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SEMIPRIME AND NILPOTENT FUZZY LIE ALGEBRAS

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Abstract – In this paper, we have introduced the concept of semiprime fuzzy Lie algebra and proved that every fuzzy Lie algebra of semiprime (nilpotent) Lie algebra is a semiprime (nilpotent).

Keywords – *Fuzzy Lie Algebra, Semiprime Fuzzy Lie Algebra, Nilpotent Fuzzy Lie Algebra.*

1 Introduction

Zadeh (1965) introduced the concept of fuzzy sets in [1]. Then Liu (1982) extended this concept to rings in [2]. Nanda (1990) defined the notion of fuzzy algebras over fuzzy fields [3]. Yehia (1996) defined the concept of fuzzy Lie algebras over fields [4]. Finally, fuzzy Lie algebras over fuzzy fields are defined by Lilly and Antony (2009) in [5]. In this paper, we first have introduced some basic definitions of fuzzy sets, fuzzy rings and fuzzy Lie algebras which will be used throughout this work. We then have introduced the concept of semiprime fuzzy Lie algebra and proved that every fuzzy Lie algebra of semiprime (nilpotent) Lie algebra is a semiprime (nilpotent).

2 Preliminaries

The following definitions and results are required.

Definition 2.1. [6] A ***Lie algebra*** is a vector space V over a field F on which a product operation $[x \quad y]$ is defined and satisfies the following axioms

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1) $[x \;\; x] = 0 \;\; \forall x \in V$,

2) $[\lambda.x \;\; y] = \lambda[x \;\; y] = [x \;\; \lambda.y] \;\; \forall x \in V, \forall \lambda \in F$,

3) $[[x \;\; y] \;\; z] + [[y \;\; z] \;\; x] + [[z \;\; x] \;\; y] = 0 \;\; \forall x, y, z \in V$.

We note that the multiplication in a Lie algebra is not associative. But it is anti commutative.

Definition 2.2. [6] Let $\mathscr{G}$ be a Lie algebra over a field F and B be a vector space over F . Then B is called ***left ideal*** if $[\mathscr{G} \;\; B] \subseteq B$.

Clearly, every left ideal (right ideal) of $\mathscr{G}$ is ideal.

Definition 2.3. [6] Let $\mathscr{G}$ be a Lie algebra over a field F . The set

$$Z(\mathscr{G}) = \{x \in \mathscr{G} : [x \;\; \mathscr{G}] = \{0\}\}$$

is called ***center*** of $\mathscr{G}$. We note that $Z(\mathscr{G})$ is ideal of $\mathscr{G}$.

Definition 2.4. [7] Let $\mathscr{G}$ be a Lie algebra over a field F . Then $\mathscr{G}$ is called ***semiprime*** if $I^2 \neq \{0\}$ for all non-zero ideal I of $\mathscr{G}$, where:

$$I^2 = \left\{ x \in \mathscr{G} \; : \; x = \sum_{i=1}^{n} [y_i \;\; z_i] \; ; \; y_i, z_i \in I \quad \forall i = 1, \ldots, n \right\}.$$

Definition 2.5. [8] Let $\mathscr{G}$ be a Lie algebra over a field F . Then $\mathscr{G}$ is called ***nilpotent*** if there exists a positive integer n such that $\mathscr{G}^n = \{0\}$, where

$$\mathscr{G}^1 = \mathscr{G} \;\&\; ... \;\&\; \mathscr{G}^n = [\mathscr{G}^{n-1} \;\; \mathscr{G}].$$

We note that:

$$\mathscr{G} = \mathscr{G}^1 \supseteq \mathscr{G}^2 \supseteq \cdots \supseteq \mathscr{G}^n \supseteq \cdots$$

Definition 2.6. [1] Let $X \neq \varnothing$ be a set. A ***fuzzy set*** μ of X is a function from X into $[0,1]$, where $([0,1], \leq, \wedge, \vee)$ is distributional complete lattice that has the minimum element 0 and the maximum element 1.

Definition 2.7. [9] Let μ, λ be two fuzzy sets of a $X \neq \varnothing$. Then,

1) $\mu = \lambda \;\Leftrightarrow\; \mu(x) = \lambda(x) \;\; \forall x \in X$.

2) $\mu \subseteq \lambda \;\Leftrightarrow\; \mu(x) \leq \lambda(x) \;\; \forall x \in X$.

Definition 2.8. [9] Let G be a group. The fuzzy set μ of G is called ***fuzzy group*** of G if it satisfies the following axiom

$$\mu(x-y) \geq \mu(x) \wedge \mu(y) \;\; \forall x, y \in G.$$

Definition 2.9. [9] Let G be a group. The mapping $E : G \rightarrow [0,1]$ which defined by

$$E(x) = \begin{cases} 1 & if \;\; x = 0, \\ 0 & if \;\; x \neq 0. \end{cases}$$

is called ***the null fuzzy group*** of G.

Definition 2.10. [9] Let $\{\mu_i\}_{i \in I}$ be a family of fuzzy sets of a $X \neq \emptyset$. Then,

1) $(\bigcap_{i \in I} \mu_i)(x) = \wedge_{i \in I} \mu_i(x) \;\; \forall x \in X$.

2) $(\bigcup_{i \in I} \mu_i)(x) = \vee_{i \in I} \mu_i(x) \;\; \forall x \in X$.

If X is a commutative ring with identity, then

3) $(\sum_{i \in I} \mu_i)(x) = \vee_{x = \sum_{i \in I} x_i} \left(\wedge_{i \in I} \mu_i(x_i) \right) \;\; \forall x \in X \;\; ; \; x_i \in X$.

If μ, φ are two fuzzy sets of the commutative ring X with identity, then

4) $(\mu.\varphi)(x) = \vee_{x = \sum_{i=1}^{n} a_i . b_i} \left(\wedge_{i=1}^{n} \left(\mu(a_i) \wedge \varphi(b_i) \right) \right) \;\; \forall x \in X \;\; ; \; a_i, b_i \in X$.

If μ is a fuzzy group of the commutative ring X with identity, then

5) $\mu(x) = \mu(-x) \;\; \forall x \in X$.

Definition 2.11. [10] Let $\mathcal{G}$ be a Lie algebra over a field F. The fuzzy set μ of $\mathcal{G}$ is called ***fuzzy Lie algebra*** of $\mathcal{G}$ if the following axioms are satisfied

1) $\mu(x-y) \geq \mu(x) \wedge \mu(y) \;\; \forall x, y \in \mathcal{G}$,

2) $\mu(\alpha.x) \geq \mu(x) \;\; \forall x \in \mathcal{G}, \; \forall \alpha \in F$,

3) $\mu([x \;\; y]) \geq \mu(x) \wedge \mu(y) \;\; \forall x, y \in \mathcal{G}$.

We let $F(\mathcal{G})$ denote the set of all fuzzy Lie algebras of $\mathcal{G}$.

Definition 2.12. [10] Let $\mathcal{G}$ be a Lie algebra over a field F. The fuzzy set μ of $\mathcal{G}$ is called a ***fuzzy Lie ideal*** of $\mathcal{G}$ if the following axioms are satisfied

1) $\mu(x-y) \geq \mu(x) \wedge \mu(y) \ \forall x, y \in \mathcal{G}$,
2) $\mu(\alpha.x) \geq \mu(x) \ \forall x \in \mathcal{G}, \forall \alpha \in F$,
3) $\mu([x \ \ y]) \geq \mu(x) \ \forall x, y \in \mathcal{G}$.

We will define semiprime fuzzy Lie algebra.

Definition 2.13. Let $\mathcal{G}$ be a Lie algebra over a field F. Then $\mu \in F(\mathcal{G})$ is called a ***semiprime*** if for any fuzzy Lie ideal $\lambda \subseteq \mu$ of $\mathcal{G}$, then

$$\lambda^2 = E \ \Rightarrow \ \lambda = E.$$

Definition 2.14. [11] Let $\mathcal{G}$ be a Lie algebra over a field F. Then $\mu \in F(\mathcal{G})$ is called a ***nilpotent*** if there exists a positive integer n such that $\mu^n = E$, where

$$\mu^1 = \mu \ \& \dots \& \ \mu^n = \mu^{n-1}.\mu.$$

We note that:

$$\mu = \mu^1 \supseteq \mu^2 \supseteq \dots \supseteq \mu^n \supseteq \dots,$$

where $\mu^n.\mu$ is defined as in (definition 1.10).

3 The Results

Theorem 3.1. Let $\mathcal{G}$ be a semiprime Lie algebra over a field F, and let $E \neq \mu \in F(\mathcal{G})$. Then μ is semiprime.

Proof: Let $\lambda \subseteq \mu$ be a fuzzy Lie ideal of $\mathcal{G}$ such that $\lambda^2 = E$. Then $\lambda = E$, because if we suppose that $\lambda \neq E$, then

$$\exists \ x \in \mathcal{G} \ ; \ \lambda(x) \neq E(x).$$

- If $x = 0$, then $\lambda(x) < 1$. Since

$$1 = \left(\lambda^2\right)(0) = \bigvee_{0=\sum_{i=1}^{n}[a_i \ \ b_i]} \left(\bigwedge_{i=1}^{n} \left(\lambda(a_i) \wedge \lambda(b_i) \right) \right),$$

so there one of forms of 0, for example $0 = \sum_{i=1}^{n}[a_i' \ \ b_i']$ such as

$$1 = \overset{n}{\underset{i=1}{\wedge}} \left(\lambda(a_i') \wedge \lambda(b_i') \right),$$

this implies that

$$1 = \lambda(a_i') = \lambda(b_i') \;\; ; \;\; i = 1,\ldots,n,$$

therefore

$$\lambda(0) = \lambda\left(\sum_{i=1}^{n} [a_i' \;\; b_i'] \right) \geq \overset{n}{\underset{i=1}{\wedge}} \lambda\left([a_i' \;\; b_i']\right) \geq \overset{n}{\underset{i=1}{\wedge}} \lambda\left(a_i'\right) = 1.$$

This is a contradiction to be $\lambda(0) < 1$.

- If $x \neq 0$, then $\lambda(x) > 0$, so $[x \;\; \mathscr{G}]$ is ideal of $\mathscr{G}$, because

$$[[x \;\; \mathscr{G}] \;\; \mathscr{G}] = -[[\mathscr{G} \;\; \mathscr{G}] \;\; x] - [[\mathscr{G} \;\; x] \;\; \mathscr{G}] \Rightarrow$$

$$[[x \;\; \mathscr{G}] \;\; \mathscr{G}] + [[\mathscr{G} \;\; x] \;\; \mathscr{G}] = -[[\mathscr{G} \;\; \mathscr{G}] \;\; x] \Rightarrow$$

$$[[x \;\; \mathscr{G}] \;\; \mathscr{G}] = -[[\mathscr{G} \;\; \mathscr{G}] \;\; x] \subseteq [\mathscr{G} \;\; x].$$

Also, $[x \;\; \mathscr{G}] \neq \{0\}$ because, if

$$[x \;\; \mathscr{G}] = \{0\} \;\Rightarrow\; x \in Z(\mathscr{G}) \;\Rightarrow\; Z(\mathscr{G}) \neq \{0\},$$

this implies that $Z(\mathscr{G})$ is non-zero ideal of $\mathscr{G}$ such that $[Z(\mathscr{G}) \;\; Z(\mathscr{G})] = \{0\}$, which is a contradiction to the hypothesis that $\mathscr{G}$ is semiprime. Since $[x \;\; \mathscr{G}]$ is non-zero ideal of $\mathscr{G}$ and $\mathscr{G}$ is semiprime, it implies that

$$[[x \;\; \mathscr{G}] \;\; [x \;\; \mathscr{G}]] \neq \{0\}.$$

Therefore:

$$\exists\, t \in [[x \;\; \mathscr{G}] \;\; [x \;\; \mathscr{G}]] \;\; ; \;\; 0 \neq t = \sum_{i=1}^{n} [x_i \;\; y_i] \;\; ; \;\; x_i, y_i \in [x \;\; \mathscr{G}] \quad \forall i \in \{1,\ldots,n\},$$

where:

$$x_i, y_i \in [x \;\; \mathscr{G}] \quad \forall i \in \{1,\ldots,n\}.$$

Thus:

$$x_i = [x \;\; z_i] \;\&\; y_i = [x \;\; z_i'] \;\; ; \;\; z_i, z_i' \in \mathscr{G} \quad \forall i \in \{1,\ldots,n\},$$

this implies that

$$\lambda(x_i) = \lambda([x \;\; z_i]) \geq \lambda(x) \;\;\&\;\; \lambda(y_i) = \lambda([y \;\; z_i']) \geq \lambda(y) \quad \forall i \in \{1,\ldots,n\},$$

Therefore

$$\left(\lambda^2\right)(t) = \bigvee_{t=\sum_{i=1}^{n}[x_i \ y_i]} \left(\bigwedge_{i=1}^{n} \left(\lambda(x_i) \wedge \lambda(y_i)\right)\right) \geq \bigwedge_{i=1}^{n} \left(\lambda(x_i) \wedge \lambda(y_i)\right)$$

$$\geq \lambda(x) \wedge \lambda(y) > 0.$$

Since $t \in \mathscr{G} - \{0\}$, then $\left(\lambda^2\right)(t) = E(t) = 0$, this is a contradiction.

Lemma 3.2 Let $\mathscr{G}$ be a Lie algebra over a field F, $\mu \in F(\mathscr{G})$ and $\lambda(0) = 1$ for all $\lambda \in F\left(\mathscr{G}\right)$, such as $\lambda \subseteq \mu$. If μ have a series

$$\mu = \mu_1 \supseteq \mu_2 \supseteq \cdots \supseteq \mu_n = E \quad ; \quad \mu_i \in F(\mathscr{G}) \quad \forall 1 \leq i \leq n,$$

such that:

$$\mu_i . \mu_1 \subseteq \mu_{i+1} \quad \forall 1 \leq i \leq n-1,$$

then μ is nilpotent.

Proof: First, we show by induction that:

$$\mu^i \subseteq \mu_i \quad \forall 1 \leq i \leq n$$

For $n = 1$:

$$\mu^1 = \mu = \mu_1 \Rightarrow \mu^1 \subseteq \mu_1,$$

therefore, the result holds for $n = 1$.

Now, we assume that $\mu^k \subseteq \mu_k$ for some $k = 1, ..., n-1$, then

$$\mu^{k+1} = \mu^k . \mu \subseteq \mu_k . \mu \subseteq \mu_{k+1}.$$

Now, since $\mu^i \subseteq \mu_i \quad \forall 1 \leq i \leq n$, then $\mu^n \subseteq \mu_n$, this implies that $E \subseteq \mu^n \subseteq \mu_n = E$, thus $\mu^n = E$. This shows that μ is nilpotent.

Theorem 3.3. Let $\mathscr{G}$ be a nilpotent Lie algebra over a field F, $E \neq \mu \in F(\mathscr{G})$ and $\lambda(0) = 1$ for all $\lambda \in F\left(\mathscr{G}\right)$, such as $\lambda \subseteq \mu$. Then μ is nilpotent.

Proof: Since $\mathscr{G}$ is nilpotent, it follows that there is an integer n that satisfies $\mathscr{G}^n = \{0\}$. For every $1 \leq i \leq n$, we define a fuzzy set λ_i by

$$\lambda_i(x) = \begin{cases} \mu(x) & if \quad x \in \mathscr{G}^i, \\ 0 & \text{otherwise.} \end{cases}$$

Thus $\lambda_i \in F(\mathscr{G})$ for $1 \leq i \leq n$.

This is because

$$1)\ \lambda_i(x) \wedge \lambda_i(y) = \begin{cases} \mu(x) & if \quad x \in \mathcal{G}^i, \\ 0 & \text{otherwise.} \end{cases} \wedge \begin{cases} \mu(y) & if \quad x \in \mathcal{G}^i, \\ 0 & \text{otherwise.} \end{cases}$$

$$= \begin{cases} \mu(x) \wedge \mu(y) & if \quad x, y \in \mathcal{G}^i, \\ 0 & \text{otherwise.} \end{cases}$$

$$= \begin{cases} \mu(x) \wedge \mu(y) & if \quad x + y \in \mathcal{G}^i, \\ 0 & \text{otherwise.} \end{cases}$$

$$\leq \begin{cases} \mu(x + y) & if \quad x + y \in \mathcal{G}^i, \\ 0 & \text{otherwise.} \end{cases}$$

$$= \lambda_i(x + y) \quad \forall x, y \in \mathcal{G}.$$

$$2)\ \lambda_i(x) = \begin{cases} \mu(x) & if \quad x \in \mathcal{G}^i, \\ 0 & \text{otherwise} \end{cases}$$

$$\leq \begin{cases} \mu(\alpha.x) & if \quad \alpha.x \in \mathcal{G}^i, \\ 0 & \text{otherwise} \end{cases}$$

$$= \lambda_i(\alpha.x) \quad \forall x \in \mathcal{G},\ \forall \alpha \in F.$$

$$3)\ \lambda_i(x) \wedge \lambda_i(y) = \begin{cases} \mu(x) & if \quad x \in \mathcal{G}^i, \\ 0 & \text{otherwise.} \end{cases} \wedge \begin{cases} \mu(y) & if \quad x \in \mathcal{G}^i, \\ 0 & \text{otherwise.} \end{cases}$$

$$= \begin{cases} \mu(x) \wedge \mu(y) & if \quad x, y \in \mathcal{G}^i, \\ 0 & \text{otherwise.} \end{cases}$$

$$\leq \begin{cases} \mu([x \quad y]) & if \quad [x \quad y] \in [\mathcal{G}^i \quad \mathcal{G}^i] = [\mathcal{G}^i \quad \mathcal{G}] \subseteq \mathcal{G}^{i+1} \subseteq \mathcal{G}^i, \\ 0 & \text{otherwise.} \end{cases}$$

$$= \lambda_i([x \quad y]) \quad \forall x, y \in \mathcal{G}.$$

Now, we will prove that $\lambda_n = E$. Thus

$$\lambda_n(x) = \begin{cases} \mu(x) & if \quad x \in \mathcal{G}^n = \{0\}, \\ 0 & \text{otherwise.} \end{cases}$$

$$= \begin{cases} \mu(0) & if \quad x = 0, \\ 0 & \text{otherwise.} \end{cases}$$

$$= \begin{cases} 1 & if \quad x = 0, \\ 0 & \text{otherwise.} \end{cases}$$

$$= E(x) \quad \forall x \in \mathcal{G}.$$

Thus $\lambda_n = E$.

Let $a,b \in \mathscr{G}$ and $1 \le i \le n-1$. Then:

- If $a \notin \mathscr{G}^i$, then $\lambda_i(a) \wedge \lambda_1(b) = 0 \le \lambda_{i+1}([a \ b])$.
- If $a \in \mathscr{G}^i$, then

$$\begin{aligned}\lambda_i(a) \wedge \lambda_1(b) &= \mu(a) \wedge \mu(b) \\ &\le \mu([a \ b]) \\ &= \lambda_{i+1}([a \ b]) \quad ; \quad [a \ b] \in [\mathscr{G}^i \ \ \mathscr{G}] = \mathscr{G}^{i+1},\end{aligned}$$

thus:

$$\lambda_i(a) \wedge \lambda_1(b) \le \lambda_{i+1}([a \ b]) \quad \forall 1 \le i \le n-1 \quad ...(*).$$

Hence $\lambda_i . \lambda_1 \subseteq \lambda_{i+1}$ for $1 \le i \le n-1$. This is because:

$$\left(\lambda_i . \lambda_1\right)(x) = \bigvee_{x = \sum_{j=1}^{m} [a'_j \ \ b'_j]} \left(\bigwedge_{j=1}^{m} \left(\lambda_i(a'_j) \wedge \lambda_1(b'_j) \right) \right),$$

so there one of forms of x , for example $x = \sum_{j=1}^{m} [a_j \ \ b_j]$ such as

$$\begin{aligned}\left(\lambda_i . \lambda_1\right)(x) &= \bigwedge_{j=1}^{m} \left(\lambda_i(a_j) \wedge \lambda_1(b_j) \right) \\ &\overset{(*)}{\le} \bigwedge_{j=1}^{m} \left(\lambda_{i+1}\left([a_j \ \ b_j] \right) \right) \\ &\le \lambda_{i+1}\left(\sum_{j=1}^{n} [a_j \ \ b_j] \right) \\ &= \lambda_{i+1}(x) \ \ \forall x \in \mathscr{G}.\end{aligned}$$

Hence there is a series

$$\mu = \lambda_1 \supseteq \lambda_2 \supseteq \cdots \supseteq \lambda_n = E \quad ; \quad \lambda_i \in F(\mathscr{G}) \ \ \forall 1 \le i \le n,$$

such that:

$$\lambda_i . \lambda_1 \subseteq \lambda_{i+1} \quad \forall 1 \le i \le n-1,$$

thus μ is nilpotent (Lemma 3.2).

4 Conclusions

In this paper, we have discussed the concepts of semiprime fuzzy Lie algebra over a field and nilpotent fuzzy Lie algebra over a field. Also, we expected that several results about Lie algebras can be extended to the concept of fuzzy Lie algebras over field.

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Q-SINGLE VALUED NEUTROSOPHIC SOFT SETS

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Abstract - In this paper, we have introduced the concept of Q-single value neutrosophic soft set, multi Q-single valued neutrosophic set and defined some basic results and related properties. We have also defined the idea of Q-single valued neutrosophic soft set, which is the genralizations of Q-fuzzy set, Q-intuitionistic fuzzy set, multi Q-fuzzy set , Multi Q-intuitionistic fuzzy set, Q-fuzzy soft set, Q-intuitionistic fuzzy soft set. We have also defined and discussed some properties and operations of Q-single valued neutrosophic soft set.

Keywords - *Neutrosophic set,single valued neutrosophic set, Q-single valued neutrosophic set, Multi Q-Single valued neutrosophic set, Q-single valued neutrosophic soft set.*

1 Introduction

In 1965 L. A. Zadeh was the first person who presented theory of fuzzy set [23], whose fundamental component is just a degree of membership. After the introduction of fuzzy sets, the idea of intuitionistic fuzzy sets (IFS) was given by K. Attanassov in 1986 [6], whose basic components are the grade of membership and the grade of non-membership under the restrictions that the sum of the two degrees does not surpass one. Attanassov's IFS is more suitable mathematical tool to handle real life application. But in some cases Attanasove's IFS is difficult to apply because in IFS we cannot define degree of indeterminacy independently. To surmount this difficulty Samarandache introduced the concept of neutrosophic sets [21], which not only genrelized the Zadeh's fuzzy set and Attanassov's IFS but also generalized Gau's vague sets [17] philosophically. In neutrosophic set degree of indeterminacy is defined independently. Neutrosophic set contains degree of truth-membership function $T_A(a): X \to]0^-, 1^+[$, indeterminacy membership function $I_A(a): X \to]0^-, 1^+[$ *and* falsity membership function $F_A(a): X \to]0^-, 1^+[$, with for

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each $a \in X$ and satisfy the condition $0^{-} \leq SupT_A(a) + SupI_A(a) + SupF_A(a) \leq 3^{+}$. In neutrosophic sets degree of indeterminacy is defined independently but it is hard to sue it in real life and engineering problem, because it contains non-standard intervals. It is important to overcome this practical difficulty. Wang et al [22] present the concept of Single Valued Neutrosophic Sets (SVNS) to overcome this difficulty. SVNS is a primary class of neutrosophic sets. SVNS is easy to apply in real life and engineering problems because it contains a single points in the standard unit interval $]0^{-},1^{+}[$ instead of non-standard intervals of $]0^{-},1^{+}[$.

Firstly Molodstov [20] introduced the concept of soft set theory. Maji et al. [18] gave some operations and primary properties on theory of soft set. Ali et al. [7] pointed out that the operations defined for soft sets are not correct and due to these operations many mathematical results leads to wrong answers. Fuzzy soft sets theory and fuzzy parametarized soft set theories was studied by Cagman et al [16] . Fuzzy soft set theory was studied by Ahmad, B., and Athar Kharal [5] . F.Adam and N. Hassan [4] introduced the concept of multi Q-fuzzy sets, multi Q-parameterized soft sets and defined some basic properties and operation such as complement, equality, union, intersection F. Adam and N. Hassan [2, 3] also introduced Q-fuzzy soft set,defined some basic operations and defined Q-fuzzy soft aggregation operators that allows constructing more efficient decision making methods. S. Broumi [9, 10] established the notion of Q- intuitionistic fuzzy set (Q-IFS), Q-intuitionistic fuzzy soft set (Q-IFSS) and defined some basic properties with illustrative examples, and also defined some basic operation for Q-IFSand Q-IFSS such as union, intersection, AND and OR operations. Broumi. S. et. al. [8] presented the concept of intuitionistic neutrosophic soft rings by applying intuitionistic neutrosophic soft set to ring theory.

Broumi S. et. al. [8, 11, 12, 13, 14, 15] presented concepts of single valued neutrosophic graphs, interval neutrosophic graphs, on bipolar single valued neutrosophic graphs, and also presented an introduction to bipolar single valued neutrosophic graph.

This article is arranged as proceed, Section 2 contains basic definitions of soft sets, Q-fuzzy sets, multi Q-fuzzy sets, Q-fuzzy soft sets, neutrosophic sets and SVNS are defined. In section 3 Q-SVNS and some basic operations are defined. In section 4 multi Q-SVNS and some basic operations such as union, intersection etc are defined. In section 5 we introduce the concept of Q-single valued neutrosophic soft set(Q-SVNSS) and defined some basic opertions and related results are discussed. At the end conclusion and references are given.

2 Preliminaries

2.1. Definition. **[20]** Let X be a universal set , E be a set of parameters and $A \subseteq E$. A pair (F, A) is said to be soft set over the universal set X, if and only if F is a mapping from A to the power set of X.

2. 2. Definition. [2,3]. Assume X be a universal set and $Q \neq \emptyset$. A Q −fuzzy subset N of X is a function $X \times X \to [0,1]$." The union of two Q −fuzzy subsets N and M is defined as

$$N \cup M = \{\max(\mu_N(\hat{\theta}, \hat{u}), \mu_M(\hat{\theta}, \hat{u})) : \hat{\theta} \in X, \hat{u} \in Q\}$$

The intersection of two Q −fuzzy subsets N and M is defined as

$$N \cap M = \{\min(\mu_N(\hat{\theta}, \hat{u}), \mu_M(\hat{\theta}, \hat{u})) : \hat{\theta} \in X, \hat{u} \in Q\}$$

2. 3. Definition[3]. Let I be unit interval $[0,1]$, $k \in Z^+$ (positive integer), X be universal set and $Q \neq \emptyset$. A multi Q −fuzzy set N_Q in X and Q is a set of ordered sequences,

$$N_Q = \{\max(\mu_j(\hat{\theta}, \hat{u})) : \hat{\theta} \in X, \hat{u} \in Q\}$$

Where $\mu_j : X \times Q \to I^K$ The function $\mu_j(\theta, \hat{u})$ is termed as membership function of multi Q-fuzzy set N_Q, and $\sum_{j=0}^{k} \mu_j(\hat{\theta}, \hat{u}) \leq 1, for\ j = 1,2,3, \dots, k. k$ is the dimension of multi Q fuzzy set N_Q.The set of all multi- Q − fuzzy set of dimension k in X and Q is denoted by $M^k FQ(X)$."

2. 4. Definition[4]. Let X be a universal set, E be the set of parameters, $Q \neq \emptyset$. Let $M^k FQ(X)$ is the power set of all multi Q −fuzzy subsets of X with dimension $k = 1$. Let $D \subseteq E$. A pair $(F_{Q,} D)$ is referred as Q −fuzzy soft set (in short QF −soft set)over X where $F_{Q,}$ is defined by

$$F_Q : D \to M^k FQ(X) \ \ such\ that\ (F_Q(\hat{\theta})) = \emptyset\ \ if\ \hat{\theta} \notin D$$

Here a Q −fuzzy soft set can be represented by the set of ordered pairs

$$(F_{Q,} D) = \{\hat{\theta}, F_Q(\hat{\theta}) : \hat{\theta} \in X, F_Q(\hat{\theta}) \in M^k FQ(X)\}$$

The set of all Q −fuzzy soft sets over X will be denoted by $QFS(X)$

2. 5. Definition. [22] Let X be a space of points (objects), with a generic element in X denoted by $\hat{\theta}$. A SVNS N in X has the features truth-membership function T_N, indeterminacy-membership function I_N, and falsity-membership function F_N. For each point $\tilde{\theta}$ in X,$T_N(\tilde{\theta}), I_N(\tilde{\theta}), F_N(\tilde{\theta}) \in [0,1]$.

Mathematically single valued neutrosophic is expressed as follows:

$$N = \{\langle \tilde{\theta}, \left(T_N(\tilde{\theta}), I_N(\tilde{\theta}), F_N(\tilde{\theta})\right) \rangle \,|\, \tilde{\theta} \in X\}$$

3 Q −Single Valued Neutrosophic Sets

3.1. Definition. Let X be a universal set and $Q \neq \emptyset$. A Q −SVNS $\tilde{N}_Q$ in X and Q is an object of the form

$$\tilde{N}_Q = \{(\tilde{\theta}, \hat{u}), \mu_{\tilde{N}}(\tilde{\theta}, \hat{u}), \boldsymbol{\nu}_{\tilde{N}}(\tilde{\theta}, \hat{u}), \boldsymbol{\lambda}_{\tilde{N}}(\tilde{\theta}, \hat{u}) : \tilde{\theta} \in X, \hat{u} \in Q\}$$

Where $\mu_{\tilde{N}_Q} : X \times Q \to [0,1]$, $\boldsymbol{\nu}_{\tilde{N}_Q} : X \times Q \to [0,1]$, $\boldsymbol{\lambda}_{\tilde{N}_Q} : X \times Q \to [0,1]$, are respectively truth-membership, indeterminacy-membership and falsity membership functions for every $\tilde{\theta} \in X \ and\ \hat{u} \in Q$ and satisfy the condition $0 \leq \mu_{\tilde{N}_Q}(\tilde{\theta}, \hat{u}) + \boldsymbol{\nu}_{\tilde{N}_Q}(\tilde{\theta}, \hat{u}) + \boldsymbol{\lambda}_{\tilde{N}_Q}(\tilde{\theta}, \hat{u}) \leq 3$.

3.2. Example. Let $X = \{\boldsymbol{p}_1, \boldsymbol{p}_1, \boldsymbol{p}_3\}$ and $Q = \{\hat{u}, \hat{v}\}$, then Q −SVNS $\tilde{N}_Q$ is defined below,

$$\begin{aligned}\tilde{N}_Q = \{< &(\boldsymbol{p}_1, \hat{u}), (0.4,0.3,0.5), (\boldsymbol{p}_1, \hat{v}), (0.2,0.4,0.6), (\boldsymbol{p}_2, \hat{u}), (0.6,0.1,0.3), \\ &(\boldsymbol{p}_2, \hat{v}), (0.7,0.2,0.1), (\boldsymbol{p}_3, \hat{u}), (0.3,0.6,0.4), (\boldsymbol{p}_3, \hat{v}), (0.5,0.4,0.6) >\}\end{aligned}$$

Now we define some basic operations for $\boldsymbol{Q}$ −SVNS.

3.3. Definition. Let X be a universal set, $Q \neq \emptyset$and $\tilde{N}_Q$ be a Q −SVNS. The complement of $\tilde{N}_Q$ is denoted and defined as follows

$$\tilde{N}_Q^C = \{(\tilde{\theta}, \hat{u}), \boldsymbol{\lambda}_{\tilde{N}_Q}(\tilde{\theta}, \hat{u}), 1 - \boldsymbol{\nu}_{\tilde{N}_Q}(\tilde{\theta}, \hat{u}), \mu_{\tilde{N}_Q}(\tilde{\theta}, \hat{u}) : \tilde{\theta} \in X, \hat{u} \in Q\}$$

3.4. Definition. Let $\tilde{A}_Q$ and $\tilde{N}_Q$be two Q −SVNS. Then the union and intersection is denoted and defined by

$$\begin{aligned}\tilde{A}_Q \cup \tilde{N}_Q = \{&(\tilde{\theta}, \hat{u}), \max(\mu_{\tilde{A}_Q}(\tilde{\theta}, \hat{u}), \mu_{\tilde{N}_Q}(\tilde{\theta}, \hat{u})), \min(\boldsymbol{\nu}_{\tilde{A}_Q}(\tilde{\theta}, \hat{u}), \boldsymbol{\nu}_{\tilde{N}_Q}(\tilde{\theta}, \hat{u})), \\ &\min(\boldsymbol{\lambda}_{\tilde{A}_Q}(\tilde{\theta}, \hat{u}), \boldsymbol{\lambda}_{\tilde{N}_Q}(\tilde{\theta}, \hat{u})) : \tilde{\theta} \in X, \hat{u} \in Q\}\end{aligned}$$

$$\tilde{A}_Q \cap \tilde{N}_Q = \{(\tilde{\theta}, \hat{u}), \min(\mu_{\tilde{A}_Q}(\tilde{\theta}, \hat{u}), \mu_{\tilde{N}_Q}(\tilde{\theta}, \hat{u})), \max(\boldsymbol{\nu}_{\tilde{A}_Q}(\tilde{\theta}, \hat{u}), \boldsymbol{\nu}_{\tilde{N}_Q}(\tilde{\theta}, \hat{u})),$$

$$\max(\boldsymbol{\lambda}_{\tilde{A}_Q}(\tilde{\theta},\hat{u}),\boldsymbol{\lambda}_{\tilde{N}_Q}(\tilde{\theta},\hat{u}))\}$$

3.5. Definition. Let $\tilde{A}_Q$ and $\tilde{N}_Q$ be two Q −SVNSs over two non-empty universal sets G and H respectively and Q be any non-empty set. Then the product of $\tilde{A}_Q$ and $\tilde{N}_Q$ is denoted by $\tilde{A}_Q \times \tilde{N}_Q$ and defined as

$$\tilde{A}_Q \times \tilde{N}_Q = \{< ((\tilde{\theta},b),\hat{u}), \mu_{\tilde{A}_Q\times \tilde{N}_Q}((\tilde{\theta},b),\hat{u}), \nu_{\tilde{A}_Q\times \tilde{N}_Q}((\tilde{\theta},b),\hat{u}), \lambda_{\tilde{A}_Q\times \tilde{N}_Q}((\tilde{\theta},b),\hat{u}) >: \tilde{\theta} \in G, b \in H, \hat{u} \in Q\}$$

Where

$$\mu_{\tilde{A}_Q\times \tilde{N}_Q}((\tilde{\theta},b),\hat{u}) = \min\{\mu_{\tilde{A}_Q}(\tilde{\theta},\hat{u}), \mu_{\tilde{N}_Q}(b,\hat{u})\}$$

$$\boldsymbol{\nu}_{\tilde{A}_Q\times \tilde{N}_Q}((\tilde{\theta},b),\hat{u}) = \max\{\boldsymbol{\nu}_{\tilde{A}_Q}(\tilde{\theta},\hat{u}), \boldsymbol{\nu}_{\tilde{N}_Q}(b,\hat{u})\}$$

$$\boldsymbol{\lambda}_{\tilde{A}_Q\times \tilde{N}_Q}\left((\tilde{\theta},b),\hat{u}\right) = \max\left\{\boldsymbol{\lambda}_{\tilde{A}_Q}(\tilde{\theta},\hat{u}), \boldsymbol{\lambda}_{\tilde{N}_Q}(b,\hat{u})\right\}$$

$For\ all\ \tilde{\theta}, b\ in\ G\ and\ \hat{u} \in Q.$

3.6. Definition. Let $\tilde{A}_Q$ a Q −single valued neutrosophic subset in a set G, the strongest Q −single valued neutrosophic relation on G, that is a Q −single valued neutrosophic relation on $\tilde{A}_Q$ is H given by

$$\mu_H((\tilde{\theta},b),\hat{u}) = \min\{\mu_{\tilde{A}_Q}(\tilde{\theta},\hat{u}), \mu_{\tilde{N}_Q}(b,\hat{u})\}$$

$$\boldsymbol{\nu}_H((\tilde{\theta},b),\hat{u}) = \max\{\boldsymbol{\nu}_{\tilde{A}_Q}(\tilde{\theta},\hat{u}), \boldsymbol{\nu}_{\tilde{N}_Q}(b,\hat{u})\}$$

$$\boldsymbol{\lambda}_H\left((\tilde{\theta},b),\hat{u}\right) = \max\left\{\boldsymbol{\lambda}_{\tilde{A}_Q}(\tilde{\theta},\hat{u}), \boldsymbol{\lambda}_{\tilde{N}_Q}(b,\hat{u})\right\}$$

$For\ all\ \tilde{\theta}, b\ in\ G\ and\ \hat{u} \in Q.$

4. Multi Q −Single Valued Neutrosophic Sets

4.1. Definition. Let X be a non-empty set and Q be any non-empty set , l be any positive integer and I be a unit interval $[0,1]$ .A multi Q −SVNS $\tilde{A}_Q$ in $\boldsymbol{X}$ and $\boldsymbol{Q}$ is a set of ordered sequences

$$\tilde{A}_Q = \{(\tilde{\theta},\hat{u}), \mu_j(\tilde{\theta},\hat{u}), \boldsymbol{\nu}_j(\tilde{\theta},\hat{u}), \boldsymbol{\lambda}_j(\tilde{\theta},\hat{u}) : \tilde{\theta} \in X, \hat{u} \in Q\ for\ all\ j = 1,2,..,l\}$$

Where $\mu_j: X \times Q \to I^K$, $\boldsymbol{\nu}_j: X \times Q \to I^K$,$\boldsymbol{\lambda}_j: X \times Q \to I^K$,$for\ all\ j = 1,2, \dots, l$

and are respectively truth-membership, indeterminacy-membership and falsity membership functions for each $\tilde{\theta} \in X$ *and* $\hat{u} \in Q$ and satisfy the condition

$$0 \leq \mu_j(\tilde{\theta},\hat{u}) + \boldsymbol{\nu}_j(\tilde{\theta},\hat{u}) + \boldsymbol{\lambda}_j(\tilde{\theta},\hat{u}) \leq 3, for\ all\ j = 1,2,\ldots,l$$

The functions $\mu_j(\tilde{\theta},\hat{u}), \boldsymbol{\nu}_j(\tilde{\theta},\hat{u}), \boldsymbol{\lambda}_j(\tilde{\theta},\hat{u})$ *for all* $j = 1,2,\ldots,l$ are called the "truth-membership , indeterminacy-membership and falsity-membership" functions respectively of the multi Q −SVNS $\tilde{A}_Q$ and satisfy the condition

$$0 \leq \mu_j(\tilde{\theta},\hat{u}) + \boldsymbol{\nu}_j(\tilde{\theta},\hat{u}) + \boldsymbol{\lambda}_j(\tilde{\theta},\hat{u}) \leq 3, for\ all\ j = 1,2,\ldots,l$$

l is called the dimension of the Q −SVNS $\tilde{A}_Q$.The set of all Q −SVNS is denoted by $Z^k QSVN(X)$.

4.2. Example. Let $X = \{\boldsymbol{p}_1, \boldsymbol{p}_2, \boldsymbol{p}_3\}$ be a universal set and $Q = \{\hat{u}, v\}$ be a non-empty set and $l = 2$ be a positive integer. If $\tilde{A}_Q: X \times Q \to I^2$,Then the set

$$\tilde{A}_Q = \{< ((\boldsymbol{p}_1,\hat{u}),(0.2,0.3,0.6),(0.6,0.2,0.3)), ((\boldsymbol{p}_1,\hat{v}),(0.5,0.1,0.3),(0.4,0.4,0.5)), ((\boldsymbol{p}_2,\hat{u}),(0.4,0.3,0.5),(0.6,0.1,0.3)),((\boldsymbol{p}_2,\hat{v}),(0.7,0.2,0.1),(0.2,0.4,0.8)) >\}$$

is a multi Q −SVNS in X and Q.

4.3. Remark. Note that if $\boldsymbol{\nu}_j(\tilde{\theta},\hat{u}) = 0$ *and* $\boldsymbol{\lambda}_j(\tilde{\theta},\hat{u}) = 0$ then multi Q −SVNS reduces to multi Q −fuzzy set.

4.4. Definition. Let $\tilde{A}_Q$ be a Q −SVNS. The the complement of $\tilde{A}_Q$ is denoted and defined as follows

$$\tilde{A}_Q^c = \{(\tilde{\theta},\hat{u}), \boldsymbol{\lambda}_j(\tilde{\theta},\hat{u}), 1 - \boldsymbol{\nu}_j(\tilde{\theta},\hat{u}), \mu_j(\tilde{\theta},\hat{u}) : \tilde{\theta} \in X\ and\ \hat{u} \in Q, for\ all\ j = 1,2,\ldots,l\}$$

4.5. Definition. Let $\tilde{A}_Q$ and A_Q ***and*** B_Q be two Q −SVNSs, and l be a positive integer such that

A=$\{(\tilde{\theta},\hat{u}), \mu_j(\tilde{\theta},\hat{u}), \boldsymbol{\nu}_j(\tilde{\theta},\hat{u}), \boldsymbol{\lambda}_j : \tilde{\theta} \in X\ and\ \hat{u} \in Q\ for\ all\ j = 1,2,\ldots,l\}$ and

B=$\{(\tilde{\theta},\hat{u}), \mu_j^*(\tilde{\theta},\hat{u}), \boldsymbol{\nu}_j^*(\tilde{\theta},\hat{u}), \boldsymbol{\lambda}_j^*(\tilde{\theta},\hat{u}) : \tilde{\theta} \in X\ and\ \hat{u} \in Q\ for\ all\ j = 1,2,\ldots,l\}$

Then we define the following basic operations for Q −SVNSs.

1. $A \subset B$ iff $\mu_j(\hat{\theta},\hat{u}) \leq \mu_j^*(\hat{\theta},\hat{u}), \boldsymbol{\nu}_j(\hat{\theta},\hat{u}) \geq \boldsymbol{\nu}_j^*(\hat{\theta},\hat{u})$ and $\boldsymbol{\lambda}_j(\hat{\theta},\hat{u}) \geq \boldsymbol{\lambda}_j^*(\hat{\theta},\hat{u})$ $for\ all\ j = 1,2,\ldots,l.$
2. $A = B$ iff $\mu_j(\hat{\theta},\hat{u}) = \mu_j^*(\hat{\theta},\hat{u}), \boldsymbol{\nu}_j(\hat{\theta},\hat{u}) = \boldsymbol{\nu}_j^*(\hat{\theta},\hat{u})$ and $\boldsymbol{\lambda}_j(\hat{\theta},\hat{u}) = \boldsymbol{\lambda}_j^*(\hat{\theta},\hat{u})$ $for\ all\ j = 1,2,\ldots,l.$
3. $A \cup B = \{(\hat{\theta},\hat{u}), \max(\mu_j(\hat{\theta},\hat{u}), \mu_j^*(\hat{\theta},\hat{u})), \min\left(\boldsymbol{\nu}_j(\hat{\theta},\hat{u}), \boldsymbol{\nu}_j^*(\hat{\theta},\hat{u})\right), \min(\boldsymbol{\lambda}_j(\hat{\theta},\hat{u}), \boldsymbol{\lambda}_j^*(\hat{\theta},\hat{u}))\}$
4. $A \cap B = \{(\hat{\theta},\hat{u}), \min(\mu_j(\hat{\theta},\hat{u}), \mu_j^*(\hat{\theta},\hat{u})), \max\left(\boldsymbol{\nu}_j(\hat{\theta},\hat{u}), \boldsymbol{\nu}_j^*(\hat{\theta},\hat{u})\right), \max(\boldsymbol{\lambda}_j(\hat{\theta},\hat{u}), \boldsymbol{\lambda}_j^*(\hat{\theta},\hat{u}))\}$

5. Q −Single Valued Neutrosophic Soft Sets

In this section we introduce the concept of Q −SVNSSs by combining soft sets and Q − SVNS. We also define some basic operations and properties of Q − SVNSSs.

5.1. Definition. Let X be a universal set, Q be any non-empty set and E be the set of parameters. Let $Z^l QSVN(X)$ denote the set of all multi Q −single valued neutrosophic subsets of X with dimension $l = 1$ Let $K \subset E$ A pair (F_Q, K) is called Q − SVNSS over X where F_Q is a mapping given

$$F_Q : K \longrightarrow Z^l QSVN(X) \quad such\ that\ (F_{Q,}(\hat{\theta})) = \emptyset\ \ if\ \hat{\theta} \notin K$$

A Q − SVNSS can be represented by the set of ordered pairs

$$(F_Q, K) = \{\hat{\theta}, F_Q(\hat{\theta}) : \hat{\theta} \in X, F_Q(\hat{\theta}) \in Z^l QSVN(X)\}$$

5.2. Example. Let $X = \{\boldsymbol{p}_1, \boldsymbol{p}_2, \boldsymbol{p}_3, \boldsymbol{p}_4\}$ be a universal set, $E = \{\boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3, \boldsymbol{k}_4\}$ and $Q = \{\hat{u}, \hat{v}\}$ be a non-empty set. If $K = \{\boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3\} \subset E$,

$F_{Q,}(\boldsymbol{k}_1) = \{((\boldsymbol{p}_1,\hat{u}),(0.3,0.4,0.6)),((\boldsymbol{p}_1,\hat{v}),(0.2,0.3,0.5)),((\boldsymbol{p}_2,\hat{u}),(0.6,0.2,0.4))\}$

$F_{Q,}(\boldsymbol{k}_1) = \{((\boldsymbol{p}_1,\hat{u}),(0.5,0.3,0.4)),((\boldsymbol{p}_1,\hat{v}),(0.4,0.1,0.7)),((\boldsymbol{p}_3,\hat{u}),(0.8,0.1,0.2))\}$

$F_{Q,}(\boldsymbol{k}_3) = \{((\boldsymbol{p}_1,\hat{u}),(0.9,0.1,0.1)),((\boldsymbol{p}_1,\hat{v}),(0.8,0.2,0.3)),((\boldsymbol{p}_3,\hat{v}),(0.4,0.3,0.6))\}$

Then

$(F_Q,K) = \{(\boldsymbol{k}_1, ((\boldsymbol{p}_1,\hat{u}),(0.3,0.4,0.6)), ((\boldsymbol{p}_1,\hat{v}),(0.2,0.3,0.5)), ((\boldsymbol{p}_2,\hat{u}),(0.6,0.2,0.4)),$
$\boldsymbol{k}_2, ((\boldsymbol{p}_1,\hat{u}),(0.5,0.3,0.4)), ((\boldsymbol{p}_1,\hat{v}),(0.4,0.1,0.7)), ((\boldsymbol{p}_3,\hat{u}),(0.8,0.1,0.2)),$
$\boldsymbol{k}_3, ((\boldsymbol{p}_1,\hat{u}),(0.9,0.1,0.1)), ((\boldsymbol{p}_1,\hat{v}),(0.8,0.2,0.3)), ((\boldsymbol{p}_3,\hat{v}),(0.4,0.3,0.6))\}$

is a Q −SVNSS.

5.3. Definition. Let $(F_Q,K) \in QSVNSS(X)$. If $F_Q(\hat{\theta}) = \emptyset$ for all $\hat{\theta} \in E$ then (F_Q,K) is called a null Q −SVNSS denoted by $(\emptyset, K)$.

5.4. Example. Let X, E *and* Q be defined in the above example 5.2 then

$(\emptyset,K) = \{(\boldsymbol{k}_1, ((\boldsymbol{p}_1,\hat{u}),(0,1,1)), ((\boldsymbol{p}_1,\hat{v}),(0,1,1)), ((\boldsymbol{p}_2,\hat{u}),(0,1,1)), \boldsymbol{k}_2,$
$((\boldsymbol{p}_1,\hat{u}),(0,1,1)), ((\boldsymbol{p}_1,\hat{v}),(0,1,1)), ((\boldsymbol{p}_3,\hat{u}),(0,1,1)),$
$\boldsymbol{k}_3, ((\boldsymbol{p}_1,\hat{u}),(0,1,1)), ((\boldsymbol{p}_1,\hat{v}),(0,1,1)), ((\boldsymbol{p}_3,\hat{v}),(0,1,1))\}$

5.5. Definition. Let $(F_Q,K) \in QSVNSS(X)$, If $F_Q(\hat{\theta}) = X$ for all $\hat{\theta} \in E$ then (F_Q,K) is called a null $Q - SVNSS$ denoted by (X, K).

5.6. Example. Let X, E *and* Q be defined in the above example 5.2 then

$(X,K) = \{(\boldsymbol{k}_1, ((\boldsymbol{p}_1,\hat{u}),(1,0,0)), ((\boldsymbol{p}_1,\hat{v}),(1,0,0)), ((\boldsymbol{p}_2,\hat{u}),(1,0,0)),$
$\boldsymbol{k}_2, ((\boldsymbol{p}_1,\hat{u}),(1,0,0)), ((\boldsymbol{p}_1,\hat{v}),(1,0,0)), ((\boldsymbol{p}_3,\hat{u}),(1,0,0)),$
$\boldsymbol{k}_3, ((\boldsymbol{p}_1,\hat{u}),(1,0,0)), ((\boldsymbol{p}_1,\hat{v}),(1,0,0)), ((\boldsymbol{p}_3,\hat{v}),(1,0,0))\}$

5.7. Definition. Let $(F_Q,K), (G_Q,L) \in QSVNS(X)$. Then (F_Q,K) is Q −SVNSS subset of (G_Q,L) , denoted by $(F_Q,K) \subset (G_Q,L)$ if $K \subset L$ and $F_Q(\hat{\theta}) \subset G_Q(\hat{\theta})$ for all $\theta \in X$.

5.8. Proposition. Let $(F_Q,K), (G_Q,L), (M_Q,N) \in QSVNS(X)$. Then

1. $(F_Q,K) \subset (G_Q,E)$
2. $(\emptyset,K) \subset (G_Q,L)$
3. $(F_Q,K) \subset (G_Q,L)$ *and* $(G_Q,L) \subset (M_Q,N)$ then $(F_Q,K) \subset (M_Q,N)$.
4. If $(F_Q,K) = (G_Q,L)$ and $(G_Q,L) = (M_Q,N)$ *then* $(F_Q,K) = (M_Q,N)$

Proof : Straightforward.

5.9. Definition. Let $(F_Q, K) \in QSVNS(X)$, Then the complement of $Q-$ SVNSS set is written as $(F_Q, K)^C$ and is defined by $(F_Q, K)^C = (F_Q^C, \neg K)$ where

$$F_Q^C : \neg K \to QSVNS(X)$$

is the mapping given by F_Q^C(e) Q −single valued neutrosophic complement for each $e \in K$.

5.10. Proposition. Let $(F_Q, K) \in QSVNS(X)$, Then

1. $((F_Q, K)^C)^C = (F_Q, K)$
2. $(\emptyset, K)^C = (X, E)$
3. $(X, E)^C = (\emptyset, E)$

Proof . 1. Let $k \in K$. Then

$$(F_Q, K) = F_Q(k) = \{(\boldsymbol{p}_1, \hat{u}), \left(\mu_{F_{Q(k)}}(\boldsymbol{p}_1, \hat{u}), \boldsymbol{\nu}_{F_{Q(k)}}(\boldsymbol{p}_1, \hat{u}), \boldsymbol{\lambda}_{F_{Q(k)}}(\boldsymbol{p}_1, \hat{u})\right) : \hat{u} \in Q, \boldsymbol{p}_1 \in X\}$$

$$(F_Q, K)^C = (F_Q(k))^C = \left\{(\boldsymbol{p}_1, \hat{u}), \left(\boldsymbol{\lambda}_{F_{Q(k)}}(\boldsymbol{p}_1, \hat{u}), 1 - \boldsymbol{\nu}_{F_{Q(k)}}(\boldsymbol{p}_1, \hat{u}), \mu_{F_{Q(k)}}(\boldsymbol{p}_1, \hat{u})\right)\right\}$$

$$((F_Q, K)^C)^C = \left\{(\boldsymbol{p}_1, \hat{u}), \left(\mu_{F_{Q(k)}}(\boldsymbol{p}_1, \hat{u}), 1 - \left(1 - \boldsymbol{\nu}_{F_{Q(k)}}(\boldsymbol{p}_1, \hat{u})\right), \boldsymbol{\lambda}_{F_{Q(k)}}(\boldsymbol{p}_1, \hat{u})\right)\right\}$$

$$((F_Q, K)^C)^C = \left\{(\boldsymbol{p}_1, \hat{u}), \left(\mu_{F_{Q(k)}}(\boldsymbol{p}_1, \hat{u}), \boldsymbol{\nu}_{F_{Q(k)}}(\boldsymbol{p}_1, \hat{u}), \boldsymbol{\lambda}_{F_{Q(k)}}(\boldsymbol{p}_1, \hat{u})\right)\right\}$$

$$((F_Q, K)^C)^C = (F_Q, K)$$

2. *Let* $(\emptyset, K) = (F_Q, K)$, Than for all $k \in K$

$$F_Q(k) = \{(\boldsymbol{p}_1, \hat{u}), \left(\mu_{F_{Q(k)}}(\boldsymbol{p}_1, \hat{u}), \boldsymbol{\nu}_{F_{Q(k)}}(\boldsymbol{p}_1, \hat{u}), \boldsymbol{\lambda}_{F_{Q(k)}}(\boldsymbol{p}_1, \hat{u})\right) : \hat{u} \in Q, \boldsymbol{p}_1 \in X\}$$
$$= \{(\boldsymbol{p}_1, \hat{u}), (0,1,1) : \hat{u} \in Q, \boldsymbol{p}_1 \in X\}$$

$$(\emptyset, K)^C = (F_Q, K)^C = (F_Q(k))^C = \{(\boldsymbol{p}_1, \hat{u}), (1, 1-1, 0) : \hat{u} \in Q, \boldsymbol{p}_1 \in X\}$$
$$= \{(\boldsymbol{p}_1, \hat{u}), (1,0,0) : \hat{u} \in Q, \boldsymbol{p}_1 \in X\}$$
$$= (X, E)$$

3. *Let* $(X, E) = (F_Q, E)$, Then for all $k \in K$

$$F_Q(k) = \{(\boldsymbol{p}_1, \hat{u}), \left(\mu_{F_{Q(k)}}(\boldsymbol{p}_1, \hat{u}), \boldsymbol{\nu}_{F_{Q(k)}}(\boldsymbol{p}_1, \hat{u}), \boldsymbol{\lambda}_{F_{Q(k)}}(\boldsymbol{p}_1, \hat{u})\right) : \hat{u} \in Q, \boldsymbol{p}_1 \in X\}$$

$= \{(\boldsymbol{p_1}, \hat{u}), (1,0,0) : \hat{u} \in Q, \boldsymbol{p_1} \in X\}$

$(X, E)^C = (F_Q, E)^C = (F_Q(k))^C = \{(\boldsymbol{p_1}, \hat{u}), (0,1-0,1) : \hat{u} \in Q, \boldsymbol{p_1} \in X\}$

$= \{(\boldsymbol{p_1}, \hat{u}), (0,1,1) : \hat{u} \in Q, \boldsymbol{p_1} \in X\}$

$= (\emptyset, E)$

5.11. Definition. Let (F_Q, K) and $(G_Q, L) \in QSVNS(X)$ .Then the union of two Q −SVNSSs (F_Q, K) and (G_Q, L) is the Q −SVNSS, (M_Q, N) written as $(F_Q, K) \cup (G_Q, L) = (M_Q, N)$ where $N = K \cup L$ for all $l \in N$ and

$$(M_Q, N) = \begin{cases} F_Q(l) & if\ l \in K - L \\ G_Q(l) & if\ l \in L - K \\ F_Q(l) \cup G_Q(l) & if\ l \in K \cap L \end{cases}$$

5.12. Example. Let $X = \{\boldsymbol{p_1}, \boldsymbol{p_2}, \boldsymbol{p_3}, \boldsymbol{p_4}, \boldsymbol{p_5}\}$ be a universal set, $E = \{\boldsymbol{a_1}, \boldsymbol{a_2}, \boldsymbol{a_3}, \boldsymbol{a_4}, \boldsymbol{a_5}\}$ be a set of parameters and $Q = \{\hat{u}, \hat{v}, w\}$ be a non-empty set. Let $N = \{\boldsymbol{a_1}, \boldsymbol{a_3}, \boldsymbol{a_4}\} \subset E, and\ M = \{\boldsymbol{a_1}, \boldsymbol{a_2}, \boldsymbol{a_3}\}$

$(F_Q, N) = \{(\boldsymbol{a_1}, ((\boldsymbol{p_1}, \hat{u}), (0.3,0.4,0.5)), ((\boldsymbol{p_1}, \hat{v}), (0.5,0.3,0.4)), ((\boldsymbol{p_1}, w), (0.6,0.1,0.2)))$
$(\boldsymbol{a_3}, ((\boldsymbol{p_1}, \hat{u}), (0.2,0.3,0.4)), ((\boldsymbol{p_1}, \hat{v}), (0.4,0.2,0.3)), ((\boldsymbol{p_1}, w), (0.6,0.2,0.4)), ((\boldsymbol{p_3}, \hat{u}), (0.7,0.1,0.2)),$
$((\boldsymbol{p_3}, \hat{v}), (0.8,0.2,0.2)), ((\boldsymbol{p_3}, w), (0.2,0.4,0.6))\}), (\boldsymbol{a_4}, \{((\boldsymbol{p_2}, \hat{u}), (0.6,0.2,0.1)), ((\boldsymbol{p_2}, \hat{v}), (0.4,0.2,0.5))$
$, ((\boldsymbol{p_2}, w), (0.5,0.4,0.4))\},$

and

$(G_Q, M) = \{(\boldsymbol{a_1}, ((\boldsymbol{p_1}, \hat{u}), (0.4,0.3,0.5)), ((\boldsymbol{p_1}, \hat{v}), (0.3,0.3,0.4)), ((\boldsymbol{p_1}, w), (0.4,0.2,0.3))),$
$(\boldsymbol{a_2}, ((\boldsymbol{p_2}, \hat{u}), (0.4,0.5,0.2)), ((\boldsymbol{p_2}, \hat{v}), (0.7,0.1,0.1)), ((\boldsymbol{p_2}, w), (0.6,0.2,0.3)),$
$(\boldsymbol{a_3}, \{((\boldsymbol{p_1}, \hat{u}), (0.4,0.3,0.5), (\boldsymbol{p_1}, \hat{v}), (0.2,0.2,0.4), (\boldsymbol{p_1}, w), (0.4,0.1,0.4))$
$, ((\boldsymbol{p_3}, \hat{v}), (0.6,0.1,0.2)), ((\boldsymbol{p_3}, w), (0.7,0.2,0.3))\},$

Then

$(K_Q, L) = \{(\boldsymbol{a_1}, \{((\boldsymbol{p_1}, \hat{u}), (0.4,0.3,0.5)), ((\boldsymbol{p_1}, \hat{v}), (0.5,0.3,0.4)), ((\boldsymbol{p_1}, w), (0.6,0.1,0.2))\}),$
$\boldsymbol{a_2}, ((\boldsymbol{p_2}, \hat{u}), (0.4,0.5,0.2)), ((\boldsymbol{p_2}, \hat{v}), (0.7,0.1,0.1)), ((\boldsymbol{p_2}, w), (0.6,0.2,0.3)),$
$\boldsymbol{a_3}, ((\boldsymbol{p_1}, \hat{u}), (0.4,0.3,0.4)), ((\boldsymbol{p_1}, \hat{v}), (0.4,0.2,0.3)), ((\boldsymbol{p_1}, w), (0.6,0.1,0.4)), (\boldsymbol{p_3}, \hat{u}), (0.8,0.1,0.1),$
$(\boldsymbol{p_3}, \hat{v}), (0.8,0.1,0.2),$

$(\boldsymbol{p}_3, w), (0.7,0.2,0.3))\})$, $\boldsymbol{a}_4, \{((\boldsymbol{p}_2, \hat{u}), (0.6,0.2,0.1)), ((\boldsymbol{p}_2, \hat{v}), (0.4,0.2,0.5)), ((\boldsymbol{p}_2, w), (0.5,0.4,0.4))\}$.

5.13. Definition. Let (F_Q, K) *and* $(G_Q, L) \in QSVNSS(X)$ .Then the intersection of two Q −SVNSSs, (F_Q, K) *and* (G_Q, L) is the $Q-$ SVNSS (M_Q, N) written as $(F_Q, K) \cap (G_Q, L) = (M_Q, N)$ where $N = K \cap L$ for all $l \in N$ and

$$(M_Q, N) = \{e, \min\left(\mu_{F_Q}(\hat{\theta}, \hat{u}), \mu_{G_Q}(\hat{\theta}, \hat{u})\right), \max\left(\boldsymbol{\nu}_{F_Q}(\hat{\theta}, \hat{u}), \boldsymbol{\nu}_{G_Q}(\hat{\theta}, \hat{u})\right),$$

$$\max\left(\boldsymbol{\lambda}_{F_Q}(\hat{\theta}, \hat{u}), \boldsymbol{\lambda}_{G_Q}(\hat{\theta}, \hat{u})\right) : \hat{\theta} \in X, \hat{u} \in Q \; and \; j = 1,2, \dots, l\}$$

5.14. Example. Let $X = \{\boldsymbol{p}_1, \boldsymbol{p}_2, \boldsymbol{p}_3, \boldsymbol{p}_4, \boldsymbol{p}_5\}$ be a universal set, $E = \{\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3, \boldsymbol{a}_4, \boldsymbol{a}_5\}$ be a set of parameters and $Q = \{\hat{u}, \hat{v}, w\}$ be a non-empty set. Let $N = \{\boldsymbol{a}_1, \boldsymbol{a}_3, \boldsymbol{a}_4\} \subset E, and \; M = \{\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3\}$

$(F_Q, N) = \{(\boldsymbol{a}_1, ((\boldsymbol{p}_1, \hat{u}), (0.3,0.4,0.5)), ((\boldsymbol{p}_1, \hat{v}), (0.5,0.3,0.4)), ((\boldsymbol{p}_1, w), (0.6,0.1,0.2)))$
$(\boldsymbol{a}_3, ((\boldsymbol{p}_1, \hat{u}), (0.2,0.3,0.4)), ((\boldsymbol{p}_1, \hat{v}), (0.4,0.2,0.3)), ((\boldsymbol{p}_1, w), (0.6,0.2,0.4)), ((\boldsymbol{p}_3, \hat{u}), (0.7,0.1,0.2)),$
$((\boldsymbol{p}_3, \hat{v}), (0.8,0.2,0.2)), ((\boldsymbol{p}_3, w), (0.2,0.4,0.6))\}), (\boldsymbol{a}_4, \{((\boldsymbol{p}_2, \hat{u}), (0.6,0.2,0.1)), ((\boldsymbol{p}_2, \hat{v}), (0.4,0.2,0.5))$
$, ((\boldsymbol{p}_2, w), (0.5,0.4,0.4))\},$

and

$(G_Q, M) = \{(\boldsymbol{a}_1, ((\boldsymbol{p}_1, \hat{u}), (0.4,0.3,0.5)), ((\boldsymbol{p}_1, \hat{v}), (0.3,0.3,0.4)), ((\boldsymbol{p}_1, w), (0.4,0.2,0.3))),$
$(\boldsymbol{a}_2, ((\boldsymbol{p}_2, \hat{u}), (0.4,0.5,0.2)), ((\boldsymbol{p}_2, \hat{v}), (0.7,0.1,0.1)), ((\boldsymbol{p}_2, w), (0.6,0.2,0.3)),$
$(\boldsymbol{a}_3, \{((\boldsymbol{p}_1, \hat{u}), (0.4,0.3,0.5), (\boldsymbol{p}_1, \hat{v}), (0.2,0.2,0.4), (\boldsymbol{p}_1, w), (0.4,0.1,0.4))$
$, ((\boldsymbol{p}_3, \hat{v}), (0.6,0.1,0.2)), ((\boldsymbol{p}_3, w), (0.7,0.2,0.3))\},$

Then

$(K_Q, L) = \{(\boldsymbol{a}_1, \{((\boldsymbol{p}_1, \hat{u}), (0.3,0.4,0.5)), ((\boldsymbol{p}_1, \hat{v}), (0.3,0.3,0.4)), ((\boldsymbol{p}_1, w), (0.4,0.2,0.3))\}),$
$\boldsymbol{a}_3, ((\boldsymbol{p}_1, \hat{u}), (0.2,0.3,0.5)), ((\boldsymbol{p}_1, \hat{v}), (0.2,0.2,0.4)), ((\boldsymbol{p}_1, w), (0.4,0.2,0.4)), (\boldsymbol{p}_3, \hat{u}), (0.7,0.2,0.2),$
$(\boldsymbol{p}_3, \hat{v}), (0.6,0.2,0.2), (\boldsymbol{p}_3, w), (0.2,0.4,0.6))\})$

5.15 Proposition. Let $(F_Q, K), (M_Q, N)$ *and* $(G_Q, L) \in QSVNSS(X)$. Then

1. $(F_Q, K) \cup (\emptyset, K) = (F_Q, K)$
2. $(F_Q, K) \cup (X, K) = (X, K)$
3. $(F_Q, K) \cup (F_Q, K) = (F_Q, K)$

4. $(F_Q,K) \cup (G_Q,L) = (G_Q,L) \cup (F_Q,K)$
5. $(F_Q,K) \cup ((G_Q,L) \cup (M_Q,N)) = ((G_Q,L) \cup (F_Q,K))) \cup (M_Q,N)$

Proof. 1. We have

$(F_Q,K) = \{(\boldsymbol{p}_1,\hat{u}),\left(\mu_{F_{Q(k)}}(\boldsymbol{p}_1,\hat{u}),\boldsymbol{\nu}_{F_{Q(k)}}(\boldsymbol{p}_1,\hat{u}),\boldsymbol{\lambda}_{F_{Q(k)}}(\boldsymbol{p}_1,\hat{u})\right):\hat{u}\in Q,\boldsymbol{p}_1\in X$

$(\emptyset,K) = \{(\boldsymbol{p}_1,\hat{u}),(0{,}1{,}1) : \hat{u}\in Q,\boldsymbol{p}_1\in X\}$

$(F_Q,K) \cup (\emptyset,K)$

$= \{k,\left(\left\{(\boldsymbol{p}_1,\theta\hat{u}),\max\left(\mu_{F_{Q(k)}}(\boldsymbol{p}_1,\hat{u}),0\right),\min\left(\boldsymbol{\nu}_{F_{Q(k)}}(\boldsymbol{p}_1,\hat{u}),1\right),\min\left(\boldsymbol{\lambda}_{F_{Q(k)}}(\boldsymbol{p}_1,\hat{u}),1\right)\right\}\right)\}$

$=\{(\boldsymbol{p}_1,\hat{u}),\left(\mu_{F_{Q(k)}}(\boldsymbol{p}_1,\hat{u}),\boldsymbol{\nu}_{F_{Q(k)}}(\boldsymbol{p}_1,\hat{u}),\boldsymbol{\lambda}_{F_{Q(k)}}(\boldsymbol{p}_1,\tilde{\hat{u}})\right):\hat{u}\in Q,\boldsymbol{p}_1\in X$

$=(F_Q,K)$

2. Let $(X,K) = (G_Q,K)$ then

$(F_Q,K) = \{(\boldsymbol{p}_1,\tilde{\hat{u}}),\left(\mu_{F_{Q(k)}}(\boldsymbol{p}_1,\hat{u}),\boldsymbol{\nu}_{F_{Q(k)}}(\boldsymbol{p}_1,\hat{u}),\boldsymbol{\lambda}_{F_{Q(k)}}(\boldsymbol{p}_1,\hat{u})\right):\hat{u}\in Q,\boldsymbol{p}_1\in X\}$

$(G_Q,L) = \{(\boldsymbol{p}_1,\hat{u}),(1{,}0{,}0) : \hat{u}\in Q,\boldsymbol{p}_1\in X\}$

$(F_Q,K) \cup (G_Q,K)$

$= \{k,\left(\left\{(\boldsymbol{p}_1,\hat{u}),\max\left(\mu_{F_{Q(k)}}(\boldsymbol{p}_1,\hat{u}),1\right),\min\left(\boldsymbol{\nu}_{F_{Q(k)}}(\boldsymbol{p}_1,\hat{u}),0\right),\min\left(\boldsymbol{\lambda}_{F_{Q(k)}}(\boldsymbol{p}_1,\hat{u}),0\right)\right\}\right)\}$

$=\{(\boldsymbol{p}_1,\hat{u}),(1{,}0{,}0):\hat{u}\in Q,\boldsymbol{p}_1\in X\}$

$=(G_Q,K) = (X,K)$

3. Let

$(F_Q,K) = \{(\boldsymbol{p}_1,\hat{u}),\left(\mu_{F_{Q(k)}}(\boldsymbol{p}_1,\hat{u}),\boldsymbol{\nu}_{F_{Q(k)}}(\boldsymbol{p}_1,\hat{u}),\boldsymbol{\lambda}_{F_{Q(k)}}(\boldsymbol{p}_1,\hat{u})\right):\hat{u}\in Q,\boldsymbol{p}_1\in X$

$(F_Q,K) \cup (F_Q,K)$

$$= \{k,\left(\left\{\begin{matrix}(\boldsymbol{p}_1,\hat{u}),\max\left(\mu_{F_{Q(k)}}(\boldsymbol{p}_1,\hat{u}),\mu_{F_{Q(k)}}(\boldsymbol{p}_1,\hat{u})\right),\min\left(\boldsymbol{\nu}_{F_{Q(k)}}(\boldsymbol{p}_1,\hat{u}),\boldsymbol{\nu}_{F_{Q(k)}}(\boldsymbol{p}_1,\hat{u})\right),\\ \min\left(\boldsymbol{\lambda}_{F_{Q(k)}}(\boldsymbol{p}_1,\hat{u}),\boldsymbol{\lambda}_{F_{Q(k)}}(\boldsymbol{p}_1,\hat{u})\right):\hat{u}\in Q,\boldsymbol{p}_1\in X\}\end{matrix}\right\}\right)\}$$

$= \{(\boldsymbol{p}_1,\hat{u}),\left(\mu_{F_{Q(k)}}(\boldsymbol{p}_1,\hat{u}),\boldsymbol{\nu}_{F_{Q(k)}}(\boldsymbol{p}_1,\hat{u}),\boldsymbol{\lambda}_{F_{Q(k)}}(\boldsymbol{p}_1,\hat{u})\right):\hat{u}\in Q,\boldsymbol{p}_1\in X$

$=(F_Q,K)$

4 and 5 can be proved easily in a similar way.

5.16. Proposition. Let $(F_Q,K),(M_Q,N)$ *and* $(G_Q,L)\in QSVNSS(X)$. Then

1. $(F_Q,K)\cap(\emptyset,K)=(\emptyset,K)$
2. $(F_Q,K)\cap(X,K)=(F_Q,K)$
3. $(F_Q,K)\cap(F_Q,K)=(F_Q,K)$
4. $(F_Q,K)\cap(G_Q,L)=(G_Q,L)\cap(F_Q,K)$
5. $(F_Q,K)\cap((G_Q,L)\cap(M_Q,N))=((G_Q,L)\cap(F_Q,K))\cap(M_Q,N)$

Proof. 1. We have

$(F_Q,K)=\{(\boldsymbol{p}_1,\hat{u}),\left(\mu_{F_{Q(k)}}(\boldsymbol{p}_1,\hat{u}),\boldsymbol{\nu}_{F_{Q(k)}}(\boldsymbol{p}_1,\hat{u}),\boldsymbol{\lambda}_{F_{Q(k)}}(\boldsymbol{p}_1,\hat{u})\right):\hat{u}\in Q,\boldsymbol{p}_1\in X$

$(\emptyset,K)=\{(\boldsymbol{p}_1,\hat{u}),(0,1,1):\hat{u}\in Q,\boldsymbol{p}_1\in X\}(F_Q,K)\cap(\emptyset,K)$

$=\{k,\left(\left\{(\boldsymbol{p}_1,\hat{u}),\min\left(\mu_{F_{Q(k)}}(\boldsymbol{p}_1,\hat{u}),0\right),\max\left(\boldsymbol{\nu}_{F_{Q(k)}}(\boldsymbol{p}_1,\hat{u}),1\right),\max\left(\boldsymbol{\lambda}_{F_{Q(k)}}(\boldsymbol{p}_1,\hat{u}),1\right)\right\}\right)\}$

$=\{(\boldsymbol{x}_1,\hat{u}),(0,1,1):\hat{u}\in Q,\boldsymbol{p}_1\in X$

$=(\emptyset,K)$

2. Let $(X,K)=(G_Q,L)$ then

$(F_Q,K)=\{(\boldsymbol{p}_1,\hat{u}),\left(\mu_{F_{Q(k)}}(\boldsymbol{p}_1,\hat{u}),\boldsymbol{\nu}_{F_{Q(k)}}(\boldsymbol{p}_1,\hat{u}),\boldsymbol{\lambda}_{F_{Q(k)}}(\boldsymbol{p}_1,\hat{u})\right):\hat{u}\in Q,\boldsymbol{p}_1\in X\}$

$(G_Q,L)=\{(\boldsymbol{p}_1,\tilde{u}),(1,0,0):\tilde{u}\in Q,\boldsymbol{p}_1\in X\}$

$(F_Q,K)\cap(G_Q,L)$

$=\{k,\left(\left\{(\boldsymbol{p}_1,\hat{u}),\min\left(\mu_{F_{Q(k)}}(\boldsymbol{p}_1,\hat{u}),1\right),\max\left(\boldsymbol{\nu}_{F_{Q(k)}}(\boldsymbol{p}_1,\hat{u}),0\right),\max\left(\boldsymbol{\lambda}_{F_{Q(k)}}(\boldsymbol{p}_1,\hat{u}),0\right)\right\}\right)\}$

$=\{(\boldsymbol{p}_1,\hat{u}),\left(\mu_{F_{Q(k)}}(\boldsymbol{p}_1,\hat{u}),\boldsymbol{\nu}_{F_{Q(k)}}(\boldsymbol{p}_1,\hat{u}),\boldsymbol{\lambda}_{F_{Q(k)}}(\boldsymbol{p}_1,\hat{u})\right):\hat{u}\in Q,\boldsymbol{p}_1\in X\}$

$=(F_Q,K)$

3. Let

$(F_Q,K)=\{(\boldsymbol{p}_1,\hat{u}),\left(\mu_{F_{Q(k)}}(\boldsymbol{p}_1,\hat{u}),\boldsymbol{\nu}_{F_{Q(k)}}(\boldsymbol{p}_1,\hat{u}),\boldsymbol{\lambda}_{F_{Q(k)}}(\boldsymbol{p}_1,\hat{u})\right):\hat{u}\in Q,\boldsymbol{p}_1\in X\}$

$(F_Q,K)\cap(F_Q,K)=\{(\boldsymbol{p}_1,\hat{u}),(min\left(\mu_{F_{Q(k)}}(\boldsymbol{p}_1,\hat{u}),\mu_{F_{Q(k)}}(\boldsymbol{p}_1,\hat{u})\right),$

$\min\left(\boldsymbol{\nu}_{F_{Q(k)}}(\boldsymbol{p}_1,\hat{u}),\boldsymbol{\nu}_{F_{Q(k)}}(\boldsymbol{p}_1,\hat{u})\right),\boldsymbol{min}(\boldsymbol{\lambda}_{F_{Q(k)}}(\boldsymbol{p}_1,\hat{u}),\boldsymbol{\lambda}_{F_{Q(k)}}(\boldsymbol{p}_1,\hat{u})):\hat{u}\in Q,\boldsymbol{p}_1\in X\}$

$=\{(\boldsymbol{p}_1,\hat{u}),\left(\mu_{F_{Q(k)}}(\boldsymbol{p}_1,\hat{u}),\boldsymbol{\nu}_{F_{Q(k)}}(\boldsymbol{p}_1,\hat{u}),\boldsymbol{\lambda}_{F_{Q(k)}}(\boldsymbol{p}_1,\hat{u})\right):\hat{u}\in Q,\boldsymbol{p}_1\in X\}$

$=(F_Q,K)$

4 and 5 can be proved easily in a similar way.

5.17. Proposition. Let (F_Q,K) and $(G_Q,L) \in QSVNSS(X)$. Then

1. $((F_Q,K) \cup (G_Q,L))^C = (F_Q,K)^C \cap (G_Q,L)^C$
2. $((F_Q,K) \cap (G_Q,L))^C = (F_Q,K)^C \cup (G_Q,L)^C$

Proof. Straightforward

5.18. Proposition. Let $(F_Q,K), (M_Q,N)$ and $(G_Q,L) \in QSVNSS(X)$. **Then**

$(F_Q,K) \cap ((G_Q,L) \cup (M_Q,N)) = ((F_Q,K) \cap (G_Q,L)) \cup ((F_Q,K)) \cap (M_Q,N))$

$(F_Q,K) \cup ((G_Q,L) \cap (M_Q,N)) = ((F_Q,K) \cup (G_Q,L)) \cap ((F_Q,K) \cup (M_Q,N)$

Proof. Straightforward.

5.19. Definition. Let $(F_Q,K), (M_Q,N)$ and $(G_Q,L) \in QSVNSS(X)$. Then the "AND" operation of two $Q-$ SVNSSs (F_Q,K) and $(G_Q,L)$is the $Q-$ SVNSS denoted by $(F_Q,K) \wedge (G_Q,L)$ and is defined by

$$(F_Q,K) \wedge (G_Q,L) = (M_Q,K \times L)$$

Where $M_Q(\gamma,\delta) = F_Q(\gamma) \cap G_Q(\delta)$ for all $\gamma \in K,\ \delta \in L$ is the intersection of two $Q-$ SVNSSs.

5.20. Definition. Let $(F_Q,K), (M_Q,N)$ and $(G_Q,L) \in QSVNS(X)$.Then the "OR" operation of two $Q-$ SVNSSs (F_Q,K) and (G_Q,L) is the $Q-$ SVNSS denoted by $(F_Q,K) \vee (G_Q,L)$ and is defined by

$$(F_Q,K) \vee (G_Q,L)) = (M_Q,K \times L)$$

Where $M_Q(\gamma,\delta) = F_Q(\gamma) \cup G_Q(\delta)$ for all $\gamma \in K,\ \delta \in L$ is the union of two $Q-$SVNSSs.

Conclusion

In this paper we have inaugurated the concept of Q-SVNS, Multi Q-SVNS. We also gave the concept of Q- SVNSS and studied some related properties with associate proofs. The equality, subset, complement, union, intersection, AND or OR operations have been defined on the Q- SVNSS. This new wing will be more useful than Q-fuzzy soft set, Q-intuitionistic fuzzy soft set and provide a substantial addition to existing theories for handling uncertainties, and pass to possible areas of further research and relevant applications.

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SOME INEQUALITIES OF THE HERMITE HADAMARD TYPE FOR PRODUCT OF TWO FUNCTIONS

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Abstaract − In this paper, we shall establish some new inequalities of the Hermite Hadamard type for product of two functions to belong to the class of s-convex functions and the class of h-convex functions. Some results of product of two functions that belong to two classes of different functions are also given.

Keywords − *Hadamard's inequality, Godunova - Levin functions, P-functions, convex functions, s-convex functions, h-convex functions.*

1 Introduction

A real-valued function $f : I \subset \mathbb{R} \to \mathbb{R}$ is called *convex* if and only if the following inequality holds

$$f(ta + (1-t)b) \leq tf(a) + (1-t)f(b) \tag{1}$$

for all $a, b \in I$ and $t \in [0, 1]$. If (1) is reversed, then f is called *concave*. In particular, if f is a convex function defined on I, then for all $a, b \in I$ with $a < b$ the following well-known double inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2} \tag{2}$$

is known in the literature as Hermite Hadamard's inequality. Inequality (2) is reversed whenever f is concave. This famous integral inequality is generalized, improved and extended by many mathematicians (see [3, 4, 5, 12] and [14]).

In 2003, the first time B. G. Pachpatte [9] established two new Hermite Hadamard type inequalities for product of positive convex functions as follows.

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Theorem 1.1 ([9])**.** Let f and g be real-valued, non-negative, and convex functions on $[a,b]$ with $a<b$. Then the following inequalities hold

$$\frac{1}{b-a}\int_a^b f(x)g(x)dx \le M(a,b)/3+N(a,b)/6, \tag{3}$$

$$2f\Big(\frac{a+b}{2}\Big)g\Big(\frac{a+b}{2}\Big) \le \frac{1}{b-a}\int_a^b f(x)g(x)dx + M(a,b)/6+N(a,b)/3, \tag{4}$$

where $M(a,b)=f(a)g(a)+f(b)g(b)$ and $N(a,b)=f(a)g(b)+f(b)g(a)$.

These results were refined by Feixiang Chen [2] in 2013. In the same year, A. Witkowskim [15] proved the following two theorems for convex functions.

Theorem 1.2 ([15])**.** If $f,g:I\subset\mathbb{R}\to\mathbb{R}$ are of the same convexity (i.e. both convex or both concave), then for all $a,b\in I$ with $a<b$ the following inequality holds

$$\frac{1}{(b-a)^2}\int_a^b (b-x)[f(a)g(x)+f(x)g(a)]dx+\frac{1}{(b-a)^2}\int_a^b (x-a)[f(b)g(x)+f(x)g(b)]dx$$
$$\le \frac{1}{b-a}\int_a^b f(x)g(x)dx + M(a,b)/3+N(a,b)/6, \tag{5}$$

where $M(a,b)$ and $N(a,b)$ are as in Theorem 1.1. If f and g are of the opposite convexity, then (5) is reversed.

Theorem 1.3 ([15])**.** Let $f,g:I\subset\mathbb{R}\to[0,\infty)$ be convex functions. Then the following inequality holds for all $a,b\in I$ with $a<b$,

$$\frac{1}{b-a}\int_a^b \Big[f\Big(\frac{a+b}{2}\Big)g(x)+g\Big(\frac{a+b}{2}\Big)f(x)\Big]dx$$
$$\le f\Big(\frac{a+b}{2}\Big)g\Big(\frac{a+b}{2}\Big)+\frac{1}{2(b-a)}\int_a^b f(x)g(x)+M(a,b)/12+N(a,b)/6, \tag{6}$$

where $M(a,b)$ and $N(a,b)$ are as in Theorem 1.1.

By the early year 2014, M. Tunç [13] advanced two new results for product of a s-convex function with a positive h-convex function as follows.

Theorem 1.4 ([13])**.** Let $h:[0,1]\to\mathbb{R}$ be a positive function, $a,b\in[0,\infty)$ with $a<b$, $f,g:[a,b]\to\mathbb{R}$ functions and $fg\in L_1([a,b]), h\in L_1([0,1])$. If f is h-convex and g is s-convex in the second sense for some fixed $s\in(0,1]$, then

$$\frac{1}{b-a}\int_a^b f(x)g(x)dx \le M(a,b)\int_0^1 h(t)t^s ds + N(a,b)\int_0^1 h(1-t)t^s dt, \tag{7}$$

where $M(a,b)$ and $N(a,b)$ are as in Theorem 1.1.

Theorem 1.5 ([13])**.** Let $h:[0,1]\to\mathbb{R}$ be a positive function, $a,b\in[0,\infty)$ with $a<b$, $f,g:[a,b]\to\mathbb{R}$ functions and $fg\in L_1([a,b]), h\in L_1([0,1])$. If f is h-convex on $[a,b]$ and g is s-convex in the second sense on $[a,b]$ for some fixed $s\in(0,1]$, then

$$\frac{2^{s-1}}{h(1/2)}f\Big(\frac{a+b}{2}\Big)g\Big(\frac{a+b}{2}\Big)-\frac{1}{b-a}\int_a^b f(x)g(x)dx$$
$$\le M(a,b)\int_0^1 h(1-t)t^s ds + N(a,b)\int_0^1 h(t)t^s dt, \tag{8}$$

where $M(a,b)$ and $N(a,b)$ are as in Theorem 1.1.

The main aim of this paper is to give some new inequalities which are similar to the above results for the classes of s-convex (concave) functions and h-convex (concave) functions. As consequences, we also obtain some results for product of two functions belonging to two different classes of functions.

2 Inequalities for the class of s-convex functions

Before stating our main results, we shall recall some notions and definitions. The first notion was introduced by E. K. Godunova and V. I. Levin in 1985 (see [3]).

Definition 2.1 (see [3]). We say that $f : I \subset \mathbb{R} \to \mathbb{R}$ is a Godunova - Levin function, or that f belongs to the class $Q(I)$, if f is non-negative and for all $a, b \in I$ and $t \in (0, 1)$, the following inequality holds

$$f(ta + (1-t)b) \leq \frac{f(a)}{t} + \frac{f(b)}{1-t}. \tag{9}$$

Restricting of the class of functions $Q(I)$ is the class $P(I)$ as follows.

Definition 2.2 (see [3]). We say that $f : I \subset \mathbb{R} \to \mathbb{R}$ is a P-function, or that f belongs to the class $P(I)$, if f is non-negative and for all $a, b \in I$ and $t \in [0, 1]$, we have

$$f(ta + (1-t)b) \leq f(a) + f(b). \tag{10}$$

The next concept is s-convex. It was introduced and investigated by Breckner in 1978 as a generalization of convex function.

Definition 2.3 (see [11]). Let $s \in (0, 1]$ be a real number and I be an interval on $[0, \infty)$. A function $f : I \to [0, \infty)$ is said to be s-convex (in the second sense), if

$$f(at + (1-t)b) \leq t^s f(a) + (1-t)^s f(b) \tag{11}$$

for all $a, b \in I$ and $t \in [0, 1]$. If (11) is reversed, then f is called to be s-concave.

A more general notion than the above notions is h-convex given in the following definition.

Definition 2.4 (see [11]). Let $h : J \subset \mathbb{R} \to \mathbb{R}$ be a non-negative function. We say that $f : I \subset \mathbb{R} \to \mathbb{R}$ is h-convex, or f belongs to the class $SX(h, I)$, if f is non-negative and for all $a, b \in I$ and $t \in (0, 1)$, we have

$$f(ta + (1-t)b) \leq h(t)f(a) + h(1-t)f(b). \tag{12}$$

If inequality (12) is reversed, then f is said to be h-concave, or shortly $f \in SV(h, I)$.

In Definition 2.4, if we choose $h(t) = t$, then f is an ordinary convex function; if $h(t) = 1/t$, then f belongs to the class $Q(I)$; if $h(t) = 1$, then f belongs to the class $P(I)$; and if $h(t) = t^s$ for some fixed $s \in (0, 1]$, then f belongs to the class of s-convex functions.

A number of properties and inequalities concerning these classes of functions can be referred to [3, 4, 5] for the classes $Q(I), P(I)$, and [1, 7, 8, 13, 10, 11] for the classes s-convex and h-convex.

We can now state the first main result as follows.

Theorem 2.1. Let $f, g : I \to \mathbb{R}$ be of the same s-convexity (i.e. both s-convex or both s-concave) and $fg \in L_1(I)$. Then, for all $a, b \in I$ with $a < b$, the following inequality holds

$$\begin{aligned} M(a,b)\frac{1}{2s+1} + N(a,b)\frac{\Gamma(s+1)^2}{\Gamma(2s+2)} + \frac{1}{b-a}\int_a^b f(x)g(x)dx \\ \geq \frac{1}{(b-a)^{s+1}}\int_a^b [(b-x)^s f(a) + (x-a)^s f(b)]g(x)dx \\ + \frac{1}{(b-a)^{s+1}}\int_a^b [(b-x)^s g(a) + (x-a)^s g(b)]f(x)dx. \quad (13) \end{aligned}$$

where $\Gamma(\cdot)$ is the Gamma function and $M(a,b), N(a,b)$ are as in Theorem 1.1. If f and g are of the opposite s-convexity, then (13) is reversed.

Proof. According to (11), for all $t \in [0,1]$, we find that the inequality

$$[t^s f(a)+(1-t)^s f(b)-f(ta+(1-t)b)][t^s g(a)+(1-t)^s g(b)-g(ta+(1-t)b)] \geq 0 \quad (14)$$

holds if f and g are of the same s-convexity, else (14) is reversed. Inequality (14) is equivalent to

$$\begin{aligned} [t^s f(a) + (1-t)^s f(b)][t^s g(a) + (1-t)^s g(b)] + f(ta + (1-t)b)g(ta + (1-t)b) \\ \geq [t^s f(a) + (1-t)^s f(b)]g(ta + (1-t)b) \\ + [t^s g(a) + (1-t)^s g(b)]f(ta + (1-t)b). \end{aligned}$$

Integrating the above inequality with respect to t over $[0,1]$, we get

$$\begin{aligned} \int_0^1 [t^s f(a)+(1-t)^s f(b)][t^s g(a)+(1-t)^s g(b)]dt + \int_0^1 f(ta+(1-t)b)g(ta+(1-t)b)dt \\ \geq \int_0^1 [t^s f(a) + (1-t)^s f(b)]g(ta + (1-t)b)dt \\ + \int_0^1 [t^s g(a) + (1-t)^s g(b)]f(ta + (1-t)b)dt. \quad (15) \end{aligned}$$

Directly computing, we obtain

$$\begin{aligned} \int_0^1 [t^s f(a) + (1-t)^s f(b)][t^s g(a) + (1-t)^s g(b)]dt \\ = \int_0^1 t^{2s} f(a)g(a)dt + \int_0^1 (1-t)^{2s} f(b)g(b)dt \\ + \int_0^1 t^s(1-t)^s[f(a)g(b) + f(b)g(a)]dt \\ = M(a,b)\frac{1}{2s+1} + N(a,b)\int_0^1 t^s(1-t)^s dt. \end{aligned}$$

By formulas (1.5.2) and (1.5.5) in [6], we have

$$\int_0^1 t^s(1-t)^s dt = B(s+1, s+1) = \frac{\Gamma(s+1)^2}{\Gamma(2s+2)},$$

where $B(\cdot,\cdot)$ is the Beta function, and so

$$\int_0^1 [t^s f(a)+(1-t)^s f(b)][t^s g(a)+(1-t)^s g(b)]dt = M(a,b)\frac{1}{2s+1}+N(a,b)\frac{\Gamma(s+1)^2}{\Gamma(2s+2)}. \tag{16}$$

Moreover, by substituting $x = ta + (1-t)b$, it is easy to see that

$$\int_0^1 f(ta+(1-t)b)g(ta+(1-tb))dt = \frac{1}{b-a}\int_a^b f(x)g(x)dx, \tag{17}$$

and

$$\int_0^1 [t^s f(a) + (1-t)^s f(b)]g(ta + (1-t)b)dt$$
$$= \frac{1}{(b-a)^{s+1}}\int_a^b [(b-x)^s f(a) + (x-a)^s f(b)]g(x)dx, \tag{18}$$

$$\int_0^1 [t^s g(a) + (1-t)^s g(b)]f(ta + (1-t)b)dt$$
$$= \frac{1}{(b-a)^{s+1}}\int_a^b [(b-x)^s g(a) + (x-a)^s g(b)]g(x)dx. \tag{19}$$

Substituting (16), (17), (18) and (19) in (15), we get the desired result. □

Theorem 2.2. Let $f, g : I \to \mathbb{R}$ be of the same s-convexity and $fg \in L_1(I)$. Then, for all $a, b \in I$ with $a < b$, the following inequality holds

$$\frac{1}{b-a}\int_a^b f(x)[g(x)+g(a+b-x)]dx + 2^{2s-1} f\Big(\frac{a+b}{2}\Big)g\Big(\frac{a+b}{2}\Big)$$
$$\geq \frac{2^s}{b-a}\int_a^b \Big[f\Big(\frac{a+b}{2}\Big)g(x) + g\Big(\frac{a+b}{2}\Big)f(x)\Big]dx. \tag{20}$$

If f and g are of the opposite s-convexity, then (20) is reversed.

Proof. Putting $x_0 = (a+b)/2$ and according to (11), for all $a < x < b$, we have the inequality

$$[f(x) + f(a+b-x) - 2^s f(x_0)][g(x) + g(a+b-x) - 2^s g(x_0)] \geq 0 \tag{21}$$

holds if f and g are of the same s-convexity, else (21) is reversed. Inequality (21) is equivalent to

$$[f(x) + f(a+b-x)][g(x) + g(a+b-x)] + 2^{2s} f(x_0)g(x_0)$$
$$\geq 2^s f(x_0)[g(x) + g(a+b-x)] + 2^s g(x_0)[f(x) + f(a+b-x)].$$

Integrating the above inequality with respect to x over $[a,b]$, we have

$$\frac{1}{b-a}\int_a^b [f(x) + f(a+b-x)][g(x) + g(a+b-x)]dx + 2^{2s} f(x_0)g(x_0)$$
$$\geq \frac{2^s}{b-a}\int_a^b g(x_0)[f(x) + f(a+b-x)]dx + \frac{2^s}{b-a}\int_a^b f(x_0)[g(x) + g(a+b-x)]dx$$
$$= \frac{2^{s+1}}{b-a}\int_a^b g(x_0)f(x)dx + \frac{2^{s+1}}{b-a}\int_a^b f(x_0)g(x)dx$$
$$= \frac{2^{s+1}}{b-a}\int_a^b [g(x_0)f(x) + f(x_0)g(x)]dx. \tag{22}$$

Besides, we have

$$\begin{aligned}\int_a^b [f(x)+f(a+b-x)][g(x)+g(a+b-x)]dx \\ &= \int_a^b f(x)g(x)dx + \int_a^b f(a+b-x)g(a+b-x)dx \\ &\quad + \int_a^b f(x)g(a+b-x)dx + \int_a^b f(a+b-x)g(x)dx \\ &= 2\int_a^b f(x)g(x)dx + 2\int_a^b f(x)g(a+b-x)dx \\ &= 2\int_a^b f(x)[g(x)+g(a+b-x)]dx. \qquad (23)\end{aligned}$$

Combining (22) and (23), we obtain inequality (20). □

A direct corollary of Theorem 2.2 when we require g is symmetric about $(a+b)/2$ as follows.

Corollary 2.3. Let $f, g : [a, b] \subset [0, \infty) \to \mathbb{R}$ be functions and g is symmetric about $(a+b)/2$. If f and g are of the same s-convexity, then the following inequality holds

$$\begin{aligned}\frac{2}{b-a}\int_a^b f(x)g(x)dx + 2^{2s-1} f\Big(\frac{a+b}{2}\Big) g\Big(\frac{a+b}{2}\Big) \\ \geq \frac{2^s}{b-a}\int_a^b \Big[f\Big(\frac{a+b}{2}\Big)g(x) + g\Big(\frac{a+b}{2}\Big)f(x)\Big]dx. \qquad (24)\end{aligned}$$

If f and g are of the opposite s-convexity, then (24) is reversed.

Proof. By the symmetric about $(a+b)/2$ of g, we find that

$$g(x) = g(a+b-x),$$

for all $x \in [a, b]$. Hence, inequality (20) reduces to inequality (24). □

Corollary 2.4. Let $f, g : [a, b] \subset [0, \infty) \to [0, \infty)$ be two s-concave functions. Then the following inequalities hold

$$\begin{aligned}\frac{1}{b-a}\int_a^b f(x)[g(x)+g(a+b-x)]dx + 2^{2s-1} f\Big(\frac{a+b}{2}\Big) g\Big(\frac{a+b}{2}\Big) \\ \geq \frac{2^s}{b-a}\int_a^b \Big[f\Big(\frac{a+b}{2}\Big)g(x) + g\Big(\frac{a+b}{2}\Big)f(x)\Big]dx \\ \geq \frac{2^s}{s+1} f\Big(\frac{a+b}{2}\Big)[g(a)+g(b)] + \frac{2^s}{s+1} g\Big(\frac{a+b}{2}\Big)[f(a)+f(b)]. \qquad (25)\end{aligned}$$

Proof. The first inequality in (25) follows immediately from Theorem 2.2. In order to prove the second inequality in (25), we remark that

$$x = \frac{b-x}{b-a}a + \frac{x-a}{b-a}b,$$

for all $a < x < b$. By s-concavity of f, we get

$$f(x) = f\Big(\frac{b-x}{b-a}a + \frac{x-a}{b-a}b\Big) \geq \Big(\frac{b-x}{b-a}\Big)^s f(a) + \Big(\frac{x-a}{b-a}\Big)^s f(b).$$

Therefore,

$$\begin{aligned}\frac{1}{b-a}\int_a^b f(x)dx &\geq \frac{1}{(b-a)^{s+1}}\int_a^b [(b-x)^s f(a) + (x-a)^s f(b)]dx \\ &= \frac{1}{s+1}[f(a)+f(b)]. \end{aligned} \tag{26}$$

Analogously, we can point out that

$$\frac{1}{b-a}\int_a^b g(x)dx \geq \frac{1}{s+1}[g(a)+g(b)]. \tag{27}$$

Since the non-negative of f and g, combining (26) and (27), we obtain

$$\begin{aligned}&\frac{2^s}{b-a}\int_a^b \Big[f\Big(\frac{a+b}{2}\Big)g(x) + g\Big(\frac{a+b}{2}\Big)f(x)\Big]dx \\ &= \frac{2^s}{b-a}\int_a^b f\Big(\frac{a+b}{2}\Big)g(x)dx + \frac{2^s}{b-a}\int_a^b g\Big(\frac{a+b}{2}\Big)f(x)dx \\ &\geq \frac{2^s}{s+1}g\Big(\frac{a+b}{2}\Big)[f(a)+f(b)] + \frac{2^s}{s+1}f\Big(\frac{a+b}{2}\Big)[g(a)+g(b)].\end{aligned}$$

This proves the desired results. □

3 Inequalities for the class of h-convex functions

The main purpose of this section is to establish some inequalities for product of two functions to belong to the class of h-convex functions. To do this, we will first denote by I a nonempty interval of the set of real numbers.

Theorem 3.1. Let $h_1, h_2 : [0,1] \to \mathbb{R}$ be positive functions satisfying $h_1, h_2, h_1h_2 \in L_1([0,1])$. Suppose that $f, g : I \to \mathbb{R}$ are of the same h-convexity (i.e. $f \in SX(h_1, I)$ and $g \in SX(h_2, I)$ or $f \in SV(h_1, I)$ and $g \in SV(h_2, I)$) and $fg \in L_1(I)$. Then, for all $a, b \in I$ with $a < b$, the following inequality holds

$$\begin{aligned}M(a,b)\int_0^1 h_1(t)h_2(t)dt + N(a,b)\int_0^1 h_1(t)h_2(1-t)dt + \frac{1}{b-a}\int_a^b f(x)g(x)dx \\ \geq \int_0^1 [h_1(t)f(a) + h_1(1-t)f(b)]g(ta+(1-t)b)dt \\ + \int_0^1 [h_2(t)g(a) + h_2(1-t)g(b)]f(ta+(1-t)b)dt,\end{aligned} \tag{28}$$

where $M(a,b)$ and $N(a,b)$ are as in Theorem 1.1. If f and g are of the opposite h-convexity, then (28) is reversed.

Proof. According to (12), for all $t \in (0,1)$, we find that the inequality

$$[h_1(t)f(a)+h_1(1-t)f(b)-f(ta+(1-t)b)][h_2(t)g(a)+h_2(1-t)g(b)-g(ta+(1-t)b)] \geq 0 \tag{29}$$

holds if f and g are of the same h-convexity, else (29) is reversed. Inequality (29) is equivalent to

$$\begin{aligned}[h_1(t)f(a)+h_1(1-t)f(b)][h_2(t)g(a)+h_2(1-t)g(b)]+f(ta+(1-t)b)g(ta+(1-t)b)\\ \geq f(ta+(1-t)b)[h_2(t)g(a)+h_2(1-t)g(b)]\\ + g(ta+(1-t)b)[h_1(t)f(a)+h_1(1-t)f(b)].\end{aligned} \tag{30}$$

By integrating (30) with respect to t over $[0,1]$ with noting that

$$\int_0^1 h_1(t)h_2(t)dt = \int_0^1 h_1(1-t)h_2(1-t)dt$$

and

$$\int_0^1 h_1(t)h_2(1-t)dt = \int_0^1 h_1(1-t)h_2(t)dt,$$

we get the desired result. □

Theorem 3.2. Let $h_1, h_2 : [0,1] \to \mathbb{R}$ be positive functions satisfying $h_1, h_2, h_1h_2 \in L_1([0,1])$. Suppose that $f, g : I \to \mathbb{R}$ are of the same h-convexity (i.e. $f \in SX(h_1, I)$ and $g \in SX(h_2, I)$ or $f \in SV(h_1, I)$ and $g \in SV(h_2, I)$) and $fg \in L_1(I)$. Then, for all $a, b \in I$ with $a < b$, the following inequality holds

$$\begin{aligned}\frac{1}{b-a}\int_a^b f(x)[g(x)+g(a+b-x)]dx + \frac{1}{2h_1(1/2)h_2(1/2)} f\Big(\frac{a+b}{2}\Big) g\Big(\frac{a+b}{2}\Big)\\ \geq \frac{1}{b-a}\int_a^b \Big[f\Big(\frac{a+b}{2}\Big)\frac{g(x)}{h_1(1/2)} + g\Big(\frac{a+b}{2}\Big)\frac{f(x)}{h_2(1/2)}\Big]dx.\end{aligned} \tag{31}$$

If f and g are of the opposite h-convexity, then (31) is reversed.

Proof. The proof runs as in the proof of Theorem 2.2. Here, in order to obtain the desired result, we start with observing that the following inequality

$$[h_1(1/2)f(x)+h_1(1/2)f(a+b-x)-f(x_0)][h_2(1/2)g(x)+h_2(1/2)g(a+b-x)-g(x_0)] \geq 0,$$

where $x_0 = (a+b)/2$, holds for all $a < x < b$ if f and g are of the same h-convexity, else the above inequality is reversed. □

Corollary 3.3. For the same hypotheses as in Theorem 3.2. If we require that g is symmetric about $(a+b)/2$, then inequality (31) reduces to

$$\begin{aligned}\frac{2}{b-a}\int_a^b f(x)g(x)dx + \frac{1}{2h_1(1/2)h_2(1/2)} f\Big(\frac{a+b}{2}\Big) g\Big(\frac{a+b}{2}\Big)\\ \geq \frac{1}{b-a}\int_a^b \Big[f\Big(\frac{a+b}{2}\Big)\frac{g(x)}{h_1(1/2)} + g\Big(\frac{a+b}{2}\Big)\frac{f(x)}{h_2(1/2)}\Big]dx.\end{aligned} \tag{32}$$

Proof. The above corollary is obtained from Theorem 3.2 with noting that g is symmetric about $(a+b)/2$. □

Corollary 3.4. Let $h_1, h_2 : [0,1] \to \mathbb{R}$ be positive functions with $h_1, h_2, h_1h_2 \in L_1([0,1])$. Suppose that $f : I \to \mathbb{R}$ is non-negative h_1-concave function and $g : I \to \mathbb{R}$ is non-negative h_2-concave function satisfying $fg \in L_1(I)$. Then, for all $a, b \in I$ with $a < b$, the following inequalities hold

$$\begin{aligned}\frac{1}{b-a}\int_a^b f(x)[g(x)+g(a+b-x)]dx + \frac{1}{2h_1(1/2)h_2(1/2)}f\Big(\frac{a+b}{2}\Big)g\Big(\frac{a+b}{2}\Big)\\ \geq \frac{1}{b-a}\int_a^b \Big[f\Big(\frac{a+b}{2}\Big)\frac{g(x)}{h_1(1/2)} + g\Big(\frac{a+b}{2}\Big)\frac{f(x)}{h_2(1/2)}\Big]dx\\ \geq f\Big(\frac{a+b}{2}\Big)\frac{g(a)+g(b)}{h_1(1/2)}\int_0^1 h_2(t)dt + g\Big(\frac{a+b}{2}\Big)\frac{f(a)+f(b)}{h_2(1/2)}\int_0^1 h_1(t)dt. \quad (33)\end{aligned}$$

Proof. The first inequality in (33) is similar to Theorem 3.2. The second inequality in (33) is proved as follows. For all $a < x < b$, we have

$$g(x) = g\Big(\frac{b-x}{b-a}a + \frac{x-a}{b-a}b\Big) \geq h_2\Big(\frac{b-x}{b-a}\Big)g(a) + h_2\Big(\frac{x-b}{b-a}\Big)g(b).$$

Integrating the above inequality over $[a,b]$ and substituting $t = (x-a)/(b-a)$, we get

$$\begin{aligned}\frac{1}{b-a}\int_a^b g(x)dx &\geq \frac{1}{b-a}\int_a^b \Big(h_2\Big(\frac{b-x}{b-a}\Big)g(a) + h_2\Big(\frac{x-b}{b-a}\Big)g(b)\Big)dx\\ &= \int_0^1 [h_2(1-t)g(a) + h_2(t)g(b)]dt\\ &= g(a)\int_0^1 h_2(1-t)dt + g(b)\int_0^1 h_2(t)dt\\ &= [g(a)+g(b)]\int_0^1 h_2(t)dt. \quad (34)\end{aligned}$$

Multiplying (34) by non-negative quantity $\frac{1}{h_1(1/2)}f(\frac{a+b}{2})$, we obtain

$$\frac{1}{b-a}\int_a^b f\Big(\frac{a+b}{2}\Big)\frac{g(x)}{h_1(1/2)}dx \geq f\Big(\frac{a+b}{2}\Big)\frac{g(a)+g(b)}{h_1(1/2)}\int_0^1 h_2(t)dt. \quad (35)$$

Analogously, we can point out that

$$\frac{1}{b-a}\int_a^b g\Big(\frac{a+b}{2}\Big)\frac{f(x)}{h_2(1/2)}dx \geq g\Big(\frac{a+b}{2}\Big)\frac{f(a)+f(b)}{h_2(1/2)}\int_0^1 h_1(t)dt. \quad (36)$$

Combining (35) and (36) reduces to the desired result. □

4 Inequalities for product of different kinds of convex functions

In this section, we shall give some inequalities for product of different kinds of convex functions as corollaries of Theorem 3.1 and 3.2 in the previous section.

Proposition 4.1. Let $h : [0,1] \to \mathbb{R}$ be an integrable positive function and $s \in (0,1]$ is a real number. Suppose that $f : [a,b] \subset [0,\infty) \to \mathbb{R}$ is a h-convex (concave) function and $g : [a,b] \subset [0,\infty) \to \mathbb{R}$ is a s-convex (concave, respectively) function satisfying $fg \in L_1([a,b])$. Then the following inequality holds

$$\begin{aligned} M(a,b)\int_0^1 h(t)t^s dt + N(a,b)\int_0^1 h(t)(1-t)^s dt + \frac{1}{b-a}\int_a^b f(x)g(x)dx \\ \geq \int_0^1 [h(t)f(a) + h(1-t)f(b)]g(ta+(1-t)b)dt \\ + \int_0^1 [t^s g(a) + (1-t)^s g(b)]f(ta+(1-t)b)dt, \quad (37) \end{aligned}$$

where $M(a,b)$ and $N(a,b)$ are as in Theorem 1.1. If f is h-convex (concave) and g is s-concave (convex, respectively), then (37) is reversed.

Proof. If choosing $h_1(t) = h(t)$ and $h_2(t) = t^s$ for all $t \in [0,1]$ in Theorem 3.1, then we obtain the desired result. □

Proposition 4.2. Let $h : [0,1] \to \mathbb{R}$ be an integrable positive function satisfying $h(t) = h(1-t)$ for all $t \in [0,1/2]$. If $f : I \subset \mathbb{R} \to \mathbb{R}$ is h-convex and $g : I \to \mathbb{R}$ is P-function satisfying $fg \in L_1(I)$. Then, for all $a,b \in I$ with $a < b$, the following inequality holds

$$\begin{aligned} [M(a,b)+N(a,b)]\int_0^1 h(t)dt + \frac{1}{b-a}\int_a^b f(x)g(x)dx \\ \geq \frac{g(a)+g(b)}{b-a}\int_a^b f(x)dx + [f(a)+f(b)]\int_0^1 h(t)g(ta+(1-t)b)dt, \quad (38) \end{aligned}$$

where $M(a,b)$ and $N(a,b)$ are as in Theorem 1.1. If f is h-concave and g is P-function, then (38) is reversed.

Proof. If choosing $h_1(t) = h(t)$ and $h_2(t) = 1$ for all $t \in [0,1]$ in Theorem 3.1, then we obtain the desired result. □

Proposition 4.3. Let $h : [0,1] \to \mathbb{R}$ be a integrable positive function and $s \in (0,1]$ is a real number. If $f : I \subset [0,\infty) \to \mathbb{R}$ is h-convex (concave) and $g : I \to \mathbb{R}$ is s-convex (concave, respectively) satisfying $fg \in L_1(I)$. Then, for all $a,b \in I$ with $a < b$, the following inequality holds

$$\begin{aligned} \frac{1}{b-a}\int_a^b f(x)[g(x)+g(a+b-x)]dx + \frac{2^{s-1}}{h(1/2)} f\Big(\frac{a+b}{2}\Big) g\Big(\frac{a+b}{2}\Big) \\ \geq \frac{1}{b-a}\int_a^b \Big[f\Big(\frac{a+b}{2}\Big)\frac{g(x)}{h(1/2)} + 2^s g\Big(\frac{a+b}{2}\Big)f(x)\Big]dx. \quad (39) \end{aligned}$$

If f is h-convex (concave) and g is s-concave (convex, respectively), then (40) is reversed.

Proof. If choosing $h_1(t) = h(t)$ and $h_2(t) = t^s$ for all $t \in [0,1]$ in Theorem 3.2, then we obtain the desired result. □

Proposition 4.4. If $f : I \subset \mathbb{R} \to \mathbb{R}$ is convex and $g : I \to \mathbb{R}$ is P-function satisfying $fg \in L_1(I)$. Then, for all $a, b \in I$ with $a < b$, the following inequality holds

$$\frac{1}{b-a}\int_a^b f(x)[g(x)+g(a+b-x)]dx + f\Big(\frac{a+b}{2}\Big)g\Big(\frac{a+b}{2}\Big) \geq \frac{1}{b-a}\int_a^b \Big[2f\Big(\frac{a+b}{2}\Big)g(x)+g\Big(\frac{a+b}{2}\Big)f(x)\Big]dx. \quad (40)$$

If f is concave and g is P-function, then (40) is reversed.

Proof. If choosing $h_1(t) = t$ and $h_2(t) = 1$ for all $t \in [0, 1]$ in Theorem 3.2, then we obtain the desired result. □

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INTERSECTİONAL (α,A)-SOFT NEW-İDEALS İN PU-ALGEBRAS

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Abstract – The new notions, intersectional A-soft *new*-ideals and intersectional (α, A)-soft *new*-ideals in PU-algebras are introduced and their properties are investigated. The relations between an intersectional A-soft *new*-ideals and an intersectional (α, A)-soft *new*-ideals are provided. The homomorphic image of an intersectional (α, A)-soft *new*-ideals is studied.

Keywords – PU-algebra, Soft PU-algebra, fuzzy soft PU-algebra, homomorphic image of an intersectional (α, A)-soft *new*-ideals

1 Introduction

Imai and Is´eki [6] in 1966 introduced the notion of a BCK-algebra. Is´eki [7] introduced BCI-algebras as a super class of the class of BCK-algebras. In [4,5], Hu and Li introduced a wide class of abstract algebras, BCH-algebras. They are shown that the class of BCI-algebras is a proper subclass of the class of BCH-algebras. Megalai and Tamilarasi[15] introduced the notion of a TM-algebra, which is a generalization of BCK/BCI/BCH-algebras and several results are presented. Mostafa et al, in [17] introduced a new algebraic structure called PU-algebra, which is a dual for TM-algebra and they investigated severed basic properties. Moreover, they derived new view of several ideals on PU-algebra. The concept of fuzzy sets was introduced by Zadeh [22]. In 1991, Xi [20] applied the concept of fuzzy sets to BCI, BCK, MV -algebras. Since its inception, the theory of fuzzy sets, ideal theory and its fuzzification has been developed in many directions and applied to a

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wide variety of fields. Mostafa et al [18,19] introduced the notion of α-fuzzy and $(\tilde{\alpha},\alpha)$-cubic new-ideal of P U -algebra. They discussed the homomorphic image (pre image) of α-fuzzy and $(\tilde{\alpha},\alpha)$-cubic new-ideal of P U -algebra under homomorprhism of P U -algebras. Molodtsov [16] introduced the concept of soft set as a new mathematical tool for dealing with uncertainties that is free form the difficulties that have troubled the usual theoretical approaches. Maji et al [12,13,14] described the application of soft theory and studied several operations on the soft sets. Many Mathematicians have studied the concept of soft set of some algebraic structures. The algebraic structure of set theories dealing with uncertainties has been studied by some authors. Çagman et al. [1, 2, 3] introduced fuzzy parameterized (FP) soft sets and their related properties. They proposed a decision making method based on FP-soft set theory, and provided an example which shows that the method can be successfully applied to the problems that contain uncertainties. Jun [8] applied Molodtsovs notion of soft sets to the theory of BCK/BCI-algebras and introduced the notion of soft BCK/BCI-algebras and soft subalgebras and then investigated their basic properties. Jun and Park [9] dealt with the algebraic structure of BCK/BCI-algebras by applying soft set theory. They introduced the notion of soft ideals and idealistic soft BCK/BCI-algebras and gave several examples. Jun et al. [10] introduced the notion of soft p-ideals and p-idealistic soft BCI-algebras and investigated their basic properties. Using soft sets, they gave characterization of (fuzzy) p-ideals in BCI-algebras. Moreover, Jun et al. [11] applied a fuzzy soft set introduced by Maji et al. [12] as a generalization of the standard soft sets for dealing with several kinds of theories in BCK/BCI-algebras. They defined the notions of fuzzy soft BCK/BCI-algebras, (closed) fuzzy soft ideals, and fuzzy soft p-ideals, and investigated related properties. Yang et al. [21] introduced the concept of the interval-valued fuzzy soft set; they studied the algebraic properties of the concept and they analyzed a decision problem by using an interval-valued fuzzy soft set.

In this paper, we introduce the notions of soft PU-algebras , A-soft *new*-ideals , (α, A)-soft *new*-ideals and discuss various operations introduced in on these concepts. Using soft sets, we give characterizations of (α, A)-soft *new*-ideals in PU-algebras. The relations between A-soft *new*-ideals and (α, A)-soft *new*-ideals in PU-algebras is provided. The homomorphic image of an intersectional α-soft *new*-ideals are studied.

2 Preliminaries

Now, we will recall some known concepts related to PU-algebra from the literature, which will be helpful in further study of this article.

Definition 2.1. [17] A PU-algebra is a non-empty set X with a constant $0 \in X$ and a binary operation $*$ satisfying the following conditions:
(I) $0 * x = x$,
(II) $(x * z) * (y * z) = y * x$ for any $x, y, z \in X$.

On X we can define a binary relation "$\leq$" by: $x \leq y$ if and only if $y * x = 0$.

Example 2.2. [17] Let X = {0, 1, 2, 3, 4} be a set and $*$ is defined by

$*$	0	1	2	3	4
0	0	1	2	3	4
1	4	0	1	2	3
2	3	4	0	1	2
3	2	3	4	0	1
4	1	2	3	4	0

Then (X, $*$, 0) is a PU-algebra.

Proposition 2.3. [17] In a **PU**-algebra (X,$*$, 0) the following hold, for all x, y, z $\in X$

(a) $x * x = 0$.
(b) $(x * z) * z = x$.
(c) $x * (y * z) = y * (x * z)$.
(d) $x * (y * x) = y * 0$.
(e) $(x * y) * 0 = y * x$.
(f) If $x \leq y$, then $x * 0 = y * 0$.
(g) $(x * y) * 0 = (x * z) * (y * z)$.
(h) $x * y \leq z$ if and only if $z * y \leq x$.
(i) $x \leq y$ if and only if $y * z \leq x * z$.
(j) In a **PU**-algebra (X, $*$, 0) , the following are equivalent:

(1) $x = y$, (2) $x * z = y * z$, (3) $z * x = z * y$.

(k) The right and the left cancellation laws hold in X .
(l) $(z * x) * (z * y) = x * y$,
(m) $(x * y) * z = (z * y) * x$.
(n) $(x * y) * (z * u) = (x * z) * (y * u)$ for all x, y, z and u $\in X$.

Lemma 2.4.[17] If (X, $*$, 0) is a PU-algebra, then (X, $\leq$) is a partially ordered set.

Definition 2.5. [17] A non-empty subset S of a **PU**-algebra (X, $*$, 0) is called a sub-algebra of X if $x * y \in S$ whenever $x, y \in S$.

Definition 2.6. [17] A non-empty subset I of a **PU**-algebra (X, $*$, 0) is called a **new**-ideal of X if,
(i) $0 \in I$,
(ii) $(a * (b * x)) * x \in I$, for all a, b $\in$ I and $x \in X$.

Theorem 2.7[17] Any sub-algebra S of a **PU**-algebra X is a **new**-ideal of X.

Example 2.8[17] Let X = {0, a, b, c} be a set with $*$ is defined by the following table:

$*$	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

Then (X , $*$, 0) is a **PU**-algebra. It is easy to show that I_1 = {0, a}, I_2 = {0, b}, I_3 = {0, c} are **new-**ideals of X.

3 Basic Results on Soft Sets

Molodtsov [16] defined the notion of a soft sets as follows. Let U be an initial universe and E be the set of parameters. The parameters are usually "attributes, characteristics or properties of an object". Let P(U) denote the power set of U and A is a subset of E.

Definition 3.1 [16]**.** A pair (F , A) is called a soft set over U, where F is a mapping given by F : A → P(U).In other words, a soft set over a universe is a U parameterized family of subsets of the universe U . For $e \in A$, $F(e)$ may be considered as the set of e-elements or e- approximate elements of the soft set (F, A).Thus $(F, A) = \{F(e) \in P(U) : e \in A \subseteq E\}$.

Definition 3.2 [12]**.** Let (F , C) and (G, D) be two soft sets over a common universe U. The soft set (F , C) is called a soft subset of (G, D), if C ⊆ D and for all ε ∈ C, $F(\varepsilon) \subseteq G(\varepsilon)$. This relationship is denoted by (F , C)$\subseteq$ (G, D). Similarly (F , C) is called a soft superset of (G, D), if (G, D) is soft subset of (F , C). This relationship is denoted by (F , C)$\supseteq$ (G, D). Two soft sets (F , C) and (G, D) over U are said to be equal, if (F , C) is a soft subset of (G, D) and (G, D) is a soft subset of (F , C).

Definition 3.3 [12]**.** Let (F , C) and (G, D) be any two soft sets over U.

(1) The intersection (H, E) of two soft sets (F , C) and (G, D) is defined as the soft set (H, E) = (F , C) $\tilde{\bigcap}$ (G, D), where E = C ∩ D and for all ε ∈ C□

$$H(C) = \begin{cases} F(\varepsilon) & if \ \varepsilon \in C \setminus D \\ G(\varepsilon) & if \ \varepsilon \in D \setminus C \\ F(\varepsilon) \bigcap G(\varepsilon) & if \ \varepsilon \in C \bigcap D \end{cases}$$

(2) The union (H, E) of two soft sets (F , C) and (G, D) is defined as the soft set (H, E) = (F , C) $\tilde{\bigcup}$ (G, D), where E = C ∪ D and for all ε ∈ C

$$H(C)=\begin{cases} F(\varepsilon) & if \ \varepsilon \in C \setminus D \\ G(\varepsilon) & if \ \ \varepsilon \in D \setminus C \\ F(\varepsilon)\bigcup G(\varepsilon) & if \ \varepsilon \in C \bigcap D \end{cases}$$

(3) The AND operation (F , C) AND (G, D) of two soft sets (F , C) and (G, D) is defined as the soft set (H, E) = (F , C) $\tilde{\wedge}$ (G, D), where H[α, β] = F [α] ∩ G[β] for all

(α, β) ∈ C × D.

(4) The OR operation (F , C) OR (G, D) of two soft sets (F , C) and (G, D) is defined as the soft set (H, E) = (F , C) $\tilde{\vee}$ (G, D), where H[α, β] = F [α] ∪ G[β] for all

(α, β) ∈ C × D.

Definition 3.4 [12]. Let (F , C) and (G, D) be two soft sets over U. Then,

(1) The $\wedge$-intersection of two soft sets (F , C) and (G, D) is defined as the soft set (H, E) = (F , C) $\wedge$ (G, D) over U, where E = C × D, where H[α, β] = F [α] ∩ G[β] for all (α, β) ∈ C × D.

(2) The $\vee$-union of two soft sets (F , C) and (G, D) is defined as the soft set (H, E) = (F , C) $\vee$ (G, D) over U, where E = C × D, where H[α, β] = F [α] ∪ G[β] for all (α, β) ∈ C × D.

(3) Let (F , C) and (G, D) be two soft sets over G and K, respectively.The Cartesian product of the soft sets (F , C) and (G, D), denoted by (F , C) × (G, D), is defined as (F , C) × (G, D) = (U, A × B), where U[α, β] = F [α] × G[β] for all (α, β) ∈ C × D.

4 Soft PU-algebras

In this section, we introduce the notion of soft PU-algebras. Let X and A be a PU-algebra and a nonempty set, respectively. A pair (F, A) is called a soft set over X if and only if F is a mapping from a set of A into the power set of X . That is, $F : A \to P(X)$ such that $F(x)=\phi$ if $x \notin A$. A soft set over X can be represented by the set of ordered pairs

$$\{(x, F(x)) : x \in A, F(x) \in P(X)\}.$$

It is clear to see that a soft set is a parameterized family of subsets of the set X .

Definition 4.1. Let (F, A) be a soft set over X . Then (F, A) is called a soft PU-algebra over X , if $F(x)$ is a **new**-ideal of X , for all $x \in A$.

Example 4.2. Let X = {0, a, b, c} be a set in Example 2.8. Define a mapping $F : X \to P(X)$ by: F (0) = {0}, F (a) = {0, a}, F (b) = {0, b} and F (c) = {0, c}. It is clear that (F, X) is a soft PU-algebra over X .

Definition 4.3. Let (F,A) and (G,B) be two soft PU-algebras over X. Then (F,A) is called a soft PU-subalgebra of (G,B), denoted by $(F,A) \prec (G,B)$, if it satisfies:
(i) $A \subset B$,
(ii) $F(x)$ is subalgebra of $G(x)$, for all $x \in A$.

Proposition 4.4. A soft set (F,A) over X is a soft PU-algebra, if and only if each $\Phi \neq F\ [\varepsilon]$ is a **new-**ideal of X, for all $\varepsilon \in A$.

Proof. Let (F,A) be a soft PU-algebra over X. Then by above definition, $F\ [\varepsilon]$ is a **new-**ideal of X, for all $\varepsilon \in A$. It follows that for all $\varepsilon \in A$, $F\ [\varepsilon] \neq \Phi$ is a **new-**ideal of X. Conversely, let us consider that (F,A) is a soft set over X such that for all $\varepsilon \in A$, $F\ [\varepsilon] \neq \Phi$ is a **new-**ideal of X, whenever $F\ [\varepsilon] \neq \Phi$. Since $F\ [\varepsilon]$ is a **new-**ideal of X. Hence (F,A) is a soft PU-algebra over X.

Theorem 4.5. Let (F,A) and (G,B) be two soft PU-algebras over X. If $A \cap B \neq \phi$,then the intersection $(F,A) \tilde{\cap} (G,B)$ is a soft PU-algebra over X.

Proof. We can write $(F,A) \tilde{\cap} (G,B) = (H,E)$, where $E = A \cap B$ and for all $\varepsilon \in E$, it is defined as

$$H(\varepsilon) = \begin{cases} F(\varepsilon) & \text{if } \varepsilon \in A \setminus B \\ G(\varepsilon) & \text{if } \varepsilon \in B \setminus A \\ F(\varepsilon) \cup G(\varepsilon) & \text{if } \varepsilon \in A \cap B \end{cases}$$

Since for all $\varepsilon \in E$ either $\varepsilon \in A \setminus B$ or $\varepsilon \in B \setminus A$, $\varepsilon \in A \cap B$. If $\varepsilon \in A \setminus B$, then H[ε]=$F$ [ε]. As F [ε] is a new-ideal over X, then H[ε] is a new-ideal over X. If $\varepsilon \in B \setminus A$, then H[ε] = G[ε]. As G[ε] is a new-ideal over X, then H[ε] is a new-ideal over X. If, then H[ε] = F [ε] ∩ G[ε]. As F [ε] and G[ε] are both new-ideals over X, then H[ε] is a new-ideal over X. In all cases, H[ε] is a new-ideal over X. Hence (H, E) = (F, A) $\tilde{\cap}$ (G, B) is a soft PU-algebra over X. □

5 Intersectional (α,A)-soft New PU- ideals

In what follow let X and A be a PU -algebra and a non empty set, respectively.

Definition 5.1. Let $E = X$ be a PU-algebra. Given a subalgebra A of E,let F_A be an A-soft set over U. Then F_A is called an intersectional A-soft PU-subalgebra over U if it satisfies the following condition:

$$F_A(x * y) \supseteq F_A(x) \cap F_A(y), \text{ for all } x, y \in X.$$

Definition 5.2 Let (X, $*$, 0) be a **PU**-algebra. F_A is called intersectional A -soft **new**-ideal over U if it satisfies the following conditions:

(F_1) $F_A(0) \supseteq F_A(x)$,
(F_2) $F_A((x*(y*z))*z) \supseteq F_A(x) \cap F_A(y)$, for all $x, y, z \in X$.

Example 5.3 . Consider the PU-algebra $(Z;*,0)$ as the initial universeset U , where $a*b = b-a \ \forall a,b \in Z$. Let $E = X = \{0,a,b,c\}$ be a PU-algebra with the following Cayley table:

Define a soft set (F_A, X) over U by

$$F_X : X \to P(U) \quad x \mapsto \begin{cases} Z & if \ x \in \{0,a\} \\ 2Z & if \ x \in \{b,c\} \end{cases}$$

Then F_A is an intersectional A-soft (subalgebra) new ideal over U .

Definition 5.4 [3] . The complement of a soft set (F, A) is denoted by (F^C, A) and is defined by $(F, A)^C$, where $F^C : A \to P(U)$ is a mapping given by

$$F^C(x) = U - F(x) \ \forall \ x \in X \ .$$

Definition 5.5 Let F_A be an intersectional A-soft PU- subalgebra over U and $\alpha \in \bigcap_{x \in X} F_A^C \ (x)$.Then F_A^{α} is called intersectional (α, A)-soft PU- subalgebra over U (w.r.t. F_A) and is defined by $F_A^{\alpha}(x) = F_A(x) \cup \alpha$, for all $x \in X$.

Lemma 5.6 If F_A is intersectional A-soft PU- subalgebra over U and $\alpha \in \bigcap_{x \in X} F_A^C \ (x)$, then $F_A^{\alpha}(x*y) \supseteq F_A^{\alpha}(x) \cap F_A^{\alpha}(y)$, for all $x, y \in X$.

Proof: Let X be a **PU**-algebra and $\alpha \in \bigcap_{x \in X} F_A^C \ (x)$.Then by Definitions (5.1, 5.5), we have

$$\begin{aligned} F_A^{\alpha}(x*y) = F_A(x*y) \cup \alpha\} &\supseteq \{F_A(x) \cap F_A(y)\} \cup \alpha \\ &= \{F_A(x) \cup \alpha\} \cap \{F_A(y) \cup \alpha\} \\ &= F_A^{\alpha}(x) \cap F_A^{\alpha}(y), \text{for all } x, y \in X. \end{aligned}$$

Definition 5.7 Let X be a **PU**-algebra, F_A^{α} is called intersectional (α, A)-soft PU-subalgebra of X if $F_A^{\alpha}(x*y) \supseteq F_A^{\alpha}(x) \cap F_A^{\alpha}(y)\}$, for all $x, y \in X$.

It is clear that intersectional (α, A)-soft PU- subalgebra over U is a generalization of intersectional A-soft PU- subalgebra over U.

Example 5.9 Let X = {0, 1, 2, 3} in which $*$ is defined by the following table:

$*$	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

Then (X, $*$, 0) is a **PU**-algebra. Define an $F_A{}^{\alpha}$ by

$$F_X{}^{\alpha}(x) = \begin{cases} Z_4 \cup \{3\} & if \quad x \in \{0,1\} \\ Z_8 \cup \{3\} & otherwise \end{cases}$$

Routine calculations give that $F_X{}^{\alpha}$ is an intersectional (α, A)-soft subalgebra over U.

Lemma 5.10 Let $F_A{}^{\alpha}$ be an intersectional (α, A)-soft new ideal over U. If the inequality $x * y \le z$ holds in X, then $F_A{}^{\alpha}(y) \supseteq F_A{}^{\alpha}(x) \cap F_A{}^{\alpha}(z)$.

Proof: Assume that the inequality $x * y \le z$ holds in X, then $z * (x * y) = 0$ and by (F_2^{α}) $F_A{}^{\alpha}(\overbrace{(z * (x * y))}^{0} * y) \supseteq F_A{}^{\alpha}(x) \cap F_A{}^{\alpha}(z)$. Since $F_A{}^{\alpha}(y) = F_A{}^{\alpha}(0 * y)$, then we have that $F_A{}^{\alpha}(y) \supseteq F_A{}^{\alpha}(x) \cap F_A{}^{\alpha}(z)$.

Corollary 5.11 Let F_A be intersectional A-soft new ideal over U. If the inequality $x * y \le z$ holds in X, then $F_A(y) \supseteq F_A(x) \cap F_A(z)$.

Lemma 5.12 If $F_A{}^{\alpha}$ is intersectional (α, A)-soft new ideal over U and if $x \le y$, then $F_A{}^{\alpha}(x) = F_A{}^{\alpha}(y)$.

Proof: If $x \le y$, then $y * x = 0$. Hence by the definition of **PU**-algebra and its properties we have $F_A{}^{\alpha}(x) = F_A(x) \cup \alpha = F_A(0 * x) \cup \alpha = F_A((y * x) * x) \cup \alpha = F_A(y) \cup \alpha\} = F_A{}^{\alpha}(y)$.

Corollary 5.13 If F_A is intersectional A-soft PU-new ideal over U and if $x \le y$, then $F_A(x) = F_A(y)$.

Lemma 5.14 If F_A is intersectional A-soft new ideal over U and $\alpha \in \bigcap_{x \in X} F_A^C(x)$,then

(F_1^{α}) $F^{\alpha}(0) \supseteq F^{\alpha}(x)$,

(F_2^{α}) $F^{\alpha}((x * (y * z)) * z) \supseteq F^{\alpha}(x) \cap F^{\alpha}(y)$, for all $x, y, z \in X$.

Proof: Let X be a **PU**-algebra and $\alpha \in \bigcap_{x\in X} F_A^C(x)$.Then by Definitions (5.2, 5.5), we have:

$$F_A{}^{\alpha}(0) = F_A(0) \bigcup \alpha \supseteq F_A(x) \bigcup \alpha = F_A{}^{\alpha}(x), \text{ for all } x \in X.$$

$$\begin{aligned} F_A{}^{\alpha}((x*(y*z))*z) &= F_A((x*(y*z))*z) \bigcup \alpha \\ &\supseteq \{F_A(x) \bigcap F_A(y)\} \bigcup \alpha\} \\ &= \{F_A(x) \bigcup \alpha\} \bigcap \{F_A(y) \bigcup \alpha\}\} \\ &= \{F_A{}^{\alpha}(x) \bigcap F_A{}^{\alpha}(y)\}, \text{ for all } x, y, z \in X. \end{aligned}$$

Definition 5.8 Let X be a **PU**-algebra, $F_A{}^{\alpha}$ is called intersectional (α, A)-soft new ideal of X if

$(F_1 I)$ $F_A^{\alpha}(0) \supseteq F_A^{\alpha}(x)$,

$(F_2 I)$ $F_A^{\alpha}((x*(y*z))*z) \supseteq F_A^{\alpha}(x) \bigcap F_A^{\alpha}(y)$, for all $x, y, z \in X$.

Remark. In what follows, denote by S(U) the set of all soft sets over U by C¸ aˇgman et al. [2,3].

Definition 5.15 Let f be a mapping from X to Y, F_X , $F_Y \in S(U)$

(1) The soft set $f^{-1}(F_Y) = \left\{ (x, f^{-1}(F_Y)(x) : x \in X, f^{-1}(F_Y)(x) \in P(U) \right\}$, where $f^{-1}(F_Y)(x) = F_Y(f(x))$,is called the soft pre-image of F_Y under f.

(2) The soft se $f\ (F_X) = \left\{ (y, f(F_X)(y) : y \in Y, f(F_X)(y) \in P(U) \right\}$
where

$$f(F_X)(y) = \begin{cases} \bigcup_{x \in f^{-1}(y)} F_X(x) & \text{if } f^{-1}(y) \neq \Phi \\ \Phi & \text{otherwise} \end{cases}$$

is called the soft image of F_X under f.

Proposition 5.16. For any PU-algebras X and Y, let $f : X \to Y$ be a function. Then

$$(\forall F_X \in S(U))\ (F_X^{\alpha} \tilde{\subseteq} f^{-1}(f(F_X)) \qquad \textbf{(*)}$$

Proof: Since $f^{-1}(f(x)) \neq \Phi \quad \forall\ x \in X$. Hence

$$F_X^{\alpha}(x) \subseteq \bigcup_{\beta \in f^{-1}(f(x))} F_X^{\alpha}(\beta) = f(F_X^{\alpha})(f(x)) = f^{-1}(f(F_X^{\alpha}))(x) \quad \forall x \in X\ ,$$

and therefore (*) is valid.

Theorem 5.17 Let $(X,*,0)$ and $(Y,*^{\backslash},0^{\backslash})$ be **PU**-algebras and $f: X \rightarrow Y$ be a homomorphism. If $F_Y^{\alpha} \in S(U)$ is α-intersectional A-soft PU-new ideal over Y, then the soft pre-imag $(f^{-1})(F_Y^{\alpha})$ of F_Y^{α} is intersectional (α, A)-soft PU-new ideal over Y, of X.

Proof: Since $((f^{-1}))(F_Y^{\alpha})$ is soft pre-image of F_Y under f. then $f^{-1}(F_Y^{\alpha})(x) = F_Y^{\alpha}(f(x))$ for all $x \in X$. Let $x \in X$, then $((f^{-1}))(F_Y^{\alpha})(0) \supseteq ((f^{-1}))(F_Y^{\alpha})(x)$

Now let $x, y, z \in X$, then

$$\begin{aligned}((f^{-1}))(F_Y^{\alpha})((x*(y*z))*z) &= F_Y^{\alpha}(f((x*(y*z))*z)) \\ &= F_Y^{\alpha}(f(x*(y*z))*^{\backslash} f(z)) \\ &= F_Y^{\alpha}((f(x)*^{\backslash} f(y*z))*^{\backslash} f(z)) \\ &= F_Y^{\alpha}((f(x)*^{\backslash} (f(y)*^{\backslash} f(z)))*^{\backslash} f(z)) \\ &\supseteq F_Y^{\alpha}(f(x)) \cap F_Y^{\alpha}(f(y))\} \\ &= ((f^{-1}))(F_Y^{\alpha})(x) \cap ((f^{-1}))(F_Y^{\alpha})(y)\end{aligned}$$

and the proof is completed.

Theorem 5.18 Let $(X,*,0)$ and $(Y,*^{\backslash},0^{\backslash})$ be **PU**-algebras, $f: X \rightarrow Y$ be a injective homomorphism, F_X^{α} be an (α, X)-intersectional soft of X, $f(F_X^{\alpha})$ be the image of F_X^{α} under f and $F_X^{\alpha}(y) = F_X^{\alpha}(f(x))$ for all $x \in X$. If F_X^{α} is an (α, X)-intersectional soft new ideal over X, then $f(F_X^{\alpha})$ is an (α, Y)-intersectional soft new ideal over Y.

Proof: If at least one of $f^{-1}(x^{\backslash})$ and $f^{-1}(y^{\backslash})$, $f^{-1}(z^{\backslash})$ is empty, then

$$f(F_X{}^{\alpha}((x^{\backslash} *^{\backslash} (y^{\backslash} *^{\backslash} z^{\backslash})) *^{\backslash} z^{\backslash})) \supseteq f(F_X{}^{\alpha}(x^{\backslash})) \cap f(F_X{}^{\alpha}(y^{\backslash})\}.$$

is clear.

Assume that $f^{-1}(x^{\backslash}) \neq \phi$, $f^{-1}(y^{\backslash}) \neq \phi$ and $f^{-1}(z^{\backslash}) \neq \phi$.Then

$$f(F_X{}^{\alpha}(x^{\backslash})) \cap f(F_X{}^{\alpha}(y^{\backslash})) = \Big(\bigcup_{x_1 \in f^{-1}(x^{\backslash})} F_X{}^{\alpha}(x_1)\Big) \cap \Big(\bigcup_{y_1 \in f^{-1}(y^{\backslash})} F_X{}^{\alpha}(y_1)\Big) =$$

$$\bigcup_{\substack{x_1 \in f^{-1}(x^{\backslash}), \\ y_1 \in f^{-1}(y^{\backslash})}} (F_X{}^{\alpha}(x_1)) \cap (F_X{}^{\alpha}(y_1)) \subseteq \bigcup_{\substack{x_1 \in f^{-1}(x^{\backslash}), \\ y_1 \in f^{-1}(y^{\backslash}) \\ z_1 \in f^{-1}(z^{\backslash})}} F_X{}^{\alpha}(x_1 * (y_1 * z_1)) * z_1) = \bigcup_{x \in f^{-1}((x^{\backslash} * (y^{\backslash} * z^{\backslash})) * z^{\backslash})} F_X{}^{\alpha}(x)$$

Therefore $f(F_X{}^{\alpha})$ is a (α, Y)-intersectional soft new ideal.

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$\mathcal{I}^*_{*g}$-CLOSED SETS

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Abstaract — In this paper, we introduce the notion of $\mathcal{I}^*_{*g}$-closed sets and prove that this class of sets is stronger than the class of gs$^*_{\mathcal{I}}$-closed sets as well as the class of $\mathcal{I}_g$-closed sets. Characterizations and properties of $\mathcal{I}^*_{*g}$-closed sets and $\mathcal{I}^*_{*g}$-open sets are given. A characterization of normal spaces is given in terms of $\mathcal{I}^*_{*g}$-open sets.

Keywords — **g-closed set, $\mathcal{I}^*_{*g}$-closed set, gs$^*_{\mathcal{I}}$-closed set, weakly $\mathcal{I}_{rg}$-closed set*

1 Introduction and Preliminaries

An ideal $\mathcal{I}$ on a topological space (X, τ) is a nonempty collection of subsets of X which satisfies

(i) A $\in \mathcal{I}$ and B $\subseteq$ A $\Rightarrow$ B $\in \mathcal{I}$ and
(ii) A $\in \mathcal{I}$ and B $\in \mathcal{I} \Rightarrow$ A $\cup$ B $\in \mathcal{I}$ [12].

Given a topological space (X, τ) with an ideal $\mathcal{I}$ on X and if $\wp$(X) is the set of all subsets of X, a set operator $(.)^*$: $\wp$(X) $\rightarrow \wp$(X), called a local function [12] of A with respect to τ and $\mathcal{I}$ is defined as follows: for A $\subseteq$ X, $A^*(\mathcal{I},\tau)$={x $\in$ X | U $\cap$ A $\notin \mathcal{I}$ for every U $\in \tau$(x)} where τ(x)={U $\in \tau$ | x $\in$ U}. We will make use of the basic facts about the local functions [[11], Theorem 2.3] without mentioning it explicitly. A Kuratowski closure operator cl*(.) for a topology $\tau^*(\mathcal{I},\tau)$, called the $\star$-topology, finer than τ is defined by cl*(A)=A $\cup$ $A^*(\mathcal{I},\tau)$ [22]. When there is no chance for confusion, we will simply write A^* for $A^*(\mathcal{I},\tau)$ and τ^* for $\tau^*(\mathcal{I},\tau)$. If $\mathcal{I}$ is an ideal on X, then (X, τ, $\mathcal{I}$) is called an ideal space or an ideal topological space. N is the ideal of all nowhere dense subsets in (X, τ). A subset A of an ideal space (X, τ, $\mathcal{I}$) is $\star$-closed [11] (resp. $\star$-dense in itself [9]) if $A^* \subseteq$ A (resp. A $\subseteq A^*$). A subset A

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of an ideal space (X, τ, $\mathcal{I}$) is $\mathcal{I}_g$-closed [2, 16] if $A^* \subseteq U$ whenever $A \subseteq U$ and U is open.

By a space, we always mean a topological space (X, τ) with no separation properties assumed. If $A \subseteq X$, cl(A) and int(A) will, respectively, denote the closure and interior of A in (X, τ) and int*(A) will denote the interior of A in (X, τ^*).

A subset A of a space (X, τ) is an α-open [19] (resp. regular open [21], semi-open [13], preopen [15]) set if $A \subseteq$ int(cl(int(A))) (resp. A = int(cl(A)), $A \subseteq$ cl(int(A)), A $\subseteq$ int(cl(A))). The family of all α-open sets in (X, τ), denoted by τ^α, is a topology on X finer than τ. The closure of A in (X, τ^α) is denoted by cl_α(A). A subset A of a space (X, τ) is said to be g-closed [14] if cl(A) $\subseteq$ U whenever A $\subseteq$ U and U is open. A subset A of a space (X, τ) is said to be $\hat{g}$-closed [23] if cl(A) $\subseteq$ U whenever A $\subseteq$ U and U is semi-open. A subset A of a space (X, τ) is said to be $\hat{g}$-open [23] if its complement is $\hat{g}$-closed. A subset A of a topological space (X, τ) is said to be *g-closed [10] if cl(A) $\subseteq$ U whenever A $\subseteq$ U and U is $\hat{g}$-open in X. The complement of *g-closed set is said to be *g-open. The intersection of all *g-closed sets of X containing a subset A of X is denoted by $^*g$cl(A). An ideal $\mathcal{I}$ is said to be codense [3] or τ-boundary [18] if $\tau \cap \mathcal{I}=\{\emptyset\}$. $\mathcal{I}$ is said to be completely codense [3] if PO(X) $\cap\, \mathcal{I}=\{\emptyset\}$, where PO(X) is the family of all preopen sets in (X, τ). Every completely codense ideal is codense but not the converse [3].

The following Lemmas will be useful in the sequel.

Lemma 1.1. *Let (X, τ, $\mathcal{I}$) be an ideal space and $A \subseteq X$. If $A \subseteq A^*$, then A^* =cl(A^*) =cl(A)=cl*(A) [[20], Theorem 5].*

Lemma 1.2. *Let (X, τ, $\mathcal{I}$) be an ideal space. Then $\mathcal{I}$ is codense if and only if $G \subseteq G^*$ for every semi-open set G in X [[20], Theorem 3].*

Lemma 1.3. *Let (X, τ, $\mathcal{I}$) be an ideal space. If $\mathcal{I}$ is completely codense, then $\tau^* \subseteq \tau^\alpha$ [[20], Theorem 6].*

Result 1.4. *If (X, τ) is a topological space, then every closed set is *g-closed but not conversely [10].*

Lemma 1.5. *If (X, τ, $\mathcal{I}$) is a $T_\mathcal{I}$ ideal space and A is an $\mathcal{I}_g$-closed set, then A is a $\star$-closed set [[16], Corollary 2.2].*

Lemma 1.6. *Every g-closed set is $\mathcal{I}_g$-closed but not conversely [[2], Theorem 2.1].*

Definition 1.7. *A subset G of an ideal topological space (X, τ, $\mathcal{I}$) is said to be*

1. *$\mathcal{I}_g$-closed [2] if $G^* \subseteq H$ whenever $G \subseteq H$ and H is open in (X, τ, $\mathcal{I}$).*
2. *$\mathcal{I}_{rg}$-closed [17] if $G^* \subseteq H$ whenever $G \subseteq H$ and H is regular open in (X, τ, $\mathcal{I}$).*
3. *pre$^*_\mathcal{I}$-open [4] if $G \subseteq$ int*(cl(G)).*
4. *pre$^*_\mathcal{I}$-closed [4] if $X \backslash G$ is pre$^*_\mathcal{I}$-open.*
5. *$\mathcal{I}$-R closed [1] if G = cl*(int(G)).*
6. *$*$-closed [11] if G = cl*(G) or $G^* \subseteq G$.*

Remark 1.8. *[5] In any ideal topological space, every $\mathcal{I}$-R closed set is $*$-closed but not conversely.*

Definition 1.9. *[5] Let $(X, \tau, \mathcal{I})$ be an ideal topological space. A subset G of X is said to be a weakly $\mathcal{I}_{rg}$-closed set if $(int(G))^* \subseteq H$ whenever $G \subseteq H$ and H is a regular open set in X.*

Definition 1.10. *[5] Let $(X, \tau, \mathcal{I})$ be an ideal topological space. A subset G of X is said to be a weakly $\mathcal{I}_{rg}$-open set if $X \backslash G$ is a weakly $\mathcal{I}_{rg}$-closed set.*

Remark 1.11. *[5] Let $(X, \tau, \mathcal{I})$ be an ideal topological space. The following diagram holds for a subset $G \subseteq X$:*

$$\begin{array}{ccccc}
\mathcal{I}_g\text{-closed} & \longrightarrow & \mathcal{I}_{rg}\text{-closed} & \longrightarrow & \text{weakly } \mathcal{I}_{rg}\text{-closed} \\
\uparrow & & & & \uparrow \\
*\text{-closed} & & & & pre^*_{\mathcal{I}}\text{-closed} \\
\uparrow & & & & \\
\mathcal{I}\text{-}R\text{-closed} & & & &
\end{array}$$

These implications are not reversible.

Definition 1.12. *[7, 8] A subset K of an ideal topological space $(X, \tau, \mathcal{I})$ is said to be*

1. *semi*-$\mathcal{I}$-open if $K \subseteq cl(int^*(K))$,*
2. *semi*-$\mathcal{I}$-closed if its complement is semi*-$\mathcal{I}$-open.*

Definition 1.13. *[7] The semi*-$\mathcal{I}$-closure of a subset K of an ideal topological space $(X, \tau, \mathcal{I})$, denoted by $s^*_{\mathcal{I}}cl(K)$, is defined by the intersection of all semi*-$\mathcal{I}$-closed sets of X containing K.*

Theorem 1.14. *[7] For a subset K of an ideal topological space $(X, \tau, \mathcal{I})$, $s^*_{\mathcal{I}}cl(K) = K \cup int(cl^*(K))$.*

Definition 1.15. *[6] Let $(X, \tau, \mathcal{I})$ be an ideal topological space and $K \subseteq X$. K is called*

1. *generalized semi*-$\mathcal{I}$-closed ($gs^*_{\mathcal{I}}$-closed) in $(X, \tau, \mathcal{I})$ if $s^*_{\mathcal{I}}cl(K) \subseteq O$ whenever $K \subseteq O$ and O is an open set in $(X, \tau, \mathcal{I})$.*
2. *generalized semi*-$\mathcal{I}$-open ($gs^*_{\mathcal{I}}$-open) in $(X, \tau, \mathcal{I})$ if $X \backslash K$ is a $gs^*_{\mathcal{I}}$-closed set in $(X, \tau, \mathcal{I})$.*

2 $\mathcal{I}^*_{*g}$-closed Sets

Definition 2.1. *A subset A of an ideal space $(X, \tau, \mathcal{I})$ is said to be $\mathcal{I}^*_{*g}$-closed if $A^* \subseteq U$ whenever $A \subseteq U$ and U is $^*g$-open.*

Definition 2.2. *A subset A of an ideal space $(X, \tau, \mathcal{I})$ is said to be $\mathcal{I}^*_{*g}$-open if $X-A$ is $\mathcal{I}^*_{*g}$-closed.*

Theorem 2.3. *If $(X, \tau, \mathcal{I})$ is any ideal space, then every $\mathcal{I}^*_{*g}$-closed set is $\mathcal{I}_g$-closed but not conversely.*

Proof. It follows from the fact that every open set is *g-open. □

Example 2.4. *If $X = \{a, b, c\}$, $\tau = \{\emptyset, X, \{c\}\}$ and $\mathcal{I} = \{\emptyset, \{a\}\}$, then $\{b\}$ is $\mathcal{I}_g$-closed set but not $\mathcal{I}^*_{*g}$-closed set.*

The following Theorem gives characterizations of $\mathcal{I}^*_{*g}$-closed sets.

Theorem 2.5. *If $(X, \tau, \mathcal{I})$ is any ideal space and $A \subseteq X$, then the following are equivalent.*
(a) A is $\mathcal{I}^*_{*g}$-closed.
*(b) $cl^*(A) \subseteq U$ whenever $A \subseteq U$ and U is *g-open in X.*
(c) For all $x \in cl^*(A)$, ${}^*gcl(\{x\}) \cap A \neq \emptyset$.
*(d) $cl^*(A) - A$ contains no nonempty *g-closed set.*
*(e) $A^* - A$ contains no nonempty *g-closed set.*

Proof. (a) ⇒ (b) If A is $\mathcal{I}^*_{*g}$-closed, then $A^* \subseteq U$ whenever $A \subseteq U$ and U is *g-open in X and so $cl^*(A) = A \cup A^* \subseteq U$ whenever $A \subseteq U$ and U is *g-open in X. This proves (b).

(b) ⇒ (c) Suppose $x \in cl^*(A)$. If ${}^*gcl(\{x\}) \cap A = \emptyset$, then $A \subseteq X - {}^*gcl(\{x\})$. By (b), $cl^*(A) \subseteq X - {}^*gcl(\{x\})$, a contradiction, since $x \in cl^*(A)$.

(c) ⇒ (d) Suppose $F \subseteq cl^*(A) - A$, F is *g-closed and $x \in F$. Since $F \subseteq X - A$, then $A \subseteq X - F$, ${}^*gcl(\{x\}) \cap A = \emptyset$. Since $x \in cl^*(A)$ by (c), ${}^*gcl(\{x\}) \cap A \neq \emptyset$. Therefore $cl^*(A) - A$ contains no nonempty *g-closed set.

(d) ⇒ (e) Since $cl^*(A) - A = (A \cup A^*) - A = (A \cup A^*) \cap A^c = (A \cap A^c) \cup (A^* \cap A^c) = A^* \cap A^c = A^* - A$, therefore $A^* - A$ contains no nonempty *g-closed set.

(e) ⇒ (a) Let $A \subseteq U$ where U is *g-open set. Therefore $X - U \subseteq X - A$ and so $A^* \cap (X - U) \subseteq A^* \cap (X - A) = A^* - A$. Therefore $A^* \cap (X - U) \subseteq A^* - A$. Since A^* is always closed set, so $A^* \cap (X - U)$ is a *g-closed set contained in $A^* - A$. Therefore $A^* \cap (X - U) = \emptyset$ and hence $A^* \subseteq U$. Therefore A is $\mathcal{I}^*_{*g}$-closed. □

Theorem 2.6. *Every $\star$-closed set is $\mathcal{I}^*_{*g}$-closed but not conversely.*

Proof. Let A be a $\star$-closed, then $A^* \subseteq A$. Let $A \subseteq U$ where U is *g-open. Hence $A^* \subseteq U$ whenever $A \subseteq U$ and U is *g-open. Therefore A is $\mathcal{I}^*_{*g}$-closed. □

Example 2.7. *If $X = \{a, b, c\}$, $\tau = \{\emptyset, X, \{a\}, \{b, c\}\}$ and $\mathcal{I} = \{\emptyset, \{c\}\}$, then $\{a, b\}$ is $\mathcal{I}^*_{*g}$-closed set but not $\star$-closed set.*

Theorem 2.8. *Let $(X, \tau, \mathcal{I})$ be an ideal space. For every $A \in \mathcal{I}$, A is $\mathcal{I}^*_{*g}$-closed.*

Proof. Let $A \subseteq U$ where U is *g-open set. Since $A^* = \emptyset$ for every $A \in \mathcal{I}$, then $cl^*(A) = A \cup A^* = A \subseteq U$. Therefore, by Theorem 2.5, A is $\mathcal{I}^*_{*g}$-closed. □

Theorem 2.9. *If $(X, \tau, \mathcal{I})$ is an ideal space, then A^* is always $\mathcal{I}^*_{*g}$-closed for every subset A of X.*

Proof. Let $A^* \subseteq U$ where U is *g-open. Since $(A^*)^* \subseteq A^*$ [11], we have $(A^*)^* \subseteq U$ whenever $A^* \subseteq U$ and U is *g-open. Hence A^* is $\mathcal{I}^*_{*g}$-closed. □

Theorem 2.10. *Let (X, τ, $\mathcal{I}$) be an ideal space. Then every $\mathcal{I}^*_{*g}$-closed, *g-open set is $\star$-closed set.*

Proof. Since A is $\mathcal{I}^*_{*g}$-closed and *g-open, then A* $\subseteq$ A whenever A $\subseteq$ A and A is *g-open. Hence A is $\star$-closed. □

Corollary 2.11. *If (X, τ, $\mathcal{I}$) is a $T_{\mathcal{I}}$ ideal space and A is an $\mathcal{I}^*_{*g}$-closed set, then A is $\star$-closed set.*

Corollary 2.12. *Let (X, τ, $\mathcal{I}$) be an ideal space and A be an $\mathcal{I}^*_{*g}$-closed set. Then the following are equivalent.*
a) A is a $\star$-closed set.
b) cl*(A)$-$A is a *g-closed set.
c) A$^*-$A is a *g-closed set.

Proof. (a) $\Rightarrow$ (b) If A is $\star$-closed, then A* $\subseteq$ A and so cl*(A)$-$A=(A $\cup$ A*)$-$A= $\emptyset$. Hence cl*(A)$-$A is *g-closed set.

(b) $\Rightarrow$ (c) Since cl*(A)$-$A=A$^*-$A and so A$^*-$A is *g-closed set.

(c) $\Rightarrow$ (a) If A$^*-$A is a *g-closed set, since A is $\mathcal{I}^*_{*g}$-closed set, by Theorem 2.5, A$^*-$A= $\emptyset$ and so A is $\star$-closed. □

Definition 2.13. *A subset A of a topological space (X, τ) is said to be $^*g^*$-closed if cl(A) $\subseteq$ U whenever A $\subseteq$ U and U is *g-open in X.*

Theorem 2.14. *Every closed set is $^*g^*$-closed but not conversely.*

Example 2.15. *In Example 2.7, {a, b} is $^*g^*$-closed set but not closed set.*

Theorem 2.16. *Every $^*g^*$-closed set is g-closed but not conversely.*

Proof. It follows from the fact that every open set is *g-open. □

Example 2.17. *In Example 2.4, {a} is g-closed set but not $^*g^*$-closed.*

Theorem 2.18. *Let (X, τ, $\mathcal{I}$) be an ideal space. Then every $^*g^*$-closed set is an $\mathcal{I}^*_{*g}$-closed set but not conversely.*

Proof. Let A be a $^*g^*$-closed set. Then cl(A) $\subseteq$ U whenever A $\subseteq$ U and U is *g-open. We have cl*(A) $\subseteq$ cl(A) $\subseteq$ U whenever A $\subseteq$ U and U is *g-open. Hence A is $\mathcal{I}^*_{*g}$-closed. □

Example 2.19. *In Example 2.4, {a} is $\mathcal{I}^*_{*g}$-closed set but not $^*g^*$-closed.*

Theorem 2.20. *If (X, τ, $\mathcal{I}$) is an ideal space and A is a $\star$-dense in itself, $\mathcal{I}^*_{*g}$-closed subset of X, then A is $^*g^*$-closed.*

Proof. Suppose A is a $\star$-dense in itself, $\mathcal{I}^*_{*g}$-closed subset of X. Let A $\subseteq$ U where U is *g-open. Then by Theorem 2.5 (b), cl*(A) $\subseteq$ U whenever A $\subseteq$ U and U is *g-open. Since A is $\star$-dense in itself, by Lemma 1.1, cl(A)=cl*(A). Therefore cl(A) $\subseteq$ U whenever A $\subseteq$ U and U is *g-open. Hence A is $^*g^*$-closed. □

Corollary 2.21. *If (X, τ, $\mathcal{I}$) is any ideal space where $\mathcal{I}$={$\emptyset$}, then A is $\mathcal{I}^*_{*g}$-closed if and only if A is $^*g^*$-closed.*

Proof. From the fact that for $\mathcal{I}$={∅}, A*=cl(A) ⊇ A. Therefore A is ⋆-dense in itself. Since A is $\mathcal{I}^*_{*g}$-closed, by Theorem 2.20, A is $^*g^*$-closed. Conversely, by Theorem 2.18, every $^*g^*$-closed set is $\mathcal{I}^*_{*g}$-closed set. □

Corollary 2.22. *If (X, τ, $\mathcal{I}$) is any ideal space where $\mathcal{I}$ is codense and A is a semi-open, $\mathcal{I}^*_{*g}$-closed subset of X, then A is $^*g^*$-closed.*

Proof. By Lemma 1.2, A is ⋆-dense in itself. By Theorem 2.20, A is $^*g^*$-closed. □

Example 2.23. *If* $X = \{a, b, c, d\}$, $\tau = \{\emptyset, X, \{a\}, \{a, b, c\}\}$ *and* $\mathcal{I} = \{\emptyset, \{a\}\}$, *then* $\{a\}$ *is* $\mathcal{I}^*_{*g}$*-closed set but not g-closed.*

Example 2.24. *In Example 2.4,* $\{b\}$ *is g-closed set but not* $\mathcal{I}^*_{*g}$*-closed.*

Remark 2.25. *By Example 2.23 and Example 2.24, g-closed sets and* $\mathcal{I}^*_{*g}$*-closed sets are independent.*

Example 2.26. *In Example 2.4,* $\{a\}$ *is ⋆-closed set but not* $^*g^*$*-closed.*

Example 2.27. *In Example 2.7,* $\{a, b\}$ *is* $^*g^*$*-closed set but not ⋆-closed.*

Remark 2.28. *By Example 2.26 and Example 2.27,* $^*g^*$*-closed sets and ⋆-closed sets are independent.*

Remark 2.29. *We have the following implications for the subsets stated above.*

$$\begin{array}{ccccc} \text{closed} & \rightarrow & ^*g^*\text{-closed} & \rightarrow & g\text{-closed} \\ \downarrow & & \downarrow & & \downarrow \\ \star\text{-closed} & \rightarrow & \mathcal{I}^*_{*g}\text{-closed} & \rightarrow & \mathcal{I}_g\text{-closed} \end{array}$$

Theorem 2.30. *Let (X, τ, $\mathcal{I}$) be an ideal space and A ⊆ X. Then A is* $\mathcal{I}^*_{*g}$*-closed if and only if A=F−N where F is ⋆-closed and N contains no nonempty *g-closed set.*

Proof. If A is $\mathcal{I}^*_{*g}$-closed, then by Theorem 2.5 (e), N=A*−A contains no nonempty *g-closed set. If F=cl*(A), then F is ⋆-closed such that $F-N=(A \cup A^*)-(A^*-A)=(A \cup A^*) \cap (A^* \cap A^c)^c=(A \cup A^*) \cap ((A^*)^c \cup A)=(A \cup A^*) \cap (A \cup (A^*)^c)=A \cup (A^* \cap (A^*)^c)=A$.

Conversely, suppose A=F−N where F is ⋆-closed and N contains no nonempty *g-closed set. Let U be a *g-open set such that A ⊆ U. Then F−N ⊆ U and F ∩ (X−U) ⊆ N. Now A ⊆ F and F* ⊆ F then A* ⊆ F* and so A* ∩ (X−U) ⊆ F* ∩ (X−U) ⊆ F ∩ (X−U) ⊆ N. By hypothesis, since A* ∩ (X−U) is *g-closed, A* ∩ (X−U)=∅ and so A* ⊆ U. Hence A is $\mathcal{I}^*_{*g}$-closed. □

Theorem 2.31. *Let (X, τ, $\mathcal{I}$) be an ideal space and A ⊆ X. If A ⊆ B ⊆ A*, then A*=B* and B is ⋆-dense in itself.*

Proof. Since A ⊆ B, then A* ⊆ B* and since B ⊆ A*, then B* ⊆ (A*)* ⊆ A*. Therefore A*=B* and B ⊆ A* ⊆ B*. Hence proved. □

Theorem 2.32. *Let (X, τ, $\mathcal{I}$) be an ideal space. If A and B are subsets of X such that A ⊆ B ⊆ cl*(A) and A is* $\mathcal{I}^*_{*g}$*-closed, then B is* $\mathcal{I}^*_{*g}$*-closed.*

Proof. Since A is $\mathcal{I}^*_{*g}$-closed, then by Theorem 2.5 (d), cl*(A)−A contains no nonempty *g-closed set. Since cl*(B)−B ⊆ cl*(A)−A and so cl*(B)−B contains no nonempty *g-closed set. Hence B is $\mathcal{I}^*_{*g}$-closed. □

Corollary 2.33. *Let (X, τ, $\mathcal{I}$) be an ideal space. If A and B are subsets of X such that $A \subseteq B \subseteq A^*$ and A is $\mathcal{I}^*_{*g}$-closed, then A and B are $^*g^*$-closed sets.*

Proof. Let A and B be subsets of X such that $A \subseteq B \subseteq A^* \Rightarrow A \subseteq B \subseteq A^* \subseteq cl^*(A)$ and A is $\mathcal{I}^*_{*g}$-closed. By the above Theorem, B is $\mathcal{I}^*_{*g}$-closed. Since $A \subseteq B \subseteq A^*$, then $A^*=B^*$ and so A and B are ⋆-dense in itself. By Theorem 2.20, A and B are $^*g^*$-closed. □

The following Theorem gives a characterization of $\mathcal{I}^*_{*g}$-open sets.

Theorem 2.34. *Let (X, τ, $\mathcal{I}$) be an ideal space and A ⊆ X. Then A is $\mathcal{I}^*_{*g}$-open if and only if F ⊆ int*(A) whenever F is *g-closed and F ⊆ A.*

Proof. Suppose A is $\mathcal{I}^*_{*g}$-open. If F is *g-closed and F ⊆ A, then X−A ⊆ X−F and so cl*(X−A) ⊆ X−F by Theorem 2.5 (b). Therefore F ⊆ X−cl*(X−A)=int*(A). Hence F ⊆ int*(A).

Conversely, suppose the condition holds. Let U be a *g-open set such that X−A ⊆ U. Then X−U ⊆ A and so X−U ⊆ int*(A). Therefore cl*(X−A) ⊆ U. By Theorem 2.5 (b), X−A is $\mathcal{I}^*_{*g}$-closed. Hence A is $\mathcal{I}^*_{*g}$-open. □

Corollary 2.35. *Let (X, τ, $\mathcal{I}$) be an ideal space and A ⊆ X. If A is $\mathcal{I}^*_{*g}$-open, then F ⊆ int*(A) whenever F is closed and F ⊆ A.*

Theorem 2.36. *Let (X, τ, $\mathcal{I}$) be an ideal space and A ⊆ X. If A is $\mathcal{I}^*_{*g}$-open and int*(A) ⊆ B ⊆ A, then B is $\mathcal{I}^*_{*g}$-open.*

Proof. Since A is $\mathcal{I}^*_{*g}$-open, then X−A is $\mathcal{I}^*_{*g}$-closed. By Theorem 2.5 (d), cl*(X−A)−(X−A) contains no nonempty *g-closed set. Since int*(A) ⊆ int*(B) which implies that cl*(X−B) ⊆ cl*(X−A) and so cl*(X−B)−(X−B) ⊆ cl*(X−A)−(X−A). Hence B is $\mathcal{I}^*_{*g}$-open. □

The following Theorem gives a characterization of $\mathcal{I}^*_{*g}$-closed sets in terms of $\mathcal{I}^*_{*g}$-open sets.

Theorem 2.37. *Let (X, τ, $\mathcal{I}$) be an ideal space and A ⊆ X. Then the following are equivalent.*
(a) A is $\mathcal{I}^*_{*g}$-closed.
(b) A ∪ (X−A) is $\mathcal{I}^*_{*g}$-closed.*
(c) A−A is $\mathcal{I}^*_{*g}$-open.*

Proof. (a) ⇒ (b) Suppose A is $\mathcal{I}^*_{*g}$-closed. If U is any *g-open set such that A ∪ (X−A*) ⊆ U, then $X-U \subseteq X-(A \cup (X-A^*))=X \cap (A \cup (A^*)^c)^c=A^* \cap A^c=A^*-A$. Since A is $\mathcal{I}^*_{*g}$-closed, by Theorem 2.5 (e), it follows that X−U=∅ and so X=U. Now A ∪ (X−A*) ⊆ X and so $(A \cup (X-A^*))^* \subseteq X^* \subseteq X=U$. Hence A ∪ (X−A*) is $\mathcal{I}^*_{*g}$-closed.

(b) ⇒ (a) Suppose A ∪ (X−A*) is $\mathcal{I}^*_{*g}$-closed. If F is any *g-closed set such that F ⊆ A*−A, then F ⊆ A* and F ⊈ A. Hence X−A* ⊆ X−F and A ⊆ X−F. Therefore

A $\cup$ (X$-$A*) $\subseteq$ A $\cup$ (X$-$F)= X$-$F and X$-$F is *g-open. Since (A $\cup$ (X$-$A*))* $\subseteq$ X$-$F $\Rightarrow$ A* $\cup$ (X$-$A*)* $\subseteq$ X$-$F and so A* $\subseteq$ X$-$F $\Rightarrow$ F $\subseteq$ X$-$A*. Since F $\subseteq$ A*, it follows that F=$\emptyset$. Hence A is $\mathcal{I}^*_{*g}$-closed.

(b) $\Leftrightarrow$ (c) Since X$-$(A*$-$A)=X $\cap$ (A* $\cap$ A^c)^c=X $\cap$ ((A*)^c $\cup$ A)=(X $\cap$ (A*)^c) $\cup$ (X $\cap$ A)=A $\cup$ (X$-$A*), it is obvious. □

Theorem 2.38. *Let (X, τ, $\mathcal{I}$) be an ideal space. Then every subset of X is $\mathcal{I}^*_{*g}$-closed if and only if every *g-open set is $\star$-closed.*

Proof. Suppose every subset of X is $\mathcal{I}^*_{*g}$-closed. If U $\subseteq$ X is *g-open, then U is $\mathcal{I}^*_{*g}$-closed and so U* $\subseteq$ U. Hence U is $\star$-closed. Conversely, suppose that every *g-open set is $\star$-closed. If U is *g-open set such that A $\subseteq$ U $\subseteq$ X, then A* $\subseteq$ U* $\subseteq$ U and so A is $\mathcal{I}^*_{*g}$-closed. □

The following Theorem gives a characterization of normal spaces in terms of $\mathcal{I}^*_{*g}$-open sets.

Theorem 2.39. *Let (X, τ, $\mathcal{I}$) be an ideal space where $\mathcal{I}$ is completely codense. Then the following are equivalent.*

(a) X is normal.

(b) For any disjoint closed sets A and B, there exist disjoint $\mathcal{I}^*_{*g}$-open sets U and V such that A $\subseteq$ U and B $\subseteq$ V.

(c) For any closed set A and open set V containing A, there exists an $\mathcal{I}^*_{*g}$-open set U such that A $\subseteq$ U $\subseteq$ cl*(U) $\subseteq$ V.

Proof. (a) $\Rightarrow$ (b) The proof follows from the fact that every open set is $\mathcal{I}^*_{*g}$-open.

(b) $\Rightarrow$ (c) Suppose A is closed and V is an open set containing A. Since A and X$-$V are disjoint closed sets, there exist disjoint $\mathcal{I}^*_{*g}$-open sets U and W such that A $\subseteq$ U and X$-$V $\subseteq$ W. Since X$-$V is *g-closed and W is $\mathcal{I}^*_{*g}$-open, X$-$V $\subseteq$ int*(W) and so X$-$int*(W) $\subseteq$ V. Again U $\cap$ W=$\emptyset$ $\Rightarrow$ U $\cap$ int*(W)=$\emptyset$ and so U $\subseteq$ X$-$int*(W) $\Rightarrow$ cl*(U) $\subseteq$ X$-$int*(W) $\subseteq$ V. U is the required $\mathcal{I}^*_{*g}$-open sets with A $\subseteq$ U $\subseteq$ cl*(U) $\subseteq$ V.

(c) $\Rightarrow$ (a) Let A and B be two disjoint closed subsets of X. By hypothesis, there exists an $\mathcal{I}^*_{*g}$-open set U such that A $\subseteq$ U $\subseteq$ cl*(U) $\subseteq$ X$-$B. Since U is $\mathcal{I}^*_{*g}$-open, A $\subseteq$ int*(U). Since $\mathcal{I}$ is completely codense, by Lemma 1.3, $\tau^* \subseteq \tau^\alpha$ and so int*(U) and X$-$cl*(U) are in τ^α. Hence A $\subseteq$ int*(U) $\subseteq$ int(cl(int(int*(U))))=G and B $\subseteq$ X$-$cl*(U) $\subseteq$ int(cl(int(X$-$cl*(U))))=H. G and H are the required disjoint open sets containing A and B respectively, which proves (a). □

Remark 2.40. *Let (X, τ, $\mathcal{I}$) be an ideal topological space. By Remark 1.11, Definition 1.15, Definition 2.1 and Theorem 2.3, the following diagram holds for a subset G $\subseteq$ X:*

$$\begin{array}{ccccccc} gs^*_{\mathcal{I}}\text{-closed} & & & & & & \\ \uparrow & & & & & & \\ \mathcal{I}^*_{*g}\text{-closed} & \longrightarrow & \mathcal{I}_g\text{-closed} & \longrightarrow & \mathcal{I}_{rg}\text{-closed} & \longrightarrow & \text{weakly } \mathcal{I}_{rg}\text{-closed} \end{array}$$

These implications are not reversible.

Example 2.41. *If X = {a, b, c, d}, τ = {$\emptyset$, X, {a}, {b, c}, {a, b, c}} and $\mathcal{I}$ = {$\emptyset$, {d}}, then {a} is $gs^*_{\mathcal{I}}$-closed set but not $\mathcal{I}^*_{*g}$-closed.*

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SOME RESULTS ON SEMI OPEN SETS IN FUZZIFYING BITOPOLOGICAL SPACES

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Abstaract − In the present paper, we introduce and study the concepts of $(i, j)-$semi open set and $(i, j)-$semi neighborhood system in fuzzifying bitopological spaces. Also, the concepts of $(i, j)-$semi derived set and $(i, j)-$semi closure, $(i, j)-$semi interior, $(i, j)-$semi exterior, $(i, j)-$semi boundary operators in fuzzifying bitopological spaces are introduced and studied. Furthermore, we introduce and study the concepts of $(i, j)-$semi convergence of nets and $(i, j)-$semi convergence of filters in fuzzifying bitopological spaces.

Keywords − *Semiopen sets, Fuzzifying topology, fuzzifying bitopological space.*

1 Introduction

In 1965 [13], Zadeh introduced the fundamental concept of fuzzy sets which to formed the backbone of fuzzy mathematics. Since Chang introduced fuzzy sets theory into topology in 1968 [1]. Wong, Lowen, Hutton, Pu and Liu, etc., discussed respectively various aspects of fuzzy topology [3, 7, 8].

In 1991-1993 [10, 11, 12], Ying introduced the concept of the fuzzifying topology with the sematic method of continuous valued logic. In 1999 Khedr et al. [6] introduced the concept of semiopen sets and semicontinuity in fuzzifying topology.

The study of bitopological spaces was first initiated by Kelley [5] in 1963. In 2003 Zhang et al. [14], studied the concept of fuzzy $\theta_{i,j}$-closed, $\theta_{i,j}$-open sets in fuzzifying bitopological spaces. Also in [2], Gowrisankar et al. studied the concepts of $(i, j)-$pre open sets in fuzzifying bitopological spaces.

The contains of this paper are arranged as follows: In section (3) we introduce the concepts of $(i, j)-$semiopen sets in fuzzifying bitopological spaces. In section (4) we introduce and study the concepts of $(i, j)-$semi neighborhood system in fuzzifying bitopological spaces. In section (5) we introduce and study the concepts of

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$(i,j)-$semi derived sets and $(i,j)-$semi closure operator in fuzzifying bitopological space. In section (6) we introduce and study the concepts of $(i,j)-$semi interior and $(i,j)-$semi exterior, and $(i,j)-$semi boundary operators in fuzzifying bitopological spaces. In section (7) we introduce and study $(i,j)-$semi convergence of nets in fuzzifying bitopological spaces. Finally in section (8) we study $(i,j)-$semi convergence of filters in fuzzifying bitopological spaces.

2 Preliminary

Firstly, we display the fuzzy logical and corresponding set-theoretical notations used in this paper.

For formula φ, the symbol $[\varphi]$ means the truth of φ, where the set of truth values is the unit interval $[0,1]$. A formula φ is valid, we write $\models \varphi$ if and only if $[\varphi]=1$ for every interpretation.

(1) $[\alpha] := \alpha$ $(\alpha \in [0,\ 1])$; $[\alpha \wedge \beta] = \min([\alpha],[\beta])$; $[\alpha \to \beta] = \min(1, 1-[\alpha]+[\beta])$, $[\forall x\ \alpha(x)] = \inf\limits_{x\in X}[\alpha(x)]$, where X is the universe of discourse.

(2) If $\widetilde{A} \in \Im(X)$, where $\Im(X)$ is the family of fuzzy sets of X, then $[x \in \widetilde{A}] := \widetilde{A}(x)$.

(3) If X is the universe of discourse, then $[\forall x\ \alpha(x)] = \inf\limits_{x\in X}[\alpha(x)]$.

In addition, the following derived formulae are given:

(1) $[\neg\alpha] := [\alpha \to 0] = 1-[\alpha]$.

(2) $[\alpha \vee \beta] := [\neg(\neg\alpha \wedge \neg\beta)] = \max([\alpha],[\beta])$.

(3) $[\alpha \leftrightarrow \beta] := [(\alpha \to \beta) \wedge (\beta \to \alpha)]$.

(4) $[\alpha \dot{\wedge} \beta] := [\neg(\alpha \to \neg\beta)] = \max(0, [\alpha]+[\beta]-1)$.

(5) $[\alpha \dot{\vee} \beta] := [\neg\alpha \to \beta] = \min(1, [\alpha]+[\beta])$.

(6) $[\exists x\ \alpha(x)] := [\neg(\forall x\ \neg\alpha(x))]$.

(7) If $\widetilde{A}, \widetilde{B} \in \Im(X)$, then

(a) $[\widetilde{A} \subseteq \widetilde{B}] := [\forall x(x \in \widetilde{A} \to x \in \widetilde{B})] = \inf\limits_{x\in X}\min(1, 1-\widetilde{A}(x)+\widetilde{B}(x))$;

(b) $[A \equiv B] := [(\widetilde{A} \subseteq \widetilde{B}) \wedge (\widetilde{B} \subseteq \widetilde{A})]$.

Secondly, we give the following definitions which are used in the sequel.

Definition 2.1. [10] Let X be a universe of discourse, $P(X)$ is the family of subsets of X and $\tau \in \Im(P(X))$ satisfy the following conditions:
(1) $\tau(X)=1$ and $\tau(\phi)=1$;
(2) for any $A, B, \tau(A \cap B) \geq \tau(A) \wedge \tau(B)$;
(3) for any $\{A_\lambda : \lambda \in \Lambda\}, \tau(\bigcup\limits_{\lambda\in\Lambda} A_\lambda) \geq \bigwedge\limits_{\lambda\in\Lambda} \tau(A_\lambda)$.
Then τ is a fuzzifying topology and (X,τ) a fuzzifying topological space.

Definition 2.2. [10] The family of fuzzifying closed sets is denoted by $F \in \Im(P(X))$, and defined as $A \in F := X \sim A \in \tau$,where $X \sim A$ is the complement of A.

Definition 2.3. [10] Let $x \in X$. The neighborhood system of x is denoted by $N_x \in \Im(P(X))$ and defined as $N_x(A) = \sup_{x\in B\subseteq A} \tau(B)$.

Definition 2.4. [10] The closure $cl(A)$ of A is defined as $cl(A)(x) = 1 - N_x(X \sim A)$. In Theorem 5.3 [10], M.S. Ying proved that the closure $cl : P(X) \to \Im(X)$ is a fuzzifying closure operator (see Definition 5.3 [10]) since its extension
$cl : \Im(X) \to \Im(X)$, $cl(\widetilde{A}) = \bigcup_{\alpha\in[0,1]} \alpha cl(\widetilde{A}_\alpha)$, $\widetilde{A} \in \Im(X)$ satisfies the following Kuratowski closure axioms:
(1) $\models cl(\phi) \equiv \phi$;
(2) for any $\widetilde{A} \in \Im(X)$, $\models \widetilde{A} \subseteq cl(\widetilde{A})$;
(3) for any $\widetilde{A}$, $\widetilde{B} \in \Im(X)$, $\models cl(\widetilde{A} \cup \widetilde{B}) = cl(\widetilde{A}) \cup cl(\widetilde{B})$;
(4) for any $\widetilde{A} \in \Im(X)$, $\models cl(cl(\widetilde{A})) \subseteq cl(\widetilde{A})$.
Where $\widetilde{A}_\alpha = \{x : \widetilde{A}(x) \geq \alpha\}$ is the α-cut of $\widetilde{A}$ and $\alpha\widetilde{A}(x) = \alpha \wedge \widetilde{A}(x)$.

Definition 2.5. [11] For any $A \in P(X)$, the interior of A is denoted by $int(A) \in \Im(P(X))$ and defined as follows: $int(A)(x) = N_x(A)$.

Lemma 2.6. [6] Let (X, τ) be a fuzzifying topological space.If $[A \subseteq B] = 1$. Then (1) $\models int(A) \subseteq int(B)$; (2) $\models cl(A) \subseteq cl(B)$.

Definition 2.7. [14] Let (X, τ_1) and (X, τ_2) be two fuzzifying topological spaces. Then a system (X, τ_1, τ_2) consisting of a universe of discourse X with two fuzzifying topologies τ_1 and τ_2 on X is called a fuzzifying bitopological space.

3 (i,j)-semiopen Sets in Fuzzifying Bitopological Spaces

Definition 3.1. Let (X, τ_1, τ_2) be a fuzzifying bitopological space. Then
(1) The family of fuzzifying $(i, j)-$semiopen sets, denoted by $s\tau_{(i,j)} \in \Im(P(X))$, is defined as follows:

$$A \in s\tau_{(i,j)} := \forall x\big(x \in A \to x \in cl_j(int_i(A))\big)$$

i.e., $s\tau_{(i,j)}(A) = \inf_{x\in A} cl_j\big(int_i(A)\big)(x)$.
(2) The family of fuzzifying $(i, j)-$semiclosed sets, denoted by $sF_{(i,j)} \in \Im(P(X))$, is defined as follows:

$$A \in sF_{(i,j)} := X \sim A \in s\tau_{(i,j)}$$

Lemma 3.2. Let (X, τ_1, τ_2) be a fuzzifying bitopological space.If $[A \subseteq B] = 1$, then $\models cl_j(int_i(A)) \subseteq cl_j(int_i(B))$.

Proof. It is obtained from Lemma (2.6) (1) and (2) .

Lemma 3.3. Let (X, τ_1, τ_2) be a fuzzifying bitopological space and $A \subseteq X$. Then
(1) $\models X \sim \big(cl_j(int_i(A))\big) \equiv int_j(cl_i(X \sim A))$;
(2) $\models X \sim \big(int_j(cl_i(A))\big) \equiv cl_j(int_i(X \sim A))$.

Proof. From Theorem 2.2 (5) in [11], we have
(1) $\Big(X \sim \big(cl_j(int_i(A))\big)\Big)(x) = \big(int_j(X \sim int_i(A))\big)(x) = \big(int_j(cl_i(X \sim A))\big)(x)$.
(2) $\Big(X \sim \big(int_j(cl_i(A))\big)\Big)(x) = \big(cl_j(X \sim cl_i(A))\big)(x) = \big(cl_j(int_i(X \sim A))\big)(x)$.

Theorem 3.4. Let (X, τ_1, τ_2) be a fuzzifying bitopological space. Then
(1) $s\tau_{(i,j)}(X) = 1$, $s\tau_{(i,j)}(\phi) = 1$;
(2) For any $\{A_\lambda : \lambda \in \Lambda\}$, $s\tau_{(i,j)}(\bigcup_{\lambda\in\Lambda} A_\lambda) \geq \bigwedge_{\lambda\in\Lambda} s\tau_{(i,j)}(A_\lambda)$.

Proof. The proof of (1) is straightforward.
(2) From Lemma (3.2), we have $\models cl_j(int_i(A_\lambda)) \subseteq cl_j(int_i(\bigcup_{\lambda\in\Lambda} A_\lambda))$. So

$$\begin{aligned} s\tau_{(i,j)}(\bigcup_{\lambda\in\Lambda} A_\lambda) &= \inf_{x\in(\bigcup_{\lambda\in\Lambda} A_\lambda)} cl_j(int_i(\bigcup_{\lambda\in\Lambda} A_\lambda))(x) \\ &= \inf_{\lambda\in\Lambda} \inf_{x\in A_\lambda} cl_j(int_i(\bigcup_{\lambda\in\Lambda} A_\lambda))(x) \\ &\geq \inf_{\lambda\in\Lambda} \inf_{x\in A_\lambda} cl_j(int_i(A_\lambda))(x) = \bigwedge_{\lambda\in\Lambda} s\tau_{(i,j)}(A_\lambda)(x). \end{aligned}$$

Theorem 3.5. Let (X, τ_1, τ_2) be a fuzzifying bitopological space. Then
(1) $sF_{(i,j)}(X) = 1$, $sF_{(i,j)}(\phi) = 1$;
(2) For any $\{A_\lambda : \lambda \in \Lambda\}$, $sF_{(i,j)}(\bigcap_{\lambda\in\Lambda} A_\lambda) \geq \bigwedge_{\lambda\in\Lambda} sF_{(i,j)}(A_\lambda)$.

Proof. From Theorem (3.4) the proof is obtained.

Lemma 3.6. Let (X, τ_1, τ_2) be a fuzzifying bitopological space. Then
(1) $\models \tau_i \subseteq s\tau_{(i,j)}$; (2)$\models F_i \subseteq sF_{(i,j)}$.

Proof. (1) From Theorem 2.2 (3) in [11], we have

$$\begin{aligned} [A \in \tau_i] &= [A \equiv int_i(A)] \\ &= [A \subseteq int_i(A)] \wedge [int_i(A) \subseteq A] \\ &= [A \subseteq int_i(A)] \leq [A \subseteq cl_j(int_i(A))] = [A \in s\tau_{(i,j)}]. \end{aligned}$$

(2) From (1) above the proof is obtained.

Remark 3.7. The following example shows that
(1) $s\tau_i \subseteq s\tau_{(i,j)}$, (2) $s\tau_j \subseteq s\tau_{(i,j)}$, (3) $\tau_j \subseteq s\tau_{(i,j)}$ and (4) $s\tau_{(i,j)} \subseteq s\tau_{(j,i)}$ may not be true for any (X, τ_1, τ_2) fuzzifying bitopological space.

Example 3.8. Let $X = \{a, b, c\}$,$A = \{a, b\}$ and τ_1, τ_2 be two fuzzifying topologies on X defined as follow:

$$\tau_1(A) = \begin{cases} 1 & \text{if} \quad A \in \{\phi, X, \{a\}, \{a, c\}\}, \\ 1/4 & \text{if} \quad A \in \{\{c\}, \{b, c\}\}, \\ 0 & \text{if} \quad A \in \{\{b\}, \{a, b\}\}. \end{cases}$$

$$\tau_2(A) = \begin{cases} 1 & \text{if} \quad A \in \{\phi, X, \{b\}, \{a, c\}\}, \\ 1/4 & \text{if} \quad A \in \{\{a\}, \{a, b\}\}, \\ 0 & \text{if} \quad A \in \{\{c\}, \{b, c\}\}. \end{cases}$$

We have $int_1(A)(a) = 1$, $int_1(A)(b) = int_1(A)(c) = 0$,
$cl_1(int_1(A))(a) = 1$, $cl_1(int_1(A))(b) = cl_1(int_1(A))(c) = 3/4$;
$s\tau_1(A) = 3/4$ and $int_2(A)(a) = 1/4$, $int_2(A)(b) = 1$, $int_2(A)(c) = 0$,
$cl_2(int_2(A))(a) = cl_2(int_2(A))(c) = 1/4$, $cl_2(int_2(A))(b) = 1$; $s\tau_2(A) = 1/4$.
So $cl_2(int_1(A))(a) = cl_2(int_1(A))(c) = 1$, $cl_2(int_1(A))(b) = 0$, $s\tau_{(1,2)}(A) = 0$. Also $cl_1(int_2(A))(a) = 1/4 = cl_1(int_2(A))(c)$, $cl_1(int_2(A))(b) = 1$; $s\tau_{(2,1)}(A) = 1/4$.
Therefore $s\tau_2 \nsubseteq s\tau_{(1,2)}$, $s\tau_1 \nsubseteq s\tau_{(1,2)}$, $\tau_2 \nsubseteq s\tau_{(1,2)}$ and $s\tau_{(2,1)} \nsubseteq s\tau_{(1,2)}$.

Theorem 3.9. Let (X, τ_1, τ_2) be a fuzzifying bitopological space. Then
(1) $\models cl_j(A) \equiv cl_j(int_i(A)) \longleftrightarrow A \in s\tau_{(i,j)}$;
(2) $\models int_j(A) \equiv int_j(cl_i(A)) \longleftrightarrow A \in sF(i, j)$.

Proof. (1) $[cl_j(A) \equiv cl_j(int_i(A))] = [cl_j(A) \subseteq cl_j(int_i(A))] \wedge [cl_j(int_i(A)) \subseteq cl_j(A)]$.
We know that $[int_i(A) \subseteq A] = 1$, so $[cl_j(int_i(A)) \subseteq cl_j(A)] = 1$. Then
$[cl_j(A) \equiv cl_j(int_i(A))] = [cl_j(A) \subseteq cl_j(int_i(A))] \leq [A \subseteq cl_j(int_i(A))] = [A \in s\tau_{(i,j)}]$.
Conversely, $[A \in s\tau_{(i,j)}] = [A \subseteq cl_j(int_i(A))] \leq [cl_j(A) \subseteq cl_j(cl_j(int_i(A)))]$.
From Definition (2.4) (4), we have $[cl_j(cl_j(int_i(A)))] \subseteq cl_j(int_i(A))] = 1$. Therefore

$$\begin{aligned}[A \in s\tau_{(i,j)}] &\leq [cl_j(A) \subseteq cl_j(int_i(A))] \\ &= [cl_j(A) \subseteq cl_j(int_i(A))] \wedge [cl_j(int_i(A)) \subseteq cl_j(A)] \\ &= [cl_j(A) \equiv cl_j(int_i(A))].\end{aligned}$$

(2) From (1) above and Lemma (3.3) (2), the proof is obtained.

Theorem 3.10. Let (X, τ_1, τ_2) be a fuzzifying bitopological space. Then
(1) $\models A \in s\tau_{(i,j)} \leftrightarrow \forall x(x \in A \to \exists B(B \in s\tau_{(i,j)} \wedge x \in B \subseteq A))$;
(2) $\models A \in sF_{(i,j)} \leftrightarrow \forall x(x \in int_j(cl_i(A)) \to x \in A)$.

Proof. (1) $[\forall x(x \in A \to \exists B(B \in s\tau_{(i,j)} \wedge x \in B \subseteq A))] = \inf_{x\in A}\sup_{x\in B\subseteq A} s\tau_{(i,j)}(B)$.
First, we have $\inf_{x\in A}\sup_{x\in B\subseteq A} s\tau_{(i,j)}(B) \geq s\tau_{(i,j)}(A)$.
In the other hand, let $\beta_x = \{B : x \in B \subseteq A\}$.Then for any $f \in \prod_{x\in A}\beta_x$, we have $\bigcup_{x\in A} f(x) = A$, $s\tau_{(i,j)}(A) = s\tau_{(i,j)}(\bigcup_{x\in A} f(x)) \geq \inf_{x\in A} s\tau_{(i,j)}(f(x))$, and so

$$s\tau_{(i,j)}(A) \geq \sup_{f\in \prod_{x\in A}\beta_x} \inf_{x\in A} s\tau_{(i,j)}(f(x)) = \inf_{x\in A} \sup_{f\in \prod_{x\in A}\beta_x} s\tau_{(i,j)}(f(x)) = \inf_{x\in A}\sup_{x\in B\subseteq A} s\tau_{(i,j)}(B).$$

(2) From Lemma (3.3) (2), we have

$$\begin{aligned}[\forall x(x \in int_j(cl_i(A)) \to x \in A)] &= [\forall x(x \in X \sim A \to x \in X \sim int_j(cl_i(A)))] \\ &= \inf_{x\in X\sim A} \big(X \sim int_j(cl_i(A))\big)(x) \\ &= \inf_{x\in X\sim A} \big(cl_j(int_i(X \sim A))\big)(x) \\ &= [X \sim A \in s\tau_{(i,j)}] = [A \in sF_{(i,j)}].\end{aligned}$$

Lemma 3.11. Let (X, τ_1, τ_2) be a fuzzifying bitopological space.Then
(1) $\models B \dot{\equiv} int_i(A) \longrightarrow B \subseteq A$;
(2) $\models B \dot{\equiv} int_i(A) \wedge A \in s\tau_{(i,j)} \longrightarrow A \subseteq cl_j(B)$.

Proof. (1) $[B\dot{\equiv} int_i(A)] = [(B \subseteq int_i(A))\dot{\wedge}(int_i(A) \subseteq B)]$. If $[B \subseteq A] = 0$, then $[B \subseteq int_i(A)] = 0$. Therefor $[B\dot{\equiv} int_i(A)] = 0$.

$$\begin{aligned}(2)[(B\dot{\equiv} int_i(A)) \wedge A \in s\tau_{(i,j)}] &= [(B\dot{\equiv} int_i(A)) \wedge A \subseteq cl_j(int_i(A)]\\ &\leq [(int_i(A) \subseteq B) \wedge (A \subseteq cl_j(int_i(A)))]\\ &\leq [(cl_j(int_i(A)) \subseteq cl_j(B)) \wedge (A \subseteq cl_j(int_i(A)))]\\ &\leq [A \subseteq cl_j(B)].\end{aligned}$$

Theorem 3.12. Let (X, τ_1, τ_2) be a fuzzifying bitopological space.Then
(1) $\models \exists U(U \in \tau_i \wedge U \subseteq A \subseteq cl_j(U)) \longrightarrow A \in s\tau_{(i,j)}$;
(2) $\models \exists V(V \in F_i \wedge int_j(V) \subseteq A \subseteq V) \longrightarrow A \in sF_{(i,j)}$.

Proof. (1) From Theorem 2.2 (3) [11],we have

$$\begin{aligned}[\exists U(U \in \tau_i \wedge U \subseteq A \subseteq cl_j(U))] &= \sup_{U\in P(X)}\big([U \in \tau_i] \wedge [U \subseteq A] \wedge [A \subseteq cl_j(U)]\big)\\ &= \sup_{U\subseteq A}\big([U \subseteq int_i(U)] \wedge [U \subseteq A] \wedge [A \subseteq cl_j(U)]\big)\\ &\leq \sup_{U\subseteq A}\big([U \subseteq int_i(U)] \wedge [int_i(U) \subseteq int_i(A)] \wedge [A \subseteq cl_j(U)]\big)\\ &\leq \sup_{U\subseteq A}\big([U \subseteq int_i(A)] \wedge [A \subseteq cl_j(U)]\big)\\ &\leq \sup_{U\subseteq A}\big([cl_j(U) \subseteq cl_j(int_i(A))] \wedge [A \subseteq cl_j(U))]\big)\\ &\leq \sup_{U\subseteq A}[A \subseteq cl_j(int_i(A))] = [A \in s\tau_{(i,j)}].\end{aligned}$$

(2) From (1) above and Theorem (2.2) (5) in [11], we have

$$\begin{aligned}[A \in sF_{(i,j)}] &= [X \sim A \in s\tau_{(i,j)}]\\ &\geq \big[\exists U(U \in \tau_i \wedge U \subseteq X \sim A \subseteq cl_j(U))\big]\\ &= \big[\exists U\big(U \in \tau_i \wedge X \sim cl_j(U) \subseteq A \subseteq X \sim U\big)\big]\\ &= \big[\exists U\big(U \in \tau_i \wedge int_j(X \sim U) \subseteq A \subseteq X \sim U\big)\big]\\ &= \big[\exists V\big(V \in F_i \wedge int_j(V) \subseteq A \subseteq V\big)\big].\end{aligned}$$

Remark 3.13. The proof of the inverse direction of Theorem (3.12) can be obtained by assuming that $[U\dot{\equiv} int_i(A)] = 1$, but the following example shows that even without the proposed requirement the proof is true. So the proof may be can obtained without the proposed requirement.

Example 3.14. From Example (3.8), $A = \{a, b\}$, $s\tau_{(2,1)}(A) = 1/4$ and
$int_2(A)(a) = 1/4$, $int_2(A)(b) = 1$, $int_2(A)(c) = 0$.
The family of all subsets of A is $\big\{\{a\}, \{b\}, \{a, b\}\big\}$ and $cl_1(\{a\})(a) = 1$,
$cl_1(\{a\})(b) = 3/4$, $cl_1(\{a\})(c) = 3/4$. Then $[A \subseteq cl_1(\{a\})] = \inf_{x\in A} cl_1(\{a\})(x) = 3/4$.
So $[\tau_2(\{a\}) \wedge A \subseteq cl_1(\{a\})] = \min(1/4, 3/4) = 1/4$.
By the same way, we have $[\tau_2(\{b\}) \wedge A \subseteq cl_1(\{b\})] = \min(1, 0) = 0$ and
$[\tau_2(\{a, b\}) \wedge A \subseteq cl_1(\{a, b\})] = \min(1/4, 1) = 1/4$.
Therefore $[\exists U(U \in \tau_2 \wedge U \subseteq A \subseteq cl_1(U))] = 1/4 = s\tau_{(2,1)}(A)$.

Note that $[U\dot{\equiv} int_2(A)] = [U \subseteq int_2(A)]\dot{\wedge}[int_2(A) \subseteq U]$ and
$[U \subseteq int_2(A)] = \inf_{x\in U} int_2(A)(x)$, $[int_2(A) \subseteq U] = \inf_{x\in X\sim U}\big(1 - int_2(A)(x)\big)$.
$[\{a\}\dot{\equiv} int_2(A)] = \max(0, 1/4+0-1) = 0$. $[\{b\}\dot{\equiv} int_2(A)] = \max(0, 1+3/4-1) = 3/4$
$[\{a, b\}\dot{\equiv} int_2(A)] = \max(0, 1 + 1/4 - 1) = 1/4$.

4 (i,j)-semi Neighborhood System in Fuzzifying Bitopological Spaces

Definition 4.1. Let (X, τ_1, τ_2) be a fuzzifying bitopological space and $x \in X$. Then the $(i,j)-$semi neighborhood system of x is denoted by $sN_x^{(i,j)} \in \Im(P(X))$ and defined as

$$A \in sN_x^{(i,j)} := \exists B(B \in s\tau_{(i,j)} \wedge x \in B \subseteq A)$$

i.e., $sN_x^{(i,j)}(A) = \sup_{x \in B \subseteq A} s\tau_{(i,j)}(B)$.

Theorem 4.2. Let (X, τ_1, τ_2) be a fuzzifying bitopological space and $A \in P(X)$. Then
(1) $\models A \in s\tau_{(i,j)} \longleftrightarrow \forall x\big(x \in A \to \exists B(B \in sN_x^{(i,j)} \wedge B \subseteq A)\big)$;
(2) $N_x^i(A) \leq sN_x^{(i,j)}(A)$.

Proof. (1) From Theorem (3.10) (1), we have

$$\begin{aligned}[\forall x\big(x \in A \to \exists B(B \in sN_x^{(i,j)} \wedge B \subseteq A)\big)] &= \inf_{x \in A} \sup_{B \subseteq A} sN_x^{(i,j)}(B)\\ &= \inf_{x \in A} \sup_{B \subseteq A} \sup_{x \in C \subseteq B} s\tau_{(i,j)}(C)\\ &= \inf_{x \in A} \sup_{x \in C \subseteq A} s\tau_{(i,j)}(C) = s\tau_{(i,j)}(A).\end{aligned}$$

(2) From Lemma (3.6) (1), we have
$sN_x^{(i,j)}(A) = \sup_{x \in B \subseteq A} s\tau_{(i,j)}(B) \geq \sup_{x \in B \subseteq A} \tau_i(B) = N_x^i(A)$.

Corollary 4.3. $s\tau_{(i,j)}(A) = \inf_{x \in A} sN_x^{(i,j)}(A)$.

Theorem 4.4. Let (X, τ_1, τ_2) be a fuzzifying bitopological space. The mapping $sN^{(i,j)} : X \to \Im^N(P(X))$, $x \mapsto sN_x^{(i,j)}$, where $\Im^N(P(X))$ is the set of all normal fuzzy subset of $P(X)$, has the following properties:
(1) $\models A \in sN_x^{(i,j)} \to x \in A$;
(2) $\models A \subseteq B \to (A \in sN_x^{(i,j)} \to B \in sN_x^{(i,j)})$;
(3) $\models A \in sN_x^{(i,j)} \to \exists H(H \in sN_x^{(i,j)} \wedge H \subseteq A \wedge \forall y(y \in H \to H \in sN_y^{(i,j)}))$.

Proof. (1) If $[A \in sN_x^{(i,j)}] = 0$, then (1) is obtain.
If $[A \in sN_x^{(i,j)}] = \sup_{x \in B \subseteq A} s\tau_{(i,j)}(B) > 0$, then there exists B_0 such that $x \in B_0 \subseteq A$.
Now we have $[x \in A] = 1$. Therefore $[A \in sN_x^{(i,j)}] \leq [x \in A]$.
(2) Immediate.
(3) $[\exists H(H \in sN_x^{(i,j)} \wedge H \subseteq A \wedge \forall y(y \in H \to H \in sN_y^{(i,j)}))]$

$$\begin{aligned}&= \sup_{H \subseteq A}(sN_x^{(i,j)}(H) \wedge \inf_{y \in H} sN_y^{(i,j)}(H))\\ &= \sup_{H \subseteq A}(sN_x^{(i,j)}(H) \wedge s\tau_{(i,j)}(H))\\ &= \sup_{H \subseteq A} s\tau_{(i,j)}(H) \geq \sup_{x \in H \subseteq A} s\tau_{(i,j)}(H) = sN_x^{(i,j)}(A) = [A \in sN_x^{(i,j)}].\end{aligned}$$

5 (i,j)-semi Derived Sets and (i,j)-semi Closure Operator in Fuzzifying Bitopological Spaces

Definition 5.1. Let (X, τ_1, τ_2) be a fuzzifying bitopological space. The $(i,j)-$semi derived set $sd_{(i,j)}(A)$ of A is defined as follows:

$$x \in sd_{(i,j)}(A) := \forall B(B \in sN_x^{(i,j)} \rightarrow B \cap (A \sim \{x\}) \neq \phi)$$

i.e., $sd_{(i,j)}(A)(x) = \inf\limits_{B\cap(A\sim\{x\})=\phi}(1 - sN_x^{(i,j)}(B))$.

Lemma 5.2. $sd_{(i,j)}(A)(x) = 1 - sN_x^{(i,j)}((X \sim A) \cup \{x\})$.

Proof.

$$\begin{aligned} sd_{(i,j)}(A)(x) &= 1 - \sup_{B\cap A\sim\{x\}=\phi} sN_x^{(i,j)}(B) = 1 - \sup_{B\subseteq(X\sim A)\cup\{x\}} \sup_{x\in C\subseteq B} s\tau_{(i,j)}(C) \\ &= 1 - \sup_{x\in C\subseteq(X\sim A)\cup\{x\}} s\tau_{(i,j)}(C) = 1 - sN_x^{(i,j)}((X \sim A) \cup \{x\}). \end{aligned}$$

Theorem 5.3. Let (X, τ_1, τ_2) be a fuzzifying bitopological space and $A, B \in P(X)$. Then
(1) $\models sd_{(i,j)}(\phi) \equiv \phi$;
(2) $\models A \subseteq B \longrightarrow sd_{(i,j)}(A) \subseteq sd_{(i,j)}(B)$;
(3) $\models A \in sF_{(i,j)} \longleftrightarrow sd_{(i,j)}(A) \subseteq A$;
(4) $\models sd_{(i,j)}(A) \subseteq d_i(A)$.

Proof. (1) From Lemma (5.2), we have

$$\begin{aligned} sd_{(i,j)}(\phi)(x) &= 1 - sN_x^{(i,j)}((X \sim \phi) \cup \{x\}) \\ &= 1 - sN_x^{(i,j)}(X) = 1 - 1 = 0. \end{aligned}$$

(2) Let $A \subseteq B$, then From Lemma (5.2) and Theorem (4.4) (2), we have

$$\begin{aligned} sd_{(i,j)}(A)(x) &= 1 - sN_x^{(i,j)}((X \sim A) \cup \{x\}) \\ &\leq 1 - sN_x^{(i,j)}((X \sim B) \cup \{x\}) = sd_{(i,j)}(B)(x). \end{aligned}$$

(3) From Lemma (5.2) and Theorem (4.2) (1), we have

$$\begin{aligned} [sd_{(i,j)}(A) \subseteq A] &= \inf_{x\in X\sim A}(1 - sd_{(i,j)}(A)(x)) = \inf_{x\in X\sim A} sN_x^{(i,j)}((X \sim A) \cup \{x\}) \\ &= \inf_{x\in X\sim A} sN_x^{(i,j)}(X \sim A) = \inf_{x\in X\sim A} \sup_{x\in B\subseteq X\sim A} s\tau_{(i,j)}(B) \\ &= s\tau_{(i,j)}(X \sim A) = sF_{(i,j)}(A) = [A \in sF_{(i,j)}]. \end{aligned}$$

(4) From Theorem (4.2) (2) and Lemma (5.1) in [10], we have
$sd_{(i,j)}(A)(x) = 1 - sN_x^{(i,j)}((X \sim A) \cup \{x\}) \leq 1 - N_x^i((X \sim A) \cup \{x\}) = d_i(A)(x)$.

Definition 5.4. Let (X, τ_1, τ_2) be a fuzzifying bitopological space. The Fuzzifying $(i,j)-$semi closure of A, is denoted and defined as follows:
$x \in scl_{(i,j)}(A) := \forall B((B \supseteq A) \wedge (B \in sF_{(i,j)}) \rightarrow x \in B)$,
i.e., $scl_{(i,j)}(A)(x) = \inf\limits_{x\notin B\supseteq A}(1 - sF_{(i,j)}(B))$.

Lemma 5.5. [6] For any $A \in P(X)$ and $\widetilde{B} \in \Im(X)$, then $[\widetilde{B} \subseteq A] = [\widetilde{B} \cup A \subseteq A]$.

Theorem 5.6. Let (X, τ_1, τ_2) be a fuzzifying bitopological space, $A, B \in P(X)$ and $x \in X$. Then
(1) $scl_{(i,j)}(A)(x) = 1 - sN_x^{(i,j)}(X \sim A)$;
(2) $\models scl_{(i,j)}(\phi) = \phi$;
(3) $\models A \subseteq scl_{(i,j)}(A)$;
(4) $\models scl_{(i,j)}(A) \equiv sd_{(i,j)}(A) \cup A$;
(5) $\models x \in scl_{(i,j)}(A) \longleftrightarrow \forall B(B \in sN_x^{(i,j)} \longrightarrow A \cap B \neq \phi)$;
(6) $\models A \equiv scl_{(i,j)}(A) \longleftrightarrow A \in sF_{(i,j)}(A)$;
(7) $\models scl_{(i,j)}(A) \subseteq cl_i(A)$;
(8) $\models A \subseteq B \longrightarrow scl_{(i,j)}(A) \subseteq scl_{(i,j)}(B)$;
(9) $\models B \dot{\equiv} scl_{(i,j)}(A) \longrightarrow B \in sF_{(i,j)}$.

Proof.

$$\begin{aligned}(1)\ scl_{(i,j)}(A)(x) &= \inf_{x \notin B \supseteq A} (1 - sF_{(i,j)}(B)) \\ &= \inf_{x \notin B \supseteq A} (1 - s\tau_{(i,j)}(X \sim B)) \\ &= 1 - \sup_{x \in X \sim B \subseteq X \sim A} s\tau_{(i,j)}(X \sim B) = 1 - sN_x^{(i,j)}(X \sim A).\end{aligned}$$

(2) $scl_{(i,j)}(\phi)(x) = 1 - sN_x^{(i,j)}(X \sim \phi) = 1 - sN_x^{(i,j)}(X) = 0$.
(3) Let $A \in P(X)$ and for any $x \in X$. If $x \notin A$, then $[x \in A] \leq [x \in scl_{(i,j)}(A)]$. If $x \in A$, then $scl_{(i,j)}(A)(x) = 1 - sN_x^{(i,j)}(X \sim A) = 1 - 0 = 1$.
So $[x \in A] \leq [x \in scl_{(i,j)}(A)]$. Therefore $[A \subseteq scl_{(i,j)}(A)] = 1$.

(4) From Lemma (5.2) and (3) above, for any $x \in X$ we have
$[x \in (sd_{(i,j)}(A) \cup A)] = \max\big(1 - sN_x^{(i,j)}((X \sim A) \cup \{x\}), A(x)\big)$.
If $x \in A$, then $[x \in (sd_{(i,j)}(A) \cup A)] = A(x) = 1 = [x \in scl_{(i,j)}(A)]$. If $x \notin A$, then $[x \in sd_{(i,j)}(A) \cup A] = 1 - sN_x^{(i,j)}(X \sim A) = [x \in scl_{(i,j)}(A)]$.
Therefore $[scl_{(i,j)}(A)] = [sd_{(i,j)}(A) \cup A]$.

$$\begin{aligned}(5)\qquad [\ \forall B\ (\ B \in sN_x^{(i,j)} \rightarrow A \cap B \neq \phi\ )\ ] &= \inf_{B \subseteq X \sim A} (\ 1 - sN_x^{(i,j)}(B)\ ) \\ &= 1 - sN_x^{(i,j)}(X \sim A) \\ &= [x \in scl_{(i,j)}(A)].\end{aligned}$$

(6) From Theorem (5.3) (3), Lemma (5.5), (4) above and since
$[A \subseteq sd_{(i,j)}(A) \cup A] = 1$, we have

$$\begin{aligned}sF_{(i,j)}(A) &= [sd_{(i,j)}(A) \subseteq A] = [sd_{(i,j)}(A) \cup A \subseteq A] \\ &= [sd_{(i,j)}(A) \cup A \subseteq A] \wedge [A \subseteq sd_{(i,j)}(A) \cup A] \\ &= [sd_{(i,j)}(A) \cup A \equiv A] = [A \equiv scl_{(i,j)}(A)].\end{aligned}$$

(7) From Lemma (3.6) (2), we have
$$scl_{(i,j)}(A)(x) = \inf_{x \notin B \supseteq A} (1 - sF_{(i,j)}(B)) \leq \inf_{x \notin B \supseteq A} (1 - F_i(B)) = cl_i(A).$$

(8) Let $A \subseteq B$, then $X \sim B \subseteq X \sim A$. From (1) above and Theorem (4.4) (2), we have $scl_{(i,j)}(A)(x) = 1 - sN_x^{(i,j)}(X \sim A) \leq 1 - sN_x^{(i,j)}(X \sim B) = scl_{(i,j)}(B)(x)$.

(9) If $[A \subseteq B] = 0$, then $[B \dot{\equiv} scl_{(i,j)}(A)] = 0$. Now suppose that $[A \subseteq B] = 1$. We have $[B \subseteq scl_{(i,j)}(A)] = 1 - \sup_{x \in B \sim A} sN_x^{(i,j)}(X \sim A)$ and $[scl_{(i,j)}(A) \subseteq B] = \inf_{x \in X \sim B} sN_x^{(i,j)}(X \sim A)$. Therefore $[B \dot{\equiv} scl_{(i,j)}(A)] = \max(0, \inf_{x \in X \sim B} sN_x^{(i,j)}(X \sim A) - \sup_{x \in B \sim A} sN_x^{(i,j)}(X \sim A))$.
Let $[B \dot{\equiv} scl_{(i,j)}(A)] > t$. Then $\inf_{x \in X \sim B} sN_x^{(i,j)}(X \sim A) > t + \sup_{x \in B \sim A} sN_x^{(i,j)}(X \sim A)$. For any $x \in X \sim B$, we have $sN_x^{(i,j)}(X \sim A) > t + \sup_{x \in B \sim A} sN_x^{(i,j)}(X \sim A)$. Therefore $\sup_{x \in C \subseteq X \sim A} s\tau_{(i,j)}(C) > t + \sup_{x \in B \sim A} sN_x^{(i,j)}(X \sim A)$, i.e., there exists C_x such that $x \in C_x \subseteq X \sim A$ and $s\tau_{(i,j)}(C_x) > t + \sup_{x \in B \sim A} sN_x^{(i,j)}(X \sim A)$. Now we want to prove $C_x \subseteq X \sim B$. If not, then there exists $x' \in C_x$ and $x' \in B \sim A$. Hence we obtain $\sup_{x \in B \sim A} sN_x^{(i,j)}(X \sim A) \geq sN_{x'}^{(i,j)}(X \sim A) \geq s\tau_{(i,j)}(C_x) > t + \sup_{x \in B \sim A} sN_x^{(i,j)}(X \sim A)$, a contradiction. Therefore $sF_{(i,j)}(B) = s\tau_{(i,j)}(X \sim B) = \inf_{x \in X \sim B} sN_x^{(i,j)}(X \sim B) \geq \inf_{x \in X \sim B} s\tau_{(i,j)}(C_x) \geq s\tau_{(i,j)}(C_x) > t + \sup_{x \in B \sim A} sN_x^{(i,j)}(X \sim A) > t$. Since t is arbitrary, it holds that $[B \dot{\equiv} scl_{(i,j)}(A)] \leq [B \in sF_{(i,j)}]$.

6 (i,j)-semi Interior, (i,j)-semi Exterior and (i,j)-semi Boundary Operators in Fuzzifying Bitopological Spaces

Definition 6.1. Let (X, τ_1, τ_2) be a fuzzifying bitopological space and $A \in P(X)$, the $(i,j)-$semi interior of A is defined as follows:

$$sint_{(i,j)}(A)(x) = sN_x^{(i,j)}(A)$$

Theorem 6.2. Let (X, τ_1, τ_2) be a fuzzifying bitopological space, $A, B \in P(X)$ and $x \in X$.Then
(1) $\models sint_{(i,j)}(X) \equiv X$;
(2) $\models sint_{(i,j)}(A) \subseteq A$;
(3) $\models int_i(A) \subseteq sint_{(i,j)}(A)$;
(4) $\models B \in s\tau_{(i,j)} \wedge B \subseteq A \longrightarrow B \subseteq sint_{(i,j)}(A)$;
(5) $\models A \equiv sint_{(i,j)}(A) \longleftrightarrow A \in s\tau_{(i,j)}$;
(6) $\models A \subseteq B \longrightarrow sint_{(i,j)}(A) \subseteq sint_{(i,j)}(B)$;
(7) $\models sint_{(i,j)}(A) \equiv X \sim scl_{(i,j)}(X \sim A)$;
(8) $\models sint_{(i,j)}(A) \equiv A \cap (X \sim sd_{(i,j)}(X \sim A))$;
(9) $\models B \dot{\equiv} sint_{(i,j)}(A) \longrightarrow B \in s\tau_{(i,j)}$.

Proof. (1) $sint_{(i,j)}(X)(x) = sN_x^{(i,j)}(X) = 1$. Therefore $sint_{(i,j)}(X) \equiv X$.
(2) Let $A \in P(X)$ and $x \in X$. If $x \notin A$, then $sint_{(i,j)}(A)(x) = sN_x^{(i,j)}(A) = 0$. Therefore $sint_{(i,j)}(A) \subseteq A$.
(3) From Theorem (4.2) (2), we have $int_i(A)(x) = N_x^i(A) \leq sN_x^{(i,j)}(A) = sint_{(i,j)}(A)(x)$.

(4) If $B \nsubseteq A$, then $[(B \in s\tau_{(i,j)}) \wedge (B \subseteq A)] = 0$. If $B \subseteq A$, then

$$\begin{aligned}[B \subseteq sint_{(i,j)}(A)] &= \inf_{x \in B} sint_{(i,j)}(A)(x) \\ &= \inf_{x \in B} sN_x^{(i,j)}(A) \\ &\geq \inf_{x \in B} sN_x^{(i,j)}(B) = s\tau_{(i,j)}(B) = [(B \in s\tau_{(i,j)}) \wedge (B \subseteq A)].\end{aligned}$$

(5)
$$\begin{aligned}[\ A \equiv sint_{(i,j)}(A)\ ] &= \min\Big(\inf_{x \in A} sint_{(i,j)}(A)(x)\ ,\ \inf_{x \in X \sim A}(1 - sint_{(i,j)}(A)(x))\Big) \\ &= \min\big(\inf_{x \in A} sN_x^{(i,j)}(A)\ ,\ \inf_{x \in X \sim A}(1 - sN_x^{(i,j)}(A))\big) \\ &= \inf_{x \in A} sN_x^{(i,j)}(A) = s\tau_{(i,j)}(A) = [A \in s\tau_{(i,j)}].\end{aligned}$$

(6) From Definition (6.1) and Theorem (4.4) (2), the proof is straightforward.

(7) From Theorem (5.6) (1), we have
$\big(X \sim scl_{(i,j)}(X \sim A)\big)(x) = 1 - (1 - sN_x^{(i,j)}(A)) = sN_x^{(i,j)}(A) = sint_{(i,j)}(A)(x)$.

(8) From Lemma (5.2), we have
$[A \cap (X \sim sd_{(i,j)}(X \sim A))] = \min(A(x), sN_x^{(i,j)}(A \cup \{x\}))$
If $x \notin A$, then $[A \cap (X \sim sd_{(i,j)}(X \sim A))] = 0 = sN_x^{(i,j)}(A) = sint_{(i,j)}(A)(x)$.
If $x \in A$, then $[A \cap (X \sim sd_{(i,j)}(X \sim A))] = sN_x^{(i,j)}(A) = sint_{(i,j)}(A)(x)$.

(9) From Theorem (5.6) (9) and (7) above, we have
$[B \doteqdot sint_{(i,j)}(A)] = [X \sim B \doteqdot scl_{(i,j)}(X \sim A)] \leq [X \sim B \in sF_{(i,j)}] = [B \in s\tau_{(i,j)}]$.

Definition 6.3. Let (X, τ_1, τ_2) be a fuzzifying bitopological space and $A \subseteq X$. The $(i,j)-$semi exterior of A is defined as follows:
$x \in sext_{(i,j)}(A) :=\ x \in sint_{(i,j)}(X \sim A)$,
i.e., $sext_{(i,j)}(A)(x) = sint_{(i,j)}(X \sim A)(x)$.

Theorem 6.4. For any A
(1) $\models sext_{(i,j)}(\phi) \equiv X$;
(2) $\models sext_{(i,j)}(A) \subseteq X \sim A$;
(3) $\models ext_i(A) \subseteq sext_{(i,j)}(A)$;
(4) $\models A \in sF_{(i,j)} \longleftrightarrow sext_{(i,j)}(A) \equiv X \sim A$;
(5) $\models B \in sF_{(i,j)} \wedge A \subseteq B \longrightarrow X \sim B \subseteq sext_{(i,j)}(A)$;
(6) $\models B \subseteq A \longrightarrow sext_{(i,j)}(B) \subseteq sext_{(i,j)}(A)$;
(7) $\models sext_{(i,j)}(A) \equiv (X \sim A) \cap (X \sim sd_{(i,j)}(A))$;
(8) $\models sext_{(i,j)}(A) \equiv X \sim scl_{(i,j)}(A)$;
(9) $\models x \in sext_{(i,j)}(A) \longleftrightarrow \exists B(x \in B \in s\tau_{(i,j)} \wedge B \cap A = \phi)$.

Proof. From Theorem (6.2), we obtain (1),(2),(3),(4),(5),(6),(7) and (8).

(9)
$$\begin{aligned}[\exists B(\ x \in B \in s\tau_{(i,j)} \wedge B \cap A = \phi\ )] &= \sup_{x \in B \subseteq X \sim A} s\tau_{(i,j)}(B) = sN_x^{(i,j)}(X \sim A) \\ &= sint_{(i,j)}(X \sim A)(x).\end{aligned}$$

Definition 6.5. Let (X, τ_1, τ_2) be a fuzzifying bitopological space and $A \subseteq X$. The $(i,j)-$semi boundary of A is defined as follows:

$x \in sb_{(i,j)}(A) := \big(x \notin sint_{(i,j)}(A)\big) \wedge \big(x \notin sint_{(i,j)}(X \sim A)\big)$,
i.e., $sb_{(i,j)}(A)(x) = \min\big(1 - sint_{(i,j)}(A)(x), 1 - sint_{(i,j)}(X \sim A)(x)\big)$.

Lemma 6.6. Let (X, τ_1, τ_2) be a fuzzifying bitopological space, $A \in P(X)$ and $x \in X$. Then $\models x \in sb_{(i,j)}(A) \longleftrightarrow \forall B\big(B \in sN_x^{(i,j)} \rightarrow (B \cap A \neq \phi) \wedge (B \cap (X \sim A) \neq \phi)\big)$.

Proof. $[\forall B\big(B \in sN_x^{(i,j)} \rightarrow (B \cap A \neq \phi) \wedge (B \cap (X \sim A) \neq \phi)\big)]$

$$= min\big(\inf_{B \subseteq A}(1 - sN_x^{(i,j)}(B)), \inf_{B \subseteq X \sim A}(1 - sN_x^{(i,j)}(B))\big)$$
$$= min\big(1 - sN_x^{(i,j)}(A), 1 - sN_x^{(i,j)}(X \sim A)\big)$$
$$= min\big(1 - sint_{(i,j)}(A)(x), 1 - sint_{(i,j)}(X \sim A)(x)\big) = [x \in sb_{(i,j)}(A)].$$

Theorem 6.7. Let (X, τ_1, τ_2) be a fuzzifying bitopological space and $A \in P(X)$. Then
(1) $\models sb_{(i,j)}(A) \equiv scl_{(i,j)}(A) \cap scl_{(i,j)}(X \sim A)$;
(2) $\models sb_{(i,j)}(A) \equiv sb_{(i,j)}(X \sim A)$;
(3) $\models X \sim sb_{(i,j)}(A) \equiv sint_{(i,j)}(A) \cup sint_{(i,j)}(X \sim A)$;
(4) $\models scl_{(i,j)}(A) \equiv A \cup sb_{(i,j)}(A)$;
(5) $\models sb_{(i,j)}(A) \subseteq A \longleftrightarrow A \in sF_{(i,j)}$;
(6) $\models sint_{(i,j)}(A) \equiv A \cap (X \sim sb_{(i,j)}(A))$;
(7) $\models (sb_{(i,j)}(A) \cap A \equiv \phi) \longleftrightarrow A \in s\tau_{(i,j)}$;
(8) $\models sb_{(i,j)}(A) \subseteq b_i(A)$;
(9) $\models X \sim sb_{(i,j)}(A) \equiv sint_{(i,j)}(A) \cup sext_{(i,j)}(A)$.

Proof. (1) From Theorem (6.2) (7), we obtain

$$\begin{aligned}\big(scl_{(i,j)}(A) \cap scl_{(i,j)}(X \sim A)\big)(x) &= \min\big(scl_{(i,j)}(A)(x), scl_{(i,j)}(X \sim A)(x)\big)\\ &= \min\big(1 - sint_{(i,j)}(X \sim A)(x), 1 - sint_{(i,j)}(A)(x)\big)\\ &= sb_{(i,j)}(A)(x).\end{aligned}$$

(2) Straightforward.
(3) From (1) above and Theorem (6.2) (7), we obtain

$$\begin{aligned}X \sim sb_{(i,j)}(A) &\equiv X \sim \big(scl_{(i,j)}(A) \cap scl_{(i,j)}(X \sim A)\big)\\ &\equiv \big(X \sim scl_{(i,j)}(A)\big) \cup \big(X \sim scl_{(i,j)}(X \sim A)\big)\\ &\equiv sint_{(i,j)}(X \sim A) \cup sint_{(i,j)}(A).\end{aligned}$$

(4) If $x \in A$, then $scl_{(i,j)}(A)(x) = 1 = (A \cup sb_{(i,j)}(A))(x)$.
If $x \notin A$, then

$$\begin{aligned}(A \cup sb_{(i,j)}(A))(x) &= sb_{(i,j)}(A)(x)\\ &= min\big(1 - sint_{(i,j)}(A)(x), 1 - sint_{(i,j)}(X \sim A)(x)\big)\\ &= 1 - sint_{(i,j)}(X \sim A)(x) = scl_{(i,j)}(A)(x).\end{aligned}$$

(5) From Theorem (5.3) (3), Theorem (5.6) (4), Lemma (5.5) and (4) above, we obtain

$$\begin{aligned}A \in sF_{(i,j)} &\longleftrightarrow sd_{(i,j)}(A) \subseteq A\\ &\longleftrightarrow A \cup sd_{(i,j)}(A) \subseteq A\\ &\longleftrightarrow scl_{(i,j)}(A) \subseteq A\\ &\longleftrightarrow A \cup sb_{(i,j)}(A) \subseteq A\\ &\longleftrightarrow sb_{(i,j)}(A) \subseteq A.\end{aligned}$$

(6) From Theorem (6.2) (7) and (4) above, we obtain

$$\begin{aligned} sint_{(i,j)}(A) &\equiv X \sim scl_{(i,j)}(X \sim A) \\ &\equiv X \sim \big(X \sim A \cup sb_{(i,j)}(X \sim A)\big) \\ &\equiv A \cap \big(X \sim sb_{(i,j)}(X \sim A)\big) \equiv A \cap \big(X \sim sb_{(i,j)}(A)\big). \end{aligned}$$

(7) From Theorem (6.2) (5) and (6) above, we obtain

$$\begin{aligned} sb_{(i,j)}(A) \cap A \equiv \phi &\longleftrightarrow (X \sim sb_{(i,j)}(A)) \cup (X \sim A) \equiv X \\ &\longleftrightarrow A \subseteq X \sim sb_{(i,j)}(A) \\ &\longleftrightarrow A \cap (X \sim sb_{(i,j)}(A)) \equiv A \\ &\longleftrightarrow sint_{(i,j)}(A) \equiv A \longleftrightarrow A \in s\tau_{(i,j)}. \end{aligned}$$

(8) From Theorem (6.2) (3), we have

$$\begin{aligned} sb_{(i,j)}(A)(x) &= \min\big(1 - sint_{(i,j)}(A)(x), 1 - sint_{(i,j)}(X \sim A)(x)\big) \\ &\leq \min\big(1 - int_i(A)(x), 1 - int_i(X \sim A)(x)\big) = b_i(A)(x). \end{aligned}$$

(9) From (3) above, we have
$X \sim sb_{(i,j)}(A) \equiv sint_{(i,j)}(A) \cup sint_{(i,j)}(X \sim A) \equiv sint_{(i,j)}(A) \cup sext_{(i,j)}(A)$.

7 (i,j)-semi Convergence of Nets in Fuzzifying Bitopological Spaces

Definition 7.1. :Let (X, τ_1, τ_2) be a fuzzifying bitopological space. The class of all nets in X is denoted by $N(X) = \{S | S : D \to X, \text{where } (D, \geq) \text{ is a directed set}\}$.

Definition 7.2. Let (X, τ_1, τ_2) be a fuzzifying bitopological space.The binary fuzzy predicates $\rhd^s_{(i,j)}$, $\propto^s_{(i,j)} \in \Im(N(X) \times X)$, are defined as follows:
$S \rhd^s_{(i,j)} x := \forall A(A \in sN^{(i,j)}_x \to S \subsetneqq A)$,
$S \propto^s_{(i,j)} x := \forall A(A \in sN^{(i,j)}_x \to S \eqslantless A)$, $S \in N(X)$,
where $S \rhd^s_{(i,j)} x$, $S \propto^s_{(i,j)} x$ stand for "S is $(i,j)-$semi converges to x", "x is $(i,j)-$semi accumulation point of S". Also, $\subsetneqq$ and $\eqslantless$ are the binary crisp predicates "almost in" and "often in", respectively.

Definition 7.3. The fuzzy sets,
$lim^s_{(i,j)} T(x) = [T \rhd^s_{(i,j)} x]$;
$adh^s_{(i,j)} T(x) = [T \propto^s_{(i,j)} x]$,
where $T \in N(X)$, are called $(i,j)-$semi limit and $(i,j)-$semi adherence of T, respectively.

Theorem 7.4. Let (X, τ_1, τ_2) be a fuzzifying bitopological space, $x \in X$, $A \in P(X)$ and $S \in N(X)$.Then
(1) $\models \exists S((S \subseteq A \sim \{x\}) \wedge (S \rhd^s_{(i,j)} x)) \longrightarrow x \in sd_{(i,j)}(A)$;
(2) $\models \exists S((S \subseteq A) \wedge (S \rhd^s_{(i,j)} x)) \longrightarrow x \in scl_{(i,j)}(A)$;
(3) $\models A \in sF_{(i,j)} \longrightarrow \forall S(S \subseteq A \to lim^s_{(i,j)} S \subseteq A)$;
(4) $\models \exists T((T < S) \wedge (T \rhd^s_{(i,j)} x)) \longrightarrow S \propto^s_{(i,j)} x$,
where $S \subseteq A$ and $T < S$ stand for "S is all in A", "T is a subnet of S", respectively.

Proof. (1) We know that, $[S \rhd^s_{(i,j)} x] = \inf_{S \not\subseteq A}(1 - sN^{(i,j)}_x(A))$.Also,
$[\exists S((S \subseteq A \sim \{x\}) \wedge (S \rhd^s_{(i,j)} x))] = \sup_{S\subseteq A\sim\{x\}} \inf_{S \not\subseteq B}(1 - sN^{(i,j)}_x(B))$.
First, for any $S \in N(X)$ such that $S \subseteq A \sim \{x\}$, we have $S \not\subseteq (X \sim A) \cup \{x\}$. Therefore, $\inf_{S \not\subseteq B}(1 - sN^{(i,j)}_x(B)) \leq 1 - sN^{(i,j)}_x((X \sim A) \cup \{x\}) = [x \in sd_{(i,j)}(A)]$.

(2) If $x \in A$, then from Theorem (5.6) (1) we can prove this similar (1) above. If $x \notin A$, then $A \sim \{x\} = A$ from Theorem (5.6) (1) and (1) above we have,

$$\begin{aligned}[\exists S((S \subseteq A) \wedge (S \rhd^s_{(i,j)} x))] &= [\exists S((S \subseteq A \sim \{x\}) \wedge (S \rhd^s_{(i,j)} x))] \\ &\leq 1 - sN^{(i,j)}_x(X \sim A) = scl_{(i,j)}(A)(x) = [x \in scl_{(i,j)}(A)].\end{aligned}$$

(3) $$\begin{aligned}[\ \forall S(S \subseteq A \to lim^s_{(i,j)}S \subseteq A)\ ] &= \inf_{S\subseteq A}\ \inf_{x\notin A}\Big(\ 1 - \inf_{S \not\subseteq B}(\ 1 - sN^{(i,j)}_x(B)\ )\ \Big) \\ &= \inf_{S\subseteq A}\inf_{x\notin A}\sup_{S \not\subseteq B} sN^{(i,j)}_x(B).\end{aligned}$$

In the other hand, from Theorem (5.6) (6) and (2) above, we have

$$\begin{aligned}[A \in sF_{(i,j)}] &= [A \equiv scl_{(i,j)}(A)] = [scl_{(i,j)}(A) \subseteq A] \wedge [A \subseteq scl_{(i,j)}(A)] \\ &= [scl_{(i,j)}(A) \subseteq A] = [X \sim A \subseteq X \sim scl_{(i,j)}(A)] \\ &= \inf_{x\in X\sim A}(1 - scl_{(i,j)}(A)(x)) \\ &\leq \inf_{x\in X\sim A}(1 - \sup_{S\subseteq A}\inf_{S \not\subseteq B}(1 - sN^{(i,j)}_x(B))) \\ &= \inf_{x\notin A}\inf_{S\subseteq A}\sup_{S \not\subseteq B} sN^{(i,j)}_x(B) = [\forall S(S \subseteq A \to lim^s_{(i,j)}S \subseteq A)].\end{aligned}$$

(4) $[S \propto^s_{(i,j)} x] = \inf_{S \not\subseteq A}(1 - sN^{(i,j)}_x(A))$,
$[\exists T((T < S) \wedge (T \rhd^s_{(i,j)} x))] = \sup_{T<S}\inf_{T \not\subseteq A}(1 - sN^{(i,j)}_x(A))$.
Set $\mathcal{A}_S = \{A|S \not\subseteq A\}$, $\mathcal{B}_T = \{A|T \not\subseteq A\}$. Then for any $T < S$, we have $\mathcal{A}_S \subseteq \mathcal{B}_T$. In fact, suppose $T = S \circ K$. If $S \not\subseteq A$, then there exists $\sigma_0 \in \mathcal{D}_S$ such that $S(\sigma) \notin A$ when $\sigma \geq \sigma_0$. Now, we will show that $T \not\subseteq A$. If not, then there exists $\mu_0 \in \mathcal{D}_T$ such that $T(\mu) \in A$, when $\mu \geq \mu_0$. Moreover, there exists $\mu_1 \in \mathcal{D}_T$ such that $K(\mu_1) \geq \sigma_0$ because $T < S$, and there exists $\mu_2 \in \mathcal{D}_T$ such that $\mu_2 \geq \mu_0, \mu_1$ because $\mathcal{D}_T$ is directed. In this way, $K(\mu_2) \geq \sigma_0$, $S(K(\mu_2)) \notin A$ and $S(K(\mu_2)) = T(\mu_2) \in A$, a contradiction. Therefore,

$$\begin{aligned}[\exists T((T < S) \wedge (T \rhd^s_{(i,j)} x))] &= \sup_{T<S}\inf_{A\in\mathcal{B}_T}(1 - sN^{(i,j)}_x(A)) \\ &\leq \inf_{A\in\mathcal{A}_S}(1 - sN^{(i,j)}_x(A)) = [S \propto^s_{(i,j)} x].\end{aligned}$$

Theorem 7.5. Let (X, τ_1, τ_2) be a fuzzifying bitopological space. If T is a universal net, then $\models lim^s_{(i,j)}T = adh^s_{(i,j)}T$.

Proof. For any net $T \in N(X)$ and any $A \subseteq X$ one can obtain that if $T \not\subseteq A$, then $T \not\subseteq A$. Suppose T is a universal net in X and $T \not\subseteq A$. Then, $T \subsetneq X \sim A$. So $T \not\subseteq A$ (Indeed, $T \subsetneq X \sim A$ if and only if there exists $m \in D$ such that for every $n \in D$, $n \geq m$, $T(n) \in X \sim A$ if and only if there exists $m \in D$ such that for every $n \in D$,

$n \geq m$, $T(n) \notin A$ if and only if $T \not\subsetneq A$.). Hence for any universal net T in X, we have
$lim^s_{(i,j)}T(x) = \inf_{T \not\subsetneq A}(1 - sN^{(i,j)}_x(A)) = \inf_{T \not\subsetneq A}(1 - sN^{(i,j)}_x(A)) = adh^s_{(i,j)}T(x)$.

Lemma 7.6. Let (X, τ_1, τ_2) be a fuzzifying bitopological space.
$\models (T \rhd^s_{(i,j)} x)) \longleftrightarrow \forall A(x \in A \in s\tau_{(i,j)} \to T \subsetneq A)$.

Proof. If $B \subseteq A$ and $T \not\subsetneq A$, then $T \not\subsetneq B$

$$\begin{aligned}[T \rhd^s_{(i,j)} x)] &= \inf_{T \not\subsetneq A}(1 - sN^{(i,j)}_x(A)) \\ &= 1 - \sup_{T \not\subsetneq A}\sup_{x \in B \subseteq A} s\tau_{(i,j)}(B) \\ &\geq 1 - \sup_{T \not\subsetneq B, x \in B} s\tau_{(i,j)}(B) \\ &= \inf_{T \not\subsetneq B, x \in B}(1 - s\tau_{(i,j)}(B)) = [\forall A(x \in A \in s\tau_{(i,j)} \to T \subsetneq A)].\end{aligned}$$

Conversely, since

$$\begin{aligned}[\forall A(x \in A \in s\tau_{(i,j)} \to T \subsetneq A)] &= \inf_{T \not\subsetneq A, x \in A}(1 - s\tau_{(i,j)}(A)) \\ &= \inf_{T \not\subsetneq A, x \in A}(1 - \inf_{x \in A}\sup_{B \subseteq A} sN^{(i,j)}_x(B)) \\ &\geq 1 - \sup_{T \not\subsetneq B, x \in B} sN^{(i,j)}_x(B) \\ &= \inf_{T \not\subsetneq B, x \in B}(1 - sN^{(i,j)}_x(B)) = [T \rhd^s_{(i,j)} x].\end{aligned}$$

8 (i,j)-semi Convergence of Filters in Fuzzifying Bitopological Spaces

Definition 8.1. Let (X, τ_1, τ_2) be a fuzzifying bitopological space and $F(X)$ be the set of all filters on X. The binary fuzzy predicates $\rhd^s_{(i,j)}$, $\propto^s_{(i,j)} \in \Im(F(X) \times X)$ are defined as follows:
$K \rhd^s_{(i,j)} x := \forall A(A \in sN^{(i,j)}_x \to A \in K)$,
$K \propto^s_{(i,j)} x := \forall A(A \in K \to x \in scl_{(i,j)}(A))$, where $K \in F(X)$.

Definition 8.2. The fuzzy sets,
$lim^s_{(i,j)}K(x) = [K \rhd^s_{(i,j)} x]$;
$adh^s_{(i,j)}K(x) = [K \propto^s_{(i,j)} x]$,
are called $(i,j)-$semi limit and $(i,j)-$semi adherence sets K, respectively.

Theorem 8.3. Let (X, τ_1, τ_2) be a fuzzifying bitopological space.
(1) If $T \in N(X)$ and K^T is the filter corresponding to T, i.e., $K^T = \{A | T \subsetneq A\}$, then
(a) $\models lim^s_{(i,j)}K^T = lim^s_{(i,j)}T$;
(b) $\models adh^s_{(i,j)}K^T = adh^s_{(i,j)}T$.
(2) If $K \in F(X)$ and T^K is the net corresponding to K, i.e., $T^K : D \to X$, $(x, A) \mapsto x, (x, A) \in D$, where $D = \{(x, A) | x \in A \in K\}$, $(x, A) \geq (y, B)$ iff $A \subseteq B$, then
(a) $\models lim^s_{(i,j)}T^K = lim^s_{(i,j)}K$;
(b) $\models adh^s_{(i,j)}T^K = adh^s_{(i,j)}K$.

Proof. (1) For any $x \in X$, we have

(a) $lim^s_{(i,j)} K^T(x) = \inf_{A \notin K^T} (1 - sN^{(i,j)}_x(A)) = \inf_{T \not\lessapprox A} (1 - sN^{(i,j)}_x(A)) = lim^s_{(i,j)} T(x)$.

$$(b)\ adh^s_{(i,j)} K^T(x) = \inf_{A \in K^T} scl_{(i,j)}(A)(x) = \inf_{T \lessapprox A} (1 - sN^{(i,j)}_x(X \sim A))$$
$$= \inf_{T \not\lessapprox X \sim A} (1 - sN^{(i,j)}_x(X \sim A)) = adh^s_{(i,j)} T(x).$$

(2) (a) First we prove that $T^K \lessapprox A$ if and only if $A \in K$. If $A \in K$, then $A \neq \phi$ and there exists at least an element $x \in A$. So for $(x, A) \in D$ and $(y, B) \in D$ such that $(y, B) \geq (x, A)$, then $B \subseteq A$ and so $T^K(y, B) = y \in B \subseteq A$. Thus $T^K \lessapprox A$.

Conversely, suppose $T^K \lessapprox A$, then there exists $(y, B) \in D$, for all $(z, C) \in D$, such that $(z, C) \geq (y, B)$ and we have $T^K(z, C) \in A$. So for every $z \in B$, $(z, B) \geq (y, B)$ and $T^K(z, B) = z \in A$ implies $B \subseteq A$. Then $A \in K$. Thus $T^K \not\lessapprox A$ if and only if $A \notin K$. Now,

$$lim^s_{(i,j)} T^K(x) = [T^K \rhd^s_{(i,j)} x] = \inf_{T^K \not\lessapprox A} (1 - sN^{(i,j)}_x(A))$$
$$= \inf_{A \notin K} (1 - sN^{(i,j)}_x(A)) = lim^s_{(i,j)} K(x).$$

(b) First we prove that $X \sim A \in K$ if and only if $T^K \not\lessapprox A$. Suppose $T^K \not\lessapprox A$, then there exists $(z, B) \in D$ such that for every $(y, C) \in D$ with $(y, C) \geq (z, B), T^K(y, C) \notin A$. Now for every $x \in B, (x, B) \geq (z, B)$ and $T^K(x, B) = x \notin A$, i.e., $B \cap A = \phi$ so $B \subseteq X \sim A$ and then $X \sim A \in K$.

Conversely, suppose $X \sim A \in K$, then $X \sim A \neq \phi$ and thus it contains at least an element x. Now, for any $(z, C) \in D$ such that $(z, C) \geq (x, X \sim A)$, one can have that $T^K(z, C) = z \notin A$. Hence, $T^K \not\lessapprox A$. Now,

$$adh^s_{(i,j)} T^K(x) = [T^K \propto^s_{(i,j)} x] = \inf_{T^K \not\lessapprox A} (1 - sN^{(i,j)}_x(A))$$
$$= \inf_{X \sim A \in K} scl_{(i,j)}(X \sim A) = \inf_{B \in K} scl_{(i,j)}(B) = adh^s_{(i,j)} K(x).$$

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EFFECTS OF NEW METROPOLITAN LAW NO.6360 TO RURAL DEVELOPMENT IN TURKEY

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Abstract – Actually implementing of Law No. 6360 by the year 2014 has brought many changes in urban and rural areas. The villages which lost their legal entity and turned into a neighborhood are considered to be affected more. At point of rural development, positive and negative effects of these changes on the local people are important. Local people living in rural are unable to grasp the impact of the law. Positively, in the future; since the villages will turn to the neighborhoods the social structures are likely to change and the level of consciousness will be increased. Negatively, agricultural production is expected to be stopped and financial difficulties will be seen. Besides, the term "rural" starts to disappear, thus traditional, folkloric and cultural heritage values will be lost. In order to eliminate the negative effects of this law, decision makers shall implement necessary plans and programs urgently. If no measures are taken, it will result in direct and indirect destruction in the future.

Keywords – *New metropolitan law, Law No. 6360, rural development, legal entity of villages, Turkey.*

1. Introduction

Rural areas are prioritized in the formulation of development policies for sustainable development. In Turkey, with respect to distance from rural and town centers, due to geographical, economic and political barriers, rural areas are failed to show enough progress. At the point of bringing the service to the villages, role of local administration is important. There seems a tripartite structure composed of special provincial administrations, local government and villages in large metropolis lack of county municipalities.

Increasing problems such as industrialization, transportation and environmental problems are not possible to be solved by the local administrations lack of adequate financial resources and production capacities. This situation prevents the effective and appropriate

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use of resources, in addition to the small settlements it leads to serious management problems in the metropolitan population.

Since there is no appropriate size and strong local government to serve, public can not be met in the quality of public services from local government and lack of coordination in the delivery of public services raises problems. Because there are many authorized local government in a specific geographical area, there occurs planning and coordination problems and insufficient use of scale economies which leads to waste of resources.

In terms of local economy, municipalities are seen to the leading institutions according to requirements of employment and investment power among the region. The expenses of the municipalities was 41.1 billion TL in 2011. Municipalities are important institutions in terms of employment. The number of personnel in the municipalities make up about one-fifth of public employment. Effective use of municipal resources, is of great importance in terms of the development of the local economy and it is expected to use limited public resources [1]. By Law No. 6360, in addition to 16 metropolises in the existing structure, 14 new metropolitans were added and Turkey has totally 30 metropolises. This amendment is expected to be both positive and negative effects. However, the effects of this law are very different to the village who lost their legal personality. [11]; One of the law's most radical arrangements is the elimination of 16 thousand of the village, about half of that, one of Turkey's oldest traditional local government units. Thus, common goods used by the villagers for centuries, meadows and pastures are now being taken into metropolitan boundaries. To what extent it would be effective transitional provisions for the protection of forests and pastures are controversial. Since it is new as of the date of implementation of the law, scientific studies on the point of impact on the countryside is insufficient. Also indirectly aspect of this law, Turkey's food supply safety is considered to be affected. Because the source of the village's agricultural production has gained a new dimension in the framework of the law. This side is an issue that should be examined in the interests of the country. The purpose of the study is to examine the positive and negative effects on social and economic aspects among the agricultural production, agricultural areas, manufacturer, country these days and future of the village-turned-neighborhood by Law 6360. Section 2 gives account of previous work. Our new and exciting results are described in Section 3. Finally, Section 4 gives the conclusions.

2. New Metropolitan Law and Changes

Main objective of the economy is the best way to use scarce resources. Ideal method for the public to meet with the public services is searched. Services need to be carried out under one authority in terms of efficiency, quality and coordination. Therefore "New Metropolitan" law was introduced with date of 2012 and numbered 6360. This new administrative configuration was found to have a significant impact on the number of cities in Turkey. When the criteria 10,000 considered, main city number 66 in 1927, reached 104 in 1950, and reached 470 in 2000 while it was 454 in 1990 [3]. In Turkey, despite the rapid increase in the urban population, the number of cities was decreased to 388 as a result of greater border settlements near metropolitan applications in 2012. [8]. Metropolitan combined with municipal status winning 30 provinces and towns that are connected to the municipality of the village with the transformation of the neighborhood by removing the entities district centers in the neighborhood, as a result of the creation of the metropolitan administrative areas the number of urban settlements in 2013 declined to 212 [5]. In the year after the law is adopted it caused controversy in many disciplines. Because radical changes in positive and negative sense of the law is thought to bring. Sustainable development is an essential element of management is in place. Because it is known that the disadvantages according to the government's decentralization from the center. Today local governments are an idea that is being considered by many countries.

According to Law No. 6360 dated March 30, 2014 about structural point of Local Government Elections [18];

- 14 provinces became metropolitan municipalities,
- 559 municipalities with a population under 2,000, converted into a village,
- After entering into force of Law No. 5747, two stages (metropolitan and district municipalities) are established and functioning in the current metropolitan municipality model, irrespective of the 2,000 population requirement, this dual model for the new metropolitan municipality within the boundaries 1076 village municipalities and 16,500 village lost its legal personality and turned to neighborhoods,
- Instead of the municipal prerequisite that population exceeds 750,000, it changed the form of the provincial population is over 750,000
- At 14 new metropolitan municipality 25 new districts were established and total number of Metropolitan Municipality has increased by 519 from 143,
- Population of Metropolitan Municipality to be taken by the district municipality's from 50,000 to 20,000.
- All other 'value-added' as in the process of closing the neighborhood converted to municipalities and villages, and sharing foreseen liquidation commissions,
- Thirty metropolitan municipality’s special provincial administrations were removed,
- Instead of the removed special provincial administration in the province, Investment

Monitoring and Coordination Departments was established depending on governor.

The benefits of the expansion of the city administration; The growth of the service delivery capacity, decrease in service costs per unit, effective use of resources, increasing the human resources capacity, can be made of integrated development plans, the implementation of more large-scale project to cover more than one district were expressed as to facilitate the provision of services that require co-ordination [6]. This construction will lead to problems of urbanization as well as the advantages of the city administration. According to Bookchin concept of urbanization [7]; "In fact, today the city and country is that it is under a siege that threaten the natural environment of the location of the humanity. Urbanization is destroying both, consisting of the tradition they have and where diversity and identity are threatened by urbanization. Urbanization not only in rural areas, are swept away in the city. only the town and village life fed by the agricultural relationship values, culture, and not the institution, fed by the civic association of city life values and culture institutions also swallow. Anonymity, homogeneity and corporate as stifling features urbanization, the closeness between people, neighborhoods and human in the unique qualities as scale swallow urban area which contains a policy of closeness to nature, eliminating the sacred aid understanding and hosting rural area close family relationships, "in the form represents. the new metropolitan structure, metropolitan expanded municipal boundaries and a new model was formed. District municipalities, towns and rural areas have been incorporated into the metropolitan boundaries, combined with the administrative provinces.

The ground of the law as an efficient, effective, citizen-driven, accountable, participatory and transparent in the framework of a local management approach is stated that the objectives pursued continuing t reform of public administration. Improve the quality of services, increase citizen satisfaction, implementation of integrated urban plans in all metropolitan and provide a framework to be considered in determining the macro policies; the scale of the utilization of the provision and coordination of the economy are considered among the main reasons [9]. New Metropolitan Law services, transfers and financial changes can be listed as follows [13];

- Metropolitan district received as part of the master plan will be held by the Municipality Council, implementation plans and Article 18 of the applications will be approved by the Municipality Council.
- Public transport services in the metropolitan boundaries (including district centers) will be carried out by the Municipality.
- The road that connects the town center to the neighborhood (village roads) construction, maintenance, cleaning, snow-fighting duties will be carried out by the Municipality.
- Water sewerage services within the boundaries of the Metropolitan Municipality will be carried out by the Water and Sewage Administration under Metropolitan Municipality
- Metropolitan municipalities in its province investments and services of public institutions and organizations, emergency, disaster, introduction of provincial representation, ceremony, the execution of reward and protocol services, are carried out by Monitoring Investment Coordination Directorate depending on the governor
- Municipalities will be able to cash assistance to amateur sports clubs will be regulated by law to worship, to the maintenance task will be given to the municipalities. There will also be given free or discounted water to mosques.
- Disaster risk of life and property or to evacuate the building and constitute a threat to the security of destruction will be assigned to the district municipality.
- Opening female guest house for metropolitan municipality with a population of over 100,000 has been compulsory.
- Significant increases in parallel increases of the income of the responsibility with the increasing responsibilities of the municipalities are outstanding.
- Metropolitan boundaries of municipalities outside of revenues, decreased to 1.50% from 2.85%, revenue from the Metropolitan to a district municipality has increased from 2.50% to 4.50%
- Metropolitan Municipality has received 5% interest rate increased to 6% of the budget. This share is 60% directly and the remaining 40% to 70% of the population, while 30% will be distributed according to the surface area basis.
- Previously way of municipalities required to collect the sewage and water facilities will be left to the discretion of the administrative expenditure of the contribution collection.
- In the villages returned to the neighborhood, grocery store, grocery store, barber shop, bakery, coffee shops, restaurants and the like shall be deemed licensed.
- County upon request of municipalities, metropolitan municipalities in accordance with the zoning regulations in force for the neighborhood turned into structures in the villages of commercial purposes will be the local traditional, cultural and architectural features of the appropriate type of architectural projects do or make the provision was introduced.

- Legal entity that needs to be removed due to the Law on Municipal Revenues in the village taxes, fees and contributions shall be for a period of 5 years. Drinking water in the lowest wage rate will be for 5 years at the rate of 25%.
- Legal personality removed municipalities and villages existing staff, movable and immovable, work equipment or receivables of other vehicles to public institutions and organizations from the date of the Law of the debt issued will be reported to the county council will participate in 1 month.

Due to be implementation of Law No. 6360 after 2014 local elections, all these changes is expected to be more clearly observed effects in later years. But the main thing is thought to be created in rural as well as how the impact of these effects. Because at those 14 provinces instead of concept of rurality the concept urbanity brought up. The rural people have some difficulty in adapting to these changes. Because the adoption of rural innovation is not easy.

3. Effects of New Metropolitan Law No. 6360 to the existing Rural and Agricultural Structure

The last point of the village came under this law in terms of agricultural production is important [12]; socio-economic structures of these centers is based on agriculture; All existing planning, services and investment priorities were constructed considering these features. However, as almost none of the locations and functions of metropolitan municipalities do not have any serious and dimensional policies and plans for agriculture. In this case, it is clear that rural development is hampered and harmed.

Table 1. Distribution of Business and Agricultural Lands in Turkey according to Agricultural Business Size and Land Management (%)

Business Size (decares)	TOTAL	Businesses that own land				Businesses who do not own land							
		Only operating its own land		Both its own land and else's land		Only operating land with rent		Operating land only sharecropping		Other forms of land operating		Operating with two or more land-saving	
	A B	A	B	A	B	A	B	A	B	A	B	A	B
Total	100.0	85.1	71.4	12.7	26.4	1.6	1.5	0.4	0.4	0.1	0.1	0.1	0.2
-5	100.0	95.2	94.7	2.1	2.3	2.2	2.3	0.6	0.7	-	-	-	-
5-9	100.0	96.2	95.9	2.2	2.4	1.3	1.4	0.2	0.2	0.0	0.0	0.1	0.1
10-19	100.0	94.2	94.0	4.1	4.3	1.3	1.3	0.3	0.2	0.1	0.1	0.0	0.1
20-49	100.0	89.9	89.1	0	8.9	1.5	1.4	0.5	0.5	0.0	0.0	0.1	0.1
50-99	100.0	82.7	81.2	14.6	16.0	2.0	2.1	0.5	0.5	0.1	0.1	0.1	0.1
100-199	100.0	72.7	70.9	24.7	26.5	1.9	1.9	0.5	0.5	0.1	0.1	0.1	0.1
200-499	100.0	64.7	62.2	334	35.8	1.4	1.5	0.4	0.3	0.0	0.0	0.1	0.1
500-999	100.0	55.2	53.1	42.0	44.0	1.6	1.5	0.2	0.2	-	-	1.0	1.2
1000-2499	100.0	59.7	59.3	40.0	40.4	0.1	0.2	0.0	0.0	0.0	0.0	0.1	0.1
2500-4999	100.0	65.7	63.7	33.8	35.7	0.2	0.2	0.3	0.4	0.1	0.1	-	-
5000+	100.0	83.3	92.1	14.3	6.3	1.4	0.5	0.6	0.1	0.5	1.0	-	-

Source: [14] **A:** Business **B:** Agricultural Land

The main effect of the new Metropolitan Law to the agricultural sector is thought to be shrinking agricultural land. Because with the new law, villages turned to neighborhood by taking the urban planning are coming to the operating position and will be opened to construction by time. Farmers are not satisfied with the agricultural policies applied in

Turkey. The high price and shortage of agricultural inputs, especially in the marketing of agricultural products, are pushing manufacturers to stop production. If this is added on top of the effects of these laws, it is considered to be inevitable in this case. Agricultural areas are fragmented and small structure size in Turkey. This disadvantage limits the agricultural production. According to farm size and shape of the savings business and distribution of agricultural land has been given in Table 1.

Referring to Table 1, the proportion of businesses which only operates their land is 85.1%. Here it is seen that family farming is still in the form of agricultural production in Turkey. Many of the small size family business are seemed to make a small amount in terms of agricultural production and production for the market in the coming years. As result of the rapid increase in population in the World, developed countries make plan for the food safety and supply. Rapid population growth in Turkey and limited food supply are likely to be dangerous in terms of security and food production. In addition, with the Law no: 6360, food supply security issues will arise indirectly from agricultural production. Agricultural policies with the high cost of trying to maintain economic difficulties of living under the low income clamp economically and more will be forced to dispose of as they encountered as a result of land [10]. The effects of this law is also seem not short term but long term. In Table 2, the rural population has seen to decrease from 2012 while the law was adopted.

Table 2. Distribution of Population of Cities, towns and villages in terms of years in Turkey

	Cities and towns				Districts and Villages			
Year	**Total %**	**Total**	**Male**	**female**	**Total %**	**Total**	**Male**	**Female**
2010	76.3	56.222.356	28.308.856	27.913.500	23.7	17.500.632	8.734.326	8.766.306
2011	76.8	57.385.706	28.853.575	28.532.131	23.2	17.338.563	8.679.379	8.659.184
2012	77.3	58.448.431	29.348.230	29.100.201	22.7	17.178.953	8.607.938	8.571.015
2013	91.3	70.034.413	35.135.795	34.898.618	8.7	6.633.451	3.337.565	3.295.886
2014	91.8	71.286.182	35.755.990	35.530.192	8.2	6.409.722	3.228.312	3.181.410

Source: [15]

With the adoption of the Law, the rural population is declined to 8.2% from 22.7%, in 2014. There was a decrease of 63% a total waist and the rural population in 2014 compared to 2012. There has been an increase in the population of the province and district as a result of the decline of the rural population. The population increased by 82% provincial and district centers in 2014 compared to 2012. This information has been confirmed in the following table (Table 3).

Table 3. Distribution of the numbers of towns and villages, districts in terms of years in Turkey

Year	Number of towns	Number of Districts	Number of Villages
2010	892	1.977	34.402
2011	892	1.977	34.425
2012	892	1.977	34.434
2013	919	394	18 214
2014	919	396	18.340

Source: [16]

In Table 3, with the adoption of the law, there is a decrease of 47% at number of villages in 2014 while compared to 2012. The number of villages was reduced number of cities was increased. This step towards urbanization is likely to trigger a migration problem in 14 metropolitan municipalities. Contraction of the agricultural areas, city's attractive opportunities, To increase opportunities for travel services will force the local population to migrate. Migration data of 14 metropolitan municipalities under the Law No: 6360 are presented in Table 4.

Table 4. Metropolitan Immigration Status of Cities Under the scope of the Law No. 6360

Period	Cities	Total Population	remigration	migration	Net migration	Net migration speed (‰)
2013/2014	Aydın	1.041.979	45 842	32.396	13,446	13.0
	Balıkesir	1.189.057	57 551	39.918	17,633	14.9
	Denizli	978.700	28 279	24.771	3,508	3.6
	Hatay	1.519.836	32 678	39.181	-6,503	-4.3
	Malatya	769.544	29 285	31.476	-2,191	-2.8
	Manisa	1.367.905	35 570	38.432	-2,862	-2.1
	Kahramanmaraş	1.089.038	27 619	30.903	-3,284	-3.0
	Mardin	788.996	22 207	30.796	-8,589	-10.8
	Muğla	894.509	48 219	29.671	18,548	21.0
	Ordu	724.268	28 555	39.937	-11,382	-15.6
	Tekirdağ	906.732	52 994	31.266	21,728	24.3
	Trabzon	766.782	31 847	29.741	2,106	2.8
	Şanlıurfa	1.845.667	35 670	49.030	-13,360	-7.2
	Van	1.085.542	27 587	44.435	-16,848	-15.4
TOTAL			**503.903**	**491.953**		
2012/2013	Aydın	1.020.957	34.688	32.338	2,350	2.3
	Balıkesir	1.162.761	38.710	39.688	-978	-0.8
	Denizli	963.464	27.088	24.039	3,049	3.2
	Hatay	1.503.066	29.067	39.315	-10,248	-6.8
	Malatya	762.538	25.876	33.194	-7,318	-9.6
	Manisa	1.359.463	36.257	36.989	-732	-0.5
	Kahramanmaraş	1.075.706	24.560	29.322	-4,762	-4.4
	Mardin	779.738	22.596	29.525	-6,929	-8.8
	Muğla	866.665	35.246	30.687	4,559	5.3
	Ordu	731.452	30.792	46.332	-15,540	-21.0
	Tekirdağ	874.475	45.313	31.681	13,632	15.7
	Trabzon	758.237	25.115	29.988	-4,873	-6.4
	Şanlıurfa	1.801.980	33.383	47.429	-14,046	-7.8
	Van	1. 070.113	32.118	38.507	-6,389	-6.0
TOTAL			**440.809**	**489.034**		
2011/2012	Aydın	1.006.541	32.412	29.623	2789	2.8
	Balıkesir	1.160.731	34.922	35.315	-393	-0.3
	Denizli	950.557	24.446	21.992	2454	2.6
	Hatay	1.483.674	27.260	35.139	-7879	-5.3
	Malatya	762.366	24.270	28.545	-4275	-5.6
	Manisa	1.346.162	32.211	34.054	-1843	-1.4
	Kahramanmaraş	1.063.174	19.908	29.467	-9559	-9.0
	Mardin	773.026	21.676	30.299	-8623	-11.1
	Muğla	851.145	33.213	28.301	4912	5.8
	Ordu	741.371	48.240	26.595	21645	29.6
	Tekirdağ	852.321	42.155	28.042	14113	16.7
	Trabzon	757.898	21.864	25.478	-3614	-4.8
	Şanlıurfa	1.762.075	31.890	44.878	-12988	-7.3
	Van	1.051.975	50.003	46.639	3364	3.2
TOTAL			**444.470**	**444.367**		

Source: [17]

The adoption of the Law No. 6360 and the date of implementation will take time to see the effects. However, the migration is observed after the implementation of the law (Table 4). Regarding the migration movements, migration of 14 cities is 444 367 in the period of 2011/2012 while 491 953 migration in the 2013/2014 period with a rise of 10%. Due to the narrowing of the income elements at the rural area, migration is likely to increase to urban area.

In terms of effects of the law; on Agricultural Production and Producers;

- Traditional agricultural products will be reduced and industrial production will be carried out instead.
- Agricultural areas are likely to become available for different purposes.
- Local flavor will occur to disappear.
- Food inflation will increase in 14 cities, depending on the agricultural production.
- Exports of agricultural products will be reduced at provinces such as Muğla, Ordu and Trabzon.
- Input costs will increase because for producers will work with limited circumstances when compared with old conditions.
- Types of tourism such as rural tourism, ecotourism, agroturizm, farm tourism would be impossible to be performed in this area.
- Input costs will increase because producers may be charged with the name of irrigation of agricultural areas, participation of water in springs campus to share at the livestock, price and so on.

In terms of the social effects of the law;

- Urban culture will be widespread and rural culture will be lost by the time.
- Awareness of local residents will increase.
- There will be a shift towards a consumption society from producer society.
- Folklore, cultural values and traditions will be forgotten and will not be passed on to future generations.
- People who live in the village, will get away from their traditional ways of life and production resources and will be directed to a new and unfamiliar way of life. [4].
- Local people living in difficulty will increase the potential to create a new wave of immigration.
- Employment will reduce.

In terms of natural resources effects of the law;

- Biological richness of flora and fauna of the villages will disappear in time.
- Rural areas which are the centers of ecological balance, cycle (nitrogen, carbon, water) problems will occur in time.
- The traditional architecture at the villages turned into the neighborhood will be destroyed unless of cultural heritage protects
- Meadows and coastal fronts will be ready to misuse.

In terms of the financial changes effects of the law [4].;

- Eliminates the authority of village headman's specific methods of obtaining revenue, and preparation of documents such as birth and death.
- Income resources of village and district municipalities are taken and given to other big municipalities
- A contribution is requested from services such as infrastructure, road construction made by the Village Service Unions without any contribution of villagers before,
- Property taxes for unpaid building, land, fields and so on will not be asked for five years. After five years, these taxes will be taken from people living in the villages.
- Cost of waste disposal and sanitation tax will be taken and also producers will pay for construction and building use permits.
- Building permits will be issued by the district municipalities and metropolitan municipalities. In this case, producers will pay fees for new construction and building use permits.
- Agricultural production in restricted areas increase agricultural production costs and increase impoverishment in rural areas.

Turkey as a country with a large rural geography and population, has an important economic and human resource potential qualified to accelerate national development. Mobilizing that potential in rural areas; constraints and needs of rural areas, co-ordination of services and investment make a multi-sectoral and integrated planning mandatory. In this context, taking into account rural development changes in the area of new approaches in World, EU integration process, a Rural Development Plan that considers local terms, our country needs, priorities is aimed to establish [2]. In accordance with Article No. 674 of the Ninth Development Plan, " Rural Development Plan will be prepared and put into practice in line with National Rural Development Strategy " is presented but it will be difficult to accomplish this in 14 metropolitan cities' rural areas. If rurality is considered to disappear, it gives a clue about negative implications for the development in rural.

4. Result

By turning boundaries of metropolitan municipalities to city boundaries, the rural-urban differentiation is disappeared with regard to sociological and managerial aspects. Instead of the traditional and natural agricultural production, commercialized industrial agricultural production will become widespread. Because instead of concept “rural”, concept of cities and towns location takes place, rural development phenomenon will disappear. Both the cost of production for the rural population and living costs in the city will increase in parallel. Local people whose lifestyle is intervened and production resources are limited, are pushed to live in more difficult conditions such as employment, education, health and housing problems in the cities. It also paves the way for an unplanned urbanization. In developed countries a flow from the city to village is planned, however, the opposite flow will occur with the migration by opening the front of urbanization in Turkey. New Metropolitan Law expressed in the study is likely to create adverse effects on the countryside, but the provisions of the law give a clue for the future years. Because this is a new application the effects are thought to be occurred in time.

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PREDICTION OF THE RELATIONSHIP BETWEEN THE BIST 100 INDEX AND ADVANCED STOCK MARKET INDICES USING ARTIFICIAL NEURAL NETWORK: (2011-2015)

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Abstract – Prediction techniques and models are significant for people and organizations who wish to make prediction at the stage of investment and decision-making. For investors who want to achieve high earnings from investments, the stock market indexes are extremely important. Price movements in the stock such as political and social ones are affected by many factors. In the studies conducted on İstanbul Stock Exchange Index (BIST-100), the estimation generally of foreign exchange rates, interest rates, gold prices, GNP (Gross Nation Product), CPI (Consumer Price Index) and the relationship with macroeconomic variables such as the rate regard to traditional statistical prediction models were used. In this study, international advanced BIST100 Index of the estimation with Artificial Neural Network (ANN) method will be used as input instead of traditional macroeconomic variables (independent variables) and also stock market index data sets will be used. From January 2011 to December 2015 period, daily closing price of some international advanced stock market indices and BIST 100 Index data were used as data set. Data analysis were carried out through Multilayer Neural Network (MLNN) method, which is an ANN model widely used in MATLAB and the successful rate was %96,92.

Keywords – *Investment, BIST-100, International stock market indices, Artificial neural networks, Prediction.*

1. Introduction

In line with the globalization agenda evolving since the beginning of the 21st century, there have been significant developments in the financial markets in parallel with the stock markets, which can be considered barometers of the economy. A need has been arisen to

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analyze stock markets with not only the domestic factors but also with the stock market indices and factors of the countries that they have an interaction with. Forecasting is to generate ideas and scenarios about the future based on existing or historical data. In other words, forecasting can be defined as any attempt made with the aim of predicting the future. Forecasting techniques and models are important for individuals and organizations who wish to make predictions during the decision making and investment phase. Predicting the stock market index is very important for investors who want to achieve higher earnings from their investments.

The index is an indicator for measuring the proportional change, which consists of variation of one or more variables. Indices are instruments that can provide approximate information about events by reducing complex events to a single digit [1]. The prediction of the BIST 100 index, which is the prediction of stock returns, is an important issue in the field of finance [2]. The value of stocks traded on the securities market is influenced by internal and external factors. Internal factors are the company's estimated earnings and changes in the financial structure of the company. And, the external factors can be caused by factors outside the firm. These macroeconomic factors are variables such as exchange rates, interest rates, gold prices, GDP and CPI rate. The prediction of the BIST100 index has attracted much interest, and pretty much study has been conducted in this regard. However, in most of these studies, only the relationship between the BIST100 index and several macroeconomic variables has been analyzed, and statistical estimation techniques and time series models have been used [3,4].

Artificial neural networks has become widespread only recently, in studies regarding the prediction of the BIST100 index [5 - 11]. In this study, instead of traditional macroeconomic variables, the dataset on the international advanced stock market indices was used as the input (independent variables) in order to predict the BIST 100 index with the ANN method. In addition, as a limitation of the study, it should be noted that unpredictable factors such as political uncertainty, "insider trading" in Turkey, natural disasters and wars directly affect the index, making the prediction difficult.

The paper is organized as follows: section 2 describes advanced stock market indices and the structure of prediction method, section 3 presents some applications on real datasets to verify the effectiveness of the proposed method, and finally, section 4 gives the conclusions.

2. Material and Method

2.1 International Stock Market Indices

Economists can make assessments about the course of economy by analyzing the trends in index. For this reason, first the changes in the indices of advanced stock markets in the United States and Europe are evaluated, and then the indices of developing countries are analyzed when performing stock assessments. In this study, the BIST 100 index and some of the advanced international stock market daily closing data of indices from January 2011 to December 2015 period were used as the data set. These datasets were obtained from website [12]. The international stock market indices used were NASDAQ 100 (USA), NIKKEI 225 (Japan), FTSE (UK), DAX (Germany), CAC 40 (France), DJ 30 (USA) and S&P 500 (USA), and are given in Table 1.

Table 1. Indices used in the study

Item Number	Indices	Index of the Stock Exchange
1	BIST 100 Index	İstanbul Stock Exchange - Turkey
2	NASDAQ 100 Index	NASDAQ Stock Exchange - USA
3	DJ30 Index	New York Stock Exchange (NYSE) - USA
4	S&P 500 Index	New York Stock Exchange (NYSE) - USA
5	DAX 30 Index	Frankfurt Stock Exchange - Germany
6	FTSE 100 Index	Londra Stock Exchange - UK
7	NIKKEI 225 Index	Tokyo Stock Exchange - Japan
8	CAC 40 Index	Euronext Paris Stock Exchange - France

The NASDAQ-100 index consists of the top 100 securities, except financial and investment companies, traded on the NASDAQ stock exchange. The NASDAQ stock market, which includes the shares of technology companies such as Facebook, Twitter, Google, Microsoft, IBM, Apple and Cisco, is referred to as the technology stock market of the world. In addition, it is regarded as the world's second largest stock exchange in terms of trading volume and market value. As one of the major stock exchanges of the World, Japan's Nikkei 225 index consists of the 225 largest Japanese companies traded on the Tokyo Stock Exchange. In terms of its calculation, NIKKEI 225 index is a weighted average index. It is one of the most basic of indicators of the Japan's economy. Toshiba, Bridgestone, Kikkoman, Mitsubishi, Tokyo Electric, Kawasaki, and Hitachi are among the companies included in the index. The FTSE 100 (The Financial Times Stock Exchange 100) Index is considered the main index of stock exchange of the UK, and consists of the top 100 companies, in terms of market value, listed on the London Stock Exchange. Approximately 65% of the index consists of domestic companies, whereas 35% is the global companies.

It is among the European stock exchanges with highest returns and trading volume. The DAX 30 index includes companies such as British Airways, British Petroleum, HSBC, Standard Chartered, Tesco Lloyd Banking Group and Unilever. The DAX 30 is an index traded on the Frankfurt Stock Exchange, representing the 30 largest German companies with high liquidity, and is considered as Germany's most important stock market index. It also includes companies such as Adidas, Deutsche Bank, SAP, Siemens as well as number of car manufacturers such as Volkswagen, and BMW. The CAC 40 index is the main index of the Euronext stock exchange in Paris. It consists of the weighted average of the 40 largest French stocks, in terms of market value. It's one of the most important indicators of France's economy. It includes companies such as Renault, Air France, Michelin, Alcatel-Lucent and L'Oreal. DJ 30 (Dow Jones Industrial Average) index is among the most used and important stock indices in the US securities markets. It is traded on the New York Stock Exchange, which is considered the world’s largest stock exchange. This index measures the performances of 30 stocks, which have higher market values, including the leading companies such as IBM, Apple, Microsoft, American Express, Intel, General Electric, Boeing, 3M, and Coca-Cola; and, it's calculated on the basis of weighted average price. And, the S&P 500 index is prepared by US-based international credit rating agency Standard&Poor's (S&P), and includes 500 largest American companies. It covers approximately 75% of the American equity market. Of the shares that make up the index, 93% is traded on the New York Stock Exchange. This index is regarded as one of the most important indicators of the US economy [13]. The time series graphs of stock market indices used in the study are shown in Figure 1.

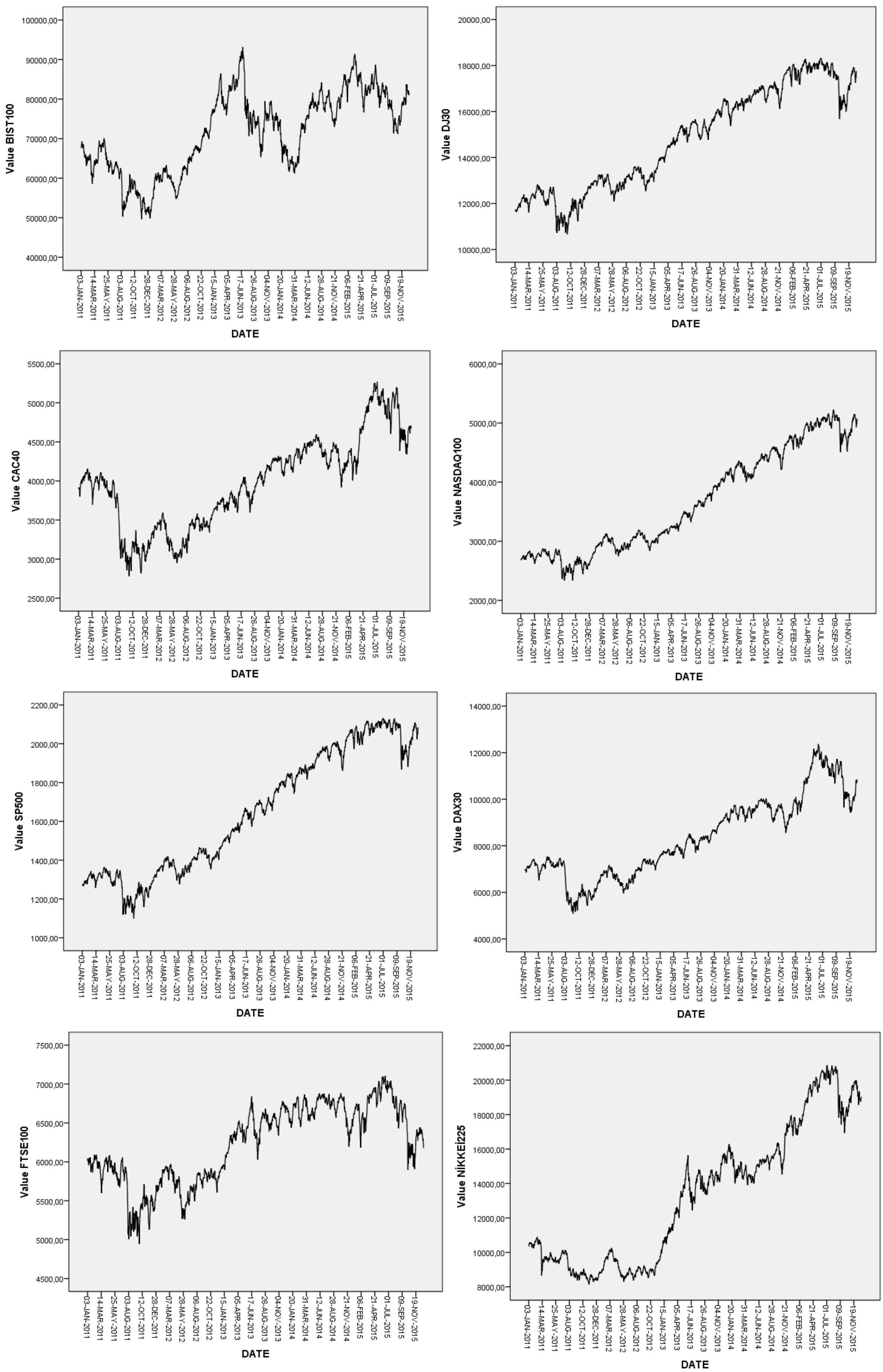

Figure 1. The time-series graphs of stock market indices over the 2011-2015 study period

The behavior of the indices was investigated by examining the time-series graphs in Figure 1. It is demonstrated that time series graphs with the purpose of seeing the price movements shown by data and relationship between indices in the period 2011-2015. It is addressed that BIST 100 indice's volatility is more than the other developed stock exchange indices. As a cause of this, if there is a sovereign, it can be interpreted as being more of capital inflows and outflows because of foreign capital ratio is very high in Borsa Istanbul. One of the interesting point in the graphics is that BİST 100 indice showed a sharp drop unlike other indices in the year 2013. This situation can be considered to be due to political uncertainty in the country, the high regional risks and more hot money flows in this period. Furthermore as determined from correlation matrix, time series graphics also support that S&P 500 and NASDAQ 100 indices act very close together and have quite a high positive correlation.

2.2 Artificial Neural Network (ANN)

The process that allows making comments about newly encountered situations on the basis of available examples with known results is called prediction. In this study, Multilayer Neural Network (MLNN) was used as the method of prediction since it's the most widely used artificial neural network (ANN) model. The overall structure of the neurons in the MLNN is shown in Figure 2.

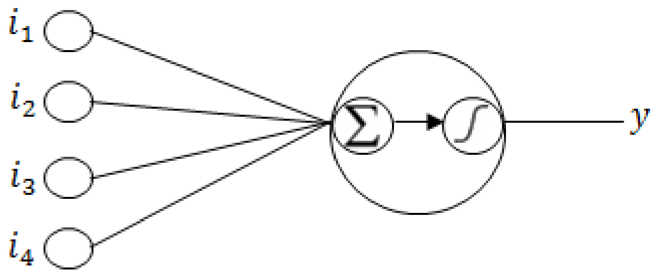

Figure 2. The neuron structures used in MLNN networks

The neuron in the MLNN model shown in Figure 2 has four inputs and one output. Each input (i_1, i_2, i_3, i_4) of the data is multiplied by the separate weight values (w_1, w_2, w_3, w_4) in the neuron, and the generated y value is provided at the output. The neuron has two processing units where linear and nonlinear operations are carried out. In the linear unit, each input is multiplied by the weight value, summed, and the result is sent to the second unit. The operation in the linear unit, which is the n value is shown in Equation 1 [14].

$$n = \sum_k i_k w_k \tag{1}$$

In the second unit, where nonlinear operations are carried out, the resulting total value passes through activation function, and the value obtained is sent to the output of the neuron. The y value at the neuron output is calculated as given in Equation 2.

$$y_{\text{out}} = f(n) \tag{2}$$

where, f in Equation 2 is the selected activation function. The commonly used activation functions are sigmoid, hyperbolic tangent, and step functions. In general, variants of back propagation algorithm are used in MLNN as the training algorithm. The weights of connections between neurons are adjusted with the help of training algorithm in order to optimize the output of the network.

The accuracy of the value obtained as a result of MLNN is based on the Mean Squared Error (MSE), and hence minimizing the Root Mean Squared Error (RMSE).

$$\mathrm{MSE} = \frac{1}{N}\sum_{i=1}^{N}(d_i - \bar{d_i})^2 \tag{3}$$

$$\mathrm{RMSE} = \sqrt{\mathrm{MSE}} \tag{4}$$

The $d_i - \bar{d_i}$ expression in the Equation 3 gives the error (distance) between each target variable and calculated linear equation. And, the RMSE expressed in the Equation 4 is used to determine the error rate between the predictions and the measured values. RMSE closer to zero indicates and increasing prediction capability of the system [15].

In the study, MLNN model was used to predict the BIST 100 index by providing international advanced stock market indices as the input datasets. This model is shown in Figure 3.

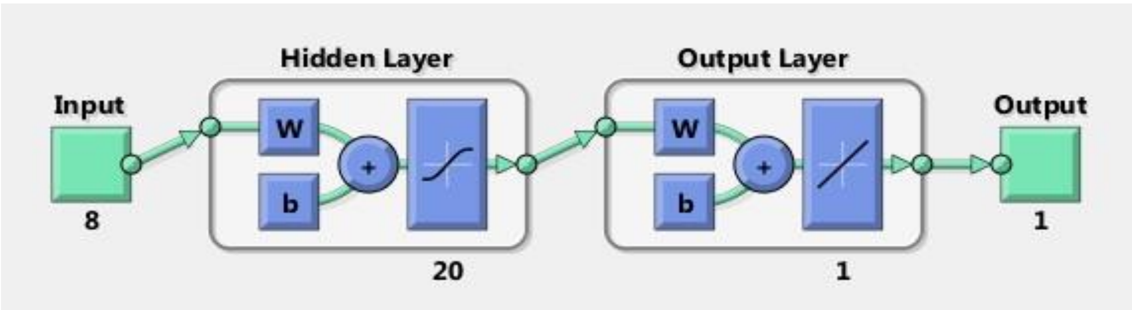

Figure 3. The ANN model used

As seen in Figure 3, there are 20 neurons in the hidden layer in this model, and the sigmoid function was used for activation in neurons as well as the Levenberg-Marquardt back-propagation algorithm, used for training. The study was carried out in Matlab 2014.

3. Results and Discussion

In the experiments carried out using the datasets and techniques described in the previous section, a dataset of international advanced stock market indices (NASDAQ-100, NICKEI-225, FTSE 100, DAX 30, CAC 40, DJ-30, S&P 500) were used to predict the BIST 100 Index.

The normality tests of the daily closing data of the index over the 2011-2015 study period were performed with the help of the SPSS 22.0 software, and the results shown in Table 2 were obtained.

Table 2. Normality test results of the stock market indices

	Kolmogorov-Smirnova			Shapiro-Wilk		
	Statistic	Df	Sig.	Statistic	df	Sig.
BIST100	,078	1229	,000	,969	1229	,000
DJ30	,135	1229	,000	,923	1229	,000
FTSE100	,106	1229	,000	,951	1229	,000
NİKKEİ225	,155	1229	,000	,904	1229	,000
DAX30	,100	1229	,000	,966	1229	,000
CAC40	,051	1229	,000	,979	1229	,000
SP500	,140	1229	,000	,908	1229	,000
NASDAQ100	,152	1229	,000	,905	1229	,000

The normal distribution of the data was tested with Kolmogorov-Smirnov test if the number of data is 29 and above, and Shapiro-Wilk test was used if it's lesser than 29 [12]. According to Kolmogorov-Smirnov test, all the daily closing data of the indices had non-normal distribution since the significance values were below 0.05 for all indices ($p<0.05$). And, the results obtained using SPSS 22.0 statistical package program according to Spearman correlation used in data that have non-parametric distribution are shown in the correlation matrix given in Table 3, which presents the relationship between stock exchange indices datasets.

Table 3. Correlation Matrix

	BIST100	**NASDAQ100**	**S&P500**	**DJ30**	**FTSE100**	**NIKKEI225**	**DAX30**	**CAC40**
BIST100	*1,000*	*0,764*	*0,792*	*0,784*	*0,750*	*0,770*	*0,780*	*0,657*
NASDAQ100	*0,764*	1,000	0,992	0,984	0,839	0,931	0,946	0,860
S&P500	*0,792*	0,992	1,000	0,995	0,862	0,932	0,947	0,861
DJ30	*0,784*	0,984	0,995	1,000	0,865	0,917	0,931	0,835
FTSE100	*0,750*	0,839	0,862	0,865	1,000	0,854	0,878	0,832
NIKKEI225	*0,780*	0,931	0,932	0,917	0,854	1,000	0,946	0,916
DAX30	*0,770*	0,946	0,947	0,931	0,878	0,946	1,000	0,947
CAC40	*0,657*	0,860	0,861	0,835	0,832	0,916	0,947	1,000

Based on the correlation matrix given in Table 3, the S&P 500 index of the United States has the strongest relationship with BIST100 index. This is followed by DJ 30 index of the United States, and Germany's DAX 30 index. The fact that the US and Germany are the top two countries that make investments in Turkey, and both countries are the top two countries that have the largest foreign trade volume with Turkey in the America and Europe regions, can be the reason behind the strong correlation between these two countries and the BIST 100 index.

In this study, the random selection cross-validity method was applied to test the MLNN model without any bias in the dataset prepared for the prediction of the BIST 100 index. In this method, the available data is randomly distributed into training, validation and test groups. In the study, the distribution of the data was 50% training, 15% validity, and 35% test respectively. In order to normalize the results, the dataset is rearranged with the same distribution ratios, and was subjected to 10 different training phases. The training, validity, test and mean prediction accuracy of the first of these experiments are shown in Figure 4.

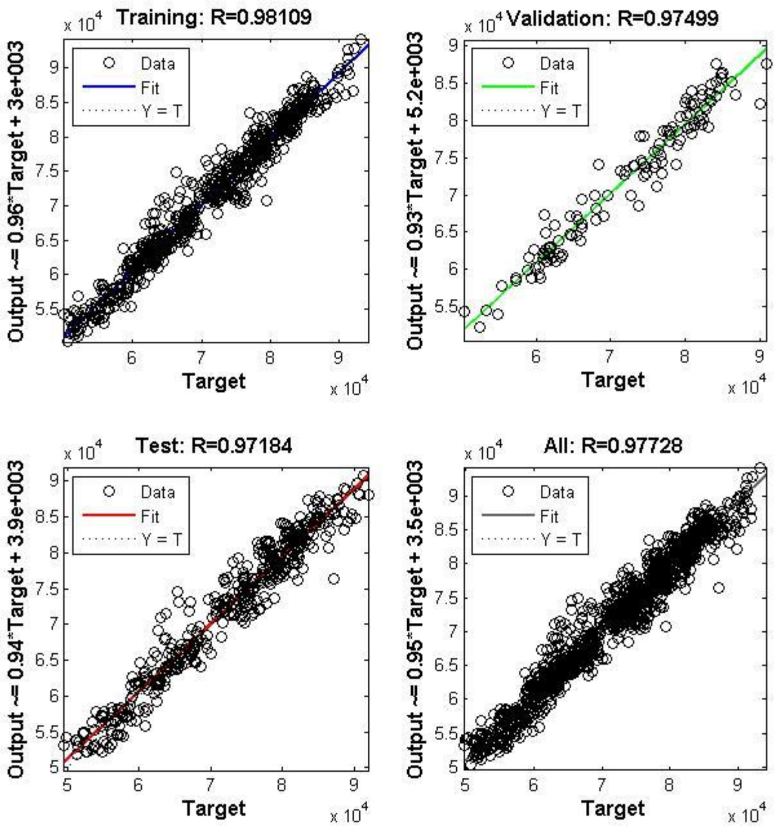

Figure 4. The success graph of MLNN training of the Experiment-1

In addition, prediction accuracy for each experiment as well as the MSE values are shown in Figure 5.

Results

	Samples	MSE	R
Training:	676	3801134.58826e-0	9.81094e-1
Validation:	123	4556329.37347e-0	9.74988e-1
Testing:	430	5750575.53445e-0	9.71835e-1

Figure 5. The MSE values and success rates of MLNN training of the Experiment-1

As provided in Figure 5, the prediction accuracy values of each experiment and RMSE values are presented in Table 4. The minimization of RMSE value ensures minimization of the error. The average of these values gives the success rate of our model.

Table 4. The prediction accuracy and RMSE values obtained after MLNN application on the model

Experiment Number	The rate of prediction accuracy (%)	Root mean square error (RMSE)
1	97,18	2398,04
2	97,26	2332,02
3	96,78	2576,12
4	97,28	2295,82
5	97,15	2409,12
6	96,14	2692,95
7	96,66	2551,81
8	97,31	2287,74
9	96,44	2615,64
10	96,91	2527,84
Mean	**96,92**	**2480,72**

As shown in Table 4, the model to be applied for the prediction of the BIST 100 index has a 96.92% prediction accuracy and 2480.72 RMSE.

4. Conclusion

Forecasting techniques and models are important for individuals and organizations who wish to make predictions during the decision making and investment phase. Predicting the stock market index is very important for investors who want to achieve higher earnings from their investments. Price movements in the stock market are affected by many factors such as political, and social effects. In studies on the prediction of the BIST 100 index, generally its relationship with traditional macroeconomic variables such as exchange rates, interest rates, gold prices, GDP and CPI rate was investigated, and statistical prediction models were utilized. In this study, the dataset on the international advanced stock market index was used as the input (independent variables), instead of traditional macroeconomic variables. These index values are the variables that may affect the value of the BIST 100 index, as an alternative investment instrument for international investors and arbitragers. The MLNN method, which is the most widely used ANN model, was used to predict the value of the BIST100 index.

Nowadays, artificial neural networks can be applied to many financial problems such as macroeconomic forecasts, the assessment of bank loans and insurance policies, predictions of bonds, stocks and exchange rates, and risk analysis. ANN is used in the financial sector as well as in many areas, thanks to its ease of design, quick adaptation to the problems, and ability to provide successful results, despite the limited data.

As a result of the study, it was observed that there is a strong relationship between the BIST 100 index and indices of advanced countries. The BIST 100 index was predicted with a high success rate of 96.92% thanks to the artificial neural networks model. An optimal forecasting model can be tested in future studies by incorporating indices of developing countries along with domestic macroeconomic factors as well as the indices of developed countries.

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COMMON FIXED POINT THEOREMS FOR f-CONTRACTION MAPPINGS IN TVS-VALUED CONE METRIC SPACE

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Abstract – We generalize the result of Abbas and Rhoades [1] and obtained some common fixed point results for two Banach pair of mappings which satisfies f-contraction condition on Topological vector space Valued Cone metric space (TVS-CMS) without the notion of normality condition of the cone.

***Keywords* –** *Fixed* point, TVS-CMS, f-contraction

1 Introduction.

Haung and Zhang [5] generalized the concept of metric Space, replacing the set of real numbers by an ordered Banach space, and obtained some fixed point theorems of contractive mappings on complete cone metric space with the assumption of the normality of the cone. Subsequently , various authors have generalized the result of Haung and Zhang and have studied fixed point theorems for normal and non-normal cones. In 2009, Beg et al [2] and in 2010 Du [4] generalized cone metric spaces to topological vector space valued cone metric spaces (TVS-CMS). In this approach ordered topological vector spaces are used as the co domain of the metric, instead of Banach spaces. While Beg et al used Hausdorff TVS, Du used locally convex Hausdorff TVS. However, a result in [10] shows that if the underlying cone of an ordered TVS is solid and normal it must be an ordered normed space. So, proper generalizations from Banach space valued cone metric space to TVS-CMS can be obtained only in the case of non normal cones. In [8] many authors have proved some common fixed point theorems for a Banach pair of mappings satisfying $f-$

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Hardy- Rogers type contraction condition in cone metric space. Also Morales and Rojas [6-7] have extended the notion of $f-$contraction mappings to Cone metric space by proving fixed point theorems for $f-$Kannan, $f-$Zamfirescu, $f-$weakly contraction mappings. In this paper, we recall the definition which was introduced and called Banach operator of type k by Subrahmanyam [9] . Chen and Li [3] also proved some best approximation result using common fixed point theorems for $f-$non expensive mappings. Here we generalize the contraction mappings which were given in [1] and obtained some common fixed point results for two Banach pair of mappings which satisfies f-contraction condition on TVS-CMS without the notion of normality condition of the cone.

2 Preliminaries

Definition 2.1. A vector space V over a field K ($\mathbb{R}$ or $\mathbb{C}$) is said to be TVS over K if it is furnished with a topology τ such that the vector space operation are continuous. i.e,

The addition operation $(x,y)\to x+y$ as a function from $V\times V\to V$ is continuous.

The scalar multiplication operation $(a,x)\to a.x$ as a function from $K\times V\to V$ is continuous.

In this case, one says that τ is a vector topology or a linear topology on the vector space V, or that τ is compatible with the linear structure of V.

Definition 2.2 [2-4] Let X be a non empty set. A mapping $d:X\times X\to E$ satisfying
1. $d(x,y)\geq 0$ and $d(x,y)=0\Leftrightarrow x=y.$
2. $d(x,y)=d(y,x)$, and
3. $d(x,y)\leq d(x,z)+d(z,y)$ $\forall\, x,y,z\in X$

Is called a TVS-valued cone metric on X. The pair (X,d) is called TVS-CMS.

Definition 2.3. Let (X,d) be a TVS-CMS, and let $x\in X$ and $\{x_n\}_{n\geq 1}$ be a sequence in X. Then

i) $\{x_n\}_{n\geq 1}$ converges to x whenever for every $c\in E$ with $0<<c$ there is a natural number N such that $d(x_n,x)\ll c$ for all $n\geq N$. We denote this by $\lim_{n\to\infty} x_n = x$.

ii) $\{x_n\}_{n\geq 1}$ is a Cauchy sequence whenever for every $c\in E$ with $0\ll c$ there is a natural number N such that $d(x_n,x_m)\ll c$ for all $n,m\geq N$.

iii) (X,d) is a complete TVS-CMS if every Cauchy sequence is convergent.

Definition 2.4. A self mapping f of a metric space (X,d) is called contraction if for any fixed constant r, $0\leq r<1$ and for all $x,y\in X$,

$$d(fx,fy)\leq r\,d(x,y)$$

Definition 2.5. [11] Let (X,d) be a metric space and $T, f: X \to X$ be two function. A mapping T is said to be $f-$contraction if there exist $0 \le r < 1$ and for all $x, y \in X$

$$d(fTx, fTy) \le d(fx, fy).$$

Example 2.6. Let $X = [0,\infty)$ be with the usual metric. Let define two mappings $T, S: X \to X$ by,

$$Tx = \alpha x, \alpha > 1$$

$$Sx = \frac{\beta}{x^2}, \beta \in \mathbb{R}.$$

It is clear that , T is not a contraction but it is $f-$contraction since,

$$d(STx, STy) = \left| \frac{\beta}{\alpha^2 x^2} - \frac{\beta}{\alpha^2 y^2} \right| = \frac{1}{\alpha^2} |Sx - Sy|.$$

Definition 2.7. [9] Let f be a self mapping of a normed space X. Then f is called a Banach operator of type k if,

$$\| f^2 x - fx \| \le k \| fx - x \|$$

for some $k \ge 0$ and all $x \in X$.

Definition 2.8. [3] Let f and g be self mappings of a non empty subset M of a normed linear space X. Then (f,g) is a Banach operator pair, if any one of the following conditions is satisfied.

1. $f[F(g)] \subseteq F(g)$.
2. $gfx = fx \;\; \forall \; x \in F(g)$
3. $fgx = gfx \;\; \forall \; x \in F(g)$
4. $\| fgx - gx \| \le r \| gx - x \|$ for some $r \ge 0$.

3 The results

Theorem 3.1. Let f, S and T be continuous self mappings of a complete TVS-cone metric space (X,d). Assume that f is injective mapping. If the mapping f, T and S satisfy,

$$d(fSx, fTy) \le \alpha d(fx, fy) + \beta[d(fx, fSx) + d(fy, fTy)] + \gamma[d(fx, fTy) + d(fy, fSx)] \quad (1)$$

$\forall \; x, y \in X$ and $\alpha, \beta, \gamma \ge 0$, $\alpha + 2\beta + 2\gamma < 1$, then T and S have a unique common fixed point in X . Moreover, if (f,T) and (f,S) are Banach pairs, then f,T and S have a unique common fixed point.

Proof: Let $x_0 \in X$ be arbitrary element and define the sequences $x_{2n+1} = Sx_{2n}$ and $x_{2n+2} = Tx_{2n+1}$ $\forall n \geq 0$ then by using (1) and triangle inequality,

$$d(fx_{2n+1}, fx_{2n}) = d(fSx_{2n}, fTx_{2n-1})$$

$$\leq \alpha d(fx_{2n}, fx_{2n-1}) + \beta[d(fx_{2n}, fSx_{2n}) + d(fx_{2n-1}, fTx_{2n-1})]$$
$$+\gamma[d(fx_{2n}, fTx_{2n-1}) + d(fTx_{2n-1}, fSx_{2n})]$$

$$\leq \alpha d(fx_{2n}, fx_{2n-1}) + \beta[d(fx_{2n}, fx_{2n+1}) + d(fx_{2n-1}, fx_{2n})]$$
$$+\gamma[d(fx_{2n}, fx_{2n}) + d(fx_{2n}, fx_{2n+1})]$$

$$\leq \alpha d(fx_{2n}, fx_{2n-1}) + \beta[d(fx_{2n-1}, fx_{2n}) + d(fx_{2n}, fx_{2n-1})]$$
$$+\gamma[d(fx_{2n}, fx_{2n-1}) + d(fx_{2n}, fx_{2n+1})]$$

$$d(fx_{2n+1}, fx_{2n}) = \frac{\alpha+\beta+\gamma}{1-(\beta+\gamma)} d(fx_{2n}, fx_{2n-1})$$

Similarly,

$$d(fx_{2n+3}, fx_{2n+2}) = d(fSx_{2n+2}, fTx_{2n+1})$$

$$\leq \alpha d(fx_{2n+2}, fx_{2n+1}) + \beta[d(fx_{2n+2}, fSx_{2n+2}) + d(fx_{2n+1}, fTx_{2n+1})]$$
$$+\gamma[d(fx_{2n+2}, fTx_{2n+1}) + d(fTx_{2n+1}, fSx_{2n+2})]$$

$$\leq \alpha d(fx_{2n+2}, fx_{2n+1}) + \beta[d(fx_{2n+2}, fx_{2n+3}) + d(fx_{2n+1}, fx_{2n+2})]$$
$$+\gamma[d(fx_{2n+2}, fx_{2n+2}) + d(fx_{2n+2}, fx_{2n+3})]$$

$$\leq \alpha d(fx_{2n+2}, fx_{2n+1}) + \beta[d(fx_{2n+2}, fx_{2n+3}) + d(fx_{2n+1}, fx_{2n+2})]$$
$$+\gamma[d(fx_{2n+2}, fx_{2n+1}) + d(fx_{2n+2}, fx_{2n+3})]$$

$$d(fx_{2n+3}, fx_{2n+2}) = \frac{\alpha+\beta+\gamma}{1-(\beta+\gamma)} d(fx_{2n+2}, fx_{2n+1})$$

Thus

$$d(fx_{n+1}, fx_n) \leq \lambda d(fx_n, fx_{n-1}) \leq \cdots\cdots\cdots \leq \lambda^n d(fx_1, fx_0)$$

for all $n \geq 0$ where,

$$\frac{\alpha+\beta+\gamma}{1-(\beta+\gamma)} < 1$$

Now for $n \geq m$ we get

$$d(fx_n, fx_m) \leq d(fx_n, fx_{n-1}) + d(fx_{n-1}, fx_{n-2}) + \cdots\cdots + d(fx_{m+1}, fx_m)$$

$$\leq (\lambda^{n-1} + \lambda^{n-2} + \cdots\cdots + \lambda^m) d(fx_1, fx_0)$$

$$\leq \frac{\lambda^m}{1-\lambda} d(fx_1, fx_0)$$

Let $0 \ll c$ be given. Choose $\delta > 0$ such that

$$c + N_\delta(0) \subseteq K \quad \text{,where}$$

$$N_\delta(0) = \{y \in E : \| y \| < \delta\}$$

Also, choose a natural number N_1 such that $\frac{\lambda^m}{1-\lambda} d(fx_1, fx_0) \in N_\delta(0)$, for all $m \geq N_1$.

$$\frac{\lambda^m}{1-\lambda} d(fx_1, fx_0) \ll c \text{ for all } m \geq N_1$$

Thus

$$d(fx_n, fx_m) \leq \frac{\lambda^m}{1-\lambda} d(fx_1, fx_0)$$

and

$$\frac{\lambda^m}{1-\lambda} d(fx_1, fx_0) \ll c$$

for all $m > n$. Then we get $d(fx_n, fx_m) \ll c \ \forall n > m$. Therefore, $\{fx_n\}$ is a Cauchy sequence in (X, d). As X is complete, there exist $q \in X$ such that, $\lim_{n\to\infty} fx_n = q$. Since f is subsequentially convergent, $\{x_n\}$ has a convergent subsequence $\{x_m\}$ such that

$\lim_{n\to\infty} x_m = u$. As f is continuous, $\lim_{m\to\infty} fx_m = fu$. By the uniqueness of limit, $q = fu$
.
Since T and S is continuous, $\lim_{m\to\infty} Tx_m = Tu$ and $\lim_{n\to\infty} Sx_m = Su$.

Again Since f is continuous, $\lim_{m\to\infty} fTx_m = fTu$ and $\lim_{m\to\infty} fSx_m = fSu$. Therefore, if m is odd, then $\lim_{n\to\infty} fTx_{2n+1} = fTu$.

Choose a natural number N_2 such that,

$$d(fx_{2n+1}, fu) \ll \left[\frac{c}{2}\left(\frac{\alpha+\beta+\gamma}{1-(\beta+\gamma}\right)\right] \quad \text{for all } n \geq N_2. \text{ Now consider,}$$

$$d(fu, fTu) \leq d(fu, fx_{2n+1}) + d(fx_{2n+1}, fTu)$$

$$= d(fu, fSx_{2n}) + d(fSx_{2n}, fTu)$$

$$\leq d(fu, fSx_{2n}) + \alpha d(fx_{2n}\, fu) + \beta(d(fx_{2n}, fSx_{2n}) + d(fu, fTu))$$

$$+\gamma(d(fx_{2n}, fTu) + d(fu, fSx_{2n}))$$

$$\leq d(fu, fx_{2n+1}) + \alpha d(fx_{2n}\, fu) + \beta(d(fx_{2n}, fx_{2n+1}) + d(fu, fTu))$$

$$+\gamma(d(fx_{2n}, fu) + d(fu, fTu) + d(fu, fx_{2n+1}))$$

$$\leq d(fu, fx_{2n+1}) + \alpha d(fx_{2n}\, fu) + \beta(d(fu, fx_{2n+1}) + d(fu, fTu))$$

$$+\gamma(d(fx_{2n}, fu) + d(fu, fTu) + d(fu, fx_{2n+1}))$$

$$= (1+\beta+\gamma)d(fu, fx_{2n+1}) + (\alpha+\gamma)d(fx_{2n}, fu) + (\beta+\gamma)d(fu, fTu)$$

So,

$$d(fu, fTu) \leq \left[\frac{1+\beta+\gamma}{1-\beta-\gamma}\right]d(fu, fx_{2n+1}) + \left[\frac{\alpha+\gamma}{1-\beta-\gamma}\right]d(fx_{2n}, fu) \ll c$$

for all $n \geq N_2$. Therefore, $d(fu, fTu) \ll \frac{c}{i}$ for all $i \geq 1$. Hence,

$$\frac{c}{i} - d(fu, fTu) \in K \ \forall\, i \geq 1$$

Since K is closed, $-d(fu, fTu) \in K$ and so $d(fu, fTu) = 0$. Hence, $fu = fTu$. As f is injective, $u = Tu$. Thus u is fixed point of T. Similarly, again by using (1) and triangular inequality , the rest can be proved if m is even, then we have,

$$\lim_{n\to\infty} fSx_{2n} = fSu\,.$$

For the uniqueness, suppose that v is another common fixed point of T and S. So,

$d(fu, fv) = d(fSu, fTv)$

$\leq \alpha d(fu, fv) + \beta(d(fu, fSv) + d(fv, fTv)) + \gamma(d(fu, fTv) + d(fv, fSu))$

$d(fu, fv) \leq (\alpha + 2\gamma) d(fu, fv)$

Since $(\alpha + 2\gamma) < 1$, $d(fu, fv) = 0$ which implies that $fu = fv$. So $u = v$ is unique common fixed point of S and T.

Corollary 3.2. Let f and S be continuous self mappings of a complete TVS-cone metric space (X, d). Assume that f is injective mapping. If the mapping f and S satisfy,

$$d(fSx, fSy) \leq \alpha d(fx, fy) + \beta[d(fx, fSx) + d(fy, fSy)] + \gamma[d(fx, fSy) + d(fy, fSx)]$$

$\forall\ x, y \in X$ and $\alpha, \beta, \gamma \geq 0$, $\alpha + 2\beta + 2\gamma < 1$, then S has a unique fixed point in X.

Corollary 3.3. Let f and S be continuous self mappings of a complete TVS-cone metric space (X, d). If the mapping f and S satisfy,

$$d(fSx, fSy) \leq \alpha d(fx, fy) + \beta[d(fx, fSx) + d(fy, fSy)]$$

$\forall\ x, y \in X$ and $\alpha, \beta \geq 0$, $\alpha + 2\beta < 1$. Then S has a unique fixed point in X.

Corollary 3.4. Let f and S be continuous self mappings of a complete TVS-cone metric space (X, d). If the mapping f and S satisfy,

$$d(fSx, fSy) \leq \beta(d(fx, fSx) + d(fy, fSy))$$

$\forall\ x, y \in X$ and $\beta \in [0, \frac{1}{2})$. Then S has a unique fixed point in X.

Corollary 3.5. Let f and S be continuous self mappings of a complete TVS-cone metric space (X, d). If the mapping f and S satisfy,

$$d(fSx, fSy) \leq \gamma(d(fx, fSy) + d(fy, fSx))$$

$\forall\ x, y \in X$ and $\gamma \in [0, \frac{1}{2})$. Then S has a unique fixed point in X.

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