Number 10 Year 2016

Journal of New Theory

ISSN: 2149-1402

www.dergipark.org.tr/en/pub/jnt

Journal of New Theory (abbreviated by J. New Theory or JNT) is a mathematical journal focusing on new mathematical theories or the applications of a mathematical theory to science.

JNT founded on 18 November 2014 and its first issue published on 27 January 2015.

ISSN: 2149-1402

Editor-in-Chief: Naim Çağman

Email: journalofnewtheory@gmail.com

Language: English only.

Article Processing Charges: It has no processing charges.

Publication Frequency: Quarterly

Publication Ethics: The governance structure of J. New Theory and its acceptance procedures are transparent and designed to ensure the highest quality of published material. Journal of New Theory adheres to the international standards developed by the Committee on Publication Ethics (COPE).

Aim: The aim of the Journal of New Theory is to share new ideas in pure or applied mathematics with the world of science.

Scope: Journal of New Theory is an international, online, open access, and peer-reviewed journal. Journal of New Theory publishes original research articles, reports, reviews, editorial, letters to the editor, technical notes etc. from all branches of science that use the theories of mathematics.

Journal of New Theory concerns the studies in the areas of, but not limited to:

· Fuzzy Sets,

· Soft Sets,

· Neutrosophic Sets,

· Decision-Making

· Algebra

· Number Theory

· Analysis

· Theory of Functions

· Geometry

· Applied Mathematics

· Topology

· Fundamental of Mathematics

· Mathematical Logic

· Mathematical Physics

You can submit your manuscript in any style or JNT style as pdf. However, you should send your paper in JNT style if it would be accepted. The manuscript preparation rules, article template (LaTeX) and article template (Microsoft Word) can be accessed from the following links.

- Manuscript Preparation Rules
- Article Template (Microsoft Word.DOC) (Version 2019)
- Article Template (LaTeX) (Version 2019)

Alaa Mohamed Abd El-latif

Department of Mathematics, Faculty of Arts and Science, Northern Border University, Rafha, Saudi Arabia
email: alaa_8560@yahoo.com

Kalyan Mondal

Department of Mathematics, Jadavpur University, Kolkata, West Bengal 700032, India
email: kalyanmathematic@gmail.com

Jun Ye

Department of Electrical and Information Engineering, Shaoxing University, Shaoxing, Zhejiang, P.R. China
email: yehjun@aliyun.com

Ayman Shehata

Department of Mathematics, Faculty of Science, Assiut University, 71516-Assiut, Egypt
email: drshehata2009@gmail.com

İdris Zorlutuna

Department of Mathematics, Cumhuriyet University, Sivas, Turkey
email: izorlu@cumhuriyet.edu.tr

Murat Sarı

Department of Mathematics, Yıldız Technical University, İstanbul, Turkey
email: sarim@yildiz.edu.tr

Daud Mohamad

Faculty of Computer and Mathematical Sciences, University Teknologi Mara, 40450 Shah Alam, Malaysia
email: daud@tmsk.uitm.edu.my

Tanmay Biswas

Research Scientist, Rajbari, Rabindrapalli, R. N. Tagore Road, P.O.- Krishnagar Dist-Nadia, PIN-741101, West Bengal, India
email: tanmaybiswas_math@rediffmail.com

Kadriye Aydemir

Department of Mathematics, Amasya University, Amasya, Turkey
email: kadriye.aydemir@amasya.edu.tr

Ali Boussayoud

LMAM Laboratory and Department of Mathematics, Mohamed Seddik Ben Yahia University, Jijel, Algeria
email: alboussayoud@gmail.com

Muhammad Riaz

Department of Mathematics, Punjab University, Quaid-e-Azam Campus, Lahore-54590, Pakistan
email: mriaz.math@pu.edu.pk

Contact

Editor-in-Chief
Name: Prof. Dr. Naim Çağman
Email: journalofnewtheory@gmail.com
Phone: +905354092136
Address: Departments of Mathematics, Faculty of Arts and Sciences, Tokat Gaziosmanpaşa University, Tokat, Turkey

Editors
Name: Assoc. Prof. Dr. Faruk Karaaslan
Email: karaaslan.faruk@gmail.com
Phone: +905058314380
Address: Departments of Mathematics, Faculty of Arts and Sciences, Çankırı Karatekin University, 18200, Çankırı, Turkey

Name: Assoc. Prof. Dr. İrfan Deli
Email: irfandeli@kilis.edu.tr
Phone: +905426732708
Address: M.R. Faculty of Education, Kilis 7 Aralık University, Kilis, Turkey

Name: Asst. Prof. Dr. Serdar Enginoğlu
Email: serdarenginoglu@gmail.com
Phone: +905052241254
Address: Departments of Mathematics, Faculty of Arts and Sciences, Çanakkale Onsekiz Mart University, 17100, Çanakkale, Turkey

CONTENT

http://www.newtheory.org ISSN: 2149-1402

Journal of New Theory

Received: 14.04.2015
Published: 04.01.2016
Year: **2016**, *Number*: **10**, *Pages*: **1-11**
Original Article[*]

(λ, μ)-FUZZY IDEALS OF ORDERED Γ-SEMIRINGS

Debabrata Mandal <dmandaljumath@gmail.com>

Department of Mathematics, Raja Peary Mohan College, Uttarpara, Hooghly-712258, India

Abstaract − The notions of (λ, μ)-fuzzy ideals, (λ, μ)-fuzzy k-ideals, (λ, μ)-fuzzy k-bi-ideals, (λ, μ)-fuzzy k-quasi-ideals of an ordered Γ-semirings are introduced and some related properties are investigated. The concepts of k-regularity, k-intra-regularity are studied along with some of their characterizations.

Keywords − *Cartesian product, (λ, μ)-fuzzy ideal, homomorphism, k-intra-regular, k-regular, ordered Γ-semiring.*

1 Introduction

Uncertainties, which could be caused by information incompleteness, data randomness limitations of measuring instruments, etc., are pervasive in many complicated problems in biology, engineering, economics, environment, medical science and social science. We cannot successfully use the classical methods for these problems. To solve this, the concept of fuzzy sets was introduced by Zadeh [12] in 1965 where each element have a degree of membership and has been extensively applied to many scientific fields.

Semirings [3] which provide a common generalization of rings and distributive lattices arise naturally in such diverse areas of mathematics as combinatorics, functional analysis, graph theory, automata theory, mathematical modelling and parallel computation systems etc.(for example, see [3], [4]). Semirings have also been proved to be an important algebraic tool in theoretical computer science, see for instance [4], for some detail and example. Many of the semirings have an order structure in addition to their algebraic structure and indeed the most interesting results concering them make use of the interplay between these two structures.

Ideals of semirings play an important role in the structure theory of ordered semirings and useful for many purposes. In this paper, like ordered semigroup [2, 5, 6, 8], it is an attempt to study how similar is the theory in terms of fuzzy for the case of ordered Γ-semiring [10], a generalization of ordered semirings [9] , since nowadays fuzzy research concerns standardization, axiomatization, extensions to lattice-valued fuzzy sets, critical comparison of the different so-called soft computing models that have

[*] *Edited by* Oktay Muhtaroğlu *(Area Editor) and* Naim Çağman *(Editor-in-Chief).*

been launched during the past three decennia for the representation and processing of incomplete information [7].

Here we first introduce (λ, μ)-fuzzy ideals of ordered Γ-semirings. After that we define intersection, cartesian product and composition of (λ, μ)-fuzzy ideals and use these to study regular (resp. intra-regular) ordered Γ-semirings.

2 Preliminaries

We recall the following definitions for subsequent use.

Definition 2.1. Let S and Γ be two additive commutative semigroups with zero. Then S is called a Γ-semiring if there exists a mapping $S \times \Gamma \times S \to S$ ((a,α,b) $\mapsto$ aαb) satisfying the following conditions:

(i) $(a + b)\alpha c = a\alpha c + b\alpha c$

(ii) $a\alpha(b + c) = a\alpha b + a\alpha c$

(iii) $a(\alpha + \beta)b = a\alpha b + a\beta b$

(iv) $a\alpha(b\beta c) = (a\alpha b)\beta c$

(v) $0_S\alpha a = 0_S = a\alpha 0_S$

(vi) $a0_\Gamma b = 0_S = b0_\Gamma a$

where $a, b, c \in S$, $\alpha, \beta \in \Gamma$, 0_S is the zero element of S and 0_Γ is the zero element of Γ.

For simplification we write 0 instead of 0_S and 0_Γ.

Definition 2.2. A left ideal I of Γ-semiring S is a nonempty subset of S satisfying the following conditions:

(i) If $a, b \in I$ then $a + b \in I$

(ii) If $a \in I$, $s \in S$ and $\gamma \in \Gamma$ then $s\gamma a \in I$

(iii) $I \neq S$.

A right ideal of S is defined in an analogous manner and an ideal of S is a nonempty subset which is both a left ideal and a right ideal of S.

Definition 2.3. An ordered semiring is a Γ-semiring S equipped with a partial order $\leq$ such that the operation is monotonic and constant 0 is the least element of S.

Definition 2.4. Let R, S be two Γ-semirings and $a, b \in R$, $\gamma \in \Gamma$. A function $f : R \to S$ is said to be a homomorphism if

(i) $f(a + b) = f(a) + f(b)$

(ii) $f(a\gamma b) = f(a)\gamma f(b)$

(iii) $f(0_R) = 0_S$ where 0_R and 0_S are the zeroes of R and S respectively.

Now we recall the definition and example of ordered ideal from [1]

Definition 2.5. A left (resp. right) ideal I of S is called a left (resp. right) ordered ideal, if for any $a \in S$, $b \in I$, $a \leq b$ implies $a \in I$ (i.e. $(I] \subseteq I$). I is called an ordered ideal of S if it is both a left and a right ordered ideal of S.

Example 2.6. Let $S = ([0,1], \vee, \cdot, 0)$ where [0,1] is the unit interval $a \vee b = \max\{a, b\}$ and $a \cdot b = (a + b - 1) \vee 0$ for $a, b \in [0,1]$. Then it is easy to verify that S equipped with the usual ordering $\leq$ is an ordered semiring and $I = [0, \frac{1}{2}]$ is an ordered ideal of S.

Definition 2.7. [12] Let S be a non-empty set. A mapping $f : S \to [0,1]$ is called a fuzzy subset of S.

Definition 2.8. The union and intersection of two fuzzy subsets f and σ of a set S, denoted by $f \cup \sigma$ and $f \cap \sigma$ respectively, are defined by

$$(f \cup \sigma)(x) = f(x) \vee \sigma(x) \text{ for all } x \in S$$

$$(f \cap \sigma)(x) = f(x) \wedge \sigma(x) \text{ for all } x \in S.$$

3 (λ, μ)-fuzzy ideals with some operations

Throughout this paper unless otherwise mentioned S denote the ordered Γ-semiring with identity 1, χ_S denote its characteristic function and we will always assume that $0 \leq \lambda < \mu \leq 1$.

Definition 3.1. Let f and g be two fuzzy subsets of an ordered Γ-semiring S. We define two compositions of f and g as follows:

$$\begin{aligned} f \circ_1 g(x) &= \vee\{\underset{x+y_1\alpha z_1 \leq y_2\beta z_2}{\wedge} \{f(y_1), f(y_2), g(z_1), g(z_2)\}\} \\ &= 0, \text{ if } x \text{ cannot be expressed as } x + y_1\alpha z_1 \leq y_2\beta z_2 \\ &\quad \text{where } x, y_1, y_2, z_1, z_2 \in S \text{ and } \alpha, \beta \in \Gamma \end{aligned}$$

and

$$\begin{aligned} f o_2 g(x) &= \vee[\underset{i}{\wedge}\{\wedge\{f(a_i), f(c_i), g(b_i), g(d_i)\}\}] \\ &\qquad x + \sum_{i=1}^{n} a_i\alpha_i b_i \leq \sum_{i=1}^{n} c_i\beta_i d_i \\ &= 0, \text{ if } x \text{ cannot be expressed as above} \\ &\quad \text{where } x, z, a_i, b_i, c_i, d_i \in S \text{ and } \alpha_i, \beta_i \in \Gamma. \end{aligned}$$

Definition 3.2. Let f be a non-empty fuzzy subset of an ordered Γ-semiring S (i.e. $f(x) \neq 0$ for some $x \in S$). Then f is called a (λ, μ)-fuzzy left ideal [resp. (λ, μ)-fuzzy right ideal] of S if

(i) $f(x + y) \vee \lambda \geq f(x) \wedge f(y) \wedge \mu$

(ii) $f(x\gamma y) \vee \lambda \geq f(y) \wedge \mu$ [resp. $f(x\gamma y) \vee \lambda \geq f(x) \wedge \mu$] and

(iii) $x \leq y$ implies $f(x) \vee \lambda \geq f(y) \wedge \mu$.

for all $x, y \in S$ and $\gamma \in \Gamma$.
A (λ, μ)-fuzzy ideal of an ordered Γ-semiring S is a non-empty (λ, μ)-fuzzy subset of S which is a (λ, μ)-fuzzy left ideal as well as a (λ, μ)-fuzzy right ideal of S.
Note that for (λ, μ)-fuzzy k-ideal the following additional relation must be holds:
For $x, a, b \in S$ with $x + a \leq b \Rightarrow f(x) \vee \lambda \geq f(a) \wedge f(b) \wedge \mu$.

Example 3.3. Let $S = \Gamma = \{0, a, b\}$ with the ordered relation $0 \prec b \prec a$. Define operations on S by following:

$\oplus$	0	a	b
0	0	a	b
a	a	a	b
b	b	b	b

and

$\odot$	0	a	b
0	0	0	0
a	0	a	a
b	0	b	b

Then $(S, \oplus, \odot)$ forms an ordered semiring.
We now define a fuzzy subset μ of S by $\mu(0) = 1,\ \mu(b) = 0.2,$ and $\mu(a) = 0.1$, then μ will be a $(0.5, 0.8)$-fuzzy right ideal of S.

Theorem 3.4. Let S be an ordered Γ-semiring and f be a (λ, μ)-fuzzy right (resp. left) ideal of S. Then $I_a = \{b \in S | f(b) \vee \lambda \geq f(a) \wedge \mu\}$ is a right (resp. left) ideal of S for every $a \in S$.

Proof. Let f be a (λ, μ)-fuzzy right ideal of S and $a \in S$. Then $I_a \neq \phi$ because $a \in I_a$ for every $a \in S$. Let $b, c \in I_a$, $\gamma \in \Gamma$ and $x \in S$. Since $b, c \in I_a$, $f(b) \vee \lambda \geq f(a) \wedge \mu$ and $f(c) \vee \lambda \geq f(a) \wedge \mu$. Now

$$\begin{aligned} f(b+c) \vee \lambda &\geq f(b) \wedge f(c) \wedge \mu \ [\because f \text{ is a } (\lambda, \mu)\text{-fuzzy right ideal}] \\ &\geq f(a) \wedge \mu. \end{aligned}$$

which implies $b + c \in I_a$.
Also $f(b\gamma x) \vee \lambda \geq f(b) \wedge \mu \geq f(a) \wedge \mu$ i.e. $b\gamma x \in I_a$.
Let $b \in I_a$ and $S \ni x \leq b$. Then $f(x) \vee \lambda \geq f(b) \wedge \mu \geq f(a) \wedge \mu \Rightarrow x \in I_a$.
Thus I_a is a right ideal of S.
Similarly we can prove the result for left ideal also. □

Proposition 3.5. Intersection of a non-empty collection of (λ, μ)-fuzzy right (resp. left) ideals is also a (λ, μ)-fuzzy right (resp. left) ideal of S.

Proof. Let $\{f_i : i \in I\}$ be a non-empty family of (λ, μ)-fuzzy right ideals of S and $x, y \in S$, $\gamma \in \Gamma$.
Then

$$\begin{aligned} (\bigcap_{i\in I} f_i)(x+y) \vee \lambda &= \bigwedge_{i\in I}\{f_i(x+y) \vee \lambda\} \geq \bigwedge_{i\in I}\{f_i(x) \wedge f_i(y) \wedge \mu\} \\ &= \wedge\{\bigwedge_{i\in I} f_i(x), \bigwedge_{i\in I} f_i(y)\} \wedge \mu = (\bigcap_{i\in I} f_i)(x) \wedge (\bigcap_{i\in I} f_i)(y) \wedge \mu. \end{aligned}$$

Again

$$(\bigcap_{i\in I} f_i)(x\gamma y) \vee \lambda = \bigwedge_{i\in I}\{f_i(x\gamma y) \vee \lambda\} \geq \bigwedge_{i\in I} f_i(x) \wedge \mu = (\bigcap_{i\in I} f_i)(x) \wedge \mu.$$

Suppose $x \leq y$. Then $f_i(x) \vee \lambda \geq f_i(y) \wedge \mu$ for all $i \in I$ which implies $(\bigcap_{i\in I} f_i)(x) \vee \lambda \geq (\bigcap_{i\in I} f_i)(y) \wedge \mu$. Hence $\bigcap_{i\in I} f_i$ is a (λ, μ)-fuzzy right ideal of S.
Similarly we can prove the result for (λ, μ)-fuzzy left ideal also. □

Proposition 3.6. Let $f : R \to S$ be a morphism of ordered Γ-semirings i.e. Γ-semiring homomorphism satisfying additional condition $a \leq b \Rightarrow f(a) \leq f(b)$. Then if ϕ is a (λ, μ)-fuzzy left ideal of S, then $f^{-1}(\phi)$ [11] is also a (λ, μ)-fuzzy left ideal of R.

Proof. Let $f : R \to S$ be a morphism of ordered semirings.
Let ϕ be a (λ, μ)-fuzzy left ideal of S.
Now $f^{-1}(\phi)(0_R) \vee \lambda = \phi(0_S) \vee \lambda \geq \phi(x^{'}) \neq 0$ for some $x^{'} \in S$.
Therefore $f^{-1}(\phi)$ is non-empty.
Now, for any $r, s \in R$ and $\gamma \in \Gamma$

$$\begin{aligned} f^{-1}(\phi)(r+s) \vee \lambda &= \phi(f(r+s)) \vee \lambda = \phi(f(r)+f(s)) \vee \lambda \\ &\geq \phi(f(r)) \wedge \phi(f(s)) \wedge \mu = (f^{-1}(\phi))(r) \wedge (f^{-1}(\phi))(s) \wedge \mu. \end{aligned}$$

Again

$$\begin{aligned} (f^{-1}(\phi))(r\gamma s) \vee \lambda &= \phi(f(r\gamma s)) \vee \lambda = \phi(f(r)\gamma f(s)) \vee \lambda \\ &\geq \phi(f(s)) \wedge \mu = (f^{-1}(\phi))(s) \wedge \mu. \end{aligned}$$

Also if $r \leq s$, $f(r) \leq f(s)$. Then

$$(f^{-1}(\phi))(r) \vee \lambda = \phi(f(r)) \vee \lambda \geq \phi(f(s)) \wedge \mu = (f^{-1}(\phi))(s) \wedge \mu.$$

Thus $f^{-1}(\phi)$ is a (λ, μ)-fuzzy left ideal of R.

□

Definition 3.7. Let f and g be fuzzy subsets of X. The cartesian product of f and g is defined by $(f \times g)(x, y) = f(x) \wedge g(y)$ for all $x, y \in X$.

Theorem 3.8. Let f and g be (λ, μ)-fuzzy left ideals of an ordered Γ-semiring S. Then $f \times g$ is a (λ, μ)-fuzzy left ideal of $S \times S$.

Proof. Let $(x_1, x_2), (y_1, y_2) \in S \times S$ and $\gamma \in \Gamma$. Then

$$\begin{aligned} (f \times g)((x_1, x_2) + (y_1, y_2)) \vee \lambda &= (f \times g)(x_1 + y_1, x_2 + y_2) \vee \lambda \\ &= (f(x_1 + y_1) \wedge g(x_2 + y_2)) \vee \lambda \\ &= (f(x_1 + y_1) \vee \lambda) \wedge (g(x_2 + y_2) \vee \lambda) \\ &\geq (f(x_1) \wedge f(y_1) \wedge \mu) \wedge (g(x_2) \wedge g(y_2) \wedge \mu) \\ &= (f \times g)(x_1, x_2) \wedge (f \times g)(y_1, y_2) \wedge \mu \end{aligned}$$

and

$$\begin{aligned} (f \times g)((x_1, x_2)\gamma(y_1, y_2)) \vee \lambda &= (f \times g)(x_1\gamma y_1, x_2\gamma y_2) \vee \lambda \\ &= (f(x_1\gamma y_1) \wedge g(x_2\gamma y_2)) \vee \lambda \\ &= (f(x_1\gamma y_1) \vee \lambda) \wedge (g(x_2\gamma y_2) \vee \lambda) \\ &\geq f(y_1) \wedge g(y_2) \wedge \mu = (f \times g)(y_1, y_2) \wedge \mu. \end{aligned}$$

Also if $(x_1, x_2) \leq (y_1, y_2)$, then

$$\begin{aligned} (f \times g)(x_1, x_2) \vee \lambda &= (f(x_1) \wedge g(x_2)) \vee \lambda = (f(x_1) \vee \lambda) \wedge (g(x_2) \vee \lambda) \\ &\geq (f(y_1) \wedge \mu) \wedge (g(y_2) \wedge \mu) = f(y_1) \wedge g(y_2) \wedge \mu \\ &= (f \times g)(y_1, y_2) \wedge \mu. \end{aligned}$$

Therefore $f \times g$ is a (λ, μ)-fuzzy left ideal of $S \times S$. □

Theorem 3.9. Let f be a (λ,μ)-fuzzy subset in an ordered Γ-semiring S. Then f is a (λ,μ)-fuzzy left ideal of S if and only if $f\times f$ is a (λ,μ)-fuzzy left ideal of $S\times S$.

Proof. Assume that f is a (λ,μ)-fuzzy left ideal of S. Then by Theorem 3.8, $f\times f$ is a (λ,μ)-fuzzy left ideal of $S\times S$.
Conversely, suppose that $f\times f$ is a (λ,μ)-fuzzy left ideal of $S\times S$. Let $x_1,x_2,y_1,y_2\in S$ and $\gamma\in\Gamma$. Then

$$\begin{aligned}(f(x_1+y_1)\wedge f(x_2+y_2))\vee\lambda &= (f\times f)(x_1+y_1,x_2+y_2)\vee\lambda\\ &= (f\times f)((x_1,x_2)+(y_1,y_2))\vee\lambda\\ &\geq (f\times f)(x_1,x_2)\wedge(f\times f)(y_1,y_2)\wedge\mu\\ &= (f(x_1)\wedge f(x_2)\wedge\mu)\wedge(f(y_1)\wedge f(y_2)\wedge\mu).\end{aligned}$$

Now, putting $x_1=x,\ x_2=0,\ y_1=y$ and $y_2=0$, in this inequality and noting that $f(0)\geq f(x)$ for all $x\in S$, we obtain $f(x+y)\vee\lambda\geq f(x)\wedge f(y)\wedge\mu$.
Next, we have

$$\begin{aligned}(f(x_1\gamma y_1)\wedge f(x_2\gamma y_2))\vee\lambda &= (f\times f)(x_1\gamma y_1,x_2\gamma y_2)\vee\lambda\\ &= (f\times f)((x_1,x_2)\gamma(y_1,y_2))\vee\lambda\\ &\geq (f\times f)(y_1,y_2)\wedge\mu\\ &= f(y_1)\wedge f(y_2)\wedge\mu.\end{aligned}$$

Taking $x_1=x,\ y_1=y$ and $y_2=0$, we obtain $f(x\gamma y)\vee\lambda\geq f(y)\wedge\mu$.
Also if $(x_1,x_2)\leq(y_1,y_2)$, then $(f(x_1)\wedge f(x_2))\vee\lambda\geq f(y_1)\wedge f(y_2)\wedge\mu$. Now, putting $x_1=x,\ x_2=0,\ y_1=y$ and $y_2=0$, in this inequality we have $f(x)\vee\lambda\geq f(y)\wedge\mu$.
Hence f is a (λ,μ)-fuzzy left ideal of S. □

Theorem 3.10. If $f_1,\ f_2$ be any two (λ,μ)-fuzzy k-ideals of an ordered semiring S then $f_1o_2f_2$ is a (λ,μ)-fuzzy ideal of S.

Proof. Let $f_1,\ f_2$ be any two (λ,μ)-fuzzy k-ideals of an ordered semiring S and $x,\ y\in S$ and $\gamma\in\Gamma$. Then

$$\begin{aligned}&(f_1o_2f_2)(x+y)\vee\lambda\\ &= \bigvee_{x+y+\sum a_i\alpha_ib_i\leq\sum c_i\beta_id_i}\{\wedge\{f_1(a_i),f_2(b_i),f_1(c_i),f_2(d_i)\}\}\vee\lambda\\ &\geq \bigvee_{x+y+\sum x_{1i}\alpha_{1i}x_{2i}+\sum y_{1i}\alpha_{2i}y_{2i}\leq\sum x_{3i}\beta_{1i}x_{4i}+\sum y_{3i}\beta_{2i}y_{4i}}\{\wedge\{f_1(x_{1i}),f_1(x_{3i}),f_1(y_{1i}),f_1(y_{3i}),\quad f_2(x_{2i}),f_2(x_{4i}),f_2(y_{2i}),f_2(y_{4i})\}\}\vee\lambda\\ &\geq \wedge\{\bigvee_{x+\sum x_{1i}\alpha_{1i}x_{2i}\leq\sum x_{3i}\beta_{1i}x_{4i}}\{\wedge\{f_1(x_{1i}),f_1(x_{3i}),f_2(x_{2i}),f_2(x_{4i})\}\},\bigvee_{y+\sum y_{1i}\alpha_{2i}y_{2i}\leq\sum y_{3i}\beta_{2i}y_{4i}}\{\wedge\{f_1(y_{1i}),f_1(y_{3i}),f_2(y_{2i}),f_2(y_{4i})\}\}\}\wedge\mu\\ &= (f_1o_2f_2)(x)\wedge(f_1o_2f_2)(y)\wedge\mu.\end{aligned}$$

Now assuming $f_1,\ f_2$ are as (λ,μ)-fuzzy right ideals we have

$$\begin{aligned}(f_1o_2f_2)(x\gamma y)\vee\lambda &= \bigvee_{x\gamma y+\sum a_i\gamma_ib_i\leq\sum c_i\delta_id_i}\{\wedge\{f_1(a_i),f_2(b_i),f_1(c_i),f_2(d_i)\}\}\vee\lambda\\ &\geq \bigvee_{x\gamma y+\sum x_{1i}\alpha_ix_{2i}\gamma y\leq\sum x_{3i}\beta_ix_{4i}\gamma y}\{\wedge\{f_1(x_{1i}),f_1(x_{3i}),\quad f_2(x_{2i}\gamma y),f_2(x_{4i}\gamma y)\}\}\vee\lambda\\ &\geq \bigvee_{x+\sum x_{1i}\alpha_ix_{2i}\leq\sum x_{3i}\beta_ix_{4i}}\{\wedge\{f_1(x_{1i}),f_1(x_{3i}),f_2(x_{2i}),f_2(x_{4i})\}\}\wedge\mu\\ &= (f_1o_2f_2)(x)\wedge\mu.\end{aligned}$$

Similarly, assuming $f_1,\ f_2$ are as (λ,μ)-fuzzy left k-ideals we can show that $(f_1o_2f_2)(x\gamma y)\geq(f_1o_2f_2)(y)$.

Now suppose $x \leq y$. Then $f_1(x) \vee \lambda \geq f_1(y) \wedge \mu$ and $f_2(x) \vee \lambda \geq f_2(y) \wedge \mu$.

$$\begin{aligned}(f_1 o_2 f_2)(x) \vee \lambda &= \underset{x+\sum x_{1i}\alpha_i x_{2i} \leq \sum x_{3i}\beta_i x_{4i}}{\vee}\{\wedge\{f_1(x_{1i}), f_1(x_{3i}), f_2(x_{2i}), f_2(x_{4i})\}\} \vee \lambda \\ &\geq \underset{x+\sum y_{1i}\gamma_{1i}y_{2i} \leq y+\sum y_{1i}\gamma_{2i}y_{2i} \leq \sum y_{3i}\gamma_{3i}y_{4i}}{\vee}\{\wedge\{f_1(y_{1i}), f_1(y_{3i}), f_2(y_{2i}), f_2(y_{4i})\}\} \wedge \mu \\ &= \underset{y+\sum y_{1i}\gamma_{1i}y_{2i} \leq \sum y_{3i}\gamma_{3i}y_{4i}}{\vee}\{\wedge\{f_1(y_{1i}), f_1(y_{3i}), f_2(y_{2i}), f_2(y_{4i})\}\} \wedge \mu \\ &= (f_1 o_2 f_2)(y) \wedge \mu.\end{aligned}$$

Hence $f_1 o_2 f_2$ is a (λ, μ)-fuzzy ideal of S. □

4 (λ, μ)-fuzzy ideals of regular ordered Γ-semiring

Definition 4.1. A fuzzy subset f of an ordered semiring S is called (λ, μ)-fuzzy bi-ideal if for all $x, y \in S$ and $\alpha, \ \beta \in \Gamma$ we have

(i) $f(x + y) \vee \lambda \geq f(x) \wedge f(y) \wedge \mu$

(ii) $f(x\alpha y) \vee \lambda \geq f(x) \wedge f(y) \wedge \mu$

(iii) $f(x\alpha y\beta z) \vee \lambda \geq f(x) \wedge f(z) \wedge \mu$

(iv) $x \leq y \Rightarrow f(x) \vee \lambda \geq f(y) \wedge \mu$.

Definition 4.2. A (λ, μ)-fuzzy subset f of an ordered Γ-semiring S is called (λ, μ)-fuzzy quasi-ideal if for all $x, y \in S$ we have

(i) $f(x + y) \vee \lambda \geq f(x) \wedge f(y) \wedge \mu$

(ii) $((f o_2 \chi_S) \cap (\chi_S o_2 f))(x) \wedge \mu \leq f(x) \vee \lambda$

(iii) $x \leq y \Rightarrow f(x) \vee \lambda \geq f(y) \wedge \mu$.

Proposition 4.3. Intersection of a non-empty collection of (λ, μ)-fuzzy bi-ideals of S is also a (λ, μ)-fuzzy bi-ideal of S.

Proof. The proof follows by routine verifications. □

Proposition 4.4. Let $\{f_i : i \in I\}$ be a family of bi-ideals of S such that $f_i \subseteq f_j$ or $f_j \subseteq f_i$ for $i, j \in I$. Then $\underset{i \in I}{\cup} f_i$ is a (λ, μ)-fuzzy bi-ideal of S.

Proof. Straightforward. □

Lemma 4.5. In an ordered Γ-semiring every (λ, μ)-fuzzy quasi ideals are (λ, μ)-fuzzy bi-ideals.

Proof. Let f be a (λ, μ)-fuzzy quasi ideal of S. It is sufficient to prove that $f(x\alpha y\beta z) \vee \lambda \geq f(x) \wedge f(z) \wedge \mu$ for all $x, \ y, \ z \in S$ and $\alpha, \ \beta \in \Gamma$. Since f is a (λ, μ)-fuzzy quasi ideal of S, we have

$$\begin{aligned}f(x\alpha y\beta z) \vee \lambda &\geq ((f o_2 \chi_S) \cap (\chi_S o_2 \mu))(x\alpha y\beta z) \wedge \mu \\ &= \{(f o_2 \chi_S)(x\alpha y\beta z) \wedge (\chi_S o_2 f)(x\alpha y\beta z)\} \wedge \mu \\ &= \wedge\{ \underset{x\alpha y\beta z+\sum_{i=1}^{n} a_i\gamma_i b_i \leq \sum_{i=1}^{n} c_i\delta_i d_i}{\vee(f(a_i) \wedge f(c_i))}, \underset{x\alpha y\beta z+\sum_{i=1}^{n} a_i\gamma_i b_i \leq \sum_{i=1}^{n} c_i\delta_i d_i}{\vee(f(b_i) \wedge f(d_i))} \} \wedge \mu \\ &\geq f(0) \wedge f(x) \wedge f(0) \wedge f(z)) \wedge \mu (\text{since } x\alpha y\beta z + 0\gamma 0 + 0 = x\alpha y\beta z + 0) \\ &= f(x) \wedge f(z) \wedge \mu.\end{aligned}$$

Similarly, we can show that $f(x\alpha y) \vee \lambda \geq f(x) \wedge f(y) \wedge \mu$ for all $x, y \in S$ and $\alpha \in \Gamma$. □

Definition 4.6. An ordered Γ-semiring S is said to be k-regular if for each $x \in S$, there exist $a, b \in S$ and $\alpha, \beta, \gamma, \delta \in \Gamma$ such that $x + x\alpha a\beta x \leq x\gamma b\delta x$.

Definition 4.7. An ordered Γ-semiring S is said to be k-intra-regular if for each $x \in S$, there exist $z, a_i, a_i', b_i, b_i' \in S$, $\alpha_{1i}, \alpha_{2i}, \alpha_{3i}, \beta_{1i}, \beta_{2i}, \beta_{3i} \in \Gamma$, $i \in \mathbf{N}$, the set of natural numbers, such that $x + \sum_{i=1}^{n} a_i\alpha_{1i}x\alpha_{2i}x\alpha_{3i}a_i' \leq \sum_{i=1}^{n} b_i\beta_{1i}x\beta_{2i}x\beta_{3i}b_i'$.

Theorem 4.8. Let S be a k-regular ordered semiring and $x \in S$. Then

(i) $f(x) \wedge \mu \leq (fo_2\chi_S o_2 f)(x) \vee \lambda$ for every (λ, μ)-fuzzy k-bi-ideal f of S.

(ii) $f(x) \wedge \mu \leq (fo_2\chi_S o_2 f)(x) \vee \lambda$ for every (λ, μ)-fuzzy k-quasi-ideal f of S.

Proof. Let S be a k-regular ordered semiring and x be any element of S. Suppose f be any (λ, μ)-fuzzy k-bi-ideal of S. Since S is k-regular there exist $a, b \in S$ and $\alpha, \beta, \gamma, \delta \in \Gamma$ such that $x + x\alpha a\beta x \leq x\gamma b\delta x$. Now

$$
\begin{aligned}
&(fo_2\chi_S o_2 f)(x) \vee \lambda \\
&= \bigvee_{x+\sum_{i=1}^{n} a_i\alpha_i b_i \leq \sum_{i=1}^{n} c_i\beta_i d_i} (\wedge\{(fo_2\chi_S)(a_i), (fo_2\chi_S)(c_i), f(b_i), f(d_i)\}) \vee \lambda \\
&\geq \wedge\{(fo_2\chi_S)(x\alpha a), (fo_2\chi_S)(x\gamma b), f(x)\} \wedge \mu \\
&= \wedge\{ \bigvee_{x\alpha a+\sum_{i=1}^{n} a_i\alpha_{1i} b_i \leq \sum_{i=1}^{n} c_i\beta_{1i} d_i}(\wedge\{(f(a_i), f(c_i))\}), \bigvee_{x\gamma b+\sum_{i=1}^{n} a_i\alpha_{2i} b_i \leq \sum_{i=1}^{n} c_i\beta_{2i} d_i}(\wedge\{(f(a_i), f(c_i))\}), f(x)\} \wedge \mu \\
&\geq \wedge\{f(x), f(x), f(x)\} \wedge \mu \\
&\qquad (\text{since } x\alpha a + x\alpha a\beta x\alpha a \leq x\gamma b\delta x\alpha a \text{ and } x\gamma b + x\alpha a\beta x\gamma b \leq x\gamma b\delta x\gamma b) \\
&= f(x) \wedge \mu.
\end{aligned}
$$

This implies that $f(x) \wedge \mu \leq (fo_2\chi_S o_2 f)(x) \vee \lambda$.
$(i) \Rightarrow (ii)$ is straight forward from Lemma 4.5. □

Theorem 4.9. Let S be a k-regular ordered semiring and $x \in S$. Then

(i) $(f \cap g)(x) \wedge \mu \leq (fo_2 go_2 f)(x) \vee \lambda$ for every (λ, μ)-fuzzy k-bi-ideal f and every (λ, μ)-fuzzy k-ideal g of S.

(ii) $(f \cap g)(x) \wedge \mu \leq (fo_2 go_2 f)(x) \vee \lambda$ for every (λ, μ)-fuzzy k-quasi-ideal f and every (λ, μ)-fuzzy k-ideal g of S.

Proof. Assume that S is a k-regular ordered semiring. Let f and g be any (λ, μ)-fuzzy k-bi-ideal and (λ, μ)-fuzzy k-ideal of S, respectively and x be any element of S. Since S is k-regular, there exist $a, b \in S$ and $\alpha, \beta, \gamma, \delta \in \Gamma$ such that $x + x\alpha a\beta x \leq x\gamma b\delta x$.

Then

$$\begin{aligned}
&(fo_2go_2f)(x)\vee\lambda\\
&=\bigvee_{x+\sum_{i=1}^{n}a_i\alpha_ib_i\le\sum_{i=1}^{n}c_i\beta_id_i}(\wedge\{(fo_2g)(a_i),(fo_2g)(c_i),f(b_i),f(d_i)\})\vee\lambda\\
&\ge\wedge\{(fo_2g)(x\alpha a),(fo_2g)(x\gamma b),f(x)\}\wedge\mu\\
&=\wedge\{\bigvee_{x\alpha a+\sum_{i=1}^{n}a_i\alpha_ib_i\le\sum_{i=1}^{n}c_i\beta_id_i}(\wedge\{(f(a_i),f(c_i),g(b_i),g(d_i))\}),\bigvee_{x\gamma b+\sum_{i=1}^{n}a_i\gamma_ib_i\le\sum_{i=1}^{n}c_i\delta_id_i}(\wedge\{(f(a_i),f(c_i),g(b_i),g(d_i))\}),f(x)\}\wedge\mu\\
&\ge\wedge\{\wedge\{f(x),f(a\beta x\alpha a),g(b\delta x\alpha a)\},\wedge\{f(x),g(a\beta x\gamma b),g(b\delta x\gamma b)\},f(x)\}\wedge\mu\\
&\qquad(\text{since } x\alpha a+x\alpha a\beta x\alpha a\le x\gamma b\delta x\alpha a \text{ and } x\gamma b+x\alpha a\beta x\gamma b\le x\gamma b\delta x\gamma b)\\
&\ge f(x)\wedge g(x)\wedge\mu=(f\cap g)(x)\wedge\mu.
\end{aligned}$$

(i)⇒(ii) is straight forward from Lemma 4.5. □

Theorem 4.10. Let S is both k-regular and k-intra-regular an ordered semiring and $x\in S$. Then

(i) $f(x)\wedge\mu=(fo_2f)(x)\vee\lambda$ for every (λ,μ)-fuzzy k-bi-ideal f of S

(ii) $f(x)\wedge\mu=(fo_2f)(x)\vee\lambda$ for every (λ,μ)-fuzzy k-quasi-ideal f of S.

Proof. Suppose S is both k-regular and k-intra-regular ordered semiring. Let $x\in S$ and f be any fuzzy k-bi-ideal of S. Since S is both k-regular and k-intra-regular there exist $a_i,b_i,c_i,d_i\in S$, $\alpha_{1i},\alpha_{2i},\alpha_{3i},\alpha_{4i},\alpha_{5i},\beta_{1i},\beta_{2i},\beta_{3i},\beta_{4i},\beta_{5i}\in\Gamma$, $i\in\mathbf{N}$ such that $x+\sum_{i=1}^{n}x\alpha_{1i}a_i\alpha_{2i}x\alpha_{3i}x\alpha_{4i}b_i\alpha_{5i}x\le\sum_{i=1}^{n}x\beta_{1i}c_i\beta_{2i}x\beta_{3i}x\beta_{4i}d_i\beta_{5i}x$. Therefore

$$\begin{aligned}
(fo_2f)(x)\vee\lambda&=\bigvee_{x+\sum_{i=1}^{n}a_i\alpha_ib_i\le\sum_{i=1}^{n}c_i\beta_id_i}[\bigwedge_i\{\wedge\{f(a_i),f(c_i),f(b_i),f(d_i)\}\}]\vee\lambda\\
&\ge\bigwedge_i[\wedge\{f(x\alpha_{1i}a_i\alpha_{2i}x),f(x\alpha_{4i}b_i\alpha_{5i}x),f(x\beta_{1i}c_i\beta_{2i}x),f(x\beta_{4i}d_i\beta_{5i}x)\}]\wedge\mu\\
&\qquad x+\sum_{i=1}^{n}x\alpha_{1i}a_i\alpha_{2i}x\alpha_{3i}x\alpha_{4i}b_i\alpha_{5i}x\le\sum_{i=1}^{n}x\beta_{1i}c_i\beta_{2i}x\beta_{3i}x\beta_{4i}d_i\beta_{5i}x\\
&\ge f(x)\wedge\mu.
\end{aligned}$$

Now $(fo_2f)(x)\vee\lambda\le(fo_2\chi_S)(x)\wedge\mu\le f(x)\wedge\mu$. Hence $f(x)\wedge\mu=(fo_2f)(x)\vee\lambda$ for every (λ,μ)-fuzzy k-bi-ideal f of S.
$(i)\Rightarrow(ii)$ is straightforward from the Lemma 4.5. □

Theorem 4.11. Let S is both k-regular and k-intra-regular ordered semiring and $x\in S$. Then

(i) $(f\cap g)(x)\wedge\mu\le(fo_2g)(x)\vee\lambda$ for all (λ,μ)-fuzzy k-bi-ideals f and g of S.

(ii) $(f\cap g)(x)\wedge\mu\le(fo_2g)(x)\vee\lambda$ for every (λ,μ)-fuzzy k-bi-ideals f and every (λ,μ)-fuzzy k-quasi-ideal g of S.

(iii) $(f \cap g)(x) \wedge \mu \leq (f o_2 g)(x) \vee \lambda$ for every (λ, μ)-fuzzy k-quasi-ideals f and every (λ, μ)-fuzzy k-bi-ideal g of S.

(iv) $(f \cap g)(x) \wedge \mu \leq (f o_2 g)(x) \vee \lambda$ for all (λ, μ)-fuzzy k-quasi-ideals f and g of S.

Proof. Assume that S is both k-regular and k-intra-regular ordered semiring. Let $x \in S$ and f, g be any (λ, μ)-fuzzy k-bi-ideals of S. Since S is both k-regular and k-intra-regular there exist $a_i, b_i, c_i, d_i \in S$, $\alpha_{1i}, \alpha_{2i}, \alpha_{3i}, \alpha_{4i}, \alpha_{5i}$, $\beta_{1i}, \beta_{2i}, \beta_{3i}, \beta_{4i}, \beta_{5i} \in \Gamma$, $i \in \mathbf{N}$ such that $x + \sum_{i=1}^{n} x\alpha_{1i}a_i\alpha_{2i}x\alpha_{3i}x\alpha_{4i}b_i\alpha_{5i}x \leq \sum_{i=1}^{n} x\beta_{1i}c_i\beta_{2i}x\beta_{3i}x\beta_{4i}d_i\beta_{5i}x$. Therefore

$$
\begin{aligned}
&(f o_2 g)(x) \vee \lambda \\
&= \vee[\wedge_i \{\wedge\{f(a_i), f(c_i), g(b_i), g(d_i)\}\}] \vee \lambda \\
&\qquad {\scriptstyle x+\sum_{i=1}^{n} a_i\alpha_i b_i \leq \sum_{i=1}^{n} c_i\beta_i d_i} \\
&\geq \wedge_i [\wedge\{f(x\alpha_{1i}a_i\alpha_{2i}x), g(x\alpha_{4i}b_i\alpha_{5i}x), f(x\beta_{1i}c_i\beta_{2i}x), g(x\beta_{4i}d_i\beta_{5i}x\}] \wedge \mu \\
&\geq f(x) \wedge g(x) \wedge \mu = (f \cap g)(x) \wedge \mu.
\end{aligned}
$$

$(i) \Rightarrow (ii) \Rightarrow (iv)$ and $(i) \Rightarrow (iii) \Rightarrow (iv)$ are obvious from Lemma 4.5. □

5 Conclusion

In this paper, the concept of (λ, μ)-fuzzy ideals and k-ideals of an ordered Γ-semiring is introduced and studied along with some operation on them and some of their characterizations are obtained. Actually the main aim of studying the concept of (λ, μ)-fuzzy set is to restrict ourself to the interval (λ, μ) with $0 \leq \lambda < \mu \leq 1$. The case for which $\lambda = 0$ and $\mu = 1$, the results will coincide with the fuzzy ideals of ordered Γ-semiring. We can similarly obtain the parallel results for fuzzy h-ideal also. The future work may be focused on prime(semiprime) fuzzy ideal, prime(semiprime) fuzzy h-ideal, fuzzy h-bi(quasi, interior)-ideal, prime(semiprime)fuzzy h-bi(quasi, interior)-ideal etc..

Acknowledgement

The author is thankful to the Referees and the Editors for their valuable comments.

References

[1] A.P. Gan, Y.L. Jiang, *On ordered ideals in ordered semiring*, Journal of Mathematical Research and Exposition, 31(6) (2011), 989-996.

[2] Y. Feng, P. Corsini, (λ, μ)- *fuzzy ideals of ordered semigroups*,Annals of Fuzzy Mathematics and Informatics, 4(1) (2012) 123-129.

[3] J.S. Golan, *Semirings and their applications*, Kluwer Academic Publishers, Dodrecht, 1999.

[4] U. Hebisch, J. Weinert, *Semirings algebraic theory and application in computer science*, World Scientific, 1998.

[5] N. Kehayopulu, M. Tsingelis, *Fuzzy bi-ideals in ordered semigroups*, Information Sciences, 171(1-3) (2005) 13-28.

[6] N. Kehayopulu, M. Tsingelis, *Fuzzy ideal in ordered semigroups*, Quasigroups and Related Systems, 15(2007) 279-289.

[7] E.E. Kerre, *The impact of fuzzy set theory on contemporary mathematics*, Appl. Comput. Math., 10(1) (2011) 20-34.

[8] B. Li, Y. Feng, *Intuitionistic* (λ, μ)*- fuzzy sets in* Γ*-semigroups*, Journal of Inequalities and Applications, (2013) doi:10.1186/1029-242X-2013-107.

[9] D. Mandal, *Fuzzy Ideals and Fuzzy Interior Ideals in Ordered Semirings*, Fuzzy Inf. Eng. , 6(1) (2014), 101-114.

[10] M.M.K. Rao, Γ*-semirings 1*, Southeast Asian Bull. of Math., 19(1995) 49-54.

[11] A. Rosenfeld, *Fuzzy groups*, J. Math. Anal. Appl., 35(1971) 512-517.

[12] L.A. Zadeh, *Fuzzy Sets*, Information and Control, 8(1965) 338-353.

http //www.newtheory.org **ISSN 2149-1402**

Journal of New Theory

Received 30.07.2015 *Year*: **2016**, *Number*: **10**, *Pages*: **12-18**
Published 05.01.2016 *Original Article*[**]

COMMON FIXED POINT THEOREMS IN G-FUZZY METRIC SPACES

Madurai Veeran Jeyaraman[1,*] <jeya.math@gmail.com >
Rengasamy Muthuraj[2] <rmr1973@yahoo.co.in>
Mangaiyarkarasu Sornavalli [3] <sornavalliv7@gmail.com>
Muthukaruppan Jeyabharathi[1] <bharathi050814@gmail.com>

[1]*PG and Research Department of Mathematics, Raja Doraisingam Govt.Arts College, Sivagangai, India.*
[2]*PG and Research Department of Mathematics, H.H. The Rajah's College, Pudukkottai, India.*
[3]*Department of Mathematics, Velammal College of Engineering & Technology, Madurai, India.*

Abstract – In this paper, we obtain a unique common fixed point theorem for six weakly compatible mappings in G-fuzzy metric spaces.

Keywords – *G-metric Spaces, compatible mappings, G-fuzzy metric spaces*

1 Introduction

Mustafa and Sims [3] introduced a *G-metric* space and obtained some fixed point theorems in it. Some interesting references in *G-metric* spaces are [2-6,8]. We have generalized the result of Rao et al. [7]. Before giving our main results, we obtain a unique common fixed point theorem for six weakly compatible mappings in G-fuzzy metric spaces.

Definition 1.1 Let X be a nonempty set and let $G \; X \times X \times X \to [0,\infty)$ be a function satisfying the following properties

(G1) $G(x, y, z) = 0$ if $x = y = z$,
(G2) $0 < G(x, x, y)$ for all $x, y \in X$ with $x \neq y$,
(G3) $G(x, x, y) \leq G(x, y, z)$ for all $x, y, z \in X$ with $y \neq z$,
(G4) $G(x, y, z) = G(x, z, y) = G(y, z, x) = \cdots$, symmetry in all three variables,
(G5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ for all $x, y, z, a \in X$.

*******Edited by Oktay Muhtaroğlu (Area Editor) and* Naim Çağman *(Editor-in-Chief).*
[*]*Corresponding Author.*

Then, the function G is called a generalized metric or a *G-metric* on X and the pair (X, G) is called a *G-metric* space.

Definition 1.2 The G-metric space (X, G) is called symmetric if $G(x, x, y) = G(x, y, y)$ for all $x, y \in X$.

Definition 1.3 A 3-tuple (X, G, *) is called a G- fuzzy metric space if X is an arbitrary nonempty set, * is a continuous t-norm, and G is a fuzzy set on $X^3 \times (0, \infty)$ satisfying the following conditions for each $t, s > 0$

(i) $G(x, x, y, t) > 0$ for all $x, y \in X$ with $x \neq y$,
(ii) $G(x, x, y, t) \geq G(x, y, z, t)$ for all $x, y, z \in X$ with $y \neq z$,
(iii) $G(x, y, z, t) = 1$ if and only if $x = y = z$,
(iv) $G(x, y, z, t) = G(p(x, y, z), t)$, where p is a permutation function,
(v) $G(x, y, z, t + s) \geq G(a, y, z, t) * G(x, a, a, s)$ for all $x, y, z, a \in X$,
(vi) $G(x, y, z, \cdot)\ (0,\infty) \rightarrow [0, 1]$ is continuous.

Definition 1.4 A G- fuzzy metric space (X,G,*) is said to be symmetric if

$$G(x, x, y, t) = G(x, y, y, t)$$

for all $x, y \in X$ and for each $t > 0$.

Example 1.5 Let X be a nonempty set and let G be a G- fuzzy metric on X. Denote $a*b=ab$ for all $a, b \in [0, 1]$. For each $t > 0$,

$$G(x, y, z, t) = \frac{t}{t+G(x,y,z,t)}$$

is a G- fuzzy metric on X. Let (X, G,*) be a G - fuzzy metric space. For $t > 0$, $0 < r < 1$, and $x \in X$, the set

$$B_G(x, r, t) = \{ y \in X\ \ G(x, y, y, t) > 1 - r\}$$

is called an open ball with center x and radius r. A subset A of X is called an open set if for each $x \in X$, there exist $t > 0$ and $0 < r < 1$ such that $B_G(x, r, t) \subseteq A$. A sequence $\{x_n\}$ in G-fuzzy metric space X is said to be G- convergent to $x \in X$ if $G(x_n, x_n, x, t) \rightarrow 1$ as $n \rightarrow \infty$ or each $t > 0$. It is called a G- Cauchy sequence if $G(x_n, x_n, x_m, t) \rightarrow 1$ as $n, m \rightarrow \infty$ for each $t > 0$. X is called G- complete if every G- Cauchy sequence in X is G- convergent in X.

Lemma 1.6 Let (X, G,*) be a G- fuzzy metric space. Then, G(x, y, z, t) is nondecreasing with respect to t for all $x, y, z \in X$.

Lemma 1.7 Let (X, G, *) be a G- fuzzy metric space. If there exists $k \in (0, 1)$ such that

$$\min \{G(x, y, z, kt), G(u, v,w, kt)\} \geq \min \{G (x, y, z, t), G(u, v,w, t)\} \quad (1)$$

for all $x, y, z, u, v, w \in X$ and $t > 0$, then $x = y = z$ and $u = v = w$.

2 Main Result

Let Φ denote the set of all continuous non decreasing functions ϕ $[0,\infty) \to [0,\infty)$ such that $\phi^n(t) \to 0$ as $n \to \infty$ for all $t > 0$. It is clear that $\phi(t) < t$ for all $t > 0$ and $\phi(0) = 0$.

Theorem 2.1 Let $(X, G, *)$ be a G- fuzzy metric space and S, T, R, f, g, h $X \to X$ be satisfying

(i) $S(X) \subseteq g(X)$, $T(X) \subseteq h(X)$ and $R(X) \subseteq f(X)$,

(ii) One of $f(X)$, $g(X)$ and $h(X)$ is a complete subspace of X,

(iii) The pairs (S, f), (T, g) and (R, h) are weakly compatible, and

(iv)
$$G(Sx, Ty, Rz, t) \geq \phi\left(\min \begin{Bmatrix} G(fx, gy, hz, t) \\ \frac{1}{3}[G(fx, Sx, Ty, t) + G(gy, Ty, Rz, t) + G(hz, Rz, Sx, t)], \\ \frac{1}{4}[G(fx, Ty, hz, t) + G(Sx, gy, hz, t) + G(fx, gy, Rz, t)] \end{Bmatrix} \right)$$

for all $x, y, z \in X$, where $\phi \in \Phi$.

Then either one of the pairs (S, f), (T, g), and (R, h) has a coincidence point or the maps S,T, R, f, g and h have a unique common fixed point in X.

Proof: Choose $x_0 \in X$. By (i), there exist x_1, x_2, x_3, $\in$ X such that $Sx_0 = gx_1 = y_0$, $Tx_1 = hx_2 = y_1$ and $Rx_2 = fx_3 = y_2$. Inductively, there exist sequences $\{x_n\}$ and $\{y_n\}$ in X such that $y_{3n} = Sx_{3n} = gx_{3n+1}$, $y_{3n+1} = Tx_{3n+1} = hx_{3n+2}$ and $y_{3n+2} = Rx_{3n+2} = fx_{3n+3}$, where $n = 0,1,\ldots$

If $y_{3n} = y_{3n+1}$ then x_{3n+1} is a coincidence point of g and T.

If $y_{3n+1} = y_{3n+2}$ then x_{3n+2} is a coincidence point of h and R.

If $y_{3n+2} = y_{3n+3}$ then x_{3n+3} is a coincidence point of f and S.

Now assume that $y_n \neq y_{n+1}$ for all n. Denote $d_n = G(y_n, y_{n+1}, y_{n+2}, t)$. Putting $x = x_{3n}$, $y = x_{3n+1}$, $z = x_{3n+2}$ in (iv), we get

$$\begin{aligned} d_{3n} &= G(y_{3n}, y_{3n+1}, y_{3n+2}, t) \\ &= G(Sx_{3n}, Tx_{3n+1}, Rx_{3n+2}, t) \\ &\geq \phi\left(\min \begin{Bmatrix} G(fx_{3n}, gx_{3n+1}, hx_{3n+2}, t), \frac{1}{3}[G(fx_n, Sx_{3n}, Tx_{3n+1}, t) + \\ G(gx_{3n+1}, Tx_{3n+1}, Rx_{3n+2}, t) + G(hx_{3n+2}, Rx_{3n+2}, Sx_{3n,}, t)] \\ \frac{1}{4}[G(fx_{3n}, Tx_{3n+1}, hx_{3n+2}, t) + G(Sx_{3n}, gx_{3n+1}, hx_{3n+2}, t) \\ + G(fx_{3n}, gx_{3n+1}, Rx_{3n+2}, t)] \end{Bmatrix} \right) \end{aligned}$$

$$\geq \phi\left(\min\left\{\begin{matrix} G(y_{3n-1}, y_{3n}, y_{3n+1}, t), \frac{1}{3}[G(y_{3n-1}, y_{3n}, y_{3n+1}, t) + \\ G(y_{3n}, y_{3n+1}, y_{3n+2}, t) + G(fy_{3n+1}, y_{3n+2}, y_{3n}, t)], \\ \frac{1}{4}[G(fy_{3n-1}, y_{3n+1}, y_{3n+1}, t) + G(y_{3n}, y_{3n}, y_{3n+1}, t) \\ +G(y_{3n-1}, y_{3n}, y_{3n+2}, t)] \end{matrix}\right\}\right)$$

$$\geq \phi\left(\min\left\{\begin{matrix} d_{3n-1}, \frac{1}{3}[d_{3n-1} + d_{3n} + d_{3n}], \\ \frac{1}{4}[d_{3n-1} + d_{3n} + (d_{3n-1} + d_{3n})] \end{matrix}\right\}\right) \quad (2)$$

If $d_{3n} \leq d_{3n-1}$ then from (1), we have $d_{3n} \geq \phi(d_{3n}) > d_{3n}$. It is a contradiction. Hence $d_{3n} \geq d_{3n-1}$. Now from (1), $d_{3n} \geq \phi(d_{3n-1})$. Similarly, by putting $x = x_{3n+3}$, $y = x_{3n+1}$, $z = x_{3n+2}$ and $x = x_{3n+3}$, $y = x_{3n+4}$, $z = x_{3n+2}$ in (iv), we get

$d_{3n+1} \geq \phi(d_{3n})$ and (3)

$d_{3n+2} \geq \phi(d_{3n+1})$ (4)

Thus from (1), (2) and (3), we have

$$\begin{aligned} G(y_n, y_{n+1}, y_{n+2}, t) &\geq \phi(G(y_{n-1}, y_n, y_{n+1}, t)) \\ &\geq \phi^2(G(y_{n-2}, y_{n-1}, y_n, t)) \\ &\;\;\vdots \\ &\geq \phi^n(G(y_0, y_1, y_2, t)) \end{aligned} \quad (5)$$

we have $G(y_n, y_n, y_{n+1}, t) \geq G(y_n, y_{n+1}, y_{n+2}, t) \geq \phi^n(G(y_0, y_1, y_2, t))$. Now for $m > n$, we have

$$\begin{aligned} G(y_n, y_n, y_m, t) &\geq G(y_n, y_n, y_{n+1}, t) + G(y_{n+1}, y_{n+1}, y_{n+2}, t) + \ldots + G(y_{m-1}, y_{m-1}, y_m, t) \\ &\geq \phi^n(G(y_0, y_1, y_2, t)) + \phi^{n+1}(G(y_0, y_1, y_2, t)) + \ldots + \phi^{m-1}(G(y_0, y_1, y_2, t)) \\ &\to 1 \text{ as } n \to \infty, \end{aligned}$$

Since $\phi^n(t) \to 1$ as $n \to \infty$ for all $t > 0$. Hence $\{y_n\}$ is G- Cauchy. Suppose f(X) is G-complete. Then there exist p, $t \in X$ such that $y_{3n+2} \to p = f t$. Since $\{y_n\}$ is G- Cauchy, it follows that $y_{3n} \to p$ and $y_{3n+1} \to p$ as $n \to \infty$.

$G(St, Tx_{3n+1}, Rx_{3n+2}, t)$

$$\geq \phi\left(\min\left\{\begin{matrix} G(ft, gx_{3n}, hx_{3n+2}, t), \frac{1}{3}[G(ft, St, Tx_{3n+1}, t) + \\ G(gx_{3n+1}, Tx_{3n+1}, Rx_{3n+2}, t) + G(hx_{3n+2}, Rx_{3n+2}, St, t)], \\ \frac{1}{4}G(ft, Tx_{3n+1}, hx_{3n+2}, t) + G(St, gx_{3n+1}, hx_{3n+2}, t) \\ +G(ft, gx_{3n+1}, Rx_{3n+2}, t)] \end{matrix}\right\}\right)$$

Letting $n \to \infty$, we get

$$G(Sp, p, p, t) \geq \phi\left(\min\begin{Bmatrix} 1, \frac{1}{3}[G(p, St, p, t) + 1 + G(p, p, St, t)] \\ \frac{1}{4}[1 + G(St, p, p, t) + 1)] \end{Bmatrix}\right)$$

$G(St, p, p, t) \geq \phi$ ($G(St, p, p, t)$, since ϕ is non decreasing. Hence St = p. Thus p = f t = St. Since the pair (S, f) is weakly compatible, we have fp = Sp. Putting x = p, y = x_{3n+1}, z=x_{3n+2} in (iv), we get

$$G(Sp, Tx_{3n+1}, Rx_{3n+2}, t)$$
$$\geq \phi\left(\min\begin{Bmatrix} G(fp, gx_{3n+1}, hx_{3n+2}, t), \frac{1}{3}[G(fp, Sp, Tx_{3n+1}, t) + \\ G(gx_{3n+1}, Tx_{3n+1}, Rx_{3n+2}, t) + G(hx_{3n+2}, Rx_{3n+2}, Sp, t)], \\ \frac{1}{4}[G(fp, Tx_{3n+1}, hx_{3n+2}, t) + G(Sp, gx_{3n+1}, hx_{3n+2}, t) \\ +G(fp, gx_{3n+1}, Rx_{3n+2}, t)] \end{Bmatrix}\right)$$

Letting $n \to \infty$, we have

$$G(Sp, p, p, t) \geq \phi\left(\min\begin{Bmatrix} G(Sp, p, p, t), \frac{1}{3}[G(Sp, Sp, p, t) + 0 + G(p, p, Sp, t)], \\ \frac{1}{4}[G(Sp, p, p, t) + G(Sp, p, p, t) + G(Sp, p, p, t)] \end{Bmatrix}\right)$$

Since $G(Sp, Sp, p, t) \geq 2G(Sp, p, p, t)$, we have $G(Sp, p, p, t) \geq \phi(G(Sp, p, p, t))$. Thus Sp = p. Hence

$$f p = Sp = p. \tag{6}$$

Since p = Sp ϵ g(X), there exists v ϵ X such that p = gv. Putting x = p, y = v, z = x_{n+2} in (iv), we get

$$G(Sp, Tv, Rx_{n+2}, t) \geq \phi\left(\min\begin{Bmatrix} G(fp, gv, hx_{3n+2}, t), \frac{1}{3}[G(fp, Sp, Tv, t) + \\ G(gv, Tv, Rx_{3n+2}, t) + G(hx_{3n+2}, Rx_{3n+2}, Sp, t)], \\ \frac{1}{4}[G(fp, Tv, hx_{3n+2}, t) + G(Sp, gv, hx_{3n+2}, t) \\ +G(fp, gv, Rx_{3n+2}, t)] \end{Bmatrix}\right)$$

Letting $n \to \infty$, we deduce that

$$G(p, Tv, p, t) \geq \phi\left(\min\begin{Bmatrix} 1, \frac{1}{3}[G(p, p, Tv, t) + G(p, Tv, p, t) + 1], \\ \frac{1}{4}[G(p, Tv, p, t) + 1 + 1] \end{Bmatrix}\right)$$
$$\geq \phi\,(G(p, Tv, p, t)),$$

since ϕ is non decreasing. Thus Tv = p, so that p = Tv = gv. Since the pair (T, g) is weakly compatible, we have Tp = gp.

$$G(Sp, Tp, Rx_{3n+2}, t) \geq \phi\left(\min\left\{\begin{array}{c} G(fp, gp, hx_{3n+2}, t), \frac{1}{3}[G(fp, Sp, Tp, t) + \\ G(gp, Tp, Rx_{3n+2}, t) + G(hx_{3n+2}, Rx_{3n+2}, Sp, t)], \\ \frac{1}{4}[G(fp, Tp, hx_{3n+2}, t) + G(Sp, gp, hx_{3n+2}, t) \\ +G(fp, gp, Rx_{3n+2}, t)] \end{array}\right\}\right)$$

Letting $n \to \infty$, we have

$$G(p, Tp, p, t) \geq \phi\left(\min\left\{\begin{array}{c} G(p, Tp, p, t), \frac{1}{3}[G(p, p, Tp, t) + G(Tp, Tp, p) + 1], \\ \frac{1}{4}[G(p, Tp, p, t) + G(p, Tp, p, t) + G(p, Tp, p, t)] \end{array}\right\}\right)$$

Since $G(Tp, Tp, p, t) \geq 2G(Tp, p, p, t)$, we have ,$G(p, Tp, p, t) \geq \phi$ $(G(p, Tp, p, t))$. Thus $Tp = p$. Hence

$$gp = Tp = p. \qquad (7)$$

Since $p = Tp \in h(X)$, there exists $w \in X$ such that $p = hw$. Putting $x = p$, $y = p$, $z = w$ in (iv), we get

$$G(Sp, Tp, Rw, t) \geq \phi\left(\min\left\{\begin{array}{c} G(fp, gp, hw, t), \frac{1}{3}[G(fp, Sp, Tp, t) + \\ G(gp, Tp, Rw, t) + G(hw, Rw, Sp, t)], \\ \frac{1}{4}[G(fp, Tp, hw, t) + G(Sp, gp, hw, t) \\ +G(fp, gp, Rw, t)] \end{array}\right\}\right)$$

$$G(p, p, Rw, t) \geq \phi\left(\min\left\{\begin{array}{c} 1, \frac{1}{3}[1 + G(p, p, Rw, t) + G(p, Rw, p, t)], \\ \frac{1}{4}[1 + 1 + G(p, p, Rw, t)] \end{array}\right\}\right) \geq \phi\,(G(p, p, Rw, t)),$$

since ϕ is non decreasing. Thus $Rw = p$, so that $p = hw = Rw$. Since the pair (R, h) is weakly compatible, we have $Rp = hp$. Putting $x = p$, $y = p$, $z = p$ in (iv), we get,

$$G(p, p, Rp, t) = G(Sp, Tp, Rp, t) \geq \phi\left(\min\left\{\begin{array}{c} G(fp, gp, Rp, t), \frac{1}{3}[1 + \\ G(p, p. Rp, t) + G(Rp, Rp, p, t)], \\ \frac{1}{4}[G(p, p, Rp, t) + G(p, p, Rp, t) \\ +G(p, p, Rp, t)] \end{array}\right\}\right)$$

Since $G(Rp, Rp, p, t) \geq 2G(p, p, Rp, t)$, we have

$$G(p, p, Rp, t) \geq \phi(G(p, p, Rp, t)). \qquad (8)$$

Thus $Rp = p$, so that $Rp = hp = p$. From (6), (7) and (8), it follows that p is a common fixed point of S, T, R, f, g and h. Uniqueness of common fixed point follows easily from (iv). Similarly, we can prove the theorem when $g(X)$ or $h(X)$ is a complete subspace of X.

Corollary 2.2 Let (X, G, *) be a G -fuzzy metric space and S, T, R, f, g, h, $X \to X$ be satisfying

(i) $S(X) \subseteq g(X)$, T(X) and $R(X) \subseteq f(X)$,
(ii) One of f(X) , g(X) and h(X) is a complete subspace of X,
(iii) The pairs (S, f) ,(T, g) and (R, h) ate weakly compatible and
(iv) $G(Sx, Ty, Rz, t) \geq \phi\ (G(fx, gy, hz, t))$ for all x, y, z ϵ X, where $\phi\ \epsilon\ \Phi$.

Then the maps S, T, R, f, g and h have a unique fixed point in X.

Corollary 2.3 Let(X, G, *) be a complete G - fuzzy metrics space and S, T, R $X \to X$ be satisfying $G(Sx, Ty, Rz, t) \geq \phi\ (G(x, y, z, t))$ for all x, y, z ϵ X, where $\phi \in \Phi$. Then the maps S, T and R have a unique common fixed point, p ϵ X and S,T and R are G-continuous at p.

Proof: There exists p ϵ X such that p is the unique common fixed point of S, T and R as in Theorem 2.1. Let $\{y_n\}$ be any sequence in X which G –converges to p. Then

$G(Sy_n, Sp, Sp, t) = G(Sy_n, Tp, Rp, t) \leq \phi\ (G(y_n, p, p, t)) \to 1$ as $n \to \infty$.

Hence S is G - continuous at p. Similarly, we can show that T and R are also G-continuous at p.

References

[1] M. Abbas, M. Ali Khan, and S. Radenovic, "Common Coupled fixed point theorems in cone metric spaces for W- compatible mappings", *Applied Mathematics and Computation*, vol. 217, no.1,pp. 195-202,2010.

[2] R. Chugh, T. Kadian, A. Rani, and B. E. Rhoades, "Property P in G-metric spaces," *Fixed Point Theory and applications*, vol.2010, Article ID 401684, 12 pages, 2010.

[3] Z. Mustafa and B. Sims, " Fixed point theorems for contractive mappings in complete G-metric spaces," *Fixed point Theory and Applications*, vol. 2009, Article ID 917175, 10 pages, 2009.

[4] Z. Mustafa and H. Obiedat, " A fixed point theorem of Reich in G-metric spaces, " *Cubo A Mathematical Journal*, vol. 12, no.1, pp. 83-93, 2010.

[5] Z. Mustafa and H. Obiedat, and F. Awawdeh, " Some fixed point theorem for mapping on complete G-metric spaces," *Fixed point Theory and Applications*, vol. 2008, Article IF 189870, 12 pages, 2008.

[6] Z. Mustafa , W. Shatanawi, and M. Bataineh, " Existence of fixed point results in G-metric spaces," *International Journal of Mathematics and Mathematical Sciences*, vol. 2009, Article ID 283028, 10 Pages, 2009.

[7] K.P.R. Rao, I. Altun, and S. Hima Bindu, " Common Coupled Fixed-Point Theorems in Generalized Fuzzy Metric Spaces" *Advances in Fuzzy Systems,* volume 2011, Article ID 986748, 6 pages.

[8] W. Shatanawi, Fixed point theory and contractive mappings satisfying Φ-maps in G-metric spaces," *Fixed point Theory and Applications,* Vol. 2010, Article ID 181650, 9 pages, 2010.

[9] G. Sun and K. Yang, " Generalized fuzzy metric spaces with properties," *Research journal of Applied Sciences, Engineering and Technology*, vol.2, no.7, pp.673-678.

http://www.newtheory.org ISSN: 2149-1402

Journal of New Theory

Received: 29.10.2015 *Year*: **2016**, *Number*: **10**, *Pages*: **19-29**
Published: 05.01.2016 *Original Article*[**]

ON (k, h)-CONVEX STOCHASTIC PROCESSES

Lysis González[1,2,*] <lysis.gonzalez@gmail.com>
Nelson Merentes[1,2] <nmerucv@gmail.com>
Maira Valera-López[1,2] <maira.valera@ciens.ucv.ve>

[1]Escuela de Matemática, Universidad Central de Venezuela, 1010 Caracas, Venezuela.
[2]Banco Central de Venezuela, 1010 Caracas, Venezuela.

Abstaract − We introduce the class of (k, h)-convex stochastic processes and we generalize results given for (k, h)-convex functions in [10] and h-convex stochastic process in [1], among them, Hermite-Hadamard and Fejér-type inequalities.

Keywords − *(k, h)-convex stochastic processes, $h-$convex stochastic processes, converse Jensen-type inequality, Fejér-type inequality, Hermite-Hadamard-type inequality.*

1 Introduction

In 1980, Nikodem [11] stated the line of investigation on stochastic convexity and later, several types of convex stochastic processes have been studied [1, 2, 4, 5, 6, 7, 8, 11, 12, 14] based in the classical convex notions for functions.

Micherda and Rajba, introduced in [10] the family of $(k, h)-$convex functions as the solutions of the functional inequality

$$f(k(t)x + k(1-t)y) \leq h(t)f(x) + h(1-t)f(y),$$

where $k, h : (0, 1) \to \mathbb{R}$ are given. The notion of (k, h)-convexity generalizes s-Orlicz convexity [3], subaditivity [9] and h-convexity [13].

In this paper, we introduce the notion of (k, h)-convex stochastic processes as a counterpart of the (k, h)-convex functions and a generalization of h-convex stochastic processes defined in [1]. Also, we prove properties of (k, h)-convex stochastic processes, among them, Hermite-Hadamard and Fejér-type inequalities.

Now, we would like to recall the context where the stochastic convexity is studied.

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space. A function $X : \Omega \to \mathbb{R}$ is a *random variable* if it is $\mathcal{A}$-measurable. A function $X : I \times \Omega \to \mathbb{R}$, where $I \subseteq \mathbb{R}$ is an interval, is a *stochastic process* if for every $t \in I$ the function $X(t, \cdot)$ is a random variable.

[**] *Edited by* Kazimierz Nikodem *and* Naim Çağman *(Editor-in-Chief).*
[*] *Corresponding Author.*

If $h : (0, 1) \to \mathbb{R}$ is a non-negative function, $h \not\equiv 0$, a stochastic process $X : I \times \Omega \to \mathbb{R}$ is *h-convex*, if for every $t_1, t_2 \in I$ and $\lambda \in (0, 1)$, the following inequality holds

$$X(\lambda t_1 + (1 - \lambda)t_2, \cdot) \leq h(\lambda)X(t_1, \cdot) + h(1 - \lambda)X(t_2, \cdot), \quad (a.e.).$$

When h is equal to the identity function, X is said to be *convex*, and additionally, if $\lambda = \frac{1}{2}$ then X is *Jensen-convex.*

Some examples and properties related with convex, Jensen-convex and h-convex stochastic processes can be readed in [1, 2, 8, 11, 14].

Now, for calculation, we need to introduce additional definitions:

Let $X : I \times \Omega \to \mathbb{R}$ be a stochastic process such that $\mathbb{E}[X(t)]^2 < \infty$ for all $t \in I$, where $\mathbb{E}[X(t)]^2 < \infty$ denotes the expectation value of $X(t, \cdot)$. The stochastic process X is

1. *continuous in probability* in the interval I, if for all $t_0 \in I$, we have

$$P - \lim_{t \to t_0} X(t, \cdot) = X(t_0, \cdot),$$

where $P - \lim$ denotes the limit in probability.

2. *mean-square continuous* in the interval I, if for all $t_0 \in I$

$$\lim_{t \to t_0} \mathbb{E}[(X(t) - X(t_0))^2] = 0.$$

Is important to note that mean-square continuity implies continuity in probability, but the converse implication is not true.

We say that the stochastic process X is *mean-square integrable* in $[a, b] \subseteq I$, if there exists a random variable Y such that for all normal sequence of partions of the interval $[a, b]$, $a = t_0 < t_1 < ... < t_n = b$, holds

$$\lim_{n \to \infty} \mathbb{E} \left[\sum_{k=1}^{n} X(\theta_k) \cdot (t_k - t_{k-1}) - Y \right]^2 = 0.$$

The random variable $Y : \Omega \to \mathbb{R}$ is the mean-square integral of the process X on $[a, b]$ and we can also write

$$Y(\cdot) = \int_a^b X(s, \cdot)ds, \quad (a.e).$$

Definition and properties of mean-square integral can be readed in [15].

2 (k, h)-convex Stochastic Processes

In order to extend the definition of h-convexity for stochastic processes, we introduce the notion of (k, h) stochastic convexity.

Given a function $k : (0, 1) \to \mathbb{R}$, a set $D \subseteq \mathbb{R}$ is *k-convex* if $k(\lambda)t_1 + k(1-\lambda)t_2 \in D$ for all t_1, $t_2 \in D$ and $t \in (0, 1)$.

In [10], k-convex sets were defined in real linear spaces and some examples for chosen functions k are given.

Definition 2.1. Let k, $h : (0,1) \to \mathbb{R}$ be two given functions and $D \subset \mathbb{R}$ a k-convex set. A stochastic process $X : D \times \Omega \to \mathbb{R}$ is (k,h)-convex if, for all t_1, $t_2 \in D$ and $\lambda \in (0,1)$,

$$X(k(\lambda)t_1 + k(1-\lambda)t_2, \cdot) \leq h(\lambda)X(t_1,\cdot) + h(1-\lambda)X(t_2,\cdot) \qquad (a.e.). \tag{1}$$

If in (1) the equality holds, the stochastic process X is called (k,h)-affine.

This definition coincides in many important cases with other ones previously introduced, some of which are listed bellow.

Example 2.2. 1. For $k(\lambda) = \lambda$, the notion of (k,h)-convexity matches with the h-convexity one given in [1] (without the additional assumption of non negativity).

2. For $k(\lambda) = h(\lambda) = 1$, the class of (k,h)-convex stochastic processes consists in all stochastic process which are subadditive.

3. If $k(\lambda) = h(\lambda) = 1/2$ for all λ, then (1) gives the family of Jensen-convex stochastic processes.

4. Let k be defined by the formula

$$k(\lambda) = \begin{cases} 2\lambda, & \lambda \leq 1/2, \\ 0, & \lambda > 1/2. \end{cases}$$

Then X is a (k,k)-convex stochastic process if and only if it is starshaped, i.e., $X(\lambda t,\cdot) \leq \lambda X(t,\cdot)$ almost everywhere, for all $\lambda \in (0,1)$ and $t \in D$. In fact, fix t_1, $t_2 \in D$ and choose $\lambda \in (0,1)$. Then, assuming that X is a (k,k)-convex stochastic process, we get

$$X(\lambda t,\cdot) = X\left(k\left(\frac{\lambda}{2}\right)t + k\left(1-\frac{\lambda}{2}\right)t,\cdot\right) \leq \lambda X(t,\cdot),$$

and

$$X(0,\cdot) = X\left(k\left(\frac{\lambda}{2}\right)t + k\left(\frac{\lambda}{2}\right)t,\cdot\right) = 0,$$

almost everywhere.

On the other hand, if X is starshaped, for anyone $t_1, t_2 \in D, \lambda \in (0,1)$ we obtain

$$X(k(\lambda)t_1 + k(1-\lambda)t_2,\cdot) = \begin{cases} X(2\lambda t_1,\cdot) \leq 2\lambda X(t_1,\cdot), & \lambda \in (0,1/2), \\ X(0,\cdot) \leq 0, & \lambda = 1/2, \\ X((2-2\lambda)t_2,\cdot) \leq (2-2\lambda)X(t_2,\cdot), & \lambda \in (1/2,1). \end{cases}$$

Hence, (1) is satisfied for all $t \in D$ and $\lambda \in (0,1)$.

Hereinafter, we keep the notation used in the definition (2.1) for D, k and h.

3 Properties of (k,h)-convex Stochastic Processes

Many of the well-known properties of convex stochastic processes are satisfied by (k,h)-convex stochastic processes too. In the following propositions we present some basic properties for (k,h)-convex stochastic processes.

Proposition 3.1. If X, $Y : D \times \Omega \to \mathbb{R}$ be a (k,h)-convex stochastic processes and $c \geq 0$, then $X+Y$ and cX are also (k,h)-convex stochastic processes.

Proof. Let be t_1, $t_2 \in D$, $\lambda \in (0,1)$ and $c \geq 0$. Then,

$$\begin{aligned} (X+Y)(k(\lambda)t_1 &+ k(1-\lambda)t_2,\cdot) \\ &= X(k(\lambda)t_1 + k(1-\lambda)t_2,\cdot) + Y(k(\lambda)t_1 + k(1-\lambda)t_2,\cdot) \\ &\leq h(\lambda)(X+Y)(t_1,\cdot) + h(1-\lambda)(X+Y)(t_2,\cdot), \quad (a.e). \end{aligned}$$

Also,

$$\begin{aligned} c(X(k(\lambda)t_1 + k(1-\lambda)t_2,\cdot)) &\leq c[h(\lambda)X(t_1,\cdot) + h(1-\lambda)X(t_2,\cdot)] \\ &\leq h(\lambda)(cX)(t_1,\cdot) + h(1-\lambda)(cX)(t_2,\cdot), \quad (a.e). \end{aligned}$$

Proposition 3.2. Let k, h_1, $h_2 : (0,1) \to \mathbb{R}$ be non negative functions and X, $Y : D \times \Omega \to \mathbb{R}$ non-negative stochastic processes such that:

$$(X(t_1,\cdot) - X(t_2,\cdot))(Y(t_1,\cdot) - Y(t_2,\cdot)) \geq 0, \tag{2}$$

for all t_1, $t_2 \in D$. If X is (k,h_1)-convex, Y is (k,h_2)-convex and $h(\lambda) + h(1-\lambda) \leq c$ for all $\lambda \in (0,1)$, where $h(\lambda) = \max\{h_1(\lambda), h_2(\lambda)\}$ and c is a fixed positive number, then the product XY is a (k,ch)-convex stochastic process.

Proof. Fix t_1, $t_2 \in D$ and λ, $\beta \in (0,1)$ such that $\lambda + \beta = 1$. First, note that if $(X(t_1,\cdot) - X(t_2,\cdot))(Y(t_1,\cdot) - Y(t_2,\cdot)) \geq 0$ holds almost everywhere, then:

$$X(t_1,\cdot)Y(t_2,\cdot) + Y(t_1,\cdot)X(t_2,\cdot) \leq X(t_1,\cdot)Y(t_1,\cdot) + Y(t_2,\cdot)X(t_2,\cdot), \quad (a.e).$$

Hence,

$$\begin{aligned} (XY)(k(\lambda)t_1 + k(1-\lambda)t_2,\cdot) &\leq (h(\lambda)X(t_1,\cdot) + h(1-\lambda)X(t_2,\cdot)) \\ &\qquad \cdot(h(\lambda)Y(t_1,\cdot) + h(1-\lambda)Y(t_2,\cdot)) \\ &\leq (h(\lambda))^2(XY)(t_1,\cdot) \\ &\quad + h(\lambda)h(1-\lambda)[(XY)(t_1,\cdot) + (XY)(t_2,\cdot)] \\ &\quad + (h(1-\lambda))^2(XY)(t_2,\cdot) \\ &= (h(\lambda) + h(1-\lambda)) \\ &\quad \cdot[h(\lambda)(XY)(t_1,\cdot) + h(1-\lambda)XY(t_2,\cdot)] \\ &\leq ch(\lambda)(XY)(t_1,\cdot) + ch(1-\lambda)X(t_2,\cdot)], \quad (a.e). \end{aligned}$$

Proposition 3.3. Let $X : I \times \Omega \to \mathbb{R}$ be a (k,h)-convex stochastic process and $f : \mathbb{R} \to \mathbb{R}$ an increasing (h,h)-convex function. Then, $f \circ X : I \times \Omega \to \mathbb{R}$ is a (k,h)-convex stochastic process.

Proof. For arbitrary $t_1, t_2 \in I$ and $\lambda \in (0,1)$, we have

$$\begin{aligned} f(X(k(\lambda)t_1 + k(1-\lambda)t_2,\cdot)) &\leq f(h(\lambda)X(t_1,\cdot) + h(1-\lambda)X(t_2,\cdot)) \\ &\leq h(\lambda)f(X(t_1,\cdot)) + h(1-\lambda)f(X(t_2,\cdot)) \quad (a.e) \end{aligned}$$

In [8], Kotrys and Nikodem defined for every stochastic process X and random variable A, the sublevel set as follows

$$L_A = \{t \in D : X(t, \cdot) \leq A(\cdot), \quad (a.e.)\}.$$

In the following proposition we present a condition for h in way to the sublevel set L_A be k-convex for given (k, h)-convex stochastic process X and random variable A.

Proposition 3.4. Let $X : D \times \Omega \to \mathbb{R}$ be a (k, h)-convex stochastic process, with h a positive function. For every random variable $A : \Omega \to \mathbb{R}$, the sublevel set L_A is k-convex if the inequality $h(\lambda) + h(1 - \lambda) \leq 1$ holds for every $\lambda \in (0, 1)$.

Proof. Since X is (k, h)-convex, for t_1, $t_2 \in L_A$ and $\lambda \in (0, 1)$, we have:

$$\begin{aligned} X(k(\lambda)t_1 + k(1-\lambda)t_2, \cdot) &\leq h(\lambda)X(t_1, \cdot) + h(1-\lambda)X(t_2, \cdot) \\ &\leq h(\lambda)A(\cdot) + h(1-\lambda)A(\cdot) \\ &= (h(\lambda) + h(1-\lambda))A(\cdot) \leq A(\cdot), \quad (a.e.). \end{aligned}$$

Therefore, L_A is k-convex set.

Example 3.5. Considering $h(\lambda) = \lambda$ in the previous proposition, the result holds.

The proof of the following proposition follows immediately from the definitions.

Proposition 3.6. If h_1, h_2 are functions such that $h_2 \geq h_1$, then every non-negative (k, h_1)-convex stochastic process is also (k, h_2)-convex stochastic process.

Remark 3.7. Note that if D is a k-convex subset of X and $X : D \times \Omega \to \mathbb{R}$ is a (k, h)-affine stochastic process, then the image of X not necessarily is an h-convex set in $\mathbb{R}$. For instance, if $D = \Omega = [0, 1]$, k, h are the identity function and X is defined by

$$X(t, \omega) = \begin{cases} 0, & \text{if} \quad t \neq \omega, \\ \\ 1, & \text{if} \quad t = \omega. \end{cases}$$

then $X(D \times \Omega) = \{0, 1\}$ is not an h-convex subset of $\mathbb{R}$.

In the following theorem we present conditions under the inequality

$$X(k(\lambda)t_1 + k(\beta)t_2, \cdot) \leq h(\lambda)X(t_1, \cdot) + h(\beta)X(t_2, \cdot),$$

holds almost everywhere, for all λ, $\beta > 0$ such that $\lambda + \beta \leq 1$.

In the following theorem definitions of supermultiplicative and submultiplicative functions are needed. We recall these notions:

Definition 3.8. A function $f : (0, 1) \to \mathbb{R}$ is said to be supermultiplicative if for all $x, y \in (0, 1)$,

$$f(x)f(y) \leq f(xy), \tag{3}$$

If inequality (3) is reversed, then f is a submultiplicative function. Moreover, if the equality holds in (3), f is multiplicative.

Theorem 3.9. Let be $k, h : (0,1) \to \mathbb{R}$ non-negative functions and $D \subseteq \mathbb{R}$ a $k-$convex set such that $0 \in D$. If k is submultiplicative, h is supermultiplicative and $X : D \times \Omega \to \mathbb{R}$ is a (k, h)-convex and non-decreasing stochastic process such that $X(0, \cdot) = 0$, then the inequality

$$X(k(\lambda)t_2 + k(\beta)t_2, \cdot) \leq h(\lambda)X(t_1, \cdot) + h(\beta)X(t_2, \cdot),$$

hold almost everywhere, for all λ, $\beta > 0$ such that $\lambda + \beta \leq 1$.

Proof. If $\lambda + \beta = 1$, the inequality holds from (k, h)-convex stochastic process definition. Let λ, $\beta > 0$ be numbers such that $\lambda + \beta = \gamma$ with $\gamma < 1$. Let us define numbers $a := \frac{\lambda}{\gamma}$ and $b := \frac{\beta}{\gamma}$. Then, $a + b = 1$ and fixed $t_1, t_2 \in D$, we have the following inequality:

$$\begin{aligned}
X(k(a\gamma)t_1 + k(b\gamma)t_2, \cdot) &\leq X(k(a)k(\gamma)t_1 + k(b)k(\gamma)t_2, \cdot) \\
&\leq h(a)X(k(\gamma)t_1, \cdot) + h(b)X(k(\gamma)t_2, \cdot) \\
&= h(a)X(k(\gamma)t_1 + k(1-\gamma)0, \cdot) \\
&\quad +h(b)X(k(\gamma)t_1 + k(1-\gamma)0, \cdot) \\
&\leq h(a)[h(\gamma)X(t_1, \cdot) + h(1-\gamma)X(0, \cdot)] \\
&\quad +h(b)[h(\gamma)X(t_1, \cdot) + h(1-\gamma)X(0, \cdot)] \\
&= h(a)h(\gamma)X(t_1, \cdot) + h(b)h(\gamma)X(t_2, \cdot) \\
&\leq h(a\gamma)X(t_1, \cdot) + h(b\gamma)X(t_2, \cdot) \\
&= h(\lambda)X(t_1, \cdot) + h(\beta)X(t_2, \cdot), \quad (a.e).
\end{aligned}$$

Theorem 3.10. Let k, h be non-negative functions and $D \subseteq \mathbb{R}$ a k-convex set such that $0 \in D$. If $X : D \times \Omega \to \mathbb{R}$ is a non-negative stochastic process such that

$$X(k(\lambda)t_1 + k(\beta)t_2, \cdot) \leq h(\lambda)X(t_1, \cdot) + h(\beta)X(t_2, \cdot) \;\; (a.e), \tag{4}$$

holds for any t_1, $t_2 \in D$ and $\lambda, \beta > 0$ with $\lambda + \beta \leq 1$ and $h(\lambda) < \frac{1}{2}$ for some $\lambda \in (0, \frac{1}{2})$, then $X(0, \cdot) = 0$.

Proof. Let us suppose that exists $w \in \Omega$ with $X(0, \omega) \neq 0$, then $X(0, \omega) > 0$ and putting $t_1 = t_2 = 0$ in the inequality (4), we get

$$X(0, \omega) \leq h(\lambda)X(0, \omega) + h(\beta)X(0, \omega),$$

for λ, $\beta > 0$ such that $\lambda + \beta \leq 1$. Putting $\lambda = \beta$, $\lambda \in (0, \frac{1}{2})$ and dividing by $X(0, \omega)$, we obtain $1 \leq h(\lambda) + h(\lambda) = 2h(\lambda)$ for all $\lambda \in (0, \frac{1}{2})$. That is, $\frac{1}{2} \leq h(\lambda)$ for all $\lambda \in (0, \frac{1}{2})$, what is a contradiction with the assumption of theorem.

In the following proposition we present a Schur-type inequality.

Proposition 3.11. If k, $h : (0,1) \to \mathbb{R}$ are non-negative functions, with $k(\lambda) \geq \lambda$, h submultiplicative and $X : D \times \Omega \to \mathbb{R}$ is a non-decreasing (k, h)-convex stochastic process, then the following inequality holds:

$$h(t_3 - t_2)X(t_1, \cdot) - h(t_3 - t_1)X(t_2, \cdot) + h(t_2 - t_1)X(t_3, \cdot) \geq 0, \;\; (a.e.), \tag{5}$$

for $t_1, t_2, t_3 \in D$, such that $t_1 < t_2 < t_3$ and $t_3 - t_1, t_3 - t_2, t_2 - t_1 \in D$.

Proof. Consider $t_1, t_2, t_3 \in D$ be numbers wich satisfy assumptions of the proposition. Then,

$$\frac{t_3 - t_2}{t_3 - t_1}, \frac{t_2 - t_1}{t_3 - t_1} \in (0, 1),$$

and

$$\frac{t_3 - t_2}{t_3 - t_1} + \frac{t_2 - t_1}{t_3 - t_1} = 1.$$

Also, since h is supermultiplicative and non-negative, we have

$$h(t_3 - t_2) = h\left(\frac{t_3 - t_2}{t_3 - t_1}.(t_3 - t_1)\right) \geq h\left(\frac{t_3 - t_2}{t_3 - t_1}\right) h(t_3 - t_1),$$

$$h(t_2 - t_1) = h\left(\frac{t_2 - t_1}{t_3 - t_1}.(t_3 - t_1)\right) \geq h\left(\frac{t_2 - t_1}{t_3 - t_1}\right) h(t_3 - t_1),$$

Let $h(t_3 - t_1) > 0$. Because $k(\lambda) \geq \lambda$, X is non-decreasing and (k, h)-convex, X satisfies:

$$X(\lambda z_1 + (1-\lambda) z_2, \cdot) \leq X(k(\lambda) z_1 + k(1-\lambda) z_2, \cdot) \leq h(\lambda) X(z_1, \cdot) + h(1-\lambda) X(z_2, \cdot), \quad (a.e),$$

for all $z_1, z_2 \in D, \lambda \in (0, 1)$. In particular, for $\lambda = \frac{t_3 - t_2}{t_3 - t_1}$, $z_1 = t_1, z_2 = t_3$, we have $t_2 = \lambda z_1 + (1 - \lambda) z_2$ and

$$\begin{aligned} X(t_2, \cdot) &\leq h\left(\frac{t_3 - t_2}{t_3 - t_1}\right) X(t_1, \cdot) + h\left(\frac{t_2 - t_1}{t_3 - t_1}\right) X(t_3, \cdot) \qquad (6) \\ &\leq \frac{h(t_3 - t_2)}{h(t_3 - t_1)} X(t_1, \cdot) + \frac{h(t_2 - t_1)}{h(t_3 - t_1)} X(t_3, \cdot), \quad (a.e). \end{aligned}$$

Finally, multiplying by $h(t_3 - t_1)$, we obtain the following

$$h(t_3 - t_1) X(t_2, \cdot) \leq h(t_3 - t_2) X(t_1, \cdot) + h(t_2 - t_1) X(t_3, \cdot), \quad (a.e).$$

That is,

$$0 \leq h(t_3 - t_2) X(t_1, \cdot) - h(t_3 - t_1) X(t_2, \cdot) + h(t_2 - t_1) X(t_3, \cdot), \quad (a.e).$$

The following theorem is an converse Jensen-type inequality.

Theorem 3.12. Let $\lambda_1, \lambda_2, ..., \lambda_n$ be positive real numbers such that $\sum_{i=1}^{n} \lambda_i = 1$ and $(m, M) \subseteq I$. If $k, h : (0, 1) \to \mathbb{R}$ is a non negative with $k(\lambda) \geq \lambda$ and h supermultiplicative function, and $X : I \times \Omega \to \mathbb{R}$ is an (k, h)-convex stochastic process, then for any $t_1, t_2, ..., t_n \in [m, M]$, the following inequality holds almost everywhere

$$\begin{aligned} \sum_{i=1}^{n} h(\lambda_i) X(t_i, \cdot) \leq \; & X(m, \cdot) \sum_{i=1}^{n} h(\lambda_i) h\left(\frac{M - t_i}{M - m}\right) \\ & + X(M, \cdot) \sum_{i=1}^{n} h(\lambda_i) h\left(\frac{t_i - m}{M - m}\right). \end{aligned}$$

Proof. Fix $i \in \{1, ..., n\}$. Putting $t_1 = m, t_2 = t_i, t_3 = M$ and $\lambda = \left(\frac{M-t_i}{M-m}\right) \in [0,1]$ in the inequality (6), we get

$$X(t_i,\cdot) \leq h\left(\frac{M-t_i}{M-m}\right) X(m,\cdot) + h\left(\frac{t_i-m}{M-m}\right) X(M,\cdot), \quad (a.e).$$

Since h is non negative, we have that multiplying by $h(\lambda_i)$:

$$\begin{aligned} h(\lambda_i)X(t_i,\cdot) &\leq h(\lambda_i)h\left(\frac{M-t_i}{M-m}\right) X(m,\cdot) \\ &\quad +h(\lambda_i)h\left(\frac{t_i-m}{M-m}\right) X(M,\cdot). \end{aligned}$$

Adding all inequalities for $i = 1, ..., n$, we complete the proof.

4 Main Results

We will prove the main results of this paper which consists in some new Fejér and Hermite-Hadamard-type inequalities for (k, h)-convex stochastic processes. From now, we suppose that all mean-square integrals considered bellow exist.

Theorem 4.1. (First Fejér-type inequality) If there are $X : D \times \Omega \to \mathbb{R}$ a (k, h)-convex stochastic process with $h(1/2) > 0$, $a < b$ such that $[a, b] \subset D$ and $G : [a, b] \times \Omega \to \mathbb{R}$ a non-negative and symmetric respect $\frac{a+b}{2}$ mean-square integrable stochastic process, then the following inequality holds almost everywhere:

$$\frac{X(k(1/2)(a+b),\cdot)}{2h(1/2)} \int_a^b G(t,\cdot)dt \leq \int_a^b X(t,\cdot)G(t,\cdot)dt, \quad (a.e). \tag{7}$$

Proof. From the definition with $\lambda = 1/2$, $t_1 = wa + (1-w)b$ and $t_2 = (1-w)a + wb$ with $w \in [0,1]$, then

$$\begin{aligned} X\left(k\left(\frac{1}{2}\right)(a+b),\cdot\right) &= X\left(k\left(\frac{1}{2}\right)t_1 + k\left(\frac{1}{2}\right)t_2,\cdot\right) \\ &= X\left(k\left(\frac{1}{2}\right)(wa+(1-w)b) + k\left(\frac{1}{2}\right)((1-w)a+wb),\cdot\right) \\ &\leq h\left(\frac{1}{2}\right) X(wa+(1-w)b,\cdot) \\ &\quad +h\left(\frac{1}{2}\right) X((1-w)a+wb,\cdot), \quad (a.e). \end{aligned} \tag{8}$$

Multiplying both sides of the inequality (8) for $G(t_1,\cdot) = G(t_2,\cdot)$, almost everywhere and integrate it with respect to w, getting:

$$\begin{aligned} &X\left(k\left(\frac{1}{2}\right)(a+b),\cdot\right) \cdot \int_0^1 G(wa+(1-w)b,\cdot)dw \\ &\quad \leq h\left(\frac{1}{2}\right)\left[\int_0^1 X(wa+(1-w)b,\cdot)G(wa+(1-w)b,\cdot)dw \right. \\ &\qquad \left. + \int_0^1 X((1-w)a+wb,\cdot)G((1-w)a+wb,\cdot)dw\right], \end{aligned}$$

almost everywhere. This implies

$$X\left(k\left(\frac{1}{2}\right)(a+b),\cdot\right)\cdot\frac{1}{b-a}\int_a^b G(t,\cdot)dt \;\le\; h\left(\frac{1}{2}\right)\cdot 2\cdot\frac{1}{b-a}\int_a^b X(t,\cdot)G(t,\cdot)dt,$$

which completes the proof.

Some important results are obtained as consequence of the previous result, among them, a Hermite-Hadamard-type inequality for (k,h)-convex stochastic processes, as the following corollary shows.

Corollary 4.2. Let $X : D\times\Omega\to\mathbb{R}$ be a (k,h)- convex stochastic process with $h(1/2)>0$ and fixed $a<b$ such that $[a,b]\subset D$. Then

$$\frac{X(k(1/2)(a+b),\cdot)}{2h(1/2)}\le\frac{1}{b-a}\int_a^b X(t,\cdot)dt,\quad (a.e). \tag{9}$$

Remark 4.3. 1. If X is an h-convex stochastic process, then (7) gives the following inequality

$$\frac{1}{2h(1/2)}X\left(\frac{a+b}{2},\cdot\right)\int_a^b G(t,\cdot)dt\le\int_a^b X(t,\cdot)G(t,\cdot)dt.$$

2. For every convex stochastic process X the following Fejér-type inequality is valid by Theorem 4.1,

$$X\left(\frac{a+b}{2},\cdot\right)\int_a^b G(t,\cdot)dt\le\int_a^b X(t,\cdot)G(t,\cdot)dt.$$

In particular, for $G(t,\cdot)=1$ we get the Hermite-Hadamard inequality

$$X\left(\frac{a+b}{2},\cdot\right)\le\frac{1}{b-a}\int_a^b X(t,\cdot)dt.$$

3. From (7) and (9) we recover the left-hand sides of the classical Fejér and Hermite-Hadamard-type inequalities for Jensen-convex stochastic processes.

Theorem 4.4. (Second Fejér-type inequality) Let be $k,h:(0,1)\to\mathbb{R}$ given functions such that $h(1/2)>0$ and $k(w)+k(1-w)=0$ for all $w\in[0,1]$. If $X : D\times\Omega\to\mathbb{R}$ is a (k,h)-convex stochastic, a, $b\in D$, $a<b$ and $G:[a,b]\times\Omega\to\mathbb{R}$ is a non-negative and symmetric respect to $\frac{a+b}{2}$ mean-square integrable stochastic process, then the following inequality holds almost everywhere:

$$\begin{aligned}\frac{1}{h\left(\frac{1}{2}\right)}\int_0^1 X\left(k\left(\frac{1}{2}\right)[k(t)+k(1-t)](a+b),\cdot\right)G(ta+(1-t)b,\cdot)dt\\ \le\int_0^1 X(k(t)a+k(1-t)b,\cdot)G(at+(1-t)b,\cdot)dt\\ \le[X(a,\cdot)+X(b,\cdot)]\int_0^1 h(t)G(at+(1-t)b,\cdot)dt.\end{aligned} \tag{10}$$

Proof. By definition (1) with $t_1 = k(w)a + k(1-w)b$, $t_2 = k(1-w)a + k(w)b$ and $t = 1/2$, we have the following inequality almost everywhere:

$$X\left(k\left(\frac{1}{2}\right)[k(w)+k(1-w)]\cdot(a+b),\cdot\right) = X\left(k\left(\frac{1}{2}\right)t_1 + k\left(\frac{1}{2}\right)t_2,\cdot\right)$$
$$\leq h\left(\frac{1}{2}\right)[X(k(w)a+k(1-w)b,\cdot)+X(k(1-w)a+k(w)b,\cdot)]. \quad (11)$$

As in the proof of the previous theorem, we multiply both sides of the inequality (11) by $G(wa+(1-w)b,\cdot) = G((1-w)a+wb,\cdot)$, and we integrate the new inequality over $(0,1)$, getting

$$\int_0^1 X\left(k\left(\frac{1}{2}\right)[k(w)+k(1-w)]\cdot(a+b),\cdot\right)G(wa+(1-w)b,\cdot)dt$$
$$\leq h\left(\frac{1}{2}\right)\left[\int_0^1 X(k(w)a+k(1-w)b,\cdot)G(wa+(1-w)b,\cdot)dw\right.$$
$$\left.+\int_0^1 X(k(1-w)a+k(w)b,\cdot)G(wa+(1-w)b,\cdot)dw\right]$$
$$\leq 2h\left(\frac{1}{2}\right)\cdot\int_0^1 X(k(1-w)a+k(w)b,\cdot)G(wa+(1-w)b,\cdot)dw, \quad (a.e).$$

From this we obtain the first desired inequality.

To prove the second one, we need to use the definition of (k,h)-convexity with $x = a$ and $y = b$. Namely, we have:

$$X(k(t)a+k(1-t)b,\cdot) \leq h(t)X(a,\cdot)+h(1-t)X(b,\cdot), \quad (a.e),$$

witch, by symmetry of $G(t,\cdot)$, implies

$$\int_0^1 X(k(t)a+k(1-t)b,\cdot)G(ta+(1-t)b,\cdot)dt$$
$$\leq X(a,\cdot)\int_0^1 h(t)G(wa+(1-w)b,\cdot)dw$$
$$+X(b,\cdot)\int_0^1 h(1-t)G((1-w)a+wb,\cdot)dw$$
$$=[X(a,\cdot)+X(b,\cdot)]\int_0^1 h(t)G(wa+(1-w)b,\cdot)dw, \quad (a.e),$$

and the proof is complete.

As a corollary, we obtain the second Hermite-Hadamard inequality for (k,h)-convex stochastic processes.

Corollary 4.5. Let $X : D\times\Omega \to \mathbb{R}$ be a (k,h)-convex stochastic process where $h(1/2) > 0$ and choose $a, b \in D$ such that $a < b$. Then

$$\frac{1}{h(1/2)}\int_0^1 X\left(k\left(\frac{1}{2}\right)[k(t)+k(1-t)](a+b),\cdot\right)dt$$
$$\leq \int_0^1 X(k(t)a+k(1-t)b,\cdot)\,dt \leq [X(a,\cdot)+X(b,\cdot)]\int_0^1 h(t)dt.$$

Acknowledgement

This research has been partially supported by Central Bank of Venezuela (B.C.V). We want to give thanks to the library staff of B.C.V. for compiling the references. Also, we are grateful with the Referees and Editors for their comments and suggestions on this work, specially on the Remark 3.7.

References

[1] D. Barráez, L. González, N. Merentes, A. Moros, *On h-convex stochastic processes*, Mathematica Aeterna, 5 (4) (2015) 571-581.

[2] L. González, D. Kotrys, K. Nikodem, *Separation by convex and strongly convex stochastic processes*, (2015) (subbmited for publication).

[3] H. Hudzik, L. Maligranda, *Some remarks on $s-$convex functions*, Aequationes Math. 48 (1) (1994) 100-111.

[4] D. Kotrys, *Remarks on strongly convex stochastic processes*, Aequat. Math. 86 (2012) 91-98.

[5] D. Kotrys, *Hermite-Hadamard inequality for convex stochastic processes*, Aequat. Math. 83 (2012) 143-151.

[6] D. Kotrys, *Some characterizations of strongly convex stochastic processes*, Mathematica Aeterna 4 (8) (2014), 855-861.

[7] D. Kotrys, *Remarks on Jensen, Hermite-Hadamard and Fejér inequalities for strongly convex stochastic processes*, Mathematica Aeterna 5 (1) (2015), 95-104.

[8] D. Kotrys, K. Nikodem, *Quasiconvex stochastic processes and a separation theorem*, Aequat. Math. 89 (2015), 41-48.

[9] J. Matkowski, T. Światkowski, *On subadditive functions*, Proc. Amer. Math. Soc, 119 (1993), 187-197.

[10] B. Micherda, T. Rajba, *On some Hermite-Hadamard inequalities for (k,h)-convex functions*, Mathematical Inequalities & Applications, 12 (4) (2012) 931-940.

[11] K. Nikodem, *On convex stochastic processes*, Aequat. Math. 20 (1980) 184-197.

[12] S. Maden, E. Set, M. Tomar, *s-convex stochastic processes in the second sense*, Turkish Journal of Analysis and Number Theory 2 (6) (2014) 202-207.

[13] S. Varošanec, *On $h-$convexity*, J. Math. Anal. Appl. 326 (2007) 303-311.

[14] M. Shaked, J. G. Shanthikumar, *Stochastic convexity and its applications*, Adv. in Appl.Prob. 20, (1988) 427-446.

[15] K. Sobczyk, *Stochastic differential equations with applications to physics and engineering*, Kluwer Academic Publishers B. V. (1991).

http://www.newtheory.org ISSN: 2149-1402

Journal of New Theory

Received: 11.06.2015 *Year*: **2016**, *Number*: **10**, *Pages*:**30-38**
Published: 06.01.2016 *Original Article*[**]

STABILITY OF THE FUNCTIONAL EQUATION IN FUZZY BANACH SPACES

Pratap Mondal[1,*] <pratapmondal111@gmail.com>
Nabin Chandra Kayal[2] <kayalnabin82@gmail.com>
Tapas Kumar Samanta[3] <mumpu_tapas5@yahoo.co.in>

[1]Department of Mathematics, Bijoy Krishna Girls' College, Howrah, West Bengal, India−711101
[2]Department of Mathematics, Moula Netaji Vidyalaya, Howrah, West Bengal, India−711312
[3]Department of Mathematics, Uluberia College, Uluberia, Howrah, West Bengal, India−711315

Abstaract − Different kind of stability have been studied concerning several areas of mathematics and fuzziness of such concepts, which is an extension of the former, are being introduced in recent times. The object of the present paper is to appraise generalization of the Hyers-Ulam-Rassias stability theorem for the functional equation

$$f(2x+y)+f(x+2y)=4f(x+y)+f(x)+f(y)$$

in fuzzy Banach spaces.

Keywords − *Fuzzy norm, functional equation, Hyers-Ulam stability, fuzzy Banach spaces.*

1 Introduction

In 1940, Ulam [18] first formulated stability for functional equation concerning group homomorphism and that was partially solved by Hyers [8] in the next year for Cauchy functional equations in Banach spaces and thereafter it was further generalized by Aoki [1]. The stability came in this way was known to be Hyers-Ulam stability. Later on the Hyers-Ulam stability was further generalized by Rassias [15]. The idea of such stability (which subsequently came to be known as Hyers-Ulam-Rassias stability) was generalized and extended to several areas of mathematics over the years. For instances, such stabilities were considered for differential equations [9], functional equations [7], isometries [5] etc.

After the introduction fuzzy set theory, it has been brought quick inroads to deal with uncertainty and vagueness for various problems in many branches of mathematics including functional analysis. In fact, when fuzzy norm on a linear space was first

[**]*Edited by* Oktay Muhtaroğlu *(Area Editor) and* Naim Çağman *(Editor-in-Chief).*
[*]*Corresponding Author.*

introduced by Katsaras [10], a great amendment has come forward in mathematical analysis and specially in functional analysis. Thereafter a few mathematicians have introduced and analyzed several notions of fuzzy norm from different points of views [3, 6, 13, 14, 16]. In particular, in 2003, Bag and Samanta [2] gave an idea of fuzzy norm in such a manner that the corresponding fuzzy metric is of Kramosil and Michalek [12] type. Thus the notion of fuzzy Banach space came in this way and since then it is being used extensively to study the stability of functional equations, differential equation etc.

The functional equation

$$f(x+y)+f(x-y)=2f(x)+2f(y) \tag{1}$$

is known as quadratic functional equation, since it is satisfied by the quadratic function $f(x)=cx^2$. The stability problem for the quadratic functional equation has been extensively investigated by a number of mathematicians [4, 11, 15, 17]. In this paper we now consider the functional equation

$$f(2x+y)+f(x+2y)=4f(x+y)+f(x)+f(y) \tag{2}$$

which is also satisfied by the quadratic function $f(x)=cx^2$ but different from the functional equation (1). Here we like to deal with Hyers-Ulam-Rassias stability for the functional equation (2) in fuzzy Banach spaces.

2 Preliminary

We adopted some definitions and notations of fuzzy norm which will be needed in the sequel.

Definition 2.1. Let X be a real linear space. A function $N: X\times\mathbb{R}\to[0,1]$ is said to be a fuzzy norm on X if for all $x, y\in X$ and all $s, t\in\mathbb{R}$
(N1) $N(x,c)=0$ for $c\leq 0$;
(N2) $x=0$ if only if $N(x,c)=1$ for all $c>0$;
(N3) $N(cx,t)=N(x,\frac{t}{|c|})$ if $c\neq 0$;
(N4) $N(x+y,s+t)\geq\min\{N(x,s),N(y,t)\}$;
(N5) $\lim_{t\to\infty}N(x,t)=1$.
Then the pair (X,N) is called a **fuzzy normed linear space**.

Example 2.2. Let $(X,\|.\|)$ be a normed linear space. then

$$\begin{aligned} N(x,t) &= \frac{t}{t+k\|x\|}\ , t>0 \\ &= 0\ , t\leq 0 \end{aligned}$$

is a fuzzy norm on X.

Definition 2.3. Let (X,N) be a fuzzy normed linear space. A sequence $\{x_n\}$ in X is said to be **convergent** if there exists $x\in X$ such that
$\lim_{n\to\infty}N(x_n-x,t)=1$ for all $t>0$ and we denote it by $N-\lim_{n\to\infty}x_n=x$.

Definition 2.4. A sequence $\{x_n\}$ in a fuzzy normed space (X, N) is said to be **Cauchy** if for each $\varepsilon > 0$ and each $t > 0$, we can find some n_0 such that for all $n \geq n_0$ and all $p > 0$ we have $N(x_{n+p} - x_n, t) > 1 - \varepsilon$.

Now we know that every convergent sequence in a fuzzy normed space is Cauchy. If each Cauchy sequence is convergent, then the fuzzy norm is said to be complete and a complete fuzzy normed space is called a **fuzzy Banach space**.

3 Hyers-Ulam-Rassias Stability for the Functional Equation

Theorem 3.1. Let X be a linear space and f be a mapping from X to a fuzzy Banach space (Y, N) such that $f(0) = 0$. Suppose that ϕ is a function from X to a fuzzy normed space (Z, N') such that

$$N(f(2x + y) + f(x + 2y) - 4f(x + y) - f(x) - f(y), t + s) \geq \min\{N'(\phi(x), t), N'(\phi(y), s)\} \tag{3}$$

for all $x, y \in X$ and positive real numbers t, s. If $\phi(3x) = \alpha\phi(x)$ for some real number α with $0 < \alpha < 9$ then there exits a unique quadratic mapping $Q : X \to Y$ define by $Q(x) = \lim_{n\to\infty} \left(\frac{f(3^n x)}{9^n}\right)$ and satisfying

$$N(f(x) - Q(x), t) \geq M\left(x, t\frac{9 - \alpha}{18}\right) \tag{4}$$

Where

$$M(x, t) = \min\left\{N'\left(\phi(x), \frac{9t}{5}\right), N'\left(\phi(x), \frac{9t}{5}\right), N'\left(\phi(x), \frac{9t}{5}\right), N'\left(\phi(0), \frac{9t}{5}\right)\right\}.$$

Proof. Putting $y = x$ and $s = t$ in (3), we get

$$N(2f(3x) - 4f(2x) - 2f(x), 2t) \geq \min\{N'(\phi(x), t), N'(\phi(x), t)\}$$

$i.e.,\ N(f(3x) - 2f(2x) - f(x), t) \geq \min\{N'(\phi(x), t), N'(\phi(x), t)\}.$

Again putting $y = 0$ in (3), we get

$$N(f(2x) - 4f(x), 2t) \geq \min\{N'(\phi(x), t), N'(\phi(0), t)\}.$$

Now

$$\begin{aligned}
&N(f(3x) - 9f(x), 5t)\\
&= N(f(3x) - 2f(2x) - f(x) + 2f(2x) - 8f(x), t + 4t)\\
&\geq \min\{N(f(3x) - 2f(2x) - f(x), t), N(f(2x) - 4f(x), 2t)\}
\end{aligned}$$

$\geq \min\{N'(\phi(x), t), N'(\phi(x), t), N'(\phi(x), t), N'(\phi(0), t)\}$

or, $N\left(f(x) - \frac{f(3x)}{9}, \frac{5t}{9}\right)$

$\geq \min\{N'(\phi(x), t), N'(\phi(x), t), N'(\phi(x), t), N'(\phi(0), t)\}$,

$i.e.,\ N\left(f(x) - \frac{f(3x)}{9}, t\right)$

$\geq \min\left\{N'(\phi(x), \frac{9t}{5}), N'(\phi(x), \frac{9t}{5}), N'(\phi(x), \frac{9t}{5}), N'(\phi(0), \frac{9t}{5})\right\}$

$= M(x, t)$ (5)

Where

$$M(x, t) = \min\left\{N'\left(\phi(x), \frac{9t}{5}\right), N'\left(\phi(x), \frac{9t}{5}\right), N'\left(\phi(x), \frac{9t}{5}\right), N'\left(\phi(0), \frac{9t}{5}\right)\right\}$$

Now from our assumption,

$$M(3x, t) = M\left(x, \frac{t}{\alpha}\right) \tag{6}$$

Replacing x by 3^x in (5) and using (6) we have

$N\left(\frac{f(3^n x)}{9^n} - \frac{f(3^{n+1} x)}{9^{n+1}}, \frac{\alpha^n t}{9^n}\right)$

$= N\left(f(3^n x) - \frac{f(3^{n+1} x)}{9}, \alpha^n t\right)$

$\geq M(3^n x, \alpha^n t) = M(x, t)$

Since

$$\frac{f(3^n x)}{9^n} - f(x) = \sum_{k=0}^{n-1}\left(\frac{f(3^{k+1} x)}{9^{k+1}} - \frac{f(3^k x)}{9^k}\right)$$

then we have

$N\left(\frac{f(3^n x)}{9^n} - f(x), t\sum_{k=0}^{n-1}\frac{\alpha^k}{9^k}\right)$

$= N\left(\sum_{k=0}^{n-1}\left(\frac{f(3^{k+1} x)}{9^{k+1}} - \frac{f(3^k x)}{9^k}\right), t\sum_{k=0}^{n-1}\frac{\alpha^k}{9^k}\right)$

$= N\left(\frac{f(3x)}{9} - f(x) + \sum_{k=1}^{n-1}\left(\frac{f(3^{k+1} x)}{9^{k+1}} - \frac{f(3^k x)}{9^k}\right), t + t\sum_{k=1}^{n-1}\frac{\alpha^k}{9^k}\right)$

$\geq \min\left\{M(x, t), N\left(\sum_{k=1}^{n-1}\left(\frac{f(3^{k+1} x)}{9^{k+1}} - \frac{f(3^k x)}{9^k}\right), t\sum_{k=1}^{n-1}\frac{\alpha^k}{9^k}\right)\right\}$

$$\geq \min \left\{ M(x,t), N\left(\frac{f(3^2x)}{9^2} - \frac{f(3x)}{9}, t\frac{\alpha}{9}\right), N\left(\frac{f(3^3x)}{9^3} - \frac{f(3^2x)}{9^2}, \frac{t\alpha^2}{9^2}\right), \right.$$

$$\left. N\left(\frac{f(3^4x)}{9^4} - \frac{f(3^3x)}{9^3}, \frac{t\alpha^3}{9^3}\right), ..., N\left(\frac{f(3^kx)}{9^k} - \frac{f(3^{k-1}x)}{9^{k-1}}, \frac{t\alpha^{k-1}}{9^{k-1}}\right)\right\}$$

$$\geq \min \{M(x,t), M(x,t), M(x,t), M(x,t), ..., M(x,t)\}$$

$$= M(x,t)$$

Therefore

$$N\left(\frac{f(3^nx)}{9^n} - f(x), t\right) \geq M\left(x, t\sum_{k=0}^{n-1}\frac{9^k}{\alpha^k}\right) \tag{7}$$

Replacing x by $3^m x$ in (7) we get

$$N\left(\frac{f(3^{n+m}x)}{9^{m+n}} - \frac{f(3^mx)}{9^m}, t\right) \geq M\left(x, t\sum_{k=m}^{m+n-1}\frac{9^k}{\alpha^k}\right) \tag{8}$$

Since $\lim_{t\to\infty} M(x,t) = 1$, taking limit $m \to \infty$,
the R. H. S. of (8) tends to 1 as $m \to \infty$.

Therefore $\left\{\frac{f(3^nx)}{9^n}\right\}$ is a Cauchy sequence in (Y, N). Since (Y, N) is a complete fuzzy normed space, the sequence converges to some point $Q(x) \in Y$. So we can define a mapping $Q : X \to Y$ by $Q(x) := N \lim_{n\to\infty} \frac{f(3^nx)}{9^n}$ for all $n \in N$.
Also

$$N(Q(x) - f(x), t) = N\left(Q(x) - \frac{f(3^nx)}{9^n} + \frac{f(3^nx)}{9^n} - f(x), \frac{t}{2} + \frac{t}{2}\right)$$

$$\geq \min\left\{N\left(Q(x) - \frac{f(3^nx)}{9^n}, \frac{t}{2}\right), N\left(\frac{f(3^nx)}{9^n} - f(x), \frac{t}{2}\right)\right\}$$

$$\geq M\left(x, \frac{t}{2\sum\limits_{k=0}^{\infty}\left(\frac{\alpha}{9}\right)^k}\right) = M\left(x, \frac{t}{2\left(\frac{1}{1-\frac{\alpha}{9}}\right)}\right) = M\left(x, \frac{t(9-\alpha)}{18}\right)$$

To show that Q satisfies the functional equation (2), we replacing x by $3^n x$ and y by $3^n y$ in (3)

$$N\,(f(3^n(2x+y)) + f(3^n(x+2y)) - 4f(3^n(x+y)) - f(3^n(x)) - f(3^n(y)), t)$$

$$\geq \min\left\{N'(\phi(3^nx), \frac{t}{2}), N'(\phi(3^ny), \frac{t}{2})\right\}$$

$$\text{or, } N\left(\frac{f(3^n(2x+y))}{9^n} + \frac{f(3^n(x+2y))}{9^n} - \frac{4f(3^n(x+y))}{9^n} - \frac{f(3^n(x))}{9^n} - \frac{f(3^n(y))}{9^n}, \frac{t}{9^n}\right)$$

$$\geq \min\left\{ N'(\phi(3^n x), \frac{t}{2}), N'(\phi(3^n y), \frac{t}{2}) \right\}$$

$$\text{or}, \; N\Bigg(\frac{f(3^n(2x+y))}{9^n} + \frac{f(3^n(x+2y))}{9^n} - \frac{4f(3^n(x+y))}{9^n} - \frac{f(3^n(x))}{9^n} - \frac{f(3^n(y))}{9^n}, t \Bigg)$$

$$\geq \min\left\{ N'(\phi(3^n x), \frac{9^n t}{2}), N'(\phi(3^n y), \frac{9^n t}{2}) \right\}$$

$$= \min\left\{ N'\left(\phi(x), \left(\frac{9}{\alpha}\right)^n \frac{t}{2}\right), N'\left(\phi(y), \left(\frac{9}{\alpha}\right)^n \frac{t}{2}\right) \right\}$$

for all $x, y \in X, t > 0$.
As $0 < \alpha < 9$, taking limit $n \to \infty$ we get

$$N(Q(2x+y)+Q(x+2y)-4Q(x+y)-Q(x)-Q(y), t) = 1$$

Therefore

$$Q(2x+y)+Q(x+2y) = 4Q(x+y)+Q(x)+Q(y)$$

Hence Q satisfies (2).
Uniqueness : Let $T : X \to Y$ be an another quadratic mapping which satisfies (3). Since $Q(2x) = 4Q(x)$ and $Q(3x) = 2Q(2x) + Q(x) = 9Q(x)$. Therefore it can be proved by induction that $Q(3^n x) = 9^n Q(x)$. Now fix $x \in X$ and using $Q(3^n x) = 9^n Q(x)$ and $T(3^n x) = 9^n T(x)$ for all $x \in X$. Now

$$N(Q(x)-T(x), t) = N\left(\frac{Q(3^n x)}{9^n} - \frac{T(3^n x)}{9^n}, t \right)$$

$$= N(Q(3^n x) - T(3^n x), 9^n t)$$

$$\geq \min\left\{ N\left(Q(3^n x) - f(3^n x), \frac{9^n t}{2} \right), N\left(T(3^n x) - f(3^n x), \frac{9^n t}{2} \right) \right\}$$

$$\geq \min\left\{ M\left(3^n x, \frac{9^n t(9-\alpha)}{2 \times 18} \right), M\left(3^n x, \frac{9^n t(9-\alpha)}{2 \times 18} \right) \right\}$$

$$= \min\left\{ M\left(x, \frac{t(9-\alpha)}{2 \times 18} \left(\frac{9}{\alpha}\right)^n \right), M\left(x, \frac{t(9-\alpha)}{2 \times 18} \left(\frac{9}{\alpha}\right)^n \right) \right\}$$

$$= M\left(x, \frac{t(9-\alpha)}{2 \times 18} \left(\frac{9}{\alpha}\right)^n \right)$$

for all $x \in X$ and $t > 0$. Since $0 < \alpha < 9$, and $\lim_{n\to\infty} \left(\frac{9}{\alpha}\right)^n = \infty$ therefore right hand side of the inequality tend to 1 as $n \to \infty$. Hence $Q(x) = T(x)$ for all $x \in X$. This completes the proof of the theorem. □

Corollary 3.2. Let $\delta > 0$ and X be a linear space, (Y, N') be a fuzzy Banach space. If let $f : X \to Y$ be a mapping and z_0 is a fixed vector of a fuzzy normed space (Z, N'') such that

$$N'(f(2x+y) + f(x+2y) - 4f(x+y) - f(x) - f(y), t+s)$$
$$\geq \min\{N''(\delta z_0, t), N''(\delta z_0, s)\} \tag{9}$$

for all $x, y \in X$ and positive real numbers t, s. Then there exits a unique quadratic mapping $Q : X \to Y$ define by $Q(x) = \lim_{n\to\infty} \left(\frac{f(3^n x)}{9^n}\right)$ and satisfying

$$N'(f(x) - Q(x), t) \geq N''\left(z_0, \frac{9t}{5\delta}\right) \tag{10}$$

Proof. Define $\phi(x) = \delta z_0$, then the proof is followed by the previous Theorem. □

Corollary 3.3. Let $\varepsilon \geq 0$ and X be a linear space, (Y, N'') be a fuzzy Banach space. If let $f : X \to Y$ be a mapping such that

$$N''(f(2x+y) + f(x+2y) - 4f(x+y) - f(x) - f(y), t) \geq \epsilon$$

for all $x, y \in X$ and positive real numbers t. Then there exits a unique quadratic mapping $Q : X \to Y$ define by $Q(x) = \lim_{n\to\infty} \left(\frac{f(3^n x)}{9^n}\right)$ and satisfying

$$N''(f(x) - Q(x), \frac{5}{8}t) \geq \epsilon$$

Proof. The proof is same as that of the previous theorem. □

Example 3.4. Let X be a normed algebra. Define $f : (X, N) \to (X, N')$ by $f(x) = x^2 + \|x\|x_0$, and
$\phi(x, y) = (\|2x+y\| + \|x+2y\| - 4\|x+y\| - \|x\| - \|y\|)x_0$ where x_0 is a unit vector in X. Then

$$N(f(2x+y) + f(x+2y) - 4f(x+y) - f(x) - f(y), t+s)$$
$$\geq \min\{N'(\phi(x), t), N'(\phi(y), s)\}$$

Also $\phi(3x, 3y) = 3\phi(x, y)$ for each $x, y \in X$. Hence all the conditions of Theorem (3.1) holds for $\alpha = 1$. Therefore fuzzy difference between $Q(x) = \lim_{n\to\infty} \left(\frac{f(3^n x)}{9^n}\right) = x^2$ and $f(x)$ is equal to [using Example 2.2]

$$N(f(x) - Q(x), t) = N(\|x\|x_0, t)$$
$$= \frac{t}{t + \|x\|} = N(x, t) \geq N'(x, t) \geq M\left(x, \frac{4}{9}t\right).$$

4 Conclusion

In this article, to establish Hyers-Ulam-Rassias stability we have used the functional equation (2) having quadratic function as its one particular solution. But this equation is not known as quadratic functional equation. So, natural question arises, whether the functional equation (2) can be derived from the quadratic functional equation (1) or the equation (1) can be derived from the equation (2). In fact, what could be the general solution of the equation (2). Can we establish the Theorem (3.1) for complex valued function f of complex variable satisfying the equation (2)? So, in our view, this article has good prospect for future work.

Acknowledgement

The authors are grateful to the chief editor and reviewers for giving us the opportunity to publish this article.

References

[1] T. Aoki, *On the stability of the linear transformation in Banach spaces*, J. Math. Soc. Japan 2 (1950) 64-66.

[2] T. Bag and S. K. Samanta, *Finite Dimensinal Fuzzy Normed Linear Space*, J. Fuzzy Math., 11 (2003), 687-705.

[3] S. C. Cheng and J. N. Moderson, *Fuzzy Linear Operator and Fuzzy Normed Linear Space*, Bull. Cal.Math. Soc., 86 (1994), 429-438.

[4] S. Czerwik, *On the stability of the quadratic mappings in normed spaces*, Abh. Math. Sem. Univ. Hamburg, 62 (1992), 59-64.

[5] Yunbai Dong, On approximate isometries and application to stability of a function, J. Math. Anal. Appl., 426 (2) (2015), 125-137.

[6] C. Felbin, *Finite dimensional fuzzy normed linear space*, Fuzzy Sets Systems 48 (1992), 239-248.

[7] A. Grabiec, The generalized Hyers-Ulam stability of a class of functional equations, Publ. Math. Debrecen, 48 (1996), 217-235.

[8] D. H. Hyers, *On the stability of the linear functional equation*, Proc. Nat.Acad.Sci. U.S.A. 27 (1941), 222-224.

[9] S. M. Jung, Hyers-Ulam stability of linear differential equations of first order, II, App. Math. Lett. 19 (2006), 854..858.

[10] A. K. Katsaras, *Fuzzy topological vector spaces.II*, Fuzzy Sets and Systems 12(1984), no. 2, 143-154.

[11] N. C. Kayal, P. Mondal, T. K. Samanta, *The Generalized Hyers-Ulam-Rassias Stability of a Quadratic Functional Equation in Fuzzy Banach Spaces*, Journal of New Results in Science, 5 (2014) 83-95.

[12] I. Kramosil and J. Michalek, *Fuzzy metric and statistical metric spaces*, Kybernetica, 11 (1975), 326-334.

[13] S.V. Krishna and K. K. M. Sarma, Separation of fuzzy normed linear spaces, Fuzzy Sets and Systems 63 (1994), 207-217.

[14] A. K. Mirmostafee, M. S. Moslehian, *Fuzzy versions of Hyers-Ulam-Rassias theorem, Fuzzy sets and systems*, 159 (2008), 720-729.

[15] Th. M. Rassias, On the stability of the linear additive mapping in Banach space, Proc. Amer. Mathematical Society, 72(2) (1978), 297-300.

[16] T. K. Samanta and Iqbal H. Jebril, Finite dimentional intuitionistic fuzzy normed linear space, Int. J. Open Problems Compt. Math., 2(4) (2009), 574-591.

[17] T. K. Samanta, P. Mondal, N. C. Kayal, *The generalized Hyers-Ulam-Rassias stability of a quadratic functional equation in fuzzy Banach spaces*, Annals of Fuzzy Mathematics and Informatics, Volume 6, No. 2, (2013), pp. 59-68.

[18] S. M. Ulam, *Problems in Modern Mathematics*, Chapter vi, Science Editions, Wiley, New York, 1964.

http://www.newtheory.org ISSN: 2149-1402

Journal of New Theory

Received: 02.09.2015 *Year*: **2016,** *Number*: **10**, *Pages*: **39-44**
Published: 07.01.2016 *Original Article*[**]

MISFIRE FAULT DIAGNOSIS OF GASOLINE ENGINES USING THE COSINE MEASURE OF SINGLE-VALUED NEUTROSOPHIC SETS

Xiaoqi Wang <1308250937@qq.com>
Jinling Wei <1731676691@qq.com>
Jun Ye[*] <yehjun@aliyun.com>

Department of Electrical and Information Engineering, Shaoxing University, 508 Huancheng West Road, Shaoxing, Zhejiang Province 312000, P.R. China

Abstract – Single-valued neutrosophic set (SVNS) is very suitable for expressing indeterminate and inconsistent information in fault diagnosis problems, and then its cosine measure is a useful mathematical tool for handling the decision making, pattern recognition, and fault diagnosis problems. However, due to the lack of engineering applications of SVNSs in fault diagnoses, the paper develops a cosine measure-based fault diagnosis method and applies it to the misfire fault diagnosis of gasoline engines with SVNS information. Through the cosine measure between each fault pattern and a real-testing sample, according to the largest cosine measure value, we can determine that the testing sample should belong to the fault pattern. Finally, we provide nine real-testing samples to illustrate the misfire fault diagnoses of gasoline engines. All diagnosis results are in accordance with actual fault types. The results demonstrate the effectiveness and rationality of the proposed diagnosis method.

***Keywords* –** *Cosine measure, single-valued neutrosophic set, misfire fault diagnosis, gasoline engine.*

1 Introduction

Neutrosophic set proposed by Smarandache [1] is a powerful tool to deal with incomplete, indeterminate and inconsistent information in real world. It is a generalization of the theory of fuzzy sets, vague sets, intuitionistic fuzzy sets and interval-valued intuitionistic fuzzy sets, then the neutrosophic set is characterized by a truth-membership degree, an indeterminacy-membership degree and a falsity-membership degree independently, which are within the real standard or nonstandard unit interval $]^{-}0, 1^{+}[$. Therefore, if their range is restrained within the real standard unit interval [0, 1], the neutrosophic set is easily applied to engineering problems. For this purpose, Wang et al. [2] introduced the concept of a single-valued neutrosophic set (SVNS) as a subclass of the neutrosophic set.

[**]*Edited by* Said Broumi *(Area Editor) and* Naim Çağman *(Editor-in-Chief).*
[*]*Corresponding Author.*

In a fault diagnosis problem, various symptoms usually imply a lot of incomplete, uncertainty and inconsistent information for a fault, which characterizes a relation between symptoms and a fault. Thus we work with the uncertainties and inconsistencies to lead us to proper fault diagnosis. Hence, SVNSs are very suitable for expressing incomplete, indeterminate and inconsistent information comprehensively in fault diagnosis problems. However, similarity measure is an important mathematical tool in fault diagnoses. Recently, Ye [3] proposed cotangent similarity measures between SVNSs based on cotangent function and successfully applied them to the fault diagnosis of steam turbine under a single-valued neutrosophic environment. Because misfire fault problems of gasoline engines are usually produced in operating process [4], they can affect the operating power and working performance of gasoline engines and increase fuel consumption. To find out misfire fault problems in gasoline engines, extension set theory and neutrosophic numbers have been applied respectively to the misfire fault diagnosis of gasoline engines [4, 5]. However, till now SVNSs have been not applied to fault diagnoses of gasoline engines. To extend existing fault diagnosis methods, the main purposes of this paper are to propose a fault diagnosis method based on the cosine measure of SVNSs and to apply it to the misfire fault diagnosis of gasoline engines with single-valued neutrosophic information.

The remainder of this paper is organized as follows. Section 2 briefly describes some basic concepts and cosine measure of SVNSs. Section 3 establishes a fault diagnosis method using the cosine measure of SVNSs and applies it to the misfire fault diagnosis of gasoline engines under a single-valued neutrosophic environment to demonstrate the effectiveness and nationality of the developed method. Section 4 contains conclusions and future research direction.

2 Some Concepts and Cosine Measure of SVNSs

Smarandache [1] firstly proposed the concept of the neutrosophic set from a philosophical viewpoint. Then, it is difficult to apply the neutrosophic set to engineering applications. Consequently, Wang et al. [2] introduced the definition of a SVNS, which is a subclass of the neutrosophic set.

Definition 2.1. [2]. Let X be a universal of discourse. A SVNS N in X is characterized by a truth-membership function $t_N(x)$, an indeterminacy-membership function $i_N(x)$ and a falsity-membership function $f_N(x)$. Then, a SVNS N can be denoted by the following form:

$$N = \{\langle x, t_N(x), i_N(x), f_N(x)\rangle \mid x \in X\},$$

where $t_N(x)$, $i_N(x)$, $f_N(x) \in [0, 1]$ for each point x in X. Obviously, the sum of $t_N(x)$, $i_N(x)$ and $f_N(x)$ is $0 \le t_N(x) + i_N(x) + f_N(x) \le 3$.

Let $N = \{\langle x, t_N(x), i_N(x), f_N(x)\rangle \mid x \in X\}$ and $M = \{\langle x, t_M(x), i_M(x), f_M(x)\rangle \mid x \in X\}$ be two SVNSs. Then there are the following relations [2]:

(1) Complement: $N^c = \{\langle x, f_N(x), 1 - i_N(x), t_N(x)\rangle \mid x \in X\}$;

(2) Inclusion: $N \subseteq M$ if and only if $t_N(x) \le t_M(x)$, $i_N(x) \ge i_M(x)$ and $f_N(x) \ge f_M(x)$ for any x in X;
(3) Equality: $N = M$ if and only if $N \subseteq M$ and $M \subseteq N$.

Based on cosine function, Ye [6] proposed an improved cosine measure between SVNSs and gave the following definition.

Definition 2.2. [6] Let two SVNSs N and M in the universe of discourse $X = \{x_1, x_2, \ldots, x_n\}$ be $N = \{\langle x_j, t_N(x_j), i_N(x_j), f_N(x_j)\rangle \mid x_j \in X\}$ and $M = \{\langle x_j, t_M(x_j), i_M(x_j), f_M(x_j)\rangle \mid x_j \in X\}$. Then, a cosine measure between SVNSs N and M is defined as

$$C(N,M) = \frac{1}{n}\sum_{j=1}^{n} \cos\left\{\frac{\pi}{6}\left(\left|t_N(x_j) - t_M(x_j)\right| + \left|i_N(x_j) - i_M(x_j)\right| + \left|f_N(x_j) - f_M(x_j)\right|\right)\right\}, \quad (1)$$

The cosine measure $C(N, M)$ satisfies the following properties (1)-(4) [6]:

(1) $0 \le C(N, M) \le 1$;
(2) $C(N, M) = 1$ if and only if $N = M$;
(3) $C(N, M) = C(M, N)$;
(4) If P is a SVNS in X and $N \subseteq M \subseteq P$, then $C(N, P) \le C(N, M)$ and $C(N, P) \le C(M, P)$.

Considering the importance of elements in the universe of discourse, one needs to give the weight w_j of the element x_j (j = 1, 2,…, n) with $w_j \in [0, 1]$ and $\sum_{j=1}^{n} w_j = 1$. Then, the weighted cosine measure between SVNSs N and M can be introduced as follows [6]:

$$W(N,M) = \sum_{j=1}^{n} w_j \cos\left\{\frac{\pi}{6}\left(\left|t_N(x_j) - t_M(x_j)\right| + \left|i_N(x_j) - i_M(x_j)\right| + \left|f_N(x_j) - f_M(x_j)\right|\right)\right\}, \quad (2)$$

3 Application of the Cosine Measure in Misfire Fault Diagnosis of Gasoline Engines

3.1 Fault Diagnosis Method Based on the Cosine Measure

In general, a set of m fault patterns (fault knowledge) $P = \{P_1, P_2, \ldots, P_m\}$ and a set of n characteristics (attributes) $A = \{A_1, A_2, \ldots, A_n\}$ should be established in a fault diagnosis problem. Then the fault information of each fault pattern P_k (k = 1, 2, …, m) with respect to characteristics of A_j (j = 1, 2, …, n) can be expressed by a set of single-valued neutrosophic values (SVNVs) $P_k = \{p_{k1}, p_{k2}, \ldots, p_{kn}\}$, where $p_{kj} = \langle t_{kj}, i_{kj}, f_{kj}\rangle$ is a SVNV, which is a basic component in the SVNS P_k, for $0 \le t_{kj} + i_{kj} + f_{kj} \le 3$ (k = 1, 2, …, m; j = 1, 2, …, n). Then, the information of a testing sample is expressed by a set of SVNVs $S_t = \{s_{t1}, s_{t2}, \ldots, s_{tn}\}$, where $s_{tj} = \langle t_{tj}, i_{tj}, f_{tj}\rangle$ is a SVNV in the SVNS S_t for $0 \le t_{tj} + i_{tj} + f_{tj} \le 3$ (t = 1, 2, …, q; j = 1, 2, …, n).

Then, the cosine measure between a testing sample S_t and each fault pattern P_k (k =1,2,…, m) can be calculated by the following formula:

$$W(S_t, P_k) = \sum_{j=1}^{n} w_j \cos\left\{\frac{\pi}{6}\left(\left|t_{tj} - t_{kj}\right| + \left|i_{tj} - i_{kj}\right| + \left|f_{tj} - f_{kj}\right|\right)\right\}. \quad (3)$$

According to the largest measure value of $W(S_t, P_k)$, we can determine that the testing sample S_t should belong to the fault pattern P_k.

3.2 Application in the Misfire Fault Diagnosis of Gasoline Engines

In this subsection, we apply the fault diagnosis method based on the cosine measure to the misfire fault diagnosis of gasoline engines to show the effectiveness and rationality of the proposed diagnosis method.

Misfire fault problems are usually produced in operating process of gasoline engines [4, 5]. Thus, they can reduce the operating power and working performance of gasoline engines and increase fuel consumption so that they aggravate the pollution of exhaust emission when the burning quality of mixture gases descends in the combustion chamber of gasoline engines. To keep better working performance of gasoline engines, we have to find out and eliminate the affected factors of low burning quality in gasoline engines. Then, the main components of HC, NO_x, CO, CO_2, O_2, water vapor etc included in the exhaust emission of gasoline engines can affect the burning quality of mixture gases in the engines. Under different burning conditions in the engines, the exhaust emission content can be changed in some variable ranges corresponding to the change of operating status or the occurrences of various mechanical and electronic faults in the engines. We have discovered the relation between the misfire fault and the content of the components in the exhaust emission of gasoline engines [4]. Hence, we can judge the operating status of the engines by analyzing the change of exhaust emission content.

Let us investigate the misfire fault diagnosis problem of the gasoline engine EQ6102 [4, 5]. In general, the misfire faults of the engine can be classified into three fault types: no misfire (normal work), slight misfire and severe misfire to indicate the operating status of the engine. The slight misfire indicates the decline in the performance of ignition capacitance or the ignition delay, or the spark plug misfire in one of six cylinders; while the severe misfire indicates the spark plug misfire in two of six cylinders. According to real-testing data [4, 5], we can obtain three kinds of fault patterns: no misfire (P_1), slight misfire (P_2), severe misfire (P_3), which are denoted by a set $P = \{P_1, P_2, P_3\}$, with respect to five characteristics (HC, NO_x, CO, CO_2, O_2) denoted by a set $A = \{A_1, A_2, A_3, A_4, A_5\}$, as shown in Table 1.

In Table 1, $\varphi_{HC}\times10^{-2}$, φ_{CO2}, $\varphi_{NOx}\times10$, $\varphi_{CO}\times10^{-1}$ and φ_{O2} in the characteristic set $A = \{A_1, A_2, A_3, A_4, A_5\}$ indicate the exhaust emission concentration of the five components HC, CO_2, NO_x, CO and O_2 expressed by volume percentage [4, 5], and then the characteristic values of A_j (j = 1, 2, 3, 4, 5) are expressed as SVNVs in Table 1.

To illustrate the effectiveness of the misfire fault diagnosis of the engine, we introduce the nine sets of real-testing samples for the engine EQ6102 from [4, 5], and then the characteristic values in the real-testing samples are expressed by SVNVs, which are shown in Table 2.

Table 1. Fault knowledge expressed bySVNVs for the engine EQ6102 [5]

	A_1 ($\varphi_{HC}\times10^{-2}$)	A_2 (φ_{CO2})	A_3 ($\varphi_{NOx}\times10$)	A_4 ($\varphi_{CO}\times10^{-1}$)	A_5 (φ_{O2})
P_1 (Normal work)	<0.03, 0.05, 0.92>	<0.51, 0.42, 0.07>	<0.03, 0.05, 0.92>	<0.3, 0.2, 0.5>	<0.062, 0.028, 0.91>
P_2 (Slight misfire)	<0.01, 0.036, 0.954>	<0.428, 0.41, 0.16>	<0.04, 0.08, 0.88>	<0.29, 0.21, 0.5>	<0.04, 0.07, 0.89>
P_3 (Severe misfire)	<0.2, 0.3, 0.5>	<0.3, 0.4, 0.3>	<0.1, 0.2, 0.7>	<0.1, 0.2, 0.7>	<0.07, 0.08, 0.85>

Table 2. Real-testing samples of exhaust emission

	A_1 ($\varphi_{HC}\times10^{-2}$)	A_2 (φ_{CO2})	A_3 ($\varphi_{NOx}\times10$)	A_4 ($\varphi_{CO}\times10^{-1}$)	A_5 (φ_{O2})	Actual fault type
S_1	<0.0455, 0, 0.9545>	<0.047, 0, 0.953>	<0.033, 0, 0.967>	<0.48, 0, 0.52>	<0.0527, 0, 0.9473>	P_2
S_2	<0.0572, 0, 0.9428>	<0.075, 0, 0.925>	<0.062, 0, 0.938>	<0.42, 0, 0.58>	<0.0751, 0, 0.9249>	P_1
S_3	<0.0261, 0, 0.9739>	<0.065, 0, 0.935>	<0.086, 0, 0.914>	(0.453, 0, 0.547>	<0.0431, 0, 0.9569>	P_2
S_4	<0.0312, 0, 0.9688>	<0.062, 0, 0.938>	<0.051, 0, 0.949>	<0.287, 0, 0.713>	<0.1064, 0, 0.8936>	P_2
S_5	<0.3761, 0, 0.6239>	<0.045, 0, 0.955>	<0.139, 0, 0.861>	<0.179, 0, 0.821>	<0.1025, 0, 0.8975>	P_3
S_6	<0.422, 0, 0.578>	<0.052, 0, 0.948>	<0.188, 0, 0.812>	<0.194, 0, 0.806>	<0.0931, 0, 0.9069)	P_3
S_7	<0.0189, 0, 0.9811>	<0.081, 0, 0.919>	<0.091, 0, 0.909>	<0.459, 0, 0.541>	<0.0377, 0, 0.9623>	P_2
S_8	<0.0555, 0, 0.9445>	<0.086, 0, 0.914>	<0.057, 0, 0.943>	<0.39, 0, 0.61>	<0.0736, 0, 0.9264>	P_1
S_9	<0.0551, 0, 0.9449>	<0.085, 0, 0.915>	<0.05, 0, 0.95>	<0.386, 0, 0.614>	<0.0789, 0, 0.9211>	P_1

Table 3. Fault diagnosis results of the nine real-testing samples

	P_1	P_2	P_3	Diagnosis result	Actual fault type
S_1	0.9552	**0.9616**	0.9364	P_2	P_2
S_2	**0.9618**	0.9617	0.9358	P_1	P_1
S_3	0.9616	**0.9618**	0.9421	P_2	P_2
S_4	0.9611	**0.9616**	0.9490	P_2	P_2
S_5	0.9402	0.9512	**0.9560**	P_3	P_3
S_6	0.9482	0.9493	**0.9560**	P_3	P_3
S_7	0.9615	**0.9617**	0.9232	P_2	P_2
S_8	**0.9618**	0.9586	0.9170	P_1	P_1
S_9	**0.9618**	0.9602	0.9187	P_1	P_1

Then, the importance of the five characteristics (five components) is considered by the weight vector $\boldsymbol{W} = (w_1, w_2, w_3, w_4, w_5) = (0.05, 0.35, 0.3, 0.2, 0.1)$ [4]. By using Eq. (3), the diagnosis results are shown in Table 3. From Table 3, all the fault diagnosis results are in accordance with all the actual fault types.

Therefore, the proposed fault diagnosis method for the gasoline engine is effective. Compared with the fault diagnosis method for the gasoline engine in [4, 5], the fault diagnosis method proposed in this paper is simpler and easier than the fault diagnosis method by using extension set theory [4], and then the fault diagnosis method with SVNSs in this paper contains more information (including truth information, indeterminacy information and falsity information) than the fault diagnosis method using the neutrosophic numbers [5] which consist of the determinate part and indeterminate part.

4 Conclusions

This paper proposed a fault diagnosis method based on the cosine measure of SVNSs and applied it to the misfire fault diagnosis of gasoline engines under a single-valued neutrosophic environment. The fault diagnosis results of the gasoline engine demonstrated the effectiveness and rationality of the proposed fault diagnosis method. The diagnosis method proposed in this paper is an extension of existing fault diagnosis methods and provides a new way for fault diagnoses of gasoline engines. In the future, the developed diagnosis method will be extended to other fault diagnoses, such as vibration faults of turbines, aircraft engines and gearboxes.

References

[1] F. Smarandache, *Neutrosophy: Neutrosophic probability, set, and logic*, American Research Press, Rehoboth, USA, 1998.

[2] H. Wang, F. Smarandache, Y.Q. Zhang, R. Sunderraman, *Single valued neutrosophic sets*, Multispace and Multistructure 4 (2010) 410-413.

[3] J. Ye, *Single valued neutrosophic similarity measures based on cotangent function and their application in the fault diagnosis of steam turbine*, Soft Computing (2015) DOI: 10.1007/s00500-015-1818-y.

[4] J. Ye, *Application of extension theory in misfire fault diagnosis of gasoline engines*, Expert Systems with Applications 36(2) (2009) 1217-1221.

[5] L. Kong, Y. Wu, J. Ye, *Misfire fault diagnosis method of gasoline engines using the cosine similarity measure of neutrosophic numbers*, Neutrosophic Sets and Systems 8 (2015) 43-46.

[6] J. Ye, *Improved cosine similarity measures of simplified neutrosophic sets for medical diagnoses*, Artificial Intelligence in Medicine 63(3) (2015) 171–179.

http://www.newtheory.org ISSN: 2149-1402

Journal of New Theory

Received: 27.10.2015
Published: 22.01.2016
Year: **2016**, *Number*: **10**, *Pages*: **45-53**
Original Article[**]

PERFECTLY ω-IRRESOLUTE FUNCTIONS

İdris Zorlutuna <izorlu@cumhuriyet.edu.tr>

Department of Mathematics, Cumhuriyet University, 58140 Sivas, Turkey

Abstract — In this paper, we investigate a new form of continuity called perfect ω-irresoluteness and we use functions which have this type of continuity as a tool to set new characterizations of some properties of topological spaces.

Keywords — *ω-open, ω-irresolute, perfectly ω-irresolute.*

1 Introduction

An ω-closed set is a set which contains all its condensation points [5]. Since the advent of this notion, lots of topologist have studied on it and most of topological notions such as continuity, compactness, connectedness were generalized. Especially, some new strong and weak forms of continuity have been arised during the last years. One of these is ω-irresoluteness introduced by Al-Zoubi [4]. On the other hand, in 1984, Noiri [8] introduced and investigated the notion of perfect continuity of functions between topological spaces.

This paper devoted to investigate a new type of continuity is stronger than ω-irresoluteness and perfect continuity. In section 3, definition and fundamental properties are given. In section 4, we use perfectly ω-irresolute functions as a tool to set new characterizations of connectedness. Moreover some separation axioms related to ω-open sets are investigated. The last section deals with graphs of perfectly ω-irresolute functions.

2 Preliminaries

Throughout the present paper, spaces always mean topological spaces on which no separation axiom is assumed unless explicitly stated and $f : (X, \tau) \to (Y, \sigma)$ (or simply $f : X \to Y$) denotes a function f from a topological space (X, τ) into a topological space (Y, σ). Let A be a subset of a space X. A point $x \in X$ is called a condensation point of A if for each open set U with $x \in U$, the set $U \cap A$ is uncountable. A is called ω-closed [5] if it contains all its condensation points. The

[**] *Edited by* Takashi Noiri *and* Naim Çağman *(Editor-in-Chief).*

complement of an ω-closed set is called ω-open. The family of all ω-open subsets of (X,τ) is denoted by τ_ω. It is known that τ_ω is a topology for X and $\tau \subset \tau_\omega$. For a subset A of (X,τ), the closure of A and the interior of A denoted by $Cl(A)$ and $Int(A)$, respectively. The closure of A with respect to τ_ω denoted by $\omega Cl(A)$. A is called regular closed [9] if $A = Cl(Int(A))$.

Let us recall the following definitions which we shall require later.

Definition 2.1. A function $f : X \to Y$ is called perfectly continuous [8] if $f^{-1}(V)$ is clopen in X for every open set V of Y.

Definition 2.2. A function $f : X \to Y$ is called ω-irresolute [4] if $f^{-1}(V)$ is ω-open in X for every ω-open set V of Y.

3 Perfectly ω-irresolute Functions

Definition 3.1. A function $f : X \to Y$ is said to be perfectly ω-irresolute if $f^{-1}(V)$ is clopen in X for every ω-open set V of Y.

Theorem 3.2. For a function $f : (X,\tau) \to (Y,\sigma)$, the followings are equivalent:

(1) f is perfectly ω-irresolute;

(2) for every ω-closed subset F of Y, $f^{-1}(F)$ is clopen in X;

(3) $f : (X,\tau) \to (Y,\sigma_\omega)$ is perfectly continuous.

Proof. (1)⇒(2). Let F be a ω-closed subset of Y. Then $Y \backslash F$ is an ω-open subset and by (1), $f^{-1}(Y \backslash F) = X \backslash f^{-1}(F)$ is clopen in X. Hence $f^{-1}(F)$ is also clopen in X.

(2)⇒(3). Let $V \in \sigma_\omega$. Then $Y \backslash V$ is an ω-closed in Y and by (2), $f^{-1}(Y \backslash V) = X \backslash f^{-1}(V)$ is clopen in X. Hence $f^{-1}(V)$ is also clopen in X.

(3)⇒(1). It can be shown easily. Recal that a space (X,τ) is said to be ω-space [1] if every ω-open set is open and is said to be locally ω-indiscrete [1] if every ω-open set is closed in X.

Then we have the following theorem which gives a characterization of locally ω-indiscrete ω-space. Its proof is clear.

Theorem 3.3. A space X is ω-space and locally ω-indiscrete if and only if the identity map of X is perfectly ω-irresolute.

Theorem 3.4. For a function $f : X \to Y$, the following are true.

(1) If f is perfectly ω-irresolute and $A \subseteq X$, then $f \mid_A : A \to Y$ is perfectly ω-irresolute.

(2) If $\{G_\alpha : \alpha \in I\}$ is a locally finite clopen cover of X and if for each α, $f_\alpha = f \mid_{G_\alpha}$ is perfectly ω-irresolute, then f is perfectly ω-irresolute.

Proof. The proof of (1) is clear. We will only prove (2).

Let F be a ω-open subset of Y. Since each f_α is perfectly ω-irresolute, each $f_\alpha^{-1}(F)$ is clopen in G_α and hence in X. Thus $f^{-1}(F) = \cup\{f_\alpha^{-1}(F) : \alpha \in I\}$ is open in X. On the other hand, since the family $\{G_\alpha : \alpha \in I\}$ is locally finite, $\{f_\alpha^{-1}(F) : \alpha \in I\}$ is a locally finite family of closed sets in X. Hence $f^{-1}(F)$ being the union of a locally finite collection of closed sets is closed in X. Consequently, $f^{-1}(F)$ is clopen in X.

Definition 3.5. A function $f : X \to Y$ is called

(1) ω-continuous [6] $f^{-1}(V)$ is ω-open in X for every open set V of Y.

(2) slightly ω-continuous [7] $f^{-1}(V)$ is ω-open in X for every clopen set V of Y.

(3) contra ω-irresolute if $f^{-1}(V)$ is ω-closed in X for every ω-open set V of Y.

Theorem 3.6. The followings hold for functions $f : X \to Y$ and $g : Y \to Z$:

(1) If $f : X \to Y$ is perfectly ω-irresolute and $g : Y \to Z$ is ω-irresolute, then $g \circ f : X \to Z$ is perfectly ω-irresolute.

(2) If $f : X \to Y$ is perfectly ω-irresolute and $g : Y \to Z$ is ω-continuous, then $g \circ f : X \to Z$ is perfectly continuous.

(3) If $f : X \to Y$ is slightly ω-continuous and $g : Y \to Z$ is perfectly ω-irresolute, then $g \circ f : X \to Z$ is ω-irresolute.

(4) If $f : X \to Y$ is perfectly ω-irresolute and $g : Y \to Z$ is contra ω-irresolute, then $g \circ f : X \to Z$ is perfectly ω-irresolute.

Proof. (1) Let V be any ω-open set in Z. By the ω-irresoluteness of g, $g^{-1}(V)$ is ω-open. Since f is perfectly ω-irresolute, $f^{-1}(g^{-1}(V)) = (g \circ f)^{-1}(V)$ is clopen in X. Therefore, $g \circ f$ is perfectly ω-irresolute.

The others can be proved similarly.

Theorem 3.7. If $f : X \to Y$ is a surjective open and closed function and $g : Y \to Z$ is a function such that $gof : X \to Z$ is perfectly ω-irresolute function, then g is perfectly ω-irresolute function.

Proof. Let V be any ω-open set in Z. Since gof is perfectly ω-irresolute, $(gof)^{-1}(V)$ is clopen in X. Since f is surjective open and closed, $f((g \circ f)^{-1}(V)) = f((f^{-1}(g^{-1}(V))) = g^{-1}(V)$ is clopen in Y. Therefore, g is perfectly ω-irresolute.

It is easy to show that perfect ω-irresoluteness implies perfect continuity and ω-irresoluteness. The following theorems are about reverse of these implications and they can be proved directly.

Theorem 3.8. Let X be a locally ω-indiscrete and ω-space. Then for any topological space Y, a function $f : X \to Y$ is perfectly ω-irresolute if and only if f is ω-irresolute.

Theorem 3.9. Let Y be an ω-space. Then for any topological spaces X, a function $f : X \to Y$ is perfectly ω-irresolute if and only if f is perfectly continuous.

Theorem 3.10. For a function $f : X \to Y$, the following properties are equivalent:

(1) f is contra ω-irresolute;

(2) for every ω-closed F of Y, $f^{-1}(F)$ is ω-open in X;

(3) for every $x \in X$ and for every ω-closed set F containing $f(x)$, there exists an ω-open set U containing x such that $f(U) \subseteq F$.

Proof. (1)⇔(2). These follow from equality $f^{-1}(Y \backslash F) = X \backslash f^{-1}(F)$ for each subset F of Y.

(2)⇒(3). Let F be an ω-closed set containing $f(x)$. Then by (2), $f^{-1}(F)$ is ω-open in X containing x. If we choose $U = f^{-1}(F)$, proof is completed.

(3)⇒(2). Obvious.

A space X is called anti locally countable (see [3]) if every nonempty open set is uncountable. It is shown in [7] that in an anti locally countable space X, U is clopen in X iff U is ω-open and ω-closed in X. Then we have the following corollary.

Corollary 3.11. Let (X, τ) be an anti locally countable space. Then for any topological spaces (Y, σ), a function $f : (X, \tau) \to (Y, \sigma)$ is perfectly ω-irresolute if and only if f is ω-irresolute and contra ω-irresolute.

A space X is called ω-regular [7] if for each closed set F and each point $x \in X - F$, there exist disjoint ω-open sets U and V such that $x \in U$ and $F \subseteq V$. It is shown in [2] that a space X is ω-regular if and only if for every point x of X and every open set V containing x, there exists an ω-open set U such that $x \in U \subseteq \omega Cl(U) \subseteq V$.

Theorem 3.12. Let X be an anti local countable space and let Y be an ω-regular space. For a function $f : X \to Y$, the following properties are equivalent:
(1) f is perfectly ω-irresolute;
(2) for every ω-open set V in Y, $f^{-1}(V)$ is regular closed in X;
(3) for every ω-open set V in Y, $f^{-1}(V)$ is closed in X;
(4) f is contra-ω-irresolute.

Proof. The implications (1)⇒(2)⇒(3)⇒(4) are trivial. If we show that f is ω-irresolute by Corollary 3.11, we have the proof of the implication (4)⇒(1). Let $x \in X$ be an arbitrary point and V be an ω-open set of Y containing $f(x)$. Since Y is ω-regular, there exists an ω-open set W in Y such that $f(x) \in \omega Cl(W) \subseteq V$. Since f is contra-ω-irresolute, there exists an ω-open set U_x containing x such that $f(U_x) \subseteq \omega Cl(W)$. Then $U_x \subseteq f^{-1}(V)$. This shows that $f^{-1}(V)$ is ω-open in X.

4 Applications

Note that (X, τ_ω) is always a T_1-space for any given space (X, τ) [3]. Hence we have the following results.

Theorem 4.1. If $f : (X, \tau) \to (Y, \sigma)$ is a perfectly ω-irresolute function, then f is constant on each component of X.

Proof. Let a and b be two points of X that lie in same component C of X. Assume that $f(a) \neq f(b)$. Since (Y, σ_ω) is T_1- space, there exists an $U \in \sigma_\omega$ containing say $f(a)$ but not $f(b)$. By perfect ω-irresoluteness of f, $f^{-1}(U)$ and $X - f^{-1}(U)$ are disjoint clopen sets containing a and b, respectively. This is a contradiction with the fact that C is a component containing a and b. Hence we have the result.

Corollary 4.2. If $f : X \to Y$ is a perfectly ω-irresolute function and if A is non-empty connected subset of X, then $f(A)$ is a single point.

Theorem 4.3. A space X is connected if and only if every perfectly ω-irresolute function from a space X into any space Y is constant.

Proof. The first part of the proof is clear by Theorem 4.1. For the seconf part, assume that X is not connected. Then there exists a proper non-empty clopen subset A of X. Let $Y = \{u, v\}$ and σ be discrete topology on Y. Then the function $f : X \to Y$ defined by $f(x) = u$ if $x \in A$, $f(x) = v$ if $x \notin A$ is non-constant and perfectly ω-irresolute. This is a contradiction by Theorem 4.1. Hence X must be connected.

Note that the topological space consisting of two points with the discrete topology is usually denoted by 2.

Corollary 4.4. For a topological space X, the following are equivalent :
(1) X is connected;
(2) Every perfectly ω-irresolute function $f : X \to 2$ is constant;
(3) There is no perfectly ω-irresolute function $f : X \to 2$ is surjective.

Definition 4.5. A space X is said to be ultra Hausdorff [9] (resp. ω-T_2 [3]) if every two distinct points of X can be separated by disjoint clopen (resp. ω-open) sets.

Theorem 4.6. If $f : X \to Y$ is a perfectly ω-irresolute injection, then X is ultra Hausdorff.

Proof. Let $x_1,\ x_2 \in X$ and $x_1 \neq x_2$. Then $f(x_1) \neq f(x_2)$. Since Y is always $\omega - T_1$, there exists an ω-open set U containing say $f(x_1)$ but not $f(x_2)$. By perfect ω-irresoluteness of f, $f^{-1}(U)$ and $X - f^{-1}(U)$ are disjoint clopen sets containing x_1 and x_2, respectively. Thus X is ultra Hausdorff.

The quasi-topology denoted by τ_q on X is the topology having as base the clopen subsets of (X, τ). A subset A of X is called quasi open if $A \in \tau_q$. The complement of a quasi-open set is called quasi-closed [9].

Theorem 4.7. Let Y be ω-T_2 space.
(1) If $f,\ g : X \to Y$ are perfectly ω-irresolute functions, then the set $A = \{x \in X : f(x) = g(x)\}$ is quasi-closed in X.
(2) If $f : X \to Y$ is perfectly ω-irresolute function, then the subset $E = \{(x, y) : f(x) = f(y)\}$ is quasi-closed in $X \times X$.

Proof. (1). Let $x \notin A$. Then $f(x) \neq g(x)$. Since Y is ω-T_2, there exist disjoint ω-open sets V_1 and V_2 in Y such that $f(x) \in V_1$ and $g(x) \in V_2$. Since f and g are perfectly ω-irresolute, $f^{-1}(V_1)$ and $g^{-1}(V_2)$ are clopen sets. Put $U = f^{-1}(V_1) \cap g^{-1}(V_2)$. Then U is clopen set containing x and $U \cap A = \varnothing$. Hence we have $U \subseteq X - A$. This shows that $X - A$ is quasi-open or equivalently A is quasi-closed.

(2). Let $(x, y) \notin E$. Then $f(x) \neq f(y)$. Since Y is ω-T_2, there exist disjoint ω-open sets V_1 and V_2 containing $f(x)$ and $f(y)$ respectively. Since f is perfectly ω-irresolute, $f^{-1}(V_1)$ and $f^{-1}(V_2)$ are clopen sets. Then for the clopen set $U = f^{-1}(V_1) \times f^{-1}(V_2)$ containing (x, y), we have $U \cap E = \varnothing$ i.e. $U \subseteq (X \times X) - E$. This shows that $(X \times X) - E$ is quasi-open or equivalently E is quasi-closed.

Definition 4.8. [5]A function $f : X \to Y$ is called
(a) ω-closed if for each closed set K in X, $f(K)$ is ω-closed in Y.
(b) ω-open if for each open set U in X, $f(U)$ is ω-open in Y.

Theorem 4.9. A function $f : X \to Y$ is ω-closed (resp. ω-open) if and only if for each subset S of Y and for each open (resp. closed) subset U of X with $f^{-1}(S) \subseteq U$, there exists an ω-open (resp. ω-closed) set V of Y such that $S \subseteq V$ and $f^{-1}(V) \subseteq U$.

Proof. We only prove for the ω-closedness. The other is entirely analogous.

($\Rightarrow$): Suppose that f is ω-closed. Let S be any subset of Y and U be an open subset of X with $f^{-1}(S) \subseteq U$. Since f is ω-closed, $Y - f(X - U)$ is an ω-open set in Y. Then for the set $V = Y - f(X - U)$, we have $S \subseteq V$ and $f^{-1}(V) = f^{-1}(Y - f(X - U)) = X - f^{-1}(f(X - U)) \subseteq U$.

($\Leftarrow$): Let K be any closed subset of X and $S = Y - f(K)$. Then $f^{-1}(S) \subseteq X - K$. By hypothesis, there exists an ω-open set V in Y containing S such that $f^{-1}(V) \subseteq X - K$. Then, we have $K \subseteq X - f^{-1}(V)$ and $Y - V = f(K)$. Since, $Y - V$ is ω-closed, $f(K)$ is ω-closed and thus f is an ω-closed function. A topological space

X is called ω-normal [7] if for every pair of disjoint closed subsets F_1 and F_2 of X, there exists disjoint ω-open sets U and V such that $F_1 \subseteq U$ and $F_2 \subseteq V$.

Theorem 4.10. If $f : X \to Y$ is a continuous ω-closed surjection and if X is normal space, then Y is ω-normal.

Proof. Let F_1 and F_2 be disjoint closed sets of Y. Since f is continuous and X is normal, there exist disjoint open sets U and V such that $f^{-1}(F_1) \subseteq U$ and $f^{-1}(F_2) \subseteq V$. By Theorem 4.9, there exist ω-open sets G and H such that $F_1 \subseteq G$, $F_2 \subseteq H$ and $f^{-1}(G) \subseteq U$ and $f^{-1}(H) \subseteq V$. Then we have $f^{-1}(G) \cap f^{-1}(H) = \varnothing$ and hence $G \cap H = \varnothing$. This shows that Y is ω-normal. The following theorem shows that we can get same result under different hypothesis.

Theorem 4.11. If $f : X \to Y$ is perfectly ω-irresolute, ω-open bijection and X is a normal space, then Y is ω-normal.

Proof. Let F_1 and F_2 be disjoint closed sets in Y. Since F_1 and F_2 are also ω-closed and f is perfectly ω-irresolute, $f^{-1}(F_1)$ and $f^{-1}(F_2)$ are disjoint clopen and so closed sets in X. By normality of X, there exist disjoint open sets U and V such that $f^{-1}(F_1) \subseteq U$ and $f^{-1}(F_2) \subseteq V$. Then we obtain that $F_1 \subseteq f(U)$ and $F_2 \subseteq f(V)$ such that $f(U)$ and $f(V)$ are disjoint ω-open sets. Thus Y is ω-normal.

Theorem 4.12. If $f : X \to Y$ is a continuous, ω-open, ω-closed surjection and if X is regular, then Y is ω-regular.

Proof. Let $y \in Y$ and V be an open set in Y with $y \in V$. Take $y = f(x)$. Since f is continuous and X is regular, there exist an open set U such that $x \in U \subseteq Cl(U) \subseteq f^{-1}(V)$. Then $y \in f(U) \subseteq f(Cl(U)) \subseteq V$. By assumptions, $f(U)$ is ω-open and $f(Cl(U))$ is ω-closed set in Y. Therefore, we have $y \in f(U) \subseteq \omega Cl f(U) \subseteq V$. This shows that Y is ω-regular.

Theorem 4.13. If $f : X \to Y$ is perfectly ω-irresolute, ω-open bijection and if X is regular, then Y is ω-regular.

Proof. It is similar to that of Theorem 4.11.

Definition 4.14. A space (X, τ) is called

(1) mildly compact [9] (resp. ω-compact [1]) if every clopen (resp. ω-open) cover of X has a finite subcover.

(2) mildly Lindelöf [9] if every cover of X by clopen sets has a countable subcover.

Theorem 4.15. Let $f : X \to Y$ be a perfectly ω-irresolute surjection. If X is mildly compact, then Y is ω-compact.

Proof. Let f be a perfectly ω-irresolute surjection and let X be a mildly compact space. If $\{V_i\}_{i\in I}$ is an ω-open cover of Y, by perfect ω-irresoluteness of f, $\{f^{-1}(V_i)\}_{i\in I}$ is a clopen cover of X and so there is a finite subset I_0 of I such that $X = \cup_{i\in I_0} f^{-1}(V_i)$. Therefore, we have $Y = \cup_{i\in I_0} V_i$ since f is surjective. Thus Y is ω-compact.

Theorem 4.16. [6]For a topological space (X, τ), (X, τ) Lindelöf if and only if (X, τ_ω) Lindelöf.

Theorem 4.17. Let $f : X \to Y$ be a perfectly ω-irresolute surjection. If X is mildly Lindelöf, then Y is Lindelöf.

Proof. It is similar to that of Theorem 4.15. We notice that a subspace A of a space X is mildly Lindelöf relative to X if for every cover $\{V_i : i \in I\}$ of A by clopen sets of X, there exists a countable subset I_0 of I such that $\{V_i : i \in I_0\}$ covers A.

Theorem 4.18. Let $f : X \to Y$ be an ω-closed surjection such that $f^{-1}(\{y\})$ is a mildly Lindelöf relative to X for each $y \in Y$. If Y is Lindelöf, then X is mildly Lindelöf.

Proof. Let $\{U_i : i \in I\}$ be an clopen cover of X. Since $f^{-1}(\{y\})$ is a mildly Lindelöf relative to X for each $y \in Y$, there exists a countable subset I_y of I such that $f^{-1}(\{y\}) \subseteq \cup\{U_i : i \in I_y\}$. Put $U_y = \cup\{U_i : i \in I_y\}$. Then since f is ω-closed, $V_y = Y - f(X - U_y)$ is an ω-open set containing y such that $f^{-1}(V_y) \subseteq U_y$. Again since $\{V_y : y \in Y\}$ is an ω-open cover of the Lindelöf space Y, by Theorem 4.16, there exist countable points of Y, says, $y_1, y_2, ..., y_n, ...$ such that $Y = \cup_{n\in\mathbb{N}} V_{y_n}$. Therefore, we have $X = f^{-1}(\cup_{n\in\mathbb{N}} V_{y_n}) = \cup_{n\in\mathbb{N}} f^{-1}(V_{y_n}) \subseteq \cup_{n\in\mathbb{N}} U_{y_n} = \cup_{n\in\mathbb{N}}(\cup\{U_i : i \in I_{y_n}\}) = \cup\{U_i : i \in I_{y_n}, n \in \mathbb{N}\}$. This completes the proof.

Corollary 4.19. Let $f : X \to Y$ be an perfectly ω-irresolute and ω-closed surjection such that $f^{-1}(\{y\})$ is a mildly Lindelöf relative to X for each $y \in Y$. Then X is mildly Lindelöf if and only if Y is Lindelöf.

5 Graphs of Perfectly ω-irresolute Functions

Recall that for a function $f : X \to Y$, the subset $\{(x, f(x)) : x \in X\} \subset X \times Y$ is called the graph of f and is denoted by $G(f)$.

Definition 5.1. The graph $G(f)$ of a function $f : X \to Y$ is said to be quasi-ω-closed if for each $(x, y) \in (X \times Y) - G(f)$, there exist a clopen set U containing x and an ω-open set V containing y such that $(U \times V) \cap G(f) = \varnothing$.

The proof of the following lemma is clear.

Lemma 5.2. The graph $G(f)$ of a function $f : X \to Y$ is quasi-ω-closed in $X \times Y$ if and only if for each $(x, y) \in (X \times Y) - G(f)$, there exist a clopen set U containing x and an ω-open set V containing y such that $f(U) \cap V = \varnothing$.

Theorem 5.3. If $f : X \to Y$ is perfectly ω-irresolute and Y is ω-T_2, then $G(f)$ is quasi-ω-closed.

Proof. Let $(x, y) \notin G(f)$, then $y \neq f(x)$. Since Y is ω-T_2, there exist disjoint ω-open sets V_1 and V_2 containing $f(x)$ and y, respectively. Again since f is perfectly ω-irresolute, $f^{-1}(V_1)$ is clopen set containing x. If we choose $U = f^{-1}(V_1)$, then we have $f(U) \cap V_2 = \varnothing$ and hence $G(f)$ is quasi-ω-closed. A subset A of a space X is said to be mildly compact (resp. ω-compact) relative to X if for every cover $\{V_i : i \in I\}$ of A by clopen (resp. ω-open) sets of X, there exists a finite subset I_0 of I such that $A \subseteq \cup\{V_\omega : \omega \in I_0\}$.

Theorem 5.4. If a function $f : (X, \tau) \to (Y, \sigma)$ has quasi-ω-closed graph, then the followings are true.

(1) $f(E)$ is ω-closed in Y for every subset E which is mildly compact relative to X.

(2) $f^{-1}(K)$ is quasi-closed in X for every subset K which is ω-compact relative to Y.

Proof. (1) Let E be mildly compact relative to X and $y \notin f(E)$. Then we have $(x, y) \in (X \times Y) - G(f)$ for each $x \in E$ and by Lemma 5.2, there exist a clopen set U_x and ω-open set V_x containing x and y respectively, such that $f(U_x) \cap V_x = \varnothing$. Since the family of $\{U_x : x \in E\}$ is a cover of E by clopen sets of X, there exists a finite number of points, say, $x_1, x_2, ..., x_n$ of E such that $E \subseteq \cup\{U_{x_i} : i = 1, 2, ..., n\}$. Set $V = \cap\{V_{x_i} : i = 1, 2, ..., n\}$, then V is an ω-open set containing y and $f(E) \cap V \subseteq (\cup\{f(U_{x_i}) : i = 1, 2, ..., n\}) \cap V = \varnothing$. Therefore, we have, $y \notin \omega Cl(f(E))$. This shows that $f(E)$ is ω-closed in Y.

(2) It is similar.

Theorem 5.5. Let $f : (X, \tau) \to (Y, \sigma)$ have the quasi-ω-closed graph. If f is injective, then (X, τ_q) is T_1.

Proof. Let x_1, $x_2 \in X$ and $x_1 \neq x_2$. Then, we have $f(x_1) \neq f(x_2)$ and so $(x_1, f(x_2)) \in (X \times Y) - G(f)$. By quasi-$\omega$-closedness of graph $G(f)$, there exist a clopen set U and an ω-open set V containing $x_1$and $f(x_2)$ respectively, such that $f(U) \cap V = \varnothing$, and hence $U \cap f^{-1}(V) = \varnothing$. Since $x_2 \in f^{-1}(V)$, clopen sets U and $X - U$ are desired sets. This completes the proof.

Theorem 5.6. If $f : X \to Y$ is an injection with quasi-ω-closed graph, then X is ultra Hausdorff.

Proof. Let x_1 and x_2 be distinct points in X. Then $f(x_1) \neq f(x_2)$ and so $(x_1, f(x_2)) \notin G(f)$. Therefore, there exist a clopen set U and an ω-open set V such that $(x_1, f(x_2)) \in U \times V$ and $U \cap f^{-1}(V) = \varnothing$. Hence we have disjoint clopen sets U and $X \backslash U$ containing x_1 and x_2 respectively. This shows that X is ultra Hausdorff.

Theorem 5.7. Let $f : X \to Y$ have the quasi-ω-closed graph. If f is a surjective ω-open function, then Y is ω-T_2.

Proof. Let y_1 and y_2 be any distinct points of Y. Since f is surjective, $f(x) = y_1$ for some $x \in X$ and $(x, y_2) \in (X \times Y) - G(f)$. By quasi-$\omega$-closedness of graph $G(f)$, there exist a clopen set U and an ω-open set V such that $(x, y_2) \in (U \times V)$ and $f(U) \cap V = \varnothing$. Since f is ω-open, then $f(U)$ is ω-open such that $f(x) = y_1 \in f(U)$. This shows that Y is ω-T_2.

Definition 5.8. A topological space X is said to be hyperconnected [10] if every pair nonempty open sets of X has nonempty intersection.

Theorem 5.9. Let X be hyperconnected. If $f : X \to Y$ is a perfectly ω-irresolute function with quasi-ω-closed graph, then f is constant.

Proof. Suppose that f is not constant. Then there exist two point x_1 and x_2 of X such that $f(x_1) \neq f(x_2)$. Then we have $(x_1, f(x_2)) \notin G(f)$. Since $G(f)$ is quasi-ω-closed, there exist a clopen set U and an ω-open set V such that $(x_1, f(x_2)) \in U \times V$ and $f(U) \cap V = \varnothing$. Therefore, we have $U \cap f^{-1}(V) = \emptyset$. This is a contradiction with the hyperconnectedness of X since $f^{-1}(V)$ is non-empty open set in X.

6 Acknowledgement

This work is supported by the Scientific Research Project Fund of Cumhuriyet University under the project number F-437.

References

[1] A. Al-OMARI, M.S.M. NOORANI, *Contra-ω-continuous and almost contra-ω-continuous*, Int. J. Math. Math. Sci. (2007) ID 40469.

[2] A. Al-OMARI, T. NOIRI, M.S.M. NOORANI, *Weak and strong forms of ω-continuous functions*, Int. J. Math. Math. Sci. (2009) ID 174042.

[3] K. Al-ZOUBI, B. Al-NASHEF, *The topology of ω-open subsets*, Al-Manarah 9(2) (2003) 169-179.

[4] K. Al-ZOUBI, *Semi ω-continuous functions*, Abhath Al-Yarmouk 12(1) (2003) 119-131.

[5] H.Z. HDEIB, *ω-closed mappings,* Revista Colombiana de Matem´aticas 16(1-2) (1982) 65-78.

[6] H.Z. HDEIB, *ω-continuous functions*, Dirasat Journal 16(2) (1989) 136-142.

[7] T. NOIRI, A. Al-OMARI, M.S.M. NOORANI, *Slightly ω-continuous functions*, Fasciculi Mathematici 41 (2009) 97-106.

[8] T. NOIRI, *Super continuity and some strong forms of continuity*, Indian J. Pure Appl. Math. 15 (3) (1984) 241-250.

[9] R. STAUM, *The algebra of bounded continuous functions into a non archimedian field*, Pacific J. Math. 50 (1974) 169-185.

[10] L.A. STEEN, J.A. SEEBACH, *Counterexamples in topology*, Holt, Rinehart and Winston New york 1970.

http://www.newtheory.org ISSN: 2149-1402

Journal of New Theory

Received: 27.11.2015
Published: 08.02.2016

Year: **2016**, *Number*: **10**, *Pages*: **54-65**
*Original Article**

HERMITE-HADAMARD'S INEQUALITIES FOR FUNCTIONS WHOSE FIRST DERIVATIVES ARE (s, m)-PREINVEX IN THE SECOND SENSE

Badreddine Meftah <badrimeftah@yahoo.fr>

Department of Mathematics, University of 8 Mai 1945 Guelma, Algeria.

Abstaract — In this paper, we introduce a new class of convex functions which is called (s, m)-preinvex functions in the second sense then we establish some new Hermite-Hadamard's inequalities whose modulus of the first derivatives are in this novel class.

Keywords — *Hermite-Hadamard inequality, Hölder's inequality, power mean inequality.*

1 Introduction

One of the most well-known inequalities in mathematics for convex functions is so called Hermite-Hadamard integral inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}, \tag{1}$$

where f is a real continuous convex function on the finite interval $[a, b]$. If the function f is concave, then (1) holds in the reverse direction (see [18]).

The Hermite-Hadamard inequality play an important role in nonlinear analysis and optimization. The above double inequality has attracted many researchers, various generalizations, refinements, extensions and variants of (1) have appeared in the literature, we can mention the works [1, 4, 5, 6, 8, 9, 13, 14, 15, 16, 17] and the references cited therein.

**Edited by* Uzair Awan *and* Naim Çağman *(Editor-in-Chief)*.

In recent years, lot of efforts have been made by many mathematicians to generalize the classical convexity. Hanson in [7], introduced a new class of generalized convex functions, called invex functions. In [2], the authors gave the concept of preinvex functions which is special case of invexity. Pini [19], Noor [11, 12], Yang and Li [23] and Weir [22], have studied the basic properties of preinvex functions and their role in optimization, variational inequalities and equilibrium problems.

In [5], Dragomir and Agarwal established the following Hermite-Hadamard's inequalities for differentiable convex functions:

Theorem 1.1. [Theorem 2.2] Let $f : I^{\circ} \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, $a, b \in I^{\circ}$ with $a < b$. If $|f'|$ is convex on $[a, b]$, then the following inequality holds:

$$\left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx \right| \leq \frac{(b-a)\left(|f'(a)| + |f'(b)|\right)}{8}. \quad (2)$$

Theorem 1.2. [Theorem 2.3] Let $f : I^{\circ} \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, $a, b \in I^{\circ}$ with $a < b$ and let $p > 1$. If the new mapping $|f'|^{\frac{p}{p-1}}$ is convex on $[a, b]$, then the following inequality holds:

$$\begin{aligned} &\left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx \right| \\ &\leq \frac{(b-a)}{2(p+1)^{\frac{1}{p}}} \left(\frac{|f'(a)|^{\frac{p}{p-1}} + |f'(b)|^{\frac{p}{p-1}}}{2} \right)^{\frac{p-1}{p}}. \end{aligned} \quad (3)$$

In [17], Pearce and Pečarić generalized Theorem 2.3 from [5] as follows:

Theorem 1.3. [Theorem 1] Let $f : I^{\circ} \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, $a, b \in I^{\circ}$ with $a < b$ and let $q \geq 1$. If the mapping $|f'|^q$ is convex on $[a, b]$, then

$$\left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx \right| \leq \frac{(b-a)}{4} \left(\frac{|f'(a)|^q + |f'(b)|^q}{2} \right)^{\frac{1}{q}}. \quad (4)$$

In [8], Kirmaci et al. gave a variant of Theorem 1.1 from [13] for functions whose first derivatives in absolute values are s-convex in the second sense as follows:

Theorem 1.4. [Theorem 1] Let $f : I \to \mathbb{R}$, $I \subset [0, \infty)$ be a differentiable mapping on I° such that $f' \in L([a, b])$, where $a, b \in I$, $a < b$, If $|f'|^q$ is s-convex on $[a, b]$ for some fixed $s \in (0, 1)$ and $q \geq 1$, then

$$\begin{aligned} \left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx \right| &\leq \frac{b-a}{2}\left(\frac{1}{2}\right)^{\frac{q-1}{q}} \left(\frac{s + \left(\frac{1}{2}\right)^s}{(s+1)(s+2)} \right)^{\frac{1}{q}} \\ &\quad \times \left(|f'(a)|^q + |f'(b)|^q\right)^{\frac{1}{q}}. \end{aligned} \quad (5)$$

In [1], Barani et al. obtained similar results given in [5] for differentiable preinvex functions as follows:

Theorem 1.5. [Theorem 2.1] Let $A \subseteq R$ be an open invex subset with respect to $\eta : A \times A \to \mathbb{R}$. Suppose that $f : A \to \mathbb{R}$ is a differentiable function. If $|f'|$ is preinvex on A then, for every $a, b \in A$ with $\eta(a,b) \neq 0$ the following inequality holds

$$\left| \frac{f(a) + f(a+\eta(b,a))}{2} - \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)dx \right| \leq \frac{|\eta(b,a)|}{8} \left[|f'(a)| + |f'(b)| \right]. \tag{6}$$

Theorem 1.6. [Theorem 2.2] Let $A \subseteq R$ be an open invex subset with respect to $\eta : A \times A \to \mathbb{R}$. Suppose that $f : A \to \mathbb{R}$ is a differentiable function. Assume that $p \in \mathbb{R}$ with $p > 0$. If $|f'|^{\frac{p}{p-1}}$ is preinvex on A then, for every $a, b \in A$ with $\eta(a,b)$ $\neq 0$ the following inequality holds

$$\left| \frac{f(a) + f(a+\eta(b,a))}{2} - \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)dx \right| \leq \frac{|\eta(b,a)|}{2(p+1)^{\frac{1}{p}}} \left(\frac{|f'(a)|^{\frac{p}{p-1}} + |f'(b)|^{\frac{p}{p-1}}}{2} \right)^{\frac{p-1}{p}}. \tag{7}$$

In [9], Latif and Shoaib established the following Hermite-Hadamard's inequalities for functions whose first derivatives in absolute values are m-preinvex as follows:

Theorem 1.7. [Theorem 4] Let $K \subseteq [0, b^*]$, $b^* > 0$ be an open invex subset with respect to $\eta : K \times K \to \mathbb{R}$ and $a, b \in K$ with $a < a + \eta(b,a)$. Suppose $f : K \to \mathbb{R}$ is differentiable mapping on K such that $f' \in L([a, a+\eta(b,a)])$. If $|f'|$ is m-preinvex on K, then the following inequalities holds

$$\left| \frac{f(a) + f(a+\eta(b,a))}{2} - \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)dx \right| \leq \frac{\eta(b,a)}{8} \left[|f'(a)| + m \left| f'(\frac{b}{m}) \right| \right]. \tag{8}$$

Theorem 1.8. [Theorem 5] Let $K \subseteq [0, b^*]$, $b^* > 0$ be an open invex subset with respect to $\eta : K \times K \to \mathbb{R}$ and $a, b \in K$ with $a < a + \eta(b,a)$. Suppose $f : K \to \mathbb{R}$ is differentiable mapping on K such that $f' \in L([a, a+\eta(b,a)])$. If $|f'|^q$ is m-preinvex on K for $q > 1$, then we have the following inequalities holds

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2}-\frac{1}{\eta(b,a)}\int_{a}^{a+\eta(b,a)} f(x)dx\right|$$
$$\leq \frac{\eta(b,a)}{2(p+1)^{\frac{1}{p}}}\left(\frac{|f'(a)|^{q}+m\left|f'(\frac{b}{m})\right|^{q}}{2}\right)^{\frac{1}{q}}, \tag{9}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Theorem 1.9. [Theorem 6] Let $K \subseteq [0,b^*]$, $b^* > 0$ be an open invex subset with respect to $\eta : K \times K \to \mathbb{R}$ and $a, b \in K$ with $a < a + \eta(b,a)$. Suppose $f : K \to \mathbb{R}$ is differentiable mapping on K such that $f' \in L([a, a+\eta(b,a)])$. If $|f'|^q$ is m-preinvex on K for $q \geq 1$, then we have the following inequalities holds

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2}-\frac{1}{\eta(b,a)}\int_{a}^{a+\eta(b,a)} f(x)dx\right|$$
$$\leq \frac{\eta(b,a)}{4}\left(\frac{|f'(a)|^{q}+m\left|f'(\frac{b}{m})\right|^{q}}{2}\right)^{\frac{1}{q}}. \tag{10}$$

Motivated by the above results, in this paper we first define a novel class of convex functions called (s,m)-preinvexity in the second sense then we establish some new Hermite-Hadamard's inequalities based on this new definition.

2 Preliminaries

In this section we recall some concepts of convexity that are well known in the literature. Throughout this section I is an interval of $\mathbb{R}$.

Definition 2.1. [18] A function $f : I \to \mathbb{R}$ is said to be convex, if

$$f(tx+(1-t)y) \leq tf(x)+(1-t)f(y),$$

holds for all $x, y \in I$ and all $t \in [0,1]$.

Definition 2.2. [3] A nonnegative function $f : I \subset [0,\infty) \to \mathbb{R}$ is said to be s-convex in the second sense, if the following inequality

$$f(tx+(1-t)y) \leq t^s f(x)+(1-t)^s f(y),$$

holds, for all $x, y \in I$ and $t \in [0,1]$.

Definition 2.3. [20] A function $f : [0, b^*] \to \mathbb{R}$, is said to be m-convex function where $m \in [0, 1]$ and $b^* > 0$, if

$$f(tx + m(1-t)y) \leq tf(x) + m(1-t)f(y),$$

holds for all $x, y \in [0, b^*]$ and $t \in [0, 1]$.

Definition 2.4. [6] A nonnegative function $f : [0, b^*] \to \mathbb{R}$, is said to be (s, m)-convex function in the second sense where $s, m \in (0, 1]$ and $b^* > 0$, if

$$f(tx + m(1-t)y) \leq t^s f(x) + m(1-t)^s f(y),$$

holds for all $x, y \in [0, b^*]$ and $t \in [0, 1]$.

Let K be a subset in $\mathbb{R}^n$ and let $f : K \to \mathbb{R}$ and $\eta : K \times K \to \mathbb{R}^n$ be continuous functions.

Definition 2.5. [22] A set K is said to be invex at x with respect to η, if

$$x + t\eta(y, x) \in K,$$

holds for all $x, y \in K$ and $t \in [0, 1]$.

K is said to be an invex set with respect to η if K is invex at each $x \in K$.

Definition 2.6. [22] A function f on the invex set K is said to be preinvex with respect to η, if

$$f(x + t\eta(y, x)) \leq (1-t) f(x) + tf(y),$$

holds for all $x, y \in K$ and $t \in [0, 1]$.

Definition 2.7. [13, 21] A nonnegative function f on the invex set $K \subseteq [0, \infty)$ is said to be s-preinvex in the second sense with respect to η, if

$$f(x + t\eta(y, x)) \leq (1-t)^s f(x) + t^s f(y),$$

holds for all $x, y \in K$ and $t \in [0, 1]$.

Definition 2.8. [9] Let K be an invex set with $K \subseteq [0, b^*]$, $b^* > 0$. A function $f : K \to \mathbb{R}$ is said to be m-preinvex function with respect to η, where $m \in (0, 1]$ and $b^* > 0$, if

$$f(x + t\eta(y, x)) \leq (1-t) f(x) + mtf(\frac{y}{m}),$$

holds for all $x, y \in K$, $t \in [0, 1]$.

Lemma 2.9. [10] The Hypergeometric function is defined as follows:

$$_2F_1(a, b; c; z) = \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}\, dt,$$

for $t \in [0, 1]$ and $\mathrm{Re}c > \mathrm{Re}b > 0$ and $|\arg(1-z)| < \pi$.

Lemma 2.10. [1] Let $K \subseteq \mathbb{R}$ be an open invex subset with respect to $\eta : K \times K \to \mathbb{R}$ and $a, b \in K$ with $a < a + \eta(b,a)$. Suppose $f : K \to \mathbb{R}$ is a differentiable function such that $f' \in L([a, a + \eta(b,a)])$, then the following equality holds

$$\begin{aligned} &\frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)dx - \frac{f(a) + f(a + \eta(b,a))}{2} \\ = &\frac{\eta(b,a)}{2} \int_0^1 (1-2t) f'(a + t\eta(b,a))dt. \end{aligned}$$

3 Main Results

We will start with the following definition.

Definition 3.1. Let K be an invex set with $K \subseteq [0, b^*]$, $b^* > 0$. A nonnegative function $f : K \to \mathbb{R}$ is said to be (s,m)-preinvex function in the second sense with respect to η, where $(s,m) \in (0,1]^2$, if

$$f(x + t\eta(y,x)) \leq (1-t)^s f(x) + mt^s f(\frac{y}{m}),$$

holds for all $x, y \in K$ and $t \in [0,1]$.

Remark 3.2. Definition 3.1 recapture all definitions cited above with the exception of Definition 2.5 for well-chosen values of $\eta(.,.)$, s and m.

Now, we can state our results.

Theorem 3.3. Let $K \subseteq [0, b^*]$, $b^* > 0$ be an invex subset with respect to η, $a, b \in K^\circ$ (interior of K) with $\eta(b,a) > 0$. Let $f : K \to (0, \infty)$ be differentiable mapping on K such that $f' \in L([a, a + \eta(b,a)])$. If $|f'|$ is (s,m)-preinvex in the second sense on K for some fixed $s, m \in (0,1]$, then the following inequality:

$$\begin{aligned} &\left| \frac{f(a) + f(a + \eta(b,a))}{2} - \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)dx \right| \\ \leq &\frac{\eta(b,a)}{2} \frac{s + \left(\frac{1}{2}\right)^s}{(s+1)(s+2)} \left[|f'(a)| + m \left| f'(\frac{b}{m}) \right| \right], \end{aligned} \tag{11}$$

holds for all $x \in [a, a + \eta(b,a)]$.

Proof. From Lemma 2.10 and properties of modulus, we have

$$\begin{aligned} &\left| \frac{f(a) + f(a + \eta(b,a))}{2} - \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)dx \right| \\ \leq &\frac{\eta(b,a)}{2} \int_0^1 |1-2t| \, |f'(a + t\eta(b,a))| \, dt, \end{aligned} \tag{12}$$

since $|f'|$ is (s,m)-preinvex function in the second sense, we have

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2}-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)dx\right|$$

$$\leq \frac{\eta(b,a)}{2}\int_0^1 |1-2t|\left[(1-t)^s|f'(a)|+mt^s\left|f'(\frac{b}{m})\right|\right]dt$$

$$= \frac{\eta(b,a)}{2}\left(|f'(a)|\int_0^1 |1-2t|(1-t)^s dt\right.$$

$$\left.+m\left|f'(\frac{b}{m})\right|\int_0^1 |1-2t|t^s dt\right)$$

$$= \frac{\eta(b,a)}{2}\frac{s+\left(\frac{1}{2}\right)^s}{(s+1)(s+2)}\left(|f'(a)|+m\left|f'(\frac{b}{m})\right|\right), \quad (13)$$

where we use the facts that

$$\int_0^1 |1-2t|(1-t)^s dt=\int_0^1 |1-2t|t^s dt=\frac{s+\left(\frac{1}{2}\right)^s}{(s+1)(s+2)}. \quad (14)$$

The proof is completed. □

Remark 3.4. In Theorem 3.3, if we choose $s=1,(s,m)=(1,1)$ or $(\eta(b,a),s,m)=(b-a,1,1)$, then (11) reduce to (8), (6) and (2) respectively.

Theorem 3.5. Let $K\subseteq[0,b^*]$, $b^*>0$ be an invex subset with respect to η, $a,b\in K^\circ$ (interior of K) with $\eta(b,a)>0$. Let $f:K\to(0,\infty)$ be differentiable mapping on K such that $f'\in L([a,a+\eta(b,a)])$ and let $q>1$ with $\frac{1}{p}+\frac{1}{q}=1$. If $|f'|^q$ is (s,m)-preinvex in the second sense on K for some fixed $s,m\in(0,1]$, then the following inequality:

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2}-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)dx\right|$$

$$\leq \frac{\eta(b,a)}{2(p+1)^{\frac{1}{p}}(s+1)^{\frac{1}{q}}}\left(\left(|f'(a)|^q+m\left|f'(\frac{b}{m})\right|^q\right)\right)^{\frac{1}{q}}, \quad (15)$$

holds for all $x\in[a,a+\eta(b,a)]$.

Proof. From Lemma 2.10, properties of modulus, Hölder's inequality and (s,m)-preinvexity of $|f'|^q$, we obtain

$$\begin{aligned} &\left|\frac{f(a)+f(a+\eta(b,a))}{2}-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)dx\right| \\ &\le \frac{\eta(b,a)}{2}\left(\int_0^1 |1-2t|^p\,dt\right)^{\frac{1}{p}}\left(\int_0^1 |f'(a+t\eta(b,a))|^q\,dt\right)^{\frac{1}{q}} \\ &= \frac{\eta(b,a)}{2(\ p+1)^{\frac{1}{p}}}\left(\int_0^1 |f'(a+t\eta(b,a))|^q\,dt\right)^{\frac{1}{q}} \\ &\le \frac{\eta(b,a)}{2(\ p+1)^{\frac{1}{p}}}\left(|f'(a)|^q\int_0^1 (1-t)^s\,dt+m\left|f'(\frac{b}{m})\right|^q\int_0^1 t^s dt\right)^{\frac{1}{q}} \\ &= \frac{\eta(b,a)}{2(s+1)^{\frac{1}{q}}(\ p+1)^{\frac{1}{p}}}\left(|f'(a)|^q+m\left|f'(\frac{b}{m})\right|^q\right)^{\frac{1}{q}}, \end{aligned}$$

where we have use the facts that

$$\int_0^1 (1-t)^s\,dt=\int_0^1 t^s dt=\frac{1}{s+1},$$

and

$$\int_0^1 |1-2t|^p\,dt=\frac{1}{p+1}.$$

The proof is achieved. □

Remark 3.6. In Theorem 3.5, if we choose $s=1,(s,m)=(1,1)$ or $(\eta(b,a),s,m)=(b-a,1,1)$, then (15) reduce to (9), (7) and (3) respectively.

Theorem 3.7. Let $K\subseteq[0,b^*], b^*>0$ be an invex subset with respect to η, $a,b\in K^\circ$ (interior of K) with $\eta(b,a)>0$. Let $f:K\to(0,\infty)$ be differentiable mapping on K such that $f'\in L([a,a+\eta(b,a)])$ and let $q\ge 1$. If $|f'|^q$ is (s,m)-preinvex in the second sense on K for some fixed $s,m\in(0,1]$, then the following inequality:

$$\begin{aligned} &\left|\frac{f(a)+f(a+\eta(b,a))}{2}-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)dx\right| \\ &\le \frac{\eta(b,a)}{2^{\,2-\frac{1}{q}}}\left(\frac{s+\left(\frac{1}{2}\right)^s}{(s+1)(s+2)}\right)^{\frac{1}{q}}\left[|f'(a)|^q+m\left|f'(\frac{b}{m})\right|^q\right]^{\frac{1}{q}}, \end{aligned} \tag{16}$$

holds for all $x\in[a,a+\eta(b,a)]$.

Proof. From Lemma 2.10, properties of modulus and power mean inequality, we have

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2}-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}f(x)dx\right|$$

$$\leq \frac{\eta(b,a)}{2}\left(\int_0^1|1-2t|\,dt\right)^{1-\frac{1}{q}}\left(\int_0^1|1-2t|\,|f'(a+t\eta(b,a))|^q\,dt\right)^{\frac{1}{q}}$$

$$= \frac{\eta(b,a)}{2^{\,2-\frac{1}{q}}}\left(\int_0^1|1-2t|\,|f'(a+t\eta(b,a))|^q\,dt\right)^{\frac{1}{q}}, \tag{17}$$

since $|f'|^q$ is (s,m)-preinvex function in the second sense, we have

$$\int_0^1|1-2t|\,|f'(a+t\eta(b,a))|^q\,dt$$

$$\leq |f'(a)|^q\int_0^1|1-2t|\,(1-t)^s+m\left|f'(\frac{b}{m})\right|^q\int_0^1 t^s\,|1-2t|\,dt$$

$$= \frac{s+\left(\frac{1}{2}\right)^s}{(s+1)(s+2)}\left[|f'(a)|^q+m\left|f'(\frac{b}{m})\right|^q\right], \tag{18}$$

where we have use (14). Substituting (18) into (17), we get the desired inequality in (16). □

Remark 3.8. In Theorem 3.7, if we choose $s=1$, $(\eta(b,a),s,m)=(b-a,s,1)$ and $(\eta(b,a),s,m)=(b-a,1,1)$, then (16) reduce to (10), (5) and (4) respectively.

Theorem 3.9. Suppose that all the assumptions of Theorem 3.7 are satisfied. Then the following inequality:

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2}-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}f(x)dx\right|\leq\frac{\eta(b,a)}{2}$$

$$\times\left(\frac{1}{2^{s+1}}B(s+1,q+1)+\frac{{}_2F_1\left(1,-s;q+2;\frac{1}{2}\right)}{q+2}\right)^{\frac{1}{q}}\left(|f'(a)|^q+m\left|f'(\frac{b}{m})\right|^q\right)^{\frac{1}{q}}, \tag{19}$$

holds for all $x\in[a,a+\eta(b,a)]$ where ${}_2F_1(.,.;.;.)$ is the Hypergeometric function and $B(.,.)$ is the beta function.

Proof. From Lemma 2.10, properties of modulus and power mean inequality, we have

$$\begin{aligned}
&\left|\frac{f(a)+f(a+\eta(b,a))}{2}-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)dx\right| \\
\leq\ & \frac{\eta(b,a)}{2}\left(\int_0^1 dt\right)^{1-\frac{1}{q}}\left(\int_0^1 |1-2t|^q\,|f'(a+t\eta(b,a))|^q\,dt\right)^{\frac{1}{q}} \\
=\ & \frac{\eta(b,a)}{2}\left(\int_0^1 |1-2t|^q\,|f'(a+t\eta(b,a))|^q\,dt\right)^{\frac{1}{q}}, \qquad (20)
\end{aligned}$$

using (s,m)-preinvexity of $|f'|^q$, (20) gives

$$\begin{aligned}
&\left|\frac{f(a)+f(a+\eta(b,a))}{2}-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)dx\right| \\
\leq\ & \frac{\eta(b,a)}{2}\left(\int_0^1 |1-2t|^q\left((1-t)^s\,|f'(a)|^q+mt^s\left|f'(\frac{b}{m})\right|^q\right)dt\right)^{\frac{1}{q}} \\
=\ & \frac{\eta(b,a)}{2}\left(|f'(a)|^q\left(\int_0^{\frac{1}{2}}(1-t)^s(1-2t)^q\,dt+\int_{\frac{1}{2}}^1(1-t)^s(2t-1)^q\,dt\right)\right. \\
& \left.+m\left|f'(\frac{b}{m})\right|^q\left(\int_0^{\frac{1}{2}}t^s(1-2t)^q\,dt+\int_{\frac{1}{2}}^1 t^s(2t-1)^q\,dt\right)\right)^{\frac{1}{q}} \\
=\ & \frac{\eta(b,a)}{2}\left(|f'(a)|^q\left(\frac{1}{2}\int_0^1(1-t)^q\left(1-\frac{1}{2}t\right)^s dt+\frac{1}{2^{s+1}}\int_0^1 t^s(1-t)^q\,dt\right)\right. \\
& \left.+m\left|f'(\frac{b}{m})\right|^q\left(\frac{1}{2^{s+1}}\int_0^1 t^s(1-t)^q\,dt+\frac{1}{2}\int_0^1(1-t)^q\left(1-\frac{1}{2}t\right)^s dt\right)\right)^{\frac{1}{q}} \\
=\ & \frac{\eta(b,a)}{2}\left(\frac{1}{2^{s+1}}B(s+1,q+1)+\frac{{}_2F_1\left(1,-s;q+2;\frac{1}{2}\right)}{2q+4}\right)^{\frac{1}{q}} \\
& \times\left(|f'(a)|^q+m\left|f'(\frac{b}{m})\right|^q\right)^{\frac{1}{q}}.
\end{aligned}$$

Which is the desired result. □

References

[1] A. Barani, A. G. Ghazanfari and S. S. Dragomir, *Hermite-Hadamard inequality for functions whose derivatives absolute values are preinvex*, J. Inequal. Appl. 2012, 2012:247, 9 pp.

[2] A. Ben-Israel and B. Mond, *What is invexity?,* J. Austral. Math. Soc. Ser. B 28 (1986), no. 1, 1–9.

[3] W. W. Breckner, *Stetigkeitsaussagen für eine Klasse verallgemeinerter konvexer Funktionen in topologischen linearen Räumen*, (German) Publ. Inst. Math. (Beograd) (N.S.) 23(37) (1978), 13–20.

[4] S. S. Dragomir and N. M. Ionescu, *On some inequalities for convex-dominated functions*, Anal. Numer. Theor. Approx. 19 (1990), no. 1, 21–27.

[5] S. S. Dragomir and R. P. Agarwal, *Two inequalities for differentiable mappings and applications to special means of real numbers and to trapezoidal formula*, Appl. Math. Lett. 11 (1998), no. 5, 91–95.

[6] N. Eftekhari, *Some remarks on (s, m)-convexity in the second sense*, J. Math. Inequal. 8 (2014), no. 3, 489–495.

[7] M. A. Hanson, *On sufficiency of the Kuhn-Tucker conditions*, J. Math. Anal. Appl. 80 (1981), no. 2, 545–550.

[8] U. S. Kirmaci, M. K. Bakula, M. E. Özdemir and J. E. Pečarić, *Hadamard-type inequalities for s-convex functions,* Appl. Math. Comput. 193 (2007), no. 1, 26–35.

[9] M. A. Latif and M. Shoaib, *Hermite-Hadamard type integral inequalities for differentiable m-preinvex and (α, m)-preinvex functions,* J. Egyptian Math. Soc. 23 (2015), no. 2, 236–241.

[10] Y. L. Luke and Y. L. Luke, *The special functions and their approximations*, Vol. 1. New York: Academic press, 1969.

[11] M. A. Noor, *Variational-like inequalities*, Optimization 30 (1994), no. 4, 323–330.

[12] M. A. Noor, *Invex equilibrium problems*, J. Math. Anal. Appl. 302 (2005), no. 2, 463–475.

[13] M. A. Noor, K. I. Noor, M. U. Awan, J. Li, *On Hermite-Hadamard type inequalities for h-preinvex functions*, Filomat. 28(7), 1463-1474, (2014).

[14] M. A. Noor, K. I. Noor, M. U. Awan, S. Khan, *Hermite-Hadamard inequalities for s-Godunova-Levin preinvex functions*, Journal of Advanced Mathematical Studies. 7(2), 12-19, (2014).

[15] M. A. Noor, K. I. Noor, M. U. Awan, S. Khan, *Hermite-Hadamard inequalities for differentiable h_φ-preinvex functions*, Arabian Journal of Mathematics. 4, 63-76, (2015).

[16] M. A. Noor, K. I. Noor, M. U. Awan, F. Qi, *Hermite-Hadamard inequalities for logarithmic h-preinvex functions*, Cogent Mathematics. 2, (2015).

[17] C.E.M. Pearce and J. Pečarić, *Inequalities for differentiable mappings with application to special means and quadrature formulæ*, Appl. Math. Lett. 13 (2000), no. 2, 51–55.

[18] J. Pečarić, F. Proschan and Y. L. Tong, *Convex functions, partial orderings, and statistical applications*, Mathematics in Science and Engineering, 187. Academic Press, Inc., Boston, MA, 1992.

[19] R. Pini, *Invexity and generalized convexity*, Optimization 22 (1991), no. 4, 513–525.

[20] G. H. Toader, *Some generalizations of the convexity*, Proceedings of the colloquium on approximation and optimization (Cluj-Napoca, 1985), 329–338.

[21] Y. Wang, S-H. Wang and F. Qi, *Simpson type integral inequalities in which the power of the absolute value of the first derivative of the integrand is s-preinvex*, Facta Univ. Ser. Math. Inform. 28 (2013), no. 2, 151–159.

[22] T. Weir and B. Mond, *Pre-invex functions in multiple objective optimization*, J. Math. Anal. Appl. 136 (1988), no. 1, 29–38.

[23] X. -M. Yang and D. Li, *On properties of preinvex functions*, J. Math. Anal. Appl. 256 (2001), no. 1, 229–241.

http://www.newtheory.org ISSN: 2149-1402

Journal of New Theory

Received: 20.08.2015 *Year*: **2016,** *Number*: **10**, *Pages*: **66-75**
Published: 09.02.2016 *Original Article*[**]

RESULTS ON INTUITIONISTIC FUZZY SOFT MULTI SETS AND IT'S APPLICATION IN INFORMATION SYSTEM

Anjan Mukherjee[1,*] <anjan2002_m@yahoo.co.in>
Ajoy Kanti Das[2] <ajoykantidas@gmail.com>

[1]*Department of Mathematics, Tripura University, Agartala-799022, Tripura, India*
[2]*Department of Mathematics, Iswar Chandra Vidyasagar College, Belonia -799155, Tripura, India*

Abstract – In this paper, we review some operations in intuitionistic fuzzy soft multi set theory in different approach and show that the De Morgan's types of results hold in intuitionistic fuzzy soft multi set theory for these operations in our way. Basic supporting tools in information system are also defined. Application of intuitionistic fuzzy soft multi sets in information system are presented and discussed. Also we show that every intuitionistic fuzzy soft multi set is an intuitionistic fuzzy multi valued information system.

Keywords – *Soft sets, fuzzy soft multi sets, ıntuitionistic fuzzy soft multi sets, information system.*

1 Introduction

Most of the problems in engineering, computer science, medical science, economics, environments etc. have various uncertainties. In 1999, Molodstov [12] initiated the concept of soft set theory as a mathematical tool for dealing with uncertainties. Later on Maji et al.[11] presented some new definitions on soft sets such as subset, union, intersection and complements of soft sets and discussed in details the application of soft set in decision making problem. Based on the analysis of several operations on soft sets introduced in [12], Ali et al. [2] presented some new algebraic operations for soft sets and proved that certain De Morgan's law holds in soft set theory with respect to these new definitions. Combining soft sets [12] with fuzzy sets [15] and intuitionistic fuzzy sets [5], Maji et al. [[9], [10]] defined fuzzy soft sets and intuitionistic fuzzy soft sets, which are rich potential for solving decision making problems. Alkhazaleh and others [[1], [4], [6], [7], [14]] as a generalization of Molodtsov's soft set, presented the definition of a soft multi set and its basic operations such as complement, union, and intersection etc and thereafter Balami and others [[7], [8]] discussed the application of soft multiset and multi-soft set in information

[**]*Edited by* Oktay Muhtaroğlu *(Area Editor) and* Naim Çağman *(Editor-in-Chief).*
[*]*Anjan Mukherjee.*

system. In 2012 Alkhazaleh and Salleh [3] introduced the concept of fuzzy soft multi set theory and studied the application of these sets and recently, Mukherjee and Das [13] introduced the concepts of intuitionistic fuzzy soft multi sets and studied intuitionistic fuzzy soft multi topological spaces in details.

In this paper, we review some new operations in intuitionistic fuzzy soft multi set theory and applications of intuitionistic fuzzy soft multi sets in information system are presented and discussed. Also we show that every intuitionistic fuzzy soft multi set is an intuitionistic fuzzy multi valued information system.

2 Preliminary Notes

In this section, we recall some basic notions in soft set theory, fuzzy soft multi set theory and intuitionistic fuzzy soft multi set theory. Molodstov defined soft set in the following way. Let U be an initial universe and E be a set of parameters. Let P(U) denotes the power set of U and A⊆E.

Definition 2.1. [12] A pair (*F*, *A*) is called a soft set over U, where F is a mapping given by *F: A→ P(U).* In other words, soft set over U is a parameterized family of subsets of the universe U.

Definition 2.2. [3] Let $\{U_i : i \in I\}$ be a collection of universes such that $\bigcap_{i\in I} U_i = \phi$ and let $\{E_{U_i} : i \in I\}$ be a collection of sets of parameters. Let $U = \prod_{i\in I} FS(U_i)$ where $FS(U_i)$ denotes the set of all fuzzy subsets of U_i, $E = \prod_{i\in I} E_{U_i}$ and $A \subseteq E$. A pair (*F*, *A*) is called a fuzzy soft multi set over U, where F is a mapping given by $\mathrm{F: A \to U}$.

Definition 2.3. [3] For any fuzzy soft multi set (F, A), a pair $\left(e_{U_i,j}, F_{e_{U_i,j}}\right)$ is called a U_i – fuzzy soft multiset part $\forall e_{U_i,j} \in a_k$ and $F_{e_{U_i,j}} \subseteq F(A)$ is a fuzzy approximate value set, where $a_k \in A$, $\mathrm{k} \in \{1,2,3,..,\mathrm{m}\}$, $i \in \{1,2,3,..,\mathrm{n}\}$ and $j \in \{1,2,3,..,\mathrm{r}\}$.

Definition 2.4. [13] Let $\{U_i : i \in I\}$ be a collection of universes such that $\bigcap_{i\in I} U_i = \phi$ and let $\{E_i : i \in I\}$ be a collection of sets of parameters. Let $U = \prod_{i\in I} IFS(U_i)$ where $IFS(U_i)$ denotes the set of all intuitionistic fuzzy subsets of U_i, $E = \prod_{i\in I} E_{U_i}$ and $A \subseteq E$. A pair (*F*, *A*) is called an intuitionistic fuzzy soft multi set (briefly, IFSM-set) over U, where F is a mapping given by $F : A \to U.$

Definition 2.5. [13] The complement of an IFSM-set (F, A) over U is denoted by $(\mathrm{F, A})^{\mathrm{c}}$ and is defined by $(\mathrm{F, A})^{\mathrm{c}} = (\mathrm{F^c, A})$, where $\mathrm{F^c} : A \to U$ is a mapping given by $\mathrm{F^c}(\alpha) = c\left(\mathrm{F}(\alpha)\right), \ \forall \alpha \in A$, where c is an intuitionistic fuzzy complement.

Definition 2.6. [13] An IFSM-set (F, A) over U is called an absolute IFSM-set, denoted by $(\mathrm{F, A})_U$, if $\left(e_{U_i,j}, F_{e_{U_i,j}}\right) = U_i, \ \forall i.$

Definition 2.7. [13] A null IFSM-set $(\mathrm{F, A})_\phi$ over U is an IFSM-set in which all the IFSM-set parts equals ϕ.

Definition 2.8. [13] Union of two IFSM-sets (F, *A*) and (G, *B*) over *U* is an IFSM-set (H, D), where $D = A \cup B$ and $\forall e \in D$,

$$H(e) = \begin{cases} F(e), & if \ e \in A - B \\ G(e), & if \ e \in B - A \\ \bigcup\left(F(e), G(e)\right), & if \ e \in A \cap B, \end{cases}$$

where $\bigcup\left(F(e), G(e)\right) = F_{e_{U_i,j}} \tilde{\cup} G_{e_{U_i,j}} \ \forall i \in \{1,2,3,..,\mathrm{m}\}, \ j \in \{1,2,3,..,\mathrm{n}\}$ with $\tilde{\cup}$ as an intuitionistic fuzzy union and is written as $(\mathrm{F, A}) \tilde{\cup} (\mathrm{G, B}) = (\mathrm{H,D})$.

Definition 2.9. [13] Intersection of two IFSM-sets (F, *A*) and (G, *B*) over *U* is an IFSM-set (H, D) where $D = A \cap B$ and $\forall e \in D$,

$$H(e) = \begin{cases} F(e), & if \ e \in A - B \\ G(e), & if \ e \in B - A \\ \bigcap\left(F(e), G(e)\right), & if \ e \in A \cap B \end{cases}$$

where $\bigcap\left(F(e), G(e)\right) = F_{e_{U_i,j}} \tilde{\cap} G_{e_{U_i,j}}, \forall i \in \{1,2,3,..,\mathrm{n}\}, \ \ j \in \{1,2,3,..,\mathrm{n}\}$ with $\tilde{\cap}$ as an intuitionistic fuzzy intersection and is written as $(\mathrm{F, A}) \tilde{\cap} (\mathrm{G, B}) = (\mathrm{H,C})$.

3 Results on Intuitionistic Fuzzy Soft Multisets

In this section, we review some operation on IFSM-sets in different approach and show that the De Morgan's types of results hold in intuitionistic fuzzy soft multi set theory for these operations in our way.

Let $\{U_i : i \in I\}$ be a collection of universes such that $\bigcap_{i \in I} U_i = \phi$ and let $\{E_i : i \in I\}$ be a collection of sets of parameters. Let $U = \prod_{i \in I} IFS(U_i)$, where $IFS(U_i)$ denotes the set of all intuitionistic fuzzy subsets of U_i, $E = \prod_{i \in I} E_{U_i}$ and $A \subseteq E$.

Definition 3.1. A pair (*F, A*) is called an intuitionistic fuzzy soft multi set (brıefly, IFSM-set)over U, where F is a mapping given by $F: A \to U$, such that $\forall e \in A$,

$$F(e) = \left(\left\{ \frac{u}{\left(\mu_{F(e)}(u), \nu_{F(e)}(u)\right)} : u \in U_i \right\} : i \in I \right).$$

Definition 3.2. For any IFSM-set (F, A), a $U_i -$intuitionistic fuzzy soft multiset part (briefly, $U_i -$IFSMS-part) of (*F, A*) over U, is of the form

$$\left\{ \frac{u}{\left(\mu_{F(e)}(u), \nu_{F(e)}(u)\right)} : u \in U_i, \ e \in A \right\}.$$

Definition 3.3. The union of two IFSM-sets (F, A) and (G, B) over a common universe U is an IFSM-set (H, C), where $\text{C=A} \cup \text{B}$ and $\forall \text{e} \in \text{C}, \ \text{u} \in \text{U}$

$$\mu_{H(e)}(u) = \begin{cases} \mu_{F(e)}(u), & \text{if } e \in A\text{-}B \\ \mu_{G(e)}(u), & \text{if } e \in B\text{-}A \\ \max\left(\mu_{F(e)}(u), \mu_{G(e)}(u)\right), & \text{if } e \in A \cap B \end{cases}$$

and

$$\nu_{H(e)}(u) = \begin{cases} \nu_{F(e)}(u), & \text{if } e \in A\text{-}B \\ \nu_{G(e)}(u), & \text{if } e \in B\text{-}A \\ \min\left(\nu_{F(e)}(u), \nu_{G(e)}(u)\right), & \text{if } e \in A \cap B \end{cases}$$

We write $(F, \ A) \tilde{\cup} (G, \ B) = (H, \ C)$.

Definition 3.4. The intersection of two IFSM-sets $(\text{F, A})$and (G, B) over a common universe U is an IFSM-set (H, C), where $\text{C=A} \cup \text{B}$ and $\forall \text{e} \in \text{C}, \ \text{u} \in \text{U}$

$$\mu_{H(e)}(u) = \begin{cases} \mu_{F(e)}(u), & \text{if } e \in A\text{-}B \\ \mu_{G(e)}(u), & \text{if } e \in B\text{-}A \\ \min\left(\mu_{F(e)}(u), \mu_{G(e)}(u)\right), & \text{if } e \in A \cap B \end{cases}$$

and

$$\nu_{H(e)}(u) = \begin{cases} \nu_{F(e)}(u), & \text{if } e \in A\text{-}B \\ \nu_{G(e)}(u), & \text{if } e \in B\text{-}A \\ \max\left(\nu_{F(e)}(u), \nu_{G(e)}(u)\right), & \text{if } e \in A \cap B \end{cases}$$

We write $(F, \ A) \tilde{\cap} (G, \ B) = (H, \ C)$.

Definition 3.5. The complement of an IFSM-set *(F, A)* over U is denoted by $(F, A)^c$ and is defined by $(F, A)^c = (F^c, A)$, where

$$F^c(e)=\left(\left\{\frac{u}{\left(\nu_{F(e)}(u),\mu_{F(e)}(u)\right)}:u\in U_i\right\}:i\in I\right),\ \forall e\in A.$$

Proposition 3.6. For two IFSM-sets *(F, A)* and *(G, B)* over U, then we have

1. $\left((F,\ A)\tilde{\cup}(G,\ B)\right)^c \tilde{\subseteq} (F,\ A)^c\,\tilde{\cup}(G,\ B)^c$
2. $(F,\ A)^c\,\tilde{\cap}(G,\ B)^c \tilde{\subseteq} \left((F,\ A)\tilde{\cap}(G,\ B)\right)^c$

Proof. Let $(F,\ A)\tilde{\cup}(G,\ B)=(H,C),$ where $C=A\cup B$ and $\forall e\in C,$

$$\mu_{H(e)}(u)=\begin{cases}\mu_{F(e)}(u), & \text{if } e\in A\text{-}B\\ \mu_{G(e)}(u), & \text{if } e\in B\text{-}A\\ \max\left(\mu_{F(e)}(u),\mu_{G(e)}(u)\right), & \text{if } e\in A\cap B\end{cases}$$

and

$$\nu_{H(e)}(u)=\begin{cases}\nu_{F(e)}(u), & \text{if } e\in A\text{-}B\\ \nu_{G(e)}(u), & \text{if } e\in B\text{-}A\\ \min\left(\nu_{F(e)}(u),\nu_{G(e)}(u)\right), & \text{if } e\in A\cap B\end{cases}$$

Thus $\left((F,\ A)\tilde{\cup}(G,\ B)\right)^c=(H,C)^c=(H^c,C),$ where $C=A\cup B$ and $\forall e\in C,$

$$\mu_{H^c(e)}(u)=\nu_{H(e)}(u)=\begin{cases}\nu_{F(e)}(u), & \text{if } e\in A\text{-}B\\ \nu_{G(e)}(u), & \text{if } e\in B\text{-}A\\ \min\left(\nu_{F(e)}(u),\nu_{G(e)}(u)\right), & \text{if } e\in A\cap B\end{cases}$$

and

$$\nu_{H^c(e)}(u)=\mu_{H(e)}(u)=\begin{cases}\mu_{F(e)}(u), & \text{if } e\in A\text{-}B\\ \mu_{G(e)}(u), & \text{if } e\in B\text{-}A\\ \max\left(\mu_{F(e)}(u),\mu_{G(e)}(u)\right), & \text{if } e\in A\cap B\end{cases}$$

Again, $(F,\ A)^c\,\tilde{\cup}(G,\ B)^c=(F^c,\ A)\tilde{\cup}(G^c,\ B)=\left(K,D\right)$. Where $D=A\cup B$ and $\forall e\in D,$

$$\mu_{K(e)}(u)=\begin{cases}\nu_{F(e)}(u), & \text{if } e\in A\text{-}B\\ \nu_{G(e)}(u), & \text{if } e\in B\text{-}A\\ \max\left(\nu_{F(e)}(u),\nu_{G(e)}(u)\right), & \text{if } e\in A\cap B\end{cases}$$

i.e. $\mu_{H^c(e)}(u)\le\mu_{K(e)}(u)$

and

$$\nu_{K(e)}(u)=\begin{cases}\mu_{F(e)}(u), & \text{if } e\in A\text{-}B\\ \mu_{G(e)}(u), & \text{if } e\in B\text{-}A\\ \min\left(\mu_{F(e)}(u),\mu_{G(e)}(u)\right), & \text{if } e\in A\cap B\end{cases}$$

i.e. $\nu_{H^c(e)}(u)\ge\nu_{K(e)}(u)$.

We see that C=D and $\forall e\in C,\ H^c(e)\subseteq K(e)$. Thus $\left((F,\ A)\tilde{\cup}(G,\ B)\right)^c\tilde{\subseteq}(F,\ A)^c\tilde{\cup}(G,\ B)^c$.

The other can be proved similarly.

Proposition 3.7. If (F,A) and (G,A) are two FSM-sets in $FSMS_A(\mathrm{F},A)$, then we have the following

(i). $\left((F, A)\tilde{\cup}(G, A)\right)^c=(F, A)^c\tilde{\cap}(G, A)^c$

(ii). $\left((F, A)\tilde{\cap}(G, A)\right)^c=(F, A)^c\tilde{\cup}(G, A)^c$

Proof. (i). Let $(F,\ A)\tilde{\cup}(G,\ B)=(H,C)$, where $C=A\cup B$ and $\forall e\in C$,

$$\mu_{H(e)}(u)=\begin{cases}\mu_{F(e)}(u), & \text{if } e\in A\text{-}B\\ \mu_{G(e)}(u), & \text{if } e\in B\text{-}A\\ \max\left(\mu_{F(e)}(u),\mu_{G(e)}(u)\right), & \text{if } e\in A\cap B\end{cases}$$

$$\nu_{H(e)}(u)=\begin{cases}\nu_{F(e)}(u), & \text{if } e\in A\text{-}B\\ \nu_{G(e)}(u), & \text{if } e\in B\text{-}A\\ \min\left(\nu_{F(e)}(u),\nu_{G(e)}(u)\right), & \text{if } e\in A\cap B\end{cases}$$

Thus $\left((F,\ A)\tilde{\cup}(G,\ B)\right)^c=(H,C)^c=(H^c,C)$, where $C=A\cup B$ and $\forall e\in C$,

$$\mu_{H^c(e)}(\mathrm{u}) = \nu_{\mathrm{H(e)}}(\mathrm{u}) = \begin{cases} \nu_{\mathrm{F(e)}}(\mathrm{u}), & \text{if } \mathrm{e} \in \mathrm{A\text{-}B} \\ \nu_{\mathrm{G(e)}}(\mathrm{u}), & \text{if } \mathrm{e} \in \mathrm{B\text{-}A} \\ \min\left(\nu_{\mathrm{F(e)}}(\mathrm{u}), \nu_{\mathrm{G(e)}}(\mathrm{u})\right), & \text{if } \mathrm{e} \in \mathrm{A} \cap \mathrm{B} \end{cases}$$

$$\nu_{H^c(e)}(\mathrm{u}) = \mu_{\mathrm{H(e)}}(\mathrm{u}) = \begin{cases} \mu_{\mathrm{F(e)}}(\mathrm{u}), & \text{if } \mathrm{e} \in \mathrm{A\text{-}B} \\ \mu_{\mathrm{G(e)}}(\mathrm{u}), & \text{if } \mathrm{e} \in \mathrm{B\text{-}A} \\ \max\left(\mu_{\mathrm{F(e)}}(\mathrm{u}), \mu_{\mathrm{G(e)}}(\mathrm{u})\right), & \text{if } \mathrm{e} \in \mathrm{A} \cap \mathrm{B} \end{cases}$$

Also, let $(F, A)^c \tilde{\cap} (G, B)^c = (F^c, A) \tilde{\cap} (G^c, B) = (K, D)$. Where $D = A \cup B$ and $\forall e \in D$,

$$\mu_{K(e)}(\mathrm{u}) = \begin{cases} \nu_{\mathrm{F(e)}}(\mathrm{u}), & \text{if } \mathrm{e} \in \mathrm{A\text{-}B} \\ \nu_{\mathrm{G(e)}}(\mathrm{u}), & \text{if } \mathrm{e} \in \mathrm{B\text{-}A} \\ \min\left(\nu_{\mathrm{F(e)}}(\mathrm{u}), \nu_{\mathrm{G(e)}}(\mathrm{u})\right), & \text{if } \mathrm{e} \in \mathrm{A} \cap \mathrm{B} \end{cases}$$

$$= \mu_{H^c(e)}(\mathrm{u}),$$

and

$$\nu_{K(e)}(\mathrm{u}) = \begin{cases} \mu_{\mathrm{F(e)}}(\mathrm{u}), & \text{if } \mathrm{e} \in \mathrm{A\text{-}B} \\ \mu_{\mathrm{G(e)}}(\mathrm{u}), & \text{if } \mathrm{e} \in \mathrm{B\text{-}A} \\ \max\left(\mu_{\mathrm{F(e)}}(\mathrm{u}), \mu_{\mathrm{G(e)}}(\mathrm{u})\right), & \text{if } \mathrm{e} \in \mathrm{A} \cap \mathrm{B} \end{cases}$$

$$= \nu_{H^c(e)}(\mathrm{u}).$$

We see that C=D and $\forall e \in C,\ H^c(e) = K(e)$. Hence the result.

The other can be proved similarly.

Definition 3.8. A IFSM-set (F, A) over U is called an absolute IFSM-set, denoted by $(F, A)_U$, if $\forall e \in A,\ \mu_{F(e)}(u) = 1$ and $\nu_{F(e)}(u) = 0,\ \forall u \in U_i,\ \ i \in I.$

Definition 3.9. A null IFSM-set $(F, A)_U$ over U is an IFSM-set in which $\forall e \in A,\ \mu_{F(e)}(u) = 0$ and $\nu_{F(e)}(u) = 1,\ \forall u \in U_i,\ i \in I.$

Proposition 3.10. If (F, *A*) be any IFSM-set in over U, then

(*i*) $(F, A) \tilde{\cap} (G, B)_\phi = (G, B)_\phi,$

(ii) $(F,A)\tilde{\cap}(H,C)_U = (F,A)$,

(d) $(F,A)\tilde{\cup}(G,B)_\phi = (F,A)$,

(iv) $(F,A)\tilde{\cup}(H,C)_U = (H,C)_U$.

4 An Application of IFSM-set in information system

Definition 4.1. An intuitionistic fuzzy multi-valued information system is a quadruple $Inf_{system} = (X, A, f, V)$ where X is a non empty finite set of objects, A is a non empty finite set of attribute, $V = \underset{a\in A}{\cup} V_a$, where V is the domain (an intuitionistic fuzzy set,) set of attribute, which has multi value and $f : X \times A \to V$ is a total function such that $f(x,a) \in V_a$ for every $(x,a) \in X \times A$.

Proposition 4.2. If (F,A) is an IFSM-set over universe U, then (F,A) is an intuitionistic fuzzy multi-valued information system.

Proof. Let $\{U_i : i \in I\}$ be a collection of universes such that $\bigcap_{i\in I} U_i = \phi$ and let $\{E_i : i \in I\}$ be a collection of sets of parameters. Let $U = \prod_{i\in I} IFS(U_i)$ where $IFS(U_i)$ denotes the set of all intuitionistic fuzzy subsets of U_i, $E = \prod_{i\in I} E_{U_i}$ and $A \subseteq E$. Let (F,A) be an IFSM-set over U and $X = \underset{i\in I}{\cup} U_i$. We define a mapping *f* where $f : X \times A \to V$, defined as

$$f(x,a) = \frac{x}{\left(\mu_{F(a)}(x), \nu_{F(a)}(x)\right)}.$$

Hence $V = \underset{a\in A}{\cup} V_a$ where V_a is the set of all counts of in $F(a)$ and $\cup$ represent the classical set union. Then the intuitionistic fuzzy multi-valued information system (X, A, f, V) represents the IFSM-set (F,A).

Example 4.3. Let us consider there are two universes U_1 *and* U_2. Let $U_1 = \{h_1, h_2, h_3\}$ and $U_2 = \{c_1, c_2\}$. Let $\{E_{U_1}, E_{U_2}\}$ be a collection of sets of decision parameters related to the above universes, where

$E_{U_1} = \{e_{U_1,1} = \text{expensive},\ e_{U_1,2} = \text{wooden}\}$, $E_{U_2} = \{e_{U_2,1} = \text{sporty}, e_{U_2,2} = \text{cheap},\ e_{U_2,3} = \text{2010 model}\}$.

Let $U = \prod_{i=1}^{2} IFS(U_i)$, $E = \prod_{i=1}^{2} E_{U_i}$ and $A = \{e_1 = (e_{U_1,1}, e_{U_2,1}),\ e_2 = (e_{U_1,1}, e_{U_2,2}), e_3 = (e_{U_1,1}, e_{U_2,3})\}$.

Let

$$F(e_1)=\left(\left\{\frac{h_1}{(.2,.7)},\frac{h_2}{(.4,.5)},\frac{h_3}{(0,1)}\right\},\left\{\frac{c_1}{(.8,.1)},\frac{c_2}{(0,1)}\right\}\right),$$
$$F(e_2)=\left(\left\{\frac{h_1}{(0,1)},\frac{h_2}{(.7,.2)},\frac{h_3}{(1,0)}\right\},\left\{\frac{c_1}{(0,1)},\frac{c_2}{(.6,.3)}\right\}\right),$$
$$F(e_3)=\left(\left\{\frac{h_1}{(0,1)},\frac{h_2}{(.8,.1)},\frac{h_3}{(0,1)}\right\},\left\{\frac{c_1}{(0.5,0.3)},\frac{c_2}{(.6,.3)}\right\}\right)$$

Then the IFSM-set (F,A) defined above describes the conditions of some "house" and "car" in a state. Then the quadruple (X,A,f,V) corresponding to the IFSM-set given above is an intuitionistic fuzzy multi-valued information system.

Where $X=\bigcup_{i=1}^{2}U_i$ and A is the set of parameters in the IFSM-set and

$$V_{e_1}=\left\{\frac{h_1}{(.2,.7)},\frac{h_2}{(.4,.5)},\frac{h_3}{(0,1)},\frac{c_1}{(.8,.1)},\frac{c_2}{(0,1)}\right\},$$
$$V_{e_2}=\left\{\frac{h_1}{(0,1)},\frac{h_2}{(.7,.2)},\frac{h_3}{(1,0)},\frac{c_1}{(0,1)},\frac{c_2}{(.6,.3)}\right\},$$
$$V_{e_3}=\left\{\frac{h_1}{(0,1)},\frac{h_2}{(.8,.1)},\frac{h_3}{(0,1)},\frac{c_1}{(0.5,0.3)},\frac{c_2}{(.6,.3)}\right\}$$

For the pair (h_1,e_1) we have $f(h_1,e_1)$ = (0.2, 0.7), for (h_2,e_1), we have $f(h_2,e_1)=$ (0.4, 0.5). Continuing in this way we obtain the values of other pairs. Therefore, according to the result above, it is seen that IFSM-sets are intuitionistic fuzzy soft multi-valued information systems. Nevertheless, it is obvious that intuitionistic fuzzy soft multi-valued information systems are not necessarily IFSM-sets. We can construct an information table representing IFSM-set (F,A) defined above as in Table 1.

Table 1. The information table representing IFSM-set (F,A).

	e_1	e_2	e_3
h_1	(0.2,0.7)	(0,1)	(0,1)
h_2	(0.4,0.5)	(0.7,0.2)	(0.8,0.1)
h_3	(0,1)	(1,0)	(0,1)
c_1	(0,1)	(0,1)	(0.5,0.3)
c_2	(0,1)	(0.6,0.3)	(0.6,0.3)

5 Conclusion

In this paper, we have made an investigation on existing basic notions and results on IFSM-sets. Some new results have been stated in our work. Here we shall present the application of IFSM-set in information system and show that every IFSM-set is an intuitionistic fuzzy multi valued information system.

References

[1]. K. Alhazaymeh, N. Hassan, *Vague soft multiset theory*, Int. J. Pure and Applied Math., 93 (2014) 511-523.

[2]. M.I. Ali, F. Feng, X. Liu, W.K. Min, M. Shabir, *On some new operations in soft set theory* Comp. Math. Appl., 57 (2009) 1547-1553.

[3]. S. Alkhazaleh, A.R. Salleh, *Fuzzy soft multi sets theory*, Abstract and Applied Analysis, Vol. 2012, Article ID 350603, 20 pages, doi: 10.1155/2012/350603, Hindawi Publishing Corporation (2012).

[4]. S. Alkhazaleh, A.R. Salleh, N. Hassan, Soft multi sets theory, Appl. Math. Sci. 5 (2011) 3561– 3573.

[5]. K. Atanassov, *Intuitionistic fuzzy sets*, Fuzzy Sets and Systems, 20 (1986) 87-96.

[6]. K.V. Babitha, S.J. John, *On soft multi sets*, Ann. Fuzzy Math. Inform., 5 (2013) 35-44.

[7]. H.M. Balami, A.M. Ibrahim, *Soft multiset and its application in information system,* International Journal of scientific research and management, 1 (2013) 471-482.

[8]. T. Herawan, M.D. Mustafa, *On multi-soft set construction in information system*, 5th International Conference on Intelligent Computing, Lecture Notes in Computer Science. vol. 5755, Springer, Ulsan, Republic of Korea (2009).

[9]. P.K. Maji, R. Biswas, A.R. Roy, *Fuzzy soft sets*, J. Fuzzy Math. 9 (2001) 589-602.

[10]. P.K. Maji, R. Biswas, A.R. Roy, *Intuitionistic fuzzy soft sets* J. Fuzzy Math. 12 (2001) 677-692.

[11]. P.K. Maji, R. Biswas, A.R. Roy, *Soft set theory*, Comp. Math. Appl. 45 (2003) 555-562.

[12]. D. Molodtsov, *Soft set theory-first results*, Comp. Math. Appl. 37 (1999) 19-31.

[13]. A. Mukherjee, A.K. Das, *Parameterized topological space induced by an intuitionistic fuzzy soft multi topological space,* Ann. Pure and Applied Math. 7 (2014) 7-12.

[14]. D. Tokat, I. Osmanoglu, *Soft multi set and soft multi topology*, Nevsehir Universitesi Fen Bilimleri Enstitusu Dergisi Cilt. 2 (2011) 109-118.

[15]. L.A. Zadeh, *Fuzzy sets,* Inform. Control., 8 (1965) 338-353.

http://www.newtheory.org ISSN: 2149-1402

Journal of New Theory

Received: 16.08.2015 *Year*: **2016**, *Number*: **10**, *Pages*: **76-85**
Published: 09.02.2016 *Original Article*[**]

ON NEW INEQUALITIES OF HERMITE-HADAMARD TYPE FOR FUNCTIONS WHOSE FOURTH DERIVATIVE ABSOLUTE VALUES ARE QUASI-CONVEX WITH APPLICATIONS

Imran Abbas Baloch[1,*] <iabbasbaloch@gmail.com>
Basharat Rehman Ali[1] <basharatrwp@gmail.com>

[1]Abdus Salam School of Mathematical Sciences, GC University, Lahore 54600, Pakistan

Abstaract − We establish some new inequalities of Hermite-Hadamard type for functions whose fourth derivatives absolute values are quasi-convex. Further, we give new identity.Using this new identity, we establish similar inequalities for left-hand side of Hermite-Hadamard result.Also, we present applications to special means.

Keywords − *Hermite-Hadamard type inequalities, Quasi-convex function, Power mean inequality.*

1 Introduction

A function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is called convex function if $f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y)$ for all $x, y \in I$ and $\lambda \in [0,1]$. Geometrically, this means that if P, Q and R are three distinct points on graph of f with Q between P and R, then Q is on or below chord PR. There are many results associated with convex functions in the area of inequalities, but one of them is the classical Hermite-Hadamard inequalities:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2},$$

for all $a, b \in I$, with $a < b$.

Recently, numerous authors [1-7] developed and discussed Hermite-Hadamard's inequalities in terms of refinements, counter-parts, generalizations and new Hemitte-Hadamard's type inequalities.

The notion of quasi-convex function which is generalization of convex function is defined as:

**Edited by* Oktay Muhtaroğlu *(Area Editor) and* Naim Çağman *(Editor-in-Chief).*
* *Corresponding Author.*

Definition 1.1. A function $F : [a, b] \to \mathbb{R}$ is called quasi-convex on $[a, b]$, if

$$f(\lambda x + (1 - \lambda)y) \leq \max\{f(x), f(y)\}, \ \ \forall x, y \in [a, b].$$

Any convex function is quasi-convex but converse is not true in general(See for example [3]). D.A Ion [6] established inequalities of right hand side of Hermite-Hadamard's type inequality for functions whose derivatives in absolute values are quasi-convex functions. These inequalities appear in the following theorems:

Theorem 1.2. Let $f : I^\circ \subseteq \mathbb{R} \to \mathbb{R}$ be a function, differentiable on I° with $a, b \in I^\circ$ and $a < b$. If $|f'|$ is quasi-convex on $[a, b]$, then we have:

$$\left|\frac{f(a) + f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{(b-a)}{4}\max\{|f'(a)|, |f'(b)|\}.$$

Theorem 1.3. Let $f : I^\circ \subseteq \mathbb{R} \to \mathbb{R}$ be a function, differentiable on I° with $a, b \in I^\circ$ and $a < b$. If $|f'|^{\frac{p}{p-1}}$ is quasi-convex on $[a, b]$, then we have:

$$\left|\frac{f(a) + f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{(b-a)}{2(p+1)^{\frac{1}{p}}}(\max\{|f'(a)|^{\frac{p}{p-1}}, |f'(b)|^{\frac{p}{p-1}}\})^{\frac{p-1}{p}}.$$

Theorem 1.4. Let $f : I^\circ \subseteq \mathbb{R} \to \mathbb{R}$ be a function, twice differentiable on I° with $a, b \in I^\circ$ and $a < b$. If $|f''|$ is quasi-convex on $[a, b]$, then we have:

$$\left|\frac{f(a) + f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{(b-a)^2}{12}\max\{|f''(a)|, |f''(b)|\}.$$

In paper [8], S.Qaisar, S.Hussain, C. He established new refined inequalities of right hand side of Hermite-Hadamard result for the class of functions whose third derivatives at certain powers are quasi-convex functions as follow:

Theorem 1.5. Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be thrice differentiable mapping on I° such that $f''' \in L[a, b]$, where $a, b \in I$ with $a < b$. If $|f'''|$ is quasi-convex on $[a, b]$, then we have the following inequality:

$$\left|\frac{f(a) + f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx - \frac{b-a}{12}[f'(b) - f'(a)]\right| \leq \frac{(b-a)^3}{192}\max\{|f'''(a)|, |f'''(b)|\}.$$

Theorem 1.6. Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be three time differentiable mapping on I° such that $f''' \in L[a, b]$, where $a, b \in I$ with $a < b$. If $|f'''|^{\frac{p}{p-1}}$ is quasi-convex on $[a, b]$, and $p > 1$, then we have the following inequality:

$$\left|\frac{f(a) + f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx - \frac{b-a}{12}[f'(b) - f'(a)]\right|$$
$$\leq \frac{(b-a)^3}{96}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}(\max\{|f'''(a)|^q, |f'''(b)|^q\})^{\frac{1}{q}}.$$

Theorem 1.7. Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be thrice differentiable mapping on I° such that $f''' \in L[a, b]$, where $a, b \in I$ with $a < b$. If $|f'''|^q$ is quasi-convex on $[a, b]$, and $q \geq 1$, then we have following inequality:

$$\left|\frac{f(a) + f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx - \frac{b-a}{12}[f'(b) - f'(a)]\right|$$
$$\leq \frac{(b-a)^3}{192}(\max\{|f'''(a)|^q, |f'''(b)|^q\})^{\frac{1}{q}}.$$

In this paper, we establish new refined inequalities of the right hand side of Hermite-Hadamard result for the class of functions whose fourth derivative at certain powers are quasi-convex functions. Further, we establish new identity using which, we establish new refined inequalities of left hand side of Hermit-Hadamard result for the same class of functions considered earlier.

2 Main Results

For establishing new inequalities of right hand side of Hermite-Hadamard result for the functions whose fourth derivative at certain powers are quasi-convex, we need the following identity:

Lemma 2.1. Let Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be four times differentiable mapping on I° such that $f^{(iv)} \in L[a, b]$, where $a, b \in I$ with $a < b$, then

$$\frac{1}{b-a}\int_a^b f(x)dx + \frac{b-a}{12}[f'(b) - f'(a)] - \frac{f(a)+f(b)}{2}$$
$$= \frac{(b-a)^4}{24}\int_0^1 (\lambda(1-\lambda))^2 f^{(iv)}(a\lambda + (1-\lambda)b)d\lambda$$

Theorem 2.2. Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a four times differentiable mapping on I° such that $f^{(iv)} \in L[a, b]$, where $a, b \in I$ with $a < b$. If $|f^{iv}|$ is quasi-convex on $[a, b]$, then we have the following inequality:

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx - \frac{b-a}{12}[f'(b) - f'(a)]\right|$$
$$\leq \frac{(b-a)^4}{720}\max\{|f^{(iv)}(a)|, |f^{(iv)}(b)|\}. \tag{1}$$

Proof. Using Lemma 2.1 and quasi-convexity of $|f^{(iv)}|$, we get

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx - \frac{b-a}{12}[f'(b) - f'(a)]\right|$$
$$\leq \frac{(b-a)^4}{24}\int_0^1 (\lambda(1-\lambda))^2 \left|f^{(iv)}(a\lambda + (1-\lambda)b)\right| d\lambda$$
$$\leq \frac{(b-a)^4}{24}\max\{|f^{(iv)}(a)|, |f^{(iv)}(a)|\}\int_0^1 (\lambda(1-\lambda))^2 d\lambda$$
$$= \frac{(b-a)^4}{720}\max\{|f^{(iv)}(a)|, |f^{(iv)}(a)|\}$$

the proof is completed. □

Theorem 2.3. Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a four times differentiable mapping on I° such that $f^{(iv)} \in L[a, b]$, where $a, b \in I$ with $a < b$. If $|f^{(iv)}|^{\frac{p}{p-1}}$ is quasi-convex on $[a, b]$, and $p > 1$, then we have the following inequality:

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx - \frac{b-a}{12}[f'(b) - f'(a)]\right|$$

$$\leq \frac{(b-a)^4}{24}\beta^{\frac{1}{p}}(2p+1,2p+1)(\max\{|f^{(iv)}(a)|^q,|f^{(iv)}(b)|^q\})^{\frac{1}{q}}, \tag{2}$$

where $q = \frac{p}{p-1}$.

Proof. Using Lemma 2.1, Holder's inequality and quasi-convexity of $|f^{(iv)}|^{\frac{p}{p-1}}$, we get

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx - \frac{b-a}{12}[f'(b)-f'(a)]\right|$$
$$\leq \frac{(b-a)^4}{24}\int_0^1 (\lambda(1-\lambda))^2 \big|f^{(iv)}(a\lambda+(1-\lambda)b)\big|d\lambda$$
$$\leq \frac{(b-a)^4}{24}\left(\int_0^1 (\lambda(1-\lambda))^{2p}d\lambda\right)^{\frac{1}{p}}\left(\int_0^1 |f^{(iv)}(a\lambda+(1-\lambda)b)|^q\right)^{\frac{1}{q}}$$
$$\leq \frac{(b-a)^4}{24}\left(\int_0^1 (\lambda(1-\lambda))^{2p}d\lambda\right)^{\frac{1}{p}}(\max\{|f^{(iv)}(a)|^q,|f^{(iv)}(b)|^q\})^{\frac{1}{q}}$$

It is easy to note that

$$\beta(2p+1,2p+1) = \int_0^1 (\lambda(1-\lambda))^{2p}d\lambda$$

which completes the proof. □

Theorem 2.4. Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a four times differentiable mapping on I° such that $f^{(iv)} \in L[a,b]$, where $a, b \in I$ with $a < b$. If $|f^{iv}|^q$ is quasi-convex on $[a,b]$, and $q \geq 1$, then we have following inequality:

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx - \frac{b-a}{12}[f'(b)-f'(a)]\right|$$
$$\leq \frac{(b-a)^4}{720}(\max\{|f^{(iv)}(a)|^q,|f^{((iv))}(b)|^q\})^{\frac{1}{q}}. \tag{3}$$

Proof. Using Lemma 2.1, power mean inequality and quasi-convexity of $|f^{(iv)}|^q$, we get

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx - \frac{b-a}{12}[f'(b)-f'(a)]\right|$$
$$\leq \frac{(b-a)^4}{24}\int_0^1 (\lambda(1-\lambda))^2 \big|f^{(iv)}(a\lambda+(1-\lambda)b)\big|d\lambda$$
$$\leq \frac{(b-a)^4}{24}\left(\int_0^1 (\lambda(1-\lambda))^2 d\lambda\right)^{1-\frac{1}{q}}\left(\int_0^1 (\lambda(1-\lambda))^2\big|f^{(iv)}(a\lambda+(1-\lambda)b)|^q\right)^{\frac{1}{q}}$$
$$\leq \frac{(b-a)^4}{24}\left(\frac{1}{30}\right)^{1-\frac{1}{q}}\left(\frac{1}{30}\max\{|f^{(iv)}(a)|^q,|f^{(iv)}(b)|^q\}\right)^{\frac{1}{q}}$$
$$= \frac{(b-a)^4}{720}(\max\{|f^{(iv)}(a)|^q,|f^{(iv)}(b)|^q\})^{\frac{1}{q}}.$$

which completes the proof. □

Now, to develop new refined inequalities of left hand side of Hermite-Hadamard result for the class of functions whose third derivatives at certain powers are quasi-convex, we need the following identity:

Lemma 2.5. Let Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be three times differentiable mapping on I° such that $f^{(''')} \in L[a, b]$, where $a, b \in I$ with $a < b$, then

$$f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)dx + \frac{b-a}{24}[f'(b) - f'(a)]$$

$$= \frac{(b-a)^3}{24}\left[\int_0^{\frac{1}{2}} \lambda(1-2\lambda)(1+2\lambda)f'''(\lambda a + (1-\lambda)b)d\lambda - \int_0^{\frac{1}{2}} \lambda(1-2\lambda)(1+2\lambda)f'''(\lambda b + (1-\lambda)a)d\lambda\right]$$

Proof. Integrating by parts, we have

$$\int_0^{\frac{1}{2}} \lambda(1-2\lambda)(1+2\lambda)f'''(\lambda a + (1-\lambda)b)d\lambda$$

$$= \frac{1}{b-a}\int_0^{\frac{1}{2}} (1-12\lambda^2)f''(\lambda a + (1-\lambda)b)d\lambda$$

$$= \frac{2}{(b-a)^2}f'\left(\frac{a+b}{2}\right) + \frac{f'(b)}{(b-a)^2} - \frac{24}{(b-a)^2}\int_0^{\frac{1}{2}} \lambda f'(\lambda a + (1-\lambda)b)d\lambda$$

$$= \frac{2}{(b-a)^2}f'\left(\frac{a+b}{2}\right) + \frac{f'(b)}{(b-a)^2} + \frac{12}{(b-a)^3}f\left(\frac{a+b}{2}\right) + \frac{24}{(a-b)^4}\int_b^{\frac{a+b}{2}} f(x)dx$$

and

$$\int_0^{\frac{1}{2}} \lambda(1-2\lambda)(1+2\lambda)f'''(\lambda b + (1-\lambda)a)d\lambda$$

$$= \frac{-1}{b-a}\int_0^{\frac{1}{2}} (1-12\lambda^2)f''(\lambda b + (1-\lambda)a)d\lambda$$

$$= \frac{2}{(b-a)^2}f'\left(\frac{a+b}{2}\right) + \frac{f'(a)}{(b-a)^2} - \frac{24}{(b-a)^2}\int_0^{\frac{1}{2}} \lambda f'(\lambda b + (1-\lambda)a)d\lambda$$

$$= \frac{24}{(b-a)^2}f'\left(\frac{a+b}{2}\right) + \frac{f'(a)}{(b-a)^2} - \frac{12}{(b-a)^3}f\left(\frac{a+b}{2}\right) + \frac{2}{(b-a)^4}\int_a^{\frac{a+b}{2}} f(x)dx$$

this ends the proof. □

Theorem 2.6. Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a three time differentiable mapping on I° such that $f''' \in L[a, b]$, where $a, b \in I$ with $a < b$. If $|f'''|$ is quasi-convex on $[a, b]$, then we have following inequality:

$$\left|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)dx + \frac{b-a}{24}[f'(b) - f'(a)]\right|$$

$$\leq \frac{(b-a)^3}{192}\left(\max\{|f'''(a)|, |f'''(b)|\}\right). \quad (4)$$

Proof. Using Lemma 2.5 and quasi-convexity of $|f'''|$, we get

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx+\frac{b-a}{24}[f'(b)-f'(a)]\right|$$

$$\leq \frac{(b-a)^3}{24}\Bigg[\int_0^{\frac{1}{2}} \lambda(1-2\lambda)(1+2\lambda)|f'''(\lambda a+(1-\lambda)b)|d\lambda$$

$$+\int_0^{\frac{1}{2}} \lambda(1-2\lambda)(1+2\lambda)|f'''(\lambda b+(1-\lambda)a)|d\lambda\Bigg]$$

$$\leq \frac{(b-a)^3}{24}\Big(\max\{|f'''(a)|,|f'''(b)|\}\Big)$$

$$\times\Bigg[\int_0^{\frac{1}{2}} \lambda(1-2\lambda)(1+2\lambda)d\lambda+\int_0^{\frac{1}{2}} \lambda(1-2\lambda)(1+2\lambda)d\lambda\Bigg]$$

$$= \frac{(b-a)^3}{192}\Big(\max\{|f'''(a)|,|f'''(b)|\}\Big)$$

this complete the proof. □

Theorem 2.7. Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a three time differentiable mapping on I° such that $f''' \in L[a,b]$, where $a,b \in I$ with $a<b$. If $|f'''|^{\frac{p}{p-1}}$ is quasi-convex on $[a,b]$, and $p>1$, then we have following inequality:

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx+\frac{b-a}{24}[f'(b)-f'(a)]\right|$$

$$\leq \frac{(b-a)^3}{96}\Big(\frac{1}{p+1}\Big)^{\frac{1}{p}}\Big(\max\{|f'''(a)|^q,|f'''(b)|^q\}\Big)^{\frac{1}{q}}, \quad (5)$$

where $q=\frac{p}{p-1}$.

Proof. Using Lemma 2.5, Holder's inequality and quasi-convexity of $|f'''|^{\frac{p}{p-1}}$, we get

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx+\frac{b-a}{24}[f'(b)-f'(a)]\right|$$

$$\leq \frac{(b-a)^3}{24}\Bigg[\int_0^{\frac{1}{2}} \lambda(1-2\lambda)(1+2\lambda)|f'''(\lambda a+(1-\lambda)b)|d\lambda$$

$$+\int_0^{\frac{1}{2}} \lambda(1-2\lambda)(1+2\lambda)|f'''(\lambda b+(1-\lambda)a)|d\lambda\Bigg]$$

$$\leq \frac{(b-a)^3}{24}\Bigg[\left(\int_0^{\frac{1}{2}} \lambda(1-2\lambda)^p(1+2\lambda)^p d\lambda\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}} \lambda|f'''(\lambda a+(1-\lambda)b)|^q d\lambda\right)^{\frac{1}{q}}$$

$$+\left(\int_0^{\frac{1}{2}} \lambda(1-2\lambda)^p(1+2\lambda)^p d\lambda\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}} \lambda|f'''(\lambda b+(1-\lambda)a)|^q d\lambda\right)^{\frac{1}{q}}\Bigg]$$

$$\leq \frac{(b-a)^3}{24}\Big(\max\{|f'''(a)|^q,|f'''(b)|^q\}\Big)^{\frac{1}{q}}\Bigg[\Big(\int_0^{\frac{1}{2}}\lambda(1-2\lambda)^p(1+2\lambda)^p d\lambda\Big)^{\frac{1}{p}}\Big(\int_0^{\frac{1}{2}}\lambda d\lambda\Big)^{\frac{1}{q}}$$

$$+\Big(\int_0^{\frac{1}{2}}\lambda(1-2\lambda)^p(1+2\lambda)^p d\lambda\Big)^{\frac{1}{p}}\Big(\int_0^{\frac{1}{2}}\lambda d\lambda\Big)^{\frac{1}{q}}\Bigg]$$

$$=\frac{(b-a)^3}{96}\Big(\frac{1}{p+1}\Big)^{\frac{1}{p}}\Big(\max\{|f'''(a)|^q,|f'''(b)|^q\}\Big)^{\frac{1}{q}}$$

□

Theorem 2.8. Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a three time differentiable mapping on I° such that $f''' \in L[a,b]$, where $a, b \in I$ with $a < b$. If $|f'''|^q$ is quasi-convex on $[a,b]$, and $q \geq 1$, then we have following inequality:

$$\left|f\Big(\frac{a+b}{2}\Big)-\frac{1}{b-a}\int_a^b f(x)dx+\frac{b-a}{24}[f'(b)-f'(a)]\right|$$

$$\leq \frac{(b-a)^3}{192}\big(\max\{|f'''(a)|^q,|f'''(b)|^q\}\big)^{\frac{1}{q}}. \tag{6}$$

Proof. Using Lemma 2.5, power mean inequality and quasi-convexity of $|f'''|^q$, we get

$$\left|f\Big(\frac{a+b}{2}\Big)-\frac{1}{b-a}\int_a^b f(x)dx+\frac{b-a}{24}[f'(b)-f'(a)]\right|$$

$$\leq \frac{(b-a)^3}{24}\Bigg[\int_0^{\frac{1}{2}}\lambda(1-2\lambda)(1+2\lambda)|f'''(\lambda a+(1-\lambda)b)|d\lambda$$

$$+\int_0^{\frac{1}{2}}\lambda(1-2\lambda)(1+2\lambda)|f'''(\lambda b+(1-\lambda)a)|d\lambda\Bigg]$$

$$\leq \frac{(b-a)^3}{24}\Bigg[\Big(\int_0^{\frac{1}{2}}\lambda(1-2\lambda)(1+2\lambda)d\lambda\Big)^{1-\frac{1}{q}}\Big(\int_0^{\frac{1}{2}}\lambda(1-2\lambda)(1+2\lambda)|f'''(\lambda a+(1-\lambda)b)|^q d\lambda|\Big)^{\frac{1}{q}}$$

$$+\Big(\int_0^{\frac{1}{2}}\lambda(1-2\lambda)(1+2\lambda)d\lambda\Big)^{1-\frac{1}{q}}\Big(\int_0^{\frac{1}{2}}\lambda(1-2\lambda)(1+2\lambda)|f'''(\lambda b+(1-\lambda)a)|^q d\lambda\Big)^{\frac{1}{q}}\Bigg]$$

$$\leq \frac{(b-a)^3}{24}\Big(\max\{|f'''(a)|^q,|f'''(b)|^q\}\Big)^{\frac{1}{q}}$$

$$\times\Bigg[\Big(\int_0^{\frac{1}{2}}\lambda(1-2\lambda)(1+2\lambda)d\lambda\Big)^{1-\frac{1}{q}}\Big(\int_0^{\frac{1}{2}}\lambda(1-2\lambda)(1+2\lambda)d\lambda|\Big)^{\frac{1}{q}}$$

$$+\Big(\int_0^{\frac{1}{2}}\lambda(1-2\lambda)(1+2\lambda)d\lambda\Big)^{1-\frac{1}{q}}\Big(\int_0^{\frac{1}{2}}\lambda(1-2\lambda)(1+2\lambda)d\lambda\Big)^{\frac{1}{q}}\Bigg]$$

$$=\frac{(b-a)^3}{192}\Big(\max\{|f'''(a)|^q,|f'''(b)|^q\}\Big)^{\frac{1}{q}}$$

the proof is so completed. □

3 Application to Some Special Means

We now consider the application of our theorem to the special means.
For positive numbers $a > 0$ and $b > 0$, define $A(a,b) = \frac{a+b}{2}$ and

$$L_p(a,b) = \begin{cases} \left[\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right], & p \neq -1, 0 \\ \frac{b-a}{\ln b - \ln a}, & p = -1 \\ \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}}, & p = 0 \end{cases}$$

We know that A and L_p respectively are called the arithmetic and generalized logarithmic means of two positive numbers a and b. By applying Hermite-Hadamard type inequalities established in Section 2, we are in a position to construct some inequalities for special means A and L_P. Consider the following function:

$$f(x) = \frac{x^{\alpha+4}}{(\alpha+1)(\alpha+2)(\alpha+3)(\alpha+4)} \tag{7}$$

for $0 < \alpha \leq 1$ and $x > 0$. Since $f^{(iv)}(x) = x^\alpha$ and $(\lambda x + (1-\lambda)y)^\alpha \leq \lambda^\alpha x^\alpha + (1-\lambda)^\alpha y^\alpha$ for all $x, y > 0$ and $\lambda \in [0,1]$, then $f^{(iv)}(x) = x^\alpha$ is α-convex function on $\mathbb{R}^+$ and

$$\frac{f(a)+f(b)}{2} = \frac{1}{(\alpha+1)(\alpha+2)(\alpha+3)(\alpha+4)} A(a^{\alpha+4}, b^{\alpha+4}),$$

$$\frac{1}{b-a}\int_a^b f(x)dx = \frac{1}{(\alpha+1)(\alpha+2)(\alpha+3)(\alpha+4)} L_{\alpha+4}(a,b),$$

$$f'(b) - f'(a) = \frac{b-a}{(\alpha+1)(\alpha+2)} L_{\alpha+2}(a,b)$$

Theorem 3.1. For positive numbers a and b such that $b > a$ and $0 < \alpha \leq 1$, we have

$$\left|12A(a^{\alpha+4}, b^{\alpha+4}) - 12L_{\alpha+4}(a,b) - (b-a)^2(\alpha+3)(\alpha+4)(\alpha+4)L_{\alpha+2}(a,b)\right|$$

$$\leq \frac{(b-a)^4}{60}(\alpha+1)(\alpha+2)(\alpha+3)(\alpha+4)\max\{|a^\alpha|, |b^\alpha|\}$$

Proof. The assertion follows from inequality (1) applied to mapping (7). □

Theorem 3.2. For positive numbers a and b such that $b > a$ and $0 < \alpha \leq 1$, we have

$$\left|12A(a^{\alpha+4}, b^{\alpha+4}) - 12L_{\alpha+4}(a,b) - (b-a)^2(\alpha+3)(\alpha+4)(\alpha+4)L_{\alpha+2}(a,b)\right|$$

$$\leq \frac{(b-a)^4}{2}\beta^{\frac{1}{p}}(2p+1, 2p+1)(\alpha+1)(\alpha+2)(\alpha+3)(\alpha+4)(\max\{|a^\alpha|^q, |b^\alpha|^q\})^{\frac{1}{q}}.$$

Proof. The assertion follows from inequality (2) applied to the mapping (7). □

Theorem 3.3. For positive numbers a and b such that $b > a$ and $0 < \alpha \leq 1$, we have

$$\left|12A(a^{\alpha+4}, b^{\alpha+4}) - 12L_{\alpha+4}(a,b) - (b-a)^2(\alpha+3)(\alpha+4)(\alpha+4)L_{\alpha+2}(a,b)\right|$$

$$\leq \frac{(b-a)^4}{60}(\alpha+1)(\alpha+2)(\alpha+3)(\alpha+4)(\max\{|a^\alpha|^q, |b^\alpha|^q\})^{\frac{1}{q}}.$$

Proof. The assertion follows from inequality (3) applied to the mapping (7). □

Theorem 3.4. For positive numbers a and b such that $b > a$ and $0 < \alpha \leq 1$, we have

$$\left|12A^{\alpha+4}(a^{\alpha+4}, b^{\alpha+4}) - 12L_{\alpha+4}(a,b) - (b-a)^2(\alpha+3)(\alpha+4)(\alpha+4)L_{\alpha+2}(a,b)\right|$$

$$\leq \frac{(b-a)^3}{16}(\alpha+1)(\alpha+2)(\alpha+3)(\alpha+4)(\max\{|a^\alpha|, |b^\alpha|\}).$$

Proof. The assertion follows from inequality (4) applied to the mapping (7). □

Theorem 3.5. For positive numbers a and b such that $b > a$ and $0 < \alpha \leq 1$, we have

$$\left|12A^{\alpha+4}(a^{\alpha+4}, b^{\alpha+4}) - 12L_{\alpha+4}(a,b) - (b-a)^2(\alpha+3)(\alpha+4)(\alpha+4)L_{\alpha+2}(a,b)\right|$$

$$\leq \frac{(b-a)^3}{8}(\frac{1}{p+1})^{\frac{1}{p}}(\alpha+1)(\alpha+2)(\alpha+3)(\alpha+4)\big(\max\{|a^\alpha|^q, |b^\alpha|^q\}\big)^{\frac{1}{q}}$$

Proof. The assertion follows from inequality (5) applied to the mapping (7). □

Theorem 3.6. For positive numbers a and b such that $b > a$ and $0 < \alpha \leq 1$, we have

$$\left|12A^{\alpha+4}(a^{\alpha+4}, b^{\alpha+4}) - 12L_{\alpha+4}(a,b) - (b-a)^2(\alpha+3)(\alpha+4)(\alpha+4)L_{\alpha+2}(a,b)\right|$$

$$\leq \frac{(b-a)^3}{16}(\alpha+1)(\alpha+2)(\alpha+3)(\alpha+4)\big(\max\{|a^\alpha|^q, |b^\alpha|^q\}\big)^{\frac{1}{q}}.$$

Proof. The assertion follows from inequality (6) applied to the mapping (7). □

Acknowledgement

The authors are grateful for financial support from Higher Education Commission of Pakistan.

References

[1] M. Alomari, M. Darus, U.S. Kirmaci, Refinements of Hadamard-type inequalities for quasi-convex functions with applications to trapezoidal formula and to special mean, Comp.Math.Appl.59(2010)225-232.

[2] M. Alomari, M. Darus, Some Ostrowski type inequalities for quasi-convex functions with applications to special means, RGMIA 13(2) (2010). Article No. 3 Preprint.

[3] M. Alomari, M. Darus,On Hermite-Hadamard type inequalities for log-convex functions on coordinates,J. Ineq. Appl. Volume 2009, Article ID 283147, 13pp. doi:http//dx.doi.org/10.1155/2009/283147.

[4] S.S. Dragomir,Two mappings in connection to Hadamard's inequalities, J. Math. Anal. Appl. 167(1992) 49-56.

[5] S.S. Dragomir,Y.J.Cho, S.S. Kim,Inequalities of Hadamard's type for Lipschitzian mappings and their applications, J. Math. Anal. Appl. 245(2000)289-501.

[6] D.A. Ion, Some estimates on the Hermite-Hadamard inequalty through quasi-convex functions, Ann. Univ. Craiova Math. Com. Sci. Ser. 34(2007) 82-87.

[7] C.E.M. Peearce, J. Pecaric, Inequalities for differentiable mappings with application to special means and quadrature formula, Appl. Math. Lett. 13 (2000) 51-55.

[8] S. Qaisar, S. Hussain, C. He, On new inequalities of Hermite-Hadamard type for functions whose third derivative absolute values are quasi-convex with applications, Journal of the Egyptian Mathematical Sociaty 22 (2014) 19-22.

http://www.newtheory.org ISSN: 2149-1402

Journal of New Theory

Received: 23.01.2016 *Year*: **2016**, *Number*: **10**, *Pages*: **86-101**
Published: 09.02.2016 *Original Article*[**]

SINGLE VALUED NEUTROSOPHIC GRAPHS

Said Broumi [1,*] <broumisaid78@gmail.com>
Mohamed Talea[1] <taleamohamed@yahoo.fr>
Assia Bakali [2] <assiabakali@yahoo.fr>
Florentin Smarandache[3] <fsmarandache@gmail.com>

[1]*Laboratory of Information Processing, Faculty of Science Ben M'Sik, University Hassan II, B.P 7955, Sidi Othman, Casablanca, Morocco.*
[2]*Ecole Royale Navale-Boulevard Sour Jdid, B.P 16303 Casablanca-Morocco,*
[3]*Department of Mathematics, University of New Mexico, 705 Gurley Avenue, Gallup, NM 87301, USA*

Abstract - The notion of single valued neutrosophic sets is a generalization of fuzzy sets, intuitionistic fuzzy sets. We apply the concept of single valued neutrosophic sets, an instance of neutrosophic sets, to graphs. We introduce certain types of single valued neutrosophic graphs (SVNG) and investigate some of their properties with proofs and examples.

Keywords - *Single valued neutrosophic set, single valued neutrosophic graph, strong single valued neutrosophic graph, constant single valued neutrosophic graph, complete single valued neutrosophic graph.*

1. Introduction

Neutrosophic sets (NSs) proposed by Smarandache [12, 13] is a powerful mathematical tool for dealing with incomplete, indeterminate and inconsistent information in real world. they are a generalization of the theory of fuzzy sets [24], intuitionistic fuzzy sets [21, 23] and interval valued intuitionistic fuzzy sets [22]. The neutrosophic sets are characterized by a truth-membership function (t), an indeterminacy-membership function (i) and a falsity-membership function (f) independently, which are within the real standard or nonstandard unit interval $]^{-}0, 1^{+}[$. In order to practice NS in real life applications conveniently, Wang et al.[16] introduced the concept of a single-valued neutrosophic sets (SVNS), a subclass of the neutrosophic sets. The SVNS is a generalization of intuitionistic fuzzy sets, in which three membership functions are independent and their value belong to the unit interval [0, 1]. Some more work on single valued neutrosophic sets and their extensions may be found on [2, 3, 4, 5,15, 17, 19, 20, 27, 28, 29, 30].

[**]*Edited by* Irfan Deli *(Area Editor) and* Naim Çağman *(Editor-in-Chief).*

Graph theory has now become a major branch of applied mathematics and it is generally regarded as a branch of combinatorics. Graph is a widely used tool for solving combinatorial problems in different areas such as geometry, algebra, number theory, topology, optimization and computer science. Most important thing which is to be noted is that, when we have uncertainty regarding either the set of vertices or edges or both, the model becomes a fuzzy graph. Lots of works on fuzzy graphs and intuitionistic fuzzy graphs [6, 7, 8, 25, 27] have been carried out and all of them have considered the vertex sets and edge sets as fuzzy and /or intuitionistic fuzzy sets. But, when the relations between nodes(or vertices) in problems are indeterminate, the fuzzy graphs and intuitionistic fuzzy graphs are failed. For this purpose, Samarandache [9, 10, 11, 14, 34] have defined four main categories of neutrosophic graphs, two based on literal indeterminacy (I), which called them; I-edge neutrosophic graph and I-vertex neutrosophic graph, these concepts are studied deeply and has gained popularity among the researchers due to its applications via real world problems [1, 33, 35]. The two others graphs are based on (t, i, f) components and called them; The (t, i, f)-Edge neutrosophic graph and the (t, i, f)-vertex neutrosophic graph, these concepts are not developed at all. In the literature the study of single valued neutrosophic graphs (SVN-graph) is still blank, we shall focus on the study of single valued neutrosophic graphs in this paper.

In this paper, some certain types of single valued neutrosophic graphs are developed and some interesting properties are explored.

2. Preliminaries

In this section, we mainly recall some notions related to neutrosophic sets, single valued neutrosophic sets, fuzzy graph and intuitionistic fuzzy graph relevant to the present work. See especially [6, 7, 12, 13, 16] for further details and background.

Definition 2.1 [12]. Let X be a space of points (objects) with generic elements in X denoted by x; then the neutrosophic set A (NS A) is an object having the form

$$A = \{< x: T_A(x), I_A(x), F_A(x)>, x \in X\}$$

where the functions T, I, F: $X \to]^{-}0,1^{+}[$ define respectively the a truth-membership function, an indeterminacy-membership function, and a falsity-membership function of the element $x \in X$ to the set A with the condition

$$^{-}0 \leq T_A(x)+ I_A(x)+ F_A(x) \leq 3^{+} \quad (1)$$

The functions $T_A(x)$, $I_A(x)$ and $F_A(x)$ are real standard or nonstandard subsets of $]^{-}0,1^{+}[$.

Since it is difficult to apply NSs to practical problems, Wang et al. [16] introduced the concept of a SVNS, which is an instance of a NS and can be used in real scientific and engineering applications.

Definition 2.2 [16]. Let X be a space of points (objects) with generic elements in X denoted by x. A single valued neutrosophic set A (SVNS A) is characterized by truth-membership function $T_A(x)$, an indeterminacy-membership function $I_A(x)$, and a falsity-membership function $F_A(x)$. For each point x in X $T_A(x), I_A(x), F_A(x) \in [0, 1]$. A SVNS A can be written as

$$A = \{< x: T_A(x), I_A(x), F_A(x)>, x \in X\} \tag{2}$$

Definition 2.3 [6]. A fuzzy graph is a pair of functions G = (σ, μ) where σ is a fuzzy subset of a non empty set V and μ is a symmetric fuzzy relation on σ. i.e σ : V → [0,1] and μ: VxV→[0,1] such that $\mu(uv) \leq \sigma(u) \wedge \sigma(v)$ for all u, v ∈ V where uv denotes the edge between u and v and $\sigma(u) \wedge \sigma(v)$ denotes the minimum of σ(u) and σ(v). σ is called the fuzzy vertex set of V and μ is called the fuzzy edge set of E.

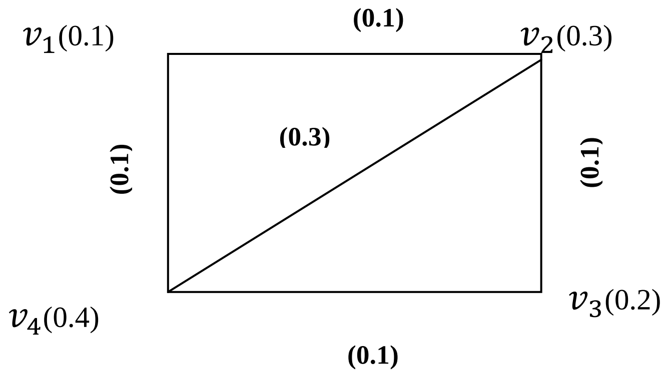

Figure 1: Fuzzy Graph

Definition 2.4 [6]. The fuzzy subgraph H = (τ, ρ) is called a fuzzy subgraph of G = (σ, μ) If $\tau(u) \leq \sigma(u)$ for all u ∈ V and $\rho(u, v) \leq \mu(u, v)$ for all u, v ∈ V.

Definition 2.5 [7]. An Intuitionistic fuzzy graph is of the form G =(V, E) where

i. V={$v_1, v_2, \ldots, v_n$} such that μ_1: V→ [0,1] and γ_1: V → [0,1] denote the degree of membership and nonmembership of the element $v_i \in V$, respectively, and
$0 \leq \mu_1(v_i) + \gamma_1(v_i)) \leq 1$ for every $v_i \in V$, (i = 1, 2, ……. n),

ii. E ⊆ V x V where μ_2: VxV→[0,1] and γ_2: VxV→ [0,1] are such that
$\mu_2(v_i, v_j) \leq \min[\mu_1(v_i), \mu_1(v_j)]$ and $\gamma_2(v_i, v_j) \geq \max[\gamma_1(v_i), \gamma_1(v_j)]$
and $0 \leq \mu_2(v_i, v_j) + \gamma_2(v_i, v_j) \leq 1$ for every $(v_i, v_j) \in E$, (i, j = 1,2, ……. n)

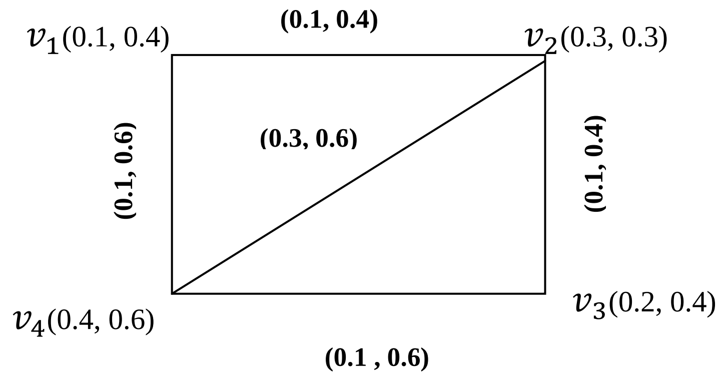

Figure 2: Intuitionistic Fuzzy Graph

Definition 2.6 [31]. Let $A = (T_A,\ I_A,\ F_A)$ and $B = (T_B,\ I_B,\ F_B)$ be single valued neutrosophic sets on a set *X*. If $A = (T_A,\ I_A,\ F_A)$ is a single valued neutrosophic relation on a set *X*, then $A = (T_A,\ I_A,\ F_A)$ is called a single valued neutrosophic relation on $B = (T_B,\ I_B,\ F_B)$ if

$T_B(x, y) \le \min(T_A(x), T_A(y))$
$I_B(x, y) \ge \max(I_A(x), I_A(y))$ and
$F_B(x, y) \ge \max(F_A x), F_A(y))$ for all $x, y \in X$.

A single valued neutrosophic relation *A* on *X* is called symmetric if

$T_A(x, y) = T_A(y, x)$,
$I_A(x, y) = I_A(y, x)$, $F_A(x, y) = F_A(y, x)$ and
$T_B(x, y) = T_B(y, x)$, $I_B(x, y) = I_B(y, x)$ and
$F_B(x, y) = F_B(y, x)$ for all $x, y \in X$.

3. Single Valued Neutrosophic Graphs

Throught this paper, we denote G^* = (V, E) a crisp graph, and G =(A, B) a single valued neutrosophic graph.

Definition 3.1. A single valued neutrosophic graph (SVN-graph) with underlying set V is defined to be a pair G= (A, B) where

1.The functions T_A:V→[0, 1], I_A:V→[0, 1] and F_A:V→[0, 1] denote the degree of truth-membership, degree of indeterminacy-membership and falsity-membership of the element $v_i \in$ V, respectively, and

$$0 \le T_A(v_i) + I_A(v_i) + F_A(v_i) \le 3 \text{ for all } v_i \in V\ (i=1, 2, \dots, n)$$

2. The functions T_B: E ⊆ V x V →[0, 1], I_B:E ⊆ V x V →[0, 1] and F_B: E ⊆ V x V →[0, 1] are defined by

$T_B(\{v_i, v_j\}) \le \min\ [T_A(v_i), T_A(v_j)]$,
$I_B(\{v_i, v_j\}) \ge \max\ [I_A(v_i), I_A(v_j)]$ and
$F_B(\{v_i, v_j\}) \ge \max\ [F_A(v_i), F_A(v_j)]$

Denotes the degree of truth-membership, indeterminacy-membership and falsity-membership of the edge $(v_i, v_j) \in$ E respectively, where

$$0 \le T_B(\{v_i, v_j\}) + I_B(\{v_i, v_j\}) + F_B(\{v_i, v_j\}) \le 3 \text{ for all } \{v_i, v_j\} \in E\ (i, j = 1, 2, \dots, n)$$

We call A the single valued neutrosophic vertex set of V, B the single valued neutrosophic edge set of E, respectively, Note that B is a symmetric single valued neutrosophic relation on A. We use the notation (v_i, v_j) for an element of E Thus, G = (A, B) is a single valued neutrosophic graph of G^*= (V, E) if

$T_B(v_i, v_j) \leq \min\,[T_A(v_i), T_A(v_j)]$,
$I_B(v_i, v_j) \geq \max\,[I_A(v_i), I_A(v_j)]$ and
$F_B(v_i, v_j) \geq \max\,[F_A(v_i), F_A(v_j)]$ for all $(v_i, v_j) \in$ E

Example 3.2. Consider a graph G^* such that V= $\{v_1, v_2, v_3, v_4\}$, E=$\{v_1v_2, v_2v_3, v_3v_4, v_4v_1\}$. Let A be a single valued neutrosophic subset of V and let B a single valued neutrosophic subset of E denoted by

	v_1	v_2	v_3	v_4
T_A	0.5	0.6	0.2	0.4
I_A	0.1	0.3	0.3	0.2
F_A	0.4	0.2	0.4	0.5

	v_1v_2	v_2v_3	v_3v_4	v_4v_1
T_B	0.2	0.3	0.2	0.1
I_B	0.3	0.3	0.3	0.2
F_B	0.4	0.4	0.4	0.5

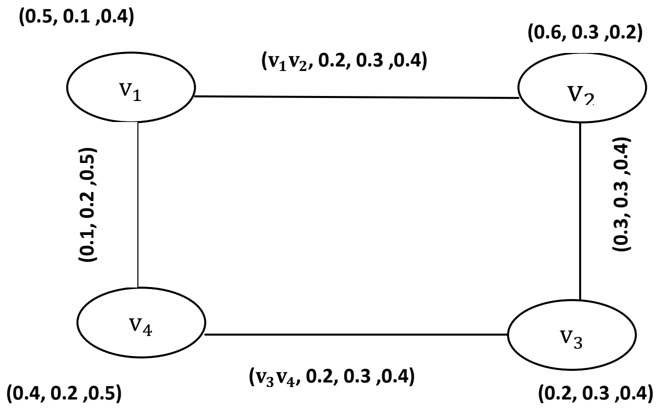

Figure 3: G: Single valued neutrosophic graph

In figure 3, (i) (v_1,0.5, 0.1,0.4) is a single valued neutrosophic vertex or SVN-vertex

(ii) (v_1v_2, 0.2, 0.3, 0.4) is a single valued neutrosophic edge or SVN-edge

(iii) (v_1, 0.5, 0.1, 0.4) and (v_2, 0.6, 0.3, 0.2) are single valued neutrosophic adjacent vertices.

(iv) (v_1v_2, 0.2, 0.3, 0.4) and (v_1v_4, 0.1, 0.2, 0.5) are a single valued neutrosophic adjacent edge.

Note 1. (i) When $T_{Bij} = I_{Bij} = F_{Bij}$ for some i and j, then there is no edge between v_i and v_j .
Otherwise there exists an edge between v_i and v_j .
(ii)If one of the inequalities is not satisfied in (1) and (2), then G is not an SVNG

The single valued neutrosophic graph G depicted in figure 3 is represented by the following adjacency matrix $\boldsymbol{M_G}$

$$M_G = \begin{bmatrix} (0.5, 0.1, 0.4) & (0.2, 0.3, 0.4) & (0, \ 0, 0) & (0.1, 0.2, 0.5) \\ (0.2, 0.3, 0.4) & (0.6, 0.3, 0.2) & (0.3, 0.3, 0.4) & (0, 0, 0) \\ (0, 0, 0) & (0.3, 0.3, 0.4) & (0.2, 0.3, 0.4) & (0.2, 0.3, 0.4) \\ (0.1, 0.2, 0.5) & (0, 0, 0) & (0.2, 0.3, 0.4) & (0.4, 0.2, 0.5) \end{bmatrix}$$

Definition 3.3. A partial SVN-subgraph of SVN-graph G= (A, B) is a SVN-graph H = (V', E') such that

(i) $V' \subseteq V$, where $T'_{Ai} \leq T_{Ai}$, $I'_{Ai} \geq I_{Ai}$, $F'_{Ai} \geq F_{Ai}$ for all $v_i \in V$.
(ii) $E' \subseteq E$, where $T'_{Bij} \leq T_{Bij}$, $I'_{Bij} \geq I_{Bij}$, $F'_{Bij} \geq F_{Bij}$ for all $(v_i\, v_j) \in E$.

Definition 3.4. A SVN-subgraph of SVN-graph G= (V, E) is a SVN-graph H = (V', E') such that

(i) $V' = V$, where $T'_{Ai} = T_{Ai}$, $I'_{Ai} = I_{Ai}$, $F'_{Ai} = F_{Ai}$ for all v_i in the vertex set of V'.
(ii) $E' = E$, where $T'_{Bij} = T_{Bij}$, $I'_{Bij} = I_{Bij}$, $F'_{Bij} = F_{Bij}$ for every $(v_i\, v_j) \in E$ in the edge set of E'.

Example 3.5. $\mathbf{G_1}$ in Figure 4 is a SVN-graph . $\mathbf{H_1}$ in Figure 5 is a partial SVN-subgraph and $\mathbf{H_2}$ in Figure 6 is a SVN-subgraph of $\mathbf{G_1}$

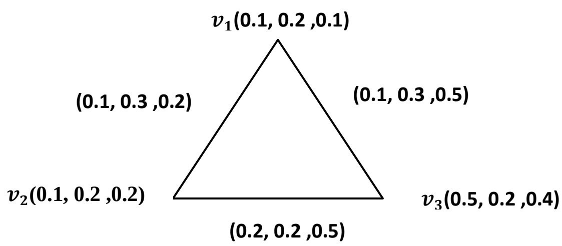

Figure 4: G_1, a single valued neutrosophic graph

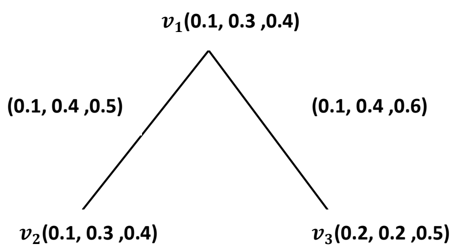

Figure 5: H_1, a partial SVN-subgraph of G_1

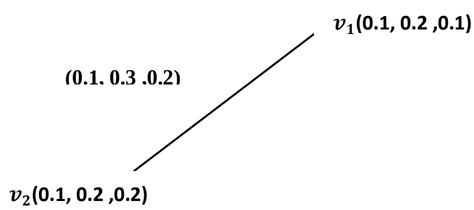

Figure 6: H_2, a SVN-subgraph of G_1.

Definition 3.6. The two vertices are said to be adjacent in a single valued neutrosophic graph G= (**A**, **B**) if $T_B(v_i, v_j) = \min [T_A(v_i), T_A(v_j)]$, $I_B(v_i, v_j) = \max [I_A(v_i), I_A(v_j)]$ and $F_B(v_i, v_j) = \max [F_A(v_i), F_A(v_j)]$. In this case, v_i and v_j are said to be neighbours and (v_i, v_j) is incident at v_i and v_j also.

Definition 3.7. A path P in a single valued neutrosophic graph G= (**A**, **B**) is a sequence of distinct vertices v_0, v_1, v_3,… v_n such that $T_B(v_{i-1}, v_i) > 0$, $I_B(v_{i-1}, v_i) > 0$ and $F_B(v_{i-1}, v_i) > 0$ for $0 \leq i \leq 1$. Here n ≥ 1 is called the length of the path P. A single node or vertex v_i may also be considered as a path. In this case the path is of the length (0, 0, 0). The consecutive pairs (v_{i-1}, v_i) are called edges of the path.We call P a cycle if $v_0 = v_n$ and n≥3.

Definition 3.8. A single valued neutrosophic graph G= (A, B) is said to be connected if every pair of vertices has at least one single valued neutrosophic path between them, otherwise it is disconnected.

Definition 3.9. A vertex $v_j \in V$ of single valued neutrosophic graph G= (**A**, **B**) is said to be an isolated vertex if there is no effective edge incident at v_j.

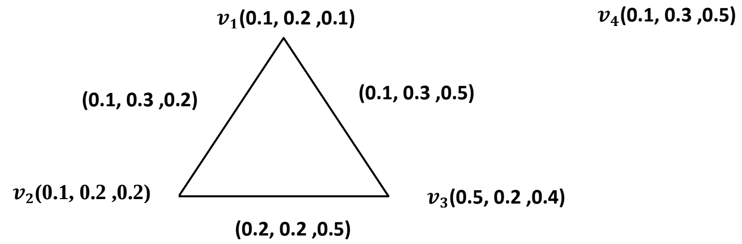

Figure 7: Example of single valued neutrosophic graph

In figure 7, the single valued neutrosophic vertex v_4 is an isolated vertex.

Definition 3.10. A vertex in a single valued neutrosophic G= (**A**, **B**) having exactly one neighbor is called ***a pendent vertex***. Otherwise, it is called ***non-pendent vertex***. An edge in a single valued neutrosophic graph incident with a pendent vertex is called a ***pendent edge***. Otherwise it is called ***non-pendent edge***. A vertex in a single valued neutrosophic graph adjacent to the pendent vertex is called **a support** of the pendent edge

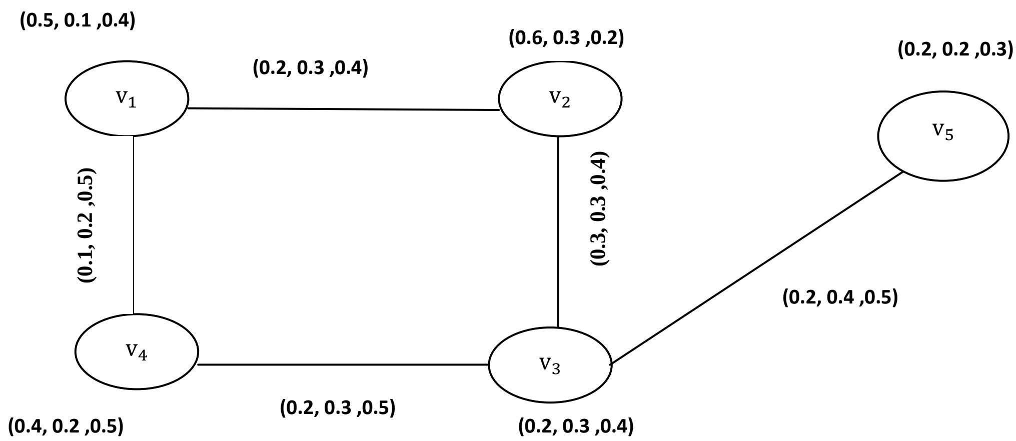

Figure 8 : Incident SVN-graph.

Definition 3.11. A single valued neutrosophic graph G= (A, B) that has neither self loops nor parallel edge is called **simple single valued neutrosophic graph.**

Definition 3.12. When a vertex $\mathbf{v_i}$ is end vertex of some edges ($\mathbf{v_i}$, $\mathbf{v_j}$) of any SVN-graph G= (A, B). Then $\mathbf{v_i}$ and ($\mathbf{v_i}$, $\mathbf{v_j}$) are said to be **incident** to each other. **In Figure 8, $\mathbf{v_2v_3}$, $\mathbf{v_3v_4}$ and $\mathbf{v_3v_5}$ are incident on $\mathbf{v_3}$.**

Definition 3.13. Let G= (**A**, **B**) be a single valued neutrosophic graph. Then the degree of any vertex **v** is sum of degree of truth-membership, sum of degree of indeterminacy-membership and sum of degree of falsity-membership of all those edges which are incident on vertex **v** denoted by d(v)= ($d_T(v)$, $d_I(v)$, $d_F(v)$) where

$d_T(v)$=$\sum_{u\neq v} T_B(u,v)$ denotes degree of truth-membership vertex.
$d_I(v)$=$\sum_{u\neq v} I_B(u,v)$ denotes degree of indeterminacy-membership vertex.
$d_F(v)$=$\sum_{u\neq v} F_B(u,v)$ denotes degree of falsity-membership vertex.

Example 3.14. Let us consider a single valued neutrosophic graph G= (A, B) of G^* = (V, E) where V ={v_1, v_2, v_3, v_4} and E={v_1v_2, v_2v_3, v_3v_4 , v_4v_1}.

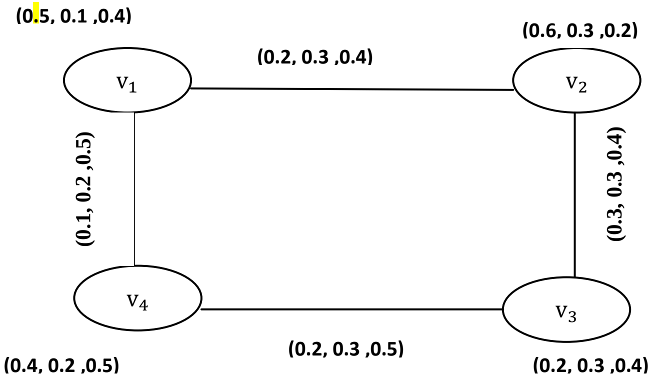

Figure 9: Degree of vertex of single valued neutrosophic graph

We have, the degree of each vertex as follows:

d(v_1)= (0.3, 0.5, 0.9), d(v_2)= (0.5, 0.6, 0.8), d(v_3)= (0.5, 0.6, 0.9), d(v_4)= (0.3, 0.5, 1)

Definition 3.15 . A single valued neutrosophic graph G= (A, B) is called constant if degree of each vertex is k =(k_1, k_2, k_3). That is, d(v) = (k_1, k_2, k_3) for all $v \in$ V.

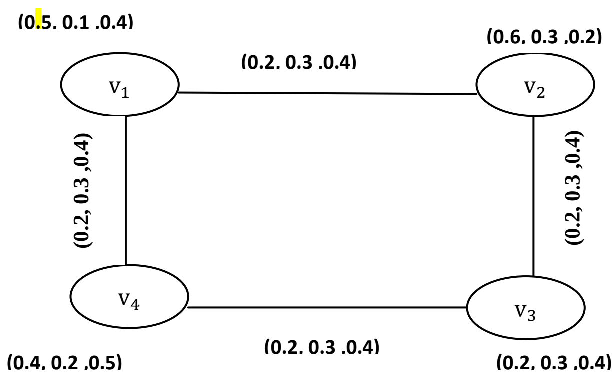

Figure 10: Constant SVN-graph.

Example 3.16. Consider a single valued neutrosophic graph G such that V ={v_1, v_2, v_3, v_4} and E={v_1v_2, v_2v_3, v_3v_4 , v_4v_1}.

Clearly, as it is seen in Figure 10, G is constant SVN-graph since the degree of $\mathbf{v_1}$, $\mathbf{v_2}$, $\mathbf{v_3}$ and $\mathbf{v_4}$ is (0.4, 0.6, 0.8).

Definition 3.17. A single valued neutrosophic graph G=(A, B) of G^*= (V, E) is called strong single valued neutrosophic graph if

$T_B(v_i, v_j) = \min\,[T_A(v_i),\ T_A(v_j)]$
$I_B(v_i, v_j) = \max\,[I_A(v_i),\ I_A(v_j)]$
$F_B(v_i, v_j) = \max\,[F_A(v_i), F_A(v_j)]$
For all $(v_i, v_j) \in$ E.

Example 3.18. Consider a graph G^* such that V= {v_1, v_2, v_3, v_4}, E={v_1v_2, v_2v_3, v_3v_4, v_4v_1}. Let A be a single valued neutrosophic subset of V and let B a single valued neutrosophic subset of E denoted by

	v_1	v_2	v_3	v_4
T_A	0.5	0.6	0.2	0.4
I_A	0.1	0.3	0.3	0.2
F_A	0.4	0.2	0.4	0.5

	v_1v_2	v_2v_3	v_3v_4	v_4v_1
T_B	0.5	0.2	0.2	0.4
I_B	0.3	0.3	0.3	0.2
F_B	0.4	0.4	0.5	0.5

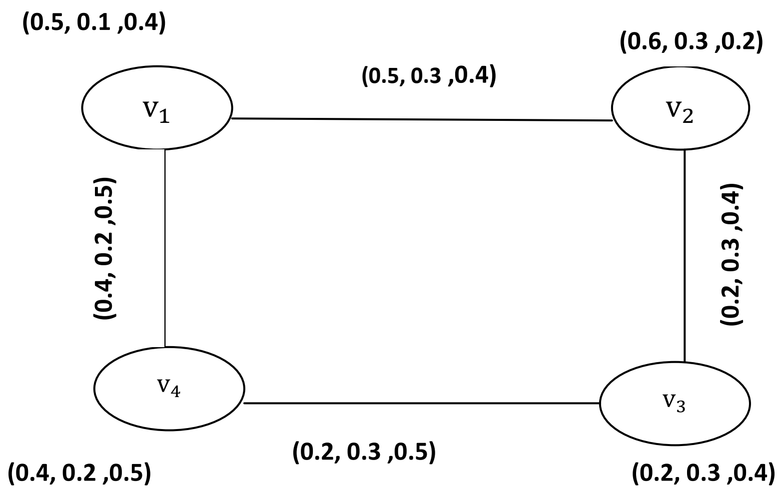

Figure 11: Strong SVN-graph

By routing computations, it is easy to see that G is a strong single valued neutrosophic of G^*.

Proposition 3.19. A single valued neutrosophic graph is the generalization of fuzzy graph

Proof: Suppose G= (V, E) be a single valued neutrosophic graph. Then by setting the indeterminacy- membership and falsity- membership values of vertex set and edge set equals to zero reduces the single valued neutrosophic graph to fuzzy graph.

Proposition 3.20. A single valued neutrosophic graph is the generalization of intuitionistic fuzzy graph.

Proof: Suppose G=(V, E) be a single valued neutrosophic graph. Then by setting the indeterminacy- membership value of vertex set and edge set equals to zero reduces the single valued neutrosophic graph to intuitionistic fuzzy graph.

Definition 3.21. The complement of a single valued neutrosophic graph G (A, B) on G^* is a single valued neutrosophic graph $\bar{G}$ on G^* where:

1. $\bar{A}$ =A
2. $\overline{T_A}(v_i)$= $T_A(v_i)$, $\overline{I_A}(v_i)$= $I_A(v_i)$, $\overline{F_A}(v_i)$ = $F_A(v_i)$, for all $v_j \in$ V.
3. $\overline{T_B}(v_i, v_j)$= min $[T_A(v_i), T_A(v_j)]$ -$T_B(v_i, v_j)$

$\overline{I_B}(v_i, v_j)$= max $[I_A(v_i), I_A(v_j)]$ -$I_B(v_i, v_j)$ and

$\overline{F_B}(v_i, v_j)$= max $[F_A(v_i), F_A(v_j)]$ -$F_B(v_i, v_j)$, For all $(v_i, v_j) \in$ E

Remark 3.22. if G= (V, E) is a single valued neutrosophic graph on G^*. Then from above definition, it follow that $\bar{\bar{G}}$ is given by the single valued neutrosophic graph $\bar{\bar{G}}$ =($\bar{\bar{V}}$, $\bar{\bar{E}}$) on G^* where

$\bar{\bar{V}}$ =V and $\overline{\overline{T_B}}(v_i, v_j)$= min $[T_A(v_i), T_A(v_j)]$-$T_B(v_i, v_j)$,

$\overline{\overline{I_B}}(v_i, v_j)$= min $[I_A(v_i), I_A(v_j)]$-$I_B(v_i, v_j)$,and

$\overline{\overline{F_B}}(v_i, v_j)$ = min $[F_A(v_i), F_A(v_j)]$-$F_B(v_i, v_j)$ For all $(v_i, v_j) \in$ E.

Thus $\overline{\overline{T_B}}$ =T_B, $\overline{\overline{I_B}}$ =I_B, and $\overline{\overline{F_B}}$ =F_B on V, where E =(T_B, I_B, F_B) is the single valued neutrosophic relation on V. For any single valued neutrosophic graph G, $\bar{G}$ is strong single valued neutrosophic graph and G ⊆ $\bar{G}$.

Proposition 3.23. G= $\bar{\bar{G}}$ if and only if G is a strong single valued neutrosophic graph.

Proof. it is obvious.

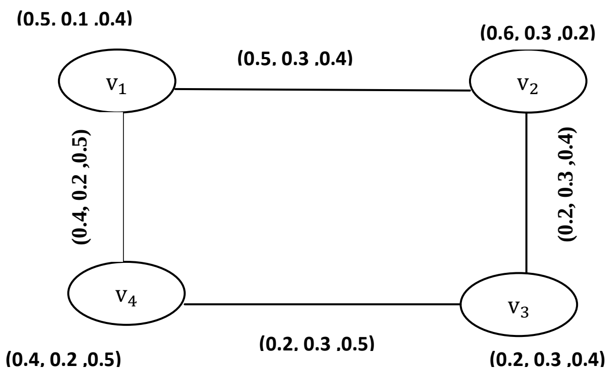

Figure 12: G: Strong SVN- graph

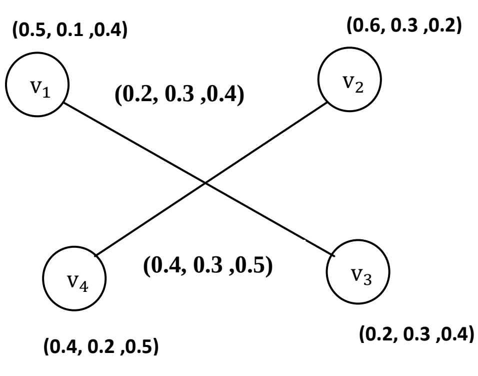

Figure 13: $\bar{G}$ Strong SVN- graph

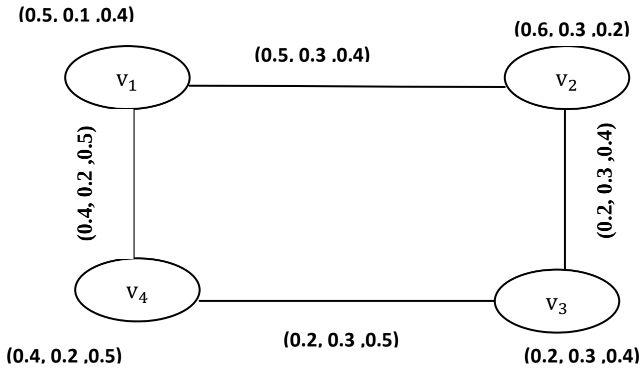

Figure 14: $\bar{\bar{G}}$ Strong SVN- graph

Definition 3.24. A strong single valued neutrosophic graph G is called self complementary if G≅ $\bar{G}$. Where $\bar{G}$ is the complement of single valued neutrosophic graph G.

Example 3.25. Consider a graph G^* = (V, E) such that V ={v_1, v_2, v_3, v_4}, E={v_1v_2, v_2v_3, v_3v_4, v_1v_4}. Consider a single valued neutrosophic graph G as in Figure 12 and 13,

Clearly, as it is seen in Figure 14, G≅ $\bar{\bar{G}}$. Hence G is self complementary.

Proposition 3.26. Let G=(A, B) be a ***strong*** single valued neutrosophic graph. If

$T_B(v_i, v_j) = \min\,[T_A(v_i), T_A(v_j)]$,
$I_B(v_i, v_j) = \max[I_A(v_i), I_A(v_j)]$ and
$F_B(v_i, v_j) = \max\,[F_A(v_i), F_A(v_j)]$ for all $v_i, v_j \in$ V.

Then G is self complementary.

Proof. Let G=(A, B) be a strong single valued neutrosophic graph such that

$T_B(v_i, v_j) = \min\,[T_A(v_i), T_A(v_j)]$
$I_B(v_i, v_j) = \max\,[I_A(v_i), I_A(v_j)]$
$F_B(v_i, v_j) = \max\,[F_A(v_i), F_A(v_j)]$

For all $v_i, v_j \in$ V. Then G ≈ $\bar{\bar{G}}$ under the identity map I: V → V. Hence G is self complementary .

Proposition 3.27. Let G be a self complementary single valued neutrosophic graph. Then

$\sum_{v_i \neq v_j} T_B(v_i, v_j) = \frac{1}{2} \sum_{v_i \neq v_j} \min\,[T_A(v_i), T_A(v_j)]$
$\sum_{v_i \neq v_j} I_B(v_i, v_j) = \frac{1}{2} \sum_{v_i \neq v_j} \max\,[I_A(v_i), I_A(v_j)]$
$\sum_{v_i \neq v_j} F_B(v_i, v_j) = \frac{1}{2} \sum_{v_i \neq v_j} \max\,[F_A(v_i), F_A(v_j)]$

Proof. If G be a self complementary single valued neutrosophic graph. Then there exist an isomorphism $f: V_1 \rightarrow V_1$ satisfying

$\overline{T_{V_1}}(f(v_i)) = T_{V_1}(f(v_i)) = T_{V_1}(v_i)$
$\overline{I_{V_1}}(f(v_i)) = I_{V_1}(f(v_i)) = I_{V_1}(v_i)$
$\overline{\overline{F_{V_1}}}(f(v_i)) = F_{V_1}(f(v_i)) = F_{V_1}(v_i)$ for all $v_i \in V_1$.

And

$\overline{T_{E_1}}(f(v_i), f(v_j)) = T_{E_1}(f(v_i), f(v_j)) = T_{E_1}(v_i, v_j)$
$\overline{I_{E_1}}(f(v_i), f(v_j)) = I_{E_1}(f(v_i), f(v_j)) = I_{E_1}(v_i, v_j)$
$\overline{F_{E_1}}(f(v_i), f(v_j)) = F_{E_1}(f(v_i), f(v_j)) = F_{E_1}(v_i, v_j)$ for all $(v_i, v_j) \in E_1$

We have

$\overline{T_{E_1}}(f(v_i), f(v_j)) = \min [\overline{T_{V_1}}(f(v_i)), \overline{T_{V_1}}(f(v_j))] - T_{E_1}(f(v_i), f(v_j))$
i.e, $T_{E_1}(v_i, v_j) = \min [T_{V_1}(v_i), T_{V_1}(v_j)] - T_{E_1}(f(v_i), f(v_j))$
$T_{E_1}(v_i, v_j) = \min [T_{V_1}(v_i), T_{V_1}(v_j)] - T_{E_1}(v_i, v_j)$

That is

$\sum_{v_i \neq v_j} T_{E_1}(v_i, v_j) + \sum_{v_i \neq v_j} T_{E_1}(v_i, v_j) = \sum_{v_i \neq v_j} \min [T_{V_1}(v_i), T_{V_1}(v_j)]$
$\sum_{v_i \neq v_j} I_{E_1}(v_i, v_j) + \sum_{v_i \neq v_j} I_{E_1}(v_i, v_j) = \sum_{v_i \neq v_j} \max [I_{V_1}(v_i), I_{V_1}(v_j)]$
$\sum_{v_i \neq v_j} F_{E_1}(v_i, v_j) + \sum_{v_i \neq v_j} F_{E_1}(v_i, v_j) = \sum_{v_i \neq v_j} \max [F_{V_1}(v_i), F_{V_1}(v_j)]$
$2 \sum_{v_i \neq v_j} T_{E_1}(v_i, v_j) = \sum_{v_i \neq v_j} \min [T_{V_1}(v_i), T_{V_1}(v_j)]$
$2 \sum_{v_i \neq v_j} I_{E_1}(v_i, v_j) = \sum_{v_i \neq v_j} \max [I_{V_1}(v_i), I_{V_1}(v_j)]$
$2 \sum_{v_i \neq v_j} F_{E_1}(v_i, v_j) = \sum_{v_i \neq v_j} \max [F_{V_1}(v_i), F_{V_1}(v_j)]$

From these equations, Proposition 3.27 holds

Proposition 3.28. Let G_1 and G_2 be strong single valued neutrosophic graph, $\overline{G_1} \approx \overline{G_2}$ (isomorphism)

Proof. Assume that G_1 and G_2 are isomorphic, there exist a bijective map *f*: $V_1 \rightarrow V_2$ satisfying

$T_{V_1}(v_i) = T_{V_2}(f(v_i))$,
$I_{V_1}(v_i) = I_{V_2}(f(v_i))$,
$F_{V_1}(v_i) = F_{V_2}(f(v_i))$ for all $v_i \in V_1$.

And

$T_{E_1}(v_i, v_j) = T_{E_2}(f(v_i), f(v_j))$,
$I_{E_1}(v_i, v_j) = I_{E_2}(f(v_i), f(v_j))$,
$F_{E_1}(v_i, v_j) = F_{E_2}(f(v_i), f(v_j))$ for all $(v_i, v_j) \in E_1$

By definition 3.21, we have

$\overline{T_{E_1}}(v_i, v_j)$= min $[T_{V_1}(v_i), T_{V_1}(v_j)] - T_{E_1}(v_i, v_j)$
= min $[T_{V_2}(f(v_i)),\ T_{V_2}(f(v_j))] - T_{E_2}(f(v_i), f(v_j))$,
=$\overline{T_{E_2}}(f(v_i), f(v_j))$,

$\overline{I_{E_1}}(v_i, v_j)$= max $[I_{V_1}(v_i),\ I_{V_1}(v_j)] - I_{E_1}(v_i, v_j)$
= max$[I_{V_2}(f(v_i)), I_{V_2}(f(v_j))] - I_{E_2}(f(v_i), f(v_j))$,
=$\overline{I_{E_2}}(f(v_i), f(v_j))$,

$\overline{F_{E_1}}(v_i, v_j)$= min $[F_{V_1}(v_i), F_{V_1}(v_j)] - F_{E_1}(v_i, v_j)$
= min $[F_{V_2}(f(v_i)), F_{V_2}(f(v_j))] - F_{E_2}(f(v_i), f(v_j))$,
=$\overline{F_{E_2}}(f(v_i), f(v_j))$,

For all $(v_i, v_j) \in E_1$. Hence $\overline{G_1} \approx \overline{G_2}$. The converse is straightforward.

4. Complete Single Valued Neutrosophic Graphs

For the sake of simplicity we denote $T_A(v_i)$ by T_{Ai} , $I_A(v_i)$ by I_{Ai}, and $I_A(v_i)$ by I_{Ai}. Also $T_B(v_i, v_j)$ by T_{Bij} , $I_B(v_i, v_j)$ by I_{Bij} and $F_B(v_i, v_j)$ by F_{Bij}.

Definition 4.1. A single valued neutrosophic graph G= (A, B) is called complete if T_{Bij}= min(T_{Ai}, T_{Aj}), I_{Bij}= max(I_{Ai}, I_{Aj}) and F_{Bij}= max(F_{Ai}, F_{Aj}) for all $v_i, v_j \in$ V.

Example 4.2. Consider a graph G^* = (V, E) such that V ={ v_1 , v_2 , v_3 , v_4 }, E={v_1v_2, v_1v_3, v_2v_3, v_1v_4, v_3v_4 , v_2v_4}. Then G= (A, B) is a complete single valued neutrosophic graph of G^*.

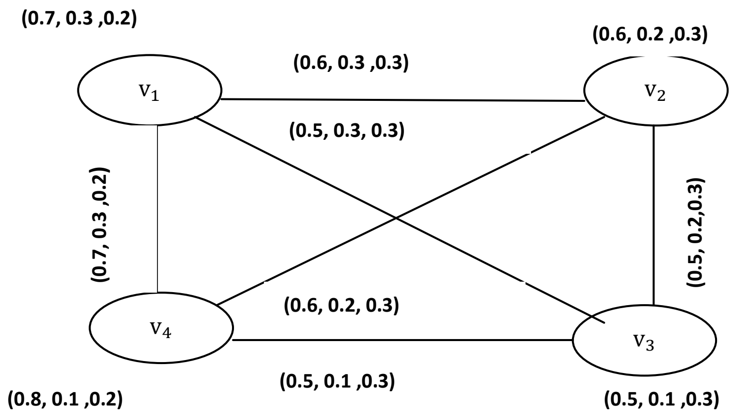

Figure 13: Complete single valued neutrosophic graph

Definition 4.3. The complement of a complete single valued neutrosophic graph G =(A, B) of G^*= (V, E) is a single valued neutrosophic complete graph $\bar{G}$= ($\bar{A}$, $\bar{B}$) on G^*= (V, $\bar{E}$) where

1. $\bar{V}$ =V
2. $\overline{T_A}(v_i)$= $T_A(v_i)$, $\overline{I_A}(v_i)$= $I_A(v_i)$, $\overline{F_A}(v_i)$= $F_A(v_i)$, for all $v_j \in$ V.
3. $\overline{T_B}(v_i, v_j)$= min $\left[T_A(v_i), T_A(v_j)\right] - T_B(v_i, v_j)$

$\overline{I_B}(v_i, v_j)$= max $[I_A(v_i), I_A(v_j)]$ $- I_B(v_i, v_j)$ and
$\overline{F_B}(v_i, v_j)$= max $[F_A(v_i), F_A(v_j)]$ $- F_B(v_i, v_j)$ for all $(v_i, v_j) \in$ E

Proposition 4.4. The complement of complete SVN-graph is a SVN-graph with no edge. Or if G is a complete then in $\bar{G}$ the edge is empty.

Proof. Let G= (A, B) be a complete SVN-graph. So

T_{Bij}= min(T_{Ai}, T_{Aj}), I_{Bij}= max(I_{Ai}, I_{Aj}) and F_{Bij}= max(F_{Ai}, F_{Aj}) for all $v_i, v_j \in$ V

Hence in $\bar{G}$,

$\bar{T}_{Bij}$= min $[T_{Ai}, T_{Aj}] - T_{Aij}$ for all i, j,…..,n
= min $[T_{Ai}, T_{Aj}]$ − min $[T_{Ai}, T_{Aj}]$ for all i, j,…..,n
= 0 for all i, j,…..,n

and

$\bar{I}_{Bij}$= max $[I_{Ai}, I_{Aj}] - I_{Bij}$ for all i, j,…..,n
= max $[I_{Ai}, I_{Aj}]$ − max $[I_{Ai}, I_{Aj}]$ for all i, j,…..,n
= 0 for all i, j,…..,n
Also

$\bar{F}_{Bij}$= max $[F_{Ai}, F_{Aj}] - F_{Bij}$ for all i, j,…..,n
= max $[F_{Ai}, F_{Aj}]$ − max $[F_{Ai}, F_{Aj}]$ for all i, j,…..,n
= 0 for all i, j,…..,n

Thus ($\bar{T}_{Bij}$, $\bar{I}_{Bij}$, $\bar{F}_{Bij}$) = (0 , 0, 0)

Hence, the edge set of $\bar{G}$ is empty if G is a complete SVN-graph.

4. Conclusion

Neutrosophic sets is a generalization of the notion of fuzzy sets and intuitionistic fuzzy sets. Neutrosophic models gives more precisions, flexibility and compatibility to the system as compared to the classical, fuzzy and/or intuitionistic fuzzy models. In this paper, we have introduced certain types of single valued neutrosophic graphs, such as strong single valued neutrosophic graph, constant single valued neutrosophic graph and complete single valued neutrosophic graphs. In future study, we plan to extend our research to regular and irregular single valued neutrosophic graphs, bipolar single valued neutrosophic graphs, interval valued neutrosophic graphs, strong interval valued neutrosophic, regular and irregular interval valued neutrosophic.

References

1. A. V. Devadoss, A. Rajkumar & N. J. P .Praveena,.A Study on Miracles through Holy Bible using Neutrosophic Cognitive Maps (NCMS).International Journal of Computer Applications, 69(3) (2013).

2. A. Aydoğdu, On Similarity and Entropy of Single Valued Neutrosophic Sets, Gen. Math. Notes, Vol. 29, No. 1, July 2015, pp. 67-74.
3. A. Q. Ansari, R.Biswas & S. Aggarwal, (2012). Neutrosophic classifier: An extension of fuzzy classifier. Elsevier- *Applied Soft Computing*, 13 (2013) 563-573, *http://dx.doi.org/10.1016/j.asoc.2012.08.002*
4. A. Q. Ansari, R.Biswas & S. Aggarwal. (Poster Presentation) Neutrosophication of Fuzzy Models, IEEE Workshop On Computational Intelligence: Theories, Applications and Future Directions (hostedby IIT Kanpur), 14th July'13.
5. A. Q. Ansari, R.Biswas & S. Aggarwal Extension to fuzzy logic representation: Moving towards neutrosophic logic - A new laboratory rat, Fuzzy Systems (FUZZ), 2013 IEEE International Conference, 1–8,DOI:10.1109/FUZZ-IEEE.2013.6622412.
6. A. Nagoor Gani . and M.Basheer Ahamed, Order and Size in Fuzzy Graphs, Bulletin of Pure and Applied Sciences, Vol 22E (No.1) 2003; p.145-148.
7. A. Nagoor Gani. A and S. Shajitha Begum, Degree, Order and Size in Intuitionistic Fuzzy Graphs, International Journal of Algorithms, Computing and Mathematics,(3)3 (2010).
8. A. Nagoor Gani and S.R Latha, On Irregular Fuzzy Graphs, Applied Mathematical Sciences, Vol.6, 2012, no.11,517-523.
9. F. Smarandache. Refined Literal Indeterminacy and the Multiplication Law of Sub-Indeterminacies, Neutrosophic Sets and Systems, Vol. 9, 58-2015,
10. F. Smarandache, *Types of Neutrosophic Graphs and neutrosophic Algebraic Structures together with their Applications in Technology*, seminar, Universitatea Transilvania din Brasov, Facultatea de Design de Produs si Mediu, Brasov, Romania 06 June 2015.
11. F. Smarandache, *Symbolic Neutrosophic* Theory, Europanova asbl, Brussels, 195p, 2015.
12. F. Smarandache , Neutrosophic set - a generalization of the intuitionistic fuzzy set, Granular Computing, 2006 IEEE International Conference, 38 – 42,2006,DOI: 10.1109/GRC.2006.1635754.
13. F. Smarandache, A geometric interpretation of the neutrosophic set — A generalization of the intuitionistic fuzzy set Granular Computing (GrC), 2011 IEEE International Conference , 602 – 606, 2011, DOI 10.1109/GRC.2011.6122665.
14. Gaurav Garg, Kanika Bhutani, Megha Kumar and Swati Aggarwal, *Hybrid model for medical diagnosis using Neutrosophic Cognitive Maps with* Genetic Algorithms, FUZZ-IEEE 2015(IEEE International conference onfuzzy systems).
15. H .Wang,. Y. Zhang, R. Sunderraman, Truth-value based interval neutrosophic sets, Granular Computing, 2005 IEEE International Conference , P274 - 277 Vol. 1,2005DOI: 10.1109/GRC.2005.1547284.
16. H. Wang, F. Smarandache, Y. Zhang, and R. Sunderraman, Single valued Neutrosophic Sets, Multisspace and Multistructure 4 (2010) 410-413.
17. I. Deli, M. Ali, F. Smarandache , Bipolar neutrosophic sets and their application based on multi-criteria decision making problems, Advanced Mechatronic Systems (ICAMechS), 2015 International Conference, 249 - 254, DOI: 10.1109/ICAMechS.2015.7287068.
18. I. Turksen, Interval valued fuzzy sets based on normal forms, Fuzzy Sets and Systems, vol. 20, p. 191-210 (1986).
19. J. Ye, vector similarity measures of simplified neutrosophic sets and their application in multicriteria decision making, *International Journal of Fuzzy Systems, Vol. 16, No. 2*, p.204-211 (2014).
20. J. Ye, Single-Valued Neutrosophic Minimum Spanning Tree and Its Clustering Method, Journal of Intelligent Systems 23(3): 311–324,(2014).
21. K. Atanassov, "Intuitionistic fuzzy sets," Fuzzy Sets and Systems, vol. 20, p. 87-96 (1986).
22. K. Atanassov and G. Gargov, "Interval valued intuitionistic fuzzy sets," Fuzzy Sets and Systems, vol.31, pp. 343-349 (1989).
23. K. Atanassov. Intuitionistic fuzzy sets: theory and applications. Physica, New York, 1999.
24. L. Zadeh, Fuzzy sets, Inform and Control, 8(1965), 338-353

25. P. Bhattacharya, Some remarks on fuzzy graphs, Pattern Recognition Letters 6: 297-302, 1987.
26. R. Parvathi and M. G. Karunambigai, Intuitionistic Fuzzy Graphs, Computational Intelligence, Theory and applications, International Conference in Germany, Sept 18 -20, 2006.
27. R. Rıdvan, A. Küçük, Subsethood measure for single valued neutrosophic sets, Journal of Intelligent & Fuzzy Systems, vol. 29, no. 2, pp. 525-530, 2015,DOI: 10.3233/IFS-141304
28. S. Aggarwal, R. Biswas, A. Q. Ansari, Neutrosophic modeling and control , Computer and Communication Technology (ICCCT), 2010 International Conference , 718 – 723, DOI:10.1109/ICCCT.2010.5640435
29. S. Broumi, F. Smarandache, New distance and similarity measures of interval neutrosophic sets, Information Fusion (FUSION), 2014 IEEE 17th International Conference, 2014,p 1 – 7.
30. S. Broumi, F. Smarandache, Single valued neutrosophic trapezoid linguistic aggregation operators based multi-attribute decision making, Bulletin of Pure & Applied Sciences-Mathematics and Statistics,(2014)135-155, DOI : 10.5958/2320-3226.2014.00006.X
31. S. Broumi, M. Talea, F. Smarandache, Single Valued Neutrosophic Graphs: Degree, Order and Size, IEEE WCCI 2016, under process.
32. Y. Hai-Long, G. She, Yanhonge, L. Xiuwu, On single valued neutrosophic relations, Journal of Intelligent & Fuzzy Systems, vol. Preprint, no. Preprint, p. 1-12 (2015)
33. W. B. Vasantha Kandasamy and F. Smarandache, Fuzzy Cognitive Maps and Neutrosophic Congtive Maps,2013.
34. W. B. Vasantha Kandasamy, K. Ilanthenral and Florentin Smarandache, Neutrosophic Graphs: A New Dimension to Graph Theory, Kindle Edition, 2015.
35. W.B. Vasantha Kandasamy and F. Smarandache “Analysis of social aspects of migrant laborers living with HIV/AIDS using Fuzzy Theory and Neutrosophic Cognitive Maps”, Xiquan, Phoenix (2004).

http://www.newtheory.org ISSN: 2149-1402

Journal of New Theory

Year: **2016**, *Number*: **10**, *Pages*: **102**

EDITORIAL

We are happy to inform you that Number 10 of the Journal of New Theory (JNT) is completed with 10 articles.

JNT publishes original research articles, reports, reviews and commentaries that are based on a theory of mathematics. However, the topics are not limited to only mathematics, but also include statistics, computer science, physics, engineering, chemistry, biology, economics or social sciences that use a theory of mathematics.

JNT is a refereed, electronic, open access and international journal.

Papers in JNT are published free of charge.

We would like to express our deepest thanks to all of the members of the editorial board and reviewers of the papers in this issue who are U. Orhan, A. Filiz, A. Fenercioğlu, A. Sarı, A. Yıldırım, A. S. Sezer, B. Mehmetoğlu, B.H. Çadırcı, C. Kaya, Ç. Çekiç, D. Mohamad, E. Altuntaş, E. Turgut, F. Karaaslan, F. Smarandache, G. Erdal, H. Aktaş, H.M. Doğan, H. Günal, H. Kızılaslan, H. Önen, H. Şimşek, İ. Zorlutuna, İ. Deli, İ. Gökce, İ. Türkekul, İ. Parmaksız, J. Zhan, J. Ye, H. Kızılaslan, M. Akar, M. Akdağ, M.I. Ali, M. Ali, M. Çavuş, M. Demirci, N. Çağman, N. Sağlam, N. Yeşilayer, N. Kızılaslan, O. Muhtaroğlu, P.K. Maji, R. Yayar, S. Broumi, S. Karaman, S. Tarhan, S. Enginoğlu, S. Demiriz, S. Karataş, S. Öztürk, S. Eğri, Ş. Sözen, Y. Budak, K. Nikodem, T. Noiri, U. Awan.

Pleases, write any original idea. If it is true, it gives an opportunity to use. If it is incomplete, it gives an opportunity to complete. If it is incorrect, it gives an opportunity to correct.

You can reach us from journal homepage at http://www.newtheory.org. To receive further information and to send your recommendations and remarks, or to submit articles for consideration, please e-mail us at jnt@newtheory.org

We hope you will enjoy this issue of JNT. We are looking forward to hearing your feedback and receiving your contributions.

Happy reading!

10 February 2016

Prof. Dr. Naim Çağman
Editor-in-Chief
Journal of New Theory
http://www.newtheory.org