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FUZZY SOFT SET PREFERENCE RELATION

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Abstract – In this paper, at first we define fuzzy strict preference relation in our way motivated from strict preference relation in crisp concept and then define fuzzy weak preference relation, fuzzy indifference relation. Hence we discuss some properties like fuzzy semi-symmetric, fuzzy negatively transitive, fuzzy connectedness and give supporting examples. Thereafter we introduce the notion of fuzzy soft set strict preference relation and define fuzzy soft set weak preference relation, fuzzy soft set indifference relation. Also we verify some properties with suitable examples.

Keywords – *Fuzzy strict preference relation, Fuzzy weak preference relation, Fuzzy soft set strict preference relation, Fuzzy soft set weak preference relation.*

1 Introduction

In real life situation, almost all objects have an ambiguous status with respect to belongingness in a particular class. To reduce this ambiguity, in 1965, Zadeh[13] introduced fuzzy set with a continuum of grades of membership. Fuzzy sets and relations

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have applications in diverse types of areas, for example in data bases, pattern recognition, neural networks, fuzzy modelling, economics, medicine, multicriteria decision making etc. But fuzzy set theory, probability theory etc. have inherent difficulties to deal with uncertainties in above mentioned areas. To deal with uncertainties free from some difficulties, in 1999, Molodtsov[12] proposed a new parameterized mathematical tool named as soft set. Thereafter in 2001, Maji[11] et al. introduced the notion of fuzzy soft set as hybrid structure of fuzzy set and soft set. Then gradually so many contributions comes from several authors [2, 5, 7, 8, 10] in the area of soft set and fuzzy soft set. On the other hand, Preference modelling is an inevitable step in a variety of field like economics, sociology, psychology, mathematical programming, medicine, decision analysis etc. In decision making problem, procedures are usually based on pair comparisons in the sense that process are linked to some degree of credibility of preference. But preference value can not be express accurately. Hence the use of fuzzy preference is needed. Some papers concerning preference relation and fuzzy preference relation have been published; see, e.g., [1, 6, 9, 15, 16]. Here we have been motivated to introduce fuzzy preference relation in our way following the notion of strict preference relation in crisp method [14]. Also, to introduce fuzzy soft set preference relation, we have considered soft set relation which was recently introduced by Babitha et al. [4] in 2010, as a soft subset of cartesian product of the soft sets. The rest of this paper is organized as follows. In section 2, we recollect basic definitions and notations for later section. In section 3, we redefine fuzzy strict preference relation by extending the concept of preference relation[14] in crisp method and hence define fuzzy weak preference relation, fuzzy indifference relation and study some of their properties. In section 4, we define fuzzy soft set strict preference, fuzzy soft set weak preference, fuzzy soft set indifference relation and examine their properties with supporting examples.

2 Preliminary

Throughout this paper, let U be the initial universe, E be the set of parameters and A, B, C are subsets of E. We denote $\max\{x, y\}$ by $x \vee y$ and $\min\{x, y\}$ by $x \wedge y$. Let $P(U)$ be the collection of all subsets of U and I^U, $I^{U\times U}$ denote the collection of all fuzzy subsets of U, $U \times U$ respectively.

Definition 2.1. Let μ, ν be two fuzzy subsets of U. Then μ is called a fuzzy subset of ν if $\mu(x) \leq \nu(x)$, $\forall\, x \in U$. We write $\mu \subseteq \nu$.

Definition 2.2. [13] A fuzzy binary relation μ on U is a fuzzy subset of $U \times U$ i.e. $\mu : U \times U \to [0, 1]$.

Definition 2.3. A fuzzy subset of $U \times U$ is said to be a null fuzzy set, denoted by $\widetilde{0}_{U\times U}$ and defined by $\widetilde{0}_{U\times U}(x,y) = 0$ for all $(x,y) \in U \times U$. A fuzzy subset of $U \times U$ is said to be a absolute fuzzy set, denoted by $\widetilde{1}_{U\times U}$ and defined by $\widetilde{1}_{U\times U}(x,y) = 1$ for all $(x,y) \in U \times U$.

Definition 2.4. [3] The Cartesian product of two fuzzy subsets μ, ν of U is denoted by $\mu \times \nu$ and defined by

$$(\mu \times \nu)(x,y) = \mu(x) \wedge \nu(y), \ \ \forall\, x,\, y \in U.$$

Definition 2.5. [13] Let μ, ν be two fuzzy relation on U. Then for all $(x,y) \in U \times U$,
(i) union of μ, ν is denoted by $\mu \cup \nu$ and defined by

$$(\mu \cup \nu)(x,y) = \mu(x,y) \ \vee \ \nu(x,y);$$

(ii) intersection of μ, ν is denoted by $\mu \cap \nu$ and defined by

$$(\mu \cap \nu)(x,y) = \mu(x,y) \ \wedge \ \nu(x,y);$$

(iii) complement of μ is denoted by μ^c and defined by

$$\mu^c(x,y) = 1 - \mu(x,y);$$

(iv) algebraic product of μ, ν is denoted by $\mu.\nu$ and defined by

$$(\mu.\nu)(x,y) = \mu(x,y).\nu(x,y);$$

(v) algebraic sum of μ, ν is denoted by $\mu \oplus \nu$ and defined by

$$(\mu \oplus \nu)(x,y) = \mu(x,y) + \nu(x,y) - (\mu.\nu)(x,y).$$

Definition 2.6. [12] Let $A \subseteq E$. A pair (F,A) is called a soft set over U, where F is a mapping given by $F : A \to P(U)$.

Definition 2.7. [4] Let (F,A) and (G,B) be two soft sets over U, then the cartesian product of (F,A) and (G,B) is defined as, $(F,A) \times (G,B) = (H, A \times B)$, where $H : A \times B \to P(U \times U)$ and $H(a,b) = F(a) \times G(b)$, for all $(a,b) \in A \times B$, i.e.

$$H(a,b) = \{(h_i, h_j);\ \text{where}\, h_i \in F(a)\, \text{and}\, h_j \in G(b)\}.$$

Definition 2.8. [4] Let (F,A) and (G,B) be two soft sets over U, then a soft set relation from (F,A) to (G,B) is a soft subset of $(F,A) \times (G,B)$, i.e., a soft set relation from (F,A) to (G,B) is of the form (H_1, C) where $C \subseteq A \times B$ and

$H_1(a,b) = H(a,b),\ \forall (a,b) \in C,$ where $(H, A\times B) = (F,A)\times(G,B)$ as defined in Definition 2.7. Any soft subset of $(F,A)\times(F,A)$ is called a soft set relation on (F,A). In an equivalent way, the soft set relation R on the soft set (F,A) in the parameterized form are as follows:

If $(F,A) = \{F(a),\ F(b), ...\}$, then $F(a)RF(b)$ iff $F(a)\times F(b) \in R$.

Definition 2.9. [11] Let $A \subseteq E$. A pair $(\mathcal{F}, A)$ is called a fuzzy soft set over U, where $\mathcal{F}$ is a mapping given by $\mathcal{F} : A \to I^U$.

Definition 2.10. [11] Let $(\mathcal{F},A)$, $(\mathcal{G},B)$ be two fuzzy soft set over U. Then we say that $(\mathcal{F},A)$ is a fuzzy soft subset of $(\mathcal{G},B)$ if

(i) $A \subseteq B$,
(ii) $\forall\, a \in A,\ \mathcal{F}(a) \subseteq \mathcal{G}(a)$.

We write $(\mathcal{F},A)\,\widetilde{\subseteq}\,(\mathcal{G},B)$, if $(\mathcal{F},A)$ is fuzzy soft subset of $(\mathcal{G},B)$.

Definition 2.11. [11] The intersection of two fuzzy soft set $(\mathcal{F},A)$ and $(\mathcal{G},B)$ over common universe U, denoted by $(\mathcal{F},A)\,\widetilde{\cap}\,(\mathcal{G},B)$, is defined as the fuzzy soft set $(\mathcal{H},C)$, where $C = A\cap B$ and for all $e \in C$, $\mathcal{H}(e) = \mathcal{F}(e)\cap\mathcal{G}(e)$.

Definition 2.12. [11] The union of two fuzzy soft set $(\mathcal{F},A)$ and $(\mathcal{G},B)$ over common universe U, denoted by $(\mathcal{F},A)\,\widetilde{\cup}\,(\mathcal{G},B)$, is defined as the fuzzy soft set $(\mathcal{H},C)$, where $C = A\cap B$ and for all $e \in C$, $\mathcal{H}(e) = \mathcal{F}(e)\cup\mathcal{G}(e)$.

Definition 2.13. The complement of a fuzzy soft set $(\mathcal{F},A)$ over U is denoted by $(\mathcal{F},A)^c$ and defined by $(\mathcal{F},A)^c = (\mathcal{F}^c, A)$, where $\mathcal{F}^c : A \to I^U$ is given by $\mathcal{F}^c(e) = [\mathcal{F}(e)]^c$ for all $e \in A$.

Definition 2.14. The cartesian product of two fuzzy soft set $(\mathcal{F},A)$, $(\mathcal{G},B)$ over U is defined as $(\mathcal{F},A)\times(\mathcal{G},B) = (\mathcal{H}, A\times B)$, where $\mathcal{H} : A\times B \to I^{U\times U}$ and $\mathcal{H}(a,b) = \mathcal{F}(a)\times\mathcal{G}(b),\ \forall\,(a,b) \in A\times B$.

Definition 2.15. Let $(\mathcal{F},A)$, $(\mathcal{G},B)$ be two fuzzy soft set over U. Then a fuzzy soft set relation from $(\mathcal{F},A)$ to $(\mathcal{G},B)$ is a fuzzy soft subset of $(\mathcal{F},A)\times(\mathcal{G},B)$, i.e., a fuzzy soft set relation from $(\mathcal{F},A)$ to $(\mathcal{G},B)$ is of the form $(\mathcal{R},C)$, where $C \subseteq A\times B$ and $\mathcal{R}(a,b) \subseteq \mathcal{H}(a,b)$, $\forall\,(a,b) \in C$, where $(\mathcal{H}, A\times B) = (\mathcal{F},A)\times(\mathcal{G},B)$ as defined in Definition 2.14.

If $(\mathcal{R},C)$ is a fuzzy soft subset of $(\mathcal{F},A)\times(\mathcal{F},A)$, then $(\mathcal{R},C)$ is called a fuzzy soft set relation on $(\mathcal{F},A)$. Fuzzy soft set relation $(\mathcal{R},C)$ may be denoted simply by $\mathcal{R}$.

Definition 2.16. Let $(\mathcal{F}, A)$ be a fuzzy soft set over U and $\mathcal{R}$ be a fuzzy soft set relation on $(\mathcal{F}, A)$. Then for all $a, b \in A$,

(i) $\mathcal{R}$ is called reflexive if $\mathcal{R}(a,a) = \widetilde{1}_{U\times U}$;
(ii) $\mathcal{R}$ is called irreflexive if $\mathcal{R}(a,a) = \widetilde{0}_{U\times U}$;
(iii) $\mathcal{R}$ is called symmetric if $\mathcal{R}(a,b) = \mathcal{R}(b,a)$;
(iv) $\mathcal{R}$ is called asymmetric if $\mathcal{R}(a,b) \supset \widetilde{0}_{U\times U} \Rightarrow \mathcal{R}(b,a) = \widetilde{0}_{U\times U}$.

3 Fuzzy Strict Preference Relation

Here we give the definition of fuzzy strict preference relation and then define fuzzy weak preference relation and fuzzy indifference relation with the help of fuzzy strict preference relation, motivated from the notion of strict preference relation in crisp method[14].

Definition 3.1. [14] A binary relation P on U, i.e. $P \subseteq U \times U$ is said to be strict preference relation if (i) P is irreflexive i.e. $(x,x) \notin P, \forall\, x \in U$,

(ii) P is asymmetric i.e. $(x,y) \in P \Rightarrow (y,x) \notin P$, where $x, y \in U$.

Given a strict preference relation P on U, two new relations on U, called indifference relation (denoted by I) and weak preference relation (denoted by W) are as follows: For all $x, y \in U$,

(i) $(x,y) \in I \Leftrightarrow (x,y) \notin P$ and $(y,x) \notin P$,
(ii) $(x,y) \in W \Leftrightarrow$ either $(x,y) \in P$ or, $(x,y) \in I$.

Definition 3.2. A fuzzy binary relation μ on U, i.e. $\mu : U \times U \to [0,1]$ is said to be fuzzy strict preference relation if

(i) μ is fuzzy irreflexive i.e. $\mu(x,x) = 0, \ \forall\, x \in U$,
(ii) μ is fuzzy asymmetric i.e. $\mu(x,y) > 0 \Rightarrow \mu(y,x) = 0, \ \forall\, x, y \in U$.

Given a fuzzy strict preference relation μ on U, we can define two new fuzzy relations called fuzzy indifference relation (denoted by μ_I) and fuzzy weak preference relation (denoted by μ_W) as follows:

(i) $\mu_I(x,y) > 0 \Leftrightarrow \mu(x,y) = 0$ and $\mu(y,x) = 0$,
(ii) $\mu_W(x,y) > 0 \Leftrightarrow$ either $\mu(x,y) > 0$ or, $\mu_I(x,y) > 0, \ \forall\, x, y \in U$.

Note 3.3. Let μ, ν be two fuzzy strict preference relation on U. Then $\mu \cup \nu$ may or may not be fuzzy strict preference relation on U.

Example 3.4. Let $U = \{1, 2\}$. Then $U \times U = \{(1, 1), (1, 2), (2, 1), (2, 2)\}$. Also let $\mu : U \times U \to [0, 1]$ is defined by $\mu(1, 1) = 0$, $\mu(1, 2) = 0.1$, $\mu(2, 1) = 0$, $\mu(2, 2) = 0$ and $\nu : U \times U \to [0, 1]$ is defined by $\nu(1, 1) = 0$, $\nu(1, 2) = 0$, $\nu(2, 1) = 0.2$, $\nu(2, 2) = 0$. Then by Definition 3.2, μ, ν are obviously fuzzy strict preference relation on U. By definition 2.5, $(\mu \cup \nu)(1, 1) = 0$, $(\mu \cup \nu)(1, 2) = 0.1$, $(\mu \cup \nu)(2, 1) = 0.2$, $(\mu \cup \nu)(2, 2) = 0$. Obviously, $(\mu \cup \nu)$ is fuzzy irreflexive. Now $(\mu \cup \nu)(1, 2) = 0.1 > 0$ but $(\mu \cup \nu)(2, 1) = 0.2 \neq 0$. Hence, $(\mu \cup \nu)$ is not fuzzy asymmetric. Therefore $(\mu \cup \nu)$ is not fuzzy strict preference relation on U.

Example 3.5. Let $U = \{1, 2\}$. Define $\mu : U \times U \to [0, 1]$ by $\mu(1, 1) = 0$, $\mu(1, 2) = 0.1$, $\mu(2, 1) = 0$, $\mu(2, 2) = 0$ and define $\nu : U \times U \to [0, 1]$ by $\nu(1, 1,) = 0$, $\nu(1, 2) = 0.2$, $\nu(2, 1) = 0$, $\nu(2, 2) = 0$. Then by definition 3.2, $\mu, \nu, \mu \cup \nu$ are fuzzy strict preference relation on U.

Theorem 3.6. Let μ, ν be two fuzzy strict preference relation on U. Then $\mu \cap \nu$ is also a fuzzy strict preference relation on U.

Proof. Let μ, ν be two fuzzy strict preference relation on U. Then

$$\forall\, (x, x) \in U \times U,\ \mu(x, x) = 0 = \nu(x, x)$$

and $\mu(x, y) > 0 \Rightarrow \mu(y, x) = 0$, $\nu(u, v) > 0 \Rightarrow \nu(v, u) = 0$ for all (x, y), $(u, v) \in U \times U$.

Then by Definition 2.5, $(\mu \cap \nu)(x, x) = \mu(x, x) \wedge \nu(x, x) = 0$.

Now let, for some $(x, y) \in U \times U$, $(\mu \cap \nu)(x, y) > 0$. This implies $\mu(x, y) \wedge \nu(x, y) > 0$.

(i) If $\mu(x, y) > 0$ then $\mu(y, x) = 0$.
(ii) If $\nu(x, y) > 0$ then $\nu(y, x) = 0$.

In both cases $(\mu \cap \nu)(y, x) = \mu(y, x) \wedge \nu(y, x) = 0$. So, $\mu \cap \nu$ is a fuzzy strict preference on U. □

Note 3.7. If μ is a fuzzy strict preference relation on U then μ is fuzzy irreflexive i.e. $\mu(x, x) = 0$, $\forall\, (x, x) \in U \times U$. Hence $\mu^c(x, x) = 1 - \mu(x, x) = 1$. So, μ^c is a fuzzy reflexive relation. Therefore μ^c is not a fuzzy strict preference relation on U.

Theorem 3.8. If μ, ν are fuzzy strict preference relation on U then $\mu.\nu$ is also a fuzzy strict preference relation on U.

Proof. Proof is straightforward. □

Note 3.9. If μ, ν are fuzzy strict preference relation on U then $\mu \oplus \nu$ may or may not be fuzzy strict preference relation on U.

Example 3.10. Let $U = \{1, 2\}$ and define two fuzzy strict preference relation μ, ν as in Example 3.4. Then by Definition 2.5, we have $(\mu \oplus \nu)(1, 1) = 0 = (\mu \oplus \nu)(2, 2)$, $(\mu \oplus \nu)(1, 2) = 0.1$, $(\mu \oplus \nu)(2, 1) = 0.2$.

This implies that $\mu \oplus \nu$ is fuzzy irreflexive relation but it is not fuzzy asymmetric. Hence $\mu \oplus \nu$ is not fuzzy strict preference relation on U.

Example 3.11. Let $U = \{1, 2\}$ and define two fuzzy strict preference relation μ, ν as in Example 3.5. Then by Definition 2.5, we have $(\mu \oplus \nu)(1, 1) = 0 = (\mu \oplus \nu)(2, 2)$, $(\mu \oplus \nu)(1, 2) = 0.28$, $(\mu \oplus \nu)(2, 1) = 0$. This implies that $\mu \oplus \nu$ is fuzzy strict preference relation on U.

Theorem 3.12. Let μ be a fuzzy strict preference relation on U. Then $\mu^{-1}(r) = \{(x, y) \in U \times U\ : \mu(x, y) = r\}$, where $r \in (0, 1]$, is a strict preference relation on U.

Proof. Take an element $r \in (0, 1]$ and fixed. As $\mu^{-1}(r) \subseteq U \times U$, $\mu^{-1}(r)$ is a binary relation on U. Since μ is a fuzzy strict preference relation on U, then $\mu(x, x) = 0$, $\forall\, x \in U$. Therefore $(x, x) \notin \mu^{-1}(r)$, $\forall\, x \in U$. So, $\mu^{-1}(r)$ is irreflexive.

Again, let $(x, y) \in \mu^{-1}(r)$, where $x, y \in U$. Then $\mu(x, y) = r > 0$. This implies $\mu(y, x) = 0$. Therefore $(y, x) \notin \mu^{-1}(r)$. Hence, $\mu^{-1}(r)$ is asymmetric. So, $\mu^{-1}(r)$ is strict preference relation on U for each $r \in (0, 1]$. □

Now we define fuzzy semi-reflexive relation, fuzzy semi-symmetric relation, fuzzy connected, negatively fuzzy transitive and fuzzy transitive relation as follows:

Definition 3.13. Let μ be a fuzzy binary relation on U. Then for all $x, y, z \in U$,

(i) μ is called fuzzy semi-reflexive if $\mu(x, x) > 0$;
(ii) μ is called fuzzy semi-symmetric if $\mu(x, y) > 0 \Rightarrow \mu(y, x) > 0$;
(iii) μ is called fuzzy connected if either $\mu(x, y) > 0$ or $\mu(y, x) > 0$;
(iv) μ is called negatively fuzzy transitive if
$\mu(x, y) = 0 = \mu(y, z) \Rightarrow \mu(x, z) = 0$;
(v) μ is called fuzzy transitive if
$\mu(x, y) > 0$ and $\mu(y, z) > 0 \Rightarrow \mu(x, z) > 0$.

Theorem 3.14. If μ is a fuzzy strict preference relation on U then fuzzy indifference relation μ_I on U is a fuzzy semi-reflexive and fuzzy semi-symmetric on U.

Proof. Since μ is a fuzzy strict preference relation then $\mu(x,x) = 0 \ \ \forall\, x \in U$. Hence by definition of μ_I we have $\mu_I(x,x) > 0$. This implies μ_I is fuzzy semi-reflexive.
Again $\mu_I(x,y) > 0 \Leftrightarrow \mu(x,y) = 0$ and $\mu(y,x) = 0 \Leftrightarrow \mu_I(y,x) > 0$. This implies μ_I is fuzzy semi-symmetric on U. □

Theorem 3.15. If μ is a fuzzy strict preference relation on U, then fuzzy weak preference relation μ_W on U is a fuzzy semi-reflexive and fuzzy connected on U.

Proof. $\mu_W(x,y) > 0 \Leftrightarrow \mu(x,y) > 0$ or $\mu_I(x,y) > 0 \ \ \forall\, x,\, y \in U$. Since $\mu_I(x,x) > 0 \ \ \forall\, x \in U$, as μ_I is fuzzy semi-reflexive.

Hence $\mu_W(x,x) > 0 \ \ \forall\, x \in U$. This implies μ_W is fuzzy semi-reflexive.
Now we are going to prove μ_W is fuzzy connected.

Since, μ is fuzzy strict preference relation, then for all $x, y \in U$, either $\mu(x,y) > 0$ or $\mu(y,x) > 0$ or $\mu(x,y) = \mu(y,x) = 0$.

Case 1: Let $\mu(x,y) > 0$. Then $\mu(y,x) = 0$, since μ is fuzzy asymmetric relation on X. Hence $\mu_I(y,x) \ngtr 0$. This implies $\mu_W(x,y) > 0$ but $\mu_W(y,x) \ngtr 0$.

Case 2: Let $\mu(y,x) > 0$. Then similarly as in case 1, we can prove that $\mu_W(y,x) > 0$ but $\mu_W(x,y) \ngtr 0$.

Case 3: Let $\mu(x,y) = \mu(y,x) = 0$. Then by Definition 3.2, we have $\mu_I(x,y) > 0$. This implies $\mu_W(x,y) > 0$.
Hence for all $x, y \in U$, either $\mu_W(x,y) > 0$ or $\mu_W(y,x) > 0$. So, by Definition 3.13, μ_W is fuzzy connected on U. □

Theorem 3.16. Let μ is a fuzzy strict preference relation on U. Then for $x, y \in U$, $\mu_W(x,y) > 0$ and $\mu_W(y,x) > 0 \ \Leftrightarrow \mu_I(x,y) > 0$.

Proof. Suppose for $x, y \in U$, $\mu_W(x,y) > 0$ and $\mu_W(y,x) > 0$. Then $\mu_W(x,y) > 0 \Rightarrow \mu(x,y) > 0$ or $\mu_I(x,y) > 0$. Let $\mu(x,y) > 0$. Then $\mu(y,x) = 0$, since μ is fuzzy asymmetric relation.

Again $\mu_W(y,x) > 0 \ \Rightarrow \ \mu(y,x) > 0$ or $\mu_I(y,x) > 0$. But $\mu(y,x) = 0$, as proved earlier. Hence $\mu_I(y,x) > 0$. This implies $\mu_I(x,y) > 0$, since μ_I is fuzzy semi-symmetric relation, by Theorem 3.14.

Conversely, let $\mu_I(x,y) > 0$. Then by Definition 3.2, $\mu_W(x,y) > 0$. Since μ_I is fuzzy semi-symmetric relation, then $\mu_I(x,y) > 0 \Rightarrow \mu_I(y,x) > 0 \Rightarrow \mu_W(y,x) > 0$. □

Note 3.17. Fuzzy weak preference relation μ_W on U is fuzzy semi-symmetric if and only if $\mu_I(x,y) > 0$ for all $x,y \in U$.

Theorem 3.18. If fuzzy strict preference relation μ on U is a negatively fuzzy transitive then μ, μ_W, μ_I are all fuzzy transitive relation on U.

Proof. Let μ be a negatively fuzzy transitive relation on U.

1. To prove μ_W is fuzzy transitive relation on U, suppose that there exist $x,y,z \in U$ such that $\mu_W(x,y) > 0$ and $\mu_W(y,z) > 0$. By Definition 3.2,

$\mu_W(x,y) > 0 \Rightarrow \mu(x,y) > 0$ or $\mu_I(x,y) > 0$;

$\mu_W(y,z) > 0 \Rightarrow \mu(y,z) > 0$ or $\mu_I(y,z) > 0$.

Case 1: Let $\mu(x,y) > 0$ and $\mu(y,z) > 0$. This implies $\mu(y,x) = 0$ and $\mu(z,y) = 0$, by Definition 3.2; $\Rightarrow \mu(z,x) = 0$, since μ is negatively fuzzy transitive.

Now if $\mu(x,z) > 0$ then $\mu_W(x,z) > 0$. Hence μ_W is fuzzy transitive.

If $\mu(x,z) = 0$ then $\mu(x,z) = 0 = \mu(z,x) \Rightarrow \mu_I(x,z) > 0$. This implies $\mu_W(x,z) > 0$. Hence μ_W is fuzzy transitive on U.

Case 2: Let $\mu(x,y) > 0$ and $\mu_I(y,z) > 0$. This implies $\mu(y,x) = 0$ and $\mu(z,y) = 0$, by Definition of $\mu, \mu_I \Rightarrow \mu(z,x) = 0$, since μ is negatively fuzzy transitive.
Hence, as in case 1, μ_W is fuzzy transitive.

Case 3: Let $\mu_I(x,y) > 0$ and $\mu(y,z) > 0$. This implies $\mu(y,x) = 0$ and $\mu(z,y) = 0$, by Definition of $\mu, \mu_I \Rightarrow \mu(z,x) = 0$, since μ is negatively fuzzy transitive.
Then we can prove similarly as in Case 1 that μ_W is fuzzy transitive.

Case 4: Let $\mu_I(x,y) > 0$ and $\mu_I(y,z) > 0$. This implies $\mu(y,x) = 0$ and $\mu(z,y) = 0$, by Definition of $\mu_I \Rightarrow \mu(z,x) = 0$, since μ is negatively fuzzy transitive, which implies that μ_W is fuzzy transitive.

2. To prove μ is fuzzy transitive on U, suppose that there exist $x, y, z \in U$ such that $\mu(x, y) > 0$ and $\mu(y, z) > 0$ but $\mu(x, z) = 0$.

Now $\mu(x, y) > 0$ and $\mu(y, z) > 0$

$\Rightarrow \mu(y, x) = 0$ and $\mu(z, y) = 0$, by Definition of 3.2

$\Rightarrow \mu(z, x) = 0$, since μ is negatively fuzzy transitive.

Now $\mu(z, x) = 0$ and $\mu(x, z) = 0$

$\Rightarrow \mu_I(z, x) > 0 \Rightarrow \mu_W(z, x) > 0$.

Again given that $\mu(y, z) > 0$. This implies $\mu_W(y, z) > 0$.

Hence $\mu_W(y, z) > 0$ and $\mu_W(z, x) > 0$

$\Rightarrow \mu_W(y, x) > 0$, since μ_W is fuzzy transitive

$\Rightarrow \mu(y, x) > 0$ or $\mu_I(y, x) > 0$.

If $\mu(y, x) > 0$ then it implies $\mu(x, y) = 0$, which contradicts our assumption.

If $\mu_I(y, x) > 0$ then it implies $\mu(x, y) = 0$ and $\mu(y, x) = 0$, which also contradicts our assumption. So, $\mu(x, z) > 0$. Hence μ is fuzzy transitive on U.

3. To prove μ_I is fuzzy transitive on U, suppose that there exist $x, y, z \in U$ such that $\mu_I(x, y) > 0$ and $\mu_I(y, z) > 0$.

Now $\mu_I(x, y) > 0$ and $\mu_I(y, z) > 0$

$\Rightarrow \mu(x, y) = 0, \mu(y, x) = 0$ and $\mu(y, z) = 0, \mu(z, y) = 0$

$\Rightarrow \mu(x, y) = 0, \mu(y, z) = 0$ and $\mu(y, x) = 0, \mu(z, y) = 0$

$\Rightarrow \mu(x, z) = 0$ and $\mu(z, x) = 0$, by transitivity of μ

$\Rightarrow \mu_I(x, z) > 0$, by Definition of μ_I.

So, μ_I is a fuzzy transitive relation on U. □

Theorem 3.19. If for $x, y, z \in U$, $\mu_W(x, y) > 0$ and $\mu(y, z) > 0$, then $\mu(x, z) > 0$.

Proof. Suppose there exist $x, y, z \in U$ such that $\mu_W(x, y) > 0$ and $\mu(y, z) > 0$ but $\mu(x, z) = 0$.
Now $\mu_W(x, y) > 0 \Rightarrow \mu(x, y) > 0$ or $\mu_I(x, y) > 0$.

(i) If $\mu(x, y) > 0$ then by Definition 3.2, we have $\mu(y, x) = 0$.
Since μ is negatively fuzzy transitive then $\mu(y, x) = 0$ together with $\mu(x, z) = 0$ implies $\mu(y, z) = 0$, which contradicts our assumption $\mu(y, z) > 0$.

(ii) If $\mu_I(x, y) > 0$ then it implies $\mu(x, y) = 0$ and $\mu(y, x) = 0$.
Now $\mu(y, x) = 0$ together with $\mu(x, z) = 0$ implies that $\mu(y, z) = 0$, which contradicts our assumption $\mu(y, z) > 0$.

Hence our assumption $\mu(x, z) = 0$ is wrong. Therefore $\mu(x, z) > 0$. □

4 Fuzzy Soft Set Strict Preference Relation

In this section, at first we define fuzzy soft set strict preference relation, then we define fuzzy soft set weak preference relation and fuzzy soft set indifference relation with the help of fuzzy soft set strict preference relation.

Definition 4.1. Let $(\mathcal{F}, A)$ be a fuzzy soft set over U and $\mathcal{R}$ be a fuzzy soft set relation on $(\mathcal{F}, A)$. Then $\mathcal{R}$ is said to be a fuzzy soft set strict preference relation on $(\mathcal{F}, A)$ if

(i) $\mathcal{R}$ is irreflexive, i.e. $\mathcal{R}(a, a) = \widetilde{0}_{U\times U}$, $\forall\, a \in A$.
(ii) $\mathcal{R}$ is asymmetric, i.e. $\mathcal{R}(a, b) \supset \widetilde{0}_{U\times U} \Rightarrow \mathcal{R}(b, a) = \widetilde{0}_{U\times U}$, $\forall\, a, b \in A$.

Given a fuzzy soft set strict preference relation $\mathcal{R}$ on $(\mathcal{F}, A)$, we can define two fuzzy soft set relation on $(\mathcal{F}, A)$ called fuzzy soft set indifference relation (denoted by $\mathcal{R}_I$) and fuzzy soft set weak preference relation (denoted by $\mathcal{R}_W$) as follows:
For all $a, b \in A$,

(i) $\mathcal{R}_I(a, b) \supset \widetilde{0}_{U\times U} \Leftrightarrow \mathcal{R}(a, b) = \widetilde{0}_{U\times U}$ and $\mathcal{R}(b, a) = \widetilde{0}_{U\times U}$,
(ii) $\mathcal{R}_W(a, b) \supset \widetilde{0}_{U\times U} \Leftrightarrow \mathcal{R}(a, b) \supset \widetilde{0}_{U\times U}$ or $\mathcal{R}_I(a, b) \supset \widetilde{0}_{U\times U}$.

Theorem 4.2. Let $(\mathcal{F}, A)$ be a fuzzy soft set over U and $\mathcal{R}$, $\mathcal{S}$ be two fuzzy soft set strict preference relation on $(\mathcal{F}, A)$. Then $\mathcal{R} \widetilde{\cap} \mathcal{S}$ is a fuzzy soft set strict preference relation on $(\mathcal{F}, A)$.

Proof. Since $\mathcal{R}$, $\mathcal{S}$ are fuzzy soft set strict preference relation, then $\mathcal{R}(a,a) = \widetilde{0}_{U\times U}$, $\mathcal{S}(a,a) = \widetilde{0}_{U\times U}$, $\forall\,(a,a) \in A \times A$. Therefore $(\mathcal{R}\widetilde{\cap}\mathcal{S})(a,a) = \mathcal{R}(a,a) \cap \mathcal{S}(a,a) = \widetilde{0}_{U\times U}$, $\forall\,(a,a) \in A \times A$. So, $\mathcal{R}\widetilde{\cap}\mathcal{S}$ is irreflexive.

Now let, for some $(a,b) \in A \times A$, $(\mathcal{R}\widetilde{\cap}\mathcal{S})(a,b) \supset \widetilde{0}_{U\times U}$

$\Rightarrow \mathcal{R}(a,b) \cap \mathcal{S}(a,b) \supset \widetilde{0}_{U\times U}$

$\Rightarrow \mathcal{R}(a,b) \supset \widetilde{0}_{U\times U}$, $\mathcal{S}(a,b) \supset \widetilde{0}_{U\times U}$

$\Rightarrow \mathcal{R}(b,a) = \widetilde{0}_{U\times U}$, $\mathcal{S}(b,a) = \widetilde{0}_{U\times U}$, since $\mathcal{R}$, $\mathcal{S}$ is asymmetric

$\Rightarrow (\mathcal{R}\widetilde{\cap}\mathcal{S})(b,a) = \widetilde{0}_{U\times U}$.

Since $(a,b) \in A \times A$ is arbitrary, then $\mathcal{R}\widetilde{\cap}\mathcal{S}$ is asymmetric.

Hence $\mathcal{R}\widetilde{\cap}\mathcal{S}$ is a fuzzy soft set strict preference relation on $(\mathcal{F}, A)$. □

Note 4.3. Given two fuzzy soft set strict preference relation $\mathcal{R}$, $\mathcal{S}$ on $(\mathcal{F}, A)$. Then $\mathcal{R}\widetilde{\cup}\mathcal{S}$ may or may not be fuzzy soft set strict preference relation on $(\mathcal{F}, A)$.

Example 4.4. Let U denotes the set of selected students in a school,

i.e. $U = \{s_1,\ s_2,\ s_3\}$.

Let A denotes different subjects.

Take $A = \{bengali,\ english,\ mathematics\}$,

i.e. $A = \{b, e, m\}$.

If a student get 95 marks out of 100 in a particular subject, then take the score of this student in that particular subject is 0.95.

Let a fuzzy soft set $(\mathcal{F}, A)$ over U describe students having different scores in different subjects in a particular examination and is given by

$F(b) = \{(s_1, 0.6),\ (s_2, 0.7),\ (s_3, 0.65)\}$;

$F(e) = \{(s_1, 0.59), (s_2, 0.75), (s_3, 0.6)\};$

$F(m) = \{(s_1, 0.8), (s_2, 0.82), (s_3, 0.9)\};$

Then

$A \times A = \{(b, b), (b, e), (b, m), (e, b), (e, e), (e, m), (m, b), (m, e), (m, m)\}.$

Then by Definition 2.14, the elements of the cartesian product

$(\mathcal{H}, A \times A) = (\mathcal{F}, A) \times (\mathcal{F}, A)$ are as follows:

$$\mathcal{H}(b,b) = \left\{ \frac{(s_1,s_1)}{0.6}, \frac{(s_1,s_2)}{0.6}, \frac{(s_1,s_3)}{0.6}, \frac{(s_2,s_1)}{0.6}, \frac{(s_2,s_2)}{0.7}, \frac{(s_2,s_3)}{0.65}, \frac{(s_3,s_1)}{0.6}, \frac{(s_3,s_2)}{0.65}, \frac{(s_3,s_3)}{0.65} \right\};$$

$$\mathcal{H}(b,e) = \left\{ \frac{(s_1,s_1)}{0.59}, \frac{(s_1,s_2)}{0.6}, \frac{(s_1,s_3)}{0.6}, \frac{(s_2,s_1)}{0.59}, \frac{(s_2,s_2)}{0.7}, \frac{(s_2,s_3)}{0.6}, \frac{(s_3,s_1)}{0.59}, \frac{(s_3,s_2)}{0.65}, \frac{(s_3,s_3)}{0.6} \right\};$$

$$\mathcal{H}(b,m) = \left\{ \frac{(s_1,s_1)}{0.6}, \frac{(s_1,s_2)}{0.6}, \frac{(s_1,s_3)}{0.6}, \frac{(s_2,s_1)}{0.7}, \frac{(s_2,s_2)}{0.7}, \frac{(s_2,s_3)}{0.7}, \frac{(s_3,s_1)}{0.65}, \frac{(s_3,s_2)}{0.65}, \frac{(s_3,s_3)}{0.65} \right\};$$

$$\mathcal{H}(e,b) = \left\{ \frac{(s_1,s_1)}{0.59}, \frac{(s_1,s_2)}{0.59}, \frac{(s_1,s_3)}{0.59}, \frac{(s_2,s_1)}{0.6}, \frac{(s_2,s_2)}{0.7}, \frac{(s_2,s_3)}{0.65}, \frac{(s_3,s_1)}{0.6}, \frac{(s_3,s_2)}{0.6}, \frac{(s_3,s_3)}{0.6} \right\};$$

$$\mathcal{H}(e,e) = \left\{ \frac{(s_1,s_1)}{0.59}, \frac{(s_1,s_2)}{0.59}, \frac{(s_1,s_3)}{0.59}, \frac{(s_2,s_1)}{0.59}, \frac{(s_2,s_2)}{0.75}, \frac{(s_2,s_3)}{0.6}, \frac{(s_3,s_1)}{0.59}, \frac{(s_3,s_2)}{0.6}, \frac{(s_3,s_3)}{0.6} \right\};$$

$$\mathcal{H}(e,m) = \left\{ \frac{(s_1,s_1)}{0.59}, \frac{(s_1,s_2)}{0.59}, \frac{(s_1,s_3)}{0.59}, \frac{(s_2,s_1)}{0.75}, \frac{(s_2,s_2)}{0.75}, \frac{(s_2,s_3)}{0.75}, \frac{(s_3,s_1)}{0.6}, \frac{(s_3,s_2)}{0.6}, \frac{(s_3,s_3)}{0.6} \right\};$$

$$\mathcal{H}(m,b) = \left\{ \frac{(s_1,s_1)}{0.6}, \frac{(s_1,s_2)}{0.7}, \frac{(s_1,s_3)}{0.65}, \frac{(s_2,s_1)}{0.6}, \frac{(s_2,s_2)}{0.7}, \frac{(s_2,s_3)}{0.65}, \frac{(s_3,s_1)}{0.6}, \frac{(s_3,s_2)}{0.7}, \frac{(s_3,s_3)}{0.65} \right\};$$

$$\mathcal{H}(m,e) = \left\{ \frac{(s_1,s_1)}{0.59}, \frac{(s_1,s_2)}{0.75}, \frac{(s_1,s_3)}{0.6}, \frac{(s_2,s_1)}{0.59}, \frac{(s_2,s_2)}{0.75}, \frac{(s_2,s_3)}{0.6}, \frac{(s_3,s_1)}{0.59}, \frac{(s_3,s_2)}{0.75}, \frac{(s_3,s_3)}{0.6} \right\};$$

$$\mathcal{H}(m,m) = \left\{ \frac{(s_1,s_1)}{0.8}, \frac{(s_1,s_2)}{0.8}, \frac{(s_1,s_3)}{0.8}, \frac{(s_2,s_1)}{0.8}, \frac{(s_2,s_2)}{0.82}, \frac{(s_2,s_3)}{0.82}, \frac{(s_3,s_1)}{0.8}, \frac{(s_3,s_2)}{0.82}, \frac{(s_3,s_3)}{0.9} \right\}.$$

Define a fuzzy soft set relation $(\mathcal{R}, C)$ on $(\mathcal{F}, A)$ as follows:

Let $(x, y) \in C \subseteq A \times A$ if and only if either both x, y are art subjects or both are science subjects.

i.e. $C = \{(b, b), (b, e), (e, b), (e, e), (m, m)\}$, and take

$\mathcal{R}(b, b) = \mathcal{R}(e, e) = \mathcal{R}(m, m) = \mathcal{R}(e, b) = \widetilde{0}_{U\times U}$;

$$\mathcal{R}(b, e) = \left\{\frac{(s_1, s_1)}{0.59}, \frac{(s_1, s_2)}{0.6}, \frac{(s_1, s_3)}{0.6}, \frac{(s_2, s_1)}{0.59}, \frac{(s_2, s_2)}{0.7}, \frac{(s_2, s_3)}{0.6}, \frac{(s_3, s_1)}{0.59}, \frac{(s_3, s_2)}{0.65}, \frac{(s_3, s_3)}{0.6}\right\}.$$

Define another fuzzy soft set relation $(\mathcal{S}, C)$ on $(\mathcal{F}, A)$ as follows:

$\mathcal{S}(b, b) = \mathcal{S}(e, e) = \mathcal{S}(m, m) = \mathcal{S}(b, e) = \widetilde{0}_{U\times U}$;

$$\mathcal{S}(e, b) = \left\{\frac{(s_1, s_1)}{0.59}, \frac{(s_1, s_2)}{0.59}, \frac{(s_1, s_3)}{0.59}, \frac{(s_2, s_1)}{0.6}, \frac{(s_2, s_2)}{0.7}, \frac{(s_2, s_3)}{0.65}, \frac{(s_3, s_1)}{0.6}, \frac{(s_3, s_2)}{0.6}, \frac{(s_3, s_3)}{0.6}\right\}.$$

Obviously, $\mathcal{R}$, $\mathcal{S}$ are fuzzy soft set strict preference relations on $(\mathcal{F}, A)$. Then $(\mathcal{R}\widetilde{\cap}\mathcal{S}, C)$, $(\mathcal{R}\widetilde{\cup}\mathcal{S}, C)$ are as follows:

$(\mathcal{R}\widetilde{\cap}\mathcal{S})(b, b) = (\mathcal{R}\widetilde{\cap}\mathcal{S})(e, e) = (\mathcal{R}\widetilde{\cap}\mathcal{S})(m, m) = \widetilde{0}_{U\times U}$;

$(\mathcal{R}\widetilde{\cap}\mathcal{S})(b, e) = \widetilde{0}_{U\times U} = (\mathcal{R}\widetilde{\cap}\mathcal{S})(e, b)$ and

$(\mathcal{R}\widetilde{\cup}\mathcal{S})(b, b) = (\mathcal{R}\widetilde{\cup}\mathcal{S})(e, e) = (\mathcal{R}\widetilde{\cup}\mathcal{S})(m, m) = \widetilde{0}_{U\times U}$;

$$(\mathcal{R}\widetilde{\cup}\mathcal{S})(b, e) = \left\{\tfrac{(s_1,s_1)}{0.59}, \tfrac{(s_1,s_2)}{0.6}, \tfrac{(s_1,s_3)}{0.6}, \tfrac{(s_2,s_1)}{0.59}, \tfrac{(s_2,s_2)}{0.7}, \tfrac{(s_2,s_3)}{0.6}, \tfrac{(s_3,s_1)}{0.59}, \tfrac{(s_3,s_2)}{0.65}, \tfrac{(s_3,s_3)}{0.6}\right\};$$

$$(\mathcal{R}\widetilde{\cup}\mathcal{S})(e, b) = \left\{\tfrac{(s_1,s_1)}{0.59}, \tfrac{(s_1,s_2)}{0.59}, \tfrac{(s_1,s_3)}{0.59}, \tfrac{(s_2,s_1)}{0.6}, \tfrac{(s_2,s_2)}{0.7}, \tfrac{(s_2,s_3)}{0.65}, \tfrac{(s_3,s_1)}{0.6}, \tfrac{(s_3,s_2)}{0.6}, \tfrac{(s_3,s_3)}{0.6}\right\}.$$

This shows that $\mathcal{R}\widetilde{\cap}\mathcal{S}$ is a fuzzy soft set strict preference relation on $(\mathcal{F}, A)$ but $\mathcal{R}\widetilde{\cup}\mathcal{S}$ is not a fuzzy soft set strict preference relation on $(\mathcal{F}, A)$. Because $(\mathcal{R}\widetilde{\cup}\mathcal{S})(b, e) \supset \widetilde{0}_{U\times U}$, $(\mathcal{R}\widetilde{\cup}\mathcal{S})(e, b) \supset \widetilde{0}_{U\times U}$ implies $\mathcal{R}\widetilde{\cup}\mathcal{S}$ is not asymmetric on $(\mathcal{F}, A)$.

Example 4.5. As a continuation of Example 4.4, we define another fuzzy soft set relation $(\mathcal{T}, C)$ on $(\mathcal{F}, A)$ as follows:

$\mathcal{T}(b, b) = \mathcal{T}(e, e) = \mathcal{T}(m, m) = \mathcal{T}(e, b) = \widetilde{0}_{U\times U}$;

$$\mathcal{T}(b, e) = \left\{\frac{(s_1, s_1)}{0.59}, \frac{(s_1, s_2)}{0.59}, \frac{(s_1, s_3)}{0.59}, \frac{(s_2, s_1)}{0.59}, \frac{(s_2, s_2)}{0.7}, \frac{(s_2, s_3)}{0.6}, \frac{(s_3, s_1)}{0.59}, \frac{(s_3, s_2)}{0.6}, \frac{(s_3, s_3)}{0.6}\right\}.$$

Then obviously, $\mathcal{T}$ is a fuzzy soft set strict preference relation on $(\mathcal{F}, A)$.
Now $(\mathcal{R}\,\widetilde{\cap}\,\mathcal{T}, C)$, $(\mathcal{R}\,\widetilde{\cup}\,\mathcal{T}, C)$ are as follows:

$(\mathcal{R}\,\widetilde{\cap}\,\mathcal{T})(b,b) = (\mathcal{R}\,\widetilde{\cap}\,\mathcal{T})(e,e) = (\mathcal{R}\,\widetilde{\cap}\,\mathcal{T})(m,m) = (\mathcal{R}\,\widetilde{\cap}\,\mathcal{T})(e,b) = \widetilde{0}_{U\times U}$;

$(\mathcal{R}\,\widetilde{\cap}\,\mathcal{T})(b,e) = \left\{\frac{(s_1,s_1)}{0.59}, \frac{(s_1,s_2)}{0.59}, \frac{(s_1,s_3)}{0.59}, \frac{(s_2,s_1)}{0.59}, \frac{(s_2,s_2)}{0.7}, \frac{(s_2,s_3)}{0.6}, \frac{(s_3,s_1)}{0.59}, \frac{(s_3,s_2)}{0.6}, \frac{(s_3,s_3)}{0.6}\right\}$;

and

$(\mathcal{R}\,\widetilde{\cup}\,\mathcal{T})(b,b) = (\mathcal{R}\,\widetilde{\cup}\,\mathcal{T})(e,e) = (\mathcal{R}\,\widetilde{\cup}\,\mathcal{T})(m,m) = (\mathcal{R}\,\widetilde{\cup}\,\mathcal{T})(e,b) = \widetilde{0}_{U\times U}$;

$(\mathcal{R}\,\widetilde{\cup}\,\mathcal{T})(b,e) = \left\{\frac{(s_1,s_1)}{0.59}, \frac{(s_1,s_2)}{0.6}, \frac{(s_1,s_3)}{0.6}, \frac{(s_2,s_1)}{0.59}, \frac{(s_2,s_2)}{0.7}, \frac{(s_2,s_3)}{0.6}, \frac{(s_3,s_1)}{0.59}, \frac{(s_3,s_2)}{0.65}, \frac{(s_3,s_3)}{0.6}\right\}$.

This shows that $\mathcal{R}\,\widetilde{\cap}\,\mathcal{T}$ and $\mathcal{R}\,\widetilde{\cup}\,\mathcal{T}$ are both fuzzy soft set strict preference relation on $(\mathcal{F}, A)$.

Note 4.6. Let $(\mathcal{F}, A)$ be a fuzzy soft set over U and $\mathcal{R}$ be a fuzzy soft set strict preference relation on $(\mathcal{F}, A)$. Then $\mathcal{R}(a,a) = \widetilde{0}_{U\times U}$, $\forall\,(a,a) \in A\times A$. Hence $\mathcal{R}^c(a,a) = \widetilde{1}_{U\times U}$, $\forall\,(a,a) \in A\times A$. So, $\mathcal{R}^c$ is fuzzy soft set reflexive relation on $(\mathcal{F}, A)$. Therefore $\mathcal{R}^c$ is not a fuzzy soft set strict preference relation on $(\mathcal{F}, A)$.

Definition 4.7. Let $(\mathcal{F}, A)$ be a fuzzy soft set over U and $\mathcal{R}$, $\mathcal{S}$ be two fuzzy soft set relation on $(\mathcal{F}, A)$. The algebraic product of $\mathcal{R}$, $\mathcal{S}$ is denoted by $\mathcal{R}.\mathcal{S}$ and defined by

$$(\mathcal{R}.\mathcal{S})(a,b) = \mathcal{R}(a,b).\,\mathcal{S}(a,b),\ \forall\,(a,b) \in A\times B.$$

Theorem 4.8. Let $(\mathcal{F}, A)$ be a fuzzy soft set over U and $\mathcal{R}$, $\mathcal{S}$ be two fuzzy soft set strict preference relation on $(\mathcal{F}, A)$. Then $\mathcal{R}.\mathcal{S}$ is a fuzzy soft set strict preference relation on $(\mathcal{F}, A)$.

Proof. This theorem can be easily proved with the help of Definition 4.1 and Definition 4.7. □

Definition 4.9. Let $(\mathcal{F}, A)$ be a fuzzy soft set over U and $\mathcal{R}$ be a fuzzy soft set relation on $(\mathcal{F}, A)$. Then for all $a, b, c \in A$,

(i) $\mathcal{R}$ is called semi-reflexive if $\mathcal{R}(a,a) \supset \widetilde{0}_{U\times U}$;
(ii) $\mathcal{R}$ is called semi-symmetric if $\mathcal{R}(a,b) \supset \widetilde{0}_{U\times U} \Rightarrow \mathcal{R}(b,a) \supset \widetilde{0}_{U\times U}$;
(iii) $\mathcal{R}$ is called connected if either $\mathcal{R}(a,b) \supset \widetilde{0}_{U\times U}$ or $\mathcal{R}(b,a) \supset \widetilde{0}_{U\times U}$;
(iv) $\mathcal{R}$ is called negatively transitive if $\mathcal{R}(a,b) = \widetilde{0}_{U\times U} = \mathcal{R}(b,c) \Rightarrow \mathcal{R}(a,c) = \widetilde{0}_{U\times U}$;
(v) $\mathcal{R}$ is called transitive if $\mathcal{R}(a,b) \supset \widetilde{0}_{U\times U}$ and $\mathcal{R}(b,c) \supset \widetilde{0}_{U\times U} \Rightarrow \mathcal{R}(a,c) \supset \widetilde{0}_{U\times U}$.

Theorem 4.10. Let $(\mathcal{F}, A)$ be a fuzzy soft set over U and $\mathcal{R}$ be a fuzzy soft set strict preference relation on $(\mathcal{F}, A)$. Then fuzzy soft set indifference relation $\mathcal{R}_I$ on $(\mathcal{F}, A)$ is semi-reflexive and semi-symmetric on $(\mathcal{F}, A)$ and fuzzy soft set weak preference relation $\mathcal{R}_W$ is semi-reflexive on $(\mathcal{F}, A)$.

Proof. Since $\mathcal{R}$ is fuzzy soft set strict preference relation on $(\mathcal{F}, A)$, then $\mathcal{R}$ is irreflexive. Hence for all $a \in A$, $\mathcal{R}(a,a) = \widetilde{0}_{U\times U}$

$\Rightarrow \mathcal{R}_I(a,a) \supset \widetilde{0}_{U\times U}$, by Definition 4.1

$\Rightarrow \mathcal{R}_I$ is semi-reflexive, by Definition 4.9.

Now by Definition 4.1, for all $a,\ b \in A$,

$\mathcal{R}_I(a,b) \supset \widetilde{0}_{U\times U} \Rightarrow \mathcal{R}(a,b) = \widetilde{0}_{U\times U}$ and $\mathcal{R}(b,a) = \widetilde{0}_{U\times U} \Rightarrow \mathcal{R}_I(b,a) \supset \widetilde{0}_{U\times U}$

Hence, by Definition 4.9, $\mathcal{R}_I$ is semi-symmetric.

Since $\mathcal{R}_I$ is semi-reflexive, $\mathcal{R}_I(a,a) \supset \widetilde{0}_{U\times U}$, $\forall\, a \in A$.

Therefore by Definition 4.1, $\mathcal{R}_W(a,a) \supset \widetilde{0}_{U\times U}$, $\forall\, a \in A$. Hence, $\mathcal{R}_W$ is semi-reflexive. □

Theorem 4.11. Let $\mathcal{R}$ be a fuzzy soft set strict preference relation on $(\mathcal{F}, A)$. Then for all $a, b \in A$,

$\mathcal{R}_W(a,b) \supset \widetilde{0}_{U\times U}$ and $\mathcal{R}_W(b,a) \supset \widetilde{0}_{U\times U} \Leftrightarrow \mathcal{R}_I(a,b) \supset \widetilde{0}_{U\times U}$.

Proof. At first let, for all $a, b \in A$, $\mathcal{R}_W(a,b) \supset \widetilde{0}_{U\times U}$ and $\mathcal{R}_W(b,a) \supset \widetilde{0}_{U\times U}$. Now $\mathcal{R}_W(a,b) \supset \widetilde{0}_{U\times U} \Rightarrow \mathcal{R}(a,b) \supset \widetilde{0}_{U\times U}$ or $\mathcal{R}_I(a,b) \supset \widetilde{0}_{U\times U}$.

Suppose $\mathcal{R}(a,b) \supset \widetilde{0}_{U\times U}$. This implies $\mathcal{R}(b,a) = \widetilde{0}_{U\times U}$.

Again, $\mathcal{R}_W(b,a) \supset \widetilde{0}_{U\times U} \Rightarrow \mathcal{R}(b,a) \supset \widetilde{0}_{U\times U}$ or $\mathcal{R}_I(b,a) \supset \widetilde{0}_{U\times U}$. But $\mathcal{R}(b,a) = \widetilde{0}_{U\times U}$. So, we must have $\mathcal{R}_I(b,a) \supset \widetilde{0}_{U\times U}$. Since $\mathcal{R}_I$ is semi-symmetric, hence $\mathcal{R}_I(a,b) \supset \widetilde{0}_{U\times U}$.

Conversely, let $\mathcal{R}_I(a,b) \supset \widetilde{0}_{U\times U}$ for all $a,\ b \in A$. Then by Definition 4.1, $\mathcal{R}_W(a,b) \supset \widetilde{0}_{U\times U}$. Since $\mathcal{R}_I$ is semi-symmetric. Hence, $\mathcal{R}_I(b,a) \supset \widetilde{0}_{U\times U}$. This implies $\mathcal{R}_W(b,a) \supset \widetilde{0}_{U\times U}$. □

Note 4.12. A fuzzy soft set weak preference relation $\mathcal{R}_W$ on $(\mathcal{F}, A)$ is semi-symmetric if and only if $\mathcal{R}_I(a,b) \supset \widetilde{0}_{U\times U},\ \forall\, a,b \in A$.

Note 4.13. The fuzzy soft set weak preference relation $\mathcal{R}_W$ on $(\mathcal{F}, A)$ may not be connected on $(\mathcal{F}, A)$, which is reflected in the following example.

Example 4.14. Take the fuzzy soft set $(\mathcal{F}, A)$ on U as in Example 4.4. Define a fuzzy soft set relation $\mathcal{P}$ on $(\mathcal{F}, A)$ as follows:

$$\mathcal{P}(b,b) = \mathcal{P}(e,e) = \mathcal{P}(m,m) = \widetilde{0}_{U\times U};$$

$$\mathcal{P}(b,e) \supset \widetilde{0}_{U\times U}, \mathcal{P}(e,b) = \widetilde{0}_{U\times U};$$

$$\mathcal{P}(e,m) \supset \widetilde{0}_{U\times U}, \mathcal{P}(m,e) = \widetilde{0}_{U\times U};$$

$$\mathcal{P}(b,m) = \left\{\tfrac{(s_1,s_1)}{0.6}, \tfrac{(s_1,s_2)}{0.6}, \tfrac{(s_1,s_3)}{0.6}, \tfrac{(s_2,s_1)}{0.7}, \tfrac{(s_2,s_2)}{0.7}, \tfrac{(s_2,s_3)}{0}, \tfrac{(s_3,s_1)}{0}, \tfrac{(s_3,s_2)}{0}, \tfrac{(s_3,s_3)}{0.65}\right\}$$

$$\mathcal{P}(m,b) = \widetilde{0}_{U\times U}.$$

Then by Definition 4.1, $\mathcal{P}$ is a fuzzy soft set strict preference relation on $(\mathcal{F}, A)$. Now we can define a fuzzy soft set weak preference relation $\mathcal{P}_W$ on $(\mathcal{F}, A)$ with the help of Definition 4.1. Hence we have $\mathcal{P}_W(b,b)$, $\mathcal{P}_W(e,e)$, $\mathcal{P}_W(m,m)$, $\mathcal{P}_W(b,e)$, $\mathcal{P}_W(e,m) \supset \widetilde{0}_{U\times U}$.

Since $\mathcal{P}(b,e) \supset \widetilde{0}_{U\times U}$, $\mathcal{P}(e,b) = \widetilde{0}_{U\times U}$, then $\mathcal{P}_W(e,b) \not\supset \widetilde{0}_{U\times U}$. Similarly $\mathcal{P}_W(m,e) \not\supset \widetilde{0}_{U\times U}$.

Again, Since $\mathcal{P}(b,m) \not\supset \widetilde{0}_{U\times U}$ and $\mathcal{P}(m,b) = \widetilde{0}_{U\times U}$, then by Definition 4.1, $\mathcal{P}_W(b,m) \not\supset \widetilde{0}_{U\times U}$ and $\mathcal{P}_W(m,b) \not\supset \widetilde{0}_{U\times U}$. So, the fuzzy soft set weak preference relation $\mathcal{P}_W$ is not connected on $(\mathcal{F}, A)$.

Theorem 4.15. Let $(\mathcal{F}, A)$ be a fuzzy soft set over U and $\mathcal{R}$ be a fuzzy soft set strict preference relation on $(\mathcal{F}, A)$. If $\mathcal{R}$ is negatively transitive then $\mathcal{R}_I$ is transitive relation on $(\mathcal{F}, A)$.

Proof. Suppose there exist $a, b, c \in A$, such that $\mathcal{R}_I(a,b) \supset \widetilde{0}_{U\times U}$ and $\mathcal{R}_I(b,c) \supset \widetilde{0}_{U\times U}$. Now by Definition 4.1, this implies

$$\mathcal{R}(a,b) = \widetilde{0}_{U\times U} = \mathcal{R}(b,a) \text{ and } \mathcal{R}(b,c) = \widetilde{0}_{U\times U} = \mathcal{R}(c,b)$$

$\Rightarrow \mathcal{R}(a,b) = \widetilde{0}_{U\times U} = \mathcal{R}(b,c)$ and $\mathcal{R}(b,a) = \widetilde{0}_{U\times U} = \mathcal{R}(c,b)$

$\Rightarrow \mathcal{R}(a,c) = \widetilde{0}_{U\times U}$ and $\mathcal{R}(c,a) = \widetilde{0}_{U\times U}$, since $\mathcal{R}$ is negatively transitive

$\Rightarrow \mathcal{R}_I(a,c) \supset \widetilde{0}_{U\times U}$, by Definition of $\mathcal{R}_I$.

So, $\mathcal{R}_I$ is transitive relation on $(\mathcal{F}, A)$. □

Note 4.16. If a fuzzy soft set strict preference relation $\mathcal{R}$ on $(\mathcal{F}, A)$ is negatively transitive, then $\mathcal{R}$ and $\mathcal{R}_W$ may not be transitive on $(\mathcal{F}, A)$.

Example 4.17. Take the fuzzy soft set $(\mathcal{F}, A)$ on U and the fuzzy soft set strict preference relation $\mathcal{P}$ on $(\mathcal{F}, A)$ as in Example 4.14.

By Definition 4.9, we conclude that $\mathcal{P}$ is negatively transitive. But $\mathcal{P}(b,e) \supset \widetilde{0}_{U\times U}$, $\mathcal{P}(e,m) \supset \widetilde{0}_{U\times U}$ and $\mathcal{P}(b,m) \not\supset \widetilde{0}_{U\times U}$ implies $\mathcal{P}$ is not transitive on $(\mathcal{F}, A)$.

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ON SOFT γ- OPERATIONS IN SOFT TOPOLOGICAL SPACES

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Abstract – In this paper,the notion of soft γ-operation on soft topological spaces is introduced and studied. Also,the concepts of soft γ-open set,soft γ-interior, soft γ-closure, soft γ-regular operation, soft γ-regular space, soft γ^*-regular space are defined and studied.The notion of soft $\gamma - T_i$ spaces are introduced,which generalizes the notion of soft T_i-spaces ($i = 0, 1/2, 1, 2$) and some of their properties are studied.

Keywords – *Soft γ-operation, soft γ-open set,soft γ-interior, soft γ-closure, soft γ-regular operation, soft γ-regular space, soft γ^*-regular space*

1 Introduction

In 1999, Molodtsov [17] introduced the soft set theory and showed how soft set theory is free from the parametrization inadequacy syndrome of fuzzy set theory, rough set theory and game theory. Based on the work of Molodtsov [17], Maji et al [14], [15] initiated the theoretical study of soft set theory which includes several basic definitions and basic operations of soft sets.Further, Shabir and Naz [19] introduced soft topological spaces which are defined over an initial universe with a fixed set of parameters and studied the basic notions such as soft open sets, soft closed sets, soft closure,soft separation axioms. Hussain and Ahmad [10] and Cagman et al [6] have continued the study of properties of soft topological spaces. Further, Benchalli et al [4] studied

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the properties of soft regular spaces and soft normal spaces. The study of soft sets and related aspects was also undertaken in [2], [3], [9], [16], [18], [20],[23]. Recently, many researchers have introduced various weaker forms of soft open sets and soft closed sets in soft topological spaces and studied their properties in [1], [5], [7, 8], [11], [12], [13], [21], [22]. In this paper,the notion of soft γ-operations on soft topological spaces is introduced and studied.The concepts of soft γ-open set,soft γ-interior, soft γ-closure, soft γ-regular operation, soft γ-regular space, soft γ^*-regular space are defined and studied. The notions of soft $\gamma - T_i$ spaces are introduced,which generalizes the notion of soft T_i-spaces ($i = 0, 1/2, 1, 2$) and some of their properties are studied.

2 Preliminary

The following definitions and results are required.

Definition 2.1. [17] Let U be an initial universe and E be a set of parameters. Let $P(U)$ denote the power set of U and A be a non-empty subset of E . A pair (F, A) is called a soft set over U , where F is a mapping given by $F : A \to P(U)$.
In other words, a soft set over U is a parameterized family of subsets of the universe U. For $e \in A, F(e)$ may be considered as the set of e-approximate elements of the soft set (F, A) . Clearly, a soft set need not be a set.

Definition 2.2. [14] For two soft sets (F, A) and (G, B) over a common universe U , we say that (F, A) is a soft subset of (G, B) if (i)$A \subset B$ and (ii) for all $e \in A, F(e)$ and $G(e)$ are identical approximations.

Definition 2.3. [19] Let τ be the collection of soft sets over X. Then τ is said to be a soft topology on X if
(1) $\emptyset, \widetilde{X}$ belongs to τ.
(2) The union of any number of soft sets in τ belongs to τ.
(3) The intersection of any two soft sets in τ belongs to τ.
The triplet (X, τ, E) is called a soft topological space.

Here the members of τ are called soft open sets in X and the relative complements of soft open sets are called as soft closed sets.

Theorem 2.4. [19] Arbitrary union of soft open sets is a soft open set and finite intersection of soft closed sets is a soft closed set.

Definition 2.5. [19] Let (X, τ, E) be a soft space over X and (F, E) be a soft set over X. Then,the soft closure of (F, E) denoted by $\overline{(F, E)}$ is the intersection of all soft closed super sets of (F, E).Clearly, $\overline{(F, E)}$ is the smallest soft closed set over X contains (F, E).

The soft neighbourhood, soft relative topology, soft T_0- space, soft T_1- space and soft T_2- space are defined by Shabir and Naz in [19].

Definition 2.6. [23] The soft interior of (G, E) is the soft set defined as $(G, E)^o = \text{int}(G, E) = \bigcup\{(S, E) : (S, E) \text{ is soft open and } (S, E) \subseteq (G, E)\}$. Here $(G, E)^o$ is largest soft open set contained in (G, E).

Throughout the study, $Cl(A, E)$ and $Int(A, E)$ means soft closure and soft interior of a soft set (A, E) respectively, in the soft topological space (X, τ, E).

Definition 2.7. [19] The difference (H, E) of two soft sets (F, E) and (G, E) over X, denoted by $(F, E) \setminus (G, E)$, is defined as $H(e) = F(e) - G(e)$, for all $e \in E$.

Definition 2.8. Let (A, E) and (B, E) be two soft sets. Then, $(A, E) \setminus (B, E) = (A, E) \cap (B, E)'$

Definition 2.9. [19] Let $x \in X$.Then (x, E) is the soft set over x for which $x(e) = \{x\}$, for all $e \in E$. Clearly, $x \in (x, E)$.

Definition 2.10. [19] Let X be an initial universe set, E be the set of parameters and $\tau = \{\phi, X\}$. Then, τ is called the soft indiscrete topology on X and (X, τ, E) is called a soft indiscrete space over X.

3 Soft γ-operations

Definition 3.1. Let (X, τ, E) be a soft topological space. An operation γ on the soft topology τ is a mapping from τ into the power set $P(X)$ of X such that $(V, E) \subset (V, E)^\gamma$, for each $(V, E) \in \tau$, where $(V, E)^\gamma = \gamma(V, E)$. It is denoted by $\gamma : \tau \to P(X)$.

Definition 3.2. A subset (A, E) of a soft topological space (X, τ, E) is called a soft γ-open set of (X, τ, E), if for each $x \in (A, E)$ there exists a soft open set (U, E) such that $x \in (U, E) \subset (U, E)^\gamma \subset (A, E)$. τ_γ will denote the set of all soft γ-open sets. Clearly, we have $\tau_\gamma \subset \tau$.
A subset (B, E) of (X, τ, E) is called soft γ-closed if $(B, E)'$ is soft γ-open in (X, τ, E).

Definition 3.3. A point $x \in X$ is called a soft γ-closure point of (A, E), if $(U, E)^\gamma \cap (A, E) \neq \phi$ for each soft open neighborhood (nbd) (U, E) of x. The set of soft γ-closure points is called the soft γ-closure of (A, E) and is denoted by $Cl_\gamma(A, E)$. For the family τ_γ, we define a soft set $\tau_\gamma - Cl(A, E)$ as,
$\tau_\gamma - Cl(A, E) = \cap\{(F, E)/(F, E) \supset (A, E) and (F, E)' \in \tau\}$

Definition 3.4. Let (A, E) be a soft set. A point $x \in (A, E)$ is said to be a soft γ-interior point of (A, E) if and only if there exist a soft open nbd (N, E) of x such that $(N, E)^\gamma \subseteq (A, E)$.That is $(N, E)^\gamma \cap (A, E)' = \phi$. We denote the set of all such points by $Int_\gamma(A, E)$.
Thus, $Int_\gamma(A, E) = \{x \in (A, E)/x \in (N, E) \in \tau, (N, E)^\gamma \subseteq (A, E)\} \subseteq (A, E)$.

Definition 3.5. An operation γ on τ is said to be soft open if for every soft nbd (U, E) of each of $x \in X$, there exists a soft γ-open set (B, E) such that $x \in (B, E) \subseteq (U, E)^\gamma$.

Definition 3.6. An operation γ on τ is said to be soft regular if for any soft nbds (U, E) and (V, E) of $x \in X$, there exists soft open nbd (W, E) of x such that $(W, E)^\gamma \subseteq (U, E)^\gamma \cap (V, E)^\gamma$.

Definition 3.7. A soft topological space (X, τ, E) is called soft γ-regular if for each soft open nbd (U, E) of x in X, there exists a soft open nbd (V, E) of x such that $(V, E)^\gamma \subseteq (U, E)$.

Proposition 3.8. Let $\gamma : \tau \to P(X)$ be an operation on a soft topological space (X, τ, E). Then, (X, τ, E) is a soft γ-regular space iff $\tau = \tau_\gamma$ holds.

Proof. Suppose that $\gamma : \tau \to P(X)$ be an operation on a soft topological space (X, τ, E) and (X, τ, E) is a soft γ-regular space. We have $\tau_\gamma \subset \tau$. Thus, it is sufficient to prove that $\tau \subset \tau_\gamma$. Let $(A, E) \in \tau$. Then for any $x \in (A, E)$ there exist a soft nbd (U, E) of x such that $x \in (U, E) \subset (A, E)$. Then, by definition 3.7, there exists a soft open nbd (W, E) of x such that $(W, E)^\gamma \subseteq (U, E)$. Thus, for each $x \in (A, E)$, we have $x \in (W, E) \subset (W, E)^\gamma \subset (A, E)$. Then, (A, E) is soft γ-open. Thus, $(A, E) \in \tau_\gamma$. Hence, $\tau = \tau_\gamma$. Conversely, for each $x \in X$ and for each soft nbd (V, E) of x, since $(V, E) \in \tau = \tau_\gamma$, there exists soft open nbd (W, E) of x such that $(W, E)^\gamma \subset (V, E)$. This implies (X, τ, E) is soft γ-regular.

Example 3.9. Let $X = \{a, b, c\}, E = \{e_1, e_2\}$ and $\tau = \{\phi, X, (A, E), (B, E), (C, E)\}$ be a soft topology on X.
Here $(A, E) = \{(e_1, \{a\}), (e_2, \{a\})\}, (B, E) = \{(e_1, \{b\}), (e_2, \{b\})\}, (C, E) = \{(e_1, \{a, b\}), (e_2, \{a, b\})\}$.
Let $\gamma : \tau \to P(X)$ be an operation defined by $\gamma(V, E) = Cl(V, E)$ and let $\delta : \tau \to P(X)$ be an operation defined by $\delta(V, E) = Int(Cl(V, E))$. Then, we have $\tau_\gamma = \{\phi, X\}$ and $\tau_\delta = \tau$. Then we see that, γ is soft regular but not soft open on (X, τ, E) and δ is soft regular and soft open on (X, τ, E).

Example 3.10. Let $X = \{a, b, c\}, E = \{e_1, e_2\}$ and
$\tau = \{\phi, X, (A, E), (B, E), (C, E), (D, E)\}$ be a soft topology on X.
where,$(A, E) = \{(e_1, \{a\}), (e_2, \{a\})\}, (B, E) = \{(e_1, \{b\}), (e_2, \{b\})\}$
$(C, E) = \{(e_1, \{a, b\}), (e_2, \{a, b\})\}, (D, E) = \{(e_1, \{a, c\}), (e_2, \{a, c\})\}$.
For $b \in X$ we define an operation $\gamma : \tau \to P(X)$ by

$$\gamma(F, E) = (F, E)^\gamma = \begin{cases} (F, E) & \text{if} \quad b \in (F, E) \\ Cl(F, E) & \text{if} \quad b \notin (F, E) \end{cases}$$

Then, the operation γ is not soft γ-regular on τ. Because, if (A, E) and (C, E) are soft open nbds of point a then $(A, E)^\gamma \cap (C, E)^\gamma = Cl(A, E) \cap (C, E) = \{(e_1, \{a\}), (e_2, \{a\})\}$. And there is no soft open nbd (W, E) of a such that $(W, E)^\gamma \subset (A, E)^\gamma \cap (C, E)^\gamma$. Therefore, γ is not a soft regular operation. But we can easily verify that γ is soft open operation.

Proposition 3.11. Let $\gamma : \tau \to P(X)$ be a soft regular operation on τ. Then,
(i) if (A, E) and (B, E) are soft γ-open sets then $(A, E) \cap (B, E)$ is soft γ-open.
(ii) τ_γ is a soft topology on X.

Proof (i). Let (A, E) and (B, E) be two soft γ-open sets. By definition, for each $x \in (A, E), (B, E)$, there exist soft open nbds $(U, E), (V, E)$ such that $x \in (U, E) \subset (U, E)^\gamma \subset (A, E)$ and $x \in (V, E) \subset (V, E)^\gamma \subset (B, E)$. Now, (U, E) and (V, E) are soft nbds of x, since γ is soft regular , there exists soft open nbd (W, E) of x such that $(W, E)^\gamma \subset (U, E)^\gamma \cap (V, E)^\gamma \subset (A, E) \cap (B, E)$. Therefore, $x \in (W, E) \subset (W, E)^\gamma \subset (A, E) \cap (B, E)$. Thus, $(A, E) \cap (B, E)$ is a soft γ-open set.
(ii). Proof follows from proposition 3.8.

Remark 3.12. If γ is not soft regular, then proposition 3.11 is not true in general by the space (X, τ, E) and the operation γ as in example 3.10. Here we get $\tau_\gamma = \{\phi, X, (F, E), (G, E), (H, E)\}$, where

$(F, E) = \{(e_1, \{b\}), (e_2, \{b\})\}$,
$(G, E) = \{(e_1, \{a, b\}), (e_2, \{a, b\})\}$,
$(H, E) = \{(e_1, \{a, c\}), (e_2, \{a, c\})\}$.

Proposition 3.13. For a point $x \in X, x \in \tau_\gamma - Cl(A,E)$ if and only if $(V,E) \cap (A,E) \neq \phi$ for any $(V,E) \in \tau_\gamma$ such that $x \in (V,E)$.

Proof. Let $(F_0,E) = \{(x,E)/(V,E) \cap (A,E) \neq \phi, (V,E) \in \tau_\gamma, x \in (V,E)\}$. Let $x \in \tau_\gamma - Cl(A,E)$.It can be seen that $(F_0,E)'$ is a soft γ-open set and $(A,E) \subset \tau_\gamma - Cl(A,E) \subset (F_0,E)$, that is $(A,E) \subset (F_0,E)$. Thus, $\tau_\gamma - Cl(A,E) \subset (F_0,E)$. Conversely, let (F,E) be a soft set such that $(A,E) \subset (F,E)$ and $(F,E)' \in \tau_\gamma$. If $x \notin (F,E)$ then $x \in (F,E)'$ and $(A,E) \cap (F,E)' = \phi$. Then, $x \notin (F_0,E)$ implies $(F_0,E) \subset (F,E)$ and $(F_0,E) \subset \tau_\gamma - Cl(A,E)$ by definition.Hence, the proof.

Remark 3.14. It can be easily shown that for any soft set (A,E) of (X,τ,E),
$(A,E) \subset Cl(A,E) \subset Cl_\gamma(A,E) \subset \tau_\gamma - Cl(A,E)$.

Theorem 3.15. Let $\gamma : \tau \to P(X)$ be an operation on τ and (A,E) be a soft subset of X.Then, the following are true:
(i) The subset $Cl_\gamma(A,E)$ is soft closed in (X,τ,E).
(ii) If (X,τ,E) is soft γ-regular then $Cl_\gamma(A,E) = Cl(A,E)$ holds.
(iii) If γ is soft open then $Cl_\gamma(A,E) = \tau_\gamma - Cl(A,E)$ and $Cl_\gamma(Cl_\gamma(A,E)) = Cl_\gamma(A,E)$ hold, and $Cl_\gamma(A,E)$ is soft γ-closed.

Proof (i). Proof follows from the definition of soft γ-closure.
(ii). By remark 3.14, it is sufficient to prove that $Cl_\gamma(A,E) \subset Cl(A,E)$. Let $x \in Cl_\gamma(A,E)$ and (U,E) be any soft open nbd of x. By definition of soft γ-regularity, there exists a soft open nbd (V,E) of x such that $(V,E)^\gamma \subset (U,E)$. Since, $x \in Cl_\gamma(A,E)$, we have $(V,E)^\gamma \cap (A,E) \neq \phi$. This implies $(U,E) \cap (A,E) \neq \phi$. Thus, $x \in Cl(A,E)$. Therefore, $Cl_\gamma(A,E) \subset (A,E)$. Hence, $Cl_\gamma(A,E) = Cl(A,E)$.
(iii). Suppose that $x \notin Cl_\gamma(A,E)$. Then there exists a soft open set (U,E) such that $x \in (U,E)$ and $(U,E)^\gamma \cap (A,E) = \phi$. Since,$\gamma$ is soft open, for (U,E) and $x \in (U,E)$, there exists a soft γ-open set (S,E) such that $x \in (S,E) \subset (U,E)^\gamma$. Then $(S,E) \cap (A,E) = \phi$. From proposition 3.13, it shows that $x \notin \tau_\gamma - Cl(A,E)$ and hence $Cl_\gamma(A,E) \supset \tau_\gamma - Cl(A,E)$. By remark 3.14, we have $Cl_\gamma(A,E) \subset \tau_\gamma - Cl(A,E)$. Thus, $Cl_\gamma(A,E) = \tau_\gamma - Cl(A,E)$. Then we obtain, $Cl_\gamma(Cl_\gamma(A,E)) = \tau_\gamma - Cl(\tau_\gamma - Cl(A,E)) = \tau_\gamma - Cl(A,E) = Cl_\gamma(A,E)$.

Theorem 3.16. For a subset (A,E) of X, the following statements are equivalent.
(i) (A,E) is soft γ-open in (X,τ,E).
(ii)$Cl_\gamma((A,E)') = (A,E)'$
(iii)$\tau_\gamma - Cl(A,E)' = (A,E)'$ holds.
(iv) $(A,E)'$ is soft γ-closed.

Proof (i)⇒(ii). Suppose (A,E) is soft γ-open. Let $x \notin (A,E)'.Then x \in (A,E)$, and there exists a soft open nbd (U,E) of x such that $(U,E)^\gamma \subset (A,E)$, which implies $(U,E)^\gamma \cap (A,E)' = \phi$, then $x \notin Cl_\gamma(A,E)'$.Thus, $Cl_\gamma(A,E)' \subset (A,E)'$. We have $(A,E)' \subset Cl_\gamma(A,E)'$ is always true. Thus, statement (ii) holds.

(ii)⇒(iii). We prove that ,$\tau_\gamma - Cl(A,E)' \subset (A,E)'$. Let $x \notin (A,E)'$.Then,$x \notin Cl_\gamma(A,E)'$.Thus, there exists a soft open nbd (U,E) of x such that $(U,E)^\gamma \cap (A,E)' = \phi$. This implies $(U,E)^\gamma \subset (A,E)$.Then, (A,E) is soft γ-open. Thus, we have $(A,E) \cap (A,E)' = \phi$ and hence $x \notin \tau_\gamma - Cl(A,E)'$. Thus,$\tau_\gamma - Cl(A,E)' \subset (A,E)'$ and from remark 3.14, we have $(A,E)' \subset \tau_\gamma - Cl(A,E)'$. Therefore, statement (iii) holds.

(iii)⇒(iv). We prove that $((A,E)')' = (A,E) \in \tau_\gamma$. Let $x \notin (A,E)'$ then $x \notin \tau_\gamma - Cl(A,E)'$.Then, by proposition 3.13, there exists a soft γ-open set (U,E) such that $x \in (U,E)$ and $(U,E)\cap(A,E)' = \phi$. Since, $x \in (U,E) \in \tau_\gamma$, there exists a soft nbd (V,E) of x such that $(V,E)^\gamma \subset (U,E)$.Then, we have $x \in (V,E) \subset (V,E)^\gamma \subset (U,E) \subset (A,E)$. Thus, (A,E) is soft γ-open. That is $(A,E) \in \tau_\gamma$.Therefore, statement (iv) holds.

(iv)⇒(i). The proof is straight forward from the definition.

Proposition 3.17. If γ is soft regular then $Cl_\gamma((A,E) \cup (B,E)) = Cl_\gamma(A,E) \cup Cl_\gamma(B,E)$.

Proof. Proof is straight forward.

Theorem 3.18. (A,E) is soft γ-open if and only if $(A,E) = Int_\gamma(A,E)$.

Proof. Proof follows from the definitions of soft γ-open set and soft $Int_\gamma(A,E)$.

4 Soft $\gamma - T_i$ Spaces (i=0,1/2,1,2)

Let $\gamma : \tau \to P(X)$ be an operation on a soft topology τ.

Definition 4.1. A space (X,τ,E) is called a soft $\gamma - T_0$-space if for each distinct points $x, y \in X$ there exist a soft open set (U,E) such that either $x \in (U,E)$ and $y \notin (U,E)^\gamma$ or $y \in (U,E)$ and $x \notin (U,E)^\gamma$.

Definition 4.2. A space (X,τ,E) is called a soft $\gamma - T_1$-space if for each distinct points $x, y \in X$ there exists soft open sets $(U,E), (V,E)$ containing x and y respectively such that $y \notin (U,E)^\gamma$ and $x \notin (V,E)^\gamma$.

Definition 4.3. A space (X,τ,E) is called a soft $\gamma - T_2$-space if for each distinct points $x, y \in X$ there exists soft open sets $(U,E), (V,E)$ such that $x \in (U,E), y \in (v,E)$ and $(U,E)^\gamma \cap (V,E)^\gamma = \phi$.

To define soft $\gamma - T_{1/2}$- space we introduce the notion of soft γ-g-closed sets.

Definition 4.4. A subset (A,E) of (X,τ,E) is called soft γ-g-closed if $Cl_\gamma(A,E) \subset (U,E)$, whenever $(A,E) \subset (U,E)$ and (U,E) is soft γ-open in (X,τ,E).

Remark 4.5. Every soft γ-closed set is soft γ-g-closed set.

Proposition 4.6. Let $\gamma : \tau \to P(X)$ be an operation and (A,E) be a soft set in (X,τ,E). Then,the following results are hold good:
(i) If $\tau_\gamma - Cl((x,E)) \cap (A,E) \neq \phi$ holds for every $x \in Cl_\gamma(A,E)$ then (A,E) is soft γ-g-closed in (X,τ,E).
(ii) If γ is a soft regular operation, then the converse of (i) is true.

Proof (i). Let (U,E) be any soft γ-open set such that $(A,E) \subset (U,E)$. Let $x \in Cl_\gamma(A,E)$. By assumption, there exists a point x such that $x \in \tau_\gamma - Cl(x,E)$ and $x \in (A,E) \subset (U,E)$. It follows from the proposition 3.13 that $(U,E) \cap (x,E) \neq \phi$ and hence $x \in (U,E)$. Therefore $Cl_\gamma(A,E) \subset (U,E)$, whenever $(A,E) \subset (U,E)$ and (U,E) is soft γ-open. Hence,by definition (A,E) is soft γ-g-closed in (X,τ,E).
(ii). Let (A,E) be a soft γ-g-closed set in (X,τ,E). Suppose that there exist a point

$x \in Cl_\gamma(A,E)$ such that $(\tau_\gamma - Cl(x,E)) \cap (A,E) = \phi$. Since, γ is soft regular operation, then τ_γ is a soft topology on X by proposition 3.11. Then, $\tau_\gamma - Cl(x,E)$ is soft τ_γ-closed and the soft complement $(\tau_\gamma - Cl(x,E))'$ is soft τ_γ-open,by theorem 3.16. Since $(A,E) \subset (\tau_\gamma - Cl(x,E))'$ and (A,E) is soft γ-g-closed, we have $Cl_\gamma(A,E) \subset (\tau_\gamma - Cl(x,E))'$. Thus, $x \notin Cl_\gamma(A,E)$.This is a contradiction. Hence, if γ is a regular operation then the converse of statement (i) is true.

Theorem 4.7. Let $(A,E),(B,E)$ be soft sets of (X,τ,E). Then we have the following;
(i) $Int_\gamma(Int_\gamma(A,E)) = Int_\gamma(A,E)$
(ii) $Int_\gamma((A,E) \cup (B,E)) \supseteq Int_\gamma(A,E) \cup Int_\gamma(B,E)$
(iii) $Int_\gamma((A,E) \cap (B,E)) = Int_\gamma(A,E) \cap Int_\gamma(B,E)$, if γ is soft regular operation.

Proof. Statements (i) and (ii) follows from the definition of soft γ interior.
(iii). Note that if $(A,E) \subseteq (B,E)$ then $Int_\gamma(A,E) \subseteq Int_\gamma(B,E)$ follows from the definition. Thus, $Int_\gamma((A,E) \cap (B,E)) \subseteq Int_\gamma(A,E) \cap Int_\gamma(B,E)$. Now , let $x \in Int_\gamma(A,E) \cap Int_\gamma(B,E)$. Then, $x \in Int_\gamma(A,E), x \in Int_\gamma(B,E)$. This implies, there exists soft open nbds (U,E) and (V,E) of x such that ,$(U,E)^\gamma \subseteq (A,E)$ and $(V,E)^\gamma \subseteq (B,E)$.This implies $(U,E)^\gamma \cap (V,E)^\gamma \subseteq (A.E) \cap (B,E)$. Since γ is regular, therefore there exists a soft open nbd (W,E) of x such that $(W,E)^\gamma \subseteq (U,E)^\gamma \cap (V,E)^\gamma$. Thus, $(W,E)^\gamma \subseteq (A,E) \cap (B,E)$. Thus, $x \in Int_\gamma((A,E) \cap (B,E))$. Thus, we get $Int_\gamma((A,E) \cap (B,E)) \subseteq Int_\gamma(A,E) \cap Int_\gamma(B,E)$. Hence, the result.

Remark 4.8. The following example shows that, the equality does not hold if γ is not soft regular operation.

Example 4.9. Consider the example 3.10. Here, γ is not soft regular. Let $(A,E) = \{(e_1,\{a,b\}),(e_2,\{a,b\})\}$,
$(B,E) = \{(e_1,\{a,c\}),(e_2,\{a,c\})\}$.Here, $Int_\gamma((A,E) \cap (B,E)) \subset Int_\gamma(A,E) \cap Int_\gamma(B,E)$, but $Int_\gamma(A,E) \cap Int_\gamma(B,E) \nsubseteq Int_\gamma((A,E) \cap (B,E))$.

Theorem 4.10. The following results are true, in any soft topological space.
(a) $Int_\gamma(A,E)' = (Cl_\gamma(A,E))'$
(b)$Cl_\gamma(A,E)' = (Int_\gamma(A,E))'$
(c)$Int_\gamma(A,E) = (Cl_\gamma(A,E)')'$

Proof(a). Let $x \in Int_\gamma(A,E)'$.Then, there exists soft open nbd (U,E) of x such that $(U,E)^\gamma \subseteq (A,E)'$. This implies $(U,E)^\gamma \cap (A,E) = \phi$. This gives $x \notin Cl_\gamma(A,E)$. That is $x \in (Cl_\gamma(A,E))'$. Thus, $Int_\gamma(A,E)' \subset (Cl_\gamma(A,E))'$.Similarly we can easily prove the converse by reversing these steps. Hence, the result.
(b). Let $x \notin Cl_\gamma(A,E)'$. Then, there exists soft open nbd (U,E) of x such that $(U,E)^\gamma \cap (A,E)' = \phi$, which implies $(U,E)^\gamma \subset (A,E)$. Thus, $x \in Int_\gamma(A,E)$, implies $x \notin (Int_\gamma(A,E))'$. Thus, $(Int_\gamma(A,E))' \subset Cl_\gamma(A,E)'$. Similarly we can easily prove the converse by reversing these steps.
(c). Let $x \in (Cl_\gamma(A,E)')'$ then $x \notin Cl_\gamma(A,E)'$. Thus, there exists soft open nbd (U,E) of x such that $(U,E)^\gamma \subset (A,E)$. Thus, $x \in Int_\gamma(A,E)$. Hence, $(Cl_\gamma(A,E)')' \subset Int_\gamma(A,E)$. Similarly the converse can be proved by reversing these steps.

Definition 4.11. The soft γ- exterior of (A,E) is defined as the soft γ-interior of $(A,E)'$. That is $ext_\gamma(A,E) = Int_\gamma(A,E)'$.

Definition 4.12. The soft γ-boundary of (A, E), denoted by $bd_\gamma(A, E)$, is defined as the set of all points which do not belong to the soft γ-interior or soft γ-exterior of (A, E).

Theorem 4.13. In any soft topological spaces (X, τ, E) the following are equivalent:
(a) $(bd_\gamma(A, E))' = Int_\gamma(A, E) \cup Int_\gamma(A, E)'$
(b) $Cl_\gamma(A, E) = Int_\gamma(A, E) \cup bd_\gamma(A, E)$
(c) $bd_\gamma(A, E) = Cl_\gamma(A, E) \cap Cl_\gamma(A, E)' = Cl_\gamma(A, E) - Int_\gamma(A, E)$

Proof (a)$\Rightarrow$ (b). We have, $(ext_\gamma(A, E))' = Int_\gamma(A, E) \cup bd_\gamma(A, E)$.Which implies, $(Int_\gamma(A, E)')' = Int_\gamma(A, E) \cup bd_\gamma(A, E)$, from definition 4.11.Thus, $((Cl_\gamma(A, E))')' = Int_\gamma(A, E) \cup bd_\gamma(A, E)$, from theorem 4.10. This implies $Cl_\gamma(A, E) = Int_\gamma(A, E) \cup bd_\gamma(A, E)$. Thus,(b) holds.

(b)$\Rightarrow$(c). We have $(bd_\gamma(A, E))' = Int_\gamma(A, E) \cup ext_\gamma(A, E) = (Cl_\gamma(A, E)')' \cup (Cl\gamma(A, E))' = (Cl_\gamma(A, E) \cap Cl_\gamma(A, E)')'$. Thus, $bd_\gamma(A, E) = Cl_\gamma(A, E) \cap Cl_\gamma(A, E)' = Cl_\gamma(A, E) \cap (Int_\gamma(A, E))'$, from theorem 4.10. Hence, $bd_\gamma(A, E) = Cl_\gamma(A, E) - Int_\gamma(A, E)$. Thus, (c) holds.

(c)$\Rightarrow$(a). Consider, $Int_\gamma(A, E) \cup Int_\gamma(A, E)'$
$= ((Int_\gamma(A, E))')' \cup ((Int_\gamma(A, E)')')'$
$= [(Int_\gamma(A, E))' \cap (Int_\gamma(A, E)')']'$
$= (Cl_\gamma(A, E)' \cap Cl_\gamma(A, E))'$, from theorem 4.10
$=(bd_\gamma(A, E))'$. Thus, (a) holds.

Remark 4.14. From theorem 4.13(c), we have $bd_\gamma(A, E) = bd_\gamma(A, E)'$

Proposition 4.15. For a soft set (A, E) of X, we have the following:
(a) (A, E) is soft γ-open if and only if $(A, E) \cap bd_\gamma(A, E) = \phi$
(b) (A, E) is soft γ-closed if and only if $bd_\gamma(A, E) \subseteq (A, E)$.

Proof (a). Let (A, E) be soft γ-open set. Then, $(A, E)'$ is soft γ-closed. Therefore,by theorem 3.16, $Cl_\gamma(A, E)' = (A, E)'$. Now, $(A, E) \cap bd_\gamma(A, E) = (A, E) \cap [Cl_\gamma(A, E) \cap Cl_\gamma(A, E)'] = (A, E) \cap Cl_\gamma(A, E) \cap (A, E)' = \phi$. Conversely, let $(A, E) \cap bd_\gamma(A, E) = \phi$. Then, $(A, E) \cap Cl_\gamma(A, E) \cap Cl_\gamma(A, E)' = \phi$ or $(A, E) \cap Cl_\gamma(A, E)' = \phi$. This implies $Cl_\gamma(A, E)' \subseteq (A, E)'$ and $(A, E)' \subseteq Cl_\gamma(A, E)'$ is always true. Thus, $(A, E)'$ is soft γ-closed and hence (A, E) is soft γ-open.
(b). Let (A, E) be a soft γ-closed set. Then, $Cl_\gamma(A, E) = (A, E)$. Now, $bd_\gamma(A, E) = Cl_\gamma(A, E) \cap Cl_\gamma(A, E)' \subseteq Cl_\gamma(A, E) = (A, E)$. That is, $bd_\gamma(A, E) \subseteq (A, E)$. Conversely, let $bd_\gamma(A, E) \subseteq (A, E)$.Then, $bd_\gamma(A, E) \cap (A, E)' = \phi$.Since from remark 4.14, $bd_\gamma(A, E) = bd_\gamma(A, E)'$. We have $bd_\gamma(A, E)' \cap (A, E)' = \phi$. By (a), $(A, E)'$ is soft γ-open set and hence (A, E) is soft γ-closed.

Theorem 4.16. The following hold in any soft topological space (X, τ, E).
(a) $bd_\gamma(A, E) \cap Int_\gamma(A, E) = \phi$
(b) $Int_\gamma(A, E) = (A, E) - bd_\gamma(A, E)$

Proof (a). Consider $bd_\gamma(A, E) \cap Int_\gamma(A, E)$
$= Cl_\gamma(A, E) \cap Cl_\gamma(A, E)' \cap Int_\gamma(A, E)$
$= Cl_\gamma(A, E) \cap (Int_\gamma(A, E))' \cap Int_\gamma(A, E)$, from theorem 4.10(b).
$= \phi$

(b). Consider $(A, E) - bd_\gamma(A, E)$
$= (A, E) - [Cl_\gamma(A, E) \cap Cl_\gamma(A, E)']$
$= (A, E) \cap [(Cl_\gamma(A, E) \cap Cl_\gamma(A, E)')']$
$= (A, E) \cap [(Cl_\gamma(A, E))' \cup (Cl_\gamma(A, E)')']$
$= (A, E) \cap [(Cl_\gamma(A, E))' \cup (Int_\gamma(A, E))]$, from theorem 4.10(c).
$= [(A, E) \cap (Cl_\gamma(A, E))'] \cup [(A, E) \cap Int_\gamma(A, E)]$
$= Int_\gamma(A, E)$

Theorem 4.17. For any two soft sets $(A, E), (B, E)$ of X, if γ is soft regular operation, then we have the following:
(a) $ext_\gamma((A, E) \cup (B, E)) = ext_\gamma(A, E) \cap ext_\gamma(B, E)$
(b) $bd_\gamma((A, E) \cup (B, E)) = [bd_\gamma(A, E) \cap Cl_\gamma(B, E)'] \cup [bd_\gamma(B, E) \cap Cl_\gamma(A, E)']$
(c)$bd_\gamma((A, E) \cap (B, E)) = [bd_\gamma(A, E) \cap Cl_\gamma(B, E)] \cup [bd_\gamma(B, E) \cap Cl_\gamma(A, E)]$

Proof (a). Consider $ext_\gamma[(A, E) \cup (B, E)]$
$= Int_\gamma[(A, E) \cup (B, E)]'$
$= Int_\gamma[(A, E)' \cap (B, E)']$
$= Int_\gamma(A, E)' \cap Int_\gamma(B, E)'$, since γ is soft regular.
$= ext_\gamma(A, E) \cap ext_\gamma(B, E)$
(b). Consider, $bd_\gamma((A, E) \cup (B, E))$
$= Cl_\gamma((A, E) \cup (B, E)) \cap Cl_\gamma((A, E) \cup (B, E))'$, from theorem 4.13(c)
$= (Cl_\gamma(A, E) \cup Cl_\gamma(B, E)) \cap ((Cl_\gamma(A, E)' \cap (B, E)')$
$= (Cl_\gamma(A, E) \cup Cl_\gamma(B, E)) \cap ((Cl_\gamma(A, E)' \cap Cl_\gamma(B, E)')$
$= [Cl_\gamma(A, E) \cap Cl_\gamma(A, E)' \cap Cl_\gamma(B, E)'] \cup [Cl_\gamma(B, E) \cap Cl_\gamma(A, E)' \cap Cl_\gamma(B, E)'] = [bd_\gamma(A, E) \cap Cl_\gamma(B, E)'] \cup [bd_\gamma(B, E) \cap Cl_\gamma(A, E)']$
(c). Consider, $bd_\gamma((A, E) \cap (B, E))$
$= Cl_\gamma((A, E) \cap (B, E)) \cap Cl_\gamma((A, E) \cap (B, E))'$
$= [Cl_\gamma(A, E) \cap Cl_\gamma(B, E)] \cap [Cl_\gamma((A, E)' \cup (B, E)')]$
$= [Cl_\gamma(A, E) \cap Cl_\gamma(B, E)] \cap [Cl_\gamma(A, E)' \cup Cl_\gamma(B, E)']$
$= [Cl_\gamma(A, E) \cap Cl_\gamma(B, E) \cap Cl_\gamma(A, E)'] \cup [Cl_\gamma(A, E) \cap Cl_\gamma(B, E) \cap Cl_\gamma(B, E)']$
$= [bd_\gamma(A, E) \cap Cl_\gamma(B, E)] \cup [Cl_\gamma(A, E) \cap bd_\gamma(B, E)]$

Theorem 4.18. For any soft sets $(A, E), (B, E)$ in soft topological space (X, τ, E) the following hold:
(a) $Cl_\gamma[(A, E) - (B, E)] \supseteq Cl_\gamma(A, E) - Cl_\gamma(B, E)$
(b) $Int_\gamma[(A, E) - (B, E)] \subseteq Int_\gamma(A, E) - Int_\gamma(B, E)$
(c)If (A, E) is soft γ-open, then $(A, E) \cap Cl_\gamma(B, E) \subseteq Cl_\gamma(B, E) \subseteq Cl_\gamma((A, E) \cap (B, E))$

Proof (a). Let $x \in Cl_\gamma(A, E) - Cl_\gamma(B, E)$. Then, $x \in Cl_\gamma(A, E)$ and $x \notin Cl_\gamma(B, E)$.Then, there exists soft open nbd (U, E) of x such that $(U, E)^\gamma \cap (A, E) \neq \phi$ and $(U, E)^\gamma \cap (B, E) = \phi$. This implies $(U, E)^\gamma \cap ((A, E) - (B, E)) \neq \phi$. That is $x \in Cl_\gamma((A, E) - (B, E))$. Thus it proves (a).

(b). Let $x \notin Int_\gamma(A, E) - Int_\gamma(B, E)$. Then, $x \notin Int_\gamma(A, E), x \in Int_\gamma(B, E)$. Thus, there exists a soft open nbd (U, E) of x such that $(U, E)^\gamma \cap (A, E)' \neq \phi$ and $(U, E)^\gamma \cap (B, E)' = \phi$. Thus, $(U, E)^\gamma \cap ((A, E) - (B, E)) = \phi$. Therefore, $x \notin Int_\gamma((A, E) - (B, E))$. Hence, $Int_\gamma[(A, E) - (B, E)] \subseteq Int_\gamma(A, E) - Int_\gamma(B, E)$.

(c). Since (A, E) is soft γ-open, then $(A, E) = Int_\gamma(A, E)$.
Now, $(A, E) \cap Cl_\gamma(B, E)$

$= Cl_\gamma(B,E) \cap Int_\gamma(A,E)$
$= Cl_\gamma(B,E) - (Int_\gamma(A,E))'$
$= Cl_\gamma(B,E) - Cl_\gamma(A,E)'$, from theorem 4.10(b).
$\subseteq Cl_\gamma[(B,E) - (A,E)']$, from theorem 4.18(a).
$= Cl_\gamma[(B,E) \cap (A,E)] = Cl_\gamma[(A,E) \cap (B,E)]$
Therefore, $(A,E) \cap Cl_\gamma(B,E) \subseteq Cl_\gamma((A,E) \cap (b,E))$.

Definition 4.19. An operation $\gamma : \tau \to P(X)$ is said to be strictly soft regular, if for any soft open nbds $(U,E),(V,E)$ of x there exists soft open nbd (W,E) of x such that $(U,E)^\gamma \cap (V,E)^\gamma = (W,E)^\gamma$.

Definition 4.20. An operation $\gamma : \tau \to P(X)$ is said to be soft γ-open if $(V,E)^\gamma$ is soft γ-open for each $(V,E) \in \tau$.

Example 4.21. Let $X = \{a,b,c\}, E = \{e_1,e_2\}, \tau = \{\phi, X, (A,E),(B,E),(C,E)\}$. where

$(A,E) = \{(e_1,\{a\}),(e_2,\{a\})\}$,
$(B,E) = \{(e_1,\{b\}),(e_2,\{b\})\}$,
$(C,E) = \{(e_1,\{a,b\}),(e_2,\{a,b\})\}$.

Let us define an operation $\gamma : \tau \to P(X)$ by $\gamma(A,E) = IntCl(A,E)$. Then, soft γ-open sets are only $\phi, X, (A,E),(B,E),(C,E)$. We can easily verify that γ is strictly soft regular and soft γ-open on (X,τ,E).

Example 4.22. Consider (X,τ,E) be same as in above example 4.21. Let us define an operation γ by $\gamma(A,E) = Cl(A,E)$. Then, the soft γ-open sets are only ϕ, X. We can verify that γ is strictly soft regular but not soft γ-open on (X,τ,E).

Theorem 4.23. If (X,τ,E) is a soft $\gamma - T_2$ space then for any two distinct points $a,b \in X$, there are soft γ-closed sets (F,E) and (G,E) such that $a \in (F,E), b \notin (F,E)$ and $a \notin (G,E), b \in (G,E)$ and $\widetilde{X} = (F,E) \cup (G,E)$, where γ is soft γ operation.

Proof. Since (X,τ,E) is soft $\gamma - T_2$ space then for any $a,b \in X$ there exist soft open sets $(U,E),(V,E)$ such that $a \in (U,E), b \in (V,E)$ and $(U,E)^\gamma \cap (V,E)^\gamma = \phi$. Therefore, $(U,E)^\gamma \subseteq ((V,E)^\gamma)'$ and $(V,E)^\gamma \subseteq ((U,E)^\gamma)'$. Hence, $a \in ((V,E)^\gamma)'$ and $b \in ((U,E)^\gamma)'$. Let $((V,E)^\gamma)' = (F,E)$ and $((U,E)^\gamma)' = (G,E)$. This gives, $a \in (F,E), b \notin (F,E)$ and $a \notin (G,E), b \in (G,E)$. Also, $(F,E) \cup (G,E) = ((V,E)^\gamma)' \cup ((U,E)^\gamma)' = [(V,E)^\gamma \cap (U,E)^\gamma]' = (\phi)' = \widetilde{X}$.

Theorem 4.24. If (X,τ,E) is a soft $\gamma - T_2$ space, then for every point x of X, $(x,E) = \cap(C,E)_x$, where $(C,E)_x$ is a soft γ-closed set containing soft open set (U,E) which contains x, where γ is soft γ-open operation.

Proof. Since (X,τ,E) is a soft $\gamma - T_2$ space, then for any $x,y \in X$ with $x \neq y$, there exist soft open sets (U,E) and (V,E) such that $x \in (U,E), y \in (V,E)$ and $(U,E)^\gamma \cap (V,E)^\gamma = \phi$. Thus,$(U,E)^\gamma \subseteq ((V,E)^\gamma)'$. Since $((V,E)^\gamma)'$ is a soft γ-closed and $(U,E)^\gamma \subseteq ((V,E)^\gamma)' = (C,E)_x$ is a soft γ-closed nbd of x and $y \notin ((V,E)^\gamma)' = (C,E)_x$. Thus, x is the only point which is in every soft γ-closed nbd of x. i.e. $(x,E) = \cap(C,E)_x$.

Definition 4.25. A soft topological space (X,τ,E) is said to be soft γ^*-regular space if for any soft γ-closed set (A,E) and $x \notin (A,E)$, there exist disjoint soft γ-open sets $(U,E),(V,E)$ such that $x \in (U,E), (A,E) \subseteq (V,E)$.

Example 4.26. The soft indiscrete space is a soft γ^*-regular space, for if any soft γ-closed set (A, E) and any point $x \notin (A, E)$ there are soft γ-open sets $(U, E), (V, E)$ such that $x \in (U, E), (A, E) \subseteq (V, E)$ and $(U, E) \cap (V, E) = \phi$.

Theorem 4.27. If (X, τ, E) is soft γ^*-regular space then for any soft γ-open set (U, E) in (X, τ, E) and $x \in (U, E)$, there is a soft γ-open set (V, E) containing x such that $x \in Cl_\gamma(V, E) \subseteq (U, E)$.

Proof. Let (X, τ, E) be soft γ^*-regular space and (U, E) be a soft γ-open set and $x \in (U, E)$. Then, $(U, E)'$ is a soft γ-closed set such that $x \notin (U, E)'$. By the definition of soft γ^*-regularity, there are soft γ-open sets $(V, E), (W, E)$ such that $x \in (V, E), (U, E)' \subseteq (W, E)$ and $(V, E) \cap (W, E) = \phi$. Clearly, $(W, E)' \subseteq (U, E)$ and $(W, E)'$ is a soft γ-closed set. Now, $(V, E) \subseteq (W, E)' \subseteq (U, E)$. This gives, $Cl_\gamma(V, E) \subseteq (W, E)' \subseteq (U, E)$. Thus, $x \in (V, E)$ and $Cl_\gamma(V, E) \subseteq (U, E)$.

Definition 4.28. Let (X, τ, E) is soft topological space and (U, E) be a soft subset. Then, the class of soft γ-open sets in(A, E) is defined in a natural way as:
$\tau_{\gamma_{(A,E)}} = \{(A, E) \cap (O, E) : (O, E) \in \tau_\gamma\}$
where τ_γ is the set of soft γ-open sets of X. That is (G, E) is soft γ-open in (A, E) iff $(G, E) = (A, E) \cap (O, E)$, where (O, E) is a soft γ-open in (X, τ, E).

Theorem 4.29. Every soft subspace of soft γ^*-regular space is soft γ^*-regular space.

Proof. Let (Y, τ, E) be a soft subspace of soft γ^*-regular space (X, τ, E). Suppose (A, E) is a soft γ-closed set in (Y, τ, E) and $y \in Y$ such that $y \notin (A, E)$. Then, $(A, E) = (B, E) \cap \widetilde{Y}$, where (B, E) is soft γ-closed in (X, τ, E). Then, $y \notin (B, E)$. Since, (X, τ, E) is soft γ^*-regular space, there exist disjoint soft γ-open sets $(U, E), (V, E)$ in (X, τ, E) such that $y \in (U, E), (B, E) \subseteq (V, E)$. Then, $(U, E) \cap \widetilde{Y}$ and $(V, E) \cap \widetilde{Y}$ are disjoint soft γ-open sets in (Y, τ, E) such that $y \in (U, E) \cap \widetilde{Y}$ and $(A, E) \subseteq (V, E) \cap \widetilde{Y}$. Thus, (Y, τ, E) is a soft γ^*-regular space.

Theorem 4.30. A soft topological space (X, τ, E) is soft γ^*-regular if and only if for each $x \in X$ and a soft γ-closed set (A, E) such that $x \notin (A, E)$, there exist soft γ-open sets $(U, E), (V, E)$ in (X, τ, E) such that $x \in (U, E), (A, E) \subseteq (V, E)$ and $Cl_\gamma(U, E) \cap Cl_\gamma(V, E) = \phi$.

Proof. For each $x \in X$ and a soft γ-closed set (A, E) such that $x \notin (A, E)$, that is $x \in (A, E)'$ and $(A, E)'$ is soft γ-open set, by theorem 4.27, there exist a soft γ-open set (W, E) such that $x \in (W, E)$ and $Cl_\gamma(W, E) \subseteq (A, E)'$. Again by theorem 4.27, there exists a soft γ-open set (U, E) containing x such that $Cl_\gamma(U, E) \subseteq (W, E)$. Let $(V, E) = (Cl_\gamma(W, E))'$. Then, $Cl_\gamma(U, E) \subseteq (W, E) \subseteq Cl_\gamma(W, E) \subseteq (A, E)'$. This implies $(A, E) \subseteq (Cl_\gamma(W, E))' = (V, E)$. Also, $Cl_\gamma(U, E) \cap Cl_\gamma(V, E) = Cl_\gamma(U, E) \cap Cl_\gamma(Cl_\gamma(W, E))' \subseteq (W, E) \cap Cl_\gamma(Cl_\gamma(W, E))' \subseteq Cl_\gamma[(W, E) \cap (Cl_\gamma(W, E))']$, (from theorem 4.18(c))$= Cl_\gamma(\phi) = \phi$. The converse is straight forward from the definition. Hence, this completes the proof.

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ASSOCIATED PROPERTIES OF α-$\pi g\alpha$-CLOSED FUNCTIONS

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Abstract – The concept of α-open sets was introduced by [17]. The primary purpose of this paper is to introduce and study pre-$\pi g\alpha$-closed functions by using α-open sets.

Keywords – *α-open sets, pre-α-closed function, pre-$\pi g\alpha$-closed functions, α-$\pi g\alpha$-closed functions.*

1 Introduction

Generalized open sets play a very important role in General Topology and they are now the research topics of many topologists worldwide. Indeed a significant theme in General Topology and Real analysis concerns the variously modified forms of continuity, separation axioms etc by utilizing generalized open sets. One of the most well known notions and also an inspiration source is the notion of α-open sets introduced in [17].

In 1970, Levine [9] defined and studied generalized closed sets in topological spaces. In 1982, Malghan [15] defined generalized closed functions and obtained some preservation theorems of normality and regularity. In 1990, Arya and Nour [5] defined generalized semi-open sets and used them to obtain characterizations of s-normal spaces due to Maheshwari and Prasad [10]. In 1993, Devi et.al. [6] defined and studied generalized semi-closed functions and showed that the continuous generalized semi-closed surjective image of a normal space is s-normal. In 1998, Noiri et.al. [18] defined generalized pre closed sets and introduced generalized pre closed functions and showed that the

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continuous generalized pre closed surjective image of normal space is prenormal [20](or p-normal [21]). Recently, Tahiliani [23] has defined generalized β-closed functions and has shown that the continuous generalized β-closed surjective images of normal (resp. regular) spaces are β-normal [11] (resp. β-regular [2]). Further, it has shown that β-regularity is preserved under continuous pre-β-open [11] β-$g\beta$-closed [23] surjections. Recently, Arockiarani et.al [4] has defined $\pi g\alpha$-closed sets and studied properties and characterizations of them.

2 Preliminaries

Throughout this paper, X and Y refer always to topological spaces on which no separation axioms are assumed unless otherwise mentioned. For a subset A of X, $cl(A)$ and $int(A)$ denote the closure of A and the interior of A in X, respectively.

A subset A of X is said to be regular open [22] (resp. regular closed [22]) if $A = int(cl(A))$ (resp. $A = cl(int(A))$). The finite union of regular open sets is said to be π-open [24]. The complement of a π-open set is said to be π-closed [24].

A subset A of X is said to be β-open [1] (= semi pre-open [3]) if $A \subseteq cl(int(cl(A)))$.

A subset A of X is said to be α-open [17] if $A \subseteq int(cl(int(A)))$.

The complement of α-open (resp. regular open) set is called α-closed (resp. regular closed).

The intersection of all α-closed sets of X containing A is called the α-closure of A and is denoted by $\alpha cl(A)$.

It is evident that a set A is α-closed if and only if $\alpha cl(A) = A$.

The α-interior of A, $\alpha int(A)$, is the union of all α-open sets contained in A.

A subset A of X is said to be α-clopen if it is α-open and α-closed.

The family of all α-open (resp. α-closed, α-clopen, β-open, regular open, regular closed) sets of X is denoted by $\alpha O(X)$ (resp. $\alpha C(X)$, $\alpha CO(X)$, $\beta O(X)$, $RO(X)$, $RC(X)$).

The family of all α-open sets of X containing a point $x \in X$ is denoted by $\alpha O(X, x)$.

A subset A of a topological space (X, τ) is called $\pi g\alpha$-closed [4] set of X if $\alpha cl(A) \subseteq U$ holds whenever $A \subseteq U$ and U is π-open in X.

A will be called $\pi g\alpha$-open if $X \backslash A$ is $\pi g\alpha$-closed.

Theorem 2.1. *[3] For any subset A of a topological space X, the following conditions are equivalent:*

1. $A \in \beta O(X)$;
2. $A \subseteq cl(int(cl(A)))$;
3. $cl(A) \in RC(X)$.

3 Pre-$\pi g\alpha$-closed Functions

Lemma 3.1. *A subset A of a space X is $\pi g\alpha$-open in X if and only if $F \subseteq \alpha int(A)$ whenever $F \subseteq A$ and F is π-closed in X.*

Remark 3.2. *Every α-open set is $\pi g\alpha$-open but not conversely.*

Example 3.3. *Let* $X = \{a, b, c\}$ *and* $\tau = \{\phi, X, \{a\}, \{a, c\}\}$. *Then* $\{b\}$ *is* $\pi g\alpha$*-open set but not* α*-open.*

Definition 3.4. *A function* $f : X \to Y$ *is said to be pre-*$\pi g\alpha$*-closed (=* α*-*$\pi g\alpha$*-closed) (resp. regular* $\pi g\alpha$*-closed, almost* $\pi g\alpha$*-closed) if for each* $F \in \alpha C(X)$ *(resp.* $F \in \alpha CO(X)$, $F \in RC(X)$*),* $f(F)$ *is* $\pi g\alpha$*-closed in* Y.

Definition 3.5. *A function* $f : X \to Y$ *is said to be* $\pi g\alpha$*-closed if for each closed set* F *of* X, $f(F)$ *is* $\pi g\alpha$*-closed in* Y.

Remark 3.6. *From the above definitions, we obtain the following diagram:*

$$\begin{array}{ccc} \textbf{pre-}\pi g\alpha\textbf{-closed} & \longrightarrow & \textbf{regular } \pi g\alpha\textbf{-closed} \\ \downarrow & & \uparrow \\ \pi g\alpha\textbf{-closed} & \longrightarrow & \textbf{almost } \pi g\alpha\textbf{-closed} \end{array}$$

None of all implications in the above diagram is reversible as the following examples show.

Example 3.7. *Let* $X = Y = \{a, b, c\}$, $\tau = \{\phi, X, \{c\}\}$ *and* $\sigma = \{\phi, Y, \{b\}, \{c\}, \{b, c\}\}$. *Let* $f : (X, \tau) \to (Y, \sigma)$ *be the identity function. Then* f *is regular* $\pi g\alpha$*-closed but it is not pre-*$\pi g\alpha$*-closed.*

Example 3.8. *Let* $X = Y = \{a, b, c\}$, $\tau = \{\phi, X, \{b\}, \{b, c\}\}$ *and* $\sigma = \{\phi, Y, \{b\}, \{c\}, \{b, c\}\}$. *Let* $f : (X, \tau) \to (Y, \sigma)$ *be the identity function. Then* f *is* $\pi g\alpha$*-closed but not pre-*$\pi g\alpha$*-closed.*

Example 3.9. *Let* $X = Y = \{a, b, c\}$, $\tau = \{\phi, X, \{a, b\}\}$ *and* $\sigma = \{\phi, Y, \{b\}, \{c\}, \{b, c\}\}$. *Let* $f : (X, \tau) \to (Y, \sigma)$ *be the identity function. Then* f *is almost* $\pi g\alpha$*-closed but not* $\pi g\alpha$*-closed.*

Example 3.10. *Let* $X = Y = \{a, b, c\}$, $\tau = \{\phi, X, \{a\}, \{b\}, \{a, b\}\}$ *and* $\sigma = \{\phi, Y, \{b\}, \{c\}, \{b, c\}\}$. *Let* $f : (X, \tau) \to (Y, \sigma)$ *be the identity function. Then* f *is regular* $\pi g\alpha$*-closed but not almost* $\pi g\alpha$*-closed.*

The proof of the following Lemma follows using a standard technique and thus omitted.

Lemma 3.11. *A surjective function* $f : X \to Y$ *is pre-*$\pi g\alpha$*-closed (resp. regular* $\pi g\alpha$*-closed) if and only if for each subset* B *of* Y *and each* $U \in \alpha O(X)$ *(resp.* $U \in \alpha CO(X)$*) containing* $f^{-1}(B)$*, there exists an* $\pi g\alpha$*-open set* V *of* Y *such that* $B \subseteq V$ *and* $f^{-1}(V) \subseteq U$.

Corollary 3.12. *If a surjective function* $f : X \to Y$ *is pre-*$\pi g\alpha$*-closed (resp. regular* $\pi g\alpha$*-closed), then for each* π*-closed set* K *of* Y *and each* $U \in \alpha O(X)$ *(resp.*$U \in \alpha CO(X)$*) containing* $f^{-1}(K)$*, there exists* $V \in \alpha O(Y)$ *containing* K *such that* $f^{-1}(V) \subseteq U$.

Proof. Suppose that $f : X \to Y$ is pre-$\pi g\alpha$-closed (resp. regular $\pi g\alpha$-closed). Let K be any π-closed set of Y and $U \in \alpha O(X)$ (resp. $U \in \alpha CO(X)$) containing $f^{-1}(K)$. By Lemma 3.11, there exists an $\pi g\alpha$-open set G of Y such that $K \subseteq G$ and $f^{-1}(G) \subseteq U$. Since K is π-closed, by Lemma 3.1, $K \subseteq \alpha int(G)$. Put $V = \alpha int(G)$. Then, $K \subseteq V \in \alpha O(Y)$ and $f^{-1}(V) \subseteq U$.

Definition 3.13. *[7] A function $f : X \to Y$ is said to be*

1. *π-irresolute if $f^{-1}(F)$ is π-closed in X for every π-closed set F of Y.*
2. *m-π-closed if $f(F)$ is π-closed in Y for every π-closed set F of X.*

Lemma 3.14. *A function $f : X \to Y$ is π-irresolute if and only if $f^{-1}(F)$ is π-open in X for every π-open set F of Y.*

Theorem 3.15. *If $f : X \to Y$ is π-irresolute pre-$\pi g\alpha$-closed bijection, then $f(H)$ is $\pi g\alpha$-closed in Y for each $\pi g\alpha$-closed set H of X.*

Proof. Let H be any $\pi g\alpha$-closed set of X and V an π-open set of Y containing $f(H)$. Since $f^{-1}(V)$ is an π-open set of X containing H, $\alpha cl(H) \subseteq f^{-1}(V)$ and hence $f(\alpha cl(H)) \subseteq V$. Since f is pre-$\pi g\alpha$-closed and $\alpha cl(H) \in \alpha C(X)$, $f(\alpha cl(H))$ is $\pi g\alpha$-closed in Y. We have $\alpha cl(f(H)) \subseteq \alpha cl(f(\alpha cl(H))) \subseteq V$. Therefore, $f(H)$ is $\pi g\alpha$-closed in Y.

Definition 3.16. *A function $f : X \to Y$ is said to be pre-$\pi g\alpha$-continuous or α-$\pi g\alpha$-continuous if $f^{-1}(K)$ is $\pi g\alpha$-closed in X for every $K \in \alpha C(Y)$.*

It is obvious that a function $f : X \to Y$ is pre-$\pi g\alpha$-continuous if and only if $f^{-1}(V)$ is $\pi g\alpha$-open in X for every $V \in \alpha O(Y)$.

Theorem 3.17. *If $f : X \to Y$ is m-π-closed pre-$\pi g\alpha$-continuous bijection, then $f^{-1}(K)$ is $\pi g\alpha$-closed in X for each $\pi g\alpha$-closed set K of Y.*

Proof. Let K be $\pi g\alpha$-closed set of Y and U an π-open set of X containing $f^{-1}(K)$. Put $V = Y - f(X - U)$, then V is an π-open in Y, $K \subseteq V$ and $f^{-1}(V) \subseteq U$. Therefore, we have $\alpha cl(K) \subseteq V$ and hence $f^{-1}(K) \subseteq f^{-1}(\alpha cl(K)) \subseteq f^{-1}(V) \subseteq U$. Since f is pre-$\pi g\alpha$-continuous and $\alpha cl(K)$ is α-closed in Y, $f^{-1}(\alpha cl(K))$ is $\pi g\alpha$-closed in X and hence $\alpha cl(f^{-1}(K)) \subseteq \alpha cl(f^{-1}(\alpha cl(K))) \subseteq U$. This shows that $f^{-1}(K)$ is $\pi g\alpha$-closed in X.

Recall that a function $f : X \to Y$ is said to be α-irresolute [14] if $f^{-1}(V) \in \alpha O(X)$ for every $V \in \alpha O(Y)$.

Remark 3.18. *Every α-irresolute function is pre-$\pi g\alpha$-continuous but not conversely.*

Proof. Let $A \in \alpha O(Y)$. Since f is α-irresolute, $f^{-1}(A) \in \alpha O(X)$. Since α-open set is $\pi g\alpha$-open, $f^{-1}(A)$ is $\pi g\alpha$-open in X. Hence f is pre-$\pi g\alpha$-continuous.

Example 3.19. *Let $X = Y = \{a, b, c\}$, $\tau = \{\phi, X, \{b\}, \{b, c\}\}$ and $\sigma = \{\phi, Y, \{b\}, \{c\}, \{b, c\}\}$. Let $f : (X, \tau) \to (Y, \sigma)$ be the identity function. Then f is pre-$\pi g\alpha$-continuous but not α-irresolute.*

Corollary 3.20. *If $f : X \to Y$ is m-π-closed α-irresolute bijection, then $f^{-1}(K)$ is $\pi g\alpha$-closed in X for each $\pi g\alpha$-closed set K of Y.*

Proof. It is obtained from Theorem 3.17 and Remark 3.18.

For the composition of pre-$\pi g\alpha$-closed functions, we have the following Theorems.

Theorem 3.21. *Let $f : X \to Y$ and $g : Y \to Z$ be functions. Then the composition $gof : X \to Z$ is pre-$\pi g\alpha$-closed if f is pre-$\pi g\alpha$-closed and g is π-irresolute pre-$\pi g\alpha$-closed bijection.*

Proof. The proof follows immediately from Theorem 3.15.

Theorem 3.22. *Let $f : X \to Y$ and $g : Y \to Z$ be functions and let the composition $gof : X \to Z$ be pre-$\pi g\alpha$-closed. Then the following hold:*

1. *If f is an α-irresolute surjection, then g is pre-$\pi g\alpha$-closed;*
2. *If g is a m-π-closed pre-$\pi g\alpha$-continuous injection, then f is pre-$\pi g\alpha$-closed.*

Proof. (1) Let $K \in \alpha C(Y)$. Since f is α-irresolute and surjective, $f^{-1}(K) \in \alpha C(X)$ and $(gof)(f^{-1}(K)) = g(K)$. Therefore, $g(K)$ is $\pi g\alpha$-closed in Z and hence g is pre-$\pi g\alpha$-closed.

(2) Let $H \in \alpha C(X)$. Then $(gof)(H)$ is $\pi g\alpha$-closed in Z and $g^{-1}((gof)(H)) = f(H)$. By Theorem 3.17, $f(H)$ is $\pi g\alpha$-closed in Y and hence f is pre-$\pi g\alpha$-closed.

The following Lemma is analogous to Lemma 3.11, the straightforward proof is omitted.

Lemma 3.23. *A surjective function $f : X \to Y$ is almost $\pi g\alpha$-closed if and only if for each subset B of Y and each $U \in RO(X)$ containing $f^{-1}(B)$, there exists an $\pi g\alpha$-open set V of Y such that $B \subseteq V$ and $f^{-1}(V) \subseteq U$.*

Corollary 3.24. *If a surjective function $f : X \to Y$ is almost $\pi g\alpha$-closed, then for each π-closed set K of Y and each $U \in RO(X)$ containing $f^{-1}(K)$, there exists $V \in \alpha O(Y)$ such that $K \subseteq V$ and $f^{-1}(V) \subseteq U$.*

Proof. The proof is similar to that of Corollary 3.12.

Recall that a topological space (X, τ) is said to be quasi-normal [24] if for every disjoint π-closed sets A and B of X, there exist disjoint sets $U, V \in \tau$ such that $A \subseteq U$ and $B \subseteq V$.

Definition 3.25. *A topological space (X, τ) is said to be quasi-α-normal if for every disjoint π-closed sets A and B of X, there exist disjoint sets U, $V \in \alpha O(X)$ such that $A \subseteq U$ and $B \subseteq V$.*

Theorem 3.26. *Let $f : X \to Y$ be a π-irresolute almost $\pi g\alpha$-closed surjection. If X is quasi-normal, then Y is quasi-α-normal.*

Proof. Let K_1 and K_2 be any disjoint π-closed sets of Y. Since f is π-irresolute, $f^{-1}(K_1)$ and $f^{-1}(K_2)$ are disjoint π-closed sets of X. By the quasi-normality of X, there exist disjoint open sets U_1 and U_2 such that $f^{-1}(K_i) \subseteq U_i$, where $i = 1, 2$. Now, put $G_i = int(cl(U_i))$ for $i = 1, 2$, then $G_i \in RO(X)$, $f^{-1}(K_i) \subseteq U_i \subseteq G_i$ and $G_1 \cap G_2 = \phi$. By Corollary 3.24, there exists $V_i \in \alpha O(Y)$ such that $K_i \subseteq V_i$ and $f^{-1}(V_i) \subseteq G_i$, $i = 1, 2$. Since $G_1 \cap G_2 = \phi$, f is surjective we have $V_1 \cap V_2 = \phi$. This shows that Y is quasi-α-normal.

Definition 3.27. *A function $f : X \to Y$ is said to be α-open [16] (resp. α-closed [19]), if $f(U) \in \alpha O(Y)$ (resp. $f(U) \in \alpha C(Y)$) for every open (resp. closed) set U of X.*

Definition 3.28. *[13] A subset A of X is said to be $g\alpha$-closed if $\alpha cl(A) \subseteq U$ whenever $A \subseteq U$ and U is α-open in X.*

Definition 3.29. *[12] A function $f : X \to Y$ is said to be $g\alpha$-closed if $f(U)$ is $g\alpha$-closed in Y for every closed set U of X.*

Remark 3.30. *For a function of a topological space, the following hold:*

$$\textbf{closed} \longrightarrow \alpha\textbf{-closed} \longrightarrow g\alpha\textbf{-closed} \longrightarrow \pi g\alpha\textbf{-closed}$$

The reverse implications are not true.

The following four Corollaries are immediate consequences of Theorem 3.26.

Corollary 3.31. *If $f : X \to Y$ is a π-irresolute $\pi g\alpha$-closed surjection and X is quasi-normal, then Y is quasi-α-normal.*

Corollary 3.32. *If $f : X \to Y$ is a π-irresolute $g\alpha$-closed surjection and X is quasi-normal, then Y is quasi-α-normal.*

Corollary 3.33. *If $f : X \to Y$ is a π-irresolute α-closed surjection and X is quasi-normal, then Y is quasi-α-normal.*

Corollary 3.34. *If $f : X \to Y$ is a π-irresolute closed surjection and X is quasi-normal, then Y is quasi-α-normal.*

Definition 3.35. *[4, 17] A function $f : X \to Y$ is said to be pre-α-closed (resp. pre-α-open) if for each $F \in \alpha C(X)$ (resp. $F \in \alpha O(X)$), $f(F) \in \alpha C(Y)$ (resp. $f(F) \in \alpha O(Y)$).*

Remark 3.36. *Every pre-α-closed function is α-closed but not conversely.*

Proof. Let A be a closed set of X. Then A is α-closed set of X. Since f is pre-α-closed, f(A) $\in \alpha$C(Y). Hence f is α-closed.

Example 3.37. *In Example 3.19, f is α-closed but not pre-α-closed.*

Remark 3.38. *Every pre-α-closed function is pre-$\pi g\alpha$-closed but not conversely.*

Proof. Let $F \in \alpha C(X)$. Since f is pre-α-closed, $f(F) \in \alpha C(Y)$. Since α-closed set is $\pi g\alpha$-closed, $f(F)$ is $\pi g\alpha$-closed in Y. Hence f is pre-$\pi g\alpha$-closed.

Example 3.39. *Let $X = Y = \{a, b, c\}$, $\tau = \{\phi, X, \{b\}, \{c\}, \{b, c\}\}$ and $\sigma = \{\phi, Y, \{b\}, \{b, c\}\}$. Let $f : (X, \tau) \to (Y, \sigma)$ be the identity function. Then f is pre-$\pi g\alpha$-closed but not pre-α-closed.*

Theorem 3.40. *Let $f : X \to Y$ be a m-π-closed pre-$\pi g\alpha$-continuous injection. If Y is quasi-α-normal, then X is quasi-α-normal.*

Proof. Let H_1 and H_2 be any disjoint π-closed sets of X. Since f is a m-π-closed injection, $f(H_1)$ and $f(H_2)$ are disjoint π-closed sets of Y. By the quasi-α-normality of Y, there exist disjoint sets $V_1, V_2 \in \alpha O(Y)$ such that $f(H_i) \subseteq V_i$, for $i = 1, 2$. Since f is pre-$\pi g\alpha$-continuous $f^{-1}(V_i)$ and $f^{-1}(V_i)$ are disjoint $\pi g\alpha$-open sets of X and $H_i \subseteq f^{-1}(V_i)$ for $i = 1, 2$. Now, put $U_i = \alpha int(f^{-1}(V_i))$ for $i = 1, 2$. Then $U_i \in \alpha O(X)$, $H_i \subseteq U_i$ and $U_1 \cap U_2 = \phi$. This shows that X is quasi-α-normal.

Corollary 3.41. *If $f : X \to Y$ is a m-π-closed α-irresolute injection and Y is quasi-α-normal, then X is quasi-α-normal.*

Proof. This is an immediate consequence of Theorem 3.40, since every α-irresolute function is pre-$\pi g\alpha$-continuous.

Definition 3.42. *A topological space X is said to be quasi-regular if for each π-closed set F and each point $x \in X{-}F$, there exist disjoint U, $V \in \tau$ such that $x \in U$ and $F \subseteq V$.*

Theorem 3.43. *For a topological space X, the following properties are equivalent:*

1. *X is quasi-regular;*
2. *For each π-open set U in X and each $x \in U$, there exists $V \in \tau$ such that $x \in V \subseteq cl(V) \subseteq U$;*
3. *For each π-open set U in X and each $x \in U$, there exists a clopen set V such that $x \in V \subseteq U$.*

Proof. (1) $\Rightarrow$ (2): Let U be an π-open set of X containing x. Then $X \backslash U$ is a π-closed set not containing x. By (1), there exist disjoint $X \backslash cl(V)$, V $\in \tau$ such that x $\in$ V and X$\backslash$U $\subseteq$ X$\backslash$cl(V). Then we have V $\in \tau$ such that x $\in$ V $\subseteq$ cl(V) $\subseteq$ U.

(2) $\Rightarrow$ (3): Let U be an π-open set of X containing x. By (2), there exists V $\in \tau$ such that x $\in$ V $\subseteq$ cl(V) $\subseteq$ U. Take V = cl(V). Thus V is closed and so V is clopen. Hence we have V is clopen set such that x $\in$ V $\subseteq$ U.

(3) $\Rightarrow$ (1): Let F = X$\backslash$U be a π-closed set not containing x. Then U is an π-open set of X containing x. By (3), there exists a clopen set V such that x $\in$ V $\subseteq$ U. Then there exist disjoint G = X$\backslash$V, V $\in \tau$ such that x $\in$ V and F = X$\backslash$U $\subseteq$ G = X$\backslash$V. Hence X is quasi-regular.

Definition 3.44. *A topological space X is said to be quasi-α-regular if for each π-closed set F and each point $x \in X{-}F$, there exist disjoint U, $V \in \alpha O(X)$ such that $x \in U$ and $F \subseteq V$.*

Theorem 3.45. *For a topological space X, the following properties are equivalent:*

1. *X is quasi-α-regular;*
2. *For each π-open set U in X and each $x \in U$, there exists $V \in \alpha O(X)$ such that $x \in V \subseteq \alpha cl(V) \subseteq U$;*
3. *For each π-open set U in X and each $x \in U$, there exists $V \in \alpha CO(X)$ such that $x \in V \subseteq U$.*

Theorem 3.46. *Let $f : X \to Y$ be an π-irresolute α-open almost $\pi g\alpha$-closed surjection. If X is quasi-regular, then Y is quasi-α-regular.*

Proof. Let $y \in Y$ and V be an π-open neighbourhood of y. Take a point $x \in f^{-1}(y)$. Then $x \in f^{-1}(V)$ and $f^{-1}(V)$ is π-open in X. By the quasi-regularity of X, there exists an π-open set U of X such that $x \in U \subseteq cl(U) \subseteq f^{-1}(V)$. Then $y \in f(U) \subseteq f(cl(U)) \subseteq V$. Also, since U is open set of X and f is α-open, $f(U) \in \alpha O(Y)$. Moreover, since U is β-open, by Theorem 2.1, $cl(U)$ is regular closed set of X. Since f is almost $\pi g\alpha$-closed, $f(cl(U))$ is $\pi g\alpha$-closed in Y. Therefore, we obtain $y \in f(U) \subseteq \alpha cl(f(U)) \subseteq \alpha cl(f(cl(U))) \subseteq V$. It follows from Theorem 3.45 that Y is quasi-α-regular.

Corollary 3.47. *If $f : X \to Y$ is an π-irresolute α-open $\pi g\alpha$-closed surjection and X is quasi-regular, then Y is quasi-α-regular.*

Corollary 3.48. *If $f : X \to Y$ is an π-irresolute α-open α-closed surjection and X is quasi-regular, then Y is quasi-α-regular.*

4 Conclusion

Topology is an area of Mathematics concerned with the properties of space that are preserved under continuous deformations including stretching and bending, but not tearing. By the middle of the 20th century, topology had become a major branch of Mathematics.

Topology as a branch of Mathematics can be formally defined as the study of qualitative properties of certain objects that are invariant under a certain kind of transformation especially those properties that are invariant under a certain kind of equivalence and it is the study of those properties of geometric configurations which remain invariant when these configurations are subjected to one-to-one bicontinuous transformations or homeomorphisms. Topology operates with more general concepts than analysis. Differential properties of a given transformation are nonessential for topology but bicontinuity is essential. As a consequence, topology is often suitable for the solution of problems to which analysis cannot give the answer.

In this paper, the concept of α-open sets introduced by [17] is used to introduce and study pre-$\pi g\alpha$-closed functions. The associated functions of pre-$\pi g\alpha$-closed functions are widely investigated.

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CODING THEORY APPLIED TO KU-ALGEBRAS

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Abstract - The notion of a KU-valued function on a set is introduced and related properties are investigated. Codes generated by KU-valued functions are established and some interesting results are obtained.

Keywords - *KU-valued function, binary block code of KU-valued functions.*

1. Introduction

BCK-algebras form an important class of logical algebras introduced by Iseki [5,6,7] and were extensively investigated by several researchers. The class of all *BCK*-algebras is a quasivariety. Iseki posed an interesting problem (solved by Wro*n*ski [13]) whether the class of *BCK*-algebras is a variety. In connection with this problem, Komori [9] introduced a notion of *BCC*-algebras and Dudek [1] redefined the notion of *BCC*-algebras by using a dual form of the ordinary definition in the sense of Komori. Dudek and Zhang [2] introduced a new notion of ideals in *BCC*-algebras and described connections between such ideals and congruences. C.Prabpayak and U.Leerawat ([11], [12]) introduced a new algebraic structure which is called KU-algebra. They gave the concept of homomorphisms of KU - algebras and investigated some related properties. These algebras form an important class of logical algebras and have many applications to various domains of mathematics, such as, group theory, functional analysis, fuzzy sets theory, probability theory, topology, etc. Coding theory is a very young mathematical topic. It started on the basis of transferring information from one place to another. For instance, suppose we are using electronic devices to transfer information (telephone, television, etc.). Here, information is converted into bits of 1's and 0's and sent through a channel, for example a cable or via satellite. Afterwards, the 1's and

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0's are reconverted into information again. Due to technical problems, one can assume that while the bits are sent through the channel, there is a positive probability p that single bits are being changed. Thus the received bits could be wrong. The idea of coding theory is to give a method of how to convert the information into bits, such that there are no mistakes in the received information, or such that at least some of them are corrected. On this account, encoding and decoding algorithms are used to convert and reconvert these bits properly. One of the recent applications of BCK-algebras was given in the Coding theory [3,8 ,12]. In Coding Theory, a block code is an error-correcting code which encodes data in blocks. In the paper [8], the authors introduced the notion of BCK-valued functions and investigate several properties. Moreover,they established block-codes by using the notion of BCK-valued functions. they show that every finite BCK-algebra determines a block-code constructed a finite binary block-codes associated to a finite BCK-algebra. In [3,12] provided an algorithm which allows to find a BCK-algebra starting from a given binary block code.

In [12] the authors presented some new connections between BCK- algebras and binary block codes.

In this paper, we apply the code theory to KU- algebras and obtain some interesting results.

2. Preliminaries

Now, we will recall some known concepts related to KU-algebra from the literature which will be helpful in further study of this article.

Definition 2.1 [10,11] Algebra(X, $*$, 0) of type (2, 0) is said to be a KU -algebra, if it satisfies the following axioms:

(ku_1) $0*x=x$

(ku_2) $x*y=0 \Rightarrow (y*z)*(x*z)=0, (z*x)*(z*y)=0$

(ku_3) $x*(y*z)=y*(x*z)$

(ku_4) $(x*y)*[(y*z)*(x*z)]=0$,

Example 2.2 Let X = {0, 1, 2, 3, 4} in which * is defined by the following table

*	0	1	2	3	4
0	0	1	2	3	4
1	0	0	2	3	4
2	0	1	0	3	3
3	0	0	2	0	2
4	0	0	0	0	0

It is easy to show that X is KU-algebra.

In a KU-algebra, the following identities are true : If we put in (ku_4) y = x = 0 we get $(0 * 0) * [(0 * z) * (0 * z)] = 0$ and it follows that : (Ku_5) $z * z = 0$, if we put y = 0 in (ku_4), we get (p_1) $z * (x * z) = 0$.

A subset S of KU-algebra X is called sub-algebra of X if $x * y \in S$, whenever $x, y \in S$.

A non empty subset A of a KU-algebra X is called a KU-ideal of X if it satisfies the following conditions:

(I_1) $0 \in A$,
(I_2) $x * (y * z) \in A$, $y \in A$ implies $x * z \in A$, for all x , y , $z \in X$.

Lemma 2.3 [9] In a KU-algebra (X, $*$, 0), the following hold:

$$x \leq y \text{ imply } y * z \leq x * z .$$

Lemma 2.4 [10] If X is KU-algebra then $y * [(y * x) * x] = 0$.

3. KU-valued Functions

In what follows let A and X denote a nonempty set and a KU-algebra respectively, unless otherwise specified.

Definition 3.1 A mapping $\tilde{A} : A \to X$ is called a KU-valued function (briefly, KU-function) on A.

Definition 3.2 A cut function of $\tilde{A}$, for $q \in X$ is defined to be a mapping

$\tilde{A}_q : A \to \{0,1\}$ such that $(\forall x \in A)\tilde{A}_q(x) = 1 \Leftrightarrow \tilde{A}(x) * q = 0$.

Obviously, $\tilde{A}_q$ is the characteristic function of the following subset of A, called a cut subset or a q-cut of $\tilde{A} : \tilde{A}_q(x) := \{ x \in A : \tilde{A}(x) * q = 0\}$.

Example 3.3 Let $A = \{x, y, z\}$ and let $X = \{0, a, b, c, d\}$ is a KU-algebra with the following Cayley table:

*	0	a	b	c	d
0	0	a	b	c	d
a	0	0	b	b	a
b	0	a	0	a	d
c	0	0	0	0	a
d	0	0	b	b	0

The function $\widetilde{A}: A \to X$ given by $\widetilde{A} = \begin{pmatrix} x & y & z \\ a & b & c \end{pmatrix}$ is a KU-function on A, and its cut subsets are

$$A_0 = \Phi \quad , \quad A_a = \{x\} \ , \ A_b = \{y\} \, , \, A_c = A \ , \, A_d = \{x\}.$$

Lemma 3.4 On KU-algebra (*X; *;* 0). We define a binary relation $\leq$ on *X* by putting $x \leq y$ if and only if $y*x = 0$. Then (X; $\leq$) is a partially ordered set and 0 is its smallest element.

Proof. Let X be KU-algebra $\forall a,b,c \in X$, we have

1. $\leq$ is reflexive as $a \leq a.$
2. if $a \leq b, b \leq a$,then $a = b$. Hence $\leq$ is anti-symmetric.
3. if $a \leq b, b \leq c$,then we want to prove that $a \leq c.$

Since $c * a = 0 * (c * a) = (c * b) * (c * a) \leq b * a = 0$,we have $c * a = 0 \Rightarrow a \leq c,$ then $\leq$ is transitive. Hence $(X, \leq)$ is partial order set.

Proposition 3.5 Every KU-function $\widetilde{A}: A \to X$ on A is represented by the infimum of the set $\{ \ q \in X, \ \ A_q(x) = 1\}$, that is $\forall x \in X \ \ : \widetilde{A}(x) = \inf \{ \ q \in X \, , \widetilde{A}_q(x) = 1 \ \}$.

Proof. For any $x \in A$. Let $\widetilde{A}(x) = q \in X$, then $\widetilde{A}(x) * q = 0$ *and so* $\widetilde{A}_q(x) = 1 \quad \forall q \in X$.

Assume that $\widetilde{A}_r(x) = 1$ *for* $r \in X$, then $\widetilde{A}(x) * r = 0 = q * r$, *i.e* $r \leq q$.

Since $q \in \{ \ r \in X \ , \widetilde{A}_r(x) = 1\}$, for $x \in A$, $r \in X$,it follows that

$$\widetilde{A}(x) = q = \inf \{ \ r \in X, \widetilde{A}_r(x) = 1\}.$$

This completes the proof.

Proposition 3.6 Let $\widetilde{A}: A \to X$ be a KU-function on A . If $q * p = 0$ for all $p \, , q \in X$, we get $A_p \subseteq A_q$.

Proof. Let $p \, , q \in X$, be such that $q * p = 0$ and $x \in A_p$, then $\widetilde{A}(x) * p = 0$

Using (ku_1) and (ku_2), we have

$$0 = \overbrace{(q * p) * (\widetilde{A}(x) * P)}^{(\text{KU}_2)} = (\widetilde{A}(x) * q)$$, and so $x \in A_q$. Therefore $A_p \subseteq A_q$.

This completes the proof.

Proposition 3.7 Let $\widetilde{A}: A \to X$ be KU-function on A. Then

$$(\forall x, y \in A)(\widetilde{A}(x) \neq \widetilde{A}(y) \Leftrightarrow A_{\widetilde{A}(x)} \neq A_{\widetilde{A}(y)}$$

Proof. (1) The sufficiency is obvious. Assume that $A_{\widetilde{A}(x)} \neq A_{\widetilde{A}(y)}$ for all $x, y \in A$. Then

$$A_{\widetilde{A}(y)} * A_{\widetilde{A}(x)} \neq 0 \ \ or \ \ A_{\widetilde{A}(x)} * A_{\widetilde{A}(y)} \neq 0 .$$

Thus

$$A_{\widetilde{A}(x)} = \{ \ z \in A, \ \widetilde{A}(z) * \widetilde{A}(x) = 0 \ \} \neq \{ \ z \in A, \widetilde{A}(z) * \widetilde{A}(y) = 0 \ \} = A_{\widetilde{A}(y)}$$

Corollary 3.8 Let $\widetilde{A}: A \to X$ be KU-function on A. Then

$$(\forall x, y \in A)(\widetilde{A}(x) * \widetilde{A}(y) = 0 \Leftrightarrow A_{\widetilde{A}(y)} \subseteq A_{\widetilde{A}(x)}) .$$

Proof. Straightforward.

For a KU-function $\widetilde{A}: A \to X$, consider the following sets:

$$A_x = \{ \ A_q : q \in X \}, \quad \widetilde{A}_x = \{ \ \widetilde{A}_q : q \in X \} \ .$$

Proposition 3.9 Let $\widetilde{A}: A \to X$ be KU-function on A. Then

$$(\forall Y \subseteq X) \quad A_{\inf(\ q:\ q \in Y)} = \bigcup \{ A_q : q \in Y \}.$$

Proof. Let $(Y \subseteq X)$, $x \in A_{\inf(\ q:\ q \in Y)}$. We have

$x \in A_{\inf(\ q:\ q \in Y)} \Leftrightarrow \widetilde{A}(x) * \inf \{ q : q \in Y \} = 0 \Leftrightarrow (\forall r \in Y)(\widetilde{A}(x) * r = 0) \Leftrightarrow (\forall r \in Y)(x \in A_r) \Leftrightarrow$
$x \in \bigcup \{ A_q : q \in Y \}$.This completes the proof.

Corollary 3.10 Let $\widetilde{A}: A \to X$ be KU-function on A, where X is a bounded KU-algebra, then

$$\forall S \subseteq X \ \ , \ A_{\inf(\ q:\ q \in S)} = \bigcup \{ A_q : q \in S \} \ .$$

Corollary 3.11 Let $\widetilde{A}: A \to X$ be KU-function on A, assume that for any $Y \subseteq X$, there exists a infimum of Y such that ($\forall p, q \in Y$) , we have $A_p \bigcup A_q \in A_X$.

The following example shows that the converse of the corollary 3.10 may not true in general.

Example 3.12. Let $A=\{x,y\}$ *be a set and let* $X=\{0,a,b,c,d\}$ be a KU-algebra with the following Cayley table:

*	0	a	b	c	d
0	0	a	b	c	d
a	0	0	b	b	a
b	0	a	o	a	d
c	0	0	0	0	a
d	0	0	b	b	0

The function $\tilde{A}:A\to X$ given by $\tilde{A}=\begin{pmatrix} x & y \\ a & b \end{pmatrix}$ is a KU-function on A, then

	x	y
	a	b
$\tilde{A}_0$	0	0
$\tilde{A}_a$	1	0
$\tilde{A}_b$	0	1
$\tilde{A}_c$	1	1
$\tilde{A}_d$	1	0

And its cut subsets are

$$A_0=\Phi,\ A_a=\{x\},\ A_b=\{y\}\ ,\ A_c=\{x,y\}\ ,\ A_d=\{x\}$$

Note that $A_a\cup A_b=\{x\}\cup\{y\}\notin A_X$, but $\inf\{a,b\}$ does exists in X .

Proposition 3.13 Let $\tilde{A}:A\to X$ be KU-function on A, then

$$\bigcap\{A_q|q\in X\}=A$$

Proof. Obviously, $\bigcap\{A_q|q\in X\}\subseteq A$ .For every $x\in A$, let $\tilde{A}(x)=q\in X$ .Then $x\in A_q$ and hence

$x\in\bigcap\{A_q|q\in X\}$. Thus $A\subseteq\bigcap\{A_q|q\in X\}$.

Therefore the result is valid.

Proposition 3.14 Let $\tilde{A}:A\to X$ be KU-function on A, then

$$(\forall x\in A)(\bigcup\{A_q\,|\,x\in A_q\}\in A_X)$$

Proof. Note that for any $x \in A, x \in A_q \Leftrightarrow \tilde{A}_q(x) = 1$,

From Proposition 3.7 we get the following

$$\bigcup \{A_q | x \in A_q\} = \bigcup \{A_q | \tilde{A}_q(x) = 1\} = A_{\inf\{q|\tilde{A}_q(x)=1\}} \in A_q .$$

This completes the proof.

Let $\tilde{A} : A \to X$ be KU-function on A and Θ be a binary operation on X defined by

$\forall p, q \in X (p\Theta q \Leftrightarrow A_p = A_q)$. Then Θ is clearly an equivalence relation on X.

Let $\tilde{A}(A) = \{ q \in X | \tilde{A}(x) = q \text{ for some } x \in A\}$ and for $q \in X$, $(q] = \{ x \in X | x * q = 0 \}$.

Proposition 3.15 For a KU-function $\tilde{A} : A \to X$ on A, we have

$$\forall p, q \in X (p\Theta q \Leftrightarrow (p] \cup \tilde{A}(A) = (q] \cup \tilde{A}(A))$$

Proof. We have $p\Theta q \Leftrightarrow A_p = A_q$

$$\Leftrightarrow (\forall x \in A)\{\tilde{A}(x) * p = 0 \Leftrightarrow \tilde{A}(x) * q = 0\}$$
$$\Leftrightarrow \{x \in A | \tilde{A}(x) \in (p]\} = \{x \in A | \tilde{A}(x) \in (q]\}$$
$$\Leftrightarrow (p] \cup \tilde{A}(A) = (q] \cup \tilde{A}(A).$$

This completes the proof.

Example 3.16 Let $X = \{a_n; n = 1,2,3,......,9\}$ and define a binary operation $*$ on X as follows $(\forall a_i, a_j \in X)$ $(a_i * a_j = a_k)$, where $k = \dfrac{j}{(i,j)}$ *and* (i, j) is the least common divisor of i *and* j . Then $(X;*,a_i)$ is a KU-algebra. Its Cayley table is as follows:

$*$	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9
a_1	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9
a_2	a_1	a_1	a_3	a_2	a_5	a_3	a_7	a_4	a_9
a_3	a_1	a_2	a_1	a_4	a_5	a_2	a_7	a_8	a_3
a_4	a_1	a_1	a_3	a_1	a_5	a_3	a_7	a_2	a_9
a_5	a_1	a_2	a_3	a_4	a_1	a_6	a_7	a_8	a_9
a_6	a_1	a_1	a_1	a_2	a_5	a_1	a_7	a_4	a_3
a_7	a_1	a_2	a_3	a_4	a_5	a_6	a_1	a_8	a_9
a_8	a_1	a_1	a_3	a_1	a_5	a_3	a_7	a_1	a_9
a_9	a_1	a_2	a_1	a_4	a_5	a_2	a_7	a_8	a_1

Let $A=\{a,b,c,d,e\}$ and $\tilde{A}:A\rightarrow X$ be a KU-function defined by

$$\tilde{A}=\begin{pmatrix} a & b & c & d & e \\ a_4 & a_6 & a_7 & a_1 & a_2 \end{pmatrix}.$$

Then

*	a a_4	b a_6	c a_7	d a_1	e a_2
$\tilde{A}_{a_1}$	0	0	0	1	0
$\tilde{A}_{a_2}$	0	0	0	1	1
$\tilde{A}_{a_3}$	0	0	0	1	0
$\tilde{A}_{a_4}$	1	0	0	1	1
$\tilde{A}_{a_5}$	0	0	0	1	0
$\tilde{A}_{a_6}$	0	1	0	1	1
$\tilde{A}_{a_7}$	0	0	1	1	0
$\tilde{A}_{a_8}$	1	0	0	1	1
$\tilde{A}_{a_9}$	0	0	0	1	0

and cut sets of Ã are as follows:

$$\tilde{A}_{a_1}=\tilde{A}_{a_3}=\tilde{A}_{a_5}=\tilde{A}_{a_9}=\{d\},\tilde{A}_{a_2}=\{d,e\},\tilde{A}_{a_4}=\tilde{A}_{a_8}=\{a,d,e\},\tilde{A}_{a_6}=\{b,d,e\},\tilde{A}_{a_7}=\{c,d\}.$$

4. Codes Generates by KU-functions

Let $x/\Theta=\{\ y\in A;\ \ x\Theta y\ \}$; for any $x\in A$, x/Θ is called equivalence class containing x.

Lemma 4.1 Let $\tilde{A}:A\rightarrow X$ be a KU- function on A. For every $x\in A$, we have $\tilde{A}(x)=\inf\{x/\Theta\}$, that is $\tilde{A}(x)$ the least element of the Θ to which it belongs.

Proof. Straightforward.

Let $A=\{1,2,3,\ldots..,n\}$ and X be a finite KU-algebra. Then every KU-function

$\tilde{A}:A\rightarrow X$ on A determines a binary block code V of length n in the following way: To every x/Θ, where $x\in A$, there corresponds a codeword $V_x=x_1x_2\ldots..x_n$

Such that

$$x_i = x_j \Leftrightarrow \tilde{A}_x(i) = j \;\; for \;\; i \in A \;\; and \;\; j \in \{0,1\}.$$

Let $V_x = x_1 x_2 .....x_n$, $V_y = y_1 y_2 .....y_n$ be two code words belonging to a binary block-code V .

Define an order relation $\leq_c$ on the set of code words belonging to a binary block- code V as follows: $V_x \leq_c V_y \Leftrightarrow x_i \leq y_i \;\; for \;\; i = 1,2,....,n$ …… (4.1)

Example 4.2 Let $X = \{0, a, b, c\}$ be a KU-algebra with the following Cayley table**:**

*	0	a	b	c
0	0	a	b	c
a	0	0	a	c
b	0	0	0	c
c	0	a	b	0

Let $\tilde{A} : X \to X$ be a KU-function on X given by

$\tilde{A} = \begin{pmatrix} 0 & a & b & c \\ 0 & a & b & c \end{pmatrix}$. Then

$\tilde{A}_x$	0	a	b	c
$\tilde{A}_0$	1	0	0	0
$\tilde{A}_a$	1	1	0	0
$\tilde{A}_b$	1	1	1	0
$\tilde{A}_c$	1	0	0	1

$V = \{1000, 1100, 1110, 1001\}$. See Figure (1)

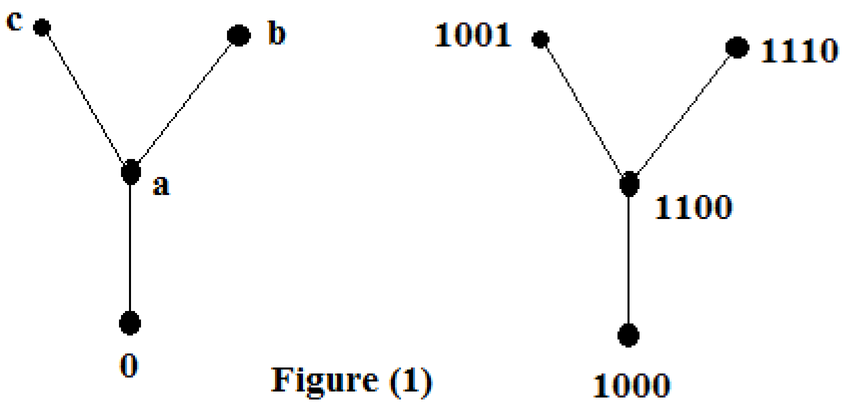

Figure (1)

$(X, \leq)$ $(V, \leq)$

Generally, we have the following theorem.

Theorem 4.3 Every finite KU-algebra X determines a block-code V such that $(X,\leq)$ is isomorphic to $(X,\leq_c)$.

Proof. Let $X=\{a_i; i=1,2,3,......,n\}$ be a finite KU-algebra in which a_1 is the least element and let $\tilde{A}: X \to X$ be identify KU-function on X .The decomposition of $\tilde{A}$ provides a family $\{\tilde{A}_q | q \in X\}$ which is the desired code under the order

$$V_x \leq_c V_y \Leftrightarrow x_i \leq y_i \quad for \quad i=1,2,....,n$$

Let $f: X \to \{\tilde{A}_q; q \in X\}$ be a function defined by $f(q)=\tilde{A}_q$ for all $q \in X$ .By lemma 4.1, every Θ class contains exactly one element .So, f is one to one. Let $x,y \in X$ be such that $y*x=a_1$ *i.e* $x \leq y$. Then $A_x \subseteq A_y$ (by Proposition 3.5), which means that $\tilde{A}_x \subseteq \tilde{A}_y$.

Therefore f is an isomorphism.

This completes the proof.

Example 4.4 Consider a KU-algebra $X=\{a_n; n=1,2,3,......,9\}$ which is considered in example 3.15.

Let $\tilde{A}: X \to X$ be a KU-function on X given by

$$\tilde{A}=\begin{pmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 & a_8 & a_9 \\ a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 & a_8 & a_9 \end{pmatrix}.$$

Then

*	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9
$\tilde{A}_{a_1}$	1	0	0	0	0	0	0	0	0
$\tilde{A}_{a_2}$	1	1	0	0	0	0	0	0	0
$\tilde{A}_{a_3}$	1	0	1	0	0	0	0	0	0
$\tilde{A}_{a_4}$	1	1	0	1	0	0	0	0	0
$\tilde{A}_{a_5}$	1	0	0	0	1	0	0	0	0
$\tilde{A}_{a_6}$	1	1	1	0	0	1	0	0	0
$\tilde{A}_{a_7}$	1	0	0	0	0	0	1	0	0
$\tilde{A}_{a_8}$	1	1	0	1	0	0	0	1	0
$\tilde{A}_{a_9}$	1	0	1	0	0	0	0	0	1

Thus

V= {100000000, 110000000, 101000000, 110100000, 100010000, 11100100,100000100, 110100010, 101000001}. See Figure (2)

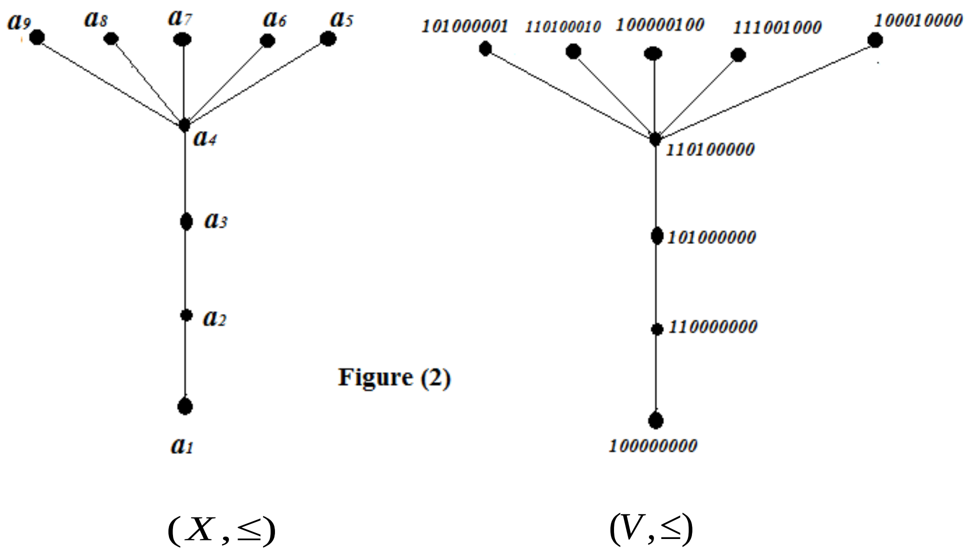

Figure (2)

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FUZZY OSTROWSKI TYPE INEQUALITIES FOR $(\alpha, m)-$CONVEX FUNCTIONS

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Abstract − Let $f : I \to \mathbb{R}$, where $I \subseteq \mathbb{R}$ is an interval, be a mapping differentiable in the interior I° of I, and let $a, b \in I^\circ$ with $a < b$. If $|f'(x)| \le M$ for all $x \in [a, b]$, then the following inequality holds:

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)dt \right| \le M(b-a)\left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2}\right] \tag{1}$$

for all $x \in [a, b]$ .This inequality is well known in the literature as the Ostrowski inequality. In this paper, we established new Ostrowski type inequalities for (α, m) −convex functions via fuzzy Riemann integrals.

Keywords − *$(\alpha, m)-$convex function, Ostrowski inequality, Fuzzy Riemann integral.*

1 Introduction

In 1938, A. M. Ostrowski (see [1]) proved the following inequality, estimating the absolute value of deviation of a differentiable function by its integral mean as:

$$\left| \frac{1}{b-a}\int_a^b f(y)\,dy - f(x) \right| \le \left(\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2}\right)(b-a)\,||f'||_\infty$$

where $f : I \subset \mathbb{R} \to \mathbb{R}$ be a diferentiable mapping on (a, b) whose derivative $f' : (a, b) \to \mathbb{R}$ is bounded on (a, b) that is $||f'||_\infty = \sup_{t\in(a,b)} |f'| < \infty$.

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[*]*Corresponding Author.*

Since that time when Ostrowski proved this inequality, many mathematicians have been working on it and have been applying it in numerical analysis and probability, etc. For some applications of Ostrowski's inequality see [2]-[5] and for recent results and generalizations concerning Ostrowski's inequality see [2]-[9].

Let I be an interval in $\mathbb{R}$. Then $f : I \to \mathbb{R}$ is said to be convex if for all $x, y \in I$ and $\alpha \in [0, 1]$

$$f(\alpha x + (1-\alpha)y) \leq \alpha f(x) + (1-\alpha) f(y)$$

(see [10], Page1). Geometrically, this means that if A, B, and C are three distinct points on the graph of with B between A and C, then B is on or below chord AB.

In [11] , Miheşan defined (α, m) -convexity as in the following:

The function $f : [0, b] \to \mathbb{R}$, $b > 0$, is said to be $(\alpha, m)-$convex, where $(\alpha, m) \in [0, 1]^2$, if one has

$$f(tx + m(1-t)y) \leq t^{\alpha} f(x) + m(1-t^{\alpha}) f(y)$$

for all $x, y \in [0, b]$ and $t \in [0, 1]$.

Since fuzziness is a natural reality different than randomness and determinism, Anastassiou extends Ostrowski type inequalities into the fuzzy setting in 2003 [12].

The concepts of fuzzy Riemann integrals were introduced by Wu [13]. Fuzzy Riemann integral is a closed interval whose end points are the classical Riemann integrals.

2 Notations and Preliminaries

In this section we point out some basic definitions and notations which would help us in this work, we begin with:

Definition 2.1. [13] If $u : \mathbb{R} \to [0, 1]$ satisfies the following properties, then u is called fuzzy number.

i. u is normal (i.e, there exists an $x_0 \in \mathbb{R}$ such that $u(x_0) = 1$)

ii. u is a convex fuzzy set, ie., $u(x\lambda + (1-\lambda)y) \geq \min\{u(x), u(y)\}$, for any $x, y \in \mathbb{R}$, $\lambda \in [0, 1]$. (u is called a convex fuzzy subset.)

iii. u is upper semi continuous on $\mathbb{R}$, i.e, $\forall x_0 \in \mathbb{R}$ and $\forall \epsilon > 0$, $\exists$ neighborhood $V(x_0) : u(x) \leq u(x_0) + \epsilon$, $\forall x \in V(x_0)$.

iv. The set $[u]^0 = \overline{\{x \in \mathbb{R} : u(x) > 0\}}$ is compact where $\overline{A}$ denotes the closure of A.

Denote the set of all fuzzy numbers with $\mathbb{R}_{\mathcal{F}}$. For $\alpha \in (0, 1]$ and $u \in \mathbb{R}_{\mathcal{F}}$, $[u]^{\alpha} = \{x \in \mathbb{R} : u(x) \geq \alpha\}$. Then, from $(1) - (4)$ it follows that the $\alpha-$level set $[u]^{\alpha}$ is a closed interval for all $\alpha \in [0, 1]$. Moreover, $[u]^{\alpha} = [u_-^{(\alpha)}, u_+^{(\alpha)}]$ for all $\alpha \in [0, 1]$, where $u_-^{(\alpha)} \leq u_+^{(\alpha)}$ and $u_-^{(\alpha)}, u_+^{(\alpha)} \in \mathbb{R}$, i.e, $u_-^{(\alpha)}$ and $u_+^{(\alpha)}$ are the endpoints of $[u]^{\alpha}$.

Definition 2.2. [14] Let $u, v \in \mathbb{R}_{\mathcal{F}}$ and $k \in \mathbb{R}$. Then, the addition and scalar multiplication are defined by the equations, respectively.

i. $[u \oplus v]^\alpha = [u]^\alpha + [v]^\alpha$

ii. $[k \odot u]^\alpha = k[u]^\alpha$

for all $\alpha \in [0,1]$ where $[u]^\alpha + [v]^\alpha$ means the usual addition of two intervals (as subsets of $\mathbb{R}$) and $k[u]^\alpha$ means the usual product between a scalar and a subset of $\mathbb{R}$.

Proposition 2.3. [15, 16] Let $u, v \in \mathbb{R}_{\mathcal{F}}$ and $k \in \mathbb{R}$. Then, the following properties are valid.

i. $1 \odot u = u$

ii. $u \oplus v = v \oplus u$

iii. $k \odot u = u \odot k$

iv. $[u]^{\alpha_1} \subseteq [u]^{\alpha_2}$ whenever $0 \leq \alpha_2 \leq \alpha_1 \leq 1$

v. For any α_n converging increasingly to $\alpha \in (0,1]$, $\bigcap_{n=1}^{\infty} [u]^{\alpha_n} = [u]^\alpha$.

Definition 2.4. [14] Let $D : \mathbb{R}_{\mathcal{F}} \times \mathbb{R}_{\mathcal{F}} \to \mathbb{R}_+ \cup \{0\}$ be a function defined by the equation

$$D(u,v) := \sup_{\alpha \in [0,1]} \max \left\{ |u_-^{(\alpha)} - v_-^{(\alpha)}|, |u_+^{(\alpha)} - v_+^{(\alpha)}| \right\}$$

for all $u, v \in \mathbb{R}_{\mathcal{F}}$. Then, D is a metric on $\mathbb{R}_{\mathcal{F}}$.

Now, using the results of [13, 16], for all $u, v, v, w, e \in \mathbb{R}_{\mathcal{F}}$ and $k \in \mathbb{R}$ we have that

i. $(\mathbb{R}_{\mathcal{F}}, D)$ is a complete metric space

ii. $D(u \oplus w, v \oplus w) = D(u,v)$

iii. $D(k \odot u, k \odot v) = |k| d(u,v)$

iv. $D(u \oplus v, w \oplus e) = D(u,w) + D(v,e)$

v. $D(u \oplus v, \tilde{0}) \leq D(u, \tilde{0}) + D(v, \tilde{0})$

vi. $D(u \oplus v, w) \leq D(u,w) + D(v, \tilde{0})$

where $\tilde{0} \in \mathbb{R}_{\mathcal{F}}$ is defined $\tilde{0}(x) = 0$ for all $x \in \mathbb{R}$.

Definition 2.5. [14] Let $x, y \in \mathbb{R}_{\mathcal{F}}$. If there exists a $z \in \mathbb{R}_{\mathcal{F}}$ such that $x = y \oplus z$, then we call z the H-difference of x and y, denoted by $z = x \ominus y$.

Definition 2.6. [14] Let $T := [x_0, x_0 + \beta] \subseteq \mathbb{R}$, with $\beta > 0$. A function $f : T \to \mathbb{R}_{\mathcal{F}}$ is H-differentiable at $x \in T$ if there exists a $f'(x) \in \mathbb{R}_{\mathcal{F}}$ such that the limits (with respect to the metric D)

$$\lim_{h \to 0^+} \frac{f(x+h) \ominus f(x)}{h}, \lim_{h \to 0^+} \frac{f(x) \ominus f(x-h)}{h}$$

exist and are equal to $f'(x)$. We call f' the derivative or H-derivative of f at x. If f is H-differentiable at any $x \in T$, we call f differentiable or H-differentiable and it has H-derivative over T the function f'.

We use a particular case of the Fuzzy Henstock integral ($\delta(x) = \frac{\delta}{2}$) introduced in [14], Definition 2.1. That is,

Definition 2.7. [18] Let $f : [a, b] \to \mathbb{R}_{\mathcal{F}}$. We say that f is Fuzzy-Riemann integrable to $I \in \mathbb{R}_{\mathcal{F}}$ if for any $\varepsilon > 0$, there exists $\delta > 0$ such that for any division $P = \{[u, v]; \xi\}$ of $[a, b]$ with the norms $\Delta(P) < \delta$, we have

$$D\left(\sum_P^{*}(v-u) \odot f(\xi, I)\right) < \varepsilon$$

where $\sum^{*}$ denotes the fuzzy summation. We choose to write

$$I := (FR)\int_a^b f(x)dx$$

We also call an f as above (FR)-integrable.

For some recent results connected with Fuzzy-Riemann integrals, see ([17]).
The main purpose of the this paper is to establish fuzzy Ostrowski type inequalties for fuzzy Riemann integral and (α, m)-convex functions.

3 Main Results

In order to establish our main results we need the following lemma.

Lemma 3.1. Let $f : I \subset \mathbb{R} \to \mathbb{R}_{\mathcal{F}}$ be differentiable mapping on I° where $ma, mb \in I$ with $ma < mb$. If $f' \in C_F[ma, mb] \cap L_F[ma, mb]$, then we have the equality for differentiable function as follow:

$$\begin{aligned} & \frac{1}{b-a} \odot (FR)\int_{ma}^{mb} f(x)\,dx \oplus \frac{(x-ma)^2}{b-a} \odot (FR)\int_0^1 t \odot f'(tx + m(1-t)a)\,dt \\ = \; & m \odot f(x) \oplus \frac{(mb-x)^2}{b-a} \odot (FR)\int_0^1 t \odot f'(tx + m(1-t)b)\,dt \end{aligned}$$

for $x \in (ma, mb)$.

Proof. By integration by parts and using properties of α-cut of fuzzy numbers, we have following identities

$$\begin{aligned} & \left[\frac{(mb-x)^2}{b-a}\odot(FR)\int_0^1 t\odot f'(tx+m(1-t)b)dt\right]^\alpha \qquad (2)\\ = & \frac{(mb-x)^2}{b-a}\left[\left(\int_0^1 tf'(tx+m(1-t)b)dt\right)_\alpha^-, \left(\int_0^1 tf'(tx+m(1-t)b)dt\right)_\alpha^+\right] \\ = & \frac{mb-x}{a-b}\left[\left(f(x)-\frac{1}{x-mb}\int_{mb}^x f(u)du\right)_\alpha^-, \left(f(x)-\frac{1}{x-mb}\int_{mb}^x f(u)du\right)_\alpha^+\right] \\ = & \frac{mb-x}{a-b}\left[f(x)\oplus\frac{1}{mb-x}\odot(FR)\int_{mb}^x f(u)du\right]^\alpha \end{aligned}$$

and

$$\begin{aligned} & \left[\frac{(x-ma)^2}{b-a}\odot(FR)\int_0^1 t\odot f'(tx+m(1-t)a)dt\right]^\alpha \qquad (3)\\ = & \frac{(x-ma)^2}{b-a}\left[\left(\int_0^1 tf'(tx+m(1-t)a)dt\right)_\alpha^-, \left(\int_0^1 tf'(tx+m(1-t)a)dt\right)_\alpha^+\right] \\ = & \frac{x-ma}{b-a}\left[\left(f(x)-\frac{1}{ma-x}\int_{ma}^x f(u)du\right)_\alpha^-, \left(f(x)-\frac{1}{ma-x}\int_{ma}^x f(u)du\right)_\alpha^+\right] \\ = & \frac{x-ma}{b-a}\left[f(x)\oplus\frac{1}{ma-x}\odot(FR)\int_{ma}^x f(u)du\right]^\alpha . \end{aligned}$$

By adding (2) and (3) we have

$$\begin{aligned} & \left[\frac{(mb-x)^2}{b-a}\odot(FR)\int_0^1 t\odot f'(tx+m(1-t)b)dt\right]^\alpha \\ & +\left[\frac{(x-ma)^2}{b-a}\odot(FR)\int_0^1 t\odot f'(tx+m(1-t)b)dt\right]^\alpha \\ & =\left[m\odot f(x)\oplus\frac{1}{a-b}\odot(FR)\int_{ma}^{mb} f(u)du\right]^\alpha \end{aligned}$$

which the proof is completed. □

Theorem 3.2. Let $f : I\subset\mathbb{R}\to\mathbb{R}_{\mathcal{F}}$ be differentiable mapping on I such that $f'\in C_F[ma,mb]\cap L_F[ma,mb]$, where $ma, mb\in I$ with $ma<mb$. If $D(f'(x),0)$ is (α,m) −convex on $[ma,mb]$ for $(\alpha,m)\in[0,1]\times[0,1]$ and $D\left(f'(x),\tilde{0}\right)\le M$, then the following inequality holds:

$$\begin{aligned} & D\left(m\odot f(x), \frac{1}{b-a}\odot(FR)\int_{ma}^{mb} f(x)\,dx\right) \\ \le\; & M\frac{\alpha m+2}{2(\alpha+2)}\left(\frac{(x-ma)^2+(mb-x)^2}{b-a}\right) \end{aligned}$$

for each $x\in[ma,mb]$.

Proof. From Lemma 3.1

$$
\begin{aligned}
& D\left(m \odot f(x), \frac{1}{b-a} \odot (FR)\int_{ma}^{mb} f(x)\,dx\right) \\
= \; & D\left(m \odot f(x) \oplus \frac{(mb-x)^2}{b-a} \odot (FR)\int_0^1 t \odot f'(tx+m(1-t)b)\,dt,\right. \\
& \frac{1}{b-a} \odot (FR)\int_{ma}^{mb} f(x)\,dx \oplus \frac{(mb-x)^2}{b-a} \\
& \left. \odot (FR)\int_0^1 t \odot f'(tx+m(1-t)b)\,dt\right) \\
= \; & D\left(\frac{1}{b-a} \odot (FR)\int_{ma}^{mb} f(x)\,dx \oplus \frac{(x-ma)^2}{b-a}\right. \\
& \odot (FR)\int_0^1 t \odot f'(tx+m(1-t)a)\,dt, \\
& \frac{1}{b-a} \odot (FR)\int_{ma}^{mb} f(x)\,dx \oplus \frac{(mb-x)^2}{b-a} \\
& \left. \odot (FR)\int_0^1 t \odot f'(tx+m(1-t)b)\,dt\right) \\
= \; & D\left(\frac{(x-ma)^2}{b-a} \odot (FR)\int_0^1 tf'(tx+m(1-t)a)\,dt,\right. \\
& \left. \frac{(mb-x)^2}{b-a} \odot (FR)\int_0^1 tf'(tx+m(1-t)b)\,dt\right) \\
\leq \; & D\left(\frac{(x-ma)^2}{b-a} \odot (FR)\int_0^1 tf'(tx+m(1-t)a)\,dt, \widetilde{0}\right) \\
& + D\left(\frac{(mb-x)^2}{b-a} \odot (FR)\int_0^1 tf'(tx+m(1-t)b)\,dt, \widetilde{0}\right) \\
= \; & \frac{(x-ma)^2}{b-a} D\left((FR)\int_0^1 tf'(tx+m(1-t)a)\,dt, \widetilde{0}\right) \\
& + \frac{(mb-x)^2}{b-a} D\left((FR)\int_0^1 tf'(tx+m(1-t)b)\,dt, \widetilde{0}\right) \\
\leq \; & \frac{(x-ma)^2}{b-a} \int_0^1 tD\left(f'(tx+m(1-t)a), \tilde{0}\right) dt \\
& + \frac{(mb-x)^2}{b-a} \int_0^1 tD\left(f'(tx+m(1-t)b), \tilde{0}\right) dt \qquad (4)
\end{aligned}
$$

Since $D\left(f'(x),\tilde{0}\right)$ is (α,m) −convex and $D\left(f'(x),\tilde{0}\right)\leq M$, then we have

$$\begin{aligned} D\left(f'(tx+m(1-t)a),\tilde{0}\right) &\leq t^{\alpha}D\left(f'(x),\widetilde{0}\right)+m(1-t^{\alpha})D\left(f'(a),\tilde{0}\right)\\ &\leq M\left[t^{\alpha}+m(1-t^{\alpha})\right] \end{aligned} \tag{5}$$

$$\begin{aligned} D\left(f'(tx+m(1-t)b),\tilde{0}\right) &\leq t^{\alpha}D\left(f'(x),\widetilde{0}\right)+m(1-t^{\alpha})D\left(f'(b),\tilde{0}\right)\\ &\leq M\left[t^{\alpha}+m(1-t^{\alpha})\right] \end{aligned} \tag{6}$$

By using (5)and (6) in (4), we get

$$\begin{aligned} &D\left(m\odot f(x),\frac{1}{b-a}\odot(FR)\int_{ma}^{mb}f(x)\,dx\right)\\ \leq\ & M\frac{\alpha m+2}{2(\alpha+2)}\left(\frac{(x-ma)^2+(mb-x)^2}{b-a}\right) \end{aligned}$$

□

Theorem 3.3. Let $f:I\subset\mathbb{R}\to\mathbb{R}_{\mathcal{F}}$ be differentiable mapping on I such that $f'\in C_F[ma,mb]\cap L_F[ma,mb]$, where $ma,mb\in I$ with $ma<mb$. If $\left[D\left(f'(x),0\right)\right]^q$ is (α,m) −convex on $[ma,mb]$ for $(\alpha,m)\in[0,1]\times[0,1]$, $p,q>1,\frac{1}{p}+\frac{1}{q}=1$ and $D\left(f'(x),\widetilde{0}\right)\leq M$, then the following inequality holds:

$$\begin{aligned} &D\left(m\odot f(x),\frac{1}{b-a}\odot(FR)\int_{ma}^{mb}f(x)\,dx\right)\\ \leq\ & \left(\frac{1}{p+1}\right)^{1/p}M\left(\frac{1+\alpha m}{\alpha+1}\right)\left(\frac{(x-ma)^2+(mb-x)^2}{b-a}\right) \end{aligned}$$

for each $x\in[ma,mb]$.

Proof. From Lemma 3.1 and Hölder's inequality, we have

$$\begin{aligned} &D\left(m\odot f(x),\frac{1}{b-a}\odot(FR)\int_{ma}^{mb}f(x)\,dx\right)\\ =\ & D\left(m\odot f(x)\oplus\frac{(mb-x)^2}{b-a}\odot(FR)\int_0^1 tf'(tx+m(1-t)b)\,dt,\right.\\ &\frac{1}{b-a}\odot(FR)\int_{ma}^{mb}f(x)\,dx\oplus\frac{(mb-x)^2}{b-a}\\ &\left.\odot(FR)\int_0^1 tf'(tx+m(1-t)b)\,dt\right)\\ =\ & D\left(\frac{1}{b-a}\odot(FR)\int_{ma}^{mb}f(x)\,dx\oplus\frac{(x-ma)^2}{b-a}\odot(FR)\int_0^1 tf'(tx+m(1-t)a)\,dt,\right. \end{aligned}$$

$$\frac{1}{b-a}\odot(FR)\int_{ma}^{mb}f(x)\,dx\oplus\frac{(mb-x)^2}{b-a}\odot(FR)\int_0^1 tf'(tx+m(1-t)b)\,dt\Bigg)$$

$$= D\left(\frac{(x-ma)^2}{b-a}\odot(FR)\int_0^1 tf'(tx+m(1-t)a)\,dt,\right.$$

$$\left.\frac{(mb-x)^2}{b-a}\odot(FR)\int_0^1 tf'(tx+m(1-t)b)\,dt\right)$$

$$\begin{aligned}
&\leq D\left(\frac{(x-ma)^2}{b-a}\odot(FR)\int_0^1 tf'(tx+m(1-t)a)\,dt,\widetilde{0}\right)\\
&\quad +D\left(\frac{(mb-x)^2}{b-a}\odot(FR)\int_0^1 tf'(tx+m(1-t)b)\,dt,\widetilde{0}\right)\\
&= \frac{(x-ma)^2}{b-a}D\left((FR)\int_0^1 tf'(tx+m(1-t)a)\,dt,\widetilde{0}\right)\\
&\quad +\frac{(mb-x)^2}{b-a}D\left((FR)\int_0^1 tf'(tx+m(1-t)b)\,dt,\widetilde{0}\right)\\
&\leq \frac{(x-ma)^2}{b-a}\int_0^1 tD\left(f'(tx+m(1-t)a),\tilde{0}\right)dt\\
&\quad +\frac{(mb-x)^2}{b-a}\int_0^1 tD\left(f'(tx+m(1-t)b),\tilde{0}\right)dt\\
&\leq \frac{(x-ma)^2}{b-a}\left(\int_0^1 t^p dt\right)^{1/p}\left(\int_0^1\left[D\left(f'(tx+m(1-t)a),\tilde{0}\right)\right]^q dt\right)^{1/q}\\
&\quad +\frac{(mb-x)^2}{b-a}\left(\int_0^1 t^p dt\right)^{1/p}\left(\int_0^1\left[D\left(f'(tx+m(1-t)b),\tilde{0}\right)\right]^q dt\right)^{1/q}
\end{aligned}$$

Since $\left[D\left(f'(x),\tilde{0}\right)\right]^q$ is $(\alpha,m)-$convex and $D\left(f'(x),\tilde{0}\right)\leq M$, then we have

$$\begin{aligned}
\left[D\left(f'(tx+m(1-t)a),\tilde{0}\right)\right]^q &\leq t^\alpha D\left(f'(x),\tilde{0}\right)^q+m(1-t^\alpha)D\left(f'(a),\tilde{0}\right)^q\\
&\leq M^q\left[t^\alpha+m(1-t^\alpha)\right]
\end{aligned}\tag{7}$$

$$\begin{aligned}
\left[D\left(f'(tx+m(1-t)b),\tilde{0}\right)\right]^q &\leq t^\alpha D\left(f'(x),\tilde{0}\right)^q+m(1-t^\alpha)D\left(f'(b),\tilde{0}\right)^q\\
&\leq M^q\left[t^\alpha+m(1-t^\alpha)\right]
\end{aligned}\tag{8}$$

By using (5)and (6) in (4), we get

$$\begin{aligned}
&D\left(m\odot f(x),\frac{1}{b-a}\odot(FR)\int_{ma}^{mb}f(x)\,dx\right)\\
\leq\ &\frac{(x-ma)^2}{b-a}\left(\frac{1}{p+1}\right)^{1/p}\left(M^q\int_0^1\left[t^\alpha+m\left(1-t^\alpha\right)\right]dt\right)^{1/q}\\
&+\frac{(mb-x)^2}{b-a}\left(\frac{1}{p+1}\right)^{1/p}\left(M^q\int_0^1\left[t^\alpha+m\left(1-t^\alpha\right)\right]dt\right)^{1/q}\\
=\ &M\left(\frac{1}{p+1}\right)^{1/p}\left(\frac{1+\alpha m}{\alpha+1}\right)^{1/q}\left(\frac{(x-ma)^2+(mb-x)^2}{b-a}\right)
\end{aligned}$$

□

Theorem 3.4. Let $f : I\subset\mathbb{R}\to R_F$ be differentiable mapping on I such that $f'\in C_F[ma,mb]\cap L_F[ma,mb]$, where $ma,mb\in I$ with $ma<mb$.If $\left[D\left(f'(x),\tilde{0}\right)\right]^q$ is (α,m) −convex on $[ma,mb]$ for $(\alpha,m)\in[0,1]\times(0,1]$, $q\in[1,\infty)$ and $D\left(f'(x),\tilde{0}\right)\leq M$, then the following inequality holds:

$$\begin{aligned}
&D\left(m\odot f(x),\frac{1}{b-a}\odot(FR)\int_{ma}^{mb}f(x)\,dx\right)\\
\leq\ &M\left(\frac{2+\alpha m}{\alpha+2}\right)^{1/q}\frac{(x-ma)^2+(mb-x)^2}{2(b-a)}
\end{aligned}$$

Proof. From Lemma 3.1 and using the well-known power-mean inequality, we get

$$\begin{aligned}
&D\left(m\odot f(x),\frac{1}{b-a}\odot(FR)\int_{ma}^{mb}f(x)\,dx\right)\\
=\ &D\left(m\odot f(x)\oplus\frac{(mb-x)^2}{b-a}\odot(FR)\int_0^1 tf'(tx+m(1-t)b)\,dt,\right.\\
&\frac{1}{b-a}\odot(FR)\int_{ma}^{mb}f(x)\,dx\oplus\frac{(mb-x)^2}{b-a}\\
&\left.\odot(FR)\int_0^1 tf'(tx+m(1-t)b)\,dt\right)\\
=\ &D\left(\frac{1}{b-a}\odot(FR)\int_{ma}^{mb}f(x)\,dx\oplus\frac{(x-ma)^2}{b-a}\right.\\
&\odot(FR)\int_0^1 tf'(tx+m(1-t)a)\,dt,\frac{1}{b-a}\odot(FR)\int_{ma}^{mb}f(x)\,dx
\end{aligned}$$

$$
\begin{aligned}
& \oplus \frac{(mb-x)^2}{b-a} \odot (FR) \int_0^1 tf'(tx+m(1-t)b)\,dt \Bigg) \\
= \; & D\Bigg(\frac{(x-ma)^2}{b-a} \odot (FR) \int_0^1 tf'(tx+m(1-t)a)\,dt, \\
& \frac{(mb-x)^2}{b-a} \odot (FR) \int_0^1 tf'(tx+m(1-t)b)\,dt \Bigg) \\
\leq \; & D\left(\frac{(x-ma)^2}{b-a} \odot (FR) \int_0^1 tf'(tx+m(1-t)a)\,dt, \widetilde{0} \right) \\
& + D\left(\frac{(mb-x)^2}{b-a} \odot (FR) \int_0^1 tf'(tx+m(1-t)b)\,dt, \widetilde{0} \right) \\
= \; & \frac{(x-ma)^2}{b-a} D\left((FR) \int_0^1 tf'(tx+m(1-t)a)\,dt, \widetilde{0} \right) \\
& + \frac{(mb-x)^2}{b-a} D\left((FR) \int_0^1 tf'(tx+m(1-t)b)\,dt, \widetilde{0} \right)
\end{aligned}
$$

$$
\begin{aligned}
\leq \; & \frac{(x-ma)^2}{b-a} \int_0^1 tD\left(f'(tx+m(1-t)a), \tilde{0}\right) dt \\
& + \frac{(mb-x)^2}{b-a} \int_0^1 tD\left(f'(tx+m(1-t)b), \tilde{0}\right) dt \\
\leq \; & \frac{(x-ma)^2}{b-a} \left(\int_0^1 tdt \right)^{1-1/q} \left(\int_0^1 t\left[D\left(f'(tx+m(1-t)a), \tilde{0}\right)\right]^q dt \right)^{1/q} \\
& + \frac{(mb-x)^2}{b-a} \left(\int_0^1 tdt \right)^{1-1/q} \left(\int_0^1 t\left[D\left(f'(tx+m(1-t)b), \tilde{0}\right)\right]^q dt \right)^{1/q}
\end{aligned}
$$

$\left[D\left(f'(x), \tilde{0}\right)\right]^q$ is (α, m) −convex and $D\left(f'(x), \tilde{0}\right) \leq M$, so we know

$$
\begin{aligned}
& \int_0^1 t\left[D\left(f'(tx+m(1-t)a), \tilde{0}\right)\right]^q dt \\
\leq \; & \int_0^1 t[t^\alpha \left(D\left(f'(x), \tilde{0}\right)\right)^q + m(1-t^\alpha)\left(D\left(f'(a), \tilde{0}\right)\right)^q]dt \\
\leq \; & \frac{M^q}{\alpha+2}\left(1+\frac{\alpha m}{2}\right)
\end{aligned}
$$

$$
\begin{aligned}
& \int_0^1 t\left[D\left(f'(tx+m(1-t)b), \tilde{0}\right)\right]^q dt \\
\leq \; & \int_0^1 t[t^\alpha \left(D\left(f'(x), \tilde{0}\right)\right)^q + m(1-t^\alpha)\left(D\left(f'(b), \tilde{0}\right)\right)^q]dt \\
\leq \; & \frac{M^q}{\alpha+2}\left(1+\frac{\alpha m}{2}\right)
\end{aligned}
$$

Therefore, we know

$$\begin{aligned} & D\left(m \odot f(x), \frac{1}{b-a} \odot (FR) \int_{ma}^{mb} f(x)\, dx\right) \\ \leq \quad & M\left(\frac{2+\alpha m}{\alpha+2}\right)^{1/q} \frac{(x-ma)^2+(mb-x)^2}{2(b-a)} \end{aligned}$$

□

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COMMUTATIVE SOFT INTERSECTION GROUPS

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Abstract − In this paper, we first present the soft sets and soft intersection groups. We then define commutative soft sets, commutative soft intersection groups and investigate their properties.

Keywords − *Soft sets, Soft intersection groups, Commutative soft intersection groups.*

1 Introduction

In 1999, Molodtsov [23] defined the notion of soft sets to deal with uncertainties. After that the operations of soft sets have been studied Maji *et al.* [22], Ali *et al.* [3] and Çağman *et al.* [13]. By using these operations, some researchers have applied soft sets theory to many different areas, such as decison making [6, 13, 15], algebras [2, 4, 8, 14], topology [9, 25], fuzzy sets [5, 10, 11, 29] and matrix theory [7, 12].

To start the algebraic structures on soft set theory, Aktaş and Çağman [2] defined soft groups in 2007. Afterward, soft intersection groups [8, 19], soft rings [1, 21], soft fields and modules [4], soft semirings [14], soft BCK/BCI-algebras [16], soft p-ideals of soft BCI-algebras [17], soft WS-algebras [24] and soft intersection near-rings [27, 28] have been studied. In this paper, we first present the soft sets and soft intersection groups. We then define commutative soft sets, commutative soft intersection groups and investigate their properties.

2 Soft Sets

In this section, we present basic definitions of soft sets and their operations. For more detailed explanations of the soft sets, we refer to the earlier studies [13, 22, 23].

Definition 2.1. [23] Let U and E be two non empty set and $P(U)$ is the power set of U. Then, a soft set f over U is a function defined by

$$f : E \to P(U),$$

where U refer to an initial universe and E is a set of parameters.

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In other words, the soft set is a parametrized family of subsets of the set U. Every set $f(e)$, $e \in E$, from this family may be considered as the set of e-elements of the soft set f, or as the set of e-approximate elements of the soft set.

As an illustration, let us consider the following examples.

A soft set f describes the attractiveness of the houses which Mr. X is going to buy.

U - is the set of houses under consideration.

E - is the set of parameters. Each parameter is a word or a sentence.

E = {expensive; beautiful; wooden; cheap; in the green surroundings; modern; in good repair; in bad repair }

In this case, to define a soft set means to point out *expensive* houses, *beautiful* houses, and so on.

It is worth noting that the sets $f(e)$ may be arbitrary. Some of them may be empty, some may have nonempty intersection.

A soft set over U can be represented by the set of ordered pairs

$$f = \{(x, f(x)) : x \in E\}$$

Note that the set of all soft sets over U will be denoted by $S_E(U)$. From here on, "soft set" will be used without over U.

Definition 2.2. [13] Let $f \in S_E(U)$. Then,

- f is called an empty soft set, denoted by Φ_E, if $f(x) = \phi$, for all $x \in E$.
- f is called a universal soft set, denoted by $f_{\widetilde{E}}$, if $f(x) = U$, for all $x \in E$.
- The set $\mathrm{Im}(f) = \{f(x) : x \in E\}$ is called image of f.

Definition 2.3. [13] Let $f, g \in S_E(U)$. Then,

- f is a soft subset of g, denoted by $f \widetilde{\subseteq} g$, if $f(x) \subseteq g(x)$ for all $x \in E$.
- f and g are soft equal, denoted by $f = g$, if and only if $f(x) = g(x)$ for all $x \in E$.

Definition 2.4. [13] Let $f, g \in S_E(U)$. Then,

- the set $(f \widetilde{\cup} g)(x) = f(x) \cup g(x)$ for all $x \in E$ is called union of f and g.
- the set $(f \widetilde{\cap} g)(x) = f(x) \cap g(x)$ for all $x \in E$ is called intersection of f and g.
- the set $f^c(x) = U \setminus f(x)$ for all $x \in E$ is called complement of f.

3 Soft Intersection Groups

In this section, we introduce the concepts of soft intersection groups (soft int-groups) and soft product with their basic properties. For more detailed explanations of the soft int-groups, we refer to the earlier studies [8, 19].

Definition 3.1. [8] Let G be a group and $f \in S_G(U)$. Then, f is called a soft intersection groupoid over U if $f(xy) \supseteq f(x) \cap f(y)$ for all $x, y \in G$ and is called a soft intersection group over U if it satisfies $f(x^{-1}) = f(x)$ for all $x \in G$ as well.

Throughout this paper, G denotes an arbitrary group with identity element e and the set of all soft int-groups with parameter set G over U will be denoted by $S_G^g(U)$, unless otherwise stated. For short, instead of "f is a soft int-group with the parameter set G over U" we say "f is a soft int-group".

Theorem 3.2. [8] Let $f \in S_G^g(U)$. Then, $f(e) \supseteq f(x)$ for all $x \in G$.

Definition 3.3. [8] Let $A, B \subseteq E$, φ be a function from A into B and $f, g \in S_E(U)$. Then, soft image $\varphi(f)$ of f under φ is defined by

$$\varphi(f)(y) = \begin{cases} \cup\{f(x) : x \in A, \varphi(x) = y\}, & \text{for } y \in \varphi(A) \\ \emptyset, & \text{otherwise} \end{cases}$$

and soft pre-image (or soft inverse image) of g under φ is $\varphi^{-1}(g) = f$ such that $f(x) = g(\varphi(x))$ for all $x \in A$.

Theorem 3.4. [19] Let $f \in S_G^*(U)$ and $x, y \in G$. If $f(xy^{-1}) = f(e)$, then $f(x) = f(y)$.

Definition 3.5. [19] Let G be a group and $f, g \in S_G(U)$. Then, soft product $(f * g)$ of f and g is defined by

$$(f * g)(x) = \bigcup\{f(u) \cap g(v) : uv = x,\ u, v \in G\}$$

and inverse f^{-1} of f is defined by

$$f^{-1}(x) = f(x^{-1})$$

for all $x \in G$.

Definition 3.6. [20] Let G be a group. If $f \in S_G^g(U)$, then the set $N(f)$ defined by

$$N(f) = \{x \in G : f(xy) = f(yx) \text{ for all } y \in G\}$$

is called normalizer of f in G.

4 Commutative Soft Intersection Groups

In this section, we first define the notion of commutative soft sets and then define commutative soft intersection groups. We also investigate their related properties.

Definition 4.1. Let H be a semigroup and $f \in S_H(U)$. Then the set

$$Z(f) = \{x \in H : y, z \in H, f(xy) = f(yx), f(xyz) = f(yxz)\}$$

is called centralizer of f in H.

Here, if the semigroup H has right identity then the equality $f(xyz) = f(yxz)$ is reduced to $f(xy) = f(yx)$ for $z = e$, so the condition $f(xy) = f(yx)$ is redundant.

Definition 4.2. Let H be a semigroup and $f \in S_H(U)$. Then f is called commutative in H if $Z(f) = H$.

Definition 4.3. Let H be a group and f be a soft intersection group. Then f is called commutative soft intersection group in H if $Z(f) = H$.

Theorem 4.4. Let G be a group and $f \in S_G(U)$. Then, $Z(G) \subseteq Z(f) \subseteq N(f)$.

Proof. The proof is straightforward. □

Now, we can give an exempla for $Z(G) \neq Z(f) \neq N(f)$ as follows.

Example 4.5. Let $D_3 = \{e, \sigma, \sigma^2, \tau, \tau\sigma, \tau\sigma^2\}$ be the symmetric group and a soft set $f \in S_{D_3}(U)$ is defined as,

$$f(x) = \begin{cases} \alpha_0, & \text{for } x \in \{e, \tau\} \\ \alpha_1, & \text{otherwise} \end{cases}$$

for $\alpha_1, \alpha_0 \in P(U)$.

Now we show that $Z(f) \neq N(f)$.

$$\left.\begin{array}{l} f(\sigma\tau) = f(\tau\sigma^2) = \alpha_1 \\ f(\tau\sigma) = \alpha_1 \end{array}\right\} \Rightarrow f(\sigma\tau) = f(\tau\sigma)$$

$$\left.\begin{array}{l} f\left(\tau\sigma^2\right)=\alpha_1 \\ f\left(\sigma^2\tau\right)=f\left(\tau\sigma\right)=\alpha_1 \end{array}\right\} \Rightarrow f\left(\tau\left(\tau\sigma\right)\right)=f\left(\left(\tau\sigma\right)\tau\right)$$

$$\left.\begin{array}{l} f\left(\tau\left(\tau\sigma\right)\right)=f\left(\sigma\right)=\alpha_1 \\ f\left(\left(\tau\sigma\right)\tau\right)=f\left(\sigma^2\right)=\alpha_1 \end{array}\right\} \Rightarrow f\left(\tau\sigma^2\right)=f\left(\sigma^2\tau\right)$$

$$\left.\begin{array}{l} f\left(\tau\left(\tau\sigma^2\right)\right)=f\left(\sigma^2\right)=\alpha_1 \\ f\left(\left(\tau\sigma^2\right)\tau\right)=f\left(\sigma\right)=\alpha_1 \end{array}\right\} \Rightarrow f\left(\tau\left(\tau\sigma^2\right)\right)=f\left(\left(\tau\sigma^2\right)\tau\right)$$

so $\tau \in N(f)$. But,

$$\left.\begin{array}{l} f\left(\tau\sigma\sigma^2\right)=f\left(\tau\right)=\alpha_o \\ f\left(\sigma\tau\sigma^2\right)=f\left(\tau\sigma^2\sigma^2\right)=f\left(\tau\sigma\right)=\alpha_1 \end{array}\right\} \Rightarrow f\left(\tau\sigma\sigma^2\right)\neq f\left(\sigma\tau\sigma^2\right)$$

so $\tau \notin Z(f)$. Thus $Z(f) \neq N(f)$.

Theorem 4.6. Let H be a semigroup and $f \in S_H(U)$. Then,

$$x \in Z(f) \Leftrightarrow f(xy_1y_2...y_n) = f(y_1xy_2...y_n) = ... = f(y_1y_2...y_nx)$$

for all $y_1, y_2, ..., y_n \in H$.

Proof. Proof is by induction on n. Suppose $x \in Z(f)$. Then, for all $y_{1,}y \in H$

$$f(xy_1y_2) = f(y_1xy_2)$$

by the definition $Z(f)$. Assume,

$$f(xy_1y_2 \dots y_n) = f(y_1xy_2...y_n) = ... = f(y_1y_2...y_nx)$$

for all $y_{1,}y_{2,}..., y_n \in H$. Then,

$$f(xy_1y_2...(y_ny_{n+1})) = f(y_1xy_2...(y_ny_{n+1})) = ... = f(y_1y_2...(y_ny_{n+1})x) \quad (1)$$

for all $y_1, y_2, ..., y_n, y_{n+1} \in H$. This can be done for any successive two y's in (1). So the proof is completed by hypothesis. □

Theorem 4.7. Let H be a semigroup and $f \in S_H(U)$. Then, f is commutative in H if and only if $x_{1,}x_{2,}..., x_n \in H$ and $f(x_1x_2 \cdots x_n) = f(x_{\sigma(1)}x_{\sigma(2)} \cdots x_{\sigma(n)})$ for all $n \in N$ and for any permutation σ of $\{1, 2, ..., n\}$.

Proof. The proof is easy consequence of Theorem 4.6 □

Theorem 4.8. Let H be a semigroup and $f \in S_H(U)$. Then,

1. if $Z(f)$ is nonempty, then $Z(f)$ is a subsemigroup of H.
2. if H is a group, then $Z(f)$ is a normal subgroup of H.

Proof. 1. Let $x_1, x_2 \in Z(f)$. Then for all $y, z \in H$, we have

$$\begin{aligned} f((x_1x_2)yz) &= f(x_1(x_2y)z) \\ &= f((x_2y)x_1z) \\ &= f(x_2(yx_1)z) \\ &= f((yx_1)x_2z) \\ &= f(y(x_1x_2)z) \end{aligned}$$

by Lemma 4.6 and clearly $f((x_1x_2)y) = f(y(x_1x_2))$. Hence $x_1x_2 \in Z(f)$. Thus $Z(f)$ is a subsemigroup of H If $Z(f)$ is nonempty.

2. Suppose H is a group. Then $Z(f)$ is nonempty since $e \in Z(f)$. If $x \in Z(f)$, then

$$\begin{aligned} f(x^{-1}yz) &= f(x^{-1}y(xx^{-1})z) \\ &= f((x^{-1}y)x(x^{-1}z)) \\ &= f(x(x^{-1}y)(x^{-1}z)) \\ &= f(yx^{-1}z) \end{aligned}$$

for all $y, z \in H$ and so $x^{-1} \in Z(f)$. Hence $Z(f) \leq H$.
Next, let $x \in Z(f)$ and $x \in H$. Then for all $y, z \in H$,

$$\begin{aligned} f((g^{-1}xg)yz) &= f(g^{-1}x(gyz)) \\ &= f(xg^{-1}(gyz)) \\ &= f(xyz) \\ &= f(xy(g^{-1}g)z) \\ &= f(y(g^{-1}xg)z) \end{aligned}$$

by Lemma 4.6 and so $g^{-1}xg \in Z(f)$. Thus $Z(f) \lhd H$, if H is a group. □

Theorem 4.9. Let G and H be two semigroups, $\varphi : G \to H$ be an epimorphism and $f \in S_G(U)$. Then,

$$\varphi(Z(f)) \subseteq Z(\varphi(f)).$$

Proof. Let $x \in \varphi(Z(f))$. Then, there exists $u \in Z(f)$ such that $\varphi(u) = x$. So for all $y \in H$,

$$\begin{aligned} \varphi(f)(xy) &= \cup\{f(a) : \varphi(a) = xy, a \in G\} \\ &= \cup\{f(uv) : a = uv, \varphi(v) = y \text{ and } a, v \in G\} \\ &= \cup\{f(vu) : b = vu, \varphi(v) = y \text{ and } v, b \in G\} \\ &= \cup\{f(b) : \varphi(b) = yx \text{ and } b \in G\} \\ &= \varphi(f)(yx) \end{aligned}$$

Similarly, for all $y, z \in H$, we obtain

$$\begin{aligned} \varphi(f)(xyz) &= \cup\{f(a) : \varphi(a) = xyz, a \in G\} \\ &= \cup\{f(uvw) : a = uvw, \varphi(v) = y, \varphi(w) = z \text{ and } v, w \in G\} \\ &= \cup\{f(vuw) : b = vuw, \varphi(v) = y, \varphi(w) = z \text{ and } v, w \in G\} \\ &= \cup\{f(b) : \varphi(b) = yxz\} \\ &= \varphi(f)(yxz) \end{aligned}$$

Thus $x \in Z(\varphi(f))$ and the result follows. □

Theorem 4.10. Let G and H be two semigroups, $\varphi : G \to H$ be an epimorphism and $f \in S_H(U)$. Then,

$$\varphi^{-1}(Z(f)) = Z(\varphi^{-1}(f)).$$

Proof. Let $x \in \varphi^{-1}(Z(f))$. Then for all $y, z \in G$,

$$\begin{aligned} \left(\varphi^{-1}(f)\right)(xyz) &= f(\varphi(xyz)) \\ &= f(\varphi(x)\varphi(y)\varphi(z)) \\ &= f(\varphi(y)\varphi(x)\varphi(z)) \\ &= f(\varphi(yxz)) \\ &= \left(\varphi^{-1}(f)\right)(yxz) \end{aligned}$$

and we have,

$$\left(\varphi^{-1}(f)\right)(xyz) = \left(\varphi^{-1}(f)\right)(yxz). \tag{2}$$

Similarly, $\left(\varphi^{-1}(f)\right)(xy) = \left(\varphi^{-1}(f)\right)(yx)$ and so $x \in Z\left(\varphi^{-1}(f)\right)$. Hence $\varphi^{-1}(Z(f)) \subseteq Z\left(\varphi^{-1}(f)\right)$.

On the other hand, let $x \in Z\left(\varphi^{-1}(f)\right)$ and $\varphi(x) = u$. Then for all $v, w \in H$,

$$\begin{aligned} f(uvw) &= f(\varphi(x)\varphi(y)\varphi(z)) \\ &= f(\varphi(xyz)) \\ &= \left(\varphi^{-1}(f)\right)(xyz) \\ &= \left(\varphi^{-1}(f)\right)(yxz) \ \text{(by 2)} \\ &= f(\varphi(yxz)) \\ &= f(\varphi(y)\varphi(x)\varphi(z)) \\ &= f(vuw) \end{aligned}$$

where $y, z \in G$ are such that $\varphi(y) = v$ and $\varphi(z) = w$. Similarly, $f(uv) = f(vu)$. Thus $u \in Z(f)$, so $x \in \varphi^{-1}(Z(f))$. Hence $Z(\varphi^{-1}(f)) \subseteq \varphi^{-1}(Z(f))$ and the result follows. $\square$

Theorem 4.11. Let G and H be two groups, $\varphi : G \to H$ be an epimorphism and $f \in S_G(U)$. Then, if f is commutative in G, then $\varphi(f)$ is commutative in H.

Proof. Let $x \in H$. Then there exists $u \in G$ such that $\varphi(u) = x$.

$$\begin{aligned} \varphi(f)(xyz) &= \cup\{f(a) : \varphi(a) = xyz\} \\ &= \cup\{f(uvw) : v, w \in G,\ \varphi(v) = y,\ \varphi(w) = z\} \\ &= \cup\{f(vuw) : v, w \in G,\ \varphi(v) = y,\ \varphi(w) = z\} \\ &= \cup\{f(b) : \varphi(b) = yxz\} \\ &= \varphi(f)(yxz) \end{aligned}$$

for all $y, z \in H$.
Similarly $\varphi(f)(xy) = \varphi(f)(yx)$.

So $x \in Z(\varphi(f))$ and $H \subseteq Z(\varphi(f))$. Thus $H = Z(\varphi(f))$ and $\varphi(f)$ is commutative in H by Definition 4.2. $\square$

The next example shows that converse of Theorem 4.11 do not hold.

Example 4.12. Let $D_4 = \{e, \sigma, \sigma^2, \sigma^3, \tau, \tau\sigma, \tau\sigma^2, \tau\sigma^3\}$ be the dihedral group, $N = \{e, \sigma^2\}$ and $\varphi : D_4 \longrightarrow D_4/N$ be the naturel homomorfizm. Let $f \in S_{D_4}(U)$ as,

$$f(x) = \begin{cases} \alpha_0, & \text{for } x \in \{e, \tau\} \\ \alpha_1, & \text{otherwise} \end{cases}$$

for $\alpha_0, \alpha_1 \in P(U)$. D_4/N is a commutative group. Then,

$$D_4/N = Z(\varphi(f))$$

so $\varphi(f)$ commutative in D_4/N. But for $\sigma, \sigma^3 \in D_4$,

$$\left.\begin{array}{l} f(\sigma(\tau\sigma)) = f(\tau) = \alpha_0 \\ f((\tau\sigma)\sigma) = f(\tau\sigma^2) = \alpha_1 \end{array}\right\} \Rightarrow f(\sigma(\tau\sigma)) \neq f((\tau\sigma)\sigma)$$

$$\left.\begin{array}{l} f(\sigma^3(\tau\sigma^3)) = f(\tau) = \alpha_0 \\ f((\tau\sigma^3)\sigma^3) = f(\tau\sigma^2) = \alpha_1 \end{array}\right\} \Rightarrow f(\sigma^3(\tau\sigma^3)) \neq f((\tau\sigma^3)\sigma^3)$$

so $\sigma, \sigma^3 \notin Z(f)$. Thus

$$D_4 \neq Z(f).$$

That is, f is not commutative in G.

Theorem 4.13. Let G and H be two semigroups, $\varphi : G \to H$ be an epimorphism and $f \in S_H(U)$. Then, if f is commutative in H, then $\varphi^{-1}(f)$ is commutative in G.

Proof. Let $x \in \varphi^{-1}(H) = G$. Then for all $y, z \in G$,

$$\begin{aligned} \varphi^{-1}(f)(xyz) &= f(\varphi(xyz)) \\ &= f(\varphi(x)\varphi(y)\varphi(z)) \\ &= f(\varphi(y)\varphi(x)\varphi(z)) \\ &= f(\varphi(yxz)) \\ &= (\varphi^{-1}(f))(yxz) \end{aligned}$$

Similarly, $\varphi^{-1}(f)(xy) = (\varphi^{-1}(f))(yx)$. So $x \in Z(\varphi^{-1}(f))$ and $G \subseteq Z(\varphi^{-1}(f))$. Thus $G = Z(\varphi^{-1}(f))$ and so $\varphi^{-1}(f)$ is commutative in G by Definition 4.2. $\square$

Theorem 4.14. Let $f \in S_G^*(U)$. Then the set defined by

$$T = \{x \in G\ : f(xyx^{-1}y^{-1}) = f(e) \text{ for all } y \in G\}$$

is equal to $Z(f)$.

Proof. Let $x \in T$. Then, for any $y \in G$, we have

$$\begin{aligned} f(e) &= f\left(xyx^{-1}y^{-1}\right) \\ &= f\left((xy)(yx)^{-1}\right) \end{aligned}$$

and $f(xy) = f(yx)$ by Theorem 3.4. Now, for all $y, z \in G$, we have

$$\begin{aligned} f\left((xyz)(yxz)^{-1}\right) &= f\left(xyzz^{-1}x^{-1}y^{-1}\right) \\ &= f\left(xyx^{-1}y^{-1}\right) \\ &= f(e) \end{aligned}$$

and by Theorem 3.4, we obtain $f(xyz) = f(yxz)$ and so $x \in Z(f)$. Therefore, $T \subseteq Z(f)$.

Conversely, if $x \in Z(f)$, then for all $y \in G$

$$\begin{aligned} f\left(xyx^{-1}y^{-1}\right) &= f\left(yxx^{-1}y^{-1}\right) \\ &= f(e) \end{aligned}$$

by Lemma 4.6. Thus $x \in T$ and so $Z(f) \subseteq T$. Hence $Z(f) = T$. □

Theorem 4.15. Let H be a semigroup and $f, g \in S_H(U)$. Then,

$$Z(f) \cap Z(g) \subseteq Z\left(f \widetilde{\cap} g\right)$$

Proof. Let $x \in Z(f) \cap Z(g)$. Then $x \in Z(f)$ and $x \in Z(g)$. For all $y \in H$,

$$\begin{aligned} \left(f \widetilde{\cap} g\right)(xy) &= f(xy) \cap g(xy) \\ &= f(yx) \cap g(yx) \\ &= \left(f \widetilde{\cap} g\right)(yx) \end{aligned}$$

and for all $y, z \in H$,

$$\begin{aligned} \left(f \widetilde{\cap} g\right)(xyz) &= f(xyz) \cap g(xyz) \\ &= f(yxz) \cap g(yxz) \\ &= \left(f \widetilde{\cap} g\right)(yxz) \end{aligned}$$

Thus $x \in Z\left(f \widetilde{\cap} g\right)$. □

Theorem 4.16. Let H be a semigroup and $f, g \in S_H(U)$. If f and g are commutative, then $f \widetilde{\cap} g$ is commutative.

Proof. The proof is straightforward. □

Theorem 4.17. Let $f, g \in S_G^*(U)$ such that $f(e) = g(e)$. Then,

$$Z(f) \cap Z(g) = Z\left(f \widetilde{\cap} g\right).$$

Proof. By Lemma 4.14, for all $y \in G$,

$$\begin{aligned} x &\in Z\left(f \widetilde{\cap} g\right) \\ &\Longleftrightarrow \left(f \widetilde{\cap} g\right)(e) = \left(f \widetilde{\cap} g\right)\left(xyx^{-1}y^{-1}\right) \\ &\Longleftrightarrow f(e) = g(e) = \left(f \widetilde{\cap} g\right)(e) = f\left(xyx^{-1}y^{-1}\right) \cap g\left(xyx^{-1}y^{-1}\right) \\ &\Longleftrightarrow f(e) = f\left(xyx^{-1}y^{-1}\right) \text{ and } g(e) = g\left(xyx^{-1}y^{-1}\right) \\ &\Longleftrightarrow x \in Z(f) \text{ and } x \in Z(g) \\ &\Longleftrightarrow x \in Z(f) \cap Z(g) \end{aligned}$$

Thus, $Z(f) \cap Z(g) = Z\left(f \widetilde{\cap} g\right)$. □

Theorem 4.18. Let $f, g \in S_G^*(U)$ such that $f(e) = g(e)$. If f and g are commutative in G if and only if $f \widetilde{\cap} g$ is commutative in G.

Proof. The proof is straightforward. □

Theorem 4.19. If $f, g \in S_G(U)$, then $Z(f)\,Z(g) \subseteq\ Z(f*g)$.

Proof. Let $x_1 \in Z(f)$ and $x_2 \in Z(g)$. Then for all $y, z \in G$

$$\begin{aligned}
(f*g)((x_1x_2)\,yz) &= \cup\{f(a)\cap g(b) : ab = x_1x_2yz,\ a,b\in G\}\\
&= \cup\left\{f\left(x_1x_2yzb^{-1}\right)\cap g(b) : b\in G\right\}\\
&= \cup\left\{f\left(x_2yx_1zb^{-1}\right)\cap g(b) : b\in G\right\}\\
&= \cup\{f(c)\cap g(b) : cb = x_2yx_1z,\ c,b\in G\}\\
&= \cup\left\{f(c)\cap g\left(c^{-1}x_2yx_1z\right) : c\in G\right\}\\
&= \cup\left\{f(c)\cap g\left(c^{-1}yx_1x_2z\right) : c\in G\right\}\\
&= \cup\{f(c)\cap g(d) : cd = yx_1x_2z,\ c,d\in G\}\\
&= (f*g)(y(x_1x_2)\,z)
\end{aligned}$$

by Theorem 4.6. Similarly, $(f*g)((x_1x_2)\,y) = (f*g)(y\,(x_1x_2))$. Hence $x_1x_2 \in Z(f*g)$ and $Z(f)\,Z(g) \subseteq\ Z(f*g)$.

□

Theorem 4.20. Let $f, g \in S_G(U)$. If either f or g is commutative in G, then $f*g$ is commutative in G.

Proof. Let f is commutative in G. Then we have $Z(f) = G$. Now,

$$\begin{aligned}
G &= G(Z(g))\\
&= (Z(f))(Z(g))\\
&\subseteq Z(f*g)\\
&\subseteq G
\end{aligned}$$

by Proposition 4.19, so $Z(f*g) = G$. Thus $f*g$ is commutative in G. □

The Theorem 4.19 , and the converse of Theorem 4.20, in general, do not hold, even if f and g are soft int-groups, as the next example demonstrate.

Example 4.21. Let $S_3 = \left\{e, \sigma, \sigma^2, \tau, \tau\sigma, \tau\sigma^2\right\}$ symmetric group and $f, g \in S_{S_3}(U)$. defined, respectively as, for $\alpha_1 \subset \alpha_0 \subset U$,

$$f(x) = \begin{cases} \alpha_0, & \text{for } x \in \left\{e, \sigma, \sigma^2\right\} \\ \alpha_1, & \text{otherwise} \end{cases}$$

$$g(x) = \begin{cases} \alpha_0, & \text{for } x \in \{e, \tau\sigma\} \\ \alpha_1, & \text{otherwise .} \end{cases}$$

Theorem 4.22. If $f, g \in S_G^*(U)$ such that $f \subseteq g$ and $f(e) = g(e)$, then

$$Z(f) \subseteq Z(g).$$

Proof. Let $x \in Z(f)$, $f \subseteq g$ and $f(e) = g(e)$. Then for all $y \in G$,

$$\begin{aligned}
f(e) &= f\left(xyx^{-1}y^{-1}\right)\\
&\subseteq g\left(xyx^{-1}y^{-1}\right)\\
&\subseteq f(e) = f(e)
\end{aligned}$$

Hence $g\left(xyx^{-1}y^{-1}\right) = g(e)$ so $x \in Z(g)$ by Theorem 4.14. Thus $Z(f) \subseteq Z(g)$. □

Theorem 4.23. Let $f, g \in S_G^*(U)$ such that $f \subseteq g$ and $f(e) = g(e)$. If g is commutative in G, then f is commutative in G.

Proof. The proof is easy. □

Theorem 4.24. Let $f \in S_G(U)$. Then, $f\left(xyx^{-1}y^{-1}\right) = f(e)$ for all $x, y \in G$ if and only if f is commutative in G.

Proof. The proof is easy by Theorem 4.14. □

5 Conculusions

In this paper, we defined commutative soft int-groups and study some of its properties. As a future works, by using this study one can develop the nilpotent and the solvable groups.

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SEMI-COMPACT SOFT MULTI SPACES

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Abstract – In this paper, first we introduce the concepts of semi-open soft mset and semi-closed soft mset. Then, we discuss some relationships about those concepts. Finally, we introduce the notion of soft multi semi-compactness as a generalization to soft multi compactness and study their properties and theorems.

Keywords – *Soft multisets, Soft mset topology, Semi-open soft msets, Semi-closed soft msets, Semi-compact soft multi space.*

1 Introduction

The notion of a multiset is well established both in mathematics and computer science [1, 2, 4, 5, 12, 16, 19, 20]. In mathematics, a multiset is considered to be the generalization of a set. In classical set theory, a set is a well-defined collection of distinct objects. If repeated occurrences of any object is allowed in a set, then a mathematical structure, that is known as multiset (mset, for short), is obtained [3, 13, 16, 17, 18]. For the sake of convenience a mset is written as $\{k_1/x_1, k_2/x_2, ..., k_n/x_n\}$ in which the element x_i occurs k_i times. We observe that each multiplicity k_i is a positive integer. The number of occurrences of an object x in an mset A, which is finite in most of the studies that involve msets, is called its multiplicity or characteristic value, usually denoted by $m_A(x)$ or $C_A(x)$ or simply by A(x). One of the most natural and simplest examples is the mset of prime factors of a positive integer n. The number 504 has the factorization $504 = 2^3 3^2 7^1$ which gives the mset $M = \{3/x, 2/y, 1/z\}$ where $C_M(x) = 3$, $C_M(y) = 2$, $C_M(z) = 1$.

Classical set theory states that a given element can appear only once in a set, it assumes that all mathematical objects occur without repetition. Thus there is only one number four, one field of complex numbers, etc. So, the only possible relation

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between two mathematical objects is either they are equal or they are different. The situation in science and in ordinary life is not like this. In the physical world it is observed that there is enormous repetition. For instance, there are many hydrogen atoms, many water molecules, many strands of DNA, etc. Coins of the same denomination and year, electrons or grains of sand appear similar, despite being obviously separate. This leads to three possible relations between any two physical objects; they are different, they are the same but separate or they coincide and are identical. For the sake of definiteness we say that two physical objects are the same or equal, if they are indistinguishable, but possibly separate, and identical if they physically coincide.

The concept of soft msets which is combining soft sets and msets can be used to solve some real life problems. Also, this concept can be used in many areas, such as data storage, computer science, information science, medicine, engineering, etc. The concept of soft msets was introduced in [7]. Also, [6] soft multi connectedness was given. D. Tokat [14] was introduced compact soft multi spaces.

In this paper, we introduce the concept of semi-open and semi-closed sets in soft mset theory. In classical set theory, semi-open and semi-closed sets were first studied by N. Levine [15]. Since its introduction, semi-closed and semi-open sets have been studied by different authors [8, 9, 11, 21].

This paper begins with the initiation of semi-open soft msets and semi-closed soft msets in soft mset topology. Then, we focus on the study of various set theoretic properties of semi-open and semi-closed soft msets. Further we introduce the concept of semi-compactness in soft mset topological space along with certain characterizations.

2 Preliminary

Definition 2.1. [6] Let U be an universal mset, E be a set of parameters and $A \subseteq E$. Then, an order pair (F,A) is called a soft mset where F is a mapping given by $F : A \to P^*(U)$. For all $e \in A$, $F(e)$ mset represent by count function $C_{F(e)} : U^* \to N$ where N represents the set of non-negative integers and U^* represents the support set of U.

Let $U = \{2/x, 3/y, 1/z\}$ be a mset. Then, the support set of U is $U^* = \{x, y, z\}$.

Definition 2.2. [6] For two soft msets (F, A) and (G, B) over U, we say that (F, A) is a sub soft mset of (G, B) if:

1. $A \subseteq B$.
2. $C_{F(e)}(x) \leqslant C_{G(e)}(x)$, $\forall x \in U^*$, $e \in A \cap B$.

We write $(F, A)\widetilde{\subseteq}(G, B)$.

Definition 2.3. [6] Two soft msets (F, A) and (G, B) over U are said to be soft multi equal if (F, A) is a sub soft mset of (G, B) and (G, B) is a sub soft mset of (F, A).

Definition 2.4. [6] The union of two soft msets of (F, A) and (G, B) over U is the soft mset (H, C), where $C = A \cup B$ and $C_{H(e)}(x) = max\{C_{F(e)}(x), C_{G(e)}(x)\}$, $\forall e \in A \cup B$, $\forall x \in U^*$. We write $(F, A)\widetilde{\cup}(G, B)$.

Definition 2.5. [6] The intersection of two soft msets of (F, A) and (G, B) over U is the soft mset (H, C), where $C = A \cap B$ and $C_{H(e)}(x) = min\{C_{F(e)}(x), C_{G(e)}(x)\}$, $\forall e \in A \cap B$, $\forall x \in U^*$. We write $(F, A)\widetilde{\cap}(G, B)$.

Definition 2.6. [6] A soft mset (F, A) over U is said to be a null soft mset denoted $\widetilde{\phi}$ if for all $e \in A$, $F(e) = \phi$.

Definition 2.7. [6] A soft mset (F, A) over U is said to be an absolute soft mset denoted $\widetilde{A}$ if for all $e \in A$, $F(e) = U$.

Definition 2.8. [6] Let V be a non-empty submset of U, then $\widetilde{V}$ denotes the soft mset (H, E) over U for which $H(e) = V$, for all $e \in E$.

In particular, (U, E) will be denoted by $\widetilde{U}$.

Definition 2.9. [6] The difference (H, E) between two soft msets (F, E) and (G, E) over U, denoted by $(F, E)\backslash(G, E)$, is defined as $H(e) = F(e)\backslash G(e)$ for all e∈E where $C_{H(e)}(x) = max\{C_{F(e)}(x) - C_{G(e)}(x), 0\}$, $\forall x \in U^*$.

Remark 2.1. [6] Let (F, E) be soft mset over U. If for all $e \in E$ and $a \in U^*$, $C_{F(e)}(a) = n$ $(n \geqslant 1)$ then we will write $a \in F(e)$ instead of $a \in^n F(e)$.

Definition 2.10. [6] Let (F, E) be a soft mset over U and a$\in U^*$. We say that $a \in (F, E)$ read as a belongs to the soft mset (F, E) whenever $a \in F(e)$ for all $e \in E$.

Note that for any $a \in U^*$, $a \notin (F, E)$, if $a \notin F(e)$ for some $e \in E$.

Definition 2.11. [6] Let $a \in U^*$, then (a, E) denotes the soft mset over U for which $a(e) = \{a\}$, for all $e \in E$.

Definition 2.12. [6] Let (F, E) be a soft mset over U and V be a non-empty submset of U. Then, the sub soft mset of (F, E) over V denoted by $(^V F, E)$, is defined as follows:
$^V F(e) = V \cap F(e)$, for all $e \in E$ where $C_{^V F(e)}(x) = min\{C_V(x), C_{F(e)}(x)\}$, $\forall x \in U^*$.

In other words $(^V F, E) = \widetilde{V}\widetilde{\cap}(F, E)$.

Definition 2.13. [6] The complement of a soft mset (F, A) is denoted by $(F, A)^c$ and is defined by $(F, A)^c = (F^c, A)$ where $F^c : A \rightarrow P^*(U)$ is a mapping given by $F^c(e) = U\backslash F(e)$ for all $e \in A$ where $C_{F^c(e)}(x) = C_U(x) - C_{F(e)}(x)$, $\forall x \in U^*$.

Definition 2.14. [6] Let X be an universal mset and E be a set of parameters. Then, the collection of all soft msets over X with parameters from E is called a soft multi class and is denoted as $SMS(X_E)$.

Definition 2.15. [6] Let $\tau \subseteq SMS(X_E)$, then τ is said to be a soft multi topology on X if the following conditions hold:

1. $\widetilde{\phi}, \widetilde{X}$ belong to τ.
2. The union of any number of soft msets in τ belongs to τ.
3. The intersection of any two soft msets in τ belongs to τ.

τ is called a soft multi topology over X and the triple (X, τ, E) is called a soft multi topological space over X. Also, The members of τ are said to be open soft msets in X.

A soft mset (F, E) in $SMS(X_E)$ is said to be a closed soft mset in X, if its complement $(F, E)^c$ belongs to τ.

Definition 2.16. [6] Let X be universal mset, E be the set of parameters. Then:

- $\tau = \{\widetilde{\phi}, \widetilde{X}\}$ is called the indiscrete soft multi topology on X and (X, τ, E) is said to be an indiscrete soft multi space over X.
- Let τ be the collection of all soft msets over X. Then, τ is called the discrete soft multi topology on X and (X, τ, E) is said to be a discrete soft multi space over X.

Definition 2.17. [6] Let (X, τ, E) be a soft multi topological space over X and Y be a non-empty submset of X. Then,

$$\tau_Y = \{({}^Y F, E) : (F, E) \in \tau\}$$

is said to be the soft multi topology on Y and (Y, τ_Y, E) is called a soft multi subspace of (X, τ, E).

Definition 2.18. [7] Let (X, τ, E) be a soft multi topological space over X and (F, E) be a soft mset over X. Then, the soft multi closure of (F, E), denoted by $cl(F, E)$ [or $\overline{(F, E)}$] is the intersection of all closed soft mset containing (F, E).

Definition 2.19. [7] Let (X, τ, E) be a soft multi topological space over X and (F, E) be a soft mset over X. Then, the soft multi interior of (F, E), denoted by $int(F, E)$ [or $(F, E)^o$] is the union of all open soft mset contained in (F, E).

Definition 2.20. [14] Let (X, τ_1, E) and (Y, τ_2, K) be two soft mset topological spaces. Let $\varphi : X^* \to Y^*$ and $\psi : E \to K$ be two functions. Then the pair (φ, ψ) is called a soft multi function and denoted by $f = (\varphi, \psi) : (X, E) \to (Y, K)$ is defined as follows:
Let $(F, E) \widetilde{\subseteq} \widetilde{X}$. Then the image of (F,E) under soft multi function f is soft mset in $\widetilde{Y}$ defined by f(F,E), where for $k \in \psi(E) \subseteq K$ and $y \in Y^*$,

$$C_{f(F,E)(k)}(y) = \begin{cases} sup_{e\in\psi^{-1}(k)\cap E, x\in\varphi^{-1}(y)} C_{F(e)}(x), & \text{if } \psi^{-1}(k) \neq \phi, \varphi^{-1}(y) \neq \phi; \\ 0, & \text{otherwise.} \end{cases}$$

Let (G, K) be a soft mset in $\widetilde{Y}$. Then the inverse image of (G, K) under soft multi function f is soft mset in $\widetilde{X}$ defined by $f^{-1}(G, K)$, where for $e \in \psi^{-1}(K) \subseteq E$ and $x \in X^*$,

$$C_{f^{-1}(G,K)(e)}(x) = C_{G(\psi(e))}(\varphi(x)).$$

Theorem 2.1. [14] Let $f : X_E \to Y_K$ be a soft multi function, (F_i, A) soft msets in X_E and (G_i, B) soft msets in Y_K. Then:

1. $f(\widetilde{\bigcup}_{i\in I}(F_i, A_i)) = \widetilde{\bigcup}_{i\in I} f(F_i, A_i)$.
2. $f^{-1}(\widetilde{\bigcup}_{i\in I}(G_i, B)) = \widetilde{\bigcup}_{i\in I} f^{-1}(G_i, B)$.

3 Semi-open soft msets and semi-closed soft msets

Definition 3.1. A soft mset (S, E) in a soft mset topology (X,τ,E) is said to be semi open soft mset iff there exists an open soft mset (F, E) such that:
$C_{F(e)}(x) \leqslant C_{S(e)}(x) \leqslant C_{cl(F)(e)}(x)$ for all $x \in X^*$, $e \in E$.

Definition 3.2. A soft mset (S, E) in a soft mset topology (X,τ,E) is said to be semi closed soft mset iff there exist a closed soft mset (F, E) such that:
$C_{int(F)(e)}(x) \leqslant C_{S(e)}(x) \leqslant C_{F(e)}(x)$ for all $x \in X^*$, $e \in E$.

Note that the complement of semi-open soft mset is semi-closed soft mset.

Example 3.1. Let $X = \{2/x, 3/y, 1/z\}$ be a mset, $E = \{e_1, e_2\}$ be a set of parameters and $\tau = \{\widetilde{\phi}, \widetilde{X}, (F_1, E), (F_2, E), (F_3, E), (F_4, E), (F_5, E), (F_6, E)\}$, where:
$F_1(e_1) = \{2/x\}$, $F_1(e_2) = \{1/y\}$;
$F_2(e_1) = \{2/x, 1/y\}$, $F_2(e_2) = \{2/x, 1/y\}$;
$F_3(e_1) = X$, $F_3(e_2) = \{2/x\}$;
$F_4(e_1) = X$, $F_4(e_2) = \{2/x, 1/y\}$;
$F_5(e_1) = \{2/x, 1/y\}$, $F_5(e_2) = \{2/x\}$;
$F_6(e_1) = \{2/x\}$, $F_6(e_2) = \phi$.
Let (G, E) be a sub soft mset of X such that $G(e_1) = \{2/x, 2/y\}$, $G(e_2) = \{2/x, 2/y\}$. Then, $C_{F_1(e)}(x) \leqslant C_{G(e)}(x) \leqslant C_{cl(F_1)(e)}(x)$ for all $x \in X^*$, $e \in E$. Hence, (G, E) is semi-open soft mset.

Definition 3.3. Let (X,τ,E) be a soft mset topology. Then:

1. The semi closure of a soft mset (G, E) is denoted by $scl(G, E)$ and defined as $scl(G, E) = \widetilde{\cap}\{(F, E) : (G, E)\widetilde{\subseteq}(F, E),\ (F, E)\ is\ semi-closed\ soft\ mset\}$, where $C_{(scl(G))(e)}(x) = min\{C_{F(e)}(x) : C_{G(e)}(x) \leqslant C_{F(e)}(x),\ (F, E)\ is\ semi-closed\ soft\ mset\}$; for all $x \in X^*$, $e \in E$.

2. The semi interior of a soft mset (G, E) is denoted by $sint(G, E)$ and defined as $sint(G, E) = \widetilde{\cup}\{(F, E) : (F, E)\widetilde{\subseteq}(G, E), (F, E)\ is\ semi-open\ soft\ mset\}$, where
$C_{(sint(G))(e)}(x) = max\{C_{F(e)}(x) : C_{F(e)}(x) \leqslant C_{G(e)}(x),\ (F, E)\ is\ semi-open\ soft\ mset\}$;
for all $x \in X^*$, $e \in E$.

Theorem 3.1. Let (X,τ,E) be a soft mset topology. Then, arbitrary union of semi-open soft msets is a semi-open soft mset.

Proof. Let $\{(T_\lambda, E) : \lambda \in \Lambda\}$ be a collection of semi-open soft msets. Since, (T_λ, E) is a semi-open soft mset, then there exists an open soft mset (O_λ, E) for each λ such that $C_{O_\lambda(e)}(x) \leqslant C_{T_\lambda(e)}(x) \leqslant C_{cl(O_\lambda)(e)}(x)$ for all $x \in X^*$, $e \in E$ and $\lambda \in \Lambda$. Now, taking arbitrary union over λ, $C_{\cup_\lambda O_\lambda(e)}(x) \leqslant C_{\cup_\lambda T_\lambda(e)}(x) \leqslant C_{\cup_\lambda cl(O_\lambda)(e)}(x) = C_{cl(\cup_\lambda O_\lambda)(e)}(x)$ for all $x \in X^*$, $e \in E$. This imply $(\cup_\lambda T_\lambda, E)$ is a semi-open soft mset, because $(\cup_\lambda O_\lambda, E)$ is an open soft mset being the arbitrary union of open soft msets.

Remark 3.1. Finite intersection of semi-open soft msets may not be semi-open soft mset. As shown in the following example.

Example 3.2. Let $X = \{3/x, , 3/y, 2/z, 1/d\}$ be a mset, $E = \{e_1, e_2\}$ be a set of parameters and $\tau = \{\widetilde{\phi}, \widetilde{X}, (F_1, E), (F_2, E), (F_3, E), (F_4, E)\}$, where:
$F_1(e_1) = \{2/x, 1/y\}$, $F_1(e_2) = \{2/x, 1/y\}$;
$F_2(e_1) = \{2/z, 1/d\}$, $F_2(e_2) = \{1/x, 3/y\}$;
$F_3(e_1) = \{2/x, 1/y, 2/z, 1/d\}$, $F_3(e_2) = \{2/x, 3/y\}$;
$F_4(e_1) = \phi$, $F_4(e_2) = \{1/x, 1/y\}$.
Let (G, E), (F, E) be two sub soft mset of X such that $G(e_1) = \{2/x, 2/y\}$, $G(e_2) = \{2/x, 2/y\}$ and $F(e_1) = \{2/z, 1/d\}$, $F(e_2) = \{3/x, 3/y\}$. Then, (G, E), (F, E) are semi-open soft msets. But, $(G \cap F)(e_1) = \phi$, $(G \cap F)(e_2) = \{2/x, 2/y\}$. Hence, $(G \cap F, E)$ is not semi-open soft mset.

Remark 3.2. The collection of all semi-open msets doesn't form a soft mset topology since intersection of two semi-open soft msets may not be semi-open soft mset.

Theorem 3.2. The union of a semi-open soft mset with an open soft mset is aslo a semi-open soft mset.

Proof. Let (O, E) be an open soft mset and (S, E) be a semi-open soft mset. So, there exist an open soft mset (F, E) such that $C_{F(e)}(x) \leqslant C_{S(e)}(x) \leqslant C_{cl(F)(e)}(x)$ for all $x \in X^*$, $e \in E$. Therefore, $C_{(F\cup O)(e)}(x) \leqslant C_{(S\cup O)(e)}(x) \leqslant C_{(cl(F)\cup O)(e)}(x)$. Now, we have $C_{(cl(F)\cup O)(e)}(x) \leqslant C_{(cl(F)\cup cl(O))(e)}(x) = C_{(cl(F\cup O))(e)}(x)$. Hence, $C_{(F\cup O)(e)}(x) \leqslant C_{(S\cup O)(e)}(x) \leqslant C_{(cl(F\cup O))(e)}(x)$. Then, $(S \cup O, E)$ is a semi-open soft mset.

Corollary 3.1. Let (X,τ,E) be a soft mset topological space, then arbitrary intersection of semi-closed soft msets is a semi-closed soft mset.

Proof. Immediate.

Remark 3.3. Union of two semi-closed soft msets may not be a semi-closed soft mset. As shown in the following example.

Example 3.3. From Example 3.2, Let (G, E), (F, E) be two sub soft mset of X such that $G(e_1) = \{1/x, 1/y, 2/z, 1/d\}$, $G(e_2) = \{1/x, 1/y, 2/z, 1/d\}$ and $F(e_1) = \{3/x, 3/y\}$, $F(e_2) = \{2/z, 1/d\}$. Then, (G, E), (F, E) are two semi-closed soft msets. Moreover, $(G \cup F)(e_1) = X$, $(G \cup F)(e_2) = \{1/x, 1/y, 2/z, 1/d\}$ but $(G \cup F, E)$ is not semi-closed soft mset.

Theorem 3.3. Every open soft mset is a semi-open soft mset.

Proof. Immediate.

Note that the converse of Theorem 3.3 is not true as shown in this example.

Example 3.4. From Example 3.1, (G, E) is semi-open soft mset but it is not open soft mset.

Theorem 3.4. If (S, E) is a semi-open soft mset such that $C_{S(e)}(x) \leqslant C_{N(e)}(x) \leqslant C_{cl(S)(e)}(x)$ for all $x \in X^*$, $e \in E$. Then, the soft mset (N, E) is also a semi-open soft mset.

Proof. As (S, E) is a semi-open soft mset, there exists an open soft mset (O, E) such that $C_{O(e)}(x) \leqslant C_{S(e)}(x) \leqslant C_{cl(O)(e)}(x)$ for all $x \in X^*$, $e \in E$. Then by hypothesis, $C_{O(e)}(x) \leqslant C_{N(e)}(x) \leqslant C_{cl(S)(e)}(x) \leqslant C_{cl(O)(e)}(x)$ for all $x \in X^*$, $e \in E$. Hence, (N, E) is a semi-open soft mset.

Corollary 3.2. If (F, E) is a semi-closed soft mset in a soft mset topology (X,τ,E) such that $C_{int(F)(e)}(x) \leqslant C_{S(e)}(x) \leqslant C_{F(e)}(x)$ for all $x \in X^*$, $e \in E$. Then the soft mset (S, E) is also a semi-closed soft mset.

Proof. Immediate.

Theorem 3.5. For a soft mset topology, the following conditions are equivalent:

1. (S,E) is a semi-open soft mset.
2. $C_{S(e)}(x) \leqslant C_{cl(int(S))(e)}(x)$.
3. $C_{int(cl(S^c))(e)}(x) \leqslant C_{S^c(e)}(x)$, where S^c is the complement of S.
4. (S^c, E) is a semi-closed soft mset.

Proof. (1⇒2) Let (S, E) be a semi-open soft mset. So, there exist an open soft mset (O, E) such that $C_{O(e)}(x) \leqslant C_{S(e)}(x) \leqslant C_{cl(O)(e)}(x)$ for all $x \in X^*$, $e \in E$. Since, (O, E) is an open soft mset, then $C_{S(e)}(x) \leqslant C_{cl(int(O))(e)}(x)$. Since $C_{O(e)}(x) \leqslant C_{S(e)}(x)$, then $C_{cl(int(O))(e)}(x) \leqslant C_{cl(int(S))(e)}(x)$. Thus, we have $C_{S(e)}(x) \leqslant C_{cl(int(S))(e)}(x)$.

(2⇒3) Taking complement of (2). Then, $C_{int(cl(S^c))(e)}(x) \leqslant C_{S^c(e)}(x)$.

(3⇒4) Since, $(cl(S^c), E)$ is a closed soft mset such that $C_{int(cl(S^c)(e)}(x) \leqslant C_{S^c(e)}(x) \leqslant C_{cl(S^c)(e)}(x)$ for all $x \in X^*$, $e \in E$. So, (S^c, E) is a semi-closed soft mset.

(4⇒1) Since, (S^c, E) is a semi-closed soft mset. Then, there exist a closed soft mset (F, E) such that $C_{int(F)(e)}(x) \leqslant C_{S^c(e)}(x) \leqslant C_{F(e)}(x)$. Therefore, $C_{F^c(e)}(x) \leqslant C_{S(e)}(x) \leqslant C_{cl(F^c)(e)}(x)$ for all $x \in X^*$, $e \in E$. Then, (S, E) is a semi-open soft mset.

4 Semi-Compactness

Definition 4.1. A collection $\{(T_\lambda, E) : \lambda \in \Lambda\}$ of soft msets is said to be a cover of a soft mset (F, E) if $C_{F(e)}(x) \leqslant C_{\tilde{\bigcup}_\lambda T_\lambda(e)}(x)$ for all $x \in X^*$, $e \in E$. Then, we say (F, E) is covered by $\{(T_\lambda, E) : \lambda \in \Lambda\}$. Also, If each (T_λ, E) is a semi-open soft mset, then the cover is said to be a semi open cover.

If $C_{\widetilde{X}(e)}(x) \leqslant C_{\widetilde{\bigcup}_\lambda T_\lambda(e)}(x)$ for all $x \in X^*$, $e \in E$. Then, we say $\widetilde{X}$ is covered by $\{(T_\lambda, E) : \lambda \in \Lambda\}$.

Definition 4.2. Any subcollection of a semi-open cover is said to be semi subcover if it covers $\widetilde{X}$.

Definition 4.3. Any subcollection of a semi open cover where each element is a whole sub soft mset is said to be semi whole subcover if it covers $\widetilde{X}$.

Definition 4.4. A soft mset topology (X,τ,E) is said to be a semi compact space if every semi open cover of $\widetilde{X}$ has a finite semi open subcover i.e., for any collection $\{(T_\lambda, E) : \lambda \in \Lambda\}$ of semi-open soft msets covering $\widetilde{X}$, there exist a finite subcollection $\{(T_{\lambda_i}, E) : i = 1, 2, 3, ..., n\}$ such that $C_{\widetilde{X}(e)}(x) \leqslant C_{\widetilde{\cup} T_{\lambda_i}(e)}(x)$ for all $x \in X^*$, $e \in E$, $i = 1, 2, 3, ..., n$.

Example 4.1.

1. Every finite soft mset topological space is a semi-compact soft multi space.
2. Any indiscrete soft mset topological space is a semi-compact soft multi space.

Remark 4.1. Every compact soft multi space is semi-compact soft multi space.

Definition 4.5. A soft mset topological space (X,τ,E) is said to be:

1. a semi-whole compact soft multi space if every semi open cover of $\widetilde{X}$ has a finite semi-whole subcover.
2. a semi-partial whole compact soft multi space if every semi open cover of $\widetilde{X}$ has a finite semi-partial whole subcover.
3. a semi-full compact soft multi space if every semi open cover of $\widetilde{X}$ has a finite semi-full subcover.

Definition 4.6. An arbitrary collection $S = \{(O_1, E), (O_2, E), ...\}$ of soft msets is said to have finite intersection property (FIP) if intersection of elements of every finite subcollection $\{(O_1, E), (O_2, E), ..., (O_n, E)\}$ of S is non-empty.
i.e., $C_{\widetilde{\cap} O_i(e)}(x) \neq C_{\widetilde{\phi}(e)}(x)$, $x \in X^*$, $e \in E$, $i = 1, 2, 3, ..., n$.

Theorem 4.1. Let (X,τ,E) be a soft mset topology. Therefore, (X,τ,E) is semi compact iff every collection $C = \{(T_\lambda, E) : \lambda \in \Lambda\}$ of semi closed soft msets in $\widetilde{X}$ having the FIP is such that $C_{\widetilde{\bigcap} T_\lambda(e)}(x) \neq C_{\widetilde{\phi}(e)}(x)$, $x \in X^*$, $e \in E$, $\lambda \in \Lambda$.

Proof. ($\Rightarrow$) Let (X,τ,E) be a semi compact space and $\{(T_\lambda, E) : \lambda \in \Lambda\}$ be a collection of semi-closed soft msets with FIP such that $C_{(\widetilde{\bigcap}_{\lambda \in \Lambda} T_\lambda)(e)}(x) = C_{\widetilde{\phi}(e)}(x)$. Then, $C_{(\widetilde{\bigcup}_{\lambda \in \Lambda} T^c_\lambda)(e)}(x) = C_{\widetilde{X}(e)}(x)$. Therefore, $\{(T^c_\lambda, E) : \lambda \in \Lambda\}$ forms semi open cover of $\widetilde{X}$. So, there exist $\{(T^c_{\lambda_i}, E) : i = 1, 2, 3, ..., n\}$ such that $C_{(\widetilde{\bigcup}_{i=1,..,n} T^c_{\lambda_i})(e)}(x) = C_{\widetilde{X}(e)}(x)$. Thus, $C_{(\widetilde{\bigcap}_{i=1,..,n} T_{\lambda_i})(e)}(x) = C_{\widetilde{\phi}(e)}(x)$ for all $x \in X^*$, $e \in E$, which is a contradiction.

($\Leftarrow$) Let every collection $C = \{(T_\lambda, E) : \lambda \in \Lambda\}$ of semi-closed soft msets having FIP be such that $C_{(\widetilde{\bigcap}_{\lambda\in\Lambda} T_\lambda)(e)}(x) \neq C_{\widetilde{\phi}(e)}(x)$. Assume that (X,$\tau$,E) is not semi compact space. Then, there exist a semi open cover $\{(S_\lambda, E) : \lambda \in \Lambda\}$ of $\widetilde{X}$ which has no finite subcover of $\widetilde{X}$. Hence, $C_{\widetilde{X}(e)}(x) > C_{(\widetilde{\bigcup}_{i=1,..,n} S_{\lambda_i})(e)}(x)$. Therefore, $C_{\widetilde{\phi}(e)}(x) \leqslant C_{(\widetilde{\bigcap}_{i=1,..,n} S^c_{\lambda_i})(e)}(x)$, which is a contradiction with hypothesis.

Theorem 4.2. Let (X,τ,E) be a soft mset topology. Therefore, (X,τ,E) is semi compact iff every collection $C = \{(T_\lambda, E) : \lambda \in \Lambda\}$ of soft msets in $\widetilde{X}$ having the FIP is such that $C_{(\widetilde{\bigcap}_{\lambda\in\Lambda} scl(T_\lambda))(e)}(x) \neq C_{\widetilde{\phi}(e)}(x)$.

Proof. ($\Rightarrow$) Let (X,τ,E) be a semi-compact. Assume that $C = \{(T_\lambda, E) : \lambda \in \Lambda\}$ be a collection of soft msets in $\widetilde{X}$ having the FIP be such that $C_{(\widetilde{\bigcap}_{\lambda\in\Lambda} scl(T_\lambda))(e)}(x) = C_{\widetilde{\phi}(e)}(x)$. Then, $C_{(\widetilde{\bigcup}_{\lambda\in\Lambda} scl(T_\lambda)^c)(e)}(x) = C_{\widetilde{X}(e)}(x)$. Therefore, $\{(scl(T_\lambda, E))^c : \lambda \in \Lambda\}$ forms a semi open cover of $\widetilde{X}$. Since, (X,τ,E) is semi compact. Then, there exist a finite subcover $\{(scl(T_{\lambda_i}, E))^c : i = 1, 2, 3, ..., n\}$ such that $C_{[\widetilde{\bigcup}_{i=1,..,n} (scl(T_{\lambda_i}))^c](e)}(x) = C_{\widetilde{X}(e)}(x)$. Then, $C_{[\widetilde{\bigcap}_{i=1,..,n} scl(T_{\lambda_i})](e)}(x) = C_{\widetilde{\phi}(e)}(x)$. Therefore, $C_{(\widetilde{\bigcap}_{i=1,..,n} T_{\lambda_i})(e)}(x) \leqslant C_{\widetilde{\phi}(e)}(x)$, which is a contradiction with the FIP.

($\Leftarrow$) Sufficiency. Assume that (X,τ,E) is not a semi-compact. Then, there exist a semi open cover $\{(T_\lambda, E) : \lambda \in \Lambda\}$ which has no finite subcover. So, for all finite subcollection $\{(T_{\lambda_i}, E) : i = 1, 2, 3, ..., n\}$, we have $C_{(\widetilde{\bigcup}_{i=1,..,n} T_{\lambda_i})(e)}(x) < C_{\widetilde{X}(e)}(x)$. Thus, $C_{(\widetilde{\bigcap}_{i=1,..,n} {T_{\lambda_i}}^c)(e)}(x) \geqslant C_{\widetilde{\phi}(e)}(x)$. Hence, $\{({T_\lambda}^c, E) : \lambda \in \Lambda\}$ is a family of semi-closed soft msets with FIP. Now, $C_{(\widetilde{\bigcup}_{\lambda\in\Lambda} T_\lambda)(e)}(x) \geqslant C_{\widetilde{X}(e)}(x)$. Then, $C_{(\widetilde{\bigcap}_{\lambda\in\Lambda} {T_\lambda}^c)(e)}(x) = C_{\widetilde{\phi}(e)}(x)$. Therefore, $C_{[\widetilde{\bigcap}_{\lambda\in\Lambda} scl({T_\lambda}^c)](e)}(x) = C_{\widetilde{\phi}(e)}(x)$, which is a contradiction with hypothesis.

Remark 4.2. The Theorems 4.1 and 4.2 hold for semi whole (resp. partial whole and full) compact spaces .

Theorem 4.3. Let (X,τ,E) be a soft mset topology and (Y,τ_Y,E) be its subspace. Let (A, E) be a soft mset such that $C_{A(e)}(x) \leqslant C_{\widetilde{Y}(e)}(x) \leqslant C_{\widetilde{X}(e)}(x)$. Then, (A, E) is τ-semi compact iff (A, E) is τ_Y-semi compact.

Proof. ($\Rightarrow$) Let (A, E) be a τ-semi compact and $\{(K_\lambda, E) : \lambda \in \Lambda\}$ be τ_Y-semi open cover of (A, E). So, there exist τ-semi open soft msets $\{(S_\lambda, E) : \lambda \in \Lambda\}$ such that $C_{K_\lambda(e)}(x) = C_{(\widetilde{Y}\widetilde{\bigcap} S_\lambda)(e)}(x)$ for all $x \in X^*$, $e \in E$, $\lambda \in \Lambda$. Now, $C_{A(e)}(x) \leqslant C_{\widetilde{Y}(e)}(x) \leqslant C_{\widetilde{\bigcup}_{\lambda\in\Lambda} K_\lambda(e)}(x) \leqslant C_{\widetilde{\bigcup}_{\lambda\in\Lambda} S_\lambda(e)}(x)$. Therefore, $\{(S_\lambda, E) : \lambda \in \Lambda\}$ forms a τ-semi open cover of (A, E). So, there exist a finite subcover $\{(S_{\lambda_i}, E) : i = 1, 2, 3, ..., n\}$ such that $C_{A(e)}(x) \leqslant C_{(\widetilde{\bigcup}_{i=1,..,n} S_{\lambda_i})(e)}(x)$. Since, $C_{A(e)}(x) \leqslant C_{\widetilde{Y}(e)}(x)$, then $C_{A(e)}(x) \leqslant C_{[\widetilde{Y}\widetilde{\bigcap}(\widetilde{\bigcup}_{i=1,..,n} S_{\lambda_i})](e)}(x) = C_{[\widetilde{\bigcup}_{i=1,..,n} (\widetilde{Y}\widetilde{\bigcap} S_{\lambda_i})](e)}(x) = C_{(\widetilde{\bigcup}_{i=1,..,n} K_{\lambda_i})(e)}(x)$. Thus, (A, E) is τ_Y-semi compact.

($\Leftarrow$) Let $\{(S_\lambda, E) : \lambda \in \Lambda\}$ be τ-semi open cover of (A, E). Putting $C_{G_\lambda(e)}(x) = C_{(\widetilde{Y}\widetilde{\bigcap}S_\lambda)(e)}(x)$ for all $x \in X^*$, $e \in E$, $\lambda \in \Lambda$. Since, $C_{A(e)}(x) \leqslant C_{\widetilde{Y}(e)}(x)$ and $C_{A(e)}(x) \leqslant C_{\widetilde{\bigcup}_{\lambda\in\Lambda}S_\lambda(e)}(x)$, then $C_{A(e)}(x) \leqslant C_{(\widetilde{Y}\widetilde{\bigcap}(\widetilde{\bigcup}_{\lambda\in\Lambda}S_\lambda))(e)}(x) = C_{\widetilde{\bigcup}_{\lambda\in\Lambda}(\widetilde{Y}\widetilde{\bigcap}S_\lambda)(e)}(x) = C_{\widetilde{\bigcup}_{\lambda\in\Lambda}G_\lambda(e)}(x)$. So, $\{(G_\lambda, E) : \lambda \in \Lambda\}$ is τ_Y-semi open cover of (A, E). By hypothesis, there exist a finite subcollection $\{(G_{\lambda_i}, E) : i = 1, 2, 3, ..., n\}$ such that $C_{A(e)}(x) \leqslant C_{(\widetilde{\bigcup}_{i=1,..,n}G_{\lambda_i})(e)}(x) = C_{[\widetilde{\bigcup}_{i=1,..,n}(\widetilde{Y}\widetilde{\bigcap}S_{\lambda_i})](e)}(x) \leqslant C_{(\widetilde{\bigcup}_{i=1,..,n}S_{\lambda_i})(e)}(x)$. This implies that (A, E) is τ-semi compact .

Definition 4.7. A soft multi function $f : (X, \tau_1, E) \to (Y, \tau_2, K)$ is said to be irresolute soft multi function if $f^{-1}(G, K)$ is τ_1-semi open (resp. closed) for every (G, K) is τ_2-semi open (resp. closed).

Theorem 4.4. Let $f : (X, \tau_1, E) \to (Y, \tau_2, K)$ be a bijection irresolute soft multi function. If (H, E) is a τ_1-semi compact, then $f(H, E)$ is a τ_2-semi compact.

Proof. Assume that $(H, E)\widetilde{\subseteq}\widetilde{X}$ is a τ_1-semi compact. Let $\{(G_\lambda, K) : \lambda \in \Lambda\}$ be a τ_2-semi open cover of $f(H, E)$ i.e., $C_{f(H)(k)}(y) \leqslant C_{(\widetilde{\bigcup}_{\lambda\in\Lambda}G_\lambda)(k)}(y)$ for all $y \in Y^*$, $k \in K$. Therefore, $C_{H(e)}(x) = C_{(f^{-1}(f(H)))(e)}(x) \leqslant C_{(f^{-1}(\widetilde{\bigcup}_{\lambda\in\Lambda}G_\lambda))(e)}(x) = C_{(\widetilde{\bigcup}_{\lambda\in\Lambda}f^{-1}(G_\lambda))(e)}(x)$ for all $x \in X^*$, $e \in E$. Since, (G_λ, K) is τ_2-semi open cover of $f(H, E)$ and f is irresolute soft multi function, then $f^{-1}(G_\lambda, K)$ is τ_1-semi open cover of (H, E). Thus, $C_{H(e)}(x) \leqslant C_{(\widetilde{\bigcup}_{i=1,2,...,n}f^{-1}(G_{\lambda_i}))(e)}(x)$ for all $x \in X^*$, $e \in E$. Then, $C_{f(H)(k)}(y) \leqslant C_{f(\widetilde{\bigcup}_{i=1,2,..,n}f^{-1}(G_{\lambda_i}))(k)}(y) = C_{(\widetilde{\bigcup}_{i=1,2,..,n}G_{\lambda_i})(k)}(y)$ for all $y \in Y^*$, $k \in K$. Hence, $f(H, E)$ is a τ_2-semi compact.

5 Conclusion

Topology is an important and major area of mathematics and it can give many relationships between other scientific areas and mathematical models. Recently, many scientists have studied and improved the mset theory and easily applied to many problems having uncertainties from social life. This paper begins with the initiation of semi-open soft msets and semi-closed soft msets in soft mset topology. Then, we focus on the study of various set theoretic properties of semi-open and semi-closed soft msets. Further, we introduce the concept of semi-compactness in soft mset topological space along with certain characterizations. Also, we discuss some important results about semi-compact soft multi space.

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ON NEUTROSOPHIC REFINED SETS AND THEIR APPLICATIONS IN MEDICAL DIAGNOSIS

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Abstract – In this paper, we present some definitions of neutrosophic refined sets such as; union, intersection, convex and strongly convex in a new way to handle the indeterminate information and inconsistent information. Also we have examined some desired properties of neutrosophic refined sets based on these definitions. Then, we give distance measures of neutrosophic refined sets with properties. Finally, an application of neutrosophic refined set is given in medical diagnosis problem (heart disease diagnosis problem) to illustrate the advantage of the proposed approach.

Keywords – *Neutrosophic sets, neutrosophic refined sets, distance measures, decision making*

1 Introduction

Recently, several theories have been proposed to deal with uncertainty, imprecision and vagueness. Theory of probability, fuzzy set theory [46], intuitionistic fuzzy sets [7], rough set theory [27] etc. are consistently being utilized as efficient tools for dealing with diverse types of uncertainties and imprecision embedded in a system. However, all these above theories failed to deal with indeterminate and inconsistent information which exist in beliefs system. In 1995, Smarandache [39] developed a new concept called neutrosophic set (NS) which generalizes probability set, fuzzy set and intuitionistic fuzzy set. NS can be described by membership degree, indeterminacy degree and non-membership degree. This theory and their hybrid structures has proven useful in many different fields such as control theory [1], databases [3, 2],

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medical diagnosis problem [4], decision making problem [5, 6, 9, 10, 11, 13, 12, 14, 17, 19, 20, 23, 25], physics [28], topology [24] etc.

Yager [43] firstly introduced a new theory, is called theory of bags, which is a multiset. Then, the concept of multisets were originally proposed by Blizard [8] and Calude et al. [15], as useful structures arising in many area of mathematics and computer sciences such as database queries. Several authors from time to time made a number of generalization of set theory. Since then, several researcher [18, 26, 35, 36, 37, 41, 42] discuussed more properties on fuzzy multiset. Shinoj and John [38] made an extension of the concept of fuzzy multisets by an intuitionstic fuzzy set, which called intuitionstic fuzzy multisets (IFMS). Since then in the study on IFMS , a lot of excellent results have been achieved by researcher [22, 29, 30, 31, 32, 33, 34]. The concepts of FMS and IFMS fails to deal with indeterminacy. Therefore, Smarandache[40] give n-valued refined neutrosophic logic and its applications. Then, Ye and Ye [44] gave single valued neutrosophic sets and operations laws. Ye et al. [45] presented generalized distance measure and its similarity measures between single valued neutrosophic multi sets. Also they applied the measure to a medical diagnosis problem with incomplete, indeterminate and inconsistent information. Chatterjee et al.[16] developed single valued neutrosophic multi sets in detail.

Combining neutrosophic set models with other mathematical models has attracted the attention of many researchers. Maji et al. presented the concept of neutrosophic soft set [25] which is based on a combination of the neutrosophic set and soft set models. Broumi and Smarandache introduced the concept of the intuitionistic neutrosophic soft set [9, 12] by combining the intuitionistic neutrosophic set and soft set.

This paper is arranged in the following manner. In section 2, some definitions and notion about intuitionstic fuzzy set, intuitionstic fuzzy multisets and neutrosophic set theory. These definitions will help us in later section. In section 3 we study the concept of neutrosophic refined (multi) sets and their operations. In section 4, we present an application of neutrosophic multisets in medical diagnosis. Finally we conclude the paper.

2 Preliminary

In this section, we give the basic definitions and results of intuitionistic fuzzy set [7], intuitionistic fuzzy multiset [29] and neutrosophic set theory [39] that are useful for subsequent discussions.

Definition 2.1. [7] Let E be a universe. An intuitionistic fuzzy set I on E can be defined as follows:

$$I = \{< x, \mu_I(x), \gamma_I(x) >: \; x \in E\}$$

where, $\mu_I : E \to [0,1]$ and $\gamma_I : E \to [0,1]$ such that $0 \leq \mu_I(x) + \gamma_I(x) \leq 1$ for any $x \in E$.

Definition 2.2. [29] Let E be a universe. An intuitionistic fuzzy multiset K on E can be defined as follows:

$$K = \{< x, (\mu_K^1(x), \mu_K^2(x), ..., \mu_K^P(x)), (\gamma_K^1(x), \gamma_K^2(x), ..., \gamma_K^P(x)) >: \; x \in E\}$$

where, $\mu_K^1(x), \mu_K^2(x), ..., \mu_K^P(x) : E \to [0,1]$ and $\gamma_K^1(x), \gamma_K^2(x), ..., \gamma_K^P(x) : E \to [0,1]$ such that $0 \leq \mu_K^i(x) + \gamma_K^i(x) \leq 1 (i = 1, 2, ..., P)$ and $\mu_K^1(x) \leq \mu_K^2(x) \leq ... \leq \mu_K^P(x)$ for any $x \in E$.

Here, $(\mu_K^1(x), \mu_K^2(x), ..., \mu_K^P(x))$ and $(\gamma_K^1(x), \gamma_K^2(x), ..., \gamma_K^P(x))$ is the membership sequence and non-membership sequence of the element x, respectively.

We arrange the membership sequence in decreasing order but the corresponding non membership sequence may not be in decreasing or increasing order.

Definition 2.3. [39] Let U be a space of points (objects), with a generic element in U denoted by u. A neutrosophic set (N-set) A in U is characterized by a truth-membership function T_A, an indeterminacy-membership function I_A and a falsity-membership function F_A. $T_A(x)$, $I_A(x)$ and $F_A(x)$ are real standard or nonstandard subsets of $[0, 1]$. It can be written as

$$A = \{< u, (T_A(x), I_A(x), F_A(x)) >: x \in E, T_A(x), I_A(x), F_A(x) \in [0, 1]\}.$$

There is no restriction on the sum of $T_A(x)$; $I_A(x)$ and $F_A(x)$, so $0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3$.

Definition 2.4. [21] t-norms are associative, monotonic and commutative two valued functions t that map from $[0, 1] \times [0, 1]$ into $[0, 1]$. These properties are formulated with the following conditions: $\forall a, b, c, d \in [0, 1]$,

1. $t(0, 0) = 0$ and $t(a, 1) = t(1, a) = a$,
2. If $a \leq c$ and $b \leq d$, then $t(a, b) \leq t(c, d)$
3. $t(a, b) = t(b, a)$
4. $t(a, t(b, c)) = t(t(a, b), c)$

Definition 2.5. [21] t-conorms (s-norm) are associative, monotonic and commutative two placed functions s which map from $[0, 1] \times [0, 1]$ into $[0, 1]$. These properties are formulated with the following conditions: $\forall a, b, c, d \in [0, 1]$,

1. $s(1, 1) = 1$ *and* $s(a, 0) = s(0, a) = a$,
2. if $a \leq c$ and $b \leq d$, then $s(a, b) \leq s(c, d)$
3. $s(a, b) = s(b, a)$
4. $s(a, s(b, c)) = s(s(a, b), c)$

t-norm and t-conorm are related in a sense of lojical duality. Typical dual pairs of non parametrized t-norm and t-conorm are complied below:

1. Drastic product:

$$t_w(a, b) = \begin{cases} min\{a, b\}, & max\{ab\} = 1 \\ 0, & otherwise \end{cases}$$

2. Drastic sum:

$$s_w(a, b) = \begin{cases} max\{a, b\}, & min\{ab\} = 0 \\ 1, & otherwise \end{cases}$$

3. Bounded product:
$$t_1(a,b) = max\{0, a+b-1\}$$
4. Bounded sum:
$$s_1(a,b) = min\{1, a+b\}$$
5. Einstein product:
$$t_{1.5}(a,b) = \frac{a.b}{2-[a+b-a.b]}$$
6. Einstein sum:
$$s_{1.5}(a,b) = \frac{a+b}{1+a.b}$$
7. Algebraic product:
$$t_2(a,b) = a.b$$
8. Algebraic sum:
$$s_2(a,b) = a+b-a.b$$
9. Hamacher product:
$$t_{2.5}(a,b) = \frac{a.b}{a+b-a.b}$$
10. Hamacher sum:
$$s_{2.5}(a,b) = \frac{a+b-2.a.b}{1-a.b}$$
11. Minumum:
$$t_3(a,b) = min\{a,b\}$$
12. Maximum:
$$s_3(a,b) = max\{a,b\}$$

3 Neutrosophic Refined Sets

In this section, we present some definitions of neutrosophic refined sets with operations. Also we have examined some desired properties of neutrosophic refined sets based on these definitions and operations. Some of it is quoted from [29, 32, 38, 39, 40].

In the following, some definition and operatios on intuitionistic fuzzy multiset defined in [18, 29], we extend this definition to NRS by using [20, 40].

Definition 3.1. [40, 44] Let E be a universe. A neutrosophic refined set (NRS) A on E can be defined as follows:

$$\begin{aligned} A &= \{< x, (T_A^1(x), T_A^2(x), ..., T_A^P(x)), (I_A^1(x), I_A^2(x), ..., I_A^P(x)), \\ &\qquad (F_A^1(x), F_A^2(x), ..., F_A^P(x)) >: \ x \in E\} \end{aligned}$$

where, $T_A^1(x), T_A^2(x), ..., T_A^P(x) : E \to [0,1]$, $I_A^1(x), I_A^2(x), ..., I_A^P(x) : E \to [0,1]$ and $F_A^1(x), F_A^2(x), ..., F_A^P(x) : E \to [0,1]$ such that $0 \leq T_A^i(x)+I_A^i(x)+F_A^i(x) \leq 3 (i = 1,2,...,P)$ and $T_A^1(x) \leq T_A^2(x) \leq ... \leq T_A^P(x)$ for any $x \in E$. $(T_A^1(x), T_A^2(x), ..., T_A^P(x))$,

$(I_A^1(x), I_A^2(x), ..., I_A^P(x))$ and $(F_A^1(x), F_A^2(x), ..., F_A^P(x))$ is the truth membership sequence, indeterminacy membership sequence and falsity membership sequence of the element x, respectively. Also, P is called the dimension of NRS A.

In [44] truth membership sequences are increase and other sequences (indeterminacy membership, falsity membership) are not increase or decrease. But throughout this paper the truth membership sequences, indeterminacy membership sequences , falsity membership sequences are not increase or decrease. The set of all Neutrosophic refined sets on E is denoted by NRS(E).

Definition 3.2. [44] Let $A, B \in NRS(E)$. Then,

1. A is said to be NM subset of B is denoted by $A \tilde{\subseteq} B$ if $T_A^i(x) \leq T_B^i(x)$, $I_A^i(x) \geq I_B^i(x)$,$F_A^i(x) \geq F_B^i(x)$, $\forall x \in E$.

2. A is said to be neutrosophic equal of B is denoted by $A = B$ if $T_A^i(x) = T_B^i(x)$, $I_A^i(x) = I_B^i(x)$,$F_A^i(x) = F_B^i(x)$, $\forall x \in E$.

3. the complement of A denoted by $A^{\tilde{c}}$ and is defined by

$$A^{\tilde{c}} = \{< x, (F_A^1(x), F_A^2(x), ..., F_A^P(x)), (1 - I_A^1(x), 1 - I_A^2(x), ..., 1 - I_A^P(x)), (T_A^1(x), T_A^2(x), ..., T_A^P(x)) >: x \in E\}$$

In the following, some definitions and operations with properties on neutrosophic multi set defined in [16, 44, 45], we generalized these definitions.

Definition 3.3. Let $A, B \in NRS(E)$. Then,

1. If $T_A^i(x) = 0$ and $I_A^i(x) = F_A^i(x) = 1$ for all $x \in E$ and $i = 1, 2, ..., P$ then A is called null ns-set and denoted by $\tilde{\Phi}$.

2. If $T_A^i(x) = 1$ and $I_A^i(x) = F_A^i(x) = 0$ for all $x \in E$ and $i = 1, 2, ..., P$, then A is called universal ns-set and denoted by $\tilde{E}$.

Definition 3.4. Let $A, B \in NRS(E)$. Then,

1. the union of A and B is denoted by $A \tilde{\cup} B = C_1$ and is defined by

$$C = \{< x, (T_C^1(x), T_C^2(x), ..., T_C^P(x)), (I_C^1(x), I_C^2(x), ..., I_C^P(x)), (F_C^1(x), F_C^2(x), ..., F_C^P(x)) >: x \in E\}$$

where $T_C^i = s\{T_A^i(x), T_B^i(x)\}$, $I_C^i = t\{I_A^i(x), I_B^i(x)\}$,$F_C^i = t\{F_A^i(x), F_B^i(x)\}$, $\forall x \in E$ and $i = 1, 2, ..., P$.

2. the intersection of A and B is denoted by $A \tilde{\cap} B = D$ and is defined by

$$D = \{< x, (T_D^1(x), T_D^2(x), ..., T_D^P(x)), (I_D^1(x), I_D^2(x), ..., I_D^P(x)), (F_D^1(x), F_D^2(x), ..., F_D^P(x)) >: x \in E\}$$

where $T_D^i = t\{T_A^i(x), T_B^i(x)\}$, $I_D^i = s\{I_A^i(x), I_B^i(x)\}$,$F_D^i = s\{F_A^i(x), F_B^i(x)\}$, $\forall x \in E$ and $i = 1, 2, ..., P$.

Proposition 3.5. Let $A, B, C \in NRS(E)$. Then,

1. $A \tilde{\cup} B = B \tilde{\cup} A$ and $A \tilde{\cap} B = B \tilde{\cap} A$

2. $A\widetilde{\cup}(B\widetilde{\cup}C) = (A\widetilde{\cup}B)\widetilde{\cup}C$ and $A\widetilde{\cap}(B\widetilde{\cap}C) = (A\widetilde{\cap}B)\widetilde{\cap}C$

Proof: The proofs can be easily made.

Proposition 3.6. Let $A, B, C \in NRS(E)$. Then,

1. $A\widetilde{\cup}A = A$ *and* $A\widetilde{\cap}A = A$
2. $A\widetilde{\cap}\Phi = \tilde{\Phi}$ *and* $A\widetilde{\cap}E = A$
3. $A\widetilde{\cup}\Phi = A$ *and* $A\widetilde{\cup}E = \tilde{E}$
4. $A\widetilde{\cap}(B\widetilde{\cup}C) = (A\widetilde{\cap}B)\widetilde{\cup}(A\widetilde{\cap}C)$ *and* $A\widetilde{\cup}(B\widetilde{\cap}C) = (A\widetilde{\cup}B)\widetilde{\cap}(A\widetilde{\cup}C)$
5. $(A^{\tilde{c}})^{\tilde{c}} = A.$

Proof. It is clear from Definition 3.3-3.4.

Theorem 3.7. Let $A, B \in NRS(E)$. Then, De Morgan's law is valid.

1. $(A\widetilde{\cup}B)^{\tilde{c}} = A^{\tilde{c}}\widetilde{\cap}B^{\tilde{c}}$
2. $(A\widetilde{\cap}B)^{\tilde{c}} = A^{\tilde{c}}\widetilde{\cup}B^{\tilde{c}}$

Proof. $A, B \in NRS(E)$ is given. *From Definition 3.2 and Definition 3.4, we have*

1.

$$\begin{aligned}
(A\widetilde{\cup}B)^{\tilde{c}} &= \{< x, (s\{T_A^1(x), T_B^1(x)\}, s\{T_A^2(x), T_B^2(x)\}, ..., s\{T_A^P(x), T_B^P(x)\}),\\
&\quad (t\{I_A^1(x), I_B^1(x)\}, t\{I_A^2(x), I_B^2(x)\}, ..., t\{I_A^P(x), I_B^P(x)\}),\\
&\quad (t\{F_A^1(x), F_B^1(x)\}, t\{F_A^2(x), F_B^2(x)\}, ..., t\{F_A^P(x), F_B^P(x)\}) >: \ x \in E\}^{\tilde{c}}\\
&= \{< x, (, t\{F_A^1(x), F_B^1(x)\}, t\{F_A^2(x), F_B^2(x)\}, ..., t\{F_A^P(x), F_B^P(x)\})\\
&\quad (1 - t\{I_A^1(x), I_B^1(x)\}, 1 - t\{I_A^2(x), I_B^2(x)\}, ..., 1 - t\{I_A^P(x), I_B^P(x)\}),\\
&\quad (s\{T_A^1(x), T_B^1(x)\}, s\{T_A^2(x), T_B^2(x)\}, ..., s\{T_A^P(x), T_B^P(x)\}) >: \ x \in E\}\\
&= \{< x, (, t\{F_A^1(x), F_B^1(x)\}, t\{F_A^2(x), F_B^2(x)\}, ..., t\{F_A^P(x), F_B^P(x)\})\\
&\quad (s\{1 - I_A^1(x), 1 - I_B^1(x)\}, s\{1 - I_A^2(x), 1 - I_B^2(x)\}, ...,\\
&\quad s\{1 - I_A^P(x), 1 - I_B^P(x)\}),\\
&\quad (s\{T_A^1(x), T_B^1(x)\}, s\{T_A^2(x), T_B^2(x)\}, ..., s\{T_A^P(x), T_B^P(x)\}) >: \ x \in E\}\\
&= A^{\tilde{c}}\widetilde{\cap}B^{\tilde{c}}.
\end{aligned}$$

2.

$$\begin{aligned}
(A\widetilde{\cap}B)^{\widetilde{c}} &= \{< x, (t\{T_A^1(x), T_B^1(x)\}, t\{T_A^2(x), T_B^2(x)\}, ..., t\{T_A^P(x), T_B^P(x)\}),\\
&\quad (s\{I_A^1(x), I_B^1(x)\}, s\{I_A^2(x), I_B^2(x)\}, ..., s\{I_A^P(x), I_B^P(x)\}),\\
&\quad (s\{F_A^1(x), F_B^1(x)\}, s\{F_A^2(x), F_B^2(x)\}, ..., s\{F_A^P(x), F_B^P(x)\}) >: x \in E\}^{\widetilde{c}}\\
&= \{< x, (, s\{F_A^1(x), F_B^1(x)\}, s\{F_A^2(x), F_B^2(x)\}, ..., s\{F_A^P(x), F_B^P(x)\})\\
&\quad (1 - s\{I_A^1(x), I_B^1(x)\}, 1 - s\{I_A^2(x), I_B^2(x)\}, ..., 1 - s\{I_A^P(x), I_B^P(x)\}),\\
&\quad (t\{T_A^1(x), T_B^1(x)\}, t\{T_A^2(x), T_B^2(x)\}, ..., t\{T_A^P(x), T_B^P(x)\}) >: x \in E\}\\
&= \{< x, (, s\{F_A^1(x), F_B^1(x)\}, s\{F_A^2(x), F_B^2(x)\}, ..., s\{F_A^P(x), F_B^P(x)\})\\
&\quad (t\{1 - I_A^1(x), 1 - I_B^1(x)\}, t\{1 - I_A^2(x), 1 - I_B^2(x)\}, ...,\\
&\quad t\{1 - I_A^P(x), 1 - I_B^P(x)\}),\\
&\quad (t\{T_A^1(x), T_B^1(x)\}, t\{T_A^2(x), T_B^2(x)\}, ..., t\{T_A^P(x), T_B^P(x)\}) >: x \in E\}\\
&= A^{\widetilde{c}}\widetilde{\cap}B^{\widetilde{c}}.
\end{aligned}$$

Theorem 3.8. Let P be the power set of all NRS defined in the universe E. Then $(P, \widetilde{\cap}, \widetilde{\cup})$ is a distributive lattice.

Proof: The proofs can be easily made by showing properties; idempotency, commutativity, associativity and distributivity

Definition 3.9. Let E is a real Euclidean space E^n. Then, a NRS A is convex if and only if

$$T_A^i(ax + (1-a)y) \geq T_A^i(x) \wedge T_A(y), I_A^i(ax + (1-a)y) \leq I_A^i(x) \vee I_A^i(y)$$

$$F_A^i(ax + (1-a)y) \leq F_A^i(x) \vee F_A^i(y)$$

for every $x, y \in E$, $a \in I$ and $i = 1, 2, ..., P$.

Definition 3.10. Let E is a real Euclidean space E^n. Then, a NRS A is strongly convex if and only if

$$T_A^i(ax + (1-a)y) > T_A^i(x) \wedge T_A(y), I_A^i(ax + (1-a)y) < I_A^i(x) \vee I_A^i(y)$$

$$F_A^i(ax + (1-a)y) < F_A^i(x) \vee F_A^i(y)$$

for every $x, y \in E$, $a \in I$ and $i = 1, 2, ..., P$.

Theorem 3.11. Let $A, B \in NRS(E)$. Then, $A\widetilde{\cap}B$ is a convex(strongly convex) when both A and B are convex(strongly convex).

Proof. It is clear from Definition 3.9-3.10.

Definition 3.12. [16] Let $A, B \in NRS(E)$. Then,

1. Hamming distance $d_{HD}(A, B)$ between A and B, defined by;

$$d_{HD}(A, B) = \sum_{j=1}^{P}\sum_{i=1}^{n}(|T_A^j(x_i) - T_B^j(x_i)| + |I_A^j(x_i) - I_B^j(x_i)| + |F_A^j(x_i) - F_B^j(x_i)|)$$

2. Normalized hamming distance $d_{NHD}(A, B)$ between A and B, defined by;

$$d_{NHD}(A, B) = \frac{1}{3nP}\sum_{j=1}^{P}\sum_{i=1}^{n}(|T_A^j(x_i) - T_B^j(x_i)| + |I_A^j(x_i) - I_B^j(x_i)| + |F_A^j(x_i) - F_B^j(x_i)|)$$

3. Euclidean distance $d_{ED}(A,B)$ between A and B, defined by;

$$d_{ED}(A,B)=\sum_{j=1}^{P}\sum_{i=1}^{n}\sqrt{\begin{array}{r}(T_A^j(x_i)-T_B^j(x_i))^2+(I_A^j(x_i)-I_B^j(x_i))^2+\\(F_A^j(x_i)-F_B^j(x_i))^2\end{array}}$$

4. Normalized euclidean distance $d_{NED}(A,B)$ between A and B, defined by;

$$d_{NED}(A,B)=\frac{1}{3n.P}\sum_{j=1}^{P}\sum_{i=1}^{n}\sqrt{\begin{array}{r}(T_A^j(x_i)-T_B^j(x_i))^2+(I_A^j(x_i)-I_B^j(x_i))^2+\\(F_A^j(x_i)-F_B^j(x_i))^2\end{array}}$$

4 Medical Diagnosis Via NRS Theory

In the following, the example on intuitionistic fuzzy multiset given in [18, 31, 33, 38], we extend this definition to NRS.

Let P={P_1,P_2,P_3,P_4} be a set of patients, D={Viral Fever, Tuberculosis, Typhoid, Throat disease}be a set of diseases and S={Temperature, cough, throat pain,headache, body pain} be a set of symptoms. In Table I each symptom S_i is described by three numbers: Membership T, non-membership F and indeterminacy I.

	Viral Fever	Tuberculosis	Typhoid	Throat disease
Temperature	(0.8,0.2,0.1)	(0.3,0.4,0.2)	(0.4,0.6,0.3)	(0.5,0.7,0.1)
Cough	(0.2,0.3,0.7)	(0.2,0.5,0.3)	(0.4,0.5,0.4)	(0.8,0.3,0.2)
Throat Pain	(0.3,0.4,0.5)	(0.4,0.4,0.3)	(0.3,0.6,0.4)	(0.6,0.5,0.4)
Headache	(0.5,0.3,0.3)	(0.5,0.2,0.3)	(0.5,0.6,0.2)	(0.4,0.3,0.5)
Body Pain	(0.5,0.2,0.4)	(0.4,0.5,0.3)	(0.6,0.5,0.3)	(0.2,0.6,0.4)

Table I -NRS R: The relation among Symptoms and Diseases

The results obtained different time intervals such as: 8:00 am 12:00 am and 4:00 pm in a day as Table II;

	Temparature	Cough	Throat pain	Headache	Body Pain
P_1	(0.1, 0.3, 0.7) (0.2, 0.4, 0.6) (0.1, 0.1, 0.9)	(0.3, 0.2, 0.6) (0.2, 0.4, 0) (0.1, 0.3, 0.7)	(0.8, 0.5, 0) (0.7, 0.6, 0.1) (0.8, 0.3, 0.1)	(0.3, 0.3, 0.6) (0.2, 0.4, 0.7) (0.2, 0.3, 0.6)	(0.4, 0.4, 0.4) (0.3, 0.2, 0.7) (0.2, 0.3, 0.7)
P_2	(0.5, 0.3, 0.3) (0.3, 0.4, 0.5) (0.4, 0.2, 0.6)	(0.7, 0.3, 0.6) (0.6, 0.4, 0.3) (0.4, 0.1, 0.7)	(0.8, 0.6, 0.1) (0.6, 0.3, 0.1) (0.7, 0.5, 0.1)	(0.4, 0.2, 0.6) (0.5, 0.4, 0.7) (0.4, 0.3, 0.6)	(0.6, 0.2, 0.4) (0.5, 0.4, 0.6) (0.6, 0.3, 0.6)
P_3	(0.7, 0.4, 0.6) (0.4, 0.5, 0.3) (0.3, 0.3, 0.5)	(0.7, 0.2, 0.5) (0.6, 0.5, 0.1) (0.4, 0.2, 0.2)	(0.5, 0.8, 0.4) (0.6, 0.4, 0.4) (0.7, 0.6, 0.3)	(0.6, 0.3, 0.4) (0.5, 0.3, 0.4) (0.4, 0.4, 0.5)	(0.6, 0.3, 0.3) (0.6, 0.5, 0.4) (0.6, 0.2, 0.8)
P_4	(0.3, 0.4, 0.6) (0.6, 0.3, 0.3) (0.4, 0.2, 0.5)	(0.5, 0.4, 0.4) (0.6, 0.5, 0.3) (0.4, 0.2, 0.2)	(0.5, 0.6, 0.31) (0.7, 0.5, 0.6) (0.8, 0.5, 0.3)	(0.7, 0.4, 0.2) (0.4, 0.3, 0.4) (0.3, 0.6, 0.5)	(0.3, 0.3, 0.5) (0.7, 0.5, 0.2) (0.3, 0.5, 0.4)

Table II -NRS Q: the relation Beween Patient and Symptoms.

The normalized Hamming distance between Q and R is computed as;

	Viral Fever	Tuberculosis	Typhoid	Throat disease
P_1	0.266	**0.23**	0.28	0.25
P_2	0.213	0.202	0.206	**0.19**
P_3	0.206	0.173	**0.16**	0.166
P_4	0.22	0.155	**0.146**	0.157

Table III :The normalized Hamming distance between Q and R

The lowest distance from the table III gives the proper medical diagnosis. Patient P_1 suffers from Tuberculosis, Patient P_2 suffers from Throat diseas, Patient P_3 suffers from Typhoid disease and Patient P_4 suffers from Typhoid

5 Conclusion

In this paper, we firstly defined some definitions on neutrosophic refined sets and investigated some of their basic properties. The concept of neutrosophic refined (NRS) generalizes the fuzzy multisets and intuitionstic fuzzy multisets. Then, an application of NRS in medical diagnosis is discussed. In the proposed method, we measured the distances of each patient from each diagnosis by considering the symptoms of that particular disease.

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ON RANKING OF TRAPEZOIDAL INTUITIONISTIC FUZZY NUMBERS AND ITS APPLICATION TO MULTI ATTRIBUTE GROUP DECISION MAKING

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Abstract – This paper focuses on the study of two characteristics of trapezoidal intuitionistic fuzzy number (TRIFN), viz., Value index and Ambiguity index. Based on these two indexes, we develop an algorithm for ranking of trapezoidal intuitionistic fuzzy number (TRIFN). Furthermore, we present an application of this ranking method in multi attribute group decision making problem. An illustrative numerical example demonstrate our approach to multi attribute group decision making problem.

Keywords – *Value, ambiguity, ranking, trapezoidal intuitionistic fuzzy numbers, multi attribute group decision making.*

1 Introduction

The theory of intuitionistic fuzzy sets were introduced by Atanassov [1] as a generalization of fuzzy set theory proposed by Zadeh [14]. The notion of fuzzy numbers were extended to develop the concept of intuitionistic fuzzy numbers by adding an additional non-membership function which is able to express more abundant and flexible information as compared to fuzzy numbers. Various definitions of intuitionistic fuzzy numbers and ranking methods have been proposed over last few years. Mitchell [7] introduced a ranking method for intuitionistic fuzzy number considering intuitionistic fuzzy numbers as an ensemble of fuzzy numbers. Chen and Hwang [2] introduced a ranking method based on scorings of intuitionistic fuzzy numbers. The concept of Chen and Hwang

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have been later generalized by Nayagam et.al [4] to formulate a new process of ranking called method of IF scorning. Wang [8] gave the definition of intuitionistic trapezoidal fuzzy number and interval intuitionistic fuzzy number. Further Wang and Zhang [9] defined the trapezoidal intuitionistic fuzzy numbers and gave a ranking method which transformed the ranking of trapezoidal intuitionistic fuzzy number in to ranking of interval numbers. Li [6] developed a ratio ranking method for triangular intuitionistic fuzzy numbers and applied to multi attribute decision making. The application of trapezoidal intuitionistic fuzzy numbers in decision making problems are abundant in literature [[5],[6],[8]- [11]]. Since ranking of alternative plays an efficient role in decision making problems,ranking of trapezoidal intuitionistic fuzzy number has become a task of outmost importance when we deal with decision making problems based on intuitionistic fuzzy information. In this article we have paid attention to the formulation of a ranking algorithm based on linear sum of value and ambiguity indexes and the procedure has been applied to rank the alternatives in multi attribute group decision making (MAGDM) problems. However, to solve the MAGDM problem, we have adopted the method suggested by Wu and Cao [10] which is based on intuitionistic trapezoidal fuzzy weighted geometric operator (ITFWG) and intuitionistic trapezoidal fuzzy hybrid geometric operator (ITFHG).

The rest of the paper is set out as follows: Section 2 includes basic definitions and operations of trapezoidal intutionistic fuzzy numbers. Section 3 consist of the algorithm which have been developed for ranking of trapezoidal intutionistic fuzzy numbers. In section 4 and 5, application of the formulated algorithm have been illustrated by giving a suitable numerical example. Section 6 contains the conclusion of this article.

2 Preliminaries

We collect some basic definitions and notations related to trapezoidal intitionistic fuzzy number.

Definition 2.1. [8] A TRIFN $\tilde{a} = \langle (a_1, a_2, a_3, a_4); w_{\tilde{a}}, u_{\tilde{a}} \rangle$ is a special Intuitionistic Fuzzy set on a set of real number $\mathbb{R}$, whose membership function and non membership function are defined as follows:

$$\mu_{\tilde{a}}(x) = \begin{cases} \dfrac{(x-a_1)}{(a_2-a_1)} w_{\tilde{a}} & a_1 \leq x \leq a_2 \\ w_{\tilde{a}} & a_2 \leq x \leq a_3 \\ \dfrac{(a_4-x)}{(a_4-a_3)} w_{\tilde{a}} & a_3 \leq x \leq a_4 \\ 0 & a_4 < x or a_1 > x \end{cases} \tag{1}$$

$$\nu_{\tilde{a}}(x) = \begin{cases} \dfrac{(a_2-x)+u_{\tilde{a}}(x-a_1)}{(a_2-a_1)} & a_1 \leq x \leq a_2 \\ u_{\tilde{a}} & a_2 \leq x \leq a_3 \\ \dfrac{(x-a_3)+u_{\tilde{a}}(a_4-x)}{(a_4-a_3)} & a_3 \leq x \leq a_4 \\ 1 & a_4 < x or a_1 > x \end{cases} \tag{2}$$

respectively.

The values $w_{\tilde{a}}$ and $u_{\tilde{a}}$ represents the maximum degree of membership and minimum degree of non membership, respectively, such that the conditions $0 \leq w_{\tilde{a}} \leq 1$, $0 \leq u_{\tilde{a}} \leq 1$ and $0 \leq w_{\tilde{a}} + u_{\tilde{a}} \leq 1$ are satisfied. The parameters $w_{\tilde{a}}$ and $u_{\tilde{a}}$ reflects the confidence level and non confidence level of the TRIFN $\tilde{a} = \langle (a_1, a_2, a_3, a_4); w_{\tilde{a}}, u_{\tilde{a}} \rangle$,respectively.

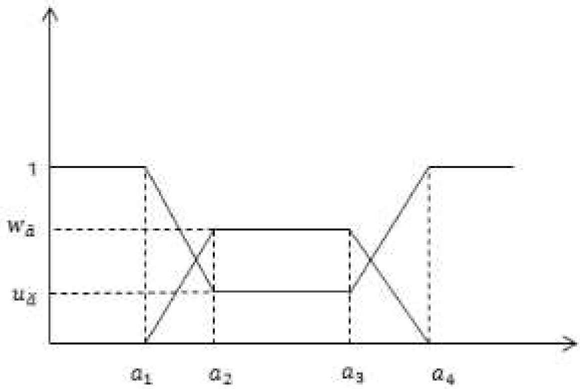

Figure 1: Trapezoidal Intuitionistic fuzzy numbers(TRIFN)

The function $\pi_{\tilde{a}}(x) = 1 - \mu_{\tilde{a}}(x) - \nu_{\tilde{a}}(x)$ is called an IF index of an element x in $\tilde{a}$.It is the degree of the indeterminacy membership of the element x in $\tilde{a}$.

Arithmatical operations of trapezoidal intuitionistic fuzzy number

Definition 2.2. [9] Let $\tilde{a} = \langle (a_1, a_2, a_3, a_4); w_{\tilde{a}}, u_{\tilde{a}} \rangle$ and $\tilde{b} = \langle (b_1, b_2, b_3, b_4); w_{\tilde{b}}, u_{\tilde{b}} \rangle$ be two TRIFNs and λ be a real number. The arithmetical operations are listed as follows:

- $\tilde{a} \oplus \tilde{b} = \langle (a_1 + b_1, a_2 + b_2, a_3 + b_3, a_4 + b_4); w_{\tilde{a}} + w_{\tilde{b}} - w_{\tilde{a}} w_{\tilde{b}}, u_{\tilde{a}} u_{\tilde{b}} \rangle$
- $\tilde{a} \otimes \tilde{b} = \langle (a_1 b_1, a_2 b_2, a_3 b_3, a_4 b_4); w_{\tilde{a}} w_{\tilde{b}}, u_{\tilde{a}} + u_{\tilde{b}} - u_{\tilde{a}} u_{\tilde{b}} \rangle$
- $\lambda \tilde{a} = \langle (\lambda a_1, \lambda a_2, \lambda a_3, \lambda a_4); 1 - (1 - w_{\tilde{a}})^\lambda, u_{\tilde{a}}^\lambda \rangle$
- $\tilde{a}^\lambda = \langle (a_1^\lambda, a_2^\lambda, a_3^\lambda, a_4^\lambda); w_{\tilde{a}}^\lambda, 1 - (1 - u_{\tilde{a}})^\lambda \rangle$

Definition 2.3. A α-cut set of a TRIFN $\tilde{a} = \langle (a_1, a_2, a_3, a_4); w_{\tilde{a}}, u_{\tilde{a}} \rangle$ is a crisp sub set of $\mathbb{R}$, denoted and defined as $\tilde{a}_\alpha = \{x | \mu_{\tilde{a}}(x) \geq \alpha\}$ where $0 \leq \alpha \leq w_{\tilde{a}}$. The α-cut set of a TRIFN $\tilde{a}$ can be represented as the closed interval $[L_{\tilde{a}}(\alpha), R_{\tilde{a}}(\alpha)] = [a_1 + \frac{\alpha(a_2 - a_1)}{w_{\tilde{a}}}, a_4 - \frac{\alpha(a_4 - a_3)}{w_{\tilde{a}}}]$.

Definition 2.4. A β-cut set of a TRIFN $\tilde{a} = \langle (a_1, a_2, a_3, a_4); w_{\tilde{a}}, u_{\tilde{a}} \rangle$ is a crisp sub set of $\mathbb{R}$, denoted and defined as $\tilde{a}_\beta = \{x | \nu_{\tilde{a}}(x) \leq \beta\}$ where $u_{\tilde{a}} \leq \beta \leq 1$. The β-cut set of a TRIFN $\tilde{a}$ can be represented as the closed interval$[L_{\tilde{a}}(\beta), R_{\tilde{a}}(\beta)] = [\frac{(1-\beta)a_2 + (\beta - u_{\tilde{a}})a_1}{1 - u_{\tilde{a}}}, \frac{(1-\beta)a_3 + (\beta - u_{\tilde{a}})a_4}{1 - u_{\tilde{a}}}]$.

Value and ambiguity of a trapezoidal intuitionistic fuzzy number

The value and ambiguity of a trapezoidal intuitionistic fuzzy number can be defined similarly to those of a triangular intuitionistic fuzzy number(TIFNs) introduced by D.F.Li [6].

Definition 2.5. [3] Let $\tilde{a}_\alpha$ and $\tilde{a}_\beta$ be an α-cut set and a β-cut set of a trapezoidal intuitionistic fuzzy number $\tilde{a} = \langle (a_1, a_2, a_3, a_4); w_{\tilde{a}}, u_{\tilde{a}} \rangle$, respectively. Then the values of the membership function $\mu_{\tilde{a}}(x)$ and the non-membership function $\nu_{\tilde{a}}(x)$ for the trapezoidal intuitionistic fuzzy number $\tilde{a}$ are defined as follows:

$$V_\mu(\tilde{a}) = \int_0^{w_{\tilde{a}}} \frac{L_{\tilde{a}}(\alpha) + R_{\tilde{a}}(\alpha)}{2} f(\alpha) d\alpha \tag{3}$$

$$V_\nu(\tilde{a}) = \int_{u_{\tilde{a}}}^1 \frac{L_{\tilde{a}}(\beta) + R_{\tilde{a}}(\beta)}{2} g(\beta) d\beta \tag{4}$$

respectively,where the function $f(\alpha)$ is a non-negative and non-decreasing function on the interval $[0, w_{\tilde{a}}]$ with $f(0) = 0$ and $\int_0^{w_{\tilde{a}}} f(\alpha)d\alpha = w_{\tilde{a}}$; the function $g(\beta)$ is a non-negative and non-increasing function on the interval $[u_{\tilde{a}}, 1]$ with $g(1) = 0$ and $\int_{u_{\tilde{a}}}^1 g(\beta)d\beta = 1 - u_{\tilde{a}}$.

Definition 2.6. [3] Let $\tilde{a}_\alpha$ and $\tilde{a}_\beta$ be an α-cut set and a β-cut set of a trapezoidal intuitionistic fuzzy number $\tilde{a} = \langle (a_1, a_2, a_3, a_4); w_{\tilde{a}}, u_{\tilde{a}} \rangle$, respectively. Then the ambiguities of the membership function $\mu_{\tilde{a}}(x)$ and the non-membership function $\nu_{\tilde{a}}(x)$ for the trapezoidal intuitionistic fuzzy number $\tilde{a}$ are defined as follows:

$$A_\mu(\tilde{a}) = \int_0^{w_{\tilde{a}}} (R_{\tilde{a}}(\alpha) - L_{\tilde{a}}(\alpha)) f(\alpha) d\alpha \tag{5}$$

$$A_\nu(\tilde{a}) = \int_{u_{\tilde{a}}}^1 (R_{\tilde{a}}(\beta) - L_{\tilde{a}}(\beta)) g(\beta) d\beta \tag{6}$$

respectively.

Remark 2.7. The weight functions $f(\alpha)$ and $g(\beta)$ can be chosen according to decision maker's choice. We shall choose $f(\alpha) = \alpha/w_{\tilde{a}}$ and $g(\beta) = (1-\beta)/(1-u_{\tilde{a}})$ in this paper. The value of membership and non-membership can be calculated substituting these $f(\alpha)$ and $g(\beta)$ in equation (2) and (3) as follows:

$$\begin{aligned} V_\mu(\tilde{a}) &= \int_0^{w_{\tilde{a}}} \left[a_1 + \frac{\alpha(a_2 - a_1)}{w_{\tilde{a}}} + a_4 - \frac{\alpha(a_4 - a_3)}{w_{\tilde{a}}} \right] \frac{\alpha}{2w_{\tilde{a}}} d\alpha \\ &= \left[\frac{a_1 + a_4}{4w_{\tilde{a}}} \alpha^2 \right]_0^{w_{\tilde{a}}} + \left[\frac{(a_2 - a_1 - a_4 + a_3)}{6(w_{\tilde{a}})^2} \alpha^3 \right]_0^{w_{\tilde{a}}} \\ &= \frac{a_1 + a_4 + 2(a_2 + a_3)}{12} w_{\tilde{a}} \end{aligned}$$

$$\begin{aligned}
V_\nu(\tilde{a}) &= \int_{u_{\tilde{a}}}^{1}\left[\frac{(1-\beta)a_2+(\beta-u_{\tilde{a}})a_1}{1-u_{\tilde{a}}}+\frac{(1-\beta)a_3+(\beta-u_{\tilde{a}})a_4}{1-u_{\tilde{a}}}\right]\frac{(1-\beta)}{2(1-u_{\tilde{a}})}d\beta \\
&= \int_{u_{\tilde{a}}}^{1}\frac{(a_2+a_3-a_1-a_4)(1-\beta)^2+(1-u_{\tilde{a}})(a_1+a_4)(1-\beta)}{2(1-u_{\tilde{a}})}d\beta \\
&= -\left[\frac{(a_2+a_3-a_1-a_4)(1-\beta)^3}{6(1-u_{\tilde{a}})^2}\right]_{u_{\tilde{a}}}^{1}-\left[\frac{(a_1+a_4)(1-u_{\tilde{a}})(1-\beta)^2}{4(1-u_{\tilde{a}})^2}\right]_{u_{\tilde{a}}}^{1} \\
&= \frac{[a_1+a_4+2(a_2+a_3)](1-u_{\tilde{a}})}{12}
\end{aligned}$$

Similarly, the ambiguity of membership and non-membership can be calculated by substituting the values of $f(\alpha)$ and $g(\beta)$ in equation (4) and (5) as follows:

$$\begin{aligned}
A_\mu(\tilde{a}) &= \int_0^{w_{\tilde{a}}}\left[a_4-\frac{\alpha(a_4-a_3)}{w_{\tilde{a}}}-a_1-\frac{\alpha(a_2-a_1)}{w_{\tilde{a}}}\right]\frac{\alpha}{w_{\tilde{a}}}d\alpha \\
&= \left[\frac{a_4-a_1}{w_{\tilde{a}}}\alpha^2\right]_0^{2w_{\tilde{a}}}-\left[\frac{(a_2-a_1+a_4-a_3)}{3(w_{\tilde{a}})^2}\alpha^3\right]_0^{w_{\tilde{a}}} \\
&= \frac{(a_4-a_1)-2(a_2-a_3)}{6}w_{\tilde{a}}
\end{aligned}$$

$$\begin{aligned}
A_\nu(\tilde{a}) &= \int_{u_{\tilde{a}}}^{1}\left[\frac{(1-\beta)a_3+(\beta-u_{\tilde{a}})a_4}{1-u_{\tilde{a}}}-\frac{(1-\beta)a_2+(\beta-u_{\tilde{a}})a_1}{1-u_{\tilde{a}}}\right]\frac{(1-\beta)}{1-u_{\tilde{a}}}d\beta \\
&= \int_{u_{\tilde{a}}}^{1}\frac{[-(a_2-a_3-a_1+a_4)(1-\beta)^2+(1-u_{\tilde{a}})(a_4-a_1)(1-\beta)]}{(1-u_{\tilde{a}})^2}d\beta \\
&= \left[\frac{(a_4-a_1+a_2-a_3)(1-\beta)^3}{3(1-u_{\tilde{a}})^2}\right]_{u_{\tilde{a}}}^{1}-\left[\frac{(a_4-a_1)(1-u_{\tilde{a}})(1-\beta)^2}{2(1-u_{\tilde{a}})^2}\right]_{u_{\tilde{a}}}^{1} \\
&= \frac{(a_4-a_1)-2(a_2-a_3)}{6}(1-u_{\tilde{a}})
\end{aligned}$$

3 Ranking of Trapezoidal Intuitionistic Fuzzy Number

The algorithm for ranking of trapezoidal intuitionistic fuzzy numbers is as follows:

- **Step-1:** Compute value and ambiguity of a trapezoidal intuitionistic fuzzy number as follows:

  (i) Evaluate value of membership $V_\mu(\tilde{a})$ and value of non-membership $V_\nu(\tilde{a})$ of a trapezoidal intutionistic fuzzy number using the following formulas:

$$V_\mu(\tilde{a})=\frac{(a_1+a_4)+2(a_2+a_3)}{12}w_{\tilde{a}}$$

$$V_\nu(\tilde{a}) = \frac{(a_1 + a_4) + 2(a_2 + a_3)}{12}(1 - u_{\tilde{a}})$$

With the condition that $0 \leq w_{\tilde{a}} + u_{\tilde{a}} \leq 1$,it follows that $V_\mu(\tilde{a}) \leq V_\nu(\tilde{a})$. Thus, the values of membership and non-membership functions of a TRIFN $\tilde{a}$ may be concisely expressed as an interval $[V_\mu(\tilde{a}), V_\nu(\tilde{a})]$.

(ii) The ambiguity of membership $A_\mu(\tilde{a})$ and ambiguity of non-membership $A_\nu(\tilde{a})$ of a trapezoidal intutionistic fuzzy number using the following formulas:

$$A_\mu(\tilde{a}) = \frac{(a_4 - a_1) - 2(a_2 - a_3)}{6}w_{\tilde{a}}$$

$$A_\nu(\tilde{a}) = \frac{(a_4 - a_1) - 2(a_2 - a_3)}{6}(1 - u_{\tilde{a}})$$

With the condition that $0 \leq w_{\tilde{a}} + u_{\tilde{a}} \leq 1$,it follows that $A_\mu(\tilde{a}) \leq A_\nu(\tilde{a})$. Thus, the ambiguities of membership and non-membership functions of a TRIFN $\tilde{a}$ may be concisely expressed as an interval $[A_\mu(\tilde{a}), A_\nu(\tilde{a})]$.

- **Step-2:** Compute value and ambiguity indices of a TRIFN given by the formulae:

$$V(\tilde{a}) = \frac{V_\mu(\tilde{a}) + V_\nu(\tilde{a})}{2}$$

$$A(\tilde{a}) = \frac{A_\mu(\tilde{a}) + A_\nu(\tilde{a})}{2}$$

- **Step-3:** Next we define ranking function for trapezoidal intuitionistic fuzzy number $\tilde{a}$ as

$$R(\tilde{a}) = V(\tilde{a}) + A(\tilde{a})$$

4 Application of Ranking Method to Multi Attribute Group Decision Making (MAGDM)

In this section we shall apply the above discussed ranking method to MAGDM using trapezoidal intuitionistic fuzzy numbers. To solve MAGDM problem we shall employ the method based on ITFWG and ITFHG operators defined by Wu and Cao [10].

We collect some basic notations and definitions of different types of operators.

Definition 4.1. [10] Let $\tilde{\alpha}_j (j = 1, 2, ..., n)$ be a collection of trapezoidal intuitionistic fuzzy numbers, and let ITFWG:$\Omega^n \to \Omega$, if

$$ITFWG_\omega(\tilde{\alpha}_1, \tilde{\alpha}_2, ..., \tilde{\alpha}_n) = \tilde{\alpha}_1^{\omega_1} \otimes \tilde{\alpha}_2^{\omega_2} ...... \otimes \tilde{\alpha}_n^{\omega_n}$$

then ITFWG is called intuitionistic trapezoidal fuzzy weighted geometric operator of dimension n, where $\omega = (\omega_1, \omega_2, ...\omega_n)$ is the weight vector of $\tilde{\alpha}_j (j = 1, 2, ..., n)$, with $\omega_j \in [0, 1]$ and $\sum_{j=1}^{n} \omega_j = 1$. Especially, if $\omega = (1/n, 1/n, ..., 1/n)^T$, then the ITFWG operator is reduced to an intuitionistic trapezoidal fuzzy heometric averaging(ITFGA) operator of dimension n. which is defined as follows:

$$ITFGA_\omega(\tilde{\alpha}_1, \tilde{\alpha}_2, ..., \tilde{\alpha}_n) = (\tilde{\alpha}_1 \otimes \tilde{\alpha}_2 ...... \otimes \tilde{\alpha}_n)^{1/n}$$

Definition 4.2. [10] Let $\tilde{\alpha}_j (j = 1, 2, ..., n)$ be a collection of trapezoidal intuitionistic fuzzy numbers. An intuitionistic trapezoidal fuzzy hybrid geometric (ITFHG) operator of dimension n is a mapping ITFHG:$\Omega^n \rightarrow \Omega$, that has an associated vector $\omega = (\omega_1, \omega_2, ..., \omega_n)^T$ such that $\omega_j \in [0, 1]$ and $\sum_{j=1}^{n} \omega_j = 1$.

$$ITFHG_{\omega,w}(\tilde{\alpha}_1, \tilde{\alpha}_2, ..., \tilde{\alpha}_n) = \tilde{\alpha}_{\sigma(1)}^{\omega_1} \otimes \tilde{\alpha}_{\sigma(2)}^{\omega_2} ...... \otimes \tilde{\alpha}_{\sigma(n)}^{\omega_n}$$

where $\tilde{\alpha}_{\sigma(j)}$ is the largest of the weighted intuitionistic trapezoidal fuzzy numbers $\dddot{\tilde{\alpha}}_j (\dddot{\tilde{\alpha}}_j = \tilde{\alpha}_j^{nw_j}, j = 1, 2, ..., n)$. $w = (w_1, w_2, ..., w_n)^T$ is the weight vector of the $\tilde{\alpha}_j$ with $w_j in$ [0,1] and $\sum_{j=1}^{n} w_j = 1$, and n is the balancing coefficient, which plays a role of balance in a such a case, if the vector $w = (w_1, w_2, ..., w_n)^T$ approaches $(1/n, 1/n, ..., 1/n)^T$, then the vector $(\tilde{\alpha}_1^{nw_1}, \tilde{\alpha}_2^{nw_2}, ..., \tilde{\alpha}_n^{nw_n})^T$ approaches $(\tilde{\alpha}_1, \tilde{\alpha}_2,$ ....., $\tilde{\alpha}_n)^T$.

Illustration of MAGDM problem

Let $A = \{A_1, A_2, ..., A_m\}$ be a discrete set of alternatives, and $U = \{U_1, U_2, ..., U_n\}$ be the set of attributes, $\psi = \{\psi_1, \psi_2, ..., \psi_n\}$ is the weighting vector of the attribute $U_j (j = 1, 2, ..., n)$, where $\psi > 0$, $\sum_{j=1}^{n} \psi_j = 1$. Let $D = \{d_1, d_2, ..., d_t\}$ be the set of decision makers, $w = (w_1, w_2, ..., w_t)^T$ be the weight vector of decision makers, with $w_k \in$ [0,1] and $\sum_{j=1}^{n} w_k = 1$. Suppose that $\tilde{R}^{(k)} = \left(\tilde{r}_{ij}^{(k)}\right)_{m \times n}$ $=\left(\left[a_{1_{ij}}^{(k)}, a_{2_{ij}}^{(k)}, a_{3_{ij}}^{(k)}, a_{4_{ij}}^{(k)}\right]; w_{\tilde{a}_{ij}}^{(k)}, u_{\tilde{a}_{ij}}^{(k)}\right)$ is the intuitionistic trapezoidal fuzzy decision matrix, $w_{\tilde{a}_{ij}}^{(k)} \subset$ [0,1],$u_{\tilde{a}_{ij}}^{(k)} \subset$ [0,1], $w_{\tilde{a}_{ij}}^{(k)} + u_{\tilde{a}_{ij}}^{(k)} \leq$ 1,j=1,2,....,n, i=1,2,....,m, k= 1,2,....,t.

The algorithm for solving MAGDM problem [10] is as follows:

1. First we apply the weights of attribute, and the ITFWG operator

$$\tilde{r}_i^{(k)} = ITFWG\left(\tilde{r}_{i1}^{(k)}, \tilde{r}_{i2}^{(k)}, ..., \tilde{r}_{in}^{(k)}\right), i = 1, 2, ..., m, k = 1, 2, ..., t,$$

   to derive the individual overall preference intuitionistic trapezoidal fuzzy values $\tilde{r}_i^{(k)}$ of the alternative A_i.

2. Utilizing the ITFHG operator we derive colletive overall preference intuitionistic trapezoidal fuzzy values $\tilde{r}_i$ $(i = 1, 2, ..., m)$ of the alternative A_i:

$$\tilde{r}_i = ([a_{1_i}, a_{2_i}, a_{3_i}, a_{4_i}]; w_{\tilde{r}_i}, u_{\tilde{r}_i}) = ITFHG_{\omega,W}\left(\tilde{r}_{ij}^{(1)}, \tilde{r}_{ij}^{(2)}, ..., \tilde{r}_{ij}^{(t)}\right)$$

   where $w = (w_1, w_2, ..., w_t)^T$ is the weight vector of decision makers, with $w_k \in$ [0,1] and $\sum_{j=1}^{n} w_k = 1$; $\omega = (\omega_1, \omega_2, ..., \omega_n)^T$ is the associated weight vector of the ITFHG operator,$\omega_k \in$ [0,1] and $\sum_{j=1}^{n} \omega_k = 1$.

3. Next we shall calculate value and ambiguity indices using (3.1) and (3.2), and evaluate R(.) for each alternative.

4. The best one will be choosen among the alternatives depending on their respective ranks. The greater the value of $R(\tilde{r}_i)$, the better the alternative A_i (i=1,2,....,m) will be.

5 Numerical Example

We shall consider a numerical example [10] with four alternatives $A_i (i = 1, 2, 3, 4)$ and four attributes U_1,U_2, U_3, U_4 have been considered with weighting vector $\psi = (0.22, 0.20, 0.28, 0.30)^T$. A group of four decision makers $D = \{d_1, d_2, d_3, d_4\}$ with weight vector $w = (0.20, 0.30, 0.35, 0.15)^T$ has been assigned to find the best alternative. The initial decision matrices $\tilde{R}^{(k)} = \left(\tilde{r}_{ij}^{(k)}\right)_{4\times 4}$ (k=1,2,3,4) are as follows:

$$\tilde{R}^{(1)} = \begin{bmatrix} ([0.6, 0.7, 0.8, 0.9]; 0.6, 0.2) & ([0.3, 0.4, 0.5, 0.6]; 0.6, 0.4) & ([0.5, 0.6, 0.7, 0.9]; 0.3, 0.4) & ([0.6, 0.7, 0.8, 0.9]; 0.5, 0.3) \\ ([0.7, 0.8; 0.9, 1.0]; 0.7, 0.3) & ([0.5, 0.6, 0.7, 0.8]; 0.5, 0.3) & ([0.4, 0.5, 0.7, 0.8]; 0.7, 0.3) & ([0.5, 0.7, 0.8, 0.9]; 0.4, 0.6) \\ ([0.4, 0.5, 0.6, 0.7]; 0.4, 0.2) & ([0.3, 0.4, 0.5, 0.7]; 0.6, 0.3) & ([0.2, 0.4, 0.5, 0.6]; 0.5, 0.3) & ([0.6, 0.7, 0.8, 0.9]; 0.5, 0.2) \\ ([0.1, 0.2, 0.4, 0.5]; 0.6, 0.2) & ([0.3, 0.4, 0.6, 0.7]; 0.5, 0.2) & ([0.5, 0.6, 0.7, 0.8]; 0.4, 0.3) & ([0.4, 0.5, 0.6, 0.7]; 0.5, 0.1) \end{bmatrix}$$

$$\tilde{R}^{(2)} = \begin{bmatrix} ([0, 1, 0.3, 0.4, 0.5], 0.7, 0.2) & ([0.2, 0.4, 0.6, 0.8]; 0.5, 0.2) & ([0.2, 0.5, 0.6, 0.8]; 0.4, 0.3) & ([0.1, 0.4, 0.5, 0.6]; 0.7, 0.1) \\ ([0.2, 0.4, 0.6, 0.8]; 0.6, 0.3) & ([0.4, 0.6, 0.8, 0.9]; 0.7, 0.2) & ([0.4, 0.6, 0.8, 1.0]; 0.5, 0.3) & ([0.3, 0.5, 0.6, 0.7]; 0.8, 0.1) \\ ([0.2, 0.4, 0.6, 1.0]; 0.6, 0.2) & ([0.2, 0.4, 0.6, 0.8]; 0.8, 0.2) & ([0.1, 0.2, 0.6, 0.8]; 0.6, 0.2) & ([0.1, 0.2, 0.3, 0.5]; 0.6, 0.4) \\ ([0.2, 0.3, 0.4, 0.7]; 0.5, 0.3) & ([0.1, 0.2, 0.3, 0.5]; 0.6, 0.2) & ([0.1, 0.3, 0.5, 0.7]; 0.7, 0.2) & ([0.1, 0.2, 0.4, 0.5]; 0.5, 0.3) \end{bmatrix}$$

$$\tilde{R}^{(3)} = \begin{bmatrix} ([0.6, 0.7, 0.8, 0.9]; 0.7, 0.2) & ([0.1, 0.6, 0.4, 0.3]; 0.5, 0.2) & ([0.3, 0.5, 0.6, 0.7]; 0.4, 0.3) & ([0.1, 0.2, 0.4, 0.5]; 0.7, 0.1) \\ ([0.5, 0.7, 0.8, 0.9]; 0.3, 0.5) & ([0.4, 0.5, 0.6, 0.8]; 0.7, 0.3) & ([0.4, 0.6, 0.7, 0.8]; 0.3, 0.1) & ([0.3, 0.5, 0.6, 0.8]; 0.5, 0.3) \\ ([0.7, 0.8, 0.9, 1.0]; 0.6, 0.2) & ([0.3, 0.4, 0.6, 0.7]; 0.5, 0.2) & ([0.1, 0.2, 0.6, 0.8]; 0.5, 0.3) & ([0.1, 0.2, 0.4, 0.5]; 0.6, 0.3) \\ ([0.4, 0.5, 0.7, 0.9]; 0.5, 0.3) & ([0.1, 0.2, 0.3, 0.4]; 0.4, 0.1) & ([0.1, 0.3, 0.5, 0.6]; 0.6, 0.2) & ([0.1, 0.2, 0.3, 0, 5]; 0.5, 0.2) \end{bmatrix}$$

$$\tilde{R}^{(4)} = \begin{bmatrix} ([0.4, 0.5, 0.7, 0.8]; 0.4, 0.5) & ([0.4, 0.5, 0.6, 0.7]; 0.6, 0.4) & ([0.5, 0.6, 0.7, 0.9]; 0.3, 0.4) & ([0.4, 0.7, 0.8, 0.9]; 0.3, 0.6) \\ ([0.5, 0.6, 0.7, 0.9]; 0.3, 0.5) & ([0.5, 0.6, 0.7, 0.8]; 0.4, 0.3) & ([0.4, 0.5, 0.7, 0.8]; 0.7, 0.3) & ([0.5, 0.6, 0.8, 0.9]; 0.5, 0.6) \\ ([0.3, 0.5, 0.6, 0.8]; 0.4, 0.2) & ([0.2, 0.4, 0.5, 0.8]; 0.6, 0.2) & ([0.2, 0.4, 0.5, 0.6]; 0.5, 0.3) & ([0.3, 0.5, 0.6, 0.8]; 0.4, 0.2) \\ ([0.1, 0.2, 0.4, 0.6]; 0.6, 0.3) & ([0.3, 0.5, 0.6, 0.7]; 0.5, 0.1) & ([0.5, 0.6, 0.7, 0.8]; 0.4, 0.3) & ([0.2, 0.4, 0.6, 0.7]; 0.5, 0.1) \end{bmatrix}$$

Solution of the given MAGDM problem consist of following steps:

- Based on the information given in the decision matrix and utilizing ITFWG operator we first evaluate the individual overall preference intuitionistic trapeziodal fuzzy numbers $\tilde{r}_i^{(k)}$ of the alternatives A_i.

$\tilde{r}_1^{(1)} = ([0.50, 0.60, 0.70, 0.83]; 0.47, 0.33)$
$\tilde{r}_1^{(2)} = ([0.14, 0.40, 0.52, 0.66]; 0.56, 0.20)$
$\tilde{r}_1^{(3)} = ([0.20, 0.37, 0.55, 0.65]; 0.50, 0.22)$
$\tilde{r}_1^{(4)} = ([0.43, 0.58, 0.71, 0.83]; 0.37, 0.65)$
$\tilde{r}_2^{(1)} = ([0.51, 0.64, 0.77, 0.87]; 0.55, 0.48)$
$\tilde{r}_2^{(2)} = ([0.32, 0.52, 0.69, 0.84]; 0.64, 0.22)$
$\tilde{r}_2^{(3)} = ([0.39, 0.57, 0.67, 0.82]; 0.44, 0.23)$
$\tilde{r}_2^{(4)} = ([0.47, 0.57, 0.73, 0.85]; 0.47, 0.56)$
$\tilde{r}_3^{(1)} = ([0.35, 0.50, 0.60, 0.72]; 0.49, 0.25)$
$\tilde{r}_3^{(2)} = ([0.13, 0.27, 0.49, 0.73]; 0.64, 0.27)$
$\tilde{r}_3^{(3)} = ([0.19, 0.31, 0.58, 0.71]; 0.47, 0.24)$
$\tilde{r}_3^{(4)} = ([0.25, 0.45, 0.55, 0.74]; 0.46, 0.57)$
$\tilde{r}_4^{(1)} = ([0.30, 0.41, 0.57, 0.67]; 0.49, 0.20)$
$\tilde{r}_4^{(2)} = ([0.12, 0.24, 0.40, 0.73]; 0.57, 0.25)$
$\tilde{r}_4^{(3)} = ([0.47, 0.57, 0.73, 0.85]; 0.47, 0.56)$
$\tilde{r}_4^{(4)} = ([0.24, 0.40, 0.57, 0.70]; 0.49, 0.71)$

- Applying ITFHG operator we evaluate the collective overall preference intuitionistic trapeziodal fuzzy numbers $\tilde{r}_i$ and let $\omega = (0.155, 0.345, 0.345, 0.155)^T$.

  $\tilde{r}_1 = ([0.24, 0.46, 0.60, 0.72]; 0.50, 0.34)$
  $\tilde{r}_2 = ([0.39, 0.43, 0.71, 0.81]; 0.55, 0.34)$
  $\tilde{r}_3 = ([0.20, 0.35, 0.55, 0.74]; 0.55, 0.33)$
  $\tilde{r}_4 = ([0.19, 0.31, 0.47, 0.63]; 0.55, 0.37)$

- Next we evaluate value and ambiguity indices of $\tilde{r}_i$ as follows:

  $V(\tilde{r}_1) = 0.1488$ $A(\tilde{r}_1) = 0.0734$ $R(\tilde{r}_1) = 0.2223$
  $V(\tilde{r}_2) = 0.1754$ $A(\tilde{r}_2) = 0.0988$ $R(\tilde{r}_2) = 0.2733$
  $V(\tilde{r}_3) = 0.1392$ $A(\tilde{r}_3) = 0.0955$ $R(\tilde{r}_3) = 0.2347$
  $V(\tilde{r}_4) = 0.1152$ $A(\tilde{r}_4) = 0.0747$ $R(\tilde{r}_4) = 0.1899$

- Rank all the alternatives $A_i(i = 1, 2, 3, 4)$ according to the descending order of $R(\tilde{r}_i)$ i.e.,$A_2 \succ A_3 \succ A_1 \succ A_4$ and thus most desirable alternative is A_2.

6 Conclusion

In the present article we studied two characteristics of trapezoidal intuitionistic fuzzy number (TRIFN), viz., Value index and Ambiguity index. An algorithm for ranking of TRIFNs has been developed in this paper which is based on these two characteristics. The application of the above method have been stated through a Multi attribute group decision making problem where the alternatives derived in the decision making process have been ranked using the proposed method. This ranking method can be applied to the computation of shortest path in a fuzzy weighted network characterized by intuitionistic fuzzy numbers (IFNs) Transportation Problem, Assignment Problem, Multi objective optimization problems etc., where the ranking of alternatives(or variables) plays a significant role.

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