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SYMMETRIC IDENTITIES INVOLVING CARLITZ'S-TYPE TWISTED (*h,q*)-TANGENT-TYPE POLYNOMIALS UNDER S_5

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Abstract – In [11], Ryoo introduced the Carlitz's-type twisted *(h,q)*-Tangent numbers and polynomials. In this paper, we consider some new symmetric identities involving Ryoo's Carlitz's-type twisted *(h,q)* Tangent-type polynomials arising from the fermionic *p*-adic invariant integral on $\mathbb{Z}_p$ under S_5 termed symmetric group of degree five.

Keywords ***–*** *Symmetric identities; Carlitz's-type twisted (h,q)-Tangent-type polynomials; Fermionic p-adic invariant integral on* $\mathbb{Z}_p$*; Invariant under* S_5.

1 Introduction

In the complex plane, the Euler polynomials are defined by

$$\sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!} = \frac{2}{e^t+1}e^{xt}, \quad (|t| < \pi).$$

When $x = 0$, then we get $E_n(0) := E_n$ is called the n-th Euler numbers, see [5], [7], [14]. As well-known that the Tangent numbers T_{2n-1} $(n \geq 1)$ are defined as the coefficients of the Taylor expansion of $\tan x$:

$$\tan x = \sum_{n=1}^{\infty} \frac{T_{2n-1}}{(2n-1)!} x^{2n-1} \quad (\text{see } [10,14]).$$

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Kim *et al.* [10] obtained the following relation between Tangent numbers and Euler numbers:

$$E_{2n-1} = (-1)^n \frac{T_{2n-1}}{2^{2n-1}}. \tag{1.1}$$

Ryoo [14] introduced Tangent-type polynomial $\mathcal{T}_n(x)$ which is different from original definition, as follows:

$$\sum_{n=0}^{\infty} \mathcal{T}_n(x) \frac{t^n}{n!} = \frac{2}{e^{2t}+1} e^{xt}, \quad (|t| < \frac{\pi}{2}). \tag{1.2}$$

Letting $x = 0$ in the Eq. (1.2) reduces to $\mathcal{T}_n(0) := \mathcal{T}_n$ that is called n-th Tangent-type number (see, e.g., [11], [14]).

Ryoo's Tangent polynomial holds the following equality (see [14])

$$E_{2n-1} = \frac{\mathcal{T}_{2n-1}}{2^{2n-1}}. \tag{1.3}$$

Note that the Eq. (1.3) is different from the Eq. (1.1). Further we have

$$\mathcal{T}_{2n-1} = (-1)^n T_{2n-1}. \tag{1.4}$$

Because of (1.4), we call $\mathcal{T}_n(x)$ and $\mathcal{T}_n$ as Tangent-type polynomials and Tangent-type numbers, respectively.

Let p be chosen as a fixed odd prime number. Along this paper $\mathbb{Z}_p$, $\mathbb{Q}$, $\mathbb{Q}_p$ and $\mathbb{C}_p$ will denote topological closure of $\mathbb{Z}$, the field of rational numbers, topological closure of $\mathbb{Q}$ and the field of p-adic completion of an algebraic closure of $\mathbb{Q}_p$, respectively. Let $\mathbb{N} = \{1, 2, 3, \cdots\}$ and $\mathbb{N}^* = \mathbb{N} \cup \{0\}$.

For d an odd positive number with $(d, p) = 1$, let

$$X := X_d = \lim_{\overleftarrow{n}} \mathbb{Z}/dp^N\mathbb{Z} \text{ and } X_1 = \mathbb{Z}_p$$

and

$$t + dp^N\mathbb{Z}_p = \{x \in X \mid x \equiv t \left(\operatorname{mod} dp^N\right)\}$$

where $t \in \mathbb{Z}$ lies in $0 \le t < dp^N$. See, for more details, $[1-11]$.

The normalized absolute value according to the theory of p-adic analysis is given by $|p|_p = p^{-1}$. The notation "q" can be considered as an indeterminate, a complex number

$q \in \mathbb{C}$ with $|q|<1$, or a p-adic number $q \in \mathbb{C}_p$ with $|q-1|_p < p^{-1/(p-1)}$ and $q^x = \exp(x \log q)$ for $|x|_p \le 1$. It is always clear in the content of the paper.

For any x, let us introduce the following notation (see [1-14])

$$[x]_q = \frac{1-q^x}{1-q} \quad (q \neq 1) \tag{1.5}$$

known as q-number of x. Note that as $q \to 1$, the notation $[x]_q$ reduces to the x. For

$$f \in UD(\mathbb{Z}_p) = \{ f \mid g : \mathbb{Z}_p \to \mathbb{C}_p \text{ is uniformly differentiable function} \},$$

Kim [7] defined the *p*-adic invariant integral on $\mathbb{Z}_p$ as follows:

$$I_{-1}(f) = \int_{\mathbb{Z}_p} f(x) \, d\mu_{-1}(x) = \lim_{N \to \infty} \sum_{x=0}^{p^N - 1} f(x)(-1)^x. \tag{1.6}$$

From Eq. (1.6), we get

$$I_{-1}(f_n) = (-1)^n I_{-1}(f) + 2 \sum_{k=0}^{n-1} (-1)^{n-k-1} f(k)$$

where $f_n(x)$ means $f(x+n)$. For more details about the p-adic invariant integral on $\mathbb{Z}_p$, see the references, e.g., [5], [7], [11], [12], [13], [14].

Let $h \in \mathbb{Z}$ and $T_p = \bigcup_{N \ge 1} C_{p^N} = \lim_{N \to \infty} C_{p^N}$, where $C_{p^N} = \{ w : w^{p^N} = 1 \}$ is the cyclic group of order p^N. For $w \in T_p$, we indicate by $\phi_w : \mathbb{Z}_p \to C_p$ the locally constant function $x \to w^x$. For $q \in C_p$ with $|1-q|_p < 1$ and $w \in T_p$, the h-extension of Carlitz's-type twisted q-Tangent-type polynomials are defined by the following p-adic invariant integral on $\mathbb{Z}_p$, with respect to μ_{-1}, in [11]:

$$\int_{\mathbb{Z}_p} w^y q^{hy} [2y + x]_q^n \, d\mu_{-1}(y) = T_{n,q,w}^{(h)}(x) \quad (n \ge 0). \tag{1.7}$$

If we let $x = 0$ into the Eq. (1.7), we then have $T_{n,q,w}^{(h)}(0) := T_{n,q,w}^{(h)}$ called n-th h-extension of Carlitz's-type twisted q-Tangent-type number. These numbers can be generated by the following recurrence relation:

$$q^h w (q^2 T_{q,w}^{(h)} + [2]_q)^n + T_{n,q,w}^{(h)} = \begin{cases} 2, & \text{if } n = 0 \\ 0, & \text{if } n \neq 0 \end{cases}$$

with the usual convention about replacing $\left(T_{q,w}^{(h)}\right)^n$ by $T_{n,q,w}^{(h)}$.

When $q \to 1^-$ and $w = 1$ in the Eq. (1.7), it gives

$$T_{n,q,w}^{(h)}(x) \to T_n(x) := \int_{\mathbb{Z}_p} (2y+x)^n \, d\mu_{-1}(y).$$

Recently, symmetric identities on some special polynomials, e.g. Bernoulli polynomials, Euler polynomails, Genocchi polynomials etc., have been studied by many mathematicians. For instance, Agyuz *et al.* [1] obtained a further investigation for the q-Genocchi numbers and polynomials of higher order under third Dihedral group D_3 and established some closed formulae of the symmetric identities. They also established some known identities for the classical Genocchi numbers and polynomials by using fermionic p-adic q-integral on $\mathbb{Z}_p$. Duran *et al.* [2] investigated some new symmetric identities for q-Genocchi polynomials which are derived from the fermionic p-adic q-integral on $\mathbb{Z}_p$. Duran *et al.* [3] derived symmetric identities involving weighted q-Genocchi polynomials using the fermionic p-adic q-integral on $\mathbb{Z}_p$. Araci *et al.* [5] performed to get some new symmetric identities for q-Frobenius-Euler polynomials under symmetric group of degree five, which are derived from the fermionic p-adic q-integral over the p-adic numbers field. Kim *et al.* [9] introduced new symmetry identities for Carlitz's q-Bernoulli polynomials under symmetric group of degree five. Kim *et al.* [7] investigated some new properties of symmetry for the Carlitz's-type q-Euler polynomials invariant under the symmetric group of degree five. Kim [8] considered new properties of symmetry for the higher-order Carlitz's q-Bernoulli polynomials which derived from p-adic q-integral on $\mathbb{Z}_p$ under the symmetric group of degree five.

In the present paper, we investigate some not only new but also interesting identities for h-extension of Carlitz's-type twisted q-Tangent-type polynomials arising from the fermionic p-adic invariant integral on $\mathbb{Z}_p$ symmetric group of degree five.

2 Symmetric Identities Involving $T_{n,q,w}^{(h)}(x)$ under S_5

For $w_i \in \mathbb{N}$ with $w_i \equiv 1 (mod 2)$ with $i \in \{1,2,3,4,5\}$, by the Eqs. (1.6) and (1.7), we get

$$\int_{\mathbb{Z}_p} w^{w_1w_2w_3w_4y} q^{hw_1w_2w_3w_4y} \times e^{\left[w_1w_2w_3w_4 2y + w_1w_2w_3w_4w_5x + w_5w_4w_2w_3 i + w_5w_4w_1w_3 j + w_5w_4w_1w_2 k + w_5w_3w_1w_2 s\right]_q t} d\mu_{-1}(y) \quad (2.1)$$

$$= \lim_{N\to\infty} \sum_{y=0}^{p^N-1} (-1)^y w^{w_1w_2w_3w_4y} q^{hw_1w_2w_3w_4y} \times e^{\left[w_1w_2w_3w_4 2y + w_1w_2w_3w_4w_5x + w_5w_4w_2w_3 i + w_5w_4w_1w_3 j + w_5w_4w_1w_2 k + w_5w_3w_1w_2 s\right]_q t}$$

$$= \lim_{N\to\infty} \sum_{l=0}^{w_5-1} \sum_{y=0}^{p^N-1} (-1)^{l+y} w^{w_1w_2w_3w_4(l+w_5y)} q^{hw_1w_2w_3w_4(l+w_5y)}$$

$$\times e^{\left[w_1w_2w_3w_4 2(l+w_5y)+w_1w_2w_3w_4w_5x+w_5w_4w_2w_3i+w_5w_4w_1w_3j+w_5w_4w_1w_2k+w_5w_3w_1w_2s\right]_q t}.$$

Taking

$$\sum_{i=0}^{w_1-1}\sum_{j=0}^{w_2-1}\sum_{k=0}^{w_3-1}\sum_{s=0}^{w_4-1}(-1)^{i+j+k+s} w^{w_5w_4w_2w_3i+w_5w_4w_1w_3j+w_5w_4w_1w_2k+w_5w_3w_1w_2s} \times q^{h\left(w_5w_4w_2w_3i+w_5w_4w_1w_3j+w_5w_4w_1w_2k+w_5w_3w_1w_2s\right)}$$

on the both sides of Eq. (2.1) gives

$$\begin{aligned}
&\sum_{i=0}^{w_1-1}\sum_{j=0}^{w_2-1}\sum_{k=0}^{w_3-1}\sum_{s=0}^{w_4-1}(-1)^{i+j+k+s} w^{w_5w_4w_2w_3i+w_5w_4w_1w_3j+w_5w_4w_1w_2k+w_5w_3w_1w_2s} \qquad (2.2)\\
&\times q^{h\left(w_5w_4w_2w_3i+w_5w_4w_1w_3j+w_5w_4w_1w_2k+w_5w_3w_1w_2s\right)}\int_{\mathbb{Z}_p} w^{w_1w_2w_3w_4y} q^{hw_1w_2w_3w_4y}\\
&\times e^{\left[w_1w_2w_3w_4 2y+w_1w_2w_3w_4w_5x+w_5w_4w_2w_3i+w_5w_4w_1w_3j+w_5w_4w_1w_2k+w_5w_3w_1w_2s\right]_q t} d\mu_{-1}(y)\\
&= \lim_{N\to\infty}\sum_{i=0}^{w_1-1}\sum_{j=0}^{w_2-1}\sum_{k=0}^{w_3-1}\sum_{s=0}^{w_4-1}\sum_{l=0}^{w_5-1}\sum_{y=0}^{p^N-1}(-1)^{i+j+k+s+y+l} \times w^{w_1w_2w_3w_4\left(l+w_5y\right)+w_5w_4w_2w_3i+w_5w_4w_1w_3j+w_5w_4w_1w_2k+w_5w_3w_1w_2s}\\
&\times q^{h\left(w_1w_2w_3w_4\left(l+w_5y\right)+w_5w_4w_2w_3i+w_5w_4w_1w_3j+w_5w_4w_1w_2k+w_5w_3w_1w_2s\right)}\\
&\times e^{\left[w_1w_2w_3w_4 2\left(l+w_5y\right)+w_1w_2w_3w_4w_5x+w_5w_4w_2w_3i+w_5w_4w_1w_3j+w_5w_4w_1w_2k+w_5w_3w_1w_2s\right]_q t}.
\end{aligned}$$

Note that the Eq. (2.2) is invariant for any permutation $\sigma \in S_5$. Therefore, we obtain the following theorem.

Theorem 1 *Let* $w_i \in \mathbb{N}$ *with* $w_i \equiv 1(mod 2)$ *and* $i \in \{1,2,3,4,5\}$. *Then the following*

$$\begin{aligned}
&\sum_{i=0}^{w_{\sigma(1)}-1}\sum_{j=0}^{w_{\sigma(2)}-1}\sum_{k=0}^{w_{\sigma(3)}-1}\sum_{s=0}^{w_{\sigma(4)}-1}(-1)^{i+j+k+s}\\
&\times w^{w_{\sigma(5)}w_{\sigma(4)}w_{\sigma(2)}w_{\sigma(3)}i+w_{\sigma(5)}w_{\sigma(4)}w_{\sigma(1)}w_{\sigma(3)}j+w_{\sigma(5)}w_{\sigma(4)}w_{\sigma(1)}w_{\sigma(2)}k+w_{\sigma(5)}w_{\sigma(3)}w_{\sigma(1)}w_{\sigma(2)}s}\\
&\times q^{h\left(w_{\sigma(5)}w_{\sigma(4)}w_{\sigma(2)}w_{\sigma(3)}i+w_{\sigma(5)}w_{\sigma(4)}w_{\sigma(1)}w_{\sigma(3)}j+w_{\sigma(5)}w_{\sigma(4)}w_{\sigma(1)}w_{\sigma(2)}k+w_{\sigma(5)}w_{\sigma(3)}w_{\sigma(1)}w_{\sigma(2)}s\right)}\\
&\times\int_{\mathbb{Z}_p} w^{w_{\sigma(1)}w_{\sigma(2)}w_{\sigma(3)}w_{\sigma(4)}\left(l+w_{\sigma(5)}y\right)} q^{hw_{\sigma(1)}w_{\sigma(2)}w_{\sigma(3)}w_{\sigma(4)}\left(l+w_{\sigma(5)}y\right)}\\
&\times\exp\Big([w_{\sigma(1)}w_{\sigma(2)}w_{\sigma(3)}w_{\sigma(4)}2y+w_{\sigma(1)}w_{\sigma(2)}w_{\sigma(3)}w_{\sigma(4)}w_{\sigma(5)}x+w_{\sigma(5)}w_{\sigma(4)}w_{\sigma(2)}w_{\sigma(3)}i\\
&+w_{\sigma(5)}w_{\sigma(4)}w_{\sigma(1)}w_{\sigma(3)}j+w_{\sigma(5)}w_{\sigma(4)}w_{\sigma(1)}w_{\sigma(2)}k+w_{\sigma(5)}w_{\sigma(3)}w_{\sigma(1)}w_{\sigma(2)}s]_q t\Big)d\mu_{-1}(y)
\end{aligned}$$

holds true for any $\sigma \in S_5$.

By Eq. (1.5), one can easily see that

$$\left[w_1w_2w_3w_4 2y+w_1w_2w_3w_4w_5x+w_5w_4w_2w_3i+w_5w_4w_1w_3j+w_5w_4w_1w_2k+w_5w_3w_1w_2s\right]_q \qquad (2.3)$$

$$=\left[w_1w_2w_3w_4\right]_q\left[2y+w_5x+\frac{w_5}{w_1}i+\frac{w_5}{w_2}j+\frac{w_5}{w_3}k+\frac{w_5}{w_4}s\right]_{q^{w_1w_2w_3w_4}}.$$

From Eqs. (2.1) and (2.3), we obtain

$$\int_{\mathbb{Z}_p} w^{w_1w_2w_3w_4y}q^{hw_1w_2w_3w_4y}e^{\left[w_1w_2w_3w_4 2y+w_1w_2w_3w_4w_5x+w_5w_4w_2w_3i+w_5w_4w_1w_3j+w_5w_4w_1w_2k+w_5w_3w_1w_2s\right]_q t}d\mu_{-1}(y) \quad (2.4)$$

$$=\sum_{n=0}^{\infty}\left[w_1w_2w_3w_4\right]_q^n T^{(h)}_{n,q^{w_1w_2w_3w_4},w^{w_1w_2w_3w_4}}\left(w_5x+\frac{w_5}{w_1}i+\frac{w_5}{w_2}j+\frac{w_5}{w_3}k+\frac{w_5}{w_4}s\right)\frac{t^n}{n!}.$$

By Eq. (2.4), we have

$$\int_{\mathbb{Z}_p} w^{w_1w_2w_3w_4y}q^{hw_1w_2w_3w_4y} \quad (2.5)$$

$$\times\left[w_1w_2w_3w_4 2y+w_1w_2w_3w_4w_5x+w_5w_4w_2w_3i+w_5w_4w_1w_3j+w_5w_4w_1w_2k+w_5w_3w_1w_2s\right]_q^n d\mu_{-1}(y)$$

$$=\left[w_1w_2w_3w_4\right]_q^n T^{(h)}_{n,q^{w_1w_2w_3w_4},w^{w_1w_2w_3w_4}}\left(w_5x+\frac{w_5}{w_1}i+\frac{w_5}{w_2}j+\frac{w_5}{w_3}k+\frac{w_5}{w_4}s\right),(n\geq 0).$$

Thus, from Theorem 1 and (2.5), we have the following theorem.

Theorem 2 *For* $w_i\in\mathbb{N}$ *with* $w_i\equiv 1(mod 2)$ *with* $i\in\{1,2,3,4,5\}$, *the following*

$$\left[w_{\sigma(1)}w_{\sigma(2)}w_{\sigma(3)}w_{\sigma(4)}\right]_q^n\sum_{i=0}^{w_{\sigma(1)}-1}\sum_{j=0}^{w_{\sigma(2)}-1}\sum_{k=0}^{w_{\sigma(3)}-1}\sum_{s=0}^{w_{\sigma(4)}-1}(-1)^{i+j+k+s}$$

$$\times w^{w_{\sigma(5)}w_{\sigma(4)}w_{\sigma(2)}w_{\sigma(3)}i+w_{\sigma(5)}w_{\sigma(4)}w_{\sigma(1)}w_{\sigma(3)}j+w_{\sigma(5)}w_{\sigma(4)}w_{\sigma(1)}w_{\sigma(2)}k+w_{\sigma(5)}w_{\sigma(3)}w_{\sigma(1)}w_{\sigma(2)}s}$$

$$\times q^{h\left(w_{\sigma(5)}w_{\sigma(4)}w_{\sigma(2)}w_{\sigma(3)}i+w_{\sigma(5)}w_{\sigma(4)}w_{\sigma(1)}w_{\sigma(3)}j+w_{\sigma(5)}w_{\sigma(4)}w_{\sigma(1)}w_{\sigma(2)}k+w_{\sigma(5)}w_{\sigma(3)}w_{\sigma(1)}w_{\sigma(2)}s\right)}$$

$$\times T^{(h)}_{n,q^{w_{\sigma(1)}w_{\sigma(2)}w_{\sigma(3)}w_{\sigma(4)}},w^{w_{\sigma(1)}w_{\sigma(2)}w_{\sigma(3)}w_{\sigma(4)}}}\left(w_{\sigma(5)}x+\frac{w_{\sigma(5)}}{w_{\sigma(1)}}i+\frac{w_{\sigma(5)}}{w_{\sigma(2)}}j+\frac{w_{\sigma(5)}}{w_{\sigma(3)}}k+\frac{w_{\sigma(5)}}{w_{\sigma(4)}}s\right)$$

holds true for any $\sigma\in S_5$.

It is easy to show by using the definition of $[x]_q$ that

$$\left[2y+w_5x+\frac{w_5}{w_1}i+\frac{w_5}{w_2}j+\frac{w_5}{w_3}k+\frac{w_5}{w_4}s\right]^n_{q^{w_1w_2w_3w_4}} \quad (2.6)$$

$$=\sum_{m=0}^{n}\binom{n}{m}\left(\frac{\left[w_5\right]_q}{\left[w_1w_2w_3w_4\right]_q}\right)^{n-m}\left[w_2w_3w_4i+w_1w_3w_4j+w_1w_2w_4k+w_1w_2w_3s\right]_{q^{w_5}}^{n-m}$$

$$\times q^{m\left(w_5w_4w_2w_3i+w_5w_4w_1w_3j+w_5w_4w_1w_2k+w_5w_3w_1w_2s\right)}\left[2y+w_5x\right]^m_{q^{w_1w_2w_3w_4}}.$$

Taking $\int_{\mathbb{Z}_p} w^{w_1w_2w_3w_4y} q^{hw_1w_2w_3w_4y} d\mu_{-1}(y)$ on the both sides of Eq. (2.6) gives

$$\int_{\mathbb{Z}_p} w^{w_1w_2w_3w_4y} q^{hw_1w_2w_3w_4y} \left[2y + w_5x + \frac{w_5}{w_1}i + \frac{w_5}{w_2}j + \frac{w_5}{w_3}k + \frac{w_5}{w_4}s\right]^n_{q^{w_1w_2w_3w_4}} d\mu_{-1}(y) \quad (2.7)$$

$$= \sum_{m=0}^{n} \binom{n}{m} \left(\frac{[w_5]_q}{[w_1w_2w_3w_4]_q} \right)^{n-m} [w_2w_3w_4i + w_1w_3w_4j + w_1w_2w_4k + w_1w_2w_3s]^{n-m}_{q^{w_5}}$$

$$\times q^{m(w_5w_4w_2w_3i + w_5w_4w_1w_3j + w_5w_4w_1w_2k + w_5w_3w_1w_2s)} \int_{\mathbb{Z}_p} w^{w_1w_2w_3w_4y} q^{hw_1w_2w_3w_4y} [2y + w_5x]^m_{q^{w_1w_2w_3w_4}} d\mu_{-1}(y)$$

$$= \sum_{m=0}^{n} \binom{n}{m} \left(\frac{[w_5]_q}{[w_1w_2w_3w_4]_q} \right)^{n-m} [w_2w_3w_4i + w_1w_3w_4j + w_1w_2w_4k + w_1w_2w_3s]^{n-m}_{q^{w_5}}$$

$$\times q^{m(w_5w_4w_2w_3i + w_5w_4w_1w_3j + w_5w_4w_1w_2k + w_5w_3w_1w_2s)} T^{(h)}_{m,q^{w_1w_2w_3w_4}, w^{w_1w_2w_3w_4}}(w_5x).$$

By the Eq. (2.7), we have

$$[w_1w_2w_3w_4]^n_q \sum_{i=0}^{w_1-1}\sum_{j=0}^{w_2-1}\sum_{k=0}^{w_3-1}\sum_{s=0}^{w_4-1} (-1)^{i+j+k+s} w^{w_5w_4w_2w_3i + w_5w_4w_1w_3j + w_5w_4w_1w_2k + w_5w_3w_1w_2s} \quad (2.8)$$

$$\times q^{h(w_5w_4w_2w_3i + w_5w_4w_1w_3j + w_5w_4w_1w_2k + w_5w_3w_1w_2s)}$$

$$\times \int_{\mathbb{Z}_p} w^{w_1w_2w_3w_4y} q^{hw_1w_2w_3w_4y} \left[2y + w_5x + \frac{w_5}{w_1}i + \frac{w_5}{w_2}j + \frac{w_5}{w_3}k + \frac{w_5}{w_4}s\right]^n_{q^{w_1w_2w_3w_4}} d\mu_{-1}(y)$$

$$= \sum_{m=0}^{n} \binom{n}{m} [w_1w_2w_3w_4]^m_q [w_5]^{n-m}_q T^{(h)}_{m,q^{w_1w_2w_3w_4}, w^{w_1w_2w_3w_4}}(w_5x) \sum_{i=0}^{w_1-1}\sum_{j=0}^{w_2-1}$$

$$\sum_{k=0}^{w_3-1}\sum_{s=0}^{w_4-1} (-1)^{i+j+k+s} w^{w_5w_4w_2w_3i + w_5w_4w_1w_3j + w_5w_4w_1w_2k + w_5w_3w_1w_2s}$$

$$\times q^{(m+h)(w_5w_4w_2w_3i + w_5w_4w_1w_3j + w_5w_4w_1w_2k + w_5w_3w_1w_2s)} \times [w_2w_3w_4i + w_1w_3w_4j + w_1w_2w_4k + w_1w_2w_3s]^{n-m}_{q^{w_5}}$$

$$= \sum_{m=0}^{n} \binom{n}{m} [w_1w_2w_3w_4]^m_q [w_5]^{n-m}_q T^{(h)}_{m,q^{w_1w_2w_3w_4}, w^{w_1w_2w_3w_4}}(w_5x) C_{n,q^{w_5},w^{w_5}}(w_1, w_2, w_3, w_4 \mid m),$$

where

$$C_{n,q,w}(w_1, w_2, w_3, w_4 \mid m) \quad (2.9)$$

$$= \sum_{i=0}^{w_1-1}\sum_{j=0}^{w_2-1}\sum_{k=0}^{w_3-1}\sum_{s=0}^{w_4-1} (-1)^{i+j+k+s} w^{w_2w_3w_4i + w_1w_3w_4j + w_1w_2w_4k + w_1w_2w_3s}$$

$$\times q^{(m+h)(w_2w_3w_4i + w_1w_3w_4j + w_1w_2w_4k + w_1w_2w_3s)} [w_2w_3w_4i + w_1w_3w_4j + w_1w_2w_4k + w_1w_2w_3s]^{n-m}_q .$$

As a result, by (2.9), we arrive at the following theorem.

Theorem 3 *Let* $w_i \in \mathbb{N}$ *with* $w_i \equiv 1(mod\,2)$ *with* $i \in \{1,2,3,4,5\}$. *For* $n \geq 0$, *the following expression*

$$\sum_{m=0}^{n} \binom{n}{m} \left[w_{\sigma(1)} w_{\sigma(2)} w_{\sigma(3)} w_{\sigma(4)} \right]_q^m \left[w_{\sigma(5)} \right]_q^{n-m}$$

$$\times T^{(h)}_{m,q^{w_{\sigma(1)}w_{\sigma(2)}w_{\sigma(3)}w_{\sigma(4)}},w^{w_{\sigma(1)}w_{\sigma(2)}w_{\sigma(3)}w_{\sigma(4)}}} \left(w_{\sigma(5)} x \right) C_{n,q^{w_{\sigma(5)}},w^{w_{\sigma(5)}}} (w_{\sigma(1)}, w_{\sigma(2)}, w_{\sigma(3)}, w_{\sigma(4)} \mid m)$$

holds true for some $\sigma \in S_5$.

References

[1] E. Ağyüz, M. Acikgoz and S. Araci, *A symmetric identity on the q-Genocchi polynomials of higher order under third Dihedral group* D_3, Proc. Jangjeon Math. Soc. 18 (**2015**), No. 2, pp. 177-187.

[2] U. Duran, M. Acikgoz, A. Esi, S. Araci, *Some new symmetric identities involving q-Genocchi polynomials under* S_4, Journal of Mathematical Analysis, Vol 6, Issue 4 (**2015**).

[3] U. Duran, M. Acikgoz, S. Araci, *Symmetric identities involving weighted q-Genocchi polynomials under* S_4, Proceedings of the Jangjeon Mathematical Society,18 (**2015**), No. 4, pp 455-465.

[4] S. Araci, M. Acikgoz, E. S en, *On the extended Kim's p-adic q-deformed fermionic integrals in the p-adic integer ring*, J. Number Theory 133 (**2013**) 3348-3361.

[5] S. Araci, U. Duran, M. Acikgoz, *Symmetric identities involving q-Frobenius-Euler polynomials under Sym (5)*, Turkish J. Anal. Number Theory, Vol. 3, No. 3, pp. 90-93 (**2015)**.

[6] T. Kim, *q-Volkenborn integration*, Russian Journal of Mathematical Physics 9.3 (**2002**): pp. 288-299.

[7] T. Kim and J. J. Seo, *Some identities of symmetry for Carlitz-type q-Euler polynomials invariant under symmetric group of degree five*, International Journal of Mathematical Analysis Vol. 9 (**2015)**, no. 37, 1815-1822.

[8] T. Kim, *Some New identities of symmetry for higher-order Carlitz q-Bernoulli polynomials arising from p-adic q-integral on* $\mathbb{Z}_p$ *under the symmetric group of degree five,* Applied Mathematical Sciences, Vol. 9 (**2015)**, no. 93, 4627-4634.

[9] T. Kim, J. J. Seo, *New identities of symmetry for Carlitz's-type q-Bernoulli polynomials under symmetric group of degree five,* International Journal of Mathematical Analysis, Vol. 9 (**2015)**, no. 35, 1707 - 1713.

[10] D. S. Kim, T. Kim, W. J. Kim, and D. V. Dolgy, *A Note on Eulerian Polynomials,* Abstract and Applied Analysis, vol. 2012, Article ID 269640, 10 pages, **2012**. doi:10.1155/2012/269640.

[11] C. S. Ryoo, *Carlitz's type twisted (h,q)-Tangent numbers and polynomials,* Applied Mathematical Sciences, Vol. 9 (**2015)**, no. 30, 1475 - 1482.

[12] C. S. Ryoo, *Symmetric identities for Carlitz's type twisted Tangent polynomials using twisted Tangent Zeta function*, Applied Mathematical Sciences, Vol. 9 (**2015)**, no. 30, 1483 - 1489.

[13] C. S. Ryoo, *On the symmetric properties for the generalized twisted Tangent polynomials*, Advanced Studies in Theoretical Physics, Vol. 8 (**2014)**, no. 23, 1043 - 1049.

[14] C. S. Ryoo, *A note on the tangent numbers and polynomials*, Advanced Studies in Theoretical Physics, Vol. 7 (**2013**), no. 9, 447 - 454.

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DETERMINANTAL IDENTITIES FOR k LUCAS SEQUENCE

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Abstaract − In this paper, we defined new relationship between k Lucas sequences and determinants of their associated matrices, this approach is different and never tried in k Fibonacci sequence literature.

Keywords − *k-Fibonacci sequence, k-Lucas sequence, Recurrence relation.*

1 Introduction

The Fibonacci sequence is a source of many nice and interesting identities. Many identities have been documented in [9],[10],[11],[12],[16],[2],[3]. A similar interpretation exists for k Fibonacci and k Lucas numbers. Many of these identities have been documented in the work of Falcon and Plaza [6],[7],[8], where they are proved by algebraic means, many of another interesting algebraic identities are also proved in [1],[4]. In this paper determinantal techniques are used to obtain several k Lucas identities.

2 Preliminary

Definition 2.1. The $k-$Fibonacci sequence $\{F_{k,n}\}_{n=1}^{\infty}$ is defined as, $F_{k,n+1} = k \cdot F_{k,n} + F_{k,n-1}$, with $F_{k,0} = 0, F_{k,1} = 1$, for $n \geq 1$

Definition 2.2. The $k-$ Lucas sequence $\{L_{k,n}\}_{n=1}^{\infty}$ is defined as, $L_{k,n+1} = k \cdot L_{k,n} + L_{k,n-1}$, with $L_{k,0} = 2, L_{k,1} = k$, for $n \geq 1$

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Characteristic equation of the initial recurrence relation is,

$$r^2 - k \cdot r - 1 = 0 \tag{1}$$

Characteristic roots are

$$r_1 = \frac{k + \sqrt{k^2 + 4}}{2} \tag{2}$$

and

$$r_2 = \frac{k - \sqrt{k^2 + 4}}{2} \tag{3}$$

Characteristic roots verify the properties

$$r_1 - r_2 = \sqrt{k^2 + 4} = \sqrt{\Delta} = \delta \tag{4}$$

$$r_1 + r_2 = k \tag{5}$$

$$r_1 . r_2 = -1 \tag{6}$$

Binet forms for $F_{k,n}$ and $L_{k,n}$ are

$$F_{k,n} = \frac{r_1^n - r_2^n}{r_1 - r_2} \tag{7}$$

and

$$L_{k,n} = r_1^n + r_2^n \tag{8}$$

2.1 First 11 k Fibonacci sequences as numbered in the Encyclopedia of Integer Sequences

$F_{k,n}$	**Classification**
$F_{1,n}$	*A*000045
$F_{2,n}$	*A*000129
$F_{3,n}$	*A*006190
$F_{4,n}$	*A*001076
$F_{5,n}$	*A*052918
$F_{6,n}$	*A*005668
$F_{7,n}$	*A*054413
$F_{8,n}$	*A*041025
$F_{9,n}$	*A*099371
$F_{10,n}$	*A*041041
$F_{11,n}$	*A*049666

3 Determinantal Identities

Theorem 3.1. If n, i, j, t, m are positive integers with $0 < t < i$, $i + 1 < m$, $j = 1$, then

$$det\begin{bmatrix} L_{k,n+t}^2 + 4L_{k,n-i}^2 & L_{k,n+i+j} & L_{k,n+i+j} \\ L_{k,n+t} & 4L_{k,n+i}^2 + L_{k,n+i+j}^2 & L_{k,n+t} \\ L_{k,n+i} & 2L_{k,n+i} & \dfrac{L_{k,n+i+j}^2 + L_{k,n+t}^2}{2L_{k,n+i}} \end{bmatrix} = 8L_{k,n+i}L_{k,n+t}L_{k,n+i+j} \tag{9}$$

Proof. Let

$$\aleph_1 = det\begin{bmatrix} L_{k,n+t}^2 + 4L_{k,n-i}^2 & L_{k,n+i+j} & L_{k,n+i+j} \\ L_{k,n+t} & 4L_{k,n+i}^2 + L_{k,n+i+j}^2 & L_{k,n+t} \\ L_{k,n+i} & 2L_{k,n+i} & \dfrac{L_{k,n+i+j}^2 + L_{k,n+t}^2}{2L_{k,n+i}} \end{bmatrix} \tag{10}$$

Assume that

$$L_{k,n+t} = \phi$$
$$L_{k,n+i} = \varphi$$

Then

$$L_{k,n+i+j} = k\varphi + \phi$$

Now,

$$\aleph_1 = det\begin{bmatrix} \dfrac{\phi^2 + \varphi^2}{k\varphi + \phi} & k\varphi + \phi & k\varphi + \phi \\ \phi & \frac{\varphi^2 + (k\varphi+\phi)^2}{\phi} & \phi \\ \varphi & \varphi & \dfrac{\phi^2 + (k\varphi + \phi)^2}{\varphi} \end{bmatrix}$$

Making the row operations $\dfrac{1}{(k\varphi + \phi)}[(k\varphi + \phi)R_1]$, $\dfrac{1}{\phi}[\phi R_2]$, $\dfrac{1}{\varphi}[\varphi R_3]$, gives

$$\aleph_1 = \frac{1}{\phi\varphi(k\varphi + \phi)}det\begin{bmatrix} \phi^2 + \varphi^2 & (k\varphi + \phi)^2 & (k\varphi + \phi)^2 \\ \phi^2 & \varphi^2 + (k\varphi + \phi)^2 & \phi^2 \\ \varphi^2 & \varphi^2 & \phi^2 + (k\varphi + \phi)^2 \end{bmatrix} \tag{11}$$

making row operations $R_1 + R_2 + R_3 \to R_1$, $R_3 - R_1 \to R_3$ and $R_2 - R_1 \to R_2$, gives

$$\aleph_1 = \frac{1}{\phi\varphi(k\varphi+\phi)} det \begin{bmatrix} \phi^2+\varphi^2 & \varphi^2+(k\varphi+\phi)^2 & \phi^2+(k\varphi+\phi)^2 \\ -\varphi^2 & 0 & -(k\varphi+\phi)^2 \\ \phi^2 & -(k\varphi+\phi)^2 & 0 \end{bmatrix}$$

Expanding we get

$$\aleph_1 = 8\phi\varphi(k\varphi+\phi)$$

Putting

$$L_{k,n+t} = \phi$$
$$L_{k,n+i} = \varphi$$
$$L_{k,n+i+j} = k\varphi+\phi$$

Gives

$$\aleph_1 = 8L_{k,n+i}L_{k,n+t}L_{k,n+i+j}$$

□

Theorem 3.2. If n, i, j, t, m are positive integers with $0 < t < i$, $i+1 < m$, $j = 1$, then

$$det \begin{bmatrix} L^2_{k,n+t} & 2L_{k,n+i}L_{k,n+i+j} & L_{k,n+t}L_{k,n+i+j} + L_{k,n+i+j} \\ L^2_{k,n+t} + 2L_{k,n+i}L_{k,n+t} & 4L^2_{k,n+i} & L_{k,n+t}L_{k,n+i+j} \\ 2L_{k,n+i}L_{k,n+t} & 4L^2_{k,n+i} + 2L_{k,n+i}L_{k,n+i+j} & L^2_{k,n+i+j} \end{bmatrix} \quad (12)$$

$= [4L_{k,n+i}L_{k,n+i+j}]^2$

Proof. Let

$$\aleph_2 = det \begin{bmatrix} L^2_{k,n+t} & 2L_{k,n+i}L_{k,n+i+j} & L_{k,n+t}L_{k,n+i+j} + L_{k,n+i+j} \\ L^2_{k,n+t} + 2L_{k,n+i}L_{k,n+t} & 4L^2_{k,n+i} & L_{k,n+t}L_{k,n+i+j} \\ 2L_{k,n+i}L_{k,n+t} & 4L^2_{k,n+i} + 2L_{k,n+i}L_{k,n+i+j} & L^2_{k,n+i+j} \end{bmatrix} \quad (13)$$

Assume that

$$L_{k,n+t} = \phi$$
$$L_{k,n+i} = \varphi$$

Then

$$\aleph_2 = det \begin{bmatrix} \phi^2 & \varphi(k\varphi+\phi) & \phi(k\varphi+\phi)+(k\varphi+\phi)^2 \\ \phi^2+\phi\varphi & \varphi^2 & \phi(k\varphi+\phi) \\ \phi\varphi & \varphi^2+\varphi(k\varphi+\phi) & (k\varphi+\phi)^2 \end{bmatrix}$$

Making the row operations $R_2 \to R_2 - (R_1 + R_3))$, gives

$$\aleph_1 = \frac{1}{\phi\varphi(k\varphi+\phi)} det \begin{bmatrix} \phi & (k\varphi+\phi) & \phi+(k\varphi+\phi) \\ 0 & -2(k\varphi+\phi) & -2(k\varphi+\phi) \\ \varphi & \varphi+(k\varphi+\phi) & (k\varphi+\phi) \end{bmatrix} \tag{14}$$

making Column operations $C_2 \to C_2 - C_3$ and expanding gives

$$\aleph_2 = 4\left[2\phi\varphi(k\varphi+\phi)\right]^2$$

Putting

$$L_{k,n+t} = \phi$$
$$L_{k,n+i} = \varphi$$
$$L_{k,n+i+j} = k\varphi + \phi$$

Gives

$$\aleph_2 = \left[4L_{k,n+i}L_{k,n+i+j}\right]^2$$

□

Corollary 3.3. If n, i, j, t, m are positive integers with $0 < t < i$, $i + 1 < m$, $j = 1$, then

$$det \begin{bmatrix} -L^2_{k,n+t} & 2L_{k,n+i}L_{k,n+t} & L_{k,n+t}L_{k,n+i+j} \\ 2L_{k,n+i}L_{k,n+t} & -4L^2_{k,n+i} & 2L_{k,n+i}L_{k,n+i+j} \\ L_{k,n+t}L_{k,n+i+j} & 2L_{k,n+i}L_{k,n+i+j} & -L^2_{k,n+i+j} \end{bmatrix} = \left[4L_{k,n+i}L_{k,n+t}L_{k,n+i+j}\right]^2 \tag{15}$$

Corollary 3.4. If n, i, j, t, m are positive integers with $0 < t < i$, $i + 1 < m$, $j = 1$, then

$$det \begin{bmatrix} 4L^2_{k,n+i} + L^2_{k,n+i+j} & 2L_{k,n+i}L_{k,n+t} & L_{k,n+t}L_{k,n+i+j} \\ 2L_{k,n+i}L_{k,n+t} & L^2_{k,n+t} & 2L_{k,n+i}L_{k,n+i+j} \\ L_{k,n+t}L_{k,n+i+j} & 2L_{k,n+i}L_{k,n+i+j} & 4L^2_{k,n+i} + L^2_{k,n+t} \end{bmatrix} = \left[4L_{k,n+i}L_{k,n+t}L_{k,n+i+j}\right]^2 \tag{16}$$

Corollary 3.5. If n, i, j, t, m are positive integers with $0 < t < i$, $i + 1 < m$, $j = 1$, then

$$det \begin{bmatrix} 2L_{k,n+i+j} + 2L_{k,n+i} + L_{k,n+t} & L_{k,n+t} & 2L_{k,n+i} \\ L_{k,n+i+j} & 2L_{k,n+t} + 2L_{k,n+i} + L_{k,n+i+j} & 2L_{k,n+i} \\ L_{k,n+i+j} & 2L_{k,n+t} & 4L_{k,n+i} + L_{k,n+t} + L_{k,n+i+j} \end{bmatrix}$$

$$= 2\left[2L_{k,n+i} + L_{k,n+t} + L_{k,n+i+j}\right]^3$$

Corollary 3.6. If n, i, j, t, m are positive integers with $0 < t < i$, $i+1 < m$, $j = 1$, then

$$det\begin{bmatrix} 1+L_{k,n+t} & 1 & 1 \\ 1 & 1+2L_{k,n+i} & 1 \\ 1 & 1 & 1+L_{k,n+i+j} \end{bmatrix}$$

$$= \{2L_{k,n+i}L_{k,n+t}L_{k,n+i+j}\}\{\frac{1}{L_{k,n+t}} + \frac{1}{2L_{k,n+i}} + \frac{1}{L_{k,n+i+j}} + 1\}$$
$$\{2L_{k,n+i}L_{k,n+t}L_{k,n+i+j} + 2L_{k,n+i}L_{k,n+i+j} + L_{k,n+t}L_{k,n+t}L_{k,n+i+j} + 2L_{k,n+i}L_{k,n+t}\}$$

4 Conclusion

In this paper we described determinantal identities for k Lucas sequence; same identities can be derived for k Fibonacci sequence.

References

[1] A. D. Godase, M. B. Dhakne, *On the properties of k Fibonacci and k Lucas numbers*, International Journal of Advances in Applied Mathematics and Mechanics, 02 (2014), 100 - 106.

[2] A. D. Godase, M. B. Dhakne,*Fundamental Properties of Multiplicative Coupled Fibonacci Sequences of Fourth Order Under Two Specific Schemes*, International Journal of Mathematical Archive , 04, No. 6 (2013), 74 - 81.

[3] A. D. Godase, M. B. Dhakne, *Recurrent Formulas of The Generalized Fibonacci Sequences of Fifth Order*, "International Journal of Mathematical Archive" , 04, No. 6 (2013), 61 - 67.

[4] A. D. Godase, M. B. Dhakne, *Summation Identities for k Fibonacci and k Lucas Numbers using Matrix Methods*, "International Journal of Mathematics and Scientific Computing" , 05, No. 2 (2015), 75 - 78.

[5] A. F. Horadam, *Basic Properties of a Certain Generalized Sequence of Numbers*, The Fibonacci Quarterly, 03, No. 3 (1965), 61 - 76.

[6] S. Falcon, A. Plaza, *On the k Fibonacci numbers*, Chaos, Solitons and Fractals, 05, No. 32 (2007), 1615 - 1624.

[7] S. Falcon, A. Plaza, *The k Fibonacci sequence and the Pascal* 2 *triangle*, Chaos, Solitons and Fractals, 33, No. 1 (2007), 38 - 49.

[8] S. Falcon, A. Plaza, *The k Fibonacci hyperbolic functions*, Chaos, Solitons and Fractals Vol. 38, No.2 (2008), 409 - 20.

[9] P. Filipponi, A. F. Horadam, *A Matrix Approach to Certain Identities*, The Fibonacci Quarterly, 26, No. 2 (1988), 115 - 126.

[10] H. W. Gould, *A History of the Fibonacci q - Matrix and a Higher - Dimensional Problem*, The Fibonacci Quarterly , 19, No. 3 (1981), 250 - 57.

[11] A. F. Horadam, *Jacobstal Representation of Polynomials*, The Fibonacci Quarterly, 35, No. 2 (1997) 137 - 148.

[12] T. Koshy, *Fibonacci and Lucas numbers with applications*, Wiley - Intersection Publication (2001).

[13] N. J. Sloane, The on-line encyclopedia of integer sequences 2006.

[14] G. Strang, *Introduction to Linear Algebra*, Wellesley - Cambridge:Wellesley, M. A. Publication (2003).

[15] S. Vajda, *Fibonacci and Lucas numbers and the Golden Section: Theory and applications*, Chichester: Ellis Horwood, (1989).

[16] M. E. Waddill , *Matrices and Generalized Fibonacci Sequences*, The Fibonacci Quarterly, 55, No. 12 (1974) 381 - 386.

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A STUDY ON PRE-m_X CONTINUOUS FUNCTION

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Abstract – The aim of this paper is to introduce the concept of pre m_X continuous function and to show some of its application. Also the concept of pre m_X open mapping and pre m_X homeomorphism is studied. The concept of pre m_X open set has already been introduced by the authors in 2011.In this paper a topology is considered which is generated from m_X structure and it is denoted as T_{m_X}. The concept of pre m_X continuous function is discussed in the topological space (X, T_{m_X}) generated from (X, m_X).

Keywords – Pre m_X continuous function, Pre m_X open mapping, Topology generated by m_X structure.

1. Introduction and Preliminaries

The concept of m_X-open set has been introduced by H. Maki in 1996.[8] and the concept of preopen set has been introduced by Mashour et al [9] .Lots of applications of preopen set and m_X structure in ordinary topological space has been introduced by various researchers.[1][2][3]. The concept of m_X pre-open set has been introduced by Ennis Rosas, Neelamegarajan Rajesh, Carlos Carpintero[17]. And the concept of Pre m_X open set has been introduced by the authors in 2011[4]. In this paper the concept of Pre m_X continuous function, Pre m_X irresolute continuous function, Pre m_X open mapping, Introduction. Pre m_X irresolute mapping, Pre m_X homeomorphism etc are introduced and some properties are discussed.

In the second section the concept of pre m_X-continuous function, pre m_X irresolute continuous function is discussed.

In the third section, the concept of pre m_X open mapping etc is introduced and their connection are shown. Lastly the concept of pre m_X homeomorphism is introduced and some of its utility is studied.

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Let us rememorize some of the basic concepts used by various researchers.

Defintion 1.1. [8] A structure is said to be a m_X structure iff $\phi \in m_X$, $X \in m_X$. From this structure the following operators may be defined as below:

For any subset A of X

$$m_X \, IntA = \cup\{G: G \subseteq A \text{ , G is a } m_X \text{ open set in X}\}$$
$$m_X \, ClA = \cap\{G: G \supseteq A \text{ , G is a } m_X \text{ closed set in X}\}$$

The subset A of X is said to be a

1.[8] open m_{X-} set in a m_X structure if m_X intA=A

2. [9] Preopen set in ordinary topological space if $A \subset int(cl(A))$

3. [14] m_X-regular open set in m_X structure if A= m_{X-}int m_{X-}clA.

4. [8] m_X-generalized closed set in m_X structure if there exist a m_X-open set containing A such that $m_X ClA \subset U$ whenever $A \subset U$.

5. [17] m_X--preopen set in X if $A \subseteq m_X Int(m_X Cl(A))$

6. [4] Pre-m_X open set on an m_X structure if $A \subseteq Int(m_{X-}Cl(A)))$.

From the above definitions a connection between the sets are shown in the following figüre

m_X- dense
↓
m_X- open → m_X-pre open → pre-m_X open → b-$m_X$open
↑
regular m_X-open

Definition 1.2. A mapping f : X→Y is said to be a

1. [9] pre continuous function in an ordinary topological space if $f^{-1}(A) \subset PO(X)$ for every open set A in Y.

2. [14] m_X-regular continuous function in a m_X structure if $f^{-1}(A)$ is a m_X regular open set in X for every m_X-regular open set A in Y.

3. [13] m_X-generalized continuous function in a m_X structure if $f^{-1}(A)$ is a m_X closed set in X for every m_X-closed set A in Y.

4. [8] m_X-continuous function in a m_X structure if $f^{-1}(A)$ is a m_X open set in X whenever A is an m_X open set in Y.

5. [9] Preopen mapping in an ordinary topological space if the image of each open set in X is a preopen set in Y.

6. [8] m_X-open mapping in a m_X structure if image of each m_X -open set in X is a m_X -open set in Y.

7. [14] m_X- regular-open mapping in a m_X structure if the image of each m_X -open set in X is a m_X – regular open set in Y.

8.[9] pre irresolute continous function in an ordinary topological space if $f^{-1}(U) \subset PO(X)$ for every $U \subset PO(Y)$,

9. [17] m_X pre irresolute continuous function in a m_X structure if the inverse image of every m_X pre open set in Y is a m_X pre open set in X.

Definition 1.3 [9] A bijective mapping $f:(X,\tau) \rightarrow (Y,\sigma)$ from X to Y is called a pre homeomorphism if both f and f^{-1} are pre irresolute mappings.

Throughout this paper we are considering the topological space as the structure formed by introducing the missing elements in m_X structure i.e. along with the elements of m_X structure we are also introducing the elements which are essentially needed for a topological space .Let us name this type of topological space as a topological space generated by an m_X structure and denote it as T_{m_X} .

Let X = {a,b,c} and the corresponding m_X structure be {ϕ, X, {a,b},{b,c}}. It is not a topology since finite intersection of the elements in m_X is not in m_X. Now T_{m_X} ={ϕ, X, {a,b},{b,c}, {b}}.This is a topology generated by an m_X structure.

For a topology generated by m_X structure let us denote the interior as Int_{Tm_X} and the closure as Cl_{Tm_X} .Now since $m_X \subseteq T_{m_X}$, $m_X\ Int \le Int_{Tm_X} \le Cl_{Tm_X} \le m_X\ Cl$.

2. Pre m_X Continuous Function

In this section the concept of pre m_X continuous function, pre m_X irresolute continuous mapping, pre m_X open mapping, pre m_X homeomorphism are introduced and their properties are studied.

Definition 2.1.A function $f:(X, T_{m_X}) \rightarrow (Y, T_{m_Y})$ is said to be a pre m_X-continuous function if the inverse image of each m_X-open set in Y is a pre m_X -open set in X.

Example 2.2. Let X = {a, b, c, d} and the m_X structure be m_X = {ϕ, X, {a,b}, {c}}, Tm_X={ϕ, X, {a,b}, {c}, {a,b,c}}.

Let Y = {x,y,z, t} then m_X structure is $m_X(y)$={ϕ, Y, {x}, {y}} and Tm_X={ϕ,Y,{x},{y},{x,y}}

Let us consider a mapping f : $(X,T_{m_X}) \rightarrow (Y,T_{m_Y})$ such that f(a) = x, f(b)=y, f(c)=z, f(d)=t. Now the inverse image of each m_X open set in Y are respectively ϕ, X, {a}, {b}. Now a subset A of X is said to be a Pre-m_X open set on an m_X structure if $A \subseteq Int_{Tm_X}(m_X\text{-}Cl(A)))$. Here ϕ, X, {a},{b} are all pre m_X open set. Hence f is a pre m_X continuous function.

Theorem 2.3. Let f $:(X,T_{m_X}) \rightarrow (Y,T_{m_Y})$ be a mapping from X to Y. Every m_X continuous function f is also a pre m_X –continuous function.

Proof: Let $x \in X$ and V be any m_X open set containing f(x).Since f is a m_X – continuous function there exist $U \in m_X(X)$ containing x such that $f^{-1}(V)$ is m_X- open in X. By the figure indicating the connection of the set ,it is shown that every m_X open set is a pre m_X open set, thus $f^{-1}(V)$ is a pre m_X –open set. Hence the proof.

Remark 2.4. The converse of the theorem is not true, which follows from the example 2.2. Here the function is a pre m_X continuous function but not a m_X continuous function since the inverse image of {x}, {y} are respectively {a}, {b} which are not a m_X open set in X.

Theorem 2.5. Let f $:(X,T_{m_X}) \rightarrow (Y,T_{m_Y})$ be a mapping from X to Y. Every m_X -preirresolute continuity is pre m_X-continuous.

Proof: Let V be a m_X-open set in Y. Since every m_X open set in Y is also a m_X pre open set in Y thus V is a m_X pre open set in Y and f being m_X pre irresolute continuous function from definition 1.1(9), $f^{-1}(V)$ is a m_X- preopen set in X i.e. inverse image of a m_X open set in Y is a m_X–preopen set in X. Again since m_X–preopen set is a pre m_X–open set in X. Hence f is a pre m_X-continuous

Remark 2.6. The converse of the theorem is not true which follows from the following example: Let

$X=\{a,b,c,d\}$,
$m_X= \{\phi, X, \{a\},\{b\},\{a,c\},\{b,c\}\}$,
$Tm_X =\{\phi,X,\{a\},\{b\},\{c\},\{a,b,c\}\}$,
$Y= \{m,n,l\}$ and $m_Y = \{\phi,Y,\{m\}, \{l\},\{n,l\},\{m,n\}\}$,
$Tm_Y=\{\phi,Y,\{m\}, \{l\}, \{n\}, \{m,l\}, \{n,l\}, \{m,n\}\}$.

Let f: $X \rightarrow Y$ be a mapping defined by f(a)= m, f(b)=l, f(c) = f(d)= n. Then clearly f is pre m_X- continuous but it is not a m_X-preirresolute continuity. Since

$$f^{-1}(\{m,n\}) = \{a,d\} \not\subset m_X\text{-}PO(X).$$

Theorem 2.7. Let f $:(X, T_{m_X}) \rightarrow (Y, T_{m_Y})$. Every m_X - regular continuity is pre m_X-continuity.

Proof: Let $x \in X$ and V be any m_X open set of Y containing f(x). Since f is m_X – regular continuous there exist $U \in m_X$ containing x such that $f^{-1}(V)$ is m_X- regular open in X. By figure indicating connections between various set**,** $f^{-1}(V)$ is pre m_X- open in X. Hence the proof.

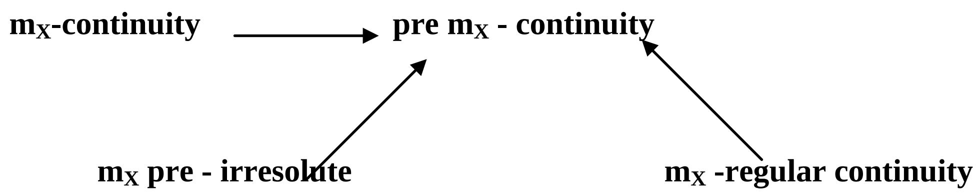
Remark 2.8. The converse of the theorem is not true, which follows from the following example : Let

$X=\{a,b,c,d\}$,
$m_X=\{\phi,X,\{d\},\{b\},\{c\},\{a,b\},\{a,c\}\}$,
$Tm_X=\{\phi,X,\{d\},\{b\},\{c\},\{a\}, \{a,b\}, \{a,c\} \{b,d\}, \{d,c\},\{a,b,c\},\{a,b,d\},\{a,c,d\}\}$and
$Y=\{m,n,l\}$and $m_Y=\{\phi,Y,\{l\},\{m,n\},\{n,l\}\}$ and $Tm_Y =\{\phi,Y,\{l\},\{n\},\{m,n\},\{n,l\}\}$.

Let $f:(X, Tm_X)\rightarrow(Y,Tm_Y)$ be a function defined by f(a) = m, f(b)=l, f(c) = f(d)= n. Then clearly f is pre m_X-continuous but it is not a m_X - regular continuous. Since

$f^{-1}(\{m,n\}) = \{a,d\}\not\subset Tm_X$

We denote the relation discussed above by a figure below.



Definition 2.9. Let (X, Tm_X) be a space with a m_X –structure. For $A\subseteq X$, the pre-m_X-closure and the pre- m_X-interior of A, denoted by $Pm_XCl(A)$ and $Pm_XInt(A)$ respectively are defined as the following:

$Pm_XCl(A)=\cap\{F\subseteq X:A\subseteq F$, F is Pre m_X-closed in X$\}$and
$Pm_XInt(A)=\cup\{U\subseteq X:U\subseteq A$,U is Pre-$m_X$ open in X$\}$.

Theorem 2.10.

(1) A is a pre-m_X-open set iff $Pm_XInt(A)= A$
(2) A is a pre-m_X-closed set iff $Pm_XCl(A)= A$

Proof : (1) Let if possible A be a pre-m_X-open set then obviously $Pm_XInt(A)= A$
Conversely let $Pm_XInt(A)= A$, then

$Pm_XInt(A)= A= \cup\{U\subseteq X:U\subseteq A$,U is Pre-$m_X$ open in X$\}$.

Since arbitrary union of pre-m_X-open set is a pre-m_X-open set[From theorem 3.3 of [17], and A being the arbitrary union of pre-m_X-open set, A is a pre-m_X-open set.This proves the theorem.

(2) can be proved similarly.

Lemma 2.11. For any subset A, B of X the following properties hold.

(i) $Pm_X Int(\phi) = \phi$, $Pm_X Int(X) = X$, $Pm_X Cl(\phi) = \phi$, $Pm_X Cl(X) = X$
(ii) $Pm_X Int\, Pm_X Int(A) = Pm_X Int(A)$, $Pm_X Cl Pm_X Cl(A) = Pm_X Cl(A)$
(iii) $Pm_X Int(A) \subseteq A \subseteq Pm_X Cl(A)$
(iv) $Pm_X Int(A) \subseteq Pm_X Int(B)$, $Pm_X Cl(A) \subseteq Pm_X Cl(B)$ whenever $A \subseteq B$
(v) $Pm_X Int(\cup A_i: i \in I) \supseteq \cup \{Pm_X Int(A_i): i \in I\}$,
$Pm_X Cl(\cap A_i: i \in I) \subseteq \cap \{Pm_X Cl(A_i): i \in I\}$
(vi) $Pm_X Cl(\cup A_i: i \in I) \supseteq \cup \{Pm_X Cl(A_i): i \in I\}$,
$Pm_X Int(\cap A_i: i \in I) \subseteq \cap \{Pm_X Int(A_i): i \in I\}$
(vii) $Pm_X Int(X-A) = X - Pm_X Cl(A)$.

Proof : (i), (iii), (iv), (v), (vi) and (vii) are obvious.

To prove (ii)

From (iii) , $Pm_X Int(A) \subseteq A$ and from (iv), $Pm_X Int Pm_X Int(A) \subseteq Pm_X Int(A)$

Now we have to prove that

$$Pm_X Int Pm_X Int(A) \supseteq Pm_X Int(A)$$

From definition it follows that,

$$Pm_X Int(A) = \cup\{U \subseteq X : U \subseteq A, U \text{ is Pre-}m_X \text{ open in } X\} \supseteq U$$

So $Pm_X Int\, Pm_X Int(A) \supseteq Pm_X Int\,(U) = U$, U is a Pre-$m_X$ open set in X

Thus $Pm_X Int\, Pm_X Int(A) \supseteq \cup\{U \subseteq X : U \subseteq A, U \text{ is Pre-}m_X \text{ open in } X\} = Pm_X Int(A)$

Thus $Pm_X Int\, Pm_X Int(A) = Pm_X Int\,(A)$

Remark 2.12: From Lemma 2.11(ii) and theorem 2.10, it is obvious that $Pm_X Int(A)$ is a Pre m_X open set and $Pm_X Cl(A)$ is a Pre m_X Closed set

Theorem 2.13: Let $f:(X,Tm_X)\to(Y,Tm_Y)$ be a function from X to Y . Then the followings are equivalent.

i) f is a pre m_X-continuous function.
ii) for each m_X open set V in Y,$f^{-1}(V)$ is pre m_X open.
iii) for each m_X closed set B in Y,$f^{-1}(B)$ is pre m_X closed.
iv) $f(p\, m_X\, Cl(A)) \subseteq m_X Cl(f(A))$ for $A \subseteq X$.
v) $p\, m_X\, Cl(f^{-1}(B)) \subseteq f^{-1}(m_X Cl(B))$ for $B \subseteq Y$.
vi) $f^{-1}(m_X Int(B)) \subseteq p\, m_X\, Int(f^{-1}(B))$ for $B \subseteq Y$.

Proof: (i) ⇔ (ii). Obvious.

(ii) ⇒ (iii). Obvious.

(iii) ⇒ (iv). For $A\subseteq X$.

$f^{-1}(m_XCl(f(A)))=f^{-1}(\cap\{F\subseteq Y:f(A)\subseteq F$ and F is m_X closed in Y$\})$
$\supseteq \cap\{f^{-1}(F)\subseteq X: A \subseteq f^{-1}(F)$ and $f^{-1}(F)$ is pre m_X closed in X$\}$
[since every m_X closed in X is a pre m_X closed set in X, so arbitrary intersection of m_X closed set in X containing f(A) is a superset of intersection of Pre m_X closed set in X containing f(A). And f being pre m_X-continuous function, $f^{-1}(F)$ is pre m_X closed in X whenever F is a m_X closed in Y]
$= p\, m_X\, Cl(A)$

implies $f^{-1}(m_XCl(f(A))) \supseteq p\, m_X\, Cl(A)$

i.e. $f(f^{-1}(m_XCl(f(A)))) \supseteq f(p\, m_X\, Cl(A))$

i.e. $m_XCl(f(A)) \supseteq f(f^{-1}(m_XCl(f(A)))) \supseteq f(p\, m_X\, Cl(A))$

i.e. $m_XCl(f(A)) \supseteq f(p\, m_X\, Cl(A))$

(iv) ⇒ (v). Let $A=f^{-1}(B)$ then $f(A) =ff^{-1}(B) \subseteq B$. From (iv)

$f(p\, m_X\, Cl(A)) = f(p\, m_X\, Cl(f^{-1}(B))) \subseteq m_XCl(f(A)) \subseteq m_XCl((B))$
$\Rightarrow f^{-1}f (p\, m_X\, Cl(f^{-1}(B))) \subseteq f^{-1}m_XCl((B))$
$\Rightarrow pm_XCl(f^{-1}(B))) \subseteq f^{-1}f (p\, m_X\, Cl(f^{-1}(B))) \subseteq f^{-1}m_XCl((B))$.

(v) ⇒ (vi). from (v) $X - Pm_X\, Cl(f^{-1}(B)))\supseteq X - f^{-1} (Cl((B))\Rightarrow Pm_XInt(f^{-1}(B))\supseteq f^{-1}(Int(B))$.

(vi) ⇒ (i). For $x\in X$ and for each m_X open set V containing f(x),from (vi),it follows

$x\in f^{-1}(V)=f^{-1}(m_X\, Int(V)) \subseteq pm_XInt(f^{-1}(V))$

From lemma 2.11(iii), $pm_XInt(f^{-1}(V)) \subseteq f^{-1}(V)$. So $pm_XInt(f^{-1}(V)) = f^{-1}(V)$. Thus $f^{-1}(V)$ is a m_X open set in X. This implies that f is a pre m_X continuous function.

Theorem 2.14. Let $f : (X, Tm_X)\to(Y, Tm_Y)$ be a pre m_X-continuous function. Then the following statements holds:

(i) $f^{-1}(V)\subseteq Pm_XInt(m_XCl(f^{-1}(V)))$ for each m_X-open set V in Y.
(ii) $Pm_XCl(m_XInt(f^{-1} (G))) \subseteq f^{-1}(G)$ for each m_X-closed set G in Y.
(iii) $f(Pm_XCl(m_XInt(A)))\subseteq m_XCl(f(A))$ for $A\subseteq X$.
(iv) $Pm_XCl(m_XInt(f^{-1}(B)))\subseteq f^{-1}(m_XCl(B))$ for $B\subseteq Y$.
(v) $f^{-1}(m_XInt(C))\subseteq Pm_XInt(m_XCl(f^{-1}(C)))$ for $C\subseteq Y$.

Proof: To Prove (i) Let V be a m_X open set in Y. Since f is a pre m_X-continuous function, $f^{-1}(V)$ is pre m_X-open in X. Therefore $f^{-1}(V) =Pm_XInt(f^{-1}(V) \subseteq Pm_XInt(m_XCl(f^{-1} (V)))$.

(i) ⇒ (ii). Let G =Y –V be a m_X-closed set in Y.From (ii)

$X - f^{-1}(V) \supseteq X - Pm_XInt(m_XCl(f^{-1} (V)))$

$\Rightarrow f^{-1}(G) \supseteq Pm_XCl(m_XInt(X - f^{-1}(V)))$
$\Rightarrow f^{-1}(G) \supseteq Pm_XCl(m_XInt(f^{-1}(G)))$.

(ii) ⇒ (iii). Let $A= f^{-1}(G)$ then from (iii)

$$Pm_XCl(m_XInt(A)) \subseteq A \Rightarrow f(Pm_XCl(m_XInt(A))) \subseteq f(A) \subseteq m_XCl(f(A)).$$

(iii) ⇒ (iv). Let $f(A)=B \Rightarrow A \subseteq f^{-1}(B)$ then from (iv)

$$f(Pm_XCl(m_XInt(A))) \subseteq f(Pm_XCl(m_XInt(f^{-1}(B)))) \subseteq m_XCl(B)$$
$$\Rightarrow Pm_XCl(m_XInt(f^{-1}(B))) \subseteq f^{-1}f(Pm_XCl(m_XInt(A))) \subseteq f^{-1}(m_XCl(B)).$$

(iv) ⇒ (v). it is obvious.

Definition 2.15.A function f: $(X, Tm_X) \rightarrow (Y, Tm_Y)$ is said to be a pre m_X irresolute continuous function iff the inverse image of each pre-m_X-open set in Y is a pre m_X open set in X.

Theorem 2.16. Consider a function f: $(X, Tm_X) \rightarrow (Y, Tm_Y)$ .Every pre m_X -irresolute continuous function is a pre m_X –continuous function.

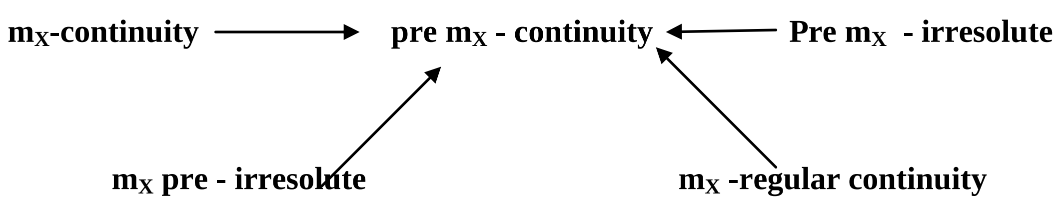
Proof: Let $x \in X$ and V be any m_X open in Y. Then we have V is a pre m_X-open in Y containing f(x). Since f is pre m_X irresolute map then $f^{-1}(V)$ is pre m_X -open in X. Hence the theorem.

Remark 2.17. The converse of the theorem is not true, which follows from the following example: Let

$X=\{a,b,c,d\}$,
$m_X=\{\phi,X,\{a,b\},\{b,c\}, \{a,c,d\}\}$,
$Tm_X=\{\phi,X, \{a,b\}, \{b,c\}, \{a,b,c\}, \{a,c,d\},\{b\}\}$,
$Y=\{x,y,z,t\}$
$m_Y= \{\phi,Y,\{x,y\},\{y,z\}\}$
$Tm_Y=\{\phi,Y, \{x,y\},\{y,z\},\{x,y,z\},\{y\}\}$.

Let f: $X \rightarrow Y$ be a mapping defined by f(a)=x, f(b)=y, f(c)=z, f(d)=t. Then clearly f is pre m_X - continuous, but it is not a pre m_X -irresolute map. since $f^{-1}(\{y\}) = \{b\}$ is not a pre m_X open set in X.

We denote the relation discussed above by a figure below.



Theorem 2.18. The following statements are equivalent for a function

$$f :(X, Tm_X) \rightarrow (Y, Tm_Y)$$

(i) f is pre m_X irresolute.
(ii) For each point x of X and each pre m_X neighborhood V of f(x), there exists a pre m_X - neighborhood U of x such that $f(U) \subseteq V$.
(iii) For each $x \in X$ and each $V \subset Pm_XO(Y)$, there exists $U \subset P\, m_XO(X)$ such that $f(U) \subset V$.

Proof. **(i) ⇒ (ii).** Assume that $x \in X$ and V is a pre m_X- open set in Y containing f(x).Since f is a pre m_X - irresolute and let $U = f^{-1}(V)$ be a pre m_X - open set in X containing x and hence $f(U) = f f^{-1} (V) \subseteq V$.

(ii) ⇒(iii). Assume that $V \subseteq Y$ is a pre m_X open set containing f(x). Then by (ii),there exists a pre m_X open set G such that $x \in G \subseteq f^{-1}(V)$. Therefore, $x \in f^{-1}(V)$. This shows that $f^{-1}(V)$ is a pre m_X neighborhood of x.

(iii)⇒ (i). Let V be a pre m_X-open set in Y, then $f^{-1}(V)$ is pre m_X neighborhood each x of X. Thus, for each x is a pre m_X interior point of $f^{-1}(V)$ which implies that $f^{-1}(V) \subset Int(m_X\text{-}Cl(f^{-1}(V))$. Therefore $f^{-1}(V)$ is a pre m_X open set in X and hence f is a pre m_X -reirresolute.

Theorem 2.19. The following are equivalent for a function $f : (X, Tm_X) \rightarrow (Y, Tm_Y)$

(i) f is pre m_X-irresolute continuous.
(ii) $f(Pm_XCl(v)) \subseteq Pm_X\text{-}Clf(v)$.
(iii) $Pm_XCl(f^{-1}(B)) \subseteq f^{-1}(Pm_X\text{-}Cl(B))$.
(iv) $Pm_X\text{-}Int(f^{-1}(A)) \supseteq f^{-1}(Pm_XInt(A))$.
(v) $f(Pm_X\text{-}Int(B)) \supseteq Pm_X\text{-}Intf(B)$ if f is bijective.

Proof: (i) ⇒ (ii). Let $x \in X$ and $V \subseteq X$ then

$$Pm_XCl(v) \subseteq Pm_XCl(f^{-1}(f(v)) \subseteq Pm_X\text{-}Cl(f^{-1}(Pm_X\text{-}Cl(f(v))) = f^{-1}(Pm_X\text{-}Clf(v))$$
$$\Rightarrow f(Pm_X\text{-}Cl(v)) \subseteq ff^{-1}(Pm_X\text{-}Cl(f(v)) \subseteq Pm_X\text{-}Cl(f(v)).$$

Therefore $f(Pm_XCl(v)) \subseteq Pm_X\text{-}Clf(v)$.

(ii) ⇒ (iii). Let $x \in X$ and $V \subseteq X$ and $B \subseteq Y$ such that $V = f^{-1}(B)$ then

$$f(Pm_X\text{-}Cl(f^{-1}(B))) \subseteq Pm_XCl\, ff^{-1}(B) \subseteq Pm_XCl\,(B)$$
$$\Rightarrow f^{-1}f(Pm_XCl(f^{-1}(B))) \subseteq f^{-1}(Pm_XCl\,(B)) \Rightarrow Pm_XCl\, f^{-1}((B)) \subseteq f^{-1}(Pm_XCl(B)).$$

(iii) ⇒(iv) Let A be any subset of Y such that $B^C=A$. By (iii)

$$X - Pm_X\text{-}Cl(f^{-1}(B)) \supseteq X - f^{-1}(Pm_X\text{-}Cl(B))$$
$$\Rightarrow Pm_XIntf^{-1}(B^C) \supseteq f^{-1}(Pm_XInt\,(B^C))$$
$$\Rightarrow Pm_XIntf^{-1}(A) \supseteq f^{-1}(Pm_XInt\,(A)).$$

(iv) ⇒ (i) Let C be any sub set of Y such that A=Pm_XIntC. By (iv)

$$Pm_X Intf^{-1}(Pm_X IntC) \supseteq f^{-1}(Pm_X Int(C)) \supseteq Pm_X Intf^{-1}(Pm_X IntC)$$

Therefore $f^{-1}(Pm_X Int(C))=Pm_X intf^{-1}(Pm_X IntC)$.

Therefore f is a pre m_X irresolute continuous.

(ii)⇔(v) Let A be a subset of X and f is a bijective then

$f(X – A)= X – f(A)$ and $X – A=A^C=B$ (say)

Now,

$$f(Pm_X cl(A)) \subseteq Pm_X\text{-}clf(A)$$
$$\Rightarrow X\text{-}f(Pm_X cl(A)) \supseteq X\text{-}Pm_X\text{-}clf(A)$$
$$\Rightarrow f(Pm_X int(B)) \supseteq Pm_X Int(f(B))$$

Converse part holds similarly
Hence the statements are equivalent is proved as follows

(i) ⇒(ii) ⇒(iii) ⇒(iv) ⇒(i) ,
↕
(v)

Theorem 2.20.
(1) If f :(X, Tm_X) →(Y, Tm_Y) is pre m_X irresolute and g:(Y, Tm_Y) →(Z, Tm_Z) is pre m_X continuous then gof is pre m_X continuous.
(2) If f :(X, Tm_X) →(Y, Tm_Y) is pre m_X irresolute and g:(Y, Tm_Y) →(Z, Tm_Z) is m_X continuous then gof is pre m_X continuous.
(3) If f :(X, Tm_X) →(Y, Tm_Y) is pre m_X continuous and g:(Y, Tm_Y) →(Z, Tm_Z) is m_X continuous then gof is pre m_X continuous.
(4) If f :(X, Tm_X) →(Y, Tm_Y) is pre m_X irresolute continuous and g:(Y, Tm_Y) →(Z, Tm_Z) is pre m_X irresolute continuous then gof is pre m_X irresolute continuous.

Proof: To Prove (1) Let W be any m_X-open set of Z. since f is pre m_X irresolute then

$$(gof)^{-1}(w)=f^{-1}(g^{-1}(w))$$

is pre m_X open in X and hence gof is a pre m_X continuous function.

The other can be proved similarly.

3. Pre m_X Open Mapping

In this section the concept of Pre m_X open mapping is introduced and also the concept of Pre m_X irresolute mapping is introduced and some of its properties were discussed.

Definition 3.1. A function f : (X, Tm_X) →(Y, Tm_Y) is said to be a pre m_X -open mapping if the image of each Pre m_X open set in X is a m_X -open set in Y.

Example 3.2. Let X = {a,b,c} and Y = {x,y,z}. Let m_X = {ϕ,X,{a,b},{c,b}}. Then Tm_X={ϕ,X,{a,b}, {b,c},{b}}. Here the pre m_X open sets are ϕ,X,{a,b},{c,b}, {b}. Let

m_Y= {ϕ,Y,{x,y},{y,z},{y}} and Tm_Y={ϕ,X,{ x,y},{y,z},{y}}.

Let f : (X, Tm_X) →(Y, Tm_Y) be a mapping such that f(a)=x, f(b)=y, f(c)=z. Then the mapping is a pre m_X open mapping .

Theorem 3.3. Consider a function f : (X, Tm_X) →(Y, Tm_Y) .Every pre m_X open map is a open map.

Proof: Let A be a open set in (X, Tm_X) then A is a pre m_X open set in (X,Tm_X). Since f is a pre m_X open map, f(A) is a m_X open set in (Y, Tm_Y). Since every m_X open set in (Y, Tm_Y) is also a open set . So f is a open map

Remark 3.4. The converse of the theorem is not true which follows from the following example : Let

X={x,y,z,t},
m_X={ϕ,X,{x,y},{y,z}}and
Tm_X={ϕ,X,{x,y},{y,z},{x,y,z},{y}}.

Let

Y={a,b,c,d},
m_Y={ϕ,Y, {a,b},{b,c}, {a,c,d}}
Tm_Y ={ϕ,Y,{a,b},{b,c},{a,c,d},{b},{a,b,c}}.

Let f : (X, Tm_X) →(Y, Tm_Y) is a map defined by f(x)= a, f(y)=b and f(z)=c, f(t)=d. Here f is a open map but not a pre m_X open mapping

Definition 3.5. A function f : (X, Tm_X) →(Y, Tm_Y) is said to be a pre m_X -irresolute mapping if the image of each Pre m_X open set in X is a pre m_X -open set in Y.

Example 3.6. The example 3.2 is also an example of Pre m_X -irresolute mapping

Theorem 3.7. Consider a function f : (X, Tm_X) →(Y, Tm_Y) . Every Pre m_X – open map is also a Pre m_X –irresolute map

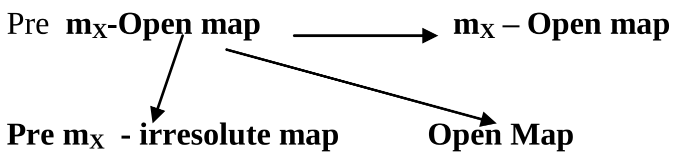
Proof: Let A be a Pre m_X –open set in X . Since f is a Pre m_X –open map, f(A) is m_X –open set in Y. Every m_X –open set is also an open set and a Pre m_X –open set. Thus f(A) is a Pre m_X –open set. This proves that f is a Pre m_X –irresolute mapping.

Remark 3.8. The converse of the above theorem need not be true which follows from the following example : Let

$X = \{a,b,c,d\}$ and $Y=\{x,y,z,t\}$,
$m_X= \{\phi, X, \{a\}, \{b\}, \{c\}\}$ and
$Tm_X= \{\phi, X, \{a\}, \{b\}, \{c\},\{a,b\}, \{a,c\}, \{b,c\}\}$,
$m_Y= \{\phi, Y, \{x\}, \{y\}, \{z\}\}$ and
$Tm_Y= \{\phi, Y, \{x\}, \{y\}, \{z\},\{x,y\}, \{x,z\}, \{y,z\}\}$,

Let $f : (X, Tm_X) \to (Y, Tm_Y)$ is a map defined by $f(x)= a$, $f(y)=b$ and $f(z)=c$, $f(t)=d$. Then f is a pre m_X irresolute map but not a Pre m_X open map.

We denote the relation discussed above by a figure below.



Theorem 3.9. The following are equivalent for a function $f : (X, Tm_X) \to (Y, Tm_Y)$
(i) f is pre- m_X irresolute mapping.
(ii) $f^{-1}(Pm_XInt(v)) \supseteq Pm_X Int(f^{-1}(v))$
(iii) f^{-1} **(**$Pm_X Cl(v)) \subseteq Pm_XCl(f^{-1}(v))$
(iv) $Pm_XIntf(A) \supseteq f (Pm_XInt(A))$
(v) $f(Pm_XCl(B)) \supseteq Pm_XClf(B)$ if f is bijective.

Proof : (i) ⇒(ii). Let $x\in X$ and $V\subseteq X$ then

$$Pm_XInt(v) \supseteq Pm_XIntff^{-1}(v)) \supseteq Pm_XIntf(Pm_XIntf^{-1}(v)) =f(Pm_XIntf^{-1}(v))$$
$$\Rightarrow f^{-1}(Pm_XInt(v))\supseteq f^{-1}f(Pm_XIntf^{-1}(v))\supseteq Pm_XInt(f^{-1}(v)).$$

Therefore

$f^{-1} (Pm_XInt(v))\supseteq Pm_XInt(f^{-1}(v))$.

(ii) ⇔(iii). From (ii),

$$X - f^{-1}(Pm_Xint(v)) \subseteq X- Pm_Xint(f^{-1}(v)) \Rightarrow f^{-1}(Pm_Xclv) \subseteq Pm_Xcl(f^{-1}(v)).$$

The converse part may be proved similarly.

(ii) ⇒(iv). Let $x\in X$ and $V\subseteq X$ and let $f^{-1}(v)=A$. From (ii),

$$f^{-1} (Pm_Xintf(A)) \supseteq Pm_Xint(A)$$

Therefore $Pm_Xintf(A))\supseteq f(Pm_Xint(A))$.

(iv)⇒i) Let $A=Pm_Xint(C)$.From (iv),

$$Pm_Xintf(Pm_Xint(C))\supseteq f(Pm_Xint(Pm_Xint(C))=f(Pm_Xint(C)) \supseteq Pm_X intf(Pm_Xint(C))$$

Therefore $f(Pm_X int(C))$ is a pre-$m_X$open i.e. the image of a pre m_X open set is a pre m_X open set

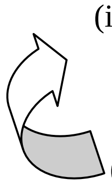
(iv) ⇔(v) Let A be any subset of X and f is a bijective mapping then $f(X – A)=X- f(A)$ and $X – A=B$(say).Therefore from (iv)

$$f(Pm_X intB) \subseteq Pm_X intf(B)$$
$$\Rightarrow Y\text{-}f(Pm_X intB) \supseteq Y\text{-} Pm_X int(f(B))$$
$$\Rightarrow f(Y - Pm_X int(B)) \supseteq Pm_X clf(B)$$
$$\Rightarrow f(Pm_X cl(B)) \supseteq Pm_X clf(B).$$

Converse part can be proved similarly. The equivalence relation is proved as below

(i) ⇒ (ii) ⇔ (iii)

⇓



(iv) ⇔ (v)

4. Pre m_X Homeomorphism

In this section we introduce the concept of Pre m_X homeomorphism and study some of its properties.

Definition 4.1: A bijective mapping $f:(X,m_X) \rightarrow (Y,Tm_Y)$ from a space X into a space Y is called pre-m_X homeomorphism if f and f^{-1} are pre m_X-irresolute mapping.

Theorem 4.2:Let $f:(X,m_X) \rightarrow (Y,m_Y)$ be a bijective mapping from a m_X structure(X,m_X) to a topological space (Y, Tm_Y).The following statements are equivalent.

(i) f is a pre m_X homeomorphism.
(ii) f^{-1} is a pre m_X homeomorphism.
(iii) f is a pre m_X irresolute mapping and a pre m_X irresolute continuous .
(iv) The image of a pre m_X open set in X is a pre m_X open set in Y and a pre m_X continuous mapping.
(v) $f^{-1}(P\, m_X Int(v)) = Pm_X Int(f^{-1}(v))$.
(vi) $f^{-1}(Pm_X Cl(B))= Pm_X\, cl(f^{-1}(B))$.
(vii) $Pm_X Intf(A)= f(Pm_X Int(A))$.
(viii) $f(Pm_X Cl(B)) =Pm_X Clf(B)$.

Proof: (i) ⇔(ii). it follows from the definition.

(i) ⇔(iii). Let f be a pre m_X homeomorphism implies that f and f^{-1} are pre m_X irresolute mapping .Now f^{-1} is a pre m_X irresolute mapping implies that $(f^{-1})^{-1}(A)$ i.e $f(A)$ is a pre m_X open for each A being a pre m_X open set in X. Therefore f is a pre m_X irresolute mapping and a pre m_X irresolute continuous.

Converse: since f is a pre m_X irresolute mapping then f^{-1} is a pre m_X irresolute continuous. Hence f and f^{-1} are pre m_X irresolute continuous mapping. Then obviously f is a pre m_X homeomorphism.

(iii) ⇔(iv). Let f be a pre m_X irresolute mapping then for each pre m_X open set A of X , f(A) is a pre m_X open and f is also pre m_X irresolute continuous then by theorem **2.5** we say that image of a pre m_X open set in X is a pre m_X open set in Y and hence f is a pre m_X irresolute continuous mapping.

(iii) ⇒ (v). Let Let $x \in X$ and $V \subseteq X$, if f is pre m_X irresolute continuous then from theorem 3.7(iv)

$$Pm_X \, Intf^{-1}(A) \supseteq f^{-1}(Pm_X Int(A)) \ldots\ldots (a)$$

and if f is pre m_X irresolute mapping then from theorem 3.8(ii)

$$f^{-1}(Pm_X Int(v)) \subseteq Pm_X Int(f^{-1}(v)) \ldots\ldots\ldots\ldots (b).$$

Combining (a) and (b) we get the result.

(v) ⇒(vi) since f is bijective and from (v)

$$X - f^{-1}(Pm_X int(v)) = X - Pm_X int(f^{-1}(v))$$
$$\Rightarrow f^{-1}(X - Pm_X int(v)) = Pm_X \, Cl(f^{-1}(v))$$
$$\Rightarrow f^{-1}(Pm_X Cl(v)) = Pm_X Cl(f^{-1}(v))$$

(vi) ⇒ (v). It is obvious.

(v) ⇒ (vii). Let $x \in X$ and $V \subseteq X$ and let $f^{-1}(v) = A$ then from(v),

$$Pm_X Int(v) = f(Pm_X Int(f^{-1}(v)) \Rightarrow Pm_X \, intf(A) = f(Pm_X int(A)).proof.$$

(vii) ⇒ (viii). It is obvious.

References

[1] M. Alimohammady a,b,*, M. Roohi c, Fuzzy Um-sets and fuzzy (U;m)-continuous functions,Chaos, Solitons and Fractals 28 (2006) 20–25

[2] M. Alimohammady a,b,*, M. Roohi c, Separation of fuzzy sets in fuzzy minimal spaces Chaos, Solitons and Fractals 31 (2007) 155–161

[3] M. Alimohammady a,b,*, M. Roohi c, Fuzzy minimal structure and fuzzy minimal vector spaces, Chaos, Solitons and Fractals 27 (2006) 599–605.

[4] Sharmistha Bhattacharya (Halder) and Gour Pal, Study on Pre- m_X open Set International Mathematical forum ,Vol.6,2011,no.45,2231-2237..

[5] A.Csaszar, generalized open set in generalized topologies, Acta Math, Hungar,106(2005),53-66.

[6] Won Keun Min and Young Key Kim, On minimal precontinuous functions,Journal of the Chungoheong Mathematical, Society,Vol-22,no.1,December 2009.

[7] N. Levine, Generalized closed sets in topology, Rend. circ. Mat .palermo(2) 19(197

[8] H. Maki ; On generalizing semi open and preopen sets ,Report of meeting of topological spaces theory and application ;Yazsushiro college of Technology; August-1996. 13-18

[9] A.S. Mashhour, M.E. Abd El-Monsef and S.N. El-Deeb (On pre-continuous and weak Pre continuous mappings, Proc. Math. Phys. Soc. Egypt, 53 (1982), 47–53

[10] A.S.Mashour,M.E Abd El-Monsef and S.N El-Deep, on pre-continuous and week precontinuous,proc,Math phys, soc, Egypt 53(1982)47-53.

[11] G.B.Navalagi; on Semi-precontinuous fuction and properties of generalized Semi-pre closed Sets inTopological Spaces,IJMMS 29(2002)85-98.Hindawi Publishing Corp.

[12] T.Nori and V.pora ,on upper and lower M-continuous multifunction ,Filomat 14(200),73-86.

[13] Popa and T.Nori ; on the definition of some generalized forms of continuity under minimal conditions,mem.Fac sci Kochi Univ(Math),22(2001),9-18

[14] V.popa and T.Noiri, on M-continuous functions, Anal. Univ. “Dunarea de jos” Galati, ser. Mat. Fiz. Mec. Teor.(2),18(23)(2000),31-41.

[15] I.L.Reilly and M.K.Vamanamurthy on alpha –continuity in topological space Acta Math.Hungar.45(1985),no1-2,27-32.

[16] Ennis Rosas, Neelamegarajan Rajesh, Carlos Carpintero, Some New Types of Open and Closed Sets in Minimal Structures-II[1], International Mathematical Forum, 4, 2009, no. 44, 2185 – 2198

[17] D.Sivaraj, semi-homeomorphism, Acta Math. Hungar,48(1986),no.1-2,139-145.

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COMPACTIFICATION OF SOFT TOPOLOGICAL SPACES

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Abstaract — In this work, we define dense soft set and compact softsubset. We then define one point compactification on soft topological spaces.

Keywords — *Soft sets, soft topology, soft compactification.*

1 Introduction

Many problems in economics, engineering, environmental scinece and social science are highly dependent on the task of modelling uncertain data, but modelling uncertain data is usually highly complicated and difficult to characterize. There are several theories which can be used for dealing these difficulties. Some of these theories are probability theory, fuzzy set theory, rough set theory and the interval mathematics. However, these theories have their own difficulties. In 1999, the soft set theory was introduced as a new mathematical tool to solve these diffuculties by Molodtsov [17]. Following his work Maji et.al. [14] gave several basic notions and the first practical application of soft sets in decision making problems. After that, Pei Miao [18] and Chen [9] improved the work of Maji et. al.. Many researchers applied this concept on topological spaces [7, 19, 21, 3], group theory, ring theory [1, 4, 12, 11, 13], and also decison making problems [5, 6, 9, 15].

Recently, Shabir and Naz [19] introduced the soft topological spaces. They defined soft open sets, soft closed sets, soft subspace, soft closure, soft nhood, soft speration axioms and their several properties. In 2012, Zorlutuna et. al. [21] initiated the soft continuity of soft functions, soft compactness and studied some properties. Then, many reasarchers [2, 10, 8, 20, 16] improved to concept of soft topological spaces.

In this paper, we introduce a notion of one point compactification on soft topological spaces.

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2 Preliminary

Throughout this paper X denotes initial universe, E denotes the set of all possible parameters which are attributes, characteristic or properties of the objects in X, and the set of all subsets of X will be denoted by $P(X)$.

Definition 2.1. [17] Let X be the initial universe set and E be the set of parameters. A pair (F, A) is called a soft set over X where F is a mapping given by $F : A \to P(X)$ and $A \subseteq E$.

In the other words, the soft set is a parametrized family of subsets of the set X. Every set $F(e)$, for every $e \in A$, from this family may be considered as the set of e-elements of the soft set (F, A).

From now on, the set of all soft sets over X will be denoted by $S(X, E)$**.**

Definition 2.2. [5]Let $A \subseteq E$. A soft set F_A over universe X is mapping from the parameter set E to $P(X)$, i.e., $F_A : E \to P(X)$, where $F_A(e) \neq \varnothing$ if $e \in A \subset E$ and $F_A(e) = \varnothing$ if $e \notin A$.

Definition 2.3. [5] The soft set $F_E \in S(X, E)$ is called null soft set, denoted by $F_\emptyset$, if for all $e \in E$, $F_E(e) = \varnothing$.

Definition 2.4. [5]Let $F_E \in S(X, E)$. The soft set F_E is called universal soft set, denoted by $F_{\widetilde{E}}$, if for all $e \in E$, $F_E(e) = X$.

Definition 2.5. [5]Let $F_A, G_B \in S(X, E)$. F_A is called a soft subset of G_B if $F_A(e) \subset G_B(e)$ for every $e \in E$ and we write $F_A \widetilde{\subset} G_B$.

Definition 2.6. [5]Let $F_A, G_B \in S(X, E)$. F_A and G_B are said to be equal, denoted by $F_A = G_B$ if $F_A \widetilde{\subset} G_B$ and $G_B \widetilde{\subset} F_A$.

Definition 2.7. [5]Let $F_A, G_B \in S(X, E)$. Then the union of F_A and G_B is also a soft set H_C, defined by $H_C(e) = F_A(e) \cup G_B(e)$ for all $e \in E$, where $C = A \cup B$. Here we write $H_C = F_A \widetilde{\cup} G_B$.

Definition 2.8. [5]Let $F_A, G_B \in S(X, E)$. Then the intersection of F_A and G_B is also a soft set H_C, defined by $H_C(e) = F_A(e) \cap G_B(e)$ for all $e \in E$, where $C = A \cap B$. Here we write $H_C = F_A \widetilde{\cap} G_B$.

Definition 2.9. [5]Let $F_A \in S(X, E)$. The complement of F_A, denoted by F_A^c, is a soft set defined by $F_A^c(e) = X - F_A(e)$ for every $e \in E$.

Let us call F_A^c to be soft complement function of F_A. Clearly $(F_A^c)^c = F_A$, $(F_{\widetilde{E}})^c = F_\emptyset$ and $(F_\emptyset)^c = F_{\widetilde{E}}$.

Definition 2.10. Let $F_A \in S(X, E)$ and $x \in X$. Then $F_A \widetilde{\cup} x$ is soft set in $S(X, E)$, defined by $(F_A \widetilde{\cup} x)(e) = F_A(e) \cup \{x\}$ for all $e \in E$.

Example 2.11. Let $E = \{e_1, e_2, e_3\}$, $X = \{x_1, x_2, x_3\}$ and $F_A = \{(e_1, \{x_1\}), (e_3, \{x_2, x_3\})\}$. Then $F_A \widetilde{\cup} x_2 = \{(e_1, \{x_1,\ x_2\}),(e_2, \{x_2\})\ (e_3, \{x_2, x_3\})\}$.

Definition 2.12. (see [19]) A soft topological space is a triple (X, τ, E) where X is a nonempty set and τ is a family of soft sets over X satisfying the following properties:
(1) $F_{\widetilde{E}}, F_{\varnothing} \in \tau$
(2) If F_A, $G_B \in \tau$, then $F_A \widetilde{\cap} G_B \in \tau$
(3) If $F_{A_i} \in \tau$, $\forall i \in J$, then $\underset{i\in I}{\widetilde{\cup}} F_{A_i} \in \tau$.
Then τ is called a topology of soft sets on X. Every member of τ is called soft open. G_B is called soft closed in (X, τ, E) if $(G_B)^c \in \tau$.

Example 2.13. Let $E = \{e_1, e_2, ..., e_k\}$ set of parameter, $X = [0, 1)$,

$$F_{An} = \{(e_i, [0, 1 - \frac{1}{n})) : e_i \in E, n \in \mathbb{N}\backslash\{0, 1\}\}$$

and $\tau = \{F_{A_n}\}_{n\in\mathbb{N}\backslash\{0,1\}} \cup F_{\varnothing} \cup F_{[0,1)}$ Then (X, τ, E) soft topological space on X.

Definition 2.14. Let (X, τ, E) be a soft topological space and $F_A \in S(X, E)$. Then $\tau_{F_A} = \{F_A \widetilde{\cap} G_B : G_B \in \tau\}$.

Example 2.15. Let $E = \{e_1, e_2, e_3\}$, $X = \{x_1, x_2, x_3\}$, $F_A = \{(e_1, \{x_1\}), (e_3, \{x_2, x_3\})\}$ and $\tau = \{F_{\varnothing},\ F_E,\ G_B\}$, where $G_B = \{(e_1, \{x_1, x_2\}), (e_2, X)\}$. Then $\tau_{F_A} = \{F_{\varnothing},\ F_A,\ F_A \widetilde{\cap} G_B\}$, where $F_A \widetilde{\cap} G_B = \{(e_1, \{x_1\})\}$.

Definition 2.16. [19]Let (X, τ, E) be a soft topological space and $F_A \in S(X, E)$. The soft closure of F_A denoted by $\overline{F_A}$ is the intersection of all soft closed supersets of F_A.

Clearly, $\overline{F_A}$ is the smallest soft closed set over X which contains F_A.

Definition 2.17. Let (X, τ, E) be a soft topological space and $F_A \in S(X, E)$. F_A is called dense soft set in X if $\overline{F_A} = F_E$.

Definition 2.18. Let (X, τ, E) be a soft topological space and $\mathcal{U} = \{F_{A_i} : i \in I\}$. A family $\mathcal{U}$ of soft sets is a cover of a soft set F_A if $F_A \widetilde{\subset} \widetilde{\cup} \{F_{A_i} : i \in I\}$.

Definition 2.19. A soft topological space (X, τ, E) is compact if each soft open cover of $F_{\widetilde{E}}$ has a finite subcover.

Example 2.20. Let us consider the soft topological space (X, τ, E) in example 2.13. Then (X, τ, E) is not compact topological space because $\{F_{A_n}\}_{n\in\mathbb{N}\backslash\{0,1\}}$ is soft open cover of $F_{\widetilde{E}}$ but there is no finite subcover.

Definition 2.21. Let (X, τ, E) be a soft topological space and $F_A \in S(X, E)$. F_A is called compact soft subset if $(F_A, \tau_{F_A}.E)$ is compact.

Proposition 2.22. Let (X, τ, E) be a soft topological space and $F_A \in S(X, E)$. F_A is a compact if and only if each soft open cover of F_A has a finite subcover.

Proof. Let F_A be a compact and $\mathcal{U}^* = \{G_{B_i} : G_{B_i} \in \tau_{F_A}, i \in I\}$ be a soft open cover of F_A. Since $G_{B_i} \in \tau_{F_A}$, then there exist F_{A_i} soft open sets such that $G_{B_i} = F_{A_i} \widetilde{\cap} F_A$. Since F_A is compact, F_A has a finite subcover of U^*. □

Theorem 2.23. Soft closed set of the compact soft topological space is compact.

Proof. Let (X, τ, E) be a soft topological space, F_A is a soft closed set in X and $\mathcal{U} = \{F_{A_i} : i \in I\}$ soft open cover of F_A. Then $F_A \widetilde{\subset} \widetilde{\bigcup\limits_{i\in I}} F_{A_i}$. Since F_A^c soft open set, $\mathcal{U}^* = \mathcal{U}\widetilde{\cup}(F_A^c)$ is a soft open cover of $F_{\widetilde{E}}$. Again since, (X, τ, E) is a compact soft topological space, then $\mathcal{U}^*$ has a finite subfamily such that $F_{\widetilde{E}} = \widetilde{\bigcup\limits_{i=1}^{n}} F_{A_i}$, hence $F_A \widetilde{\subset} \widetilde{\bigcup\limits_{i=1}^{n}}(F_{A_i} \widetilde{\cap} F_A) = \widetilde{\bigcup\limits_{i=1}^{n}} F_{A_i}$. Thus F_A is compact. □

Proposition 2.24. Let (X, τ, E) be a noncompact soft topological space, $X^* = X \cup \{x\}$ and $\mathcal{W} = \{F_A \widetilde{\cup} x : F_A^c \text{ compact}, F_A \in \tau\}$. Then $\tau^* = \tau \cup \mathcal{W}$ is soft topology on X^*.

Proof. T1) Since $F_\varnothing$ and $F_{\widetilde{E}}$ elements of τ, then $F_\varnothing, F^*_{\widetilde{E}} \in \tau^*$.

T2) Let $F_{A_1}, F_{A_2} \in \tau^*$. Then

Case I. If $F_{A_1}, F_{A_2} \in \tau$, then the proof is clear.

Case II. If $F_{A_1} \in \tau$ and $F_{A_2} \in \mathcal{W}$, then there exists $G_B \in \tau$ such that $F_{A_2} = G_B \widetilde{\cup} x$ and G_B^c is compact. Since $F_{A_1} \widetilde{\cap} F_{A_2} = F_{A_1} \widetilde{\cap} (G_B \widetilde{\cup} x) = F_{A_1} \widetilde{\cap} G_B$, then $F_{A_1} \widetilde{\cap} F_{A_2} \in \tau$. Thus, we have $F_{A_1} \widetilde{\cap} F_{A_2} \in \tau^*$.

Case III. If $F_{A_1}, F_{A_2} \in \mathcal{W}$, then there exist $F_{A_1} = G_{B_1} \widetilde{\cup} x$ and $F_{A_2} = G_{B_2} \widetilde{\cup} x$ such that $G_{B_1}, G_{B_2} \in \tau$ and $G^c_{B_1}, G^c_{B_2}$ are compact. Since $F_{A_1} \widetilde{\cap} F_{A_2} = (G_{B_1} \widetilde{\cup} x) \widetilde{\cap} (G_{B_2} \widetilde{\cup} x) = (G_{B_1} \widetilde{\cap} G_{B_2}) \widetilde{\cup} x$, $(G_{B_1} \widetilde{\cap} G_{B_2})^c$ is compact, then $F_{A_1} \widetilde{\cap} F_{A_2} \in \tau^*$.

T3) Let I be an arbitrary index set and $F_{A_i} \in \tau^*$ for all $i \in I$. Then

Case I. If $F_{A_i} \widetilde{\in} \tau$ for all $i \in I$, then $\widetilde{\bigcup\limits_{i\in I}} F_{A_i} \widetilde{\in} \tau$.

Case II. If $F_{A_{i_0}} \widetilde{\in} \mathcal{W}$ for some $i_0 \in I$, then there exists $G_{B_{i_0}} \in \tau$ such that $F_{A_{i_0}} = G_{B_{i_0}} \widetilde{\cup} x$ and $G^c_{B_{i_0}}$ is compact. Therefore, we have $\widetilde{\bigcup\limits_{i\in I}} F_{A_i} = (\widetilde{\bigcup\limits_{i\neq i_0}} F_{A_i}) \widetilde{\cup} (G_{B_{i_0}} \widetilde{\cup} x) = ((\widetilde{\bigcup\limits_{i\neq i_0}} F_{A_i}) \widetilde{\cup} G_{B_{i_0}}) \widetilde{\cup} x$. Then $((\widetilde{\bigcup\limits_{i\neq i_0}} F_{A_i}) \widetilde{\cup} G_{B_{i_0}})^c = (\widetilde{\bigcap\limits_{i\neq i_0}} F^c_{A_i}) \widetilde{\cap} (G^c_{B_{i_0}})$. Since $\widetilde{\bigcap\limits_{i\neq i_0}} F^c_{A_i}$ is soft closed and $G^c_{B_{i_0}}$ is compact, $((\widetilde{\bigcup\limits_{i\neq i_0}} F_{A_i}) \widetilde{\cup} G_{B_{i_0}})^c$ is compact.

Case III. If $F_{A_i} \widetilde{\in} \mathcal{W}$ for all $i \in I$, then there exist $G_{B_i} \in \tau$ such that $F_{A_i} = G_{B_i} \widetilde{\cup} x$ and $G^c_{B_i}$ is compact. Therefore, we have $\widetilde{\bigcup\limits_{i\in I}} F_{A_i} = \widetilde{\bigcup\limits_{i\in I}} (G_{B_i} \widetilde{\cup} x) = (\widetilde{\bigcup\limits_{i\in I}} G_{B_i}) \widetilde{\cup} x$. Then $((\widetilde{\bigcup\limits_{i\in I}} G_{B_i}) \widetilde{\cup} x)^c = (\widetilde{\bigcap\limits_{i\in I}} G^c_{B_i})$. Since $G^c_{B_i}$ is compact for all $i \in I$, then $\widetilde{\bigcap\limits_{i\in I}} G^c_{B_i}$ is compact. Hence $\widetilde{\bigcup\limits_{i\in I}} F_{A_i} \widetilde{\in} \tau^*$. □

Proposition 2.25. (X^*, τ^*, E) soft topological space is compact.

Proof. Let $\mathcal{U} = \{F_{A_i} : i \in I\}$ be a cover of $F^*_{\widetilde{E}}$. Since $x \in X^*$, then there exists $i_0 \in I$ such that $x \in F_{A_{i_0}} \in \mathcal{U}$. Then there exists $G_B \in \tau$ such that $F_{A_{i_0}} = G_B \widetilde{\cup} x$ where G_B^c is compact. Since G_B^c is compact, then there exist $F_{A_1}, F_{A_2}, ..., F_{A_n} \in \mathcal{U}$ such that $G_B^c \widetilde{\subset} F_{A_1} \widetilde{\cup} F_{A_2} \widetilde{\cup} ... \widetilde{\cup} F_{A_n}$. Then $F^*_{\widetilde{E}} = (F_{\widetilde{E}} \widetilde{\setminus} G_B) \widetilde{\cup} F_{A_{i_0}} \widetilde{\subset} F_{A_1} \widetilde{\cup} F_{A_2} \widetilde{\cup} ... \widetilde{\cup} F_{A_n} \widetilde{\cup} F_{A_{i_0}}$. Hence (X^*, τ^*, E) topological space is compact. □

Proposition 2.26. $F_{\widetilde{E}}$ is dense soft subset in (X^*, τ^*, E) topological space.

Proof. Since $\overline{F_{\widetilde{E}}}$ is the intersection of all soft closed supersets of $F_{\widetilde{E}}$ in $S(X^*, E)$, $\overline{F_{\widetilde{E}}} = F^*_{\widetilde{E}}$. Hence we have $F_{\widetilde{E}}$ is dense soft subset in (X^*, τ^*, E). □

Example 2.27. Let us consider the soft topological space (X, τ, E) in example 2.13 and $X^* = X \cup \{1\} = [0, 1]$. Since $F_\varnothing$ only compact soft set in τ, then

$$\mathcal{W} = \{F_A \widetilde{\cup} x : F_A^c compact, F_A \in \tau\} = \{F_E \widetilde{\cup} x\}$$

Again since $\tau^* = \tau \cup \mathcal{W}$, then (X^*, τ^*, E) soft compact. Hence (X^*, τ^*, E) soft compactification of (X, τ, E).

3 Conclusion

In the present work, we have continued to study soft topological spaces. We introduce soft compactifiation. We hope that the findings in this paper will help researcher enhance and promote the further study soft topology to carry out a general framework for their applications in practical life.

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References

[1] Acar, U., Koyuncu, F. and Tanay, B. *Soft sets and soft rings*, Comput. Math. Appl. **59**,(2010) 3458-3463.

[2] B. Ahmad, S. Hussain, *On some structures of soft topology*, Mathematical Sciences, **6** (2012) 69.

[3] A. Aygunoglu and H. Aygun, *Some notes on soft topological spaces*, Neural. Comput. Appl. (2011), 1-7.

[4] H. Aktaş, N. Çağman, *Soft sets and soft groups*, Inform. Sci., **177(13)**(2007), 2726-2735.

[5] N. Çagman and S. Enginoğlu, *Soft set theory and uni-int decision making*, European J. Oper. Res. **207** (2010) 848–855.

[6] Çağman, N. and Enginoglu, S. *Soft matrix theory and its decision making*, Comput. Math. Appl. **59**, (2010) 3308-3314.

[7] N. Çağman, S. Karataş and S. Enginoğlu, *Soft topology*, Comput. Math. Appl. **62** (2011) 351–358.

[8] B. Chen, *Soft semi-open sets and related properties in soft topological spaces,* Appl. Math. Inf. Sci. **1** (2013) 287-294.

[9] D. Chen, E. C. C. Tsong, D. S. Young and X. Wong, *The parametrization reduction of soft sets and its applications*, Comput. Math. Appl. **49** (2005), 757-763.

[10] H Hazra, P Majumdar, SK Samanta, *Soft topology*, Fuzzy Inf. Eng. **1** (2012) 105-115.

[11] F. Feng, Y. B. Jun and X. Zhao, *Soft semirings*, Comput. Math. Appl. **56** (2008) 2621–2628.

[12] Y. C. Jiang, Y. Tang, Q. M. Chen, J. Wang and S. Q. Tang, *Extending soft sets with description logics*, Knowledge-Based Systems. **24** (2011) 1096–1107.

[13] Y.B. Jun and C. H. Park, *Applications of soft sets in ideal theory of BCK/BCI-algebras*, Inform. Sci. **178** (2008) 2466–2475.

[14] P. K. Maji, R. Biswas, A. R. Roy, *Soft set theory*, Comput. Math. Appl. **45** (2003), 555-562.

[15] P. K. Maji, A. R. Roy, R. Biswas, *An application of soft sets in desicion making problem,* Comput. Math. Appl. **44** (2002), 1077-1083.

[16] W. K. Min, *A note on soft topological spaces*, Comput. Math. Appl. **62** (2011), 3524–3528.

[17] D. Molodtsov, *Soft set theory-First results*, Comput. Math. Appl. **37 (4/5)** (1999), 19-31.

[18] D. Pei and D. Miao, "*From soft sets to information systems*" in Proceedings of the IEEE International Conference on Granular Computing, **2** (2005), 617–621.

[19] M. Shabir and M. Naz, *On soft topological spaces*, Comput. Math. Appl. **61** (2011) 1786–1799.

[20] B.P. Varol, H. Aygun, *On soft Hausdorff spaces*, Annals of Fuzzy Mathematics and Informatics **5** (2013) 15-24.

[21] I. Zorlutuna, M. Akdag, W.K. Min and S. Atmaca, Remarks on soft topological spaces, Annals of Fuzzy Mathematics and Informatics **3 (2)** (2011) 171-185.

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DECOMPOSITIONS OF TOPOLOGICAL FUNCTIONS

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Abstaract − We obtain new classes of sets by using λ-closed sets in topological spaces and study their basic properties; and their connections with other kind of topological sets. Moreover new decompositions of topological functions are obtained.

Keywords − *λ-α-closed set, λ-s-closed set, λ-p-closed set, λ-β-closed set, λ-b-closed set.*

1 Introduction

In 1986, Maki [24] introduced the notion of Λ-sets in topological spaces. A Λ-set is a set A which is equal to its kernel (= saturated set) i.e to the intersection of all open supersets of A. Arenas et al. [4] introduced and investigated the notion of λ-closed sets by involving Λ-sets and closed sets. In 1965, Njastad [29] introduced α-open sets which have been considered as an important research tool in the field of topology.

In this paper, we introduce generalized λ-closed sets in topological spaces. In Section 3, we obtain characterizations of generalized λ-closed sets. In Section 4, we obtain some decompositions of topological functions.

2 Preliminaries

Throughout this paper (X, τ) and (Y, σ) (or X and Y) represent topological spaces on which no separation axioms are assumed unless otherwise mentioned. For a subset A of a space (X, τ), cl(A) and int(A) denote the closure of A and the interior of A respectively.

We recall the following definitions and remark which are useful in the sequel.

Definition 2.1. *A subset A of a topological space (X, τ) is called*

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1. α-open [29] if $A \subseteq int(cl(int(A)))$;
2. preopen [26] if $A \subseteq int(cl(A))$;
3. semi-open [22] if $A \subseteq cl(int(A))$;
4. β-open [1] if $A \subseteq cl(int(cl(A)))$;
5. b-open [3] if $A \subseteq int(cl(A)) \cup cl(int(A))$;

The complements of the above mentioned open sets are called their respective closed sets.

The collection of all α-open (resp. semi-open, preopen, β-open, b-open) sets is denoted by $\alpha O(X)$ (resp. $SO(X)$, $PO(X)$, $\beta O(X)$, $bO(X)$).

The preclosure [31] (resp. semi-closure [14], α-closure [27], β-closure [1], b-closure [3]) of a subset A of X, denoted by $pcl(A)$ (resp. $scl(A)$, $\alpha cl(A)$, $\beta cl(A)$, $bcl(A)$), is defined to be the intersection of all preclosed (resp. semi-closed, α-closed, β-closed, b-closed) sets of (X, τ) containing A. It is known that $pcl(A)$ (resp. $scl(A)$, $\alpha cl(A)$, $\beta cl(A)$, $bcl(A)$) is a preclosed (resp. semi-closed, α-closed, β-closed, b-closed) set.

Definition 2.2. *A subset A of a topological space (X, τ) is called*

1. generalized closed (briefly g-closed) [23] if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is open.
2. α-generalized closed (briefly αg-closed) [25] if $\alpha cl(A) \subseteq U$ whenever $A \subseteq U$ and U is open.
3. a generalized semiclosed (briefly gs-closed) [7] if $scl(A) \subseteq U$ whenever $A \subseteq U$ and U is open.
4. a generalized preclosed (briefly gp-closed) [8] if $pcl(A) \subseteq U$ whenever $A \subseteq U$ and U is open.
5. a generalized semi-preclosed (briefly gsp-closed) [15] if $\beta cl(A) \subseteq U$ whenever $A \subseteq U$ and U is open.
6. a generalized b-closed (briefly gb-closed) [17] if $bcl(A) \subseteq U$ whenever $A \subseteq U$ and U is open.

The complements of the above mentioned closed sets are called their respective open sets.

Definition 2.3. *A subset A of a topological space (X, τ) is called*

1. Λ-set if $A = A^\wedge$ where $A^\wedge = \cap \{G : A \subseteq G,\ G \in \tau\}$ [24].
2. Λ_α-set if $A = \Lambda_\alpha(A)$ where $\Lambda_\alpha(A) = \cap \{G : A \subseteq G,\ G \in \alpha O(X)\}$[13].
3. Λ_s-set if $A = \Lambda_s(A)$ where $\Lambda_s(A) = \cap \{G : A \subseteq G,\ G \in SO(X)\}$[12].
4. Λ_p-set if $A = \Lambda_p(A)$ where $\Lambda_p(A) = \cap \{G : A \subseteq G,\ G \in PO(X)\}$[19].
5. Λ_β-set (= Λ_{sp}-set [30]) if $A = \Lambda_{sp}(A)$ where $\Lambda_{sp}(A) = \cap \{G : A \subseteq G,\ G \in \beta O(X)\}$.

6. Λ_b-set if $A = \Lambda_b(A)$ where $\Lambda_b(A) = \cap \{G : A \subseteq G, G \in bO(X)\}$[11].

Remark 2.4. *In a topological space, every α-closed set is αg-closed but not conversely [25].*

Definition 2.5. *A subset A of a topological space (X, τ) is called*

1. *locally closed set (briefly lc-set)[18] if $A = L \cap F$, where L is open and F is closed.*
2. *αlc^*-set [21] if $A = L \cap F$, where L is open and F is α-closed.*
3. *slc^*-set [5] if $A = L \cap F$, where L is open and F is semi-closed.*
4. *λ-closed set [4] if $A = L \cap F$, where L is Λ-set and F is closed.*

Definition 2.6. *A function $f : X \to Y$ is called*

1. *continuous [9] if $f^{-1}(V)$ is closed in X for every closed subset V of Y.*
2. *α-continuous [27] if $f^{-1}(V)$ is an α-closed in X for every closed subset V of Y.*
3. *αg-continuous [20] if $f^{-1}(V)$ is an αg-closed in X for every closed subset V of Y.*
4. *αlc^*-continuous [21] if $f^{-1}(V)$ is αlc^*-set in X for every closed subset V of Y.*
5. *semi-continuous [22] if $f^{-1}(V)$ is semi-closed in X for every closed subset V of Y.*
6. *gs-continuous [32] if $f^{-1}(V)$ is gs-closed in X for every closed subset V of Y.*
7. *slc^*-continuous [5] if $f^{-1}(V)$ is slc^*-set in X for every closed subset V of Y.*
8. *precontinuous [26] if $f^{-1}(V)$ is preclosed in X for every closed subset V of Y.*
9. *gp-continuous [6] if $f^{-1}(V)$ is gp-closed in X for every closed subset V of Y.*
10. *gsp-continuous [15] if $f^{-1}(V)$ is gsp-closed in X for every closed subset V of Y.*
11. *gb-continuous [17] if $f^{-1}(V)$ is gb-closed in X for every closed subset V of Y.*
12. *β-continuous [1] if $f^{-1}(V)$ is β-closed in X for every closed subset V of Y.*
13. *b-continuous [16] if $f^{-1}(V)$ is b-closed in X for every closed subset V of Y.*

3 Characterizations of generalized λ-closed sets

Definition 3.1. *A subset A of a topological space (X, τ) is called*

1. *αg^*-closed [28] if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is α-open.*
2. *sg^*-closed [28] if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is semi-open.*
3. *pg^*-closed [28] if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is preopen.*

4. *βg^*-closed [28] if cl(A) $\subseteq$ U whenever A $\subseteq$ U and U is β-open.*

5. *bg*-closed if cl(A) $\subseteq$ U whenever A $\subseteq$ U and U is b-open.*

Definition 3.2. *A subset A of a topological space (X, τ) is called*

1. *αlc-set [2] if $A = L \cap F$ where L is α-open and F is closed.*

2. *slc-set [10] if $A = L \cap F$ where L is semi-open and F is closed.*

3. *plc-set [10] if $A = L \cap F$ where L is preopen and F is closed.*

4. *βlc-set [10] if $A = L \cap F$ where L is β-open and F is closed.*

5. *blc-set if $A = L \cap F$ where L is b-open and F is closed.*

Definition 3.3. *A subset A of a topological space (X, τ) is called λ-α-closed if $A = L \cap F$, where L is Λ-set and F is an α-closed set.*

Proposition 3.4. *Every λ-closed set is λ-α-closed but not conversely.*

Example 3.5. *Let $X = \{a, b, c\}$ with $\tau = \{\emptyset, \{a\}, X\}$. Then $\{b\}$ is λ-α-closed but not λ-closed.*

Lemma 3.6. *For a subset A of a topological space (X, τ), the following conditions are equivalent.*

1. *A is λ-α-closed.*

2. *$A = L \cap \alpha cl(A)$ where L is a Λ-set.*

3. *$A = A^{\wedge} \cap \alpha cl(A)$.*

Lemma 3.7. *In a space X, the following statements hold.*

1. *Every α-closed set is λ-α-closed but not conversely.*

2. *Every Λ-set is λ-α-closed but not conversely.*

3. *Every α-closed set is αlc^*-set but not conversely.*

4. *Every αlc^*-set is λ-α-closed.*

Example 3.8. *Let X and τ be as in Example 3.5. Then*

1. *$\{a\}$ is λ-α-closed but not α-closed.*

2. *$\{b\}$ is λ-α-closed but not Λ-set.*

3. *$\{a\}$ is αlc^*-set but not α-closed.*

Lemma 3.9. *A subset $A \subset (X, \tau)$ is αg-closed if and only if $\alpha cl(A) \subset A^{\wedge}$.*

Theorem 3.10. *For a subset A of a topological space (X, τ), the following conditions are equivalent.*

1. *A is α-closed.*

2. *A is αg-closed and αlc^*-set.*

3. *A is αg-closed and λ-α-closed.*

Proof. (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (3) : Obvious.

(3) $\Rightarrow$ (1) : Since A is αg-closed, by Lemma 3.9, αcl(A) $\subset$ A^. Since A is λ-α-closed, by Lemma 3.6, A = A^ $\cap$ αcl(A) = αcl(A). Hence A is α-closed.

Remark 3.11. *The following Example shows that the concepts of αg-closed set and αlc^*-set are independent of each other.*

Example 3.12. *Let X and τ be as in Example 3.5. Then $\{a, b\}$ is αg-closed but not αlc^*-set in (X, τ). Moreover, $\{a\}$ is αlc^*-set but not αg-closed in (X, τ).*

Remark 3.13. *The following Example shows that the concepts of αg-closed set and λ-α-closed set are independent of each other.*

Example 3.14. *Let $X = \{a, b, c\}$ with $\tau = \{\emptyset, \{a, b\}, X\}$. Then $\{a, c\}$ is αg-closed but not λ-α-closed in (X, τ). Moreover, $\{a, b\}$ is λ-α-closed but not αg-closed in (X, τ).*

Definition 3.15. *A subset A of a topological space (X, τ) is called*

1. *λ-s-closed if $A = L \cap F$, where L is Λ-set and F is semi-closed.*

2. *λ-p-closed if $A = L \cap F$, where L is Λ-set and F is preclosed.*

3. *λ-β-closed if $A = L \cap F$, where L is Λ-set and F is β-closed.*

4. *λ-b-closed if $A = L \cap F$, where L is Λ-set and F is b-closed.*

Definition 3.16. *A subset A of a topological space (X, τ) is called*

1. *plc*-set if $A = L \cap F$, where L is open and F is preclosed.*

2. *βlc^*-set if $A = L \cap F$, where L is open and F is β-closed.*

3. *blc*-set if $A = L \cap F$, where L is open and F is b-closed.*

Lemma 3.17. *A subset $A \subset (X, \tau)$ is*

1. *gs-closed if and only if scl(A) $\subset$ A^.*

2. *gp-closed if and only if pcl(A) $\subset$ A^.*

3. *gsp-closed if and only if βcl(A) $\subset$ A^.*

4. *gb-closed if and only if bcl(A) $\subset$ A^.*

Corollary 3.18. *For a subset A of a topological space (X, τ), the following conditions are equivalent.*

1. *(a) A is semi-closed.*

   (b) A is gs-closed and slc-set.*

   (c) A is gs-closed and λ-s-closed.

2. (a) *A is preclosed.*
   (b) *A is gp-closed and plc*-set.*
   (c) *A is gp-closed and λ-p-closed.*
3. (a) *A is β-closed.*
   (b) *A is gsp-closed and βlc^*-set.*
   (c) *A is gsp-closed and λ-β-closed.*
4. (a) *A is b-closed.*
   (b) *A is gb-closed and blc*-set.*
   (c) *A is gb-closed and λ-b-closed.*

Proof. The proof is similar to that of Lemma 3.6, Lemma 3.17 and Theorem 3.10.

Remark 3.19. *The following Examples show that the concepts of*

1. *gs-closed set and slc*-set are independent of each other.*
2. *gs-closed set and λ-s-closed set are independent of each other.*
3. *gp-closed set and plc*-set are independent of each other.*
4. *gp-closed set and λ-p-closed set are independent of each other.*
5. *gsp-closed set and βlc^*-set are independent of each other.*
6. *gsp-closed set and λ-β-closed set are independent of each other.*
7. *gb-closed set and blc*-set are independent of each other.*
8. *gb-closed set and λ-b-closed set are independent of each other.*

Example 3.20. *Let X and τ be as in Example 3.14. Then*

1. *$\{a, c\}$ is gs-closed but not slc*-set in (X, τ). Moreover, $\{a, b\}$ is slc*- set but not gs-closed in (X, τ).*
2. *$\{b, c\}$ is gs-closed but not λ-s-closed in (X, τ). Moreover, $\{a, b\}$ is λ-s-closed but not gs-closed in (X, τ).*

Example 3.21. *Let $X = \{a, b, c\}$ with $\tau = \{\emptyset, \{a\}, \{a, c\}, X\}$. Then*

1. *$\{a, b\}$ is gp-closed but not plc*-set in (X, τ). Moreover, $\{a, c\}$ is plc*-set but not gp-closed in (X, τ).*
2. *$\{a, b\}$ is gp-closed but not λ-p-closed in (X, τ). Moreover, $\{a\}$ is λ-p-closed but not gp-closed in (X, τ).*

Example 3.22. *Let $X = \{a, b, c\}$ with $\tau = \{\emptyset, \{b\}, \{a, b\}, X\}$. Then*

1. *$\{b, c\}$ is gsp-closed but not βlc^*-set in (X, τ). Moreover, $\{b\}$ is βlc^*- set but not gsp-closed in (X, τ).*

2. *$\{b, c\}$ is gsp-closed but not λ-β-closed in (X, τ). Moreover, $\{a, b\}$ is λ-β-closed but not gsp-closed in (X, τ).*

3. *$\{b, c\}$ is gb-closed but not blc*-set in (X, τ). Moreover, $\{a, b\}$ is blc*-set but not gb-closed in (X, τ).*

4. *$\{b, c\}$ is gb-closed but not λ-b-closed in (X, τ). Moreover, $\{b\}$ is λ-b-closed but not gb-closed in (X, τ).*

Remark 3.23. *We have the following diagrams for the subsets we stated above:*

Diagram 1.

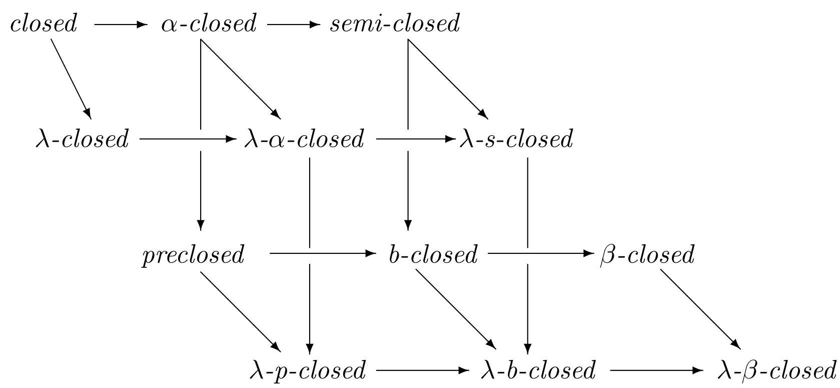
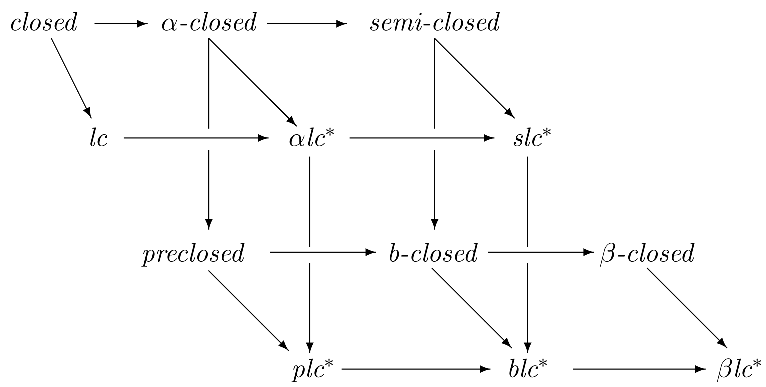


Diagram 2.



Definition 3.24. *A subset A of a topological space (X, τ) is called*

1. *λ-αg^*-closed if $A = L \cap F$, where L is a Λ_α-set and F is closed.*

2. *λ-sg*-closed if $A = L \cap F$, where L is a Λ_s-set and F is closed.*

3. *λ-pg*-closed if $A = L \cap F$, where L is a Λ_p-set and F is closed.*

4. *λ-βg*-closed if $A = L \cap F$, where L is a Λ_{sp}-set and F is closed.*

5. *λ-bg*-closed if $A = L \cap F$, where L is a Λ_b-set and F is closed.*

Lemma 3.25. 1. *Every αlc-set (resp. slc-set, plc-set, βlc-set, blc-set) is λ-αg*-closed (resp. λ-sg*-closed, λ-pg*-closed, λ-βg*-closed, λ-bg*-closed).*

2. *Every Λ_α-set (resp. Λ_s-set, Λ_p-set, Λ_{sp}-set, Λ_b-set) is λ-αg*-closed (resp. λ-sg*-closed, λ-pg*-closed, λ-βg*-closed, λ-bg*-closed).*

Lemma 3.26. 1. *A subset $A \subset (X, \tau)$ is αg*-closed if and only if $cl(A) \subset \Lambda_\alpha(A)$.*

2. *A subset $A \subset (X, \tau)$ is sg*-closed if and only if $cl(A) \subset \Lambda_s(A)$.*

3. *A subset $A \subset (X, \tau)$ is pg*-closed if and only if $cl(A) \subset \Lambda_p(A)$.*

4. *A subset $A \subset (X, \tau)$ is βg*-closed if and only if $cl(A) \subset \Lambda_\beta(A)$.*

5. *A subset $A \subset (X, \tau)$ is bg*-closed if and only if $cl(A) \subset \Lambda_b(A)$.*

Lemma 3.27. *For a subset A of a topological space (X, τ), the following conditions are equivalent.*

1. *A is λ-αg*-closed.*

2. *$A = L \cap cl(A)$ where L is a Λ_α-set.*

3. *$A = \Lambda_\alpha(A) \cap cl(A)$.*

Theorem 3.28. *For a subset A of a topological space (X, τ), the following conditions are equivalent.*

1. (a) *A is closed.*
   (b) *A is αg*-closed and αlc-set.*
   (c) *A is αg*-closed and λ-αg*-closed.*

2. (a) *A is closed.*
   (b) *A is sg*-closed and slc-set.*
   (c) *A is sg*-closed and λ-sg*-closed.*

Remark 3.29. *The following Examples show that the concepts of*

1. *αg*-closed set and αlc-set are independent of each other.*

2. *αg*-closed set and λ-αg*-closed set are independent of each other.*

3. *sg*-closed set and slc-set are independent of each other.*

4. *sg*-closed set and λ-sg*-closed set are independent of each other.*

Example 3.30. *Let X and τ be as in Example 3.14. Then*

1. *$\{a,\ c\}$ is αg*-closed but it is neither αlc-set nor λ-αg*-closed in X.*

2. *$\{a,\ b\}$ is both αlc-set and λ-αg*-closed but not αg*-closed in X.*

Example 3.31. *Let $X = \{a,\ b,\ c\}$ with $\tau = \{\emptyset,\ \{b,\ c\},\ X\}$. Then*

1. *$\{a,\ b\}$ is sg*-closed but it is neither slc-set nor λ-sg*-closed in X.*

2. *$\{b,\ c\}$ is both slc-set and λ-sg*-closed but not sg*-closed in X.*

Remark 3.32. *We have the following diagrams for the subsets we stated above:*

Diagram 3.

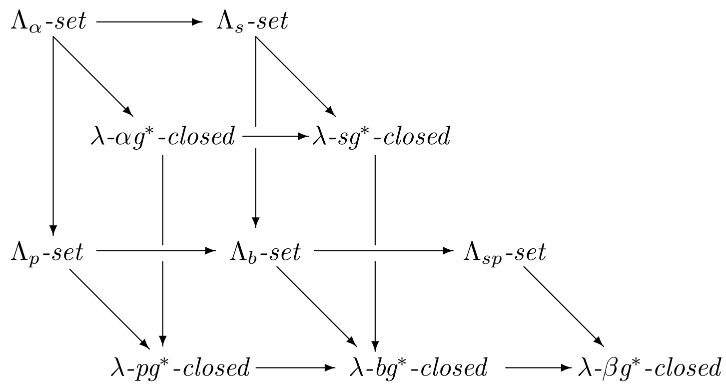
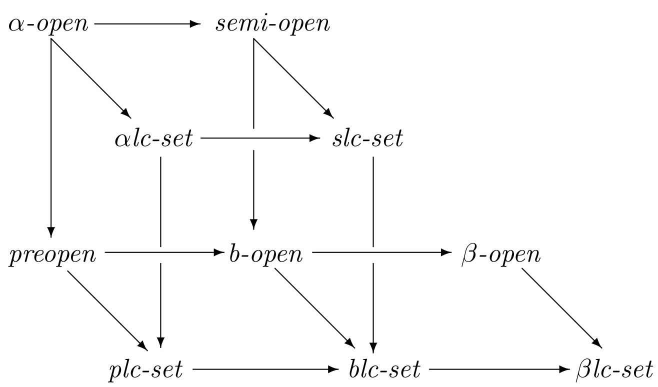


Diagram 4.



4 Decompositions of Topological Functions

Definition 4.1. *A function $f : (X,\ \tau) \to (Y,\ \sigma)$ is called*

1. *λ-α-continuous if $f^{-1}(V)$ is a λ-α-closed set in X for every closed subset V of Y.*

2. λ-s-*continuous if* $f^{-1}(V)$ *is a* λ-s-*closed set in* X *for every closed subset* V *of* Y.

3. λ-p-*continuous if* $f^{-1}(V)$ *is a* λ-p-*closed set in* X *for every closed subset* V *of* Y.

4. λ-β-*continuous if* $f^{-1}(V)$ *is a* λ-β-*closed set in* X *for every closed subset* V *of* Y.

5. λ-b-*continuous if* $f^{-1}(V)$ *is a* λ-b-*closed set in* X *for every closed subset* V *of* Y.

Definition 4.2. *A function* $f : (X, \tau) \to (Y, \sigma)$ *is called*

1. αg^*-*continuous if* $f^{-1}(V)$ *is an* αg^*-*closed set in* X *for every closed subset* V *of* Y.

2. sg^*-*continuous if* $f^{-1}(V)$ *is a* sg^*-*closed set in* X *for every closed subset* V *of* Y.

3. αlc-*continuous if* $f^{-1}(V)$ *is an* αlc-*set in* X *for every closed subset* V *of* Y.

4. slc-*continuous if* $f^{-1}(V)$ *is a* slc-*set in* X *for every closed subset* V *of* Y.

5. λ-αg^*-*continuous if* $f^{-1}(V)$ *is an* λ-αg^*-*closed set in* X *for every closed subset* V *of* Y.

6. λ-sg^*-*continuous if* $f^{-1}(V)$ *is a* λ-sg^*-*closed set in* X *for every closed subset* V *of* Y.

7. plc^*-*continuous if* $f^{-1}(V)$ *is a* plc^*-*set in* X *for every closed subset* V *of* Y.

8. βlc^*-*continuous if* $f^{-1}(V)$ *is a* βlc^*-*set in* X *for every closed subset* V *of* Y.

9. blc^*-*continuous if* $f^{-1}(V)$ *is a* blc^*-*set in* X *for every closed subset* V *of* Y.

We have the following decompositions of topological functions.

Theorem 4.3. *Let* $f : X \to Y$ *be a function. Then the following are equivalent.*

1. f *is* α-*continuous.*

2. f *is* αg-*continuous and* αlc^*-*continuous.*

3. f *is* αg-*continuous and* λ-α-*continuous.*

Proof. It follows from Theorem 3.10.

Theorem 4.4. *Let* $f : X \to Y$ *be a function. Then the following are equivalent.*

1. f *is semi-continuous.*

2. f *is* gs-*continuous and* slc^*-*continuous.*

3. f *is* gs-*continuous and* λ-s-*continuous.*

Proof. It follows from Corollary 3.18 (1).

Theorem 4.5. *Let $f : X \to Y$ be a function. Then the following are equivalent.*

1. *f is precontinuous.*
2. *f is gp-continuous and plc*-continuous.*
3. *f is gp-continuous and λ-p-continuous.*

Proof. It follows from Corollary 3.18(2).

Theorem 4.6. *Let $f : X \to Y$ be a function. Then the following are equivalent.*

1. *f is β-continuous.*
2. *f is gsp-continuous and βlc^*-continuous.*
3. *f is gsp-continuous and λ-β-continuous.*

Proof. It follows from Corollary 3.18(3).

Theorem 4.7. *Let $f : X \to Y$ be a function. Then the following are equivalent.*

1. *f is b-continuous.*
2. *f is gb-continuous and blc*-continuous.*
3. *f is gb-continuous and λ-b-continuous.*

Proof. It follows from Corollary 3.18(4).

Theorem 4.8. *Let $f : X \to Y$ be a function. Then the following are equivalent.*

1. *f is continuous.*
2. *f is αg^*-continuous and αlc-continuous.*
3. *f is αg^*-continuous and λ-αg^*-continuous.*

Proof. It follows from Theorem 3.28(1).

Theorem 4.9. *Let $f : X \to Y$ be a function. Then the following are equivalent.*

1. *f is continuous.*
2. *f is sg*-continuous and slc-continuous.*
3. *f is sg*-continuous and λ-sg*-continuous.*

Proof. It follows from Theorem 3.28(1).

Remark 4.10. *The following Examples show that the concepts of the following are independent of each other.*

1. *αg-continuity and αlc^*-continuity.*
2. *αg-continuity and λ-α-continuity.*
3. *gs-continuity and slc*-continuity.*

4. *gs-continuity and* λ-*s-continuity.*
5. *gp-continuity and plc*-continuity.*
6. *gp-continuity and* λ-*p-continuity.*
7. *gsp-continuity and* βlc^*-*continuity.*
8. *gsp-continuity and* λ-β-*continuity.*
9. *gb-continuity and blc*-continuity.*
10. *gb-continuity and* λ-*b-continuity.*
11. αg^*-*continuity and* αlc-*continuity.*
12. αg^*-*continuity and* λ-αg^*-*continuity.*
13. *sg*-continuity and slc-continuity.*
14. *sg*-continuity and* λ-*sg*-continuity.*

Example 4.11. *Let* $X = Y = \{a, b, c\}$, $\tau = \{\emptyset, \{a, b\}, X\}$ *and* $\sigma = \{\emptyset, \{b\}, \{a, b\}, Y\}$. *Then the identity function* $f : (X, \tau) \to (Y, \sigma)$ *is* αg-*continuous but it is neither* αlc^*-*continuous nor* λ-α-*continuous.*

Example 4.12. *Let* $X = Y = \{a, b, c\}$, $\tau = \{\emptyset, \{a\}, \{a, b\}, X\}$ *and* $\sigma = \{\emptyset, \{c\}, \{b, c\}, Y\}$. *Then the identity function* $f : (X, \tau) \to (Y, \sigma)$ *is both* αlc^*-*continuous and* λ-α-*continuous but not* αg-*continuous.*

Example 4.13. *Let* X, Y, τ *and* σ *be as in Example 4.11. Then the identity function* $f : (X, \tau) \to (Y, \sigma)$ *is* αg^*-*continuous but it is neither* αlc-*continuous nor* λ-αg^*-*continuous.*

Example 4.14. *Let* $X = Y = \{a, b, c\}$, $\tau = \{\emptyset, \{a, b\}, X\}$ *and* $\sigma = \{\emptyset, \{c\}, Y\}$. *Then the identity function* $f : (X, \tau) \to (Y, \sigma)$ *is both* αlc-*continuous and* λ-αg^*-*continuous but not* αg^*-*continuous.*

Example 4.15. *Let* $X = Y = \{a, b, c\}$, $\tau = \{\emptyset, \{a\}, X\}$ *and* $\sigma = \{\emptyset, \{c\}, \{a, c\}, Y\}$. *Then the identity function* $f : (X, \tau) \to (Y, \sigma)$ *is gs-continuous but it is neither slc*-continuous nor* λ-*s-continuous.*

Example 4.16. *Let* $X = Y = \{a, b, c\}$, $\tau = \{\emptyset, \{a\}, X\}$ *and* $\sigma = \{\emptyset, \{a\}, \{b, c\}, Y\}$. *Then the identity function* $f : (X, \tau) \to (Y, \sigma)$ *is both slc*-continuous and* λ-*s-continuous but not gs-continuous.*

Example 4.17. *Let* $X = Y = \{a, b, c\}$, $\tau = \{\emptyset, \{b, c\}, X\}$ *and* $\sigma = \{\emptyset, \{c\}, \{b, c\}, Y\}$. *Then the identity function* $f : (X, \tau) \to (Y, \sigma)$ *is sg*-continuous but it is neither slc-continuous nor* λ-*sg*-continuous.*

Example 4.18. *Let* $X = Y = \{a, b, c\}$, $\tau = \{\emptyset, \{b, c\}, X\}$ *and* $\sigma = \{\emptyset, \{a\}, Y\}$. *Then the identity function* $f : (X, \tau) \to (Y, \sigma)$ *is both slc-continuous and* λ-*sg*-continuous but not sg*-continuous.*

Example 4.19. *Let $X = Y = \{a, b, c\}$, $\tau = \{\emptyset, \{b\}, X\}$ and $\sigma = \{\emptyset, \{c\}, \{b, c\}, Y\}$. Then the identity function $f : (X, \tau) \to (Y, \sigma)$ is gp-continuous but it is neither plc*-continuous nor λ-p-continuous.*

Example 4.20. *Let $X = Y = \{a, b, c\}$, $\tau = \{\emptyset, \{b\}, X\}$ and $\sigma = \{\emptyset, \{a, c\}, Y\}$. Then the identity function $f : (X, \tau) \to (Y, \sigma)$ is both plc*-continuous and λ-p-continuous but not gp-continuous.*

Example 4.21. *In Example 4.19, f is gsp-continuous but it is neither βlc*-continuous nor λ-β-continuous*

Example 4.22. *In Example 4.18, f is both βlc*-continuous and λ-β-continuous but not gsp-continuous.*

Example 4.23. *In Example 4.20, f is gb-continuous but it is neither blc*-continuous nor λ-b-continuous.*

Example 4.24. *Let X, Y and τ be as in Example 4.15 and $\sigma = \{\emptyset, \{b, c\}, Y\}$. Then the identity function $f : (X, \tau) \to (Y, \sigma)$ is both blc*-continuous and λ-b-continuous but not gb-continuous.*

5 Conclusion

Topology is an area of Mathematics concerned with the properties of space that are preserved under continuous deformations including stretching and bending, but not tearing. By the middle of the 20th century, topology had become a major branch of Mathematics.

Topology as a branch of Mathematics can be formally defined as the study of qualitative properties of certain objects that are invariant under a certain kind of transformation especially those properties that are invariant under a certain kind of equivalence and it is the study of those properties of geometric configurations which remain invariant when these configurations are subjected to one-to-one bicontinuous transformations or homeomorphisms. Topology operates with more general concepts than analysis. Differential properties of a given transformation are nonessential for topology but bicontinuity is essential. As a consequence, topology is often suitable for the solution of problems to which analysis cannot give the answer.

Though the concept of topology has been identified as a difficult territory in Mathematics, we have taken it up as a challenge and cherishingly worked out this research study. It can also further up the understanding of basic structure of classical mathematics and offers new methods and results in obtaining significant results of classical mathematics. Moreover it also has applications in some important fields of Science and Technology.

In this paper, we obtained new classes of sets by using λ-closed sets in topological spaces and studied their basic properties; and their connections with other kind of topological sets. Moreover new decompositions of topological functions are obtained.

References

[1] M. E. Abd El-Monsef, S. N. El-Deeb and R. A. Mahmoud, β-open sets and β-continuous mappings, Bull. Fac. Sci. Assuit. Univ., 12 (1983), 77-90.

[2] B. Al-Nashef, A decomposition of α-continuity and semi-continuity, Acta Math. Hungar., 97 (2002), 115-120.

[3] D. Andrijevic, On b-open sets, Mat. Vesnik, 48 (1996), 59-64.

[4] F. G. Arenas, J. Dontchev and M. Ganster, On λ-sets and dual of generalized continuity, Questions Answers Gen. Topology, 15 (1997), 3-13.

[5] I. Arokiarani, Studies on generalizations of generalized closed sets and maps in topological spaces, Ph. D Thesis, Bharathiar University, Coimbatore, 1997.

[6] I. Arokiarani, K. Balachandran and J. Dontchev, Some characterizations of gp-irresolute and gp-continuous maps between topological spaces, Mem. Fac. Sci. Kochi Univ. Ser. A. Math., 20 (1999), 93-104.

[7] S. P. Arya and T. M. Nour, Characterizations of s-normal spaces, Indian J. Pure Appl. Math., 21(8) (1990), 717-719.

[8] K. Balachandran and I. Arokiarani, On generalized preclosed sets (preprint).

[9] K. Balachandran, P. Sundaram and H. Maki, On generalized continuous maps in topological spaces, Mem. Fac. Sci. Kochi Univ. Ser. A. Math., 12 (1991), 5-13.

[10] Y. Beceren, T. Noiri, M. C. Fidanci and K. Arslan, On some generalizations of locally closed sets and lc-continuous functions, Far East J. Math. Sci., (FJMS) 22 (2006), 333-344.

[11] M. Caldas, S. Jafari and T. Noiri, On Λ_b-sets and the associated topology τ^{Λ_b}, Acta Math. Hungar., 110(4) (2006), 337-345.

[12] M. Caldas and J. Dontchev, G.$\wedge_S$-sets and G.$\vee_S$-sets, Mem. Fac. Sci. Kochi. Univ. (Math), 21 (2000), 21-30.

[13] M. Caldas, D. N. Georgiou and S. Jafari, Study of (Λ,α)-closed sets and the related notions in topological spaces, Bull. Malays. Math. Sci., (2) 30(1) (2007), 23-36.

[14] S. G. Crossley and S. K. Hildebrand, Semi-closure, Texas J. Sci., 22 (1971), 99-112.

[15] J. Dontchev, On generalizing semi-preopen sets, Mem. Fac. Sci. Kochi. Univ. Ser. A. Math., 16 (1995), 35-45.

[16] A. A. El-Atik, A study on some types of mappings on topological spaces, M. Sc.,Thesis, Egypt, Tanta University, 1997.

[17] M. Ganster and M. Steiner, On bτ-closed sets, Appl. Gen. Topol., 8(2) (2007), 243-247.

[18] M. Ganster and I. L. Reilly, Locally closed sets and LC-continuous functions, Internat. J. Math. Math. Sci., 12(3) (1989), 417-424.

[19] M. Ganster, S. Jafari and T. Noiri, On pre-$\wedge$-sets and pre-$\vee$-sets, Acta Math. Hungar., 95(4) (2002), 337-343.

[20] Y. Gnanambal, On generalized preregular closed sets in topological spaces, Indian J. Pure Appl. Math., 28(3) (1997), 351-360.

[21] Y. Gnanambal, Studies on generalized pre-regular closed sets and generalizations of locally closed sets, Ph. D Thesis, Bharathiar University, Coimbatore, 1998.

[22] N. Levine, Semi-open sets and semi-continuity in topological spaces, Amer. Math. Monthly, 70 (1963), 36-41.

[23] N. Levine, Generalized closed sets in topology, Rend. Circ. Mat. Palermo, 19(2) (1970), 89-96.

[24] H. Maki, Generalized Λ-sets and the associated closure operator, The special issue in commemoration of Prof. Kazusada IKEDA' Retirement, 1. Oct. (1986), 139-146.

[25] H. Maki, R. Devi and K. Balachandran, Associated topologies of generalized α-closed sets and α-generalized closed sets, Mem. Fac. Sci. Kochi. Ser. A., 15 (1994), 57-63.

[26] A. S. Mashhour, M. E. Abd El-Monsef and S. N. El-Deeb, On precontinuous and weak precontinuous mappings, Proc. Math. Phys. Soc. Egypt, 51 (1982), 47-53.

[27] A. S. Mashhour, I. A. Hasanein and S. N. El-Deeb, α-continuous and α-open mappings, Acta Math. Hungar., 41 (1983), 213-218.

[28] M. Murugalingam, A study of semi-generalized Topology, Ph. D Thesis, Manonmaniam Sundaranar University, Tirunelveli, 2005.

[29] O. Njastad, On some classes of nearly open sets, Pacific J. Math., 15 (1965), 961-970.

[30] T. Noiri and E. Hatir, Λ_{sp}-sets and some weak separation axioms, Acta Math. Hungar., 103(3) (2004), 225-232.

[31] T. Noiri, H. Maki and J. Umehara, Generalized preclosed functions, Mem. Fac. Sci. Kochi Univ. Math, 19 (1998), 13-20.

[32] A. Pushpalatha and K. Anitha, g*s-closed sets in topological spaces, Int. J. Contemp. Math. Sciences, 6(19) (2011), 917-929.

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ON SOME BITOPOLOGICAL SEPARATION AXIOMS

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Abstaract − Fletcher et al. [1] introduced the concept of pairwise compactness for bitopological spaces. Reilly extended this concept to a larger class of bitopological spaces, called pairwise Lindelöf spaces. In this paper we prove some results on the bitopological spaces which have well known topological analogues.

Keywords − *Bitopological space; pairwise Lindelöf; pairwise countably compact.*

1 Introduction

In 1963, Kelly [2] introduced the notion of bitopological spaces. Such spaces equipped with its two (arbitrary) topologies. The reader is suggested to refer [2] for the detail definitions and notations. Furthermore, Kelly was extended some of the standard results of separation axioms in a topological space to a bitopological space. Such extensions are pairwise regular, pairwise Hausdorff and pairwise normal. There are several works [1] dedicated to the investigation of bitopologies, i.e., pairs of topologies on the same set; most of them deal with the theory itself but very few with applications. We are concerned in this paper with the idea of pairwise Lindelöf in bitopological spaces and give some results.

2 Preliminary

Throughout this paper, all spaces (X, τ) and (X, τ_1, τ_2) (or simply X) are always mean topological spaces and bitopological spaces, respectively. Let F be a subset

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of (X, τ_1, τ_2), $\tau_1 - cl(F)$ and $\tau_2 - cl(F)$ represent the τ_1-closure and τ_2-closure of F with respect to τ_1 and τ_2, respectively. The open (respectively closed) sets in X with respect to τ_1 is denoted by τ_1-open (respectively τ_1-closed), and the open (respectively closed) sets in X with respect to τ_2 is denoted by τ_2-open (respectively τ_2-closed).

Definition 2.1. *A bitopological space (X, τ_1, τ_2) is said to be pairwise-compact if the topological space (X, τ_1) and (X, τ_2) are both compact. Equivalently, (X, τ_1, τ_2) is pairwise-compact if every τ_1-open cover of X can be reduced to a finite τ_1-open cover and every τ_2-open cover of X can be reduced to a finite τ_2-open cover.*

In [5], it was mentioned that Birsan has given definitions of pairwise compactness which do allow Tychonoff product theorems. According to Birsan, a bitopological space (X, τ_1, τ_2) is said to be pairwise compact (denote p_1-compact) if every τ_1-open cover of X can be reduced to a finite τ_2-open cover and every τ_2-open cover of X can be reduced to a finite τ_1-open cover. We will generalize it to pairwise Lindelöf in Section 4.

We shall sometimes say that a bitopological space (X, τ_1, τ_2) has a particular topological property, without referring specifically to τ_1 or τ_2, and we shall then mean that both (X, τ_1) and (X, τ_2) have the property; for instance, (X, τ_1, τ_2) is said to satisfy second axiom of countability if both (X, τ_1) and (X, τ_2) do so.

Definition 2.2. *Let (X, τ_1, τ_2) be a bitopological space.*

(a) A set G is said to be pairwise open if G are both τ_1-open and τ_2-open in X,

(b) A set F is said to be pairwise closed if F are both τ_1-closed and τ_2-closed in X.

(c) A cover of a bitopological space (X, τ_1, τ_2) is called pairwise open if its elements are members of τ_1 and τ_2 and if contains at least one non-empty member of each τ_1 and τ_2.

3 Bitopological Separation Axioms

Definition 3.1. *[2] In a bitopological space (X, τ_1, τ_2), τ_1 is said to be regular with respect to τ_2 if, for each point $x \in X$, there is a τ_1-neighbourhood base of τ_2-closed sets, or, as is easily seen to be equivalent, if, for each point $x \in X$ and each τ_1-closed set F such that $x \notin F$,there are a τ_1-open set U and a τ_2-open set V such that*

$$x \in U,\ F \subseteq V,\ \text{and}\ U \cap V = \emptyset.$$

(X, τ_1, τ_2) is, or τ_1 and τ_2 are, pairwise regular if τ_1 is regular with respect to τ_2 and vice versa.

Theorem 3.1. *In a bitopological space (X, τ_1, τ_2), τ_1 is regular with respect to τ_2 if and only if for each point $x \in X$ and τ_1-open set H containing x, there exists a τ_1-open set U such that*

$$x \in U \subseteq \tau_2 - cl(U) \subseteq H.$$

Proof. (Necessity) suppose τ_1 is regular with respect to τ_2. Let $x \in X$ and H is a τ_1-open set containing x. Then $G = X \backslash H$ is a τ_1-closed set which $x \notin G$. Since τ_1 is

regular with respect to τ_2, then there are τ_1-open set U and τ_2-open set V such that $x \in U, G \subseteq V$ and $U \cap V = \emptyset$. Since $U \subseteq X \backslash V$, then $\tau_2 - cl(U) \subseteq \tau_2 - cl(X \backslash V) = X \backslash V \subseteq X \backslash G = H$. Thus, $x \in U \subseteq \tau_2 - cl(U) \subseteq H$ as desired.
(Sufficiency)Suppose the condition holds. Let $x \in X$ and F is a τ_1-closed set such that $x \notin F$. Then $x \in X \backslash F$, and by hypothesis there exists a τ_1-open set U such that $x \in U \subseteq \tau_2 - cl(U) \subseteq X \backslash F$. It follows that $x \in U, F \subseteq X \backslash \tau_2 - cl(U)$ and $U \cap (X \backslash \tau_2 - cl(U)) = \emptyset$. This completes the proof. □

Remark 3.1. *In other words, Theorem 3.1 stated that τ_1 is regular with respect to τ_2 if, for each point $x \in X$, there is a τ_1-neighbourhood base of τ_2-closed sets containing x. This is equivalent definition in Definition 3.1.*

If τ_2 is also regular with respect to τ_1, we have the similar result as previous theorem and stated in the following corollary. By these reason we obtain a pairwise regular space.

Corollary 3.1. *In a space bitopological space (X, τ_1, τ_2), τ_2 is regular with respect to τ_1 if and only if for each point $x \in X$ and τ_2-open set H containing x, there exists a τ_2-open set U such that $x \in U \subseteq \tau_1 - cl(U) \subseteq H$.*

If $Y \subseteq X$, then the collections $(\tau_1)_Y = \{A \cap Y : A \in \tau_1\}$ and $(\tau_2)_Y = \{B \cap Y : B \in \tau_2\}$ are the relative topology on Y . A bitopological space $(Y, (\tau_1)_Y, (\tau_2)_Y)$ is then called a subspace of (X, τ_1, τ_2). Moreover, Y is said to be pairwise closed subspace of X if Y is both $(\tau_1)_Y$-closed and $(\tau_2)_Y$-closed in X. The pairwise open subspace is defined in the similar way.
the following theorem shows that, pairwise regular spaces satisfy the hereditary property.

Theorem 3.2. *Every subspace of a pairwise regular bitopological space (X, τ_1, τ_2) is pairwise regular.*
Proof. Let (X, τ_1, τ_2) be a pairwise regular space and let $(Y, (\tau_1)_Y, (\tau_2)_Y)$ be a subspace of (X, τ_1, τ_2). Furthermore, let F be a $(\tau_1)_Y$-closed set in Y, then$F = A \cap Y$ where A is a τ_1-closed set in X. Now if $y \in Y$ and $y \notin F$, then $y \notin A$, so there are τ_1-open set U and τ_2-open set V such that

$$y \in U, \quad A \subseteq V \text{ and } U \cap V = \emptyset.$$

But $U \cap Y$ and $V \cap Y$ are $(\tau_1)_Y$ -open set and $(\tau_2)_Y$ -open set in Y, respectively. Also $y \in U \cap Y$, $F \subseteq V \cap Y$and $(U \cap Y) \cap (V \cap Y) = (U \cap V) \cap Y = \emptyset$.

Similarly, let G be a $(\tau_2)_Y$-closed set in Y , then $G = B \cap Y$ where B is a τ_2-closed set in X. Now if $y \in Y$ and $Y \notin G$, then $y \notin B$, so there are τ_2-open set U and τ_2-open set V such that

$$y \in U, \quad B \subseteq V \text{ and } U \cap V = \emptyset.$$

But $U \cap Y$ and $V \cap Y$ are $(\tau_2)_Y$-open set and $(\tau_1)_Y$-open set in Y, respectively. Also $y \in U \cap Y$, $G \subseteq V \cap Y$ and $(U \cap Y) \cap (V \cap Y) = \emptyset$. This completes the proof. □

Definition 3.2. *(Kelly, 1963). A bitopological space (X, τ_1, τ_2) is said to be pairwise normal if, given a τ_1-closed set A and a τ_2-closed set B with $A \cap B = \emptyset$, there exist a τ_2-open set U and a τ_1-open set V such that $A \subseteq U, B \subseteq V$ and $U \cap V = \emptyset$.*

Equivalently, (X, τ_1, τ_2) *is pairwise normal if, given a* τ_2*-closed set* C *and a* τ_1*-open set* D *such that* $C \subseteq D$*, there are a* τ_1*-open set* G *and* τ_2*-closed set* F *such that* $C \subseteq G \subseteq F \subseteq D$*.*

We shall prove the equivalent definition above in the following theorem.

Theorem 3.3. *A bitopological space* (X, τ_1, τ_2) *is pairwise normal if and only if given a* τ_2*-closed set* C *and a* τ_1*-open set* D *such that* $C \subseteq D$*, there are a* τ_1*-open set* G *and a* τ_2*-closed set* F *such that* $C \subseteq G \subseteq F \subseteq D$*.*
Proof. (Necessity) Suppose (X, τ_1, τ_2) *is pairwise normal. Let* C *be a* τ_2*-closed set and* D *a* τ_1*-open set such that* $C \subseteq D$*. Then* $K = X \backslash D$ *is a* τ_1*-closed set with* $K \cap C = \emptyset$*. Since* (X, τ_1, τ_2) *is pairwise normal, there exists a* τ_2*-open set* U *and a* τ_1*-open set* V *such that* $K \subseteq U, C \subseteq G$ *and* $U \cap G = \emptyset$*. Hence* $G \subseteq X \backslash U \subseteq X \backslash K = D$*. Thus* $C \subseteq G \subseteq X \backslash U \subseteq D$ *and the result follows by taking* $X \backslash U = F$*.*
(Sufficiency)Suppose the condition holds. Let A *be a* τ_1*-closed set and* B *a* τ_2*-closed set with* $A \cap B = \emptyset$*. Then* $D = X \backslash A$ *is a* τ_1*-open set with* $B \subseteq D$*. By hypothesis, there are a* τ_1*-open set* G *and a* τ_2*-closed set* F *such that* $B \subseteq G \subseteq F \subseteq D$*. It follows that* $A = X \backslash D \subseteq X \backslash F, B \subseteq G$ *and* $(X \backslash F) \cap G = \emptyset$*. where* $X \backslash F$ *is* τ_2*-open set and* G *is* τ_1*-open set. This completes the proof.* □

Theorem 3.4. *A bitopological space* (X, τ_1, τ_2) *is pairwise normal if and only if given a* τ_1*-closed set* C *and a* τ_2*-open set* D *such that* $C \subseteq D$*, there are a* τ_2*-open set* U *and a* τ_1*-closed set* F *such that* $C \subseteq U \subseteq F \subseteq D$*.*
Proof. (Necessity) Suppose (X, τ_1, τ_2) *is pairwise normal. Let* C *be a* τ_1*-closed set and* D *a* τ_2*-open set such that* $C \subseteq D$*. Then* $K = X - D$ *is a* τ_2*-closed set with* $C \cap K = \emptyset$*. Since* (X, τ_1, τ_2) *is pairwise normal, there exists a* τ_2*-open set* U *and a* τ_1*-open set* V *such that* $C \subseteq U, K \subseteq V$ *, and* $U \cap V = \emptyset$*. Hence* $U \subseteq X \backslash V \subseteq X \backslash K = D$*. Thus* $C \subseteq U \subseteq X \backslash V \subseteq D$ *and the result follows by taking* $X \backslash V = F$*.*
(Sufficiency)Suppose the condition holds. Let A *be a* τ_1*-closed set and* B *a* τ_2*-closed set with* $A \cap B = \emptyset$*. Then* $D = X - B$ *is a* τ_2*-open set with* $A \subseteq D$*. By hypothesis, there are a* τ_2*-open set* U *and a* τ_1*-closed set* F *such that* $A \subseteq U \subseteq F \subseteq D$*. It follows that* $B = X \backslash D \subseteq X \backslash F, A \subseteq U$ *and* $(X \backslash F) \cap U = \emptyset$*. where* $X \backslash F$ *is* τ_2*-open set and* U *is* τ_2*-open set. This completes the proof.* □

Now we define a new weaker form of pairwise normal bitopological spaces.

Definition 3.3. *A space* (X, τ_1, τ_2) *is said to be pairwise weak normal if, given* A *and* B *are pairwise closed sets with* $A \cap B = \emptyset$*, there exist a* τ_2*-open set* U *and a* τ_1*-open set* V *such that* $A \subseteq U, B \subseteq V$ *, and* $U \cap V = \emptyset$*.*

Theorem 3.5. *A bitopological space* (X, τ_1, τ_2) *is pairwise weak normal if and only if given a pairwise closed set* C *and a pairwise open set* D *such that* $C \subseteq D$*, there are a* τ_1*-open set* G *and a* τ_2*-closed set* F *such that* $C \subseteq G \subseteq F \subseteq D$*.*
Proof. (Necessity) Suppose (X, τ_1, τ_2) *is pairwise weak normal. Let* C *be a pairwise closed set and* D *a pairwise open set such that* $C \subseteq D$*. Then* $K = X \backslash D$ *is a pairwise closed set with* $K \cap C = \emptyset$*. Since* (X, τ_1, τ_2) *is pairwise weak normal, there exists a* τ_2*-open set* U *and a* τ_1*-open set* G *such that* $K \subseteq U, C \subseteq G$ *and* $U \cap G = \emptyset$*. Hence*

$G \subseteq X \backslash U \subseteq X \backslash K = D$. Thus $C \subseteq G \subseteq X \backslash U \subseteq D$ and the result follows by taking $X \backslash U = F$.
(Sufficiency)Suppose the condition holds. Let A and B are pairwise closed sets with $A \cap B = \emptyset$. Then $D = X \backslash A$ is a pairwise open set with $B \subseteq D$. By hypothesis, there are a τ_1-open set G and a τ_2-closed set F such that $B \subseteq G \subseteq F \subseteq D$. It follows that $A = X \backslash D \subseteq X \backslash F, B \subseteq G$ and $(X \backslash F) \cap G = \emptyset$. where $X \backslash F$ is τ_2-open set and G is τ_1-open set. This completes the proof. □

Example 3.1. *Consider $X = \{a, b, c\}$ with topologies $\tau_1 = \{\emptyset, \{b\}, \{c\}, \{b, c\}, X\}$ and $\tau_2 = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{b, c\}, X\}$ defined on X. Observe that τ_1-closed subsets of X are $\emptyset, \{a, c\}, \{a, b\}, \{a\}$, and X and τ_2-closed subsets of X are $\emptyset, \{b, c\}, \{a, c\}, \{c\}, \{a\}$ and X. It follows that (X, τ_1, τ_2) does satisfy the condition in definition of pairwise normal. One of them we can take $A = \{a\}, B = \{b, c\}, U = \{a\}$ and $V = \{b, c\}$ in the definition, we can checks for the other. Hence (X, τ_1, τ_2) is pairwise normal, and hence pairwise weak normal.*

It is clear from definition that every pairwise normal space is pairwise weak normal. The converse is not true in general as shown in the following counter-example.

Example 3.2. *Consider $X = \{a, b, c, d\}$ with topologies $\tau_1 = \{\emptyset, \{a, b\}, X\}$ and $\tau_2 = \{\emptyset, \{a\}, \{b\}, \{b, c, d\}, X\}$ defined on X. Observe that τ_1-closed subsets of X are $\emptyset, \{c, d\}$ and X and τ_2-closed subsets of X are $\emptyset, \{b, c, d\}, \{a\}$ and X is pairwise weak normal as we can checks since the only pairwise closed sets of X are $\emptyset$ and X. However (X, τ_1, τ_2) is not pairwise normal since the τ_1-closed set $A = \{c, d\}$ and τ_2-closed set $B = \{a\}$ satisfy $A \cap B = \emptyset$, but do not exist the τ_2-open set U and τ_1-open set V such that $A \subseteq U, B \subseteq V$ and $U \cap V = \emptyset$.*

Naturally, any result stated in terms of τ_1 and τ_2 has a dual, in terms of τ_2 and τ_1. The definitions of separation properties of two topologies τ_1 and τ_2, such as pairwise regularity, of course reduce to the usual separation properties of one topology τ_1, such as regularity, when we take $\tau_1 = \tau_2$, and the theorems quoted above then yield as corollaries of the classical results of which they are generalizations.

4 Pairwise Lindelöf Spaces

According to Definition 2.1, we generalize pairwise compact spaces to pairwise Lindelöf as the following.

Definition 4.1. *A bitopological space (X, τ_1, τ_2) is said to be pairwisw Lindelöf if the topological space (X, τ_1) and (X, τ_2) are both Lindelöf. Equivalently, (X, τ_1, τ_2) is pairwisw Lindelöf if every τ_1-open cover of X can be reduced to a countable τ_1-open cover and every τ_2-open cover of X can be reduced to a countable τ_2-open cover. Equivalently, (X, τ_1, τ_2) is pairwise Lindelöf if every pairwise open cover of (X, τ_1, τ_2) be a countable subcover.*

Recall that, the relation between compactness and Lindelöfness is very strong, where every pairwise compact space is pairwise Lindelöf but not the converse, and hence the relation between pairwise compactness and pairwise Lindelöfness is very strong also.

Example 4.1. *Let* $X = [0, \Omega]$, τ_1 *be the discrete topology on* X *and* τ_2 *be the topology* $\{\emptyset, X, (a, \Omega)\}$ *for each* $a \in X$. *Then Reilly in [4] proved that* (X, τ_1, τ_2) *is pairwise Lindelöf. Furthermore,* (X, τ_1, τ_2) *is not pairwise compact.*

Theorem 4.1. *If* (X, τ_1, τ_2) *is second countable bitopological space, then* (X, τ_1, τ_2) *is pairwise Lindelöf.*

Proof. *In bitopological space* (X, τ_1, τ_2), *let* $\{B_n\}$ *and* $\{C_n\}$, $n = 1, 2, ...$ *be countable bases for* τ_1 *and* τ_2 *respectively. Let* $\mathcal{U} = \{U_\alpha : \alpha \in \nabla\}$ *be a* τ_1*-open cover of* X, *then for every* $x \in X$, *there exists* $U_x \in \mathcal{U}$ *such that* $x \in U_x$. *From hypothesis* (X, τ_1, τ_2) *is second countable, then so is* (X, τ_1). *Since* $\{B_n\}$ *is a base for* τ_1, *for each* $x \in U_x$ *and* $U_x \in \mathcal{U}$, *there is* $B_x \in \{B_n\}$ *such that* $x \in B_x \subseteq U_x$. *Hence* $X = \bigcup\{B_x : x \in X\}$. *But* $\{B_x : x \in X\} \subseteq \{B_n\}$, *so it is countable and hence* $\{B_x : x \in X\} = \{B_n : n \in \mathbb{N}\}$. *For each* $n \in \mathbb{N}$, *choose one set* $B_n \in \{B_n\}$ *such that* $B_n \subseteq U_n$. *Then* $X = \bigcup\{B_n : n \in \mathbb{N}\} = \{U_n : n \in \mathbb{N}\}$ *and so* $\{U_n : n \in \mathbb{N}\}$ *is a countable subcover of* X. *Thus* (X, τ_1) *is a Lindelöf space. Similarly* (X, τ_2) *is also a Lindelöf space. Therefore* (X, τ_1, τ_2) *is pairwise Lindelöf.* □

Proposition 4.1. *Every pairwise closed subset of a pairwise Lindelöf bitopological space* (X, τ_1, τ_2) *is pairwise Lindelöf.*

Proof. *Let* (X, τ_1, τ_2) *be a pairwise Lindelöf bitopological space and let* F *be a pairwise closed subset of* X. *Then* (X, τ_1) *and* (X, τ_2) *are Lindelöf, and* F *are* τ_1*-closed and* τ_2*-closed subset of* X. *If* $\{U_\alpha : \alpha \in \nabla\}$ *is a* τ_1*-open cover of* F, *then* $X = \{\cup U_\alpha : \alpha \in \nabla\} \cup (X \backslash F)$. *Hence the collection* $\{U_\alpha : \alpha \in \nabla\}$ *and* $X \backslash F$ *form a* τ_1*-open cover of* X. *Since* (X, τ_1) *is Lindelöf, there will be a countable subcover,* $\{X \backslash F, U_{\alpha 1}, U_{\alpha 2}, ...\}$. *But* F *and* $X \backslash F$ *are disjoint; hence the subcollection of* τ_1*-open set* $\{U_{\alpha i} : i \in \mathbb{N}\}$ *also cover* F, *and so* $\{U_\alpha : \alpha \in \nabla\}$ *has a countable subcover.* □

Definition 4.2. *[3] A bitopological space* (X, τ_1, τ_2) *is called pairwise countably compact if every countable pairwise open cover of* (X, τ_1, τ_2) *has a finite subcover.*

The proof of the following two results are straightforward.

Proposition 4.2. *In a pairwise Lindelöf space, pairwise countable compactness, is equivalent to pairwise compactness.*

Proposition 4.3. *The pairwise continuous image of a pairwise Lindelöf space is pairwise Lindelöf.*

Theorem 4.2. *If* A *is a proper subset of a pairwise Lindelöf bitopological space* (X, τ_1, τ_2) *which is* τ_1*-closed, then* A *is pairwise Lindelöf and* τ_2*-Lindelöf.*

Proof. *Let* β *be any pairwise open cover of a bitopological space* $(A, \tau_1|A, \tau_2|A)$. *Then* $\beta \cup \{(X \backslash A)\}$ *induces a pairwise open cover of a bitopological space* (X, τ_1, τ_2) *which has a countable subcover and hence so does* β. *Let* β^* *be any* τ_2*-open cover of* A. *Then* $\beta^* \cup \{(X \backslash A)\}$ *is a pairwise open cover of* (X, τ_1, τ_2) *which has a countable subcover and hence so does* β^*.

Proposition 4.4. *In a bitopological space (X, τ_1, τ_2), let τ_1 be Lindelöf with respect to τ_2. Then τ_1-closed subset of (X, τ_1, τ_2) is also τ_1-Lindelöf with respect to τ_2.*
Proof. *Let F be a τ_1-closed subset of (X, τ_1, τ_2) and let $\{U_\alpha : \alpha \in \nabla\}$ be a τ_1-open cover of F, then $X = (\cup\{U_\alpha : \alpha \in \nabla\}) \cup (X \backslash F)$, hence the collection $\{U_\alpha : \alpha \in \nabla\}$ form a τ_1-open cover of X. Since τ_1 is Lindelöf with respect to τ_2, then the τ_1-open cover of X can be reduced to a countable τ_2-open cover $\{X \backslash F, U_{\alpha 1}, U_{\alpha 2}, ...\}$. But for $X \backslash F$ are disjoint, hence the subcollection of τ_2-open set $\{U_{\alpha i} : i \in \mathbb{N}\}$ also cover F and so $\{U_\alpha : \alpha \in \nabla\}$ can be reduced to a countable τ_2-open cover. This shows that F is τ_1-Lindelöf with respect to τ_2.*

Corollary 4.1. *If τ_2 is Lindelöf with respect to τ_1, then τ_2-closed subset of a bitopological space (X, τ_1, τ_2) is τ_2-Lindelöf with respect to τ_1.*

5 Conclusion

For the following separation axioms, we can apply the results established in Sections 3 and 4:
(1) Spaces defined in Definition 3.3.
(2) Spaces defined in Definition 4.1.

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References

[1] P. Fletche; H.B. Hoyle and C.W. Patty, The comparsion of topologies, Duke Math.T., 36(1969), 325-331.

[2] J.C. Kelly, Bitopological spaces, Proc. London Math. Soc. (3)13(1963), 71-89.

[3] D.H. Pahk and B.D. Choi, Notes on pairwise compactness, KyungTook Math. J. II(1971), 52-54.

[4] I.L. Reilly, Pairwise Lindelöf bitopological spaces, Kyungpouk Math.J. Vol.13, Nu.1(1973), 1-3.

[5] M.J. Saegrove, Pairwise complete regularity and compactification in bitopological spaces, J. London Math. Soc. (2)7(1973), 286-290.

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ON L-FUZZY INTERIOR (CLOSURE) SPACES

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Abstaract − The aim of this paper is to introduce the concept of L-fuzzy interior (closure) spaces and the L-fuzzy topological space in a complete residuated lattice. We study some relationships among those structures. Finally, we give their examples.

Keywords − *Complete residuated lattice, L-fuzzy interior operator, L-fuzzy closure operator, L-fuzzy topological space and continuous maps.*

1 Introduction

Since Chang [6] introduced fuzzy set theory to topology, many researchers have successfully generalized the theory of general topology to the fuzzy setting with crisp methods. In Chang's I-topology on a set X, each open set was fuzzy, while the topology itself was a crisp subset of the family of all fuzzy subsets of X.

From a different direction, the fundamental idea of a topology itself being fuzzy was first defined by Höhle [14] in 1980, then was independently generalized be each of Kubiak [17] and Sôstak [25] in 1985 and independently rediscovered by Ying [26, 27] in Höhle's original setting in 1991 in Höhle's approach a topology was an L-subset of a traditional powerset.

In 1999, the axioms of many-valued L-fuzzy topological spaces and L-fuzzy continuous mappings are given a lattice-theoretical foundation by Höhle and Sôstak and a categorical foundation by Rodabaugh [23]. Sôstak [25] introduced the fuzzy

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topology as an extension of Chang's fuzzy topology, Ramadan and his colleagues [21] called it smooth topology.

Closure and interior operators on ordinary sets belongs to the very fundamental mathematical structure with direct applications, both mathematical (topology, logic, for instance) and extra mathematical (e.g. data mining, knowledge representation). In fuzzy set theory, several particular cases as well as general theory of closure operators which operate with fuzzy sets (so called fuzzy closure operators) are studied (Mashour and Ghanim [19], Bandler and Kohout [1], Bêlohàvek [2, 3], Gerla [11]).

Interior operators, however, have appeared in a few studies only (Bandler and Kohout [1], Dubois and Prade [7], Bodenhofer et al [5]), and it seem that no general theory of interior operators appeared so far. In ordinary set theory, closure and interior operators on a set in a bijective correspondence.

In this paper is, we investigate the concept of L-fuzzy interior (closure) operators using the definition of the L-fuzzy topology, which deduced an L-fuzzy (interior) closure spaces and vise versa. Continuity property and examples of those spaces are also discussed.

2 Preliminary

Definition 2.1. [4, 15] An algebra $(L, \wedge, \vee, \odot, \rightarrow, \perp, \top)$ is called a complete residuated lattice if it satisfies the following conditions

(C1) $L = (L, \leq, \vee, \wedge, \perp, \top)$ is a complete lattice with the greatest element $\top$ and the least element $\perp$;
(C2) $(L, \odot, \top)$ is a commutative monoid;
(C3) $x \odot y \leq z$ iff $x \leq y \rightarrow z$ for $x, y, z \in L$.

An operator $^* : L \rightarrow L$ defined by $a^* = a \rightarrow 0$ is called a *strong negation* if $a^{**} = a$.
For $\alpha \in L,\ \lambda \in L^X$, we denote $(\alpha \rightarrow \lambda),\ (\alpha \odot \lambda),\ \alpha_X,\ \top_x \in L^X$ as

$$(\alpha \rightarrow \lambda)(x) = \alpha \rightarrow \lambda(x),\ (\alpha \odot \lambda)(x) = \alpha \odot \lambda(x),\ \alpha_X(x) = \alpha,$$

$$\top_x(y) = \begin{cases} \top, & \text{if } y = x, \\ \perp, & \text{otherwise.} \end{cases}$$

In this paper, we assume that $(L, \vee, \wedge, \odot, \rightarrow, {}^*, \perp, \top)$ be a complete residuated lattice with a strong negation *.

Lemma 2.2. [4, 15, 24] For each $x, y, z, x_i, y_i \in L$, the following properties hold.

(1) $x \rightarrow y = \top$ iff $x \leq y,\ x \rightarrow \top = \top$ and $\top \rightarrow x = x$,
(2) If $y \leq z$, then $x \rightarrow y \leq x \rightarrow z,\ z \rightarrow x \leq y \rightarrow x,\ x \oplus y \leq x \oplus z$ and $x \odot y \leq x \odot z$,

(3) $x \odot y \leq x \oplus y$,
(4) $x \odot (\bigvee_{i\in\Gamma} y_i) = \bigvee_{i\in\Gamma}(x \odot y_i)$ and $x \odot (\bigwedge_{i\in\Gamma} y_i) \leq \bigwedge_{i\in\Gamma}(x \odot y_i)$,
(5) $x \oplus (\bigvee_{i\in\Gamma} y_i) = \bigvee_{i\in\Gamma}(x \oplus y_i)$ and $(\bigvee_{i\in\Gamma} x_i) \oplus y = \bigvee_{i\in\Gamma}(x_i \oplus y)$,
(6) $x \to (\bigwedge_{i\in\Gamma} y_i) = \bigwedge_{i\in\Gamma}(x \to y_i)$ and $(\bigwedge_{i\in\Gamma} x_i) \to y \geq \bigvee_{i\in\Gamma}(x_i \to y)$,
(7) $x \to (\bigvee_{i\in\Gamma} y_i) \geq \bigvee_{i\in\Gamma}(x \to y_i)$ and $(\bigvee_{i\in\Gamma} x_i) \to y = \bigwedge_{i\in\Gamma}(x_i \to y)$,
(8) $\bigvee_{i\in\Gamma} x_i \to \bigvee_{i\in\Gamma} y_i \geq \bigwedge_{i\in\Gamma}(x_i \to y_i)$ and $\bigwedge_{i\in\Gamma} x_i \to \bigwedge_{i\in\Gamma} y_i \geq \bigwedge_{i\in\Gamma}(x_i \to y_i)$,
(9) $(x \to y) \odot x \leq y$ and $(x \to y) \odot (y \to z) \leq (x \to z)$,
(10) $x \to y \leq (y \to z) \to (x \to z)$, $x \to y \leq (z \to x) \to (z \to y)$ and $y \to z \leq x \odot y \to x \odot z$,
(11) $(x \odot y) \to z = x \to (y \to z) = y \to (x \to z)$,
(12) $x \odot (y \to z) \leq y \to (x \odot z)$,
(13) $(x \to y) \odot (z \to w) \leq (x \odot z) \to (y \odot w)$,
(14) $(x \to y) \odot (z \to w) \leq (x \oplus z) \to (y \oplus w)$,
(15) $(x \to y) \oplus (z \to w) \leq (x \odot z) \to (y \oplus w)$,
(16) $x^* \to y^* = y \to x$,
(17) $\bigwedge_{i\in\Gamma} x_i^* = (\bigvee_{i\in\Gamma} x_i)^*$ and $\bigvee_{i\in\Gamma} x_i^* = (\bigwedge_{i\in\Gamma} x_i)^*$,
(18) $(x \odot y)^* = x \to y^*$ and $(x \to y)^* = x \odot y^*$,
(19) $x \odot (x^* \oplus y^*) \leq y^*$.

Definition 2.3. [2, 3] Let X be a set. A function $R : X \times X \to L$ is called an *L-partial order* if it satisfies the following conditions
(E1) reflexive if $R(x, x) = \top$ for all $x \in X$,
(E2) transitive if $R(x, y) \odot R(y, z) \leq R(x, z)$ for all $x, y, z \in X$,
(E3) if $R(x, y) = R(y, x) = \top$, then $x = y$.

Lemma 2.4. [2, 3] For a given set X, define a binary mapping $S : L^X \times L^X \to L$ by

$$S(\lambda, \mu) = \bigwedge_{x\in X} (\lambda(x) \to \mu(x)).$$

Then, for each $\lambda, \mu, \rho, \nu \in L^X$ and $\alpha \in L$ the following properties hold.
(1) S is an L-partial order on L^X,
(2) $\lambda \leq \mu$ iff $S(\lambda, \mu) \geq \top$,
(3) If $\lambda \leq \mu$, then $S(\rho, \lambda) \leq S(\rho, \mu)$ and $S(\lambda, \rho) \geq S(\mu, \rho)$ for each $\rho \in L^X$,
(4) $S(\lambda, \mu) \odot S(\nu, \rho) \leq S(\lambda \odot \nu, \mu \odot \rho)$,
(5) $S(\lambda, \mu) \odot S(\nu, \rho) \leq S(\lambda \oplus \nu, \mu \oplus \rho)$,
(6) $S(\lambda, \alpha \to \mu) = S(\alpha \odot \lambda, \mu) = \alpha \to S(\lambda, \mu)$ and $\alpha \odot S(\lambda, \mu) \leq S(\lambda, \alpha \odot \mu)$,
(7) $\mu \odot S(\mu, \lambda) \leq \lambda$, $S(\mu, \lambda) \to \lambda \geq \mu$ and $S(\lambda, \mu) = S(\mu^*, \lambda^*)$.

Proof. We need to prove (5) by Lemma 2.2(8),(14), we have

$$\begin{aligned} S(\lambda \oplus \nu, \mu \oplus \rho) &= \bigwedge_{x\in X} \big((\lambda \oplus \nu)(x) \to (\mu \oplus \rho)(x)\big) \\ &\geq \bigwedge_{x\in X} \big((\lambda \to \mu)(x) \odot (\nu \to \rho)(x)\big) \\ &\geq \big(\bigwedge_{x\in X} (\lambda \to \mu)(x)\big) \odot \big(\bigwedge_{x\in X} (\nu \to \rho)(x)\big) \\ &= S(\lambda, \mu) \odot S(\nu, \rho). \end{aligned}$$

Lemma 2.5. [2, 3] Let $\phi : X \to Y$ be an ordinary mapping. Define $\phi^{\to} : L^X \to L^Y$ and $\phi^{\leftarrow} : L^Y \to L^X$ by

$$\phi^{\to}(\lambda)(y) = \bigvee_{\phi(x)=y} \lambda(x) \;\; \forall\, \lambda \in L^X,\; y \in Y,$$

$$\phi^{\leftarrow}(\mu)(x) = \mu(\phi(x)) = \mu \circ \phi(x) \;\; \forall\, \mu \in L^Y.$$

Then for $\lambda, \mu \in L^X$ and $\rho, \nu \in L^Y$,

$$S(\lambda, \mu) \le S(\phi^{\to}(\lambda), \phi^{\to}(\mu)), \qquad S(\rho, \nu) \le S(\phi^{\leftarrow}(\rho), \phi^{\leftarrow}(\nu)),$$

and the equalities hold if ϕ is bijective.

Definition 2.6. [15] A map $\mathcal{T} : L^X \to L$ is called an L-fuzzy topology on X if it satisfies the following conditions.

(LO1) $\mathcal{T}(\perp_X) = \mathcal{T}(\top_X) = \top$,
(LO2) $\mathcal{T}(\lambda \odot \mu) \ge \mathcal{T}(\lambda) \odot \mathcal{T}(\mu), \;\; \forall\, \lambda, \mu \in L^X$,
(LO3) $\mathcal{T}(\bigvee_i \lambda_i) \ge \bigwedge_i \mathcal{T}(\lambda_i), \;\; \forall\, \{\lambda_i\}_{i\in\Gamma} \subseteq L^X$.

An L-fuzzy topology is enriched if (R) $\mathcal{T}(\alpha \odot \lambda) \ge \mathcal{T}(\lambda)$ for all $\lambda \in L^X, \alpha \in L$.

The pair $(X, \mathcal{T})$ is called an L-fuzzy topological space. Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be two L-fuzzy topological spaces. A mapping $\phi : X \to Y$ is said to be LF-fuzzy continuous iff for each $\lambda \in L^Y$, we have

$$\mathcal{T}_Y(\lambda) \le \mathcal{T}_X(\phi^{\leftarrow}(\lambda)).$$

Definition 2.7. [15] A map $\mathcal{F} : L^X \to L$ is called an L-fuzzy co-topology on X if it satisfies the following conditions.

(LF1) $\mathcal{F}(\perp_X) = \mathcal{F}(\top_X) = \top$,
(LF2) $\mathcal{F}(\lambda \oplus \mu) \ge \mathcal{F}(\lambda) \odot \mathcal{F}(\mu), \;\; \forall\, \lambda, \mu \in L^X$,
(LF3) $\mathcal{F}(\bigwedge_i \lambda_i) \le \bigvee_i \mathcal{F}(\lambda_i), \;\; \forall\, \{\lambda_i\}_{i\in\Gamma} \subseteq L^X$.

The pair $(X, \mathcal{F})$ is called an L-fuzzy co-topological space. An L-fuzzy co-topology is called enriched if (S) $\mathcal{F}(\alpha \to \lambda) \ge \mathcal{F}(\lambda)$ for all $\lambda \in L^X$ and $\alpha \in L$.

Let $(X, \mathcal{F}_X)$ and $(Y, \mathcal{F}_Y)$ be two L-fuzzy co-topological spaces. A mapping $\phi : X \to Y$ is said to be LF-fuzzy continuous iff for each $\lambda \in L^Y$, we have

$$\mathcal{F}_Y(\lambda) \le \mathcal{F}_X(\phi^{\leftarrow}(\lambda)).$$

Definition 2.8. [22] A map $\mathcal{I} : L^X \times L_\perp \to L^X,\; L_\perp = L - \{\perp\}$ is called an L-fuzzy interior operator on X if $\mathcal{I}$ satisfies the following conditions

(I1) $\mathcal{I}(\top_X, r) = \top_X$,
(I2) $\mathcal{I}(\lambda, r) \le \lambda$, or equivalently, $S(\mathcal{I}(\lambda, r), \lambda) \ge \top$ for all $\lambda \in L^X$,

(I3) $S(\lambda,\mu) \le S(\mathcal{I}(\lambda,r),\mathcal{I}(\mu,r))$ for all $\lambda,\mu \in L^X$,
(I4) If $r \le s$, then $\mathcal{I}(\lambda,s) \le \mathcal{I}(\lambda,r)$,
(I5) $\mathcal{I}(\lambda \odot \mu, r \odot s) \ge \mathcal{I}(\lambda,r) \odot \mathcal{I}(\mu,s)$.

The pair $(X,\mathcal{I})$ is called an L-fuzzy interior space. An L-fuzzy interior space $(X,\mathcal{I})$ is topological if
(T) $\mathcal{I}(\mathcal{I}(\lambda,r),r) = \mathcal{I}(\lambda,r)\ \ \forall\, \lambda \in L^X, r \in L_\perp$.

Let $(X,\mathcal{I}_X)$ and $(X,\mathcal{I}_Y)$ be two L-fuzzy interior spaces. A map $\phi : X \to Y$ is called $\mathcal{I}$-map if

$$\phi^{\leftarrow}(\mathcal{I}_Y(\mu,r)) \le \mathcal{I}_X(\phi^{\leftarrow}(\mu),r)\ \ \forall\, \mu \in L^Y, r \in L_\perp.$$

Lemma 2.9. Let $\mathcal{I} : L^X \times L_\perp \to L^X$, $L_\perp = L - \{\perp\}$ be a map. It satisfies $S(\lambda,\mu) \le S(\mathcal{I}(\lambda,r),\mathcal{I}(\mu,r))$ for all $\lambda,\mu \in L^X$ iff $\mathcal{I}(\alpha \odot \lambda, r) \ge \alpha \odot \mathcal{I}(\lambda,r)$ and $\mathcal{I}(\lambda,r) \le \mathcal{I}(\mu,r)$ if $\lambda \le \mu$.

Proof. If $\lambda \le \mu$, $\top = S(\lambda,\mu) \le S(\mathcal{I}(\lambda,r),\mathcal{I}(\mu,r))$, then $\mathcal{I}(\lambda,r) \le \mathcal{I}(\mu,r)$. Moreover, $S(\mathcal{I}(\lambda,r),\mathcal{I}(\alpha \odot \lambda,r)) \ge S(\lambda,\alpha \odot \lambda) \ge \alpha$. That is,

$$\alpha \odot \mathcal{I}(\lambda,r) \le \mathcal{I}(\alpha \odot \lambda,r).$$

On the other hand, put $\alpha = S(\lambda,\mu)$, then

$$S(\lambda,\mu) \odot \mathcal{I}(\lambda,r) \le \mathcal{I}(S(\lambda,\mu) \odot \lambda,r) \le \mathcal{I}(\mu,r).$$

Hence, $S(\lambda,\mu) \le S(\mathcal{I}(\lambda,r),\mathcal{I}(\mu,r))$.

Definition 2.10. A map $\mathcal{C} : L^X \times L_\perp \to L^X$ is called an L-fuzzy closure operator on X if $\mathcal{C}$ satisfies the following conditions

(C1) $\mathcal{C}(\perp_X,r) = \perp_X$,
(C2) $\mathcal{C}(\lambda,r) \ge \lambda$, or equivalently, $S(\lambda,\mathcal{C}(\lambda,r)) = \top_X$ for all $\lambda \in L^X$,
(C3) $S(\lambda,\mu) \le S(\mathcal{C}(\lambda,r),\mathcal{C}(\mu,r))$ for all $\lambda,\mu \in L^X$,
(C4) If $r \le s$, then $\mathcal{C}(\lambda,r) \le \mathcal{C}(\lambda,s)$,
(C5) $\mathcal{C}(\lambda \oplus \mu, r \odot s) \le \mathcal{C}(\lambda,r) \oplus \mathcal{C}(\mu,s)$.

The pair $(X,\mathcal{C})$ is called an L-fuzzy closure space. An L-fuzzy closure space $(X,\mathcal{C})$ is topological if
(T) $\mathcal{C}(\mathcal{C}(\lambda,r),r) = \mathcal{C}(\lambda,r)\ \forall\, \lambda \in L^X, r \in L_\perp$.

Let $(X,\mathcal{C}_X)$ and $(X,\mathcal{C}_Y)$ be two L-fuzzy closure spaces. A map $\phi : X \to Y$ is called a $\mathcal{C}$-map if $\phi^{\leftarrow}(\mathcal{C}_Y(\lambda,r)) \ge \mathcal{C}_X(\phi^{\leftarrow}(\lambda),r),\ \ \forall\, \lambda \in L^Y, r \in L_\perp$.

Lemma 2.11. Let $\mathcal{C} : L^X \times L_\perp \to L^X$, $L_\perp = L - \{\perp\}$ be a map. It satisfies $S(\lambda,\mu) \le S(\mathcal{C}(\lambda,r),\mathcal{C}(\mu,r))$ for all $\lambda,\mu \in L^X$ iff $\mathcal{C}(\alpha \odot \lambda,r) \ge \alpha \odot \mathcal{C}(\lambda,r)$ and $\mathcal{C}(\lambda,r) \le \mathcal{C}(\mu,r)$ if $\lambda \le \mu$.

3 L-fuzzy Interior Space Induced by L-fuzzy Topological Space

Theorem 3.1. Let $(X, \mathcal{T})$ be an L-fuzzy topological space. Define the mapping $\mathcal{I}_{\mathcal{T}} : L^X \times L_{\perp} \to L^X$ as follows

$$\mathcal{I}_{\mathcal{T}}(\lambda, r) = \bigvee_{\mu} \{\mu \odot S(\mu, \lambda) \mid \mathcal{T}(\mu) \geq r\}.$$

Then we have the following properties.

(1) $(X, \mathcal{I}_{\mathcal{T}})$ is an L-fuzzy interior space,

(2) If $(X, \mathcal{T})$ is enriched, then $(X, \mathcal{I}_{\mathcal{T}})$ is a strong L-fuzzy interior space,

(3) $\mathcal{I}_{\mathcal{T}}(\lambda, r) \leq \bigvee\{\mu \mid \mu \leq \lambda, \mathcal{T}(\mu) \geq r\}$,

(4) If $(X, \mathcal{T})$ is enriched, then the equality in (3) holds.

Proof. (1) (I1) For each $\mathcal{T}(\mu) \geq r$, $S(\top_X, \top_X) = \top$. Thus, $\mathcal{I}_{\mathcal{T}}(\top_X, r) \geq \top_X \odot \top = \top_X$. Therefore, $\mathcal{I}_{\mathcal{T}}(\top_X, r) = \top_X$.

(I2) By Lemma 2.4(7), we have $\mathcal{I}_{\mathcal{T}}(\lambda, r) = \bigvee_{\mu}\{\mu \odot S(\mu, \lambda) \mid \mathcal{T}(\mu) \geq r\} \leq \lambda$ for all $\lambda \in L^X$.

(I3) Using Lemma 2.2(8),(10), we can get

$$\begin{aligned} S(\mathcal{I}_{\mathcal{T}}(\lambda, r), \mathcal{I}_{\mathcal{T}}(\mu, r)) &= \bigwedge_{x \in X} \big(\mathcal{I}_{\mathcal{T}}(\lambda, r)(x) \to \mathcal{I}_{\mathcal{T}}(\mu, r)(x)\big) \\ &= \bigwedge_{x \in X} \Big(\bigvee_{\mathcal{T}(\rho) \geq r} \rho(x) \odot S(\nu, \lambda) \to \bigvee_{\mathcal{T}(\rho) \geq r} \rho(x) \odot S(\rho, \mu)\Big) \\ &\geq \bigwedge_{x \in X} \bigwedge_{\mathcal{T}(\rho) \geq r} \big(\rho(x) \odot S(\rho, \lambda) \to \rho(x) \odot S(\rho, \mu)\big) \\ &\geq \bigwedge_{x \in X} \bigwedge_{\mathcal{T}(\rho) \geq r} (S(\rho, \lambda) \to S(\rho, \mu)) \geq S(\lambda, \mu). \end{aligned}$$

(I4) If $r \leq s$, then

$$\mathcal{I}_{\mathcal{T}}(\lambda, s) = \bigvee_{\mathcal{T}(\mu) \geq s} \mu \odot S(\mu, \lambda) \leq \bigvee_{\mathcal{T}(\mu) \geq r} \mu \odot S(\mu, \lambda) = \mathcal{I}_{\mathcal{T}}(\lambda, r).$$

(I5) By Lemma 2.4(4), we have

$$\begin{aligned} \mathcal{I}_{\mathcal{T}}(\lambda, r) \odot \mathcal{I}_{\mathcal{T}}(\mu, s) &= \bigvee_{\mathcal{T}(\rho_1) \geq r} \rho_1 \odot S(\rho_1, \lambda) \odot \bigvee_{\mathcal{T}(\rho_2) \geq s} \rho_2 \odot S(\rho_2, \mu) \\ &= \bigvee_{\mathcal{T}(\rho_1) \geq r} \bigvee_{\mathcal{T}(\rho_2) \geq s} (\rho_1 \odot \rho_2) \odot S(\rho_1, \lambda) \odot S(\rho_2, \mu) \\ &\leq \bigvee_{\mathcal{T}(\rho_1) \odot \mathcal{T}(\rho_2) \geq r \odot s} (\rho_1 \odot \rho_2) \odot S(\rho_1 \odot \rho_2, \lambda \odot \mu) \\ &= \mathcal{I}_{\mathcal{T}}(\lambda \odot \mu, r \odot s). \end{aligned}$$

(2) Since $\mathcal{T}$ is enriched, $\mathcal{T}(\mathcal{I}_{\mathcal{T}}(\lambda, r)) \geq r$. Thus,

$$\begin{aligned}\mathcal{I}_{\mathcal{T}}(\mathcal{I}_{\mathcal{T}}(\lambda, r), r) &= \bigvee_{\mathcal{T}(\mu)\geq r} \mu \odot S(\mu, \mathcal{I}_{\mathcal{T}}(\lambda, r)) \\ &\geq \mathcal{I}_{\mathcal{T}}(\lambda, r) \odot S(\mathcal{I}_{\mathcal{T}}(\lambda, r), \mathcal{I}_{\mathcal{T}}(\lambda, r)) = \mathcal{I}_{\mathcal{T}}(\lambda, r).\end{aligned}$$

(3) For each $\mathcal{T}(\mu) \geq r$ with $\mu \leq \lambda$, we have $\mu = \top \odot \mu \leq S(\mu, \lambda) \odot \mu$, it follows that

$$\bigvee_{\mathcal{T}(\mu)\geq r} \{\mu \mid \mu \leq \lambda\} \leq \bigvee_{\mathcal{T}(\mu)\geq r} S(\mu, \lambda) \odot \mu = \mathcal{I}_{\mathcal{T}}(\lambda, r).$$

(4) For any $\mathcal{T}(\mu) \geq r$, $\mathcal{T}(S(\mu, \lambda) \odot \mu) \geq \mathcal{T}(\mu) \geq r$, because $\mathcal{T}$ is enriched. Thus, $\mathcal{I}_{\mathcal{T}}(\lambda, \mu) = \bigvee_{\mathcal{T}(\mu)\geq r} S(\mu, \lambda) \odot \mu \leq \bigvee_{\mathcal{T}(\mu)\geq r}\{\mu \mid \mu \leq \lambda\}$.

Theorem 3.2. Let $(X, \mathcal{I})$ be an L-fuzzy interior space. Define the mapping $\mathcal{T}_{\mathcal{I}} : L^X \to L$ by

$$\mathcal{T}_{\mathcal{I}}(\lambda) = \bigvee\{r \in L \mid S(\lambda, \mathcal{I}(\lambda, r)) = \top\}.$$

Then, $\mathcal{T}_{\mathcal{I}}$ is an enriched L-fuzzy topology on X.

Proof. (LO1) $\mathcal{T}_{\mathcal{I}}(\top_X) = \bigvee\{r \in L \mid S(\top_X, \mathcal{I}(\top_X, r)) = \top\}$, and
$\mathcal{T}_{\mathcal{I}}(\bot_X) = \bigvee\{r \in L \mid S(\bot_X, \mathcal{I}(\bot_X, r)) = \top\}$.

(LO2) By Lemma 2.4(4) and Definition 2.8(I5), we have

$$\begin{aligned}S(\lambda_1, \mathcal{I}(\lambda_1, r)) \odot S(\lambda_2, \mathcal{I}(\lambda_2, s)) &\leq S(\lambda_1 \odot \lambda_2, \mathcal{I}(\lambda_1, r) \odot \mathcal{I}(\lambda_2, s)) \\ &\leq S(\lambda_1 \odot \lambda_2, \mathcal{I}(\lambda_1 \odot \lambda_2, r \odot s)).\end{aligned}$$

If $S(\lambda_1, \mathcal{I}(\lambda_1, r)) = \top$ and $S(\lambda_2, \mathcal{I}(\lambda_2, s)) = \top$, then
$S(\lambda_1 \odot \lambda_2, \mathcal{I}(\lambda_1 \odot \lambda_2, r \odot s)) = \top$. Thus, $\mathcal{T}_{\mathcal{I}}(\lambda_1 \odot \lambda_2) \geq \mathcal{T}_{\mathcal{I}}(\lambda_1) \odot \mathcal{T}_{\mathcal{I}}(\lambda_2)$.

(LO3) For a family of $\{\lambda_i \mid i \in I\} \subseteq L^X$, we have

$$\begin{aligned}\mathcal{T}_{\mathcal{I}}(\bigvee_{i\in I} \lambda_i) &= \bigvee\{r \in L \mid S(\bigvee_{i\in I} \lambda_i, \mathcal{I}(\bigvee_{i\in I} \lambda_i, r)) = \top\} \\ &\geq \bigwedge_{i\in I} \bigvee\{r \in L \mid S(\lambda_i, \mathcal{I}(\bigvee_{i\in I} \lambda_i, r)) = \top\} \\ &\geq \bigwedge_{i\in I} \bigvee\{r \in L \mid S(\lambda_i, \mathcal{I}(\lambda_i, r)) = \top\} = \bigwedge_{i\in I} \mathcal{T}_{\mathcal{I}}(\lambda_i).\end{aligned}$$

Finally, for $\alpha \in L_\bot$ and $\lambda \in L^X$, we have

$$\begin{aligned}\mathcal{T}_{\mathcal{I}}(\alpha \odot \lambda) &= \bigvee\{r \in L \mid S(\alpha \odot \lambda, \mathcal{I}(\alpha \odot \lambda, r)) = \top\} \\ &\geq \bigvee\{r \in L \mid S(\alpha \odot \lambda, \alpha \odot \mathcal{I}(\lambda, r)) = \top\} \\ &\geq \bigvee\{r \in L \mid S(\lambda, \mathcal{I}(\lambda, r)) = \top\} = \mathcal{T}_{\mathcal{I}}(\lambda).\end{aligned}$$

Hence, $\mathcal{T}_{\mathcal{I}}$ is an enriched L-fuzzy topology on X.

Theorem 3.3. (1) If $(X, \mathcal{I})$ is an L-fuzzy interior space, then $\mathcal{I}_{\mathcal{T}_{\mathcal{I}}} \leq \mathcal{I}$.
(2) If $(X, \mathcal{T})$ is an L-fuzzy topological space, then $\mathcal{T}_{\mathcal{I}_{\mathcal{T}}} \geq \mathcal{T}$.

Proof. (1) By Lemma 2.4(7), we have

$$\begin{aligned}\mathcal{I}_{\mathcal{T}_{\mathcal{I}}}(\lambda, r) &= \bigvee_{\mu}\{\mu \odot S(\mu, \lambda) \mid \mathcal{T}_{\mathcal{I}}(\mu) \geq r\} \\ &= \bigvee_{\mu}\{\mu \odot S(\mu, \lambda) \odot S(\lambda, \mathcal{I}(\lambda, r)) \mid \mathcal{T}_{\mathcal{I}}(\mu) \geq r\} \\ &\leq \bigvee_{\mu}\{\mu \odot S(\mu, \mathcal{I}(\lambda, r)) \mid \mathcal{T}_{\mathcal{I}}(\mu) \geq r\} \leq \mathcal{I}(\lambda, r).\end{aligned}$$

(2) Let $\mathcal{T}(\lambda) \geq r$. Then, $\mathcal{I}_{\mathcal{T}}(\lambda, r) = \lambda$. Thus, $\mathcal{T}_{\mathcal{I}_{\mathcal{T}}}(\lambda) \geq r$. Hence, $\mathcal{T}_{\mathcal{I}_{\mathcal{T}}} \geq \mathcal{T}$.

Theorem 3.4. Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be two L-fuzzy topological spaces. If $\phi : (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$ is an LF-continuous map, then $\phi : (X, \mathcal{I}_{\mathcal{T}_X}) \to (Y, \mathcal{I}_{\mathcal{T}_Y})$ is an I-map.

Proof. By Lemma 2.5 and Definition 2.6, we have

$$\begin{aligned}\phi^{\leftarrow}(\mathcal{I}_{\mathcal{T}_Y}(\lambda, r)) &= \phi^{\leftarrow}\big(\bigvee_{\mu}\{\mu \odot S(\mu, \lambda) \mid \mathcal{T}_Y(\mu) \geq r\}\big) \\ &= \bigvee_{\phi^{\leftarrow}(\mu)}\{\phi^{\leftarrow}(\mu) \odot S(\mu, \lambda) \mid \mathcal{T}_Y(\mu) \geq r\} \\ &\leq \bigvee_{\phi^{\leftarrow}(\mu)}\{\phi^{\leftarrow}(\mu) \odot S(\phi^{\leftarrow}(\mu), \phi^{\leftarrow}(\lambda)) \mid \mathcal{T}_X(\phi^{\leftarrow}(\mu)) \geq r\} \\ &\leq \bigvee_{\rho}\{\rho \odot S(\rho, \phi^{\leftarrow}(\lambda)) \mid \mathcal{T}_X(\rho) \geq r\} = \mathcal{I}_{\mathcal{T}_X}(\phi^{\leftarrow}(\lambda), r).\end{aligned}$$

Theorem 3.5. Let $(X, \mathcal{I}_X)$ and $(Y, \mathcal{I}_Y)$ be two L-fuzzy interior spaces. If $\phi : (X, \mathcal{I}_X) \to (Y, \mathcal{I}_Y)$ is an I-map, then $\phi : (X, \mathcal{T}_{\mathcal{I}_X}) \to (Y, \mathcal{T}_{\mathcal{I}_Y})$ is LF-continuous.

Proof. From Theorem 3.4 and Lemma 2.5, we have

$$S(\phi^{\leftarrow}(\lambda), \mathcal{I}_X(\phi^{\leftarrow}(\lambda), r)) \geq S(\phi^{\leftarrow}(\lambda), \phi^{\leftarrow}(\mathcal{I}_Y(\lambda, r))) \geq S(\lambda, \mathcal{I}_Y(\lambda, r)).$$

So, $\mathcal{T}_{\mathcal{I}_X}(\phi^{\leftarrow}(\lambda)) \geq \mathcal{T}_{\mathcal{I}_Y}(\lambda)$.

4 L-fuzzy Closure Space Induced by L-fuzzy Co-topological Space

Theorem 4.1. Let $(X, \mathcal{F})$ be an L-fuzzy co-topological space. Define the mapping $\mathcal{C}_{\mathcal{F}} : L^X \times L_{\perp} \to L^X$ by

$$\mathcal{C}_{\mathcal{F}}(\lambda, r)(x) = \bigwedge_{\mathcal{F}(\mu) \geq r} \big(S(\lambda, \mu) \to \mu(x)\big).$$

Then we have the following properties.

(1) $(X, \mathcal{C}_{\mathcal{F}})$ is an L-fuzzy closure space,
(2) If $(X, \mathcal{F})$ is enriched, then $(X, \mathcal{C}_{\mathcal{F}})$ is a topological L-fuzzy closure space,
(3) $\mathcal{C}^*_{\mathcal{F}}(\lambda^*, r) = \mathcal{I}_{\mathcal{T}}(\lambda, r)$,
(4) $\mathcal{C}_{\mathcal{F}}(\lambda, r) \leq \bigwedge_{\mathcal{F}(\mu)\geq r}\{\mu \mid \lambda \leq \mu\}$,
(5) If $(X, \mathcal{F})$ is enriched, $\mathcal{C}_{\mathcal{F}}(\lambda, r) = \bigwedge_{\mathcal{F}(\mu)\geq r}\{\mu \mid \lambda \leq \mu\}$.

Proof. (1) (C1) By Lemma 2.4(7), we have

$$\mathcal{C}_{\mathcal{F}}(\perp_X, r)(x) = \bigwedge_{\mathcal{F}(\mu)\geq r} \big(S(\perp_X, \mu) \to \mu(x)\big) \geq \perp_X(x) = \perp.$$

(C2) By Lemma 2.2(11), we have

$$\begin{aligned} S(\lambda, \mathcal{C}_{\mathcal{F}}(\lambda, r)) &= \bigwedge_{x\in X} \big(\lambda(x) \to \mathcal{C}_{\mathcal{F}}(\lambda, r)(x)\big) \\ &= \bigwedge_{x\in X} (\lambda(x) \to \bigwedge_{\mathcal{F}(\mu)\geq r} \big(S(\lambda, \mu) \to \mu(x))\big) \\ &= \bigwedge_{x\in X} \bigwedge_{\mathcal{F}(\mu)\geq r} \big(\lambda(x) \to ((\bigwedge_{x\in X} \lambda(x) \to \mu(x)) \to \mu(x))\big) \\ &\geq \bigwedge_{x\in X} \bigwedge_{\mathcal{F}(\mu)\geq r} \big(\lambda(x) \to ((\lambda(x) \to \mu(x)) \to \mu(x))\big) \\ &= \bigwedge_{x\in X} \bigwedge_{\mathcal{F}(\mu)\geq r} \big((\lambda(x) \to \mu(x)) \to (\lambda(x) \to \mu(x))\big) = \top. \end{aligned}$$

(C3) By Lemma 2.2(10), we have

$$\begin{aligned} S(\mathcal{C}_{\mathcal{F}}(\lambda, r), \mathcal{C}_{\mathcal{F}}(\rho, r)) &= \bigwedge_{x\in X} \big(\mathcal{C}_{\mathcal{F}}(\lambda, r)(x) \to \mathcal{C}_{\mathcal{F}}(\rho, r)(x)\big) \\ &= \bigwedge_{x\in X} \big((\bigwedge_{\mathcal{F}(\mu)\geq r} S(\lambda, \mu) \to \mu(x)) \to (\bigwedge_{\mathcal{F}(\mu)\geq r} S(\rho, \mu) \to \mu(x))\big) \\ &\geq \bigwedge_{x\in X} \bigwedge_{\mathcal{F}(\mu)\geq r} \big((S(\lambda, \mu) \to \mu(x)) \to (S(\rho, \mu) \to \mu(x))\big) \\ &\geq \bigwedge_{\mathcal{F}(\mu)\geq r} \big(S(\rho, \mu) \to S(\lambda, \mu)\big) \geq S(\lambda, \rho). \end{aligned}$$

(C4) It follows from the definition of $\mathcal{C}_{\mathcal{F}}$.

(C5) By Lemma 2.4(5) and Lemma 2.2(15), we have

$$\begin{aligned}
\mathcal{C}_{\mathcal{F}}(\lambda,r)(x)\oplus\mathcal{C}_{\mathcal{F}}(\rho,s)(x) &= \Big(\bigwedge_{\mathcal{F}(\mu_1)\geq r} S(\lambda,\mu_1)\to\mu_1(x)\Big)\oplus\Big(\bigwedge_{\mathcal{F}(\mu_2)\geq s} S(\rho,\mu_2)\to\mu_2(x)\Big)\\
&= \bigwedge_{\mathcal{F}(\mu_1)\geq r}\ \bigwedge_{\mathcal{F}(\mu_2)\geq s}\Big((S(\lambda,\mu_1)\to\mu_1(x))\oplus(S(\rho,\mu_2)\to\mu_2(x))\Big)\\
&\geq \bigwedge_{\mathcal{F}(\mu_1\oplus\mu_2)\geq r\odot s}\Big((S(\lambda,\mu_1)\odot S(\rho,\mu_2))\to(\mu_1\oplus\mu_2)(x)\Big)\\
&\geq \bigwedge_{\mathcal{F}(\mu_1\oplus\mu_2)\geq r\odot s}\Big(S(\lambda\oplus\rho,\mu_1\oplus\mu_2)\to(\mu_1\oplus\mu_2)(x)\Big)\\
&= \mathcal{C}_{\mathcal{F}}(\lambda\oplus\rho,r\odot s)(x).
\end{aligned}$$

(2) Since $\mathcal{F}$ is enriched, then $\mathcal{F}(\mathcal{C}_{\mathcal{F}}(\lambda,r)\geq r$. Thus,

$$\begin{aligned}
\mathcal{C}_{\mathcal{F}}(\mathcal{C}_{\mathcal{F}}(\lambda,r),r)(x) &= \bigwedge_{\mathcal{F}(\mu)\geq r}\Big(S(\mathcal{C}_{\mathcal{F}}(\lambda,r),\mu)\to\mu(x)\Big)\\
&\leq \bigwedge_{\mathcal{F}(\mathcal{C}_{\mathcal{F}}(\lambda,r))\geq r}\Big(S(\mathcal{C}_{\mathcal{F}}(\lambda,r),\mathcal{C}_{\mathcal{F}}(\mu,r))\to\mathcal{C}_{\mathcal{F}}(\lambda,r)(x)\Big)\\
&= \mathcal{C}_{\mathcal{F}}(\lambda,r)(x).
\end{aligned}$$

(3)

$$\begin{aligned}
\mathcal{C}^*_{\mathcal{F}}(\lambda^*,r) &= \Big\{\bigwedge_{\mathcal{F}(\mu^*)\geq r}\Big(S(\lambda^*,\mu^*)\to\mu^*\Big)\Big\}^*\\
&= \bigvee_{\mathcal{F}(\mu^*)\geq r}\Big(S(\lambda^*,\mu^*)\odot\mu\Big) = \bigvee_{\mathcal{T}(\mu)\geq r}\mu\odot S(\mu,\lambda) = \mathcal{I}_{\mathcal{T}}(\lambda,r).
\end{aligned}$$

(4) If $\mu\leq\lambda$, then $S(\lambda,\mu)=\top$ and $S(\lambda,\mu)\to\mu\leq\mu$. Thus,

$$\bigwedge_{\mathcal{F}(\mu)\geq r}\Big(S(\lambda,\mu)\to\mu\Big)\leq\bigwedge_{\mathcal{F}(\mu)\geq r}\{\mu\mid\lambda\leq\mu\}.$$

(5) For any $\mathcal{F}(\mu)\geq r$, $\mathcal{F}\big(S(\lambda,\mu)\to\mu\big)\geq\mathcal{F}(\mu)$, i.e., $\mathcal{F}\big(S(\lambda,\mu)\to\mu\big)\geq r$, because $\mathcal{F}$ is enriched. Thus,

$$\bigwedge_{\mathcal{F}(\mu)\geq r}\Big(S(\lambda,\mu)\to\mu\Big)\geq\bigwedge_{\mathcal{F}(\mu)\geq r}\{\mu\mid\lambda\leq\mu\}.$$

Theorem 4.2. If $\mathcal{C}:L^X\times L_\perp$ is an L-fuzzy closure operator. Define the mapping $\mathcal{F}_{\mathcal{C}}:L^X\to L$ by

$$\mathcal{F}_{\mathcal{C}}(\lambda)=\bigvee\{r\in L\mid S(\mathcal{C}(\lambda,r),\lambda)=\top\}.$$

Then, $\mathcal{F}_{\mathcal{C}}$ is an enriched L-fuzzy co-topology on X.

Proof. (LF1) $\mathcal{F}_{\mathcal{C}}(\top_X) = \bigvee\{r \in L \mid S(\mathcal{C}(\top_X, r), \top_X) = \top\}$ by (C2), and $\mathcal{F}_{\mathcal{C}}(\bot_X) = \bigvee\{r \in L \mid S(\mathcal{C}(\bot_X, r), \bot_X) = \top\}$ by (C1).

(LF2) By Lemma 2.4(5) and (C4), we have

$$\begin{aligned} S(\mathcal{C}(\lambda_1, r), \lambda_1) \odot S(\mathcal{C}(\lambda_2, r), \lambda_2) &\leq S(\mathcal{C}(\lambda_1, r) \oplus \mathcal{C}(\lambda_2, r), \lambda_1 \oplus \lambda_2) \\ &\leq S(\mathcal{C}(\lambda_1 \oplus \lambda_2, r), \lambda_1 \oplus \lambda_2). \end{aligned}$$

If $S(\mathcal{C}(\lambda_1, r), \lambda_1) = \top$ and $S(\mathcal{C}(\lambda_2, r), \lambda_2) = \top$, then $S(\mathcal{C}(\lambda_1 \oplus \lambda_2, r), \lambda_1 \oplus \lambda_2) = \top$. Thus, $\mathcal{F}_{\mathcal{C}}(\lambda_1 \oplus \lambda_2) \geq \mathcal{F}_{\mathcal{C}}(\lambda_1) \odot \mathcal{F}_{\mathcal{C}}(\lambda_2)$.

(LF3) For a family of $\{\lambda_i \mid i \in I\} \subseteq L^X$, we have

$$\begin{aligned} \mathcal{F}_{\mathcal{C}}(\bigwedge_{i\in I} \lambda_i) &= \bigvee\{r \in L \mid S(\mathcal{C}(\bigwedge_{i\in I} \lambda_i, r), \bigwedge_{i\in I} \lambda_i) = \top\} \\ &\leq \bigvee_{i\in I} \bigvee\{r \in L \mid S(\bigwedge_{i\in I} \mathcal{C}(\lambda_i, r), \lambda_i) = \top\} \\ &\leq \bigvee_{i\in I} \bigvee\{r \in L \mid S(\mathcal{C}(\lambda_i, r), \lambda_i) = \top\} = \bigvee_{i\in I} \mathcal{F}_{\mathcal{C}}(\lambda_i). \end{aligned}$$

Hence, $\mathcal{F}_{\mathcal{C}}$ is an L-fuzzy co-topology on X. By Lemma 2.4(3), (6), we have

$$\begin{aligned} \mathcal{F}_{\mathcal{C}}(\alpha \to \lambda) &= \bigvee\{r \in L \mid S(\mathcal{C}(\alpha \to \lambda, r), \alpha \to \lambda) = \top\} \\ &= \bigvee\{r \in L \mid S(\alpha \odot \mathcal{C}(\alpha \to \lambda, r), \lambda) = \top\} \\ &\geq \bigvee\{r \in L \mid S(\mathcal{C}(\alpha \odot (\alpha \to \lambda), r), \lambda) = \top\} \\ &\geq \bigvee\{r \in L \mid S(\mathcal{C}(\lambda, r), \lambda) = \top\} = \mathcal{F}_{\mathcal{C}}(\lambda). \end{aligned}$$

Theorem 4.3. Let $(X, \mathcal{C}_{\mathcal{F}})$ be an L-fuzzy closure space, then $\mathcal{C}_{\mathcal{F}_{\mathcal{C}}} \geq \mathcal{C}$.

Proof. By Lemma 2.4(7), we have

$$\begin{aligned} \mathcal{C}_{\mathcal{F}_{\mathcal{C}}}(\lambda, r) = \bigwedge_{\mathcal{F}_{\mathcal{C}}(\mu)\geq r} \big(S(\lambda, \mu) \to \mu\big) &= \bigwedge_{\mathcal{F}_{\mathcal{C}}(\mu)\geq r} \big((S(\mathcal{C}(\lambda, r), \lambda) \odot S(\lambda, \mu)) \to \mu\big) \\ &\geq \bigwedge_{\mathcal{F}_{\mathcal{C}}(\mu)\geq r} \big(S(\mathcal{C}(\lambda, r), \mu) \to \mu\big) \geq \mathcal{C}(\lambda, r). \end{aligned}$$

Theorem 4.4. Let $(X, \mathcal{F}_X)$ and $(Y, \mathcal{F}_Y)$ be two L-fuzzy co-topological spaces. If $\phi : (X, \mathcal{F}_X) \to (Y, \mathcal{F}_Y)$ is an LF-continuous map, then $\phi : (X, \mathcal{C}_{\mathcal{F}_X}) \to (Y, \mathcal{C}_{\mathcal{F}_Y})$ is a C-map.

Proof. By Lemma 2.11, we have

$$\begin{aligned} \phi^{\leftarrow}(\mathcal{C}_{\mathcal{F}_Y}(\lambda, r)) = \phi^{\leftarrow}\big(\bigwedge_{\mathcal{F}_Y(\mu)\geq r} (S(\lambda, \mu) \to \mu)\big) &= \bigwedge_{\mathcal{F}_Y(\mu)\geq r} (S(\lambda, \mu) \to \phi^{\leftarrow}(\mu)) \\ \geq \bigwedge_{\mathcal{F}_X(\phi^{\leftarrow}(\mu))\geq r} \big(S(\phi^{\leftarrow}(\lambda), \phi^{\leftarrow}(\mu)) \to \phi^{\leftarrow}(\mu)\big) &= \mathcal{C}_{\mathcal{F}_X}(\phi^{\leftarrow}(\lambda), r). \end{aligned}$$

Theorem 4.5. Let $(X, \mathcal{C}_X)$ and $(Y, \mathcal{C}_Y)$ be two L-fuzzy closure spaces. If $\phi : (X, \mathcal{C}_X) \to (Y, \mathcal{C}_Y)$ is a C-map, then $\phi : (X, \mathcal{F}_{\mathcal{C}_X}) \to (Y, \mathcal{F}_{\mathcal{C}_Y})$ is LF-continuous.

Proof. From Theorem 4.3, we have

$$\begin{aligned}
\mathcal{F}_{\mathcal{C}_X}(\phi^{\leftarrow}(\lambda)) &= \bigvee\{r \in L \mid S(\mathcal{C}_X(\phi^{\leftarrow}(\lambda), r), \phi^{\leftarrow}(\lambda)) = \top\} \\
&\geq \bigvee\{r \in L \mid S(\phi^{\leftarrow}(\mathcal{C}_Y(\lambda, r)), \phi^{\leftarrow}(\lambda)) = \top\} \\
&= \bigvee\{r \in L \mid \bigwedge_{x \in X} \big(\mathcal{C}_Y(\lambda, r)(\phi(x)) \to \lambda(\phi(x))\big) = \top\} \\
&\geq \bigvee\{r \in L \mid \bigwedge_{y \in Y} \big(\mathcal{C}_Y(\lambda, r)(y) \to \lambda(y)\big) = \top\} \\
&= \bigvee\{r \in L \mid S(\mathcal{C}_Y(\lambda, r), \lambda) = \top\} = \mathcal{F}_{\mathcal{C}_Y}(\lambda).
\end{aligned}$$

Example 4.6. Let $(L = [0, 1], \odot, \to, *)$ be a complete residuated lattice defined as

$$x \odot y = (x + y - 1) \vee 0, \ \ x \to y = (1 - x + y) \wedge 1, \ \ x* = 1 - x.$$

Let $X = \{x, y, z\}$ be a set and let $\mu \in [0, 1]^X$ be a fuzzy set as follow

$$\mu(x) = 0.5, \quad \mu(y) = 0.3, \quad \mu(z) = 0.6.$$

We define the $[0, 1]$-fuzzy topology $\mathcal{T} : [0, 1]^X \to [0, 1]$ as follows

$$\mathcal{T}(\lambda) = \begin{cases} 1, & \text{if } \lambda = \bot_X \ or \ \top_X, \\ 0.3, & \text{if } \lambda = \mu \odot \mu, \\ 0.6, & \text{if } \lambda = \mu, \\ 0, & \text{otherwise.} \end{cases}$$

Also, we define the $[0, 1]$-fuzzy co-topology $\mathcal{F} : [0, 1]^X \to [0, 1]$ as follows

$$\mathcal{F}(\lambda) = \begin{cases} 1, & \text{if } \lambda = \bot_X \ or \ \top_X, \\ 0.2, & \text{if } \lambda = \mu \oplus \mu, \\ 0.6, & \text{if } \lambda = \mu, \\ 0, & \text{otherwise.} \end{cases}$$

(1) By Theorem 3.1, we have $\mathcal{I}_{\mathcal{T}} : [0, 1]^X \times (0, 1] \to [0, 1]^X$ as a $[0, 1]$-fuzzy interior space as follows

$$\mathcal{I}_{\mathcal{T}}(\lambda, r) = \begin{cases} (\bigwedge \lambda(x)), & \text{if } r > 0.6, \\ (\bigwedge \lambda(x)) \vee (\mu \odot S(\mu, \lambda)), & \text{if } 0.3 < r \leq 0.6, \\ (\bigwedge \lambda(x)) \vee (\mu \odot S(\mu, \lambda)), & \text{if } 0 < r \leq 0.3, \\ \vee(\mu \odot \mu \odot S(\mu \odot \mu, \lambda)). & \end{cases}$$

For $\lambda = (0.1, 0, 2, 0, 3)$, we have

$$\mathcal{I}_{\mathcal{T}}(\lambda, 0.5) = (\bigwedge \lambda(x)) \vee (\mu \odot S(\mu, \lambda)) = (0.1, 0.1, 0.2).$$

Since $\mathcal{I}_{\mathcal{T}}((0.1, 0.1, 0.2), r) = (0.1, 0.1, 0.2)$ for $0 < r \leq 0.6$, then we have

$$\mathcal{T}(\mathcal{I}_{\mathcal{T}}(0.1, 0.1, 0.2)) = 0.6.$$

(2) By Theorem 4.1, we have $\mathcal{C}_{\mathcal{F}} : [0,1]^X \times (0,1] \to [0,1]^X$ as a $[0,1]$-fuzzy closure space as follows

$$\mathcal{C}_{\mathcal{F}}(\lambda, r) = \begin{cases} \bigvee_{x \in X} \lambda(x), & \text{if } r > 0.6, \\ (\bigvee \lambda(x)) \wedge (S(\lambda, \mu) \to \mu), & \text{if } 0.3 < r \leq 0.6, \\ (\bigvee \lambda(x)) \wedge (S(\lambda, \mu) \to \mu), & \text{if } 0 < r \leq 0.3, \\ \wedge(S(\lambda, \mu \oplus \mu) \to \mu \oplus \mu), & \end{cases}$$

because $S(\lambda, 0) \to 0 = \bigwedge_{x \in X}(\lambda^*(x)) \to 0 = \bigvee_{x \in X} \lambda(x)$.

For $\lambda = (0.7, 0, 6, 0, 8)$, $\mathcal{C}_{\mathcal{F}}(\lambda, 0.5) = (\bigvee \lambda(x)) \wedge (S(\lambda, \mu) \to \mu) = (0.8, 0.8, 0.9)$. Since $(0.9, 0.8, 0.9) = \mathcal{C}_{\mathcal{F}}(\mathcal{C}_{\mathcal{F}}(\lambda, 0.5), 0.5) \neq \mathcal{C}_{\mathcal{F}}(\lambda, 0.5) = (0.8, 0.8, 0.9)$.

5 Conclusion

In this paper, we managed to deduce a new form of an L-fuzzy interior space (L-fuzzy closure space) through an L-fuzzy topological space (L-fuzzy co-topological space) and vise versa in a complete residuated lattice. We gave an example on [0,1] interval and finally we proved that the continuity property is compatible with the introduced spaces.

References

[1] Bandler W., Kohout L., *Special properties, closures and interiors of crisp and fuzzy relations*, Fuzzy sets and Systems 26(3) (1988) 317-331.

[2] Bělohlávek R., *Fuzzy closure operators I*, J. Math. Anal. Appl. 262 (2001) 473-489.

[3] Bělohlávek R., *Fuzzy closure operators II*, Soft Comput. 7(1) (2002) 53-64.

[4] Bělohlávek R., *Fuzzy Relational Systems*, Kluwer Academic Publishers, New York 156 (2002) 369.

[5] Bodenhofer U., De Cock M., Kerre E. E., *Openings and closures of fuzzy preorderings:theoretical basics and applications to fuzzy rule-based systems*, International Journal of General Systems 32(4) (2003) 343-360.

[6] Chang C.L., *Fuzzy topological spaces*, J.Math.Anal.Appl. 24 (1968) 182-190.

[7] Dubois D., Prade H., *"Putting rough sets and fuzzy sets together". In : Slowinsiki, R. ed., Intelligent Decision Support*, Handbook of Applications and Advances of the Rough Set Theory (Kluwer, Dordecht) 996 (1990) 203-232.

[8] Fang J., *I-fuzzy Alexandrov topologies and specialization orders*, Fuzzy Sets and Systems 161 (2007) 2359-2374.

[9] Fang J., Yue Y., *L-fuzzy closure systems*, Fuzzy Sets and Systems 161 (2010) 1242-1252.

[10] Fang J., *The relationship between L-ordered convergence structures and strong L-topologies*, Fuzzy Sets and Systems 161 (2010) 2923-2944.

[11] Gerla G., *Fuzzy Logic Mathematical Tools for Approximate Reasoning*, Kluwer, Dordecht (2001).

[12] Gutierrez Garcia J., Mardones Perez I., Burton M. H., *The relationship between various filter notions on a GL-monoid*, J. Math. Anal. Appl. 230 (1999) 291-302.

[13] Hajek P., *Metamathematices of Fuzzy Logic*, Kluwer Academic Publishers, Dordrecht (1998).

[14] Hohle U., *Upper semi continuous fuzzy sets and applications*, J.Math.Anal.Appl. 78 (1980) 659-673.

[15] Hohle U., Rodabaugh S. E., *Mathematics of Fuzzy Sets, Logic, Topology and Measure Theory*, The Handbooks of Fuzzy Sets Series, Kluwer Academic Publishers, Dordrecht 3 (1999) 273-388.

[16] Kotze W., *Uniform spaces*, in: Hohle U., Rodabaugh S. E.(Eds.), Mathematics of Fuzzy Sets: Logic, Topology and Measure Theory, Handbook Series, Kluwer Academic Publishers, Boston, Dordrecht, London, Chapter 8, 3 (1999) 553-580.

[17] Kubiak T., *On fuzzy topologies*, Ph.D. Thesis, Adam Mickiewicz Uniformity, Poznan, Poland (1985).

[18] Lai H., Zhang D., *Fuzzy preorder and fuzzy topology*, Fuzzy Sets and Systems 157 (2006) 1865-1885.

[19] Mashour A. S., Ghanim M. H., *Fuzzy closure spaces*, J.Math. Anal. Appl. 106 (1985) 154-170.

[20] Radzikowska A. M., Kerre E. E., *A comparative study of fuzzy rough sets*, Fuzzy Sets and Systems 126 (2002) 137-155.

[21] Ramadan A. A., *Smooth topological Spaces*, Fuzzy Sets and Systems 48(3) (1992) 371-357.

[22] Ramadan A. A., *L-fuzzy interior systems*, Comp. and Math. with Appl. 62 (2011) 4301-4307.

[23] Rodabaugh S. E., *Categorial foundations of variable-basis fuzzy topology*, In: Hohle U., Rodabaugh S. E.(Eds.), Mathematics of Fuzzy Sets: Logic, Topology and Measure Theory, Handbook series, Kluwer Academic Publishers, Chapter 4 (1999).

[24] Rodabaugh S. E., Klement E. P., *Topological and Algebraic Structures In Fuzzy Sets*, The Handbook of Recent Developments in the Mathematics of Fuzzy Sets, Kluwer Academic Publishers, Boston, Dordrecht, London (2003) 467 pp.

[25] Sostak A., *On a fuzzy topological structure*, Suppl. Rend Circ. Matem. Palermo, Ser. II [11] (1985) 125-186.

[26] Ying M.S., *A new approach to fuzzy topology*, Part I, Fuzzy Sets Syst. 39 (1991) 303-321.

[27] Ying M.S., *A new approach to fuzzy topology*, Part II, Fuzzy Sets Syst. 47 (1992) 221-232.

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BOUNDARY AND EXTERIOR OF A MULTISET TOPOLOGY

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Abstract − The concepts of exterior and boundary in multiset topological space are introduced. We further established few relationships between the concepts of boundary, closure, exterior and interior of an M- set. These concepts have been pigeonholed by other existing notions viz., open sets, closed sets, clopen sets and limit points. The necessary and sufficient condition for a multiset to have an empty exterior is also discussed.

Keywords − *Boundary, exterior, M-sets, M-topology.*

1 Introduction

The theory of sets is indispensable to the world of mathematics. But in set theory where repetitions of objects are not allowed it often become difficult to complex systems. If one considers those complex systems where repetitions of objects become certainly inevitable, the set theoretical concepts fails and thus one need more sophisticated tools to handle such situations. This led to the initiation of multiset (M-set) theory by Blizard [1] in 1989 as a generalization of set theory. Multiset theory was further studied by Dedekind [3] by considering each element in the range of a function to have a multiplicity equal to the number of elements in the domain that are mapped to it. The theory of multisets have been studied by many other authors in different senses [8], [10], [13], [16], [17], [18] and [24].

Since its inception M-set theory have been receiving considerable attention from researchers and wide application of the same can be found in literature [[17],[18], [23] etc]. Algebraic structures for multiset space have been constructed by Ibrahim *et al.* in [11]. In [15], use of multisets in colorings of graphs have been discussed by Okamota *et al.* Application of M-set theory in decision making can be seen in [23]. Syropoulos [20], presented a categorical approach to multisets along with partially ordered multisets. Venkateswaran [22] found a large new class of multisets Wilf equivalent pairs which

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is claimed to be the most general multiset Wilf equivalence result to date. In 2012, Girish and John [7] introduced multiset topologies induced by multiset relations. The same authors further studied the notions of open sets, closed sets, basis, sub basis, closure and interior, continuity and related properties in M-topological spaces in [9]. Further the concepts of semi open, semi closed multisets were introduced in [14], which were then used to study semicompactness in multiset topology.

In this paper, we introduce the concept of exterior and boundary in multiset topological space. We begin with preliminary notions and definitions of multiset theory in Section 2. Section 3 which contains main results forms the most fundamental part of the paper and it is followed by Section 4 which contains the concluding remarks.

2 Preliminaries

Below are some definitions and results as discussed in [7], which are required throughout the paper.

Definition 2.1. An M-set M drawn from the set X is represented by a function Count M or $C_M : X \longrightarrow W$, where W represents the set of whole numbers.

Here $C_M(x)$ is the number of occurrences of the element x in the M-set M. We represent the M-set M drawn from the set $X = \{x_1, ..., x_n\}$ as $M = \{m_1/x_1, m_2/x_2, ..., m_n/x_n\}$ where m_i is the number of occurrences of the element $x_i, i = 1, 2, ..., n$ in the M-set M. Those elements which are not included in the M-set have zero count.

Note: Since the count of each element in an M-set is always a non-negative integer so we have taken W as the range space instead of N.

Example 2.2. Let $X = \{a, b, c\}$. Then $M = \{3/a, 5/b, 1/c\}$ represents an M-set drawn from X.

Various operations on M-sets are defined as follows:
If M and N are two M-sets drawn from the set X, then

- $M = N \Leftrightarrow C_M(x) = C_N(x)\ \forall x \in X$.
- $M \subseteq N \Leftrightarrow C_M(x) \leq C_N(x)\ \forall x \in X$.
- $P = M \cup N \Leftrightarrow C_P(x) = max\{C_M(x), C_N(x)\}\ \forall x \in X$.
- $P = M \cap N \Leftrightarrow C_P(x) = min\{C_M(x), C_N(x)\}\ \forall x \in X$.
- $P = M \oplus N \Leftrightarrow C_P(x) = C_M(x) + C_N(x)\ \forall x \in X$.
- $P = M \ominus N \Leftrightarrow C_P(x) = max\{C_M(x) - C_N(x), 0\}\ \forall x \in X$, where $\oplus$ and $\ominus$ represents M-set addition and M-set subtraction respectively.

Operations under collections of M-sets: Let $[X]^w$ be an M-space and $\{M_i \mid i \in I\}$ be a collection of M-sets drawn from $[X]^w$. Then the following operations are defined

- $\bigcup_{i \in I} M_i = \{C_{M_i}(x)/x \mid C_{M_i}(x) = max\{C_{M_i}(x) \mid x \in X\}\}$.
- $\bigcap_{i \in I} M_i = \{C_{M_i}(x)/x \mid C_{M_i}(x) = min\{C_{M_i}(x) \mid x \in X\}\}$.

Definition 2.3. The support set of an M-set M, denoted by M^* is a subset of X and is defined as $M^* = \{x \in X \mid C_M(x) > 0\}$. M^* is also called root set.

Definition 2.4. An M-set M is called an empty M-set if $C_M(x) = 0,\ \ \forall\ x \in X$.

Definition 2.5. A domain X, is defined as the set of elements from which M-set are constructed. The M-set space $[X]^w$ is the set of all M-sets whose elements are from X such that no element occurs more than w times.

Remark 2.6. It is clear that the definition of the operation of M-set addition is not valid in the context of M-set space $[X]^w$, hence it was refined as
$C_{M_1 \oplus M_2}(x) = min\{w, C_{M_1}(x) + C_{M_2}(x)\}$ for all $x \in X$.

In multisets the number of occurrences of each element is allowed to be more than one which leads to generalization of the definition of subsets in classical set theory. So, in contrast to classical set theory, there are different types of subsets in multiset theory.

Definition 2.7. A subM-set N of M is said to be a whole subM-set if and only if $C_N(x) = C_M(x)$ for every $x \in N$.

Definition 2.8. A subM-set N of M is said to be a partial whole subM-set if and only if $C_N(x) = C_M(x)$ for some $x \in N$.

Definition 2.9. A subM-set N of M is said to be a full subM-set if and only if $C_N(x) \leq C_M(x)$ for every $x \in N$.

As various subset relations exist in multiset theory, the concept of power M-set can also be generalized as follows:

Definition 2.10. Let $M \in [X]^w$ be an M-set.

- The power M-set of M denoted by $\mathcal{P}(M)$ is defined as the set of all subM-sets of M.
- The power whole M-set of M denoted by $\mathcal{PW}(M)$ is defined as the set of all whole subM-sets of M.
- The power full M-set of M denoted by $\mathcal{PF}(M)$ is defined as the set of all full subM-sets of M.

The power set of an M-set is the support set of the power M-set and is denoted by $\mathcal{P}^*(M)$.

Definition 2.11. Let $M \in [X]^w$ and $\tau \subseteq \mathcal{P}^*(M)$. Then τ is called an M-topology if it satisfies the following properties:

- The M-set M and the empty M-set ϕ are in τ.
- The M-set union of the elements of any subcollection of τ is in τ.
- The M-set intersection of the elements of any finite subcollection of τ is in τ.

The elements of τ are called open M-set and their complements are called closed M-sets.

Definition 2.12. Given a subM-set A of an M-topological space M in $[X]^w$

- The interior of A is defined as the M-set union of all open M-sets contained in A and is denoted by $int(A)$ i.e., $C_{int(A)}(x) = C_{\cup G}(x)$ where G is an open M-set and $G \subseteq A$.
- The closure of A is defined as the M-set intersection of all closed M-sets containing A and is denoted by $cl(A)$ i.e., $C_{cl(A)}(x) = C_{\cap K}(x)$ where G is a closed M-set and $A \subseteq K$.

Definition 2.13. If M is an M-set, then the M-basis for an M-topology in $[X]^w$ is a collection $\mathcal{B}$ of subM-sets of M such that

- For each $x \in^m M$, for some $m > 0$ there is at least one M-basis element $B \in \mathcal{B}$ containing m/x.

- If m/x belongs to the intersection of two M -basis elements P and Q, then $\exists$ an M- basis element R containing m/x such that $R \subseteq P \cap Q$ with $C_R(x) = C_{P\cap Q}(x)$ and $C_R(y) \leq C_{P\cap Q}(y)$ $\forall y \neq x$.

Definition 2.14. Let (M, τ) be an M-topological space in $[X]^w$ and A is a subM-set of M. If k/x is an element of M then k/x is a limt point of an M-set when every neighborhood of k/x intersects A in some point (point with non-zero multiplicity) other than k/x itself.

Definition 2.15. Let (M, τ) be an M-topological space and N is a subM-set of M. The collection $\tau_N = \{N \cap U : U \in \tau\}$ is an M-topology on N, called the subspace M-topology. With this M-topology, N is called a subspace of M.

Throughout the paper we shall follow the following definition of complement in an M- topological space.

Definition 2.16. [14] The M-complement of a subM-set N in an M-topological space (M, τ) is denoted and defined as $N^c = M \ominus N$.

3 Exterior and Boundary of Multisets

The notions of interior and closure of an M-set in M-topology have been introduced and studied by Jacob et al. [7]. The other topological structures like exterior and boundary have remain untouched. In this section, we introduce the concepts of exterior and boundary in multiset topology.
Consider an M-topological space (M, τ) in $[X]^w$.

Definition 3.1. The exterior of an M-set A in M is defined as the interior of M-complement of A and is denoted by ext(A), i.e.,

$$C_{ext(A)}(x) = C_{int(A^c)}(x) \text{ for all } x \in X.$$

Example 3.2. Let $X = \{a, b\}$,$w = 3$ and $M = \{2/a, 3/b\}$. We consider the topology $\tau = \{\phi, M, \{1/a\}, \{2/b\}, \{1/a, 2/b\}\}$ on M. Then exterior of the M-set $A = \{1/a, 3/b\}$ is $\{1/a\}$.

Remark 3.3. *ext*(A) *is the largest open subM-set contained in* A^c.

Definition 3.4. The boundary of an M-set A is the M-set of elements which does not belong to the interior or the exterior of A. In other words, the boundary of an M-set A is the M-set of elements which belongs to the intersection of closure of A and closure of M-complement of A. It is denoted by bd(A).

$$C_{bd(A)}(x) = C_{cl(A)\cap cl(A^c)}(x) \text{ for all } x \in X.$$

Example 3.5. Let $X = \{a, b, c, d\}$, $w = 5$ and $M = \{5/a, 3/b, 5/c, 5/d\}$. We consider the topology $\tau = \{\phi, M, \{1/a, 2/b, 3/c, 2/d\}, \{1/a, 3/c\}, \{2/b, 5/d\}, \{1/a, 2/b, 3/c, 5/d\}, \{2/b, 2/d\}\}$ on M. Then for any set $A = \{3/a, 3/b, 3/c, 3/d\}$ we have $cl(A) = M$ and $cl(A^c) = \{4/a, 1/b, 2/c, 3/d\}$. Hence, $bd(A) = \{4/a, 1/b, 2/c, 3/d\}$.

Remark 3.6. *bd*(A) *is the smallest closed subM-set containing* A^c.

Remark 3.7. A *and* A^c *both have same boundary.*

Theorem 3.8. *Let* (M, τ) *be an M-topological space. Then*

(i) $C_{ext(A\cup B)}(x) = C_{ext(A)\cap ext(B)}(x)$, $\forall x \in X$.

(ii) $C_{ext(A\cap B)}(x) \geq C_{ext(A)\cup ext(B)}(x)$, $\forall x \in X$.

Proof. (i) From the definition of exterior,

$$\begin{aligned} C_{ext(A\cup B)}(x) &= C_{int(A\cup B)^c}(x) \\ &= C_{int(A^c\cap B^c)}(x) \\ &= C_{int(A^c)\cap int(B^c)}(x) \\ &= C_{ext(A)\cap ext(B)}(x), \forall x \in X. \end{aligned}$$

(ii)

$$\begin{aligned} C_{ext(A\cap B)}(x) &= C_{int(A\cap B)^c}(x) \\ &= C_{int(A^c\cup B^c)}(x) \\ &\geq C_{int(A^c)\cup int(B^c)}(x) \\ &= C_{ext(A)\cup ext(B)}(x), \forall x \in X. \end{aligned}$$

□

Theorem 3.9. Let (M, τ) be an M-topological space in $[X]^w$. For any two M-sets A and B in M, the following results hold:

(i) $C_{(bd(A))^c}(x) = C_{int(A)\cup int(A^c)}(x) = C_{int(A)\cup ext(A)}(x)$

(ii) $C_{cl(A)}(x) = C_{int(A)\cup bd(A)}(x)$

(iii) $C_{bd(A)}(x) = C_{cl(A)\ominus int(A)}(x)$

(iv) $C_{int(A)}(x) = C_{A\ominus bd(A)}(x)$

Proof.

$$\begin{aligned} (i) C_{(bd(A))^c}(x) &= C_{(cl(A)\cap cl(A^c))^c}(x) \\ &= C_{(cl(A))^c\cup(cl(A^c))^c}(x) \\ &= C_{int(A^c)\cup int(A)}(x) \\ &= C_{int(A)\cup ext(A)}(x). \end{aligned}$$

$$\begin{aligned} (ii) C_{int(A)\cup bd(A)}(x) &= C_{int(A)\cup(cl(A)\cap cl(A^c))}(x) \\ &= C_{(int(A)\cup(cl(A))\cap(int(A)\cup cl(A^c))}(x) \\ &= C_{cl(A)\cap(int(A)\cup(int(A))^c)}(x) \\ &= C_{cl(A)}(x). \end{aligned}$$

(iii) We have $C_{cl(A)\ominus int(A)}(x) = max\{\ C_{cl(A)}(x) - C_{int(A)}(x), 0\}$

- Case 1: max is 0
  So we must have $C_{cl(A)}(x) = C_{int(A)}(x)$.
  Then $C_{bd(A)}(x) = C_{int(A)\cap cl(A^c)}(x) = C_{int(A)\cap(int(A))^c}(x) = C_{\phi}(x)$.
- Case 2: max is $C_{cl(A)}(x) - C_{int(A)}(x)$. Then

$$\begin{aligned} C_{bd(A)}(x) &= C_{cl(A)\cap cl(A^c)}(x) \\ &= C_{cl(A)\cap(int(A))^c}(x) \\ &= C_{cl(A)}(x) - C_{int(A)}(x). \end{aligned}$$

(iv) We have $C_{A\ominus bd(A)}(x) = max\{\ C_A(x) - C_{bd(A)}(x), 0\}$

- Case 1: When max is 0 proof is trivial.
- Case 2: When max is $C_A(x) - C_{bd(A)}(x)$, we have

$$\begin{aligned} C_{A\ominus bd(A)}(x) &= C_{A\cap(bd(A))^c}(x) \\ &= C_{A\cap(int(A)\cup ext(A))}(x) \\ &= C_{A\cap(int(A)\cup int(A^c))}(x) \\ &= C_{(A\cap int(A))\cup(A\cap int(A^c))}(x) \\ &= C_{int(A)\cup\phi}(x) \\ &= C_{int(A)}(x). \end{aligned}$$

□

The following three theorems characterize the open and closed M-sets in terms of boundary.

Theorem 3.10. Let A be a subM-sets in an M-topology (M, τ). Then A is open if and only if $C_{A\cap bd(A)}(x) = 0$, $\forall x \in X$.

Proof. Let A be an open M-set. Then $C_{int(A)}(x) = C_A(x)$, $\forall x \in X$. Now, $C_{A\cap bd(A)}(x) = C_{int(A)\cap bd(A)}(x) = 0$.

Conversely, let A be an M-set such that $C_{A\cap bd(A)}(x) = 0 \Rightarrow C_{A\cap(cl(A)\cap cl(A^c))}(x) = 0 \Rightarrow C_{A\cap cl(A^c)}(x) = 0 \Rightarrow C_{cl(A^c)}(x) \leq C_{A^c}(x) \Rightarrow A^c$ is closed M-set $\Rightarrow A$ is open M-set. □

Theorem 3.11. Let A be a subM-sets in an M-topology (M, τ). Then$A$ is closed if and only if $C_{bd(A)}(x) \leq C_A(x)$, $\forall x \in X$.

Proof. Let A be a closed M-set. Then $C_{cl(A)}(x) = C_A(x)$, $\forall x \in X$. Now, $C_{bd(A)}(x) = C_{cl(A)\cap cl(A^c)}(x) \leq C_{cl(A)}(x) = C_A(x)$, $\forall x \in X$.

Conversely, let $C_{bd(A)}(x) \leq C_A(x) \Rightarrow C_{bd(A)\cap A^c}(x) = 0 \Rightarrow C_{bd(A^c)\cap A^c}(x) = 0$. Therefore, A^c is an open M-set. Hence, A is a closed M-set. □

Theorem 3.12. Let A be a subM-sets in an M-topology (M, τ). Then A is clopen if and only if $C_{bd(A)}(x) = 0$.

Proof. Let $C_{bd(A)}(x) = 0 \Rightarrow C_{cl(A)\cap cl(A^c)}(x) = 0 \Rightarrow C_{cl(A)}(x) \leq C_{(cl(A^c))^c}(x) \Rightarrow C_{cl(A)}(x) \leq C_{int(A)}(x) \leq C_A(x) \Rightarrow A$ is a closed M-set.

Again, $C_{cl(A)\cap cl(A^c)}(x) = 0 \Rightarrow C_{cl(A)\cap(int(A))^c}(x) \Rightarrow C_{A\cap(int(A))^c}(x) \Rightarrow C_A(x) \leq C_{int(A)}(x) \Rightarrow A$ is an open M-set.

Conversely, let A be both open and closed M-set.

Then $C_{bd(A)}(x) = C_{cl(A)\cap cl(A^c)}(x) = C_{cl(A)\cap(int(A))^c}(x) = C_{A\cap A^c}(x) = 0$. □

Theorem 3.13. For any two M-sets A and B in (M, τ) the followings hold true:

(i) $C_{bd(A\cup B)}(x) \leq C_{bd(A)\cup bd(B)}(x)$, $\forall x \in X$.

Proof. Let A and B be any two M-sets in (M,τ). Then,

$$\begin{aligned}
C_{bd(A\cup B)}(x) &= C_{cl(A\cup B)\cap cl(A\cup B)^c}(x)\\
&= C_{[cl(A)\cup cl(B)]\cap[cl(A^c)\cap cl(B^c)]}(x)\\
&= C_{[(cl(A)\cap cl(A^c))\cap(cl(A)\cap cl(B^c))]\cup[(cl(B)\cap cl(A^c))\cap(cl(B)\cap cl(B^c))]}(x)\\
&= C_{[bd(A)\cap(cl(A)\cap cl(B^c))]\cup[(cl(B)\cap cl(A^c))\cap bd(B)]}(x)\\
&\leq C_{bd(A)\cup bd(B)}(x).
\end{aligned}$$

□

(ii) $\mathrm{C}_{bd(A\cap B)}(x) \leq \mathrm{C}_{bd(A)\cap bd(B)}(x)$, $\forall x \in X$.

Proof. Let A and B be any two M-sets in (M,τ). Then,

$$\begin{aligned}
C_{bd(A\cap B)}(x) &= C_{cl(A\cap B)\cap cl(A\cap B)^c}(x)\\
&= C_{[cl(A)\cap cl(B)]\cap[cl(A^c)\cup cl(B^c)]}(x)\\
&= C_{[(cl(A)\cap cl(A^c))\cap(cl(A)\cap cl(B^c))]\cap[(cl(B)\cap cl(A^c))\cap(cl(B)\cap cl(B^c))]}(x)\\
&= C_{[bd(A)\cap(cl(A)\cap cl(B^c))]\cap[(cl(B)\cap cl(A^c))\cap bd(B)]}(x)\\
&\leq C_{bd(A)\cap bd(B)}(x).
\end{aligned}$$

□

Theorem 3.14. In an M-topological space, for any M-set A $bd(bd(A))$ is a closed M-set.

Proof. Let $bd(A) = B$. Then

$$\begin{aligned}
C_{cl(bd(bd(A)))}(x) &= C_{cl(bd(B))}(x)\\
&= C_{cl(cl(B)\cap cl(B^c))}(x)\\
&\leq C_{cl(cl(B))\cap cl(cl(B^c))}(x)\\
&= C_{cl(B)\cap cl(B^c)}(x)\\
&= C_{bd(B)}(x)\\
&= C_{bd(bd(A))}(x).
\end{aligned}$$

i.e., closure of $bd(bd(A))$ is contained in itself and hence is a closed M-set. □

Theorem 3.15. In an M-topological space, for any M-set A we have the following:

(i) $\mathrm{C}_{bd(bd(A))}(x) \leq \mathrm{C}_{bd(A)}(x)$, $\forall x \in X$.

Proof.

$$\begin{aligned}
C_{bd(bd(A))}(x) &= C_{bd(cl(A)\cap cl(A^c))}(x)\\
&= C_{[cl(cl(A)\cap cl(A^c))]\cap[cl(cl(A)\cap cl(A^c))^c]}(x)\\
&\leq C_{[cl(A)\cap cl(A^c)]\cap[cl(int(A^c)\cup int(A))]}(x)\\
&= C_{bd(A)\cap cl(M)}(x)\\
&= C_{bd(A)\cap M}(x)\\
&= C_{bd(A)}(x).
\end{aligned}$$

□

(ii) $\mathrm{C}_{bd(bd(bd(A)))}(x) = \mathrm{C}_{bd(bd(A))}(x)$, $\forall x \in X$.

Proof.

$$\begin{aligned} C_{bd(bd(bd(A)))}(x) &= C_{cl(bd(bd(A)))\cap cl(bd(bd(A)))^c}(x) \quad (1)\\ &= C_{bd(bd(A))\cap cl(bd(bd(A)))^c}(x). \quad (2) \end{aligned}$$

$$\begin{aligned} Now, C_{(bd(bd(A)))^c}(x) &= C_{[cl(bd(A))\cap cl(bd(A))^c]^c}(x)\\ &= C_{[bd(A)\cap cl(bd(A))^c]^c}(x)\\ &= C_{(bd(A))^c\cup[cl(bd(A))^c]^c}(x) \end{aligned}$$

Taking closure on both sides and considering $cl(bd(A))^c = B$, we have

$$\begin{aligned} C_{cl(bd(bd(A)))^c}(x) &= C_{B\cup cl(B^c)}(x)\\ &\geq C_{B\cup B^c}(x)\\ &= C_M(x). \end{aligned}$$

Now, substituting this in equation(1)

$$\begin{aligned} C_{bd(bd(bd(A)))}(x) &= C_{cl(bd(bd(A)))\cap M}(x)\\ &= C_{bd(bd(A))}(x). \end{aligned}$$

□

The following theorem decomposes boundary of an M-set.

Theorem 3.16. $\mathrm{C}_{bd(A)}(x) = \mathrm{C}_{int(bd(A))\cup bd(bd(A))}(x)$, $\forall x \in X$.

Proof. From theorem 3.12(i) and the property of interior i.e.,$C_{int(bd(A))}(x) \leq C_{bd(A)}(x)$, its obvious that $C_{int(bd(A))\cup bd(bd(A))}(x) \leq C_{bd(A)}(x)$, $\forall x \in X$. □

Following is a theorem to characterize boundary of an M-set in terms of limit points of the set.

Theorem 3.17. An M-set A in an M-topology (M, τ) contains all its boundary points if and only if it contains all its limit points.

Proof. Suppose A contains all its boundary points and if possible let $k/x \in A^c$ be a limit point of A. Since every neighborhood of k/x contains both a point of A^c and a point of A, we have $k/x \in bd(A) \subseteq cl(A)$, which is a contradiction since A contains all its boundary points.
Conversely, let A contains all its limit points. If $k/x \in A \ominus bd(A)$ and N is a neighborhood of k/x then N contains a point of A which cannot be equal to k/x since $k/x \notin A$. Therefore, k/x is a limit point of A and is not contained in A. Hence, A contains all its boundary points. □

Theorem 3.18. Let A be an M-set in an M-topology (M, τ). Then $ext(A)$ is empty if and only if every nonempty open M-set in M contains a point of A.

Proof. Let every non empty open M- set in M, τ contains a point of A. Then, every $k/x \in A \subseteq M$ is a limit point if A. So,

$$k \leq C_{cl(A)(x)} \Rightarrow C_{M(x)} \leq C_{cl(A)(x)} \tag{3}$$

$$\begin{aligned} Now,\ to\ show\ that\ C_{ext(A)}(x) &= C_\phi(x) \\ \Leftrightarrow C_{int(A^c)}(x) &= C_\phi(x) \\ \Leftrightarrow C_{(cl(A))^c}(x) &= C_\phi(x) \\ \Leftrightarrow C_{cl(A)}(x) &= C_M(x) \end{aligned}$$

But then we have,

$$C_{cl(A)}(x) \leq C_M(x), \forall x. \tag{4}$$

So, (3) and (4) imply that $ext(A)$ is empty.

Conversely, let $C_{ext(A)}(x) = C_\phi(x)$. Let O be any open M -set in (M, τ). To show that O contains a point of A.
Let $k/x \in O$. Since $ext(A)$ is empty so no neighborhood of k/x is contained in A^c, i.e., all neighborhoods of k/x are contained in A. Therefore we have, $C_{O \cap A}(x) \neq C_\phi(x)$. □

4 Conclusion

The notions of exterior and boundary in context of multiset theory have been introduced and studied in this paper.Some properties of the introduced notions are studied along with their characterization and decomposition. Further, boundary is characterized in terms of open sets, closed sets, clopen sets. Theorem 3.17 characterizes boundary in terms of limit points. The necessary and sufficient condition for an M- set to have empty exterior is contemplated by Theorem 3.18.

Topological and topology-based data are useful for detecting and correcting digitizing errors which occurs in spatial analysis. Keeping this in view, applications of the initiated concepts in those models which are designed using multiset theory can be considered for future work.

References

[1] Blizard, W.D., Multiset Theory, Notre Dame Journal of Logic, 30, 36–65, 1989.

[2] Cantor, G., Contributions to the Founding of the Theory of Transfinite Numbers, Translation, Introduction, and notes by P. Jourdain, Dover, New York, 1955.

[3] Dedekind, R., Essays on the Theory of Numbers, translated by W. W. Beman, Dover, New York, 1963.

[4] Dorsett, Ch., Semi compactness, semi separation axioms, and product spaces, Bull. Malaysian Math. Soc., 2(4), 21–28, 1981.

[5] Dorsett, Ch., Semi convergence and semi compactness, Indian J. Mech. Math. 19(1), 11–17, 1982.

[6] Ganster, M., Some Remarks on Strongly Compact and Semi compact Spaces, Bull. Malaysian Math. Soc., 10(2), 67–81, 1987.

[7] Girish, K.P., Sunil Jacob John, Multiset topologies induced by multiset relations, Information Sciences, 188(0), 298–313, 2012.

[8] Girish, K.P. and Sunil Jacob John, Relations and functions in multiset context. Inf. Sci. 179(6), 758–768, 2009.

[9] Girish, K.P. and Sunil Jacob John, On Mutiset Topologies, Theory and Applications of Mathematics and Computer Science, 2(1), 37–52, 2012.

[10] Hallett, M.L., Cantorian Set Theory and Limitation of Size, Oxford Logic Guide 10, Clarendon Press, Oxford, 1984.

[11] Ibrahim, A.M., Singh, D., Singh, J.N., An Outline of Multiset Space Algebra, International Journal of Algebra, 5(31), 1515 –1525, 2011.

[12] Levine, N., Semi-open sets and semi-continuity in topological spaces, The American Mathematical Monthly, 70, 36–41, 1963.

[13] Levy, A., Basic Set Theory, Springer-Verlag, Berlin, 1979.

[14] Mahanta, J., Das, D., Semi Compactness in Multiset Topology, arXiv:1403.5642v2 [math.GM] 21 Nov 2014.

[15] Okamoto, F., Salehi, E., Zhang, P., On Multiset Colorings of Graphs, Discussiones Mathematicae Graph Theory, 30, 137–153, 2010.

[16] Singh, D., A note on the development of multiset theory. Modern Logic 4(4), 405–406, 1994.

[17] Singh, D., Ibrahim, A.M., Yohanna, T., Singh, J.N., An overview of the applications of multisets, Novi Sad J. Math,37(2), 73–92, 2007.

[18] Singh, D., Singh, J.N., Some combinatorics of multisets. International Journal of Mathematical Education in Science and Technology, 34(4), 489–499, 2003.

[19] Singh, D., Ibrahim, A.M., Yohanna, T., Singh, J.N., A systematization of fundamentals of multisets, Lecturas Matematicas, 29, 33–48, 2008.

[20] Syropoulos, A., Mathematics of Multisets, Multiset Processing, LNCS 2235, Springer-Verlag Berlin Heidelberg, 347–358, 2001.

[21] Thompson, T., S-closed spaces, Proc. Amer. Math. Soc. 60, 335–338, 1976.

[22] Venkateswaran, V., A new class of multiset Wilf equivalent pairs, Discrete Mathematics 307, 2508–2513, 2007.

[23] Yang, Y., Tan, X., Meng, C., The multi-fuzzy soft set and its application in decision making, Applied Mathematical Modelling, 37, 4915–4923, 2013.

[24] Weyl, H., Philosophy of Mathematics and Natural Science, Revised and Augmented English Edition, Atheneum, New York, 1963.

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BRIEF DISCUSSION ON NEUTROSOPHIC h-IDEALS OF Γ-HEMIRINGS

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Abstaract — The concept of neutrosophic h-bi-ideals and neutrosophic h-quasi-ideals of a Γ-hemiring are introduced and some of their related properties are investigated. The notions of h-hemiregularity, h-intra-hemiregularity of a Γ-hemiring are studied and some of their characterizations in terms of neutrosophic h-ideals are also obtained.

Keywords — *Γ-hemiring, neutrosophic h-ideal, neutrosophic h-bi-ideal, neutrosophic h-quasi-ideal, h-hemiregular, h-intra-hemiregular.*

1 Introduction

Semiring is a well known universal algebra. This is a generalization of an associative ring $(R, +, .)$. If $(R, +)$ becomes a semigroup instead of a group then $(R, +, .)$ reduces to a semiring. Semiring has been found very useful for solving problems in different areas of applied mathematics and information sciences, since the structure of a semiring provides an algebraic framework for modelling and studying the key factors in these applied areas. Ideals of semiring play a central role in the structure theory and useful for many purposes. However they do not in general coincide with the usual ring ideals and for this reason, their use is somewhat limited in trying to obtain analogues of ring theorems for semiring. To ammend this gap Henriksen [12] defined a more restricted class of ideals, which are called k-ideals. A still more restricted class of ideals in hemirings are given by Iizuka [14], which are called h-ideals. LaTorre [18], investigated h-ideals and k-ideals in hemirings in an effort to obtain analogues of ring Results for hemiring and to amend the gap between ring ideals and semiring ideals. The theory of Γ-semiring was introduced by Rao [24]. These concepts are extended by Dutta and Sardar [10].

The theory of fuzzy sets, proposed by Zadeh [29], has provided a useful mathematical tool for describing the behavior of the systems that are too complex or illdefined to admit precise mathematical analysis by classical methods and tools.

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The study of fuzzy algebraic structure has started by Rosenfeld [26]. Since then many researchers developed this ideas.

As a generalization of fuzzy sets, the intuitionistic fuzzy set was introduced by Atanassov [1] in 1986, where besides the degree of membership of each element there was considered a degree of non-membership with (membership value + non-membership value) ≤ 1.

There are also several well-known theories, for instances, rough sets, vague sets, interval-valued sets etc. which can be considered as mathematical tools for dealing with uncertainties. In 1995, inspired from the sport games (winning/tie/ defeating), votes, from (yes/NA/no), from decision making(making a decision/ hesitating/not making), from (accepted/pending/rejected) etc. and guided by the fact that the law of excluded middle did not work any longer in the modern logics, F. Smarandache [23] combined the non-standard analysis [8, 25] with a tri-component logic/set/probability theory and with philosophy and introduced *Neutrosophic set* which represents the main distinction between *fuzzy* and *intuitionistic fuzzy* logic/set. Here he included the middle component. i.e. the neutral/ indeterminate/ unknown part (besides the truth/membership and falsehood/non-membership components that both appear in fuzzy logic/set) to distinguish between 'absolute membership and relative membership' or 'absolute non-membership and relative non-membership'(see, [16, 27]). There are also several authors, for example [3, 4, 5, 6, 7] who have enriched the theory of neutrosophic sets.

Inspired from the above idea and motivated by the fact that 'semirings arise naturally in combinatorics, mathematical modelling, graph theory, automata theory, parallel computation system etc.', in the paper, we have used that to study the h-ideals, h-bi-ideals, h-quasi-ideals [13, 15, 19, 21, 22, 28, 30] of Γ-semirings [24] - a generalization of semirings [11] and obtain some of its characterizations.

2 Preliminaries

We recall the following preliminaries for subsequent use.

Definition 2.1. [11] A hemiring [respectively semiring] is a non-empty set S on which operations addition and multiplication have been defined such that the following conditions are satisfied:

(i) $(S, +)$ is a commutative monoid with identity 0.
(ii) $(S, .)$ is a semigroup [respectively monoid with identity 1_S].
(iii) Multiplication distributes over addition from either side.
(iv) $0s = 0 = s0$ for all $s \in S$.
(v) $1_S \neq 0$

Definition 2.2. [21] Let S and Γ be two additive commutative semigroups with zero. Then S is called a $\Gamma-$hemiring if there exists a mapping
$S \times \Gamma \times S \to S$ ((a,α,b) $\mapsto$ aαb) satisfying the following conditions:

(i) $(a + b)\alpha c = a\alpha c + b\alpha c$,
(ii) $a\alpha(b + c) = a\alpha b + a\alpha c$,
(iii) $a(\alpha + \beta)b = a\alpha b + a\beta b$,
(iv) $a\alpha(b\beta c) = (a\alpha b)\beta c$.

(v) $0_S \alpha a = 0_S = a\alpha 0_S$,
(vi)$a0_\Gamma b = 0_S = b0_\Gamma a$

for all $a, b, c \in S$ and for all $\alpha, \beta \in \Gamma$.

For simplification we write 0 instead of 0_S and 0_Γ.

Throughout this paper, unless otherwise mentioned S denotes a $\Gamma-$hemiring and χ_S be its characteristic function.

A subset A of a Γ-hemiring S is called a left(resp. right) ideal of S if A is closed under addition and $S\Gamma A \subseteq A$ (resp. $A\Gamma S \subseteq A$). A subset$A$ of a hemiring S is called an ideal if it is both left and right ideal of S

A subset A of a Γ-hemiring S is called a quasi-ideal of S if A is closed under addition and $S\Gamma A \cap A\Gamma S \subseteq A$.

A subset A of a Γ-hemiring S is called a bi-ideal(resp. interior ideal) if A is closed under addition and $A\Gamma S\Gamma A \subseteq A$(resp. $S\Gamma A\Gamma S \subseteq A$).

A left ideal A of S is called a left h-ideal if $x, z \in S$, $a, b \in A$ and $x + a + z = b + z$ implies $x \in A$. A right h-ideal is defined analoguesly.

The h-closure $\overline{A}$ of A in S is defined as $\overline{A} = \{x \in S \mid x + a + z = b + z, \text{ for some } a, b \in A \text{ and } z \in S\}$.

Now if A is a left (right) ideal of S, then $\overline{A}$ is the smallest left (right) h-ideal containing A.

A quasi-ideal(resp. bi-ideal) A of S is called an h-quasi-ideal(resp. h-bi-ideal) of S if $\overline{S\Gamma A} \cap \overline{A\Gamma S}$(resp. $\overline{A\Gamma S\Gamma A}$) $\subseteq A$ and $x + a + z = b + z$ implies $x \in A$ for all $x, z \in S$ and $a, b \in A$.

Definition 2.3. [29] A fuzzy subset of a nonempty set X is defined as a function $\mu : X \to [0, 1]$.

Definition 2.4. [23] A neutrosophic set A on the universe of discourse X is defined as $A = \{< x, A^T(x), A^I(x), A^F(x) >, x \in X\}$, where $A^T, A^I, A^F : X \to]^-0, 1^+[$ and $^-0 \leq A^T(x) + A^I(x) + A^F(x) \leq 3^+$. From philosophical point of view, the neutrosophic set takes the value from real standard or non-standard subsets of $]^-0, 1^+[$. But in real life application in scientific and engineering problems it is difficult to use neutrosophic set with value from real standard or non-standard subset of $]^-0, 1^+[$. Hence we consider the neutrosophic set which takes the value from the subset of [0, 1].

3 Neutrosophic h-ideals in Γ-hemiring

Using the above concepts, we now define neutrosophic left(right) h-ideal, neutrosophic h-bi-ideal, neutrosophic h-quasi-ideal and several operations such as composition, cartesian product, intersection etc. on them and use these to study some of their related properties. At the time of investigation we may see that the obtained results are parallel to that of Γ-hemiring and by routine verification we can proof them. So, after giving one introductory proof, I omit all the proof.

Definition 3.1. Let $\mu = (\mu^T, \mu^I, \mu^F)$ be a non empty neutrosophic subset of a Γ-semiring S (i.e. anyone of $\mu^T(x)$, $\mu^I(x)$ or $\mu^F(x)$ not equal to zero for some $x \in S$). Then μ is called a neutrosophic left ideal of S if

(i) $\mu^T(x+y) \geq \min\{\mu^T(x), \mu^T(y)\}$, $\mu^T(x\gamma y) \geq \mu^T(y)$

(ii) $\mu^I(x+y) \geq \frac{\mu^I(x)+\mu^I(y)}{2}$, $\mu^I(x\gamma y) \geq \mu^I(y)$

(iii) $\mu^F(x+y) \leq \max\{\mu^F(x), \mu^F(y)\}$, $\mu^F(x\gamma y) \leq \mu^F(y)$.

for all $x, y \in S$ and $\gamma \in \Gamma$.

A neutrosophic left ideal is called neutrosophic left h-ideal if for $x, a, b, z \in S$ with $x + a + z = b + z$ implies

(i) $\mu^T(x) \geq \min\{\mu^T(a), \mu^T(b)\}$,

(ii) $\mu^I(x) \geq \frac{\mu^I(a)+\mu^I(b)}{2}$,

(iii) $\mu^F(x) \leq \max\{\mu^F(a), \mu^F(b)\}$.

Similarly we can define neutrosophic right h-ideal of S.

Result 3.2. Intersection of a nonempty collection of neutrosophic left h-ideals is a neutrosophic left h-ideal of S.

Proof. Let $\{\mu_i : i \in I\}$ be a non-empty family of neutrosophic left h-ideals of S and $x, y \in S$ and $\gamma \in \Gamma$. Then

$$\begin{aligned}
(\bigcap_{i\in I}\mu_i^T)(x+y) &= \inf_{i\in I} \mu_i^T(x+y) \geq \inf_{i\in I} \{\min\{\mu_i^T(x), \mu_i^T(y)\}\} \\
&= \min\{\inf_{i\in I} \mu_i^T(x), \inf_{i\in I} \mu_i^T(y)\} \\
&= \min\{(\bigcap_{i\in I}\mu_i^T)(x), (\bigcap_{i\in I}\mu_i^T)(y)\}
\end{aligned}$$

$$\begin{aligned}
(\bigcap_{i\in I}\mu_i^I)(x+y) &= \inf_{i\in I} \mu_i^I(x+y) \geq \inf_{i\in I} \frac{\mu_i^I(x)+\mu_i^I(y)}{2} \\
&= \frac{\inf_{i\in I} \mu_i^I(x) + \inf_{i\in I} \mu_i^I(y)}{2} \\
&= \frac{\bigcap_{i\in I}\mu_i^I(x) + \bigcap_{i\in I}\mu_i^I(y)}{2}
\end{aligned}$$

$$\begin{aligned}
(\bigcap_{i\in I}\mu_i^F)(x+y) &= \sup_{i\in I} \mu_i^F(x+y) \leq \sup_{i\in I} \{\max\{\mu_i^F(x), \mu_i^F(y)\}\} \\
&= \max\{\sup_{i\in I} \mu_i^F(x), \sup_{i\in I} \mu_i^F(y)\} \\
&= \max\{(\bigcap_{i\in I}\mu_i^F)(x), (\bigcap_{i\in I}\mu_i^F)(y)\}
\end{aligned}$$

$$(\bigcap_{i\in I}\mu_i^T)(x\gamma y) = \inf_{i\in I} \mu_i^T(x\gamma y) \geq \inf_{i\in I} \mu_i^T(y) = (\bigcap_{i\in I}\mu_i^T)(y).$$

$$(\bigcap_{i\in I}\mu_i^I)(x\gamma y) = \inf_{i\in I} \mu_i^I(x\gamma y) \geq \inf_{i\in I} \mu_i^I(y) = (\bigcap_{i\in I}\mu_i^I)(y).$$

$$(\bigcap_{i\in I}\mu_i^F)(x\gamma y) = \sup_{i\in I} \mu_i^F(x\gamma y) \leq \sup_{i\in I} \mu_i^F(y) = (\bigcap_{i\in I}\mu_i^F)(y).$$

Hence $\bigcap_{i\in I}\mu_i$ is a neutrosophic left ideal of S.
Now suppose $x, a, b, z \in S$ with $x + a + z = b + z$. Then

$$\begin{aligned}
(\bigcap_{i\in I}\mu_i^T)(x) &= \inf_{i\in I} \mu_i^T(x) \geq \inf_{i\in I} \min\{\mu_i^T(a), \mu_i^T(b)\} \\
&= \min\{\inf_{i\in I} \mu_i^T(a), \inf_{i\in I} \mu_i^T(b)\} = \min\{(\bigcap_{i\in I}\mu_i^T)(a), (\bigcap_{i\in I}\mu_i^T)(a)\}.
\end{aligned}$$

$$\begin{aligned}(\underset{i\in I}{\cap}\mu_i^I)(x) &= \inf_{i\in I}\ \mu_i^I(x) \geq \inf_{i\in I}\ \tfrac{\mu_i^I(y)+\mu_i^I(b)}{2} \\ &= \tfrac{\inf_{i\in I}\mu_i^I(y)+\inf_{i\in I}\mu_i^I(b)}{2} = \tfrac{\underset{i\in I}{\cap}\mu_i^I(y)+\underset{i\in I}{\cap}\mu_i^I(b)}{2}.\end{aligned}$$

$$\begin{aligned}(\underset{i\in I}{\cap}\mu_i^F)(x) &= \sup_{i\in I}\ \mu_i^F(x) \leq \sup_{i\in I}\ \max\{\mu_i^F(a), \mu_i^F(b\} \\ &= \max\{\sup_{i\in I}\ \mu_i^F(a), \sup_{i\in I}\ \mu_i^F(b)\} = \max\{(\underset{i\in I}{\cap}\mu_i^F)(a), (\underset{i\in I}{\cap}\mu_i^F)(a)\}.\end{aligned}$$

Therefore $\underset{i\in I}{\cap}\mu_i$ is a neutrosophic left h-ideal of S. □

Definition 3.3. Let μ and θ be two neutrosophic sets of a Γ-hemiring S. Now h-product of μ and θ denoted by $\mu o_h \theta$ and defined as

$$\mu^T o_h \theta^T(x) = \sup[\min_i\{\mu^T(a_i), \mu^T(c_i), \theta^T(b_i),\ \theta^T(d_i)\}\}] \quad {\scriptstyle x+\sum_{i=1}^{n} a_i\gamma_i b_i + z = \sum_{i=1}^{n} c_i\delta_i d_i + z}$$
$$= 0, \text{ if } x \text{ cannot be expressed as above}$$

$$\mu^I o_h \theta^I(x) = \sup[\min_i\{\tfrac{1}{4}[\mu^I(a_i) + \mu^I(c_i) + \theta^I(b_i)+\theta^I(d_i)]\}\}] \quad {\scriptstyle x+\sum_{i=1}^{n} a_i\gamma_i b_i + z = \sum_{i=1}^{n} c_i\delta_i d_i + z}$$
$$= 0, \text{ if } x \text{ cannot be expressed as above}$$

$$\mu^F o_h \theta^F(x) = \inf[\max_i\{[\mu^F(a_i), \mu^F(c_i), \theta^F(b_i),\ \theta^F(d_i)]\}\}] \quad {\scriptstyle x+\sum_{i=1}^{n} a_i\gamma_i b_i + z = \sum_{i=1}^{n} c_i\delta_i d_i + z}$$
$$= 0, \text{ if } x \text{ cannot be expressed as above}$$

where $x, z, a_i, b_i, c_i, d_i \in S$ and $\gamma_i, \delta_i \in \Gamma$, for $i = 1, ..., n$.

Result 3.4. If μ and ν be two neutrosophic left h-ideals of S then $\mu o \nu$ is also a neutrosophic left h-ideal of S.

Result 3.5. Let μ_1, μ_2 be two neutrosophic h-ideal of a Γ-hemiring S. Then $\mu_1 o_h \mu_2 \subseteq \mu_1 \cap \mu_2 \subseteq \mu_1, \mu_2$.

Result 3.6. Let S be a Γ-hemiring and $A, B \subseteq S$. Then we have

(i) $A \subseteq B$ if and only if $\chi_A \subseteq \chi_B$.
(ii) $\chi_A \cap \chi_B = \chi_{A\cap B}$
(iii) $\chi_A o_h \chi_B = \chi_{A\Gamma B}$

Definition 3.7. Let μ and ν be two neutrosophic subsets of S. The cartesian product of μ and ν is defined by

$$(\mu^T \times \nu^T)(x, y) = \min\{\mu^T(x), \nu^T(y)\}$$
$$(\mu^I \times \nu^I)(x, y) = \frac{\mu^I(x) + \nu^I(y)}{2}$$
$$(\mu^F \times \nu^F)(x, y) = \max\{\mu^F(x), \nu^F(y)\}$$

for all $x, y \in S$.

Result 3.8. Let μ and ν be two neutrosophic left h-ideals of S. Then $\mu \times \nu$ is a neutrosophic left h-ideal of $S \times S$.

Definition 3.9. A neutrosophic subset μ of a Γ-hemiring S is called neutrosophic h-bi-ideal if for all $x, y, z, a, b \in S$ and $\alpha, \beta \in \Gamma$ we have

(i) $\mu^T(x+y) \geq \min\{\mu^T(x), \mu^T(y)\}$

(ii) $\mu^T(x\alpha y) \geq \min\{\mu^T(x), \mu^T(y)\}$

(iii) $\mu^T(x\alpha y\beta z) \geq \min\{\mu^T(x), \mu^T(z)\}$

(iv) $x + a + z = b + z \ \Rightarrow \mu^T(x) \geq \min\{\mu^T(a), \mu^T(b)\}$

(v) $\mu^I(x+y) \geq \frac{\mu^I(x)+\mu^I(y)}{2}$

(vi) $\mu^I(x\alpha y) \geq \frac{\mu^I(x)+\mu^I(y)}{2}$

(vii) $\mu^I(x\alpha y\beta z) \geq \frac{\mu^I(x)+\mu^I(z)}{2}$

(viii) $x + a + z = b + z \ \Rightarrow \mu^T(x) \geq \frac{\mu^I(a)+\mu^I(b)}{2}$

(ix) $\mu^F(x+y) \leq \max\{\mu^F(x), \mu^F(y)\}$

(x) $\mu^F(x\alpha y) \leq \max\{\mu^F(x), \mu^F(y)\}$

(xi) $\mu^F(x\alpha y\beta z) \leq \max\{\mu^F(x), \mu^F(z)\}$

(xii) $x + a + z = b + z \ \Rightarrow \mu^F(x) \leq \max\{\mu^F(a), \mu^F(b)\}$

Definition 3.10. A neutrosophic subset μ of a Γ-hemiring S is called neutrosophic h-quasi-ideal if for all $x, y, z, a, b \in S$ we have

(i) $\mu^T(x+y) \geq \min\{\mu^T(x), \mu^T(y)\}$

(ii) $\mu^I(x+y) \geq \frac{\mu^I(x)+\mu^I(y)}{2}$

(iii) $\mu^F(x+y) \leq \max\{\mu^F(x), \mu^F(y)\}$

(iv) $(\mu^T o_h \chi_S^T) \cap (\chi_S^T o_h \mu^T) \subseteq \mu^T$

(v) $(\mu^I o_h \chi_S^I) \cap (\chi_S^I o_h \mu^I) \subseteq \mu^I$

(vi) $(\mu^F o_h \chi_S^F) \cap (\chi_S^F o_h \mu^F) \supseteq \mu^T$

(vii) $x + a + z = b + z \ \Rightarrow \mu^T(x) \geq \min\{\mu^T(a), \mu^T(b)\}$

(viii) $x + a + z = b + z \ \Rightarrow \mu^T(x) \geq \frac{\mu^I(a)+\mu^I(b)}{2}$

(ix) $x + a + z = b + z \ \Rightarrow \mu^F(x) \leq \max\{\mu^F(a), \mu^F(b)\}$

For any neutrosophic subset in a set X and any $t \in [0,1]$, define level subsets of μ by $\{\mu_t^T := \{x \in S : \mu^T(x) \geq t, \ t \in [0,1]\}, \ \mu_t^I := \{x \in S : \mu^I(x) \geq t, \ t \in [0,1]\}$ and $\mu_t^F := \{x \in S : \mu^F(x) \leq t, \ t \in [0,1]\}\}$. In [17], Kondo et al. introduced the Transfer Principle in fuzzy set theory, from which a neutrosophic set can be characterized by its level subsets. For any algebraic system $\mathcal{U} = (X, F)$, where F is a family of operations defined on X, the Transfer Principle can be formulated as follows:

Result 3.11. A fuzzy subset defined on $\mathcal{U}$ has the property $\mathcal{P}$ if and only if all non-empty level subset μ_t have the property $\mathcal{P}$.

As a direct consequence of the above Result, the following two results can be obtained.

Result 3.12. Let S be a Γ-hemiring. Then the following conditions hold:

(i) μ is a neutrosophic left(resp. right) h-ideal of S if and only if all non-empty level subsets μ_t are left (resp. right) h-ideals of S.

(ii) μ is a neutrosophic h-bi-ideal of S if and only if all non-empty level subsets μ_t are h-bi-ideals of S.

(iii) μ is a neutrosophic h-quasi-ideal of S if and only if all non-empty level subsets μ_t are h-quasi-ideals of S.

Result 3.13. Let S be a Γ-hemiring and $A \subseteq S$. Then the following conditions hold:

(i) A is a left(resp. right) h-ideal of S if and only if χ_A is a neutrosophic left (resp. right) h-ideal of S.

(ii) A is an h-bi-ideal of S if and only if χ_A is a neutrosophic h-bi-ideal of S.

(iii) A is an h-quasi-ideal of S if and only if χ_A is a neutrosophic h-quasi-ideal of S.

Result 3.14. Any neutrosophic h-quasi-ideal of S is a neutrosophic h-bi-ideal of S.

Definition 3.15. [19] A Γ-hemiring S is said to be h-hemiregular if for each $x \in S$, there exist $a, b \in S$ and $\alpha, \beta, \gamma, \delta \in \Gamma$ such that $x + x\alpha a\beta x + z = x\gamma b\delta x + z$.

Result 3.16. A Γ-hemiring S is h-hemiregular if and only if for any neutrosophic right h-ideal μ and any neutrosophic left h-ideal ν of S we have $\mu o_h \nu = \mu \cap \nu$.

Now we obtain the following characterizations of h-hemiregular Γ-hemirings. Note that for any two neutrosophic subsets μ and ν of S, $\mu \sqsubseteq \nu$ implies $\mu^T \subseteq \nu^T$, $\mu^I \subseteq \nu^I$ and $\mu^F \supseteq \nu^F$.

Result 3.17. Let S be a Γ-hemiring. Then the following conditions are equivalent.
(i) S is h-hemireglar.
(ii) $\mu \sqsubseteq \mu o_h \chi_S o_h \mu$ for every neutrosophic h-bi-ideal μ of S.
(iii) $\mu \sqsubseteq \mu o_h \chi_S o_h \mu$ for every neutrosophic h-quasi-ideal μ of S.

Result 3.18. Let S is a Γ-hemiring. Then the following conditions are equivalent.
(i) S is h-hemiregular.
(ii) $\mu \cap \nu \sqsubseteq \mu o_h \nu o_h \mu$ for every neutrosophic h-bi-ideal μ and every neutrosophic h-ideal ν of S.
(iii) $\mu \cap \nu \sqsubseteq \mu o_h \nu o_h \mu$ for every neutrosophic h-quasi-ideal μ and every neutrosophic h-ideal ν of S.

Result 3.19. Let S is a Γ-hemiring. Then the following conditions are equivalent.

(i) S is h-hemiregular.
(ii) $\mu \cap \nu \sqsubseteq \mu o_h \nu$ for every neutrosophic h-bi-ideal μ and every neutrosophic left h-ideal ν of S.
(iii) $\mu \cap \nu \sqsubseteq \mu o_h \nu$ for every neutrosophic h-quasi-ideal μ and every neutrosophic left h-ideal ν of S.
(iv) $\mu \cap \nu \sqsubseteq \mu o_h \nu$ for every neutrosophic right h-ideal μ and every neutrosophic h-bi-ideal ν of S.
(v) $\mu \cap \nu \sqsubseteq \mu o_h \nu$ for every neutrosophic right h-ideal μ and every neutrosophic h-quasi-ideal ν of S.
(vi) $\mu \cap \nu \cap \omega \sqsubseteq \mu o_h \nu o_h \omega$ for every neutrosophic right h-ideal μ, for every neutrosophic h-bi-ideal ν and for every neutrosophic left h-ideal ω of S.
(vii) $\mu \cap \nu \cap \omega \sqsubseteq \mu o_h \nu o_h \omega$ for every neutrosophic right h-ideal μ, for every neutrosophic h-quasi-ideal ν and for every neutrosophic left h-ideal ω of S.

Result 3.20. If a Γ-hemiring S is h-hemiregular then any neutrosophic right h-ideal μ and neutrosophic left h-ideal ν are idempotent and $\mu o_h \nu$ is an quasi-ideal of S.

Definition 3.21. A Γ-hemiring S is said to be h-intra-hemiregular if for each $x \in S$, there exist $z, a_i, a_i', b_i, b_i' \in$S, and $\alpha_i, \beta_i, \gamma_i, \delta_i, \eta \in \Gamma$, i$\in$ **N**, the set of natural numbers, such that $x + \sum_{i=1}^{n} a_i \alpha_i x \eta x \beta_i a_i' + z = \sum_{i=1}^{n} b_i \gamma_i x \eta x \delta b_i' + z$.

Result 3.22. Let S be a Γ-hemiring. Then S is h-intra-hemiregular if and only if $\mu \cap \nu \sqsubseteq \mu o_h \nu$ for every neutrosophic left h-ideal μ and every neutrosophic right h-ideal ν of S.

Result 3.23. Let S be a Γ-hemiring and $x \in S$. Then S is h-intra-hemiregular if and only if $\mu(x) = \mu(x\gamma x)$, for all neutrosophic h-ideal μ of S and for all $x \in S$ and $\gamma \in \Gamma$.

Result 3.24. Let S be a Γ-hemiring. Then the following conditions are equivalent.

(i) S is both h-hemiregular and h-intra-hemiregular.
(ii) $\mu = \mu o_h \mu$ for every h-bi-ideal μ of S.
(iii) $\mu = \mu o_h \mu$ for every h-quasi-ideal μ of S.

Result 3.25. Let S be a Γ-hemiring. Then the following conditions are equivalent.

(i) S is both h-hemiregular and h-intra-hemiregular.
(ii) $\mu \cap \nu \sqsubseteq \mu o_h \nu$ for all neutrosophic h-bi-ideals μ and ν of S.
(iii) $\mu \cap \nu \sqsubseteq \mu o_h \nu$ for every neutrosophic h-bi-ideals μ and every neutrosophic h-quasi-ideal ν of S.
(iv) $\mu \cap \nu \sqsubseteq \mu o_h \nu$ for every neutrosophic h-quasi-ideals μ and every neutrosophic h-bi-ideal ν of S.
(v) $\mu \cap \nu \sqsubseteq \mu o_h \nu$ for all neutrosophic h-quasi-ideals μ and ν of S.

Conclusion: Since I have studied the results in case of Γ-hemiring $-$ a general setting of hemiring, the obtained results are also true for hemiring along with some parallel changes. In a similar way, neutrosophic k-ideals of Γ-semiring can be studied.

References

[1] K. Atanassov, *Intuitionistic fuzzy sets*, Fuzzy Sets and Systems, 20 (1986) 87 - 96.

[2] S. Broumi, F. Smarandache and P. K. Maji, Intuitionistic neutrosophic soft set over rings, Mathematics and Statistics 2(3): (2014) 120-126, DOI: 10.13189/ms.2014.020303.

[3] S. Broumi, M. Talea, A. Bakali, F. Smarandache, "Single Valued Neutrosophic Graphs," Journal of New Theory, N 10, 2016, pp. 86-101.

[4] S. Broumi, M. Talea, A. Bakali, F. Smarandache, "On Bipolar Single Valued Neutrosophic Graphs," Journal of New Theory, N11, 2016, pp.84-102.

[5] S. Broumi, M. Talea, A. Bakali, F. Smarandache, Interval Valued Neutrosophic Graphs, SISOM (2016) in press.

[6] S. Broumi, M. Talea, F. Smarandache, A. Bakali, Single Valued Neutrosophic Graphs: Degree, Order and Size, FuzzIEEE, 8 pages (2016) accepted

[7] S Broumi, F. Smarandache, M. Talea , A. Bakali, An Introduction to Bipolar Single Valued Neutrosophic Graph Theory, Applied Mechanics and Materials, Vol. 841, pp 184-191, doi:10.4028/www.scientific.net/AMM.841.184

[8] N. J. Cutland, *Nonstandard Analysis and its Applications*, Cambridge University Press, Cambridge, 1988.

[9] T.K.Dutta, B.K. Biswas, *Ph.D. dissertation*, University of Calcutta, India

[10] T.K. Dutta, S.K. Sardar, "On Operator Semiring of a $\Gamma-$semiring", *Southest Asian Bull of Mathematics, Springer-Vergal*, 26(2002)203-213

[11] J.S.Golan, *Semirings and their applications*, Kluwer Academic Publishers,1999.

[12] M.Henriksen, "Ideals in Semirings with Commutative Addition", *Am.Math.Soc.Notices* 6(1958)321

[13] X.Huang, H.Li and Y.Yin, "The h-hemiregular Fuzzy Duo Hemirings", *Int. J. Fuzzy Systems*, vol. 9, no. 2(2007) 105-109

[14] K.Iizuka, "On the Jacobson Radical of Semiring", *Tohoku Math.J.*11(2) (1959)409-421

[15] Y.B.Jun, M.A.Öztürk, S.Z.Song, "On Fuzzy h-ideals in hemiring", *Information sciences* 162(2004)211-226.

[16] G. J. Klir and B. Yuan, *Fuzzy Sets and Fuzzy logic- Theory and Applications*, Prentice Hail P T R Upper Saddle River, New Jersey, 1995.

[17] M.Kando, W.A.Dudek, "On Transfer Principle in Fuzzy Theory", *Mathware Soft Comput.* 12(2005)41-55.

[18] D.R.La Torre, "On h-ideals and k-ideals in hemirings", *Publ. Math. Debrecen* 12(1965)219-226.

[19] X.Ma, J.Zahn, "Fuzzy h-ideals in h-hemiregular and h-semisimple Γ-hemirings", *Neural Comput and Applic* (2010)19:477-485

[20] D.Mandal, Neutrosophic Ideals of Γ-Semirings, *Journal of New Results in Science*, No. 6(2014), 51-61

[21] S.K.Sardar, D.Mandal, "On fuzzy h-ideal in Γ-hemiring", *Int.J.Pure.Appl.Math*, Vol. 56, No. 3(2009),439-450

[22] S.K.Sardar and D.Mandal, On fuzzy h-ideals in h-regular Γ-hemiring and h-duo Γ-hemiring, *Gen. Math. Notes*, Vol. 2, No. 1(2011), 64-85

[23] F. Smarandache, *Neutrosophic set- a generalization of the intuitionistic fuzzy sets*, Int. J. Pure Appl. Math., 24 (2005) 287 - 297.

[24] M.M.K.Rao, "$\Gamma-$semirings-1", *Southeast Asian Bull. of Math.*, 19(1995)49-54

[25] A. Robinson, *Nonstandard Analysis*, Princeton University Press, Princeton, 1996.

[26] A. Rosenfeld, "Fuzzy groups", *J. Math. Anal. Appl.* 35(1971) 512-517

[27] T. J. Ross, J. M. Booker, V. J. Parkinson, *Fuzzy logic and probability : Bridiging the gap*, ASA-SIAM Series on Statistics and Applied Probability, Alexandria, 2002.

[28] Y.Yin and H.Li, "The characterization of h-hemiregular hemirings and h-intra-hemiregular hemirings", *Information Sciences*, 178(2008)3451-3464

[29] L.A.Zadeh, "Fuzzy Sets", *Information and Control* 8(1965)338-353.

[30] J.Zhan and W.A.Dudek, "Fuzzy h-ideals of hemirings",*Information sciences* 177(2007)876-886.

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SOME NEW CONCEPTS IN TOPOLOGICAL GROUPS

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Abstaract − In this study, we define a new boundedness concept different from existing definitions. Also we give some theorems and results in topological groups. The new definition more general than boundedness definition in topological vector spaces.

Keywords − *Topological Groups, Boundedness.*

1 Introduction

There exists some works with regards to boundedness of topological groups. Bruguera, Tkachenko and Hejcman have presented another boundedness definitions in topological groups [1], [2]. In 1991, Atkin gave the boundedness concept in uniform spaces which are more general structures than topological groups [3]. Then Hernandez presented Pontryagin duality for topological abelian groups in [4]. If a set is absorbed by every neighbourhood of 0 the set is called as a bounded set in a topological vector space. That is, there exists a number $\varepsilon > 0$ for each neighbourhood U of 0 such that $tA \subseteq U$ for every $|t| < \varepsilon$. The operation of scalar multiplication tA is very important in this definition. There isn't exist this operation in groups so it cannot be applied directly to the topological groups. We know that every topological vector space has an additive topological group structure so the boundedness definition is also generalization of current available boundedness definition in topological vector spaces. Therefore we present a kind of boundedness definition in topological groups so similar to those in topological vector spaces. The new definition is not a generalization of existing boundedness definitions for topological groups.

2 Preliminaries

Let G be an abstract group, A and B be two subsets of G. Then AB is the set of all elements of xy such that $x \in A$ and $y \in B$. The definition of A^2 and $A^m = A^{m-1}A$

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is clear by taking $B = A$ for some $m \in \mathbb{N}$. Further, $A^{-1} = \{a^{-1} : a \in A\}$, $A^{-m} = (A^{-1})^m$ and $A^0 = \{e\}$ for the unit element e of G. Given $x \in A^m$ there exist some $a_1, a_2, ..., a_m \in A$ such that $x = a_1 a_2 ... a_m$. If $x^m \in A^m$ and $e \notin A$ then x^n may not be an element of A^m, for $n < m$. Hence we define the set $A^{\leq m}$ by $x = a_1 a_2 ... a_n$ for $m, n \in \mathbb{N}$, $a_1, a_2, ..., a_n \in A$ and some $n \leq m$. It is clear that $A^m \subseteq A^{\leq m}$ and $A^m = A^{\leq m}$ whenever e is contained by A.

It is known as every topological vector space is an additive group, it is written mU instead of U^m. Then a set B is bounded if and only if there exists a positive integer m depend on U for every symmetrical neighbourhood U such that $B \subseteq U$ or $\frac{1}{m}B \subseteq U$. This is known as boundedness definition in topological vector spaces.

Now we mention that some definitions and propositions in topological groups. Since a topological group has a local basis of symmetrical neighbourhood of the unit element e, a connected topological group G is generated by a neighbourhood U of e i.e. all elements of G is denoted by finite multiplication of elements belong to U [5]. A set S is called as precompact set in a topological group if there exists a finite set F for each neighbourhood U of e such that $S \subseteq FU$. We have known that if a set is bounded then it is metrically bounded i.e. boundedness with respect to the semimetric in a topological vector space. But opposite of this proposition is not correct [6].

Let G be a group and $p : G \to \mathbb{R}$ be a function. p is called an absolute value function on G if satisfies the following properties for each $x, y, a \in G$

(i) $p(x) \geq 0$,

(ii) $p(e) = 0$ and $p(x^{-1}) = p(x)$,

(iii) $p(xy) \leq p(x) + p(y)$,

(iv) If $p(x_n) \to 0$ then $p(a x_n a^{-1}) \to 0$ for every sequence (x_n).

Last condition is unnecessary for abelian groups. The equality $d(x, y) = p(x^{-1}y)$ defines a semimetric generating group topology on G. d is called a left invariant semimetric if $d(ax, ay) = d(x, y)$ for every $x, y, a \in G$. The topology of a topological group first countable comes from a left invariant semimetric [7].

Let G is a topological group and $B \subseteq G$. If the set B absorbs every bounded set then B is called a bornivorous.

Let G is a topological group. If every bornivorous in G is a neighbourhood of e then G is called bornological group [5].

3 Main Results

In this section, we will give some new definitions and results in topological groups.

Definition 3.1. Let G be a topological group and $A \subseteq G$. The set A is called as absorbing set if there exist a finite set $F_x \subseteq G$ and a number $m \in \mathbb{N}$ for every $x \in G$ such that $x \in F_x A^m$.

Definition 3.2. Let G be a topological group and $A \subseteq G$. The set A is called as a bounded set if the set is absorbed by every neighbourhood of the unit element e of G i.e. there exist a finite set F and a number $m \in \mathbb{N}$ for every $U \in N_e$ such that $A \subseteq FU^m = \bigcup_{x \in F} \{xU^m\}$.

Proposition 3.3. According to this (boundedness) definition, boundedness of a set A in a topological group $(X, +)$ is equivalent to boundedness of A in the topological vector space X.

Proof. Now we take a subset A is bounded in the topological vector space X. There exists a number $\lambda > 0$ for every $U \in N_0$ such that $A \subseteq \lambda U$. Therefore we get $A \subseteq ([[\lambda]] + 1)U$. If we select $F = \{0\}$ and $([[\lambda]] + 1) = m$ then

$$A \subseteq ([[\lambda]] + 1)U = F + mU = F + U^m$$

Thus the subset A is bounded in the topological group $(X, +)$.

On the contrary, $A \subseteq F + U^m = F + mU$. If we take $F = \{0\}$ and $m = \lambda$ then $A \subseteq \lambda U$. □

Theorem 3.4. Every singleton is bounded in a topological group.

Proof. If we take $F = \{a\}$ and $m = 1$ then $\{a\} \subset \{a\}U = \{aU\}$. This completes the proof, easily. □

Theorem 3.5. Union of two bounded sets is also bounded in a topological group.

Proof. Let A and B be two bounded subsets in a topological group X. There exists a finite set $F \subseteq X$ and a number $m \in \mathbb{N}$ for every $U \in N_e$ such that

$$A \cup B \subseteq FU^n = \underset{x\in F}{\cup}\{xU^m\}$$

We suppose that the above inclusion isn't true. Thus $A \cup B$ isn't covered by FU^m for every finite set $F \subseteq X$ and every number $m \in \mathbb{N}$. Then A isn't covered by FU^m or B isn't covered by FU^m. This contradict with our hypothesis. □

Corollary 3.6. Every subset of a bounded set is bounded in a topological group.

Corollary 3.7. Intersection of two bounded sets is bounded in a topological group.

Theorem 3.8. Every finite set is bounded in a topological group.

Proof. It is easily seen that union of finite number of bounded sets is bounded by induction method since we know that every set is written by union of singletons. □

Theorem 3.9. Every precompact set is bounded in a topological group.

Proof. Let G be a topological group, S be a precompact set in G, U be any neighbourhood of e and V be an other neighbourhood of e such that $VV \subset U$. There exists a finite set F such that $S \subset FV$ by hypothesis then F is bounded. Thus there exist a number $n \in \mathbb{N}$ and a finite set G such that $F \subset GV^n$. Then

$$S \subset FV \subset GV^nV \subset GV^nV^n = G(VV)^n \subset GU^n$$

i.e. S is a bounded set. □

Corollary 3.10. Every compact set is bounded in a topological group.

Lemma 3.11. Let X be a topological group and $x \in X$. Then $xD_r(e) = D_r(x)$.

Proof. $y \in xD_r(e) \Leftrightarrow$ if and only if there exists a point $a \in D_r(e)$ such that $y = xa$. Thus

$$\begin{aligned} a \in D_r(e) &\Leftrightarrow d(e, a) < r \\ &\Leftrightarrow d(e, \tfrac{y}{x}) < r \\ &\Leftrightarrow d(e, x^{-1}y) < r \end{aligned}$$

and also since $d(e, x^{-1}y) = d(xe, xx^{-1}y) = d(x, y)$ then $y \in D_r(x)$. □

Lemma 3.12. Let X be a topological group and $x \in X$ then $xD_r(e)^m \subseteq (xD_r(e))^m$.

Lemma 3.13. Let X be a topological group and $x \in X$ then $D_r(x)^m \subseteq D_{rm}(x)$.

Proof. If $y \in D_r(x)^m$ there exist $a_1, a_2, ..., a_m \in D_r(x)$ such that $y = a_1 a_2 ... a_m$. Hence

$$\begin{aligned} d(y,x) &= d(a_1a_2...a_m, x) \\ &< d(x,a_1) + d(x,a_2) + ... + d(x,a_m) \\ &< r + r + ... + r \\ &= mr \end{aligned}$$

Thus $y \in D_{rm}(x)$. □

Theorem 3.14. Let G be a semimetric group and $A \subseteq G$ be a bounded set then A is a metrically bounded.

Proof. Let G be a semimetric group and $A \subseteq G$. A set A is bounded if and only if there exists a number $m \in \mathbb{N}$ and a finite set F such that $A \subseteq FD_r(e)^m$. Thus

$$A \subseteq FD_r(e)^m \Leftrightarrow A \subseteq \underset{x \in F}{\cup} \{xD_r(e)^m\} \subseteq D_{rm}(x).$$

This completes the proof. □

Proposition 3.15. A set is absorbed by each member of a local basis of neighbourhoods of e if and only if this set is bounded.

Proof. Let $B = \{U_\alpha : \alpha \in I\}$ be a basis of neighbourhoods of e in a topological group G. It is easily seen that a subset $A \subseteq G$ is absorbed for every neighbourhood U_α. On the contrary, if every $U \in N_e$ then $U_\alpha \subseteq U$ for every $\alpha \in I$. The set A is absorbed by U_α for $\alpha \in A$ if and only if there exists a finite set F_{U_α} and a number $m \in \mathbb{N}$ such that $A \subseteq F_{U_\alpha} U_\alpha^m \subseteq F_{U_\alpha} U^m$. Thus the set A is bounded. □

Proposition 3.16. Every bounded subset of a topological group is contained by the set $\overline{\{e\}}$.

Proof. Let G be a topological group and S be a bounded subset of G. Now we show that $S \subseteq \overline{\{e\}}$. We assume that $x \in \overline{\{e\}}$ is wrong. $U \cap \{e\} = \emptyset$ for a neighbourhood $U \in N_x$ if and only if $U \subseteq \{e\}^c$ or $\{e\} \subseteq U^c$. There exists a finite set F and $m \in \mathbb{N}$ such that $x \in S \subseteq FW^m$ because S is bounded and $x \in S$ for every $W \in N_e$. There exists $f \in F$ and $w \in W^m$ such that $x = fw$. Then $FW^m \in N_x$. That is $FW^m \cap \{e\} = \underset{f \in F}{\cup} \{fW^m\} \cap \{e\} \neq \emptyset$. This is a contradiction. □

Theorem 3.17. A set B is bounded if and only if every countable subset of B is bounded in a topological space.

Proof. It is obvious that every countable subset of this set is bounded since if a set is bounded then every subset of this set is bounded.

On the contrary, we assume that every countable subset of B is bounded, but B isn't bounded. There exists a neighbourhood U of e such that B isn't included by FU^m for every number $m \in \mathbb{N}$ and a finite set F. Now, we construct the sequence $\{x_m\}_{m=1}^\infty$ such that

$$x_1 \in B \backslash FU,\ x_2 \in B \backslash FU^2,\ ..., x_i \in B \backslash FU^i, ...$$

Obviously the sequence $\{x_m\}_{m=1}^\infty$ isn't absorbed by U i.e. $\{x_m\}_{m=1}^\infty \subset B$ isn't bounded. This contradict with our hypothesis. □

Now we give definitions of bounded mapping, bornological group and then we prove some theorems connected with these concepts.

Definition 3.18. If a mappings is conserved bounded sets between topological groups then this mapping is called as bounded mapping.

Lemma 3.19. Let f be any homomorphism and $m \in \mathbb{N}$ then $\{f^{-1}(V)\}^m \subseteq f^{-1}(V^m)$.

Proof. For all $z \in f^{-1}(V)^m$ there exist $a_1, a_2, ..., a_m \in f^{-1}(V)$ such that $z = a_1 a_2 ... a_m$.

$f(z) = f(a_1 a_2 ... a_m) = f(a_1) f(a_2) ... f(a_m)$ and $f(a_i) \in V$ for all $1 \leq i \leq m$. $f(z) \in V^m$ then $z \in f^{-1}(V^m)$. Since $f(S) \subseteq \bigcup_{x \in F} \{f(x) f(\{f^{-1}(V^m)\})\}$ and $\{f^{-1}(V)\}^m \subseteq f^{-1}(V^m)$ then $f(S) \subseteq \bigcup_{x \in F} \{f(x) f(f^{-1}(V^m))\}$. Thus

$$f(S) \subseteq \bigcup_{x \in F} \{f(x) V^m\} = f(F) V^m$$

□

Theorem 3.20. Every continuous homomorphism between topological groups must be bounded.

Proof. Let G and G' be two topological groups, $f : G \to G'$ be a homomorphism and $S \subseteq G$ be bounded. Also let e and e' be unit elements of G and G', respectively. Since S is bounded there exists a number $m \in \mathbb{N}$ and a finite set F for every neighbourhood U of e such that $S \subseteq FU^m = \bigcup_{x \in F} \{xU^m\}$.

$xU^m \in N_x$ because $U^m \in N_e$. If we take $V \in N_{e'}$ then $f^{-1}(V) \in N_e$ and

$$S \subseteq F\left\{f^{-1}(V)\right\}^m = \bigcup_{x \in F} \left\{x f^{-1}(V)^m\right\}$$

then

$$f\left(\bigcup_{x \in F} \left\{x f^{-1}(V)^m\right\}\right) = \bigcup_{x \in F} \left\{f(x) f\left(f^{-1}(V)^m\right)\right\}.$$

Hence

$$f(S) \subseteq \bigcup_{x \in F} \left\{f(x) f(\{f^{-1}(V)\}^m)\right\}$$

and

$$f(S) \subseteq \bigcup_{x \in F} \{f(x) V^m\} = f(F) V^m.$$

□

Definition 3.21. Let G be a topological group and $B \subseteq G$. If the set B absorbs every bounded set then B is called as a bornivorous.

Definition 3.22. Let G be a topological group. If every bornivorous in G is a neighbourhood of e then G is called as a bornological group.

Theorem 3.23. Every bornivorous in a semimetric group G is a neighbourhood of e.

Proof. Let B be a bornivorous in G. We assume that B isn't a neighbourhood of e. In this case, the set B^n isn't also a neighbourhood of e for every number $n \in \mathbb{N}$.

The open sphere $D_{\frac{1}{n}}(e) = \{x : d(x,e) < \frac{1}{n}\}$ isn't contained by B, for every number n. So this sphere isn't contained the sets B^n because they aren't also neighbourhood of e. Then $\{D_{\frac{1}{n}}(e)\} \backslash B^n \neq \phi$ for every number n. The sequence $\{x_m\}_{m=1}^{\infty}$ which is constructed the style that

$$x_1 \in \{D_1(e)\} \backslash B,\ x_2 \in \{D_{\frac{1}{2}}(e)\} \backslash B^{\leq 2}, ...$$

isn't absorbed by the set B. But the sequence is bounded since $\{x_m\}_{m=1}^{\infty}$ is absorbed by neighbourhood $D_1(e)$ of e. This case is contrary to the fact that B is a bornivorous. □

Remark 3.24. Obviously every neighbourhood of e is a bornivorous. Also it is understand that every semimetric group is a bornological group by above theorem.

Proposition 3.25. Let G and H be two topological groups, $f : G \to H$ be a bounded homomorphism. If $A \subseteq G$ is a bornivorous, then $f(A)$ is also a bornivorous in H.

Proof. Let we take $y \in f(S)$. Then

$$\begin{aligned} y \in f(S) &\Rightarrow f(x) \in f(S) \\ &\Rightarrow x \in S \\ &\Rightarrow x \in FA^n \end{aligned}$$

Thus $f(S) \subseteq f(FA^n) = f(\underset{x\in F}{\cup}\{xA^n\}) = f(\underset{x\in F}{\cup}\{x\}))f(A^n)$. $f(A)$ is a bornivorous in H because $f(F)$ is a finite set. □

Proposition 3.26. Let G and H be two topological groups, $f : G \to H$ be a bounded homomorphism. If $B \subseteq f(G)$ is a bornivorous in H, then $f^{-1}(B)$ is also a bornivorous in G.

Theorem 3.27. Let G be a bornological group. In this case, every bounded homomorphism f which is defined from G into any topological group H is continuous.

Proof. Let U be a neighbourhood of e in H then the set U absorbs every bounded set in H. Thus the set U is a bornivorous. $f^{-1}(U)$ is a bornivorous in G by above proposition and G is also a neighbourhood of e by hypothesis i.e. f is continuous on e. So f is continuous in everywhere. □

Proposition 3.28. Let (X, τ) and $(Y, \tau^{'})$ be any topological groups and $f : X \to Y$ be a continuous homomorphism. If $A \subseteq X$ is bounded then $(f(\bar{A})) \subseteq Y$ is bounded.

Proof. Let we take any $V \in N_{e^{'}}$ so $f^{-1}(V) \in N_e$. Since A is bounded set, there exists a finite set F and a number $m \in \mathbb{N}$ such that $A \subseteq Ff^{-1}(V)^m$. Thus $f(A) \subseteq f(F)f(f^{-1}(V)^m) \subseteq f(F)f(f^{-1}(V^m)) \subseteq f(F)V^m$ and then $(f(\bar{A})) \subseteq f(F)V^{m+1}$ i.e. $(f(\bar{A}))$ is bounded. □

Proposition 3.29. Let $(X_i, \tau_i)_{i\in I}$ is any family of topological groups, $X = \underset{i\in I}{\Pi} X_i$ and $\Pi_i : X \to X_i$ be the projection. $A \subseteq X$ is bounded if and only if $\Pi_i(A) \subseteq X_i$ is bounded for every $i \in I$.

Proof. If A is bounded in (X, τ) there exists a finite set F and $m \in \mathbb{N}$ such that $A \subseteq \left(\Pi_i^{-1}(V_i)\right)^m F$. $\Pi_i(A) \subseteq \Pi_i\left(\Pi_i^{-1}(V_i)^m\right)\Pi_i(F)$. Then

$$\Pi_i(A) \subseteq \Pi_i\left(\{\Pi_i^{-1}(V_i)\}^m\right)\Pi_i(F) \subseteq \Pi_i\left(\Pi_i^{-1}(V_i^m)\right)\Pi_i(F) \subseteq V_i^m \Pi_i(F).$$

On the contrary, let $\Pi_i(A)$ is bounded in (X_i, τ_i) for every $i \in I$.

We take any $V \in N_e$. For every $i \in I$ and $V_i \in N_i(e_i)$, $V = \prod_{i\in I} V_i$. There exists a finite set F_i and $m \in \mathbb{N}$ such that $\Pi_i(A) \subseteq V_i^m F_i$ because $\Pi_i(A)$ is bounded for every $i \in I$. Let we take $\Pi_i(A) = A_i$. Therefore $\prod_{i\in I} A_i \subseteq \prod_{i\in I}(V_i^m F_i) = \left(\prod_{i\in I} V_i\right)^m \prod_{i\in I} F_i$. Thus $A \subseteq V^m F$ i.e. $A \subseteq X$ is bounded. □

References

[1] M. Bruguera, And M. Tkachenko, Bounded Sets in Topological Groups and Embeddings, Topology and Its Appl., 154, (2007) 1298-1306.

[2] J. Hejcman, Boundedness in Uniform Spaces and Topological Groups, Czechoslovak Mathematical Journal, 09, (4) (1959) 544-563.

[3] C. J. Atkin, Boundedness in Uniform Spaces, Topological Groups and Homogeneous Spaces, Acta Math. Hungarica, 57 (3-4) (1991) 213-232.

[4] S. Hernandez, Pontryagin Duality for Topological Abelian Groups, Mathematische Zeitschrift, 238 (2001) 493-503.

[5] L. Pontrjagin, Topological Groups, Princeton University Press, 1946.

[6] A. Wilansky, Modern Methods in Topological Vector Spaces, McGraw Hill, New York, 1978.

[7] A. Wilansky, Topology for Analysis, R.E.Krieger Pub. Company Inc., Malabar, Florida 1983.

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