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A NOTE ON "APPLICATION OF FUZZY SOFT SETS TO INVESTMENT DECISION MAKING PROBLEM"

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Abstract – In this paper we give a new method on investment decision making problem by means of the notions of "period", "soft set" and "matrix form".

Keywords – *Soft sets, Fuzzy soft sets, Soft matrix theory, Period.*

1 Introduction

Kalaichelvi and Malini studied application of fuzzy soft sets to investment decision making problem based on the data collected from female employees working in both government and private sector undertakings located in Coimbatore, Tamil Nadu, India [2]. Clearly it is very important the notion of "period" (daily, weekly, monthly or annually) for investment decision making problems. The aim of this paper is to introduce a new method to include the notion of period using the soft set and matrix form theories.

2 Main Results

We recall the following definition which is needed throughout the paper.

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Definition 2.1. [3] Let U be an initial universe, $P(U)$ be the power set of U and E be the set of all parameters. A soft set (F, E) on the universe U is defined

$$(F, E) = \{(e, F(e)) : e \in E, F(e) \in P(U)\},$$

where $F : E \rightarrow P(U)$.

In [1] it was introduced soft matrix theory and used this theory to construct a soft max-min decision making method. In this paper we use different matrix forms obtained soft sets and new characteristic functions. Using the parameters given in [2], we introduce a new method by means of the notions of "period", "soft set" and "matrix form".

The following informations were given by Kalaichelvi and Malini in [2]. But the notion of period is not used in this application.

Factors Influencing Investment Decision:

$\mathbf{P}_1$ - Safety of funds
$\mathbf{P}_2$ - Liquidity of funds
$\mathbf{P}_3$ - High returns
$\mathbf{P}_4$ - Maximum profit in minimum period
$\mathbf{P}_5$ - Stable return
$\mathbf{P}_6$ - Easy accessibility
$\mathbf{P}_7$ - Tax concession
$\mathbf{P}_8$ - Minimum risk of possession

Investment Avenues:

$\mathbf{I}_1$ - Bank Deposit
$\mathbf{I}_2$ - Insurance
$\mathbf{I}_3$ - Postal Savings
$\mathbf{I}_4$ - Shares and Stocks
$\mathbf{I}_5$ - Mutual Fund
$\mathbf{I}_6$ - Gold
$\mathbf{I}_7$ - Real Estate

Let $U = \{I_1, I_2, I_3, I_4, I_5, I_6, I_7\}$ be the initial universal set and $E = \{P_1, P_2, P_3, P_4, P_5, P_6, P_7, P_8\}$ be the set of all parameters. We consider the universal set and the set of parameters, respectively as follows:

$$\begin{aligned} U &= \{I_i : 1 \leq i \leq n\}, \\ E &= \{P_j : 1 \leq j \leq m\}. \end{aligned}$$

Let the soft set (F, E) be defined as follows:

$$(F, E) = \{(P_j, A_j) : 1 \leq j \leq m, A_j \in P(U)\}.$$

Now we define new characteristic function of the soft set (F, E) as follows:

$$f(P_j)_i = \begin{cases} 1 & \text{if } I_i \in A_j \\ 0 & \text{if } I_i \notin A_j \end{cases}.$$

The matrix form of the soft set (F, E) is obtained as follows:

$$M = \begin{bmatrix} f(P_1)_1 & f(P_1)_2 & \cdots & f(P_1)_n \\ \vdots & \vdots & \ddots & \vdots \\ f(P_m)_1 & f(P_m)_2 & \cdots & f(P_m)_n \end{bmatrix}.$$

We create a matrix which shows selectional investment avenues using a parameter set E and defined the function h as follows:

$$h(P_{j_1}, P_{j_2}, ..., P_{j_n}) = \begin{cases} 1 & \text{if } f_1(P_{j_1})_i = ... = f_n(P_{j_n})_i = 1 \\ 0 & \text{if others} \end{cases},$$

where $i \in \{1, 2, 3, 4, 5, 6, 7\}$ and $j_1, ..., j_n \in \{1, 2, 3, 4, 5, 6, 7, 8\}$.

Now we introduce our method as an application of profit situations via soft sets on different periods.

Since it was not given any information about the period of data for private and government sector in [2], we consider investment avenues I_i, $1 \leq i \leq n$ with $\mu_{F_j(P_j)}(I_i) \geq 0.5$ as the *first period*. Then we obtain the following soft set:

$$\begin{gathered} (F, E) = \{(P_1, A_1 = \{I_1, I_2, I_3, I_6\}), (P_2, A_2 = U), (P_3, A_3 = U), \\ (P_4, A_4 = \{I_4, I_5, I_6, I_7\}), (P_5, A_5 = \{I_1, I_2, I_3\}), (P_6, A_6 = U), \\ (P_7, A_7 = \{I_1, I_2, I_3\}), (P_8, A_8 = \{I_1, I_2, I_3, I_4, I_5\})\}. \end{gathered}$$

Let us write a matrix form of (F, E) using the characteristic function $f(P_j)_i$ of the set (F, E) as follows:

$$f(P_j)_i = \begin{cases} 1 & \text{if } I_i \in A_j \\ 0 & \text{if } I_i \notin A_j \end{cases},$$

where $i \in \{1, 2, 3, 4, 5, 6, 7\}$ and $j \in \{1, 2, 3, 4, 5, 6, 7, 8\}$.

Then we obtain the following matrix:

$$M = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 \end{bmatrix},$$

where columns show investment avenues (resp. $I_1, I_2, I_3, I_4, I_5, I_6, I_7$) and rows show parameters (resp. $P_1, P_2, P_3, P_4, P_5, P_6, P_7, P_8$).

We consider three cases which are given in [2]. For these cases, we give the matrix forms using soft sets in the first period.

Case 1:

Let $E_1 = \{P_1, P_3\}$ be a parameter set. Then we have the following soft set (F_1, E_1) in the first period:

$$(F_1, E_1) = \{(P_1, A_1 = \{I_1, I_2, I_3, I_6\}), (P_3, A_3 = U)\}.$$

Then the matrix form of the soft set (F_1, E_1) is obtained using the characteristic function $f_1(P_j)_i$ as follows:

$$M_1 = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix},$$

where columns show investment avenues (resp. $I_1, I_2, I_3, I_4, I_5, I_6, I_7$) and rows show parameters (resp. P_1, P_3). The matrix form M_1 shows us that I_1, I_2, I_3 and I_6 are the best investment avenues for investor.

Case 2:

Let $E_2 = \{P_2, P_4, P_8\}$ be a parameter set. Then we have the following soft set (F_2, E_2) in the first period:

$$(F_2, E_2) = \{(P_2, A_2 = U), (P_4, A_4 = \{I_4, I_5, I_6, I_7\}), (P_8, A_8 = \{I_1, I_2, I_3, I_4, I_5\})\}.$$

Then the matrix form of the soft set (F_2, E_2) is obtained using the characteristic function $f_2(P_j)_i$ as follows:

$$M_2 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 \end{bmatrix},$$

where columns show investment avenues (resp. $I_1, I_2, I_3, I_4, I_5, I_6, I_7$) and rows show parameters (resp. P_2, P_4, P_8). The matrix form M_2 shows us that I_4 and I_5 are the best investment avenues for investor.

We give a matrix form for M_1 and M_2 using a parameter set $E_{1,2} = \{P_1, P_2, P_3, P_4, P_8\}$ and the function h defined as follows:

$$h(P_{j_1}, P_{j_2}) = \begin{cases} 1 & \text{if } f_1(P_{j_1})_i = f_2(P_{j_2})_i = 1 \\ 0 & \text{if others} \end{cases},$$

where $i \in \{1, 2, 3, 4, 5, 6, 7\}$ and $j_1, j_2 \in \{1, 2, 3, 4, 8\}$. Then

$$M_{1,2} = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 \end{bmatrix},$$

where columns show avenues (resp. $I_1, I_2, I_3, I_4, I_5, I_6, I_7$) and rows show parameter pairs (resp. (P_1, P_2), (P_1, P_4), (P_1, P_8), (P_3, P_2), (P_3, P_4), (P_3, P_8)). For example, the parameter pair (P_1, P_2) shows factors influencing investment decision which are both safety of funds and liquidity of funds.

Case 3:

Let $E_3 = \{P_2, P_5, P_6, P_7\}$ be a parameter set. Then we have the following soft set (F_3, E_3) in the first period:

$$(F_3, E_3) = \{(P_2, A_2 = U), (P_5, A_5 = \{I_1, I_2, I_3\}), (P_6, A_6 = U), (P_7, A_7 = \{I_1, I_2, I_3\})\}.$$

Then the matrix form of the soft set (F_3, E_3) is obtained using the characteristic function $f_3(P_j)_i$ as follows:

$$M_3 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \end{bmatrix},$$

where columns show investment avenues (resp. $I_1, I_2, I_3, I_4, I_5, I_6, I_7$) and rows show parameters (resp. P_2, P_5, P_6, P_7). The matrix form M_3 shows us that I_1, I_2 and I_3 are the best investment avenues for investor.

We give a matrix form for M_1, M_2 and M_3 using a parameter set $E_{1,2,3} = \{P_1, P_2, P_3, P_4, P_5, P_6, P_7, P_8\}$ and the function h defined as follows:

$$h(P_{j_1}, P_{j_2}, P_{j_3}) = \begin{cases} 1 & \text{if } f_1(P_{j_1})_i = f_2(P_{j_2})_i = f_3(P_{j_3})_i = 1 \\ 0 & \text{if others} \end{cases},$$

where $i \in \{1, 2, 3, 4, 5, 6, 7\}$ and $j_1, j_2, j_3 \in \{1, 2, 3, 4, 5, 6, 7, 8\}$. Then the matrix form $M_{1,2,3}$ is obtained similarly $M_{1,2}$.

We consider investment avenues I_i, $1 \leq i \leq n$ with $\mu_{F_j(P_j)}(I_i) \geq 0.75$ as the *second period*. Then we obtain the following soft set:

$$\begin{aligned}(F', E) = \{&(P_1, A_1 = \{I_1, I_2, I_3, I_6\}), (P_2, A_2 = \{I_1, I_4, I_5, I_6\}), (P_3, A_3 = \{I_6, I_7\}), \\ &(P_4, A_4 = \{I_4\}), (P_5, A_5 = \{I_1, I_2, I_3\}), (P_6, A_6 = \{I_1, I_2, I_3, I_6\}), \\ &(P_7, A_7 = \{I_2, I_3\}), (P_8, A_8 = \{I_1, I_2, I_3, I_4, I_5\})\}.\end{aligned}$$

Let us write a matrix form of (F', E) using the characteristic function $f'(P_j)_i$ of the set (F', E) as follows:

$$f'(P_j)_i = \begin{cases} 1 & \text{if } I_i \in A_j \\ 0 & \text{if } I_i \notin A_j \end{cases},$$

where $i \in \{1, 2, 3, 4, 5, 6, 7\}$ and $j \in \{1, 2, 3, 4, 5, 6, 7, 8\}$.

Then we obtain the following matrix:

$$M' = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 \end{bmatrix},$$

where columns show investment avenues (resp. $I_1, I_2, I_3, I_4, I_5, I_6, I_7$) and rows show parameters (resp. $P_1, P_2, P_3, P_4, P_5, P_6, P_7, P_8$).

Using the above similar arguments, for the second period we only consider **Case1** and **Case2** which is given in [2]. For these cases, we give matrix form using soft set in the second period. **Case3** can be shown by similar arguments.

Case 1:

Let $E_1 = \{P_1, P_3\}$ be a parameter set. Then we have the following soft set (F_1', E_1) in the second period:

$$(F_1', E_1) = \{(P_1, A_1 = \{I_1, I_2, I_3, I_6\}), (P_3, A_3 = \{I_6, I_7\})\}.$$

Then the matrix form of the soft set (F_1', E_1) is obtained using the characteristic function $f_1'(P_j)_i$ as follows:

$$M_1' = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix},$$

where columns show investment avenues (resp. $I_1, I_2, I_3, I_4, I_5, I_6, I_7$) and rows show parameters (resp. P_1, P_3). The matrix form M_1' shows us that I_6 is the best investment avenue for investor. Hence we see that our result coincides with the result of **Case1** given in [2].

Case 2:

Let $E_2 = \{P_2, P_4, P_8\}$ be a parameter set. Then we have the following soft set (F_2', E_2) in the second period:

$$(F_2', E_2) = \{(P_2, A_2 = \{I_1, I_4, I_5, I_6\}), (P_4, A_4 = \{I_4\}), (P_8, A_8 = \{I_1, I_2, I_3, I_4, I_5\})\}.$$

Then the matrix form of the soft set (F_2', E_2) is obtained using the characteristic function $f_2'(P_j)_i$ as follows:

$$M_2' = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 \end{bmatrix},$$

where columns show investment avenues (resp. $I_1, I_2, I_3, I_4, I_5, I_6, I_7$) and rows show parameters (resp. P_2, P_4, P_8). The matrix form M_2' shows us that I_4 is the best investment avenue for investor. Again our result coincides with the result of **Case2** given in [2].

We give a matrix form for M_1' and M_2' using a parameter set $E_{1,2} = \{P_1, P_2, P_3, P_4, P_8\}$ and the function h defined as follows:

$$h(P_{j_1}, P_{j_2}) = \begin{cases} 1 & \text{if } f_1'(P_{j_1})_i = f_2'(P_{j_2})_i = 1 \\ 0 & \text{if others} \end{cases},$$

where $i \in \{1,2,3,4,5,6,7\}$ and $j_1, j_2 \in \{1,2,3,4,8\}$. Then

$$M_{1,2}' = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

where columns show avenues (resp. $I_1, I_2, I_3, I_4, I_5, I_6, I_7$) and rows show parameter pairs (resp. (P_1, P_2), (P_1, P_4), (P_1, P_8), (P_3, P_2), (P_3, P_4), (P_3, P_8)).

Finally we consider investment avenues I_i, $1 \le i \le n$ with $\mu_{F_j(P_j)}(I_i) = 1$ as the *third period*. Then we obtain the following soft set:

$$\begin{gathered}(F'', E) = \{(P_1, A_1 = \{I_1, I_3\}), (P_2, A_2 = \{I_1, I_6\}), (P_3, A_3 = \emptyset), \\ (P_4, A_4 = \emptyset), (P_5, A_5 = \{I_1, I_2, I_3\}), (P_6, A_6 = \{I_1, I_3, I_6\}), \\ (P_7, A_7 = \emptyset), (P_8, A_8 = \{I_1, I_2, I_3\})\}.\end{gathered}$$

Let us write a matrix form of (F'', E) using the characteristic function $f''(P_j)_i$ of the set (F'', E) as follows:

$$f''(P_j)_i = \begin{cases} 1 & \text{if } I_i \in A_j \\ 0 & \text{if } I_i \notin A_j \end{cases},$$

where $i \in \{1,2,3,4,5,6,7\}$ and $j \in \{1,2,3,4,5,6,7,8\}$.

Then we obtain the following matrix:

$$M'' = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \end{bmatrix},$$

where columns show investment avenues (resp. $I_1, I_2, I_3, I_4, I_5, I_6, I_7$) and rows show parameters (resp. $P_1, P_2, P_3, P_4, P_5, P_6, P_7, P_8$).

We only consider **Case1** and **Case2** which is given in [2]. For these cases, we give matrix form using soft set in the third period.

Case 1:

Let $E_1 = \{P_1, P_3\}$ be a parameter set. Then we have the following soft set (F_1'', E_1) in the third period:

$$(F_1'', E_1) = \{(P_1, A_1 = \{I_1, I_3\}), (P_3, A_3 = \emptyset)\}.$$

Then the matrix form of the soft set (F_1'', E_1) is obtained using the characteristic function $f_1''(P_j)_i$ as follows:

$$M_1'' = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

where columns show investment avenues (resp. $I_1, I_2, I_3, I_4, I_5, I_6, I_7$) and rows show parameters (resp. P_1, P_3). The matrix form M_1'' shows us that I_1 and I_3 are the best investment avenues for investor.

Case 2:

Let $E_2 = \{P_2, P_4, P_8\}$ be a parameter set. Then we have the following soft set (F_2'', E_2) in the third period:

$$(F_2'', E_2) = \{(P_2, A_2 = \{I_1, I_6\}), (P_4, A_4 = \emptyset), (P_8, A_8 = \{I_1, I_2, I_3\})\}.$$

Then the matrix form of the soft set (F_2'', E_2) is obtained using the characteristic function $f_2''(P_j)_i$ as follows:

$$M_2'' = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \end{bmatrix},$$

where columns show investment avenues (resp. $I_1, I_2, I_3, I_4, I_5, I_6, I_7$) and rows show parameters (resp. P_2, P_4, P_8). The matrix form M_2'' shows us that I_1 is the best investment avenue for investor.

We give a matrix form for M_1'' and M_2'' using a parameter set $E_{1,2} = \{P_1, P_2, P_3, P_4, P_8\}$ and the function h defined as follows:

$$h(P_{j_1}, P_{j_2}) = \begin{cases} 1 & \text{if } f_1''(P_{j_1})_i = f_2''(P_{j_2})_i = 1 \\ 0 & \text{if others} \end{cases},$$

where $i \in \{1, 2, 3, 4, 5, 6, 7\}$ and $j_1, j_2 \in \{1, 2, 3, 4, 8\}$. Then

$$M_{1,2}'' = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

where columns show avenues (resp. $I_1, I_2, I_3, I_4, I_5, I_6, I_7$) and rows show parameter pairs (resp. (P_1, P_2), (P_1, P_4), (P_1, P_8), (P_3, P_2), (P_3, P_4), (P_3, P_8)).

Now we show these investment results for **Case1** and **Case2** on the table.

FIRST PERIOD

	I_1	I_2	I_3	I_4	I_5	I_6	I_7		I_1	I_2	I_3	I_4	I_5	I_6	I_7
P_1	1	1	1	0	0	1	0	(P_1, P_2)	1	1	1	0	0	1	0
P_3	1	1	1	1	1	1	1	(P_1, P_4)	0	0	0	0	0	1	0
P_2	1	1	1	1	1	1	1	(P_1, P_8)	1	1	1	0	0	0	0
P_4	0	0	0	1	1	1	1	(P_3, P_2)	1	1	1	1	1	1	1
P_6	1	1	1	1	1	0	0	(P_3, P_4)	0	0	0	1	1	1	1
								(P_3, P_8)	1	1	1	1	1	0	0

SECOND PERIOD

	I_1	I_2	I_3	I_4	I_5	I_6	I_7		I_1	I_2	I_3	I_4	I_5	I_6	I_7
P_1	1	1	1	0	0	1	0	(P_1, P_2)	1	0	0	0	0	1	0
P_3	0	0	0	0	0	1	1	(P_1, P_4)	0	0	0	0	0	0	0
P_2	1	0	0	1	1	1	0	(P_1, P_8)	1	1	1	0	0	0	0
P_4	0	0	0	1	0	0	0	(P_3, P_2)	0	0	0	0	0	1	0
P_6	1	1	1	1	1	0	0	(P_3, P_4)	0	0	0	0	0	0	0
								(P_3, P_8)	0	0	0	0	0	0	0

THIRD PERIOD

	I_1	I_2	I_3	I_4	I_5	I_6	I_7		I_1	I_2	I_3	I_4	I_5	I_6	I_7
P_1	1	0	1	0	0	0	0	(P_1, P_2)	1	0	0	0	0	0	0
P_3	0	0	0	0	0	0	0	(P_1, P_4)	0	0	0	0	0	0	0
P_2	1	0	0	0	0	1	0	(P_1, P_8)	1	0	1	0	0	0	0
P_4	0	0	0	0	0	0	0	(P_3, P_2)	0	0	0	0	0	0	0
P_6	1	1	1	0	0	0	0	(P_3, P_4)	0	0	0	0	0	0	0
								(P_3, P_8)	0	0	0	0	0	0	0

From above table, we see that investment avenues change according to period. Consequently if we have period data then fuzzy soft set is not necessary for investment decision making problem. In this paper we construct the notion of period using membership value since we did not have real period data for government and private sector.

3 Conclusion

In this paper we introduce a new method for investment decision making problems. Our method needs the periods of the data (so it is very attractive) and uses only soft set theory (instead of fuzzy soft set theory). Investor decides more accurate investment avenue according to the factors influencing his investment decision.

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SEPARATION AXIOMS ON MULTISET TOPOLOGICAL SPACE

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Abstract – The aim of this paper is to introduce separation axioms on multiset topological spaces and study some of their properties. Characterization of these spaces and some examples have investigated. Furthermore, we show that such separation axioms are preserved under hereditary properties.

Keywords – *Multiset, Power multiset, M-point, M-singleton, Multiset function, M-topology, Continuous multiset function.*

1 Introduction

The notion of a multiset is well established both in mathematics and computer science [1, 2, 7, 10, 11, 14, 15]. In mathematics, a multiset is considered to be the generalization of a set. In classical set theory, a set is a well-defined collection of distinct objects. If repeated occurrences of any object is allowed in a set, then a mathematical structure, that is known as multiset (mset, for short), is obtained [3, 8, 11, 12, 13, 16]. For the sake of convenience a mset is written as $\{k_1/x_1, k_2/x_2, ..., k_n/x_n\}$ in which the element x_i occurs k_i times. We observe that each multiplicity k_i is a positive integer. The number of occurrences of an object x in a mset A, which is finite in most of the studies that involve msets, is called its multiplicity or characteristic value, usually denoted by $m_A(x)$ or $C_A(x)$ or simply by A(x). One of the most natural and simplest examples is the mset of prime factors of a positive integer n. The number 504 has the factorization $504 = 2^3 3^2 7^1$ which gives the mset $M = \{3/x, 2/y, 1/z\}$ where $C_M(x) = 3$, $C_M(y) = 2$, $C_M(z) = 1$.

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Classical set theory states that a given element can appear only once in a set, it assumes that all mathematical objects occur without repetition. So, the only possible relation between two mathematical objects is either they are equal or they are different. The situation in science and in ordinary life is not like this. In the physical world it is observed that there is enormous repetition. For instance, there are many hydrogen atoms, many water molecules, many strands of DNA, etc. Coins of the same denomination and year, electrons or grains of sand appear similar, despite being obviously separate. This leads to three possible relations between any two physical objects; they are different, they are the same but separate or they coincide and are identical. For the sake of definiteness we say that two physical objects are the same or equal, if they are indistinguishable, but possibly separate, and identical if they physically coincide.

A wide application of msets can be found in various branches of mathematics. Algebraic structures for multiset space have been constructed by Ibrahim et al. in [9]. Application of mset theory in decision making can be seen in [17]. In 2012, Girish and Sunil [5] introduced multiset topologies induced by multiset relations. The same authors further studied the notions of open sets, closed sets, basis, sub-basis, closure, interior, continuity and related properties in M-topological spaces in [6]. In 2015, El-Sheikh et al. [4] introduced some types of generalized open msets and their properties.

In this paper, we extend the separation axioms $T_i(i = 0, 1, 2, 3, 4, 5, 2\frac{1}{2})$ on multiset topological space (M, τ) and study some of their properties. The behaviour of these separation axioms under the hereditary property is investigated.

2 Preliminaries

Definition 2.1. [10] A mset M drawn from the set X is represented by a count function C_M defined as $C_M : X \to N$ where N represents the set of non-negative integers.

Here C_M(x) is the number of occurrences of the element x in the mset M. We present the mset M drawn from the set $X = \{x_1, x_2, x_3, ..., x_n\}$ as
$M = \{m_1/x_1, m_2/x_2, m_3/x_3, ..., m_n/x_n\}$ where m_i is the number of occurreneces of the element x_i, $i = 1, 2, 3, ..., n$ in the mset M. However, those elements which are not included in the mset M have zero count.

Definition 2.2. [10] A domain X, is defined as a set of elements from which msets are constructed. The mset space $[X]^w$ is the set of all msets whose elements are in X such that no element in the mset occurs more than w times.

The mset space $[X]^\infty$ is the set of all msets over a domain X such that there is no limit on the number of occurrences of an element in a mset. If $X = \{x_1, x_2, ..., x_k\}$, then $[X]^w = \{\{m_1/x_1, m_2/x_2, ..., m_k/x_k\} : for\ i = 1, 2, ..., k;\ m_i \in \{0, 1, 2, ..., w\}\}$.

Definition 2.3. [10] Let M and N be two msets drawn from a set X. Then:

1. $M = N$ if $C_M(x) = C_N(x)$ for all x $\in$ X.

2. M $\subseteq$ N if $C_M(x) \leq C_N(x)$ for all x $\in$ X.

3. P = M $\cup$ N if $C_P(x)$ = Max $\{C_M(x), C_N(x)\}$ for all x $\in$ X.

4. P = M $\cap$ N if $C_P(x)$ = Min $\{C_M(x), C_N(x)\}$ for all x $\in$ X.

5. P = M $\oplus$ N if $C_P(x)$ = Min $\{C_M(x) + C_N(x)$, w$\}$ for all x $\in$ X.

6. P = M $\ominus$ N if $C_P(x)$ = Max $\{C_M(x)$ - $C_N(x)$, 0$\}$ for all x $\in$ X where $\oplus$ and $\ominus$ represent mset addition and mset subtraction respectively.

Definition 2.4. [10] Let M be a mset drawn from the set X and if C_M(x)=0 $\forall$x$\in$X. Then, M is called empty set and denoted by ϕ i.e., ϕ(x)=0 $\forall x \in X$.

Definition 2.5. [5] (Whole submset) A submset N of M is a whole submset of M with each element in N having full multiplicity as in M i.e., $C_N(x) = C_M(x)$ for every x in N.

Definition 2.6. [5] (Partial Whole submset) A submset N of M is a partial whole submset of M with at least one element in N having full multiplicity as in M i.e., $C_N(x) = C_M(x)$ for some x in N.

Definition 2.7. [5] (Full submset) A submset N of M is a full submset of M if each element in M is an element in N with the same or lesser multiplicity as in M i.e., $M^* = N^*$ with $C_N(x) \leqslant C_M(x)$ for every x in N.

Remark 2.1. [5] Empty set ϕ is a whole submset of every mset but it is neither a full submset nor a partial whole submset of any nonempty mset M.

Example 2.1. [5] Let $M = \{2/x, 3/y, 5/z\}$ be a mset. Then:

1. A submset {2/x,3/y} is whole submset and partial whole submset of M but it is not full submset of M.

2. A submset {1/x,3/y,2/z} is partial whole submset and full submset of M but it is not a whole submset of M.

3. A submset {1/x,3/y} is partial whole submset of M which is neither whole submset nor full submset of M.

Definition 2.8. [1] (Power Whole Mset) Let $M \in [X]^w$ be a mset. The power whole mset of M denoted by $PW(M)$ is defined as the set of all whole submsets of M i.e., for constructing power whole submsets of M, every element of M with its full multiplicity behaves like an element in a classical set. The cardinality of $PW(M)$ is 2^n where n is the cardinality of the support set (root set) of M.

Definition 2.9. [5] (Power Full Mset) Let $M \in [X]^w$ be a mset. The power full mset of M denoted by $PF(M)$ is defined as the set of all full submsets of M. The cardinality of $PF(M)$ is the product of the counts of the elements in M.

Remark 2.2. [5] $PW(M)$ and $PF(M)$ are ordinary sets whose elements are msets.

If M is an ordinary set with n distinct elements, then the power set $P(M)$ of M contains exactly 2^n elements. If M is a multiset with n elements (repetitions counted), then the power set $P(M)$ contains strictly less than 2^n elements because singleton submsets do not repeat in $P(M)$. In classical set theory, Cantor's power set theorem fails for msets. It is possible to formulate the following reasonable definition of a power mset of M for finite mset M that preserves Cantor's power set theorem.

Definition 2.10. [5] (Power Mset) Let $M \in [X]^w$ be a mset. The power mset $P(M)$ of M is the set of all submsets of M. We have $N \in P(M)$ if and only if $N \subseteq M$. If $N = \phi$, then N$\in^1$P(M). If N$\neq \phi$, then $N \in^k P(M)$ where $k = \prod_z \begin{pmatrix} \mid [M]_z \mid \\ \mid [N]_z \mid \end{pmatrix}$, the product $\prod_z$ is taken over by distinct elements of z of the mset N and $\mid [M]_z \mid= m$ iff z $\in^m$M, $\mid [N]_z \mid= n$ iff $z \in^n N$, then $\begin{pmatrix} \mid [M]_z \mid \\ \mid [N]_z \mid \end{pmatrix} = \begin{pmatrix} m \\ n \end{pmatrix} = \frac{m!}{n!(m-n)!}$

The power set of a mset is the support set of the power mset and is denoted by $P^*(M)$. The following theorem shows the cardinality of the power set of a mset.

Theorem 2.1. [12] Let $P(M)$ be a power mset drawn from the mset $M = \{m_1/x_1, m_2/x_2, ..., m_n/x_n\}$ and $P^*(M)$ be the power set of a mset M. Then, $Card(P^*(M)) = \Pi_{i=1}^n$ $(1{+}m_i)$.

Definition 2.11. [5] Let $M \in [X]^w$ and $\tau \subseteq P^*(M)$. Then, τ is called a multiset topology on M if τ satisfies the following properties:

1. The mset M and the empty mset ϕ are in τ.
2. The mset union of the elements of any subcollection of τ is in τ.
3. The mset intersection of the elements of any finite subcollection of τ is in τ.

Hence, (M, τ) is called M-topological space. Each element in τ is called open mset.

Definition 2.12. [6] Let (M, τ) be a M-topological space and N is a submset of M. The collection $\tau_N = \{U^* : U^* = N \cap U, U \in \tau\}$ is a M-topology on N, called the subspace M-topology.

Definition 2.13. [6] A submset N of a M-topological space M in $[X]^w$ is said to be closed if the mset $M \ominus N$ is open.

Remark 2.3. [5] The complement of any submset N in a M-topological space (M, τ) is mset subtraction from M i.e., $N^c = M \ominus N$.

Definition 2.14. [6] Given a submset A of a M-topological space M in $[X]^w$, the interior of A is defined as the mset union of all open msets contained in A and is denoted by $Int(A)$.
i.e., $Int(A) = \cup\{G \subseteq M : G \textit{ is an open mset and } G \subseteq A\}$
and $C_{Int(A)}(x) = max\{C_G(x) : G \subseteq A\}$.

Definition 2.15. [6] Given a submset A of a M-topological space M in $[X]^w$, the closure of A is defined as the mset intersection of all closed msets containing A and is denoted by $Cl(A)$.
i.e., $Cl(A) = \cap\{K \subseteq M : K \textit{ is a closed mset and } A \subseteq K\}$
and $C_{Cl(A)}(x) = min\{C_K(x) : A \subseteq K\}$.

Definition 2.16. [13] Two msets A and B are said to be similar msets if for all x $(x \in A \Leftrightarrow x \in B)$, where x is an object. Thus, similar msets have equal root sets but need not be equal themselves.

Definition 2.17. [1] Let M be a mset and if $x \in^m M$, $x \in^n M$. Then, $m = n$.

3 Separation Axioms on Multiset Topological Space

3.1 M-T_o-space

Definition 3.1. A mset M is called a whole M-singleton and denoted by $\{k/x\}$ if $C_M : X \to N$ such that $C_M(x) = k$ and $C_M(x') = 0 \; \forall \; x' \in X - \{x\}$.

Note that if $x \in^k M$ means $C_M(x) = k$, so $\{k/x\}$ is called whole M-singleton submset of M and $\{m/x\}$ is called M-singleton where $0 < m < k$.

Definition 3.2. Let (M, τ) be a M-topological space. If for every two M-singletons $\{k_1/x_1\}$, $\{k_2/x_2\} \subseteq M$ such that $x_1 \neq x_2$, then there exist $V, U \in \tau$ such that $(\{k_1/x_1\} \subseteq V$ and $\{k_2/x_2\} \nsubseteq V)$ or $(\{k_1/x_1\} \nsubseteq U$ and $\{k_2/x_2\} \subseteq U)$. Hence, (M, τ) is M-T_o-space. i.e., there exists τ-open mset which contains one of the msets $\{k_1/x_1\}$, $\{k_2/x_2\}$ but not the other.

Theorem 3.1. The property of being M-T_o-space is a hereditary property.

Proof. Let (M, τ) be a M-T_o-space and N $\subseteq$ M s.t. (N, τ_N) is M-topology on N where $\tau_N = \{N \cap G : G \in \tau\}$. Now, we want to prove that (N, τ_N) is M-T_o-space. Let $\{k_1/x_1\}, \{k_2/x_2\} \subseteq N$ s.t. $x_1 \neq x_2$, then $\{k_1/x_1\}, \{k_2/x_2\} \subseteq M$ such that $x_1 \neq x_2$. Since, (M, τ) is M-T_o-space. Then, there exist $H, G \in \tau$ such that $(\{k_1/x_1\} \subseteq H$, $\{k_2/x_2\} \nsubseteq H)$ or $(\{k_1/x_1\} \nsubseteq G$, $\{k_2/x_2\} \subseteq G)$. Therefore, $(\{k_1/x_1\} \subseteq N \cap H$, $\{k_2/x_2\} \nsubseteq N \cap H)$ or $(\{k_1/x_1\} \nsubseteq N \cap G$, $\{k_2/x_2\} \subseteq N \cap G)$ and $N \cap H, N \cap G \in \tau_N$. Hence, (N, τ_N) is M-T_o-space.

Example 3.1. Let $M = \{2/a, 3/b, 1/c\}$ be a mset and $\tau = \{\phi, M, \{2/a\}, \{3/b\}, \{2/a, 3/b\}, \{2/a, 1/c\}\}$. It's clear that, (M, τ) is M-T_o-space. Let $N = \{1/a, 2/b\} \subseteq M$. Then, $\tau_N = \{\phi, N, \{1/a\}, \{2/b\}\}$. Hence, (N, τ_N) is M-T_o-space.

Theorem 3.2. If (M,τ_1) is M-T_o-space and $\tau_1 \leqslant \tau_2$, then (M, τ_2) is also a M-T_o-space.

Proof. Immediate.

3.2 M-T_1-space

Definition 3.3. Let (M, τ) be a M-topological space. If for every two M-singletons $\{k_1/x_1\}$, $\{k_2/x_2\} \subseteq M$ s.t. $x_1 \neq x_2$, then there exist $G, H \in \tau$ s.t. $\{k_1/x_1\} \subseteq H$, $\{k_2/x_2\} \nsubseteq H$ and $\{k_1/x_1\} \nsubseteq G$, $\{k_2/x_2\} \subseteq G$. Hence, (M, τ) is M-T_1-space.

Theorem 3.3. Every M-T_1-space is M-T_o-space.

Proof. Straightforward.

Remark 3.1. The converse of Theorem 3.3 is not true in general as shown in the following example.

Example 3.2. Let $M = \{2/a, 3/b, 1/c\}$ be a mset and $\tau = \{\phi, M, \{2/a\}, \{3/b\}, \{2/a, 3/b\}, \{2/a, 1/c\}\}$, it's clear that (M,$\tau$) is M-$T_o$-space, but not M-$T_1$-space, because $\exists\{2/a\}, \{1/c\} \subseteq M$ s.t. $a \neq c$ and all open msets contain $1/c$, contain $2/a$ in the same time.

Theorem 3.4. The property of being M-T_1-space is a hereditary property.

Proof. Similar to the proof of Theorem 3.1.

Theorem 3.5. Let (M, τ) be a M-topological space. If $\{k/x\}$ is τ-closed $\forall\, x \in M^*$, $k = C_M(x)$. Then, (M, τ) is M-T_1-space. i.e., if every whole M-singleton is closed, then (M, τ) is M-T_1-space.

Proof. Let $\{l_1/x_1\}, \{l_2/x_2\} \subseteq M$ s.t. $x_1 \neq x_2$, by hypothesis $\{k_1/x_1\}, \{k_2/x_2\}$ are τ-closed msets on M where $k_1 = C_M(x_1)$, $k_2 = C_M(x_2)$. Then, $\{k_1/x_1\}^c, \{k_2/x_2\}^c \in \tau$ s.t. $\{l_1/x_1\} \subseteq \{k_2/x_2\}^c$, $\{l_2/x_2\} \nsubseteq \{k_2/x_2\}^c$ and $\{l_1/x_1\} \nsubseteq \{k_1/x_1\}^c$, $\{l_2/x_2\} \subseteq \{k_1/x_1\}^c$. Hence, (M, τ) is M-T_1-space.

Corollary 3.1. Let (M, τ) be a M-topological space. If every finite whole submset of M is τ-closed mset, then (M, τ) is M-T_1-space.

Proof. Clear (by using the above theorem).

Remark 3.2. Every discrete M-topology $(M, P^*(M))$ is M-T_1-space. But, if M is a finite mset and (M, τ) is M-T_1-space $\nRightarrow \tau = P^*(M)$ [Discrete M-topology]. As shown in example 3.3.

Example 3.3. Let $M = \{2/a, 3/b, 1/c\}$, $\tau = \{\phi, M, \{2/a\}, \{3/b\}, \{1/c\}, \{2/a, 3/b\}, \{2/a, 1/c\}, \{3/b, 1/c\}\} \neq P^*(M)$. But, (M, τ) is M-T_1-space.

Remark 3.3. For a finite mset M, the smallest M-T_1-space is $PW(M)$.

Theorem 3.6. If (M, τ_1) is a M-T_1-space and $\tau_1 \leqslant \tau_2$, then (M, τ_2) is a M-T_1-space.

Proof. Immediate.

3.3 M-T_2-space

Definition 3.4. Let (M, τ) be a M-topological space. If for every two M-singletons $\{k_1/x_1\}$, $\{k_2/x_2\} \subseteq M$ s.t. $x_1 \neq x_2$, then there exist $G, H \in \tau$ s.t. $\{k_1/x_1\} \subseteq G$, $\{k_2/x_2\} \subseteq H$ and $G \cap H = \phi$. Hence, (M, τ) is M-T_2-space.

Example 3.4. Every discrete M-topology $(M, P^*(M))$ is M-T_2-space.

Example 3.5. Every indiscrete M-topology (M, τ) is not M-T_2-space where M has more than or equal two different M-points.

Theorem 3.7. The property of being M-T_2-space is a hereditary property.

Proof. Let (M,τ) be a M-T_2-space, $N \subseteq M$ and let (N, τ_N) be a subspace of (M, τ). Now, we want to prove that (N, τ_N) is M-T_2-space. Let $\{k_1/n_1\}, \{k_2/n_2\} \subseteq N$ s.t. $n_1 \neq n_2$. Since, (M, τ) is M-T_2-space. Then, there exist $G, H \in \tau$ s.t. $\{k_1/n_1\} \subseteq G$, $\{k_2/n_2\} \subseteq H$ and $G \cap H = \phi$. By a definition of the subspace, we have: $N \cap G$, $N \cap H \in \tau_N$. Therefore, $\{k_1/n_1\} \subseteq N \cap G$ and $\{k_2/n_2\} \subseteq N \cap H$. Since, $G \cap H = \phi$. Then, $(G \cap N) \cap (H \cap N) = (G \cap H) \cap N = \phi \cap N = \phi$. Hence, (N, τ_N) is M-T_2-space.

Theorem 3.8. If (M, τ_1) is a M-T_2-space and $\tau_1 \leqslant \tau_2$, then (M, τ_2) is also a M-T_2-space.

Proof. Immediate.

Theorem 3.9. Every M-T_2-space is a M-T_1-space.

Proof. Let (M, τ) be a M-T_2-space. Also, assume that $\{k_1/m_1\}, \{k_2/m_2\} \subseteq M$ s.t. $m_1 \neq m_2$. Then, $\exists\, G, H \in \tau$ s.t. $\{k_1/m_1\} \subseteq G$, $\{k_2/m_2\} \subseteq H$ and $G \cap H = \phi$. Since, $G \cap H = \phi$, $\{k_1/m_1\} \subseteq G$, $\{k_2/m_2\} \subseteq H$ then $\{k_1/m_1\} \nsubseteq H$ and $\{k_2/m_2\} \nsubseteq G$. Consequently, we have $G, H \in \tau$ s.t. $\{k_1/m_1\} \subseteq G$,$\{k_2/m_2\} \nsubseteq G$ and $\{k_1/m_1\} \nsubseteq H$,$\{k_2/m_2\} \subseteq H$. Hence, (M,τ) is M-T_1-space.

3.4 M-T_3-space

Definition 3.5. Let (M, τ) be a M-topological space. If for all $F \in \tau^c$, $\forall$ $\{k/x\} \nsubseteq F$, then there exist $G, H \in \tau$ s.t. $F \subseteq G$, $\{k/x\} \subseteq H$ and $G \cap H = \phi$. Hence, (M, τ) is M-regular space.

Definition 3.6. A M-topological space (M, τ) is said to be a M-T_3-space if:

1. (M,τ) is M-regular space.
2. (M,τ) is M-T_1-space.

Example 3.6. Every discrete M-topology $(M, P^*(M))$ is M-T_3-space.

Theorem 3.10. The property of being M-regular space is a hereditary property.

Proof. Let (M, τ) be a M-regular space and $N \subseteq M$. Let (N, τ_N) be subspace of (M, τ). Now, we want to prove that (N, τ_N) is M-regular space. Let B be τ_N-closed submset of N and $\{k/n\} \subseteq N$ s.t. $\{k/n\} \nsubseteq B$, then there exists F is τ-closed submset of M s.t. $B = F \cap N$. Since, $\{k/n\} \nsubseteq B$. Then, $\{k/n\} \nsubseteq F$. As (M, τ) be M-regular space, $\exists\, G, H \in \tau$ s.t. $F \subseteq G$, $\{k/n\} \subseteq H$ and $G \cap H = \phi$. Then, $F \cap N \subseteq G \cap N$ [i.e., $B \subseteq G \cap N$], $\{k/n\} \subseteq H \cap N$ and $(G \cap N) \cap (H \cap N) = (G \cap H) \cap N = \phi \cap N = \phi$. Hence, (N, τ_N) is M-regular space.

Corollary 3.2. Every subspace of M-T_3-space is also a M-T_3-space.

Theorem 3.11. Every M-T_3-space is a M-regular space.

Proof. Clear by using the definition 3.6.

Remark 3.4. The converse of Theorem 3.11 is not true in general as shown by the following example.

Example 3.7. Let $M = \{2/a, 3/b, 1/c\}$ and $\tau = \{\phi, M, \{3/b\}, \{2/a, 1/c\}\}$. Then, $\tau^c = \{\phi, M, \{2/a, 1/c\}, \{3/b\}\}$. Hence, (M, τ) is a M-regular space but not a M-T_1-space.

3.5 M-T_4-space

Definition 3.7. Let (M, τ) be a M-topological space. If for all $F_1, F_2 \in \tau^c$ s.t. $F_1 \cap F_2 = \phi$, then there exist $G, H \in \tau$ s.t. $F_1 \subseteq G$, $F_2 \subseteq H$ and $G \cap H = \phi$. Hence, (M, τ) is M-normal space.

Definition 3.8. A M-topological space (M, τ) is said to be a M-T_4-space if:

1. (M,τ) is M-normal space.
2. (M,τ) is M-T_1-space.

Theorem 3.12. Every closed subspace of M-normal space is also a M-normal space.

Proof. Let (M, τ) be a M-normal space. Also, assume that $N \subseteq M$ s.t. N is τ-closed submset of M. Now, we want to prove that (N, τ_N) is also M-normal space. Let B_1, B_2 be τ_N-closed submsets of N s.t. $B_1 \cap B_2 = \phi$. Then, there exist τ-closed submsets F_1, F_2 of M s.t. $B_1 = F_1 \cap N$, $B_2 = F_2 \cap N$. Since, F_1, F_2, N are τ-closed submsets of M. Then, $F_1 \cap N$, $F_2 \cap N$ are τ-closed submsets of M i.e., B_1 *and* B_2 are τ-closed submsets of M s.t. $B_1 \cap B_2 = \phi$. Since, (M, τ) is a M-normal space. Thus, $\exists\ G, H \in \tau$ s.t. $B_1 \subseteq G$, $B_2 \subseteq H$ and $G \cap H = \phi$. Since, $B_1 \subseteq N$ and $B_1 \subseteq G$. Therefore, $B_1 \subseteq G \cap N$ and similarly $B_2 \subseteq H \cap N$. But, $G, H \in \tau$. Then, $G \cap N, H \cap N \in \tau_N$. Consequently, $\exists\ G \cap N, H \cap N \in \tau_N$ s.t. $B_1 \subseteq G \cap N$, $B_2 \subseteq H \cap N$ and $(G \cap N) \cap (H \cap N) = (G \cap H) \cap N = \phi \cap N = \phi$. Hence, (N, τ_N) is a M-normal space.

Corollary 3.3. The property of being M-T_4-space is topological property.

3.6 M-T_5-space

Definition 3.9. Let (M, τ) be a M-topological space and let $A, B \subseteq M$ be two non-empty msets. Then, we say that: A, B are separated msets if $A \cap \overline{B} = \phi$, $\overline{A} \cap B = \phi$.

Definition 3.10. A M-topological space (M, τ) is said to be M-completely normal space iff for any two separated submsets A, B of M there exist $G, H \in \tau$ s.t. $A \subseteq G$, $B \subseteq H$ and $G \cap H = \phi$.

Theorem 3.13. Every M-completely normal space is M-normal space.

Proof. Let (M, τ) be a M-completely normal space. Now, we want to show that (M, τ) is a M-normal space. Let A, B be any two τ-closed submsets of M s.t. $A \cap B = \phi$. Then, $\overline{A} = A$ and $\overline{B} = B$. Hence, $\overline{A} \cap B = \phi$ and $A \cap \overline{B} = \phi$. Consequently, A, B are separated msets. Since, (M, τ) is a M-completely normal space and A, B are separated msets. Therefore, there exist $G, H \in \tau$ s.t. $A \subseteq G$, $B \subseteq H$ and $G \cap H = \phi$. Hence, (M, τ) is a M-normal space.

Theorem 3.14. The property of being M-completely normal space is a hereditary property.

Proof. Immediate.

Definition 3.11. A M-topological space (M, τ) is said to be a M-T_5-space if:

1. (M,τ) is M-completely normal space.
2. (M,τ) is M-T_1-space.

Theorem 3.15. Every M-T_5-space is a M-T_4-space.

Proof. Straightforward.

Theorem 3.16. The property of being M-T_5-space is a hereditary property.

Proof. Immediate.

3.7 M-$T_{2\frac{1}{2}}$-space

Definition 3.12. Let (M, τ) be a M-topological space. If for every two M-singleton $\{k_1/x_1\}$, $\{k_2/x_2\} \subseteq M$ such that $x_1 \neq x_2$, then there exist $G, H \in \tau$ such that $\{k_1/x_1\} \subseteq G$, $\{k_2/x_2\} \subseteq H$ and $\overline{G} \cap \overline{H} = \phi$. Hence, (M, τ) is M-$T_{2\frac{1}{2}}$-space.

Example 3.8. Every discrete M-topology $(M, P^*(M))$ is M-$T_{2\frac{1}{2}}$-space.

Theorem 3.17. The property of being M-$T_{2\frac{1}{2}}$-space is a hereditary property.

Proof. Let (M, τ) be a M-$T_{2\frac{1}{2}}$-space, $N \subseteq M$ and let (N, τ_N) be a subspace of (M, τ). Now, we want to prove that (N, τ_N) is M-$T_{2\frac{1}{2}}$-space. Let $\{k_1/n_1\}, \{k_2/n_2\} \subseteq N$ such that $n_1 \neq n_2$. Since, $N \subseteq M$ and (M, τ) is M-$T_{2\frac{1}{2}}$-space. Then, there exist $G, H \in \tau$ such that $\{k_1/n_1\} \subseteq G$, $\{k_2/n_2\} \subseteq H$ and $\overline{G} \cap \overline{H} = \phi$. By a definition of the subspace, we have: $(N \cap G)$, $(N \cap H) \in \tau_N$. Therefore, $\{k_1/n_1\} \subseteq N \cap G$ and $\{k_2/n_2\} \subseteq N \cap H$. Also, $(\overline{G \cap N}) \cap (\overline{H \cap N}) \subseteq (\overline{G} \cap \overline{N}) \cap (\overline{H} \cap \overline{N}) = (\overline{G} \cap \overline{H}) \cap \overline{N} = \phi \cap \overline{N} = \phi$. Thus, $(\overline{G \cap N}) \cap (\overline{H \cap N}) = \phi$. Hence, (N, τ_N) is M-$T_{2\frac{1}{2}}$-space.

Theorem 3.18. If (M, τ_1) is a M-$T_{2\frac{1}{2}}$-space and $\tau_1 \leqslant \tau_2$, then (M, τ_2) is also a M-$T_{2\frac{1}{2}}$-space.

Proof. Straightforward.

Theorem 3.19. Every M-$T_{2\frac{1}{2}}$-space is a M-T_2-space.

Proof. Let (M, τ) be a M-$T_{2\frac{1}{2}}$-space and assume that $\{k_1/m_1\}, \{k_2/m_2\} \subseteq M$ such that $m_1 \neq m_2$. Then, there exist $G, H \in \tau$ such that $\{k_1/m_1\} \subseteq G$, $\{k_2/m_2\} \subseteq H$ and $\overline{G} \cap \overline{H} = \phi$. Since, $G \subseteq \overline{G}$, $H \subseteq \overline{H}$. Therefore, $G \cap H = \phi$. Hence, (M, τ) is M-T_2-space.

4 Conclusion

In this paper we introduce the separation axioms on mset topological spaces based on the singleton mset $\{m/x\}$. This approach contains all multi-points which considered as a submset. The behavior of these axioms under some types of mapping have obtained. In the future, we study another topological property such as connected, some types of submsets and mappings on these spaces.

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ON THE ZEROS OF THE DERIVATIVES OF FIBONACCI AND LUCAS POLYNOMIALS

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Abstract − The purpose of this article is to derive some functions which map the zeros of Fibonacci polynomials to the zeros of Lucas polynomials. Also we find some equations which are satisfied by $F_n^{'}(x)$ and so $L_n^{''}(x)$. To obtain these equations, formulizations which are made up of hyperbolic functions for Fibonacci and Lucas polynomials are used.

Keywords − *Fibonacci polynomial, Lucas polynomial.*

1 Introduction

As it is well known, studying zeros of polynomials plays an increasingly important role in Mathematical research. Fibonacci polynomials $F_n(x)$ are defined recursively by

$$F_n(x) = xF_{n-1}(x) + F_{n-2}(x), \tag{1}$$

by initial conditions $F_1(x) = 1$, $F_2(x) = x$. Similarly, Lucas polynomials $L_n(x)$ are defined by

$$L_n(x) = xL_{n-1}(x) + L_{n-2}(x), \tag{2}$$

with the initial values $L_1(x) = x$ and $L_2(x) = x^2 + 2$ (see [7]). In [6], V. E. Hoggat and M. Bicknell found the zeros of these polynomials using hyperbolic trigonometric

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functions defined by $\sinh z = \frac{e^z - e^{-z}}{2}$ and $\cosh z = \frac{e^z + e^{-z}}{2}$. They found the general form of Fibonacci polynomials as follows:

$$F_{2n}(x) = \frac{\sinh 2nz}{\cosh z} \tag{3}$$

and

$$F_{2n+1}(x) = \frac{\cosh(2n+1)z}{\cosh z}, \tag{4}$$

where $x = 2\sinh z$ (see [6] for more details).

Furthermore, in [4], M. X. He, P. E. Ricci and D. Simon determined the location and distribution of the zeros of the Fibonacci polynomials.

There are several papers on the derivatives of the Fibonacci and Lucas polynomials (see [1], [2], [8], [10] and [11]). For any fixed n, in [10], Jun Wang proved the following equation

$$L_n^{(t)}(x) = nF_n^{(t-1)}(x), n \geq 1. \tag{5}$$

For the first order derivatives we have $L_n^{'}(x) = nF_n(x)$. Thus we have the zeros of $L_{2n}^{'}(x)$ and $L_{2n+1}^{'}(x)$ as follows:

$$\pm 2i \sin\frac{k\pi}{2n}, k = 0, 1, 2, ..., n-1 \tag{6}$$

and

$$\pm 2i \sin\left(\frac{2k+1}{2n+1}\right)\frac{\pi}{2}, k = 1, 2, ..., n, \tag{7}$$

respectively in [6]. It will be very nice to obtain the formulization of the zeros of $L_n^{(t)}(x)$ for $t \geq 2$. Then using (5) same formulization would also be adapted to the zeros of $F_n^{(t)}(x)$ for $t > 1$. But the successively application of chain rule makes computations difficult as t increases. The computations get difficult even for second order derivative. Our aim is to obtain the zeros of $L_n^{(t)}(x)$ as iterates of some function which map the zeros of $L_n(x)$ to the zeros of $L_n^{(t)}(x)$. In this paper we get some results for $t = 1$. Also we prove some equations which are satisfied by $F_n^{'}(x)$ and so $L_n^{''}(x)$.

2 Mapping from the Modulus of the Zeros of $L_n(x)$ to the Modulus of the Zeros of $L_n^{'}(x)$

First we give some functions which map the modulus of the zeros of $L_{2n}(x)$ and $L_{2n+1}(x)$ to the modulus of the zeros of $L_{2n}^{'}(x)$ and $L_{2n+1}^{'}(x)$, respectively. It is known that the zeros of Fibonacci and Lucas polynomials are pure imaginary and come in complex conjugates. Hence we consider their magnitudes.

For the polynomials $L_{2n}^{'}(x)$, we denote the modulus of the zeros of these polynomials by x_n $(x_1 = 0 < x_2 < ... < x_n)$ and the modulus of the zeros of $L_{2n}(x)$ by y_n $(y_1 < y_2 < ... < y_n)$.

We know that the number of positive zeros of $L_{2n}^{'}(x)$ is n including 0 and the number of positive zeros of $L_{2n}(x)$ is n. In the following theorem, we give a function which map the modulus of the zeros of $L_{2n}(x)$ to the modulus of the zeros of $L_{2n}^{'}(x)$ in ascending order.

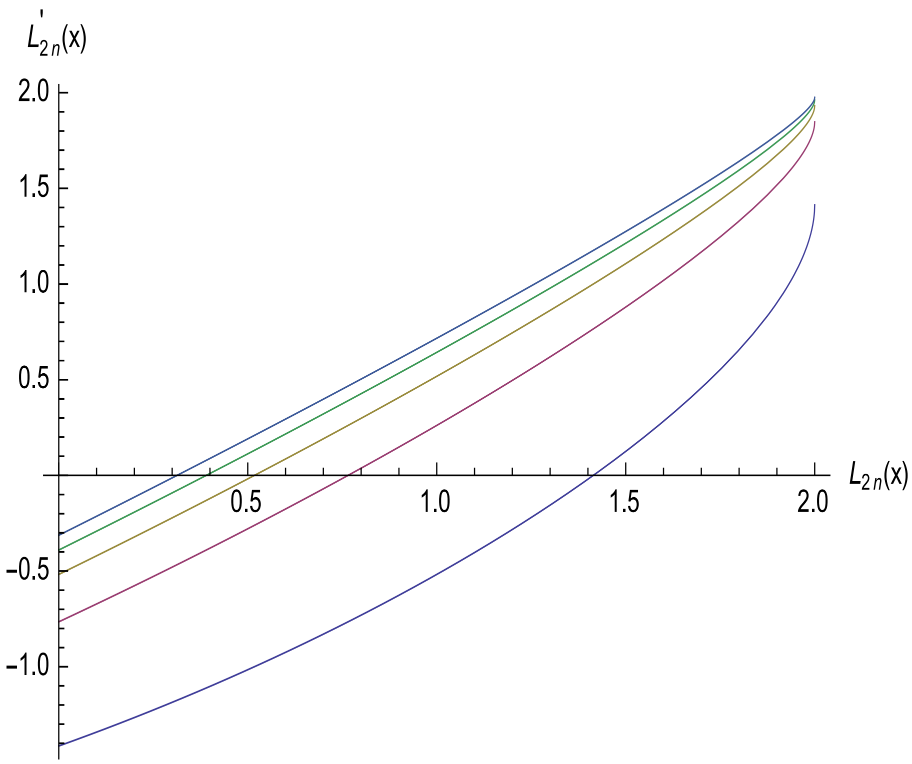

Figure 1: The graph of the function $\alpha_n(x)$ from $n = 1$ to 5.

Theorem 2.1. The modulus of the zeros of $L_{2n}(x)$ are mapped to the modulus of the zeros of $L_{2n}^{'}(x)$ by the following function

$$\alpha_n(x) = 2\sin(\arcsin\frac{x}{2} - \frac{\pi}{4n} + k\pi),$$

where k is an integer.

Proof. From [6], we know that the magnitudes of zeros of $L_{2n}^{'}(x)$ and $L_{2n}(x)$ are

$$x = 2\sin\frac{k\pi}{2n}, k = 0, 1, ..., n-1$$

and

$$y = 2\sin\left(\frac{2k+1}{2n}\right)\frac{\pi}{2}, k = 0, 1, ..., n-1,$$

respectively. For $\theta = \frac{k\pi}{2n}$, if we write $x = 2\sin\theta$ and $y = 2\sin\left(\theta + \frac{\pi}{4n}\right)$, the equality of the values of θ gives us

$$\arcsin(\frac{y}{2}) - \frac{\pi}{4n} + \pi k_1 = \arcsin(\frac{x}{2}) + \pi k_2,$$

where $k_1$and k_2 are integers. Then, taking $k = k_1 - k_2$, the desired function can be found easily. □

The results have been controlled numerically for $L_2(x)$ through $L_{10}(x)$ (see Figure 1).

Now we focus on the magnitudes of the zeros of odd indexed Lucas polynomials. We denote them by x_n ($x_1 = 0 < x_2 < ... < x_n$). The zeros of $L'_{2n+1}(x)$ are denoted by y_n ($y_1 < y_2 < ... < y_{n-1} < y_n$). As well as being well known that the number of the modulus of the zeros of $L_{2n+1}(x)$ is n, the number of the modulus of the zeros of $L'_{2n+1}(x)$ is n (see [3]).

Theorem 2.2. The modulus of the zeros of $L_{2n+1}(x)$ are mapped to the modulus of the zeros of $L'_{2n+1}(x)$, arranging from small to large, by the following function

$$\beta_n(x) = 2\sin(\arcsin\frac{x}{2} + \frac{\pi}{2(2n+1)} + \pi k),$$

where k is an integer.

Proof. Since we know that the magnitudes of zeros of $L_{2n+1}(x)$ and $L'_{2n+1}(x)$ are

$$x = 2\sin\frac{k\pi}{2n+1}, k = 0, 1, ..., n-1$$

and

$$y = 2\sin\left(\frac{2k+1}{2n+1}\right)\frac{\pi}{2}, k = 0, 1, ..., n-1,$$

respectively. If we take $\theta = \frac{k\pi}{2n+1}$ then we find

$$\arcsin(\frac{x}{2}) + \frac{\pi}{2(2n+1)} + \pi k_1 = \arcsin(\frac{y}{2}) + \pi k_2,$$

where $k_1$and k_2 are integers. Then, taking $k = k_1 - k_2$, the desired function can be found easily. □

Figure 2 shows the graphics of the functions $\beta_n(x)$ for $1 \leq n \leq 5$.

By (5), we know that the zeros of $F_n(x)$ are identical with the zeros of $L'_n(x)$. The functions $\beta_n(x)$ and $\alpha_n(x)$ map the zeros of $L_n(x)$ to the zeros of both $F_n(x)$ and $L'_n(x)$.

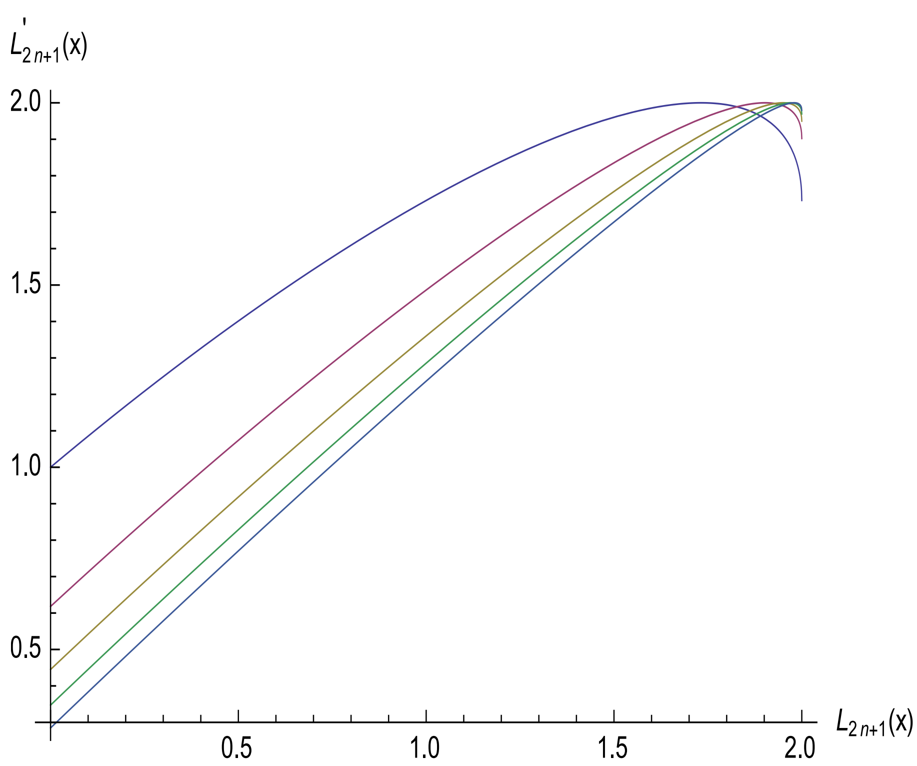

Figure 2: The graph of the function $\beta_n(x)$ from $n = 1$ to 5.

If we use the following well-known equations $\cosh z = \cos(iz)$ and $\sinh z = -i\sin(iz)$ we can rewrite the equations (3) and (4) as follows:

$$F_{2n}(x) = -i\frac{\sin(2niz)}{\cos(iz)} \tag{8}$$

and

$$F_{2n+1}(x) = \frac{\cos\left[i(2n+1)z\right]}{\cos(iz)}, \tag{9}$$

where $x = 2\sinh z$.

Then we can give the following theorems.

Theorem 2.3. The zeros of $F'_{2n}(x)$, and so the zeros of $L''_{2n}(x)$, satisfy the following equation

$$2n = -\tan(2niz)\tan(iz), \tag{10}$$

where $x = 2\sinh z$.

Proof. Using the equation (8), from the equation $F'_{2n}(x) = 0$, we can obtain

$$2n(\cos(2niz))(\cos iz) = -(\sin(2niz))(\sin iz), \tag{11}$$

where $\cos(iz) \neq 0$ and $x = 2\sinh z$. Then by rearranging the equation (11) we find the desired result (10). $\square$

Example 2.4. Let us find the zeros of the polynomial $F_6^{'}(x) = 5x^4 + 12x^2 + 3$. By (10), we obtain

$$6 = -\tan(6iz)\tan(iz).$$

Using half angle formulas we have

$$3 = -\frac{\tan(3iz)\tan(iz)}{1-\tan^2(3iz)}. \tag{12}$$

The solutions of the equation (12), the zeros of $F_6^{'}(x)$ and $L_6^{''}(x)$, can be found easily using the equation $x = 2\sinh z$. So we have the roots

$$-i\sqrt{\frac{1}{5}(6-\sqrt{21})}, i\sqrt{\frac{1}{5}(6-\sqrt{21})}, -i\sqrt{\frac{1}{5}(6+\sqrt{21})}, -i\sqrt{\frac{1}{5}(6+\sqrt{21})}.$$

Theorem 2.5. The zeros of $F_{2n+1}^{'}(x)$, and so the zeros of $L_{2n+1}^{''}(x)$, satisfy the following equation

$$2n+1 = \tan(iz)\cot(i(2n+1)z), \tag{13}$$

where $x = 2\sinh z$.

Proof. The proof follows from (9). $\square$

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THE USE OF USER CENTERED AND MODEL BASED APPROACHES IN THE SOLUTION OF USABILITY PROBLEMS OF WEB INTERFACE

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Abstract – As the computers are included in our daily lives more and more, the importance of comfortable and easy usage of interfaces in computer media has been increasing day by day. There are different approaches which are used to evaluate the usability of these interfaces. The aim of this study is to investigate the effect of the data obtained by applying the user centered and model based approaches on usability in terms of the solution of usability problems that the web interface has. Within this context, with the help of 12 academicians who work at Gaziosmanpaşa University Faculty of Medicine, the user centered approach was conducted on "Faculty of Medicine Assessment and Evaluation" system. The data obtained from CogTool was used for Model Based approach. As a result of this study, it was concluded that both approaches increased the usability, but when they were used separately, they were incompetent to diagnose some of the usability problems. In the light of these results, it can be said that the use of both models together can contribute increasing usability much more.

Keywords – *Usability, user centered approach, model based approach, CogTool.*

1. Introduction

The knowledge produced by humanity makes it possible for societies to have a better place and makes the lives of people easier through technology. It can be said that these developments in technology can make the lives of people easier only by their appropriate use[1]. These rapid changes in information technologies provide opportunities such as immediate communication; obtaining, storing and processing huge amount of data, and presentation of these huge data with other media tools [2]. All of these changes in Information technologies enabled the studies related to the interaction between humans and computers to be conducted from 1960s till today [3].

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The human-computer interaction can be summarized as an interdisciplinary field which investigates not only the users and the processes that these users want to conduct, but also the interaction and usability of computer interfaces [4, 5]. It is not a coincidence that the common point in different definitions by different people is “usability”. Usability is defined as a measure about how easy the users complete the specific tasks and the achievement of specific tasks at a targeted level easily in appropriate environment conditions by practitioners who were trained and supported technically [6-8].The things that a user want to do and the responds of the system is called as Gulf approach [9]. This approach can be summarized as the decrease of the effort needed to use the system [10]. The Gulf approach is composed of two parts called as “Gulf of Execution” and “Gulf of Evaluation”. The gulf of evaluation refers to the understanding level of messages by the user, and the Gulf of execution refers to the situation where the user understands the messages but has no idea what to do [11, 12].

There are different methods which aim the measurement of the usability of systems. Some of these methods are design guide, heuristic, experimental approach and model based approach. The methods are varied in terms of the style of practice. For example, the usability study can be conducted through eye tracking or think-aloud both on PC and on paper [12]. Among the methods mentioned, it can be said that the model based approach differs from the rest. The cognitive and physical behaviors of the users are tried to be modeled, and it is investigated how to use these models faster in model based approach [12]. It can be said that the limited area where the model based approaches can be used is the handicap of this model.

The number of participants is another issue that should be considered during usability studies based on user centered approach. It is stated that such studies should have at least five participants, but this number is not sufficient for 100% achievement[12, 13].The user centered approach generally reveals the most important problem, making it possible for smaller problems to be ignored or missed out, and this approach doesn’t offer a solution for the problems detected. These points are seen as the handicaps of user centered approach [12]. Therefore, using it with model based approach, which helps obtaining the data most about usability problems, can be a guide way in detecting usability problems and providing a solution.

Problem

Considering the current literature, this study seeks an answer to the question “what is the effect of improvement made by the data obtained through user centered and model based approaches in terms of the solution of usability problems of a web interface on the usability?

Research Questions

Within the scope of the research, two questions below are tried to be answered.

1. What kind of usability problems of the relevant web interface are revealed by user centered and model based approaches?

2. What are the effects of user centered and model based approach usage on the gulf of execution and gulf of evaluation?

2. Method

2.1. Research Design

Considering the volunteered participants, the research design was determined as one-sample pretest-posttest model. With one-sample pretest-posttest model, the higher scores on posttest (P_2) are assumed to source from the changes made in the system [14]. The research design is summarized in Table 1.

Table 1. Research Design, One-Sample Pretest-Posttest Model

Sample	Pretest	Practice	Posttest
G_1	P_1	X (Change of Design)	P_2

As the pretest score, the time spent on completion of present tasks was tried to be determined. As for the posttest score, the time spent on completion of the tasks on which the changes were made was tried to be determined.

The practice steps, the approaches used during this process and information about the participants are presented in Table 2. The data obtained from CogTool and users were used in order to determine the usability problems of "*Faculty of Medicine Assessment and Evaluation*" interface, which is in use at Gaziosmanpaşa University and to evaluate the newly developed interface within the scope of model based approach.

Table 2. The approaches used in the study and information about the participants

	Stage	Practice	The approach used	Participants	Goal
1	**Pretest P_1**	Determination of Needs	User Centered	12Academicians	The determination of usability problems
			Model Based	Researchers/Expert	
2	**Practice X**	Designing new interface	Design	Researchers	Interface design in accordance with the data
3	**Posttest P_2**	Testing the new interface	User centered	12Academicians	The determination of usability problems Evaluation
			Model based	Researchers/Expert	

The study was conducted in three steps as can be seen in Table 2. The scores related to the present situation were obtained and the needs were determined in Step 2 through pretest. The new interface was designed in the light of data obtained in Step 2. The data related to the new interface were obtained in Step 4 through posttest. A model based practice was conducted in Step 1 and 3 using Cog

Tool in order to gather data about cognitive processes. Along with this, a user centered experimental study was conducted as it is the strongest method in revealing the usability problems [12, 15].

2.2. Sample

Considering the volunteers, the convenience sampling method, which is one of the non-random sampling methods, was preferred as a part of the selection of participants [16]. The sample was composed of 12 academicians who used “Gaziosmanpaşa University Faculty of Medicine Assessment and Evaluation web interface” and taught courses in *Semester 1*. The researchers were also served as field expert in the conduction of model based approach. The demographic information about the participants can be viewed in Table 3.

Table 3. Demographics of Participants.

Participants	P1	P2	P3	P4	P5	P6	P7	P8	P9	P10	P11	P12
Female / Male	M	F	M	M	M	M	M	M	F	F	M	F
Age	41	34	36	35	43	42	45	35	35	37	35	32

2.3. Practice

Both user centered one sample experimental practice and model based approach were used separately during the process of practice. The steps of each practice are explained. The practice process is presented in Figure 1.

2.3.1. The Practice Process of User Centered Approach

The user centered approach was conducted as a pre-experiment model with a pretest and posttest. In this step, all of the academicians completed the specific tasks twice; before (pretest) and after the design (posttest). The tasks were presented clearly and in the same operation sequence. The tasks and the steps to be taken during the tasks are presented in Table 4. There was a three-week period between the first and second practice. In order to determine what the participants thought during the practices, they were asked to think aloud. The process of task completion was recorded both in audio and video formats using computer tools.

2.3.2. Web Based System Evaluation Questionnaire

In order to determine the usability of the system and the user satisfaction, a questionnaire was used. The questionnaire model is five point likert and it has 17 question. The participants were asked to complete the questionnaire twice during pretest and posttest. The items of the questionnaire were created benefiting from the items from Çağıltay's (2011) System Usability Scale.

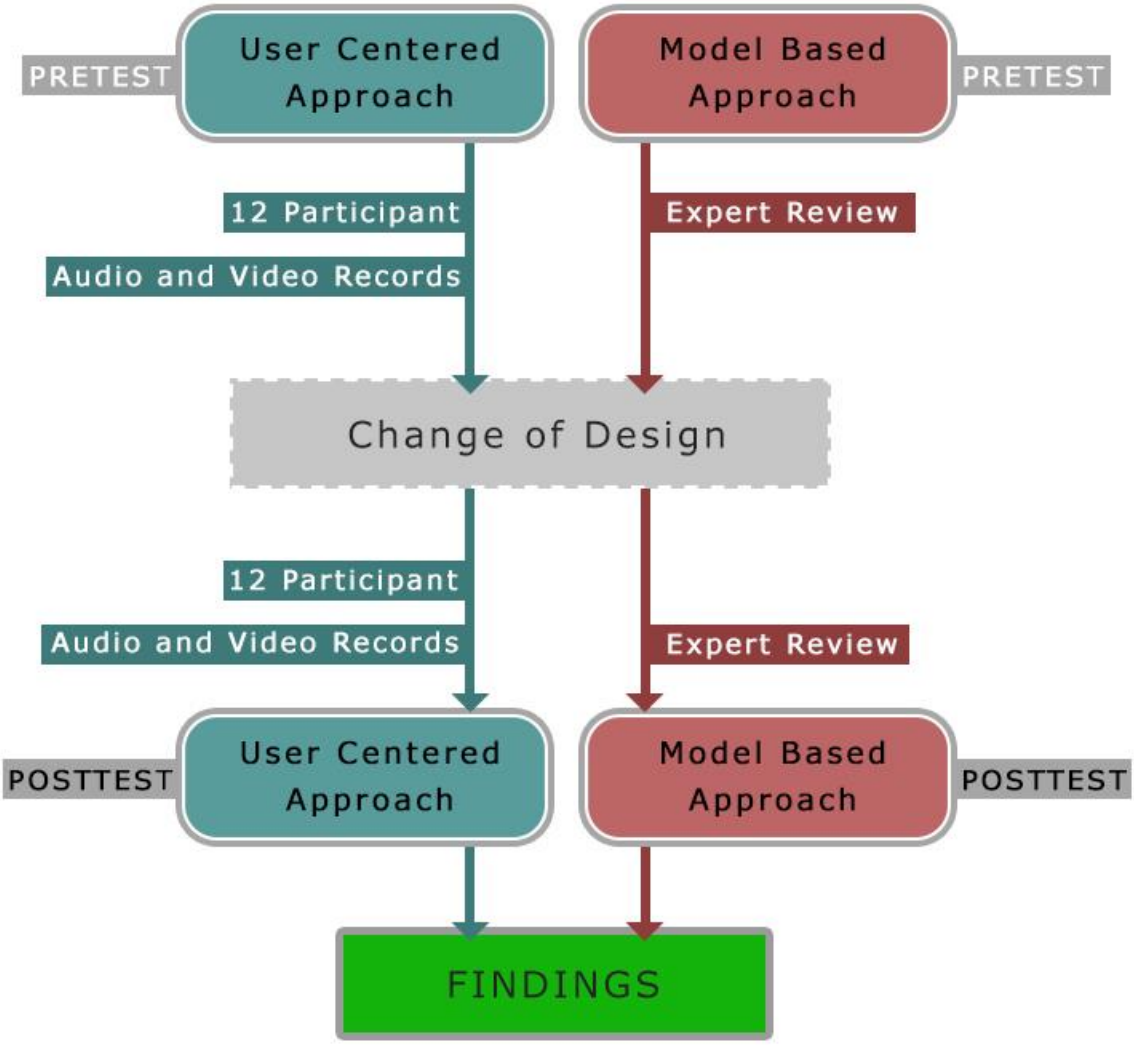

Figure 1. The Practice Process

2.3.3. The Practice Process of Model Based Approach

It is indicated that the studies with model based approach should be conducted by individuals who are experienced both on human-computer interaction/usability and the related system [17]. For that purpose, this model based study was conducted by the researcher who both has studies on human-computer interaction and is the administer of Faculty of Medicine Assessment and Evaluation System. The practices were performed twice before (pretest) and after (posttest) the design.

2.3.4. Data Collection

The data were collected from three different sources. These are the data from user centered approach, model based approach and usability questionnaire. By the user centered approach, the data about the duration of practice were obtained in terms of seconds. By the model based approach, the estimated duration of completing the practice steps were obtained in terms of seconds. By the Usability questionnaire, the satisfaction of individuals using the system was determined. The responses to the adverse items were reversed so that the questionnaire could produce one way values.

Table 4. The Tasks and the Steps of Tasks

Task 1	**Uploading Questions**
Steps in Task 1	Clicking on Question Link Clicking on Normal Question Link Pasting Question Text Pasting Alternatives Marking the Correct Alternative Uploading Question Information Clicking on Save Button
Task 2	**Saving Council Questions**
Steps in Task 2	Shifting to Council Question Records Selecting the Council on Which the operation will take place Selecting the Department on Which the operation will take place Finding the Related Question (Which question is this?) Clicking on the Use Button
Task 3	**Exploring the Question Statistics**
Steps in Task 3	Question Statistics Entering the Statistics Menu Clicking on the Question Statistics Option Uploading Council Information

3. Findings

In this part, the data were presented in parallel with the two research questions.

3.1. What kind of usability problems of the relevant web interface are revealed by user centered and model based approaches?

While answering this question, the think aloud technique in user centered approach and web based system evaluation questionnaire was conducted. The findings about the model based approach were obtained through the CogTool. The findings are as follows.

3.2. The Findings about User Centered Approach

The data were collected in two different ways in user centered approach. The first one is think aloud technique and the second one is the system usability questionnaire.

3.2.1. The Findings Obtained Through the Think Aloud Technique

What the participants aimed at doing on the steps of the practice was tried to be determined using the think aloud technique. Within this scope, the recordings of the participants were examined repeatedly and their opinions of the system were tried to be determined. The opinions determined are presented in Table 5.

Table 5. The Findings obtained through participant observations.

Participant	Observation Data
P1	Doesn't know how to return to the Council question records after saving the question.
P1	Doesn't know the function of category based question search.
P1	Doesn't know which option to select during category based question search.
P1	Can't find the return to Question statistics menu after the Council question records.
P1	After saving the question to the Council, can't get a feedback even though clicking on the "Use" button. This situation makes the participant assume that "it's probably used".
P1	Can't find where the question statistics menu.
P3	Doesn't know which option to select about the question entering type.
P3	Doesn't know how to access the council question records menu.
P5	Can't find where the question statistics menu.
P6	Indicates that the question difficulty level is unnecessary to enter.
P1	After saving a question to the council and clicking on the "Use" button, thinks that another operation should be run. Expresses this situation as "I think I should find somewhere to confirm this".
P7	While entering the question, indicates that the right button of the mouse is useless for pasting the question from Word. Instead of this, uses the paste button in the Editor.
P7	The notice informing that the question was uploaded pops up on the top of the page but that feedback isn't enough since the user can't see it. It raises doubts about whether the question was uploaded.
P8	Can't find where the question statistics menu.
P9	Can't find where the question statistics menu. Tells that he/she looked for it earlier but couldn't find, so he/she gives up.
P10	Tried to paste by clicking on the right mouse button. When couldn't, complained about the complexity of the editor.
P12	Told that he/she knew where the question statistics menu since he/she had problems about the question statistics menu earlier.

3.2.2. The Findings Obtained through the User Centered Experimental Method

By examining the video recordings of each participant, the completion duration of steps of the tasks was tried to be determined. The completion duration of steps of the tasks by each academician is presented in Table 6 in terms of seconds.

The findings of the study were dealt with separately on the basis of practices. Obtained through the e-user centered experimental method, the findings about the "question uploading, council question records and question statistics" tasks are as follows.

Task 1: Wilcoxon test results, which was conducted in order to determine whether there was a significant difference between total amount of time spent on completion of "uploading questions" task are presented in Table 7.

Table 6. The completion duration of the tasks by participants (in terms of seconds)

	Tasks	P1		P2		P3		P4		P5		P6		P7		P8		P9		P10		P11		P12	
		P_{11}	P_{12}	P_{11}	P_{12}	P_{11}	P_{12}	P_{11}	P_{12}	P_{11}	P_{12}	P_{11}	P_{12}	P_{11}	P_{12}	P_{11}	P_{12}	P_{11}	P_{12}	P_{11}	P_{12}	P_{11}	P_{12}	P_{11}	P_{12}
Question Uploading Window	Clicking on the question link	3	3	2	3	2	3	5	3	8	3	3	2	3	3	5	3	3	3	4	3	3	2	4	3
	Clicking on the normal question link	2	2	2	2	2	2	2	3	2	2	2	2	10	4	5	2	7	2	2	3	2	2	2	2
	Pasting the question text	37	19	31	21	16	6	52	5	11	11	14	7	52	16	29	10	13	9	48	14	76	10	15	5
	Pasting the alternatives	84	39	44	38	48	27	50	41	43	34	15	36	59	46	70	23	47	39	61	30	38	31	34	25
	Marking the right alternative	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
	Uploading Question Information	7	19	4	11	5	6	14	9	12	7	17	14	24	13	15	8	20	13	17	18	16	8	15	5
	Clicking on Save Button	1	2	1	2	1	2	1	2	1	1	1	1	1	1	1	1	1	1	1	2	1	1	1	1
Council Question Records	Shifting to Council Question Records	12	6	2	2	9	2	9	2	11	2	9	3	X	2	4	2	19	4	11	2	15	4	13	2
	Selecting the Council on Which the operation will take place	3	3	5	4	2	2	2	4	3	4	2	3	7	3	2	5	12	2	2	3	2	2	2	2
	Selecting the Department on Which the operation will take place	2	3	2	3	2	2	2	3	2	3	2	3	2	2	1	5	4	2	2	3	2	2	2	2
	Finding the Related Question (Which question is this?)	21	11	7	8	16	6	21	9	24	8	10	5	12	10	2	12	17	11	21	13	19	6	19	14
	Clicking on the Use Button	2	2	2	2	2	2	2	2	2	2	2	2	1	2	2	2	2	2	2	2	2	2	2	2
Question Statistics	Question Statistics	39	3	103	3	16	4	12	3	61	3	5	4	X	2	115	3	14	2	80	3	32	2	5	3
	Entering the Statistics Menu	21	X	X	X	3	X	2	X	X	X	2	X	X	X	5	X	6	X	X	X	5	X	2	X
	Clicking on the Question Statistics Option	9	3	X	3	9	3	6	2	X	3	4	2	X	3	9	2	18	3	X	2	24	1	14	1
	Uploading Council Information	4	6	X	7	6	4	16	8	X	5	X	7	X	11	X	5	X	13	X	5	X	7	X	2

X: Couldn't be observed.
P: Participant
P11: Pretest measurement
P12: Posttest measurement

As can be seen in Table 7, the p<0,01level of pretest and posttest about the "uploading questions" task is significant (Z=-4,637 p<.00). This finding shows that the 31,99 secs pretest completion time has improved to 14,68 secs after the changes were made.

Table 7. Wilcoxon test results about the "uploading questions" task.

Group	**N**	**Average Duration (seconds)**	**Total Duration (seconds)**	**Z**	**p**
Pretest	12	31,99 secs	1279.5 secs	-4,637	.00
Posttest		14,68 secs	205.5 secs		

Task 2: Wilcoxon test results, which was conducted in order to determine whether there was a significant difference between total amount of time spent on the completion of "council question records" task are presented in Table 8.

Table 8. Wilcoxon test results about the "council question records" task.

Group	N	Average Duration (seconds)	Total Duration (seconds)	Z	p
Pretest	12	26,13 secs	627 secs	-3,32	.01
Posttest		10,20 secs	153 secs		

As can be seen in Table 8, the completion time of pretest and posttest about the "council question records" task is significant at p<0,05level (Z=-3,32 p<.01). This finding shows that the 26,13secs pretest completion time has improved to 10,20 secs after the changes were made.

Task 3: Wilcoxon test results, which was conducted in order to determine whether there was a significant difference between total amount of time spent on the completion of "question statistics" task are presented in Table 9.

Table 9. Wilcoxon test results about the "question statistics" task.

Group	N	Average Duration (seconds)	Total Duration (seconds)	Z	p
Pretest	12	11,88 secs	249 secs	-3,996	.00
Posttest		3,5 secs	3,5 secs		

As can be seen in Table 9, the completion time of pretest and posttest about the "question statistics" task is significant at p<0,01level (Z=-3,996 p<.00). This finding shows that the 11,88secs pretest completion time has improved to 3,5 secs after the changes were made.

3.2.3. The Findings Obtained through the Web Based System Evaluation Questionnaire

After the user centered approach, the scores obtained from the questionnaire which was presented to the participants twice as pretest and posttest were calculated. The mean scores of 12 participants on each item are presented in Table 10.

When the pretest-posttest responses of the participants were examined, it was observed that the user satisfaction changed positively except for only one item.

Table 10. The Pretest-Posttest Average Scores of System Usability Questionnaire

ITEMS	Mean Score of Pretest (secs)	Mean Score of Posttest (secs)
The use of system was clear and understandable.	3,62	4
I was able to use the system comfortably.	3,77	4,38
Learning how to use the system was easy.	3,54	4,38
I am satisfied about the use of the system.	3,69	3,69
I think that I will use the system frequently.	3,85	4,38
I think that the system is unnecessarily complicated.	3,31	4,08
I think that the use of system was easy.	3,23	4,46
I think that I can use the system without needing foreknowledge.	2,69	2,15
I think that the system has too many inconsistencies.	3,31	3,85
I think that the use of the system is too slow.	3,77	4,08
I had to learn much before using the system.	3,23	3,69
I think that using the system will be beneficial.	4	4,46
I regard the system as a waste of time.	4,38	4,46
I think that the evaluations made by using the system will make the lessons more quality.	3,69	3,69
I think about sharing the system with my colleagues.	3,77	4,23
I think that I can guide my colleagues about using the system.	3,46	4,15
I have the intention to use the system in my courses in the future.	3,46	4,31
Mean	**3,58**	**4,03**

3.3. The Findings Obtained through the Model Based Approach

The findings about the "question uploading, council question records and question statistics" tasks obtained through the CogTool using model based approach are as follows.

Task 1: The before and after design process steps of the "Uploading Questions" task were simulated by CogTool. The time spent on before and after design process steps are presented in Table 11 as a result of evaluation conducted using CogTool. Moreover, CogTool Screenshot of the practice can be seen in Figure 2.

When the Table 11 is examined, how much time the windows opening one after another and the links clicked on took cognitively and physically can be viewed. When the data of pretest and posttest is examined, it can be observed that instead of opening on new windows, presenting the related functions with a menu by a link below made it possible for the operations to take less time.

When the Figure 2 is examined, it can be observed that the 15,9 secs duration of participants' accessing to the question uploading screen improved to 11 secs thanks to the changes on the system. The function of pop-up menu was added to the system in the light of the data obtained from the pretest, which was conducted through expert review and user centered approach.

Table 11. The Time spent on process steps of "Question Uploading" practice.

Procedure	Pretest	Posttest
Thinking on the operation to be made.	Mr=1.2 secs	Mr=1.2 secs
+ Accessing memory	Mo=0.05 secs	Mo=0.05 secs
Thinking about moving the cursor	Mr=0.05 secs	Mr=0.05 secs
Moving the cursor	K=0.43 secs	K=0.43 secs
Thinking on Clicking on the mouse	Mr=0,05 secs	Mr=0,05 secs
Clicking on the **Upload** button	Mo=0.15 secs	Mo=0.15 secs
Thinking on the operation to be made	Mr=1.2 secs	-
Perceiving the opening window (wait)	K=3.8 secs	K=4.1 secs
+ Accessing memory	Mr=0,1 secs	Mr=0,1 secs
Thinking about moving the cursor	Mr=0.05 secs	Mr=0.05 secs
Moving the cursor	K=0.28 secs	K=0.42 secs
Thinking on Clicking on the mouse	Mr=0,05 secs	Mr=0,05 secs
Clicking on the **Question** button	Mo=0.15 secs	Mo=0.15 secs
Thinking on the operation to be made	Mr=1.2 secs	-
Perceiving the opening window (wait)	K= 1.8 secs	-
+ Accessing memory	Mr=0,1 secs	Mr=0,1 secs
Thinking about moving the cursor	Mr=0.05 secs	Mr=0.05 secs
Moving the cursor	K=0.2 secs	K=0.204 secs
Thinking on Clicking on the mouse	Mr=0.05 secs	Mr=0.05 secs
Clicking on the **Normal Question Upload** button	Mo=0.15 secs	Mo=0.15 secs
Perceiving the operation (wait)	K= 5 secs	K= 2.4 secs
TOTAL	**16.11** secs	**11** secs

Mr : Mental reach.
Mo : Show item with mouse.
K : Pressing a button on keyboard.

Figure 2. CogTool Screenshot of the Question Uploading Practice

Task 2: The before and after design process steps of the “Council Question Records” task were simulated by CogTool. The time spent on before and after design process steps are presented in Table 12 as a result of evaluation conducted using CogTool. Moreover, CogTool Screenshot of the practice can be seen in Figure 3.

Table 12. The Time spent on process steps of “Council Question Records” practice.

Procedure	Pretest	Posttest
Thinking on the operation to be made.	Mr=1.2 secs	Mr=1.2 secs
+ Accessing memory	Mo=0.05 secs	Mo=0.05 secs
Hand on Mouse	Mr=0.357 secs	Mr=0.357 secs
Thinking about moving the cursor	Mr=0.05 secs	Mr=0.05 secs
Moving the cursor	K=0.458 secs	K=0.392 secs
Thinking on Clicking on the mouse	Mr=0,05 secs	Mr=0,05 secs
Clicking on the **Enter** button	Mo=0.15 secs	Mo=0.15 secs
Perceiving the opening window (wait)	K=2.0 secs	K=1.2 secs
+ Accessing memory	Mr=0,1 secs	-
Thinking about moving the cursor	Mr=0.05 secs	-
Moving the cursor	K=0.474 secs	-
Thinking on Clicking on the mouse	Mr=0,05 secs	-
Clicking on the **Question** link	Mo=0.15 secs	-
Thinking on the operation to be made.	Mr=1.2 secs	-
+ Accessing memory	Mr=0,1 secs	-
Thinking about moving the cursor	Mr=0.05 secs	-
Moving the cursor	K=0.294 secs	-
Thinking on Clicking on the mouse	Mr=0,05 secs	-
Clicking on the **Council Question Records** button	Mo=0.15 secs	-
Thinking on the operation to be made.	Mr=1.2 secs	-
+ Accessing memory	Mr=0,1 secs	Mr=0,1 secs
Thinking about moving the cursor	Mr=0.05 secs	Mr=0.05 secs
Moving the cursor	K=0.313 secs	K=0.409 secs
Thinking on Clicking on the mouse	Mr=0,05 secs	Mr=0,05 secs
Clicking on the **Save** button	Mo=0.15 secs	Mo=0.15 secs
TOTAL	**8.84 secs**	**4. 208 secs**

Mr : Mental reach.
Mo : Show item with mouse.
K : Pressing a button on keyboard.

When the Table 12 is examined, how much time the participants spent on council question records cognitively and physically can be viewed. When the data of pretest and posttest is examined, it can be observed that removing the “council question records” from the top menus and fixing it on the left menu made it possible for the operations to take less time.

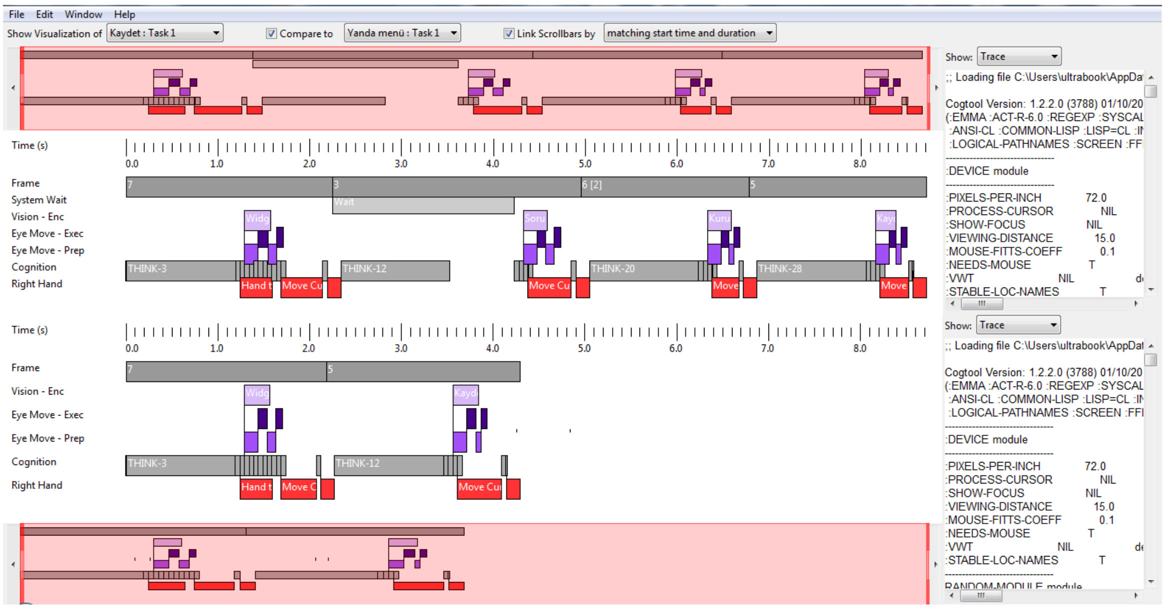

Figure 3. CogTool Screenshot of the Council Question Records Practice

When the Figure 3 is examined, it can be observed that the 8,8 secs duration of participants' accessing to the council question records screen improved to 4,3 secs thanks to the changes on the system. The relevant link on the left menu was placed on the edge of the page moving together with the scrolling and was made visible all the time in the light of the data obtained from the pretest, which was conducted through expert review and user centered approach. Thus, both the number of process steps was reduced and leaving without saving the council was prevented.

Task 3: The before and after design process steps of the "Question Statistics" task were simulated by CogTool. The time spent on before and after design process steps are presented in Table 13 as a result of evaluation conducted using CogTool. Moreover, CogTool Screenshot of the practice can be seen in Figure 4.

When the Table 13 is examined, how much time the participants spent on question statistics cognitively and physically can be viewed. When the data of pretest and posttest is examined, it can be observed that replacing the "question statistics" link made it possible for the operations to take less time.

When the Figure 4 is examined, it can be observed that the 7,5 secs duration of participants' accessing to question statistics screen improved to 4,3 secs thanks to the changes on the system. The question statistics link was added to the pop-up menu in the light of the data obtained from the pretest, which was conducted through expert review and user centered approach.

Table 13. The Time spent on process steps of "Question Statistics" practice.

Procedure	Pretest	Posttest
Thinking on the operation to be made.	Mr=1.2 secs	Mr=1.2 secs
+ Accessing memory	Mo=0.05 secs	Mo=0.05 secs
Thinking about moving the cursor	Mr=0.05 secs	Mr=0.05 secs
Moving the cursor	K=0.419 secs	K=0.419 secs
Thinking on Clicking on the mouse	Mr=0,05 secs	Mr=0,05 secs
Clicking on the **Enter** button	Mo=0.15 secs	Mo=0.15 secs
Searching Academician link in the new window	Mr=1.2 secs	-
Searching statistics link in the new window	-	Mr=1.2 secs
+ Accessing memory	Mr=0,05 secs	Mr=0,05 secs
Thinking about moving the cursor	Mr=0.05 secs	Mr=0.05 secs
Moving the cursor	K=0.319 secs	K=0.348 secs
Thinking on Clicking on the mouse	Mr=0,05 secs	Mr=0,05 secs
Clicking on the Academician link	Mo=0.15 secs	-
Clicking on the **Statistics** link	-	Mo=0.15 secs
Finding the academician on the menu	Mr=1.2 secs	-
Finding the question link on the menu	-	-
+ Accessing memory	Mr=0,1 secs	-
Thinking about moving the cursor	Mr=0.05 secs	Mr=0.05 secs
Moving the cursor	K=0.319 secs	K=0.204 secs
Thinking on Clicking on the mouse	Mr=0,05 secs	Mr=0,05 secs
Clicking on the **Select** button	Mo=0.15 secs	Mo=0.15 secs
Finding question statistics link on the menu	Mr=1.2 secs	-
+ Accessing memory	Mr=0,1 secs	-
Thinking about moving the cursor	Mr=0.05 secs	-
Moving the cursor	K=0.204 secs	-
Thinking on Clicking on the mouse	Mr=0,05 secs	-
Clicking on the **Question Statistics** button	Mo=0.15 secs	-
TOTAL	**7.36 secs**	**4.22 secs**

Mr : Mental reach.
Mo : Show item with mouse.
K : Pressing a button on keyboard.

3.4. What are the effects of user centered and model based approach usage on the gulf of execution and gulf of evaluation?

The data obtained from the think aloud technique and web based system evaluation in user centered approach were used while answering this research question. Moreover, the data about the Norman's gulf of execution were obtained through CogTool. The findings about the gulf of execution and evaluation are as follows.

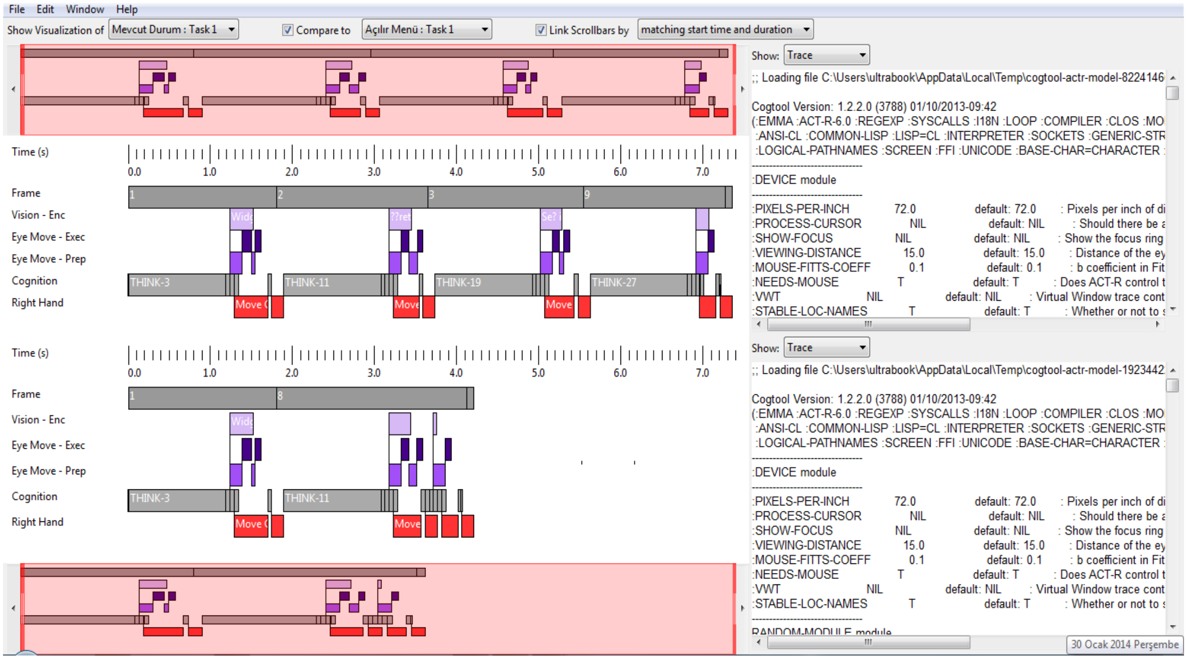

Figure 4. CogTool Screenshot of the Question Statistics Practice

3.5. The Findings about the Gulf of Evaluation

The findings about the targets and aims of the users are presented in this section. Thus, the reactions of the users against what the physical system presented were tried to be determined. Within this context, the data obtained through especially the think aloud technique are presented in six items below:

1) The function of category based question search is unknown.
2) The function of question uploading options is unknown.
3) The options in category based question search are incomprehensible.
4) The functions of buttons on the editor are incomprehensible.
5) Where the question statistics menu is unknown.
6) How to go to the council question records is unknown.

3.6. The Findings about the Gulf of Execution

The findings about the reactions of the users against the aims are presented in this section. Thus, the physical reactions of the users against the things on their mind were tried to be determined. Within this context, the data obtained through think aloud technique and model based approach are presented below.

1) The lack of feedback after the operation is completed makes the users have a suspicion.
2) The lack of feedback on the completed operations makes the users feel anxious.
3) It is indicated that the right click function of the mouse doesn't work in uploading question in the text editor.

Along with the think aloud technique, the prospective process time in the old system and the newly designed system was determined by CogTool. Thus, the duration of the task completion in the gulf of execution was tried to be minimized. The related findings are presented in Table 11.

Table 11. The Estimated Task Completion Durations by Model Based Approach

	The Old System	The Newly Designed System
Task 1	15,9 secs.	11,0 secs
Task 2	8,8 secs.	4,3 secs.
Task 3	7,5 secs.	4,3 secs.

4. Conclusion and Discussion

The ultimate goal of the interface designers is to design usable interfaces. There are different methods to assess the usability of these interfaces. The most common of these methods is the think aloud method. The data obtained through the think aloud technique is useful in the identification of the problems at endpoints [17]. Think aloud is a technique that is used with user centered approaches. Within this context, it was tried to identify the usability problems using both user centered and model based approaches in the research. Three tasks were defined in order to identify the problems, and both approaches were used within the scope of these tasks.

The estimated pretest completion durations were estimated to be 31,99 seconds, 26,13 seconds and 11,88 seconds, respectively, as part of the three tasks of user centered approach. During these time periods, the problems that the users faced with were identified with the help of think aloud technique and the necessary alternations were made. After the modifications on the system, the completion durations of three tasks were estimated to be 14,68 seconds, 10,20 seconds, and 3,5 seconds, respectively. These results show that the a total of 66 seconds pretest completion duration decreased to 28,28 seconds. The similar studies conducted using user centered approach indicate that the modifications in accordance with the data increases the usability of the interface and the satisfaction of the users [18,19].

The estimated pretest completion durations were estimated to be 15,9 seconds, 8,8 seconds and 7,5 seconds, respectively, as part of the three tasks of model based approach. After the modifications on the system as a result of the data obtained through expert review and the pretest conducted with user centered approach, the completion durations of three tasks were estimated to be 11,0 seconds, 4,3 seconds, and 4,3 seconds, respectively. These findings show that the assessment of newly designed interfaces by CogTool can contribute to usability. In relation with this situation, John (2010) states that CogTool is one of the best tools that can be used for the duration estimations of operations in the steps of the specific tasks [20].

The data of the gulf of execution and evaluation were obtained using the both methods. It can be said that the user centered approach presented more efficient data about the gulf of evaluation since the messages from the system couldn’t be understood by the users. On the other hand, both of the methods presented efficient data about the level of response from the system to what the users wanted to do, which is called the gulf of execution. By the user centered approach, the data about the problems and expectations of the users about these gulfs. On the other hand, by model based approach, the data about how the problems and expectations of the users can be fulfilled were obtained.

By using the both methods, some usability problems in the old system were determined. However, when the findings of the methods were examined, it was observed that the two approaches revealed different usability problems. CogTool is widely used in model based approach in order to determine whether the system is congenial with the users' cognitive structures [21]. This tool makes it possible for the designers to analyze their preliminary designs quickly [22]. However, the data about the solutions corresponding to the users' expectations couldn't be obtained although it presented data about whether the system is congenial with the users' cognitive structures. When the literature was reviewed, it was concluded that the use of cognitive methods in understanding the usability problems are efficient in the phase of setting actions [23].

In the light of the findings, it can be indicated that using different usability methods together can give efficient results for the solution of usability problems.

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CHARACTERIZATIONS OF FUZZY SOFT PRE SEPARATION AXIOMS

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Abstract – The notions of fuzzy pre open soft sets and fuzzy pre closed soft sets were introduced by Abd El-latif et al. [2]. In this paper, we continue the study on fuzzy soft topological spaces and investigate the properties of fuzzy pre open soft sets, fuzzy pre closed soft sets and study various properties and notions related to these structures. In particular, we study the relationship between fuzzy pre soft interior fuzzy pre soft closure. Moreover, we study the properties of fuzzy soft pre regular spaces and fuzzy soft pre normal spaces, which are basic for further research on fuzzy soft topology and will fortify the footing of the theory of fuzzy soft topological space.

Keywords – *Fuzzy soft topological space, Fuzzy pre open soft, Fuzzy pre closed soft, Fuzzy pre continuous soft functions, Fuzzy soft pre separation axioms, Fuzzy soft pre regular, Fuzzy soft pre normal.*

1 Introduction

In real life situation, the problems in economics, engineering, social sciences, medical science etc. do not always involve crisp data. So, we cannot successfully use the traditional classical methods because of various types of uncertainties presented in these problems. To exceed these uncertainties, some kinds of theories were given like theory of fuzzy set, intuitionistic fuzzy set, rough set, bipolar fuzzy set, i.e. which we can use as mathematical tools for dealings with uncertainties. But, all these theories have their inherent difficulties. The reason for these difficulties Molodtsov [35] initiated the concept of soft set theory as a new mathematical tool for dealing with uncertainties which is free from the above difficulties. In [35, 36], Molodtsov successfully applied the soft theory in several directions, such as smoothness of functions, game theory, operations research, Riemann integration, Perron integration, probability, theory of measurement, and so on. After presentation of the operations of soft sets [33], the properties and applications of soft set theory have been studied increasingly [7, 28, 36].

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Xiao et al. [46] and Pei and Miao [39] discussed the relationship between soft sets and information systems. They showed that soft sets are a class of special information systems. In recent years, many interesting applications of soft set theory have been expanded by embedding the ideas of fuzzy sets [5, 6, 9, 17, 26, 31, 32, 33, 34, 36, 37, 49]. To develop soft set theory, the operations of the soft sets are redefined and a uni-int decision making method was constructed by using these new operations [10].
Recently, in 2011, Shabir and Naz [43] initiated the study of soft topological spaces. They defined soft topology on the collection τ of soft sets over X. Consequently, they defined basic notions of soft topological spaces such as open soft and closed soft sets, soft subspace, soft closure, soft nbd of a point, soft separation axioms, soft regular spaces and soft normal spaces and established their several properties. Min in [45] investigate some properties of these soft separation axioms. In [18], Kandil et. al. introduced some soft operations such as semi open soft, pre open soft, α-open soft and β-open soft and investigated their properties in detail. Kandil et al. [25] introduced the notion of soft semi separation axioms. In particular they study the properties of the soft semi regular spaces and soft semi normal spaces. The notion of soft ideal was initiated for the first time by Kandil et al.[21]. They also introduced the concept of soft local function. These concepts are discussed with a view to find new soft topologies from the original one, called soft topological spaces with soft ideal $(X, \tau, E, \tilde{I})$. Applications to various fields were further investigated by Kandil et al. [19, 20, 22, 23, 24, 27]. The notion of supra soft topological spaces was initiated for the first time by El-sheikh and Abd El-latif [13]. They also introduced new different types of subsets of supra soft topological spaces and study the relations between them in detail. The notion of $b-$open soft sets was initiated by El-sheikh and Abd El-latif [12] and extended in [40]. An applications on $b-$open soft sets were introduced in [3, 14].
Maji et. al. [31] initiated the study involving both fuzzy sets and soft sets. In [8], the notion of fuzzy soft set was introduced as a fuzzy generalization of soft sets and some basic properties of fuzzy soft sets are discussed in detail. Then, many scientists such as X. Yang et. al. [47], improved the concept of fuzziness of soft sets. In [4], Karal and Ahmed defined the notion of a mapping on classes of fuzzy soft sets, which is fundamental important in fuzzy soft set theory, to improve this work and they studied properties of fuzzy soft images and fuzzy soft inverse images of fuzzy soft sets. Chang [11], introduced the concept of fuzzy topology on a set X by axiomatizing a collection $\mathfrak{T}$ of fuzzy subsets of X. Tanay et. al. [44] introduced the definition of fuzzy soft topology over a subset of the initial universe set while Roy and Samanta [42] gave the definition of fuzzy soft topology over the initial universe set. Some fuzzy soft topological properties based on fuzzy semi (resp. β-) open soft sets, were introduced in [1, 16, 17, 26].

In the present paper, we investigate more properties of the concepts of fuzzy pre open soft sets, fuzzy pre closed soft sets, fuzzy pre soft interior, fuzzy pre soft closure and fuzzy soft pre separation axioms in fuzzy soft topological spaces. In particular, we study the relationship between fuzzy pre soft interior and fuzzy pre soft closure. Also, we study the properties of fuzzy soft pre regular spaces and fuzzy soft pre normal spaces. Moreover, we show that if every fuzzy soft point f_e is fuzzy pre closed soft set in a fuzzy soft topological space $(X, \mathfrak{T}, E)$, then $(X, \mathfrak{T}, E)$ is fuzzy soft pre T_1- (resp. T_2-) space. We hope that, the findings in this paper will help researcher enhance and promote the further study on fuzzy soft topology to carry out a general framework for their applications in practical life.

2 Preliminary

In this section, we present the basic definitions and Theorems related to fuzzy soft set theory.

Definition 2.1. [48] A fuzzy set A in a non-empty set X is characterized by a membership function $\mu_A : X \longrightarrow [0,1] = I$ whose value $\mu_A(x)$ represents the "degree of membership" of x in A for $x \in X$. The family of all fuzzy sets is denoted by I^X.

Definition 2.2. [31] Let $A \subseteq E$. A pair (f, A), denoted by f_A, is called a fuzzy soft set over X , where f is a mapping given by $f : A \to I^X$ defined by $f_A(e) = \mu^e_{f_A}$ where $\mu^e_{f_A} = \overline{0}$ if $e \notin A$ and $\mu^e_{f_A} \neq \overline{0}$ if $e \in A$ where $\overline{0}(e) = 0 \; \forall \; x \in X$. The family of all these fuzzy soft sets over X denoted by $FSS(X)_A$.

Definition 2.3. [41]. Let $\mathfrak{T}$ be a collection of fuzzy soft sets over a universe X with a fixed set of parameters E, then $\mathfrak{T}$ is called a fuzzy soft topology on X if

(1) $\tilde{1}_E, \tilde{0}_E \in \mathfrak{T}$, where $\tilde{0}_E(e) = \overline{0}$ and $\tilde{1}_E(e) = \overline{1}$, $\forall e \in E$,

(2) The union of any members of $\mathfrak{T}$, belongs to $\mathfrak{T}$,

(3) The intersection of any two members of $\mathfrak{T}$, belongs to $\mathfrak{T}$.

The triplet $(X, \mathfrak{T}, E)$ is called a fuzzy soft topological space over X. Also, each member of $\mathfrak{T}$ is called a fuzzy open soft in $(X, \mathfrak{T}, E)$. We denote the set of all fuzzy open soft sets by $FOS(X, \mathfrak{T}, E)$, or $FOS(X)$.

Definition 2.4. [41] Let $(X, \mathfrak{T}, E)$ be a fuzzy soft topological space. A fuzzy soft set f_A over X is said to be fuzzy closed soft set in X, if its relative complement f^c_A is fuzzy open soft set. We denote the set of all fuzzy closed soft sets by $FCS(X, \mathfrak{T}, E)$, or $FCS(X)$.

Definition 2.5. [38] Let $(X, \mathfrak{T}, E)$ be a fuzzy soft topological space and $f_A \in FSS(X)_E$. The fuzzy soft closure of f_A, denoted by $Fcl(f_A)$ is the intersection of all fuzzy closed soft super sets of f_A. i.e.,

$Fcl(f_A) = \sqcap\{h_D : h_D \; is \; fuzzy \; closed \; soft \; set \; and \; f_A \sqsubseteq h_D\}$.

The fuzzy soft interior of g_B, denoted by $Fint(g_B)$ is the fuzzy soft union of all fuzzy open soft subsets of g_B.i.e.,

$Fint(g_B) = \sqcup\{h_D : h_D \; is \; fuzzy \; open \; soft \; set \; and \; h_D \sqsubseteq g_B\}$.

Definition 2.6. [30] The fuzzy soft set $f_A \in FSS(X)_E$ is called fuzzy soft point if there exist $x \in X$ and $e \in E$ such that $\mu^e_{f_A}(x) = \alpha$ $(0 < \alpha \leq 1)$ and $\mu^e_{f_A}(y) = \overline{0}$ for each $y \in X - \{x\}$, and this fuzzy soft point is denoted by x^e_α or f_e.

Definition 2.7. [30] The fuzzy soft point x^e_α is said to be belonging to the fuzzy soft set (g, A), denoted by $x^e_\alpha \tilde{\in} (g, A)$, if for the element $e \in A$, $\alpha \leq \mu^e_{g_A}(x)$.

Theorem 2.1. [30] Let $(X, \mathfrak{T}, E)$ be a fuzzy soft topological space and f_e be a fuzzy soft point. Then, the following properties hold:

(1) If $f_e \tilde{\in} g_A$, then $f_e \tilde{\notin} g^c_A$;

(2) $f_e \tilde{\in} g_A \nRightarrow f_e^c \tilde{\in} g_A^c$;

(3) Every non-null fuzzy soft set f_A can be expressed as the union of all the fuzzy soft points belonging to f_A.

Definition 2.8. [30] Let $(X, \mathfrak{T}, E)$ be a fuzzy soft topological space and $Y \subseteq X$. Let h_E^Y be a fuzzy soft set over (Y, E) such that $h_E^Y : E \to I^Y$ such that $h_E^Y(e) = \mu^e_{h_E^Y}$,

$$\mu^e_{h_E^Y}(x) = \begin{cases} 1, & x \in Y, \\ 0, & x \notin Y. \end{cases}$$

Let $\mathfrak{T}_Y = \{h_E^Y \sqcap g_B : g_B \in \mathfrak{T}\}$, then the fuzzy soft topology $\mathfrak{T}_Y$ on $(Y, E)$is called fuzzy soft subspace topology for (Y, E) and $(Y, \mathfrak{T}_Y, E)$ is called fuzzy soft subspace of $(X, \mathfrak{T}, E)$. If $h_E^Y \in \mathfrak{T}$ (resp. $h_E^Y \in \mathfrak{T}^c$), then $(Y, \mathfrak{T}_Y, E)$ is called fuzzy open (resp. closed) soft subspace of $(X, \mathfrak{T}, E)$.

Definition 2.9. [38] Let $FSS(X)_E$ and $FSS(Y)_K$ be families of fuzzy soft sets over X and Y, respectively. Let $u : X \to Y$ and $p : E \to K$ be mappings. Then, the map f_{pu} is called a fuzzy soft mapping from X to Y and denoted by $f_{pu} : FSS(X)_E \to FSS(Y)_K$ such that,

(1) If $f_A \in FSS(X)_E$. Then, the image of f_A under the fuzzy soft mapping f_{pu} is the fuzzy soft set over Y defined by $f_{pu}(f_A)$, where $\forall\, k \in p(E)$, $\forall\, y \in Y$,

$$f_{pu}(f_A)(k)(y) = \begin{cases} \bigvee_{u(x)=y} \; [\vee_{p(e)=k}(f_A(e))](x) & if\ x \in u^{-1}(y), \\ 0 & otherwise. \end{cases}$$

(2) If $g_B \in FSS(Y)_K$, then the pre-image of g_B under the fuzzy soft mapping f_{pu} is the fuzzy soft set over X defined by $f_{pu}^{-1}(g_B)$, where $\forall\, e \in p^{-1}(K)$, $\forall\, x \in X$,

$$f_{pu}^{-1}(g_B)(e)(x) = \begin{cases} g_B(p(e))(u(x)) & for\ p(e) \in B, \\ 0 & otherwise. \end{cases}$$

The fuzzy soft mapping f_{pu} is called surjective (resp. injective) if p and u are surjective (resp. injective), also it is said to be constant if p and u are constant.

Definition 2.10. [38] Let $(X, \mathfrak{T}_1, E)$ and $(Y, \mathfrak{T}_2, K)$ be two fuzzy soft topological spaces and $f_{pu} : FSS(X)_E \to FSS(Y)_K$ be a fuzzy soft mapping. Then, f_{pu} is called

(1) Fuzzy continuous soft if $f_{pu}^{-1}(g_B) \in \mathfrak{T}_1 \;\forall\, (g_B) \in \mathfrak{T}_2$.

(2) Fuzzy open soft if $f_{pu}(g_A) \in \mathfrak{T}_2 \forall\, (g_A) \in \mathfrak{T}_1$.

Theorem 2.2. [4] Let $FSS(X)_E$ and $FSS(Y)_K$ be two families of fuzzy soft sets. For the fuzzy soft function $f_{pu} : FSS(X)_E \to FSS(Y)_K$, the following statements hold,

(a) $f_{pu}^{-1}((g, B)^c) = (f_{pu}^{-1}(g, B))^c \forall\, (g, B) \in FSS(Y)_K$.

(b) $f_{pu}(f_{pu}^{-1}((g, B))) \sqsubseteq (g, B) \forall\, (g, B) \in FSS(Y)_K$. If f_{pu} is surjective, then the equality holds.

(c) $(f, A) \sqsubseteq f_{pu}^{-1}(f_{pu}((f, A))) \forall\, (f, A) \in FSS(X)_E$. If f_{pu} is injective, then the equality holds.

(d) $f_{pu}(\tilde{0}_E) = \tilde{0}_K$, $f_{pu}(\tilde{1}_E) \sqsubseteq \tilde{1}_K$. If f_{pu} is surjective, then the equality holds.

(e) $f_{pu}^{-1}(\tilde{1}_K) = \tilde{1}_E$ and $f_{pu}^{-1}(\tilde{0}_K) = \tilde{0}_E$.

(f) If $(f, A) \sqsubseteq (g, A)$, then $f_{pu}(f, A) \sqsubseteq f_{pu}(g, A)$.

(g) If $(f, B) \sqsubseteq (g, B)$, then $f_{pu}^{-1}(f, B) \sqsubseteq f_{pu}^{-1}(g, B)\ \forall\ (f, B), (g, B) \in FSS(Y)_K$.

(h) $f_{pu}^{-1}(\sqcup_{j\in J}(f, B)_j) = \sqcup_{j\in J} f_{pu}^{-1}(f, B)_j$ and $f_{pu}^{-1}(\sqcap_{j\in J}(f, B)_j) = \sqcap_{j\in J} f_{pu}^{-1}(f, B)_j, \forall\ (f, B)_j \in FSS(Y)_K$.

(I) $f_{pu}(\sqcup_{j\in J}(f, A)_j) = \sqcup_{j\in J} f_{pu}(f, A)_j$ and $f_{pu}(\sqcap_{j\in J}(f, A)_j) \sqsubseteq \sqcap_{j\in J} f_{pu}(f, A)_j\ \forall\ (f, A)_j \in FSS(X)_E$. If f_{pu} is injective, then the equality holds.

Theorem 2.3. [26] Let $(X, \mathfrak{T}, E)$ be a fuzzy soft topological space and $f_A \in FSS(X)_E$. Then:

(1) $f_A \in FSOS(X)$ if and only if $Fcl(f_A) = Fcl(Fint(f_A))$.

(2) If $g_B \in \mathfrak{T}$, then $g_B \sqcap Fcl(f_A) \sqsubseteq Fcl(g_B \sqcap g_B)$.

Definition 2.11. [18] Let (X, τ, E) be a soft topological space and $F_A \in SS(X)_E$. If $F_A \tilde{\subseteq} int(cl(F_A))$, then F_A is called pre open soft set. We denote the set of all pre open soft sets by $POS(X, \tau, E)$, or $POS(X)$ and the set of all pre closed soft sets by $PCS(X, \tau, E)$, or $PCS(X)$.

Definition 2.12. [2] Let $(X, \mathfrak{T}, E)$ be a fuzzy soft topological space and $f_A \in FSS(X)_E$. If $f_A \sqsubseteq Fint(Fcl(f_A))$, then f_A is called fuzzy pre open soft set. We denote the set of all fuzzy pre open soft sets by $FPOS(X, \mathfrak{T}, E)$, or $FPOS(X)$ and the set of all fuzzy pre closed soft sets by $FPCS(X, \mathfrak{T}, E)$, or $FPCS(X)$.

Definition 2.13. [2] Let $(X, \mathfrak{T}, E)$ be a fuzzy soft topological space, $f_A \in FSS(X)_E$ and $f_e \tilde{\in} FSS(X)_E$. Then,

(1) f_e is called fuzzy pre interior soft point of f_A if $\exists\ g_B \in FPOS(X)$ such that $f_e \tilde{\in} g_B \sqsubseteq f_A$. The set of all fuzzy pre interior soft points of f_A is called the fuzzy pre soft interior of f_A and is denoted by $FPint(f_A)$ consequently, $FPint(f_A) = \sqcup\{g_B : g_B \sqsubseteq f_A,\ g_B \in FPOS(X)\}$.

(2) f_e is called fuzzy pre closure soft point of f_A if $f_A \sqcap h_C \neq \tilde{0}_E\ \forall\ h_D \in FPOS(X)$. The set of all fuzzy pre closure soft points of f_A is called fuzzy pre soft closure of f_A and denoted by $FPcl(f_A)$. Consequently, $FPcl(f_A) = \sqcap\{h_D : h_D \in FPCS(X),\ f_A \sqsubseteq h_D\}$.

3 Fuzzy Pre Open (Closed) Soft Sets

The notions of fuzzy pre open soft sets and fuzzy pre closed soft sets were introduced by Abd El-latif et al. [2]. In this section, we investigate more properties of the notions of fuzzy pre open soft sets, fuzzy pre closed soft sets and study various properties and notions related to these structures.

Theorem 3.1. Let $(X, \mathfrak{T}, E)$ be a fuzzy soft topological space and $f_A \in FPOS(X)$.Then

(1) Arbitrary fuzzy soft union of fuzzy pre open soft sets is fuzzy pre open soft.

(2) Arbitrary fuzzy soft intersection of fuzzy pre closed soft sets is fuzzy pre closed soft.

Proof.

(1) Let $\{(f, A)_j : j \in J\} \subseteq FPOS(X)$. Then, $\forall\ j \in J$, $(f, A)_j \sqsubseteq Fint(Fcl((f, A)_j))$. It follows that, $\sqcup_j (f, A)_j \sqsubseteq \sqcup_j (Fint(Fcl((f, A)_j))) \sqsubseteq Fint(\sqcup_j Fcl(f, A)_j) = Fint(Fcl(\sqcup_j (f, A)_j)$. Hence, $\sqcup_j (f, A)_j \in FPOS(X)\ \forall\ j \in J$.

(2) By a similar way.

Theorem 3.2. Let $(X, \mathfrak{T}, E)$ be a fuzzy soft topological space and $f_A \in FSS(X)_E$. Then, $f_A \in FPOS(X)$ if and only if $Fcl(f_A) = Fint(Fcl(f_A))$.

Proof. Immediate.

Theorem 3.3. Let $(X, \mathfrak{T}, E)$ be a fuzzy soft topological space and $f_A, g_B \in FSS(X)_E$. Then, the following properties are satisfied for the fuzzy pre interior operator, denoted by $FPint$.

(1) $FPint(\tilde{1}_E) = \tilde{1}_E$ and $FPint(\tilde{0}_E)) = \tilde{0}_E$.

(2) $FPint(f_A) \sqsubseteq (f_A)$.

(3) $FPint(f_A)$ is the largest fuzzy pre open soft set contained in f_A.

(4) If $f_A \sqsubseteq g_B$, then $FPint(f_A) \sqsubseteq FPint(g_B)$.

(5) $FPint(FPint(f_A)) = FPint(f_A)$.

(6) $FPint(f_A) \sqcup FPint(g_B) \sqsubseteq FPint[(f_A) \sqcup (g_B)]$.

(7) $FPint[(f_A) \sqcap (g_B)] \sqsubseteq FPint(f_A) \sqcap FPint(g_B)$.

Proof. Obvious.

Theorem 3.4. Let $(X, \mathfrak{T}, E)$ be a fuzzy soft topological space and $f_A, g_B \in FSS(X)_E$. Then, the following properties are satisfied for the fuzzy pre closure operator, denoted by $FPcl$.

(1) $FPcl(\tilde{1}_E) = \tilde{1}_E$ and $FPcl(\tilde{0}_E) = \tilde{0}_E$.

(2) $(f_A) \sqsubseteq FPcl(f_A)$.

(3) $FPcl(f_A)$ is the smallest fuzzy pre closed soft set contains f_A.

(4) If $f_A \sqsubseteq g_B$, then $FPcl(f_A) \sqsubseteq FPcl(g_B)$.

(5) $FPcl(FPcl(f_A)) = FPcl(f_A)$.

(6) $FPcl(f_A) \sqcup FPcl(g_B) \sqsubseteq FPcl[(f_A) \sqcup (g_B)]$.

(7) $FPcl[(f_A) \sqcap (g_B)] \sqsubseteq FPcl(f_A) \sqcap FPcl(g_B)$.

Proof. Immediate.

Lemma 3.1. Every fuzzy open (resp. closed) soft set in a fuzzy soft topological space $(X, \mathfrak{T}, E)$ is fuzzy pre open (resp. closed) soft.

Proof. Let $f_A \in FOS(X)$. Then, $Fint(f_A) = f_A$. Since $f_A \sqsubseteq Fcl(f_A)$, then $f_A \sqsubseteq Fint(Fcl(f_A))$. Thus, $f_A \in FPOS(X)$.

Remark 3.1. The converse of Lemma 3.1 is not true in general as shown in the following example.

Example 3.1. Let $X = \{a, b, c\}$, $E = \{e_1, e_2, e_3\}$ and $A, B, C, D \subseteq E$ where $A = \{e_1, e_2\}$, $B = \{e_2, e_3\}$, $C = \{e_1, e_3\}$ and $D = \{e_2\}$. Let $\mathfrak{T} = \{\tilde{1}_E, \tilde{0}_E, f_{1A}, f_{2B}, f_{3D}, f_{4E}, f_{5B}, f_{6D}\}$ where $f_{1A}, f_{2B}, f_{3D}, f_{4E}, f_{5B}, f_{6D}$ are fuzzy soft sets over X defined as follows:

$\mu^{e_1}_{f_{1A}} = \{a_{0.5}, b_{0.75}, c_{0.4}\}$, $\mu^{e_2}_{f_{1A}} = \{a_{0.3}, b_{0.8}, c_{0.7}\}$,
$\mu^{e_2}_{f_{2B}} = \{a_{0.4}, b_{0.6}, c_{0.3}\}$, $\mu^{e_3}_{f_{2B}} = \{a_{0.2}, b_{0.4}, c_{0.45}\}$,
$\mu^{e_2}_{f_{3D}} = \{a_{0.3}, b_{0.6}, c_{0.3}\}$,
$\mu^{e_1}_{f_{4E}} = \{a_{0.5}, b_{0.75}, c_{0.4}\}$, $\mu^{e_2}_{f_{4E}} = \{a_{0.4}, b_{0.8}, c_{0.7}\}$, $\mu^{e_3}_{f_{4E}} = \{a_{0.2}, b_{0.4}, c_{0.45}\}$,
$\mu^{e_2}_{f_{5B}} = \{a_{0.4}, b_{0.8}, c_{0.7}\}$, $\mu^{e_3}_{f_{5B}} = \{a_{0.2}, b_{0.4}, c_{0.45}\}$,
$\mu^{e_2}_{f_{6D}} = \{a_{0.3}, b_{0.8}, c_{0.7}\}$.

Then $\mathfrak{T}$ defines a fuzzy soft topology on X. Then, the fuzzy soft set k_E where:

$\mu^{e_1}_{k_E} = \{a_{0.5}, b_{0.75}, c_{0.45}\}$, $\mu^{e_2}_{k_E} = \{a_{0.9}, b_{0.8}, c_{0.7}\}$, $\mu^{e_3}_{k_E} = \{a_{0.25}, b_{0.7}, c_1\}$.

is fuzzy pre open soft set of $(X, \mathfrak{T}, E)$, but it is not fuzzy open soft.

Theorem 3.5. Let $(X, \mathfrak{T}, E)$ be a fuzzy soft topological space and $f_A \in FSS(X)$. Then,

(1) $FPint(f^c_A) = \tilde{1} - [FPcl(f_A)]$.

(2) $FPcl(f^c_A) = \tilde{1} - [FPint(f_A)]$.

Proof.

(1) Since $FPcl(f_A) = \sqcap\{h_D : h_D \in FPCS(X),\ f_A \sqsubseteq h_D\}$. Then, $\tilde{1} - FPcl(f_A) = \sqcup\{h^c_D : h^c_D \in FPOS(X),\ h^c_D \sqsubseteq f^c_A\} = FPint(f^c_A)$.

(2) By a similar way.

Theorem 3.6. Let $(X, \mathfrak{T}, E)$ be a fuzzy soft topological space, $f_A \in FOS(X)$ and $g_B \in FPOS(X)$. Then, $f_A \sqcap g_B \in FPOS(X)$.

Proof. Let $f_A \in FOS(X)$ and $g_B \in FPOS(X)$. Then, $f_A \sqcap g_B \sqsubseteq Fint(f_A) \sqcap Fint(Fcl(g_B)) = Fint[Fcl(f_A) \sqcap (g_B)] \sqsubseteq Fint(Fcl[(f_A) \sqcap (g_B)))$ from Theorem 2.3 (2). Hence, $f_A \sqcap g_B \in FPOS(X)$.

Theorem 3.7. Let $(X, \mathfrak{T}, E)$ be a fuzzy soft topological space and $f_A \in FSS(X)_E$. Then, $f_A \in FPCS(X)$ if and only if $Fcl(Fint(f_A)) \sqsubseteq f_A$.

Proof. Obvious.

Corollary 3.1. Let $(X, \mathfrak{T}, E)$ be a fuzzy soft topological space and $f_A \in FSS(X)_E$. Then, $f_A \in FPCS(X)$ if and only if $f_A = f_A \sqcup Fcl(Fint(f_A))$.

4 Fuzzy Pre Continuous Soft Functions

In [4], Karal et al. defined the notion of a mapping on classes of fuzzy soft sets, which is fundamental important in fuzzy soft set theory, to improve this work and they studied properties of fuzzy soft images and fuzzy soft inverse images of fuzzy soft sets. Kandil et al. [25] introduced some types of soft function in soft topological spaces. Here, we introduce the notions of fuzzy pre soft function in fuzzy soft topological spaces and study its basic properties.

Definition 4.1. Let $(X, \mathfrak{T}_1, E)$, $(Y, \mathfrak{T}_2, K)$ be fuzzy soft topological spaces and $f_{pu} : FSS(X)_E \to FSS(Y)_K$ be a fuzzy soft function. Then, the function f_{pu} is called;

(1) Fuzzy pre continuous soft if $f_{pu}^{-1}(g_B) \in FPOS(X) \forall\, g_B \in \mathfrak{T}_2$.

(2) Fuzzy pre open soft if $f_{pu}(g_A) \in FPOS(Y) \forall\, g_A \in \mathfrak{T}_1$.

(3) Fuzzy pre closed soft if $f_{pu}(f_A) \in FPCS(Y) \forall\, f_A \in \mathfrak{T}_1^c$.

(4) Fuzzy pre irresolute soft if $f_{pu}^{-1}(g_B) \in FPOS(X) \forall\, g_B \in FPOS(Y)$.

(5) Fuzzy pre irresolute open soft if $f_{pu}(g_A) \in FPOS(Y) \forall\, g_A \in FPOS(X)$.

(6) Fuzzy pre irresolute closed soft if $f_{pu}(f_A) \in FPCS(Y) \forall\, f_A \in FPCS(Y)$.

Example 4.1. Let $X = Y = \{a, b, c\}$, $E = \{e_1, e_2, e_3\}$ and $A \subseteq E$ where $A = \{e_1, e_2\}$. Let $f_{pu} : (X, \mathfrak{T}_1, E) \to (Y, \mathfrak{T}_2, K)$ be the constant soft mapping where $\mathfrak{T}_1$ is the indiscrete fuzzy soft topology and $\mathfrak{T}_2$ is the discrete fuzzy soft topology such that $u(x) = a \ \forall\, x \in X$ and $p(e) = e_1 \ \forall\, e \in E$. Let f_A be fuzzy soft set over Y defined as follows:

$\mu_{f_A}^{e_1} = \{a_{0.1}, b_{0.5}, c_{0.6}\}$, $\mu_{f_A}^{e_2} = \{a_{0.6}, b_{0.2}, c_{0.5}\}$.

Then $f_A \in \mathfrak{T}_2$. Now, we find $f_{pu}^{-1}(f_A)$ as follows:

$f_{pu}^{-1}(f_A)(e_1)(a) = f_A(p(e_1))(u(a)) = f_A(e_1)(a) = 0.5$,
$f_{pu}^{-1}(f_A)(e_1)(b) = f_A(p(e_1))(u(b)) = f_A(e_1)(a) = 0.5$,
$f_{pu}^{-1}(f_A)(e_1)(c) = f_A(p(e_1))(u(c)) = f_A(e_1)(a) = 0.5$,
$f_{pu}^{-1}(f_A)(e_2)(a) = f_A(p(e_2))(u(a)) = f_A(e_1)(a) = 0.5$,
$f_{pu}^{-1}(f_A)(e_2)(b) = f_A(p(e_2))(u(b)) = f_A(e_1)(a) = 0.5$,
$f_{pu}^{-1}(f_A)(e_2)(c) = f_A(p(e_2))(u(c)) = f_A(e_1)(a) = 0.5$,
$f_{pu}^{-1}(f_A)(e_3)(a) = f_A(p(e_3))(u(a)) = f_A(e_1)(a) = 0.5$,
$f_{pu}^{-1}(f_A)(e_3)(b) = f_A(p(e_3))(u(b)) = f_A(e_1)(a) = 0.5$,

$f_{pu}^{-1}(f_A)(e_3)(c) = f_A(p(e_3))(u(c)) = f_A(e_1)(a) = 0.5.$

Hence, $f_{pu}^{-1}(f_A) \notin FPOS(X)$. Therefore, f_{pu} is not fuzzy pre continuous soft function.

Theorem 4.1. Every fuzzy continuous soft function is fuzzy pre continuous soft.

Proof. Immediate from Lemma 3.1.

Theorem 4.2. Let $(X, \mathfrak{T}_1, E)$, $(Y, \mathfrak{T}_2, K)$ be fuzzy soft topological spaces and f_{pu} be a soft function such that $f_{pu} : FSS(X)_E \rightarrow FSS(Y)_K$. Then, the following are equivalent:

(1) f_{pu} is a fuzzy pre continuous soft function.

(2) $f_{pu}^{-1}(h_B) \in FPCS(X)\ \forall\ h_B \in FCS(Y)$.

(3) $f_{pu}(FPcl(g_A) \sqsubseteq Fcl_{\mathfrak{T}_2}(f_{pu}(g_A))\ \forall\ g_A \in FSS(X)_E$.

(4) $FPcl(f_{pu}^{-1}(h_B)) \sqsubseteq f_{pu}^{-1}(Fcl_{\mathfrak{T}_2}(h_B))\ \forall\ h_B \in FSS(Y)_K$.

(5) $f_{pu}^{-1}(Fint_{\mathfrak{T}_2}(h_B)) \sqsubseteq FPint(f_{pu}^{-1}(h_B))\ \forall\ h_B \in FSS(Y)_K$.

Proof.

(1) $\Rightarrow$ (2) Let h_B be a fuzzy closed soft set over Y. Then, $h_B^c \in FOS(Y)$ and $f_{pu}^{-1}(h_B^c) \in FPOS(X)$ from Definition 4.1. Since $f_{pu}^{-1}(h_B^c) = (f_{pu}^{-1}(h_B))^c$ from Theorem 2.2. Thus, $f_{pu}^{-1}(h_B) \in FPCS(X)$.

(2) $\Rightarrow$ (3) Let $g_A \in FSS(X)_E$. Since $g_A \sqsubseteq f_{pu}^{-1}(f_{pu}(g_A)) \sqsubseteq f_{pu}^{-1}(Fcl_{\mathfrak{T}_2}(f_{pu}(g_A))) \in FPCS(X)$ from (2) and Theorem 2.2. Then, $g_A \sqsubseteq FPcl(g_A) \sqsubseteq f_{pu}^{-1}(Fcl_{\mathfrak{T}_2}(f_{pu}(g_A)))$. Hence, $f_{pu}(FPcl(g_A)) \sqsubseteq f_{pu}(f_{pu}^{-1}(Fcl_{\mathfrak{T}_2}(f_{pu}(g_A)))) \sqsubseteq Fcl_{\mathfrak{T}_2}(f_{pu}(g_A)))$ from Theorem 2.2. Thus, $f_{pu}(FPcl(g_A)) \sqsubseteq Fcl_{\mathfrak{T}_2}(f_{pu}(g_A))$.

(3) $\Rightarrow$ (4) Let $h_B \in FSS(Y)_K$ and $g_A = f_{pu}^{-1}(h_B)$. Then, $f_{pu}(FPclf_{pu}^{-1}(h_B)) \sqsubseteq Fcl_{\mathfrak{T}_2}(f_{pu}(f_{pu}^{-1}(h_B)))$ From (3). Hence, $FPcl(f_{pu}^{-1}(h_B)) \sqsubseteq f_{pu}^{-1}(f_{pu}(FPcl(f_{pu}^{-1}(h_B))))$ $\sqsubseteq f_{pu}^{-1}(Fcl_{\mathfrak{T}_2}(f_{pu}(f_{pu}^{-1}(h_B)))) \sqsubseteq f_{pu}^{-1}(Fcl_{\mathfrak{T}_2}(h_B))$ from Theorem 2.2. Thus, $FPcl(f_{pu}^{-1}(h_B)) \sqsubseteq f_{pu}^{-1}(Fcl_{\mathfrak{T}_2}(h_B))$.

(4) $\Rightarrow$ (2) Let h_B be a fuzzy closed soft set over Y. Then, $FPcl(f_{pu}^{-1}(h_B)) \sqsubseteq f_{pu}^{-1}(Fcl_{\mathfrak{T}_2}(h_B))\ \forall\ h_B \in FSS(Y)_K$ from (4). But clearly, $f_{pu}^{-1}(h_B) \sqsubseteq FPcl(f_{pu}^{-1}(h_B))$. This means that, $f_{pu}^{-1}(h_B) = FPcl(f_{pu}^{-1}(h_B))$, and consequently $f_{pu}^{-1}(h_B) \in FPCS(X)$.

(1) $\Rightarrow$ (5) Let $h_B \in FSS(Y)_K$. Then, $f_{pu}^{-1}(Fint_{\mathfrak{T}_2}(h_B)) \in FPOS(X)$ from (1). Hence, $f_{pu}^{-1}(Fint_{\mathfrak{T}_2}(h_B)) = FPint(f_{pu}^{-1}Fint_{\mathfrak{T}_2}(h_B)) \sqsubseteq FPint(f_{pu}^{-1}(h_B))$. Thus, $f_{pu}^{-1}(Fint_{\mathfrak{T}_2}(h_B)) \sqsubseteq FPint(f_{pu}^{-1}(h_B))$.

(5) $\Rightarrow$ (1) Let h_B be a fuzzy open soft set over Y. Then, $Fint_{\mathfrak{T}_2}(h_B) = h_B$ and $f_{pu}^{-1}(Fint_{\mathfrak{T}_2}(h_B)) = f_{pu}^{-1}((h_B)) \sqsubseteq FPint(f_{pu}^{-1}(h_B))$ from (5). But, we have $FPint(f_{pu}^{-1}(h_B)) \sqsubseteq f_{pu}^{-1}(h_B)$. This means that, $FPint(f_{pu}^{-1}(h_B)) = f_{pu}^{-1}(h_B) \in FPOS(X)$. Thus, f_{pu} is a fuzzy pre continuous soft function.

Theorem 4.3. Let $(X, \mathfrak{T}_1, E)$ and $(Y, \mathfrak{T}_2, K)$ be fuzzy soft topological spaces and f_{pu} be a soft function such that $f_{pu} : FSS(X)_E \rightarrow FSS(Y)_K$. Then, the following are equivalent,

(1) f_{pu} is a fuzzy pre open soft function.

(2) $f_{pu}(Fint_{\mathfrak{T}_1}(g_A)) \sqsubseteq FPint(f_{pu}(g_A)) \; \forall \; g_A \in FSS(X)_E$.

Proof.

(1) $\Rightarrow$ (2) Let $g_A \in FSS(X)_E$. Since $Fint_{\mathfrak{T}_1}(g_A) \in \mathfrak{T}_1$. Then, $f_{pu}(Fint_{\mathfrak{T}_1}(g_A)) \in FPOS(Y) \; \forall \; g_A \in \mathfrak{T}_1$ by (1). It follow that,

$$f_{pu}(Fint_{\mathfrak{T}_1}(g_A)) = FPint(f_{pu}Fint_{\mathfrak{T}_1}(g_A)) \sqsubseteq FPint(f_{pu}(g_A))$$

Therefore, $f_{pu}(Fint_{\mathfrak{T}_1}(g_A)) \sqsubseteq FPint(f_{pu}(g_A)) \; \forall \; g_A \in FSS(X)_E$.

(2) $\Rightarrow$ (1) Let $g_A \in \mathfrak{T}_1$. By hypothesis, $f_{pu}(Fint_{\mathfrak{T}_1}(g_A)) = f_{pu}(g_A) \sqsubseteq FPint(f_{pu}(g_A)) \in FPOS(Y)$, but $FPint(f_{pu}(g_A)) \sqsubseteq f_{pu}(g_A)$. So, $FPint(f_{pu}(g_A)) = f_{pu}(g_A) \in FPOS(Y) \; \forall \; g_A \in \mathfrak{T}_1$. Hence, f_{pu} is a fuzzy pre open soft function.

Theorem 4.4. Let $f_{pu} : FSS(X)_E \rightarrow FSS(Y)_K$ be a fuzzy pre open soft function. If $k_D \in FSS(Y)_K$ and $l_C \in \mathfrak{T}_1^c$ such that $f_{pu}^{-1}(k_D) \sqsubseteq l_C$, then there exists $h_B \in FPCS(Y)$ such that $k_D \sqsubseteq h_B$ and $f_{pu}^{-1}(h_B) \sqsubseteq l_C$.

Proof. Let $k_D \in FSS(Y)_K$ and $l_C \in \mathfrak{T}_1^c$ such that $f_{pu}^{-1}(k_D) \sqsubseteq l_C$. Then, $f_{pu}(l'_C) \sqsubseteq k_D^c$ from Theorem 2.2 where $l_C^c \in \mathfrak{T}_1$. Since f_{pu} is fuzzy pre open soft function. Then, $f_{pu}(l_C^c) \in FPOS(Y)$. Take $h_B = [f_{pu}(l_C^c)]^c$. Hence, $h_B \in FPCS(Y)$ such that $k_D \sqsubseteq h_B$ and $f_{pu}^{-1}(h_B) = f_{pu}^{-1}([f_{pu}(l_C^c)]^c) \sqsubseteq f_{pu}^{-1}(k_D^c)^c = f_{pu}^{-1}(k_D) \sqsubseteq l_C$. This completes the proof.

Theorem 4.5. Let $(X, \mathfrak{T}_1, E)$ and $(Y, \mathfrak{T}_2, K)$ be fuzzy soft topological spaces and f_{pu} be a soft function such that $f_{pu} : FSS(X)_E \rightarrow FSS(Y)_K$. Then, the following are equivalent:

(1) f_{pu} is a fuzzy pre closed soft function.

(2) $FPcl(f_{pu}(h_A)) \sqsubseteq f_{pu}(Fcl_{\mathfrak{T}_1}(h_A)) \; \forall \; h_A \in FSS(X)_E$.

Proof. It follows immediately from Theorem 4.3.

5 Fuzzy Soft Pre Separation Axioms

Soft separation axioms in soft topological spaces were introduced by Shabir et al. [43]. Kandil et al. [25] introduced and studied the notions of soft semi separation axioms in soft topological spaces. Here, we introduce the notions of fuzzy soft pre separation axioms in fuzzy soft topological spaces and study some of its basic properties in detail.

Definition 5.1. A fuzzy soft topological space $(X, \mathfrak{T}, E)$ is said to be a fuzzy soft pre T_o-space if for every pair of distinct fuzzy soft points f_e, g_e there exists a fuzzy pre open soft set containing one of the points but not the other.

Examples 5.1. **(1)** Let $X = \{a, b, c\}$, $E = \{e_1, e_2, e_3\}$ and $\mathfrak{T}$ be the discrete fuzzy soft topology on X. Then, $(X, \mathfrak{T}, E)$ is fuzzy soft pre T_o-space.

(2) Let $X = \{a, b\}$, $E = \{e_1, e_2\}$ and $\mathfrak{T}$ be the indiscrete fuzzy soft topology on X. Then, $\mathfrak{T}$ is not fuzzy soft pre T_o-space.

Theorem 5.1. A soft subspace $(Y, \mathfrak{T}_Y, E)$ of a fuzzy soft pre T_o-space $(X, \mathfrak{T}, E)$ is fuzzy soft pre T_o.

Proof. Let h_e, g_e be two distinct fuzzy soft points in (Y, E). Then, these fuzzy soft points are also in (X, E). Hence, there exists a fuzzy pre open soft set f_A in $\mathfrak{T}$ containing one of the fuzzy soft points but not the other. Thus, $h_E^Y \sqcap f_A$ is a fuzzy pre open soft set in $(Y, \mathfrak{T}_Y, E)$ containing one of the fuzzy soft points but not the other from Definition 2.8. Therefore, $(Y, \mathfrak{T}_Y, E)$ is fuzzy soft pre T_o.

Definition 5.2. A fuzzy soft topological space $(X, \mathfrak{T}, E)$ is said to be a fuzzy soft pre T_1-space if for every pair of distinct fuzzy soft points f_e, g_e there exist fuzzy pre open soft sets f_A and g_B such that $f_e \tilde{\in} f_A$, $g_e \tilde{\notin} f_A$; and $f_e \tilde{\notin} g_B$, $g_e \tilde{\in} g_B$.

Example 5.1. Let $X = \{a, b\}$, $E = \{e_1, e_2, e_3\}$ and $\mathfrak{T}$ be the discrete fuzzy soft topology on X. Then, $(X, \mathfrak{T}, E)$ is fuzzy soft pre T_1-space.

Theorem 5.2. A fuzzy soft subspace $(Y, \mathfrak{T}_Y, E)$ of a fuzzy soft pre T_1-space $(X, \mathfrak{T}, E)$ is fuzzy soft pre T_1.

Proof. It is similar to the proof of Theorem 5.1.

Theorem 5.3. If every fuzzy soft point of a fuzzy soft topological space $(X, \mathfrak{T}, E)$ is fuzzy pre closed soft, then $(X, \mathfrak{T}, E)$ is fuzzy soft pre T_1.

Proof. Suppose that f_e and g_e be two distinct fuzzy soft points of (X, E). By hypothesis, f_e and g_e are fuzzy pre closed soft sets. Hence, f_e^c and g_e^c are distinct fuzzy pre open soft sets where $f_e \tilde{\in} g_e^c$, $g_e \tilde{\notin} g_e^c$; and $f_e \tilde{\notin} f_e^c$, $g_e \tilde{\in} f_e^c$. Therefore, $(X, \mathfrak{T}, E)$ is fuzzy soft pre T_1.

Definition 5.3. A fuzzy soft topological space $(X, \mathfrak{T}, E)$ is said to be a fuzzy soft pre T_2-space if for every pair of distinct fuzzy soft points f_e, g_e there exist disjoint fuzzy pre open soft sets f_A and g_B such that $f_e \tilde{\in} f_A$ and $g_e \tilde{\in} g_B$.

Example 5.2. Let $X = \{a, b, c, d\}$, $E = \{e_1, e_2\}$ and $\mathfrak{T}$ be the discrete fuzzy soft topology on X. Then, $(X, \mathfrak{T}, E)$ is fuzzy soft pre T_2-space.

Proposition 5.1. For a fuzzy soft topological space $(X, \mathfrak{T}, E)$ we have:
fuzzy soft pre T_2-space $\Rightarrow$ fuzzy soft pre T_1-space $\Rightarrow$ fuzzy soft pre T_o-space.

Proof.

(1) Let $(X, \mathfrak{T}, E)$ be a fuzzy soft pre T_2-space and f_e, g_e be two distinct fuzzy soft points. Then, there exist disjoint fuzzy pre open soft sets f_A and g_B such that $f_e \tilde{\in} f_A$ and $g_e \tilde{\in} g_B$. Since $f_A \sqcap g_B = \tilde{0}_E$. Then, $f_e \tilde{\notin} g_B$ and $g_e \tilde{\notin} f_A$. Therefore, there exist fuzzy pre open soft sets f_A and g_B such that $f_e \tilde{\in} f_A$, $g_e \tilde{\notin} f_A$; and $f_e \tilde{\notin} g_B$, $g_e \tilde{\in} g_B$. Thus, $(X, \mathfrak{T}, E)$ is fuzzy soft pre T_1-space.

(2) Let $(X, \mathfrak{T}, E)$ be a fuzzy soft pre T_1-space and f_e, g_e be two distinct fuzzy soft points. Then, there exist fuzzy pre open soft sets f_A and g_B such that $f_e\tilde{\in}f_A$, $g_e\tilde{\notin}f_A$; and $f_e\tilde{\notin}g_B$, $g_e\tilde{\in}g_B$. Then, we have a fuzzy pre open soft set containing one of the fuzzy soft point but not the other. Thus, $(X, \mathfrak{T}, E)$ is fuzzy soft pre T_o-space.

Theorem 5.4. Let $(X, \mathfrak{T}, E)$ be a fuzzy soft topological space . If $(X, \mathfrak{T}, E)$ is fuzzy soft pre T_2-space, then for every pair of distinct fuzzy soft points f_e, g_e there exists a fuzzy pre closed soft set k_A such that containing one of the fuzzy soft points $g_e\tilde{\in}k_A$, but not the other $f_e\tilde{\notin}k_A$ and $g_e\tilde{\notin}FPcl(k_A)$.

Proof. Let f_e, g_e be two distinct fuzzy soft points. By assumption, there exists disjoint fuzzy pre open soft sets b_A and h_B such that $f_e\tilde{\in}b_A$, $g_e\tilde{\in}h_B$. Hence, $g_e\tilde{\in}b_A^c$ and $f_e\tilde{\notin}b_A^c$ from Theorem 2.1. Thus, b_A^c is a fuzzy pre closed soft set containing g_e but not f_e and $f_e\tilde{\notin}FPcl(b_A^c) = b_A^c$.

Theorem 5.5. A fuzzy soft subspace $(Y, \mathfrak{T}_Y, E)$ of fuzzy soft pre T_2-space $(X, \mathfrak{T}, E)$ is fuzzy soft pre T_2.

Proof. Let j_e, k_e be two distinct fuzzy soft points in (Y, E). Then, these fuzzy soft points are also in (X, E). Hence, there exist disjoint fuzzy pre open soft sets f_A and g_B in $\mathfrak{T}$ such that $j_e \in f_A$ and $k_e \in g_B$. Thus, $h_E^Y \sqcap f_A$ and $h_E^Y \sqcap g_B$ are disjoint fuzzy pre open soft sets in $\mathfrak{T}_Y$ such that $j_e \in h_E^Y \sqcap f_A$ and $k_e \in h_E^Y \sqcap g_B$. So, $(Y, \mathfrak{T}_Y, E)$ is fuzzy soft pre T_2.

Theorem 5.6. If every fuzzy soft point of a fuzzy soft topological space $(X, \mathfrak{T}, E)$ is fuzzy pre closed soft, then $(X, \mathfrak{T}, E)$ is fuzzy soft pre T_2.

Proof. It similar to the proof of Theorem 5.3.

Definition 5.4. Let $(X, \mathfrak{T}, E)$ be a fuzzy soft topological space, h_C be a fuzzy pre closed soft set and g_e be a fuzzy soft point such that $g_e\tilde{\notin}h_C$. If there exist disjoint fuzzy pre open soft sets f_S and f_W such that $g_e\tilde{\in}f_S$ and $g_B \sqsubseteq f_W$. Then, $(X, \mathfrak{T}, E)$ is called fuzzy soft pre regular space. A fuzzy soft pre regular T_1-space is called a fuzzy soft pre T_3-space.

Proposition 5.2. Let $(X, \mathfrak{T}, E)$ be a fuzzy soft topological space, h_C be a fuzzy pre closed soft set and g_e be a fuzzy soft point such that $g_e\tilde{\notin}h_C$. If $(X, \mathfrak{T}, E)$ is fuzzy soft pre regular space, then there exists a fuzzy pre open soft set f_A such that $g_e\tilde{\in}f_A$ and $f_A \sqcap h_C = \tilde{0}_E$.

Proof. Obvious from Definition 5.4.

Theorem 5.7. Let $(X, \mathfrak{T}, E)$ be a fuzzy soft pre regular space and be a fuzzy pre open soft set g_B such that $f_e\tilde{\in}g_B$. Then, there exists a fuzzy pre open soft set f_S such that $f_e\tilde{\in}f_S$ and $FPcl(f_S) \sqsubseteq g_B$.

Proof. Let g_B be a fuzzy pre open soft set containing a fuzzy soft point f_e in a fuzzy soft pre regular space $(X, \mathfrak{T}, E)$. Then, g_B^c is a fuzzy pre closed soft such that $f_e\tilde{\notin}g_B^c$ from Theorem 2.1. By hypothesis, there exist disjoint fuzzy pre open soft sets f_S and f_W such that $f_e\tilde{\in}f_S$ and $g_B^c \sqsubseteq f_W$. It follows that, $f_W^c \sqsubseteq g_B$ and $f_S \sqsubseteq f_W^c$. Thus, $FPcl(f_S) \sqsubseteq f_W^c \sqsubseteq g_B$. So, we have a fuzzy pre open soft set f_S containing f_e such that $FPcl(f_S) \sqsubseteq g_B$.

Theorem 5.8. Every fuzzy soft pre T_3-space, in which every fuzzy soft point is fuzzy pre closed soft, is fuzzy soft pre T_2-space.

Proof. Let f_e, g_e be two distinct fuzzy soft points of a fuzzy soft pre T_3-space $(X, \mathfrak{T}, E)$. By hypothesis, g_e is fuzzy pre closed soft set and $f_e \tilde{\notin} g_e$. From the fuzzy soft pre regularity, there exist disjoint fuzzy pre open soft sets k_A and h_B such that $f_e \tilde{\in} k_A$ and $g_e \sqsubseteq h_B$. Thus, $f_e \tilde{\in} k_A$ and $g_e \tilde{\in} h_B$. Therefore, $(X, \mathfrak{T}, E)$ is fuzzy soft pre T_2-space.

Theorem 5.9. A fuzzy soft subspace $(Y, \mathfrak{T}_Y, E)$ of a fuzzy soft pre T_3-space $(X, \mathfrak{T}, E)$ is fuzzy soft pre T_3.

Proof. By Theorem 5.2, $(Y, \mathfrak{T}_Y, E)$ is fuzzy soft pre T_1-space. Now, we want to prove that $(Y, \mathfrak{T}_Y, E)$ is fuzzy soft pre regular space. Let k_E be a fuzzy pre closed soft set in (Y, E) and g_e be a fuzzy soft point in (Y, E) such that $g_e \tilde{\notin} k_E$. Then, $k_E = h_E^Y \sqcap g_B$ for some fuzzy pre closed soft set g_B in (X, E). Hence, $g_e \tilde{\notin} h_E^Y \sqcap g_B$. But $g_e \tilde{\in} h_E^Y$, so $g_e \tilde{\notin} g_B$. Since $(X, \mathfrak{T}, E)$ is fuzzy soft pre T_3. Then, there exist disjoint fuzzy pre open soft sets f_S and f_W in $\mathfrak{T}$ such that $g_e \tilde{\in} f_S$ and $g_B \sqsubseteq f_W$. It follows that, $h_E^Y \sqcap f_S$ and $h_E^Y \sqcap f_W$ are disjoint fuzzy pre open soft sets in $\mathfrak{T}_Y$ such that $g_e \tilde{\in} h_E^Y \sqcap f_S$ and $k_E \sqsubseteq h_E^Y \sqcap f_W$. Therefore, $(Y, \mathfrak{T}_Y, E)$ is fuzzy soft pre T_3.

Definition 5.5. Let $(X, \mathfrak{T}, E)$ be a fuzzy soft topological space and h_C, g_B be disjoint fuzzy pre closed soft sets. If there exist disjoint fuzzy pre open soft sets f_S and f_W such that $h_C \sqsubseteq f_S$, $g_B \sqsubseteq f_W$. Then, $(X, \mathfrak{T}, E)$ is called fuzzy soft pre normal space. A fuzzy soft pre normal T_1-space is called a fuzzy soft pre T_4-space.

Theorem 5.10. Let $(X, \mathfrak{T}, E)$ be a fuzzy soft topological space. Then, the following are equivalent:

(1) $(X, \mathfrak{T}, E)$ is a fuzzy soft pre normal space.

(2) For every fuzzy pre closed soft set h_C and fuzzy pre open soft set g_B such that $h_C \sqsubseteq g_B$, there exists a fuzzy pre open soft set f_S such that $h_C \sqsubseteq f_S$, $FPcl(f_S) \sqsubseteq g_B$.

Proof.

(1) ⇒ (2) Let h_C be a pre closed soft set and g_B be a fuzzy pre open soft set such that $h_C \sqsubseteq g_B$. Then, h_C, g_B^c are disjoint fuzzy pre closed soft sets. It follows by (1), there exist disjoint fuzzy pre open soft sets f_S and f_W such that $h_C \sqsubseteq f_S$, $g_B^c \sqsubseteq f_W$. Now, $f_S \sqsubseteq f_W^c$, so $FPcl(f_S) \sqsubseteq FPcl f_W^c = f_W^c$, where g_B is fuzzy pre open soft set. Also, $f_W^c \sqsubseteq g_B$. Hence, $FPcl(f_S^c) \sqsubseteq f_W^c \sqsubseteq g_B$. Thus, $h_C \sqsubseteq f_S$, $FPcl(f_S) \sqsubseteq g_B$.

(2) ⇒ (1) Let g_A and g_B be disjoint fuzzy pre closed soft sets. Then, $g_A \sqsubseteq g_B^c$. By hypothesis, there exists a fuzzy pre open soft set f_S such that $g_A \sqsubseteq f_S$, $FPcl(f_S) \sqsubseteq g_B^c$. So $g_B \sqsubseteq [FPcl(f_S)]^c$, $g_A \sqsubseteq f_S$ and $[FPcl(f_S)]^c \sqcap f_S = \tilde{0}_E$, where f_S and $[FPcl(f_S)]^c$ are fuzzy pre open soft sets. Thus, $(X, \mathfrak{T}, E)$ is fuzzy soft pre normal space.

Theorem 5.11. A fuzzy pre closed fuzzy soft subspace $(Y, \mathfrak{T}_Y, E)$ of a fuzzy soft pre normal space $(X, \mathfrak{T}, E)$ is fuzzy soft pre normal.

Proof. Let g_A and g_B be disjoint fuzzy pre closed soft sets in $\mathfrak{T}_Y$. Then, $g_A = h_E^Y \sqcap f_C$ and $g_B = h_E^Y \sqcap f_D$ for some fuzzy pre closed soft sets f_C, f_D in (X, E). Hence, f_C, f_D are disjoint fuzzy pre closed soft sets in $\mathfrak{T}$. Since $(X, \mathfrak{T}, E)$ is fuzzy soft pre normal. Then, there exist disjoint fuzzy pre open soft sets f_S and f_W in $\mathfrak{T}$ such that $f_C \sqsubseteq f_S$, $f_D \sqsubseteq f_W$. It follows that, $h_E^Y \sqcap f_S$ and $h_E^Y \sqcap f_W$ are disjoint fuzzy pre open soft sets in $\mathfrak{T}_Y$ such that $g_A = h_E^Y \sqcap f_C \sqsubseteq h_E^Y \sqcap f_S$ and $g_B = h_E^Y \sqcap f_D \sqsubseteq h_E^Y \sqcap f_W$. Therefore, $(Y, \mathfrak{T}_Y, E)$ is fuzzy soft pre normal.

Theorem 5.12. Let $(X, \mathfrak{T}_1, E)$ and $(Y, \mathfrak{T}_2, K)$ be fuzzy soft topological spaces and $f_{pu} : SS(X)_E \rightarrow SS(Y)_K$ be a fuzzy soft function which is bijective, fuzzy pre irresolute soft and fuzzy pre irresolute open soft. If $(X, \mathfrak{T}_1, E)$ is a fuzzy soft pre normal space, then $(Y, \mathfrak{T}_2, K)$ is also a fuzzy soft pre normal space.

Proof. Let f_A, g_B be disjoint fuzzy pre closed soft sets in Y. Since f_{pu} is fuzzy pre irresolute soft, then $f_{pu}^{-1}(f_A)$ and $f_{pu}^{-1}(g_B)$ are fuzzy pre closed soft set in X such that $f_{pu}^{-1}(f_A) \sqcap f_{pu}^{-1}(g_B) = f_{pu}^{-1}[f_A \sqcap g_B] = f_{pu}^{-1}[\tilde{0}_K] = \tilde{0}_E$ from Theorem 2.2. By hypothesis, there exist disjoint fuzzy pre open soft sets k_C and h_D in X such that $f_{pu}^{-1}(f_A) \sqsubseteq k_C$ and $f_{pu}^{-1}(g_B) \sqsubseteq h_D$. It follows that, $f_A = f_{pu}[f_{pu}^{-1}(f_A)] \sqsubseteq f_{pu}(k_C)$,$g_B = f_{pu}[f_{pu}^{-1}(g_B)] \sqsubseteq f_{pu}(h_D)$ from Theorem 2.2 and $f_{pu}(k_C) \sqcap f_{pu}(h_D) = f_{pu}[k_C \sqcap h_D] = f_{pu}[\tilde{0}_E] = \tilde{0}_K$ from Theorem 2.2. Since f_{pu} is fuzzy pre irresolute open soft function. Then, $f_{pu}(k_C), f_{pu}(h_D)$ are fuzzy pre open soft sets in Y. Thus, $(Y, \mathfrak{T}_2, K)$ is a fuzzy soft pre normal space.

6 Conclusion

Therefore, we introduce some properties of the notions of fuzzy pre open soft sets, fuzzy pre closed soft sets, fuzzy pre soft interior, fuzzy pre soft closure and fuzzy pre separation axioms and have established several interesting properties. Since the authors introduced topological structures on fuzzy soft sets [8, 15, 44], so the pre topological properties is generalized here to the fuzzy soft sets which will be useful in the fuzzy systems. Because there exists compact connections between soft sets and information systems [46, 39], we can use the results deducted from the studies on fuzzy soft topological space to improve these kinds of connections. We hope that the findings in this paper will help researcher enhance and promote the further study on fuzzy soft topology to carry out a general framework for their applications in practical life.

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EXTENDED HAUSDORFF DISTANCE AND SIMILARITY MEASURES FOR NEUTROSOPHIC REFINED SETS AND THEIR APPLICATION IN MEDICAL DIAGNOSIS

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Abstract - In this paper we present a new distance measure between neutrosophic refined sets on the basis of extended Hausdorff distance of neutrosophic set and we study some of their basic properties. Finally, using the extended Hausdorff distance and/or similarity measures, an application to medical diagnosis is presented.

Keywords - Neutrosophic sets, similarity measure, neutrosophic refined sets, extended Hausdorff distance.

1. Introduction

The neutrosophic set theory (NS) proposed in 1995 by Smarandache [6] was the generalization of the (FS for short) [21], intuituionitic fuzzy set (IFS for short) [20] and so on. In fuzzy set, the object, partially belong to a set with a membership degree (T) between 0 and 1 whereas in the IFS represent the uncertainty with respect to both membership ($T \in [0, 1]$) and non membership ($F \in [0, 1]$) such that $0 \leq T + F \leq 1$. Here, the number $I = 1 - T - F$ is called the hesitation degree or intuitionistic index. In neutrosophic set, indeterminacy is quantified explicitly and truth-membership, indeterminacy-membership and falsity-membership are independent. From scientific or engineering point of view, the

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neutrosophic set and set- theoretic view, operators need to be specified. Otherwise, it will be difficult to apply in the real applications. Therefore, H. Wang et al [8] defined a single valued neutrosophic set (SVNS) and then provided the set theoretic operations and various properties of single valued neutrosophic sets. Many researches on neutrosophic set on neutrosophic set theory and its applications in various fields are progressing rapidly [e.g., 2, 3, 4, 7, 8, 10, 11, 12, 15, 16, 17, 18, 19, 22, 23, 25, 31, 32, 34, 35, 36, 37, 38, 39].

The study of distance and similarity measure of NSs gives lots of measures, each representing specific properties and behavior in real-life decision making and pattern recognition works. For measuring the degree of similarity between neutrosophic sets, Broumi et al. [33] proposed several similarity measures and investigated some of their basic properties. Ye [16] presented the Hamming, Euclidean distance and similarity measures between interval neutrosophic sets. The generalized weighted distance and similarity measures between neutrosophic sets were given by J.Ye [15]. Also, the same author proposed three vector similarity measures for simplified neutrosophic sets (SNSs for short), including the Jaccard, Dice, and cosine similarity measures for SVNS and applied them to multicriteria decision-making problems with simplified neutrosophic information. Therefore, S.Broumi et al. [42] extended generalized weighted distance between neutrosophic sets (NSs) to the case of interval neutrosophic sets. Hanafy et al. [13, 14] presented the correlation measure neutrosophic sets.

The multi set introduced by Yager [30] allows the repeated occurrences of any element and hence the fuzzy multi set (FMS for short) can occur more than once with the possibly of the same or the different membership values.

Recently, based on [8], the new concept neutrosophic refined set(neutrosophic multisets) NRS was proposed by Broumi et al. [40] which allows the repeated occurrences of different truth membership, indeterminacy and non membership functions. Later on, Broumi et al. [40] studied correlation measure for neutrosophic refined sets and gave an application in decision making. The same author [41] defined the similarity measure between neutrosophic refined sets based on cosine function. The concept of NRS is a generalization of fuzzy multisets [42] and intuitionistic fuzzy multisets [43].

In this paper we extend the Hausdorff distance between neutrosophic sets to the case of neutrosophic refined sets (NRSs).

The organization of this paper is as follows: In section 2, the neutrosophic sets, neutrosophic refined sets and extended Hausdorff distance of neutrosophic sets are presented. The section 3 deals with the proposed extended Hausdorff distance and similarity measure for neutrosophic refined sets. Section 4, present an application of extended Hausdorff distance and similarity measure to medical diagnosis.

2. Preliminaries

This section gives a brief overview of concepts of neutrosophic set [6], and neutrosophic refined sets [8], Hausdorff distance and extended Hausdorff distance between NSs [33].

2.1 Neutrosophic Sets

Definition 2.1[6] Let X be an universe of discourse, with a generic element in X denoted by x, the neutrosophic (NS) set is an object having the form

$$A = \{< x: T_A(x), I_A(x), F_A(x)>, x \in X\}$$

where the functions T, I, F : X→ $]^-0, 1^+[$ define respectively the degree of membership (or Truth), the degree of indeterminacy, and the degree of non-membership (or Falsehood) of the element x ∈X to the set A with the condition.

$$^-0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3^+ \quad (1)$$

From philosophical point of view, the neutrosophic set takes the value from real standard or non-standard subsets of $]^-0, 1^+[$. So instead of $]-0, 1^+[$ we need to take the interval [0, 1] for technical applications, because $]^-0, 1^+[$ will be difficult to apply in the real applications such as in scientific and engineering problems. For two NS,

$$A_{NS} = \{ <x, T_A(x), I_A(x), F_A(x)> \mid x \in X \} \quad (2)$$

And $B_{NS} = \{<x, T_B(x), I_B(x), F_B(x)> \mid x \in X \}$ the two relations are defined as follows:

(1) $A_{NS} \subseteq B_{NS}$ if and only if $T_A(x) \leq T_B(x)$, $I_A(x) \geq I_B(x)$, $F_A(x) \geq F_B(x)$
(2) $A_{NS} = B_{NS}$ if and only if $T_A(x) = T_B(x)$, $I_A(x) = I_B(x)$, $F_A(x) = F_B(x)$

2.2 Neutrosophic Refined Sets

In 2013, Smarandache [8] extended the neutrosophic set to n-valued refined neutrosophic set.

Definition 2.2 [8] Let E be a universe, a neutrosophic sets on E can be defined as follows:

$$A = \{<x, (T_A^1(x), T_A^2(x), ..., T_A^p(x)), (I_A^1(x), I_A^2(x), ..., I_A^p(x)), (F_A^1(x), F_A^2(x), \ldots, F_A^p(x))>: x \in X\}$$

Where

$T_A^1(x), T_A^2(x), ..., T_A^p(x)$: E→ [0 ,1],
$I_A^1(x), I_A^2(x), ..., I_A^p(x)$: E→ [0 ,1], and
$F_A^1(x), F_A^2(x), ..., F_A^p(x)$: E→ [0 ,1]

such that $0 \leq T_A^i(x) + I_A^i(x) + F_A^i(x) \leq 3$ for i=1 ,2,…,p for any x∈ X ,

$$(T_A^1(x), T_A^2(x), \ldots, T_A^p(x)),\ (I_A^1(x), I_A^2(x), \ldots, I_A^p(x)) \text{ and } (F_A^1(x), F_A^2(x), \ldots, F_A^p(x))$$

is the truth-membership sequence, indeterminacy-membership sequence and falsity-membership sequence of the element x, respectively. Also, P is called the dimension of neutrosophic refined sets (NRS) A.

2.3 Hausdorff distance

Definition 2.3 The Hausdorff distance defined as follows

$$d_H(A, B) = H(A, B) = \max\{|a_1 - b_1|, |a_2 - b_2|\} \quad (3)$$

For the two intervals A=$[a_1, a_2]$ and B=$[b_1, b_2]$ in areal space R.

Definition 2.4: The Hausdorff distance $d_H(A, B)$ between A and B satisfies the following properties (D1-D4):

(D1) $1 \leq d_H(A, B) \leq 1$.
(D2) $d_H(A, B) = 0$ if and only if A = B; for all A, B
(D3) $d_H(A, B) = d_H(B, A)$.
(D4) If A⊆B⊆C, for A, B, C, then $d_H(A, C) \geq d_H(A, B$ and $d_H(A, C) \geq d_H(B, C)$

2.4 Extended Hausdorff distance between neutrosophic sets

Definition2.5 [33] Based on the Hausdorff metric, Szmidt and Kacprzykdefined new distance between intuitionistic fuzzy sets and/or interval-valued fuzzy sets in [5], taking into account three parameter representation (membership, non-membership values, and the hesitation margins) of A-IFSs which fulfill the properties of the Hausdorff distances. Their definition is defined by

$$H_{A-IFS}(A, B) = \frac{1}{n}\sum_{i=1}^{n} max\{|\mu_A(x_i) - \mu_B(x_i)|, |\nu_A(x_i) - \nu_B(x_i)|, |\pi_A(x_i) - \pi_B(x_i)|\} \quad (4)$$

Where A = {< x, $\mu_A(x)$, $\nu_A(x)$, $\pi_A(x)$>} and B = {< x, $\mu_B(x)$, $\nu_B(x)$, $\pi_B(x)$>}.

The terms and symbols used in [5] are changed so that they are consistent with those in this section.

Let X= $\{x_1, x_2, \ldots, x_n\}$ be a discrete finite set. Consider a neutrosophic set A in X where $T_A(x_i)$, $I_A(x_i)$, $F_A(x_i) \in [0,1]$ for every $x_i \in X$, represent its membership, indeterminacy, and non-membership values respectively denoted by A = {< x, $T_A(x_i)$, $I_A(x_i)$, $F_A(x_i)$>}.

Then, the distance between A ∈NSs and B∈ NSs is defined as follows.

$$d_{HNS}(A, B) = \frac{1}{n}\sum_{i=1}^{n} \max\{|T_A(x_i) - T_B(x_i)|, |I_A(x_i) - I_B(x_i)|, |F_A(x_i) - F_B(x_i)|\} \quad (5)$$

Where $d_{HNS}(A, B)$= H(A, B)denote the extended Hausdorff distance between two neutrosophic sets(NS) A and B.

Let A, B and C be three neutrosophic sets

$$d_{HNS}(A, B) = H(A, B) = \max\{|T_A(x_i) - T_B(x_i)|, \left|I_A(x_i) - I_{B(x_i)}\right|, |F_A(x_i) - F_B(x_i)|\} \quad (6)$$

The same between A and C is written as:

$$H(A, C)= \max\{|T_A(x_i) - T_C(x_i)|, |I_A(x_i) - I_C(x_i)\ |, |F_A(x_i) - F_C(x_i)|\} \quad (7)$$

And between B and C is written as:

$$H(B, C)= \max\{|T_B(x_i) - T_C(x_i)|, |I_B(x_i) - I_C(x_i)\ |, |F_B(x_i) - F_C(x_i)|\} \quad (8)$$

Hung and yang [45] presented their similarity measures based on Hausdorff distance as follows

Definition2.6 The similarity measure based on the Hausdroff distance is

s_H^1(A,B)= 1-$d_H(A,B)$.
s_H^2(A,B)= $(e^{-d_H(A,B)} - e^{-1})/(1 - e^{-1})$.
s_H^3(A,B)= $(1 - d_H(A,B))/(1 - d_H(A,B))$.

Where d_H (A, B) was the Hausdorff distance

3. Extended Hausdorff Distance and Similarity Measures for Neutrosophic Refined Sets

3.1 Extended Hausdorff Distance Measures for Neutrosophic Refined Sets

On the basis of the extended Hausdorff distance between two neutrosophic set defined by Broumi et al. in [33], defined as follows:

$$d_{HNS}(A,B) = \frac{1}{n}\sum_{i=1}^{n} \max\{|T_A(x_i) - T_B(x_i)|, |I_A(x_i) - I_B(x_i)|, |F_A(x_i) - F_B(x_i)|\} \quad (9)$$

Where $d_{HNS}(A,B)$ = H(A, B)denote the extended Hausdorff distance between two neutrosophic sets A and B.

We extend the above equation (9) distance to the case of neutrosophic refined set between A and B as follows:

when the sets A and B do not have the same number of subcomponents for T, I, F. If

A = {<x,($T_A^1(x)$,$T_A^2(x)$,...,$T_A^a(x)$), ($I_A^1(x)$,$I_A^2(x)$,...,$I_A^b(x)$), ($F_A^1(x)$,$F_A^2(x)$,…,$F_A^c(x)$)>: x ∈ X}

and

B = {<x,($T_B^1(x)$,$T_B^2(x)$,...,$T_B^p(x)$), ($I_B^1(x)$,$I_B^2(x)$,...,$I_B^r(x)$), ($F_B^1(x)$,$F_B^2(x)$,…,$F_B^q(x)$)>: x ∈ X}

then we take m = max{a, b, c, p, r, q} and transform A and B into refined neutrosophic sets where all components T, I, F have each of them m subcomponents, i.e.

$$A = \{<x,(T_A^1(x),T_A^2(x),...,T_A^m(x)), (I_A^1(x),I_A^2(x),...,I_A^m(x)), (F_A^1(x),F_A^2(x),\ldots,F_A^m(x))>: x \in X\}$$

and

$$B = \{<x,(T_B^1(x),T_B^2(x),...,T_B^m(x)), (I_B^1(x),I_B^2(x),...,I_B^m(x)), (F_B^1(x),F_B^2(x),\ldots,F_B^m(x))>: x \in X\}.$$

$$d_{HNRS}(A,B) = \frac{1}{m}\sum_{j=1}^{m}\left\{\frac{1}{n}\sum_{i=1}^{n}\max\left\{\left|T_A^j(x_i) - T_B^j(x_i)\right|, \left|I_A^j(x_i) - I_B^j(x_i)\right|, \left|F_A^j(x_i) - F_B^j(x_i)\right|\right\}\right\} \quad (10)$$

Proposition 3.1 The defined distance $d_{HNRS}(A,B)$ between NRSs A and B satisfies the following properties (D1-D4):

(D1) $d_{HNRS}(A,B) \geq 0$.
(D2) $d_{HNRS}(A,B) = 0$ if and only if A = B; for all A, B ∈ NRSs.
(D3) $d_{HNRS}(A,B) = d_{HNRS}(B,A)$.
(D4) If $A \subseteq B \subseteq C$, for A, B, C ∈ NRSs, then $d_{HNRS}(A, C) \geq d_{HNRS}(A,B)$ and $d_{HNRS}(A,C) \geq d_{HNRS}(B,C)$

Proof: (D1) $d_{HNRS}(A,B) \geq 0$.
As truth-membership, indeterminacy-membership and falsity-membership functions of the NRSs lies between 0 and 1, the distance measure based on these function also lies between 0 to 1.

(D2) $d_{HNRS}(A, B) = 0$ if and only if A = B.

(i) Let the two NRS A and B be equal(i.e) A=B

This implies for any $T_A^j(x_i) = T_B^j(x_i)$, $I_A^j(x_i) = I_B^j(x_i)$ and $F_A^j(x_i) = F_B^j(x_i)$ which states that

$$\left|T_A^j(x_i) - T_B^j(x_i)\right|, \left|I_A^j(x_i) - I_B^j(x_i)\right| \text{ and } \left|F_A^j(x_i) - F_B^j(x_i)\right|$$

Hence $d_{HNRS}(A,B) = 0$.

(ii) Let the $d_{HNRS}(A,B) = 0$.

The zero distance measure is possible only if both

$$\left|T_A^j(x_i) - T_B^j(x_i)\right|, \left|I_A^j(x_i) - I_B^j(x_i)\right| \text{ and} \left|I_A^j(x_i) - I_B^j(x_i)\right| = 0$$

as the extended Hausdorff distance measure concerns with maximum truth-membership, indeterminacy-membership and falsity-membership differences.This refers that

$$T_A^j(x_i) = T_B^j(x_i),\ I_A^j(x_i) = I_B^j(x_i) \text{ and } F_A^j(x_i) = F_B^j(x_i)$$

for all i, j values. Hence A=B

(D3) $d_{\mathrm{HNRS}}(A,B)$ = $d_{\mathrm{HNRS}}(B,A)$

It is obvious that

$$T_A^j(x_i) - T_B^j(x_i) \neq T_B^j(x_i) - T_A^j(x_i)\ , I_A^j(x_i) - I_B^j(x_i) \neq I_B^j(x_i) - I_A^j(x_i)$$

and

$$F_A^j(x_i) - T_B^j(x_i) \neq T_B^j(x_i) - T_A^j(x_i)$$

But

$$\left|T_A^j(x_i) - T_B^j(x_i)\right| = \left|T_B^j(x_i) - T_A^j(x_i)\right|, \left|I_A^j(x_i) - I_B^j(x_i)\right| = \left|I_B^j(x_i) - I_A^j(x_i)\right|,$$

and

$$\left|F_A^j(x_i) - F_B^j(x_i)\right| = \left|F_A^j(x_i) - F_B^j(x_i)\right|$$

Hence

$$\begin{aligned} d_{\mathrm{HNRS}}(A,B) &= \frac{1}{m}\sum_{j=1}^{m}\left\{\frac{1}{\mathrm{n}}\sum_{\mathrm{i}=1}^{\mathrm{n}} \max\left\{\left|T_A^j(x_i) - T_B^j(x_i)\right|, \left|I_A^j(x_i) - I_B^j(x_i)\right|, \left|F_A^j(x_i) - F_B^j(x_i)\right|\right\}\right\} \\ &= \frac{1}{m}\sum_{j=1}^{m}\left\{\frac{1}{\mathrm{n}}\sum_{\mathrm{i}=1}^{\mathrm{n}} \max\left\{\left|T_B^j(x_i) - T_A^j(x_i)\right|, \left|I_B^j(x_i) - I_A^j(x_i)\right|, \left|F_B^j(x_i) - F_A^j(x_i)\right|\right\}\right\} \\ &= d_{\mathrm{HNRS}}(B,A). \end{aligned}$$

Remark: Let A, B ∈ NRSs , A⊆ B if and only if

$T_A^j(x_i) \le T_B^j(x_i)$,$I_A^j(x_i) \ge I_B^j(x_i)$,$F_A^j(x_i) \ge F_B^j(x_i)$ for i=1,2,…,p, for every x_i ∈ X .

(D4) If A⊆B⊆C, for A, B, C ∈ NRSs, then $d_{\mathrm{HNRS}}(A,C) \ge d_{\mathrm{HNRS}}(A,B)$ and $d_{\mathrm{HNRS}}(A,C) \ge d_{\mathrm{HNRS}}(B,C)$

Let A⊆ B ⊆C,then the assumption is

$$T_A^j(x_i) \le T_B^j(x_i) \le T_C^j(x_i), I_A^j(x_i) \ge I_B^j(x_i) \ge I_C^j(x_i), F_A^j(x_i) \ge F_B^j(x_i) \ge F_C^j(x_i)$$

Case (i) we prove that $d_H(A,B) \le d_H(A,C)$

$$\alpha\text{- } if \left|T_A^j(\boldsymbol{x_i}) - T_C^j(\boldsymbol{x_i})\right| \ge \left|I_A^j(\boldsymbol{x_i}) - I_C^j(\boldsymbol{x_i})\right| \ge \left|F_A^j(\boldsymbol{x_i}) - F_C^j(\boldsymbol{x_i})\right|$$

then

H(A,C)=$\left|T_A^j(\mathrm{x_i}) - T_C^j(\mathrm{x_i})\right|$but we have for all x_i∈ X

(i) $\left|\mathrm{I}_\mathrm{A}^\mathrm{j}(\mathrm{x_i}) - \mathrm{I}_\mathrm{B}^\mathrm{j}(\mathrm{x_i})\right| \le \left|\mathrm{I}_\mathrm{A}^\mathrm{j}(\mathrm{x_i}) - \mathrm{I}_\mathrm{C}^\mathrm{j}(\mathrm{x_i})\right|$for all x_i ∈ X

$$\le \left|\mathrm{T}_\mathrm{A}^\mathrm{j}(\mathrm{x_i}) - \mathrm{T}_\mathrm{C}^\mathrm{j}(\mathrm{x_i})\right|$$

And $\left|F_A^j(x_i) - F_B^j(x_i)\right| \leq \left|F_A^j(x_i) - F_C^j(x_i)\right|$ for all $x_i \in X$
$\leq \left|T_A^j(x_i) - T_C^j(x_i)\right|$

(ii) $\left|I_B^j(x_i) - I_C^j(x_i)\right| \leq \left|I_A^j(x_i) - I_C^j(x_i)\right|$ for all $x_i \in X$
$\leq \left|T_A^j(x_i) - T_C^j(x_i)\right|$

And $\left|F_B^j(x_i) - F_C^j(x_i)\right| \leq \left|F_A^j(x_i) - F_C^j(x_i)\right|$ for all $x_i \in X$
$\leq \left|T_A^j(x_i) - T_C^j(x_i)\right|$ for all $x_i \in X$

On the other hand we have

(iii) $\left|T_A^j(x_i) - T_B^j(x_i)\right| \leq \left|T_A^j(x_i) - T_C^j(x_i)\right|$

and $\left|T_B^j(x_i) - T_C^j(x_i)\right| \leq \left|T_A^j(x_i) - T_C^j(x_i)\right|$ for all $x_i \in X$

Combining (i) and (iii) we obtain therefore

$$\frac{1}{m}\sum_{j=1}^{m}\left\{\frac{1}{n}\sum_{i=1}^{n}\max\{\left|T_A^j(x_i) - T_B^j(x_i)\right|, \left|I_A^j(x_i) - I_B^j(x_i)\right|, \left|F_A^j(x_i) - F_B^j(x_i)\right|\}\right\}$$
$$\leq \frac{1}{m}\sum_{j=1}^{m}\left\{\frac{1}{n}\sum_{i=1}^{n}\max\{\left|T_A^j(x_i) - T_C^j(x_i)\right|, \left|I_A^j(x_i) - I_C^j(x_i)\right|, \left|F_A^j(x_i) - F_C^j(x_i)\right|\}\right\}$$
$$\frac{1}{m}\sum_{j=1}^{m}\left\{\frac{1}{n}\sum_{i=1}^{n}\max\{\left|T_B^j(x_i) - T_C^j(x_i)\right|, \left|I_B^j(x_i) - I_C^j(x_i)\right|, \left|F_B^j(x_i) - F_C^j(x_i)\right|\}\right\}$$
$$\leq \frac{1}{m}\sum_{j=1}^{m}\left\{\frac{1}{n}\sum_{i=1}^{n}\max\{\left|T_A^j(x_i) - T_C^j(x_i)\right|, \left|I_A^j(x_i) - I_C^j(x_i)\right|, \left|F_A^j(x_i) - F_C^j(x_i)\right|\}\right\}$$

That is

$$d_{\mathrm{HNRS}}(A,B) \leq d_{\mathrm{HNRS}}(A,C) \text{ AND } d_{\mathrm{HNRS}}(B,C) \leq d_{\mathrm{HNRS}}(A,C)$$

β - If $\left|T_A^j(\mathbf{x_i}) - T_C^j(\mathbf{x_i})\right| \leq \left|F_A^j(\mathbf{x_i}) - F_C^j(\mathbf{x_i})\right| \leq \left|I_A^j(\mathbf{x_i}) - I_C^j(\mathbf{x_i})\right|$

Then

H(A,C)=$\left|I_A^j(x_i) - I_C^j(x_i)\right|$ But we have for all $x_i \in X$

(a) $\left|T_A^j(x_i) - T_B^j(x_i)\right| \leq \left|T_A^j(x_i) - T_C^j(x_i)\right|$ for all $x_i \in X$
$\leq \left|I_A^j(x_i) - I_C^j(x_i)\right|$

And $\left|F_A^j(x_i) - F_B^j(x_i)\right| \leq \left|F_A^j(x_i) - F_C^j(x_i)\right|$ for all $x_i \in X$

$$\leq \left|\mathrm{I}_A^j(x_i) - \mathrm{I}_C^j(x_i)\right|$$

(b) $\left|\mathrm{T}_B^j(x_i) - \mathrm{T}_C^j(x_i)\right| \leq \left|\mathrm{T}_A^j(x_i) - \mathrm{T}_C^j(x_i)\right|$ for all $x_i \in X$

$$\leq \left|\mathrm{I}_A^j(x_i) - \mathrm{I}_C^j(x_i)\right|$$

and $\left|\mathrm{F}_B^j(x_i) - \mathrm{F}_C^j(x_i)\right| \leq \left|\mathrm{F}_A^j(x_i) - \mathrm{F}_C^j(x_i)\right|$ for all $x_i \in X$

$$\leq \left|\mathrm{I}_A^j(x_i) - \mathrm{I}_C^j(x_i)\right|$$

On the other hand we have

(c) $\left|\mathrm{I}_A^j(x_i) - \mathrm{I}_B^j(x_i)\right| \leq \left|\mathrm{I}_A^j(x_i) - \mathrm{I}_C^j(x_i)\right|$ and

$\left|\mathrm{I}_B^j(x_i) - \mathrm{I}_C^j(x_i)\right| \leq \left|\mathrm{I}_A^j(x_i) - \mathrm{I}_C^j(x_i)\right|$ for all $x_i \in X$.

Combining (a) and (c) we obtain, therefore

$$\frac{1}{m}\sum_{j=1}^{m}\left\{\frac{1}{n}\sum_{i=1}^{n}\max\left\{\left|\mathrm{T}_A^j(x_i) - \mathrm{T}_B^j(x_i)\right|, \left|\mathrm{I}_A^j(x_i) - \mathrm{I}_B^j(x_i)\right|, \left|\mathrm{F}_A^j(x_i) - \mathrm{F}_B^j(x_i)\right|\right\}\right\}$$
$$\leq \frac{1}{m}\sum_{j=1}^{m}\left\{\frac{1}{n}\sum_{i=1}^{n}\max\left\{\left|T_A^j(x_i) - T_C^j(x_i)\right|, \left|I_A^j(x_i) - I_C^j(x_i)\right|, \left|F_A^j(x_i) - F_C^j(x_i)\right|\right\}\right\}$$
$$\frac{1}{m}\sum_{j=1}^{m}\left\{\frac{1}{n}\sum_{i=1}^{n}\max\left\{\left|T_B^j(x_i) - T_C^j(x_i)\right|, \left|I_B^j(x_i) - I_C^j(x_i)\right|, \left|F_B^j(x_i) - F_C^j(x_i)\right|\right\}\right\}$$
$$\leq \frac{1}{m}\sum_{j=1}^{m}\left\{\frac{1}{n}\sum_{i=1}^{n}\max\left\{\left|T_A^j(x_i) - T_C^j(x_i)\right|, \left|I_A^j(x_i) - I_C^j(x_i)\right|, \left|F_A^j(x_i) - F_C^j(x_i)\right|\right\}\right\}$$

That is

$$d_{\mathrm{HNRS}}(A,B) \leq d_{\mathrm{HNRS}}(A,C) \text{ AND } d_{\mathrm{HNRS}}(B,C) \leq d_{\mathrm{HNRS}}(A,C)$$

γ. **If** $\left|\mathrm{T}_A^j(\mathbf{x_i}) - \mathrm{T}_C^j(\mathbf{x_i})\right| \leq \left|\mathrm{I}_A^j(\mathbf{x_i}) - \mathrm{I}_C^j(\mathbf{x_i})\right| \leq \left|\mathrm{F}_A^j(\mathbf{x_i}) - \mathrm{F}_C^j(\mathbf{x_i})\right|$

Then

$\mathrm{H(A,C)}=|\mathrm{F}_A(x_i) - \mathrm{F}_C(x_i)|$ But we have for all $x_i \in X$

(a) $\left|\mathrm{T}_A^j(\mathbf{x_i}) - \mathrm{T}_B^j(\mathbf{x_i})\right| \leq \left|\mathrm{T}_A^j(\mathbf{x_i}) - \mathrm{T}_C^j(\mathbf{x_i})\right|$ for all $x_i \in X$

$$\leq \left|\mathrm{F}_A^j(\mathbf{x_i}) - \mathrm{F}_C^j(\mathbf{x_i})\right|$$

And $\left|\mathrm{I}_A^j(x_i) - \mathrm{I}_B^j(x_i)\right| \leq \left|\mathrm{I}_A^j(x_i) - \mathrm{I}_C^j(x_i)\right|$ for all $x_i \in X$

$$\leq \left|\mathrm{F}_A^j(x_i) - \mathrm{F}_C^j(x_i)\right|$$

(b) $\left|\mathrm{T}_B^j(\mathbf{x_i}) - \mathrm{T}_C^j(\mathbf{x_i})\right| \leq \left|\mathrm{T}_A^j(\mathbf{x_i}) - \mathrm{T}_C^j(\mathbf{x_i})\right|$ for all $x_i \in X$

$$\leq \left|F_A^j(\mathbf{x_i}) - F_C^j(\mathbf{x_i})\right|$$

And $\left|I_B^j(\mathbf{x_i}) - I_C^j(\mathbf{x_i})\right| \leq \left|I_A^j(\mathbf{x_i}) - I_C^j(\mathbf{x_i})\right|$ for all $x_i \in X$

$$\leq \left|F_A^j(\mathbf{x_i}) - F_C^j(\mathbf{x_i})\right|$$

On the other hand we have

(c) $\left|F_A^j(\mathbf{x_i}) - F_B^j(\mathbf{x_i})\right| \leq \left|F_A^j(\mathbf{x_i}) - F_C^j(\mathbf{x_i})\right|$ and $\left|F_B^j(\mathbf{x_i}) - F_C^j(\mathbf{x_i})\right| \leq \left|F_A^j(\mathbf{x_i}) - F_C^j(\mathbf{x_i})\right|$ for all $x_i \in X$

Combining (a) and (c) we obtain

$$\frac{1}{m}\sum_{j=1}^{m}\left\{\frac{1}{n}\sum_{i=1}^{n}\max\left\{\left|T_A^j(x_i) - T_B^j(x_i)\right|, \left|I_A^j(x_i) - I_B^j(x_i)\right|, \left|F_A^j(x_i) - F_B^j(x_i)\right|\right\}\right\}$$
$$\leq \frac{1}{m}\sum_{j=1}^{m}\left\{\frac{1}{n}\sum_{i=1}^{n}\max\left\{\left|T_A^j(x_i) - T_C^j(x_i)\right|, \left|I_A^j(x_i) - I_C^j(x_i)\right|, \left|F_A^j(x_i) - F_C^j(x_i)\right|\right\}\right\}$$

$$\frac{1}{m}\sum_{j=1}^{m}\left\{\frac{1}{n}\sum_{i=1}^{n}\max\left\{\left|T_B^j(x_i) - T_C^j(x_i)\right|, \left|I_B^j(x_i) - I_C^j(x_i)\right|, \left|F_B^j(x_i) - F_C^j(x_i)\right|\right\}\right\}$$
$$\leq \frac{1}{m}\sum_{j=1}^{m}\left\{\frac{1}{n}\sum_{i=1}^{n}\max\left\{\left|T_A^j(x_i) - T_C^j(x_i)\right|, \left|I_A^j(x_i) - I_C^j(x_i)\right|, \left|F_A^j(x_i) - F_C^j(x_i)\right|\right\}\right\}$$

That is

$$d_{\mathrm{HNRS}}(A,B) \leq d_{\mathrm{HNRS}}(A,C) \text{ AND } d_{\mathrm{HNRS}}(B,C) \leq d_{\mathrm{HNRS}}(A,C).$$

From γ, α and β, we can obtain the property (D4).

3.2 Weighted Extended Hausdorff Distance between Neutrosophic Refined Sets

In many situation the weight of the element $x_i \in X$ should be taken into account. Usually the elements have different importance .We need to consider the weight of the element so that we have the following weighted distance between NRSs. Assume that the weight of $x_i \in X$ is w_i where $X=\{x_1, x_2,..,x_n\}$,$w_i \in [0, 1]$, i= 1, 2, 3, .., n and $\sum_1^n w_i = 1$. Then the weighted extended hausdorff distance between NRSs A and B is defined as:

$$d_{\mathrm{WHNRS}}(A,B) = \sum_1^n w_i\, d_{\mathrm{HNRS}}(A(x_i), B(x_i)) \qquad (11)$$

It is easy to check that $d_{\mathrm{WHNRS}}(A,B)$ satisfies the four properties D1-D4 defined above.

It is well known that similarity measure can be generated from distance measure. Therefore we may use the proposed distance measure to define similarity measures.

Based on the relationship of similarity measure and distance we can define some similarity measures between NRSs A and B as follows.

Definition The similarity measure based on the extended Hausdorff distance is

$s^1_{\mathrm{HNRS}}(A, B) = 1 - d_{\mathrm{HNRS}}(A, B)$
$s^2_{\mathrm{HNRS}}(A, B) = (e^{-d_{\mathrm{HNRS}}(A,B)} - e^{-1})/(1 - e^{-1})$
$s^3_{\mathrm{HNRS}}(A, B) = (1 - d_{\mathrm{HNRS}}(A, B))/(1 + d_{\mathrm{HNRS}}(A, B))$

$s_{\mathrm{HNRS}}(A, B)$ is said to be the similarity measure between A an B, where A ,B∈ NRS, as $s_{\mathrm{HNRS}}(A, B)$ satisfies the following properties

(P1) S(A,B) = S(B, A).
(P2) S(A,B) = (1, 0, 0)=$\underline{1}$ if A=B for all A,B ∈ NVNSs.
(P3) $S(A, B) \in [0, 1]$
(P4) If A⊆B⊆C for all A, B, C ∈ NRSs then S(A, B)≥S(A, C) and S(B, C) ≥ S(A, C)

4. Medical Diagnosis Using Extended Hausdorff Distance of NRS.

In what follows, let us consider an illustrative example adopted from [40].

"As Medical diagnosis contains lots of uncertainties and increased volume of information available to physicians from new medical technologies, the process of classifying different set of symptoms under a single name of disease becomes difficult. In some practical situations, there is the possibility of each element having different truth membership, indeterminate and false membership functions. The proposed correlation measure among the patients Vs symptoms and symptoms Vs diseases gives the proper medical diagnosis. The unique feature of this proposed method is that it considers multi truth membership, indeterminate and false membership. By taking one time inspection, there may be error in diagnosis. Hence, this multi time inspection, by taking the samples of the same patient at different times gives best diagnosis".

Now, an example of a medical diagnosis will be presented.

Example 1 Let P={P_1, P_2, P_3} be a set of patients, D={Viral Fever, Tuberculosis, Typhoid, Throat disease} be a set of diseases and S={Temperature, cough, throat pain, headache, body pain} be a set of symptoms. Our solution is to examine the patient at different time intervals (three times a day), which in turn give arise to different truth membership, indeterminate and false membership function for each patient

Table I -3-valued neutrosophic set: The Relation between Patient and Symptoms

Q	Temperature	Cough	Throat	Head Ache	Body Pain
P_1	(0.4, 0.3, 0.4) (0.2, 0.5, 0.5) (0.3, 0.4, 0.6)	(0.5, 0.4, 0.4) (0.4, 0.1, 0.3) (0.3, 0.4, 0.5)	(0.3, 0.5, 0.5) (0.2, 0.6, 0.4) (0.1, 0.6, 0.3)	(0.5, 0.3, 0.4) (0.5, 0.4, 0.7) (0.3, 0.3, 0.6)	(0.5, 0.2, 0.4) (0.2, 0.3, 0.5) (0.1, 0.4, 0.3)
P_2	(0.6, 0.3, 0.5) (0.5, 0.5, 0.2) (0.4, 0.4, 0.5)	(0.6, 0.3, 0.7) (0.4, 0.4, 0.2) (0.2, 0.4, 0.5)	(0.6, 0.3, 0.3) (0.3, 0.5, 0.4) (0.1, 0.4, 0.5)	(0.6, 0.3, 0.1) (0.4, 0.5, 0.8) (0.2, 0.4, 0.3)	(0.4, 0.4, 0.5) (0.3, 0.2, 0.7) (0.1, 0.5, 0.5)
P_3	(0.8, 0.3, 0.5) (0.7, 0.5, 0.4) (0.6, 0.4, 0.4)	(0.5, 0.5, 0.3) (0.1, 0.6, 0.4) (0.3, 0.4, 0.3)	(0.3, 0.3, 0.6) (0.2, 0.5, 0.7) (0.1, 0.4, 0.5)	(0.6, 0.2, 0.5) (0.5, 0.3, 0.6) (0.2, 0.2, 0.6)	(0.6, 0.4, 0.5) (0.3, 0.3, 0.4) (0.2, 0.2, 0.6)

Let the samples be taken at three different timing in a day (morning, noon and night)

Table II – neutrosophic refined set:The Relation between Patient and Symptoms

Q	Viral Fever	Tuberculosis	Typhoid	Throat disease
Temperature	(0.2, 0.5, 0.6)	(0.4, 0.6, 0.5)	(0.4, 0.3, 0.5)	(0.3, 0.7, 0.8)
Cough	(0.6, 0.4, 0.6)	(0.8, 0.2, 0.3)	(0.3, 0.2, 0.6)	(0.2, 0.4, 0.1)
Throat	(0.5, 0.2, 0.3)	(0.4, 0.5, 0.3)	(0.4, 0.5, 0.5)	(0.2, 0.6, 0.2)
Head Ache	(0.6, 0.8, 0.2)	(0.2, 0.3, 0.6)	(0.1, 0.6, 0.3)	(0.2, 0.5, 0.5)
Body Pain	(0.7, 0.5, 0.4)	(0.2, 0.3, 0.4)	(0.2, 0.3, 0.4)	(0.2, 0.2, 0.3)

Distance measure (d)

Proposed distance measure	Viral Fever	Tuberculosis	Typhoid	Throat disease
P_1	0.34667	0.2533	0.22	0.2733
P_2	0.366	0.3	0.266	0.346
P_3	0.426	0.326	0.3067	0.34

S=1-d

Proposed similarity measure(s^1_{HNVNS}(A,B))	Viral Fever	Tuberculosis	Typhoid	Throat disease
P_1	0.6534	0.7467	**0.78**	0.7267
P_2	0.634	0.7	**0.733**	0.654
P_3	0.574	0.674	**0.693**	0.66

Proposed similarity measure(s^3_{HNVNS}(A,B)	Viral Fever	Tuberculosis	Typhoid	Throat disease
P_1	0.4852	0.5957	**0.6393**	0.5707
P_2	0.4641	0.5384	**0.5797**	0.4858
P_3	0.4025	0.5082	**0.5306**	0.4925

The highest similarity measure from table gives the proper medical diagnosis

Patient P1 suffers from Tuberculosis and Typhoid, Patient P2 suffers from Typhoid and Patient P3 suffers Typhoid

5 Conclusions

In this paper we have presented a new distance measure between NRS on the basis of extended Hausdorff distance of neutrosophic set, then we proved their properties. Finally we used this distance measure in an application of medical diagnosis. It's hoped that our findings will help enhancing this study on neutrosophic set for researchers. In future work, we will extended this distance to the case of interval neutrosophic refined sets.

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THE NEW SOLUTION OF TIME FRACTIONAL WAVE EQUATION WITH CONFORMABLE FRACTIONAL DERIVATIVE DEFINITION

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Abstract – In this paper, we used new fractional derivative definition, the conformable fractional derivative, for solving two and three dimensional time fractional wave equation. This definition is simple and very effective in the solution procedures of the fractional differential equations that have complicated solutions with classical fractional derivative definitions like Caputo, Riemann-Liouville and etc. The results show that conformable fractional derivative definition is usable and convenient for the solution of higher dimensional fractional differential equations.

***Keywords* –** *Time fractional wave equation, conformable fractional derivative.*

1 Introduction

Fractional differential equations sometimes called as extraordinary differential equations because of their nature and easily find in various fields of applied sciences [1-8]. So the scientific and engineering problems which involve fractional calculus are very large and still very effective. In recent years, scientists have proposed many efficient and powerful methods to obtain exact or numerical solutions of fractional differential equations [9-10].

In addition to this, many researchers have been trying to form a new definition of fractional derivative. Most of these definitions include integral form for fractional derivatives. Two of these definitions which are most popular:

i. Riemann-Liouville definition:

If n is a positive integer and $\alpha \in [n-1, n)$, α derivative of f is given by

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$$D_a^{\alpha}(f)(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t \frac{f(x)}{(t-x)^{\alpha-n+1}} dx.$$

ii. *Caputo definition:*

If n is a positive integer and $\alpha \in [n-1, n)$, α derivative of f is given by

$$D_a^{\alpha}(f)(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{f^{(n)}(x)}{(t-x)^{\alpha-n+1}} dx.$$

In [11] R.Khalil and et al. give a new definition of fractional derivative called "conformable fractional derivative".

Definition 1.1. Let $f : [0,\infty) \to \mathbb{R}$ be a function. α^{th} order *"conformable fractional derivative"* of f is defined by

$$T_{\alpha}(f)(t) = \lim_{\varepsilon \to 0} \frac{f(t + \varepsilon t^{1-\alpha}) - f(t)}{\varepsilon}$$

for all $t > 0$, $\alpha \in (0,1)$. If f is α-differentiable in some $(0,a)$, $a > 0$, and $\lim_{t \to 0^+} f^{(\alpha)}(t)$ exists, then define $f^{(\alpha)}(0) = \lim_{t \to 0^+} f^{(\alpha)}(t)$.

This new definition satisfies the properties which are given in the following theorem [11].

Theorem 1.1. Let $\alpha \in (0,1]$ and f, g be $\alpha-$differentiable at point $t > 0$. Then

a) $T_{\alpha}(cf + dg) = cT_{\alpha}(f) + dT_{\alpha}(g)$, for all $a, b \in \mathbb{R}$,
b) $T_{\alpha}(t^p) = pt^{p-\alpha}$ for all $p \in \mathbb{R}$,
c) $T_{\alpha}(\lambda) = 0$ for all constant functions $f(t) = \lambda$,
d) $T_{\alpha}(fg) = fT_{\alpha}(g) + gT_{\alpha}(f)$,
e) $T_{\alpha}\left(\frac{f}{g}\right) = \frac{gT_{\alpha}(g) - fT_{\alpha}(f)}{g^2}$,
f) If, in addition to f is differentiable, then $T_{\alpha}(f)(t) = t^{1-\alpha} \frac{df}{dt}$.

2 Time Fractional Wave Equation in Rectangular Domain

In this section we investigate the solution of two dimensional wave equation with conformable fractional derivative,

$$\frac{\delta^{\alpha}}{\delta t^{\alpha}} \frac{\partial^{\alpha} u}{\partial t^{\alpha}} = \kappa^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right), \; 0 < x < a, \; 0 < y < b, t > 0 \tag{1}$$

in the rectangular domain $D = \{(x, y) : 0 < x < a,\ 0 < y < b, x, y \in \mathbb{R}\}$, which describes vibrating membrane or plate with the conditions

$$u(0, y, t) = u(a, y, t) = u(x, 0, t) = u(x, b, t) = 0 \tag{2}$$

$$\begin{aligned} u(x, y, 0) &= f(x, y);\ 0 \le x \le a,\ 0 \le y \le b \\ u_t(x, y, 0) &= g(x, y); 0 \le x \le a,\ 0 \le y \le b \end{aligned} \tag{3}$$

where $0 < \alpha \le 1$.

Let $u(x, y, t) = P(x)Q(y)R(t)$. Then taking necessary derivatives in equation (1) and dividing all sides by $\kappa^2 P(x)Q(y)R(t)$ we have,

$$\frac{1}{\kappa^2 R(t)} \frac{\delta^\alpha}{\delta t^\alpha} \frac{\partial^\alpha R}{\partial t^\alpha} = \frac{1}{P(x)} \frac{\partial^2 P}{\partial x^2} + \frac{1}{Q(y)} \frac{\partial^2 Q}{\partial y^2} = -\mu^2. \tag{4}$$

If we handle the right side of the equation we get,

$$\frac{1}{P(x)} \frac{\partial^2 P}{\partial x^2} = -\frac{1}{Q(y)} \frac{\partial^2 Q}{\partial y^2} - \mu^2. \tag{5}$$

For $0 < x < a$ the right side and for $0 < y < b$ the left side of the equation (5) does not change. So we obtain,

$$\frac{1}{P(x)} \frac{\partial^2 P}{\partial x^2} = -\frac{1}{Q(y)} \frac{\partial^2 Q}{\partial y^2} - \mu^2 = -\vartheta^2 \tag{6}$$

where ϑ is separation constant. From equations (4) and (6) we have the following equalities,

$$\begin{aligned} &\frac{\delta^\alpha}{\delta t^\alpha} \frac{\partial^\alpha R}{\partial t^\alpha} + \kappa^2 \mu^2 R(t) = 0, \\ &\frac{\partial^2 P}{\partial x^2} + \vartheta^2 P = 0, \\ &\frac{\partial^2 Q}{\partial y^2} + \iota^2 Q = 0. \end{aligned}$$

By using (f) in Theorem 1.1. [11] and Eq.(29) in [12] which states the sequential conformable fractional derivative the first equation becomes,

$$(1-\alpha)t^{1-2\alpha}R'(t)+t^{2-2\alpha}R''(t)+\kappa^2\mu^2R(t)=0.$$

The solution of above equation can be easily obtained as,

$$R(t)=A\cos\left(\frac{\kappa\mu t^{\alpha}}{\alpha}\right)+B\sin\left(\frac{\kappa\mu t^{\alpha}}{\alpha}\right)$$

and the solutions of other equalities can be obtained as,

$$\begin{aligned}P(x)&=C\cos(\vartheta x)+D\sin(\vartheta x)\\ Q(y)&=E\cos(\iota y)+F\sin(\iota y)\end{aligned}$$

and the conditions (2) force these equalities to be

$$P(x)=\sum_{n=1}^{\infty}D_n\sin\left(\frac{n\pi x}{a}\right), \tag{7}$$

$$Q(y)=\sum_{m=1}^{\infty}F_m\sin\left(\frac{m\pi y}{b}\right). \tag{8}$$

Hence from our assumption $u(x,y,t)$ can be obtained as

$$u(x,y,t)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left[\phi_{mn}\cos\left(\frac{\eta_{mn}\kappa t^{\alpha}}{\alpha}\right)+\gamma_{mn}\sin\left(\frac{\eta_{mn}\kappa t^{\alpha}}{\alpha}\right)\right]\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{b}\right) \tag{9}$$

where $\eta_{mn}=\frac{m^2\pi^2}{b^2}+\frac{n^2\pi^2}{a^2}$.

Now with the help of conditions (3), the coefficients in equation (9) can be found as,

$$\phi_{mn}=\frac{4}{ab}\int_0^a\int_0^b f(x,y)\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{b}\right)dydx, \tag{10}$$

$$\gamma_{mn}=\frac{4}{\eta_{mn}ab\kappa}\int_0^a\int_0^b g(x,y)\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{b}\right)dydx. \tag{11}$$

So the solution of equation (1) obtained as,

$$u(x,y,t)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left[\left[\frac{4}{ab}\int_0^a\int_0^b f(x,y)\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{b}\right)dydx\right]\cos\left(\frac{\eta_{mn}\kappa t^{\alpha}}{\alpha}\right)\right.$$
$$\left.+\left[\frac{4}{\eta_{mn}ab\kappa}\int_0^a\int_0^b g(x,y)\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{b}\right)dydx\right]\sin\left(\frac{\eta_{mn}\kappa t^{\alpha}}{\alpha}\right)\right]\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{b}\right).$$

3 Three Dimensional Time Fractional Wave Equation

In three dimensional space, when an object or gas makes free vibration motion in a prism, without an outer effect, this motion corresponds to three dimensional wave equation. In fractional form this equation can be expressed as

$$\frac{\delta^{\alpha}}{\delta t^{\alpha}}\frac{\partial^{\alpha}u}{\partial t^{\alpha}}=\kappa^2\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right),\ 0<x<a,\ 0<y<b, 0<z<d, t>0 \qquad (12)$$

with the conditions

$$u(0,y,z,t)=u(a,y,z,t)=u(x,0,z,t)=u(x,b,z,t)=u(x,y,0,t)=u(x,y,d,t)=0 \qquad (13)$$

$$\begin{aligned} u(x,y,z,0) &= f(x,y);\ 0\le x\le a,\ 0\le y\le b, 0\le z\le d \\ u_t(x,y,z,0) &= g(x,y); 0\le x\le a,\ 0\le y\le b, 0\le z\le d \end{aligned} \qquad (14)$$

where $0<\alpha\le 1$ and the derivative is conformable fractional derivative. Assume that $u(x,y,t)=P(x)Q(y)R(z)T(t)$. Then taking derivatives which are necessary in equation (12) and dividing all sides by $\kappa^2 P(x)Q(y)R(z)T(t)$ we have,

$$\frac{1}{\kappa^2 T(t)}\frac{\delta^{\alpha}}{\delta t^{\alpha}}\frac{\partial^{\alpha}T}{\partial t^{\alpha}}=\frac{1}{P(x)}\frac{\partial^2 P}{\partial x^2}+\frac{1}{Q(y)}\frac{\partial^2 Q}{\partial y^2}+\frac{1}{R(z)}\frac{\partial^2 R}{\partial z^2}=-\mu^2. \qquad (15)$$

If we repeat the same procedure which is given in Section 2, we obtain the following equations,

$$\frac{\delta^{\alpha}}{\delta t^{\alpha}}\frac{\partial^{\alpha} T}{\partial t^{\alpha}}+\kappa^2\mu^2 T(t)=0,$$
$$\frac{\partial^2 P}{\partial x^2}+\vartheta^2 P=0,$$
$$\frac{\partial^2 Q}{\partial y^2}+\iota^2 Q=0,$$
$$\frac{\partial^2 R}{\partial z^2}+\rho^2 R=0.$$

By using (f) in Theorem 1.1. [11] and Eq.(29) in [12] which states the sequential conformable fractional derivative the first equation becomes,

$$(1-\alpha)t^{1-2\alpha}T'(t)+t^{2-2\alpha}T''(t)+\kappa^2\mu^2 T(t)=0$$

The solution of above equation can be easily obtained as,

$$T(t)=A\cos\left(\frac{\kappa\mu t^{\alpha}}{\alpha}\right)+B\sin\left(\frac{\kappa\mu t^{\alpha}}{\alpha}\right). \tag{16}$$

Additionally solutions of other equations can be obtained as,

$$P(x)=C\cos(\vartheta x)+D\sin(\vartheta x)$$
$$Q(y)=E\cos(\iota y)+F\sin(\iota y)$$

and the conditions (13) force these equalities to be,

$$P(x)=\sum_{n=1}^{\infty}D_n\sin\left(\frac{n\pi x}{a}\right), \tag{17}$$
$$Q(y)=\sum_{m=1}^{\infty}F_m\sin\left(\frac{m\pi y}{b}\right), \tag{18}$$
$$R(z)=\sum_{r=1}^{\infty}K_r\sin\left(\frac{r\pi z}{d}\right). \tag{19}$$

Thus, from our assumption, we have $u(x,y,z,t)$ as,

$$\begin{aligned}u(x,y,t)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\sum_{r=1}^{\infty}\ &\phi_{mnr}\cos\left(\frac{\eta_{mnr}\kappa t^{\alpha}}{\alpha}\right)\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{b}\right)\sin\left(\frac{r\pi z}{d}\right)\\ &+\gamma_{mnr}\sin\left(\frac{\eta_{mnr}\kappa t^{\alpha}}{\alpha}\right)\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{b}\right)\sin\left(\frac{r\pi z}{d}\right)\end{aligned} \tag{20}$$

By using conditions (14) we have the coefficients ϕ_{mnr} and γ_{mnr} as,

$$\phi_{mn} = \frac{8}{abd}\int_0^a\int_0^b\int_0^d f(x,y,z)\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{b}\right)\sin\left(\frac{r\pi z}{d}\right)dzdydx, \qquad (21)$$

$$\gamma_{mn} = \frac{8}{\eta_{mnr}abd\kappa}\int_0^a\int_0^b\int_0^d g(x,y,z)\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{b}\right)\sin\left(\frac{r\pi z}{d}\right)dzdydx. \qquad (22)$$

4 Conclusions

In this paper we discuss about the solution of two and three dimensional time fractional wave equation. With the help of conformable fractional derivative definition we can easily transform fractional differential equations to the known classical differential equations. By this solution procedure there is no need any other complex methods or complex definitions to get the exact solution of the problem. From the solution procedure and the results it is easily seen that this definition is convenient and applicable for the solution of higher dimensional partial fractional differential equations.

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ON HYPERSPACES OF SOFT SETS

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Abstract – In this paper first, we introduce the soft sets families$(V,K)^{+}$, $(V,K)^{-}$ and we investigate basic properties of them. Second, by use these soft families, we introduce some hyperspaces of soft sets, called upper (lower) and Vietoris soft topological space, which defined on classes of soft sets $2^{S(Y,K)}$. Third, we define the upper and lower Vietoris continuity of soft multifunction and we give the relationship between Vietoris continuity of soft multifunction and continuity of soft mapping.

Keywords – *Soft hyperspaces, soft Vietoris topological spaces, soft sets, soft multifunction, soft continuity.*

1. Introduction

There are many complex problems in the several fields of sciences that involve uncertainties in data. Several set theories can be regarded as mathematical tools for dealing with these uncertainties, but these theories sometimes fail to handle uncertainty properly. This limitation was pointed by Molodtsov [7]. He introduced the concept of soft set theory. Çağman et al [4] defined a soft topological space. Shabir and Naz [10] introduced the notions of soft topological spaces. Then, Zorlutuna and et al. [11] studied the properties of soft topological spaces. After that, Kharal and Ahmad [5] defined a mapping on soft classes and studied properties of these mappings. Then Akdağ and Erol [1,2] introduced the concept of soft multifunction and studied their properties. In this paper we define and study the hyperspaces of soft sets.

2. Preliminaries

Definition 2.1. [7] Let X be an initial universe and E be a set of parameters. Let $P(X)$ denote the power set of X and A be a non-empty subset of E. A pair (F,A) is called a soft set over X, where F is a mapping given by $F: A \rightarrow P(X)$. In other words, a soft set over X is

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a parameterized family of subsets of the universe X. For $\varepsilon \in A$, $F(\varepsilon)$ may be considered as the set of ε-approximate elements of the soft set (F, A).

Definition 2.2. [6] A soft set (F, A) over X is called a null soft set, denoted by Φ_A, if $F(e) = \emptyset$ for all $e \in A$. If $A = E$, then the null soft set denoted by Φ.

Definition 2.3. [6] A soft set (F, A) over X is called an absolute soft set, denoted by $\widetilde{X_A}$, if $F(e) = X$ for all $e \in A$. If $A = E$, then the A-universal soft set is called a universal soft set, denoted by $\tilde{X}$.

Definition 2.4. [6] The union of two soft sets of (F, A) and (G, B) over the common universe X is the soft set (H, C), where $C = A \cup B$ and for all $e \in C$,

$$H(e) = \begin{cases} \mathrm{F(e)}, & \text{if } \mathrm{e} \in \mathrm{A} - \mathrm{B}, \\ \mathrm{G(e)}, & \text{if } \mathrm{e} \in \mathrm{B} - \mathrm{A}, \\ \mathrm{F(e)} \cup \mathrm{G(e)}, & \text{if } \mathrm{e} \in \mathrm{A} \cap \mathrm{B}. \end{cases}$$

We write $(F, A)\ \tilde{\cup}\ (G, B) = (H, C)$.

Definition 2.5. [6] Let (F, A) and (G, B) be two soft sets over a common universe X. The soft intersection (F, A) and (G, B) is also a soft set $(H, C) = (F, A)\ \tilde{\cap}\ (G, B)$ and defined as $H(e) = F(e) \cap G(e)$ for all $e \in C$, where $C = A \cap B$.

Definition 2.6. [6] Let (F, A) and (G, B) be two soft sets over a common universe X. (F, A) is soft subset of (G, B), if $A \subset B$ and $F(e) \subset G(e)$ for all $e \in A$. Then we write $(F, A)\ \tilde{\subseteq}\ (G, B)$.

Definition 2.7. [3] For a soft set (F, E) over X the relative complement of (F, E) is denoted by $(F, E)^c$ and is defined by $(F, E)^c = (F^c, E)$, where $F^c: E \to P(X)$ is a mapping given by $F^c(e) = X - F(e)$ for all $e \in E$.

Definition 2.8. [10] Let τ be the collection of soft sets over X, then τ is said to be a soft topology on X if satisfies the following axioms.

(1) $\Phi, \tilde{X}$ belong to τ,
(2) the union of any number of soft sets in τ belongs to τ,
(3) the intersection of any two soft sets in τ belongs to τ.

The triplet (X, τ, E) is called a soft topological space over X and the members of τ are said soft open sets in X. A soft set (F, E) over X is said soft closed set in X, if its relative complement $(F, E)^c$ belongs to τ.

Definition 2.9. [8] Let (X, τ, E) be a soft topological space. A sub-collection β of τ is said to be a soft open base of τ if every member of τ can be expressed as the union of some members of β.

Example 2.10. [8] Let $X = \{x_1, x_2, x_3, x_4, x_5\}$, $E = \{e_1, e_2\}$ and
$\tau = \{\ \Phi, \tilde{X},\ \{(e_1, \{x_1\}), (e_2, \{x_2\})\},\ \{(e_1, \{x_4\}), (e_2, \{x_3\})\},\ \{(e_1, \{x_2, x_3\}), (e_2, \{x_1, x_4\})\},$
$\{(e_1, \{x_1, x_4\}), (e_2, \{x_2, x_3\})\},\ \{(e_1, \{x_1, x_2, x_3\}), (e_2, \{x_1, x_2, x_4\})\},$

$\{(e_1, \{x_2, x_3, x_4\}), (e_2, \{x_1, x_3, x_4\})\}$, $\{(e_1, \{x_1, x_2, x_3, x_4\}), (e_2, \{x_1, x_2, x_3, x_4\})\}\}$. Then τ is a soft topology over X.

Let $\beta = \{\Phi, \tilde{X}, \{(e_1, \{x_1\}), (e_2, \{x_2\})\}, \{(e_1, \{x_4\}), (e_2, \{x_3\})\}, \{(e_1, \{x_2, x_3\}), (e_2, \{x_1, x_4\})\}\}$.

Then β forms an soft open base of τ.

Proposition 2.11. [8] A collection β of soft open sets of a soft topological space (X, τ, E) forms an open base of τ iff $\forall (F, E) \in \tau$ and $\forall E_e{}^x \tilde{\in} (F, E)$, $\exists (G, E) \in \beta$ such that $E_e{}^x \tilde{\in} (G, E) \tilde{\subset} (F, E)$.

Proposition 2.12. [8] A collection β of soft subsets over X forms an open base of a soft topology over X iff the following conditions are satisfied.

(i) $\Phi \in \beta$
(ii) $\tilde{X}$ is union of the members of β
(iii) If $(F, E), (G, E) \in \beta$ then $(F, E) \tilde{\cap} (G, E)$ is union of some members of β, i.e. $(F, E), (G, E) \in \beta$ and $E_e{}^x \tilde{\in} (F, E) \tilde{\cap} (G, E)$ then $\exists (H, E) \in \beta$ such that $E_e{}^x \tilde{\in} (H, E) \tilde{\subset} (F, E) \tilde{\cap} (G, E)$.

Definition 2.13. [9] A collection β of some soft subsets of (F, A) is called a soft open base or simply a base for some soft topology on (F, A) if the following conditions hold:

(i) $\Phi \in \beta$
(ii) $\cup \beta = (F, A)$ i.e., for each $e \in A$ and $x \in (F, A)(e)$, there exists $(G, B) \in \beta$ such that $x \in (G, B)(e)$, where $B \subseteq A$.
(iii) If $(G, B), (H, C) \in \beta$ then for each $e \in B \cap C$ and $x \in ((G, B) \cap (H, C))(e) = (G, B)(e) \cap (H, C)(e)$, there exists $(I, D) \in \beta$ such that $(I, D) \tilde{\subset} (G, B) \tilde{\cap} (H, C)$ and $x \in (I, D)(e)$, where $D \subseteq B \cap C$.

Example 2.14. [9] Let $X = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}\}$, $E = \{e_1, e_2, e_3, e_4, e_5\}$, $A = \{e_1, e_2, e_3, e_4\}$ and $(F, A) = \{(e_1, \{x_1, x_5, x_8\}), (e_2, \{x_2, x_6, x_9\}), (e_3, \{x_3, x_7, x_9\}), (e_4, \{x_4, x_7, x_{10}\})\}$ be a soft set.

Now let us consider the collection

$\tau = \{\Phi, (F, A), \{(e_2, \{x_2\})\}, \{(e_4, \{x_4\})\}, \{(e_1, \{x_1\}), (e_3, \{x_3\})\}, \{(e_2, \{x_2\}), (e_4, \{x_4\})\},$
$\{(e_2, \{x_2, x_9\}), (e_4, \{x_4, x_7\})\}, \{(e_1, \{x_1\}), (e_2, \{x_2\}), (e_3, \{x_3\})\},$
$\{(e_1, \{x_1\}), (e_3, \{x_3\}), (e_4, \{x_4\})\}, \{(e_1, \{x_1, x_5\}), (e_2, \{x_2, x_6\}), (e_3, \{x_3, x_7\})\},$
$\{(e_1, \{x_1, x_8\}), (e_3, \{x_3, x_9\}), (e_4, \{x_4, x_{10}\})\}, \{(e_1, \{x_1\}), (e_2, \{x_2\}), (e_3, \{x_3\}), (e_4, \{x_4\})\},$
$\{(e_1, \{x_1\}), (e_2, \{x_2, x_9\}), (e_3, \{x_3\}), (e_4, \{x_4, x_7\})\},$
$\{(e_1, \{x_1, x_8\}), (e_2, \{x_2\}), (e_3, \{x_3, x_9\}), (e_4, \{x_4, x_{10}\})\},$
$\{(e_1, \{x_1, x_5\}), (e_2, \{x_2, x_6\}), (e_3, \{x_3, x_7\}), (e_4, \{x_4\})\},$
$\{(e_1, \{x_1, x_8\}), (e_2, \{x_2, x_9\}), (e_3, \{x_3, x_9\}), (e_4, \{x_4, x_7, x_{10}\})\},$
$\{(e_1, \{x_1, x_5\}), (e_2, \{x_2, x_6, x_9\}), (e_3, \{x_3, x_7\}), (e_4, \{x_4, x_7\})\},$
$\{(e_1, \{x_1, x_5, x_8\}), (e_2, \{x_2, x_6\}), (e_3, \{x_3, x_7, x_9\}), (e_4, \{x_4, x_{10}\})\}\}$

of some soft subsets of (F, A). Then obviously, τ forms a soft topology on a soft set (F, A).

If we take

$\beta = \{\Phi, \{(e_2, \{x_2\})\}, \{(e_4, \{x_4\})\}, \{(e_1, \{x_1\}), (e_3, \{x_3\})\}, \{(e_2, \{x_2, x_9\}), (e_4, \{x_4, x_7\})\},$
$\{(e_1, \{x_1, x_5\}), (e_2, \{x_2, x_6\}), (e_3, \{x_3, x_7\})\}, \{(e_1, \{x_1, x_8\}), (e_3, \{x_3, x_9\}), (e_4, \{x_4, x_{10}\})\}\}.$

Then obviously, β forms a soft base for the topology τ on (F, A).

Theorem 2.15. [9] Let β be a soft base for a soft topology on (F, A). Suppose τ_β consists of those soft subset (G, B) of (F, A) for which corresponding to each $e \in B$ and $x \in (G, B)(e)$, there exists $(H, C) \in \beta$ such that $(H, C) \tilde{\subseteq} (G, B)$ and $x \in (H, C)(e)$, where $C \subseteq B$. Then $\tau_\beta$is a soft topology on (F, A).

Definition 2.16. [9] Suppose β is a soft base for a soft topology on (F, A). Then τ_β, described in above theorem, is called the soft topology generated by β and β is called the soft base for τ_β.

Theorem 2.17. [9] Let β be a soft base for a soft topology on (F, A). Then $(G, B) \in \tau_\beta$ if and only if $(G, B) = \bigcup(G_i, B)$, where $(G_i, B) \in \beta$ for each $i \in \Lambda$ and Λ an index set.

Theorem 2.18. [9] Let $((F, A), \tau)$ be a soft topological space and β be a sub collection of τ such that every member of τ is a union of some members of β. Then β is a soft base for the soft topology τ on (F, A).

Definition 2.19. [9] A collection Ω the members of a soft topology τ is said to be subbase for τ if and only if the collection of all finite intersections of members of Ω is a base for τ.

Theorem 2.20. [9] A collection Ω of soft subsets of (F, A) is a subbase for a suitable soft topology τ on (F, A) if and only if
(i) $\Phi \in \Omega$ or Φ is the intersection of a finite number of members of Ω
(ii) $(F, A) = \bigcup \Omega$.

3. Hyperspaces of Soft Sets

Definition 3.1. Let (G, K) be a soft open set in a soft topological space (Y, τ, K) and $S(Y, K)$ is the family of soft set on X. Then the soft set families $(G, K)^+$ and $(G, K)^-$ are defined as follows:

$(G, K)^+ = \{(T, K) \in S(Y, K): (T, K) \tilde{\subseteq} (G, K)\},$
$(G, K)^- = \{(T, K) \in S(Y, k): (T, K) \tilde{\cap} (G, K) \neq \Phi\}.$

Proposition 3.2. Let (Y, τ, K) be a soft topological space. For a non null soft sets (G, K) and (H, K) the following statements are true;

(a) $(G, K)^+ \cap (H, K)^+ = ((G, K) \tilde{\cap} (H, K))^+$
(b) $(G, K)^+ \cup (H, K)^+ \subset ((G, K) \tilde{\cup} (H, K))^+$
(c) $((G, K) \tilde{\cap} (H, K))^- \subset (G, K)^- \cap (H, K)^-$
(d) $(G, K)^- \cup (H, K)^- = ((G, K) \tilde{\cup} (H, K))^-$
(e) $(G, K) \tilde{\subseteq} (H, K)$ if and only if $(G, K)^+ \subset (H, K)^+$

(f) $(G,K) \tilde{\subseteq} (H,K)$ if and only if $(G,K)^{-} \subset (H,K)^{-}$
(g) $(G,K) \tilde{\cap} (H,K) = \Phi$ if and only if $(G,K)^{+} \cap (H,k)^{+} = \Phi$
(h) $(G,K)^{-} \cap (H,K)^{-} \neq \Phi$.

Proof. (a) Let $(T,K) \in (G,K)^{+} \cap (H,K)^{+}$. *Then* $(T,K) \in (G,K)^{+}$ *and* $(T,K) \in (H,K)^{+}$. *Thus* $(T,K) \tilde{\subseteq} (G,K)$ *and* $(T,K) \tilde{\subseteq} (H,K)$. *Hence* $(T,K) \tilde{\subseteq} (G,K) \tilde{\cap} (H,K)$ *and thus* $(T,K) \in ((G,K) \cap (H,K))^{+}$.

Conversely, let $(T,K) \in ((G,K) \cap (H,K))^{+}$. *Then* $(T,K) \tilde{\subseteq} (G,K) \tilde{\cap} (H,K)$. *Thus* $(T,K) \tilde{\subseteq} (G,K)$ *and* $(T,K) \tilde{\subseteq} (H,K)$. *Hence* $(T,K) \in (G,K)^{+}$ *and* $(T,K) \in (H,K)^{+}$. *Therefore* $(T,K) \in (G,K)^{+} \cap (H,K)^{+}$.

(b) Let $(T,K) \in (G,K)^{+} \cup (H,K)^{+}$, *then* $(T,K) \in (G,K)^{+}$ *and* $(T,K) \in (H,K)^{+}$. *Thus* $(T,K) \tilde{\subseteq} (G,K)$ *and* $(T,K) \tilde{\subseteq} (H,K)$. *Hence* $(T,K) \tilde{\subseteq} (G,K) \tilde{\cup} (H,K)$ *and* $(T,K) \in ((G,K) \tilde{\cup} (H,K))^{+}$.

(c) Let $(T,K) \in ((G,K) \tilde{\cap} (H,K))^{-}$, *then* $(T,K) \tilde{\cap} ((G,K) \tilde{\cap} (H,K)) \neq \Phi$. *Thus* $(T,K) \tilde{\cap} (G,K) \neq \Phi$ *and* $(T,K) \tilde{\cap} (H,K) \neq \Phi$. *Then* $(T,K) \in (G,K)^{-}$ *and* $(T,K) \in (H,K)^{-}$. *Therefore* $(T,K) \in (G,K)^{-} \cap (H,K)^{-}$.

(d) Let $(T,K) \in ((G,K) \tilde{\cup} (H,K))^{-}$, *then we have* $(T,K) \tilde{\cap} ((G,K) \tilde{\cup} (H,K)) \neq \Phi$. *Thus* $((T,K) \tilde{\cap} (G,K)) \tilde{\cup} ((T,K) \tilde{\cap} (H,K)) \neq \Phi$. *Hence* $(T,K) \tilde{\cap} (G,k) \neq \Phi$ *or* $(T,K) \tilde{\cap} (H,K) \neq \Phi$. *Then* $(T,K) \in (G,K)^{-}$ *or* $(T,K) \in (H,K)^{-}$. *Thus we have* $(T,K) \in (G,K)^{-} \cup (H,K)^{-}$.

Conversely, let $(T,K) \in (G,K)^{-} \cup (H,K)^{-}$. *Then* $(T,K) \in (G,K)^{-}$ *or* $(T,K) \in (H,K)^{-}$ *and thus we have* $(T,K) \tilde{\cap} (G,K) \neq \Phi$ *or* $(T,K) \tilde{\cap} (H,K) \neq \Phi$. *Hence* $((T,K) \tilde{\cap} (G,K)) \tilde{\cup} ((T,K) \tilde{\cap} (H,K)) \neq \Phi$ *and* $(T,K) \tilde{\cap} ((G,K) \tilde{\cup} (H,K)) \neq \Phi$. *Thus* $(T,K) \in ((G,K) \tilde{\cup} (H,K))^{-}$.

(e) Let ${E_e}^{x} \tilde{\in} (G,K)$. *Then we have* ${E_e}^{x} \in (G,K)^{+}$. *Since* $(G,K)^{+} \subset (H,K)^{+}$, *then we have* ${E_e}^{x} \in (H,K)^{+}$ *and thus* ${E_e}^{x} \tilde{\in} (H,K)$. *Therefore,* $(G,K) \tilde{\subseteq} (H,K)$.

Conversely, let $(T,K) \in (G,K)^{+}$. *Then* $(T,K) \tilde{\subseteq} (G,K)$. *Since* $(G,K) \tilde{\subseteq} (H,K)$ *then we have* $(T,K) \tilde{\subseteq} (H,K)$ *and thus* $(T,K) \in (H,K)^{+}$.

(f) Let $(T,K) \in (G,K)^{-}$. *Then* $(T,K) \tilde{\cap} (G,K) \neq \Phi$. *Since* $(G,K) \tilde{\subseteq} (H,K)$, *then* $(T,K) \tilde{\cap} (H,K) \neq \Phi$. *Thus* $(T,K) \in (H,K)^{-}$. *Therefore,* $(G,K)^{-} \subset (H,K)^{-}$.

Conversely, ${E_e}^{x} \tilde{\in} (G,K)$. *Then* ${E_e}^{x} \in (G,K)^{-}$. *Since* $(G,K)^{-} \subset (H,K)^{-}$, *then* ${E_e}^{x} \in (H,K)^{-}$ *and thus* ${E_e}^{x} \tilde{\cap} (H,K) \neq \Phi$. *Therefore,* ${E_e}^{x} \tilde{\in} (H,K)$ *and* $(G,K) \tilde{\subseteq} (H,K)$.

(g) Let $(G,K) \tilde{\cap} (H,K) = \Phi$, then $\left((G,K) \tilde{\cap} (H,K)\right)^{+} = \Phi$. Since $(G,K)^{+} \cap (H,K)^{+} = \left((G,K) \tilde{\cap} (H,K)\right)^{+}$ thus we have $(G,K)^{+} \cap (H,k)^{+} = \Phi$.

Conversely, let $(G,K)^{+} \cap (H,K)^{+} = \Phi$. Then $\left((G,K) \tilde{\cap} (H,K)\right)^{+} = \Phi$ and thus $(G,K) \tilde{\cap} (H,K) = \Phi$.

(h) Since $\tilde{X} \tilde{\cap} (G,K) = \Phi$ *and* $\tilde{X} \tilde{\cap} (H,K) = \Phi$ *then* $\tilde{X} \in (G,K)^-$ *and* $\tilde{X} \in (H,K)^-$. *Thus we have* $\tilde{X} \in (G,K)^- \cap (H,K)^-$. Therefore $(G,E)^- \cap (H,E)^- \neq \Phi$.

Theorem 3.3. Let (Y,τ,K) be a soft topological space. Then the soft set families

$\beta_{SV^+} = \{(G,K)^+ : (G,K)\ soft\ open\ set\}$,
$\mathcal{S}_{SV^-} = \{(G,K)^- : (G,K)\ soft\ open\ set\}$

are soft base and soft sub base for a different soft topological spaces on $2^{S(Y,K)}$, respectively.

Proof. For $\tilde{Y} \in \tau$, $\tilde{Y}^+ = 2^{S(Y,K)} \subset \beta_{SV^+}$ and $2^{S(Y,K)} = \bigcup_{(B,K)^+ \in \beta_{SV^+}} (B,K)^+$.

Also let $(G_1,K)^+, (G_2,K)^+ \in \beta_{SV^+}$ and $(H,K) \in (G_1,K)^+ \cap (G_2,K)^+$. Since (G_1,K) and (G_2,K) are soft open sets, then $(G_3,K) = (G_1,K) \tilde{\cap} (G_2,K)$ is soft open set. Since $(G_1,K)^+ \cap (G_2,K)^+ = \big((G_1,K) \tilde{\cap} (G_2,K)\big)^+ = (G_3,K)^+$ then we have $(H,K) \in (G_3,K)^+$, $(G_3,K)^+ \in \beta_{SV^+}$ and $(G_3,K)^+ \subset (G_1,K)^+ \cap (G_2,K)^+$. Thus β_{SV^+} is soft base for a soft topology.

Definition 3.4. Let (Y,τ,K) be a soft topological space and $2^{S(Y,K)}$ be family of all non null soft sets over Y.

i) The soft topological space, which accepts β_{SV^+} a base, defined on $2^{S(Y,K)}$ is called soft upper Vietoris and denoted by τ_{SV^+}.

ii) The soft topological space, which accepts $\mathcal{S}_{SV^-}$ a subbase, defined on $2^{S(Y,K)}$ is called soft lower Vietoris and denoted by τ_{SV^-}.

iii) The soft Vietoris topological space is denoted by τ_{SV} and defined as $\tau_{SV} = \tau_{SV^+} \cup \tau_{SV^-}$. Let $(U,K),(V_1,K),(V_2,K),\dots,(V_n,K)$ be soft open sets. Then a element of soft base for soft Vietoris topological space is denoted by $B\big((U,K),(V_1,K),(V_2,K),\dots,(V_n,K)\big) = \big\{(T,K) \in 2^{S(Y,K)} : (T,K) \tilde{\subseteq} (U,K), (T,K) \tilde{\cap} (V_i,K) \neq \Phi,\ i = 1,2,\dots,n\big\}$.

Example 3.5. Let (Y,τ,K) be a soft topological space with $Y = \{y_1,y_2\}$, $K = \{k_1,k_2\}$ and $\tau = \{\Phi, \tilde{Y}, (F,K), (G,K), (H,K)\}$. Where, $(F,K) = \{(k_1,\{y_1\})\}$, $(G,K) = \{(k_2,\{y_2\})\}$ and $(H,K) = \{(k_1,\{y_1\}),(k_2,\{y_2\})\}$.

Then

$(F,K)^+ = \{\{(k_1,\{y_1\})\}\}$
$(G,K)^+ = \{\{(k_2,\{y_2\})\}\}$
$(H,K)^+ = \{\{(k_1,\{y_1\})\},\ \{(k_2,\{y_2\})\},\ \{(k_1,\{y_1\}),(k_2,\{y_2\})\}\}$
$\tilde{Y}^+ = 2^{S(Y,K)} = S(Y,K) - \{\phi\}$.

Thus

$\beta_{SV^+} = \{(F,K)^+,\ (G,K)^+, (H,K)^+, \tilde{Y}^+\}$

is a base for a soft topological space on $2^{S(Y,K)}$. Because,

$\tilde{Y}^{+} = 2^{S(Y,K)}$ and
$(F,K)^{+} \cap (G,K)^{+} = \emptyset$,
$(F,K)^{+} \cap (H,K)^{+} = (F,K)^{+} \in \beta_{SV^{+}}$,
$(F,K)^{+} \cap \tilde{Y}^{+} = (F,K)^{+} \in \beta_{SV^{+}}$,
$(G,K)^{+} \cap (H,K)^{+} = (G,K)^{+} \in \beta_{SV^{+}}$,
$(G,K)^{+} \cap \tilde{Y}^{+} = (G,K)^{+} \in \beta_{SV^{+}}$,
$(H,K)^{+} \cap \tilde{Y}^{+} = (H,K)^{+} \in \beta_{SV^{+}}$.

This topology (called upper soft Vietoris topology) is

$$\tau_{SV^{+}} = \left\{(F,K)^{+},\ (G,K)^{+}, (H,K)^{+}, (F,K)^{+} \cup (G,K)^{+}, \tilde{Y}^{+}, \{\phi\}\right\}.$$

Example 3.6. Let (Y, τ, K) be a soft topological space in Example 3.5. Then,
$(F,K)^{-} = \{\{(k_1,\{y_1\})\},\ \{(k_1,Y)\},\ \{(k_1,\{y_1\}),(k_2,\{y_1\})\},\ \{(k_1,\{y_1\}),(k_2,\{y_2\})\},$
$\{(k_1,\{y_1\}),(k_2,Y)\},\ \{(k_1,Y),(k_2,\{y_1\})\},\ \{(k_1,Y),(k_2,\{y_2\})\},\tilde{Y}\}$

$(G,K)^{-} = \{\{(k_2,\{y_2\})\},\ \{(k_2,Y)\},\ \{(k_1,\{y_1\}),(k_2,\{y_2\})\},\ \{(k_1,\{y_1\}),(k_2,Y\},$
$\{(k_1,\{y_2\}),(k_2,\{y_2\})\},\ \{(k_1,\{y_2\}),(k_2,Y\},\ \{(k_1,Y),(k_2,\{y_2\})\},\ \tilde{Y}\}$

$(H,K)^{-} = \{\{(k_1,\{y_1\})\},\ \{(k_2,\{y_2\})\},\ \{(k_1,Y)\},\ \{(k_2,Y)\},\ \{(k_1,\{y_1\}),(k_2,\{y_1\})\},$
$\{(k_1,\{y_1\}),(k_2,\{y_2\})\},\ \{(k_1,\{y_1\}),(k_2,Y)\},\ \{(k_1,\{y_2\}),(k_2,\{y_2\})\},\ (k_1,\{y_2\}),(k_2,Y)\},$
$\{(k_1,Y),(k_2,\{y_1\})\},\ \{(k_1,Y),(k_2,\{y_2\})\},\ \tilde{Y}\}$

$\tilde{Y}^{-} = 2^{S(Y,K)} = S(Y,K) - \{\phi\}$.

Thus the family $\mathcal{S}_{SV^{-}} = \{(F,K)^{-},\ (G,K)^{-},(H,K)^{-},\tilde{Y}^{-}\}$ is a sub base for a soft topological space on $2^{S(Y,K)}$.

Because,

$(D,K)^{-} = \{\{(k_1,\{y_1\}),(k_2,\{y_2\})\},\ \{(k_1,\{y_1\}),(k_2,Y)\},\ \{(k_1,Y),(k_2,\{y_2\})\},\ \tilde{Y}\}$

the family

$\beta_{SV^{-}} = \{(F,K)^{-},\ (G,K)^{-},(H,K)^{-},(D,K)^{-},\tilde{Y}^{-}\}$

is a base for a soft topological space on $2^{S(Y,K)}$. This topology (called lower soft Vietoris topology) is

$$\tau_{SV^{-}} = \left\{(F,K)^{-},\ (G,K)^{-},\ (H,K)^{-},\ (D,K)^{-},\ (F,K)^{-} \cup (G,K)^{-},\ \tilde{Y}^{-},\{\phi\}\right\}.$$

Example 3.7. Let (Y, τ, K) be a soft topological space in Example 3.5. Then, the soft vietoris topology is

$\tau_{SV} = \Big\{(F,K)^{+},\ (G,K)^{+},(H,K)^{+},(F,K)^{+} \cup (G,K)^{+},\tilde{Y}^{+},(F,K)^{-},\ (G,K)^{-},\ (H,K)^{-},$
$(D,K)^{-},\ (F,K)^{-} \cup (G,K)^{-},\ \tilde{Y}^{-},\{\phi\}\Big\}.$

4. Vietoris Soft Continuous Multifunction

Definition 4.1. [2] Let $S(X,E)$ and $S(Y,K)$ be two classes of soft sets. Let $u: X \to Y$ be multifunction and $p: E \to K$ be mapping. Then a soft multifunction $F: S(X,E) \to S(Y,K)$ is defined as follows: for $k \in K$,

$$F(G)(k) = \begin{cases} \bigcup\limits_{e \in p^{-1}(k) \cap E} u\big(G(e)\big), & p^{-1}(k) \cap E \neq \emptyset \\ \emptyset, & \text{otherwise} \end{cases}$$

For a soft set (G,E) in $S(X,E)$, $(F(G,E),K)$ is a soft set in $S(Y,K)$ and $(F(G,E),K)$ is called a soft image of a soft set (G,E). Moreover, $F(G,E) = \tilde{\cup}\,\{F(E_e{}^{x}): E_e{}^{x}\ \tilde{\in}\ (G,E)\}$ for a soft subset (G,E) of X.

Definition 4.2. [2] Let $F: S(X,E) \to S(Y,K)$ be a soft multifunction. The soft upper inverse image of (H,K) denoted by $F^{+}(H,K)$ and the soft lower inverse image of (H,K) denoted by $F^{-}(H,K)$ defined as follows, respectively;

$F^{+}(H,K) = \{E_e{}^{x}\ \tilde{\in}\ \tilde{X}: F(E_e{}^{x})\ \tilde{\subset}\ (H,K)\}$
$F^{-}(H,K) = \{E_e{}^{x}\ \tilde{\in}\ \tilde{X}: F(E_e{}^{x})\ \tilde{\cap}\ (H,K) \neq \Phi\}.$

Also, $F(\tilde{X}) = \bigcup F(E_e{}^{x})$.

Definition 4.3. Let $F,G: X \to Y$ be two soft multifunctions. For $E_e{}^{x}\ \tilde{\in}\ \tilde{X}$, the union and intersection of F and G is denoted by

$(F \cup G)(E_e{}^{x}) = F(E_e{}^{x}) \cup G(E_e{}^{x}),$
$(F \cap G)(E_e{}^{x}) = F(E_e{}^{x}) \cap G(E_e{}^{x}).$

Definition 4.4. [2] Let $F: S(X,E) \to S(Y,K)$ and $G: S(X,E) \to S(Y,K)$ be two soft multifunctions. Then, F equal to G if $F(E_e{}^{x}) = G(E_e{}^{x})$, for each $E_e{}^{x} \in X$.

Definition 4.5. [2] The soft multifunction $F: S(X,E) \to S(Y,K)$ is called surjective if p and u are surjective.

Theorem 4.6. [2] Let $F: S(X,E) \to S(Y,K)$ be a soft multifunction. Then, for soft sets (F,E), (G,E) and for a family of soft sets $(G_i,E)_{i\in I}$ in the soft class $S(X,E)$ the following statements are hold:

(a) $F(\Phi) = \Phi$
(b) $F(\tilde{X})\ \tilde{\subset}\ \tilde{Y}$
(c) $F((G,A)\ \tilde{\cup}\ (H,B)) = F(G,A)\ \tilde{\cup}\ F(H,B)$ in general $F(\tilde{\cup}_i\ (G_i,E)) = \tilde{\cup}_i\ F(G_i,E)$
(d) $F((G,A)\ \tilde{\cap}\ (H,B))\ \tilde{\subset}\ F(G,A)\ \tilde{\cap}\ F(H,B)$ in general $F(\cap_i\ (G_i,E))\ \tilde{\subset} \tilde{\cap}_i\ F(G_i,E)$

(e) If $(G,E)\tilde{\subseteq}(H,E)$, then $F(G,E)\tilde{\subseteq}F(H,E)$.

Theorem 4.7. [2] Let $F: S(X,E)\to S(Y,K)$ be a soft multifunction. Then the following statements are true:

(a) $F^{-}(\Phi)=\Phi$ and $F^{+}(\Phi)=\Phi$
(b) $F^{-}(\tilde{Y})=\tilde{X}$ and $F^{+}(\tilde{Y})=\tilde{X}$
(c) $F^{-}((G,K)\tilde{\cup}(H,K))= F^{-}(G,K)\tilde{\cup}F^{-}(H,K)$
(d) $F^{+}(G,K)\tilde{\cup}F^{+}(H,K)\tilde{\subseteq}F^{+}((G,K)\tilde{\cup}(H,K))$
(e) $F^{-}((G,K)\tilde{\cap}(H,K))\tilde{\subseteq} F^{-}(G,K)\tilde{\cap}F^{-}(H,K)$
(f) $F^{+}(G,K)\tilde{\cap}F^{+}(H,K)=F^{+}((G,K)\tilde{\cap}(H,K))$
(g) If $(G,K)\tilde{\subseteq}(H,K)$, then $F^{-}(G,K)\tilde{\subseteq}F^{-}(H,K)$ and $F^{+}(G,K)\tilde{\subseteq}F^{+}(H,K)$.

Definition 4.8 Let $F: S(X,E)\to S(Y,K)$ and $G: S(Y,K)\to S(Z,L)$ be two soft multifunction. The combination of F ad G denoted by $GoF: S(X,E)\to S(Z,L)$ is a soft multifunction and defined as $(GoF)(E_e{}^x)=G\big(F(E_e{}^x)\big)$.

Proposition 4.9. [2] Let $F: S(X,E)\to S(Y,K)$ be a soft multifunction. Then the following statements are true:

(a) $(G,A)\tilde{\subseteq}F^{+}(F(G,A))\tilde{\subseteq}F^{-}(F(G,A))$ for a soft subset (G,A) in X. If F is surjectice then $(G,A)=F^{+}(F(G,A))=F^{-}(F(G,A))$
(b) $F(F^{+}(H,B))\tilde{\subseteq}(H,B)\tilde{\subseteq}F(F^{-}(H,B))$ for a soft subset (H,B) in Y.
(c) For two soft subsets (H,B) and (U,C) in Y such that $(H,B)\tilde{\cap}(U,C)=\Phi$ then $F^{+}(H,B)\tilde{\cap}F^{-}(U,C)=\Phi$.

Proposition 4.10. [2] Let $F:(X,\tau,E)\to (Y,\sigma,K)$ and $G:(Y,\sigma,K)\to (Z,\eta,L)$ be two soft multifunction. Then the follows are true:

(a) $(F^{-})^{-}=F$
(b) For a soft subset (T,C) in Z, $(GoF)^{-}(T,C)=F^{-}(G^{-}(T,C))$ and $(GoF)^{+}(T,C)=F^{+}(G^{+}(T,C))$.

Proposition 4.11. Let $F,G: X\to Y$ be two soft multifunctions. For a soft set (H,K) in Y the following statements are hold:

(i) $(F\cup G)^{-}(H,K)=F^{-}(H,K)\cup G^{-}(H,K)$.
(ii) $(F\cup G)^{+}(H,K)=F^{+}(H,K)\cup G^{+}(H,K)$
(iii) $(F\cap G)^{-}(H,K)\subset F^{-}(H,K)\cap G^{-}(H,K)$
(iv) $F^{+}(H,K)\cap G^{+}(H,K)\subset(F\cap G)^{+}(H,K)$

Proof. (i) Let $E_e{}^x\in(F\cup G)^{-}(H,K)$, then $(F\cup G)(E_e{}^x)\tilde{\cap}(H,K)\neq\Phi$ implies that $(F(E_e{}^x)\cup G(E_e{}^x))\tilde{\cap}(H,K)\neq\Phi$. Thus $(F(E_e{}^x)\tilde{\cap}(H,K))\tilde{\cup}(G(E_e{}^x)\tilde{\cap}(H,K))\neq\Phi$. Thus $F(E_e{}^x)\tilde{\cap}(H,K)\neq\Phi$ or $G(E_e{}^x)\tilde{\cap}(H,K)\neq\Phi$. Hence $E_e{}^x\in F^{-}(H,K)$ or $E_e{}^x\in G^{-}(H,K)$. Thus $E_e{}^x\in F^{-}(H,K)\cup G^{-}(H,K)$.

Conversely, the proof is similar.

(iv) Let ${E_e}^x \in F^+(H,K) \cap G^+(H,K)$ then $F({E_e}^x) \tilde{\subset} (H,K)$ and $G({E_e}^x) \tilde{\subset} (H,K)$. Thus $F({E_e}^x) \tilde{\cap}\, G({E_e}^x) \tilde{\subset} (H,K)$ and $(F \cap G)({E_e}^x) \tilde{\subset} (H,K)$. Thus ${E_e}^x \in (F \cap G)^+(H,K)$.

The proof is (ii) and (iii) is similar.

Proposition 4.12. Let $F: X \to Y$ be a soft multifunction. Then the following statements are hold:

i) $(\cup F_i)^-(H,K) = \cup {F_i}^-(H,K)$
ii) $(\cup F_i)^+(H,K) = \cup {F_i}^+(H,K)$
iii) $(\cap F_i)^-(H,K) \subset \cap {F_i}^-(H,K)$
iv) $\cap {F_i}^+(H,K) \subset (\cap F_i)^+(H,K)$

Proof. Obvious.

Proposition 4.13. [2] Let (G,K) be a soft set over Y. Then the followings are true for a soft multifunction $F: (X,\tau,E) \to (Y,\sigma,K)$:

(a) $F^+(\tilde{Y} - (G,K)) = \tilde{X} - F^-(G,K)$
(b) $F^-(\tilde{Y} - (G,K)) = \tilde{X} - F^+(G,K)$.

Definition 4.14. Let $(X,\tau,E),(Y,\sigma,K)$ be two soft topological space, ${E_e}^x$ be a soft point in X and $F: (X,\tau,E) \to (Y,\sigma,K)$ be a soft multifunction.

(i) F is Vietoris soft upper continuous at a ${E_e}^x$ if for each soft open set (H,K) with $F({E_e}^x) \tilde{\subset} (H,K)$, there exists (P,E) a soft open neighbourhood of ${E_e}^x$ such that $F({E_e}^z) \tilde{\subset} (H,K)$ for all ${E_e}^z \tilde{\in} (P,E)$.

(ii) F is Vietoris soft lower continuous at a ${E_e}^x$ if for each soft open set (H,K) with $F({E_e}^x) \tilde{\cap} (H,K) \neq \Phi$, there exists (P,E) a soft open neighbourhood of ${E_e}^x$ such that $F({E_e}^z) \tilde{\cap} (H,K) \neq \emptyset$ for all ${E_e}^z \tilde{\in} (P,E)$.

(iii) If F is Vietoris soft upper continuous and Vietoris soft lower continuous at ${E_e}^x$ then F is called Vietoris soft continuous at ${E_e}^x$.

(iv) F is Vietoris soft upper continuous (resp. Vietoris soft lower continuous, Vietoris soft continuous) if F has this property at every ${E_e}^x$ soft point of X.

Theorem 4.15. Let $(X,\tau,E),(Y,\sigma,K)$ be two soft topological space and $F: (X,\tau,E) \to (Y,\sigma,K)$ be soft multifunction. We define a soft mapping $f: (X,\tau,E) \to (2^{S(Y,K)}, \tau_{SV^+}, K)$, $f({E_e}^x) = F({E_e}^x)$ for each soft point ${E_e}^x$ in X. Then the soft multifunction F is Vietoris soft upper continuous if and only if the soft mapping f is soft continuous.

Proof. ($\Rightarrow$) Let $F: X \to Y$ be Vietoris soft upper continuous at ${E_e}^{x_0}$ and let $f({E_e}^{x_0}) \in (G,K)^+ \in \beta_1$. Since $(G,K)^+ = \{(H,K): (H,K) \tilde{\subset} (G,K)\}$ then we have $f({E_e}^{x_0}) \tilde{\in} (G,K)$ and thus $F({E_e}^{x_0}) \tilde{\subset} (G,K)$. Since $F: X \to Y$ be Vietoris soft upper continuous, then there exists (P,E) is soft open neighborhood of ${E_e}^{x_0}$ such that $F({E_e}^x) \tilde{\subset} (G,K)$ for every ${E_e}^x \tilde{\in} (P,E)$. Therefore $f({E_e}^x) \tilde{\in} (G,K)$. Hence $f: X \to (2^{S(Y,K)}, \tau_{SV^+}, K)$ is soft continuous at ${E_e}^{x_0}$.

($\Leftarrow$) Let $f: X \rightarrow (2^{S(Y,K)}, \tau_{SV}{}^{+}, K)$ is soft continuous at $E_e{}^{x_0}$ and let (G, K) be soft open set such that $F(E_e{}^{x_0}) \tilde{\subset} (G, K)$. Then $f(E_e{}^{x_0}) \in (G, K)^+ \in \beta_1$. Since f is soft continuous to $\tau_{SV}{}^{+}$ there exists (P, E) a soft open neigborhood of $E_e{}^{x_0}$ such that $F(P, E) \tilde{\subset} (G, K)$. Then we have $f(E_e{}^{x}) \in (G, K)^+$ for $E_e{}^{x} \tilde{\in} (P, E)$. Thus $F(E_e{}^{x}) \tilde{\subset} (G, K)$. This implies that $F: X \rightarrow Y$ is Vietoris soft upper continuous at $E_e{}^{x_0}$.

Theorem 4.16. Let $(X, \tau, E), (Y, \sigma, K)$ be two soft topological space and $F: (X, \tau, E) \rightarrow (Y, \sigma, K)$ be soft multifunction. We define a soft mapping $f: (X, \tau, E) \rightarrow (2^{S(Y,K)}, \tau_{SV^-}, K)$, $f(E_e{}^{x}) = F(E_e{}^{x})$ for each soft point $E_e{}^{x}$ in X. Then the soft multifunction F is Vietoris soft lower continuous if and only if the soft mapping f is soft continuous.

Proof. It can be show that similarly to Theorem 4.15.

5. Conclusions

Recently, many researcher have studied the soft set theory, which is applied to many problems having uncertainties. In this paper, we define the concept of Vietoris soft topological space one of the hyperspaces of soft sets. Then we define the Vietoris continuity of soft multifunction. Finally, we give the relationship between Vietoris continuity of soft multifunction and continuity of soft mapping. We hope that this paper is going to help researcher to enhance the further study on soft set theory.

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Soft $\wedge_\beta$-CLOSED SETS IN SOFT TOPOLOGICAL SPACES

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Abstract – In this paper, we introduce the notions of soft β-kernel of soft sets, S$\wedge_\beta$-closed sets and S$\wedge_\beta$-open sets in soft topological spaces. The concept of S$\wedge_\beta$-sets, as a generalization to the class of soft β-open sets, is defined. Some of their fundamental properties that illustrated by examples are studied. Further, the soft topologies defined by families soft β-kernel of soft sets, S$\wedge_\beta$-closed sets are investigated. We show that, the collection of S$\wedge_\beta$-sets is Alexandroff soft space.

Keywords – *Soft sets, Soft topology, Soft β-open sets, Soft regular-closed sets, Soft $\wedge_\beta$-sets.*

1 Introduction

Molodtsov [13] initiated the concept of soft sets, which is a completely new approach for modeling vagueness and uncertainty. Also, it is free of the difficulties present in the theories of fuzzy sets, rough set, vague, etc. In recent years, researchers have been formulated the theoretical bases for further applications of topology on soft sets that lead to the development of information system and various fields in digital topology, engineering, economics, social science, medical science, etc. Maji et al. [11] proposed several operations on soft sets, and some basic properties of these operations have been revealed. Shabir et al. [15] introduced the notion of soft topological spaces that are defined to be over an initial universe with a fixed set of parameters. Hussain et al. [6] investigated the properties of soft open (closed) sets, soft neighbourhood and soft closure. In addition, Arockiarani et al. [3] introduced soft gβ-closed sets and soft gsβ-closed sets in soft topological spaces and they obtained some properties in the light of these defined sets. The concept of $\wedge_\beta$-sets has been introduced (known as $\wedge_{sp}$-sets) with some of their properties in [14].

In this paper, we generalize the concept $\wedge_\beta$-sets to soft setting. Soft $\wedge_\beta$-closed sets, soft $\wedge_\beta$-open sets on soft topological spaces will be introduced. Some of their properties on soft topology will be studied. The collection of S$\wedge_\beta$-sets is Alexandroff space will be presented. Furthermore, the soft topologies defined by families soft β-kernel of soft sets, S$\wedge_\beta$-closed sets will be investigated.

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* *Corresponding Author.*

2 Preliminary

For the definitions and results on soft set theory, we refer to [1, 3, 4, 6, 11, 12, 13, 15, 16, 17]. However, we recall some definitions and results on soft set theory and soft topology.

Let X be an initial universe set and E be a set of parameters. Let $\mathcal{P}$(X) be the power set of X and A be a non empty subset of E. F_A is called a soft set over X [13], where F is a mapping given by F: A$\longrightarrow\mathcal{P}$(X). For two soft sets F_A, H_B over common universe X, we say that F_A is a subset of H_B [11], if A$\subseteq$B and F(e)$\subseteq$H(e), for all e$\in$A. In this case, we write $F_A \sqsubseteq H_B$. Two soft sets F_A, H_B over a common universe X are said to be equal, if $F_A \sqsubseteq H_B$ and $H_B \sqsubseteq F_A$. A soft set F_A over X is called a null (resp., an absolute) soft set, denoted by $\tilde{\emptyset}_A$ (resp., $\tilde{X}_A$), if for each e$\in$A, F(e)=$\emptyset$ (resp., F(e)=X)[11]. In particular, (X, A) will be denoted by $\tilde{X}_A$.

The union of two soft sets of F_A and H_B over the common universe X [11] is the soft set K_C, where C=A$\cup$B and for all e$\in$C, K(e)=F(e), if e$\in$(A - B); K(e)=H(e), if e$\in$(B - A), and K(e)=F(e)$\cup$H(e), if e$\in$(A$\cap$B). We write $F_A \sqcup H_B = K_C$. Moreover, the intersection K_C of two soft sets F_A and H_B over a common universe X [15], denoted $F_A \sqcap H_B$, is defined as C=A$\cap$B, and K(e)=F(e)$\cap$H(e) for all e$\in$C. The difference between two soft sets F_A and H_A over X [15], denoted by $K_A = (F_A - H_A)$, is defined as K(e)=F(e) - H(e) for all e$\in$A. The relative complement of a soft set F_A [1], is denoted by $(F_A)^c$ and is defined by $(F_A)^c = \tilde{X}_A - F_A$, where F^c: A$\longrightarrow\mathcal{P}$(X) is a mapping given by $(F)^c$(e)= X - F(e) for all e$\in$A. Moreover, $((F_A)^c)^c = F_A$.

In order to efficiently discuss, we consider only soft sets F_A over a universe X in which all the parameters set A are the same.

In this paper for convenience, let $\mathcal{SS}$(X, A) be the family of soft sets over X with set of parameters A.

Definition 2.1. [15] Let F_A be a soft set over X and x$\in$X. x$\in F_A$ whenever x$\in$F(e) for all e$\in$A. Note that for any x$\in$X, x$\notin F_A$, if x$\notin$F(e) for some e$\in$A.

Definition 2.2. [15] Let x$\in$X, then x_A denotes the soft set over X for which x(e)={x} for all e$\in$A.

Definition 2.3. [15] Let τ be the collection of soft sets over X with the fixed set of parameters A. Then, τ is said to be a soft topology on X, if it satisfies the following axioms:
(i) $\tilde{\emptyset}_A$, $\tilde{X}_A$ belong to τ.
(ii) The union of any number of soft sets in τ belongs to τ.
(iii) The intersection of any two soft sets in τ belongs to τ.

The triplet (X, A, τ) is called a soft topological space over X or a soft space. Every member of τ is called a soft open set. The complement of a soft open set is called the soft closed set [6] in (X, A, τ).

Definition 2.4. Let (X, A, τ) be a soft topological space and F_A be a soft set. Then,

(i) [15] the soft closure of F_A, denoted by cl F_A, is the intersection of all soft closed sets that contain F_A, i.e., cl $F_A = \sqcap\{M_A \in \tau^c \mid F_A \sqsubseteq M_A\}$. Clearly, x$\in$cl F_A if and only if for every soft open nbd. U_A of x; $F_A \sqcap U_A \neq \tilde{\emptyset}_A$.

(ii) [6] the soft interior of F_A, denoted by int F_A, is the union of all soft open sets contained in F_A i.e, int $F_A = \sqcup\{M_A \in \tau \mid M_A \sqsubseteq F_A\}$.

Definition 2.5. A soft set F_A in a soft topological space (X, A, τ) is said to be
((i) [8] Soft regular closed, if cl(int(F_A))=F_A.

(ii)[7] Soft β-open, if $F_A \sqsubseteq cl(int(cl(F_A)))$ (resp., soft β-closed, if $int(cl(int(F_A))) \sqsubseteq F_A$).
(iii) [10] Soft g-closed set, if $cl(F_A) \sqsubseteq U_A$ whenever $F_A \sqsubseteq U_A$ and U_A is soft open set.
(iv) [3] Soft βg-closed set, if $cl(F_A) \sqsubseteq U_A$ whenever $F_A \sqsubseteq U_A$ and U_A is soft β-open set.

The family of all soft β-open sets in a soft topological space (X, A, τ) will be denote by Sβ(X, A).

Lemma 2.6. [7] (i) Arbitrary intersection of soft β-closed sets is soft β-closed but the union of two soft β-closed sets need not be soft β-closed set.
(ii) Arbitrary union of soft β-open sets is soft β-open but the intersection of two soft β-open sets need not be soft β-open set.

Definition 2.7. [7] Let (X, A, τ) be a soft topological space and F_A be a soft set. Then,
(i) A soft β-closure of F_A, denoted by Scl_β F_A, is the intersection of all soft β-closed sets that contain F_A, i.e., $Scl_\beta F_A = \sqcap\{M_A \mid M_A$ is a soft β-closed and $F_A \sqsubseteq M_A\}$.
(ii) A soft β-interior of F_A, denoted by $Sint_\beta$, is the union of all soft β-open sets contained in F_A i.e, $Sint_\beta F_A = \sqcup\{M_A \mid M_A$ is a soft β-open and $M_A \sqsubseteq F_A\}$.

Definition 2.8. [5] Let τ be the collection of soft sets over X; then, τ is said to be a supra soft topology on X, if it satisfies the following axioms
(i) $\tilde{\emptyset}_A$, $\tilde{X}_A$ belong to τ.
(ii) the union of any number of soft sets in τ belongs to τ.

3 Soft $\wedge_\beta$-closed Sets

Definition 3.1. The soft β-kernel of a soft set F_A, denoted by $S\wedge_\beta(F_A)$, is the intersection of all soft β-open superset of F_A i.e $S\wedge_\beta(F_A) = \sqcap\{U_A \in S\beta(X, A) \mid F_A \sqsubseteq U_A\}$.

Example 3.2. Let X={x, y, z}, A={e_1, e_2} and the soft topology τ over (X, A) is

$\tau = \{F_{iA} \mid i = 1, 2, \ldots, 7\} \sqcup \{\tilde{\emptyset}_A, \tilde{X}_A\}$.

We consider the following soft sets F_{iA}, i = 1,2, . . . ,7 over X defined as follows:

F_{1A}={{x}, {x}},
F_{2A}={{y}, {z}},
F_{3A}={{y}, ∅},
F_{4A}={{x, y}, {x}},
F_{5A}={{x, y}, {x, z}},
F_{6A}={X, {x}},
F_{7A}={X, {x, z}}.

A soft set {{z}, {y}} is $S\wedge_\beta$-set, but {{x, z}, X} is not.

Some of fundamental properties of soft β-kernel of soft sets will be shown in the next theorem.

Lemma 3.3. For soft sets F_A, H_A and F_{iA} ($i \in \Gamma$) of a soft topological space (X, A, τ), the following properties hold:
(i) $S\wedge_\beta(\tilde{\emptyset}_A) = \tilde{\emptyset}_A$ and $S\wedge_\beta(\tilde{X}_A) = \tilde{X}_A$.
(ii) $F_A \sqsubseteq S\wedge_\beta(F_A)$.
(iii) $F_A \sqsubseteq H_A$, then $S\wedge_\beta(F_A) \sqsubseteq S\wedge_\beta(H_A)$.
(iv) $S\wedge_\beta(S\wedge_\beta(F_A)) = S\wedge_\beta(F_A)$.
(v) $F_A \in S\beta(X, A)$, then $F_A = S\wedge_\beta(F_A)$.
(vi) $S\wedge_\beta(\sqcup_{i\in\Gamma} F_{iA}) = \sqcup_{i\in\Gamma} S\wedge_\beta(F_{iA})$.
(vii) $S\wedge_\beta(\sqcap_{i\in\Gamma} F_{iA}) \sqsubseteq \sqcap_{i\in\Gamma} S\wedge_\beta(F_{iA})$.

Proof. We prove only (vi) and the rest of the proof follows directly from Definition 3.1.
(vi) It is obvious that $S\wedge_\beta(F_{iA})\sqsubseteq S\wedge_\beta(\sqcup_{i\in\Gamma}F_{iA})$ for each $i\in\Gamma$ and so $\sqcup_{i\in\Gamma}(S\wedge_\beta(F_{iA}))\sqsubseteq S\wedge_\beta(\sqcup_{i\in\Gamma}F_{iA})$. Conversely, suppose $x\notin\sqcup_{i\in\Gamma}(S\wedge_\beta(F_{iA}))$, then $x\notin(S\wedge_\beta(F_{iA}))$ for each $i\in\Gamma$. Therefore, for each $i\in\Gamma$there exists $U_{iA}\in S\beta(X, A)$ such that $x\notin U_{iA}$ and $F_{iA}\sqsubseteq U_{iA}$. Thus $\sqcup_{i\in\Gamma}(F_{iA})\sqsubseteq\sqcup_{i\in\Gamma}(U_{iA})$ and $\sqcup_{i\in\Gamma}(U_{iA})\in S\beta(X, A)$ which does not contain x. Which implies that $x\notin S\wedge_\beta(\sqcup_{i\in\Gamma}F_{iA})$. Consequently, $S\wedge_\beta(\sqcup_{i\in\Gamma}F_{iA})\sqsubseteq\sqcup_{i\in\Gamma}S\wedge_\beta(F_{iA})$.

The equality of (vii) of Lemma 3.3 does not hold as indicated in the following example.

Example 3.4. Let (X, A, τ) be a soft topological space as in Example 3.2. Let U_A={{z}, X}, V_A={X, {x, y}}, then $S\wedge_\beta(U_A)=\tilde{X}_A$, $S\wedge_\beta(V_A)$={X, {x, y}} and so
$S\wedge_\beta(U_A)\sqcap S\wedge_\beta(V_A)$={X, {x, y}}, but $S\wedge_\beta(U_A\sqcap V_A)$={{z}, {x, y}}.

Corollary 3.5. (i) $Sint_\beta(F_A)\sqsubseteq S\wedge_\beta(F_A)$.
(ii) If $F_A\in S\beta(X, A)$, then $Sint_\beta(F_A)=S\wedge_\beta(F_A)$.
(iii) If $F_A\in S\beta(X, A)$, then $S\wedge_\beta(F_A)\sqsubseteq Scl_\beta(F_A)\sqsubseteq cl(F_A)$.
(iv) If F_A is soft β-closed, then $Scl_\beta(F_A)\sqsubseteq S\wedge_\beta(F_A)$.

In view of Lemma 3.3, the next theorem hold.

Theorem 3.6. The collection of all soft β-kernel of soft sets is a supra soft topological space.

Definition 3.7. The concept of $S\wedge_\beta$-sets in a soft topological space (X, A, τ) is the soft sets that coincide with their soft β-kernel. In other words, a soft set F_A is called $S\wedge_\beta$-set, if $F_A=S\wedge_\beta(F_A)$.

Theorem 3.8. Let (X, A, τ) be a soft topological space. Then, for a soft set F_A the following statements hold:
(i) $\tilde{\emptyset}_A$, $\tilde{X}_A$ are $S\wedge_\beta$-sets.
(ii) $S\wedge_\beta(F_A)$ is $S\wedge_\beta$-sets.
(iii) Every soft β-open is $S\wedge_\beta$-set.
(iv) The union of $S\wedge_\beta$-sets is $S\wedge_\beta$-set.
(v) The intersection of $S\wedge_\beta$-sets is $S\wedge_\beta$-set.

Proof. Follows directly from Lemma 3.3 and Definition 3.7.

Corollary 3.9. The class of all $S\wedge_\beta$-sets is stronger than soft β-open sets.

The converse of this corollary is not true as shown of the next example.

Example 3.10. Let (X, A, τ) be a soft topological space as in Example 3.2. Then, a soft set {{z}, {y}} is $S\wedge_\beta$-set and it is not soft β-open.

Definition 3.11. A soft topological space (X, A, τ) is an Alexandroff soft space, if arbitrary intersections of soft sets in τ belongs to τ.

Theorem 3.12. Let (X, A, τ) be a soft topological space, we put $\tau^{\wedge_\beta}$={F_A | F_A is $S\wedge_\beta$-set}. Then, (X, A, $\tau^{\wedge_\beta}$) is an Alexandroff soft space.

Proof. Immediate from Theorem 3.8 (i), (iv).

Corollary 3.13. (i) Sβ(X, A)$\sqsubseteq\tau^{\wedge\beta}$.
(ii) $\tau\sqsubseteq\tau^{\wedge\beta}$.

Definition 3.14. A soft topological space (X, A, τ) is called
(i) β-disconnected, if there exist nonempty soft β-open sets G_A, H_A such that $G_A\sqcup H_A=\tilde{X}_A$, $G_A\sqcap H_A=\tilde{\emptyset}_A$.
(ii) S$\wedge_\beta$-disconnected, if there exist $\tilde{\emptyset}_A\neq G_A$, $H_A\in\tau^{\wedge\beta}$ such that $G_A\sqcup H_A=\tilde{X}_A$, $G_A\sqcap H_A=\tilde{\emptyset}_A$.

Theorem 3.15. If (X, A, τ) is soft β-disconnected, then (X, A, $\tau^{\wedge\beta}$) is soft disconnected.

Proof. Immediate.

Lemma 3.16. A soft set F_A of a soft topological space (X, A, τ) is a soft βg-closed if and only if cl(F_A)$\sqsubseteq$S$\wedge_\beta$(F_A).

Proof. Let x$\notin$S$\wedge_\beta$(F_A), then there is soft β-open set U_A such that $F_A\sqsubseteq U_A$ and x$\notin U_A$. Since F_A is a soft βg-closed, then cl$F_A\sqsubseteq U_A$. Hence, x$\notin$cl F_A. On the other hand, let $F_A\sqsubseteq U_A$ and U_A be soft β-open set. Then, cl$F_A\sqsubseteq$S$\wedge_\beta$(F_A)$\sqsubseteq$S$\wedge_\beta$(U_A)=U_A. Hence, F_A is soft βg-closed set.

Lemma 3.17. A soft set F_A of a soft topological space (X, A, τ) is soft βg-closed and soft β-open set if and only if it is soft regular closed.

Proof. Immediate.

Corollary 3.18. (i) cl(F_A)=S$\wedge_\beta$(F_A), if F_A is soft regular closed set.
(ii) Scl$_\beta$(F_A)$\sqsubseteq$S$\wedge_\beta$(F_A), if F_A is a soft βg-closed .
(iii) cl(F_A) is S$\wedge_\beta$-set, if $F_A\in$Sβ(X, A).

Proof. Immediate from Lemmas 3.3, 3.16, 3.17 and Corollary 3.5.

Lemma 3.19. A soft set F_A of a soft topological space (X, A, τ) is soft βg-closed if and only if cl(F_A) - F_A does not contain any nonempty soft β-closed set.

Proof. Let F_A be soft βg-closed set over X and suppose that U_A is soft β-closed set such that $U_A\sqsubseteq$cl(F_A) - F_A, then $U_A\sqsubseteq$cl(F_A) - $F_A\sqsubseteq\tilde{X}_A$ - F_A. Which implies that $F_A\sqsubseteq\tilde{X}_A$ - U_A and ($\tilde{X}_A$ - U_A) is soft β-open set. Since F_A is soft βg-closed set, then cl(F_A)$\sqsubseteq$($\tilde{X}_A$ - U_A). Hence, $U_A\sqsubseteq\tilde{X}_A$ - cl(F_A), which is a contradiction. Consequently, $U_A=\tilde{\emptyset}_A$. On the other hand, let $F_A\sqsubseteq U_A$ and U_A be soft β-open set. Suppose cl(F_A)$\sqcap$($\tilde{X}_A$ - U_A)$\neq\tilde{\emptyset}_A$, since cl(F_A)$\sqcap$($\tilde{X}_A$ - U_A)$\sqsubseteq$cl(F_A) - F_A. Then, cl(F_A)$\sqcap$($\tilde{X}_A$ - U_A) is nonempty soft β-closed set, which is a contradiction. Hence, cl$F_A\sqsubseteq U_A$.

Definition 3.20. A soft set F_A of a soft topological space (X, A, τ) is called S$\wedge_\beta$-closed set, if $F_A=G_A\sqcap H_A$, where G_A is a S$\wedge_\beta$-set and H_A is a soft closed set. A soft set is said to be a S$\wedge_\beta$-open set, if its complement is a S$\wedge_\beta$-closed.

In view of $\tilde{X}_A$ is both S$\wedge_\beta$-set and soft closed set, then proof of next lemma is immediate.

Lemma 3.21. (i) Every S$\wedge_\beta$-set is S$\wedge_\beta$-closed set.
(ii) Every soft closed set is S$\wedge_\beta$-closed set.

Lemma 3.22. For a soft set F_A of a soft topological space (X, A, τ), the following statements are equivalent
(i) F_A is $S\wedge_\beta$-closed set.
(ii) $F_A=G_A\sqcap cl(F_A)$, where G_A is a $S\wedge_\beta$-set.
(iii) $F_A=S\wedge_\beta(F_A)\sqcap cl(F_A)$.

Proof. (i)$\Longrightarrow$(ii) Let F_A be $S\wedge_\beta$-closed set, then there exist $S\wedge_\beta$-set G_A and soft closed set H_A such that $F_A=G_A\sqcap H_A$. Since $F_A\sqsubseteq H_A$, then $cl(F_A)\sqsubseteq H_A$ and so $F_A\sqsubseteq G_A\sqcap cl(F_A)\sqsubseteq G_A\sqcap H_A=F_A$. Hence, $F_A=G_A\sqcap cl(F_A)$.

(ii)$\Longrightarrow$(iii) Let $F_A=G_A\sqcap cl\ (F_A)$ and G_A is a $S\wedge_\beta$-set. Since $F_A\sqsubseteq G_A$, then $S\wedge_\beta(F_A)\sqsubseteq S\wedge_\beta(G_A)=G_A$ follows from Lemma 3.3. Therefore, $F_A\sqsubseteq S\wedge_\beta(F_A)\sqcap cl(F_A)\ \sqsubseteq G_A\sqcap cl(F_A)=F_A$ and so $F_A=S\ \wedge_\beta(F_A)\sqcap cl(F_A)$.

(iii)$\Longrightarrow$(i) Straightforward.

Corollary 3.23. A soft set F_A in a soft topological space (X, A, τ) is soft closed if and only if it is $S\wedge_\beta$-closed and soft βg-closed.

Proof. Immediate from Lemmas 3.16, 3.22.

Theorem 3.24. The intersection of $S\wedge_\beta$-closed sets is a $S\wedge_\beta$-closed set.

Proof. Let F_{iA} be $S\wedge_\beta$-closed sets and $i\in\Gamma$, then for each $i\in\Gamma$ there exist $S\wedge_\beta$-sets G_{iA} and soft closed sets H_{iA} such that $F_{iA}=G_{iA}\sqcap H_{iA}$. Hence, $\sqcap_{i\in\Gamma}F_{iA}=\sqcap_{i\in\Gamma}(G_{iA}\sqcap H_{iA})=(\sqcap_{i\in\Gamma}G_{iA})\sqcap(\sqcap_{i\in\Gamma}H_{iA})$. In view of Theorem 3.8, $\sqcap_{i\in\Gamma}G_{iA}$ is $S\wedge_\beta$-set and $\sqcap_{i\in\Gamma}H_{iA}$ is a soft closed set. Consequently, $\sqcap_{i\in\Gamma}F_{iA}$ is $S\wedge_\beta$-closed set.

Corollary 3.25. The union of $S\wedge_\beta$-open sets is a $S\wedge_\beta$-open set.

The union of $S\wedge_\beta$-closed sets need not to be a $S\wedge_\beta$-closed set as shown by the following example.

Example 3.26. The soft topological space (X, A, τ) is the same as in Example 3.2. Then, H_A={{y, z}, {x, y}}, K_A={{z}, {z}} are $S\wedge_\beta$-closed sets but $H_A\sqcup K_A$={{y, z}, X} does not $S\wedge_\beta$-closed set.

Theorem 3.27. The class of $S\wedge_\beta$-closed sets is a supra soft topology over X, which is finer than $\tau^{\wedge_\beta}$.

Definition 3.28. For a soft set F_A of a soft topological space (X, A, τ)
(i) $S\vee_\beta(F_A)=\sqcup\{H_A \mid (H_A)^c\in S\beta(X, A)$ and $H_A\sqsubseteq F_A\}$.
(ii) $S\coprod_\beta(F_A)=S\wedge_\beta(F_A)\sqcap S\wedge_\beta(F_A)^c$.

Several results which are similar to Lemma 3.3 for $S\vee_\beta(\ )$ will be obtained.

Lemma 3.29. For soft sets F_A, H_A and F_{iA} ($i\in\Gamma$) of a soft topological space (X, A, τ), the following properties hold:
(i) $S\vee_\beta(\tilde{\emptyset}_A)=\tilde{\emptyset}_A$ and $S\vee_\beta(\tilde{X}_A)=\tilde{X}_A$.
(ii) $S\vee_\beta(F_A)\sqsubseteq F_A$.
(iii) $F_A\sqsubseteq H_A$, then $S\vee_\beta(F_A)\sqsubseteq S\vee_\beta(H_A)$.
(iv) $S\vee_\beta(S\vee_\beta(F_A))=S\vee_\beta(F_A)$.
(v) If F_A is soft β-closed, then $F_A=S\vee_\beta(F_A)$.
(vi) $S\vee_\beta(\sqcap_{i\in\Gamma}F_{iA})=\sqcap_{i\in\Gamma}S\vee_\beta(F_{iA})$.

(vii) $\sqcup_{i\in\Gamma}$S$\vee_\beta$(F$_{iA}$)$\sqsubseteq$S$\vee_\beta$($\sqcup_{i\in\Gamma}$F$_{iA}$).

The proof of next theorems are obvious and omitted.

Theorem 3.30. For a soft set F$_A$ of a soft topological space (X, A, τ), the following statements hold:
(i) S$\vee_\beta$(F$_A$)^c=$\tilde{X_A}$ - S$\wedge_\beta$(F$_A$).
(ii) S$\wedge_\beta$(F$_A$)^c=$\tilde{X_A}$ - S$\vee_\beta$(F$_A$).
(iii) S$\coprod_\beta$(F$_A$)=S$\coprod_\beta$(F$_A$)^c.
(iv) S$\vee_\beta$(F$_A$)$\sqcup$S$\coprod_\beta$(F$_A$)=S$\wedge_\beta$(F$_A$).
(v) S$\vee_\beta$(F$_A$)$\sqcap$S$\coprod_\beta$(F$_A$)=$\tilde{\emptyset_A}$.
(vi) S$\coprod_\beta$(F$_A$)=S$\wedge_\beta$(F$_A$) - S$\vee_\beta$(F$_A$).
(vii) S$\vee_\beta$(F$_A$)=F$_A$ - S$\coprod_\beta$(F$_A$).
(viii) S$\wedge_\beta$S$\coprod_\beta$(F$_A$)=S$\coprod_\beta$(F$_A$).
(ix) S$\coprod_\beta$S$\coprod_\beta$(F$_A$)$\sqsubseteq$S$\coprod_\beta$(F$_A$).
(x) S$\coprod_\beta$(S$\vee_\beta$(F$_A$))$\sqsubseteq$S$\coprod_\beta$(F$_A$).

Theorem 3.31. For soft sets F$_A$, H$_A$ of a soft topological space (X, A, τ), the following statements hold:
(i) S$\wedge_\beta$(F$_A$ - H$_A$)$\sqsubseteq$S$\wedge_\beta$(F$_A$) - S$\vee_\beta$(H$_A$).
(ii) S$\vee_\beta$(F$_A$ - H$_A$)=S$\vee_\beta$(F$_A$) - S$\wedge_\beta$(H$_A$).
(iii) S$\coprod_\beta$(F$_A$$\sqcup$H$_A$)$\sqsubseteq$S$\coprod_\beta$(F$_A$)$\sqcup$S$\coprod_\beta$(H$_A$).

4 Conclusion

In this paper, we generalized the concept $\wedge_\beta$-sets to soft setting and introduced soft $\wedge_\beta$-closed sets, soft $\wedge_\beta$-open sets on soft topological spaces. Some of their properties were studied. The collection of S$\wedge_\beta$-sets is Alexandroff space was presented. Furthermore, the soft topologies defined by families, soft β-kernel of soft sets, S$\wedge_\beta$-closed were investigated.

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JNT publishes original research articles, reports, reviews and commentaries that are based on a theory of mathematics. However, the topics are not limited to only mathematics, but also include statistics, computer science, physics, engineering, chemistry, biology, economics or social sciences that use a theory of mathematics.

JNT is a refereed, electronic, open access and international journal.

Papers in JNT are published free of charge.

We would like to express our deepest thanks to all of the members of the editorial board and reviewers of the papers in this issue who are A. Filiz, A. Fenercioğlu, A. Sarı, A. Yıldırım, A. S. Sezer, B. Mehmetoğlu, B.H. Çadırcı, C. Kaya, Ç. Çekiç, D. Mohamad, E. Altuntaş, E. Turgut, F. Karaaslan, F. Smarandache, G. Erdal, H. Aktaş, H.M. Doğan, H. Günal, H. Kızılaslan, H. Önen, H. Şimşek, İ. Zorlutuna, İ. Deli, İ. Gökce, İ. Türkekul, İ. Parmaksız, J. Zhan, J. Ye, H. Kızılaslan, M. Akar, M. Akdağ, M.I. Ali, M. Ali, M. Çavuş, M. Demirci, N. Çağman, N. Sağlam, N. Yeşilayer, N. Kızılaslan, O. Muhtaroğlu, P.K. Maji, R. Yayar, S. Broumi, S. Karaman, S. Tarhan, S. Enginoğlu, S. Demiriz, S. Karataş, S. Öztürk, S. Eğri, Ş. Sözen, U. Orhan, Y. Budak, Y. Karadağ, K. Kaygısız, A. Kandil, A.M. A. El-latif, O. Tantawy, M. Hosny.

Pleases, write any original idea. If it is true, it gives an opportunity to use. If it is incomplete, it gives an opportunity to complete. If it is incorrect, it gives an opportunity to correct.

You can reach us from journal homepage at http://www.newtheory.org. To receive further information and to send your recommendations and remarks, or to submit articles for consideration, please e-mail us at jnt@newtheory.org

We hope you will enjoy this issue of JNT. We are looking forward to hearing your feedback and receiving your contributions.

Happy reading!

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