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ON SOFT e-OPEN SETS AND SOFT e-CONTINUITY IN SOFT TOPOLOGICAL SPACE

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Abstract: In this paper, a new class of generalized soft open sets in soft topological spaces, called soft *e-open* set is focused and investigated some properties of them. Then focused the relationships among soft δ-pre open sets, soft δ-semi open sets, soft pre-open sets and soft *e-open* sets. We also investigated the concepts of soft *e-open* functions, soft *e*-continuous, soft *e*-irresolute and soft *e*-homeomorphism on soft topological space and discussed their relations with existing soft continuous and other weaker forms of soft continuous functions. Further soft *e*-separation axioms have been introduced and investigated with the help of soft *e*-open sets. Finally, we observed that the collection S*e*r-h(X,τ,E) form a soft group.

Keywords: Soft e-open (S*e*-open) sets, Soft *e*-closed (S*e*-closed) sets, Soft *e*-continuous, soft *e*-irresolute and soft *e*-homomorphism.

1. Introduction

Molodtsov [1] initiated a novel concept of soft set theory, which is completely a new approach for modeling vagueness and uncertainty. He successfully applied the soft set theory into several directions such as smoothness of functions, game theory, Riemann Integration, theory of measurement, and so on. Soft set theory and its applications have shown great development in recent years. This is because of the general nature of parametrization expressed by a soft set. Shabir and Naz [2] introduced the notion of soft topological spaces which are defined over an initial universe with a fixed set of parameters. Later, Zorlutuna et al.[3], Aygunoglu and Aygun [4] and Hussain et al are continued to study the properties of soft topological space. They got many important results in soft topological spaces. Weak forms of soft open sets were first studied by Chen [5].He investigated soft semi-open sets in soft topological spaces and studied some properties of it. Yumak and Kaymakci [10] are defined soft β-open sets and continued to study weak forms

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of soft open sets in soft topological space. Later, Akdag and Ozkan [6] [7] defined soft b-open (soft b-closed) sets and soft α-open(soft α-closed) sets respectably.
In the present study, first of all, we have focused some new concepts such as soft *e*-open sets, soft *e*-closed sets, soft *e*-interior, soft *e*-closure in soft topological spaces and investigated some of their properties. Secondly, we have defined the concepts of soft *e*-continuous, soft *e*-open, soft *e*-irresolute mappings and soft *e*-homeomorphism on soft topological spaces and obtained some characterizations of these mappings. We have also studied the relationships among soft δ-semi-continuity, soft δ-pre-continuity and soft *e*-continuity and with the help of counter examples we have shown the non-coincidence of these various types of mappings. Further soft *e*-separation axioms have been introduced and investigated with the help of soft *e*-open sets. Finally, we have observed that the collection S*e*r-h(X, τ,E) form a soft group.

2. Preliminaries

Throughout the paper, the space X and Y stand for soft topological spaces with (X,τ, E) and (Y,ν, K) assumed unless otherwise stated. Moreover, throughout this paper, a soft mapping f : X →Y stands for a mapping, where f : (X,τ, E) → (Y,ν, K), u : X →Y and p : E →K are assumed mappings unless otherwise stated.

Definition: 2.1[1]. Let X be an initial universe and E be a set of parameters. Let *P*(X) denotes the power set of X and *A* be a non-empty subset of E. A pair (F,*A*) is called a soft set over X, where F is a mapping given by F: *A*→*P*(X) defined by F(e)∈*P*(X) ∀e∈A. In other words, a soft set over X is a parameterized family of subsets of the universe X. For e∈A, F(e) may be considered as the set of e-approximate elements of the soft set (F,A).

Definition 2.2[11]. A soft set (F,A) over X is called a null soft set, denoted by $\tilde{\varphi}$, if e∈A,F(e)=φ.

Definition 2.3[11]. A soft set (F,A) over X is called an absolute soft set, denoted by $\tilde{A}$, if e∈A, F(e)=X. If A=E, then the A-universal soft set is called a universal soft set, denoted by $\tilde{X}$.

Theorem 2.4[2]. Let Y be a non-empty subset of X, then $\tilde{Y}$ denotes the soft set (Y,E) over X for which Y(e)=Y, for all e∈E.

Definition 2.5 [11]. The union of two soft sets (F,A) and (G,B) over the common universe X is the soft set (H,C), where C=A∪B and for all e∈C,

$$H(e)=\begin{cases} F(e), if e\in A-B \\ G(e), if e\in B-A \\ F(e)\cup G(e), if e\in A\cap B \end{cases}$$

We write (F,A) $\tilde{\cup}$ (G,B) = (H,C)

Definition 2.6 [11]. The intersection (H,C) of two soft sets (F,A) and (G,B) over a common universe X, denoted by (F,A) $\tilde{\cap}$ (G,B), is defined as C=A∩B and H(e)= F(e) ∩ G(e) for all e∈C.

Definition 2.7[2]. Let (F,A) be a soft set over a soft topological space (X,τ,E). We say that x∈(F,E) read as x belongs to the set (F,E) whenever x∈F(e) for all e∈E. Note that for any x∈X, x∉(F,E), if x∉F(e) for some e∈E.

Definition 2.8 [11]. Let (F,A) and (G,B) be two soft sets over a common universe X. Then (F,A) $\tilde{\subseteq}$ (G,B) if A ⊆ B, and F(e) ⊆ G(e) for all e∈A.

Definition 2.9 [2]. Let τ be the collection of soft sets over X, then τ is said to be a soft topology on X if it satisfies the following axioms.

(1) $\tilde{\varphi}, \tilde{X}$ belong to τ.
(2) The union of any number of soft sets in τ belongs to τ.
(3) The intersection of any two soft sets in τ belongs to τ.

The triplet (X,τ,E) is called a soft topological space over X. Let (X,τ,E) be a soft topological space over X, then the members of τ are said to be soft open sets in X. A soft set (F,A) over X is said to be a soft closed set in X, if its relative complement $(F,A)^c$ belongs to τ.

Definition 2.10 [12]. For a soft set (F,A) over X, the relative complement of (F,A) is denoted by $(F,A)^c$ and is defined by $(F,A)^c$ = (F^c,A), where F^c : A→*P*(X) is a mapping given by $F^c(e)$ =X-F(e), for all e∈A.

Definition 2.11. A soft set (F,A) in a soft topological space X is called
(i) soft regular open (resp.soft regular closed) set [13] if (F,A) = Int(Cl(F,A)) [resp. (F,A) = Cl(Int (F,A))].
(ii) soft semi-open (resp.soft semi-closed) set [5] if (F,A) $\tilde{\subseteq}$ Cl(Int(F,A)) [resp. Int(Cl(F,A)) $\tilde{\subseteq}$ (F,A).
(iii) soft pre-open (resp.soft pre-closed)[13] if (F,A) $\tilde{\subseteq}$ Int(Cl(F,A)) [resp. Cl(Int(F,A)) $\tilde{\subseteq}$ (F,A).
(iv) soft α-open (resp.soft α-closed)[13] if (F,A) $\tilde{\subseteq}$ Int(Cl(Int(F,A))) [resp. Cl(Int(Cl(F,A))) $\tilde{\subseteq}$ (F,A)].
(v) soft β-open (resp.soft β-closed) set [13] if (F,A) $\tilde{\subseteq}$ Cl(Int(Cl(F,A))) [resp. Int(Cl(Int(F,A))) $\tilde{\subseteq}$ (F,A)].
(vi) soft γ-open (resp.soft γ-closed) set [6] if (F,A) $\tilde{\subseteq}$ [Int(Cl(F,A)) $\tilde{\cup}$ Cl(Int(F,A))] [resp. Int(Cl(F,A)) $\tilde{\cap}$ Cl(Int(F,A)) $\tilde{\subseteq}$ (F,A)].

Definition 2.12[15]. The soft set (F,A) in a soft topological space (X,τ,E) is called a soft point in X, denoted by P_λ^F , if for λ∈A, F(λ) ≠ φ and F(β) = φ, for β∉A.

3. Soft *e*-open Sets and Soft *e*-closed Sets

In this section we introduce soft δ-open, soft δ-semi open, soft δ-pre open and soft *e*-open sets in soft topological spaces and study some of their properties.

Definition 3.1. A soft point P_λ^F in a soft topological space (X,τ,E) is called a soft δ-cluster point of a soft Set (G,A) if for each soft regular open set (U,A) containing P_λ^F, (G,A) $\tilde{\cap}$ (U,A) ≠ $\tilde{\varphi}$.

The set of all soft δ-cluster points of (G,A) is called soft δ-closure of (G,A) and is denoted by $[(G,A)]_\delta$ or $SCl_\delta(G,A)$.Soft δ-interior of a soft set (F,A) denoted by $SInt_\delta(F,A)$ ={ P_λ^F ∈X: for some soft open subset (G,A) of X, P_λ^F ∈(G,A) $\tilde{\subseteq}$ Int(Cl(G,A)) $\tilde{\subseteq}$ (F,A)}.

Definition 3.2. A soft set (G,A) in a soft topological space (X,τ,E) is called soft δ-closed set iff (G,A) = $[(G,A)]_\delta$ and it's compliment $\tilde{X}$ - (G,A) is called soft δ-open sets in X.
Or, equivalently, if (G,A) is the union of soft regular open sets, then (G,A) is said to be soft δ-open sets in X.

The collection of all soft δ-open sets & soft δ-closed sets are respectably, denoted by SδOS(X) & SδCS(X).

Definition 3.3. A soft set (F,A) in a soft topological space (X,τ,E) is called
- (i) soft δ-semi open (Sδ-semi open) set iff (F,A) $\tilde{\subseteq}$ $Cl(Int_\delta(F,A))$.
- (ii) soft δ-semi closed (Sδ-semi closed) set iff $Int(Cl_\delta(F,A))$ $\tilde{\subseteq}$ (F,A).

The union of all soft δ-semi open sets contained in a soft set (F,A) in a soft topological space X is called the soft δ-semi interior of (F,A) and it is denoted by $SSInt_\delta(F,A)$. The intersection of all soft δ-semi closed sets containing a soft set (F,A) in a soft topological space X is called the soft δ-semi closure of (F,A) and it is denoted by $SSCl_\delta(F,A)$.

Definition 3.4. A soft set (F,A) in a soft topological space (X,τ,E) is called
- (i) soft δ-pre open (Sδ-pre open) set iff (F,A) $\tilde{\subseteq}$ $Int(Cl_\delta(F,A))$.
- (ii) soft δ-pre closed (Sδ-pre closed) set iff $Cl(Int_\delta(F,A))$ $\tilde{\subseteq}$ (F,A).

The union of all soft δ-pre open sets contained in a soft set (F,A) in a soft topological space X is called the soft δ-pre interior of (F,A) and it is denoted by $SPInt_\delta(F,A)$. The intersection of all soft δ-pre closed sets containing a soft set (F,A) in a soft topological space X is called the soft δ-pre closure of (F,A) and it is denoted by $SPCl_\delta(F,A)$.

Definition 3.5. A soft set (F,A) in a soft topological space (X,τ,E) is called
- (i) soft *e*-open (s*e*-open) set iff (F,A) $\tilde{\subseteq}$ $[Int(Cl_\delta(F,A)) \tilde{\cup} Cl(Int_\delta(F,A))]$
- (ii) soft *e*-closed (s*e*-closed) set iff (F,A) $\tilde{\supseteq}$ $[Int(Cl_\delta(F,A)) \tilde{\cap} Cl(Int_\delta(F,A))]$

Example 3.6. Let X ={x_1, x_2, x_3, x_4}, E ={e_1, e_2, e_3} and τ ={ $\tilde{\varphi}$, $\tilde{X}$, (G,E)} where, (G,A) = {(e_1,{ x_1}), (e_2,{ x_2,x_4}), (e_3,{x_2})}. Then, (X,τ,E) is a soft topological space and (G,A) = { (e_1,{ x_1}),(e_2,{x_2, x_4}),(e_3,{x_2})} is a soft *e*-open set.

Theorem 3.7. For a soft set (F,A) in a soft topological space (X,τ,E)

(i) (F,A) is a soft *e*-open set iff $(F,A)^c$ is a soft *e* -closed set.

(ii) (F,A) is a soft *e*-closed set iff $(F,A)^c$ is a soft *e*-open set.

Proof. Obvious from the Definition 3.5.

Definition 3.8. Let (X,τ,E) be a soft topological space and (F,A) be a soft set over X.

(i) Soft *e*-closure of a soft set (F,A) in X is denoted by S*e*-Cl(F,A)= $\tilde{\cap}$ {(H,A) $\tilde{\supseteq}$ (F,A): (H,A) is a soft *e*-closed set of X}.

(ii) Soft *e*-interior of a soft set (F,A) in X is denoted by S*e*-Int(F,A)= $\tilde{\cup}$ {(G,A) $\tilde{\subseteq}$ (F,A): (G,A) is a soft *e*-open set of X}.

Clearly, S*e*-Cl(F,A) is the smallest soft *e*-closed set over X which contains (F,A) and S*e*-Int(F,A) is the largest soft *e*-open set over X which is contained in (F,A).

Theorem 3.9. Let (F,A) be any soft set in a soft topological space X. Then,

(i) S*e*-Cl$(F,A)^c$ = $\tilde{X}$ -S*e*-Int(F,A).

(ii) S*e*-Int$(F,A)^c$ = $\tilde{X}$ -S*e*-Cl(F,A).

Proof. **(i)** Let soft *e*-open set (G,A) $\tilde{\subseteq}$ (F,A) and soft *e*-closed set (H,A) $\tilde{\supseteq}$ $(F,A)^c$. Then S*e*-Int(F,A)= $\tilde{\cup}$ {$(H,A)^c$: (H,A) is soft *e*-closed set and (H,A) $\tilde{\supseteq}$ $(F,A)^c$ } = $\tilde{X}$ - $\tilde{\cap}$ {(H,A): (H,A) is soft *e*-closed set and (H,A) $\tilde{\supseteq}$ $(F,A)^c$ }= $\tilde{X}$ - S*e*-Cl$(F,A)^c$.So,S*e*-Cl$(F,A)^c$ = $\tilde{X}$ -S*e*-Int(F,A).

(ii) Let (G,A) be a soft *e*-open set. Then for a soft *e*-closed set $(G,A)^c$ $\tilde{\supseteq}$ (F,A), (G,A) $\tilde{\subseteq}$ $(F,A)^c$. Now, S*e*-Cl(F,A)= $\tilde{\cap}$ {$(G,A)^c$: (G,A) is soft *e*-open set and (G,A) $\tilde{\subseteq}$ $(F,A)^c$} = $\tilde{X}$ - $\tilde{\cup}$ {(G,A): (G,A) is soft *e*-open set and (G,A) $\tilde{\subseteq}$ $(F,A)^c$ }= $\tilde{X}$ - S*e*-Int$(F,A)^c$.
So, S*e*-Int$(F,A)^c$ = $\tilde{X}$ -S*e*-Cl(F,A).

Theorem 3.10. In a soft topological space X, (F,A) be a soft *e*-closed (resp. soft *e*-open) if and only if (F,A) = S*e*-Cl(F,A) (resp. (F,A)= S*e*-Int(F,A).

Proof. Suppose (F,A) = S*e*-Cl(F,A)= $\tilde{\cap}$ {(H,A) $\tilde{\supseteq}$ (F,A): (H,A) is a soft *e*-closed set of X}.This means (F,A)∈ $\tilde{\cap}$ {(H,A) $\tilde{\supseteq}$ (F,A): (H,A) is a soft *e*-closed set of X} and hence (F,A) is soft *e*-closed set.
Conversely, suppose (F,A) be a soft *e*-closed in X. Then we have (F,A)∈ {(H,A) $\tilde{\supseteq}$ (F,A): (H,A) is a soft *e*-closed set of X}. Hence, (F,A) $\tilde{\subseteq}$ (H,A) implies (F,A) = $\tilde{\cap}$ {(H,A) $\tilde{\supseteq}$ (F,A): (H,A) is a soft *e*-closed set of X }= S*e*-Cl(F,A).
Similarly for (F,A)= S*e*-Int(F,A)).

Theorem 3.11. In a soft topological space X, the following holds for soft *e*-closure and soft *e*-interiors.

(i) S*e*-Cl ($\tilde{\varphi}$) = $\tilde{\varphi}$.

(ii) S*e*-Int($\tilde{\varphi}$) = $\tilde{\varphi}$.

(iii) S*e*-Cl(F,A) is a soft *e*-closed set in X.

(iv) S*e*-Int(F,A) is a soft *e*-open set in X.

(v) S*e*-Cl(F,A) $\tilde{\subseteq}$ S*e*-Cl(G,A) if (F,A) $\tilde{\subseteq}$ (G,A).

(vi) S*e*-Int(F,A) $\tilde{\subseteq}$ S*e*-Int(G,A) if (F,A) $\tilde{\subseteq}$ (G,A).

(vii) Se-Cl(Se-Cl(F,A)) = Se-Cl(F,A).
(viii) Se-Int(Se-Int(F,A)) = Se-Int(F,A).

Theorem 3.12. In a soft topological space X, we have
(i) Se-Cl ((F,A) $\tilde{\cup}$ (G,A)) $\tilde{\supseteq}$ Se-Cl(F,A) $\tilde{\cup}$ Se-Cl(G,A).
(ii) Se-Cl((F,A) $\tilde{\cap}$ (G,A)) $\tilde{\subseteq}$ Se-Cl(F,A) $\tilde{\cap}$ Se-Cl(G,A).

Proof. **(i)** (F,A) $\tilde{\subseteq}$ ((F,A) $\tilde{\cup}$ (G,A)) or (G,A) $\tilde{\subseteq}$ ((F,A) $\tilde{\cup}$ (G,A)) this implies Se-Cl(F,A) $\tilde{\subseteq}$ Se-Cl((F,A) $\tilde{\cup}$ (G,A)) or Se-Cl (G,A) $\tilde{\subseteq}$ Se-Cl((F,A) $\tilde{\cup}$ (G,A)).Therefore Se-Cl((F,A) $\tilde{\cup}$ (G,A)) $\tilde{\supseteq}$ Se-Cl(F,A) $\tilde{\cup}$ Se-Cl(G,A).

(ii) Similar to the proof of (i).

Theorem 3.13. In a soft topological space X, we have
(i) Se-Int((F,A) $\tilde{\cup}$ (G,A)) $\tilde{\supseteq}$ Se-Int(F,A) $\tilde{\cup}$ Se-Int(G,A)
(ii) Se-Int((F,A) $\tilde{\cap}$ (G,A)) $\tilde{\subseteq}$ Se-Int(F,A) $\tilde{\cap}$ Se-Int(G,A)

Proof. Same as the proof of theorem 3.12.

Theorem 3.14. Let (F,A) be soft e-open set,
(i) If Int_δ(F,A) = $\tilde{\varphi}$, then (F,A) is soft δ-preopen set.
(ii) If Cl_δ(F,A) = $\tilde{\varphi}$, then (F,A) is soft δ-semiopen set.

Proof. Ovious from definition 3.5.

Lemma 3.15. Let (F,A) be a soft subset of X, then
(i) $SSCl_\delta$(F,A) = (F,A) $\tilde{\cup}$ Int(Cl_δ(F,A)) and $SSInt_\delta$(F,A) = (F,A) $\tilde{\cap}$ Cl(Int_δ(F,A))
(ii) $SPCl_\delta$(F,A) = (F,A) $\tilde{\cup}$ Cl(Int_δ(F,A))and $SPInt_\delta$(F,A) = (F,A) $\tilde{\cap}$ Int(Cl_δ(F,A)).

Proof. **(i)** $SSCl_\delta$(F,A) $\tilde{\supseteq}$ Int(Cl_δ($SSCl_\delta$(F,A))) $\tilde{\supseteq}$ Int(Cl_δ(F,A))
$\Rightarrow$ (F,A) $\tilde{\cup}$ $SSCl_\delta$(F,A) $\tilde{\supseteq}$ (F,A) $\tilde{\cup}$ Int(Cl_δ(F,A))
So, (F,A) $\tilde{\cup}$ Int(Cl_δ(F,A)) $\tilde{\subseteq}$ $SSCl_\delta$(F,A)…………......(i)
Also, (F,A) $\tilde{\subseteq}$ $SSCl_\delta$(F,A)
$\Rightarrow$ Int(Cl_δ(F,A)) $\tilde{\subseteq}$ Int(Cl_δ($SSCl_\delta$(F,A))) $\tilde{\subseteq}$ $SSCl_\delta$(F,A)
$\Rightarrow$ (F,A) $\tilde{\cup}$ Int(Cl_δ(F,A)) $\tilde{\subseteq}$ $SSCl_\delta$(F,A) $\tilde{\cup}$ $SSCl_\delta$(F,A)= $SSCl_\delta$(F,A)……(ii)
Hence, from (i) and (ii), $SSCl_\delta$(F,A) = (F,A) $\tilde{\cup}$ Int(Cl_δ(F,A)).

(ii) Follows immediately from (i) by taking the complements.

Theorem 3.16. For any soft subset (F,A) of a space X, (F,A) is soft e-open if and only if (F,A) = $SPInt_\delta$(F,A) $\tilde{\cup}$ $SSInt_\delta$(F,A).

Proof. Let (F,A) be soft e-open. Then (F,A) $\tilde{\subseteq}$ Int(Cl_δ(F,A)) $\tilde{\cup}$ Cl(Int_δ(F,A))).By above lemma 3.15, we have, $SPInt_\delta$(F,A) $\tilde{\cup}$ $SSInt_\delta$(F,A) = [(F,A) $\tilde{\cap}$ Int(Cl_δ(F,A))] $\tilde{\cup}$ [(F,A) $\tilde{\cap}$ Cl(Int_δ(F,A))]= (F,A) $\tilde{\cap}$ (Int(Cl_δ(F,A))] $\tilde{\cup}$ Cl(Int_δ(F,A))) = (F,A).

Conversely, if $(F,A) = SPInt_\delta(F,A) \tilde{\cup} SSInt_\delta(F,A)$, then by above lemma 3.15, $(F,A) = SPInt_\delta(F,A) \tilde{\cup} SSInt_\delta(F,A) = [(F,A) \tilde{\cap} Int(Cl_\delta(F,A))] \tilde{\cup} [(F,A) \tilde{\cap} Cl(Int_\delta(F,A))] = (F,A) \tilde{\cap} (Int(Cl_\delta(F,A))) \tilde{\cup} Cl(Int_\delta(F,A))) \tilde{\subseteq} (Int(Cl_\delta(F,A))) \tilde{\cup} Cl(Int_\delta(F,A)))$
and hence (F,A) is soft *e*-open set.

Theorem 3.17. Let (F,A) be a soft subset of a space X, then, $Se\text{-}Cl(F,A) = SPCl_\delta(F,A) \tilde{\cap} SSCl_\delta(F,A)$.

Proof. It is obvious that, $Se\text{-}Cl(F,A) \tilde{\subseteq} SPCl_\delta(F,A) \tilde{\cap} SSCl_\delta(F,A)$.
Conversely, from definition we have, $Se\text{-}Cl(F,A) \tilde{\supseteq} [Int(Cl_\delta(Se\text{-}Cl(F,A)) \tilde{\cap} Cl(Int_\delta(Se\text{-}Cl(F,A))] \tilde{\supseteq} Cl(Int_\delta(F,A)) \tilde{\cap} Int(Cl_\delta((F,A))$. Since $Se\text{-}Cl(F,A)$ is soft e-closed, by lemma 3.15, we have, $SPCl_\delta(F,A) \tilde{\cap} SSCl_\delta(F,A) = [(F,A) \tilde{\cup} Cl(Int_\delta(F,A))] \tilde{\cap} [(F,A) \tilde{\cup} Int(Cl_\delta(F,A))] = [(F,A) \tilde{\cup} (Cl(Int_\delta(F,A)) \tilde{\cap} Int(Cl_\delta(F,A)))] = (F,A) \tilde{\subseteq} Se\text{-}Cl(F,A)$.

Theorem 3.18. Let (F,A) be a soft subset of a space X, then, $Se\text{-}Int(F,A)= SPInt_\delta(F,A) \tilde{\cap} SSInt_\delta(F,A)$.

Proof. It is similar to the above proof.

Theorem 3.19. In a Soft topological space X, we have the followings:
(i) Arbitrary union of Soft *e*-open sets is a Soft *e*-open set, and
(ii) Arbitrary intersection of Soft *e*-closed sets is a Soft *e*-closed set.

Proof. **(i)** Let $\{(F,A)_\alpha : \alpha \in \Lambda, \text{an index set}\}$ be a collection of Soft *e*-open sets. Then for each α, $(F,A)_\alpha \tilde{\subseteq} [Int(Cl_\delta((F,A)_\alpha)) \tilde{\cup} Cl(Int_\delta((F,A)_\alpha))]$.
Taking union of all such relations we get,
$\tilde{\cup}\{(F,A)_\alpha\} \tilde{\subseteq} \tilde{\cup} [Int(Cl_\delta((F,A)_\alpha)) \tilde{\cup} Cl(Int_\delta((F,A)_\alpha))]$
$\tilde{\subseteq} [Int(Cl_\delta(\tilde{\cup} (F,A)_\alpha)) \tilde{\cup} Cl(Int_\delta(\tilde{\cup} (F,A)_\alpha))]$.
Thus $\tilde{\cup}(F,A)_\alpha$ is Soft *e*-open set.
(ii) Follows immediately from (i) by taking the complements.

Definition 3.20 [10]. Let τ_S be the collection of soft sets over X, then τ_S is said to be a soft supra topology on X if it satisfies the following axioms.
(1) $\tilde{\varphi}, \tilde{X}$ belong to τ_S.
(2) The union of any number of soft sets in τ_S belongs to τ_S.
The triplet (X,τ_S,E) is called a soft supratopological space over X.
Now we give the following property for soft *e*-open sets.

Proposition 3.21. The collection $\tau_S = S.e.OS(X)$ of all soft *e*-open sets form a soft supra topology over a soft space (X,τ,E).

Proof. **(1)** is obvious.

(2) Let $(F,E)_\alpha \in \tau_S$, $\forall \alpha \in \Lambda = \{1, 2, 3.....\}$. Then, for $\forall \alpha \in \Lambda$, $(F,E)_\alpha \tilde{\subseteq} Int(Cl_\delta((F,A)_\alpha)) \tilde{\cup} Cl(Int_\delta((F,A)_\alpha))$.Taking union of all such relations we get,
$\tilde{\cup}\{(F,A)_\alpha\} \tilde{\subseteq} \tilde{\cup} [Int(Cl_\delta((F,A)_\alpha)) \tilde{\cup} Cl(Int_\delta((F,A)_\alpha))]$
$\tilde{\subseteq} [Int(Cl_\delta(\tilde{\cup}(F,A)_\alpha)) \tilde{\cup} Cl(Int_\delta(\tilde{\cup}(F,A)_\alpha))]$.

This implies that $\tilde{\cup}(F,A)_\alpha$ is Soft *e*-open set and hence, $\tilde{\cup}(F,A)_\alpha \in \tau_S$.

Remark 3.22. In a soft topological space it is obvious that
(i) Every soft regular open set is soft δ-open set.
(iii) Every soft δ-open set is both soft δ-semi-open and soft δ-pre-open set.
(iii) Every soft δ-semi-open set and every soft δ-pre-open set is soft *e*-open set.

Let (X,τ,E) be a soft topological space. Then, the family of all soft *e*-open set (resp. soft open,soft regular open, soft δ-open, soft δ-semi-open, soft δ-pre-open) sets in X may be denoted by S*e*-OS(X) (resp. SOS(X), SROS(X), SδOS(X), SδSOS(X), SδPOS(X)).The family of all soft regular closed (resp. soft δ-closed, soft δ-semi-closed, soft δ-pre-closed) sets in X may be denoted by S*e*-CS(X) (resp.SRCS(X), SδCS(X), SδSCS(X) , SδPCS(X). Thus we have implications as shown in Figure 1.

Soft regular open
↓
Soft δ-open → Soft open → soft semi-open → soft γ-open → soft β-open
↓ ↓
Soft δ-pre open   Soft δ-semi open
↓ ↓
Soft *e*-open set

Figure-1

The examples given below show that the converses of these implications are not true.

Example 3.23. Let X={x_1, x_2, x_3, x_4},E ={e_1, e_2, e_3} and τ ={ $\tilde{\varphi}$, $\tilde{X}$, (F_1,E),(F_2,E), (F_3,E), (F_4,E), (F_5,E), (F_6,E), (F_7,E), (F_8,E), (F_9,E), (F_{10},E), (F_{11},E), (F_{12},E), (F_{13},E)} where, (F_1,E),(F_2,E), (F_3,E), (F_4,E), (F_5,E), (F_6,E), (F_7,E), (F_8,E), (F_9,E), (F_{10},E), (F_{11},E), (F_{12},E), and (F_{13},E) are soft sets over X, defined as follows:
$(F_1,E) =\{(e_1,\{ x_1\}),(e_2,\{x_2,x_3\}),(e_3,\{x_1,x_3\})\}$;
$(F_2,E) =\{(e_1,\{x_2,x_4\}),(e_2,\{ x_1,x_3,x_4\}),(e_3,\{ x_1, x_2, x_4\})\}$;
$(F_3,E) =\{ (e_1, \varphi),(e_2,\{x_3\}),(e_3,\{ x_1\})\}$;
$(F_4,E) =\{ (e_1, \{x_1,x_2,x_4\}),(e_2, X),(e_3,X)\}$;
$(F_5,E) =\{ (e_1,\{x_1,x_3\}),(e_2,\{x_2, x_4\}),(e_3,\{x_2\})\}$;
$(F_6,E) = \{(e_1,\varphi),(e_2,\{x_2\}),(e_3, \varphi)\}$;
$(F_7,E) =\{ (e_1,\{ x_1,x_3\}),(e_2,\{ x_2, x_3, x_4\}),(e_3,\{ x_1, x_2,x_3\})\}$;
$(F_8,E) =\{ (e_1,\{x_3\}),(e_2,\{ x_4\}),(e_3,\{x_2\})\}$;
$(F_9,E) =\{ (e_1,X),(e_2,X),(e_3,\{ x_1, x_2, x_3\})\}$;
$(F_{10},E) =\{ (e_1,\{ x_1,x_3\}),(e_2,\{ x_2, x_3, x_4\}),(e_3,\{ x_1, x_2\})\}$;
$(F_{11},E) =\{ (e_1,\{ x_3\}),(e_2,\{ x_2, x_4\}),(e_3,\{x_2\})\}$;
$(F_{12},E) =\{ (e_1,\{x_1\}),(e_2,\{x_2\}),(e_3, \varphi)\}$;
$(F_{13},E) = \{ (e_1,\{x_1\}),(e_2,\{ x_2, x_3\}),(e_3,\{x_1\})\}$.

Then, τ defines a soft topology on X, and thus (X,τ, E) is a soft topological space over X. Clearly, soft closed sets are $\tilde{\varphi}$, $\tilde{X}$, $(F_1, E)^c$, $(F_2, E)^c$, $(F_3, E)^c$, $(F_4, E)^c$, $(F_5, E)^c$, $(F_6, E)^c$, $(F_7,E)^c$, $(F_8,E)^c$, $(F_9, E)^c$, $(F_{10},E)^c$, $(F_{11}, E)^c$, $(F_{12}, E)^c$ and $(F_{13}, E)^c$.

Now, consider the soft set (G,E)={ $(e_1,\{x_1,x_2\}),(e_2,\{x_2,x_3\}),(e_3,\{x_1,x_3\})$}, then, $[Int(Cl_\delta(G,E)) \tilde{\cup} Cl(Int_\delta(G,E))]=(F_8,E)^c \tilde{\supseteq} (G,E)$.So,$(G,E) \tilde{\subseteq} Int(Cl_\delta(G,E)) \tilde{\cup} Cl(Int_\delta(G,E))$. Thus, (G,E) is soft *e*-open set but since, $Int(Cl_\delta(G,E))=(F_1,E)$ which does not contain (G,E). So (G,E) is not soft δ-pre open set. Also it is clear that (G,E) is neither soft δ-open nor soft regular open nor soft open nor soft semi-open nor γ-open nor β-open set.

Again, consider the soft set (F,E) $=\{(e_1,\{x_4\}),(e_2,\{x_1, x_3\}),(e_3,\{x_1,x_2\})\}$, then $[Int(Cl_\delta(F,E)) \tilde{\cup} Cl(Int_\delta(F,E))]=\{(e_1,\{x_2,x_4\}), (e_2,\{x_1,x_3,x_4\}),(e_3,X)\}$ and so,$(F,A) \tilde{\subseteq} [Int(Cl_\delta(F,E)) \tilde{\cup} Cl(Int_\delta(F,E))]$. Thus, (F,E) is soft *e*-open set but since $Cl(Int_\delta(F,E)) = (F_5,E)^c$ which does not contain (F,E). So (F,E) is not soft δ-semi open set. Also it is clear that (F,E) is neither soft δ-open nor soft regular open nor soft open nor soft semi-open nor γ-open nor β-open set.

Remark 3.24. The intersection of two soft *e*-open sets need not be soft *e*-open set as is illustrated by the following example.

Example 3.25. Let (X,τ,E) be a soft topological space defined in Example 3.23. Now we consider two soft sets (G,E) and (H,E) in (X,τ,E) defined as follows:
$(G,E)=\{(e_1,\{x_1\}),(e_2,\{x_2\})\}$;$(H,E)=\{(e_1,\{x_1\}),(e_2,\{x_3\}),(e_3,\{x_3\})\}$.
Then, (G,E) and (H,E) are soft *e*-open sets over X, therefore, $(G,E) \tilde{\cap} (H,E) =\{(e_1,\{x_1\})\}= (K,E)$ and $Int(Cl_\delta(K,E)) \tilde{\cup} Cl(Int_\delta(K,E)) =\{(e_1,\{x_1\}), (e_2,\{x_2\})\} \tilde{\not\supseteq} (K,E)$. Hence, K,E) is not a soft *e*-open set.

Remark 3.26. The union of two soft *e*-closed sets need not be soft *e*-closed set as is illustrated by the following example.

Example 3.27. Let (X,τ,E) be a soft topological space defined in Example 3.23. Now we consider two soft sets (A,E) and (B,E) in (X,τ,E) defined as follows:

$(A,E)=\{(e_1,\{x_2,x_3,x_4\}),(e_2,\{x_1, x_3, x_4\}),(e_3,X)\}$;$(B,E)=\{(e_1,\{x_2, x_3, x_4\}),(e_3,\{x_1, x_2,x_4\}),(e_3, \{x_1, x_2,x_4\})\}$.Then, (A,E) and (B,E) are soft *e*-closed sets over X , therefore, $(A,E) \tilde{\cup} (B,E) = \{(e_1,\{x_2, x_3, x_4\}),(e_3,X), (e_3, X)\} = (C,E)$ and $Int(Cl_\delta(C,E)) \tilde{\cap} Cl(Int_\delta(C,E) = X \tilde{\not\subset} (C,E)$. Hence, (C,E) is not a soft *e*-closed set.

Theorem 3.28. In a soft topological space X,
(i) Every soft δ-pre-open set is soft *e*-open set.
(ii) Every soft δ-semi-open set is *e*-open set.

***Proof*. (i)** Let (F,A) be a soft δ-pre-open set in a soft topological space X.
Then, $(F,A) \tilde{\subseteq} Int(Cl_\delta(F,A))$ which implies that
$(F,A) \tilde{\subseteq} [Int(Cl_\delta(F,A)) \tilde{\cup} Int_\delta(F,A))] \tilde{\subseteq} [Int(Cl_\delta(F,A)) \tilde{\cup} Cl(Int_\delta(F,A))]$
Thus (F,A) is soft *e*-open set.

(ii) Let (F,A) be a soft δ-semi-open set in a soft topological space X.
Then, $(F,A) \tilde{\subseteq} Cl(Int_\delta(F,A))$ which implies that
$(F,A) \tilde{\subseteq} [Cl(Int_\delta(F,A)) \tilde{\cup} Int(F,A))] \tilde{\subseteq} [Cl(Int_\delta(F,A)) \tilde{\cup} Int(Cl_\delta(F,A))]$
Thus (F,A) is soft *e*-open set.

4. Soft *e*-continuity and Soft *e*-homeomorphisms

In this section, we introduce soft *e*-continuous maps, soft *e*-irresolute maps, soft e-closed maps, soft e-open maps and soft *e*-homeomorphisms. We also study some of their properties and separation axioms with the help of soft *e*-open sets.

Definition 4.1 [14]. Let (X,E) and (Y,K) be two soft classes. Let u : $X \to Y$ and p : E → K be mappings.Then a mapping *f* : (X,E) → (Y,K) is defined as follows: for a soft set (F,A) in (X,E) , (*f*(F,A),B), B = p(A) $\tilde{\subseteq}$ K is a soft set in (Y,K) given by

f(F,A)(β) = u$\begin{pmatrix} \cup F(\alpha) \\ \alpha \in p^{-1}(\beta) \cap A \end{pmatrix}$ β∈K and (*f*(F,A),B) is called a soft image of a soft set (F,A) . If B=K, then we will write (*f(F,A)*,K) as *f(F,A)*.

Definition 4.2 [14]. Let *f* : (X,E) → (Y,K) be a mapping from a soft class (X,E) to another soft class (Y,K), and (G,C) a soft set in soft class (Y,K), where, C ⊆ K. Let u : $X \to Y$ and p : E →K be mappings.Then (f^{-1}(G,C),D), D=p^{-1}(C), is a soft set in the soft classes(X,E), defined as (f^{-1}(G,C)(α) = u^{-1}(G(p(α))) for α∈D ⊆ E. (f^{-1}(G,C),D) is called a soft inverse image of (G,C). Hereafter, we shall write, (f^{-1}(G,C),E) as (f^{-1}(G,C).

Theorem 4.3 [14]. Let *f* : (X,E) → (Y,K) ; u : $X \to Y$ and p : E →K be mappings. Then for soft sets (F,A), (G,B) and a family of soft sets {(F_α,A_α):α∈Λ,an index set} in the soft class (X,E), we have:

(1) $f(\tilde{\varphi}) = \tilde{\varphi}$,
(2) $f(\tilde{X}) = \tilde{Y}$,
(3) f((F,A) $\tilde{\cup}$ (G,B)) = f(F,A) $\tilde{\cup}$ f(G,B), in general,
$f(\bigcup_{\alpha\in\Lambda}(F_\alpha, A_\alpha)) = (\bigcup_{\alpha\in\Lambda} f(F_\alpha, A_\alpha))$,
(4) f((F,A) $\tilde{\cap}$ (G,B)) $\tilde{\subseteq}$ f(F,A) $\tilde{\cap}$ f(G,B), in general,
$f(\bigcap_{\alpha\in\Lambda}(F_\alpha, A_\alpha)) \tilde{\cap} (\bigcap_{\alpha\in\Lambda} f(F_\alpha, A_\alpha))$,
(5) If (F,A) $\tilde{\subseteq}$ (G,B) then, f(F,A) $\tilde{\subseteq}$ f(G,B),
(6) $f^{-1}(\tilde{\varphi}) = \tilde{\varphi}$,
(7) $f^{-1}(\tilde{Y}) = \tilde{X}$,
(8) f^{-1}((F,A) $\tilde{\cup}$ (G,B))=f^{-1}(F,A) $\tilde{\cup}$ f^{-1}(G,B),ingeneral,
$f^{-1}(\bigcup_{\alpha\in\Lambda}(F_\alpha, A_\alpha))=(\bigcup_{\alpha\in\Lambda} f^{-1}(F_\alpha, A_\alpha))$,
(9) f^{-1}((F,A) $\tilde{\cap}$ (G,B))=f^{-1}(F,A) $\tilde{\cap}$ f^{-1}(G,B),ingeneral,
$f^{-1}(\bigcap_{\alpha\in\Lambda}(F_\alpha, A_\alpha)) \tilde{\subseteq} (\bigcap_{\alpha\in\Lambda} f^{-1}(F_\alpha, A_\alpha))$,
(10) If (F,A) $\tilde{\subseteq}$ (G,B) then, f^{1} (F,A) $\tilde{\subseteq}$ f^{1} (G,B),

Definition 4.4. A mapping f :(X,τ_1,E) → (Y,τ_2,K) is said to be a soft δ-pre continuous if f^{-1}(F,A) is soft δ-pre open in X for every soft open set (F,A) in Y.

Definition 4.5. A mapping f :(X,τ_1,E) → (Y,τ_2,K) is said to be a soft δ-semi continuous if f^{-1}(F,A) is soft δ-semi open in X for every soft open set (F,A) in Y.

Definition 4.6. A mapping f :(X,τ_1,E) $\rightarrow$ (Y,τ_2,K) is said to be a soft *e*-continuous if f^{-1}(F,A) is soft *e*-open in X for every soft open set (F,A) in Y.

Definition 4.7. A mapping f :(X,τ_1,E) $\rightarrow$ (Y,τ_2,K) is said to be a soft *e*-irresolute if f^{-1}(F,A) is soft *e*-open in X for every soft *e*-open set (F,A) in Y.

Remark 4.8. It is clear that every soft δ-pre continuous map and soft δ-semi continuous map is soft *e*-continuous. Thus we have implications as shown in Figure 2.
The converses of these implications are not necessarily true, which is clear from the following examples.

Soft δ-pre continuous   Soft δ-semi continuous
↓   ↓
Soft *e*-continuous

Figure-2

Example 4.9. Let X={x_1, x_2, x_3, x_4}, Y={y_1, y_2, y_3, y_4}, E ={e_1, e_2, e_3}, K={k_1, k_2, k_3}and (X,τ,E) and (Y,ν,K) be soft topological spaces. Let f_{up}: (X,τ,E) $\rightarrow$ (Y,ν,K) be a soft mapping. Define u : X $\rightarrow$ Y and P : E $\rightarrow$ K as u(x_1) = y_2, u(x_2) = y_3, u(x_3) = y_4, u(x_4) = y_1, and p(e_1) = k_2, p(e_2) = k_1, p(e_3) = k_3;

Let us consider the soft topology τ in X given in Example 3.23; that is,τ ={ $\tilde{\varphi}$, $\tilde{X}$, (F_1,E), (F_2, E), (F_3,E), (F_4,E), (F_5,E), (F_6,E), (F_7,E), (F_8,E), (F_9,E), (F_{10},E), (F_{11},E), (F_{12},E), (F_{13},E)} and soft topology ν ={ $\tilde{\varphi}$, $\tilde{Y}$, (G,K) = { (k_1,{ y_1,y_2}),(k_2,{ y_2, y_3}),(k_3,{y_1, y_3})}} in Y. Then (G,K) is a soft open in Y and f_{up}^{-1}(G,K) = {(e_1,{x_1,x_2}), (e_2,{x_2, x_3}),(e_3,{ x_1, x_3})} is soft *e*-open but not soft δ-pre open in X. Therefore, f_{up} is a soft *e*-continuous but not soft δ-pre continuous function.

Example 4.10. Let X={x_1, x_2, x_3, x_4}, Y={y_1, y_2, y_3, y_4}, E ={e_1, e_2, e_3}, K={k_1, k_2, k_3}and (X,τ,E) and (Y,ν,K) be soft topological spaces. Let f_{up}: (X,τ,E) $\rightarrow$ (Y,ν,K) be a soft mapping. Define u : X $\rightarrow$ Y and P : E $\rightarrow$ K as u(x_1) = y_3, u(x_2) = y_1, u(x_3) = y_4, u(x_4) = y_2, and p(e_1) = k_2, p(e_2) = k_1, p(e_3) = k_3;

Let us consider the soft topology τ in X given in Example 3.23; that is,τ ={ $\tilde{\varphi}$, $\tilde{X}$, (F_1,E), (F_2,E), (F_3,E), (F_4,E), (F_5,E), (F_6,E), (F_7,E), (F_8,E), (F_9,E), (F_{10},E), (F_{11},E), (F_{12},E), (F_{13},E)} and soft topology ν={ $\tilde{\varphi}$, $\tilde{Y}$, (H,K) = {(k_1,{ y_3,y_4}),(k_2,{ y_2}), (k_3,{ y_1,y_3})}} in Y. Then (H,K) is a soft open in Y and f_{up}^{-1}(H,K)= {(e_1,{ x_4}),(e_2,{ x_1, x_3}),(e_3,{ x_1, x_2})} is soft *e*-open but not soft δ-semi open in X. Therefore, f_{up} is a soft *e*-continuous but not soft δ-semi continuous function.

Theorem 4.11. For a mapping f :(X,τ_1,E) $\rightarrow$ (Y,τ_2,K), the following statements are equivalent
(i) f is a soft *e*-continuous.

(ii) For every soft singleton $P_{\lambda}^{F} \in$ X and every soft open set (F,A) in Y such that f(P_{λ}^{F}) $\tilde{\subseteq}$(F,A), ∃ a soft e-open set (G,A) in X such that $P_{\lambda}^{F} \in$(G,A) and f((G,A))$\tilde{\subseteq}$ (F,A).
(iii) f^{-1}(F,A) = Int(Cl$_{\delta}$(f^{-1}(F,A)))$\tilde{\cup}$Cl(Int$_{\delta}$(f^{-1}(F,A))) for each soft open set (F,A) in Y.
(iv) The inverse image of each soft closed set in Y is soft *e*-closed.
(v) Int(Cl$_{\delta}$(f^{-1}(F,A)))$\tilde{\cap}$Cl(Int$_{\delta}$(f^{-1}(F,A))) $\tilde{\subseteq}$ f^{-1}((Cl(F,A))) for each soft set (F,A) $\tilde{\subseteq}$Y.
(vi) f[Cl(Int$_{\delta}$(G,A))$\tilde{\cap}$Int(Cl$_{\delta}$(G,A))]$\tilde{\subseteq}$Cl(f(G,A)) for every soft set (G,A) in X.

Proof. **(i)⇒(ii):** Let the singleton set P_{λ}^{F} in X and every soft open set (F,A) in Y such that f(P_{λ}^{F})∈ (F,A). Since f is soft *e*-continuous. Then $P_{\lambda}^{F} \in f^{-1}$(f(P_{λ}^{F}))$\tilde{\subseteq}$ f^{-1}(F,A) .
Let (G,A) = f^{-1}(F,A) which is a soft *e*-open set in X. So, we have $P_{\lambda}^{F} \in$ (G,A). Now f(G,A) = f(f^{-1}(F,A)) $\tilde{\subseteq}$ (F,A).

(ii)⇒(iii): Let (F,A) be any soft open set in Y. Let P_{λ}^{F} be any soft point in X such that f(P_{λ}^{F})$\tilde{\subseteq}$(F,A). Then $P_{\lambda}^{F} \in f^{-1}$(F,A). By(ii), there exists a soft e-open set (G,A) in X such that $P_{\lambda}^{F} \in$ (G,A) and f((G,A)) $\tilde{\subseteq}$ (F,A). Therefore, $P_{\lambda}^{F} \in$(G,A)$\tilde{\subseteq}$ f^{-1}(f((G,A)))$\tilde{\subseteq}f^{-1}$(F,A) $\tilde{\subseteq}$ Int(Cl$_{\delta}$(f^{-1}(F,A)) $\tilde{\cup}$ Cl(Int$_{\delta}$(f^{-1}(F,A)).

(iii)⇒(iv): Let (F,A) be any soft closed set in Y. Then $\tilde{Y}$ -(F,A) be a soft open set in Y. By (iii), (f^{-1}($\tilde{Y}$- (F,A))) $\tilde{\subseteq}$ Int(Cl$_{\delta}$(f^{-1}($\tilde{Y}$- (F,A)))) $\tilde{\cup}$ Cl(Int$_{\delta}$(f^{-1}($\tilde{Y}$-(F,A)))).This implies $\tilde{X}$ -(f^{-1}(F,A)) $\tilde{\subseteq}$ Int(Cl$_{\delta}$($\tilde{X}$ -f^{-1}(F,A)))$\tilde{\cup}$Cl(Int$_{\delta}$($\tilde{X}$ -f^{-1}(F,A))) $\tilde{\subseteq}$ Int($\tilde{X}$ - Cl$_{\delta}$(f^{-1}(F,A))) $\tilde{\cup}$ Cl($\tilde{X}$ -Int$_{\delta}$(f^{-1}(F,A))) $\tilde{\subseteq}$ [$\tilde{X}$ -Int(Cl$_{\delta}$(f^{-1}(F,A))] $\tilde{\cup}$ [$\tilde{X}$ -Cl(Int$_{\delta}$(f^{-1}(F,A))] and hence $\tilde{X}$ -(f^{-1}(F,A)) $\tilde{\subseteq}$ $\tilde{X}$ -[Int(Cl$_{\delta}$(f^{-1}(F,A))$\tilde{\cap}$Cl(Int$_{\delta}$(f^{-1}(F,A))].Hence (f^{-1}(F,A)) $\tilde{\supseteq}$[Int (Cl$_{\delta}$(f^{-1}(F,A))$\tilde{\cap}$Cl(Int$_{\delta}$(f^{-1}(F,A))] and this implies that f^{-1}(F,A) is soft *e*-closed in X.

(iv)⇒(v): Let (F,A) $\tilde{\subseteq}$Y. Then f^{-1}(Cl(F,A)) is soft *e*-closed in X. Now, [Int(Cl$_{\delta}$(f^{-1} (F,A)))$\tilde{\cap}$Cl(Int$_{\delta}$(f^{-1}(F,A)))]$\tilde{\subseteq}$[Int(Cl$_{\delta}$(f^{-1}(Cl(F,A))))$\tilde{\cap}$Cl(Int$_{\delta}$(f^{-1}(Cl(F,A))))]$\tilde{\subseteq}f^{-1}$(Cl (F,A)).

(v)⇒(vi): Let (G,A) $\tilde{\subseteq}$ X. Put (F,A) = f(G,A) in **(v)**. Then, [Int(Cl$_{\delta}$(f^{-1}(f(G,A)))) $\tilde{\cap}$ Cl (Int$_{\delta}$(f^{-1}(f(G,A))))] $\tilde{\subseteq}f^{-1}$(Cl(f(G,A))).This implies that [Int(Cl$_{\delta}$(G,A))$\tilde{\cap}$Cl(Int$_{\delta}$(G,A))] $\tilde{\subseteq}$ f^{-1} (Cl(f((G,A)))) ⇒ f[Int(Cl$_{\delta}$(G,A))$\tilde{\cap}$Cl(Int$_{\delta}$(G,A))] $\tilde{\subseteq}$ Cl(f(G,A)).

(vi)⇒(i): Let (G,A) $\tilde{\subseteq}$ Y be soft open set. Put (G,A) = f^{-1}(F,A) and (F,A) =$\tilde{Y}$-(G,A) then f[Int(Cl$_{\delta}$(f^{-1}(F,A)))$\tilde{\cap}$Cl(Int$_{\delta}$(f^{-1}(F,A)))] $\tilde{\subseteq}$ Cl(f(f^{-1}(F,A))) $\tilde{\subseteq}$ Cl(F,A) = (F,A). That is, f^{-1}(F,A) is soft *e*-closed in X, so f is soft *e*-continuous.

Theorem 4.12. Every soft *e*-irresolute mapping is soft *e*-continuous mapping.

Proof. Let f :(X,τ_1,E) → (Y,τ_2,K) is soft *e*-irresolute mapping. Let (F,K) be a soft closed set in Y, then (F,K) is soft *e*-closed set in Y. Since f is soft *e*-irresolute mapping, f^{1}(F,K) is a soft *e*-closed set in X. Hence, f is soft *e*-continuous mapping.

Theorem 4.13. If f : $(X,\tau_1,E) \to (Y,\tau_2,K)$ be soft *e*-continuous function and g : $(Y,\tau_2,K) \to (Z,\tau_3,L)$ be soft continuous function. Then gof: $(X,\tau_1,E) \to (Z,\tau_3,L)$ is also soft *e*-continuous function.

Proof. Let (F,A) be a soft open set in Z. Now, $(gof)^{-1}(F,A) = (f^{-1}og^{-1})(F,A) = (f^{-1}(g^{-1}(F,A))$.Since g is soft continuous, $g^{-1}(F,A)$ is soft open & then $(gof)^{-1}(F,A) = f^{1}$(soft open in Y). But f being soft *e*-continuous $(gof)^{-1}(F,A)$ is soft *e*-open set in X. Thus gof is soft *e*-continuous function.

Theorem 4.14. If f :$(X,\tau_1,E) \to (Y,\tau_2,K)$ be soft *e*-irresolute function and g : $(Y,\tau_2,K) \to (Z,\tau_3,L)$ be soft *e*-continuous function. Then gof: $(X,\tau_1,E) \to (Z,\tau_3,L)$ is also soft *e*-continuous function.

Proof. Let (F,A) be a soft open set in Z. Now, $(gof)^{-1}(F,A) = (f^{-1}og^{-1})(F,A) = (f^{-1}(g^{-1}(F,A))$.Since g is soft *e*-continuous, $g^{-1}(F,A)$ is soft *e*-open & then $(gof)^{-1}(F,A) = f^{-1}$(soft *e*-open in Y). But f being *e*-irresolute, $(gof)^{-1}(F,A)$ is soft *e*-open set in X. Thus gof is soft *e*-continuous function.

Theorem 4.15. Composition of two soft *e*-irresolute function is again a soft *e*-irresolute function.

Proof. Straight forward.

Definition 4.16. A mapping $f : X \to Y$ is said to be soft *e*-open (briefly s*e*-open) map if the image of every soft open set in X is soft *e*-open set in Y.

Definition 4.17. A mapping $f : X \to Y$ is said to be soft *e*-closed (briefly s*e*-closed) map if the image of every soft closed set in X is soft *e*-closed set in Y.

Theorem 4.18. If $f : X \to Y$ is soft closed function and $g : Y \to Z$ is soft *e*-closed function, then *gof* is soft *e*-closed function.

Proof. For a soft closed set (F,A) in X, f (F,A) is soft closed set in Y. Since g : Y → Z is soft *e*-closed function, $g(f$ (F,A)) is soft *e*-closed set in Z. $g(f$ (F,A)) = (gof) (F,A) is soft *e*-closed set in Z. Therefore, gof is soft *e*-closed function.

Theorem 4.19. A map $f : X \to Y$ is soft *e*-closed if and only if for each soft set (H,K) of Y and for each soft open set (F,A) such that $f^{1}(H,K) \tilde{\subseteq} (F,A)$, there is a soft *e*-open set (G,K) of Y such that $(H,K) \tilde{\subseteq} (G,K)$ and $f^{1}(G,K) \tilde{\subseteq} (F,A)$.

Proof. Suppose f is soft *e*-closed map. Let (H,K) be a soft set of Y, and (F,A) be a soft open set of X, such that $f^{1}(H,K) \tilde{\subseteq} (F,A)$. Then $(G,K) = (f((F,A)^c))^c$ is a soft *e*-open set in Y such that $(H,K) \tilde{\subseteq} (G,K)$ and $f^{1}(G,K) \tilde{\subseteq} (F,A)$.

Conversely, suppose that (F,B) is a soft closed set of X.Then $f^{1}(f((F,B))^c) \tilde{\subseteq} (F,B)^c$, and $(F,B)^c$ is soft open set. By hypothesis, there is a soft *e*-open set (G,K) of Y such that $(f((F,A)^c))^c \tilde{\subseteq} (G,K)$ and $f^{1}(G,K) \tilde{\subseteq} (F,B)$, Thus $(F,B) \tilde{\subseteq} f^{1}(G,K)$. Hence $(G,K)^c \tilde{\subseteq} f(G,K) \tilde{\subseteq} f(f^{1}(G,K))^c) \tilde{\subseteq} (G,K)$ which implies $f(F,B)= (G,K)^c$. Since $(G,K)^c$ is soft *e*-closed set, f (F,B) is sb-closed set. So, f is a soft *e*-closed map.

Theorem 4.20. Let $f : X \to Y$, $g : Y \to Z$ be two maps such that $gof : X \to Z$ is sb-closed map.
(i) If f is soft continuous and surjective, then g is soft e-closed map.
(ii) If g is soft e-irresolute and injective, then f is soft e-closed map.

Proof. **(i)** Let (H,K) be a soft closed set of Y. Then, $f^{1}(H,K)$ is soft closed set in X as f is soft continuous. Since gof is soft e-closed map, $(gof)\ (f^{1}(H,K))= g(H,K)$ is soft e-closed set in Z. Hence $g : Y \to Z$ soft e-closed map.

(ii) Let (F,A) be a soft closed set in X. Then, $(gof)\ (F,A)$ is soft e-closed set in Z, and so $g^{-1}(gof)\ (F,A) = f(F,A)$ is soft e-closed set in Y. Since g is soft e-irresolute and injective. Hence, f is a soft e-closed map.

5. Applications in Separation Axioms and in Soft Group Theory

In this section e-separation axioms has been introduced and investigated with the help of soft e-open sets. Finally, we have shown that the collection Ser-h(X,τ,E) form a soft group.

Definition 5.1. A soft topological space (X,τ,E) is said to be soft e-T_1 if for each pair of distinct soft points P_λ^F and P_μ^G of X, there exists soft e-open sets (U,A) and (V,B) such that $P_\lambda^F \in (U,A)$ and $P_\mu^G \in (V,B)$, $P_\lambda^F \notin (V,B)$ and $P_\mu^G \notin (U,A)$.

Theorem 5.2. If f : X → Y is soft e-continuous injective function and Y is soft T_1, then X is soft e-T_1.

Proof. Suppose that Y is soft T_1. For any two distinct soft points P_λ^F and P_μ^G of X, there exists soft open sets (U,A) and (V,A) in Y such that $f(P_\lambda^F) \in$ (U,A), $f(P_\mu^G) \in$ (V,A), $f(P_\lambda^F) \notin$ (V,A) and $f(P_\mu^G) \notin$ (U,A). Since f is injective soft e-continuous function, we have f^{1}(U,A) and f^{1}(V,A) are soft e-open sets in X. Hence by definition X is soft e-T_1.

Definition 5.3. A soft topological space (X,τ,E) is said to be soft e-T_2 (i.e., soft e-Hausdorff) if for each pair of distinct soft points P_λ^F and P_μ^G of X, there exists disjoint soft e-open sets (U,A) and (V,B) such that $P_\lambda^F \in$ (U,A) and $P_\mu^G \in$ (V,B).

Theorem 5.4. If $f : (X,\tau_1, E) \to (Y,\tau_2, E)$ is soft e-continuous injective function and Y is soft T_2 then X is soft e-T_2.

Proof. Suppose that Y is soft T_2 space. For any two distinct soft points P_λ^F and P_μ^G of X, there exists disjoint soft open sets (U,A) and (V,B) in Y such that $f(P_\lambda^F) \in$ (U,A) , $f(P_\mu^G) \in$ (V,B), $f(P_\lambda^F) \notin$ (V,B) and $f(P_\mu^G) \notin$ (U,A). Since f is injective soft e-continuous function, we have f^{1}(U,A) and f^{1}(V,B) are disjoint soft e-open sets in X. Hence by definition, X is soft e-T_2.

Definition 5.5. A soft topological space (X,τ,E) is said to be soft *e*-normal if for every two disjoint soft closed sets (F,A) and (H,B) of X, there exist two disjoint soft *e*-open sets (U,A) and (V,B) such that (F,A)$\tilde{\subseteq}$(U,A) and (H,B)$\tilde{\subseteq}$(V,B) and (U,A)$\tilde{\cap}$(H,B) $= \tilde{\varphi}$.

Theorem 5.6. If f : (X,τ_1, E) $\rightarrow$ (Y,τ_2, E) is soft *e*-continuous closed injective function and Y is soft normal then X is soft *e*-normal.

Proof. Suppose that Y is soft normal. Let *(F,A)* and *(H,B)* be soft closed sets in X such that $(F,A)\tilde{\cap}(H,B)=\tilde{\varphi}$. Since *f* is soft closed injection *f(F,A)* and *f(H,B)* are soft closed in Y and $f(F,A)\tilde{\cap}\ f(H,B)=\tilde{\varphi}$. Since Y is normal, there exists soft open sets *(U,A)* and *(V,B)* in Y such that $f(F,A)\tilde{\subseteq}U$, $f(H,B)\tilde{\subseteq}V$ and $U\tilde{\cap}V = \tilde{\varphi}$. Therefore we obtain, $(F,A)\tilde{\subseteq}\ f^{1}\ (U)$ and $(H,B)\tilde{\subseteq}\ f^{1}(V)$ and $f^{1}(U\ \tilde{\cap}V)=\tilde{\varphi}$. Since *f* is soft *e*-continuous, $f^{1}(U)$ and $f^{1}(V)$ are soft *e*-open sets. Hence by definition X is soft *e*-normal.

Definition 5.7. A space X is said to be soft *e*-regular if for each soft closed set (F, A) of X and each soft point $P_{\lambda}^{F} \in$X- (F,A), there exist disjoint soft *e*-open sets (U,A) and (V,B) such that $P_{\lambda}^{F} \in$(U,A) and (F,A)$\tilde{\subseteq}$(V,B).

Theorem 5.8. If $f:(X,\tau_1,E)\rightarrow(Y,\tau_2,E)$ is soft *e*-continuous closed injective function and Y is soft regular then X is soft *e*-regular.

Proof. Let (F,A) be soft closed set in Y with a soft point $P_{\mu}^{G}\notin$(F,A) Take $P_{\mu}^{G}=f\,(P_{\lambda}^{F})$. Since Y is soft regular, there exists disjoint soft open sets (U,A) and (V,B) such that $P_{\lambda}^{F}\in$(U,A) and P_{μ}^{G}=$f\,(P_{\lambda}^{F})\in f$(U,A) and (F,A) $\tilde{\subseteq}$ f(V,B) such that f(U,A) and f(V,B) are disjoint soft open sets. Therefore, we obtain that,f^{1}(F,A)$\tilde{\subseteq}$(V,B). Since *f* is soft *e*-continuous, f^{1}(F,A) is soft *e*-closed set in X and $P_{\lambda}^{F}\notin f^{1}$(F,A). Hence by definition X is soft *e*-regular.

Theorem 5.9. If *(F,A)* is soft *e*-closed set in X and $f:X\rightarrow Y$ is bijective, soft continuous and soft *e*-closed , then *f(F,A)* is soft *e*-closed set in Y.

Proof. Let $f(F,A)\tilde{\subseteq}(G,B)$ where *(G,B)* is a soft open set in Y. Since *f* is soft continuous, $f^{1}(G,B)$ is a soft open set containing *(F,A)*. Hence, S*e*-Cl$(F,A)\tilde{\subseteq}\ f^{1}(G,B)$ as *(F,A)* is soft *e*-closed set. Since *f* is soft *e*-closed, *f(Se-Cl(F,A))* is soft *e*-closed set contained in the soft open set *(G,B)*, which implies S*e*-Cl $f(Se\text{-}Cl(F,A))\tilde{\subseteq}(G,B)$ and hence S*e*-Cl *f(F,A))* $\tilde{\subseteq}(G,B)$. So *f(F,A))* is soft *e*-closed set in Y.

Definition 5.10. A soft subset (F,A) of a soft topological space (X,τ,E) is soft *e*-connected iff (F,A) can't be expressed as the union of two non empty disjoint soft *e*-open sets.

Theorem 5.11. Let $f: X\rightarrow Y$ is soft *e*-continuous and surjection map. If *(H,A)* is soft *e*-connected, then *f(H,A)* is soft connected.

Proof. Suppose that *f(H,A)* is not soft connected space. Then, $\exists$ non empty soft open sets (F,K) and (G,K) in Y such that *f(H,A)* = (F,A)$\tilde{\cup}$(G,A). Since *f* is soft *e*-continuous, $f^{1}(F,A)$ and $f^{1}(G,A)$ are soft *e*-open set in X and (H,A) $=f^{1}[(F,A)\tilde{\cup}(G,A)]=f^{1}(F,A)\ \tilde{\cup}\ f$

$^{1}(\mu)$. It is clear that f^{-1}(F,A) and f^{1}(G,A) are soft *e*-open set in X. Therefore, (H,A) is not soft *e*-connected, which is a contradiction to the given hypothesis. Hence, *f(H,A)* is soft connected.

Definition 5.12. A function $f: (X,\tau_1, E) \rightarrow (Y,\tau_2, E)$ is called soft *e*-homeomorphism (resp. soft *e*r-homeomorphism) if f is a soft *e*-continuous bijection (resp. soft *e*-irresolute bijection) and $f^{-1}: (Y,\tau_2, E) \rightarrow (X, \tau_1, E)$ is a soft *e*-continuous (resp.soft *e*-irresolute).
Now we can give the following definition by taking the soft space (X,τ,E) instead of the soft space (Y,τ_2, E).

Definition 5.13. For a soft topological space (X,τ,E), we define the following two collections of functions:

(a) S*e*-h(X,τ,E) $=\{f \mid f: (X,\tau,E) \rightarrow (X,\tau,E)$ is a soft *e*-continuous bijection, $f^{-1}: (X,\tau,E) \rightarrow (X,\tau,E)$ is soft *e*-continuous$\}$.

(b) S*er*-h(X,τ,E) $=\{ f \mid f: (X,\tau,E) \rightarrow (X,\tau,E)$ is a soft *e*-irresolute bijection, $f^{-1}: (X,\tau,E) \rightarrow (X,\tau,E)$ is soft *e*-irresolute$\}$.

Theorem 5.14. For a soft topological space (X,τ,E), S-h(X,τ,E) $\tilde{\subseteq}$ S*er*-h(X,τ,E) $\tilde{\subseteq}$ S*e*-h(X,τ,E), where, S-h(X,τ,E) $=\{ f \mid f: (X,\tau,E) \rightarrow (X,\tau,E)$ is a soft continuous bijection, $f^{-1}: (X,\tau,E) \rightarrow (X,\tau,E)$ is soft continuous i.e. f is soft homeomorphisms$\}$.

Proof. First we show that every soft-homeomorphism $f: (X,\tau_1, E) \rightarrow (Y,\tau_2, E)$ is a soft *e*-r-homeomorphism. Let $(G,A) \in$ S*e*-OS(Y), then $(G,A) \tilde{\subseteq} Int(Cl_\delta(G,A)) \tilde{\cup} Cl(Int_\delta(G,A))$. Hence, $f^{-1}(G,A) \tilde{\subseteq} f^{-1}[Int(Cl_\delta(G,A)) \tilde{\cup} Cl(Int_\delta(G,A))] = Int(Cl_\delta(f^{-1}(G,A)) \tilde{\cup} Cl(Int_\delta(f^{-1}(G,A))$ and so $f^{-1}(G,A) \in$ S*e*-OS(X). Thus, f is soft *e*-irresolute. In a similar way, it can be shown that f^{-1} is soft *e*-irresolute. Hence, we have, S-h(X,τ,E) $\tilde{\subseteq}$ S*er*-h(X,τ,E).
Finally, it is obvious that S*er*-h(X,τ,E) $\tilde{\subseteq}$ S*e*-h(X,τ,E), because every soft *e*-irresolute function is soft *e*-continuous.

Theorem 5.15. For a soft topological space (X,τ,E), the collection S*er*-h(X,τ,E) forms a group under the composition of functions.

Proof. If $f: (X,\tau_1,E) \rightarrow (Y,\tau_2,E)$ and $g: (Y,\tau_2,E) \rightarrow (Z,\tau_3,E)$ are soft *e*r-homeomorphism, then their composition $gof: (X,\tau_1,E) \rightarrow (Z,\tau_3,E)$ is a soft *e*r-homeomorphism.It is obvious that for a bijective soft *e*r-homeomorphism $f: (X,\tau_1,E) \rightarrow (Y,\tau_2,E)$, $f^{-1}: (Y,\tau_2,E) \rightarrow (X,\tau_1,E)$ is also a soft *e*r-homeomorphism and the identity function $I: (X,\tau_1,E) \rightarrow (X,\tau_1,E)$ is a soft *e*r-homeomorphism. A binary operation α : S*er*-h$(X,\tau,E)\times$S*er*-h$(X,\tau,E) \rightarrow$ S*er*-h(X,τ,E) is well defined by $\alpha(a,b) = boa$, where a,b$\in$S*er*-h(X,τ,E) and boa is the composition of a and b. By using the above properties, the set S*er*-h(X,τ,E) forms a group under composition of function.

Theorem 5.16. The group S-h(X,τ,E) of all soft homeomorphisms on (X,τ,E) is a subgroup of S*er*-h(X,τ,E).

Proof. For any a,b$\in$S-h(X,τ,E), we have, $\alpha(a,b^{-1}) = b^{-1}oa \in$ S-h(X,τ,E) and $I_X \in$ S-h $(X,\tau,E) \neq \varphi$. Thus, using (Theorem 4.14) and (Theorem 4.15), it is obvious that the group S-h(X,τ,E) is a subgroup of S*er*-h(X,τ,E).

For a soft topological space (X,τ,E), we can construct a new group S*e*r-h(X,τ,E) satisfying the property: If there exists a homeomorphism $(X,\tau,E) \cong (Y,\tau,E)$, then there exists a group isomorphism S*e*r-h$(X,\tau,E) \cong$ S*e*r-h(X,τ,E).

Corollary 5.17. Let $f : (X,\tau_1,E) \to (Y,\tau_2,E)$ and $g : (Y,\tau_2,E) \to (Z,\tau_3,E)$ be two functions between soft topological spaces.
(i) For a soft *e*r-homeomorphism $f : (X,\tau_1,E) \to (Y,\tau_2,E)$, there exists an isomorphism, say, $f_* :$ S*e*r-h$(X,\tau,E) \to$ S*e*r-h(X,τ,E), defined $f_*(a) = f \circ a \circ f^{-1}$, for any element $a \in$ S*e*r-h(X,τ,E).
(ii) For two soft *e*r-homeomorphisms $f : (X,\tau_1,E) \to (Y,\tau_2,E)$ and $g : (Y,\tau_2,E) \to (Z,\tau_3,E)$, $(g \circ f)_* = g_* \circ f_* :$ S*e*r-h$(X,\tau_1,E) \to$ S*e*r-h(Z,τ_3,E) holds.
(iii) For the identity function $I_X : (X,\tau,E) \to (X,\tau,E)$, $(I_X)_* = I :$ S*e*r-h$(X,\tau,E) \to$ S*e*r-h(X,τ,E) holds where I denotes the identity isomorphism.

Proof. Straightforward.

6. Conclusion

In this work we introduced the concept of soft *e-open* set and investigated some properties of them. Then focused the relationships among soft δ-pre open sets, soft δ-semi open sets, soft pre-open sets and soft *e-open* sets. We also investigated the concepts of soft *e-open* functions, soft *e*-continuous, soft *e*-irresolute and soft *e*-homomorphism on soft topological space and discussed their relations with existing soft continuous and other weaker forms of soft continuous functions. Further soft *e*-separation axioms have been introduced and investigated with the help of soft *e*-open sets. Finally, we observed that the collection S*e*r-h(X,τ,E) form a soft group. We hope that the findings in this work will help researcher enhance and promote the further study on soft topological spaces to carry out a general framework for their applications in separation axioms, connectedness, compactness etc. and also in practical life.

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A NEW TYPE OF CONVERGENCE IN INTUITIONISTIC FUZZY NORMED LINEAR SPACES

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Abstaract − In the present paper, we introduce a new type of convergence, called standard convergence (or std-convergence), in an intuitionistic fuzzy normed linear space (IFNLS). We have also introduced the concept of std-Cauchyness and proved that these notions are stronger than usual convergence and usual Cauchyness in an IFNLS. Further, we have shown that these two notions are not directly compatible with each other and hence, defined the notion of strong std-convergence which is compatible with std-Cauchy sequences.

Keywords − *Intuitionistic fuzzy normed linear space; std-convergence; strong std-convergence; std-Cauchy sequence.*

1 Introduction

The concepts of convergence and Cauchyness of sequences lay the foundation of structure of any metric space and as such, the study of these concepts are of greatest importance in analysis. Therefore, the study of both weaker and stronger concepts than the usual convergence has always been a well motivated area of research. Some very important work in this direction in connection with fuzzy metric spaces may be found in [9, 11, 14, 17].

Recently, Ricarte and Romaguera [23] have established relationships between the theory of complete fuzzy metric spaces and domain theory by introducing a stronger notion than Cauchy sequence called standard Cauchy sequence. They proved that the famous result due to Edalat and Heckmann [8] which gives a characterization of complete metric spaces with the help of continuous domains could be obtained from their results in fuzzy metrics and in fact, could not be obtained from classical metric. More recently, answer to two well posed questions by Morillas and Sapena [18] was

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given by Gregori and Minana [12] by establishing what conditions must be included in the definition of standard convergence in fuzzy metric spaces so that it remains compatible with the concept of Cauchyness.

The theory of intuitionistic fuzzy sets was introduced by Atanassov [2] which has been extensively used in decision making problems [1] and in E-infinity theory of high energy physics [21]. The concept of intuitionistic fuzzy metric space was introduced by Park [22]. Furthermore, Saadati and Park [24] gave the notion of intuitionistic fuzzy normed space. Some works related to the convergence of sequences in several normed linear spaces in fuzzy setting can be found in [3, 4, 5, 6, 7, 10, 13, 15, 16, 19, 20, 25, 26, 27].

Due to its successful application in connection with fuzzy metric spaces and domain theory [23], in the current paper, we introduce and generalize the notions of standard (std-) convergence and standard (std-) Cauchy sequences in an IFNLS.

2 Preliminary

Throughout the paper $\mathbb{N}$ will denote the set of all natural numbers and $\mathbb{R}$ will denote the set of real numbers. First we collect some preliminary existing definitions in literature.

Definition 2.1. [24] The 5-tuple $(X, \mu, \nu, *, \circ)$ is said to be an IFNLS if X is a linear space, $*$ is a continuous t-norm, $\circ$ is a continuous t-conorm, and μ, ν fuzzy sets on $X \times (0, \infty)$ satisfying the following conditions for every $x, y \in X$ and $s, t > 0$:

(a) $\mu(x,t) + \nu(x,t) \leq 1$,

(b) $\mu(x,t) > 0$,

(c) $\mu(x,t) = 1$ if and only if $x = 0$,

(d) $\mu(\alpha x, t) = \mu(x, \frac{t}{|\alpha|})$ for each $\alpha \neq 0$,

(e) $\mu(x,t) * \mu(y,s) \leq \mu(x+y, t+s)$,

(f) $\mu(x,t) : (0,\infty) \to [0,1]$ is continuous in t,

(g) $\lim_{t\to\infty} \mu(x,t) = 1$ and $\lim_{t\to 0} \mu(x,t) = 0$,

(h) $\nu(x,t) < 1$,

(i) $\nu(x,t) = 0$ if and only if $x = 0$,

(j) $\nu(\alpha x, t) = \nu(x, \frac{t}{|\alpha|})$ for each $\alpha \neq 0$,

(k) $\nu(x,t) \circ \nu(y,s) \geq \nu(x+y, t+s)$,

(l) $\nu(x,t) : (0,\infty) \to [0,1]$ is continuous in t,

(m) $\lim_{t\to\infty} \nu(x,t) = 0$ and $\lim_{t\to 0} \nu(x,t) = 1$.

In this case (μ, ν) is called an intuitionistic fuzzy norm (IFN). When no confusion arises, an IFNLS will be denoted simply by X.

Example 2.2. Let $(V, ||\cdot||)$ be a normed linear space. Let $a * b = ab$ and $a \circ b = \min\{a+b, 1\}$ for all $a, b \in [0,1]$ and μ_0, ν_0 be fuzzy sets on $X \times (0, \infty)$ defined as $\mu_0(x,t) = \frac{t}{t+||x||}$, $\nu_0(x,t) = \frac{||x||}{t+||x||}$ for all $t \in (0,\infty)$. Then $(X, \mu_0, \nu_0, *, \circ)$ is an IFNLS.

Remark 2.3. Let $(X, \mu_0, \nu_0, *, \circ)$ be an IFNLS. For $t > 0$, the open ball $B_r^t(x)$ with center x and radius $r \in (0,1)$ is defined as

$$B_r^t(x) = \{y \in V : \mu(x-y,t) > 1-r, \nu(x-y,t) < r\}.$$

Considering these open balls as base, the IFN (μ, ν) induces a topology $\tau_{(\mu,\nu)}$ on X.

Definition 2.4. [24] Let X be an IFNLS. A sequence $x = \{x_k\}$ in X is said to be convergent to $\xi \in X$ with respect to the intuitionistic fuzzy norm (μ, ν) if, for every $\epsilon \in (0,1)$ and $t > 0$, there exists $k_0 \in \mathbb{N}$ such that $\mu(x_k - \xi, t) > 1 - \epsilon$ and $\nu(x_k - \xi, t) < \epsilon$ for all $k \geq k_0$. It is denoted by $(\mu, \nu) - \lim x = \xi$.

Definition 2.5. [24] Let X be an IFNLS. A sequence $x = \{x_k\}$ in X is said to be a Cauchy sequence with respect to the intuitionistic fuzzy norm (μ, ν) if, for every $\epsilon \in (0,1)$ and $t > 0$, there exists $k_0 \in \mathbb{N}$ such that $\mu(x_k - x_m, t) > 1 - \epsilon$ and $\nu(x_k - x_m, t) < \epsilon$ for all $k, m \geq k_0$.

3 std-Convergence and std-Cauchy Sequences in IFNLS

Now we are ready to introduce the notions of std-Convergence and std-Cauchy sequences in IFNLS.

Definition 3.1. Let X be an IFNLS. A sequence $x = \{x_k\}$ in X is said to be std-convergent to $\xi \in X$ with respect to the intuitionistic fuzzy norm (μ, ν) if, for every $\epsilon \in (0,1)$, there exists $k_\epsilon \in \mathbb{N}$ such that $\mu(x_k - \xi, t) > \frac{t}{t+\epsilon}$ and $\nu(x_k - \xi, t) < \frac{\epsilon}{t+\epsilon}$ for all $k \geq k_\epsilon$ and for all $t > 0$. We denote it by $(\mu, \nu) \overset{std}{-} \lim x = \xi$.

Definition 3.2. Let X be an IFNLS. A sequence $x = \{x_k\}$ in X is said to be std-Cauchy with respect to the intuitionistic fuzzy norm (μ, ν) if, for every $\epsilon \in (0,1)$, there exists $k_\epsilon \in \mathbb{N}$ such that $\mu(x_k - x_m, t) > \frac{t}{t+\epsilon}$ and $\nu(x_k - x_m, t) < \frac{\epsilon}{t+\epsilon}$ for all $k, m \geq k_\epsilon$ and for all $t > 0$. We call X std-complete if every std-Cauchy sequence is std-convergent in X.

Our first two results show that the notions of std-convergence and std-Cauchy are both stronger than usual convergence and usual Cauchy respectively in an IFNLS.

Theorem 3.3. Let X be an IFNLS and the sequence $x = \{x_k\}$ in X be std-convergent to $\xi \in X$. Then $\{x_k\}$ converges to ξ with respect to the IFN (μ, ν).

Proof. Let $(\mu,\nu) \overset{std}{-} \lim x = \xi$. Then for given $\epsilon > 0$, there exists $k_\epsilon \in \mathbb{N}$ such that $\mu(x_k - \xi, t) > \frac{t}{t+\epsilon}$ and $\nu(x_k - \xi, t) < \frac{\epsilon}{t+\epsilon}$ for all $k \geq k_\epsilon$ and all $t > 0$.

Now since $\frac{\epsilon}{t+\epsilon} < \epsilon$ for all $t > 0$, we have that $\frac{t}{t+\epsilon} = 1 - \frac{\epsilon}{t+\epsilon} > 1 - \epsilon$. Consequently, $\mu(x_k - \xi, t) > \frac{t}{t+\epsilon} > 1 - \epsilon$. In a similar way it can be proved that $\nu(x_k - \xi, t) < \epsilon$ for all $k \geq k_\epsilon$. This proves that $(\mu,\nu) - \lim x = \xi$. □

The following can be proved using techniques in 3.3.

Theorem 3.4. Let X be an IFNLS and the sequence $x = \{x_k\}$ in X be std-Cauchy. Then $\{x_k\}$ is Cauchy with respect to the IFN (μ, ν).

Next we give an example of a sequence in an IFNLS which is std-convergent but not std-Cauchy.

Example 3.5. Consider the usual norm $|\cdot|$ on $\mathbb{R}$ restricted to $[0,\infty)$ and the IFN (μ_0, ν_0) as defined in Example 2.2. Let $X = [0,\infty)$ and define on $X \times (0,\infty)$ the functions μ, ν as

$$\mu(x-y,t) = \begin{cases} 1, & \text{if } x = y \\ \mu_0(x-0,t)\mu_0(0-y,t), & \text{if } x \neq y, \end{cases}$$

and $\nu(x-y,t) = 1 - \mu(x-y,t)$. Then it is a routine verification to check that $(X, \mu, \nu, *, \circ)$ is an IFNLS.

Consider the sequence $\{x_k\}$ in X where $x_k = \frac{1}{k}$ for all $k \in \mathbb{N}$. Let $\epsilon \in (0,1)$. We can choose $k_\epsilon \in \mathbb{N}$ such that $k_\epsilon > \frac{1}{\epsilon}$ and hence $\mu(x_k - 0, t) = \frac{t}{t+\frac{1}{k}} > \frac{t}{t+\epsilon}$ for all $k \geq k_\epsilon$ and all $t > 0$. In a similar fashion it can be proved that $\mu(x_k - 0, t) < \frac{\epsilon}{t+\epsilon}$. So, $\{x_k\}$ is std-convergent to 0 in X.

Now if we assume $\{x_k\}$ to be std-Cauchy, then for each $\epsilon \in (0,1)$, there exists $k_\epsilon \in \mathbb{N}$ such that $\mu(x_k - x_m, t) = \frac{t}{t+\frac{1}{k}} \cdot \frac{t}{t+\frac{1}{m}} > \frac{t}{t+\frac{1}{\epsilon}}$ for all $k, m \geq k_\epsilon$ and all $t > 0$. Thus we have $\frac{t}{(t+\frac{1}{k_\epsilon})(t+\frac{1}{k_\epsilon})} > \frac{1}{t+\epsilon}$ for all $t > 0$. But then we have $\lim_{t\to 0} \frac{t}{(t+\frac{1}{k_\epsilon})(t+\frac{1}{k_\epsilon})} > \lim_{t\to 0} \frac{1}{t+\epsilon}$. This implies that $0 > \frac{1}{\epsilon}$, which is a contradiction. This proves that $\{x_k\}$ can not be std-Cauchy.

From Example 3.5 we observe that the concept of std-convergence and std-Cauchy are not compatible with each other. To settle this, we define the notion of strong std-convergence so that every strong std-convergent sequence is std-Cauchy as well.

Definition 3.6. A sequence $\{x_k\}$ in an IFNLS $(X, \mu, \nu, *, \circ)$ is said to be strong std-convergent if it is both convergent and std-Cauchy with respect to the IFN (μ, ν).

A question naturally arises, whether every strong std-convergent sequence is std-convergent or not. An affirmative answer to this question is given by our next result.

Theorem 3.7. Let $(X, \mu, \nu, *, \circ)$ be an IFNLS and $\{x_k\}$ be a strong std-convergent sequence in X. Then $\{x_k\}$ is std-convergent with respect to the IFN (μ, ν).

Proof. Let $\epsilon \in (0,1)$ and $t > 0$. We assume that $\{x_k\}$ converges to $\xi \in X$ with respect to (μ,ν). Since $\mu(x, \_)$ is continuous for all $x \in X$, we have that $\lim_{m\to\infty} \mu(x_k - x_m, t) = \mu(x_k - \xi, t)$ for all $k \in \mathbb{N}$.

Again since $\{x_k\}$ is std-Cauchy, we have that for $\delta \in (0,\epsilon)$, there exists $k_\delta \in \mathbb{N}$ such that $\mu(x_k - x_m, t) > \frac{t}{t+\delta} > \frac{t}{t+\epsilon}$ for all $k, m \geq k_\delta$ and all $t > 0$. This again implies that $\mu(x_k - \xi, t) = \lim_{m\to\infty} \mu(x_k - x_m, t) \geq \frac{t}{t+\delta} > \frac{t}{t+\epsilon}$ for all $k \geq k_\delta$ and all $t > 0$.

In a similar fashion, it can be proved that $\nu(x_k - \xi, t) < \frac{\epsilon}{t+\epsilon}$ for all $k \geq k_\delta$ and all $t > 0$. Hence $\{x_k\}$ is std-convergent. □

Next we show that the notion of strong std-convergence is free from any ambiguity by showing that strong std-convergent sequences have unique limit. To show this, it is sufficient to prove that a std-convergent sequence has a unique limit, which is the aim of our next result.

Theorem 3.8. Let $(X, \mu, \nu, *, \circ)$ be an IFNLS. If a sequence $\{x_k\}$ in X is std-convergent with respect to the IFN (μ,ν), then $(\mu,\nu) \overset{std}{-} \lim x_k$ is unique.

Proof. Let $(\mu,\nu) \overset{std}{-} \lim x_k = \xi_1$ and $(\mu,\nu) \overset{std}{-} \lim x_k = \xi_2$. Given $\epsilon > 0$ and $t > 0$ choose $\gamma \in (0,1)$ such that $(\frac{t}{t+\gamma}) * (\frac{t}{t+\gamma}) > \frac{t}{t+\epsilon}$.

Since $(\mu,\nu) \overset{std}{-} \lim x_k = \xi_1$, there exists $k_1 \in \mathbb{N}$ such that $\mu(x_k - \xi_1, \frac{t}{2}) > \frac{t}{t+\gamma}$ for all $k \geq k_1$ and all $t > 0$. Also since $(\mu,\nu) \overset{std}{-} \lim x_k = \xi_2$, there exists $k_2 \in \mathbb{N}$ such that $\mu(x_k - \xi_2, \frac{t}{2}) > \frac{t}{t+\gamma}$ for all $k \geq k_2$ and all $t > 0$.

Let $k_0 = \max\{k_1, k_2\}$. Then both of the above two conditions hold together for all $k \geq k_0$ and all $t > 0$.

Now we have, for all $k \geq k_0$ and all $t > 0$,

$$\begin{aligned} \mu(\xi_1 - \xi_2, t) &\geq \mu(x_k - \xi_1, \frac{t}{2}) * \mu(x_k - \xi_2, \frac{t}{2}) \\ &> (\frac{t}{t+\gamma}) * (\frac{t}{t+\gamma}) \\ &> \frac{t}{t+\epsilon}. \end{aligned}$$

Since ϵ was chosen arbitrarily, we must have $\mu(\xi_1 - \xi_2, t) = 1$ for all $t > 0$. Hence we must have $\xi_1 - \xi_2 = 0$, i.e., $\xi_1 = \xi_2$. □

4 Conclusion

In this paper, the concept of std-convergence and std-Cauchy sequences have been introduced. Another concept, called strong std-convergence has also been introduced which is directly compatible with std-Cauchy sequences. These new concepts are stronger than their usual counterparts and as such, they constitute a well motivated area of research. The study of std-statistical convergence, std-ideal convergence, std-lacunary statistical convergence may be suggested as some important future work in this new setting.

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$\Omega - \mathscr{N}$-FILTERS ON CI-ALGEBRA

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Abstaract − This paper deals the notion of $\Omega - \mathscr{N}$-structured subalgebras and $\Omega - \mathscr{N}$-structured Filters on CI-algebra. Further some of the properties and results using the idea of $\Omega - \mathscr{N}$-function on CI-algebra also established.

Keywords − *CI-algebra, Subalgebra, Filter,* $\Omega - \mathscr{N}$*-filter*

1 Introduction

After the initiation of the two classes of abstract algebras: BCK-algebras and BCI-algebras by Y. Imai and K. Iseki [2] , B. L. Meng[4][5], introduced the notion of a CI-algebra . K. H. Kim [3] also dealt about some concepts on CI-algebras. Zadeh. L. A. [9], introduced Fuzzy Sets for classifying the uncertainty. Then many researches used the notion of fuzzy in various algebraic structures. Samy. M. Mostafa [8] dealt fuzzification of ideals in CI-algebra and Intuitionistic (T, S)-fuzzy CI-algebras were discussed by A. Borumand Saeid et. al [1]. Also in [6] and [7] the authors introduced $\mathscr{N}$-ideals of a BF-algebras and $\mathscr{N}$-filters of CI-algebras. Motivated by these, this paper, intends to discuss $\Omega - \mathscr{N}$-structured filter of a CI-algebra and establish some simple, elegant and interesting results.

2 Preliminaries

This section deals with the basic definition of $\mathscr{N}$-function,$\Omega - \mathscr{N}$-function, CI-algebra, subalgebra and Filter of a CI-algebra.

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Definition.2.1. [6][7] *$\mathscr{N}$-structure and $\mathscr{N}$-function*
Consider a non-empty Set S. Denote the collection of functions from S to $[-1,0]$ by $\mathcal{F}(S,[-1,0])$. It is said that a member of $\mathcal{F}(S,[-1,0])$ is a negative valued function from S to $[-1,0]$, briefly $\mathscr{N}$-function and by an $\mathscr{N}$-structure on S, it means that an ordered pair (S,η) of S and $\mathscr{N}$-function η on S.

Definition.2.2. $\Omega-\mathscr{N}$*-function:*
A $\Omega-\mathscr{N}$-function η in a non-empty set S is a function $\eta : S\times\Omega \rightarrow [-1,0]$, where Ω is any non-empty set. The set of all $\Omega-\mathscr{N}$-functions from $S\times\Omega$ to $[-1,0]$ is denoted by $\mathcal{F}(S\times\Omega,[-1,0])$ and by the term $\Omega-\mathscr{N}$-Structure(Ω-NS) on S, it means that an ordered pair $(S\times\Omega,\eta)$ of $S\times\Omega$ and $\Omega-\mathscr{N}$-function η on $S\times\Omega$.

Definition.2.2. Consider the $\Omega-\mathscr{N}$-structure $(S\times\Omega,\eta)$ on a non-empty S. The negative Ω-Level subset η_t of η is defined as follows:
For some $t\in[-1,0]$, $\eta_t=\{x\in S \;:\; \eta(x,q)\geq t \;\forall q \in\Omega\}$.

Definition 2.3. [3][4] A CI-algebra is a non-empty set X with a consonant 1 and a single binary operation $*$ satisfying the following axioms:
(i)$x*x=1$
(ii)$1*x=x$
(iii)$x*(y*z)=y*(x*z)$ for all $x,y\in X$

Example 2.4.[3][4][5] Let $X=\{1,a,b,c\}$ and $Y=\{1,a,b,c,d\}$ be a set with the following tables

*	1	a	b	c
1	1	a	b	c
a	1	1	a	c
b	1	1	1	c
c	1	a	b	1

*	1	2	3	4	5
1	1	2	3	4	5
2	1	1	1	4	4
3	1	1	1	4	4
4	4	5	1	1	2
5	4	4	4	1	1

Then $\{X,*,1\}$ and $\{Y,*,1\}$ are CI-algebra.

Example 2.5.[5] Let X be the set of all positive real numbers. Then X becomes a CI-algebra by defining $x*y=\frac{y}{x}$ for all $x,y\in X$.

Definition 2.6.[3][4] A partial ordering $\leq$ on a CI-algebra $(X,*,1)$ can be defined as $x\leq y$ if, and only if, $x*y=1$.

Definition 2.7. [3][4] A non-empty subset S of a CI-algebra X is said to be a subalgebra if $x*y\in S$ for all $x,y\in S$.

Definition 2.8. [3] A non-empty subset F of a CI-algebra X is said to be a Filter of X if (i) $1 \in F$ and (ii) $x * y \in F$ and $x \in F$ then $y \in F$ for all $x, y \in X$.

Definition 2.9. An Filter F of X is called closed if $x * 1 \in F$ for all $x \in F$.

3 $\Omega - \mathscr{N}$-Subalgebra and $\Omega - \mathscr{N}$-Filter on a CI-algebra

This section introduces, the notion of $\Omega - \mathscr{N}$-subalgebra and $\Omega - \mathscr{N}$-Filter on a CI-algebra and discuss some of its results. In the rest of the paper, X represents a CI-algebra, Ω is any non-empty set and η is a $\Omega - \mathscr{N}$ function from $X \times \Omega$ to $[-1, 0]$ unless otherwise specified.

Definition 3.1. An $\Omega - \mathscr{N}$-structure $(X \times \Omega, \eta)$, on a CI-algebra X is called an $\Omega - \mathscr{N}$-subalgebra on X if $\eta((x*y), q) \leq \eta(x, q) \vee \eta(y, q)$ for all $x, y \in X$ and $q \in \Omega$.

Example 3.2. Consider the CI-algebra $X = (\{1, a, b, c, d\}, *, 1)$ given below.

*	1	a	b	c	d
1	1	a	b	c	d
a	1	1	b	b	d
b	1	a	1	a	d
c	1	1	1	1	d
d	d	d	d	d	1

The $\Omega - \mathscr{N}$-structure $(X \times \Omega, \eta)$ defined by, $\forall\ q \in \Omega$

$$\eta(x, q) = \begin{cases} -0.8 & ; \quad x = 1 \\ -0.7 & ; \quad x = a \\ -0.5 & ; \quad x = b \\ -0.3 & ; \quad x = c \\ -0.3 & ; \quad x = d \end{cases}$$

is an $\Omega - \mathscr{N}$-subalgebra on X.

Proposition 3.3. If (X, η) is an $\Omega - \mathscr{N}$-subalgebra on X then
$\eta(1, q) \leq \eta(x * 1, q) \leq \eta(x, q)$ for all $x \in X$ and $q \in \Omega$.
Proof. Let $x \in X$.
Then $\eta(1, q) = \eta((x * 1) * (x * 1), q) \leq \eta(x * 1, q) \vee \eta(x * 1, q) = \eta(x * 1, q)$ and $\eta(x * 1, q) \leq \eta(x, q) \vee \eta(1, q) = \eta(x, q) \vee \eta(x * x, q) = \eta(x, q)$. ■

Proposition 3.4. If (X, η) is an N-subalgebra of X then negative Level subset η_t of X is either empty or subalgebra of X, for all $t \in [-1, 0]$.
Proof. Let $t \in [-1, 0]$ and η_t be nonempty.

Take $x, y \in \eta_t \Rightarrow \eta(x,q) \leq t$ and $\eta(y,q) \leq t$.
Then $\eta(x*y,q) \leq \eta(x,q) \vee \eta(y,q) \leq t \vee t = t \Rightarrow x*y \in \eta_t$. ■

Definition.3.5. An Ω-NS on a CI-algebra X is said to be $\Omega-\mathscr{N}$-structured filter ($\Omega-\mathscr{N}$*-filter*) on X if (i) $\eta(1,q) \leq \eta(x,q)$ and
(ii) $\eta(y,q) \leq \eta(x*y,q) \vee \eta(x,q)$ for all $x, y \in X$ and $q \in \Omega$

Definition.3.6. An Ω-NS on a CI-algebra X is said to be $\Omega-\mathscr{N}$-structured closed filter ($\Omega-\mathscr{N}$*c-filter*) on X if (i) $\eta(y,q) \leq \eta(x*y,q) \vee \eta(x,q)$ and (ii) $\eta(x*1,q) \leq \eta(1,q)$ for all $x, y \in X$ and $q \in \Omega$.

Example.3.7. The $\Omega-\mathscr{N}$-structure (X,η) on the CI-algebra in Example.2.5 defined by, $\forall\ q \in \Omega$

$$\eta(x,q) = \begin{cases} -0.8 & ; \quad x = 1 \\ -0.7 & ; \quad x = 2^n \quad ; n \in \mathrm{N} \\ -0.5 & ; \quad \text{otherwise} \end{cases}$$

is an $\Omega-\mathscr{N}$-filter but not $\Omega-\mathscr{N}c$-filter on X.

Example.3.8. The $\Omega-\mathscr{N}$-structure (X,η) on the CI-algebra in Example.2.5 defined by $\forall\ q \in \Omega$

$$\eta(x,q) = \begin{cases} -0.8 & ; \quad x = 1 \\ -0.7 & ; \quad x = 2^n \quad ; n \in Z^+ \\ -0.5 & ; \quad \text{otherwise} \end{cases}$$

is an $\Omega-\mathscr{N}c$-filter on X.

Proposition.3.9. If (X,η) is an $\Omega-\mathscr{N}$-filter on X with $x \leq y$ for all $x, y \in X$, and $q \in \Omega$ then $\eta(x,q) \geq \eta(y,q)$ that is η is order-reversing.

Proof. Let $x, y \in X$ and $q \in \Omega$ such that $x \leq y$.
Then by the partial ordering $\leq$ defined in X, we have $x*y = 1$. Thus $\eta(y,q) \leq \eta(x*y,q) \vee \eta(x,q) = \eta(1,q) \vee \eta(x,q) \leq \eta(x,q)$. This completes the proof. ■

Proposition.3.10. If (X,η) is an $\Omega-\mathscr{N}$-filter on X with $x \leq y*z$ for all $x, y, z \in X$, and $q \in \Omega$ then $\eta(z,q) \leq \eta(x,q) \vee \eta(y,q)$.

Proof. Let $x, y, z \in X$ such that $x \leq y*z$.
Then by the partial ordering $\leq$ defined in X, we have $x*(y*z) = 1$.
Then $\eta(z,q) \leq \eta(y*z,q) \vee \eta(y,q)$
$\leq (\eta((x*(y*z),q)) \vee \eta(x,q)) \vee \eta(y,q)$
$= (\eta(1,q) \vee \eta(x,q)) \vee \eta(y,q)$
$= \eta(x,q) \vee \eta(y,q)$. ■

Remark.3.11. The terms $\Omega-\mathscr{N}$-subalgebra and $\Omega-\mathscr{N}$-filter on X are independent to each other. The following examples give the illustration.

Example.3.12. Consider the $\Omega-\mathscr{N}$-filter in Example 3.7.
Here

$$\eta\left(\left(2^4 * 2^2\right), q\right) = \eta\left(\frac{1}{4}, q\right) = -0.5 > -0.7 = \eta\left(2^4, q\right) \vee \eta\left(2^2, q\right)$$

, which is not an $\Omega-\mathscr{N}$-subalgebra.

Example.3.13. Consider the $\Omega-\mathscr{N}$-subalgebra in Example 3.2.
Here

$$\eta(c, q) = -0.3 > -0.5 = -0.7 \vee -0.5 = \eta(b * c, q) \vee \eta(b, q),$$

which is not an $\Omega-\mathscr{N}$-filter.

The following gives a sufficient condition for an $\Omega-\mathscr{N}$-subalgebra to be an $\Omega-\mathscr{N}$-filter.

Theorem.3.14. In a $\Omega-\mathscr{N}$-subalgebra (X, η), If $\eta(x * y, q) \leq \eta(y * x, q)$ $\forall\ x, y \in X$ and $q \in \Omega$ then (X, η) is an $\Omega-\mathscr{N}$-filter of X.

Proof. Let (X, η) be a $\Omega-\mathscr{N}$-subalgebra of X with
$\eta(x * y, q) \leq \eta(y * x, q)\ \forall\ x, y \in X$ and $q \in \Omega$.
Then $\eta(y, q) = \eta(1 * y, q) \leq \eta(y * 1, q)$
$= \eta((y * (x * x), q))$
$\leq \eta((x * (y * x), q))$
$\leq \eta(x, q) \vee \eta(y * x, q)$.
Hence (X, η) is an $\Omega-\mathscr{N}$-filter of X. ■

Theorem.3.15. If the $\Omega-\mathscr{N}$-structure (X, η) of X is a $\Omega-\mathscr{N}c$-filter of X, then the set $K = \{x \in X; \eta(x, q) = \eta(1, q) \forall\ q \in \Omega\}$ is a filter of X.

Proof. Clearly, K is nonempty (since $1 \in K$). Let $x, x * y \in K$.
Then $\eta(x * y, q) = \eta(x, q) = \eta(1, q)$
$\Rightarrow \eta(y, q) \leq \eta(x * y, q) \vee \eta(x, q)$
$= \eta(1, q) \vee \eta(1, q)$
$= \eta(1, q)$.
But $\eta(1, q) \leq \eta(y, q) \Rightarrow \eta(y, q) = \eta(1, q)$.
Thus $y \in K$. Hence K is a filter of X. ■

The following theorem shows the arbitrary union of family of $\Omega-\mathscr{N}c$-filters of X is also an $\Omega-\mathscr{N}c$-filter of X.

Theorem.3.16. Let $\{\eta_i : i \in I\}$ be the family of $\Omega-\mathscr{N}c$-filter of X. Then $\bigcup_i \eta_i$ is also $\Omega-\mathscr{N}c$-filter of X.

Proof. Let $x * y \in X$.
Since $\{\eta_i : i \in I\}$ is the family of $\Omega - \mathscr{N}c$-filter of X, for any $i \in I$ we have,

$$(i)\ \eta_i(y,q) \leq \eta_i(x * y, q) \ \vee \ \eta_i(x,q) \ and \ (ii)\ \eta_i(x * 1, q) \leq \eta_i(x,q).$$

Now $\bigcup_i \eta_i(y,q) = sup\{\eta_i : i \in I\}$
$\leq sup\{\eta_i(x * y, q) \vee \eta_i(x,q) : i \in I\}$
$= sup\{\eta_i(x * y, q) : i \in I\} \vee sup\{\eta_i(x,q) : i \in I\}$
$= \bigcup_i \eta_i(x * y, q) \vee \bigcup_i \eta_i(x,q)$
and $\bigcup_i \eta_i(x * 1, q) = sup\{\eta_i(x * 1, q) : i \in I\} \leq sup\{\eta_i(x,q) : i \in I\} = \bigcup_i \eta_i(x,q)$
Hence $\bigcup_i \eta_i$ is an $\Omega - \mathscr{N}c$-filter of X. ■

Conclusion

In this paper, the notion of $\Omega - \mathscr{N}$-subalgebra and $\Omega - \mathscr{N}$-filter on a CI-algebra are introduced and some of the results have been discussed. In future it is planned to extend these ideas to homomorphism on $\Omega - \mathscr{N}$-filters,Cartesian products on $\Omega - \mathscr{N}$-filters and translation on $\Omega - \mathscr{N}$-filters.

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NORMS OVER FUZZY LIE ALGEBRA

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Abstaract − In this paper we introduce the concept of fuzzy Lie ideal and anti fuzzy Lie ideal by using a t-norm T and a t-conorm C, respectively. Next we introduce the concept of quotient fuzzy Lie ideal with respect to t-norm T. We investigate some their properties and obtain new results.

Keywords − *Lie algebra, ideals, fuzzy set theory, t-norm.*

1 Introduction

Lie algebras were first discovered by Sophus Lie (1842-1899) when he attempted to classify certain "smooth" subgroups of general linear groups. Lie algebra is applied in different domains of physics and mathematics, such as spectroscopy of molecules, atoms, nuclei, hadrons, hyperbolic and stochastic differential equations. The notion of fuzzy sets was first introduced by Zadeh[4]. Fuzzy and anti fuzzy Lie ideals in Lie algebras have been studied in[1, 2, 3]. In this paper we have tried apply the concepts of norms to fuzzy Lie algebras and fuzzy Lie ideals.

2 Preliminary

In this section, we first review some elementary aspects that are necessary for this paper.
A Lie algebra is a vector space L over a field F (equal to R or C) on which $L \times L \to L$ denoted by $(x, y) \to [x, y]$ is defined satisfying the following axioms:
(1) $[x, y]$ is bilinear,
(2) $[x, x] = 0$ for all $x \in L$,
(3) $[[x, y], z] + [[y, z], x] + [[z, x], y] = 0$ (Jacobi identity),
for all $x, y, z \in L$.

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In this paper by L will be denoted a Lie algebra. We note that the multiplication in a Lie algebra is not associative, i.e., it is not true in general that $[[x, y], z] = [x, [y, z]]$. But it is anti commutative, i.e., $[x, y] = -[y, x]$. A subspace H of L closed under $[,]$ will be called a Lie subalgebra. A subspace I of L with the property $[I, L] \subseteq I$ will be called a Lie ideal of L. Obviously, any Lie ideal is a subalgebra.
A t-norm T is a function $T : [0, 1]\times[0, 1] \to [0, 1]$ having the following four properties:
(T1) $T(x, 1) = x$ (neutral element),
(T2) $T(x, y) \leq T(x, z)$ if $y \leq z$ (monotonicity),
(T3) $T(x, y) = T(y, x)$ (commutativity),
(T4) $T(x, T(y, z)) = T(T(x, y), z)$ (associativity),
for all $x, y, z \in [0, 1]$. Replacing 1 by 0 in condition (T1), we obtain the concept of t-conorm C. If T be a t-norm, then

$$T(T(x, y), T(w, z)) = T(T(x, w), T(y, z)),$$

for all $x, y, w, z \in [0, 1]$ and we can replace t-conorm C by t-norm T. Recall that $T(C)$ is idempotent if for all $x \in [0, 1]$, $T(x, x) = x(C(x, x) = x)$.
Let L_1 and L_2 be Lie algebras over a field F. A linear transformation $f : L_1 \to L_2$ is called a Lie homomorphism if $f([x, y]) = [f(x), f(y)]$ for all $x, y \in L_1$.
Let T and C be t-norm and t-conorm, respectively. For all $x, y \in [0, 1]$, we say T and C are dual when

$$T(x, y) = 1 - C(1 - x, 1 - y),$$
$$C(x, y) = 1 - T(1 - x, 1 - y).$$

Let $\mu : L \to [0, 1]$. The complement of μ, denoted by μ^c is the fuzzy set in L given by $\mu^c(x) = 1 - \mu(x)$ for all $x \in L$.

3 Fuzzy Lie Subalgebra with Respect to a t-norm

In this section, we define the notion of fuzzy Lie subalgebra of L with respect to a t-norm T and investigate some related properties.

Definition 3.1. Let μ be a fuzzy set on L, i.e., a map $\mu : L \to [0, 1]$. A fuzzy set $\mu : L \to [0, 1]$ is called a fuzzy Lie subalgebra of L with respect to a t-norm T if
(1) $\mu(x + y) \geq T(\mu(x), \mu(y))$,
(2) $\mu(\alpha x) \geq \mu(x)$,
(3) $\mu([x, y]) \geq T(\mu(x), \mu(y))$
hold for all $x, y \in L$ and $\alpha \in F$.
A fuzzy subset $\mu : L \to [0, 1]$ satisfying (1), (2) and
(4) $\mu([x, y]) \geq \mu(x)$
is called a fuzzy Lie ideal of L with respect to a t-norm T.

Example 3.2. Let $L = R^3$ and $[x, y] = x \times y$, where $\times$ is cross product, for all $x, y \in L$. By routine calculations, it is clear that L is a Lie algebra over a field R.

Define $\mu : L \to [0,1]$ by

$$\mu(x_1, x_2, x_3) = \begin{cases} 1 & \text{if } x_1 = x_2 = x_3 = 0 \\ 0.50 & \text{if } x_1 = x_2 = 0 \text{ and } x_3 \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

if $T(a,b) = \min\{a,b\}$ for all $a, b \in [0,1]$, then μ is a fuzzy Lie subalgebra of L with respect to a t-norm T.

Lemma 3.3. Let μ be a fuzzy Lie subalgebra of L with respect to a t-norm T.
(1) If T be idempotent, then for all $x \in L$ we have that $\mu(0) \geq \mu(x)$.
(2) $\mu([x,y]) = \mu([y,x])$.

Proof. (1) Let μ be a fuzzy Lie subalgebra of L with respect to a t-norm T and $x \in L$. Then $\mu(0) = \mu(x + (-x)) \geq T(\mu(x), \mu(-x)) \geq T(\mu(x), \mu(x)) = \mu(x)$.
(2) $\mu([x,y]) = \mu(-[y,x]) \geq \mu([y,x]) = \mu(-[x,y]) \geq \mu([x,y])$.

Proposition 3.4. Let μ be a fuzzy Lie ideal in a Lie algebra L with respect to a t-norm T such that T be idempotent. Then for all $t \in [0,1]$ the set $L(\mu, t) = \{x \in L \mid \mu(x) \geq t\}$ is a Lie ideal of L.

Proof. Let $x, y \in L(\mu, t)$ and $\alpha \in F$. Then $\mu(x+y) \geq T(\mu(x), \mu(y) \geq T(t,t) = t$ and $\mu(\alpha x) \geq \mu(x) \geq t$. Therefore $x + y, \alpha x \in L(\mu, t)$. Also if $x \in L(\mu, t)$ and $y \in L$, then from $\mu([x,y]) \geq \mu(x) \geq t$ we have that $[x,y] \in L(\mu, t)$. This completes the proof.

Definition 3.5. Let $f : L_1 \to L_2$ be an epimorphism of Lie algebras. Let $\mu : L_1 \to [0,1]$ and $\nu : L_2 \to [0,1]$ be two fuzzy sets of L_1 and L_2 respectively. For all $x \in L_1$ and $y \in L_2$ define

$$f(\mu)(y) = \begin{cases} \sup\{\mu(x) \mid x \in L_1, f(x) = y\} & \text{if } f^{-1}(y) \neq \emptyset \\ 0 & \text{if } f^{-1}(y) = \emptyset \end{cases}$$

and $f^{-1}(\nu)(x) = \nu(f(x))$.

Proposition 3.6. Let $f : L_1 \to L_2$ be an epimorphism of Lie algebras. If μ is a fuzzy Lie ideal of L_1 with respect to a t-norm T, then $f(\mu)$ is a fuzzy Lie ideal of L_2 with respect to a t-norm T.

Proof. Let $x_1, x_2 \in L_1$ and $y_1, y_2 \in L_2$. If $y_1 = f(x_1)$ and $y_2 = f(x_2)$, then
(1)

$$\begin{aligned} f(\mu)(y_1 + y_2) &= \sup\{\mu(x_1 + x_2) \mid y_1 = f(x_1), y_2 = f(x_2)\} \\ &\geq \sup\{T(\mu(x_1), \mu(x_2)) \mid y_1 = f(x_1), y_2 = f(x_2)\} \\ &= T(\sup\{\mu(x_1) \mid y_1 = f(x_1)\}, \sup\{\mu(x_2) \mid y_2 = f(x_2)\}) \\ &= T(f(\mu)(y_1), f(\mu)(y_2)). \end{aligned}$$

(2) $f(\mu)(\alpha y_1) = \sup\{\mu(\alpha x_1) \mid \alpha y_1 = f(\alpha x_1) = \alpha f(x_1)\} \geq \sup\{\mu(x_1) \mid x_1 = f(y_1)\} = f(\mu)(y_1)$.
(3) $f(\mu)([y_1, y_2]) = \sup\{\mu([x_1, x_2]) \mid y_1 = f(x_1), y_2 = f(x_2)\} \geq \sup\{\mu(x_1) \mid y_1 = f(x_1)\} = f(\mu)(y_1)$.

Proposition 3.7. Let $f : L_1 \to L_2$ be an epimorphism of Lie algebras. If ν is a fuzzy Lie ideal of L_2 with respect to a t-norm T, then $f^{-1}(\nu)$ is a fuzzy Lie ideal of L_1 with respect to a t-norm T.

Proof. Let $x, y \in L_1$ and $\alpha \in F$. Then

$$\begin{aligned} f^{-1}(\nu)(x+y) &= \nu(f(x+y)) = \nu(f(x)+f(y)) \\ &\geq T(\nu(f(x)), \nu(f(y))) = T(f^{-1}(\nu)(x), f^{-1}(\nu)(y)), \end{aligned}$$

$f^{-1}(\nu)(\alpha x) = \nu(f(\alpha x)) = \nu(\alpha f(x)) \geq \nu(f(x)) = f^{-1}(\nu)(x)$, and
$f^{-1}(\nu)([x,y]) = \nu(f([x,y]) = \nu([f(x), f(y)]) \geq \nu(f(x)) = f^{-1}(\nu)(x)$.
Thus $f^{-1}(\nu)$ is a fuzzy Lie ideal of L_1 with respect to a t-norm T.

Definition 3.8. let μ and ν be fuzzy Lie ideals of a Lie algebra L with respect to a t-norm T. Define the intersection of μ and ν the function $\mu \cap \nu : L \to [0,1]$ such that $(\mu \cap \nu)(x) = T(\mu(x), \nu(x))$ for all $x \in L$.

Proposition 3.9. let μ and ν be two fuzzy Lie ideals of a Lie algebra L with respect to a t-norm T such that T be idempotent. Then $\mu \cap \nu$ be a fuzzy Lie ideal in a Lie algebra L with respect to a t-norm T.

Proof. Let $x, y \in L$ and $\alpha \in F$. Then
(1)

$$(\mu \cap \nu)(x+y) = T(\mu(x+y), \nu(x+y)) \geq T(T(\mu(x), \mu(y)), T(\nu(x), \nu(y)))$$

$$= T(T(\mu(x), \nu(x)), T(\mu(y), \nu(y))) = T((\mu \cap \nu)(x), (\mu \cap \nu)(y)).$$

(2) $(\mu \cap \nu)(\alpha x) = T(\mu(\alpha x), \nu(\alpha x)) \geq T(\mu(x), \nu(x)) = (\mu \cap \nu)(x)$.
(3) $(\mu \cap \nu)([x,y]) = T(\mu([x,y]), \nu([x,y])) \geq T(\mu(x), \nu(x)) = (\mu \cap \nu)(x)$.
Hence $\mu \cap \nu$ be a fuzzy Lie ideal in a Lie algebra L with respect to a t-norm T.

Next we will introduce the concept of quotient fuzzy Lie ideal.

Definition 3.10. Let L be a Lie algebra, $\mu : L \to [0,1]$ and I be an ideal of L. Define $\mu_{L/I} : L/I \to [0,1]$ by

$$\mu_{L/I}(x+I) = \begin{cases} T(\mu(x), \mu(i)) & \text{if } x \neq i \\ 1 & \text{if } x = i \end{cases}$$

for all $x \in L$ and $i \in I$.

Proposition 3.11. Let μ be a fuzzy Lie ideal of L with respect to a t-norm T. If T be idempotent, then $\mu_{L/I}$ will be a fuzzy Lie ideal of L/I with respect to a t-norm T.

Proof. Let $x+I, y+I \in L/I$ and $i \in I$ such that $x \neq i \neq y$.
(1)

$$\begin{aligned}\mu_{L/I}((x+I)+(y+I)) &= \mu_{L/I}((x+y)+I) = T(\mu(x+y), \mu(i)) \\ &\geq T(T(\mu(x), \mu(y)), \mu(i)) = T(T(\mu(x), \mu(y)), T(\mu(i), \mu(i))) \\ &= T(T(\mu(x), \mu(i)), T(\mu(y), \mu(i))) = T(\mu_{L/I}(x+I), \mu_{L/I}(y+I)).\end{aligned}$$

(2) $\mu_{L/I}(\alpha(x+I)) = \mu_{L/I}(\alpha x+I) = T(\mu(\alpha x), \mu(i)) \geq T(\mu(x), \mu(i)) = \mu_{L/I}(x+I)$.
(3) $\mu_{L/I}([x,y]+I) = T(\mu([x,y]), \mu(i)) \geq T(\mu(x), \mu(i)) = \mu_{L/I}(x+I)$.

4 Anti Fuzzy Lie Subalgebra with Respect to a t-conorm

Definition 4.1. A fuzzy set $\mu : L \to [0,1]$ is called an anti fuzzy Lie subalgebra of L with respect to a t-conorm C if
(1) $\mu(x+y) \leq C(\mu(x), \mu(y))$,
(2) $\mu(\alpha x) \leq \mu(x)$,
(3) $\mu([x,y]) \leq T(\mu(x), \mu(y))$
hold for all $x, y \in L$ and $\alpha \in F$.
A fuzzy subset $\mu : L \to [0,1]$ satisfying (1), (2) and
(4) $\mu([x,y]) \leq \mu(x)$
is called an anti fuzzy Lie ideal of L with respect to a t-conorm C.

Proposition 4.2. Let μ be an anti fuzzy Lie ideal in a Lie algebra L with respect to a t-conorm C such that C be idempotent. Then for all $t \in [0,1]$ the set $L(\mu, t) = \{x \in L \mid \mu(x) \leq t\}$ is a Lie ideal of L.

Proof. Let $x, y \in L(\mu, t)$ and $\alpha \in F$. Then $\mu(x+y) \leq C(\mu(x), \mu(y) \leq C(t,t) = t$ and $\mu(\alpha x) \leq \mu(x) = t$. Therefore $x+y, \alpha x \in L(\mu, t)$. Also if $x \in L(\mu, t)$ and $y \in L$, then from $\mu([x,y]) \leq \mu(x) \leq t$ we have that $[x,y] \in L(\mu, t)$. Thus $L(\mu, t)$ will be a Lie ideal of L.

Definition 4.3. let μ and ν be anti fuzzy Lie ideals of a Lie algebra L with respect to a t-conorm C. Define the union of μ and ν the function $\mu \cup \nu : L \to [0,1]$ such that $(\mu \cup \nu)(x) = C(\mu(x), \nu(x))$ for all $x \in L$.

Proposition 4.4. let μ and ν be two anti fuzzy Lie ideals of a Lie algebra L with respect to a t-conorm C such that C be idempotent. Then $\mu \cup \nu$ be an anti fuzzy Lie ideal in a Lie algebra L with respect to a t-conorm C.

Proof. Let $x, y \in L$ and $\alpha \in F$. Then
(1)

$$(\mu \cup \nu)(x+y) = C(\mu(x+y), \nu(x+y)) \leq C(C(\mu(x), \mu(y)), C(\nu(x), \nu(y)))$$

$$= C(C(\mu(x), \nu(x)), C(\mu(y), \nu(y))) = C((\mu \cup \nu)(x), (\mu \cup \nu)(y)).$$

(2) $(\mu \cup \nu)(\alpha x) = C(\mu(\alpha x), \nu(\alpha x)) \leq C(\mu(x), \nu(x)) = (\mu \cup \nu)(x)$.
(3) $(\mu \cup \nu)([x, y]) = C(\mu([x, y]), \nu([x, y])) \leq C(\mu(x), \nu(x)) = (\mu \cup \nu)(x)$.
Hence $\mu \cup \nu$ be an anti fuzzy Lie ideal in a Lie algebra L with respect to a t-conorm C.

Proposition 4.5. Let $f : L_1 \to L_2$ be an epimorphism of Lie algebras. If ν be an anti fuzzy Lie ideal of L_2 with respect to a t-conorm C, then $f^{-1}(\nu)$ will be an anti fuzzy Lie ideal of L_1 with respect to a t-conorm C.

Proof. Let $x, y \in L_1$ and $\alpha \in F$. Then

$$f^{-1}(\nu)(x + y) = \nu(f(x + y)) = \nu(f(x) + f(y))$$
$$\leq C(\nu(f(x)), \nu(f(y))) = C(f^{-1}(\nu)(x), f^{-1}(\nu)(y)),$$

$f^{-1}(\nu)(\alpha x) = \nu(f(\alpha x)) = \nu(\alpha f(x)) \leq \nu(f(x)) = f^{-1}(\nu)(x)$, and
$f^{-1}(\nu)([x, y]) = \nu(f([x, y]) = \nu([f(x), f(y)]) \leq \nu(f(x)) = f^{-1}(\nu)(x)$.
Therefore $f^{-1}(\nu)$ is an anti fuzzy Lie ideal of L_1 with respect to a t-conorm C.

Proposition 4.6. Let L be a Lie algebra and $\mu : L \to [0, 1]$. Then μ be a fuzzy Lie ideal of L with respect to a t-norm T if and only if μ^c be an anti fuzzy Lie ideal of L with respect to a t-conorm C.

Proof. Let μ be a fuzzy Lie ideal of L with respect to a t-norm T and $x, y \in L$ and $\alpha \in F$.
(1) From $\mu(x + y) \geq T(\mu(x), \mu(y))$ we have

$$1 - \mu^c(x + y) \geq T(1 - \mu^c(x), 1 - \mu^c(y)),$$

which implies that

$$\mu^c(x + y) \leq 1 - T(1 - \mu^c(x), 1 - \mu^c(y))$$
$$= C(\mu^c(x), \mu^c(y)).$$

(2) $\mu^c(\alpha x) = 1 - \mu(\alpha x) \leq 1 - \mu(x) = \mu^c(x)$.
(3) $\mu^c([x, y]) = 1 - \mu([x, y]) \leq 1 - \mu(x) = \mu^c(x)$.
Hence μ^c will be an anti fuzzy Lie ideal of L with respect to a t-conorm C.
Converse also can be proved similarly.

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NATURAL TRANSFORM AND SPECIAL FUNCTIONS

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Abstaract − The branch of integral transform attract many researcher in this field and hence various types of integral transforms are introduced. Natural transform is one of the newly defined transform which has wide range of applications in science and engineering field. In this paper we derived the Natural transform of some special functions.

Keywords − *Bessel's function, Hermite Polynomial, Hypergeometric function, Legendre Polynomials, Leguerre Polynomial,Natural Transform.*

1 Introduction

The Natural transform was established by Khan and Khan[1] as N - transform who studied its properties and application as unsteady fluid flow problem over a plane wall.Later on Belgacem [2, 3] defined the inverse Natural transform and studied some properties and applications of Natural transforms.In the literature survey we can see the further applications of Natural transform.[4, 5, 6, 7] The specialty of Natural transform is that it can converges to Laplace transform and Sumudo transform [8] just by changing the parameter.Natural transform is the theoretical dual of Laplace transform.We can derive Laplace, Sumudu, Fourier and Mellin transform from Natural transform.[9] Natural transform plays as a source for other transform and hence can be used to solve many complicated problems in engineering, fluid mechanics and other scientific discipline like Physics, Chemistry and Dynamics etc.

1.1 Preliminary Definition of Natural Transform

The Natural transform of the function $f(t) \in \Re^2$ is given by the following integral equation [3]

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$$\mathbb{N}[f(t)] = R(s,u) = \int_0^\infty e^{-st} f(ut)dt \tag{1}$$

where $Re(s) > 0$, u $\in (\tau_1,\ \tau_2)$ provided the function $f(t) \in \Re^2$ is defined in the set

A=$[f(t)/\exists$ M,$\tau_1,\ \tau_2 > 0$,|f(t)| $<$ M $e^{\frac{|t|}{\tau_j}}$, if t $\in (-1)^j \times [0,\infty)$]

The inverse Natural transform related with Bromwich contour integral[2, 3] is defined by

$$\mathbb{N}^{-1}[R(s,u)] = f(t) = \lim_{T\to\infty} \frac{1}{2\Pi i} \int_{\gamma-iT}^{\gamma+iT} e^{\frac{st}{u}} R(s,u)ds \tag{2}$$

1.2 Some Standard Result of Natural Transform

In this section we assume that all the considered functions are such that their Natural tranform exists.[1],[3]

1. $\mathbb{N}[1] = \frac{1}{s}$
2. $\mathbb{N}[t] = \frac{u}{s^2}$
3. $\mathbb{N}[t^n]=\frac{u^n}{s^{n+1}} n!$
4. $\mathbb{N}[e^{at}] = \frac{1}{s-au}$
5. $\mathbb{N}[\frac{sin(at)}{a}] = \frac{u}{s^2+s^2u^2}$
6. $\mathbb{N}[cos(at)] = \frac{s}{s^2+s^2u^2}$
7. $\mathbb{N}[\frac{t^{n-1}e^{at}}{(n-1)!}] = \frac{u^{n-1}}{(s-au)^2}$
8. $\mathbb{N}[f^{(n)}(t)] = \frac{s^n}{u^n}.R(s,u) - \sum_{n=0}^{\infty} \frac{s^{n-(k+1)}}{u^{n-k}}.u^{(k)}(0)$
   where $f^{(n)}(t) = \frac{d^n f}{dt^n}$
9. The Convolution Theorem

   If F(s,u) and G(s,u) are the Natural transforms of respective functions $f(t)$ and $g(t)$ both defined in set A then ,
   $\mathbb{N}[(f * g)]=u.F(s,u)G(s,u)$

1.3 Pochhamber Symbol

The pochhamber symbol denoted by $(\alpha)_n$ is defined by[10] the equation

$$(\alpha)_n = \alpha(\alpha+1)(\alpha+2)....(\alpha+n+1)$$
$$= \prod_{m=1}^{n} (\alpha+m+1) \qquad for \quad n \geq 1$$

In particular $(\alpha)_0 = 1 \quad for \quad \alpha \neq 0 \, , \qquad (1)_n = n!$

1.4 Some Standard Results

1. if n is positive integer, then

$$\frac{\Gamma n}{\Gamma n+1} = (\alpha)_n$$

where α is neither zero nor a negative integer.

2. If α is not an integer,

$$\frac{\Gamma 1-\alpha-n}{\Gamma 1-\alpha} = \frac{(-1)^n}{(\alpha)_n}$$

3.

$$(1-Z)^{-\alpha} = \sum_{n=0}^{\infty} \frac{(\alpha)_n Z^n}{n!}$$

4.

$$(\alpha)_{n-k} = \frac{(\alpha)_n(-1)^k}{(1-\alpha-n)_k} \quad 0 \leq k \leq n$$

5. If $\alpha = 1$, then

$$(-1)_{n-k} = (n-k)! = \frac{n!(-1)^k}{(-n)_k} \quad 0 \leq k \leq n$$

6.

$$(\alpha)_{2n} = 2^{2n}(\frac{\alpha}{2})_n(\frac{\alpha+1}{2})_n$$

7. The function $f(a,b,c;Z)$ is written as $F\left[\begin{array}{cc} a,b & ; \\ c & ; \end{array} \mid Z\right]$ and is defined as

$$f(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n z^n}{(c)_n n!}$$

8. The Hypergeometric function ${}_pF_q$ is defined by

$${}_pF_q\left[\begin{array}{cc} a_1,a_2,.......,a_p & : \\ a_1,a_2,.......,a_p & : \end{array} \mid Z\right] = \sum_{n=0}^{\infty} \frac{\prod_{k=1}^{p}(a_k)_n z^n}{\prod_{m=1}^{q}(b_m)_n n!}$$

2 Well known special functions

1. The Bessel's Function is defined by

$$J_n(t) = \sum_{n=0}^{\infty} \frac{(-1)^k t^{2k+1}}{2^{2k+n} k! \Gamma 1 + n - k} \tag{3}$$

2 The Legendre polynomial is defined by

$$P_n(t) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(-1)^k (\frac{1}{2})_{n-k} (2t)^{n-2k}}{(n-2k)!k!} \tag{4}$$

3 The Hermite polynomial is defined by

$$H_n(t) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(-1)^k n! (2t)^{n-2k}}{(n-2k)!k!} \tag{5}$$

4 The Leguerre polynomial is defined by

$$L_n(\alpha)_t = \sum_{k=0}^{\infty} \frac{(-1)^k (1+\alpha)_n t^k}{(n-k)!k!(1+\alpha)_k} \tag{6}$$

3 Main Result

3.1 The Natural transform of Hypergeometric function

$$\begin{aligned}
\mathbb{N}\{{}_pF_q \left[\begin{array}{cc} a_1, a_2, ..., a_p & ; \\ a_1, a_2, ..., a_q & ; \end{array} \mid t \right] \} &= \int_0^{\infty} e^{-st} \sum_{n=0}^{\infty} \frac{\prod_{k=1}^{p} (a_k)_n (ut)^n}{\prod_{m=1}^{q} (b_m)_n n!} dt \\
&= \sum_{n=0}^{\infty} \frac{\prod_{k=1}^{p} (a_k)_n}{\prod_{m=1}^{q} (b_m)_n n!} \int_0^{\infty} e^{-st} (ut)^n dt \\
&= \sum_{n=0}^{\infty} \frac{\prod_{k=1}^{p} (a_k)_n}{\prod_{m=1}^{q} (b_m)_n n!} \mathbb{N}\{t^n\} \\
&= \sum_{n=0}^{\infty} \frac{\prod_{k=1}^{p} (a_k)_n}{\prod_{m=1}^{q} (b_m)_n n!} \frac{u^n}{s^{n+1}} n! \\
\mathbb{N}\{{}_pF_q \left[\begin{array}{cc} a_1, a_2, ..., a_p & ; \\ a_1, a_2, ..., a_q & ; \end{array} \mid t \right] \} &= \frac{n!}{s} {}_pF_q \left[\begin{array}{cc} a_1, a_2, ..., a_p & ; \\ a_1, a_2, ..., a_q & ; \end{array} \mid \frac{u}{s} \right]
\end{aligned}$$

In particular,

$$
\begin{aligned}
\mathbb{N}\{{}_2F_1\left[\begin{array}{cc} a,b & ; \\ 1 & ; \end{array} \mid t\right]\} &= \mathbb{N}\{\sum_{n=0}^{\infty}\frac{(a)_n(b)_n t^n}{(1)_n n!}\} \\
&= \int_0^{\infty} e^{-st}\sum_{n=0}^{\infty}\frac{(a)_n(b)_n(ut)^n}{(1)_n n!}dt \\
&= \sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(1)_n n!}\int_0^{\infty} e^{-st}(ut)^n dt \\
&= \sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(1)_n n!}\mathbb{N}\{t^n\} \\
&= \sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(1)_n n!}\frac{u^n}{s^{n+1}}n! \\
&= {}_2F_0\left[\begin{array}{cc} a,b & ; \\ - & ; \end{array} \mid \frac{u}{s}\right]\frac{1}{s}
\end{aligned}
$$

3.2 The Natural transform of Bessel's function

$$
\begin{aligned}
\mathbb{N}\{J_n(t)\} &= \mathbb{N}\{\sum_{n=0}^{\infty}\frac{(-1)^k t^{2k+1}}{2^{2k+n}k!\Gamma 1+n-k}\} \\
&= \int_0^{\infty} e^{-st}\sum_{n=0}^{\infty}\frac{(-1)^k(ut)^{2k+n}}{2^{2k+n}k!\Gamma 1+n-k}dt \\
&= \sum_{n=0}^{\infty}\frac{(-1)^k}{2^{2k+n}k!\Gamma 1+n-k}\int_0^{\infty} e^{-st}(ut)^{2k+n}dt \\
&= \sum_{n=0}^{\infty}\frac{(-1)^k}{2^{2k+n}k!\Gamma 1+n-k}\mathbb{N}\{t^{2k+n}\} \\
&= \sum_{n=0}^{\infty}\frac{(-1)^k}{2^{2k+n}k!\Gamma 1+n-k}\frac{u^{2k+n}}{s^{2k+n+1}}(2k+n)! \\
&= \{\sum_{n=0}^{\infty}\frac{(-1)^k\Gamma n+1}{2^{2k}k!\Gamma 1+n-k\Gamma n+1}\frac{u^{2k}}{s^{2k}}(2k+n)!\}\frac{u^n}{2^n s^{n+1}} \\
&= \{\sum_{n=0}^{\infty}\frac{(-1)^k(1+n)_{2k}}{2^{2k}k!(1+n)_k}\frac{u^{2k}}{s^{2k}}\}\frac{u^n}{2^n s^{n+1}} \\
&= \{\sum_{n=0}^{\infty}\frac{(-1)^k(\frac{1+n}{2})_k(1+\frac{n}{2})_k}{k!(1+n)_k}\frac{u^{2k}}{s^{2k}}\}\frac{u^n}{2^n s^{n+1}} \\
\therefore \mathbb{N}\{J_n(t)\} &= {}_2F_1\left[\begin{array}{cc} \frac{n+1}{2},1+\frac{n}{2} & ; \\ 1+n & ; \end{array} \mid -\frac{u^2}{s^2}\right]\frac{u^n}{2^n s^{n+1}}
\end{aligned}
$$

3.3 The Natural transform of Legendre Polynomial

$$
\begin{aligned}
\mathbb{N}\{P_n(t)\} &= \mathbb{N}\{\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(-1)^k(\frac{1}{2})_{n-k}(2t)^{n-2k}}{(n-2k)!k!}\} \\
&= \int_0^\infty e^{-st} \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(-1)^k(\frac{1}{2})_{n-k}(2ut)^{n-2k}}{(n-2k)!k!} dt \\
&= 2^n \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(-1)^k(\frac{1}{2})_n(-n)_{2k}}{k!(1-1/2-n)_k(-1)^{2k}n!2^{2k}} \int_0^\infty e^{-st}(ut)^{n-2k} dt \\
&= 2^n \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(-1)^k(\frac{1}{2})_n(-n)_{2k}}{k!(1-1/2-n)_k(-1)^{2k}n!2^{2k}} \mathbb{N}\{t^{n-2k}\} \\
&= 2^n \frac{(\frac{1}{2})_n}{n!} \{\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(-n)_{2k}}{(1/2-n)_k k! 2^{2k}} \frac{u^{n-2k}}{s^{n-2k+1}} \Gamma n - 2k + 1\} \\
&= 2^n \frac{(\frac{1}{2})_n u^n}{s^{n+1} n!} \{\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(-n)_{2k}}{(1/2-n)_k k! 2^{2k}} \frac{s^{2k}}{u^{2k}} \Gamma n - 2k + 1\} \\
&= 2^n \Gamma n + 1 \frac{(\frac{1}{2})_n u^n}{s^{n+1} n!} \{\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(-\frac{n}{2})_k(\frac{-n+1}{2})_k 2^{2k}}{(1/2-n)_k 2^{2k} k!} \frac{s^{2k}}{u^{2k}} \frac{\Gamma 1 - (-n) - 2k}{\Gamma 1 - (-n)}\} \\
&= 2^n \frac{(\frac{1}{2})_n u^n}{s^{n+1}} \{\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(-\frac{n}{2})_k(\frac{-n+1}{2})_k(-1)^{2k}}{(1/2-n)_k(-n)_{2k} k!} \frac{s^{2k}}{u^{2k}}\} \\
&= 2^n \frac{(\frac{1}{2})_n u^n}{s^{n+1}} \{\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(-\frac{n}{2})_k(\frac{-n+1}{2})_k(-1)^{2k}}{(1/2-n)_k(-\frac{n}{2})_k(\frac{-n+1}{2})_k k! 2^{2k}} \frac{s^{2k}}{u^{2k}}\} \\
&= 2^n \frac{(\frac{1}{2})_n u^n}{s^{n+1}} \{\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(-1)^{2k}}{(1/2-n)_k k!} \left(\frac{s^2}{4u^2}\right)^k\} \\
\therefore \mathbb{N}\{P_n(t)\} &= {}_0F_1 \left[\begin{matrix} - & ; \\ (\frac{1}{2}-n) & ; \end{matrix} \mid \frac{s^2}{2u^2} \right] 2^n \frac{(\frac{1}{2})_n u^n}{s^{n+1}}
\end{aligned}
$$

3.4 The Natural transform of Hermite Polynomial

$$
\begin{aligned}
\mathbb{N}\{H_n(t)\} &= \mathbb{N}\{\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(-1)^k n!(2t)^{n-2k}}{(n-2k)!k!}\} \\
&= \int_0^\infty e^{-st} \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(-1)^k n!(2ut)^{n-2k}}{(n-2k)!k!} dt \\
&= \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(-1)^k n!(2)^n}{(n-2k)!k!2^{2k}} \int_0^\infty e^{-st}(ut)^{n-2k} dt \\
&= \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(-1)^k n!(2)^n}{(n-2k)!k!2^{2k}} \mathbb{N}\{t^{n-2k}\} \\
&= \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(-1)^k n!(2)^n}{(n-2k)!k!2^{2k}} \frac{u^{n-2k}}{s^{n-2k+1}} \Gamma n - 2k + 1 \\
&= 2^n \frac{u^n}{s^{n+1}} \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(-1)^k n!(-n)_{2k}}{(-1)^{2k} k! 2^{2k}} \frac{s^{2k}}{u^{2k}} \Gamma n - 2k + 1 \\
&= 2^n \frac{u^n}{s^{n+1}} \Gamma n + 1 \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(-1)^k (-n)_{2k}}{(-1)^{2k} k!} \left(\frac{s^2}{4u^2} \right)^k \frac{\Gamma n - 2k + 1}{\Gamma n + 1} \\
&= 2^n \frac{u^n}{s^{n+1}} n! \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(-1)^k}{k!} \left(\frac{s^2}{4u^2} \right)^k \\
\therefore \mathbb{N}\{H_n(t)\} &= {}_0F_0 \left[\begin{matrix} - & ; \\ - & ; \end{matrix} \mid -\frac{s^2}{4u^2} \right] 2^n \frac{u^n}{s^{n+1}} n!
\end{aligned}
$$

3.5 The Natural transform of Leguerre Polynomial

$$\begin{aligned}
\mathbb{N}\{L_n(\alpha)_t\} &= \mathbb{N}\{\sum_{k=0}^{\infty}\frac{(-1)^k(1+\alpha)_n t^k}{(n-k)!k!(1+\alpha)_k}\} \\
&= \int_0^{\infty} e^{-st}\sum_{k=0}^{\infty}\frac{(-1)^k(1+\alpha)_n t^k}{(n-k)!k!(1+\alpha)_k}dt \\
&= (1+\alpha)_n\sum_{k=0}^{\infty}\frac{(-1)^k}{(n-k)!k!(1+\alpha)_k}\int_0^{\infty} e^{-st}(ut)^k dt \\
&= (1+\alpha)_n\sum_{k=0}^{\infty}\frac{(-1)^k}{(n-k)!k!(1+\alpha)_k}\mathbb{N}\{t^k\} \\
&= (1+\alpha)_n\sum_{k=0}^{\infty}\frac{(-1)^k}{(n-k)!k!(1+\alpha)_k}\frac{u^k}{s^{k+1}}\Gamma k+1 \\
&= (1+\alpha)_n\sum_{k=0}^{\infty}\frac{(-1)^k(-n)_k}{(-1)^k k!(1+\alpha)_k n!}\frac{u^k}{s^{k+1}}\Gamma k+1 \\
&= \{\sum_{k=0}^{\infty}\frac{(1)_k(-n)_k}{k!(1+\alpha)_k}\left(\frac{u^k}{s^k}\right)^k\}\frac{(1+\alpha)_n}{sn!} \\
\therefore \mathbb{N}\{L_n(\alpha)_t\} &= {}_2F_1\left[\begin{array}{lll}(-n),1 & ; & \\ (1+\alpha) & ; & \end{array} \mid \frac{u}{s}\right]\frac{(1+\alpha)_n}{sn!}
\end{aligned}$$

Note that throughout the discussion, we assume that the validity of integration term by term in the summation.

4 Conclusion

In this paper we have find the Natural transform of some special well known functions in terms of Hypergeometric function.These special functions are useful in solving differential equations.

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IDENTITIES FOR MULTIPLICATIVE COUPLED FIBONACCI SEQUENCES OF R^{TH} ORDER

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Abstaract − Many author studied coupled Fibonacci sequences and multiplicative coupled Fibonacci sequences of lower order two, three and four etc. In this paper we defined multiplicative coupled Fibonacci Sequences of r^{th} order under 2^r different schemes. Some new identities for these sequences are established under one specific scheme.

Keywords − *Fibonacci Sequence, Coupled Fibonacci Sequence, Recurrence Relation.*

1 Introduction

The Fibonacci sequence is a source of intresting identities. Many identities have been documented in [14], [15], [16], [17], [21]. A similar interpretation exists for k Fibonacci and k Lucas numbers, many of these identities have been documented in the work of Falcon and Plaza [3], [6], [7], [11], [12], [13]. Many authors defined coupled and multiplicative coupled Fibonacci sequences by varying initial conditions and recurrence relation. Properties of these sequences are documented in[1], [2], [9], [4], [5]. Many authors defined coupled and multiplicative coupled Fibonacci sequences by varying initial conditions and recurrence relation. Properties of these sequences are documented in[1], [2], [9], [4], [5]. In this paper we defined multiplicative coupled Fibonacci sequences of r^{th} order by varying recurrence relation and some identities for these mentioned sequences are also obtained under $2^{r-1^{th}}$ scheme.
Coupled Fibonacci sequences involve two sequences of integers in which the elements of one sequence are part of the generalization of the other and vice versa. K. T. Atanassov [1] was first introduced coupled Fibonacci sequences of second order

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in additive form and also discussed many curious properties and new direction of generalization of Fibonacci sequence in his series of papers on coupled Fibonacci sequences. He defined and studied about four different ways to generate coupled sequences and called them coupled Fibonacci sequences (or 2-F sequences). The multiplicative Fibonacci Sequences studied by singh-shikwal [**?**]. K. T. Atanassov [2] notifies four different schemes in multiplicative form for coupled Fibonacci sequences. The analogue of the standard Fibonacci sequence in this form is $x_0 = a$, $x_1 = b$, $x_{n+2} = x_{n+1} \cdot x_n \quad (n \geq 0)$.

Attanasov [1] introduced a new view of generalized Fibonacci sequences by taking a pair of sequences $\{X_i\}_{i=0}^{i=\infty}$ and $\{Y_i\}_{i=0}^{i=\infty}$ and which can be generated by famous Fibonacci formula and gave various identities involving Fibonacci sequence called the coupled Fibonacci sequences.

In this paper we defined multiplicative coupled Fibonacci sequences of r^{th} order by varying recurrence relation and some identities for these mentioned sequences are also obtained under $2^{r-1^{th}}$ scheme.

2 Preliminary and Notations

Definition 2.1. Multiplicative Coupled Fibonacci sequences of third order: Let $\{X_i\}_{i=0}^{i=\infty}$ and $\{Y_i\}_{i=0}^{i=\infty}$ be two infinite sequences and six arbitrary real numbers x_0, x_1, x_2, y_0, y_1, y_2 are given. The Multiplicative coupled Fibonacci sequences of 3^{rd} order are generated by the following eight different ways:

First scheme

$$X_{n+3} = Y_{n+2} \cdot Y_{n+1} \cdot Y_n, \quad n \geq 0$$
$$Y_{n+3} = X_{n+2} \cdot X_{n+1} \cdot X_n, \quad n \geq 0$$

Second scheme

$$X_{n+3} = X_{n+2} \cdot X_{n+1} \cdot X_n, \quad n \geq 0$$
$$Y_{n+3} = Y_{n+2} \cdot Y_{n+1} \cdot Y_n, \quad n \geq 0$$

Third scheme

$$X_{n+3} = Y_{n+2} \cdot Y_{n+1} \cdot X_n, \quad n \geq 0$$
$$Y_{n+3} = X_{n+2} \cdot X_{n+1} \cdot Y_n, \quad n \geq 0$$

Fourth scheme

$$X_{n+3} = Y_{n+2} \cdot X_{n+1} \cdot Y_n, \quad n \geq 0$$
$$Y_{n+3} = X_{n+2} \cdot Y_{n+1} \cdot X_n, \quad n \geq 0$$

Fifth scheme

$$X_{n+3} = Y_{n+2} \cdot X_{n+1} \cdot X_n, \quad n \geq 0$$
$$Y_{n+3} = X_{n+2} \cdot Y_{n+1} \cdot Y_n, \quad n \geq 0$$

Sixth scheme

$$X_{n+3} = X_{n+2} \cdot X_{n+1} \cdot Y_n, \quad n \geq 0$$
$$Y_{n+3} = Y_{n+2} \cdot Y_{n+1} \cdot X_n, \quad n \geq 0$$

Seventh scheme

$$X_{n+3} = X_{n+2} \cdot Y_{n+1} \cdot Y_n, \quad n \geq 0$$
$$Y_{n+3} = Y_{n+2} \cdot X_{n+1} \cdot X_n, \quad n \geq 0$$

Eighth scheme

$$X_{n+3} = X_{n+2} \cdot Y_{n+1} \cdot X_n, \quad n \geq 0$$
$$Y_{n+3} = Y_{n+2} \cdot X_{n+1} \cdot Y_n, \quad n \geq 0$$

Definition 2.2. Multiplicative Coupled Fibonacci sequences of r^{th} order: Let $\{X_i\}_{i=0}^{i=\infty}$ and $\{Y_i\}_{i=0}^{i=\infty}$ be two infinite sequences and $2r$ arbitrary real numbers x_0, x_1, x_2, x_3, ..., x_{r-1} and y_0, y_1, y_2, y_3, ..., y_{r-1} are given. The Multiplicative coupled Fibonacci sequences of r^{th} order are generated by the following 2^r different ways:

First scheme

$$X_{n+r} = Y_{n+r-1} \cdot Y_{n+r-2} \cdot Y_{n+r-3} \cdots Y_n, n \geq 0$$
$$Y_{n+r} = X_{n+r-1} \cdot X_{n+r-2} \cdot X_{n+r-3} \cdots X_n, n \geq 0$$

Second scheme

$$X_{n+r} = X_{n+r-1} \cdot Y_{n+r-2} \cdot Y_{n+r-3} \cdots Y_n, n \geq 0$$
$$Y_{n+r} = Y_{n+r-1} \cdot X_{n+r-2} \cdot X_{n+r-3} \cdots X_n, n \geq 0$$

$$\vdots$$

$(2^{r-1})^{th}$ **scheme**

(a) **If r is even**,

$$X_{n+r} = X_{n+r-1} \cdot Y_{n+r-2} \cdot X_{n+r-3} \cdots Y_n, n \geq 0$$
$$Y_{n+r} = Y_{n+r-1} \cdot X_{n+r-2} \cdot Y_{n+r-3} \cdots X_n, n \geq 0$$

(b) **If r is odd**,

$$X_{n+r} = X_{n+r-1} \cdot Y_{n+r-2} \cdot X_{n+r-3} \cdots X_n, n \geq 0$$
$$Y_{n+r} = Y_{n+r-1} \cdot X_{n+r-2} \cdot Y_{n+r-3} \cdots Y_n, n \geq 0$$

$$\vdots$$

$(2^r)^{th}$ **scheme**

$$X_{n+r} = X_{n+r-1} \cdot X_{n+r-2} \cdot X_{n+r-3} \cdots X_n, n \geq 0$$
$$Y_{n+r} = Y_{n+r-1} \cdot Y_{n+r-2} \cdot Y_{n+r-3} \cdots Y_n, n \geq 0$$

n	Y_{n+r}	X_{n+r}
0	$y_0y_1y_2y_3\ldots\ldots\ y_{r-1}$	$x_0x_1x_2x_3\ldots\ldots\ x_{r-1}$
1	$x_0x_1x_2x_3\ldots\ldots\ x_{r-1}y_1y_2y_3\ldots\ldots\ y_{r-1}$	$x_1x_2x_3\ldots\ldots\ x_{r-1}y_0y_1y_2y_3\ldots\ldots\ y_{r-1}$
2	$x_0x_1^2x_2^2x_3^2\ldots\ldots\ x_{r-1}^2y_0y_1y_2^2y_3^2\ldots\ldots\ y_{r-1}^2$	$x_0x_1x_2^2x_3^2\ldots\ldots\ x_{r-1}^2y_0y_1^2y_2^2y_3^2\ldots\ldots\ y_{r-1}^2$
3	$x_0^2x_1^3x_2^4x_3^4\ldots\ldots\ x_{r-1}^4y_0^2y_1^3y_2^3y_3^4\ldots\ldots\ y_{r-1}^4$	$x_0^2x_1^3x_2^3x_3^4\ldots\ldots\ x_{r-1}^4y_0^2y_1^3y_2^4y_3^4\ldots\ldots\ y_{r-1}^4$
4	$x_0^4x_1^5x_2^7x_3^8\ldots\ldots\ x_{r-1}^8y_0^4y_1^6y_2^7y_3^7y_4^8\ldots\ldots\ y_{r-1}^8$	$x_0^4x_1^5x_2^7x_3^7x_4^8\ldots\ldots\ x_{r-1}^8y_0^4y_1^6y_2^7y_3^8\ldots\ldots\ y_{r-1}^8$

Table 1. **First few terms of these sequences under $(2^{r-1})^{th}$(a) scheme**

n	X_{n+r}	Y_{n+r}
0	$y_0y_1y_2y_3\ldots\ldots\ y_{r-1}$	$x_0x_1x_2x_3\ldots\ldots\ x_{r-1}$
1	$x_0x_1x_2x_3\ldots\ldots\ x_{r-1}y_1y_2y_3\ldots\ldots\ y_{r-1}$	$x_1x_2x_3\ldots\ldots\ x_{r-1}y_0y_1y_2y_3\ldots\ldots\ y_{r-1}$
2	$x_0x_1^2x_2^2x_3^2\ldots\ldots\ x_{r-1}^2y_0y_1y_2^2y_3^2\ldots\ldots\ y_{r-1}^2$	$x_0x_1x_2^2x_3^2\ldots\ldots\ x_{r-1}^2y_0y_1^2y_2^2y_3^2\ldots\ldots\ y_{r-1}^2$
3	$x_0^2x_1^3x_2^4x_3^4\ldots\ldots\ x_{r-1}^4y_0^2y_1^3y_2^3y_3^4\ldots\ldots\ y_{r-1}^4$	$x_0^2x_1^3x_2^3x_3^4\ldots\ldots\ x_{r-1}^4y_0^2y_1^3y_2^4y_3^4\ldots\ldots\ y_{r-1}^4$
4	$x_0^4x_1^5x_2^7x_3^8\ldots\ldots\ x_{r-1}^8y_0^4y_1^6y_2^7y_3^7y_4^8\ldots\ldots\ y_{r-1}^8$	$x_0^4x_1^5x_2^7x_3^7x_4^8\ldots\ldots\ x_{r-1}^8y_0^4y_1^6y_2^7y_3^8\ldots\ldots\ y_{r-1}^8$

Table 2. **First few terms of these sequences under $(2^{r-1})^{th}$(b) scheme**

Godase-Dhakne [8] obtained many interesting properties of multiplicative coupled Fibonacci sequences of r^{th} order under $(2^r)^{th}$ scheme, some of these are listed below, for every integer $n \geq 0$ and $r \geq 0$

$$X_{n(r+1)} \cdot Y_0 = Y_{n(r+1)} \cdot X_0 \tag{1}$$
$$X_{n(r+1)+1} \cdot Y_1 = Y_{n(r+1)+1} \cdot X_1 \tag{2}$$
$$X_{n(r+1)+2} \cdot Y_2 = Y_{n(r+1)+2} \cdot X_2 \tag{3}$$
$$X_{n(r+1)+3} \cdot Y_3 = Y_{n(r+1)+3} \cdot X_3 \tag{4}$$
$$X_{n(r+1)+m} \cdot Y_m = Y_{n(r+1)+m} \cdot X_m \tag{5}$$
$$\prod_{i=1}^{i=n} X_{ri+1} = \prod_{i=1}^{i=rn} Y_i \tag{6}$$
$$\prod_{i=1}^{i=n} Y_{ri+1} = \prod_{i=1}^{i=rn} X_i \tag{7}$$

3 Methodology

For this research it was decided to consider multiplicative coupled Fibonacci sequences of r^{th} order. When investigating the properties of the mentioned sequences, it is necessary to take into consideration that reader will need special skills and abilities of recurrence relations, master the method of mathematical induction, knowledge on a Fibonacci sequence. All results in this research are proved only using methods of mathematical induction.

4 Main Results

In this section identities for multiplicative coupled Fibonacci sequences of r^{th} order under $(2^{r-1})^{th}$ scheme are established.

Theorem 4.1. For every integer $n \geq 0, r \geq 0$

$$X_{2n(r+1)} \cdot Y_0 = Y_{2n(r+1)} \cdot X_0 \tag{8}$$

Proof. : **Case:(a)**
If r is an even, then

$$X_{n+r} = X_{n+r-1} \cdot Y_{n+r-2} \cdot X_{n+r-3} \cdots Y_n, n \geq 0$$
$$Y_{n+r} = Y_{n+r-1} \cdot X_{n+r-2} \cdot Y_{n+r-3} \cdots X_n, n \geq 0$$

Using Induction Method, For $n = 0$, the result is true because

$$X_o \cdot Y_0 = Y_0 \cdot X_0$$

Now assume that the result is true for some integer $n \geq 1$

$$X_{n+r} = X_{n+r-1} \cdot Y_{n+r-2} \cdot X_{n+r-3} \cdots Y_n, n \geq 0 \tag{9}$$
$$Y_{n+r} = Y_{n+r-1} \cdot X_{n+r-2} \cdot Y_{n+r-3} \cdots X_n, n \geq 0 \tag{10}$$

Now we prove for $n+1$

$$\begin{aligned}
X_{2n(r+1)+2r+2}\cdot Y_0 &= \left[X_{2n(r+1)+2r+1}\cdot Y_{2n(r+1)+2r}\cdot X_{2n(r+1)+2r-1}\cdots\cdot Y_{2n(r+1)+r+2}\right]\cdot Y_0\\
&= \left[X_{2n(r+1)+2r}\cdot Y_{2n(r+1)+2r-1}\cdot X_{2n(r+1)+2r-2}\cdots\cdot Y_{2n(r+1)+r+1}\right]\\
&\cdot\left[Y_{2n(r+1)+2r}\cdot X_{2n(r+1)+2r-1}\cdots\cdot Y_{2n(r+1)+r+2}\right]\cdot Y_0\\
&= \left[X_{2n(r+1)+2r}\cdot Y_{2n(r+1)+2r-1}\cdot X_{2n(r+1)+2r-2}\cdots\cdot Y_{2n(r+1)+r+2}\right]\\
&\cdot\left[Y_{2n(r+1)+r}\cdot X_{2n(r+1)+r-1}\cdot Y_{2n(r+1)+r-2}\cdots\cdot Y_{2n(r+1)+1}\right]\\
&\cdot\left[Y_{2n(r+1)+2r}\cdot X_{2n(r+1)+2r-1}\cdots\cdot Y_{2n(r+1)+r+2}\right]\cdot Y_0\\
&= \left[X_{2n(r+1)+2r}\cdot Y_{2n(r+1)+2r-1}\cdot X_{2n(r+1)+2r-2}\cdots\cdot X_{2n(r+1)+r+2}\right]\\
&\cdot\left[Y_{2n(r+1)+r-1}\cdot X_{2n(r+1)+r-2}\cdot Y_{2n(r+1)+r-3}\cdots\cdot Y_{2n(r+1)}\right]\\
&\cdot\left[X_{2n(r+1)+r-1}Y_{2n(r+1)+r-1}X_{2n(r+1)+r-2}\cdots\cdot X_{2n(r+1)+r+1}\right]\\
&\cdot\left[Y_{2n(r+1)+2r}X_{2n(r+1)+2r-1}\cdots\cdot Y_{2n(r+1)+r+2}\right]Y_0\\
&= [X_{2n(r+1)+2r}\cdot Y_{2n(r+1)+2r-1}\cdot X_{2n(r+1)+2r-2}\cdots\cdot X_{2n(r+1)+r+2}]\\
&\cdot\left[Y_{2n(r+1)+r-1}\cdot X_{2n(r+1)+r-2}\cdot Y_{2n(r+1)+r-3}\cdots\cdot Y_{2n(r+1)+1}\right]\\
&\cdot\left[X_{2n(r+1)+r-1}Y_{2n(r+1)+r-1}X_{2n(r+1)+r-2}\cdots\cdot X_{2n(r+1)+r+1}\right]\\
&\cdot\left[Y_{2n(r+1)+2r}X_{2n(r+1)+2r-1}\cdots\cdot Y_{2n(r+1)+r+2}\right]\left[Y_0X_{2n(r+1)}\right]
\end{aligned}$$

Using induction hypothesis 9,10.

$$\begin{aligned}
&= \left[X_{2n(r+1)+2r}\cdot Y_{2n(r+1)+2r-1}\cdot X_{2n(r+1)+2r-2}\cdots\cdot X_{2n(r+1)+r+2}\right]\\
&\cdot\left[Y_{2n(r+1)+r-1}\cdot X_{2n(r+1)+r-2}\cdot Y_{2n(r+1)+r-3}\cdots\cdot Y_{2n(r+1)+1}\right]\\
&\cdot\left[X_{2n(r+1)+r-1}Y_{2n(r+1)+r-1}X_{2n(r+1)+r-2}\cdots\cdot X_{2n(r+1)+r+1}\right]\\
&\cdot\left[Y_{2n(r+1)+2r}X_{2n(r+1)+2r-1}\cdots\cdot Y_{2n(r+1)+r+2}\right]\left[X_0Y_{2n(r+1)}\right]\\
&= \left[X_{2n(r+1)+2r}Y_{2n(r+1)+2r-1}\cdot X_{2n(r+1)+2r-2}\cdots\cdot X_{2n(r+1)+r+2}\right]\\
&\cdot\left[Y_{2n(r+1)+r-1}\cdot X_{2n(r+1)+r-2}\cdot Y_{2n(r+1)+r-3}\cdots\cdot Y_{2n(r+1)+1}\right]\\
&\cdot\left[X_{2n(r+1)+r-1}\cdot Y_{2n(r+1)+r-1}\cdot X_{2n(r+1)+r-2}\cdots\cdot X_{2n(r+1)+1}\cdot Y_{2n(r+1)}\right]\\
&\cdot\left[Y_{2n(r+1)+2r}\cdot X_{2n(r+1)+2r-1}\cdots\cdot Y_{2n(r+1)+r+2}\right]\cdot X_0\\
&= \left[X_{2n(r+1)+2r}\cdot Y_{2n(r+1)+2r-1}\cdot X_{2n(r+1)+2r-2}\cdots\cdot X_{2n(r+1)+r+2}\right]\\
&\cdot\left[X_{2n(r+1)+r}\cdot Y_{2n(r+1)+r-1}\cdot X_{2n(r+1)+r-2}\cdot Y_{2n(r+1)+r-3}\cdots\cdot Y_{2n(r+1)+1}\right]\\
&\cdot\left[Y_{2n(r+1)+2r}\cdot X_{2n(r+1)+2r-1}\cdots\cdot Y_{2n(r+1)+r+2}\right]\cdot X_0\\
&= \left[X_{2n(r+1)+2r}\cdot Y_{2n(r+1)+2r-1}\cdot X_{2n(r+1)+2r-2}\cdots\cdot X_{2n(r+1)+r+2}\right]\\
&\cdot\left[Y_{2n(r+1)+2r}\cdot X_{2n(r+1)+2r-1}\cdots\cdot Y_{2n(r+1)+r+2}\cdot X_{2n(r+1)+r+1}\right]\cdot X_0\\
&= \left[Y_{2n(r+1)+2r+1}\cdot X_{2n(r+1)+2r}\cdot Y_{2n(r+1)+2r-1}\cdot X_{2n(r+1)+2r-2}\cdots\cdot X_{2n(r+1)+r+2}\right]\cdot X_0\\
&= Y_{2n(r+1)+2r+2}\cdot X_0
\end{aligned}$$

Case:(b)

If r is an odd, then

$$X_{n+r} = X_{n+r-1} \cdot Y_{n+r-2} \cdot X_{n+r-3} \cdots X_n, n \geq 0$$
$$Y_{n+r} = Y_{n+r-1} \cdot X_{n+r-2} \cdot Y_{n+r-3} \cdots Y_n, n \geq 0$$

Using Induction Method,for $n = 0$, the result is true because

$$X_o \cdot Y_0 = Y_0 \cdot X_0$$

Assume that the result is true for some integer $n \geq 1$

$$X_{n+r} = X_{n+r-1} \cdot Y_{n+r-2} \cdot X_{n+r-3} \cdots X_n, n \geq 0 \tag{11}$$
$$Y_{n+r} = Y_{n+r-1} \cdot X_{n+r-2} \cdot Y_{n+r-3} \cdots Y_n, n \geq 0 \tag{12}$$

Now we prove for, $n + 1$

$$\begin{aligned}
X_{2n(r+1)+2r+2} \cdot Y_0 &= \left[X_{2n(r+1)+2r+1} \cdot Y_{2n(r+1)+2r} \cdot X_{2n(r+1)+2r-1} \cdots X_{2n(r+1)+r+2}\right] \cdot Y_0 \\
&= \left[X_{2n(r+1)+2r} \cdot Y_{2n(r+1)+2r-1} \cdot X_{2n(r+1)+2r-2} \cdots X_{2n(r+1)+r+1}\right] \\
&\cdot \left[Y_{2n(r+1)+2r} \cdot X_{2n(r+1)+2r-1} \cdots X_{2n(r+1)+r+2}\right] \cdot Y_0 \\
&= \left[X_{2n(r+1)+2r} \cdot Y_{2n(r+1)+2r-1} \cdot X_{2n(r+1)+2r-2} \cdots Y_{2n(r+1)+r+2}\right] \\
&\cdot \left[X_{2n(r+1)+r} \cdot X_{2n(r+1)+r-1} \cdot Y_{2n(r+1)+r-2} \cdots X_{2n(r+1)+1}\right] \\
&\cdot \left[Y_{2n(r+1)+2r} \cdot X_{2n(r+1)+2r-1} \cdots X_{2n(r+1)+r+2}\right] \cdot Y_0 \\
&= \left[X_{2n(r+1)+2r} \cdot Y_{2n(r+1)+2r-1} \cdot X_{2n(r+1)+2r-2} \cdots X_{2n(r+1)+r+2}\right] \\
&\cdot \left[X_{2n(r+1)+r-1} \cdot Y_{2n(r+1)+r-2} \cdot X_{2n(r+1)+r-3} \cdots X_{2n(r+1)}\right] \\
&\cdot \left[Y_{2n(r+1)+r-1} \cdot X_{2n(r+1)+r-1} \cdot Y_{2n(r+1)+r-2} \cdots X_{2n(r+1)+r+1}\right] \\
&\cdot \left[Y_{2n(r+1)+2r} \cdot X_{2n(r+1)+2r-1} \cdots X_{2n(r+1)+r+2}\right] \cdot Y_0 \\
&= \left[X_{2n(r+1)+2r} \cdot Y_{2n(r+1)+2r-1} \cdot X_{2n(r+1)+2r-2} \cdots Y_{2n(r+1)+r+2}\right] \\
&\cdot \left[X_{2n(r+1)+r-1} \cdot Y_{2n(r+1)+r-2} \cdot X_{2n(r+1)+r-3} \cdots X_{2n(r+1)+1}\right] \\
&\cdot \left[Y_{2n(r+1)+r-1} \cdot X_{2n(r+1)+r-1} \cdot Y_{2n(r+1)+r-2} \cdots X_{2n(r+1)+r+1}\right] \\
&\cdot \left[Y_{2n(r+1)+2r} \cdot X_{2n(r+1)+2r-1} \cdots X_{2n(r+1)+r+2}\right] \cdot \left[Y_0 \cdot X_{2n(r+1)}\right]
\end{aligned}$$

Using induction hypothesis 11,12

$$= \left[X_{2n(r+1)+2r} \cdot Y_{2n(r+1)+2r-1} \cdot X_{2n(r+1)+2r-2} \cdots Y_{2n(r+1)+r+2}\right]$$
$$\cdot \left[X_{2n(r+1)+r-1} \cdot Y_{2n(r+1)+r-2} \cdot X_{2n(r+1)+r-3} \cdots Y_{2n(r+1)+1}\right]$$
$$\cdot \left[Y_{2n(r+1)+r-1} \cdot X_{2n(r+1)+r-1} \cdot Y_{2n(r+1)+r-2} \cdots X_{2n(r+1)+r+1}\right]$$
$$\cdot \left[Y_{2n(r+1)+2r} \cdot X_{2n(r+1)+2r-1} \cdots Y_{2n(r+1)+r+2}\right] \cdot \left[X_0 \cdot Y_{2n(r+1)}\right]$$
$$= \left[X_{2n(r+1)+2r} \cdot Y_{2n(r+1)+2r-1} \cdot X_{2n(r+1)+2r-2} \cdots Y_{2n(r+1)+r+2}\right]$$
$$\cdot \left[X_{2n(r+1)+r-1} \cdot Y_{2n(r+1)+r-2} \cdot X_{2n(r+1)+r-3} \cdots Y_{2n(r+1)+1}\right]$$
$$\cdot \left[Y_{2n(r+1)+r-1} \cdot X_{2n(r+1)+r-1} \cdot Y_{2n(r+1)+r-2} \cdots X_{2n(r+1)+1} \cdot Y_{2n(r+1)}\right]$$
$$\cdot \left[Y_{2n(r+1)+2r} \cdot X_{2n(r+1)+2r-1} \cdots Y_{2n(r+1)+r+2}\right] \cdot X_0$$
$$= \left[X_{2n(r+1)+2r} \cdot Y_{2n(r+1)+2r-1} \cdot X_{2n(r+1)+2r-2} \cdots Y_{2n(r+1)+r+2}\right]$$
$$\cdot \left[Y_{2n(r+1)+r} \cdot X_{2n(r+1)+r-1} \cdot Y_{2n(r+1)+r-2} \cdot X_{2n(r+1)+r-3} \cdots Y_{2n(r+1)+1}\right]$$
$$\cdot \left[Y_{2n(r+1)+2r} \cdot X_{2n(r+1)+2r-1} \cdots Y_{2n(r+1)+r+2}\right] \cdot X_0$$
$$= \left[X_{2n(r+1)+2r} \cdot Y_{2n(r+1)+2r-1} \cdot X_{2n(r+1)+2r-2} \cdots Y_{2n(r+1)+r+1}\right]$$
$$\cdot \left[Y_{2n(r+1)+2r} \cdot X_{2n(r+1)+2r-1} \cdots Y_{2n(r+1)+r+2} \cdot X_{2n(r+1)+r+1}\right] \cdot X_0$$
$$= \left[Y_{2n(r+1)+2r+1} \cdot X_{2n(r+1)+2r} \cdot Y_{2n(r+1)+2r-1} \cdot X_{2n(r+1)+2r-2} \cdots Y_{2n(r+1)+r+2}\right] \cdot X_0$$
$$= Y_{2n(r+1)+2r+2} \cdot X_0$$

Hence proof. □

Theorem 4.2. For every integer $n \geq 0, \quad r \geq 0$

$$X_{2n(r+1)+1} \cdot Y_1 = Y_{2n(r+1)+1} \cdot X_1 \tag{13}$$
$$X_{2n(r+1)+2} \cdot Y_2 = Y_{2n(r+1)+2} \cdot X_2 \tag{14}$$
$$X_{2n(r+1)+3} \cdot Y_3 = Y_{2n(r+1)+3} \cdot X_3 \tag{15}$$

Proof. Proof is similar to theorem 4.1 □

Theorem 4.3. For every integer $n \geq 0, r \geq 0$ and $m \geq 0$

$$X_{2n(r+1)+m} \cdot Y_m = Y_{2n(r+1)+m} \cdot X_m \tag{16}$$

Proof. **Case:(a)**
If r is an even, then

$$X_{n+r} = X_{n+r-1} \cdot Y_{n+r-2} \cdot X_{n+r-3} \cdots Y_n, n \geq 0$$
$$Y_{n+r} = Y_{n+r-1} \cdot X_{n+r-2} \cdot Y_{n+r-3} \cdots X_n, n \geq 0$$

Using induction method, for $n = 0$ the result is true because

$$X_m \cdot Y_m = Y_m \cdot X_m$$

Assume that the result is true for some integer $n \geq 1$

$$X_{n+r} = X_{n+r-1} \cdot Y_{n+r-2} \cdot X_{n+r-3} \cdots Y_n, n \geq 0 \tag{17}$$
$$Y_{n+r} = Y_{n+r-1} \cdot X_{n+r-2} \cdot Y_{n+r-3} \cdots X_n, n \geq 0 \tag{18}$$

Now, we prove for $n+1$

$$\begin{aligned}
X_{2n(r+1)+m+2r+2} \cdot Y_m &= \big[X_{2n(r+1)+m+2r+1} \cdot Y_{2n(r+1)+m+2r} \cdot X_{2n(r+1)+m+2r-1} \cdots Y_{2n(r+1)+r+2}\big] \cdot Y_m \\
&= \big[X_{2n(r+1)+m+2r} \cdot Y_{2n(r+1)+m+2r-1} \cdot X_{2n(r+1)+m+2r-2} \cdots Y_{2n(r+1)+m+r+1}\big] \\
&\cdot \big[Y_{2n(r+1)+m+2r} \cdot X_{2n(r+1)+m+2r-1} \cdots Y_{2n(r+1)+m+r+2}\big] \cdot Y_m \\
&= \big[X_{2n(r+1)+m+2r} \cdot Y_{2n(r+1)+m+2r-1} \cdot X_{2n(r+1)+m+2r-2} \cdots Y_{2n(r+1)+m+r+2}\big] \\
&\cdot \big[Y_{2n(r+1)+m+r} \cdot X_{2n(r+1)+m+r-1} \cdot Y_{2n(r+1)+m+r-2} \cdots Y_{2n(r+1)+m+1}\big] \\
&\cdot \big[Y_{2n(r+1)+m+2r} \cdot X_{2n(r+1)+m+2r-1} \cdots Y_{2n(r+1)+m+r+2}\big] \cdot Y_m \\
&= \big[X_{2n(r+1)+m+2r} \cdot Y_{2n(r+1)+m+2r-1} \cdot X_{2n(r+1)+m+2r-2} \cdots X_{2n(r+1)+m+r+2}\big] \\
&\cdot \big[Y_{2n(r+1)+m+r-1} \cdot X_{2n(r+1)+m+r-2} \cdot Y_{2n(r+1)+m+r-3} \cdots Y_{2n(r+1)+m}\big] \\
&\cdot \big[X_{2n(r+1)+m+r-1} \cdot Y_{2n(r+1)+m+r-1} \cdot X_{2n(r+1)+m+r-2} \cdots X_{2n(r+1)+m+r+1}\big] \\
&\cdot \big[Y_{2n(r+1)+m+2r} \cdot X_{2n(r+1)+m+2r-1} \cdots Y_{2n(r+1)+m+r+2}\big] \cdot Y_m \\
&= \big[X_{2n(r+1)+m+2r} \cdot Y_{2n(r+1)+m+2r-1} \cdot X_{2n(r+1)+m+2r-2} \cdots X_{2n(r+1)+m+r+2}\big] \\
&\cdot \big[Y_{2n(r+1)+m+r-1} \cdot X_{2n(r+1)+m+r-2} \cdot Y_{2n(r+1)+m+r-3} \cdots Y_{2n(r+1)+m+1}\big] \\
&\cdot \big[X_{2n(r+1)+m+r-1} \cdot Y_{2n(r+1)+m+r-1} \cdot X_{2n(r+1)+m+r-2} \cdots X_{2n(r+1)+m+r+1}\big] \\
&\cdot \big[Y_{2n(r+1)+m+2r} \cdot X_{2n(r+1)+m+2r-1} \cdots Y_{2n(r+1)+m+r+2}\big] \cdot \big[Y_m \cdot X_{2n(r+1)+m}\big]
\end{aligned}$$

Using induction hypothesis 17, 18

$$\begin{aligned}
&= \big[X_{2n(r+1)+m+2r} \cdot Y_{2n(r+1)+m+2r-1} \cdot X_{2n(r+1)+m+2r-2} \cdots X_{2n(r+1)+m+r+2}\big] \\
&\cdot \big[Y_{2n(r+1)+m+r-1} \cdot X_{2n(r+1)+m+r-2} \cdot Y_{2n(r+1)+m+r-3} \cdots Y_{2n(r+1)+m+1}\big] \\
&\cdot \big[X_{2n(r+1)+m+r-1} \cdot Y_{2n(r+1)+m+r-1} \cdot X_{2n(r+1)+m+r-2} \cdots X_{2n(r+1)+m+r+1}\big] \\
&\cdot \big[Y_{2n(r+1)+m+2r} \cdot X_{2n(r+1)+m+2r-1} \cdots Y_{2n(r+1)+m+r+2}\big] \cdot \big[X_m \cdot Y_{2n(r+1)} + m\big] \\
&= \big[X_{2n(r+1)+m+2r} \cdot Y_{2n(r+1)+m+2r-1} \cdot X_{2n(r+1)+m+2r-2} \cdots X_{2n(r+1)+m+r+2}\big] \\
&\cdot \big[Y_{2n(r+1)+m+r-1} \cdot X_{2n(r+1)+m+r-2} \cdot Y_{2n(r+1)+m+r-3} \cdots Y_{2n(r+1)+m+1}\big] \\
&\cdot \big[X_{2n(r+1)+m+r-1} \cdot Y_{2n(r+1)+m+r-1} \cdot X_{2n(r+1)+m+r-2} \cdots X_{2n(r+1)+m+1} \cdot Y_{2n(r+1)+m}\big] \\
&\cdot \big[Y_{2n(r+1)+m+2r} \cdot X_{2n(r+1)+m+2r-1} \cdots Y_{2n(r+1)+m+r+2}\big] \cdot X_m \\
&= \big[X_{2n(r+1)+m+2r} \cdot Y_{2n(r+1)+m+2r-1} \cdot X_{2n(r+1)+m+2r-2} \cdots X_{2n(r+1)+m+r+2}\big] \\
&\cdot \big[X_{2n(r+1)+m+r} \cdot Y_{2n(r+1)+m+r-1} \cdot X_{2n(r+1)+m+r-2} \cdot Y_{2n(r+1)+m+r-3} \cdots Y_{2n(r+1)+m+1}\big] \\
&\cdot \big[Y_{2n(r+1)+m+2r} \cdot X_{2n(r+1)+m+2r-1} \cdots Y_{2n(r+1)+m+r+2}\big] \cdot X_m \\
&= \big[X_{2n(r+1)+m+2r} \cdot Y_{2n(r+1)+m+2r-1} \cdot X_{2n(r+1)+m+2r-2} \cdots X_{2n(r+1)+m+r+2}\big] \\
&\cdot \big[Y_{2n(r+1)+m+2r} \cdot X_{2n(r+1)+m+2r-1} \cdots Y_{2n(r+1)+m+r+2} \cdot X_{2n(r+1)+m+r+1}\big] \cdot X_m \\
&= \big[Y_{2n(r+1)+m+2r+1} \cdot X_{2n(r+1)+m+2r} \cdot Y_{2n(r+1)+m+2r-1} \cdot X_{2n(r+1)+m+2r-2} \cdots X_{2n(r+1)+m+r+2}\big] \cdot X_m \\
&= Y_{2n(r+1)+m+2r+2} \cdot X_m
\end{aligned}$$

Case:(b)
If r is odd, then

$$X_{n+r} = X_{n+r-1} \cdot Y_{n+r-2} \cdot X_{n+r-3} \cdots \cdot X_n, n \geq 0$$
$$Y_{n+r} = Y_{n+r-1} \cdot X_{n+r-2} \cdot Y_{n+r-3} \cdots \cdot Y_n, n \geq 0$$

Using induction method, if $n = 0$, the result is true because

$$X_m \cdot Y_m = Y_m \cdot X_m$$

Assume that the result is true for some integer $n \geq 1$

$$X_{n+r} = X_{n+r-1} \cdot Y_{n+r-2} \cdot X_{n+r-3} \cdots \cdot X_n, n \geq 0 \tag{19}$$
$$Y_{n+r} = Y_{n+r-1} \cdot X_{n+r-2} \cdot Y_{n+r-3} \cdots \cdot Y_n, n \geq 0 \tag{20}$$

Now, we prove for $n + 1$

$$\begin{aligned}
X_{2n(r+1)+m+2r+2} \cdot Y_m &= \left[X_{2n(r+1)+m+2r+1} \cdot Y_{2n(r+1)+m+2r} \cdot X_{2n(r+1)+m+2r-1} \cdots \cdot X_{2n(r+1)+m+r+2}\right] \cdot Y_m \\
&= \left[X_{2n(r+1)+m+2r} \cdot Y_{2n(r+1)+m+2r-1} \cdot X_{2n(r+1)+m+2r-2} \cdots \cdot X_{2n(r+1)+m+r+1}\right] \\
&\cdot \left[Y_{2n(r+1)+m+2r} \cdot X_{2n(r+1)+m+2r-1} \cdots \cdot X_{2n(r+1)+m+r+2}\right] \cdot Y_m \\
&= \left[X_{2n(r+1)+m+2r} \cdot Y_{2n(r+1)+m+2r-1} \cdot X_{2n(r+1)+m+2r-2} \cdots \cdot Y_{2n(r+1)+m+r+2}\right] \\
&\cdot \left[X_{2n(r+1)+m+r} \cdot X_{2n(r+1)+m+r-1} \cdot Y_{2n(r+1)+m+r-2} \cdots \cdot X_{2n(r+1)+m+1}\right] \\
&\cdot \left[Y_{2n(r+1)+m+2r} \cdot X_{2n(r+1)+m+2r-1} \cdots \cdot X_{2n(r+1)+m+r+2}\right] \cdot Y_m \\
&= \left[X_{2n(r+1)+m+2r} \cdot Y_{2n(r+1)+m+2r-1} \cdot X_{2n(r+1)+m+2r-2} \cdots \cdot X_{2n(r+1)+m+r+2}\right] \\
&\cdot \left[X_{2n(r+1)+m+r-1} \cdot Y_{2n(r+1)+m+r-2} \cdot X_{2n(r+1)+m+r-3} \cdots \cdot X_{2n(r+1)+m}\right] \\
&\cdot \left[Y_{2n(r+1)+m+r-1} \cdot X_{2n(r+1)+m+r-1} \cdot Y_{2n(r+1)+m+r-2} \cdots \cdot X_{2n(r+1)+m+r+1}\right] \\
&\cdot \left[Y_{2n(r+1)+m+2r} \cdot X_{2n(r+1)+m+2r-1} \cdots \cdot X_{2n(r+1)+m+r+2}\right] \cdot Y_m \\
&= \left[X_{2n(r+1)+m+2r} \cdot Y_{2n(r+1)+m+2r-1} \cdot X_{2n(r+1)+m+2r-2} \cdots \cdot Y_{2n(r+1)+m+r+2}\right] \\
&\cdot \left[X_{2n(r+1)+m+r-1} \cdot Y_{2n(r+1)+m+r-2} \cdot X_{2n(r+1)+m+r-3} \cdots \cdot X_{2n(r+1)+m+1}\right] \\
&\cdot \left[Y_{2n(r+1)+m+r-1} \cdot X_{2n(r+1)+m+r-1} \cdot Y_{2n(r+1)+m+r-2} \cdots \cdot X_{2n(r+1)+m+r+1}\right] \\
&\cdot \left[Y_{2n(r+1)+m+2r} \cdot X_{2n(r+1)+m+2r-1} \cdots \cdot X_{2n(r+1)+m+r+2}\right] \cdot \left[Y_m \cdot X_{2n(r+1)+m}\right]
\end{aligned}$$

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$$= \left[X_{2n(r+1)+m+2r} \cdot Y_{2n(r+1)+m+2r-1} \cdot X_{2n(r+1)+m+2r-2} \cdots Y_{2n(r+1)+m+r+2}\right]$$
$$\cdot \left[X_{2n(r+1)+m+r-1} \cdot Y_{2n(r+1)+m+r-2} \cdot X_{2n(r+1)+m+r-3} \cdots Y_{2n(r+1)+m+1}\right]$$
$$\cdot \left[Y_{2n(r+1)+m+r-1} \cdot X_{2n(r+1)+m+r-1} \cdot Y_{2n(r+1)+m+r-2} \cdots X_{2n(r+1)+m+r+1}\right]$$
$$\cdot \left[Y_{2n(r+1)+m+2r} \cdot X_{2n(r+1)+m+2r-1} \cdots Y_{2n(r+1)+m+r+2}\right] \cdot \left[X_m \cdot Y_{2n(r+1)+m}\right]$$
$$= \left[X_{2n(r+1)+m+2r} \cdot Y_{2n(r+1)+m+2r-1} \cdot X_{2n(r+1)+m+2r-2} \cdots Y_{2n(r+1)+m+r+2}\right]$$
$$\cdot \left[X_{2n(r+1)+m+r-1} \cdot Y_{2n(r+1)+m+r-2} \cdot X_{2n(r+1)+m+r-3} \cdots Y_{2n(r+1)+m+1}\right]$$
$$\cdot \left[Y_{2n(r+1)+m+r-1} \cdot X_{2n(r+1)+m+r-1} \cdot Y_{2n(r+1)+m+r-2} \cdots X_{2n(r+1)+m+1} \cdot Y_{2n(r+1)+m}\right]$$
$$\cdot \left[Y_{2n(r+1)+m+2r} \cdot X_{2n(r+1)+m+2r-1} \cdots Y_{2n(r+1)+m+r+2}\right] \cdot X_m$$
$$= \left[X_{2n(r+1)+m+2r} \cdot Y_{2n(r+1)+m+2r-1} \cdot X_{2n(r+1)+m+2r-2} \cdots Y_{2n(r+1)+m+r+2}\right]$$
$$\cdot \left[Y_{2n(r+1)+m+r} \cdot X_{2n(r+1)+m+r-1} \cdot Y_{2n(r+1)+m+r-2} \cdot X_{2n(r+1)+m+r-3} \cdots Y_{2n(r+1)+m+1}\right]$$
$$\cdot \left[Y_{2n(r+1)+m+2r} \cdot X_{2n(r+1)+m+2r-1} \cdots Y_{2n(r+1)+m+r+2}\right] \cdot X_m$$
$$= \left[X_{2n(r+1)+m+2r} \cdot Y_{2n(r+1)+m+2r-1} \cdot X_{2n(r+1)+m+2r-2} \cdots Y_{2n(r+1)+m+r+1}\right]$$
$$\cdot \left[Y_{2n(r+1)+m+2r} \cdot X_{2n(r+1)+m+2r-1} \cdots Y_{2n(r+1)+m+r+2} \cdot X_{2n(r+1)+m+r+1}\right] \cdot X_m$$
$$= \left[Y_{2n(r+1)+m+2r+1} \cdot X_{2n(r+1)+m+2r} \cdot Y_{2n(r+1)+m+2r-1} \cdot X_{2n(r+1)+m+2r-2} \cdots Y_{2n(r+1)+m+r+2}\right] \cdot X_m$$
$$= Y_{2n(r+1)+m+2r+2} \cdot X_m$$

Hence proof. □

Theorem 4.4. For every integer $n \geq 0, \quad r \geq 0$

$$\prod_{i=1}^{i=n} X_{ri+1} \cdot Y_{ri+1} = \prod_{i=1}^{i=rn} Y_i \cdot X_i$$

Proof. Using induction method, for $n = 1$, the result is true because

$$\begin{aligned} X_{r+1} \cdot Y_{r+1} &= [Y_r \cdot Y_{r-1} \cdot Y_{r-2} \cdots Y_1] \cdot [X_r \cdot X_{r-1} \cdot X_{r-2} \cdots X_1] \\ &= [Y_r \cdot X_r] \cdot [Y_{r-1} \cdot X_{r-1}] \cdot [Y_{r-2} \cdot X_{r-2}] \cdots [Y_1 \cdot X_1] \\ &= \prod_{i=1}^{i=r} Y_i \cdot X_i \end{aligned}$$

Assume that the result is true for some integer $n \geq 1$ (a) If r is an even, then

$$X_{n+r} = \qquad X_{n+r-1} \cdot Y_{n+r-2} \cdot X_{n+r-3} \cdots Y_n, n \geq 0 \tag{21}$$
$$Y_{n+r} = \qquad Y_{n+r-1} \cdot X_{n+r-2} \cdot Y_{n+r-3} \cdots X_n, n \geq 0 \tag{22}$$

(b) If r is an odd, then

$$X_{n+r} = X_{n+r-1} \cdot Y_{n+r-2} \cdot X_{n+r-3} \cdots X_n, n \geq 0 \tag{23}$$
$$Y_{n+r} = Y_{n+r-1} \cdot X_{n+r-2} \cdot Y_{n+r-3} \cdots Y_n, n \geq 0 \tag{24}$$

Now, we prove for $n+1$

$$\prod_{i=1}^{i=n+1} X_{ri+1} \cdot Y_{ri+1} = \prod_{i=1}^{i=n} [X_{ri+1} \cdot Y_{ri+1}] \cdot \left[X_{r(n+1)+1} \cdot X_{r(n+1)+1}\right]$$

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$$= \prod_{i=1}^{i=rn} [X_i \cdot Y_i] \cdot [X_{rn+r+1} \cdot Y_{rn+r+1}]$$
$$= \prod_{i=1}^{i=rn} [X_i \cdot Y_i] \cdot [Y_{rn+r} \cdot X_{rn+r}] \cdot [Y_{rn+r-1} \cdot X_{rn+r-1}] \cdot [Y_{rn+r-2} \cdot X_{rn+r-2}] \cdots [Y_{rn+1} \cdot X_{rn+1}]$$
$$= \prod_{i=1}^{i=rn+r} [Y_i \cdot X_i]$$

Hence proof. □

5 Conclusion

Identities of multiplicative coupled Fibonacci sequences of r^{th} order under $2^{r-1^{th}}$ scheme are described in this paper, this idea can be extended for other schemes and multiplicative coupled Fibonacci sequences of r^{th} order with negative integers.

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APPROXIMATION ON AN INTUITIONISTIC FUZZY NEAR-RINGS

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Abstaract − This paper proposes the notion of a rough near-ring with respect to an ideal of an intuitionistic fuzzy near-ring. Further the properties of rough ideals on near-rings are characterized.

Keywords − *Rough intuitionistic fuzzy ideals, Rough intuitionistic fuzzy near-ring, Rough intuitionistic fuzzy N-subgroup.*

1 Introduction

Rough set theory, proposed by Pawlak [25] is a new mathematical tool that supports uncertainty reasoning.The basic assumption of rough set theory is every knowledge in universe depends upon their capability of its classification. So that equivalennce relations are considered to define rough sets. It may be seen as an extension of classical set theory and has been successfully applied to machine learning, intelligent systems, inductive reasoning, pattern recognition, image processing, signal analysis, knowledge discovery, decision analysis, expert systems and many other fields. In 1965, Zadeh[32] initiated the novel concept of fuzzy set theory. There have been attempts to fuzzify various mathematical structures like topological spaces, groups, rings, etc., also concepts like relations measure, probability and automata etc. Biswas and Nanda[6] in 1994 introduced the concept of rough ideal in semi group. Based on an equivalence relation in 1990, Dubois and Prade[12] introduced the lower and upper approximations of fuzzy sets in Pawlak approximation space to obtain an extended notion called rough fuzzy sets. In 2008, Kazanci and Davaaz[16] introduced rough prime ideals and rough fuzzy prime ideals in commutative rings. Recently

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Jayanta Ghosh and T.K. Samantha[15] introduced rough intuitionistic fuzzy sets in semigroups.Yong Ho Yon et.al[29] introduced the concept of intuitionistic fuzzy R-subgroups of near-rings. In this paper we define rough intuitionistic fuzzy ideals of a near-ring based on its lower and upper approximation. Some interesting properties are established.

2 Preliminary

Definition 2.1. [22] By a near-ring we mean a nonempty set R with two binary operations " + " and "." satisfying the following axioms:

(i) $(R, +)$ is a group.

(ii) $(R, .)$ is a semigroup.

(iii) $x.(y + z) = x.y + x.z$ for all $x, y, z \in R$.

Definition 2.2. [22] An ideal of a near-ring R is a subset I of R such that

(i) $(I, +)$ is a normal subgroup of $(R, +)$.

(ii) $RI \subseteq I$.

(iii) $(x + i)y - xy \in I$ for all $i \in I$ and$x, y \in R$.

Definition 2.3. [3] An intuitionistic fuzzy set (IFS in short) A in X is an object having the form $A = \{\langle x, \mu_A(x), \nu_A(x)/x \in X\rangle\}$ where the function $\mu : X \to [0, 1]$ and $\nu : X \to [0, 1]$ denote the degree of membership (namely $\mu_A(x)$) and the degree of non membership (namely $\nu_A(x)$) of each element $x \in X$ to the set A, respectively and $0 \leq \mu_A(x) + \nu_A(x) \leq 1$ for each $x \in X$. Denote by IFS(X) the set of all intuitionistic fuzzy set in X.

Definition 2.4. [3] Let A and B be IFS's of the form $A = \{\langle x, \mu_A(x), \nu_A(x)/x \in X\rangle\}$ and $B = \{\langle x, \mu_B(x), \nu_B(x)/x \in X\rangle\}$. Then

1. $A \subseteq B$ if and only if $\mu_A(x) \leq \mu_B(x)$ and $\nu_A(x) \geq \nu_B(x)$ for all $x \in X$.
2. $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$.
3. $\overline{A} = \{\langle x, \nu_A(x), \mu_A(x)/x \in X\rangle\}$.(Complement of A)
4. $A \cap B = \{\langle x, \mu_A(x) \wedge \mu_B(x), \nu_A(x) \vee \nu_B(x)/x \in X\rangle\}$.
5. $A \cup B = \{\langle x, \mu_A(x) \vee \mu_B(x), \nu_A(x) \wedge \nu_B(x)/x \in X\rangle\}$.

For the sake of simplicity we use the notion $A = \langle x, \mu_A, \nu_A\rangle$ instead of $A = \{\langle x, \mu_A(x), \nu_A(x)/x \in X\rangle\}$. The intuitionistic fuzzy set $0 \sim= \{\langle x, 0 \sim, 1 \sim\rangle /x \in X\}$ and $1 \sim= \{\langle x, 1 \sim, 0 \sim\rangle /x \in X\}$ are respectively the empty set and the whole set of X.

Definition 2.5. [33] An intuitionistic fuzzy set $A = (\mu_A, \nu_A)$ is called an intuitionistic fuzzy ideal of near-ring of R if for all x,y,i $\in R$.

(IF1) $\mu_A(x - y) \geq \mu_A(x) \wedge \mu_A(y)$ and $\nu_A(x - y) \leq \nu_A(x) \vee \nu_A(y)$

(IF2) $\mu_A(y + x - y) \geq \mu_A(x)$ and $\nu_A(y + x - y) \leq \nu_A(x)$

(IF3) $\mu_A(xy) \geq \mu_A(y)$ and $\nu_A(xy) \leq \nu_A(y)$

(IF4) $\mu_A(i(x + y) - ix) \geq \mu_A(y)$ and $\nu_A(i(x + z) - ix) \leq \nu_A(y)$

Definition 2.6. [29] An IFS $A = (\mu_A, \nu_A)$ in R is called an intuitionistic fuzzy subnear ring of R if for all $x, y \in R$

(i) $\mu_A(x - y) \geq \mu_A(x) \wedge \mu_A(y)$ and $\nu_A(x - y) \leq \nu_A(x) \vee \nu_A(y)$

(ii) $\mu_A(xy) \geq \mu_A(y)$ and $\nu_A(xy) \leq \nu_A(y)$.

Definition 2.7. [16] An equivalence relation θ on R is a reflexive, symmetric and transitive binary relation on R. If θ is an equivalence relation on R then the equivalence class of $x \in R$ is the set $\{y \in R | (x, y) \in \theta$.We write it as $[x]_\theta$.

Definition 2.8. [16] Let θ be an equivalence relation on R, then θ is called a full congruence relation if $(a, b) \in \theta$ implies $(a + x, b + x), (ax, bx)$ and $(xa, xb) \in \theta$ for all $x \in R$

Theorem 2.9. [16] Let θ be a full congruence relation on R, then $(a, b) \in \theta$ and $(c, d) \in \theta$ imply $(a + c, b + d) \in \theta, (ca, bd) \in \theta$ and $(-a, -b) \in \theta$ for all $a, b, c, d \in R$

Definition 2.10. [15] Let us consider θ to be a congruence relation of S. If X is a nonempty subset of S then the sets $\theta_*(X) = \{x \in S | [x]_\theta \subseteq X\}$ and $\theta^*(X) = \{x \in S | [x]_\theta \cap X \neq \phi\}$ are respectively called the θ-lower and θ-upper approximation of the set X and $\theta(X) = (\theta_*(X), \theta^*(X))$ is called rough set with respect to θ if $\theta_*(X) \neq \theta^*(X)$. If $A = (\mu_A, \nu_A)$ be IFS of S. Then the IFS $\theta_*(A) = (\theta_*(\mu_A, \theta_*\nu_A))$ and $\theta^*(A) = (\theta^*(\mu_A, \theta^*\nu_A))$ are respectively called $\theta-$lower and $\theta-$upper approximation of the IFS $A = (\mu_A, \nu_A)$ where for all $x \in S$

$$\theta_*(\mu_A)(x) = \wedge_{a \in [x]_\theta} \mu_A(a), \theta_*(\nu_A)(x) = \vee_{a \in [x]_\theta} \nu_A(a)$$
$$\theta^*(\mu_A)(x) = \vee_{a \in [x]_\theta} \mu_A(a), \theta_*(\nu_A)(x) = \wedge_{a \in [x]_\theta} \nu_A(a)$$

For an IFS $A = (\mu_A, \nu_A)$ of S, $\theta(A) = (\theta_*(A), \theta^*(A))$ is called rough intuitionistic fuzzy set with respect to θ if $\theta_*(A) \neq \theta^*(A)$

Definition 2.11. [29] An IFS $A = (\mu_A, \nu_A)$ in R is called an intuitionistic fuzzy N-subgroup of R if for all $x, y, n \in R$

(i) $\mu_A(x - y) \geq \mu_A(x) \wedge \mu_A(y)$ and $\nu_A(x - y) \leq \nu_A(x) \vee \nu_A(y)$.

ii $\mu_A(nx) \geq \mu_A(x)$ and $\nu_A(nx) \leq \nu_A(x)$.

iii $\mu_A(xn) \geq \mu_A(x)$ and $\nu_A(xn) \leq \nu_A(x)$.

3 ROUGH INTUITIONISTIC FUZZY IDEALS IN NEAR-RINGS

Throughout N denotes an abelian near ring and R denotes a near ring.

Lemma 3.1. If an IFS $A = (\mu_A, \nu_A)$ in R satisfies the condition (IF1) of definition [2.5], then

(i) $\mu_A(0) \geq \mu_A(x)$ and $\nu_A(0) \leq \nu_A(x)$,

(ii) $\mu_A(-x) = \mu_A(x)$and $\nu_A(-x) = \nu_A(x)$.

for all $x \in R$.

Lemma 3.2. If an IFS $A = (\mu_A, \nu_A)$ in R satisfies the condition (IF1) of definition [2.5], then

(i) $\mu_A(x - y) = \mu_A(0) \Rightarrow \mu_A(x) = \mu_A(y)$,

(ii) $\nu_A(x - y) = \nu_A(0) \Rightarrow \nu_A(x) = \nu_A(y)$.

for all $x, y \in R$.
The proof of 3.1 and 3.2 are immediate.

Theorem 3.3. Let θ be a full congruence relation on R. If $A = (\mu_A, \nu_A)$ is an intuitionistic fuzzy subnear-ring of R then so is $\theta^*(A) = (\theta^*(\mu_A), \theta^*(\nu_A))$.
Proof: Since θ is a congruence relation of R. Then for all $x, y \in R$,

(i) a) $\theta^*(\mu_A)(x - y) = \bigvee_{z \in [x-y]_\theta} \mu_A(z)$
$= \bigvee_{z \in [x]_\theta - [y]_\theta} \mu_A(z)$
$= \bigvee_{a-b \in [x]_\theta - [y]_\theta} \mu_A(a - b)$
$\geq \bigvee_{a \in [x]_\theta, b \in [y]_\theta} [\mu_A(a) \wedge \mu_A(b)]$
$= [\bigvee_{a \in [x]_\theta} \mu_A(a)] \vee [\bigvee_{b \in [y]_\theta} \mu_A(b)] = \theta^*(\mu_A)(x) \wedge \theta^*(\mu_A)(y)$
b) $\theta^*(\nu_A)(x - y) = \bigwedge_{z \in [x-y]_\theta} \nu_A(z)$
$= \bigwedge_{z \in [x]_\theta - [y]_\theta} \nu_A(z)$
$= \bigwedge_{a-b \in [x]_\theta - [y]_\theta} \nu_A(a - b)$
$\leq \bigwedge_{a \in [x]_\theta, b \in [y]_\theta} [\nu_A(a) \vee \nu_A(b)]$
$= [\bigwedge_{a \in [x]_\theta} \nu_A(a)] \vee [\bigwedge_{b \in [y]_\theta} \nu_A(b)] = \theta^*(\nu_A)(x) \vee \theta^*(\nu_A)(y)$

(ii) a) $\theta^*(\mu_A)(xy) = \bigvee_{z\in[xy]_\theta} \mu_A(z)$

$\geq \bigvee_{z\in[x]_\theta[y]_\theta} \mu_A(z)$

$= \bigvee_{ab\in[x]_\theta[y]_\theta} \mu_A(ab)$

$\geq \bigvee_{a\in[x]_\theta b\in[y]_\theta} \mu_A(ab)$

$\geq \bigvee_{b\in[y]_\theta} \mu_A(b) = \theta^*(\mu_A)(y)$

b) $\theta^*(\nu_A)(xy) = \bigwedge_{z\in[yx]_\theta} \nu_A(z)$

$\leq \bigwedge_{z\in[x]_\theta[y]_\theta} \nu_A(z)$

$= \bigwedge_{ab\in[x]_\theta[y]_\theta} \nu_A(ab)$

$\leq \bigwedge_{a\in[x]_\theta b\in[y]_\theta} \nu_A(ab)$

$\leq \bigwedge_{b\in[y]_\theta} \nu_A(b) = \theta^*(\nu_A)(y)$ This shows that $\theta^*(A)$ is an intuitionistic fuzzy subnear-ring of R. Therefore A is an upper rough intuitonistic fuzzy subnear-ring of R.

Theorem 3.4. Let θ be a full congruence relation on R. If $A = (\mu_A, \nu_A)$ is an intuitionistic fuzzy subnear-ring of R then so is $(\theta)_*(A) = (\theta_*(\mu_A), \theta_*(\nu_A))$.

Proof. Since θ is a congruence relation of R. Then for all $x, y \in R$

(i) a) $\theta_*(\mu_A)(x-y) = \bigwedge_{z\in[x-y]_\theta} \mu_A(z)$

$= \bigwedge_{z\in[x]_\theta-[y]_\theta} \mu_A(z)$

$\geq \bigwedge_{z\in[x]_\theta[y]_\theta} \mu_A(z)$

$= \bigwedge_{a-b\in[x]_\theta-[y]_\theta} \mu_A(a-b)$

$\geq \bigwedge_{a\in[x]_\theta, b\in[y]_\theta} [\mu_A(a) \wedge \mu_A(b)]$

$= [\bigwedge_{a\in[x]_\theta} \mu_A(a)] \wedge [\bigwedge_{b\in[y]_\theta} \mu_A(b)] = \theta_*(\mu_A)(x) \wedge \theta_*(\mu_A)(y)$

b) $\theta_*(\nu_A)(x-y) = \bigvee_{z\in[x-y]_\theta} \nu_A(z)$

$= \bigvee_{z\in[x]_\theta-[y]_\theta} \nu_A(z)$

$= \bigvee_{a-b\in[x]_\theta-[y]_\theta} \nu_A(a-b)$

$\leq \bigvee_{a\in[x]_\theta, b\in[y]_\theta} [\nu_A(a) \vee \nu_A(b)]$

$= [\bigvee_{a\in[x]_\theta} \nu_A(a)] \vee [\bigvee_{b\in[y]_\theta} \nu_A(b)] = \theta_*(\nu_A)(x) \vee \theta_*(\nu_A)(y)$

(ii) a) $\theta_*(\mu_A)(xy) = \bigwedge_{z\in[xy]_\theta} \mu_A(z)$

$= \bigwedge_{z\in[x]_\theta[y]_\theta} \mu_A(z)$
$= \bigwedge_{ab\in[x]_\theta[y]_\theta} \mu_A(ab)$
$\geq \bigwedge_{a\in[x]_\theta b\in[y]_\theta} \mu_A(ab)$
$= \bigvee_{b\in[y]_\theta} \mu_A(b) = \theta_*(\mu_A)(y)$
b) $\theta_*(\nu_A)(xy) = \bigvee_{z\in[yx]_\theta} \nu_A(z)$
$= \bigvee_{z\in[x]_\theta[y]_\theta} \nu_A(z)$
$= \bigvee_{ab\in[x]_\theta[y]_\theta} \nu_A(ab)$
$\leq \bigvee_{a\in[x]_\theta b\in[y]_\theta} \nu_A(ab)$
$\leq \bigvee_{b\in[y]_\theta} \nu_A(b) = \theta_*(\nu_A)(y)$ This shows that $\theta_*(A)$ is an intuitionistic fuzzy subnear-ring of R. Therefore A is an lower rough intuitonistic fuzzy subnear-ring of R.

Corollary 3.5. If $A = (\mu_A, \nu_A)$ is an intuitionistic fuzzy subnear-ring of R then $\theta(A) = (\theta_*(A), \theta^*(A))$ is a rough intuitionistic fuzzy subnear-ring of R.

Example 3.6. Let $R = \{0, a, b, c\}$ be a set with two binary operations as follows

+	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

.	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

Then $(R, +, .)$ is a near-ring. Let θ be a congruence relation on R such that the θ-congruence classes are the subsets $\{0\}, \{c\}, \{a, b\}$. Let $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle | x \in R\}$ be an intuitionistic fuzzy subset of R defined by

$$A = \{\langle 0, 0.4, 0.4\rangle, \langle a, 0.1, 0.5\rangle, \langle b, 0.4, 0.6\rangle, \langle c, 0.4, 0.4\rangle\}$$

Since for every $x \in R, \theta^*(\mu_A)(x) = \bigvee_{\alpha\in[x]_\theta} \mu_A(\alpha)$ and $\theta^*(\nu_A)(x) = \bigwedge_{\alpha\in[x]_\theta} \nu_A(\alpha)$, so the upper approximation $\theta^*(A) = \{\langle x, \theta^*(\mu_A(x)), \theta^*(\nu_A(x))\rangle | x \in T\}$ is given by

$$\theta^*(A) = \{\langle 0, 0.4, 0.4\rangle, \langle a, 0.4, 0.5\rangle, \langle b, 0.4, 0.5\rangle, \langle c, 0.4, 0.4\rangle\}$$

then it can be easily verified that

$$\theta^*(\mu_A)(x - y) \geq \theta^*(\mu_A)(x) \wedge \theta^*(\mu_A)(y)$$
$$\theta^*(\nu_A)(x - y) \leq \theta^*(\nu_A)(x) \vee \theta^*(\nu_A)(y)$$

and

$$\theta^*(\mu_A)(x.y) \geq \theta^*(\mu_A)(y)$$
$$\theta^*(\nu_A)(x.y) \leq \theta^*(\nu_A)(y)$$

for all $x, y \in R$. Therefore $\theta^*(A)$ is an intuitionistic fuzzy subnear-ring of R. Hence A is an upper rough intuitionistic fuzzy subnear-ring of R.

Theorem 3.7. Let θ be a congruence relation on R then, if A is an intuitionistic fuzzy N-subgroup of R, then A is an upper rough intuitionistic fuzzy N-subgroup of R.

Proof. Since θ is a congruence relation on R $[a]_\theta[b]_\theta \subseteq [ab]_\theta \forall a, b \in R.$
Let $A = (\mu_A, \nu_A)$ be an intuitionistic fuzzy N-subgroup of N and $x \in N$. Now $\theta^*(A) = (\theta^*(\mu_A), \theta^*(\nu_A))$. Thus

(i) a) $\theta^*(\mu_A)(x-y) \geq \theta^*(\mu_A)(x) \wedge \theta^*(\mu_A)(y)$
b) $\theta^*(\nu_A)(x-y) \leq \theta^*(\nu_A)(x) \vee \theta^*(\nu_A)(y)$

(ii) a) $\theta^*(\mu_A)(nx) = \bigvee_{z\in[nx]_\theta} \mu_A(z)$
$\geq \bigvee_{z\in[n]_\theta[x]_\theta} \mu_A(z)$
$= \bigvee_{ab\in[n]_\theta[x]_\theta} \mu_A(ab)$
$\geq \bigvee_{a\in[n]_\theta b\in[x]_\theta} \mu_A(ab)$
$\geq \bigvee_{b\in[x]_\theta} \mu_A(b) = \theta^*(\mu_A)(x)$
b) $\theta^*(\nu_A)(nx) = \bigwedge_{z\in[nx]_\theta} \nu_A(z)$
$\leq \bigwedge_{z\in[n]_\theta[x]_\theta} \nu_A(z)$
$= \bigwedge_{ab\in[n]_\theta[x]_\theta} \nu_A(ab)$
$\leq \bigwedge_{a\in[n]_\theta b\in[x]_\theta} \nu_A(ab)$
$\leq \bigwedge_{b\in[x]_\theta} \nu_A(b) = \theta^*(\nu_A)(x)$

(iii) a) $\theta^*(\mu_A)(xn) \geq \theta^*(\nu_A)(x)$
b) $\theta^*(\nu_A)(xn) \leq \theta^*(\nu_A)(x)$
Proof is analogs to (ii).

Theorem 3.8. Let θ be a complete congruence relation on R then, if A is an intuitionistic fuzzy N-subgroup of R, then A is an lower rough intuitionistic fuzzy N-subgroup of R.

Proof. Since θ is a congruence relation on R $[a]_\theta[b]_\theta = [ab]_\theta \forall a, b \in R.$
Let $A = (\mu_A, \nu_A)$ be an intuitionistic fuzzy N-subgroup of N and $x, \in N$. Now $\theta_*(A) = (\theta_*(\mu_A), \theta_*(\nu_A))$. Thus

(i) a) $\theta_*(\mu_A)(x-y) \geq \theta_*(\mu_A)(x) \vee \theta_*(\mu_A)(y)$
b) $\theta_*(\nu_A)(x-y) \leq \theta_*(\nu_A)(x) \wedge \theta_*(\nu_A)(y)$
Proof of (i) is similar to subnear-rings.

(ii) a) $\theta_*(\mu_A)(nx) = \bigwedge_{z\in[nx]_\theta} \mu_A(z)$
$= \bigwedge_{z\in[n]_\theta[x]_\theta} \mu_A(z)$
$= \bigwedge_{ab\in[n]_\theta[x]_\theta} \mu_A(ab)$
$\geq \bigwedge_{a\in[n]_\theta b\in[x]_\theta} \mu_A(ab)$
$= \bigwedge_{b\in[x]_\theta} \mu_A(b) = \theta_*(\mu_A)(x)$
b) $\theta_*(\nu_A)(nx) = \bigvee_{z\in[nx]_\theta} \nu_A(z)$
$= \bigvee_{z\in[n]_\theta[x]_\theta} \nu_A(z)$
$= \bigvee_{ab\in[n]_\theta[x]_\theta} \nu_A(ab)$
$\leq \bigvee_{a\in[n]_\theta b\in[x]_\theta} \nu_A(ab)$
$= \bigvee_{b\in[x]_\theta} \nu_A(b) = \theta_*(\nu_A)(x)$

(iii) a) $\theta_*(\mu_A)(xn) \geq \theta_*(\nu_A)(x)$
b) $\theta_*(\nu_A)(xn) \leq \theta_*(\nu_A)(x)$
Proof is analogous to (ii).

Theorem 3.9. Let θ be a congruence relation of R, then if A is an intuitionistic fuzzy ideal of R then A is an upper rough intuitionistic fuzzy ideal of R.

Proof. (i) a. $\theta^*(\mu_A(x-y)) \geq \theta^*(\mu_A)(x) \wedge \theta^*(\mu_A)(y)$.
b. $\theta^*(\nu_A(x-y)) \leq \theta^*(\nu_A)(x) \vee \theta^*(\nu_A)(y)$.

(ii) a. $\theta^*(\mu_A(xn)) \geq \theta^*(\mu_A)(x)$
b. $\theta^*(\nu_A(xn)) \leq \theta^*(\nu_A)(x)$

(iii) a. $\theta^*(\mu_A)(y+x-y) = \bigvee_{z\in[y+x-y]_\theta} \mu_A(z)$
$= \bigvee_{z\in[y+x]_\theta-[y]_\theta} \mu_A(z)$
$= \bigvee_{a-d\in[y+x]_\theta-[y]_\theta} \mu_A(a-d)$
$= \bigvee_{a\in[y+x]_\theta, d\in[y]_\theta} \mu_A(a-d)$
$= \bigvee_{d+c\in[y+x]_\theta, d\in[y]_\theta} \mu_A(d+c-d)$
$= \bigvee_{c\in[x]_\theta, d\in[y]_\theta} \mu_A(d+c-d)$
$= \bigvee_{c\in[x]_\theta, d\in[y]_\theta} \mu_A(c)$

$= \bigvee_{c\in[x]_\theta} \mu_A(c) = \theta^*(\mu_A)(x)$.

b. $\theta^*(\nu_A)(y+x-y) = \bigwedge_{z\in[y+x-y]_\theta} \mu_A(z)$

$= \bigwedge_{z\in[y+x]_\theta-[y]_\theta} \nu_A(z)$

$= \bigwedge_{a-d\in[y+x]_\theta-[y]_\theta} \nu_A(a-d)$

$= \bigwedge_{a\in[y+x]_\theta,d\in[y]_\theta} \nu_A(a-d)$

$= \bigwedge_{d+c\in[y+x]_\theta,d\in[y]_\theta} \nu_A(d+c-d)$

$= \bigwedge_{c\in[x]_\theta,d\in[y]_\theta} \nu_A(d+c-d)$

$= \bigwedge_{c\in[x]_\theta,d\in[y]_\theta} \nu_A(c)$

$= \bigwedge_{c\in[x]_\theta} \nu_A(c) = \theta^*(\nu_A)(x)$.

Theorem 3.10. Let θ be a complete congruence relation of R, then if A is an intuitionistic fuzzy ideal of R then A is an lower rough intuitionistic fuzzy ideal of R.

Proof. (i) a. $\theta_*(\mu_A(x-y)) \geq \theta_*(\mu_A)(x) \vee \theta_*(\mu_A)(y)$.

b. $\theta_*(\nu_A(x-y)) \leq \theta_*(\nu_A)(x) \wedge \theta_*(\nu_A)(y.)$

(ii) a. $\theta_*(\mu_A(xn)) \geq \theta_*(\mu_A)(x)$

b. $\theta_*(\nu_A(xn)) \leq \theta_*(\nu_A)(x)$.

(iii) a. $\theta_*(\mu_A)(y+x-y) = \bigwedge_{z\in[y+x-y]_\theta} \mu_A(z)$

$= \bigwedge_{z\in[y+x]_\theta-[y]_\theta} \mu_A(z)$

$= \bigwedge_{a-d\in[y+x]_\theta-[y]_\theta} \mu_A(a-d)$

$= \bigwedge_{a\in[y+x]_\theta,d\in[y]_\theta} \mu_A(a-d)$

$= \bigwedge_{d+c\in[y+x]_\theta,d\in[y]_\theta} \mu_A(d+c-d)$

$= \bigwedge_{c\in[x]_\theta,d\in[y]_\theta} \mu_A(d+c-d)$

$= \bigwedge_{c\in[x]_\theta,d\in[y]_\theta} \mu_A(c)$

$= \bigwedge_{c\in[x]_\theta} \mu_A(c) = \theta_*(\mu_A)(x)$. b. $\theta_*(\nu_A)(y+x-y) = \bigvee_{z\in[y+x-y]_\theta} \mu_A(z)$

$= \bigvee_{z\in[y+x]_\theta-[y]_\theta} \nu_A(z)$

$= \bigvee_{a-d\in[y+x]_\theta-[y]_\theta} \nu_A(a-d)$

$= \bigvee_{a\in[y+x]_\theta,d\in[y]_\theta} \nu_A(a-d)$

$= \bigvee_{d+c\in[y+x]_\theta,d\in[y]_\theta} \nu_A(d+c-d)$

$$= \bigvee_{c\in[x]_\theta, d\in[y]_\theta} \nu_A(d+c-d)$$
$$= \bigvee_{c\in[x]_\theta, d\in[y]_\theta} \nu_A(c)$$
$$= \bigvee_{c\in[x]_\theta} \nu_A(c) = \theta_*(\nu_A)(x).$$

Let N denote an abelian near-ring.

Definition 3.11. Let A be an intuitionistic fuzzy ideal of R. For each $\alpha, \beta \in [0,1]$ with $\alpha+\beta \leq 1$, the set $A^\beta_\alpha = \{(a,b) \in R \times R | \mu_A(a-b) \leq \alpha, \nu_A(a-b) \geq \beta\}$ is called a (α,β)level relation of A.

Proposition 3.12. If A and B be two intuitionistic fuzzy sets of an universal set X, then the following holds

(i) $A^\beta_\alpha \subseteq A^\delta_\gamma$ if $\alpha \geq \gamma$ and $\beta \leq \delta$.

(ii) $A^\beta_{1-\beta} \subseteq A^\beta_\alpha \subseteq A^{1-\alpha}_\alpha$.

(iii) $A \subseteq B \Rightarrow A^\beta_\alpha \subseteq B^\beta_\alpha$.

(iv) $(A \cap B)^\beta_\alpha = A^\beta_\alpha \cap B^\beta_\alpha$.

(v) $(A \cup B)^\beta_\alpha \supseteq A^\beta_\alpha \cup B^\beta_\alpha$.

Proof. (i) Let $(x,y) \in A^\beta_\alpha \Rightarrow \mu_A(x-y) \geq \alpha$ and $\nu_A(x-y) \leq \beta$. Since $\gamma \leq \alpha$ and $\delta \geq \beta \Rightarrow \mu_A(x-y) \geq \alpha \geq \gamma$ and $\nu_A(x-y) \leq \beta \leq \delta \Rightarrow \mu_A(x-y) \geq \gamma$ and $\nu_A(x-y) \leq \delta$. Hence $(x,y) \in A^\delta_\gamma$.

(ii) Since $\alpha+\beta \leq 1 \Rightarrow 1-\beta \geq \alpha$ and $\beta \leq \beta$. Thus $A^\beta_{1-\beta} \subseteq A^\beta_\alpha$. Also $\alpha \geq \alpha$ and $\beta \leq 1-\alpha$. Thus $A^\beta_\alpha \subseteq A^{1-\alpha}_\alpha$.

(iii) Let $(x,y) \in A^\beta_\alpha \Rightarrow \mu_A(x-y) \geq \alpha$ and $\nu_A(x-y) \leq \beta$. As $A \subseteq B \Rightarrow \mu_B(x-y) \geq \alpha$ and $\nu_B(x-y) \leq \beta$ and so $(x,y) \in B^\beta_\alpha$.

(iv) Since $A \cap B \subseteq A$ and $A \cap B \subseteq B$. Thus $(A \cap B)^\beta_\alpha \subseteq A^\beta_\alpha$ and $(A \cap B)^\beta_\alpha \subseteq B^\beta_\alpha$. Thus $(A \cap B)^\beta_\alpha \subseteq A^\beta_\alpha \cap B^\beta_\alpha$. Let $(x,y) \in A^\beta_\alpha \cap B^\beta_\alpha \Rightarrow (x,y) \in A^\beta_\alpha$ and $(x,y) \in B^\beta_\alpha$
$\Rightarrow \mu_A(x-y) \geq \alpha \, and \, \nu_A(x-y) \leq \beta \;\; and \;\; \mu_B(x-y) \geq \alpha \, and \, \nu_B(x-y) \leq \beta$
$\Rightarrow \mu_A(x-y) \geq \alpha \, and \, \mu_B(x-y) \geq \alpha \;\; and \;\; \nu_A(x-y) \leq \beta \, and \, \nu_B(x-y) \leq \beta$
$\Rightarrow \mu_A(x-y) \wedge \mu_B(x-y) \geq \alpha \;\; and \;\; \nu_A(x-y) \vee \nu_B(x-y) \leq \beta,$
$\Rightarrow (\mu_A \cap \mu_B)(x-y) \geq \alpha$ and $(\nu_A \cup \nu_B)(x-y) \leq \beta,$
$\Rightarrow (x,y) \in (A \cap B)^\beta_\alpha$.

(iv) Since $A \subseteq A \cup B$ and $B \subseteq A \cup B \Rightarrow A^\beta_\alpha \subseteq (A \cup B)^\beta_\alpha$ and $B^\beta_\alpha \subseteq (A \cup B)^\beta_\alpha$. Thus $A^\beta_\alpha \cup B^\beta_\alpha \subseteq (A \cup B)^\beta_\alpha$ Now if $\alpha+\beta = 1$ then $(A \cup B)^\beta_\alpha \subseteq A^\beta_\alpha \cup B^\beta_\alpha$. Let $(x,y) \in (A \cup B)^\beta_\alpha \Rightarrow (\mu_A \cup \mu_B)(x-y) \geq \alpha$ and $(\nu_A \cup \nu_B)(x-y) \leq \beta \Rightarrow \mu_A(x-y) \vee \mu_B(x-y) \geq \alpha$ and $\nu_A(x-y) \vee \nu_B(x-y)) \leq \beta$. If $\mu_A(x-y) \geq \alpha$, then $\nu_A(x-y) \leq 1-\mu_A(x-y) \leq 1-\alpha = \beta \Rightarrow (x,y) \in A_\alpha\beta \subseteq A^\beta_\alpha \cup B^\beta_\alpha$. Similarly if $\mu_B(x-y) \geq \alpha$, then $\nu_B(x-y) \leq 1-\mu_B(x-y) \leq 1-\alpha = \beta \Rightarrow (x,y) \in B_\alpha\beta \subseteq A^\beta_\alpha \cup B^\beta_\alpha$. Thus $(A \cup B) = A^\beta_\alpha \cup B^\beta_\alpha$.

Remark 3.13. Every rough ring is a rough near-ring.

Lemma 3.14. Let A be an intuitionistic fuzzy ideal of an abelian near-ring N, and let $\alpha, \beta \in [0,1]$ with $\alpha + \beta \leq 1$. Then A_α^β is a congruence relation on N.

Proof. For any element $a \in N$, $\mu_A(a - a) = \mu_A(0) \geq \alpha$ and so $(a,a) \in A_\alpha^\beta$. If $(a,b) \in A_\alpha^\beta$, then $\mu_A(a-b) \geq \alpha$ and $\nu_A(a-b) \leq \beta$. Since A is an ideal of abelian near-ring N, $\mu_A(b-a) = \mu_A(-(a-b)) = \mu_A(a-b) \geq \alpha, \nu_A(b-a) = \nu_A(-(a-b)) = \nu_A(a-b) \leq \beta \Rightarrow (b,a) \in A_\alpha^\beta$. If $(a,b) \in A_\alpha^\beta$ and $(b,c) \in A_\alpha^\beta$, then since A is a fuzzy ideal of N, $\mu_A(a-c) = \mu_A((a-b)+(b-c)) \geq min\{\mu_A(a-b), \mu_A(b-c)\} \geq min\{\alpha,\alpha\} = \alpha$, and so $(a,c) \in A_\alpha^\beta$. Therefore A_α^β is an equivalence relation on N. Now, let $(a,b) \in A_\alpha^\beta$ and x be any element of N. Then since $\mu_A(a-b) \geq \alpha$ and $\nu_A(a-b) \leq \beta$ $\mu_A((a+x)-(b+x)) = \mu_A((a+x)+(-x-b)) = \mu_A(a+(x-x)-b) = \mu_A(a+0-b), \mu_A(a-b) \geq \alpha, \nu_A((A+x)-(b+x)) = \nu_A((a+x)+(-x-b)) = \nu_A(a+(x-x)-b) = \nu_A(a+0-b)\nu_A(a-b) \leq \beta \Rightarrow (a+x, b+x) \in A_{\alpha,\beta}$. Since (N,+) is an abelian group, we have $(x+a, x+b) \in A_{\alpha,\beta}$. Therefore $A_{\alpha,\beta}$ is a congruence relation on R.

Definition 3.15. Let A be an intuitionisic fuzzy ideal of an abelian near-ring N and $\alpha, \beta \in [0,1]$ with $\alpha+\beta \leq 1$. We know that A_α^β is an equivalence relation(congruence relation) on N. Therefore when the universe is an abelian near-ring then (N, A_α^β) can be used instead of the approximation space (U, θ).

Let A be an intuitionistic fuzzy ideal of N and A_α^β be an (α,β)-level congruence relation of A on N. Let X be a non-empty subset of N. Then the sets

$$\underline{R}(A_\alpha^\beta, X) = \{x \in N | [x]_{A_\alpha^\beta} \subseteq X\}$$
$$\overline{R}(A_\alpha^\beta, X) = \{x \in N | [x]_{A_\alpha^\beta} \cap \neq \phi\}.$$

Proposition 3.16. Let A be an intuitionistic fuzzy ideal of an abelian near-ring N. Then for any $\alpha, \beta \in [0,1]$ with $\alpha+\beta \leq 1$. If B is an ideal of an abelian near-ring R, then A is an upper rough left ideal of N.

Proof. **Proof:** Let $a, b \in \underline{R}(A_\alpha^\beta, B)$. Then $[a]_{A_\alpha^\beta} \subseteq B$ and $[b]_{A_\alpha^\beta} \subseteq B$. Since A_α^β is a congruence relation

$$\begin{aligned} [a+b]_{A_\alpha^\beta} &= [a]_{A_\alpha^\beta} + [b]_{A_\alpha^\beta} \\ &\subseteq B + B \subseteq B \\ &\Rightarrow a+b \in \underline{R}(A_\alpha^\beta, B). \end{aligned}$$

Let a be any element of $\underline{R}(A_\alpha^\beta, B)$. Then $[a]_{A_\alpha^\beta} \subseteq B$. Let x be any element of $[-a]_{A_\alpha^\beta}$. Then $(x, -a) \in A_\alpha^\beta$ and so $(-x, a) \in A_\alpha^\beta$. Thus $-x \in [a]_{A_\alpha^\beta} \subseteq B$.. Since A is an intuitionistic fuzzy ideal of N it is a subgroup of N, thus $x \in A$ and so $[-a]_{A_\alpha^\beta} \subseteq A$. Thus $-a \in \underline{R}(A_\alpha^\beta, B)$. Let a and x be any element of $\underline{R}(A_\alpha^\beta, B)$. Then $[a]_{A_\alpha^\beta} \subseteq B$. Let z be any element of $[x+a-x]_{A_\alpha^\beta}$. Then $(z, (x+a-x)) \in A_\alpha^\beta$. Since A_α^β is a congruence relation on N, $(-x+z+x, a) \in A_\alpha^\beta$ and so $-x+z+x \in [a]_{A_\alpha^\beta} \subseteq B$.Thus

$-x + z + x = b$ for some $b \in B$. Since B is normal $z = x + b - x \in x + B - x \in B$ and so we have $[x + b - x]_{A_\alpha^\beta} \in B$. Therefore $x + b - x \in \underline{R}(A_\alpha^\beta, B)$, which means $\underline{R}(A_\alpha^\beta, B)$ is normal subgroup of N.
Let $r \in R$ and $a \in \underline{R}(A_\alpha^\beta, B)$, then $[a]_{A_\alpha^\beta} \subseteq B$. Let a be any element of $[x]_{A_\alpha^\beta}$. Then $(x, a) \in A_\alpha^\beta$

$$\begin{aligned} &\Rightarrow (rx, ra) \in A_\alpha^\beta \\ &\Rightarrow rx \in [ra]_{A_\alpha^\beta} \subseteq B \\ &\Rightarrow ra \in \underline{R}(A_\alpha^\beta, B). \end{aligned}$$

Lemma 3.17. Let A be an intuitionistic fuzzy ideal of an abelian near-ring N and $\alpha, \beta \in [0, 1]$ where $\alpha + \beta \leq 1$. If $\underline{R}(A_\alpha^\beta, X)$ is a non-empty set then $[0]_{A_\alpha^\beta} \subseteq A$.

Proposition 3.18. Let A be an intuitionistic fuzzy ideal of an abelian near-ring N and $\alpha, \beta \in [0, 1]$ where $\alpha+\beta \leq 1$. Let B be an ideal of N. If $\underline{R}(A_\alpha^\beta, X)$ is a non-empty set then it is equal to A.

Corollary 3.19. If A is an intuitionistic fuzzy ideal of an abelian near-ring then $(\underline{R}(A_\alpha^\beta, X), \overline{R}(A_\alpha^\beta, X))$ is a rough left ideal of N.

4 Conclusion

The idea of rough intuitionistic fuzzy ideals of a near-ring based on its lower and upper approximation is discussed. Some interesting properties are established which hold directly for a ring.We hope that by the extension of this work further we would get new results satisfing more properties for a ring.

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ANOTHER QUASI μs -OPEN AND QUASI μs-CLOSED FUNCTIONS

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Abstaract − The purpose of this paper is to give a new type of open functions called quasi μs-open function. Also, we obtain its characterizations and its basic properties.

Keywords − *Topological spaces, μs-open set, μs-closed set, μs-interior, μs-closure, quasi μs-open function, quasi μs-closed function.*

1 Introduction

Functions and of course open functions stand among the most important notions in the whole of mathematical science. Many different forms of open functions have been introduced over the years. Various interesting problems arise when one considers openness. Its importance is significant in various areas of mathematics and related sciences.

Recently, as a generalization of closed sets, the notion of μs-closed sets were introduced and studied by Veera Kumar [7]. In this paper, we will continue the study of related functions by involving μs-open sets. We introduce and characterize the concept of quasi μs-open functions.

2 Preliminaries

Throughout this paper (X, τ),(Y, σ)and (Z, η) (or X, Y and Z)represent topological spaces on which no separation axioms are assumed unless otherwise mentioned. For a subset A of a space (X,τ), cl(A), int(A) and A^C denote the closure of A, the interior of A and complement of A respectively.

We recall the following definitions which are useful in the sequel.

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Definition 2.1. *A subset A of a space (X, τ) is called:*

1. *α-open set [4] if A$\subseteq$ int(cl(int(A))).*

2. *semi-open set [2] if A$\subseteq$ cl(int(A)).*

The complements of the above mentioned open sets are called their respective closed sets.

The α-closure [4](resp.semi-closure [1]) of a subset A of X, denoted by αcl(A) (resp.scl(A)) is defined to be the intersection of all α-closed (resp. semi-closed) sets of (X, τ) containing A.

Definition 2.2. *A subset A of a space (X, τ) is called:*

1. *a gα*-closed set [3, 5] if αcl(A)$\subseteq$int(U) whenever A$\subseteq$U and U is α-open in (X, τ). The complement of gα*-closed set is called gα*-open set.*

2. *a μ-closed set [6] if cl(A)$\subseteq$U whenever A$\subseteq$U and U is gα*-open in (X, τ). The complement of μ-closed set is called μ-open set.*

3. *a μs-closed set [7] if scl(A)$\subseteq$U whenever A$\subseteq$U and U is gα*-open in (X, τ). The complement of μs-closed set is called μs-open set.*

The union (resp. intersection) of all μs-open (resp. μs-closed) sets, each contained in (resp. containing) a set A in a space X is called the μs-interior(resp. μs-closure) of A and is denoted by μs-int(A)(resp. μs-cl(A)).

Definition 2.3. *[7] A function f : (X, τ)$\rightarrow$(Y, σ) is called a μs-irresolute (resp. μs-continuous)if f^{-1}(V) is is μs-closed in X for every μs-closed (resp. closed) subset V of Y.*

Definition 2.4. *A function f : (X, τ)$\rightarrow$(Y, σ) is called a μs-open (resp. μs-closed) if f(V) is μs-open (resp. μs-closed) in Y for every open (resp. closed) subset V of X.*

3 Quasi μs-open Functions

We introduce a new definitions as follows.

Definition 3.1. *A function f : X $\rightarrow$ Y is said to be quasi μs-open if the image of every μs-open set in X is open in Y.*
It is evident that, the concepts quasi μs-openness and μs-continuity coincide if the function is a bijection.

Theorem 3.2. *A function f : X $\rightarrow$ Y is quasi μs-open if and only if for every subset U of X, f(μs-int(U))$\subset$ int(f(U)).*

Proof: Let f be a quasi μs-open function. Now, we have int(U)$\subset$ U and μs-int(U) is a μs-open set. Hence, we obtain that f(μs-int(U))$\subset$ f(U). As f (μs-int(U)) is open, f (μs-int(U))$\subset$ int(f(U)).

Conversely, assume that U is a μs-open set in X. Then, f(U) = f(μs-int(U)) $\subset$ int(f(U)) but int(f(U))$\subset$ f(U). Consequently, f(U) = int(f(U)) and hence f is quasi μs-open.

Lemma 3.3. *If a function $f : X \rightarrow Y$ is quasi μs-open, then μs-int($f^{-1}(G)$) $\subset$ f^{-1}(int(G)) for every subset G of Y.*

Proof: Let G be any arbitrary subset of Y. Then, μs-int(f^{-1}(G)) is a μs-open set in X and f is quasi μs-open, then f(μs-int(f^{-1}(G)))$\subset$ int(f(f^{-1}(G))) $\subset$ int(G). Thus, μs-int(f^{-1}(G)) $\subset$ f^{-1}(int(G)).

Definition 3.4. *A subset S of a space (X, τ) is called a μs-neighbourhood of a point x of X if there exists a μs-open set U such that $x \in U \subset S$.*

Theorem 3.5. *For a function $f : X \rightarrow Y$, the following are equivalent: (i) f is quasi μs-open; (ii) For each subset U of X, f(μs-int(U))$\subset$ int(f(U)); (iii) For each $x \in X$ and each μs-neighbourhood U of x in X, there exists a neighbourhood f(U) of f(x) in Y such that f(V) $\subset$ f(U).*

(i) $\Rightarrow$ (ii): It follows from Theorem 3.2.

(ii) $\Rightarrow$(iii): Let x $\in$ X and U be an arbitrary μs-neighbourhood of x in X. Then there exists a μs-open set V in X such that x $\in$ V $\subset$ U. Then by (ii), we have f (V) = f(μs-int(V)) $\subset$ int(f(V)) and hence f(V) = int(f(V)). Therefore, it follows that f(V) is open in Y such that f(x)$\in$ f (V)$\subset$ f (U).

(iii)$\Rightarrow$ (i): Let U be an arbitrary μs-open set in X. Then for each y $\in$ f (U), by (iii) there exists a neighbourhood Vy of y in Y such that Vy $\subset$ f (U). As Vy is a neighbourhood of y, there exists an open set Wy in Y such that y $\in$ Wy $\subset$ Vy . Thus f (U) = $\cup$ {Wy : y $\in$ f (U)} which is an open set in Y. This implies that f is quasi μs-open function.

Theorem 3.6. *A function $f : X \rightarrow Y$ is quasi μs-open if and only if for any subset B of Y and for any μs-closed set F of X containing f^{-1}(B), there exists a closed set G of Y containing B such that f^{-1}(G) $\subset$ F.*

Proof: Suppose f is quasi μs-open. Let B $\subset$ Y and F be a μs-closed set of X containing f^{-1}(B). Now, put G = Y - f (X - F). It is clear that f^{-1}(B) $\subset$ F implies B $\subset$ G. Since f is quasi μs-open, we obtain G as a closed set of Y. Moreover, we have f^{-1}(G)$\subset$ F.

Conversely, let U be a μs-open set of X and put B = Y \f (U). Then X \U is a μs-closed set in X containing f^{-1}(B). By hypothesis, there exists a closed set F of Y such that B $\subset$ F and f^{-1}(F) $\subset$ X \U. Hence, we obtain f(U) $\subset$ Y \F. On the other hand, it follows that B $\subset$ F, Y \F $\subset$ Y \B = f(U). Thus, we obtain f(U) = Y \F which is open and hence f is a quasi μs-open function.

Theorem 3.7. *A function $f : X \rightarrow Y$ is quasi μs-open if and only if f^{-1}(cl(B)) $\subset$ μs-cl(f^{-1}(B)) for every subset B of Y.*

Proof: Suppose that f is quasi μs-open. For any subset B of Y , f^{-1}(B)$\subset$ μs-cl(f^{-1}(B)). Therefore by Theorem 3.6, there exists a closed set F in Y such that B $\subset$ F and f^{-1}(F)$\subset$ μs-cl(f^{-1}(B)). Therefore, we obtain f^{-1}(cl(B)) $\subset$ f^{-1}(F)$\subset$ μs-cl(f^{-1}(B)).

Conversely, let B $\subset$ Y and F be a μs-closed set of X containing f^{-1}(B). Put W = cl_Y(B), then we have B $\subset$ W and W is closed and f^{-1}(W)$\subset$ μs-cl(f^{-1}(B))$\subset$ F. Then by Theorem 3.6, f is quasi μs-open.

Lemma 3.8. *Let $f : X \to Y$ and $g : Y \to Z$ be two functions. Given that $g \circ f : X \to Z$ is quasi μs-open. If g is continuous injective, then f is quasi μs-open.*

Proof: Let U be a μs-open set in X. Then (g o f)(U) is open in Z since g o f is quasi μs-open. Again g is an injective continuous function, f (U) = g^{-1}(g o f (U)) is open in Y . This shows that f is quasi μs-open.

4 Quasi μs-Closed Functions

Definition 4.1. *A function $f : X \to Y$ is said to be quasi μs-closed if the image of each μs-closed set in X is closed in Y.*

Clearly, every quasi μs-closed function is closed as well as μs-closed.

Remark 4.2. *Every μs-closed (resp. closed) function need not be quasi μs-closed as shown by the following example.*

Example 4.3. *Let $X = Y = \{a, b, c\}$ with $\tau = \{\phi, \{a, b\}, X\}$ and $\sigma = \{\phi, \{a\}, \{b, c\}, Y\}$. Let $f : (X, \tau) \to (Y, \sigma)$ by f(a)=b, f(b)=c and f(c)=a. Then clearly f is μs-closed as well as closed but not quasi μs-closed.*

Lemma 4.4. *If a function $f : X \to Y$ is quasi μs-closed, then $f^{-1}(int(B)) \subset$ μs-int($f^{-1}(B)$) for every subset B of Y.*

Proof: This proof is similar to the proof of Lemma 3.3.

Theorem 4.5. *A function $f : X \to Y$ is quasi μs-closed if and only if for any subset B of Y and for any μs-open set G of X containing $f^{-1}(B)$, there exists an open set U of Y containing B such that $f^{-1}(U) \subset G$.*

Proof: This proof is similar to that of Theorem 3.6.

Definition 4.6. *A function $f : X \to Y$ is called μs*-closed if the image of every μs-closed subset of X is μs-closed in Y.*

Theorem 4.7. *If $f : X \to Y$ and $g : Y \to Z$ are two quasi μs-closed functions, then $g \circ f: X \to Z$ is a quasi μs-closed function.*

Proof. Obvious.
Furthermore, we have the following theorem

Theorem 4.8. *Let $f : X \to Y$ and $g : Y \to Z$ be any two functions. Then: (i) if f is μs-closed and g is quasi μs-closed, then g o f is closed; (ii) if f is quasi μs-closed and g is μs-closed, then g o f is μs* -closed; (iii) if f is μs* -closed and g is quasi μs-closed, then g o f is quasi μs-closed.*

Proof. Obvious.

Theorem 4.9. *Let $f : X \to Y$ and $g : Y \to Z$ be two functions such that $g \circ f : X \to Z$ is quasi μs-closed. Then: (i) if f is μs-irresolute surjective, then g is closed. (ii) if g is μs-continuous injective, then f is μs*-closed.*

Proof: (i) Suppose that F is an arbitrary closed set in Y. As f is μs-irresolute, $f^{-1}(F)$ is μs-closed in X. Since g o f is quasi μs-closed and f is surjective, (g o f(f^{-1}(F))) = g(F), which is closed in Z. This implies that g is a closed function. (ii) Suppose F is any μs-closed set in X. Since g o f is quasi μs-closed, (g o f)(F) is closed in Z. Again g is a μs-continuous injective function, g^{-1}(g o f(F)) = f (F), which is μs-closed in Y. This shows that f is μs*-closed.

Theorem 4.10. *Let X and Y be topological spaces. Then the function $g : X \to Y$ is a quasi μs-closed if and only if g(X) is closed in Y and $g(V) \setminus g(X \setminus V)$ is open in g(X) whenever V is μs-open in X.*

Proof: Necessity: Suppose $g : X \to Y$ is a quasi μs-closed function. Since X is μs-closed, g(X) is closed in Y and $g(V) \setminus g(X \setminus V) = g(V) \cap g(X) \setminus g(X \setminus V)$ is open in g(X) when V is μs-open in X.
Sufficiency: Suppose g(X) is closed in Y, $g(V) \setminus g(X \setminus V)$ is open in g(X) when V is μs-open in X, and let C be closed in X. Then $g(C) = g(X) \setminus (g(X \setminus C) \setminus g(C))$ is closed in g(X) and hence, closed in Y.

Corollary 4.11. *Let X and Y be topological spaces. Then a surjective function $g : X \to Y$ is quasi μs-closed if and only if $g(V) \setminus g(X \setminus V)$ is open in Y whenever V is μs-open in X.*

Proof: Obvious.

Corollary 4.12. *Let X and Y be topological spaces and let $g : X \to Y$ be a μs-continuous quasi μs-closed surjective function. Then the topology on Y is $\{g(V) \setminus g(X \setminus V) : V$ is μs-open in X$\}$.*

Proof: Let W be open in Y. Then $g^{-1}(W)$ is μs-open in X, and $g(g^{-1}(W)) \setminus g(X \setminus g^{-1}(W)) = W$. Hence, all open sets in Y are of the form $g(V) \setminus g(X \setminus V)$, V is μs-open in X. On the other hand, all sets of the form $g(V) \setminus g(X \setminus V)$, V is μs-open in X, are open in Y from Corollary 4.11.

Definition 4.13. *A topological space (X, τ) is said to be μs*-normal if for any pair of disjoint μs-closed subsets F_1 and F_2 of X, there exist disjoint open sets U and V such that $F_1 \subset U$ and $F_2 \subset V$.*

Theorem 4.14. *Let X and Y be topological spaces with X is μs*-normal. If $g : X \to Y$ is a μs-continuous quasi μs-closed surjective function, then Y is normal.*

Proof: Let K and M be disjoint closed subsets of Y . Then $g^{-1}(K)$, $g^{-1}(M)$ are disjoint μs-closed subsets of X. Since X is μs*-normal, there exist disjoint open sets V and W such that $g^{-1}(K) \subset V$ and $g^{-1}(M) \subset W$. Then $K \subset g(V) \setminus g(X \setminus V)$ and $M \subset g(W) \setminus g(X \setminus W)$. Further by Corollary 4.11, $g(V) \setminus g(X \setminus V)$ and $g(W) \setminus g(X \setminus W)$ are open sets in Y and clearly $(g(V) \setminus g(X \setminus V)) \cap (g(W) \setminus g(X \setminus W)) = \phi$. This shows that Y is normal.

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BIPOLAR FUZZY HYPER *KU*–IDEALS (SUB ALGEBRAS)

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Abstract – In this paper, using the notion of bipolar-valued fuzzy set, we establish the bipolar fuzzification the notion of (strong, weak, s-weak) hyper KU-ideals in hyper KU-algebras, and investigate some of their properties.

Keywords – *KU-algebra, hyper KU-algebra, fuzzy hyper KU-ideal.*

1. Introduction

Prabpayak and Leerawat [13,14] introduced a new algebraic structure which is called KU-algebras. They studied ideals and congruences in KU-algebras. Also, they introduced the concept of homomorphism of KU-algebra and investigated some related properties. Moreover, they derived some straightforward consequences of the relations between quotient KU-algebras and isomorphism. Mostafa et. al. [10] introduced the notion of fuzzy KU-ideals of KU-algebras and then they investigated several basic properties which are related to fuzzy KU-ideals. The hyper structure theory (called also multi-algebras) is introduced in 1934 by Marty [9] at the 8th congress of Scandinvian Mathematiciens. Around the 40's, several authors worked on hyper groups, especially in France and in the United States, but also in Italy, Russia and Japan. Hyper structures have many applications to several sectors of both pure and applied sciences. Jun and Xin [3,6] considered the fuzzification of the notion of a (weak, strong, reflexive) hyper BCK-ideal, and investigated the relations among them. Mostafa et. al. [11] applied the hyper structures to KU- algebras and introduced the concept of a hyper KU-algebra which is a generalization of a KU-algebra, and investigated some related properties. They also introduced the notion of a

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hyper KU-ideal, a weak hyper KU-ideal and gave relations between hyper KU-ideals and weak hyper KU-ideals. In 1956, Zadeh [10] introduced the notion of fuzzy sets. At present this concept has been applied to many mathematical branches. There are several kinds of fuzzy sets extensions in the fuzzy set theory, for example, intuitionistic fuzzy sets, interval valued fuzzy sets, vague sets etc.[see 1,3.5,6,12]. Mostafa et al.[12] , stated and proved more several theorems of hyper KU-algebras and studied fuzzy set theory to the hyper KU-sub algebras (ideals). Lee [8] introduced an extension of fuzzy sets named bipolar-valued fuzzy sets. Bipolar-valued fuzzy sets are an extension of fuzzy sets whose membership degree range is enlarged from the interval [0, 1] to [-1, 1]. The authors in [1, 2, 6 and 9], introduced bipolar-valued fuzzy set on different algebraic structures. In this paper, the bipolar fuzzy set theory to the (s-weak-strong) hyper KU-ideals in hyper KU-algebras are applied and discussed.

2. Preliminaries

Let H be a nonempty set and $P^*(H) = P(H)\setminus\{\phi\}$ the family of the nonempty subsets of H. A multi valued operation (said also hyper operation) "$\circ$" on H is a function, which associates with every pair $(x,y)\in H\times H = H^2$ a non empty subset of H denoted $x\circ y$ .An algebraic hyper structure or simply a hyper structure is a non empty set H endowed with one or more hyper operations.

Definition 2.1 **[11,12]** Let H be a nonempty set and "$\circ$" a hyper operation on H, such that $\circ : H\times H \to P^*(H)$. Then H is called a hyper KU-algebra if it contains a constant "0 " and satisfies the following axioms: for all $x,y,z\in H$

(HKU_1) $[(y\circ z)\circ(x\circ z)] << x\circ y$

(HKU_2) $x\circ 0 = \{0\}$

(HKU_3) $0\circ x = \{x\}$

(HKU_4) *if* $x << y,\ y << x$ *implies* $x = y$

where x << y is defined by $0\in y\circ x$ and for every $A,B\subseteq H$, $A << B$ is defined by $\forall a\in A, \exists b\in B$ *such that* $a << b$. In such case, we call “<<” the hyper order in H .

We shall use the $x\circ y$ instead of $x\circ\{y\}$, $\{x\}\circ y$ or $\{x\}\circ\{y\}$. Note if $A,B\subseteq H$, then by $A\circ B$ we mean the subset $\bigcup_{a\in A, b\in B} a\circ b$ *of* H .

Example 2.2. (A) Let $H = \{0,1,2\}$ be a set. Define hyper operation $\circ$ on H as follows:

$\circ$	0	1	2
0	{0}	{1}	{2}
1	{0}	{0,1}	{1,2}
2	{0}	{0,1}	{0,1,2}

Then $(H,\circ,0)$ is a hyper KU-algebra.

In what follows, H denotes a hyper KU-algebra unless otherwise specified.

Lemma 2.3. **[11,12]** For all $x, y \in H$ *and* $A \subseteq H$

(i) $A \circ (y \circ x) = y \circ (A \circ x)$
(ii) $(0 \circ x) \circ x = \{0\}$

Proposition 2.4. **[12]** In any hyper KU-algebra H, $0 \circ x = \{x\}\ \forall\ x \in H$

Theorem 2.5. **[12]** For all $x, y, z \in H$ *and* $A, B, C \subseteq H$

(i) $x \circ y << z \Rightarrow z \circ y << x$
(ii) $x \circ y << y$
(iii) $x << 0 \circ x$
(iv) $A << B$, $B << C \Rightarrow A << C$
(v) $x \circ A << A$
(vi) $A \circ x << z \Leftrightarrow z \circ x << A$.
(vii) $A << B \Rightarrow C \circ A << C \circ B$ *and* $B \circ C << A \circ C$
(viii) $A << 0 \circ A$
(ix) $x \in 0 \circ x$
(x) $x \in 0 \circ 0 \Leftrightarrow x = 0$
(xi) $x \circ x = \{x\} \Leftrightarrow x = 0$

Lemma 2.6. **[11]** In hyper KU-algebra $(H, \circ, 0)$, we have

$$z \circ (y \circ x) = y \circ (z \circ x) \text{ for all } x, y, z \in H .$$

Definition2.7. **[12]** Let S be a non-empty subset of a hyper KU-algebra H. Then S is said to be a hyper sub-algebra of H if $S_2 : x \circ y \subseteq S, \forall x, y \in S$

Proposition 2.8. **[12]** Let S be a non-empty subset of a hyper KU-algebra $(H, \circ, 0)$. If y ∘ x ⊆ S for all x, y ∈ S, then 0 ∈ S.

Theorem 2.9. **[12]** Let S be a non-empty subset of a hyper KU-algebra $(H, \circ, 0)$. Then S is a hyper subalgebra of H if and only if y ∘ x ⊆ S for all x, y ∈ S.

Definition 2.10 **[11].** Let I be a non-empty subset of a hyper KU-algebra H and $0 \in I$. Then

(1) I is said to be a weak hyper KU- ideal of H if $x \circ (y \circ z) \subseteq I$ and $x \in I$ imply $y \circ z \in I$, for all $x, y, z \in H$,
(2) I is said to be hyper KU-ideal of H if $x \circ (y \circ z) << I$ and $x \in I$ imply $y \circ z \in I$, for all $x, y, z \in H$
(3) I is said a strong hyper KU-ideal of H if $x \circ (y \circ z) \cap I \neq \Phi$ and $x \in I$ imply $y \circ z \in I$, for all $x, y, z \in H$.
(4) I is said to be reflexive if $x \circ x \subseteq I$ for all $x \in H$.

Definition 2.11. **[11].** Let A be a non-empty subset of a hyper KU-algebra H. Then A is said to be a hyper ideal of H if

(HI_1) $0 \in A$,
(HI_2) $y \circ x << A$ and $y \in A$ imply $x \in A$ for all $x, y \in H$.

Definition 2.12. **[11]** A non-empty set A of a hyper KU-algebra H is called a distributive hyper ideal if it satisfies (HI_1) and

(HI_3) $(z \circ y) \circ (z \circ (z \circ x)) << A$ and $y \in A$ imply $x \in A$.

Definition 2.13. **[11,12]** Let I be a non-empty subset of a hyper KU-algebra H and $0 \in I$. Then,

(1) I is called a weak hyper ideal of H if $y \circ x \subseteq I$ and $y \in I$ imply that $x \in I$, for all $x, y \in H$.
(2) I is called a strong hyper ideal of H if $(y \circ x) \cap I \neq \phi$ and $y \in I$ imply that $x \in I$, for all $x, y \in H$.

Lemma 2.14. **[12]** Let A be a subset of a hyper KU -algebra H . If I is a hyper ideal of H such that $A \ll I$ then $A \subseteq I$.

Lemma 2.15. **[12]** In hyper KU-algebra $(H, \circ, 0)$, we have :

(i) Any strong hyper KU- ideal of *H* is a hyper ideal of *H*.
(ii) Any weak hyper KU-ideal of *H* is a weak ideal of *H*.

Definition2.7. **[8]** A bipolar valued fuzzy subset B in a nonempty set X is an object having the form $\phi = (H, \mu_\Phi{}^P, \mu_\Phi{}^N)$ where $\mu^N : X \to [-1,0]$ and $\mu^P : X \to [0,1]$ are mappings. The positive membership degree $\mu^P(x)$ denotes the satisfaction degree of an element x to the property corresponding to a bipolar-valued fuzzy set $\phi = (H, \mu_\Phi{}^P, \mu_\Phi{}^N)$, and the negative membership degree $\mu^N(x)$ denotes the satisfaction degree of x to some implicit counter-property of a bipolar-valued fuzzy set $\phi = (H, \mu_\Phi{}^P, \mu_\Phi{}^N)$. For simplicity, we shall use the symbol $\phi = (\mu_\Phi{}^P, \mu_\Phi{}^N)$ for bipolar fuzzy set $\phi = (H, \mu_\Phi{}^P, \mu_\Phi{}^N)$, and use the notion of bipolar fuzzy sets instead of the notion of bipolar-valued fuzzy sets.

3. Bipolar Fuzzy hyper KU – subalgebras (ideals)

Now some fuzzy logic concepts are reviewed .A fuzzy set μ in a set H is a function $\mu : H \to [0,1]$. A fuzzy set μ in a set H is said to satisfy the inf (resp. sup) property if for any subset T of H there exists $x_0 \in T$ such that $\mu(x_0) = \inf_{x \in T} \mu(x)$ (resp. $\mu(x_0) = \sup_{x \in T} \mu(x)$).

Definition 3.1. A fuzzy set $\phi = (H, \mu_\Phi{}^P, \mu_\Phi{}^N)$ in H is said to be bipolar fuzzy hyper KU-subalgebra of H if it satisfies the following inequalities:

(1) $\inf_{z \in x \circ y} \mu_\Phi{}^P(z) \geq \min\{\mu_\Phi{}^P(x), \mu_\Phi{}^P(y)\}$.

(2) $\sup_{w \in x \circ y} \mu_\Phi{}^N(w) \leq \max\{\mu_\Phi{}^N(x), \mu_\Phi{}^N(y)\} \forall x, y \in H$.

Proposition 3.2. Let $\phi = (H, \mu_\Phi{}^P, \mu_\Phi{}^N)$ be a bipolar fuzzy hyper KU-sub-algebra of H. Then $\mu_\Phi{}^P(0) \geq \mu_\Phi{}^P(x)$ *and* $\mu_\Phi{}^N(0) \leq \mu_\Phi{}^N(x)$ for all $\forall x \in H$

Proof. Using Proposition 2.5 (xi) , we see that $0 \in x \circ x$ for all $x \in H$. Hence

$$\inf_{0 \in x \circ x} \mu_\Phi{}^P(0) \geq \min\{\mu_\Phi{}^P(x), \mu_\Phi{}^P(x)\} = \mu_\Phi{}^P(x)$$

and

$$\sup_{0 \in x \circ x} \mu_\Phi{}^N(0) \leq \max\{\mu_\Phi{}^N(x), \mu_\Phi{}^N(x)\} = \mu_\Phi{}^N(x) \quad \text{for all } x \in H .$$

Example 3.3 **.**Let $H = \{0,1,2,3\}$ be a set. The hyper operations $\circ$ on H are defined as follows.

$\circ_2$	0	1	2	3
0	{0}	{1}	{2}	{3}
1	{0}	{0}	{1}	{3}
2	{0}	{0}	{0,1}	{0,3}
3	{0}	{0}	{1}	{0,3}

Then $(H, \circ, 0)$ is a hyper KU-algebra. Define $\mu^N : X \to [-1,0]$ and $\mu^P : X \to [0,1]$ by

	0	1	2	3
μ^N	**-0.7**	**-0.7**	**0.6**	**0.4**
μ^P	**0.6**	**0.5**	**0.3**	**0.3**

By routine calculations, we know that $\Phi = (H, \mu^N, \mu^P)$ is bipolar fuzzy hyper sub-algebra of H .

Definition 3.4. For a “hyper KU-algebra” H , a “a bipolar fuzzy set” $\phi = (H, \mu_\Phi{}^P, \mu_\Phi{}^N)$ in H is called:

- BHFI: Bipolar fuzzy hyper ideal of H ,if

$$F_1 : x << y \text{ implies } \mu_\Phi{}^P(x) \geq \mu_\Phi{}^P(y), \quad \mu_\Phi{}^N(x) \leq \mu_\Phi{}^N(y)$$

and

$$F_2 : \mu_\Phi{}^P(z) \geq \min\left\{ \inf_{u\in((y\circ z)} \mu_\Phi{}^P \mu(u), \mu_\Phi{}^P(y) \right\}$$

$$F_3 : \mu_\Phi{}^N(w) \leq \max\left\{ \sup_{w\in((y\circ z)} \mu_\Phi{}^N(w), \mu_\Phi{}^N(y) \right\}$$

- B *FWH* :Bipolar fuzzy weak hyper ideal of H if, for any y; z $\in H$

$$\mu_\Phi{}^P(0) \geq \mu_\Phi{}^P(z) \geq \min\left\{ \inf_{u\in\,(y\circ z)} \mu_\Phi{}^P(u), \mu_\Phi{}^P(y) \right\}$$

and

$$\mu_\Phi{}^N(0) \leq \mu_\Phi{}^N(w) \leq \max\left\{ \sup_{w\in\,(y\circ z)} \mu_\Phi{}^N(w), \mu_\Phi{}^N(y) \right\}$$

- B *FS* H : Bipolar fuzzy strong hyper ideal of H if, for any y; z $\in H$

$$\inf_{u\in(y\circ z)} \mu_\Phi{}^P(u) \geq \mu_\Phi{}^P(z) \geq \min\left\{ \sup_{u\in(y\circ z)} \mu_\Phi{}^P(u), \mu_\Phi{}^P(y) \right\}$$

and

$$\sup_{w\in(y\circ z)} \mu_\Phi{}^N(w) \leq \mu_\Phi{}^N(z) \leq \max\left\{ \inf_{w\in(y\circ z)} \mu_\Phi{}^N(w), \mu_\Phi{}^N(y) \right\}$$

Definition 3.5. For a "hyper KU-algebra" H , a "bipolar fuzzy set" $\phi = (H, \mu_\Phi{}^P, \mu_\Phi{}^N)$ in H is called :

(I) Bipolar fuzzy hyper KU-ideal of H ,if

$$x << y \text{ implies } \mu_\Phi{}^P(x) \geq \mu_\Phi{}^P(y), \quad \mu_\Phi{}^N(x) \leq \mu_\Phi{}^N(y) ,$$
$$\mu_\Phi{}^P(x\circ z) \geq \min\left\{ \inf_{u\in(x\circ(y\circ z)} \mu_\Phi{}^P(u) , \mu_\Phi{}^P(y) \right\}$$

and

$$\mu_\Phi{}^N(x\circ z) \leq \max\left\{ \sup_{w\in(x\circ(y\circ z)} \mu_\Phi{}^N(w) , \mu_\Phi{}^N(y) \right\}$$

(II) Bipolar fuzzy weak hyper KU-ideal of H , if for any x; y; z $\in H$

$$\mu_\Phi{}^P(0) \geq \mu_\Phi{}^P(x\circ z) \geq \min\left\{ \inf_{u\in(x\circ(y\circ z)} \mu_\Phi{}^P(u), \mu_\Phi{}^P(y) \right\}$$

and

$$\mu_\Phi{}^N(0) \le \mu_\Phi{}^N(x \circ z) \le \max\left\{ \sup_{w\in(x\circ(y\circ z)} \mu_\Phi{}^N(w), \mu_\Phi{}^N(y) \right\}$$

(III) Bipolar fuzzy strong hyper KU-ideal of H if, for any x; y; z ∈ H

$$\inf_{u\in x\circ(y\circ z)} \mu_\Phi{}^P(u) \ge \mu_\Phi{}^P(x \circ z) \ge \min\left\{ \sup_{u\in x\circ(y\circ z)} \mu_\Phi{}^P(u), \mu_\Phi{}^P(y) \right\}$$

and

$$\sup_{w\in x\circ(y\circ z)} \mu_\Phi{}^N(w) \le \mu_\Phi{}^N(x \circ z) \le \max\left\{ \inf_{w\in x\circ(y\circ z)} \mu_\Phi{}^N(w), \mu_\Phi{}^N(y) \right\}$$

Example 3.6. (1) Consider the hyper KU -algebra in Example 2.2. Define bipolar fuzzy set $\phi = (H, \mu_\Phi{}^P, \mu_\Phi{}^N)$ in H by

	0	1	2
μ^N	**- 0.7**	**- 0.7**	**- 0.6**
μ^P	**1**	**0.5**	**0**

Then we can see that $\phi = (H, \mu_\Phi{}^P, \mu_\Phi{}^N)$ is bipolar fuzzy(bipolar fuzzy weak) hyper KU -ideal of *H*.

Example 3.7. Consider the hyper KU -algebra H

$\circ$	0	1	2
0	$\{0\}$	$\{1\}$	$\{2\}$
1	$\{0\}$	$\{0\}$	$\{2\}$
2	$\{0\}$	$\{1\}$	$\{0,2\}$

Define bipolar fuzzy set $\phi = (H, \mu_\Phi{}^P, \mu_\Phi{}^N)$ in H by

	0	1	2
μ^N	**- 0.8**	**- 0.6**	**- 0.2**
μ^P	**0.9**	**0.5**	**0.3**

It is easily verified that $\phi = (H, \mu_\Phi{}^P, \mu_\Phi{}^N)$ is bipolar fuzzy strong hyper KU -ideal of H.

Theorem 3.8. Any bipolar fuzzy (weak, strong) hyper KU-ideal is a bipolar fuzzy (weak, strong) hyper ideal.

Proof. Let $\phi = (H, \mu_\Phi{}^P, \mu_\Phi{}^N)$ be a bipolar fuzzy weak hyper KU-ideal of *H*, we get for any x; y; z $\in H$

$$\mu_\Phi{}^P(0) \geq \mu_\Phi{}^P(x \circ z) \geq \min\left\{ \inf_{u \in x \circ (y \circ z)} \mu_\Phi{}^P(u), \mu_\Phi{}^P(y) \right\} \quad \text{(a)}$$

$$\mu_\Phi{}^N(0) \leq \mu_\Phi{}^N(x \circ z) \leq \max\left\{ \sup_{w \in (x \circ (y \circ z)} \mu_\Phi{}^N(w), \mu_\Phi{}^N(y) \right\} \quad \text{(b)}$$

Put x = 0 in (a) and (b) , we get

$$\mu_\Phi{}^P(0) \geq \mu_\Phi{}^P(0 \circ z) \geq \min\left\{ \inf_{u \in 0 \circ (y \circ z)} \mu_\Phi{}^P(u), \mu_\Phi{}^P(y) \right\} \Rightarrow$$
$$\mu_\Phi{}^P(0) \geq \mu_\Phi{}^P(z) \geq \min\left\{ \inf_{u \in (y \circ z)} \mu_\Phi{}^P(u), \mu_\Phi{}^P(y) \right\}$$

and

$$\mu_\Phi{}^N(0) \leq \mu_\Phi{}^N(0 \circ z) \leq \max\left\{ \sup_{w \in 0 \circ (y \circ z)} \mu_\Phi{}^N(w), \mu_\Phi{}^N(y) \right\} \Rightarrow$$
$$\mu_\Phi{}^N(0) \leq \mu_\Phi{}^N(z) \leq \max\left\{ \sup_{w \in (y \circ z)} \mu_\Phi{}^N(w), \mu_\Phi{}^N(y) \right\}.$$

Similarly we can prove that , every bipolar fuzzy strong hyper KU-ideal of *H* is bipolar fuzzy strong hyper ideal of *H.* Ending the proof.

Definition 3.9. A bipolar fuzzy set $\phi = (H, \mu_\Phi{}^P, \mu_\Phi{}^N)$ in *H* is called bipolar fuzzy s-weak hyper KU-ideal of H if

(i) $\mu_\Phi{}^P(0) \geq \mu_\Phi{}^P(x)$, $\mu_\Phi{}^N(0) \leq \mu_\Phi{}^N(x)$, $\forall x \in H$
(ii) for every $x, y, z \in H$ there exists $a, b \in x \circ (y \circ z)$ such that

$$\mu_\Phi{}^P(x \circ z) \geq \min\{\mu_\Phi{}^P(a), \mu_\Phi{}^P(y)\} \text{ and } \mu_\Phi{}^N(x \circ z) \leq \max\{\mu_\Phi{}^N(b), \mu_\Phi{}^N(y)\}$$

Theorem 3.10. Every bipolar fuzzy s- weak hyper KU-ideal of *H* is bipolar fuzzy weak hyper KU-ideal of *H*.

Proof. Let $\phi = (H, \mu_\Phi{}^P, \mu_\Phi{}^N)$ be a bipolar fuzzy s-weak hyper KU-ideal of *H*, and let x; y; z $\in H$,then there exist $a, b \in x \circ (y \circ z)$ such that

$$\mu_\Phi{}^P(x \circ z) \geq \min\{\mu_\Phi{}^P(a), \mu_\Phi{}^P(y)\} \text{ and } \mu_\Phi{}^N(x \circ z) \leq \max\{\mu_\Phi{}^N(b), \mu_\Phi{}^N(y)\}$$

Since $\mu_\Phi{}^P(0) \geq \inf_{c\in(y\circ z)} \mu_\Phi{}^P(c)$ and $\mu_\Phi{}^N(0) \leq \sup_{d\in(y\circ z)} \mu_\Phi{}^N(d)$, it follows that

$$\mu_\Phi{}^P(x\circ z) \geq \min\left\{ \inf_{c\in x\circ(y\circ z)} \mu_\Phi{}^P(c) \, , \, \mu_\Phi{}^P(y) \right\}$$

and

$$\mu_\Phi{}^N(x\circ z) \leq \max\left\{ \sup_{d\in x\circ(y\circ z)} \mu_\Phi{}^N(d) \, , \, \mu_\Phi{}^N(y) \right\}.$$

Hence $\phi = (H, \mu_\Phi{}^P, \mu_\Phi{}^N)$ is a bipolar fuzzy weak hyper KU-ideal of *H*

Proposition 3.11. If $\phi = (H, \mu_\Phi{}^P, \mu_\Phi{}^N)$ is bipolar fuzzy weak hyper KU-ideal of *H*. satisfying the inf-sup property, then $\phi = (H, \mu_\Phi{}^P, \mu_\Phi{}^N)$ is a bipolar fuzzy s-weak hyper KU -ideal of *H*.

Proof. Since $\phi = (H, \mu_\Phi{}^P, \mu_\Phi{}^N)$ satisfies the inf-sup property, there exists $a_0, b_0 \in x\circ(y\circ z)$, such that $\mu_\Phi{}^P(a_0) = \inf_{a\in x\circ(y\circ z)} \mu_\Phi{}^P(a)$ and $\mu_\Phi{}^N(b_0) = \sup_{b\in x\circ(y\circ z)} \mu_\Phi{}^N(b)$. i.e

$$\mu_\Phi{}^P(a) \geq \inf_{a\in x\circ(y\circ z)} \mu_\Phi{}^P(a) \text{ and } \mu_\Phi{}^N(b) \leq \sup_{b\in x\circ(y\circ z)} \mu_\Phi{}^N(b)$$

It follows that

$$\mu_\Phi{}^P(x\circ z) \geq \min\left\{ \inf_{a\in x\circ(y\circ z)} \mu_\Phi{}^P(a) \, , \, \mu_\Phi{}^P(y) \right\} \geq \min\left\{ \mu_\Phi{}^P(a) \, , \, \mu_\Phi{}^P(y) \right\}$$

and

$$\mu_\Phi{}^N(x\circ z) \leq \max\left\{ \sup_{b\in x\circ(y\circ z)} \mu_\Phi{}^N(b) \, , \, \mu_\Phi{}^N(y) \right\} \leq \max\left\{ \mu_\Phi{}^N(b) \, , \, \mu_\Phi{}^N(y) \right\}.$$

For every $a, b \in x\circ(y\circ z)$ .Hence $\phi = (H, \mu_\Phi{}^P, \mu_\Phi{}^N)$ is bipolar fuzzy s-weak hyper KU -ideal of *H*. Ending the proof.

Proposition 3.12. Let $\phi = (H, \mu_\Phi{}^P, \mu_\Phi{}^N)$ be bipolar fuzzy strong hyper KU-ideal of *H* and let x; y; z $\in$ H .Then

(i) $\mu_\Phi{}^P(0) \geq \mu_\Phi{}^P(x), \quad \mu_\Phi{}^N(0) \leq \mu_\Phi{}^N(x) \quad , \forall x \in H$

(ii) $x << y \Rightarrow \mu_\Phi{}^P(x) \geq \mu_\Phi{}^P(y) \text{ and } \mu_\Phi{}^N(x) \leq \mu_\Phi{}^N(y)$.

(iii) $\mu_\Phi{}^P(x\circ z) \geq \min\{\mu_\Phi{}^P(a), \mu_\Phi{}^P(y)\}, \forall a \in x\circ(y\circ z)$,

$$\mu_\Phi{}^N(x\circ z)\le \max\{\mu_\Phi{}^N(b),\mu_\Phi{}^N(y)\},\forall b\in x\circ(y\circ z)$$

Proof. (i) Since $0\in x\circ x\ \forall x\in H$,we have

$$\mu_\Phi{}^P(0)\ge \inf_{a\in x\circ x}\mu_\Phi{}^P(a)\ge \mu_\Phi{}^P(x),\ \mu_\Phi{}^N(0)\le \sup_{a\in x\circ x}\mu_\Phi{}^N(a)\le \mu_\Phi{}^N(x).$$

Which proves (i).

(ii) Let x; y$\in$ H be such that $x<<y$. Then $0\in y\circ x\forall x,y\in H$ and so

$$\sup_{b\in(y\circ x)}\mu_\Phi{}^P(b)\ge \mu_\Phi{}^P(0),\ \inf_{w\in(y\circ x)}\mu_\Phi{}^N(w)\le \mu_\Phi{}^N(0)$$

It follows from (i) that

$$\mu_\Phi{}^P(0\circ x)=\mu_\Phi{}^P(x)\ge \min\left\{\sup_{a\in y\circ x}\mu_\Phi{}^P(a),\mu_\Phi{}^P(y)\right\}\ge \min\{\mu_\Phi{}^P(0),\mu_\Phi{}^P(y)\}=\mu_\Phi{}^P(y)$$

and

$$\mu_\Phi{}^N(0\circ x)=\mu_\Phi{}^P(x)\le \max\left\{\inf_{a\in y\circ x}\mu_\Phi{}^P(a),\mu_\Phi{}^P(y)\right\}\le \max\{\mu_\Phi{}^P(0),\mu_\Phi{}^P(y)\}=\mu_\Phi{}^P(y)$$

(iii) $\mu_\Phi{}^P(x\circ z)\ge \min\left\{\sup_{a\in x\circ(y\circ z)}\mu_\Phi{}^P(a),\mu_\Phi{}^P(y)\right\}\ge \min\{\mu_\Phi{}^P(a),\mu_\Phi{}^P(y)\}\forall a\in x\circ(y\circ z)$

and

$$\mu_\Phi{}^N(x\circ z)\le \max\left\{\inf_{b\in x\circ(y\circ z)}\mu_\Phi{}^N(b),\mu_\Phi{}^N(y)\right\}\le \max\{\mu_\Phi{}^N(b),\mu_\Phi{}^N(y)\}\forall b\in x\circ(y\circ z)$$

we conclude that (iii) is true. Ending the proof.

Note that, in a finite hyper KU-algebra, every bipolar fuzzy set satisfies inf -sup property.Hence the concept of bipolar fuzzy weak hyper KU -ideals and bipolar fuzzy s-weak hyper KU-ideals coincide in a finite hyper KU -algebra.

Proposition 3.13 . Let $\phi=(H,\mu_\Phi{}^P,\ \mu_\Phi{}^N)$ be a bipolar fuzzy hyper KU-ideal of *H,* then:

$$\mu_\Phi{}^P(0)\ge\mu_\Phi{}^P(x),\quad \mu_\Phi{}^N(0)\le\mu_\Phi{}^N(x)\ \text{, If }\ \phi=(H,\mu_\Phi{}^P,\ \mu_\Phi{}^N)$$

satisfies the inf-sup property , then

$$\mu_\Phi{}^P(x\circ z)\ge \min\{\mu_\Phi{}^P(a),\mu_\Phi{}^P(y)\}\text{ and }\mu_\Phi{}^N(x\circ z)\le \max\{\mu_\Phi{}^N(b),\mu_\Phi{}^N(y)\}$$

for every $a,b\in x\circ(y\circ z)$.

Proof. Since $0 << x \quad \forall x \in H$, it follows from Definition 3.5. (I) that $\mu_\Phi{}^P(0) \geq \mu_\Phi{}^P(x)$ *and* $\mu_\Phi{}^N(0) \leq \mu_\Phi{}^N(x)$

Since $\phi = (H, \mu_\Phi{}^P, \mu_\Phi{}^N)$ satisfies the inf-sup property there exists $a_0, b_0 \in x \circ (y \circ z)$, such that $\mu_\Phi{}^P(a_0) = \inf_{a \in x \circ (y \circ z)} \mu_\Phi{}^P(a)$ and $\mu_\Phi{}^N(b_0) = \sup_{b \in x \circ (y \circ z)} \mu_\Phi{}^N(b)$. Hence

$$\mu_\Phi{}^P(x \circ z) \geq \min\left\{ \inf_{a \in x \circ (y \circ z)} \mu_\Phi{}^P(a), \mu_\Phi{}^P(y) \right\} \geq \min\left\{ \mu_\Phi{}^P(a_0), \mu_\Phi{}^P(y) \right\}$$

$$\mu_\Phi{}^N(x \circ z) \leq \max\left\{ \sup_{b \in x \circ (y \circ z)} \mu_\Phi{}^N(b), \mu_\Phi{}^N(y) \right\} \leq \max\left\{ \mu_\Phi{}^N(b_0), \mu_\Phi{}^N(y) \right\}$$

Corollary 3.14. (1) Every bipolar fuzzy hyper KU-ideal is a bipolar fuzzy weak hyper KU-ideal**.**
(2) If $\phi = (H, \mu_\Phi{}^P, \mu_\Phi{}^N)$ bipolar fuzzy hyper KU-ideal satisfies the inf-sup property , then $\phi = (H, \mu_\Phi{}^P, \mu_\Phi{}^N)$ is bipolar fuzzy s-weak hyper KU -ideal of *H*.

Theorem3.15. Let $\phi = (H, \mu_\Phi{}^P, \mu_\Phi{}^N)$ be bipolar fuzzy set ,then $\phi = (H, \mu_\Phi{}^P, \mu_\Phi{}^N)$ is bipolar fuzzy weak hyper KU -ideal of *H* if and only if the positive level set Φ_t^P and negative level set Φ_s^N for every $(\alpha, \beta) \in [0,1] \times [-1,0]$, are weak hyper KU -ideal of *H*, where the sets $\Phi_s^N = \{x \in H : \mu^N(x) \leq s\}$ and $\Phi_t^P = \{x \in H : \mu^+(x) \geq t\}$ are called the negative level set and the positive level set of $\phi = (H, \mu_\Phi{}^P, \mu_\Phi{}^N)$, respectively.

Proof. Assume that $\phi = (H, \mu_\Phi{}^P, \mu_\Phi{}^N)$ is bipolar fuzzy weak hyper KU -ideal of *H* and $\Phi_t^P \neq \Phi \neq \Phi_s^N$ for every $(\alpha, \beta) \in [0,1] \times [-1,0]$. It clear from

$$\mu_\Phi{}^P(0) \geq \mu_\Phi{}^P(x \circ z) \geq \min\left\{ \inf_{u \in x \circ (y \circ z)} \mu_\Phi{}^P(u), \mu_\Phi{}^P(y) \right\} \quad \text{(a)}$$

$$\mu_\Phi{}^N(0) \leq \mu_\Phi{}^N(x \circ z) \leq \max\left\{ \sup_{w \in (x \circ (y \circ z)} \mu_\Phi{}^N(w), \mu_\Phi{}^N(y) \right\} \quad \text{(b)}$$

That $0 \in \Phi_t^P \cap \Phi_s^N$. Let x; y; z $\in$ *H* be such that $x \circ (y \circ z) \subseteq \Phi_t^P$ and $y \in \Phi_t^P$.

Then for any $a \in x \circ (y \circ z), a \in \Phi_t^P$ .It follows that $\mu_\Phi{}^P(a) \geq \alpha$ so that $\inf_{a \in x \circ (y \circ z)} \mu_\Phi{}^P(a) \geq \alpha$, thus $\mu_\Phi{}^P(x \circ z) \geq \min\left\{ \inf_{a \in x \circ (y \circ z)} \mu_\Phi{}^P(a), \mu_\Phi{}^P(y) \right\} \geq \alpha$ and so $x \circ z \subseteq \Phi_t^P$, there for Φ_t^P is weak hyper KU -ideal of *H*.

Now let x; y; z $\in$ *H* be such that $x \circ (y \circ z) \subseteq \Phi_s^N$ and $y \in \Phi_s^N$. Then for any $b \in x \circ (y \circ z), b \in \Phi_{ts}^N$ .It follows that $\mu_\Phi{}^N(b) \leq \beta$, so that $\sup_{b \in x \circ (y \circ z)} \mu_\Phi{}^N(b) \leq \beta$. Using

$\mu_\Phi{}^N(x\circ z)\le \max\left\{\sup_{w\in(x\circ(y\circ z)}\mu_\Phi{}^N(w)\,,\mu_\Phi{}^N(y)\right\}\le\alpha$, which implies that $x\circ z\subseteq\Phi_s^N$. Consequently Φ_s^N is weak hyper KU -ideal of *H.*

Theorem 3.16 . Let $\phi=(H,\mu_\Phi{}^P,\ \mu_\Phi{}^N)$ be bipolar fuzzy set ,then $\phi=(H,\mu_\Phi{}^P,\ \mu_\Phi{}^N)$ is bipolar fuzzy hyper KU -ideal of H if and only if the positive level set Φ_t^P and negative level set Φ_s^N for every $(\alpha,\beta)\in[0,1]\times[-1,0]$, are hyper KU -ideal of H .

Proof. Assume that $\phi=(H,\mu_\Phi{}^P,\ \mu_\Phi{}^N)$ is bipolar fuzzy hyper KU -ideal of H and $\Phi_t^P\ne\Phi\ne\Phi_s^N$ for every $(\alpha,\beta)\in[0,1]\times[-1,0]$. It clear that $0\in\Phi_t^P\cap\Phi_s^N$. Let x; y; z $\in H$ be such that $x\circ(y\circ z)\subseteq\Phi_t^P$ and $y\in\Phi_t^P$.

Then for any $a\in x\circ(y\circ z), a\in\Phi_t^P$ .It follows that $\mu_\Phi{}^P(a)\ge\alpha$ so that $\inf_{a\in x\circ(y\circ z)}\mu_\Phi{}^P(a)\ge\alpha$, thus $\mu_\Phi{}^P(x\circ z)\ge\min\left\{\inf_{a\in x\circ(y\circ z)}\mu_\Phi{}^P(a),\mu_\Phi{}^P(y)\right\}\ge\alpha$ and so $x\circ z\subseteq\Phi_t^P$, there for Φ_t^P is hyper KU -ideal of H .

Now let x; y; z $\in H$ be such that $x\circ(y\circ z)\subseteq\Phi_s^N$ and $y\in\Phi_s^N$. Then for any $b\in x\circ(y\circ z), b\in\Phi_{ts}^N$ .It follows that $\mu_\Phi{}^N(b)\le\beta$, so that $\sup_{b\in x\circ(y\circ z)}\mu_\Phi{}^N(b)\le\beta$. Using $\mu_\Phi{}^N(x\circ z)\le \max\left\{\sup_{w\in(x\circ(y\circ z)}\mu_\Phi{}^N(w)\,,\mu_\Phi{}^N(y)\right\}\le\beta$, which implies that $x\circ z\subseteq\Phi_s^N$.

Consequently Φ_s^N is hyper KU -ideal of *H.*

Conversely, suppose that the nonempty positive and negative level sets Φ_t^P, Φ_s^N are is hyper KU -ideals of *H* for every $(\alpha,\beta)\in[0,1]\times[-1,0]$. Let

$\mu_\Phi{}^P(x)=\alpha$, $\mu_\Phi{}^N(x)=\beta$ for $x\in H$, then by $0\in\Phi_t^P$, $0\in\Phi_s^N$, It follows that. $\mu_\Phi{}^P(0)\ge\alpha$, $\mu_\Phi{}^N(0)\le\beta$ and so $\mu_\Phi{}^P(0)\ge\mu_\Phi{}^P(x)$ *and* $\mu_\Phi{}^N(0)\le\mu_\Phi{}^N(x)$. Now let

$$\min\left\{\inf_{a\in x\circ(y\circ z)}\mu_\Phi{}^P(a),\mu_\Phi{}^P(y)\right\}=\alpha \text{ and } \max\left\{\sup_{w\in(x\circ(y\circ z)}\mu_\Phi{}^N(w)\,,\mu_\Phi{}^N(y)\right\}=\beta$$

Note that, in a finite hyper KU-algebra, every bipolar fuzzy set satisfies inf -sup property. Hence the concept of bipolar fuzzy weak hyper KU -ideals and bipolar fuzzy s-weak hyper KU-ideals coincide in a finite hyper KU -algebra.

Corollary e 3.17. Every bipolar fuzzy strong hyper KU-ideal is both a bipolar fuzzy s-weak hyper KU-ideal (a bipolar fuzzy weak hyper ideal) and bipolar fuzzy hyper KU -ideal.

Proof. Straight forward.

Proposition 3.18. Let Let $\phi = (H, \mu_\Phi^P, \mu_\Phi^N)$ be bipolar fuzzy hyper KU -ideal of H and let $x, y, z \in H$. Then

(i) $\mu_\Phi^P(0) \geq \mu_\Phi^P(x), \quad \mu_\Phi^N(0) \leq \mu_\Phi^N(x)$
(ii) if $\phi = (H, \mu_\Phi^P, \mu_\Phi^N)$ satisfies the inf - sup property, then

$$\mu_\Phi^P(x \circ z) \geq \min\{ \mu_\Phi^P(a), \mu_\Phi^P(y) \}, \textit{ for some } \quad a \in x \circ (y \circ z)$$

and

$$\mu_\Phi^N(x \circ z) \leq \max\{ \mu_\Phi^N(w), \mu_\Phi^N(y) \}, \textit{ for some } \quad w \in x \circ (y \circ z)$$

Proof. (i) Since 0 << x for each $x \in H$; we have $\mu_\Phi^P(0) \geq \mu_\Phi^P(x), \quad \mu_\Phi^N(0) \leq \mu_\Phi^N(x)$ by Definition 3.11(i) and hence (i) holds.

(ii) Since $\phi = (H, \mu_\Phi^P, \mu_\Phi^N)$ satisfies the inf-sup property, there is $a_0, w_0 \in x \circ (y \circ z)$,such that $\mu(a_0) = \inf_{a \in x \circ (y \circ z)} \mu(a)$ and $\mu(w_0) = \sup_{w \in x \circ (y \circ z)} \mu(w)$. Hence

$$\mu_\Phi^P(x \circ z) \geq \min\left\{ \inf_{a \in x \circ (y \circ z)} \mu_\Phi^P(a), \mu_\Phi^P(y) \right\} = \min\{ \mu_\Phi^P(a_0), \mu_\Phi^P(y) \}$$

$$\mu_\Phi^N(x \circ z) \leq \max\left\{ \sup_{w \in x \circ (y \circ z)} \mu_\Phi^N(w), \mu_\Phi^N(y) \right\} = \min\{ \mu_\Phi^N(w_0), \mu_\Phi^N(y) \}$$

which implies that (ii) is true. The proof is complete.

Corollary 3.19. (i) Every bipolar fuzzy hyper KU -ideal of *H* is bipolar fuzzy weak hyper KU -idealof *H*.
(ii) If $\phi = (H, \mu_\Phi^P, \mu_\Phi^N)$ is bipolar fuzzy hyper KU -ideal of H satisfying *inf –sup* property, then $\phi = (H, \mu_\Phi^P, \mu_\Phi^N)$ is bipolar fuzzy s-weak Hyper KU -ideal of *H*.

Proof. Straightforward.

The following example shows that the converse of Corollary 3.17 and 3.19 (i) may not be true.

Example 3.20. (1) Consider the hyper KU -algebra H

$\circ$	0	1	2
0	$\{0\}$	$\{1\}$	$\{2\}$
1	$\{0\}$	$\{0,1\}$	$\{1,2\}$
2	$\{0\}$	$\{0,1\}$	$\{0,1,2\}$

Define bipolar fuzzy set μ in H by

	0	1	2
μ^N	**- 0.7**	**- 0.7**	**- 0.6**
μ^P	**1**	**0.5**	**0**

Then we can see that $\phi = (H, \mu_\Phi{}^P, \mu_\Phi{}^N)$ is bipolar fuzzy hyper KU -ideal of *H*. and hence it is also bipolar fuzzy weak hyper KU -ideal of *H*. But $\phi = (H, \mu_\Phi{}^P, \mu_\Phi{}^N)$ is not bipolar fuzzy strong hyper KU -ideal of *H* since

$$\min\left\{ \sup_{a \in 0 \circ (1 \circ 2)} \mu_\Phi{}^P(a), \mu_\Phi{}^P(y) \right\} \geq \min\{\mu_\Phi{}^P(1), \mu_\Phi{}^P(1)\} = 1/2 \geq 0 = \mu_\Phi{}^P(2), \forall a \in 0 \circ (1 \circ 2)$$

(2) Consider the hyper KU-algebra H in Example 3.14. Define bipolar fuzzy set $\phi = (H, \mu_\Phi{}^P, \mu_\Phi{}^N)$ in H by

	0	1	2
$\mu_\Phi{}^N$	**- 0.7**	**- 0.7**	**- 0.6**
$\mu_\Phi{}^P$	**1**	**0**	**0.5**

Then $\phi = (H, \mu_\Phi{}^P, \mu_\Phi{}^N)$ is bipolar fuzzy weak hyper KU-ideal of H but it is not a bipolar fuzzy hyper KU-ideal of H since 1 << 2 but $\mu_\Phi{}^P(1) \not\geq \mu_\Phi{}^P(2)$.

Theorem 3.21. If $\phi = (H, \mu_\Phi{}^P, \mu_\Phi{}^N)$ is bipolar fuzzy strong hyper KU-ideal of H, then the set $\mu_{t,s} = \{x \in H, \mu_\Phi{}^P(x) \geq t, \mu^N(x) \leq s\}$ is a strong hyper KU-ideal of H,when $\mu_{t,s} \neq \Phi,\ for\ t \in [0,1]$, $s \in [-1,0]$.

Proof. Let $\phi = (H, \mu_\Phi{}^P, \mu_\Phi{}^N)$ be a fuzzy strong hyper KU-ideal of H and $\mu_{t,s} \neq \Phi,\ for\ t \in [0,1]$. $s \in [-1,0]$.Then there $a, b \in \mu_{t,s}$ and so $\mu_\Phi{}^P(a) \geq t, \mu^N(b) \leq s$. By Proposition 3.12 (i), $\mu_\Phi{}^P(0) \geq \mu_\Phi{}^P(a) \geq t$, $\mu_\Phi{}^N(0) \leq \mu_\Phi{}^N(b) \leq s$ and so $0 \in \mu_{t,s}$.

Let $x, y, z \in H$ such that $x \circ (y \circ z) \cap \mu_{t,s} \neq \Phi\ and\ y \in \mu_{t,s}$. Then there exist $a_0, b_0 \in x \circ (y \circ z) \cap \mu_{t,s}\ and\ hence\ \mu_\Phi{}^P(a_0) \geq t, \mu^N(b_0) \leq s$. By definition 3.5 (iii), we have

$$\mu_\Phi{}^P(x \circ z) \geq \min\left\{ \sup_{a \in x \circ (y \circ z)} \mu_\Phi{}^P(a), \mu_\Phi{}^P(y) \right\} \geq \min\{\mu_\Phi{}^P(a_0), \mu(y)\} \geq \min\{t, t\} = t$$

and

$$\mu_\Phi{}^N(x \circ z) \leq \max\left\{ \inf_{a \in x \circ (y \circ z)} \mu_\Phi{}^N(a), \mu_\Phi{}^N(y) \right\} \leq \max\{\mu_\Phi{}^N(b_0), \mu_\Phi{}^N(y)\} \leq \max\{s, s\} = s$$

So $(x \circ z) \in \mu_{t,s}$. It follows that $\mu_{t,s}$ is a strong hyper KU -ideal of H .

Theorem 3.22. Let $\phi = (H, \mu_\Phi{}^P, \mu_\Phi{}^N)$ is bipolar fuzzy in H satisfying the inf- sup property,. If the set $\mu_{t,s} = \{x \in H, \mu_\Phi{}^P(x) \geq t, \mu^N(x) \leq s\} \neq \Phi$ is a strong hyper KU -ideal of H for all $t \in [0,1]$. $s \in [-1,0]$, then $\phi = (H, \mu_\Phi{}^P, \mu_\Phi{}^N)$ is bipolar fuzzy strong hyper KU-ideal of H.

Proof. Assume that $\mu_{t,s} \neq \Phi$ is a strong hyper KU-ideal of H for all $t \in [0,1]$. $s \in [-1,0]$. Then there is $x \in \mu_{t,s}$ such that $x \circ x << x \in \mu_{t,s}$ .Using Proposition 2.8, we have $x \circ x \subseteq \mu_{t,s}$. Thus for $a, b \in x \circ x$, *we have* $a, b \in \mu_{t,s}$ and hence $\mu_\Phi{}^P(a) \geq t, \mu^N(b) \leq s$. It follows that $\inf_{a \in x \circ (y \circ z)} \mu_\Phi{}^P(a) \geq t = \mu_\Phi{}^P(x)$ and $\sup_{b \in x \circ (y \circ z)} \mu_\Phi{}^N(b) \leq s = \mu_\Phi{}^N(x)$. Moreover let $x, y, z \in H$ and $\mu_{\alpha', \beta'}$, where

$$\alpha' = \min\left\{ \sup_{a \in x \circ (y \circ z)} \mu_\Phi{}^P(a), \mu_\Phi{}^P(y) \right\} \text{ , } \beta' = \max\left\{ \inf_{b \in x \circ (y \circ z)} \mu_\Phi{}^N(b), \mu_\Phi{}^N(y) \right\}\}$$

By hypothesis $\mu_{\alpha', \beta'}$ is a strong hyper KU-ideal of H.

Since $\phi = (H, \mu_\Phi{}^P, \mu_\Phi{}^N)$ satisfies the ***inf-sup*** property there is $a_0, b_0 \in x \circ (y \circ z)$,such that $\mu_\Phi{}^P(a_0) = \sup_{a \in x \circ (y \circ z)} \mu_\Phi{}^P(a)$, $\mu_\Phi{}^N(b_0) = \inf_{b \in x \circ (y \circ z)} \mu_\Phi{}^P(b)$. Thus

$$\mu_\Phi{}^P(a_0) = \sup_{a \in x \circ (y \circ z)} \mu_\Phi{}^P(a) \geq \min\left\{ \sup_{a \in x \circ (y \circ z)} \mu_\Phi{}^P(a), \mu_\Phi{}^P(y) \right\} = \alpha'$$

and

$$\mu_\Phi{}^N(b_0) = \inf_{b \in x \circ (y \circ z)} \mu_\Phi{}^N(b) \leq \max\left\{ \inf_{b \in x \circ (y \circ z)} \mu_\Phi{}^N(b), \mu_\Phi{}^N(y) \right\} = \beta'$$

This shows that $a_0, b_0 \in \mu_{\alpha', \beta'}$, $a_0, b_0 \in x \circ (y \circ z) \cap \mu_{\alpha', \beta'}$ and hence $x \circ (y \circ z) \cap \mu_{\alpha', \beta'} \neq \Phi$. Combining $y \in \mu_{\alpha', \beta'}$ and noticing that any bipolar fuzzy (weak, strong) hyper KU-ideal is a bipolar fuzzy (weak, strong) hyper ideal., we get $x \circ z \in \mu_{\alpha', \beta'}$. Hence

$$\mu_\Phi{}^P(x \circ z) \geq \min\left\{ \sup_{a \in x \circ (y \circ z)} \mu_\Phi{}^P(a), \mu_\Phi{}^P(y) \right\}, \ \mu_\Phi{}^N(x \circ z) \leq \max\left\{ \inf_{b \in x \circ (y \circ z)} \mu_\Phi{}^N(b), \mu_\Phi{}^N(y) \right\}$$

Therefore $\phi = (H, \mu_\Phi{}^P, \mu_\Phi{}^N)$ is bipolar fuzzy strong hyper K U-ideal of *H*.

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Conflicts of Interest

State any potential conflicts of interest here or "The author declare no conflict of interest".

4. Conclusion

In the present work the bipolar fuzzy hyper structure in KU-algebras is introduced .The concepts of bipolar fuzzy weakly (s-weakly strong) hyper KU-ideals and bipolar fuzzy hyper weakly (s-weakly strong) hyper KU-ideals are studied and their properties are characterized.

The main purpose of our future work is to investigate the following:

- bipolar fuzzy folding theory applied to some types of positive implicative hyper KU-ideals in hyper KU-algebras
- On bipolar fuzzy strong implicative hyper ku-ideals of hyper KU-algebras.
- On bipolar fuzzy positive implicative hyper KU-ideals.
- Super Implicative hyper KU-Algebras.
- bipolar Intuitionistic fuzziness of strong hyperKU-ideals.
- bipolar fuzzy filter theory on hyper KU-algebras.
- On Intuitionistic Fuzzy Implicative Hyper KU-Ideals of Hyper KU-algebras.
- On intuitionistic fuzzy commutative hyper KU-ideals.
- On interval-valued intuitionistic fuzzy Hyper KU-ideals of hyper KU- algebras.
- On cubic Implicative Hyper KU-Ideals of Hyper KU-algebras .

Algorithm for hyper KU-algebras

Input (X : set, $\circ$ hyper operation)
Output (" X is a hyper KU-algebra or not")
Begin
If $X = \phi$ then go to (1.);
End If
If $0 \notin X$ then go to (1.);
End If
Stop: =false;
$i := 1$;
While $i \leq |X|$ and not (Stop) do
If $0 \notin x_i \circ x_i$ then
Stop: = true;
End If
$j := 1$

While $j \le |X|$ *and not (Stop) do*

If $0 \notin x_i \circ (y_j \circ x_i)$ *or* $0 \in x_i \circ y_j$ *and* $0 \in (y_j \circ x_i)$ *and* $x_i \ne y_j$*,then*

Stop: = true;
End If
End If
$k := 1$
While $k \le |X|$ *and not (Stop) do*

If $0 \notin (x_i * y_j) \circ ((y_j * z_k) \circ (x_i * z_k))$ *then*

Stop: = true;
End If
End While
End While
End While
If Stop then
Output (" X is not hyper KU-algebra")
Else
Output (" X is hyper KU-algebra")
End If
End.

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