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WEIGHTED NEUTROSOPHIC SOFT SETS APPROACH IN A MULTI-CRITERIA DECISION MAKING PROBLEM

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Abstract – The paramount importance of decision making problem in an imprecise environment is becoming very much significant in recent years. In this paper we have studied weighted neutrosophic soft sets which are a hybridization of neutrosophic sets with soft sets corresponding to weighted parameters. We have considered here a multicriteria decision making problem as an application of weighted neutrosophic soft sets.

Keywords – *Soft sets, neutrosophic sets, neutrosophic soft sets, weighted neutrosophic soft sets.*

1 Introduction

In 1999, Molodtsov initiated the novel concept, the concept of 'soft set theory' [1] which has been proved as a generic mathematical tool to deal with problems involving uncertainties. Due to the inadequacy of parametrization in the theory of fuzzy sets [2], rough sets [3], vague sets [4], probability theory etc. we become handicapped to use them successfully. Consequently Molodtsov has shown that soft set theory has a potential to use in different fields [1]. Recently, the works on soft set theory is growing very rapidly with all its potentiality and is being used in different fields [5 - 10]. A detailed theoretical study may be found in [10]. Depending on the characteristics of the parameters involved in soft set different hybridization viz. fuzzy soft sets [11], soft rough sets [12], intuitionistic fuzzy soft sets [13], vague soft sets [14], neutrosophic soft sets [15] etc. have been introduced. The soft set theory is now being used in different fields as an application of it. Some of them have been investigated in [6 -10, 16]. Based soft set [1] and neutrosophic sets [17] a hybrid structure 'neutrosophic soft sets' has been initiated [15]. The parameters considered here are neutrosophic in nature. Imposing the weights on the parameters (may be in a particular parameter also) a weighted neutrosophic soft sets has been introduced [18]. In this paper we

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use this concept to solve a multi-criteria decision making problem. In section 2 of this paper we briefly recall some relevant preliminaries centered around our problem. Some basic definitions on weighted neutrosophic soft sets relevant to this work are available in section 3. A decision making problem has been discussed and solved in section 4. Conclusions are there in the concluding Section 5.

2 Preliminaries

Most of the real life problems in the fields of medical sciences, economics, engineering etc. the data involve are imprecise in nature. The classical mathematical tools are not capable to handle such problems. The novel concept 'soft set theory' initiated by Molodtsov [1] is a new mathematical tool to deal with such problems. For better understanding we now recapitulate some preliminaries relevant to the work.

Definition 2.1 [1] Let U be an initial universe set and E be a set of parameters. Let P(U) denotes the power set of U. Consider a nonempty set A, $A \subset E$.
A pair (F, A) is called a soft set over U, where F is a mapping given by $F : A \rightarrow P(U)$.
A soft set over U is a parameterized family of subsets of the universe U. For $\varepsilon \in A$, $F(\varepsilon)$ may be considered as the set of ε - approximate elements of the soft set (F, A).

Definition 2.2 [10] For two soft sets (F, A) and (G, B) over a common universe U, we say that (F, A) is a soft subset of (G, B) if

(i) $A \subset B$, and
(ii) $\forall\ \varepsilon \in A$, $F(\varepsilon)$ and $G(\varepsilon)$ are identical approximations.

We write (F, A) $\tilde{\subset}$ (G, B).

(F, A) is said to be a soft super set of (G, B), if (G, B) is a soft subset of (F, A). We denote it by (F, A) $\tilde{\supset}$ (G, B).

Let A and B be two subsets of E, the set of parameters. Then $A \times B \subset E \times E$. Now we are in the position to define 'AND', 'OR' operations on two soft sets over a common universe.

Definition 2.3 [10] If (F, A) and (G, B) be two soft sets over a common universe U then '(F, A) AND (G, B)' denoted by (F, A) ^ (G, B) is defined by

$$(F, A) \wedge (G, B) = (H,\ A \times B),$$

where $H(\alpha, \beta) = F(\alpha) \cap G(\beta)$, $\forall\ (\alpha, \beta) \in A \times B$.

Definition 2.4 [10] If (F, A) and (G, B) be two soft sets over a common universe U then '(F, A) OR (G, B)' denoted by (F, A) $\vee$ (G, B) is defined by

$$(F, A) \vee (G, B) = (O, A\times B),$$

where, $O(\alpha, \beta) = F(\alpha) \cup G(\beta)$, $\forall (\alpha,\beta)\in A\times B$.

The non-standard analysis was introduced by Abraham Robinson in 1960. The non-standard analysis is a formalization of analysis and a branch of mathematical logic that rigorously defines the infinitesimals. Informally, an infinitesimal is an infinitely small number. Formally, x is said to be infinitesimal if and only if for all positive integers n one has $|x| < \frac{1}{n}$. Let $\varepsilon > 0$ be a such infinitesimal number. Let's consider the non-standard finite numbers $1^+ = 1+ \varepsilon$, where '1' is its standard part and 'ε' its non-standard part, and $^-0 = 0 - \varepsilon$, where '0' is its standard part and 'ε' its non-standard part.

Definition 2.5 [17] A neutrosophic set A on the universe of discourse X is defined as

$$A = \{< x, T_A(x), I_A(x), F_A(x) > x \in X\},$$

where $T_A, I_A, F_A: X \to]^-0, 1^+[$ and $^-0 \le T_A + I_A + F_A \le 3^+$.

Here T_A, I_A, F_A are respectively the true membership, indeterministic membership and false membership function of an object $x \in X$.

From philosophical point of view, the neutrosophic set takes the value from real standard or non-standard subsets of $]^-0, 1^+[$. But in real life applications in scientific and engineering problems it is difficult to use neutrosophic set with value from real standard or non-standard subset of $]^-0, 1^+[$. Hence we consider the neutrosophic set which takes the value from the subset of [0, 1].

Definition 2.6 [15] Let U be an initial universe set and E be a set of parameters which is of neutrosophic in nature. Consider $A \subset E$. Let P(U) denotes the set of all neutrosophic sets of U.

The collection (F, A) is termed to be the neutrosophic soft set (N S S) over U, where F is a mapping given by $F : A \to P (U)$.

For an illustration we consider the following example.

Example 2.7 Let U be the set of objects under consideration and E is the set of parameters. Each parameter is a neutrosophic word or sentence involving neutrosophic words. Consider E= {beautiful, large, very large, small, average large, costly, cheap, brick build }. In this case to define a neutrosophic soft set means to point out beautiful objects, large objects, very large objects etc. and so on. Suppose that there are five objects in the universe U given by $U = \{ o_1, o_2, o_3, o_4, o_5 \}$ and the set of parameters $A = \{ e_1, e_2, e_3, e_4 \}$ where e_1 stands for the parameter 'large', e_2 stands for the parameter 'very large', e_3 stands for the parameter 'small' and e_4

stands for the parameter 'average'. Suppose that the NSS (F, A) describes the length of the objects under consideration for which,

F(large) = {< o_1, 0.6, 0.4, 0.7 >, < o_2, 0.5, 0.6, 0.8 >, < o_3, 0.8, 0.7, 0.7 >, < o_4, 0.6, 0.4, 0.8 >, < o_5, 0.8, 0.6, 0.7 > },

F(very large) = {< o_1, 0.5, 0.3, 0.6 >, < o_2, 0.8, 0,5, 0.7 >, < o_3, 0.9, 0.7, 0.8 >, < o_4, 0.7, 0.6, 0.7 >, < o_5, 0.6, 0.7, 0.9 > },

F(small) = {< o_1, 0.3, 0.8, 0.9 >, < o_2, 0.4, 0.6, 0.8 >, < o_3, 0.6, 0.8, 0.4 >, < o_4, 0.7, 0.7, 0.6 >, < o_5, 0.6, 0.7, 0.9 > },

F(average) = {< o_1, 0.8, 0.3, 0.4 >, < o_2, 0.9, 0.6, 0.8 >, < o_3,0.8, 0.7, 0.8 >, < o_4, 0.6, 0.7, 0.>, < o_5, 0.7, 0.6, 0.8 > }.

So, F(large) means large objects, F(small) means the objects having small length etc. For the purpose of storing a neutrosophic soft set in a computer, we could represent it in the form of a table as shown below (corresponding to the neutrosophic soft set in the above example). In this table, the entries c_{ij} correspond to the object o_i and the parameter e_j, where c_{ij} = (true-membership value of o_i, indeterminacy-membership value of o_i, falsity-membership value of o_i) in F(e_j). The tabular representation of the neutrosophic soft set (F, A) is as follow:

Table 1. The Tabular form of the NSS (F, A).

U	e_1 = large	e_2 = very large	e_3 = small	e_4 = average
o_1	(0.6, 0.4, 0.7)	(0.5, 0.3, 0.6)	(0.3, 0.8, 0.9)	(0.8, 0.3, 0.4)
o_2	(0.5, 0.6, 0.8)	(0.8, 0.5, 0.7)	(0.4, 0.6, 0.8)	(0.9, 0.6, 0.8)
o_3	(0.8, 0.7, 0.7)	(0.9, 0.7, 0.8)	(0.6, 0.8, 0.4)	(0.8, 0.7, 0.8)
o_4	(0.6, 0.4, 0.8)	(0.7, 0.6, 0.7)	(0.7, 0.7, 0.6)	(0.6, 0.7, 0.5)
o_5	(0.8, 0.6, 0.7)	(0.6, 0.7, 0.9)	(0.6, 0.7, 0.9)	(0.7, 0.6, 0.8)

Definition 2.8 [15] Let (F, A) and (G, B) be two neutrosophic soft sets over the common universe U. (F, A) is said to be neutrosophic soft subset of (G, B) if A $\subset$ B and $T_{F(e)}(x) \leq T_{G(e)}(x)$, $I_{F(e)}(x) \leq I_{G(e)}(x)$, $F_{F(e)}(x) \geq F_{G(e)}(x)$, $e \in A$.

We denote it by (F, A) $\subset$ (G, B). (F, A) is said to be neutrosophic soft super set of (G, B) if (G, B)is a neutrosophic soft subset of (F, A).

Definition 2.9 [15] AND operation on two neutrosophic soft sets.

Let (H, A) and (G, B) be two NSSs over the same universe U. Then the 'AND' operation on them is denoted by '(H, A) ^ (G, B)' and is defined by (H, A) ^ (G, B) = (K, $A \times B$), where the truth-membership value, indeterminacy-membership value and falsity-membership value of (K, $A \times B$) are as follows:

$T_{K(\alpha,\beta)}(m) = \min(T_{H(\alpha)}(m), T_{G(\beta)}(m))$,

$$I_{K(\alpha,\beta)}(m) = \frac{I_{H(\alpha(}(m) + I_{G(\beta(}(m)}{2}, \text{ and}$$

$F_{K(\alpha,\beta)}(m) = \max(F_{H(\alpha)}(m), F_{G(\beta)}(m))$, $\forall\, \alpha \in A$, $\forall\, \beta \in B$.

The decision maker may not have equal choice for all the parameters. He/she may impose some conditions to choose the parameters for which the decision will be taken. The conditions may be imposed in terms of weights (positive real numbers ≤ 1). This imposition motivates us to define weighted neutrosophic soft sets.

3 Weighted Neutrosophic Soft Sets

Definition 3.1 [18] A neutrosophic soft set is termed to be a weighted neutrosophic soft sets (WNSS) if the weights (w_i, a real positive number ≤ 1) be imposed on the parameters of it. The entries of the weighted neutrosophic soft set $d_{ij} = w_i \times c_{ij}$, where c_{ij} is the ij-th entry in the table of neutrosophic soft set.

For an illustration we consider the following example.

Example 3.2 Consider the example 2.7 . Suppose that the decision maker has no equal preference for each of the parameters. He may impose the weights of preference for the parameters 'e_1= large' as '$w_1 = 0.8$', 'e_2= very large' as '$w_2 = 0.4$', 'e_3= small' as '$w_3 = 0.5$', 'e_4= average large' as '$w_4 = 0.6$'. Then the weighed neutrosophic soft set obtained from (F, A) denoted as (H, A^w) and its tabular representation is as below:

Table 2: Tabular form of the weighted NSS (H, A^w).

U	$e_1,w_1 = 0.8$	$e_2, w_2 = 0.4$	$e_3, w_3 = 0.5$	$e_4,w_4 = 0.6$
o_1	(0.48, 0.32, 0.56)	(0.20, 0.12, 0.24)	(0.15, 0.40, 0.45)	(0.48, 0.18, 0.24)
o_2	(0.40, 0.48, 0.64)	(0.32, 0.20, 0.28)	(0.20, 0.30, 0.40)	(0.54, 0.36, 0.48)
o_3	(0.64, 0.56, 0.56)	(0.36, 0.28,0.32)	(0.30,0.40,0.20)	(0.48,0.42,0.48)
o_4	(0.48, 0.32, 0.64)	(0.28, 0.24,0.28)	(0.35,0.35,0.30)	(0.36,0.42,0.30)
o_5	(0.64, 0.48, 0.56)	(0.24, 0.28,0.36)	(0.30,0.35,0.45)	(0.42,0.36,0.48)

Definition 3.3 [18] AND operation on two weighted neutrosophic soft sets.

Let (H, A^{w_1}) and (G, B^{w_2}) be two WNSSs over the same universe U. Then the 'AND' operation on them is denoted by '(H, A^{w_1}) ^ (G, B^{w_2})' and is defined by (H, A^{w_1}) ^ (G, B^{w_2}) = (K, $A^{w_1} \times B^{w_2}$), where the truth-membership value, indeterminacy-membership value and falsity-membership value of (K, $A^{w_1} \times B^{w_2}$) are as follows:

$$T_{K(\alpha^{w_1}, \beta^{w_2})}(m) = \min(w_1, w_2).\min(T_{H(\alpha)}(m), T_{G(\beta)}(m)), \ \forall \alpha \in A, \ \forall \beta \in B,$$

$$I_{K(\alpha^{w_1}, \beta^{w_2})}(m) = \frac{I_{H(\alpha^{w_1})}(m) + I_{G(\beta^{w_2})}(m)}{2}, \forall \alpha \in A, \ \forall \beta \in B,$$

$$F_{K(\alpha^{w_1}, \beta^{w_2})}(m) = \max(w_1, w_2).\max(F_{H(\alpha)}(m), F_{G(\beta)}(m)), \ \forall \alpha \in A, \ \forall \beta \in B.$$

Definition 3.4 Comparison Matrix. It is a matrix whose rows are labelled by n object o_1, o_2, ..., o_n and the columns are labelled by m weighted parameters e_1, e_2, ..., e_m. The entries c_{ij} of the comparison matrix are evaluated by c_{ij} = a + b - c, where 'a' is the positive integer calculated as 'how many times T_{oi} (e_j) exceeds or equal to T_{ok} (e_j)', for i ≠ k, ∀ i = 1, 2, ..., n, 'b' is the positive integer calculated as 'how many times I_{oi} (e_j) exceeds or equal to I_{ok} (e_j)', for i ≠ k and ∀ i = 1, 2, ..., n and 'c' is the integer 'how many times F_{oi} (e_j) exceeds or equal to F_{ok} (e_j)', for i ≠ k and ∀ i = 1, 2, ..., n.

Definition 3.5 Score of an Object. The score of an object o_i is S_i and is calculated as

$$S_i = \sum_j c_{ij} \ , \ \forall \ i = 1, 2, ..., n.$$

Here we consider a problem to choose an object from a set of given objects with respect to a set of choice parameters P. We follow an algorithm to identify an object based on multiobserver (considered here three observers with their own choices) input data characterized by colours (F, A^w), size (G, B^w) and surface textures (H, C^w) features. The algorithm to choose an appropriate object depending upon the choice parameters is given below.

3.6 Algorithm

1. input the neutrosophic soft sets (H, A), (G, B) and (H, C) (for three observers)
2. input the weights (w_i) for the parameters A, B and C
3. compute weighted neutrosophic soft sets (H, A^w), (G, B^w) and (H, C^w) corresponding to

the NSSs (H, A), (G, B) and (H, C) respectively

4. input the parameter set P as preferred by the decision maker
5. compute the corresponding NSS (S, P) from the WNSSs (H, A^W), (G, B^W) and (H, C^W) and place in tabular form
6. compute the comparison matrix of the NSS (S, P)
7. compute the score S_i of o_i, $\forall$ i = 1, 2, ..., n
8. the decision is o_k if $S_k = \max_i S_i$
9. if k has more than one values then any one of o_i may be chosen.

Based on the above algorithm we consider the following multi-criteria decision making problem.

4 Application in a Decision Making Problem

Let U = { o_1, o_2, o_3, o_4, o_5 } be the set of objects characterized by different lengths, colours and surface texture. Consider the parameter set, E = { blackish, dark brown, yellowish, reddish, large, small, very small, average, rough, very large, coarse, moderate, fine, smooth, extra fine }. Also consider A = { very large, small, average large }, B = { reddish, yellowish, blackish } and C = { smooth, rough, moderate } be three subsets of the set of parameters E. Let the NSSs (F, A), (G, B) and (H, C) describe the objects 'having different lenghts', 'objects having different colours' and 'surface structure features of the objects' respectively. These NSSs as computed by the three observers Mr. X, Mr. Y and Mr. Z respectively, are given below in their respective tabular forms in table 3, 4 and 5. Now suppose that the decision maker imposes the weights on the parameters A, B and C and the repective weighted neutrosophic soft sets are (F, A^W), (G, B^W) and (H, C^W). The WNSS (F, A^W) describes the 'objects having different lengths', the WNSS (G, B^W) describes the 'different colours of the objects' and the WNSS (H, C^W) describes the 'surface structure feature of the objects'. We consider the problem to identify an object from U based on the multiobservers neutrosophic data, specified by different observers (we consider here three observers), in terms of WNSSs (F, A^W), (G, B^W) and (H, C^W) as described above.

Table 3: Tabular form of the WNSS (F, A^w).

U	a_1 = very large	a_2 = small	a_3 = average large
o_1	(0.5, 0.6, 0.8)	(0.7, 0.3, 0.5)	(0.6, 0.7, 0.3)
o_2	(0.6, 0.8, 0.7)	(0.3, 0.6, 0.4)	(0.8, 0.3, 0.5)
o_3	(0.3, 0.5, 0.8)	(0.8, 0.3, 0.2)	(0.3, 0.2, 0.6)
o_4	(0.8, 0.3, 0.5)	(0.3, 0.5, 0.3)	(0.6, 0.7, 0.3)
o_5	(0.7, 0.3, 0.6)	(0.4, 0.6, 0.8)	(0.8, 0.3, 0.8)
weight	w_1 = 0.5	w_2 = 0.6	w_3 = 0.3
o_1	(0.25, 0.30, 0.40)	(0.42, 0.18, 0.30)	(0.18, 0.21, 0.09)
o_2	(0.30, 0.40, 0.35)	(0.18, 0.36, 0.24)	(0.24, 0.09, 0.15)
o_3	(0.15, 0.25, 0.40)	(0.48, 0.18, 0.12)	(0.09, 0.06, 0.18)
o_4	(0.40, 0.15, 0.25)	(0.18, 0.30, 0.18)	(0.18, 0.21, 0.09)
o_5	(0.35, 0.15, 0.30)	(0.24, 0.36, 0.48)	(0.24, 0.09, 0.24)

Table 4: Tabular form of the WNSS (G, B^w).

U	b_1 = reddish	b_2 = yellowish	b_3 = blackish
o_1	(0.5, 0.7, 0.3)	(0.7, 0.8, 0.6)	(0.8, 0.3, 0.4)
o_2	0.6, 0.7, 0.3)	(0.8, 0.5, 0.7)	(0.6, 0.7, 0.3)
o_3	(0.8, 0.5, 0.6)	(0.7, 0.3, 0.6)	(0.8, 0.3, 0.5)
o_4	(0.7, 0.2, 0.6)	(0.8, 0.6, 0.5)	(0.6, 0.7, 0.3)
o_5	(0.8, 0.4, 0.7)	(0.6, 0.5, 0.8)	(0.7, 0.4, 0.2)
weight	w_1 = 0.6	w_2 = 0.4	w_3 = 0.7
o_1	(0.30, 0.42, 0.18)	(0.28, 0.32, 0.24)	(0.56, 0.21, 0.28)
o_2	(0.36, 0.42, 0.18)	(0.32, 0.20, 0.28)	(0.42, 0.49, 0.21)
o_3	(0.48, 0.30, 0.36)	(0.28, 0.12, 0.24)	(0.56, 0.21, 0.35)
o_4	(0.42, 0.12, 0.36)	(0.32, 0.24, 0.20)	(0.42, 0.49, 0.21)
o_5	(0.48, 0.24, 0.42)	(0.24, 0.20, 0.32)	(0.49, 0.28, 0.14)

Table 5: Tabular form of the WNSS (H, C^w).

U	c_1 = smooth	c_2 = rough	c_3 = moderate
o_1	(0.8, 0.5, 0.6)	(0.8, 0.7, 0.3)	(0.8, 0.6, 0.4)
o_2	(0.7, 0.6, 0.7)	(0.7, 0.5, 0.6)	(0.7, 0.5, 0.6)
o_3	(0.8, 0.7, 0.6)	(0.6, 0.3, 0.7)	(0.8, 0.2, 0.4)
o_4	(0.7, 0.5, 0.7)	(0.8, 0.7, 0.4)	(0.7, 0.8, 0.7)
o_5	(0.8, 0.7, 0.4)	(0.7, 0.4, 0.8)	(0.8, 0.6, 0.5)
weight	w_1 = 0.6	w_2 = 0.8	w_3 = 0.5
o_1	(0.48, 0.30, 0.36)	(0.64, 0.56, 0.24)	(0.40, 0.30, 0.20)
o_2	(0.42, 0.36, 0.42)	(0.56, 0.40, 0.48)	(0.35, 0.25, 0.30)
o_3	(0.48, 0.42, 0.36)	(0.48, 0.24, 0.56)	(0.40, 0.10, 0.20)
o_4	(0.42, 0.30, 0.42)	(0.64, 0.56, 0.32)	(0.35, 0.40, 0.35)
o_5	(0.48, 0.42, 0.24)	(0.56, 0.32, 0.64)	(0.40, 0.30, 0.25)

In the above two WNSSs (F, A^w) and (G, B^w) given in their respective tabular form in 3 and 4, if the evaluator wants to perform the operation '(F, A^w) AND (G, B^w)' then we will have 3×3 = 9 parameters of the form e_{ij}, where $e_{ij} = a_i \wedge b_j$, for i= 1, 2, 3 and j = 1, 2, 3 and $e_{ij} \in E \times E$. On the basis of the choice parameters of the evaluator if we consider the WNSS with parameters R = { e_{11}, e_{21}, e_{22}, e_{31}, e_{32} } we have the WNSS (K, R^w) obtained from the WNSSs (F, A^w) and (G, B^w). So e_{11} = (very large, reddish), e_{22} = (small, yellowish) etc. Computing '(F, A^w) AND (G, B^w)' for the choice parameters R, we have the tabular representation of the WNSS (K, R^w) as below:

Table 6: Tabular form of the WNSS (K, R^w).

U	e_{11}	e_{21}	e_{22}	e_{31}	e_{32}
o_1	(0.25, 0.36, 0.48)	(0.30, 0.30, 0.30)	(0.28, 0.25, 0.36)	(0.15, 0.615, 0.18)	(0.18, 0.265, 0.24)
o_2	(0.30, 0.41, 0.56)	(0.18, 0.39, 0.24)	(0.12, 0.28, 0.42)	(0.18, 0.255, 0.30)	(0.24, 0.145, 0.28)
o_3	(0.15, 0.275, 0.48)	(0.48, 0.24, 0.36)	(0.28, 0.15, 0.36)	(0.09, 0.18, 0.36)	(0.09, 0.09, 0.24)
o_4	(0.35, 0.135, 0.36)	(0.18, 0.21, 0.36)	(0.12, 0.27, 0.30)	(0.18, 0.165, 0.36)	(0.18, 0.175, 0.20)
o_5	(0.35, 0.195, 0.42)	(0.24, 0.30, 0.48)	(0.16, 0.28, 0.48)	(0.24, 0.285, 0.48)	(0.18, 0.145, 0.32)

Computing the WNSS (S, P) from the WNSSs (K, R^W) and (H, C^W) for the specified parameters P = { $e_{11} \wedge c_1$, $e_{21} \wedge c_2$, $e_{21} \wedge c_3$, $e_{31} \wedge c_1$ }, where the parameter $e_{11} \wedge c_1$ means (very large, reddish, smooth), $e_{21} \wedge c_2$ means (small, reddish, rough) etc. The tabular form of the WNSS (S, P) is as below:

Table 7: Tabular form of the WNSS (S, P).

U	$e_{11} \wedge c_1$	$e_{21} \wedge c_2$	$e_{21} \wedge c_3$	$e_{31} \wedge c_1$
o_1	(0.25, 0.4375, 0.48)	(0.30, 0.58, 0.40)	(0.25, 0.425, 0.30)	(0.15, 0.45, 0.36)
o_2	(0.30, 0.6675, 0.42)	(0.18, 0.488, 0.48)	(0.15, 0.388, 0.36)	(0.18, 0.455, 0.42)
o_3	(0.15, 0.51, 0.48)	(0.36, 0.295, 0.56)	(0.40, 0.20, 0.36)	(0.09, 0.4975, 0.36)
o_4	(0.35, 0.3375, 0.42)	(0.18, 0.542, 0.48)	(0.15, 0.488, 0.42)	(0.18, 0.388, 0.42)
o_5	(0.35, 0.4725, 0.42)	(0.24, 0.385, 0.64)	(0.20, 0.425, 0.48)	(0.24, 0.4725, 0.48)

Then the tabular form of the comparison matrix for the WNSS (S, P) is as below:

Table 8: Tabular form of the comparison matrix of the WNSS (S, Q).

U	$e_{11} \wedge c_1$	$e_{21} \wedge c_2$	$e_{21} \wedge c_3$	$e_{31} \wedge c_1$
o_1	**-2**	7	6	1
o_2	4	1	0	2
o_3	**-1**	1	2	3
o_4	2	2	2	**-2**
o_5	4	**-1**	1	3

Computing the score for each of the objects we have the respective scores as below:

U	Score
o_1	**12**
o_2	7
o_3	5
o_4	4
o_5	7

Clearly, the maximum score is 12 and scored by the object o_1. The selection will be in favour of the object o_1. The second choice will be in favour of either o_2 or o_5 as they have the same score 7. Next the decision maker may choose the objects o_3 and o_4 as the score 5 and 4 are scored by them respectively.

5 Conclusion

Since its initiation the soft set theory is being used in variety of many fields involving imprecise and uncertain data. In this paper we present an application of weighted neutrosophic soft sets for selection of an object. Here the selection is based on multicriteria input data of neutrosophic in nature. We also introduce an algorithm to select an appropriate object from a set of objects based on some specified parameters.

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SOME INEQUALITIES FOR q AND (q, k) DEFORMED GAMMA FUNCTIONS

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Abstract – In this short paper, the authors establish some inequalities involving the q and (q, k) deformed Gamma functions by employing some basic analytical techniques.

Keywords – *Gamma function, q-deformation, (q, k)-deformation, q-addition, inequality.*

1 Introduction

Let $\Gamma(x)$ be the classical Gamma function and $\psi(x)$ be the classical Psi or Digamma function defined for $x \in R^+$ as:

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\,dt,$$

$$\psi(x) = \frac{d}{dx}\ln\Gamma(x) = \frac{\Gamma'(x)}{\Gamma(x)}.$$

It is common knowledge in literature that the Gamma function satisfies the following properties.

$$\Gamma(n+1) = n!, \quad n \in Z^+,$$
$$\Gamma(x+1) = x\Gamma(x), \quad x \in R^+.$$

Also, let $\Gamma_q(x)$ be the q-deformed Gamma function (also known as the q-Gamma function or the q-analogue of the Gamma function) and $\psi_q(x)$ be the q-deformed Psi function defined for $q \in (0, 1)$ and $x \in R^+$ as (See [6], [7] and the references therein):

$$\Gamma_q(x) = (1-q)^{1-x}\prod_{n=1}^{\infty}\frac{1-q^n}{1-q^{x+n}} \quad \text{and} \quad \psi_q(x) = \frac{d}{dx}\ln\Gamma_q(x)$$

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with $\Gamma_q(x)$ satisfying the properties:

$$\Gamma_q(n+1) = [n]_q! \quad n \in Z^+, \tag{1}$$

$$\Gamma_q(x+1) = [x]_q\Gamma_q(x) \quad x \in R^+. \tag{2}$$

where $[x]_q = \frac{1-q^x}{1-q}$ and $[x+y]_q = [x]_q + q^x[y]_q$ for $x, y \in R^+$. See [2].

Similarly, let $\Gamma_{(q,k)}(x)$ be the (q,k)-deformed Gamma function and $\psi_{(q,k)}(x)$ be the (q,k)-deformed Psi function defined for $q \in (0,1)$, $k > 0$ and $x \in R^+$ as (See [2], [8], [10] and the references therein):

$$\Gamma_{(q,k)}(x) = \frac{(1-q^k)_{q,k}^{\frac{x}{k}-1}}{(1-q)^{\frac{x}{k}-1}} = \frac{(1-q^k)_{q,k}^{\infty}}{(1-q^x)_{q,k}^{\infty}.(1-q)^{\frac{x}{k}-1}} \quad \text{and} \quad \psi_{(q,k)}(x) = \frac{d}{dx}\ln\Gamma_{(q,k)}(x)$$

where $(x+y)_{q,k}^n = \prod_{j=0}^{n-1}(x+q^{jk}y)$ with $\Gamma_{(q,k)}(x)$ satisfying the following property:

$$\Gamma_{(q,k)}(x+k) = [x]_q\Gamma_{(q,k)}(x), \quad x \in R^+. \tag{3}$$

The q-addition (otherwise known as the q-analogue or q-deformation of the ordinary addition) can be defined in the following two ways:

The Nalli-Ward-Alsalam q-addition, $\oplus_q$ is defined (See [11], [1], [3]) as:

$$(a \oplus_q b)^n := \sum_{k=1}^{n}\binom{n}{k}_q a^k b^{n-k} \quad \text{for} \quad a, b \in R,\, n \in N. \tag{4}$$

where $\binom{n}{k}_q := \frac{[n]_q!}{[k]_q![n-k]_q!}$ is the q-binomial coefficient.

The Jackson-Hahn-Cigler q-addition, $\boxplus_q$ is defined (See [4], [5], [3]) as:

$$(a \boxplus_q b)^n := \sum_{k=1}^{n}\binom{n}{k}_q q^{\frac{k(k-1)}{2}} a^{n-k} b^k \quad \text{for} \quad a, b \in R,\, n \in N. \tag{5}$$

Notice that both $\oplus_q$ and $\boxplus_q$ reduce to the ordinary addition, $+$ when $q = 1$.

In a recent paper [9], the inequalities:

$$\frac{\Gamma(m+n+1)}{\Gamma(m+1)\Gamma(n+1)} < \frac{(m+n)^{m+n}}{m^m n^n}, \qquad m, n \in Z^+ \tag{6}$$

$$\frac{\Gamma(x+y+1)}{\Gamma(x+1)\Gamma(y+1)} \le \frac{(x+y)^{x+y}}{x^x y^y}, \qquad x, y \in R^+ \tag{7}$$

which occur in the study of probability theory were presented together with some other results. In this paper, the objective is to establish related inequalities for the q and (q,k) deformed Gamma functions. The results are presented in the following section.

2 Main Results

Theorem 2.1. Let $q \in (0,1)$ and $m, n \in Z^+$. Then, the inequality:

$$\frac{\Gamma_q(m+n+1)}{\Gamma_q(m+1)\Gamma_q(n+1)} \le \frac{(m \oplus_q n)^{m+n}}{m^m n^n} \tag{8}$$

holds true.

Proof. By equation (4) we obtain;

$$(m \oplus_q n)^{m+n} \geq \binom{m+n}{m}_q m^m n^n$$

since the binomial expansion of $(m \oplus_q n)^{m+n}$ includes the term $\binom{m+n}{m}_q m^m n^n$ as well as some other terms. That implies,

$$\frac{[m+n]_q!}{[m]_q![n]_q!} \leq \frac{(m \oplus_q n)^{m+n}}{m^m n^n}.$$

Now using relation (1) yields,

$$\frac{\Gamma_q(m+n+1)}{\Gamma_q(m+1)\Gamma_q(n+1)} \leq \frac{(m \oplus_q n)^{m+n}}{m^m n^n}$$

completing the proof.

□

Theorem 2.2. Let $q \in (0,1)$ and $m, n \in Z^+$. Then, the inequality:

$$\frac{\Gamma_q(m+n+1)}{\Gamma_q(m+1)\Gamma_q(n+1)} \leq \frac{(m \boxplus_q n)^{m+n} q^{\frac{n(1-n)}{2}}}{m^m n^n} \tag{9}$$

holds true.

Proof. Similarly, by equation (5) we obtain;

$$(m \boxplus_q n)^{m+n} \geq \binom{m+n}{n}_q q^{\frac{n(n-1)}{2}} m^m n^n.$$

Implying,

$$\frac{[m+n]_q!}{[m]_q![n]_q!} \leq \frac{(m \boxplus_q n)^{m+n} q^{\frac{n(1-n)}{2}}}{m^m n^n}.$$

By relation (1), we obtain;

$$\frac{\Gamma_q(m+n+1)}{\Gamma_q(m+1)\Gamma_q(n+1)} \leq \frac{(m \boxplus_q n)^{m+n} q^{\frac{n(1-n)}{2}}}{m^m n^n}$$

concluding the proof.

□

Lemma 2.3. If $q \in (0,1)$ and $x \in (0,1)$ then,

$$\ln(1-q^x) - \ln(1-q) < 0. \tag{10}$$

Proof. We have $q^x > q$ for all $q \in (0,1)$ and $x \in (0,1)$. That implies, $1 - q^x < 1 - q$. Taking the logarithm of both sides concludes the proof.

□

Theorem 2.4. Let $q \in (0,1)$ fixed, $x \in (0,1)$ and $y \in (0,1)$ be such that $\psi_q(x+1) > 0$. Then, the inequality:

$$\frac{\Gamma_q(x+y+1)}{\Gamma_q(x+1)\Gamma_q(y+1)} \geq \frac{[x+y]_q^{[x+y]_q}}{[x]_q^{[x]_q}[y]_q e^{q^x [y]_q}\Gamma_q(y)} \tag{11}$$

holds true.

Proof. Let Q and T be defined for $q \in (0,1)$ fixed, $x \in (0,1)$ and $y \in (0,1)$ by,

$$Q(x) = \frac{e^{[x]_q}\Gamma_q(x+1)}{[x]_q^{[x]_q}} \quad \text{and} \quad T(x,y) = \frac{Q(x+y)}{Q(x)Q(y)}.$$

Let $\mu(x) = \ln Q(x)$. That is,

$$\begin{aligned}\mu(x) &= [x]_q + \ln\Gamma_q(x+1) - [x]_q \ln[x]_q. \quad \text{Then,}\\ \mu(x)' &= \psi_q(x+1) + (\ln q)\frac{q^x}{1-q}\ln[x]_q\\ &= \psi_q(x+1) + (\ln q)\frac{q^x}{1-q}\left(\ln(1-q^x) - \ln(1-q)\right) > 0\end{aligned}$$

This is as a result of Lemma 2.3 and the fact that $\ln q < 0$ for $q \in (0,1)$. Hence $Q(x)$ is increasing.

Next, we have,

$$T(x,y) = \frac{Q(x+y)}{Q(x)Q(y)} = \frac{Q(x+y)}{Q(x)}.\frac{1}{Q(y)} \geq \frac{1}{Q(y)} = \frac{[y]_q^{[y]_q}}{e^{[y]_q}[y]_q\Gamma_q(y)}$$

since $Q(x)$ is increasing and $\Gamma_q(y+1) = [y]_q\Gamma(y)$. That implies,

$$\begin{aligned}T(x,y) &= \frac{[x]_q^{[x]_q}[y]_q^{[y]_q}}{[x+y]_q^{[x+y]_q}}.\frac{e^{[x+y]_q}}{e^{[x]_q+[y]_q}}.\frac{\Gamma_q(x+y+1)}{\Gamma_q(x+1)\Gamma_q(y+1)}\\ &= \frac{[x]_q^{[x]_q}[y]_q^{[y]_q}}{[x+y]_q^{[x+y]_q}}.\frac{e^{[x]_q+q^x[y]_q}}{e^{[x]_q+[y]_q}}.\frac{\Gamma_q(x+y+1)}{\Gamma_q(x+1)\Gamma_q(y+1)} \geq \frac{[y]_q^{[y]_q}}{e^{[y]_q}[y]_q\Gamma_q(y)}\end{aligned}$$

yielding the results as in (11). □

Remark 2.5. Let $B_q(x,y) = \frac{\Gamma_q(x)\Gamma_q(y)}{\Gamma_q(x+y)}$ be the q-deformation of the classical Beta function. Then, inequality (11) can be rearranged as follows.

$$B_q(x,y) \leq \frac{[x]_q^{[x]_q-1}e^{q^x[y]_q}\Gamma_q(y)}{[x+y]_q^{[x+y]_q-1}}.$$

Theorem 2.6. Let $q \in (0,1)$ fixed, $k > 0$ and $x \in (0,1)$ be such that $\psi_{(q,k)}(x+k) > 0$. Then, the inequality:

$$\frac{\Gamma_{(q,k)}(x+y+k)}{\Gamma_{(q,k)}(x+k)\Gamma_{(q,k)}(y+k)} \geq \frac{[x+y]_q^{[x+y]_q}}{[x]_q^{[x]_q}[y]_q e^{q^x[y]_q}\Gamma_{(q,k)}(y)} \tag{12}$$

is valid.

Proof. Let G and H be defined for $q \in (0,1)$ fixed, $k > 0$, $x \in (0,1)$ and $y \in (0,1)$ by,

$$G(x) = \frac{e^{[x]_q}\Gamma_{(q,k)}(x+k)}{[x]_q^{[x]_q}} \quad \text{and} \quad H(x,y) = \frac{G(x+y)}{G(x)G(y)}.$$

In a similar fashion, let $\lambda(x) = \ln G(x)$. That is,

$$\begin{aligned}\lambda(x) &= [x]_q + \ln\Gamma_{(q,k)}(x+k) - [x]_q \ln[x]_q. \quad \text{Then,}\\ \lambda(x)' &= \psi_{(q,k)}(x+k) + (\ln q)\frac{q^x}{1-q}\left(\ln(1-q^x) - \ln(1-q)\right) > 0.\end{aligned}$$

Hence $G(x)$ is increasing.

Next, observe that,

$$H(x,y)=\frac{G(x+y)}{G(x)G(y)}=\frac{G(x+y)}{G(x)}.\frac{1}{G(y)}\geq\frac{1}{G(y)}=\frac{[y]_q^{[y]_q}}{e^{[y]_q}[y]_q\Gamma_{(q,k)}(y)}$$

since $G(x)$ is increasing and $\Gamma_{(q,k)}(y+k)=[y]_q\Gamma_{(q,k)}(y)$. That implies,

$$H(x,y)=\frac{[x]_q^{[x]_q}[y]_q^{[y]_q}}{[x+y]_q^{[x+y]_q}}.\frac{e^{[x]_q+q^x[y]_q}}{e^{[x]_q+[y]_q}}.\frac{\Gamma_{(q,k)}(x+y+k)}{\Gamma_{(q,k)}(x+k)\Gamma_{(q,k)}(y+k)}\geq\frac{[y]_q^{[y]_q}}{e^{[y]_q}[y]_q\Gamma_{(q,k)}(y)}$$

establishing the results as in (12). □

Remark 2.7. Let $B_{(q,k)}(x,y)=\frac{\Gamma_{(q,k)}(x)\Gamma_{(q,k)}(y)}{\Gamma_{(q,k)}(x+y)}$ be the (q,k)-deformation of the classical Beta function. Then, inequality (12) can be written as follows.

$$B_{(q,k)}(x,y)\leq\frac{[x]_q^{[x]_q-1}e^{q^x[y]_q}\Gamma_{(q,k)}(y)}{[x+y]_q^{[x+y]_q-1}}.$$

3 Concluding Remarks

Some new inequalities related to (6) and (7) have been established for the q and (q,k) deformed Gamma functions. In particular, if we allow $q\to1$ in either inequality (8) or (9), then, inequality (6) is restored as a special case. Also, by allowing $q\to1$ in (12), then we obtain the k-analogue of inequality (11).

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HOMOMORPHISM IN ROUGH LATTICE

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Abstract – In this paper, we introduce the concept of set-valued homomorphism of a lattice which is the generalization of ordinary lattice homomorphism. We construct generalized rough lower and upper approximations operators, by means of a set-valued mapping, which are also the generalized form of lower and upper approximations of a lattice, and the corresponding properties are investigated and finally we cite an example to show usefulness of the paper.

Keywords – *Rough Set, Lower and Upper approximations, Lattice, Equivalence Relation, Homomorphism.*

1 Introduction

In this section, we give some basic notions and results about generalized rough sets and lattices. The concept of rough sets was introduced by Pawlak [13], a mathematical tool for dealing with uncertainty or vagueness ([14],[20]). In rough set theory, rough set can be described by a pair of ordinary sets called lower and upper approximations. The theory of rough set is an extension of set theory. The study of the algebraic structure of the mathematical theory proves itself effective in making the applications more efficient. Such researches may not only provide more insight into rough set theory, but also hopefully developed methods for applications. Rough set has been studied from algebraic view points by many researchers. Pomykala [15] showed that the set of rough set forms a stone algebra. Iwinski [4] suggested a lattice theoretic approach to the rough set. Liu and Zhu [8] presented the structures of the approximations based on arbitrary binary relation. The generalized rough sets over fuzzy lattices have been explored by Liu [7]. Algebraic structure of T-rough set and corresponding lattice theory are explored in ([5], [2]) respectively. In mathematics, a lattice is a partially ordered set in which any two elements have a unique supremum and infimum. Lattice can also be characterized as algebraic structures satisfying certain axiomatic identities. Since the two definitions are equivalent, lattice theory draws on both order theory and universal algebra. In this paper, we consider a lattice as a universal set and study the rough sets in a lattice.

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2 Preliminaries

Definition 2.1. Let U and V be two non-empty universes. Let f be a set-valued mapping given by $f : U \longrightarrow P(V)$. Then the triple (U, V, f) is referred to as a generalized approximation space or generalized rough set. Any set-valued function from U to $P(V)$ defines a binary relation from U to V by setting $R_f = \{(x, y) : y \in f(x)\}$. Obviously, if R is an arbitrary relation from U to V, then it can be defined a set-valued mapping $f_R : U \longrightarrow P(V)$ by $f_R(x) = \{y \in V : (x, y) \in R\}$ where $x \in U$. For any set $A \subseteq V$, the lower and upper rough approximations $A \downarrow f$ and $A \uparrow f$, are defined by

$$A \downarrow_f \;=\; \{x \in U : f(x) \subseteq X\} \quad (1)$$
$$and \quad A \uparrow_f \;=\; \{x \in U : f(x) \cap A \neq \emptyset\}. \quad (2)$$

The pair $(A \downarrow_f, A \uparrow_f)$ is referred to as a generalized rough set. If a subset $A \subseteq V$satisfies that $A \downarrow_f = A \uparrow_f$, then A is called a definable set of (U, V, f). We denote all the definable sets of (U, V, f) by $Def(f)$.

Definition 2.2. Let L be a lattice and $\emptyset \neq A \subseteq L$. Then A is called a sublattice if $a, b \in A$ implies $a \vee b \in A$ and $a \wedge b \in A$.

Definition 2.3. Let L and K be two complete lattices. A mapping $f : L \longrightarrow P(K)$ is called a complete set-valued homomorphism if
$\bigvee_{i\in I} f(a_i) \subseteq f(\bigvee_{i\in I} a_i)$ $\bigwedge_{i\in I} f(a_i) \subseteq f(\bigwedge_{i\in I} a_i)$.
A set-valued mapping f is called a strong complete set-valued homomorphism
if $\bigvee_{i\in I} f(a_i) = f(\bigvee_{i\in I} a_i)$ and $\bigwedge_{i\in I} f(a_i) = f(\bigwedge_{i\in I} a_i)$.

Definition 2.4. A non-empty sub-set A of L is called a sublattice of L if $a \vee b \in A, a \wedge b \in A$ for all $a, b \in A$

Definition 2.5. A non-empty sub-set A of L is called a convex sublattice of L if $[a \vee b, a \wedge b] \subseteq A$ for any $a, b \in A$.

Definition 2.6. An equivalence relation E on L is a reflexive, symmetric, and transitive binary relation on L. If E is an equivalence relation on L then the equivalence class of $x \in L$ is the set $\{y \in L : (x, y) \in E\}$. We write it as $[x]_E$.

Let us illustrate this definition using the following example.
Example (1) Let $L = \{a, b, c, d\}$ such that $c \vee b = b, c \vee d = d, b \vee d = a$ and $K = \{u, v, x, y, z\}$ such that $z \vee x = x, y \vee z = y, x \vee y = v, u \vee v = u$. Consider the set-valued mapping $f : L \longrightarrow P(K)$ defined by $f(a) = \{u, v\}, f(b) = \{v, x\}, f(c) = y, f(d) = \{y, z\}$. Then f is set valued homomorphism but not a strong set valued homomorphism for $f(b) \vee f(c) = v \neq f(b \vee c) = f(a) = \{u, v\}$.
(2) Let g be a lattice homomorphism from L to K. Then the set valued mapping $f : L \longrightarrow P(K)$ defined by $f(a) = \{g(a) : a \in L\}$ is a strong set valued homomorphism. If L and K are complete, then f is a strong complete set valued homomorphism.

3 Rough Lattice and Set-valued Homomorphism

Rough set was originally proposed by Pawlak [13] with the consideration of an equivalence relation. An equivalence relation is sometimes difficult to be obtained in real-world problems due to vagueness and incompleteness of human knowledge. From this point of view, in this section, we introduce the concept of set-valued isomorphism of lattices. Let L and K be two lattices. A mapping f is a set-valued mapping from L to $P(K)$. Where $P(K)$ represents the set of all non-empty sub set of K. For $a, b \in L$, we define $f(a) \vee f(b) = \{x \vee y : x \in f(a), y \in f(b)\}$.
$f(a) \wedge f(b) = \{x \wedge y : x \in f(a), y \in f(b)\}$.

Definition 3.1. Let L and K be two lattices. A mapping $f : L \longrightarrow P(K)$ is called a set-valued homomorphism if
$f(a) \bigvee f(b) \subseteq f(a \bigvee b)$
$f(a) \bigwedge f(b) \subseteq f(a \bigwedge b)$ for all $a, b \in L$.
A set-valued mapping f is called a strong set-valued homomorphism if
$f(a) \bigvee f(b) = f(a \bigvee b)$
$f(a) \bigwedge f(b) = f(a \bigwedge b)$ for all $a, b \in L$.

Theorem 3.2. Let L, K be lattices and $f : L \longrightarrow P(K)$ be a strong set valued homomorphism. If A, B are two non empty subset of K, then $(1) f \downarrow (A) \vee f \downarrow (B) \subseteq f \downarrow (A \vee B)$
$(2) f \downarrow (A) \wedge f \downarrow (B) \subseteq f \downarrow (A \wedge B)$

Proof: (1) Assume that $x \in f \downarrow (A) \vee f \downarrow (B)$, then there exist $y \in f \downarrow (A), z \in f \downarrow (B)$ such that $x = y \vee z$. Since f is a strong set-valued homomorphism, we have $f(x) = f(y \vee z) = f(y) \vee f(z) \subseteq A \vee B$. So $x \in f \downarrow (A \vee B)$.
(2) Again we assume that $x \in f \downarrow (A) \wedge f \downarrow (B)$, then there exist $y \in f \downarrow (A), z \in f \downarrow (B)$ such that $x = y \wedge z$. Since f is a strong set-valued homomorphism, we have $f(x) = f(y \wedge z) = f(y) \wedge f(z) \subseteq A \wedge B$. So $x \in f \downarrow (A \wedge B)$.

Theorem 3.3. Let L, K be two lattices. Then (1) Let $f : L \longrightarrow P(K)$ be a set-valued homomorphism. If A is a sublattice of K and $f \uparrow (A)$ is non-empty subset of L, then $f \uparrow (A)$ is a sublattice of L
(2) Let $f : L \longrightarrow P(K)$ be a strong set-valued homomorphism. If A is a sublattice of K and $f \downarrow (A)$ is non-empty subset of L, then $f \downarrow (A)$ is a sublattice of L.

Proof: (1) Assume that $x, y \in f \uparrow (A)$ there exist $a, b \in A$ such that $a \in f(x), b \in f(y)$. Since f is a set-valued homomorphism and A is a sublattice, we have $a \vee b \in f(x) \vee f(y) \subseteq f(x \vee y)$ and $a \vee b \in A$. So $a \vee b \in f(x \vee y) \cap A$ which implies that $x \vee y \in f \uparrow (A)$. Similarly, we have $x \wedge y \in f \uparrow (A)$
(2) Assume that $x, y \in f \downarrow (A)$, we have $f(x) \subseteq A, f(y) \subseteq A$. Since f is a strong set-valued homomorphism and A is a sublattice, we have $f(x \wedge y) = f(x) \wedge f(y)$ which implies that $x \vee y \in f \downarrow (A)$. Similarly, we have $x \wedge y \in f \downarrow (A)$.

Theorem 3.4. Let $f : L \longrightarrow P(K)$ be a set-valued homomorphism of lattices. If A, B are down-sets of K, then $f(A \cap B) = fA) \cap f(B)$.

Proof: Assume that $x \in f \uparrow (A) \cap f \uparrow (B)$, there exist $y \in A, z \in B$ such that $y.z \in f(x)$. Since A, B are down-sets, we have $y \wedge z \in A \cap B$. f is a set-valued homomorphism, we have $y \wedge z \in f(x) \cap f(x) \subseteq f(x \wedge x) = f(x)$. So $y \wedge z \in f(x) \cap (A \cap B)$ which implies $x \in f(A \cap B)$. We also know that $f \uparrow (A \cap B) \subseteq f \uparrow (A) \cap f \uparrow (B)$. Thus, we get the conclusion easily.

Definition 3.5. [5] Let E be an equivalence relation on L, then E is called a full congruence relation if $(a, b) \in E$ and $(c, d) \in E \Leftrightarrow (a \vee c, b \vee d) \in E$ and $(a \wedge c, b \wedge d) \in E$ for all $a, b, c, d \in L$.

Definition 3.6. Let E be an equivalence relation on L, then $(a, b) \in E \Leftrightarrow (a \vee x, b \vee x), (a \wedge x, b \wedge x) \in E$ for all $x \in L$. If A, B are non empty subset of L, for any $a \in L$ we define $A \vee B = \{x \vee y : x \in A, y \in B\}$
$a \wedge A = \{x \wedge y : x \in A\}$
$A \wedge B = \{x \wedge y : x \in A, y \in B\}$.

Theorem 3.7. Let E be an equivalence relation on $L$and if $a, b \in L$, then
(1) $[a]_E \vee [b]_E \subseteq [a \vee b]_E$
(2) $[a]_E \wedge [b]_E \subseteq [a \wedge b]_E$

Proof: Suppose $z \in [a]_E \vee [b]_E$ then there exist $x \in [a]_E$ and $y \in [b]_E$ such that $z = x \vee y$. Since $(a, x) \in E$ and $(b, y) \in E$, we have $(a \vee b, x \vee y) \in E$, namely $(a \vee b, z) \in E$, so $z \in [a \vee b]_E$
Again, suppose $z \in [a]_E \wedge [b]_E$ then there exist $x \in [a]_E$ and $y \in [b]_E$ such that $z = x \wedge y$. Since $(a, x) \in E$ and $(b, y) \in E$, we have $(a \wedge b, x \wedge y) \in E$, namely $(a \wedge b, z) \in E$, so $z \in [a \wedge b]_E$.

Definition 3.8. [5] Let E be a full congruence relation, then E is called a complete full congruence relation if $[a]_E \vee [b]_E = [a \vee b]_E$ and $[a]_E \wedge [b]_E = [a \wedge b]_E$ for all $a, b \in L$.

Theorem 3.9. Let E be a complete equivalence relation on L, if A, B are non- empty subsets of L, then $E \uparrow (A) \vee E \uparrow (B) \subseteq E \uparrow (A \vee B)$ and $E \uparrow (A) \wedge E \uparrow (B) \subseteq E \uparrow (A \wedge B)$

Proof: Let us suppose $z \in E \uparrow (A) \vee E \uparrow (B)$, then there exist $x \in E \uparrow (A)$ and $y \in E \uparrow (B)$ such that $z = x \vee y$. Then $[x]_E \cap A \neq \emptyset$ and $[y]_E \cap B \neq \emptyset$, so there exist $a \in [x]_E, a \in A$ and $b \in [y]_E, b \in B$. Then $a \vee b \in [x]_E \vee [y]_E \subseteq [x \vee y]_E = [z]_E$ and $a \vee b \in A \vee B$. Thus $[z]_E \cap (A \vee B) \neq \emptyset$ and hence $z \in E \uparrow (A \vee B)$.
Next suppose $z \in E \uparrow (A) \wedge E \uparrow (B)$, then there exist $x \in E \uparrow (A)$ and $y \in E \uparrow (B)$ such that $z = x \wedge y$. Then $[x]_E \cap A \neq \emptyset$ and $[y]_E \cap B \neq \emptyset$, so there exist $a \in [x]_E, a \in A$ and $b \in [y]_E, b \in b$. Then $a \wedge b \in [x]_E \wedge [y]_E \subseteq [x \wedge y]_E = [z]_E$ and $a \wedge b \in A \wedge B$. Thus $[z]_E \cap (A \wedge B) \neq \emptyset$ and hence $z \in E \uparrow (A \wedge B)$.

Theorem 3.10. Let E be a complete equivalence relation on L, if A, B are non empty subsets of L, then $E \downarrow (A) \vee E \downarrow (B) \subseteq E \uparrow (A \vee B)$ and
$E \downarrow (A) \wedge E \downarrow (B) \subseteq E \downarrow (A \wedge B)$

Proof: Suppose $z \in E \downarrow (A) \vee E \downarrow (B)$, then there exist $x \in E \downarrow (A)$ and $y \in E \downarrow (B)$ such that $z = x \vee y$. Then $[x]_E \subseteq$ and $[y]_E \subseteq B$ then $[x]_E \vee [y]_E \subseteq A \vee B$. Since E is a full complete equivalence relation on L, we have $[x]_E \vee [y]_E = [x \vee y]_E = [z]_E$, namely $[z]_E \subseteq A \vee B$. Hence $z \in f \downarrow (A \vee B)$.
Again suppose $z \in E \downarrow (A) \wedge E \downarrow (B)$, then there exist $x \in E \downarrow (A)$ and $y \in E \downarrow (B)$ such that $z = x \wedge y$. Then $[x]_E \subseteq$ and $[y]_E \subseteq B$ then $[x]_E \wedge [y]_E \subseteq A \wedge B$. Since E is a complete equivalence relation on L, we have $[x]_E \wedge [y]_E = [x] \wedge y]_E = [z]_E$, namely $[z]_E \subseteq A \wedge B$. Hence $z \in f \downarrow (A \wedge B)$.

Theorem 3.11. Suppose E_1, E_2 are two complete equivalence relations on L, A is a non empty subset of L, then $(E_1 \cap E_2) \uparrow (A) \subseteq E_1 \uparrow (A) \cap E_2 \uparrow (A)$

Proof: Let us suppose that $x \in (E_1 \cap E_2) \uparrow (A)$, then $[x]_{E_1 \cap E_2} \neq \emptyset$. So there exists $a \in [x]_{E_1 \cap E_2} \cap A$. Since $(a, x) \in E_1 \cap E_2$, then $(a, x) \in E_1$ and $(a, x) \in E_2$. Thus $a \in [x]_{E_1}$ and $a \in [x]_{E_2}$. Then $[x]_{E_1} \cap A \neq \emptyset$ and $[x]_{E_2} \cap A \neq \emptyset$. Therefore, $x \in E_1 \uparrow (A)$ and $x \in E_2 \uparrow (A)$. Hence $x \in E_1 \uparrow (A) \cap E_2 \uparrow (A)$. Thus $(E_1 \cap E_2) \uparrow (A) \subseteq E_1 \uparrow (A) \cap E_2 \uparrow (A)$.

Definition 3.12. Let L be a complete lattice and let $k \in L$. k is said to be compact if, for every subset S of $L, k \leq \bigvee S \Rightarrow k \leq \bigvee T$ for some finite subset T of S. The set of all compact elements of L is denoted $K(L)$. A complete lattice L is said to be algebraic if, for each $a \in L$; $a = \bigvee \{k \in k(L) : k \leq a\}$.

Definition 3.13. Let $(L, \leq)$ and $(K, \leq)$ be two lattices and $A \in P(K)$ where $P(K)$ denotes the set of all non-empty subsets of K. Let $f : L \to P(K)$ be a set-valued mapping. The lower and upper approximations of A under f are defined by $f \downarrow (A) = \{x \in L : f(x) \subseteq A\} f \uparrow (A) = \{x \in L : f(x) \cap A \neq \emptyset\}$.

Definition 3.14. The pair $(f \downarrow (A), f \uparrow (A))$ is referred to as the generalized rough set with respect to A, induced by f or f- rough set with respect to A.

Example Let (L, E) be an approximation space and $f : L \to P(L)$ be a set-valued mapping where $f(x) = [x]_E$ for all $x \in L$, then for any $A \subseteq L, f \downarrow (A)$ and $f \uparrow (A)$ are lower and upper approximations respectively.

Proposition 3.15. Let L and K be two lattices and $A, B \in P(K)$. Let $f : L \to P(K)$ be a set-valued mapping. Then the following assertions hold:
$(i) f \uparrow (A \cup B) = f \uparrow (A) \cup f \uparrow (B)$;
$(ii) f \downarrow (A \cap B) = f \downarrow (A) \cap f \downarrow (B)$;
$(iii) A = B$ implies $f \downarrow (A) = f \downarrow (B)$ and $f \uparrow (A) = f \uparrow (B)$;
$(iv) f \downarrow (A) \cup f \downarrow (B) = f \downarrow (A \cup B)$; $(v) f \uparrow (A \cap B) = f \uparrow (A) \cap f \uparrow (B)$.

Definition 3.16. A non-empty subset K of L is a sublattice of the lattice $(L, \bigwedge, \bigvee)$ if $a \vee b, a \wedge b \in K$ for all $a, b \in K$.

Definition 3.17. If A and B are non-empty subsets of L, we define $A \bigwedge B$ and $A \bigvee B$ as follows: $A \bigwedge B = \{a \wedge b : a \in A, b \in B\}; A \bigvee B = \{a \vee b : a \in A, b \in B\}$.

Definition 3.18. Let L and K be two lattices and $f : L \to P(K)$ be a set- valued mapping. f is called a set-valued homomorphism if $(i)f(x \wedge y) = f(x) \bigwedge f(y); (ii)f(x \vee y) = f(x) \bigvee f(y)$, for all $x, y \in L$.

Lemma 3.19. Let L and K be two lattices and $f : L \to P(K)$ be a set-valued homomorphism. If S is a sublattice of K and $f \downarrow (S) \neq \emptyset$, and $f \uparrow (S) \neq \emptyset$, then $f \downarrow (S)$ and $f \uparrow (S)$ are sublattices of L.

Proof: Let $x, y \in f \downarrow (S)$, by definition we have $f(x), f(y) \subseteq S$. Since S is a sublattice of K, we have $f(x \vee y) = f(x) \bigvee f(y) \subseteq S$ and $f(x \wedge y) = f(x) \bigwedge f(y) = S$. It shows that $x \vee y; x \wedge y \in f \downarrow (S)$. Moreover, let $x, y \in f \uparrow (S)$, by definition, $f(x) \bigcap S \neq \emptyset$ and $f(y) \bigcap S \neq \emptyset$. Suppose $a \in f(x) \cap S$ and $b \in f(y) \cap S$. Since S is a sublattice of K, we have $a \vee b \in S$ and $a \wedge b \in f(x) \bigvee f(y) = f(x \vee y)$. It implies that $a \vee b \in f(x \vee y) \bigcap S$. Hence $f(x \vee y) \bigcap S \neq \emptyset$. It means that $x \vee y \in \uparrow (S)$. Again, $a \wedge b \in S$ and $a \wedge b \in f(x) \bigvee f(y)$. So that $a \wedge b \in f(x \wedge y) f \cap S$. Therefore $f(x \wedge y) \bigcap S \neq \emptyset$. It means that $x \wedge y \in f \uparrow (S)$.

Lemma 3.20. Let L and K be two lattices and $f : L \to P(K)$ be a set- valued homomorphism. If S is a sublattice of K and $f \downarrow (S) \neq \emptyset \neq f \uparrow (S)$, then $(f \downarrow (S), f \uparrow (S))$ is a rough sublattice of L.

Proposition 3.21. Let L and K be two lattices and $f : L \to P(K)$ be a set-valued homomorphism. If A, B be non-empty subsets of K , then $(1) f \downarrow (A) \bigvee f \downarrow (B) \subseteq f \downarrow (A \bigvee B); (2) f \downarrow (A) \bigwedge f \downarrow (B) \subseteq f \downarrow (A \bigwedge B)$.

Proof: Suppose z be any element of $f \downarrow (A) \bigvee f \downarrow (B)$. Then $z = a \vee b$ for some $a \in f \downarrow (A)$ and $b \in f \downarrow (B)$. By definition, $f(a) \subseteq A$ and $f(b) \subseteq B$. Since $f(a \vee b) = f(a) \bigvee f(b) = \{x \vee y : x \in f(a), y \in f(b)\} \subseteq \{x \vee y : x \in A, y \in B\} = A \bigvee B$, we imply that $a \vee b \in f \downarrow (A \bigvee B)$ and so $z \in f \downarrow (A \bigvee B)$. The proof of (2) is similar to the proof of (1).
The following examples show that the converse of above proposition is not true.
Example (1): Let $L = \{x_0, x_1, x_2, \ldots, x_7\}$, where $x_0 < x_1 < x_2 < x_3 < x_4 < x_5 < x_6 < x_7$. Let $a \vee b = max\{a, b\}$ and $a \wedge b = min\{a, b\}$ for all $a, b \in L$. Then $(L, \vee, \wedge)$ is a lattice. If we consider equivalence classes $[x_0] = \{x_0, x_1, x_2\}, [x_3] = \{x_3, x_4\}, [x_5] = \{x_5, x_6, x_7\}$ and $f : L \to P(L)$ be a set-valued homomorphism with $f(x) = [x]$ for all $x \in L$. Let $A = \{x_3, x_4, x_5, x_7\}, B = \{x_0, x_1, x_2, x_3, x_6\}$. Then $A \bigvee B = \{x_3, x_4, x_5, x_6, x_7\}, f \downarrow (A \bigvee B) = \{x_3, x_4, x_5, x_6, x_7\}, f \downarrow (A) = \{x_3, x_4\}, f \downarrow (B) = \{x_0, x_1, x_2\}$ and $f \downarrow (A) \bigvee f \downarrow (B) = \{x_3, x_4\}$. And so $f \downarrow (A \bigvee B) \not\subseteq f \downarrow (A) \bigvee f \downarrow (B)$.
(2): Let $L = [0, 1]$ and $f : L \to P(L)$ be a set-valued homomorphism with $f(x) = [0, x]$ for all $x \in L$. And let $A = \{0, \frac{1}{2}\}, B = \{\frac{1}{3}, \frac{1}{2}\}$. Then $f \downarrow (A) = \{0\}, f \downarrow (B) = \emptyset, f \downarrow (A \bigwedge B) = \{0\}, f \downarrow (A) \bigwedge f \downarrow (B) = \emptyset$. Therefore $f \downarrow (A \bigwedge B) \not\subseteq f \downarrow (A) \bigwedge f \downarrow (B)$.

Proposition 3.22. [3] Let L and K be two lattices and $f : L \to P(K)$ be a set-valued homomorphism. If A, B be non-empty subsets of K, then
(1) $f \uparrow (A) \bigvee f \uparrow (B) \subseteq f \uparrow (A \bigvee B)$; (2)$f \uparrow (A) \bigwedge f \uparrow (B) \subseteq f \uparrow (A \bigwedge B)$.

Proof: (1) Let us suppose that $z \in f \uparrow (A) \bigvee f \uparrow (B)$. Then $z = a \vee b$ for some $a \in f \uparrow (A)$ and $b \in f \uparrow (B)$. Hence, $f(a) \bigcap A \neq \emptyset$ and $f(b) \bigcap B \neq \emptyset$ and so there exist $x \in f(a) \bigcap A$ and $y \in f(b) \bigcap B$. Therefore, $x \vee y \in A \bigvee B$ and $x \vee y \in f(a) \bigvee f(b) = f(a \vee b)$. Thus $x \vee y \in f(a \vee b) \bigcap (A \bigvee B)$ which implies that $f(a \vee b) \bigcap f(A \bigvee B) \neq \emptyset$. So $z = a \vee b \in f \uparrow (A \bigvee B)$.
(2). The proof is obvious as that of (1).
The following examples show that the converse of above proposition is not true.
Example (i) Let $L = \{0, x_1, x_2, x_3, x_4, x_5, 1\}$ be the lattice and $f : L \to P(L)$ be a set-valued homomorphism with $f(x) = \{x_5\}$ for all $x \in L$. And let $A = \{x_2, x_3\}, B = \{x_1, x_2\}$. Then $f \uparrow (A) =$

$\emptyset, f \uparrow (B) = \emptyset, f \uparrow (A \bigvee B) = L, f \uparrow (A) \bigvee f \uparrow (B) = \emptyset$. Therefore $f \uparrow (A \bigvee B) \subseteq f \uparrow (A) \bigvee f \uparrow (B)$.

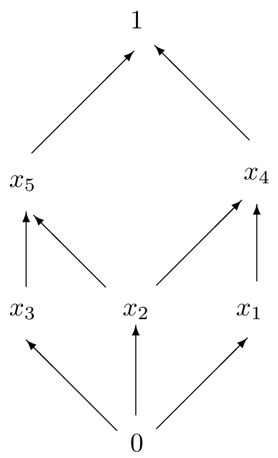

Figure-1: Lattice Structure of L

(ii) Let L be the above lattice and $f : L \rightarrow P(L)$ be a set-valued homomorphism with $f(x) = \{x_2\}$ for all $x \in L$. And let $A = \{x_4\}, B = \{x_5\}$. Then $f \uparrow (A) = \emptyset, f \uparrow (B) = \emptyset, f \uparrow (A \bigwedge B) = L, f \uparrow (A) V f \uparrow (B) = \emptyset$. Therefore $f \uparrow (A \bigwedge B) \nsubseteq f \uparrow (A) \bigwedge f \uparrow (B)$.

4 Conclusion

The paper is devoted to the application of rough lattice determined by Pawlak's Information System in which the concept of upper and lower approximations of a subset in a lattice are considered and studied their algebraic properties. By some indiscernibility relation, we have shown that the entries of the indiscernibility relation of an information system forms a rough lattice. Some properties of set valued homomorphism are obtained which shall be very practical in the theory and application of rough lattice.

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MAPPINGS ON NEUTROSOPHIC SOFT EXPERT SETS

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Abstract - In this paper we introduced mapping on neutrosophic soft expert sets through which we can study the images and inverse images of neutrosophic soft expert sets. Further, we investigated the basic operations and other related properties of mapping on neutrosophic soft expert sets in this paper.

***Keywords* -** Neutrosophic soft expert set, neutrosophic soft expert images, neutrosophic soft expert inverse images, mapping on neutrosophic soft expert set.

1. Introduction

Neutrosophy has been introduced by Smarandache [14, 15, 16] as a new branch of philosophy. Smarandache using this philosophy of neutrosophy to initiate neutrosophic sets and logics which is the generalization of fuzzy logic, intuitionistic fuzzy logic, paraconsistent logic etc. Fuzzy sets [42] and intuitionistic fuzzy sets [36] are characterized by membership functions, membership and non-membership functions, respectively. In some real life problems for proper description of an object in uncertain and ambiguous environment, we need to handle the indeterminate and incomplete information. Fuzzy sets and intuitionistic fuzzy sets are not able to handle the indeterminate and inconsistent information. Thus neutrosophic set (NS in short) is defined by Samarandache [15], as a new mathematical tool for dealing with problems involving incomplete, indeterminacy, inconsistent knowledge. In NS, the indeterminacy is quantified explicitly and truth-membership, indeterminacy membership, and false-membership are completely independent. From scientific or engineering point of view, the neutrosophic set and set-theoretic view, operators need to be defined. Otherwise, it will be difficult to apply in the

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real applications. Therefore, H. Wang et al [19] defined a single valued neutrosophic set (SVNS) and then provided the set theoretic operations and various properties of single valued neutrosophic sets. Recent research works on neutrosophic set theory and its applications in various fields are progressing rapidly. A lot of literature can be found in this regard in [3, 6, 7, 8, 9, 10, 11, 12, 13, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 61, 62,70, 73, 76, 80, 83,84, 85, 86].

In other hand, Molodtsov [12] initiated the theory of soft set as a general mathematical tool for dealing with uncertainty and vagueness and how soft set theory is free from the parameterization inadequacy syndrome of fuzzy set theory, rough set theory, probability theory. A soft set is a collection of approximate descriptions of an object. Later Maji et al.[58] defined several operations on soft set. Many authors [37, 41, 44, 47, 49, 50, 51, 52, 53, 54, 55, 56, 57, 60] have combined soft sets with other sets to generate hybrid structures like fuzzy soft sets, generalized fuzzy soft sets, rough soft sets, intuitionistic fuzzy soft sets, possibility fuzzy soft sets, generalized intuitionistic fuzzy softs, possibility vague soft sets and so on. All these research aim to solve most of our real life problems in medical sciences, engineering, management, environment and social sciences which involve data that are not crisp and precise. But most of these models deal with only one opinion (or) with only one expert. This causes a problem with the user when questionnaires are used for the data collection. Alkhazaleh and Salleh in 2011 [65] defined the concept of soft expert set and created a model in which the user can know the opinion of the experts in the model without any operations and give an application of this concept in decision making problem. Also, they introduced the concept of the fuzzy soft expert set [64] as a combination between the soft expert set and the fuzzy set. Based on [15], Maji [53] introduced the concept of neutrosophic soft set a more generalized concept, which is a combination of neutrosophic set and soft set and studied its properties. Various kinds of extended neutrosophic soft sets such as intuitionistic neutrosophic soft set [68, 70, 79], generalized neutrosophic soft set [61, 62], interval valued neutrosophic soft set [23], neutrosophic parameterized fuzzy soft set [72], Generalized interval valued neutrosophic soft sets [75], neutrosophic soft relation [20, 21], neutrosophic soft multiset theory [24] and cyclic fuzzy neutrosophic soft group [61] were studied. The combination of neutrosophic soft sets and rough sets [77, 81, 82] is another interesting topic.

Recently, Broumi and Smaranadache [88] introduced, a more generalized concept, the concept of the intuitionistic fuzzy soft expert set as a combination between the soft expert set and the intuitionistic fuzzy set. The same authors defined the concept of single valued neutrosophic soft expert set [87] and gave the application in decision making problem. The concept of single valued neutrosophic soft expert set deals with indeterminate and inconsistent data. Also, Sahin et al. [91] presented the concept of neutrosophic soft expert sets. The soft expert models are richer than soft set models since the soft set models are created with the help of one expert where as the soft expert models are made with the opinions of all experts. Later on, many researchers have worked with the concept of soft expert sets and their hybrid structures [1, 2, 17, 18, 24, 38, 39, 46, 48, 87, 91, 92].

The notion of mapping on soft classes are introduced by Kharal and Ahmad [4]. The same authors presented the concept of a mapping on classes of fuzzy soft sets [5] and studied the properties of fuzzy soft images and fuzzy soft inverse images of fuzzy soft sets, and supported them with examples and counter inconsistency in examples. In neutrosophic environment, Alkazaleh et al [67] studied the notion of mapping on neutrosophic soft classes.

Until now, there is no study on mapping on the classes of neutrosophic soft expert sets, so there is a need to develop a new mathematical tool called “Mapping on neutrosophic soft expert set”.

In this paper, we introduce the notion of mapping on neutrosophic soft expert classes and study the properties of neutrosophic soft expert images and neutrosophic soft expert inverse images of neutrosophic soft expert sets. Finally, we give some illustrative examples of mapping on neutrosophic soft expert for intuition.

2. Preliminaries

In this section, we will briefly recall the basic concepts of neutrosophic sets, soft sets, neutrosophic soft sets, soft expert sets, fuzzy soft expert sets, intutionistic fuzzy soft expert sets and neutrosophic soft expert sets.

Let U be an initial universe set of objects and E is the set of parameters in relation to objects in U. Parameters are often attributes, characteristics or properties of objects. Let P (U) denote the power set of U and A ⊆ E.

2.1. Neutrosophic Set

Definition 2.1 [15] Let U be an universe of discourse, Then the neutrosophic set A is an object having the form $A = \{< x: T_A(x), I_A(x), F_A(x)>, x \in U\}$,where the functions $T_A(x)$, $I_A(x)$, $F_A(x) : U \to]^-0,1^+[$ define respectively the degree of membership, the degree of indeterminacy, and the degree of non-membership of the element $x \in X$ to the set A with the condition.

$$^-0 \leq sup T_A(x) + sup I_A(x) + sup F_A(x)) \leq 3^+.$$

From philosophical point of view, the neutrosophic set takes the value from real standard or non-standard subsets of $]^-0,1^+[$. So instead of $]^-0,1^+[$ we need to take the interval [0,1] for technical applications, because $]^-0,1^+[$ will be difficult to apply in the real applications such as in scientific and engineering problems.
For two NS,

$A_{NS} = \{<x, T_A(x), I_A(x), F_A(x)> \mid x \in X\}$

And

$B_{NS} = \{<x, T_B(x), I_B(x), F_B(x)> \mid x \in X\}$

We have,

1. $A_{NS} \subseteq B_{NS}$ if and only if

$T_A(x) \leq T_B(x), I_A(x) \geq I_B(x), F_A(x) \geq F_B(x)$.

2. $A_{NS} = B_{NS}$ if and only if ,

$T_A(x) = T_B(x)$, $I_A(x) = I_B(x)$, $F_A(x) = F_B(x)$ for all $x \in X$.

3. The complement of A_{NS} is denoted by A^o_{NS} and is defined by

A^o_{NS}= {<x, $F_A(x), 1 - T_A(x), T_A(x)$| x ∈ X }

4. A∩B = {<x, $\min\{T_A(x), T_B(x)\}$,$\max\{I_A(x), I_B(x)\}$, $\max\{F_A(x), F_B(x)\}$>:x ∈ X }

5. A∪B = {<x, $\max\{T_A(x), T(x)\}$,$\min\{I_A(x), I_B(x)\}$, $\min\{F_A(x), F_B(x)\}$>:x ∈ X }

As an illustration, let us consider the following example.

Example 2.2. Assume that the universe of discourse U={x_1, x_2, x_3, x_4}. It may be further assumed that the values of x_1, x_2, x_3 and x_4 are in [0, 1], then A is a neutrosophic set (NS) of U such that,

A= {< x_1,0.4, 0.6, 0.5 >,< x_2,0.3, 0.4, 0.7>, < x_3,0.4, 0.4, 0.6>,<x_4,0.5, 0.4, 0.8 >}

2.2. Soft Set

Definition 2.3 [12] Let U be an initial universe set and E be a set of parameters. Let P(U) denote the power set of U. Consider a nonempty set A, A ⊂ E. A pair (K, A) is called a soft set over U, where K is a mapping given by K : A → P(U).

As an illustration, let us consider the following example.

Example 2.4 Suppose that U is the set of houses under consideration, say U = {h_1, h_2, . . ., h_5}. Let E be the set of some attributes of such houses, say E = {e_1, e_2, . . ., e_8}, where e_1, e_2, . . ., e_8 stand for the attributes “beautiful”, “costly”, “in the green surroundings’”, “moderate”, respectively.

In this case, to define a soft set means to point out expensive houses, beautiful houses, and so on. For example, the soft set (K, A) that describes the “attractiveness of the houses” in the opinion of a buyer, say Thomas, may be defined like this:

A={e_1, e_2, e_3, e_4, e_5};

K(e_1) = {h_2, h_3, h_5}, K(e_2) = {h_2, h_4}, K(e_3) = {h_1}, K(e_4) = U, K(e_5) = {h_3, h_5}.

2.3 Neutrosophic Soft Sets

Definition 2.5 [59] Let U be an initial universe set and $A \subset E$ be a set of parameters. Let NS(U) denotes the set of all neutrosophic subsets of U. The collection (F, A) is termed to be the neutrosophic soft set over U, where F is a mapping given by $F: A \to NS(U)$.

Example 2.6 Let U be the set of houses under consideration and E is the set of parameters.

Each parameter is a neutrosophic word or sentence involving neutrosophic words. Consider E ={beautiful, wooden, costly, very costly, moderate, green surroundings, in good repair, in bad repair, cheap, expensive}. In this case, to define a neutrosophic soft set means to point out beautiful houses, wooden houses, houses in the green surroundings and so on. Suppose that, there are five houses in the universe Ugiven by$U = \{h_1, h_2, \ldots, h_5\}$ and the set of parameters

$A = \{e_1, e_2, e_3, e_4\}$,where e_1 stands for the parameter `beautiful', $e_2$ stands for the parameter `wooden', e_3 stands for the parameter `costly' and the parameter $e_4$stands for `moderate'. Then the neutrosophic set (F, A) is defined as follows:

$$(F,A) = \left\{ \begin{array}{l} \left(e_1\left\{\frac{h_1}{(0.5,0.6,0.3)}, \frac{h_2}{(0.4,0.7,0.6)}, \frac{h_3}{(0.6,0.2,0.3)}, \frac{h_4}{(0.7,0.3,0.2)}, \frac{h_5}{(0.8,0.2,0.3)}\right\}\right) \\ \left(e_2\left\{\frac{h_1}{(0.6,0.3,0.5)}, \frac{h_2}{(0.7,0.4,0.3)}, \frac{h_3}{(0.8,0.1,0.2)}, \frac{h_4}{(0.7,0.1,0.3)}, \frac{h_5}{(0.8,0.3,0.6)}\right\}\right) \\ \left(e_3\left\{\frac{h_1}{(0.7,0.4,0.3)}, \frac{h_2}{(0.6,0.7,0.2)}, \frac{h_3}{(0.7,0.2,0.5)}, \frac{h_4}{(0.5,0.2,0.6)}, \frac{h_5}{(0.7,0.3,0.4)}\right\}\right) \\ \left(e_4\left\{\frac{h_1}{(0.8,0.6,0.4)}, \frac{h_2}{(0.7,0.9,0.6)}, \frac{h_3}{(0.7,0.6,0.4)}, \frac{h_4}{(0.7,0.8,0.6)}, \frac{h_5}{(0.9,0.5,0.7)}\right\}\right) \end{array} \right\}$$

Definition 2.7 [59] Let (H, A) and (G, B) be two NSs over the common universe U. Then the union of (H, A) and (G, B), is denoted by " (H, A)$\tilde{\cup}$(G, B)" and is defined by(H, A) $\tilde{\cup}$ (G, B)= (K, C), where C= A∪B and the truth-membership, indeterminacy-membership and falsity-membership of (K, C) are as follows:

$$T_{K(e)}(m) = \begin{cases} T_{H(e)}(m)\ if e \in A - B \\ T_{G(e)}(m)\ if e \in B - A \\ \max\left(T_{H(e)}, T_{G(e)}(m)\right)\ if e \in A \cap B \end{cases}$$

$$I_{K(e)}(m) = \begin{cases} I_{H(e)}(m)\ if e \in A - B \\ I_{G(e)}(m)\ if e \in B - A \\ \frac{\left(I_{H(e)}(m) + I_{G(e)}(m)\right)}{2}\ if e \in A \cap B \end{cases}$$

$$F_{K(e)}(m) = \begin{cases} F_{H(e)}(m)\ if e \in A - B \\ F_{G(e)}(m)\ if e \in B - A \\ \min\left(F_{H(e)}, F_{G(e)}(m)\right)\ if e \in A \cap B \end{cases}$$

Definition 2.8 [59] Let (H, A) and (G, B) be two NSs over the common universe U. Then the intersectionof (H, A) and (G, B), is denoted by " (H, A)$\tilde{\cap}$(G, B)" and is defined by(H, A) $\tilde{\cap}$ (G, B)= (K, C), where C= A ∩ B and the truth-membership, indeterminacy-membership and falsity-membership of (K, C) are as follows:

$$T_{K(e)}(m)=\begin{cases} T_{H(e)}(m)\ if e \in A-B \\ T_{G(e)}(m)\ if e \in B-A \\ \min\left(T_{H(e)}, T_{G(e)}(m)\right)\ if e \in A\cap B \end{cases}$$

$$I_{K(e)}(m)=\begin{cases} I_{H(e)}(m)\ if e \in A-B \\ I_{G(e)}(m)\ if e \in B-A \\ \frac{\left(I_{H(e)}(m)+I_{G(e)}(m)\right)}{2} if e \in A\cap B \end{cases}$$

$$F_{K(e)}(m)=\begin{cases} F_{H(e)}(m)\ if e \in A-B \\ F_{G(e)}(m)\ if e \in B-A \\ \max\left(F_{H(e)}, F_{G(e)}(m)\right)\ if e \in A\cap B \end{cases}$$

2.4. Soft expert sets

Definition 2.9 [65] Let U be a universe set, E be a set of parameters and X be a set of experts (agents). Let O={1=agree, 0=disagree} be a set of opinions. Let Z= E × X × O and A ⊆ Z.

A pair (F, E) is called a soft expert set over U, where F is a mapping given by F: A → P(U) and P(U) denote the power set of U.

Definition 2.10 [65] An agree-soft expert set $(F, \mathrm{A})_1$ over U, is a soft expert subset of (F,A) defined as :

$(F, \mathrm{A})_1$ = {F(α):α ∈ E × X ×{1}}.

Definition 2.11 [65] A disagree- soft expert set $(F, \mathrm{A})_0$ over U, is a soft expert subset of (F,A) defined as :

$(F, \mathrm{A})_0$= {F(α):α ∈ E × X ×{0}}.

2.5. Fuzzy Soft expert sets

Definition 2.12 [64] A pair (F, A) is called a fuzzy soft expert set over U, where F is a mapping given by

F : A→ I^U, and I^Udenote the set of all fuzzy subsets of U.

2.6. Intuitionitistic Fuzzy Soft Expert sets

Definition 2.13 [88] Let U={ $u_1, u_2, u_3, \ldots, u_n$} be a universal set of elements, E={ e_1, $e_2, e_3, \ldots, e_m$} be a universal set of parameters, X={ $x_1, x_2, x_3, \ldots, x_i$} be a set of experts (agents) and O= {1=agree, 0 = disagree} be a set of opinions. Let Z= { E ×

X × Q } and A ⊆ Z. Then the pair (U, Z) is called a soft universe. Let F: Z → $(I \times I)^U$ where $(I \times I)^U$ denotes the collection of all intuitionistic fuzzy subsets of U. Suppose F: Z → $(I \times I)^U$ be a function defined as:

$$F(z) = F(z)(u_i), \text{ for all } u_i \in U.$$

Then $F(z)$ is called an intuitionistic fuzzy soft expert set (IFSES in short) over the soft universe (U, Z)

For each $z_i \in Z$. $F(z)$ = F(z_i)(u_i) where F(z_i) represents the degree of belongingnessand non-belongingness of the elements of U in F(z_i). Hence $F(z_i)$ can be written as:

$$F(z_i) = \{(\frac{u_1}{F(z_1)(u_1)}), \ldots, (\frac{u_i}{F(z_i)(u_i)})\} \text{ ,for i=1,2,3,…n}$$

where F(z_i)(u_i) = < $\mu_{F(z_i)}(u_i), \omega_{F(z_i)}(u_i)$> with $\mu_{F(z_i)}(u_i)$ and $\omega_{F(z_i)}(u_i)$ representing the membership function and non-membership function of each of the elements $u_i \in U$ respectively.

Sometimes we write F as (F, Z). If A ⊆ Z. we can also have IFSES (F, A).

2.7 Neutrosophic Soft Expert Sets

Definition 2.14 [89] A pair (F, A) is called a neutrosophic soft expert set over U, where F is a mapping given by

$$F : A \to P(U)$$

where P(U) denotes the power neutrosophic set of U.

3. Mapping on Neutrosophic Soft Expert Set

In this paper, we introduce the mapping on neutrosophic soft expert classes. Neutrosophic soft expert classes are collections of neutrosophic soft expert sets. We also define and study the properties of neutrosophic soft expert images and neutrosophic soft expert inverse images of neutrosophic soft expert sets, and support them with examples and theorems.

Definition 3.1 Let $(\widetilde{U,Z})$ and $(\widetilde{Y,Z'})$ be neutrosophic soft expert classes. Let r: U→Y and s: Z→ Z' be mappings.

Then a mapping f:$(\widetilde{U,Z}) \to (\widetilde{Y,Z'})$ is defined as follows :

For a neutrosophic soft expert set (F, A) in $(\widetilde{U,Z})$, f(F, A) is a neutrosophic soft expert set in $(\widetilde{Y,Z'})$, where

$$f(\text{F, A})(\beta)(\text{y}) = \begin{cases} \vee_{x\in r^{-1}(y)}(\vee_{\alpha} F(\alpha)) if \;\; r^{-1}(y) \; and \; s^{-1}(\beta) \cap A \neq \emptyset, \\ 0 \qquad\qquad\qquad\qquad otherwise \end{cases}$$

for $\beta \in$ s(Z) ⊆ Z', y ∈ Y and $\forall\alpha \in s^{-1}(\beta) \cap A$, f(F, A) is called a neutrosophic soft expert image of the neutrosophic soft expert set (F, A).

Definition 3.2 Let $(\widetilde{U,Z})$ *and* $(\widetilde{Y,Z'})$ be the neutrosophic soft expert classes. Let r: U→Y and s: Z→ Z' be mappings. Then a mapping $f^{-1}:(\widetilde{Y,Z'}) \to (\widetilde{U,Z})$ is defined as follows :
For a neutrosophic soft expert set (G, B) in $(\widetilde{Y,Z'})$, $f^{-1}(G,$ B) is a neutrosophic soft expert set in $(\widetilde{U,Z})$,

$$f^{-1}(G, \text{B})(\alpha)(\text{u}) = \begin{cases} G(s(\alpha))(r(u)) & , if s(\alpha) \in B \\ 0 & otherwise \end{cases}$$

For $\alpha \in s^{-1}(\beta) \subseteq Z$ and u ∈ U. $f^{-1}(G,$ B) is called a neutrosophic soft expert inverse image of the neutrosophic soft expert set (F, A).

Example 3.3. Let U={u_1,u_2 , u_3}, Y={y_1,y_2,y_3} and let A ⊆ Z = {(e_1, p, 1), (e_2, p, 0), (e_3, p,1)}, and $A' \subseteq Z'$={(e_1', p',1), (e_2', p',0), (e_1', q',1)}.

Suppose that $(\widetilde{U,A})$ and $(\widetilde{Y,A'})$ are neutrosophic soft expert classes. Define r : U → Y and s: A → A' as follows :

r(u_1) = y_1, r(u_2) = y_3, r(u_3) = y_2,

s (e_1, p, 1) = (e_2', p',0) , s (e_2, p, 0) = (e_1', p',1), s (e_3, p, 1) = (e_1', q',1),

Let (F, A) and (G, A') be two neutrosophic soft experts over U and Y respectively such that.

(F, A) =

$$\left\{\left((e_1,p,1),\left\{\frac{u_1}{(0.4,0.3,0.6)},\frac{u_2}{(0.3,0.6,0.4)},\frac{u_3}{(0.3,0.5,0.5)}\right\}\right),\right.$$
$$\left((e_3,p,1),\left\{\frac{u_1}{(0.3,0.3,0.2)},\frac{u_2}{(0.5,0.4,0.4)},\frac{u_3}{(0.6,0.4,0.3)}\right\}\right),$$
$$\left.\left((e_2,p,0),\left\{\frac{u_1}{(0.5,0.6,0.3)},\frac{u_2}{(0.5,0.3,0.6)},\frac{u_3}{(0.6,0.4,0.7)}\right\}\right)\right\},$$

(G, A') =

$$\left\{\left((e_1',p',1),\left\{\frac{y_1}{(0.3,0.2,0.1)},\frac{y_2}{(0.5,0.6,0.4)},\frac{y_3}{(0.3,0.5,0.1)}\right\}\right),\right.$$
$$\left((e_1',q',1),\left\{\frac{y_1}{(0.5,0.7,0.4)},\frac{y_2}{(0.5,0.2,0.3)},\frac{y_3}{(0.6,0.5,0.1)},\right\}\right),$$
$$\left.\left((e_2',p',0),\left\{\frac{y_1}{(0.3,0.2,0.4)},\frac{y_2}{(0.1,0.7,0.5)},\frac{y_3}{(0.1,0.4,0.2)},\right\}\right)\right\}$$

Then we define the mapping from $f:(\widetilde{U,Z}) \to (\widetilde{Y,Z'})$ as follows :

For a neutrosophic soft expert set (F, A) in (U, Z), (f (F, A), K) is neutrosophic soft expert set in (Y, Z') where

K= s(A)={(e_1', p',1), (e_2', p',0), (e_1', q',1)} and is obtained as follows:

f (F, A) (e_1', p',1) (y_1) = $\vee_{x\in r^{-1}(y_1)}(\vee_\alpha F(\alpha)) = \vee_{x\in\{u_1\}}\left(\vee_{\alpha\in\{(e_2,\text{p},0),(e_3,\text{p},1)\}} F(\alpha)\right)$
$= (0.5, 0.6,0.3) \cup (0.3, 0.3,0.2)$
$=(0.5, 0.45, 0.2)$

f (F, A) (e_1', p',1) (y_2) = $\vee_{x\in r^{-1}(y_2)}(\vee_\alpha F(\alpha)) = \vee_{x\in\{u_3\}}\left(\vee_{\alpha\in\{(e_2,\text{p},0),(e_3,\text{p},1)\}} F(\alpha)\right)$
$=(0.6, 0.4, 0.7) \cup (0.6, 0.4, 0.3)$
$=(0.6, 0.4, 0.3)$

f (F, A) (e_1', p',1) (y_3) = $\vee_{x\in r^{-1}(y_3)}(\vee_\alpha F(\alpha)) = \vee_{x\in\{u_2\}}\left(\vee_{\alpha\in\{(e_2,\text{p},0),(e_3,\text{p},1)\}} F(\alpha)\right)$
$=(0.5, 0.3, 0.6) \cup (0.5, 0.4, 0.4)$
$=(0.5, 0.35, 0.4)$

Then,

f (F, A) (e_1', p',1) = $\left\{\frac{y_1}{(0.5,0.45,0.2)}, \frac{y_2}{(0.6,0.4,0.3)}, \frac{y_3}{(0.5,0.35,0.4)}\right\}$

f (F, A) (e_2', p',0) (y_1) = $\vee_{x\in r^{-1}(y_1)}(\vee_\alpha F(\alpha)) = \vee_{x\in\{u_1\}}\left(\vee_{\alpha\in\{(e_1,\text{p},1)\}} F(\alpha)\right)$
$= (0.4, 0.3, 0.6)$

f (F, A) (e_2', p',0) (y_2) = $\vee_{x\in r^{-1}(y_2)}(\vee_\alpha F(\alpha)) = \vee_{x\in\{u_3\}}\left(\vee_{\alpha\in\{(e_1,\text{p},1)\}} F(\alpha)\right)$
$= (0.3, 0.5, 0.5)$

f (F, A) (e_2', p',0) (y_3) = $\vee_{x\in r^{-1}(y_3)}(\vee_\alpha F(\alpha)) = \vee_{x\in\{u_2\}}\left(\vee_{\alpha\in\{(e_1,\text{p},1)\}} F(\alpha)\right)$
$= (0.3, 0.6, 0.4)$

Next,

f (F, A) ((e_2', p',0)= $\left\{\frac{y_1}{(0.4,0.3,0.6)}, \frac{y_2}{(0.3,0.5,0.5)}, \frac{y_3}{(0.3,0.6,0.4)}\right\}$

f (F, A) (e_1', q',1) (y_1) = $\vee_{x\in r^{-1}(y_1)}(\vee_\alpha F(\alpha)) = \vee_{x\in\{u_1\}}\left(\vee_{\alpha\in\{(e_3,\text{p},1)\}} F(\alpha)\right)$
$= (0.3, 0.3, 0.2)$

f (F, A) (e_1', q',1) (y_2) = $\vee_{x\in r^{-1}(y_2)}(\vee_\alpha F(\alpha)) = \vee_{x\in\{u_3\}}\left(\vee_{\alpha\in\{(e_3,\text{p},1)\}} F(\alpha)\right)$
$= (0.6, 0.4, 0.3)$

f (F, A) (e_1', q',1) (y_3) = $\vee_{x\in r^{-1}(y_3)}(\vee_\alpha F(\alpha)) = \vee_{x\in\{u_2\}}\left(\vee_{\alpha\in\{(e_3,\text{p},1)\}} F(\alpha)\right)$
$= (0.5, 0.4, 0.4)$

Also.

f (F, A) ((e'_1, q',1)= $\left\{\frac{y_1}{(0.3,0.3,0.2)}, \frac{y_2}{(0.6,0.4,0.3)}, \frac{y_3}{(0.5,0.4,0.4)}\right\}$

Hence,

$$(f\ (F,A),K)=\left\{\left((e'_1,p',1),\left\{\frac{y_1}{(0.5,0.45,0.2)}, \frac{y_2}{(0.6,0.4,0.3)}, \frac{y_3}{(0.5,0.35,0.4)}\right\}\right),\right.$$
$$\left((e'_2,p',0),\left\{\frac{y_1}{(0.4,0.3,0.6)}, \frac{y_2}{(0.3,0.5,0.5)}, \frac{y_3}{(0.3,0.6,0.4)}\right\}\right),$$
$$\left.\left((e'_1,q',1),\left\{\frac{y_1}{(0.3,0.3,0.2)}, \frac{y_2}{(0.6,0.4,0.3)}, \frac{y_3}{(0.5,0.4,0.4)}\right\}\right)\right\}$$

Next, for the neutrosophic soft expert set inverse images, we have the following:
For a neutrosophic soft expert set (G, A') in (Y, Z'), (f^{-1} (G, A'), D) is a neutrosophic soft expert set in (U, Z), where

D= $s^{-1}(A')$= {(e_1, p, 1), (e_2, p, 0), (e_3, p,1)}, and is obtained as follows:

f^{-1} (G, B) (e_1, p, 1) (u_1) = $G(\mathrm{s}(e_1,\mathrm{p},1))(r(u_1))$ = $G((e'_2,p',0))(y_1)$ =(0.3, 0.2, 0.4)
f^{-1} (G, B) (e_1, p, 1) (u_2) = $G(\mathrm{s}(e_1,\mathrm{p},1))(r(u_2))$ = $G((e'_2,p',0))(y_3)$ =(0.1, 0.4, 0.2)
f^{-1} (G, B) (e_1, p, 1) (u_3) = $G(\mathrm{s}(e_1,\mathrm{p},1))(r(u_3))$ = $G((e'_2,p',0))(y_2)$=(0.1, 0.7, 0.5)

Then

f^{-1} (G, B) (e_1, p, 1) = $\left\{\frac{u_1}{(0.3,0.2,0.4)}, \frac{u_2}{(0.1,0.4,0.2)}, \frac{u_3}{(0.1,0.7,0.5)}\right\}$
f^{-1} (G, B) (e_2, p, 0) (u_1) = $G(\mathrm{s}(e_2,\mathrm{p},0))(r(u_1))$ = $G((e'_1,p',1))(y_1)$ =(0.3, 0.2, 0.1)
f^{-1}G, B) (e_2, p, 0) (u_2) = $G(\mathrm{s}(e_2,\mathrm{p},0))(r(u_2))$ = $G((e'_1,p',1))(y_3)$ =(0.3, 0.5, 0.1)
f^{-1} (G, B) (e_2, p, 0) (u_3) = $G(\mathrm{s}(e_2,\mathrm{p},0))(r(u_3))$ = $G((e'_1,p',1))(y_2)$=(0.5, 0.6, 0.4)

Then,

f^{-1} (G, B) (e_2, p, 0)= $\left\{\frac{u_1}{(0.3,0.2,0.1)}, \frac{u_2}{(0.3,0.5,0.1)}, \frac{u_3}{(0.5,0.6,0.4)}\right\}$
f^{-1} (G, B) (e_3, p,1) (u_1) = $G(s(e_3,\mathrm{p},1))(r(u_1))$ = $G((e'_1,q',1))(y_1)$ =(0.5, 0.7,0.4)
f^{-1} (G, B) (e_3, p,1) (u_2) = $G(s(e_3,\mathrm{p},1))(r(u_2))$ = $G((e'_1,q',1))(y_3)$ =(0.6, 0.5, 0.1)
f^{-1} (G, B) (e_3, p,1) (u_3) = $G(\mathrm{s}(e_3,\mathrm{p},1))(r(u_3))$ = $G((e'_1,q',1))(y_2)$=(0.5, 0.2, 0.3)

Then

f^{-1} (G, B) (e_3, p,1)= $\left\{\frac{u_1}{(0.5,0.7,0.4)}, \frac{u_2}{(0.6,0.5,0.1)}, \frac{u_3}{(0.5,0.2,0.4)}\right\}$

Hence

$$(f^{-1}\ (G,A'),D)=\left\{\left((e_1,\mathrm{p},1),\left\{\frac{u_1}{(0.3,0.2,0.4)}, \frac{u_2}{(0.1,0.4,0.2)}, \frac{u_3}{(0.1,0.7,0.5)}\right\}\right),\right.$$

$\left((e_2, \mathrm{p}, 0), \left\{\frac{\mathrm{u}_1}{(0.3,0.2,0.1)}, \frac{\mathrm{u}_2}{(0.3,0.5,0.1)}, \frac{\mathrm{u}_3}{(0.5,0.6,0.4)}\right\}\right)$,
$\left((e_3, \mathrm{p}, 1), \left\{\frac{\mathrm{u}_1}{(0.5,0.7,0.4)}, \frac{\mathrm{u}_2}{(0.6,0.5,0.1)}, \frac{\mathrm{u}_3}{(0.5,0.2,0.4)}\right\}\right)\}$

Definition 3.4 Let f: $(\widetilde{\mathrm{U}, \mathrm{Z}}) \rightarrow (\widetilde{\mathrm{Y}, \mathrm{Z}'})$ be a mapping and (F, A) and (G, B) a neutrosophic soft expert sets in $(\widetilde{\mathrm{U}, \mathrm{E}})$. Then for $\beta \in \mathrm{Z}'$, $\mathrm{y} \in \mathrm{Y}$ the union and intersection of neutrosophic soft expert images (F, A) and (G, B) are defined as follows :

$$\left(f(\mathrm{F}, \mathrm{A}) \widetilde{\vee} f(\mathrm{G}, \mathrm{B})\right)(\beta)(\mathrm{y}) = f(\mathrm{F}, \mathrm{A})(\beta)(\mathrm{y}) \widetilde{\vee} f(\mathrm{G}, \mathrm{B})(\beta)(\mathrm{y}).$$
$$\left(f(\mathrm{F}, \mathrm{A}) \widetilde{\wedge} f(\mathrm{G}, \mathrm{B})\right)(\beta)(\mathrm{y}) = f(\mathrm{F}, \mathrm{A})(\beta)(\mathrm{y}) \widetilde{\wedge} f(\mathrm{G}, \mathrm{B})(\beta)(\mathrm{y}).$$

Definition 3.5 Let f: $(\widetilde{\mathrm{U}, \mathrm{Z}}) \rightarrow (\widetilde{\mathrm{Y}, \mathrm{Z}'})$ be a mapping and (F, A) and (G, B) a neutrosophic soft expert sets in $(\widetilde{\mathrm{U}, \mathrm{E}})$. Then for $\alpha \in \mathrm{Z}$, $\mathrm{u} \in \mathrm{U}$, the union and intersection of neutrosophic soft expert inverse images (F, A) and (G, B) are defined as follows :

$$\left(f^{-1}(\mathrm{F}, \mathrm{A}) \widetilde{\vee} f^{-1}(\mathrm{G}, \mathrm{B})\right)(\alpha)(\mathrm{u}) = f^{-1}(\mathrm{F}, \mathrm{A})(\alpha)(\mathrm{u}) \widetilde{\vee} f^{-1}(\mathrm{G}, \mathrm{B})(\alpha)(\mathrm{u}).$$
$$\left(f^{-1}(\mathrm{F}, \mathrm{A}) \widetilde{\wedge} f^{-1}(\mathrm{G}, \mathrm{B})\right)(\alpha)(\mathrm{u}) = f^{-1}(\mathrm{F}, \mathrm{A})(\alpha)(\mathrm{u}) \widetilde{\wedge} f^{-1}(\mathrm{G}, \mathrm{B})(\alpha)(\mathrm{u}).$$

Theorem 3.6 Let f: $(\widetilde{\mathrm{U}, \mathrm{Z}}) \rightarrow (\widetilde{\mathrm{Y}, \mathrm{Z}'})$ be a mapping. Then for neutrosophic soft expert sets (F, A) and (G, B) in the neutrosophic soft expert class $(\widetilde{\mathrm{U}, \mathrm{Z}})$.

1. $f(\emptyset)=\emptyset$
2. $f(\mathrm{Z}) \subseteq \mathrm{Y}$.
3. $f\left((\mathrm{F}, \mathrm{A}) \widetilde{\vee} (\mathrm{G}, \mathrm{B})\right) = f(\mathrm{F}, \mathrm{A}) \widetilde{\vee} f(\mathrm{G}, \mathrm{B})$
4. $f\left((\mathrm{F}, \mathrm{A}) \widetilde{\wedge} (\mathrm{G}, \mathrm{B})\right) = f(F, A) \widetilde{\wedge} f(\mathrm{G}, \mathrm{B})$
5. If $(\mathrm{F}, \mathrm{A}) \subseteq (\mathrm{G}, \mathrm{B})$, then $f(\mathrm{F}, \mathrm{A}) \subseteq f(\mathrm{G}, \mathrm{B})$.

Proof: For (1) ,(2) and (5) the proof is trivial, so we just give the proof of (3) and (4).
(3). For $\beta \in \mathrm{Z}'$ and $\mathrm{y} \in \mathrm{Y}$, we want to prove that

$$\left(f(\mathrm{F}, \mathrm{A}) \widetilde{\vee} f(\mathrm{G}, \mathrm{B})\right)(\beta)(\mathrm{y}) = f(\mathrm{F}, \mathrm{A})(\beta)(\mathrm{y}) \widetilde{\vee} f(\mathrm{G}, \mathrm{B})(\beta)(\mathrm{y})$$

For left hand side, consider $f\left((\mathrm{F}, \mathrm{A}) \widetilde{\vee} (\mathrm{G}, \mathrm{B})\right)(\beta)(\mathrm{y}) = f(\mathrm{H}, \mathrm{A} \cup \mathrm{B})(\beta)(\mathrm{y})$. Then

$$f(\mathrm{H}, \mathrm{A} \cup \mathrm{B})(\beta)(\mathrm{y}) = \begin{cases} \vee_{\mathrm{x} \in \mathrm{r}^{-1}(\mathrm{y})}(\vee_{\alpha} \mathrm{H}(\alpha)) & \text{if } \mathrm{r}^{-1}(\mathrm{y}) \text{and } \mathrm{s}^{-1}(\beta) \cap (\mathrm{A} \cup \mathrm{B}) \neq \emptyset, \\ 0 & \text{otherwise} \end{cases} \quad \textbf{(1,1)}$$

such that $\mathrm{H}(\alpha) = \mathrm{F}(\alpha) \widetilde{\cup} \mathrm{G}(\alpha)$ where $\widetilde{\cup}$ denotes neutrosophic union.
Considering only the non-trivial case, Then equation 1.1 becomes:

$$f(\mathrm{H}, \mathrm{A} \cup \mathrm{B})(\beta)(\mathrm{y}) = \vee_{\mathrm{x} \in \mathrm{r}^{-1}(\mathrm{y})}(\vee(\mathrm{F}(\alpha) \widetilde{\cup} \mathrm{G}(\alpha))) \quad \textbf{(1,2)}$$

For right hand side and by using definition 3.4, we have

$\left(f(\mathrm{F},\mathrm{A})\tilde{\vee}f(\mathrm{G},\mathrm{B})\right)(\beta)(\mathrm{y})=f(\mathrm{F},\mathrm{A})(\beta)(\mathrm{y})\vee f(\mathrm{G},\mathrm{B})(\beta)(\mathrm{y})$

$=\left(\vee_{x\in r^{-1}(y)}\left(\vee_{\alpha\in s^{-1}(\beta)\cap A}\mathrm{F}(\alpha)\right)(\mathrm{x})\right)\vee\left(\vee_{x\in r^{-1}(y)}\left(\vee_{\forall\alpha\in s^{-1}(\beta)\cap B}\mathrm{F}(\alpha)\right)(\mathrm{x})\right)$

$= \vee_{x\in r^{-1}(y)} \quad \vee_{\alpha\in s^{-1}(\beta)\cap(A\cup B)}(\mathrm{F}(\alpha)\vee\mathrm{G}(\alpha))$

$=\vee_{x\in r^{-1}(y)}(\vee(\mathrm{F}(\alpha)\,\tilde{\cup}\,\mathrm{G}(\alpha)))$ **(1,3)**

From equation (1.1) and (1.3) we get (3)

(4). For $\beta \in Z'$ and $y \in Y$, and using definition 3.4, we have

$f\left((\mathrm{F},\mathrm{A})\;\tilde{\wedge}\;(\mathrm{G},\mathrm{B})\right)(\beta)(\mathrm{y})$

$=f(\mathrm{H},\mathrm{A}\cup\mathrm{B})(\beta)(\mathrm{y})$

$=\vee_{x\in r^{-1}(y)}\left(\vee_{\alpha\in s^{-1}(\beta)\cap(A\cup B)}\mathrm{H}(\alpha)\right)(\mathrm{x})$

$=\vee_{x\in r^{-1}(y)}\left(\vee_{\alpha\in s^{-1}(\beta)\cap(A\cup B)}\mathrm{F}(\alpha)\;\tilde{\cap}\;\mathrm{G}(\alpha)\right)(\mathrm{x})$

$=\vee_{x\in r^{-1}(y)}\left(\vee_{\alpha\in s^{-1}(\beta)\cap(A\cup B)}\mathrm{F}(\alpha)(\mathrm{x})\;\tilde{\cap}\;\mathrm{G}(\alpha)\,(\mathrm{x})\right)$

$\subseteq\left(\bigvee_{x\in r^{-1}(y)}\left(\bigvee_{\alpha\in s^{-1}(\beta)\cap A}\mathrm{F}(\alpha)\right)\right)\wedge\bigvee_{x\in r^{-1}(y)}\left(\bigvee_{\alpha\in s^{-1}(\beta)\cap B}\mathrm{G}(\alpha)\right)$

$= f\left((\mathrm{F},\mathrm{A})(\beta)(\mathrm{y})\wedge(\mathrm{G},\mathrm{B})(\beta)(\mathrm{y})\right)$

$=\left(f(\mathrm{F},\mathrm{A})\;\tilde{\wedge}\;\mathrm{f}(\mathrm{G},\mathrm{B})\right)(\beta)(\mathrm{y})$

This gives (4).

Theorem 3.7 Let $f^{-1}: (\widetilde{U,Z}) \to (\widetilde{Y,Z'})$ be a an inverse mapping. Then for neutrosophic soft expert sets (F, A) and (G, B) in the neutrosophic soft expert class $(\widetilde{U,Z})$.

1. $f^{-1}(\emptyset)=\emptyset$
2. $f^{-1}(X)\subseteq X$.
3. $f^{-1}\left((\mathrm{F},\mathrm{A})\tilde{\vee}(\mathrm{G},\mathrm{B})\right)=f^{-1}(\mathrm{F},\mathrm{A})\tilde{\vee}f^{-1}(\mathrm{G},\mathrm{B})$
4. $f^{-1}\left((\mathrm{F},\mathrm{A})\;\tilde{\wedge}\;(\mathrm{G},\mathrm{B})\right)=f^{-1}(\mathrm{F},\mathrm{A})\;\tilde{\wedge}\;f^{-1}(\mathrm{G},\mathrm{B})$
5. If $(\mathrm{F},\mathrm{A}) \subseteq (\mathrm{G},\mathrm{B})$, Then $f^{-1}(\mathrm{F},\mathrm{A}) \subseteq f^{-1}(\mathrm{G},\mathrm{B})$.

Proof. The proof is straightforward**.**

4. Conclusion

In this paper, we studied mappings on neutrosophic soft expert classes and their basic properties. We also give some illustrative examples of mapping on neutrosophic soft expert set. We hope these fundamental results will help the researchers to enhance and promote the research on neutrosophic soft set theory.

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$\ddot{g}$-LOCALLY CLOSED SETS IN TOPOLOGICAL SPACES

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Abstract − The aim of this paper is to introduce and study the classes of $\ddot{g}$-locally closed sets, $\ddot{g}$-lc* sets and $\ddot{g}$-lc** sets which are weaker forms of the class of locally closed sets. Furthermore the relations with other notions connected with the forms of locally closed sets are investigated.

Keywords − *Topological space, g-closed set, $\ddot{g}$-closed set, $\ddot{g}$-lc* set and $\ddot{g}$-lc** set.*

1 Introduction

The first step of locally closedness was done by Bourbaki [4]. He defined a set A to be locally closed if it is the intersection of an open set and a closed set. In literature many general topologists introduced the studies of locally closed sets. Extensive research on locally closedness and generalizing locally closedness were done in recent years. Stone [20] used the term FG for a locally closed set. Ganster and Reilly used locally closed sets in [7] to define LC-continuity and LC-irresoluteness. Balachandran et al [2] introduced the concept of generalized locally closed sets. Veera Kumar [23] (Sheik John [19]) introduced $\hat{g}$-locally closed sets (=ω-locally closed sets) respectively.

In this paper, we introduce three forms of locally closed sets called $\ddot{g}$-locally closed sets, $\ddot{g}$-lc* sets and $\ddot{g}$-lc** sets. Properties of these new concepts are studied as well as their relations to the other classes of locally closed sets will be investigated.

2 Preliminaries

Throughout this paper (X, τ) (or X) represents topological spaces on which no separation axioms are assumed unless otherwise mentioned. For a subset A of a space (X, τ), cl(A), int(A) and A^c denote the closure of A, the interior of A and the complement of A, respectively.

We recall the following definitions, Corollary and Remarks which are useful in the sequel.

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Definition 2.1. *A subset A of a space (X, τ) is called:*

1. *semi-open set [10] if A⊆cl(int(A));*
2. *α-open set [11] if A⊆int(cl(int(A)));*
3. *regular open set [21] if A=int(cl(A)).*

The complements of the above mentioned open sets are called their respective closed sets.

The semi-closure [5] of a subset A of X, denoted by scl(A), is defined to be the intersection of all semi-closed sets of (X, τ) containing A. It is known that scl(A) is a semi-closed set.

Definition 2.2. *A subset A of a space (X, τ) is called*

1. *a generalized closed (briefly g-closed) set [9] if cl(A)⊆U whenever A⊆U and U is open in (X, τ). The complement of g-closed set is called g-open set;*
2. *a semi-generalized closed (briefly sg-closed) set [3] if scl(A)⊆U whenever A⊆U and U is semi-open in (X, τ). The complement of sg-closed set is called sg-open set;*
3. *a regular generalized closed (briefly rg-closed) set [12] if cl(A)⊆U whenever A⊆U and U is regular open in (X, τ). The complement of rg-closed set is called rg-open set;*
4. *a $\hat{g}$-closed set [22] (=ω-closed set [19]) if cl(A)⊆U whenever A⊆U and U is semi-open in (X, τ). The complement of $\hat{g}$-closed set is called $\hat{g}$-open set;*
5. *a $\ddot{g}$-closed set [15] if cl(A)⊆U whenever A⊆U and U is sg-open in (X, τ). The complement of $\ddot{g}$-closed set is called $\ddot{g}$-open set.*

Remark 2.3. *The collection of all $\ddot{g}$-closed (resp. ω-closed, g-closed, rg-closed, sg-closed) sets in X is denoted by $\ddot{G}C(X)$ (resp. $\omega C(X)$, GC(X), RGC(X), SGC(X)).*

The collection of all $\ddot{g}$-open (resp. ω-open, g-open, rg-open, sg-open) sets in X is denoted by $\ddot{G}O(X)$ (resp. $\omega O(X)$, GO(X), RGO(X), SGO(X)).

We denote the power set of X by P(X).

Definition 2.4. *A subset S of a space (X, τ) is called:*

1. *locally closed (briefly lc) [7] if S=U∩F, where U is open and F is closed in (X, τ).*
2. *generalized locally closed (briefly glc) [2] if S=U∩F, where U is g-open and F is g-closed in (X, τ).*
3. *semi-generalized locally closed (briefly sglc) [13] if S=U∩F, where U is sg-open and F is sg-closed in (X, τ).*
4. *regular-generalized locally closed (briefly rg-lc) [1] if S=U∩F, where U is rg-open and F is rg-closed in (X, τ).*
5. *generalized locally semi-closed (briefly glsc) [8] if S=U∩F, where U is g-open and F is semi-closed in (X, τ).*
6. *locally semi-closed (briefly lsc) [8] if S=U∩F, where U is open and F is semi-closed in (X, τ).*
7. *α-locally closed (briefly α-lc) [8] if S=U∩F, where U is α-open and F is α-closed in (X, τ).*
8. *ω-locally closed (briefly ω-lc) [19] if S=U∩F, where U is ω-open and F is ω-closed in (X, τ).*
9. *$sglc^\star$ [13] if S=U∩F, where U is sg-open and F is closed in (X, τ).*

The class of all locally closed (resp. generalized locally closed, generalized locally semi-closed, locally semi-closed, ω-locally closed) sets in X is denoted by LC(X) (resp. GLC(X), GLSC(X), LSC(X), ω-LC(X)).

Definition 2.5. *[16] For any A⊆X, $\ddot{g}$-int(A) is defined as the union of all $\ddot{g}$-open sets contained in A. i.e., $\ddot{g}$-int(A)=∪{G : G⊆A and G is $\ddot{g}$-open}.*

Definition 2.6. *[16] For every set $A \subseteq X$, we define the $\ddot{g}$-closure of A to be the intersection of all $\ddot{g}$-closed sets containing A. ie., $\ddot{g}$-cl(A)=$\cap\{F : A \subseteq F \in \ddot{G}C(X)\}$.*

Definition 2.7. *[17] A space (X, τ) is called a $T_{\ddot{g}}$-space if every $\ddot{g}$-closed set in it is closed.*

Recall that a subset A of a space (X, τ) is called dense if cl(A)=X.

Definition 2.8. *A topological space (X, τ) is called:*

1. *submaximal [6, 23] if every dense subset is open.*
2. *$\hat{g}$ (or ω)-submaximal [19, 23] if every dense subset is ω-open.*
3. *g-submaximal [2] if every dense subset is g-open.*
4. *rg-submaximal [12] if every dense subset is rg-open.*

Remark 2.9. *For a topological space X, the following statements hold:*

1. *Every closed set is $\ddot{g}$-closed but not conversely [15].*
2. *Every $\ddot{g}$-closed set is ω-closed but not conversely [15].*
3. *Every $\ddot{g}$-closed set is g-closed but not conversely [15].*
4. *Every $\ddot{g}$-closed set is sg-closed but not conversely [15].*
5. *Every $\ddot{g}$-open set is ω-open but not conversely [18].*
6. *A subset A of X is $\ddot{g}$-closed if and only if $\ddot{g}$-cl(A)=A [16].*
7. *A subset A of X is $\ddot{g}$-open if and only if $\ddot{g}$-int(A)=A [16].*

Corollary 2.10. *[15] If A is a $\ddot{g}$-closed set and F is a closed set, then $A \cap F$ is a $\ddot{g}$-closed set.*

Theorem 2.11. *[23] Let (X, τ) be a topological space.*

1. *If X is submaximal, then X is $\hat{g}$-submaximal.*
2. *If X is $\hat{g}$-submaximal, then X is g-submaximal.*
3. *If X is g-submaximal, then X is rg-submaximal.*
4. *The respective converses of the above need not be true in general.*

3 $\ddot{g}$-locally Closed Sets

We introduce the following definition.

Definition 3.1. *A subset A of (X, τ) is called $\ddot{g}$-locally closed (briefly $\ddot{g}$-lc) if $A=S \cap G$, where S is $\ddot{g}$-open and G is $\ddot{g}$-closed in (X, τ).*

The class of all $\ddot{g}$-locally closed sets in X is denoted by $\ddot{G}$LC(X).

Proposition 3.2. *Every $\ddot{g}$-closed (resp. $\ddot{g}$-open) set is $\ddot{g}$-lc set but not conversely.*

Proof. It follows from Definition 3.1.

Example 3.3. *Let X=$\{a, b, c\}$ with $\tau=\{\emptyset, \{b\}, X\}$. Then the set $\{b\}$ is $\ddot{g}$-lc set but it is not $\ddot{g}$-closed and the set $\{a, c\}$ is $\ddot{g}$-lc set but it is not $\ddot{g}$-open in (X, τ).*

Proposition 3.4. *Every lc set is $\ddot{g}$-lc set but not conversely.*

Proof. It follows from Remark 2.9 (1).

Example 3.5. *Let X=$\{a, b, c\}$ with $\tau=\{\emptyset, \{b, c\}, X\}$. Then the set $\{b\}$ is $\ddot{g}$-lc set but it is not lc set in (X, τ).*

Proposition 3.6. *Every $\ddot{g}$-lc set is a (1) ω-lc set, (2) glc set and (3) sglc set. However the separate converses are not true.*

Proof. It follows from Remark 2.9 (2), (3) and (4).

Example 3.7. *Let X=$\{a, b, c\}$ with $\tau=\{\emptyset, \{a\}, X\}$. Then the set $\{b\}$ is g-lc set but it is not $\ddot{g}$-lc set in (X, τ). Moreover, the set $\{c\}$ is sg-lc set but it is not $\ddot{g}$-lc set in (X, τ).*

Example 3.8. *Let X=$\{a, b, c\}$ with $\tau=\{\emptyset, \{b\}, \{a, c\}, X\}$. Then the set $\{a\}$ is ω-lc set but it is not $\ddot{g}$-lc set in (X, τ).*

Remark 3.9. *The concepts of α-lc set and $\ddot{g}$-lc set are independent of each other.*

Example 3.10. *The set $\{b, c\}$ in Example 3.3 is α-lc set but it is not a $\ddot{g}$-lc set in (X, τ) and the set $\{a, b\}$ in Example 3.5 is $\ddot{g}$-lc set but it is not an α-lc set in (X, τ).*

Remark 3.11. *The concepts of lsc set and $\ddot{g}$-lc set are independent of each other.*

Example 3.12. *The set $\{a\}$ in Example 3.3 is lsc set but it is not a $\ddot{g}$-lc set in (X, τ) and the set $\{a, b\}$ in Example 3.5 is $\ddot{g}$-lc set but it is not a lsc set in (X, τ).*

Remark 3.13. *The concepts of $\ddot{g}$-lc set and glsc set are independent of each other.*

Example 3.14. *The set $\{b, c\}$ in Example 3.3 is glsc set but it is not a $\ddot{g}$-lc set in (X, τ) and the set $\{a, b\}$ in Example 3.5 is $\ddot{g}$-lc set but it is not a glsc set in (X, τ).*

Remark 3.15. *The concepts of $\ddot{g}$-lc set and sglc* set are independent of each other.*

Example 3.16. *The set $\{b, c\}$ in Example 3.3 is sglc* set but it is not a $\ddot{g}$-lc set in (X, τ) and the set $\{a, b\}$ in Example 3.5 is $\ddot{g}$-lc set but it is not a sglc* set in (X, τ).*

Theorem 3.17. *For a $T_{\ddot{g}}$-space (X, τ), the following properties hold:*

1. *$\ddot{G}$LC(X)=LC(X).*
2. *$\ddot{G}$LC(X)$\subseteq$GLC(X).*
3. *$\ddot{G}$LC(X)$\subseteq$GLSC(X).*
4. *$\ddot{G}$LC(X)$\subseteq\omega$-LC(X).*

Proof. (1) Since every $\ddot{g}$-open set is open and every $\ddot{g}$-closed set is closed in (X, τ), $\ddot{G}$LC(X)$\subseteq$LC(X) and hence $\ddot{G}$LC(X)=LC(X).

(2), (3) and (4) follows from (1), since for any space (X, τ), LC(X)$\subseteq$GLC(X), LC(X)$\subseteq$GLSC(X) and LC(X)$\subseteq\omega$-LC(X).

Corollary 3.18. *If GO(X)=τ, then $\ddot{G}$LC(X)$\subseteq$GLSC(X)$\subseteq$LSC(X).*

Proof. GO(X)=τ implies that (X, τ) is a $T_{\ddot{g}}$-space and hence by Theorem 3.17, $\ddot{G}$LC(X) $\subseteq$GLSC(X). Let A$\in$GLSC(X). Then A=U$\cap$F, where U is g-open and F is semi-closed. By hypothesis, U is open and hence A is a lsc-set and so A$\in$LSC(X).

Definition 3.19. *A subset A of a space (X, τ) is called*

1. *$\ddot{g}$-lc* set if A=S$\cap$G, where S is $\ddot{g}$-open in (X, τ) and G is closed in (X, τ).*
2. *$\ddot{g}$-lc** set if A=S$\cap$G, where S is open in (X, τ) and G is $\ddot{g}$-closed in (X, τ).*

The class of all $\ddot{g}$-lc* (resp. $\ddot{g}$-lc**) sets in a topological space (X, τ) is denoted by $\ddot{G}$LC*(X) (resp. $\ddot{G}$LC**(X)).

Proposition 3.20. *Every lc set is $\ddot{g}$-lc* set but not conversely.*

Proof. It follows from Definitions 2.4 (1) and 3.19 (1).

Example 3.21. *The set $\{b\}$ in Example 3.5 is $\ddot{g}$-lc* set but it is not a lc set in (X, τ).*

Proposition 3.22. *Every lc set is $\ddot{g}$-lc** set but not conversely.*

Proof. It follows from Definitions 2.4 (1) and 3.19 (2).

Example 3.23. *The set $\{a, c\}$ in Example 3.5 is $\ddot{g}$-lc** set but it is not a lc set in (X, τ).*

Proposition 3.24. *Every $\ddot{g}$-lc* set is $\ddot{g}$-lc set but not conversely.*

Proof. It follows from Definitions 3.1 and 3.19 (1).

Example 3.25. *The set $\{a, b\}$ in Example 3.5 is $\ddot{g}$-lc set but it is not a $\ddot{g}$-lc* set in (X, τ).*

Proposition 3.26. *Every $\ddot{g}$-lc** set is $\ddot{g}$-lc set but not conversely.*

Proof. It follows from Definitions 3.1 and 3.19 (2).

Question 1. Give an example for a set which is $\ddot{g}$-lc set but not $\ddot{g}$-lc** set.

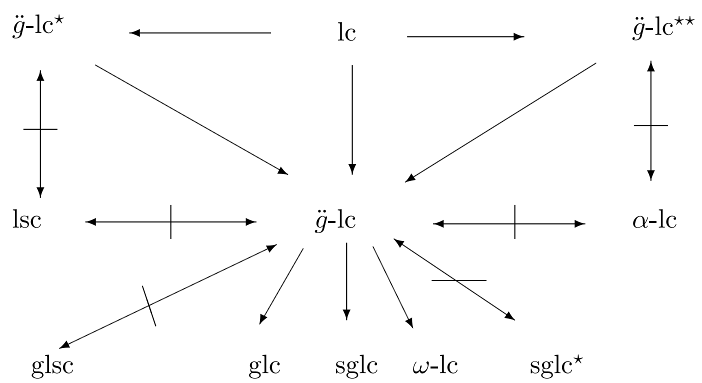
Remark 3.27. *The concepts of $\ddot{g}$-lc* set and lsc set are independent of each other.*

Example 3.28. *The set $\{c\}$ in Example 3.5 is $\ddot{g}$-lc* set but it is not a lsc set in (X, τ) and the set $\{a\}$ in Example 3.3 is lsc set but it is not a $\ddot{g}$-lc* set in (X, τ).*

Remark 3.29. *The concepts of $\ddot{g}$-lc** set and α-lc set are independent of each other.*

Example 3.30. *The set $\{a, b\}$ in Example 3.5 is $\ddot{g}$-lc** set but it is not an α-lc set in (X, τ) and the set $\{a, b\}$ in Example 3.3 is α-lc set but it is not a $\ddot{g}$-lc** set in (X, τ).*

Remark 3.31. *From the above discussions we have the following implications where A $\rightarrow$ B (resp. A $\nleftrightarrow$ B) represents A implies B but not conversely (resp. A and B are independent of each other).*



Proposition 3.32. *If GO(X)=τ, then $\ddot{G}LC(X)=\ddot{G}LC^{\star}(X)=\ddot{G}LC^{\star\star}(X)$.*

Proof. For any space (X, τ), $\tau\subseteq\ddot{G}O(X)\subseteq$GO(X). Therefore by hypothesis, $\ddot{G}O(X)=\tau$. i.e., (X, τ) is a $T_{\ddot{g}}$-space and hence $\ddot{G}$LC(X)=$\ddot{G}$LC*(X)=$\ddot{G}$LC**(X).

Remark 3.33. *The converse of Propositions 3.32 need not be true.*

For the topological space (X, τ) in Example 3.3. $\ddot{G}$LC(X)=$\ddot{G}$LC*(X)=$\ddot{G}$LC**(X). However GO(X)={$\emptyset$, {a}, {b}, {c}, {a, b}, {b, c}, X}$\neq\tau$.

Proposition 3.34. *Let (X, τ) be a topological space. If GO(X)$\subseteq$LC(X), then $\ddot{G}LC(X)=\ddot{G}LC^{\star\star}(X)$.*

Proof. Let A$\in\ddot{G}$LC(X). Then A=S$\cap$G where S is $\ddot{g}$-open and G is $\ddot{g}$-closed. Since $\ddot{G}O(X)\subseteq$GO(X) and by hypothesis GO(X)$\subseteq$LC(X), S is locally closed. Then S=P$\cap$Q, where P is open and Q is closed. Therefore, A=P$\cap$(Q$\cap$G). By Corollary 2.10, Q$\cap$G is $\ddot{g}$-closed and hence A$\in\ddot{G}$LC**(X). i.e., $\ddot{G}$LC(X)$\subseteq\ddot{G}$LC**(X). For any topological space, $\ddot{G}$LC**(X)$\subseteq\ddot{G}$LC(X) and so $\ddot{G}$LC(X)=$\ddot{G}$LC**(X).

Remark 3.35. *The converse of Proposition 3.34 need not be true in general.*

For the topological space (X, τ) in Example 3.3, then $\ddot{G}$LC(X)=$\ddot{G}$LC**(X)={$\emptyset$, {b}, {a, c}, X}. But GO(X)={$\emptyset$, {a}, {b}, {c}, {a, b}, {b, c}, X}$\nsubseteq$LC(X)={$\emptyset$, {b}, {a, c}, X}.

Corollary 3.36. *Let (X, τ) be a topological space. If ωO(X)$\subseteq$LC(X), then $\ddot{G}$LC(X) =$\ddot{G}$LC**(X).*

Proof. It follows from the fact that ωO(X)$\subseteq$GO(X) and Proposition 3.34.

Remark 3.37. *The converse of Corollary 3.36 need not be true in general.*

For the topological space (X, τ) in Example 3.8, then $\ddot{G}$LC(X)=$\ddot{G}$LC**(X)={$\emptyset$, {b}, {a,c}, X}. But ωO(X)=P(X)$\nsubseteq$LC(X)={$\emptyset$, {b}, {a, c}, X}.

The following results are characterizations of $\ddot{g}$-lc sets, $\ddot{g}$-lc* sets and $\ddot{g}$-lc** sets.

Theorem 3.38. *For a subset A of (X, τ) the following statements are equivalent:*

1. *A$\in\ddot{G}$LC(X),*
2. *A=S$\cap\ddot{g}$-cl(A) for some $\ddot{g}$-open set S,*
3. *$\ddot{g}$-cl(A)$-$A is $\ddot{g}$-closed,*
4. *A$\cup$($\ddot{g}$-cl(A))^c is $\ddot{g}$-open,*
5. *A$\subseteq\ddot{g}$-int(A$\cup$($\ddot{g}$-cl(A))^c).*

Proof. (1) $\Rightarrow$ (2). Let A$\in\ddot{G}$LC(X). Then A=S$\cap$G where S is $\ddot{g}$-open and G is $\ddot{g}$-closed. Since A$\subseteq$G, $\ddot{g}$-cl(A)$\subseteq$G and so S$\cap\ddot{g}$-cl(A)$\subseteq$A. Also A$\subseteq$S and A$\subseteq\ddot{g}$-cl(A) implies A$\subseteq$S$\cap\ddot{g}$-cl(A) and therefore A=S$\cap\ddot{g}$-cl(A).

(2) $\Rightarrow$ (3). A=S$\cap\ddot{g}$-cl(A) implies $\ddot{g}$-cl(A)$-$A=$\ddot{g}$-cl(A)$\cap$S^c which is $\ddot{g}$-closed since S^c is $\ddot{g}$-closed and $\ddot{g}$-cl(A) is $\ddot{g}$-closed.

(3) $\Rightarrow$ (4). A$\cup$($\ddot{g}$-cl(A))^c=($\ddot{g}$-cl(A)$-$A)^c and by assumption, ($\ddot{g}$-cl(A)$-$A)^c is $\ddot{g}$-open and so is A$\cup$($\ddot{g}$-cl(A))^c.

(4) $\Rightarrow$ (5). By assumption, A$\cup$($\ddot{g}$-cl(A))^c=$\ddot{g}$-int(A$\cup$($\ddot{g}$-cl(A))^c) and hence A$\subseteq\ddot{g}$-int(A$\cup$($\ddot{g}$-cl(A))^c).

(5) $\Rightarrow$ (1). By assumption and since A$\subseteq\ddot{g}$-cl(A), A=$\ddot{g}$-int(A$\cup$($\ddot{g}$-cl(A))^c)$\cap\ddot{g}$-cl(A). Therefore, A$\in\ddot{G}$LC(X).

Theorem 3.39. *For a subset A of (X, τ), the following statements are equivalent:*

1. *A$\in\ddot{G}$LC*(X),*
2. *A=S$\cap$cl(A) for some $\ddot{g}$-open set S,*
3. *cl(A)$-$A is $\ddot{g}$-closed,*
4. *A$\cup$(cl(A))^c is $\ddot{g}$-open.*

Proof. (1) $\Rightarrow$ (2). Let A$\in\ddot{G}$LC*(X). There exist an $\ddot{g}$-open set S and a closed set G such that A=S$\cap$G. Since A$\subseteq$S and A$\subseteq$cl(A), A$\subseteq$S$\cap$cl(A). Also since cl(A)$\subseteq$G, S$\cap$cl(A)$\subseteq$S$\cap$G=A. Therefore A=S$\cap$cl(A).

(2) $\Rightarrow$ (1). Since S is $\ddot{g}$-open and cl(A) is a closed set, A=S$\cap$cl(A)$\in\ddot{G}$LC*(X).

(2) $\Rightarrow$ (3). Since cl(A)$-$A=cl(A)$\cap$S^c, cl(A)$-$A is $\ddot{g}$-closed by Corollary 2.10.

(3) $\Rightarrow$ (2). Let S=(cl(A)$-$A)^c. Then by assumption S is $\ddot{g}$-open in (X, τ) and A=S$\cap$cl(A).

(3) $\Rightarrow$ (4). Let G=cl(A)$-$A. Then G^c=A$\cup$(cl(A))^c and A$\cup$(cl(A))^c is $\ddot{g}$-open.

(4) $\Rightarrow$ (3). Let S=A$\cup$(cl(A))^c. Then S^c is $\ddot{g}$-closed and S^c=cl(A)$-$A and so cl(A)$-$A is $\ddot{g}$-closed.

Theorem 3.40. *Let A be a subset of (X, τ). Then A$\in\ddot{G}$LC**(X) if and only if A=S$\cap\ddot{g}$-cl(A) for some open set S.*

Proof. Let A$\in\ddot{G}$LC**(X). Then A=S$\cap$G where S is open and G is $\ddot{g}$-closed. Since A$\subseteq$G, $\ddot{g}$-cl(A)$\subseteq$G. We obtain A=A$\cap\ddot{g}$-cl(A)=S$\cap$G$\cap\ddot{g}$-cl(A)=S$\cap\ddot{g}$-cl(A).

Converse part is trivial.

Corollary 3.41. *Let A be a subset of (X, τ). If A$\in\ddot{G}$LC**(X), then $\ddot{g}$-cl(A)$-$A is $\ddot{g}$-closed and A$\cup$($\ddot{g}$-cl(A))^c is $\ddot{g}$-open.*

Proof. Let A$\in\ddot{G}$LC**(X). Then by Theorem 3.40, A=S$\cap\ddot{g}$-cl(A) for some open set S and $\ddot{g}$-cl(A)$-$A=$\ddot{g}$-cl(A)$\cap$S^c is $\ddot{g}$-closed in (X, τ). If G=$\ddot{g}$-cl(A)$-$A, then G^c=A$\cup$($\ddot{g}$-cl(A))^c and G^c is $\ddot{g}$-open and so is A$\cup$($\ddot{g}$-cl(A))^c.

4 $\ddot{g}$-dense Sets and $\ddot{g}$-submaximal Spaces

We introduce the following definition.

Definition 4.1. *A subset A of a space (X, τ) is called $\ddot{g}$-dense if $\ddot{g}$-cl(A)=X.*

Example 4.2. *Consider the topological space (X, τ) in Example 3.5. Then the set A=$\{b, c\}$ is $\ddot{g}$-dense in (X, τ).*

Proposition 4.3. *Every $\ddot{g}$-dense set is dense.*

Proof. Let A be an $\ddot{g}$-dense set in (X, τ). Then $\ddot{g}$-cl(A)=X. Since $\ddot{g}$-cl(A)⊆cl(A), we have cl(A)=X and so A is dense.

The converse of Proposition 4.3 need not be true as can be seen from the following example.

Example 4.4. *The set $\{a, c\}$ in Example 3.5 is a dense in (X, τ) but it is not $\ddot{g}$-dense in (X, τ).*

Definition 4.5. *A topological space (X, τ) is called $\ddot{g}$-submaximal if every dense subset in it is $\ddot{g}$-open in (X, τ).*

Proposition 4.6. *Every submaximal space is $\ddot{g}$-submaximal.*

Proof. Let (X, τ) be a submximal space and A be a dense subset of (X, τ). Then A is open. But every open set is $\ddot{g}$-open and so A is $\ddot{g}$-open. Therefore (X, τ) is $\ddot{g}$-submaximal.

The converse of Proposition 4.6 need not be true as can be seen from the following example.

Example 4.7. *For the topological space (X, τ) of Example 3.5, every dense subset is $\ddot{g}$-open and hence (X, τ) is $\ddot{g}$-submaximal. However, the set A=$\{a, b\}$ is dense in (X, τ), but it is not open in (X, τ). Therefore (X, τ) is not submaximal.*

Proposition 4.8. *Every $\ddot{g}$-submaximal space is ω-submaximal.*

Proof. Let (X, τ) be an $\ddot{g}$-submaximal space and A be a dense subset of (X, τ). Then A is $\ddot{g}$-open. But every $\ddot{g}$-open set is ω-open [Remark 2.9 (5)] and so A is ω-open. Therefore is (X, τ) is ω-submaximal.

The converse of Proposition 4.8 need not be true as can be seen from the following example.

Example 4.9. *Consider the topological space (X, τ) in Example 3.8. Then (X, τ) is ω-submaximal but it is not $\ddot{g}$-submaximal, because the set A=$\{b, c\}$ is a dense set in (X, τ) but it is not $\ddot{g}$-open in (X, τ).*

Remark 4.10. *From Propositions 4.6, 4.8 and Theorem 2.11, we have the following diagram:*

$$\begin{array}{ccccccc} \textit{submaximal} & \longrightarrow & \ddot{g}\textit{-submaximal} & \longrightarrow & \omega\textit{-submaximal} & \longrightarrow & g\textit{-submaximal} \\ & & & & & & \downarrow \\ & & & & & & rg\textit{-submaximal} \end{array}$$

Theorem 4.11. *A space (X, τ) is $\ddot{g}$-submaximal if and only if P(X)=$\ddot{G}LC^{\star}$(X).*

Proof. Necessity. Let A∈P(X) and let V=A∪(cl(A))^c. This implies that cl(V)=cl(A)∪ (cl(A))^c=X. Hence cl(V)=X. Therefore V is a dense subset of X. Since (X, τ) is $\ddot{g}$-submaximal, V is $\ddot{g}$-open. Thus A∪(cl(A))^c is $\ddot{g}$-open and by Theorem 3.39, we have A∈$\ddot{G}$LC*(X).

Sufficiency. Let A be a dense subset of (X, τ). This implies A∪(cl(A))^c=A∪X^c=A∪∅ =A. Now A∈$\ddot{G}$LC*(X) implies that A=A∪(cl(A))^c is $\ddot{g}$-open by Theorem 3.39. Hence (X, τ) is $\ddot{g}$-submaximal.

Proposition 4.12. *For subsets A and B in (X, τ), the following are true:*

1. *If A, B∈$\ddot{G}LC$(X), then A∩B∈$\ddot{G}LC$(X).*
2. *If A, B∈$\ddot{G}LC^{\star}$(X), then A∩B∈$\ddot{G}LC^{\star}$(X).*
3. *If A, B∈$\ddot{G}LC^{\star\star}$(X), then A∩B∈$\ddot{G}LC^{\star\star}$(X).*
4. *If A∈$\ddot{G}LC$(X) and B is $\ddot{g}$-open (resp. $\ddot{g}$-closed), then A∩B∈$\ddot{G}LC$(X).*

5. *If $A\in\ddot{G}LC^{\star}(X)$ and B is $\ddot{g}$-open (resp. closed), then $A\cap B\in\ddot{G}LC^{\star}(X)$.*
6. *If $A\in\ddot{G}LC^{\star\star}(X)$ and B is $\ddot{g}$-closed (resp. open), then $A\cap B\in\ddot{G}LC^{\star\star}(X)$.*
7. *If $A\in\ddot{G}LC^{\star}(X)$ and B is $\ddot{g}$-closed, then $A\cap B\in\ddot{G}LC(X)$.*
8. *If $A\in\ddot{G}LC^{\star\star}(X)$ and B is $\ddot{g}$-open, then $A\cap B\in\ddot{G}LC(X)$.*
9. *If $A\in\ddot{G}LC^{\star\star}(X)$ and $B\in\ddot{G}LC^{\star}(X)$, then $A\cap B\in\ddot{G}LC(X)$.*

Proof. By Remark 2.9 and Corollary 2.10., (1) to (8) hold.

(9). Let A=S∩G where S is open and G is $\ddot{g}$-closed and B=P∩Q where P is $\ddot{g}$-open and Q is closed. Then A∩B=(S∩P)∩(G∩Q) where S∩P is $\ddot{g}$-open and G∩Q is $\ddot{g}$-closed, by Corollary 2.10. Therefore $A\cap B\in\ddot{G}LC(X)$.

Remark 4.13. *Union of two $\ddot{g}$-lc sets (resp. $\ddot{g}$-lc* sets, $\ddot{g}$-lc** sets) need not be an $\ddot{g}$-lc set (resp. $\ddot{g}$-lc* set, $\ddot{g}$-lc** set) as can be seen from the following examples.*

Example 4.14. *Let X={a, b, c} with τ={∅, {a}, {a, b}, X}. Then $\ddot{G}LC(X)$={∅, {a}, {b}, {c}, {a, b}, {b, c}, X}. Then the sets {a} and {c} are $\ddot{g}$-lc sets, but their union {a, c}$\notin\ddot{G}LC(X)$.*

Example 4.15. *Let X={a, b, c} with τ ={∅, {b}, {a, b}, X}. Then $\ddot{G}LC^{\star}(X)$={∅, {a}, {b}, {c}, {a, b}, {a, c}, X}. Then the sets {b} and {c} are $\ddot{g}$-lc* sets, but their union {b, c}$\notin\ddot{G}LC^{\star}(X)$.*

Example 4.16. *Let X={a, b, c} with τ={∅, {b}, {b, c}, X}. Then $\ddot{G}LC^{\star\star}(X)$={∅, {a}, {b}, {c}, {a, c}, {b, c}, X}. Then the sets {a} and {b} are $\ddot{g}$-lc** sets, but their union {a, b}$\notin\ddot{G}LC^{\star\star}(X)$.*

We introduce the following definition.

Definition 4.17. *Let A and B be subsets of (X, τ). Then A and B are said to be $\ddot{g}$-separated if A∩$\ddot{g}$-cl(B)=∅ and $\ddot{g}$-cl(A)∩B=∅.*

Example 4.18. *For the topological space (X, τ) of Example 3.5. Let A={b} and let B={c}. Then $\ddot{g}$-cl(A)={a, b} and $\ddot{g}$-cl(B)={a, c} and so the sets A and B are $\ddot{g}$-separated.*

Proposition 4.19. *For a topological space (X, τ), the followings are true:*

1. *Let A, $B\in\ddot{G}LC(X)$. If A and B are $\ddot{g}$-separated then $A\cup B\in\ddot{G}LC(X)$.*
2. *Let A, $B\in\ddot{G}LC^{\star}(X)$. If A and B are separated (i.e., A∩cl(B)=∅ and cl(A)∩B=∅), then $A\cup B\in\ddot{G}LC^{\star}(X)$.*
3. *Let A, $B\in\ddot{G}LC^{\star\star}(X)$. If A and B are $\ddot{g}$-separated then $A\cup B\in\ddot{G}LC^{\star\star}(X)$.*

Proof. (1) Since A, $B\in\ddot{G}LC(X)$, by Theorem 3.38, there exist $\ddot{g}$-open sets U and V of (X, τ) such that A=U∩$\ddot{g}$-cl(A) and B=V∩$\ddot{g}$-cl(B). Now G=U∩(X−$\ddot{g}$-cl(B)) and H=V∩(X−$\ddot{g}$-cl(A)) are $\ddot{g}$-open subsets of (X, τ). Since A∩$\ddot{g}$-cl(B)=∅, A⊆($\ddot{g}$-cl(B))^c. Now A=U∩$\ddot{g}$-cl(A) becomes A∩($\ddot{g}$-cl(B))^c=G∩$\ddot{g}$-cl(A). Then A=G∩$\ddot{g}$-cl(A). Similarly B=H∩$\ddot{g}$-cl(B). Moreover G∩$\ddot{g}$-cl(B)=∅ and H∩$\ddot{g}$-cl(A)=∅. Since G and H are $\ddot{g}$-open sets of (X, τ), G∪H is $\ddot{g}$-open. Therefore A∪B=(G∪H)∩$\ddot{g}$-cl(A∪B) and hence $A\cup B\in\ddot{G}LC(X)$.

(2) and (3) are similar to (1), using Theorems 3.39 and 3.40.

Remark 4.20. *The assumption that A and B are $\ddot{g}$-separated in (1) of Proposition 4.19 cannot be removed. In the topological space (X, τ) in Example 4.14, the sets {a} and {c} are not $\ddot{g}$-separated and their union {a, c}$\notin\ddot{G}LC(X)$.*

5 Conclusion

Topology is an area of Mathematics concerned with the properties of space that are preserved under continuous deformations including stretching and bending, but not tearing. By the middle of the 20th century, topology had become a major branch of Mathematics.

Topology as a branch of Mathematics can be formally defined as the study of qualitative properties of certain objects that are invariant under a certain kind of transformation especially those properties that are invariant under a certain kind of equivalence and it is the study of those properties of geometric configurations which remain invariant when these configurations are subjected to one-to-one bicontinuous transformations or homeomorphisms. Topology operates with more general concepts than analysis. Differential properties of a given transformation are nonessential for topology but bicontinuity is essential. As a consequence, topology is often suitable for the solution of problems to which analysis cannot give the answer.

Though the concept of topology has been identified as a difficult territory in Mathematics, we have taken it up as a challenge and cherishingly worked out this research study. It can also further up the understanding of basic structure of classical mathematics and offers new methods and results in obtaining significant results of classical mathematics. Moreover it also has applications in some important fields of Science and Technology.

In this paper we introduced and studied the classes of $\ddot{g}$-locally closed sets, $\ddot{g}$-lc* sets and $\ddot{g}$-lc** sets which are weaker forms of the class of locally closed sets. Furthermore the relations with other notions connected with the forms of locally closed sets are investigated.

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SOFT π-OPEN SETS IN SOFT GENERALIZED TOPOLOGICAL SPACES

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Abstract – The main purpose of this paper is to study some interesting properties of the soft mapping $\pi : S(U)_E \rightarrow S(U)_E$ which satisfy the condition $\pi F_B \subset \pi F_D$ whenever $F_B \subset F_D \subset F_{\tilde{E}}$. A new class of generalized soft open sets, called soft π-open sets is introduced and studied their basic properties. A soft set $F_G \subset F_{\tilde{E}}$ is said to be a soft π-open set iff $F_G \subset \pi F_G$. The notions of soft interior and soft closure are generalized using these sets. We then introduce the concepts of soft π-interior $i_\pi F_G$, soft π-closure $c_\pi F_G$, soft $\pi^* F_G$ of a soft set $F_G \subset F_{\tilde{E}}$. Under suitable conditions on π, the soft π-interior $i_\pi F_G$ and the soft π-closure $c_\pi F_G$ of a soft set $F_G \subset F_{\tilde{E}}$ are easily obtained by explicit formulas. The soft μ-semi-open sets, soft μ-pre-open sets, soft μ-α-open sets and soft μ-β-open sets for a given Soft Generalized Topological Space $(F_{\tilde{E}}, \mu)$ can be obtained from soft π-open sets which are important for further research on soft generalized topology.

Keywords – *Soft sets, soft generalized topology, soft mapping, soft π-open sets, soft π-interior, soft π-closure.*

1 Introduction

The concept of soft set theory was introduced by Molodtsov [19] in 1999 as a mathematical tool for modeling uncertainties. Molodtsov successfully applied the soft set theory in several directions such as game theory, probability, Perron and Riemann Integration, theory of measurements [20]. Maji et al [17] and Naim Cagman et al. [5] have further modified the theory of soft sets which is similar to that of Molodtsov. After the introduction of the notion of soft sets, several researchers improved this concept. Cagman [6] presented the soft matrix theory and set up the maximum decision making method. D. Pei and D Miao [21] showed that soft sets are a class of special information systems. Babitha and Sunil [4] studied the soft set relation and discussed some related concepts. Kharal et al. [16] introduced soft functions over classes of soft sets. The notion of soft ideal is initiated for

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the first time by Kandil et al. [13]. Feng et al. [9] worked on soft semi rings, soft ideals and idealistic soft semi rings.

It is known that topology is an important area of mathematics, with many applications in the domain of computer science and physical sciences. Topological structure of soft sets was also studied by many researchers. Shabir and Naz [22] and Cagman [7] initiated the study of soft topology and soft topological spaces independently. Shabir and Naz defined soft topology on the collection of soft sets over an initial universe with a fixed set of parameters. On the other hand, Cagman et al. [7] introduced soft topology on a soft set and defined soft topological space. The notion of soft topology by Cagman is more general than that by Shabir and Naz. B Ahmad and S Hussain [1] explored the structures of soft topology using soft points. Weak forms of soft open sets were first studied by Chen [8]. He investigated soft semi-open sets in soft topological spaces and studied some properties of it. Arockiarani and Lancy [3] are defined soft β-open sets and continued to study weak forms of soft open sets in soft topological space. Akdag and Ozkan [2], defined soft α-open and soft α-closed sets in soft topological spaces and studied many important results and some properties of it. Soft pre-open sets were introduced by [3]. Kandil et al. [14] introduced a unification of some types of different kinds of subsets of soft topological spaces using the notion of γ-operations. Kandil et al. [15] generalize this unification of types of different kinds of subsets of soft topological spaces using the notion of γ-oprations to supra topological spaces. Soft generalized topology is relatively new and promising domain which can lead to the development of new mathematical models and innovative approaches that will significantly contribute to the solution of complex problems in engineering and environment. Jyothis and Sunil [10] introduced the notion of soft generalized topology (SGT) on a soft set and studied basic concepts of soft generalized topological spaces (SGTS). It is showed that a soft generalized topological space gives a parameterized family of generalized topological space. They also define and discuss the properties of soft generalized separation axioms which are important for further research on soft topology [12]. Jyothis and Sunil [11] introduced the concept of soft μ-compactness in soft generalized topological spaces as a generalization of compact spaces.

This paper is organized as follows. In section 2, we begin with the basic definitions and important results related to soft set theory which are useful for subsequent sections. In section 3, the definitions and basic theorems of soft generalized topology on an initial soft set are given. Finally in section 4, we study some interesting properties of the soft mapping $\pi : S(U)_E \to S(U)_E$ which satisfy the condition $\pi F_B \subset \pi F_D$ whenever $F_B \subset F_D \subset F_{\tilde{E}}$. We introduce the concept of soft π-open sets and study their basic properties. The most important special cases are obtained if μ is a SGT, i_μ and c_μ denote the soft μ-interior and soft μ-closure respectively, and $\pi = c_\mu i_\mu$, $\pi = i_\mu c_\mu$, $\pi = i_\mu c_\mu i_\mu$ and $\pi = c_\mu i_\mu c_\mu$. The corresponding soft π-open sets are called the soft μ-semi-open sets, soft μ-pre-open sets, soft μ-α-open sets and soft μ-β-open sets. Under suitable conditions on π, the soft π-interior $i_\pi F_G$ and the soft π-closure $c_\pi F_G$ of a soft set $F_G \subset F_{\tilde{E}}$ are easily obtained by explicit formulas.

2 Preliminaries

In this section we recall some definitions and results defined and discussed in [5, 10, 11, 16]. Throughout this paper U denotes the initial universe, E denotes the set of all possible parameters, $\mathcal{P}(U)$ is the power set of U and A is a nonempty subset of E.

Definition 2.1. A soft set F_A on the universe U is defined by the set of ordered pairs $F_A = \{(e, f_A(e)) \,/\, e \in E, f_A(e) \in \mathcal{P}(U)\}$, where $f_A : E \to \mathcal{P}(U)$ such that $f_A(e) = \emptyset$ if $e \notin A$. Here f_A is called an approximate function of the soft set F_A. The value of $f_A(e)$ may be arbitrary. Some of them may be empty, some may have nonempty intersection. The set of all soft sets over U with E as the parameter set will be denoted by $S(U)_E$ or simply S(U).

Definition 2.2. Let $F_A \in S(U)$. If $f_A(e) = \emptyset$ for all $e \in E$, then F_A is called an empty soft set, denoted by $F_\emptyset$. $f_A(e) = \emptyset$ means that there is no element in U related to the parameter e in E. Therefore we do not display such elements in the soft sets as it is meaningless to consider such parameters.

Definition 2.3. Let $F_A \in S(U)$. If $f_A(e) = U$ for all $e \in A$, then F_A is called an A-universal soft set, denoted by $F_{\tilde{A}}$. If A = E, then the A-universal soft set is called an universal soft set, denoted by $F_{\tilde{E}}$.

Definition 2.4. Let $F_A, F_B \in S(U)$. Then F_B is a soft subset of F_A (or F_A is a soft superset of F_B), denoted by $F_B \subseteq F_A$, if $f_B(e) \subseteq f_A(e)$, for all $e \in E$.

Definition 2.5. Let $F_A, F_B \in S(U)$. Then F_B and F_A are soft equal, denoted by $F_B = F_A$, if $f_B(e) = f_A(e)$, for all $e \in E$.

Definition 2.6. Let $F_A, F_B \in S(U)$. Then, the soft union of F_A and F_B, denoted by $F_A \cup F_B$, is defined by the approximate function $f_{A\cup B}(e) = f_A(e) \cup f_B(e)$.

Definition 2.7. Let $F_A, F_B \in S(U)$. Then, the soft intersection of F_A and F_B, denoted by $F_A \cap F_B$, is defined by the approximate function $f_{A\cap B}(e) = f_A(e) \cap f_B(e)$.

Definition 2.8. Let $F_A, F_B \in S(U)$. Then, the soft difference of F_A and F_B, denoted by $F_A \setminus F_B$, is defined by the approximate function $f_{A\setminus B}(e) = f_A(e) \setminus f_B(e)$.

Definition 2.9. Let $F_A \in S(U)$. Then, the soft complement of F_A, denoted by $(F_A)^c$, is defined by the approximate function $f_{A^c}(e) = (f_A(e))^c$, where $(f_A(e))^c$ is the complement of the set $f_A(e)$, that is, $(f_A(e))^c = U \setminus f_A(e)$ for all $e \in E$.

Cleary $((F_A)^c)^c = F_A$, $(F_\emptyset)^c = F_{\tilde{E}}$, and $(F_{\tilde{E}})^c = F_\emptyset$.

Definition 2.10. Let $F_A \in S(U)$. The soft power set of F_A, denoted by $\mathcal{P}(F_A)$, is defined by $\mathcal{P}(F_A) = \{F_{A_i} \,/\, F_{A_i} \subseteq F_A, i \in J \subseteq N\}$.

Theorem 2.11. Let $F_A, F_B, F_C \in S(U)$. Then,

(1) $F_A \cup F_A = F_A$.
(2) $F_A \cap F_A = F_A$.
(3) $F_A \cup F_\emptyset = F_A$.
(4) $F_A \cap F_\emptyset = F_\emptyset$.
(5) $F_A \cup F_{\tilde{E}} = F_{\tilde{E}}$.
(6) $F_A \cap F_{\tilde{E}} = F_A$.
(7) $F_A \cup (F_A)^c = F_{\tilde{E}}$.
(8) $F_A \cap (F_A)^c = F_\emptyset$.
(9) $F_A \cup F_B = F_B \cup F_A$.

(10) $F_A \cap F_B = F_B \cap F_A$.
(11) $(F_A \cup F_B)^c = (F_A)^c \cap (F_B)^c$.
(12) $(F_A \cap F_B)^c = (F_A)^c \cup (F_B)^c$.
(13) $(F_A \cup F_B) \cup F_C = F_A \cup (F_B \cup F_C)$.
(14) $(F_A \cap F_B) \cap F_C = F_A \cap (F_B \cap F_C)$.
(15) $F_A \cup (F_B \cap F_C) = (F_A \cup F_B) \cap (F_A \cup F_C)$.
(16) $F_A \cap (F_B \cup F_C) = (F_A \cap F_B) \cup (F_A \cap F_C)$.

Definition 2.12. [16] Let $S(U)_E$ and $S(V)_K$ be the families of all soft sets over U and V, respectively. Let $\varphi : U \to V$ and $\chi : E \to K$ be two mappings. The soft mapping

$$\varphi_\chi : S(U)_E \to S(V)_K$$

is defined as:

(1) Let F_A be a soft set in $S(U)_E$. The image of F_A under the soft mapping φ_χ is the soft set over V, denoted by $\varphi_\chi(F_A)$ and is defined by

$$\varphi_\chi(f_A)(k) = \begin{cases} \bigcup_{e \in \chi^{-1}(k) \cap A} \varphi\big(f_A(e)\big), & \text{if } \chi^{-1}(k) \cap A \neq \emptyset; \\ \emptyset, & \text{otherwise} \end{cases}$$

for all $k \in K$.

(2) Let G_B be a soft set in $S(V)_K$. The inverse image of G_B under the soft mapping φ_χ is the soft set over U, denoted by $\varphi_\chi{}^{-1}(G_B)$ and is defined by

$$\varphi_\chi{}^{-1}(g_B)(e) = \begin{cases} \varphi^{-1}(g_B(\chi(e))), & \text{if } \chi(e) \in B; \\ \emptyset, & \text{otherwise} \end{cases}$$

for all $e \in E$.

The soft mapping φ_χ is called injective, if φ and χ are injective. The soft mapping φ_χ is called surjective, if φ and χ are surjective.

The soft mapping from $S(U)_E$ to itself is denoted by $\varphi: S(U)_E \to S(U)_E$

Definition 2.13. Let $\varphi_\chi: S(U)_E \to S(V)_K$ and $\tau_\sigma: S(V)_K \to S(W)_L$, then the soft composition of the soft mappings φ_χ and τ_σ, denoted by $\tau_\sigma \circ \varphi_\chi$, is defined by $\tau_\sigma \circ \varphi_\chi = (\tau \circ \varphi)_{(\sigma \circ \chi)}$.

3 Soft Generalized Topological Spaces

Definition 3.1. [10] Let $F_A \in S(U)$. A Soft Generalized Topology (SGT) on F_A, denoted by μ or μ_{F_A} is a collection of soft subsets of F_A having the following properties:

(1) $F_\emptyset \in \mu$
(2) Any soft union of members of μ belongs to μ.

The pair (F_A, μ) is called a Soft Generalized Topological Space (SGTS)
Observe that $F_A \in \mu$ must not hold.

Definition 3.2. [10] A soft generalized topology μ on F_A is said to be strong if $F_A \in \mu$.

Definition 3.3. [10] Let (F_A, μ) be a SGTS. Then, every element of μ is called a soft μ–open set.

Definition 3.4. [10] Let (F_A, μ) be a SGTS and $F_B \subseteq F_A$. Then the collection $\mu_{F_B} = \{F_D \cap F_B \,/\, F_D \in \mu\}$ is called a Subspace Soft Generalized Topology (SSGT) on F_B. The pair $(F_B, \ \mu_{F_B})$ is called a Soft Generalized Topological Subspace (SGTSS) of F_A.

Definition 3.5. [10] Let (F_A, μ) be a SGTS and $F_B \subseteq F_A$. Then the soft μ-interior of F_B denoted by $i_\mu(F_B)$ is defined as the soft union of all soft μ-open subsets of F_B.
Note that $i_\mu(F_B)$ is the largest soft μ-open set that is contained in F_B.

Theorem 3.6. [10] Let (F_A, μ) be a SGTS and $F_B \subseteq F_A$. Then F_B is a soft μ-open set if and only if $F_B = i_\mu(F_B)$.

Theorem 3.7. [10] Let (F_A, μ) be a SGTS and $F_G, F_H \subseteq F_A$. Then

(1) $i_\mu(i_\mu(F_G)) = i_\mu(F_G)$
(2) $F_G \subseteq F_H \Rightarrow i_\mu(F_G) \subseteq i_\mu(F_H)$
(3) $i_\mu(F_G) \cap i_\mu(F_H) \supseteq i_\mu(F_G \cap F_H)$
(4) $i_\mu(F_G) \cup i_\mu(F_H) \subseteq i_\mu(F_G \cup F_H)$
(5) $i_\mu(F_G) \subseteq F_G$.

Definition 3.8. [10] Let (F_A, μ) be a SGTS and $F_B \subseteq F_A$. Then F_B is said to be a soft μ-closed set if its soft complement $(F_B)^c$ is a soft μ-open set.

Theorem 3.9. [10] Let (F_A, μ) be a SGTS and $F_B \subseteq F_A$. Then the following conditions hold:

(1) The universal soft set $F_{\tilde{E}}$ is soft μ–closed.
(2) Arbitrary soft intersections of the soft μ–closed sets are soft μ–closed.

Definition 3.10. [10] Let (F_A, μ) be a SGTS and $F_B \subseteq F_A$. Then the soft μ-closure of F_B, denoted by $c_\mu(F_B)$ is defined as the soft intersection of all soft μ-closed super sets of F_B.
Note that $c_\mu(F_B)$ is the smallest soft μ-closed superset of F_B.

Theorem 3.11. [10] Let (F_A, μ) be a SGTS and $F_B \subseteq F_A$. F_B is a soft μ-closed set if and only if $F_B = c_\mu(F_B)$.

Theorem 3.12. [10] Let (F_A, μ) be a SGTS and $F_B \subseteq F_A$. Then $i_\mu(F_B) \subseteq F_B \subseteq c_\mu(F_B)$

Theorem 3.13. [10] Let (F_A, μ) be a SGTS and $F_G, F_H \subseteq F_A$. Then

(1) $c_\mu(c_\mu(F_G)) = c_\mu(F_G)$
(2) $F_G \subseteq F_H \Rightarrow c_\mu(F_G) \subseteq c_\mu(F_H)$
(3) $c_\mu(F_G) \cap c_\mu(F_H) \supseteq c_\mu(F_G \cap F_H)$

(4) $c_\mu(F_G) \cup c_\mu(F_H) \subseteq c_\mu(F_G \cup F_H)$

4 Soft π-Open Sets

Consider the soft mapping $\pi : S(U)_E \to S(U)_E$ possessing the property of monotony, i.e, $F_B \subset F_D$ imply $\pi F_B \subset \pi F_D$. We denote the collection of all soft mapping having this property by Π. Consider the following conditions for a soft mapping $\pi \in \Pi$, $F_B \subset F_{\tilde{E}}$

(Π0) $\pi F_\emptyset = F_\emptyset$

(Π1) $\pi F_{\tilde{E}} = F_{\tilde{E}}$

(Π2) $\pi^2 F_B = \pi\pi F_B = \pi F_B$

(Π3) $F_B \subset \pi F_B$,

(Π4) $\pi F_B \subset F_B$,

(Π5) $\pi^2 F_B \subset \pi F_B$,

Example 4.1. The soft identity mapping *id*: $S(U)_E \to S(U)_E \in$ (Π0), (Π1), (Π2), (Π3), (Π4).

Let $(F_{\tilde{E}}, \mu)$ be a SGTS and $i_\mu : S(U)_E \to S(U)_E$ and $c_\mu : S(U)_E \to S(U)_E$ be the soft μ-interior and soft μ-closure operators respectively. If $\pi = i_\mu$, then $\pi \in$ (Π0), (Π2), (Π4). If $\pi = c_\mu$, then $\pi \in$ (Π1), (Π2), (Π3).

Definition 4.2. A soft set $F_G \subset F_{\tilde{E}}$ is said to be a soft π-open set iff $F_G \subset \pi F_G$.

Example 4.3. The following are some examples of soft π-open sets:

1. $F_\emptyset$ is always soft π-open for any $\pi \in \Pi$
2. $F_{\tilde{E}}$ is soft π-open iff $\pi \in$ (Π1)
3. Every soft set of the form πF_G is soft π-open if $\pi \in$ (Π2)
4. Every soft subset of $F_{\tilde{E}}$ is soft π-open if $\pi \in$ (Π3)
5. If $\pi \in$ (Π4), then F_G is soft π-open iff $F_G = \pi F_G$

Note: Let $(F_{\tilde{E}}, \mu)$ be a SGTS. Then F_G is soft i_μ-open (i.e, if $\pi = i_\mu$) iff $F_G \subset i_\mu F_G$. But $i_\mu F_G \subset F_G$. Thus F_G is soft i_μ-open iff $F_G = i_\mu F_G$ iff F_G is soft μ-open by theorem 3.6. Hence soft i_μ-open set coincides with the soft μ-open sets.

Theorem 4.4. Any soft union of soft π–open sets is soft π–open.

Proof. Let $\{F_{Bj}\}_{j\in J}$ be a collection of soft π-open sets. i.e, $F_{Bj} \subset \pi F_{Bj}$ $\forall$ $j \in J$. Let $F_B = \bigcup_{j\in J} F_{Bj}$. Now $F_{Bj} \subset F_B$ imply $\pi F_{Bj} \subset \pi F_B$ $\forall$ $j \in J$. Therefore $F_B = \bigcup_{j\in J} F_{Bj} \subset \bigcup_{j\in J} \pi F_{Bj} \subset \pi F_B$. i.e, $F_B \subset \pi F_B$. Hence F_B is soft π-open.■

Theorem 4.5. The collection of all soft π-open sets is a SGT.

Theorem 4.6. If μ is a SGT on $F_{\tilde{E}}$, then there is a soft mapping $\pi \in$ (Π0), (Π2), (Π4) such that μ is the collection of all soft π-open sets.

Proof. Define πF_G to be the soft union of all $F_H \in \mu$ satisfying $F_H \subset F_G$. Then clearly $\pi F_G \in \mu$ and $\pi F_G \subset F_G$, $\pi F_\emptyset = F_\emptyset$. Now $F_H \in \mu \Rightarrow \pi F_H = F_H \supset F_H$ so that the elements of μ are soft π-open, while $F_G \subset \pi F_G \Rightarrow \pi F_G = F_G$ and $F_G \in \mu$. Finally $\pi F_G \in \mu \Rightarrow \pi\pi F_G = \pi F_G$.■

Definition 4.7. Let $F_B \subset F_{\tilde{E}}$. The soft union of all soft π-open subsets of the soft set F_B is called the soft π-interior of F_B, and is denoted by $i_\pi F_B$.

Theorem 4.8. The soft set $i_\pi F_B$ is the largest soft π-open subset of F_B.

Note: Let $(F_{\tilde{E}}, \mu)$ be a SGTS and suppose $\pi = i_\mu$, then the soft set $i_{i\mu} F_B$ is the largest soft i_μ-open subset of F_B. Since soft i_μ-open sets are soft μ-open sets, $i_{i\mu} F_B$ is the largest soft μ-open subset of F_B. Hence $i_{i\mu} = i_\mu$.

Theorem 4.9. For any $\pi \in \Pi$ and $F_B \subset F_{\tilde{E}}$,

i) $i_\pi F_\emptyset = F_\emptyset$
ii) $i_\pi F_B = i_\pi i_\pi F_B$
iii) $i_\pi F_B \subset F_B$, and
iv) $i_\pi F_{\tilde{E}} = F_{\tilde{E}}$ iff $\pi F_{\tilde{E}} = F_{\tilde{E}}$

i.e, $i_\pi \in (\Pi 0)$, $(\Pi 2)$ and $(\Pi 4)$ for any $\pi \in \Pi$; $i_\pi \in (\Pi 1)$ iff $\pi \in (\Pi 1)$
Conversely if $\pi \in (\Pi 0)$, $(\Pi 2)$and $(\Pi 4)$, then $\pi = i_\pi$.

Proof. First show that i_π possess the property of monotony. Suppose $F_G \subset F_H$. By definition of i_π and by theorem 4.8, $i_\pi F_G \subset F_G$ and $i_\pi F_H \subset F_H$. $i_\pi F_H$ is the largest soft π-open subset of F_H. Hence $i_\pi F_G \subset i_\pi F_H$. Clearly $i_\pi F_\emptyset = F_\emptyset$. i.e, $i_\pi \in (\Pi 0)$. By definition 4.7, $i_\pi F_G \subset F_G$ for any $F_G \subset F_{\tilde{E}}$. i.e, $i_\pi \in (\Pi 4)$. By theorem 4.8, $i_\pi F_G$ is soft π –open, so $i_\pi(i_\pi F_G)$ = largest soft π-open subset of $i_\pi F_G = i_\pi F_G$. i.e, $i_\pi \in (\Pi 2)$. Again $i_\pi F_{\tilde{E}}$ = largest soft π-open subset of $F_{\tilde{E}} = F_{\tilde{E}}$ $\Leftrightarrow$ $F_{\tilde{E}}$ is a soft π-open set $\Leftrightarrow \pi \in (\Pi 1)$.

Conversely, assume that $\pi \in (\Pi 0)$, $(\Pi 2)$ and $(\Pi 4)$. $\pi \in (\Pi 2) \Rightarrow \pi(\pi F_G) = \pi F_G \Rightarrow \pi F_G$ is soft π-open. $\pi \in (\Pi 4) \Rightarrow \pi F_G \subset F_G$ for any $F_G \subset F_{\tilde{E}}$. Therefore πF_G is a soft π-open subset of F_G. Next if $F_H \subset F_G$ is soft π-open, then $F_H \subset \pi F_H \subset \pi F_G$. So πF_G = largest soft π-open subset of F_G. Hence $i_\pi = \pi$.■

Theorem 4.10. A soft set F_G is soft i_π-open iff $F_G = i_\pi F_G$ iff F_G is soft π-open.

Proof. i_π possess the property of monotony. i.e, if $F_G \subset F_H$, then $i_\pi F_G \subset i_\pi F_H$. Also $i_\pi F_G \subset F_G$ for any $F_G \subset F_{\tilde{E}}$. Now F_G is soft i_π-open iff $F_G \subset i_\pi F_G$ iff $F_G = i_\pi F_G$ iff F_G is soft π-open by theorem 4.8.■

Definition 4.11. A soft set $F_G \subset F_{\tilde{E}}$ is soft π-closed iff its soft complement $(F_G)^c$ is soft π-open.

Note: 1) Since $F_\emptyset$ is always soft π-open, $F_{\tilde{E}}$ is always soft π-closed, for any $\pi \in \Pi$
2) $F_\emptyset$ is soft π-closed iff $F_{\tilde{E}}$ is soft π-open iff $\pi \in (\Pi 1)$
3) If $\pi \in (\Pi 3)$, every soft subset of $F_{\tilde{E}}$ is soft π-closed.

Theorem 4.12. Any soft intersection of soft π-closed sets is soft π-closed.

Proof. Suppose $\{F_{Gj}\}_{j\in J}$ be a collection of soft π-closed sets. Then $\{(F_{Gj})^c\}_{j\in J}$ is a collection of soft π-open sets. By theorem 4.4, $\bigcup_{j\in J}(F_{Gj})^c$ is soft π-open $\Rightarrow$ $(\bigcap_{j\in J}F_{Gj})^c$ is soft π-open $\Rightarrow$ $\bigcap_{j\in J}F_{Gj}$ is soft π-closed.■

Theorem 4.13. Let ξ be the collection of all soft π-closed sets. Then the following conditions hold.

1. The universal soft set $F_{\tilde{E}} \in \xi$.
2. Arbitrary soft intersection of members of ξ belongs to ξ.

Definition 4.14. The soft intersection of all soft π-closed supersets of F_G is called the soft π-closure of F_G and is denoted by $c_\pi F_G$.

Theorem 4.15. The soft set $c_\pi F_G$ is the smallest soft π-closed super set of F_G.

Note: Let $(F_{\tilde{E}}, \mu)$ be a SGTS and if $\pi = i_\mu$, then F_G is soft i_μ-closed set $\Leftrightarrow$ $(F_G)^c$ is soft i_μ-open $\Leftrightarrow$ $(F_G)^c$ is soft μ-open $\Leftrightarrow$ F_G is soft μ-closed. Hence soft i_μ-closed sets coincides with the soft μ-closed ones and $c_{i\mu} = c_\mu$

Definition 4.16. For any $\pi \in \Pi$ and $F_G \subset F_{\tilde{E}}$, $\pi^*F_G = [\pi(F_G)^c]^c$.

Theorem 4.17. For any $\pi \in \Pi$, the following conditions hold:

$\pi^* \in \Pi$, $(\pi^*)^* = \pi$, $\pi \in (\Pi0) \Leftrightarrow \pi^* \in (\Pi1)$, $\pi \in (\Pi1) \Leftrightarrow \pi^* \in (\Pi0)$, $\pi \in (\Pi2) \Leftrightarrow \pi^* \in (\Pi2)$, $\pi \in (\Pi3) \Leftrightarrow \pi^* \in (\Pi4)$, $(i_\pi)^* = c_\pi$.

Proof . Assume that $\pi \in \Pi$, i.e, if $F_G \subset F_H$, then $\pi F_G \subset \pi F_H$. Now $F_G \subset F_H \Rightarrow (F_G)^c \supset (F_H)^c \Rightarrow \pi(F_G)^c \supset \pi(F_H)^c \Rightarrow (\pi(F_G)^c)^c \subset (\pi(F_H)^c)^c \Rightarrow \pi^*F_G \subset \pi^*F_H$. Hence $\pi^* \in \Pi$. $\pi^*F_G = (\pi(F_G)^c)^c$. $\therefore (\pi^*)^* F_G = [\pi^*(F_G)^c]^c = [(\pi F_G)^c]^c = \pi F_G$. Hence $(\pi^*)^* = \pi$. $\pi \in (\Pi0) \Leftrightarrow \pi F_\emptyset = F_\emptyset \Leftrightarrow (\pi F_\emptyset)^c = F_{\tilde{E}} \Leftrightarrow (\pi(F_{\tilde{E}})^c)^c = F_{\tilde{E}} \Leftrightarrow \pi^*F_{\tilde{E}} = F_{\tilde{E}} \Leftrightarrow \pi^* \in (\Pi1)$. $\pi \in (\Pi1) \Leftrightarrow \pi F_{\tilde{E}} = F_{\tilde{E}} \Leftrightarrow (\pi F_{\tilde{E}})^c = F_\emptyset \Leftrightarrow (\pi(F_\emptyset)^c)^c = F_\emptyset \Leftrightarrow \pi^*F_\emptyset = F_\emptyset \Leftrightarrow \pi^* \in (\Pi0)$. $\pi \in (\Pi2) \Leftrightarrow \pi(\pi(F_G)^c) = \pi(F_G)^c \Leftrightarrow [\pi(\pi(F_G)^c)]^c = [\pi(F_G)^c]^c \Leftrightarrow [\pi(\pi^*F_G)^c]^c = \pi^*F_G \Leftrightarrow \pi^*(\pi^*F_G) = \pi^*F_G \Leftrightarrow \pi^* \in (\Pi2)$. $\pi \in (\Pi3) \Leftrightarrow (F_G)^c \subset \pi(F_G)^c \Leftrightarrow F_G \supset (\pi(F_G)^c)^c \Leftrightarrow F_G \supset \pi^*F_G \Leftrightarrow \pi^* \in (\Pi4)$. $(i_\pi)^*F_G = (i_\pi(F_G)^c)^c$. By theorem 4.8, $i_\pi(F_G)^c$ is the largest soft π-open subset of $(F_G)^c$. Hence its soft complement coincides with the smallest soft π-closed super set of F_G. i.e, $(i_\pi)^*F_G = c_\pi F_G$ for any $F_G \subset F_{\tilde{E}}$. Hence $(i_\pi)^* = c_\pi$.■

Theorem 4.18. Let $(F_{\tilde{E}}, \mu)$ be a SGTS. Then $(i_\mu)^* = c_\mu$.

Proof. Take $\pi = i_\mu$ and since $i_{i\mu} = i_\mu$, the proof follows from theorem 4.17.■

Theorem 4.19. A soft set $F_G \subset F_{\tilde{E}}$ is soft π^*-closed $\Leftrightarrow \pi F_G \subset F_G$.

Proof. F_G is soft π^*-closed $\Leftrightarrow (F_G)^c$ is soft π^*-open $\Leftrightarrow (F_G)^c \subset \pi^*(F_G)^c \Leftrightarrow (F_G)^c \subset (\pi F_G)^c \Leftrightarrow \pi F_G \subset F_G$.■

Theorem 4.20. For any $\pi \in \Pi$, $c_\pi \in (\Pi1), (\Pi2), (\Pi3)$; $c_\pi \in (\Pi0)$ iff $\pi \in (\Pi1)$. Conversely, if $\pi \in (\Pi1), (\Pi2), (\Pi3)$, then $\pi = c_{\pi^*}$.

Proof. Assume that $F_G \subset F_H \subset F_{\tilde{E}}$. By theorem 4.15, $c_\pi F_H$ is the smallest soft π-closed super set of F_H. But $F_H \supset F_G$. $\therefore c_\pi F_H$ is a soft π-closed super set of F_G. Again by theorem

4.15, $c_\pi F_G$ is the smallest soft π-closed super set of F_G. Hence $c_\pi F_H \supset c_\pi F_G \Rightarrow c_\pi \in \Pi$. Since $F_{\tilde{E}}$ is a soft π-closed set, $c_\pi F_{\tilde{E}} = F_{\tilde{E}} \Rightarrow c_\pi \in (\Pi 1)$. By theorem 4.15, $c_\pi F_G$ is soft π-closed for any $F_G \subset F_{\tilde{E}}$. Therefore $c_\pi(c_\pi F_G) = c_\pi F_G \Rightarrow c_\pi \in (\Pi 2)$. By theorem 4.15, $c_\pi F_G$ is the smallest soft π-closed super set of F_G, $c_\pi F_G \supset F_G \Rightarrow c_\pi \in (\Pi 3)$. $c_\pi F_\emptyset = F_\emptyset \Leftrightarrow F_\emptyset$ is soft π-closed set $\Leftrightarrow F_{\tilde{E}}$ is soft π-open set $\Leftrightarrow F_{\tilde{E}} = \pi F_{\tilde{E}}$. Hence $c_\pi \in (\Pi 0)$ iff $\pi \in (\Pi 1)$.

Conversely, assume that $\pi \in (\Pi 1), (\Pi 2), (\Pi 3)$. Since $\pi \in (\Pi 2)$, $\pi(\pi F_G) = \pi F_G \Rightarrow \pi F_G$ is soft π^*-closed by theorem 4.19. Since $\pi \in (\Pi 3)$, πF_G is soft π^*-closed super set of F_G, for any $F_G \subset F_{\tilde{E}}$. If $F_H \supset F_G$ is a soft π^*-closed set, then by theorem 4.19, $\pi F_H \subset F_H$, so $F_H \supset \pi F_H \supset \pi F_G \supset F_G$. i.e, πF_G is the smallest soft π^*-closed super set of F_G. Hence $\pi = c_{\pi^*}$.■

Theorem 4.21. Any soft set F_G is soft i_π-closed iff $F_G = c_\pi F_G$ iff F_G is soft π-closed.

Proof. By theorem 4.9, $i_\pi \in \Pi$. By theorem 4.17 and 4.19, F_G is soft i_π-closed $\Leftrightarrow F_G$ is soft $((i_\pi)^*)^*$-closed $\Leftrightarrow F_G$ is soft $(c_\pi)^*$-closed $\Leftrightarrow c_\pi F_G \subset F_G \Leftrightarrow c_\pi F_G = F_G \Leftrightarrow F_G$ is soft π-closed by theorem 4.15.■

Theorem 4.22. If $\pi_1, \pi_2 \in \Pi$, $\pi_2\pi_1 \in \Pi$. If π_1 and $\pi_2 \in (\Pi 0), (\Pi 1), (\Pi 3), (\Pi 4)$, then $\pi_2\pi_1 \in (\Pi 0), (\Pi 1), (\Pi 3), (\Pi 4)$ and $(\pi_2\pi_1)^* = \pi_2^*\pi_1^*$.

Suppose the soft mappings $\theta, \sigma \in (\Pi 2)$. We will consider the soft mappings π that are the products of factors θ or σ. Only the products of alternating factors θ, σ need be taken into consideration.

Theorem 4.23. If $\theta, \sigma \in (\Pi 2)$, $\theta\sigma F_G \subset \sigma F_G$, $\theta\sigma F_G \subset \sigma\theta\sigma F_G$, and $\theta F_G \subset \sigma\theta F_G$ for any $F_G \subset F_{\tilde{E}}$. Then $\pi \in (\Pi 2)$ if π is a product of alternating factors θ and σ.

Proof. Clearly $\pi \in \Pi$ by theorem 4.22. Since $\theta\sigma F_G \subset \sigma F_G$, $\theta\sigma(\theta F_G) \subset \sigma(\theta F_G)$ and hence $\sigma\theta\sigma\theta(F_G) \subset \sigma\theta F_G$. Again since $\theta\sigma F_G \subset \sigma F_G$ and $\theta F_G \subset \sigma\theta F_G$, $\theta F_G \subset \sigma\theta F_G \Rightarrow \theta\theta F_G \subset \theta\sigma\theta F_G \Rightarrow \theta F_G \subset \theta\sigma\theta F_G \Rightarrow \sigma\theta F_G \subset \sigma\theta\sigma\theta F_G$. Hence $\sigma\theta\sigma\theta = \sigma\theta \Rightarrow \sigma\theta \in (\Pi 2)$. Since $\theta\sigma F_G \subset \sigma F_G \Rightarrow \sigma\theta\sigma F_G \subset \sigma\sigma F_G \Rightarrow \sigma\theta\sigma F_G \subset \sigma F_G \Rightarrow \theta\sigma\theta\sigma F_G \subset \theta\sigma F_G$. Again $\theta\sigma F_G \subset \sigma\theta\sigma F_G \Rightarrow \theta\theta\sigma F_G \subset \theta\sigma\theta\sigma F_G \Rightarrow \theta\sigma F_G \subset \theta\sigma\theta\sigma F_G$. Hence $\theta\sigma\theta\sigma = \theta\sigma \Rightarrow \theta\sigma \in (\Pi 2)$. Further, since $\theta\sigma\theta\sigma = \theta\sigma$, $(\theta\sigma\theta)(\theta\sigma\theta) = (\theta\sigma\theta\sigma)\theta = \theta\sigma\theta \Rightarrow \theta\sigma\theta \in (\Pi 2)$. $(\sigma\theta\sigma)(\sigma\theta\sigma) = \sigma(\theta\sigma\theta\sigma) = \sigma\theta\sigma \Rightarrow \sigma\theta\sigma \in (\Pi 2)$. $(\theta\sigma\theta\sigma)(\theta\sigma\theta\sigma) = (\theta\sigma)(\theta\sigma) \Rightarrow \theta\sigma\theta\sigma \in (\Pi 2)$. $(\sigma\theta\sigma\theta)(\sigma\theta\sigma\theta) = \sigma(\theta\sigma\theta\sigma)(\theta\sigma\theta) = \sigma(\theta\sigma)(\theta\sigma\theta) = \sigma(\theta\sigma\theta\sigma)\theta = \sigma(\theta\sigma)\theta \Rightarrow \sigma\theta\sigma\theta \in (\Pi 2)$. Again any alternating products of $k \geq 5$ factors is equal to another such product of $(k - 2)$ factor and the statement holds for it.■

Theorem 4.24. If $\theta \in (\Pi 2), (\Pi 4)$ and $\sigma \in (\Pi 2), (\Pi 3)$, then $\pi \in (\Pi 2)$ if π is any product of the factors θ and σ.

Proof. If $\theta \in (\Pi 2), (\Pi 4)$ and $\sigma \in (\Pi 2), (\Pi 3)$, then $\theta\sigma F_G \subset \sigma F_G$, $\theta\sigma F_G \subset \sigma\theta\sigma F_G$, and $\theta F_G \subset \sigma\theta F_G$ for any $F_G \subset F_{\tilde{E}}$. The proof follows from theorem 4.23.

Note: Let $(F_{\tilde{E}}, \mu)$ be a SGTS. Clearly the soft mappings $\theta = i_\mu \in (\Pi 2), (\Pi 4)$ and $\sigma = c_\mu \in (\Pi 2), (\Pi 3)$, so by theorem 4.24 any product of factor i_μ and c_μ is idempotent. In particular $i_\mu c_\mu i_\mu c_\mu = i_\mu c_\mu$ and $c_\mu i_\mu c_\mu i_\mu = c_\mu i_\mu$ so that any product of this kind is equal to one of the mappings i_μ, c_μ, $i_\mu c_\mu$, $c_\mu i_\mu$, $i_\mu c_\mu i_\mu$, $c_\mu i_\mu c_\mu$.

Theorem 4.25. Let $(F_{\tilde{E}}, \mu)$ be a SGTS. Then the soft mappings i_μ, c_μ, $i_\mu c_\mu$, $c_\mu i_\mu$, $i_\mu c_\mu i_\mu$, $c_\mu i_\mu c_\mu$ are all belong to $(\Pi 2)$.

Proof. Take $\pi_1 = i_\mu$ and $\pi_2 = c_\mu$, where $i_\mu F_G$ be the soft μ-interior of the soft set F_G and $c_\mu F_G$ be the soft μ-closure of the soft set F_G w.r. t. the SGT μ. Clearly the soft mappings $\pi_1 = i_\mu \in$ (Π2), (Π4) and $\pi_2 = c_\mu$ (Π2), (Π3). So by theorem 4.24, the soft mappings i_μ, c_μ, $i_\mu c_\mu$, $c_\mu i_\mu$, $i_\mu c_\mu i_\mu$, $c_\mu i_\mu c_\mu$ are all belong to (Π2)■

Definition 4.26. Let $(F_{\tilde{E}}, \mu)$ be a SGTS. Then a soft set $F_G \subset F_{\tilde{E}}$ is said to be a soft μ-semi-open set iff $F_G \subset c_\mu i_\mu F_G$ (i.e, the case when $\pi = c_\mu i_\mu$). The class of all soft μ-semi-open sets is denoted by $\delta_{(\mu)}$ or δ_μ.

Definition 4.27. Let $(F_{\tilde{E}}, \mu)$ be a SGTS. Then a soft set $F_G \subset F_{\tilde{E}}$ is said to be a soft μ-pre-open set iff $F_G \subset i_\mu c_\mu F_G$ (i.e, the case when $\pi = i_\mu c_\mu$). The class of all soft μ-pre-open sets is denoted by $\rho_{(\mu)}$ or ρ_μ.

Definition 4.28. Let $(F_{\tilde{E}}, \mu)$ be a SGTS. Then a soft set $F_G \subset F_{\tilde{E}}$ is said to be a soft μ-α-open set iff $F_G \subset i_\mu c_\mu i_\mu F_G$ (i.e, the case when $\pi = i_\mu c_\mu i_\mu$). The class of all soft μ-α-open set is denoted by $\alpha_{(\mu)}$ or α_μ.

Definition 4.29. Let $(F_{\tilde{E}}, \mu)$ be a SGTS. Then a soft set $F_G \subset F_{\tilde{E}}$ is said to be a soft μ-β-open sets iff $F_G \subset c_\mu i_\mu c_\mu F_G$ (i.e, the case when $\pi = c_\mu i_\mu c_\mu$). The class of all soft μ-β-open set is denoted by $\beta_{(\mu)}$ or β_μ.

Example 4.30. Let $U = \{h_1, h_2, h_3\}$, $E = \{e_1, e_2\}$ and $\mu = \{F_\emptyset, F_A, F_{\tilde{E}}\}$ where $F_A = \{(e_1, \{h_3\}), (e_2, \{h_1\})\}$. Then $(F_{\tilde{E}}, \mu)$ is a SGTS. The Soft set $F_G = \{(e_1, \{h_1, h_3\}), (e_2, \{h_1\})\}$ is a soft μ-semi-open sets

Example 4.31. Let $U = \{h_1, h_2, h_3\}$, $E = \{e_1, e_2\}$ and $\mu = \{F_\emptyset, F_B, F_{\tilde{E}}\}$ where $F_B = \{(e_1, \{h_1, h_2\}), (e_2, \{h_1, h_3\})\}$. Then $(F_{\tilde{E}}, \mu)$ is a SGTS. The Soft sets $F_G = \{(e_1, \{h_2, h_3\}), (e_2, \{h_2\})\}$, $F_H = \{(e_1, \{h_1, h_3\}), (e_2, \{h_2\})\}$ are soft μ-pre-open sets

Example 4.32. Let $U = \{h_1, h_2, h_3\}$, $E = \{e_1, e_2\}$ and $\mu = \{F_\emptyset, F_D, F_{\tilde{E}}\}$ where $F_D = \{(e_1, \{h_1\}), (e_2, \{h_2\})\}$. Then $(F_{\tilde{E}}, \mu)$ is a SGTS. The Soft set $F_G = \{(e_1, \{h_1, h_2\}), (e_2, \{h_2\})\}$ is a soft μ-α-open sets

Example 4.33. Let $U = \{h_1, h_2, h_3, h_4\}$, $E = \{e_1\}$ and $\mu = \{F_\emptyset, F_P, F_Q, F_R, F_S, F_{\tilde{E}}\}$ where $F_P = \{(e_1, \{h_4\})\}$, $F_Q = \{(e_1, \{h_1\})\}$, $F_R = \{(e_1, \{h_1, h_4\})\}$, $F_S = \{(e_1, \{h_1, h_3, h_4\})\}$. Then $(F_{\tilde{E}}, \mu)$ is a SGTS. The Soft set $F_G = \{(e_1, \{h_3, h_4\})\}$ is a soft μ-β-open sets

Theorem 4.34. Let $(F_{\tilde{E}}, \mu)$ be a SGTS. Then δ_μ, ρ_μ, α_μ and β_μ are SGT's.

Proof. Follows from theorem 4.5.■

Now consider the soft mappings $\pi \in$ (Π5)

Theorem 4.35. If $\pi \in \Pi$, then every soft π-open set is soft π^n–open and $\pi F_G \subset \pi^n F_G$ for $n \in \mathbb{N}$.

Proof. Suppose F_G is soft π-open. Then $F_G \subset \pi F_G$. Now $F_G \subset \pi F_G \Rightarrow \pi^m F_G \subset \pi^{m+1} F_G \Rightarrow F_G \subset \pi F_G \subset \pi^n F_G \Rightarrow F_G$ is soft π^n–open.■

Theorem 4.36. If $\pi \in$ (Π5), then $\pi^n F_G \subset \pi F_G$ and soft π-open sets and soft π^n–open sets coincide, $n \in \mathbb{N}$.

Proof. Suppose $\pi \in (\Pi5)$, then $\pi^2F_G \subset \pi F_G \Rightarrow \pi^{m+1}F_G \subset \pi^m F_G$ and $\pi^n F_G \subset \pi F_G$. Hence by theorem 4.35, $\pi F_G = \pi^n F_G$. Threfore soft π-open sets and soft π^n–open sets coincide.■

Theorem 4.37. If $\theta, \sigma \in (\Pi5)$ satisfies $\theta\sigma F_G \subset \sigma F_G$, then any product of factors θ and σ belong to $(\Pi5)$. If both θ and σ occur among the factors of a product of this kind, then

(1) soft $\theta\pi'\sigma$-open $\Rightarrow$ soft $\theta\sigma$-open
(2) soft $\theta\pi'\theta$-open $\Rightarrow$ soft $\theta\sigma\theta$-open
(3) soft $\sigma\pi'\theta$-open $\Rightarrow$ soft $\sigma\theta$-open
(4) soft $\sigma\pi'\sigma$-open $\Rightarrow$ soft $\sigma\theta\sigma$-open

The converse implication is true if no factor θ is immediately followed by another such factor.

Proof. Since $\theta, \sigma \in (\Pi5)$, by theorem 4.36, we have $\theta^n F_G \subset \theta F_G$ and $\sigma^n F_G \subset \sigma F_G$ for $n \in N$ and since $\theta\sigma F_G \subset \sigma F_G$, $(\theta\sigma)^n F_G \subset \sigma^n F_G \subset \sigma F_G$. $\therefore$ $\theta^n\theta^n F_G \subset \theta\theta^n F_G \subset \theta^2\theta^{n-1}F_G \subset \theta\theta^{n-1}F_G \subset \theta^n F_G$. Hence $\pi \in (\Pi5)$ if $\pi = \theta^n$. Suppose π is a product of factors θ and σ, containing atleast one factor σ. Then π can be written in the form $\pi_1\sigma\pi_2$, where π_1 and π_2 (may be empty) are products of factors θ and σ. Then $\pi\pi F_G = \pi_1\sigma\pi_2\pi_1\sigma\pi_2 F_G$. Since $\theta^n F_G \subset \theta F_G$, $\sigma^n F_G \subset \sigma F_G$ and $(\theta\sigma)^n F_G \subset \sigma F_G$, in the product $\sigma\pi_2\pi_1\sigma$, each group of factors θ^p can be replaced by θ, each group of factors σ^q can be replaced by σ, and then $(\theta\sigma)^r$ can be replaced by σ. Therefore $\pi\pi F_G = \pi_1\sigma\pi_2\pi_1\sigma\pi_2 F_G \subset \pi_1\sigma\sigma\pi_2 F_G \subset \pi_1\sigma\pi_2 F_G = \pi F_G$. Hence $\pi \in (\Pi5)$.

Consider (1). Suppose F_G is soft $\theta\pi'\sigma$-open, where π' is any product of both the factors θ and σ. Then $F_G \subset \theta\pi'\sigma F_G$, Now consider the product $\theta\pi'\sigma$, by theorem 4.36, each group of factors θ^p can be replaced by θ and each group of factors σ^q can be replaced by σ, so we can write $\theta\pi'\sigma F_G \subset (\theta\sigma)^n F_G$. $\therefore$ $F_G \subset \theta\pi'\sigma F_G \subset (\theta\sigma)^n F_G = \theta\sigma(\theta\sigma)^{n-1}F_G \subset \theta\sigma\sigma^{n-1}F_G \subset \theta\sigma^n F_G \subset \theta\sigma F_G$ for a suitable $n \in N$. Hence F_G is soft $\theta\sigma$-open. Conversely suppose that F_G is soft $\theta\sigma$-open and no factor θ is followed by another one in $\pi = \theta\pi'\sigma$. Then $F_G \subset \theta\sigma F_G \Rightarrow F_G \subset (\theta\sigma)^m F_G$ by theorem 4.35, where m is the number of the factors σ in the product π. Apply the condition $\theta\sigma F_G \subset \sigma F_G$ repeatedly, then it is easy to show that $(\theta\sigma)^m F_G \subset \theta\pi'\sigma F_G$. Hence F_G is soft $\theta\pi'\sigma$-open.

Consider (2). Suppose F_G is soft $\theta\pi'\theta$-open, where π' is any product of both the factors θ and σ. Then $F_G \subset \theta\pi'\theta F_G$. By theorem 4.36, we can write $\theta\pi'\theta F_G \subset (\theta\sigma)^k\theta F_G$. Since $(\theta\sigma)^n F_G \subset \theta\sigma F_G$, $(\theta\sigma)^k\theta F_G \subset \theta\sigma\theta F_G$. Hence $F_G \subset \theta\sigma\theta F_G \Rightarrow F_G$ is soft $\theta\sigma\theta$-open. Conversely assume that F_G is soft $\theta\sigma\theta$-open and no factor θ is followed by another one in $\pi = \theta\pi'\theta$. Then $F_G \subset \theta\sigma\theta F_G \Rightarrow (\theta\sigma)^j\theta F_G \subset (\theta\sigma)^j\theta\theta\sigma\theta F_G \subset (\theta\sigma)^j\theta\sigma\theta F_G = (\theta\sigma)^{j+1}\theta F_G$ for $j \in N$. i.e, $F_G \subset \theta\sigma\theta F_G \subset (\theta\sigma)^{j+1}\theta F_G$ and as above $\theta\sigma\theta F_G \subset (\theta\sigma)^m\theta F_G \subset \theta\pi'\theta F_G$ by the repeated application of the condition $\theta\sigma F_G \subset \sigma F_G$. Hence F_G is soft $\theta\pi'\theta$-open.

Consider (3). Suppose F_G is soft $\sigma\pi'\theta$-open, where π' is any product of both the factors θ and σ. Then $F_G \subset \sigma\pi'\theta F_G$. Since $(\theta\sigma)^n F_G \subset \theta\sigma F_G$ and $\sigma\theta \in (\Pi5)$, we can write $\sigma\pi'\theta F_G \subset (\sigma\theta)^k F_G = \sigma(\theta\sigma)^{k-1}\theta F_G \subset \sigma(\theta\sigma)\theta F_G \subset \sigma\theta F_G$ for some $k \in N$. Hence $F_G \subset \sigma\theta F_G \Rightarrow F_G$ is soft $\sigma\theta$-open. Conversely assume that F_G is soft $\sigma\theta$-open and no factor θ is followed by another one in $\pi = \sigma\pi'\theta$. Then $F_G \subset \sigma\theta F_G \Rightarrow F_G \subset (\sigma\theta)^m F_G$, by theorem 4.35. Since $\theta\sigma F_G \subset \sigma F_G$, it is easy to show that $(\sigma\theta)^m F_G \subset \sigma\pi'\theta F_G$. Hence $F_G \subset \sigma\pi'\theta F_G \Rightarrow F_G$ is soft $\sigma\pi'\theta$-open.

Consider (4). Suppose F_G is soft $\sigma\pi'\sigma$-open, then $F_G \subset \sigma\pi'\sigma F_G$. Since $(\sigma\theta)^k F_G \subset \sigma\theta F_G$, we can write $\sigma\pi'\sigma F_G \subset (\sigma\theta)^k\sigma F_G \subset \sigma\theta\sigma F_G$. Thus $F_G \subset \sigma\pi'\sigma F_G \Rightarrow F_G \subset \sigma\theta\sigma F_G \Rightarrow F_G$ is soft $\sigma\theta\sigma$-open. Conversely assume that F_G is soft $\sigma\theta\sigma$-open and no factor θ is followed by another one in $\pi = \sigma\pi'\sigma$. Then $F_G \subset \sigma\theta\sigma F_G \Rightarrow (\sigma\theta)^j\sigma F_G \subset (\sigma\theta)^j\sigma\sigma\theta\sigma F_G \subset (\sigma\theta)^j\sigma\theta\sigma F_G = (\sigma\theta)^{j+1}\sigma F_G$. Hence $\sigma\theta\sigma F_G \subset (\sigma\theta)^{m-1}\sigma F_G$. Since $\theta\sigma F_G \subset \sigma F_G$, it is easy to show that $(\sigma\theta)^{m-1}\sigma F_G \subset \sigma\pi'\sigma F_G$. Hence $F_G \subset \sigma\pi'\sigma F_G \Rightarrow F_G$ is soft $\sigma\pi'\sigma$-open.■

Theorem 4.38. If $\theta \in (\Pi 4)$ and $\sigma \in (\Pi 5)$ then the statements of theorem 4.37 are valid; moreover, soft $\theta\sigma\theta$–open $\Leftrightarrow$ (soft $\theta\sigma$-open and soft $\sigma\theta$–open) $\Rightarrow$ (soft $\theta\sigma$-open or soft $\sigma\theta$-open) $\Rightarrow$ soft $\sigma\theta\sigma$-open $\Rightarrow$ soft σ-open.

Proof. If $\theta \in (\Pi 4)$, then $\theta F_G \subset F_G \Rightarrow \theta\theta F_G \subset \theta F_G \Rightarrow \theta \in (\Pi 5)$ and also $\theta\sigma F_G \subset \sigma F_G$. Now the hypotheses of theorem 4.37 are fulfilled. Further, $\theta\sigma(\theta F_G) \subset \theta\sigma F_G$ and $\theta(\sigma\theta F_G) \subset \sigma\theta F_G$; i.e F_G is soft $\theta\sigma\theta$–open $\Rightarrow F_G \subset \theta\sigma\theta F_G \subset \theta\sigma F_G$ and $F_G \subset \theta\sigma\theta F_G \subset \sigma\theta F_G \Rightarrow F_G$ is both soft $\theta\sigma$-open and soft $\sigma\theta$–open. Conversely assume that F_G is both soft $\theta\sigma$-open and soft $\sigma\theta$-open. Then $F_G \subset \theta\sigma F_G$ and $F_G \subset \sigma\theta F_G \Rightarrow F_G \subset \theta\sigma F_G \cap \sigma\theta F_G \Rightarrow F_G \subset \theta\sigma F_G \subset \theta\sigma(\sigma\theta F_G) \subset \theta\sigma\theta F_G \Rightarrow F_G$ is soft $\theta\sigma\theta$–open. Again, F_G is soft $\theta\sigma$-open or soft $\sigma\theta$-open $\Rightarrow F_G \subset \theta\sigma F_G$ or $F_G \subset \sigma\theta F_G \Rightarrow F_G \subset \theta\sigma\theta\sigma F_G$ or $F_G \subset \sigma\theta\sigma\theta F_G$ respectively. Hence $F_G \subset \sigma\theta\sigma F_G$ by $\theta \in (\Pi 4)$ $\Rightarrow F_G$ is soft $\sigma\theta\sigma$-open. And F_G is soft $\sigma\theta\sigma$-open $\Rightarrow F_G \subset \sigma\theta\sigma F_G \subset \sigma F_G$ by $\theta \in (\Pi 4)$ and $\sigma \in (\Pi 5) \Rightarrow F_G$ is soft σ-open.■

Note: Let $(F_{\tilde{E}}, \mu)$ be a SGTS. Then we can say that a soft set F_G is soft $i_\mu c_\mu i_\mu$–open iff it is both soft $i_\mu c_\mu$–open and soft $c_\mu i_\mu$–open.

Theorem 4.39. If $\pi \in (\Pi 5)$ and F_G is soft π-open then $c_{\pi*}F_G = \pi F_G$.

Proof. Since $\pi \in (\Pi 5)$, $\pi\pi F_G \subset \pi F_G \Rightarrow \pi F_G$ is soft π^*-closed by theorem 4.19. If $F_H \supset F_G$ is soft π^*-closed, then $F_H \supset \pi F_H \supset \pi F_G$. Hence $\pi F_G \supset F_G$ is the smallest soft π^*-closed super set of F_G. Hence $c_{\pi*}F_G = \pi F_G$.■

Theorem 4.40. For any $\pi \in \Pi$ and $F_G \subset F_{\tilde{E}}$, we have $i_\pi F_G \subset F_G \cap \pi F_G$.

Proof. Suppose $F_H \subset F_G$ is soft π-open. Then $F_H \subset \pi F_H \subset \pi F_G$ so that $F_H \subset F_G \cap \pi F_G$. Hence $i_\pi F_G \subset F_G \cap \pi F_G$.■

Theorem 4.41. Let $(F_{\tilde{E}}, \mu)$ be a SGTS and if $\pi = c_\mu i_\mu$ or $\pi = i_\mu c_\mu i_\mu$, then $i_\pi F_G = F_G \cap \pi F_G$ for any $F_G \subset F_{\tilde{E}}$.

Proof. Clearly $i_\mu F_G \subset c_\mu i_\mu F_G$ for $F_G \subset F_{\tilde{E}}$ and $i_\mu F_G \subset c_\mu i_\mu F_G \Rightarrow i_\mu i_\mu F_G \subset i_\mu c_\mu i_\mu F_G \Rightarrow i_\mu F_G \subset i_\mu c_\mu i_\mu F_G$. Therefore $i_\mu F_G \subset F_G \cap c_\mu i_\mu F_G$ and $i_\mu F_G \subset F_G \cap i_\mu c_\mu i_\mu F_G$. Hence $i_\mu i_\mu F_G \subset i_\mu(F_G \cap c_\mu i_\mu F_G)$ and $i_\mu i_\mu F_G \subset i_\mu(F_G \cap i_\mu c_\mu i_\mu F_G)$. i.e, $i_\mu F_G \subset i_\mu(F_G \cap c_\mu i_\mu F_G)$ and $i_\mu F_G \subset i_\mu(F_G \cap i_\mu c_\mu i_\mu F_G)$. Therefore $c_\mu i_\mu F_G \subset c_\mu i_\mu(F_G \cap c_\mu i_\mu F_G)$ and $i_\mu c_\mu i_\mu F_G \subset i_\mu c_\mu i_\mu(F_G \cap i_\mu c_\mu i_\mu F_G)$. Hence $F_G \cap c_\mu i_\mu F_G \subset c_\mu i_\mu(F_G \cap c_\mu i_\mu F_G)$ and $F_G \cap i_\mu c_\mu i_\mu F_G \subset i_\mu c_\mu i_\mu(F_G \cap i_\mu c_\mu i_\mu F_G)$. ie, $F_G \cap \pi F_G \subset \pi(F_G \cap \pi F_G)$ for $\pi = c_\mu i_\mu$ or $\pi = i_\mu c_\mu i_\mu$. i.e, $F_G \cap \pi F_G$ is soft π-open for $\pi = c_\mu i_\mu$ or $\pi = i_\mu c_\mu i_\mu$. Thus $F_G \cap \pi F_G \subset i_\pi F_G$ in these two cases. But by theorem 4.40, $i_\pi F_G \subset F_G \cap \pi F_G$. Hence $i_\pi F_G = F_G \cap \pi F_G$ for $\pi = c_\mu i_\mu$ or $\pi = i_\mu c_\mu i_\mu$.■

Theorem 4.42. For $\pi \in \Pi$ and $F_G \subset F_{\tilde{E}}$, $i_\pi F_G = F_G \cap \pi F_G$ for $F_G \subset F_{\tilde{E}}$ is true iff $c_\pi F_G = F_G \cup \pi^* F_G$.

Proof. Suppose $i_\pi F_G = F_G \cap \pi F_G$ is true. Then by theorem 4.17, $c_\pi F_G = (i_\pi)^* F_G = [i_\pi(F_G)^c]^c = [(F_G)^c \cap \pi(F_G)^c]^c = F_G \cup [\pi(F_G)^c]^c = F_G \cup \pi^* F_G$. Conversely, suppose that $c_\pi F_G = F_G \cup \pi^* F_G$. Then $i_\pi F_G = (c_\pi)^* F_G = [c_\pi(F_G)^c]^c = [F_G{}^c \cup \pi^*(F_G)^c]^c = F_G \cap [\pi^*(F_G)^c]^c = F_G \cap \pi F_G$, by theorem 4.17.■

Theorem 4.43. Let $(F_{\tilde{E}}, \mu)$ be a SGTS. Then $c_\pi F_G = F_G \cup \pi^* F_G$ is true if $\pi = c_\mu i_\mu$ or $\pi = i_\mu c_\mu i_\mu$.

Proof. The proof follows from theorem 4.41 and 4.42.

Conclusion

In the present work, we mainly study some interesting properties of the soft mapping $\pi : S(U)_E \to S(U)_E$ which satisfy the condition $\pi F_B \subset \pi F_D$ whenever $F_B \subset F_D \subset F_{\tilde{E}}$. The concept of soft π-open set is introduced and established some of their properties. The notions of soft interior and soft closure are generalized using these sets and under suitable conditions on π, the soft π-interior $i_\pi F_G$ and the soft π-closure $c_\pi F_G$ of a soft set $F_G \subset F_{\tilde{E}}$ are easily obtained by explicit formulas. We expect that results in this paper will be a basis for applications of soft π-open sets in soft set theory and will promote the further study on soft generalized topology to carry out general frame work for the applications in practical life.

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Λ_g-CLOSED SETS WITH RESPECT TO AN IDEAL

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Abstract – In this paper, the notions of $\mathcal{I}_{\Lambda_g}$-closed sets and $\mathcal{I}_{\Lambda_g}$-open sets are introduced. Characterizations and properties of such notions are obtained. Suitable examples are given to substantiate each established notions.

Keywords – *Topological space, open set, λ-closed set, Λ_g-closed set, $\mathcal{I}_g$-closed set, $\mathcal{I}_{\pi g}$-closed set, ideal.*

1 Introduction and Preliminaries

In 1986, Maki [12] introduced the notion of Λ-sets in topological spaces. A Λ-set is a set A which is equal to its kernel (= saturated set) i.e to the intersection of all open supersets of A. Arenas et al [1] introduced and investigated the notion of λ-closed sets by involving Λ-sets and closed sets.

The notion of closed set is fundamental in the study of topological spaces. In 1970, Levine [11] introduced the concept of generalized closed sets in a topological space by comparing the closure of a subset with its open supersets. He defined a subset A of a topological space X to be generalized closed (briefly, g-closed) if cl(A) $\subseteq$ U whenever A $\subseteq$ U and U is open. This notion has been studied extensively in recent years by many topologists. After advent of g-closed sets, many generalizations of g-closed sets are being introduced and investigated by modern topologists.

An ideal on a set X is a non empty collection of subsets of X with heredity property which is also closed under finite unions. Quite Recently, Jafari and Rajesh [8] have introduced and studied the notion of generalized closed (g-closed) sets with respect to an ideal. Many generalizations of g-closed sets are being introduced and investigated by modern researchers. One among them is Λ_g-closed sets which were introduced by Caldas et al [2]. In this paper, we introduce and investigate the concept of Λ_g-closed sets with respect to an ideal.

Indeed ideals are very important tools in General Topology. It was the works of Newcomb [13], Rancin [14], Samuels [16] and Hamlett and Jankovic (see [4, 5, 6, 7, 9]) which motivated the research in applying topological ideals to generalize the most basic properties in General Topology. A nonempty

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collection $\mathcal{I}$ of subsets on a topological space (X, τ) is called a topological ideal [10] if it satisfies the following two conditions:

1. If A $\in \mathcal{I}$ and B $\subseteq$ A implies B $\in \mathcal{I}$ (heredity)
2. If A $\in \mathcal{I}$ and B $\in \mathcal{I}$, then A $\cup$ B $\in \mathcal{I}$ (finite additivity)

If A is a subset of a topological space (X, τ), cl(A) and int(A) denote the closure of A and the interior of A, respectively. Let A $\subseteq$ B $\subseteq$ X. Then cl_B(A) (resp. int_B(A)) denotes closure of A (resp. interior of A) with respect to B.

In this paper, we introduce and study the concept of Λ_g-closed sets with respect to an ideal, which is the extension of the concept of Λ_g-closed sets.

The following Definitions, Result, Lemma and Remarks are useful in the sequel.

Definition 1.1. *A subset A of a topological space (X, τ) is called regular open [17] if A = int(cl(A)).*

Definition 1.2. *The finite union of regular open sets is called π-open [18]. The complement of π-open set is called π-closed [18].*

Definition 1.3. *A subset A of a topological space (X, τ) is called*

1. *λ-closed [1] if A = L $\cap$ D, where L is a Λ-set and D is a closed set.*
2. *λ-open [1] if its complement is λ-closed.*
3. *Λ_g-closed [2] if cl(A)$\subseteq$U whenever A$\subseteq$U and U is λ-open.*
4. *π-generalized closed (briefly, πg-closed) [3] if cl(A) $\subseteq$ U whenever A $\subseteq$ U and U is π-open.*

Definition 1.4. *Let (X, τ) be a topological space and $\mathcal{I}$ be an ideal on X. A subset A of X is said to be generalized closed with respect to an ideal (briefly $\mathcal{I}_g$-closed) [8] if and only if cl(A)$-$B $\in \mathcal{I}$, whenever A $\subseteq$ B and B is open.*

Result 1.5. *For a subset of a topological space, the following properties hold:*

1. *Every closed set is Λ_g-closed but not conversely [2].*
2. *Every Λ_g-closed set is g-closed but not conversely [2].*
3. *Every closed set is λ-closed but not conversely [1, 2].*

Remark 1.6. *[8] Every g-closed set is $\mathcal{I}_g$-closed but not conversely.*

Definition 1.7. *[15] Let (X, τ) be a topological space and $\mathcal{I}$ be an ideal on X. A subset A of X is said to be π-generalized closed with respect to an ideal (briefly $\mathcal{I}_{\pi g}$-closed) if and only if cl(A)$-$B $\in \mathcal{I}$, whenever A $\subseteq$ B and B is π-open.*

Remark 1.8. *[15] For several subsets defined above, we have the following implications.*

$$\begin{array}{ccccc} & & \mathcal{I}_g\text{-closed set} & \longrightarrow & \mathcal{I}_{\pi g}\text{-closed set} \\ & & \uparrow & & \uparrow \\ \text{closed set} & \longrightarrow & g\text{-closed set} & \longrightarrow & \pi g\text{-closed set} \end{array}$$

The reverse implications are not true.

Lemma 1.9. *[1] Let $A_i (i \in \mathcal{I})$ be subsets of a topological space (X, τ). The following properties hold:*

1. *If A_i is λ-closed for each $i \in I$, then $\cap_{i\in I} A_i$ is λ-closed.*
2. *If A_i is λ-open for each $i \in I$, then $\cup_{i\in I} A_i$ is λ-open.*

Recall that the intersection of a λ-closed set and a closed set is λ-closed.

Definition 1.10. *[2] A function f : (X, τ) $\rightarrow$ (Y, σ) is called λ-irresolute if the inverse image of λ-open set of Y is λ-open in X.*

2 Λ_g-Closed Sets with Respect to an Ideal

Definition 2.1. *Let (X, τ) be a topological space and $\mathcal{I}$ be an ideal on X. A subset A of X is said to be Λ_g-closed with respect to an ideal (briefly $\mathcal{I}_{\Lambda_g}$-closed) if and only if cl(A)$-$B $\in \mathcal{I}$, whenever A $\subseteq$ B and B is λ-open.*

Remark 2.2. *Every Λ_g-closed set is $\mathcal{I}_{\Lambda_g}$-closed, but the converse need not be true, as this may be seen from the following Example.*

Example 2.3. *Let $X = \{a, b, c\}$, $\tau = \{\phi, X, \{a, c\}\}$ and $\mathcal{I} = \{\phi, \{b\}\}$. Then $\{a\}$ is $\mathcal{I}_{\Lambda_g}$-closed but not Λ_g-closed.*

The following Theorem gives a characterization of $\mathcal{I}_{\Lambda_g}$-closed sets.

Theorem 2.4. *A set A is $\mathcal{I}_{\Lambda_g}$-closed in (X, τ) if and only if F $\subseteq$ cl(A)$-$A and F is λ-closed in X implies F $\in \mathcal{I}$.*

Proof. Assume that A is $\mathcal{I}_{\Lambda_g}$-closed. Let F $\subseteq$ cl(A)$-$A. Suppose F is λ-closed. Then A $\subseteq$ X$-$F. By our assumption, cl(A)$-$(X$-$F) $\in \mathcal{I}$. But F $\subseteq$ cl(A)$-$(X$-$F) and hence F $\in \mathcal{I}$.

Conversely, assume that F $\subseteq$ cl(A)$-$A and F is λ-closed in X implies that F $\in \mathcal{I}$. Suppose A $\subseteq$ U and U is λ-open. Then cl(A)$-$U = cl(A) $\cap$ (X$-$U) is a λ-closed set in X, that is contained in cl(A)$-$A. By assumption, cl(A)$-$U $\in \mathcal{I}$. This implies that A is $\mathcal{I}_{\Lambda_g}$-closed.

Theorem 2.5. *If A and B are $\mathcal{I}_{\Lambda_g}$-closed sets of (X, τ), then their union A $\cup$ B is also $\mathcal{I}_{\Lambda_g}$-closed.*

Proof. Suppose A and B are $\mathcal{I}_{\Lambda_g}$-closed sets in (X, τ). If A $\cup$ B $\subseteq$ U and U is λ-open, then A $\subseteq$ U and B $\subseteq$ U. By assumption, cl(A)$-$U $\in \mathcal{I}$ and cl(B)$-$U $\in \mathcal{I}$ and hence cl(A $\cup$ B)$-$U = (cl(A)$-$U) $\cup$ (cl(B)$-$U) $\in \mathcal{I}$. That is A $\cup$ B is $\mathcal{I}_{\Lambda_g}$-closed.

Remark 2.6. *The intersection of two $\mathcal{I}_{\Lambda_g}$-closed sets need not be an $\mathcal{I}_{\Lambda_g}$-closed as shown by the following Example.*

Example 2.7. *Let $X = \{a, b, c, d\}$, $\tau = \{\phi, X, \{a\}, \{d\}, \{a, d\}\}$ and $\mathcal{I} = \{\phi, \{c\}\}$. Then $A = \{a, b\}$ and $B = \{a, c\}$ are $\mathcal{I}_{\Lambda_g}$-closed but their intersection $A \cap B = \{a\}$ is not $\mathcal{I}_{\Lambda_g}$-closed.*

Remark 2.8. *Every $\mathcal{I}_{\Lambda_g}$-closed set is $\mathcal{I}_g$-closed but not conversely.*

Example 2.9. *Let $X = \{a, b, c\}$, $\tau = \{\phi, X, \{b\}, \{b, c\}\}$ and $\mathcal{I} = \{\phi\}$. Then $\{a, b\}$ is $\mathcal{I}_g$-closed but not $\mathcal{I}_{\Lambda_g}$-closed.*

Remark 2.10. *For several subsets defined above, we have the following implications.*

$$\begin{array}{ccccccc} & & \mathcal{I}_{\Lambda_g}\text{-closed set} & \longrightarrow & \mathcal{I}_g\text{-closed set} & \longrightarrow & \mathcal{I}_{\pi g}\text{-closed set} \\ & & \uparrow & & \uparrow & & \uparrow \\ \text{closed set} & \longrightarrow & \Lambda_g\text{-closed set} & \longrightarrow & g\text{-closed set} & \longrightarrow & \pi g\text{-closed set} \end{array}$$

The reverse implications are not true.

Theorem 2.11. *If A is $\mathcal{I}_{\Lambda_g}$-closed and A $\subseteq$ B $\subseteq$ cl(A) in (X, τ), then B is $\mathcal{I}_{\Lambda_g}$-closed in (X, τ).*

Proof. Suppose A is $\mathcal{I}_{\Lambda_g}$-closed and A $\subseteq$ B $\subseteq$ cl(A) in (X, τ). Suppose B $\subseteq$ U and U is λ-open. Then A $\subseteq$ U. Since A is $\mathcal{I}_{\Lambda_g}$-closed, we have cl(A)$-$U $\in \mathcal{I}$. Now B $\subseteq$ cl(A). This implies that cl(B)$-$U $\subseteq$ cl(A)$-$U $\in \mathcal{I}$. Hence B is $\mathcal{I}_{\Lambda_g}$-closed in (X, τ).

Theorem 2.12. *Let A $\subseteq$ Y $\subseteq$ X and suppose that A is $\mathcal{I}_{\Lambda_g}$-closed in (X, τ). Then A is $\mathcal{I}_{\Lambda_g}$-closed relative to the subspace Y of X, with respect to the ideal $\mathcal{I}_Y = \{F \subseteq Y : F \in \mathcal{I}\}$.*

Proof. Suppose A $\subseteq$ U $\cap$ Y and U is λ-open in (X, τ), then A $\subseteq$ U. Since A is $\mathcal{I}_{\Lambda_g}$-closed in (X, τ), we have cl(A)$-$U $\in \mathcal{I}$. Now (cl(A) $\cap$ Y)$-$(U$\cap$Y) = (cl(A)$-$U) $\cap$ Y $\in \mathcal{I}$, whenever A $\subseteq$ U $\cap$ Y and U is λ-open. Hence A is $\mathcal{I}_{\Lambda_g}$-closed relative to the subspace Y.

Theorem 2.13. *Let A be an $\mathcal{I}_{\Lambda_g}$-closed set and F be a closed set in (X, τ), then A $\cap$ F is an $\mathcal{I}_{\Lambda_g}$-closed set in (X, τ).*

Proof. Let A ∩ F ⊆ U and U is λ-open. Then A ⊆ U ∪ (X−F). Since A is $\mathcal{I}_{\Lambda_g}$-closed, we have cl(A)−(U ∪ (X−F)) ∈ $\mathcal{I}$. Now, cl(A ∩ F) ⊆ cl(A) ∩ F = (cl(A) ∩ F)−(X−F). Therefore, cl(A ∩ F)−U ⊆ (cl(A) ∩ F)−(U ∩ (X−F)) ⊆ cl(A)−(U ∪ (X−F)) ∈ $\mathcal{I}$. Hence A ∩ F is $\mathcal{I}_{\Lambda_g}$-closed in (X, τ).

Definition 2.14. *Let (X, τ) be a topological space and $\mathcal{I}$ be an ideal on X. A subset A ⊆ X is said to be Λ_g-open with respect to an ideal (briefly $\mathcal{I}_{\Lambda_g}$-open) if and only if X−A is $\mathcal{I}_{\Lambda_g}$-closed.*

Theorem 2.15. *A set A is $\mathcal{I}_{\Lambda_g}$-open in (X, τ) if and only if F− U ⊆ int(A), for some U ∈ $\mathcal{I}$, whenever F ⊆ A and F is λ-closed.*

Proof. Suppose A is $\mathcal{I}_{\Lambda_g}$-open. Suppose F ⊆ A and F is λ-closed. We have X−A ⊆ X−F. By assumption, cl(X−A) ⊆ (X−F) ∪ U, for some U ∈ $\mathcal{I}$. This implies X−((X−F) ∪ U) ⊆ X− (cl(X−A)) and hence F−U ⊆ int(A).

Conversely, assume that F ⊆ A and F is λ-closed. Then F−U ⊆ int(A), for some U ∈ $\mathcal{I}$. Consider an λ-open set G such that X−A ⊆ G. Then X−G ⊆ A. By assumption, (X−G)−U ⊆ int(A) = X−cl(X−A). This gives that X−(G ∪ U) ⊆ X−cl(X−A). Then, cl(X−A) ⊆ G ∪ U, for some U ∈ $\mathcal{I}$. This shows that cl(X−A)−G ∈ $\mathcal{I}$. Hence X−A is $\mathcal{I}_{\Lambda_g}$-closed.

Recall that the sets A and B are said to be separated if cl(A) ∩ B = φ and A ∩ cl(B) = φ.

Theorem 2.16. *If A and B are separated $\mathcal{I}_{\Lambda_g}$-open sets in (X, τ), then A ∪ B is $\mathcal{I}_{\Lambda_g}$-open.*

Proof. Suppose A and B are separated $\mathcal{I}_{\Lambda_g}$-open sets in (X, τ) and F be a λ-closed subset of A ∪ B. Then F ∩ cl(A) ⊆ (A ∪ B) ∩ cl(A) = (A ∩ cl(A)) ∪ (B ∩ cl(A)) = A ∪ φ = A and F ∩ cl(B) ⊆ (A ∪ B) ∩ cl(B) = (A ∩ cl(B)) ∪ (B ∩ cl(B)) = φ ∪ B = B. By assumption and by Theorem 2.15, (F ∩ cl(A))−U_1 ⊆ int(A) and (F ∩ cl(B))−U_2 ⊆ int(B), for some U_1, U_2 ∈ $\mathcal{I}$. It means that ((F ∩ cl(A))−int(A)) ∈ $\mathcal{I}$ and ((F ∩ cl(B))−int(B)) ∈ $\mathcal{I}$. Then ((F ∩ cl(A))−int(A)) ∪ ((F ∩ cl(B))−int(B)) ∈ $\mathcal{I}$. Hence (F ∩ (cl(A) ∪ cl(B))−(int(A) ∪ int(B))) ∈ $\mathcal{I}$. But F = F ∩ (A ∪ B) ⊆ F ∩ cl(A ∪ B), and we have F−int(A ∪ B) ⊆ (F ∩ cl(A ∪ B))−int(A ∪ B) ⊆ (F ∩ cl(A ∪ B))−(int(A) ∪ int(B)) ∈ $\mathcal{I}$. Hence, F−U ⊆ int(A ∪ B), for some U ∈ $\mathcal{I}$. This proves that A ∪ B is $\mathcal{I}_{\Lambda_g}$-open.

Corollary 2.17. *Let A and B be $\mathcal{I}_{\Lambda_g}$-closed sets and suppose X−A and X−B are separated in (X, τ). Then A ∩ B is $\mathcal{I}_{\Lambda_g}$-closed.*

Corollary 2.18. *If A and B are $\mathcal{I}_{\Lambda_g}$-open sets in (X, τ), then A ∩ B is $\mathcal{I}_{\Lambda_g}$-open.*

Proof. If A and B are $\mathcal{I}_{\Lambda_g}$-open, then X−A and X−B are $\mathcal{I}_{\Lambda_g}$-closed. By Theorem 2.5, X−(A ∩ B) is $\mathcal{I}_{\Lambda_g}$-closed, which implies A ∩ B is $\mathcal{I}_{\Lambda_g}$-open.

Theorem 2.19. *If int(A) ⊆ B ⊆ A and A is $\mathcal{I}_{\Lambda_g}$-open in (X, τ), then B is $\mathcal{I}_{\Lambda_g}$-open in X.*

Proof. Suppose int(A) ⊆ B ⊆ A and A is $\mathcal{I}_{\Lambda_g}$-open. Then X−A ⊆ X−B ⊆ cl(X−A) and X−A is $\mathcal{I}_{\Lambda_g}$-closed. By Theorem 2.11, X−B is $\mathcal{I}_{\Lambda_g}$-closed and hence B is $\mathcal{I}_{\Lambda_g}$-open.

Theorem 2.20. *Let (X, τ) be a topological space. Then a set A is $\mathcal{I}_{\Lambda_g}$-closed in X if and only if cl(A)−A is $\mathcal{I}_{\Lambda_g}$-open in X.*

Proof. Necessity: Suppose F ⊆ cl(A)−A and F be λ-closed. Then by Theorem 2.4, F ∈ $\mathcal{I}$. This implies that F−U = φ, for some U ∈ $\mathcal{I}$. Clearly, F−U ⊆ int(cl(A)−A). By Theorem 2.15, cl(A)−A is $\mathcal{I}_{\Lambda_g}$-open.

Sufficiency: Suppose A ⊆ G and G is λ-open in (X, τ). Then cl(A) ∩ (X−G) ⊆ cl(A) ∩ (X−A) = cl(A)−A. By hypothesis and by Theorem 2.15, (cl(A) ∩ (X−G))−U ⊆ int(cl(A)−A) = φ, for some U ∈ $\mathcal{I}$. This implies that cl(A) ∩ (X−G) ⊆ U ∈ $\mathcal{I}$ and hence cl(A)−G ∈ $\mathcal{I}$. Thus, A is $\mathcal{I}_{\Lambda_g}$-closed.

Theorem 2.21. *Let f : (X, τ) → (Y, σ) be λ-irresolute and closed. If A ⊆ X is $\mathcal{I}_{\Lambda_g}$-closed in X, then f(A) is $f(\mathcal{I})_{\Lambda_g}$-closed in (Y, σ), where f($\mathcal{I}$) = {f(U) : U ∈ $\mathcal{I}$}.*

Proof. Suppose A ⊆ X and A is $\mathcal{I}_{\Lambda_g}$-closed. Suppose f(A) ⊆ G and G is λ-open in Y. Then A ⊆ f^{-1}(G). By definition, cl(A)−f^{-1}(G) ∈ $\mathcal{I}$ and hence f(cl(A))−G ∈ f($\mathcal{I}$). Since f is closed, cl(f(A)) ⊆ cl(f(cl(A))) = f(cl(A)). Then cl(f(A))−G ⊆ f(cl(A))−G ∈ f($\mathcal{I}$) and hence f(A) is $f(\mathcal{I})_{\Lambda_g}$-closed in Y.

3 Conclusion

Topology is an area of Mathematics concerned with the properties of space that are preserved under continuous deformations including stretching and bending, but not tearing. By the middle of the 20th century, topology had become a major branch of Mathematics.

Topology as a branch of Mathematics can be formally defined as the study of qualitative properties of certain objects that are invariant under a certain kind of transformation especially those properties that are invariant under a certain kind of equivalence and it is the study of those properties of geometric configurations which remain invariant when these configurations are subjected to one-to-one bicontinuous transformations or homeomorphisms. Topology operates with more general concepts than analysis. Differential properties of a given transformation are nonessential for topology but bicontinuity is essential. As a consequence, topology is often suitable for the solution of problems to which analysis cannot give the answer.

Though the concept of topology has been identified as a difficult territory in Mathematics, we have taken it up as a challenge and cherishingly worked out this research study. It can also further up the understanding of basic structure of classical mathematics and offers new methods and results in obtaining significant results of classical mathematics. Moreover it also has applications in some important fields of Science and Technology.

In this paper, the notions of $\mathcal{I}_{\Lambda_g}$-closed sets and $\mathcal{I}_{\Lambda_g}$-open sets are introduced. Furthermore the relations with other notions connected with the notions of $\mathcal{I}_{\Lambda_g}$-closed sets and $\mathcal{I}_{\Lambda_g}$-open are investigated.

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FUZZY ALMOST CONTRA θ-SEMIGENERALIZED-CONTINUOUS FUNCTIONS

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Abstract – The aim of this paper is to introduce new notion of the fuzzy almost contra θ-semigeneralized-continuous functions using fuzzy θ-semigeneralized-closed set and to investigate properties and relationships of fuzzy functions.

Keywords – *Fuzzy θsg-closed set, Fuzzy almost contra θsg-continuous, FTSGO-connected space, FTSGO-compact space, fuzzy θsg-normal space, fuzzy $\theta sg - T_1$, fuzzy $\theta sg - T_2$.*

1 Introduction

The concept of fuzzy sets due to Zadeh [10] naturally plays important role in the study of fuzzy topological space which has been introduced by Chang [2]. In 2013, Zabidin Salleh et al introduced and studied the notion of θ-semi-generalized-closed sets in fuzzy topological spaces. Ekici and Kerre [4] introduced the concept of fuzzy contra continuous functions. The purpose of this paper is to introduce the forms of fuzzy almost contra θsg-continuous functions and to investigate properties and relationships of fuzzy functions. We have also defined fuzzy θsg-compact and fuzzy θsg-connected spaces.

2 Preliminary

Throughout this paper X be a set and I the unit interval. A fuzzy set in X is an element of the set of all functions from X to I. The family of all fuzzy sets in X is denoted by I^X. A fuzzy singleton x_α is a fuzzy set in X define by $x_\alpha(x) = \alpha$, $x_\alpha(y) = 0$ for all $y \neq x, x \in (0, 1]$. The set of all fuzzy singletons in X is denoted by $S(X)$. For every $x_\alpha \in S(X)$ and $\mu \in I^X$, we define $x_\alpha \in \mu$ if and only if $x_\alpha \leq \mu(x)$. The members of τ are called fuzzy open sets and their complements are fuzzy closed sets. Spaces (X, τ) and (Y, σ)(or simply, X and Y) always mean fuzzy topological spaces in the sense of Chang [2]. By 1_X and 0_X, we mean fuzzy sets with constant function 1 (unit function) and 0 (zero function), respectively.

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For a fuzzy set μ of X, fuzzy closure and fuzzy interior of μ denoted by $cl(\mu)$ and $int(\mu)$, respectively. The operators fuzzy closure and fuzzy interior are defined by $cl(\mu) = \wedge\{\lambda : \lambda \geq \mu, 1-\mu \in \tau\}$ where λ is fuzzy closed set in X and $int\mu = \wedge\{\eta : \eta \leq \mu, \eta \in \tau$ [9] where η is fuzzy open set in X. Fuzzy semi-closure [9] of μ denoted by $scl(\mu) = \wedge\{\eta : \mu \leq \eta, \eta \in FSC(X)\}$ and fuzzy θ-closure of μ denoted by $cl_\theta = \wedge\{cl(\eta) : \mu \leq \eta, \eta \in \tau\}$[3]. θ-semi-generalized closed set in fuzzy topology is introduced by Z.Salleh et al [8].

Definition 2.1. A subset A of a space X is called
(1) Fuzzy semi-open(briefly, Fs-open) set [1] if $A \leq cl(int(A))$.
(2) Fuzzy semi-closed (briefly, Fs-closed)set[1] if $int(cl(A)) \leq A$.
(3) Fuzzy regular closed [1] if $cl(int(A)) = A$ and fuzzy regular open if $int(cl(A)) = A$. The family of all fuzzy semi open and fuzzy semi closed sets in X will be denoted by FSO(X)and FSC(X), respectively.

Definition 2.2. [8] Let X be a fuzzy topological space and μ be a fuzzy set of X. Then the operators semi-θ-closure of μ denoted by $scl_\theta(\mu)$ and fuzzy semi-θ-interior of μ is denoted by $sint_\theta(\mu)$ are defined as follows,
$scl_\theta(\mu) = \wedge\{scl(\eta) : \mu \leq \eta, \eta \in FSO(X)\}$,
$sint_\theta(\mu) = \vee\{sint(\eta) : \mu \geq \eta, \eta \in FSC(X)\}$.

Definition 2.3. A fuzzy set μ in X is called
(1) fuzzy θ-generalized closed [3] (briefly, f-θg-closed set) if $cl_\theta(\mu) \leq \eta$ whenever $\mu \leq \eta$ and η is fuzzy open
(2)fuzzy θ-semigeneralized-closed set [8] (briefly, f-θsg-closed set) if $scl_\theta(\mu) \leq \eta$ whenever $\mu \leq \eta$ and η is fuzzy semiopen. The coplement of fuzzy θ-semi-generalized-closed set is fuzzy θ-semi-generalized-open set (briefly, f-θsg-open set). The family of all Fθsg-closed sets in X are denoted by $F\theta$SGC(X)and The family of all f-θsg-open sets in X are denoted by $F\theta$SGO(X)

Definition 2.4. [8] A function $f : X \to Y$ is said to be
(1) fuzzy θ-semi-generalized continuous(briefly, f-θsg-continuous) if $f^{-1}(\lambda)$ is f-θsg-closed in X for each fuzzy semi-closed set λ in Y.
(2)fuzzy θ-semi-generalized irresolute(briefly, f-θsg-irresolute) if $f^{-1}(\lambda)$ is f-θsg-closed in X for each f-θsg-closed set λ in Y.
(3) fuzzy θ-semi-generalized open(briefly, f-θsg-open) if $f(\lambda)$ of Y and for each f-θsg-open in Y for every fuzzy semi-open set λ in X.

3 Fuzzy Almost Contra θ-Semigeneralized-Continuous Functions

In this section, the notion of fuzzy almost contra θsg-continuous functions via f-θsg-closed set is introduced.

Definition 3.1. Let X and Y be fuzzy topological spaces. A fuzzy function $f : X \to Y$ is said to be fuzzy almost θ-semigeneralized-continuous (briefly, fuzzy almost contra θsg-continuous) if inverse image of each fuzzy regular open set in Y is f-θsg-closed in X.

Example 3.2. Let $X = Y = \{a, b, c\}$. A, B, C are fuzzy sets of X defined as A(a) =0, A(b) = 1, A(c) = 0, B(a) = 0, B(b)= 0, B(c)= 1 and C(a)=0, C(b)=1, C(c)=1 and D be a fuzzy set of Y defined as D(a)=1, D(b)=0, D(c)=0. Then $\tau = \{0, 1, A, B, C\}$and $\mu = \{0, 1, D\}$ be fuzzy toplogies on sets X and Y respectively. The identity function $f : X \to Y$ is fuzzy almost contra-θsg-continuous fucntion.

Theorem 3.3. For a function $f : X \to Y$, the following statements are equivalent:
(i) f is fuzzy almost contra θsg-continuous.
(ii) For every fuzzy regular closed set μ in Y, $f^{-1}(\mu)$ is f-θsg-open.
(iii) For each $x \in X$ and each fuzzy regular closed set λ in Y containing f(x), there exists a f-θsg-open set η in X containing x such that $f(\eta) \leq \lambda$.
(iv) For each $x \in X$ and fuzzy regular open set μ in Y containing f(x), there exists a f-θsg-open set ω in X containing x such that $f^{-1}(\mu) \leq \omega$.

Proof:$(i) \Rightarrow (ii)$. Let μ be a fuzzy regular closed set in Y, then Y-μ is fuzzy regular open set in Y. By (i) $f^{-1}(Y-\mu) = X - f^{-1}(\mu)$ is f-θsg-closed set in X. This implies that $f^{-1}(\mu)$ is f-θsg-open set in X. Therefore (ii) holds.
$(ii) \Rightarrow (i)$. Let G be a fuzzy regular open set of Y. Then Y-G be a fuzzy regular closed set in Y. By (ii) $f^{-1}(Y-G)$ is f-θsg-open set in X. This implies that $X - f^{-1}(G)$ is f-θsg-open in X, which implies $f^{-1}(G)$ is f-θsg-closed set in X. Therefore (i) holds.
$(ii) \Rightarrow (iii)$. Let λ be a fuzzy regular closed set of Y containing f(x). By (ii) $f^{-1}(\lambda)$ is f-θsg-open set in X and $x \in f^{-1}(\lambda)$. Take $\eta = f^{-1}(\lambda)$. Then $f(\eta) \leq \lambda$.
$(iii) \Rightarrow (ii)$. Let λ be a fuzzy regular closed set of Y and $x \in f^{-1}(\lambda)$. From (iii), there exists a f-θsg-open set η in X containing x suxh that $\eta \leq f^{-1}(\lambda)$. We have $f^{-1}(\lambda) = \vee_{x \in f^{-1}(\lambda)} \eta$. Thus $f^{-1}(\lambda)$ is f-θsg-open set in X.
$(iii) \Rightarrow (iv)$. Let μ be a fuzzy regular open set in Y not containing f(x). Then and $1-\mu$ is a fuzzy regular closed set containing f(x). By (iii), there exists a f-θsg-open set η in X containing x suxh that $f(\eta) \leq 1-\mu$. Hence $\eta \leq f^{-1}(1-\mu) \leq 1 - f^{-1}(\mu)$ and then $f^{-1}(\mu) \leq 1-\eta$. Take $\omega = 1-\eta$. Therefore we obtain that ω is a f-θsg-open set in X not containing x. The converse can be be shown easily.

Theorem 3.4. Let $f : X \to Y$ be a function and let $g : X \to X \times Y$ be the fuzzy graph function of f defined by $g(x_\in) = (x_\in, f(x_\in))$ for every $x_\in \in X$. If g is fuzzy almost contra θsg-continuous, then f is fuzzy almost contra θsg-continuous.

Proof: Let μ be a fuzzy regular closed set in Y, then $X \times \mu$ is fuzzy regular closed set in $X \times Y$. Since g is fuzzy almost contra θsg-continuous, then $f^{-1}(\mu) = g^{-1}(X \times \mu)$ is f-θsg-open in X. Thus, f is fuzzy almost contra θsg-continuous.

Definition 3.5. A fuzzy filter base Λ is said to be fuzzy θsg-convergent to a fuzzy singleton $x_\in$ in X if for any f-θsg-open set μ in X containing $x_\in$, there exists a fuzzy set $\eta \in \Lambda$ such that $\eta \leq \mu$.

Definition 3.6. A fuzzy filter base Λ is said to be fuzzy rc-convergent[5] to a fuzzy singleton $x_\in$ in X if for any fuzzy regular closed set μ in X containing $x_\in$, there exists a fuzzy set $\eta \in \Lambda$ such that $\eta \leq \mu$.

Theorem 3.7. If a function $f : X \to Y$ is fuzzy almost contra θsg-continuous, then for each fuzzy singleton $x_\in \in X$ and each filter base Λ in X fuzzy θsg-converging to $x_\in$, the fuzzy filter base $f(\Lambda)$ is fuzzy rc-convergent to $f(x_\in)$.

Proof: Let $x_\in \in X$ and Λ be any fuzzy filter base in fuzzy θsg-converging to $x_\in$. Since f is fuzzy almost contra θsg-continuous, then for any fuzzy regular closed set λ in Y containing $f(x_\in)$, there exists a Fθsg-open set $\mu \in X$ containing $x_\in$ such that $f(\mu) \leq \lambda$. Since Λ is fuzzy θsg-converging to $x_\in$, there exists a $A \in \Lambda$ such that $A \leq \mu$. This means that $f(A) \leq \mu$ and therefore the fuzzy filter base $f(\Lambda)$ is fuzzy rc-convergent to $f(x_\in)$.

4 Fuzzy θ-Semigeneralized-Connectedness

In this section we introduce fuzzy θ-semigeneralized-connected (briefly, FTSGO-connected) and fuzzy θ-semigeneralized-normal spaces.

Definition 4.1. A fuzzy topological space X is called Fuzzy θ-semigeneralized-connected(briefly,FTSGO-Connected) if X is not the union of two disjoint nonempty f-θsg-open sets.

Definition 4.2. A fuzzy topological space X is called fuzzy connected [7] if X is not the union of two disjoint nonempty fuzzy open sets.

Theorem 4.3. If $f : X \to Y$ is fuzzy almost contra θsg-continuous surjection and X is FTSGO-connected, then Y is fuzzy connected.

Proof: Suppose Y is not fuzzy connected. Then there exist nonempty disjoint fuzzy open sets μ_1 and μ_2 such that $Y = \mu_1 \wedge \mu_2$. Therefore,μ_1 and μ_2 are fuzzy clopen in Y. Since f is fuzzy almost contra θsg-continuous , $f^{-1}(\mu_1)$ and $f^{-1}(\mu_2)$ are f-θsg-open in in X. Moreover, $f^{-1}(\mu_1)$ and $f^{-1}(\mu_2)$ are nonempty disjoint and $X = f^{-1}(\mu_1) \wedge f^{-1}(\mu_2)$. This shows that X is not FTSGO-connected. This contradicts the fact that Y is not Fuzzy connected assumed. Hence Y is fuzzy connected.

Definition 4.4. A fuzzy space X is said to be fuzzy θsg-normal (briefly,f-θsg-normal)if every pair of nonempty disjoint fuzzy closed sets can be separated by disjoint f-θsg-open sets.

Definition 4.5. A fuzzy space X is said to be fuzzy strongly θsg-normal if every pair of nonempty disjoint fuzzy closed sets A and B there exist disjoint f-θsg-open sets U and V such that $A \leq U$, $B \leq V$ and $cl(A) \wedge cl(B) = \phi$.

Theorem 4.6. If Y is fuzzy strongly θsg-normal and $f : X \to Y$ is fuzzy almost contra θsg-continuous closed surjection, then X is f-θsg-normal.

Proof: Let A and B be disjoint nonempty fuzzy closed sets of X. Since f is injective and closed, f(A) and f(B) are disjoint fuzzy closed sets. Since Y is fuzzy strongly θsg-normal, then there exist f-θsg-open sets U and V such that $f(A) \leq U$ and $f(B) \leq V$ and $cl(A) \wedge cl(B) = \phi$. Then, since cl(A) and cl(B) are regular closed and f is fuzzy almost contra θsg-continuous, $f^{-1}(cl(A))$ and $f^{-1}(cl(B))$ are f-θsg-open sets. Since, $U \leq f^{-1}(cl(A))$, $V \leq f^{-1}(cl(B))$ and $f^{-1}(cl(A))$ and $f^{-1}(cl(B))$ are disjoint, X is f-θsg-normal.

Definition 4.7. A fuzzy space X is said to be fuzzy $\theta sg - T_1$ if for each pair of distinct fuzzy singletons x and y in X, there exist f-θsg-open sets U and V containing x and y respectively, such that $y \notin U$ and $x \notin V$.

Definition 4.8. A fuzzy space X is said to be fuzzy $\theta sg - T_2$ if for each pair of distinct fuzzy points x and y in X, there exist f-θsg-open set U containing x and f-θsg-open set V containinig y such that $U \wedge V = \phi$.

Theorem 4.9. If $f : X \to Y$ is a fuzzy almost contra θsg-continuous injection and Y is fuzzy Urysohn, then X is fuzzy $\theta sg - T_2$.

Proof: Let Y is fuzzy Urysohn. By the injectivity of f, it follows that $f(x) \neq f(y)$ for any distinct fuzzy singletons x and y in X. Since Y is fuzzy Urysohn, then there exist fuzzy open sets U and V such that $f(x) \in U$ and $f(y) \in V$ and $cl(U) \wedge cl(V) = \phi$. Since f is fuzzy almost contra θsg-continuous, then there exist fuzzy open sets W and Z in X containing x and y, respectively, such that $f(W) \leq cl(U)$ and $f(Z) \leq cl(V)$. Hence $W \wedge Z = \phi$. This shows that X is fuzzy $\theta sg - T_2$.

Definition 4.10. A fuzzy space X is said to be fuzzy weakly T_2 [5] if each element of X is an intersection of fuzzy regular closed sets.

Theorem 4.11. If $f : X \to Y$ is a fuzzy almost contra θsg-continuous injection and Y is fuzzy weakly T_2, then X is fuzzy $\theta sg - T_1$.

Proof: Suppose that Y is fuzzy weakly T_2. For any distinct points x and y in X, there exist fuzzy regular closed sets U , V in Y such that $f(x) \in U, f(y) \notin U$, $f(x) \notin V$ and $f(y) \in V$. Since f is fuzzy almost contra θsg-continuous , by Theorem 3.2(ii), $f^{-1}(U)$ and $f^{-1}(V)$ are f-θsg-open subsets of X such that $x \in f^{-1}(U), y \notin f^{-1}(U)$ and $x \notin f^{-1}(V), y \in f^{-1}(V)$. This shows that X is fuzzy $\theta sg - T_1$.

Definition 4.12. The fuzzy graph G(f) of a fuzzy function $f : X \to Y$ is said to be fuzzy strongly contra-θsg-closed if for each $(x, y) \in (X \times Y)$ - G(f), there exist a f-θsg-open set U in x containing x and a fuzzy regular closed set V in Y containing y, such that $(U \times V) \wedge G(f) = \phi$.

Lemma 4.13. The following properties are equivalent for the fuzzy graph G(f) of a fuzzy fuction f:
(i) G(f) is fuzzy strongly contra-θsg-closed.
(ii) For each $(x, y) \in (X \times Y)$-G(f), there exist a f-θsg-open set U in X containing x and a fuzzy regular closed set V containing y such that $f(U) \wedge V = \phi$.

Theorem 4.14. If $f : X \to Y$ is fuzzy almost contra θsg-continuous and Y is fuzzy Urysohn, G(f) is fuzzy strongly contra-θsg-closed set in $X \times Y$.

Proof: Let Y is fuzzy Urysohn. Let $(x, y) \in (X \times Y)$ G(f). It follows that $f(x) \neq y$. Sinc Y is fuzzy Urysohn, then there exist fuzzy open sets U and V such that $f(x) \in U, y \in V$ and $cl(U) \wedge cl(V) = \phi$. Since f is fuzzy almost contra θsg-continuous, then there exists a f-θsg-open set μ in X containing x such that $f(\mu) \leq cl(U)$. Therefore, $f(\mu) \wedge cl(V) = \phi$ and G(f) is fuzzy strongly contra-θsg-closed in $X \times Y$.

Theorem 4.15. Let $f : X \to Y$ is fuzzy strongly contra-θsg-closed graph. If f is injective, then X is fuzzy $\theta sg - T_1$.

Proof: Let x and y be any two distinct points of X. Then, we have $(x, f(y)) \in (X \times Y)$-G(f). By Lemma 4.13, there exist a f-θsg-open set μ containing x and a fuzyy regular closed set η in Y containing f(y) such that $f(\mu) \wedge \eta = \phi$;hence $\mu \wedge f^{-1}(\eta) = \phi$. Therefore, we have $y \notin \mu$. This implies that X is fuzzy $\theta sg - T_1$.

5 Fuzzy Weakly Almost Contra-θ-Semigeralized-Continuous Functions

In this section, Fuzzy weakly almost contra-θ-semigeneralized-continuous function is introduced. The relationships between fuzzy almost contra-θsg-continuous functions and other forms are investigated. Also introduced the concept of Fuzzy θ-semigeneralized-compact (briefly, FTSGO-Compact)space.

Definition 5.1. A function $f : X \to Y$ is called fuzzy weakly almost contra-θsg-continuous if for each $x \in X$ and each fuzzy regular closed set η of Y containing f(x), there exists f-θsg-open set μ in X containing x, such that $int(f(\mu)) \leq \eta$.

Definition 5.2. A function $f : X \to Y$ is called fuzzy(θsg,s)-open if the image of each f-θsg-open set is Fs-open.

Theorem 5.3. If a function $f : X \to Y$ is fuzzy weakly almost contra-θsg-continuous and fuzzy (θsg,s)-open, then f is fuzzy almost contra-θsg-continuous.

Proof: Let $x \in X$ and η be a fuzzy regular closed set containing f(x). Since f is fuzzy weakly almost contra-θsg-continuous, there exists a f-θsg-open set μ in X containig x such that $int(f(\mu)) \leq \eta$. Since f is fuzzy (θsg, s)-open, $f(\mu)$ is a Fs-open set in Y and $f(\mu) \leq cl(int(f(\mu))) \leq \eta$. This shows that f is fuzzy almost contra-θsg-continuous.

Definition 5.4 (5)**.** A fuzzy space is said to be fuzzy P_Σ if for any fuzzy open set μ of X and each $x_\Sigma \in mu$, there exists fuzzy regular closed set ρ containing $x_\in$ such that $x_\in \in \rho \leq \mu$.

Theorem 5.5. Let $f : X \to Y$ ba a fuzzy function. Then, if f is fuzzy almost contra-θsg-continuous and Y is fuzzy P_Σ, then f is fuzzy almost contra-θsg-continuous.

Proof: Let μ be a fuzzy ope set in Y. Since Y is fuzzy P_Σ, there exists a family Ψ whose members are fuzzy regular closed set of Y such that $\mu = \wedge \{\rho : \rho \in \Psi\}$. Since f is fuzzy almost contra-θsg-continuous, $f^{-1}(\rho)$ is f-θsg-open in X for each $\rho \in \Psi$ and $f^{-1}(\mu)$ is f-θsg-open in X. Therefore, f is fuzzy almost contra-θsg-continuous.

Definition 5.6 (5)**.** A fuzzy space is said to be fuzzy weakly P_Σ if for any fuzzy regular open set μ of X and each $x_\Sigma \in mu$, there exists fuzzy regular closed set ρ containing $x_\in$ such that $x_\in \in \rho \leq \mu$.

Definition 5.7. A function $f : X \to Y$ is said to be fuzzy almost θsg-continuous at $x_\in \in \mu$ if for each fuzzy open set η containing $f(x_\in)$, there exists a f-θsg-open set μ containing $x_\in$ such that $f(\mu) \leq int(cl(\eta))$.

Theorem 5.8. Let $f : X \to Y$ be a fuzzy almost contra-θsg-continuous function. If Y is fuzzy weakly P_Σ, then f is fuzzy almost θsg-continuous.

Proof: Let μ be any fuzzy regular open set of Y. Since Y is fuzzy weakly P_Σ, there exists a family Ψ whose members are fuzzy regular closed set of Y such that $\mu = \wedge \{\rho : \rho \in \Psi\}$. Since f is fuzzy almost contra-θsg-continuous, $f^{-1}(\mu)$ is f-θsg-open in X. Hence, f is fuzzy almost θsg-continuous.

Theorem 5.9. Let X, Y, Z be fuzzy topological spaces and let $f : X \to Y$ and $g : Y \to Z$ be fuzzy functions. If f is fuzzy θsg-irresolute and g is fuzzy almost contra-θsg-continuous, then $g \circ f : X \to Z$ is a fuzzy almost contra-θsg-continuous function.

Proof: Let $\mu \leq Z$ be any fuzzy regular closed set and let $(g \circ f)(x_{\in}) \in \mu$. Then $g(f(x_{\in})) \in \mu$. Since g is fuzzy almost contra-θsg-continuous fuction, it follows that there exists a f-θsg-open set ρ containing $f(x_{\in})$ such that $g(\rho) \leq \mu$. Since f is fuzzy θsg-irresolute fuction, it follows that there exists a f-θsg-open set η containiing $x_{\in}$ such that $f(\eta) \leq \rho$. From here we obtain that $(g \circ f)(\eta) = g(f(\eta)) \leq \ g(\rho) \leq \mu$. Thus we have shown that $g \circ f$ is fuzzy almost contra-θsg-continuous function.

Theorem 5.10. If $f : X \to Y$ is a surjective fuzzy θsg-open function and $g : Y \to Z$ is a fuzzy function such that $g \circ f : X \to Z$ is fuzzy almost contra-θsg-continuous, then g is fuzzy almost contra-θsg-continuous.

Proof: Suppose that $x_{\in}$ is a fuzzy singleton in X. Let η be regular closed set in Z containing $(g \circ f)(x_{\in}$. Then there exists a f-θsg-open set μ in X containing $x_{\in}$ such that $g(f(\mu)) \leq \eta$. Since f is f-θsg-open, $f(\mu)$ is a f-θsg-open set in Y containing $f(x_{\in})$ such that $g(f(\mu)) \leq \eta$. This implies that g is fuzzy almost contra-θsg-continuous.

Corollary 5.11. If $f : X \to Y$ be a surjective f-θsg-irresolute and f-θsg-open function and let $g : Y \to Z$ is a fuzzy function. Then $g \circ f : X \to Z$ is fuzzy almost contra-θsg-continuous if and only if g is fuzzy almost contra-θsg-continuous.

Proof: It can be obtained from Theorem 5.9 and Theorem 5.10.

Definition 5.12. A space X is said to be fuzzy θsg-compact(briefly, FTSGO-Compact) if every f-θsg-open cover of X has a finite subcover.

Definition 5.13. A space X is said to be fuzzy θsg-closed-compact if every f-θsg-closed cover of X has a finite subcover.

Definition 5.14 (6)**.** A space X is said to be fuzzy nearly compact if every fuzzy regular open cover of X has a finite subcover.

Theorem 5.15. The fuzzy almost contra-θsg-continuous images of fuzzy θsg-closed-compact spaces are fuzzy nearly compact.

Proof: Suppose that $f : X \to Y$ is a fuzzy almost contra-θsg-continuous surjection. Let $\{\eta_i : i \in I\}$ be any fuzzy regular open cover of Y. Since f is fuzzy almost contra-θsg-continuous, then $\left\{f^{-1}(\eta_i) : i \in I\right\}$ uzzy is a f-θsg-closed cover of X. Since X is fuzzy θsg-closed-compact, there exists a finite subset I_o of I such that $X = \wedge \left\{f^{-1}(\eta_i) : i \in I_o\right\}$. Thus, we have $Y = \wedge \{\eta_i : i \in I_o\}$ and Y is nearly compact.

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Q-INTUITIONISTIC FUZZY SOFT SETS

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Abstract - In this paper, we first present the concept of Q- intuitionistic fuzzy soft sets which combine Q-intuitionistic fuzzy sets and soft sets. Basic properties are introduced along with illustrative examples. We also define its basic operations, namely equality, subset, complement, union, intersection, AND and OR, and study some related properties with supporting proofs. This concept is a generalization of Q-fuzzy soft sets.

***Keywords* -** *Intuitionistic fuzzy sets, Q-intuitionistic fuzzy sets, Q-intuitionistic fuzzy soft sets.*

1. Introduction

The concept of fuzzy sets was introduced by Zadeh [10] whose basic component is only a degree of membership. Atanassov [7] generalized this idea to intuitionistic fuzzy set (IFS in short) using a degree of membership and a degree of non-membership, under the constraint that the sum of the two degrees does not exceed one. The conception of IFS can be viewed as an appropriate /alternative approach in case where available information is not sufficient to define the impreciseness by the conventional fuzzy set. The idea of “intuitionistic Q-fuzzy set” was first published by Atanassov [8], as a generalization of the notion of fuzzy set.

In many fields, such as economics, engineering, environment, involve data that contain uncertainties. To understand and manipulate the uncertainties, there are many approaches such as probability theory, fuzzy set theory [10], intuitionistic fuzzy sets [7], rough set theory [20], etc. Each of these theories has its own difficulties as pointed out in [1]. To address these difficulties, Molodtsov[1] introduced the concept of soft set as a new mathematical tool for dealing with uncertainties that is free from difficulties. The main advantage of soft set theory in data analysis is that it does not need any grade of membership as in the fuzzy set theory. In soft set theory there is no limited condition to the description of the objects; so researchers can choose the form of parameter they need which greatly simplifies the decision making process and make the process more efficient in the

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absence of partial information. After Molodtsov's work, Maji et al.[14] introduced the concept of fuzzy soft set, a more generalized concept, which is a combination of fuzzy set and soft set and studied its properties and also discussed their properties. Also, Maji et al.[15] devoted the concept of intuitionistic fuzzy soft sets by combining intuitionistic fuzzy sets with soft sets.By using this definition of intuitionistic fuzzy soft sets many interesting applications of soft set theory have been expanded by some researchers [9, 11, 12 , 13, 16, 17,18]. Recently Adam et al. [5] defined a new concept called Q-fuzzy soft set which combine Q-fuzzy set and soft set. The same authors introduced the concept of multi Q-fuzzy set and a multi Q-fuzzyparameterized soft set [2], studied their operations and gave an application in decision making. Based on [5] and [8], we presented the concept of Q-intuitionistic fuzzy soft sets as ageneralization of Q-fuzzy soft sets.

The rest structure of this paper is as follows: part 2 presents some definitions which will be used in the sequel. Part 3 discusses the concept of Q-intuitionistic fuzzy soft set. Part 4 introduced the union, intersection, AND and OR operations on a Q-intuitionistic fuzzy soft set. Part 5 gives the conclusion.

2. Preliminaries

In this section we present the basic definitions of soft set theory, Q-fuzzy set, multi Q-fuzzy set, Q-fuzzy soft set, intuitionistic fuzzy set,Q- intuitionistic fuzzy set, intuitionistic fuzzy soft set and multi-Q intuitionistic fuzzy set required in this paper.

2.1. Soft Sets

Definition 2.1[1] A pair (F, E) is called a soft set over U, if and only if F is a mapping of E into the set of all subsets of the set U. In other words, the soft set is parameterized family of subsets of the set U.

As an illustration, let us consider the following example.

Example 2.2. Suppose that U is the set of houses under consideration, say $U = \{h_1, h_2,...,h_5\}$. Let E be the set of some attributes of such houses, say $E = \{e_1, e_2, . . ., e_8\}$, where e_1, e_2, . . ., e_8 stand for the attributes "beautiful", "costly", "in the green surroundings", "moderate", respectively.

In this case, to define a soft set means to point out expensive houses, beautiful houses, and so on. For example, the soft set (K, A) that describes the "attractiveness of the houses" in the opinion of a buyer, say Thomas, may be defined like this:

$$A=\{e_1,e_2,e_3,e_4,e_5\};$$

$$K(e_1) = \{h_2, h_3, h_5\}, K(e_2) = \{h_2, h_4\}, K(e_3) = \{h_1\}, K(e_4) = U, K(e_5) = \{h_3, h_5\}.$$

2.2 Q-fuzzy Sets

Definition 2.3 Let X be a non-empty set and Q be a non-empty set. A Q-fuzzy subset A of X is a function

$$A : XxQ \rightarrow [\ 0, 1\ ].$$

Definition 2.4: The union of two Q-fuzzy subsets A and B of a set X is defined by

$$(A \cup B)(x,q) = \max \{\ A(x, q), B(x, q)\ \}$$

for all x in X and q in Q.

Definition 2.5: The intersection of two Q-fuzzy subsets A and B of a set X is defined by

$$(A \cap B)(x, q) = \min \{\ A(x, q), B(x, q)\ \}$$

for all x in X and q in Q.

2.3 Multi Q-fuzzy Sets

Definition 2.5 [2] Let I be a unit interval [0, 1], k be a positive integer. U be a universal set and Q be a non-empty set. A multi Q-fuzzy set $\tilde{A}_Q$ in U and q is a set of ordered sequences

$$\tilde{A}_Q=\{(u,q), \mu_i(u, q): u\in U, q \in Q\}$$

where μ_i: U ×Q → I^k.The function μ_1(u, q),μ_2(u, q),…,μ_k(u, q) is called membership function of multi Q-fuzzy set $\tilde{A}_Q$; and μ_1(u, q)+μ_2(u, q)+…+μ_k(u, q) ≤ 1, k is called the dimension of $\tilde{A}_Q$. The set of all multi Q-fuzzy sets of dimension k in U and Q is denoted by M^kQF(U).

2.4 Q-fuzzy Soft Sets

Definition 2.6 [5] Let U be a universal set, E be a set of parameters, and Q be a non-empty set. LetM^kQF(U) denote the power set of all multi Q-fuzzy subset of U with dimension k= 1. Let A ⊆ E. A pair (F_Q,A) is called a Q-fuzzy soft set (in short QF-soft set) over U where F_Q is a mapping given by

$$F_Q: A \rightarrow M^k QF(U) \text{ such that } F_Q(x) = \emptyset \text{ if } x \notin A.$$

Here a Q-fuzzy soft set can be represented by the set of ordered pairs

$$(F_Q,A)= \{(x,F_Q(x)): x \in U,\ F_Q(x) \in M^k QF(U)\ \}$$

Note that the set of all Q-fuzzy soft set over U will be denoted by QFS(U).

Definition 2.7 [5] Let (F_Q, A) and (G_Q, B) ∈QFS(U). The union of two Q-fuzzy soft sets (F_Q, A) and (G_Q, B), is the Q-fuzzy soft set (H_Q, C), written as (F_Q, A) ⋃(G_Q, B) = (H_Q, C), where C = A⋃ B for all e ∈ C and

$$H_Q(e)=\begin{cases} F_Q(e) if \ \ e \in A-B\ ; \\ G_Q(e) if \ \ e \in B-A; \\ F_Q(e) \cup G_Q(e)\ \ if \ \ e \in A \cap B. \end{cases}$$

Definition 2.8 [5] Let (F_Q, A) and (G_Q, B) ∈QFS(U). The intersection of two Q-fuzzy soft sets (F_Q, A) and (G_Q, B), is the Q-fuzzy soft set (H_Q, C), written as (F_Q, A) ∩(G_Q, B) = (H_Q, C), where C = A∩ B for all e ∈ C,

$$(H_Q, \mathrm{C})= \{\mathrm{e}, \min \{\mu_{\mathrm{i}_{F_Q}}(\mathrm{x, q}), \mu_{\mathrm{i}_{G_Q}}(\mathrm{y, q})\} : \mathrm{u} \in \mathrm{U}, \mathrm{q} \in \mathrm{Q}\} \text{and i}=1, 2,\ldots,\mathrm{k}.$$

2.5. Intuitionistic Fuzzy Sets

Definition 2.9 [7] Let U be an universe of discourse then the intuitionistic fuzzy set A is an object having the form A = {< x, μ_A(x), ω_A(x)>,x ∈ U},where the functions μ_A(x),

$$\omega_A(\mathrm{x}) : \mathrm{U} \rightarrow [0,1]$$

define respectively the degree of membership, and the degree of non-membership of the element x ∈ X to the set A with the condition.

$$0 \le \mu_A(\mathrm{x})+ \omega_A(\mathrm{x}) \le 1.$$

For two IFS,

$$A_{\mathrm{IFS}}= \{<\mathrm{x}, \mu_A(\mathrm{x}), \omega_A(\mathrm{x})> | \ \mathrm{x} \in \mathrm{X}\ \}$$

And

$$B_{\mathrm{IFS}}= \{<\mathrm{x}, \mu_B(\mathrm{x}), \omega_B(\mathrm{x})> | \ \mathrm{x} \in \mathrm{X}\ \}$$

Then,

1. $A_{\mathrm{IFS}} \subseteq B_{\mathrm{IFS}}$ if and only if

$$\mu_A(\mathrm{x}) \le \mu_B(\mathrm{x}), \omega_A(\mathrm{x}) \ge \omega_B(\mathrm{x})$$

2. $A_{\mathrm{IFS}} = B_{\mathrm{IFS}}$ if and only if ,

$$\mu_A(\mathrm{x}) = \mu_B(\mathrm{x})\ , \omega_A(\mathrm{x}) = \omega_B(\mathrm{x}) \text{ for any x} \in \mathrm{X}.$$

3. The complement of A_{IFS} is denoted by A^o_{IFS} and is defined by

$$A^{o}_{IFS} = \{<x, \omega_A(x), \mu_A(x) \mid x \in X\}$$

4. $A \cap B = \{<x, \min\{\mu_A(x), \mu_B(x)\}, \max\{\omega_A(x), \omega_B(x)\}>: x \in X\}$

5. $A \cup B = \{<x, \max\{\mu_A(x), \mu_B(x)\}, \min\{\omega_A(x), \omega_B(x)\}>: x \in X\}$

6. $0_{IFS} = (0, 1)$ and $1_{IFS} = (1, 0)$

As an illustration, let us consider the following example.

Example 2.10. Assume that the universe of discourse $U=\{x_1, x_2, x_3, x_4\}$. It may be further assumed that the values of x_1, x_2, x_3 and x_4 are in [0, 1] Then, A is an intuitionistic fuzzy set (IFS) of U, such that,

$$A = \{< x_1, 0.4, 0.6>, < x_2, 0.3, 0.7>, < x_3, 0.2, 0.8>, < x_4, 0.2, 0.8>\}$$

2.6. Q-intuitionistic Fuzzy Sets

Definition 2.11 [8] A Q-intuitionistic fuzzy subset A in X is defined as an object of the form

$$A = \{< (x, q), \mu_A(x, q), \nu_A(x, q) > / \ x \in X \text{ and } q \text{ in } Q\}$$

where $\mu_A: X \times Q \to [0, 1]$ and $\nu_A: X \times Q \to [0, 1]$ define the degree of membership and the degree of non-membership of the element x in X and q in A respectively and for every x in X and q in Q satisfying

$$0 \le \mu_A(x, q) + \nu_A(x, q) \le 1.$$

Definition 2.12 [8] If A is a Q-intuitionistic fuzzy subset A of X, then the complement of A, denoted A^C is the Q-intuitionistic fuzzy set of X, given by

$$A^C(x, q) = \{< (x, q), \nu_A(x, q), \mu_A(x, q) > / \ x \in X \text{ and } q \text{ in } Q\}.$$

Definition 2.13 [8] Let A and B be Q-intuitionistic fuzzy subsets of sets G and H respectively. The product of A and denoted by $A \times B$ is defined as

$$A \times B = \{< ((x, y), q), \mu_{A \times B}((x, y), q), \nu_{A \times B}((x, y), q) > / \ x \text{ in } G \text{ and } y \text{ in } H \text{ and } q \text{ in } Q\},$$

where

$$\mu_{A \times B}((x, y), q) = \min\{\mu_A(x, q), \mu_B(y, q)\} \text{ and } \nu_{A \times B}((x, y), q) = \max\{\nu_A(x, q), \nu_B(y, q)\}.$$

Definition 2.14 [8] Let A be a Q-intuitionistic fuzzy subset in a set S, the strongest Q-intuitionistic fuzzy relation on S, that is a Q-intuitionistic fuzzy relation on A is V given by

$$\mu_V((x, y), q) = \min\{\mu_A(x, q), \mu_B(y, q)\} \text{ and } \nu_V((x, y), q) = \max\{\nu_A(x, q), \nu_B(y, q)\},$$

for all x and y in S and q in Q.

2.7 Intuitionistic Fuzzy Soft Sets

Definition 2.15 [15] Let U be an initial universe set and $A \subset E$ be a set of parameters. Let IFS(U) denotes the set of all intuitionistic fuzzy subsets of U. The collection (F, A) is termed to be the intuitionistic fuzzy soft set over U, where F is a mapping given by

$$\boldsymbol{F}: \boldsymbol{A} \rightarrow \boldsymbol{IFS(U)}.$$

Example 2.16 Let U be the set of houses under consideration and E is the set of parameters. Each parameter is a word or sentence involving intuitionistic fuzzy words. Consider E ={beautiful, wooden, costly, very costly, moderate, green surroundings, in good repair, in bad repair, cheap, expensive}. In this case, to define an intuitionistic fuzzy soft set means to point out beautiful houses, wooden houses, houses in the green surroundings and so on. Suppose that, there are five houses in the universe U given by $U = \{h_1, h_2, \ldots, h_5\}$ and the set of parameters $A = \{e_1, e_2, e_3, e_4\}$,where e_1 stands for the parameter `beautiful', $e_2$ stands for the parameter `wooden', e_3 stands for the parameter `costly' and the parameter $e_4$ stands for `moderate'. Then the intuitionistic fuzzy set (F, A) is defined as follows:

$$(F,A) = \left\{\begin{array}{l} \left(e_1\left\{\frac{h_1}{(0.1,0.6)}, \frac{h_2}{(0.2,0.7)}, \frac{h_3}{(0.6,0.2)}, \frac{h_4}{(0.7,0.3)}, \frac{h_5}{(0.2,0.3)}\right\}\right) \\ \left(e_2\left\{\frac{h_1}{(0.3,0.5)}, \frac{h_2}{(0.2,0.4)}, \frac{h_3}{(0.1,0.2)}, \frac{h_4}{(0.1,0.3)}, \frac{h_5}{(0.3,0.6)}\right\}\right) \\ \left(e_3\left\{\frac{h_1}{(0.4,0.3)}, \frac{h_2}{(0.6,0.3)}, \frac{h_3}{(0.2,0.5)}, \frac{h_4}{(0.2,0.6)}, \frac{h_5}{(0.7,0.3)}\right\}\right) \\ \left(e_4\left\{\frac{h_1}{(0.1,0.6)}, \frac{h_2}{(0.3, 0.6)}, \frac{h_3}{(0.6,0.4)}, \frac{h_4}{(0.4,0.2)}, \frac{h_5}{(0.5,0.3)}\right\}\right) \end{array}\right\}$$

2.8. Multi Q-intuitionistic Fuzzy Sets

Definition 2.17 [19] Let I be a unit interval [0, 1], k be a positive integer. U be a universal set and Q be a non-empty set. A multi Q-intuitionistic fuzzy set $\tilde{A}_Q$ in U and q is a set of ordered sequences

$$\tilde{A}_Q=\{(u,q), \mu_i(u, q),\nu_i(u, q): u\in U, q \in Q\}$$

where μ_i:U ×Q → I^k and ν_i:U ×Q → $I^k$and . The functions μ_1(u, q),μ_2(u, q),…,μ_k(u, q) is called membership function of multi Q-fuzzy set $\tilde{A}_Q$ and the functions ν_1(u, q),ν_2(u, q),…,ν_k(u, q) is called non-membership function of multi Q-intuitionistic fuzzy set $\tilde{A}_Q$; and 0 $\leq \mu_i$(x, q) +ν_i(x, q)$\leq$ 1, for i=1,2,….k. k is called the dimension of $\tilde{A}_Q$. The set of all multi Q- intuitionistic fuzzy sets of dimension k in U and Q is denoted by M^kQIF(U).

Example 2.18 [19] Let U= {u_1, u_2, u_3, u_4} be a universal set, Q ={p, q } be a non- empty set , and k = 3 be a positive integer. If $\tilde{A}_Q$ is a function from U ×Q to I^3 then the set $\tilde{A}_Q$={((u_1, q),(0.2, 0.3),(0.4, 0.5),(0.4, 0.6)), ((u_1, p),(0.4, 0.5), (0.1, 0.2),(0.2, 0.4)), ((u_2,q), (0.3, 0.5),(0.2, 0.5), (0.3, 0.4))} is a multi Q- intuitionistic fuzzy sets in U and Q.

Remark 2.19: If ν_i(u, q) = 0, then the multi Q intuitionistic fuzzy set $\tilde{A}_Q$={(u,q), μ_i(u, q),ν_i(u, q): u ∈ U, q ∈ Q} degenerate to the multi Q fuzzy set $\tilde{A}_Q$={(u,q), μ_i(u, q) : u∈ U, q ∈ Q}

3. Q-Intuitionistic Fuzzy Soft Sets

In this section we introduce the concept Q- intuitionistic fuzzy soft set and define some properties of a Q- intuitionistic fuzzy soft set namely, null, absolute, subset, equality and complement, and give an example of Q- intuitionistic fuzzy soft set.

Definition 3.1 Let U be a universal set, E be a set of parameters, and Q be a non-empty set. LetM^kQIF(U) denote the power set of all multi Q- intuitionistic fuzzy subset of U with dimension k= 1. Let A ⊆ E. A pair (F_Q,A) is called a Q- intuitionistic fuzzy soft set (in short QIF-soft set) over U where F_Q is a mapping given by

$$F_Q: A \rightarrow M^k\text{QIF(U) such that } F_Q(x) = \emptyset \text{ if } x \notin A.$$

Here a Q- intuitionistic fuzzy soft set can be represented by the set of ordered pairs

$$(F_Q,A)= \{(x,F_Q(x)): x \in U,\ F_Q(x) \in M^k\text{QIF(U)} \}$$

Note that the set of all Q- intuitionistic fuzzy soft set over U will be denoted by QIFS(U).

Example 3.2 Let U= {u_1, u_2, u_3, u_4, u_5} be a universal set, Q ={p, q, r} be a non- empty set , and E= {e_1, e_2, e_3, e_4, e_5} be a set of parameters. If

A={e_1, e_2, e_3} ⊂ E,
$F_Q(e_1)$={((u_1,p),(0.2, 0.3)), ((u_1,q),(0.4, 0.5)), ((u_1,r),(0.3, 0.5))}
$F_Q(e_2)$={((u_1,p),(0.1, 0.4)), ((u_1,q),(0.2, 0.3)), ((u_1,r),(0.3, 0.5)), ((u_4,p),(0.2, 0.6)), ((u_4,q),(0.3, 0.4)), ((u_4,r),(0.2, 0.3))}
$F_Q(e_3)$={((u_1,p),(0.6, 0.3)), ((u_1,q),(0.4, 0.3)), ((u_1,r),(0.3, 0.2))},

then

(F_Q,A)={(e_1, {((u_1,p),(0.2, 0.3)), ((u_1,q),(0.4, 0.5)), ((u_1,r),(0.3, 0.5))}), (e_2, {((u_1,p),(0.1, 0.4)), ((u_1,q),(0.2, 0.3)), ((u_1,r),(0.3, 0.5)), ((u_4,p),(0.2, 0.6)), ((u_4,q),(0.3, 0.4)), ((u_4,r),(0.2, 0.3))}), (e_3, {((u_1,p),(0.6, 0.3)), ((u_1,q),(0.4, 0.3)), ((u_1,r),(0.3, 0.2))})}

is Q- intuitionistic fuzzy soft set.

Definition 3.3 Let (F_Q, A) ∈ QIFS(U). If F_Q(x) = Φ for all x ∈ E then (F_Q, A) is called a null QIF-S-set denoted by (Φ, A).

Example 3.4 (Φ,A)={(e_1, {((u_1,p),(0, 1)), ((u_1,q),(0, 1)), ((u_1,r),(0, 1))}), (e_2, {((u_1,p),(0, 1)), ((u_1,q),(0, 1)), ((u_1,r),(0, 1)), ((u_4,p),(0, 1)), ((u_4,q),(0, 1)), ((u_4,r),(0, 1))}), (e_3, {((u_1,p),(0, 1)), ((u_1,q),(0, 1)), ((u_1,r),(0, 1))})}.

Definition3.5 Let (F_Q, A) ∈ QIFS(U). If F_Q(x) = U for all x ∈ E then (F_Q, A) is called an absolute QIF-soft set denoted by (U, A).

Example 3.6 (U, A)={(e_1, {((u_1,p),(1, 0)), ((u_1,q),(1, 0)), ((u_1,r),(1, 0))}), (e_2, {((u_1,p),(1, 0)), ((u_1,q),(1, 0)), ((u_1,r),(1, 0)), ((u_4,p),(1, 0)), ((u_4,q),(0, 1)), ((u_4,r),(1, 0))}), (e_3, {((u_1,p),(1, 0)), ((u_1,q),(1, 0)), ((u_1,r),(1, 0))})}.

Definition 3.7 Let (F_Q, A), (H_Q, B) ∈ QIFS(U). then we say that (F_Q, A) is a QIF-soft subset of (H_Q,B), denoted by (F_Q, A) ⊆ (H_Q, B), if A ⊆ B and F_Q(x) ⊆ H_Q(x) for all x ∈ U.

Proposition 3.8 Let (F_Q, A), (H_Q, B) ∈ QIFS(U).Then
1. (F_Q, A) ⊆ (U, E)
2. (Φ, A) ⊆ (F_Q, A)
3. If (F_Q, A) ⊆ (H_Q, B) and (H_Q, B) ⊆ (G_Q, C), then (F_Q, A) ⊆ (G_Q, C)

Proof. The proof can be easily obtained from Definition 3.7

Proposition 3.9 Let (F_Q, A), (H_Q, B) ∈ QIFS(U). If (F_Q, A) = (H_Q, B) and (H_Q, B) =(G_Q, C) Then (F_Q, A) = (G_Q, C).

Proof. The proof can be easily obtained from Definition 3.7

Definition 3.10 Let (F_Q, A) ∈ QIFS(U). Then, the complement of QIF-soft set denoted by $(F_Q,\text{A})^{\text{C}}$ is defined by $(F_Q,\text{A})^{\text{C}}$ = (F_Q^c, ¬A) where

$$F_Q^c : \neg\text{A} \rightarrow \text{QIF(U)}$$

is the mapping given by $F_Q^c(e)$ = Q - intuitionistic fuzzy complement for every e ∈ A.

Example 3.11 Consider example 3.2

(F_Q,A)={(e_1, {((u_1,p),(0.2, 0.3)), ((u_1,q),(0.4, 0.5)), ((u_1,r),(0.3, 0.5))}), (e_2, {((u_1,p),(0.1, 0.4)), ((u_1,q),(0.2, 0.3)), ((u_1,r),(0.3, 0.5)), ((u_4,p),(0.2, 0.6)), ((u_4,q),(0.3, 0.4)), ((u_4,r),(0.2, 0.3))}), (e_3, {((u_1,p),(0.6, 0.3)), ((u_1,q),(0.4, 0.3)), ((u_1,r),(0.3, 0.2))})}

Then

$(F_Q,\text{A})^{\text{C}}$={($e_1$, {(($u_1$,p),(0.3, 0.2)), (($u_1$,q),(0.5, 0.4)), (($u_1$,r),(0.5, 0.3))}), ($e_2$, {(($u_1$,p),(0.4, 0.1)), (($u_1$,q),(0.3, 0.2)), (($u_1$,r),(0.5, 0.3)), (($u_4$,p),(0.6, 0.2)), (($u_4$,q),(0.4, 0.3)), (($u_4$,r),(0.3, 0.2))}), ($e_3$, {(($u_1$,p),(0.3, 0.6)), (($u_1$,q),(0.3, 0.4)), (($u_1$,r),(0.2, 0.3))})}

Proposition 3.12 Let (F_Q, A) ∈ QIFS(U).Then
1. $\left((F_Q, \text{A})^c\right)^c$ = (F_Q, A)
2. $(\Phi, \text{A})^c$ = (U, E)
3. $(U, \text{E})^c$ = (Φ, E)

Proof. The proof can be easily obtained from Definition 3.7

4. Union and Intersection of Q-intuitionistic Fuzzy Soft Set.

In this section we introduce the union, intersection, AND and OR operations on a Q-intuitionistic fuzzy soft set.

Definition 4.1 Let (F_Q, A) and (G_Q, B) ∈ QIFS(U). The union of two Q- intuitionistic fuzzy soft sets (F_Q, A) and (G_Q, B), is the Q- intuitionistic fuzzy soft set (H_Q, C), written as (F_Q, A) ∪(G_Q, B) = (H_Q, C), where C = A∪ B for all e ∈ C and

$$H_Q(e)=\begin{cases} F_Q(e) if \ \ e \in A - B\,; \\ G_Q(e) if \ \ e \in B - A; \\ F_Q(e) \cup G_Q(e) \ \ if \ \ e \in A \cap B. \end{cases}$$

Example 4.2 : Let U= $\{u_1, u_2, u_3, u_4, u_5\}$ be a universal set, Q ={p, q,r} be a non- empty set , and E= $\{e_1, e_2, e_3, e_4, e_5\}$ be a set of parameters. If A=$\{e_1, e_2, e_3\}$ ⊂ E, and B=$\{e_1, e_2, e_4\}$ ⊂ E

(F_Q,A)={(e_1, {((u_1,p),(0.2, 0.3)), ((u_1,q),(0.4, 0.5)), ((u_1,r),(0.3, 0.5))}), (e_2, {((u_1,p),(0.1, 0.4)), ((u_1,q),(0.2, 0.3)), ((u_1,r),(0.3, 0.5)), ((u_4,p),(0.2, 0.6)), ((u_4,q),(0.3, 0.4)), ((u_4,r),(0.2, 0.3))}), (e_3, {((u_1,p),(0.6, 0.3)), ((u_1,q),(0.4, 0.3)), ((u_1,r),(0.3, 0.2))})}

And

(G_Q,B)={(e_1, {((u_1,p),(0.4, 0.5)), ((u_1,q),(0.3, 0.2)), ((u_1,r),(0.2, 0.4))}), (e_2, {((u_1,p),(0.3, 0.5)), ((u_1,q),(0.3, 0.6)), ((u_1,r),(0.4, 0.5)), ((u_4,p),(0.3, 0.6)), ((u_4,q),(0.2, 0.3)), ((u_4,r),(0.3, 0.5))}), (e_4, {((u_1,p),(0.6, 0.3)), ((u_1,q),(0.4, 0.3)), ((u_1,r),(0.3, 0.2))})}

Then

(H_Q, C)= {(e_1, {((u_1,p),(0.4, 0.3)), ((u_1,q),(0.4, 0.2)), ((u_1,r),(0.3, 0.4))}), (e_2, {((u_1,p),(0.3, 0.4)), ((u_1,q),(0.3, 0.3)), ((u_1,r),(0.4, 0.5)), ((u_4,p),(0.3, 0.6)), ((u_4,q),(0.3, 0.3)), ((u_4,r),(0.3, 0.3))}), (e_3, {((u_1,p),(0.6, 0.3)), ((u_1,q),(0.4, 0.3)), ((u_1,r),(0.3, 0.2))}), (e_4, {((u_1,p),(0.6, 0.3)), ((u_1,q),(0.4, 0.3)), ((u_1,r),(0.3, 0.2))})}

Definition 4.3 Let (F_Q, A) and (G_Q, B) ∈ QIFS(U). The intersection of two Q-intuitionistic fuzzy soft sets (F_Q, A) and (G_Q, B), is the Q- intuitionistic fuzzy soft set (H_Q, C), written as (F_Q, A) ∩(G_Q, B) = (H_Q, C), where C = A∩ B for all e ∈ C,

(H_Q, C)= {e, (min {$\mu_{i_{F_Q}}$(x, q), $\mu_{i_{G_Q}}$(y, q), max({$\nu_{i_{F_Q}}$(x, q), $\nu_{i_{G_Q}}$(y, q)))} : u∈ U, q ∈ Q} and i=1, 2,…,k.

Example 4.4 : Let U= $\{u_1, u_2, u_3, u_4, u_5\}$ be a universal set, Q ={p, q,r} be a non- empty set , and E= $\{e_1, e_2, e_3, e_4, e_5\}$ be a set of parameters. If A=$\{e_1, e_2, e_3\} \subset$ E, and B=$\{e_1, e_2, e_4\} \subset$ E

$(F_Q$,A)=$\{(e_1$, {((u_1,p),(0.2, 0.3)), ((u_1,q),(0.4, 0.5)), ((u_1,r),(0.3, 0.5))}), (e_2, {((u_1,p),(0.1, 0.4)), ((u_1,q),(0.2, 0.3)), ((u_1,r),(0.3, 0.5)), ((u_4,p),(0.2, 0.6)), ((u_4,q),(0.3, 0.4)), ((u_4,r),(0.2, 0.3))}), (e_3, {((u_1,p),(0.6, 0.3)), ((u_1,q),(0.4, 0.3)), ((u_1,r),(0.3, 0.2))})}

And

(G_Q,B)={(e_1, {((u_1,p),(0.4, 0.5)), ((u_1,q),(0.3, 0.2)), ((u_1,r),(0.2, 0.4))}), (e_2, {((u_1,p),(0.3, 0.5)), ((u_1,q),(0.3, 0.6)), ((u_1,r),(0.4, 0.5)), ((u_4,p),(0.3, 0.6)), ((u_4,q),(0.2, 0.3)), ((u_4,r),(0.3, 0.5))}), (e_4, {((u_1,p),(0.6, 0.3)), ((u_1,q),(0.4, 0.3)), ((u_1,r),(0.3, 0.2))})}

Then

(H_Q, C)= {(e_1, {((u_1,p),(0.2, 0.5)), ((u_1,q),(0.3, 0.5)), ((u_1,r),(0.2, 0.5))}), (e_2, {((u_1,p),(0.1, 0.5)), ((u_1,q),(0.2, 0.6)), ((u_1,r),(0.3, 0.5)), ((u_4,p),(0.2, 0.6)), ((u_4,q),(0.2, 0.4)), ((u_4,r),(0.2, 0.5))})}

Proposition 4.5 Let (F_Q, A), (G_Q, B) and (H_Q, C) ∈ QIFS(U).Then
1. (F_Q, A) ∪(Φ, A) = (F_Q, A)
2. (F_Q, A) ∪(U, A) = (U, A)
3. (F_Q, A) ∪(F_Q, A) = (F_Q, A)
4. (F_Q, A) ∪(G_Q, B) = (G_Q, B) ∪(F_Q, A)
5. (F_Q, A) ∪((G_Q, B) ∪ (H_Q, C)) = ((F_Q, A) ∪(G_Q, B)) ∪ (H_Q, C)

Proof. The proof can be easily obtained from Definition 4.1

Proposition 4.6 Let (F_Q, A), (G_Q, B) and (H_Q, C) ∈ QIFS(U).Then
1. (F_Q, A) ∩(Φ, A) = (Φ, A)
2. (F_Q, A) ∩(U, A) = (F_Q, A)
3. (F_Q, A) ∩(F_Q, A) = (F_Q, A)
4. (F_Q, A) ∩(G_Q, B) = (G_Q, B) ∩(F_Q, A)
5. (F_Q, A) ∩((G_Q, B) ∩(H_Q, C)) = ((F_Q, A) ∩(G_Q, B)) ∩ (H_Q, C)

Proof. The proof are straightforward.

Proposition 4.7 Let (F_Q, A), (G_Q, B) and (H_Q, C) ∈ QIFS(U).Then
1. $((F_Q, A) \cap (G_Q, B))^c = (F_Q, A)^c \cup (G_Q, B)^c$
2. $((F_Q, A) \cup (G_Q, B))^c = (F_Q, A)^c \cap (G_Q, B)^c$

The proof are straightforward by using the properties of a multi Q–intuitionistic fuzzy sets.

Proposition 4.8 Let (F_Q, A), (G_Q, B) and (H_Q, C) $\in$ QIFS(U) .Then

1. (F_Q, A) $\cap$((G_Q, B) $\cup$(H_Q, C)) = ((F_Q, A) $\cap$(G_Q, B)) $\cup$ ((F_Q, A) $\cap$(H_Q, C))
2. (F_Q, A) $\cup$((G_Q, B) $\cap$(H_Q, C)) = ((F_Q, A) $\cup$(G_Q, B)) $\cap$ ((F_Q, A) $\cup$(H_Q, C))

Definition 4.9 Let (F_Q, A) and (G_Q, B) $\in$ QIFS(U). Then (F_Q, A) AND (G_Q, B) is the Q-intuitionistic fuzzy soft set denoted by (F_Q, A) $\wedge$ (G_Q, B) and defined by

$$(F_Q, A) \wedge (G_Q, B) = (H_Q, A\times B)$$

where $H_Q(\alpha ,\beta) = F_Q(\alpha) \cap G_Q(\beta)$ for all $\alpha \in$ A and $\beta \in$ B, is the operation of intersection of two Q-intuitionistic fuzzy sets.

Definition 4.10 Let (F_Q, A) and (G_Q, B) $\in$ QIFS(U). Then (F_Q, A) OR (G_Q, B) is the Q-intuitionistic fuzzy soft set denoted by (F_Q, A) $\vee$ (G_Q, B) and defined by

$$(F_Q, A) \vee (G_Q, B) = (H_Q, A\times B)$$

where $H_Q(\alpha ,\beta) = F_Q(\alpha) \cup G_Q(\beta)$ for all $\alpha \in$ A and $\beta \in$ B, is the operation of union of two Q-intuitionistic fuzzy sets.

Conclusion

In this paper we have introduced the concept of Q-intuitionistic fuzzy soft sets and studied some related properties with supporting proofs. The equality, subset, complement, union, intersection, AND or OR operations have been defined on the Q-intuitionistic fuzzy soft sets. This new extension will provide a significant addition to existing theories for handling uncertainties, and lead to potential areas of further research and pertinent applications.

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THE HERMITE-HADAMARD TYPE INEQUALITIES FOR OPERATOR m-CONVEX FUNCTIONS IN HILBERT SPACE

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Abstract – In this paper, we first define operators m-convex functions for positive, bounded, self-adjoint operators in Hilbert space via m-convex functions. Secondly, we establish some new theorems for them. Finally, we obtain the Hermite-Hadamard type inequalities for the product two operators m-convex functions in Hilbert space.

Keywords – *The Hermite-Hadamard inequality, m-convex functions, operator m-convex functions, selfadjoint operator, inner product space, Hilbert space.*

1 Introduction

The following inequality holds for any convex function f define on $\mathbb{R}$ and $a, b \in \mathbb{R}$, with $a < b$

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_0^1 f(x)dx \leq \frac{f(a)+f(b)}{2} \tag{1}$$

both inequalities hold in the reversed direction if f is concave.

The inequality (1) is known in the literature as the Hermite-Hadamard's inequality. The Hermite-Hadamard's inequality may be regarded as a refinement of the concept of convexity and it follows easily from Jensen's inequality. The classical Hermite-Hadamard inequality provides estimates of the mean value of a continuous convex function $f : [a, b] \to \mathbb{R}$. In this paper, Firstly we defined for bounded positive self-adjoint operator m-convex functions in Hilbert space, secondly established some new

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theorems for them and finally Hermite-Hadamard type inequalities for product two bounded positive selfadjoint operators m-convex set up in Hilbert space.

2 Preliminary

First, we review the operator order in $B(H)$ and the continuous functional calculus for a bounded selfadjoint operator. For selfadjoint operators $A, B \in B(H)$ we write, for every $x \in H$

$$A \leq B (\text{or } B \geq A) \, \text{if} \, \langle Ax, x\rangle \leq \langle Bx, x\rangle (\text{or } \langle Bx, x\rangle \geq \langle Ax, x\rangle)$$

we call it the operator order.

Let A be a selfadjoint linear operator on a complex Hilbert space $(H, \langle ., .\rangle)$ and $C(Sp(A))$ the C^* -algebra of all continuous complex-valued functions on the spectrum A. The Gelfand map establishes a $*$-isometrically isomorphism Φ between $C(Sp(A))$ and the C^*-algebra $C^*(A)$ generated by A and the identity operator 1_H on H as follows [1].

For any $f, g \in C(Sp(A))$ and any $\alpha, \beta \in \mathbb{C}$ we have

i. $\Phi(\alpha f + \beta g) = \alpha\Phi(f) + \beta\Phi(g)$;

ii. $\Phi(fg) = \Phi(f)\Phi(g)$ and $\Phi(f^*) = \Phi(f)^*$;

iii. $\|\Phi(f)\| = \|f\| := sup_{t\in Sp(A)}|f(t)|$;

iv. $\Phi(f_0) = 1$ and $\Phi(f_1) = A$, where $f_0(t) = 1$ and $f_1(t) = t$, for $t \in Sp(A)$

If f is a continuous complex-valued functions on $C(Sp(A)$, the element $\Phi(f)$ of $C^*(A)$ is denoted by $f(A)$, and we call it the continuous functional calculus for a bounded selfadjoint operator A.

If A is bounded selfadjoint operator and f is real valued continuous function on $Sp(A)$, then $f(t) \geq 0$ for any $t \in Sp(A)$ implies that $f(A) \geq 0$, i.e $f(A)$ is a positive operator on H. Moreover, if both f and g are real valued functions on $Sp(A)$ such that $f(t) \leq g(t)$ for any $t \in Sp(A)$, then $f(A) \leq f(B)$ in the operator order $B(H)$.

A real valued continuous function f on an interval I is said to be operator convex (operator concave) if

$$f((1-\lambda)A + \lambda B) \leq (\geq)(1-\lambda)f(A) + \lambda f(B)$$

in the operator order in $B(H)$, for all $\lambda \in [0, 1]$ and for every bounded self-adjoint operator A and B in $B(H)$ whose spectra are contained in I.

We denoted by $B(H)^+$ the set of all positive operators in $B(H)$.

G.H. Toader [2] defines the m-convexity, on intermediate between the usual convexity and starshaped property.

Definition 2.1. [2] The function $f : [a, b] \to \mathbb{R}$ is said to be m-convex, where $m \in [0, 1]$, if for $x, y \in [a, b]$ and $t \in [0, 1]$ we have $f(tx + m(1-t)y) \leq tf(x) + m(1-t)f(y)$

Denote by $K_m(b)$ the set of the m-convex functions on [a,b] for which $f(0) \leq 0$. Note that, for $m = 1$, we recapture the concept of convex functions defined on $[a, b]$ and for $m = 0$ we get the concept of starshaped functions on $[a, b]$. We recall that $f : [a, b] \to \mathbb{R}$ is starshaped if $f(tx) \leq tf(x)$, for all $t \in [0, 1]$ and $x \in [a, b]$.

3 The Hermite-Hadamard Type Inequalities for Operator m-convex Functions in Hilbert Space

3.1 Operator m-convex Functions in Hilbert Space

The following definition is firstly defined by Yeter Erdaş

Definition 3.1. Let I be an interval in $\mathbb{R}$ and K be convex subset of $B(H)^+$. A continuous function $f : I \subseteq [0,\infty) \to \mathbb{R}$ is said to be operator m-convex on I for operators in K if

$$f(tA + m(1-t)A) \leq tf(A) + m(1-t)f(A)$$

in the operator order in $B(H)^+$, for all $m, t \in [0,1]$ and for every positive operators A and B in K whose spectra are contained in I.

Lemma 3.2. If f is operator m-convex on $[0,\infty)$ for operator in K, then $f(A)$ is positive for every $A \in K$.

Proof. For $A \in K$, we have

$$\begin{aligned} f(A) &= f\left(\frac{tA + m(1-t)A + (1-t)A + mtA}{2}\right) \\ &\leq f(tA + m(1-t)A + (1-t)A + mtA) \\ &\leq tf(A) + m(1-t)f(A) + (1-t)f(A) + mtf(A) \\ &= tf(A) + mf(A) - mtf(A) + f(A) - tf(A) + mtf(A)f(A) \\ &\leq f(A)(m+1) \\ &\leq mf(A) \end{aligned}$$

This implies that $f(A) \geq 0$.

Moslehian and Najafi [3] proved the following theorem for positive operators as follows:

Theorem 3.3. [3] Let $A, B \in B(H)^+$. Then $AB + BA$ is positive if and only if $f(A+B) \leq f(A) + f(B)$ for all non-negative operator functions f on $[0,\infty)$.

Dragomir in [4] has proved a Hermite-Hadamard type inequality for operator convex function as following

Theorem 3.4. [4] Let $f : I \to \mathbb{R}$ be an operator convex function on the interval I. Then for all selfadjoint operators A and B with spectra in I we have the inequality

$$\left(f\left(\frac{A+B}{2}\right) \leq \right)\frac{1}{2}\left[f\left(\frac{3A+B}{4}\right) + f\left(\frac{A+3B}{4}\right)\right]$$

$$\leq \int_0^1 f\Big((1-t)A + tB\Big)\Big)dt$$

$$\leq \frac{1}{2}\left[f(\frac{A+B}{2}\right) + \frac{f(A)+f(B)}{2}\right]$$

$$\left(\leq \frac{f(A)+f(B)}{2}\right)\Big]$$

Let X be a vector space, $x, y \in X, x \neq y$. Define the segment

$$[x, y] := (1-t)x + ty; t \in [0, 1].$$

We consider the function $f : [x, y] :\rightarrow \mathbb{R}$ and the associated function

$$g(x, y) : [0, 1] \rightarrow \mathbb{R}$$

$$g(x, y)(t) := f((1-t)x + ty), t \in [0, 1].$$

Note that f is convex on $[x, y]$ if and only if $g(x, y)$ is convex on $[0, 1]$. For any convex function defined on a segment $[x, y] \in X$, we have the Hermite-Hadamard integral inequality

$$f\left(\frac{x+y}{2}\right) \leq \int_0^1 f((1-t)x + ty)dt \leq \frac{f(x) + f(y)}{2}$$

which can be derived from the classical Hermite-Hadamard inequality for the convex $g(x, y) : [0, 1] \rightarrow \mathbb{R}$.

Lemma 3.5. Let $f : I \subseteq [0, \infty) \rightarrow \mathbb{R}$ be a continuous function on the interval I. Then for every two positive operators $A, B \in K \subseteq B(H)^+$ with spectra in I the function f is operator m-convex for operators in

$$[A, B] := \{(1-t)A + mtB : t \in [0, 1]\}$$

if and only if the function $\varphi_{x,A,B} : [0, 1] \rightarrow \mathbb{R}$ defined by

$$\varphi_{x,A,B}(t) = \langle f((1-t)A + mtB)x, x\rangle$$

is m-convex on $[0, 1]$ for every $x \in H$ with $\|x\| = 1$.

Proof. Let f be operator m-convex for operators in $[A, B]$ then for any $t_1, t_2 \in [0, 1]$ and $\lambda, \gamma \geq 0$ with $\lambda + \gamma = 1$ we have

$$\varphi_{x,A,B}(\lambda t_1 + \gamma t_2) = \langle f((1-(\lambda t_1 + \gamma t_2)A) + m(\lambda t_1 + \gamma t_2)B)x, x\rangle$$

$$\begin{aligned} &= \langle f(\lambda A + \gamma A - \lambda A t_1 - \gamma A t_2 + m\lambda t_1 B + m\gamma t_2 B)x, x\rangle \\ &= \langle f(\lambda[(1-t_1)A + mt_1 B] + \gamma[(1-t_2)A + mt_2 B])x, x\rangle \\ &\leq \lambda\varphi_{x,A,B}(t_1)\varphi_{x,A,B}(t_2) \end{aligned}$$

showing that $\varphi_{x,A,B}$ is a m-convex function on $[0, 1]$. Let now $\varphi_{x,A,B}$ be m-convex on $[0, 1]$, we show that f is operator convex for operators in $[A, B]$. For every $C :=$

$(1-t_1)A + mt_1B$ and $D := (1-t_2)A + mt_2B$ we have

$$\begin{aligned}
\langle f((1-\lambda)C + m\lambda D)x, x\rangle &= \langle f((1-\lambda)[(1-t_1)A + mt_1B] \\
&\quad + m\lambda[(1-t_2)A + mt_2B])x, x\rangle \\
&= \langle f(A - t_1A + mt_1B - \lambda A + \lambda t_1 A - m\lambda t_1 B \\
&\quad + m\lambda A - m\lambda t_2 A + m^2\lambda t_2 B)x, x\rangle \\
&= \langle f(A(1-t_1) - \lambda A(1-t_1) + m\lambda A(1-t_2) \\
&\quad + mt_1B + m^2\lambda t_2 B - m\lambda t_1 B)x, x\rangle \\
&= \langle f(-\lambda((1-t_1)A + mt_1B) + A(1-t_1) \\
&\quad + mt_1B + m\lambda(A(1-t_2) + mt_2B))x, x\rangle \\
&= \langle f((1-\lambda)((1-t_1)A + mt_1B) \\
&\quad + m\lambda((1-t_2)A + mt_2B))x, x\rangle \\
&\leq (1-\lambda)\langle f(C)x, x\rangle + m\lambda\langle f(D)x, x\rangle
\end{aligned}$$

Theorem 3.6. Let $f : I \to \mathbb{R}$ be an operator m-convex function on the interval $I \subseteq [0, \infty)$ for operators in $K \subseteq B(H)^+$. Then for all positive operators A and B in K with spectra in I we have the inequality

$$\begin{aligned}
f(\frac{A+B}{2}) &\leq \int_0^1 \Big[tf(A) + m(1-t)f(A) + tf(B) + m(1-t)f(B)\Big]dt \\
&\leq (m+1)(f(A) + f(B))
\end{aligned}$$

Proof. For $x \in H$ with $\|x\| = 1$ and $t \in [0, 1]$, we have

$$\langle [tA + m(1-t)B]x, x\rangle = t\langle Ax, x\rangle + m(1-t)\langle Bx, x\rangle \in I \tag{2}$$

since $\langle Ax, x\rangle \in Sp(A) \subseteq I$ and $\langle Bx, x\rangle \in Sp(B) \subseteq I$, (2) imply that the operator-valued integral $\int_0^1 f(tA + (1-t)B)dt$ exists. Since f is operator m-convex, therefore for t in $[0, 1]$ and $A, B \in K$ we have

$$f(tA + m(1-t)B) \leq tf(A) + m(1-t)f(B) \tag{3}$$

integrating both sides of (3) over $[0, 1]$ we get the following inequality

$$
\begin{aligned}
&\int_0^1 \Big[f(tA+m(1-t)B)\Big]dt \leq \int_0^1 \Big[tf(A)+m(1-t)f(B)\Big]dt \\
= \;& f(A)+mf(B)-f(B) \\
= \;& f(A)+(m-1)Bf(\frac{A+B}{2}) \\
= \;& f\Big(\frac{tA+m(1-t)A+(1-t)A+mtA+tB+m(1-t)B+(1-t)B+mtB}{2(m+1)}\Big) \\
\leq \;& f\Big(\frac{t(A+B)+m(1-t)(A+B)+(1-t)(A+B)+mt(A+B)}{2}\Big) \\
\leq \;& tf(A)+tf(B)+m(1-t)f(A)+m(1-t)f(B)+f(A)+f(B) \\
& -tf(A)-tf(B)+mtf(A)+mtf(B) \\
= \;& (m+1)[f(A)+f(B)]\int_0^1 f(tA+m(1-t)B)dt \\
= \;& \int_0^1 f((1-t)A+mtB)dt
\end{aligned}
$$

4 The Hermite-Hadamard Type Inequalites for Product Two Operators m-convex Functions

Let $f : I \to \mathbb{R}$ be operator m-convex and $g : I \to \mathbb{R}$ operator m-convex function on the interval I. Then for all positive operators A and B on a Hilbert space H with spectra in I, we define real functions $K(A)(x)$, $L(A,B)(x)$, $R(A,B)(x)$, $S(B)(x)$, $M(A,B)(x)$, $N(A,B)(x)$ on H by

$$
\begin{aligned}
K(A)(x) &= \langle f(A)x,x\rangle\langle g(A)x,x\rangle \\
L(A,B)(x) &= \langle f(A)x,x\rangle\langle g(B)x,x\rangle \\
R(A,B)(x) &= \langle f(B)x,x\rangle\langle g(A)x,x\rangle \\
S(B)(x) &= \langle f(B)x,x\rangle\langle g(B)x,x\rangle \\
M(A,B)(x) &= \langle f(A)x,x\rangle\langle g(A)x,x\rangle + \langle f(B)x,x\rangle\langle g(B)x,x\rangle \\
N(A,B)(x) &= \langle f(A)x,x\rangle\langle g(B)x,x\rangle + \langle f(B)x,x\rangle\langle g(A)x,x\rangle.
\end{aligned}
$$

Theorem 4.1. Let $f : I \to \mathbb{R}$ be operator m_1-convex and $g : I \to \mathbb{R}$ operator m_2-convex function on the interval I for operators in $K \subseteq B(H)^+$. Then for all positive operators A and B in K with spectra in I, the inequality

$$
\int_0^1 \Big[\langle f(tA+m_1(1-t)B)x,x\rangle\langle g(tA+m_2(1-t)B)x,x\rangle\Big]dt
$$

$$\leq \Big(\frac{K}{3}\Big) + \Big(\frac{m_2 L}{6}\Big) + \Big(\frac{m_1 R}{6}\Big) - \Big(\frac{m_1 m_2 S}{3}\Big)$$

Proof. For $x \in H$ with $\|x\| = 1$ and $t \in [0,1]$ we have

$$\langle [tA + m(1-t)B]x, x\rangle = t\langle Ax, x\rangle + m(1-t)\langle Bx, x\rangle \in I \tag{4}$$

since $\langle Ax, x\rangle \in Sp(A) \subseteq I$ and $\langle Bx, x\rangle \in Sp(B) \subseteq I$. Continuity of f, g and (4) imply that the operator valued integrals $\int_0^1 f(tA+m_1(1-t)B)dt$, $\int_0^1 g(tA+m_2(1-t)B)dt$ and $\int_0^1 (fg)(tA + m(1-t)B)dt$ exist. Since f, g are operator convex, therefore for $t \in [0,1]$ and $t \in [0,1]$ we have

$$\langle f(tA + m_1(1-t)B)x, x\rangle \leq t\langle f(A)x, x\rangle + m_1(1-t)\langle f(B)x, x\rangle$$

$$\langle g(tA + m_2(1-t)B)x, x\rangle \leq t\langle g(A)x, x\rangle + m_2(1-t)\langle g(B)x, x\rangle$$

$$\Big(\langle f(tA + m_1(1-t)B)x, x\rangle\Big)\Big(\langle g(tA + m_2(1-t)B)x, x\rangle\Big)$$

$$\leq t^2\langle f(A)x, x\rangle\langle g(A)x, x\rangle + tm_2(1-t)\langle f(A)x, x\rangle\langle g(B)x, x\rangle \tag{5}$$

$$+tm_1(1-t)\langle f(B)x, x\rangle\langle g(A)x, x\rangle$$

$$+m_1m_2(1-t)^2\langle f(B)x, x\rangle\langle g(B)x, x\rangle$$

Integrating both sides of (5) over $[0,1]$, we get the following inequality

$$\int_0^1 \Big[\langle f(tA + m_1(1-t)B)x, x\rangle\langle g(tA + m_2(1-t)B)x, x\rangle\Big]dt \leq$$

$$\Big(\frac{K}{3}\Big) + \Big(\frac{m_2 L}{6}\Big) + \Big(\frac{m_1 R}{6}\Big) - \Big(\frac{m_1 m_2 S}{3}\Big)$$

Theorem 4.2. Let $f : I \to \mathbb{R}$ be operator m_1-convex and $g : I \to \mathbb{R}$ operator m_2-convex function on the interval I for operators in $K \subseteq B(H)^+$. Then for all positive operators A and B in K with spectra in I, the inequality

$$\langle f\Big(\frac{A+B}{2}\Big)x, x\rangle\langle g\Big(\frac{A+B}{2}\Big)x, x\rangle$$

$$\leq \Big[\frac{1 - m_1m_2}{3} + \frac{m_1 + m_2}{6}\Big]M(A,B)(x)N(A,B)(x)$$

Since f is operator m_1-convex and g is operator m_2-convex, for any $t \in I$ and any $x \in H$ with $\|x\| = 1$ we observe that

$$\begin{aligned}
&\langle f\Big(\frac{A+B}{2}\Big)x, x\rangle \\
\leq\ & \langle [tf(A) + m_1(1-t)f(A) + tf(B) + m_1(1-t)f(B)]x, x\rangle
\end{aligned}$$

$$\begin{aligned}
&\langle g\Big(\frac{A+B}{2}\Big)x, x\rangle \\
\leq\ & \langle [tg(A) + m_2(1-t)g(A) + tg(B) + m_2(1-t)g(B)]x, x\rangle
\end{aligned}$$

$$\begin{aligned}
&\Big(\langle f\Big(\frac{A+B}{2}\Big)x, x\rangle\Big)\Big(\langle g\Big(\frac{A+B}{2}\Big)x, x\rangle\Big) \\
\leq\ & t^2\langle f(A)x, x\rangle\langle g(A)x, x\rangle + tm_2(1-t)\langle f(A)x, x\rangle\langle g(A)x, x\rangle \\
+\ & t^2\langle f(A)x, x\rangle\langle g(B)x, x\rangle \\
+\ & tm_2(1-t)\langle f(A)x, x\rangle\langle g(B)x, x\rangle \\
+\ & tm_1(1-t)\langle f(A)x, x\rangle\langle g(A)x, x\rangle + m_1m_2(1-t)^2\langle f(A)x, x\rangle\langle g(A)x, x\rangle \\
+\ & tm_1(1-t)\langle f(A)x, x\rangle\langle g(B)x, x\rangle + m_1m_2(1-t)^2\langle f(A)x, x\rangle\langle g(B)x, x\rangle \\
+\ & t^2\langle f(B)x, x\rangle\langle g(A)x, x\rangle + tm_2(1-t)\langle f(B)x, x\rangle\langle g(A)x, x\rangle \\
+\ & t^2\langle f(B)x, x\rangle\langle g(B)x, x\rangle + tm_2(1-t)\langle f(B)x, x\rangle\langle g(B)x, x\rangle \\
+\ & tm_1(1-t)\langle f(B)x, x\rangle\langle g(A)x, x\rangle + m_1m_2(1-t)^2\langle f(B)x, x\rangle\langle g(A)x, x\rangle \\
+\ & tm_1(1-t)\langle f(B)x, x\rangle\langle g(B)x, x\rangle + m_1m_2(1-t)^2\langle f(B)x, x\rangle\langle g(B)x, x\rangle
\end{aligned}$$

Integrating both sides of (6) over $[0, 1]$ we get the following inequality

$$\begin{aligned}
&\langle f\Big(\frac{A+B}{2}\Big)x, x\rangle\langle g\Big(\frac{A+B}{2}\Big)x, x\rangle \\
\leq\ & \Big[\frac{1-m_1m_2}{3} + \frac{m_1+m_2}{6}\Big]M(A, B)(x)N(A, B)(x)
\end{aligned}$$

and this finishes the proof.

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