Number 11 Year 2016

Journal of New Theory

ISSN: 2149-1402

Editor-in-Chief
Naim Çağman

www.dergipark.org.tr/en/pub/jnt

Journal of New Theory (abbreviated by J. New Theory or JNT) is a mathematical journal focusing on new mathematical theories or the applications of a mathematical theory to science.

JNT founded on 18 November 2014 and its first issue published on 27 January 2015.

ISSN: 2149-1402

Editor-in-Chief: Naim Çağman

Email: journalofnewtheory@gmail.com

Language: English only.

Article Processing Charges: It has no processing charges.

Publication Frequency: Quarterly

Publication Ethics: The governance structure of J. New Theory and its acceptance procedures are transparent and designed to ensure the highest quality of published material. Journal of New Theory adheres to the international standards developed by the Committee on Publication Ethics (COPE).

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http://www.newtheory.org ISSN: 2149-1402

Journal of New Theory

Received: 01.09.2015 *Year*: **2016**, *Number*: **11**, *Pages*: **01-15**
Published: 12.02.2016 *Original Article*[**]

FUZZY SOFT CONNECTEDNESS BASED ON FUZZY b-OPEN SOFT SETS

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Abstract – As a continuation of study fuzzy soft topological spaces, we are going to introduce and investigate fuzzy soft b-connected sets, fuzzy soft b-separated sets and fuzzy soft b s-connected sets and have established several interesting properties supported by examples. Moreover, we show that a fuzzy soft b-disconnectedness property is not hereditary property in general.

Keywords – *Soft sets, Fuzzy soft sets, Fuzzy soft connectedness, Fuzzy b-open soft sets.*

1 Introduction

Uncertain or imprecise data are inherent and pervasive in many important applications in the areas such as business management, computer science, engineering, environment, social science and medical science. Uncertain data in those applications could be caused by data randomness, information incompleteness, limitations of measuring instrument, delayed data updates, and so forth. Due to the importance of those applications and the rapidly increasing amount of uncertain data collected and accumulated, research on effective and efficient techniques that are dedicated to modeling uncertain data and tackling uncertainties has attracted much interest in recent years and yet remained challenging at large. There have been a great amount of research and applications in the literature concerning some special tools like probability theory, (intuitionistic) fuzzy set theory, rough set theory, vague set theory, and interval mathematics. However, all of these have theirs advantages as well as inherent limitations in dealing with uncertainties. One major problem shared by those theories is their incompatibility with the parameterizations tools. Soft set theory [33] was firstly proposed by a Russian researcher, Molodtsov, in 1999 to overcome these difficulties. At present, work on the extension of soft set theory is progressing rapidly. Fuzzy soft set is a hybridization of fuzzy sets and soft sets, in which soft set

[**] *Edited by* Oktay Muhtaroğlu *(Area Editor) and* Naim Çağman *(Editor-in-Chief).*
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is defined over fuzzy set. Maji et al. proposed the concept of fuzzy soft set [31] and then gave its application. Roy and Maji presented a method of object recognition from an imprecise multi observer data [41]. Yuksel et al. [48] applied soft set theory to determine cancer risk. Yao et al. proposed the concept of fuzzy sets and defined some operations on fuzzy soft sets [46]. From the above discussion, we can see that all of these works are based on Zadeh̀s fuzzy sets theory [49]. Recently, many interesting applications of soft set theory have been expanded by embedding the ideas of fuzzy sets [4, 5, 8, 16, 25, 29, 30, 31, 32, 34, 35, 50]. To develop soft set theory, the operations of the soft sets are redefined and a uni-int decision making method was constructed by using these new operations [9]. In the year 2011, Shabir and Naz [42] introduced soft topology, soft relative topology and studied some introductory results. In the same year Cagman et al. [7] introduced soft topology in a different approach. Till then various researchers have studied various foundational results in soft topology. Min in [44] investigate some properties of these soft separation axioms. In [17], Kandil et al. introduced some soft operations such as semi open soft, pre open soft, α-open soft and β-open soft and investigated their properties in detail. Kandil et al. [24] introduced the notion of soft semi separation axioms. In particular they study the properties of the soft semi regular spaces and soft semi normal spaces. The notion of soft ideal was initiated for the first time by Kandil et al. [20]. They also introduced the concept of soft local function. These concepts are discussed with a view to find new soft topologies from the original one, called soft topological spaces with soft ideal $(X, \tau, E, \tilde{I})$. Applications to various fields were further investigated by Kandil et al. [18, 19, 21, 22, 23, 26]. The notion of supra soft topological spaces was initiated for the first time by El-sheikh and Abd El-latif [12]. They also introduced new different types of subsets of supra soft topological spaces and study the relations between them in detail. The notion of $b-$open soft sets was initiated by El-sheikh and Abd El-latif [11] and extended in [38]. An applications on $b-$open soft sets were introduced in [1, 13]. It is well known to us that fuzzy soft topology is playing crucial roles in mathematics, economics, data reduction, image processing, genotype-phenotype mapping of DNA etc. In 2001-2003 Maji et al. [29, 30, 31] who studied and worked on some mathematical aspects of soft sets and fuzzy soft sets. In [6] the notion of fuzzy soft set was introduced as a fuzzy generalization of soft sets and some basic properties of fuzzy soft sets are discussed in detail. Then, many scientists such as X. Yang et al. [47], improved the concept of fuzziness of soft sets. In [3], Karal and Ahmed defined the notion of a mapping on classes of fuzzy soft sets, which is fundamental important in fuzzy soft set theory, to improve this work and they studied properties of fuzzy soft images and fuzzy soft inverse images of fuzzy soft sets. Chang [10] introduced the concept of fuzzy topology on a set X by axiomatizing a collection $\mathfrak{T}$ of fuzzy subsets of X. Tanay et al. [43] proposed the definition of fuzzy soft topology over a subset of the initial universe set while Roy and Samanta [40] gave the definition of fuzzy soft topology over the initial universe set and discussed some properties of fuzzy topological spaces related fuzzy soft base. Some fuzzy soft topological properties based on fuzzy semi (resp. β-) open soft sets, were introduced in [2, 15, 16, 25]. Connectedness is one of the important notions of topology. F. Lin [27] defined the notions of soft connectedness in soft topological spaces. Mahanta et al.[28] introduced and studied the fuzzy soft connectedness in fuzzy soft topological spaces.

As a continuation of study fuzzy soft topological spaces, we are going to introduce and investigate the properties and behaviours of fuzzy soft b-connected sets, fuzzy soft b-separated sets and fuzzy soft bs-connected sets and have established several interesting properties supported by examples, which are basic for further research on fuzzy soft topology and will fortify the footing of the theory of fuzzy soft topological space. Moreover, we show that a fuzzy soft b-disconnectedness property is not hereditary property in general. Finally, we show that the fuzzy b-irresolute surjective soft image of fuzzy soft b-connected (resp. fuzzy soft bs-connected) is also fuzzy soft b-connected (resp. fuzzy soft b-s-connected).

2 Preliminary

In this section, for the sake of completeness, we first cite some useful definitions and results.
Throughout this paper, X refers to an initial universe, E is the set of all parameters for X and I^X is the set of all fuzzy sets on X (where, $I = [0, 1]$).

Definition 2.1. [29] Let $A \subseteq E$. Then the mapping $f : A \to I^X$ defined by $f_A(e) = \mu^e_{f_A}$ (a fuzzy subset of X) is called fuzzy soft set over (X, E), where $\mu^e_{f_A} = \overline{0}$ if $e \notin A$ and $\mu^e_{f_A} \neq \overline{0}$ if $e \in A$, where $\overline{0}(x) = 0 \;\forall\; x \in X$. The family of all these fuzzy soft sets over X denoted by $FSS(X)_E$.

Several basic properties for fuzzy soft sets are given by [29, 39]

Definition 2.2. [39] Let $\mathfrak{T}$ be a collection of fuzzy soft sets over a universe X with a fixed set of parameters E, then $\mathfrak{T} \subseteq FSS(X)_E$ is called fuzzy soft topology on X if

(i) $\tilde{1}_E, \tilde{0}_E \in \mathfrak{T}$, where $\tilde{0}_E(e) = \overline{0}$ and $\tilde{1}_E(e) = \overline{1}$, $\forall e \in E$,

(ii) the union of any members of $\mathfrak{T}$ belongs to $\mathfrak{T}$,

(iii) the intersection of any two members of $\mathfrak{T}$ belongs to $\mathfrak{T}$.

The triplet $(X, \mathfrak{T}, E)$ is called fuzzy soft topological space over X. Also, each member of $\mathfrak{T}$ is called fuzzy open soft in $(X, \mathfrak{T}, E)$. We denote the set of all open soft sets by $FOS(X, \mathfrak{T}, E)$, or $FOS(X)$. A fuzzy soft set f_A over X is said to be fuzzy closed soft set in X [39], if its relative complement f'_A is fuzzy open soft set. We denote the set of all fuzzy closed soft sets by $FCS(X, \mathfrak{T}, E)$, or $FCS(X)$.

Definition 2.3. [36] Let $(X, \mathfrak{T}, E)$ be a fuzzy soft topological space and $f_A \in FSS(X)_E$.

(i) The fuzzy soft closure of f_A, denoted by $Fcl(f_A)$ is the intersection of all fuzzy closed soft super sets of f_A. i.e.,

$$Fcl(f_A) = \sqcap\{h_D : h_D \; is \; fuzzy \; closed \; soft \; set \; and \; f_A \sqsubseteq h_D\}.$$

(ii) The fuzzy soft interior of f_A, denoted by $Fint(f_A)$ is the fuzzy soft union of all fuzzy open soft subsets of f_A.i.e.,

$$Fint(f_A) = \sqcup\{h_D : h_D \; is \; fuzzy \; open \; soft \; set \; and \; h_D \sqsubseteq f_A\}.$$

Definition 2.4. [28] The fuzzy soft set $f_A \in FSS(X)_E$ is called fuzzy soft point if there exist $x \in X$ and $e \in E$ such that $\mu^e_{f_A}(x) = \alpha$ $(0 < \alpha \leq 1)$ and $\mu^e_{f_A}(y) = \overline{0}$ for each $y \in X - \{x\}$, and this fuzzy soft point is denoted by x^e_α or f_e.

Definition 2.5. [28] The fuzzy soft point x^e_α is said to be belonging to the fuzzy soft set (g, A), denoted by $x^e_\alpha \tilde{\in} (g, A)$, if for the element $e \in A$, $\alpha \leq \mu^e_{g_A}(x)$.

Theorem 2.6. [28] Let $(X, \mathfrak{T}, E)$ be a fuzzy soft topological space and f_e be a fuzzy soft point. Then, the following properties hold:

(i) If $f_e \tilde{\in} g_A$, then $f_e \tilde{\notin} g'_A$;

(ii) $f_e \tilde{\in} g_A \nRightarrow f'_e \tilde{\in} g'_A$;

(iii) Every non-null fuzzy soft set f_A can be expressed as the union of all the fuzzy soft points belonging to f_A.

Definition 2.7. [28] A fuzzy soft set g_B in a fuzzy soft topological space $(X, \mathfrak{T}, E)$ is called fuzzy soft neighborhood of the fuzzy soft point x^e_α if there exists a fuzzy open soft set h_C such that $x^e_\alpha \tilde{\in} h_C \sqsubseteq g_B$. A fuzzy soft set g_B in a fuzzy soft topological space $(X, \mathfrak{T}, E)$ is called fuzzy soft neighborhood of the soft set f_A if there exists a fuzzy open soft set h_C such that $f_A \sqsubseteq h_C \sqsubseteq g_B$. The fuzzy soft neighborhood system of the fuzzy soft point x^e_α, denoted by $N_{\mathfrak{T}}(x^e_\alpha)$, is the family of all its fuzzy soft neighborhoods.

Definition 2.8. [28] Let $(X, \mathfrak{T}, E)$ be a fuzzy soft topological space and $Y \subseteq X$. Let h^Y_E be a fuzzy soft set over (Y, E) such that $h^Y_E : E \to I^Y$ such that $h^Y_E(e) = \mu^e_{h^Y_E}$,

$$\mu^e_{h^Y_E}(x) = \begin{cases} 1 & x \in Y, \\ 0, & x \notin Y. \end{cases}$$

Let $\mathfrak{T}_Y = \{h^Y_E \sqcap g_B : g_B \in \mathfrak{T}\}$, then the fuzzy soft topology $\mathfrak{T}_Y$ on $(Y, E)$is called fuzzy soft subspace topology for (Y, E) and $(Y, \mathfrak{T}_Y, E)$ is called fuzzy soft subspace of $(X, \mathfrak{T}, E)$. If $h^Y_E \in \mathfrak{T}$ (resp. $h^Y_E \in \mathfrak{T}'$), then $(Y, \mathfrak{T}_Y, E)$ is called fuzzy open (resp. closed) soft subspace of $(X, \mathfrak{T}, E)$.

Definition 2.9. [36] Let $FSS(X)_E$ and $FSS(Y)_K$ be families of fuzzy soft sets over X and Y, respectively. Let $u : X \to Y$ and $p : E \to K$ be mappings. Then, the map f_{pu} is called fuzzy soft mapping from X to Y and denoted by $f_{pu} : FSS(X)_E \to FSS(Y)_K$ such that,

(1) If $f_A \in FSS(X)_E$. Then, the image of f_A under the fuzzy soft mapping f_{pu} is the fuzzy soft set over Y defined by $f_{pu}(f_A)$, where $\forall\ k \in p(E)$, $\forall\ y \in Y$,

$$f_{pu}(f_A)(k)(y) = \begin{cases} \bigvee_{u(x)=y} \ [\vee_{p(e)=k}(f_A(e))](x) & if\ x \in u^{-1}(y), \\ 0 & otherwise. \end{cases}$$

(2) If $g_B \in FSS(Y)_K$, then the pre-image of g_B under the fuzzy soft mapping f_{pu} is the fuzzy soft set over X defined by $f^{-1}_{pu}(g_B)$, where $\forall\ e \in p^{-1}(K)$, $\forall\ x \in X$,

$$f^{-1}_{pu}(g_B)(e)(x) = \begin{cases} g_B(p(e))(u(x)) & for\ p(e) \in B, \\ 0 & otherwise. \end{cases}$$

The fuzzy soft mapping f_{pu} is called surjective (resp. injective) if p and u are surjective (resp. injective), also it is said to be constant if p and u are constant.

Definition 2.10. [36] Let $(X, \mathfrak{T}_1, E)$ and $(Y, \mathfrak{T}_2, K)$ be two fuzzy soft topological spaces and $f_{pu} : FSS(X)_E \rightarrow FSS(Y)_K$ be a fuzzy soft mapping. Then, f_{pu} is called

(i) Fuzzy continuous soft if $f_{pu}^{-1}(g_B) \in \mathfrak{T}_1 \ \forall \ (g_B) \in \mathfrak{T}_2$.

(ii) Fuzzy open soft if $f_{pu}(g_A) \in \mathfrak{T}_2 \forall \ (g_A) \in \mathfrak{T}_1$.

Several properties and characteristics for f_{pu} and f_{pu}^{-1} are reported in detail in [3].

Definition 2.11. [16] Two fuzzy soft sets f_A, g_B are said to be disjoint, denoted by $f_A \sqcap g_B = \tilde{0}_E$, if $A \cap B = \emptyset$ and $\mu_{f_A}^e \cap \mu_{g_B}^e = \bar{0}$ for every $e \in E$

Definition 2.12. [28] Let $(X, \mathfrak{T}, E)$ be a fuzzy soft topological space. A fuzzy soft separation of $\tilde{1}_E$ is a pair of non null proper fuzzy open soft sets g_B, h_C such that $g_B \sqcap h_C = \tilde{0}_E$ and $\tilde{1}_E = g_B \sqcup h_C$.

Definition 2.13. [28] A fuzzy soft topological space $(X, \mathfrak{T}, E)$ is said to be fuzzy soft connected if and only if there is no fuzzy soft separations of $\tilde{X}$. Otherwise, $(X, \mathfrak{T}, E)$ is said to be fuzzy soft disconnected space.

3 Fuzzy Soft b-Connectedness

This section is devoted to define and discuss the fuzzy soft b-connectedness in fuzzy soft topological space and investigate its behaviours.

Definition 3.1. Let $(X, \mathfrak{T}, E)$ be a fuzzy soft topological space and $f_A \in FSS(X)_E$. If $f_A \sqsubseteq Fint(Fcl(f_A)) \sqcup Fcl(Fint(f_A))$, then f_A is called fuzzy b-open soft set. A fuzzy soft set f_A is fuzzy b-closed, if its complement is fuzzy b-open soft set.

We denote the set of all fuzzy b-open soft sets by $FBOS(X, \mathfrak{T}, E)$, or $FBOS(X)$ and the set of all fuzzy b-closed soft sets by $FBCS(X, \mathfrak{T}, E)$, or $FBCS(X)$.

Definition 3.2. Let $(X, \mathfrak{T}, E)$ be a fuzzy soft topological space, $f_A \in FSS(X)_E$ and $f_e \tilde{\in} FSS(X)_E$. Then,

(i) f_e is called fuzzy b-interior soft point of f_A if there exists $g_B \in FBOS(X)$ such that $f_e \tilde{\in} g_B \sqsubseteq f_A$. The set of all fuzzy b-interior soft points of f_A is called the fuzzy b-soft interior of f_A and is denoted by $FBint(f_A)$ consequently, $FBint(f_A) = \sqcup\{g_B : g_B \sqsubseteq f_A, \ g_B \in FBOS(X)\}$.

(ii) f_e is called fuzzy b-closure soft point of f_A if $f_A \sqcap h_D \neq \tilde{0}_E$ for every $h_D \in FBOS(X)$. The set of all fuzzy b-closure soft points of f_A is called fuzzy b-soft closure of f_A and denoted by $FBcl(f_A)$. Consequently, $FBcl(f_A) = \sqcap\{h_D : h_D \in FBCS(X), \ f_A \sqsubseteq h_D\}$.

Definition 3.3. Let $(X, \mathfrak{T}, E)$ be a fuzzy soft topological space. A fuzzy soft b-separation on $\tilde{1}_E$ is a pair of non null proper fuzzy b-open soft sets f_A, g_B such that $f_A \sqcap g_B = \tilde{0}_E$ and $\tilde{1}_E = f_A \sqcup g_B$.

Definition 3.4. A fuzzy soft topological space $(X, \mathfrak{T}, E)$ is said to be fuzzy soft b-connected if there is no fuzzy soft b-separations of $\tilde{1}_E$. Otherwise, $(X, \mathfrak{T}, E)$ is said to be fuzzy soft b-disconnected space.

Definition 3.5. A fuzzy soft topological space $(X, \mathfrak{T}, E)$ is said to be fuzzy soft b-disconnected if $(X, \mathfrak{T}, E)$ has a proper fuzzy b-open soft and fuzzy b-closed soft set in X.

Examples 3.6. (i) The discrete fuzzy soft topological space $(X, \mathfrak{T}, E)$ is not fuzzy soft b-connected i.e fuzzy soft b-disconnected .

(ii) Any space with indiscrete fuzzy soft topology is fuzzy soft connected but not it is fuzzy soft b-connected because soft b-open sets establish a discrete topology.

Example 3.7. Let $X = \{a, b\}$, $E = \{e_1, e_2, e_3\}$ and $A = \{e_1, e_2\} \subseteq E$. Let $\mathfrak{T} = \{\tilde{1}_E, \tilde{0}_E, f_A\}$ where f_A is fuzzy soft sets over X defined as follows:
$\mu^{e_1}_{f_A} = \{a_{0.4}, b_{0.2}\}$, $\mu^{e_2}_{f_A} = \{a_{0.3}, b_{0.5}\}$, $\mu^{e_3}_{f_A} = \{a_0, b_0\}$,
Consider the fuzzy soft sets g_E, h_E
$\mu^{e_1}_{g_E} = \{a_1, b_0\}$, $\mu^{e_2}_{g_E} = \{a_0, b_1\}$, $\mu^{e_3}_{g_E} = \{a_1, b_0\}$,
$\mu^{e_1}_{h_E} = \{a_0, b_1\}$, $\mu^{e_2}_{h_E} = \{a_1, b_0\}$, $\mu^{e_3}_{h_E} = \{a_0, b_1\}$,
Then, g_E, h_E be a pair of non null proper fuzzy b-open soft sets such that $g_E \sqcap h_E = \tilde{0}_E$ and $\tilde{1}_E = g_E \sqcup h_E$. Hence, $(X, \mathfrak{T}, E)$ is not fuzzy soft b-connected.

Theorem 3.8. If $(X, \mathfrak{T}_2, E)$ is a fuzzy soft b-connected space and $\mathfrak{T}_1 \subseteq \mathfrak{T}_2$, then $(X, \mathfrak{T}_1, E)$ is also a fuzzy soft b-connected.

Proof. Let f_A, g_B be fuzzy soft b-separation on $(X, \mathfrak{T}_1, E)$. Then, $f_A, g_B \in \mathfrak{T}_1$. Since $\mathfrak{T}_1 \subseteq \mathfrak{T}_2$. Then, $f_A, g_B \in \mathfrak{T}_2$ such that f_A, g_B is fuzzy soft b-separation on $(X, \mathfrak{T}_2, E)$, which is a contradiction with the fuzzy soft b-connectedness of $(X, \mathfrak{T}_2, E)$. Hence, $(X, \mathfrak{T}_1, E)$ is fuzzy soft b-connected.

Remark 3.9. The converse of Theorem 3.8 is not true in general, as shown in the following example.

Example 3.10. Let $X = \{a, b, c\}$, $E = \{e_1, e_2, e_3, e_4\}$ and $A, B \subseteq E$ where $A = \{e_1, e_2\}$ and $B = \{e_3, e_4\}$. Let $\mathfrak{T}_1$ be the indiscrete fuzzy soft topology, then $\mathfrak{T}_1$ is fuzzy soft b-connected, on the other hand, let $\mathfrak{T}_2 = \{\tilde{1}_E, \tilde{0}_E, f_A, g_A, k_B, h_B, s_E, v_E\}$ where $f_A, g_A, k_B, h_B, s_E, v_E$ are fuzzy soft sets over X defined as follows:
$\mu^{e_1}_{f_A} = \{a_1, b_1, c_1\}$, $\mu^{e_2}_{f_A} = \{a_1, b_1, c_1\}$,
$\mu^{e_1}_{g_A} = \{a_{0.2}, b_{0.5}, c_{0.8}\}$, $\mu^{e_2}_{g_A} = \{a_{0.1}, b_{0.6}, c_{0.7}\}$,
$\mu^{e_3}_{k_B} = \{a_1, b_1, c_1\}$, $\mu^{e_4}_{k_B} = \{a_1, b_1, c_1\}$,
$\mu^{e_3}_{h_B} = \{a_{0.5}, b_0, c_{0.3}\}$, $\mu^{e_4}_{h_B} = \{a_1, b_{0.8}, c_{0.3}\}$,
$\mu^{e_1}_{s_E} = \{a_{0.2}, b_{0.5}, c_{0.8}\}$, $\mu^{e_2}_{s_E} = \{a_{0.1}, b_{0.6}, c_{0.7}\}$, $\mu^{e_3}_{s_E} = \{a_1, b_1, c_1\}$, $\mu^{e_4}_{s_E} = \{a_1, b_1, c_1\}$,
$\mu^{e_1}_{v_E} = \{a_1, b_1, c_1\}$, $\mu^{e_2}_{v_E} = \{a_1, b_1, c_1\}$, $\mu^{e_3}_{v_E} = \{a_{0.5}, b_0, c_{0.3}\}$, $\mu^{e_4}_{v_E} = \{a_1, b_{0.8}, c_{0.3}\}$,
Then, $\mathfrak{T}_2$ defines a fuzzy soft topology on X such that $\mathfrak{T}_1 \subseteq \mathfrak{T}_2$. Now, f_A and k_B are fuzzy b-open soft sets in which form a fuzzy soft b-separation of $(X, \mathfrak{T}_2, E)$ where $f_A \sqcap k_B = \tilde{0}_E$ and $\tilde{1}_E = f_A \sqcup k_B$. Hence, $(X, \mathfrak{T}_2, E)$ is fuzzy soft b-disconnected.

Theorem 3.11. A fuzzy soft topological space $(X, \mathfrak{T}, E)$ is a fuzzy soft b-connected space if and only if there exist no non-zero fuzzy b-open soft sets f_A and g_B such that $f_A = g^c_B$.

Proof. Let f_A and g_B be two fuzzy b-open soft sets in $(X, \mathfrak{T}, E)$ such that $f_A \neq \tilde{0}_E$, $g^c_B \neq \tilde{0}_E$ and $f_A = g^c_B$. Therefore g^c_B is a fuzzy b-closed soft set. Since $f_A \neq \tilde{0}_E$, $g_B \neq \tilde{1}_E$. This implies that g_B is a proper fuzzy soft set which is both fuzzy b-open soft and fuzzy b-closed soft in $(X, \mathfrak{T}, E)$. Hence $(X, \mathfrak{T}, E)$ is a fuzzy soft b-disconnected space. But this is a contradiction to our hypothesis, then there exist no non-zero fuzzy b-open soft sets f_A and g_B in $(X, \mathfrak{T}, E)$ such that $f_A = g_B^c$. Conversely, Let f_A be both fuzzy b-open soft and fuzzy b-closed soft set in $(X, \mathfrak{T}, E)$ such that $f_A \neq \tilde{0}_E$, $f_A \neq \tilde{1}_E$. Let $f_A^c = g_B$, then g_B is a fuzzy b-open soft set and $g_B^c \neq \tilde{1}_E$. This implies that $g_B = f_A^c \neq \tilde{0}_E$, which is a contradiction to our hypothesis. Hence $(X, \mathfrak{T}, E)$ is a fuzzy soft b-connected space.

Theorem 3.12. A fuzzy soft topological space $(X, \mathfrak{T}, E)$ is a fuzzy soft b-connected space if and only if there exist no non-zero fuzzy b-open soft sets f_A and g_B in such that $f_A = g_B^c$, $g_B = (FBcl(f_A))^c$ and $f_A = (FBcl(g_B))^c$.

Proof. Assume that there exists fuzzy soft sets f_A and g_B such that $f_A \neq \tilde{0}_E$, $g_B^c \neq \tilde{0}_E$, $f_A = g_B^c$, $g_B = (FBcl(f_A))^c$ and $f_A = (FBcl(g_B))^c$. Since $(FBcl(f_A))^c$ and $(FBcl(g_B))^c$ are fuzzy b-open soft sets in $(X, \mathfrak{T}, E)$, f_A and g_B are fuzzy b-open soft sets in $(X, \mathfrak{T}, E)$. This implies $(X, \mathfrak{T}, E)$ is a fuzzy soft b-disconnected space, which is a contradiction. Thus there exist no non-zero fuzzy b-open soft sets f_A and g_B in such that $f_A = g_B^c$, $g_B = (FBcl(f_A))^c$ and $f_A = (FBcl(g_B))^c$. Conversely, Let f_A be both fuzzy b-open soft and fuzzy b-closed soft set in $(X, \mathfrak{T}, E)$ such that $f_A \neq \tilde{0}_E$, $f_A \neq \tilde{1}_E$. Now by taking $f_A^c = g_B$, we obtain a contradiction to our hypothesis. Hence $(X, \mathfrak{T}, E)$ is a fuzzy soft b-connected space.

Definition 3.13. A fuzzy soft subspace $(Y, \mathfrak{T}_Y, E)$ of fuzzy soft topological space $(X, \mathfrak{T}, E)$ is said to be fuzzy b-open soft (resp. b-closed soft, soft b-connected) subspace if $h_E^Y \in FBOS(X)$ (resp. $h_E^Y \in FBCS(X)$, h_E^Y is fuzzy soft b-connected).

Theorem 3.14. Let $(Y, \mathfrak{T}_Y, E)$ be a fuzzy soft b-connected subspace of fuzzy soft topological space $(X, \mathfrak{T}, E)$ such that $h_E^Y \sqcap g_A \in FBOS(X)$ $g_A \in FBOS(X)$. If $\tilde{1}_E$ has a fuzzy soft b-separations f_A, g_B. Then, either $h_E^Y \sqsubseteq f_A$, or $h_E^Y \sqsubseteq g_B$.

Proof. Let f_A, g_B be fuzzy soft b-separation on $\tilde{1}_E$. By hypothesis, $f_A \sqcap h_E^Y \in FBOS(X)$, $g_B \sqcap h_E^Y \in FBOS(X)$ and $[g_B \sqcap h_E^Y] \sqcup [f_A \sqcap h_E^Y] = h_E^Y$. Since h_E^Y is fuzzy soft b-connected. Then, either $g_B \sqcap h_E^Y = \tilde{0}_E$, or $f_A \sqcap h_E^Y = \tilde{0}_E$. Therefore, either $h_E^Y \sqsubseteq f_A$, or $h_E^Y \sqsubseteq g_B$.

Theorem 3.15. A fuzzy soft subspace $(Y, \mathfrak{T}_Y, E)$ of fuzzy soft b-disconnectedness space $(X, \mathfrak{T}, E)$ is fuzzy soft b-disconnected if $h_E^Y \sqcap g_A \in FBOS(X)$ $\forall$ $g_A \in FBOS(X)$.

Proof. Let $(Y, \mathfrak{T}_Y, E)$ be fuzzy soft b-connected space. Since $(X, \mathfrak{T}, E)$ is fuzzy soft b-disconnected. Then, there exist fuzzy soft b-separation f_A, g_B on $(X, \mathfrak{T}, E)$. By hypothesis, $f_A \sqcap h_E^Y \in FBOS(X)$, $g_B \sqcap h_E^Y \in FBOS(X)$ and $[g_B \sqcap h_E^Y] \sqcup [f_A \sqcap h_E^Y] = h_E^Y$, which is a contradiction with the fuzzy soft b-connectedness of $(Y, \mathfrak{T}_Y, E)$. Therefore, $(Y, \mathfrak{T}_Y, E)$ is fuzzy soft b-disconnected.

Remark 3.16. A fuzzy soft b-disconnectedness property is not hereditary property in general, as in the following example.

Example 3.17. In Example 3.10, let $Y = \{a, b\} \subseteq X$. We consider the fuzzy soft set h_E^Y over (Y, E) defined as follows:
$\mu_{h_E^Y}^{e_1} = \{a_1, b_1, c_0\}$, $\mu_{h_E^Y}^{e_2} = \{a_1, b_1, c_0\}$, $\mu_{h_E^Y}^{e_3} = \{a_1, b_1, c_0\}$, $\mu_{h_E^Y}^{e_4} = \{a_1, b_1, c_0\}$.
Then, we find $\mathfrak{T}_Y$ as follows, $\mathfrak{T}_Y = \{h_E^Y \sqcap z_E : z_E \in \mathfrak{T}\}$ where
$h_E^Y \sqcap \tilde{0}_E = \tilde{0}_E$, $h_E^Y \sqcap \tilde{1}_E = h_E^Y$, $h_E^Y \sqcap f_A = h_C$, where

$\mu^{e_1}_{h_C} = \{a_1, b_1, c_0\}$, $\mu^{e_2}_{h_C} = \{a_1, b_1, c_0\}$,

$h^Y_E \sqcap g_A = h_W$, where
$\mu^{e_1}_{h_W} = \{a_{0.2}, b_{0.5}, c_0\}$, $\mu^{e_2}_{h_W} = \{a_{0.1}, b_{0.6}, c_0\}$,

$h^Y_E \sqcap k_B = h_R$, where
$\mu^{e_3}_{h_R} = \{a_1, b_1, c_0\}$, $\mu^{e_4}_{h_R} = \{a_1, b_1, c_0\}$,

$h^Y_E \sqcap h_B = h_T$, where
$\mu^{e_3}_{h_T} = \{a_{0.5}, b_0, c_0\}$, $\mu^{e_4}_{h_T} = \{a_1, b_{0.8}, c_0\}$,

$h^Y_E \sqcap s_E = h_P$, where
$\mu^{e_1}_{h_P} = \{a_{0.2}, b_{0.5}, c_0\}$, $\mu^{e_2}_{h_P} = \{a_{0.1}, b_{0.6}, c_0\}$, $\mu^{e_3}_{h_P} = \{a_1, b_1, c_0\}$, $\mu^{e_4}_{h_P} = \{a_1, b_1, c_0\}$.
Thus, the collection $\mathfrak{T}_Y = \{h^Y_E \sqcap z_E : z_E \in \mathfrak{T}\}$ is a fuzzy soft topology on (Y, E) in which there is no fuzzy soft b-separation on $(Y, \mathfrak{T}_Y, E)$. Therefore, $(Y, \mathfrak{T}_Y, E)$ is fuzzy soft b-connected at the time that $(X, \mathfrak{T}, E)$ is fuzzy soft b-disconnected as shown in Example 3.10.

Definition 3.18. Let $(X, \mathfrak{T}_1, E)$, $(Y, \mathfrak{T}_2, K)$ be fuzzy soft topological spaces and $f_{pu} : FSS(X)_E \to FSS(Y)_K$ be a soft function. Then, f_{pu} is called;

(i) Fuzzy b-irresolute soft if $f^{-1}_{pu}(g_B) \in FBOS(X)\ \forall\ g_B \in FBOS(Y)$.

(ii) Fuzzy b-irresolute open soft if $f_{pu}(g_A) \in FBOS(Y)\ \forall\ g_A \in FBOS(X)$.

(iii) Fuzzy b-irresolute closed soft if $f_{pu}(f_A) \in FBCS(Y)\ \forall\ f_A \in FBCS(Y)$.

The following Theorem shows that the fuzzy soft b-connectedness is an invariant property under a fuzzy b-irresolute surjective soft function.

Theorem 3.19. Let $(X_1, \mathfrak{T}_1, E)$ and $(X_2, \mathfrak{T}_2, K)$ be fuzzy soft topological spaces and $f_{pu} : (X_1, \mathfrak{T}_1, E) \to (X_2, \mathfrak{T}_2, K)$ be a fuzzy b-irresolute surjective soft function. If $(X_1, \mathfrak{T}_1, E)$ is fuzzy soft b-connected, then $(X_2, \mathfrak{T}_2, K)$ is also a fuzzy soft b-connected.

Proof. Let $(X_2, \mathfrak{T}_2, K)$ be a fuzzy soft b-disconnected space. Then, there exist f_A, g_B pair of non null proper fuzzy b-open soft subsets of $\tilde{1}_K$ such that $f_A \sqcap g_B = \tilde{0}_K$ and $\tilde{1}_K = f_A \sqcup g_B$. Since f_{pu} is fuzzy b-irresolute soft function, then $f^{-1}_{pu}(f_A), f^{-1}_{pu}(g_B)$ are pair of non null proper fuzzy b-open soft subsets of $\tilde{1}_E$ such that $f^{-1}_{pu}(f_A) \sqcap f^{-1}_{pu}(g_B) = f^{-1}_{pu}(f_A \sqcap g_B) = f^{-1}_{pu}(\tilde{0}_K) = \tilde{0}_E$ and $f^{-1}_{pu}(f_A) \sqcup f^{-1}_{pu}(g_B) = f^{-1}_{pu}(f_A \sqcup g_B) = f^{-1}_{pu}(\tilde{1}_K) = \tilde{1}_E$. This means that, $f^{-1}_{pu}(f_A), f^{-1}_{pu}(g_B)$ forms a fuzzy soft b-separation of $\tilde{1}_E$, which is a contradiction with the fuzzy soft b-connectedness of $(X_1, \mathfrak{T}_1, E)$. Therefore, $(X_2, \mathfrak{T}_2, K)$ is fuzzy soft b-connected.

4 Fuzzy Soft b-s-Connected Spaces

In this section, we introduce the notions of fuzzy soft b-separated sets and use it to introduce the notions of fuzzy b-s-connectedness in fuzzy soft topological spaces and study some of its fundamental properties.

Definition 4.1. A non null fuzzy soft subsets f_A, g_B of fuzzy soft topological space $(X, \mathfrak{T}, E)$ are said to be fuzzy soft b-separated sets if $FBcl(f_A) \sqcap g_B = FBcl(g_B) \sqcap f_A = \tilde{0}_E$.

Theorem 4.2. Let $f_A \sqsubseteq g_B$, $h_C \sqsubseteq k_D$ and g_B, k_D are soft fuzzy soft b-separated subsets of fuzzy soft topological space $(X, \mathfrak{T}, E)$. Then, f_A, h_C are fuzzy soft b-separated sets.

Proof. Let $f_A \sqsubseteq g_B$, then $FBcl(f_A) \sqsubseteq FBcl(g_B)$. It follows that, $FBcl(f_A) \sqcap h_C \sqsubseteq FBcl(f_A) \sqcap k_D \sqsubseteq FBcl(g_B) \sqcap k_D = \tilde{0}_E$. Also, since $h_C \sqsubseteq k_D$. Then, $FBcl(h_C) \sqsubseteq FBcl(k_D)$. Hence, $f_A \sqcap FBcl(h_C) \sqsubseteq FBcl(k_D) \sqcap g_B = \tilde{0}_E$. Thus, f_A, h_C are fuzzy soft b-separated sets.

Theorem 4.3. Two fuzzy b-closed soft subsets of fuzzy soft topological space $(X, \mathfrak{T}, E)$ are fuzzy soft b-separated sets if and only if they are disjoint.

Proof. Let f_A, g_B are fuzzy soft b-separated sets. Then, $FBcl(g_B) \sqcap f_A = g_B \sqcap FBcl(f_A) = \tilde{0}_E$. Since f_A, g_B are fuzzy b-closed soft sets. Then, $f_A \sqcap g_B = \tilde{0}_E$. Conversely, let f_A, g_B are disjoint fuzzy b-closed soft sets. Then, $g_B \sqcap FBcl(f_A) = f_A \sqcap g_B = \tilde{0}_E$ and $FBcl(g_B) \sqcap f_A = f_A \sqcap g_B = \tilde{0}_E$. It follows that, f_A, g_B are fuzzy soft b-separated sets.

Theorem 4.4. Let g_B and h_C be non null fuzzy b-open soft sets of a fuzzy soft topological space $(X, \mathfrak{T}, E)$. If $u_D = g_B \sqcap (\tilde{1}_E - h_C)$ and $v_A = h_C \sqcap (\tilde{1}_E - g_B)$, then u_D and v_A are fuzzy soft b-separated sets.

Proof. Let g_B and h_C be fuzzy b-open soft sets, then $(\tilde{1}_E - g_B)$ and $(\tilde{1}_E - h_C)$ are fuzzy b-closed soft sets. Let $u_D = g_B \sqcap (\tilde{1}_E - h_C)$ and $v_A = h_C \sqcap (\tilde{1}_E - g_B)$, then $u_D \sqsubseteq (\tilde{1}_E - h_C)$ and $v_A \sqsubseteq (\tilde{1}_E - g_B)$. Hence, $FBcl(u_D) \sqsubseteq (\tilde{1}_E - h_C) \sqsubseteq (\tilde{1}_E - v_A)$ and $FBcl(v_A) \sqsubseteq (\tilde{1}_E - g_B) \sqsubseteq (\tilde{1}_E - u_D)$. Consequently, $FBcl(u_D) \sqcap v_A = \tilde{0}_E$ and $v_A \sqcap FBcl(u_D) = \tilde{0}_E$ and so u_D and v_A are fuzzy soft b-separated.

Definition 4.5. A fuzzy soft topological space $(X, \mathfrak{T}, E)$ is said to be fuzzy soft b-s-connected if and only if $\tilde{1}_E$ can not expressed as the fuzzy soft union of two fuzzy soft b-separated sets in $(X, \mathfrak{T}, E)$.

Theorem 4.6. Let $(Z, \mathfrak{T}_Z, E)$ be a fuzzy soft subspace of fuzzy soft topological space $(X, \mathfrak{T}, E)$ and f_A, $g_B \sqsubseteq z_E \sqsubseteq \tilde{1}_E$. Then, f_A and g_B are fuzzy soft b-separated on $\mathfrak{T}_Z$ if and only if f_A and g_B are fuzzy soft b-separated on $\mathfrak{T}$, where $\mathfrak{T}_Z$ is the fuzzy soft subspace for z_E.

Proof. Suppose that f_A and g_B are fuzzy soft b-separated on $\mathfrak{T}_Z$, then $FBcl_{\mathfrak{T}_Z}(f_A) \sqcap g_B = \tilde{\phi}$ and $f_A \sqcap FBcl_{\mathfrak{T}_Z}(g_B) = \tilde{0}_E$. Hence, $[FBcl_{\mathfrak{T}}(f_A) \sqcap z_E] \sqcap g_B = FBcl_{\mathfrak{T}}(f_A) \sqcap g_B = \tilde{0}_E$ and $[FBcl_{\mathfrak{T}}(g_B) \sqcap z_E] \sqcap f_A = FBcl_{\mathfrak{T}}(g_B) \sqcap f_A = \tilde{0}_E$. Consequently, f_A and g_B are fuzzy soft b-separated sets on $\mathfrak{T}$.

Theorem 4.7. Let z_E be a fuzzy soft subset of fuzzy soft topological space $(X, \mathfrak{T}, E)$. Then, z_E is fuzzy soft b-s-connected with respect to $(X, \mathfrak{T}, E)$ if and only if z_E is fuzzy soft b-s-connected with respect to $(Z, \mathfrak{T}_Z, E)$.

Proof. Suppose that z_E is not fuzzy soft b-s-connected with respect to $(Z, \mathfrak{T}_Z, E)$. Then, $z_E = f_{1A} \sqcup f_{2B}$, where f_{1A} and f_{2B} are fuzzy soft b-separated sets on $\mathfrak{T}_Z$. So $z_E = f_{1A} \sqcup f_{2B}$, where f_{1A} and f_{2B} are fuzzy soft b-separated on $\mathfrak{T}_Z$ from Theorem 4.6 . Consequently, z_E is not fuzzy soft b-s-connected with respect to $(X, \mathfrak{T}, E)$.

Theorem 4.8. Let $(Z, \mathfrak{T}_Z, E)$ be a fuzzy soft b-s-connected subspace of fuzzy soft topological space $(X, \mathfrak{T}, E)$ and $f_A,\ g_B$ be fuzzy soft b-separated of $\tilde{1}_E$ with $z_E \sqsubseteq f_A \sqcup g_B$. Then, either $z_E \sqsubseteq f_A$, or $z_E \sqsubseteq g_B$.

Proof. Let $z_E \sqsubseteq f_A \sqcup g_B$ for some fuzzy soft b-separated subsets $f_A,\ g_B$ of $\tilde{1}_E$. Since $z_E = (z_E \sqcap f_A) \sqcup (z_E \sqcap g_B)$. Then, $(z_E \sqcap f_A) \sqcap FBcl_{\mathfrak{T}}(z_E \sqcap g_B) \sqsubseteq (f_A \sqcap FBcl_{\mathfrak{T}} g_B) = \tilde{0}_E$. Also, $FBcl_{\mathfrak{T}}(z_E \sqcap f_A) \sqcap (z_E \sqcap g_B) \sqsubseteq FBcl_{\mathfrak{T}}(f_A) \sqcap g_B = \tilde{0}_E$. Since $(Z, \mathfrak{T}_Z, E)$ is fuzzy soft b-s-connected. Thus, either $z_E \sqcap f_A = \tilde{0}_E$ or $z_E \sqcap g_B = \tilde{0}_E$. It follows that, $z_E = z_E \sqcap f_A$ or $z_E = z_E \sqcap g_B$. This implies that, $z_E \sqsubseteq f_A$ or $z_E \sqsubseteq g_B$.

Theorem 4.9. Let $(Z, \mathfrak{T}_Z, N)$ and $(Y, \mathfrak{T}_Y, M)$ be fuzzy soft b-s-connected subspaces of fuzzy soft topological space $(X, \mathfrak{T}, E)$ such that none of them is fuzzy soft b-separated. Then, $z_N \sqcup y_M$ is fuzzy soft b-s-connected.

Proof. Let $(Z, \mathfrak{T}_Z, N)$ and $(Y, \mathfrak{T}_Y, M)$ be fuzzy soft b-s-connected subspaces of $\tilde{1}_E$ such that $z_N \sqcup y_M$ is not fuzzy soft b-s-connected. Then, there exist two non null fuzzy soft b-separated sets k_D and h_C of $\tilde{1}_E$ such that $z_N \sqcup y_M = k_D \sqcup h_C$. Since z_N, y_M are fuzzy soft b-s-connected, $z_N, y_M \sqsubseteq z_N \sqcup f_A = k_D \sqcup h_C$. By Theorem 4.8, either $z_N \sqsubseteq k_D$ or $z_N \sqsubseteq h_C$, also, either $y_M \sqsubseteq k_D$ or $y_M \sqsubseteq h_C$. If $z_N \sqsubseteq k_D$ or $z_N \sqsubseteq h_C$. Then, $z_N \sqcap h_C \sqsubseteq k_D \sqcap h_C = \tilde{0}_E$ or $z_N \sqcap k_D \sqsubseteq z_N \sqcap k_D = \tilde{0}_E$. Therefore, $[z_N \sqcup y_M] \sqcap k_D = [z_N \sqcap k_D] \sqcup [y_M \sqcup k_D] = [y_M \sqcap k_D] \sqcup \tilde{0}_E = y_M \sqcap k_D = y_M$ since $y_M \sqsubseteq k_D$. Similarly, if $y_M \sqsubseteq k_D$ or $y_M \sqsubseteq h_C$. we get $[z_N \sqcup y_M] \sqcap h_C = z_N$.

Now, $[(z_N \sqcup y_M) \sqcap h_C] \sqcap FBcl[(z_N \sqcup y_M) \sqcap k_D] \sqsubseteq [(z_N \sqcup y_M) \sqcap h_C] \sqcap [FBcl(z_N \sqcup y_M) \sqcap FBcl(k_D)] = [z_N \sqcup y_M] \sqcap [h_C \sqcap FBcl(k_D)] = \tilde{0}_E$ and $FBcl[(z_N \sqcup y_M) \sqcap h_C] \sqcap [(z_N \sqcup y_M) \sqcap k_D] \sqsubseteq [FBcl(z_N \sqcup y_M) \sqcap FBcl(h_C)] \sqcap [(z_N \sqcup y_M) \sqcap k_D] = [z_N \sqcup y_M] \sqcap [FBcl(h_C) \sqcap k_D] = \tilde{0}_E$. It follows that, $[z_N \sqcup y_M] \sqcap k_D = z_N$ and $[z_N \sqcup y_M] \sqcap h_C = y_M$ are fuzzy soft b-separated, which is a contradiction. Hence, $z_N \sqcup y_M$ is fuzzy soft b- s-connected.

Theorem 4.10. Let $(Z, \mathfrak{T}_Z, N)$ be a fuzzy soft b-s-connected subspace of fuzzy soft topological space $(X, \mathfrak{T}, E)$ and $S_M \in SS(X)_E$. If $z_N \sqsubseteq S_M \sqsubseteq FBcl(z_N)$. Then, $(S, \mathfrak{T}_S, M)$ is fuzzy soft b-s-connected subspace of $(X, \mathfrak{T}, E)$.

Proof. Suppose that $(S, \mathfrak{T}_S, M)$ is not fuzzy soft b-s-connected subspace of $(X, \mathfrak{T}, E)$. Then, there exist fuzzy soft b-separated sets f_A and g_B on $\mathfrak{T}$ such that $S_M = f_A \sqcup g_B$. So, we have z_N is fuzzy soft b-s-connected subset of fuzzy soft b-s-disconnected space. By Theorem 4.8, either $z_N \sqsubseteq f_A$ or $z_N \sqsubseteq g_B$. If $z_N \sqsubseteq f_A$. Then, $FBcl(z_N) \sqsubseteq FBcl(f_A)$. It follows $FBcl(z_N) \sqcap g_B \sqsubseteq FBcl(f_A) \sqcap g_B = \tilde{0}_E$. Hence, $g_B = FBcl(z_N) \sqcap g_B = \tilde{0}_E$ which is a contradiction. If $z_N \sqsubseteq g_B$. By a similar way, we can get $f_A = \tilde{0}_E$, which is a contradiction. Hence, $(S, \mathfrak{T}_S, M)$ is fuzzy soft b- s-connected subspace of $(X, \mathfrak{T}, E)$.

Corollary 4.11. If $(Z, \mathfrak{T}_Z, N)$ is fuzzy soft b-s-connected subspace of fuzzy soft topological space $(X, \mathfrak{T}, E)$. Then, $FBcl(z_N)$ is fuzzy soft b-s-connected.

Proof. It is obvious from Theorem 4.10.

Theorem 4.12. If for all pair of distinct fuzzy soft point f_e, g_e, there exists a fuzzy soft b-s-connected set $z_N \sqsubseteq \tilde{1}_E$ with $f_e, g_e \tilde{\in} z_N$, then $\tilde{1}_E$ is fuzzy soft b-s-connected.

Proof. Suppose that $\tilde{1}$ is fuzzy soft b-s-disconnected. Then, $\tilde{1}_E = f_A \sqcup g_B$, where f_A, g_B are fuzzy soft b-separated sets. It follows $f_A \sqcap g_B = \tilde{0}_E$. So, $\exists\ f_e \tilde{\in} f_A$ and $g_e \tilde{\in} g_B$. Since $f_A \sqcap g_B = \tilde{0}_E$. Then, f_e, g_e are distinct fuzzy soft point in $\tilde{1}_E$. By hypothesis, there exists a fuzzy soft b-s-connected set z_N such that $f_e, g_e \tilde{\in} z_N \sqsubseteq \tilde{1}_E$ and $f_e, g_e \tilde{\in} z_N$. Moreover, we have z_N is fuzzy soft b-s-connected subset of a a fuzzy soft b-s-disconnected space. It follows by Theorem 4.8, either $z_N \sqsubseteq f_A$ or $z_N \sqsubseteq g_B$ and both cases is a contradiction with the hypothesis. Therefore, $\tilde{1}_E$ is fuzzy soft b-s-connected.

Theorem 4.13. Let $\{(Z_j, \mathfrak{T}_{Z_j}, N) : j \in J\}$ be a non null family of fuzzy soft b-s-connected subspaces of fuzzy soft topological space $(X, \mathfrak{T}, E)$. If $\sqcap_{j\in J}(z_j, N) \neq \tilde{0}_E$, then $(\sqcup_{j\in J} Z_j, \mathfrak{T}_{\sqcup_{j\in J} Z_j}, N)$ is also a fuzzy soft b-s-connected fuzzy subspace of $(X, \mathfrak{T}, E)$.

Proof. Suppose that $(Z, \mathfrak{T}_Z, N) = (\sqcup_{j\in J} Z_j, \mathfrak{T}_{\sqcup_{j\in J} Z_j}, N)$ is fuzzy soft b-s-disconnected. Then, $z_N = f_A \sqcup g_B$ for some fuzzy soft b-separated subsets f_A, g_B of $\tilde{1}_E$. Since $\sqcap_{j\in J}(z_j, N) \neq \tilde{0}_E$. Then, $\exists\ f_e \tilde{\in} \sqcap_{j\in J}(z, N)_j$. It follows that, $f_e \tilde{\in} z_N$. So, either $f_e \tilde{\in} f_A$ or $f_e \tilde{\in} g_B$. Suppose that $f_e \tilde{\in} f_A$. Since $f_e \tilde{\in} (z, N)_j\ \forall j \in J$ and $(z, N)_j \sqsubseteq z_N$. So, we have $(z, N)_j$ is fuzzy soft b-s-connected subset of fuzzy soft b-s-disconnected set z_N. By Theorem 4.8, either $(z, N)_j \sqsubseteq f_A$ or $(z, N)_j \sqsubseteq g_B\ \forall j \in J$. If $(z, N)_j \sqsubseteq f_A\ \forall j \in J$. Then, $z_N \sqsubseteq f_A$. This implies that, $g_B = \tilde{0}_E$, which is a contradiction. Also, if $(z, N)_j \sqsubseteq g_B\ \forall j \in J$. Also, if $f_e \tilde{\in} g_B$, by a similar way, we get $f_A = \tilde{0}_E$, which is a contradiction. Therefore, $(Z, \mathfrak{T}_Z, N) = (\sqcup_{j\in J} Z_j, \mathfrak{T}_{\sqcup_{j\in J} Z_j}, N)$ is fuzzy soft b-s-connected.

Theorem 4.14. Let $\{(Z_j, \mathfrak{T}_{Z_j}, N) : j \in J\}$ be a family of fuzzy soft b-s-connected subspaces of fuzzy soft topological space $(X, \mathfrak{T}, E)$ such that one of the members of the family intersects every other members, then $(\sqcup_{j\in J} Z_j, \mathfrak{T}_{\sqcup_{j\in J} Z_j}, N)$ is fuzzy subspace of $(X, \mathfrak{T}, E)$.

Proof. Let $(Z, \mathfrak{T}_Z, N) = (\sqcup_{j\in J} Z_j, \mathfrak{T}_{\sqcup_{j\in J} Z_j}, N)$ and $(z, N)_{jo} \in \{(z, N)_j : j \in J\}$ such that $(z, N)_{jo} \sqcap (z, N)_j \neq \tilde{0}_E\ \ \forall j \in J$. Then, $(z, N)_{jo} \sqcup (z, N)_j$ is fuzzy soft b-s-connected $\forall j \in J$ by Theorem 4.9. Hence, the collection $\{(z, N)_{jo} \sqcup (z, N)_j : j \in J\}$ is a collection of fuzzy soft b-s-connected subsets of $\tilde{1}$, which having a non null fuzzy soft intersection. Therefore, $(Z, \mathfrak{T}_Z, N) = (\sqcup_{j\in J} Z_j, \mathfrak{T}_{\sqcup_{j\in J} Z_j}, N)$ is fuzzy soft b-s-connected subspace of $(X, \mathfrak{T}, E)$ by Theorem 4.9.

Theorem 4.15. Let $(X_1, \mathfrak{T}_1, E)$ and $(X_2, \mathfrak{T}_2, K)$ be fuzzy soft topological spaces and $f_{pu} : (X_1, \mathfrak{T}_1, E) \rightarrow (X_2, \mathfrak{T}_2, K)$ be a fuzzy b-irresolute surjective soft function. If $(X_1, \mathfrak{T}_1, E)$ is fuzzy soft b-s-connected, then $(X_2, \mathfrak{T}_2, K)$ is also a fuzzy soft b-s-connected.

Proof. Let $(X_2, \mathfrak{T}_2, K)$ be fuzzy soft b-disconnected space. Then, there exist f_A, g_B pair of non null proper fuzzy soft b-separated sets such that $\tilde{1}_K = f_A \sqcup g_B$, $FBcl(f_A) \sqcap g_B = FBcl(g_B) \sqcap f_A = \tilde{0}_E$. Since f_{pu} is fuzzy b-irresolute soft function, then $f_{pu}^{-1}(f_A), f_{pu}^{-1}(g_B)$ are pair of non null proper fuzzy b-open soft subsets of $\tilde{1}_E$ such that $FBcl(f_{pu}^{-1}(f_A)) \sqcap f_{pu}^{-1}(g_B) \sqsubseteq f_{pu}^{-1}(FBcl(f_A)) \sqcap f_{pu}^{-1}(g_B) = f_{pu}^{-1}(f_A \sqcap g_B) = f_{pu}^{-1}(\tilde{0}_K) = \tilde{0}_E$, $f_{pu}^{-1}(f_A) \sqcap FBcl(f_{pu}^{-1}(g_B)) \sqsubseteq f_{pu}^{-1}(f_A) \sqcap f_{pu}^{-1}(FBcl(g_B)) = f_{pu}^{-1}(f_A \sqcap g_B) = f_{pu}^{-1}(\tilde{0}_K) = \tilde{0}_E$ and $f_{pu}^{-1}(f_A) \sqcup f_{pu}^{-1}(g_B)) = f_{pu}^{-1}(f_A \sqcup g_B) = f_{pu}^{-1}(\tilde{1}_K) = \tilde{1}_E$ from [[25], Theorem 4.2]. This means that, $f_{pu}^{-1}(f_A), f_{pu}^{-1}(g_B)$ are pair of non null proper fuzzy soft b-separated sets of $\tilde{1}_E$, which is a contradiction of the fuzzy soft b-s-connectedness of $(X_1, \mathfrak{T}_1, E)$. Therefore, $(X_2, \mathfrak{T}_2, K)$ is fuzzy soft b-s-connected.

5 Conclusion

As a continuation of study fuzzy soft topological spaces, fuzzy soft b-connected sets, fuzzy soft b-separated sets and fuzzy soft b s-connected sets have been introduced and several interesting properties supported by examples have been established. Moreover, a fuzzy soft b-disconnectedness property is not hereditary property in general have been showed.

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http://www.newtheory.org ISSN: 2149-1402

Journal of New Theory

Received: 02.10.2015
Published: 18.02.2016

Year: **2016**, *Number*: **11**, *Pages*: **16-28**
Original Article[**]

SOME VARIATIONS OF JANOWSKI TYPE FUNCTIONS ASSOCIATED WITH M-SYMMETRIC POINTS

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Abstaract − In the present paper, we introduce a new subclass $S_s^m(b, \gamma, A, B)$, of starlike functions with respect to m-symmetric points. Some basic properties, Integral representations, first Hankel determinant and convolution properties for the functions belonging to this class are investigated.

Keywords − *Janowski type functions, convolution, symmetric points.*

1 Introduction

Let $\mathcal{A}$ denote the class of functions $f(z)$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1}$$

which are analytic in the open unit disc $E = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$. Denote by E, the class of functions

$$w(z) = \sum_{k=1}^{\infty} c_k z^k, \tag{2}$$

which are analytic in the open unit disk E and satisfies the conditions $w(0) = 0$ and $|w(z)| < 1$. For two functions $f(z)$ and $g(z)$ analytic in E, we say that $f(z)$ is subordinate to $g(z)$, denoted by $f \prec g$ or $f(z) \prec g(z)$, if there exists an analytic function $w(z)$ with $|w(z)| \leq |z|$ such that $f(z) = g(w(z))$. If $g(z)$ is univalent in E then $f(z) \prec g(z)$ if and only if $f(0) = g(0)$ and $f(E) \subset g(E)$. The idea of subordinations goes back to Lindelöf [8]. Subordination was more formally introduced and studied by Littelwood [9] and later by Rogosinski [15] and [16]. The concept of subordination was considered by Miller [12] and further investigated by Noor et al [13] and many others see [8, 17].

[**]*Edited by* Ali Yakar *and* Naim Çağman *(Editor-in-Chief).*
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Sakaguchi [18] introduced a class of functions starlike with respect to symmetric points, it consists of functions $f(z) \in S$, satisfying the inequality

$$\mathfrak{Re}\left\{\frac{zf'(z)}{f(z)-f(-z)}\right\} > 0, \qquad (z \in E). \tag{3}$$

Chand and Singh [2] introduced a class $\mathcal{S}_s^m$ of functions starlike with respect to m-symmetric points, which consists of functions $f(z) \in S$ satisfying the inequality

$$\mathfrak{Re}\left\{\frac{zf'(z)}{f_m(z)}\right\} > 0, \qquad (z \in E), \tag{4}$$

where

$$f_m(z) = \frac{1}{m}\sum_{\upsilon=0}^{m-1} \epsilon^{-\upsilon} f(\epsilon^{\upsilon} z), \quad (\epsilon^m = 1 : m \in \mathbb{N}). \tag{5}$$

From (5), we can write

$$f_m(z) = \frac{1}{m}\sum_{\upsilon=0}^{m-1} \epsilon^{-\upsilon} f(\epsilon^{\upsilon} z) = \frac{1}{m}\sum_{\upsilon=0}^{m-1} \epsilon^{-\upsilon}\left[\epsilon^{\upsilon} z + \sum_{n=2}^{\infty} a_n (\epsilon^{\upsilon} z)^n\right] \tag{6}$$

$$= z + \sum_{n=2}^{\infty} a_n \psi_n z^n, \tag{7}$$

where,

$$\psi_n = \frac{1}{m}\sum_{\upsilon=0}^{m-1} \epsilon^{(n-1)\upsilon}, \quad (m \in \mathbb{N};\ n \geq 2;\ \epsilon^m = 1). \tag{8}$$

Note that the following identities follow directly from the above definition (5),

$$f_m(\epsilon^{\upsilon} z) = \epsilon^{\upsilon} f_m(z), \tag{9}$$

$$f'_m(\epsilon^{\upsilon} z) = f_m(z) = \frac{1}{m}\sum_{\upsilon=0}^{m-1} f(\epsilon^{\upsilon} z), \quad (z \in E). \tag{10}$$

Using the concept of subordination we introduce a subclass $\mathcal{S}_s^m(b, \gamma, A, B)$ as follows.

Definition 1.1. A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{S}_s^m(b, \gamma, A, B)$, if it satisfies the following subordination condition:

$$1 + \frac{1}{b}\left(\frac{zf'(z)}{f_m(z)} - 1\right) \prec \left(\frac{1+Az}{1+Bz}\right)^{\gamma}, \quad (z \in E), \tag{11}$$

where $b \in \mathbb{C} \setminus \{0\}$, $-1 \leq B < A \leq 1$ and $0 < \gamma \leq 1$.

Where $f_m(z)$ is defined in equation (5). To avoid repetitions, it is admitted once that $b \in \mathbb{C} \setminus \{0\}$, $-1 \leq B < A \leq 1$ and $0 < \gamma \leq 1$.

Special Cases;

(i) For $b = 1$, $\gamma = 1$, we obtain the class studied by Al-Shaqsi and Darus [1].

(ii) For $b = 1$, $\gamma = 1, A = \beta, B = -\alpha\beta$, we obtain the class studied by Gao and Zhou [3].

(iii) For $m=1, b=1, \gamma=1$, we obtain the class studied by Janowski [4].
(iv) For $m=1, b=e^{-i\lambda}\lambda, \gamma=1, A=1-2\delta, B=-1$, we obtain the class studied by Keogh and Markes [5].
(v) For $m=1, b=(1-\rho)e^{-i\beta}, \gamma=1, A=1, B=-1$, we obtain the class studied by Libera [7].
(vi) For $b=1, A=1, B=-1$, we obtain the class studied by Ming-Sheng Liu and Yu-Can Zhu [10].
(vii) For $m=2, b=1, A=1, B=-1$, we obtain the class studied by V. Ravichandran [14].
$(viii)$ For $m=2, b=1, \gamma=1, A=1, B=-1$, we obtain the class studied by Sakaguchi [18].

2 Preliminary

To prove our main results we need the following Lemmas.

Lemma 2.1. [11] If $p(z)=1+c_1z+c_2z^2+\ldots$ is analytic function with positive real part in E and υ is a complex number, then

$$\left|c_2-\upsilon c_1^2\right|\leq 2\max\{1,|2\upsilon-1|\}.$$

The result is sharp for the functions given by

$$p(z)=\frac{1+z}{1-z},\quad p(z)=\frac{1+z^2}{1-z^2}.$$

Lemma 2.2. [11] If $p(z)=1+c_1z+c_2z^2+\ldots$ is analytic function with positive real part in E, then

$$\left|c_2-\upsilon c_1^2\right|\leq\begin{cases}-4\upsilon+2, & (\upsilon\leq 0),\\ 2, & (0\leq\upsilon\leq 1),\\ 4\upsilon-2, & (\upsilon\geq 1).\end{cases}$$

When $\upsilon<0$ or $\upsilon>0$, equality holds if and only if $p(z)=\frac{1+z}{1-z}$ or one of its rotations. If $0<\upsilon<1$, then the equality holds if and only if $p(z)=\frac{1+z^2}{1-z^2}$ or one of its rotations. If $\upsilon=0$, the equality holds if and only if

$$p(z)=\left(\frac{1+\xi}{2}\right)\left(\frac{1+z}{1-z}\right)+\left(\frac{1-\xi}{2}\right)\left(\frac{1-z}{1+z}\right),\quad (0\leq\xi\leq 1),$$

or one of its rotations. For $\upsilon=1$, equality holds if and only if $p(z)$ is the reciprocal of one of the functions such that equality holds in the case of $\upsilon=0$. Although the above upper bound is sharp, it can be improved as follows when $0<\upsilon<1$:

$$\begin{aligned}\left|c_2-\upsilon c_1^2\right|+\upsilon\left|c_1\right|^2 &\leq 2,\quad (0<\upsilon\leq\frac{1}{2}),\\ \left|c_2-\upsilon c_1^2\right|+(1-\upsilon)\left|c_1\right|^2 &\leq 2,\quad (\frac{1}{2}\leq\upsilon<1).\end{aligned}$$

In this paper, we investigate integral representation, Feketo-Szegö inequality and convolution properties for the class $\mathcal{S}_s^m(b,\gamma,A,B)$. The motivation of this paper is to improve and generalize previously known results.

3 Main Results

3.1 Integral Representation

First we give meaningful conclusion about the class $\mathcal{S}_s^m(b,\gamma,A,B)$.

Theorem 3.1. Let $f(z)\in\mathcal{S}_s^m(b,\gamma,A,B)$, then $f_m(z)\in\mathcal{S}^*(b,\gamma,A,B)\subset\mathcal{S}$.

Proof. For $f(z)\in\mathcal{S}_s^m(b,\gamma,A,B)$, we can obtain

$$1+\frac{1}{b}\left(\frac{zf'(z)}{f_m(z)}-1\right)\prec\left(\frac{1+Az}{1+Bz}\right)^{\gamma},\quad (z\in E).\tag{12}$$

Substituting z by $\epsilon^{\mu}z$ respectively $(\upsilon=0,1,2,3,\ldots,m-1)$, we have

$$1+\frac{1}{b}\left(\frac{\epsilon^{\upsilon}zf'(\epsilon^{\upsilon}z)}{f_m(\epsilon^{\upsilon}z)}-1\right)\prec\left(\frac{1+A(\epsilon^{\upsilon}z)}{1+B(\epsilon^{\upsilon}z)}\right)^{\gamma}\prec\left(\frac{1+Az}{1+Bz}\right)^{\gamma},\quad (z\in E).\tag{13}$$

By definition of $f_m(z)$ and $\epsilon=\exp\left(\frac{2\pi}{m}\right)$, we know $\epsilon^{-\upsilon}f_m(\epsilon^{\upsilon}z)=f_m(z)$. Then equation (13) becomes

$$1+\frac{1}{b}\left(\frac{zf'(\epsilon^{\upsilon}z)}{f_m(\epsilon^{\upsilon}z)}-1\right)\prec\left(\frac{1+Az}{1+Bz}\right)^{\gamma},\quad (z\in E).\tag{14}$$

Let $\upsilon=0,1,2,3,\ldots,m-1$ in (14), respectively and sum them we can get

$$1+\frac{1}{b}\left(\frac{zf_m'(z)}{f_m(z)}-1\right)=1+\frac{1}{b}\left(\frac{1}{m}\sum_{\mu=0}^{m-1}\frac{zf'(\epsilon^{\upsilon}z)}{f_m(z)}-1\right)\prec\left(\frac{1+Az}{1+Bz}\right)^{\gamma},\quad (z\in E).$$

That is, $f_m(z)\in\mathcal{S}_s^m(b,\gamma,A,B)\subset\mathcal{S}$. □

Putting $b=1,\gamma=1$ in Theorem 3.1, we can obtain the following result obtained by O. Kwon and Y. Sim [6].

Corollary 3.2. Let $f(z)\in\mathcal{S}_s^m(A,B)$, then $f_m(z)\in\mathcal{S}^*(A,B)\subset\mathcal{S}$.

Putting $b=1$, $\gamma=1$, $A=1$, $B=-1$ and $m=2$, in Theorem 3.1, we can obtain the Corollary 3.3, below which is comparable to the corollary of O. Kwon and Y. Sim [6, Cor.2.2].

Corollary 3.3. Let $f(z)\in\mathcal{S}_s^m$, defined as (3). Then the odd function,

$$\frac{1}{2}\left(f(z)-f(-z)\right),$$

is a starlike function.

Now we give the integral representations of the functions belonging to the class $\mathcal{S}_s^m(b,\gamma,A,B)$.

Theorem 3.4. Let $f(z)\in\mathcal{S}_s^m(b,\gamma,A,B)$. Then

$$f_m(z)=z\cdot\exp\int_0^{\epsilon^{\upsilon}z}\left\{\frac{b}{m}\sum_{\upsilon=0}^{m-1}\frac{1}{t}\left(\left(\frac{1+Aw(t)}{1+Bw(t)}\right)^{\gamma}-1\right)\right\}\mathrm{d}t.\tag{15}$$

where $f_m(z)$ is define by (5), $w(z)$ is analytic in E with $w(0)=0$, and $|w(z)|<1$.

Proof. Suppose that $f(z) \in \mathcal{S}_s^m(b, \gamma, A, B)$. It is easy to know that condition (11), can be written as

$$1 + \frac{1}{b}\left(\frac{zf'(z)}{f_m(z)} - 1\right) = \left(\frac{1 + Aw(z)}{1 + Bw(z)}\right)^{\gamma}, \tag{16}$$

where $w(z)$ is analytic in E and $w(0) = 0$, $|w(z)| < 1$. Substituting z by $\epsilon^{\upsilon} z$ respectively $(\upsilon = 0, 1, 2, \ldots, m-1)$, we have

$$\frac{zf'(\epsilon^{\upsilon} z)}{\epsilon^{-\upsilon} f_m(\epsilon^{\upsilon} z)} = b\left(\left(\frac{1 + Aw(\epsilon^{\upsilon} z)}{1 + Bw(\epsilon^{\upsilon} z)}\right)^{\gamma} - 1\right) + 1. \tag{17}$$

Letting $\upsilon = 0, 1, 2, \ldots, m-1$ in (17), respectively and summing them we can obtain

$$\frac{zf_m'(z)}{f_m(z)} = \frac{1}{m}\sum_{\upsilon=0}^{m-1}\left\{b\left(\left(\frac{1 + Aw(\epsilon^{\upsilon} z)}{1 + Bw(\epsilon^{\upsilon} z)}\right)^{\gamma} - 1\right) + 1\right\}. \tag{18}$$

From the equality (18), we can obtain

$$\frac{f_m'(z)}{f_m(z)} - \frac{1}{z} = \frac{1}{m}\sum_{\upsilon=0}^{m-1}\frac{1}{z}\left\{b\left(\left(\frac{1 + Aw(\epsilon^{\upsilon} z)}{1 + Bw(\epsilon^{\upsilon} z)}\right)^{\gamma} - 1\right) + 1\right\} - \frac{1}{z}. \tag{19}$$

Integrating equality (19), we have

$$\begin{aligned} \log\frac{f_m(z)}{z} &= \int_0^z \left\{\frac{1}{m}\sum_{\upsilon=0}^{m-1}\frac{1}{\zeta}\left\{b\left(\left(\frac{1 + Aw(\epsilon^{\upsilon}\zeta)}{1 + Bw(\epsilon^{\upsilon}\zeta)}\right)^{\gamma} - 1\right) + 1\right\} - \frac{1}{\zeta}\right\} \mathrm{d}\zeta \\ &= \int_0^{\epsilon^{\upsilon} z} \left\{\frac{1}{m}\sum_{\upsilon=0}^{m-1}\frac{1}{t}\left\{b\left(\left(\frac{1 + Aw(t)}{1 + Bw(t)}\right)^{\gamma} - 1\right) + 1\right\} - \frac{1}{t}\right\} \mathrm{d}t. \end{aligned} \tag{20}$$

That is

$$f_m(z) = z \cdot \exp\int_0^{\epsilon^{\upsilon} z}\left\{\frac{1}{m}\sum_{\upsilon=0}^{m-1}\frac{b}{t}\left(\left(\frac{1 + Aw(t)}{1 + Bw(t)}\right)^{\gamma} - 1\right)\right\} \mathrm{d}t.$$

Hence the proof of the Theorem 3.4 is complete. □

Putting $b = 1$ and $\gamma = 1$ in Theorem 3.4, we can obtain the following result obtained by O. Kwon and Y. Sim [6].

Corollary 3.5. Let $f(z) \in \mathcal{S}_m(A, B)$. Then

$$f_m(z) = z \cdot \exp\left\{\frac{A - B}{m}\sum_{\upsilon=0}^{m-1}\int_0^{\epsilon^{\upsilon} z}\frac{w(t)}{t(1 + Bw(t))}\mathrm{d}t\right\},$$

where $f_m(z)$ is define by (5) and $w(z)$ is analytic in E with $w(0) = 0$, $|w(z)| < 1$.

Theorem 3.6. Let $f(z) \in \mathcal{S}_s^m(b, \gamma, A, B)$. Then

$$f(z) = \int_0^z \left\{\begin{array}{c} \exp\int_0^{\epsilon^{\upsilon} z}\left\{\frac{b}{m}\sum_{\upsilon=0}^{m-1}\frac{1}{t}\left(\left(\frac{1+Aw(t)}{1+Bw(t)}\right)^{\gamma} - 1\right)\right\}\mathrm{d}t \\ \times\left(1 + b\left(\left(\frac{1+Aw(\zeta)}{1+Bw(\zeta)}\right)^{\gamma} - 1\right)\right) \end{array}\right\} \mathrm{d}\zeta, \tag{21}$$

where $f_m(z)$ is define by (5) and $w(z)$ is analytic in E with $w(0) = 0$, $|w(z)| < 1$.

Proof. Let $f(z) \in \mathcal{S}_s^m(b,\gamma,A,B)$. Then from equalities (15) and (16) we have

$$f'(z) = \frac{f_m(z)}{z} \cdot \left(1 + b\left(\left(\frac{1+Aw(z)}{1+Bw(z)}\right)^\gamma - 1\right)\right)$$

$$\begin{aligned} f'(z) &= \exp \int_0^{\epsilon^\upsilon z} \left\{ \frac{1}{m} \sum_{\upsilon=0}^{m-1} \frac{1}{t} \left\{ b\left(\left(\frac{1+Aw(t)}{1+Bw(t)}\right)^\gamma - 1\right) + 1 \right\} - \frac{1}{t} \right\} dt \\ &\quad \times \left(1 + b\left(\left(\frac{1+Aw(z)}{1+Bw(z)}\right)^\gamma - 1\right)\right). \end{aligned}$$

Integrating the above equality, we can obtain (21). Hence the proof of the Theorem 3.6 is complete. □

Putting $b = 1$ and $\gamma = 1$ in Theorem 3.6, we can obtain the following result obtained by O. Kwon and Y. Sim [6].

Corollary 3.7. Let $f(z) \in S_s^m(A,B)$. Then

$$f(z) = \int_0^z \exp\left\{ \frac{A-B}{m} \sum_{\upsilon=0}^{m-1} \int_0^{\epsilon^\upsilon \zeta} \frac{w(t)}{t(1+Bw(t))} dt \right\} \cdot \left(\frac{1+Aw(\zeta)}{1+Bw(\zeta)}\right) d\zeta,$$

where $f_m(z)$ is define by (5) and $w(z)$ is analytic in E with $w(0) = 0$, $|w(z)| < 1$.

3.2 Coefficient Problems

A typical problem in geometric function theory is to study a functional made up of combinations of the coefficients of the original function. Usually, there is a parameter over which the extremal value of the functional is needed. Here we deals with one important functional of this type the Fekete-Szegö functional. In this sections, we proved the first Hankel determinant and Feketo-Szegö inequality for the functions belonging to the class $S_s^m(b,\gamma,A,B)$.

Theorem 3.8. Let $f(z) \in S_s^m(b,\gamma,A,B)$. Then

$$\left|a_3 - \mu a_2^2\right| \le \frac{|b|\gamma(A-B)}{(3-\psi_3)} \max\left\{1, \left| \begin{array}{c} \frac{2}{(2-\psi_2)^2}\{(2-\psi_2^2)\left[B + \frac{1}{2}(1-\gamma)(A-B)\right] - b\gamma(A-B) \\ ((2-\psi_2)\psi_2 - (3-\psi_3)\mu) - (2-\psi_2)^2\} \end{array} \right| \right\},$$

where ψ_n, is given in (8). The result is sharp.

Proof. Suppose that $f(z) \in S_s^m(b,\gamma,A,B)$. It is easy to know that condition (11), can be written as

$$1 + \frac{1}{b}\left(\frac{zf'(z)}{f_m(z)} - 1\right) = \left(\frac{1+Aw(z)}{1+Bw(z)}\right)^\gamma, \tag{22}$$

where $w(z)$ is analytic in E and $w(0) = 0$, $|w(z)| < 1$. We can write equation (22), as

$$\frac{zf'(z)}{f_m(z)} = 1 + b\left(\left(\frac{1+Aw(z)}{1+Bw(z)}\right)^\gamma - 1\right). \tag{23}$$

By expanding the L.H.S of (23), we can obtain

$$1 + (2-\psi_2)\, a_2 z + \left((3-\psi_3)\, a_3 + \left(\psi_2^2 - 2\psi_2\right) a_2^2\right) z^2 + \cdots , \tag{24}$$

similarly expanding the R.H.S of (23), we can obtain

$$1+b\times(\gamma\,(A-B)\,c_1 z$$

$$+(\gamma\,(A-B)\,c_2 - \gamma\,(A-B)\,[B+\frac{1}{2}\,(1-\gamma)\,(A-B)]c_1^2)z^2+\cdots). \tag{25}$$

Equating the coefficients of z and z^2 in (24) and (25) we have

$$a_2 = \frac{b\gamma\,(A-B)}{(2-\psi_2)} c_1, \tag{26}$$

and

$$\begin{aligned} a_3 \;&=\; \frac{b\gamma\,(A-B)}{(3-\psi_3)} c_2 - \frac{b\gamma\,(A-B)}{(3-\psi_3)\,(2-\psi_2)} \\ &\times \left((2-\psi_2)\,[B+\frac{1}{2}\,(1-\gamma)\,(A-B)] - b\psi_2\gamma\,(A-B)\right) c_1^2. \end{aligned} \tag{27}$$

For any complex number μ, we have

$$a_3-\mu a_2^2 = \frac{b\gamma\,(A-B)}{(3-\psi_3)} c_2 - \frac{b\gamma\,(A-B)}{(2-\psi_2)^2\,(3-\psi_3)}$$

$$\times\left\{(2-\psi_2)^2\,[B+\frac{1}{2}\,(1-\gamma)\,(A-B)] - b\gamma\,(A-B)\,((2-\psi_2)\,\psi_2 - (3-\psi_3)\,\mu)\right\} c_1^2$$

or

$$\begin{aligned} a_3-\mu a_2^2 \;&=\; \frac{b\gamma\,(A-B)}{3-\psi_3} \\ &\times\left[c_2 - \frac{1}{(2-\psi_2)^2}\left\{\begin{array}{c} (2-\psi_2)^2\,[B+\frac{1}{2}\,(1-\gamma)\,(A-B)] \\ -b\gamma\,(A-B)\,((2-\psi_2)\,\psi_2-(3-\psi_3)\,\mu) \end{array}\right\} c_1^2\right], \end{aligned}$$

we can write

$$a_3-\mu a_2^2 = \frac{b\gamma\,(A-B)}{(3-\psi_3)}\left\{c_2 - \nu c_1^2\right\},$$

where

$$\nu = \frac{1}{(2-\psi_2)^2}$$

$$\times\left\{(2-\psi_2)^2 \times [B+\frac{1}{2}\,(1-\gamma)\,(A-B)] - b\gamma\,(A-B)\,((2-\psi_2)\,\psi_2-(3-\psi_3)\,\mu)\right\},$$

our result now follows directly by an applications of Lemma 2.1. Equality can be attained by the function

$$1+\frac{1}{b}\left(\frac{zf'(z)}{f_m(z)}-1\right) = \left(\frac{1+Az}{1+Bz}\right)^{\gamma},$$

or

$$1+\frac{1}{b}\left(\frac{zf'(z)}{f_m(z)}-1\right)=\left(\frac{1+Az^2}{1+Bz^2}\right)^{\gamma}.$$

Hence the proof of the Theorem 3.8 is complete. □

Putting $b=1, \gamma=1, A=1$ and $B=-1$ in Theorem 3.8, we can obtain the following Corollary.

Corollary 3.9. Let $f(z)\in S_s^m$. Then

$$\left|a_3-\mu a_2^2\right|\le\frac{2}{(3-\psi_3)}$$

$$\max\left\{1,\left|\frac{2}{(2-\psi_2)^2}\left\{\left(2-\psi_2^2\right)+2\left((2-\psi_2)\psi_2-(3-\psi_3)\mu\right)\right\}+1\right|\right\},$$

where ψ_n, is given by (8). The result is sharp.

Setting $m=1$, $b=1$, $A=1$ and $B=-1$ in Theorem 3.8, we can obtain Corollary 2.2, below which is comparable to the result obtain by Cho and Owa [19, Th. 2.1,$\alpha=0$].

Corollary 3.10. Let $f(z)\in\widetilde{\mathcal{S}}_{\gamma}^{*}$. Then

$$\left|a_3-\mu a_2^2\right|\le\gamma\max\left\{1\ ,\left|(3-4\mu)\gamma\right|\right\},$$

Putting $b=1, \gamma=1, \mu=1, A=1$ and $B=-1$ in Theorem 3.8, we can obtain the following Corollary.

Corollary 3.11. Let $f(z)\in S_s^m$. Then

$$\left|a_3-a_2^2\right|\le\frac{2}{(3-\psi_3)}$$

$$\max\left\{1,\left|\frac{2}{(2-\psi_2)^2}\left\{\left(2-\psi_2^2\right)+2\left((2-\psi_2)\psi_2-(3-\psi_3)\right)\right\}+1\right|\right\},$$

where ψ_n, is given by (8). The result is sharp.

Theorem 3.12. Let $f(z)\in S_s^m(b,\gamma,A,B)$. If $b>0$, then

$$\left|a_3-\mu a_2^2\right|\le$$
$$\begin{cases}\frac{2b\gamma(A-B)}{(3-\psi_3)(2-\psi_2)^2}[2b\gamma(A-B)\left((2-\psi_2)\psi_2-(3-\psi_3)\mu\right)\\ \qquad -2(2-\psi_2)^2\left(B+\frac{1}{2}(1-\gamma)(A-B)\right)+(2-\psi_2)^2], & (\mu\le\sigma_1),\\ \qquad \frac{4b\gamma(A-B)}{(3-\psi_3)}, & (\sigma_1\le\mu\le\sigma_2),\\ \frac{2b\gamma(A-B)}{(3-\psi_3)(2-\psi_2)^2}[2(2-\psi_2)^2\left[\left(B+\frac{1}{2}(1-\gamma)(A-B)\right)\right.\\ \qquad -2b\gamma(A-B)\left((2-\psi_2)\psi_2-(3-\psi_3)\mu\right)-(2-\psi_2)^2], & (\mu\ge\sigma_2),\end{cases}$$

where

$$\sigma_1 = \frac{b\gamma (A-B)(2-\psi_2)\psi_2 - (2-\psi_2)^2 \left[B + \frac{1}{2}(1-\gamma)(A-B)\right]}{b\gamma (A-B)(3-\psi_3)}, \tag{28}$$

and

$$\sigma_2 = \frac{(2-\psi_2)^2 + b\gamma (A-B)(2-\psi_2)\psi_2 - (2-\psi_2)^2 \left[B + \frac{1}{2}(1-\gamma)(A-B)\right]}{b\gamma (A-B)(3-\psi_3)}, \tag{29}$$

where ψ_n, is given by (8). The result is sharp.

Proof. Since

$$\left|a_3 - \mu a_2^2\right| \leq \frac{\gamma b (A-B)}{(3-\psi_3)}$$

$$\times \left[c_2 - \frac{1}{(2-\psi_2)^2} \left\{ \begin{array}{c} (2-\psi_2)^2 \left(B + \frac{1}{2}(1-\gamma)(A-B)\right) \\ -b\gamma (A-B)((2-\psi_2)\psi_2 - (3-\psi_3)\mu) \end{array} \right\} c_1^2 \right],$$

therefore using Lemma (2.2), we can get the required result. To show that the bounds are sharp, we defined the functions $F(z)$, as follows:

$$1 + \frac{1}{b}\left(\frac{zF'(z)}{F_m(z)} - 1\right) = \left(\frac{1+Az}{1+Bz}\right)^{\gamma}, \text{ if } \mu < \sigma_1, \mu > \sigma_2,$$

$$1 + \frac{1}{b}\left(\frac{zF'(z)}{F_m(z)} - 1\right) = \left(\frac{1+Az^2}{1+Bz^2}\right)^{\gamma}, \text{ if } \sigma_1 < \mu < \sigma_1,$$

$$1 + \frac{1}{b}\left(\frac{zF'(z)}{F_m(z)} - 1\right) = \left(\frac{1+A\phi(z)}{1+B\phi(z)}\right)^{\gamma}, \text{ if } \mu = \sigma_1,$$

$$1 + \frac{1}{b}\left(\frac{zF'(z)}{F_m(z)} - 1\right) = \left(\frac{1-A\phi(z)}{1-B\phi(z)}\right)^{\gamma}, \text{ if } \mu = \sigma_2,$$

where $\phi(z) = \frac{z(z+\eta)}{1+\eta z}$ with $0 \leq \eta \leq 1$. □

Setting $m = 1$, $b = 1$, $A = 1$ and $B = -1$ in Theorem 3.12, we can obtain Corollary 3.13, below which is comparable to the result obtain by Cho and Owa [19, Th.2.1, $\alpha = 0$].

Corollary 3.13. Let $f(z) \in \widetilde{\mathcal{S}}^*_{\gamma}$. Then

$$\left|a_3 - \mu a_2^2\right| \leq \begin{cases} (3-4\mu)\gamma^2 & \left(\mu \leq \frac{3\gamma-1}{4\gamma}\right), \\ \gamma, & \left(\frac{3\gamma-1}{4\gamma} \leq \mu \leq \frac{3\gamma+1}{4\gamma}\right), \\ (4\mu-3)\gamma^2 & \left(\mu \geq \frac{3\gamma+1}{4\gamma}\right). \end{cases}$$

The result is sharp.

Theorem 3.14. Let $f(z) \in S_s^m(b,\gamma,A,B)$. If $b>0$, then

$$\left|a_3-\mu a_2^2\right|+\left(\mu-\frac{b\gamma(A-B)(2-\psi_2)\psi_2-(2-\psi_2)^2\left[B+\frac{1}{2}(1-\gamma)(A-B)\right]}{b\gamma(A-B)(3-\psi_3)}\right)|a_2|^2$$

$$\leq \frac{4b\gamma(A-B)}{(3-\psi_3)}, \qquad (\sigma_1 \leq \mu \leq \sigma_3),$$

$$\left|a_3-\mu a_2^2\right|$$

$$+\left(\frac{(2-\psi_2)^2+b\gamma(A-B)(2-\psi_2)\psi_2-(2-\psi_2)^2\left[B+\frac{1}{2}(1-\gamma)(A-B)\right]}{b\gamma(A-B)(3-\psi_3)}-\mu\right)|a_2|^2$$

$$\leq \frac{4b\gamma(A-B)}{(3-\psi_3)}, \qquad (\sigma_3 \leq \mu \leq \sigma_2),$$

where σ_1 and σ_2 are given by (28) and (29) and

$$\begin{aligned}\sigma_3 &= \frac{(2-\psi_2)^2}{2b\gamma(A-B)(3-\psi_3)}\\ &\quad\times\left(b\gamma(A-B)(2-\psi_2)\psi_2-(2-\psi_2)^2\left[B+\frac{1}{2}(1-\gamma)(A-B)\right]\right).\end{aligned}$$

The result is sharp.

Proof. The proof of Theorem 3.14 is similar to the proof of Theorem 3.12 so the details are omitted. □

Putting $b=1$, $\gamma=1$, $A=1$ and $B=-1$ in Theorem (3.14), we can obtain the following Corollary.

Corollary 3.15. Let $f(z) \in S_s^m(b,\gamma,A,B)$. If $b>0$, then

$$\left|a_3-\mu a_2^2\right|+\left(\mu-\frac{2(2-\psi_2)\psi_2+(2-\psi_2)^2}{2(3-\psi_3)}\right)|a_2|^2$$

$$\leq \frac{8}{(3-\psi_3)}, \qquad (\sigma_1 \leq \mu \leq \sigma_3),$$

$$\left|a_3-\mu a_2^2\right|+\left(\frac{(2-\psi_2)^2+2(2-\psi_2)\psi_2+(2-\psi_2)^2}{2(3-\psi_3)}-\mu\right)|a_2|^2$$

$$\leq \frac{8}{(3-\psi_3)}, \qquad (\sigma_3 \leq \mu \leq \sigma_2),$$

where σ_1 and σ_2 are given by (28) and (29) and

$$\sigma_3=\frac{(2+\psi_2)(2-\psi_2)^3}{4(3-\psi_3)}.$$

The result is sharp.

3.3 Convolution Properties

In this sections the necessary and sufficient conditions are given in terms of convolution operators for a function to be in the class $S_s^m(b,\gamma,A,B)$.

Theorem 3.16. A function $f(z)\in S_s^m(b,\gamma,A,B)$, if and only if

$$\frac{1}{z}\left\{f(z)*\left(\frac{z}{(1-z)^2}\left(\left(1+B(e^{i\theta})\right)^\gamma\right)-\left(\left(1+A(e^{i\theta})\right)^\gamma\right)h(z)\right)\right\}\neq 0, \tag{30}$$

for all $z\in E$ and $0\le\theta<2\pi$, where $h(z)$ is given by (36).

Proof. Assume that $f(z)\in S_s^m(b,\gamma,A,B)$, then we have

$$\frac{zf'(z)}{f_m(z)}\prec\left(\frac{1+Az}{1+Bz}\right)^\gamma,\qquad (z\in E). \tag{31}$$

If and only if

$$\frac{zf'(z)}{f_m(z)}\neq\left(\frac{1+A(e^{i\theta})}{1+B(e^{i\theta})}\right)^\gamma, \tag{32}$$

for all $z\in E$, and $0\le\theta<2\pi$. The condition (32), can be written as

$$\frac{1}{z}\left\{zf'(z)\left[\left(1+B(e^{i\theta})\right)^\gamma\right]-f_m(z)\left[\left(1+A(e^{i\theta})\right)^\gamma\right]\right\}\neq 0. \tag{33}$$

On the other hand it is well known that

$$zf'(z)=f(z)*\frac{z}{(1-z)^2}. \tag{34}$$

And from the definition of $f_m(z)$, we have

$$f_m(z)=z+\sum_{n=2}^{\infty}a_n\psi_n z^n=(f*h)(z), \tag{35}$$

where

$$h(z)=z+\sum_{n=2}^{\infty}\psi_n z^n, \tag{36}$$

for

$$\psi_n=\begin{cases}1, & n=lm+1\\ 0, & n\neq lm+1\end{cases}\qquad (l\in\mathbb{N}_0),$$

substituting (34) and (35) in (33) we can get (30). This completes the proof of the Theorem 3.16. □

4 Conclusion

In this paper, we have used the techniques of differential subordination and convolution to obtain inclusion theorems and subordination theorems. Many interesting particular cases of the main theorems are emphasized in the form of corollaries. The ideas and techniques of this work may motivate and inspire the others to explore this interesting field further.

Acknowledgement

The authors express deep gratitude to Dr. S. M. Junaid Zaidi, Rector, CIIT, for his support and providing excellent research facilities. This research is carried out under the HEC project grant No. NRPU No. 20-1966/R&D/11-2553.

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http://www.newtheory.org ISSN: 2149-1402

Journal of New Theory

Received: 18.09.2015 *Year*: **2016**, *Number*: **11**, *Pages*: **29-53**
Published: 02.03.2016 *Original Article*[**]

MULTIPLE SETS: A UNIFIED APPROACH TOWARDS MODELLING VAGUENESS AND MULTIPLICITY

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Abstaract − Multiple set is a new mathematical model to represent vagueness together with multiplicity. Multiple sets are generalization of fuzzy sets, multisets, fuzzy multisets and multi fuzzy sets. In this paper, a modified version for the definition of multiple sets is given and it is shown that the revised definition also satisfies all fundamental properties satisfied by the earlier definition. The notion of $\alpha_i - cut$ and strong $\alpha_i - cut$ are defined and their properties are studied. Finally, multiple complement function is defined and some results related to it are obtained.

Keywords − *Multiple sets, Fuzzy sets, Multi sets, Multiple complement function.*

1 Introduction

Sets are the fundamental ideas of Mathematics. The development of set theory was mainly due to a German Mathematician Cantor (1845-1918) and it has become the language of science. A set is a well defined collection of distinct objects. The objects that make up a set can be anything: numbers, people, letters of the alphabet, other sets and so on. In this theory, a sharp, crisp and unambiguous distinction exists between a member and a nonmember for any well defined set of entities and there is a very precise and clear boundary to indicate whether an entity belongs to the set or not.

Fuzzy sets have been introduced by Zadeh [21] in 1965 as an extension of the classical notion of a set. It was specifically designed to represent uncertainty and vagueness, mathematically and to provide formalized tools for dealing with the imprecision intrinsic to many problems. In real world, there exists much fuzzy knowledge like vague, imprecise, uncertain, ambiguous, inexact etc. Since its inception, the theory of fuzzy sets has advanced in a variety of ways and in many disciplines.

[**]*Edited by* Oktay Muhtaroğlu *(Area Editor) and* Naim Çağman *(Editor-in-Chief).*
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Applications of this theory can be found in artificial intelligence, computer science, medicine, control engineering, decision theory, expert systems, logic, management science, operation research, pattern recognition and robotics.

After this, a lot of new mathematical constructions and theories treating imprecision, inexactness, ambiguity and uncertainty have been developed. Some of these constructions and theories are extensions of fuzzy set theory, while others try to mathematically model imprecision and uncertainty in a different way. The diversity of such constructions and corresponding theories includes: L-fuzzy sets by Goguen [8] in 1967, Multisets by Cerf et al. [4] in 1971, Rough sets by Pawlak [15] in 1982, Intuitionistic fuzzy sets by Atanassov [1] in 1983, Fuzzy multisets by Yager [20] in 1986, Genuine sets by Demirci [6] in 1999, Soft sets by Molodtsov [14] in 1999, Multi fuzzy sets by Sebastian and Ramakrisnan [18] in 2011 etc.

The notion of multiset (or bag) is a generalization of the notion of set in which members are allowed to appear more than once. A set takes no account of multiple occurrence of any one of its members, so when one think of the set of roots of a polynomial f(x) or the spectrum of a linear operator, we need multisets. Multiset theory was introduced by Cerf et al.[4] in 1971. Peterson [16] and Yager [20] made further contributions to it. The naive concept of multiset was formalised by Blizard [2] in 1989. Multisets have become an important tool in databases, for instance, multisets are often used to implement relations in database systems. Multisets also play an important role in computer science. A complete account of the development of multiset theory can be seen in [9, 5, 3, 7].

Fuzzy multisets were first discussed by Yager [20] as a generalization of multisets. In fuzzy multisets an element of X may occur more than once with possibly same or different membership values. Later, Miyamoto established more results on fuzzy multiset theory and discussed applications of fuzzy multisets in his papers [10, 12].

The concept of multi fuzzy set was introduced by S. Sebastian and Ramakrishnan [18] in 2011. Theory of multi fuzzy sets is a generalization of theories of fuzzy sets, L-fuzzy sets and intuitionistic fuzzy sets. Theory of multi fuzzy sets deals with the multi level fuzziness and multi dimensional fuzziness. Multi fuzzy set theory is useful to characterize the problems in the fields of image processing, taste recognition, pattern recognition, decision making and approximation of vague data. Further study was carried on by the same author in his paper [17].

Fuzzy sets are useful in dealing with uncertainty of only one kind, that is only one membership function is possible. Multi fuzzy sets were introduced to handle more membership functions representing various types of uncertainties. Multisets handles repetition of elements or quantitative nature of objects. Fuzzy multisets handles quantitative and qualitative aspects together. All these ideas were developed independently and proved to be quite useful in their respective contexts. Motivated by all these concepts, one may think about a unified structure which represents all these aspects simultanously. As an attempt towards this, multiple sets [19] are introduced to model imperfect knowledge from which all the above discussed cases can be derived as particular cases. In multiple sets, multiple occurrences of elements are permitted in which each occurrence has a finite number of same or different membership values. That is, in multiple set theory, a multiple set of order (n, k) gives nk membership grades to each element x in the universal set X.

In this paper, a revised definition of multiple sets is given and it is examined that new definition satisfies all the properties satisfied by the old definition. $\alpha_i - cut$ and strong $\alpha_i - cut$ are defined and their properties are studied. Then special multiple sets and strong special multiple sets are defined and representations of multiple sets in terms of special multiple sets and strong special multiple sets are mentioned as three Decomposition Theorems. Finally, multiple complement function is defined and Characterization Theorems of multiple complements are discussed.

2 Preliminary

2.1 Fuzzy Sets

The word "fuzzy" means "vagueness". Fuzzy set is very convenient method for representing some form of uncertainty or vagueness. Fuzzy set theory permits the gradual assessment of the membership of elements in a set, described with the aid of a membership function valued in the real unit interval [0, 1].

Definition 2.1. [21] Let X be a given universal set, which is always a crisp set. A fuzzy set A on X is characterized by a function $A : X \rightarrow [0, 1]$ called fuzzy membership function, which assigns to each object a grade of membership ranging between zero and one. A *fuzzy set* A is defined as

$$A = \{(x, A(x)); x \in X\}$$

where $A(x)$ is the fuzzy membership value of x in X.

Each fuzzy set is completely and uniquely defined by one particular membership function. Words like young, tall, good or high are fuzzy.

2.2 Multisets

A multiset is an unordered collection of objects in which elements may occur more than once. In other words, a multiset is a collection in which objects may appear more than once and each individual occurrence of an object is called its element. All duplicates of an object in a multiset are indistinguishable. The objects of a multiset are the distinguishable or distinct elements of the multiset.

Definition 2.2. [9] Let X be a non empty set, called universe. A *multiset* M drawn from X is represented by a count function $C_M : X \rightarrow \mathbb{N} \cup \{0\}$, where $\mathbb{N}$ is the set of positive integers. For each $x \in X$, $C_M(x)$ indicates the number of occurrences of the element x in M. Then a multiset M can be expressed as $\{C_M(x)/x; x \in X\}$.

The number of distinct elements in a multiset M (which need not be finite) and their multiplicities jointly determine its cardinality, denoted by $C(M)$. In other words, the cardinality of a multiset is the sum of multiplicities of all its elements. A multiset M is called finite if the number of distinct elements in M and their multiplicities are both finite, it is infinite otherwise. Thus, a multiset M is infinite if either the number of elements in M is infinite or the multiplicity of one or more of its elements is infinite. A multiset corresponds to an ordinary set if the multiplicity of every element is one.

Operations on Multisets [9]

Let M_1 and M_2 be two multisets drawn from X.

1. **Submultiset**: M_1 is a sub multiset of M_2, denoted by $M_1 \subseteq M_2$, if $C_{M_1}(x) \leq C_{M_2}(x)$ for every $x \in X$.

2. **Equal**: M_1 and M_2 are equal, denoted by $M_1 = M_2$, if $M_1 \subseteq M_2$ and $M_2 \subseteq M_1$.

3. **Union**: The union of M_1 and M_2 is a multiset, denoted by $M = M_1 \cup M_2$, with the count function $C_M(x) = \max\{C_{M_1}(x), C_{M_2}(x)\}$, for every $x \in X$.

4. **Intersection**: The intersection of M_1 and M_2 is a multiset, denoted by $M = M_1 \cap M_2$, with the count function $C_M(x) = \min\{C_{M_1}(x), C_{M_2}(x)\}$, for every $x \in X$.

2.3 Fuzzy Multiset

In fuzzy multisets an element of X may occur more than once with possibly the same or different membership values.

Definition 2.3. [13] For $x \in X$, the membership sequence of x is defined as a non increasing sequence of membership values of x and it is denoted by $(\mu_A^1(x), \mu_A^2(x), ..., \mu_A^k(x))$, such that $\mu_A^1(x) \geq \mu_A^2(x) \geq ... \geq \mu_A^k(x)$, where μ_A is a membership function and μ_A^j, $j = 1, 2, ..., k$ are values(same or different) of membership function μ_A . A *fuzzy multiset* is a collection of all x together with its membership sequence.

2.4 Multi Fuzzy Sets

Multi fuzzy sets are defined in terms of ordered sequences of membership functions.

Definition 2.4. [18] Let X be a non empty set and let $\{L_i; i \in \mathbb{N}\}$ be a family of complete lattices where $\mathbb{N}$ is the set of positive integers. A *multi fuzzy set* A in X is a set of ordered sequences

$$A = \{(x, \mu_1(x), \mu_2(x), ...); x \in X\}$$

where $\mu_i \in L_i^X$ for $i \in \mathbb{N}$. The function $\mu_A = (\mu_1, \mu_2, ...)$ is called a multi membership function of multi fuzzy set A.

If the sequences of the membership function have only k terms, k is called dimension of A. Let $L_i = [0, 1]$ for $i = 1, 2, ..., k$, then the set of all multi fuzzy sets in X of dimension k is denoted by $M^kFS(X)$

3 Multiple Sets

Multiple sets are defined in [19]. Definition of multiple sets is modified and it is investigated that new definition satisfies all the properties satisfied by the old definition.

Definition 3.1. Let X be a non-empty crisp set called the universal set. A *multiple set* A of order (n,k) over X is an object of the form $\{(x, A(x)); x \in X\}$, where for each $x \in X$ its membership value is an $n \times k$ matrix

$$A(x) = \begin{bmatrix} A_1^1(x) & A_1^2(x) & \cdots & A_1^k(x) \\ A_2^1(x) & A_2^2(x) & \cdots & A_2^k(x) \\ & \cdots & & \\ A_n^1(x) & A_n^2(x) & \cdots & A_n^k(x) \end{bmatrix}$$

where $A_1, A_2, ..., A_n$ are fuzzy membership functions and for each $i = 1, 2, ..., n$, $A_i^1(x)$, $A_i^2(x), ..., A_i^k(x)$ are membership values of the membership function A_i for the element $x \in X$, written in decreasing order.

The universal multiple set X of order (n,k) is a multiple set of order (n,k) over X forwhich the membership matrix for each $x \in X$ is an $n \times k$ matrix with all entries one. The empty multiple set Φ of order (n,k) is a multiple set of order (n,k) over X forwhich the membership matrix for each $x \in X$ is an $n \times k$ matrix with all entries zero.

The set of all multiple sets of order (n,k) over X is denoted by $MS_{(n,k)}(X)$. It is noticed that a multiple set A of order (n,k) over X can be viewed as a function $A : X \to \mathbb{M}$, where $\mathbb{M} = \mathbb{M}_{n \times k}([0,1])$ is the set of all matrices of order $n \times k$ with entries from $[0,1]$, which maps each $x \in X$ to its $n \times k$ membership matrix $A(x)$. It is proved that a multiple set can be viewed as a generalization of fuzzy sets, multi fuzzy sets, fuzzy multisets and multisets. The standard set operations namely, subset, intersection, union and complement are defined on multiple sets. It is proved that multiple sets satisfies the following fundemental properties of the set operations.

1. **Involution**: $\overline{\overline{A}} = A$
2. **Commutativity**: $A \cup B = B \cup A$ and $A \cap B = B \cap A$
3. **Associativity**: $(A \cup B) \cup C = A \cup (B \cup C)$ and $(A \cap B) \cap C = A \cap (B \cap C)$
4. **Distributivity**: $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ and $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
5. **Idempotence**: $A \cup A = A$ and $A \cap A = A$
6. **Absorption**: $A \cup (A \cap B) = A$ and $A \cap (A \cup B) = A$
7. **Absorption by X and Φ**: $A \cup X = X$ and $A \cap \Phi = \Phi$
8. **Identity:** $A \cup \Phi = A$ and $A \cap X = A$
9. **De Morgan's laws**: $\overline{A \cup B} = \overline{A} \cap \overline{B}$ and $\overline{A \cap B} = \overline{A} \cup \overline{B}$
10. $A \subseteq A \cup B$ and $B \subseteq A \cup B$
11. $A \cap B \subseteq A$ and $A \cap B \subseteq B$

Finally, it is noticed that law of contradiction and law of excluded middle are violated for multiple sets.

Example 3.2. Suppose $X = \{x_1, x_2, x_3\}$ is the universal set of students under consideration and there is a panel consisting of three experts evaluating the students under the criteria of intelligence, extra curricular activities, communication skill and personality. The membership functions A_1, A_2, A_3 and A_4 represents criteria intelligence, extra curricular activities, communication skill and personality respectively. For each $i = 1, 2, 3, 4$, membership values $A_i^1(x), A_i^2(x), A_i^3(x)$ of the membership function A_i for the element $x \in X$ are the values given by the three experts, written in decreasing order. Then the performance of the students can be represented by a multiple set of order (4,3) as follows:

$$A = \{(x_1, A(x_1)), (x_2, A(x_2)), (x_3, A(x_3))\}$$

where $A(x_i)$ for $i = 1, 2, 3$ are 4×3 matrices given as follows;

$$A(x_1) = \begin{bmatrix} 0.7 & 0.6 & 0.5 \\ 0.6 & 0.5 & 0.4 \\ 0.7 & 0.5 & 0.3 \\ 0.9 & 0.9 & 0.8 \end{bmatrix}$$

$$A(x_2) = \begin{bmatrix} 0.8 & 0.6 & 0.6 \\ 0.6 & 0.5 & 0.4 \\ 0.7 & 0.5 & 0.4 \\ 0.9 & 0.8 & 0.7 \end{bmatrix}$$

$$A(x_3) = \begin{bmatrix} 0.8 & 0.7 & 0.5 \\ 0.7 & 0.6 & 0.4 \\ 0.7 & 0.4 & 0.4 \\ 0.8 & 0.8 & 0.7 \end{bmatrix}$$

Here, for the student x_1 the membership values corresponding to intelligence are 0.7, 0.6 and 0.5, corresponding to extra curricular activities are 0.6, 0.5 and 0.4 and so on.

Notation: Suppose I denotes the closed interval $[0, 1]$ and I^n denotes the cartesian product $[0, 1] \times [0, 1] \times ... \times [0, 1]$ $(n - times)$. A new notation can be introduced for the purpose of notational simplicity: $(\alpha_i)_1^n$ denotes the n-tuple $(\alpha_1, \alpha_2, ..., \alpha_n)$.

3.1 $\alpha_i - cut$ and strong $\alpha_i - cut$

In this section, the concept of $\alpha_i - cut$ and strong $\alpha_i - cut$ of multiple sets are introduced. They play a principle role in the relationship between multiple sets and crisp multisets.

Definition 3.3. Let $A \in MS_{(n,k)}(X)$ and $(\alpha_i)_1^n \in I^n$. An $\alpha_i - cut$ of A is a crisp multiset $A_{[\alpha_i]} = \{C_{A_{[\alpha_i]}}(x)/x; x \in X\}$, where $C_{A_{[\alpha_i]}}(x)$ is the count of x in $A_{[\alpha_i]}$, given by

$$C_{A_{[\alpha_i]}}(x) = \begin{cases} 0 & \text{if } A_i^1(x) < \alpha_i \text{ for some } i = 1, 2, ..., n. \\ j & \text{if } A_i^j(x) \geq \alpha_i \text{ for every } i = 1, 2, ..., n \text{ and} \\ & A_i^{j+1}(x) < \alpha_i \text{ for some } i = 1, 2, ..., n. \\ k & \text{if } A_i^k(x) \geq \alpha_i \text{ for every } i = 1, 2, ..., n. \end{cases}$$

A strong $\alpha_i - cut$ of A is a crisp multiset $A_{[\alpha_i]+} = \{C_{A_{[\alpha_i]+}}(x)/x; x \in X\}$, where $C_{A_{[\alpha_i]+}}(x)$ is the count of x in $A_{[\alpha_i]+}$, given by

$$C_{A_{[\alpha_i]+}}(x) = \begin{cases} 0 & \text{if } A_i^1(x) \leq \alpha_i \text{ for some } i = 1, 2, ..., n. \\ j & \text{if } A_i^j(x) > \alpha_i \text{ for every } i = 1, 2, ..., n \text{ and} \\ & A_i^{j+1}(x) \leq \alpha_i \text{ for some } i = 1, 2, ..., n. \\ k & \text{if } A_i^k(x) > \alpha_i \text{ for every } i = 1, 2, ..., n. \end{cases}$$

Example 3.4. Let A be a multiple set of order $(4, 3)$ over the universal set $X = \{x, y, z\}$, given by the membership matrices

$$A(x) = \begin{bmatrix} 0.7 & 0.6 & 0.5 \\ 0.6 & 0.5 & 0.4 \\ 0.7 & 0.5 & 0.4 \\ 0.9 & 0.8 & 0.7 \end{bmatrix}$$

$$A(y) = \begin{bmatrix} 0.8 & 0.7 & 0.5 \\ 0.7 & 0.6 & 0.4 \\ 0.7 & 0.6 & 0.4 \\ 0.8 & 0.8 & 0.7 \end{bmatrix}$$

$$A(z) = \begin{bmatrix} 0.7 & 0.5 & 0.3 \\ 0.6 & 0.6 & 0.4 \\ 0.4 & 0.3 & 0.3 \\ 0.8 & 0.7 & 0.1 \end{bmatrix}$$

For $(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = (0.5, 0.5, 0.3, 0.4) \in I^4$, the $\alpha_i - cut$ and the strong $\alpha_i - cut$ are

$$A_{[\alpha_i]} = \{2/x, 2/y, 2/z\}$$
$$A_{[\alpha_i]+} = \{1/x, 2/y, 1/z\}$$

3.2 Properties of $\alpha_i - cut$ and strong $\alpha_i - cut$

The various properties of $\alpha_i - cut$ and strong $\alpha_i - cut$ of multiple set are expressed in terms of theorems.

Theorem 3.5. Let $A, B \in MS_{(n,k)}(X)$ and $(\alpha_i)_1^n, (\beta_i)_1^n \in I^n$. Then

1. $A_{[\alpha_i]+} \subseteq A_{[\alpha_i]}$
2. If $\alpha_i \leq \beta_i$ for every $i = 1, 2, ..., n$, then $A_{[\beta_i]} \subseteq A_{[\alpha_i]}$ and $A_{[\beta_i]+} \subseteq A_{[\alpha_i]+}$

3. $(A \cap B)_{[\alpha_i]} = A_{[\alpha_i]} \cap B_{[\alpha_i]}$
4. $(A \cup B)_{[\alpha_i]} = A_{[\alpha_i]} \cup B_{[\alpha_i]}$
5. $(A \cap B)_{[\alpha_i]+} = A_{[\alpha_i]+} \cap B_{[\alpha_i]+}$
6. $(A \cup B)_{[\alpha_i]+} = A_{[\alpha_i]+} \cup B_{[\alpha_i]+}$

Proof: Let $A, B \in MS_{(n,k)}(X)$.

1. Let $(\alpha_i)_1^n \in I^n$. To prove $A_{[\alpha_i]+} \subseteq A_{[\alpha_i]}$, it is enough to prove $C_{A_{[\alpha_i]+}}(x) \leq C_{A_{[\alpha_i]}}(x)$ for every $x \in X$. Let $x \in X$. There are three cases;

   Case 1: $C_{A_{[\alpha_i]+}}(x) = 0$. Then $C_{A_{[\alpha_i]+}}(x) \leq C_{A_{[\alpha_i]}}(x)$ trivially.

   Case 2: $C_{A_{[\alpha_i]+}}(x) = j$. Then $A_i^j(x) > \alpha_i$ for every $i = 1, 2, ..., n$. Thus $C_{A_{[\alpha_i]}}(x) \geq j$ and hence $C_{A_{[\alpha_i]+}}(x) \leq C_{A_{[\alpha_i]}}(x)$.

   Case 3: $C_{A_{[\alpha_i]+}}(x) = k$, then $A_i^k(x) > \alpha_i$ for every $i = 1, 2, ..., n$. Thus $C_{A_{[\alpha_i]}}(x) = k$ and hence $C_{A_{[\alpha_i]+}}(x) = C_{A_{[\alpha_i]}}(x)$

   These three cases prove that $C_{A_{[\alpha_i]+}}(x) \leq C_{A_{[\alpha_i]}}(x)$ for every $x \in X$. Therefore $A_{[\alpha_i]+} \subseteq A_{[\alpha_i]}$.

2. Let $(\alpha_i)_1^n, (\beta_i)_1^n \in I^n$ such that $\alpha_i \leq \beta_i$ for every $i = 1, 2, ..., n$. To prove $A_{[\beta_i]} \subseteq A_{[\alpha_i]}$, it is enough to prove $C_{A_{[\beta_i]}}(x) \leq C_{A_{[\alpha_i]}}(x)$ for every $x \in X$. Let $x \in X$. There are three cases;

   Case 1: $C_{A_{[\alpha_i]}}(x) = 0$. Then there exist some $i \in \{1, 2, ..., n\}$ such that $A_i^1(x) < \alpha_i$. Since $\alpha_i \leq \beta_i$ for every $i = 1, 2, ..., n$, we get $A_i^1(x) < \beta_i$ for some i. Thus $C_{A_{[\beta_i]}}(x) = 0$ and hence $C_{A_{[\beta_i]}}(x) = C_{A_{[\alpha_i]}}(x)$.

   Case 2: $C_{A_{[\alpha_i]}}(x) = j$. Then there exist some $i \in \{1, 2, ..., n\}$ such that $A_i^{j+1}(x) < \alpha_i$. Since $\alpha_i \leq \beta_i$ for every $i = 1, 2, ..., n$, we get $A_i^{j+1} < \beta_i$ for some i. Thus $C_{A_{[\beta_i]}}(x) \leq j$ and hence $C_{A_{[\beta_i]}}(x) \leq C_{A_{[\alpha_i]}}(x)$.

   Case 3: $C_{A_{[\alpha_i]}}(x) = k$. Then $C_{A_{[\beta_i]}}(x) \leq C_{A_{[\alpha_i]}}(x)$ trivially.

   These three cases prove that $C_{A_{[\beta_i]}}(x) \leq C_{A_{[\alpha_i]}}(x)$ for every $x \in X$. Therefore $A_{[\beta_i]} \subseteq A_{[\alpha_i]}$. The proof of $A_{[\beta_i]+} \subseteq A_{[\alpha_i]+}$ is analogous.

3. Let $(\alpha_i)_1^n \in I^n$. To prove $(A \cap B)_{[\alpha_i]} = A_{[\alpha_i]} \cap B_{[\alpha_i]}$, it is enough to prove $C_{(A \cap B)_{[\alpha_i]}}(x) = C_{A_{[\alpha_i]} \cap B_{[\alpha_i]}}(x)$ for every $x \in X$. Let $x \in X$. There are three cases;

   Case 1: $C_{(A \cap B)_{[\alpha_i]}}(x) = 0$. Then there exist some $i \in \{1, 2, ..., n\}$ such that $(A \cap B)_i^1 < \alpha_i$. This means that $\min\{A_i^1(x), B_i^1(x)\} < \alpha_i$ for some i. Then either $A_i^1(x) < \alpha_i$ or $B_i^1(x) < \alpha_i$ for some i. This implies that either

$C_{A_{[\alpha_i]}}(x) = 0$ or $C_{B_{[\alpha_i]}}(x) = 0$. Thus $\min\{C_{A_{[\alpha_i]}}(x), C_{B_{[\alpha_i]}}(x)\} = 0$ and hence $C_{A_{[\alpha_i]} \cap B_{[\alpha_i]}}(x) = 0$. Therefore $C_{(A \cap B)_{[\alpha_i]}}(x) = C_{A_{[\alpha_i]} \cap B_{[\alpha_i]}}(x)$.

Case 2: $C_{(A \cap B)_{[\alpha_i]}}(x) = j$. Then there exist some $i \in \{1, 2, ..., n\}$ such that $(A \cap B)_i^{j+1}(x) < \alpha_i$. This means that $\min\{A_i^{j+1}(x), B_i^{j+1}(x)\} < \alpha_i$ for some i. Then either $A_i^{j+1}(x) < \alpha_i$ or $B_i^{j+1}(x) < \alpha_i$ for some i. This implies that $C_{A_{[\alpha_i]}}(x) \leq j$ or $C_{B_{[\alpha_i]}}(x) \leq j$. Then $\min\{C_{A_{[\alpha_i]}}(x), C_{B_{[\alpha_i]}}(x)\} \leq j$ and hence $C_{A_{[\alpha_i]} \cap B_{[\alpha_i]}}(x) \leq j$. Therefore $C_{(A \cap B)_{[\alpha_i]}}(x) \geq C_{A_{[\alpha_i]} \cap B_{[\alpha_i]}}(x)$.

Also, $(A \cap B)_i^j(x) \geq \alpha_i$ for every $i = 1, 2, ..., n$. This means that $\min\{A_i^j(x), B_i^j(x)\} \geq \alpha_i$ for every $i = 1, 2, ..., n$. Then $A_i^j(x) \geq \alpha_i$ and $B_i^j(x) \geq \alpha_i$ for every $i = 1, 2, ..., n$. This implies that both $C_{A_{[\alpha_i]}}(x) \geq j$ and $C_{B_{[\alpha_i]}}(x) \geq j$. Then $\min\{C_{A_{[\alpha_i]}}(x), C_{B_{[\alpha_i]}}(x)\} \geq j$ and hence $C_{A_{[\alpha_i]} \cap B_{[\alpha_i]}}(x) \geq j$. Therefore $C_{(A \cap B)_{[\alpha_i]}}(x) \leq C_{A_{[\alpha_i]} \cap B_{[\alpha_i]}}(x)$. From two inqualities, we have $C_{(A \cap B)_{[\alpha_i]}}(x) = C_{A_{[\alpha_i]} \cap B_{[\alpha_i]}}(x)$.

Case 3: $C_{(A \cap B)_{[\alpha_i]}}(x) = k$. Then $(A \cap B)_i^k(x) \geq \alpha_i$ for every $i = 1, 2, ..., n$. This means that $\min\{A_i^k(x), B_i^k(x)\} \geq \alpha_i$ for every $i = 1, 2, ..., n$. This implies that both $A_i^k(x) \geq \alpha_i$ and $B_i^k(x) \geq \alpha_i$ for every $i = 1, 2, ..., n$. Thus both $C_{A_{[\alpha_i]}}(x) = k$ and $C_{B_{[\alpha_i]}}(x) = k$. Then $\min\{C_{A_{[\alpha_i]}}(x), C_{B_{[\alpha_i]}}(x)\} = k$ and hence $C_{A_{[\alpha_i]} \cap B_{[\alpha_i]}}(x) = k$. Therefore $C_{(A \cap B)_{[\alpha_i]}}(x) = C_{A_{[\alpha_i]} \cap B_{[\alpha_i]}}(x)$.

These three cases prove that $C_{(A \cap B)_{[\alpha_i]}}(x) = C_{A_{[\alpha_i]} \cap B_{[\alpha_i]}}(x)$ for every $x \in X$. Therefore $(A \cap B)_{[\alpha_i]} = A_{[\alpha_i]} \cap B_{[\alpha_i]}$. The proof of (4), (5) and (6) are analogous.

Theorem 3.6. Let $A_t \in MS_{(n,k)}(X)$ for all $t \in T$, where T is an index set and let $(\alpha_i)_1^n \in I^n$. Then

1. $\bigcup_{t \in T} (A_t)_{[\alpha_i]} \subseteq \left(\bigcup_{t \in T} A_t \right)_{[\alpha_i]}$
2. $\bigcap_{t \in T} (A_t)_{[\alpha_i]} \supseteq \left(\bigcap_{t \in T} A_t \right)_{[\alpha_i]}$
3. $\bigcup_{t \in T} (A_t)_{[\alpha_i]+} \supseteq \left(\bigcup_{t \in T} A_t \right)_{[\alpha_i]+}$
4. $\bigcap_{t \in T} (A_t)_{[\alpha_i]+} \subseteq \left(\bigcap_{t \in T} A_t \right)_{[\alpha_i]+}$

Proof: Let $A_t \in MS_{(n,k)}(X)$ for all $t \in T$ and let $(\alpha_i)_1^n \in I^n$.

1. To prove $\bigcup_{t \in T} (A_t)_{[\alpha_i]} \subseteq \left(\bigcup_{t \in T} A_t \right)_{[\alpha_i]}$, it is enough to prove $C_{\bigcup_{t \in T} (A_t)_{[\alpha_i]}}(x) \leq C_{(\bigcup_{t \in T} A_t)_{[\alpha_i]}}(x)$ for every $x \in X$. Let $x \in X$. There are three cases;

Case 1: $C_{(\bigcup\limits_{t\in T} A_t)_{[\alpha_i]}}(x) = 0$. Then there exist some $i \in \{1, 2, ..., n\}$ such that $\left(\bigcup\limits_{t\in T} A_t\right)_i^1 (x)$
$< \alpha_i$. This means that $\sup\limits_{t\in T} \{(A_t)_i^1(x)\} < \alpha_i$ for some i. This implies that $(A_t)_i^1(x) < \alpha_i$ for some i and for every $t \in T$. Thus $C_{(A_t)_{[\alpha_i]}}(x) = 0$ for every $t \in T$. This implies that $\sup\limits_{t\in T} \left\{C_{(A_t)_{[\alpha_i]}}(x)\right\} = 0$ and thus $C_{\bigcup\limits_{t\in T}(A_t)_{[\alpha_i]}}(x) = 0$. Therefore $C_{\bigcup\limits_{t\in T}(A_t)_{[\alpha_i]}}(x) = C_{(\bigcup\limits_{t\in T} A_t)_{[\alpha_i]}}(x)$.

Case 2: $C_{(\bigcup\limits_{t\in T} A_t)_{[\alpha_i]}}(x) = j$. Then there exist some $i \in \{1, 2, ..., n\}$ such that $\left(\bigcup\limits_{t\in T} A_t\right)_i^{j+1} (x) < \alpha_i$. This means that $\sup\limits_{t\in T} \{(A_t)_i^{j+1}(x)\} < \alpha_i$ for some i. This implies that $(A_t)_i^{j+1}(x) < \alpha_i$ for some i and for every $t \in T$. Thus $C_{(A_t)_{[\alpha_i]}}(x) \leq j$ for every $t \in T$. This implies that $\sup\limits_{t\in T} \left\{C_{(A_t)_{[\alpha_i]}}(x)\right\} \leq j$ and thus $C_{\bigcup\limits_{t\in T}(A_t)_{[\alpha_i]}}(x) \leq j$. Therefore $C_{\bigcup\limits_{t\in T}(A_t)_{[\alpha_i]}}(x) \leq C_{(\bigcup\limits_{t\in T} A_t)_{[\alpha_i]}}(x)$.

Case 3: $C_{(\bigcup\limits_{t\in T} A_t)_{[\alpha_i]}}(x) = k$. Then $C_{\bigcup\limits_{t\in T}(A_t)_{[\alpha_i]}}(x) \leq C_{(\bigcup\limits_{t\in T} A_t)_{[\alpha_i]}}(x)$ trivially.

These three cases prove that $C_{\bigcup\limits_{t\in T}(A_t)_{[\alpha_i]}}(x) \leq C_{(\bigcup\limits_{t\in T} A_t)_{[\alpha_i]}}(x)$ for every $x \in X$. Therefore, $\bigcup\limits_{t\in T} (A_t)_{[\alpha_i]} \subseteq \left(\bigcup\limits_{t\in T} A_t\right)_{[\alpha_i]}$.

2. To prove $\bigcap\limits_{t\in T} (A_t)_{[\alpha_i]} \supseteq \left(\bigcap\limits_{t\in T} A_t\right)_{[\alpha_i]}$, it is enough to prove $C_{\bigcap\limits_{t\in T}(A_t)_{[\alpha_i]}}(x) \geq C_{(\bigcap\limits_{t\in T} A_t)_{[\alpha_i]}}(x)$, for every $x \in X$. Let $x \in X$. There are three cases;

   Case 1: $C_{(\bigcap\limits_{t\in T} A_t)_{[\alpha_i]}}(x) = 0$. Then $C_{\bigcap\limits_{t\in T}(A_t)_{[\alpha_i]}}(x) \geq C_{(\bigcap\limits_{t\in T} A_t)_{[\alpha_i]}}(x)$ trivially.

   Case 2: $C_{(\bigcap\limits_{t\in T} A_t)_{[\alpha_i]}}(x) = j$. Then $\left(\bigcap\limits_{t\in T} A_t\right)_i^j (x) \geq \alpha_i$ for every $i = 1, 2, ..., n$. This means that $\inf\limits_{t\in T} \{(A_t)_i^j(x)\} \geq \alpha_i$ for every $i = 1, 2, ..., n$. This implies that $(A_t)_i^j(x) \geq \alpha_i$ for every $i = 1, 2, ..., n$ and for every $t \in T$. Thus $C_{(A_t)_{[\alpha_i]}}(x) \geq j$ for every $t \in T$. This implies that $\inf\limits_{t\in T} \left\{C_{(A_t)_{[\alpha_i]}}\right\} \geq j$ and thus $C_{\bigcap\limits_{t\in T}(A_t)_{[\alpha_i]}}(x) \geq j$. Therefore $C_{\bigcap\limits_{t\in T}(A_t)_{[\alpha_i]}}(x) \geq C_{(\bigcap\limits_{t\in T} A_t)_{[\alpha_i]}}(x)$.

   Case 3: $C_{(\bigcap\limits_{t\in T} A_t)_{[\alpha_i]}}(x) = k$. Then $\left(\bigcap\limits_{t\in T} A_t\right)_i^k (x) \geq \alpha_i$ for every $i = 1, 2, ..., n$. This means that $\inf\limits_{t\in T} \{(A_t)_i^k(x)\} \geq \alpha_i$ for every $i = 1, 2, ..., n$. This implies that $(A_t)_i^k(x) \geq \alpha_i$ for every $i = 1, 2, ..., n$ and for every $t \in T$. Thus $C_{(A_t)_{[\alpha_i]}}(x) = k$ for every $t \in T$. This implies that $\inf\limits_{t\in T} \left\{C_{(A_t)_{[\alpha_i]}}\right\} = k$ and thus $C_{\bigcap\limits_{t\in T}(A_t)_{[\alpha_i]}}(x) =$

k. Therefore $C_{\bigcap\limits_{t\in T}(A_t)_{[\alpha_i]}}(x) = C_{(\bigcap\limits_{t\in T} A_t)_{[\alpha_i]}}(x)$.

These three cases prove that $C_{\bigcap\limits_{t\in T}(A_t)_{[\alpha_i]}}(x) \geq C_{(\bigcap\limits_{t\in T} A_t)_{[\alpha_i]}}(x)$ for every $x \in X$. Therefore, $\bigcap\limits_{t\in T}(A_t)_{[\alpha_i]} \supseteq \left(\bigcap\limits_{t\in T} A_t\right)_{[\alpha_i]}$.

The proof of (3) and (4) are analogous.

Theorem 3.7. Let $A, B \in MS_{(n,k)}(X)$ and let $(\alpha_i)_1^n \in I^n$. Then

1. $A \subseteq B$ iff $A_{[\alpha_i]} \subseteq B_{[\alpha_i]}$
2. $A \subseteq B$ iff $A_{[\alpha_i]+} \subseteq B_{[\alpha_i]+}$
3. $A = B$ iff $A_{[\alpha_i]} = B_{[\alpha_i]}$
4. $A = B$ iff $A_{[\alpha_i]+} = B_{[\alpha_i]+}$

Proof: Let $A, B \in MS_{(n,k)}(X)$ and let $(\alpha)_n \in I^n$.

1. Suppose $A \subseteq B$. Then $A_i^j(x) \leq B_i^j(x)$ for every $i = 1, 2, ..., n$, $j = 1, 2, ..., k$ and $x \in X$. To prove $A_{[\alpha_i]} \subseteq B_{[\alpha_i]}$, it is enough to prove $C_{A_{[\alpha_i]}}(x) \leq C_{B_{[\alpha_i]}}(x)$ for every $x \in X$. Let $x \in X$. There are three cases;

   Case 1: $C_{B_{[\alpha_i]}}(x) = 0$. Then there exist some $i \in \{1, 2, ..., n\}$ such that $B_i^1(x) < \alpha_i$. This implies that $A_i^1(x) \leq B_i^1(x) < \alpha_i$ for some i. Therefore $C_{A_{[\alpha_i]}}(x) = 0$. Thus $C_{A_{[\alpha_i]}}(x) = C_{B_{[\alpha_i]}}(x)$.

   Case 2: $C_{B_{[\alpha_i]}}(x) = j$. Then there exist some $i \in \{1, 2, ..., n\}$ such that $B_i^{j+1}(x) < \alpha_i$. This implies that $A_i^{j+1}(x) \leq B_i^{j+1}(x) < \alpha_i$ for some i. Therefore $C_{A_{[\alpha_i]}}(x) \leq j$. Thus $C_{A_{[\alpha_i]}}(x) \leq C_{B_{[\alpha_i]}}(x)$.

   Case 3: Suppose $C_{B_{[\alpha_i]}}(x) = k$. Then $C_{A_{[\alpha_i]}}(x) \leq C_{B_{[\alpha_i]}}(x)$ trivially.

   These three cases prove that $C_{A_{[\alpha_i]}}(x) \leq C_{B_{[\alpha_i]}}(x)$ for every $x \in X$. Therefore $A_{[\alpha_i]} \subseteq B_{[\alpha_i]}$.

   Conversely, suppose that $A_{[\alpha_i]} \subseteq B_{[\alpha_i]}$. This means that $C_{A_{[\alpha_i]}}(x) \leq C_{B_{[\alpha_i]}}(x)$ for every $x \in X$. Let $x \in X$ and $j \in \{1, 2, ..., k\}$. Take $\alpha_i = A_i^j(x)$ for $i = 1, 2, ..., n$. Then $A_i^j(x) \geq \alpha_i$ for every $i = 1, 2, ..., n$. This implies $B_i^j(x) \geq \alpha_i$ for every $i = 1, 2, ..., n$, since $C_{A_{[\alpha_i]}}(x) \leq C_{B_{[\alpha_i]}}(x)$. That is $B_i^j(x) \geq A_i^j(x)$ for every $i = 1, 2, ..., n$. Therefore $A \subseteq B$.

2. Suppose $A \subseteq B$. Then $A_i^j(x) \leq B_i^j(x)$ for every $i = 1, 2, ..., n$, $j = 1, 2, ..., k$ and $x \in X$. To prove $A_{[\alpha_i]+} \subseteq B_{[\alpha_i]+}$, it is enough to prove $C_{A_{[\alpha_i]+}}(x) \leq C_{B_{[\alpha_i]+}}(x)$ for every $x \in X$. Let $x \in X$. There are three cases;

   Case 1: $C_{B_{[\alpha_i]+}}(x) = 0$. Then there exist some $i \in \{1, 2, ...n\}$ such that $B_i^1(x) \leq \alpha_i$. This implies that $A_i^1(x) \leq B_i^1(x) \leq \alpha_i$ for some i. Therefore

$C_{A_{[\alpha_i]+}}(x) = 0$. Thus $C_{A_{[\alpha_i]+}}(x) = C_{B_{[\alpha_i]+}}(x)$.

Case 2: $C_{B_{[\alpha_i]+}}(x) = j$. Then there exist some $i \in \{1, 2, ..., n\}$ such that $B_i^{j+1}(x) \leq \alpha_i$. This implies that $A_i^{j+1}(x) \leq B_i^{j+1}(x) \leq \alpha_i$ for some i. Therefore $C_{A_{[\alpha_i]+}}(x) \leq j$. Thus $C_{A_{[\alpha_i]+}}(x) \leq C_{B_{[\alpha_i]+}}(x)$.

Case 3: $C_{B_{[\alpha_i]+}}(x) = k$. Then $C_{A_{[\alpha_i]+}}(x) \leq C_{B_{[\alpha_i]+}}(x)$ trivially.

These three cases prove that $C_{A_{[\alpha_i]+}}(x) \leq C_{B_{[\alpha_i]+}}(x)$ for every $x \in X$. Hence $A_{[\alpha_i]+} \subseteq B_{[\alpha_i]+}$.

Conversely, suppose that $A_{[\alpha_i]+} \subseteq B_{[\alpha_i]+}$. This means that $C_{A_{[\alpha_i]+}}(x) \leq C_{B_{[\alpha_i]+}}(x)$ for every $x \in X$. Let $x \in X$ and $j \in \{1, 2, ..., k\}$. Take $\alpha_i = A_i^j(x) - \epsilon$ for any $\epsilon > 0$ and for every $i = 1, 2, ..., n$. Then $A_i^j(x) > \alpha_i$ for every $i = 1, 2, ..., n$. This implies that $B_i^j(x) > \alpha_i$ for every $i = 1, 2, ..., n$, since $C_{A_{[\alpha_i]+}}(x) \leq C_{B_{[\alpha_i]+}}(x)$. That is $B_i^j(x) > A_i^j(x) - \epsilon$ for any $\epsilon > 0$ and for every $i = 1, 2, ..., n$. This implies that $B_i^j(x) \geq A_i^j(x)$ for every $i = 1, 2, ..., n$. Therefore $A \subseteq B$.

3. Suppose $A = B$. Then $A_i^j(x) = B_i^j(x)$ for every $i = 1, 2, ..., n$, $j = 1, 2, ..., k$ and $x \in X$. To prove $A_{[\alpha_i]} = B_{[\alpha_i]}$, it is enough to prove $C_{A_{[\alpha_i]}}(x) = C_{B_{[\alpha_i]}}(x)$ for every $x \in X$. Let $x \in X$. there are three cases;

   Case 1: $C_{B_{[\alpha_i]}}(x) = 0$. Then there exist some $i \in \{1, 2, ..., n\}$ such that $B_i^1(x) < \alpha_i$. This implies that $A_i^1(x) = B_i^1(x) < \alpha_i$ for some i. Therefore $C_{A_{[\alpha_i]}}(x) = 0$. Thus $C_{A_{[\alpha_i]}}(x) = C_{B_{[\alpha_i]}}(x)$.

   Case 2: $C_{B_{[\alpha_i]}}(x) = j$. Then $B_i^j(x) \geq \alpha_i$ for every $i = 1, 2, ..., n$. This implies that $A_i^j(x) \geq \alpha_i$ for every $i = 1, 2, ..., n$. Thus $C_{A_{[\alpha_i]}}(x) \geq j$. Also, there exist some $i \in \{1, 2, ..., n\}$ such that $B_i^{j+1}(x) < \alpha_i$. This implies that $A_i^{j+1}(x) = B_i^{j+1}(x) < \alpha_i$ for some i. Therefore $C_{A_{[\alpha_i]}}(x) \leq j$. Thus $C_{A_{[\alpha_i]}}(x) = C_{B_{[\alpha_i]}}(x)$.

   Case 3: $C_{B_{[\alpha_i]}}(x) = k$. Then $B_i^k(x) \geq \alpha_i$ for every i=1,2,...,n. This implies that $A_i^k(x) \geq \alpha_i$. Thus $C_{A_{[\alpha_i]}}(x) = k$ and therefore $C_{A_{[\alpha_i]}}(x) = C_{B_{[\alpha_i]}}(x)$.

   These three cases prove that $C_{A_{[\alpha_i]}}(x) = C_{B_{[\alpha_i]}}(x)$ for every $x \in X$. Hence $A_{[\alpha_i]} = B_{[\alpha_i]}$.

   Conversely, suppose that $A_{[\alpha_i]} = B_{[\alpha_i]}$. This means that $C_{A_{[\alpha_i]}}(x) = C_{B_{[\alpha_i]}}(x)$ for every $x \in X$. Let $x \in X$ and $j \in \{1, 2, ..., k\}$. Take $\alpha_i = A_i^j(x)$ for every $i = 1, 2, ..., n$. Then $A_i^j(x) \geq \alpha_i$ for every $i = 1, 2, ..., n$. This implies that $B_i^j(x) \geq \alpha_i$ for every $i = 1, 2, ..., n$, since $C_{A_{[\alpha_i]}}(x) = C_{B_{[\alpha_i]}}(x)$. That is $B_i^j(x) \geq A_i^j(x)$ for every $i = 1, 2, ..., n$. Also, take $\alpha_i = B_i^j(x)$ for every $i = 1, 2, ..., n$. Then $B_i^j(x) \geq \alpha_i$ for every $i = 1, 2, ..., n$. This implies $A_i^j(x) \geq \alpha_i$ for every $i = 1, 2, ..., n$, since $C_{A_{[\alpha_i]}}(x) = C_{B_{[\alpha_i]}}(x)$. That is $A_i^j(x) \geq B_i^j(x)$ for every $i = 1, 2, ..., n$. Therefore $A_i^j(x) = B_i^j(x)$ for every $i = 1, 2, ..., n$, $j = 1, 2, ..., k$ and $x \in X$. Thus $A = B$.

4. Suppose $A = B$. Then $A_i^j(x) = B_i^j(x)$ for every $i = 1, 2, ..., n$, $j = 1, 2, ..., k$ and $x \in X$. To prove $A_{[\alpha_i]+} = B_{[\alpha_i]+}$, it is enough to prove $C_{A_{[\alpha_i]+}}(x) = C_{B_{[\alpha_i]+}}(x)$ for every $x \in X$. Let $x \in X$. There are three cases;

   Case 1: $C_{B_{[\alpha_i]+}}(x) = 0$. Then there exist some $i \in \{1, 2, ..., n\}$ such that $B_i^1(x) \leq \alpha_i$. This implies that $A_i^1(x) = B_i^1(x) \leq \alpha_i$ for some i. Therefore $C_{A_{[\alpha_i]+}}(x) = 0$. Thus $C_{A_{[\alpha_i]+}}(x) = C_{B_{[\alpha_i]+}}(x)$.

   Case 2: $C_{B_{[\alpha_i]+}}(x) = j$. Then $B_i^j(x) > \alpha_i$ for every $i = 1, 2, ..., n$. This implies that $A_i^j(x) > \alpha_i$ for every $i = 1, 2, ..., n$. Thus $C_{A_{[\alpha_i]+}}(x) \geq j$. Also, there exist some $i \in \{1, 2, .., n\}$ such that $B_i^{j+1}(x) \leq \alpha_i$. This implies that $A_i^{j+1}(x) = B_i^{j+1}(x) \leq \alpha_i$ for some i. Therefore $C_{A_{[\alpha_i]+}}(x) \leq j$. Thus $C_{A_{[\alpha_i]+}}(x) = C_{B_{[\alpha_i]+}}(x)$.

   Case 3: $C_{B_{[\alpha_i]+}}(x) = k$. Then $B_i^k(x) > \alpha_i$ for every $i = 1, 2, ..., n$. This implies that $A_i^k(x) > \alpha_i$ for every $i = 1, 2, ..., n$. Thus $C_{A_{[\alpha_i]+}}(x) = k$ and therefore $C_{A_{[\alpha_i]+}}(x) = C_{B_{[\alpha_i]+}}(x)$.

   These three cases prove that $C_{A_{[\alpha_i]+}}(x) = C_{B_{[\alpha_i]+}}(x)$ for every $x \in X$. Hence $A_{[\alpha_i]+} = B_{[\alpha_i]+}$.

   Conversely, suppose that $A_{[\alpha_i]+} = B_{[\alpha_i]+}$. This means that $C_{A_{[\alpha_i]+}}(x) = C_{B_{[\alpha_i]+}}(x)$ for every $x \in X$. Let $x \in X$ and $j \in \{1, 2, ..., k\}$. Take $\alpha_i = A_i^j(x) - \epsilon$ for any $\epsilon > 0$ and for every $i = 1, 2, ..., n$. Then $A_i^j(x) > \alpha_i$ for every $i = 1, 2, ..., n$. This implies that $B_i^j(x) > \alpha_i$ for every $i = 1, 2, ..., n$, since $C_{A_{[\alpha_i]+}}(x) = C_{B_{[\alpha_i]+}}(x)$. That is $B_i^j(x) > A_i^j(x) - \epsilon$ for every $i = 1, 2, ..., n$. Thus $B_i^j(x) \geq A_i^j(x)$ for every $i = 1, 2, ..., n$. Also, take $\alpha_i = B_i^j(x) - \epsilon$ for any $\epsilon > 0$ and for every $i = 1, 2, ..., n$. Then $B_i^j(x) > \alpha_i$ for every $i = 1, 2, ..., n$. That implies $A_i^j(x) > \alpha_i$ for every $i = 1, 2, ..., n$, since $C_{A_{[\alpha_i]+}}(x) = C_{B_{[\alpha_i]+}}(x)$. That is $A_i^j(x) > B_i^j(x) - \epsilon$ for every $i = 1, 2, ..., n$. Thus $A_i^j(x) \geq B_i^j(x)$ for every $i = 1, 2, ..., n$. Therefore $A_i^j(x) = B_i^j(x)$ for every $i = 1, 2, ..., n$, $j = 1, 2, ..., k$ and $x \in X$. Thus $A = B$.

Theorem 3.8. For any $A \in MS_{(n,k)}(X)$ and $(\alpha_i)_1^n \in I^n$ such that $\alpha_i \neq 0$ for every $i = 1, 2, ..., n$, the following property holds:

$$A_{[\alpha_i]} = \bigcap_{\substack{(\beta_i)_1^n \in I^n \\ \beta_i < \alpha_i, i=1,2,...,n}} A_{[\beta_i]} = \bigcap_{\substack{(\beta_i)_1^n \in I^n \\ \beta_i < \alpha_i, i=1,2,...,n}} A_{[\beta_i]+}$$

Proof: Let $A \in MS_{(n,k)}(X)$ and $(\alpha_i)_1^n \in I^n$ such that $\alpha_i \neq 0$ for every $i = 1, 2, ..., n$. To prove $A_{[\alpha_i]} = \bigcap_{\substack{(\beta_i)_1^n \in I^n \\ \beta_i < \alpha_i}} A_{[\beta_i]}$, it is enough to prove $C_{A_{[\alpha_i]}}(x) = C_{\bigcap_{\substack{(\beta_i)_1^n \in I^n \\ \beta_i < \alpha_i}} A_{[\beta_i]}}(x)$ for every $x \in X$. We have $A_{[\alpha_i]} \subseteq A_{[\beta_i]}$ for any $(\beta_i)_1^n \in I^n$ such that $\beta_i < \alpha_i$, for every $i = 1, 2, ..., n$. This means that $C_{A_{[\alpha_i]}}(x) \leq C_{A_{[\beta_i]}}(x)$ for every $x \in X$. This implies that $C_{A_{[\alpha_i]}}(x) \leq \inf_{\substack{(\beta_i)_1^n \in I^n \\ \beta_i < \alpha_i}} \left\{ C_{A_{[\beta_i]}}(x) \right\}$ and therefore $C_{A_{[\alpha_i]}}(x) \leq$

$C_{\bigcap\limits_{\substack{(\beta_i)_1^n \in I^n \\ \beta_i < \alpha_i}} A_{[\beta_i]}}(x)$.

To prove the reverse inequality, it is enough to prove there exist $(\beta_i)_1^n \in I^n$ such that $\beta_i < \alpha_i$ for every $i = 1, 2, ..., n$ satisfying $C_{A_{[\beta_i]}}(x) \leq C_{A_{[\alpha_i]}}(x)$ for every $x \in X$. Then $\inf\limits_{\substack{(\beta_i)_1^n \in I^n \\ \beta_i < \alpha_i}} \left\{ C_{A_{[\beta_i]}}(x) \right\} \leq C_{A_{[\beta_i]}}(x) \leq C_{A_{[\alpha_i]}}(x)$ for every $x \in X$. Hence $C_{\bigcap\limits_{\substack{(\beta_i)_1^n \in I^n \\ \beta_i < \alpha_i}} A_{[\beta_i]}}(x) \leq C_{A_{[\alpha_i]}}(x)$ for every $x \in X$. From two inequalities, we have $C_{A_{[\alpha_i]}}(x) = C_{\bigcap\limits_{\substack{(\beta_i)_1^n \in I^n \\ \beta_i < \alpha_i}} A_{[\beta_i]}}(x)$ for every $x \in X$.

It remains to prove the existence of $(\beta_i)_1^n \in I^n$ such that $\beta_i < \alpha_i$ for every $i = 1, 2, ..., n$ satisfying $C_{A_{[\beta_i]}}(x) \leq C_{A_{[\alpha_i]}}(x)$ for every $x \in X$. Let $x \in X$. There are three cases;

Case 1: $C_{A_{[\alpha_i]}}(x) = 0$. Then there exist some $i \in \{1, 2, ..., n\}$ such that $A_i^1(x) < \alpha_i$. For those $i's$ there exist $\beta_i \in I$ such that $A_i^1(x) < \beta_i < \alpha_i$ and for other $i's$ take $\beta_i = \alpha_i/2$. Then $C_{A_{[\beta_i]}}(x) = 0$. That is, for $(\beta_i)_1^n$ we have $\beta_i < \alpha_i$ for every $i = 1, 2, ..., n$ and $C_{A_{[\beta_i]}}(x) = C_{A_{[\alpha_i]}}(x)$.

Case 2: $C_{A_{[\alpha_i]}}(x) = j$. Then there exist some $i \in \{1, 2, ..., n\}$ such that $A_i^{j+1}(x) < \alpha_i$. For those $i's$ there exist $\beta_i \in I$ such that $A_i^{j+1}(x) < \beta_i < \alpha_i$ and for other $i's$ take $\beta_i = \alpha_i/2$. Then $C_{A_{[\beta_i]}}(x) \leq j$. That is, for $(\beta_i)_1^n$ we have $\beta_i < \alpha_i$ for every $i = 1, 2, ..., n$ and $C_{A_{[\beta_i]}}(x) \leq C_{A_{[\alpha_i]}}(x)$.

Case 3: $C_{A_{[\alpha_i]}}(x) = k$. Take $\beta_i = \alpha_i/2$ for every $i = 1, 2, ..., n$. Then $C_{A_{[\beta_i]}}(x) \leq k$. That is, for $(\beta_i)_1^n$ we have $\beta_i < \alpha_i$ for every $i = 1, 2, ..., n$ and $C_{A_{[\beta_i]}}(x) \leq C_{A_{[\alpha_i]}}(x)$.

These three cases prove that, for every $x \in X$, there exist $(\beta_i)_1^n \in I^n$ such that $\beta_i < \alpha_i$ for every $i = 1, 2, ..., n$ satisfying $C_{A_{[\beta_i]}}(x) \leq C_{A_{[\alpha_i]}}(x)$.

Next, to prove $A_{[\alpha_i]} = \bigcap\limits_{\substack{(\beta_i)_1^n \in I^n \\ \beta_i < \alpha_i}} A_{[\beta_i]+}$, it is enough to prove

$$C_{A_{[\alpha_i]}}(x) = C_{\bigcap\limits_{\substack{(\beta_i)_1^n \in I^n \\ \beta_i < \alpha_i}} A_{[\beta_i]+}}(x)$$

for every $x \in X$. We have $A_{[\alpha_i]} \subseteq A_{[\beta_i]+}$ for any $(\beta_i)_1^n \in I^n$ such that $\beta_i < \alpha_i$ for every $i = 1, 2, ..., n$. This means that $C_{A_{[\alpha_i]}}(x) \leq C_{A_{[\beta_i]+}}(x)$ for every $x \in X$. This implies that $C_{A_{[\alpha_i]}}(x) \leq \inf\limits_{\substack{(\beta_i)_1^n \in I^n \\ \beta_i < \alpha_i}} \left\{ C_{A_{[\beta_i]+}}(x) \right\}$ and therefore $C_{A_{[\alpha_i]}}(x) \leq C_{\bigcap\limits_{\substack{(\beta_i)_1^n \in I^n \\ \beta_i < \alpha_i}} A_{[\beta_i]+}}(x)$.

To prove the reverse inequality, it is enough to prove there exist $(\beta_i)_1^n \in I^n$ such that $\beta_i < \alpha_i$ for every $i = 1, 2, ..., n$ satisfying $C_{A_{[\beta_i]+}}(x) \leq C_{A_{[\alpha_i]}}(x)$ for every $x \in X$. Then $\inf\limits_{\substack{(\beta_i)_1^n \in I^n \\ \beta_i < \alpha_i}} \left\{ C_{A_{[\beta_i]+}}(x) \right\} \leq C_{A_{[\beta_i]+}}(x) \leq C_{A_{[\alpha_i]}}(x)$ for every $x \in X$. Hence $C_{\bigcap\limits_{\substack{(\beta_i)_1^n \in I^n \\ \beta_i < \alpha_i}} A_{[\beta_i]+}}(x) \leq C_{A_{[\alpha_i]}}(x)$ for every $x \in X$. From two inequalities, we have

$C_{A_{[\alpha_i]}}(x) = C_{\bigcap\limits_{\substack{(\beta_i)_1^n \in I^n \\ \beta_i < \alpha_i}} A_{[\beta_i]+}}(x)$ for every $x \in X$.

It remains to prove the existence of $(\beta_i)_1^n \in I^n$ such that $\beta_i < \alpha_i$ for every $i = 1, 2, ..., n$ satisfying $C_{A_{[\beta_i]+}}(x) \leq C_{A_{[\alpha_i]}}(x)$ for every $x \in X$. Let $x \in X$. There are three cases;

Case 1: $C_{A_{[\alpha_i]}}(x) = 0$. Then there exist some $i \in \{1, 2, ..., n\}$ such that $A_i^1(x) < \alpha_i$. For those $i's$ there exist $\beta_i \in I$ such that $A_i^1(x) < \beta_i < \alpha_i$ and for other $i's$ take $\beta_i = \alpha_i/2$. Then $C_{A_{[\beta_i]+}}(x) = 0$. That is, for $(\beta_i)_1^n$ we have $\beta_i < \alpha_i$ for every $i = 1, 2, ..., n$ and $C_{A_{[\beta_i]+}}(x) = C_{A_{[\alpha_i]}}(x)$.

Case 2: $C_{A_{[\alpha_i]}}(x) = j$. Then there exist some $i \in \{1, 2, ..., n\}$ such that $A_i^{j+1}(x) < \alpha_i$. For those $i's$ there exist $\beta_i \in I$ such that $A_i^{j+1}(x) < \beta_i < \alpha_i$ and for other $i's$ take $\beta_i = \alpha_i/2$. Then $C_{A_{[\beta_i]+}}(x) \leq j$. That is, for $(\beta_i)_1^n$ we have $\beta_i < \alpha_i$ for every $i = 1, 2, ..., n$ and $C_{A_{[\beta_i]+}}(x) \leq C_{A_{[\alpha_i]}}(x)$.

Case 3: $C_{A_{[\alpha_i]}}(x) = k$. Take $\beta_i = \alpha_i/2$ for every $i = 1, 2, ..., n$. Then $C_{A_{[\beta_i]+}}(x) \leq k$. That is, for $(\beta_i)_1^n$ we have $\beta_i < \alpha_i$ for every $i = 1, 2, ..., n$ and $C_{A_{[\beta_i]+}}(x) \leq C_{A_{[\alpha_i]}}(x)$.

These three cases prove that, for every $x \in X$, there exist $(\beta_i)_1^n \in I^n$ such that $\beta_i < \alpha_i$ for every $i = 1, 2, ..., n$ satisfying $C_{A_{[\beta_i]+}}(x) \leq C_{A_{[\alpha_i]}}(x)$.

Theorem 3.9. For any $A \in MS_{(n,k)}(X)$ and $(\alpha_i)_1^n \in I^n$ such that $\alpha_i \neq 1$ for every $i = 1, 2, ..., n$, the following property holds:

$$A_{[\alpha_i]+} = \bigcup_{\substack{(\beta_i)_1^n \in I^n \\ \beta_i > \alpha_i, i=1,2,...,n}} A_{[\beta_i]} = \bigcup_{\substack{(\beta_i)_1^n \in I^n \\ \beta_i > \alpha_i, i=1,2,...,n}} A_{[\beta_i]+}$$

Proof: Let $A \in MS_{(n,k)}(X)$ and $(\alpha_i)_1^n \in I^n$ such that $\alpha_i \neq 1$ for every $i = 1, 2, ..., n$. To prove $A_{[\alpha_i]+} = \bigcup\limits_{\substack{(\beta_i)_1^n \in I^n \\ \beta_i > \alpha_i}} A_{[\beta_i]}$, it is enough to prove $C_{A_{[\alpha_i]+}}(x) =$ $C_{\bigcup\limits_{\substack{(\beta_i)_1^n \in I^n \\ \beta_i > \alpha_i}} A_{[\beta_i]}}(x)$ for every $x \in X$. We have $A_{[\beta_i]} \subseteq A_{[\alpha_i]+}$ for any $(\beta_i)_1^n \in I^n$ such that $\beta_i > \alpha_i$ for every $i = 1, 2, ..., n$. This means that $C_{A_{[\beta_i]}}(x) \leq C_{A_{[\alpha_i]+}}(x)$ for every $x \in X$. This implies that $\sup\limits_{\substack{(\beta_i)_1^n \in I^n \\ \beta_i > \alpha_i}} \left\{C_{A_{[\beta_i]}}(x)\right\} \leq C_{A_{[\alpha_i]+}}(x)$ and therefore

$$C_{\bigcup\limits_{\substack{(\beta_i)_1^n \in I^n \\ \beta_i > \alpha_i}} A_{[\beta_i]}}(x) \leq C_{A_{[\alpha_i]+}}(x)$$

To prove the reverse inequality, it is enough to prove there exist $(\beta_i)_1^n \in I^n$ such that $\beta_i > \alpha_i$ for every $i = 1, 2, ..., n$ satisfying $C_{A_{[\alpha_i]+}}(x) \leq C_{A_{[\beta_i]}}(x)$ for every $x \in X$. Then $C_{A_{[\alpha_i]+}}(x) \leq C_{A_{[\beta_i]}}(x) \leq \sup\limits_{\substack{(\beta_i)_1^n \in I^n \\ \beta_i > \alpha_i}} \left\{C_{A_{[\beta_i]}}(x)\right\}$ for every $x \in X$. Hence $C_{A_{[\alpha_i]+}}(x) \leq C_{\bigcup\limits_{\substack{(\beta_i)_1^n \in I^n \\ \beta_i > \alpha_i}} A_{[\beta_i]}}(x)$ for every $x \in X$. From two inequalities, we have $C_{A_{[\alpha_i]+}}(x) = C_{\bigcup\limits_{\substack{(\beta_i)_1^n \in I^n \\ \beta_i < \alpha_i}} A_{[\beta_i]}}(x)$ for every $x \in X$.

It remains to prove the existence of $(\beta_i)_1^n \in I^n$ such that $\beta_i > \alpha_i$ for every $i = 1, 2, ..., n$ satisfying $C_{A_{[\alpha_i]+}}(x) \leq C_{A_{[\beta_i]}}(x)$ for every $x \in X$. Let $x \in X$. There are three cases;

Case 1: $C_{A_{[\alpha_i]+}}(x) = 0$. Then there exist some $i \in \{1, 2, ..., n\}$ such that $A_i^1(x) \leq \alpha_i$. Since $\alpha_i \neq 1$ take $\beta_i \in I$ such that $\alpha_i < \beta_i < 1$. Then $C_{A_{[\beta_i]}}(x) = 0$. That is, for $(\beta_i)_1^n$ we have $\beta_i > \alpha_i$ for every $i = 1, 2, ..., n$ and $C_{A_{[\alpha_i]+}}(x) = C_{A_{[\beta_i]}}(x)$

Case 2: $C_{A_{[\alpha_i]+}}(x) = j$. Then $A_i^j(x) > \alpha_i$ for every $i = 1, 2, .., n$. Then there exist $\beta_i \in I$ such that $A_i^j(x) > \beta_i > \alpha_i$ for every $i = 1, 2, .., n$. Then $C_{A_{[\beta_i]}}(x) \geq j$. That is, for $(\beta_i)_1^n$ we have $\beta_i > \alpha_i$ for every $i = 1, 2, ..., n$ and $C_{A_{[\alpha_i]+}}(x) \leq C_{A_{[\beta_i]}}(x)$

Case 3: $C_{A_{[\alpha_i]+}}(x) = k$. Then $A_i^k(x) > \alpha_i$ for every $i = 1, 2, ..., n$. Then there exist $\beta_i \in I$ such that $A_i^j(x) > \beta_i > \alpha_i$ for every $i = 1, 2, ..., n$. Then $C_{A_{[\beta_i]}}(x) = k$. That is, for $(\beta_i)_1^n$ we have $\beta_i > \alpha_i$ for every $i = 1, 2, ..., n$ and $C_{A_{[\alpha_i]+}}(x) = C_{A_{[\beta_i]}}(x)$

These three cases prove that, for every $x \in X$, there exist $(\beta_i)_1^n \in I^n$ such that $\beta_i > \alpha_i$ for every $i = 1, 2, ..., n$ satisfying $C_{A_{[\alpha_i]+}}(x) \leq C_{A_{[\beta_i]}}(x)$.

Next, to prove $A_{[\alpha_i]+} = \bigcup_{\substack{(\beta_i)_1^n \in I^n \\ \beta_i > \alpha_i}} A_{[\beta_i]+}$, it is enough to prove

$$C_{A_{[\alpha_i]+}}(x) = C_{\bigcup_{\substack{(\beta_i)_1^n \in I^n \\ \beta_i > \alpha_i}} A_{[\beta_i]+}}(x)$$

for every $x \in X$. We have $A_{[\beta_i]+} \subseteq A_{[\alpha_i]+}$ for any $(\beta_i)_1^n \in I^n$ such that $\beta_i > \alpha_i$ for every $i = 1, 2, ..., n$. This means that $C_{A_{[\beta_i]+}}(x) \leq C_{A_{[\alpha_i]+}}(x)$ for every $x \in X$. This implies that $\sup_{\substack{(\beta_i)_1^n \in I^n \\ \beta_i > \alpha_i}} \left\{C_{A_{[\beta_i]+}}(x)\right\} \leq C_{A_{[\alpha_i]+}}(x)$ and therefore $C_{\bigcup_{\substack{(\beta_i)_1^n \in I^n \\ \beta_i > \alpha_i}} A_{[\beta_i]+}}(x) \leq C_{A_{[\alpha_i]+}}(x)$.

To prove the reverse inequality, it is enough to prove there exist $(\beta_i)_1^n \in I^n$ such that $\beta_i > \alpha_i$ for every $i = 1, 2, ..., n$ satisfying $C_{A_{[\alpha_i]+}}(x) \leq C_{A_{[\beta_i]+}}(x)$ for every $x \in X$. Then $C_{A_{[\alpha_i]+}}(x) \leq C_{A_{[\beta_i]+}}(x) \leq \sup_{\substack{(\beta_i)_1^n \in I^n \\ \beta_i > \alpha_i}} \left\{C_{A_{[\beta_i]+}}(x)\right\}$ for every $x \in X$. Hence $C_{A_{[\alpha_i]+}}(x) \leq C_{\bigcap_{\substack{(\beta_i)_1^n \in I^n \\ \beta_i < \alpha_i}} A_{[\beta_i]+}}(x)$ for every $x \in X$. From two inequalities, we have $C_{A_{[\alpha_i]+}}(x) = C_{\bigcup_{\substack{(\beta_i)_1^n \in I^n \\ \beta_i > \alpha_i}} A_{[\beta_i]+}}(x)$ for every $x \in X$.

It remains to prove the existence of $(\beta_i)_1^n \in I^n$ such that $\beta_i > \alpha_i$ for every $i = 1, 2, ..., n$ satisfying $C_{A_{[\alpha_i]+}}(x) \leq C_{A_{[\beta_i]+}}(x)$ for every $x \in X$. Let $x \in X$. There are three cases;

Case 1: $C_{A_{[\alpha_i]+}}(x) = 0$. Then there exist some $i \in \{1, 2, ..., n\}$ such that $A_i^1(x) \leq \alpha_i$. Choose $\beta_i \in I$ such that $\alpha_i < \beta_i < 1$ and then $A_i^1(x) \leq \alpha_i < \beta_i$ for some $i \in \{1, 2, ..., n\}$. Then $C_{A_{[\beta_i]+}}(x) = 0$. That is, for $(\beta_i)_1^n$ we have $\beta_i > \alpha_i$ for every $i = 1, 2, ..., n$ and $C_{A_{[\alpha_i]+}}(x) = C_{A_{[\beta_i]+}}(x)$.

Case 2: $C_{A_{[\alpha_i]+}}(x) = j$. Then $A_i^j(x) > \alpha_i$ for every $i = 1, 2, .., n$. Then there exist $\beta_i \in I$ such that $A_i^j(x) > \beta_i > \alpha_i$ for every $i = 1, 2, .., n$. Then $C_{A_{[\beta_i]+}}(x) \geq j$. That is, for $(\beta_i)_1^n$ we have $\beta_i > \alpha_i$ for every $i = 1, 2, ..., n$ and $C_{A_{[\alpha_i]+}}(x) \leq C_{A_{[\beta_i]+}}(x)$.

Case 3: $C_{A_{[\alpha_i]+}}(x) = k$. Then $A_i^k(x) > \alpha_i$ for every $i = 1, 2, ..., n$. Then there exist $\beta_i \in I$ such that $A_i^j(x) > \beta_i > \alpha_i$ for every $i = 1, 2, ..., n$. Then $C_{A_{[\beta_i]+}}(x) = k$. That is, for $(\beta_i)_1^n$ we have $\beta_i > \alpha_i$ for every $i = 1, 2, ..., n$ and $C_{A_{[\alpha_i]+}}(x) = C_{A_{[\beta_i]+}}(x)$.

These three cases prove that, for every $x \in X$, there exist $(\beta_i)_1^n \in I^n$ such that $\beta_i > \alpha_i$ for every $i = 1, 2, ..., n$ satisfying $C_{A_{[\alpha_i]+}}(x) \leq C_{A_{[\beta_i]+}}(x)$.

Definition 3.10. The *level set* of a multiple set A is a crisp set, denoted by $\Lambda(A)$, is defined as

$$\begin{aligned}\Lambda(A) \;=\; & \{(\alpha_i)_1^n \in I^n; \alpha_i = A_i^j(x), 1 \leq i \leq n, \\ & \text{for some } x \in X \text{ and for some } j \in \{1, 2, ..., k\}\}\end{aligned}$$

3.3 Representations of Multiple Sets

The principal role of $\alpha_i - cuts$ and strong $\alpha_i - cuts$ in multiple set theory is their capability to represent corresponding multiple sets. In this section, it is shown that each multiple set can uniquely be represented by either the family of all its $\alpha_i - cuts$ or the family of all its strong $\alpha_i - cuts$.

Definition 3.11. Let $A \in MS_{(n,k)}(X)$, $(\alpha_i)_1^n \in I^n$ and $A_{[\alpha_i]} = \left\{C_{A_{[\alpha_i]}}(x)/x; x \in X\right\}$ be the $\alpha_i - cut$ of A. Then the special multiple set of A with respect to $(\alpha_i)_1^n$ is a multiple set $S[A; (\alpha_i)_1^n]$ over X, with the membership matrix $S[A; (\alpha_i)_1^n](x)$ in which the first $C_{A_{[\alpha_i]}}(x)$ columns are $((\alpha_i)_1^n)^T$ and remaining are zero columns. The strong special multiple set of A with respect to $(\alpha_i)_1^n$ is a multiple set $S^+[A; (\alpha_i)_1^n]$ over X, with the membership matrix $S^+[A; (\alpha_i)_1^n](x)$ for which the first $C_{A_{[\alpha_i]+}}(x)$ columns are $((\alpha_i)_1^n)^T$ and remaining are zero columns. $((\alpha_i)_1^n)^T$ denotes the column vector $(\alpha_i)_1^n$.

The representation of an arbitrary multiple set in terms of special multiple sets or in terms of strong special multiple sets is formulated as three basic decomposition theorems of multiple sets:

Theorem 3.12. (First Decomposition Theorem for Multiple sets)
For every $A \in MS_{(n,k)}(X)$, we have

$$A = \bigcup_{(\alpha_i)_1^n \in I^n} S[A; (\alpha_i)_1^n]$$

where $S[A; (\alpha_i)_1^n]$ is the special multiple set of A with respect to $(\alpha_i)_1^n$ and $\bigcup$ denotes the standard union.

Proof: Let $A \in MS_{(n,k)}(X)$ and $(\alpha_i)_1^n \in I^n$. To prove $A = \bigcup_{(\alpha_i)_1^n \in I^n} S[A;(\alpha_i)_1^n]$, it is enough to prove $A_i^j(x) = \left(\bigcup_{(\alpha_i)_1^n \in I^n} S[A;(\alpha_i)_1^n]\right)_i^j(x)$ for every $x \in X$. Let $x \in X$, $i \in \{1,2,...,n\}$ and $j \in \{1,2,...,k\}$. Take $a = A_i^j(x)$. Then

$$\begin{aligned}\left(\bigcup_{(\alpha_i)_1^n \in I^n} S[A;(\alpha_i)_1^n]\right)_i^j(x) &= \sup_{(\alpha_i)_1^n \in I^n}\left\{(S[A;(\alpha_i)_1^n])_i^j(x)\right\}\\ &= \max\left\{\sup_{\alpha_i \in [0,a]}\left\{(S[A;(\alpha_i)_1^n])_i^j(x)\right\},\right.\\ &\qquad \left.\sup_{\alpha_i \in (a,1]}\left\{(S[A;(\alpha_i)_1^n])_i^j(x)\right\}\right\}\end{aligned}$$

Since, for every $\alpha_i \in (a,1]$, $(S[A;(\alpha_i)_1^n])_i^j(x) = 0$ and for every $\alpha_i \in [0,a]$, $(S[A;(\alpha_i)_1^n])_i^j(x) = \alpha_i$ Therefore,

$$\left(\bigcup_{(\alpha_i)_1^n \in I^n} S[A;(\alpha_i)_1^n]\right)_i^j(x) = a = A_i^j(x)$$

for every $x \in X$. Therefore,

$$A = \bigcup_{(\alpha_i)_1^n \in I^n} S[A;(\alpha_i)_1^n]$$

Theorem 3.13. (Second Decomposition Theorem for Multiple sets)
For every $A \in MS_{(n,k)}(X)$, we have

$$A = \bigcup_{(\alpha_i)_1^n \in I^n} S^+[A;(\alpha_i)_1^n]$$

where $S^+[A;(\alpha_i)_1^n]$ is the strong special multiple set of A with respect to $(\alpha_i)_1^n$ and $\bigcup$ denotes the standard union.

Proof: Let $A \in MS_{(n,k)}(X)$ and $(\alpha_i)_1^n \in I^n$. We have to prove $A = \bigcup_{(\alpha_i)_1^n \in I^n} S^+[A;(\alpha_i)_1^n]$, it is enough to prove $A_i^j(x) = \left(\bigcup_{(\alpha_i)_1^n \in I^n} S^+[A;(\alpha_i)_1^n]\right)_i^j(x)$ for every $x \in X$. For, let $x \in X$, $i \in \{1,2,...,n\}$ and $j \in \{1,2,...,k\}$. Take $a = A_i^j(x)$. Then

$$\begin{aligned}\left(\bigcup_{(\alpha_i)_1^n \in I^n} S^+[A;(\alpha_i)_1^n]\right)_i^j(x) &= \sup_{(\alpha_i)_1^n \in I^n}\left\{\left(S^+[A;(\alpha_i)_1^n]\right)_i^j(x)\right\}\\ &= \max\left\{\sup_{\alpha_i \in [0,a)}\left\{\left(S^+[A;(\alpha_i)_1^n]\right)_i^j(x)\right\},\right.\\ &\qquad \left.\sup_{\alpha_i \in [a,1]}\left\{\left(S^+[A;(\alpha_i)_1^n]\right)_i^j(x)\right\}\right\}\end{aligned}$$

Since, for every $\alpha_i \in [a,1]$, $(S^{+}[A;(\alpha_i)_1^n])_i^j(x) = 0$ and for every $\alpha_i \in [0,a)$, $(S^{+}[A;(\alpha_i)_1^n])_i^j(x) = \alpha_i$. Therefore,

$$\left(\bigcup_{(\alpha_i)_1^n \in I^n} S^{+}[A;(\alpha_i)_1^n]\right)_i^j (x) = a = A_i^j(x)$$

for every $x \in X$. Therefore,

$$A = \bigcup_{(\alpha_i)_1^n \in I^n} S^{+}[A;(\alpha_i)_1^n]$$

Theorem 3.14. (Third Decomposition Theorem for Multiple sets)
For every $A \in MS_{(n,k)}(X)$, we have

$$A = \bigcup_{(\alpha)_n \in \Lambda(A)} S[A;(\alpha_i)_1^n]$$

where $\Lambda(A)$ is the level set of A, $S[A;(\alpha_i)_1^n]$ is the special multiple set of A and $\bigcup$ denotes the standard union.

The proof is analogous to the proof of theorem 3.12.

4 Multiple Complements

Definition 4.1. [11] A *fuzzy complement function* is a function $c:[0,1] \to [0,1]$ satisfying the axioms;
(c1) Boundary conditions: $c(0) = 1$ and $c(1) = 0$.
(c2) Monotonicity: For all $a, b \in [0,1]$, if $a \leq b$ then $c(a) \geq c(b)$.
A fuzzy complement function is said to be *continuous* if it satisfies the axiom;
(c3) c is a continuous function.
and is said to be *involutive*, if it satisfies the axiom;
(c4) $c(c(a)) = a$ for every $a \in [0,1]$.

Definition 4.2. [11] The *equilibrium point* of a fuzzy complement c is defined as any value $a \in [0,1]$ for which $c(a) = a$.

Theorem 4.3. [11] Every fuzzy complement has at most one equilibrium.

Theorem 4.4. [11] Assume that a given fuzzy complement c has an equilibrium e_c, which is unique by Theorem 4.3. Then

$$a \leq c(a) \text{ if and only if } a \leq e_c$$
$$a \geq c(a) \text{ if and only if } a \geq e_c$$

Theorem 4.5. [11] If c is continuous fuzzy complement, then c has a unique equilibrium.

Definition 4.6. [11] Let c be any fuzzy complemet function and $a \in [0,1]$ be any membership grade, then any real number $d_a \in [0,1]$ such that

$$c(d_a) - d_a = a - c(a)$$

is called a dual point of a with respect to c.

Theorem 4.7. [11] If a fuzzy complement c has an equilibrium e_c, then

$$d_{e_c} = e_c$$

Theorem 4.8. [11] For each $a \in [0,1]$, $d_a = c(a)$ if and only if c is involutive.

Theorem 4.9. [11]**(First Characterization Theorem of Fuzzy Complements)** Let c be a function from $[0,1]$ to $[0,1]$. Then, c is a fuzzy complement(involutive) if and only if there exists a continuous function g from $[0,1]$ to $\mathbb{R}$ such that $g(0) = 0$, g is strictly increasing and

$$c(a) = g^{-1}(g(1) - g(a))$$

for all $a \in [0,1]$.

Theorem 4.10. [11]**(Second Characterization Theorem of Fuzzy Complements)** Let c be a function from $[0,1]$ to $[0,1]$. Then, c is a fuzzy complement if and only if there exists a continuous function f from $[0,1]$ to $\mathbb{R}$ such that $f(1) = 0$, f is strictly decreasing and

$$c(a) = f^{-1}(f(0) - f(a))$$

for all $a \in [0,1]$.

Definition 4.11. Let $\mathbb{M} = \mathbb{M}_{n\times k}([0,1])$ be the set of all matrices of order $n \times k$ with entries from $[0,1]$. A *multiple complement function* is a function $c : \mathbb{M} \to \mathbb{M}$, where c is characterized by fuzzy complement functions c_{ij} in such a way that $A = (a_{ij})$ in $\mathbb{M}$ is mapped to $B = (b_{ij})$ in $\mathbb{M}$ such that $b_{ij} = c_{ij}(a_{ij})$ for every $i = 1, 2, ..., n$ and $j = 1, 2, ..., k$. In this case we represent multiple complement function c as a matrix (c_{ij}).

Remark 4.12. Using multiple complement function we can define complement of a multiple set as follows: Given a multiple set A in $MS_{(n,k)}(X)$, we obtain the complement of A, denoted by $c(A)$, by applying function c to matrix $A(x)$ for all $x \in X$.

Definition 4.13. A multiple complement function $c = (c_{ij})$ is said to be continuous if c_{ij} is continuous and is said to be involutive if c_{ij} is involutive for every $i = 1, 2, ..., n$ and $j = 1, 2, ..., k$.

Example 4.14. 1. Threshold Type Multiple Complement: Let $t_{ij} \in [0,1)$ for every $i = 1, 2, ..., n$ and $j = 1, 2, ..., k$. The function $c = (c_{ij})$ given by

$$c_{ij}(a) = \begin{cases} 1 & \text{for } a \leq t_{ij} \\ 0 & \text{for } a > t_{ij} \end{cases}$$

for $a \in [0,1]$ is a multiple complement. The matrix $T = (t_{ij})$ is called the threshold of c. This function is neither continuous and nor involutive.

2. The function $c = (c_{ij})$, where c_{ij} is defined by $c_{ij}(a) = \frac{1}{2}(1 + \cos \pi a)$ for every $i = 1, 2, ..., n$ and $j = 1, 2, ..., k$, is a multiple complement. This is continuous, but not involutive.

3. Sugeno Class Multiple Complement: The function $c = (c_{ij})$, where c_{ij} is defined by $c_{ij}(a) = \frac{1-a}{1+\lambda_{ij}a}$, where $\lambda_{ij} \in (-1, \infty)$, is an involutive multiple complement. Clearly, Sugeno class multiple complement is characterized by the matrix $\Lambda = (\lambda_{ij})$ and is represented by c_Λ. For $\lambda_{ij} = 0$ for every $i = 1, 2, ..., n$ and $j = 1, 2, ..., k$, this function becomes the standard multiple complement.

4. Yager Class Multiple Complement: The function $c = (c_{ij})$, where c_{ij} is defined by $c_{ij}(a) = (1 - a^{\omega_{ij}})^{1/\omega_{ij}}$ where $\omega_{ij} \in (0, \infty)$, is an involutive multiple complement. Clearly, Yager class multiple complement is characterized by the matrix $\Omega = (\omega_{ij})$ and is represented by c_Ω. When $\omega_{ij} = 1$, for every $i = 1, 2, ..., n$ and $j = 1, 2, ..., k$, this function becomes the standard multiple complement.

Definition 4.15. An equilibrium matrix of a multiple complement function c is defined as any matrix A in $\mathbb{M}$ for which $c(A) = A$.

Note: If $c = (c_{ij})$, there exists an equilibrium matrix E_c for c if and only if there exist $e_{ij} \in [0, 1]$ such that $c_{ij}(e_{ij}) = e_{ij}$ (that is e_{ij} is the equilibrium point of c_{ij}) for every $i = 1, 2, ..., n$ and $j = 1, 2, ..., k$. Then $E_c = (e_{ij})$.

Example 4.16. The equilibrium matrix of the Sugeno class multiple complement c_Λ characterized by the matrix $\Lambda = (\lambda_{ij})$ is given by $E_\Lambda = (e_{ij})$, where

$$e_{ij} = \begin{cases} ((1+\lambda_{ij})^{1/2} - 1)/\lambda_{ij} & \text{for } \lambda_{ij} \neq 0 \\ 1/2 & \text{for } \lambda_{ij} = 0 \end{cases}$$

Theorem 4.17. Every multiple complement has atmost one equilibrium matrix.

Proof. By Theorem 4.3, c_{ij} has at most one equilibrium point for every $i = 1, 2, ..., n$ and $j = 1, 2, ..., k$. Hence a multiple complement has at most one equilibrium matrix.
Notation: In this context, for $A, B \in \mathbb{M}$, we say $A \leq B$, if $a_{ij} \leq b_{ij}$ for $i = 1, 2, ..., n$ and $j = 1, 2, ..., k$

Theorem 4.18. Let E_c be the equilibrium matrix of multiple complement function c. Then

$$A \leq c(A) \text{ if and only if } A \leq E_c$$
$$A \geq c(A) \text{ if and only if } A \geq E_c$$

Proof. By Theorem 4.4, for every c_{ij}, we have

$$a \leq c_{ij}(a) \text{ if and only if } a \leq e_{c_{ij}}$$
$$a \geq c_{ij}(a) \text{ if and only if } a \geq e_{c_{ij}}$$

for $i = 1, 2, ..., n$ and $j = 1, 2, ..., k$. Hence the theorem.

Theorem 4.19. If c is a continuous multiple complement, then c has a unique equilibrium matrix.

Proof. Since c_{ij} is continuous, by Theorem 4.5, c_{ij} has a unique equilibrium point, say e_{ij} for every $i = 1, 2, ..., n$ and $j = 1, 2, ..., k$.. Then the matrix $E_c = (e_{ij})$ is the unique equilibrium matrix of c.

Definition 4.20. Let c be any multiple complement function and $A \in \mathbb{M}$, then any matrix $D_A \in \mathbb{M}$ such that

$$c(D_A) - D_A = A - c(A)$$

is called a dual matrix of A with respect to c.

Note: If $c = (c_{ij})$, there exists a dual matrix D_A for $A = (a_{ij})$ with respect to c if and only if there exist $d_{ij} \in [0, 1]$ such that $c_{ij}(d_{ij}) - d_{ij} = a_{ij} - c(a_{ij})$ (that is d_{ij} is the dual point of a_{ij} with respect to c_{ij}) for every $i = 1, 2, ..., n$ and $j = 1, 2, ..., k$. Then $D_A = (d_{ij})$.

Theorem 4.21. If a multiple complement c has an equilibrium matrix E_c, then

$$D_{E_c} = E_c$$

Proof. We have $E_c = (e_{ij})$, where e_{ij} is the equilibrium point of c_{ij} for $i = 1, 2, ..., n$ and $j = 1, 2, ..., k$. Then by Theorem 4.7, $d_{ij} = e_{ij}$, where d_{ij} is the dual point of e_{ij} with respect to c_{ij} for $i = 1, 2, ..., n$ and $j = 1, 2, ..., k$. Thus

$$D_{E_c} = (d_{ij}) = (e_{ij}) = E_c$$

Theorem 4.22. For each $A \in \mathbb{M}$, $D_A = c(A)$ if and only if c is involutive.

Proof. We have, for each complement function c_{ij}, by Theorem 4.8, $d_{a_{ij}} = c_{ij}(a_{ij})$ if and only if c_{ij} is involutive, for every $i = 1, 2, ..., n$ and $j = 1, 2, ...k$. Hence the theorem.

Theorem 4.23. (First Characterization Theorem of Multiple Complements) Let c be a function from $\mathbb{M}$ to $\mathbb{M}$, where c is characterized by functions c_{ij} for $i = 1, 2, ..., n$ and $j = 1, 2, ...k$ in such a way that $A = (a_{ij})$ in $\mathbb{M}$ is mapped to $B = (b_{ij})$ in $\mathbb{M}$ such that $b_{ij} = c_{ij}(a_{ij})$ for every $i = 1, 2, ..., n$ and $j = 1, 2, ..., k$. Then c is a multiple complement(involutive) if and only if there exists a function $G : \mathbb{M} \to \mathbb{M}_{n\times k}(\mathbb{R})$ where G is characterized by continuous functions g_{ij} for $i = 1, 2, ..., n$ and $j = 1, 2, ...k$ in such a way that $A = (a_{ij})$ in $\mathbb{M}$ is mapped to $B = (b_{ij})$ in $\mathbb{M}_{n\times k}(\mathbb{R})$ such that $b_{ij} = g_{ij}(a_{ij})$, $g_{ij}(0) = 0$, g_{ij} is strictly increasing and $c_{ij}(a) = g_{ij}^{-1}(g_{ij}(1) - g_{ij}(a))$ for every $a \in [0, 1]$, $i = 1, 2, ..., n$ and $j = 1, 2, ..., k$. In this case we represent function G as a matrix (g_{ij}).

Proof. For every $i = 1, 2, ..., n$ and $j = 1, 2, ..., k$, by Theorem 4.9, c_{ij} is a fuzzy complement(involutive) if and only if there exists a continuous function g_{ij} from $[0, 1]$ to $\mathbb{R}$ such that $g_{ij}(0) = 0$, g_{ij} is strictly increasing and

$$c_{ij}(a) = g_{ij}^{-1}(g_{ij}(1) - g_{ij}(a))$$

for all $a \in [0, 1]$. Hence the theorem.

Remark 4.24. Functions G, defined in the Theorem 4.23, are usually called increasing generators. Each function G that qualifies as an increasing generator determines a multiple complement.
For standard multiple complement, the increasing generator is $G = (g_{ij})$, where g_{ij} is defined as $g_{ij}(a) = a$ for evrey $a \in [0, 1]$.
For Sugeno class of multiple complements, the increasing generators are $G = (g_{ij})$, where g_{ij} is defined as $g_{ij}(a) = \dfrac{1}{\lambda_{ij}} \ln(1 + \lambda_{ij} a)$ for every $a \in [0, 1]$ and for $\lambda_{ij} > -1$.
For Yager class of multiple complements, the increasing generators are $G = (g_{ij})$, where g_{ij} is defined as $g_{ij}(a) = a^{\omega_{ij}}$ for every $a \in [0, 1]$ and for $\omega_{ij} > 0$.

Theorem 4.25. (Second Characterization Theorem of Multiple Complements) Let c be a function from $\mathbb{M}$ to $\mathbb{M}$, where c is characterized by functions c_{ij} for $i = 1, 2, ..., n$ and $j = 1, 2, ...k$ in such a way that $A = (a_{ij})$ in $\mathbb{M}$ is mapped to $B = (b_{ij})$ in $\mathbb{M}$ such that $b_{ij} = c_{ij}(a_{ij})$ for every $i = 1, 2, ..., n$ and $j = 1, 2, ..., k$. Then c is a multiple complement if and only if there exists a function $F : \mathbb{M} \to M_{n\times k}(\mathbb{R})$ where F is characterized by continuous functions f_{ij} for $i = 1, 2, ..., n$ and $j = 1, 2, ...k$ in such a way that $A = (a_{ij})$ in $\mathbb{M}$ is mapped to $B = (b_{ij})$ in $M_{n\times k}(\mathbb{R})$ such that $b_{ij} = f_{ij}(a_{ij})$, $f_{ij}(1) = 0$, f_{ij} is strictly decreasing and $c_{ij}(a_{ij}) = f_{ij}^{-1}(f_{ij}(0) - f_{ij}(a))$ for every $a \in [0, 1]$, i=1,2,...,n and j=1,2,...,k. In this case we represent function F as a matrix (f_{ij}).

Proof. For every $i = 1, 2, ..., n$ and $j = 1, 2, ..., k$, by Theorem 4.9, c_{ij} is a fuzzy complement if and only if there exists a continuous function f_{ij} from $[0, 1]$ to $\mathbb{R}$ such that $f_{ij}(1) = 0$, f_{ij} is strictly decreasing and

$$c_{ij}(a) = f_{ij}^{-1}(f_{ij}(0) - f_{ij}(a))$$

for all $a \in [0, 1]$. Hence the theorem.

Remark 4.26. Functions F defined in the Theorem 4.24 are usually called decreasing generators. Each function F that qualifies as an decreasing generator also determines a multiple complement.
For standard multiple complement, the decreasing generator is $F = (f_{ij})$, where f_{ij} is defined as $f_{ij}(a) = -ka + k$ for every $a \in [0, 1]$, where $k > 0$.
For Yager class of multiple complements, the decreasing generators are $F = (f_{ij})$, where f_{ij} is defined as $f_{ij}(a) = 1 - a^{\omega_{ij}}$ for every $a \in [0, 1]$ and for $\omega_{ij} > 0$.

5 Conclusion

In this paper, a modified definition of multiple sets is given and it is shown that the revised definition also satisfies all fundamental properties satisfied by the earlier definition. Then, the ideas of $\alpha_i - cut$ and strong $\alpha_i - cut$ and representation of multiple set by using $\alpha_i - cut$ and strong $\alpha_i - cut$ are proposed . Then, the concept of multiple complement is introduced as an extension of fuzzy complement and its properties are discussed.

Aggregation operations on fuzzy sets are operations by which several fuzzy sets are combined in a desirable way to produce a single fuzzy set. So one can think about aggregation operations on multiple sets, which yields a single multiple set by combining two or more multiple sets. Fuzzy numbers are special type of fuzzy sets, which play an important role in many applications, including fuzzy control, decision making, approximate reasoning, optimization and statistics with imprecise probabilities. So in future work, it is possible to think multiple number as a special type of multiple set.

Acknowledgement

The first author acknowledge the financial assistance given by Ministry of Human Resource Development, Government of India and the National Institute of Technology Calicut throughout the preparation of this paper.

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http://www.newtheory.org ISSN: 2149-1402

Journal of New Theory

Received: 21.09.2015
Published: 02.03.2016
Year: **2016**, *Number*: **11**, *Pages*: **54-68**
*Original Article***

HESITANT FUZZY SUBGROUPS

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Abstaract — Hesitant fuzzy subgroup defined on a group G generalizes the idea of fuzzy subgroups. It mainly focuses on the multiplicity of values encountered when dealing with hesitant fuzzy sets. The preliminary concepts needed to build Hesitant fuzzy groups are discussed in this paper. Some results using the extension principle for hesitant fuzzy sets are discussed. The extension principle uses a new conjunction and disjunction operation on hesitant fuzzy sets. Various properties of composition and inverse operations on hesitant fuzzy sets are discussed before studying the structure of hesitant fuzzy groups. The concept of normal hesitant fuzzy subgroup of a group G is also studied in detail.

Keywords — *Hesitant fuzzy subgroups, hesitant fuzzy algebra, normal hesitant fuzzy subgroup.*

1 Introduction

The introduction of Fuzzy set theory by Lotfi A Zadeh [19] was a landmark which opened up new avenues for researchers. The use of membership functions to characterise a set meant that we can better simulate real world scenarios by mathematics. It brought along with it new challenges to researchers all over the world on how to introduce mathematical operations and structures into the new area.

A Rosenfeld introduced fuzzy groups [10] by applying fuzzy set theory to generalize some of the basic concepts of groups. Some mathematical structures which would intuitively seem to be fuzzy did not satisfy the stated definitions. Anthony and Sherwood [1] have redefined the fuzzy algebraic structures (giving examples) to meet such requirements. Seselja and Tepavcevic [11] have taken fuzzy subgroups to be mappings from a group to a partially ordered set and have studied them from a general point of view. Six kinds of fuzzy homomorphisms have been introduced in [7]. Chen and Gu [2] have dealt with fuzzy factor groups and the first fundamental theorem of isomorphism of fuzzy groups. Dixit et al. [6] have studied the union of fuzzy subgroups addressing some of the already existing queries in this area. Dib and

** *Edited by* Oktay Muhtaroğlu *(Area Editor) and* Naim Çağman *(Editor-in-Chief).*
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Hassan [5] have defined the concept of a normal fuzzy subgroup in a fuzzy group. Mordeson and Bhutani [8] have illustrated all these concepts regarding fuzzy group theory. The concept of homologous fuzzy subgroups of a group G is introduced in [20]. A new kind of fuzzy group based on fuzzy binary operations is proposed in [18].

Hesitant fuzzy set was introduced by Vicenc Torra [12],[13] which further characterised an element by a set of membership values thereby decreasing the loss of information during fuzzification. Distance measures for Hesitant Fuzzy Sets (HFSs) have been investigated in [17]. Studies on hesitant fuzzy information aggregation techniques and their application in decision making can be seen in [15],[16]. Zhu et al. [21] have defined a hesitant fuzzy geometric Bonferroni mean as an extension of the geometric Bonferroni mean to hesitant fuzzy environment. Rodriguez et al. [9] have introduced hesitant fuzzy linguistic term sets which gives flexibility in applications involving hesitant situations under qualitative settings. Wei [14] has studied the Hesitant fuzzy prioritized operators and has illustrated their application in the hesitant fuzzy multiple attribute decision making problems in which the attributes are in different priority level.

This paper mainly introduces algebraic structures on the Hesitant fuzzy domain. Some of the preliminary definitions and results regarding hesitant fuzzy groups and homomorphisms between them have been discuseed in [4]. The paper begins by discussing basic results regarding Hesitant fuzzy set theory. It then moves on to study Hesitant fuzzy alpha-cut and the extension principle in Hesitant Fuzzy set domain. The next section discusses certain properties and results regarding Hesitant fuzzy subgroups. The last section examines the notion of a normal hesitant fuzzy subgroup and studies its general structure and properties.

2 Basic Concepts

This section introduces the basic concepts in Hesitant fuzzy set theory.

Definition 2.1 ([13]). Let X be a reference set then a Hesitant fuzzy set(HFS) on X is defined in terms of a function h that when applied to X returns a subset of $[0,1]$ $h : X \to P[0,1]$ where $P[0,1]$ denotes power set of $[0,1]$.

The set of all hesitant fuzzy subsets of X is called the hesitant fuzzy power set of X and is denoted by $HF(X)$.

The empty hesitant set, the full hesitant set, the set to represent complete ignorance for x and the nonsense set are defined as follows:
empty set : $h_0(x) = \{0\} \quad \forall\, x \in X$
full set : $h_X(x) = \{1\} \quad \forall\, x \in X$
complete ignorance $h(x) = [0,1]$
set for a nonsense $x : h(x) = \phi$

Given an hesitant fuzzy set h, its lower and upper bound are defined as follows :
$h^-(x) = \min h(x)$
$h^+(x) = \max h(x)$

For convenience we call $h(x)$ a hesitant fuzzy element (HFE) [15]. Let $l(h(x))$ be the number of values in $h(x)$.

Definition 2.2 ([15])**.** Score for a HFE , $s(h) = \frac{1}{l(h)}\sum_{\gamma\in h}\gamma$ is called the score function of h.

Note : If the HFE is infinite then $s(h(x)) = \frac{1}{2}(\inf(h(x)) + \sup(h(x)))$. .

Definition 2.3. [3] Let $h \in HF(X)$. Then the set $\bigcup_{x\in X} h(x)$ is called the image of h and is denoted by $h(X)$. The set $\{x|x \in X, s(h(x)) > 0\}$, is called the support of h and is denoted by h^* . h is called finite hesitant fuzzy set if h^* is a finite set , and an infinite hesitant fuzzy set otherwise.

Definition 2.4. [4] Let $Y \subseteq X$ and $A \subseteq [0, 1]$. We define $A_Y \in HF(X)$ as follows :

$$A_Y(x) = \begin{cases} A, & \text{for } x \in Y \\ \{0\}, & \text{for } x \in X \setminus Y \end{cases}$$

If Y is a singleton, say $\{y\}$, then $A_{\{y\}}$ is called a hesitant fuzzy point (or hesitant fuzzy singleton), and is denoted by y_A. Let $\{1\}_Y$ denote the characteristic function of Y. If S is a set of hesitant fuzzy singletons, then we let $foot(S) = \{y \in X|y_A \in S\}$.

Definition 2.5 ([3])**.** Hesitant Equality : Let h_1 and h_2 be two hesitant fuzzy sets on X, then we say that h_1 is equal to h_2 (denoted $h_1 = h_2$) iff $h_1(x) = h_2(x)\ \forall x \in X$ and h_1 is hesitantly equal to h_2 (denoted $h_1 \approx h_2$) iff $s(h_1(x)) = s(h_2(x)\ \forall x \in X$.

Definition 2.6 ([3])**.** Hesitant subset :(h_1 hesitant subset of h_2) Let h_1 and h_2 be two hesitant fuzzy sets on X, then we say that h_1 is a hesitant subset of h_2 (denoted by $h_1 \preceq h_2$) iff $s(h_1(x)) \le s(h_2(x))\ \forall x \in X$.

Definition 2.7 ([3])**.** Hesitant proper subset : $h_1 \prec h_2$ if $s(h_1(x)) \le s(h_2(x)\forall x \in X$ and $s(h_1(x)) < s(h_2(x)$ for atleast one $x \in X$.

Definition 2.8 ([13])**.** Given two hesitant fuzzy sets represented by their membership functions h_1 and h_2, their union represented by $h_1 \bigcup h_2$ is defined as

$$\begin{aligned}(h_1 \bigcup h_2)(x) &= \left\{\gamma \in (h_1(x) \bigcup h_2(x)/\gamma \ge \max(h_1^-, h_2^-)\right\} \\ &= \bigcup_{\gamma_1\in h_1,\gamma_2\in h_2} \max\{\gamma_1, \gamma_2\}.\end{aligned}$$

Definition 2.9 ([13])**.** Given two hesitant fuzzy sets represented by their membership functions h_1 and h_2, their intersection represented by $h_1 \bigcap h_2$ is defined as

$$\begin{aligned}(h_1 \bigcap h_2)(x) &= \left\{\gamma \in (h_1(x) \bigcup h_2(x)/\gamma \le min(h_1^+, h_2^+)\right\}. \\ &= \bigcup_{\gamma_1\in h_1,\gamma_2\in h_2} \min\{\gamma_1, \gamma_2\}.\end{aligned}$$

Definition 2.10. (max union and min intersection). Let H and G be two subsets of $[0,1]$. Then we define the max union of H and G denoted by $\bigcup_{\max}$, as
$H \bigcup_{\max} G = \bigcup_{a \in H, b \in G} \max\{a,b\}$

The min intersection of H and G denoted by $\bigcap_{min}$ is defined as
$H \bigcap_{\min} G = \bigcap_{a \in H, b \in G} \min\{a,b\}$

For any collection , $\{h_i | i \in I\}$, of hesitant fuzzy subsets of X, where I is a nonempty index set, then $\forall x \in X$

$$(\bigcup_{i \in I}) h_i(x) = \bigcup_{i \in I} \cdot_{\max} h_i(x)$$
$$(\bigcap_{i \in I}) h_i(x) = \bigcap_{i \in I} \cdot_{\min} h_i(x)$$

Note : The above definition applies to the case where we are taking the union of h_i 's the hesitant values for the different values of x in the same set in contrast to the definition of Torra where we are taking the union over two different sets.

Definition 2.11. [4] Given two hesitant fuzzy sets represented by their membership functions h_1 and h_2, we define a score based intersection of h_1 and h_2 (denoted by $h_1 \widetilde{\wedge} h_2$)as

$$(h_1 \widetilde{\wedge} h_2)(x) = \begin{cases} h_1(x) & \text{if } h_1(x) \prec h_2(x) \\ h_2(x) & \text{if } h_2(x) \prec h_1(x) \\ h_1(x) \bigcup h_2(x) & \text{if } h_1(x) \approx h_2(x) \end{cases}$$

and a score based union of h_1 and h_2 (denoted by $h_1 \widetilde{\vee} h_2$)as

$$(h_1 \widetilde{\vee} h_2)(x) = \begin{cases} h_1(x) & \text{if } h_1(x) \succ h_2(x) \\ h_2(x) & \text{if } h_2(x) \succ h_1(x) \\ h_1(x) \bigcup h_2(x) & \text{if } h_1(x) \approx h_2(x) \end{cases}$$

For a collection, $\{h_i | i \in I\}$ of hesitant fuzzy subsets of X, where I is a non empty index set, we have $\forall x \in X$

$$(\widetilde{\vee}_{i \in I} h_i)(x) = \widetilde{\vee}_{i \in I} h_i(x)$$

$$(\widetilde{\wedge}_{i \in I} h_i)(x) = \widetilde{\wedge}_{i \in I} h_i(x)$$

Definition 2.12. Let $h_A \in HF(X)$ and $\alpha \in [0,1]$ then the $\alpha - level$ cut set of hesitant fuzzy set h_A , denoted by $h_{A\alpha}$ is $h_{A\alpha} = \{x \in X | s(h_A(x)) \geq \alpha\}$.
$h_{A\alpha+} = \{x \in X | s(h_A(x)) > \alpha\}$. is called strong $\alpha - level$ cut set of h_A . We can define $h^+_{A\alpha} = \left\{x \in X | s(h^+_A(x)) \geq \alpha\right\}$ and $h^-_{A\alpha} = \left\{x \in X | s(h^-_A(x)) \geq \alpha\right\}$. Then clearly $h^-_{A\alpha} \subseteq h_{A\alpha} \subseteq h^+_{A\alpha}$ $\forall \alpha \in [0,1]$

Note : When the set under consideration is clear we can denote $h_{A\alpha}$ by h_α , $h^+_{A\alpha}$ by h^+_α and $h^-_{A\alpha}$ by h^-_α.

Lemma 2.13. Let $h_A, h_B \in HF(X)$, then the following assertions hold,

1. $h_A \preceq h_B, \alpha \in [0,1] \Rightarrow h_{A\alpha} \subseteq h_{B\alpha}$
2. $\alpha \leq \beta; \alpha, \beta, \in [0,1] \Rightarrow h_{A\beta} \subseteq h_{A\alpha}$
3. $h_A = h_B \Leftrightarrow h_{A\alpha} = h_{B\alpha}$

Definition 2.14. Let I be a nonempty index set and let $\{X_i | I \in I\}$ be a collection of nonempty sets. Let X denote the Cartesian product of the X_i's, namely,

$$X = \prod_{i \in I} X_i = \{(x_i)_{i \in I} | x_i \in X_i, i \in I\}$$

Let $h_i \in HF(X_i)$ for all $i \in I$. Define the fuzzy subset h of X by $h(x) = \widetilde{\wedge}_{i \in I} h_i(x_i) \quad \forall x = (x_i)_{I \in I} \in X$. Then h is called the complete direct product of the h_i's and is denoted by $h = \widetilde{\prod}_{i \in I} h_i$.

Definition 2.15. [4] (Extension Principle). Let f be a function from X into Y, and let $h_1 \in HF(X)$ and $h_2 \in HF(Y)$. Define the hesitant fuzzy subsets $f(h_1) \in HF(Y)$ and $f^{-1}(h_2) \in HF(Y)$ by $\forall y \in Y$,

$$f(h_1)(y) = \begin{cases} \widetilde{\vee} \{h_1(x) | x \in X, f(x) = y\} ; \text{ if } f^{-1}(y) \neq \phi, \\ \{0\} , \text{ otherwise} \end{cases}$$

and $\forall x \in x, f^{-1}(h_2)(x) = h_2(f(x))$., Then $f(h_1)$ is called the image of h_1 under f and $f^{-1}(h_2)$ is called the pre image of h_2 under f.

Theorem 2.16. Let f be a function from X into Y and g a function from Y into Z. Then the following assertions hold.

1. $h_1 \preceq h_2 \Rightarrow f(h_1) \preceq f(h_2) \quad \forall h_1, h_2 \in HF(X)$
2. $h_1 \preceq h_2 \Rightarrow f^{-1}(h_1) \preceq f^{-1}(h_2) \quad \forall h_1, h_2 \in HF(Y)$
3. $f^{-1}(f(h_1)) \succeq h_1 \quad \forall h_1 \in HF(X)$. In particular if f is a injection, then $f^{-1}(f(h_1)) = h_1 \quad \forall h_1 \in HF(X)$.
4. $f(f^{-1}(h_2)) \preceq h_2 \quad \forall h_2 \in HF(X)$. In particular if f is a surjection, then $f(f^{-1}(h_2)) = h_2 \quad \forall h_2 \in HF(Y)$.
5. $f(h_1) \preceq h_2 \Leftrightarrow h_1 \preceq f^{-1}(h_2) \quad \forall h_1 \in HF(X)$ and $h_2 \in HF(Y)$.
6. $g(f(h_1)) = (g \circ f)(h_1) \quad \forall h_1 \in HF(X)$ and $f^{-1}(g^{-1}(h_3)) = (g \circ f)^{-1}(h_3) \quad \forall h_3 \in HF(Z)$.

Proof. 1.

$$\begin{aligned} h_1 \preceq h_2 \quad &\Rightarrow \quad s(h_1(x)) \leq s(h_2(x)) \quad \forall x \in X \qquad (1) \\ f(h_1)(y) \quad &= \quad \widetilde{\vee} \{h_1(x) | x \in X, f(x) = y\} \\ &\preceq \quad \widetilde{\vee} \{h_2(x) | x \in X, f(x) = y\} \quad \text{from (1)} \\ &= \quad f(h_2(y)) \end{aligned}$$

2. $f^{-1}(h_1)(x) = h_1(f(x)) \preceq h_2(f(x)) = f^{-1}(h_2)(x)$

3.
$$\begin{aligned} f^{-1}(f(h_1))(x) &= f(h_1)(f(x)) \\ &= \widetilde{\vee}\{h_1(x')|x' \in X, f(x') = f(x)\} \\ &\succeq h_1 \quad \forall x \in X \end{aligned}$$

   In particular if f is an injection then
   $f^{-1}(f(h_1))(x) = \widetilde{\vee}\{h_1(x')|x' \in X, f(x') = f(x)\} = h_1(x)$

4. Let $h_2 \in HF(Y)$ then
$$\begin{aligned} f(f^{-1}(h_2))(y) &= \widetilde{\vee}\left\{f^{-1}(h_2(x))|x \in X, f(x) = y\right\} \\ &= \widetilde{\vee}\{h_2(f(x))|x \in X, f(x) = y\} \\ &= \begin{cases} h_2(y), & \text{for } y \in f(X) \\ \{0\}, & \text{otherwise} \end{cases} \\ &\preceq h_2(y) \quad \forall y \in Y \end{aligned}$$

   Thus $f(f^{-1}(h_2))(y) = h_2(y) \quad \forall y \in Y$ if f is a surjection.

5. $f(h_1) \preceq h_2 \Rightarrow f^{-1}(f(h_1)) \preceq f(h_2) \quad \forall y \in Y$ from (2)
   $\Rightarrow h_1 \preceq f^{-1}(f(h_1)) \preceq f(h_2)$ from (3)
   $\Rightarrow h_1 \preceq f(h_2)$
   Conversely, $h_1 \preceq f^{-1}(h_2)$ from (1)
   $\Rightarrow f(h)_1 \preceq f(f^{-1}(h_2))$
   $\Rightarrow f(h_1) \preceq f(f^{-1}(h_2)) \preceq h_2$ from (4)
   $\Rightarrow f(h_1) \preceq h_2$

6. Consider any $h \in HF(X)$ and any $z \in Z$. Then
$$\begin{aligned} g(f(h))(z) &= \widetilde{\vee}\{f(h)(y)|y \in Y, g(y) = z\} \\ &= \widetilde{\vee}\left\{\widetilde{\vee}\{h(x)|x \in X, f(x) = Y\}|y \in Y, g(y) = z\right\} \\ &= \widetilde{\vee}\{h(x)|x \in X, (g \circ f)(x) = z\} \\ &= (g \circ f)(h)(z) \quad \forall z \in Z \end{aligned}$$

   Further, $\forall h' \in HF(Z)$ and $\forall x \in X$,
$$\begin{aligned} ((g \circ f)^{-1}(h')(x)) &= h'(g(f(x))) \\ &= g^{-1}(h')(f(x)) \\ &= f^{-1}(g^{-1}(h'))(x) \end{aligned}$$

□

3 Hesitant Fuzzy Subgroups

This section discusses the concept of composition in the case of hesitant fuzzy sets. Certain results regarding Hesitant fuzzy subgroups are discussed. Let G denote an arbitrary group with a multiplicative binary operation and identity e.

Definition 3.1. [4] We define the binary operation $\circ$ on $HF(G)$ and the unary operation $^{-1}$ on $HF(G)$ as follows : $\forall h_1, h_2 \in HF(G)$ and $\forall x \in G$,
$(h_1 \circ h_2)(x) = \widetilde{\vee}\left\{h_1(x)\widetilde{\wedge}h_2(x)|y, z \in G, yz = x\right\}$ and $h_1^{-1}(x) = h(x^{-1})$

We call $h_1 \circ h_2$ the product of h_1 and h_2, and h_1^{-1} the inverse of h_1.

Lemma 3.2. Let $h, h', h_i \in HF(G), i \in I$. Let $A = \widetilde{\vee}\{h(x)|x \in G\}$. Then the following assertions hold:

1. $(h \circ h')(x) = \widetilde{\vee}_{y\in G}(h(y)\widetilde{\wedge}h'(y^{-1}x))$
   $= \widetilde{\vee}_{y\in G}(h(xy^{-1})\widetilde{\wedge}h'(y)) \quad \forall x \in G$
2. $(A_y \circ h)(x) = h(y^{-1}x) \quad \forall x, y \in G$
3. $(h \circ A_y)(x) = h(xy^{-1}) \quad \forall x, y \in G$
4. $(h^{-1})^{-1} = h$
5. $h \preceq h^{-1} \Leftrightarrow h^{-1} \preceq h$
   $\Leftrightarrow h \approx h^{-1}$
   $\Leftrightarrow h(x) \preceq h(x^{-1}) \quad \forall x \in G$
6. $h \preceq h' \Leftrightarrow h^{-1} \preceq (h')^{-1}$
7. $(\widetilde{\vee}_{i\in I}h_i)^{-1} = \widetilde{\vee}_{i\in I}h_i^{-1}$
8. $(\widetilde{\wedge}_{i\in I}h_i)^{-1} = \widetilde{\wedge}_{i\in I}h_i^{-1}$
9. $(h \circ h')^{-1} = h^{-1} \circ (h')^{-1}$

Definition 3.3. [4] Let $h \in HF(G)$. Then h is called a hesitant fuzzy subgroup of G if

(i) $h(xy) \succeq h(x)\widetilde{\wedge}h(y) \quad \forall x, y \in G$ and

(ii) $h(x^{-1}) \succeq h(x) \quad \forall x \in G$

Denote by $HFG(G)$, the set of all Hesitant Fuzzy subgroups of G . If $h \in HFG(G)$, then let $h_* = \{x \in G|h(x) = h(e)\}$. From (i) of the above definition we have $h(x^n) \succeq h(x) \quad \forall x \in G,$ where $n \in \mathbb{N}$.

Example 3.4. Let $G = \{e, a, b, c\}$ be the Klein's 4-group. $h : G \to [0, 1]$ G be a hesitant fuzzy set with $h(e) = \{1\}, h(a) = \left\{\frac{5}{12}, \frac{11}{12}\right\}$, $h(b) = \left\{\frac{9}{20}, \frac{3}{4}, \frac{4}{5}\right\}, h(c) = \left\{\frac{2}{3}\right\}$. Then h is a hesitant fuzzy subgroup of G .

Lemma 3.5. $h \in HF(G)$ is a hesitant fuzzy subgroup iff $h(xy^{-1}) \succeq h(x)\widetilde{\wedge}h(y) \quad \forall x, y \in G$

Proof. $h(xy^{-1}) \succeq h(x)\widetilde{\wedge}h(y^{-1}) \succeq h(x)\widetilde{\wedge}h(y) \quad (\because h(y) \preceq h(y^{-1}))$
Conversely, To prove (ii)
$h(x) = h(xe) = h(xx^{-1}x) \succeq h(x)\widetilde{\wedge}h(xx^{-1}) \succeq h(x)\widetilde{\wedge}(h(x)\widetilde{\wedge}h(x^{-1}))$
$\succeq h(x)\widetilde{\wedge}h(x^{-1})$
$\Rightarrow h(x) \succeq h(x^{-1})$
Now to prove that $h(xy^{-1}) \succeq h(x)\widetilde{\wedge}h(y)$
$h(xy) \succeq h(x)\widetilde{\wedge}h(y^{-1}) \succeq h(x)\widetilde{\wedge}h(y); (\because h(y) \preceq h(y^{-1}))$ □

Lemma 3.6. Let $h \in HFG(G)$. Then $\forall x \in G$,

(i) $h(e) \succeq h(x)$

(ii) $h(x) \approx h(x^{-1})$

Proof. Let $x \in G$

(i) $h(e) = h(xx^{-1}) \succeq h(x)\widetilde{\wedge}h(x^{-1}) \succeq h(x)\widetilde{\wedge}h(x) = h(x)$

(ii) $h(x) = h((x^{-1}))^{-1} \succeq h(x^{-1})$
Hence $h(x) \approx h(x^{-1})$

□

Lemma 3.7. If $h \in HFG(G)$ and if $x, y \in G$ with $h(x) \succeq h(y)$, then

$$h(xy) \approx h(x)\widetilde{\wedge}h(y)$$

Proof. Let $h(x) \succeq h(y)$.
Then $h(y) = h(x^{-1}xy) \succeq h(x^{-1})\widetilde{\wedge}h(xy) \approx h(x)\widetilde{\wedge}h(xy)$
$h(y) \succeq h(x)\widetilde{\wedge}h(xy)$ and since $h(x) \succeq h(y)$ it follows that
$h(y) \succeq h(xy) \succeq h(x)\widetilde{\wedge}h(y) \approx h(y)$
From this it follows that
$h(xy) \succeq h(x)\widetilde{\wedge}h(y)$ and $h(x)\widetilde{\wedge}h(y) \approx h(y) \succeq h(xy)$
Hence the result □

Lemma 3.8. If h is a hesitant fuzzy subgroup of G then $H = \{x \in X : h(x) = \{1\}\}$ is either empty or is a subgroup of G.

Proof. If $x, y \in H$ then
$h(xy^{-1}) \succeq h(x)\widetilde{\wedge}h(y^{-1}) \approx h(x)\widetilde{\wedge}h(y) \approx \{1\}\widetilde{\wedge}\{1\} = \{1\}$ Therefore $xy^{-1} \in H$. Hence H is a subgroup of G. □

Lemma 3.9. If h is a hesitant fuzzy subgroup of a group G and if there is a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} s(h(x_n)) = 1$ then $s(h(e)) = 1$ where e is the identity in G.

Proof. Let $x \in G$. Then
$s(h(e)) = s(h(xx^{-1})) \geq s(h(x))\widetilde{\wedge}s(h(x^{-1})) = s(h(x))$
Therefore for each n, $s(h(e)) \geq s(h(x_n))$
Since $1 \geq s(h(e)) \geq \lim_{n\to\infty} s(h(x_n)) = 1 \Rightarrow s(h(e)) = 1$ □

Note : If $s(h(x)) = 1$ then clearly $h(x) = \{1\}$.

Lemma 3.10. Let h be a hesitant fuzzy subgroup of a group G . If $h(xy^{-1}) = \{1\}$ then $h(x) \approx h(y)$.

Proof. $h(x) = h((xy^{-1})y) \succeq h(xy^{-1})\widetilde{\wedge}h(y) \approx \{1\}\,\widetilde{\wedge}h(y) = h(y)$
$\Rightarrow h(x) \succeq h(y)$
Similarly we can prove that $h(y) \succeq h(x)$.
Hence we can conclude that $h(x) \approx h(y)$. □

Lemma 3.11. Let h be a hesitant fuzzy set on a group G. If $h(e) = \{1\}$ and $h(xy^{-1}) \succeq h(x)\widetilde{\wedge}h(y) \quad \forall x, y \in G$ then h is a hesitant fuzzy subgroup of G.

Proof. $h(y^{-1}) = h(ey^{-1}) \succeq h(e)\widetilde{\wedge}h(y) = \{1\}\,\widetilde{\wedge}h(y) = h(y)$.
Similarly $h(y) \succeq h(y^{-1}) \Rightarrow h(y) \approx h(y^{-1})$
$h(xy) = h(x(y^{-1})^{-1}) \succeq h(x)\widetilde{\wedge}h(y^{-1}) \approx h(x)\widetilde{\wedge}h(y)$
Hence h is a hesitant fuzzy subgroup of G. □

Lemma 3.12. Let $h \in HF(G)$. If h is a hesitant fuzzy subgroup of G then h_α is a subgroup of G $\forall \alpha \in [0, s(h(e))]$.

Proof. Suppose $h \in HF(G)$.
Let $\alpha \in [0, s(h(e))]$. Since $h(e) \succeq h(x)$ $\forall x \in G, e \in h_a$. Thus $h_\alpha \neq \phi$
Let $x, y \in h_\alpha$. Then $s(h(x)) \geq \alpha, s(h(y)) \geq \alpha$. Since h is a hesitant fuzzy subgroup $h(xy^{-1}) \succeq h(x)\widetilde{\wedge}h(y) \geq \alpha\widetilde{\wedge}\alpha = \alpha \Rightarrow s(h(xy^{-1})) \geq \alpha$.
Hence $xy^{-1} \in h_\alpha$ and so h_α is a subgroup of G.
Converse, given $h(xy) \succeq h(x)\widetilde{\wedge}h(y)$,
Suppose h_α is a subgroup of G $\forall \alpha \in [0, s(h(e))]$, then $\forall \alpha \in h(G)$ we must have $e \in h_\alpha$ and so it follows that $s(h(e)) \geq \alpha$. Let $x, y \in G$ and let $s(h(x)) = \alpha$ and $s(h(y)) = \beta$. Let $\gamma = \alpha \wedge \beta$. Then $x, y \in h_\gamma$ and $\gamma \leq s(h(e))$. By hypothesis h_γ is a subgroup of G and so $xy^{-1} \in h_\gamma$. Hence $s(xy^{-1}) \geq \gamma = \alpha \wedge \beta = s(h(x)) \wedge s(h(y))$. Thus h is a HF subgroup of G. □

Theorem 3.13. $h \in HFG(G)$ iff h satisfies the following conditions :

1. $h \circ h \preceq h$
2. $h^{-1} \preceq h$

Proof.

$$\begin{aligned}(h \circ h)(x) &= \widetilde{\vee}\left\{h(y)\widetilde{\wedge}h(z) | y, z \in G, yz = x\right\} \\ &= h(y_*)\widetilde{\wedge}h(z_*) \text{ for some } y_*z_* \in G \text{ such that } y_*z_* = x \\ &\preceq h(x) \quad (\because h(x) = h(yz)) \succeq h(y)\widetilde{\wedge}h(z))\end{aligned}$$

Converse, let $h \circ h \preceq h$
$\Rightarrow \widetilde{\vee}\left\{h(y)\widetilde{\wedge}h(z) | y, z \in G, yz = x\right\} \preceq h(x)$
$\Rightarrow h(y)\widetilde{\wedge}h(z) \preceq h(x) \quad \forall t, z \in G$ such that $yz = x$
$\Rightarrow h(y)\widetilde{\wedge}h(z) \preceq h(yz)$, Hence proved. □

Theorem 3.14. Let $h_1, h_2 \in HFG(G)$. Then $h_1 \circ h_2 \in HFG(G)$ iff $h_1 \circ h_2 \approx h_2 \circ h_1$.

Proof. Suppose $h_1 \circ h_2 \in HFG(G)$.
Then $h_1 \circ h_2 \approx h_1^{-1} \circ h_2^{-1} \approx (h_2 \circ h_1)^{-1} \approx h_2 \circ h_1$.
Conversely suppose that $h_1 \circ h_2 \approx h_2 \circ h_1$ then
$(h_1 \circ h_2)^{-1} \approx (h_2 \circ h_1)^{-1} \approx h_1^{-1} \circ h_2^{-1} \preceq h_1 \circ h_2$ and
$(h_1 \circ h_2) \circ (h_1 \circ h_2) = h_1 \circ (h_2 \circ h_1) \circ h_2 \approx h_1 \circ (h_1 \circ h_2) \circ h_2$
$= (h_1 \circ h_1) \circ (h_2 \circ h_2) \preceq h_1 \circ h_2$
Hence by Theorem 3.13, $h_1 \circ h_2 \in HFG(G)$. □

Theorem 3.15. For $i \in I$. Let $h_i \in HFG(G)$. Then $\widetilde{\wedge}_{i\in I} h_i \in HFG(G)$.

Proof. Let $x, y \in G$. then
$(\widetilde{\wedge}_{i\in I} h_i)(xy^{-1}) = \widetilde{\wedge}\{h_i(xy^{-1})|i \in I\} \widetilde{\wedge} \{h_i(x)\widetilde{\wedge}h_i(y)|i \in I\}$
$\approx (\widetilde{\wedge}\{h_i(x)|i \in I\})\widetilde{\wedge}(\widetilde{\wedge}\{h_i(y)|i \in I\}) \approx (\widetilde{\wedge}_{i\in I} h_i)(x)\widetilde{\wedge}(\widetilde{\wedge}_{i\in I} h_i)(y)$. □

4 Normal Hesitant Fuzzy Subgroups, Homomorphisms and Isomorphisms

This section discusses the notion of a Normal Hesitant fuzzy subgroup. Normal Hesitant fuzzy subgroup is an important concept when it comes to the study of Hesitant fuzzy group theory. This section moves on to discuss various results regarding them.

Definition 4.1. [4] Let G be a group . A hesitant fuzzy subgroup h of a group G is called normal if $h(x) \approx h(y^{-1}xy) \quad \forall x, y \in G$. Let $NHF(G)$ denote the set of all normal hesitant fuzzy subgroups of G.

Theorem 4.2. Let $h \in HF(G)$ then the following conditions are equivalent

1. $h(y) \approx h(xyx^{-1})$
2. $h \circ h' \approx h' \circ h \quad \forall h' \in HF(G)$

Proof. Let $x \in G$. We have $h(y) \approx h(xyx^{-1})$. Then
$(h \circ h')(x) \approx \widetilde{\vee}_{y\in G}(h(xy^{-1})\widetilde{\wedge}h'(y)) \quad \forall x \in G$
$\approx \widetilde{\vee}_{y\in G}(h(x^{-1}xy^{-1}x)\widetilde{\wedge}h'(y)) \approx \widetilde{\vee}_{y\in G}(h(y^{-1}x)\widetilde{\wedge}h'(y))$
$\approx \widetilde{\vee}_{y\in G}(h'(y)\widetilde{\wedge}h(y^{-1}x)) \approx (h' \circ h)(x)$
Hence $h \circ h' \approx h' \circ h$
(2) $\Rightarrow$ (1): We have $h \circ h' \approx h' \circ h \quad \forall h' \in HF(G)$
$\{1\}_{y^{-1}} \circ h \approx h \circ \{1\}_{y^{-1}} \quad y \in G$
$(\{1\}_{y^{-1}} \circ h)(x) \approx \widetilde{\vee}_{z\in G}(\{1\}_{y^{-1}}(z)\widetilde{\wedge}h(z^{-1}x))$
$\approx \{1\}\widetilde{\wedge}h(yx)$ since $\{1\}_{y^{-1}}(z) = \begin{cases} \{1\} & \text{for } x = y \\ \{0\} & \text{otherwise} \end{cases}$
$\approx h(yx)$
Similarly $(h \circ \{1\}_{y^{-1}})(x) \approx h(xy)$
Now we have that $(\{1\}_{y^{-1}} \circ h)(x) \approx (h \circ \{1\}_{y^{-1}})(x) \quad \forall x \in G$
Thus $h(yx) \approx h(xy)$ □

Note : If $h_1, h_2 \in HFG(G)$ and there exists $u \in G$ such that $h_1(x) = h_2(uxu^{-1}) \quad \forall x \in G$, , then h_1 and h_2 are called conjugate hesitant fuzzy subgroups (with respect to u) and we write $h_1 = h_2^u$ where $h_2^u(x) = h_1(uxu^{-1}) \quad \forall x \in G$.

Theorem 4.3. Let $h \in HF(G)$. Then $h \in NHF(G)$ if and only if h_α is a normal subgroup of $G \quad \forall\alpha \in [0, s(h(e))]$.

Proof. Let $h \in NHF(G)$ and $\forall\alpha \in [0, s(h(e))]$ since $h \in HFG(G)$, h_α is a subgroup of G. If $x \in G$ and $y \in h_\alpha$ it follows that $s(h(xyx^{-1})) = s(h(y)) \geq \alpha$ by definition of a normal hesitant fuzzy subgroup. Thus $xyx^{-1} \in h_\alpha$. Hence h_α is a normal subgroup of G.
Conversely, Assume that h_α is a normal subgroup of $G \quad \forall\alpha \in [0, s(h(e))]$ Then we have that $h \in HFG(G)$ by lemma [3.12] .
Let $x, y \in G$ and $\alpha = s(h(y))$. Then $y \in h_\alpha$ and so $xyx^{-1} \in h_\alpha$. Hence $s(h(xyx^{-1})) \geq \alpha = s(h(y))$ i.e., we have that $s(h(xyx^{-1})) \geq s(h(y))$ Now put $y = x^{-1}yx$ and we have $s(h(x^{-1}yx)) \leq s(h(xx^{-1}yxx^{-1})) = s(h(y))$. So we have $h(y) \approx h(xyx^{-1}) \Rightarrow h \in NHF(G)$. □

Theorem 4.4. Let $h \in NHF(G)$. Then h_* and h^* are normal subgroups of G.

Proof. $h \in HFG(G)$. It follows from lemma[3.12] that h_* and h^* are subgroups of G .
Let $x \in G$ and $y \in h_*$. $h(xyx^{-1}) \approx h(y) \approx h(e)$
$\Rightarrow xyx^{-1} \in h_*$. Hence h_* is a normal subgroup of G. Now let $x \in G$ and $y \in h_*$. Since $h(xyx^{-1}) \approx h(y) \succ \{0\}$ and $xyx^{-1} \in h^*$. Therefore h^* is a normal subgroup of G. □

Note : The converse need not be true.
Example : Let G be a group and H be a subgroup of G which is not normal. Define

$$h \in HF(G) \text{ as } h(x) = \begin{cases} \left\{\frac{3}{4}\right\}; x = e \\ \left\{\frac{7}{12}, \frac{1}{3}, \frac{1}{12}\right\}; x \in H \setminus \{e\} \\ \left\{\frac{1}{2}, \frac{1}{4}\right\}; x \in G \setminus H \end{cases}$$

Then $h \in HFG(G)$. We have $h_{\frac{1}{3}} = H$ which is not normal in G. But $h_* = \{e\}$ and $h^* = G$ are normal in G.

Theorem 4.5. Let $h \in HFG(G)$. Then $\widetilde{\wedge}_{u\in G}h^u \in HFG(G)$ and is the largest normal hesitant fuzzy group of G that is contained in h.

Proof. We have $h \in HFG(G)$
$h^u(xy) = h(uxyu^{-1}) = h(uxu^{-1}uyu^{-1}) \succeq h(uxu^{-1})\widetilde{\wedge}h(uyu^{-1})$
$\Rightarrow h^u(xy) \succeq h^u(x)\widetilde{\wedge}h^u(y)$
$\Rightarrow h^u \in HFG(G) \quad \forall u \in G$
$\Rightarrow \widetilde{\wedge}_{u\in G}h^u \in HFG(G)$

For all $x \in G$ we have $\{h^u | u \in G\} = \{h^{ux} | u \in G\}$ because each $u = vx$ for some $v \in G$. Thus $\widetilde{\wedge}_{u\in G}h^u(xyx^{-1}) = \widetilde{\wedge}_{u\in G}h(uxyx^{-1}u^{-1}) = \widetilde{\wedge}_{u\in G}h(uxy(ux)^{-1}) = \widetilde{\wedge}_{u\in G}h^{ux}(y)$.

Hence by the definition of normal hesitant fuzzy group we have $\widetilde{\wedge}_{u\in G}h^u \in NHF(G)$.
Let $h_1 \in NHF(G)$ with $h_1 \preceq h$ Then $h_1 \preceq h \Rightarrow s(h_1(x)) \leq s(h(x)) \quad \forall x \in G$
$\Rightarrow s(h_1(uxu^{-1})) \leq s(h(uxu^{-1}))$
$\Rightarrow h_1^u(x) \preceq h^u(x) \quad \forall x \in G$
$\Rightarrow h_1^u \preceq h^u$
Since $h_1 \in NHF(G)$ we have $h_1(uxu^{-1}) \approx h_1(u)$.

$\Rightarrow h_1 \approx h_1^u \preceq h^u \quad \forall u \in G$
$h_1 \preceq \widetilde{\wedge}_{u \in G} h^u$
Hence the result. □

Definition 4.6. [4] Let $h \in HFG(G)$ and $x \in G$. The hesitant fuzzy subset $h(e)_{\{x\}} \circ h$ is referred to as the left coset of h with respect to x and is written $x\widetilde{h}$ and is referred to as the right coset of h with respect to x and is written $\widetilde{h}x$.
Now,

$$\begin{aligned}(h(e)_x \circ h)(a) &= \widetilde{\vee}\left\{h(e)_x(y)\widetilde{\wedge}h(z)|y,z \in G; yz = a\right\}\\ &= \widetilde{\vee}\begin{cases}h(e)\widetilde{\wedge}h(z) \text{ for } xz = a\\ \{0\}\,\widetilde{\wedge}h(z) \text{ for } xz \neq a\end{cases}\\ &= h(e)\widetilde{\wedge}h(x^{-1}a)\\ &= h(x^{-1}a)\end{aligned}$$

Therefore $x\widetilde{h}(a) = h(x^{-1}a)$ and $\widetilde{h}x(a) = h(ax^{-1})$.

Note : We write $\widetilde{h}$ in the notation of a coset in place of the hesitant fuzzy set h so as to differentiate between the element x and the hesitant fuzzy set h. We have that if $h \in NHF(G)$ then $x\widetilde{h} = \widetilde{h}x$. Thus we call $x\widetilde{h}$ a coset of h (dropping the notion of left or right coset).

Theorem 4.7. Let $h \in HFG(G)$. Then $\forall x, y \in G$ we have $x\widetilde{h} = y\widetilde{h} \Leftrightarrow xh_* = yh_*$.

Proof. Let us take $x\widetilde{h} = y\widetilde{h}$. Then $h(e)_{\{x\}} \circ h \approx h(e)_{\{y\}} \circ h$
$\Rightarrow h(x^{-1}z) \approx h(y^{-1}z) \quad \forall z \in G$
Substituting $z = y \Rightarrow h(x^{-1}y) \approx h(y^{-1}y) \approx h(e)$.
$\Rightarrow x^{-1}y \in h_*$
Hence $xh_* = yh_*$.
Conversely Let us take $xh_* = yh_*$.
Then $x^{-1}y \in h_*$ and $y^{-1}x \in h_* \Rightarrow h(x^{-1}y) = h(e)$ and $h(y^{-1}x) = h(e)$ Hence $h(x^{-1}z) = h(x^{-1}y \cdot y^{-1}z) \succeq h(x^{-1}y)\widetilde{\wedge}h(y^{-1}z) \approx h(e)\widetilde{\wedge}h(y^{-1}z)$
$\approx h(y^{-1}z) \quad \forall z \in G$
Similarly $h(y^{-1}z) \succeq h(x^{-1}z) \quad \forall z \in G$
Therefore $h(x^{-1}z) \approx h(y^{-1}z) \quad \forall z \in G$
$\Rightarrow x\widetilde{h} = y\widetilde{h}$ □

Theorem 4.8. Let $h \in NHF(G)$ and $x, y \in G$. If $x\widetilde{h} = y\widetilde{h}$, then $h(x) \approx h(y)$

Proof. Suppose that $x\widetilde{h} = y\widetilde{h}$.
Then by Theorem[4.7] we have that $x^{-1}y \in h_*$ and $y^{-1}x \in h_*$.
$h \in NHF(G) \approx h(x) \approx h(y^{-1}xy) \succeq h(y^{-1}x)\widetilde{\wedge}h(y) \approx h(e)\widetilde{\wedge}h(y) \approx h(y)$ Similarly $h(y) \succeq h(x)$.
Therefore $h(x) \approx h(y)$. □

Theorem 4.9. Let $h \in NHF(G)$. Set $G/h = \left\{x\widetilde{h}|x \in G\right\}$ Then

1. $(x\widetilde{h} \circ (y\widetilde{h})) = (xy)\widetilde{h} \quad \forall x, y \in G$

2. $(G/h, \circ)$ is a group

Proof. 1. $\forall x, y \in G$ we have

$$\begin{aligned}
(x\widetilde{h} \circ (y\widetilde{h})) &= (h(e)_{\{x\}} \circ h) \circ (h(e)_{\{y\}} \circ h) \text{ by definition} \\
&= h(e)_{\{x\}} \circ (h \circ (h(e)_{\{y\}}) \circ h \\
&= h(e)_{\{x\}} \circ (h \circ h) \circ h(e)_{\{y\}} \\
&= (h(e)_{\{x\}} \circ h(e)_{\{y\}}) \circ h \\
&= h(e)_{\{xy\}} \circ h \\
&= (xy)\widetilde{h}
\end{aligned}$$

2. By 1), G/h is closed under the operation $\circ$. Also $\circ$ satisfies the associative law.
Now $h \circ (x\widetilde{h}) = (e\widetilde{h}) \circ (x\widetilde{h}) = (ex)\widetilde{h} = x\widetilde{h} \quad \forall x \in G$.
Hence the identity element is $(e\widetilde{h})$
and $(x^{-1}\widetilde{h}) \circ (x\widetilde{h}) = (x^{-1}x)\widetilde{h} = e\widetilde{h} = h \quad \forall x \in G$
Hence the inverse element of $x\widetilde{h}$ is $x^{-1}\widetilde{h}$.
Therefore $(G/h, \circ)$ is a group.

□

Definition 4.10. [4] The group $(G/h, \circ)$ defined in the above theorem, where $G/h = \left\{x\widetilde{h} | x \in G\right\}$, is called the quotient group or the factor group of G relative to the normal hesitant fuzzy subgroup h.

Theorem 4.11. Let $h \in HFG(G)$ and let N be a normal subgroup of G. Define $(h/N) \in HF(G/N)$ as $(h/N)(xN) = \widetilde{\vee}\{h(z)|z \in xN\} \quad \forall x \in G$ Then $(h/N) \in HFG(G/N)$.

Proof. Now

$$\begin{aligned}
(h/N)((xN)^{-1}) &= (h/N)(x^{-1}N) \\
&= \widetilde{\vee}\left\{h(z)|z \in x^{-1}N\right\} \\
&= \widetilde{\vee}\left\{h(y^{-1})|y^{-1} \in x^{-1}N\right\} \\
&= \widetilde{\vee}\{h(y)|y \in xN\} \\
&= (h/N)(xN) \quad \forall x \in G
\end{aligned}$$

$$\begin{aligned}
(h/N)(xNyN) &= \widetilde{\vee}\{h(z)|z \in xyN\} \\
&= \widetilde{\vee}\{h(uv)|u \in xN, v \in yN\} \\
&\succeq \widetilde{\vee}\left\{h(u)\widetilde{\wedge}h(v)|u \in xN, v \in yN\right\} \\
&= (\widetilde{\vee}\{h(u)|u \in xN\})\widetilde{\wedge}(\widetilde{\vee}\{h(v)|, v \in yN\}) \\
&= (h/N)(xN)\widetilde{\wedge}(h/N)(yN) \quad \forall x, y \in G
\end{aligned}$$

Hence $(h/N) \in HFG(G/N)$ □

5 Conclusion

In this paper certain properties and results regarding hesitant fuzzy groups are studied. The general structure of normal hesitant fuzzy subgroups are discussed. Some conditions for hesitant fuzzy subgroups to be normal are established. The notion of a quotient group relative to a normal hesitant fuzzy subgroup is studied.

Acknowledgement

The first author acknowledge the financial assistance given by Ministry of Human Resource Development, Government of India and the National Institute of Technology Calicut throughout the preparation of this paper.

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http://www.newtheory.org ISSN: 2149-1402

Journal of New Theory

Received: 19.06.2015 *Year*: **2016**, *Number*: **11**, *Pages*:**69-83**
Published: 02.03.2016 *Original Article*[**]

A NOTE ON INTUITIONISTIC ANTI FUZZY BI-IDEALS OF SEMIGROUPS

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Abstaract − In this paper the notions of intuitionistic anti fuzzy bi-ideal, intuitionistic anti fuzzy interior ideal, intuitionistic anti fuzzy (1,2)-ideal in semigroups are introduced and some important characterizations have been obtained.

Keywords − *Semigroup, Regular semigroup, Intuitionistic anti fuzzy bi-ideal, Intuitionistic anti fuzzy interior ideal, Intuitionistic anti fuzzy (1,2)-ideal.*

1 Introduction

After the introduction of fuzzy set by Zadeh[12] in 1965, the researchers in mathematics were trying to introduce and study the concept of fuzzyness in different mathematical systems under study. In 1986, K. T. Atanassov[1] introduced the notion of intuitionistic fuzzy sets, which is the generalization of fuzzy sets. Fuzzy set gives the degree of membership of an element in a given set, but intuitionistic fuzzy set gives both degree of membership and degree of non-membership. The degree of membership and the degree of non-membership are the real numbers between 0 and 1, having sum not greater than 1. For more details on intuitionistic fuzzy sets, we refer [1, 2].

In [8] K. H. Kim and Y. B. Jun introduced the concept of intuitionistic fuzzy ideals of semigroups. In [7] K. H. Kim and J. G. Lee gave the notion of intuitionistic fuzzy bi-ideals of semigroups. In [6] Y. B. Jun introduced intuitionistic fuzzy bi-ideals of ordered semigroups and studied natural equivalence relation on the set of all intuitionistic fuzzy bi-ideals of an ordered semigroup. A. Iampan and M. Siripitukdet[5]

[**]*Edited by* Oktay Muhtaroğlu *(Area Editor) and* Naim Çağman *(Editor-in-Chief).*
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studied the minimal and maximal left ideals in Po-Γ-semigroups and P. Dheena and B. Elavarasan[3] introduced the notion of right chain Po-Γ-semigroups. The concept of (1,2)-ideals in a semigroup was introduced by S. Lajos[10]. In [9] N. Kuroki gave some properties of fuzzy ideals and fuzzy bi-ideals in semigroups. In [4] K. Hila and E. Pisha introduced the notion of bi-ideals in ordered Γ-semigroups. T. Srinivas, T. Nagaiah and P. Narasimha Swamy[11] studied the various properties of Γ-nearrings in terms of their anti fuzzy ideals. In this paper we introduce the concept of intuitionistic anti fuzzy bi-ideals and intuitionistic anti fuzzy (1,2)-ideals in a semigroup. Here a regular semigroup has been characterized in terms of intuitionistic anti fuzzy bi-ideal.

2 Preliminary

Let S be a semigroup. Now recall the following from[1, 8].

• By a subsemigroup of S we mean a non-empty subset A of S such that $A^2 \subseteq A$.

• By a left(right) ideal of S we mean a non-empty subset A of S such that $SA \subseteq A(AS \subseteq A)$.

• By a two-sided ideal or simply ideal we mean a non-empty subset of S which is both a left and a right ideal of S.

• A subsemigroup A of a non empty subset of a semigroup S is called a bi-ideal of S if $ASA \subseteq A$.

• A subsemigroup A of S is called a (1,2)-ideal of S if $ASA^2 \subseteq A$.

• Let X be a non-empty set. A fuzzy set of X is a function $\Psi : X \to [0,1]$ and the complement of Ψ, denoted by $\overline{\Psi}$ is a fuzzy set in X given by $\overline{\Psi}(x) = 1 - \Psi(x)$ for all $x \in X$.

• A semigroup S is called regular if, for each $a \in S$, there exist $x \in S$ such that $a = axa$.

• A fuzzy set Ψ in a semigroup S is called an anti fuzzy subsemigroup of S if, $\Psi(xy) \leq \max\{\Psi(x), \Psi(y)\}$ for all $x, y \in S$

• An anti fuzzy subsemigroup Ψ of a semigroup S is called an anti fuzzy interior ideal of S if $\Psi(xay) \leq \Psi(a)$, for all $x, y, a \in S$.

• An intuitionistic fuzzy set (briefly, IFS) A in a non-empty set X is an object having the form

$$A = \{(x, \Psi_A(x), \Omega_A(x)) : x \in X\},$$

where the functions $\Psi_A : X \to [0,1]$ and $\Omega_A : X \to [0,1]$ denote the degree of membership and the degree of non-membership, respectively and

$$0 \leq \Psi_A(x) + \Omega_A(x) \leq 1$$

for all $x \in X$.

• An intuitionistic fuzzy set $A = \{(x, \Psi_A(x), \Omega_A(x)) : x \in X\}$ in X can be identified as an order pair (Ψ_A, Ω_A) in $I^X XI^X$. For the sake of simplicity, we use the symbol $A = (\Psi_A, \Omega_A)$ for IFS $A = \{(x, \Psi_A(x), \Omega_A(x)) : x \in X\}$.

• Let X be a non-empty set and let $A = (\Psi_A, \Omega_A)$ and $B = (\Psi_B, \Omega_B)$ be any two IFSs in X. Then the following operations and relations are valid [1].

(1) $A \subseteq B$ iff $\Psi_A \leq \Psi_B$ and $\Omega_A \geq \Omega_B$

(2) $A = B$ iff $A \subset B$ and $B \subset A$

(3) $\overline{A} = (\overline{\Psi}_A, \overline{\Omega}_A) = (\Omega_A, \Psi_A)$

(4) $A \cap B = (\Psi_A \wedge \Psi_B, \Omega_A \vee \Omega_B)$
(5) $A \cup B = (\Psi_A \vee \Psi_B, \Omega_A \wedge \Omega_B)$
(6) $\Box A = (\Psi_A, 1 - \Psi_A), \Diamond A = (1 - \Omega_A, \Omega_A)$.

• An IFS $A = (\Psi_A, \Omega_A)$ of a semigroup S is called an intuitionistic fuzzy subsemigroup of S if
(1) $\Psi_A(xy) \geq \min\{\Psi_A(x), \Psi_A(y)\} \forall x, y \in S$,
(2) $\Omega_A(xy) \leq \max\{\Omega_A(x), \Omega_A(y)\} \forall x, y \in S$.

• An IFS $A = (\Psi_A, \Omega_A)$ of a semigroup S is called an intuitionistic fuzzy left ideal of S if $\Psi_A(xy) \geq \Psi_A(y)$ and $\Omega_A(xy) \leq \Omega_A(y)$ for all $x, y \in S$. An intuitionistic fuzzy right ideal of S is defined in an analogous way.

• An IFS $A = (\Psi_A, \Omega_A)$ in S is called an intuitionistic fuzzy ideal of S if it is both an intuitionstic fuzzy left and intuitionistic fuzzy right ideal of S.

• An intuitionistic fuzzy subsemigroup $A = (\Psi_A, \Omega_A)$ of a semigroup S is called an intuitionistic fuzzy bi-ideal of S if
(1) $\Psi_A(xay) \geq \min\{\Psi_A(x), \Psi_A(y)\} \forall x, y, a \in S$,
(2) $\Omega_A(xay) \leq \max\{\Omega_A(x), \Omega_A(y)\} \forall x, y, a \in S$.

• An intuitionistic fuzzy subsemigroup $A = (\Psi_A, \Omega_A)$ of a semigroup S is called an intuitionistic fuzzy (1,2)-ideal of S if
(1) $\Psi_A(xa(yz)) \geq \min\{\Psi_A(x), \Psi_A(y), \Psi_A(z)\} \forall x, y, z, a \in S$,
(2) $\Omega_A(xa(yz)) \leq \max\{\Omega_A(x), \Omega_A(y), \Omega_A(z)\} \forall x, y, z, a \in S$.

• Let f be a mapping from a set X to a set Y. If $A = (\Psi_A, \Omega_A)$ and $B = (\Psi_B, \Omega_B)$ are IFSs in X and Y respectively, then the pre-image of B under f, denoted by $f^{-1}(B)$, is an IFS in X defined by $f^{-1}(B) = (f^{-1}(\Psi_B), f^{-1}(\Omega_B))$.

2.1 Intuitionistic Anti Fuzzy Bi-ideal

In this section we introduce the concepts of intuitionistic anti fuzzy bi-ideal, intuitionistic anti fuzzy interior ideal and intuitionistic anti fuzzy $(1, 2)$-ideal of a semigroup S.

Definition 2.1. An IFS $A = (\Psi_A, \Omega_A)$ of a semigroup S is called an intuitionistic anti fuzzy subsemigroup of S if
(1) $\Psi_A(xy) \leq \max\{\Psi_A(x), \Psi_A(y)\} \forall x, y \in S$,
(2) $\Omega_A(xy) \geq \min\{\Omega_A(x), \Omega_A(y)\} \forall x, y \in S$.

Definition 2.2. An intuitionistic anti fuzzy subsemigroup $A = (\Psi_A, \Omega_A)$ of a semigroup S is called an intuitionistic anti fuzzy bi-ideal of S if
(1) $\Psi_A(xay) \leq \max\{\Psi_A(x), \Psi_A(y)\}, \forall x, y, a \in S$,
(2) $\Omega_A(xay) \geq \min\{\Omega_A(x), \Omega_A(y)\}, \forall x, y, a \in S$.

Example 2.3. Let $S = \{a, b, c, d, e\}$ be a semigroup with the following cayley table.

.	a	b	c	d	e
a	a	a	a	a	a
b	a	a	a	a	a
c	a	a	c	c	e
d	a	a	c	d	e
e	a	a	a	c	e

Define an IFS $A = (\Psi_A, \Omega_A)$ in S by $\Psi_A(a) = 0.3 = \Psi_A(b)$, $\Psi_A(c) = 0.4$, $\Psi_A(d) = 0.5$, $\Psi_A(e) = 0.6$ and $\Omega_A(a) = 0.6$, $\Omega_A(b) = 0.5$, $\Omega_A(c) = 0.4$, $\Omega_A(d) = \Omega_A(e) = 0.3$. The routine calculation shows that $A = (\Psi_A, \Omega_A)$ is an intuitionistic anti fuzzy bi-ideal of a semigroup S.

Definition 2.4. An intuitionistic anti fuzzy subsemigroup $A = (\Psi_A, \Omega_A)$ of a semigroup S is called an intuitionistic anti fuzzy interior ideal of S if
(1) $\Psi_A(xay) \leq \Psi_A(a) \forall x, y, a \in S$,
(2) $\Omega_A(xay) \geq \Omega_A(a)$ for all $x, y, a \in S$.

Definition 2.5. An intuitionistic anti fuzzy subsemigroup $A = (\Psi_A, \Omega_A)$ of a semigroup S is called an intuitionistic anti fuzzy (1,2)-ideal of S if
(1) $\Psi_A(xa(yz)) \leq \max\{\Psi_A(x), \Psi_A(y), \Psi_A(z)\} \forall x, y, z, a \in S$,
(2) $\Omega_A(xa(yz)) \geq \min\{\Omega_A(x), \Omega_A(y), \Omega_A(z)\} \forall x, y, z, a \in S$.

3 Main Results

Theorem 3.1. Every intuitionistic anti fuzzy bi-ideal of a regular semigroup S is an intuitionistic anti fuzzy subsemigroup of S.

Proof. Let $A = (\Psi_A, \Omega_A)$ be an intuitionistic anti fuzzy bi-ideal of a regular semigroup S and let $a, b \in S$. Since S is regular, there exists $x \in S$ such that $b = bxb$. Then we have

$$\Psi_A(ab) = \Psi_A(a(bxb)) = \Psi_A(a(bx)b) \leq \max\{\Psi_A(a), \Psi_A(b)\}$$

and

$$\Omega_A(ab) = \Omega_A(a(bxb)) = \Omega_A(a(bx)b) \geq \min\{\Omega_A(a), \Omega_A(b)\}.$$

Hence $A = (\Psi_A, \Omega_A)$ is an intuitionistic anti fuzzy subsemigroup of S. □

Lemma 3.2. If an IFS $A = (\Psi_A, \Omega_A)$ of a semigroup S is an intuitionistic anti fuzzy bi-deal of S, then so is $\Box A := (\Psi_A, \overline{\Psi}_A)$.

Proof. Suppose an IFS $A = (\Psi_A, \Omega_A)$ of a semigroup S is an intuitionistic anti fuzzy bi-ideal of S. Then by definition it is clear that Ψ_A is an intuitionistic anti fuzzy bi-ideal. It is sufficient to show that $\overline{\Psi}_A$ is an intuitionistic anti fuzzy bi-ideal of S. For any $x, y, a \in S$, we have

$$\begin{aligned} \overline{\Psi}_A(xy) &= 1 - \Psi_A(xy) \\ &\geq 1 - \max\{\Psi_A(x), \Psi_A(y)\} \\ &= \min\{1 - \Psi_A(x), 1 - \Psi_A(y)\} \\ &= \min\{\overline{\Psi}_A(x), \overline{\Psi}_A(y)\} \end{aligned}$$

and

$$\begin{aligned} \overline{\Psi}_A(xay) &= 1 - \Psi_A(xay) \\ &\geq 1 - \max\{\Psi_A(x), \Psi_A(y)\} \\ &= \min\{1 - \Psi_A(x), 1 - \Psi_A(y)\} \\ &= \min\{\overline{\Psi}_A(x), \overline{\Psi}_A(y)\}. \end{aligned}$$

Hence $\Box A$ is an intuitionistic anti fuzzy bi-ideal of S. This completes the proof. □

Similarly we can prove the following lemma.

Lemma 3.3. If $A = (\Psi_A, \Omega_A)$ is an intuitionistic anti fuzzy bi-ideal of S then so is $\Diamond A = (\overline{\Omega}_A, \Omega_A)$.

Combining Lemmas 3.2 and 3.3 we obtain the following theorem.

Theorem 3.4. $A = (\Psi_A, \Omega_A)$ is an intuitionistic anti fuzzy bi-ideal of S, if and only if $\Box A$ and $\Diamond A$ are intuitionistic anti fuzzy bi-ideals of S.

Theorem 3.5. An IFS $A = (\Psi_A, \Omega_A)$ of a semigroup S is an intuitionistic anti fuzzy bi-deal of S if and only if $\overline{A}$ is an intuitionistic fuzzy bi-ideal of S.

Proof. Let $A = (\Psi_A, \Omega_A)$ be an intuitionistic anti fuzzy bi-ideal of S. To show that $\overline{A} = (\overline{\Psi}_A, \overline{\Omega}_A) = (\Omega_A, \Psi_A)$ is an intuitionistic fuzzy bi-ideal. For any $x, y \in S$, we have

$$\begin{aligned} \overline{\Psi}_A(xy) &= 1 - \Psi_A(xy) \\ &\geq 1 - \max\{\Psi_A(x), \Psi_A(y)\} \\ &= \min\{1 - \Psi_A(x), 1 - \Psi_A(y)\} \\ &= \min\{\overline{\Psi}_A(x), \overline{\Psi}_A(y)\} \end{aligned}$$

and

$$\begin{aligned} \overline{\Omega}_A(xy) &= 1 - \Omega_A(xy) \\ &\leq 1 - \min\{\Psi_A(x), \Omega_A(y)\} \\ &= \max\{1 - \Omega_A(x), 1 - \Omega_A(y)\} \\ &= \max\{\overline{\Omega}_A(x), \overline{\Omega}_A(y)\}. \end{aligned}$$

Hence $\overline{A} = (\overline{\Psi}_A, \overline{\Omega}_A)$ is an intuitionistic fuzzy subsemigroup of S. Let $x, y, a \in S$. Then

$$\begin{aligned} \overline{\Psi}_A(xay) &= 1 - \Psi_A(xay) \\ &\geq 1 - \max\{\Psi_A(x), \Psi_A(y)\} \\ &= \min\{1 - \Psi_A(x), 1 - \Psi_A(y)\} \\ &= \min\{\overline{\Psi}_A(x), \overline{\Psi}_A(y)\} \end{aligned}$$

and

$$\begin{aligned} \overline{\Omega}_A(xay) &= 1 - \Omega_A(xay) \\ &\leq 1 - \min\{\Omega_A(x), \Omega_A(y)\} \\ &= \max\{1 - \Omega_A(x), 1 - \Omega_A(y)\} \\ &= \max\{\overline{\Omega}_A(x), \overline{\Omega}_A(y)\}. \end{aligned}$$

Hence $\overline{A}$ is an intuitionistic fuzzy bi-ideal of S.

Conversely suppose that $\overline{A}$ is an intuitionistic fuzzy bi-ideal of S. To show that $A = (\Psi_A, \Omega_A)$ is an intuitionistic anti fuzzy bi-ideal of S. Let $x, y \in S$. Then

$$\begin{aligned} \Psi_A(xy) &= 1 - \overline{\Psi}_A(xy) \\ &\leq 1 - \min\{\overline{\Psi}_A(x), \overline{\Psi}_A(y)\} \\ &= \max\{1 - \overline{\Psi}_A(x), 1 - \overline{\Psi}_A(y)\} \\ &= \max\{\Psi_A(x), \Psi_A(y)\} \end{aligned}$$

and

$$\begin{aligned}\Omega_A(xy) &= 1-\overline{\Omega}_A(xy)\\ &\geq 1-\max\{\overline{\Omega}_A(x),\overline{\Omega}_A(y)\}\\ &= \min\{1-\overline{\Omega}_A(x),1-\overline{\Omega}_A(y)\}\\ &= \min\{\Omega_A(x),\Omega_A(y)\}.\end{aligned}$$

Hence A is an intuitionistic anti fuzzy subsemigroup of S. Let $x, y, a \in S$. Then

$$\begin{aligned}\Psi_A(xay) &= 1-\overline{\Psi}_A(xay)\\ &\leq 1-\min\{\overline{\Psi}_A(x),\overline{\Psi}_A(y)\}\\ &= \max\{1-\overline{\Psi}_A(x),1-\overline{\Psi}_A(y)\}\\ &= \max\{\Psi_A(x),\Psi_A(y)\}\end{aligned}$$

and

$$\begin{aligned}\Omega_A(xay) &= 1-\overline{\Omega}_A(xay)\\ &\geq 1-\max\{\overline{\Omega}_A(x),\overline{\Omega}_A(y)\}\\ &= \min\{1-\overline{\Omega}_A(x),1-\overline{\Omega}_A(y)\}\\ &= \min\{\Omega_A(x),\Omega_A(y)\}.\end{aligned}$$

Hence A is an intuitionistic anti fuzzy bi-ideal of S. This completes the proof. □

Theorem 3.6. If an IFS $A = (\Psi_A, \Omega_A)$ in S is an intuitionistic anti fuzzy interior ideal of S, then so is $\Box A := (\Psi_A, \overline{\Psi}_A)$ where $\overline{\Psi}_A = 1 - \Psi_A$.

Proof. Since A is an intuitionistic anti fuzzy interior ideal of S then $\Psi_A(xy) \leq \max\{\Psi_A(x), \Psi_A(y)\}$ and $\Psi_A(xay) \leq \Psi_A(a) \forall x, y, a \in S$. Now

$$\begin{aligned}\overline{\Psi}_A(xay) &= 1-\Psi_A(xay)\\ &\geq 1-\max\{\Psi_A(x),\Psi_A(y)\}\\ &= \min\{1-\Psi_A(x),1-\Psi_A(y)\}\\ &= \min\{\overline{\Psi}_A(x),\overline{\Psi}_A(y)\}\end{aligned}$$

and $\overline{\Psi}_A(xay) = 1 - \Psi_A(xay) \geq 1 - \Psi_A(a) = \overline{\Psi}_A(a)$.

Hence A is an intuitionistic anti fuzzy interior ideal of S. This completes the proof. □

Theorem 3.7. An IFS $A = (\Psi_A, \Omega_A)$ in S is an intuitionistic anti fuzzy interior ideal of S if and only if the fuzzy sets Ψ_A and $\overline{\Omega}_A$ are anti fuzzy interior ideals of S.

Proof. Let $A = (\Psi_A, \Omega_A)$ be an intuitionistic anti fuzzy interior ideal of S. Then Ψ_A is an anti fuzzy interior ideal of S. Let $x, y, a \in S$. Then

$$\begin{aligned}\overline{\Omega}_A(xy) &= 1-\Omega_A(xy)\\ &\leq 1-\min\{\Omega_A(x),\Omega_A(y)\}\\ &= \max\{1-\Omega_A(x),1-\Omega_A(y)\}\\ &= \max\{\overline{\Omega}_A(x),\overline{\Omega}_A(y)\}.\end{aligned}$$

also $\overline{\Omega}_A(xay) = 1 - \Omega_A(xay) \leq 1 - \Omega_A(a) = \overline{\Omega}_A(a)$.

Hence $\overline{\Omega}_A$ is an anti fuzzy interior ideals of S.

Conversely, suppose that Ψ_A and $\overline{\Omega}_A$ are anti fuzzy interior ideals of S. Let $x, y, a \in S$. Then we have to show that $A = (\Psi_A, \Omega_A)$ intuitionistic anti fuzzy interior ideal of S. Since Ψ_A is an anti fuzzy interior ideal then $\Psi_A(xy) \leq \max\{\Psi_A(x), \Psi_A(y)\}$ and $\Psi_A(xay) \leq \Psi_A(a)$. Now

$$\begin{aligned}
1 - \Omega_A(xy) &= \overline{\Omega}_A(xy) \\
&\leq \max\{\overline{\Omega}_A(x), \overline{\Omega}_A(y)\} \\
&= \max\{1 - \Omega_A(x), 1 - \Omega_A(y)\} \\
&= 1 - \min\{\Omega_A(x), \Omega_A(y)\} \\
\Rightarrow 1 - \Omega_A(xy) &\leq 1 - \min\{\Omega_A(x), \Omega_A(y)\} \\
\Rightarrow \Omega_A(xy) &\geq \min\{\Omega_A(x), \Omega_A(y)\}
\end{aligned}$$

Also $1 - \Omega_A(xay) = \overline{\Omega}_A(xay) \leq \overline{\Omega}_A(a) = 1 - \Omega_A(a)$. This implies $\Omega_A(xay) \geq \Omega_A(a)$. Hence $A = (\Psi_A, \Omega_A)$ is an intuitionistic anti fuzzy interior ideals of S. This completes the proof. □

Theorem 3.8. Let $\{A_i : i \in I\}$ be a collection of intuitionistic anti fuzzy bi-ideals(intuitonistic anti fuzzy (1,2)-ideals, intuitionistic anti fuzzy interior ideals) of a semigroup S then their union $\bigcup_{i\in I} A_i = (\bigvee_{i\in I} \Psi_{A_i}, \bigwedge_{i\in I} \Omega_{A_i})$ is an intuitionistic anti fuzzy bi-ideal(intuitionistic anti fuzzy $(1, 2)$-ideal, intuitionistic anti fuzzy interior ideal) of S, where

$$\bigvee_{i\in I} \Psi_{A_i}(x) = \sup\{\Psi_{A_i}(x) : i \in I, x \in S\} \text{ and } \bigwedge_{i\in I} \Omega_{A_i}(x) = \inf\{\Omega_{A_i}(x) : i \in I, x \in S\}.$$

Proof. As $\bigcup_{i\in I} A_i = (\bigvee_{i\in I} \Psi_{A_i}, \bigwedge_{i\in I} \Omega_{A_i})$, where $\bigvee_{i\in I} \Psi_{A_i}(x) = \sup\{\Psi_{A_i}(x) : i \in I, x \in S\}$ and $\bigwedge_{i\in I} \Omega_{A_i}(x) = \inf\{\Omega_{A_i}(x) : i \in I, x \in S\}$. For $x, y \in S$, we have

$$\begin{aligned}
(\bigvee_{i\in I} \Psi_{A_i})(xy) &= \sup\{\Psi_{A_i}(xy) : i \in I, xy \in S\} \\
&\leq \sup\{\max\{\Psi_{A_i}(x), \Psi_{A_i}(y)\} : i \in I, x, y \in S\} \\
&\leq \max\{\sup\{\Psi_{A_i}(x) : i \in I, x \in S\}, \sup\{\Psi_{A_i}(y) : i \in I, y \in S\}\} \\
&= \max\{\bigvee_{i\in I} \Psi_{A_i}(x), \bigvee_{i\in I} \Psi_{A_i}(y)\} \\
(\bigvee_{i\in I} \Psi_{A_i})(xy) &\leq \max\{\bigvee_{i\in I} \Psi_{A_i}(x), \bigvee_{i\in I} \Psi_{A_i}(y)\}
\end{aligned}$$

and

$$\begin{aligned}
(\bigwedge_{i\in I} \Omega_{A_i})(xy) &= \inf\{\Omega_{A_i}(xy) : i \in I, xy \in S\} \\
&\geq \inf\{\min\{\Omega_{A_i}(x), \Omega_{A_i}(y)\} : i \in I, x, y \in S\} \\
&\geq \min\{\inf\{\Omega_{A_i}(x) : i \in I, x \in S\}, \inf\{\Omega_{A_i}(y) : i \in I, y \in S\}\} \\
&= \min\{\bigwedge_{i\in I} \Omega_{A_i}(x), \bigwedge_{i\in I} \Omega_{A_i}(y)\} \\
(\bigvee_{i\in I} \Omega_{A_i})(xy) &\geq \min\{\bigwedge_{i\in I} \Omega_{A_i}(x), \bigwedge_{i\in I} \Omega_{A_i}(y)\}
\end{aligned}$$

Hence $\bigcup_{i\in I} A_i$ is an intuitionistic anti fuzzy subsemigroup of S. Also for any $x, y, a \in S$

$$\begin{aligned}(\bigvee_{i\in I}\Psi_{A_i})(xay) &= \sup\{\Psi_{A_i}(xay) : i \in I, xay \in S\}\\ &\leq \sup\{\max\{\Psi_{A_i}(x), \Psi_{A_i}(y)\} : i \in I, x, y \in S\}\\ &\leq \max\{\sup\{\Psi_{A_i}(x) : i \in I, x \in S\}, \sup\{\Psi_{A_i}(y) : i \in I, y \in S\}\}\\ &= \max\{\bigvee_{i\in I}\Psi_{A_i}(x), \bigvee_{i\in I}\Psi_{A_i}(y)\}\end{aligned}$$

and

$$\begin{aligned}(\bigwedge_{i\in I}\Omega_{A_i})(xay) &= \inf\{\Omega_{A_i}(xay) : i \in I, xay \in S\}\\ &\geq \inf\{\min\{\Omega_{A_i}(x), \Omega_{A_i}(y)\} : i \in I, x, y \in S\}\\ &\geq \min\{\inf\{\Omega_{A_i}(x) : i \in I, x \in S\}, \inf\{\Omega_{A_i}(y) : i \in I, y \in S\}\}\\ &= \min\{\bigvee_{i\in I}\Omega_{A_i}(x), \bigvee_{i\in I}\Omega_{A_i}(y)\}.\end{aligned}$$

Hence $\bigcup_{i\in I} A_i$ is an intuitionistic anti fuzzy bi-ideal of S. Similarly we can prove the other cases also. This completes the proof. □

Definition 3.9. For any $t \in [0, 1]$ and a fuzzy subset Ψ of a semigroup S, the set $U(\Psi; t) = \{x \in S : \Psi(x) \geq t\}$(resp. $L(\Psi; t) = \{x \in S : \Psi(x) \leq t\}$) is called an upper(resp. lower) t-level cut of Ψ.

Theorem 3.10. If $A = (\Psi_A, \Omega_A)$ is an intuitionistic anti fuzzy bi-ideal of a semigroup S, then the upper and lower level cuts $L(\Psi_A; t)$ and $U(\Omega_A; t)$ are bi-ideals of S, for every $t \in Im(\Psi_A) \cap Im(\Omega_A)$.

Proof. Let $t \in Im(\Psi_A) \cap Im(\Omega_A)$. Let $A = (\Psi_A, \Omega_A)$ be an intuitionistic anti fuzzy bi-ideal of S. Then $A = (\Psi_A, \Omega_A)$ is an intuitionistic anti fuzzy subsemigroup of S. Let $x, y \in L(\Psi_A; t)$. Then $\Psi_A(x) \leq t$ and $\Psi_A(y) \leq t$ whence $\max\{\Psi_A(x), \Psi_A(y)\} \leq t$. Now $\Psi_A(xy) \leq \max\{\Psi_A(x), \Psi_A(y)\}$. Hence $\Psi_A(xy) \leq t$, *i.e.*, $xy \in L(\Psi_A; t)$. Consequently, $L(\Psi_A; t)$ is a subsemigroup of S.

Now let $x, z \in L(\Psi_A; t)$. Then $\Psi_A(x) \leq t$ and $\Psi_A(z) \leq t$ whence $\max\{\Psi_A(x), \Psi_A(z)\} \leq t$. Now for $y \in S$, $\Psi_A(xyz) \leq \max\{\Psi_A(x), \Psi_A(z)\}$. Hence $\Psi_A(xyz) \leq t$ whence $xyz \in L(\Psi_A; t)$. Consequently, $L(\Psi_A; t)$ is a bi-ideal of S. Similarly we can prove the other case also. □

Theorem 3.11. If $A = (\Psi_A, \Omega_A)$ is an intuitionistic fuzzy subset of S such that the non-empty sets $L(\Psi_A; t)$ and $U(\Omega_A; t)$ are bi-ideals of S, for $t \in [0, 1]$, then $A = (\Psi_A, \Omega_A)$ is an intuitionistic anti fuzzy bi-ideal of S.

Proof. Let $L(\Psi_A; t)$ and $U(\Omega_A; t)$ be bi-ideals of S, for all $t \in [0, 1]$. Then $L(\Psi_A; t)$ and $U(\Omega_A; t)$ are subsemigroups of S. Let $x, y \in S$. Let $\Psi_A(x) = t_1$ and $\Psi_A(y) =$

t_2. Without any loss of generality suppose $t_1 \geq t_2$. Then $x, y \in L(\Psi_A; t_1)$. Then by hypothesis $xy \in L(\Psi_A; t_1)$. Hence $\Psi_A(xy) \leq t_1 = \max\{\Psi_A(x), \Psi_A(y)\}$. Now let $\Omega_A(x) = t_3$ and $\Omega_A(y) = t_4$. Without any loss of generality suppose $t_3 \geq t_4$. Then $x, y \in U(\Omega_A; t_4)$. Then by hypothesis $xy \in U(\Omega_A; t_4)$. Hence $\Omega_A(xy) \geq t_4 = \min\{\Omega_A(x), \Omega_A(y)\}$. Hence $A = (\Psi_A, \Omega_A)$ is an intuitionistic anti fuzzy subsemigroup of S.

Now let $x, y, z \in S$. Let $\Psi_A(x) = t_1, \Psi_A(z) = t_2$. Without any loss of generality suppose $t_1 \geq t_2$. Then $x, z \in L(\Psi_A; t_1)$. Hence by hypothesis $xyz \in L(\Psi_A; t_1)$ whence $\Psi_A(xyz) \leq t_1 = \max\{\Psi_A(x), \Psi_A(z)\}$. Again let $\Omega_A(x) = t_3, \Omega_A(z) = t_4$. Without any loss of generality suppose $t_3 \geq t_4$. Then $x, z \in U(\Omega_A; t_4)$. Hence by hypothesis $xyz \in U(\Omega_A; t_4)$ whence $\Omega_A(xyz) \geq t_4 = \min\{\Omega_A(x), \Omega_A(z)\}$. Hence $A = (\Psi_A, \Omega_A)$ is an intuitionistic anti fuzzy bi-ideal of S. This completes the proof. □

Theorem 3.12. Let I be a non-empty subset of a semigroup S. If two fuzzy subsets Ψ and Ω are defined on S by

$$\Psi(x) := \begin{cases} \alpha_0 & \text{if } x \in I \\ \alpha_1 & \text{if } x \in S - I \end{cases}$$

and

$$\Omega(x) := \begin{cases} \beta_0 & \text{if } x \in I \\ \beta_1 & \text{if } x \in S - I \end{cases}$$

where $0 \leq \alpha_0 < \alpha_1, 0 \leq \beta_1 < \beta_0$ and $\alpha_i + \beta_i \leq 1$ for $i = 0, 1$. Then $A = (\Psi, \Omega)$ is an intuitionistic anti fuzzy bi-ideal of S and $U(\Omega; \alpha_0) = I = L(\Psi; \beta_0)$.

Proof. Let I be a bi-ideal of S. Then I is a subsemigroup of S. Let $x, y \in S$. Then following four cases may arise:

(1) : If $x \in I$ and $y \in I$, (2) : If $x \notin I$ and $y \in I$, (3) : If $x \in I$ and $y \notin I$, (4) : If $x \notin I$ and $y \notin I$.

In Case 4, $\Psi(x) = \Psi(y) = \alpha_1$ and $\Omega(x) = \Omega(y) = \beta_1$. Then $\max\{\Psi(x), \Psi(y)\} = \alpha_1$ and $\min\{\Omega(x), \Omega(y)\} = \beta_1$. Now $\Psi(xy) = \alpha_0$ and α_1 or $\Omega(xy) = \beta_0$ and β_1 according as $xy \in I$ or $xy \notin I$. Again $\alpha_0 < \alpha_1$ and $\beta_1 < \beta_0$. Hence we see that $\Psi(xy) \leq \max\{\Psi(x), \Psi(y)\}$ and $\Omega(xy) \geq \min\{\Omega(x), \Omega(y)\}$. Hence $A = (\Psi, \Omega)$ is an intuitionistic anti fuzzy subsemigroup of S. For other cases, by using a similar argument we can deduce that $A = (\Psi, \Omega)$ is an intuitionistic anti fuzzy subsemigroup of S.

Now, let $x, y, z \in S$. Then following four cases may arise:

(1) : If $x \in I$ and $z \in I$, (2) : If $x \notin I$ and $z \in I$, (3) : If $x \in I$ and $z \notin I$, (4) : If $x \notin I$ and $z \notin I$.

In Case 4, $\Psi(x) = \Psi(z) = \alpha_1$ and $\Omega(x) = \Omega(z) = \beta_1$. Then $\max\{\Psi(x), \Psi(z)\} = \alpha_1$ and $\min\{\Omega(x), \Omega(z)\} = \beta_1$. Now $\Psi(xyz) = \alpha_0$ and α_1 or $\Omega(xyz) = \beta_0$ and β_1 according as $xyz \in I$ or $xyz \notin I$. Again $\alpha_0 < \alpha_1$ and $\beta_1 < \beta_0$. Hence we see that $\Psi(xyz) \leq \max\{\Psi(x), \Psi(z)\}$ and $\Omega(xyz) \geq \min\{\Omega(x), \Omega(z)\}$. Hence $A = (\Psi, \Omega)$ is an intuitionistic anti fuzzy bi-ideal of S. For other cases, by using a similar argument we can deduce that $A = (\Psi, \Omega)$ is an intuitionistic anti fuzzy bi-ideal of S.

In order to prove the converse, we first observe that by definition of Ψ and Ω, $U(\Omega; \alpha_0) = I = L(\Psi; \beta_0)$. Then the proof follows from Theorem 3.11. □

Definition 3.13. Let S be a semigroup. Let $A = (\Psi_A, \Omega_A)$ and $B = (\Psi_B, \Omega_B)$ be two intuitionistic fuzzy subsets of S. Then the anti intuitionistic fuzzy product $A \circ B$ of A and B is defined as

$$(\Psi_{A\circ B})(x) = \begin{cases} \inf\limits_{x=uv}[\max\{\Psi_A(u), \Psi_B(v)\} : u, v \in S] \\ 1, \text{ if for any } u, v \in S, x \neq uv \end{cases}$$

and

$$(\Omega_{A\circ B})(x) = \begin{cases} \sup\limits_{x=uv}[\min\{\Omega_A(u), \Omega_B(v)\} : u, v \in S] \\ 0, \text{ if for any } u, v \in S, x \neq uv \end{cases}$$

Theorem 3.14. A non-empty intuitionistic fuzzy subset $A = (\Psi_A, \Omega_A)$ of a semigroup S is an intuitionistic anti fuzzy subsemigroup of S if and only if $A \subseteq A \circ A$.

Proof. Let $A \subseteq A \circ A$. Then for $x, y \in S$ we obtain

$$\Psi_A(xy) \leq \Psi_{A\circ A}(xy) \leq \max\{\Psi_A(x), \Psi_A(y)\}$$

and

$$\Omega_A(xy) \geq \Omega_{A\circ A}(xy) \geq \min\{\Omega_A(x), \Omega_A(y)\}.$$

So $A = (\Psi_A, \Omega_A)$ is an intuitionistic anti fuzzy subsemigroup of S.

Conversely, let $A = (\Psi_A, \Omega_A)$ be an intuitionistic anti fuzzy subsemigroup of S and $x \in S$. Suppose there exist $y, z \in S$ such that $x = yz$ then $\Psi_{A\circ A}(x) \neq 1$ and $\Omega_{A\circ A}(x) \neq 0$. By hypothesis, $\Psi_A(yz) \leq \max\{\Psi_A(y), \Psi_A(z)\}$ and $\Omega_A(yz) \geq \min\{\Omega_A(y), \Omega_A(z)\}$. Hence

$$\Psi_A(yz) \leq \inf_{x=yz} \max\{\Psi_A(y), \Psi_A(z)\} = \Psi_{A\circ A}(x)$$

and

$$\Omega_A(yz) \geq \sup_{x=yz} \min\{\Omega_A(y), \Omega_A(z)\} = \Omega_{A\circ A}(x).$$

Again if there does not exist $y, z \in S$ such that $x = yz$ then $\Psi_{A\circ A}(x) = 1 \geq \Psi_A(x)$ and $\Omega_{A\circ A}(x) = 0 \leq \Omega_A(x)$. Consequently, $\Psi_A \subseteq \Psi_{A\circ A}$ and $\Omega_A \supseteq \Omega_{A\circ A}$. Hence $A \subseteq A \circ A$. This completes the proof. □

Theorem 3.15. In a semigroup S for a non-empty intuitionistic fuzzy subset $A = (\Psi_A, \Omega_A)$ of S the following are equivalent: (1) $A = (\Psi_A, \Omega_A)$ is an intuitionistic anti fuzzy bi-ideal of S, (2) $A \subseteq A \circ A$ and $A \subseteq A \circ S \circ A$, where $S = (\Psi_S, \Omega_S)$ and Ψ_S is the characteristic function of S.

Proof. Suppose (1) holds, *i.e.*, $A = (\Psi_A, \Omega_A)$ is an intuitionistic anti fuzzy bi-ideal of S. Then $A = (\Psi_A, \Omega_A)$ is an intuitionistic anti fuzzy subsemigroup of S. So by Theorem 3.14, $A \subseteq A \circ A$. Let $a \in S$. Suppose there exists $x, y, p, q \in S$ such that $a = xy$ and $x = pq$. Since $A = (\Psi_A, \Omega_A)$ is an intuitionistic anti fuzzy bi-ideal of S, we obtain $\Psi_A(pqy) \leq \max\{\Psi_A(p), \Psi_A(y)\}$ and $\Omega_A(pqy) \geq \min\{\Omega_A(p), \Omega_A(y)\}$. Then

$$\begin{aligned} \Psi_{A\circ S\circ A}(a) &= \inf_{a=xy}[\max\{\Psi_{A\circ S}(x), \Psi_A(y)\}] \\ &= \inf_{a=xy}[\max\{\inf_{x=pq}\{\max\{\Psi_A(p), \Psi_S(q)\}\}, \Psi_A(y)\}] \\ &= \inf_{a=xy}[\max\{\inf_{x=pq}\{\max\{\Psi_A(p), 0\}\}, \Psi_A(y)\}] \\ &= \inf_{a=xy}[\max\{\Psi_A(p), \Psi_A(y)\}] \\ &\geq \Psi_A(pqy) = \Psi_A(xy) = \Psi_A(a). \end{aligned}$$

So we have $\Psi_A \subseteq \Psi_{A\circ S\circ A}$. Otherwise $\Psi_{A\circ S\circ A}(a) = 1 \geq \Psi_A(a)$. Thus $\Psi_A \subseteq \Psi_{A\circ S\circ A}$. Now

$$\begin{aligned}\Omega_{A\circ S\circ A}(a) &= \sup_{a=xy}[\min\{\Omega_{A\circ S}(x), \Omega_A(y)\}] \\ &= \sup_{a=xy}[\min\{\sup_{x=pq}\{\min\{\Omega_A(p), \Omega_S(q)\}\}, \Omega_A(y)\}] \\ &= \sup_{a=xy}[\min\{\sup_{x=pq}\{\min\{\Omega_A(p), 1\}\}, \Omega_A(y)\}] \\ &= \sup_{a=xy}[\min\{\Omega_A(p), \Omega_A(y)\}] \\ &\leq \Omega_A(pqy) = \Omega_A(xy) = \Omega_A(a).\end{aligned}$$

So we have $\Omega_A \supseteq \Omega_{A\circ S\circ A}$. Otherwise $\Omega_{A\circ S\circ A}(a) = 0 \leq \Omega_A(a)$. Thus $\Omega_A \supseteq \Omega_{A\circ S\circ A}$. Hence $A \subseteq A \circ S \circ A$.

Conversely, let us assume that (2) holds. Since $\Psi_A \subseteq \Psi_{A\circ A}$ and $\Omega_A \supseteq \Omega_{A\circ A}$ so $A = (\Psi_A, \Omega_A)$ is an intuitionistic anti fuzzy subsemigroup of S. Let $x, y, z \in S$ and $a = xyz$. Since $\Psi_A \subseteq \Psi_{A\circ S\circ A}$ and $\Omega_A \supseteq \Omega_{A\circ S\circ A}$, we have

$$\begin{aligned}\Psi_A(xyz) &= \Psi_A(a) \leq \Psi_{A\circ S\circ A}(a) \\ &= \inf_{a=xyz}[\max\{\Psi_{A\circ S}(xy), \Psi_A(z)\}] \\ &\leq \max\{\Psi_{A\circ S}(p), \Psi_A(z)\}(\text{let } p = xy) \\ &= \max[\inf_{p=xy}\{\max\{\Psi_A(x), \Psi_S(y)\}\}, \Psi_A(z)] \\ &\leq \max[\max\{\Psi_A(x), 0\}, \Psi_A(z)] \\ &= \max\{\Psi_A(x), \Psi_A(z)\}\end{aligned}$$

and

$$\begin{aligned}\Omega_A(xyz) &= \Omega_A(a) \geq \Omega_{A\circ S\circ A}(a) \\ &= \sup_{a=xyz}[\min\{\Omega_{A\circ S}(xy), \Omega_A(z)\}] \\ &\geq \min\{\Omega_{A\circ S}(p), \Omega_A(z)\}(\text{let } p = xy) \\ &= \min[\sup_{p=xy}\{\min\{\Omega_A(x), \Omega_S(y)\}\}, \Omega_A(z)] \\ &\geq \min[\min\{\Omega_A(x), 1\}, \Omega_A(z)] \\ &= \min\{\Omega_A(x), \Omega_A(z)\}\end{aligned}$$

Hence $A = (\Psi_A, \Omega_A)$ is an intuitionistic anti fuzzy bi-ideal of S. This completes the proof. □

Theorem 3.16. Every intuitionistic anti fuzzy bi-ideal is an intuitionistic anti fuzzy $(1, 2)$-ideal.

Proof. Let $A = (\Psi_A, \Omega_A)$ be an intuitionistic anti fuzzy bi-ideal of a semigroup S and let $x, y, z, a \in S$. Then

$$\begin{aligned}\Psi_A(xa(yz)) &= \Psi_A((xay)z))\\ &\leq \max\{\Psi_A(xay), \Psi_A(z)\}\\ &\leq \max\{\max\{\Psi_A(x), \Psi_A(y)\}, \Psi_A(z)\}\\ &= \max\{\Psi_A(x), \Psi_A(y), \Psi_A(z)\}\end{aligned}$$

and

$$\begin{aligned}\Omega_A(xa(yz)) &= \Omega_A((xay)z))\\ &\geq \min\{\Omega_A(xay), \Omega_A(z)\}\\ &\geq \min\{\min\{\Omega_A(x), \Omega_A(y)\}, \Omega_A(z)\}\\ &= \min\{\Omega_A(x), \Omega_A(y), \Omega_A(z)\}.\end{aligned}$$

Hence $A = (\Psi_A, \Omega_A)$ is an intuitionistic anti fuzzy (1,2)-ideal of S. □

For the converse of Theorem 3.16 we need to strengthen the condition of a semigroup S.

Theorem 3.17. If S is a regular semigroup then every intuitionistic anti fuzzy $(1,2)$-ideal of S is an intuitionistic anti fuzzy bi-ideal of S.

Proof. Let us assume that the semigroup S is regular and let $A = (\Psi_A, \Omega_A)$ be an intuitionistic anti fuzzy (1,2)-ideal of S. Let $x, y, a \in S$. Since S is regular, we have $xa \in (xsx)s \subseteq xsx$, which implies that $xa = xsx$ for some $s \in S$.

$$\begin{aligned}\Psi_A(xay) &= \Psi_A((xa)y)\\ &= \Psi_A((xsx)y)\\ &= \Psi_A(xs(xy))\\ &\leq \max\{\Psi_A(x), \Psi_A(x), \Psi_A(y)\}\\ &= \max\{\Psi_A(x), \Psi_A(y)\}\end{aligned}$$

and

$$\begin{aligned}\Omega_A(xay) &= \Omega_A((xa)y))\\ &= \Omega_A((xsx)y))\\ &= \Omega_A(xs(xy))\\ &\geq \min\{\Omega_A(x), \Omega_A(x), \Omega_A(y)\}\\ &= \min\{\Omega_A(x), \Omega_A(y)\}.\end{aligned}$$

Hence $A = (\Psi_A, \Omega_A)$ is an intuitionistic anti fuzzy bi–ideal of S. □

Theorem 3.18. Every intuitionistic anti fuzzy bi-ideal of a group S is constant.

Proof. Let $A = (\Psi_A, \Omega_A)$ be an intuitionistic anti fuzzy bi-ideal of a group S and e be the identity of S. Let $x, y \in S$. Then

$$\begin{aligned}\Psi_A(x) &= \Psi_A(exe)\\ &\leq \max\{\Psi_A(e), \Psi_A(e)\} = \Psi_A(e)\\ &= \Psi_A(ee)\\ &= \Psi_A((xx^{-1})(x^{-1}x))\\ &= \Psi_A(x(x^{-1}x^{-1})x)\\ &\leq \max\{\Psi_A(x), \Psi_A(x)\} = \Psi_A(x)\end{aligned}$$

and

$$\begin{aligned}\Omega_A(x) &= \Omega_A(exe)\\ &\geq \min\{\Omega_A(e), \Omega_A(e)\} = \Omega_A(e)\\ &= \Omega_A(ee)\\ &= \Omega_A((xx^{-1})(x^{-1}x))\\ &= \Omega_A(x(x^{-1}x^{-1})x)\\ &\geq \min\{\Omega_A(x), \Omega_A(x)\} = \Omega_A(x).\end{aligned}$$

It follows that $\Psi_A(x) = \Psi_A(e)$ and $\Omega_A(x) = \Omega_A(e)$ which means that $A = (\Psi_A, \Omega_A)$ is constant. This completes the proof. □

Theorem 3.19. Let $f : S \to T$ be an homomorphism of semigroups. If $B = (\Psi_B, \Omega_B)$ is an intuitionistic anti fuzzy bi-ideal(intuitionistic anti fuzzy $(1, 2)$-ideal) of T, then the pre-image $f^{-1}(B) = (f^{-1}(\Psi_B), f^{-1}(\Omega_B))$ of B under f is an intuitionistic anti fuzzy bi-ideal(resp. intuitionistic anti fuzzy $(1, 2)$-ideal) of S.

Proof. Let $B = (\Psi_B, \Omega_B)$ be an intuitionistic anti fuzzy bi-ideal of T and let $x, y \in S$. Then

$$\begin{aligned}f^{-1}(\Psi_B(xy)) &= \Psi_B(f(xy))\\ &= \Psi_B(f(x)f(y))\\ &\leq \max\{\Psi_B(f(x)), \Psi_B(f(y))\}\\ &= \max\{f^{-1}(\Psi_B(x)), f^{-1}(\Psi_B(y))\}\end{aligned}$$

and

$$\begin{aligned}f^{-1}(\Omega_B(xy)) &= \Omega_B(f(xy))\\ &= \Omega_B(f(x)f(y))\\ &\geq \min\{\Omega_B(f(x)), \Omega_B(f(y))\}\\ &= \max\{f^{-1}(\Omega_B(x)), f^{-1}(\Omega_B(y))\}.\end{aligned}$$

Hence $f^{-1}(B) = (f^{-1}(\Psi_B), f^{-1}(\Omega_B))$ is an intuitionistic anti fuzzy subsemigroup of S. For any $a, x, y \in S$ we have

$$\begin{aligned}f^{-1}(\Psi_B(xay)) &= \Psi_B(f(xay))\\ &= \Psi_B(f(x)f(a)f(y))\\ &\leq \max\{\Psi_B(f(x)), \Psi_B(f(y))\}\\ &= \max\{f^{-1}(\Psi_B(x)), f^{-1}(\Psi_B(y))\}\end{aligned}$$

and

$$\begin{aligned}f^{-1}(\Omega_B(xay)) &= \Omega_B(f(xay))\\ &= \Omega_B(f(x)f(a)f(y))\\ &\geq \min\{\Omega_B(f(x)), \Omega_B(f(y))\}\\ &= \min\{f^{-1}(\Omega_B(x)), f^{-1}(\Omega_B(y))\}.\end{aligned}$$

Therefore $f^{-1}(B) = (f^{-1}(\Psi_B), f^{-1}(\Omega_B))$ is an intuitionistic anti fuzzy bi-ideal of S. Similarly we can prove the other case also. This completes the proof. □

Proposition 3.20. Let θ be an endomorphism and $A = (\Psi_A, \Omega_A)$ be an intuitionistic anti fuzzy subsemigroup(intuitionistic anti fuzzy bi-ideal, intuitionistic anti fuzzy $(1, 2)$-ideal) of a semigroup S. Then $A[\theta]$ is also an intuitionistic anti fuzzy subsemigroup(resp. intuitionistic anti fuzzy bi-ideal, intuitionistic anti fuzzy $(1, 2)$-ideal) of S, where $A[\theta](x) := (\Psi_A[\theta](x), \Omega_A[\theta](x)) = (\Psi_A(\theta(x)), \Omega_A(\theta(x)))\forall x \in S$.

Proof. Let $x, y \in S$. Then

$$\begin{aligned}\Psi_A[\theta](xy) &= \Psi_A(\theta(xy)) = \Psi_A(\theta(x)\theta(y)) \\ &\leq \max\{\Psi_A(\theta(x)), \Psi_A(\theta(y))\} \\ &= \max\{\Psi_A[\theta](x), \Psi_A[\theta](y)\}\end{aligned}$$

and

$$\begin{aligned}\Omega_A[\theta](xy) &= \Omega_A(\theta(xy)) = \Omega_A(\theta(x)\theta(y)) \\ &\geq \min\{\Omega_A(\theta(x)), \Omega_A(\theta(y))\} \\ &= \min\{\Omega_A[\theta](x), \Omega_A[\theta](y)\}.\end{aligned}$$

Hence $A[\theta]$ is an intuitionistic anti fuzzy subsemigroup of S. Similarly we can prove other cases also. This completes the proof. □

Proposition 3.21. Let $\alpha \geq 0$ be a real number and $A = (\Psi_A, \Omega_A)$ be an intuitionistic anti fuzzy subsemigroup(intuitionistic anti fuzzy bi-ideal, intuitionistic anti fuzzy $(1, 2)$-ideal) of a semigroup S. Then so is $A^\alpha = (\Psi_A^\alpha, \Omega_A^\alpha)$, where $\Psi_A^\alpha(x) = (\Psi_A(x))^\alpha$ and $\Omega_A^\alpha(x) = (\Omega_A(x))^\alpha$ for all $x \in S$.

Proof. Let $x, y \in S$. Without any loss of generality, suppose $\Psi_A(x) \geq \Psi_A(y)$ and $\Omega_A(x) \leq \Omega_A(y)$. Then $\Psi_A^\alpha(x) \geq \Psi_A^\alpha(y)$ and $\Omega_A^\alpha(x) \leq \Omega_A^\alpha(y)$. Now $\Psi_A(xy) \leq \max\{\Psi_A(x), \Psi_A(y)\} = \Psi_A(x)$ and $\Omega_A(xy) \geq \min\{\Omega_A(x), \Omega_A(y)\} = \Omega_A(x)$. Then

$$\Psi_A^\alpha(xy) = (\Psi_A(xy))^\alpha \leq (\Psi_A(x))^\alpha = \Psi_A^\alpha(x) = \max\{\Psi_A^\alpha(x), \Psi_A^\alpha(y)\}$$

and

$$\Omega_A^\alpha(xy) = (\Omega_A(xy))^\alpha \geq (\Omega_A(x))^\alpha = \Omega_A^\alpha(x) = \min\{\Omega_A^\alpha(x), \Omega_A^\alpha(y)\}.$$

Consequently, $A^\alpha = (\Psi_A^\alpha, \Omega_A^\alpha)$ is an intuitionistic anti fuzzy subsemigroup of S. Similarly we can prove other cases also. This completes the proof. □

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http://www.newtheory.org ISSN: 2149-1402

Journal of New Theory

Received: 10.02.2016 *Year*: **2016,** *Number*: **11**, *Pages*: **84-102**
Accepted: 03.03.2016 *Original Article*[**]

ON BIPOLAR SINGLE VALUED NEUTROSOPHIC GRAPHS

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Abstract - In this article, we combine the concept of bipolar neutrosophic set and graph theory. We introduce the notions of bipolar single valued neutrosophic graphs, strong bipolar single valued neutrosophic graphs, complete bipolar single valued neutrosophic graphs, regular bipolar single valued neutrosophic graphs and investigate some of their related properties.

Keywords *- Bipolar neutrosophic sets, bipolar single valued neutrosophic graph, strong bipolar single valued neutrosophic graph, complete bipolar single valued neutrosophic graph.*

1. Introduction

Zadeh [32] coined the term 'degree of membership' and defined the concept of fuzzy set in order to deal with uncertainty. Atanassov [29, 31] incorporated the degree of non-membership in the concept of fuzzy set as an independent component and defined the concept of intuitionistic fuzzy set. Smarandache [12, 13] grounded the term 'degree of indeterminacy as an independent component and defined the concept of neutrosophic set from the philosophical point of view to deal with incomplete, indeterminate and inconsistent information in real world. The concept of neutrosophic sets is a generalization of the theory of fuzzy sets, intuitionistic fuzzy sets. Each element of a neutrosophic sets has three membership degrees including a truth membership degree, an indeterminacy membership degree, and a falsity membership degree which are within the real standard or nonstandard unit interval]−0, 1+[. Therefore, if their range is restrained within the real standard unit interval [0, 1], the neutrosophic set is easily applied to engineering problems. For this purpose, Wang et al. [17] introduced the concept of a single valued neutrosophic set (SVNS) as a subclass of the neutrosophic set. Recently, Deli et al. [23] defined the concept of bipolar neutrosophic as an extension of the fuzzy sets, bipolar fuzzy sets, intuitionistic fuzzy sets

[**]*Edited by* Mumtaz Ali and Naim Çağman *(Editor-in- Chief).*
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and neutrosophic sets studied some of their related properties including the score, certainty and accuracy functions to compare the bipolar neutrosophic sets. The neutrosophic sets theory of and their extensions have been applied in various part[1, 2, 3, 16, 18, 19, 20, 21, 25, 26, 27, 41, 42, 50, 51, 53].

A graph is a convenient way of representing information involving relationship between objects. The objects are represented by vertices and the relations by edges. When there is vagueness in the description of the objects or in its relationships or in both, it is natural that we need to designe a fuzzy graph Model. The extension of fuzzy graph theory [4, 6, 11] have been developed by several researchers including intuitionistic fuzzy graphs [5, 35, 44] considered the vertex sets and edge sets as intuitionistic fuzzy sets. Interval valued fuzzy graphs [32, 34] considered the vertex sets and edge sets as interval valued fuzzy sets. Interval valued intuitionistic fuzzy graphs [8, 52] considered the vertex sets and edge sets as interval valued intuitionstic fuzzy sets. Bipolar fuzzy graphs [6, 7, 40] considered the vertex sets and edge sets as bipolar fuzzy sets. M-polar fuzzy graphs [39] considered the vertex sets and edge sets as m-polar fuzzy sets. Bipolar intuitionistic fuzzy graphs [9] considered the vertex sets and edge sets as bipolar intuitionistic fuzzy sets. But, when the relations between nodes(or vertices) in problems are indeterminate, the fuzzy graphs and their extensions are failed. For this purpose, Samarandache [10, 11] have defined four main categories of neutrosophic graphs, two based on literal indeterminacy (I), which called them; I-edge neutrosophic graph and I-vertex neutrosophic graph, these concepts are studied deeply and has gained popularity among the researchers due to its applications via real world problems [7, 14, 15, 54, 55, 56]. The two others graphs are based on (t, i, f) components and called them; The (t, i, f)-Edge neutrosophic graph and the (t, i, f)-vertex neutrosophic graph, these concepts are not developed at all. Later on, Broumi et al.[46] introduced a third neutrosophic graph model. This model allows the attachment of truth-membership (t), indeterminacy–membership (i) and falsity- membership degrees (f) both to vertices and edges, and investigated some of their properties. The third neutrosophic graph model is called single valued neutrosophic graph (SVNG for short). The single valued neutrosophic graph is the generalization of fuzzy graph and intuitionistic fuzzy graph. Also the same authors [45] introduced neighborhood degree of a vertex and closed neighborhood degree of vertex in single valued neutrosophic graph as a generalization of neighborhood degree of a vertex and closed neighborhood degree of vertex in fuzzy graph and intuitionistic fuzzy graph. Also, Broumi et al.[47] introduced the concept of interval valued neutrosophic graph as a generalization fuzzy graph, intuitionistic fuzzy graph, interval valued fuzzy graph, interval valued intuitionistic fuzzy graph and single valued neutrosophic graph and have discussed some of their properties with proof and examples. In addition Broumi et al [48] have introduced some operations such as cartesian product, composition, union and join on interval valued neutrosophic graphs and investigate some their properties. On the other hand, Broumi et al [49] have discussed a sub class of interval valued neutrosophic graph called strong interval valued neutrosophic graph, and have introduced some operations such as, cartesian product, composition and join of two strong interval valued neutrosophic graph with proofs. In the literature the study of bipolar single valued neutrosophic graphs (BSVN-graph) is still blank, we shall focus on the study of bipolar single valued neutrosophic graphs in this paper. In the present paper, bipolar neutrosophic sets are employed to study graphs and give rise to a new class of graphs called bipolar single valued neutrosophic graphs. We introduce the notions of bipolar single valued neutrosophic graphs, strong bipolar single valued neutrosophic graphs, complete bipolar single valued neutrosophic graphs, regular bipolar single valued neutrosophic graphs and investigate some of their related properties. This paper is organized as follows;

In section 2, we give all the basic definitions related bipolar fuzzy set, neutrosophic sets, bipolar neutrosophic set, fuzzy graph, intuitionistic fuzzy graph, bipolar fuzzy graph, N-graph and single valued neutrosophic graph which will be employed in later sections. In section 3, we introduce certain notions including bipolar single valued neutrosophic graphs, strong bipolar single valued neutrosophic graphs, complete bipolar single valued neutrosophic graphs, the complement of strong bipolar single valued neutrosophic graphs, regular bipolar single valued neutrosophic graphs and illustrate these notions by several examples, also we described degree of a vertex, order, size of bipolar single valued neutrosophic graphs. In section 4, we give the conclusion.

2. Preliminaries

In this section, we mainly recall some notions related to bipolar fuzzy set, neutrosophic sets, bipolar neutrosophic set, fuzzy graph, intuitionistic fuzzy graph, bipolar fuzzy graph, ***N***-graph and single valued neutrosophic graph relevant to the present work. The readers are referred to [9, 12, 17, 35, 36, 38, 43, 46, 57] for further details and background.

Definition 2.1 [12]. Let U be an universe of discourse; then the neutrosophic set A is an object having the form A = {< x: $T_A(x)$, $I_A(x)$, $F_A(x)$>, x ∈ U}, where the functions T, I, F : U→]$^-0,1^+$[define respectively the degree of membership, the degree of indeterminacy, and the degree of non-membership of the element x ∈ U to the set A with the condition:

$$^-0 \leq T_A(x)+ I_A(x)+ F_A(x)\leq 3^+. \qquad (1)$$

The functions $T_A(x)$, $I_A(x)$ and $F_A(x)$ are real standard or nonstandard subsets of]−0,1+[. Since it is difficult to apply NSs to practical problems, Wang et al. [16] introduced the concept of a SVNS, which is an instance of a NS and can be used in real scientific and engineering applications.

Definition 2.2 [17]. Let X be a space of points (objects) with generic elements in X denoted by x. A single valued neutrosophic set A (SVNS A) is characterized by truth-membership function $T_A(x)$, an indeterminacy-membership function $I_A(x)$, and a falsity-membership function $F_A(x)$. For each point x in X $T_A(x), I_A(x), F_A(x) \in [0, 1]$. A SVNS A can be written as

$$A = \{< x: T_A(x), I_A(x), F_A(x)>, x \in X\} \qquad (2)$$

Definition 2.3 [9]. A bipolar neutrosophic set A in X is defined as an object of the form

$$A=\{<x, T^P(x), I^P(x), F^P(x), T^N(x), I^N(x), F^N(x)>: x \in X\},$$

where T^P, I^P, F^P:X→ [1, 0] and T^N, I^N, F^N: X→ [-1, 0] .The Positive membership degree $T^P(x)$, $I^P(x)$, $F^P(x)$ denotes the truth membership, indeterminate membership and false membership of an element ∈ X corresponding to a bipolar neutrosophic set A and the negative membership degree $T^N(x)$, $I^N(x)$, $F^N(x)$ denotes the truth membership, indeterminate membership and false membership of an element ∈ X to some implicit counter-property corresponding to a bipolar neutrosophic set A.

Example 2.4 Let X={x_1, x_2, x_3}

$$A=\left\{\begin{matrix} <x_1,0.5,0.3,0.1,-0.6,-0.4,-0.05> \\ <x_2,0.3,0.2,0.7,-0.02,-0.3,-0.02> \\ <x_3,0.8,0.05,0.4,-0.6,-0.6,-0.03> \end{matrix}\right\}$$

is a bipolar neutrosophic subset of X.

Definition 2.5 [9]. Let A_1={<x, T_1^P(x), I_1^P(x), F_1^P(x), T_1^N(x), I_1^N(x), F_1^N(x)>} and A_2={<x, T_2^P(x), I_2^P(x), F_2^P(x), T_2^N(x), I_2^N(x), F_2^N(x) >} be two bipolar neutrosophic sets . Then $A_1 \subseteq A_2$ if and only if

$$T_1^P(x) \le T_2^P(x),\ I_1^P(x) \le I_2^P(x),\ F_1^P(x) \ge F_2^P(x)$$

and

$$T_1^N(x) \ge T_2^N(x),\ I_1^N(x) \ge I_2^N(x),\ F_1^N(x) \le F_2^N(x) \text{ for all } x \in X.$$

Definition 2.6 [9]. Let A_1={<x, T_1^P(x), I_1^P(x), F_1^P(x), T_1^N(x), I_1^N(x), F_1^N(x)>} and A_2={<x, T_2^P(x), I_2^P(x), F_2^P(x), T_2^N(x), I_2^N(x), F_2^N(x) >} be two bipolar neutrosophic sets. Then $A_1 = A_2$ if and only if

$$T_1^P(x) = T_2^P(x),\ I_1^P(x) = I_2^P(x),\ F_1^P(x) = F_2^P(x)$$

and

$$T_1^N(x) = T_2^N(x),\ I_1^N(x) = I_2^N(x),\ F_1^N(x) = F_2^N(x) \text{ for all } x \in X$$

Definition 2.7 [9]. Let A_1={<x, T_1^P(x), I_1^P(x), F_1^P(x), T_1^N(x), I_1^N(x), F_1^N(x)>} and A_2={<x, T_2^P(x), I_2^P(x), F_2^P(x), T_2^N(x), I_2^N(x), F_2^N(x) >} be two bipolar neutrosophic sets . Then their union is defined as:

$$(A_1 \cup A_2)(x) = \begin{pmatrix} \max(T_1^P(x), T_2^p(x)), \frac{I_1^P(x)+I_2^P(x)}{2}, \min(T_1^P(x), T_2^P(x)) \\ \min(T_1^N(x), T_2^N(x)), \frac{I_1^N(x)+I_2^N(x)}{2}, \max(T_1^N(x), T_2^N(x)) \end{pmatrix}$$

for all x ∈ X.

Definition 2.8 [9]. Let A_1={<x, T_1^P(x), I_1^P(x), F_1^P(x), T_1^N(x), I_1^N(x), F_1^N(x)>} and A_2={<x, T_2^P(x), I_2^P(x), F_2^P(x), T_2^N(x), I_2^N(x), F_2^N(x) >} be two bipolar neutrosophic sets . Then their intersection is defined as:

$$(A_1 \cap A_2)(x) = \begin{pmatrix} \min(T_1^P(x), T_2^P(x)), \frac{I_1^P(x)+I_2^P(x)}{2}, \max(T_1^P(x), T_2^P(x)) \\ \max(T_1^N(x), T_2^N(x)), \frac{I_1^N(x)+I_2^N(x)}{2}, \min(T_1^N(x), T_2^N(x)) \end{pmatrix}$$

for all x ∈ X.

Definition 2.9 [9]. Let A_1={<x, T_1^P(x), I_1^P(x), F_1^P(x), T_1^N(x), I_1^N(x), F_1^N(x)>: x ∈ X } be a bipolar neutrosophic set in X. Then the complement of A is denoted by A^c and is defined by

$$T_{A^c}^P(x)=\{1^P\}-T_A^P(x),\quad I_{A^c}^P(x)=\{1^P\}-I_A^P(x),\quad F_{A^c}^P(x)=\{1^P\}-F_A^+(x)$$

and

$$T_{A^c}^N(x)=\{1^N\}-T_A^N(x),\quad I_{A^c}^N(x)=\{1^N\}-I_A^N(x),\quad F_{A^c}^N(x)=\{1^N\}-F_A^N(x)$$

Definition 2.10 [43]. A fuzzy graph is a pair of functions G = (σ, μ) where σ is a fuzzy subset of a non empty set V and μ is a symmetric fuzzy relation on σ. i.e σ : V → [0,1] and μ: VxV→[0,1] such that μ(uv) ≤ σ(u) ∧ σ(v) for all u, v ∈ V where uv denotes the edge between u and v and σ(u) ∧ σ(v) denotes the minimum of σ(u) and σ(v). σ is called the fuzzy vertex set of V and μ is called the fuzzy edge set of E.

Definition 2.11[38]: By a *N*-graph G of a graph G^*, we mean a pair G= (μ_1, μ_2) where μ_1 is an *N*-function in V and μ_2 is an *N*-relation on E such that μ_2(u,v) ≥ max (μ_1(u), μ_1(v)) all u, v ∈ V.

Definition 2.12[35] : An Intuitionistic fuzzy graph is of the form G = (V, E) where

i. V={$v_1, v_2, \ldots, v_n$} such that μ_1: V→ [0,1] and γ_1: V → [0,1] denote the degree of membership and non-membership of the element v_i ∈ V, respectively, and

$$0 \le \mu_1(v_i) + \gamma_1(v_i)) \le 1$$

for every v_i ∈ V, (i = 1, 2, ……. n),

ii. E ⊆ V x V where μ_2: VxV→[0,1] and γ_2: VxV→ [0,1] are such that

$$\mu_2(v_i, v_j) \le \min [\mu_1(v_i), \mu_1(v_j)] \text{ and } \gamma_2(v_i, v_j) \ge \max [\gamma_1(v_i), \gamma_1(v_j)]$$

and $0 \le \mu_2(v_i, v_j) + \gamma_2(v_i, v_j) \le 1$ for every $(v_i, v_j) \in E$, (i, j = 1,2, ……. n)

Definition 2.13 [57]. Let X be a non-empty set. A bipolar fuzzy set A in X is an object having the form A = {(x, μ_A^P(x), μ_A^N(x)) | x ∈ X}, where μ_A^P(x): X ⟶ [0, 1] and μ_A^N(x): X ⟶ [−1, 0] are mappings.

Definition 2.14 [57] Let X be a non-empty set. Then we call a mapping

$$A = (\mu_A^P, \mu_A^N): X \times X \longrightarrow [-1, 0] \times [0, 1]$$

a bipolar fuzzy relation on X such that μ_A^P(x, y) ∈ [0, 1] and μ_A^N(x, y) ∈ [−1, 0].

Definition 2.15 [36]. Let *A* = (μ_A^P, μ_A^N) and *B* = (μ_B^P, μ_B^N) be bipolar fuzzy sets on a set *X*. If *A* = (μ_A^P, μ_A^N) is a bipolar fuzzy relation on a set *X*, then *A* =(μ_A^P, μ_A^N) is called a bipolar fuzzy relation on

$$B = (\mu_B^P, \mu_B^N) \text{ if } \mu_B^P(x, y) \le \min(\mu_A^P(x), \mu_A^P(y))$$

and

$$\mu_B^N(x, y) \ge \max(\mu_A^N(x), \mu_A^N(y) \text{ for all } x, y \in X.$$

A bipolar fuzzy relation *A* on *X* is called symmetric if $\mu_A^P(x, y) = \mu_A^P(y, x)$ and $\mu_A^N(x, y) = \mu_A^N(y, x)$ for all *x*, *y* ∈ *X*.

Definition 2.16 [36]. A bipolar fuzzy graph of a graph G^*= (V, E) is a pair G = (A,B), where A = (μ_A^P, μ_A^N) is a bipolar fuzzy set in V and B = (μ_B^P, μ_B^N) is a bipolar fuzzy set on E ⊆ V x V such that μ_B^P (xy) ≤ min{ μ_A^P (x), μ_A^P (y)} for all xy ∈ E , μ_B^N (xy) ≥

$\min\{\mu_A^N(x), \mu_A^N(y)\}$ for all xy $\in E$ and $\mu_B^P(xy) = \mu_B^N(xy) = 0$ for all xy $\in \tilde{V}^2$ –E. Here A is called bipolar fuzzy vertex set of V, B the bipolar fuzzy edge set of E.

Definition 2.17 [46] A single valued neutrosophic graph (SVNG) of a graph G^*= (V, E) is a pair G = (A, B), where

1.V= $\{v_1, v_2,\ldots, v_n\}$ such that T_A:V→[0, 1], I_A:V→[0, 1] and F_A:V→[0, 1] denote the degree of truth-membership, degree of indeterminacy-membership and falsity-membership of the element $v_i \in$ V, respectively, and

$$0\leq T_A(v_i) + I_A(v_i) + F_A(v_i) \leq 3$$

for every $v_i \in$ V (i=1, 2, …,n)

2. E ⊆ V x V where T_B:V x V →[0, 1], I_B:V x V →[0, 1] and F_B:V x V →[0, 1] are such that

$$T_B(v_i, v_j) \leq \min\,[T_A(v_i), T_A(v_j)],\ I_B(v_i, v_j) \geq \max\,[I_A(v_i), I_A(v_j)]$$

and

$$F_B(v_i, v_j) \geq \max\,[F_A(v_i), F_A(v_j)]$$

and

$$0\leq T_B(v_i, v_j) + I_B(v_i, v_j) + F_B(v_i, v_j) \leq 3$$

for every $(v_i, v_j) \in$ E (i, j = 1, 2,…, n)

Definition 2.18 [46]: Let G=(V, E) be a single valued neutrosophic graph. Then the degree of a vertex v is defined by d(v)= $(d_T(v), d_I(v), d_F(v))$ where

$$d_T(v)=\sum_{u\neq v} T_B(u, v),\ d_I(v)=\sum_{u\neq v} I_B(u, v) \text{ and } d_F(v)=\sum_{u\neq v} F_B(u, v)$$

3. Bipolar Single Valued Neutrosophic Graph

Definition 3.1. Let X be a non-empty set. Then we call a mapping A = (x, T^P(x), I^P(x), F^P(x), T^N(x), I^N(x), F^N(x)):X × X → [−1, 0] × [0, 1] a bipolar single valued neutrosophic relation on X such that T_A^P(x, y) ∈ [0, 1], I_A^P(x, y) ∈ [0, 1], F_A^P(x, y) ∈ [0, 1], and T_A^N(x, y) ∈ [−1, 0], I_A^N(x, y) ∈ [−1, 0], F_A^N(x, y) ∈ [−1, 0].

Definition 3.2. Let $A = (T_A^P, I_A^P, F_A^P, T_A^N, I_A^N, F_A^N)$ and $B = (T_B^P, I_B^P, F_B^P, T_B^N, I_B^N, F_B^N)$ be bipolar single valued neutrosophic graph on a set *X*. If $B = (T_B^P, I_B^P, F_B^P, T_B^N, I_B^N, F_B^N)$ is a bipolar single valued neutrosophic relation on $A = (T_A^P, I_A^P, F_A^P, T_A^N, I_A^N, F_A^N)$ then

$T_B^P(x, y) \leq \min(T_A^P(x), T_A^P(y))$, $T_B^N(x, y) \geq \max(T_A^N(x), T_A^N(y))$

$I_B^P(x, y) \geq \max(I_A^P(x), I_A^P(y))$, $I_B^N(x, y) \leq \min(I_A^N(x), I_A^N(y))$

$F_B^P(x, y) \geq \max(F_A^P(x), F_A^P(y))$, $F_B^N(x, y) \leq \min(F_A^N(x), F_A^N(y))$ for all $x, y \in X$.

A bipolar single valued neutrosophic relation *B* on *X* is called symmetric if

$$T_B^P(x, y) = T_B^P(y, x),\ I_B^P(x, y) = I_B^P(y, x),\ F_B^P(x, y) = F_B^P(y, x)$$

and

$$T_B^N(x, y) = T_B^N(y, x),\ I_B^N(x, y) = I_B^N(y, x),\ F_B^N(x, y) = F_B^N(y, x),\ \text{for all } x, y \in X.$$

Definition 3.3. A bipolar single valued neutrosophic graph of a graph $G^*=$ (V, E) is a pair G = (A, B), where A = ($T_A^P, I_A^P, F_A^P, T_A^N, I_A^N, F_A^N$) is a bipolar single valued neutrosophic set in V and B = ($T_B^P, I_B^P, F_B^P, T_B^N, I_B^N, F_B^N$) is a bipolar single valued neutrosophic set in $\tilde{V}^2$ such that

$$T_B^P(\boldsymbol{v_i}, \boldsymbol{v_j}) \le \min(T_A^P(\boldsymbol{v_i}), T_A^P(\boldsymbol{v_j}))$$
$$I_B^P(\boldsymbol{v_i}, \boldsymbol{v_j}) \ge \max(I_A^P(\boldsymbol{v_i}), I_A^P(\boldsymbol{v_j}))$$
$$F_B^P(\boldsymbol{v_i}, \boldsymbol{v_j}) \ge \max(F_A^P(\boldsymbol{v_i}), F_A^P(\boldsymbol{v_j}))$$

and

$$T_B^N(\boldsymbol{v_i}, \boldsymbol{v_j}) \ge \max(T_A^N(\boldsymbol{v_i}), T_A^N(\boldsymbol{v_j}))$$
$$I_B^N(\boldsymbol{v_i}, \boldsymbol{v_j}) \le \min(I_A^N(\boldsymbol{v_i}), I_A^N(\boldsymbol{v_j}))$$
$$F_B^N(\boldsymbol{v_i}, \boldsymbol{v_j}) \le \min(F_A^N(\boldsymbol{v_i}), F_A^N(\boldsymbol{v_j}))$$

for all $\boldsymbol{v_i v_j} \in \tilde{V}^2$.

Notation : An edge of BSVNG is denoted by $e_{ij} \in$ E or $\boldsymbol{v_i v_j} \in$ E

Here the sextuple (v_i, $T_A^P(v_i)$, $I_A^P(v_i)$, $F_A^P(v_i)$, $T_A^N(v_i)$, $I_A^N(v_i)$, $F_A^N(v_i)$) denotes the positive degree of truth-membership, the positive degree of indeterminacy-membership, the positive degree of falsity-membership, the negative degree of truth-membership, the negative degree of indeterminacy-membership, the negative degree of falsity- membership of the vertex vi.

The sextuple (e_{ij}, T_B^P, I_B^P, F_B^P, T_B^N, I_B^N, F_B^N) denotes the positive degree of truth-membership, the positive degree of indeterminacy-membership, the positive degree of falsity-membership, the negative degree of truth-membership, the negative degree of indeterminacy-membership, the negative degree of falsity- membership of the edge relation e_{ij} = ($\boldsymbol{v_i}$, $\boldsymbol{v}_j$) on V× V.

Note 1. (i) When $T_A^P = I_A^P = F_A^P = 0$ and $T_A^N = I_A^N = F_A^N = 0$ for some i and j, then there is no edge between v_i and v_j .

Otherwise there exists an edge between v_i and v_j .

(ii) If one of the inequalities is not satisfied in (1) and (2), then G is not an BSVNG

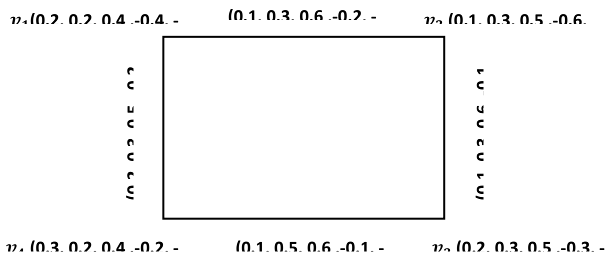


Figure 1: Bipolar single valued neutrosophic graph.

Proposition 3.5: A bipolar single valued neutrosophic graph is the generalization of fuzzy graph

Proof: Suppose G= (A, B) be a bipolar single valued neutrosophic graph. Then by setting the positive indeterminacy-membership, positive falsity-membership and negative truth-membership, negative indeterminacy-membership, negative falsity-membership values of vertex set and edge set equals to zero reduces the bipolar single valued neutrosophic graph to fuzzy graph.

Example 3.6:

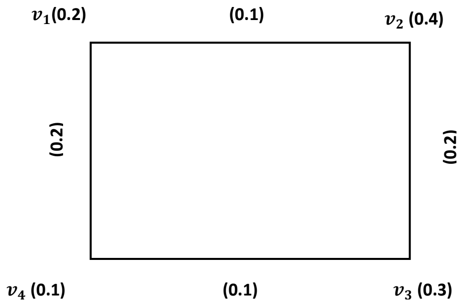
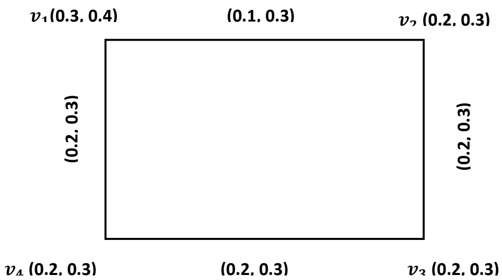


Figure 2**: Fuzzy graph**

Proposition 3.7: A bipolar single valued neutrosophic graph is the generalization of intuitionistic fuzzy graph

Proof: Suppose G= (A, B) be a bipolar single valued neutrosophic graph. Then by setting the positive indeterminacy-membership, negative truth-membership, negative indeterminacy-membership, negative falsity-membership values of vertex set and edge set equals to zero reduces the bipolar single valued neutrosophic graph to intuitionistic fuzzy graph.

Example 3.8



Figur 3: Intuitionistic fuzzy graph

Proposition 3.9: A bipolar single valued neutrosophic graph is the generalization of single valued neutrosophic graph

Proof: Suppose G= (A, B) be a bipolar single valued neutrosophic graph. Then by setting the negative truth-membership, negative indeterminacy-membership, negative falsity-membership values of vertex set and edge set equals to zero reduces the bipolar single valued neutrosophic graph to single valued neutrosophic graph.

Example 3.10

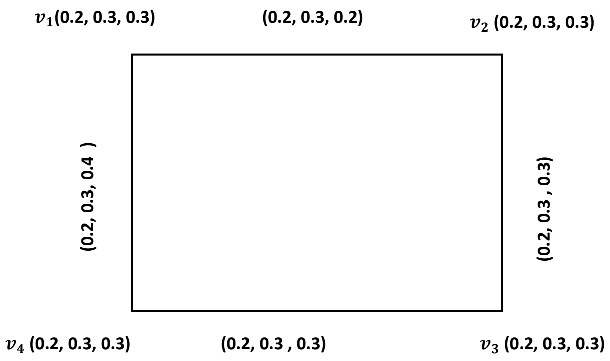


Figure 4: Single valued neutrosophic graph

Proposition 3.11: A bipolar single valued neutrosophic graph is the generalization of bipolar intuitionstic fuzz graph

Proof: Suppose G= (A, B) be a bipolar single valued neutrosophic graph. Then by setting the positive indeterminacy-membership, negative indterminacy-membership values of vertex set and edge set equals to zero reduces the bipolar single valued neutrosophic graph to bipolar intuitionstic fuzzy graph

Example 3.12

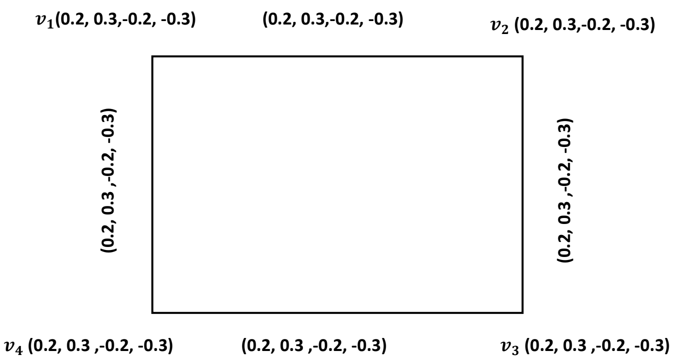


Figure 5: Bipolar intuitionistic fuzzy graph.

Proposition 3.13: A bipolar single valued neutrosophic graph is the generalization of *N*-graph

Proof: Suppose G= (A, B) be a bipolar single valued neutrosophic graph. Then by setting the positive degree membership such truth-membership, indeterminacy- membership, falsity-membership and negative indeterminacy-membership, negative falsity-membership values of vertex set and edge set equals to zero reduces the single valued neutrosophic graph to *N*-graph.

Example 3.14:

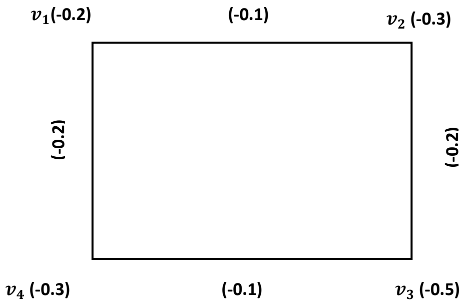


Figure 6: *N*- graph

Definition 3.15. A bipolar single valued neutrosophic graph that has neither self loops nor parallel edge is called simple bipolar single valued neutrosophic graph.

Definition 3.16. A bipolar single valued neutrosophic graph is said to be connected if every pair of vertices has at least one bipolar single valued neutrosophic graph between them, otherwise it is disconnected.

Definition 3.17. When a vertex $\mathbf{v_i}$ is end vertex of some edges $(\mathbf{v_i}, \mathbf{v_j})$ of any BSVN-graph G= (A, B). Then $\mathbf{v_i}$ and $(\mathbf{v_i}, \mathbf{v_j})$ are said to be **incident** to each other.

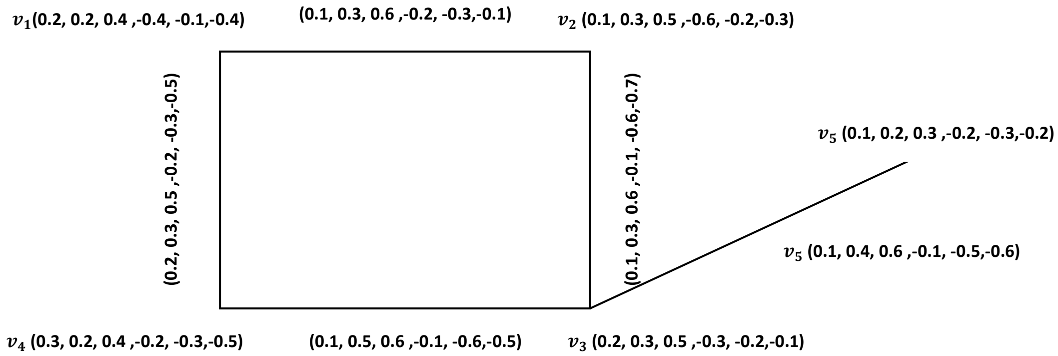


Figure 7 : Incident BSVN-graph

In this graph v_2v_3, v_3v_4 and v_3v_5 are incident on v_3.

Definition 3.18 Let G= (V, E) be a bipolar single valued neutrosophic graph. Then the degree of any vertex **v** is sum of positive degree of truth-membership, positive sum of degree of indeterminacy-membership, positive sum of degree of falsity-membership, negative degree of truth-membership, negative sum of degree of indeterminacy-membership, and negative sum of degree of falsity-membership of all those edges which are incident on vertex **v** denoted by d(v)= $(d_T^P(v), d_I^P(v), d_F^P(v), d_T^N(v), d_I^N(v), d_F^N(v))$ where

$d_T^P(v) = \sum_{u \neq v} T_B^P(u,v)$ denotes the positive T- degree of a vertex v,

$d_I^P(v) = \sum_{u \neq v} I_B^P(u,v)$ denotes the positive I- degree of a vertex v,

$d_F^P(v) = \sum_{u \neq v} F_B^P(u,v)$ denotes the positive F- degree of a vertex v,

$d_T^N(v) = \sum_{u \neq v} T_B^N(u,v)$ denotes the negative T- degree of a vertex v,

$d_I^N(v) = \sum_{u \neq v} I_B^N(u,v)$ denotes the negative I- degree of a vertex v,

$d_F^N(v) = \sum_{u \neq v} F_B^N(u, v)$ denotes the negative F- degree of a vertex v

Definition 3.19: The minimum degree of G is

$$\delta(G) = (\delta_T^P(G), \delta_I^P(G), \delta_F^P(G), \delta_T^N(G), \delta_I^N(G), \delta_F^N(G))$$

where

$\delta_T^P(G) = \wedge \{d_T^P(v) \mid v \in V\}$ denotes the minimum positive T- degree,
$\delta_I^P(G) = \wedge \{d_I^P(v) \mid v \in V\}$ denotes the minimum positive I- degree,
$\delta_F^P(G) = \wedge \{d_F^P(v) \mid v \in V\}$ denotes the minimum positive F- degree,
$\delta_T^N(G) = \wedge \{d_T^N(v) \mid v \in V\}$ denotes the minimum negative T- degree,
$\delta_I^N(G) = \wedge \{d_I^N(v) \mid v \in V\}$ denotes the minimum negative I- degree,
$\delta_F^N(G) = \wedge \{d_F^N(v) \mid v \in V\}$ denotes the minimum negative F- degree

Definition 3.20: The maximum degree of G is

$$\Delta(G) = (\Delta_T^P(G), \Delta_I^P(G), \Delta_F^P(G), \Delta_T^N(G), \Delta_I^N(G), \Delta_F^N(G))$$

where

$\Delta_T^P(G) = \vee \{d_T^P(v) \mid v \in V\}$ denotes the maximum positive T- degree,
$\Delta_I^P(G) = \vee \{d_I^P(v) \mid v \in V\}$ denotes the maximum positive I- degree,
$\Delta_F^P(G) = \vee \{d_F^P(v) \mid v \in V\}$ denotes the maximum positive F- degree,
$\Delta_T^N(G) = \vee \{d_T^N(v) \mid v \in V\}$ denotes the maximum negative T- degree,
$\Delta_I^N(G) = \vee \{d_I^N(v) \mid v \in V\}$ denotes the maximum negative I- degree,
$\Delta_F^N(G) = \vee \{d_F^N(v) \mid v \in V\}$ denotes the maximum negative F- degree

Example 3.21. Let us consider a bipolar single valued neutrosophic graph G= (A, B) of G^* = (V, E), such that V = $\{v_1, v_2, v_3, v_4\}$, E = $\{(v_1, v_2), (v_2, v_3), (v_3, v_4), (v_4, v_1)\}$

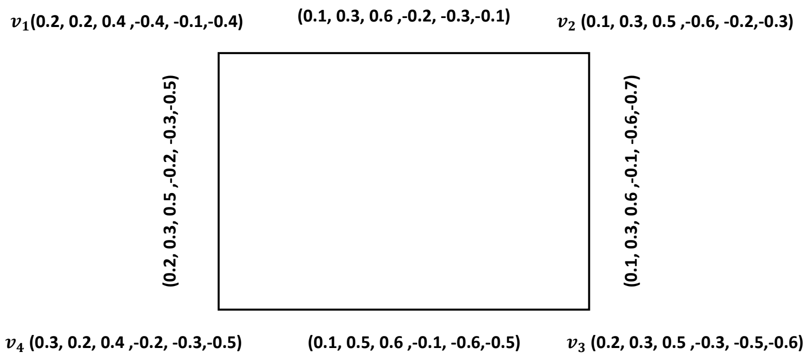


Figure 8: Degree of a bipolar single valued neutrosophic graph G.

In this example, the degree of v_1 is (0.3, 0.6, 1.1, -0.4, -0.6, -0.6). the degree of v_2 is (0.2, 0.6, 1.2, -0.3, -0.9, -0.8). the degree of v_3 is (0.2, 0.8, 1.2, -0.2, -1.2, -1.2). the degree of v_4 is (0.3, 0.8, 1.1, -0.3, -0.9, -1)

Order and size of a bipolar single valued neutrosophic graph is an important term in bipolar single valued neutrosophic graph theory. They are defined below.

Definition 3.22: Let G =(V, E) be a BSVNG. The order of G, denoted O(G) is defined as O(G)= $(O_T^p(G), O_I^p(G), O_F^p(G), O_T^N(G), O_I^N(G), O_F^N(G))$, where

$O_T^p(G) = \sum_{v \in V} T_1^p(v)$ denotes the positive T- order of a vertex v,
$O_I^p(G) = \sum_{v \in V} I_1^p(v)$ denotes the positive I- order of a vertex v,
$O_F^p(G) = \sum_{v \in V} F_1^p(v)$ denotes the positive F- order of a vertex v,
$O_T^N(G) = \sum_{v \in V} T_1^N(v)$ denotes the negative T- order of a vertex v,
$O_I^N(G) = \sum_{v \in V} I_1^N(v)$ denotes the negative I- order of a vertex v,
$O_F^N(G) = \sum_{v \in V} F_1^N(v)$ denotes the negative F- order of a vertex v.

Definition 3.23: Let G =(V, E) be a BSVNG. The size of G, denoted S(G) is defined as S(G)= ($S_T^p(G)$, $S_I^p(G)$, $S_F^p(G)$, $S_T^N(G)$, $S_I^N(G)$, $S_F^N(G)$), where

$S_T^p(G) = \sum_{u \neq v} T_2^p(u, v)$ denotes the positive T- size of a vertex v,
$S_I^p(G) = \sum_{u \neq v} I_2^p(u, v)$ denotes the positive I- size of a vertex v,
$S_F^p(G) = \sum_{u \neq v} F_2^p(u, v)$ denotes the positive F- size of a vertex v,
$S_T^N(G) = \sum_{u \neq v} T_2^N(u, v)$ denotes the negative T- size of a vertex v,
$S_I^N(G) = \sum_{u \neq v} I_2^N(u, v)$ denotes the negative I- size of a vertex v,
$S_F^N(G) = \sum_{u \neq v} F_2^N(u, v)$ denotes the negative F- size of a vertex v.

Definition 3.24 A bipolar single valued neutrosophic graph G=(V, E) is called constant if degree of each vertex is k =(k_1, k_2, k_3, k_4, k_5, k_6). That is, d(v) = (k_1, k_2, k_3, k_4, k_5, k_6) for all $v \in$ V.

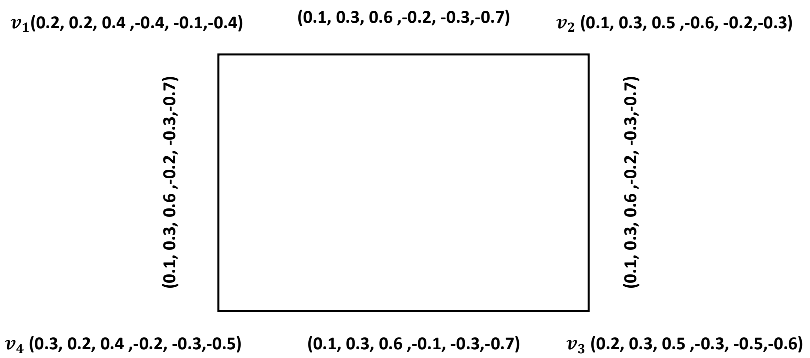


Figure 9: Constant bipolar single valued neutrosophic graph G.

In this example, the degree of v_1, v_2, v_3, v_4 is (0.2, 0.6, 1.2, -0.4, -0.6, -1.4).

O(G)=(0.8, 1, 1.8, -1.5, -1.1, -1.8)
S(G) =(0.4, 1.2, 2.4 ,-0.7, -1.2, -2.8)

Remark 3.25. G is a (k_i, k_j, k_l, k_m, k_n, k_o)-constant BSVNG iff δ = Δ= k, where k = k_i+ k_j+ k_l +k_m+ k_n+ k_o.

Definition 3.26. A bipolar single valued neutrosophic graph G= (A, B) is called strong bipolar single valued neutrosophic graph if

$$T_B^P(u, v) = \min(T_A^P(u), T_A^P(v)),$$
$$I_B^P(u, v) = \max(I_A^P(u), I_A^P(v)),$$
$$F_B^P(u, v) = \max(F_A^P(u), F_A^P(v)),$$
$$T_B^N(u, v) = \max(T_A^N(u), T_A^N(v)),$$

$$I_B^N(u,v) = \min(I_A^N(u), I_A^N(v)),$$
$$F_B^N(u,v) = \min(F_A^N(u), F_A^N(v))$$

for all (u, v) ∈ E

Example 3.27. Consider a strong BSVN-graph G such that V = $\{v_1, v_2, v_3, v_4\}$and E = $\{(v_1, v_2),(v_2, v_3), (v_3, v_4), (v_4, v_1)\}$

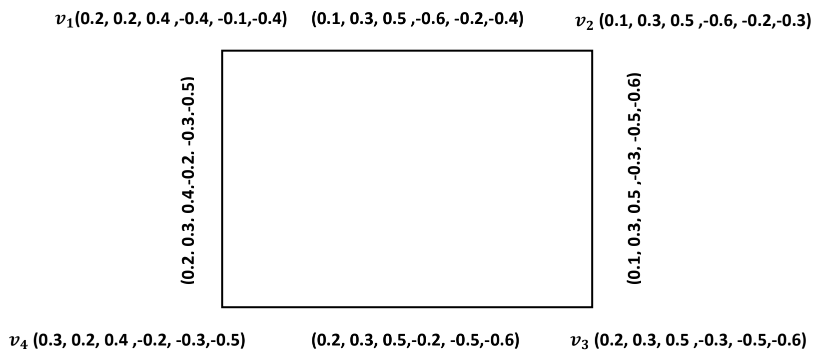


Figure 10: Strong bipolar single valued neutrosophic graph G.

Definition 3.28. A bipolar single valued neutrosophic graph G= (A, B) is called complete if

$$T_B^P(u,v) = \min(T_A^P(u), T_A^P(v)),$$
$$I_B^P(u,v) = \max(I_A^P(u), I_A^P(v)),$$
$$F_B^P(u,v) = \max(F_A^P(u), F_A^P(v)) ,$$
$$T_B^N(u,v) = \max(T_A^N(u), T_A^N(v)),$$
$$I_B^N(u,v) = \min(I_A^N(u), I_A^N(v)),$$
$$F_B^N(u,v) = \min(F_A^N(u), F_A^N(v))$$

for all u, v ∈ V.

Example 3.29. Consider a complete BSVN-graph G such that V = $\{v_1, v_2, v_3, v_4\}$and E = $\{(v_1, v_2),(v_2, v_3), (v_3, v_4), (v_4, v_1), (v_1, v_3), (v_2, v_4)\}$

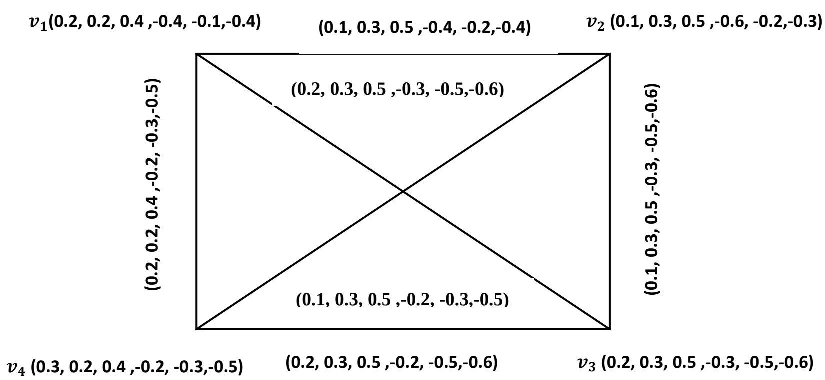


Figure 11: Complete bipolar single valued neutrosophic graph G.

$d(\boldsymbol{v_1})$= (0.5, 0.8, 1.4, -0 .9, -1, -1.5)
$d(\boldsymbol{v_2})$= (0.4, 0.9, 1.5, -1 .2, -1, -1.6)
$d(\boldsymbol{v_3})$= (0.4, 0.9, 1.5, -0 .7, -1.3, -1.7)

$$d(\boldsymbol{v_4})= (0.5, 0.8, 1.4, -0.6, -1.1, -1.6)$$

Definition 3.30. The complement of a bipolar single valued neutrosophic graph G = (A, B) of a graph G^*= (V, E) is a bipolar single valued neutrosophic graph $\bar{G}$ = ($\bar{A}$, $\bar{B}$) of $\overline{G^*}$ = (V,V ×V), where $\bar{A}$ = A = ($T_A^P, I_A^P, F_A^P, T_A^N, I_A^N, F_A^N$) and $\bar{B}$ = ($\overline{T_B^P}, \overline{I_B^P}, \overline{F_B^P}, \overline{T_B^N}, \overline{I_B^N}, \overline{F_B^N}$) is defined by

$$\bar{T}_B^P(u,v) = \min\big(T_A^P(u), T_A^P(v)\big) - T_B^P(u,v) \text{ for all } u,v \in V,\ uv \in \tilde{V}^2$$
$$\bar{I}_B^P(u,v) = \max\big(I_A^P(u), I_A^P(v)\big) - I_B^P(u,v) \text{ for all } u,v \in V,\ uv \in \tilde{V}^2$$
$$\bar{F}_B^P(u,v) = \max\big(F_A^P(u), F_A^P(v)\big) - F_B^P(u,v) \text{ for all } u,v \in V,\ uv \in \tilde{V}^2$$
$$\bar{T}_B^N(u,v) = \max\big(T_A^N(u), T_A^N(v)\big) - T_B^N(u,v) \text{ for all } u,v \in V,\ uv \in \tilde{V}^2$$
$$\bar{I}_B^N(u,v) = \min\big(I_A^N(u), I_A^N(v)\big) - I_B^N(u,v) \text{ for all } u,v \in V,\ uv \in \tilde{V}^2$$
$$\bar{F}_B^N(u,v) = \min\big(F_A^N(u), F_A^N(v)\big) - F_B^N(u,v) \text{ for all } u,v \in V,\ uv \in \tilde{V}^2$$

Proposition 3.31: The complement of complete BSVN-graph is a BSVN-graph with no edge. Or if G is a complete then in $\bar{G}$ the edge is empty.

Proof. Let G= (V, E) be a complete BSVN-graph . $T_B^P(u,v)$ =min($T_A^P(u)$, $T_A^P(v)$),
So $T_B^P(u,v)$ =min($T_A^P(u)$, $T_A^P(v)$), $T_B^N(u,v)$ =max($T_A^N(u)$, $T_A^N(v)$),

$$I_B^P(u,v) = \max(T_A^P(u), T_A^P(v)), \quad I_B^N(u,v) = \min(I_A^N(u), I_A^N(v)),$$
$$F_B^P(u,v) = \max(T_A^P(u), T_A^P(v)), \quad F_B^N(u,v) = \min(F_A^N(u), F_A^N(v)$$

for all $u,v \in$ V. Hence in $\bar{G}$,

$$\begin{aligned}\bar{T}_B^P &= \min\big(T_A^P(u), T_A^P(v)\big) - T_B^P(u,v) \text{ for all } u,v \in V\\ &= \min\big(T_A^P(u), T_A^P(v)\big) - \min\big(T_A^P(u), T_A^P(v)\big) \text{ for all } u,v \in V\\ &= 0 \quad \text{for all } u,v \in V\end{aligned}$$

and

$$\begin{aligned}\bar{I}_B^P &= \max\big(I_A^P(u), I_A^P(v)\big) - I_B^P(u,v) \text{ for all } u,v \in V\\ &= \max\big(I_A^P(u), I_A^P(v)\big) - \max\big(I_A^P(u), I_A^P(v)\big) \text{ for all } u,v \in V\\ &= 0 \quad \text{for all } u,v \in V\end{aligned}$$

Also

$$\begin{aligned}\bar{F}_B^P &= \max\big(F_A^P(u), F_A^P(v)\big) - F_B^P(u,v) \text{ for all } u,v \in V\\ &= \max\big(F_A^P(u), F_A^P(v)\big) - \max\big(F_A^P(u), F_A^P(v)\big) \text{ for all } u,v \in V\\ &= 0 \quad \text{for all } u,v \in V\end{aligned}$$

Similarly

$$\begin{aligned}\bar{T}_B^N &= \max\big(T_A^N(u), T_A^N(v)\big) - T_B^N(u,v) \text{ for all } u,v \in V\\ &= \max\big(T_A^N(u), T_A^N(v)\big) - \max\big(T_A^N(u), T_A^N(v)\big) \text{ for all } u,v \in V\\ &= 0 \quad \text{for all } u,v \in V\end{aligned}$$

and

$\bar{I}_B^P$= $\min\left(I_A^N(u), I_A^N(v)\right)$ - $I_B^N(u,v)$ for all $u, v \in$ V
= $\min\left(I_A^N(u), I_A^N(v)\right)$ - $\min\left(I_A^N(u), I_A^N(v)\right)$ for all $u, v \in$ V
= 0 for all $u, v \in$ V

Also

$\bar{F}_B^N$= $\min\left(F_A^N(u), F_A^N(v)\right)$ - $F_B^N(u,v)$ for all $u, v \in$ V
= $\min\left(F_A^N(u), F_A^N(v)\right)$ - $\min\left(F_A^N(u), F_A^N(v)\right)$ for all $u, v \in$ V
= 0 for all $u, v \in$ V

($\bar{T}_B^P$, $\bar{I}_B^P$, $\bar{F}_B^P$, $\bar{T}_B^N$, $\bar{I}_B^N$, $\bar{F}_B^N$). Thus ($\bar{T}_B^P$, $\bar{I}_B^P$, $\bar{F}_B^P$, $\bar{T}_B^N$, $\bar{I}_B^N$, $\bar{F}_B^N$)= (0 , 0, 0, 0, 0). Hence the edge set of $\bar{G}$ is empty if G is a complete BSVNG.

Definition 3.32: A regular BSVN-graph is a BSVN-graph where each vertex has the same number of open neighbors degree. d_N(v)= ($d_{NT}^P(v)$, $d_{NI}^P(v)$, $d_{NF}^P(v)$, $d_{NT}^N(v)$, $d_I^N(v)$, $d_{NF}^N(v)$).

The following example shows that there is no relationship between regular BSVN-graph and a constant BSVN-graph

Example 3.33. Consider a graph G^* such that V= $\{v_1, v_2, v_3, v_4\}$, E=$\{v_1v_2, v_2v_3, v_3v_4, v_4v_1\}$. Let A be a single valued neutrosophic subset of V and le B a single valued neutrosophic subset of E denoted by

	v_1	v_2	v_3	v_4
T_A^P	0.2	0.2	0.2	0.2
I_A^P	0.2	0.2	0.2	0.2
F_A^P	0.4	0.4	0.4	0.4
T_A^N	-0.4	-0.4	-0.4	-0.4
I_A^N	-0.1	-0.4	-0.1	-0.1
F_A^N	-0.4	-0.4	-0.4	-0.4

	v_1v_2	v_2v_3	v_3v_4	v_4v_1
T_B^P	0.1	0.1	0.1	0.2
I_B^P	0.3	0.3	0.5	0.3
F_B^P	0.6	0.6	0.6	0.5
T_B^N	-0.2	-0.1	-0.1	-0.2
I_B^N	-0.3	-0.6	-0.6	-0.3
F_B^N	-0.5	-0.7	-0.7	-0.5

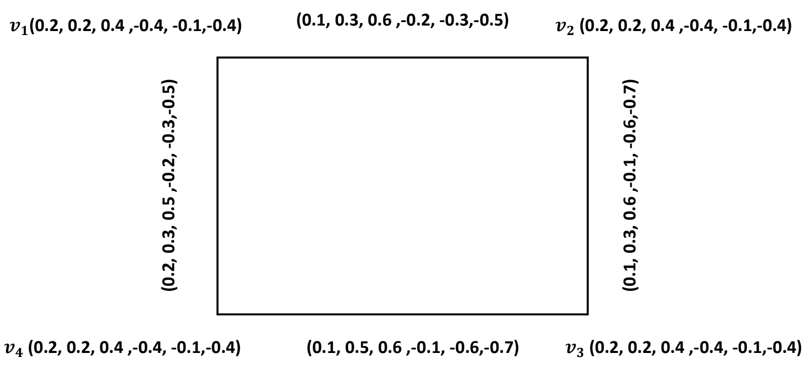


Figure 12: Regular bipolar single valued neutrosophic graph G.

By routing calculations show that G is regular BSVN-graph since each open neighbors degree is same , that is (0.4, 0.4, 0.8 ,-0.8, -0.2,-0.8). But it is not constant BSVN-graph since degree of each vertex is not same.

Definition 3.34 : Let G=(V, E) be a bipolar single valued neutrosophic graph. Then the totally degree of a vertex v ∈ V is defined by

td(v)= $(td_T^P(v), td_I^P(v), td_F^P(v), td_T^N(v), td_I^N(v), td_F^N(v))$ where
$td_T^P(v) = \sum_{u \neq v} T_B^P(u, v) + T_A^P(v)$ denotes the totally positive T- degree of a vertex v,
$td_I^P(v) = \sum_{u \neq v} I_B^P(u, v) + I_A^P(v)$ denotes the totally positive I- degree of a vertex v,
$td_F^P(v) = \sum_{u \neq v} F_B^P(u, v) + F_A^P(v)$ denotes the totally positive F- degree of a vertex v,
$td_T^N(v) = \sum_{u \neq v} T_B^N(u, v) + T_A^N(v)$ denotes the totally negative T- degree of a vertex v,
$td_I^N(v) = \sum_{u \neq v} I_B^N(u, v) + I_A^N(v)$ denotes the totally negative I- degree of a vertex v,
$td_F^N(v) = \sum_{u \neq v} F_B^N(u, v) + F_A^N(v)$ denotes the totally negative F- degree of a vertex v

If each vertex of G has totally same degree **m=(**m_1, m_2, m_3, m_4, m_5, m_6**),** then G is called a **m**-totally constant BSVN-Graph.

Example 3.35. Let us consider a bipolar single valued neutrosophic graph G= (A, B) of G^* = (V, E), such that V = {v_1, v_2, v_3, v_4}, E = {(v_1, v_2),(v_2, v_3), (v_3, v_4), (v_4, v_1)}

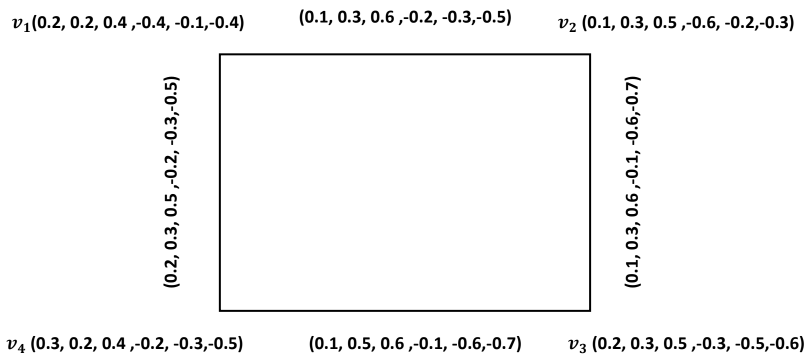


Figure 13: Totally degree of a bipolar single valued neutrosophic graph G.

In this example, the totally degree of v_1 is (0.5, 0.8, 1.4, -0.8, -0.7, -1.4). The totally degree of v_2 is (0.3, 0.9, 1.7, -0.9, -1.1, -1.5). The totally degree of v_3 is (0.4, 1.1, 1.7, -0.5, -1.7, -2). The totally degree of v_4 is (0.6, 1, 1.5, -0.5, -1.1, -1,7).

Definition 3.36: A totally regular BSVN-graph is a BSVN-graph where each vertex has the same number of closed neighbors degree, it is noted d[v].

Example 3.37. Let us consider a BSVN-graph G= (A, B) of G^* = (V, E), such that V = {v_1, v_2, v_3, v_4} and E = {(v_1, v_2),(v_2, v_3), (v_3, v_4), (v_4, v_1)}

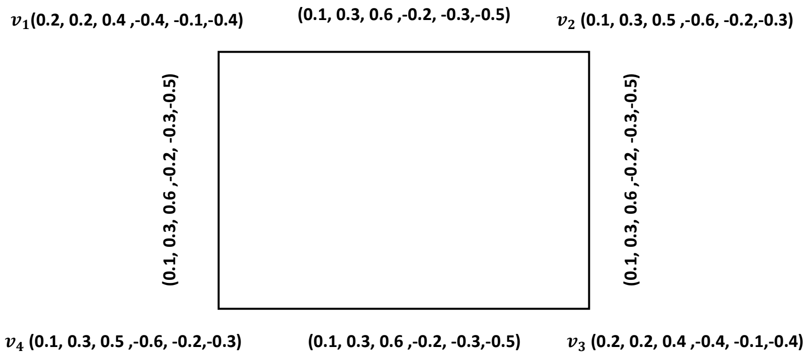


Figure 14: Degree of a bipolar single valued neutrosophic graph G.

By routing calculations we show that G is regular BSVN-graph since the degree of v_1, v_2, v_3, and v_4 is (0.2, 0.6, 1.2 , -0.4, -0.6, -1). It is neither totally regular BSVN-graph not constant BSVN-graph.

4. Conclusion

In this paper, we have introduced the concept of bipolar single valued neutrosophic graphs and described degree of a vertex, order, size of bipolar single valued neutrosophic graphs, also we have introduced the notion of complement of a bipolar single valued neutrosophic graph, strong bipolar single valued neutrosophic graph, complete bipolar single valued neutrosophic graph, regular bipolar single valued neutrosophic graph. Further, we are going to study some types of single valued neutrosophic graphs such irregular and totally irregular single valued neutrosophic graphs and bipolar single valued neutrosophic graphs.

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http://www.newtheory.org ISSN: 2149-1402

Journal of New Theory

Year: 2016, Number: 11, Pages: 103

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