proposition 2.12.
(4) $\Rightarrow$ (5): Follows from Proposition 2.15.
(5) $\Rightarrow$ (1): Let A be an $\mathcal{I}$-$^*g_t$-set and $\mathcal{I}_{*g}^*$-closed set. Since A is an $\mathcal{I}$-$^*g_t$-set, A = C $\cap$ D, where C is *g-open and D is a t-$\mathcal{I}$-set. Now A $\subseteq$ C and A is $\mathcal{I}_{*g}^*$-closed implies A* $\subseteq$ C. Also A $\subseteq$ D and D is a t-$\mathcal{I}$-set implies int(D) = int(cl*(D)) = int(D $\cup$ D*) $\supseteq$ int(D) $\cup$ int(D*). This shows that int(D*) $\subseteq$ int(D). Thus D* $\subseteq$ D and hence A* $\subseteq$ D. Therefore A* $\subseteq$ C $\cap$ D =A. Hence A is $*$-closed.

Remark 2.24.
1. *The notions of weakly-$\mathcal{I}$-LC sets and $\mathcal{I}_g^*$-closed sets are independent [10].*
2. *The notions of $^*g$-$\mathcal{I}$-LC*-sets and $\mathcal{I}_g^*$-closed sets are independent.*
3. *The notions of $^*g$-$\mathcal{I}$-LC*-sets and $\mathcal{I}_{*g}^*$-closed sets are independent.*
4. *The notions of $\mathcal{I}$-$^*g_t$-sets and $\mathcal{I}_{*g}^*$-clo sed sets are independent.*

Example 2.25. *In Example 2.22, we have weakly-$\mathcal{I}$-LC sets are $\emptyset$, X, $\{a\}$, $\{b\}$, $\{c\}$, $\{a, b\}$, $\{b, c\}$ and $\mathcal{I}_g^*$-closed sets are $\emptyset$, X, $\{a\}$, $\{a, b\}$, $\{a, c\}$. It is clear that $\{b, c\}$ is weakly-$\mathcal{I}$-LC set but it is not $\mathcal{I}_g^*$-closed and $\{a, c\}$ is $\mathcal{I}_g^*$-closed set but it is not weakly-$\mathcal{I}$-LC set.*

Example 2.26. *In Example 2.22, we have *g-$\mathcal{I}$-LC*-sets are $\emptyset$, X, $\{a\}$, $\{b\}$, $\{c\}$, $\{a, b\}$, $\{b, c\}$ and $\mathcal{I}_g^*$-closed sets are $\emptyset$, X, $\{a\}$, $\{a, b\}$, $\{a, c\}$. It is clear that $\{b, c\}$ is *g-$\mathcal{I}$-LC*-set but it is not $\mathcal{I}_g^*$-closed and $\{a, c\}$ is $\mathcal{I}_g^*$-closed set but it is not *g-$\mathcal{I}$-LC*-set.*

Example 2.27. *In Example 2.22, we have *g-$\mathcal{I}$-LC*-sets are $\emptyset$, X, $\{a\}$, $\{b\}$, $\{c\}$, $\{a, b\}$, $\{b, c\}$ and $\mathcal{I}_{*g}^*$-closed sets are $\emptyset$, X, $\{a\}$, $\{a, b\}$, $\{a, c\}$. It is clear that $\{b, c\}$ is *g-$\mathcal{I}$-LC*-set but it is not $\mathcal{I}_{*g}^*$-closed and $\{a, c\}$ is $\mathcal{I}_{*g}^*$-closed set but it is not *g-$\mathcal{I}$-LC*-set.*

Example 2.28. *In Example 2.22, we have $\mathcal{I}$-$^*g_t$-sets are $\emptyset$, X, $\{a\}$, $\{b\}$, $\{c\}$, $\{a, b\}$, $\{b, c\}$ and $\mathcal{I}_{*g}^*$-closed sets are $\emptyset$, X, $\{a\}$, $\{a, b\}$, $\{a, c\}$. It is clear that $\{b, c\}$ is $\mathcal{I}$-$^*g_t$-set but it is not $\mathcal{I}_{*g}^*$-closed and $\{a, c\}$ is $\mathcal{I}_{*g}^*$-closed set but it is not $\mathcal{I}$-$^*g_t$-set.*

3 Decompositions of $*$-continuity

Definition 3.1. *A function $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is said to be $\mathcal{I}_{*g}^*$-continuous (resp. $^*g$-$\mathcal{I}$-LC*-continuous, $\mathcal{I}$-$^*g_t$-continuous) if $f^{-1}(V)$ is $\mathcal{I}_{*g}^*$-closed (resp. $^*g$-$\mathcal{I}$-LC*-set, $\mathcal{I}$-$^*g_t$-set) in $(X, \tau, \mathcal{I})$ for every closed set V in (Y, σ).*

Remark 3.2.
1. *Every $*$-continuous function is weakly $\mathcal{I}$-LC continuous [10].*
2. *Every weakly $\mathcal{I}$-LC continuous function is $^*g$-$\mathcal{I}$-LC*-continuous.*
3. *Every $*$-continuous function is $\mathcal{I}_g^*$-continuous [10].*
4. *Every $\mathcal{I}_g^*$-continuous function is $\mathcal{I}_{*g}^*$-continuous.*

Example 3.3. *Let $X = Y = \{a, b, c\}$, $\tau = \{\emptyset, X, \{c\}, \{b, c\}\}$, $\sigma = \{\emptyset, Y, \{a\}, \{a, b\}\}$ and $\mathcal{I} = \{\emptyset, \{a\}\}$. Then weakly-$\mathcal{I}$-LC sets are $\emptyset$, X, $\{a\}$, $\{b\}$, $\{c\}$, $\{a, b\}$, $\{b, c\}$ and $*$-closed sets are $\emptyset$, X, $\{a\}$, $\{a, b\}$. Let $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ be an identity function. It is clear that $f^{-1}(\{b, c\}) = \{b, c\}$ is not $*$-closed set. Hence f is weakly $\mathcal{I}$-LC continuous but not $*$-continuous function.*

Example 3.4. *Let $X = Y = \{a, b, c\}$, $\tau = \{\emptyset, X, \{c\}\}$, $\sigma = \{\emptyset, Y, \{a\}, \{a, b\}\}$ and $\mathcal{I} = \{\emptyset, \{a\}\}$. Then weakly-$\mathcal{I}$-LC sets are $\emptyset$, X, $\{a\}$, $\{c\}$, $\{a, b\}$ and $^*g$-$\mathcal{I}$-LC*-sets are $P(X)$. Let $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ be an identity function. It is clear that $f^{-1}(\{b, c\}) = \{b, c\}$ is not weakly-$\mathcal{I}$-LC set. Hence f is $^*g$-$\mathcal{I}$-LC* continuous but not weakly $\mathcal{I}$-LC continuous function.*

Example 3.5. *Let $X = Y = \{a, b, c\}$, $\tau = \{\emptyset, X, \{c\}, \{b, c\}\}$, $\sigma = \{\emptyset, Y, \{b\}\}$ and $\mathcal{I} = \{\emptyset, \{a\}\}$. Then $*$-closed sets are $\emptyset$, X, $\{a\}$, $\{a, b\}$ and $\mathcal{I}_g^*$-closed sets are $\emptyset$, X, $\{a\}$, $\{a, b\}$, $\{a, c\}$. Let $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ be an identity function. It is clear that $f^{-1}(\{a, c\}) = \{a, c\}$ is not $*$-closed set. Hence f is $\mathcal{I}_g^*$-continuous but not $*$-continuous function.*

Example 3.6. *Let $X = Y = \{a, b, c\}$, $\tau = \{\emptyset, X, \{a\}, \{b, c\}\}$, $\sigma = \{\emptyset, Y, \{c\}, \{b, c\}\}$ and $\mathcal{I} = \{\emptyset, \{c\}\}$. Then $\mathcal{I}_{*g}^*$-closed sets are $P(X)$ and $\mathcal{I}_g^*$-closed sets are $\emptyset$, X, $\{a\}$, $\{c\}$, $\{a, c\}$, $\{b, c\}$. Let $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ be an identity function. It is clear that $f^{-1}(\{a, b\}) = \{a, b\}$ is not $\mathcal{I}_g^*$-closed set. Hence f is $\mathcal{I}_{*g}^*$-continuous but not $\mathcal{I}_g^*$-continuous function.*

Definition 3.7. *[20] A subset A of a topological space (X, τ) is said to be $^*g^*$-closed if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is $^*g$-open in X.*

Definition 3.8. *A function $f : (X, \tau) \to (Y, \sigma)$ is said to be $^*g^*$-continuous if $f^{-1}(A)$ is $^*g^*$-closed set in (X, τ) for every closed set A of (Y, σ).*

Remark 3.9.

1. *Every g^*-continuous function is $^*g^*$-continuous.*
2. *Every $^*g$-LC*-continuous function is $^*g_t$-continuous.*

Example 3.10. *Let $X = Y = \{a, b, c\}$, $\tau = \{\emptyset, X, \{c\}, \{a, b\}\}$ and $\sigma = \{\emptyset, Y, \{a, c\}\}$. Then $^*g^*$-closed sets are $P(X)$ and g^*-closed sets are $\emptyset$, X, $\{c\}$, $\{a, b\}$. Let $f : (X, \tau) \to (Y, \sigma)$ be an identity function. It is clear that $f^{-1}(\{b\}) = \{b\}$ is not g^*-closed set. Hence f is $^*g^*$-continuous but not g^*-continuous function.*

Example 3.11. *Let $X = Y = \{a, b, c, d\}$, $\tau = \{\emptyset, X, \{b\}, \{b, c, d\}\}$ and $\sigma = \{\emptyset, Y, \{b\}, \{b, d\}\}$. Then $^*g_t$-sets are $\emptyset$, X, $\{a\}$, $\{b\}$, $\{c\}$, $\{d\}$, $\{a, c\}$, $\{a, d\}$, $\{b, c\}$, $\{b, d\}$, $\{c, d\}$, $\{a, c, d\}$, $\{b, c, d\}$ and $^*g$-LC*-sets are $\emptyset$, X, $\{a\}$, $\{b\}$, $\{c\}$, $\{d\}$, $\{b, c\}$, $\{b, d\}$, $\{c, d\}$, $\{a, c, d\}$, $\{b, c, d\}$. Let $f : (X, \tau) \to (Y, \sigma)$ be an identity function. It is clear that $f^{-1}(\{a, c\}) = \{a, c\}$ is not $^*g$-LC*-set. Hence f is $^*g_t$-continuous but not $^*g$-LC*-continuous function.*

Proposition 3.12. *Let $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ be $\mathcal{I}_{*g}^*$-continuous and $g : (Y, \sigma) \longrightarrow (Z, \eta)$ be continuous. Then $g \circ f : (X, \tau, \mathcal{I}) \longrightarrow (Z, \eta)$ is $\mathcal{I}_{*g}^*$-continuous.*

Theorem 3.13. *For a function $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$, the following are equivalent:*

1. *f is $*$-continuous;*
2. *f is weakly $\mathcal{I}$-LC continuous and $\mathcal{I}_g^*$-continuous [10];*
3. *f is $^*g$-$\mathcal{I}$-LC*-continuous and $\mathcal{I}_g^*$-continuous;*
4. *f is $^*g$-$\mathcal{I}$-LC*-continuous and $\mathcal{I}_{*g}^*$-continuous;*
5. *f is $\mathcal{I}$-$^*g_t$-continuous and $\mathcal{I}_{*g}^*$-continuous.*

Proof. Immediately follows from Theorem 2.23.

Corollary 3.14. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space and $\mathcal{I} = \{\phi\}$, for a function $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$, the following are equivalent:*

1. *f is continuous;*
2. *f is LC-continuous and g^*-continuous [10];*
3. *f is $^*g$-LC*-continuous and g^*-continuous;*
4. *f is $^*g$-LC*-continuous and $^*g^*$-continuous;*
5. *f is $^*g_t$-continuous and $^*g^*$-continuous.*

4 On $\mathcal{I}^*_{*g}$-normal Spaces

Definition 4.1. *An ideal topological space (X, τ, $\mathcal{I}$) is said to be $\mathcal{I}^*_{*g}$-normal, if for any two disjoint closed sets F and G in (X, τ, $\mathcal{I}$) there exist disjoint $\mathcal{I}^*_{*g}$-open sets U and V such that F $\subseteq$ U and G $\subseteq$ V.*

Theorem 4.2. *For an ideal topological space (X, τ, $\mathcal{I}$), the following are equivalent:*

1. *(X, τ, $\mathcal{I}$) is $\mathcal{I}^*_{*g}$-normal.*
2. *For each closed set F and for each open set V containing F, there exists an $\mathcal{I}^*_{*g}$-open set U such that F $\subseteq$ U $\subseteq$ cl*(U) $\subseteq$ V.*

Proof. (1) $\Rightarrow$ (2): Let F be a closed subset of X and D be an open set such that F $\subseteq$ D. Then F and X $-$ D are disjoint closed sets in X. Therefore, by hypothesis there exist disjoint $\mathcal{I}^*_{*g}$-open sets U and V such that F $\subseteq$ U and X $-$ D $\subseteq$ V. Hence F $\subseteq$ U $\subseteq$ X $-$ V $\subseteq$ D. Now with D being open it is also *g-open and since X $-$ V is $\mathcal{I}^*_{*g}$-closed, we have F $\subseteq$ U $\subseteq$ cl*(U) $\subseteq$ cl*(X $-$ V) $\subseteq$ D.

(2) $\Rightarrow$ (1): Let F and G be two disjoint closed subsets of X. Then by hypothesis, there exists an $\mathcal{I}^*_{*g}$-open set U such that F $\subseteq$ U $\subseteq$ cl*(U) $\subseteq$ X $-$ G. If we take W = X $-$ cl*(U), then U and W are the required disjoint $\mathcal{I}^*_{*g}$-open sets containing F and G respectively. Hence (X, τ, $\mathcal{I}$) is $\mathcal{I}^*_{*g}$-normal.

Theorem 4.3. *Let (X, τ, $\mathcal{I}$) be $\mathcal{I}^*_{*g}$-normal. Then the following statements are true.*

1. *If F is closed and A is an *g-closed set such that A $\cap$ F = ϕ, then there exist disjoint $\mathcal{I}^*_{*g}$-open sets U and V such that A $\subseteq$ U and F $\subseteq$ V.*
2. *If A is closed and B is an *g-open set containing A, then there exists $\mathcal{I}^*_{*g}$-open set U such that A $\subseteq$ int*(U) $\subseteq$ U $\subseteq$ B.*
3. *If A is *g-closed and B is an open set containing A, then there exists $\mathcal{I}^*_{*g}$-open set U such that A $\subseteq$ U $\subseteq$ cl*(U) $\subseteq$ B.*

Proof. (1) Since A $\cap$ F = ϕ, A $\subseteq$ X $-$ F, where X $-$ F is open and hence ω-open. Hence by hypothesis, cl(A) $\subseteq$ X $-$ F. Since cl(A) $\cap$ F = ϕ and X is $\mathcal{I}^*_{*g}$-normal, there exist disjoint $\mathcal{I}^*_{*g}$-open sets U and V such that cl(A) $\subseteq$ U and F $\subseteq$ V. The proofs of (2) and (3) are similar.

Definition 4.4. *A function f : (X, τ, $\mathcal{I}$) $\rightarrow$ (Y, σ, $\mathcal{J}$) is said to be $\mathcal{I}^*_{*g}$-irresolute if f^{-1}(V) is $\mathcal{I}^*_{*g}$-open in (X, τ, $\mathcal{I}$) for every $\mathcal{J}^*_{*g}$-open set V in (Y, σ, $\mathcal{J}$).*

Theorem 4.5. *If f : (X, τ, $\mathcal{I}$) $\rightarrow$ (Y, σ, $\mathcal{J}$) is an $\mathcal{I}^*_{*g}$-irresolute closed injection and Y is an $\mathcal{J}^*_{*g}$-normal space, then X is $\mathcal{I}^*_{*g}$-normal.*

Proof. Let F and G be disjoint closed sets of X. Since f is a closed injection, f(F) and f(G) are disjoint closed sets of Y. Now from the $\mathcal{J}^*_{*g}$-normality of Y, there exist disjoint $\mathcal{J}^*_{*g}$-open sets U and V such that f(F) $\subseteq$ U and f(G) $\subseteq$ V. Also since, f is $\mathcal{I}^*_{*g}$-irresolute, f^{-1}(U) and f^{-1}(V) are disjoint $\mathcal{I}^*_{*g}$-open sets containing F and G respectively. Hence by Definition 4.1, it follows that X is $\mathcal{I}^*_{*g}$-normal.

5 Conclusion

Topology is an area of Mathematics concerned with the properties of space that are preserved under continuous deformations including stretching and bending, but not tearing. By the middle of the 20th century, topology had become a major branch of Mathematics.

Topology as a branch of Mathematics can be formally defined as the study of qualitative properties of certain objects that are invariant under a certain kind of transformation especially those properties that are invariant under a certain kind of equivalence and it is the study of those properties of geometric configurations which remain invariant when these configurations are subjected to one-to-one bicontinuous transformations or homeomorphisms. Topology operates with more general concepts than analysis. Differential properties of a given transformation are nonessential for topology but bicontinuity is essential. As a consequence, topology is often suitable for the solution of problems to which analysis cannot give the answer.

Though the concept of topology has been identified as a difficult territory in Mathematics, we have taken it up as a challenge and cherishingly worked out this research study. Ideal Topology is a generalization of topology in classical mathematics, but it also has its own unique characteristics. It can also further up the understanding of basic structure of classical mathematics and offers new methods and results in obtaining significant results of classical mathematics. Moreover it also has applications in some important fields of Science and Technology.

We introduce the notions of $^*g$-$\mathcal{I}$-LC*-sets, $\mathcal{I}^*_{*g}$-closed sets and $\mathcal{I}$-$^*g_t$-sets. Also we define the notions of $^*g$-$\mathcal{I}$-LC*-continuous maps, $\mathcal{I}^*_{*g}$-continuous maps, $\mathcal{I}$-$^*g_t$-continuous maps and obtain decompositions of $*$-continuity.

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ON SOME DECOMPOSITIONS OF FUZZY SOFT CONTINUITY

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Abstract – In this article, some open-like fuzzy soft sets such as fuzzy soft semi-open set, fuzzy soft pre-open set, fuzzy soft α-open set and corresponding variants of fuzzy soft continuous functions are introduced and discussed. Some other variants of fuzzy soft sets such as fuzzy soft semi-preclosed set, fuzzy soft t-set, fuzzy soft α*-set, fuzzy soft regular open set, fuzzy soft *B*-set, fuzzy soft *C*-set and fuzzy soft *D*(α)-set are defined and some properties of these sets are studied and investigated. Some continuous-like functions are introduced and we obtained some decomposition of fuzzy soft continuity.

***Keywords* –** *Soft sets, fuzzy sets, fuzzy soft sets, fuzzy soft B-sets, fuzzy soft B-continuous function, fuzzy soft C-continuous function, fuzzy soft D(α)-continuous function.*

1 Introduction

The notion of continuity is always considered as an important concept in topological study and investigations. It is seen from existing literatures that several weak forms of continuity were introduced both for general and fuzzy topology to investigate and find deep properties of continuity. Each of the weak forms of continuity is strictly weaker than continuity. Theoretically, for each weak form of continuity, there is another weak form of continuity such that both of them imply continuity. This gives rise to different decompositions of continuous function. A classical example towards decomposition of continuity is the paper of N. Levine [8]. Inception of concept of soft set of Molodtsov [10] opened different directions for subsequent rapid developments, encompassing various basic concepts and results of topology for their generalizations to soft settings.

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In 2011, Shabir and Naz [14] initiated the study of soft topological spaces. In 2001, Maji et al. [9], introduced the concept of fuzzy soft set. Analytical part of fuzzy soft set theory practically began with the work of B. Tanay et al.[15]. Recently, some researchers have worked to find some decompositions of continuity in soft topological spaces. In this paper, we proposed to define some open-like fuzzy soft sets and investigate for some decompositions of fuzzy soft continuity.

In section 2, some open-like fuzzy soft sets such as fuzzy soft semi-open set, fuzzy soft pre-open set, fuzzy soft α-open set and corresponding variants of fuzzy soft continuous functions are introduced and discussed.

In section 3, we defined fuzzy soft semi-preclosed set, fuzzy soft t-set, fuzzy soft α*-set, fuzzy soft regular open set, fuzzy soft *B*-set, fuzzy soft *C*-set and fuzzy soft *D*(α)-set. We studied these sets and investigate some properties of these sets.

In section 4, we defined some continuous-like functions and we obtained some decompositions of fuzzy soft continuity.

2 Preliminaries

Definition 2.1. [10**]** Let $A \subseteq E$. A pair (F, A) is called a soft set over U if and only if F is a mapping given by $F : A \rightarrow P(U)$ such that $F(e) = \varphi$ if $e \notin A$ and $F(e) \neq \varphi$ if $e \in A$, where φ stands for the empty set, U is an initial universe set, E is the set of parameters and $P(U)$ is the set of all subsets of U. Here F is called approximate function of the soft set (F, A) and the value $F(e)$ is a set called e–element of the soft set. In other words, the soft set is a parameterized family of subsets of the set U.

Definition 2.2. [9] Let U be an initial universe set, let E be a set of parameters, let $A \subseteq E$. A pair (F, A) is called a fuzzy soft set over U if and only if F is a mapping given by $F : A \rightarrow I^U$ such that $F(e) = 0_U$ if $e \notin A$ and $F(e) \neq 0_U$ if $e \in A$, where $0_U(u) = 0$ for all $u \in U$. Here F is called approximate function of the fuzzy soft set (F, A) and the value $F(e)$ is a fuzzy set called e–element of the fuzzy soft set (F, A). Thus a fuzzy soft set (F, A) over U can be represented by the set of ordered pairs $(F, A) = \{ (e, F(e)) : e \in A,\ F(e) \in I^U \}$. In other words, the fuzzy soft set is a parameterized family of fuzzy subsets of the set U.

Definition 2.3. [3,4] A fuzzy soft set (F, A) over U is called a *null* fuzzy soft set, denoted by $\tilde{0}_E$, if $F(e) = 0_U$ for all $e \in A \subseteq E$.

Remark 2.4. According to the definition of fuzzy soft set, i.e., $F(e) \neq 0_U$ if $e \in A \subseteq E$, 0_U does not belong to the co-domain of F. Therefore, the concept of null fuzzy soft set can be defined as follows.

Definition 2.5. A fuzzy soft set (F, A) over U is called a *null* fuzzy soft set or an *empty* fuzzy soft set, whenever $A = \varphi$.

Definition 2.6. A fuzzy soft set (F, A) over U is said to be an *A–universal* fuzzy soft set if $F(e) = 1_U$ if $e \in A$, where $1_U(u) = 1$ for all $u \in U$.

An *A–universal* fuzzy soft set is denoted by $\tilde{1}_A$.

Definition 2.7. [13] A fuzzy soft set (F, A) over U is said to be an *absolute* fuzzy soft set or a *universal* fuzzy soft set if $A = E$ and $F(e) = 1_U$ for all $e \in E$.

An *absolute* fuzzy soft set is denoted by $\tilde{1}_E$.

Definition 2.8. [9] A fuzzy soft set (F, A) is said to be a fuzzy soft subset of a fuzzy soft set (G, B) over a common universe U if $A \subseteq B$ and $F(e) \leq G(e)$ for all $e \in A$.

We redefine fuzzy soft subset as follows.

Definition 2.9. A fuzzy soft set (F, A) is said to be a fuzzy soft subset of a fuzzy soft set (G, B) over a common universe U if either $F(e) = 0_U$ for all $e \in A$ or $A \subseteq B$ and $F(e) \leq G(e)$ for all $e \in A$.

If a fuzzy soft set (F, A) is a fuzzy soft subset of a fuzzy soft set (G, B) we write $(F, A) \tilde{\subseteq} (G, B)$.

(F, A) is said to be a fuzzy soft superset of a fuzzy soft set (G, B) if (G, B) is a fuzzy soft subset of (F, A) and we write $(F, A) \tilde{\supseteq} (G, B)$.

Definition 2.10. [13] Two fuzzy soft sets (F, A) and (G, B) over a common universe are said to be equal, denoted by $(F, A) = (G, B)$, if $(F, A) \tilde{\subseteq} (G, B)$ and $(G, B) \tilde{\subseteq} (F, A)$. That is, if $F(e) \leq G(e)$ and $G(e) \leq F(e)$ for all $e \in E$.

Definition 2.11. [1,13] The intersection of two fuzzy soft sets (F, A) and (G, B) over a common universe U is the fuzzy soft set (H, C) where $C = A \cap B$ and $H(e) = F(e) \wedge G(e)$ for all $e \in C$ and we write $(H, C) = (F, A) \tilde{\cap} (G, B)$.

In particular, if $A \cap B = \varphi$ or $F(e) \wedge G(e) = 0_U$ for every $e \in A \cap B$, then $H(e) = 0_U$.

Definition 2.12. [9] The union of two fuzzy soft sets (F, A) and (G, B) over a common universe U is the fuzzy soft set (H, C) where $C = A \cup B$ and for all $e \in C$, $H(e) = F(e)$ if $e \in A - B$, $H(e) = G(e)$ if $e \in B - A$, $H(e) = F(e) \vee G(e)$ if $e \in A \cap B$. In this case we write $(H, C) = (F, A) \tilde{\cup} (G, B)$.

Definition 2.13. [9] The complement of a fuzzy soft set (F, A), denoted by $(F, A)^C$, is defined as $(F, A)^C = (F^C, \neg A)$, where $F^C : \neg A \rightarrow I^U$ is a mapping given by $F^C(e) = (F(\neg e))^C$ for all $e \in \neg A$.

Alternatively, the complement of a fuzzy soft set can be defined as follows.

Definition 2.14. [15] The fuzzy soft complement of a fuzzy soft set (F, A), denoted by $(F, A)^C$, is defined as $(F, A)^C = (F^C, A)$, where $F^C(e) = 1 - F(e)$ for every $e \in A$. Clearly, $((F, A)^C)^C = (F, A)$ and $(\tilde{1}_E)^C = \tilde{0}_E$ and $(\tilde{0}_E)^C = \tilde{1}_E$.

Proposition 2.15. Let (F, A) be a fuzzy soft set over (U, E). Then

1. $(F, A) \tilde{\cup} (F, A) = (F, A)$, $(F, A) \tilde{\cap} (F, A) = (F, A)$

2. $(F, A)\tilde{\cup}\tilde{0}_E = (F, A)$, $(F, A)\tilde{\cap}\tilde{0}_E = \tilde{0}_E$
3. $(F, A)\tilde{\cup}\tilde{1}_E = \tilde{1}_E$, $(F, A)\tilde{\cap}\tilde{1}_E = (F, A)$
4. $(F, A)\tilde{\cup}(F, A)^C = \tilde{1}_E$, $(F, A)\tilde{\cap}(F, A)^C = \tilde{0}_E$

Proposition 2.16. Let (F, A), $(G\ B)$, (H, C) be fuzzy soft sets over (U, E). Then

1. $(F, A)\tilde{\cup}(G\ B) = (G, B)\tilde{\cup}(F, A)$, $(F, A)\tilde{\cap}(G, B) = (G, B)\tilde{\cap}(F, A)$
2. $((F, A)\tilde{\cup}(G, B))^C = (G, B)^C\tilde{\cap}(F, A)^C$, $((F, A)\tilde{\cap}(G, B))^C = (G, B)^C\tilde{\cup}(F, A)^C$
3. $((F, A)\tilde{\cup}(G, B))\tilde{\cup}(H, C) = (F, A)\tilde{\cup}((G, B)\tilde{\cup}(H, C))$, $((F, A)\tilde{\cap}(G, B))\tilde{\cap}(H, C) = (F, A)\tilde{\cap}((G, B)\tilde{\cap}(H, C))$
4. $(F, A)\tilde{\cup}((G, B)\tilde{\cap}(H, C)) = ((F, A)\tilde{\cup}(G\ B))\tilde{\cap}((F, A)\tilde{\cup}(H, C))$, $(F, A)\tilde{\cap}((G, B))\tilde{\cup}(H, C)) = ((F, A)\tilde{\cap}(G, B))\tilde{\cup}((F, A)\tilde{\cap}(H, C))$

3 Fuzzy Soft Pre-open Set, Fuzzy soft α-open Set, Fuzzy Soft semi-open Set

In this section, we defined fuzzy soft pre-open set, fuzzy soft α-open set and we mentioned fuzzy soft semi-open set [5]. Then we defined the corresponding weaker forms of fuzzy soft continuous functions, namely, fuzzy soft pre-continuous, fuzzy soft α-continuous and fuzzy soft semi-continuous functions.

Let us recall the following definitions, propositions and theorems.

Definition 3.1. [13,15] A fuzzy soft topology τ on (U, E) is a family of fuzzy soft sets over (U, E), satisfying the following properties:

1. $\tilde{0}_E, \tilde{1}_E \in \tau$
2. If $(F, A), (G, B) \in \tau$ then $(F, A)\tilde{\cap}(G, B) \in \tau$.
3. If $(F, A)_\alpha \in \tau$, $\forall \alpha \in \Lambda$ then $\tilde{\bigcup}_{\alpha\in\Lambda} (F, A)_\alpha \in \tau$.

Definition 3.2. [13,15] If τ is a fuzzy soft topology on (U, E), the triple (U, E, τ) is said to be a fuzzy soft topological space. Each member of τ is called a fuzzy soft open set in (U, E, τ). The family of all Fuzzy soft open sets is denoted by $FSOS(U, E)$.

Definition 3.3. [12] Let (U, E, τ) be a fuzzy soft topological space. A fuzzy soft set is called fuzzy soft closed if its complement is a member of τ.

Proposition 3.4. [12] Let (U, E, τ) be a fuzzy soft topological space and let τ' be the collection of all fuzzy soft closed sets. Then

1. $\tilde{0}_E, \tilde{1}_E \in \tau'$
2. If $(F, A), (G\ B) \in \tau'$ then $(F, A)\tilde{\cup}(G, B) \in \tau'$.

3. If $(F, A)_\alpha \in \tau'$, $\forall \alpha \in \Lambda$ then $\tilde{\bigcap}_{\alpha \in \Lambda} (F, A)_\alpha \in \tau'$.

Definition 3.5.[12,15] Let (U, E, τ) be a fuzzy soft topological space. Let (F, A) be a fuzzy soft set over (U, E). Then the fuzzy soft closure of (F, A), denoted by $\overline{(F,A)}$, is defined as the intersection of all fuzzy soft closed sets which contain (F, A). That is, $\overline{(F,A)} = \tilde{\bigcap}\{(G, B) : (G, B)$ is fuzzy soft closed and $(F, A) \tilde{\subseteq} (G, B)\}$. Clearly, $\overline{(F,A)}$ is the smallest fuzzy soft closed set over (U, E) which contain (F, A). It is also clear that $\overline{(F,A)}$ is fuzzy soft closed and $(F, A) \tilde{\subseteq} \overline{(F,A)}$.

Theorem 3.6.[6] Let (U, E, τ) be a fuzzy soft topological space. Let (F, A) and (G, B) are fuzzy soft sets over (U, E). Then

1. $\overline{\tilde{0}_E} = \tilde{0}_E$, $\overline{\tilde{1}_E} = \tilde{1}_E$.
2. $(F, A) \tilde{\subseteq} \overline{(F,A)}$.
3. *(F, A)* is fuzzy soft closed if and only if $(F, A) = \overline{(F,A)}$.
4. $\overline{(\overline{(F,A)})} = \overline{(F,A)}$.
5. $(F, A) \tilde{\subseteq} (G, B)$ implies $\overline{(F,A)} \tilde{\subseteq} \overline{(G,B)}$.
6. $\overline{(F,A) \tilde{\cup} (G,B)} = \overline{(F,A)} \tilde{\cup} \overline{(G,B)}$.
7. $\overline{(F,A) \tilde{\cap} (G,B)} \tilde{\subseteq} \overline{(F,A)} \tilde{\cap} \overline{(G,B)}$

Definition 3.7. [12,15] Let (U, E, τ) be a fuzzy soft topological space. Let (F, A) be a fuzzy soft set over (U, E). Then the fuzzy soft interior of (F, A), denoted by $(F, A)^o$, is defined as the union of all fuzzy soft open sets contained in (F, A). That is, $(F, A)^o = \tilde{\bigcup}\{ (G, B) : (G, B)$ is fuzzy soft open and $(G, B) \tilde{\subseteq} (F, A) \}$. Clearly, $(F, A)^o$ is the largest fuzzy soft open set over (U, E) contained in (F, A). It is also clear that $(F, A)^o$ is fuzzy soft open and $(F, A)^o \tilde{\subseteq} (F, A)$.

Theorem 3.8. [6] Let (U, E, τ) be a fuzzy soft topological space. Let (F, A) and (G, B) are fuzzy soft sets over (U, E). Then

1. $(\tilde{0}_E)^o = \tilde{0}_E$ and $(\tilde{1}_E)^o = \tilde{1}_E$.
2. $(F, A)^o \tilde{\subseteq} (F, A)$.
3. $((F, A)^o)^o = (F, A)^o$.
4. (F, A) is a fuzzy soft open set if and only if $(F, A)^o = (F, A)$.
5. $(F, A) \tilde{\subseteq} (G, B)$ implies $(F, A)^o \tilde{\subseteq} (G, B)^o$.
6. $(F, A)^o \tilde{\cap} (G, B)^o = ((F, A) \tilde{\cap} (G, B))^o$.
7. $(F, A)^o \tilde{\cup} (G, B)^o \tilde{\subseteq} ((F, A) \tilde{\cup} (G, B))^o$.

We now define some open-like fuzzy soft sets.

Let us denote a family of fuzzy soft sets over (U, E) by $FSS(U, E)$.

Definition 3.9. [5] Let (U, E, τ) be a fuzzy soft topological space. Let $(F, A) \in FSS(U, E)$. Then (F, A) is said to be fuzzy soft semi-open if $(F, A) \tilde{\subseteq} \overline{(F,A)^o}$. The family of all fuzzy soft semi-open sets is denoted by $FSSOS(U, E)$.

Example 3.10. Let $U = \{p, q, r\}$, $E = \{e_1, e_2, e_3, e_4\}$. $A = \{e_1\} \subseteq E$, $B = \{e_2\} \subseteq E$. Let us consider the following fuzzy soft sets over (U, E).

$(F, A) = \{F(e_1) = \{p/0.2, q/0.7, r/0.6\}, F(e_2) = \{p/0, q/0, r/0\}, F(e_3) = \{p/0, q/0, r/0\}, F(e_4) = \{p/0, q/0, r/0\}\}$
$(G, B) = \{G(e_1) = \{p/0, q/0, r/0\}, G(e_2) = \{p/0.1, q/0.3, r/0.2\}, G(e_3) = \{p/0, q/0, r/0\}, G(e_4) = \{p/0, q/0, r/0\}\}$
Let us consider the fuzzy soft topology $\tau = \{\tilde{0}_E, \tilde{1}_E, (G, B)\}$ over (U, E). Then (F, A) is fuzzy soft semi-open set.

Definition 3.11. Let (U, E, τ) be a fuzzy soft topological space. Let $(F, A) \in FSS(U, E)$. Then (F, A) is said to be

1. Fuzzy soft pre-open if $(F, A) \tilde{\subseteq} (\overline{F,A})^o$,
2. Fuzzy soft α-open if $(F, A) \tilde{\subseteq} (\overline{(F,A)^o})^o$.

The family of all Fuzzy soft pre-open (Fuzzy soft α-open) is denoted by $FSPOS(U, E)$ ($FS\alpha OS(U, E)$).

Remark 3.12 $\tilde{0}_E$ and $\tilde{1}_E$ are always fuzzy soft pre-open.

Remark 3.13 $\tilde{0}_E$ and $\tilde{1}_E$ are always fuzzy soft α-open.

Remark 3.14 Every fuzzy soft open set is a fuzzy soft pre-open set but not conversely.

Example 3.15 Let $U = \{p, q, r\}$, $E = \{e_1, e_2, e_3, e_4\}$. $A = \{e_1\} \subseteq E$, $B = \{e_3\} \subseteq E$. Let us consider the following fuzzy soft sets over (U, E).

$(F, A) = \{F(e_1) = \{p/0.1, q/0.7, r/0.9\}, F(e_2) = \{p/0, q/0, r/0\}, F(e_3) = \{p/0, q/0, r/0\}, F(e_4) = \{p/0, q/0, r/0\}\}$
$(G, B) = \{G(e_1) = \{p/0, q/0, r/0\}, G(e_2) = \{p/0, q/0, r/0\}, G(e_3) = \{p/0.4, q/0.2, r/0.7\}, G(e_4) = \{p/0, q/0, r/0\}\}$
Let us consider the fuzzy soft topology $\tau = \{\tilde{0}_E, \tilde{1}_E, (G, B)\}$ over (U, E). Then (F, A) is fuzzy soft pre-open set but (F, A) is not a fuzzy soft open.

Remark 3.16 Every fuzzy soft open set is a fuzzy soft α-open set but not conversely.

Example 3.17 Let $U = \{p, q, r\}$, $E = \{e_1, e_2, e_3\}$. $A = \{e_2\} \subseteq E$, $B = \{e_3\} \subseteq E$. Let us consider the following fuzzy soft sets over (U, E).

$(F, A) = \{F(e_1) = \{p/0, q/0, r/0\}, F(e_2) = \{p/0.7, q/0.6, r/0.5\}, F(e_3) = \{p/0, q/0, r/0\}\}$

$(G, B) = \{G(e_1) = \{ p/0, q/0, r/0\}, G(e_2) = \{ p/0, q/0, r/0\}, G(e_3) = \{ p/0.1, q/0.3, r/0.2\}\}$

Let us consider the fuzzy soft topology $\tau = \{\tilde{0}_E, \tilde{1}_E, (G, B)\}$ over (U, E). Then (F, A) is fuzzy soft α-open set but not a fuzzy soft open set.

Theorem 3.18. Let (U, E, τ) be a fuzzy soft topological space. Let (F, A) and (G, B) are fuzzy soft sets over (U, E). If either (F, A) is a fuzzy soft semi-open set or (G, B) is a fuzzy soft semi-open set $(\overline{(F, A) \tilde{\cap} (G, B)})^o = (\overline{(F, A)})^o \tilde{\cap} (\overline{(G, B)})^o$

Definition 3.19. [7] Let FSS(U, E_1) and $FSS(V, E_2)$ be the families of all fuzzy soft sets over (U, E_1) and (V, E_2) respectively. Let $u : U \rightarrow V$ and $p : E_1 \rightarrow E_2$ be two functions. Then f_{pu} is called a fuzzy soft mapping from $FSS(U, E_1)$ to $FSS(V, E_2)$, denoted by $f_{pu} : FSS(U, E_1) \rightarrow FSS(V, E_2)$ and defined as follows:

(1) Let (F, A) be a fuzzy soft set in $FSS(U, E_1)$. Then the image of (F, A) under the fuzzy soft mapping f_{pu} is the fuzzy soft set over (V, E_2) defined by $f_{pu}((F, A))$, where

$$f_{pu}((F, A))(e_2)(y) = \begin{cases} \bigvee_{x \in u^{-1}(y)} \Big(\bigvee_{e_1 \in p^{-1}(e_2) \cap A} F(e_1) \Big)(x) & \text{if } u^{-1}(y) \neq \varphi, \text{ and } p^{-1}(e_2) \cap A \neq \varphi. \\ 0_V & \text{otherwise.} \end{cases}$$

(2) Let (G, B) be a fuzzy soft set in $FSS(V, E_2)$. Then the pre-image (inverse image) of (G, B) under the fuzzy soft mapping f_{pu} is the fuzzy soft set over (U, E_1) defined by $f^{-1}_{pu}((G, B))$, where

$$f^{-1}_{pu}((G, B))(e_1)(x) = \begin{cases} G(p(e_1))(u(x)) & \text{for } p(e_1) \in B \\ 0_U & \text{otherwise.} \end{cases}$$

Definition 3.20. If p and u are injective in definition 3.19, then the fuzzy soft mapping f_{pu} is said to be injective. If p and u are surjective then the fuzzy soft mapping f_{pu} is said to be surjective. If p and u are constant then f_{pu} is called constant.

Definition 3.21. [2] Let (U, E_1, τ_1) and (V, E_2, τ_2) be two fuzzy soft topological spaces. A fuzzy soft mapping $f_{pu} : (U, E_1, \tau_1) \rightarrow (V, E_2, \tau_2)$ is called fuzzy soft continuous if $f^{-1}_{pu}((G, B)) \in \tau_1$ for all $(G, B) \in \tau_2$.

Definition 3.22. Let (U, E_1, τ_1) *and* (V, E_2, τ_2) be two fuzzy soft topological spaces. A fuzzy soft mapping $f_{pu} : (U, E_1, \tau_1) \rightarrow (V, E_2, \tau_2)$ is called

1. fuzzy soft pre-continuous if $f^{-1}_{pu}((G, B)) \in FSPOS(U, E_1)$ for all $(G, B) \in FSOS(V, E_2)$,
2. fuzzy soft α-continuous if $f^{-1}_{pu}((G, B)) \in FS\alpha OS(U, E_1)$ for all $(G, B) \in FSOS(V, E_2)$,
3. fuzzy soft semi-continuous if $f^{-1}_{pu}((G, B)) \in FSSOS(U, E_1)$ for all $(G, B) \in FSOS(V, E_2)$.

Remark 3.23. A fuzzy soft continuous mapping is fuzzy soft pre-continuous but not conversely.

Example 3.24. Let $U = \{p, q, r\}$, $E = \{e_1, e_2, e_3, e_4\}$. $A = \{e_1\} \subseteq E$, $B = \{e_3\} \subseteq E$. Let us consider the following fuzzy soft sets over *(U, E).*

$(F, A) = \{F(e_1) = \{ p/0.1, q/0.7, r/0.9\}, F(e_2) = \{ p/0, q/0, r/0\}, F(e_3) = \{ p/0, q/0, r/0\}, F(e_4) = \{ p/0, q/0, r/0\}\}$
$(G, B) = \{G(e_1) = \{ p/0, q/0, r/0\}, G(e_2) = \{ p/0, q/0, r/0\}, G(e_3) = \{ p/0.4, q/0.2, r/0.7\}, G(e_4) = \{ p/0, q/0, r/0\}\}$

Let us consider the fuzzy soft topology $\tau_1 = \{\tilde{0}_E, \tilde{1}_E, (G, B)\}$, and $\tau_2 = \{\tilde{0}_E, \tilde{1}_E, (F, A)\}$ over (U, E). We define the fuzzy soft mapping $f_{pu} : (U, E, \tau_1) \to (U, E, \tau_2)$ where $u : U \to U$ and $p : E \to E$ be a mapping defined as $u(p) = p$, $u(q) = q$, $u(r) = r$ and $p(e_1) = e_1$, $p(e_2) = e_2$, $p(e_3) = e_3$, $p(e_4) = e_4$. Now, $f^{-1}{}_{pu}((F, A)) = (F, A) \not\in (U, E, \tau_1)$ but (F, A) is fuzzy soft pre-open set. Thus $f_{pu} : (U, E, \tau_1) \to (U, E, \tau_2)$ is fuzzy soft pre-continuous; but not fuzzy soft continuous.

Remark 3.25. A fuzzy soft continuous mapping is fuzzy soft α-continuous but not conversely.

Example 3.26. Let U = $\{p, q, r\}$, E = $\{e_1, e_2, e_3\}$. $A = \{e_2\} \subseteq E$, $B = \{e_3\} \subseteq E$. Let us consider the following fuzzy soft sets over *(U, E).*

$(F, A) = \{F(e_1) = \{ p/0, q/0, r/0\}, F(e_2) = \{ p/0.7, q/0.6, r/0.5\}, F(e_3) = \{ p/0, q/0, r/0\}\}$

$(G, B) = \{G(e_1) = \{ p/0, q/0, r/0\}, G(e_2) = \{ p/0, q/0, r/0\}, G(e_3) = \{ p/0.1, q/0.3, r/0.2\}\}$

Let us consider the fuzzy soft topology $\tau_1 = \{\tilde{0}_E, \tilde{1}_E, (G, B)\}$, and $\tau_2 = \{\tilde{0}_E, \tilde{1}_E, (F, A)\}$ over *(U, E).* We define the fuzzy soft mapping $f_{up} : (U, E, \tau_1) \to (U, E, \tau_2)$ where $u : U \to U$ and $p : E \to E$ be a mapping defined as $u(p) = p, u(q) = q, u(r) = r, p(e_1) = e_1, p(e_2) = e_2, p(e_3) = e_3$
Now, $f_{up}^{-1}(F, A) = (F, A) \tilde{\not\in} (U, E, \tau_1)$ but (F, A) is fuzzy soft α-open set.

Thus $f_{up} : (U, E, \tau_1) \to (U, E, \tau_2)$ is fuzzy soft α-continuous; but not fuzzy soft continuous.

4 Fuzzy Soft *B*-Set, Fuzzy Soft *C*-Set, Fuzzy Soft *D*(α)-Set

In this section, we defined fuzzy soft semi-preclosed set, fuzzy soft t-set, fuzzy soft α*-set, fuzzy soft regular open set, fuzzy soft *B*-set, fuzzy soft *C*-set and fuzzy soft *D*(α)-set. We studied these sets and investigate some properties of these sets.

Definition 4.1. Let (U, E, τ) be a fuzzy soft topological space. Let $(F, A) \in FSS(U, E)$. Then (F, A) is said to be

1. fuzzy soft semi-preclosed set if $(\overline{(\overline{(F,A)})^o})^o \tilde{\subseteq}$ (*F*, *A*),
2. fuzzy soft *t*-set if $(F, A)^o = (\overline{(F,A)})^o$,
3. fuzzy soft α*-set if $(\overline{((F,A))^o})^o = (F, A)^o$,
4. fuzzy soft regular open [11] if $(F, A) = (\overline{(F,A)})^o$.

Example 4.2. $\tilde{0}_E$ and $\tilde{1}_E$ are always fuzzy soft semi pre-closed set, fuzzy soft *t*-set, fuzzy soft α*-set, fuzzy soft regular open set.

Remark 4.3. It is clear from definition that in a fuzzy soft topological space (U, E, τ), every fuzzy soft regular open set is fuzzy soft open set, but the converse is not true, which follows from the following example.

Example 4.4. Let U = {*a*, *b*, *c*}, E = {e_1, e_2, e_3, e_4}. $A = \{ e_1, e_2\} \subseteq E$, $B = \{ e_1, e_2, e_3\} \subseteq E$. Let us consider the following fuzzy soft sets over (*U*, *E*).

$(F, A) = \{F(e_1) = \{ a/0.5, b/0.2, c/0\}, F(e_2) = \{ a/0.7, b/0.6, c/0.3\}, F(e_3) = \{ a/0, b/0, c/0\}, F(e_4) = \{ a/0, b/0, c/0\}\}$
$(G, B) = \{G(e_1) = \{ a/0.5, b/0.3, c/0\}, G(e_2) = \{ a/0.7, b/0.8, r/0.5\}, G(e_3) = \{ a/0.4, b/0.9, c/0.8\}, G(e_4) = \{ a/0, b/0, c/0\}\}$
Let us consider the fuzzy soft topology $\tau_1 = \{\tilde{0}_E, \tilde{1}_E, (F, A), (G, B)\}$, over (*U*, *E*). Now,
$(F, A)^C = (F^C, A) = \{ F^C(e_1) = \{ a/0.5, b/0.8, c/1\}, F^C(e_2) = \{ a/0.3, b/0.4, c/0.7\}, F^C(e_3) = \{ a/1, b/1, c/1\}, F^C(e_4) = \{ a/1, b/1, c/1\}\}$
and
$(G, B)^C = (G^C, B) = \{G^C(e_1) = \{ a/0.5, b/0.7, c/1\}, G^C(e_2) = \{ a/0.3, b/0.2, c/0.5\}, G^C(e_3) = \{ a/0.6, b/0.1, c/0.2\}, G^C(e_4) = \{ a/1, b/1, c/1\}\}$
and clearly, $(F,A)^C$ and $(G,B)^C$ are fuzzy soft closed sets.
Then the fuzzy soft closure of (F,A), is the intersection of all fuzzy soft closed sets containing (F,A). That is $\overline{(F,A)} = \tilde{1}_E$
The fuzzy soft interior of $\overline{(F,A)}$, is the union of all fuzzy soft open sets contained in $\overline{(F,A)}$.
That is $(\overline{(F,A)})^o = (\tilde{1}_E)^o = \tilde{1}_E$
Hence, (F,A) is open but not a fuzzy soft regular open set.

Remark 4.5. A fuzzy soft *t*-set and fuzzy soft α*-set may not be fuzzy soft regular open set, which follows from the following example.

Example 4.6. Let $U = \{a,b\}$, $E = \{e_1, e_2\}$,
Let us consider the following fuzzy soft sets over (U,E).
$(F,E) = \{F(e_1) = \{a/0.1, b/0.1\}, F(e_2) = \{a/0.1, b/0.2\}\}$
$(G,E) = \{G(e_1) = \{a/0.2, b/0.2\}, G(e_2) = \{a/0.1, b/0.2\}\}$
$(H,E) = \{H(e_1) = \{a/0.2, b/0.7\}, H(e_2) = \{a/0.2, b/0.7\}\}$

$(I,E)=\{I(e_1)=\{a/0.9,b/0.9\},I(e_2)=\{a/0.7,b/0.7\}\}$
$(J,E)=\{J(e_1)=\{a/0.9,b/1\},J(e_2)=\{a/0.7,b/0.9\}\}$
Let us consider the fuzzy soft topology $\tau=\{\tilde{0}_E,\tilde{1}_E,(F,E),(G,E),(H,E),(I,E),(J,E)\}$ over *(U, E)*.
Now, $(E,F)^C=\{F^C(e_1)=\{a/0.9,b/0.9\},F^C(e_2)=\{a/0.9,b/0.8\}\}$
$(G,E)^C=\{G^C(e_1)=\{a/0.8,b/0.8\},G^C(e_2)=\{a/0.9,b/0.8\}\}$
$(H,E)^C=\{H^C(e_1)=\{a/0.8,b/0.3\},H^C(e_2)=\{a/0.8,b/0.3\}\}$
$(I,E)^C=\{I^C(e_1)=\{a/0.1,b/0.1\},I^C(e_2)=\{a/0.3,b/0.3\}\}$
$(J,E)^C=\{J^C(e_1)=\{a/0.1,b/0\},J^C(e_2)=\{a/0.3,b/0.1\}\}$
Clearly, $(F,E)^C,(G,E)^C,(H,E)^C,(I,E)^C$ and $(J,E)^C$ are fuzzy soft closed sets.
Obviously, $(F,E),(G,E),(H,E),(I,E)$ are fuzzy soft α*-sets and also fuzzy soft regular open sets.
Let us consider the fuzzy soft set (K,E) over (U,E) defined as
$(K,E)=\{K(e_1)=\{a/0.4,b/0.5\},K(e_2)=\{a/0.3,b/0.4\}\}$. Then (K,E) is a fuzzy soft t-set and also fuzzy soft α*-set but not a fuzzy soft regular open set.

Definition 4.7. Let (U, E, τ) be a fuzzy soft topological space. Let (F, A) $\in FSS(U, E)$. Then (F, A) is said to be

1. fuzzy soft B-set if (F, A) = (G, B) $\tilde{\cap}$ (H, C), where (G, B) $\in \tau$ and (H, C) is a fuzzy soft t-set,
2. fuzzy soft C-set if (F, A) = (G, B) $\tilde{\cap}$ (H, C), where (G, B) $\in \tau$ and (H, C) is a fuzzy soft α*-set,
3. fuzzy soft D(α)-set if $(F, A)^o = (F, A) \tilde{\cap} (\overline{(F,A)^o})^o$.

Remark 4.8. $\tilde{0}_E$ and $\tilde{1}_E$ are always fuzzy soft B-set, fuzzy soft C-set, fuzzy soft D(α)-set.

Theorem 4.9. Let (U, E, τ) be a fuzzy soft topological space. Then the following statements are equivalent:

1. (F, A) is fuzzy soft α*-set.
2. (F, A) is fuzzy soft semi-preclosed set.
3. (F, A) is fuzzy soft regular open set.

Proof: Straight forward.

Theorem 4.10. Let (U, E, τ) be a fuzzy soft topological space. Then we have the following results:

1. A fuzzy soft semi-open set (F, A) is fuzzy soft t-set if and only if (F, A) is fuzzy soft α*-set.

2. A fuzzy soft α-open set (*F*, *A*) is fuzzy soft α*-set if and only if (*F*, *A*) is fuzzy soft regular open set.

Proof: (1) Let (*F*, *A*) be fuzzy soft semi-open and fuzzy soft *t*-set. Since (*F*, *A*) is a fuzzy soft semi-open set, $\overline{(F,A)^o} = \overline{(F,A)}$. Then $(F, A)^o$ = $\overline{(F,A)}^o = (\overline{(F,A)^o})^o$. Hence (*F*, *A*) is fuzzy soft α*-set.

Conversely, let (*F*, *A*) be fuzzy soft semi-open and fuzzy soft α*-set. Since (*F*, *A*) is a fuzzy soft semi-open set, $\overline{(F,A)^o} = \overline{(F,A)}$. Then $\overline{(F,A)}^o = (\overline{(F,A)^o})^o = (F, A)^o$. Hence (*F*, *A*) is fuzzy soft *t*-set.

(2) Let (*F*, *A*) be fuzzy soft α-open and fuzzy soft α*-set. Then by theorem 3.1, (*F*, *A*) is fuzzy soft semi-preclosed. Since (*F*, *A*) is fuzzy soft α-open, we have $(\overline{(F,A)^o})^o =$ (*F*, *A*) and so $\overline{(F,A)}^o = (\overline{(F,A)^o})^o$ = (*F*, *A*). Hence (*F*, *A*) is fuzzy soft regular open set.

Conversely, proof is obvious.

Theorem 4.11. Let (*U*, *E*, τ) be a fuzzy soft topological space. If (*F*, *A*) is fuzzy soft t-set, then (*F*, *A*) is fuzzy soft α*-set.

Proof: (1) Let (*F*, *A*) is fuzzy soft *t*-set. Then $(F, A)^o = \overline{(F,A)}^o$. We have $(\overline{(F,A)^o})^o = \overline{(F,A)}^o = (F, A)^o$. Hence is (*F*, *A*) is fuzzy soft α*-set.

Theorem 4.12. Let (*U*, *E*, τ) be a fuzzy soft topological space. Then

(1) Every fuzzy soft α*-set is fuzzy soft *C*-set.

(2) Every fuzzy soft open set is fuzzy soft *C*-set.

Proof: The proof of (1) and (2) are obvious since $\tilde{1}_E$ is both fuzzy soft open and fuzzy soft α*-set.

Theorem 4.13. Every fuzzy soft t-set in a fuzzy soft topological space (*U*, *E*, τ) is fuzzy soft *B*-set.

Proof: Let a fuzzy soft set (*F*, *A*) in a fuzzy soft topological space (*U*, *E*, τ) be fuzzy soft *t*-set Let (*G*, *B*) = $\tilde{1}_E \in \tau$. Then (*F*, *A*) = (*G*, *B*)$\tilde{\cap}$(*F*, *A*) and hence (*F*, *A*) is fuzzy soft *B*-set.

Theorem 4.14. Every fuzzy soft t-set in a fuzzy soft topological space (*U*, *E*, τ) is fuzzy soft *C*-set.

Proof: Let a fuzzy soft set (*F*, *A*) in a fuzzy soft topological space (*U*, *E*, τ) be fuzzy soft *t*-set. Then by theorem 3.5, (*F*, *A*) is fuzzy soft *B*-set. As (*F*, *A*) is fuzzy soft *B*-set, (*F*, *A*) = (*G*, *B*)$\tilde{\cap}$(*H*, *C*), where (*G*, *B*) $\in \tau$ and (*H*, *C*) is a fuzzy soft *t*-set. Then $(H, C)^o = \overline{(H,C)}^o \tilde{\supseteq} (\overline{(H,C)^o})^o \tilde{\supseteq} (H, C)^o$. Hence $(H, C)^o = (\overline{(H,C)^o})^o$. Therefore, (*F*, *A*) is fuzzy soft *C*-set.

Remark 4.15. Converse of the theorem 3.6 is not always true.

Example 4.16. Let $U = \{p, q, r\}$, $E = \{e_1, e_2, e_3, e_4\}$. $A = \{e_1\} \subseteq E$, $B = \{e_3\} \subseteq E$ and $C = \{e_4\} \subseteq E$. Let us consider the following fuzzy soft sets over (U, E).

$(F, A) = \{F(e_1) = \{p/0.3, q/0.4, r/0.4\}, F(e_2) = \{p/0, q/0, r/0\}, F(e_3) = \{p/0, q/0, r/0\}, F(e_4) = \{p/0, q/0, r/0\}\}$
$(G, B) = \{G(e_1) = \{p/0, q/0, r/0\}, G(e_2) = \{p/0, q/0, r/0\}, G(e_3) = \{p/0.4, q/0.5, r/0.5\}, G(e_4) = \{p/0, q/0, r/0\}\}$
$(H, C) = \{H(e_1) = \{p/0, q/0, r/0\}, H(e_2) = \{p/0, q/0, r/0\}, H(e_3) = \{p/0, q/0, r/0\}, H(e_4) = \{p/0.7, q/0.6, r/0.6\}\}$
Let us consider the fuzzy soft topology $\tau = \{\tilde{0}_E, \tilde{1}_E, (F, A), (G, B)\}$ over (U, E). Then (H, C) is fuzzy soft C-set but not fuzzy soft t-set.

Theorem 4.17. Let (U, E, τ) be a fuzzy soft topological space. Then (F, A) is fuzzy soft open set if and only if it is both fuzzy soft α-open and fuzzy soft C-set.

Proof: If (F, A) is fuzzy soft open set then clearly (F, A) is fuzzy soft α-open as well as fuzzy soft C-set.
Conversely, let (F, A) be both fuzzy soft α-open and fuzzy soft C-set. Since (F, A) is fuzzy soft C-set, there exist $(G, B) \in \tau$ and a fuzzy soft α*-set (H, C) such that $(F, A) = (G, B) \tilde{\cap} (H, C)$. Since (F, A) is fuzzy soft α-open, we get $(F, A) \tilde{\subseteq} (\overline{(F,A)^o})^o = ((\overline{((G,B) \tilde{\cap} (H,C))^o})^o = (\overline{(G,B)})^o \tilde{\cap} ((\overline{(H,C)^o})^o = (\overline{(G,B)})^o \tilde{\subseteq} (H, C)^o$. Therefore, $(F, A) = (G, B) \tilde{\cap} (H, C) \tilde{\subseteq} (G, B) \tilde{\cap} [(\overline{(G,B)})^o \tilde{\cap} (H, C)^o] = (G, B)) \tilde{\cap} (H, C)^o \tilde{\subseteq} (F, A)$. Consequently, $(F, A) = (G, B) \tilde{\cap} (H, C)^o$. Hence (F, A) is fuzzy soft open set.

Theorem 4.18. Let (U, E, τ) be a fuzzy soft topological space. Then (F, A) is fuzzy soft open set if and only if it is both fuzzy soft pre-open and fuzzy soft B-set.

Proof: If (F, A) is fuzzy soft open set then clearly (F, A) is fuzzy soft pre-open as well as fuzzy soft B-set.

Conversely, let (F, A) be both fuzzy soft pre-open and fuzzy soft B-set. Since (F, A) is fuzzy soft B-set, there exist $(G, B) \in \tau$ and a fuzzy soft t-set (H, C) such that $(F, A) = (G, B) \tilde{\cap} (H, C)$. Since (F, A) is fuzzy soft pre-open, we get $(F, A) \tilde{\subseteq} (\overline{(F,A)})^o = ((\overline{(G,B) \tilde{\cap} (H,C)})^o = (\overline{(G,B)})^o \tilde{\cap} (\overline{(H,C)})^o = (\overline{(G,B)})^o \tilde{\subseteq} (H, C)^o$. Therefore, $(F, A) = (G, B) \tilde{\cap} (H, C) \tilde{\subseteq} (G, B) \tilde{\cap} [(\overline{(G,B)})^o \tilde{\cap} (H, C)^o] = (G, B)) \tilde{\cap} (H, C)^o \tilde{\subseteq} (F, A)$. As a consequence, $(F, A) \in \tau$.

Theorem 4.19. Let (U, E, τ) be a fuzzy soft topological space. Then (F, A) is fuzzy soft open set if and only if it is both fuzzy soft α-open and fuzzy soft D(α)-set.

Proof: If (F, A) is fuzzy soft open set then clearly (F, A) is fuzzy soft α-open as well as fuzzy soft D(α)-set. Conversely, let (F, A) be both fuzzy soft α-open and fuzzy soft D(α)-set. Since (F, A) is fuzzy soft D(α)-set, $(F, A)^o = (F, A) \tilde{\cap} ((\overline{(F,A)^o})^o$. Since $(F,$

A) is fuzzy soft *α*-open, we have $(F, A) \tilde{\subseteq} (\overline{(F,A)^o})^o$. Then $(F, A) \tilde{\cap} (F, A) = (F, A)$ $\tilde{\subseteq} (\overline{(F,A)^o})^o \tilde{\cap} (F, A)$. Hence $(F, A) \tilde{\subseteq} (F, A)^o$. As a consequence, $(F, A) \in \tau$.

5 Decomposition of Fuzzy Soft Continuity

In this section, we obtained some decomposition of fuzzy soft continuity.

Definition 5.1. Let (U, E_1, τ_1) and (V, E_2, τ_2) be two fuzzy soft topological spaces. A fuzzy soft mapping $f_{pu} : (U, E_1, \tau_1) \to (V, E_2, \tau_2)$ is called

1. fuzzy soft *C*-continuous if $f^{-1}_{pu}((G, B))$ is fuzzy soft *C*-set for all $(G, B) \in \tau_2$,
2. fuzzy soft *B*-continuous if $f^{-1}_{pu}((G, B))$ is fuzzy soft *B*-set for all $(G, B) \in \tau_2$,
3. fuzzy soft *D*(α)-continuous if $f^{-1}_{pu}((G, B))$ is fuzzy soft *D*(α)-set for all $(G, B) \in \tau_2$.

Theorem 5.2. Let (U, E_1, τ_1) and (V, E_2, τ_2) be two fuzzy soft topological spaces. A fuzzy soft mapping $fpu : (U, E_1, \tau_1) \to (V, E_2, \tau_2)$ is fuzzy soft continuous function if and only if it is both fuzzy soft α-continuous and fuzzy soft *C*-continuous.

Proof: The proof follows from theorem 4.17.

Theorem 5.3. Let (U, E_1, τ_1) and (V, E_2, τ_2) be two fuzzy soft topological spaces. A *fuzzy soft mapping* $f_{pu} : (U, E_1, \tau_1) \to (V, E_2, \tau_2)$ is fuzzy soft continuous function if and only if it is both fuzzy soft pre-continuous and fuzzy soft *B*-continuous.

Proof: The proof follows from theorem 4.18.

Theorem 5.4. Let (U, E_1, τ_1) and (V, E_2, τ_2) be two fuzzy soft topological spaces. A fuzzy soft mapping $f_{pu} : (U, E_1, \tau_1) \to (V, E_2, \tau_2)$ is fuzzy soft continuous function if and only if it is both fuzzy soft α-continuous and fuzzy soft *D*(α)-continuous.

Proof: The proof follows from theorem 4.19.

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A NOTE ON RELATION BETWEEN POINT-LINE DISPLACEMENT AND EQUIFORM TRANSFORMATION

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Abstract – The present paper studies the relation between the point-line displacement and the equiform transformation in Euclidean 3-space $\mathbb{R}^3$. A point-line can be transformed into another point-line via an equiform transformation. Observing that a point-line is nothing but a line element when its reference point is the origin of the coordinate system, we show that this transformation can also be performed by using dual quaternions.

Keywords – *Line geometry, Line element, Dual quaternion, Equiform kinematics.*

1 Introduction

In kinematics, a point-line is represented by an oriented (directed) line and an incident point on this line. The point-line in kinematics has many implementation areas in manufacturing. Zhang and Ting [8] examine the point-line positions and displacement with the help of dual quaternion algebra. On the other hand, Odehnal, Pottmann and Wallner [1] investigate Plücker coordinates of the line elements in Euclidean three-space $\mathbb{R}^3$. Also, the relation between the point-line displacement and the equiform transformation in Minkowski 3-space is studied in [7].

Our interest in this paper is to investigate the relation between point-line representations and equiform kinematics in Euclidean 3-space $\mathbb{R}^3$. In Section 2, we give dual quaternions and some of their algebraic properties. Then in Section 3, we give the point-line operator, the equiform transformation and the Plücker coordinates of line elements in Euclidean 3-space $\mathbb{R}^3$. We examined the similarity between a point-line and a line element. Finally, we introduce the point-line operator which transforms one point-line to another.

2 Preliminaries

In this section, we give some definitions and fundamental facts about Euclidean three-space $\mathbb{R}^3$, that will be used through the paper.

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2.1 Some Properties of Euclidean 3-space $\mathbb{R}^3$

Theorem 2.1. Let $\vec{u}$, $\vec{v}$ and $\vec{w}$ be two vectors in Euclidean three-space $\mathbb{R}^3$. Then,
i. $\vec{u} \times (\vec{v} \times \vec{w}) = \langle \vec{u}, \vec{w} \rangle \vec{v} - \langle \vec{u}, \vec{v} \rangle \vec{w}$,
ii. $\langle \vec{u} \times \vec{v}, \vec{u} \times \vec{v} \rangle = \langle \vec{u}, \vec{u} \rangle \langle \vec{v}, \vec{v} \rangle - \langle \vec{u}, \vec{v} \rangle^2$,
where $\vec{u} = (u_1, u_2, u_3)$, $\vec{v} = (v_1, v_2, v_3)$ and

$$\begin{aligned} \vec{u} \times \vec{v} &= \begin{vmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} \\ &= (u_2v_3 - u_3v_2,\ u_3v_1 - u_1v_3,\ u_1v_2 - u_2v_1) \end{aligned}$$

is the *vector product* in $\mathbb{R}^3$.

Let $\mathbb{R}^m_n$ be the set of matrices of m rows and n columns.

Definition 2.2. Let $A = [a_{ij}] \in \mathbb{R}^m_n$ and $B = [b_{jk}] \in \mathbb{R}^n_p$. Matrix multiplication is defined as

$$AB = \left[\sum_{j=1}^{n} a_{ij} b_{jk} \right]. \tag{1}$$

Note that AB is an $m \times p$ matrix.

Definition 2.3. An $n \times n$ identity matrix with respect to matrix multiplication, denoted by I_n, is given by

$$I_n = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}_{n \times n}. \tag{2}$$

Note that for every $A \in \mathbb{R}^n_n$, $I_n A = A I_n = A$.

Definition 2.4. A matrix $A \in \mathbb{R}^n_n$ is called invertible if there exists an $n \times n$ matrix B such that $AB = BA = I_n$. Then B is called the inverse of A and is denoted by A^{-1}.

Definition 2.5. The transpose of a matrix $A = [a_{ij}] \in \mathbb{R}^m_n$ is denoted by A^T and defined as $A^T = [a_{ji}] \in \mathbb{R}^n_m$.

Definition 2.6. A matrix $A \in \mathbb{R}^n_n$ is called orthogonal matrix if $A^{-1} = A^T$.

2.2 Dual Quaternions

In analogy with the complex numbers, W. K. Clifford, defined [2] the dual numbers and showed that they form an algebra. As the dual numbers are defined by

$$\begin{aligned} D &= \{A = a + \varepsilon a^* \mid a, a^* \in R\} \quad (3) \\ &= \{A = (a, a^*) \mid a, a^* \in R\}, \quad (4) \end{aligned}$$

where ε is the dual symbol subjected to the rules

$$\varepsilon \neq 0,\ 0\varepsilon = \varepsilon 0 = 0,\ 1\varepsilon = \varepsilon 1 = \varepsilon,\ \varepsilon^2 = 0.$$

The set D of dual numbers is a commutative ring with the operations $(+)$ and $(\cdot)$.
The algebra

$$H = \{q = q_0 + q_1\vec{e}_1 + q_2\vec{e}_2 + q_3\vec{e}_3 \mid q_0,\ q_1,\ q_2,\ q_3 \in R\}$$

of quaternions is defined as the four-dimensional vector space over R having basis $\{1, \vec{e}_1, \vec{e}_2, \vec{e}_3\}$ with the following properties:

$$\begin{aligned} &1)\ (\vec{e}_1)^2 = (\vec{e}_2)^2 = (\vec{e}_3)^2 = 1, \\ &2)\ \vec{e}_1\vec{e}_2 = -\vec{e}_2\vec{e}_1 = \vec{e}_3,\ \vec{e}_2\vec{e}_3 = -\vec{e}_3\vec{e}_2 = \vec{e}_1,\ \vec{e}_3\vec{e}_1 = -\vec{e}_1\vec{e}_3 = \vec{e}_2. \end{aligned} \tag{5}$$

It is clear that H is an associative and not commutative algebra and 1 is the identity element of H. H is called quaternion algebra (see [4] for quaternions).

Similarly, as a consequence of this definition, a dual quaternion Q can also be written as

$$Q = q + \varepsilon q^*,$$

where q and q^* are quaternions.
A dual quaternion

$$Q = q + \varepsilon q^*$$

is characterized by the following properties in [4]:
Scalar and vector parts of a dual quaternion $Q = A_0 + A_1\vec{e}_1 + A_2\vec{e}_2 + A_3\vec{e}_3$ are denoted by $S_Q = A_0$ and $\vec{V}_Q = A_1\vec{e}_1 + A_2\vec{e}_2 + A_3\vec{e}_3$, respectively. The basis $\{1,\ \vec{e}_1,\ \vec{e}_2,\ \vec{e}_3\}$ have the same multiplication properties of basis elements in real quaternions.
Two dual quaternions Q and P obey the following multiplication rule,

$$QP = (qp) + \varepsilon\,(qp^* + pq^*)$$

where $P = p + \varepsilon p^*$, p and p^* are quaternions.
Scalar product of quaternions Q and P is given by

$$\begin{aligned} \langle Q,\ P\rangle &= \langle P,\ Q\rangle \\ &= \langle q,\ p\rangle + \varepsilon\,(\langle q,\ p^*\rangle + \langle q^*,\ p\rangle)\,. \end{aligned} \tag{6}$$

3 Point-line Displacement with Equiform Transformations of $\mathbb{R}^3$

In [1], a point-line is represented by an oriented (directed) line and an incident point on this line. Moreover, an oriented (directed) line can be represented with a unit line vector or signed Plücker coordinates. Thus, we can say the point-line representation can be built up as a dual vector or signed Plücker coordinates.

Let L be an oriented (directed) line and P be a reference point in Euclidean three-space $\mathbb{R}^3$. If we take N as the foot of the perpendicular from P to the directed line L and E is an incident on this directed line L, then the distance h from N to E depends on the location of E and the oriented (directed) line L, (see Fig. 1).

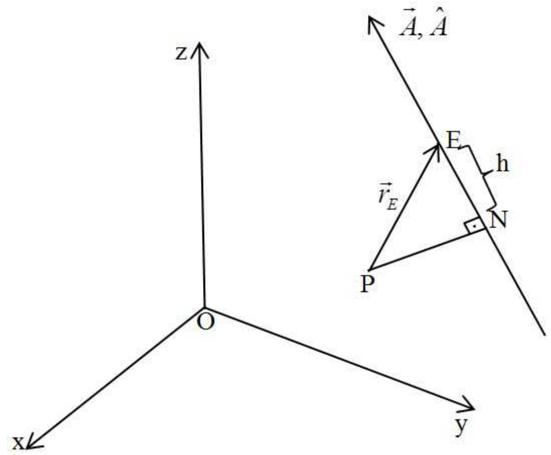

Figure 1. Point-line representation (7)

The oriented (directed) line L passing through points E and N can be represented by a unit dual vector.

Let $\vec{A} = \vec{a} + \varepsilon \overrightarrow{a}_0$ be a unit dual vector satisfying $\langle \vec{a},\ \vec{a} \rangle = 1$ and $\langle \vec{a}, \overrightarrow{a}_0 \rangle = 0$ where the vector $\overrightarrow{a}$ denotes the unit vector along the oriented line, and the vector $\overrightarrow{a}_0$ is the moment vector of the oriented line with respect to the origin of reference frame $O - xyz$.
A point-line can be represented by multiplication of a dual number $exp(\varepsilon h) = 1 + \varepsilon h$, and $\vec{A}$, namely

$$\begin{aligned} \hat{A} &= exp(\varepsilon h)\vec{A} \\ &= \left\| \hat{A} \right\| \vec{A} \\ &= \vec{a} + \varepsilon \overrightarrow{a}'_0, \end{aligned} \tag{8}$$

where $\overrightarrow{a}'_0 = \overrightarrow{a}_0 + h\vec{a}$ and $\hat{A}$ is a dual vector with dual length $exp(\varepsilon h)$.
When we have the point-line coordinates, the incident offset, the directed line, and the incident can be determined easily. Then,

$$\vec{A} = \vec{a} + \varepsilon \left(\overrightarrow{a}'_0 - h\vec{a} \right), \tag{9}$$

and

$$h = g(\vec{a}, \overrightarrow{a}'_0). \tag{10}$$

Here, the value of h changes related to the reference point. Without losing generality, if we assume that the reference point is the origin of the coordinate system, we can write the position vector of the incident E as

$$\vec{r}_E = \overrightarrow{PN} + \overrightarrow{NE},$$

where $\overrightarrow{a}_0 = \overrightarrow{PN} \times \vec{a}$ and $\overrightarrow{NE} = h\vec{a}$. Therefore, from Theorem 2.1 and $\overrightarrow{a}'_0 = \overrightarrow{a}_0 + h\vec{a}$, the position vector $\vec{r}_E$ of the incident E is

$$\begin{aligned} \vec{r}_E &= \vec{a} \times \overrightarrow{a}_0 + h\vec{a} \\ &= \vec{a} \times \overrightarrow{a}'_0 + \langle \vec{a}, \overrightarrow{a}'_0 \rangle \vec{a}, \end{aligned}$$

where $\times$ is the cross-product.

3.1 Equiform Transformations

This section describes equiform transformations, which means affine transformations whose linear part is composed from an orthogonal transformation and a homothetical transformation in Euclidean three-space $\mathbb{R}^3$.
Such an equiform transformation maps points $x \in \mathbb{R}^3$ by using

$$\begin{aligned} \varphi : \mathbb{R}^3 &\longrightarrow \mathbb{R}^3 \\ x &\longrightarrow \varphi(x) = y(t) = \alpha(t)D(t)x + b(t), \end{aligned} \tag{11}$$

where $D \in O(3)$, $b \in \mathbb{R}^3$ and α is a homothetic scale. D, α and b are differentiable functions of class C^∞ of a parameter t.
The velocity $\dot{y}(t)$ has the form

$$v(y) = \dot{D}D^T y + \frac{\dot{\alpha}}{\alpha} y - \dot{D}D^T b - \frac{\dot{\alpha}}{\alpha} b + \dot{b}, \tag{12}$$

where $v(y) = \dot{y}(t) = \frac{dy}{dt}$.
Since D is orthogonal, the matrix $\dot{D}D^T := C^\times$ is skew-symmetric and the product $C^\times x$ can be written in the form $c \times x$ in Euclidean three-space $\mathbb{R}^3$:

$$v(y) = c \times y + \gamma y + \bar{c}, \tag{13}$$

where $\gamma = \frac{\dot{\alpha}}{\alpha}$ and $\bar{c} = \dot{D}D^T b - \frac{\dot{\alpha}}{\alpha} b + \dot{b}$.
Any triple $(c, \bar{c}, \gamma) \in \mathbb{R}^7$ defines a uniform equiform motion in Euclidean three-space $\mathbb{R}^3$, uniquely *[1]*.

3.2 Plücker Coordinates of Line Elements

Let L be an oriented (directed) line in Euclidean three-space $\mathbb{R}^3$ passing through a point $\vec{x}$. In order to assign coordinates to the line element $(L, \vec{x})$, we use the familiar definition of Plücker coordinates. The triple $(\vec{a}, \overrightarrow{a}_0, h) \in \mathbb{R}^7$ is called the Plücker coordinates of the line element $(L, \vec{x})$ in $\mathbb{R}^3_1$, if $\vec{a} \neq \vec{0}$ is parallel to L, then $\overrightarrow{a}_0 = \vec{x} \times \vec{a}$, $h = \langle \vec{x}, \vec{a} \rangle$. It is easy to show that

$$\vec{x} = N(\vec{a}, \overrightarrow{a}_0) + h\vec{a}, \tag{14}$$

where $N(\vec{a}, \overrightarrow{a}_0) = \vec{a} \times \overrightarrow{a}_0$.
The point $N(\vec{a}, \overrightarrow{a}_0)$ is the foot point of the origin on the line L. We know that Plücker coordinates satisfy $\langle \vec{a}, \overrightarrow{a}_0 \rangle = 0$, and $\vec{a} \neq \vec{0}$ occurs as coordinates of lines in $\mathbb{R}^3$. Therefore, from (14) we obtain the equation

$$\vec{x} = \vec{a} \times \overrightarrow{a}_0 + h\vec{a}, \tag{15}$$

where $h = \langle \vec{x}, \vec{a} \rangle$ and $\vec{a}$ is a unit parallel vector to the line L.

If the corresponding line has an orientation, then a line element becomes oriented. The equiform transformation (11) transforms the line element $(\vec{a}, \overrightarrow{a}_0, h_1)$ into $(\vec{u}, \vec{u}_0, h_2)$ with $\vec{x}' = \alpha R\vec{x} + \vec{b}$, $\vec{u} = R\vec{a}$, $\vec{u}_0 = \vec{x}' \times \vec{u}$, $h_2 = \langle \vec{x}', \vec{u} \rangle$. In block matrix form, this transformation reads

$$\begin{bmatrix} \vec{u} \\ \vec{u}_0 \\ h_2 \end{bmatrix} = \begin{bmatrix} D & 0 & 0 \\ D^{\times} D & \alpha D & 0 \\ \vec{b}^T D & 0^T & \alpha \end{bmatrix} \begin{bmatrix} \vec{a} \\ \overrightarrow{a}_0 \\ h_1 \end{bmatrix}, \tag{16}$$

where $D \in O(3)$, $b \in \mathbb{R}^3$, α is a homothetic scale D, $D^{\times}\vec{x} = \vec{b} \times \vec{x}$, $\vec{A} = \vec{a} + \varepsilon \overrightarrow{a}_0$, $\langle \vec{a}, \vec{a} \rangle = 1$, $\langle \vec{a}, \overrightarrow{a}_0 \rangle = 0$ and $\vec{U} = \overrightarrow{u} + \varepsilon \overrightarrow{u}_0$, $\langle \overrightarrow{u}, \overrightarrow{u} \rangle = 1$, $\langle \overrightarrow{u}, \overrightarrow{u}_0 \rangle = 0$, ([1]).
Using the correspondence between line elements and point-lines we observe the following:

Conclusion 3.1. Let $\hat{A} = \left\| \hat{A} \right\| \vec{A}$ and $\hat{U} = \left\| \hat{U} \right\| \vec{U}$ be two point-lines. When the reference point is chosen as the origin of the coordinate system for a point-line, the transformations (16) transform the point-line $\hat{A}$ to the point-line $\hat{U}$ if $\vec{A}$ is a unit dual quaternion vector.

We can obtain the oriented (directed) line elements in the equation (16) by using dual quaternions. Moreover, we also can transform a point-line to another point-line by using dual quaternions with the following theorem.

Theorem 3.2. A dual quaternion Q transforms a given point-line to another given point-line and is defined by

$$Q = \frac{1}{\left\| \hat{A} \right\|^2} \left(\left\langle \hat{A}, \hat{U} \right\rangle + \left(\hat{A} \times \hat{U} \right) \right), \tag{17}$$

where $\hat{A}$ and $\hat{U}$ denoted two point-lines, $\times$ is cross product and the Q is called the point-line operator which acts on point-lines.

Proof. Let $\hat{A}$ and $\hat{U}$ be two point-lines defined by $\hat{A} = \left\| \hat{A} \right\| \vec{A}$ and
$\hat{U} = \left\| \hat{U} \right\| \vec{U}$. Here, from the Eq. (8) $\vec{A}$ and $\vec{U}$ are unit dual vectors, dual length $\left\| \hat{A} \right\| = \exp \varepsilon(h_1)$ of $\hat{A}$ and dual length $\left\| \hat{U} \right\| = \exp \varepsilon(h_2)$ of $\hat{U}$.
If we apply quaternion multiplication to the Eq. (17) with $\hat{A}$ from right-side, then we have

$$Q\hat{A} = \frac{1}{\left\| \hat{A} \right\|^2} \left[\left\langle \hat{A}, \hat{U} \right\rangle \hat{A} + (\hat{A} \times \hat{U}) \times \hat{A} \right]$$

and from Theorem 2.1 we have

$$Q\hat{A} = \frac{1}{\left\| \hat{A} \right\|^2} \left[\left\langle \hat{A}, \hat{U} \right\rangle \hat{A} + \left\langle \hat{A}, \hat{A} \right\rangle \hat{U} - \left\langle \hat{A}, \hat{U} \right\rangle \hat{A} \right]$$

and from $\left\langle \hat{A}, \hat{A} \right\rangle = 1$

$$Q\hat{A} = \hat{U}.$$

Also, since

$$\begin{aligned} \hat{A} &= \left\| \hat{A} \right\| \vec{A}, \\ \hat{U} &= \left\| \hat{U} \right\| \vec{U}, \end{aligned}$$

Eq. (17) can be modified

$$Q = \frac{\left\| \hat{U} \right\|}{\left\| \hat{A} \right\|} (\left\langle \vec{A}, \vec{U} \right\rangle + \left(\vec{A} \times \vec{U} \right)),$$

and from the Eq. (8) since $\left\| \hat{A} \right\| = \exp \varepsilon(h_1)$ and $\left\| \hat{U} \right\| = \exp \varepsilon(h_2)$, the last equation can be rewritten as

$$Q = \{\exp [\varepsilon(h_2 - h_1)]\} Q_0,$$

where $\frac{\|\hat{U}\|}{\|\hat{A}\|} = \exp [\varepsilon(h_2 - h_1)]$ is dual length of Q and $Q_0 = \left\langle \vec{A}, \vec{U} \right\rangle + \left(\vec{A} \times \vec{U} \right)$.
Because $\left\langle \vec{A}, \vec{U} \right\rangle$ is the scalar part of Q_0 and $\left(\vec{A} \times \vec{U} \right)$ is the vector part of Q_0, then Q is a dual quaternion.

Example 3.3. Let $\hat{A} = \left(0,\ \frac{1}{2},\ \frac{\sqrt{3}}{2},\ \frac{\sqrt{3}}{2},\ 0,\ 0,\ \frac{1}{2}\right)$
and $\hat{U} = \left(0,\ 0,\ 1,\ -\frac{3}{2},\ -1,\ 0,\ -\frac{\sqrt{3}}{2}\right)$ be two point-lines in $\mathbb{R}^7$. Since from the Eq. (8)

$$\hat{A} = \underbrace{\left(1 + \frac{\varepsilon}{2}\right)}_{\|\hat{A}\|} \underbrace{\left[\left(0,\ \frac{1}{2},\ \frac{\sqrt{3}}{2}\right) + \varepsilon\left(\frac{\sqrt{3}}{2},\ 0,\ 0\right)\right]}_{\vec{A}}$$

and

$$\hat{U} = \underbrace{\left(1 - \frac{\sqrt{3}}{2}\varepsilon\right)}_{\|\hat{U}\|} \underbrace{\left[(0,\ 0,\ 1) + \varepsilon\left(-\frac{3}{2},\ -1,\ 0\right)\right]}_{\vec{U}},$$

from the Eq. (17) it can be written

$$Q = \left(1 - \frac{\sqrt{3}+1}{2}\varepsilon\right) (\left(\frac{\sqrt{3}}{2} - \frac{1}{2}\varepsilon\right) + \left(\left(\frac{1}{2},\ 0,\ 0\right) + \varepsilon\left(\frac{\sqrt{3}}{2},\ -\frac{5}{4}\sqrt{3},\ \frac{3}{4}\right)\right)).$$

If we apply quaternion multiplication to Q with $\hat{A}$ from right-side, then we have

$$\begin{aligned} Q\hat{A} &= \left(1 - \frac{\sqrt{3}}{2}\varepsilon\right) \left[(0,\ 0,\ 1) + \varepsilon\left(-\frac{3}{2},\ -1,\ 0\right)\right] \\ &= \hat{U}. \end{aligned}$$

4 Conclusion

In this study, we used a block matrix to transform a given point-line to another given one that is given in [1]. We prove that dual quaternions can be used to map a given point-line to another given one. Since it is compact, free of redundancies and easier to compute compared to the matrix given in the Eq. (16), this approach has some advantages.

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r-τ_{12}-θ-GENERALIZED FUZZY CLOSED SETS IN SMOOTH BITOPOLOGICAL SPACES

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Abstract − In [34] we introduced the notion of r-(τ_i, τ_j)-θ-generalized fuzzy closed sets in smooth bitopological spaces by using $(\tau_i, \tau_j)\theta$-fuzzy closure $T^{\tau_i}_{\tau_j}$ defined in [19]. Recently, [33] we defined a new θ-fuzzy closure, denoted C^{θ}_{12} on smooth bitopological spaces by using smooth supra topological space (X, τ_{12}) which is generated from smooth bitopological space (X, τ_1, τ_2) [1], such that $C^{\theta}_{12} \leq T^{\tau_i}_{\tau_j}$. In this paper, we introduce a new class of r-θ-generalized fuzzy closed sets, namely, r-τ_{12}-θ-gfc in smooth bitopological spaces via C^{θ}_{12}-fuzzy closure operator. The basic properties of these sets are studied. Furthermore, the relationship with other notions of r-generalized fuzzy closed sets in [31, 32, 33, 34] are investigated and we give many examples for reverse. In addition, by using r-τ_{12}-θ-gfc sets, we define a new fuzzy closure operator which generates a new smooth topology. Finally, generalized fuzzy θ-continuous (resp. irresolute) and fuzzy strongly θ-continuous mappings are introduced and some of their properties are studied.

Keywords − *Smooth topology, θ-generalized fuzzy closed, generalized fuzzy closure operator, generalized fuzzy θ-continuous mapping, generalized fuzzy θ-irresolute mapping, fuzzy strongly θ- continuous mapping.*

1 Introduction

Kubiak [20] and Šostak [29] independently in (1985) introduced the fundamental concept of a fuzzy topology as an extension of both crisp topology and Chang's fuzzy topology [5]. Šostak presented some rules and showed how such an extension can be realized. Subsequently, Badard [3], introduced the concept of 'smooth topological space'. Chattopadhyay et al. [6] and Chattopadhyay and Samanta [7] have re-introduced the same concept, calling it 'gradation of openess'. Ramadan [26] and his colleagues have introduced a similar definition, namely, smooth topological space for lattice $L = [0, 1]$. Following Ramadan, several authors have re-introduced and further studied smooth topological space (cf. [6, 7, 11, 22, 28, 30]). Lee et al. [21] introduced the concept of smooth bitopological space as a generalization of smooth topological space and Kandil's defined fuzzy bitopological space [14].

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The so-called supra topology was established, by Mashhour et al. [24] (recall that a supra topology on a set X is a collection of subsets of X, which is closed under arbitrary unions). Abd El-Monsef and Ramadan [2] introduced the concept supra fuzzy topology, followed by Ghanim et al. [13] who introduced the supra fuzzy topology in Šostak sense. Abbas [1] generated the supra fuzzy topology (X, τ_{12}) from fuzzy bitopological space (X, τ_1, τ_2) in Šostak sense as an extension of supra fuzzy topology due to Kandil et al. [15].

The first attempt of generalizing closed sets was done by Levine [23]. Subsequently, Fukutake [12], generalized this concept in bitopological space. Balasubramanian and Sundaram [4], introduced the concept of generalized fuzzy closed sets within Chang's fuzzy topology. Kim and Ko [18] defined r-generalized fuzzy closed sets in smooth topological spaces. Recently, in [31], we introduced the concept of generalized fuzzy closed sets in smooth bitopological spaces. Noiri [25] and Dontchev and Maki [8] introduced another new generalization of Livine generalized closed set by utilizing the θ-closure operator. The concept of θ-generalized closed sets was applied to the digital line [9]. Khedr and Al-Saadi [16] generalized the notion of θ-generalized sets to bitopological space. El-Shafei and Zakari [10] introduced the concept of θ-generalized fuzzy closed sets in Chang's fuzzy topology. Recently, in [34], we introduced the notion of θ-generalized fuzzy closed sets in smooth bitopological spaces by utilizing the $(\tau_i, \tau_j)\theta$-fuzzy closure $T^{\tau_i}_{\tau_j}$ defined in [19]. In this paper we define another type of r-θ-generalized fuzzy closed sets in smooth bitopological spaces via C^{θ}_{12}-fuzzy closure which was established by us [33], and study its relationship with other types of r-generalized fuzzy closed sets which introduced in ([31, 32, 33, 34]). By using this new class of generalized fuzzy closed sets we define a new fuzzy closure operator which generates a new smooth topology. Finally, we define and study generalized fuzzy θ-continuous (resp. irresolute) and fuzzy strongly θ-continuous mappings.

2 Preliminary

Throughout this paper, let X be a non-empty set, $I = [0, 1]$, $I_0 = (0, 1]$. A fuzzy set μ of X is a mapping $\mu : X \longrightarrow I$, and I^X be the family of all fuzzy sets on X. For any $\mu_1, \mu_2 \in I^X$, then $(\mu_1 \wedge \mu_2)(x) = \min\{\mu_1(x), \mu_2(x) : x \in X\}$, $(\mu_1 \vee \mu_2)(x) = \max\{\mu_1(x), \mu_2(x) : x \in X\}$. The complement of a fuzzy set λ is denoted by $\bar{1} - \lambda$. For $\alpha \in I$, $\bar{\alpha}(x) = \alpha\ \forall x \in X$. By $\bar{0}$ and $\bar{1}$, we denote constant maps on X with value 0 and 1, respectively. For $x \in X$ and $t \in I_0$, the fuzzy set x_t of X whose value t at x and 0 otherwise is called the fuzzy point in X. Let $Pt(X)$ be a family of all fuzzy points in X. For $\lambda \in I^X$, $x_t \in \lambda$ if and only if $\lambda(x) \geq t$ and x_t is said to be quasi-coincident (q-coincident, for short) with λ, denoted by $x_t\ q\ \lambda$ if and only if $1 - \lambda(x) < t$. For $\mu, \lambda \in I^X$, μ is called q-coincident with λ, denoted by $\mu\ q\ \lambda$, if $\mu(x) + \lambda(x) > 1$ for some $x \in X$, otherwise we write $\mu\ \bar{q}\ \lambda$. Also, for two fuzzy sets λ_1 and $\lambda_2 \in I^X$, $\lambda_1 \leq \lambda_2$ if and only if $\lambda_1\ \bar{q}\ \bar{1} - \lambda_2$. FP (resp. FP^*) stand for fuzzy pairwise (resp. fuzzyP^*). The indices $i, j \in \{1, 2\}$ and $i \neq j$.

Definition 2.1. [3, 6, 26, 29] A smooth topology on X is a mapping $\tau : I^X \to I$ which satisfies the following properties:

(1) $\tau(\bar{0}) = \tau(\bar{1}) = 1$,

(2) $\tau(\mu_1 \wedge \mu_2) \geq \tau(\mu_1) \wedge \tau(\mu_2)$, $\forall\ \mu_1, \mu_2 \in I^X$,

(3) $\tau(\bigvee_{i \in J} \mu_i) \geq \bigwedge_{i \in J} \tau(\mu_i)$, for any $\{\mu_i : i \in J\} \subseteq I^X$.

The pair (X, τ) is called a smooth topological space. For $r \in I_0$, μ is an r-open fuzzy set of X if $\tau(\mu) \geq r$, and μ is an r-closed fuzzy set of X if $\tau(\bar{1} - \mu) \geq r$. Note, Šostak [29] used the term 'fuzzy topology' and Chattopadhyay et al. [6], the term 'gradation of openness' for a smooth topology τ.

If τ satisfies conditions (1) and (3), then τ is said to be supra smooth topology and (X, τ) is said to be a supra smooth topological space [13].

Definition 2.2. [21, 29] A triple (X, τ_1, τ_2) consisting of the set X endowed with smooth topologies τ_1 and τ_2 on X is called a smooth bitopological space (smooth bts, for short). For $\lambda \in I^X$ and $r \in I_0$, r-τ_i-open (resp. closed) fuzzy set denotes the r-open (resp. closed) fuzzy set in (X, τ_i), for $i = 1, 2$.

Subsequently, the fuzzy closure (resp. interior) for any fuzzy set in smooth topological space is given as follows:

Definition 2.3. [7] Let (X, τ) be a smooth topological space. For $\lambda \in I^X$ and $r \in I_0$, a fuzzy closure is a mapping $C_\tau : I^X \times I_0 \to I^X$ such that

$$C_\tau(\lambda, r) = \bigwedge\{\mu \in I^X |\ \mu \geq \lambda,\ \tau(\bar{1} - \mu) \geq r\}. \tag{1}$$

And, a fuzzy interior of λ is a mapping $I_\tau : I^X \times I_0 \to I^X$ defined as

$$I_\tau(\lambda, r) = \bigvee\{\mu \in I^X |\ \mu \leq \lambda,\ \tau(\mu) \geq r\}, \tag{2}$$

satisfies

$$I_\tau(\bar{1} - \lambda, r) = \bar{1} - C_\tau(\lambda, r). \tag{3}$$

Remark 2.4. If (X, τ) is a supra smooth topological space. Then the definition of fuzzy closure (resp. interior) for any fuzzy set is defined as (1) and (2) in Definition 2.3 respectively.

Definition 2.5. [7] A mapping $C : I^X \times I_0 \to I^X$ is called a fuzzy closure operator if, for $\lambda, \mu \in I^X$ and $r, s \in I_0$, the mapping C satisfies the following conditions:

(C1) $C(\bar{0}, r) = \bar{0}$,

(C2) $\lambda \leq C(\lambda, r)$,

(C3) $C(\lambda, r) \vee C(\mu, r) = C(\lambda \vee \mu, r)$,

(C4) $C(\lambda, r) \leq C(\lambda, s)$ if $r \leq s$,

(C5) $C(C(\lambda, r), r) = C(\lambda, r)$.

The fuzzy closure operator C generates a smooth topology $\tau_C : I^X \longrightarrow I$ given by

$$\tau_C(\lambda) = \bigvee\{r \in I |\ C(\bar{1} - \lambda, r) = \bar{1} - \lambda\} \tag{4}$$

If C satisfies conditions (C1),(C2),(C4),(C5) and the following inequality:
$(C3)^*$ $\quad C(\lambda, r) \vee C(\mu, r) \leq C(\lambda \vee \mu, r)$,
then C is called supra fuzzy closure operator on X [1]. and it generates a supra smooth topology $\tau_C : I^X \longrightarrow I$ as in (4)

By using (3), the definitions of fuzzy interior operator and supra fuzzy interior operator are obtained. In analogs of Definition 2.5, a fuzzy interior operator was defined.

The following theorem shows how to generate a supra fuzzy closure operator from smooth bts (X, τ_1, τ_2).

Theorem 2.6. [1] Let (X, τ_1, τ_2) be a smooth bts, for each $\lambda \in I^X$ and $r \in I_0$. Then:

(1) The mapping $C_{12} : I^X \times I_0 \to I^X$ such that $C_{12}(\lambda, r) = C_{\tau_1}(\lambda, r) \wedge C_{\tau_2}(\lambda, r)$ is a supra fuzzy closure operator, and (X, C_{12}) is a supra fuzzy closure space.

(2) The mapping $I_{12} : I^X \times I_0 \to I^X$ defined by $I_{12}(\lambda, r) = I_{\tau_1}(\lambda, r) \vee I_{\tau_2}(\lambda, r)$ is a supra fuzzy interior operator, satisfies $I_{12}(\bar{1} - \lambda, r) = \bar{1} - C_{12}(\lambda, r)$.

Theorem 2.7. [1] Let (X, τ_1, τ_2) be a smooth bts, let (X, C_{12}) be a supra fuzzy closure space. Define the mapping $\tau_S : I^X \to I$ on X by

$$\tau_S(\lambda) = \bigvee\{\tau_1(\lambda_1) \wedge \tau_2(\lambda_2) : \lambda = \lambda_1 \vee \lambda_2,\ \ \lambda_1, \lambda_2 \in I^X\}$$

where $\bigvee$ is taken over all families $\{\lambda_1, \lambda_2 \in I^X : \lambda = \lambda_1 \vee \lambda_2\}$. Then:
(1) $\tau_S = \tau_{C_{12}}$ is the coarsest smooth supra topology on X which is finer than τ_1 and τ_2.
(2) $C_{12} = C_{\tau_s} = C_{\tau_{C_{12}}}$.

Remark 2.8. In this paper we will denote to $\tau_{C_{12}}$ by τ_{12}.

Definition 2.9. Let (X, τ_1, τ_2) be a smooth bts, $\lambda \in I^X$ and $r \in I_0$. Then, a fuzzy set λ is called:

(1) an r-(τ_i, τ_j)-generalized fuzzy closed (r-(τ_i, τ_j)-gfc, for short), if $C_{\tau_j}(\lambda, s) \leq \mu$, whenever $\lambda \leq \mu$ such that $\tau_i(\mu) \geq s \ \forall\ 0 < s \leq r$. The complement of r-$(\tau_i, \tau_j)$-gfc is an r-$(\tau_i, \tau_j)$-generalized fuzzy open (r-(τ_i, τ_j)-gfo, for short) [31].

(2) an r-τ_{12}-generalized fuzzy closed (r-τ_{12}-gfc, for short) if $C_{12}(\lambda, s) \leq \mu$ whenever $\lambda \leq \mu$ and $\tau_{12}(\mu) \geq s \quad \forall\ 0 < s \leq r$. The complement of r-τ_{12}-gfc is an r-τ_{12}-generalized fuzzy open (r-τ_{12}-gfo, for short) [32].

The concepts of r-τ_{12}-gfc and r-(i, j)-gfc sets are independent.

Recall next the definitions of open Q-nbd, θ-cluster point and θ-fuzzy closure operator in smooth bts (X, τ_1, τ_2).

Definition 2.10. [19] Let (X, τ_1, τ_2) be a smooth bts, $\mu \in I^X$, $x_t \in Pt(X)$ and $r \in I_0$. Then, μ is called an r-open Q_{τ_i}-nighborhood of x_t if $x_t\ q\ \mu$ with $\tau_i(\mu) \geq r$, we denote

$$Q_{\tau_i}(x_t, r) = \{\mu \in I^X |\ x_t\ q\ \mu, \tau_i(\mu) \geq r\}.$$

Definition 2.11. [19] Let (X, τ_1, τ_2) be a smooth bts, $\lambda \in I^X$ and $r \in I_0$. Then:

(1) A fuzzy point $x_t \in Pt(X)$ is called an r-$(\tau_i, \tau_j)\theta$-cluster point of λ if for every $\mu \in Q_{\tau_i}(x_t, r)$, $C_{\tau_j}(\mu, r)\ q\ \lambda$.

(2) An $(\tau_i, \tau_j)\theta$-closure is a mapping $T^{\tau_i}_{\tau_j} : I^X \times I_0 \longrightarrow I^X$ defined as follows:

$$T^{\tau_i}_{\tau_j}(\lambda, r) = \bigvee\{x_t \in Pt(X)|\ x_t \textit{ is } r\text{-}(\tau_i, \tau_j)\theta\textit{-cluster point of } \lambda\}.$$

(3) λ is called an r-(τ_i, τ_j) fuzzy θ-closed iff $\lambda = T^{\tau_i}_{\tau_j}(\lambda, r)$. The complement of an r-(τ_i, τ_j) fuzzy θ-closed is called r-(τ_i, τ_j) fuzzy θ-open.

Theorem 2.12. [19] Let (X, τ_1, τ_2) be a smooth bts, $\lambda, \mu \in I^X$, $x_t \in Pt(X)$ and $r \in I_0$. Then:

(1) $T^{\tau_i}_{\tau_j}(\lambda, r) = \bigwedge\{\mu \in I^X |\ I_{\tau_j}(\mu, r) \geq \lambda,\ \tau_i(\bar{1} - \mu) \geq r\}$, i.e., $T^{\tau_i}_{\tau_j}(\lambda, r)$ is an r-τ_i-closed fuzzy set.

(2) x_t is an r-$(\tau_i, \tau_j)\theta$-cluster point of λ iff $x_t \in T^{\tau_i}_{\tau_j}(\lambda, r)$.

Definition 2.13. [34] Let (X, τ_1, τ_2) be a smooth bts, $\lambda \in I^X$ and $r \in I_0$. A fuzzy set λ is an r-(τ_i, τ_j)-θ-generalized fuzzy closed (r-(τ_i, τ_j)-θ-gfc, for short) if $T^{\tau_j}_{\tau_i}(\lambda, s) \leq \mu$ whenever $\lambda \leq \mu$ such that $\tau_i(\mu) \geq s \ \forall\ 0 < s \leq r$. The complement of r-(τ_i, τ_j)-θ-gfc is an r-(τ_i, τ_j)-θ-generalized fuzzy open (r-(τ_i, τ_j)-θ-gfo, for short).

Definition 2.14. [33] Let (X, τ_1, τ_2) be a smooth bts, $\lambda \in I^X$, $r \in I_0$ and $x_t \in Pt(X)$. Then:

(1) A fuzzy point x_t is said to be an r-τ_{12}-θ-cluster point of λ if and only if $C_{12}(\mu, r)$ q λ, for each $\mu \in Q_{\tau_{12}}(x_t, r)$, where $Q_{\tau_{12}}(x_t, r) = \{\mu \in I^X |\ x_t\ q\ \mu, \tau_{12}(\mu) \geq r\}$. The set of all r-τ_{12}-θ-cluster points of λ is called C^{θ}_{12}-fuzzy closure of λ, i.e. $C^{\theta}_{12} : I^X \times I_0 \longrightarrow I^X$ defined as

$$C^{\theta}_{12}(\lambda, r) = \bigvee\{x_t \in Pt(X) \mid x_t \textit{ is } r\text{-}\tau_{12}\text{-}\theta\textit{-cluster point of}\lambda\}.$$

(2) λ is said to be an r-τ_{12}-θ-closed fuzzy set iff $C^{\theta}_{12}(\lambda, r) = \lambda$. The complement of r-τ_{12}-θ-closed fuzzy set is an r-τ_{12}-θ-open fuzzy set.

Theorem 2.15. [33] Let (X, τ_1, τ_2) be a smooth bts, $\lambda \in I^X$ and $r \in I_0$. Then:

(1) $C_{12}(\lambda, r) \leq C^{\theta}_{12}(\lambda, r) \leq T^{\tau_i}_{\tau_j}(\lambda, r)$.

(2) If λ is an r-τ_{12}-open fuzzy set in X, then $C_{12}(\lambda, r) = C^{\theta}_{12}(\lambda, r)$.

Some properties of C^{θ}_{12} are given in the following proposition:

Proposition 2.16. [33] Let (X, τ_1, τ_2) be a smooth bts, $\lambda, \lambda_1, \lambda_2 \in I^X$ and $r \in I_0$. Then:

(1) $C^{\theta}_{12}(\lambda, r) = \bigwedge\{C_{12}(\rho, r) :\ \rho \geq \lambda\ , \tau_{12}(\rho) \geq r\}$.

(2) If $\lambda_1 \leq \lambda_2$, then $C^{\theta}_{12}(\lambda_1, r) \leq C^{\theta}_{12}(\lambda_2, r)$.

(3) $C^{\theta}_{12}(\lambda_1, r) \vee C^{\theta}_{12}(\lambda_2, r) = C^{\theta}_{12}(\lambda_1 \vee \lambda_2, r)$.

(4) $C^{\theta}_{12}(\lambda, r) \leq C^{\theta}_{12}(\lambda, s)$, if $r \leq s$.

(5) $C^{\theta}_{12}(\lambda_1 \wedge \lambda_2, r) \leq C^{\theta}_{12}(\lambda_1, r) \wedge C^{\theta}_{12}(\lambda_2, r)$.

(6) $C^{\theta}_{12}(\lambda, r) \leq C^{\theta}_{12}(C^{\theta}_{12}(\lambda, r), r)$.

Next we introduce the concept of I^{θ}_{12}-fuzzy interior in smooth bts (X, τ_1, τ_2).

Definition 2.17. [33] Let (X, τ_1, τ_2) be a smooth bts, $\lambda \in I^X$ and $r \in I_0$. A fuzzy point x_t is said to be an r-τ_{12}-θ-interior point of λ if there exists $\mu \in Q_{\tau_{12}}(x_t, r)$ such that $C_{12}(\mu, r)\ \bar{q}\ \bar{1} - \lambda$. The set of all r-τ_{12}-θ-interior points of λ is called I^{θ}_{12}-fuzzy interior of λ. i.e. $I^{\theta}_{12} : I^X \times I_0 \longrightarrow I^X$ defined as

$$I^{\theta}_{12}(\lambda, r) = \bigvee\{x_t \in Pt(X) \mid x_t \textit{ is } r\text{-}\tau_{12}\text{-}\theta\textit{-interior point of } \lambda\}.$$

Equivalently, I^{θ}_{12}-fuzzy interior can be stated as folllows.

Proposition 2.18. [33] Let (X, τ_1, τ_2) be a smooth bts, $\lambda \in I^X$ and $r \in I_0$. Then:

$$I^{\theta}_{12}(\lambda, r) = \bigvee\{\mu \in I^X \mid C_{12}(\mu, r) \leq \lambda, \tau_{12}(\mu) \geq r\}.$$

Throughout this paper (X, τ_{12}) and (Y, τ^*_{12}) denote the supra smooth topological spaces which are induced from smooth bitopological spaces (X, τ_1, τ_2) and (Y, τ^*_1, τ^*_2) respectively.

Definition 2.19. A mapping $f : (X, \tau_1, \tau_2) \longrightarrow (Y, \tau^*_1, \tau^*_2)$ from a smooth bts (X, τ_1, τ_2) to another one (Y, τ^*_1, τ^*_2) is said to be:

(1) FP-continuous if and only if $\tau_i(f^{-1}(\mu)) \geq \tau^*_i(\mu)$ for each $\mu \in I^Y$ and $i = 1, 2$ [17].

(2) FP^*-continuous if and only if $f : (X, \tau_{12}) \longrightarrow (Y, \tau^*_{12})$ is F-continuous [27]. That is, $\tau_{12}(f^{-1}(\mu)) \geq \tau^*_{12}$ for each $\mu \in I^Y$.

(3) FP^*-open if and only if $f : (X, \tau_{12}) \longrightarrow (Y, \tau^*_{12})$ is F-open [17]. That is, $\tau^*_{12}(f(\lambda)) \geq \tau_{12}(\lambda)$ for each $\lambda \in I^X$.

(4) generalized FP^*-continuous (GFP^*-continuous, for short) if and only if $f^{-1}(\mu)$ is an r-τ_{12}-gfc for all $\mu \in I^Y$ with $\tau^*_{12}(\bar{1} - \mu) \geq r$ [32].

(5) generalized FP^*-irresolute closed (GFP^*-irresolute closed, for short) if and only if $f(\mu)$ is an r-τ^*_{12}-gfc in Y for each r-τ_{12}-gfc μ in X [32].

3 r-τ_{12}-θ-generalized Fuzzy Closed Sets

In this section we introduce a new class of generalized fuzzy closed sets via a fuzzy closure C^{θ}_{12} defined in [33], and we study its relationship with other types of generalized fuzzy closed sets which introduced in ([31, 32, 33, 34]).

Definition 3.1. Let (X, τ_1, τ_2) be a smooth bts, $\lambda \in I^X$ and $r \in I_0$. Then:

(1) A fuzzy set λ is called an r-τ_{12}-θ-generalized fuzzy closed (r-τ_{12}-θ-gfc, for short) if $C^{\theta}_{12}(\lambda, s) \leq \mu$ whenever $\lambda \leq \mu$ and $\tau_{12}(\mu) \geq s$ for all $0 < s \leq r$.

(2) A fuzzy set λ is called an r-τ_{12}-θ-generalized fuzzy open (r-τ_{12}-θ-gfo, for short) if $\bar{1} - \lambda$ is an r-τ_{12}-θ-gfc.

Proposition 3.2. Let (X, τ_1, τ_2) be a smooth bts, $\lambda_1, \lambda_2 \in I^X$ and $r \in I_0$. Then:

(1) If λ_1, λ_2 are r-τ_{12}-θ-gfc sets, then $\lambda_1 \vee \lambda_2$ is an r-τ_{12}-θ-gfc set.

(2) If λ_1, λ_2 are r-τ_{12}-θ-gfo sets, then $\lambda_1 \wedge \lambda_2$ is an r-τ_{12}-θ-gfo set.

Proof. To prove part (1), let $\lambda_1 \vee \lambda_2 \leq \mu$ such that $\tau_{12}(\mu) \geq s$ for $0 < s \leq r$. This implies $\lambda_1 \leq \mu$ and $\lambda_2 \leq \mu$. Since λ_1 and λ_2 are r-τ_{12}-θ-gfc sets, then in view of Proposition 2.16(3) and Definition 3.1(1), we have, $C^{\theta}_{12}(\lambda_1 \vee \lambda_2, s) = C^{\theta}_{12}(\lambda_1, s) \vee C^{\theta}_{12}(\lambda_2, s) \leq \mu \vee \mu = \mu$. Hence, $\lambda_1 \vee \lambda_2$ is an r-τ_{12}-θ-gfc. The prove of part (2), follows from the duality of (1). □

Remark 3.3. The finite intersection (resp. union) of r-τ_{12}-θ-gfc (resp. gfo) sets in a smooth bts (X, τ_1, τ_2) need not to be an r-τ_{12}-θ-gfc (resp. gfo), as the following example shows.

Example 3.4. Let $X = \{a, b\}$. Define $\lambda_1, \lambda_2 \in I^X$ as follows:

$$\lambda_1 = a_{0.2} \vee b_{0.5}, \qquad \lambda_2 = a_{0.4} \vee b_{0.2}.$$

We define smooth topologies $\tau_1, \tau_2 : I^X \longrightarrow I$ as follows:

$$\tau_1(\lambda) = \begin{cases} 1 & \text{if } \lambda = \bar{0}, \bar{1}, \\ \frac{1}{4} & \text{if } \lambda = \lambda_1, \\ 0 & \textit{otherwise}. \end{cases} \quad \textit{and} \quad \tau_2(\lambda) = \begin{cases} 1 & \text{if } \lambda = \bar{0}, \bar{1} \\ \frac{3}{4} & \text{if } \lambda = \lambda_2 \\ 0 & \textit{otherwise}. \end{cases}$$

The induced supra smooth topological space of (X, τ_1, τ_2), is defined as $\tau_{12} : I^X \longrightarrow I$ such that

$$\tau_{12}(\lambda) = \begin{cases} 1 & \text{if } \lambda = \bar{0}, \bar{1}, \\ \frac{1}{4} & \text{if } \lambda = \lambda_1, \\ \frac{3}{4} & \text{if } \lambda = \lambda_2, \\ \frac{1}{4} & \text{if } \lambda = \lambda_1 \vee \lambda_2, \\ 0 & \textit{otherwise}. \end{cases}$$

Then, for $r = \frac{1}{4}$ the fuzzy sets $\eta_1 = a_{0.2} \vee b_{0.6}$ and $\eta_2 = a_{0.6} \vee b_{0.2}$ are $\frac{1}{4}$-τ_{12}-θ-gfc sets but $\eta_1 \wedge \eta_2$ is not a $\frac{1}{4}$-τ_{12}-θ-gfc. By taking the complement of η_1 and η_2 we obtain the finite union of r-τ_{12}-θ-gfo sets. This union need not to be r-τ_{12}-θ-gfo.

In the following Propositions 3.5, 3.7, 3.9, 3.10 and 3.11 with the examples following them show that the class of r-τ_{12}-θ-gfc sets is properly placed between the classes of r-τ_{12}-gfc sets and r-τ_{12}-θ-closed fuzzy sets.

Proposition 3.5. Let (X, τ_1, τ_2) be a smooth bts, $\lambda \in I^X$ and $r \in I_0$. If λ is an r-τ_{12}-θ-closed fuzzy set, then λ is an r-τ_{12}-θ-gfc set.

Proof. Let $\lambda \leq \mu$ such that $\tau_{12}(\mu) \geq s$ for $0 < s \leq r$. Since λ is an r-τ_{12}-θ-closed fuzzy set, then $C^{\theta}_{12}(\lambda, r) = \lambda$ and from Proposition 2.16(4) , for $s \leq r$ we have $C^{\theta}_{12}(\lambda, s) \leq C^{\theta}_{12}(\lambda, r) = \lambda \leq \mu$. Hence, λ is an r-τ_{12}-θ-gfc set. □

The converse of Proposition 3.5 is not true as we show in the next example.

Example 3.6. Let $X = \{a, b\}$. Define $\lambda_1, \lambda_2 \in I^X$ as follows:

$$\lambda_1 = a_{0.2} \vee b_{0.5}, \quad \lambda_2 = a_{0.5} \vee b_{0.3}.$$

We define smooth topologies $\tau_1, \tau_2 : I^X \longrightarrow I$ as follows:

$$\tau_1(\lambda) = \begin{cases} 1 & \text{if } \lambda = \bar{0}, \bar{1}, \\ \frac{1}{2} & \text{if } \lambda = \lambda_1, \\ 0 & \textit{otherwise}; \end{cases} \quad \textit{and} \quad \tau_2(\lambda) = \begin{cases} 1 & \text{if } \lambda = \bar{0}, \bar{1}, \\ \frac{1}{2} & \text{if } \lambda = \lambda_2, \\ 0 & \textit{otherwise}. \end{cases}$$

The induced supra smooth topological space of (X, τ_1, τ_2), is defined as $\tau_{12} : I^X \longrightarrow I$ such that

$$\tau_{12}(\lambda) = \begin{cases} 1 & \text{if } \lambda = \bar{0}, \bar{1}, \\ \frac{1}{2} & \text{if } \lambda = \lambda_1,\ \lambda_2,\ \lambda_1 \vee \lambda_2, \\ 0 & \textit{otherwise}. \end{cases}$$

Then, for $r = \frac{1}{2}$, the fuzzy set $\lambda = a_{0.4} \vee b_{0.4}$ is a $\frac{1}{2}$-τ_{12}-θ-gfc but is not a $\frac{1}{2}$-τ_{12}-θ-closed fuzzy set.

Proposition 3.7. Let (X, τ_1, τ_2) be a smooth bts, $\lambda \in I^X$ and $r \in I_0$. If λ is an r-τ_{12}-θ-gfc set, then λ is an r-τ_{12}-gfc set.

Proof. The proof follows directly from Theorem 2.15(1). □

The following example shows the converse of the previous proposition is not true.

Example 3.8. Let $X = \{a, b\}$. Define $\lambda_1, \lambda_2 \in I^X$ as follows:

$$\lambda_1 = a_{0.7} \vee b_{0.5}, \quad \lambda_2 = a_{0.2} \vee b_{0.9}.$$

We define smooth topologies $\tau_1, \tau_2 : I^X \longrightarrow I$ as follows:

$$\tau_1(\lambda) = \begin{cases} 1 & \text{if } \lambda = \bar{0}, \bar{1}, \\ \frac{1}{2} & \text{if } \lambda = \lambda_1, \\ 0 & \textit{otherwise}; \end{cases} \quad \textit{and} \quad \tau_2(\lambda) = \begin{cases} 1 & \text{if } \lambda = \bar{0}, \bar{1}, \\ \frac{1}{4} & \text{if } \lambda = \lambda_2, \\ 0 & \textit{otherwise}. \end{cases}$$

The induced supra smooth topological space of (X, τ_1, τ_2), is defined as $\tau_{12} : I^X \longrightarrow I$ such that

$$\tau_{12}(\lambda) = \begin{cases} 1 & \text{if } \lambda = \bar{0}, \bar{1}, \\ \frac{1}{2} & \text{if } \lambda = \lambda_1, \\ \frac{1}{4} & \text{if } \lambda = \lambda_2, \\ \frac{1}{4} & \text{if } \lambda = \lambda_1 \vee \lambda_2, \\ 0 & \textit{otherwise}. \end{cases}$$

Then for $r = \frac{1}{2}$, the fuzzy set $\lambda = a_{0.3} \vee b_{0.5}$ is a $\frac{1}{2}$-τ_{12}-gfc but is not a $\frac{1}{2}$-τ_{12}-θ-gfc.

Proposition 3.9. [32] Let (X, τ_1, τ_2) be a smooth bts, $\lambda \in I^X$ and $r \in I_0$. If λ is an r-τ_{12}-closed fuzzy set, then λ is an r-τ_{12}-gfc set.

The converse of Proposition 3.9 is not true (see [32]).

Proposition 3.10. [33] Let (X, τ_1, τ_2) be a smooth bts, $\lambda \in I^X$ and $r \in I_0$. If λ is an r-τ_{12}-θ-closed fuzzy set, then λ is an r-τ_{12}-closed fuzzy set.

The converse of Proposition 3.10 is not true (see [33]).

Proposition 3.11. Let (X, τ_1, τ_2) be a smooth bts, $\lambda \in I^X$ and $r \in I_0$. If λ is an r-(τ_j, τ_i) fuzzy θ-closed set, then λ is an r-τ_{12}-θ-closed fuzzy set.

Proof. To prove λ is an r-τ_{12}-θ-closed fuzzy set, we must prove $C^\theta_{12}(\lambda, r) = \lambda$. Clearly $\lambda \le C^\theta_{12}(\lambda, r)$. On the other hand, from Theorem 2.15(1), $C^\theta_{12}(\lambda, r) \le T^{\tau_j}_{\tau_i}(\lambda, r)$. Since λ is an r-(τ_j, τ_i) fuzzy θ-closed set, then $T^{\tau_j}_{\tau_i}(\lambda, r) = \lambda$. Consequently, $C^\theta_{12}(\lambda, r) \le \lambda$. Hence, λ is an r-τ_{12}-θ-closed fuzzy set. □

The next example shows the converse of Proposition 3.11 is not true in general.

Example 3.12. Let $X = \{a, b\}$. Define $\lambda_1, \lambda_2 \in I^X$ as follows:

$$\lambda_1 = a_{0.4} \vee b_{0.5}, \quad \lambda_2 = a_{0.5} \vee b_{0.4}.$$

We define smooth topologies $\tau_1, \tau_2 : I^X \longrightarrow I$ as follows:

$$\tau_1(\lambda) = \begin{cases} 1 & \text{if } \lambda = \bar{0}, \bar{1}, \\ \frac{1}{3} & \text{if } \lambda = \lambda_1, \\ 0 & \textit{otherwise}; \end{cases} \quad \textit{and} \quad \tau_2(\lambda) = \begin{cases} 1 & \text{if } \lambda = \bar{0}, \bar{1}, \\ \frac{1}{4} & \text{if } \lambda = \lambda_2, \\ 0 & \textit{otherwise}. \end{cases}$$

The induced supra smooth topological space of (X, τ_1, τ_2), is defined as $\tau_{12} : I^X \longrightarrow I$ such that

$$\tau_{12}(\lambda) = \begin{cases} 1 & \text{if } \lambda = \bar{0}, \bar{1}, \\ \frac{1}{3} & \text{if } \lambda = \lambda_1, \\ \frac{1}{4} & \text{if } \lambda = \lambda_2, \\ \frac{1}{4} & \text{if } \lambda = \lambda_1 \vee \lambda_2, \\ 0 & \textit{otherwise}. \end{cases}$$

Then, for $r = \frac{1}{4}$, the fuzzy set $\lambda = a_{0.5} \vee b_{0.5}$ is a $\frac{1}{4}$-τ_{12}-θ-closed fuzzy set but is not a $\frac{1}{4}$-(τ_1, τ_2)fuzzy θ-closed set.

From the above discussion we have the following diagram which is an enlargement of a Diagram from [33].

$$\begin{array}{ccccccc} \mathbf{r}\text{-}(\tau_\mathbf{i}, \tau_\mathbf{j})\text{-}\theta\text{-}\mathbf{gfc} & \Leftarrow & \mathbf{r}\text{-}(\tau_\mathbf{j}, \tau_\mathbf{i})\mathbf{fuzzy}\ \theta\text{-}\mathbf{closed} & \Rightarrow & \mathbf{r}\text{-}\tau_\mathbf{12}\text{-}\theta\text{-}\mathbf{closed\ fuzzy\ set} & \Rightarrow & \mathbf{r}\text{-}\tau_\mathbf{12}\text{-}\mathbf{closed\ fuzzy\ set} \\ \Downarrow & & \Downarrow & & \Downarrow & & \Downarrow \\ \mathbf{r}\text{-}(\tau_\mathbf{i}, \tau_\mathbf{j})\text{-}\mathbf{gfc} & \Longleftarrow & \mathbf{r}\text{-}\tau_\mathbf{j}\text{-}\mathbf{closed\ fuzzy\ set} & & \mathbf{r}\text{-}\tau_\mathbf{12}\text{-}\theta\text{-}\mathbf{gfc} & \Longrightarrow & \mathbf{r}\text{-}\tau_\mathbf{12}\text{-}\mathbf{gfc} \end{array}$$

From the above diagram one can notice that the concepts of r-(τ_i, τ_j)-θ-gfc and r-τ_{12}-θ-gfc sets are independent as the following two examples show.

Example 3.13. Let $X = \{a, b\}$. Define $\lambda_1, \lambda_2 \in I^X$ as follows:

$$\lambda_1 = a_{0.3} \vee b_{0.5}, \quad \lambda_2 = a_{0.6} \vee b_{0.2}.$$

We define smooth topologies $\tau_1, \tau_2 : I^X \longrightarrow I$ as follows:

$$\tau_1(\lambda) = \begin{cases} 1 & \text{if } \lambda = \bar{0}, \bar{1}, \\ \frac{1}{2} & \text{if } \lambda = \lambda_1, \\ 0 & \textit{otherwise}; \end{cases} \quad \textit{and} \quad \tau_2(\lambda) = \begin{cases} 1 & \text{if } \lambda = \bar{0}, \bar{1}, \\ \frac{1}{4} & \text{if } \lambda = \lambda_2, \\ 0 & \textit{otherwise}. \end{cases}$$

The induced supra smooth topological space of (X, τ_1, τ_2), is defined as $\tau_{12} : I^X \longrightarrow I$ such that

$$\tau_{12}(\lambda) = \begin{cases} 1 & \text{if } \lambda = \bar{0}, \bar{1}, \\ \frac{1}{2} & \text{if } \lambda = \lambda_1, \\ \frac{1}{4} & \text{if } \lambda = \lambda_2, \\ \frac{1}{4} & \text{if } \lambda = \lambda_1 \vee \lambda_2, \\ 0 & \textit{otherwise}. \end{cases}$$

Then for $r = \frac{1}{4}$, the fuzzy set $\lambda = a_{0.4} \vee b_{0.3}$ is a $\frac{1}{4}$-(τ_1, τ_2)-θ-gfc but is not a $\frac{1}{4}$-τ_{12}-θ-gfc set.

Example 3.14. Let $X = \{a, b\}$. Define $\lambda_1, \lambda_2 \in I^X$ as follows:

$$\lambda_1 = a_{0.4} \vee b_{0.5}, \quad \lambda_2 = a_{0.6} \vee b_{0.2}.$$

We define smooth topologies $\tau_1, \tau_2 : I^X \longrightarrow I$ as follows:

$$\tau_1(\lambda) = \begin{cases} 1 & \text{if } \lambda = \bar{0}, \bar{1}, \\ \frac{1}{2} & \text{if } \lambda = \lambda_1, \\ 0 & \textit{otherwise}; \end{cases} \quad \textit{and} \quad \tau_2(\lambda) = \begin{cases} 1 & \text{if } \lambda = \bar{0}, \bar{1}, \\ \frac{1}{3} & \text{if } \lambda = \lambda_2, \\ 0 & \textit{otherwise}. \end{cases}$$

The induced supra smooth topological space of (X, τ_1, τ_2), is defined as $\tau_{12} : I^X \longrightarrow I$ such that

$$\tau_{12}(\lambda) = \begin{cases} 1 & \text{if } \lambda = \bar{0}, \bar{1}, \\ \frac{1}{2} & \text{if } \lambda = \lambda_1, \\ \frac{1}{3} & \text{if } \lambda = \lambda_2, \\ \frac{1}{3} & \text{if } \lambda = \lambda_1 \vee \lambda_2, \\ 0 & \textit{otherwise}. \end{cases}$$

Then for $r = \frac{1}{3}$, the fuzzy set $\lambda = a_{0.1} \vee b_{0.3}$ is a $\frac{1}{3}$-τ_{12}-θ-gfc but is not a $\frac{1}{3}$-(τ_1, τ_2)-θ-gfc set.

4 Generalized C^{θ}_{12}-fuzzy Closure Operator

In this section we use the class of r-τ_{12}-θ-gfc (resp. gfo) sets to introduce a new fuzzy closure (resp. interior) operator on smooth bts (X, τ_1, τ_2). In fact this new fuzzy closure (resp. interior) operator represents a generalization of the fuzzy closure (resp. interior) operator C^{θ}_{12} (resp. I^{θ}_{12}) [32]. Some properties of these new fuzzy closure are given. We show that C^{θ}_{12} (resp. I^{θ}_{12}) generates a smooth fuzzy topology which is finer than τ^{θ}_{12}.

Definition 4.1. Let (X, τ_1, τ_2) be a smooth bts. For $\lambda \in I^X$ and $r \in I_0$, a generalized C^{θ}_{12}-fuzzy closure is a map $\mathcal{G}C^{\theta}_{12} : I^X \times I_0 \longrightarrow I^X$ define as

$$\mathcal{G}C^{\theta}_{12}(\lambda, r) = \bigwedge\{\rho \in I^X \mid \rho \geq \lambda \textit{ and } \rho \textit{ is } r\text{-}\tau_{12}\text{-}\theta\text{-}gfc \textit{ set}\}.$$

And a generalized I^{θ}_{12}-fuzzy interior of λ is a map $\mathcal{G}I^{\theta}_{12} : I^X \times I_0 \longrightarrow I^X$ define as

$$\mathcal{G}I^{\theta}_{12}(\lambda, r) = \bigvee\{\rho \in I^X \mid \rho \leq \lambda \textit{ and } \rho \textit{ is } r\text{-}\tau_{12}\text{-}\theta\text{-}gfo \textit{ set}\}.$$

Some properties of $\mathcal{G}C^{\theta}_{12}$ and $\mathcal{G}I^{\theta}_{12}$ are given next.

Proposition 4.2. Let (X, τ_1, τ_2) be a smooth bts, $\lambda, \lambda_1, \lambda_2 \in I^X$ *and* $r \in I_0$. Then:

(1) $\mathcal{G}I^{\theta}_{12}(\bar{1} - \lambda, r) = \bar{1} - \mathcal{G}C^{\theta}_{12}(\lambda, r)$.

(2) If $\lambda_1 \leq \lambda_2$, then $\mathcal{G}C^{\theta}_{12}(\lambda_1, r) \leq \mathcal{G}C^{\theta}_{12}(\lambda_2, r)$.

(3) If $\lambda_1 \leq \lambda_2$, then $\mathcal{G}I^{\theta}_{12}(\lambda_1, r) \leq \mathcal{G}I^{\theta}_{12}(\lambda_2, r)$.

(4) If λ is an r-τ_{12}-θ-gfc, then $\mathcal{G}C^{\theta}_{12}(\lambda, r) = \lambda$.

(5) If λ is an r-τ_{12}-θ-gfo, then $\mathcal{G}I^{\theta}_{12}(\lambda, r) = \lambda$.

Proof. We prove (1), using Definition 4.1:

$$\begin{aligned} \bar{1} - \mathcal{G}C^{\theta}_{12}(\lambda, r) &= \bar{1} - \bigwedge\{\rho \in I^X \mid \rho \geq \lambda,\ \rho \textit{ is } r\text{-}\tau_{12}\text{-}\theta\text{-}gfc \textit{ set}\} \\ &= \bigvee\{\bar{1} - \rho \in I^X \mid \bar{1} - \rho \leq \bar{1} - \lambda, \bar{1} - \rho \textit{ is } r\text{-}\tau_{12}\text{-}\theta\text{-}gfo \textit{ set}\} \\ &= \mathcal{G}I^{\theta}_{12}(\bar{1} - \lambda, r). \end{aligned}$$

To prove (2), suppose there exist $x \in X$ and $t \in I_0$ such that

$$\mathcal{G}C^{\theta}_{12}(\lambda_1, r)(x) > t > \mathcal{G}C^{\theta}_{12}(\lambda_2, r)(x). \tag{5}$$

Since $\mathcal{G}C^{\theta}_{12}(\lambda_2, r)(x) < t$, then there exists an r-τ_{12}-θ-gfc ρ with $\rho \geq \lambda_2$ such that $\rho(x) < t$. Since $\lambda_1 \leq \lambda_2$, then $\mathcal{G}C^{\theta}_{12}(\lambda_1, r) \leq \rho$. It follows $\mathcal{G}C^{\theta}_{12}(\lambda_1, r)(x) < t$. This contradicts (5). Hence, $\mathcal{G}C^{\theta}_{12}(\lambda_1, r) \leq \mathcal{G}C^{\theta}_{12}(\lambda_2, r)$. The proof of (3), follows from taking the complement of (2) and then using (1). The proof of (4), follows from Definition 4.1. Finally, the proof of (5) is similar to the proof of (3). □

In Proposition 4.2 the converse of (4) and (5) are not true as the following example show. The example is inspired by the one introduced in $[18, p.333]$

Example 4.3. Let $X = \{a, b\}$. Define smooth topologies $\tau_1 = \tau_2 : I^X \longrightarrow I$ as follows:

$$\tau_1(\lambda) = \tau_2(\lambda) = \begin{cases} 1 & \text{if } \lambda = \bar{0}, \bar{1}, \\ 0.3 & \text{if } \lambda = a_{0.6}, \\ 0 & \textit{otherwise.} \end{cases}$$

The induced supra smooth topological space of (X, τ_1, τ_2), is defined as $\tau_{12} : I^X \longrightarrow I$ such that

$$\tau_{12}(\lambda) = \begin{cases} 1 & \text{if } \lambda = \bar{0}, \bar{1}, \\ 0.3 & \text{if } \lambda = a_{0.6}, \\ 0 & \textit{otherwise.} \end{cases}$$

The fuzzy set $a_{0.6}$ is not a 1-τ_{12}-θ-gfc, but $\mathcal{G}C^{\theta}_{12}(a_{0.6}, 1) = a_{0.6}$. Because, $a_{0.6} \vee b_s$ is a 1-τ_{12}-θ-gfc for $s \in I_0$. Therefore,

$\mathcal{G}C^{\theta}_{12}(a_{0.6}, 1) = \bigwedge_{s \in I_0}(a_{0.6} \vee b_s) = a_{0.6} \vee \bigwedge_{s \in I_0} b_s = a_{0.6}$.

Next we show $\mathcal{G}C_{12}^{\theta}$ (resp. $\mathcal{G}I_{12}^{\theta}$) is fuzzy closure operator.

Theorem 4.4. Let (X, τ_1, τ_2) be a smooth bts, $\lambda \in I^X$ and $r \in I_0$. Then:

(1) $\mathcal{G}C_{12}^{\theta}$ (resp. $\mathcal{G}I_{12}^{\theta}$) is a fuzzy closure (resp. interior) operator.

(2) The mapping $\tau_{12}^{\mathcal{G}\theta} : I^X \longrightarrow I$ defined as

$$\tau_{12}^{\mathcal{G}\theta}(\lambda) = \bigvee\{r \in I|\, \mathcal{G}C_{12}^{\theta}(\bar{1} - \lambda, r) = \bar{1} - \lambda\}.$$

is a smooth topology on X such that $\tau_{12}^{\theta} \leq \tau_{12}^{\mathcal{G}\theta}$.

Proof. We have shown that $\mathcal{G}C_{12}^{\theta}$ is a fuzzy closure operator and in a similar way can prove that $\mathcal{G}I_{12}^{\theta}$ is a fuzzy interior operator. To prove (1), we need to satisfy conditions $(C1) - (C5)$ in Definition 2.5.

(C1) Since $\bar{0}$ is an r-τ_{12}-θ-gfc set in X, then from Proposition 4.2(4), $\mathcal{G}C_{12}^{\theta}(\bar{0}, r) = \bar{0}$.

(C2) Follows immediately from the Definition of $\mathcal{G}C_{12}^{\theta}$.

(C3) Since $\lambda \leq \lambda \vee \mu$ and $\mu \leq \lambda \vee \mu$, then from Proposition 4.2(2),

$$\mathcal{G}C_{12}^{\theta}(\lambda, r) \leq \mathcal{G}C_{12}^{\theta}(\lambda \vee \mu, r) \text{ and } \mathcal{G}C_{12}^{\theta}(\mu, r) \leq \mathcal{G}C_{12}^{\theta}(\lambda \vee \mu, r).$$

This implies, $\mathcal{G}C_{12}^{\theta}(\lambda, r) \vee \mathcal{G}C_{12}^{\theta}(\mu, r) \leq \mathcal{G}C_{12}^{\theta}(\lambda \vee \mu, r)$.
Suppose $\mathcal{G}C_{12}^{\theta}(\lambda \vee \mu, r) \nleq \mathcal{G}C_{12}^{\theta}(\lambda, r) \vee \mathcal{G}C_{12}^{\theta}(\mu, r)$. Consequently, $x \in X$ and $t \in I_0$ exist such that

$$\mathcal{G}C_{12}^{\theta}(\lambda, r)(x) \vee \mathcal{G}C_{12}^{\theta}(\mu, r)(x) < t < \mathcal{G}C_{12}^{\theta}(\lambda \vee \mu, r)(x). \tag{6}$$

Since $\mathcal{G}C_{12}^{\theta}(\lambda, r)(x) < t$ and $\mathcal{G}C_{12}^{\theta}(\mu, r)(x) < t$, then there exist r-τ_{12}-θ-gfc sets ρ_1, ρ_2 with $\lambda \leq \rho_1$ and $\mu \leq \rho_2$ such that

$$\rho_1(x) < t, \rho_2(x) < t.$$

Since $\lambda \vee \mu \leq \rho_1 \vee \rho_2$ and $\rho_1 \vee \rho_2$ is an r-τ_{12}-θ-gfc from Proposition 3.2(1), we have $\mathcal{G}C_{12}^{\theta}(\lambda \vee \mu, r)(x) \leq (\rho_1 \vee \rho_2)(x) < t$. This, however, contradicts (6). Hence, $\mathcal{G}C_{12}^{\theta}(\lambda, r) \vee \mathcal{G}C_{12}^{\theta}(\mu, r) = \mathcal{G}C_{12}^{\theta}(\lambda \vee \mu, r)$.

(C4) Let $r \leq s$, $r, s \in I_0$. Suppose $\mathcal{G}C_{12}^{\theta}(\lambda, r) \nleq \mathcal{G}C_{12}^{\theta}(\lambda, s)$. Consequently, $x \in X$ and $t \in I_0$ exist such that

$$\mathcal{G}C_{12}^{\theta}(\lambda, s)(x) < t < \mathcal{G}C_{12}^{\theta}(\lambda, r)(x). \tag{7}$$

Since $\mathcal{G}C_{12}^{\theta}(\lambda, s)(x) < t$, then there is an s-τ_{12}-θ-gfc set ρ with $\lambda \leq \rho$ such that $\rho(x) < t$. This yields $C_{12}^{\theta}(\rho, s_1) \leq \mu$, whenever $\rho \leq \mu$ and $\tau_{12}(\mu) \geq s_1$, for $0 < s_1 \leq s$. Since $r \leq s$, then $C_{12}^{\theta}(\rho, r_1) \leq \mu$ whenever $\rho \leq \mu$ and $\tau_{12}(\mu) \geq r_1$, for $0 < r_1 \leq r \leq s_1 \leq s$. This implies ρ is an r-τ_{12}-θ-gfc. From Definition 4.1, we have $\mathcal{G}C_{12}^{\theta}(\lambda, r)(x) \leq \rho(x) < t$. This contradicts (7). Hence, $\mathcal{G}C_{12}^{\theta}(\lambda, r) \leq \mathcal{G}C_{12}^{\theta}(\lambda, s)$.

(C5) Let ρ be any r-τ_{12}-θ-gfc containing λ. Then, from Definition 4.1, we have $\mathcal{G}C_{12}^{\theta}(\lambda, r) \leq \rho$. From proposition 4.2(2), we obtain $\mathcal{G}C_{12}^{\theta}(\mathcal{G}C_{12}^{\theta}(\lambda, r), r) \leq \mathcal{G}C_{12}^{\theta}(\rho, r) = \rho$. This means that $\mathcal{G}C_{12}^{\theta}(\mathcal{G}C_{12}^{\theta}(\lambda, r), r)$ is contained in every r-τ_{12}-θ-gfc set containing λ. Hence, $\mathcal{G}C_{12}^{\theta}(\mathcal{G}C_{12}^{\theta}(\lambda, r), r) \leq \mathcal{G}C_{12}^{\theta}(\lambda, r)$. However, $\mathcal{G}C_{12}^{\theta}(\lambda, r) \leq \mathcal{G}C_{12}^{\theta}(\mathcal{G}C_{12}^{\theta}(\lambda, r), r)$. Therefore, $\mathcal{G}C_{12}^{\theta}(\mathcal{G}C_{12}^{\theta}(\lambda, r), r) = \mathcal{G}C_{12}^{\theta}(\lambda, r)$. Thus $\mathcal{G}C_{12}^{\theta}$ is a fuzzy closure operator.

To prove (2), we employ (1) and Definition 2.5, we get $\tau_{12}^{\mathcal{G}\theta}(\lambda)$ is a smooth topology on X. By Proposition 3.5, $C_{12}^{\theta}(\bar{1} - \lambda, r) = \bar{1} - \lambda$ which yields $\mathcal{G}C_{12}^{\theta}(\bar{1} - \lambda, r) = \bar{1} - \lambda$. Thus, $\tau_{12}^{\theta}(\lambda) \leq \tau_{12}^{\mathcal{G}\theta}(\lambda)$ for all $\lambda \in I^X$.

□

At the end of this section we state the following proposition which is describes each r-τ_{12}-θ-gfc set in smooth topological space $(X, \tau_{12}^{\mathcal{G}\theta})$.

Proposition 4.5. Let (X, τ_1, τ_2) be a smooth bts. $\lambda \in I^X$ and $r \in I_0$. If λ is an r-τ_{12}-θ-gfc, then λ is an r-$\tau_{12}^{\mathcal{G}\theta}$-closed fuzzy set.

Proof. The proof follows from Proposition 4.2(4) and Theorem 4.4(2). □

5 GFP^*-θ-continuous and GFP^*-θ-irresolute Mappings

In this section we use the smooth supra topological space (X, τ_{12}) which is generated from smooth bts (X, τ_1, τ_2) to introduce and study the concepts of generalized FP^*-θ- continuous (resp. irresolute) and FP^*-strongly-θ-continuous mappings for the smooth bts (X, τ_1, τ_2).

Definition 5.1. A mapping $f : (X, \tau_1, \tau_2) \longrightarrow (Y, \tau_1^*, \tau_2^*)$ is called:

(1) generalized FP^*-θ- continuous (GFP^*-θ-continuous , for short) if $f^{-1}(\mu)$ is an r-τ_{12}-θ-gfc in X for each r-τ_{12}^*-closed fuzzy set μ in Y.

(2) generalized-FP^*-θ-irresolute (GFP^*-θ-irresolute, for short) if $f^{-1}(\mu)$ is an r-τ_{12}-θ-gfc in X for each r-τ_{12}^*-θ-gfc μ in Y.

(3) FP^*-strongly-θ- continuous (FP^*-S-θ-continuous, for short) if for each $x_t \in Pt(X)$ and for each $\mu \in Q_{\tau_{12}^*}(f(x_t), r)$, there exists $\nu \in Q_{\tau_{12}}(x_t, r)$ such that $f(C_{12}(\nu, r)) \leq \mu$.

Next we study the relationships between GFP^*-θ-continuous, FP^*-S-θ-continuous, GFP^*-continuous and FP^*-continuous. Next proposition give the relationship between GFP^*-θ-continuous and GFP^*-continuous.

Proposition 5.2. If $f : (X, \tau_1, \tau_2) \longrightarrow (Y, \tau_1^*, \tau_2^*)$ is GFP^*-θ-continuous, then f is GFP^*-continuous.

Proof. Let $\mu \in I^Y$ such that μ is an r-τ_{12}^*-fuzzy closed set. Since f is GFP^*-θ-continuous, then we have, $f^{-1}(\mu)$ is an r-τ_{12}-θ-gfc, and from Proposition 3.7, this yields $f^{-1}(\mu)$ is an r-τ_{12}-gfc. Hence, f is GFP^*-continuous. □

The converse of the above proposition in not true according to the following counterexample.

Example 5.3. Let $X = \{a, b\}$ and $Y = \{p, q, w\}$. Define $\lambda_1, \lambda_2 \in I^X$ and $\mu_1, \mu_2 \in I^Y$ as follows:

$$\lambda_1 = a_{\frac{2}{3}} \vee b_{\frac{1}{2}}, \quad \lambda_2 = a_{\frac{3}{4}} \vee b_{\frac{1}{4}}, \quad \mu_1 = p_{\frac{3}{4}} \vee q_{\frac{2}{3}} \vee w_{\frac{1}{2}}, \quad \mu_2 = p_{\frac{2}{3}} \vee q_{\frac{3}{4}} \vee w_{\frac{1}{2}}.$$

We define the smooth topologies $\tau_1, \tau_2 : I^X \longrightarrow I$ and $\tau_1^*, \tau_2^* : I^Y \longrightarrow I$ as follows:

$$\tau_1(\lambda) = \begin{cases} 1 & \text{if } \lambda = \bar{0}, \bar{1}, \\ \frac{1}{2} & \text{if } \lambda = \lambda_1, \\ 0 & otherwise; \end{cases} \qquad \tau_2(\lambda) = \begin{cases} 1 & \text{if } \lambda = \bar{0}, \bar{1}, \\ \frac{1}{4} & \text{if } \lambda = \lambda_2, \\ 0 & otherwise; \end{cases}$$

$$\tau_1^*(\mu) = \begin{cases} 1 & \text{if } \mu = \bar{0}, \bar{1}, \\ \frac{1}{2} & \text{if } \mu = \mu_1, \\ 0 & otherwise; \end{cases} \qquad and \qquad \tau_2^*(\mu) = \begin{cases} 1 & \text{if } \mu = \bar{0}, \bar{1}, \\ \frac{1}{3} & \text{if } \mu = \mu_2, \\ 0 & otherwise. \end{cases}$$

From the smooth bts's (X, τ_1, τ_2) and (Y, τ_1^*, τ_2^*) we can induce the supra smooth topologies τ_{12} and τ_{12}^* as follows:

$$\tau_{12}(\lambda) = \begin{cases} 1 & \text{if } \lambda = \bar{0}, \bar{1}, \\ \frac{1}{2} & \text{if } \lambda = \lambda_1, \\ \frac{1}{4} & \text{if } \lambda = \lambda_2, \\ \frac{1}{4} & \text{if } \lambda = \lambda_1 \vee \lambda_2, \\ 0 & otherwise; \end{cases} \qquad and \qquad \tau_{12}^*(\mu) = \begin{cases} 1 & \text{if } \mu = \bar{0}, \bar{1}, \\ \frac{1}{2} & \text{if } \mu = \mu_1, \\ \frac{1}{3} & \text{if } \mu = \mu_2, \\ \frac{1}{3} & \text{if } \mu = \mu_1 \vee \mu_2, \\ 0 & otherwise. \end{cases}$$

Consider the mapping $f : (X, \tau_1, \tau_2) \longrightarrow (Y, \tau_1^*, \tau_2^*)$ defined by $f(a) = q$ and $f(b) = w$. Then, f is GFP^*-continuous but is not GFP^*-θ-continuous because, there exists $\bar{1} - \mu_1$ is a $\frac{1}{2}$-τ_{12}^*-closed fuzzy set but $f^{-1}(\bar{1} - \mu_1)$ is not a $\frac{1}{2}$-τ_{12}-θ-gfc set.

Next we give the relationship between FP^*-S-θ-continuous and GFP^*-θ-continuous.

Proposition 5.4. If $f : (X, \tau_1, \tau_2) \longrightarrow (Y, \tau_1^*, \tau_2^*)$ is FP^*-S-θ-continuous, then f is GFP^*-θ-continuous.

Proof. Let $\lambda \in$ be an r-τ_{12}^*-closed fuzzy set in Y. Let $f^{-1}(\lambda) \leq \mu$ where $\tau_{12}(\mu) \geq s$ for $0 < s \leq r$. We must show $C_{12}^{\theta}(f^{-1}(\lambda), s) \leq \mu$. Let $x_t \notin \mu$ this mean, $x_t\, q\, \bar{1}-\mu$. In fact that $f^{-1}(\lambda) \leq \mu$, which implies $\bar{1}-\mu \leq \bar{1}-f^{-1}(\lambda)$, and since $x_t\, q\, \bar{1}-\mu$ this yields, $x_t\, q\, \bar{1}-f^{-1}(\lambda)$. Thus, we have $f(x_t)\, q\, \bar{1}-\lambda$ such that $\bar{1}-\lambda$ is r-τ_{12}^*-open fuzzy set in Y. That is mean $\bar{1}-\lambda \in Q_{\tau_{12}^*}(f(x_t), r)$. Since f is FP^*-S-θ-continuous. Then, there exists $\eta \in Q_{\tau_{12}}(x_t, r)$ such that $f(C_{12}(\eta, r)) \leq \bar{1}-\lambda$. This implies, $f(C_{12}(\eta, r))\, \bar{q}\, \lambda$ and then $C_{12}(\eta, r)\, \bar{q}\, f^{-1}(\lambda)$. In view of Definition 2.14, we get $x_t \notin C_{12}^{\theta}(f^{-1}(\lambda), r)$. Since $s \leq r$ then, from Proposition 2.16(4), we have $x_t \notin C_{12}^{\theta}(f^{-1}(\lambda), s)$. Hence, we obtain $C_{12}^{\theta}(f^{-1}(\lambda), s) \leq \mu$. Thus, f is GFP^*-θ-continuous. □

The converse of the above Proposition not true as seen from the following example.

Example 5.5. Let $X = \{a, b\}$ and $Y = \{p, q\}$. Define $\lambda_1, \lambda_2 \in I^X$ and $\mu_1, \mu_2 \in I^Y$ as follows:

$$\lambda_1 = a_{\frac{1}{2}} \vee b_{\frac{1}{3}}, \quad \lambda_2 = a_{\frac{1}{3}} \vee b_{\frac{1}{2}}, \quad \mu_1 = p_{\frac{1}{2}} \vee q_{\frac{1}{4}}, \quad \mu_2 = p_{\frac{1}{4}} \vee q_{\frac{1}{2}}.$$

We define the smooth topologies $\tau_1, \tau_2 : I^X \longrightarrow I$ and $\tau_1^*, \tau_2^* : I^Y \longrightarrow I$ as follows:

$$\tau_1(\lambda) = \begin{cases} 1 & \text{if } \lambda = \bar{0}, \bar{1}, \\ \frac{1}{2} & \text{if } \lambda = \lambda_1, \\ 0 & \textit{otherwise}; \end{cases} \qquad \tau_2(\lambda) = \begin{cases} 1 & \text{if } \lambda = \bar{0}, \bar{1}, \\ \frac{1}{3} & \text{if } \lambda = \lambda_2, \\ 0 & \textit{otherwise}; \end{cases}$$

$$\tau_1^*(\mu) = \begin{cases} 1 & \text{if } \mu = \bar{0}, \bar{1}, \\ \frac{1}{2} & \text{if } \mu = \mu_1, \\ 0 & \textit{otherwise}; \end{cases} \qquad \textit{and} \qquad \tau_2^*(\mu) = \begin{cases} 1 & \text{if } \mu = \bar{0}, \bar{1}, \\ \frac{1}{3} & \text{if } \mu = \mu_2, \\ 0 & \textit{otherwise}. \end{cases}$$

From the smooth bts's (X, τ_1, τ_2) and (Y, τ_1^*, τ_2^*) we can induce the supra smooth topologies τ_{12} and τ_{12}^* as follows

$$\tau_{12}(\lambda) = \begin{cases} 1 & \text{if } \lambda = \bar{0}, \bar{1}, \\ \frac{1}{2} & \text{if } \lambda = \lambda_1, \\ \frac{1}{3} & \text{if } \lambda = \lambda_2, \\ \frac{1}{3} & \text{if } \lambda = \lambda_1 \vee \lambda_2, \\ 0 & \textit{otherwise}; \end{cases} \qquad \textit{and} \qquad \tau_{12}^*(\mu) = \begin{cases} 1 & \text{if } \mu = \bar{0}, \bar{1}, \\ \frac{1}{2} & \text{if } \mu = \mu_1, \\ \frac{1}{3} & \text{if } \mu = \mu_2, \\ \frac{1}{3} & \text{if } \mu = \mu_1 \vee \mu_2, \\ 0 & \textit{otherwise}. \end{cases}$$

Consider the mapping $f : (X, \tau_1, \tau_2) \longrightarrow (Y, \tau_1^*, \tau_2^*)$ defined by $f(a) = q$ and $f(b) = p$. Then, f is GFP^*-θ-continuous but is not FP^*-S-θ-continuous because, there exists $a_{0.7} \in Pt(X)$, $r = \frac{1}{3}$ and $\mu_1 \in Q_{\tau_{12}^*}(f(a_{0.7}), \frac{1}{3})$ such that for any $\lambda \in Q_{\tau_{12}}(a_{0.7}, \frac{1}{3})$, $f(C_{12}(\lambda, \frac{1}{3})) \not\leq \mu_1$.

Proposition 5.6. [32] If $f : (X, \tau_1, \tau_2) \longrightarrow (Y, \tau_1^*, \tau_2^*)$ is FP^*-continuous, then f is GFP^*-continuous.

The converse of the proceeded proposition is not true in general (see [32]).

To discuss the relation between FP^*-S-θ-continuous and FP^*-continuous, we need to give an equivalent definition to FP^*-continuous.

Theorem 5.7. A mapping $f : (X, \tau_1, \tau_2) \longrightarrow (Y, \tau_1^*, \tau_2^*)$ is FP^*-continuous iff for each $x_t \in Pt(X)$ and for each $\mu \in Q_{\tau_{12}^*}(f(x_t), r)$, there exists $\eta \in Q_{\tau_{12}}(x_t, r)$ such that $f(\eta) \leq \mu$.

Proof. The proof is similar to the one in [[34], Theorem 5.3]. □

Proposition 5.8. If $f : (X, \tau_1, \tau_2) \longrightarrow (Y, \tau_1^*, \tau_2^*)$ is FP^*-S-θ-continuous, then f is FP^*-continuous.

Proof. Let $x_t \in Pt(X)$ and $\mu \in Q\tau_{12}^*(f(x_t), r)$. Since f is FP^*-S-θ-continuous. Then, there exists $\eta \in Q\tau_{12}(x_t, r)$ such that $f(C_{12}(\eta, r)) \leq \mu$. Since $\eta \leq C_{12}(\eta, r)$, then $f(\eta) \leq f(C_{12}(\eta, r)) \leq \mu$. Thus, in view of Theorem 5.7, f is FP^*-continuous. □

The converse of proposition 5.8 is not true as we have shown in Example 5.3. Note that Example 5.3 and Example 5.5 show that the FP^*-continuous and GFP^*-θ-continuous are independent. Therefore we have the following implications and none of them are reversible.

$$\begin{array}{ccc} \textbf{GFP}^*\text{-}\theta\textbf{-continuous} & \Longrightarrow & \textbf{GFP}^*\textbf{-continuous} \\ \Uparrow & & \Uparrow \\ \textbf{FP}^*\textbf{-S-}\theta\textbf{-continuous} & \Longrightarrow & \textbf{FP}^*\textbf{-continuous} \end{array}$$

The following theorem provides conditions to obtain GFP^*-θ-irresolute mapping from GFP^*-θ-continuous mapping.

Theorem 5.9. If $f : (X, \tau_1, \tau_2) \longrightarrow (Y, \tau_1^*, \tau_2^*)$ is bijective, FP^*-open and GFP^*-θ-continuous, then f is GFP^*-θ-irresolute.

Proof. Let ν is an r-τ_{12}^*-θ-gfc set and $f^{-1}(\nu) \leq \mu$ such that $\tau_{12}(\mu) \geq s$ for $0 < s \leq r$. Since $f^{-1}(\nu) \leq \mu$, then $\nu \leq f(\mu)$. From the fact that f is FP^*-open, we obtain $f(\mu)$ is an s-τ_{12}^*-open fuzzy set. Now, we have ν is an r-τ_{12}^*-θ-gfc and $\nu \leq f(\mu)$. From Definition 3.1(1) we get, $C_{12}^{*\theta}(\nu, s) \leq f(\mu)$ and thus, $f^{-1}(C_{12}^{*\theta}(\nu, s)) \leq \mu$. Since $C_{12}^{*\theta}(\nu, s)$ is an r-τ_{12}^*-closed fuzzy set in Y and f is GFP^*-θ-continuous. Then, $f^{-1}(C_{12}^{*\theta}(\nu, s))$ is an r-τ_{12}-θ-gfc in X. Thus, from Definition 3.1(1), $C_{12}^{\theta}(f^{-1}(C_{12}^{*\theta})(\nu, s)), s) \leq \mu$ this yields $C_{12}^{\theta}(f^{-1}(\nu), s) \leq \mu$. Therefore, $f^{-1}(\nu)$ is an r-τ_{12}-θ-gfc. Hence, f is GFP^*-θ-irresolute. □

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THE HERMITE-HADAMARD TYPE INEQUALITIES FOR OPERATOR p-CONVEX FUNCTIONS IN HILBERT SPACE

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Abstract – In this paper, we introduce operator p-convex functions and establish some Hermite-Hadamard type inequalities in which some operator p-convex functions of positive operators in Hilbert spaces are involved.

Keywords – *The Hermite-Hadamard inequality, p-convex functions, operator p-convex functions, selfadjoint operator, inner product space, Hilbert space.*

1 Introduction

The following inequality holds for any convex function f define on $\mathbb{R}$ and $a, b \in \mathbb{R}$, with $a < b$

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_0^1 f(x)dx \leq \frac{f(a)+f(b)}{2} \tag{1}$$

both inequalities hold in the reversed direction if f is concave.

The inequality (1) is known in the literature as the Hermite-Hadamard's inequality. The Hermite-Hadamard's inequality may be regarded as a refinement of the concept of convexity and it follows easily from Jensen's inequality. The classical Hermite-Hadamard inequality provides estimates of the mean value of a continuous convex function $f : [a, b] \to \mathbb{R}$.

In this paper, Firstly we defined for bounded positive selfadjoint operator p-convex functions in Hilbert space, secondly established some new theorems for them and finally Hermite-Hadamard type inequalities for product two bounded positive selfadjoint operators p-convex set up in Hilbert space.

In the paper [1] Dragomir et al. consider $P(I)$. This class is defined in the following way.

Definition 1.1. [1] We say that $f : I \to \mathbb{R}$ is a P-function, or that f belongs to the class $P(I)$, if f is a non-negative function and for all $x, y \in I, \alpha \in [0, 1]$, we have

$$f\big(\alpha x + (1-\alpha)y\big) \leq f(x) + f(y).$$

For some results about the class $P(I)$ see, e.g., [2] and [3].

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2 Preliminary

First, we review the operator order in $B(H)$ and the continuous functional calculus for a bounded selfadjoint operator. For selfadjoint operators $A, B \in B(H)$ we write, for every $x \in H$

$$A \leq B (\text{or } B \geq A) \text{ if } \langle Ax, x\rangle \leq \langle Bx, x\rangle (\text{or } \langle Bx, x\rangle \geq \langle Ax, x\rangle)$$

we call it the operator order.

Let A be a selfadjoint linear operator on a complex Hilbert space $(H, \langle ., .\rangle)$ and $C(Sp(A))$ the C^* -algebra of all continuous complex-valued functions on the spectrum A. The Gelfand map establishes a $*$-isometrically isomorphism Φ between $C(Sp(A))$ and the C^*-algebra $C^*(A)$ generated by A and the identity operator 1_H on H as follows [6].

For any $f, g \in C(Sp(A))$ and any $\alpha, \beta \in \mathbb{C}$ we have

i. $\Phi(\alpha f + \beta g) = \alpha\Phi(f) + \beta\Phi(g)$;

ii. $\Phi(fg) = \Phi(f)\Phi(g)$ and $\Phi(f^*) = \Phi(f)^*$;

iii. $\|\Phi(f)\| = \|f\| := sup_{t \in Sp(A)}|f(t)|$;

iv. $\Phi(f_0) = 1$ and $\Phi(f_1) = A$, where $f_0(t) = 1$ and $f_1(t) = t$, for $t \in Sp(A)$

If f is a continuous complex-valued functions on $C(Sp(A))$, the element $\Phi(f)$ of $C^*(A)$ is denoted by $f(A)$, and we call it the continuous functional calculus for a bounded selfadjoint operator A.

If A is bounded selfadjoint operator and f is real valued continuous function on $Sp(A)$, then $f(t) \geq 0$ for any $t \in Sp(A)$ implies that $f(A) \geq 0$, i.e $f(A)$ is a positive operator on H. Moreover, if both f and g are real valued functions on $Sp(A)$ such that $f(t) \leq g(t)$ for any $t \in Sp(A)$, then $f(A) \leq f(B)$ in the operator order $B(H)$.

A real valued continuous function f on an interval I is said to be operator convex (operator concave) if

$$f((1-\lambda)A + \lambda B) \leq (\geq)(1-\lambda)f(A) + \lambda f(B)$$

in the operator order in $B(H)$, for all $\lambda \in [0, 1]$ and for every bounded self-adjoint operator A and B in $B(H)$ whose spectra are contained in I.

3 Operator p-convex Functions in Hilbert Space

The following definition and function class are firstly defined by Seren Salaş.

Definition 3.1. Let I be interval in $\mathbb{R}$ and K be a convex subset of $B(H)^+$. A continuous function $f : I \to \mathbb{R}$ is said to be operator p-convex on I, operators in K if

$$f\big(\alpha A + (1-\alpha)B\big) \leq f(A) + f(B) \tag{2}$$

in the operator order in $B(H)$, for all $\alpha \in [0, 1]$ and for every positive operators A and B in K whose spectra are contained in I .

In the other words, if f is an operator p-convex on I, we denote by $f \in S_pO$.

Lemma 3.2. If f belongs to S_pO for operators in K, then $f(A)$ is positive for every $A \in K$.

Proof. For $A \in K$, we have

$$f(A) = f\left(\frac{A}{2} + \frac{A}{2}\right) \leq f(A) + f(A) = 2f(A).$$

This implies that $f(A) \geq 0$.

Moslehian and Najafi [4] proved the following theorem for positive operators as follows :

Theorem 3.3. [4] Let $A, B \in B(H)^+$. Then $AB+BA$ is positive if and only if $f(A+B) \leq f(A)+f(B)$ for all non-negative operator functions f on $[0, \infty)$.

Dragomir in [5] has proved a Hermite-Hadamard type inequality for operator convex function as follows:

Theorem 3.4. [5] Let $f : I \to \mathbb{R}$ be an operator convex function on the interval I. Then for all selfadjoint operators A and B with spectra in I we have the inequality

$$\begin{aligned}\left(f\left(\frac{A+B}{2}\right)\leq\right) \quad & \frac{1}{2}\left[f\left(\frac{3A+B}{4}\right)+f\left(\frac{A+3B}{4}\right)\right] \\ \leq\ & \int_0^1 f\Big(\Big(1-t)A+tB\Big)\Big)dt \\ \leq\ & \frac{1}{2}\left[f\left(\frac{A+B}{2}\right)+\frac{f(A)+f(B)}{2}\right]\left(\leq\left(\frac{f(A)+f(B)}{2}\right)\right].\end{aligned}$$

Let X be a vector space, $x, y \in X, x \neq y$. Define the segment

$$[x, y] := (1-t)x + ty; t \in [0, 1].$$

We consider the function $f : [x, y] :\to \mathbb{R}$ and the associated function

$$g(x, y) : [0, 1] \to \mathbb{R}$$

$$g(x, y)(t) := f((1-t)x + ty), t \in [0, 1].$$

Note that f is convex on $[x, y]$ if and only if $g(x, y)$ is convex on $[0, 1]$. For any convex function defined on a segment $[x, y] \in X$, we have the Hermite-Hadamard integral inequality

$$f\left(\frac{x+y}{2}\right) \leq \int_0^1 f((1-t)x+ty)dt \leq \frac{f(x)+f(y)}{2}$$

which can be derived from the classical Hermite-Hadamard inequality for the convex $g(x, y) : [0, 1] \to \mathbb{R}$.

Lemma 3.5. Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a continuous function on the interval I. Then for every two positive operators $A, B \in K \subseteq B(H)^+$ with spectra in I the function $f \in S_pO$ for operators in

$$[A, B] := (1-t)A + tB; t \in [0, 1]$$

if and only if the function $\varphi_{x,A,B} : [0, 1] \to \mathbb{R}$ defined by

$$\varphi_{x,A,B} := \big\langle f((1-t)A + tB)x, x\big\rangle$$

is operator p-convex on $[0, 1]$ for every $x \in H$ with $\|x\| = 1$.

Proof. Since $f \in S_pO$ operator in $[A, B]$, then for any $t_1, t_2 \in [0, 1]$ and $\alpha \in [0, 1]$ we have

$$\begin{aligned}\varphi_{x,A,B}(\alpha t_1 + (1-\alpha)t_2) &= \big\langle f((1-(\alpha t_1+(1-\alpha)t_2)A + (\alpha t_1+(1-\alpha)t_2)B)x, x\big\rangle \\ &= \big\langle f(\alpha[(1-t_1)A+t_1B] + (1-\alpha)[(1-t_2)A+t_2B])x, x\big\rangle \\ &\leq \big\langle f((1-t_1)A+t_1B)x, x\big\rangle + f((1-t_2)A+t_2B)x, x\big\rangle \\ &\leq \varphi_{x,A,B}(t_1) + \varphi_{x,A,B}(t_2)\end{aligned}$$

Theorem 3.6. Let $f \in S_pO$ on the interval $I \subseteq [0, \infty)$ for operators $K \subseteq B(H)^+$. Then for all positive operators A and B in K with spectra in I, we have the inequality

$$\frac{1}{2}f\left(\frac{A+B}{2}\right) \leq \int_0^1 f\big(tA+(1-t)B\big)dt \leq \big[f(A)+(B)\big] \tag{3}$$

Proof. For $x \in H$ with $\|x\| = 1$ and $t \in [0,1]$, we have

$$\big\langle ((1-t)A + tB)x, x \big\rangle = (1-t)\big\langle Ax, x\big\rangle + t\big\langle Bx, x\big\rangle \in I, \tag{4}$$

Since $\big\langle Ax, x\big\rangle \in Sp(A) \subseteq I$ and $\big\langle Bx, x\big\rangle \in Sp(B) \subseteq I$.

Continuity of f and 4 imply that the operator-valued integral $\int_0^1 f\big(tA + (1-t)B\big)dt$ exists.

Since f is operator p-convex, therefore for t in $[0,1]$, and $A, B \in K$ we have

$$f\big(tA + (1-t)B\big)dt \le f(A) + f(B) \tag{5}$$

Integrating both sides of 5 over $[0,1]$ we get the following inequality

$$\int_0^1 f\big(tA + (1-t)B\big)dt \le f(A) + f(B)$$

To prove the first inequality of 3, we observe that

$$f\left(\frac{A+B}{2}\right) \le f\big(tA + (1-t)B\big) + f\big((1-t)A + tB\big) \tag{6}$$

Integrating the inequality 6 over $t \in [0,1]$ and taking into account that

$$\int_0^1 f\big(tA + (1-t)B\big)dt = \int_0^1 f\big((1-t)A + tB\big)dt$$

then we deduce the first part of 3.

4 The Hermite-Hadamard Type Inequality for the Product Two Operators p-convex Functions

Let $f, g \in S_pO$ on the interval in I. Then for all positive operators A and B on a Hilbert space H with spectra in I, we define real functions $M(A,B)$ and $N(A,B)$ on H by

$$M(A,B)(x) = \langle f(A)x, x\rangle\langle g(A)x, x\rangle + \langle f(B)x, x\rangle\langle g(B)x, x\rangle \ (x \in H),$$
$$N(A,B)(x) = \langle f(A)x, x\rangle\langle g(B)x, x\rangle + \langle f(B)x, x\rangle\langle g(A)x, x\rangle \ (x \in H).$$

Theorem 4.1. Let $f, g \in S_pO$ be on the interval I for operators in $K \subseteq B(H)^+$. Then for all positive operators A and B in K with spectra in I, we have the inequality

$$\int_0^1 \langle f\big(tA + (1-t)B\big)x, x\rangle\langle g\big(tA + (1-t)B\big)x, x\rangle dt \le M(A,B) + N(A,B)$$

hold for any $x \in H$ with $\|x\| = 1$.

Proof. For $x \in H$ with $\|x\| = 1$ and $t \in [0,1]$, we have

$$\big\langle (A+B)x, x\big\rangle = \big\langle Ax, x\big\rangle + \big\langle Bx, x\big\rangle \in I, \tag{7}$$

since $\big\langle Ax, x\big\rangle \in Sp(A) \subseteq I$ and $\big\langle Bx, x\big\rangle \in Sp(B) \subseteq I$.

Continuity of f, g and 7 imply that the operator-valued integrals

$$\int_0^1 f\big(tA + (1-t)B\big)dt,\ \int_0^1 g\big(tA + (1-t)B\big)dt \text{ and } \int_0^1 (fg)\big(tA + (1-t)B\big)dt$$

exist.

Since $f, g \in S_pO$, therefore for t in $[0,1]$ and $x \in H$ we have

$$\langle f\big(tA+(1-t)B\big)x,x\rangle \leq \langle f(A)+f(B)x,x\rangle \tag{8}$$

$$\langle g\big(tA+(1-t)B\big)x,x\rangle \leq \langle g(A)+g(B)x,x\rangle. \tag{9}$$

From 8 and 9, we obtain

$$\begin{aligned}\langle f\big(tA+(1-t)B\big)x,x\rangle\langle g\big(tA+(1-t)B\big)x,x\rangle \leq\; & \langle f(A)x,x\rangle\langle g(A)x,x\rangle \\ & +\langle f(A)x,x\rangle\langle g(B)x,x\rangle \\ & +\langle f(B)x,x\rangle\langle g(A)x,x\rangle \\ & +\langle f(B)x,x\rangle\langle g(B)x,x\rangle \end{aligned} \tag{10}$$

Integrating both sides of 10 over $[0,1]$, we get the required inequality 7.

Theorem 4.2. Let f, g belong to S_pO on the interval I for operators in $K \subseteq B(H)^+$. Then for all positive operators A and B in K with spectra in I, we have the inequality

$$\frac{1}{2}\left\langle f\left(\frac{A+B}{2}\right)x,x\right\rangle\left\langle g\left(\frac{A+B}{2}\right)x,x\right\rangle \tag{11}$$

$$\begin{aligned} \leq\; & \int_0^1 \langle f\big(tA+(1-t)B\big)x,x\rangle\langle g\big(tA+(1-t)B\big)x,x\rangle dt \\ & +M(A,B)+N(A,B) \end{aligned} \tag{12}$$

hold for any $x \in H$ with $\|x\| = 1$.

Proof. Since $f, g \in S_pO$, therefore for any $t \in I$ and any $x \in H$ with $\|x\| = 1$, we observe that

$$\begin{aligned} & \left\langle f\left(\frac{A+B}{2}\right)x,x\right\rangle\left\langle g\left(\frac{A+B}{2}\right)x,x\right\rangle \\ \leq\; & \left\langle f\left(\frac{tA+(1-t)B}{2}+\frac{(1-t)A+tB}{2}\right)x,x\right\rangle \\ & \times\left\langle g\left(\frac{tA+(1-t)B}{2}+\frac{(1-t)A+tB}{2}\right)x,x\right\rangle \end{aligned}$$

$$\begin{aligned} \leq\; & \Big\{\langle f\big(tA+(1-t)B\big)\rangle+\langle f\big((1-t)A+tB\big)\rangle \\ & \times\langle g\big(tA+(1-t)B\big)\rangle+\langle g\big((1-t)A+tB\big)\rangle\Big\} \end{aligned}$$

$$\begin{aligned} \leq\; & \Big\{\Big[\langle f\big(tA+(1-t)B\big)x,x\rangle\langle g\big(tA+(1-t)B\big)x,x\rangle\Big] \\ & +\Big[\langle f\big((1-t)A+tB\big)x,x\rangle\langle g\big((1-t)A+tB\big)x,x\rangle\Big] \\ & +\Big[\langle f(A)x,x\rangle+\langle f(B)x,x\rangle\Big]\times\Big[\langle g(A)x,x\rangle+\langle g(B)x,x\rangle\Big] \\ & +\Big[\langle f(A)x,x\rangle+\langle f(B)x,x\rangle\Big]\times\Big[\langle g(A)x,x\rangle+\langle g(B)x,x\rangle\Big]\Big\} \end{aligned}$$

$$
\begin{aligned}
= \quad & \Big\{ \Big[\langle f\big(tA+(1-t)B\big)x,x\rangle g\big(tA+(1-t)B\big)x,x\rangle \Big] \\
& + \Big[\langle f\big((1-t)A+tB\big)x,x\rangle \langle g\big((1-t)A+tB\big)x,x\rangle \Big] \\
& +2\Big[\langle f(A)x,x\rangle \langle g(A)x,x\rangle \Big] + 2\Big[\langle f(B)x,x\rangle \langle g(B)x,x\rangle \Big] \\
& +2\Big[\langle f(A)x,x\rangle \langle g(B)x,x\rangle \Big] + 2\Big[\langle f(B)x,x\rangle \langle g(A)x,x\rangle \Big] \Big\}
\end{aligned}
$$

By integration over $[0,1]$, we obtain

$$
\begin{aligned}
& \left\langle f\left(\frac{A+B}{2}\right)x,x \right\rangle \left\langle g\left(\frac{A+B}{2}\right)x,x \right\rangle \\
\leq \quad & \int_0^1 \Big[\langle f\big((1-t)A+tB\big)x,x\rangle \langle g\big(tA+(1-t)B\big)x,x\rangle \\
& + \langle f\big(tA+(1-t)B\big)x,x\rangle \langle g\big((1-t)A+tB\big)x,x\rangle \Big] dt \\
& +2M(A,B)+2N(A,B)
\end{aligned}
$$

This implies the inequality 11.

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SOFT β-OPEN SETS AND THEIR APPLICATIONS

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Abstract − First of all, we focused on soft β-open sets, soft β-closed sets, soft β-interior and soft β-closure over the soft topological space and investigated some properties of them. Secondly, we defined the concepts soft β-continuity, soft β-irresolute and soft β-homeomorphism on soft topological spaces. We also obtained some characterizations of these mappings. Finally, we observed that the collection $S\beta r\text{-}h(X, \tau, E)$ was a soft group.

Keywords − *Soft sets, Soft topology, Soft β-open sets, Soft β-interior, Soft β-closure, Soft β-continuity.*

1 Introduction

Molodtsov [14], in 1999, presented the soft theory as a new mathematical tool for tackling with ambiguities that known mathematical tools cannot hold. He has indicated a few aplications of soft theory for finding solutions to many practical problems such as economics, social science, engineering, medical science, etc.

Recently, papers about soft sets and their applications in various fields have increased largely. With a fixed number of parameters Shabir and Naz [15] came up with some notions of soft topological spaces defined on the initial universe . The authors defined soft open sets, soft interior, soft closed sets, soft closure, and soft seperation axioms. Chen [7] presented soft semi open sets and of the some related properties. With a fixed number of parameters Gunduz Aras et al. [4] came up with some soft continuous mappings defined on the initial universe. Mahanta and Das [12] presented and classified many forms of soft functions, such as irresolute, semicontinuous and semiopen soft functions. Arockiarani and Lancy [5] presented soft $g\beta$-closed and soft $gs\beta$-closed sets in soft topological spaces and with the aid of these presented sets they found out some properties.

In the present study, firstly, we focused soft β-open sets, soft β-closed sets, soft β-interior and soft β-closure over the soft topological space and investigated some properties of them. Secondly, we defined the concepts soft β-continuity, soft β-irresolute and soft β-homeomorphism on soft topological spaces. We also obtained some characterizations of these mappings. Finally, we observed that the collection $S\beta r\text{-}h(X, \tau, E)$ was a soft group.

This study is a part of correspoinding author's MSc thesis.

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2 Preliminary

Let U be an initial universe set and E be a collection of all probable parameters with respect to U. Here the parameters are characteristics or properties of objects in U. Let $P(U)$ denote the power set of U, and let $A \subseteq E$.

Definition 2.1. [14] A pair (F, A) is called a soft set over U, where F is a mapping given by $F : A \longrightarrow P(U)$. In other words, a soft set over U is a parametrized family of subsets of the universe U. For a particular $e \in A$, $F(e)$ may be considered the set of e-approximate elements of the soft set (F, A).

Definition 2.2. [13] For two soft sets (F, A) and (G, B) over a common universe U, we say that (F, A) is a soft subset of (G, B) if (i) $A \subseteq B$, and (ii)$\forall e \in A$, $F(e) \subseteq G(e)$ are identical approximations. We write (F, A) $\widetilde{\subseteq}$ (G, B). (F, A) is said to be a soft super set of (G, B), if (G, B) is a soft subset of (F, A). We denote it by $(F, A) \widetilde{\supseteq} (G, B)$.

Definition 2.3. [13] A soft set (F, A) over U is said to be

(i) null soft set denoted by Φ, if $\forall e \in A$, $F(e) = \phi$.
(ii) absolute soft set denoted by $\widetilde{A}$, if $\forall e \in A$, $F(e) = U$.

Definition 2.4. For two soft sets (F, A) and (G, B) over a common universe U,

(i) [13] union of two soft sets of (F, A) and (G, B) is the soft set (H, C), where $C = A \cup B$, and $\forall e \in C$,

$$H(e) = \begin{cases} F(e) & , \quad \text{if } e \in A - B \\ G(e) & , \quad \text{if } e \in B - A \\ F(e) \cup G(e) & , \quad \text{if } e \in A \cap B \end{cases}$$

We write $(F, A) \,\widetilde{\cup}\, (G, B) = (H, C)$.

(ii) [9] intersection of (F, A) and (G, B) is the soft set (H, C), where $C = A \cap B$, and $\forall e \in C$, $H(e) = F(e) \cap G(e)$. We write $(F, A) \,\widetilde{\cap}\, (G, B) = (H, C)$.

Let X be an initial universe set and E be the non-empty set of parameters.

Definition 2.5. [15] Let (F, E) be a soft set over X and $x \in X$. We say that $x \in (F, E)$ is read as x belongs to the soft set (F, E) whenever $x \in F(e)$ for all $e \in E$. Note that for any $x \in X$. $x \notin (F, E)$, if $x \notin F(e)$ for some $e \in E$.

Definition 2.6. [15] Let Y be a non-empty subset of X, then $\widetilde{Y}$ denotes the soft set (Y, E) over X for which $Y(e) = Y$, for all $e \in E$. In particular, (X, E), will be denoted by $\widetilde{X}$.

Definition 2.7. [3] The relative complement of a soft set (F, E) is denoted by $(F, E)'$and is defined by $(F, E)' = (F', E)$ where $F' : E \longrightarrow P(U)$ is a mapping given by $F'(e) = U - F(e)$ for all $e \in E$.

Definition 2.8. [15] Let τ be the collection of soft sets over X, then τ is said to be soft topology on X if

(1) Φ, $\widetilde{X}$ belong to τ
(2) the union of any number of soft sets in τ belongs to τ
(3) the intersection of any two soft sets in τ belongs to τ

The triplet (X, τ, E) is called a soft topological space over X. The members of τ are said to be soft open sets in X.

We will denote all soft open sets(resp. soft closed sets) in X as $S.O(X)$ (resp. $S.C(X)$).

Definition 2.9. [15] Let (X,τ,E) be a soft topological space over X. A soft set (F,E) over X is said to be a soft closed set in X, if its relative complement $(F,E)'$ belongs to τ.

Definition 2.10. Let (X,τ,E) be a soft topological space over X and (F,E) be a soft set over X. Then

a) soft interior[10] of the soft set (F,E) is denoted by $(F,E)^\circ$ and is defined as the union of all soft open sets contained in (F,E). Thus $(F,E)^\circ$ is the largest soft open set contained in (F,E).

b) soft closure[15] of (F,E), denoted by $\overline{(F,E)}$ is the intersection of all soft closed super sets of (F,E). Clearly $\overline{(F,E)}$ is the smallest soft closed set over X which contains (F,E).

We will denote interior(resp. closure) of the soft set (F,E) as $int\,(F,E)$ (resp.$cl\,(F,E)$).

Proposition 2.11. [10] Let (X,τ,E) be a soft topological space over X and (F,E) and (G,E) be a soft set over X. Then

a) $int\,(int\,(F,E)) = int\,(F,E)$
b) $(F,E)\,\tilde{\subseteq}\,(G,E)$ imples $int\,(F,E)\,\tilde{\subseteq}\,int(G,E)$
c) $cl\,(cl\,(F,E)) = cl\,(F,E)$
d) $(F,E)\,\tilde{\subseteq}\,(G,E)$ imples $cl\,(F,E)\,\tilde{\subseteq}\,cl(G,E)$

Definition 2.12. [6] Let (F,E) be a soft set X. The soft set (F,E) is called a soft point , denoted by (x_e,E) *or* x_e , if for the element $e\in E$, $F(e)=\{x\}$ and $F(e')=\phi$ for all $e'\in E-\{e\}$.

Definition 2.13. [18] The *soft point* x_e is said to belong to the soft set (G,E), denoted by $x_e\in(G,E)$, if for the element $e\in E$, $F(e)\,\tilde{\subseteq}\,G(e)$.

Definition 2.14. [18] A soft set (G,E) in a soft topological space (X,τ,E) is called a soft neighborhood of the soft point x_e if there exists an open soft set (H,E) such that $x_e\in(H,E)\,\tilde{\subseteq}\,(G,E)$. A soft set (G, E) in a soft topological space (X,τ,E) is called a soft neighborhoood of the soft set (F,E) if there exists an open soft set (H,E) such that $(F,E)\in(H,E)\,\tilde{\subseteq}\,(G,E)$.The neighborhood system of a soft point x_e, denoted by $N_\tau(x_e)$, is the family of all its neigborhoods.

Definition 2.15. [11] Let (X,τ,E) be a soft topological space. A soft point $x_e\in cl(F,E)$ if and only if each soft neighborhood of x_e intersects (F,E).

3 Soft β-open Sets and Soft β-closed Sets

Definition 3.1. A soft set (F,E) in a soft topological space (X,τ,E) is said to be

a) soft *semi*-open[7] if $(F,E)\,\tilde{\subseteq}\,cl(int(F,E))$.
b) soft *pre*-open[5] if $(F,E)\,\tilde{\subseteq}\,int(cl(F,E))$.
c) soft α-open if[5] if $(F,E)\,\tilde{\subseteq}\,int(cl(int(F,E)))$.
d) soft β-open (soft β-closed)[5] if $(F,E)\,\tilde{\subseteq}\,cl(int(cl(F,E)))$ $(int(cl(int(F,E)))\,\tilde{\subseteq}\,(F,E))$.
e) soft regular-open (soft regular-closed)[16] if $(F,E)=int(cl(F,E))$ $((F,E)=cl(int(F,E)))$

We will denote all the soft β-open (resp. soft *semi*-open, soft *pre*-open, soft α-open, soft β-closed, soft regular-open, soft regular-closed) sets in X as $S.\beta.O(X)$ (resp. $S.S.O(X)$, $S.P.O(X)$, $S.\alpha.O(X)$, $S.\beta.C(X)$, $S.R.O(X)$, $S.R.C(X)$).

Remark 3.2. It is clear that $S.\beta.O(X)$ contains each of $S.S.O(X)$, $S.P.O(X)$ and $S.\alpha.O(X)$, and the following diagram shows this fact.

$$\begin{array}{ccccc} soft\ open\ set & \longrightarrow & soft\ \alpha\text{-}open\ set & \longrightarrow & soft\ semi\text{-}open\ set \\ & & \downarrow & & \downarrow \\ & & soft\ pre\text{-}open\ set & \longrightarrow & soft\ \beta\text{-}open\ set \end{array}$$

The converses need not be true, in general, as show in the following examples.

Example 3.3. Let $X = \{x_1, x_2, x_3\}$, $E = \{e_1, e_2\}$ and $\tau = \{\Phi, \widetilde{X}, (F_1, E), (F_2, E), ..., (F_7, E)\}$ where $(F_1, E), (F_2, E), ..., (F_7, E)$ are soft sets over X, which is defined as follows: $F_1(e_1) = \{x_1, x_2\}$, $F_1(e_2) = \{x_1, x_2\}$, $F_2(e_1) = \{x_2\}$, $F_2(e_2) = \{x_1, x_3\}$, $F_3(e_1) = \{x_2, x_3\}$, $F_3(e_2) = \{x_1\}$, $F_4(e_1) = \{x_2\}$, $F_4(e_2) = \{x_1\}$, $F_5(e_1) = \{x_1, x_2\}$, $F_5(e_2) = X$, $F_6(e_1) = X$, $F_6(e_2) = \{x_1, x_2\}$, $F_7(e_1) = \{x_2, x_3\}$, $F_7(e_2) = \{x_1, x_3\}$ [7]. Then τ defines a soft topology on X and hence (X, τ, E) is a soft topological space over X. Now we give a soft set (H, E) in (X, τ, E) is defined as follows: $H(e_1) = \phi$, $H(e_2) = \{x_1\}$. Then, (H, E) is a soft *pre*-open set but not a soft α-open set, also it is a soft β-open set but not a soft *semi*-open set .

Example 3.4. Let $X = \{x_1, x_2, x_3, x_4\}$, $E = \{e_1, e_2\}$ and $\tau = \{\Phi, \widetilde{X}, (F_1, E), (F_2, E), (F_3, E)\}$, where (F_1, E), (F_2, E), (F_3, E), are soft sets over X, defined as follows. $F_1(e_1) = \{x_1, x_3\}$, $F_1(e_2) = \phi$, $F_2(e_1) = \{x_4\}$, $F_2(e_2) = \{x_4\}$, $F_3(e_1) = \{x_1, x_3, x_4\}$, $F_3(e_2) = \{x_4\}$. Then τ defines a soft topology on X. Hence (X, τ, E) is a soft topological space over X. Now we give two soft sets (H, E) and (K, E) in (X, τ, E) are defined as follows: $H(e_1) = \{x_2, x_3\}$, $H(e_2) = \{x_3\}$, $K(e_1) = \{x_2, x_4\}$, $K(e_2) = \{x_1, x_4\}$. Then (H, E) is a soft β-open set which is not soft *pre*-open and (K, E) is a soft *semi*-open set which is not soft α-open.

Theorem 3.5. (a) For every soft open set (F, E) in a soft topological space X and every $(G, E) \widetilde{\subseteq} X$ we have $(F, E) \widetilde{\cap} cl(G, E) \widetilde{\subseteq} cl((F, E) \widetilde{\cap} (G, E))$; (b) For every soft closed set (F, E) in a soft topological space X and every $(G, E) \widetilde{\subseteq} X$ we have $int((F, E) \widetilde{\cup} (G, E)) \widetilde{\subseteq} (F, E) \widetilde{\cup} int(G, E)$.

Proof. (a) Let x_e be a soft point on (X, τ, E). $x_e \in (F, E) \widetilde{\cap} cl(G, E) \Longrightarrow x_e \in (F, E)$ and $x_e \in cl(G, E)$. $x_e \in cl(G, E) \Longleftrightarrow \forall\ (K, E) \in N_\tau(x_e)$, $(K, E) \widetilde{\cap} (G, E) \neq \Phi$. Since $(K, E) \widetilde{\cap} (F, E) \in N_\tau(x_e)$, $(K, E) \widetilde{\cap} (F, E) \widetilde{\cap} (G, E) \neq \Phi$. Then, $x_e \in cl((F, E) \widetilde{\cap} (G, E))$.

(b) It can be proved by taking the complement of $(F, E) \widetilde{\cap} cl(G, E) \widetilde{\subseteq} cl((F, E) \widetilde{\cap} (G, E))$ in (a). □

Theorem 3.6. If (F, E) is soft open and (G, E) is soft β-open, then $(F, E) \widetilde{\cap} (G, E)$ is soft β-open.

Proof. Using Theorem 3.5(a) we obtain $(F, E) \widetilde{\cap} (G, E) \widetilde{\subseteq} (F, E) \widetilde{\cap} cl(int(cl\ (G, E))) \widetilde{\subseteq} cl[(F, E) \widetilde{\cap} int\ (cl(G, E))] = cl[int((F, E) \widetilde{\cap} cl(G, E))] \widetilde{\subseteq} cl[int[cl\ ((F, E) \widetilde{\cap} (G, E))]]$ which completes the proof. □

Theorem 3.7. If (F, E) is soft closed and (G, E) is soft β-closed, then $(F, E) \widetilde{\cup} (G, E)$ is soft β-closed.

Proof. Using Theorem 3.5(b) we obtain $int[cl[int((F, E) \widetilde{\cup} (G, E))]] \widetilde{\subseteq} int[cl\ ((F, E) \widetilde{\cup} int(G, E))] = int((F, E) \widetilde{\cup} cl(int(G, E))) \widetilde{\subseteq} (F, E) \widetilde{\cup} int(cl(int\ (G, E))) \widetilde{\subseteq} (F, E) \widetilde{\cup} (G, E)$ which completes the proof. □

Theorem 3.8. $S.S.O(X) \widetilde{\cup} S.P.O(X) \widetilde{\subseteq} S.\beta.O(X)$

Proof. Let $(F, E) \in S.S.O(X)$ and $(G, E) \in S.P.O(X)$. Then, $(F, E) \widetilde{\subseteq} cl(int(F, E)) \widetilde{\subseteq} cl(int(cl(F, E)))$ and $(G, E) \widetilde{\subseteq} int(cl(G, E)) \widetilde{\subseteq} cl(int(cl\ (G, E)))$. Therefore, $(F, E) \widetilde{\cup} (G, E) \widetilde{\subseteq} cl(int(cl(F, E))) \widetilde{\cup} cl(int(cl(G, E))) = cl[int(cl(F, E)) \widetilde{\cup} int(cl(G, E))] \widetilde{\subseteq} cl[int(cl(F, E) \widetilde{\cup} cl(G, E))] = cl[int[cl\ ((F, E) \widetilde{\cup} (G, E))]]$. □

Theorem 3.9. $S.S.C(X) \widetilde{\cup} S.P.C(X) \widetilde{\subseteq} S.\beta.C(X)$

Proof. Easy. □

Now we define the notion of soft supratopology is weaker than soft topology.

Definition 3.10. [17, 8] Let τ be the collection of soft sets over X, then τ is said to be soft supratopology on X if

(1) Φ, $\widetilde{X}$ belong to τ
(2) the union of any number of soft sets in τ belongs to τ

We give the following property for soft β-open sets.

Proposition 3.11. The collection $S.\beta.O(X)$ of all soft β-open sets of a space (X, τ, E) forms a soft supratopology.

Proof. (1) is obvious

(2) Let $(F_i, E) \in S.\beta.O(X)$ for $\forall i \in I = \{1, 2, 3.....\}$. Then, for $\forall i \in I$, $(F_i, E) \tilde{\subseteq} cl(int(cl(F_i, E))) \Longrightarrow \underset{i\in I}{\tilde{\cup}}(F_i, E) \tilde{\subseteq} \underset{i\in I}{\tilde{\cup}}(cl(int(cl(F_i, E)))) = cl(\underset{i\in I}{\tilde{\cup}}(int(cl(F_i, E)))) \tilde{\subseteq} cl(int(\underset{i\in I}{\tilde{\cup}}(cl(F_i, E)))) = cl(int(cl(\underset{i\in I}{\tilde{\cup}}(F_i, E))))$ □

The intersection of two soft β-open sets need not be a soft β-open set as is illustrated by the following example.

Example 3.12. Let $X = \{x_1, x_2\}$, $E = \{e_1, e_2\}$ and $\tau = \{\Phi, \widetilde{X}, (F_1, E), (F_2, E), (F_3, E)\}$ where (F_1, E), $(F_2, E), (F_3, E)$ are soft sets over X, defined as follows. $F_1(e_1) = \{x_1\}$, $F_1(e_2) = \{x_2\}$, $F_2(e_1) = \{x_1, x_2\}$, $F_2(e_2) = \{x_2\}$, $F_3(e_1) = \{x_1\}$, $F_3(e_2) = \{x_1, x_2\}$. Then τ defines a soft topology on X and hence (X, τ, E) is a soft topological space over X. Now we give two soft sets (G, E), (H, E) in (X, τ, E) which are defined as follows: $G(e_1) = \{x_2\}$, $G(e_2) = \{x_2\}$, $H(e_1) = \{x_1, x_2\}$, $H(e_2) = \{x_1\}$. Then, (G, E) and (H, E) are soft β-open sets over X, therefore, $(G, E) \tilde{\cap} (H, E) = \{\{x_2\}, \phi\}$ and $cl(int(cl((G, E) \tilde{\cap} (H, E)))) = \Phi$. Hence, $(G, E) \tilde{\cap} (H, E)$ is not a soft β-open set.

We have the following proposition by using relative complements.

Proposition 3.13. Arbitrary intersection of soft β-closed sets is soft β-closed.

Proof. Let $(F_i, E) \in S.\beta.C(X)$ for $\forall i \in I = \{1, 2, 3.....\}$. Then, for $\forall i \in I$, $(F_i, E) \tilde{\supseteq} int(cl(int(F_i, E))) \Longrightarrow \underset{i\in I}{\tilde{\cap}}(F_i, E) \tilde{\supseteq} \underset{i\in I}{\tilde{\cap}}(int(cl(int(F_i, E)))) = int(\underset{i\in I}{\tilde{\cap}}(cl(int(F_i, E)))) \tilde{\supseteq} int(cl(\underset{i\in I}{\tilde{\cap}}(int(F_i, E)))) = int(cl(int(\underset{i\in I}{\tilde{\cap}}(F_i, E))))$. The union of two soft β-closed sets need not be soft β-closed set as is illustrated by the following example. □

Example 3.14. Let (X, τ, E) be as in Example 3.12. Now we give two soft sets (G, E), (H, E) in (X, τ, E) which are defined as follows: $G(e_1) = \{x_1\}$, $G(e_2) = \{x_1\}$, $H(e_1) = \phi$, $H(e_2) = \{x_2\}$. Then, (G, E) and (H, E) are soft β-closed sets over X , therefore, $(G, E) \tilde{\cup} (H, E) = \{\{x_1\}, \{x_1, x_2\}\}$ and $int(cl(int((G, E) \tilde{\cup} (H, E)))) = \widetilde{X}$. Hence, $(G, E) \tilde{\cup} (H, E)$ is not a soft β-closed set.

Theorem 3.15. For any soft set (F, E) of a soft topological space X the following conditions are equivalent: (a) $(F, E) \in S.\beta.O(X)$ (b) $cl(F, E) \in S.R.C(X)$.

Proof. $(a) \to (b)$ Let (F, E) be a soft β-open set. Then $(F, E) \tilde{\subseteq} cl(int(cl\ (F, E)))$. This implies $cl(F, E) = cl(int(cl(F, E)))$ that is $cl(F, E) \in S.R.C\ (X)$. $(b) \to (a)$ is obvious. □

Theorem 3.16. For any soft set (F, E) of a soft topological space X the following conditions are equivalent: (a) $(F, E) \in S.\beta.C(X)$ (b) $int(F, E) \in S.R.O(X)$.

Theorem 3.17. Each soft β-open set which is soft semi-closed is soft semi-open .

Proof. $(F, E) \in S.\beta.O(X) \Longrightarrow (F, E) \tilde{\subseteq} cl(int(cl(F, E)))$ and $(F, E) \in S.S.C(X) \Longrightarrow int(cl(F, E)) \tilde{\subseteq} (F, E)$. Then $int(cl(F, E)) \tilde{\subseteq} (F, E) \tilde{\subseteq} cl(int\ (cl(F, E)))$. Since $int(cl(F, E)) = (U, E)$ is a soft open set, we can write $(U, E) \tilde{\subseteq} (F, E) \tilde{\subseteq} cl(U, E)$. Hence (F, E) is a soft semi-open set. □

Corollary 3.18. If a soft set (F, E) in a soft topological space (X, τ, E) is soft β-closed and soft *semi*-open, then (F, E) is soft *semi*-closed.

Theorem 3.19. If (F, E) is soft α-open and (G, E) is soft β-open then $(F, E) \tilde{\cap} (G, E)$ is soft β-open.

Proof. $(F, E) \tilde{\cap} (G, E) \tilde{\subseteq} int(cl(int(F, E))) \tilde{\cap} cl(int(cl(G, E))) \tilde{\subseteq} cl[int(cl\ (int(F, E))) \tilde{\cap} int(cl(G, E))] = cl[int[cl(int(F, E)) \tilde{\cap} int(cl(G, E))]] \tilde{\subseteq} cl[int[\ cl[int(F, E) \tilde{\cap} int(cl(G, E))]]] = cl[int[int(F, E) \tilde{\cap} cl(G, E)]] \tilde{\subseteq} cl[int[cl[int\ (F, E) \tilde{\cap} (G, E)]]] \tilde{\subseteq} cl[int[cl[(F, E) \tilde{\cap} (G, E)]]]$. □

Corollary 3.20. If (F, E) is soft α-closed and (G, E) is soft β-closed then $(F, E) \tilde{\cup} (G, E)$ is soft β-closed.

Proposition 3.21. In an indiscrete soft topological space (X, τ, E), each soft β-open is soft *pre*-open.

Proof. If $(F,E) = \Phi$, then (F,E) is soft β-open and soft *pre*-open. Let $(F,E) \neq \Phi$, then, $(F,E) \in S.\beta.O(X) \Longrightarrow (F,E) \widetilde{\subseteq} cl(int(cl(F,E))) = \widetilde{X} = (int(cl(F,E))$. Hence (F,E) is soft *pre*-open. □

Theorem 3.22. A soft set (F,E) in a soft topological space (X,τ,E) is soft β-closed if and only if $cl(\widetilde{X} - cl(int(F,E))) - (\widetilde{X} - cl(F,E)) \widetilde{\supseteq} cl(F,E) - (F,E)$.

Proof. $cl(\widetilde{X}-cl(int(F,E)))-(\widetilde{X}-cl(F,E)) \widetilde{\supseteq} cl(F,E)-(F,E) \Longleftrightarrow (\widetilde{X}-int(cl(int(F,E))))-(\widetilde{X}-cl(F,E)) \widetilde{\supseteq} cl(F,E)-(F,E) \Longleftrightarrow (\widetilde{X}-int(cl(int\ (F,E)))) \widetilde{\cap}\, cl(F,E) \widetilde{\supseteq} cl(F,E)-(F,E) \Longleftrightarrow (\widetilde{X} \widetilde{\cap}\, cl(F,E))-[int(cl(int\ (F,E))) \widetilde{\cap}\, cl(F,E)] \widetilde{\supseteq} cl(F,E) - (F,E) \Longleftrightarrow cl(F,E) - int(cl(int(F,E))) \widetilde{\supseteq} cl(F,E) - (F,E) \Longleftrightarrow (F,E) \widetilde{\supseteq} int(cl(int(F,E))) \Longleftrightarrow (F,E)$ is soft β-closed. □

Theorem 3.23. Each soft β-open and soft α-closed set is soft closed.

Proof. Let $(F,E) \in S.\beta.O(X), (F,E) \widetilde{\subseteq} cl(int(cl(F,E)))$, since (F,E) is soft α-closed $cl(int(cl(F,E))) \widetilde{\subseteq} (F,E)$, then $cl(int(cl(F,E))) \widetilde{\subseteq} (F,E) \widetilde{\subseteq} cl(int(cl(F,E))), (F,E) = cl(int(cl(F,E)))$ which is soft closed. □

Corollary 3.24. Each soft β-closed and soft α-open set is soft open.

Definition 3.25. [2]Let (F,E) be a soft subset of (X,τ,E) then the soft beta-closure of (F,E), denoted by $S\beta cl(F,E)$, is the soft intersection of all soft β-closed subsets of X containing (F,E).

Theorem 3.26. Let (F,E) be a soft subset of X. Then $S\beta cl(F,E) = (F,E) \widetilde{\cup}\, int(cl(int(F,E)))$.

Proof. We observe that $int[cl[int[(F,E) \widetilde{\cup}\, int(cl(int(F,E)))]]] \widetilde{\subseteq} int[cl[int\ [(F,E) \widetilde{\cup}\, cl(int(F,E))]]] \widetilde{\subseteq} int[cl\ [int(F,E) \widetilde{\cup}\, cl(int(F,E))]] = int[cl(int\ (F,E)) \widetilde{\cup}\, cl(int(F,E)] = int(cl(int(F,E))) \widetilde{\subseteq} (F,E) \widetilde{\cup}\, int(cl(int\ (F,E)))$. Hence $(F,E) \widetilde{\cup}\, int(cl(int(F,E)))$ is soft β-closed and thus $S\beta cl(F,E) \widetilde{\subseteq} (F,E) \widetilde{\cup}\, int(cl(int(F,E)))$. On the other hand, since $S\beta cl(F,E)$ is soft β-closed, we have $int(cl(int(F,E))) \widetilde{\subseteq} int(cl(int(S\beta cl(F,E)))) \widetilde{\subseteq} S\beta cl(F,E)$ and hence $(F,E) \widetilde{\cup}\, int(cl(int(F,E))) \widetilde{\subseteq} S\beta cl(F,E)$. □

Definition 3.27. [2]Let (F,E) be a soft subset of (X,τ,E) then the soft beta-interior of (F,E), denoted by $S\beta int(F,E)$, is the soft union of all soft β-open subsets of X contained in (F,E).

Theorem 3.28. Let (F,E) be a soft subset of X. Then $S\beta int(F,E) = (F,E) \widetilde{\cap}\, cl(int(cl(F,E)))$.

Proof. We observe that $(F,E) \widetilde{\cap}\, cl(int(cl(F,E))) \widetilde{\subseteq} cl(int(cl(F,E))) = cl[int[cl(F,E) \widetilde{\cap}\, int(cl(F,E))]] \widetilde{\subseteq} cl[int[cl[(F,E) \widetilde{\cap}\, int(cl(F,E))]]] \widetilde{\subseteq} cl[int[cl\ [(F,E) \widetilde{\cap}\, cl(int(cl(F,E)))]]]$. Hence $(F,E) \widetilde{\cap}\, cl(int(cl(F,E)))$ is soft β-open and thus $(F,E) \widetilde{\cap}\, cl(int(cl(F,E))) \widetilde{\subseteq} S\beta int(F,E)$. On the other hand, since $S\beta int(F,E)$ is soft β-open, we have $S\beta int(F,E) \widetilde{\subseteq} cl(int(cl(S\beta int\ (F,E)))) \widetilde{\subseteq} cl(int(cl(F,E)))$ and hence $S\beta int(F,E) \widetilde{\subseteq} (F,E) \widetilde{\cap}\, cl(int(cl(F,E)))$. □

Corollary 3.29. (a) $S\beta int((F,E)') = (S\beta cl(F,E))'$ (b) $S\beta cl((F,E)') = (S\beta int(F,E))'$

The following theorem is an easy consequence of the definitions of soft α-open and soft β-open sets.

Theorem 3.30. $a)$ $(F,E) \in S.\alpha.O(X)$ if and only if $S\beta cl(F,E) = int(cl(int\ (F,E)))$, $b)$ $(F,E) \in S.\alpha.C(X)$ if and only if $S\beta int(F,E) = cl(int(cl(F,E)))$.

Proof. $(a) \Longrightarrow$ Let $(F,E) \in S.\alpha.O(X)$, then $(F,E) \widetilde{\subseteq} int(cl(int(F,E)))$. $S\beta cl(F,E) = (F,E) \widetilde{\cup}\, int(cl(int\ (F,E))) = int(cl(int(F,E)))$.

$\Longleftarrow S\beta cl(F,E) = int(cl(int(F,E))) = (F,E) \widetilde{\cup}\, int(cl(int(F,E)))$, then $(F,E) \widetilde{\subseteq} int(cl(int(F,E)))$.

(b) Easy □

Theorem 3.31. Let (F,E) be a soft subset of X. Then $S\beta int(S\beta cl(F,E)) = S\beta cl(S\beta int(F,E))$.

Proof. We have $S\beta int(S\beta cl(F,E)) = S\beta cl(F,E) \widetilde{\cap}\, cl(int(cl(S\beta cl(F,E)))) = [(F,E) \widetilde{\cup}\, int(cl(int(F,E)))] \widetilde{\cap}\, cl[int[cl[(F,E) \widetilde{\cup}\, int(cl(int(F,E)))]]] = [(F,E) \widetilde{\cup}\, int(cl(int(F,E)))] \widetilde{\cap}\, cl(int(cl(F,E))) = [(F,E) \widetilde{\cap}\, cl(int(cl\ (F,E)))] \widetilde{\cup}\, [int(cl(int(F,E))) \widetilde{\cap}\, cl(int(cl(F,E)))] = [(F,E) \widetilde{\cap}\, cl(int(cl(F,E)))] \widetilde{\cup}\, int(cl(int(F,E))) = [(F,E) \widetilde{\cap}\, cl(int(cl(F,E)))] \widetilde{\cup}\, int[cl[int[(F,E) \widetilde{\cap}\, cl(int(cl\ (F,E)))]]] = S\beta int(F,E) \widetilde{\cup}\, int(cl(int(S\beta int(F,E)))) = S\beta cl(S\beta int(F,E))$ □

Corollary 3.32. (a) $(F,E)\,\widetilde{\cup}\,S\beta int(S\beta cl(F,E)) = S\beta cl(F,E)$ (b) $(F,E)\,\widetilde{\cap}\,S\beta int(S\beta cl(F,E)) = S\beta int(F,E)$

Proof. (a) $(F,E)\,\widetilde{\cup}\,S\beta int(S\beta cl(F,E)) = (F,E)\,\widetilde{\cup}\,[S\beta cl(F,E)\,\widetilde{\cap}\,cl(int(cl\ (S\beta cl(F,E))))] = (F,E)\,\widetilde{\cup}\,[[(F,E)\,\widetilde{\cup}\,int(cl(int(F,E)))]\,\widetilde{\cap}\,cl[int[cl[(F,E)\,\widetilde{\cup}\,int(cl(int(F,E)))]]]] = (F,E)\,\widetilde{\cup}\,[[(F,E)\,\widetilde{\cup}\,int(cl(int(F,E)))]\,\widetilde{\cap}\,cl(int(cl\ (F,E)))] = [(F,E)\,\widetilde{\cup}\,int(cl(int(F,E)))]\,\widetilde{\cap}\,[(F,E)\,\widetilde{\cup}\,cl(int(cl(F,E)))] = [(F,E)\,\widetilde{\cup}\,int(cl(int(F,E)))] = S\beta cl(F,E)$

(b) Easy ☐

Theorem 3.33. For any soft subset (F,E) of a soft topological space X the following conditions are equivalent: (a) $(F,E) \in S.\beta.O(X)$ (b) $(F,E)\,\widetilde{\subseteq}\,S\beta int\,[\ S\beta cl(F,E)]$.

Proof. (a) →(b) Let $(F,E) \in S.\beta.O(X)$. Then $(F,E)\,\widetilde{\subseteq}\,cl(int(cl(F,E)))$. $S\beta int(S\beta cl(F,E)) = S\beta cl(F,E)\,\widetilde{\cap}\,cl(int(cl(S\beta cl(F,E)))) = [(F,E)\,\widetilde{\cup}\,int(cl\ (int(F,E)))]\,\widetilde{\cap}\,cl[int[cl[(F,E)\,\widetilde{\cup}\,int(cl(int(F,E)))]]] = [(F,E)\,\widetilde{\cup}\,int(cl(int\ (F,E)))\,\widetilde{\cap}\,cl(int(cl(F,E)))] = [(F,E)\,\widetilde{\cap}\,cl(int(cl(F,E)))]\,\widetilde{\cup}\,[int(cl(int(F,E)\ ))\,\widetilde{\cap}\,cl(int(cl(F,E)))] = (F,E)\,\widetilde{\cup}\,int(cl(int(F,E)))\,\widetilde{\supseteq}\,(F,E)$.

(b) →(a) $(F,E)\,\widetilde{\subseteq}\,S\beta int\,[\ S\beta cl(F,E)] = S\beta cl(F,E)\,\widetilde{\cap}\,cl(int(cl(S\beta cl\ (F,E)))) = [(F,E)\,\widetilde{\cup}\,int(cl(int\ (F,E)))]\,\widetilde{\cap}\,cl[int[cl[(F,E)\,\widetilde{\cup}\,int(cl(int(F,E)))\ ]]] = [(F,E)\,\widetilde{\cup}\,int(cl(int(F,E)))]\,\widetilde{\cap}\,cl(int(cl(F,E)))$. Hence $(F,E)\,\widetilde{\subseteq}\,cl(int(cl\ (F,E)))$. ☐

3.1 Soft β-continuous Mappings

We define the notion of soft β-continuity by using soft β-open sets.

Definition 3.34. Let (X,τ,E) and (Y,τ',E) be two soft topological spaces. A function $f:(X,\tau,E)\longrightarrow(Y,\tau',E)$ is said to be

a) soft *semi*-continuons[12] if $f^{-1}((G,E))$ is soft *semi*-open in (X,τ,E), for every soft open set (G,E) of (Y,τ',E).

b) soft *pre*-continuons[1] if $f^{-1}((G,E))$ is soft *pre*-open in (X,τ,E), for every soft open set (G,E) of (Y,τ',E).

c) soft α-continuons if[1] $f^{-1}((G,E))$ is soft α-open in (X,τ,E), for every soft open set (G,E) of (Y,τ',E).

d) soft β-continuons if $f^{-1}((G,E))$ is soft β-open in (X,τ,E), for every soft open set (G,E) of (Y,τ',E).

e) soft β-irresolute if $f^{-1}((G,E))$ is soft β-open in (X,τ,E), for every soft β-open set (G,E) of (Y,τ',E).

It is clear that the class of soft β-continuity contains each of classes soft *semi*-continuous and soft *pre*-continuous, the implications between them and other types of soft continuities are given by the following diagram.

$$\begin{array}{ccccc} soft\ continuity & \longrightarrow & soft\ \alpha\text{-}continuity & \longrightarrow & soft\ semi\text{-}continuity \\ & & \downarrow & & \downarrow \\ & & soft\ pre\text{-}continuity & \longrightarrow & soft\ \beta\text{-}continuity \end{array}$$

The converses of these implications do not hold, in general, as show in the following examples.

Example 3.35. Let $X = Y = \{x_1, x_2, x_3\}$, $E = \{e_1, e_2\}$ and let the soft topology on X be soft indiscrete and on Y be soft discrete. If we get the mapping $f:(X,\tau,E)\longrightarrow(Y,\tau',E)$ defined as $f(x_1) = x_2$, $f(x_2) = x_1$, $f(x_3) = x_3$ then f is soft β-continuous but not soft *semi*-continuous.

Example 3.36. Let $X = Y = \{x_1, x_2, x_3\}$, $E = \{e_1, e_2\}$. Then $\tau = \{\Phi, \widetilde{X}, (F_1,E), (F_2,E), (F_3,E)\}$ is a soft topological space over X and $\tau' = \{\Phi, \widetilde{Y}, (G_1,E), (G_2,E)\}$ is a soft topological space over Y. Here (F_1,E), $(F_2,E), (F_3,E)$ are soft sets over X and $(G_1,E), (G_2,E)$ are soft sets over Y,defined as follows: $F_1(e_1) = \{x_1\}$, $F_1(e_2) = \{x_1\}$, $F_2(e_1) = \{x_2\}$, $F_2(e_2) = \{x_2\}$, $F_3(e_1) = \{x_1, x_2\}$, $F_3(e_2) = \{x_1, x_2\}$ and $G_1(e_1) = \{x_1\}$, $G_1(e_2) = \{x_1\}$, $G_2(e_1) = \{x_1, x_2\}$, $G_2(e_2) = \{x_1, x_2\}$.

If we get the mapping $f : (X, \tau, E) \longrightarrow (Y, \tau', E)$ defined as $f(x_1) = x_1, \quad f(x_2) = x_3, \quad f(x_3) = x_2$ then f is soft β-continuous but not soft *pre*-continuous, since $f^{-1}(G_2) = \{\{x_1, x_3\}, \{x_1, x_3\}\}$ is not a soft *pre*-open set over X.

We give some characterizations of soft β-continuity.

Theorem 3.37. Let $f : (X, \tau, E) \longrightarrow (Y, \tau', E)$ be a soft mapping, then the following statements are equivalent.

a) f is soft β-continuous.

b) For each soft point (x_e, E) over X and each soft open (G, E) containing $f(x_e, E) = (f(x)_e, E)$ over Y, there exists a soft β-open set (F, E) over X containing (x_e, E) such that $f(F, E) \widetilde{\subseteq} (G, E)$.

c) The inverse image of each soft closed set in Y is soft β-closed in X.

d) $int(cl(int(f^{-1}(G, E)))) \widetilde{\subseteq} f^{-1}(cl(G, E))$ for each soft set (G, E) over Y.

e) $f(int(cl(int(F, E)))) \widetilde{\subseteq} cl(f(F, E))$ for each soft set (F, E) over X.

Proof. $(a) \Longrightarrow (b)$ Since $(G, E) \widetilde{\subseteq} Y$ containing $f(x_e, E) = (f(x)_e, E)$ is soft open, then $f^{-1}(G, E) \in S.\beta.O(X)$. Soft set $(F, E) = f^{-1}(G, E)$ which contains (x_e, E), therefore $f(F, E) \widetilde{\subseteq} (G, E)$.

$(a) \Longrightarrow (c)$ Let $(G, E) \in S.C(Y)$,then $(\widetilde{Y} - (G, E)) \in S.O(Y)$.Since f is soft β-continuous, $f^{-1}(\widetilde{Y} - (G, E)) \in S.\beta.O(X)$. Hence $[\widetilde{X} - f^{-1}(G, E)] \in S.\beta.O(X)$. Then $f^{-1}(G, E) \in S.\beta.C(X)$

$(c) \Longrightarrow (d)$ Let (G, E) be a soft set over Y, then $f^{-1}(cl(G, E)) \in S.\beta.C(X)$. $f^{-1}(cl(G, E)) \widetilde{\supseteq} int(cl(int(f^{-1}(cl(G, E))))) \widetilde{\supseteq} int(cl(int(f^{-1}(G, E))))$

$(d) \Longrightarrow (e)$ Let (F, E) be a soft set over X and $f(F, E) = (G, E)$. Then, according to (d) $int(cl(int(f^{-1}(f(F, E))))) \widetilde{\subseteq} f^{-1}(cl(f(F, E)) \Longrightarrow int(cl(int(F, E))) \widetilde{\subseteq} f^{-1}(cl(f(F, E)) \Longrightarrow f(int(cl(int(F, E)))) \widetilde{\subseteq} cl(f(F, E))$

$(e) \Longrightarrow (a)$ Let $(G, E) \in S.O(Y)$, $(H, E) = \widetilde{Y} - (G, E)$ and $(F, E) = f^{-1}(H, E)$, by (e) $f(int(cl(int(f^{-1}(H, E))))) \widetilde{\subseteq} cl(f(f^{-1}(H, E))) \widetilde{\subseteq} cl(H, E) = (H, E)$, so $int(cl(int(f^{-1}(H, E)))) \widetilde{\subseteq} f^{-1}(H, E)$. Then $f^{-1}(H, E) \in S.\beta.C(X)$, thus (by (c)) f is soft β-continuous. □

Remark 3.38. The composition of two soft β-continuous mappings need not be soft β-continuous, in general, as shown by the following example.

Example 3.39. Let $X = Z = \{x_1, x_2, x_3\}$, $Y = \{x_1, x_2, x_3, x_4\}$ and $E = \{e_1, e_2\}$. Then $\tau = \{\Phi, \widetilde{X}, (F, E)\}$ is a soft topological space over X, $\tau' = \{\Phi, \widetilde{Y}, (G, E)\}$ is a soft topological space over Y and $\tau'' = \{\Phi, \widetilde{Z}, (H_1, E), (H_2, E)\}$ is a soft topological space over Z. Here (F, E) is a soft set over X, (G, E) is a soft set over Y and $(H_1, E), (H_2, E)$ are soft sets over Z defined as follows: $F(e_1) = \{x_1\}$, $F(e_2) = \{x_1\}$, $G(e_1) = \{x_1, x_3\}$, $G(e_2) = \{x_1, x_3\}$, $H_1(e_1) = \{x_3\}$, $H_1(e_2) = \{x_3\}$, $H_2(e_1) = \{x_1, x_2\}$, $H_2(e_2) = \{x_1, x_2\}$.

If we get the identity mapping $I : (X, \tau, E) \longrightarrow (Y, \tau', E)$ and $f : (Y, \tau', E) \longrightarrow (Z, \tau'', E)$ defined as $f(x_1) = x_1$, $f(x_2) = f(x_4) = x_2$, $f(x_3) = x_3$. It is clear that each of I and f is soft β-continuous but $f \ o \ I$ is not soft β-continuous, since $(f \ o \ I)^{-1}(H_1, E) = \{\{x_3\}, \{x_3\}\}$ is not a soft β-open set over X.

Definition 3.40. A function $f : (X, \tau, E) \longrightarrow (Y, \tau', E)$ is called a soft β-homeomorphism (resp.soft βr-homeomorphism) if f is a soft β-continuous bijection (resp. sorf β-irresolute bijection) and $f^{-1} : (Y, \tau', E) \longrightarrow (X, \tau, E)$ is a soft β-continuous (soft β-irresolute).

Now we can give the following definition by taking the soft space (X, τ, E) instead of the soft space (Y, τ', E).

Definition 3.41. For a soft topological space (X, τ, E), we define the following two collections of functions:

$S\beta\text{-}h(X, \tau, E) = \{f \mid f : (X, \tau, E) \longrightarrow (X, \tau, E)$ is a softβ-continuous bijection, $f^{-1} : (X, \tau, E) \longrightarrow (X, \tau, E)$ is softβ-continuous$\}$

$S\beta r\text{-}h(X, \tau, E) = \{f \mid f : (X, \tau, E) \longrightarrow (X, \tau, E)$ is a softβ-irresolute bijection, $f^{-1} : (X, \tau, E) \longrightarrow (X, \tau, E)$ is softβ-irresolute$\}$

Theorem 3.42. For a soft topological space (X, τ, E), $S\text{-}h(X, \tau, E) \tilde{\subseteq} S\beta r\text{-}h(X, \tau, E) \tilde{\subseteq} S\beta\text{-}h(X, \tau, E)$, where $S\text{-}h(X, \tau, E) = \{f \mid f : (X, \tau, E) \longrightarrow (X, \tau, E)$ is a soft-homeomorphism$\}$.

Proof. First we show that every soft-homeomorphism $f : (X, \tau, E) \longrightarrow (Y, \tau', E)$ is a soft βr-homeomorphism. Let $(G, E) \in S.\beta.O(Y)$, then $(G, E) \tilde{\subseteq} cl(int(cl(G, E)))$. Hence, $f^{-1}((G, E)) \tilde{\subseteq} f^{-1}(cl(int(cl(G, E)))) = cl(int(cl\ (f^{-1}(G, E))))$ and so $f^{-1}((G, E)) \in S.\beta.O(X)$. Thus, f is soft β-irresolute. In a similar way, it is shown that f^{-1} is soft β-irresolute. Hence, we have that $S\text{-}h(X, \tau, E) \tilde{\subseteq} S\beta r\text{-}h(X, \tau, E)$.

Finally, it is obvious that $S\beta r\text{-}h(X, \tau, E) \tilde{\subseteq} S\beta\text{-}h(X, \tau, E)$, because every soft β-irresolute function is soft β-continuous. □

Theorem 3.43. For a soft topological space (X, τ, E), the collection $S\beta r\text{-}h(X, \tau, E)$ forms a group under the composition of functions.

Proof. If $f : (X, \tau, E) \longrightarrow (Y, \tau', E)$ and $g : (Y, \tau', E) \longrightarrow (Z, \tau'', E)$ are soft βr-homeomorphism, then their composition $gof : (X, \tau, E) \longrightarrow (Z, \tau'', E)$ is a soft βr-homeomorphism. It is obvious that for a bijective soft βr-homeomorphism $f : (X, \tau, E) \longrightarrow (Y, \tau', E)$, $f^{-1} : (Y, \tau', E) \longrightarrow (X, \tau, E)$ is also a soft βr-homeomorphism and the identity $1 : (X, \tau, E) \longrightarrow (X, \tau, E)$ is a soft βr-homeomorphism. A binary operation $\alpha : S\beta r\text{-}h(X, \tau, E) \times S\beta r\text{-}h(X, \tau, E) \longrightarrow S\beta r\text{-}h(X, \tau, E)$ is well defined by $\alpha(a, b) = boa$, where $a, b \in S\beta r\text{-}h(X, \tau, E)$ and boa is the composition of a and b. By using the above properties, the set $S\beta r\text{-}h(X, \tau, E)$ forms a group under composition of functions. □

Theorem 3.44. The group $S\text{-}h(X, \tau, E)$ of all soft homeomorphisms on (X, τ, E) is a subgroup of $S\beta r\text{-}h(X, \tau, E)$.

Proof. For any $a, b \in S\text{-}h(X, \tau, E)$, we have $\alpha(a, b^{-1}) = b^{-1}o\ a \in S\text{-}h(X, \tau, E)$ and $1_X \in S\text{-}h(X, \tau, E) \neq \varnothing$. Thus, using (Theorem 4.10) and (Theorem 4.11), it is obvious that the group $S\text{-}h(X, \tau, E)$ is a subgroup of $S\beta r\text{-}h(X, \tau, E)$. □

For a soft topological space (X, τ, E), we can construct a new group $S\beta r\text{-}h(X, \tau, E)$ satisfying the property: if there exists a homeomorphism $(X, \tau, E) \cong (Y, \tau', E)$, then there exists a group isomorphism $S\beta r\text{-}h(X, \tau, E) \cong S\beta r\text{-}h(X, \tau, E)$.

Corollary 3.45. Let $f : (X, \tau, E) \longrightarrow (Y, \tau', E)$ and $g : (Y, \tau', E) \longrightarrow (Z, \tau'', E)$ be two functions between soft topological spaces.

a) For a soft βr-homeomorphism $f : (X, \tau, E) \longrightarrow (Y, \tau', E)$, there exists an isomorphism, say $f_* : S\beta r\text{-}h(X, \tau, E) \longrightarrow S\beta r\text{-}h(X, \tau, E)$, defined $f_*(a) = f\ o\ a\ o\ f^{-1}$, for any element $a \in S\beta r\text{-}h(X, \tau, E)$.
b) For two soft βr-homeomorphisms $f : (X, \tau, E) \longrightarrow (Y, \tau', E)$ and $g : (Y, \tau', E) \longrightarrow (Z, \tau'', E)$, $(gof)_* = g_* o\ f_* : S\beta r\text{-}h(X, \tau, E) \longrightarrow S\beta r\text{-}h(Z, \tau'', E)$ holds.
c) For the identity function $1_X : (X, \tau, E) \longrightarrow (X, \tau, E)$, $(1_X)_* = 1 : S\beta r\text{-}h(X, \tau, E) \longrightarrow S\beta r\text{-}h(X, \tau, E)$ holds where 1 denotes the identity isomorphism.

Proof. Straightforward . □

4 Conclusion

We obtain some properties of two operators called soft β-interior and soft β-closure. Besides, in soft topological spaces, two new varieties of continuity via soft soft β-open and soft β-homeomorphism with soft β-irresolute homeomorphism are defined and given some characterizations of these notions. Of course, the most important the family of soft β-irresolute homeomorphism was a soft group. Therefore, one can say that this paper is applying to algebra.

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COTANGENT SIMILARITY MEASURE OF ROUGH NEUTROSOPHIC SETS AND ITS APPLICATION TO MEDICAL DIAGNOSIS

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Abstract – Similarity measure plays an important role in medical diagnosis. In this paper, a new rough cotangent similarity measure between two rough neutrosophic sets is proposed. The notion of rough neutrosophic set is used as vector representations in 3D-vector space. The rating of all elements in rough neutrosophic set is expressed with the upper and lower approximation operator and the pair of neutrosophic sets which are characterized by truth-membership degree, indeterminacy-membership degree, and falsity-membership degree. A numerical example of the medical diagnosis is provided to show the effectiveness and flexibility of the proposed method.

Keywords – *Rough cotangent similarity measure, Rough sets, Neutrosophic sets, Indeterminacy Membership degree, 3D vector space.*

1 Introduction

Similarity measure is an important research topic in the current fuzzy, rough, neutrosophic and differrnt hybrid environments. In 1965, Zadeh [48] introduced the concept of fuzzy set to deal with informational (epistemic) vagueness. Fuzzy set is capable of formalizing and reasoning of intangible internal characteristics, typically natural language-based and visual image information, as well as incomplete, unreliable, imprecise and vague performance and priority data. However, while focusing on the degree of membership of vague parameters or events, fuzzy set fails to deal with indeterminacy magnitudes of measured responses. In 1986, Atanassov [1] developed the concept of intuitionistic fuzzy set (IFS) which considers degree of membership (acceptance) and degree of non-membership (rejection) simultaneously. However, IFS cannot deal with all types of uncertainties, particularly

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paradoxes. One of the interesting generalizations of the theory of Cantor set [11], fuzzy set [48] and intuitionistic fuzzy set [1] is the theory of neutrosophic sets [37] introduced by Smarandache in the late 1990s. Neutrosophic sets [38], [39] and their specific sub-class of single-valued neutrosophic set (SVNS) [43] are characterized by the three independent functions, namely membership (truth) function, non-membership (falsity) function and indeterminacy function. Smarandache [39] stated that such formulation enables modeling of the most general ambiguity cases, including paradoxes. In the literature, some interesting applications of neutrosophic logic, neutrosophic sets and single valued neutrosophic sets are reported in different fields such as decision making [3, 4, 5, 6, 8, 20, 44, 45, 46], education [23, 25, 32], image processing [12, 16, 49], medical diagnosis [19], conflict resolution [2, 35], Robotics [40], social problem [22, 33, 41], etc.

In 1982, Pawlak [31] introduced the notion of rough set theory as the extension of the Cantor set theory [11]. Broumi et al. [10] comment that the concept of rough set is a formal tool for modeling and processing incomplete information in information systems. Rough set theory [31] is very useful to study of intelligent systems characterized by uncertain or insufficient information. Main mathematical basis of rough set theory is formed by two basic components namely, crisp set and equivalence relation. Rough set is the approximation of a pair of sets known as the lower approximation and the upper approximation. Here, the lower and upper approximation operators are equivalence relation.

In 2014, Broumi et al. [9, 10] introdced the concept of rough neutrosophic set. It is a new hybrid intelligent structure. İt is developed based on the concept of rough set theory [31] and single valued neutrosophic set theory [43] Rough neutrosophic set theory [9, 10] is the generalization of rough fuzzy sets [15, 29, 30], and rough intuitionistic fuzzy sets [42]. While the concept of single valued neutrosophic set [43] is a powerful tool to deal with the situations with indeterminacy and inconsistancy, the theory of rough neutrosophic sets [9, 10] is also a powerful mathematical tool to deal with incompleteness.

Many methods have been proposed in the literature to measure the degree of similarity between neutrosophic sets. Broumi and Smarandache [7] studied the Hausdorff distance [17] between neutrosophic sets, some distance based similarity measures and set theoretic approach and matching functions. Majumdar and Smanta [21] studied several similarity measures of SNVSs based on distance, membership grades, a matching function, and then proposed an entropy measure for a SVNS. Ye [44] proposed the distance-based similarity measure of SVNSs and applied it to the group decision making problems with single valued neutrosophic information. Ye [46] also proposed three vector similarity measure, an instance of SVNS and interval valued neutrosophic set, including the Jaccard [18], Dice [14], and cosine similarity [36] and applied them to multi-attribute decision-making problems under simplified neutrosophic environment. Ye [47] studied improved cosine similarity measures of SNSs based on cosine function, including single valued neutrosophic cosine similarity measures and interval neutrosophic cosine similarity measures and provided medical diagnosis method based on the improved cosine similarity measures. Recently, Mondal and Pramanik [28] proposed a neutrosophic similarity measure based on tangent function. Mondal and Pramanik [26] also proposed neutrosophic refined similarity measure based on cotangent function. Biswas et al. [5] studied cosine similarity measure based multi-attribute decision-making with trapezoidal fuzzy neutrosophic numbers.

Literature rewview reflects that a few studies related to decision making under rough neutrosophic environment are done. Mondal and Pramanik [24] proposed rough neutrosophic multi-attribute decision-making based on grey relational analysis [13]. Pramanik and Mondal [34] proposed cosine similarity measure under rough neutrosophic environment. Mondal and Pramanik [27] also proposed rough neutrosophic multi-attribute decision-making based on accuracy score function.

Realistic practical problems consist of more uncertainty and complexity. So, it is necessary to employ more flexible tool which can deal uncertain situation easily. In this situation, rough neutrosophic set [10] is very useful tool to uncertainty and incompleteness. In this paper, we propose cotangent similarity measure of rough neutrosophic sets and establish some of its properties. Finally, a numerical example of medical diagnosis is presented to demonstrate the applicability and effectiveness of the proposed approach.

The rest of the paper is organized as follows: In section 2, some basic definitions of single valued neutrosophic sets and rough neutrosophic sets are preseneted. Section 3 is devoted to present rough neutrosophic cotangent similarity measure and proofs of some its basic properties. In section 4, numerical example is provided to show the applicability of the proposed approach to medical diagnosis. Section 5 presents the concluding remarks.

2 Mathematical Preliminaries

Definition 2.1.1 [43] Let *X* be a universal space of points (objects) with a generic element of *X* denoted by *x*.

A single valued neutrosophic set [43] *S* is characterized by a truth membership function $T_S(x)$, a falsity membership function $F_S(x)$ and indeterminacy function $I_S(x)$ with $T_S(y), F_S(x), I_S(x) \in [0,1]$ for all *x* in *X*.

When *X* is continuous, a SNVS *S* can be written as follows:

$$S = \int_x \langle T_S(x), F_S(x), I_S(x) \rangle / x, \forall x \in X$$

and when *X* is discrete, a SVNS *S* can be written as follows:

$$S = \sum \langle T_S(x), F_S(x), I_S(x) \rangle / x, \forall x \in X$$

It should be observed that for a SVNS *S*,

$$0 \leq \sup T_S(x) + \sup F_S(x) + \sup I_S(x) \leq 3, \quad \forall x \in X$$

Definition 2.1.2 [43] The complement of a single valued neutrosophic set *S* [43] is denoted by S^c and is defined as

$$T_S{}^c(x) = F_S(x);\ I_S{}^c(x) = 1 - I_S(x);\ F_S{}^c(x) = T_S(x)$$

Definition 2.1.3 [43] A SVNS S_N is contained in the other SVNS [43] S_P, denoted as

$$S_N \subseteq S_P \text{ iff } T_{S_N}(x) \leq T_{S_P}(x);\ I_{S_N}(x) \geq I_{S_P}(x);\ F_{S_N}(x) \geq F_{S_P}(x),\ \forall x \in X.$$

Definition 2.1.4 [43] Two single valued neutrosophic sets [43] S_N and S_P are equal, i.e.

$$S_N = S_P, \text{ iff, } S_N \subseteq S_P \text{ and } S_N \supseteq S_P$$

Definition 2.1.5 [43] The union of two SVNSs [43] S_N and S_P is a SVNS S_Q , written as

$$S_Q = S_N \cup S_P .$$

Its truth membership, indeterminacy-membership and falsity membership functions are related to S_N and S_P by the following equation

$$T_{S_Q}(x) = \max\left(T_{S_N}(x), T_{S_P}(x)\right);$$
$$I_{S_Q}(x) = \max\left(I_{S_N}(x), I_{S_P}(x)\right);$$
$$F_{S_Q}(x) = \min\left(F_{S_N}(x), F_{S_P}(x)\right)$$

for all x in X.

Definition 2.1.6 [43] The intersection of two SVNSs [43] N and P is a SVNS Q, written as $Q = N \cap P$. Its truth membership, indeterminacy membership and falsity membership functions are related to N an P by the following equation

$$T_{S_Q}(x) = \min\left(T_{S_N}(x), T_{S_P}(x)\right);$$
$$I_{S_Q}(x) = \max\left(I_{S_N}(x), I_{S_P}(x)\right);$$
$$F_{S_Q}(x) = \max\left(F_{S_N}(x), F_{S_P}(x)\right), \forall x \in X$$

Distance Between Two Neutrosophic Sets

The general SVNS can be presented in the follow form

$$S = \left\{\left(x/\left(T_S(x), I_S(x), F_S(x)\right)\right): x \in X\right\}$$

Finite SVNSs can be represented as follows:

$$S = \left\{\left(x_1/\left(T_S(x_1), I_S(x_1), F_S(x_1)\right)\right), \cdots, \left(x_m/\left(T_S(x_m), I_S(x_m), F_S(x_m)\right)\right)\right\}, \forall x \in X \quad (1)$$

Definition 2.1.7 [21] Let

$$S_N = \left\{\left(x_1/\left(T_{S_N}(x_1), I_{S_N}(x_1), F_{S_N}(x_1)\right)\right), \cdots, \left(x_n/\left(T_{S_N}(x_n), I_{S_N}(x_n), F_{S_N}(x_n)\right)\right)\right\} \quad (2)$$
$$S_P = \left\{\left(x_1/\left(T_{S_P}(x_1), I_{S_P}(x_1), F_{S_P}(x_1)\right)\right), \cdots, \left(x_n/\left(T_{S_P}(x_n), I_{S_P}(x_n), F_{S_P}(x_n)\right)\right)\right\} \quad (3)$$

be two single-valued neutrosophic sets, then the Hamming distance [21] between two SNVS N and P is defined as follows:

$$d_S\left(S_N, S_P\right) = \sum_{i=1}^{n} \left\langle \left| T_{S_N}(x) - T_{S_P}(x) \right| + \left| I_{S_N}(x) - I_{S_P}(x) \right| + \left| F_{S_N}(x) - F_{S_P}(x) \right| \right\rangle \quad (4)$$

and normalized Hamming distance [21] between two SNVSs S_N and S_P is defined as follows:

$$^{N}d_S(S_N, S_P) = \frac{1}{3n}\sum_{i=1}^{n}\left\langle \left| T_{S_N}(x) - T_{S_P}(x) \right| + \left| I_{S_N}(x) - I_{S_P}(x) \right| + \left| F_{S_N}(x) - F_{S_P}(x) \right| \right\rangle \quad (5)$$

with the following properties

1. $0 \le d_S(S_N, S_P) \le 3n$ (6)
2. $0 \le {}^{N}d_S(S_N, S_P) \le 1$ (7)

2.2. Definitions

[9, 10] Rough set theory [9, 10] consists of two basic components namely, crisp set and equivalence relation. The basic idea of rough set is based on the approximation of sets by a couple of sets known as the lower approximation and the upper approximation of a set. Here, the lower and upper approximation operators are based on equivalence relation.

Definition 2.2.1 [9, 10] Let Y be a non-null set and R be an equivalence relation on Y. Let P be neutrosophic set in Y with the membership function T_P, indeterminacy function I_P and non-membership function F_P. The lower and the upper approximations of P in the approximation (Y, R) denoted by $\underline{N}(P)$ and $\overline{N}(P)$ are respectively defined as follows:

$$\underline{N}(P) = \left\langle < x, T_{\underline{N}(P)}(x), I_{\underline{N}(P)}(x), F_{\underline{N}(P)}(x) > / Y \in [x]_R, x \in Y \right\rangle \quad (8)$$

$$\overline{N}(P) = \left\langle < x, T_{\overline{N}(P)}(x), I_{\overline{N}(P)}(x), F_{\overline{N}(P)}(x) > / Y \in [x]_R, x \in Y \right\rangle \quad (9)$$

Here,

$T_{\underline{N}(P)}(x) = \wedge_z \in [x]_R\, T_P(Y)$,

$I_{\underline{N}(P)}(x) = \wedge_z \in [x]_R\, I_P(Y)$, $F_{\underline{N}(P)}(x) = \wedge_z \in [x]_R\, F_P(Y)$,

$T_{\overline{N}(P)}(x) = \vee_Y \in [x]_R\, T_P(Y)$, $I_{\overline{N}(P)}(x) = \vee_Y \in [x]_R\, T_P(Y)$,

$F_{\overline{N}(P)}(x) = \vee_Y \in [x]_R\, I_P(Y)$

So,

$0 \le T_{\underline{N}(P)}(x) + I_{\underline{N}(P)}(x) + F_{\underline{N}(P)}(x) \le 3$

$0 \le T_{\overline{N}(P)}(x) + I_{\overline{N}(P)}(x) + F_{\overline{N}(P)}(x) \le 3$

Here $\vee$ and $\wedge$ indicate "max" and "min'' operators respectively, $T_P(Y)$, $I_P(Y)$ and $F_P(Y)$ are the membership , indeterminacy and non-membership of Y with respect to P. It is easy to see that $\underline{N}(P)$ and $\overline{N}(P)$ are two neutrosophic sets in Y.

Thus NS mappings $\underline{N}, \overline{N} : N(Y) \to N(Y)$ are, respectively, refered to as the lower and upper rough NS approximation operators, and the pair $(\underline{N}(P), \overline{N}(P))$ is called the rough neutrosophic set in (Y, R).

From the above definition, it is seen that $\underline{N}(P)$ and $\overline{N}(P)$ have constant membership on the equivalence classes of R if $\underline{N}(P) = \overline{N}(P)$; i.e.

$$T_{\underline{N}(P)}(x) = T_{\overline{N}(P)}(x),\ I_{\underline{N}(P)}(x) = I_{\overline{N}(P)}(x),\ F_{\underline{N}(P)}(x) = F_{\overline{N}(P)}(x).$$

For any x $\in Y$, P is said to be a definable neutrosophic set in the approximation (Y, R). It can be easily proved that zero neutrosophic set (0_N) and unit neutrosophic sets (1_N) are definable neutrosophic sets.

Definition 2.2.2 [9, 10] If $N(P) = (\underline{N}(P), \overline{N}(P))$ is a rough neutrosophic set in (Y, R) , the rough complement of $N(P)$ is the rough neutrosophic set denoted $\sim N(P) = (\underline{N}(P)^c, \overline{N}(P)^c)$,where $\underline{N}(P)^c, \overline{N}(P)^c$ are the complements of neutrosophic sets of $\underline{N}(P), \overline{N}(P)$ respectively.

$$\underline{N}(P)^c = \left\langle < x, T_{\underline{N}(P)}(x), 1 - I_{\underline{N}(P)}(x),\ F_{\underline{N}(P)}(x) > / , x \in Y \right\rangle ,$$

and

$$\overline{N}(P)^c = \left\langle < x, T_{\underline{N}(P)}(x),\ 1 - I_{\overline{N}(P)}(x),\ F_{\overline{N}(P)}(x) > / , x \in Y \right\rangle \quad (10)$$

Definition 2.2.3 [9, 10] If $N(P)$ *and* $N(Q)$ are two rough neutrosophic sets of the neutrosophic sets respectively in Y, then the following definitions holds.

$$N(P) = N(Q) \Leftrightarrow \underline{N}(P) = \underline{N}(Q) \wedge \overline{N}(P) = \overline{N}(Q)$$
$$N(P) \subseteq N(Q) \Leftrightarrow \underline{N}(P) \subseteq \underline{N}(Q) \wedge \overline{N}(P) \subseteq \overline{N}(Q)$$
$$N(P) \cup N(Q) = < \underline{N}(P) \cup \underline{N}(Q), \overline{N}(P) \cup \overline{N}(Q) >$$
$$N(P) \cap N(Q) = < \underline{N}(P) \cap \underline{N}(Q), \overline{N}(P) \cap \overline{N}(Q) >$$
$$N(P) + N(Q) = < \underline{N}(P) + \underline{N}(Q), \overline{N}(P) + \overline{N}(Q) >$$
$$N(P) \,.\, N(Q) = < \underline{N}(P) \,.\, \underline{N}(Q), \overline{N}(P) \,.\, \overline{N}(Q) >$$

If A, B, C are rough neutrosophic sets in (Y, R), then the following proposition are stated from definitions

Proposition I [9, 10]

1. $\sim A(\sim A) = A$
2. $A \cup B = B \cup A,\ A \cup B = B \cup A$
3. $(A \cup B) \cup C = A \cup (B \cup C),\ (A \cap B) \cap C = A \cap (B \cap C)$
4. $(A \cup B) \cap C = (A \cup B) \cap (A \cup C),\ (A \cap B) \cup C = (A \cap B) \cup (A \cap C)$

Proposition II [9, 10]
De Morgan's Laws are satisfied for rough neutrosophic sets

1. $\sim(N(P)\cup N(Q)) = (\sim N(P))\cap(\sim N(Q))$
2. $\sim(N(P)\cap N(Q)) = (\sim N(P))\cup(\sim N(Q))$

Proposition III [9, 10]
If P and Q are two rough neutrosophic sets in U such that $P\subseteq Q, then\ N(P)\subseteq N(Q)$

1. $N(P\cap Q)\subseteq N(P)\cap N(Q)$
2. $N(P\cup Q)\supseteq N(P)\cup N(Q)$

Proposition IV[9, 10]
1. $\underline{N}(P) = \sim\overline{N}(\sim P)$
2. $\overline{N}(P) = \sim\underline{N}(\sim P)$
3. $\underline{N}(P)\subseteq \overline{N}(P)$

3 Cotangent Similarity Measures of Rough Neutrosophic Sets

Let $M=<\left\langle\left(\underline{T}_M(x_i), \underline{I}_M(x_i), \underline{F}_M(x_i)\right), \left(\overline{T}_M(x_i), \overline{I}_M(x_i), \overline{F}_M(x_i)\right)\right\rangle>$ and

$N=<\left\langle\left(\underline{T}_N(x_i), \underline{I}_N(x_i), \underline{F}_N(x_i)\right), \left(\overline{T}_N(x_i), \overline{I}_N(x_i), \overline{F}_N(x_i)\right)\right\rangle>$ be two rough neutrosophic numbers. Now rough cotangent similarity function which measures the similarity between two vectors based only on the direction, ignoring the impact of the distance between them. Therefore, a new cotangent similarity measure between rough neutrosophic sets is proposed in 3-D vector space.

Definition 3.1 Rough cotangent similarity measure

Assume that there are two rough neutrosophic sets
$M=\left\langle\left(\underline{T}_M(x_i), \underline{I}_M(x_i), \underline{F}_M(x_i)\right), \left(\overline{T}_M(x_i), \overline{I}_M(x_i), \overline{F}_M(x_i)\right)\right\rangle$ and
$N=\left\langle\left(\underline{T}_N(x_i), \underline{I}_N(x_i), \underline{F}_N(x_i)\right), \left(\overline{T}_N(x_i), \overline{I}_N(x_i), \overline{F}_N(x_i)\right)\right\rangle$ in $X = \{x_1, x_2, \ldots, x_n\}$. A cotangent similarity measure between rough neutrosophic sets M and N is proposed as follows:

$$COT_{RNS}(M,N)=$$
$$\left[\frac{1}{n}\sum_{i=1}^{n}\left\langle\cot\left(\frac{\pi}{12}\left(3+\left|\delta T_M(x_i)-\delta T_N(x_i)\right|+\left|\delta I_M(x_i)-\delta I_N(x_i)\right|+\left|\delta F_M(x_i)-\delta F_N(x_i)\right|\right)\right)\right\rangle\right] \tag{11}$$

Here,

$$\delta T_M(x_i)=\left(\frac{\underline{T}_M(x_i)+\overline{T}_M(x_i)}{2}\right),\ \delta T_N(x_i)=\left(\frac{\underline{T}_N(x_i)+\overline{T}_N(x_i)}{2}\right),\ \delta I_M(x_i)=\left(\frac{\underline{I}_M(x_i)+\overline{I}_M(x_i)}{2}\right),$$

$\delta I_N(x_i)=\left(\dfrac{\underline{I}_N(x_i)+\overline{I}_N(x_i)}{2}\right)$, $\delta F_M(x_i)=\left(\dfrac{\underline{F}_M(x_i)+\overline{F}_M(x_i)}{2}\right)$, $\delta F_N(x_i)=\left(\dfrac{\underline{F}_N(x_i)+\overline{F}_N(x_i)}{2}\right)$.

Proposition V

Let *M* and *N* be rough neutrosophic sets then

1. $0\le COT_{RNS}(M,N)\le 1$
2. $COT_{RNS}(M,N)=COT_{RNS}(N,M)$
3. $COT_{RNS}(M,N)=1$, iff $M=N$
4. If *P* is a RNS in *Y* and $M\subset N\subset P$ then, $COT_{RNS}(M,P)\le COT_{RNS}(M,N)$, and $COT_{RNS}(M,P)\le COT_{RNS}(N,P)$

Proof :

1.Since, $\dfrac{\pi}{4}\le\left(\dfrac{\pi}{12}\left(3+|\delta T_M(x_i)-\delta T_N(x_i)|+|\delta I_M(x_i)-\delta I_N(x_i)|+|\delta F_M(x_i)-\delta F_N(x_i)|\right)\right)\le\dfrac{\pi}{2}$, it is obvious that the cotangent function $COT_{RNS}(M,N)$ are within 0 and 1.

2. It is obvious that the proposition is true.

3. When $M=N$, then obviously $COT_{RNS}(M,N)=1$. On the other hand if $COT_{RNS}(M,N)=1$

then,

$\delta T_M(x_i)=\delta T_N(x_i)$, $\delta I_M(x_i)=\delta I_N(x_i)$, $\delta F_M(x_i)=\delta F_N(x_i)$ *ie,*

$\underline{T}_M(x_i)=\underline{T}_N(x_i)$, $\overline{T}_M(x_i)=\overline{T}_N(x_i)$, $\underline{I}_M(x_i)=\underline{I}_N(x_i)$, $\overline{I}_M(x_i)=\overline{I}_N(x_i)$, $\underline{F}_M(x_i)=\underline{F}_N(x_i)$,

$\overline{F}_M(x_i)=\overline{F}_N(x_i)$

This implies that $M=N$.

4. If $M\subset N\subset P$ then we can write $\underline{T}_M(x_i)\le\underline{T}_N(x_i)\le\underline{T}_P(x_i)$, $\overline{T}_M(x_i)\le\overline{T}_N(x_i)\le\overline{T}_P(x_i)$,
$\underline{I}_M(x_i)\ge\underline{I}_N(x_i)\ge\underline{I}_P(x_i)$, $\overline{I}_M(x_i)\ge\overline{I}_N(x_i)\ge\overline{I}_P(x_i)$, $\underline{F}_M(x_i)\ge\underline{F}_N(x_i)\ge\underline{F}_P(x_i)$,
$\overline{F}_M(x_i)\ge\overline{F}_N(x_i)\ge\overline{F}_P(x_i)$.

The cotangent function is decreasing function within the interval $\left[\dfrac{\pi}{4},\dfrac{\pi}{2}\right]$. Hence we can write $COT_{RNS}(M,P)\le COT_{RNS}(M,N)$, and $COT_{RNS}(M,P)\le COT_{RNS}(N,P)$.

Definition 3.3 Weighted rough cotangent similarity measure

If we consider the weights of each element x_i, a weighted rough cotangent similarity measure between rough neutrosophic sets *A* and *B* can be defined as follows:

$$COT_{WRNS}(M,N)=$$
$$\left[\frac{1}{n}\sum_{i=1}^{n}w_i\left\langle\cot\left(\frac{\pi}{12}\left(3+|\delta T_M(x_i)-\delta T_N(x_i)|+|\delta I_M(x_i)-\delta I_N(x_i)|+|\delta F_M(x_i)-\delta F_N(x_i)|\right)\right)\right\rangle\right]$$

$w_i\in[0,1]$, i = 1, 2,…, n and $\sum_{i=1}^{n}w_i=1$. If we take $w_i=\dfrac{1}{n}$, i = 1, 2,…, n, then $COT_{WRNS}(M,N)=COT_{RNS}(M,N)$

Proposition VI: The weighted rough cotangent similarity measure $COT_{WRNS}(M, N)$ between two rough neutrosophic sets M and N also satisfies the following properties:

1. $0 \le COT_{WRNS}(M, N) \le 1$
2. $COT_{WRNS}(M, N) = COT_{WRNS}(N, M)$
3. $COT_{WRNS}(M, N) = 1$, iff $M = N$
4. If P is a WRNS in Y and $M \subset N \subset P$ then, $COT_{WRNS}(M, P) \le COT_{WRNS}(M, N)$, and $COT_{WRNS}(M, P) \le COT_{WRNS}(N, P)$

Proof :

1.Since, $\frac{\pi}{4} \le \left(\frac{\pi}{12}\left(3 + \left| \delta T_M(x_i) - \delta T_N(x_i) \right| + \left| \delta I_M(x_i) - \delta I_N(x_i) \right| + \left| \delta F_M(x_i) - \delta F_N(x_i) \right| \right) \right) \le \frac{\pi}{2}$ and $\sum_{i=1}^{n} w_i = 1$, it is obvious that the weighted cotangent function are within 0 and 1 ie, $0 \le COT_{WRNS}(M, N) \le 1.$

2. It is obvious that the proposition is true.

3. Here, $\sum_{i=1}^{n} w_i = 1.$ When $M = N$, then obviously $COT_{WRNS}(M, N) = 1$. On the other hand if $COT_{WRNS}(M, N) = 1$ then,

$\delta T_M(x_i) = \delta T_N(x_i)$, $\delta I_M(x_i) = \delta I_N(x_i)$, $\delta F_M(x_i) = \delta F_N(x_i)$ *ie,*
$\underline{T}_M(x_i) = \underline{T}_N(x_i)$, $\overline{T}_M(x_i) = \overline{T}_N(x_i)$, $\underline{I}_M(x_i) = \underline{I}_N(x_i)$, $\overline{I}_M(x_i) = \overline{I}_N(x_i)$, $\underline{F}_M(x_i) = \underline{F}_N(x_i)$,
$\overline{F}_M(x_i) = \overline{F}_N(x_i)$

This implies that $M = N$.

4. If $M \subset N \subset P$ then we can write $\underline{T}_M(x_i) \le \underline{T}_N(x_i) \le \underline{T}_P(x_i)$, $\overline{T}_M(x_i) \le \overline{T}_N(x_i) \le \overline{T}_P(x_i)$,
$\underline{I}_M(x_i) \ge \underline{I}_N(x_i) \ge \underline{I}_P(x_i)$, $\overline{I}_M(x_i) \ge \overline{I}_N(x_i) \ge \overline{I}_P(x_i)$, $\underline{F}_M(x_i) \ge \underline{F}_N(x_i) \ge \underline{F}_P(x_i)$,
$\overline{F}_M(x_i) \ge \overline{F}_N(x_i) \ge \overline{F}_P(x_i)$.

The cotangent function is decreasing function within the interval $\left[\frac{\pi}{4}, \frac{\pi}{2} \right]$. Here, $\sum_{i=1}^{n} w_i = 1.$ Hence we can write $COT_{WRNS}(M, P) \le COT_{WRNS}(M, N)$, and $COT_{WRNS}(M, P) \le COT_{WRNS}(N, P)$.

4 Examples on Medical Diagnosis

We consider a medical diagnosis problem from practical point of view for illustration of the proposed approach. Medical diagnosis comprises of uncertainties and increased volume of information available to physicians from new medical technologies. The process of classifying different set of symptoms under a single name of a disease is very difficult task. In some practical situations, there exists possibility of each element within a lower and an upper approximation of neutrosophic sets. It can deal with the medical diagnosis involving more indeterminacy. Actually this approach is more flexible and easy to use. The proposed similarity measure among the patients versus symptoms and symptoms versus diseases will provide the proper medical diagnosis. The main feature of this proposed approach is that it

considers truth membership, indeterminate and false membership of each element between two approximations of neutrosophic sets by taking one time inspection for diagnosis.
Now, an example of a medical diagnosis is presented. Let P = {P_1, P_2, P_3} be a set of patients, D = {Viral Fever, Malaria, Stomach problem, Chest problem} be a set of diseases and S = {Temperature, Headache, Stomach pain, Cough, Chest pain.} be a set of symptoms. Our task is to examine the patient and to determine the disease of the patient in rough neutrosophic environment.

Table 1: (Relation-1) The relation between Patients and Symptoms

Relation-1	**Temperature**	**Headache**	**Stomach pain**	**cough**	**Chest pain**
P_1	$\left\langle \begin{matrix}(0.6,0.3,0.3),\\(0.8,0.3,0.1)\end{matrix}\right\rangle$	$\left\langle \begin{matrix}(0.4,0.4,0.3),\\(0.6,0.2,0.1)\end{matrix}\right\rangle$	$\left\langle \begin{matrix}(0.5,0.4,0.2),\\(0.7,0.2,0.2)\end{matrix}\right\rangle$	$\left\langle \begin{matrix}(0.6,0.3,0.3),\\(0.8,0.1,0.1)\end{matrix}\right\rangle$	$\left\langle \begin{matrix}(0.5,0.4,0.4),\\(0.5,0.2,0.2)\end{matrix}\right\rangle$
P_2	$\left\langle \begin{matrix}(0.5,0.4,0.3),\\(0.7,0.2,0.3)\end{matrix}\right\rangle$	$\left\langle \begin{matrix}(0.5,0.3,0.5),\\(0.7,0.3,0.3)\end{matrix}\right\rangle$	$\left\langle \begin{matrix}(0.5,0.2,0.4),\\(0.7,0.0,0.2)\end{matrix}\right\rangle$	$\left\langle \begin{matrix}(0.5,0.3,0.5),\\(0.9,0.3,0.3)\end{matrix}\right\rangle$	$\left\langle \begin{matrix}(0.5,0.5,0.3),\\(0.7,0.3,0.3)\end{matrix}\right\rangle$
P_3	$\left\langle \begin{matrix}(0.7,0.4,0.2),\\(0.9,0.2,0.2)\end{matrix}\right\rangle$	$\left\langle \begin{matrix}(0.5,0.3,0.2),\\(0.7,0.1,0.2)\end{matrix}\right\rangle$	$\left\langle \begin{matrix}(0.6,0.5,0.4),\\(0.8,0.3,0.2)\end{matrix}\right\rangle$	$\left\langle \begin{matrix}(0.6,0.3,0.4),\\(0.8,0.1,0.2)\end{matrix}\right\rangle$	$\left\langle \begin{matrix}(0.5,0.5,0.3),\\(0.7,0.3,0.1)\end{matrix}\right\rangle$

Table 2: (Relation-2) The relation among Symptoms and Diseases

Relation-2	**Viral Fever**	**Malaria**	**Stomach problem**	**Chest problem**
Temperature	$\left\langle \begin{matrix}(0.6,0.5,0.4),\\(0.8,0.5,0.2)\end{matrix}\right\rangle$	$\left\langle \begin{matrix}(0.3,0.4,0.5),\\(0.5,0.2,0.3)\end{matrix}\right\rangle$	$\left\langle \begin{matrix}(0.3,0.3,0.4),\\(0.5,0.1,0.2)\end{matrix}\right\rangle$	$\left\langle \begin{matrix}(0.2,0.4,0.5),\\(0.4,0.4,0.3)\end{matrix}\right\rangle$
Headache	$\left\langle \begin{matrix}(0.5,0.4,0.4),\\(0.7,0.2,0.2)\end{matrix}\right\rangle$	$\left\langle \begin{matrix}(0.4,0.3,0.5),\\(0.6,0.3,0.3)\end{matrix}\right\rangle$	$\left\langle \begin{matrix}(0.2,0.4,0.4),\\(0.4,0.2,0.2)\end{matrix}\right\rangle$	$\left\langle \begin{matrix}(0.3,0.5,0.4),\\(0.5,0.3,0.2)\end{matrix}\right\rangle$
Stomach pain	$\left\langle \begin{matrix}(0.2,0.3,0.3),\\(0.4,0.3,0.1)\end{matrix}\right\rangle$	$\left\langle \begin{matrix}(0.1,0.4,0.3),\\(0.3,0.2,0.1)\end{matrix}\right\rangle$	$\left\langle \begin{matrix}(0.4,0.4,0.4),\\(0.6,0.2,0.2)\end{matrix}\right\rangle$	$\left\langle \begin{matrix}(0.1,0.4,0.6),\\(0.3,0.2,0.2)\end{matrix}\right\rangle$
Cough	$\left\langle \begin{matrix}(0.4,0.3,0.4),\\(0.6,0.1,0.2)\end{matrix}\right\rangle$	$\left\langle \begin{matrix}(0.3,0.3,0.3),\\(0.5,0.3,0.1)\end{matrix}\right\rangle$	$\left\langle \begin{matrix}(0.1,0.6,0.6),\\(0.3,0.2,0.2)\end{matrix}\right\rangle$	$\left\langle \begin{matrix}(0.5,0.4,0.3),\\(0.7,0.2,0.1)\end{matrix}\right\rangle$
Chest pain	$\left\langle \begin{matrix}(0.2,0.4,0.3),\\(0.6,0.2,0.1)\end{matrix}\right\rangle$	$\left\langle \begin{matrix}(0.1,0.3,0.4),\\(0.3,0.1,0.2)\end{matrix}\right\rangle$	$\left\langle \begin{matrix}(0.2,0.4,0.4),\\(0.4,0.2,0.4)\end{matrix}\right\rangle$	$\left\langle \begin{matrix}(0.3,0.4,0.3),\\(0.5,0.2,0.3)\end{matrix}\right\rangle$

Table 3: The Correlation Measure between Relation-1 and Relation-2

Rough cotangent similarity measure	**Viral Fever**	**Malaria**	**Stomach problem**	**Chest problem**
P_1	**0.8726**	0.8194	0.7977	0.8235
P_2	**0.8298**	0.7968	0.8024	0.7857
P_3	**0.8382**	0.7356	0.7448	0.7536

The highest correlation measure (see the Table 3) reflects the proper medical diagnosis. Therefore, all three patients P_1, P_2, P_3 suffer from viral fever.

5. Conclusion

In this paper, we have proposed rough cotangent similarity measure of rough neutrosophic sets and proved some of their basic properties. We have presented an application of rough cotangent similarity measure of rough neutrosophic sets in medical diagnosis problems. We hope that the proposed concept can be applied in solving realistic multi-attribute decision making problems.

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