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CONTENT

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ON (L, M)-FUZZY TOPOGENOUS SPACES

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Abstract – In this paper, we introduce the concept of an (L, M)-fuzzy topogenous space, where L, M are strictly two sided commutative quantales lattices. Basic properties of (L, M)-fuzzy topogenous spaces are studied, (L, M)-fuzzy topological spaces, (L, M)-fuzzy uniform spaces and (L, M)-fuzzy proximity space are characterized in the framework of (L, M)-fuzzy topogenous spaces. We study some relationships between previous spaces and give their examples. The notion of their continuity property is investigated.

Keywords – *Complete residuated lattice, (L, M)-fuzzy topogenous order, (L, M)-fuzzy uniform space, L-fuzzy topologies.*

1 Introduction

The concepts of topogenous order and topogenous space were first introduced by Cs*ász*è*r* [8] in 1963. These concepts allow to develop a unified approach to the three spaces: topologies, proximities and uniformities. This enabled him to evolve a theory including the foundations of the three classical theories of topological spaces, uniform spaces and proximity spaces.

In the case of the fuzzy structures there are at least two notions of fuzzy topogenous structures, the first notion worked out in (Katsaras 1983 [24], 1985 [26], 1988 [27]) present a unified approach to the theories of Chang in 1968 [6] fuzzy topological spaces, Hutton fuzzy uniform spaces (Hutton, 1977 [19]), Katsaras fuzzy proximity

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spaces (Katsaras 1979 [21], 1985 [26], 1990 [28]) and Artico fuzzy proximity (Artico and Moresco 1984 [2]).

The second notion worked out in Katsaras (1990 [28],1991 [29]) agree very well with Lowen fuzzy topological spaces (Lowen 1976 [35]) and Lowen-Höhle fuzzy uniform spaces (Lowen 1981 [36]). Čimoka [7] introduced L-fuzzy topogenous structures in complete lattices. El-Dardery investigated L-fuzzy topogenous order which induced L-fuzzy topology [9].

Based on the idea of (L, M)-fuzzy topological space introduced by Kubiak [33, 34] (the motivation for this concept comes from an idea of Höhle [15] which was called fuzzifying topology in [46]).

In this paper, we introduce the concept of an (L, M)-fuzzy topogenous space, where L, M are strictly two sided commutative quantales lattices. Basic properties of (L, M)-fuzzy topogenous spaces are studied, (L, M)-fuzzy topological spaces, (L, M)-fuzzy uniform space and (L, M)-fuzzy proximity space are characterized in the framework of (L, M)-fuzzy topogenous spaces. We give some important propositions that link the previous spaces to each other. We study some relationships between previous spaces and give their examples. The notion of their continuity property is investigated.

2 Preliminary

In this paper, Let X be a non-empty set and let $L = (L, \leq, \vee, \wedge, 0, 1)$ be a completely distributive lattice with the least element 0_L and the greatest element 1_L in L.

Definition 2.1. [14, 16, 41] A complete lattice $(L, \leq, \odot)$ is called a strictly two-sided commutative quantale (stsc-quantale, for short) iff it satisfies the following properties.

(L1) $(L, \odot)$ is a commutative semigroup,

(L2) $x = x \odot 1$, for each $x \in L$ and 1 is the universal upper bound,

(L3) $\odot$ is distributive over arbitrary joins, i.e. $(\bigvee_i x_i) \odot y = \bigvee_i (x_i \odot y)$.

There exists a further binary operation $\rightarrow$ (called the implication operator or residuated) satisfying the following condition

$$x \rightarrow y = \bigvee \{z \in L | x \odot z \leq y\}.$$

Then it satisfies Galois correspondence; i.e, $(x \odot z) \leq y$ iff $z \leq (x \rightarrow y)$.

Remark 2.2. Every completely distributive lattice $(L, \leq, \wedge, \vee, ^*)$ with an order reversing involution * is a stsc-quantale $(L, \leq, \odot, \oplus, ^*)$ with an order reversing involution * where $\odot = \wedge$ and $\oplus = \vee$.

In this paper, we always assume that $(L, \leq, \odot, \oplus, ^*)$ (resp. $(M, \leq, \odot, \oplus, ^*)$) is a stsc-quantale with an order reversing involution * which is defined by

$$x \oplus y = (x^* \odot y^*)^*, \quad x^* = x \rightarrow 0$$

unless otherwise specified.

Lemma 2.3. [16, 17, 42] For each $x, y, z, x_i, y_i, w \in L$, we have the following properties.
(1) $1 \to x = x$, $0 \odot x = 0$ and $x \to 0 = x^*$,
(2) If $y \le z$, then $x \odot y \le x \odot z$, $x \oplus y \le x \oplus z$, $x \to y \le x \to z$ and $z \to x \le y \to x$,
(3) $x \odot y \le x \wedge y \le x \vee y \le x \oplus y$,
(4) $(\bigwedge_i y_i)^* = \bigvee_i y_i^*$, $(\bigvee_i y_i)^* = \bigwedge_i y_i^*$,
(5) $x \odot (\bigwedge_i y_i) \le \bigwedge_i (x \odot y_i)$,
(6) $x \oplus (\bigwedge_i y_i) = \bigwedge_i (x \oplus y_i)$, $x \oplus (\bigvee_i y_i) = \bigvee_i (x \oplus y_i)$,
(7) $x \to (\bigwedge_i y_i) = \bigwedge_i (x \to y_i)$,
(8) $(\bigvee_i x_i) \to y = \bigwedge_i (x_i \to y)$,
(9) $x \to (\bigvee_i y_i) \ge \bigvee_i (x \to y_i)$,
(10) $(\bigwedge_i x_i) \to y \ge \bigvee_i (x_i \to y)$,
(11) $(x \odot y) \to z = x \to (y \to z) = y \to (x \to z)$,
(12) $x \odot (x \to y) \le y$ and $x \to y \le (y \to z) \to (x \to z)$,
(13) $x \odot (x^* \oplus y^*) \le y^*$, $x \odot y = (x \to y^*)^*$ and $x \oplus y = x^* \to y$,
(14) $(x \to y) \odot (z \to w) \le (x \odot z) \to (y \odot w)$,
(15) $x \to y \le (x \odot z) \to (y \odot z)$ and $(x \to y) \odot (y \to z) \le x \to z$,
(16) $(x \to y) \odot (z \to w) \le (x \oplus z) \to (y \oplus w)$.

Definition 2.4. [10, 11] For a given set X, define a binary mapping $S : L^X \times L^X \to L$ by

$$S(\lambda, \mu) = \bigwedge_{x \in X} (\lambda(x) \to \mu(x)) \ \ \forall \ \lambda, \mu \in L^X,$$

then S is an L-partial order on L^X. For $\lambda, \mu \in L^X$, $S(\lambda, \mu)$ can be interpreted as the degree to which λ is a subset of μ. It is called the subsethood degree or the fuzzy inclusion order.

Lemma 2.5. [10, 11] Let S be the fuzzy inclusion order, then $\forall \ \lambda, \mu, \rho, \nu \in L^X$ and $a \in L$ the following statements hold
(1) $\mu \le \rho \Leftrightarrow S(\mu, \rho) = 1$,
(2) $S(\lambda, \rho) \odot (\rho, \mu) \le S(\lambda, \mu)$,
(3) $\mu \le \rho \Rightarrow S(\lambda, \mu) \le S(\lambda, \rho)$ and $S(\mu, \lambda) \ge S(\rho, \lambda) \ \ \forall \ \lambda \in L^X$,
(4) $S(\lambda, \mu) \odot S(\rho, \nu) \le S(\lambda \odot \rho, \mu \odot \nu)$, and $S(\lambda, \mu) \wedge S(\rho, \nu) \le S(\lambda \wedge \rho, \mu \wedge \nu)$.

Definition 2.6. [34] A map $\mathcal{T} : L^X \to M$ is called an (L, M)-fuzzy topology on X if it satisfies the following conditions.
(O1) $\mathcal{T}(0_X) = \mathcal{T}(1_X) = 1_M$,
(O2) $\mathcal{T}(\lambda_1 \odot \lambda_2) \ge \mathcal{T}(\lambda_1) \odot \mathcal{T}(\lambda_2) \ \ \forall \ \lambda_1, \lambda_2 \in L^X$,
(O3) $\mathcal{T}(\bigvee_i \lambda_i) \ge \bigwedge_i \mathcal{T}(\lambda_i) \ \forall \ \lambda_i \in L^X$, $i \in I$.

The pair $(X, \mathcal{T})$ is called an (L, M)-fuzzy topological space. Let $(X, \mathcal{T}_1)$ and $(Y, \mathcal{T}_2)$ be L-fuzzy topological spaces and $f : X \to Y$ be a map. Then f is called LF-continuous if

$$\mathcal{T}_2(\lambda) \le \mathcal{T}_1(f^{\leftarrow}(\lambda)) \ \ \forall \ \lambda \in L^Y.$$

Definition 2.7. [7] A map $\mathcal{F} : L^X \to M$ is called an (L, M)-fuzzy cotopology on X if it satisfies the following conditions.

(F1) $\mathcal{F}(0_X) = \mathcal{F}(1_X) = 1$,

(F2) $\mathcal{F}(\lambda_1 \oplus \lambda_2) \geq \mathcal{F}(\lambda_1) \odot \mathcal{F}(\lambda_2), \ \ \forall \ \lambda_1, \lambda_2 \in L^X$,

(F3) $\mathcal{F}(\bigvee_i \lambda_i) \geq \bigwedge_i \mathcal{F}(\lambda_i), \ \forall \ \lambda_i \in L^X, \ i \in I$.

The pair $(X, \mathcal{F})$ is called an (L, M)-fuzzy cotopological space. Let $(X, \mathcal{F}_1)$ and $(Y, \mathcal{F}_2)$ be (L, M)-fuzzy topological spaces and $f : X \to Y$ be a map. Then f is called LF-continuous if

$$\mathcal{F}_2(\lambda) \leq \mathcal{F}_1(f^{\leftarrow}(\lambda)), \quad \forall \lambda \in L^Y.$$

3 Perfect (L, M)-fuzzy topogenous structures and (L, M)-fuzzy topologies

Definition 3.1. A mapping $\xi : L^X \times L^X \to L$ is called an (L, M)-fuzzy semi-topogenous order on X if it satisfies the following axioms.

(ST1) $\xi(1_X, 1_X) = \xi(0_X, 0_X) = 1_M$,

(ST2) $\xi(\lambda, \mu) \leq S(\lambda, \mu)$,

(ST3) If $\lambda_1 \leq \lambda, \ \ \mu \leq \mu_1$, then $\xi(\lambda, \mu) \leq \xi(\lambda_1, \mu_1)$.

Remark 3.2. If ξ is an (L, M)-fuzzy semi-topogenous order on X. Then

(1) If $\xi(\lambda, \mu) = 1_M$, then $\lambda \leq \mu$,

(2) $\xi(1_X, \lambda) \leq \bigwedge_x \lambda(x)$ and $\xi(\lambda, 0_X) \leq \bigwedge_x \lambda^*(x)$,

(3) Define a mapping $\xi^s : L^X \times L^X \to M$ as $\xi^s(\lambda, \mu) = \xi(\mu^*, \lambda^*)$. Then ξ^s is an (L, M)-fuzzy semi-topogenous order on X.

Definition 3.3. An (L, M)-fuzzy semi-topogenous order ξ on X is called symmetric if

(S) $\xi = \xi^s$.

Definition 3.4. For every $\lambda_1, \lambda_2, \mu_1, \mu_2 \in L^X$, an (L, M)-fuzzy semi-topogenous order ξ is called

(1) (L, M)-fuzzy topogenous if

$$(T) \quad \xi(\lambda_1 \odot \lambda_2, \mu_1 \odot \mu_2) \geq \xi(\lambda_1, \mu_1) \odot \xi(\lambda_2, \mu_2),$$

(2) (L, M)-fuzzy co-topogenous if

$$(CT) \quad \xi(\lambda_1 \oplus \lambda_2, \mu_1 \oplus \mu_2) \geq \xi(\lambda_1, \mu_1) \odot \xi(\lambda_2, \mu_2),$$

(3) (L, M)-fuzzy bitopogenous if ξ are (L, M)-fuzzy topogenous and (L, M)-fuzzy cotopogenous.

Remark 3.5. Let $(L = M, \odot = \wedge, \oplus = \vee)$ be a complete lattice, then (L, M)-fuzzy bitopogenous order is an L-fuzzy topogenous in a Čimoka sense from:
(T) $\xi(\lambda \wedge \lambda, \mu_1 \wedge \mu_2) \geq \xi(\lambda, \mu_1) \wedge \xi(\lambda_2, \mu_2)$,
(CT) $\xi(\lambda_1 \vee \lambda_2, \mu \vee \mu) \geq \xi(\lambda_1, \mu) \wedge \xi(\lambda_2, \mu)$.

Definition 3.6. An (L, M)-fuzzy topogenous (resp. cotopogenous) order ξ on X is said to be L-fuzzy topogenous (resp. cotopogenous) space if $\xi \circ \xi \geq \xi$, where
(TS) $(\xi_1 \circ \xi_2)(\lambda, \mu) = \bigvee_{\rho \in L^X} \xi_1(\lambda, \rho) \odot \xi_2(\rho, \mu)$.

Definition 3.7. An (L, M)-fuzzy semi-topogenous order ξ on X is called perfect if
(ST4) $\xi(\bigvee_i \lambda_i, \mu) \geq \bigwedge_i \xi(\lambda_i, \mu)$.
An (L, M)-fuzzy semi-topogenous order ξ on X is called co-perfect if
(ST5) $\xi(\lambda, \bigwedge_i \mu_i) \geq \bigwedge_i \xi(\lambda, \mu_i)$.
An (L, M)-fuzzy semi-topogenous order ξ on X is called bi-perfect if ξ are (L, M)-fuzzy perfect and (L, M)-fuzzy co-perfect.

Theorem 3.8. Let ξ_1 and ξ_2 be (L, M)-fuzzy cotopogenous (respectively, topogenous, perfect, co-perfect) order on X. Define the composition $(\xi_1 \circ \xi_2)$ of ξ_1 and ξ_2 by

$$(\xi_1 \circ \xi_2)(\lambda, \mu) = \bigvee_{\rho \in X} \xi_1(\lambda, \rho) \odot \xi_2(\rho, \mu).$$

Then $(\xi_1 \circ \xi_2)$ is (L, M)-fuzzy cotopogenous (respectively, topogenous perfect, co-perfect) order on X.

Proof. (ST2) By Lemma 2.5 (2), we have

$$(\xi_1 \circ \xi_2)(\lambda, \mu) = \bigvee_{\rho \in X} \xi_1(\lambda, \rho) \odot \xi_2(\rho, \mu) \leq \bigvee_{\rho \in X} S(\lambda, \rho) \odot S(\rho, \mu) \leq S(\lambda, \mu).$$

(CT)

$$\begin{aligned}
&(\xi_1 \circ \xi_2)(\lambda_1, \mu_1) \odot (\xi_1 \circ \xi_2)(\lambda_2, \mu_2) \\
&= \bigvee_{\rho_1 \in L^X} (\xi_1(\lambda_1, \rho_1) \odot \xi_2(\rho_1, \mu_1)) \odot \bigvee_{\rho_2 \in L^X} (\xi_1(\lambda_2, \rho_2) \odot \xi_2(\rho_2, \mu_2)) \\
&\leq \bigvee_{\rho_1, \rho_2 \in L^X} ((\xi_1(\lambda_1, \rho_1) \odot \xi_1(\lambda_2, \rho_2)) \odot (\xi_2(\rho_1, \mu_1) \odot \xi_2(\rho_2, \mu_2))) \\
&\leq \bigvee_{\rho_1, \rho_2 \in L^X} (\xi_1(\lambda_1 \oplus \lambda_2, \rho_1 \oplus \rho_2) \odot \xi_2(\rho_1 \oplus \rho_2, \mu_1 \oplus \mu_2) \leq (\xi_1 \circ \xi_2)(\lambda_1 \oplus \lambda_2, \mu_1 \oplus \mu_2).
\end{aligned}$$

Other cases are similarly proved .

Theorem 3.9. Let ξ be a co-perfect (L, M)-fuzzy cotopogenous order, then
(1) The mapping $\mathcal{F}_\xi : L^X \to M$ defined by $\mathcal{F}_\xi(\lambda) = \xi(\lambda, \lambda)$ is an (L, M)-fuzzy cotopology on X,
(2) ξ^s is a perfect (L, M)-fuzzy topogenous order.

Proof. (1) (F1) $\mathcal{F}_\xi(0_X) = \xi(0_X, 0_X) = \xi(1_X, 1_X) = \mathcal{F}_\xi(1_X) = 1$,

(F2) $\mathcal{F}_\xi(\lambda_1 \oplus \lambda_2) = \xi(\lambda_1 \oplus \lambda_2, \lambda_1 \oplus \lambda_2) \geq \xi(\lambda_1, \lambda_1) \odot \xi(\lambda_2, \lambda_2) = \mathcal{F}_\xi(\lambda_1) \odot \mathcal{F}_\xi(\lambda_2)$,

(F3) $\mathcal{F}_\xi(\bigwedge_i \lambda_i) = \xi(\bigwedge_i \lambda_i, \bigwedge_i \lambda_i) \geq \bigwedge_i \xi(\bigwedge_i \lambda_i, \lambda_i) \geq \bigwedge_i \xi(\lambda_i, \lambda_i) = \bigwedge_i \mathcal{F}_\xi(\lambda_i)$.

(2) (T) $\quad \xi^s(\lambda_1 \odot \lambda_2, \mu_1 \odot \mu_2) = \xi((\mu_1 \odot \mu_2)^*, (\lambda_1 \odot \lambda_2)^*)$
$= \xi(\mu_1^* \oplus \mu_2^*, \lambda_1^* \oplus \lambda_2^*) \geq \xi(\mu_1^*, \lambda_1^*) \odot \xi(\mu_2^*, \lambda_2^*) \geq \xi^s(\lambda_1, \mu_1) \odot \xi^s(\lambda_2, \mu_2)$.

Other cases are easily proved.

Theorem 3.10. Let $\mathcal{F}$ be an (L, M)-fuzzy cotopology on X, then
(1) The mapping $\xi_{\mathcal{F}} : L^X \times L^X \to M$ defined by

$$\xi_{\mathcal{F}}(\lambda, \mu) = \bigvee\{\mathcal{F}(\gamma) \mid \lambda \leq \gamma \leq \mu, \gamma \in L^X\}$$

is a co-perfect L-fuzzy cotopogenous space. Moreover, $\mathcal{F}_{\xi_{\mathcal{F}}} = \mathcal{F}$,

(2) If ξ is a co-perfect (L, M)-fuzzy cotopogenous order, then $\xi_{\mathcal{F}_\xi} \leq \xi$.

Proof. (1) (ST1) $\xi_{\mathcal{F}}(0_X, 0_X) = \bigvee\{\mathcal{F}(\gamma) \mid 0_X \leq \gamma \leq 0_X, \gamma \in L^X\} = \mathcal{F}(0_X) = 1$,

$$\xi_{\mathcal{F}}(1_X, 1_X) = \bigvee\{\mathcal{F}(\gamma) \mid 1_X \leq \gamma \leq 1_X, \gamma \in L^X\} = \mathcal{F}(1_X) = 1.$$

(ST2) If $\lambda \leq \gamma$, then $S(\lambda, \mu) = 1$. If $\lambda \not\leq \gamma$, then

$$\bigvee\{\mathcal{F}(\gamma) \mid \lambda \leq \gamma \leq \mu, \gamma \in L^X\} = 0.$$

It is trivial.

(ST3) If $\lambda_1 \leq \lambda$, $\mu \leq \mu_1$, then $\lambda_1 \leq \lambda \leq \gamma \leq \mu \leq \mu_1$. So, $\lambda_1 \leq \gamma \leq \mu_1$. Thus,

$$\begin{aligned}\xi_{\mathcal{F}}(\lambda, \mu) &= \bigvee\{\mathcal{F}(\gamma) \mid \lambda \leq \gamma \leq \mu, \gamma \in L^X\}\\ &\leq \bigvee\{\mathcal{F}(\gamma) \mid \lambda_1 \leq \gamma \leq \mu_1, \gamma \in L^X\} = \xi_{\mathcal{F}}(\lambda_1, \mu_1).\end{aligned}$$

(CT)

$$\begin{aligned}\xi_{\mathcal{F}}(\lambda_1, \mu_1) \odot \xi_{\mathcal{F}}(\lambda_2, \mu_2) &= \big(\bigvee\{\mathcal{F}(\gamma_1) \mid \lambda_1 \leq \gamma_1 \leq \mu_1\}\big) \odot \big(\bigvee\{\mathcal{F}(\gamma_2) \mid \lambda_2 \leq \gamma_2 \leq \mu_2\}\big)\\ &\leq \bigvee\{\mathcal{F}(\gamma_1) \odot \mathcal{F}(\gamma_2) \mid \lambda_1 \oplus \lambda_2 \leq \gamma_1 \oplus \gamma_2 \leq \mu_1 \oplus \mu_2\}\\ &\leq \bigvee\{\mathcal{F}(\gamma_1 \oplus \gamma_2) \mid \lambda_1 \oplus \lambda_2 \leq \gamma_1 \oplus \gamma_2 \leq \mu_1 \oplus \mu_2\}\\ &\leq \bigvee\{\mathcal{F}(\gamma) \mid \lambda_1 \oplus \lambda_2 \leq \gamma \leq \mu_1 \oplus \mu_2\}\\ &= \xi_{\mathcal{F}}(\lambda_1 \oplus \lambda_2, \mu_1 \oplus \mu_2).\end{aligned}$$

(ST5)

$$\xi_{\mathcal{F}}(\lambda, \bigwedge_i \mu_i) = \bigvee\{\mathcal{F}(\gamma)|\lambda \leq \gamma \leq \bigwedge_i \mu_i\} = \bigvee\{\mathcal{F}(\gamma)|\gamma = \bigwedge_i \gamma_i,\ \lambda \leq \gamma_i \leq \mu_i\}$$
$$\geq \bigvee\{\bigwedge_i \mathcal{F}(\gamma_i)|\lambda \leq \gamma_i \leq \mu_i\} = \bigwedge_i \big(\bigvee\{\mathcal{F}(\gamma_i)|\lambda \leq \gamma_i \leq \mu_i\}\big) = \bigwedge_i \xi_{\mathcal{F}}(\lambda, \mu_i).$$

Finally, $\mathcal{F}_{\xi_{\mathcal{F}}}(\lambda) = \xi_{\mathcal{F}}(\lambda, \lambda) = \bigvee\{\mathcal{F}(\gamma) \mid \lambda \leq \gamma \leq \lambda, \gamma \in L^X\} = \mathcal{F}(\lambda).$

(2) $\xi_{\mathcal{F}_\xi}(\lambda, \mu) = \bigvee\{\mathcal{F}_\xi(\gamma) \mid \lambda \leq \gamma \leq \mu\} = \bigvee\{\xi(\gamma, \gamma) \mid \lambda \leq \gamma \leq \mu\} \leq \xi(\lambda, \mu).$

Theorem 3.11. Let ξ be a perfect (L, M) -fuzzy topogenous order, then
(1) The mapping $\mathcal{T}_\xi : L^X \to M$ defined by $\mathcal{T}_\xi(\lambda) = \xi(\lambda, \lambda)$ is an L-fuzzy topology on X,
(2) ξ^s is a coperfect (L, M)-fuzzy cotopogenous order such that $\mathcal{F}_{\xi^s}(\lambda) = \mathcal{T}_\xi(\lambda^*)$,
(3) If ξ is a symmetric bi-perfect (L, M)-fuzzy bitopogenous order, then $\mathcal{T}_\xi = \mathcal{F}_\xi$.

Proof. (1) It is similarly proved as Theorem 3.9(1).
(2) $\mathcal{T}_\xi(\lambda^*) = \xi(\lambda^*, \lambda^*) = \xi^s(\lambda, \lambda) = \mathcal{F}_{\xi^s}(\lambda)$,
(3) $\mathcal{T}_\xi(\lambda) = \xi(\lambda, \lambda) = \xi^s(\lambda, \lambda) = \mathcal{F}_{\xi^s}(\lambda)$.

Theorem 3.12. Let $\mathcal{T}$ be an (L, M)-fuzzy topology on X.
(1) The mapping $\xi_{\mathcal{T}} : L^X \times L^X \to M$ defined by

$$\xi_{\mathcal{T}}(\lambda, \mu) = \bigvee\{\mathcal{T}(\gamma) \mid \lambda \leq \gamma \leq \mu, \gamma \in L^X\}$$

is a perfect (L, M)-fuzzy topogenous space. Moreover, $\mathcal{T}_{\xi_{\mathcal{T}}} = \mathcal{T}$,
(2) If $\mathcal{F}_{\mathcal{T}}(\lambda) = \mathcal{T}(\lambda^*)$ is an (L, M)-fuzzy topology on X, then $\xi_{\mathcal{F}_{\mathcal{T}}} = \xi^s_{\mathcal{T}}$.

Proof. (1) It is similarly proved as Theorem 3.10 (1).

(2) $\xi_{\mathcal{F}_{\mathcal{T}}}(\lambda, \mu) = \bigvee\{\mathcal{F}_{\mathcal{T}}(\gamma)|\lambda \leq \gamma \leq \mu\} = \bigvee\{\mathcal{T}(\gamma^*)|\mu^* \leq \gamma^* \leq \lambda^*\} = \xi_{\mathcal{T}}(\mu^*, \lambda^*) = \xi^s_{\mathcal{T}}(\lambda, \mu).$

Example 3.13. Let $(L = M = [0, 1], \odot, \to)$ be a complete residuated lattice defined by

$$x \odot y = (x + y - 1) \vee 0, \quad x \to y = (1 - x + y) \wedge 1.$$

Let $X = \{x, y\}$ be a set and $u, v \in L^X$ such that

$$u(x) = 0.6, u(y) = 0.5, \quad v(x) = 0.4, v(y) = 0.7.$$

Define $\mathcal{T}, \mathcal{F} : L^X \to M$ as follows

$$\mathcal{T}(\lambda) = \begin{cases} 1, & \text{if } \lambda \in \{1_X, 0_X\} \\ 0.6, & \text{if } \lambda = u, \\ 0.3, & \text{if } \lambda = u \odot u, \\ 0, & \text{otherwise} \end{cases} \quad , \mathcal{F}(\lambda) = \begin{cases} 1, & \text{if } \lambda \in \{1_X, 0_X\} \\ 0.7, & \text{if } \lambda = v, \\ 0.4, & \text{if } \lambda = v \oplus v, \\ 0, & \text{otherwise.} \end{cases}$$

(1) Since $0.3 = \mathcal{T}(u \odot u) \geq \mathcal{T}(u) \odot \mathcal{T}(u) = 0.2$, $\mathcal{T}$ is an (L, M)-fuzzy topology on X. By Theorem, we obtain a perfect topogenous space $\xi_{\mathcal{T}} : L^X \times L^X \to L$ as follows

$$\xi_{\mathcal{T}}(\lambda, \rho) = \begin{cases} 1, & \text{if } \lambda = 0_X \text{ or } \rho = 1_X, \\ 0.6, & \text{if } u \odot u \not\geq \lambda \leq u \leq \rho, \\ 0.3, & \text{if } 0_X \neq \lambda \leq u \odot u \leq \rho, u \not\leq \rho, \\ 0, & \text{otherwise.} \end{cases}$$

By Theorem 3.12, we obtain a co-perfect cotopogenous space $\xi^s_{\mathcal{T}} : L^X \times L^X \to L$ as follows

$$\xi^s_{\mathcal{T}}(\lambda, \rho) = \begin{cases} 1, & \text{if } \lambda = 0_X \text{ or } \rho = 1_X \\ 0.6, & \text{if } \lambda \leq u^* \leq \rho, \rho \not\geq u^* \oplus u^* \\ 0.3, & \text{if } \lambda \leq u^* \oplus u^* \leq \rho \neq 1_X, \lambda \not\leq u^*, \\ 0, & \text{otherwise.} \end{cases}$$

Moreover, $\mathcal{F}_{\xi^s_{\mathcal{T}}}(\lambda) = \mathcal{T}(\lambda^*)$.

(2) Since $0.4 = \mathcal{F}(v \oplus v) \geq \mathcal{F}(v) \odot \mathcal{F}(v) = 0.4$, $\mathcal{F}$ is an (L, M)-fuzzy cotopology on X. By Theorem 3.10, we obtain co-perfect cotopogenous order $\xi_{\mathcal{F}} : L^X \times L^X \to M$ as follows

$$\xi_{\mathcal{F}}(\lambda, \rho) = \begin{cases} 1, & \text{if } \lambda = 0_X \text{ or } \rho = 1_X \\ 0.7, & \text{if } v \oplus v \not\geq \lambda \leq v \leq \rho, \\ 0.5, & \text{if } 0_X \neq \lambda \leq v \oplus v \leq \rho, v \not\leq \rho, \\ 0, & \text{otherwise.} \end{cases}$$

By Theorem 3.10, we obtain perfect topogenous order $\xi_{\mathcal{F}} : L^X \times L^X \to M$ as follows

$$\xi^s_{\mathcal{F}}(\lambda, \rho) = \begin{cases} 1, & \text{if } \lambda = 0_X \text{ or } \rho = 1_X \\ 0.7, & \text{if } v \oplus v \not\geq \lambda \leq v^* \leq \rho, \rho \not\geq v^* \odot v^* \\ 0.5, & \text{if } \lambda \leq v^* \odot v^* \leq \rho, \lambda \not\leq v^*, \\ 0, & \text{otherwise.} \end{cases}$$

Moreover, $\mathcal{T}_{\xi^s_{\mathcal{F}}}(\lambda) = \mathcal{F}(\lambda^*)$.

Definition 3.14. Let ξ_X and ξ_Y be two (L, M)-fuzzy semi-topogenous orders on X and Y, respectively. A mapping $f : (X, \xi_X) \to (Y, \xi_Y)$ is said to be topogenous continuous if

$$\xi_Y(\lambda, \mu) \leq \xi_X(f^{\leftarrow}(\lambda), f^{\leftarrow}(\mu)), \quad \forall\ \lambda, \mu \in L^Y.$$

Theorem 3.15. Let (X, ξ_X) and (Y, ξ_Y) be perfect (L, M)-fuzzy topogenous space. If a mapping $f : (X, \xi_X) \to (Y, \xi_Y)$ is topogenous continuous, then the mapping $f : (X, \mathcal{T}_{\xi_X}) \to (Y, \mathcal{T}_{\xi_Y})$ is LF-continuous.

Conversely, a mapping $f : (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$ is LF-continuous iff $f : (X, \xi_{\mathcal{T}_X}) \to (Y, \xi_{\mathcal{T}_Y})$ is topogenous continuous.

Proof. Since $f : (X, \xi_X) \to (Y, \xi_Y)$ is LF-topogenous continuous, then

$$\mathcal{T}_{\xi_X}(f^{\leftarrow}(\lambda)) = \xi_X(f^{\leftarrow}(\lambda), f^{\leftarrow}(\lambda)) \geq \xi_Y(\lambda, \lambda) = \mathcal{T}_{\xi_Y}(\lambda).$$

Conversely, since $f : (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$ is LF-continuous, then

$$\begin{aligned}\xi_{\mathcal{T}_Y}(\lambda, \mu) = \bigvee\{\mathcal{T}_Y(\gamma) \mid \lambda \leq \gamma \leq \mu\} &\leq \bigvee\{\mathcal{T}_X(f^{\leftarrow}(\gamma)) \mid f^{\leftarrow}(\lambda) \leq f^{\leftarrow}(\gamma) \leq f^{\leftarrow}(\mu)\} \\ &\leq \bigvee\{\mathcal{T}_X(\rho) \mid \rho = f^{\leftarrow}(\gamma),\ f^{\leftarrow}(\lambda) \leq \rho \leq f^{\leftarrow}(\mu)\} \\ &= \xi_{\mathcal{T}_X}(f^{\leftarrow}(\lambda), f^{\leftarrow}(\mu)).\end{aligned}$$

Conversely, since $\mathcal{T}_{\xi_{\mathcal{T}_X}} = \mathcal{T}_X$ and $\mathcal{T}_{\xi_{\mathcal{T}_X}} = \mathcal{T}_X$ from Theorem 3.12(1), it is trivial.

Corollary 3.16. Let (X, ξ_X) and (Y, ξ_Y) be co-perfect (L, M)-fuzzy cotopogenous space. If a mapping $f : (X, \xi_X) \to (Y, \xi_Y)$ is topogenous continuous, then the mapping $f : (X, \mathcal{F}_{\xi_X}) \to (Y, \mathcal{F}_{\xi_Y})$ is LF-continuous.

Conversely, a mapping $f : (X, \mathcal{F}_X) \to (Y, \mathcal{F}_Y)$ is LF-continuous iff $f : (X, \xi_{\mathcal{F}_X}) \to (Y, \xi_{\mathcal{F}_Y})$ is topogenous continuous.

Lemma 3.17. Let $f : X \to Y$ be a mapping, then the following inequalities hold.
(1) $(f^{\rightarrow}(\mu^*))^* \leq f^{\rightarrow}(\mu)$,
(2) $S(f^{\rightarrow}(\lambda), (f^{\rightarrow}(\mu^*))^*) \leq S(\lambda, \mu), \quad \forall\ \lambda, \mu \in L^X$,
(3) $S(\lambda, \mu) \leq S(f^{\leftarrow}(\lambda), f^{\leftarrow}(\mu)) \quad \forall\ \lambda, \mu \in L^Y$,
(4) $f^{\rightarrow}(\lambda \odot \mu) \leq f^{\rightarrow}(\lambda) \odot f^{\rightarrow}(\mu)$,
(5) $f^{\rightarrow}(\lambda \oplus \mu) \leq f^{\rightarrow}(\lambda) \oplus f^{\rightarrow}(\mu)$,
(6) $(f^{\rightarrow}((\lambda \odot \mu)^*))^* \geq (f^{\rightarrow}(\lambda^*))^* \odot (f^{\rightarrow}(\mu^*))^*$,
(7) $(f^{\rightarrow}((\lambda \oplus \mu)^*))^* \geq (f^{\rightarrow}(\lambda^*))^* \oplus (f^{\rightarrow}(\mu^*))^*$.

Proof. (1)

$$(f^{\rightarrow}(\mu^*))^*(y) = (\bigvee_{x \in f^{-1}(y)} \mu^*(x))^* = \bigwedge_{x \in f^{-1}(\{y\})} \mu(x) \leq \bigvee_{x \in f^{-1}(\{y\})} \mu(x) = (f^{\rightarrow}(\mu))(y).$$

(2) Let $y_\circ \in Y$, then

$$\begin{aligned} S(f^{\rightarrow}(\lambda), (f^{\rightarrow}(\mu^*))^*) &= \bigwedge_{y \in Y} (f^{\rightarrow}(\lambda) \to (f^{\rightarrow}(\mu^*))^*)(y) \\ &\leq f^{\rightarrow}(\lambda)(y_\circ) \to (f^{\rightarrow}(\mu^*))^*(y_\circ) \\ &= \bigvee_{x \in f^{-1}(y_\circ)} \lambda(x) \to (\bigvee_{x \in f^{-1}(y_\circ)} \mu^*(x))^* \\ &= \bigvee_{x \in f^{-1}(y_\circ)} \lambda(x) \to \bigwedge_{x \in f^{-1}(y_\circ)} \mu(x) \leq \lambda(x) \to \mu(x). \end{aligned}$$

Hence, $S(f^{\rightarrow}(\lambda), (f^{\rightarrow}(\mu^*))^*) \leq S(\lambda, \mu)$.

(3)

$$S(f^{\leftarrow}(\lambda), f^{\leftarrow}(\mu)) = \bigwedge_{x\in X}(\lambda(f(x)) \to \mu(f(x))) \geq \bigwedge_{y\in Y}(\lambda(y) \to \mu(y)) = S(\lambda,\mu).$$

(4)

$$\begin{aligned} f^{\to}(\lambda \odot \mu)(y) = \bigvee_{x\in f^{-1}(\{y\})}(\lambda\odot\mu)(x) &\leq (\bigvee_{x\in f^{-1}(\{y\})}\lambda(x)) \odot (\bigvee_{x\in f^{-1}(\{y\})}\mu(x)) \\ &\leq f^{\to}(\lambda)(y) \odot f^{\to}(\mu)(y). \end{aligned}$$

(5)

$$\begin{aligned} f^{\to}(\lambda \oplus \mu)(y) = \bigvee_{x\in f^{-1}(\{y\})}(\lambda\oplus\mu)(x) &\leq (\bigvee_{x\in f^{-1}(\{y\})}\lambda(x)) \oplus (\bigvee_{x\in f^{-1}(\{y\})}\mu(x)) \\ &\leq f^{\to}(\lambda)(y) \oplus f^{\to}(\mu)(y). \end{aligned}$$

Other cases are easily proved.

Theorem 3.18. let $f : X \to Y$ be a mapping. Let ξ be an (L, M)-fuzzy topogenous (co-topogenous, perfect, co-perfect, respectively) order on Y. We define the pre-image $f^{\leftarrow}(\xi)$ of ξ under f as

$$f^{\leftarrow}(\xi)(\lambda,\mu) = \xi(f^{\to}(\lambda), (f^{\to}(\mu^*))^*), \quad \forall\ \lambda,\mu \in L^X.$$

Then,

(1) $f^{\leftarrow}(\xi)$ is an (L, M)-fuzzy topogenous (co-topogenous, perfect, co-perfect, respectively) order on X. Moreover, if $\xi \circ \xi \leq \xi$, then $f^{\leftarrow}(\xi) \circ f^{\leftarrow}(\xi) \leq f^{\leftarrow}(\xi)$.

(2) A mapping $f : (X, \xi_X) \to (Y, \xi_Y)$ is topogenous continuous if and only if $f^{\leftarrow}(\xi) \leq \xi_X$.

Proof. (1) (ST2) By Lemma 3.17, we have

$$f^{\leftarrow}(\xi)(\lambda,\mu) = \xi(f^{\to}(\lambda), (f^{\to}(\mu^*))^*) \leq S(f^{\to}(\lambda), (f^{\to}(\mu^*))^*) \leq S(\lambda,\mu).$$

(T)

$$\begin{aligned} f^{\leftarrow}(\xi)(\lambda_1 \odot \lambda_2, \mu_1 \odot \mu_2) &= \xi(f^{\to}(\lambda_1 \odot \lambda_2), (f^{\to}((\mu_1 \odot \mu_2)^*))^*) \\ &= \xi(f^{\to}(\lambda_1) \odot f^{\to}(\lambda_2), (f^{\to}(\mu_1^*))^* \odot (f^{\to}(\mu_2^*))^*) \\ &\geq \xi(f^{\to}(\lambda_1), (f^{\to}(\mu^*))^*) \odot \xi(f^{\to}(\lambda_2), (f^{\to}(\mu^*))^*) \\ &= f^{\leftarrow}(\xi)(\lambda_1,\mu_1) \odot f^{\leftarrow}(\xi)(\lambda_2,\mu_2). \end{aligned}$$

(CT)

$$\begin{aligned} f^{\leftarrow}(\xi)(\lambda_1 \oplus \lambda_2, \mu_1 \oplus \mu_2) &= \xi(f^{\to}(\lambda_1 \oplus \lambda_2), (f^{\to}((\mu_1 \oplus \mu_2)^*))^*) \quad \text{(by Lemma 3.17)} \\ &= \xi(f^{\to}(\lambda_1) \oplus f^{\to}(\lambda_2), (f^{\to}(\mu_1^*))^* \oplus (f^{\to}(\mu_2^*))^*) \\ &\geq \xi(f^{\to}(\lambda_1), (f^{\to}(\mu^*))^*) \odot \xi(f^{\to}(\lambda_2), (f^{\to}(\mu^*))^*) \\ &= f^{\leftarrow}(\xi)(\lambda_1,\mu_1) \odot f^{\leftarrow}(\xi)(\lambda_2,\mu_2). \end{aligned}$$

If $\xi \circ \xi \leq \xi$, then $f^{\leftarrow}(\xi) \circ f^{\leftarrow}(\xi) \leq f^{\leftarrow}(\xi)$ since

$$\begin{aligned} f^{\leftarrow}(\xi) \circ f^{\leftarrow}(\xi)(\lambda, \mu) &= \bigvee_{\rho \in L^X} (f^{\leftarrow}(\xi)(\lambda, \rho) \odot f^{\rightarrow}(\xi)(\rho, \mu)) \\ &= \bigvee_{\rho \in L^X} (\xi(f^{\rightarrow}(\lambda), (f^{\rightarrow}(\rho^*))^*) \odot \xi(f^{\rightarrow}(\rho), (f^{\rightarrow}(\mu^*))^*) \ (by\ Lemma\ 3.17(1)) \\ &\leq \bigvee_{\rho \in L^X} (\xi(f^{\rightarrow}(\lambda), f^{\rightarrow}(\rho)) \odot \xi(f^{\rightarrow}(\rho), (f^{\rightarrow}(\mu^*))^*) \\ &\leq \xi(f^{\rightarrow}(\lambda), (f^{\rightarrow}(\mu^*))^*) = f^{\leftarrow}(\xi)(\lambda, \mu). \end{aligned}$$

(2) For any $\rho, \nu \in L^X$, we have

$$f^{\leftarrow}(\xi)(\rho, \nu) = \xi(f^{\rightarrow}(\rho), (f^{\rightarrow}(\nu^*)^*)) \leq \xi_X(f^{\leftarrow}(f^{\rightarrow}(\rho)), f^{\leftarrow}(f^{\rightarrow}(\nu^*)^*)) \leq \xi_X(\rho, \nu),$$

$$\xi_X(f^{\leftarrow}(\lambda), f^{\leftarrow}(\mu)) \geq f^{\leftarrow}(\xi)(f^{\leftarrow}(\lambda), f^{\leftarrow}(\mu)) = \xi(f^{\rightarrow}f^{\leftarrow}(\lambda), (f^{\rightarrow}(f^{\leftarrow}(\mu))^*)^*) \geq \xi(\lambda, \mu).$$

4 Perfect (L, M)-fuzzy topogenous space and (L, M)-fuzzy quasi-proximities

Kim et al [30] introduced the concept of L-fuzzy proximities in a strictly two sided, commutative quantales. We here reintroduce them in a slightly different way as follows.

Definition 4.1. A mapping $\delta : L^X \times L^X \to M$ is called an (L, M)-fuzzy quasi-proximity on X if it satisfies the following axioms.

(QP1) $\delta(0_X, 1_X) = \delta(1_X, 0_X) = 0_M$,

(QP2) $\delta(\lambda, \mu) \geq \bigvee_{x \in X} (\lambda \odot \mu)(x)$,

(QP3) If $\lambda_1 \leq \lambda_2, \rho_1 \leq \rho_2$, then $\delta(\lambda_1, \rho_1) \leq \delta(\lambda_2, \rho_2)\ \forall\ \rho \in L^X$,

(QP4) $\delta(\lambda_1 \odot \lambda_2, \rho_1 \oplus \rho_2) \leq \delta(\lambda_1, \rho_1) \oplus \delta(\lambda_2, \rho_2)$,

(QP5) $\delta(\lambda, \mu) \geq \bigwedge_{\rho} \{\delta(\lambda, \rho) \oplus \delta(\mu, \rho^*)\}$.

The pair (X, δ) is called an (L, M)-fuzzy quasi-proximity space. We call $\delta(\lambda, \mu)$ a gradation of nearness.

Let δ_1 and δ_2 be (L, M)-fuzzy quasi-proximities on X. Then δ_1 is called coarser than δ_2 if $\delta_2(\lambda, \mu) \leq \delta_1(\lambda, \mu)$ for all $\lambda, \mu \in L^X$.

An (L, M)-fuzzy quasi-proximity is called (L, M)-fuzzy proximity on X if it satisfies the following axiom

(P) $\delta(\lambda, \mu) = \delta(\mu, \lambda)$.

An (L, M)-fuzzy quasi-proximity is called perfect if it satisfies the following axiom

(PP) $\delta(\bigvee_{i\in\Gamma}\lambda_i,\mu)=\bigvee_{i\in\Gamma}\delta(\lambda_i,\rho)$.

An (L,M)-fuzzy quasi-proximity is called co-perfect if it satisfies the following axiom

(CPP) $\delta(\lambda,\bigvee_{i\in\Gamma}\rho_i)=\bigvee_{i\in\Gamma}\delta(\lambda,\rho_i)$.

Proposition 4.2. (1) If δ is an (L,M)-fuzzy quasi-proximity space and we define $\delta^s : L^X\times L^X\to M$ by

$$\delta^s(\lambda,\mu)=\delta(\mu^*,\lambda^*),\quad \forall\ \lambda,\mu\in L^X,$$

then δ^s is an (L,M)-fuzzy quasi-proximity space.

(2) If (X,ξ) is a perfect (L,M)-fuzzy topogenous space and we define $\delta_\xi : L^X\times L^X\to M$ by

$$\delta_\xi(\lambda,\mu)=\xi^*(\lambda,\mu^*)\ \ \forall\ \lambda,\mu\in L^X,$$

then δ_ξ is a perfect (L,M)-fuzzy quasi-proximity space on X. Moreover, if ξ is symmetric, then δ_ξ is a bi-perfect (L,M)-fuzzy proximity space on X.

(3) If (X,ξ) is a co-perfect (L,M)-fuzzy co-topogenous space and we define δ_ξ : $L^X\times L^X\to M$ by

$$\delta_\xi(\lambda,\mu)=\xi^*(\mu,\lambda^*)\ \ \forall\ \lambda,\mu\in L^X,$$

then δ_ξ is a co-perfect (L,M)-fuzzy quasi-proximity space on X. Moreover, if ξ is symmetric, then δ_ξ is a bi-perfect (L,M)-fuzzy proximity space on X.

(4) If δ is an (resp. perfect) (L,M)-fuzzy quasi-proximity space and we define $\xi_\delta : L^X\times L^X\to M$ by

$$\xi_\delta(\lambda,\mu)=\delta^*(\lambda,\mu^*)\ \ \forall\ \lambda,\mu\in L^X,$$

then ξ_δ is an (resp. perfect) (L,M)-fuzzy topogenous space such that $\delta_{\xi_\delta}=\delta$. Moreover, if ξ is an (resp. perfect) (L,M)-fuzzy topogenous space, then $\xi_{\delta_\xi}=\xi$.

(5) If δ is an (resp. co-perfect) (L,M)-fuzzy quasi-proximity space and we define $\xi_\delta : L^X\times L^X\to M$ by

$$\xi_\delta(\lambda,\mu)=\delta^*(\mu^*,\lambda)\ \ \forall\ \lambda,\mu\in L^X,$$

then ξ_δ is an (resp. co-perfect) (L,M)-fuzzy co-topogenous space such that $\delta_{\xi_\delta}=\delta$. Moreover, if ξ is an (resp. co-perfect) (L,M)-fuzzy co-topogenous space, then $\xi_{\delta_\xi}=\xi$.

Proof. (1) It is easily proved.

(2) (QP1) $\delta_\xi(1_X,0_X)=\xi^*(1_X,0_X^*)=\xi^*(1_X,0_X)=0_M$. Similarly, $\delta_\xi(0_X,1_X)=0$.

(QP2) By Definition 3.1 (ST2) and Lemma 2.3 (16), we have

$$\delta_\xi(\lambda,\mu) \geq (S(\lambda,\mu^*))^* = (\bigwedge_{x\in X}(\lambda(x)\rightarrow\mu^*(x))^* = \bigvee_{x\in X}(\lambda(x)\rightarrow\mu^*(x))^* = \bigvee_{x\in X}(\lambda\odot\mu)(x).$$

(QP3) If $\lambda \geq \mu$, then

$$\xi(\lambda,\rho^*) \leq \xi(\mu,\rho^*) \text{ iff } \xi^*(\mu,\rho^*) \leq \xi^*(\lambda,\rho^*), \text{ then } \delta_\xi(\mu,\rho) \leq \delta_\xi(\lambda,\rho).$$

(QP4)

$$\begin{aligned}\delta_\xi(\lambda_1\odot\lambda_2,\rho_1\oplus\rho_2) &= \xi^*(\lambda_1\odot\lambda_2,(\rho_1\oplus\rho_2)^*) = \xi^*(\lambda_1\odot\lambda_2,\rho_1^*\odot\rho_2^*)\\ &\leq \xi^*(\lambda_1,\rho_1^*)\oplus\xi^*(\lambda_2,\rho_2^*) = \delta_\xi(\lambda_1,\rho_1)\oplus\delta_\xi(\lambda_2,\rho_2).\end{aligned}$$

(QP5) Since $\xi\circ\xi \geq \xi$ by definition 3.7, then

$$\begin{aligned}\delta_\xi(\lambda,\mu) = \xi^*(\lambda,\mu^*) \geq (\xi\circ\xi)^*(\lambda,\mu^*) &= \big(\bigvee_{\gamma\in L^X}\xi(\lambda,\gamma)\odot\xi(\gamma,\mu^*)\big)^*\\ &= \bigwedge_{\gamma\in L^X}\xi^*(\lambda,\gamma)\oplus\xi^*(\gamma,\mu^*) = \bigwedge_{\gamma\in L^X}\delta_\xi(\lambda,\gamma^*)\oplus\delta_\xi(\gamma,\mu).\end{aligned}$$

(PP) $\delta_\xi(\bigvee_{i\in\Gamma}\lambda_i,\mu) = \xi^*(\bigvee_{i\in\Gamma}\lambda_i,\mu^*) = \bigvee_{i\in\Gamma}\xi^*(\lambda_i,\mu^*) = \bigvee_{i\in\Gamma}\delta_\xi(\lambda_i,\mu).$

Let $\xi = \xi^s$ be given, then ξ is co-perfect by

$$\xi(\lambda,\bigwedge_{i\in\Gamma}\rho_i) = \xi^s(\lambda,\bigwedge_{i\in\Gamma}\rho_i) = \xi(\bigvee_{i\in\Gamma}\rho_i^*,\lambda^*) = \bigwedge_{i\in\Gamma}\xi(\rho_i^*,\lambda^*) = \bigwedge_{i\in\Gamma}\xi^s(\lambda,\rho_i) = \bigwedge_{i\in\Gamma}\xi(\lambda,\rho_i).$$

(P) $\delta_\xi(\lambda,\mu) = \xi^*(\lambda,\mu^*) = (\xi^s)^*(\lambda,\mu^*) = \xi^*(\mu,\lambda^*) = \delta_\xi(\mu,\lambda).$

(CPP) $\delta_\xi(\lambda,\bigvee_{i\in\Gamma}\rho_i) = \xi^*(\lambda,\bigwedge_{i\in\Gamma}\rho_i^*) = \bigvee_{i\in\Gamma}\xi^*(\lambda,\rho_i^*) = \bigvee_{i\in\Gamma}\delta_\xi(\lambda,\rho_i)$. Hence δ_ξ is a biperfect (L,M)-fuzzy proximity space on X.

(3) It is similarly proved as (2).

(QP4)

$$\begin{aligned}\delta_\xi(\lambda_1\odot\lambda_2,\rho_1\oplus\rho_2) &= \xi^*(\rho_1\oplus\rho_2,(\lambda_1\odot\lambda_2)^*) = \xi^*(\rho_1\oplus\rho_2,\lambda_1^*\oplus\lambda_2^*)\\ &\leq \xi^*(\rho_1,\lambda_1^*)\oplus\xi^*(\rho_2,\lambda_2^*) = \delta_\xi(\lambda_1,\rho_1)\oplus\delta_\xi(\lambda_2,\rho_2).\end{aligned}$$

Other cases are similarly proved as (2).

(4) (ST1) $\xi_\delta(1_X,1_X) = \delta^*(1_X,1_X^*) = \delta^*(1_X,0_X) = 0^* = 1_M,$

$$\xi_\delta(0_X,0_X) = \delta^*(0_X,0_X^*) = \delta^*(0_X,1_X) = 0^* = 1_M.$$

(ST2) From Lemma 2.3 (16), we have

$$\xi_\delta(\lambda,\mu)=\delta^*(\lambda,\mu^*)\le\big(\bigvee_{x\in X}(\lambda\odot\mu^*)(x)\big)^*=\bigwedge_{x\in X}(\lambda\odot\mu^*)^*(x)$$
$$=\bigwedge_{x\in X}(\lambda(x)\to\mu(x))=S(\lambda,\mu).$$

(ST3) If $\lambda_1\le\lambda,\ \mu\le\mu_1$, then from (QP3) and (QP6)

$$\xi_\delta(\lambda,\mu)=\delta^*(\lambda,\mu^*)\ge\delta^*(\lambda_1,\mu^*)=\delta^*(\mu^*,\lambda_1)\ge\delta^*(\mu_1^*,\lambda_1)=\delta^*(\lambda_1,\mu_1^*)=\xi_\delta(\lambda_1,\mu_1).$$

(ST4) Obviously, $\xi_\delta(\lambda,\mu)=\delta^*(\lambda,\mu^*)=\delta^*(\mu^*,\lambda)=\xi_\delta(\mu^*,\lambda^*)=\xi_\delta^s(\lambda,\mu)$.

(T)

$$\xi_\delta(\lambda_1,\mu_1)\odot\xi_\delta(\lambda_2,\mu_2)=\delta^*(\lambda_1,\mu_1^*)\odot\delta^*(\lambda_2,\mu_2^*)=\big(\delta(\lambda_1,\mu_1^*)\oplus\delta(\lambda_2,\mu_2^*)\big)^*$$
$$\le\delta^*(\lambda_1\odot\lambda_2,\mu_1^*\oplus\mu_2^*)=\delta^*(\lambda_1\odot\lambda_2,(\mu_1\odot\mu_2)^*)=\xi_\delta(\lambda_1\odot\lambda_2,\mu_1\odot\mu_2).$$

$$\delta_{\xi_\delta}(\lambda,\mu)=\xi_\delta^*(\lambda,\mu^*)=\delta(\lambda,\mu),\quad \xi_{\delta_\xi}(\lambda,\mu)=\delta_\xi^*(\lambda,\mu^*)=\xi(\lambda,\mu).$$

(5) (CT)

$$\xi_\delta(\lambda_1,\mu_1)\odot\xi_\delta(\lambda_2,\mu_2)=\delta^*(\mu_1^*,\lambda_1)\odot\delta^*(\mu_2^*,\lambda_2)=\big(\delta(\mu_1^*,\lambda_1)\oplus\delta(\mu_2^*,\lambda_2)\big)^*$$
$$\le\delta^*(\mu_1^*\odot\mu_2^*,\lambda_1\oplus\lambda_2)=\delta^*((\mu_1\oplus\mu_2)^*,\lambda_1\oplus\lambda_2)=\xi_\delta(\lambda_1\oplus\lambda_2,\mu_1\oplus\mu_2).$$

$$\delta_{\xi_\delta}(\lambda,\mu)=\xi_\delta^*(\mu,\lambda^*)=\delta(\lambda,\mu),\quad \xi_{\delta_\xi}(\lambda,\mu)=\delta_\xi^*(\mu^*,\lambda)=\xi(\lambda,\mu).$$

Theorem 4.3. Let δ be an (L,M)-fuzzy quasi-proximity on X, then

(1) If δ is perfect and the mapping $\mathcal{T}_\delta: L^X\to M$ defined by $\mathcal{T}_\delta(\lambda)=\delta^*(\lambda,\lambda^*)$, then $\mathcal{T}_\delta$ is an (L,M)-fuzzy topology on X.

(2) If δ is co-perfect and the mapping $\mathcal{F}_\delta: L^X\to M$ defined by $\mathcal{F}_\delta(\lambda)=\delta^*(\lambda^*,\lambda)$, then $\mathcal{F}_\delta$ is an (L,M)-fuzzy cotopology on X.

(3) If δ is a perfect (L,M)-fuzzy proximity on X, then $\mathcal{T}_\delta(\lambda)=\mathcal{F}_\delta(\lambda)$.

Proof. (1) Let δ be a perfect (L,M)-fuzzy quasi-proximity on X and define $\xi_\delta(\lambda,\mu)=\delta^*(\lambda,\mu^*)$, then ξ_δ a perfect (L,M)-fuzzy topogenuous and

$$\mathcal{T}_\delta(\lambda)=\delta^*(\lambda,\lambda^*)=\xi_\delta(\lambda,\lambda).$$

Hence $\mathcal{T}_\delta$ is an (L,M)-fuzzy topology on X.

(2) It is easily proved as $\mathcal{T}_\delta(\lambda)=\delta^*(\lambda,\lambda^*)=\delta^*(\lambda^*,\lambda)=\mathcal{T}_\delta(\lambda)$.

Theorem 4.4. Let $\mathcal{F}$ be an (L,M)-fuzzy co-topology on X, then

(1) The mapping $\delta_{\mathcal{F}} : L^X \times L^X \to M$ defined by

$$\delta_{\mathcal{F}}(\lambda,\mu) = \bigwedge\{(\mathcal{F}(\gamma))^* \mid \mu \le \gamma \le \lambda^*\}$$

is a co-perfect (L,M)-fuzzy quasi-proximity space. Moreover, $\mathcal{F}_{\delta_{\mathcal{F}}} = \mathcal{F}$.

(2) If δ is a co-perfect (L,M)-fuzzy quasi-proximity on X, then $\delta_{\mathcal{F}_\delta} \ge \delta$.

Theorem 4.5. Let $\mathcal{T}$ be an (L,M)-fuzzy topology on X, then

(1) The mapping $\delta_{\mathcal{T}} : L^X \times L^X \to M$ defined by

$$\delta_{\mathcal{T}}(\lambda,\mu) = \bigwedge\{(\mathcal{T}(\gamma))^* \mid \lambda \le \gamma \le \mu^*\}$$

is a perfect (L,M)-fuzzy quasi-proximity space. Moreover, $\mathcal{T}_{\delta_{\mathcal{T}}} = \mathcal{T}$.

(2) If $\mathcal{F}_{\mathcal{T}}(\lambda) = \mathcal{T}(\lambda^*)$ is an (L,M)-fuzzy topology on X, then $\delta_{\mathcal{F}_{\mathcal{T}}} = \delta^s_{\mathcal{T}}$.

Example 4.6. Let ξ_i be given as Example 3.13 and since $\delta_{\xi_i}(\lambda,\rho) = \xi_i^*(\lambda,\rho^*)$, then we have

$$\delta_{\xi_1}(\lambda,\rho) = S^*(\lambda,\rho^*) = \bigvee_{x\in X}(\lambda\odot\rho)(x),$$

$$\delta_{\xi_2}(\lambda,\rho) = \begin{cases} 0, & \text{if } \lambda = 0_X, \text{ or } \rho = 0_X, \\ 1, & \text{otherwise} \end{cases}, \delta_{\xi_3}(\lambda,\rho) = \begin{cases} 0, & \text{if } \lambda \le \rho^*, \\ 1, & \text{otherwise.} \end{cases}$$

Example 4.7. Let $\mathcal{T},\mathcal{F}$ be given as Example 3.13.

(1) By Theorems 4.2(2) and 4.5, we obtain a perfect (L,M)-quasi-proximity $\delta_{\xi_{\mathcal{T}}} = \delta_{\mathcal{T}} : L^X \times L^X \to M$ as follows

$$\delta_{\xi_{\mathcal{T}}}(\lambda,\rho) = \begin{cases} 0, & \text{if } \lambda = 0_X \text{ or } \rho = 0_X \\ 0.4, & \text{if } u\odot u \not\geq \lambda \le u \le \rho^*, \\ 0.7, & \text{if } 0_X \ne \lambda \le u\odot u \le \rho^*, u \not\le \rho^*, \\ 1, & \text{otherwise.} \end{cases}$$

By Theorems 4.2(2) and 4.5, we obtain a co-perfect (L,M)-quasi-proximity $\delta_{\xi^s_{\mathcal{T}}} = \delta_{\xi_{\mathcal{T}^*}} : L^X \times L^X \to M$ with $\mathcal{T}^*(\lambda) = \mathcal{T}(\lambda^*)$ as follows

$$\delta_{\xi^s_{\mathcal{T}}}(\lambda,\rho) = \begin{cases} 0, & \text{if } \lambda = 0_X \text{ or } \rho = 0_X \\ 0.4, & \text{if } \lambda \le u^* \le \rho^*, \rho^* \not\geq u^*\oplus u^* \\ 0.7, & \text{if } \lambda \le u^*\oplus u^* \le \rho^* \ne 1_X, \lambda \not\le u^*, \\ 1, & \text{otherwise.} \end{cases}$$

Moreover, $\mathcal{F}_{\delta_{\xi^s_{\mathcal{T}}}}(\lambda) = \mathcal{T}(\lambda^*)$.

(2) By Theorems 4.2(2) and 4.4, we obtain co-perfect (L, M)-quasi-proximity $\delta_{\xi_{\mathcal{F}}} = \delta_{\mathcal{F}} : L^X \times L^X \to M$ as follows

$$\delta_{\xi_{\mathcal{F}}}(\lambda, \rho) = \begin{cases} 0, & \text{if } \lambda = 0_X \text{ or } \rho = 0_X \\ 0.3, & \text{if } v \oplus v \not\geq \lambda \leq v \leq \rho^*, \\ 0.5, & \text{if } 0_X \neq \lambda \leq v \oplus v \leq \rho^*, v \not\leq \rho^*, \\ 1, & \text{otherwise.} \end{cases}$$

By Theorems 4.2(2) and 4.4, we obtain perfect (L, M)-quasi-proximity $\delta_{\xi^s_{\mathcal{F}}} = \delta_{\mathcal{F}^*} : L^X \times L^X \to M$ with $\mathcal{F}^*(\lambda) = \mathcal{F}(\lambda^*)$ as follows

$$\delta_{\xi^s_{\mathcal{F}}}(\lambda, \rho) = \begin{cases} 0, & \text{if } \lambda = 0_X \text{ or } \rho = 0_X \\ 0.3, & \text{if } v \oplus v \not\geq \lambda \leq v^* \leq \rho^*, \rho^* \not\geq v^* \odot v^* \\ 0.5, & \text{if } \lambda \leq v^* \odot v^* \leq \rho^*, \lambda \not\leq v^*, \\ 1, & \text{otherwise.} \end{cases}$$

Moreover, $\mathcal{T}_{\delta_{\xi^s_{\mathcal{F}}}}(\lambda) = \mathcal{F}(\lambda^*)$.

Definition 4.8. Let (X, δ_X) and (Y, δ_Y) be two (L, M)-fuzzy quasi-proximity spaces. A mapping $f : (X, \delta_X) \to (Y, \delta_Y)$ is said to be L-fuzzy proximally continuous if

$$\delta_X(\lambda, \mu) \leq \delta_Y(f^{\to}(\lambda), f^{\to}(\mu)), \quad \forall\ \lambda, \mu \in L^X,$$

or equivalently, $\quad \delta_X(f^{\leftarrow}(\lambda), f^{\leftarrow}(\mu)) \leq \delta_Y(\lambda, \mu)$.

Theorem 4.9. A mapping $f : (X, \delta_X) \to (Y, \delta_Y)$ of two (L, M)-fuzzy quasi-proximity spaces is L-fuzzy proximally continuous iff the the mapping $f : (X, \xi_{\delta_X}) \to (Y, \xi_{\delta_Y})$ is topogenous continuous.

Conversely, a mapping $f : (X, \xi_X) \to (Y, \xi_Y)$ of (L, M)-fuzzy topogenous spaces is topogenous continuous iff the mapping $f : (X, \delta_{\xi_X}) \to (Y, \delta_{\xi_Y})$ of the corresponding (L, M)-fuzzy quasi-proximity spaces is L-fuzzy proximally continuous.

Proof. Since $f : (X, \delta_X) \to (Y, \delta_Y)$ is L-fuzzy proximally continuous, then

$$\begin{aligned} \xi_{\delta_X}(f^{\leftarrow}(\lambda), f^{\leftarrow}(\mu)) &= \delta^*_X(f^{\leftarrow}(\lambda), (f^{\leftarrow}(\mu))^*) \\ &= \delta^*_X(f^{\leftarrow}(\lambda), f^{\leftarrow}(\mu^*)) \leq \delta^*_Y(\lambda, \mu^*) = \xi_{\delta_Y}(\lambda, \mu). \end{aligned}$$

Conversely, Since $f : (X, \xi_X) \to (Y, \xi_Y)$ is topogenous continuous, then

$$\begin{aligned} \delta_{\xi_X}(f^{\leftarrow}(\lambda), f^{\leftarrow}(\mu)) &= \xi^*_X(f^{\leftarrow}(\lambda), (f^{\leftarrow}(\mu))^*) \\ &= \xi^*_X(f^{\leftarrow}(\lambda), f^{\leftarrow}(\mu^*)) \leq \xi^*_Y(\lambda, \mu^*) = \delta_{\xi_Y}(\lambda, \mu). \end{aligned}$$

Theorem 4.10. Let (Y, δ) be an (L, M)-fuzzy quasi-proximity space, X be a non-empty set and $f : X \to Y$ be a mapping. We define $\delta_f : L^X \times L^X \to M$ by

$$\delta_f(\lambda, \mu) = \delta(f^{\to}(\lambda), f^{\to}(\mu)), \quad \forall\ \lambda, \mu \in L^X.$$

Then,

(1) δ_f is the coarsest (L, M)-fuzzy quasi-proximity for which f is L-fuzzy proximally continuous,

(2) A mapping $g : (Z, \xi) \to (X, \delta_f)$ is L-fuzzy proximally continuous iff $f \circ g$ is L-fuzzy proximally continuous.

Proof. (QP1) $\delta_f(1_X, 0_X) = \delta(f^{\to}(1_X), f^{\to}(0_X)) \leq \delta(1_Y, 0_Y) = 0_M$. Similarly,

$$\delta_f(0_X, 1_X) = 0_M.$$

(QP2)

$$\delta_f(\lambda, \mu) = \delta(f^{\to}(\lambda), f^{\to}(\mu)) \geq \bigvee_{y \in Y} (f^{\to}(\lambda) \odot f^{\to}(\mu))(y)$$

$$\geq \bigvee_{x \in f^{\leftarrow}(y_\circ)} \lambda(x) \odot \bigvee_{x \in f^{\leftarrow}(y_\circ)} \mu(x) \geq \bigvee_{x \in X} \lambda(x) \odot \mu(x) = \bigvee_{x \in X} (\lambda \odot \mu)(x).$$

(QP3) If $\lambda \leq \mu$, then $\delta_f(\lambda, \rho) = \delta(f^{\to}(\lambda), f^{\to}(\rho)) \leq \delta(f^{\to}(\mu), f^{\to}(\rho)) = \delta_f(\mu, \rho)$.

(QP4)

$$\begin{aligned}\delta_f(\lambda_1, \rho_1) \oplus \delta_f(\lambda_2, \rho_2) &= \delta(f^{\to}(\lambda_1), f^{\to}(\rho_1)) \oplus \delta(f^{\to}(\lambda_2), f^{\to}(\rho_2)) \\ &\geq \delta(f^{\to}(\lambda_1) \odot f^{\to}(\lambda_2), f^{\to}(\rho_1) \oplus f^{\to}(\rho_2)) \\ &\geq \delta(f^{\to}(\lambda_1 \odot \lambda_2), f^{\to}(\rho_1 \oplus \rho_2)) = \delta_f(\lambda_1 \odot \lambda_2, \rho_1 \oplus \rho_2).\end{aligned}$$

(QP5) Since $\delta_f(\lambda, (f^{\leftarrow}(\rho))^*) = \delta(f^{\to}(\lambda), f^{\to}(f^{\leftarrow}(\rho^*))) \leq \delta(f^{\to}(\lambda), \rho^*)$, then we have

$$\begin{aligned}\delta_f(\lambda, \mu) = \delta(f^{\to}(\lambda), f^{\to}(\mu)) &\geq \bigwedge_{\rho \in L^X} \delta(f^{\to}(\lambda), \rho) \oplus \delta(f^{\to}(\mu), \rho^*) \\ &\geq \bigwedge_{f^{\leftarrow}(\rho) \in L^X} \delta_f(\lambda, f^{\leftarrow}(\rho)) \oplus \delta_f(\mu, (f^{\to}(\rho))^*) \\ &\geq \bigwedge_{\gamma \in L^X} \delta_f(\lambda, \gamma) \oplus \delta_f(\mu, \gamma^*).\end{aligned}$$

From the definition of δ_f, f is L-fuzzy proximally continuous. Let $f : (X, \delta_1) \to (Y, \delta)$ be L-fuzzy proximally continuous, and since

$$\delta_1(\lambda, \mu) \leq \delta(f^{\to}(\lambda), f^{\to}(\mu)) = \delta_f(\lambda, \mu).$$

Then, δ_f is coarser than δ_1.

(2) Let g be L-fuzzy proximally continuous. So,

$$\xi(\lambda, \mu) \leq \delta_f(g^{\to}(\lambda), g^{\to}(\mu)) = \delta(f^{\to}(g^{\to}(\lambda)), f^{\to}(g^{\to}(\mu))).$$

Hence, $f \circ g$ is L-fuzzy proximally continuous. Let $f \circ g$ be L-fuzzy proximally continuous, then

$$\xi(\lambda, \mu) \leq \delta(f^{\to}(g^{\to}(\lambda)), f^{\to}(g^{\to}(\mu))) = \delta_f(g^{\to}(\lambda), g^{\to}(\mu)).$$

Then g is L-fuzzy proximally continuous.

5 (L, M)-fuzzy topogenous order induced by (L, M)-fuzzy quasi uniformity

Definition 5.1. [31, 47] A mapping $\mathcal{U} : L^{X\times X} \to M$ is called an (L, M)-fuzzy quasi-uniformity on X iff it satisfies the properties.

(LU1) There exists $u \in L^{X\times X}$ such that $\mathcal{U}(u) = 1_M$,
(LU2) If $v \leq u$, then $\mathcal{U}(v) \leq \mathcal{U}(u)$,
(LU3) For every $u, v \in L^{X\times X}$, $\mathcal{U}(u \odot v) \geq \mathcal{U}(u) \odot \mathcal{U}(v)$,
(LU4) If $\mathcal{U}(u) \neq 0_M$, then $1_\triangle \leq u$, where

$$1_\triangle(x, y) = \begin{cases} 1 & \text{if x = y} \\ 0 & \text{if } x \neq y, \end{cases}$$

(LU5) $\mathcal{U} \leq \mathcal{U} \circ \mathcal{U}$, where $\mathcal{U} \circ \mathcal{U}(u) = \bigvee\{\mathcal{U}(v) \odot \mathcal{U}(w) \mid v \circ w \leq u\}$,

$$v \circ w(x, y) = \bigvee_{z\in X} (v(z, x) \odot v(x, y)), \quad \forall\ x, y \in X.$$

Remark 5.2. Let $(X,\mathcal{U})$ be an (L, M)-fuzzy quasi-uniform space, then by (LU1) and (LU2), we have $\mathcal{U}(1_{X\times X}) = 1_M$ because $u \leq 1_{X\times X}$ for all $u \in L^{X\times X}$.

Definition 5.3. [31, 47] Let $(X,\mathcal{U})$ and $(Y,\mathcal{V})$ be (L, M)-fuzzy uniform spaces, and $\phi : X \to Y$ ba a mapping. Then ϕ is said to be L-uniformly continuous if

$$\mathcal{V}(v) \leq \mathcal{U}((\phi \times \phi)^{\leftarrow}(v)),$$

for every $v \in L^{Y\times Y}$.

Lemma 5.4. [31] Let $(X,\mathcal{U})$ be an (L, M)-fuzzy quasi-uniform space. For each $u \in L^{X\times X}$ and $\lambda \in L^X$, the image $u[\lambda]$ of λ with respect to u is the fuzzy subset of X defined by

$$u[\lambda](x) = \bigvee_{y\in X} (\lambda(y) \odot u(y, x)), \quad \forall\ x \in X.$$

For each $u, v, u_1, u_2 \in L^{X\times X}$ and $\lambda, \rho, \lambda_1, \lambda_2, \lambda_i \in L^X$, we have

(1) $\lambda \leq u[\lambda]$, for each $\mathcal{U}(u) > 0_M$,
(2) $u \leq u \circ u$, for each $\mathcal{U}(u) > 0_M$,
(3) $(v \circ u)[\lambda] = v[u[\lambda]]$,
(4) $u[\bigvee_i \lambda_i] = \bigvee_i u[\lambda_i]$,
(5) $(u_1 \odot u_2)[\lambda_1 \odot \lambda_2] \leq u_1[\lambda_1] \odot u_2[\lambda_2]$,
(6) $(u_1 \odot u_2)[\lambda_1 \oplus \lambda_2] \leq u_1[\lambda_1] \oplus u_2[\lambda_2]$.

Theorem 5.5. Let $(X,\mathcal{U})$ be an (L,M)-fuzzy quasi-uniform space. Define a mapping $\xi_{\mathcal{U}} : L^X \times L^X \to M$ by

$$\xi_{\mathcal{U}}(\lambda,\mu) = \bigvee\{\mathcal{U}(u) \mid u[\lambda] \leq \mu\}.$$

Then $(X,\xi_{\mathcal{U}})$ is an (L,M)-fuzzy topogenous space.

Proof. . (ST1) Since $u[0_X] = 0_X$ and $u[1_X] = 1_X$, for $\mathcal{U}(u) = 1_M$, we have $\xi_{\mathcal{U}}(1_X,1_X) = \xi_{\mathcal{U}}(0_X,0_X) = 1_M$.

(ST2) Since for all $\mathcal{U}(u) > 0_M$, we have $\lambda \leq u[\lambda]$. Then if $\xi_{\mathcal{U}}(\lambda,\mu) = 1_M$, we have $\lambda \leq \mu$.

(ST3) If $\lambda_1 \leq \lambda,\ \mu \leq \mu_1$, then

$$\begin{aligned}\xi_{\mathcal{U}}(\lambda,\mu) = \bigvee\{\mathcal{U}(u) \mid u[\lambda] \leq \mu\} &\leq \bigvee\{\mathcal{U}(u) \mid u[\lambda] \leq \mu_1\}\\ &\leq \bigvee\{\mathcal{U}(u) \mid u[\lambda_1] \leq \mu_1\} = \xi_{\mathcal{U}}(\lambda_1,\mu_1).\end{aligned}$$

(T)

$$\begin{aligned}\xi_{\mathcal{U}}(\lambda_1,\mu_1) \odot \xi_{\mathcal{U}}(\lambda_2,\mu_2) &= \bigvee\{\mathcal{U}(u) \mid u[\lambda_1] \leq \mu_1\} \odot \bigvee\{\mathcal{U}(v) \mid v[\lambda_2] \leq \mu_2\}\\ &\leq \bigvee\{\mathcal{U}(u) \odot \mathcal{U}(v) \mid u[\lambda_1] \odot v[\lambda_2] \leq \mu_1 \odot \mu_2\}\\ &\leq \bigvee\{\mathcal{U}(u \odot v) \mid (u \odot v)[\lambda_1 \odot \lambda_2] \leq \mu_1 \odot \mu_2\}\\ &\leq \bigvee\{\mathcal{U}(w) \mid w[\lambda_1 \odot \lambda_2] \leq \mu_1 \odot \mu_2\}\\ &= \xi_{\mathcal{U}}(\lambda_1 \odot \lambda_2, \mu_1 \odot \mu_2).\end{aligned}$$

(CT)

$$\begin{aligned}\xi_{\mathcal{U}}(\lambda_1,\mu_1) \odot \xi_{\mathcal{U}}(\lambda_2,\mu_2) &= \bigvee\{\mathcal{U}(u) \mid u[\lambda_1] \leq \mu_1\} \odot \bigvee\{\mathcal{U}(v) \mid v[\lambda_2] \leq \mu_2\}\\ &\leq \bigvee\{\mathcal{U}(u) \odot \mathcal{U}(v) \mid u[\lambda_1] \oplus v[\lambda_2] \leq \mu_1 \oplus \mu_2\}\\ &\leq \bigvee\{\mathcal{U}(u \odot v) \mid u \odot v[\lambda_1 \oplus \lambda_2] \leq \mu_1 \oplus \mu_2\}\\ &= \xi_{\mathcal{U}}(\lambda_1 \oplus \lambda_2, \mu_1 \oplus \mu_2).\end{aligned}$$

(TS) For each $u \in L^{X\times X}$ such that $u[\lambda] \leq \mu$, by (LU5), we have

$$\mathcal{U}(u) = \bigvee\{\mathcal{U}(v) \odot \mathcal{U}(w) \mid v \circ w \leq u\}.$$

Thus,

$$\begin{aligned}\bigvee\{\mathcal{U}(u) \mid u[\lambda] \le \mu\} &\le \bigvee\{\mathcal{U}(v) \odot \mathcal{U}(w) \mid v \circ w[\lambda] = v[w[\lambda]] \le \mu\}\\ &\le \bigvee_{\gamma \in L^X} \{\bigvee\{\mathcal{U}(v) \odot \mathcal{U}(w) \mid w[\lambda] \le \gamma,\ v[\gamma] \le \mu\}\}\\ &\le \bigvee_{\gamma \in L^X} \{\bigvee\{\mathcal{U}(v) \mid v[\gamma] \le \mu\} \odot \bigvee\{\mathcal{U}(w) \mid w[\lambda] \le \gamma\}\}\\ &= \bigvee_{\gamma \in L^X} \xi_{\mathcal{U}}(\lambda, \gamma) \odot \xi_{\mathcal{U}}(\gamma, \mu).\end{aligned}$$

Example 5.6. Let $(L = M = [0, 1], \odot, \to)$ be a complete residuated lattice defined as

$$x \odot y = (x + y - 1) \vee 0, \quad x \to y = (1 - x + y) \wedge 1.$$

Let $X = \{x, y, z\}$ be a set and $w \in L^{X\times X}$ such that

$$w = \begin{pmatrix} 1 & 0.5 & 0.3 \\ 0.7 & 1 & 0.5 \\ 0.6 & 0.6 & 1 \end{pmatrix}, \quad w \odot w = \begin{pmatrix} 1 & 0 & 0 \\ 0.4 & 1 & 0 \\ 0.2 & 0.2 & 1 \end{pmatrix}.$$

Define $\mathcal{U} : L^{X\times X} \to M$ as follows

$$\mathcal{U}(u) = \begin{cases} 1, & \text{if } u = \top_{X\times X}, \\ 0.6, & \text{if } w \le u \ne \top_{X\times X}, \\ 0.3, & \text{if } \ w \odot w \le u \not\ge w, \\ 0, & \text{otherwise.} \end{cases}$$

Since $0.3 = \mathcal{U}(w \odot w) \ge \mathcal{U}(w) \odot \mathcal{U}(w) = 0.2$ and $w \circ w = w$, $(w \odot w) \circ (w \odot w) = (w \odot w)$, then $\mathcal{U}$ is an (L, M)-fuzzy quasi-uniformity on X.

By Theorem 5.5, we obtain (L, M)-fuzzy topogenous order $\xi_{\mathcal{U}} : L^X \times L^X \to M$ as follows

$$\xi_{\mathcal{U}}(\lambda, \rho) = \begin{cases} 1, & \text{if } \lambda \le \bigvee_{x\in X} \lambda(x) \le \rho, \\ 0.6, & \text{if } 0_X \ne \lambda \le w[\lambda] \le \rho, \bigvee_{x\in X} \lambda(x) \not\le \rho, \\ 0.3, & \text{if } \ \lambda \le (w \odot w)[\lambda] \le \rho, w[\lambda] \not\le \rho, \\ 0, & \text{otherwise.} \end{cases}$$

Definition 5.7. Let (X, ξ_X) and (Y, ξ_Y) be two (L, M)-fuzzy topogenous orders and let $f : X \to Y$ be a map. Then $f : (X, \xi_X) \to (Y, \xi_Y)$ is called an L-fuzzy open topogenous map if

$$\xi_X(\lambda, \mu) \le \xi_Y(f^{\to}(\lambda), f^{\to}(\mu)), \quad \forall\ \lambda, \mu \in L^{X\times X}.$$

Theorem 5.8. Let $(X,\mathcal{U})$ and $(Y, \mathcal{V})$ be (L, M)-fuzzy quasi-uniform spaces. If $f : (X,\mathcal{U}) \to (Y, \mathcal{V})$ is LF-uniformly continuous, then $f : (X, \xi_{\mathcal{U}}) \to (Y, \xi_{\mathcal{V}})$ is an L-fuzzy open topogenous map.

Proof. Let $v[f^{\rightarrow}(\lambda)] \leq f^{\rightarrow}(\mu)$, then

$$(f \times f)^{\leftarrow}(v)[\lambda] = f^{\leftarrow}(v[f^{\rightarrow}(\lambda)]) \leq f^{\leftarrow}f^{\rightarrow}(\mu) \leq \mu.$$

Hence,

$$\begin{aligned}\xi_{\mathcal{V}}(f^{\rightarrow}(\lambda), f^{\rightarrow}(\mu)) &= \bigvee\{\mathcal{V}(v) \mid v[f^{\rightarrow}(\lambda)] \leq f^{\rightarrow}(\mu)\} \\ &\geq \bigvee\{\mathcal{U}((f \times f)^{\leftarrow}(v)) \mid f^{\leftarrow}(v[f^{\rightarrow}(\lambda)]) \leq f^{\leftarrow}(f^{\rightarrow}(\mu))\} \\ &\geq \bigvee\{\mathcal{U}((f \times f)^{\leftarrow}(v)) \mid (f \times f)^{\leftarrow}(v)[\lambda] \leq \mu\} \\ &\geq \bigvee\{\mathcal{U}(w) \mid w[\lambda] \leq \mu\} = \xi_{\mathcal{U}}(\lambda, \mu).\end{aligned}$$

Theorem 5.9. Let $(X,\mathcal{U})$ be an (L, M)-quasi uniform space. Define a mapping $\xi_{\mathcal{U}} : L^{X\times X} \rightarrow L$ such that

$$\xi_{\mathcal{U}}(\lambda, \rho) = \bigvee_{u} \big\{\mathcal{U}(u) \odot S(u[\lambda], u[\rho^*]^*)\big\},$$

then $\xi_{\mathcal{U}}$ is an (L, M)-fuzzy topogenous order.

Proof. (ST1) Since $u[0_X] = 0_X$, and $u[1_X] = 1_X$, then

$$\xi_{\mathcal{U}}(0_X, 0_X) = \xi_{\mathcal{U}}(1_X, 1_X) = \bigvee_{u} \mathcal{U}(u) = 1_M.$$

(ST2) By (QU1) and Lemma 2.3 (16), we have

$$\xi_{\mathcal{U}}(\lambda, \mu) \leq \bigwedge_{x\in X} \big(u[\lambda] \odot u[\mu^*]\big)^*(x) = \bigwedge_{x\in X} \big(u[\lambda] \rightarrow (u[\mu^*])^*\big)(x).$$

For $\mathcal{U}(u) > 0_M$, we have $\lambda \leq u[\lambda]$ and $\mu \geq (u[\mu^*])^*$. Thus, by Lemma 2.3 (2), we have

$$\bigwedge_{x\in X} \big(u[\lambda](x) \rightarrow (u[\mu^*])^*\big)(x) \leq \bigwedge_{x\in X} \big(u[\lambda](x) \rightarrow \mu(x)\big) \leq \bigwedge_{x\in X} \big(\lambda(x) \rightarrow \mu(x)\big) = S(\lambda, \mu).$$

Since $\lambda \leq u[\lambda]$, $u[\rho^*]^* \leq \rho$,

$$\xi_{\mathcal{U}}(\lambda, \rho) = \bigvee_{u} \big\{\mathcal{U}(u) \odot S(u[\lambda], u[\rho^*]^*)\big\} \leq \bigvee_{u} \big\{\mathcal{U}(u) \odot S(\lambda, \rho)\big\} \leq S(\lambda, \rho).$$

Therefore, $\xi_{\mathcal{U}}(\lambda, \mu) \leq S(\lambda, \mu)$.

(ST3) It is obvious.

(ST4) By Lemma 2.5(3) and Lemma 5.4(5), we have

$$\begin{aligned}\xi_{\mathcal{U}}(\lambda_1\odot\lambda_2,\rho_1\odot\rho_2) &= \bigvee_u\{\mathcal{U}(u)\odot S(u[\lambda_1\odot\lambda_2],u[(\rho_1\odot\rho_2)^*]^*)\}\\ &\geq \bigvee_u\{\mathcal{U}(u)\odot S(u[\lambda_1]\odot u[\lambda_2],u[\rho_1^*]^*\odot u[\rho_2^*]^*)\}\\ &\geq \bigvee_u\{\mathcal{U}(u)\odot S(u[\lambda_1],u[\rho_1^*]^*)\}\odot\bigvee_u\{\mathcal{U}(u)\odot S(u[\lambda_2],u[\rho_2^*]^*)\}\\ &= \xi_{\mathcal{U}}(\lambda_1,\rho_1)\odot\xi_{\mathcal{U}}(\lambda_2,\rho_2).\end{aligned}$$

(T) By Lemma 2.5(3) and Lemma 5.4(6), we have

$$\begin{aligned}\xi_{\mathcal{U}}(\lambda_1\oplus\lambda_2,\rho_1\oplus\rho_2) &= \bigvee_u\{\mathcal{U}(u)\odot S(u[\lambda_1\oplus\lambda_2],u[(\rho_1\oplus\rho_2)^*]^*)\}\\ &\geq \bigvee_u\{\mathcal{U}(u)\odot S(u[\lambda_1]\oplus u[\lambda_2],u[\rho_1^*]^*\oplus u[\rho_2^*]^*)\}\\ &\geq \bigvee_u\{\mathcal{U}(u)\odot S(u[\lambda_1],u[\rho_1^*]^*)\}\odot\bigvee_u\{\mathcal{U}(u)\odot S(u[\lambda_2],u[\rho_2^*]^*)\}\\ &= \xi_{\mathcal{U}}(\lambda_1,\rho_1)\odot\xi_{\mathcal{U}}(\lambda_2,\rho_2).\end{aligned}$$

Theorem 5.10. Let $(X,\mathcal{U})$ and $(Y,\mathcal{V})$ be two (L,M)-fuzzy quasi uniform spaces and $f:X\to Y$ be LF-uniformly continuous, then $f:(X,\xi_{\mathcal{U}})\to(Y,\xi_{\mathcal{V}})$ is L- fuzzy topogenous continuous.

Proof. Since $(f\times f)^{\leftarrow}(v)[f^{\leftarrow}(\lambda)]=f^{\leftarrow}(v[f^{\rightarrow}(f^{\leftarrow}(\lambda))])\leq f^{\leftarrow}(v[\lambda])$ and by Theorem 5.7 for $u=(f\times f)^{\leftarrow}(v)$, we have for all $\lambda,\mu\in L^X$

$$\begin{aligned}\xi_{\mathcal{V}}(\lambda,\mu) &= \bigvee_v\{\mathcal{V}(v)\odot S(v[\lambda],(v[\mu^*])^*)\}\\ &\leq \bigvee_v\{\mathcal{V}(v)\odot S(f^{\leftarrow}(v[\lambda]),f^{\leftarrow}((v[\mu^*])^*))\}\\ &\leq \bigvee_u\{\mathcal{U}(u)\odot S(u[f^{\leftarrow}(\lambda)],(u[f^{\leftarrow}(\mu^*)]))^*)\}\leq\xi_{\mathcal{U}}(f^{\leftarrow}(\lambda),f^{\leftarrow}(\mu)).\end{aligned}$$

Lemma 5.11. For every $\lambda,\rho\in L^X$, we define $u_{\lambda,\rho},u^{-1}_{\lambda,\rho}:X\times X\to L$ by

$$u_{\lambda,\rho}(x,y)=\lambda(x)\to\rho(y),\quad u^{-1}_{\lambda,\rho}(x,y)=u_{\lambda,\rho}(y,x),$$

then we have the following statements

(1) $1_{X\times X}=u_{0_X,\,0_X}=u_{1_X,\,1_X}$,
(2) If $\lambda_1\leq\lambda_2$ and $\rho_1\leq\rho_2$, then $u_{\lambda_2,\rho_1}\leq u_{\lambda_1,\rho_2}$,
(3) If $\lambda\leq\rho$, then $1_{\triangle}\leq u_{\lambda,\rho}$,
(4) For every $u_{\mu,\rho}\in L^{X\times X}$ and $\lambda\in L^X$, we have $u_{\gamma,\rho}\circ u_{\lambda,\gamma}\leq u_{\lambda,\rho}$,

(5) $u_{\lambda_1,\rho_1} \odot u_{\lambda_2,\rho_2} \le u_{\lambda_1\odot\lambda_2}, u_{\rho_1\odot\rho_2}$,
(6) $u_{\lambda_1,\rho_1} \odot u_{\lambda_2,\rho_2} \le u_{\lambda_1\oplus\lambda_2}, u_{\rho_1\oplus\rho_2}$,
(7) $u^{-1}_{\lambda,\rho} = u_{\rho^*,\lambda^*}$,
(8) $u^{-1}_{\lambda_1\odot\lambda_2,\rho_1\odot\rho_2} = u_{\rho_1^*\oplus\rho_2^*,\lambda_1^*\oplus\lambda_2^*}$,
(9) $u^{-1}_{\lambda_1\oplus\lambda_2,\rho_1\oplus\rho_2} = u_{\rho_1^*\odot\rho_2^*,\lambda_1^*\odot\lambda_2^*}$.

Proof.

(1) $1_{X\times X}(x,y) = 1 = u_{0_X,0_X}(x,y) = 0_X(x) \to 0_X(y) = 1_X(x) \to 1_X(y) = u_{1_X,1_X}(x,y)$.

(2) Let $\lambda_1 \le \lambda_2$ and $\rho_1 \le \rho_2$, then

$$u_{\lambda_2,\rho_1}(x,y) = \lambda_2(x) \to \rho_1(y) \le \lambda_1(x) \to \rho_2(y) = u_{\lambda_1,\rho_2}(x,y).$$

(3) Since $1_\triangle[\lambda] = \lambda \le \rho$, then $1_\triangle \le u_{\lambda,\rho}$.

(4)

$$u_{\gamma,\rho}(x,z) \circ u_{\lambda,\gamma}(x,z) = \bigvee_{y\in X} ((\gamma(y) \to \rho(z))\odot(\lambda(x) \to \gamma(y))) \le \lambda(x) \to \rho(y) = u_{\lambda,\rho}(x,z).$$

(5)

$$\begin{aligned}(u_{\lambda_1,\rho_1} \odot u_{\lambda_2,\rho_2})(x,z) &= u_{\lambda_1,\rho_1}(x,z) \odot u_{\lambda_2,\rho_2}(x,z)\\ &\le (\lambda_1(x) \to \rho_1(y)) \odot (\lambda_2(x) \to \rho_2(y))\\ &\le \lambda_1(x) \odot \lambda_2(x) \to \rho_1(y) \odot \rho_2(y) = u_{\lambda_1\odot\lambda_2}, u_{\rho_1\odot\rho_2}(x,y).\end{aligned}$$

(6)

$$\begin{aligned}(u_{\lambda_1,\rho_1} \odot u_{\lambda_2,\rho_2})(x,y) &= u_{\lambda_1,\rho_1}(x,y) \odot u_{\lambda_2,\rho_2}(x,y)\\ &\le (\lambda_1(x) \to \rho_1(y)) \odot (\lambda_2(x) \to \rho_2(y))\\ &\le \lambda_1(x) \oplus \lambda_2(x) \to \rho_1(y) \oplus \rho_2(y) = u_{\lambda_1\oplus\lambda_2}, u_{\rho_1\oplus\rho_2}(x,y).\end{aligned}$$

(7) $\quad u^{-1}_{\lambda,\rho}(x,y) = u_{\lambda,\rho}(y,x) = \lambda(y) \to \rho(x) = \rho^*(x) \to \lambda^*(y) = u_{\rho^*,\lambda^*}(x,y)$.

(8),(9) are similarly proved.

In the following theorem, we obtain an (L,M)-fuzzy quasi uniform space from an (L,M)-fuzzy topogenous order.

Theorem 5.12. Let (X,ξ) be an (L,M)-fuzzy quasi topogenous space. Define $\mathcal{U}_\xi : L^{X\times X} \to M$ by

$$\mathcal{U}_\xi(u) = \bigvee\{\odot_{i=1}^n \xi(\mu_i,\rho_i) \mid \odot_{i=1}^n u_{\mu_i,\rho_i} \le u\},$$

where $\bigvee$ is taken over every finite family $\{u_{\mu_i,\rho_i} \mid i = 1,2,3,....,n\}$. Then,

(1) $\mathcal{U}_\xi$ is an (L,M)-fuzzy quasi uniformity on X,

(2) $\xi_{\mathcal{U}_\xi} = \xi$.

Proof. (1) (LU1) Since $\xi(0_X, 0_X) = \xi(1_X, 1_X) = 1_M$, there exists $1_{X\times X} = u_{0_X,0_X} = u_{1_X,1_X} \in L^{X\times X}$. It follows $\mathcal{U}_\xi(1_{X\times X}) = 1_M$.

(LU2) It is trivial from the definition of $\mathcal{U}_\xi$.

(LU3) For every $u, v \in L^{X\times X}$, each two families $\{u_{\mu_i,\rho_i} \mid \odot_{i=1}^n u_{\mu_i,\rho_i} \leq u\}$ and $\{u_{\nu_j,w_j} \mid \odot_{j=1}^k u_{\nu_j,w_j} \leq v\}$, we have

$$\begin{aligned}\mathcal{U}_\xi(u) \odot \mathcal{U}_\xi(v) &= \big(\bigvee\{\odot_{i=1}^n \xi(\mu_i, \rho_i) \mid \odot_{i=1}^n u_{\mu_i,\rho_i} \leq u\}\big) \odot \big(\bigvee\{\odot_{j=1}^k \xi(\nu_i, w_i) \mid \odot_{j=1}^k u_{\nu_i,w_i} \leq v\}\big)\\ &\leq \bigvee\{(\odot_{i=1}^n \xi(\mu_i, \rho_i)) \odot (\odot_{j=1}^k \xi(\nu_i, w_i)) \mid \odot_{i=1}^n u_{\mu_i,\rho_i} \leq u,\ \odot_{j=1}^k u_{\nu_i,w_i} \leq v\}\\ &\leq \bigvee\{(\odot_{i=1}^n \xi(\mu_i, \rho_i)) \odot (\odot_{j=1}^k \xi(\nu_i, w_i)) \mid (\odot_{i=1}^n u_{\mu_i,\rho_i}) \odot (\odot_{j=1}^k u_{\nu_i,w_i}) \leq u \odot v\}\\ &\leq \mathcal{U}_\xi(u \odot v).\end{aligned}$$

(LU4) If $\mathcal{U}(u) \neq 0_M$, there exists a family $\{u_{\lambda_i,\rho_i} \mid \odot_{i=1}^m u_{\lambda_i,\rho_i} \leq u\}$ such that $\odot_{j=1}^m \xi(\lambda_i, \rho_i) \neq 0_M$. Since $\xi(\lambda_i, \rho_i) \neq 0_M$, for $i = 1, 2, ..., m$, then $\lambda_i \leq \rho_i$ for $i = 1, 2, ..., m$, i.e. $1_\triangle \leq u_{\lambda_i,\rho_i}$. Thus $1_\triangle \leq \odot_{i=1}^m u_{\lambda_i,\rho_i} \leq u$.

(LU5) Suppose there exists $u \in L^{X\times X}$ such that

$$\bigvee\{\mathcal{U}_\xi(v) \odot \mathcal{U}_\xi(w) \mid v \circ w \leq u\} \not\geq \mathcal{U}_\xi(u).$$

Put $t = \bigvee\{\mathcal{U}_\xi(v) \odot \mathcal{U}_\xi(w) \mid v \circ w \leq u\}$. From the Definition of $\mathcal{U}_\xi(u)$, there exists family $\{u_{\mu_i,\rho_i} \mid \odot_{i=1}^m u_{\mu_i,\rho_i} \leq u\}$ such that

$$t \not\geq \odot_{i=1}^m \xi(\lambda_i, \rho_i).$$

Since $\xi \circ \xi \geq \xi$, $t \not\geq \odot_{i=1}^m \xi \circ \xi(\lambda_i, \rho_i) = \odot_{i=1}^m\{\bigvee_{\gamma\in L^X}\{\xi(\gamma, \rho_i) \odot (\xi(\lambda_i, \gamma))\}\}$ and L is a stsc-quantal, then there exist $\gamma_i \in L^X$ such that

$$t \not\geq \odot_{i=1}^m(\xi(\gamma_i, \rho_i) \odot \xi(\lambda_i, \gamma_i)).$$

On the other hand $v_i = u_{\gamma_i,\rho_i}, w_i = u_{\lambda_i,\gamma_i}$, then it satisfies

$$v_i \circ w_i \leq u_{\gamma_i,\ \rho_i} \circ u_{\lambda_i,\gamma_i} \leq u_{\lambda_i,\rho_i},\ \ \mathcal{U}_\xi(v_i) \geq \xi(\gamma_i, \rho_i),\ \mathcal{U}_\xi(w_i) \geq \xi(\lambda_i, \gamma_i).$$

Let $v = \odot_{i=1}^m v_i$ and $w = \odot_{i=1}^m w_i$ be given. Since $v_i \circ w_i \leq u_{\lambda_i,\rho_i}$, for each $i = 1, 2, 3, ..., m$, we have

$$\big(\odot_{i=1}^m\ v_i\big) \circ \big(\odot_{i=1}^m\ v_i\big) = \odot_{i=1}^m\ (v_i \circ w_i) \leq \odot_{i=1}^m\ u_{\lambda_i,\rho_i} \leq u.$$

Then, we have $v \circ w \leq u$, $\mathcal{U}_\xi(v) \geq \odot_{i=1}^m \mathcal{U}_\xi(v_i)$ and $\mathcal{U}_\xi(w) \geq \odot_{i=1}^m \mathcal{U}_\xi(w_i)$. Thus,

$$t = \bigvee\{\mathcal{U}_\xi(w) \odot \mathcal{U}_\xi(w) \mid w \circ w \leq u\} \geq \mathcal{U}_\xi(v) \odot \mathcal{U}_\xi(w) \geq \odot_{i=1}^m\ (\xi(\gamma_i, \rho_i) \odot \xi(\lambda_i, \gamma_i)).$$

It is a contradiction. Thus, $\mathcal{U}_\xi$ is an (L, M)-fuzzy quasi uniformity on X.

(2) Since $u[\lambda] \leq \rho$, then $u \leq u_{\lambda,\rho}$. Hence,

$$\xi_{\mathcal{U}_\xi}(\lambda, \rho) = \bigvee\{\mathcal{U}_\xi(u) \mid u[\lambda] \leq \rho\} = \mathcal{U}_\xi(u_{\lambda,\rho}) = \xi(\lambda, \rho).$$

6 Conclusion

The main purpose of this paper is to introduce concepts in fuzzy set theory, namely that an (L, M)-fuzzy semi-topogenous order, (L, M)-fuzzy topogenous space, (L, M)-fuzzy uniform space and the (L, M)-fuzzy proximity space in strictly two sided, commutative quantales. On the other hand, we study some relationships between previous spaces and we give their examples. As a special case our (L, M)-fuzzy topogenous structures contain classical Cs*ász*è*r topogenous structures, Katasaras fuzzy topogenous structures and Čimoka L-fuzzy topogenous structures.

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PRE OPEN SOFT SETS VIA SOFT GRILLS

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Abstract – This paper aims to introduce and investigate a collection of pre open soft sets in a soft topological space via soft grill G, namely pre G-open soft sets. Detailed study on pre G-open soft sets through soft sets theory with examples is carried out. Suitable condition on the collection of pre G-open soft sets to coincide with soft topology is deduced.

Keywords – *Soft sets, Soft topological spaces, Soft grill, Pre open soft sets, Semi open soft sets, Regular open soft sets.*

1 Introduction

In [25], D. Molodtsov introduced the concept of soft set theory and it has received much attention since its inception. Molodtsov presented the fundamental results of new theory and successfully applied it into several directions such as smoothness of functions, game theory, operations research, Riemann-integration, Perron integration, theory of probability etc. A soft set is a collection of approximate description of an object. He also showed how soft set theory is free from parametrization inadequacy syndrome of fuzzy set theory, rough set theory, probability theory and game theory. Soft systems provide a very general framework with the involvement of parameters. Research works on soft set theory and its applications in various fields are progressing rapidly in these years. After presentation of the operations of soft sets [23], the properties and applications of soft set theory have been studied increasingly [2, 20, 26, 27]. In recent years, many interesting applications of soft set theory have been expanded by embedding the ideas of fuzzy sets [9, 18, 21, 22, 23, 24, 26]. To develop soft set theory, the operations of the soft sets are redefined and a uni-int decision making method was constructed by using these new operations [3]. It got some stability only after the introduction of soft topology [31] in 2011. In [10], Kandil et al. introduced some soft operations such as semi open soft, pre open soft, $\alpha-$open

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soft and $\beta-$open soft and investigated their properties in detail. The notion of soft ideal was initiated for the first time by Kandil et al. [13]. They also introduced the concept of soft local function. Applications to various fields were further investigated by Kandil et al. [11, 12, 17, 14, 15, 16, 19]. The notion of $b-$open soft sets was initiated for the first time by El-sheikh and Abd El-latif [6], which is generalized to the supra soft topological spaces in [1, 8]. Properties of $b-$open soft sets in [28] are discussed. The notions of soft grill G and soft operators φ_G, ψ_G were introduced in [29]. These concepts are discussed with a view to find new soft topologies τ_G from the original one τ via soft grill G. Pei and Miao [27] showed that soft sets are a class of special information systems. Nevertheless, the idea of pre G-open soft sets in soft topological spaces and suitable conditions for the collection of pre G-open soft sets to be soft topology are not investigated, which are the aims of the current paper.

2 Preliminary

Definition 2.1. [25] Let X be an initial universe and E be a set of parameters. Let $\mathcal{P}(X)$ denote the power set of X and A be a non-empty subset of E. A pair F denoted by F_A is called a soft set over X , where F is a mapping given by $F : A \rightarrow \mathcal{P}(X)$. In other words, a soft set over X is a parametrized family of subsets of the universe X. For a particular $e \in A$, $F(e)$ may be considered the set of $e-$approximate elements of the soft set (F, A) and if $e \notin A$, then $F(e) = \emptyset$ i.e $F_A = \{F(e) : e \in A \subseteq E,\ F : A \rightarrow \mathcal{P}(X)\}$.

Definition 2.2. [4] A soft set F over X is a set valued function from E to $\mathcal{P}(X)$. It can be written a set of ordered pairs $F = \big\{(e, F(e)) : e \in E\big\}$. Note that if $F(e) = \emptyset$, then the element $(e, F(e))$ is not appeared in F. The set of all soft sets over X is denoted by $S_E(X)$.

Definition 2.3. [4] Let $F, G \in S_E(X)$. Then,

(i). If $F(e) = \emptyset$ for each $e \in E$, F is said to be a null soft set, denoted by $\tilde{\emptyset}$.

(ii). If $F(e) = X$ for each $e \in E$, F is said to be absolute soft set, denoted by $\tilde{X}$.

(iii). F is soft subset of G, denoted by $F\tilde{\subseteq}G$, if $F(e) \subseteq G(e)$ for each $e \in E$.

(iv). $F = G$, if $F\tilde{\subseteq}G$ and $G\tilde{\subseteq}F$.

(v). Soft union of F and G, denoted by $F\tilde{\cup}G$, is a soft set over X and defined by $F\tilde{\cup}G : E \rightarrow \mathcal{P}(X)$ such that $(F\tilde{\cup}G)(e) = F(e) \cup G(e)$ for each $e \in E$.

(vi). Soft intersection of F and G, denoted by $F\tilde{\cap}G$, is a soft set over X and defined by $F\tilde{\cap}G : E \rightarrow \mathcal{P}(X)$ such that $(F\tilde{\cap}G)(e) = F(e) \cap G(e)$ for each $e \in E$.

(vii). Soft difference of F and G, denoted by $F\tilde{\setminus}G$, is a soft set over U whose approximate function is defined by $F\tilde{\setminus}G : E \rightarrow \mathcal{P}(X)$ such that $(F\tilde{\setminus}G)(e)=F(e)\setminus G(e)$.

(viii). Soft complement of F is denoted by $F^{\tilde{c}}$ and defined by $F^{\tilde{c}} : E \rightarrow \mathcal{P}(X)$ such that $F^{\tilde{c}}(e) = X \setminus F(e)$ for each $e \in E$.

Definition 2.4. [33] Let Δ be an arbitrary indexed set and $\Upsilon = \{F_i \mid i \in \Delta\}$ be a subfamily of $S_E(X)$, then *(i).* The soft union of Υ is the soft set H, however, $H(e) = \tilde{\cup}\{F_i(e) \mid i \in \Delta\}$ for all $e \in E$ that can be write as $\tilde{\cup}_{i\in\Delta} F_i = H$.

(ii). The soft intersection of Υ is the soft set K, however $K(e) = \tilde{\cap}\{F_i(e) \mid i \in \Delta\}$ for all $e \in E$ that can be write as $\tilde{\cup}_{i\in\Delta} F_i = K$.

Definition 2.5. [33] The soft set $F \in S_E(X)$ is called a soft point if there exist an $e \in E$ such that $F(e) \neq \emptyset$ and $F(e') = \emptyset$ for each $e' \in E \setminus \{e\}$, and the soft point F is denoted by e_F. The soft point e_F is said to be in the soft set G, denoted by $e_F \tilde{\in} G$, if $F(e) \tilde{\subseteq} G(e)$ for the element $e \in E$.

Definition 2.6. [17, 31] The soft set F over X such that $F(e) = \{x\}\ \forall e \in E$ is called singleton soft point and denoted by x_E or (x, E).

Definition 2.7. [31] Let τ be a collection of soft sets over a universe X with a fixed set of parameters E, then $\tau \subseteq S_E(X)$ is called a soft topology on X if

(i). $\tilde{X}, \tilde{\emptyset} \in \tau$, where $\tilde{\emptyset}(e) = \emptyset$ and $\tilde{X}(e) = X$, for each $e \in E$,

(ii). The soft union of any number of soft sets in τ belongs to τ,

(iii). The soft intersection of any two soft sets in τ belongs to τ.

The triplet (X, τ, E) is called a soft topological space over X. A soft set F over X is said to be open soft set in X if $F \in \tau$, and it is said to be closed soft set in X, if its relative complement $F^{\tilde{c}}$ is an open soft set.

Definition 2.8. [31] Let (X, τ, E) be a soft topological space over X and $F \in S_E(X)$. Then, the soft interior and soft closure of F, denoted by $int(F)$ and $cl(F)$, respectively, are defined as,

$$\begin{aligned} int(F) &= \tilde{\cup}\big\{G : G \text{ is open soft set and } G\tilde{\subseteq}F\big\} \\ cl(F) &= \tilde{\cap}\big\{H : H \text{ is closed soft set and } F\tilde{\subseteq}H\big\}. \end{aligned}$$

Definition 2.9. [31] Let F be a soft set over X and $x \in X$. $x \in F$ whenever $x \in F(e)$ for all $e \in E$. Note that for any $x \in X$, $x \notin F$, if $x \notin F(e)$ for some $e \in E$.

Definition 2.10. [33] A soft set H of a soft topological space (X, τ, E) is known as a soft neighborhood (soft nbd.) of the soft point x, if there is a soft open set K such that $x \in K \tilde{\subseteq} F$.

Lemma 2.11. [23] Let (X, τ, E) be a soft topological space and F be a soft set. Then,

(i). $int(F \tilde{\cup} H) \tilde{\subseteq} int(F) \tilde{\cup} H$, if H is a τ-closed soft set.

(ii). $H \tilde{\cap} cl(F) \tilde{\subseteq} cl(H \tilde{\cup} F)$, if H is a τ-open soft set, then.

Definition 2.12. A soft set F of a soft topological space (X, τ, E) is called

(i). [10] Pre open soft, if $F \tilde{\subseteq} int\ clF$ (resp., pre closed soft, if $cl\ intF \tilde{\subseteq} F$).

(ii). [5] Semi open soft, if $F \tilde{\subseteq} cl\ intF$ (resp., semi closed soft, if $int\ clF \tilde{\subseteq} F$).

(iii). [32] Regular open soft, if $F = int\ clF$ (resp., regular closed soft, if $F = cl\ intF$).

(iv). [10] α-open soft, if $F\tilde{\subseteq}int\ cl\ intF$.

Definition 2.13. [5] Let (X, τ, E) be a soft topological space, then a semi soft closure Scl_sF of a soft set F is the intersection of all semi-closed soft supersets of F. In other words, $Scl_sF = F\tilde{\cup}int\ clF$.

Definition 2.14. [30] A soft set F is called dense soft in H (resp., dense soft), if $H\tilde{\subseteq}clF$ (resp., $clF = \tilde{X}$).

Definition 2.15. [29] A non-empty collection $G\tilde{\subseteq}S_E(X)$ of soft sets over X is known as a soft grill, if these conditions hold:

(i). $\tilde{\emptyset} \notin G$

(ii). If $F \in G$ and $F\tilde{\subseteq}H$, then $H \in G$.

(iii). If $F\tilde{\cup}H \in G$, then $F \in G$ or $H \in G$.

Definition 2.16. [29] Let G be a soft grill over a soft topological space (X, τ, E). Now consider the soft operator $\varphi_G : S_E(X) \longrightarrow S_E(X)$, given by, for every soft set F, $\varphi_G(F) = \{x \mid U\tilde{\cap}F \in G$ for every soft open nbd. U of $x\}$. Then, the soft operator $\psi_G : S_E(X) \longrightarrow S_E(X)$, defined by for every soft set F, $\psi_G(F) = F\tilde{\cup}\varphi_G(F)$ is a kuratowski's soft closure operator and hence gives rise to a new soft topology over X with the same parameters, $\tau_G = \{H \mid \psi_G(\tilde{X} - H) = (\tilde{X} - H)\}$, which is finer than τ in general.

Lemma 2.17. [29] Let G be a soft grill over a soft topological space (X, τ, E). Then, for every soft set F the following statements hold:

(i). If $F \notin G$ then, $\varphi_G(F) = \tilde{\emptyset}$. Moreover, $\varphi_G(\tilde{\emptyset}) = \tilde{\emptyset}$.

(ii). $\varphi_G\varphi_G(F)\tilde{\subseteq}\varphi_G(F) = cl\varphi_G(F)\tilde{\subseteq}clF$. Moreover, $\varphi_G(F)$ is soft τ-closed.

(iii). $\varphi_G\psi_G(F) = \psi_G\varphi_G(F) = \varphi_G(F)$.

(iv). If a soft set F is τ-closed, then $\varphi_G(F)\tilde{\subseteq}F$. Moreover, $\psi_G(F)\tilde{\subseteq}F$.

(v). A soft set F is τ_G-closed if and only if $\varphi_G(F)\tilde{\subseteq}F$.

Lemma 2.18. [29] Let G be a soft grill over a soft topological space (X, τ, E). Then, for soft sets F, H the following statements hold:

(i). $F\tilde{\subseteq}H$ implies $\varphi_G(F)\tilde{\subseteq}\varphi_G(H)$.

(ii). $\varphi_G(F\tilde{\cup}H) = \varphi_G(F)\tilde{\cup}\varphi_G(H)$ and $\varphi_G(F\tilde{\cap}H)\tilde{\subseteq}\varphi_G(F)\tilde{\cap}\varphi_G(H)$.

(iii). $\varphi_G(F) - \varphi_G(H) = \varphi_G(F - H) - \varphi_G(H)$.

(iv). If $H \notin G$, then $\varphi_G(F\tilde{\cup}H) = \varphi_G(F) = \varphi_G(F - H)$.

Lemma 2.19. [29] Let G be a soft grill over a soft topological space (X, τ, E) with $(\tau - \{\tilde{\emptyset}\})\tilde{\subseteq}G$. Then, the following statements hold:

(i). $\varphi_G(\tilde{X}) = \tilde{X}$.

(ii). $H\tilde{\subseteq}\varphi_G(H)$, for any open soft set H.

Theorem 2.20. [29] Let G be a soft grill over a soft topological space (X, τ, E) and F be soft set such that $F\tilde{\subseteq}\varphi_G\ (F)$. Then, $clF = \psi_G(F) = \tau_G - clF = cl(\varphi_G(F)) = \varphi_G(F)$.

Theorem 2.21. [29] Let G be a soft grill over a soft topological space (X, τ, E). If H is a τ-open soft set, then $H\tilde{\cap}\varphi_G(F) = H\tilde{\cap}\varphi_G(H\tilde{\cap}F)$, for every soft set F.

Lemma 2.22. [29] Let (X, τ, E) be a soft topological space and F be a soft set. Then, for soft grills G_1, G_2 over X the following statements hold:

(i). If $G_1\tilde{\subseteq}G_2$, then $\varphi_{G1}(F)\tilde{\subseteq}\varphi_{G2}(F)$.

(ii). $\varphi_{G1\tilde{\cup}G2}(F) = \varphi_{G1}(F)\tilde{\cup}\varphi_{G2}(F)$.

Lemma 2.23. [29] Let (X, τ, E) be a soft topological space with $G = P(X) - \{\tilde{\emptyset}\}$, then for any soft set F, $\varphi_G(F) = F$. Moreover, $\psi_G(F) = F$.

Definition 2.24. [30] Let G be a soft grill over a soft topological space (X, τ, E). A soft set F is called

(i). τ_G-perfect (resp., G-dense) soft, if $\varphi_G(F) = F$ (resp., $\varphi_G(F) = \tilde{X}$)

(ii). G-dense in a soft set H (resp., G-dense in itself) soft, if $H\tilde{\subseteq}\varphi_G(F)$ (resp., $F\tilde{\subseteq}\varphi_G(F)$).

Lemma 2.25. [30] Every G-dense soft is dense soft set.

3 Pre G-Open Soft Sets

Definition 3.1. Let G be a soft grill over a soft topological space (X, τ, E). A soft set F is called pre G-open soft, if $F\tilde{\subseteq}int\varphi_G(F)$. The complement of such set will be called pre G-closed soft.

Remark 3.2. *(i).* G-dense soft set $\Longrightarrow$ pre G-open soft set $\Longrightarrow$ pre open soft set.

(ii). pre G-open soft set $\Longrightarrow$ G-dense in it self soft set.

These implications are irreversible as indicated in the next example.

Example 3.3. Let $X = \{a, b, c\}$, $A = \{e_1, e_2\}$, $\tau = \{\tilde{\emptyset}, \tilde{X}, F_1, F_2, F_3, F_4, F_5, F_6, F_7, F_8, F_9\}$ and $G = \{\tilde{X}, C, D, E, K\}$ where $F_1, F_2, F_3, F_4, F_5, F_6, F_7, F_8, F_9, C, D, M, K$ are soft sets over X, explained as follow
$F_1 = \{\{a\}, \emptyset\}, F_2 = \{\{b, c\}, \{a\}\}, F_3 = \{X, \{a\}\}, F_4 = \{\{b\}, \{a, c\}\}$,
$F_5 = \{\{a, b\}, \{a, c\}\}, F_6 = \{\{b, c\}, \{a, c\}\}, F_7 = \{X, \{a, c\}\}, F_8 = \{\{b\}, \{a\}\}$,
$F_9 = \{\{a, b\}, \{a\}\}, C = \{\{a\}, \emptyset\}, D = \{\{a, c\}, \emptyset\}, M = \{\{a, b\}, \emptyset\}, K = \{X, \emptyset\}$.
It is clear that

(i). $F_8 = \{\{b\}, \{a\}\}$ is pre open soft set, but it is not pre G-open soft set.

(ii). $F_1 = \{\{a\}, \emptyset\}$ is pre G-open soft set, but it is not G-dense soft set.

The following example indicates that the notions pre G-open soft and open soft sets are independent.

Example 3.4. Let $X = \{a, b\}, A = \{e_1, e_2\}$. Define $F_1 = \{\{a\}, \{b\}\}, F_2 = \{X, \{b\}\}, F_3 = \{\{a\}, X\}$ are all soft sets on universe set X. Soft topology $\tau = \{\tilde{\emptyset}, \tilde{X}, F_1, F_2\}$ and soft grill $G = \{\tilde{X}, F_3\}$. It is clear that F_3 is pre G-open soft set, but it is not open soft. Also, F_2 is open soft set, but it is not pre G-open soft.

Example 3.5. If $G = P(X) - \{\tilde{\emptyset}\}$, then in view of Lemma 2.6, the concepts pre G-open soft and open soft sets are equivalent.

Theorem 3.6. For any soft set F in a soft topological space (X, τ, E) with grill G, the following properties hold:

(i). If $(\tau - \{\tilde{\emptyset}\})\tilde{\subseteq}G$, then each τ-open soft set is pre G-open soft.

(ii). If F is pre G-open soft and τ_G-closed soft set, then it is τ-open soft set. Moreover, $\varphi_G(F)$ is τ-open soft set.

(iii). If F is pre G-open soft and τ_G-perfect soft set, then it is τ-open soft set.

Proof. We prove only (ii) and the rest of the proof may be done straightforward. Let F be a τ_G-closed soft set, then $\varphi_G(F)\tilde{\subseteq}F$, follows directly in view of Lemma 2.2. Since F is a pre G-open soft set, then $F\tilde{\subseteq}int\varphi_G(F)$. Hence, $F\tilde{\subseteq}int(F)$ and so F is a τ-open soft set. Also, $\varphi_G(F)\tilde{\subseteq}F\tilde{\subseteq}int\varphi_G(F)$. Hence, $\varphi_G(F)$ is a τ-open soft set. The following corollary is immediate from (ii) of Theorem 3.1.

Corollary 3.7. If F is pre G-open soft and τ_G-closed soft set, then $int\varphi_G(F) = \varphi_G(intF)$.

Definition 3.8. A soft topological space (X, τ, E) is known as soft locally indiscrete, if each open soft set is closed soft.

Definition 3.9. A soft set F in a soft topological space (X, τ, E) with soft grill G is said to be semi G-open soft, if $F\tilde{\subseteq}\varphi_G(intF)$.

Theorem 3.10. Let G be a soft grill on a soft locally indiscrete topological space (X, τ, E). Therefore, F is pre G-open soft, if it is a semi G-open soft set.

Proof. Let F be a semi G-open soft set, then $F\tilde{\subseteq}\varphi_G int(F)\tilde{\subseteq}\varphi_G(F)$. Since (X, τ, E) is soft locally indiscrete space and in view of Lemma 2.2, then $F\tilde{\subseteq}\varphi_G int(F)\tilde{\subseteq}cl\ intF = intF\tilde{\subseteq}int\varphi_G(F)$. Thus, F is a pre G-open soft set.

Theorem 3.11. Let G be a soft grill on a soft topological space (X, τ, E) with $(\tau - \{\tilde{\emptyset}\})\tilde{\subseteq}G$. Then, F is a G-dense soft set with respect to τ if and only if it is a dense soft set with respect to τ_G.

Proof. Let F be a G-dense soft set, then $\tilde{X} = \varphi_G(F)\tilde{\subseteq}\psi_G(F)$. Hence, F is a dense soft set with respect to τ_G. Conversely, let F be a dense soft set with respect to τ_G, then $\tilde{X} = \psi_G(F)$. Then, $\tilde{X} = \varphi_G(F)$ follows directly since $(\tau - \{\tilde{\emptyset}\})\tilde{\subseteq}G$ and by using (iii) of Lemma 2.2, (i) of Lemma 2.3 and (i) of Lemma 2.4. Consequently, F is a G-dense soft set with respect to τ.

Theorem 3.12. Let G be a soft grill on a soft topological space (X, τ, E) with $(\tau - \{\tilde{\emptyset}\})\tilde{\subseteq}G$. Then, F is a pre G-open soft set with respect to τ, if and only if it is a pre open soft set with respect to τ_G.

Proof. It is clear that, every pre G-open soft set with respect to τ is a pre open soft set with respect to τ_G. Conversely, let F be a pre open soft set with respect to τ_G, then $F\tilde{\subseteq} int_{\tau G}\psi_G(F)$. Hence, x is a τ_G-interior soft point of $\psi_G(F)$ for $x \in F$. Consequently, there exists a τ_G-open soft set H of x which is a subset of $\psi_G(F)$ and then further there exists a soft open base member $(U - K)$ where U is τ-open soft set containing x and $K \notin G$ such that $(U - K)\tilde{\subseteq} H\tilde{\subseteq}\psi_G(F)$. Hence, $\varphi_G(U-K)\tilde{\subseteq}\varphi_G\psi_G(F)$ follows directly from Lemma 2.3. Therefore, $\varphi_G(U)\tilde{\subseteq}\varphi_G(F)$ by using Lemmas 2.2 and 2.3. Since $(\tau - \{\tilde{\emptyset}\})\tilde{\subseteq} G$, U is τ-open soft set containing x and $F\tilde{\subseteq} U$, then $U\tilde{\subseteq}\varphi_G(U)\tilde{\subseteq}\varphi_G(F)$ follows from Lemma 2.4. Consequently, $U\tilde{\subseteq} int\varphi_G(F)$. Then, $F\tilde{\subseteq} int\varphi_G(F)$ and so F is a pre G-open soft set with respect to τ.

Theorem 3.13. Let G be a soft grill on a soft topological space (X, τ, E). Then,

(i). If H is a pre G-open soft set and $F\tilde{\subseteq} H\tilde{\subseteq}\varphi_G(F)$, then F is a pre G-open soft set.

(ii). Let F be a G-dense in an open soft set H and $F\tilde{\subseteq} H$, then F is a pre G-open soft set.

Proof. We prove only (i) and the rest of the proof is obvious. Since $F\tilde{\subseteq} H\tilde{\subseteq}\varphi_G(F)$ and H is a pre G-open soft set, then $H\tilde{\subseteq} int\varphi_G(H)$ and so $F\tilde{\subseteq} H\tilde{\subseteq} int\varphi_G(H)\tilde{\subseteq} int\varphi_G\varphi_G(F)$. Then, $F\tilde{\subseteq} int\varphi_G(F)$ is obtained by using Lemma 2.2. Consequently, F is a pre G-open soft set.

The following theorem is immediate in view of Lemma 2.5.

Theorem 3.14. Let (X, τ, E) be a soft topological space with soft grills G_1, G_2 over X. Then,

(i). G_1-pre open soft set is G_2-pre open soft, if $G_1\tilde{\subseteq} G_2$.

(ii). If a soft set F is both G_1-pre open soft and G_2-pre open soft, then it is a $(G_1\tilde{\cup} G_2)$-pre open soft set.

Theorem 3.15. Let G be a soft grill on a soft topological space (X, τ, E). Then, an arbitrary union (resp., intersection) of pre G-open (resp., pre G-closed) soft sets is a pre G-open (resp., pre G-closed) soft.

Proof. Let $\{F_i \mid i \in \Gamma\}$ be a class of pre G-open soft sets, then for each $i \in \Gamma$, $F_i\tilde{\subseteq} int\varphi_G(F_i)$. Hence, $\tilde{\cup}_{i\in\Gamma}(F_i)\tilde{\subseteq}\tilde{\cup}_{i\in\Gamma}(int\varphi_G(F_i))\tilde{\subseteq} int\varphi_G(\tilde{\cup}_{i\in\Gamma}(F_i))$ and so $\tilde{\cup}_{i\in\Gamma}(F_i)$ is a pre G-open soft set. The other result follows immediately by taking complements.

The next example shows that intersection of two pre G-open soft sets may not be a pre G-open soft.

Example 3.16. Let $X = \{a, b\}$, $A = \{e_1, e_2\}$. Define $F_1 = \{\{a\}, \{b\}\}, F_2 = \{X, \{b\}\}, F_3 = \{\{a\}, X\}, F_4 = \{\{a\}, \emptyset\}, F_5 = \{X, \emptyset\}, F_6 = \{X, \{a\}\}$ are all soft sets on universe set X and $\tau = \{\tilde{\emptyset}, \tilde{X}, F_1, F_2\}$ is soft topology over X If $G = \{\tilde{X}, F_1, F_2, F_3, F_4, F_5, F_6\}$ is a soft grill over X, then it is clear that F_3, F_6 are pre G-open soft sets, but their intersection F_8 is not a pre G-open soft set.

Lemma 3.17. Let G be a soft grill on a soft topological space (X, τ, E). Then,

(i). The intersection of pre G-open soft set and open soft set is a pre G-open soft.

(ii). The intersection of G-dense soft set and open soft set is a pre G-open soft.

Proof. *(i).* Let F be a pre G-open soft set and H be an open soft set, then $F\tilde{\subseteq}int\varphi_G(F)$ and $H = intH$. Therefore, $(H\tilde{\cap}F)\tilde{\subseteq}intH\tilde{\cap}int\varphi_G(F) = int(H\tilde{\cap}\varphi_G(F))$ $\tilde{\subseteq}int\varphi_G(H\tilde{\cap}F)$, in view of Theorem 2.2. This shows that $(H\tilde{\cap}F)$ is a pre G-open soft set.

(ii). Let F be a G-dense soft set and H be an open soft set, then $\varphi_G(F) = \tilde{X}$ and $H = intH$. Hence, $(H\tilde{\cap}F)\tilde{\subseteq}H = intH = int(H\tilde{\cap}\varphi_G(F))\tilde{\subseteq}int\varphi_G(H\tilde{\cap}F)$, by using Theorem 2.2. Then, $(H\tilde{\cap}F)$ is pre G-open soft set.

Theorem 3.18. Let G be a soft grill on a soft topological space (X, τ, E) with $(\tau - \{\tilde{\emptyset}\})\tilde{\subseteq}G$. Then, A soft set is pre G-open soft if and only if it is the intersection of G-dense soft set and open soft set.

Proof. Let H be G-dense soft set and U be open soft set, then by using (ii) of Lemma 3.1 $H\tilde{\cap}U$ is pre G-open soft set. On the other hand, let F be pre G-open soft set, then $F\tilde{\subseteq}int\varphi_G(F)$. $\tilde{X} = \varphi_G\varphi_G(F)\tilde{\cup}(\tilde{X} - \varphi_G\varphi_G(F))\tilde{\subseteq}\varphi_G(F)\tilde{\cup}(\tilde{X} - \varphi_G\varphi_G(F)) = \varphi_G(F)\tilde{\cup}(\varphi_G(\tilde{X}) - \varphi_G\varphi_G(F))\tilde{\subseteq}\varphi_G(F)\tilde{\cup}\varphi_G(\tilde{X} - \varphi_G(F))\tilde{\subseteq}\varphi_G[F\tilde{\cup}(\tilde{X} - \varphi_G(F))]$, in view of Lemmas 2.2, 2.3 and 2.4. Consequently, $[F\tilde{\cup}(\tilde{X} - \varphi_G(F))]$ is G-dense soft set. It is obvious that $F = [F\tilde{\cup}(\tilde{X} - \varphi_G(F))]\tilde{\cap}int\varphi_G(F)$.

Definition 3.19. A soft topological space (X, τ, E) is called soft G-sub maximal, if each soft G-dense set is a soft open set.

In the following theorem, conditions for the collection of soft pre G-open sets to be soft topology will be deduced.

Theorem 3.20. Let G be a soft grill on a soft topological space (X, τ, E) with $(\tau - \{\tilde{\emptyset}\})\tilde{\subseteq}G$. Then, τ is the collection of pre G-open soft sets if and only if a soft topological space (X, τ, E) is soft G-sub maximal.

Proof. Let τ be a collection of soft pre G-open sets i.e every soft pre G-open soft set is open soft, then in view of Remark 3.1, the space (X, τ, E) is soft G-sub maximal. Conversely, since $(\tau - \{\tilde{\emptyset}\})\tilde{\subseteq}G$ and F is open soft set, then F is a pre G-open soft set follows from (i) of Theorem 3.1. On the other hand, let F be a pre G-open soft set; then in view of Theorem 3.8, there exist G-dense soft set H and open soft set U such that $F = H\tilde{\cap}U$. Since (X, τ, E) be a soft G-sub maximal, then H is an open soft set. Consequently, F is an soft open soft set.

Theorem 3.21. Let G be a soft grill on a soft topological space (X, τ, E). Then, for every soft set F the following statements are equivalent

(i). F is pre G-open soft set.

(ii). F is G-dense in itself soft and pre open soft set.

(iii). F is G-dense in itself soft set and $Scl_s(F) = int\ cl(F)$.

Proof. $(i) \implies (ii)$ In view of Remark 3.1, every pre G-open soft set is G-dense in itself soft and pre open soft.

$(ii) \implies (iii)$ Let F be G-dense in itself soft and pre open soft set, then in view of Definition 2.12, $Scl_s(F) = F\tilde{\cup}int\ clF = intcl(F)$.

$(iii) \Longrightarrow (i)$ Straightforward.

Corollary 3.22. Let G be a soft grill on a soft topological space (X, τ, E). If F is a semi-closed soft and pre G-open soft set, then it is a regular open soft.

Lemma 3.23. Let G be a soft grill on a soft topology space (X, τ, E). If F is a pre G-closed soft set, then $\varphi_G(intF)\tilde{\subseteq}cl\ intF\tilde{\subseteq}F$.

Proof. Let F be pre G-closed soft set, then $(F)^c$ is pre G-open soft i.e $(F)^c\tilde{\subseteq}int\varphi_G((F)^c)\tilde{\subseteq}\varphi_G((F)^c)$. $\varphi_G((F)^c) = cl((F)^c) = (int(F))^c$ follows directly by using Theorem 2.1. Therefore, $(F)^c\tilde{\subseteq}int(intF)^c = (cl\ intF)^c$. Thus, $cl\ intF\tilde{\subseteq}F$. Moreover, $\varphi_G(intF)\tilde{\subseteq}cl\ intF\tilde{\subseteq}F$ follows from Lemma 2.2.

Theorem 3.24. Let G be a soft grill on a soft topological space (X, τ, E). If F is a pre G-closed soft and α-open soft set, then it is regular clopen soft.

Proof. Let F be a soft α-open set, then it is semi open soft and pre open soft set. Therefore, $F\tilde{\subseteq}cl\ intF$ and $F\tilde{\subseteq}int\ clF$. Since F is pre G-closed soft set, then $cl\ intF\tilde{\subseteq}F$ follows from Lemma 3.2. Hence, $cl\ intF = F$ and so $F\tilde{\subseteq}int\ clF\tilde{\subseteq}clF = cl\ cl\ intF = cl\ intF = F$. Consequently, $F = cl\ intF = int\ clF$ and so F is a soft regular clopen set.

Theorem 3.25. Let G be a soft grill on a soft topological space (X, τ, E) with $(\tau - \{\tilde{\emptyset}\})\tilde{\subseteq}G$. Then, for every soft set F the following statements are equivalent

(i). F is pre G-open soft set.

(ii). There is regular open soft set U such that $F\tilde{\subseteq}U$ and $\varphi_G(F) = \varphi_G(U)$.

(iii). $F = U\tilde{\cap}H$, where U be regular open soft set and H be G-dense soft set.

(iv). $F = U\tilde{\cap}H$, where U be open soft set and H be G-dense soft set.

Proof. $(i) \Longrightarrow (ii)$ Let F be pre G-open soft set, then $F\tilde{\subseteq}int\varphi_G(F)\tilde{\subseteq}\varphi_G(F)$. Hence, $\varphi_G(F)\tilde{\subseteq}\varphi_G(int\varphi_G(F))\tilde{\subseteq}\varphi_G\varphi_G(F)\tilde{\subseteq}\varphi_G(F)$ follows from Lemmas 2.2 and 2.3. Thus, $\varphi_G(int\varphi_G(F)) = \varphi_G(F)$. Put $U = int\varphi_G(F)$, then $F\tilde{\subseteq}U$, $\varphi_G(U) = \varphi_G(F)$ and $int\varphi_G(U) = int\varphi_G(F)$. In view of Theorem 2.1, $int\ cl(U) = int\varphi_G(U) = int\varphi_G(F) = U$. Hence, U is regular open soft set.

$(ii) \Longrightarrow (iii)$ Let $F\tilde{\subseteq}U$ and U be a soft regular open set such that $\varphi_G(F) = \varphi_G(U)$. Suppose $H = F\tilde{\cup}(U)^c$, then $\varphi_G(H) = \varphi_G(F\tilde{\cup}(U)^c) = \varphi_G(F)\tilde{\cup}\varphi_G((U)^c) = \varphi_G(U)\tilde{\cup}\varphi_G((U)^c) = \varphi_G(U\tilde{\cup}(U)^c) = \varphi_G(\tilde{X}) = \tilde{X}$, follows by using Lemmas 2.3, 2.4. Hence, H is a soft G-dense set and $U\tilde{\cap}H = F$. The rest of the proof are immediate.

The collection $G_\delta = \{F \mid int\ clF \neq \tilde{\emptyset}\}$ is a soft grill on X.

Theorem 3.26. Let G_δ be a soft grill on a soft topological space (X, τ, E). F is a soft pre G_δ-open set if and only if it is a pre open soft.

Proof. Clearly, pre G_δ-open soft set is a pre open soft. Conversely, let F be a soft pre open set, subsequently $F\tilde{\subseteq}int\ clF$. Let $x \notin \varphi_{G\delta}(F)$, then there exists soft open nbd. U of x such that $(U\tilde{\cap}F) \notin G_\delta$. Hence, $int\ cl(U\tilde{\cap}F) = \tilde{\emptyset}$ and $(U\tilde{\cap}F)\tilde{\subseteq}(U\tilde{\cap}int\ clF) = int(U\tilde{\cap}clF)\tilde{\cap}int\ cl(U\tilde{\cap}F) = \tilde{\emptyset}$ follows from Lemma 1.10. Therefore, $x \notin F$ and so $F\tilde{\subseteq}\varphi_{G\delta}(F)$. $F\tilde{\subseteq}int\ cl(F) = int\varphi_{G\delta}(F)$ follows directly by using Theorem 1.19. Consequently, F is soft pre G_δ-open set.

Definition 3.27. A soft GP-interior of F, denoted by $Sint_{GP}(F)$, is defined as the largest soft pre G-open sets contained in F.

Theorem 3.28. Let G be a soft grill on a soft topological space (X, τ, E). Then, for any soft set F the following statements hold:

(i). $(F\tilde{\cap} int\varphi_G(F))$ is a soft pre G-open set.

(ii). $Sint_{GP}(F) = F\tilde{\cap} int\varphi_G(F)$.

(iii). $F \notin G$, then $Sint_{GP}(F) = \tilde{\emptyset}$.

(iv). If F is soft pre G-open set and $Sint_{GP}(F) = \tilde{\emptyset}$, then $F = \tilde{\emptyset}$

(v). $(F\tilde{\cap}\varphi_G(F))\tilde{\subseteq} Sint_{GP}(F)$.

Proof. *(i).* Since int $\varphi_G(F) = \varphi_G(F)\tilde{\cap} int\varphi_G(F)$, then in view of Theorem 2.2 int $\varphi_G(F)\tilde{\subseteq}\varphi_G(F\tilde{\cap} int\varphi_G(F))$. Hence, $(F\tilde{\cap} int\varphi_G(F))\tilde{\subseteq} int\varphi_G(F)\tilde{\subseteq} int\varphi_G(F\tilde{\cap} int\varphi_G(F))$. Thus, $(F\tilde{\cap} int\varphi_G(F))$ is soft pre G-open set.

(ii). $(F\tilde{\cap} int\varphi_G(F))$ is soft pre G-open set contained in F, From (i). Suppose H is soft pre G-open set contained in F, then $H\tilde{\subseteq} int\varphi_G(H)$ and $int\varphi_G(H)\tilde{\subseteq} int\varphi_G(F)$. Therefore, $H\tilde{\subseteq} int\varphi_G(F)$ and so $H\tilde{\subseteq}(F\tilde{\cap} int\varphi_G(F))$. Consequently, $(F\tilde{\cap} int\varphi_G(F))$ is largest soft pre G-open sets contained in F. Then, $Sint_{GP}(F) = F\tilde{\cap} int\varphi_G(F)$.

(iii). In view of Lemma 2.2 and $F \notin G$, then $Sint_{GP}(F) = F\tilde{\cap} int\varphi_G(F) = \tilde{\emptyset}$.

(iv). Straightforward.

(v). Since $(F\tilde{\cap}\varphi_G(F)) - Sint_{GP}(F)) = [(F\tilde{\cap}\varphi_G(F)) - (F\tilde{\cap} int\varphi_G(F))] = [(F\tilde{\cap}\varphi_G(F)) - int\varphi_G(F)]$, then by (ii) $Sint_{GP}[(F\tilde{\cap}\varphi_G(F)) - int\varphi_G(F)] = [(F\tilde{\cap}\varphi_G(F)) - int\varphi_G(F)]\tilde{\cap} int\varphi_G[(F\tilde{\cap}\varphi_G(F)) - int\varphi_G(F)] = \tilde{\emptyset}$.

Corollary 3.29. $Sint_{GP}(F) = F\tilde{\cap}\varphi_G(F)$

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NEUTROSOPHIC REFINED SIMILARITY MEASURE BASED ON TANGENT FUNCTION AND ITS APPLICATION TO MULTI ATTRIBUTE DECESION MAKING

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Abstract – In the paper, tangent similarity measure of neutrosophic refined set is proposed and its properties are studied. The concept of this tangent similarity measure of single valued neutrosophic refined sets is an extension of tangent similarity measure of single valued neutrosophic sets. Finally, using the propsed refined tangent similarity measure of single valued neutrosophic sets, a numerical example on medical diagnosis is presented.

Keywords – *Refined tangent similarity measure, Neutrosophic sets, Indeterminacy Membership degree, 3D vector space, decision making.*

1. Introduction

Similarity measure is now an interesting research tropic for multi attribute decision making in current neutrosophic environment. Literature review reflects that several similarity measures have been proposed by researchers to deal with different type problems. Broumi and Smarandache [1] studied the neutrosophic Hausdorff distance between neutrosophic sets. In their study, they also presented some similarity measures based on the geometric distance models, set theoretic approach, and matching function to determine the similarity degree between neutrosophic sets. Broumi and Smarandache [2] also proposed the correlation coefficient between intervals valued neutrosophic sets. Majumdar and Samanta [3] studied several distance based similarity measures of single valued neutrosophic set (SVNS), a matching function, membership grades, and then proposed an entropy measure for a SVNS. Ye [4] proposed three vector similarity measures between SVNSs as a generalization of the Jaccard, Dice, and cosine similarity measures in vector space and

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applied them to the multicriteria decision-making problem with simplified neutrosophic information. Ye [5] also proposed single-valued neutrosophic clustering methods dealing with two distance-based similarity measures of SVNSs and presented a clustering algorithm based on the similarity measures of SVNSs to cluster single-valued neutrosophic data. Ye and Ye [6] proposed Dice similarity measure and weighted Dice similarity measure for single valued neutrosophic multisets (SVNMs) and investigated their properties. The Dice similarity measure of SVNMs proposed by Ye and Ye [6] is effective in handling the medical diagnosis problems with indeterminate and inconsistent information . Ye [7] further studied multiple attribute group decision-making method with completely unknown weights based on similarity measures under single valued neutrosophic environment. In the study, Ye [7] proposed two weight models based on the similarity measures to derive the weights of the decision makers and the attributes from the decision matrices represented by the form of single valued neutrosophic numbers (SVNNs) to decrease the effect of some unreasonable evaluations. Then, he [7] introduced the weighted similarity measure between the evaluation value (SVNS) for each alternative and the ideal solution (ideal SVNS) for the ideal alternative to rank the alternatives and select the best one(s).Ye and Zhang [8] developed three similarity measures between SVNSs based on the minimum and maximum operators and investigated their properties. Then they [8] proposed weighted similarity measure of SVNS and applied them to multiple attribute decision-making problems under single valued neutrosophic environment. Ye [9] proposed improved cosine similarity measures of simplified neutrosophicsets based on cosine function, including single valued neutrosophic cosine similarity measures and interval neutrosophic cosine similarity measures and demonstrated that improved cosine similarity measures overcome some drawbacks of existing cosine similarity measures of simplified neutrosophicsets. Biswas et al. [10] studied cosine similarity measure based multi-attribute decision-making with trapezoidal fuzzy neutrosophic numbers. They [10] developed expected value theorem and cosine similarity measure of trapezoidal fuzzy neutrosophic numbers. Pramanik and Mondal [11] proposed rough cosine similarity measure in rough neutrosophic environment. Mondal and Pramanik [12] also proposed refined cotangent similarity measure in single valued neutrosophic environment. Mondal and Pramanik [13] further proposed cotangent similarity measure under rough neutrosophic environments.

The concept of multi sets, the generalization of normal set theory was introduced by Yager [14]. Sebastian and Ramakrishnan [15] studied multi fuzzy sets, which is the generalization of multi sets. Sebastian and Ramakrishnan [16] also established more properties on multi fuzzy sets. Shinoj and John [17] extended the concept of fuzzy multi sets (FMSs) intuitionistic fuzzy multi sets (IFMSs). An element of a FMS can occur more than once with possibly the same or different membership values. An element of intuitionistic fuzzy multi sets has repeated occurrences of membership and non-membership values. Practically, the concepts of FMS and IFMS are not capable of dealing with indeterminacy. Smarandache [18] extended the classical neutrosophic logic to n-valued refined neutrosophic logic. Here each neutrosophic component T, I, F refine into respectively,T_1 ,T_2 , ... T_p, and , I_1 ,I_2 , ... I_q and F_1 ,F_2 , ... F_r. Broumi and Smarandache [19] proposed neutrosophic refined similarity measure based on cosine function.

Pramanik and Mondal [20] studied weighted fuzzy similarity measure based on tangent function and provided its application to medical diagnosis. Mondal and Pramanik [21] also proposed tangent similarity measure on intuitionistic fuzzy environment. Mondal and Pramanik [22] also proposed tangent similarity measure on neutrosophic environment.

In the paper, motivated by study of Mondal and Pramanik [12], we propose a new similarity measure called "refined tangent similarity measure for single valued neutrosophic sets". The proposed refined tangent similarity measure is applied to medical diagnosis problem.

Rest of the paper is structured as follows. Section 2 presents neutrosophic preliminaries. Section 3 is devoted to introduce refined tangent similarity measure for single valued neutrosophic sets and some of its properties. Section 4 presents decision making based on refined tangent similarity measure. Section 5 presents the application of refined tangent similarity measure to the problem on medical diagnosis. Finally, section 6 presents the concluding remarks and future scope of this research.

2. Mathematical preliminaries

2.1 Neutrosophic Sets

Definition 1 [23] Let X be an universe of discourse. Then the neutrosophic set N is of the form $N = \{< x{:}T_N(x), I_N(x), F_N(x)> x \in X\}$, where the functions T, I, F: $X \rightarrow\ ]^-0,1^+[$ are defined respectively the degree of membership, the degree of indeterminacy, and the degree of non-membership of the element $x \in X$ to the set N satisfying the following the condition.

$$^-0 \leq \sup T_N(x) + \sup I_N(x) + \sup F_N(x) \leq 3^+ \quad (1)$$

For two neutrosophic sets, $N = \{< x{:}\ T_N(x), I_N(x), F_N(x)> \mid x \in X\}$ and $P = \{< x, T_P(x), I_P(x), F_P(x)> \mid x \in X\}$ the two relations are defined as follows:

(1) $N \subseteq P$ if and only if $T_N(x) \leq T_P(x)$, $I_N(x) \geq I_P(x)$, $F_N(x) \geq F_P(x)$
(2) $N = P$ if and only if $T_N(x) = T_P(x)$, $I_N(x) = I_P(x)$, $F_N(x) = F_P(x)$

2.2 Single Valued Neutrosophic sets

Definition 2.2 [24] Let X be a space of points with generic elements in X denoted by x. A SVNS N in X is characterized by a truth-membership function $T_N(x)$, an indeterminacy-membership function $I_N(x)$, and a falsity membership function $F_N(x)$, for each point x in X, $T_N(x), I_N(x), F_N(x) \in [0, 1]$. When X is continuous, a SVNS N can be written as:

$$N = \int_X \frac{<T_N(x), I_N(x), F_N(x)>}{x} : x \in X$$

When X is discrete, a SVNS N can be written as:

$$N = \sum_{i=1}^{n} \frac{<T_N(x_i), I_N(x_i), F_N(x_i)>}{x_i} : x_i \in X$$

For two SVNSs, $N_{SVNS} = \{<x{:}\ T_N(x), I_N(x), F_N(x)> \mid x \in X\}$ and $P_{SVNS} = \{<x, T_P(x), I_P(x), F_P(x)> \mid x \in X\}$ the two relations are defined as follows:(1) $N_{SVNS} \subseteq P_{SVNS}$ if and only if $T_N(x) \leq T_P(x)$, $I_N(x) \geq I_P(x)$, $F_N(x) \geq F_P(x)$

$N_{SVNS} = P_{SVNS}$ if and only if $T_N(x) = T_P(x)$, $I_N(x) = I_P(x)$, $F_N(x) = F_P(x)$ for any $x \in X$

2.3 Neutrosophic Refined Sets

Definition 2.3 [20] Let M be a neutrosophic refined set.

$M = \{< x,\ T_M^1(x_i), T_M^2(x_i), ..., T_M^r(x_i)), (I_M^1(x_i), I_M^2(x_i), ..., I_M^r(x_i)), (F_M^1(x_i), F_M^2(x_i), ..., F_M^r(x_i))>: x \in X\}$

where, $T_M^1(x_i)$, $T_M^2(x_i)$,..., $T_M^r(x_i)$: $X \in [0,1]$, $I_M^1(x_i)$, $I_M^2(x_i)$,..., $I_M^r(x_i)$: $X \in [0,1]$, and $F_M^1(x_i)$, $F_M^2(x_i)$,..., $F_M^r(x_i)$: $X \in [0,1]$, such that $0 \le \sup T_M^i(x_i) + \sup I_M^i(x_i) + \sup F_M^i(x_i) \le 3$, for i = 1, 2, …, r for any $x \in X$.

Now, $(T_M^1(x_i), T_M^2(x_i), ..., T_M^r(x_i))$, $(I_M^1(x_i), I_M^2(x_i), ..., I_M^r(x_i))$, $(F_M^1(x_i), F_M^2(x_i), ..., F_M^r(x_i))$ is the truth-membership sequence, indeterminacy-membership sequence and falsity-membership sequence of the element x, respectively. Also, r is called the dimension of neutrosophic refined sets M.

3. Tangent Similarity Measure for Single Valued Refined Neutrosophic Sets

Let N = $<x(T_N^j(x_i), I_N^j(x_i), F_N^j(x_i))>$ and P = $< x(T_P^j(x_i), I_P^j(x_i), F_P^j(x_i))>$ be two single valued refined neutrosophic numbers. Now refined tangent similarity function which measures the similarity between two vectors based only on the direction, ignoring the impact of the distance between them can be presented as:

$$T_{NRS}(N,P) = \frac{1}{p}\sum_{j=1}^{p}\left[\frac{1}{n}\sum_{i=1}^{n}\left\langle 1-\tan\left(\frac{\pi}{12}\left(\left|T_P^j(x_i)-T_Q^j(x_i)\right|+\left|I_P^j(x_i)-I_Q^j(x_i)\right|+\left|F_P^j(x_i)-F_Q^j(x_i)\right|\right)\right)\right\rangle\right] \tag{2}$$

Proposition 3.1. The defined refined tangent similarity measure $T_{NRS}(N, P)$ between NRSs N and P satisfies the following properties:

1. $0 \le T_{NRS}(N, P) \le 1$
2. $T_{NRS}(N, P) = 1$ iff $N = P$
3. $T_{NRS}(N, P) = T_{NRS}(P, N)$
4. If R is a NRS in X and $N \subset P \subset R$ then $T_{NRS}(N, R) \le T_{NRS}(N, P)$ and $T_{NRS}(N, R) \le T_{NRS}(P, R)$

Proofs: (1) The membership, indeterminacy and non-membership functions of the NRSs are within [0 ,1]. Again $0 \le \tan\left(\frac{\pi}{12}\left(\left|T_P^j(x_i)-T_Q^j(x_i)\right|+\left|I_P^j(x_i)-I_Q^j(x_i)\right|+\left|F_P^j(x_i)-F_Q^j(x_i)\right|\right)\right) \le 1$. So, refined tangent similarity function is also within [0,1]. Hence $0 \le T_{NRS}(N, P) \le 1$

(2) For any two NRS N and P if $N = P$ this implies $T_P^j(x) = T_P^j(x)$, $I_P^j(x) = I_P^j(x)$, $F_P^j(x) = F_P^j(x)$. Hence

$\left|T_N^j(x) - T_P^j(x)\right| = 0$, $\left|I_N^j(x) - I_P^j(x)\right| = 0$, $\left|F_N^j(x) - F_P^j(x)\right| = 0$, Thus $T_{NRS}(N, P) = 1$

Conversely,

If $T_{NRS}(N, P) = 1$ then $\left|T_N^j(x) - T_P^j(x)\right| = 0$, $\left|I_N^j(x) - I_P^j(x)\right| = 0$, $\left|F_N^j(x) - F_P^j(x)\right| = 0$ since tan(0)=0. So we can write $T_P^j(x) = T_P^j(x),\ I_P^j(x) = I_P^j(x),\ F_P^j(x) = F_P^j(x)$

Hence $N = P$.

(3) This proof is obvious.

(4) If $N \subset P \subset R$ then $T_N^j(x) \le T_P^j(x) \le T_R^j(x)$, $I_N^j(x) \le I_P^j(x) \le I_R^j(x)$, $F_N^j(x) \le F_P^j(x) \le F_R^j(x)$ for $x \in X$.

Now we can write the following inequalities:

$$\left|T_N^j(x) - T_P^j(x)\right| \le \left|T_N^j(x) - T_R^j(x)\right|, \left|T_P^j(x) - T_R^j(x)\right| \le \left|T_N^j(x) - T_R^j(x)\right|;$$
$$\left|I_N^j(x) - I_P^j(x)\right| \le \left|I_N^j(x) - I_R^j(x)\right|, \left|I_P^j(x) - I_R^j(x)\right| \le \left|I_N^j(x) - I_R^j(x)\right|;$$
$$\left|F_N^j(x) - F_P^j(x)\right| \le \left|F_N^j(x) - F_R^j(x)\right| \left|F_P^j(x) - F_R^j(x)\right| \le \left|F_N^j(x) - F_R^j(x)\right|$$

Thus $T_{NRS}(N, R) \le T_{NRS}(N, P)$ and $T_{NRS}(N, R) \le T_{NRS}(P, R)$, since tangent function is increasing in the interval $\left[0, \frac{\pi}{4}\right]$.

4. Decision Making Under Single Valued Refined Neutrosophic Environment Based on Tangent Similarity Measure

Let $A_1, A_2, ..., A_m$ be a discrete set of candidates, $C_1, C_2, ..., C_n$ be the set of criteria of each candidate, and $B_1, B_2, ..., B_k$ are the alternatives of each candidates. The decision-maker provides the ranking of alternatives with respect to each candidate. The ranking presents the performances of candidates A_i (i = 1, 2,..., m) against the criteria C_j (j = 1, 2, ..., n). The single valued neutrosophic values associated with the candidates and their attributes for MADM problem can be presented in the following decision matrix (see the table 1).

Table 1: The relation between candidates and attributes

	C_1	C_2	...	C_n
A_1	$\langle d_{11}^1, d_{11}^2, \cdots, d_{11}^r \rangle$	$\langle d_{12}^1, d_{12}^2, \cdots, d_{12}^r \rangle$	...	$\langle d_{1n}^1, d_{1n}^2, \cdots, d_{1n}^r \rangle$
A_2	$\langle d_{21}^1, d_{21}^2, \cdots, d_{21}^r \rangle$	$\langle d_{22}^1, d_{22}^2, \cdots, d_{22}^r \rangle$	...	$\langle d_{2n}^1, d_{2n}^2, \cdots, d_{2n}^r \rangle$
...	...	...	...	...
A_m	$\langle d_{n1}^1, d_{n1}^2, \cdots, d_{n1}^r \rangle$	$\langle d_{n2}^1, d_{n2}^2, \cdots, d_{n2}^r \rangle$	...	$\langle d_{mn}^1, d_{mn}^2, \cdots, d_{mn}^r \rangle$

The relational values between attributes and alternatives in terms of single valued neutrosophic numbers can be presented as follows (see the table 2).

Table 2: The relation between attributes and alternatives

	B_1	B_2	...	B_k
C_1	ξ_{11}	ξ_{12}	...	ξ_{1k}
C_2	ξ_{21}	ξ_{22}	...	ξ_{2k}
...	...	...	...	...
C_n	ξ_{n1}	ξ_{n2}	...	ξ_{nk}

Here d_{ij}^{r} and ξ_{ij} and are all single valued neutrosophic numbers.

The steps corresponding to refined neutrosophic similarity measure based on tangent function are presented as follows.

Step 1: Determination the relation between candidates and attributes: Each candidate A_i (i = 1, 2, ..., m) having the attribute C_j (j = 1, 2, ..., n) is presented as follows (see the table 3):

Table 3: Relation between candidates and attributes in terms of NRSs

	C_1	C_2	...	C_n
A_1	$\left\{\begin{array}{l}\langle T_{11}^{1}, I_{11}^{1}, F_{11}^{1}\rangle, \\ \langle T_{11}^{2}, I_{11}^{2}, F_{11}^{2}\rangle, \\ \dots\dots\ \dots\dots, \\ \dots\dots\ \dots\dots, \\ \langle T_{11}^{r}, I_{11}^{r}, F_{11}^{r}\rangle\end{array}\right\}$	$\left\{\begin{array}{l}\langle T_{12}^{1}, I_{12}^{1}, F_{12}^{1}\rangle, \\ \langle T_{12}^{2}, I_{12}^{2}, F_{12}^{2}\rangle, \\ \dots\dots\ \dots\dots, \\ \dots\dots\ \dots\dots, \\ \langle T_{12}^{r}, I_{12}^{r}, F_{12}^{r}\rangle\end{array}\right\}$	...	$\left\{\begin{array}{l}\langle T_{1n}^{1}, I_{1n}^{1}, F_{1n}^{1}\rangle, \\ \langle T_{1n}^{2}, I_{1n}^{2}, F_{1n}^{2}\rangle, \\ \dots\dots\ \dots\dots, \\ \dots\dots\ \dots\dots, \\ \langle T_{2n}^{r}, I_{2n}^{r}, F_{2n}^{r}\rangle\end{array}\right\}$
A_2	$\left\{\begin{array}{l}\langle T_{21}^{1}, I_{21}^{1}, F_{21}^{1}\rangle, \\ \langle T_{21}^{2}, I_{21}^{2}, F_{21}^{2}\rangle, \\ \dots\dots\ \dots\dots, \\ \dots\dots\ \dots\dots, \\ \langle T_{21}^{r}, I_{21}^{r}, F_{21}^{r}\rangle\end{array}\right\}$	$\left\{\begin{array}{l}\langle T_{22}^{1}, I_{22}^{1}, F_{22}^{1}\rangle, \\ \langle T_{22}^{2}, I_{22}^{2}, F_{22}^{2}\rangle, \\ \dots\dots\ \dots\dots, \\ \dots\dots\ \dots\dots, \\ \langle T_{22}^{r}, I_{22}^{r}, F_{22}^{r}\rangle\end{array}\right\}$	...	$\left\{\begin{array}{l}\langle T_{2n}^{1}, I_{2n}^{1}, F_{2n}^{1}\rangle, \\ \langle T_{2n}^{2}, I_{2n}^{2}, F_{2n}^{2}\rangle, \\ \dots\dots\ \dots\dots, \\ \dots\dots\ \dots\dots, \\ \langle T_{1n}^{r}, I_{1n}^{r}, F_{1n}^{r}\rangle\end{array}\right\}$
...	...	...	...	...
A_m	$\left\{\begin{array}{l}\langle T_{m1}^{1}, I_{m1}^{1}, F_{m1}^{1}\rangle, \\ \langle T_{m1}^{2}, I_{m1}^{2}, F_{m1}^{2}\rangle, \\ \dots\dots\ \dots\dots, \\ \dots\dots\ \dots\dots, \\ \langle T_{m1}^{r}, I_{m1}^{r}, F_{m1}^{r}\rangle\end{array}\right\}$	$\left\{\begin{array}{l}\langle T_{m2}^{1}, I_{m2}^{1}, F_{m2}^{1}\rangle, \\ \langle T_{m2}^{2}, I_{m2}^{2}, F_{m2}^{2}\rangle, \\ \dots\dots\ \dots\dots, \\ \dots\dots\ \dots\dots, \\ \langle T_{m2}^{r}, I_{m2}^{r}, F_{m2}^{r}\rangle\end{array}\right\}$	...	$\left\{\begin{array}{l}\langle T_{mn}^{1}, I_{mn}^{1}, F_{mn}^{1}\rangle, \\ \langle T_{mn}^{2}, I_{mn}^{2}, F_{mn}^{2}\rangle, \\ \dots\dots\ \dots\dots, \\ \dots\dots\ \dots\dots, \\ \langle T_{mn}^{r}, I_{mn}^{r}, F_{mn}^{r}\rangle\end{array}\right\}$

Step 2: Determination the relation between attributes and alternatives: The relation between attributes C_i (i = 1, 2, ..., n) and alternatives B_t (t = 1, 2, ..., k) is presented in the table 4.

Table 4: The relation between attributes and alternatives in terms of NRSs

	B_1	B_2	...	B_k
C_1	$\langle T_{11}, I_{11}, F_{11}\rangle$	$\langle T_{12}, I_{12}, F_{12}\rangle$	...	$\langle T_{1k}, I_{1k}, F_{1k}\rangle$
C_2	$\langle T_{21}, I_{21}, F_{21}\rangle$	$\langle T_{22}, I_{22}, F_{22}\rangle$	...	$\langle T_{2k}, I_{2k}, F_{2k}\rangle$
...	...	...	...	...
C_n	$\langle T_{n1}, I_{n1}, F_{n1}\rangle$	$\langle T_{n2}, I_{n2}, F_{n2}\rangle$	...	$\langle T_{nk}, I_{nk}, F_{nk}\rangle$

Step 3: Determination of the relation between attributes and alternatives: Determine the correlation measure ($T_{NRS}(N, P)$) between the table 3 and the table 4 using equation 1.

Step 4: Ranking the alternatives: Ranking of alternatives is prepared based on the descending order of correlation measures. Highest value indicates the best alternatives.

Step 5: End

5. Example on Medical Diagnosis

Let us consider an illustrative example on medical diagnosis. As medical diagnosis contains a large amount of uncertainties and increased volume of information available to physicians from new updated technologies, the process of classifying different set of symptoms under a single name of a disease. In some practical situations, there is the possibility of each element having different truth membership, indeterminate and falsity membership functions. The proposed similarity measure among the patients versus symptoms and symptoms versus diseases will give the proper medical diagnosis. The main feature of the proposed method is that it includes multi truth membership, multi-indeterminate and multi-falsity membership by taking many times inspection for diagnosis. Now, an example of a medical diagnosis will be presented. Example: Let P = {P_1, P_2, P_3, P_4} be a set of patients, D = {Viral fever, malaria, typhoid, stomach problem, chest problem} be a set of diseases and S ={Temperature, headache, stomach pain, cough, chest pain.} be a set of symptoms. The solution strategy is to examine the patient which will provide truth membership, indeterminate and false membership function for each patient regarding the relation between patient and different symptoms. Here we take three observations in a day: at 7 am, 1 pm and 6pm. (see the table 5).

Table 5: (Relation-1)The relation between patients and symptoms

	Temperature	Headache	Stomach pain	Cough	Chest pain
P_1	(0.8, 0.1, 0.1) (0.6, 0.3, 0.3) (0.6, 0.3, 0.1)	(0.6, 0.1, 0.3) (0.5, 0.2, 0.4) (0.5, 0.1, 0.2)	(0.2, 0.8, 0.0) (0.3, 0.5, 0.2) (0.2, 0.3, 0.4)	(0.6, 0.1, 0.3) (0.4, 0.4, 0.4) (0.4, 0.3, 0.3)	(0.1,0.6, 0.3) (0.3,0.4, 0.5) (0.2,0.5, 0.4)
P_2	(0.0, 0.8, 0.2) (0.2, 0.6, 0.4) (0.1, 0.6, 0.4)	(0.4, 0.4, 0.2) (0.5, 0.4, 0.1) (0.4, 0.6, 0.3)	(0.6, 0.1, 0.3) (0.4, 0.2, 0.5) (0.3, 0.2, 0.4)	(0.1, 0.7, 0.2) (0.2, 0.7, 0.5) (0.3, 0.5, 0.4)	(0.1, 0.8, 0.1) (0.3, 0.6, 0.4) (0.3, 0.6, 0.3)
P_3	(0.8, 0.1, 0.1) (0.6, 0.4, 0.1) (0.5, 0.3, 0.3)	(0.8, 0.1, 0.1) (0.6, 0.2, 0.4) (0.6, 0.1, 0.3)	(0.0, 0.6, 0.4) (0.2, 0.5, 0.5) (0.3, 0.4, 0.6)	(0.2, 0.7, 0.1) (0.2, 0.5, 0.5) (0.1, 0.6, 0.3)	(0.0, 0.5, 0.5) (0.2, 0.5, 0.3) (0.3, 0.3, 0.4)
P_4	(06, 0.1, 0.3) (04, 0.3, 0.2) (05, 0.2, 0.3)	(0.5, 0.4, 0.1) (0.4, 0.4, 0.4) (0.5, 0.2, 0.4)	(0.3, 0.4, 0.3) (0.2, 0.4, 0.5) (0.1, 0.5, 0.4)	(0.7, 0.2, 0.1) (0.5, 0.2, 0.4) (0.6, 0.4, 0.1)	(0.3, 0.4, 0.3) (0.4, 0.3, 0.4) (0.3, 0.5, 0.5)

Now the relation between symptoms and diseases in terms of single valued neutrosophic form are given below (see table 6).

Table 6: (Relation-2)The relation between symptoms and diseases

	Viral fever	Malaria	Typhoid	Stomach problem	Chest problem
Temperature	(0.6, 0.3, 0.3)	(0.2, 0.5, 0.3)	(0.2, 0.6, 0.4)	(0.1, 0.6, 0.6)	(0.1, 0.6, 0.4)
Headache	(0.4,0.5,0.3)	(0.2, 0.6, 0.4)	(0.1, 0.5, 0.4)	(0.2, 0.4, 0.6)	(0.1, 0.6, 0.4)
Stomach pain	(0.1, 0.6, 0.3)	(0.0, 0.6, 0.4)	(0.2, 0.5, 0.5)	(0.8, 0.2, 0.2)	(0.1, 0.7, 0.1)
Cough	(0.4, 0.4, 0.4)	(0.4, 0.1, 0.5)	(0.2, 0.5, 0.5)	(0.1, 0.7, 0.4)	(0.4, 0.5, 0.4)
Chest pain	(0.1, 0.7, 0.4)	(0.1, 0.6, 0.3)	(0.1, 0.6, 0.4)	(0.1, 0.7, 0.4)	(0.8, 0.2, 0.2)

Using equation (1) the tangent refined correlation measures (TRCM) between Relation-1 and Relation-2 are presented as follows (see the table 7).

Table 7: The tangent refined correlation measure between Relation-1 and Relation-2

TRSM	Viral Fever	Malaria	Typhoid	Stomach problem	Chest problem
P_1	**0.8963**	0.8312	0.8237	0.8015	0.7778
P_2	0.8404	0.8386	**0.8877**	0.8768	0.8049
P_3	**0.8643**	0.8091	0.8393	0.7620	0.7540
P_4	**0.8893**	0.8465	0.8335	0.7565	0.7959

The highest correlation measure from the Table 7 reflects the proper medical diagnosis. Therefore, patient P_1 suffers from viral fever, P_2 suffers from typhoid, P_3 suffers from viral fever and P_4 also suffers from viral fever.

6. Conclusions

In this paper, we have proposed a refined tangent similarity measure approach of single valued neutrosophic set and proved some of their basic properties. We have presented an application of tangent similarity measure of single valued neutrosophic sets in medical diagnosis. The concept presented in the paper can be applied in other practical decision making problems involving uncertainity, falsity and indeterminacy. The proposed concept can be extended to the hybrid envirobment namely, rough neutrosophic environment.

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EFFECT OF DEFUZZIFICATION METHODS IN SOLVING FUZZY MATRIX GAMES

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Abstract – This paper deals with two-person matrix games whose elements of pay-off matrix are fuzzy numbers. Then the corresponding matrix game has been converted into crisp game using different defuzzification techniques. The value of the matrix game for each player is obtained by solving corresponding crisp game problems using the existing method. Finally, to illustrate the proposed methodology, a practical and realistic numerical example has been applied for different defuzzification methods and the obtained results have been compared.

Keywords – *Fuzzy number, Fuzzy payoff, Defuzzification, Matrix Game*

1. Introduction

In many real world practical problems with competitive situation, it is required to take the decision where there are two or more opposite parties with conflicting interests and the action of one depends upon the action which is taken by the opponent. A great variety of competitive situation is commonly seen in everyday life viz., in military battles, political campaign, elections, advertisement, etc. Game theory is a mathematical way out for finding of conflicting interests with competitive situations, which includes players or decision makers (DM) who select different strategies from the set of admissible strategies.

During the past, several researchers formulated and solved matrix game considering crisp/precise payoff. This means that every probable situation to select the payoff involved in the matrix game is perfectly known in advance. In this case, it is usually assumed that there exists some complete information about the payoff matrix. However, in real-life situations, there are not sufficient data available in most of the cases where the situation is known or it exists only a market situation. It is not always possible to observe the stability

from the statistical point of view. This means that only some partial information about the situations is known. In these cases, parameters are said to be imprecise.

To handle the problem with such types of imprecise parameters, generally stochastic, fuzzy and fuzzy-stochastic approaches are applied and the corresponding problems are converted into deterministic problems for solving them. In this paper, we have treated imprecise parameters considering fuzzy sets/fuzzy numbers. In the last few years, several attempts have been made in the existing literature for solving game problem with fuzzy payoff. Fuzziness in game problem has been well discussed by Campos [1]. Sakawa and Nishizaki [2] introduced max-min solution procedure for multi-objective fuzzy games. Based on fuzzy duality theory [3, 4, 5], Bector et al. [6, 7], and Vijay et al. [8] proved that a two person zero-sum matrix game with fuzzy goals and fuzzy payoffs is equivalent to a pair of linear programming problems. Nayak and Pal [9, 10] studied the interval and fuzzy matrix games. Chen and Larbani [11] used two persons zero-sum game approach to solve fuzzy multiple attributes decision making problem. Çevikel and Ahlatçıoglu [12] presented new concepts of solutions for multi-objective two person zerosum games with fuzzy goals and fuzzy payoffs using linear membership functions. Li and Hong [13] gave an approach for solving constrained matrix games with payoffs of triangular fuzzy numbers. Bandyopadhyay et al. [14] well studied a matrix game with payoff as triangular intuitionistic fuzzy number. Very recently, Mijanur et al. [15] introduced an alternative approach for solving fuzzy matrix games.

In this paper, two person matrix games have taken into consideration. The element of payoff matrix is considered to be fuzzy number [16]. Then the corresponding problem has been converted into crisp equivalent two person matrix game using different defuzzification methods [17]. The value of the matrix game for each player is obtained by solving corresponding crisp game problems using the existing method. Finally, to illustrate the methodology, a numerical example has been applied for different defuzzification methods and the computed results have been compared.

The rest of the paper is organized as follows. Sec. 2 presents the basic definition and preliminaries of Fuzzy Numbers. Defuzzification method is presented in Sec. 3. Mathematical model of matrix game is described in Sec. 4. Solution of matrix game is presented in Sec. 5. Numerical example and Computational results are reported in Sec. 6 and a conclusion has been drawn in Sec 7.

2. Definition and Preliminaries

Definition 2.1. Let X be a non empty set. A fuzzy set $\tilde{A}$ is defined as the set of pairs $\tilde{A}=\{(x,\mu_{\tilde{A}}(x)):x\in X\}$, where $\mu_{\tilde{A}}:X\rightarrow[0,1]$ is a mapping and $\mu_{\tilde{A}}(x)$ is called the membership function of $\tilde{A}$ or grade of membership of x in $\tilde{A}$. The value $\mu_{\tilde{A}}(x)=0$ is used to represent for complete non-membership, whereas $\mu_{\tilde{A}}(x)=1$ is used to represent for complete membership. The values in between zero and one are used to represent intermediate degrees of membership.

Definition 2.2. A fuzzy set $\tilde{A}$ is called convex iff for all $x_1,\ x_2\in X$ $\mu_{\tilde{A}}(\lambda x_1+(1-\lambda)x_2)\geq\min\{\mu_{\tilde{A}}(x_1),\mu_{\tilde{A}}(x_2)\}$, where $\lambda\in[0,1]$.

Definition 2.3. The set of elements that belong to the fuzzy set $\tilde{A}$ at least to the degree α is called the α-level set or α-cut and is given by $\tilde{A}_\alpha = \{x \in X : \mu_{\tilde{A}}(x) \geq \alpha\}$. If $\tilde{A}_\alpha = \{x \in X : \mu_{\tilde{A}}(x) > \alpha\}$, it is called strong α-level set or strong α-cut.

Definition 2.4. A fuzzy set $\tilde{A}$ is called a normal fuzzy set if there exists at least one $x \in X$ such that $\mu_{\tilde{A}}(x) = 1$.

Definition 2.5. A fuzzy number $\tilde{A}$ is a fuzzy set on the real line R, must satisfy the following conditions.

(i) There exists at least one $x_0 \in R$ for which $\mu_{\tilde{A}}(x_0) = 1$.

(ii) $\mu_{\tilde{A}}(x)$ is pair wise continuous.

(iii) $\tilde{A}$ must be convex and normal.

Definition 2.6. A triangular fuzzy number (TFN) $\tilde{A}$ is a normal fuzzy number represented by the triplet (a_1, a_2, a_3) where $a_1 \leq a_2 \leq a_3$ are real numbers and its membership function $\mu_{\tilde{A}}(x) : X \to [0,1]$ is given below

$$\mu_{\tilde{A}}(x) = \begin{cases} \dfrac{x - a_1}{a_2 - a_1} & \text{if } a_1 \leq x \leq a_2 \\ 1 & \text{if } x = a_2 \\ \dfrac{a_3 - x}{a_3 - a_2} & \text{if } a_2 \leq x \leq a_3 \\ 0 & \text{otherwise} \end{cases}$$

Definition 2.7. A parabolic fuzzy number (PFN) $\tilde{A}$ is a normal fuzzy number represented by the triplet (a_1, a_2, a_3) where $a_1 \leq a_2 \leq a_3$ are real numbers and its membership function $\mu_{\tilde{A}}(x) : X \to [0,1]$ is given below

$$\mu_{\tilde{A}}(x) = \begin{cases} 1 - \left(\dfrac{a_2 - x}{a_2 - a_1}\right)^2 & \text{if } a_1 \leq x \leq a_2 \\ 1 & \text{if } x = a_2 \\ 1 - \left(\dfrac{x - a_2}{a_3 - a_2}\right)^2 & \text{if } a_2 \leq x \leq a_3 \\ 0 & \text{otherwise} \end{cases}$$

3. Defuzzification

Defuzzification is the process of producing a quantifiable result in fuzzy logic, given the fuzzy sets and the corresponding degrees of membership. There are several defuzzification techniques available in the existing literature. However, the common and useful techniques are as follows:

3.1. Centre of Area of Fuzzy Number (COA of Fuzzy Number)

This defuzzification can be expressed as

$$x_{COA} = \frac{\int_x x\mu_A(x)\,dx}{\int_x \mu_A(x)\,dx}$$

where x_{COA} is the crisp output, $\mu_A(x)$ is the membership function corresponding to the fuzzy number and x is the output variable. This method is also known as center of gravity or centroid defuzzification method.

3.2. Bisector of Area of Fuzzy Number (BOA of Fuzzy Number)

The bisector of area is the vertical line that divides the region into two sub-regions of equal area. The formula for x_{BOA} is given by

$$\int_{a_1}^{x_{BOA}} \mu_A(x)\,dx = \int_{x_{BOA}}^{a_4} \mu_A(x)\,dx.$$

It is sometimes, but not always coincident with the centroid line.

3.3. Largest of Maxima of Fuzzy Number (LOM of Fuzzy Number)

Largest of maximum x_{LOM} takes the largest amongst all x that belong to $[a_2, a_3]$ as the crisp value.

3.4. Smallest of Maxima of Fuzzy Number (SOM of Fuzzy Number)

It takes the smallest output with the maximum membership function as the crisp value and it is denoted by x_{SOM} .

3.5. Mean of Maxima of Fuzzy Number (MOM of Fuzzy Number)

In this method only active rules with the highest degree of fulfillment are taken into account. The output is computed as:

$$x_{MOM} = \frac{1}{2}(x_{LOM} + x_{SOM}).$$

3.6. Regular Weighted Point of Fuzzy Number (RWP of Fuzzy Number)

For the fuzzy number $A=(a_1,a_2,a_3)$, the $\alpha-$cut is $A_\alpha = [L_A(\alpha), R_A(\alpha)]$ and the regular weighted point for $\tilde{A}$ is given by Saneifard [18].

$$RWP(\tilde{A}) = \frac{\int_0^1 \left(\frac{L_A(\alpha)+R_A(\alpha)}{2}\right) f(\alpha)d\alpha}{\int_0^1 f(\alpha)d\alpha} = \int_0^1 \left(L_A(\alpha)+R_A(\alpha)\right) f(\alpha)d\alpha$$

where

$$f(\alpha) = \begin{cases} 1-2\alpha & \text{when } \alpha \in [0,1/2] \\ 2\alpha-1 & \text{when } \alpha \in [1/2,1] \end{cases}.$$

3.7. Graded Mean Integration Value of Fuzzy Number (GMIV of Fuzzy Number)

For the generalized fuzzy number $\tilde{A}$ with membership function $\mu_{\tilde{A}}(x)$, according to Chen et al. [19], the Graded Mean Integral Value $P_{dGw}(\tilde{A})$ of $\tilde{A}$ is given by

$$P_{dGw}(\tilde{A}) = \frac{\int_0^1 x\left\{(1-w)L^{-1}(x) + wR^{-1}(x)\right\} dx}{\int_0^1 xdx} = 2\int_0^1 x\left\{(1-w)L^{-1}(x) + wR^{-1}(x)\right\} dx$$

where the pre-assigned parameter $w \in [0,1]$ refers the degree of optimism. $w=1$ represents an optimistic point of view, $w=0$ represents a pessimistic point of view and $w=0.5$ indicates a moderately optimistic decision makers' point of view.

3.8. Centre of the Approximated Interval of Fuzzy Number (COAI of Fuzzy Number)

Let $\tilde{A}$ be a fuzzy number with interval of confidence at the level α, then the α-cut is $[A_L(\alpha), A_R(\alpha)]$. The nearest interval approximation of $\tilde{A}$ with respect to the distance metric d is

$$C_d(\tilde{A}) = \left[\int_0^1 A_L(\alpha)d\alpha, \int_0^1 A_R(\alpha)d\alpha\right],$$

where

$$d(\tilde{A},\tilde{B})=\sqrt{\int_0^1\{A_L(\alpha)-B_L(\alpha)\}^2\,d\alpha+\int_0^1\{A_R(\alpha)-B_R(\alpha)\}^2\,d\alpha}.$$

The interval approximation for the triangular fuzzy number $\tilde{A}=(a_1,a_2,a_3)$ is $\left[\frac{(a_1+a_2)}{2},\frac{(a_2+a_3)}{2}\right]$ and for the parabolic fuzzy number $\tilde{A}=(a_1,a_2,a_3)$ is $\left[\frac{1}{3}(2a_1+a_2),\frac{1}{3}(a_2+2a_3)\right]$. The defuzzified value for triangular fuzzy number is $\frac{1}{4}(a_1+2a_2+a_3)$ and for the parabolic fuzzy number is $\frac{1}{3}(a_1+a_2+a_3)$. The defuzzification values for different fuzzy numbers are listed in Table 1.

4. Mathematical Model of a Matrix Game

Let $A_i\in\{A_1,A_2,...,A_m\}$ be a pure strategy available for player A and $B_j\in\{B_1,B_2,...,B_n\}$ be a pure strategy available for player B. When player A chooses a pure strategy A_i and the player B chooses a pure strategy B_j, then g_{ij} is the payoff for player A and $-g_{ij}$ be a payoff for player B. The two-person zero-sum matrix game G can be represented as a pay-off matrix $G=\left[g_{ij}\right]_{m\times n}$.

4.1 Fuzzy Payoff matrix:

Let players A has m strategies, say, $A_1,A_2,...,A_m$ and player B has n strategies, say, $B_1,B_2,...,B_m$.

Table 1. Defuzzified values for different fuzzy numbers.

Defuzzification technique	**Defuzzified value for TFN**	**Defuzzified value for PFN**
COA	$\frac{1}{3}(a_1+a_2+a_3)$	$\frac{1}{8}(3a_1+2a_2+3a_3)$
BOA	$\frac{1}{4}(a_1+2a_2+a_3)$	$\frac{1}{3}(a_1+a_2+a_3)$
MOM	a_2	a_2
SOM	a_2	a_2
LOM	a_2	a_2
RWP	$\frac{1}{4}(a_1+2a_2+a_3)$	$a_2+\frac{2(\sqrt{2}+1)}{15}(a_3-2a_2+a_1)$
GMIV (with $w=0.5$)	$\frac{1}{6}(a_1+4a_2+a_3)$	$\frac{1}{15}(4a_1+7a_2+4a_3)$
COAI	$\frac{1}{4}(a_1+2a_2+a_3)$	$\frac{1}{3}(a_1+a_2+a_3)$

Here, it is assumed that each player has his/her choices from amongst the pure strategies. Also, it is assumed that player A is always the gainer and player B is always the loser. That is, all payoffs are assumed in terms of player A. Let $\tilde{g}_{ij}$ be the fuzzy payoff which is the gain of player A from player B if player A chooses strategy A_i where as player B chooses B_j. Then the fuzzy payoff matrix of player A and A and B is $\tilde{G} = \left[\tilde{g}_{ij}\right]_{m\times n}$.

4.2 Mixed strategy

Let us consider the fuzzy matrix game whose payoff matrix is $\tilde{G} = \left[\tilde{g}_{ij}\right]_{m\times n}$. The mixed strategy for the player-A, is denoted by $\xi = (x_1, \cdots, x_m)'$, where $x_i \geq 0$, $i = 1, 2, ..., m$ and $\sum_{i=1}^{m} x_i = 1$. It is to be noted that $e_i^m = (0, \cdots, 0, 1, 0, \cdots, 0)'$, $i = 1, 2, \cdots, m$ represent the pure strategy for the player-A and $\xi = \sum_{i=1}^{m} e_i^m x_i$. If $S_m = \left\{ \xi : x_i \geq 0,\ \sum_{i=1}^{m} x_i = 1 \right\}$ then $S_m \in E_m$.

Similarly, a mixed strategy for the player-B is denoted by $\eta = (y_1, y_2, \cdots, y_n)'$ where $y_j \geq 0,\ j = 1, 2, \cdots, n$ and $\sum_{j=1}^{n} y_j = 1$. It is to be note that $e_j^n = (0, 0, \cdots, 0, 1, 0, \cdots, 0)'$, $j = 1, 2, \cdots, n$ represent the pure strategy of the player-B and $\eta = \sum_{j=1}^{n} e_j^n y_j$. If $S_n = \left\{ \eta : y_j \geq 0,\ \sum_{j=1}^{n} y_j = 1 \right\}$, then $S_n \in E_n$. Where S_m and S_n are the spaces of mixed strategies for the player-A and player-B respectively.

4.3. Maximin-Minimax principle or Maximin-Minimax criteria of optimality for Fuzzy Payoff matrix

Let the player A's payoff matrix be $\left[\tilde{g}_{ij}\right]_{m\times n}$. If player A takes the strategy A_i then surely he/she will get at least $i = 1, 2, ..., m$ for taking any strategy by the opponent player B. Thus by the maximin-minimax criteria of optimality, the player A will choose that strategy which corresponds to the best of these worst outcomes

$$\min_j DFV(\tilde{g}_{1j}), \min_j DFV(\tilde{g}_{2j}), ..., \min_j DFV(\tilde{g}_{mj})$$

Thus the maximin value for player A is given by $\max_i \left(\min_j DFV(\tilde{g}_{ij}) \right)$

Similarly, player B will choose that strategy which corresponds to the best (minimum) of the worst outcomes (maximum losses)

$$\max_i DFV(\tilde{g}_{i1}), \max_i DFV(\tilde{g}_{i2}), ..., \max_i DFV(\tilde{g}_{in})$$

Thus the minimax value for player B is given by $\min_j \left(\max_i DFV(\tilde{g}_{ij}) \right)$

Here, $DFV(\tilde{g}_{ij})$ represents defuzzified value of the fuzzy number $\tilde{g}_{ij}$.

Theorem 4.1. If a matrix game possesses a saddle point, it is necessary and sufficient that

$$\max_i \min_j DFV(\tilde{g}_{ij}) = \min_j \max_i DFV(\tilde{g}_{ij})$$

Definition 4.2. A pair (ξ, η) of mixed strategies for the players in a matrix game is called a situation in mixed strategies. In a situation (ξ, η) of mixed strategies each usual situation (*i, j*) in pure strategies becomes a random event occurring with probabilities x_i, y_j. Since in the situation (*i, j*), player-A receives a payoff $DFV(\tilde{g}_{ij})$, the mathematical expectation of his payoff under (ξ, η) is equal to

$$E(\xi, \eta) = \sum_{i=1}^{m} \sum_{j=1}^{n} DFV(\tilde{g}_{ij}) x_i y_j$$

Theorem 4.2. Let $E(\xi, \eta)$ be such that both $\min_{\eta \in S_n} \max_{\xi \in S_m} E(\xi, \eta)$ and $\max_{\xi \in S_m} \min_{\eta \in S_n} E(\xi, \eta)$ exist, then

$$\min_{\eta \in S_n} \max_{\xi \in S_m} E(\xi, \eta) \geq \max_{\xi \in S_m} \min_{\eta \in S_n} E(\xi, \eta)$$

4.4 Saddle point of a function

Let $E(\xi, \eta)$ be a function of two variables (vectors) ξ and η in S_m and S_n respectively. The point $(\xi_\circ, \eta_\circ)$, $\xi_\circ \in S_m$, $\eta_\circ \in S_n$ is said to be the saddle point of the function $E(\xi, \eta)$ if

$$E(\xi, \eta_\circ) \leq E(\xi_\circ, \eta_\circ) \leq E(\xi_\circ, \eta)$$

Theorem 4.3. Let $E(\xi, \eta)$ be a function of two variables $\xi \in S_m$ and $\eta \in S_n$ such that $\max_\xi \min_\eta E(\xi, \eta)$ and $\min_\eta \max_\xi E(\xi, \eta)$ exist. Then the necessary and sufficient condition for the existence of a saddle point (ξ_0, η_0) of $E(\xi, \eta)$ is that

$$E(\xi_\circ, \eta_\circ) = \max_\eta \min_\xi E(\xi, \eta) = \min_\eta \max_\xi E(\xi, \eta)$$

4.4 Value of a Matrix Game

The common value of $\max_{\xi}\left\{\min_{\eta} E(\xi,\eta)\right\}$ and $\min_{\eta}\left\{\max_{\xi} E(\xi,\eta)\right\}$ is called the value of the matrix game with payoff matrix $\tilde{G}=\left[\tilde{g}_{ij}\right]$ and denoted by $v(G)$ or simply v .

Definition 4.3. Thus if $\left(\xi^{*},\eta^{*}\right)$ is an equilibrium situation in mixed strategies of the game $\left\langle S_m,S_n,E\right\rangle$, then ξ^{*},η^{*} are the optimal strategies for the players A and B respectively in the matrix game with fuzzy payoff matrix $\tilde{G}=\left[\tilde{g}_{ij}\right]_{m\times n}$. Hence ξ^{*},η^{*} are optimal strategies for the players A and B respectively iff

$$E\left(\xi,\eta^{*}\right)\le E\left(\xi^{*},\eta^{*}\right)\le E\left(\xi^{*},\eta\right)\quad \forall \xi\in S_m,\ \eta\in S_n$$

Definition4.2.

(i) $\min_{\xi} E(\xi,\eta)=E\left(\xi,\eta^{*}\right)\Rightarrow\therefore \max_{\xi}\left\{\min_{\eta} E(\xi,\eta)\right\}=\max_{\xi} E\left(\xi,\eta^{*}\right)=E\left(\xi^{*},\eta^{*}\right)$

(ii) $\max_{\xi} E(\xi,\eta)=E\left(\xi^{*},\eta\right)\Rightarrow\min_{\eta}\left\{\max_{\xi} E(\xi,\eta)\right\}=\min_{\eta} E(\xi,\eta)=E\left(\xi^{*},\eta^{*}\right)$

Theorem 4.4. $v=\max_{\xi}\left\{\min_{j} E(\xi,j)\right\}=\min_{\eta}\left\{\max_{i} E(i,\eta)\right\}$ and the outer extrema are attained at optimal strategies of players.

Theorem 4.5. $\max_{i}\left\{\min_{j} DFV(\tilde{g}_{ij})\right\}\le v\le \min_{j}\left\{\max_{i} DFV(\tilde{g}_{ij})\right\}$

Proof: By the theorem 4.4, we have $v=\max_{\xi}\left\{\min_{j} E(\xi,j)\right\}\ \forall\xi\in S_m$. But $\max_{\xi}\left\{\min_{j} E(\xi,j)\right\}\ge\min_{j} E(\xi,j)\ \ \forall\xi\in S_m$. Therefore $v\ge\min_{j} E(\xi,j)\ \ \forall\xi\in S_m$. Letting $\xi=e_i^m$ we have $v\ge\min_{j} E\left(e_i^m,j\right)=\min_{j} E(i,j)=\min_{j} DFV(\tilde{g}_{ij})$ and we get $v\ge\min_{j} DFV(\tilde{g}_{ij})$. The left side v is independent of i so that taking maximum with respect to i, we obtain $v\ge\max_{i}\left\{\min_{j} DFV(\tilde{g}_{ij})\right\}$. Proof of the second part is similar.

Theorem 4.6.

(i). If player-A possesses a pure optimal strategy i^{*}, then

$$v=\max_{i}\left(\min_{j} DFV(\tilde{g}_{ij})\right)=\min_{j} DFV(\tilde{g}_{i^{*}j})$$

(ii). If player-B possesses a pure optimal strategy j^{*}, then

$$v=\min_{j}\left(\max_{i} DFV(\tilde{g}_{ij})\right)=\max_{i} DFV(\tilde{g}_{ij^{*}})$$

Proof: $v = \max_{\xi} \min_{j} E(\xi, j) = \min_{j} E\left(e_{i^*}^m, j\right)$ as $\xi^* = e_{i^*}^m$ is optimal. Proof of the rest is similar.

5. Solution of Matrix Game

Let us consider a 2 × 2 Matrix game whose fuzzy payoff matrix $\tilde{G}$ is given by

$$\tilde{G} = \begin{bmatrix} \tilde{g}_{11} & \tilde{g}_{12} \\ \tilde{g}_{21} & \tilde{g}_{22} \end{bmatrix}$$

If $\tilde{G}$ has a saddle point, solution is obvious.

Let $\tilde{G}$ have no saddle point. Let the player-A has the strategy $\xi = (x_1, x_2)' \equiv (x, 1-x)(0 \le x \le 1)$ and the player-B has the strategy $\eta = (y, 1-y)'(0 \le y \le 1)$. Then

$$E(\xi, \eta) = \sum_{i=1}^{2} \sum_{j=1}^{2} DFV(\tilde{g}_{ij}) x_i y_j$$

If $\xi^* = \left(x^*, 1-x^*\right)'$, $\eta^* = \left(y^*, 1-y^*\right)$ be optimal strategies, then from

$$E\left(\xi, \eta^*\right) \le E\left(\xi^*, \eta^*\right) \le E\left(\xi^*, \eta\right) \quad \forall \xi \in S_2, \eta \in S_2$$

we have $E\left(x, y^*\right) \le E\left(x^*, y^*\right) \le E\left(x^*, y\right) \ \forall x \in (0,1),\ y \in (0,1)$.

From the first part of the inequality, we set that $E\left(x, y^*\right)$ regarded as a function of x has a maximum at x^* thus,

$$\left.\frac{\partial E}{\partial x}\right|_{\left(x^*, y^*\right)} = 0 \Rightarrow y^* = \frac{DFV(\tilde{g}_{22}) - DFV(\tilde{g}_{12})}{(DFV(\tilde{g}_{11}) + DFV(\tilde{g}_{22})) - (DFV(\tilde{g}_{12}) + DFV(\tilde{g}_{21}))}.$$

Provided that $(DFV(\tilde{g}_{11}) + DFV(\tilde{g}_{22})) - (DFV(\tilde{g}_{12}) + DFV(\tilde{g}_{21})) \ne 0$

Similarly, from the second part of the inequality, it is seen that $E\left(x^*, y\right)$ regard as a function of y has a minimum at y^* i.e.,

$$\left.\frac{\partial E}{\partial y}\right|_{\left(x^*, y^*\right)} = 0 \Rightarrow x^* = \frac{DFV(\tilde{g}_{22}) - DFV(\tilde{g}_{21})}{(DFV(\tilde{g}_{11}) + DFV(\tilde{g}_{22})) - (DFV(\tilde{g}_{12}) + DFV(\tilde{g}_{21}))}.$$

Provided that $(DFV(\tilde{g}_{11})+DFV(\tilde{g}_{22}))-(DFV(\tilde{g}_{12})+DFV(\tilde{g}_{21}))\neq 0$. And

$$v^* = E\left(x^*, y^*\right) = \frac{DFV(\tilde{g}_{11})DFV(\tilde{g}_{22}) - DFV(\tilde{g}_{12})DFV(\tilde{g}_{21})}{(DFV(\tilde{g}_{11})+DFV(\tilde{g}_{22}))-(DFV(\tilde{g}_{12})+DFV(\tilde{g}_{21}))}$$

It can be proved that $(DFV(\tilde{g}_{11})+DFV(\tilde{g}_{22}))-(DFV(\tilde{g}_{12})+DFV(\tilde{g}_{21}))=0$ implies that $\tilde{G}$ has a saddle point.

6. Numerical Example

To illustrate the proposed methodology, we have solved one numerical example. In this example, the elements of payoff matrix are fuzzy valued (taken from Mijanur et al. [15]). Using eight different defuzzification methods, the matrix game has been converted into eight matrix games which are shown in Table 2. Finally, we have solved all the matrix games and computed results have been presented in Table 3.

Example-1

Suppose that there are two companies A and B to enhance the market share of a new product by competing in advertising. The two companies are considering two different strategies to increase market share: strategy I (adv. by TV), II (adv. by Newspaper). Here it is assumed that the targeted market is fixed, i.e. the market share of the one company increases while the market share of the other company decreases and also each company puts all its advertisements in one. The above problem may be regarded as matrix game. Namely, the company A and B are considered as players A and B respectively.

Table 2. Converted matrix games

Defuzzification Methods	Defuzzified Pay of Matrix for TFN	Defuzzified Pay of Matrix for PFN
COA	$\begin{pmatrix} 181.67 & 154.67 \\ 90 & 181.67 \end{pmatrix}$	$\begin{pmatrix} 181.88 & 154.50 \\ 90 & 181.88 \end{pmatrix}$
BOA	$\begin{pmatrix} 181.25 & 155.00 \\ 90 & 181.25 \end{pmatrix}$	$\begin{pmatrix} 181.65 & 154.67 \\ 90 & 181.65 \end{pmatrix}$
MOM	$\begin{pmatrix} 180.00 & 156.00 \\ 90 & 180.00 \end{pmatrix}$	$\begin{pmatrix} 180.00 & 156.00 \\ 90 & 180.00 \end{pmatrix}$
SOM	$\begin{pmatrix} 180.00 & 156.00 \\ 90 & 180.00 \end{pmatrix}$	$\begin{pmatrix} 180.00 & 156.00 \\ 90 & 180.00 \end{pmatrix}$
LOM	$\begin{pmatrix} 180.00 & 156.00 \\ 90 & 180.00 \end{pmatrix}$	$\begin{pmatrix} 180.00 & 156.00 \\ 90 & 180.00 \end{pmatrix}$
RWP	$\begin{pmatrix} 181.25 & 155.00 \\ 90 & 181.25 \end{pmatrix}$	$\begin{pmatrix} 181.61 & 154.71 \\ 90 & 181.61 \end{pmatrix}$
GMIV (with $\omega = 0.5$)	$\begin{pmatrix} 180.83 & 155.33 \\ 90 & 180.83 \end{pmatrix}$	$\begin{pmatrix} 181.33 & 154.93 \\ 90 & 181.33 \end{pmatrix}$
COAI	$\begin{pmatrix} 181.25 & 155.00 \\ 90 & 181.25 \end{pmatrix}$	$\begin{pmatrix} 181.25 & 154.67 \\ 90 & 181.25 \end{pmatrix}$

The marketing research department of company A establishes the following pay-off matrix.

$$\tilde{G} = \begin{matrix} & \text{Adv. by TV} & \text{Adv. by Newspaper} \\ \text{Adv. by TV} & (175,180,190) & (150,156,158) \\ \text{Adv by Newspaper} & (80,90,100) & (175,180,190) \end{matrix}$$

Where the element (175, 180, 190) in the matrix $\tilde{G}$ indicates that the sales amount of the company A increase by "about 180" units when the company A and B use the strategy I (adv. by TV) simultaneously. The other elements in the matrix $\tilde{G}$ can be explained similarly.

Table 3. Solutions of matrix games

Defuzzification Methods	Player-A (For TFN)			For PFN (Player-A)		
	x^*	$1-x^*$	V^*	x^*	$1-x^*$	V^*
COA	0.227522	0.772478	160.81309	0.229324	0.770676	160.80972
BOA	0.223404	0.776596	160.86436	0.227430	0.772570	160.80606
MOM	0.210526	0.789474	161.05263	0.210526	0.789474	161.05263
SOM	0.210526	0.789474	161.05263	0.210526	0.789474	161.05263
LOM	0.210526	0.789474	161.05263	0.210526	0.789474	161.05263
RWP	0.223404	0.776596	160.86436	0.226985	0.773015	160.81590
GMIV(with $\omega = 0.5$)	0.219204	0.780796	160.91970	0.224242	0.775758	160.84999
COAI	0.223404	0.776596	160.86436	0.225579	0.774421	160.66590

The computational results have been shown in Table 3 for different parametric values. From Table 3, it follows that in the case of TFN values, the best game value is obtained in the cases of MOM, SOM and LOM. In case of PFN values, the best game value is obtained in cases of MOM, SOM and LOM.

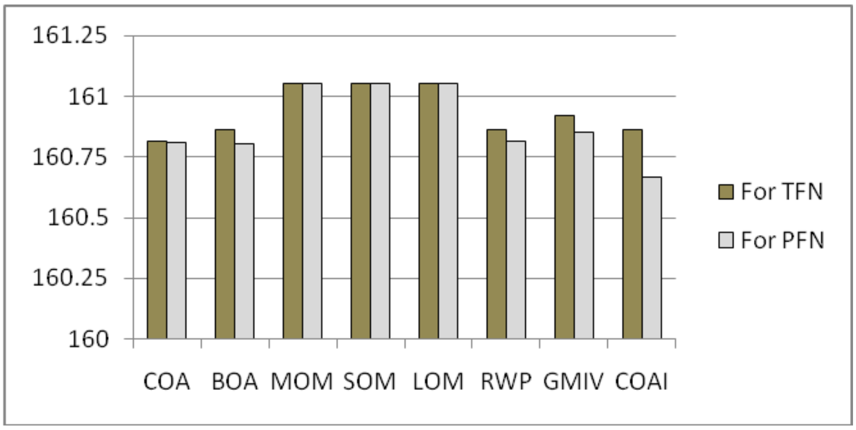

Fig. 1. Value of the game for different defuzzification methods

All the results have been shown in Fig. 1. The optimal solution sets, as obtained by the defuzzification approach, are consistent with those obtained by standard existing approach under fuzzy set up. Thus, it can be claimed that the defuzzification approach attempted in this work well to handle the matrix game with fuzzy payoff.

7. Conclusion

In this paper, a method of solving fuzzy game problem using several fuzzy defuzzification techniques of fuzzy numbers has been considered. A Numerical example is presented to illustrate the proposed methodology. Due to the choices of decision makers', the payoff value in a zero sum game might be imprecise rather than precise value. This impreciseness may be represented by various ways. In this paper, we have represented this by fuzzy number. Then the fuzzy game problem has been converted into crisp game problem after defuzzification in which all the payoff values are crisp valued. Here, several defuzzification techniques have been used to solve the fuzzy game and the corresponding crisp games with their strategies and value of the game have been presented and compared.

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Λ_g-CLOSED SETS IN IDEAL TOPOLOGICAL SPACES

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Abstract – The notion of Λ_g-closed sets is introduced in ideal topological spaces. Characterizations and properties of $\mathcal{I}_{\Lambda_g}$-closed sets and $\mathcal{I}_{\Lambda_g}$-open sets are given. A characterization of normal spaces is given in terms of $\mathcal{I}_{\Lambda_g}$-open sets. Also, it is established that an $\mathcal{I}_{\Lambda_g}$-closed subset of an $\mathcal{I}$-compact space is $\mathcal{I}$-compact.

Keywords – *λ-closed set, Λ_g-closed set, $\mathcal{I}_{\Lambda_g}$-closed set, $\mathcal{I}$-compact space.*

1 Introduction and Preliminaries

In 1986, Maki [14] introduced the notion of Λ-sets in topological spaces. A Λ-set is a set A which is equal to its kernel (= saturated set) i.e to the intersection of all open supersets of A. Arenas et al [1] introduced and investigated the notion of λ-closed sets by involving Λ-sets and closed sets. Caldas et al [2] introduced and investigated the notion of Λ_g-closed sets in topological spaces and established several properties of such sets.

In this paper, the notion of Λ_g-closed sets is introduced in ideal topological spaces. Characterizations and properties of $\mathcal{I}_{\Lambda_g}$-closed sets and $\mathcal{I}_{\Lambda_g}$-open sets are given. A characterization of normal spaces is given in terms of $\mathcal{I}_{\Lambda_g}$-open sets. Also, it is established that an $\mathcal{I}_{\Lambda_g}$-closed subset of an $\mathcal{I}$-compact space is $\mathcal{I}$-compact.

An ideal $\mathcal{I}$ on a topological space (X, τ) is a nonempty collection of subsets of X which satisfies

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1. $A \in \mathcal{I}$ and $B \subseteq A \Rightarrow B \in \mathcal{I}$ and

2. $A \in \mathcal{I}$ and $B \in \mathcal{I} \Rightarrow A \cup B \in \mathcal{I}$.

Given a topological space (X, τ) with an ideal $\mathcal{I}$ on X and if $\wp(X)$ is the set of all subsets of X, a set operator $(.)^\star : \wp(X) \rightarrow \wp(X)$, called a local function [11] of A with respect to τ and $\mathcal{I}$ is defined as follows: for $A \subseteq X$, $A^\star(\mathcal{I},\tau) = \{x \in X \mid U \cap A \notin \mathcal{I}$ for every $U \in \tau(x)\}$ where $\tau(x) = \{U \in \tau \mid x \in U\}$. We will make use of the basic facts about the local functions [[8], Theorem 2.3] without mentioning it explicitly. A Kuratowski closure operator $cl^\star(.)$ for a topology $\tau^\star(\mathcal{I},\tau)$, called the $\star$-topology, finer than τ is defined by $cl^\star(A) = A \cup A^\star(\mathcal{I},\tau)$ [24]. When there is no chance for confusion, we will simply write $A^\star$ for $A^\star(\mathcal{I},\tau)$ and $\tau^\star$ for $\tau^\star(\mathcal{I},\tau)$.

If $\mathcal{I}$ is an ideal on X, then (X, τ, $\mathcal{I}$) is called an ideal topological space. $\mathcal{N}$ is the ideal of all nowhere dense subsets in (X, τ). A subset A of an ideal topological space (X, τ, $\mathcal{I}$) is $\star$-closed [8] (resp. $\star$-dense in itself [6]) if $A^\star \subseteq A$ (resp. $A \subseteq A^\star$). A subset A of an ideal topological space (X, τ, $\mathcal{I}$) is $\mathcal{I}_g$-closed [3] if $A^\star \subseteq U$ whenever $A \subseteq U$ and U is open.

By a space, we always mean a topological space (X, τ) with no separation properties assumed. If $A \subseteq X$, cl(A) and int(A) will, respectively, denote the closure and interior of A in (X, τ) and $int^\star(A)$ will denote the interior of A in (X, $\tau^\star$).

A subset A of a space (X, τ) is an α-open [19] (resp. semi-open [12], preopen [15], regular open [23]) set if $A \subseteq int(cl(int(A)))$ (resp. $A \subseteq cl(int(A))$, $A \subseteq int(cl(A))$, $A = int(cl(A))$).

The family of all α-open sets in (X, τ), denoted by τ^α, is a topology on X finer than τ. The closure of A in (X, τ^α) is denoted by $cl_\alpha(A)$.

Definition 1.1. A subset A of a space (X, τ) is said to be

1. *g*-closed [13] if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is open.

2. *g*-open [13] if its complement is *g*-closed.

3. λ-closed [1] if $A = L \cap D$, where L is a Λ-set and D is a closed set.

4. λ-open [1] if its complement is λ-closed.

5. Λ_g-closed [2] if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is λ-open.

6. $\hat{g}$-closed [25] or ω-closed [22] or $s^\star g$-closed [10, 16, 20] if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is semi-open.

Definition 1.2. An ideal $\mathcal{I}$ is said to be

1. codense [4] or τ-boundary [18] if $\tau \cap \mathcal{I} = \{\phi\}$,

2. completely codense [4] if $PO(X) \cap \mathcal{I} = \{\phi\}$, where PO(X) is the family of all preopen sets in (X, τ).

Lemma 1.3. Every completely codense ideal is codense but not conversely [4].

The following Lemmas, Result and Definition will be useful in the sequel.

Lemma 1.4. [8] Let (X, τ, $\mathcal{I}$) be an ideal topological space and A, B subsets of X. Then the following properties hold:

1. $A \subseteq B \Rightarrow A^\star \subseteq B^\star$,
2. $A^\star = cl(A^\star) \subseteq cl(A)$,
3. $(A^\star)^\star \subseteq A^\star$,
4. $(A \cup B)^\star = A^\star \cup B^\star$,
5. $(A \cap B)^\star \subseteq A^\star \cap B^\star$.

Lemma 1.5. Let (X, τ, $\mathcal{I}$) be an ideal topological space and $A \subseteq X$. If $A \subseteq A^\star$, then $A^\star = cl(A^\star) = cl(A) = cl^\star(A)$ [[21], Theorem 5].

Lemma 1.6. Let (X, τ, $\mathcal{I}$) be an ideal topological space. Then $\mathcal{I}$ is codense if and only if $G \subseteq G^\star$ for every semi-open set G in X [[21], Theorem 3].

Lemma 1.7. Let (X, τ, $\mathcal{I}$) be an ideal topological space. If $\mathcal{I}$ is completely codense, then $\tau^\star \subseteq \tau^\alpha$ [[21], Theorem 6].

Result 1.8. For a subset of a topological space, the following properties hold:

1. Every closed set is Λ_g-closed but not conversely [2].
2. Every Λ_g-closed set is g-closed but not conversely [2].
3. Every closed set is λ-closed but not conversely [1, 2].
4. Every closed set is $\hat{g}$-closed but not conversely [25].
5. Every $\hat{g}$-closed set is g-closed but not conversely [25].

Definition 1.9. An ideal space (X, τ, $\mathcal{I}$) is said to be a $T_{\mathcal{I}}$-space [3] if every $\mathcal{I}_g$-closed subset of X is a $\star$-closed set.

Lemma 1.10. If (X, τ, $\mathcal{I}$) is a T_1-space and A is an $\mathcal{I}_g$-closed set, then A is a $\star$-closed set [[17], Corollary 2.2].

Lemma 1.11. Every g-closed set is $\mathcal{I}_g$-closed but not conversely [[3], Theorem 2.1].

Lemma 1.12. [1] Let $A_i (i \in \mathcal{I})$ be subsets of a topological space (X, τ). The following properties hold:

1. If A_i is λ-closed for each $i \in I$, then $\cap_{i \in I} A_i$ is λ-closed.
2. If A_i is λ-open for each $i \in I$, then $\cup_{i \in I} A_i$ is λ-open.

Recall that the intersection of a λ-closed set and a closed set is λ-closed.

2 Ideal Topological View of Λ_g-closed Sets

Definition 2.1. A subset A of an ideal topological space (X, τ, $\mathcal{I}$) is said to be

1. $\mathcal{I}_{\Lambda_g}$-closed if $A^\star \subseteq U$ whenever $A \subseteq U$ and U is λ-open,
2. $\mathcal{I}_{\Lambda_g}$-open if its complement is $\mathcal{I}_{\Lambda_g}$-closed.

Theorem 2.2. If (X, τ, $\mathcal{I}$) is any ideal topological space, then every $\mathcal{I}_{\Lambda_g}$-closed set is $\mathcal{I}_g$-closed but not conversely.

Proof. It follows from the fact that every open set is λ-open.

Example 2.3. Let X={a, b, c}, τ={ϕ, X, {a}, {a, b}} and $\mathcal{I}$={ϕ}. It is clear that {a, c} is $\mathcal{I}_g$-closed but it is not $\mathcal{I}_{\Lambda_g}$-closed.

The following Theorem gives characterizations of $\mathcal{I}_{\Lambda_g}$-closed sets.

Theorem 2.4. If (X, τ, $\mathcal{I}$) is any ideal topological space and $A \subseteq X$, then the following are equivalent.

1. A is $\mathcal{I}_{\Lambda_g}$-closed,
2. $cl^\star(A) \subseteq U$ whenever $A \subseteq U$ and U is λ-open in X,
3. $cl^\star(A) - A$ contains no nonempty λ-closed set,
4. $A^\star - A$ contains no nonempty λ-closed set.

Proof. (1) $\Rightarrow$ (2) Let $A \subseteq U$ where U is λ-open in X. Since A is $\mathcal{I}_{\Lambda_g}$-closed, $A^\star \subseteq U$ and so $cl^\star(A) = A \cup A^\star \subseteq U$.

(2) $\Rightarrow$ (3) Let F be a λ-closed subset such that $F \subseteq cl^\star(A) - A$. Then $F \subseteq cl^\star(A)$. Also $F \subseteq cl^\star(A) - A \subseteq X - A$ and hence $A \subseteq X - F$ where $X - F$ is λ-open. By (2) $cl^\star(A) \subseteq X - F$ and so $F \subseteq X - cl^\star(A)$. Thus $F \subseteq cl^\star(A) \cap X - cl^\star(A) = \phi$.

(3) $\Rightarrow$ (4) $A^\star - A = A \cup A^\star - A = cl^\star(A) - A$ which has no nonempty λ-closed subset by (3).

(4) $\Rightarrow$ (1) Let $A \subseteq U$ where U is λ-open. Then $X - U \subseteq X - A$ and so $A^\star \cap (X - U) \subseteq A^\star \cap (X - A) = A^\star - A$. Since $A^\star$ is always a closed subset and $X - U$ is λ-closed, $A^\star \cap (X - U)$ is a λ-closed set contained in $A^\star - A$ and hence $A^\star \cap (X - U) = \phi$ by (4). Thus $A^\star \subseteq U$ and A is $\mathcal{I}_{\Lambda_g}$-closed.

Theorem 2.5. Every $\star$-closed set is $\mathcal{I}_{\Lambda_g}$-closed but not conversely.

Proof. Let A be a $\star$-closed. To prove A is $\mathcal{I}_{\Lambda_g}$-closed, let U be any λ-open set such that $A \subseteq U$. Since A is $\star$-closed, $A^\star \subseteq A \subseteq U$. Thus A is $\mathcal{I}_{\Lambda_g}$-closed.

Example 2.6. Let X={a, b, c}, τ={ϕ, X, {a}} and $\mathcal{I}$={ϕ}. It is clear that {b} is $\mathcal{I}_{\Lambda_g}$-closed set but it is not $\star$-closed.

Theorem 2.7. Let (X, τ, $\mathcal{I}$) be an ideal topological space. For every $A \in \mathcal{I}$, A is $\mathcal{I}_{\Lambda_g}$-closed.

Proof. Let $A \in \mathcal{I}$ and let $A \subseteq U$ where U is λ-open. Since $A \in \mathcal{I}$, $A^\star = \phi \subseteq U$. Thus A is $\mathcal{I}_{\Lambda_g}$-closed.

Theorem 2.8. If (X, τ, $\mathcal{I}$) is an ideal topological space, then $A^\star$ is always $\mathcal{I}_{\Lambda_g}$-closed for every subset A of X.

Proof. Let $A^\star \subseteq U$ where U is λ-open. Since $(A^\star)^\star \subseteq A^\star$ [8], we have $(A^\star)^\star \subseteq U$. Hence $A^\star$ is $\mathcal{I}_{\Lambda_g}$-closed.

Theorem 2.9. Let (X, τ, $\mathcal{I}$) be an ideal topological space. Then every $\mathcal{I}_{\Lambda_g}$-closed, λ-open set is $\star$-closed.

Proof. Let A be $\mathcal{I}_{\Lambda_g}$-closed and λ-open. We have $A \subseteq A$ where A is λ-open. Since A is $\mathcal{I}_{\Lambda_g}$-closed, $A^\star \subseteq A$. Thus A is $\star$-closed.

Corollary 2.10. If (X, τ, $\mathcal{I}$) is a $T_{\mathcal{I}}$-space and A is an $\mathcal{I}_{\Lambda_g}$-closed set, then A is $\star$-closed set.

Proof. By assumption A is $\mathcal{I}_{\Lambda_g}$-closed in (X, τ, $\mathcal{I}$) and so by Theorem 2.2, A is $\mathcal{I}_g$-closed. Since (X, τ, $\mathcal{I}$) is a $T_{\mathcal{I}}$-space, by Definition 1.9, A is $\star$-closed.

Corollary 2.11. Let (X, τ, $\mathcal{I}$) be an ideal topological space and A be an $\mathcal{I}_{\Lambda_g}$-closed set. Then the following are equivalent.

1. A is a $\star$-closed set,
2. $cl^\star(A)-A$ is a λ-closed set,
3. $A^\star - A$ is a λ-closed set.

Proof. (1) $\Rightarrow$ (2) By (1) A is $\star$-closed. Hence $A^\star \subseteq A$ and $cl^\star(A) - A = (A \cup A^\star) - A = \phi$ which is a λ-closed set.

(2) $\Rightarrow$ (3) $A^\star - A = A \cup A^\star - A = cl^\star(A) - A$ which is a λ-closed set by (2).

(3) $\Rightarrow$ (1) Since A is $\mathcal{I}_{\Lambda_g}$-closed, by Theorem 2.4 $A^\star - A$ contains no non-empty λ-closed set. By assumption (3) $A^\star - A$ is λ-closed and hence $A^\star - A = \phi$. Thus $A^\star \subseteq A$ and A is $\star$-closed.

Theorem 2.12. Let (X, τ, $\mathcal{I}$) be an ideal topological space. Then every Λ_g-closed set is an $\mathcal{I}_{\Lambda_g}$-closed set but not conversely.

Proof. Let A be a Λ_g-closed set. Let U be any λ-open set such that $A \subseteq U$. Since A is Λ_g-closed, $cl(A) \subseteq U$. So, by Lemma 1.4, $A^\star \subseteq cl(A) \subseteq U$ and thus A is $\mathcal{I}_{\Lambda_g}$-closed.

Example 2.13. Let X={a, b, c}, τ={ϕ, X, {a}, {c}, {a, b}, {a, c}} and $\mathcal{I}$={ϕ, {a}}. It is clear that {a} is $\mathcal{I}_{\Lambda_g}$-closed set but it is not Λ_g-closed.

Theorem 2.14. If (X, τ, $\mathcal{I}$) is an ideal topological space and A is a $\star$-dense in itself, $\mathcal{I}_{\Lambda_g}$-closed subset of X, then A is Λ_g-closed.

Proof. Let $A \subseteq U$ where U is λ-open. Since A is $\mathcal{I}_{\Lambda_g}$-closed, $A^\star \subseteq U$. As A is $\star$-dense in itself, by Lemma 1.5, $cl(A) = A^\star$. Thus $cl(A) \subseteq U$ and hence A is Λ_g-closed.

Corollary 2.15. If (X, τ, $\mathcal{I}$) is any ideal topological space where $\mathcal{I}=\{\phi\}$, then A is $\mathcal{I}_{\Lambda_g}$-closed if and only if A is Λ_g-closed.

Proof. In (X, τ, $\mathcal{I}$), if $\mathcal{I} = \{\phi\}$ then $A^\star = cl(A)$ for the subset A. A is $\mathcal{I}_{\Lambda_g}$-closed $\Leftrightarrow$ $A^\star \subseteq U$ whenever $A \subseteq U$ and U is λ-open $\Leftrightarrow$ $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is λ-open $\Leftrightarrow$ A is Λ_g-closed.

Corollary 2.16. In an ideal topological space (X, τ, $\mathcal{I}$) where $\mathcal{I}$ is codense, if A is a semi-open and $\mathcal{I}_{\Lambda_g}$-closed subset of X, then A is Λ_g-closed.

Proof. By Lemma 1.6, A is $\star$-dense in itself. By Theorem 2.14, A is Λ_g-closed.

Example 2.17. In Example 2.3, it is clear that {a, c} is g-closed set but it is not $\mathcal{I}_{\Lambda_g}$-closed.

Example 2.18. In Example 2.13, it is clear that {a} is $\mathcal{I}_{\Lambda_g}$-closed set but it is not g-closed.

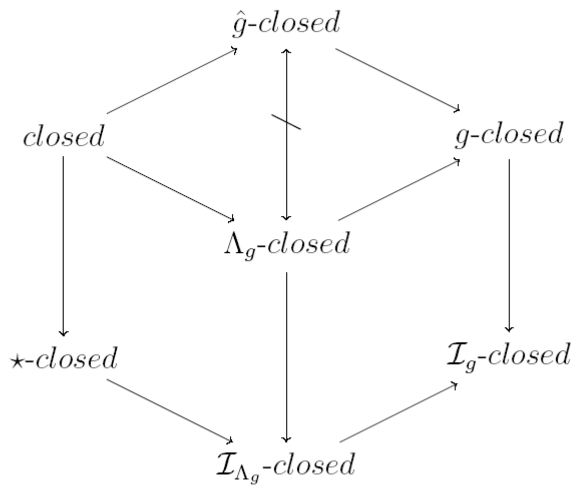
Example 2.19. In Example 2.6, it is clear that {b} is Λ_g-closed but it is not $\hat{g}$-closed.

Example 2.20. In Example 2.6, it is clear that {a} is $\hat{g}$-closed but it is not Λ_g-closed.

Remark 2.21. We see that

1. From Examples 2.17 and 2.18, g-closed sets and $\mathcal{I}_{\Lambda_g}$-closed sets are independent.
2. From Examples 2.19 and 2.20, Λ_g-closed sets and $\hat{g}$-closed sets are independent.

Remark 2.22. We have the following implications for the subsets stated above.



Theorem 2.23. Let (X, τ, $\mathcal{I}$) be an ideal topological space and A⊆X. Then A is $\mathcal{I}_{\Lambda_g}$-closed if and only if A=F−N where F is $\star$-closed and N contains no nonempty λ-closed set.

Proof. If A is $\mathcal{I}_{\Lambda_g}$-closed, then by Theorem 2.4 (4), N=$A^\star$−A contains no nonempty λ-closed set. If F=$cl^\star$(A), then F is $\star$-closed such that $F-N=(A\cup A^\star)-(A^\star-A)=(A\cup A^\star)\cap(A^\star\cap A^c)^c=(A\cup A^\star)\cap((A^\star)^c\cup A)=(A\cup A^\star)\cap(A\cup(A^\star)^c)= A\cup(A^\star\cap(A^\star)^c)=A$.

Conversely, suppose A=F−N where F is $\star$-closed and N contains no nonempty λ-closed set. Let U be an λ-open set such that A⊆U. Then F−N⊆U which implies that F∩(X−U)⊆N. Now A⊆F and $F^\star\subseteq F$ then $A^\star\subseteq F^\star$ and so $A^\star\cap(X-U)\subseteq F^\star\cap(X-U)\subseteq F\cap(X-U)\subseteq N$. Since $A^\star\cap(X-U)$ is λ-closed, by hypothesis $A^\star\cap(X-U)=\phi$ and so $A^\star\subseteq U$. Hence A is $\mathcal{I}_{\Lambda_g}$-closed.

Theorem 2.24. Let (X, τ, $\mathcal{I}$) be an ideal topological space and A⊆X. If $A\subseteq B\subseteq A^\star$, then $A^\star=B^\star$ and B is $\star$-dense in itself.

Proof. Since A⊆B, then $A^\star\subseteq B^\star$ and since $B\subseteq A^\star$, then $B^\star\subseteq(A^\star)^\star\subseteq A^\star$. Therefore $A^\star=B^\star$ and $B\subseteq A^\star\subseteq B^\star$. Hence proved.

Theorem 2.25. Let (X, τ, $\mathcal{I}$) be an ideal topological space. If A and B are subsets of X such that A$\subseteq$B$\subseteq$cl*(A) and A is $\mathcal{I}_{\Lambda_g}$-closed, then B is $\mathcal{I}_{\Lambda_g}$-closed.

Proof. Since A is $\mathcal{I}_{\Lambda_g}$-closed, then by Theorem 2.4 (3), cl*(A)$-$A contains no nonempty λ-closed set. But cl*(B)$-$B$\subseteq$cl*(A)$-$A and so cl*(B)$-$B contains no nonempty λ-closed set. Hence B is $\mathcal{I}_{\Lambda_g}$-closed.

Corollary 2.26. Let (X, τ, $\mathcal{I}$) be an ideal topological space. If A and B are subsets of X such that A$\subseteq$B$\subseteq$A* and A is $\mathcal{I}_{\Lambda_g}$-closed, then A and B are Λ_g-closed sets.

Proof. Let A and B be subsets of X such that A$\subseteq$B$\subseteq$A*. Then A$\subseteq$B $\subseteq$A*$\subseteq$cl*(A). Since A is $\mathcal{I}_{\Lambda_g}$-closed, by Theorem 2.25, B is $\mathcal{I}_{\Lambda_g}$-closed. Since A$\subseteq$B$\subseteq$A*, we have A* = B*. Hence A $\subseteq$ A* and B $\subseteq$ B*. Thus A is $\star$-dense in itself and B is $\star$-dense in itself and by Theorem 2.14, A and B are Λ_g-closed.

The following Theorem gives a characterization of $\mathcal{I}_{\Lambda_g}$-open sets.

Theorem 2.27. Let (X, τ, $\mathcal{I}$) be an ideal topological space and A$\subseteq$X. Then A is $\mathcal{I}_{\Lambda_g}$-open if and only if F$\subseteq$int*(A) whenever F is λ-closed and F$\subseteq$A.

Proof. Suppose A is $\mathcal{I}_{\Lambda_g}$-open. If F is λ-closed and F$\subseteq$A, then X$-$A$\subseteq$X$-$F and so cl*(X$-$A)$\subseteq$X$-$F by Theorem 2.4(2). Therefore F$\subseteq$X$-$cl*(X$-$A)=int*(A). Hence F$\subseteq$int*(A).

Conversely, suppose the condition holds. Let U be a λ-open set such that X$-$A$\subseteq$U. Then X$-$U$\subseteq$A and so X$-$U$\subseteq$int*(A). Therefore cl*(X$-$A)$\subseteq$U. By Theorem 2.4(2), X$-$A is $\mathcal{I}_{\Lambda_g}$-closed. Hence A is $\mathcal{I}_{\Lambda_g}$-open.

Corollary 2.28. Let (X, τ, $\mathcal{I}$) be an ideal topological space and A$\subseteq$X. If A is $\mathcal{I}_{\Lambda_g}$-open, then F$\subseteq$int*(A) whenever F is closed and F$\subseteq$A.

The following Theorem gives a property of $\mathcal{I}_{\Lambda_g}$-closed.

Theorem 2.29. Let (X, τ, $\mathcal{I}$) be an ideal topological space and A$\subseteq$X. If A is $\mathcal{I}_{\Lambda_g}$-open and int*(A)$\subseteq$B$\subseteq$A, then B is $\mathcal{I}_{\Lambda_g}$-open.

Proof. Since int*(A) $\subseteq$ B $\subseteq$ A, we have X $-$ A $\subseteq$ X $-$ B $\subseteq$ X $-$ int*(A) = cl*(X $-$ A). By assumption A is $\mathcal{I}_{\Lambda_g}$-open and so X $-$ A is $\mathcal{I}_{\Lambda_g}$-closed. Hence by Theorem 2.25, X $-$ B is $\mathcal{I}_{\Lambda_g}$-closed and B is $\mathcal{I}_{\Lambda_g}$-open.

The following Theorem gives a characterization of $\mathcal{I}_{\Lambda_g}$-closed sets in terms of $\mathcal{I}_{\Lambda_g}$-open sets.

Theorem 2.30. Let (X, τ, $\mathcal{I}$) be an ideal topological space and A$\subseteq$X. Then the following are equivalent.

1. A is $\mathcal{I}_{\Lambda_g}$-closed,
2. A$\cup$(X$-$A*) is $\mathcal{I}_{\Lambda_g}$-closed,
3. A*$-$A is $\mathcal{I}_{\Lambda_g}$-open.

Proof. (1)$\Rightarrow$(2) Let U be any λ-open set such that A$\cup$(X$-$A*)$\subseteq$U. Then U^c $\subseteq$ [A$\cup$(X$-$A*)]^c = [A $\cup$ (A*)^c]^c = A*$\cap$A^c=A*$-$A where U^c is λ-closed. Since A is $\mathcal{I}_{\Lambda_g}$-closed, by Theorem 2.4(4), U^c = ϕ and X=U. Thus X is the only λ-open set containing A$\cup$(X$-$A*) and hence A$\cup$(X$-$A*) is $\mathcal{I}_{\Lambda_g}$-closed.

(2)$\Rightarrow$(3) (A* $-$ A)^c = (A* $\cap$ A^c)^c = A $\cup$ A$^{\star c}$ = A $\cup$ (X $-$ A*) which is $\mathcal{I}_{\Lambda_g}$-closed by (2). Hence A* $-$ A is $\mathcal{I}_{\Lambda_g}$-open.

(3) ⇒ (1) Since $A^\star - A$ is $\mathcal{I}_{\Lambda_g}$-open, $(A^\star - A)^c = A \cup A^{\star c}$ is $\mathcal{I}_{\Lambda_g}$-closed. Hence by Theorem 2.4(4) $(A \cup (A^\star)^c)^\star - (A \cup A^{\star c})$ contains no nonempty λ-closed subset. But $(A \cup (A^\star)^c)^\star - (A \cup (A^\star)^c) = (A \cup (A^\star)^c)^\star \cap (A \cup (A^\star)^c)^c = (A \cup (A^\star)^c)^\star \cap (A^\star \cup A^c) = (A^\star \cup ((A^\star)^c)^\star) \cap (A^\star \cap A^c) = A^\star \cap A^c = A^\star - A$. Thus $A^\star - A$ has no nonempty λ-closed subset. Hence by Theorem 2.4(4), A is $\mathcal{I}_{\Lambda_g}$-closed.

Theorem 2.31. Let (X, τ, $\mathcal{I}$) be an ideal topological space. Then every subset of X is $\mathcal{I}_{\Lambda_g}$-closed if and only if every λ-open set is $\star$-closed.

Proof. Suppose every subset of X is $\mathcal{I}_{\Lambda_g}$-closed. Let U be λ-open in X. Then $U \subseteq U$ and U is $\mathcal{I}_{\Lambda_g}$-closed by assumption implies $U^\star \subseteq U$. Hence U is $\star$-closed.

Conversely, let $A \subseteq X$ and U be λ-open such that $A \subseteq U$. Since U is $\star$-closed by assumption, we have $A^\star \subseteq U^\star \subseteq U$. Thus A is $\mathcal{I}_{\Lambda_g}$-closed.

The following Theorem gives a characterization of normal spaces in terms of $\mathcal{I}_{\Lambda_g}$-open sets.

Theorem 2.32. Let (X, τ, $\mathcal{I}$) be an ideal topological space where $\mathcal{I}$ is completely codense. Then the following are equivalent.

1. X is normal,
2. For any disjoint closed sets A and B, there exist disjoint $\mathcal{I}_{\Lambda_g}$-open sets U and V such that $A \subseteq U$ and $B \subseteq V$,
3. For any closed set A and open set V containing A, there exists an $\mathcal{I}_{\Lambda_g}$-open set U such that $A \subseteq U \subseteq cl^\star(U) \subseteq V$.

Proof. (1)⇒(2) The proof follows from the fact that every open set is $\mathcal{I}_{\Lambda_g}$-open.

(2)⇒(3) Suppose A is closed and V is an open set containing A. Since A and X−V are disjoint closed sets, there exist disjoint $\mathcal{I}_{\Lambda_g}$-open sets U and W such that $A \subseteq U$ and $X - V \subseteq W$. Since X−V is λ-closed and W is $\mathcal{I}_{\Lambda_g}$-open, $X - V \subseteq int^\star(W)$. Then $X - int^\star(W) \subseteq V$. Again $U \cap W = \phi$ which implies that $U \cap int^\star(W) = \phi$ and so $U \subseteq X - int^\star(W)$. Then $cl^\star(U) \subseteq X - int^\star(W) \subseteq V$ and thus U is the required $\mathcal{I}_{\Lambda_g}$-open sets with $A \subseteq U \subseteq cl^\star(U) \subseteq V$.

(3)⇒(1) Let A and B be two disjoint closed subsets of X. Then A is a closed set and X − B an open set containing A. By hypothesis, there exists an $\mathcal{I}_{\Lambda_g}$-open set U such that $A \subseteq U \subseteq cl^\star(U) \subseteq X - B$. Since U is $\mathcal{I}_{\Lambda_g}$-open and A is λ-closed we have $A \subseteq int^\star(U)$. Since $\mathcal{I}$ is completely codense, by Lemma 1.7, $\tau^\star \subseteq \tau^\alpha$ and so $int^\star(U)$ and $X - cl^\star(U) \in \tau^\alpha$. Hence $A \subseteq int^\star(U) \subseteq int(cl(int(int^\star(U)))) = G$ and $B \subseteq X - cl^\star(U) \subseteq int(cl(int(X - cl^\star(U)))) = H$. G and H are the required disjoint open sets containing A and B respectively, which proves (1).

Definition 2.33. A subset A of a topological space (X, τ) is said to be an $\Lambda_{g\alpha}$-closed set if $cl_\alpha(A) \subseteq U$ whenever $A \subseteq U$ and U is λ-open. The complement of $\Lambda_{g\alpha}$-closed is said to be an $\Lambda_{g\alpha}$-open set.

If $\mathcal{I} = \mathcal{N}$, it is not difficult to see that $\mathcal{I}_{\Lambda_g}$-closed sets coincide with $\Lambda_{g\alpha}$-closed sets and so we have the following Corollary.

Corollary 2.34. Let (X, τ, $\mathcal{I}$) be an ideal topological space where $\mathcal{I} = \mathcal{N}$. Then the following are equivalent.

1. X is normal,

2. For any disjoint closed sets A and B, there exist disjoint $\Lambda_{g\alpha}$-open sets U and V such that A⊆U and B⊆V,

3. For any closed set A and open set V containing A, there exists an $\Lambda_{g\alpha}$-open set U such that A⊆U⊆cl_α(U)⊆V.

Definition 2.35. A subset A of an ideal topological space is said to be $\mathcal{I}$-compact [5] or compact modulo $\mathcal{I}$ [18] if for every open cover $\{U_\alpha \mid \alpha\in\Delta\}$ of A, there exists a finite subset Δ_0 of Δ such that $A-\cup\{U_\alpha \mid \alpha\in\Delta_0\}\in\mathcal{I}$. The space (X, τ, $\mathcal{I}$) is $\mathcal{I}$-compact if X is $\mathcal{I}$-compact as a subset.

Theorem 2.36. Let (X, τ, $\mathcal{I}$) be an ideal topological space. If A is an $\mathcal{I}_g$-closed subset of X, then A is $\mathcal{I}$-compact [[17], Theorem 2.17].

Corollary 2.37. Let (X, τ, $\mathcal{I}$) be an ideal topological space. If A is an $\mathcal{I}_{\Lambda_g}$-closed subset of X, then A is $\mathcal{I}$-compact.

Proof. The proof follows from the fact that every $\mathcal{I}_{\Lambda_g}$-closed is $\mathcal{I}_g$-closed.

3 λ-$\mathcal{I}$-locally Closed Sets

Definition 3.1. A subset A of an ideal topological space (X, τ, $\mathcal{I}$) is called a λ-$\mathcal{I}$-locally closed set (briefly, λ-$\mathcal{I}$-LC) if A=U∩V where U is λ-open and V is $\star$-closed.

Definition 3.2. [9] A subset A of an ideal topological space (X, τ, $\mathcal{I}$) is called a weakly $\mathcal{I}$-locally closed set (briefly, weakly $\mathcal{I}$-LC) if A=U∩V where U is open and V is $\star$-closed.

Proposition 3.3. Let (X, τ, $\mathcal{I}$) be an ideal topological space and A a subset of X. Then the following hold.

1. If A is λ-open, then A is λ-$\mathcal{I}$-LC-set.

2. If A is $\star$-closed, then A is λ-$\mathcal{I}$-LC-set.

3. If A is a weakly $\mathcal{I}$-LC-set, then A is a λ-$\mathcal{I}$-LC-set.

The converses of the above Proposition 3.3 need not be true as shown in the following examples.

Example 3.4. 1. In Example 2.6, it is clear that {a} is a λ-$\mathcal{I}$-LC-set but it is not $\star$-closed.

2. In Example 2.3, it is clear that {b} is a λ-$\mathcal{I}$-LC-set but it is not λ-open.

Example 3.5. In Example 2.3, it is clear that {a, c} is a λ-$\mathcal{I}$-LC-set but it is not a weakly $\mathcal{I}$-LC-set.

Theorem 3.6. Let (X, τ, $\mathcal{I}$) be an ideal topological space. If A is a λ-$\mathcal{I}$-LC-set and B is a $\star$-closed set, then A∩B is a λ-$\mathcal{I}$-LC-set.

Proof. Let B be $\star$-closed, then A∩B=(U∩V)∩B=U∩(V∩B), where V∩B is $\star$-closed. Hence A∩B is a λ-$\mathcal{I}$-LC-set.

Theorem 3.7. A subset of an ideal topological space (X, τ, $\mathcal{I}$) is $\star$-closed if and only if it is

1. weakly $\mathcal{I}$-LC and $\mathcal{I}_g$-closed [7]

2. λ-$\mathcal{I}$-LC and $\mathcal{I}_{\Lambda_g}$-closed.

Proof. (2) Necessity is trivial. We prove only sufficiency. Let A be λ-$\mathcal{I}$-LC-set and $\mathcal{I}_{\Lambda_g}$-closed set. Since A is λ-$\mathcal{I}$-LC, A=U∩V, where U is λ-open and V is $\star$-closed. So, we have A=U∩V⊆U. Since A is $\mathcal{I}_{\Lambda_g}$-closed, $A^\star \subseteq U$. Also since A = U∩V⊆V and V is $\star$-closed, we have $A^\star \subseteq V$. Consequently, $A^\star \subseteq U \cap V = A$ and hence A is $\star$-closed.

Remark 3.8. 1. The notions of weakly $\mathcal{I}$-LC-set and $\mathcal{I}_g$-closed set are independent [7].

2. The notions of λ-$\mathcal{I}$-LC-set and $\mathcal{I}_{\Lambda_g}$-closed set are independent.

Example 3.9. In Example 2.6, it is clear that {a} is λ-$\mathcal{I}$-LC-set but not $\mathcal{I}_{\Lambda_g}$-closed.

Example 3.10. In Example 2.6, it is clear that {a, c} is $\mathcal{I}_{\Lambda_g}$-closed set but not λ-$\mathcal{I}$-LC-set.

Definition 3.11. Let A be a subset of a topological space (X, τ). Then the λ-kernel of the set A, denoted by λ-ker(A), is the intersection of all λ-open supersets of A.

Definition 3.12. A subset A of a topological space (X, τ) is called Λ_λ-set if A=λ-ker(A).

Definition 3.13. A subset A of an ideal topological space (X, τ, $\mathcal{I}$) is called $\lambda^\star$-$\mathcal{I}$-closed if A=L∩F where L is a Λ_λ-set and F is $\star$-closed.

Lemma 3.14. 1. Every $\star$-closed set is $\lambda^\star$-$\mathcal{I}$-closed but not conversely.

2. Every Λ_λ-set is $\lambda^\star$-$\mathcal{I}$-closed but not conversely.

3. Every λ-$\mathcal{I}$-LC-set is $\lambda^\star$-$\mathcal{I}$-closed but not conversely.

Example 3.15. In Example 2.6, it is clear that {a} is $\lambda^\star$-$\mathcal{I}$-closed set but not $\star$-closed.

Example 3.16. Let X = {a, b, c, d}, τ = {ϕ, X, {c}, {d}, {a, c}, {c, d}, {a, c, d}} and $\mathcal{I}$ = {ϕ}. It is clear that {a} is $\lambda^\star$-$\mathcal{I}$-closed but not a Λ_λ-set.

Example 3.17. In Example 3.16, it is clear that {a} is $\lambda^\star$-$\mathcal{I}$-closed but not a λ-$\mathcal{I}$-LC-set.

Remark 3.18. The following Example supports the concepts of Λ_λ-set and $\star$-closed set are independent. Let X = {a, b, c}, τ = {ϕ, X, {b, c}} and $\mathcal{I}$ = {ϕ, {a}, {b}, {a, b}}. It is clear that {b, c} is a Λ_λ-set but not a $\star$-closed whereas {b} is $\star$-closed but not a Λ_λ-set.

Lemma 3.19. For a subset A of an ideal topological space (X, τ, $\mathcal{I}$), the following are equivalent.

1. A is $\lambda^\star$-$\mathcal{I}$-closed.

2. A=L∩cl*(A) where L is a Λ_λ-set.

3. A=λ-ker(A)∩cl*(A).

Lemma 3.20. A subset A⊆(X, τ, $\mathcal{I}$) is $\mathcal{I}_{\Lambda_g}$-closed if and only if cl*(A)⊆ λ-ker(A).

Proof. Suppose that A $\subseteq$ X is an $\mathcal{I}_{\Lambda_g}$-closed set. Suppose x $\notin$ λ-ker(A). Then there exists an λ-open set U containing A such that x $\notin$ U. Since A is an $\mathcal{I}_{\Lambda_g}$-closed set, A $\subseteq$ U and U is λ-open implies that cl*(A) $\subseteq$ U and so x $\notin$ cl*(A). Therefore cl*(A) $\subseteq$ λ-ker(A).

Conversely, suppose cl*(A) $\subseteq$ λ-ker(A). If A $\subseteq$ U and U is λ-open, then cl*(A) $\subseteq$ λ-ker(A) $\subseteq$ U. Therefore, A is $\mathcal{I}_{\Lambda_g}$-closed.

Theorem 3.21. For a subset A of an ideal topological space (X, τ, $\mathcal{I}$), the following are equivalent.

1. A is $\star$-closed.
2. A is $\mathcal{I}_{\Lambda_g}$-closed and λ-$\mathcal{I}$-LC.
3. A is $\mathcal{I}_{\Lambda_g}$-closed and $\lambda^\star$-$\mathcal{I}$-closed.

Proof. (1)$\Rightarrow$(2)$\Rightarrow$(3) Obvious.

(3)$\Rightarrow$(1) Since A is $\mathcal{I}_{\Lambda_g}$-closed, by Lemma 3.20, cl*(A)$\subseteq$$\lambda$-ker(A). Since A is $\lambda^\star$-$\mathcal{I}$-closed, by Lemma 3.19, A=λ-ker(A)$\cap$cl*(A)=cl*(A). Hence A is $\star$-closed.

The following two Examples show that the concepts of $\mathcal{I}_{\Lambda_g}$-closedness and $\lambda^\star$-$\mathcal{I}$-closedness are independent.

Example 3.22. In Example 2.6, it is clear that {b} is $\mathcal{I}_{\Lambda_g}$-closed set but not $\lambda^\star$-$\mathcal{I}$-closed.

Example 3.23. In Example 2.6, it is clear that {a} is $\lambda^\star$-$\mathcal{I}$-closed but not $\mathcal{I}_{\Lambda_g}$-closed.

4 Decompositions of $\star$-continuity

Definition 4.1. A function f : (X, τ, $\mathcal{I}$)$\rightarrow$(Y, σ) is said to be $\star$-continuous [7] (resp. $\mathcal{I}_g$-continuous [7], λ-$\mathcal{I}$-LC-continuous, $\lambda^\star$-$\mathcal{I}$-continuous, $\mathcal{I}_{\Lambda_g}$-continuous, weakly $\mathcal{I}$-LC-continuous [9]) if f^{-1}(A) is $\star$-closed (resp. $\mathcal{I}_g$-closed, λ-$\mathcal{I}$-LC-set, $\lambda^\star$-$\mathcal{I}$-closed, $\mathcal{I}_{\Lambda_g}$-closed, weakly $\mathcal{I}$-LC-set) in (X, τ, $\mathcal{I}$) for every closed set A of (Y, σ).

Theorem 4.2. A function f : (X, τ, $\mathcal{I}$)$\rightarrow$(Y, σ) is $\star$-continuous if and only if it is

1. weakly $\mathcal{I}$-LC-continuous and $\mathcal{I}_g$-continuous [7].
2. λ-$\mathcal{I}$-LC-continuous and $\mathcal{I}_{\Lambda_g}$-continuous.

Proof. It is an immediate consequence of Theorem 3.7.

Theorem 4.3. For a function f : (X, τ, $\mathcal{I}$)$\rightarrow$(Y, σ), the following are equivalent.

1. f is $\star$-continuous.
2. f is $\mathcal{I}_{\Lambda_g}$-continuous and λ-$\mathcal{I}$-LC-continuous.
3. f is $\mathcal{I}_{\Lambda_g}$-continuous and $\lambda^\star$-$\mathcal{I}$-continuous.

Proof. It is an immediate consequence of Theorem 3.21.

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TOPOLOGY SPECTRUM OF A KU-ALGEBRA

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Abstract – The aim of this paper is to study the Zariski topology of a commutative KU-algebra. Firstly, we introduce new concepts of a KU-algebra, such as KU-lattice, involutory ideal and prime ideal and investigate some basic properties of these concepts. Secondly, the notion of the topology spectrum of a commutative KU-algebra is studied and several properties of this topology are provided. Also, we study the continuous map of this topological space.

Keywords – *KU-algebra, KU-lattice, involutory ideal, prime ideal, topology spectrum.*

1. Introduction

The Zariski topology on the spectrum of prime ideals of a commutative ring is one of the main tools in Algebraic Geometry. Atiyah and Macdonald [1] introduced the spectrum $Spc(R)$ of a ring R as the following: for each ideal I of R , $V(I)=\{P\in Spec(R): I\subseteq P\}$, then the set $V(I)$ satisfy the axioms for the closed sets of a topology on $Spc(R)$, called the Zariski topology. Also, the notion of a spectrum of modules has been introduced by many authors see [2, 5, 6 and 7]. Prabpayak and Leerawat [11] introduced a new algebraic structure which is called KU-algebras. They introduced the concept of homomorphisms of KU-algebras and investigated some related properties. In [3, 4, 12 and 13], the authors introduced topologies on the set of all prime ideals by different way. In this paper, we study the relationship between a KU-algebra and topological space by the notion of the Zariski topology. We give the new concept of KU-lattice, involutory ideal and prime ideal of a KU-algebra X and discuss some properties which related to these concepts. Consequently,

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we show that $Spc(X)$ of a KU-algebra X is a compact and disconnected space. Also, we study some of separation axioms and continuous map of this topological space.

2. Preliminaries

Now we recall some known concepts related to KU-algebra from the literature which will be helpful in further study of this article.

***Definition 2.1* [11].** Let X be a nonempty set with a binary operation $*$ and a constant 0. The triple $(X,*,0)$ is called a KU-algebra, if the following axioms are satisfied. For all $x, y, z \in X$.

(ku_1) $(x*y)*[(y*z))*(x*z)]=0$.

(ku_2) $x*0=0$.

(ku_3) $0*x=x$.

(ku_4) $x*y=0$ and $y*x=0$ implies $x=y$.

(ku_5) $x*x=0$.

On a KU-algebra X, we can define a binary relation $\leq$ on X by putting $x \leq y \Leftrightarrow y*x=0$. Then $(X,\leq)$ is a partially ordered set and 0 is its smallest element. Thus $(X,*,0)$ satisfies the following conditions. For all $x, y, z \in X$, we that

$(ku_{1\backslash})$ $(y*z)*(x*z) \leq (x*y)$

$(ku_{2\backslash})$ $0 \leq x$

$(ku_{3\backslash})$ $x \leq y, y \leq x$ implies $x = y$,

$(ku_{4\backslash})$ $y*x \leq x$.

***Theorem 2.2* [8].** In a KU-algebra X. The following axioms are satisfied.
For all $x, y, z \in X$,

(1) $x \leq y$ imply $y*z \leq x*z$,

(2) $x*(y*z) = y*(x*z)$,

(3) $((y*x)*x) \leq y$.

***Definition 2.3* [11].** A non-empty subset I of a KU-algebra X is called an ideal of X if for any $x, y \in X$, then

(i) $\mathbf{0} \in I$ and

(ii) $x*y, x \in I$ imply $y \in I$.

***Definition 2.4* [9].** A KU-algebra X is said to be KU-commutative if it satisfies $(y*x)*x=(x*y)*y$, for all x, y in X.

***Lemma 2.5* [9].** If X is KU-commutative algebra, then for any distinct elements $x, y, z \in X$, $x \dot{\wedge} (y*z) = (x \dot{\wedge} y)*(x \dot{\wedge} z)$.

Definition 2.6. If there is an element E of a KU-algebra X satisfying $x \leq E$ for all $x \in X$, then the element E is called unit of X. A KU-algebra with unit is called bounded. In a bounded KU-algebra X, we denote $x * E$ by N_x. It is easy to see that $N_E = 0, N_0 = E$.

Example 2.7. Let $X = \{0, a, b, c, d\}$ be a set with a binary operation $*$ defined by the following table.

$*$	0	a	b	c	d	e
0	0	a	b	c	d	e
a	0	0	b	c	b	c
b	0	a	0	b	a	d
c	0	a	0	0	a	a
d	0	0	0	b	0	b
e	0	0	0	0	0	0

Using the algorithms in Appendix, we can prove that (X, *, 0) is a KU-algebra and by routine calculations, we can see that X is a bounded KU-algebra with unit "d".

Theorem 2.8. For a bounded KU-commutative algebra X, we denote $x \dot{\vee} y = N_{(N_x \dot{\wedge} N_y)}$ and for all $x, y \in X$, we have

(a) $N_{N_x} = x$,

(b) $N_x \dot{\wedge} N_y = N_{(x \dot{\vee} y)}$, $N_x \dot{\vee} N_y = N_{(x \dot{\wedge} y)}$,

(c) $x \leq y$ implies $N_y \leq N_x$.

(d) $E \dot{\wedge} x = x$,

(e) $x \dot{\wedge} E = E$.

Proof. The proof is straightforward. □

Definition 2.9. A partially ordered set $(L, \leq)$ is said to be a lower semilattice if every pair of elements in L has a greatest lower bound and it is called to be an upper semilattice if every pair of elements in L has a least upper bound. If L is a lattice, then we define $x \wedge y = \mathbf{glb}\{x, y\}$ and $x \vee y = \mathbf{lub}\{x, y\}$. A lattice L is said to be distributive if it satisfies the following conditions. For all $x, y, z \in L$

(1) $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$,

(2) $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$.

Theorem 2.10. Every KU-commutative algebra X is a KU-lower semilattice with respect to $(X, \leq)$.

Proof. Suppose X is a KU-commutative algebra. We know that $x \dot{\wedge} y \leq x$ and $x \dot{\wedge} y \leq y$. Let z be any element of X such that $z \leq x$ and $z \leq y$, then $x * z = y * z = 0$ (by Definition of $\leq$), so we have that $z = 0 * z = \overbrace{(x * z) * z = (z * x) * x}^{commutativity}$.

By the same reason we have $z=(z*y)*y$, and hence $z=(z*x)*x=(((z*y)*y)*x)*x\leq(y*x)*x=x\dot{\wedge}y$, thus $x\dot{\wedge}y$ is the greatest KU-lower bound and so $(X,\leq)$ is a KU-lower semilattice.□

The converse of this theorem may not be true. For example, in Example 2.7 we have that X is a lower semilattice, but $(a*c)*c=c*c=0\neq a=0*a=(c*a)*a$.

Theorem 2.11. Any bounded KU-commutative algebra X with respect to $(X,\leq)$ is a KU-lattice.

Proof. Since $N_x\dot{\wedge}N_y\leq N_x$ and $N_x\dot{\wedge}N_y\leq N_y$, from Theorem 2.8 we have that

$x=N_{N_x}\leq N_{(N_x\dot{\wedge}N_y)}=x\dot{\vee}y$ and $y=N_{N_y}\leq N_{(N_x\dot{\wedge}N_y)}=x\dot{\vee}y$.

This shows that $x\dot{\vee}y$ is a common upper bound of x and y. Now, by Theorem 2.8 if $x\leq z$ and $y\leq z$, then $N_z\leq N_x$ and $N_z\leq N_y$. It follows that $N_z\leq N_x\dot{\wedge}N_y$, therefore $N_{(N_x\dot{\wedge}N_y)}\leq N_{N_z}$ and $x\dot{\vee}y\leq z$. Hence $x\dot{\vee}y$ is a least upper bound of x and y, i.e. $(X,\leq)$ is a KU-upper semilattice. By using Theorem 2.10 and this Theorem, we obtain $(X,\leq)$ is a KU-lattice. □

Definition 2.12. Let X be a KU-algebra and A a nonempty subset of X. The ideal of X generated by A is denoted by $\langle A\rangle=\{x\in X:\exists a_1,...,a_n\in A$ such that $(a_1*(...*(a_n*x)=0\}$, if $A\neq\phi$. We have that $\langle\phi\rangle=\{0\}$.

Definition 2.13. Let X be KU-commutative algebra and A a subset of X. Then we define $A^*=\{x\in X: a\dot{\wedge}x=\mathbf{0}$ for all $a\in A\}$ and call it the KU-annihilator of A.

We write A^{**} in place of $(A^*)^*$. Note that A^* is a nonempty since $\mathbf{0}\in A^*$. Obviously we have $X^*=\{\mathbf{0}\}$ and $\{\mathbf{0}\}^*=X$. If A is an ideal it is easy to see that $A\bigcap A^*=\{\mathbf{0}\}$. We observe that if $x\in A^*$ then $a\dot{\wedge}x=\mathbf{0}$ for all $a\in A$. It follows that $(x*a)*a=0$ then $a\leq x*a$ and $a*(x*a)=0$, hence $x*a\leq a$ which implies that $a=x*a$. Thus $x\in A^*$ if and only if $a=x*a$ for all $a\in A$. Moreover if X is commutative, then $x\in A^*$ if and only if $a=x*a$ for all $a\in A$.

If $A=\{a\}$, then we write $(a)^*$ instead of $(\{a\})^*$.

Example 2.14. Let $X=\{0,a,b,c,d,e\}$ be a set with a binary operation $*$ defined by the following table.

It is easy to show that X is a bounded KU-commutative algebra. If $A=\{b,c\}$, then $A^*=\{\mathbf{0},a\}$.

Definition 2.15. An ideal A of a KU-commutative algebra X is said to be involutory if $A=A^{**}$. Moreover a KU-commutative algebra X is said to be involutory if every ideal of X is involutory.

Clearly $\{0\}$ and X are involutory ideals.

Remark 2.16. In involutory KU-commutative algebra X , for any two ideals A, B of X , we have that $(A \cap B)^* = \langle A^* \cup B^* \rangle$.

Lemma 2.17. Let X be involutory KU-commutative algebra. Then $X = \langle A \cup A^* \rangle$ for any ideal A of X .

Proof. Note that $A \cap A^* = \{0\}$. By Remark 2.16 and note X is involutory, we have $\langle A \cup A^* \rangle = \langle A^{**} \cup A^* \rangle = (A^* \cap A)^* = (0)^* = X$.

Definition 2.18. A KU-algebra X is said to be KU-positive implicative if it satisfies that $(z*x)*(z*y) = z*(x*y)$, for all x, y, z in X .

Definition 2.19. A nonempty subset I of a KU-algebra X is said to be a KU-positive implicative ideal if for all x, y, z in X , then

(1) $0 \in I$ and

(2) $z*(x*y) \in I$ and $z*x \in I$ imply $z*y \in I$.

Theorem 2.20. If we are given an ideal I of a KU-algebra X , then I is a KU-positive implicative if and only if, for any $a \in X$ the set $A_a = \{x \in X : a * x \in I\}$ is an ideal of X .

Proof. **($\Rightarrow$)** Suppose that I is positive implicative ideal and $(x*y) \in A_a$ and $x \in A_a$. Then $a*(x*y) \in I$ and $a*x \in I$. By Definition 2.19 we obtain $(a*y) \in I$ i.e. $y \in A_a$. This says A_a is an ideal.

($\Leftarrow$) Suppose that A_a is an ideal of X , for any $a \in X$. If $z*(x*y) \in I$ and $z*x \in I$, then $(x*y) \in A_z$ and $x \in A_z$. Since A_z is an ideal of X then $y \in A_z$ and $z*y \in I$. This means that I is positive implicative ideal. □

Corollary 2.21. If I is a KU-positive implicative ideal of X , then $A_a = \{x \in X : a*x \in I\}$ is the least ideal containing I and a , for any $a \in X$.

Definition 2.22. A nonempty subset I of a KU-algebra X is said to be a KU-implicative ideal if for all x, y, z in X , then

(1) $0 \in I$ and

(2) $z*((x*y)*x) \in I$ and $z \in I$ imply $x \in I$.

Definition 2.23. A proper ideal I of a KU-algebra X is called a maximal ideal if and only if $I \subseteq A \subseteq X$ implies that $I = A$ or $A = X$, for any ideal A of X .

Theorem 2.24. If I is an ideal of a KU-algebra X . Then the following statements are equivalent.

(a) I is maximal and KU-implicative ideals,

(b) I is maximal and KU-positive implicative ideals,

(c) $x, y \notin I$ implies $x*y \in I$ and $y*x \in I$ for all x, y in X .

Proof. (a)$\Rightarrow$(b). Suppose that I is KU-implicative ideal and $z*(x*y)\in I$, $z*x\in I$. Since $(z*x)*(z*(z*y))\leq x*(z*y)=z*(x*y)\in I$ then $(z*x)*(z*(z*y))\in I$ and $z*x\in I$. I is an ideal, we have that $(z*(z*y))\in I$. It follows that $((z*y)*y)*(z*y)=z*(z*y)\in I$ and $0*(((z*y)*y))*(z*y))\in I$. Combining $0\in I$ we obtain $z*y\in I$. Hence I is KU-positive implicative ideal.

(b) $\Rightarrow$(c). Let $x,y\notin I$. Since I is KU-positive implicative. By Corollary 2.21 $A_y=\{u\in X: u*y\in I\}$ is the least ideal containing I and y. Using maximality of I we have that $A_y=X$. Hence $x\in A_y$, that is $x*y\in I$. Likewise for $y*x\in I$.

(c) $\Rightarrow$(a) At first we prove that I is KU-implicative. Suppose I does not KU implicative, then there are x,y in X such that $(x*y)*x\in I$ but $x\notin I$. If $x*y\in I$, combining $(x*y)*x\in I$ we get $x\in I$. This contradicts to $x\notin I$. If $x*y\notin I$, by (c) we have $y\in I$ as $x\notin I$. By $ku_{4^\backslash}$, we have $x*y\leq y$, we get $x*y\in I$. This contradicts to $x*y\notin I$. Hence I is KU-implicative. Next we prove that I is maximal. Note that I is also KU-positive implicative. Hence it is sufficient to prove that for any $a\notin I$ we have $A_a=\{x\in X: x*a\in I\}=X$. By Corollary 2.21, A_a is the least ideal containing I and a. For all x in X, when $x\in I$ then $x\in A_a$ and when $x\notin I$, by $a\in I$ and (c) we have that $x*a\in I$ i.e. $x\in A_a$. This means that $A_a=X$. Therefore I is maximal ideal of X. □

Definition 2.25. Let X be a KU-lower semilattice and P a proper ideal of X. Then P is said to be a prime ideal if $a\dot\wedge b\in P$ implies $a\in P$ or $b\in P$, for any a,b in X.

Theorem 2.26. In a KU-lower semilattice X, a proper ideal P of X is said to be a prime if $A\bigcap B\subseteq P$ implies $A\subseteq P$ or $B\subseteq P$, for any ideals A,B in X.

Proof. Suppose that $A\bigcap B\subseteq P$, $A\not\subset P$ and $B\not\subset P$ for some two ideals A,B in X. Thus there exist a and b such that $a\in A-P$ and $b\in B-P$. From $a\dot\wedge b\leq a$ and $a\dot\wedge b\leq b$ it follows that $a\dot\wedge b\in A,B$ and $a\dot\wedge b\in A\bigcap B\subseteq P$. This contradicts to primness of P. Hence $A\subseteq P$ or $B\subseteq P$. □

Theorem 2.27. If X is a KU-implicative algebra, then each prime ideal of X is maximal.

Proof. Suppose that P is prime ideal and $a,b\notin P$. Since X is KU-implicative, then $a\dot\wedge(a*b)=((a*b)*a)*a=a*a=0\in P$. Noticing $a\notin P$, we have $a*b\in P$. By the same way we get $b*a\in P$. Hence P is maximal ideal by Theorem 2.24. □

Lemma 2.28. Let X be a KU-lower semilattice. If $a\leq x^n$ and $a\leq x^m$ for natural numbers n and m, then there exists a natural number p such that $a\leq(x\dot\wedge y)^p$, for any $x,y,a\in X$.

Proof. Since for $m\leq n$, $a\leq x^m$ implies $a\leq x^n$, it suffices to verify that when $x^n*a=y^n*a=0$, there exist a natural number p such that $(x\wedge y)^p*a=0$. We proceed by induction on n. When $n=1$, we have $x*a=y*a=0$, $a\leq x$ and $a\leq y$. Hence $a\leq x\dot\wedge y$, i.e., $(x\dot\wedge y)*a=0$.

Now suppose the assertion holds for natural number n, that is, $x^n * a = y^n * a = 0$ implies that there exists a natural number p such that $(x \dot\wedge y)^p * a = 0$.

If $x^{n+1} * a = y^{n+1} * a = 0$, then $0 = x^{n+1} * a = (x * (y^n * (x^n * a))$.

By the same argument we have $0 = y * (y^n * (x^n * a)$. In view of the first step of induction we get

$(x \dot\wedge y) * (y^n * (x^n * a) = 0$, $(y^n * (x^n * ((x \dot\wedge y) * a) = 0$, $(x * (x^{n-1} * (y^n * ((x \dot\wedge y) * a)) = 0$.

From $y^{n+1} * a = 0$. It easily follows that $(y * (x^{n-1} * (y^n * ((x \dot\wedge y) * a)) = 0$. Hence $x^{n-1} * (y^n * ((x \dot\wedge y)^2 * a) = 0$. Repeating the above procedure n times we obtain $y^n * ((x \dot\wedge y)^{n+1} * a) = 0$...........(1). By an entirely similar way we have that

$x^n * ((x \dot\wedge y)^{n+1} * a) = 0$...........(2). By the induction hypothesis and (1), (2), we know that there is a natural number p such that $(x \dot\wedge y)^p * ((x \dot\wedge y)^{n+1} * a) = 0$, $(x \dot\wedge y)^{n+p+1} * a = 0$. □

Corollary 2.29. Let X be a KU-lower semilattice and P an ideal in X. Then for any $x, y \in X$ if $x \dot\wedge y \in P$, then $\langle P \bigcup \{x\} \rangle \bigcap \langle P \bigcup \{y\} \rangle = P$.

Definition 2.30. Let X be a KU-lower semilattice. A nonempty subset S of X is said to be $\dot\wedge$-closed if $x \dot\wedge y \in S$ whenever $x, y \in S$.

Theorem 2.31. Let X be a KU-lower semilattice and S a nonempty $\dot\wedge$-closed subset of X such that $0 \notin S$, $I(X)$ denotes the set of all ideals of X then $\{I \in I(X) : I \bigcap S = \phi\}$ have a maximal ideal P such that $P \bigcap S = \phi$. Moreover P is a prime ideal.

Proof. The existence of an ideal P easily follows from Zorn's lemma. We will prove that P is a prime ideal. Let us suppose it is not the case, i.e., there exist $x, y \in X$ such that $x \dot\wedge y \in P$, $x \notin P$ and $y \notin P$. Then P is properly contained in both $\langle P \bigcup \{x\} \rangle = P_1$ and $\langle P \bigcup \{y\} \rangle = P_2$. Because of maximality of P, $P_1 \bigcap S \neq \phi$ and $P_2 \bigcap S \neq \phi$. Let $s_i \in P_i \bigcap S, i = 1,2$. We known $s_1 \dot\wedge s_2 \leq s_i, i = 1,2$ implies $s_1 \dot\wedge s_2 \in P_1 \bigcap P_2 = P$ (by Corollary 2.29). On the other hand $s_1 \dot\wedge s_2 \in S$. This is a contradiction. Hence P is a prime ideal. □

Theorem 2.32. In a KU-lower semilattice X. Any maximal ideal must be prime.

Proof. By using Theorem 2.31 and Corollary 2.29, we obtain the result.

Definition 2.33. Let I be an ideal of a KU-algebra X. We will call an ideal J of X a minimal prime ideal associated with the ideal I if J is a minimal element in the set of all prime ideals containing I.

Lemma 2.34. Let I be a proper ideal of a KU-lower semilattice X .Then
(a) I is contained in a prime ideal,
(b) Any prime ideal containing I contains a minimal prime ideal associated with the ideal I .

Proof. If I is a prime ideal, then the Lemma is true. Let us suppose that I is not a prime ideal and $a \in X - I$. Obviously, $S = \{x \in X : a \le x\}$ is a nonempty, $\dot{\curlywedge}$ -closed and $\mathbf{0} \notin S$. By Theorem 2.31, there exists a prime ideal P such that $P \cap S = \phi$. (a) holds.

To show (b) it is sufficient to show that the intersection of any chain of prime ideals is a prime ideal. Let $\{P_i : i \in \omega\}$ be a chain of prime ideals of X and $P = \bigcap\{P_i : i \in \omega\}$. Suppose that P is not a prime ideal, that is, there are $x, y \in X$ such that $x \dot{\curlywedge} y \in P, x \notin P, y \notin P$. Thus, there are $i, j \in \omega$ such that $x \notin P_i, y \notin P_j$. Without loss of generality we can assume that $P_i \subseteq P_j$, $x \notin P_i, y \notin P_i$ and $x \dot{\curlywedge} y \in P \subseteq P_i$. This contradicts to P_i being a prime.

3. Topology Spectrum of KU-commutative algebra X

In this section, we define the notion of a spectrum of KU-commutative algebra X and study some of its properties.

Definition 3.1. Let X be KU-commutative algebra and $Spec(X)$ the set of all prime ideals of X . Then for any ideal A of X , we define $W(A) = \{P \in Spec(X) \mid A \not\subset P\}$.

Proposition 3.2. Let X be KU-commutative semilattice algebra. Then
(i) $A \subseteq B$ implies that $W(A) \subseteq W(B)$, for any ideals A, B of X ,
(ii) $W(A) = W(\langle A \rangle)$.

Proof. (i) Let $L \in W(A) \Rightarrow A \not\subset L$. Since $A \subseteq B \Rightarrow L \in W(B)$. Hence $W(A) \subseteq W(B)$.
(ii) Since $A \subseteq \langle A \rangle$ from (i) we get that $W(A) \subseteq W(\langle A \rangle)$. Let $P \in W(\langle A \rangle) \Rightarrow \langle A \rangle \not\subset P$ and since $A \subseteq \langle A \rangle$ then $A \not\subset P$, $P \in W(A)$ it follows that
$W(\langle A \rangle) \subseteq W(A)$. Hence $W(A) = W(\langle A \rangle)$. □

Theorem 3.3. Let X be KU-commutative algebra. Then the family $T(X) = \{W(A)\}_{A \in I(X)}$ forms a topology on $Spec(X)$.

Proof. $W(0) = \{P \in Spec(X) : (0) \not\subset P\} = \phi$ and
$W(X) = \{P \in Spec(X) : X \not\subset P\} = Spec(X)$. For any family $\{W(A_i)\}_{i \in I}$

$$\bigcup_{i \in I} W(A_i) = \{P \in Spec(X) : A_i \not\subset P \text{ for some } A_i\} = \{P \in Spec(X) : \bigcup_{i \in I} A_i \not\subset P\}$$

$$= \{P \in Spec(X) : \langle \bigcup_{i \in I} A_i \rangle \not\subset P\} = W(\langle \bigcup_{i \in I} A_i \rangle) \text{ implies that } \bigcup_{i \in I} W(A_i) \in T(X).$$

Finally,
$$W(A) \cap W(B) = \{P \in Spec(X) : A \not\subset P\} \cap \{P \in Spec(X) : B \not\subset P\}$$
$$= \{P \in Spec(X) : A \not\subset P \text{ and } B \not\subset P\}.$$

Since P is a prime ideal, therefore can be written as

$W(A)\bigcap W(B)=\{P\in Spec(X): A\bigcap B\not\subset P\}=W(A\bigcap B)$, i.e., $W(A)\bigcap W(B)\in T(X)$. Hence $T(X)$ is a topology on $Spec(X)$, this topology will be called the spectrum topology.

Example 3.4. In Example 2.14. By using the algorithms in Appendix A, we can found that $\{X,\{0\},\{0,a\},\{0,b,c\}\}$ is the set of all ideals. Note that $\{\{0,a\}, \{0,b,c\}\}$ is the set of all prime ideals of X and $Spec(X)=\{\{0,a\},\{0,b,c\}\}$. Therefore $T(X)=\{\phi, Spec(X)\}$ this is the indiscrete topology.

Definition 3.5. For any $A\in I(X)$ we denote the complement of $W(A)$ by $V(A)$. Hence $V(A)=\{P\in spec(X)\mid A\subseteq P\}$, it follows that the set $\{V(A)\}_{A\in I(X)}$ is the family of the closed sets of a topological space $Spec(X)$.

Remark 3.6. For any $x\in A$ we denote $V(\{x\})$ by $V(x)$ and $W(\{x\})$ by $W(x)$, i.e. $V(x)=\{P\in spec(X)\mid x\in P\}$ and $W(x)=\{P\in spec(X)\mid x\notin P\}$.

Now, we give some properties of the topological space $Spec(X)$.

Theorem 3.7. Let X be a KU-commutative semilattice. The family $\{W(x)\}_{x\in A}$ is a basis for the topology of $Spec(X)$.

Proof. Let $A\subseteq X$ and $W(A)$ an open subset of $Spec(X)$, then $W(A)=W(\bigcup_{x\in A}\{x\})=\bigcup_{x\in A}W(x)$. Hence, any open set of $Spec(X)$ is union of subsets from the family $\{W(x)\}_{x\in A}$.□

Theorem 3.8. Let X be a KU-lower semilattice and A a proper ideal of X. Then A is equal to the intersection of all minimal prime ideals associated with it.

Proof. Denote $J(A)=\bigcap\{P\in I(X): P$ is a prime ideal and associated with $A\}$.

It is clearly $A\subseteq J(A)$. We will show that $J(A)\subseteq A$. Let us suppose that it is not the case, then there is $a\in J(A)$ and $a\notin A$. As in the proof of Lemma 2.34, we can show that if $S=\{x\in X: a\leq x\}$, then there exists a prime ideal P such that $A\subseteq P$ and $P\bigcap S=\phi$. The existence of such a prime ideal P contradicts to the assumptions. Hence $J(A)=A$. □

Lemma 3.9. The mapping $f: I(X)\to T(X)$ given by $f(A)=W(A)$ is a lattice isomorphism.

Proof. By Theorem 3.3 of $W(A)$, it follows that f define a lattice homomorphism. We only show that f is one to one and onto. For any ideals $A,B\in I(X)$. Suppose that $f(A)=f(B)$ then $W(A)=W(B)$ and $Spec(X)-W(A)=Spec(X)-W(B)$. Consequently, $J(A)=J(B)$, hence $A=B$, it follows that f is one to one and onto. Hence $I(X)$ and $T(X)$ are isomorphic. □

Proposition 3.10. If X is a bounded KU-commutative algebra, then $Spec(X)$ is a compact space.

Proof. Let $\{W(A_i)\}_{i\in I}$ be an open cover of $Spec(X)$. Then $Spec(X) = \bigcup_{i\in I} W(A_i) = W(\langle \bigcup_{i\in I} A_i \rangle)$. By injectiveness of W (Lemma3.9) implies that $\langle \bigcup_{i\in I} A_i \rangle = X$. Since X is a bounded $\Rightarrow$ $\mathrm{E} \in \langle \bigcup_{i\in I} A_i \rangle$ and hence $(a_1 * (a_2 * (... * (a_n * \mathrm{E})) = 0$. We may assume that $a_k \in A_i$ for $k = 1,2,...,n$, then $a_k \in \bigcup_{k=1}^{n} A_{i_k}$ for all $k = 1,2,...,n$. This implies that $\mathrm{E} \in \langle \bigcup_{k=1}^{n} A_{i_k} \rangle$ and hence $\langle \bigcup_{k=1}^{n} A_{i_k} \rangle = X$ (because no proper ideal contains E). This shows that $\bigcup_{k=1}^{n} W(A_{i_k}) = W(\bigcup_{k=1}^{n} A_{i_k}) = W(\langle \bigcup_{k=1}^{n} A_{i_k} \rangle) = W(X) = Spec(X)$. Thus we obtain a finite Sub cover and consequently, $Spec(X)$ is compact. □

Proposition 3.11. Let X be KU-commutative algebra. Then $Spec(X)$ is T_0 topological space.

Proof. Let P and Q be any two distinct prime ideals in $Spec(X)$. Then either $P \not\subset Q$ or $Q \not\subset P$. If $P \not\subset Q$, there exists $x \in P$ such that $x \notin Q$ *which implies that* $Q \in W(x)$ and $P \notin W(x)$. Therefore exists an open set $W(x)$ containing Q but not P. Similarly, if $Q \not\subset P$. There exists $x \in Q$ such that $x \notin P$, *which implies that* $Q \notin W(x)$ and $P \in W(x)$. Therefore exists an open set $W(x)$ containing P but not Q. Hence $Spec(X)$ is a T_0-space.

Proposition 3.12. If X is a KU-implicative algebra. Then $Spec(X)$ is T_1 topological space.

Proof. If $Spec(X) = \phi$, *then* $Spec(X)$ is trivial space and it is a T_1 space.
If $Spec(X) \neq \phi$, then there exist a prime ideal P of $Spec(X)$. It follows by Theorem 2.27 that P is a maximal ideal. Hence $V(P) = \{i\}$ and $\{i\}$ is closed set in $Spec(X)$, i.e. $Spec(X)$ is a T_1 space. □

Proposition 3.13. If A is an involutory ideal of X and $P \in Spec(X)$, then $P \notin W(A^*)$ if and only if $P \in W(A)$.

Proof. If $P \notin W(A^*)$, then $A^* \subseteq P$. Since A is an involutory ideal of X, therefore by Lemma 2.17 $X = \langle A \cup A^* \rangle$ and hence $A \not\subset P$. This implies that $P \in W(A)$.
Conversely, assume that $P \in W(A)$ then $A \not\subset P$. Since $A \cap A^* = \{0\} \subseteq P$ and P is a prime ideal. Therefore by Theorem 2.26 $A \subseteq P$ or $A^* \subseteq P$, but $A \not\subset P$. It follows that $A^* \subseteq P$ and consequently we have $P \notin W(A^*)$.□

Proposition 3.14. Let X be an involutory KU-algebra with at least one involutory ideal (proper). Then $Spec(X)$ is a disconnected topological space.

Proof. Let A be an involutory (proper) ideal of X. We claim that $W(A)$ and $W(A^*)$ form disconnection of $Spec(X)$. That $W(A)$ and $W(A^*)$ mutually exclusive, follows from

Proposition 3.13. We show that $Spec(X) = W(A) \bigcup W(A^*)$. Indeed A is an involutory ideal, then $X = \langle A \bigcup A^* \rangle$. This implies that

$$W(X) = W(\langle A \bigcup A^* \rangle) = W(A \bigcup A^*) = W(A) \bigcup W(A^*).$$

This means that $Spec(X) = W(A) \bigcup W(A^*)$ and consequently $Spec(X)$ is a disconnected space. □

Proposition 3.15. If X is an involutory KU-algebra, then $Spec(X)$ is Hausdorff space.

Proof. Let P and Q be any two distinct prime ideals in $Spec(X)$. Then there exists an element x in X such that $x \in P$ and $x \notin Q$. This implies that $\langle x \rangle \subseteq P$ and $\langle x \rangle \not\subset Q$. In other word $P \notin W(\langle x \rangle)$ and $Q \in W(\langle x \rangle)$. By Proposition 3.13, we have $P \in W(\langle x \rangle^*)$. Thus we obtain two open sets $W(\langle x \rangle)$ and $W(\langle x \rangle^*)$ such that $P \in W(\langle x \rangle^*)$ and $Q \in W(\langle x \rangle)$. It follows that $W(\langle x \rangle) \bigcap W(\langle x \rangle^*) = W(\langle x \rangle \bigcap \langle x \rangle^*) = W(\mathbf{0}) = \phi$. Hence $Spec(X)$ is Hausdorff space. □

Corollary 3.16. If X is a bounded involutory KU-algebra, then $Spec(X)$ is normal space.

***Definition 3.17* [4].** Let $(G,*,0)$ and $(H,\bullet,0)$ be KU-algebras. A homomorphism is a map $h: G \to H$ satisfying $h(x*y) = h(x) \bullet h(y)$ for all $x, y \in G$. An injective homomorphism is called monomorphism and a surjective homomorphism is called epimorphism.

Proposition 3.18. Let $(G,*,0)$ and $(H,\bullet,0)$ be KU-algebras and $h: G \to H$ a homomorphism map of KU-algebras, then for any prime ideal P of H. The ideal $h^{-1}(P) = \{x \in G : h(x) \in P\}$ is also a prime ideal of G.

Proof. Let $x \dot{\wedge} y \in h^{-1}(P)$ for any $x, y \in G$, then

$(y*x)*x \in h^{-1}(P) \Rightarrow h((y*x)*x) \in P(\text{by homomorphism}) \Rightarrow h(y*x) \bullet h(x) \in P \Rightarrow$

$(h(y) \bullet h(x)) \bullet h(x) \in P \Rightarrow h(x) \dot{\wedge} h(y) \in P.$

Since P is prime $\Rightarrow h(x) \in P$ or $h(y) \in P$

$\Rightarrow x \in h^{-1}(P)$ or $y \in h^{-1}(P)$. Hence $h^{-1}(P)$ is prime ideal of G. □

Theorem 3.19. Let $(G,*,0)$, $(H,\bullet,0)$ be KU-algebras and $h: G \to H$ a homomorphism map of KU-algebras. If $\sigma : SpecH \to SpecG$, define by $\sigma(P) = h^{-1}(P)$ for any $P \in SpecH$, then σ is continuous map.

Proof. Let $W(x)$ be a basic open set in $Spec(G)$, for any $x \in G$. Then

$$\begin{aligned}\sigma^{-1}(W(x)) &= \{P \in SpecH : \sigma(P) \in W(x)\} \\ &= \{P \in SpecH : h^{-1}(P) \in W(x)\} \\ &= \{P \in SpecH : x \notin h^{-1}(P)\} \\ &= \{P \in SpecH : h(x) \notin P\},\end{aligned}$$

which is open in $Spec(H)$.

Thus the inverse image of any open set in $Spec(G)$ is open in $Spec(H)$ and hence σ is a continuous map. □

4. Conclusion

This work is a study of the relationship between the KU-algebras and topological spaces. We introduced the topology spectrum of a commutative KU-algebra and we obtained some results that were different from the topology spectrum of commutative ring. However, there are differences because KU-algebras are not rings. We proved that the spectrum of KU-algebra is compact, disconnected and Hausdorff space. Also, we studied the continuous map of this topological space. The main purpose of our future work is to investigate the fuzzy topology of KU-algebras, which may have a lot of applications in different branches of mathematics.

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Appendix Algorithms

<u>Algorithm for KU-algebras</u>

Input (X :set, $*$:binary operation)
Output (“ X is a KU-algebra or not”)
Begin
If $X = \phi$ then go to (1.);
EndIf
If $0 \notin X$ then go to (1.);
EndIf
Stop: =false;
$i := 1$;
While $i \le |X|$ and not (Stop) do
If $x_i * x_i \neq 0$ then
Stop: = true;
EndIf
$j := 1$
While $j \le |X|$ and not (Stop) do
If $((y_j * x_i) * x_i) \neq 0$ then
Stop: = true;
EndIf
EndIf
$k := 1$
While $k \le |X|$ and not (Stop) do
If $(x_i * y_j) * ((y_j * z_k) * (x_i * z_k)) \neq 0$ then
Stop: = true;
EndIf

EndIf While
EndIf While
EndIf While
If Stop then

(1.) Output (" X is not a KU-algebra")
Else
Output (" X is a KU-algebra")
EndIf
End

<u>Algorithm for ideals</u>

Input (X :KU-algebra, I :subset of X);
Output ("I is an ideal of X or not");
Begin
If $I = \phi$ then go to (1.);
EndIf
If $0 \notin I$ then go to (1.);
EndIf
Stop: =false;
$i := 1$;
While $i \leq |X|$ and not (Stop) do
$j := 1$
While $j \leq |X|$ and not (Stop) do
If $(x_i * y_j) \in I$ and $x_i \in I$ then
If $y_j \notin I$ then
Stop: = true;
EndIf
EndIf
EndIf While
EndIf While
EndIf While
If Stop then
Output ("I is an ideal of X ")
Else
(1.) Output ("I is not an ideal of X ")
EndIf
End

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INVESTIGATION OF THE DYEING PROPERTIES OF *Sideritis trojana ehrend* IN THE FABRICS THAT PRE-TREATED WITH WILLOW EXTRACT

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Abstract - In this study, the usage of Sarıkız herb tea (*Sideritis trojana ehrend)* was examined in terms of textile dyeing. For this purpose, cotton and wool fabrics were treated with willow extract for 24 h, at room temperature. The pretreated samples were dyed with *S. trojana* extract in the presence of three mordants including alum ($AlK(SO_4)_2 \cdot 12H_2O$), ferrous sulfate heptahydrate ($FeSO_4 \cdot 7H_2O$) and copper sulfate pentahydrate ($CuSO_4 \cdot 5H_2O$) and using three mordanting methods (pre-mordanting, meta-mordanting and post-mordanting). Fastness properties (rubbing and light) were also determined. Generally, high fastness values were obtained. The color strength values of the wool fabrics were found to be higher than that of cotton fabrics. It is concluded that, *S. trojana ehrend* has affinity to the wool fabrics, and can be used as an alternative source in the presence of willow extract in natural dyeing.

Keywords - *Sideritis trojana ehrend, mordant, dyeing, fastness, wool, cotton*

1. Introduction

Natural coloring agents have been used since beginning of the time to color wool, silk, cotton and leather [1]. Natural dyes are widely used in textile dyeing due to their ecofriendly properties [2-4]. In addition, those pigments are anti-allergic and harmless to human and environment [5]. Natural dyes and pigments can be considered as an important alternative to the harmful synthetic dyes and generally they give soft and lustrous pastel colors. It is known that, synthetic dyes are synthesized from petrochemical sources that resulted chemical substances which are hazardous to human health and environment. Thus, there is a growing interest to natural dyes due to their biodegradable, less toxic and eco-friendly properties in recent years [6, 7]. Therefore, necessity of lowered cost natural dyeing for production was canalized people to use of dyestuff containing wastes such as food, beverage and aerial parts of the plants [8].

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To color the fiber, generally different parts of plants have been used including bark, flowers, leaves and seed. Although bark of the plant is rich with coloring agent, usage of the bark could kill the plant. Bark is preferred in dyeing because of its high percentage of coloring agent. Therefore, leaves, flowers and seeds are used for the extraction of the dyestuff from the plant. The leaves of the plant provide abundant and easy availability source for dyeing industry [9].

S. trojana ehrend belongs to Labiatae family and grows in Kaz mountains in Turkey. It is called as Sarıkız herb tea. There are 45 genus, 546 type and 730 taxa in Turkey. [10]
S. trojana exhibits antioxidant and antimicrobial activity [11] and has been using as a natural tea.

Best to our knowledge, there is no study on the dyeing properties of *S. trojana*. It is reported that, the plant contains o- methyl- izo skultelarin-7-o-[(6- o- acetyl- Beta-allopyrazonyl-(1,2)]- beta- glucopiyranoside as coloring agent (Figure 1) [12].

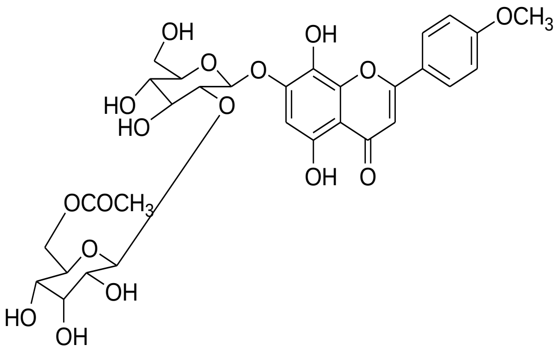

Figure.1. *O*- methyl- izo skultelarin-7-*o*-[(6- *o*- acetyl- *β*-allopyrazonyl-(1,2)]- *β*-glucopiyranoside

The dyestuff has oxochorome groups (carbonyl and hydroxide groups) which may be exhibited good dyeing properties.

2. Experimental

Fabrics

Scoured, bleached and mercerized plain weaved cotton fabric (240 g/m^2) and wool fabric (125 g/m^2) were purchased from Has Ozgen Textile Company (Tokat, Turkey).

Preparation of willow extract

Willow branches (1 kg) were soaked in distilled water (10 L) for 21 days at room temperature and then filtered. The filtrate was used in the pre-treatment processes.

Preparation of the mordant solutions and the dye-bath

To prepare wool and cotton samples for dyeing processes, the samples were treated with the water extract of willow, at room temperature for 24 h. The stem and the leaves of *S. tojana* were supplied from Plant Research Laboratory, Gaziosmanpasa University, in June, 2014.

The parts of the plant were dried and cleaned in order to remove the impurities. Soxhlet apparatus was used for the extraction of the plant. Plant material (100 g) was extracted with distilled water (1 L). Extraction was maintained at its boiling point, for 12 h. After the end of the period, the mixture was filtered and the clear solution was used as dye bath in the dyeing experiments.

Reagents and equipments

Analytically grade chemicals including alum ($AlK(SO_4)_2 \cdot 12H_2O$), ferrous sulfate heptahydrate ($FeSO_4 \cdot 7H_2O$) and copper sulfate pentahydrate ($CuSO_4 \cdot 5H_2O$) were supplied from Merck. Soxhlet apparatus was used for the extraction process. Premier Colorscan SS 6200A Spectrophotometer was used for the determination of CIELab values (L^*, a^*, b^*) and color strength (K/S) values. Kubelka-Munk equation was used for the expression of color strength values of the dyed samples as K/S values:

$$K / S = (1 - R)^2 / 2R$$

K indicates the absorption coefficient, R is the reflectance of the dyed sample and S is the scattering coefficient.

Fastness levels of the dyed fabrics were evaluated using rubbing (wet, dry) and light fastness tests and determined according to ISO 105-C06 and to CIS, respectively. For light fastness, dyed samples were exposure to bare sunlight for 200 h. After the end of the time, light fastness ratings of dyed samples were given on 1-8 grey scale. A 255 model crock-meter and Atlas Weather-ometer were used for the determination of rubbing and light fastness values, respectively. [13].

Dyeing procedures

Dyeing procedures of the wool and the cotton samples were carried out according to the dyeing diagram (Figure 2). The undyed materials were kept into willow extract for 24h, at room temperature before dyeing procedures. At the end of the time, the samples were rinsed with distilled water and dyed using pre-mordanting, meta-mordanting and post-mordanting methods.

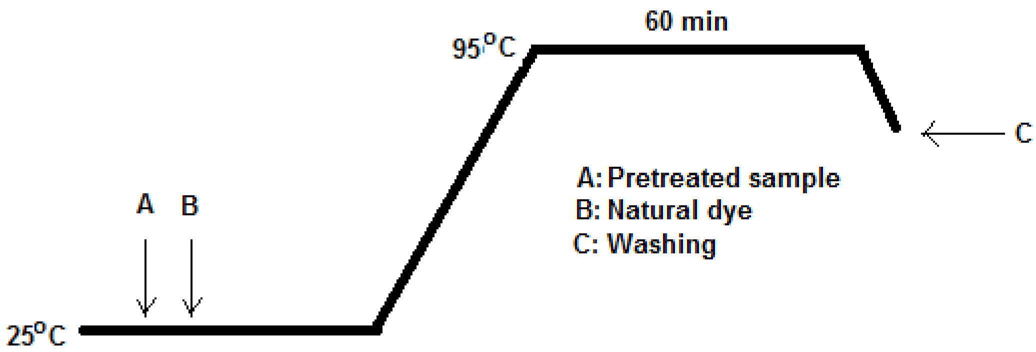

Figure 2. Dyeing diagram

Dyeing process was started at 25°C. Natural dye and the samples which were pretreated with willow extract were added and the temperature was increased to 95°C. Dyeing was continued at the same temperature for 60 min. After the dyeing process, the dyed material was cooled and rinsed with distilled water [14].

Dyeing method

For pre-mordanting method, fabrics (pre-treated and unpretreated) were soaked into mordant solution (0.1 M, 100 mL) and heated for 30 min at 95ºC. Then, it was cooled and washed with distilled water. The fabric was then placed into the dye-bath solution (100 mL) and dyed at 95ºC for 1 h. At the end of the period, the dyed material was removed, rinsed with distilled water and dried.

In meta-mordanting method, both mordant (in solid form that is equal to 0.1 M mordant solution) and the dye residue was transferred to a conical flask and the fabric was poured into the mixture. Then the mixture was heated at 95ºC until 1 h. Then it was cooled and washed with distilled water, squeezed and finally it was dried.

For post-mordanting method, the non-colored material (1 g) was firstly treated with the dye solution for 1 h at 95ºC. Then the material was cooled, washed twice with distilled water and poured into 0.1 M mordant solution (100 mL). It was heated for 30 min. at 95ºC. After the end of the process, the dyed fabrics were rinsed with distilled water [15].

3. Results and Discussion

Proposed dyeing mechanism

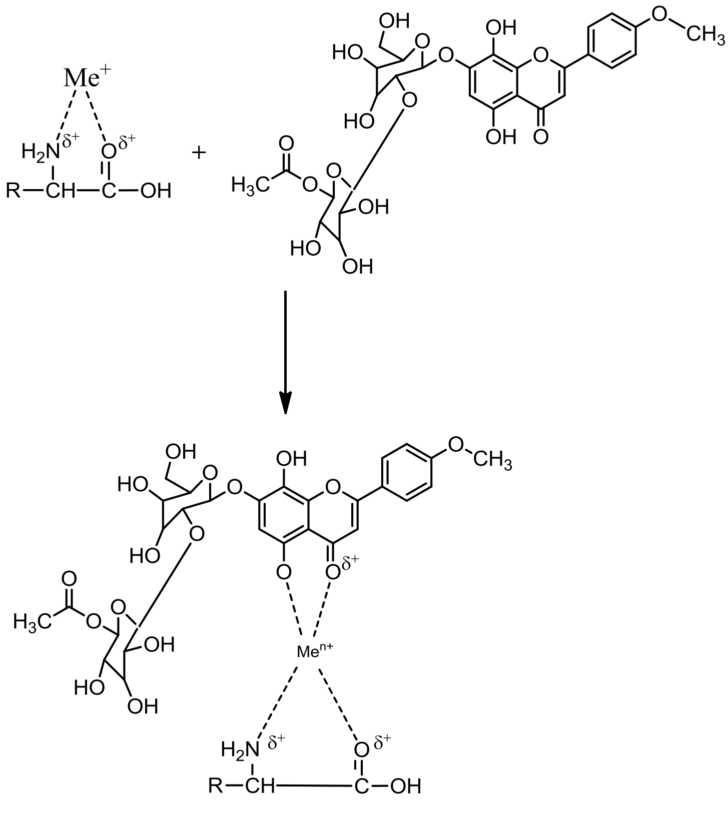

Figure 3. Proposed mordant-dye complex in the dyeing of wool fabrics

Wool structure contains both $–NH_2$ and –COOH groups. Therefore, it is expected that chemical interactions between *S. trojana* extract dye and the wool fabric occurred between –OH (hydroxyl) group of the dye molecule and oxygen and nitrogen atoms of the wool fabric via H-bonding (Figure 3).

The structure of mordant-dye complex that occurred in the dyeing of wool fabric with *S. trojana* extract can be considered as follows [15]:

Cotton consists of CH_2O- units. Due to its cellulosic structure, formation of complex is expected between CH_2O- groups of cellulose and metal cation via coordinate covalent bonding. The predicted structure is given below (Figure 4):

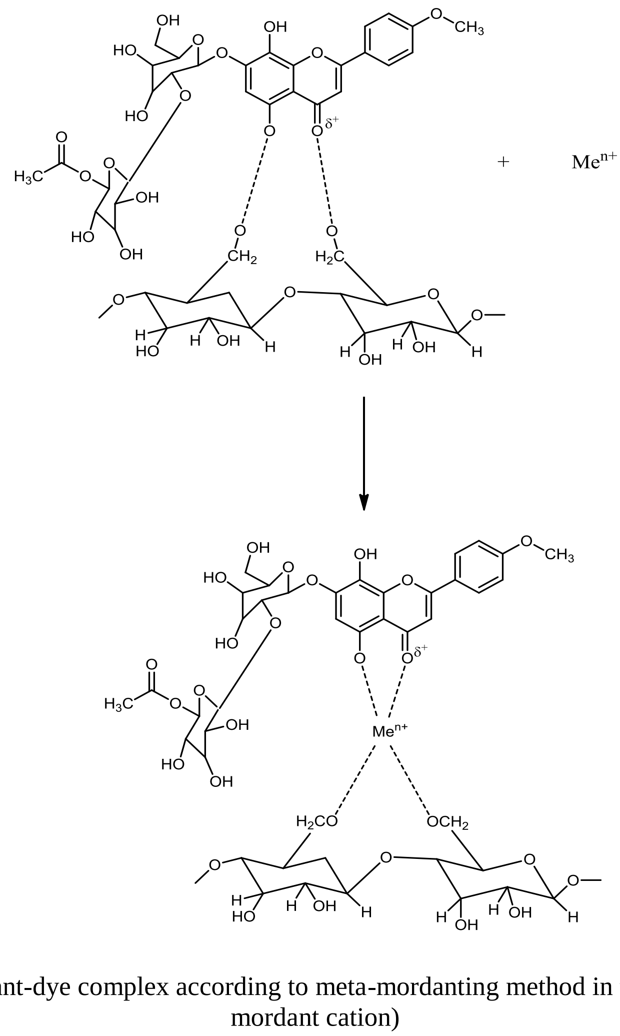

Figure 4. Proposed mordant-dye complex according to meta-mordanting method in the dyeing of cotton (Me^{n+}: mordant cation)

Fastness properties

Fastness values of the dyed fabrics are given in Table 1. Rubbing fastness of the dyed samples was determined in both dry and wet form. It is observed that rubbing fastness values were found higher in the dry form than in the wet form. Additionally, higher rubbing fastness rates were obtained with pre-mordanting method for cotton fabrics. The light fastness values of the dyed fabrics range between 3 and 7 i.e. moderate to excellent. There is no important difference between pH 4 and pH 8 in all mordanting methods and each three mordants in terms of light fastness levels. Dyeing of wool and cotton fabrics in the presence of copper sulfate mordant exhibited higher light fastness values than the other mordants (Table 1).

Table 1. Light and rubbing fastness results for the dyed cotton and the wool samples

			Light fastness			Rubbing fastness					
						Dry			Wet		
			$CuSO_4$	$FeSO_4$	$AlK(SO_4)_2$	$CuSO_4$	$FeSO_4$	$AlK(SO_4)_2$	$CuSO_4$	$FeSO_4$	$AlK(SO_4)_2$
Pre-mord.	Cotton	pH:4	6	3	3	5	4	5	5	4	4/5
		pH:8	6	4	3	5	5	5	5	5	5
	Wool	pH:4	7	5	6	5	4	5	4/5	5	4
		pH:8	7	7	6	5	5	5	4/5	4/5	4/5
Meta-mord.	Cotton	pH:4	6	5	4	5	5	5	5	4/5	4/5
		pH:8	6	5	7	4/5	4	5	4	4/5	5
	Wool	pH:4	7	4	6	5	5	4/5	5	5	4/5
		pH:8	7	7	6	4	4/5	5	3/4	4/5	5
Post mord.	Cotton	pH:4	6	4	4	5	5	5	5	5	4/5
		pH:8	6	5	5	5	4/5	5	4/5	4/5	5
	Wool	pH:4	6	5	6	5	4	5	5	4/5	5
		pH:8	7	7	7	5	4	5	5	5	5

Determination of color strength and color coordinates

Table 2. The CIELab values for the dyed cotton samples

Dyeing method	mordant	Control samples (untreated and dyed with *S. trojana* extract)			Cotton samples (pre-treated with willow extract and dyed with *S. trojana* extract)		
		*L**	*a**	*b**	*L**	*a**	*b**
Pre-mord.	$CuSO_4$	71.7	4.4	37.7	64.5	7.5	26.2
	$FeSO_4$	67.7	6.7	23.1	69.3	4.4	20.3
	$AlK(SO_4)_2$	77.4	2.0	28.6	76.1	4.1	25.9
Meta-mord.	$CuSO_4$	71.9	2.8	27.8	65.5	5.1	26.5
	$FeSO_4$	55.1	8.3	25.0	55.1	4.1	16.4
	$AlK(SO_4)_2$	82.2	0.3	27.7	81.8	-2.1	16.2
Post-mord.	$CuSO_4$	78.1	-0.4	17.8	75.1	1.5	17.2
	$FeSO_4$	68.4	10.7	34.1	69.6	3.8	20.1
	$AlK(SO_4)_2$	86.6	-0.2	10.5	83.2	2.5	16.9

Evaluation of color parameters was performed using CIELab system. Results were given in Table 2 and Table 3, respectively. Lightness-darkness values of dyed fabrics symbolized with "*L*" and these values varied between 100 and 0, representing white to black; + values of *a** and *b** indicate redness and yellowness shade, respectively. Additionally, - values of *a** and *b** refer to greenness and blueness color tones, respectively. Lightness values of the dyed fabrics were found between 55-87 and 54-84, for cotton and wool fabrics, respectively. Darker color and color tones were obtained with pre-treatment processes in the dyeing of wool fabric. Cream and brown color and color tones were obtained in the dyeing of cotton fabrics. Cream and yellow color and color shades were achieved in the dyeing of wool fabrics. Additionally, it is observed that different mordants are not only affected the hue of the color but also color strength of the dyed fabrics.

Table 3. The CIELab values for the dyed wool samples

Dyeing method	mordant	Control samples (untreated and dyed with *S. trojana* extract)			Wool samples (pre-treated with willow extract and dyed with *S. trojana* extract)		
		*L**	*a**	*b**	*L**	*a**	*b**
Pre-mord.	$CuSO_4$	69.9	2.9	34.7	59.7	5.4	30.0
	$FeSO_4$	83.7	-0.6	9.3	54.4	2.9	14.2
	$AlK(SO_4)_2$	79.9	0.4	22.0	69.4	4.5	25.6
Meta-mord.	$CuSO_4$	71.6	1.3	23.7	71.9	4.7	28.4
	$FeSO_4$	67.1	3.6	16.2	55.9	2.9	13.8
	$AlK(SO_4)_2$	79.8	-0.6	13.4	65.8	3.9	28.4
Post-mord.	$CuSO_4$	79.9	-3.7	11.9	69.6	1.7	17.8
	$FeSO_4$	68.9	10.1	33.0	60.5	7.0	25.6
	$AlK(SO_4)_2$	63.0	5.9	20.5	79.7	-0.1	13.0

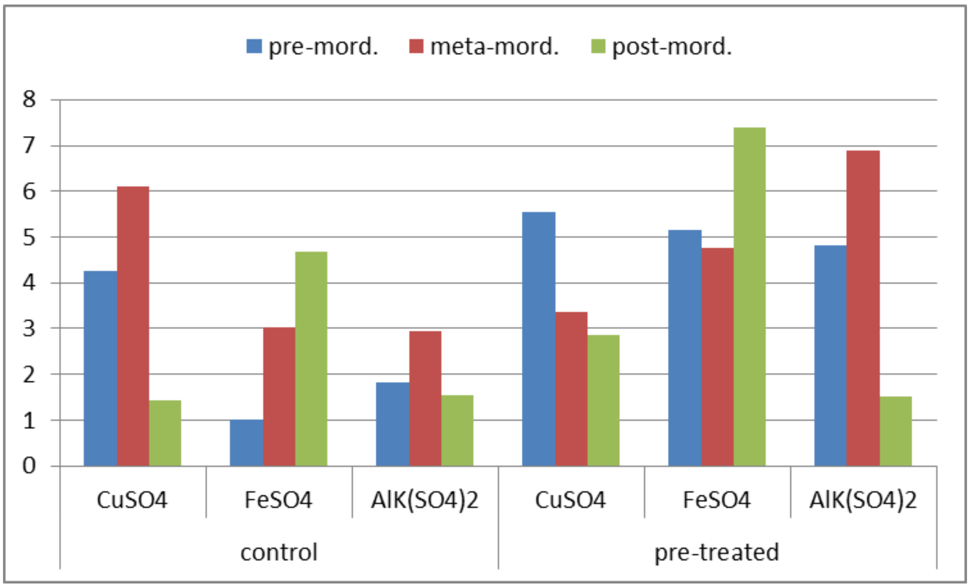

Figure 5. K/S values of the dyed wool samples

Influence of various mordants such as alum, ferrous sulfate and copper sulfate was investigated for wool and cotton fabrics (Figure 5 and Figure 6). The highest K/S value (8.5) was obtained in the presence of ferrous sulfate mordant for cotton fabric (Figure 5). The results also indicated that pre-treatment agent (willow extract) generally helps to increase the color strength of the dyed samples. It is observed that K/S values depend on the mordant type, dyeing method and pre-treatment process.

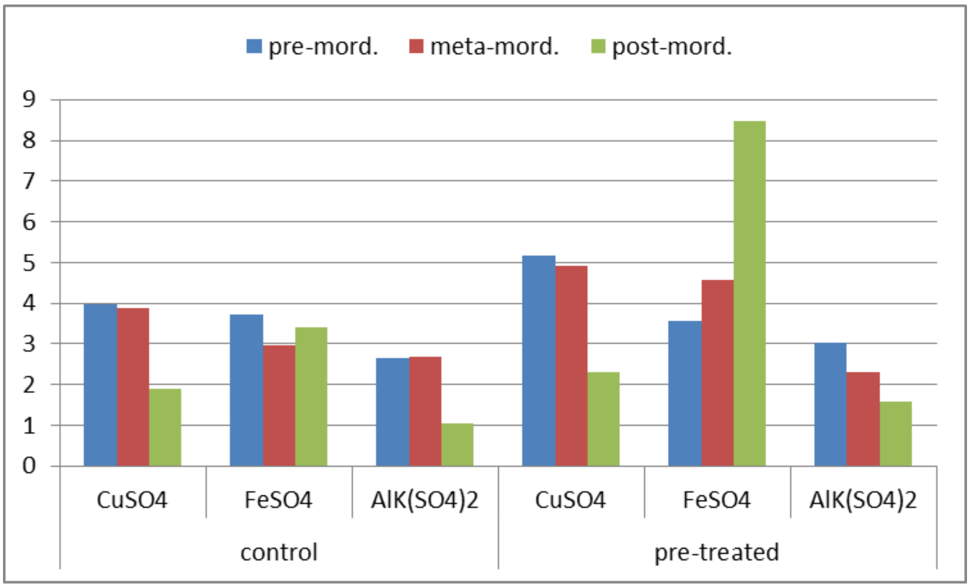

Figure 6. K/S values of the dyed cotton samples

4. Conclusions

The way to improve the quality of the dyeing is to use natural mordants such as willow extract. This extract plays an important role on the brightness of the colors. Willow extract contains salicylic acid and other tannins [16]. These components extend the pores of the fiber micelles during the pre-treatment process, and so, it facilitates to increase the affinity of the dye to the keratin. Therefore, high color fastness values were obtained in the presence of willow extract.

As a result, fastness values of the wool samples are found higher than that of cotton samples. Yellow, brown, cream color and color tones are obtained from the dyeing of the fabrics with *S. trojana* extract in the presence of willow extract pre-treatment. High fastness values are obtained for three mordanting methods with all mordants that used in the study.

Consequently, *S. trojana* is a proper natural source for dyeing of wool and cotton fabrics. Therefore, this plant may be used as a natural source in the production of the carpets and kilims.

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