Number 16 Year 2017

Journal of New Theory

ISSN: 2149-1402

Editor-in-Chief
Naim Çağman

www.dergipark.org.tr/en/pub/jnt

Journal of New Theory (abbreviated by J. New Theory or JNT) is a mathematical journal focusing on new mathematical theories or the applications of a mathematical theory to science.

JNT founded on 18 November 2014 and its first issue published on 27 January 2015.

ISSN: 2149-1402

Editor-in-Chief: Naim Çağman

Email: journalofnewtheory@gmail.com

Language: English only.

Article Processing Charges: It has no processing charges.

Publication Frequency: Quarterly

Publication Ethics: The governance structure of J. New Theory and its acceptance procedures are transparent and designed to ensure the highest quality of published material. Journal of New Theory adheres to the international standards developed by the Committee on Publication Ethics (COPE).

Aim: The aim of the Journal of New Theory is to share new ideas in pure or applied mathematics with the world of science.

Scope: Journal of New Theory is an international, online, open access, and peer-reviewed journal. Journal of New Theory publishes original research articles, reports, reviews, editorial, letters to the editor, technical notes etc. from all branches of science that use the theories of mathematics.

Journal of New Theory concerns the studies in the areas of, but not limited to:

- Fuzzy Sets,
- Soft Sets,
- Neutrosophic Sets,
- Decision-Making
- Algebra
- Number Theory
- Analysis
- Theory of Functions
- Geometry
- Applied Mathematics
- Topology
- Fundamental of Mathematics
- Mathematical Logic
- Mathematical Physics

You can submit your manuscript in any style or JNT style as pdf. However, you should send your paper in JNT style if it would be accepted. The manuscript preparation rules, article template (LaTeX) and article template (Microsoft Word) can be accessed from the following links.

- Manuscript Preparation Rules
- Article Template (Microsoft Word.DOC) (Version 2019)
- Article Template (LaTeX) (Version 2019)

Pabitra Kumar Maji

Department of Mathematics, Bidhan Chandra College, Asansol 713301, Burdwan (W), West Bengal, India.
email: pabitra_maji@yahoo.com

Harish Garg

School of Mathematics, Thapar Institute of Engineering & Technology, Deemed University, Patiala-147004, Punjab, India
email: harish.garg@thapar.edu

Jianming Zhan

Department of Mathematics, Hubei University for Nationalities, Hubei Province, 445000, P. R. C.
email: zhanjianming@hotmail.com

Surapati Pramanik

Department of Mathematics, Nandalal Ghosh B.T. College, Narayanpur, Dist- North 24 Parganas, West Bengal 743126, India
email: sura_pati@yaho.co.in

Muhammad Irfan Ali

Department of Mathematics, COMSATS Institute of Information Technology Attock, Attock 43600, Pakistan
email: mirfanali13@yahoo.com

Said Broumi

Department of Mathematics, Hassan II Mohammedia-Casablanca University, Kasablanka 20000, Morocco
email: broumisaid78@gmail.com

Mumtaz Ali

University of Southern Queensland, Darling Heights QLD 4350, Australia
email: Mumtaz.Ali@usq.edu.au

Oktay Muhtaroğlu

Department of Mathematics, Tokat Gaziosmanpaşa University, 60250 Tokat, Turkey
email: oktay.muhtaroglu@gop.edu.tr

Ahmed A. Ramadan

Mathematics Department, Faculty of Science, Beni-Suef University, Beni-Suef, Egypt
email: aramadan58@gmail.com

Sunil Jacob John

Department of Mathematics, National Institute of Technology Calicut, Calicut 673 601 Kerala, India
email: sunil@nitc.ac.in

Aslıhan Sezgin

Department of Statistics, Amasya University, Amasya, Turkey
email: aslihan.sezgin@amasya.edu.tr

Alaa Mohamed Abd El-latif

Department of Mathematics, Faculty of Arts and Science, Northern Border University, Rafha, Saudi Arabia
email: alaa_8560@yahoo.com

Kalyan Mondal

Department of Mathematics, Jadavpur University, Kolkata, West Bengal 700032, India
email: kalyanmathematic@gmail.com

Jun Ye

Department of Electrical and Information Engineering, Shaoxing University, Shaoxing, Zhejiang, P.R. China
email: yehjun@aliyun.com

Ayman Shehata

Department of Mathematics, Faculty of Science, Assiut University, 71516-Assiut, Egypt
email: drshehata2009@gmail.com

İdris Zorlutuna

Department of Mathematics, Cumhuriyet University, Sivas, Turkey
email: izorlu@cumhuriyet.edu.tr

Murat Sarı

Department of Mathematics, Yıldız Technical University, İstanbul, Turkey
email: sarim@yildiz.edu.tr

Daud Mohamad

Faculty of Computer and Mathematical Sciences, University Teknologi Mara, 40450 Shah Alam, Malaysia
email: daud@tmsk.uitm.edu.my

Tanmay Biswas

Research Scientist, Rajbari, Rabindrapalli, R. N. Tagore Road, P.O.- Krishnagar Dist-Nadia, PIN-741101, West Bengal, India
email: tanmaybiswas_math@rediffmail.com

Kadriye Aydemir

Department of Mathematics, Amasya University, Amasya, Turkey
email: kadriye.aydemir@amasya.edu.tr

Ali Boussayoud

LMAM Laboratory and Department of Mathematics, Mohamed Seddik Ben Yahia University, Jijel, Algeria
email: alboussayoud@gmail.com

Muhammad Riaz

Department of Mathematics, Punjab University, Quaid-e-Azam Campus, Lahore-54590, Pakistan
email: mriaz.math@pu.edu.pk

Contact

Editor-in-Chief
Name: Prof. Dr. Naim Çağman
Email: journalofnewtheory@gmail.com
Phone: +905354092136
Address: Departments of Mathematics, Faculty of Arts and Sciences, Tokat Gaziosmanpaşa University, Tokat, Turkey

Editors
Name: Assoc. Prof. Dr. Faruk Karaaslan
Email: karaaslan.faruk@gmail.com
Phone: +905058314380
Address: Departments of Mathematics, Faculty of Arts and Sciences, Çankırı Karatekin University, 18200, Çankırı, Turkey

Name: Assoc. Prof. Dr. İrfan Deli
Email: irfandeli@kilis.edu.tr
Phone: +905426732708
Address: M.R. Faculty of Education, Kilis 7 Aralık University, Kilis, Turkey

Name: Asst. Prof. Dr. Serdar Enginoğlu
Email: serdarenginoglu@gmail.com
Phone: +905052241254
Address: Departments of Mathematics, Faculty of Arts and Sciences, Çanakkale Onsekiz Mart University, 17100, Çanakkale, Turkey

CONTENT

http://www.newtheory.org ISSN: 2149-1402

Journal of New Theory

Received: 10.02.2016 *Year*: **2017**, *Number*: **16**, *Pages*: **01-09**
Published: 04.10.2017 *Original Article*

MULTICRITERIA DECISION MAKING BASED ON CUBIC SET

Tahir Mahmood[1] <tahirbakhat@yahoo.com>
Saleem Abdullah[2] <saleemabdullah81@yahoo.com>
Saeed-ur-Rashid[1,*] <saeedraja10@gmail.com>
Muhammad Bilal[1] <mb_bilal@yahoo.com>

[1]Department of Mathematics and Statistic, International Islamic University Islamabad, Pakistan.
[2]Department of Mathematics, Abdul Wali Khan University Mardan, Pakistan.

Abstaract − This paper solves multicriteria decision making problems based on cubic set. The whole cubic set information given by the decision maker has been presented in a matrix form along with the weights assigned to each criteria. We have applied proposed method to select best alternative among available altermatives.

Keywords − *Fuzzy sets, Cubic sets, Score function.*

1 Introduction

The idea of fuzzy sets (FSs) was first proposed by Zadeh and has achieved a huge success in many areas.The concept of fuzzy sets was generalized as intuitionistic fuzzy sets (IFSs) by Atanassov. In 2008, Xu proposed some geometric aggregation operators, like the intuitionistic fuzzy weighted geometric (IFWG) operator, the intuitionistic fuzzy ordered weighted geomtric (IFOWG) operator and the intuitionistic fuzzy hybrid geometric (IFHG) operator, and applied IFGH operator to multicriteria decision-making problems with intuitionistic fuzzy knowledge. Some of the arithmetic aggregation operators like intuitionistic fuzzy weighted averaging (IFWA) etc. were introduced by Xu (2000). Tursken (1986) and Gorzaleczany (1987) gave the idea of so-called interval-valued fuzzy sets (IVFSs) which was considered to be furthur general form of a fuzzy set, but really there is solid bond between IFSs and IVFSs. Both the IFSs and IVFSs were furthur generalized by Gargov (1989), named as interval-valued intuitionistic fuzzy sets (IVIFSs). For IVIFSs some aggregation operators, labelled as the interval-valued intuitionistic fuzzy weighted geometric aggregation (IIFWGA) operator and the interval-valued intuitionistic fuzzy weighted

*Corresponding Author.

arithematic aggregation (IIFWAA) operator were introduced, and utilized these operators to decision making problems involving multicriteria with the help of the score function of interval-valued intuitionistic fuzzy information.

In the current article we have proposed the application of cubic set instead of IVIFS to decision-making problems having multicriteria. Our proposed score function or an accuracy function does not lead to the paradox of the difficult decision to the alternatives. The remaining article is arranged as follows. In section no. 3, we briefly introduce some aggregation operators for cubic sets. In third section, we suggest a score function, and then we provide two examples to justify that the suggested function is more suitable in the process of decision-making . In section 4, we have established a algorithm to recognize the best alternative. We make the use of cubic set weighted geometric aggregation (CSWGA) and cubic set weighted aggregation (CSWAA) operators to aggregate cubic set information corresponding to each alternative, and then give ranking to the alternatives and choose the best one(s) in view of the accuracy degrees of the aggregated cubic set information corresponding to score function. We show the worth of the adopted method by presenting illustrative examples in section 5.

2 Preliminaries

A fuzzy set in a set U is a function defined by $\mu : U \to I$ where $I = [0, 1]$. The closed subinterval $\widetilde{c} = [c^-, c^+]$ of I, is called an interval number, where $0 \leq c^- \leq c^+ \leq 1$. The interval number $\widetilde{c} = [c^-, c^+]$ with $c^- = c^+$ denoted by c. For the set of all interval numbers we will use the notation $[I]$.

Let U be a nonempty set. A function $B : U \to [I]$ is called an interval-valued fuzzy set (IVF) in U. Let $[I]^U$ denote the set of all IVF sets in U. For every $B \in [I]^U$ and $u \in U$, $B(u) = [B^-(u), B^+(u)]$ is called the degree of the membership of an element u to B, where $B^- : U \to I$ and $B^+ : U \to I$ are fuzzy sets in U which are termed as lower fuzzy set and upper fuzzy set in U resp. For every $F, G \in [I]^U$, we define $F \subseteq G \iff F(u) \leq G(u)$ for all $u \in U$, and $F = G \iff F(u) = G(u)$ for all $u \in U$.

2.1 Cubic Sets

Let $U \neq \Phi$ be a set. A cubic set in U has the form, $B = \{\langle\ u, B(u), \mu(u)\rangle \mid u \in U\}$,where B is an IVF set in U and μ is a fuzzy set in U. A cubic set $B = \{\langle\ u, B(u), \lambda(u)\rangle \mid u \in U\}$ is denoted by $B = \langle B, \mu\rangle$ for simplicity.

2.1.1 Internal Cubic Set (briefly, ICS)

Let $U \neq \Phi$ be a set. A cubic set $B = \langle B, \mu\rangle$ in U is known as an internal cubic set (ICS) if $B^-(u) \leq \mu(u) \leq B^+(u)$ for all $u \in U$.

2.1.2 External Cubic Set (briefly,ECS)

Let $U \neq \Phi$ be a set. A cubic set $B = \langle B, \mu\rangle$ in U is known as an external cubic set (ECS) if $\mu(x) \notin (B^-(u), B^+(u))$ for all $u \in U$

2.1.3 Example

Let $B = \{\langle u, B(u), \mu(u)\rangle \mid u \in I\}$ be a cubic set in I. If $B(u) = [0.2, 0.5]$ and $\mu(u) = 0.4$ for all $\mu \in I$, then B is an ICS. If $B(x) = [0.2, 0.5]$ and $\mu(u) = 0.7$ for all $u \in I$, then B is an ECS. If $B(u) = [0.2, 0.5]$ and $\mu(u) = u$ for all $u \in I$, then B does not belong to the class of ICS and ECS.

3 Score Function

Before defining score function, we define two weighted aggregation operators related to CSs.

Definition 3.1. *Let $B = \langle B, \mu\rangle$ and $C = \langle C, \nu\rangle$ be cubic sets in U. Then we define (i) (Equality) $B = C \iff B = C$ and $\mu = \nu$. (ii) (P-order) $B \subseteq_p C \iff B \subseteq C$ and $\mu \leq \nu$. (ii) (R-order) $B \subseteq_R C \iff B \subseteq C$ and $\mu \geq \nu$.*

From here on we will denote by CS(U) the set of all cubic sets in U. The value of a cubic set will be conventionally denoted by $B = ([b, c], d)$.

Definition 3.2. *Let $B_j(1 \leq j \leq n) \in CS(U)$. The weighted arithematic average operator is defined by $F_w(\ B_1, B_2, ..., B_n) =$*

$$\sum_{j=1}^{n} w_j B_j = \left(\left[1 - \underset{j=1}{\overset{n}{\pi}} \left(1 - B_j^-(u)\right)^{w_j}, 1 - \underset{j=1}{\overset{n}{\pi}} \left(\left(1 - B_j^+(u)\right)^{w_j}\right)\right], \left[\underset{j=1}{\overset{n}{\pi}} \mu_j^{wj}(u)\right]\right) \tag{1}$$

where w_j is the weight of $B_j(1 \leq j \leq n)$, $w_j \in [0, 1]$ and $\sum_{j=1}^{n} w_j = 1$. Especially assume $w_j = \frac{1}{n}$ ($j = 1, 2, ..., n$) then, F_w is known as an arithematic average operator for CSs.

Definition 3.3. *Let $B_j(1 \leq j \leq n) \in CS(U)$. The weighted geometric average operator is defined by*

$$G_w(\ B_1, B_2, ..., B_n) = \underset{j=1}{\overset{n}{\pi}} B_j^{w_j} = \left(\left[\underset{j=1}{\overset{n}{\pi}} B_j^{-^{w_j}}(u), \underset{j=1}{\overset{n}{\pi}} B_j^{+^{w_j}}(u)\right], \left[1 - \underset{j=1}{\overset{n}{\pi}} \left(1 - \mu_j(u)\right)^{wj}\right]\right) \tag{2}$$

where w_j is the weight of $B_j(1 \leq j \leq n)$, $w_j \in [0, 1]$ and $\sum_{j=1}^{n} w_j = 1$. Especially assume $w_j = \frac{1}{n}$ ($j = 1, 2, ..., n$) then, G_w is known as geometric average operator for CSs.

The aggregation results F_w & G_w are still CS(U).

Let $B = ([b, c], d)$ be a CSV, a score function M of cubic set value is suggested by the formula given below

$$M(\mathbf{B}) = \frac{b + c - 1 + d}{2} \tag{3}$$

where $M(\mathbf{B}) \in [-1, +1]$. Now we consider following examples.

3.1 Example

If internal cubic set values for different alternatives are $B_1 = ([0.3, 0.5], 0.4)$ and $B_2 = ([0.5, 0.7], 0.6)$ the wanted alternative is selected in view of score function. After applying equation (3) we have

$$\begin{aligned} M(\mathbf{B}_1) &= \frac{0.3 + 0.5 - 1 + 0.4}{2} = 0.1 \\ M(\mathbf{B}_1) &= \frac{0.5 + 0.7 - 1 + 0.6}{2} = 0.4 \end{aligned}$$

Obviously the alternative B_2 has prefrence over B_1.

3.2 Example

If external cubic set values for two diffrent alternatives are $B_1 = ([0.3, 0.4], 0.5)$ and $B_2 = ([0.4, 0.5], 0.6)$ the desired alternative is choosen with the help of score function. By using equation (3) we get

$$\begin{aligned} M(\mathbf{B}_1) &= \frac{0.3 + 0.4 - 1 + 0.5}{2} = 0.10 \\ M(\mathbf{B}_2) &= \frac{0.4 + 0.5 - 1 + 0.6}{2} = 0.25 \end{aligned}$$

clearly the alternative B_2 has advantage over B_1.

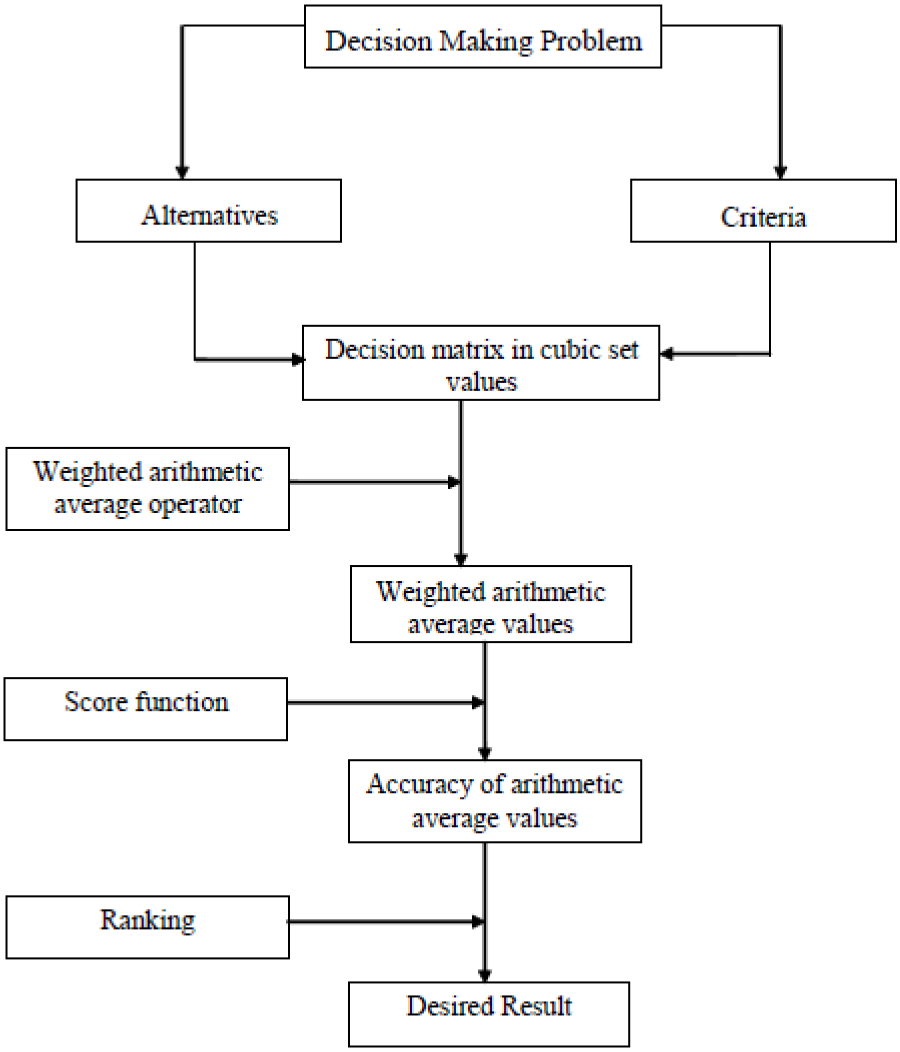
4 Multicriteria Cubic Set Decision Making Method Based on the Score Function

Here we are going to present a method for tackling of multicriteria cubic set decision-making problems along with weights. Suppose that $B = \{\mathbf{B}_1, \mathbf{B}_2, ..., \mathbf{B}_m\}$ is a collection of alternatives and also suppose that $C = \{C_1, C_2, ..., C_n\}$ is a set of criteria. Consider the criterion C_j $(1 \leq j \leq n)$, recommended by the decision-maker, has weight $w_j, w_j \in [0, 1]$ and $\sum_{j=1}^{n} w_j = 1$. In this situation, the characterstic of the alternative B_i is represented by a cubic set as

$$B_i = \left\{ \left\langle C_j, \left[B^-(C_j), B^+(C_j)\right], [\mu(C_j)] \right\rangle \mid C_j \in C \right\}.$$

The cubic set value that is the pair of IVFS and fuzzy number, i.e.

$$(\mathbf{B}_i(C_j) = [b_{ij}, c_{ij}], \mu(C_j) = d_{ij} \text{ for } C_j \in C) \text{ is denoted by } \alpha_{ij} = ([b_{ij},\ c_{ij}], d_{ij})$$



Flow chart of the proposed method.

Since $[b_{ij},\ c_{ij}] \subseteq [0,1]$ & $d_{ij} \in [0,1]$. Therefore a decision matrix of the form $D = (\alpha_{ij})$ can be formulated. The aggregating cubic set value α_i for $B_i\,(1 \leq i \leq m)$ is $\alpha_i = ([b_i,\ c_i]\,, d_i) = F_{iw}\,(\alpha_{i1}, \alpha_{i2,}..., \alpha_{in})$ or $\alpha_i = ([b_i,\ c_i]\,, d_i) = G_{iw}\,(\alpha_{i1}, \alpha_{i2,}..., \alpha_{in})$ which is obtained by using equation (1) or Eq. (2), in accordance with each row in the decision matrix. We will use Eq. (3) to calculate the accuracy $M(\alpha_i)$ of aggregating cubic set value $\alpha_i\,(1 \leq i \leq m)$ to rank the alternatives $B_i\,(1 \leq i \leq m)$ and then to choose the best one(s). Simply, the decision making process for the suggested technique can be described by the following steps.

Step(a). Obtain the weighted arithmetic average values by applying Eq. (1) if we prefer the influence of group, otherwise get the weighted geometric values with the help of Eq. (2).

Step(b). Obtain the accuracy $M(\alpha_i)$ of cubic set value $\alpha_i\,(1 \leq i \leq m)$ by the application of Eq. (3).

Step (c). Rank the alternatives $B_i\,(1 \leq i \leq m)$ and choose the best one(s) in comparison with $M(\alpha_i)\,(1 \leq i \leq m)$.

5 Illustrative Examples

This section is consisting of two examples. First example adapted from Herrera and Herrera -Viedma (2000) for a decision-making problem of alternatives along with multicriteria is used to potray the suggested fuzzy decision making method in the spectrum of reallity, as well as the validity of the effectiveness of the suggested algorithm.

Here is a set of people provided with four options to invest the money: (1) B_1 is a company of car; (2) B_2 is a company of food; (3) B_3 is a company of computer; (4) B_4 is a company of arms. The investor must have to decide by keeping in mind these three criteria: (1) C_1 is the analysis of risk; (2) C_2 is the analysis of growth; (3) C_3 is the analysis of enviromental impact. Now decider will evaluate the four possible alternatives under the above mentioned criteria, as provided in the following matrices. First we consider the matrix D_1 consisting of internal cubic set values.

$$D_1 = \begin{bmatrix} ([0.1, 0.3], 0.2) & ([0.2, 0.4], 0.3) & ([0.3, 0.6], 0.4) \\ ([0.5, 0.7], 0.5) & ([0.3, 0.4], 0.3) & ([0.7, 0.8], 0.7) \\ ([0.3, 0.5], 0.4) & ([0.7, 0.9], 0.8) & ([0.6, 0.8], 0.7) \\ ([0.4, 0.6], 0.4) & ([0.1, 0.2], 0.2) & ([0.6, 0.8], 0.7) \end{bmatrix}$$

Now assume that the weights of C_1, C_2 & C_3 are $0.35, 0.25$ and 0.40 resp. Then we use the following algorithm.

Step 1. Eq. (1) provides us the weighted arithmetic average value α_i for $B_i\,(i = 1, 2, ..., 4)$.

$\alpha_1 = ([0.2097, 0.4615], 0.2921)$

$\alpha_2 = ([0.5566, 0.6967], 0.5035)$

$\alpha_3 = ([0.5472, 0.7682], 0.5950)$

$\alpha_4 = ([0.3827, 0.5243], 0.3678)$

Step 2. By applying Eq. (3) we can compute $M(\alpha_i)$ where $i = 1, 2, 3, 4$ as $M(\alpha_1) = 0.4817$, $M(\alpha_2) = 0.3784$, $M(\alpha_3) = 0.4552$, $M(\alpha_4) = 0.1374$.

Step 3. Awarding ranks to all alternatives in view of the accuracy degree of $M(\alpha_i)$ $(i = 1, 2, 3, 4) : B_1 \succ B_3 \succ B_2 \succ B_4$, and thus the best alternative is B_1.

Now we consider the matrix D_2 consisting of external cubic set values.

$$D_2 = \begin{bmatrix} ([0.4, 0.5], 0.3) & ([0.4, 0.6], 0.2) & ([0.1, 0.3], 0.5) \\ ([0.6, 0.7], 0.2) & ([0.5, 0.7], 0.2) & ([0.4, 0.7], 0.1) \\ ([0.3, 0.6], 0.1) & ([0.5, 0.6], 0.4) & ([0.5, 0.6], 0.3) \\ ([0.7, 0.8], 0.1) & ([0.6, 0.7], 0.3) & ([0.3, 0.4], 0.2) \end{bmatrix}$$

Consider the same weights for C_1, C_2 & C_3 as mentioned above and use the following algorithm.

Step 1. Applying Eq. (1) we obtain the weighted arithmetic average value α_i for $B_i\,(i = 1, 2, ..., 4)$.

$\alpha_1 = ([0.2944, 0.4590], 0.3325)$

$\alpha_2 = ([0.5026, 0.7000], 0.1516)$

$\alpha_3 = ([0.4375, 0.6000], 0.2195)$

$\alpha_4 = ([0.5476, 0.6565], 0.1737)$

Step 2. By applying Eq. (3) we can compute $M(\alpha_i)$ where $i = 1, 2, 3, 4$ as $M(\alpha_1) = 0.0430$, $M(\alpha_2) = 0.1771$, $M(\alpha_3) = 0.1285$, $M(\alpha_4) = 0.1889$.

Step 3. By ranking all alternatives in view of the accuracy degree of $M(\alpha_i)$ $(i = 1, 2, 3, 4) : B_4 \succ B_2 \succ B_3 \succ B_1$, and thus the alternative B_4 is the best one.

Finally we consider the matrix D_3 consisting of cubic set values which are neither internal cubic set values nor external cubic set values.

$$D_3 = \begin{bmatrix} ([0.3, 0.7], 0.1) & ([0.3, 0.7], 0.2) & ([0.3, 0.7], 0.4) \\ ([0.3, 0.7], 0.4) & ([0.3, 0.7], 0.5) & ([0.4, 0.7], 0.1) \\ ([0.3, 0.7], 0.7) & ([0.3, 0.7], 0.8) & ([0.3, 0.7], 0.6) \\ ([0.2, 0.5], 1) & ([0.2, 0.5], 0.3) & ([0.2, 0.5], 0.6) \end{bmatrix}$$

Again using the similar procedure as stated above with similar weights we have $M(\alpha_1) = 0.1036$, $M(\alpha_2) = 0.1215$, $M(\alpha_3) = 0.3403$, $M(\alpha_4) = 0.3017$ so $B_3 \succ B_4 \succ B_2 \succ B_1$ and thus the alternative B_3 is the most wishful one.

Now we present another example in this section in which we want to investigate the suitability of an S-box to image encryption applications. We have been provided with nine different alternatives of S-boxes: (1) B_1 is Plain Image; (2) B_2 is Advanced Encryption Standard; (3) B_3 is Affine Power Affine; (4) B_4 is Gray; (5) B_5 is S_8; (6) B_6 is Liu; (7) B_7 is Prime; (8) B_8 is Xyi; (9) B_9 is Skipjack. We have to make the decision according to the following criterion: (1) C_1 is the entropy analysis; (2) C_2 is the contrast analysis; (3) C_3 is the average correlation analysis; (4) C_4 is the energy analysis; (5) C_5 is the homogeneity analysis; (6) C_6 is the mean of absolute deviation analysis. The nine possible alternatives are to be sorted out using the cubic set information by the decider from the given criterion as presented in the following matrix.

$$D = \begin{bmatrix} ([0.1, 0.2], 0.3) & ([0.1, 0.3], 0.2) & ([0.3, 0.4], 0.1) & ([0.4, 0.5], 0.6) & ([0.3, 0.6], 0.5) & ([0.5, 0.6], 0.4) \\ ([0.5, 0.7], 0.4) & ([0.3, 0.4], 0.2) & ([0.7, 0.8], 0.6) & ([0.4, 0.5], 0.3) & ([0.6, 0.7], 0.2) & ([0.4, 0.7], 0.1) \\ ([0.3, 0.5], 0.4) & ([0.7, 0.9], 0.8) & ([0.6, 0.8], 0.7) & ([0.5, 0.6], 0.3) & ([0.7, 0.8], 0.1) & ([0.1, 0.3], 0.5) \\ ([0.4, 0.6], 0.4) & ([0.1, 0.2], 0.2) & ([0.3, 0.6], 0.4) & ([0.3, 0.4], 0.1) & ([0.3, 0.4], 0.2) & ([0.6, 0.7], 0.3) \\ ([0.1, 0.3], 0.3) & ([0.5, 0.6], 0.7) & ([0.2, 0.4], 0.3) & ([0.6, 0.8], 0.7) & ([0.1, 0.2], 0.2) & ([0.3, 0.5], 0.1) \\ ([0.5, 0.6], 0.2) & ([0.4, 0.7], 0.6) & ([0.5, 0.7], 0.9) & ([0.8, 0.9], 0.8) & ([0.4, 0.6], 0.3) & ([0.7, 0.8], 0.2) \\ ([0.7, 0.8], 0.9) & ([0.4, 0.7], 0.5) & ([0.4, 0.6], 0.2) & ([0.7, 0.9], 0.2) & ([0.8, 0.9], 0.7) & ([0.2, 0.5], 0.4) \\ ([0.8, 0.9], 0.7) & ([0.7, 0.9], 0.8) & ([0.1, 0.2], 0.1) & ([0.3, 0.2], 0.1) & ([0.5, 0.6], 0.1) & ([0.4, 0.8], 0.6) \\ ([0.8, 0.9], 0.6) & ([0.6, 0.9], 0.7) & ([0.3, 0.5], 0.6) & ([0.4, 0.7], 0.3) & ([0.4, 0.6], 0.5) & ([0.1, 0.2], 0.3) \end{bmatrix}$$

Now we assume the same weight for each of C_1, C_2 ,..., C_6, that is 0.167 and use the following algorithm.

Step 1. We calculate the weighted arithmetic average value α_i for B_i $(i = 1, 2, ..., 9)$ with the aid of Eq. (1).

$\alpha_1 = ([0.3035, 0.4592], 0.2922)$
$\alpha_2 = ([0.5096, 0.6646], 0.2501)$
$\alpha_3 = ([0.5330, 0.7200], 0.3797)$
$\alpha_4 = ([0.3575, 0.5170], 0.2334)$
$\alpha_5 = ([0.3350, 0.5194], 0.3025)$
$\alpha_6 = ([0.5884, 0.7499], 0.4088)$
$\alpha_7 = ([0.5912, 0.7845], 0.4068)$
$\alpha_8 = ([0.5330, 0.7242], 0.2567)$
$\alpha_9 = ([0.4942, 0.7272], 0.4670)$

Step 2. By applying Eq. (3) we can compute $M(\alpha_i)$ where $i = 1, 2, ..., 9$ as $M(\alpha_1) = 0.0275$, $M(\alpha_2) = 0.2122$, $M(\alpha_3) = 0.3164$, $M(\alpha_4) = 0.0540$, $M(\alpha_5) = 0.0785$, $M(\alpha_6) = 0.3736$, $M(\alpha_7) = 0.8913$, $M(\alpha_8) = 0.7570$, $M(\alpha_9) = 0.3342$.

Step 3. After awarding ranks to all alternatives in view of the accuracy degree of $M(\alpha_i)$ $(i = 1, 2\ , ..., 9.)$: $B_7 \succ B_8 \succ B_6 \succ B_9 \succ B_3 \succ B_2 \succ B_5 \succ B_4 \succ B_1$ and thus the alternative B_7 is the most desired one.

References

[1] K. Atanassov, Intuitionistic fuzzy sets, VII ITKR's session, Sofia, June (1983).

[2] K. Atanassov, Intuitionistic fuzzy sets, Fuzzy Sets and Systems 20 (1986) 87–96.

[3] K. Atanassov, Operators over interval-valued intuitionistic fuzzy sets. Fuzzy Sets and Systems 64(2) (1994) 159–174.

[4] K. Atanassov, G. Gargov, Interval-valued intuitionistic fuzzy sets, Fuzzy Sets and Systems 31(3) (1989) 343–349.

[5] H. Bustine, P. Burillo, Vague sets are intuitionistic fuzzy sets, Fuzzy Sets and Systems 79 403–405.

[6] S. M. Chen, J. M. Tan, Handling multicriteria fuzzy decision-making problems based on vague set theory, Fuzzy Sets and Systems 67 (1994) 163–172.

[7] W. L. Gau, D. J. Buehrer, Vague sets, IEEE Transactions on Systems, Man and Cybernetics 23(2) (1993) 610–614.

[8] M. B. Gorzaleczany, A method of inference in approximate reasoning based on interval-valued fuzzy sets, Fuzzy Sets and Systems 21(1987) 1–17.

[9] F. Herrera, Herrera - E. Viedma, Linguistic decision analysis: Steps for solving decision problems under linguistic information, Fuzzy Sets and Systems 115 (2000) 67–82.

[10] D. H. Hong, C.-H. Choi, Multicriteria fuzzy decision-making problems based on vague set theory, Fuzzy Sets and Systems 114 (2000) 103–113.

[11] D. F. Li, Multiattribute decision making models and methods using intuitionistic fuzzy sets, Journal of Computer and System Sciences 70 (2005) 73–85.

[12] L. Lin, X. H.Yuan, Z. Q. Xia, Multicriteria decision-making methods based on intuitionistic fuzzy sets, Journal of Computer and System Sciences 73 (2007) 84–88.

[13] H. W. Liu, G. J. Wang, Multicriteria decision-making methods based on intuitionistic fuzzy sets, European Journal of Operational Research 179 (2007) 220–233.

[14] H. B. Mitchell, Ranking intuitionistic fuzzy numbers, International Journal of Uncertainty, Fuzziness and Knowledge Based Systems 12(3) (2004) 377–386.

[15] B. Turksen, Interval valued fuzzy sets based on normal forms, Fuzzy Sets and Systems 20 (1986) 191–210.

[16] Z. S. Xu, Intuitionistic preference relations and their application in group decision making, Information Science 177(11) (2007) 2363–2379.

[17] Z. S. Xu, Intuitionistic fuzzy aggregation operators, IEEE Transactions on Fuzzy Systems 15(6) (2007) 1179–1187.

[18] Z. S. Xu, Methods for aggregating interval-valued intuitionistic fuzzy information and their application to decision making, Control and Decision 22(2) (2007) 215–219.

[19] Z. S. Xu, J. Chen, An approach to group decision making based on interval-valued intuitionistic fuzzy judgment matrices, System Engineer – Theory and Practice 27(4) (2007) 126–133.

[20] Z. S. Xu, J. Chen, On geometric aggregation over interval-valued intuitionistic fuzzy information. In FSKD, 4th international conference on fuzzy systems and knowledge discovery (FSKD 2007) (Vol. 2, pp. 466–471).

[21] Z. S. Xu, Q. L. Da, An overview of operators for aggregating information, International Journal of Intelligent Systems 18 (2003) 953–969.

[22] Z. S. Xu, R. R. Yager, Some geometric aggregation operators based on intuitionistic fuzzy sets, International Journal of General System 35 (2006) 417–433.

[23] J. Ye, Multicriteria fuzzy decision-making based on a novel accuracy function under interval-valued intuitionistic fuzzy environment, Fuzzy Systems with Application 36 (2009) 6899-6902.

[24] Young Bae Jun, Chang Su Kim, Kim Oong Yang, Cubic sets, Annals of Fuzzy Mathematics and Informatics (2011) Vol. 4, No, 1, pp. 83-98

[25] L. A. Zadeh, Fuzzy sets. Information and Control 8(3) (1965) 338–356.

http://www.newtheory.org ISSN: 2149-1402

Journal of New Theory

Received: 14.03.2016 *Year*: **2017**, *Number*: **16**, *Pages*: **10-18**
Published: 04.10.2017 *Original Article*

JUST CHROMATIC EXCELLENCE IN FUZZY GRAPHS

Dharmalingam* <kmdharma6902@yahoo.in>
Udaya Suriya <udayasurya20@gmail.com>

Department of Mathematics, The Madura College, Madurai, India

Abstaract − Let G be a simple fuzzy graph. A family $\Gamma^f = \{\gamma_1, \gamma_2, \ldots, \gamma_k\}$ of fuzzy sets on a set V is called k-fuzzy colouring of $V = (V, \sigma, \mu)$ if i)$\cup\Gamma^f = \sigma$, ii) $\gamma_i \cap \gamma_j = \emptyset$, iii)for every strong edge (x, y)(i.e., $\mu(xy) > 0$) of G $\min\{\gamma_i(x), \gamma_i(y)\} = 0, (1 \leq i \leq k)$. The minimum number of k for which there exists a k-fuzzy colouring is called the fuzzy chromatic number of G denoted as $\chi^f(G)$. Then Γ^f is the partition of independent sets of vertices of G in which each sets has the same colour is called the fuzzy chromatic partition. A graph G is called the just χ^f-excellent if every vertex of G appears as a singleton in exactly one χ^f-partition of G. This paper aims at the study of the new concept namely Just Chromatic excellence in fuzzy graphs. Fuzzy colourful vertex is defined and studied. We explain these new concepts through examples.

Keywords − *fuzzy chromatic excellent, fuzzy just excellent,fuzzy colourful vertex*

1 Introduction

A fuzzy graph is a symmetric binary fuzzy relation on a fuzzy subset. The concept of fuzzy sets and fuzzy relations was introduced by L.A.Zadeh in 1965[1] and further studied[2]. It was Rosenfeld[5] who considered fuzzy relations on fuzzy sets and developed the theory of fuzzy graphs in 1975. The concepts of fuzzy trees, blocks, bridges and cut nodes in fuzzy graph has been studied[3]. Computing chromatic sum of an arbitrary graph introduced by Kubica [1989] is known as NP-complete problem. Graph coloring is the most studied problem of combinatorial optimization. As an advancement fuzzy coloring of a fuzzy graph was defined by authors Eslahchi and Onagh in 2004, and later developed by them as Fuzzy vertex coloring[4] in 2006.This fuzzy vertex coloring was extended to fuzzy total coloring in terms of family of fuzzy sets by Lavanya. S and Sattanathan. R[6]. In this paper we are introducing "Just Chromatic excellence in fuzzy graphs".

2 Preliminary

Definition 2.1. A fuzzy graph $G = (\sigma, \mu)$ is a pair of functions $\sigma : V \rightarrow [0, 1]$ and $\mu : V \times V \rightarrow [0, 1]$ where for all $u, v \in V$, we have $\mu(u, v) \leq \sigma(u) \wedge \sigma(v)$.

* *Corresponding Author.*

Definition 2.2. The order p and size q of a fuzzy graph $G = (\sigma, \mu)$ are defined to be $p = \sum_{x \in V} \sigma(x)$ and $q = \sum_{xy \in E} \mu(xy)$.

Definition 2.3. The degree of vertex u is defined as the sum of the weights of the edges incident at u and is denoted by $d(u)$.

Definition 2.4. The union of two fuzzy graphs G_1 and G_2 is defined as a fuzzy graph $G = G_1 \cup G_2 : ((\sigma_1 \cup \sigma_2, \mu_1 \cup \mu_2))$ defined by

$$(\sigma_1 \cup \sigma_2)(u) = \begin{cases} \sigma_1(u), \text{ if } u \in V_1 - V_2 \text{ and} \\ \sigma_2(u) \text{ if } u \in V_2 - V_1 \end{cases}$$

$$(\mu_1 \cup \mu_2)(uv) = \begin{cases} \mu_1(uv), \text{ if } uv \in E_1 - E_2 \text{ and} \\ \mu_2(uv) \text{ if } uv \in E_2 - E_1 \end{cases}$$

Definition 2.5. The join of two fuzzy graphs G_1 and G_2 is defined as a fuzzy graph $G = G_1 + G_2 : ((\sigma_1 + \sigma_2, \mu_1 + \mu_2))$ defined by

$$(\sigma_1 + \sigma_2)(u) = (\sigma_1 \cup \sigma_2)(u) \forall u \in V_1 \cup V_2$$

$$(\mu_1 + \mu_2)(uv) = \begin{cases} (\mu_1 \cup \mu_2)(uv) \text{ if } uv \in E_1 \cup E_2 \text{ and} \\ \sigma_1(u) \wedge \sigma_2(v) \text{ if } uv \in E'. \end{cases}$$

where E' is the set of all edges joining the nodes of V_1 and V_2.

Definition 2.6. The cartesian product of two fuzzy graphs G_1 and G_2 is defined as a fuzzy graph $G = G_1 \times G_2 : (\sigma_1 \times \sigma_2, \mu_1 \times \mu_2)$ on $G* : (V, E)$ where $V = V_1 \times V_2$ and $E = \{((\sigma_1, \sigma_2), (\mu_1, \mu_2))/u_1 = v_1, u_2v_2 \in E_2 \text{or} u_2 = v_2, u_1v_1 \in E_1\}$ with

$$(\sigma_1 \times \sigma_2)(u_1, v_1) = \sigma_1(u_1) \wedge \sigma_2(u_2) forall (u_1, u_2) \in V_1 \times V_2$$

$$(\mu_1 \times \mu_2)((u_1, u_2)(v_1, v_2)) = \begin{cases} \sigma_1(u_1) \wedge \mu_2(u_2, v_2), \text{ if } u_1 = v_1 \text{ and } u_2v_2 \in E_2 \\ \sigma_2(u_2) \wedge \mu_1(u_1, v_1), \text{ if } u_2 = v_2 \text{ and } u_1v_1 \in E_1 \end{cases}$$

3 Main Defintions and Results

Definition 3.1. Let G be a fuzzy graph. A family $\Gamma^f = \{\gamma_1.\gamma_2, \ldots, \gamma_k\}$ of fuzzy sets on a set V is called k-fuzzy colouring of $V = (V, \sigma, \mu)$ if

(i) $\cup \Gamma^f = \sigma$,

(ii) $\gamma_i \cap \gamma_j = \emptyset$,

(iii) for every strong edge (x, y)(i.e., $\mu(xy) > 0$) of G $\min\{\gamma_i(x), \gamma_i(y)\} = 0, (1 \leq i \leq k)$.

The minimum number of k for which there exists a k-fuzzy colouring is called the fuzzy chromatic number of G denoted as $\chi^f(G)$.

Definition 3.2. Γ^f is the partition of independent sets of vertices of G in which each sets has the same colour is called the fuzzy chromatic partition.

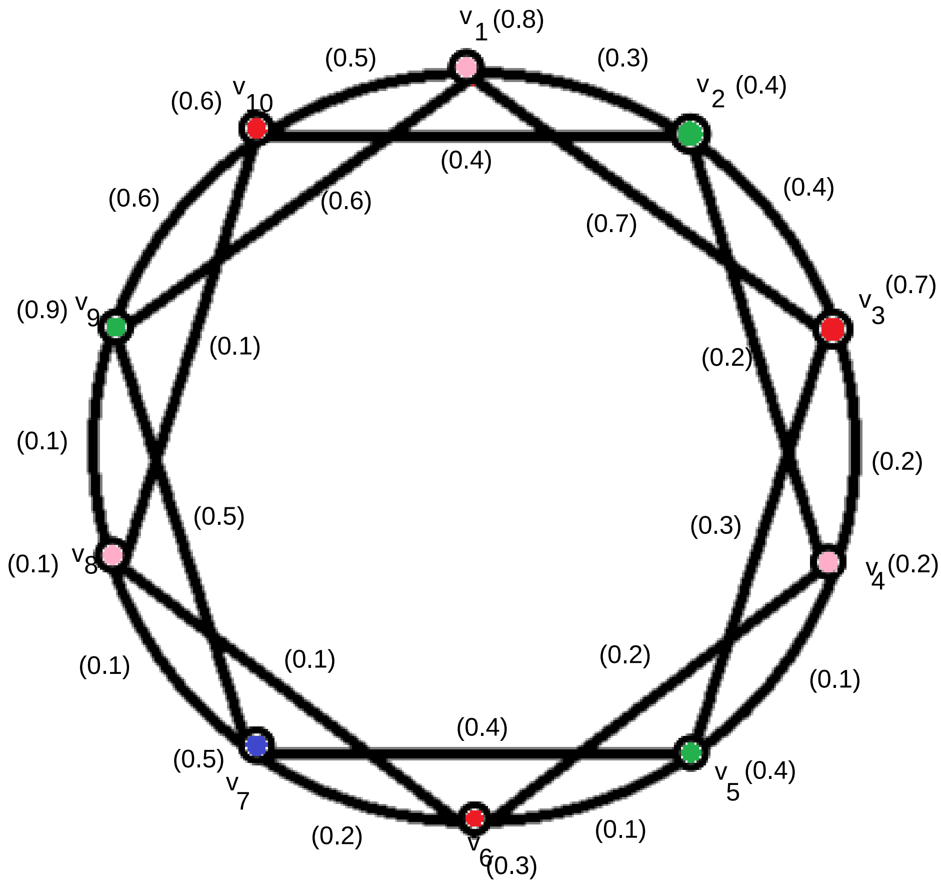


Figure 1: For Exaple 3.7

Definition 3.3. A vertex $v \in V(G)$ is called Γ^f-good if $\{v\}$ belongs to a Γ^f-partition. Otherwise v is said to be Γ^f-bad vertex.

Definition 3.4. A graph is called Γ^f-excellent fuzzy graph if every vertex of G is Γ^f-good.

Definition 3.5. A graph G is said to be Γ^f**- commendable**fuzzy graph if the number of Γ^f-good vertices is greater than the number of Γ^f-bad vertices.

A graph G is said to be Γ^f**- fair** fuzzy graph if the number of Γ^f-good vertices is equal to the number of Γ^f-bad vertices.

A graph G is said to be Γ^f**- poor** fuzzy graph if the number of Γ^f-good vertices is less than the number of Γ^f-bad vertices.

Definition 3.6. A fuzzy graph G is just χ^f-excellent if every vertex of G appears as a singleton in exactly one χ^f-partition.

Example 3.7. See Figure 1. The fuzzy colouring $\Gamma^f = \{\gamma_1, \gamma_2, \gamma_3, \gamma_4\}$
$\gamma_1(v_i) = 0.8 \quad i = 1$

$$\gamma_2(v_i) = \begin{cases} 0.4 & i = 2 \\ 0.4 & i = 5 \\ 0.1 & i = 8 \\ 0 & \text{otherwise} \end{cases}$$

$$\gamma_3(v_i) = \begin{cases} 0.7 & i = 3 \\ 0.3 & i = 6 \\ 0.9 & i = 9 \\ 0 & \text{otherwise} \end{cases}$$

$$\gamma_4(v_i) = \begin{cases} 0.2 & i = 4 \\ 0.5 & i = 7 \\ 0.6 & i = 10 \\ 0 & \text{otherwise} \end{cases}$$

For the above fuzzy graph $\chi^f(G) = 4$. Similarly, the χ^f-partitions are

$$\begin{aligned}
\Gamma_1^f &= \{\{v_1\}, \{v_2, v_5, v_8\}, \{v_3, v_6, v_9\}, \{v_4, v_7, v_{10}\}\} \\
\Gamma_2^f &= \{\{v_2\}, \{v_3, v_6, v_9\}, \{v_4, v_7, v_{10}\}, \{v_5, v_8, v_1\}\} \\
\Gamma_3^f &= \{\{v_3\}, \{v_4, v_7, v_{10}\}, \{v_5, v_8, v_1\}, \{v_6, v_9, v_2\}\} \\
\Gamma_4^f &= \{\{v_4\}, \{v_5, v_8, v_1\}, \{v_6, v_9, v_2\}, \{v_7, v_{10}, v_3\}\} \\
\Gamma_5^f &= \{\{v_5\}, \{v_6, v_9, v_1\}, \{v_7, v_{10}, v_3\}, \{v_8, v_1, v_4\}\} \\
\Gamma_6^f &= \{\{v_6\}, \{v_7, v_{10}, v_3\}, \{v_8, v_1, v_4\}, \{v_9, v_2, v_5\}\} \\
\Gamma_7^f &= \{\{v_7\}, \{v_8, v_1, v_4\}, \{v_9, v_2, v_5\}, \{v_{10}, v_3, v_6\}\} \\
\Gamma_8^f &= \{\{v_8\}, \{v_9, v_2, v_5\}, \{v_{10}, v_3, v_6\}, \{v_1, v_3, v_6\}\} \\
\Gamma_9^f &= \{\{v_9\}, \{v_{10}, v_3, v_6\}, \{v_1, v_4, v_7\}, \{v_2, v_5, v_8\}\} \\
\Gamma_{10}^f &= \{\{v_{10}\}, \{v_1, v_4, v_7\}, \{v_2, v_5, v_8\}, \{v_3, v_6, v_9\}\}
\end{aligned}$$

Therefore every vertex in above figure is appears in a singleton in exactly one χ^f-partition. Hence above figure is just χ^f-excellent.

Remark 3.8. (1) Every just χ^f-excellent fuzzy graph is χ^f-excellent graph

(2) Let G be any χ^f-excellent graph. Add a vertex u to every vertex in G such that $\mu(uv) \leq \sigma(u) \wedge \sigma(v)$ for every $v \in V(G)$. Let the resulting graph be H. Then H is χ^f-excellent but not just χ^f-excellent.

For every χ^f-partition of H contains $\{u\}$. Since G is χ^f-excellent, then for any $v \in V(G)$, there exists a χ^f-partition Γ^f of G such that $\{v\} \in \Gamma^f$. Then $\Gamma^f \cup \{u\}$ is χ^f-partition of H.

(3) If G is χ^f-excellent, then G has exactly one χ^f-partition(i.e.,G is uniquely colourable) iff G is complete.

For: If G is complete, then G is χ^f-excellent and it has exactly one χ^f-partition

Conversely, if G is χ^f-excellent and it has exactly one χ^f-partition, then every vertex in G must appear as a singleton in that χ^f-partition. Therefore G is complete.

Proposition 3.9. If G is not complete fuzzy graph and G is χ^f-excellent, then G has atleast two χ^f-partitions.

Proof. Let us take Γ^f be a χ^f-patition of G. Since G is not complete, then there exists atleast non full degree vertex say u. Let $\Gamma_1^f = \{\{u\}, V_2, \ldots, V_{\chi^f}\}$ be a χ^f-partition of G. Let $v \in V(G)$ such that u and v are not adjacent(i.e.,$\mu(uv) > \sigma(u) \wedge \sigma(v)$). Let $v \in V_i, 2 \leq i \leq \chi^f$. Then $\Gamma_2^f = \{\{V_i - \{v\}, \{u, v\}, V_3, \ldots, V_{\chi^f}\}$ is also a χ^f-partition of G not containing $\{u\}$. □

Proposition 3.10. If G is not complete fuzzy graph and G is χ^f-excellent, then G has atleast three χ^f-partitions.

Proof. We know that any χ^f-excellent non complete fuzzy graph has atleast two χ^f-partitions(from the above proposition). Suppose that G has exactly two χ^f-partitions Γ_1^f and Γ_2^f. Let $\Gamma_1^f = \{V_1, V_1, \ldots, V_{\chi^f}\}$ and $\Gamma_2^f = \{W_1, W_2, \ldots, W_{\chi^f}\}$ be the two partitions of G. Since G is χ^f-excellent and not complete, Γ_1^f has r singletons and Γ_2^f contains atleast $n - r$ singletons. Let Γ_1^f contains $\{u_1\}, \{u_2\}, \ldots, \{u_n\}$ and let $\{u_{r+1}\}, \ldots, \{u_n\}$ be the elements of Γ_2^f. Then $\langle u_1, u_2, \ldots, u_r \rangle$ is complete and also $\langle u_{r+1}, u_{r+2}, \ldots, u_n \rangle$ is complete.

Therefore in Γ_1^f there will be $\{u_1\}, \{u_2\}, \ldots, \{u_r\}, \{u_{r+1}\}, \ldots, \{u_n\}$ elements, a contradiction. Hence there are atleast three χ^f-partitions. □

Remark 3.11. A similar arguement as in the above proposition shows that there are atleast four χ^f-partitions

Remark 3.12. There exists fuzzy graphs having not full degree vertex and not just χ^f-excellent but χ^f-excellent.

Proposition 3.13. If G is just χ^f-excellent fuzzy graph and $G \neq K_n$ then $\chi^f = \lfloor \frac{n+1}{2} \rfloor$. The converse is not true.

Remark 3.14. P_n is not just χ^f-excellent fuzzy graph but is an induced subgraph of a just χ^f-excellent fuzzy graph.(If n is odd say $n = 2k + 1$, then P_n is an induced subgraph of C_{2k+3}. If n is even say $n = 2k$,then P_n is an induced subgraph of C_{2k+1}).

Remark 3.15. Let $G \neq K_n$, be a χ^f-excellent fuzzy graph with a full degree vertex. Then G is not just χ^f-excellent.

Proof. Since $G \neq K_n$, then $\chi^f(G) < n$. Let $\{u\}$ be a full degree vertex of G. Then clearly, G has atleast two χ^f-partitions. Then $\{u\}$ appears in all χ^f-partitions of G. Therefore G is not just χ^f-excellent. □

Proposition 3.16. If G is a just χ^f-excellent fuzzy graph and $G \neq K_n$,then any χ^f-partition of G can contain exactly one singleton.

Proof. Let us assume that there exists a χ^f-partition Γ^f of G containing more than one singleton. Let $\Gamma_1^f = \{\{u_1\}, \{u_2\}, V_3, \ldots, V_{\chi^f}\}$ be a partition of G. Since G is just χ^f-excellent and $G \neq K_n$, no vertex of $V(G)$ is a full degree vertex. Therefore there exists $v_1 \in V(G)$ such that u_1 and v_1 are not adjacent such that $\mu(u_1v_1) > \sigma(u_1) \wedge \sigma(v_1)$. Let $v_1 \in V_i, 3 \leq i \leq \chi^f$. Clearly, $|V_i| \geq 2$, for if $V_i = \{v_1\}$,then u_1 and v_1 are adjacent. Let $\Gamma_2^f = \{\{u_1, v_1\}, \{u_2\}, V_3, \ldots, V_i - \{v_1\}, \ldots, V_{\chi^f}\}$. Then Γ_2^f is a χ^f-partition containing $\{u_2\}$,which is a contradiction to G is just χ^f-excellent. □

Corollary 3.17. If G is just χ^f-excellent fuzzy graph and $G \neq K_n$, then $\chi^f \leq \lfloor \frac{n+1}{2} \rfloor$.

Proof. Since G is just χ^f-excellent,then any χ^f-partition contains exactly one singleton. Therefore $n \geq 1 + 2(\chi^f - 1)$. That is $n \geq 2(\chi^f - 1)$. Hence $\chi^f \leq \left\lfloor \frac{n+1}{2} \right\rfloor$. □

Remark 3.18. (1) W_6 has chromatic number $4 > \left\lfloor \frac{n+1}{2} \right\rfloor$ and W_6 is χ^f-excellent. Clearly, W_6 is not just χ^f-excellent.

(2) The bound is sharp as seen in C_5 ($\chi^f(C_5) = 3 = \frac{5+1}{2}$) and C_5 is just χ^f-excellent.

Remark 3.19. The sum of two just χ^f-excellent graphs need not be just χ^f-excellent.

For: C_5 is just χ^f-excellent but $C_5 + C_5$ is not just χ^f-excellent.

Remark 3.20. If $G + H$ is just χ^f-excellent fuzzy graph then G and H are just χ^f-excellent graph. Proof: Any chromatic partition of $G + H$ is a union of a chromatic partition of G and H.Then $G + H$ is just χ^f-excellent,then G and H are just χ^f-excellent.

Proposition 3.21. If G and H are just χ^f-excellent fuzzy graph and one of them is not complete if other is K_1 then $G + H$ is not just χ^f-excellent. Proof: Let $G = K_1$. Then H is not not complete fuzzy graph.Then $G + H$ is not complete but it has a full degree vertex. Therefore $G + H$ is not just χ^f-excellent graph.

Let $G \neq K_1$ and $h \neq K_1$. Since G and H are just χ^f-excellent, $G, H \neq \bar{K_n}$ for $n \geq 2$. Then any χ^f-partition of G and H contains atleast two elements. Then for any χ^f-partition of G with a singleton element, we can associate several χ^f-partitions of H, giving a χ^f-partition of $G + H$. Therefore $G + H$ is not just χ^f-excellent.

Proposition 3.22. Let G and H be two fuzzy graphs. $G + H$ is just χ^f-excellent if and only if both of them are complete graphs. Proof: Let us assume that G and H are complete fuzzy graph. Then $G + H$ is complete fuzzy graph and hence just χ^f-excellent.

Conversely, assume that $G + H$ are just χ^f-excellent. Therefore both G and H are just χ^f-excellent.If G or H is not complete, then using above remark, $G + H$ is not just χ^f-excellent, a contradiction. Therefore G and H are complete. Hence $G + H$ is complete fuzzy graph.

Proposition 3.23. Let $G \neq K_n$ be just χ^f-excellent graph. Let $u \in V(G)$. Let $\Gamma^f = \{\{u\}, V_2, \ldots, V_{\chi^f}$ be a χ^f-partition of G. Then for every vertex in $V_i, 2 \leq i \leq \chi^f$ is adjacent with atleast one vertex in V_j, for all j,$j \neq i, 2 \leq j \leq \chi^f$. Proof: Since G is just χ^f-excellent, $|V_i| \geq 2$ for all $i, 2 \leq i \leq \chi^f$. Let $v \in V_i$. suppose v is not adjacent to any vertex of some V_j such that $\mu(uv) > \sigma(u) \wedge \sigma(v)$,for $u \in V_j$,$j \neq i, 2 \leq j \leq \chi^f$. Then $\Gamma_1^f = \{\{u\}, V_2, \ldots, V_i - \{v\}, \ldots, V_j \cup \{v\}, \ldots, V_{\chi^f}\}$ is a χ^f-partition of G(note that $V_i - \{v\} \neq \emptyset$) different from Γ^f a contradiction.

Definition 3.24. A vertex of a fuzzy graph G with respect to a χ^f-partition Γ^f of G is called a fuzzy colourful vertex if it is adjacent to every colour class other than the one to which it belongs.

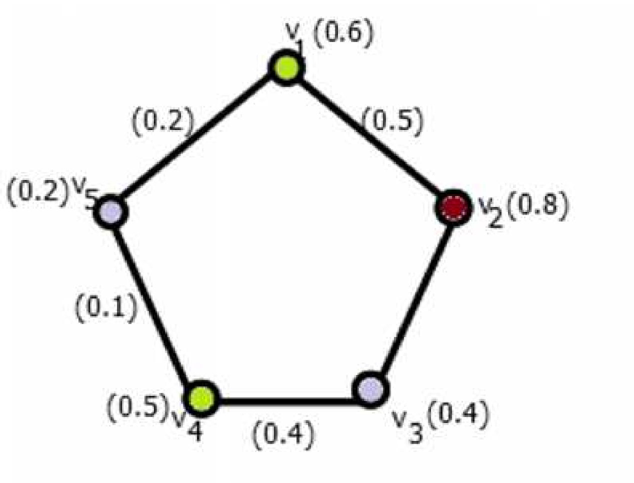


Figure 2: For Exaple 3.25

Let $\Gamma^f = \{V_1, V_2, \ldots, V_{\chi^f}\}$ be a χ^f-partition of G.Let $u \in V_i$ is said to be fuzzy colourful vertex if u is adjacent to every colour class in Γ^f-partition but not adjacent to V_i such that $\mu(uv_i) \leq \sigma(u) \wedge \sigma(v_i)$ for some vertex $v_i \in V_1, \ldots, V_{i-1}, V_{i+1}, \ldots, V_{\chi^f}$ and $\mu(uv_j) > \sigma(u) \wedge \sigma(v_j)$ for every $v_j \in V_i$

Example 3.25. See Figure 2. For the above figure $\chi^f(G) = 3$. Let

$$\Gamma^f = \{V_1 = \{v_1\}, V_2 = \{v_2, v_4\}, V_3 = \{v_3, v_5\}\}$$

be a χ^f-partition. From this partition $\{v_1\}$ is adjacent to some vertex in V_2 and V_3, $\{v_2\}$ is adjacent to V_1 and V_3, $\{v_4\}$ is adjacent to V_3 but not $\{v_1\}$, $\{v_3\}$ is adjacent to V_2 but not V_1, $\{v_5\}$ is adjacent to V_1 and V_2. Hence $\{v_1, v_2, v_5\}$ are colourful vertices with respect to the Γ^f-partition.

Corollary 3.26. (1) If G is just χ^f-excellent then every vertex in $N^f[u], u \in V(G)$ is a fuzzy colourful vertex in the χ^f-partition in which $\{u\}$ is an element. Then the number of colourful vertices is $deg^f(u) + 1$.

(2) There exists a χ^f-partition in which the number of fuzzy colourful vertices is equal to $\Delta^f(G) + 1$ which is greater than or equal to $\chi^f(G)$.

Theorem 3.27. Let G be a just χ^f-excellent fuzzy graph which is not complete.Let $u \in V(G)$ and let $\Gamma^f = \{\{u\}, V_2, \ldots, V_{\chi^f}\}$ be a χ^f-partition of G. If $|V_i| \geq 3|$ for some $2 \leq i \leq \chi^f$ then there exists a atleast some V_j with $|V_j| \geq 3$ containing a vertex not adjacent to u. Proof: Suppose let u is adjacent to every vertex in V_i with $|V_i| \geq 3(2 \leq i \leq \chi^f)$.

Case(1): $|V_i| \geq 3$ for all i, $2 \leq i \leq \chi^f$. Then u is a full degree vertex and it appears singleton in every χ^f-partition of G, which is a contradiction to G is just χ^f-excellent and $G \neq K_n$.

Case(2): Let $|V_i| \geq 3$ for all i, $2 \leq i \leq t$ and $|V_{t+1}| = 2$. Let $|V_{t+1} = \{v_1, v_2\}|$. Suppose there exists $V_{t+1}, V_{t+2}, \ldots, V_{\chi^f}$ such that $|V_{t+j}| = 2, 2 \leq j \leq \chi^f - t$(Note that no V_i,($2 \leq i \leq \chi^f$) is a singleton since G is just χ^f-excellent). Since Γ^f is a χ^f-partition, u is adjacent with atleast one vertex in each of $V_{t+1}, \ldots, V_{\chi^f}$. Suppose u

is adjacent with v_1 and not adjacent with v_2 in V_{t+1} such that $\mu(uv_1) \leq \sigma(u) \wedge \sigma(v_1)$ and $\mu(uv_2) > \sigma(u) \wedge \sigma(v_2)$for $v_1, v_2 \in V_{t+1}$.Then u is adjacent with every vertex $V_{t+j}, 2 \leq j \leq \chi^t - 1$ such that $\mu(uv_i) \leq \sigma(u) \wedge \sigma(v_i)$ for every $v_i \in V_{t+j}, 2 \leq j \leq \chi^f - 1$. For: otherwise there exists some vertex $w \in V_{t+j}$ not adjacent with u. Therefore $\Gamma_1^f = \{\{u, v_2, w\}, V_2, \ldots, V_t, \{v_1\}, \ldots, V_{t+j} - \{w\}, \ldots, V_{\chi^f}\}$ which is a contradiction to G is just χ^f-excellent. Hence u is adjacent with every vertex in $V - \{v_1\}$.(Note that if $V_{t+1} = V_{\chi^f}$ then also u is adjacent with every vertex in $V - \{v_2\}$). Since G is just χ^f-excellent there exists a χ^f-excellent $\Gamma_2^f = \{\{v_2\}, V_2', \ldots, V_{\chi^f}'\}$. Therefore $u \in V_i'$, a contradiction since u is adjacent with every vertex in $V - \{v_2\}$ such that $\mu(uv_i) \leq \sigma(u) \wedge \sigma(v_i)$ for every vertex $v_i \in V - \{v_2\}$. Hence the theorem.

Remark 3.28. Let G be a graph which is just χ^f-excellent. If there exists a χ^f-partition in which one of the element is a singleton $\{u\}$ and some other element with cardinality greater than or equal to 3, then there exists a χ^f-partition in which none of the elements is singleton. Proof: Let G be a just χ^f-excellent fuzzy graph satisfying the hypothesis. Then there exists a χ^f-partition $\Gamma^f = \{\{u\}, V_2, \ldots, V_{\chi^f}\}$ in which $|V_i| \geq 3$ for some i,$2 \leq i \leq \chi^f$ and V_i contains a non-neighbour,say, v and u. Then $\Gamma_1^f = \{\{u, v\}, V_2, \ldots, V_i - \{v\}, \ldots, V_{\chi^f}\}$ is a χ^f-partition of G in which each class contains atleast 2 vertices of G.

Remark 3.29. If G is just χ^f-excellent and $G \neq K_n$ and $\beta_0^f(G) = 2$, then the number of χ^f-partitions of G is exactly 'n'. For:

Let $V(G) = \{u_1, u_2, \ldots, u_k$, by the hypothesis there exists a χ^f-partitions $\{\{u_i, V_2, \ldots, V_k\}$ and $|V_i| = 2$ for all $2 \leq i \leq k$. Therefore $|V(G)| = 2k + 1$. Hence there can not exists a χ^f-partitions in which one of the element is a singleton.

Remark 3.30. If G is just χ^f-excellent and $G \neq K_n$, then G has exactly 'n' χ^f-partitions if and only if in those χ^f-partitions in which one element is a singleton, the cardinality of any other element of the partition is 2.

Remark 3.31. If G is just χ^f-excellent fuzzy graph, then $deg^f(u) \leq n - 3$ for any vertex $u \in V(G)$

4 Application

Fuzzy graph coloring has extensive applications in the following fields and solving different problems as follows: In Human Resource management such as assignment, job allocation, scheduling,In telecommunication process,In Bioinformatics, In traffic light problem.

5 Conclusion

In this paper we define new parameter called just chromatic excellence in fuzzy graphs. We can extend this concept to new type of fuzzy chromatic excellence and study the characteristics of this new parameter.

References

[1] Zadeh, L.A.,1971, Similarity relations and fuzzy ordering, *Information Sciences*, 3(2), pp.177-200.

[2] Kaufmann.A., 1976. *Introduction a Ia theorie des sous-ensembles flous, Io Elements thoriques de base*,Paris: Masson et cie.

[3] M.S Sunitha and Sunil Mathew , Fuzzy Graph Theory: A Survey, *Annals of Pure and Applied mathematics*, Vol. 4, No.1, 2013, 92-110.

[4] Eslahchi. C and B.N. Onagh, Vertex Strength of Fuzzy Graphs,*International Journal of Mathematics and Mathematical Sciences*,Volume 2006.

[5] Rosenfeld. A, *Fuzzy sets and their applications to cognitive and decision Process*, Academic press, New york, 1975,77-95.

[6] Lavanya. S.& Sattanathan. R,Fuzzy total coloring of fuzzy graphs,*International journal of information technology and knowledge management*, Volume 2, No.1, pp.37-39.

[7] Sambathkumar. E, Chromatically fixed, free and totally free vertices in a graph,*Journal of Combinatorics*, Information and system sciences, Vol.17, Nos.1-2,(1992), 130-138.

http://www.newtheory.org ISSN: 2149-1402

Journal of New Theory

Received: 17.09.2017 *Year*: **2017,** *Number*: **16**, *Pages*: **19-26**
Published: 05.10.2017 *Original Article*

ON SOME IDEALS OF INTUITIOISTIC FUZZY POINTS SEMIGROUPS

Essam Hamed Hamouda <ehamouda70@gmail.com>

Department of Basic Sciences, Faculty of Industrial Education, Beni-Suef University, Beni-Suef, Egypt

Abstract – In this paper, the minimal ideal A of a semigroup S is characterized by the intuitionistic characteristic function χ_A. The existence of an intuitionistic fuzzy kernel in a semigroup is explored. Finally, we consider the semigroup $\underline{S}$ of the intuitionistic fuzzy points of a semigroup S and discuss some relations between some ideals A of S and the subset $\underline{C_A}$ of the semigroup $\underline{S}$.

Keywords **–** *Semigroups; Intuitionistic fuzzy points; Intuitionistic fuzzy ideals.*

1 Introduction

Semigroups are important in many areas of mathematics, for example, coding and language theory, automata theory, combinatorics and mathematical analysis. Zadeh [16] introduced the concept of a fuzzy set for the first time and this concept was applied by Rosenfeld [14] to define fuzzy subgroups and fuzzy ideals. Based on this crucial work, Kuroki [9,10,11,12] defined a fuzzy semigroup and various kinds of fuzzy ideals in semigroups and characterized them. In [13], Kim considered the semigroup $\underline{S}$ of the fuzzy points of a semigroup S, and discussed the relation between the fuzzy interior ideals and the subsets of $\underline{S}$, also see [6, 7]. Atanassov [4, 5] introduced the notion of intuitionistic fuzzy sets as a generalization of fuzzy sets. Many concepts in fuzzy set theory were also extended to intuitionistic fuzzy set theory, such as intuitionistic fuzzy relations, intuitionistic L- fuzzy sets, intuitionistic fuzzy implications, intuitionistic fuzzy logics, intuitionistic fuzzy semigroups etc. Jun and Song [8] introduced the notion of intuitionistic fuzzy points. In [15] Sardar et al., defined some relations between the intuitionistic fuzzy ideals of a semigroup S and the set of all intuitionistic fuzzy points of S. In [3] Akram characterized intuitionistic fuzzy ideals in ternary semigroups by intuitionistic fuzzy points. Also in [2] he analyzed some relations between the intuitionistic fuzzy Γ-ideals and the sets of intuitionistic fuzzy points of these Γ-ideals of a Γ-semigroup. In this paper, we consider the semigroup $\underline{S}$ of the intuitionistic fuzzy points of a semigroup S, and discuss some relations between some ideals A of S and the subset $\underline{C_A}$ of the semigroup $\underline{S}$.

2 Basic Definitions and Results

Let S be a semigroup. A nonempty subset A of S is called a *left (resp., right) ideal* of S if $SA \subseteq A(resp., AS \subseteq A)$, and *a two-sided ideal* (or simply *ideal*) of S if A is both a left and a right ideal of S. A nonempty subset A of S is called an interior ideal of S if $\subseteq A$. An ideal A of S is called *minimal* ideal of S if A does not properly contains any other ideal of S. If the intersection K of all the ideals of a semigroup S is nonempty then we shall call K*the kernel* of S. A sub-semigroup A of S is called *a bi-ideal* of S if $ASA \subseteq A$. A function f from S to the closed interval [0, 1] is called a *fuzzy set* in S. The semigroup S itself is a fuzzy set in S such that $S(x) = 1$ for all $x \in S$, denoted also by S. Let A and B be two fuzzy sets in$S$. Then the inclusion relation $A \subseteq B$ is defined $A(x) \leq B(x)$ for all $x \in S$. $A \cap B$ and $A \cup B$ are fuzzy sets in S defined by

$$(A \cap B)(x) = A(x) \wedge B(x) = min\{A(x), B(x)\}$$

$$(A \cup B)(x) = A(x) \vee B(x) = max\{A(x), B(x)\}$$

for all $x \in S$.

For any $\alpha \in (0,1]$ and $x \in S$, a fuzzy set x_α in S is called a *fuzzy point* in S if

$$x_\alpha(y) = \begin{cases} \alpha & if\ x = y, \\ 0 & otherwise, \end{cases}$$

for all $x \in S$. The fuzzy point x_α is said to be contained in a fuzzy set A, denoted by$x_\alpha \in A$, iff $\alpha \leq A(x)$.

Definition 1. [4, 5] The intuitionistic fuzzy sets (IFS, for short) defined on a non-empty set X as objects having the form

$$A = \{< x, \mu_A(x), \gamma_A(x) >: x \in X\},$$

where the functions $\mu_A : X \to [0,1]$ and $\gamma_A : X \to [0,1]$ denote the degree of membership and the degree of non-membership of each element $x \in X$ to the set A respectively, and $0 \leq \mu_A(x) + \gamma_A(x) \leq 1$ for all $x \in X$.

For the sake of simplicity, we shall use$A = (\mu_A, \gamma_A)$for intuitionistic fuzzy set $A = \{< x, \mu_A(x), \gamma_A(x) >: x \in X\}$.

Definition 2. [15] Let $\alpha, \beta \in [0,1]$ with $\alpha + \beta \leq 1$. An intuitionistic fuzzy point, written as $x_{(\alpha,\beta)}$ is defined to be an intuitionistic fuzzy subset of S, given by

$$x_{(\alpha,\beta)}(y) = \begin{cases} (\alpha, \beta) & if\ x = y \\ (0,1) & otherwise \end{cases}$$

Definition 3. [15] A non-empty IFS $A = (\mu_A, \gamma_A)$ of a semigroup S is called an intuitionistic fuzzy subsemigroup of S if

(i) $\mu_A(xy) \geq \mu_A(x) \wedge \mu_A(y),\ \forall x, y \in S$,

(ii) $\gamma_A(xy) \leq \gamma_A(x) \vee \gamma_A(y),\ \forall x, y \in S$.

Definition 4. [15] An intuitionistic fuzzy subsemigroup $A = (\mu_A, \gamma_A)$ of a semigroup S is called an intuitionistic fuzzy interior ideal of S if

(i) $\mu_A(xay) \geq \mu_A(x)\ \ \forall x, a, y \in S$,

(ii) $\gamma_A(xay) \leq \gamma_A(a)\ \ \forall x, a, y \in S$.

Definition 5. [15] An intuitionistic fuzzy subsemigroup $A = (\mu_A, \gamma_A)$ of a semigroup S is called an intuitionistic fuzzy bi-ideal of S if

(i) $\mu_A(xwy) \geq \mu_A(x) \wedge \mu_A(y),\ \forall x, w, y \in S$,

(ii) $\gamma_A(xwy) \leq \gamma_A(x) \vee \gamma_A(y)\ \ \forall x, w, y \in S$.

Definition 6. [15] A non-empty IFS $A = (\mu_A, \gamma_A)$ of a semigroup S is called an intuitionistic fuzzy left (right) ideal of S if

(i) $\mu_A(xy) \geq \mu_A(y)$ (resp. $\mu_A(xy) \geq \mu_A(x)$) $\forall x, y \in S$,

(ii) $\gamma_A(xy) \leq \gamma_A(y)$ (resp. $\gamma_A(xy) \leq \gamma_A(y)$) $\forall x, y \in S$.

Definition 7. [15] A non-empty IFS $A = (\mu_A, \gamma_A)$ of a semigroup S is called an intuitionistic fuzzy two-sided ideal or an intuitionistic fuzzy ideal of S if it is both an intuitionistic fuzzy left and an intuitionistic fuzzy right ideal of S.

Let A be a subset of a semigroup S and A^c be the complement of A. $= (C_A, C_{A^c})$ is defined as:

$$C_A(x) = \begin{cases} 1 & if\ x \in A, \\ 0 & otherwise, \end{cases} \quad C_{A^c}(x) = \begin{cases} 0 & if\ x \in A, \\ 1 & otherwise, \end{cases}$$

for all $x \in S$.

Let $\mathcal{IF}(S)$ be the set of all intuitionistic fuzzy sets in a semigroup S. For each $A = (\mu_A, \gamma_A)$, $B = (\mu_B, \gamma_B) \in \mathcal{IF}(S)$, the product of A and B is an intuitionistic fuzzy set $A \circ B$ defined as follows:

$$A \circ B = \{< x, \mu_{A \circ B}(x),\ \gamma_{A \circ B}(x) >: x \in S\},$$

where

$$\mu_{A \circ B}(x) = \begin{cases} \bigvee_{x=uv} \mu_A(u) \wedge \mu_B(v) & if\ uv = x \\ 0 & otherwise. \end{cases}$$

$$\gamma_{A\circ B}(x) = \begin{cases} \bigwedge_{x=uv} \gamma_A(u) \vee \gamma_B(v) & \text{if } uv = x \\ 1 & \text{otherwise.} \end{cases}$$

Lemma 1. [1] *For any nonempty subsets A and B of a semigroup S, we have $A \subseteq B$ if and only if $\chi_A \subseteq \chi_B$.*

Lemma 2. [1] *Let A be a nonempty subset of a semigroup S, then A is an ideal of S if and only if χ_A is an intuitionistic fuzzy ideal of S.*

Theorem 1. *A nonempty subsetA of a semigroup S is a minimal ideal if and only if χ_A is a minimal intuitionistic fuzzy ideal of S.*

Proof .Let A be a minimal ideal of S, then by lemma 2., χ_A is an intuitionistic fuzzy ideal of S. Suppose that χ_A is not minimalintuitionistic fuzzy ideal of S, then there exists some intuitionistic fuzzy ideal χ_B of S such that $\chi_B \subseteq \chi_A$. Hence, lemma 1 implies that $B \subseteq A$, where B is an ideal of S. This is a contradiction to the fact that A is minimal ideal of S. Thus χ_A is minimalintuitionistic fuzzy ideal of S. Conversely, let χ_A be a minimalintuitionistic fuzzy ideal of S, then$A$ is an ideal of S. Suppose that A is not minimal ideal of S, then there exists some ideal B of S such that $B \subseteq A$. Now by lemma 1, $\chi_B \subseteq \chi_A$ where χ_B is an intuitionistic fuzzy ideal of S. This contradicts that χ_A is a minimalintuitionistic fuzzy ideal of S. Thus A is minimal ideal of S. ∎

Lemma 3. *If $A = (\mu_A, \gamma_A)$ is a minimal intuitionistic fuzzy ideal of a semigroup S , then A is the intuitionistic fuzzy kernel of S.*

Proof. Let $B = (\mu_B, \gamma_B)$ be any intuitionistic fuzzy ideal of S, then $B \circ A \subseteq B \cap A$. Since $B \cap A$ is an intuitionistic fuzzy ideal of S and $B \cap A \subseteq A$, it follows that $B \cap A = A$. But then$A = B \cap A \subseteq B$, so A is contained in every intuitionistic fuzzy ideal of S and hence is an intuitionistic fuzzy kernel of S. ∎

Lemma 4. *If $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy kernel of a semigroup S , then A is a simple intuitionistic fuzzy subsemigroup of S.*

Proof. Since A is *an intuitionistic fuzzy ideal of* S, so A is an intuitionistic fuzzy subsemigroup of S. To show that A is simple, let B be any intuitionistic fuzzy ideal of A, then $A \circ B \circ A$ is an intuitionistic fuzzy ideal of S, since

$$S \circ (A \circ B \circ A) = (S \circ A) \circ B \circ A \subseteq A \circ B \circ A$$

and

$$(A \circ B \circ A) \circ S = A \circ B \circ (A \circ S) \subseteq A \circ B \circ A$$

Also, $A \circ B \circ A \subseteq A \circ A \subseteq A$, but by lemma 3, A is minimal intuitionistic fuzzy ideal of S. Hence $A \circ B \circ A = A$. Also, $A \circ B \circ A \subseteq B \circ A \subseteq B$, which implies that $A \subseteq B$. Thus $A = B$, that is A is simple subsemigroup of S. ∎

Lemma 5. *Let $A = (\mu_A, \gamma_A)$ be an intuitionistic fuzzy left ideal of a semigroup S and $x_{(\alpha,\beta)}$ be any intuitionistic fuzzy point of S, then $A \circ x_{(\alpha,\beta)}$ is a minimal intuitionistic fuzzy left ideal of S.*

Proof. $A \circ x_{(\alpha,\beta)}$ is an intuitionistic fuzzy left ideal of S, since $S \circ (A \circ x_{(\alpha,\beta)}) = (S \circ A) \circ x_{(\alpha,\beta)} \subseteq A \circ x_{(\alpha,\beta)}$. Suppose B is an intuitionistic fuzzy left ideal of $A \circ x_{(\alpha,\beta)}$ and let $D = \{y_{(\gamma,\delta)} \in A: y_{(\gamma,\delta)} \circ x_{(\alpha,\beta)} \subseteq B\}$. Let $z_{(\sigma,\tau)}$ be in S and $y_{(\gamma,\delta)}$ be in D, then $z_{(\sigma,\tau)} \circ y_{(\gamma,\delta)} \in A$ and so $z_{(\sigma,\tau)} \circ y_{(\gamma,\delta)} \circ x_{(\alpha,\beta)} \subseteq B$. Hence $z_{(\sigma,\tau)} \circ y_{(\gamma,\delta)} \in D$, which implies that $S \circ D \subseteq D$. Thus D is an intuitionistic fuzzy left ideal of S contained in A and besause of minimality of A, we get $D = A$. Thus for all $y_{(\gamma,\delta)} \in A$, $y_{(\gamma,\delta)} \circ x_{(\alpha,\beta)} \in B$, which implies that $A \circ x_{(\alpha,\beta)} \subseteq B$. Hence $A \circ x_{(\alpha,\beta)} = B$ and therefore, $A \circ x_{(\alpha,\beta)}$is a minimal intuitionistic fuzzy left ideal of S. ■

3 Main Results

If S is a semigroup, then $\mathcal{IF}(S)$ is a semigroup with the product "$\circ$"[15]. Let $\underline{S}$ be the set of all intuitionistic fuzzy points in a semigroup S. Then $x_{(\alpha,\beta)} \circ y_{(\gamma,\delta)} = (xy)_{(\alpha\wedge\gamma,\beta\vee\delta)} \in \underline{S}$, for $x_{(\alpha,\beta)}, y_{(\gamma,\delta)} \in \underline{S}$ and $y_{(\alpha,\beta)} \circ (x_{(\sigma,\tau)} \circ z_{(\gamma,\delta)}) = (y_{(\alpha,\beta)} \circ x_{(\sigma,\tau)}) \circ z_{(\gamma,\delta)}$. Thus $\underline{S}$ is a subsemigroup of $\mathcal{IF}(S)$[15]. For any $A \in \mathcal{IF}(S)$, $\underline{A}$ denotes the set of all intuitionistic fuzzy points contained in A, that is, $\underline{A} = \{x_{(\alpha,\beta)} \in \underline{S}: \mu_A(x) \geq \alpha, \gamma_A(x) \leq \beta\}$. For any $\underline{A}, \underline{B} \subseteq \underline{S}$, we define the product of $\underline{A}$ and $\underline{B}$ as $\underline{A} \circ \underline{B} = \{x_{(\alpha,\beta)} \circ y_{(\gamma,\delta)}: x_{(\alpha,\beta)} \in \underline{A}, y_{(\gamma,\delta)} \in \underline{B}\}$.

Lemma 6. [15] *Let $A = (\mu_A, \gamma_A)$ and $B = (\mu_B, \gamma_B)$ be two intuitionistic fuzzy subsets of a semigroup S, then*

1) $\underline{A \cup B} = \underline{A} \cup \underline{B}$
2) $\underline{A \cap B} = \underline{A} \cap \underline{B}$.
3) $\underline{A} \circ \underline{B} \subseteq \underline{A \circ B}$.

Lemma 7. *Let A be nonempty subset of a semigroup S, we have $x_{(\alpha,\beta)} \in \underline{\chi_A}$ if and only if $x \in A$.*

Proof. Suppose that $x_{(\alpha,\beta)} \in \underline{\chi_A}$ for any $x \in S$, then $C_A(x) \geq \alpha$. Hence $C_A(x) = 1$ for any $\alpha > 0$, which implies that $x \in A$.Conversely, Let $x \in A$, then $C_A(x) = 1 \geq \alpha$ and $C_{A}(x) = 0 < \beta$ for any $\alpha, \beta > 0$. This means that $x_{(\alpha,\beta)} \in \underline{\chi_A}$. ■

Lemma 8. *For any nonempty subsets A and B of a semigroup S, we have*

1) $A \subseteq B$ *if and only if* $\underline{\chi_A} \subseteq \underline{\chi_B}$.
2) $\chi_A \subseteq \chi_B$ *if and only if* $\underline{\chi_A} \subseteq \underline{\chi_B}$.

Proof. (1)Assume that $A \subseteq B$, and let $x_{(\alpha,\beta)} \in \underline{\chi_A}$. By lemma 7, $x \in A \subseteq B$ and $x_{(\alpha,\beta)} \in \underline{\chi_B}$, this implies that $\underline{\chi_A} \subseteq \underline{\chi_B}$. Conversely, suppose that $\underline{\chi_A} \subseteq \underline{\chi_B}$. Let $x \in A$, then by lemma 7, for any $\alpha, \beta > 0 x_{(\alpha,\beta)} \in \underline{\chi_A}andx_{(\alpha,\beta)} \in \underline{\chi_B}$ which implies that $x \in B$. (2) it is obvious that if $\chi_A \subseteq \chi_B$ then $\underline{\chi_A} \subseteq \underline{\chi_B}$. Now assume that $\underline{\chi_A} \subseteq \underline{\chi_B}$ and let $x_{(\alpha,\beta)} \in \underline{\chi_A} \subseteq \underline{\chi_B}$, then $A \subseteq B$and consequently, we have $\chi_A \subseteq \chi_B$. This completes the proof. ■

Lemma 9. *Let A be a nonempty subset of a semigroup S. Then A is an ideal of S if and only if $\underline{\chi_A}$ is an ideal of $\underline{S}$.*

Proof. By lemma 2, A is an ideal of S if and only if χ_A is a fuzzy ideal of S. And from theorem 3.5[13], χ_A is a fuzzy ideal of S if and only if $\underline{\chi_A}$ is an ideal of $\underline{S}$. ∎

Theorem 2. *A nonempty subset A of a semigroup S is minimal ideal if and only if χ_A is a minimal intuitionistic fuzzy ideal of S.*

***Proof*.** Let A be a minimal ideal of S, then by lemma 2, χ_A is an intuitionistic fuzzy ideal of S. Suppose that χ_A is not minimalintuitionistic fuzzy ideal of S, then there exists some intuitionistic fuzzy ideal χ_B of S such that $\chi_B \subseteq \chi_A$. Hence, lemma 1 implies that $B \subseteq A$, where B is an ideal of S. This is a contradiction to the fact that A is minimal ideal of S. Thus χ_A is minimalintuitionistic fuzzy ideal of S. Conversely, let χ_A be a minimalintuitionistic fuzzy ideal of S, then by lemma, A is an ideal of S. Suppose that A is not minimal ideal of S, then there exists some ideal B of S such that $B \subseteq A$. Now by lemma, $\chi_B \subseteq \chi_A$ where χ_B is an intuitionistic fuzzy ideal of S. This contradicts that χ_A is a minimal intuitionistic fuzzy ideal of S. Thus A is minimal ideal of S. ∎

Theorem 3. *Let A be a nonempty subset of a semigroup S. Then A is a minimal ideal of S if and only if $\underline{\chi_A}$ is a minimal ideal of $\underline{S}$.*

***Proof*.** By theorem 1, A is a minimal ideal of S if and only if χ_A is an intuitionistic fuzzy minimal ideal of S. We only need to prove that, χ_A is a minimal intuitionistic fuzzy ideal of S if and only if $\underline{\chi_A}$ is a minimal ideal of $\underline{S}$. Let χ_A be a minimal intuitionistic fuzzy ideal of S, then $\underline{\chi_A}$ is an ideal of $\underline{S}$. Suppose that $\underline{\chi_A}$ is not minimal, then there exists some ideals $\underline{\chi_B}$ of $\underline{S}$ such that $\underline{\chi_B} \subseteq \underline{\chi_A}$ which implies that $\chi_B \subseteq \chi_A$, where χ_B is an intuitionistic fuzzy ideal of S. This is a contradiction to χ_A is a minimal intuitionistic fuzzy ideal of S. Thus $\underline{\chi_A}$ is a minimal ideal of $\underline{S}$. Conversely, assume that $\underline{\chi_A}$ is a minimal ideal of $\underline{S}$ and that χ_A is not a minimal intuitionistic fuzzy ideal of S. Then there exists an intuitionistic fuzzy ideal χ_B of S such that $\chi_B \subseteq \chi_A$. Hence $\underline{\chi_B} \subseteq \underline{\chi_A}$, where $\underline{\chi_B}$ is an ideal of $\underline{S}$. This contradicts that $\underline{\chi_A}$ is a minimal ideal of $\underline{S}$. This completes the proof of the theorem. ∎

Theorem 4. *Let A be a nonempty subset of a semigroup S. Then A is the kernel of S if and only if $\underline{\chi_A}$ is the kernel of $\underline{S}$.*

***Proof*.** Suppose that A is the kernel of S, then $A = \bigcap_i I_i$, where I_i is an ideal of S. Let $\underline{\chi_B}$ be an ideal of $\underline{S}$, then B is an ideal of S. Now we need to show that, $\underline{\chi_A} \subseteq \underline{\chi_B}$. Let $x_{(\alpha,\beta)} \in \underline{\chi_A}$, by lemma 7, $x \in A$ and also $x \in B$, since A is the kernel of S. This implies that $x_{(\alpha,\beta)} \in \underline{\chi_B}$ and hence, $\underline{\chi_A}$ is the kernel of $\underline{S}$. Conversely, Let $\underline{\chi_A}$ be the kernel of $\underline{S}$, then $\underline{\chi_A} \subseteq \underline{\chi_B}$, for every ideal $\underline{\chi_B}$ of $\underline{S}$. Thus $A \subseteq B$ and therefore, A is the kernel of S. ∎

The following lemma weakens the condition of theorem 4.

Lemma 10. *Let A be a minimal ideal of a semigroup S, then $\underline{\chi_A}$ is the kernel of $\underline{S}$.*

***Proof*.** Since A is a minimal ideal of S, then χ_A is a minimal intuitionistic fuzzy ideal of S. Also lemma 3 implies that χ_A is the fuzzy kernel of S. Now, let χ_B be an intuitionistic fuzzy ideal of S, then we have $\chi_A \subseteq \chi_B$ and hence $\underline{\chi_A} \subseteq \underline{\chi_B}$. So $\underline{\chi_A}$ is a minimal ideal contained in every ideal of $\underline{S}$. Thus $\underline{\chi_A}$ is the kernel of $\underline{S}$. ∎

Theorem 5. *Let A be a nonempty subset of a semigroup S. Then A is an interior ideal of S if and only if $\underline{\chi_A}$ is an interior ideal of $\underline{S}$.*

Proof. Let A be an interior ideal of S, and let $y_{(\alpha,\beta)}, z_{(\gamma,\delta)} \in \underline{S}$ and $x_{(\sigma,\tau)} \in \underline{\chi_A}$. Since $x \in A$, then $y_{(\alpha,\beta)} \circ x_{(\sigma,\tau)} \circ z_{(\gamma,\delta)} = (yxz)_{(\alpha\wedge\sigma\wedge\gamma,\ \beta\vee\tau\vee\delta)} \in \underline{\chi_A}$. This implies that $\underline{S} \circ \underline{\chi_A} \circ \underline{S} \subseteq \underline{\chi_A}$, thus $\underline{\chi_A}$ is an interior ideal of $\underline{S}$. Conversely, suppose that $\underline{\chi_A}$ is an interior ideal of $\underline{S}$. Let $y, z \in S$ and $x \in A$, then $x_{(\sigma,\tau)} \in \underline{\chi_A}$. Assume that, $y_{(\alpha,\beta)} \circ x_{(\sigma,\tau)} \circ z_{(\gamma,\delta)} = (yxz)_{(\alpha\wedge\sigma\wedge\gamma,\ \beta\vee\tau\vee\delta)} \in \underline{S} \circ \underline{\chi_A} \circ \underline{S} \subseteq \underline{\chi_A}$, then $yxz \in A$. This implies that $SAS \subseteq A$, and hence A is an interior ideal of S. ∎

Theorem 6. *Let A be a nonempty subset of a semigroup S. Then A is a bi- ideal of S if and only if $\underline{\chi_A}$ is a bi- ideal of $\underline{S}$.*

Proof. Let A be a bi- ideal of S, and let $y_{(\alpha,\beta)}, z_{(\gamma,\delta)} \in \underline{\chi_A}$ and $x_{(\sigma,\tau)} \in \underline{S}$. Since $y, z \in A$ and $yxz \in A$ then

$$y_{(\alpha,\beta)} \circ x_{(\sigma,\tau)} \circ z_{(\gamma,\delta)} = (yxz)_{(\alpha\wedge\sigma\wedge\gamma,\ \beta\vee\tau\vee\delta)} \in \underline{\chi_A}.$$

This implies that $\underline{\chi_A} \circ \underline{S} \circ \underline{\chi_A} \subseteq \underline{\chi_A}$, thus $\underline{\chi_A}$ is a bi-ideal of $\underline{S}$. Conversely, suppose that $\underline{\chi_A}$ is a bi-ideal of $\underline{S}$. Let $y, z \in A$ and $x \in S$, then $y_{(\alpha,\beta)}, z_{(\gamma,\delta)} \in \underline{\chi_A}$ by lemma 7. Assume that, $y_{(\alpha,\beta)} \circ x_{(\sigma,\tau)} \circ z_{(\gamma,\delta)} = (yxz)_{(\alpha\wedge\sigma\wedge\gamma,\ \beta\vee\tau\vee\delta)} \in \underline{\chi_A} \circ \underline{S} \circ \underline{\chi_A} \subseteq \underline{\chi_A}$, then $yxz \in A$. This implies that $ASA \subseteq A$, and hence A is a bi- ideal of S. ∎

Acknowledgements

The author is grateful for Editor-in-Chief and the referee for their valuable efforts.

References

[1] A. Ahn, K. Hur and H. Kay, *Intuitionistic fuzzy quasi-ideals of a semigroup,* Inter. J. Pure Applied Math. 23 (2) (2005), 207-229.

[2] M. Akram, *Characterizing Γ - semigroups by intuitionistic fuzzy points,* ARPN J. of System and Software, Vol. 2,12 (2012) 359-365.

[3] M. Akram, *Intuitionistic fuzzy points and ideals of ternary semigroups,* Int. J. of Algebra Statistics, Volume 1: 1(2012), 74-82.

[4] K. Atanassov, *Intuitionistic fuzzy sets,* Fuzzy Sets and Systems, 20(1986) 87-96.

[5] K. Atanassov, 1994, *New operations defined over the intuitionistic fuzzy sets*, Fuzzy Sets and Systems, 61(1994) 137-142.

[6] E. H. Hamouda, *On some ideals of fuzzy points semigroups*, Gen. Math. Notes Vol. 17, 2(2013) 76-80.

[7] E. H. Hamouda, *On Some Ideals of Fuzzy Points in ternary semigroups*, J. Semigroup Theory App. 3(2014) 1-9.

[8] Y. B. Jun, S. Z. Song, *Intuitionistic fuzzy semipreopen set and intuitionistic fuzzy semiprecontinuous mappings*, J. Appl. Math. Computing, 19(1-2) (2005) 467-474.
[9] N. Kuroki, *Fuzzy bi-ideals in semigroups*, Comment. Math. Univ. St. Pauli, 28: (1979)17-21.
[10] N. Kuroki, *On fuzzy ideals and bi-ideals in semigroups*, Fuzzy Sets and Systems 5(1981) 203-215.
[11] N. Kuroki, *Fuzzy semiprime ideals in semigroups*, Fuzzy Sets and Systems, 8 (1982) 71-79.
[12] N. Kuroki, *On fuzzy semigroups*, Inform. Sci., 53 (1991) 203-236.
[13] K. H. Kim, On fuzzy points in semigroups, Int. J. Math. & Math. Sc., 26(11) (2001)707-712.
[14] A. Rosenfeld, *Fuzzy groups*, J. Math. Anal. Appl., 35 (1971) 512-517.
[15] S. K. Sardar, M. Mandal and S. K. Majumder, *Intuitionistic fuzzy points in semigroups*, World Academy of Sciences, Engineering and technology75 (2011) 1107-1112.
[16] L. Zadeh, *Fuzzy sets*, Inform. & Control, 8 (1965) 338-353.

http://www.newtheory.org ISSN: 2149-1402

Journal of New Theory

Received: 25.09.2017 *Year*:**2017**, *Number*: **16**, *Pages*: **27-48**
Published: 06.10.2017 *Original Article*

SOME TOPOLOGICAL PROPERTIES OF SOFT DOUBLE TOPOLOGICAL SPACES

Osama Abd El-hamed El-Tantawy[1] <drosamat@yahoo.com>
Sobhy Ahmed Ali El-sheikh[2] <sobhyesheikh@yahoo.com>
Salama Hussien Ali Shaliel[2,*] <slamma-elarabi@yahoo.com>

[1]Mathematics Department, Faculty of Science, Zagazeg University, Zagazeg, Egypt.
[2]Mathematics Department, Faculty of Education, Ain Shams University, Cairo, Egypt.

Abstaract — In this paper, we introduce new separation axioms on soft double topological spaces and study some of their properties. Also, we define the soft double subspaces and study some related properties. Finally, we study the behaviour of the separation axioms under open (homeomorphism) mappings.

Keywords — *Soft double T_i^*-spaces ($T_i^{**}-$spaces), $(i = 0, 1, 2, 3)$, SDT_0-spaces, $SDT_{\frac{1}{2}}-$spaces, SDT_1-spaces, soft double Hausdorff spaces, soft double regular spaces, soft double R_2-spaces (SDR_2-spaces, for short), soft double subspaces, soft double open mappings, soft double closed mappings, soft double homeomorphism mappings, soft double continuous functions and separation axioms.*

1 Introduction

Atanassov [1, 2, 3, 4] introduced the concept of intuitionistic fuzzy sets as a generalization of fuzzy sets. Coker [5] generalized topological structures in intuitionistic fuzzy case. The concept of intuitionistic sets and the topology on intuitionistic sets was first given by Coker [7, 6].

In 2005, the suggestion of J. G. Garcia et al. [8] that double set is a more appropriate name than flou (intuitionistic) set, and double topology for the flou (intuitionistic) topology. Kandil et al. [11, 12] introduced the concept of double sets, double topological spaces, continuous functions between these spaces and separation axioms on double topological spaces.

After presentation of the operations of soft sets [16], the properties and applications of soft set theory have been studied increasingly [1, 14, 16, 18].

* *Corresponding Author.*

Recently, in 2011, Shabir and Naz [19] initiated the study of soft topological spaces. They defined soft topology on the collection τ of soft sets over X. Consequently, they defined basic notions of soft topological spaces such as open(closed) soft sets, soft subspace, soft separation axioms and established their several properties. Hussain and Ahmad [9] investigated the properties of soft nbds and soft closure operator.

In [21] Tantawy, et al. introduced the concept of soft double sets (SD-sets, for short), soft double points (SD-points, for short), soft double topological space (SDTS, for short) and continuous functions between these spaces.

The purpose of this paper is to introduce some separation axioms on SDTS (SD-separation axioms, for short) and some of its basic properties, soft double subspace (SD-subspace, for short) and some properties related to it, continuous function and separation axioms on SDTS. Moreover, some basic properties of these notions have obtained.

2 Preliminary

In this section, we collect some definitions and theorems which will be needed in the sequel. For more details see [9, 11, 12, 13, 15, 16, 17, 19, 20, 21, 22].

Definition 2.1. [12] Let X be a nonempty set.

1. A double set $\underline{A}$ is an ordered pair $(A_1, A_2) \in P(X) \times P(X)$ such that $A_1 \subseteq A_2$.
2. $D(X) = \{(A_1, A_2) \in P(X) \times P(X), A_1 \subseteq A_2\}$ is the family of all double sets on X.
3. Let $\eta_1, \eta_2 \subseteq P(X)$. The product of η_1 and η_2, denoted by $\eta_1 \hat{\times} \eta_2$, and defined by: $\eta_1 \hat{\times} \eta_2 = \{(A_1, A_2) \in \eta_1 \hat{\times} \eta_2 : A_1 \subseteq A_2\}$.
4. The double set $\underline{X} = (X, X)$ is called the universal double set.
5. The double set $\underline{\emptyset} = (\emptyset, \emptyset)$ is called the empty double set.
6. Let $x \in X$. Then, the double sets $\underline{x}_1 = (\{x\}, \{x\})$ and $\underline{x}_{\frac{1}{2}} = (\emptyset, \{x\})$ are said to be double points in X. The family of all double points in X, denoted by $DP(X)$ i.e, $DP(X) = \{x_t : x \in X, t \in \{\frac{1}{2}, 1\}\}$.
7. $\underline{x}_1 \underline{\in} \underline{A} \Leftrightarrow x \in A_1$ and $\underline{x}_{\frac{1}{2}} \underline{\in} \underline{A} \Leftrightarrow x \in A_2$.

Definition 2.2. [12] Let $\underline{A} = (A_1, A_2) \in D(X)$. $\underline{A}$ is called a finite double set if A_2 is a finite subset of X.

Definition 2.3. [12] Let $\underline{A} = (A_1, A_2)$, $\underline{B} = (B_1, B_2) \in D(X)$.

1. $\underline{A} \underline{\cup} \underline{B} = (A_1 \cup B_1,\ A_2 \cup B_2)$.
2. $\underline{A} \underline{\cap} \underline{B} = (A_1 \cap B_1,\ A_2 \cap B_2)$.

Definition 2.4. [11] Two double sets $\underline{A}$ and $\underline{B}$ are said to be a quasi-coincident, denoted by $\underline{A}q\underline{B}$, if $A_1 \cap B_2 \neq \emptyset$ or $A_2 \cap B_1 \neq \emptyset$. $\underline{A}$ is called a not quasi-coincident with $\underline{B}$, denoted by $\underline{A} \not{q}\underline{B}$, if $A_1 \cap B_2 = \emptyset$ and $A_2 \cap B_1 = \emptyset$.

Definition 2.5. [12] Let X be a non-empty set. The family η of double sets in X is called a double topology on X if it satisfies the following axioms:

1. $\underline{\emptyset}, \underline{X} \in \eta$,
2. If $\underline{A}, \underline{B} \in \eta$, then $\underline{A} \sqcap \underline{B} \in \eta$,
3. If $\{\underline{A}_s : s \in S\} \subseteq \eta$, then $\underline{\bigcup}_{s\in S} \underline{A}_s \in \eta$.

The pair (X, η) is called a double topological space. Each element of η is called an open double set in X. The complement of an open double set is called a closed double set.

Definition 2.6. [17] Let X be an initial universe and E be a set of parameters. Let $P(X)$ denotes the power set of X and A be a non-empty subset of E. A soft set F_A over the universal X is a mapping from the parameter set E to $P(X)$ with support A i.e., $F_A : E \longrightarrow P(X)$. In other words a soft set over X is a parameterized family of subsets of X, where $F_A(e) \neq \emptyset$ if $e \in A \subseteq E$ and $F_A(e) = \emptyset$ if $e \notin A$.
Note that, a soft set can be written in the following form, $F_A = \{(e, F_A(e)) : e \in A \subseteq E, F_A : E \longrightarrow P(X)\}$.

The family of all soft sets over X denoted by $S(X, E)$.

Definition 2.7. Let $F_E, G_E \in S(X, E)$.

1. F_E is said to be a null soft set, denoted by Φ, if $F_E(e) = \emptyset$, $\forall e \in E$. [16]
2. F_E is called absolute soft set, denoted by X_E, if $F_E(e) = X$, $\forall e \in E$. [16]

Definition 2.8. [19] Let τ be a collection of soft sets over a universal X with a fixed set of parameters E. τ is called a soft topology on X if it satisfies the following conditions:

1. $\Phi, X_E \in \tau$,
2. The union of any number of soft sets in τ belongs to τ,
3. The intersection of any two soft sets in τ belongs to τ.

The triple (X, τ, E) is called a soft topological space over X. Every element of τ is called an open soft set in X and its complement is called a closed soft set in X.

Definition 2.9. [21] Let X be an initial universe and E be a set of parameters. Let $D(X)$ denotes the family of all double sets over the universal X. A SD-set $\widetilde{F}_A$ over the universal X is a mapping from the parameter set E to $D(X)$ with support A i.e., $\widetilde{F}_A : E \longrightarrow D(X)$. In other words a SD-set over the universal X is a parameterized family of double subsets of X, where $\widetilde{F}_A(e) \neq \underline{\emptyset}$ if $e \in A \subseteq E$ and $\widetilde{F}_A(e) = \underline{\emptyset}$ if $e \notin A$.
Note that, a SD-set can be written in the following form, $\widetilde{F}_A = \{(e, \widetilde{F}_A(e)) : e \in A \subseteq E, \widetilde{F}_A : E \longrightarrow D(X)\}$.

The family of all SD-sets over X denoted by $SD(X)_E$.
In this paper we use the notation $\widetilde{F}_E$ for any SD-subset where, $\widetilde{F}_E(e) \neq \underline{\emptyset}$, $\forall e \in A$ *and* $\widetilde{F}_E(e) = \underline{\emptyset}$, $\forall e \notin A$.

Definition 2.10. Let $\widetilde{F}_E, \widetilde{G}_E \in SD(X)_E$. Then,

1. $\widetilde{F}_E$ is called a null SD-set, denoted by $\widetilde{\Phi}$, where $\widetilde{F}_E(e) = \underline{\emptyset}$, $\forall e \in E$. [21]

2. $\widetilde{F}_E$ is called an absolute SD-set, denoted by $\widetilde{X}$, where $\widetilde{F}_E(e) = \underline{X}$, $\forall e \in E$. [21]

3. $\widetilde{F}_E$ is a SD-subset of $\widetilde{G}_E$, denoted by $\widetilde{F}_E \widetilde{\subseteq} \widetilde{G}_E$, if $\widetilde{F}_E(e) \subseteq \widetilde{G}_E(e)$, $\forall e \in E$. [21]

4. $\widetilde{F}_E$ is equal to $\widetilde{G}_E$, denoted by $\widetilde{F}_E = \widetilde{G}_E$, if $\widetilde{F}_E(e) = \widetilde{G}_E(e)$, $\forall e \in E$. [21]

5. The union of $\widetilde{F}_E$ and $\widetilde{G}_E$ is a SD-set $\widetilde{H}_E$ defined by: $\widetilde{H}_E(e) = (\widetilde{F}_E \widetilde{\bigcup} \widetilde{G}_E)(e) = \widetilde{F}_E(e) \underline{\bigcup} \widetilde{G}_E(e)$, $\forall e \in E$. We write $\widetilde{F}_E \widetilde{\bigcup} \widetilde{G}_E = \widetilde{H}_E$. [21]

6. The intersection of $\widetilde{F}_E$ and $\widetilde{G}_E$ is a SD-set $\widetilde{H}_E$ defined by: $\widetilde{H}_E(e) = (\widetilde{F}_E \widetilde{\bigcap} \widetilde{G}_E)(e) = \widetilde{F}_E(e) \underline{\bigcap} \widetilde{G}_E(e)$, $\forall e \in E$. We write $\widetilde{F}_E \widetilde{\bigcap} \widetilde{G}_E = \widetilde{H}_E$. [21]

7. The difference of $\widetilde{F}_E$ and $\widetilde{G}_E$ is a SD-set $\widetilde{H}_E$ defined by: $\widetilde{H}_E(e) = \widetilde{F}_E(e) \setminus \widetilde{G}_E(e)$, $\forall e \in E$. We write $\widetilde{H}_E = \widetilde{F}_E \setminus \widetilde{G}_E$. [21]

8. The complement of $\widetilde{F}_E$, denoted by $\widetilde{F}^c_E$, defined by: $\widetilde{F}^c_E(e) = \underline{X} \setminus \widetilde{F}_E(e)$, $\forall e \in E$. and $(\widetilde{F}^c_E)^c = \widetilde{F}_E$. [21]

Definition 2.11. [21] Let $\widetilde{F}_E \in SD(X)_E$. $\widetilde{F}_E$ is called a SD-point for short over X if there exist $e \in E, x \in X$ and $t \in \{\frac{1}{2}, 1\}$ such that

$$\widetilde{F}_E(\alpha) = \begin{cases} \underline{x}_t, & \text{if } \alpha = e; \\ \underline{\emptyset}, & \text{if } \alpha \in E - \{e\}. \end{cases}$$

and we will denote $\widetilde{F}_E$ by $\widetilde{x}^e_t$.
The family of all SD-points over X will be denoted by $SDP(X)_E$.

Definition 2.12. [21] Two SD-sets $\widetilde{F}_E$ and $\widetilde{G}_E$ are said to be quasi- coincident, denoted by $\widetilde{F}_E \text{ q } \widetilde{G}_E$ if $\widetilde{F}_E(e) \text{ q } \widetilde{G}_E(e)$, for some $e \in E$. If $\widetilde{F}_E$ is not quasi- coincident with $\widetilde{G}_E$, we write $\widetilde{F}_E \not{q}\, \widetilde{G}_E$ or $\widetilde{F}_E(e) \not{q}\, \widetilde{G}_E(e)$, $\forall e \in E$.

Proposition 2.13. [21] Let $\widetilde{F}_E, \widetilde{G}_E, \widetilde{H}_E \in SD(X)_E$ *and* $\widetilde{x}^e_t \in SDP(X)_E$. Then,

1. $\widetilde{F}_E \not{q}\, \widetilde{G}_E \Leftrightarrow \widetilde{F}_E \widetilde{\subseteq} \widetilde{G}^c_E$.

2. $\widetilde{F}_E \not{q}\, \widetilde{G}_E, \widetilde{H}_E \widetilde{\subseteq} \widetilde{G}_E \Rightarrow \widetilde{F}_E \not{q}\, \widetilde{H}_E$.

3. $\widetilde{x}^e_t \not{q}\, (\widetilde{F}_E \widetilde{\bigcap} \widetilde{G}_E) \Leftrightarrow \widetilde{x}^e_t \not{q}\, \widetilde{F}_E$ *or* $\widetilde{x}^e_t \not{q}\, \widetilde{G}_E$.

Definition 2.14. [21] Let $SD(X)_E$ and $SD(Y)_K$ be the families of all SD-sets over X and Y, respectively.

1. The mapping $f_{\beta\psi} : SD(X)_E \rightarrow SD(Y)_K$ is called a soft double mapping, where $\beta : X \rightarrow Y$ and $\psi : E \rightarrow K$ are two mappings.

2. Let $\widetilde{F}_E \widetilde{\in} SD(X)_E$. Then, the image of $\widetilde{F}_E$ under the soft double mapping $f_{\beta\psi}$is the SD-set over Y, denoted by $f_{\beta\psi}(\widetilde{F}_E)$ and defined by:

$$f_{\beta\psi}(\widetilde{F}_E)(k) = \begin{cases} \beta(\underline{\bigcup}_{e\in\psi^{-1}(k)} \widetilde{F}_E(e)), & \text{if } \psi^{-1}(k) \neq \emptyset; \\ \underline{\emptyset}, & \text{otherwise.} \end{cases}$$

3. Let $\widetilde{G}_K \in SD(Y)_K$. The pre-image of $\widetilde{G}_K$ under the soft double mapping $f_{\beta\psi}$ is the SD-set over X, denoted by $f^{-1}_{\beta\psi}(\widetilde{G}_K)$ and defined by:

$$f^{-1}_{\beta\psi}(\widetilde{G}_K)(e) = \beta^{-1}(\widetilde{G}_K(\psi(e))).$$

Proposition 2.15. [21] Let $f_{\beta\psi} : SD(X)_E \rightarrow SD(Y)_K, \widetilde{F}_E,\ \widetilde{G}_E \in SD(X)_E$ and $\widetilde{H}_K, \widetilde{L}_K \in SD(Y)_K$. Then,

1. If $\widetilde{F}_E \widetilde{\subseteq} \widetilde{G}_E$, then $f_{\beta\psi}(\widetilde{F}_E) \widetilde{\subseteq} f_{\beta\psi}(\widetilde{G}_E)$.

2. If $\widetilde{H}_K \widetilde{\subseteq} \widetilde{L}_K$, then $f^{-1}_{\beta\psi}(\widetilde{H}_K) \widetilde{\subseteq} f^{-1}_{\beta\psi}(\widetilde{L}_K)$.

3. $\widetilde{F}_E \widetilde{\subseteq} f^{-1}_{\beta\psi}(f_{\beta\psi}(\widetilde{F}_E))$, the equality holds if $f_{\beta\psi}$ is an injective.

4. $f_{\beta\psi}(f^{-1}_{\beta\psi}(\widetilde{H}_K)) \widetilde{\subseteq} \widetilde{H}_K$, the equality holds if $f_{\beta\psi}$ is a surjective.

5. $f^{-1}_{\beta\psi}(\widetilde{H}^c_K) = (f^{-1}_{\beta\psi}(\widetilde{H}_K))^c$.

Definition 2.16. [21] Let $\widetilde{\tau}$ be a collection of SD-sets over X, i. e, $\widetilde{\tau} \subseteq SD(X)_E$. $\widetilde{\tau}$ is said to be a SD-topology over X if it satisfies the following conditions:

1. $\widetilde{\Phi}, \widetilde{X} \in \widetilde{\tau}$,

2. The union of any number of SD-sets in $\widetilde{\tau}$ belongs to $\widetilde{\tau}$,

3. The intersection of any two SD-sets in $\widetilde{\tau}$ belongs to $\widetilde{\tau}$.

The triple $(X, \widetilde{\tau}, E)$ is called a SDTS. Every member of $\widetilde{\tau}$ is called an open SD-set and its complement is called a closed SD-set.

The family of all closed SD-sets we denoted by $\widetilde{\tau}^c$.

Definition 2.17. [21] Let $(X, \widetilde{\tau}, E)$ be a SDTS and let $\widetilde{F}_E \in SD(X)_E$. $\widetilde{F}_E$ is called a quasi-neighborhood of a SD-point $\widetilde{x}^e_t$, if there exists $\widetilde{G}_E \in \widetilde{\tau}$ such that $\widetilde{x}^e_t q \widetilde{G}_E \widetilde{\subseteq} \widetilde{F}_E$. The family of all quasi-neighborhoods of $\widetilde{x}^e_t$ denoted by $N^q_{(\widetilde{x}^e_t)_E}$.

Definition 2.18. [21] Let $(X, \widetilde{\tau}, E)$ be a SDTS and let $\widetilde{F}_E \in SD(X)_E$. The soft double closure of $\widetilde{F}_E$, denoted by $cl_{\widetilde{\tau}}(\widetilde{F}_E)$, and defined by:
$cl_{\widetilde{\tau}}(\widetilde{F}_E) = \widetilde{\bigcap}\{\widetilde{G}_E \in \widetilde{\tau}^c : \widetilde{F}_E \widetilde{\subseteq} \widetilde{G}_E\}$.

Proposition 2.19. [21] Let $(X, \widetilde{\tau}, E)$ be a SDTS and let $\widetilde{F}_E \in SD(X)_E$. Then, $cl_{\widetilde{\tau}}(\widetilde{F}_E)$ is the smallest closed SD-set containing $\widetilde{F}_E$.

Proposition 2.20. [21] Let $\widetilde{F}_E \in SD(X)_E$ and $\widetilde{x}_t^e \in SDP(X)_E$. Then,

$$\widetilde{x}_t^e \; q \; cl_{\widetilde{\tau}}(\widetilde{F}_E) \Leftrightarrow \forall \widetilde{G}_E \in \widetilde{\tau}, \widetilde{x}_t^e \widetilde{\in} \widetilde{G}_E, \widetilde{G}_E \; q \; \widetilde{F}_E.$$

Definition 2.21. [21] Let $f_{\beta\psi} : SD(X)_E \to SD(Y)_K$, where $\beta : X \to Y$ and $\psi : E \to K$. Let $(X, \widetilde{\tau}, E)$ and $(Y, \widetilde{\sigma}, K)$ be two SDT-spaces. $f_{\beta\psi}$ is called a soft double continuous mapping, denoted by SD-continuous, if $f_{\beta\psi}^{-1}(\widetilde{H}_K) \in \widetilde{\tau}$, whenever $\widetilde{H}_K \in \widetilde{\sigma}$.

Proposition 2.22. [21] Let $(X, \widetilde{\tau}, E)$ and $(Y, \widetilde{\sigma}, K)$ be two SDT-spaces and let $f_{\beta\psi} : SD(X)_E \to SD(Y)_K$ be a mapping, $\widetilde{F}_E \in SD(X)_E$ and $\widetilde{H}_K \in SD(Y)_K$. Then, the following conditions are equivalent:

1. $f_{\beta\psi}$ is an SD-continuous,
2. $f_{\beta\psi}^{-1}(\widetilde{H}_K) \in \widetilde{\tau}^c$, $\forall \widetilde{H}_K \in \widetilde{\sigma}^c$,
3. $f_{\beta\psi}(cl_{\widetilde{\tau}}(\widetilde{F}_E)) \widetilde{\subseteq} cl_{\widetilde{\sigma}}(f_{\beta\psi}(\widetilde{F}_E))$, $\forall \widetilde{F}_E \in SD(X)_E$,
4. $cl_{\widetilde{\tau}}(f_{\beta\psi}^{-1}(\widetilde{H}_K)) \widetilde{\subseteq} f_{\beta\psi}^{-1}(cl_{\widetilde{\sigma}}(\widetilde{H}_K))$, $\forall \widetilde{H}_K \in SD(Y)_K$,

Definition 2.23. [10] A double topological space (X, η) is called $DT_{\frac{1}{2}}-$space iff for each $\underline{x}_t \in DP(X)$, either $\underline{x}_t$ is an open double set or $\underline{x}_t$ is a closed double set.

3 SD-separation axioms

Theorem 3.1. Let $\widetilde{F}_E, \widetilde{G}_E, \widetilde{H}_E \in SD(X)_E$. Then,

1. $\widetilde{F}_E \setminus \widetilde{G}_E = \widetilde{F}_E \widetilde{\bigcap} \widetilde{G}_E^c$.
2. $\widetilde{F}_E \setminus (\widetilde{G}_E \widetilde{\bigcup} \widetilde{H}_E) = (\widetilde{F}_E \setminus \widetilde{G}_E) \widetilde{\bigcap} (\widetilde{F}_E \setminus \widetilde{H}_E)$.
3. $\widetilde{F}_E \setminus (\widetilde{G}_E \widetilde{\bigcap} \widetilde{H}_E) = (\widetilde{F}_E \setminus \widetilde{G}_E) \widetilde{\bigcup} (\widetilde{F}_E \setminus \widetilde{H}_E)$.
4. $(\widetilde{F}_E \widetilde{\bigcap} \widetilde{G}_E) \setminus \widetilde{H}_E = (\widetilde{F}_E \setminus \widetilde{H}_E) \widetilde{\bigcap} (\widetilde{G}_E \setminus \widetilde{H}_E)$.

Proof.

1. $(\widetilde{F}_E \setminus \widetilde{G}_E)(e) = \widetilde{F}_E(e) \setminus \widetilde{G}_E(e) = \widetilde{F}_E(e) \underline{\bigcap} \widetilde{G}_E^c(e) = (\widetilde{F}_E \widetilde{\bigcap} \widetilde{G}_E^c)(e)$ $\forall e \in E$. Hence $\widetilde{F}_E \setminus \widetilde{G}_E = \widetilde{F}_E \widetilde{\bigcap} \widetilde{G}_E^c$.
2. $\widetilde{F}_E \setminus (\widetilde{G}_E \widetilde{\bigcup} \widetilde{H}_E) = \widetilde{F}_E \widetilde{\bigcap} (\widetilde{G}_E \widetilde{\bigcup} \widetilde{H}_E)^c = \widetilde{F}_E \widetilde{\bigcap} (\widetilde{G}_E^c \widetilde{\bigcap} \widetilde{H}_E^c) = (\widetilde{F}_E \widetilde{\bigcap} \widetilde{G}_E^c) \widetilde{\bigcap} (\widetilde{F}_E \widetilde{\bigcap} \widetilde{H}_E^c) = (\widetilde{F}_E \setminus \widetilde{G}_E) \widetilde{\bigcap} (\widetilde{F}_E \setminus \widetilde{H}_E)$.
3. It is similar to (2).
4. $(\widetilde{F}_E \widetilde{\bigcap} \widetilde{G}_E) \setminus \widetilde{H}_E = (\widetilde{F}_E \widetilde{\bigcap} \widetilde{G}_E) \widetilde{\bigcap} \widetilde{H}_E^c = (\widetilde{F}_E \widetilde{\bigcap} \widetilde{H}_E^c) \widetilde{\bigcap} (\widetilde{G}_E \widetilde{\bigcap} \widetilde{H}_E^c) = (\widetilde{F}_E \setminus \widetilde{H}_E) \widetilde{\bigcap} (\widetilde{G}_E \setminus \widetilde{H}_E)$.

Proposition 3.2. Let $\widetilde{x}_t^e, \widetilde{y}_r^{e'} \in SDP(X)_E$. Then,

1. $x \neq y \Rightarrow \widetilde{x}_t^e \not{q}\, \widetilde{y}_r^{e'}$ for every $r, t \in \{\frac{1}{2}, 1\}, e, e' \in E$.

2. $\widetilde{x}_t^e \not{q}\, \widetilde{y}_r^e \Leftrightarrow x \neq y$ *or* $x = y, t = r = \frac{1}{2}$ *and*
$\widetilde{x}_t^e q\, \widetilde{y}_r^{e'} \Leftrightarrow x = y$ *and* $t + r > 1$.

Proof. It is obvious.

Proposition 3.3. Let $(X, \widetilde{\tau}, E)$ be a SDTS and let $\widetilde{F}_E \in \tau, \widetilde{G}_E \in SD(X)_E$. Then, $\widetilde{F}_E\ q\ \widetilde{G}_E \Leftrightarrow \widetilde{F}_E\ q\ cl_{\widetilde{\tau}}(\widetilde{G}_E)$.

Proof. $\widetilde{F}_E \not{q}\, \widetilde{G}_E \Leftrightarrow \widetilde{G}_E \widetilde{\subseteq} \widetilde{F}_E^c \Leftrightarrow cl_{\widetilde{\tau}}(\widetilde{G}_E) \widetilde{\subseteq} \widetilde{F}_E^c[\ by\ Proposition\ \ 2.19] \Leftrightarrow \widetilde{F}_E \not{q}\, cl_{\widetilde{\tau}}(\widetilde{G}_E)$.

Definition 3.4. Let $\widetilde{\eta}$ be a collection of SD-sets over X, i. e, $\widetilde{\eta} \subseteq SD(X)_E$. Then, $\widetilde{\eta}$ is said to be a stratified soft double topology over X if it satisfies the following conditions:

1. $\widetilde{\Phi}, \widetilde{X}$ and $\widetilde{X}_\emptyset \in \widetilde{\eta}, \widetilde{X}_\emptyset(e) = (\emptyset, X),\ \forall\, e \in E$,

2. The union of any number of SD-sets in $\widetilde{\eta}$ belongs to $\widetilde{\eta}$,

3. The intersection of any two SD-sets in $\widetilde{\eta}$ belongs to $\widetilde{\eta}$.

The triple $(X, \widetilde{\eta}, E)$ is called a stratified soft double topological space (SSDTS). Each element of $\widetilde{\eta}$ is called an open SD-set in X. The complement of the open SD-set is called a closed SD-set.

Proposition 3.5. Let $f_{\beta\psi} : SD(X)_E \to SD(Y)_K, \widetilde{F}_E \in SD(X)_E$. Then, if $f_{\beta\psi}$ is one-one, onto, then $f_{\beta\psi}(\widetilde{F}_E^c) = (f_{\beta\psi}(\widetilde{F}_E))^c$.

Proof. Suppose that $f_{\beta\psi}$ is one-one, then $\widetilde{F}_E = f_{\beta\psi}^{-1}(f_{\beta\psi}(\widetilde{F}_E))$. Implies,

$$\widetilde{F}_E^c = (f_{\beta\psi}^{-1}(f_{\beta\psi}(\widetilde{F}_E)))^c = f_{\beta\psi}^{-1}(f_{\beta\psi}(\widetilde{F}_E))^c.$$

Since $f_{\beta\psi}$ is onto, then

$$f_{\beta\psi}(\widetilde{F}_E^c) = f_{\beta\psi}(f_{\beta\psi}^{-1}(f_{\beta\psi}(\widetilde{F}_E))^c) = (f_{\beta\psi}(\widetilde{F}_E))^c.$$

Hence, $f_{\beta\psi}(\widetilde{F}_E^c) = (f_{\beta\psi}(\widetilde{F}_E))^c$.

Definition 3.6. Let $(X, \widetilde{\tau}_1, E)$ and $(X, \widetilde{\tau}_2, E)$ be two SDTS over X.

1. If $\widetilde{\tau}_1 \subseteq \widetilde{\tau}_2$, then $\widetilde{\tau}_2$ is soft double finer than $\widetilde{\tau}_1$.

2. If $\widetilde{\tau}_1 \subset \widetilde{\tau}_2$, then $\widetilde{\tau}_2$ is soft double strictly finer than $\widetilde{\tau}_1$.

3. If $\widetilde{\tau}_1 \subseteq \widetilde{\tau}_2$ or $\widetilde{\tau}_2 \subseteq \widetilde{\tau}_1$, then $\widetilde{\tau}_1$ is comparable with $\widetilde{\tau}_2$.

Example 3.7. Let X be the universal set, E be the set of parameters.

1. If $\widetilde{\tau}$ is the collection of all SD-sets which can be defined over X. Then, $\widetilde{\tau}$ is called the discrete SD-topology on X and $(X, \widetilde{\tau}, E)$ is said to be a discrete SDTS over X.

2. $\widetilde{\tau} = \{\widetilde{\Phi}, \widetilde{X}\}$ is called the indiscrete SD-topology on X and $(X, \widetilde{\tau}, E)$ is said to be a indiscrete SDTS over X.

Definition 3.8. Let $\widetilde{F}_E \in SD(X)_E$. $\widetilde{F}_E$ is a finite SD-set if $\widetilde{F}_E(e)$ is a finite double set, $\forall e \in E$.

Example 3.9. Let X be an infinite set. The family

$$\widetilde{\tau}_\infty = \{\widetilde{\Phi}\}\widetilde{\bigcup}\{\widetilde{F}_E \widetilde{\subseteq} \widetilde{X} : \widetilde{F}^c_E \ is\ finite\ \}$$

is called a co-finite SD-topology on X.

Definition 3.10. Let $(X, \widetilde{\tau}, E)$ be a SDTS and let Y be a non-empty subset of X. $\widetilde{Y}$ denotes the SD-set over X, such that $\widetilde{Y}(e) = \underline{Y}$, $\forall\ e \in E$.

Definition 3.11. Let $(X, \widetilde{\tau}, E)$ be a SDTS and let Y be a non-empty subset of X, $\widetilde{F}_E \in SD(X)_E$. The SD-subset over Y, will denote by $\widetilde{F}^Y_E$, and defined by:

$$\widetilde{F}^Y_E(e) = \underline{Y}\underline{\bigcap}\widetilde{F}_E(e),\ \forall e \in E.$$

We write $\widetilde{F}^Y_E = \widetilde{Y}\widetilde{\bigcap}\widetilde{F}_E$.

Definition 3.12. Let $(X, \widetilde{\tau}, E)$ be a SDTS and Y be a non-empty subset of X. The soft double topology over Y, will denoted by $\widetilde{\tau}_Y$, and defined by:

$$\widetilde{\tau}_Y = \{\widetilde{F}^Y_E : \widetilde{F}_E \in \widetilde{\tau}\}.$$

$(Y, \widetilde{\tau}_Y, E)$ is called a SD-subspace of a SDTS $(X, \widetilde{\tau}, E)$.

Example 3.13. Any SD-subspace of a SD-discrete topological space is a SD-discrete. Also, any SD-subspace of a SD-indiscrete topological space is a SD-indiscrete.

Definition 3.14. A SDTS $(X, \widetilde{\tau}, E)$ is said to be:

1. SDT_0-space if $\widetilde{x}^e_t \notin \widetilde{y}^{e'}_r \Rightarrow cl_{\widetilde{\tau}}(\widetilde{x}^e_t) \notin \widetilde{y}^{e'}_r$ or $cl_{\widetilde{\tau}}(\widetilde{y}^{e'}_r) \notin \widetilde{x}^e_t$, $\forall \widetilde{x}^e_t, \widetilde{y}^{e'}_r \in SDP(X)_E$.

2. $SDT_{\frac{1}{2}}-$space if each $\widetilde{x}^e_t \in SDP(X)_E$ is either open SD-set or closed SD-set.

3. SDT^*_0-space if $\widetilde{x}^e_t \notin \widetilde{y}^{e'}_r \Rightarrow cl_{\widetilde{\tau}}(\widetilde{x}^e_t) \notin \widetilde{y}^{e'}_r$ or $cl_{\widetilde{\tau}}(\widetilde{y}^{e'}_r) \notin \widetilde{x}^e_t$, $\forall \widetilde{x}^e_t, \widetilde{y}^{e'}_r \in SDP(X)_E, x \neq y,\ \forall x, y \in X$.

4. SDT^{**}_0-space if $\widetilde{x}^e_t \notin \widetilde{y}^{e'}_r \Rightarrow cl_{\widetilde{\tau}}(\widetilde{x}^e_t) \notin \widetilde{y}^{e'}_r$ or $cl_{\widetilde{\tau}}(\widetilde{y}^{e'}_r) \notin \widetilde{x}^e_t$, $\forall \widetilde{x}^e_t, \widetilde{y}^{e'}_r \in SDP(X)_E, x = y,\ \forall x, y \in X$.

5. SDT_1-space if $\widetilde{x}^e_t \notin \widetilde{y}^{e'}_r \Rightarrow cl_{\widetilde{\tau}}(\widetilde{x}^e_t) \notin \widetilde{y}^{e'}_r$ and $cl_{\widetilde{\tau}}(\widetilde{y}^{e'}_r) \notin \widetilde{x}^e_t$, $\forall \widetilde{x}^e_t, \widetilde{y}^{e'}_r \in SDP(X)_E$.

6. SDT^*_1-space if $\widetilde{x}^e_t \notin \widetilde{y}^{e'}_r \Rightarrow cl_{\widetilde{\tau}}(\widetilde{x}^e_t) \notin \widetilde{y}^{e'}_r$ and $cl_{\widetilde{\tau}}(\widetilde{y}^{e'}_r) \notin \widetilde{x}^e_t$, $\forall \widetilde{x}^e_t, \widetilde{y}^{e'}_r \in SDP(X)_E, x \neq y,\ \forall x, y \in X$.

7. SDT^{**}_1-space if $\widetilde{x}^e_t \notin \widetilde{y}^{e'}_r \Rightarrow cl_{\widetilde{\tau}}(\widetilde{x}^e_t) \notin \widetilde{y}^{e'}_r$ and $cl_{\widetilde{\tau}}(\widetilde{y}^{e'}_r) \notin \widetilde{x}^e_t$, $\forall \widetilde{x}^e_t, \widetilde{y}^{e'}_r \in SDP(X)_E, x = y,\ \forall x, y \in X$.

8. SDT_2-space or soft double Hausdorff space if $\widetilde{x}_t^e \notin \widetilde{y}_r^{e'} \Rightarrow \exists\ \widetilde{O}_{\widetilde{x}_t^e}, \widetilde{O}_{\widetilde{y}_r^{e'}}$ such that $\widetilde{O}_{\widetilde{x}_t^e} \notin \widetilde{O}_{\widetilde{y}_r^{e'}},\ \forall \widetilde{x}_t^e, \widetilde{y}_r^{e'} \in SDP(X)_E$.

9. SDT_2^*-space if $\widetilde{x}_t^e \notin \widetilde{y}_r^{e'} \Rightarrow \exists \widetilde{O}_{\widetilde{x}_t^e}, \widetilde{O}_{\widetilde{y}_r^{e'}}$ such that $\widetilde{O}_{\widetilde{x}_t^e} \notin \widetilde{O}_{\widetilde{y}_r^{e'}},\ \forall \widetilde{x}_t^e, \widetilde{y}_r^{e'} \in SDP(X)_E, x \neq y,\ \forall x, y \in X$.

10. $SDT_2^{**}-$space if $\widetilde{x}_t^e \notin \widetilde{y}_r^{e'} \Rightarrow \exists \widetilde{O}_{\widetilde{x}_t^e}, \widetilde{O}_{\widetilde{y}_r^{e'}}$ such that $\widetilde{O}_{\widetilde{x}_t^e} \notin \widetilde{O}_{\widetilde{y}_r^{e'}},\ \forall \widetilde{x}_t^e, \widetilde{y}_r^{e'} \in SDP(X)_E, x = y,\ \forall x, y \in X$.

11. SDR_2-space if $\widetilde{x}_t^e \notin \widetilde{F} \Rightarrow \exists\ \widetilde{O}_{\widetilde{x}_t^e}, \widetilde{O}_{\widetilde{F}} \in \widetilde{\tau}$ such that $\widetilde{O}_{\widetilde{x}_t^e} \notin \widetilde{O}_{\widetilde{F}},\ \forall \widetilde{x}_t^e \in SDP(X)_E,\ \forall \widetilde{F} \in \widetilde{\tau}^c$.

12. SDT_3-space or soft double regular space if it is SDR_2 and SDT_1-spaces.

13. SDT_3^*-space if it is SDR_2 and SDT_1^*-spaces.

14. $SDT_3^{**}-$space if it is SDR_2 and $SDT_1^{**}-$spaces.

Theorem 3.15. Let $(X, \widetilde{\tau}, E)$ be a SDTS. Then,
$(X, \widetilde{\tau}, E)$ is SDT_1-space (SDT_1^*-space) iff $\forall \widetilde{x}_t^e\ \not q\ \exists\ \widetilde{O}_{\widetilde{x}_t^e}$ such that $\widetilde{y}_r^{e'}\ \not q\ \widetilde{O}_{\widetilde{x}_t^e}$ and $\exists\ \widetilde{O}_{\widetilde{y}_r^{e'}}$ such that $\widetilde{x}_t^e \notin \widetilde{O}_{\widetilde{y}_r^{e'}}$.

Proof. It follows from Proposition 2.20.

Theorem 3.16. Let $(X, \widetilde{\tau}, E)$ be a SDTS. Then,
$(X, \widetilde{\tau}, E)$ is SDT_1^*-space iff $\widetilde{x}_t^e \notin \widetilde{y}_r^{e'}, \widetilde{y}_r^{e'}, x \neq y,\ \forall x, y \in X \exists\ \widetilde{O}_{\widetilde{x}_t^e}$ such that $\widetilde{y}_r^{e'} \notin \widetilde{O}_{\widetilde{x}_t^e}$ and $\exists\ \widetilde{O}_{\widetilde{y}_r^{e'}}$ such that $\widetilde{x}_t^e \notin \widetilde{O}_{\widetilde{y}_r^{e'}}$.

Proof. It is obvious.

Theorem 3.17. Let $(X, \widetilde{\tau}, E)$ be a SDTS. Then,
$(X, \widetilde{\tau}, E)$ is SDT_1-space iff $\widetilde{x}_t^e = cl_{\widetilde{\tau}}(\widetilde{x}_t^e),\ \forall \widetilde{x}_t^e \in SDP(X)_E$.

Proof. Suppose $(X, \widetilde{\tau}, E)$ is a SDT_1-space and let $\widetilde{x}_t^e \notin \widetilde{y}_r^{e'}$. Then, $cl_{\widetilde{\tau}}(\widetilde{x}_t^e) \notin \widetilde{y}_r^{e'}$. By Theorem 3.15, there exists $\widetilde{O}_{y_r^{e'}}$ such that $\widetilde{x}_t^e \not q\ \widetilde{O}_{y_r^{e'}}$. This implies that $\widetilde{O}_{y_r^{e'}} \widetilde{\subseteq} (\widetilde{x}_t^e)^c$, thus $(\widetilde{x}_t^e)^c$ is open SD-set, $\forall \widetilde{x}_t^e \in SDP(X)_E$, i.e, $\widetilde{x}_t^e$ is closed SD-set, $\forall \widetilde{x}_t^e \in SDP(X)_E$. Conversely, Suppose that $\widetilde{x}_t^e = cl_{\widetilde{\tau}}(\widetilde{x}_t^e),\ \forall \widetilde{x}_t^e \in SDP(X)_E$ and let $\widetilde{x}_t^e \notin \widetilde{y}_r^{e'}$. Then, $\widetilde{x}_t^e$ and $\widetilde{y}_r^{e'}$ are closed SD-sets. So that, $cl_{\widetilde{\tau}}(\widetilde{x}_t^e)\ \not q\ \widetilde{y}_r^{e'}$ and $cl_{\widetilde{\tau}}(\widetilde{y}_r^{e'})\ \not q\ \widetilde{x}_t^e,\ \forall \widetilde{x}_t^e, \widetilde{y}_r^{e'} \in SDP(X)_E$. Hence, $(X, \widetilde{\tau}, E)$ is a SDT_1.

Theorem 3.18. Let $(X, \widetilde{\tau}, E)$ be a SDTS. Then,
$(X, \widetilde{\tau}, E)$ is SDT_1^*-space iff $\widetilde{x}_t^e = cl_{\widetilde{\tau}}(\widetilde{x}_t^e),\ \forall \widetilde{x}_t^e \in SDP(X)_E$.

Proof. It is obvious.

Theorem 3.19. Let $(X, \widetilde{\tau}, E)$ be a SDTS. Then,
$(X, \widetilde{\tau}, E)$ is SDT_2-space iff $\widetilde{x}_t^e = \widetilde{\bigcap}_{\widetilde{O}_{\widetilde{x}_t^e} \in N^q_{(\widetilde{x}_t^e)_E}} cl_{\widetilde{\tau}}(\widetilde{O}_{\widetilde{x}_t^e}),\ \forall \widetilde{x}_t^e \in SDP(X)_E$.

Proof. Suppose $(X, \widetilde{\tau}, E)$ is a SDT_2-space and let $\widetilde{x}_t^e \not q \widetilde{y}_r^{e'}$.
Then, $\exists\ \widetilde{O}_{x_t^e} \in N_{(\widetilde{x}_t^e)_E^q}$, $\widetilde{O}_{y_r^{e'}} \in N_{(\widetilde{y}_r^{e'})_E^q}$ such that $\widetilde{O}_{x_t^e} \not q\ \widetilde{O}_{y_r^{e'}}$. So that $\widetilde{O}_{y_r^{e'}} \not q\ \widetilde{O}_{x_t^e}$, implies $\widetilde{O}_{y_r^{e'}} \not q \widetilde{\bigcap}_{\widetilde{O}_{\widetilde{x}_t^e} \in N^q_{(\widetilde{x}_t^e)_E}} cl_{\widetilde{\tau}}(\widetilde{O}_{\widetilde{x}_t^e})$. Thus, $\widetilde{x}_t^e \widetilde{\supseteq} \widetilde{\bigcap}_{\widetilde{O}_{\widetilde{x}_t^e} \in N^q_{(\widetilde{x}_t^e)_E}} cl_{\widetilde{\tau}}(\widetilde{O}_{\widetilde{x}_t^e})$. It is clear that, $\widetilde{x}_t^e \widetilde{\subseteq} \widetilde{\bigcap}_{\widetilde{O}_{\widetilde{x}_t^e} \in N^q_{(\widetilde{x}_t^e)_E}} cl_{\widetilde{\tau}}(\widetilde{O}_{\widetilde{x}_t^e})$. Hence, $\widetilde{x}_t^e = \widetilde{\bigcap}_{\widetilde{O}_{\widetilde{x}_t^e} \in N^q_{(\widetilde{x}_t^e)_E}} cl_{\widetilde{\tau}}(\widetilde{O}_{\widetilde{x}_t^e})$.

Conversely, let $\widetilde{x}_t^e = \widetilde{\bigcap}_{\widetilde{O}_{\widetilde{x}_t^e} \in N^q_{(\widetilde{x}_t^e)_E}} cl_{\widetilde{\tau}}(\widetilde{O}_{\widetilde{x}_t^e})$, $\forall \widetilde{x}_t^e \in SDP(X)_E$ and let $\widetilde{x}_t^e \not q\ \widetilde{y}_r^{e'}$. Then, $\widetilde{x}_t^e \not q \widetilde{\bigcap}_{\widetilde{O}_{\widetilde{y}_r^{e'}} \in N^q_{(\widetilde{y}_r^{e'})_E}} cl_{\widetilde{\tau}}(\widetilde{O}_{\widetilde{y}_r^{e'}})$. This implies that, $\widetilde{x}_t^e \not q\ cl_{\widetilde{\tau}}(\widetilde{O}_{\widetilde{y}_r^{e'}})$, for some $\widetilde{O}_{y_r^{e'}} \in N_{(\widetilde{y}_r^{e'})_E^q}$. So, $\widetilde{x}_t^e \widetilde{\subseteq} (cl_{\widetilde{\tau}}(\widetilde{O}_{\widetilde{y}_r^{e'}}))^c$ and $\widetilde{O}_{x_t^e} = (cl_{\widetilde{\tau}}(\widetilde{O}_{\widetilde{y}_r^{e'}}))^c \not q\ \widetilde{O}_{\widetilde{y}_r^{e'}}$. Therefore, $(X, \widetilde{\tau}, E)$ is a SDT_2.

Theorem 3.20. Let $(X, \widetilde{\tau}, E)$ be a SDTS. Then,
$(X, \widetilde{\tau}, E)$ is SDT_2^*-space iff $\widetilde{x}_t^e = \widetilde{\bigcap}_{\widetilde{O}_{\widetilde{x}_t^e} \in N^q_{(\widetilde{x}_t^e)_E}} cl_{\widetilde{\tau}}(\widetilde{O}_{\widetilde{x}_t^e})$, $\forall \widetilde{x}_t^e \in SDP(X)_E$.

Proof. It is obvious.

Theorem 3.21. Let $(X, \widetilde{\tau}, E)$ be a SDTS. Then,
$(X, \widetilde{\tau}, E)$ is a SDT_0-space $\rightarrow (X, \widetilde{\tau}, E)$ is a SDT_0^*.

Proof. It is obvious.

Example 3.22. Let $X = \{h_1, h_2\}, E = \{e_1, e_2\}$ and let $\widetilde{\tau} = \{\widetilde{\Phi}, \widetilde{X}, \widetilde{F}_E^1, \widetilde{F}_E^2, \widetilde{F}_E^3, \widetilde{F}_E^4\}$, where
$\widetilde{F}_E^1(e_1) = \underline{\emptyset}, \widetilde{F}_E^1(e_2) = (\{h_2\}, \{h_2\})$,
$\widetilde{F}_E^2(e_1) = \underline{\emptyset}, \widetilde{F}_E^2(e_2) = \underline{X}$,
$\widetilde{F}_E^3(e_1) = (\{h_1\}, \{h_1\}), \widetilde{F}_E^3(e_2) = \underline{X}$,
$\widetilde{F}_E^4(e_1) = (\{h_2\}, \{h_2\}), \widetilde{F}_E^4(e_2) = \underline{X}$.
Then, $(X, \widetilde{\tau}, E)$ is a SDTS and SDT_0^*-space. But it is not SDT_0-space, for $\exists\ \widetilde{h}^{e_1}_{1_{\frac{1}{2}}} \in SDP(X)_E$ such that $\widetilde{h}^{e_1}_{1_{\frac{1}{2}}} \not q\ \widetilde{h}^{e_1}_{1_{\frac{1}{2}}}$, but $\widetilde{F}_E^{4c} = cl_{\widetilde{\tau}}(\widetilde{h}^{e_1}_{1_{\frac{1}{2}}}) q\ \widetilde{h}^{e_1}_{1_{\frac{1}{2}}}$.

Theorem 3.23. Let $(X, \widetilde{\tau}, E)$ be a SDTS. Then,
$(X, \widetilde{\tau}, E)$ is a $SDT_{\frac{1}{2}}-$space $\rightarrow (X, \widetilde{\tau}, E)$ is a SDT_0.

Proof. Suppose $(X, \widetilde{\tau}, E)$ is a $SDT_{\frac{1}{2}}-$space and let $\widetilde{x}_t^e \not q\ \widetilde{y}_r^{e'}$. Now, if $\widetilde{x}_t^e$ is an open SD-point, then by Proposition 3.3 $cl_{\widetilde{\tau}}(\widetilde{y}_r^{e'}) \not q\ \widetilde{x}_t^e$. On the other hand, if $\widetilde{x}_t^e$ is a closed SD-point, then $cl_{\widetilde{\tau}}(\widetilde{x}_t^e) = \widetilde{x}_t^e$. Implies, $cl_{\widetilde{\tau}}(\widetilde{x}_t^e) \not q\ \widetilde{y}_r^{e'}$. Hence, $(X, \widetilde{\tau}, E)$ is a SDT_0.

Example 3.24. Let $X = \{h_1, h_2\}, E = \{e_1, e_2\}$ and let $\widetilde{\tau} = \{\widetilde{\Phi}, \widetilde{X}, \widetilde{F}_E^1, \widetilde{F}_E^2, ..., \widetilde{F}_E^{37}\}$, where
$\widetilde{F}_E^1(e_1) = \underline{\emptyset}, \widetilde{F}_E^1(e_2) = (\{h_2\}, \{h_2\})$,
$\widetilde{F}_E^2(e_1) = \underline{\emptyset}, \widetilde{F}_E^2(e_2) = \underline{X}$,
$\widetilde{F}_E^3(e_1) = (\{h_1\}, \{h_1\}), \widetilde{F}_E^3(e_2) = \underline{X}$,
$\widetilde{F}_E^4(e_1) = (\{h_2\}, \{h_2\}), \widetilde{F}_E^4(e_2) = \underline{X}$,
$\widetilde{F}_E^5(e_1) = \underline{X}, \widetilde{F}_E^5(e_2) = (\{h_2\}, X)$,
$\widetilde{F}_E^6(e_1) = \underline{X}, \widetilde{F}_E^6(e_2) = (\{h_1\}, X)$,

$\widetilde{F}_E^7(e_1) = (\{h_1\}, X), \widetilde{F}_E^7(e_2) = \underline{X},$
$\widetilde{F}_E^8(e_1) = (\{h_2\}, X), \widetilde{F}_E^8(e_2) = \underline{X},$
$\widetilde{F}_E^9(e_1) = \underline{X}, \widetilde{F}_E^9(e_2) = (\emptyset, X),$
$\widetilde{F}_E^{10}(e_1) = (\{h_2\}, X), \widetilde{F}_E^{10}(e_2) = (\{h_2\}, X),$
$\widetilde{F}_E^{11}(e_1) = (\{h_1\}, X), \widetilde{F}_E^{11}(e_2) = (\{h_2\}, X),$
$\widetilde{F}_E^{12}(e_1) = \underline{\emptyset}, \widetilde{F}_E^{12}(e_2) = (\{h_2\}, X),$
$\widetilde{F}_E^{13}(e_1) = (\{h_1\}, \{h_1\}), \widetilde{F}_E^{13}(e_2) = (\{h_2\}, X),$
$\widetilde{F}_E^{14}(e_1) = (\{h_2\}, \{h_2\}), \widetilde{F}_E^{14}(e_2) = (\{h_2\}, X),$
$\widetilde{F}_E^{15}(e_1) = (\{h_2\}, X), \widetilde{F}_E^{15}(e_2) = (\{h_1\}, X),$
$\widetilde{F}_E^{16}(e_1) = (\{h_1\}, X), \widetilde{F}_E^{16}(e_2) = (\{h_1\}, X),$
$\widetilde{F}_E^{17}(e_1) = \underline{\emptyset}, \widetilde{F}_E^{17}(e_2) = (\emptyset, \{h_2\}),$
$\widetilde{F}_E^{18}(e_1) = \underline{\emptyset}, \widetilde{F}_E^{18}(e_2) = (\{h_1\}, X),$
$\widetilde{F}_E^{19}(e_1) = (\{h_1\}, \{h_1\}), \widetilde{F}_E^{19}(e_2) = (\{h_1\}, X),$
$\widetilde{F}_E^{20}(e_1) = (\{h_2\}, \{h_2\}), \widetilde{F}_E^{20}(e_2) = (\{h_1\}, X),$
$\widetilde{F}_E^{21}(e_1) = (\{h_2\}, X), \widetilde{F}_E^{4}(e_{21}) = (\emptyset, X),$
$\widetilde{F}_E^{22}(e_1) = (\{h_1\}, X), \widetilde{F}_E^{4}(e_{22}) = (\emptyset, X),$
$\widetilde{F}_E^{23}(e_1) = \underline{\emptyset}, \widetilde{F}_E^{23}(e_2) = (\emptyset, X),$
$\widetilde{F}_E^{24}(e_1) = (\{h_1\}, \{h_1\}), \widetilde{F}_E^{24}(e_2) = (\emptyset, X),$
$\widetilde{F}_E^{25}(e_1) = (\{h_2\}, \{h_2\}), \widetilde{F}_E^{25}(e_2) = (\emptyset, X),$
$\widetilde{F}_E^{26}(e_1) = (\emptyset, X), \widetilde{F}_E^{26}(e_2) = \underline{X},$
$\widetilde{F}_E^{27}(e_1) = (\emptyset, \{h_1\}), \widetilde{F}_E^{27}(e_2) = \underline{X},$
$\widetilde{F}_E^{28}(e_1) = (\emptyset, X), \widetilde{F}_E^{28}(e_2) = (\{h_2\}, X),$
$\widetilde{F}_E^{29}(e_1) = (\emptyset, \{h_1\}), \widetilde{F}_E^{29}(e_2) = (\{h_2\}, X),$
$\widetilde{F}_E^{30}(e_1) = (\emptyset, X), \widetilde{F}_E^{30}(e_2) = (\{h_1\}, X),$
$\widetilde{F}_E^{31}(e_1) = (\emptyset, \{h_1\}), \widetilde{F}_E^{31}(e_2) = (\{h_1\}, X),$
$\widetilde{F}_E^{32}(e_1) = (\emptyset, X), \widetilde{F}_E^{32}(e_2) = (\emptyset, X),$
$\widetilde{F}_E^{33}(e_1) = (\emptyset, \{h_1\}), \widetilde{F}_E^{33}(e_2) = (\emptyset, X),$
$\widetilde{F}_E^{34}(e_1) = (\emptyset, \{h_2\}), \widetilde{F}_E^{34}(e_2) = \underline{X},$
$\widetilde{F}_E^{35}(e_1) = (\emptyset, \{h_2\}), \widetilde{F}_E^{35}(e_2) = (\{h_2\}, X),$
$\widetilde{F}_E^{36}(e_1) = (\emptyset, \{h_2\}), \widetilde{F}_E^{36}(e_2) = (\{h_1\}, X),$
$\widetilde{F}_E^{37}(e_1) = (\emptyset, \{h_2\}), \widetilde{F}_E^{37}(e_2) = (\emptyset, X).$
Then, $(X, \widetilde{\tau}, E)$ is a SDTS and SDT_0-space. But it is not $SDT_{\frac{1}{2}}-$space, for $\exists\, \widetilde{h}_{1_1}^{e_2} \in SDP(X)_E$, such that $\widetilde{h}_{1_1}^{e_2}$ is neither open nor closed SD-point.

Theorem 3.25. Let $(X, \widetilde{\tau}, E)$ be a SDTS. Then,
$(X, \widetilde{\tau}, E)$ is a SDT_1-space $\rightarrow (X, \widetilde{\tau}, E)$ is a $SDT_{\frac{1}{2}}$.

Proof. Suppose $(X, \widetilde{\tau}, E)$ is a SDT_1-space, then every SD-point in X is a closed SD-point by Theorem 3.17. Hence, $(X, \widetilde{\tau}, E)$ is a $SDT_{\frac{1}{2}}$.

Example 3.26. Let $X = \{h_1, h_2\}, E = \{e_1, e_2\}$ and let
$\widetilde{\tau} = \{\widetilde{\Phi}, \widetilde{X}, \widetilde{F}_E^1, \widetilde{F}_E^2, \widetilde{F}_E^3, \widetilde{F}_E^4, \widetilde{F}_E^5, \widetilde{F}_E^6,$
$\widetilde{F}_E^7, \widetilde{F}_E^8, \widetilde{F}_E^9, \widetilde{F}_E^{10}, \widetilde{F}_E^{11}, \widetilde{F}_E^{12}\}, \widetilde{F}_E^{13}, \widetilde{F}_E^{14}, \widetilde{F}_E^{15}\},$

where $\widetilde{F}^1_E(e_1) = \underline{\emptyset}, \widetilde{F}^1_E(e_2) = (\{h_1\}, \{h_1\}),$
$\widetilde{F}^2_E(e_1) = \underline{\emptyset}, \widetilde{F}^2_E(e_2) = (\emptyset, \{h_1\}),$
$\widetilde{F}^3_E(e_1) = (\{h_1\}, \{h_1\}), \widetilde{F}^3_E(e_2) = \underline{\emptyset},$
$\widetilde{F}^4_E(e_1) = (\emptyset, \{h_1\}), \widetilde{F}^4_E(e_2) = \underline{\emptyset},$
$\widetilde{F}^5_E(e_1) = \underline{X}, \widetilde{F}^5_E(e_2) = (\{h_1\}, \{h_1\}),$
$\widetilde{F}^6_E(e_1) = \underline{X}, \widetilde{F}^6_E(e_2) = (\{h_1\}, X),$
$\widetilde{F}^7_E(e_1) = (\{h_1\}, \{h_1\}), \widetilde{F}^7_E(e_2) = \underline{X},$
$\widetilde{F}^8_E(e_1) = (\{h_1\}, X), \widetilde{F}^8_E(e_2) = \underline{X},$
$\widetilde{F}^9_E(e_1) = (\{h_1\}, \{h_1\}), \widetilde{F}^9_E(e_2) = (\{h_1\}, \{h_1\}),$
$\widetilde{F}^{10}_E(e_1) = (\emptyset, \{h_1\}), \widetilde{F}^{10}_E(e_2) = (\{h_1\}, \{h_1\}),$
$\widetilde{F}^{11}_E(e_1) = (\{h_1\}, \{h_1\}), \widetilde{F}^{11}_E(e_2) = (\emptyset, \{h_1\}),$
$\widetilde{F}^{12}_E(e_1) = (\emptyset, \{h_1\}), \widetilde{F}^{12}_E(e_2) = (\emptyset, \{h_1\}),$
$\widetilde{F}^{13}_E(e_1) = (\{h_1\}, X), \widetilde{F}^{13}_E(e_2) = (\{h_1\}, \{h_1\}),$
$\widetilde{F}^{14}_E(e_1) = (\{h_1\}, \{h_1\}), \widetilde{F}^{14}_E(e_2) = (\{h_1\}, X),$
$\widetilde{F}^{15}_E(e_1) = (\{h_1\}, X), \widetilde{F}^{15}_E(e_2) = (\{h_1\}, X).$

Then, $(X, \widetilde{\tau}, E)$ is a SDTS and $SDT_{\frac{1}{2}}-$space. But it is not SDT_1-space for the SD-point $\widetilde{h}^{e_1}_{1_{\frac{1}{2}}}$ is not a closed SD-point.

Theorem 3.27. Let $(X, \widetilde{\tau}, E)$ be a SDTS. Then,
$(X, \widetilde{\tau}, E)$ is a SDT_2-space $\rightarrow (X, \widetilde{\tau}, E)$ is a SDT_1.

Proof. Suppose $(X, \widetilde{\tau}, E)$ is a SDT_2-space, then $\widetilde{x}^e_t = \widetilde{\bigcap}_{\widetilde{O}_{\widetilde{x}^e_t} \in N^q_{(\widetilde{x}^e_t)_E}} cl_{\widetilde{\tau}} \widetilde{O}_{\widetilde{x}^e_t}, \ \forall \widetilde{x}^e_t \in SDP(X)$. It follows that, every SD-point in X is a closed SD-point. Hence by Theorem 3.17, $(X, \widetilde{\tau}, E)$ is a SDT_1.

Example 3.28. Let N be the set of all natural numbers. Then, the family $\widetilde{\tau}_N = \{\widetilde{\Phi}\}\widetilde{\bigcup}\{\widetilde{F}_E \widetilde{\subseteq} \widetilde{N} : \widetilde{F}^c_E \ is \ finite \}$ is a co-finite SD-topology over X, $(N, \widetilde{\tau}, E)$ is a co-finite SDTS and SDT_1-space. But it is not SDT_2-space for, $\widetilde{\bigcap}_{\widetilde{O}_{\widetilde{n}^e_t} \in N^q_{(\widetilde{n}^e_t)_E}} cl_{\widetilde{\tau}}(\widetilde{O}_{\widetilde{n}^e_t}) = \widetilde{N} \neq \widetilde{n}^e_t$.

Theorem 3.29. Let $(X, \widetilde{\tau}, E)$ be a SDTS. Then,
$(X, \widetilde{\tau}, E)$ is a SDT_3-space $\rightarrow (X, \widetilde{\tau}, E)$ is a SDT_2-space.

Proof. Suppose $(X, \widetilde{\tau}, E)$ is a SDT_3-space and let $\widetilde{x}^e_t \not q \, \widetilde{y}^{e'}_r$. Then, $\widetilde{x}^e_t = cl_{\widetilde{\tau}}(\widetilde{x}^e_t), \ \forall \widetilde{x}^e_t \in SDP(X)$ [by hypothesis]. It follows that $\exists \ \widetilde{O}_{\widetilde{y}^{e'}_r} \in N^q_{(\widetilde{y}^{e'}_r)_E}, \widetilde{O}_{\widetilde{x}^e_t} \in N^q_{(\widetilde{x}^e_t)_E}$ such that $\widetilde{O}_{\widetilde{y}^{e'}_r} \not q \, \widetilde{O}_{\widetilde{x}^e_t}$. Hence, $(X, \widetilde{\tau}, E)$ is a SDT_2-space.

Theorem 3.30. Let $(X, \widetilde{\tau}, E)$ be a SDTS. Then,
$(X, \widetilde{\tau}, E)$ is a SDT^*_1-space $\rightarrow (X, \widetilde{\tau}, E)$ is a SDT^*_0.

Proof. It is obvious.

Example 3.31. From example 3.22, we have $(X, \widetilde{\tau}, E)$ is a SDT^*_0-space. But it is not SDT^*_1-space for, $\widetilde{h}^{e_1}_{1_1} \not q \, \widetilde{h}^{e_2}_{2_1}$, but $cl(\widetilde{h}^{e_2}_{2_1}) = \widetilde{X} \ q \ \widetilde{h}^{e_1}_{1_1}$.

Theorem 3.32. Let $(X, \widetilde{\tau}, E)$ be a SDTS. Then,
$(X, \widetilde{\tau}, E)$ is a SDT_1-space $\rightarrow (X, \widetilde{\tau}, E)$ is a SDT_1^*.

Proof. It is obvious.

Example 3.33. Let $X = \{h_1, h_2\}, E = \{e_1, e_2\}$ and let $\widetilde{\tau} = \{\widetilde{\Phi}, \widetilde{X}, \widetilde{F}_E^1, \widetilde{F}_E^2, \widetilde{F}_E^3, \widetilde{F}_E^4, \widetilde{F}_E^5, \widetilde{F}_E^6, \widetilde{F}_E^7, \widetilde{F}_E^8, \widetilde{F}_E^9, \widetilde{F}_E^{10}, \widetilde{F}_E^{11}, \widetilde{F}_E^{12}\}, \widetilde{F}_E^{13}, \widetilde{F}_E^{14}\}$, where
$\widetilde{F}_E^1(e_1) = (\{h_1\}, \{h_1\}), \widetilde{F}_E^1(e_2) = (\{h_1\}, \{h_1\})$,
$\widetilde{F}_E^2(e_1) = (\{h_2\}, \{h_2\}), \widetilde{F}_E^2(e_2) = (\{h_2\}, \{h_2\})$,
$\widetilde{F}_E^3(e_1) = (\{h_1\}, \{h_1\}), \widetilde{F}_E^3(e_2) = \underline{\emptyset}$,
$\widetilde{F}_E^4(e_1) = (\{h_2\}, \{h_2\}), \widetilde{F}_E^4(e_2) = \underline{\emptyset}$,
$\widetilde{F}_E^5(e_1) = \underline{\emptyset}, \widetilde{F}_E^5(e_2) = (\{h_1\}, \{h_1\})$,
$\widetilde{F}_E^6(e_1) = \underline{\emptyset}, \widetilde{F}_E^6(e_2) = (\{h_2\}, \{h_2\})$,
$\widetilde{F}_E^7(e_1) = \widetilde{X}, \widetilde{F}_E^7(e_2) = \underline{\emptyset}$,
$\widetilde{F}_E^8(e_1) = (\{h_1\}, \{h_1\}), \widetilde{F}_E^8(e_2) = (\{h_2\}, \{h_2\})$,
$\widetilde{F}_E^9(e_1) = (\{h_2\}, \{h_2\}), \widetilde{F}_E^9(e_2) = (\{h_1\}, \{h_1\})$,
$\widetilde{F}_E^{10}(e_1) = \widetilde{X}, \widetilde{F}_E^{10}(e_2) = (\{h_1\}, \{h_1\})$,
$\widetilde{F}_E^{11}(e_1) = \widetilde{X}, \widetilde{F}_E^{11}(e_2) = (\{h_2\}, \{h_2\})$,
$\widetilde{F}_E^{12}(e_1) = \underline{\emptyset}, \widetilde{F}_E^{12}(e_2) = \underline{X}$,
$\widetilde{F}_E^{13}(e_1) = (\{h_1\}, \{h_1\}), \widetilde{F}_E^{13}(e_2) = \underline{X}$,
$\widetilde{F}_E^{14}(e_1) = (\{h_2\}, \{h_2\}), \widetilde{F}_E^{14}(e_2) = \underline{X}$.
Then, $(X, \widetilde{\tau}, E)$ is a SDTS and SDT_1^*-space. But it is not SDT_1-space for, $\widetilde{h}_{1\frac{1}{2}}^{e_1} \neq cl(\widetilde{h}_{1\frac{1}{2}}^{e_1})$.

Theorem 3.34. Let $(X, \widetilde{\tau}, E)$ be a SDTS. Then,
$(X, \widetilde{\tau}, E)$ is a SDT_2^*-space $\rightarrow (X, \widetilde{\tau}, E)$ is a SDT_1^*.

Proof. It follows from Theorem 3.16, 3.18.

Example 3.35. From example 3.28, we have $(N, \widetilde{\tau}, E)$ is a co-finite SDTS and SDT_1^*-space. But it is not SDT_2^*-space for, $\widetilde{\bigcap}_{\widetilde{O}_{\widetilde{n}_t^e} \in N^q_{(\widetilde{n}_t^e)_E}} cl_{\widetilde{\tau}}(\widetilde{O}_{\widetilde{n}_t^e}) = \widetilde{N} \neq \widetilde{n}_t^e$.

Theorem 3.36. Let $(X, \widetilde{\tau}, E)$ be a SDTS. Then,
$(X, \widetilde{\tau}, E)$ is a SDT_2-space $\rightarrow (X, \widetilde{\tau}, E)$ is a SDT_2^*.

Proof. It is obvious.

Example 3.37. From example 3.33, we have $(X, \widetilde{\tau}, E)$ is a SDTS and SDT_2^*-space. But it is not SDT_2-space, for $\widetilde{\bigcap}_{\widetilde{O}_{\widetilde{h}_1^{e_1}} \in N^q_{(\widetilde{h}_1^{e_1})_E}} cl_{\widetilde{\tau}}(\widetilde{O}_{\widetilde{h}_1^{e_1}}) = (\widetilde{F}_E^{14})^c \neq \widetilde{h}_1^{e_1}$.

Theorem 3.38. Let $(X, \widetilde{\tau}, E)$ be a SDTS. Then,
$(X, \widetilde{\tau}, E)$ is a SDT_3-space $\rightarrow (X, \widetilde{\tau}, E)$ is a SDT_3^*.

Proof. It is obvious.

Example 3.39. From example 3.33, we have $(X, \widetilde{\tau}, E)$ is a SDTS and SDT_3^*-space. But it is not SDT_3-space, since $(X, \widetilde{\tau}, E)$ is not a SDT_1-space.

Remark 3.40. 1. From example 3.26 $(X, \widetilde{\tau}, E)$ is a $SDT_{\frac{1}{2}}-$space, but it is not SDT_1^*. and from example 3.33 $(X, \widetilde{\tau}, E)$ is a SDT_1^*-space, but it is not $SDT_{\frac{1}{2}}$.

2. From example 3.28 $(X, \widetilde{\tau}, E)$ is a SDT_1-space, but it is not SDT_2^*. and from example 3.33 $(X, \widetilde{\tau}, E)$ is a SDT_2^*-space, but it is not SDT_1.

3. From example 3.33 $(X, \widetilde{\tau}, E)$ is a SDT_3^*-space, but it is not SDT_2.

Remark 3.41. Theorems 3.16, 3.18, 3.20, 3.21, 3.30, 3.32, 3.34, 3.36, 3.38 are satisfied if we replace SDT_i^* by SDT_i^{**}, $(i = 0, 1, 2, 3)$.

Remark 3.42. Let $(X, \widetilde{\tau}, E)$ be a $SDTS$. Then,

1. SDT_i^* is SDT_i, $(i = 0, 1, 3)$ iff $\forall\, x \in X$, $\widetilde{x}^e_{\frac{1}{2}} \notin cl_{\widetilde{\tau}}(\widetilde{x}^e_{\frac{1}{2}})$.

2. SDT_2^* is SDT_2 iff $\forall\, x \in X$, $\exists\, \widetilde{O}_{\widetilde{x}^e_{\frac{1}{2}}} \notin \widetilde{O}_{\widetilde{x}^e_{\frac{1}{2}}}$.

Corollary 3.43. For a $SDTS$ $(X, \widetilde{\tau}, E)$ we have the following implication:

$$
\begin{array}{ccccccccc}
SDT_3^* & \rightarrow & SDT_2^* & \rightarrow & SDT_1^* & & \rightarrow & & SDT_0^*. \\
\uparrow & & \uparrow & & \uparrow & & & \nwarrow & \\
SDT_3 & \rightarrow & SDT_2 & \rightarrow & SDT_1 & \rightarrow & SDT_{\frac{1}{2}} & \rightarrow & SDT_0. \\
\downarrow & & \downarrow & & \downarrow & & & \swarrow & \\
SDT_3^{**} & \rightarrow & SDT_2^{**} & \rightarrow & SDT_1^{**} & & \rightarrow & & SDT_0^{**}.
\end{array}
$$

4 SD-subspaces

Theorem 4.1. Let $(Y, \widetilde{\tau}_Y, E)$ be a SD-subspace of a SD-space $(X, \widetilde{\tau}, E)$ and $\widetilde{F}_E \in SD(X)_E$. Then,

1. If $\widetilde{F}_E \in \widetilde{\tau}_Y$ and $\widetilde{Y}_E \in \widetilde{\tau}$, then $\widetilde{F}_E \in \widetilde{\tau}$.

2. $\widetilde{F}_E \in \widetilde{\tau}_Y^c$ iff $\widetilde{F}_E = \widetilde{Y}_E \widetilde{\bigcap} \widetilde{G}_E$ for some $\widetilde{G}_E \in \widetilde{\tau}^c$.

Proof. 1. Let $\widetilde{F}_E \in \widetilde{\tau}_Y$. Then, $\exists \widetilde{G}_E \in \widetilde{\tau}$ such that $\widetilde{F}_E = \widetilde{Y}_E \widetilde{\bigcap} \widetilde{G}_E$. Now, if $\widetilde{Y}_E \in \widetilde{\tau}$, then $\widetilde{Y}_E \widetilde{\bigcap} \widetilde{G}_E \in \widetilde{\tau}$. Hence, $\widetilde{F}_E \in \widetilde{\tau}$.

2. Let $\widetilde{F}_E \in \widetilde{\tau}_Y^c$. Then, $\widetilde{F}_E = \widetilde{Y}_E \setminus \widetilde{G}_E, \widetilde{G}_E \in \widetilde{\tau}_Y$ and $\widetilde{G}_E = \widetilde{Y}_E \widetilde{\bigcap} \widetilde{H}_E$ for some $\widetilde{H}_E \in \widetilde{\tau}$.
Now,$\widetilde{F}_E = \widetilde{Y}_E \setminus (\widetilde{Y}_E \widetilde{\bigcap} \widetilde{H}_E) = \widetilde{Y}_E \setminus \widetilde{H}_E = \widetilde{Y}_E \widetilde{\bigcap} \widetilde{H}_E^c$, where $\widetilde{H}_E^c \in \widetilde{\tau}^c$. Therefore, $\widetilde{F}_E = \widetilde{Y}_E \widetilde{\bigcap} \widetilde{G}_E$ for some $\widetilde{G}_E \in \widetilde{\tau}^c$.
Conversely, suppose that $\widetilde{F}_E = \widetilde{Y}_E \widetilde{\bigcap} \widetilde{G}_E$ for some $\widetilde{G}_E \in \widetilde{\tau}^c$, then
$\widetilde{F}_E = \widetilde{Y}_E \widetilde{\bigcap} \widetilde{G}_E$
$= \widetilde{Y}_E \widetilde{\bigcap} (\widetilde{X} \setminus \widetilde{H}_E), (\widetilde{G}_E = \widetilde{X} \setminus \widetilde{H}_E, \widetilde{H}_E \in \widetilde{\tau})$
$= \widetilde{Y}_E \widetilde{\bigcap} \widetilde{H}_E^c$
$= \widetilde{Y}_E \setminus \widetilde{H}_E$
$= \widetilde{Y}_E \setminus (\widetilde{Y}_E \widetilde{\bigcap} \widetilde{H}_E), \widetilde{Y}_E \widetilde{\bigcap} \widetilde{H}_E \in \widetilde{\tau}_Y$.
Therefore, $\widetilde{F}_E \in \widetilde{\tau}_Y^c$. Hence, the result.

Theorem 4.2. Let $\widetilde{F}_E \in SD(X)_E, \widetilde{x}_t^e \in SDP(X)_E$ and $Y \subseteq X$. Then,
$\widetilde{x}_t^e \; q \; \widetilde{F}_E$ and $\widetilde{x}_t^e \widetilde{\in} \widetilde{Y} \Leftrightarrow \widetilde{x}_t^e \; q \; (\widetilde{F}_E \widetilde{\cap} \widetilde{Y})$.

Proof. If $t = 1$
$\widetilde{x}_1^e \; q \; \widetilde{F}_E$ and $\widetilde{x}_1^e \widetilde{\in} \widetilde{Y}$
$\Leftrightarrow \widetilde{x}_1^e(e) \; q \; \widetilde{F}_E(e)$ and $\widetilde{x}_1^e(e) \underline{\in} \widetilde{Y}(e), e \in E$
$\Leftrightarrow \underline{x}_1 \; q \; \widetilde{F}_E(e) = (A_1, A_2)$ and $\underline{x}_1 \underline{\in} \widetilde{Y}(e) = \underline{Y} = (Y, Y), \; e \in E$
$\Leftrightarrow (x \in A_1 \; or \; x \in A_2)$ and $x \in Y$
$\Leftrightarrow x \in (A_1 \cap Y) \; or \; x \in (A_2 \cap Y)$
$\Leftrightarrow \underline{x}_1 \; q \; (\widetilde{F}_E(e) \underline{\cap} \; \underline{Y})$
$\Leftrightarrow \widetilde{x}_1^e \; q \; (\widetilde{F}_E \widetilde{\cap} \widetilde{Y})$.
If $t = \frac{1}{2}$
$\widetilde{x}_{\frac{1}{2}}^e \; q \; \widetilde{F}_E$ and $\widetilde{x}_{\frac{1}{2}}^e \widetilde{\in} \widetilde{Y}$
$\Leftrightarrow \widetilde{x}_{\frac{1}{2}}^e(e) \; q \; \widetilde{F}_E(e)$ and $\widetilde{x}_{\frac{1}{2}}^e(e) \underline{\in} \widetilde{Y}(e), e \in E$
$\Leftrightarrow \underline{x}_{\frac{1}{2}} \; q \; \widetilde{F}_E(e) = (A_1, A_2)$ and $\underline{x}_{\frac{1}{2}} \underline{\in} \widetilde{Y}(e) = \underline{Y} = (Y, Y), \; e \in E$
$\Leftrightarrow (x \in A_1)$ and $x \in Y$
$\Leftrightarrow x \in (A_1 \cap Y)$
$\Leftrightarrow \underline{x}_{\frac{1}{2}} \; q \; (\widetilde{F}_E(e) \underline{\cap} \; \underline{Y})$
$\Leftrightarrow \widetilde{x}_{\frac{1}{2}}^e \; q \; (\widetilde{F}_E \widetilde{\cap} \widetilde{Y})$.
Hence, the result.

Theorem 4.3. Let $(Y, \widetilde{\tau}_Y, E)$ be a SD-subspace of a SD-space $(X, \widetilde{\tau}, E)$ and let $\widetilde{N}_E^Y \in SD(Y)_E, \widetilde{y}_r^e \in SDP(Y)_E$. Then, if $\widetilde{N}_E^Y = \widetilde{Y} \widetilde{\cap} \widetilde{N}_E$ for some $\widetilde{N}_E \in \widetilde{N}^q(\widetilde{y}_r^e)_E$, then $\widetilde{N}_E^Y \in \widetilde{N}_Y^q(\widetilde{y}_r^e)_E$ $(nbd.w.r.t(Y, \widetilde{\tau}_Y, E))$.

Proof. Let $\widetilde{N}_E^Y = \widetilde{Y} \widetilde{\cap} \widetilde{N}_E, \widetilde{N}_E \in \widetilde{N}^q(\widetilde{y}_r^e)_E$. Then, $\exists \widetilde{G}_E \in \widetilde{\tau}$ such that $\widetilde{y}_r^e \; q \; \widetilde{G}_E \widetilde{\subseteq} \widetilde{N}_E$. Thus, $\widetilde{y}_r^e \; q \; \widetilde{G}_E \widetilde{\cap} \widetilde{Y} \widetilde{\subseteq} \widetilde{N}_E \widetilde{\cap} \widetilde{Y} = \widetilde{N}_E^Y$. Therefore, $\widetilde{y}_r^e \; q \; \widetilde{G}_E^Y \widetilde{\subseteq} \widetilde{N}_E^Y$. Hence, $\widetilde{N}_E^Y \in \widetilde{N}_Y^q(\widetilde{y}_r^e)_E$.

Theorem 4.4. A SD-subspace $(Y, \widetilde{\tau}_Y, E)$ of a SDT_0^*-space $(X, \widetilde{\tau}, E)$ is a SDT_0^*.

Proof. Let $\widetilde{x}_t^e, \widetilde{y}_r^{e'} \in SDP(Y)_E, \; x \neq y$ such that $\widetilde{x}_t^e \not{q} \; \widetilde{y}_r^{e'}$. Then, $\widetilde{x}_t^e, \widetilde{y}_r^{e'} \in SDP(X)_E, \; x \neq y$ and $\widetilde{x}_t^e \not{q} \; \widetilde{y}_r^{e'}$. Implies, $\widetilde{x}_t^e \not{q} \; cl_{\widetilde{\tau}}(\widetilde{y}_r^{e'})$ or $\widetilde{y}_r^{e'} \not{q} \; cl_{\widetilde{\tau}}(\widetilde{x}_t^e)$. Thus, $\widetilde{x}_t^e \widetilde{\cap} \widetilde{Y} \not{q} \; cl_{\widetilde{\tau}}(\widetilde{y}_r^{e'}) \widetilde{\cap} \widetilde{Y}$ or $\widetilde{y}_r^{e'} \widetilde{\cap} \widetilde{Y} \not{q} \; cl_{\widetilde{\tau}}(\widetilde{x}_t^e) \widetilde{\cap} \widetilde{Y}$. Therefore, $\widetilde{x}_t^e \not{q} \; cl_{\widetilde{\tau_Y}}(\widetilde{y}_r^{e'})$ or $\widetilde{y}_r^{e'} \not{q} \; cl_{\widetilde{\tau_Y}}(\widetilde{x}_t^e)$. Hence, $(Y, \widetilde{\tau}_Y, E)$ is a SDT_0^*.

Theorem 4.5. A SD-subspace $(Y, \widetilde{\tau}_Y, E)$ of a SDT_0-space $(X, \widetilde{\tau}, E)$ is a SDT_0.

Proof. Let $\widetilde{x}_t^e, \widetilde{y}_r^{e'} \in SDP(Y)_E$ such that $\widetilde{x}_t^e \not{q} \; \widetilde{y}_r^{e'}$. Then, $\widetilde{x}_t^e, \widetilde{y}_r^{e'} \in SDP(X)_E$ and $\widetilde{x}_t^e \not{q} \; \widetilde{y}_r^{e'}$. Implies, $\widetilde{x}_t^e \not{q} \; cl_{\widetilde{\tau}}(\widetilde{y}_r^{e'})$ or $\widetilde{y}_r^{e'} \not{q} \; cl_{\widetilde{\tau}}(\widetilde{x}_t^e)$. Thus, $\widetilde{x}_t^e \widetilde{\cap} \widetilde{Y} \not{q} \; cl_{\widetilde{\tau}}(\widetilde{y}_r^{e'}) \widetilde{\cap} \widetilde{Y}$ or $\widetilde{y}_r^{e'} \widetilde{\cap} \widetilde{Y} \not{q} \; cl_{\widetilde{\tau}}(\widetilde{x}_t^e) \widetilde{\cap} \widetilde{Y}$. Therefore, $\widetilde{x}_t^e \not{q} \; cl_{\widetilde{\tau_Y}}(\widetilde{y}_r^{e'})$ or $\widetilde{y}_r^{e'} \not{q} \; cl_{\widetilde{\tau_Y}}(\widetilde{x}_t^e)$. Hence, $(Y, \widetilde{\tau}_Y, E)$ is a SDT_0.

Theorem 4.6. A SD-subspace $(Y, \widetilde{\tau}_Y, E)$ of a $SDT_{\frac{1}{2}}-$space $(X, \widetilde{\tau}, E)$ is a $SDT_{\frac{1}{2}}$.

Proof. Let $\widetilde{y}_r^e \in SDP(Y)_E$. Then, $\widetilde{y}_r^e \in SDP(X)_E$. This implies that, $\widetilde{y}_r^e$ is an open or closed SD-set in X. Therefore, $\widetilde{y}_r^e = \widetilde{y}_r^e \widetilde{\cap} \widetilde{Y}$ is an open or closed SD-set in Y. Hence, $(Y, \widetilde{\tau}_Y, E)$ is a $SDT_{\frac{1}{2}}$.

Theorem 4.7. A SD-subspace $(Y, \widetilde{\tau}_Y, E)$ of a SDT_1-space $(X, \widetilde{\tau}, E)$ is a SDT_1.

Proof. Let $\widetilde{y}_r^e \in SDP(Y)_E$. Then, $\widetilde{y}_r^e \in SDP(X)_E$. This implies that, $\widetilde{y}_r^e = cl_{\widetilde{\tau}}(\widetilde{y}_r^e)$. It follows that, $\widetilde{y}_r^e \widetilde{\cap} \widetilde{Y} = cl_{\widetilde{\tau}}(\widetilde{y}_r^e) \widetilde{\cap} \widetilde{Y}$. Therefore, $\widetilde{y}_r^e = cl_{\widetilde{\tau}_Y}(\underline{y}_r)$. Hence, $(Y, \widetilde{\tau}_Y, E)$ is a SDT_1.

Theorem 4.8. A SD-subspace $(Y, \widetilde{\tau}_Y, E)$ of a SDT_1^*-space $(X, \widetilde{\tau}, E)$ is a SDT_1^*.

Proof. Let $\widetilde{x}_t^e, \widetilde{y}_r^{e'} \in SDP(Y)_E$ such that $\widetilde{x}_t^e \not\Delta \widetilde{y}_r^{e'}$. Then, $\widetilde{x}_t^e, \widetilde{y}_r^{e'} \in SDP(X)_E$ and $\widetilde{x}_t^e \not\Delta \widetilde{y}_r^{e'}$. This implies that, $\widetilde{x}_t^e \not\in cl_{\widetilde{\tau}}(\widetilde{y}_r^{e'})$ and $\widetilde{y}_r^{e'} \not\in cl_{\widetilde{\tau}}(\widetilde{x}_t^e)$. Thus, $\widetilde{x}_t^e \widetilde{\cap} \widetilde{Y} \not\in cl_{\widetilde{\tau}}(\widetilde{y}_r^{e'}) \widetilde{\cap} \widetilde{Y}$ and $\widetilde{y}_r^{e'} \widetilde{\cap} \widetilde{Y} \not\in cl_{\widetilde{\tau}}(\widetilde{x}_t^e) \widetilde{\cap} \widetilde{Y}$. Therefore, $\widetilde{x}_t^e \not\in cl_{\widetilde{\tau_Y}}(\widetilde{y}_r^{e'})$ and $\widetilde{y}_r^{e'} \not\in cl_{\widetilde{\tau_Y}}(\widetilde{x}_t^e)$. Hence, $(Y, \widetilde{\tau}_Y, E)$ is a SDT_1^*.

Theorem 4.9. A SD-subspace $(Y, \widetilde{\tau}_Y, E)$ of a SDT_2-space $(X, \widetilde{\tau}, E)$ is a SDT_2.

Proof. Let $\widetilde{y}_r^e \in SDP(Y)_E$. Then, $\widetilde{y}_r^e \in SDP(X)_E$. Implies, $\widetilde{y}_r^e = \widetilde{\bigcap}_{\widetilde{O}_{\widetilde{y}_r^e} \in N^q_{(\widetilde{y}_r^e)_E}} cl_{\widetilde{\tau}} \widetilde{O}_{\widetilde{y}_r^e}$. It follows that, $\widetilde{y}_r^e \widetilde{\cap} \widetilde{Y} = [\widetilde{\bigcap}_{\widetilde{O}_{\widetilde{y}_r^e} \in N^q_{(\widetilde{y}_r^e)_E}} cl_{\widetilde{\tau}} \widetilde{O}_{\widetilde{y}_r^e}] \widetilde{\cap} \widetilde{Y}$. Therefore, $\widetilde{y}_r^e = \widetilde{\bigcap}_{\widetilde{O}_{\widetilde{y}_r^e} \in N^q_Y(\widetilde{y}_r^e)_E} cl_{\widetilde{\tau}_Y} \widetilde{O}_{\widetilde{y}_r^e}$. Hence, $(Y, \widetilde{\tau}_Y, E)$ is a SDT_2.

Theorem 4.10. A SD-subspace $(Y, \widetilde{\tau}_Y, E)$ of a SDT_2^*-space $(X, \widetilde{\tau}, E)$ is a SDT_2^*.

Proof. Let $\widetilde{x}_t^e, \widetilde{y}_r^{e'} \in SDP(Y)_E$ such that $\widetilde{x}_t^e \not\Delta \widetilde{y}_r^{e'}$. Then, $\widetilde{x}_t^e, \widetilde{y}_r^{e'} \in SDP(X)_E$ and $\widetilde{x}_t^e \not\Delta \widetilde{y}_r^{e'}$. This implies that, there exist $\widetilde{O}_{\widetilde{x}_t^e}, \widetilde{O}_{\widetilde{y}_r^{e'}} \widetilde{\in} \widetilde{\tau}$ such that $\widetilde{O}_{\widetilde{x}_t^e} \not\Delta \widetilde{O}_{\widetilde{y}_r^{e'}}$. It follows that, $\widetilde{O}^*_{\widetilde{x}_t^e} = \widetilde{O}_{\widetilde{x}_t^e} \widetilde{\cap} \widetilde{Y} \not\Delta \widetilde{O}_{\widetilde{y}_r^{e'}} \widetilde{\cap} \widetilde{Y} = \widetilde{O}^*_{\widetilde{y}_r^{e'}}$ and $\widetilde{O}^*_{\widetilde{x}_t^e}, \widetilde{O}^*_{\widetilde{y}_r^{e'}} \in \widetilde{\tau}_Y$. Hence, $(Y, \widetilde{\tau}_Y, E)$ is a SDT_2^*.

Theorem 4.11. A SD-subspace $(Y, \widetilde{\tau}_Y, E)$ of a SDR_2-space $(X, \widetilde{\tau}, E)$ is a SDR_2.

Proof. Let $\widetilde{y}_r^e \in SDP(Y)_E$ and $\widetilde{y}_r^e \not\Delta \widetilde{F} \widetilde{\cap} \widetilde{Y}, \widetilde{F} \in \widetilde{\tau}^c$. Then, $\widetilde{y}_r^e \not\Delta \widetilde{F}$ [by Proposition 2.13]. Implies, there exist $\widetilde{O}_{\widetilde{y}_r^e}, \widetilde{O}_{\widetilde{F}} \in \widetilde{\tau}$ such that $\widetilde{O}_{\widetilde{y}_r^e} \not\Delta \widetilde{O}_{\widetilde{F}}$. It follows that, $\widetilde{O}^Y_{\widetilde{y}_r^e} = \widetilde{O}_{\widetilde{y}_r^e} \widetilde{\cap} \widetilde{Y} \not\in \widetilde{O}_{\widetilde{F}} \widetilde{\cap} \widetilde{Y} = \widetilde{O}^Y_{\widetilde{F}}$ and $\widetilde{O}^Y_{\widetilde{y}_r^e}, \widetilde{O}^Y_{\widetilde{F}} \in \widetilde{\tau}_Y$. Hence, $(Y, \widetilde{\tau}_Y, E)$ is a SDR_2.

Theorem 4.12. A SD-subspace $(Y, \widetilde{\tau}_Y, E)$ of a SDT_3-space $(X, \widetilde{\tau}, E)$ is a SDT_3.

Proof. It follows from theorem 4.7 and theorem 4.11.

Theorem 4.13. A SD-subspace $(Y, \widetilde{\tau}_Y, E)$ of a SDT_3^*-space $(X, \widetilde{\tau}, E)$ is a SDT_3^*.

Proof. It follows from theorem 4.8 and theorem 4.11.

Theorem 4.14. A SD-subspace $(Y, \widetilde{\tau}_Y, E)$ of a $SDT_i^{**}-$space $(X, \widetilde{\tau}, E)$ is a $SDT_i^{**}, (i = 0, 1, 2, 3)$.

Proof. It is obvious.

5 Some Properties of the SD-continuous Functions

In this section, we study the behavior of the separation axioms under open (homeomorphism) mappings.

Definition 5.1. Let $(X, \widetilde{\tau}, E)$ and $(Y, \widetilde{\eta}, K)$ be two SDT-spaces and let $f_{\beta\psi} : SD(X)_E \to SD(Y)_K$ be a mapping and $\widetilde{F}_E \in SD(X)_E$.

1. $f_{\beta\psi}$ is called SD-open if $f_{\beta\psi}(\widetilde{F}_E) \in \widetilde{\eta}$, $\forall \widetilde{F}_E \in \widetilde{\tau}$.

2. $f_{\beta\psi}$ is called SD-closed if $f_{\beta\psi}(\widetilde{F}_E) \in \widetilde{\eta}^c$, $\forall \widetilde{F}_E \in \widetilde{\tau}^c$.

Theorem 5.2. Let $(X, \widetilde{\tau}, E)$ and $(Y, \widetilde{\eta}, K)$ be two SDT-spaces and let $f_{\beta\psi} : SD(X)_E \to SD(Y)_K$ be a mapping and $\widetilde{F}_E \in SD(X)_E$. Then, $f_{\beta\psi}$ is SD-closed iff $cl_{\widetilde{\eta}}(f_{\beta\psi}(\widetilde{F}_E)) \widetilde{\subseteq} f_{\beta\psi}(cl_{\widetilde{\tau}}(\widetilde{F}_E))$, $\forall \widetilde{F}_E \in SD(X)_E$.

Proof. Suppose $f_{\beta\psi}$ is SD-closed and $\widetilde{F}_E \in SD(X)_E$, then $\widetilde{F}_E \widetilde{\subseteq} cl_{\widetilde{\tau}}(\widetilde{F}_E)$, and so $cl_{\widetilde{\eta}}(f_{\beta\psi}(\widetilde{F}_E)) \widetilde{\subseteq} cl_{\widetilde{\eta}}(f_{\beta\psi}(cl_{\widetilde{\tau}}(\widetilde{F}_E))) = f_{\beta\psi}(cl_{\widetilde{\tau}}(\widetilde{F}_E)), cl_{\widetilde{\tau}}(\widetilde{F}_E) \in \widetilde{\tau}^c$.
Therefore, $cl_{\widetilde{\eta}}(f_{\beta\psi}(\widetilde{F}_E)) \widetilde{\subseteq} f_{\beta\psi}(cl_{\widetilde{\tau}}(\widetilde{F}_E))$.

Conversely, suppose $cl_{\widetilde{\eta}}(f_{\beta\psi}(\widetilde{F}_E)) \widetilde{\subseteq} f_{\beta\psi}(cl_{\widetilde{\tau}}(\widetilde{F}_E))$, $\forall \widetilde{F}_E \in SD(X)_E$. Let $\widetilde{F}_E$ be an SD-closed in X, then $cl_{\widetilde{\eta}}(f_{\beta\psi}(\widetilde{F}_E)) \widetilde{\subseteq} f_{\beta\psi}(\widetilde{F}_E)$. But $f_{\beta\psi}(\widetilde{F}_E) \widetilde{\subseteq} cl_{\widetilde{\eta}}(f_{\beta\psi}(\widetilde{F}_E))$, so that $f_{\beta\psi}(\widetilde{F}_E) = cl_{\widetilde{\eta}}(f_{\beta\psi}(\widetilde{F}_E))$. Therefore, $f_{\beta\psi}$ is SD-closed. Hence, the result.

Lemma 5.3. Let $(X, \widetilde{\tau}, E)$ and $(Y, \widetilde{\eta}, K)$ be two SDTS and let $f_{\beta\psi} : SD(X)_E \to SD(Y)_K$ be a (one-one) and onto mapping. Then:

1. If $\widetilde{y}_t^k \in SDP(Y)_K$, then $\exists\, x \in X$ and $e \in E$ such that $\beta(x) = y, \psi(e) = k, \widetilde{x}_t^e \in SDP(X)_E$ and $f(\widetilde{x}_t^e) = \widetilde{y}_t^k$.

2. If $\widetilde{y}_t^k \in SDP(Y)_K$, then $f^{-1}(\widetilde{y}_t^e) \in SDP(X)_E$.

3. If $\widetilde{y_1}_t^{k_1}, \widetilde{y_2}_r^{k_2} \in SDP(Y)_K, \widetilde{y_1}_t^{k_1} \notin \widetilde{y_2}_r^{k_2}$, then $\exists\, x_1, x_2 \in X, e_1, e_2 \in E$ such that $\beta(x_i) = y_i, \psi(e_i) = k_i, (i = 1, 2)$ and $f(\widetilde{x_1}_t^{e_1}) = \widetilde{y_1}_t^{k_1}, f(\widetilde{x_2}_r^{e_2}) = \widetilde{y_2}_r^{k_2}, \widetilde{x_1}_t^{e_1} \notin \widetilde{x_2}_r^{e_2}$.

Proof. 1. $f_{\beta\psi}(\widetilde{x}_t^e)(k)$
$= \beta(\bigcup_{e \in \psi^{-1}(k)} \widetilde{x}_t^e(e))$
$= \beta(\widetilde{x}_t^e(e))$
$= \beta(\widetilde{x}_t)$
$= (\widetilde{y}_t), \psi(e) = k$
$= \widetilde{y}_t^k(k)$.
Therefore, $f_{\beta\psi}(\widetilde{x}_t^e) = \widetilde{y}_t^k$.

2. $f_{\beta\psi}^{-1}(\widetilde{y_1}_t^k)(e_1)$
$= \beta^{-1}(\widetilde{y_1}_t^k(\psi(e_1)))$
$= \beta^{-1}(\widetilde{y_1}_t(k)), \psi(e_1) = k$
$= \widetilde{x_1}_t(e_1), \psi^{-1}(k) = e_1$

$= \widetilde{x_1}_t^{e_1}(e_1)$.
Thus, $f_{\beta\psi}^{-1}(\widetilde{y_1}_t^k) = \widetilde{x_1}_t^{e_1}$.
Hence, the result.

3. $f_{\beta\psi}(\widetilde{x_1}_t^{e_1})(k)$
$= \beta(\bigcup_{e\in\psi^{-1}(k)} \widetilde{x_1}_t^{e_1}(e))$
$= \beta(\widetilde{x}_{1t}), e = e_1$
$= (\widetilde{y}_{1t}), \psi(e_1) = k$
$= \widetilde{y_1}_t^k(k)$.
Therefore, $f_{\beta\psi}(\widetilde{x_1}_t^{e_1}) = \widetilde{y_1}_t^k$.
Similarly, we can see that $f_{\beta\psi}(\widetilde{x_2}_r^{e_2}) = \widetilde{y_2}_r^{k'}$.
Now, since $\widetilde{y_1}_t^{k_1} \not\in \widetilde{y_2}_r^{k_2}$, then $\widetilde{y_1}_t^{k_1} \widetilde{\subseteq} (\widetilde{y_2}_r^{k_2})^c$. So that, $f_{\beta\psi}^{-1}(\widetilde{y_1}_t^{k_1}) \widetilde{\subseteq} f_{\beta\psi}^{-1}((\widetilde{y_2}_r^{k_2})^c) = (f_{\beta\psi}^{-1}(\widetilde{y_2}_r^{k_2}))^c$ [by Proposition 2.15]. Thus, $\widetilde{x_1}_t^{e_1} \widetilde{\subseteq} (\widetilde{x_2}_r^{e_2})^c$. Therefore, $\widetilde{x_1}_t^{e_1} \not\in \widetilde{x_2}_r^{e_2}$.

Definition 5.4. Let $(X, \widetilde{\tau}, E)$ and $(Y, \widetilde{\eta}, K)$ be two SDT-spaces and let $f_{\beta\psi} : SD(X)_E \to SD(Y)_K$ be a mapping. $f_{\beta\psi}$ is called SD-homeomorphism if it is SD-continuous, SD-closed, one-one and onto.

Theorem 5.5. The property of being SDT_0^* is a topological property.

Proof. Suppose that $(X, \widetilde{\tau}, E)$ and $(Y, \widetilde{\eta}, K)$ be two SDT-spaces and let $f_{\beta\psi} : SD(X)_E \to SD(Y)_K$ be an SD-homeomorphism mapping.

Let $\widetilde{y_1}_t^{k_1}, \widetilde{y_2}_r^{k_2} \in SDP(Y)_K$ such that $\widetilde{y_1}_t^{k_1} \not\in \widetilde{y_2}_r^{k_2}$, $y_1 \neq y_2$. Then, by lemma 5.3 $\exists\ x_1, x_2 \in X,\ x_1 \neq x_2,\ e_1, e_2 \in E$ such that $\beta(x_i) = y_i, \psi(e_i) = k_i, (i = 1, 2)$. Also, $\widetilde{x_1}_t^{e_1} \not\in \widetilde{x_2}_r^{e_2}$ and $f(\widetilde{x_1}_t^{e_1}) = \widetilde{y_1}_t^{k_1}, f(\widetilde{x_2}_r^{e_2}) = \widetilde{y_2}_r^{k_2}$. Since $(X, \widetilde{\tau}, E)$ is SDT_0^*-space, then $\widetilde{x_1}_t^{e_1} \not\in cl_{\widetilde{\tau}}(\widetilde{x_2}_r^{e_2})$ or $\widetilde{x_2}_r^{e_2} \not\in cl_{\widetilde{\tau}}(\widetilde{x_1}_t^{e_1})$, so that $\widetilde{x_1}_t^{e_1} \widetilde{\subseteq} (cl_{\widetilde{\tau}}(\widetilde{x_2}_r^{e_2}))^c$, implies $f_{\beta\psi}(\widetilde{x_1}_t^{e_1}) \widetilde{\subseteq} f_{\beta\psi}(cl_{\widetilde{\tau}}(\widetilde{x_2}_r^{e_2}))^c = (f_{\beta\psi}(cl_{\widetilde{\tau}}(\widetilde{x_2}_r^{e_2})))^c$ [by proposition 3.5]. Thus, $\widetilde{y_1}_t^{k_1} \widetilde{\in} (cl_{\widetilde{\eta}}(f_{\beta\psi}(\widetilde{x_2}_r^{e_2})))^c$ (as $f_{\beta\psi}$ is SD-homeomorphism). It follows that, $\widetilde{y_1}_t^{k_1} \not\in cl_{\widetilde{\eta}}(\widetilde{y_2}_r^{k_2})$. similarly, we also have $\widetilde{y_2}_t^{k_2} \not\in cl_{\widetilde{\eta}}(\widetilde{y_1}_r^{k_1})$. Hence, $(Y, \widetilde{\eta}, K)$ is a SDT_0^*.

Theorem 5.6. The property of being SDT_0 is a topological property.

Proof. Suppose that $(X, \widetilde{\tau}, E)$ and $(Y, \widetilde{\eta}, K)$ be two SDTS and let $f_{\beta\psi} : SD(X)_E \to SD(Y)_K$ be an SD-homeomorphism mapping.
Let $\widetilde{y_1}_t^{k_1}, \widetilde{y_2}_r^{k_2} \in SDP(Y)_K$ such that $\widetilde{y_1}_t^{k_1} \not\in \widetilde{y_2}_r^{k_2}$. Then, by lemma 5.3 $\exists\ x_1, x_2 \in X, e_1, e_2 \in E$ such that $\beta(x_i) = y_i, \psi(e_i) = k_i, (i = 1, 2)$. Also, $\widetilde{x_1}_t^{e_1} \not\in \widetilde{x_2}_r^{e_2}$ and $f(\widetilde{x_1}_t^{e_1}) = \widetilde{y_1}_t^{k_1}, f(\widetilde{x_2}_r^{e_2}) = \widetilde{y_2}_r^{k_2}$. Since $(X, \widetilde{\tau}, E)$ is SDT_0-space, then $\widetilde{x_1}_t^{e_1} \not\in cl_{\widetilde{\tau}}(\widetilde{x_2}_r^{e_2})$ or $\widetilde{x_2}_r^{e_2} \not\in cl_{\widetilde{\tau}}(\widetilde{x_1}_t^{e_1})$. So that, $\widetilde{x_1}_t^{e_1} \widetilde{\subseteq} (cl_{\widetilde{\tau}}(\widetilde{x_2}_r^{e_2}))^c$, implies $f_{\beta\psi}(\widetilde{x_1}_t^{e_1}) \widetilde{\subseteq} f_{\beta\psi}(cl_{\widetilde{\tau}}(\widetilde{x_2}_r^{e_2}))^c = (f_{\beta\psi}(cl_{\widetilde{\tau}}(\widetilde{x_2}_r^{e_2})))^c$ [by proposition 3.5]. Thus, $\widetilde{y_1}_t^{k_1} \widetilde{\in} (cl_{\widetilde{\eta}}(f_{\beta\psi}(\widetilde{x_2}_r^{e_2})))^c$ (as $f_{\beta\psi}$ is SD-homeomorphism). It follows that, $\widetilde{y_1}_t^{k_1} \not\in cl_{\widetilde{\eta}}(\widetilde{y_2}_r^{k_2})$. similarly, we also have $\widetilde{y_2}_t^{k_2} \not\in cl_{\widetilde{\eta}}(\widetilde{y_1}_r^{k_1})$. Hence, $(Y, \widetilde{\eta}, K)$ is a SDT_0.

Theorem 5.7. The property of being a $SDT_{\frac{1}{2}}-$space is a topological property.

Proof. Suppose that $(X, \widetilde{\tau}, E)$ and $(Y, \widetilde{\eta}, K)$ be two SDT-spaces and let $f_{\beta\psi} : SD(X)_E \to SD(Y)_K$ be SD-open, SD-closed, one-one, onto.

Let $\widetilde{y}_r^k \in SDP(Y)$. Then, by lemma 5.3 $\exists\ x \in X$ and $e \in E$ such that $\beta(x) = y, \psi(e) = k$ and $f_{\beta\psi}(\widetilde{x}_t^e) = \widetilde{y}_r^k$. Since $(X, \widetilde{\tau}, E)$ is $SDT_{\frac{1}{2}}-$space, then $\widetilde{x}_t^e$ is an open or a closed SD-point in X. Since $f_{\beta\psi}$ is SD-open and SD-closed, then $f(\widetilde{x}_t^e) = \widetilde{y}_t^k$ is open SD-set and closed SD-set in Y. Hence, $(Y, \widetilde{\eta}, K)$ is $SDT_{\frac{1}{2}}$.

Theorem 5.8. The property of being a SDT_1-space is a topological property.

Proof. Suppose that $(X, \widetilde{\tau}, E)$ and $(Y, \widetilde{\eta}, K)$ be two SDT-spaces and let $f_{\beta\psi} : SD(X)_E \to SD(Y)_K$ be SD-homeomorphism mapping.

Let $\widetilde{y}_r^k \in SDP(Y)_K$. Then, by lemma 5.3 $\exists x \in X$ and $e \in E$ such that $\beta(x) = y, \psi(e) = k, \widetilde{x}_t^e \in SDP(X)_E$ and $f(\widetilde{x}_t^e) = \widetilde{y}_t^k$. Since $(X, \widetilde{\tau}, E)$ is SDT_1-space, then $\widetilde{x}_t^e = cl_{\widetilde{\tau}}(\widetilde{x}_t^e)$. Thus, $f_{\beta\psi}(\widetilde{x}_t^e) = f_{\beta\psi}(cl_{\widetilde{\tau}}(\widetilde{x}_t^e)) = cl_{\widetilde{\eta}}(f_{\beta\psi}(\widetilde{x}_t^e)) = cl_{\widetilde{\tau}}(\widetilde{y}_r^k)$ (as $f_{\beta\psi}$ is SD-homeomorphism). Therefore, $\widetilde{y}_r^k = cl_{\widetilde{\eta}}(\widetilde{y}_r^k)$. Hence, $(Y, \widetilde{\eta}, K)$ is SDT_1.

Theorem 5.9. The property of being SDT_1^*-space is a topological property.

Proof. Suppose that $(X, \widetilde{\tau}, E)$ and $(Y, \widetilde{\eta}, K)$ be two SDT-spaces and let $f_{\beta\psi} : SD(X)_E \to SD(Y)_K$ be SD-homeomorphism mapping.

Let $\widetilde{y_1}_t^{k_1}, \widetilde{y_2}_r^{k_2} \in SDP(Y)_K$ such that $\widetilde{y_1}_t^{k_1} \notin \widetilde{y_2}_r^{k_2}$. Then, by lemma 5.3 $\exists\ x_1, x_2 \in X, e_1, e_2 \in E$ such that $\beta(x_i) = y_i, \psi(e_i) = k_i, (i = 1, 2)$. Also, $\widetilde{x_1}_t^{e_1} \not{q}\ \widetilde{x_2}_r^{e_2}$ and $f(\widetilde{x_1}_t^{e_1}) = \widetilde{y_1}_t^{k_1}, f(\widetilde{x_2}_r^{e_2}) = \widetilde{y_2}_r^{k_2}$. Since $(X, \widetilde{\tau}, E)$ is SDT_1^*-space, then $\widetilde{x_1}_t^{e_1} \notin cl_{\widetilde{\tau}}(\widetilde{x_2}_r^{e_2})$ and $\widetilde{x_2}_r^{e_2} \notin cl_{\widetilde{\tau}}(\widetilde{x_1}_t^{e_1})$. So that $\widetilde{x_1}_t^{e_1} \widetilde{\subseteq} (cl_{\widetilde{\tau}}(\widetilde{x_2}_r^{e_2}))^c$, implies $f_{\beta\psi}(\widetilde{x_1}_t^{e_1}) \widetilde{\subseteq} f_{\beta\psi}(cl_{\widetilde{\tau}}(\widetilde{x_2}_r^{e_2}))^c = (f_{\beta\psi}(cl_{\widetilde{\tau}}(\widetilde{x_2}_r^{e_2})))^c$ [by proposition 3.5]. Thus, $\widetilde{y_1}_t^{k_1} \widetilde{\in} (cl_{\widetilde{\eta}}(f_{\beta\psi}(\widetilde{x_2}_r^{e_2})))^c$ (as $f_{\beta\psi}$ is SD-homeomorphism). It follows that, $\widetilde{y_1}_t^{k_1} \notin cl_{\widetilde{\eta}}(\widetilde{y_2}_r^{k_2})$. similarly, we also have $\widetilde{y_2}_t^{k_2} \notin cl_{\widetilde{\eta}}(\widetilde{y_1}_r^{k_1})$. Hence, $(Y, \widetilde{\eta}, K)$ is a SDT_1^*.

Theorem 5.10. The property of being a SDT_2-space is a topological property.

Proof. Suppose $(X, \widetilde{\tau}, E)$ and $(Y, \widetilde{\eta}, K)$ be two SDT-spaces and let $f_{\beta\psi} : SD(X)_E \to SD(Y)_K$ be SD-homeomorphism mapping.

Let $\widetilde{y}_r^k \in SDP(Y)_K$. Then, by lemma 5.3 $\exists x \in X$ and $e \in E$ such that $\beta(x) = y, \psi(e) = k, \widetilde{x}_t^e \in SDP(X)_E$ and $f(\widetilde{x}_t^e) = \widetilde{y}_t^k$. Since $(X, \widetilde{\tau}, E)$ is SDT_2-space, then $\widetilde{x}_t^e = \widetilde{\bigcap}_{\widetilde{O}_{\widetilde{x}_t^e} \in N^q_{(\widetilde{x}_t^e)_E}} cl_{\widetilde{\tau}}(\widetilde{O}_{\widetilde{x}_t^e})$.

Thus, $f_{\beta\psi}(\widetilde{x}_t^e) = f_{\beta\psi}(\widetilde{\bigcap}_{\widetilde{O}_{\widetilde{x}_t^e} \in N^q_{(\widetilde{x}_t^e)_E}} cl_{\widetilde{\tau}}(\widetilde{O}_{\widetilde{x}_t^e})) = \widetilde{\bigcap}_{\widetilde{O}_{f_{\beta\psi}(\widetilde{x}_t^e)} \in N^q_{(f_{\beta\psi}(\widetilde{x}_t^e))_K}} f_{\beta\psi}(cl_{\widetilde{\tau}}(\widetilde{O}_{\widetilde{x}_t^e})) =$
$\widetilde{\bigcap}_{\widetilde{O}_{f_{\beta\psi}(\widetilde{x}_t^e)} \in N^q_{(f_{\beta\psi}(\widetilde{x}_t^e))_K}} cl_{\widetilde{\eta}}(f_{\beta\psi}(\widetilde{O}_{f_{\beta\psi}(\widetilde{x}_t^e)})) = \widetilde{\bigcap}_{\widetilde{O}_{\widetilde{y}_r^k} \in N^q_{(\widetilde{y}_r^k)_K}} cl_{\widetilde{\eta}}(\widetilde{O}_{\widetilde{y}_r^k})$.

Therefore, $\widetilde{y}_r^k = \widetilde{\bigcap}_{\widetilde{O}_{\widetilde{y}_r^k} \in N^q_{(\widetilde{y}_r^k)_K}} cl_{\widetilde{\eta}}(\widetilde{O}_{\widetilde{y}_r^k})$. Hence, $(Y, \widetilde{\eta}, K)$ is SDT_2.

Theorem 5.11. The property of being SDT_2^*-space is a topological property.

Proof. Suppose that $(X, \widetilde{\tau}, E)$ and $(Y, \widetilde{\eta}, K)$ be two SDT-spaces and let $f_{\beta\psi} : SD(X)_E \to SD(Y)_K$ be SD-open, one-one and onto.

Let $\widetilde{y_1}_t^{k_1}, \widetilde{y_2}_r^{k_2} \in SDP(Y)_K$ such that $\widetilde{y_1}_t^{k_1} \notin \widetilde{y_2}_r^{k_2}$. Then, by lemma 5.3 $\exists\ x_1, x_2 \in X, e_1, e_2 \in E$ such that $\beta(x_i) = y_i, \psi(e_i) = k_i, (i = 1, 2)$. Also, $\widetilde{x_1}_t^{e_1} \not{q}\ \widetilde{x_2}_r^{e_2}$ and $f(\widetilde{x_1}_t^{e_1}) = \widetilde{y_1}_t^{k_1}, f(\widetilde{x_2}_r^{e_2}) = \widetilde{y_2}_r^{k_2}$. Since $(X, \widetilde{\tau}, E)$ is SDT_2^*-space, then there exist $\widetilde{F}_E, \widetilde{G}_E \in \widetilde{\tau}$ such that $\widetilde{x_1}_t^{e_1} \widetilde{\in} \widetilde{F}_E, \widetilde{x_2}_r^{e_2} \widetilde{\in} \widetilde{G}_E$ and $\widetilde{F}_E \notin \widetilde{G}_E$. Thus, $f_{\beta\psi}(\widetilde{x_1}_t^{e_1}) \widetilde{\in} f_{\beta\psi}(\widetilde{F}_E)$, $f_{\beta\psi}(\widetilde{x_2}_t^{e_2}) \widetilde{\in} f_{\beta\psi}(\widetilde{G}_E)$ and $f_{\beta\psi}(\widetilde{F}_E) \notin f_{\beta\psi}(\widetilde{G}_E)$ [by proposition 3.5]. Therefore, $\widetilde{y_1}_t^{k_1} \widetilde{\in} f_{\beta\psi}(\widetilde{F}_E), \widetilde{y_2}_r^{k_2} \widetilde{\in} f_{\beta\psi}(\widetilde{G}_E)$ and $f_{\beta\psi}(\widetilde{F}_E) \notin f_{\beta\psi}(\widetilde{G}_E), (f_{\beta\psi}(\widetilde{F}_E), f_{\beta\psi}(\widetilde{G}_E) \in \widetilde{\eta})$. Hence, $(Y, \widetilde{\eta}, K)$ is SDT_2^*.

Theorem 5.12. The property of being a SDR_2-space is a topological property.

Proof. Suppose $(X, \widetilde{\tau}, E)$ and $(Y, \widetilde{\eta}, K)$ be two SDT-Spaces and let $f_{\beta\psi} : SD(X)_E \to SD(Y)_K$ be SD-homeomorphism.

Let $\widetilde{y}_r^k \in SDP(Y)^K$ and $\widetilde{F}_K \in \widetilde{\eta}^c$ such that $\widetilde{y}_r^k \notin \widetilde{F}_K$. Then, by lemma 5.3 $\exists x \in X$ and $e \in E$ such that $\psi(e) = k, \beta(x) = y, \widetilde{x}_t^e \in SDP(X)_E, f(\widetilde{x}_t^e) = \widetilde{y}_t^k$ and $f_{\beta\psi}^{-1}(\widetilde{F}_K) = \widetilde{G}_E, \widetilde{G}_E \in \widetilde{\tau}^c)$ (as $f_{\beta\psi}$ is D-continuous). Also, $\widetilde{x}_t^e \notin \widetilde{G}_E$, $(X, \widetilde{\tau}, E)$ is SDR_2-space, then there exist $\widetilde{H}_E, \widetilde{M}_E \in \widetilde{\tau}$ such that $\widetilde{x}_t^e \widetilde{\in} \widetilde{H}_E, \widetilde{G}_E \widetilde{\subseteq} \widetilde{M}_E$ and $\widetilde{H}_E \not{q}\ \widetilde{M}_E$. Thus, $f_{\beta\psi}(\widetilde{x}_t^e) \widetilde{\in} f_{\beta\psi}(\widetilde{H}_E), f_{\beta\psi}(\widetilde{G}_E) \widetilde{\subseteq} f_{\beta\psi}(\widetilde{M}_E)$ and $f_{\beta\psi}(\widetilde{H}_E) \not{q}\ f_{\beta\psi}(\widetilde{M}_E)$ [by proposition 3.5]. Therefore, $\widetilde{y}_t^k \widetilde{\in} f_{\beta\psi}(\widetilde{H}_E), \widetilde{F}_K \widetilde{\subseteq} f_{\beta\psi}(\widetilde{M}_E)$ and $f_{\beta\psi}(\widetilde{H}_E) \not{q}\ f_{\beta\psi}(\widetilde{M}_E), (f_{\beta\psi}(\widetilde{H}_E), f_{\beta\psi}(\widetilde{M}_E) \in \widetilde{\eta})$. Hence, $(Y, \widetilde{\eta}, K)$ is SDR_2.

Theorem 5.13. The property of being a SDT_3-space is a topological property.

Proof. Suppose $(X, \widetilde{\tau}, E)$ and $(Y, \widetilde{\eta}, K)$ be two SDT-spaces and let $f_{\beta\psi} : SD(X)_E \to SD(Y)_K$ be SD-homeomorphism mapping and $(X, \widetilde{\tau}, E)$ is SDT_3-space, then $(Y, \widetilde{\eta}, K)$ is SDT_1 and SDR_2-spaces [by theorems 5.8,5.12]. Hence, $(Y, \widetilde{\eta}, K)$ is SDT_3.

Theorem 5.14. The property of being a SDT_3^*-space is a topological property.

Proof. Suppose $(X, \widetilde{\tau}, E)$ and $(Y, \widetilde{\eta}, K)$ be two SDT-spaces and let $f_{\beta\psi} : SD(X)_E \to SD(Y)_K$ be SD-homeomorphism mapping and $(X, \widetilde{\tau}, E)$ is SDT_3^*-space, then $(Y, \widetilde{\eta}, K)$ is SDT_1^* and SDR_2-spaces [by theorems 5.9,5.12]. Hence, $(Y, \widetilde{\eta}, K)$ is SDT_3^*.

Theorem 5.15. The property of being a $DT_i^{**}-$space, (i=0, 1, 2, 3) is a topological property.

Proof. Straightforward.

Acknowledgement

The authors express their sincere thanks to the reviewers for their careful checking of the details. The authors are also thankful to the editor-in-chief.

References

[1] K. Atanassov, Intuitionistic fuzzy sets, *Fuzzy sets and systems* 20 (1) (1986) 87−96.

[2] K. Atanassov, More on intuitionistic fuzzy sets, *Fuzzy sets and systems* 33 (1) (1989) 37−45.

[3] K. Atanassov, New operators defined over the intuitionistic fuzzy sets, *Fuzzy sets and systems* 61 (1993) 131−142.

[4] K. Atanassov and S. Stoeva, Intuitionistic fuzzy sets, *In Procceding of the Polish Synposium on Interval and Fuzzy Mathematics, Poznan* August (1983) 23−26.

[5] D. Coker, An introduction on intuitionistic fuzzy topological spaces, *Fuzzy sets and systems* 88 (1997) 81−89.

[6] D. Coker, An introduction to intuitionistic topological spaces, *BUSEFAL* 81 (2000), 51−56.

[7] D. Coker, Anote on intuitionistic sets and intuitionistic points, *Turkish J. Math.* 20 (3) (1996) 343−351.

[8] J. G. Garica and S. E. Rodabaugh, Order-theoretic, topological, Categorical redundancies of interval-valued sets, grey sets, vague sets, interval-valued intuitionistic sets, intuitionistic fuzzy sets and topologies, *Fuzzy sets and system* 156 (3) (2005) 445−484.

[9] S. Hussain and B. Ahmed, Some properties of soft topological spaces, *Comput. Math. Appl.* 62 (2011) 4058−4067.

[10] A. Kandil, O. A. E. Tantawy, S. A. El-Sheikh and S. Hussien, Some generalized separation axioms of double topological spaces, *Asian Journal of Mathematics and physics* submitted.

[11] A. Kandil, O. A. E. Tantawy and M. Wafaie, On flou (INTUITIONISTIC) compact space, *J. Fuzzy Math.* 17 (2) (2009) 275−294.

[12] A. Kandil, O. A. E. Tantawy and M. Wafaie, On flou (INTUITIONISTIC) topological spaces, *J. Fuzzy Math.* 15 (2) (2007) 1−23.

[13] A. Kharal and D. Ahmed , Mappings on Soft Classes, *New Mathematics and Natural Computation* 7 (3) (2011) 471−481.

[14] D. V. Kovkov, V. M. Kolbanov and D. V. Molodtsov, Soft sets theory-based optimization, *J. Comput. Syst. Sci. Int.* 46 (6) (2007) 872−880.

[15] N. Levine, Generalized closed sets in topology, *Rend. Circ. Mat. Palermo* 2 (1970), 89−96.

[16] P. K. Maji, R. Biswas and A. R. Roy, Soft set theory, *Comput. Math. Appl.* 45 (2003) 555−562.

[17] D. A. Molodtsov, Soft set theory-firs tresults, *Comput. Math. Appl.* 37 (1999) 19−31.

[18] D. Pei and D. Miao, From soft sets to information systems, *Proceedings of Granular computing, in: IEEE* 2 (2005) 617−621.

[19] M. Shabir and M. Naz, On soft topological spaces, *Comput. Math. Appl.* 61 (2011) 1786−1799.

[20] Sujoy Das and S. K. Samanta, Soft metric, *Ann. Fuzzy Math. Inform.* 6 (2013) 77−94.

[21] O. A. E. Tantawy, S. A. El-Sheikh and S. Hussien, Topology of soft double sets, *Ann. Fuzzy Math. Inform.* 12 (5) (2016) 641−657.

[22] I. Zorlutuna, M. Akdag, W. K. Min and S. Atmaca, Remarks on soft topological spaces, *Ann. Fuzzy Math. Inform.* 3 (2) (2012) 171−185.

http://www.newtheory.org ISSN: 2149-1402

Journal of New Theory

Received: 15.09.2017 *Year*: **2017,** *Number*: **16**, *Pages*: **49-51**
Published: 09.10.2017 *Original Article*

THE JACOBIAN CONJECTURE IS TRUE

Kerimbayev Rashid Konyrbayevich <ker_im@mail.ru>

Kazakh National University After al-Farabi, Mathematics Department, 050040, Almaty, Kazakhstan

Abstract – We are talking about famous the Jacobian conjecture. Let f and g be polynomials dependent from two variables over the field K zero characteristics, $f(x, y), g(x, y) \in K[x, y]$.

Keywords – *Jecobian Conjecture, Polynomial Maps.*

1 Introduction

The Jacobian conjecture consists in next: *If Jacobian* $J(f, g)(x, y)$ of *polynomials* f, g *are invertible in the ring* $K[x, y]$, then *polynomials of* f, g *gives K – automorphism of the ring* $K[x, y]$.

The Jacobian conjecture solved for a particular case. They are presented in the book [1]. Although, the problem does not fully solved. Below we will describe the solution of the problem.

2 Formal Inverse Mapping

On the line with polynomials $K[x, y]$, also important to consider and ring of formal power series $K[[x, y]]$, where K – field. Here several pitfalls. Composition of polynomials are always defined, but composition of series does not. Composition of formal power series are defined in case, when power series without free terms. In other side, formal power series, different from zero with free term, always invertible in the ring $K[[x, y]]$. So, in the book [1] has proven next theorem.

Formal inverse function theorem. *Let* $f(x, y), g(x, y) \in K[[x, y]]$ *be formal power series with next properties:*

$$f(0,0) = 0, \;\; g(0,0) = 0 \;\; \textit{and} \; J(f, g)(0,0) \in K^*.$$

Then exists formal power series $u(x, y), v(x, y) \in K[[x, y]]$ such as

$$u(0,0) = 0, \;\; v(0,0) = 0 \quad \textit{and} \;\; u(f, g) = x, \; v(f, g) = y.$$

Moreover, such formal power series unique and satisfies condition $f(u, v) = x, \;\; g(u, v) = y$. As result of the theorem immediately we can get next lemma.

Lemma. *If* $f(x, y), g(x, y) \in K[[x, y]]$ *polynomials with properties*

$$f(0,0) = 0, \;\; g(0,0) = 0 \;\; \textit{and} \; J(f, g)(x, y) \in K^*$$

Then algebraic variate of polynomials f and g consists from one zero point. Exactly,

$$V(f, g) = \{(x, y) \in K^2 \,|\, f(x, y) = 0, \; g(x, y) = 0\} = \{(0,0)\}.$$

Proof. Indeed, by the theorem of formal inverse function, exist series

$$u(x, y), v(x, y) \in K[[x, y]]$$

such as

$$u(0,0) = 0 = v(0,0) \;\; \textit{and} \;\; x = u(f(x, y), g(x, y)), \quad y = v(f(x, y), g(x, y)).$$

Then, if

$$f(a, b) = 0 = g(a, b),$$

then,

$$a = u(f(a, b), g(a, b)) = u(0,0) = 0,$$
$$b = v(f(a, b), g(a, b)) = v(0,0) = 0,$$

that is - $V(f, g) = \{(0,0)\}$.

After all, next theorem will be proven easily.

The injective function theorem. *Let* $f(x, y), g(x, y) \in K[x, y]$ *polynomials with properties* $J(f, g)(x, y) \in K^*$. *Then polynomials mapping*

$$\phi : K^2 \to K^2, \quad \phi(x, y) = (f(x, y), g(x, y))$$

is injective.

Proof. Let $\phi(a,b)=\phi(c,d)$. Consider next polynomials

$$F(x,y)=f(x+a,y+b)-f(a,b), \qquad G(x,y)=g(x+a,y+b)-g(a,b)$$

Then

$$F(0,0)=0=G(0,0) \text{ and } J(F,G)(x,y)\in K^*$$

By the lemma $V(F,G)=\{(0,0)\}$. We have $F(c-a,d-b)=0,\ G(c-a,d-b)=0$. It means

$$(c-a,d-b)\in V(F,G)=\{(0,0)\}.$$

It means $c=a, \quad d=b$. ϕ injective.

References

[1] van den Essen A. Polynomial automorphisms and the Jacobian Conjecture. Birkhauser: Progress in Mathematics, 2000.

http://www.newtheory.org ISSN: 2149-1402

Journal of New Theory

Received: 27.09.2017 *Year*:**2017**, *Number*: **16**, *Pages*: **52-58**
Published: 10.10.2017 *Original Article*

ON SOME NEW SUBSETS OF NANO TOPOLOGICAL SPACES

Ilangovan Rajasekaran[1,*] <sekarmelakkal@gmail.com>
Ochanan Nethaji[2] <jionetha@yahoo.com>

[1]5/140-B, South Street, Melakkal - 625 234, Madurai District, Tamil Nadu, India
[2]2/71, West Street, Sangampatti - 625 514, Madurai District, Tamil Nadu, India

Abstaract − In this paper, we introduce some kernels in nano topological spaces, nano $\wedge_r$-set and nano λ-closed sets investigate some of their properties.

Keywords − *nano $\wedge_r$-set, nano $\wedge_\pi$-set, nano λ-closed set and nano λ_π-closed set*

1 Introduction

Lellis Thivagar et al [4] introduced a nano kernel to nano topological space with respect to a subset X of an universe which is defined in terms of lower approximation and upper approximation and boundary region. The classical nano topological space is based on an equivalence relation on a set, but in some situation, equivalence relations are nor suitable for coping with granularity, instead the classical nano topology is extend to general binary relation based covering nano topological space

In this paper, we introduce some kernels, nano $\wedge_r$-set, nano $\wedge_\pi$-set, nano λ-closed set and nano λ_π-closed set in nano topological spaces and investigate some of their properties.

2 Preliminary

Throughout this paper $(U, \tau_R(X))$ (or X) represent nano topological spaces on which no separation axioms are assumed unless otherwise mentioned. For a subset H of a space $(U, \tau_R(X))$, $Ncl(H)$ and $Nint(H)$ denote the nano closure of H and the nano interior of H respectively. We recall the following definitions which are useful in the sequel.

*Corresponding Author.

Definition 2.1. *[5] Let U be a non-empty finite set of objects called the universe and R be an equivalence relation on U named as the indiscernibility relation. Elements belonging to the same equivalence class are said to be indiscernible with one another. The pair (U, R) is said to be the approximation space. Let* $X \subseteq U$.

1. *The lower approximation of X with respect to R is the set of all objects, which can be for certain classified as X with respect to R and it is denoted by* $L_R(X)$. *That is,* $L_R(X) = \bigcup_{x\in U}\{R(x) : R(x) \subseteq X\}$, *where R(x) denotes the equivalence class determined by x.*

2. *The upper approximation of X with respect to R is the set of all objects, which can be possibly classified as X with respect to R and it is denoted by* $U_R(X)$. *That is,* $U_R(X) = \bigcup_{x\in U}\{R(x) : R(x) \cap X \neq \phi\}$.

3. *The boundary region of X with respect to R is the set of all objects, which can be classified neither as X nor as not - X with respect to R and it is denoted by* $B_R(X)$. *That is,* $B_R(X) = U_R(X) - L_R(X)$.

Proposition 2.2. *[3] If* (U, R) *is an approximation space and* $X, Y \subseteq U$; *then*

1. $L_R(X) \subseteq X \subseteq U_R(X)$;
2. $L_R(\phi) = U_R(\phi) = \phi$ *and* $L_R(U) = U_R(U) = U$;
3. $U_R(X \cup Y) = U_R(X) \cup U_R(Y)$;
4. $U_R(X \cap Y) \subseteq U_R(X) \cap U_R(Y)$;
5. $L_R(X \cup Y) \supseteq L_R(X) \cup L_R(Y)$;
6. $L_R(X \cap Y) \subseteq L_R(X) \cap L_R(Y)$;
7. $L_R(X) \subseteq L_R(Y)$ *and* $U_R(X) \subseteq U_R(Y)$ *whenever* $X \subseteq Y$;
8. $U_R(X^c) = [L_R(X)]^c$ *and* $L_R(X^c) = [U_R(X)]^c$;
9. $U_RU_R(X) = L_RU_R(X) = U_R(X)$;
10. $L_RL_R(X) = U_RL_R(X) = L_R(X)$.

Definition 2.3. *[3] Let U be the universe, R be an equivalence relation on U and* $\tau_R(X) = \{U, \phi, L_R(X), U_R(X), B_R(X)\}$ *where* $X \subseteq U$. *Then by the Property 2.2, R(X) satisfies the following axioms:*

1. *U and* $\phi \in \tau_R(X)$,
2. *The union of the elements of any sub collection of* $\tau_R(X)$ *is in* $\tau_R(X)$,
3. *The intersection of the elements of any finite subcollection of* $\tau_R(X)$ *is in* $\tau_R(X)$.

That is, $\tau_R(X)$ is a topology on U called the nano topology on U with respect to X. We call $(U, \tau_R(X))$ as the nano topological space. The elements of $\tau_R(X)$ are called as nano open sets and $[\tau_R(X)]^c$ is called as the dual nano topology of $[\tau_R(X)]$.

Remark 2.4. *[3] If $[\tau_R(X)]$ is the nano topology on U with respect to X, then the set $B = \{U, \phi, L_R(X), B_R(X)\}$ is the basis for $\tau_R(X)$.*

Definition 2.5. *[3] If $(U, \tau_R(X))$ is a nano topological space with respect to X and if $H \subseteq U$, then the nano interior of H is defined as the union of all nano open subsets of H and it is denoted by $Nint(H)$.*

That is, $Nint(H)$ is the largest nano open subset of H. The nano closure of H is defined as the intersection of all nano closed sets containing H and it is denoted by $Ncl(H)$.

That is, $Ncl(H)$ is the smallest nano closed set containing H.

Definition 2.6. *[3] A subset H of a nano topological space $(U, \tau_R(X))$ is called nano regular-open $H = Nint(Ncl(H))$.*

The complement of the above mentioned set are called their respective closed set.

Definition 2.7. *[1] Let H be a subset of a space $(U, \tau_R(X))$ is nano π-open if the finite union of nano regular-open sets.*

Definition 2.8. *[4] Let $(U, \tau_R(X))$ be a nano topological spaces and $H \subseteq U$. The nano $Ker(H) = \bigcap\{U : H \subseteq U, U \in \tau_R(X)\}$ is called the nano kernal of H and is denoted by $\mathcal{N}Ker(H)$.*

Definition 2.9. *A subset H of a nano topological space $(U, \tau_R(X))$ is called;*

1. *nano g-closed [2] if $Ncl(H) \subseteq G$, whenever $H \subseteq G$ and G is nano open.*

2. *nano rg-closed set [6] if $Ncl(H) \subseteq G$ whenever $H \subseteq G$ and G is nano regular-open.*

3 On Some New Subsets of Nano Topological Spaces

Definition 3.1. *A subset H of a space $(U, \tau_R(X))$ is called a nano $\wedge$-set if $H = \mathcal{N}Ker(H)$.*

Example 3.2. *Let $U = \{a, b, c, d\}$ with $U/R = \{\{a\}, \{c\}, \{b, d\}\}$ and $X = \{a, b\}$. Then the nano topology $\tau_R(X) = \{\phi, \{a\}, \{b, d\}, \{a, b, d\}, U\}$. Then $\{a\}$ is nano $\wedge$-set.*

Definition 3.3. *A subset H of a space $(U, \tau_R(X))$ is called nano λ-closed if $H = L \cap F$ where L is a nano $\wedge$-set and F is nano closed.*

Example 3.4. *In Example 3.2, then $\{b, c, d\}$ is nano λ-closed.*

Lemma 3.5.

1. *Every nano $\wedge$-set is nano λ-closed.*

2. *Every nano open set is nano λ-closed.*

3. *Every nano closed set is nano λ-closed.*

Remark 3.6. *The converses of statements in Lemma 3.5 are not necessarily true as seen from the following Examples.*

Example 3.7. *In Example 3.2,*

1. *then $\{c\}$ is nano λ-closed but not nano $\wedge$-set.*
2. *then $\{a, c\}$ is nano λ-closed but not nano open.*
3. *then $\{b, d\}$ is nano λ-closed but not nano closed.*

Lemma 3.8. *For a subset H of a space $(U, \tau_R(X))$, the following conditions are equivalent.*

1. *H is nano λ-closed.*
2. *$H = L \cap Ncl(H)$ where L is a nano $\wedge$-set.*
3. *$H = \mathcal{N}Ker(H) \cap Ncl(H)$.*

Lemma 3.9. *A subset $H \subset (U, \tau_R(X))$ is nano g-closed if and only if $Ncl(H) \subset \mathcal{N}Ker(H)$.*

Definition 3.10. *Let H be a subset of a space $(U, \tau_R(X))$ is nano πg-closed if $Ncl(H) \subseteq G$, whenever $H \subseteq G$ and G is nano π-open.*

Example 3.11. *In Example 3.2, then $\{b, c, d\}$ is nano πg-closed.*

Remark 3.12. *For a subset of a space $(U, \tau_R(X))$, we have the following implications:*

nano closed $\rightarrow$ ***nano g-closed*** $\rightarrow$ ***nano πg-closed*** $\rightarrow$ ***nano rg-closed***

None of the above implications are reversible.

Theorem 3.13. *For a subset H of a space $(U, \tau_R(X))$, the following conditions are equivalent.*

1. *H is nano closed.*
2. *H is nano g-closed and nano λ-closed.*

Proof. (1) $\Rightarrow$ (2): Obvious by Remark 3.12 and (3) of Lemma 3.5.

(2) $\Rightarrow$ (1): Since H is nano g-closed, by Lemma 3.9, $Ncl(H) \subset \mathcal{N}Ker(H)$. Since H is nano λ-closed, by Lemma 3.8, $H = \mathcal{N}Ker(H) \cap Ncl(H) = Ncl(H)$. Hence H is nano closed.

Remark 3.14. *In a nano topological space, the concepts of nano g-closed sets and nano λ-closed sets are independent as seen from the following Examples.*

Example 3.15. *In Example 3.2,*

1. *then* $\{b, c\}$ *is nano g-closed set but not nano* λ*-closed.*
2. *then* $\{a\}$ *is* λ*-closed set but not nano g-closed.*

Remark 3.16. *Theorem 3.13 together with Remark 3.14 and Example 3.15 gives a decomposition of nano closed set into a nano g-closed set and a* λ*-nano closed set.*

Definition 3.17. *Let* H *be a subset of a space* $(U, \tau_R(X))$*. Then*

1. *The nano r-Kernel of the set* H*, denoted by* $\mathcal{N}r$*-Ker(H), is the intersection of all nano regular-open supersets of* H*.*
2. *The nano* π*-Kernel of the set* H*, denoted by* $\mathcal{N}\pi$*-Ker(H), is the intersection of all nano* π*-open supersets of* H*.*

Example 3.18. *In Example 3.2,*

1. *then* $\{a\}$ *is nano r-Kernel.*
2. *then* $\{b, d\}$ *is nano* π*-Kernel.*

Definition 3.19. *A subset* H *of a space* $(U, \tau_R(X))$ *is called*

1. *nano* $\wedge_r$*-set if* $H = \mathcal{N}r$*-Ker(H).*
2. *nano* $\wedge_\pi$*-set if* $H = \mathcal{N}\pi$*-Ker(H).*

Example 3.20. *In Example 3.2,*

1. *then* $\{b, d\}$ *is nano* $\wedge_r$*-set.*
2. *then* $\{a\}$ *is nano* $\wedge_\pi$*-set.*

Definition 3.21. *A subset* H *of a space* $(U, \tau_R(X))$ *is called*

1. *nano* λ_r*-closed if* $H = L \cap F$ *where* L *is a nano* $\wedge_r$*-set and* F *is nano closed.*
2. *nano* λ_π*-closed if* $H = L \cap F$ *where* L *is a nano* $\wedge_\pi$*-set and* F *is nano closed.*

Example 3.22. *In Example 3.2,*

1. *then* $\{b, c, d\}$ *is nano* λ_r*-closed.*
2. *then* $\{a, c\}$ *is nano* λ_π*-closed.*

Lemma 3.23.

1. *Every nano closed set is nano* λ_r*-closed.*
2. *Every nano* $\wedge_r$*-set is nano* λ_r*-closed.*
3. *Every nano closed set is nano* λ_π*-closed.*
4. *Every nano* $\wedge_\pi$*-set is nano nano* λ_π*-closed.*

Remark 3.24. *The converses of the statements in Lemma 3.23 are not necessarily true as seen from the following Examples.*

Example 3.25. *In Example 3.2,*

1. *then $\{b, d\}$ is nano λ_r-closed set but not nano closed.*
2. *then $\{a, c\}$ is nano λ_r-closed set but not nano $\wedge_r$-set.*
3. *then $\{b, d\}$ is nano λ_π-closed set but not nano closed.*
4. *then $\{b, c, d\}$ is nano λ_π-closed but not nano $\wedge_\pi$-set.*

Lemma 3.26. *For a subset H of a space $(U, \tau_R(X))$, the following are equivalent.*

1. (a) *H is nano λ_r-closed.*
   (b) *$H = L \cap Ncl(H)$ where L is a nano $\wedge_r$-set.*
   (c) *$H = \mathcal{N}r$-Ker(H) $\cap$ Ncl(H).*
2. (a) *H is nano λ_π-closed.*
   (b) *$H = L \cap Ncl(H)$ where L is a nano $\wedge_\pi$-set.*
   (c) *$H = \mathcal{N}\pi$-Ker(H) $\cap Ncl(H)$.*

Lemma 3.27. 1. *A subset $H \subset (U, \tau_R(X))$ is nano πg-closed if and only if $Ncl(H) \subset \mathcal{N}\pi$-Ker(H).*

2. *A subset $H \subset (U, \tau_R(X))$ is nano rg-closed if and only if $Ncl(H) \subset \mathcal{N}r$-Ker(H).*

Theorem 3.28. *For a subset H of a space $(U, \tau_R(X))$, the following are equivalent.*

1. *H is nano closed.*
2. *H is nano πg-closed and nano λ_π-closed.*

Proof. (1)⇒(2) Proof follows by Remark 3.12 and (6) of Lemma 3.23.
(2)⇒(1) By Lemma 3.27 and Lemma 3.26(2), proof follows similar to the proof of Theorem 3.13.

Remark 3.29. *In a nano topological space, the concepts of nano λ_π-closed sets and nano πg-closed sets are independent as seen from the following Examples.*

Example 3.30. *Let $U = \{a, b, c, d\}$ with $U/R = \{\{a\}, \{b\}, \{c, d\}\}$ and $X = \{b, d\}$. Then the nano topology $\tau_R(X) = \{\phi, \{b\}, \{c, d\}, \{b, c, d\}, U\}$.*

1. *then $\{a, c\}$ is nano πg-closed set but not nano λ_π-closed.*
2. *then $\{c, d\}$ is nano λ_π-closed set but not nano πg-closed.*

Remark 3.31. *Theorem 3.28 together with Remark 3.29 and Example 3.30 gives a decomposition of nano closed set into a nano λ_π-closed set and a nano πg-closed set.*

Theorem 3.32. *For a subset H of a space $(U, \tau_R(X))$, the following are equivalent.*

1. *H is nano closed.*

2. *H is nano rg-closed and nano λ_r-closed.*

Proof. (1) ⇒ (2) Proof follows by Remark 3.12 and (3) of Lemma 3.23.
(2) ⇒ (1) By Lemma 3.27 and Lemma 3.26(1), proof follows similar to the proof of Theorem 3.13.

Remark 3.33. *In a nano topological space, the concepts of nano λ_r-closed sets and nano rg-closed sets are independent as seen from the following Examples.*

Example 3.34. *In Example 3.2,*

1. *then $\{a\}$ is nano λ_r-closed set but not nano rg-closed.*

2. *then $\{a, d\}$ is nano rg-closed set but not nano λ_r-closed.*

Remark 3.35. *Theorem 3.32 together with Remark 3.33 and Example 3.34 gives a decomposition of nano closed set into a nano λ_r-closed set and a nano rg-closed set.*

Acknowledgement

The authors thank the referees for their valuable comments and suggestions for improvement of this paper.

References

[1] Adinatha C. Upadhya, *On quasi nano p-normal spaces*, International Journal of Recent Scientific Research, 8(6)2017, 17748-17751.

[2] K. Bhuvaneshwari and K. Mythili Gnanapriya, *Nano Generalizesd closed sets*, International Journal of Scientific and Research Publications, 4(5)2014,1-3.

[3] M. Lellis Thivagar and Carmel Richard, *On Nano forms of weakly open sets*, International Journal of Mathematics and Statistics Invention,1(1) 2013, 31-37.

[4] M. Lellis Thivagar, Saeid Jafari and V. Sutha Devi, *On new class of contra continuity in nano topology*, Italian Journal of Pure and Applaied Mathematics, 2017, 1-10. .

[5] Z. Pawlak, *Rough sets*, International journal of computer and Information Sciences, 11(5)(1982), 341-356.

[6] P. Sulochana Devi and K. Bhuvaneswari , *On Nano Regular Generalized and Nano Generalized Regular Closed Sets in Nano Topological Spaces* , International Journal of Engineering Trends and Technology (IJETT), 8(13) 2014, 386-390.

http://www.newtheory.org ISSN: 2149-1402

Journal of New Theory

Received: 28.06.2016 *Year*:**2017**, *Number*: **16**, *Pages*: **59-67**
Published: 12.10.2017 *Original Article*

APPLICATION OF NATURAL TRANSFORM IN CRYPTOGRAPHY

Anil Dhondiram Chindhe[1,*] <anilchindhe5@gmail.com>
Sakharam Kiwne[2] <sbkdcamath@gamil.com>

[1]Department of Mathematics, Balbhim College, Beed, Maharashtra, India
[2]Department of Mathematics, Deogiri College, Aurangabad, Maharashtra, India

Abstaract − The newly defined integral transform "Natural transform" has many application in the field of science and engineering.In this paper we described the application of Natural transform to Cryptography.This provide the algorithm for cryptography in which we use the natural transform of the exponential function for encryption of the plain text and corresponding inverse natural transform for decryption.

Keywords − *Cryptography, Data encryption, Data decryption, Natural transform.*

1 Introduction

In today's world of globalization and digitalization , the security of information (data) is the most important aspect of the society.There is a commonly and widely used technique called as cryptography for the security purpose.cryptography deals with the actual securing of digital data.It is the art and science of making a cryptosystem that is capable of providing information security. The objectives of cryptography are Confidentiality,Integrity,Non-repudiation and Authentication.Different tools and techniques are used for cryptography [12, 13, 14]. There are Mathematical technique used for the cryptography are found in [8, 9, 10].

The original information is known as plain-text, and the encrypted from as cipher text. The cipher text message contains all the information of the plain-text message,but is not in a format readable to a human or computer without the mechanism to decrypt it. Cipher are usually parametrized by a piece of auxiliary information called a key. The encryption process is varied depending the key which changes the detailed operation of the algorithm [11]. Without having the proper key it is impossible to decrypt the given text.

* *Corresponding Author.*

1.1 Natural Transform

The new integral transform Natural transform was defined by Khan and Khan [1] as N - transform who gave the properties and application of N-transform. Belgacem [2, 3] defined inverse Natural transform and studied some properties and applications. Many authors have contributed in the study of N-transform [4, 5, 6, 7]. Natural transform can be used to solve the problems in engineering, fluid mechanics and other science faculty.

1.2 Definition of Natural Transform

The Natural transform of the function $f(t) \in \Re^2$ is given by the following integral equation [3]

$$\mathbb{N}[f(t)] = G(s,u) = \int_0^\infty e^{-st} f(ut)dt \tag{1}$$

where $Re(s) > 0$, u $\in (\tau_1, \tau_2)$ provided the function $f(t) \in \Re^2$ is defined in the set

$$\text{A}=[f(t)/\exists \text{ M}, \tau_1, \tau_2 > 0 , |\text{f(t)}| < \text{M } e^{\frac{|t|}{\tau_j}} \text{ , if t } \in (-1)^j \times [0,\infty) \text{]}$$

The inverse Natural transform related with Bromwich contour integral[2, 3] is defined by

$$\mathbb{N}^{-1}[G(s,u)] = f(t) = \lim_{T\to\infty} \frac{1}{2\Pi i} \int_{\gamma - iT}^{\gamma + iT} e^{\frac{st}{u}} G(s,u)ds \tag{2}$$

1.3 Standard Result of Natural Transform

In this section we can see the Natural transform of some of the standard functions. [1, 3]

$$\mathbb{N}[1] = \frac{1}{s} \tag{3}$$

$$\mathbb{N}[t] = \frac{u}{s^2} \tag{4}$$

$$\mathbb{N}[t^n] = \frac{u^n}{s^{n+1}} n! \tag{5}$$

$$\mathbb{N}[e^{at}] = \frac{1}{s - au} \tag{6}$$

$$\mathbb{N}[\frac{sin(at)}{a}] = \frac{u}{s^2 + s^2u^2} \tag{7}$$

$$\mathbb{N}[cos(at)] = \frac{s}{s^2 + s^2u^2} \tag{8}$$

$$\mathbb{N}[\frac{t^{n-1}e^{at}}{(n-1)!}] = \frac{u^{n-1}}{(s-au)^2} \tag{9}$$

$$\mathbb{N}[f^{(n)}(t)] = \frac{s^n}{u^n}.R(s,u) - \sum_{n=0}^{\infty} \frac{s^{n-(k+1)}}{u^{n-k}}.u^{(k)}(0), \quad \text{where} f^{(n)}(t) = \frac{\mathrm{d}^n f}{\mathrm{d}t^n} \tag{10}$$

2 Main Result

2.1 Encryption Using Exponential Function

Consider the Taylor series expansion of the exponential function e^{rt} as

$$e^{rt} = 1 + \frac{rt}{1!} + \frac{(rt)^2}{2!}..... = \sum_{n=0}^{\infty} \frac{(rt)^n}{n!} \tag{11}$$

where r is constant.

$$\therefore t.e^{rt} = t + \frac{rt^2}{1!} + \frac{r^2t^3}{2!}..... = \sum_{n=0}^{\infty} \frac{r^n t^{n+1}}{n!} \tag{12}$$

Now we allocate 0 to A,1 to B and so on then Z will be 25.

consider the plain-text as "SCIENCE" which is equivalent to 18 2 8 4 13 2 4

Put$P_0 = 18, P_1 = 2, P_2 = 8, P_3 = 4, P_4 = 13, P_5 = 2, P_6 = 4, P_n = 0$ for n≥7

$$f(t) = Pt.e^{rt} = P_0 t + P_1 \frac{rt^2}{1!} + P_2 \frac{r^2t^3}{2!} + P_3 \frac{r^3t^4}{3!}..... = \sum_{n=0}^{\infty} P_n \frac{r^n t^{n+1}}{n!} \tag{13}$$

for r = 2 we have

$$f(t) = Pt.e^{2t} = P_0 t + P_1 \frac{2t^2}{1!} + P_2 \frac{2^2t^3}{2!} + P_3 \frac{2^3t^4}{3!}..... = \sum_{n=0}^{\infty} P_n \frac{2^n t^{n+1}}{n!} \tag{14}$$

$$f(t) = Pt.e^{2t} = 18t + 2\frac{2t^2}{1!} + 8\frac{2^2t^3}{2!} + 4\frac{2^3t^4}{3!} + 13\frac{2^4t^5}{4!} + 2\frac{2^5t^6}{5!} + 4\frac{2^6t^7}{6!} \tag{15}$$

Now taking the Natural transform on both sides of above equation, we get

$$\begin{aligned}
&\mathbb{N}[f(t)] = \\
&= \mathbb{N}[Pt.e^{2t}] \\
&= \mathbb{N}[18t + 2\frac{2t^2}{1!} + 8\frac{2^2t^3}{2!} + 4\frac{2^3t^4}{3!} + 13\frac{2^4t^5}{4!} + 2\frac{2^5t^6}{5!} + 4\frac{2^6t^7}{6!}] \\
&= 18.\mathbb{N}[t] + 2.\frac{2}{1!}\mathbb{N}[t^2] + 8.\frac{2^2}{2!}\mathbb{N}[t^3] + 4.\frac{2^3}{3!}\mathbb{N}[t^4] + 13.\frac{2^4}{4!}\mathbb{N}[t^5] + 2.\frac{2^5}{5!}\mathbb{N}[t^6] + 4.\frac{2^6}{6!}\mathbb{N}[t^7] \\
&= 18.\frac{u}{s^2} + 2.\frac{2}{1!}\frac{u^2}{s^3}2! + 8.\frac{2^2}{2!}\frac{u^3}{s^4}3! + 4.\frac{2^3}{3!}\frac{u^4}{s^5}4! + 13.\frac{2^4}{4!}\frac{u^5}{s^6}5! + 2.\frac{2^5}{5!}\frac{u^6}{s^7}6! + 4.\frac{2^6}{6!}\frac{u^7}{s^8}7! \\
&= 18.\frac{u}{s^2} + 8.\frac{u^2}{s^3} + 96.\frac{u^3}{s^4} + 128.\frac{u^4}{s^5} + 1040.\frac{u^5}{s^6} + 384.\frac{u^6}{s^7} + 1792.\frac{u^7}{s^8}
\end{aligned}$$

Now the key (K_i) for the cipher text is calculated by following method

$18 \equiv 18(mod26)$,$8 \equiv 8(mod26)$,$96 \equiv 18(mod26)$,$128 \equiv 24(mod26)$,

$1040 \equiv 0(mod26)$,$384 \equiv 20(mod26)$,$1792 \equiv 24(mod26)$.

Which gives the key as 0 0 3 4 40 14 68.

Let $P_i' = r_i = q_i - 26K_i$ for i = 0,1,2,3,4,5,6

$\therefore P_0' = 18, P_1' = 8, P_2' = 18, P_3' = 24, P_4' = 0, P_5' = 20, P_6' = 24, P_n' = 0$ for $n \geq 7$

Hence the given plain-text "SCIENCE" get converted into "SISYAUY".

2.2 For Decryption

Now receiver receives the message as "SISYAUY" which is equivalent to 18 8 24 0 20 24

$Since P_0' = 18, P_1' = 8, P_2' = 18, P_3' = 24, P_4' = 0, P_5' = 20, P_6' = 24, P_n' = 0 \quad forn \geq 7$ and we have the key as 0 0 3 4 40 14 68 so we can calculate $q_i = 26K_i + P_i'$ for i = 0,1,2...

$$P\frac{u}{(s-2u)^2} = 18.\frac{u}{s^2} + 8.\frac{u^2}{s^3} + 96.\frac{u^3}{s^4} + 128.\frac{u^4}{s^5} + 1040.\frac{u^5}{s^6} + 384.\frac{u^6}{s^7} + 1792.\frac{u^7}{s^8} \quad (16)$$

Now taking inverse Natural transform on both sides

$$\mathbb{N}^{-1}[P\tfrac{u}{(s-2u)^2}] = \mathbb{N}^{-1}[18.\tfrac{u}{s^2} + 8.\tfrac{u^2}{s^3} + 96.\tfrac{u^3}{s^4} + 128.\tfrac{u^4}{s^5} + 1040.\tfrac{u^5}{s^6} + 384.\tfrac{u^6}{s^7} + 1792.\tfrac{u^7}{s^8}]$$

$$\begin{aligned} f(t) &= Pte^{2t} \\ &= 18\mathbb{N}^{-1}[\frac{u}{s^2}] + 8\mathbb{N}^{-1}[\frac{u^2}{s^3}] + 96\mathbb{N}^{-1}[\frac{u^3}{s^4}] + 128\mathbb{N}^{-1}[\frac{u^4}{s^5}] + 1040\mathbb{N}^{-1}[\frac{u^5}{s^6}] \\ &+ 384\mathbb{N}^{-1}[\frac{u^6}{s^7}] + 1792\mathbb{N}^{-1}[\frac{u^7}{s^8}] \\ &= 18t + 2\frac{2t^2}{1!} + 8\frac{2^2t^3}{2!} + 4\frac{2^3t^4}{3!} + 13\frac{2^4t^5}{4!} + 2\frac{2^5t^6}{5!} + 4\frac{2^6t^7}{6!} \end{aligned}$$

Here $P_0 = 18, P_1 = 2, P_2 = 8, P_3 = 4, P_4 = 13, P_5 = 2, P_6 = 4, P_n = 0$ for $n \geq 7$

This gives the message "SISYAUY" get converted into the original message "SCIENCE".

2.2.1 More Illustrative Examples

1 The original message "SCIENCE" get converted into "SMIQNEC" with the proper key as

0 0 8 202 112 785 for r = 3

2 The original message "SCIENCE" get converted into "SGUKAQC" with the proper key as

0 1 14 39 640 472 4411 for r = 4

3 The original message "SCIENCE" get converted into "SEYQAMC" with the proper key as

0 2 180 1688 96040 248226 8108731 for r = 14

3 Encryption Using Hyperbolic Function

Consider the Taylor series expansion of hyperbolic sine function sinh(rt) as

$$sinh(rt) = \frac{rt}{1!} + \frac{r^3t^3}{3!} + \frac{r^5t^5}{5!}..... = \sum_{n=0}^{\infty} \frac{(rt)^{2n+1}}{(2n+1)!} \tag{17}$$

where r is constant.

$$\therefore t.sinh(rt) = \frac{rt^2}{1!} + \frac{r^3t^4}{3!} + \frac{r^5t^6}{5!}..... = \sum_{n=0}^{\infty} \frac{r^{2n+1}t^{2n+2}}{(2n+1)!} \tag{18}$$

Now we allocate 0 to A,1 to B and so on then Z will be 25.

consider the plain-text as "STUDENT" which is equivalent to 18 19 20 3 4 13 19

Put$P_0 = 18, P_1 = 19, P_2 = 20, P_3 = 3, P_4 = 4, P_5 = 13, P_6 = 19, P_n = 0$ for n≥7

$$f(t) = Pt.sinh(rt) = P_0\frac{rt^2}{1!} + P_1\frac{r^3t^4}{3!} + P_2\frac{r^5t^6}{5!} + ... = \sum_{n=0}^{\infty} P_n\frac{r^{2n+1}t^{2n+2}}{(2n+1)!} \tag{19}$$

for r = 2 we have

$$f(t) = Pt.sinh(2t) = P_0\frac{2t^2}{1!} + P_1\frac{2^3t^4}{3!} + P_2\frac{2^5t^6}{5!} + ... = \sum_{n=0}^{\infty} P_n\frac{2^{2n+1}t^{2n+2}}{(2n+1)!} \tag{20}$$

$$f(t) = Pt.sinh(2t) = 18\frac{2t^2}{1!} + 19\frac{2^3t^4}{3!} + 20\frac{2^5t^6}{5!} + 3\frac{2^7t^8}{7!} + 4\frac{2^9t^{10}}{9!} + 13\frac{2^{11}t^{12}}{11!} + 19\frac{2^{13}t^{14}}{13!} \tag{21}$$

Now taking the Natural transform on both sides of above equation ,we get

$$
\begin{aligned}
\mathbb{N}[f(t)] &= \mathbb{N}[Pt.sinh(2t)] \\
&= P\frac{(2s)(2u^2)}{(s^2-2^2u^2)^2} \\
&= \mathbb{N}[18\frac{2t^2}{1!}+19\frac{2^3t^4}{3!}+20\frac{2^5t^6}{5!}+3\frac{2^7t^8}{7!} \\
&+4\frac{2^9t^{10}}{9!}+13\frac{2^{11}t^{12}}{11!}+19\frac{2^{13}t^{14}}{13!}] \\
&= 18\frac{2}{1!}\mathbb{N}[t^2]+19\frac{2^3t^4}{3!}\mathbb{N}[t^4]+20\frac{2^5}{5!}\mathbb{N}[t^6]+3\frac{2^7}{7!}\mathbb{N}[t^8]+4\frac{2^9}{9!}\mathbb{N}[t^{10}] \\
&+13\frac{2^{11}}{11!}\mathbb{N}[t^{12}]+19\frac{2^{13}}{13!\mathbb{N}[t^{14}]} \\
&= 72.\frac{u^2}{s^3}+608.\frac{u^4}{s^5}+3840.\frac{u^6}{s^7}+3072.\frac{u^8}{s^9}+.20480\frac{u^{10}}{s^{11}}+319488.\frac{u^{12}}{s^{13}} \\
&+2179072.\frac{u^{14}}{s^{15}}
\end{aligned}
$$

Now the key(K_i) for the cipher text is calculated by following method

$72 \equiv 20(mod26), 608 \equiv 10(mod26), 3840 \equiv 18(mod26), 3072 \equiv 4(mod26)$

$20480 \equiv 18(mod26), 319488 \equiv 0(mod26), 2179072 \equiv 12(mod26).$

Which gives the key as 2 23 147 118 787 12288 83810.

Let $P_i' = r_i = q_i - 26K_i$ for i = 0,1,2,3,4,5,6

$\therefore P_0' = 20, P_1' = 10, P_2' = 18, P_3' = 4, P_4' = 18, P_5' = 0, P_6' = 12, P_n' = 0$ for $n \geq 7$

Hence the given plain-text " STUDENT " get converted into " UKSESAM ".

3.1 For Decryption

Now receiver receives the message as " UKSESAM " which is equivalent to 20 10 18 4 18 0 12

Since $P_0' = 20, P_1' = 10, P_2' = 18, P_3' = 4, P_4' = 18, P_5' = 0, P_6' = 12, P_n' = 0$ for $n \geq 7$ and we have the key as 2 23 147 118 787 12288 83810 so we can calculate $q_i = 26K_i + P_i'$ for i = 0,1,2...

$$
\begin{aligned}
P\frac{(2s)(2u^2)}{(s^2-2^2u^2)^2} &= 72.\frac{u^2}{s^3}+608.\frac{u^4}{s^5}+3840.\frac{u^6}{s^7}+3072.\frac{u^8}{s^9}+.20480\frac{u^{10}}{s^{11}}+319488.\frac{u^{12}}{s^{13}} \\
&+2179072.\frac{u^{14}}{s^{15}}
\end{aligned}
$$

Now taking inverse Natural transform on both sides

$$\mathbb{N}^{-1}[P\frac{(2s)(2u^2)}{(s^2-2^2u^2)^2}] = \mathbb{N}^{-1}[72.\frac{u^2}{s^3} + 608.\frac{u^4}{s^5} + 3840.\frac{u^6}{s^7} + 3072.\frac{u^8}{s^9} + 20480\frac{u^{10}}{s^{11}} + 319488.\frac{u^{12}}{s^{13}} + 2179072.\frac{u^{14}}{s^{15}}]$$

$$\begin{aligned} f(t) &= Pt.sinh(2t) \\ &= 72.\mathbb{N}^{-1}[\frac{u^2}{s^3}] + 60.\mathbb{N}^{-1}[\frac{u^4}{s^5}] + 3840.\mathbb{N}^{-1}[\frac{u^6}{s^7}] + 3072.\mathbb{N}^{-1}[\frac{u^8}{s^9}] + 20480.\mathbb{N}^{-1}[\frac{u^{10}}{s^{11}}] \\ &+ 319488.\mathbb{N}^{-1}[\frac{u^{12}}{s^{13}}] + 2179072.\mathbb{N}^{-1}[\frac{u^{14}}{s^{15}}] \\ &= 18\frac{2t^2}{1!} + 19\frac{2^3t^4}{3!} + 20\frac{2^5t^6}{5!} + 3\frac{2^7t^8}{7!} + 4\frac{2^9t^{10}}{9!} + 13\frac{2^{11}t^{12}}{11!} + 19\frac{2^{13}t^{14}}{13!} \end{aligned}$$

Here $P_0 = 18, P_1 = 19, P_2 = 20, P_3 = 3, P_4 = 4, P_5 = 13, P_6 = 19, P_n = 0 \quad forn \geq 7$

This gives the cipher text " UKSESAM " get converted into the original message " STUDENT ".

3.2 Generalization

for encryption of given plain-text in terms of P ,we consider the function

$$f(t) = Pt^j sinh(rt) \qquad forr, j \in \mathbb{N}$$

Taking Natural transform and following the procedure we can have the given messageP_i can

be converted into $P_i^{'}$ with the private key as $K_i = \frac{q_i - P_i^{'}}{26}$ for i = 0,1,2 ...

where $q_i = P_i r^{2i+1}(2i+1)(2i+3)...(2i+j)$

For dycryption for recived message (cipher text) in terms of P_i we have

$$P.u^j.(-\frac{\partial}{\partial s})^j(\frac{ru}{s^2-r^2u^2}) = \sum_{n=0}^{\infty}\frac{q_n u^{2n+1+j}}{s^{2n+2+j}}$$

Taking the inverse Natural transform,we can convert the given cipher text $P_i^{'}$ into the original message P_i as

$$P_i = \frac{26K_i + P_i^{'}}{r^{2i+1}(2i+1)(2i+3)...(2i+j)}$$

for i = 0,1,2 ...

4 Conclusion

Now a day's e-crimes such as internet banking fraud, data hacking etc. are commonly seen in the society. This paper gives a new cartographic application using Natural transform which helps to prevent such e-crimes in the society.It is too difficult for hackers or unauthorized person to find the private key by the brute farce attack or any other attack.

References

[1] Z.H.Khan and W.A.Khan.N-transform properties and applications. NUST Jour of Engg Sciences. , 1(1) pp 127–133,2008.

[2] Silambarasan, R. and Belgacem, F. B. M, *Applications of the Natural transform to Maxwell's Equations*, PIERS Suzhou, China,, Sept 12-16, pp 899–902, 2011..

[3] Belgacem, F. B. M and Silambarasan R.,*Theory of the Natural transform*, Mathematics in Engg Sci and Aerospace (MESA) journal, Vol. 3, No. 1, pp 99–124,2012..

[4] Silambarasan, R. and Belgacem, F. B. M., *Advances in the Natural transform*, 9th International Conference on Mathematical Problems in Engineering, Aerospace and Sciences AIP Conf. Proc. , 1493, 106–110, 2012.

[5] Loonker Deshna and Banerji P.K., *Natural transform for distribution and Boehmian spaces*, Math.Engg.Sci.Aerospace ,4(1), pp 69–76,2013.

[6] Loonker Deshna and Banerji P.K., *application of Natural transform to differential equations* , J.Inadian Acad Math.,35(1), pp 151–158,2013.

[7] Loonker Deshna and Banerji P.K., *Natural transform and solution of integral equations for distribution spaces* , Amer. J. Math. Sci.,8,2013.

[8] Hiwarekar AP.,*Application of Laplace Transform for Cryptography*,International Journal of engg. sci.Resch.,5(4), 129–135,2015..

[9] Dhanorkar GA, Hiwarekar AP., *A generalized Hill cipher using matrix transformation.*,International J. of Math. Sci. Engg. Appls, 5(IV), 19–23,2011.

[10] Hiwarekar AP., *A new method of cryptography using Laplace transform.*, International Journal of Mathematical Archive., 3(3), 1193–1197.2012.

[11] Hiwarekar AP., *New Mathematical Modeling for Cryptography*, Journal of Information Assurance and Security, MIR Lab USA,9,027–033,2014.

[12] Stallings W, *Cryptography and network security, 4th edition, Prentice Hall, 2005.*

[13] Barr TH, *Invitation to Cryptography, Prentice Hall, 2002.*

[14] Buchmann JA., *Introduction to Cryptography,Fourth Edn., Indian Reprint, Springer, 2009.*

http://www.newtheory.org ISSN: 2149-1402

Journal of New Theory

Received: 01.06.2016
Published: 13.10.2017
Year: **2017**, *Number*: **16**, *Pages*: **68-79**
Original Article

NEW OSTROWSKI TYPE INEQUALITIES FOR FUNCTIONS WHOSE DERIVATIVES ARE p-PREINVEX

Imran Abbas Baloch <iabbasbaloch@gmail.com>

Abdus Salam School of Mathematical Sciences, GC University, Lahore, Pakistan.

Abstaract − In this paper, making use of new an identity, we established new inequalities of Ostrowski's type for the class of p-preinvex functions and gave some midpoint type inequalities.

Keywords − *Preinvex functions, p-preinvex functions, Ostrowski type inequalities.*

1 Introduction

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$, be a mapping differentiable in I° and $a, b \in I$ with $a < b$. If $|f'(x)| \leq M$, for all $x \in [a, b]$, then the following inequality holds

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)dt \right| \leq M(b-a)\left[\frac{1}{4} + \frac{(x - \frac{a+b}{2})^2}{(b-a)^2}\right] \tag{1}$$

for $x \in [a, b]$. This inequality is known in the literature as the Ostrowski inequality ([8]), which gives an upper bound for the approximation of the integral average $\frac{1}{b-a}\int_a^b f(t)dt$ by the value $f(x)$ at the point $x \in [a, b]$. For some results which generalize, improve and extend the inequality (1), we refer the reader to recent papers (see [9, 10]) and the references therein.

Definition 1.1. A function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is said to be convex function, if

$$f(\lambda x + (1-\lambda)y)) \leq \lambda f(x) + m(1-\lambda)f(y)$$

for all $x, y \in I$ and $\lambda \in [0, 1]$. We say that f is concave if $-f$ is convex.

In recent years several extensions and generalizations have been considered for classical convexity. A significant generalization of convex function is that of invex functions introduced by Hanson in [4]. Weir and Mond [13] introduced the concept of preinvex functions and applied it to the establishment of the sufficient optimality conditions and duality in nonlinear programming. Pini [12] introduced the concept of prequasiinvex functions as a generalization of invex functions. Later, Mohan and Neogy [9] obtained some properties of generalized preinvex functions.

In [1], I. A. Baloch et. al. introduced the concept of the p-preinvex functions which is generalization of preinvex and harmonically preinvex functions. They also defined the notion of p-prequasiinvex function.

The aim of this paper is to establish some Ostrowski type inequalities for the functions whose derivative in absolute value are p-preinvex. Now, we recall some notions in invexity analysis which will be used through out the paper (see [2,8,14] and references therein).

Definition 1.2. A set $S \subseteq \mathbb{R}^n$ is said to be invex with respect to the map $\eta : S \times S \to \mathbb{R}^n$, if for every $x, y \in S$ and $t \in [0,1]$, we have

$$x + t\eta(y,x) \in S.$$

Note that definition of invex set has a clear geometric interpretation.This definition essentially says that there is a path starting from a point x which is contain in S.We do not require that the point y should be the one of the end points of path. This observation plays an important role in our analysis. Note that, if we demand that y should be an end point of the path for every pair of points, $x, y \in S$, then $\eta(y,x) = y - x$ and corresponding invexity reduces to convexity. Thus, it is true that every convex set is also an invex set with respect to $\eta(y,x) = y - x$, but converse is not necessarily true, see [15],[18] and references therein.

Definition 1.3. Let $S \subseteq \mathbb{R}^n$ be an invex set with respect to $eta : S \times S \to \mathbb{R}^n$. A function $f : S \to \mathbb{R}$ is said to be preinvex with respect to η if for every $x, y \in S$ and $t \in [0,1]$, we have

$$f(x + t\eta(y,x)) \leq tf(x) + (1-t)f(y).$$

Note that every convex function is a preinvex function, but converse is not true (see [8]). For example, $f(x) = -|x|, x \in \mathbb{R}$, is not a convex function, but it is a preinvex function with respect to

$$\eta(x,y) = \begin{cases} x - y, & xy \geq 0 \\ y - x, & xy < 0 \end{cases}$$

We also need the following assumption regarding the function η which is due to Mohan and Neogy [9].

Condition C: Let $S \subseteq \mathbb{R}$ be an open invex subset with respect to $\eta : S \times S \to \mathbb{R}$. For any $x, y \in S$ and any $t \in [0,1]$,

$$\eta(y, y + t\eta(y,x)) = -t\eta(y,x)$$

$$\eta(x, y + t\eta(y,x)) = (1-t)\eta(y,x).$$

Note that for every $x, y \in S$ and $t_1, t_2 \in [0,1]$, from Condition C, we have

$$\eta(y + t_2\eta(y,x), y + t_1\eta(y,x)) = (t_2 - t_1)\eta(y,x).$$

There are many vector functions that satisfy condition C (see [8]), besides the trivial case $\eta(x,y) = x - y$. For example, let $S = \mathbb{R}/\{0\}$ and

$$\eta(x,y) = \begin{cases} x-y, & x>0, y>0 \\ y-x, & x<0, y<0 \\ -y, & \text{otherwise.} \end{cases}$$

Then S is an invex set and η satisfies condition C.
In [3], İ.İscan established the Ostrowski type inequalities for the preinvex function as follow:

Theorem 1.4. Let $S \subset \mathbb{R}$ be an invex set with respect to $\eta : S \times S \to \mathbb{R}$ and $a, b \in S$ with $a < a + \eta(b,a)$. Suppose that $f : S \to \mathbb{R}$ is a differentiable function and $|f'|$ is preinvex function on S. If f' is integrable on $[a, a+\eta(b,a)]$. Then the following inequality holds:

$$\left| f(x) - \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(u)du \right| \leq \frac{\eta(b,a)}{6}$$

$$\times \left\{ \left[3\left(\frac{x-a}{\eta(b,a)}\right)^2 - 2\left(\frac{x-a}{\eta(b,a)}\right)^3 + \left(\frac{a+\eta(b,a)-x}{\eta(b,a)}\right)^3 \right] |f'(a)| + \left[1 - 3\left(\frac{x-a}{\eta(b,a)}\right)^2 + 4\left(\frac{x-a}{\eta(b,a)}\right)^3 \right] |f'(b)| \right\} \quad (2)$$

for all $x \in [a, a+\eta(b,a)]$. The constant $\frac{1}{6}$ is best possible in the sense that cannot be replaced by a smaller value.

Theorem 1.5. Let $S \subseteq \mathbb{R}$ be an open invex set with respect to $eta : S \times S \to \mathbb{R}$ and $a, b \in S$ with $a < a + \eta(b,a)$. Suppose that $f : S \to \mathbb{R}$ is a differentiable function such that $|f'|^q$ is preinvex on $[a, a+\eta(b,a)]$, for some fixed $q > 1$. If f' is integral on $[a, a+\eta(b,a)]$ and η satisfies condition C, then for each $x \in [a, a+\eta(b,a)]$, the following inequality holds

$$\left| f(x) - \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(u)du \right|$$

$$\leq \left(\frac{1}{p+1}\right)^{\frac{1}{p}} \left\{ \frac{(x-a)^2}{\eta(b,a)} \left(\frac{|f'(a)|^q + |f'(x)|^q}{2}\right)^{\frac{1}{q}} + \frac{\left(a+\eta(b,a)-x\right)^2}{\eta(b,a)} \left(\frac{|f'(a+\eta(b,a))|^q + |f'(x)|^q}{2}\right)^{\frac{1}{q}} \right\}, \quad (3)$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Theorem 1.6. Let $S \subseteq \mathbb{R}$ be an open invex set with respect to $eta : S \times S \to \mathbb{R}$ and $a, b \in S$ with $a < a + \eta(b,a)$. Suppose that $f : S \to \mathbb{R}$ is a differentiable function such that $|f'|^q$ is preinvex on $[a, a+\eta(b,a)]$, for some fixed $q \geq 1$. If f' is integral on $[a, a+\eta(b,a)]$ and η satisfies condition C, then for each $x \in [a, a+\eta(b,a)]$, the following inequality holds

$$\left|f(x) - \frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(u)du\right| \leq \eta(b,a)\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left\{\left(\frac{x-a}{\eta(b,a)}\right)^{2(1-\frac{1}{q})}\right.$$

$$\times\left[\frac{(x-a)^2(3\eta(b,a)-2x+2a)}{6\eta^3(b,a)}|f'(a)|^q + \frac{1}{3}\left(\frac{x-a}{\eta(b,a)}\right)^3|f'(b)|^q\right]^{\frac{1}{q}} + \left(\frac{a+\eta(b,a)-x}{\eta(b,a)}\right)^{2(1-\frac{1}{q})}$$

$$\left.\times\left[\frac{1}{3}\left(\frac{a+\eta(b,a)-x}{\eta(b,a)}\right)^3|f'(a)|^q + \left(\frac{1}{6} + \frac{(x-a)^2(2x-3\eta(b,a)-2a)}{6\eta^3(b,a)}\right)|f'(b)|^q\right]^{\frac{1}{q}}\right\} \tag{4}$$

for each $x \in [a, a+\eta(b,a)]$.

Now, we recall the class of the p-preinvex functions [1] which is a generalization of preinvex functions, harmonically preinvex functions and also recall the class of p-prequasiinvex functions :

Definition 1.7. Let $p \in \mathbb{R}/\{0\}$.The set $A_{\eta,p} \subseteq (0,\infty)$is said to be p-invex with respect to $\eta(.,.)$, if for every $x, y \in A$ and $t \in [0,1]$, we have

$$[(1-t)x^p + t(x+\eta(y,x))^p]^{\frac{1}{p}} \in A.$$

The p-invex set $A_{\eta,p}$ is also call a (p,η)-connected set.

Remark 1.8. Note that for $p = 1$, p-invex set becomes invex set and for $p = -1$, p-invex set become to harmonic invex-set.

Definition 1.9. Let $p \in \mathbb{R}/\{0\}$. The function f on the p-invex set $A_{\eta,p}$ is said to be p-preinvex function with respect to η if, where $p \in \mathbb{R}/\{0\}$, , if

$$f\left([(1-t)x^p + t(x+\eta(y,x))^p]^{\frac{1}{p}}\right) \leq tf(x) + (1-t)f(y), \tag{5}$$

for all $x, y \in A_{\eta,p}$ and $t \in [0,1]$.

Remark 1.10. Note that for $p = 1$ p-preinvex functions becomes preinvex functions and for $p = -1$, p-preinvex functions become harmonically preinvex functions.

Theorem 1.11. [1] Let $f : S = [a, a+\eta(b,a)] \to (0,\infty)$ be a p-preinvex function on the interval S° and $a, b \in S^\circ$ with $a < a+\eta(b,a)$. Then the following inequality holds:

$$f\left(\left[\frac{a^p + (a+\eta(b,a))^p}{2}\right]^{\frac{1}{p}}\right) \leq \frac{p}{[(a+\eta(b,a))^p - a^p]}\int_a^{a+\eta(b,a)} \frac{f(x)}{x^{1-p}}dx \leq \frac{f(a)+f(b)}{2}$$

Definition 1.12. Let $p \in \mathbb{R}/\{0\}$. The function f on the p-invex set $A_{\eta,p}$ is said to be p-prequasiinvex function with respect to η if, where $p \in \mathbb{R}/\{0\}$, , if

$$f\left([(1-t)x^p + t(x+\eta(y,x))^p]^{\frac{1}{p}}\right) \leq \max\{f(x), f(y)\}, \tag{6}$$

for all $x, y \in A_{\eta,p}$ and $t \in [0,1]$.

2 Main Results

Lemma 2.1. Let S be an open invex set with respect to η and $a, a+\eta(b,a) \in S$ with $a < a+\eta(b,a)$. Suppose that $f : S \to \mathbb{R}$ is differentiable function. If f' is integrable on $[a, a+\eta(b,a)]$ Then, we have following identity

$$f(x) - \frac{p}{[(a+\eta(b,a))^p - a^p]} \int_a^{a+\eta(b,a)} \frac{f(u)}{u^{1-p}} du$$

$$\begin{aligned} = \quad & \frac{(a+\eta(b,a))^p - a^p}{p} \Bigg[\int_0^{\frac{x^p-a^p}{(a+\eta(b,a))^p-a^p}} t[(1-t)a^p + t(a+\eta(b,a))^p]^{\frac{1-p}{p}} \\ & \times f'\left([(1-t)a^p + t(a+\eta(b,a))^p]^{\frac{1}{p}} \right) dt \\ + \quad & \int_{\frac{x^p-a^p}{(a+\eta(b,a))^p-a^p}}^1 (t-1)[(1-t)a^p + t(a+\eta(b,a))^p]^{\frac{1-p}{p}} \\ & \times f'\left([(1-t)a^p + t(a+\eta(b,a))^p]^{\frac{1}{p}} \right) dt \Bigg] \end{aligned}$$

for all $x \in [a, a+\eta(b,a)]$ and $p \in \mathbb{R}/\{0\}$.

Proof. Let

$$\begin{aligned} I_1 \quad = \quad & \frac{(a+\eta(b,a))^p - a^p}{p} \int_0^{\frac{x^p-a^p}{(a+\eta(b,a))^p-a^p}} t[(1-t)a^p + t(a+\eta(b,a))^p]^{\frac{1-p}{p}} \\ & \times f'\left([(1-t)a^p + t(a+\eta(b,a))^p]^{\frac{1}{p}} \right) dt \\ = \quad & t f\left([(1-t)a^p + t(a+\eta(b,a))^p]^{\frac{1}{p}} \right) \Bigg|_0^{\frac{x^p-a^p}{(a+\eta(b,a))^p-a^p}} \\ & - \int_0^{\frac{x^p-a^p}{(a+\eta(b,a))^p-a^p}} f\left([(1-t)a^p + t(a+\eta(b,a))^p]^{\frac{1}{p}} \right) dt \\ = \quad & \frac{x^p - a^p}{(a+\eta(b,a))^p - a^p} f(x) - \int_0^{\frac{x^p-a^p}{(a+\eta(b,a))^p-a^p}} f\left([(1-t)a^p + t(a+\eta(b,a))^p]^{\frac{1}{p}} \right) dt \\ = \quad & \frac{x^p - a^p}{(a+\eta(b,a))^p - a^p} f(x) - \frac{p}{(a+\eta(b,a))^p - a^p} \int_a^x \frac{f(u)}{u^{1-p}} du, \end{aligned}$$

and let

$$\begin{aligned} I_2 &= \frac{(a+\eta(b,a))^p - a^p}{p} \int_{\frac{x^p-a^p}{(a+\eta(b,a))^p-a^p}}^{1} (t-1)[(1-t)a^p + t(a+\eta(b,a))^p]^{\frac{1-p}{p}} \\ &\quad \times f'\left([(1-t)a^p + t(a+\eta(b,a))^p]^{\frac{1}{p}}\right) dt \\ &= (t-1) f\left([(1-t)a^p + t(a+\eta(b,a))^p]^{\frac{1}{p}}\right)\Bigg|_{\frac{x^p-a^p}{(a+\eta(b,a))^p-a^p}}^{1} \\ &\quad - \int_{\frac{x^p-a^p}{(a+\eta(b,a))^p-a^p}}^{1} f\left([(1-t)a^p + t(a+\eta(b,a))^p]^{\frac{1}{p}}\right) dt \\ &= \left(1 - \frac{x^p-a^p}{(a+\eta(b,a))^p-a^p}\right) f(x) - \int_{\frac{x^p-a^p}{(a+\eta(b,a))^p-a^p}}^{1} f\left([(1-t)a^p + t(a+\eta(b,a))^p]^{\frac{1}{p}}\right) dt \\ &= \left(1 - \frac{x^p-a^p}{(a+\eta(b,a))^p-a^p}\right) f(x) - \frac{p}{(a+\eta(b,a))^p-a^p} \int_x^{a+\eta(b,a)} \frac{f(u)}{u^{1-p}} du. \end{aligned}$$

Now, by adding I_1 and I_2, we get required result. □

Theorem 2.2. Let $S \subset \mathbb{R}$ be an invex set with respect to $\eta : S \times S \to \mathbb{R}$ and $a, b \in S$ with $a < a+\eta(b,a)$. Suppose that $f : S \to \mathbb{R}$ is a differentiable function and $|f'|$ is p-preinvex function on S with $p = 2k+1$ or $p = \frac{n}{m}$, $n = 2r+1$, $m = 2t+1$ where $k, r, t \in N$. If f' is integrable on $[a, a+\eta(b,a)]$. Then the following inequality holds:

$$\begin{aligned} &\left| f(x) - \frac{p}{[(a+\eta(b,a))^p - a^p]} \int_a^{a+\eta(b,a)} \frac{f(u)}{u^{1-p}} du \right| \\ &\leq \frac{(a+\eta(b,a))^p - a^p}{p} \Big[(S_1+S_3)|f'(a)| + (S_2+S_4)|f'(b)| \Big], \end{aligned} \tag{7}$$

where

$$S_1 = \int_0^{\frac{x^p-a^p}{(a+\eta(b,a))^p-a^p}} t^2 [(1-t)a^p + t(a+\eta(b,a))^p]^{\frac{1-p}{p}} dt$$

$$S_2 = \int_0^{\frac{x^p-a^p}{(a+\eta(b,a))^p-a^p}} t(1-t) [(1-t)a^p + t(a+\eta(b,a))^p]^{\frac{1-p}{p}} dt$$

$$S_3 = \int_{\frac{x^p-a^p}{(a+\eta(b,a))^p-a^p}}^{1} t(1-t) [(1-t)a^p + t(a+\eta(b,a))^p]^{\frac{1-p}{p}} dt$$

$$S_4 = \int_{\frac{x^p-a^p}{(a+\eta(b,a))^p-a^p}}^{1} (1-t)(1-t) [(1-t)a^p + t(a+\eta(b,a))^p]^{\frac{1-p}{p}} dt$$

Proof. Using Lemma 2.1 and $|f'|$ is p-preinvex on S, we have

$$\left| f(x) - \frac{p}{[(a+\eta(b,a))^p - a^p]} \int_a^{a+\eta(b,a)} \frac{f(u)}{u^{1-p}} du \right|$$

$$\leq \frac{(a+\eta(b,a))^p-a^p}{p}\Bigg[\int_0^{\frac{x^p-a^p}{(a+\eta(b,a))^p-a^p}} t[(1-t)a^p+t(a+\eta(b,a))^p]^{\frac{1-p}{p}}$$

$$\times\left|f'\left([(1-t)a^p+t(a+\eta(b,a))^p]^{\frac{1}{p}}\right)\right|dt$$

$$+\int_{\frac{x^p-a^p}{(a+\eta(b,a))^p-a^p}}^1 (t-1)[(1-t)a^p+t(a+\eta(b,a))^p]^{\frac{1-p}{p}}$$

$$\times\left|f'\left([(1-t)a^p+t(a+\eta(b,a))^p]^{\frac{1}{p}}\right)\right|dt\Bigg]$$

$$\leq \frac{(a+\eta(b,a))^p-a^p}{p}\Bigg[\int_0^{\frac{x^p-a^p}{(a+\eta(b,a))^p-a^p}} t[(1-t)a^p+t(a+\eta(b,a))^p]^{\frac{1-p}{p}}$$

$$\times\Big(t|f'(a)|+(1-t)|f'(b)|\Big)dt$$

$$+\int_{\frac{x^p-a^p}{(a+\eta(b,a))^p-a^p}}^1 (t-1)[(1-t)a^p+t(a+\eta(b,a))^p]^{\frac{1-p}{p}}\Big(t|f'(a)|+(1-t)|f'(b)|\Big)dt\Bigg]$$

$$= \frac{(a+\eta(b,a))^p-a^p}{p}\Big[(S_1+S_3)|f'(a)|+(S_2+S_4)|f'(b)|\Big].$$

This completes the proof. □

Theorem 2.3. Let $S \subseteq \mathbb{R}$ be an open invex set with respect to $eta : S\times S \to \mathbb{R}$ and $a, b \in S$ with $a < a+\eta(b,a)$. Suppose that $f : S \to \mathbb{R}$ is a differentiable function such that $|f'|^q$ is p-preinvex on $[a, a+\eta(b,a)]$ with $p = 2k+1$ or $p = \frac{n}{m}$, $n = 2r+1$, $m = 2t+1$ where $k, r, t \in N$, for some fixed $q > 1$. If f' is integral on $[a, a+\eta(b,a)]$ and η satisfies condition C, then for each $x \in [a, a+\eta(b,a)]$, the following inequality holds

$$\left|f(x)-\frac{p}{[(a+\eta(b,a))^p-a^p]}\int_a^{a+\eta(b,a)}\frac{f(u)}{u^{1-p}}du\right|$$

$$\leq \frac{[(a+\eta(b,a))^p-a^p]}{2^{1+\frac{1}{q}}(q+1)^{\frac{1}{q}}}\Bigg[\left(S_5|f(a)|^{\frac{q}{q-1}}+S_6|f(b)|^{\frac{q}{q-1}}\right)^{\frac{q-1}{q}}$$

$$+\left(S_7|f(a)|^{\frac{q}{q-1}}+S_8|f(b)|^{\frac{q}{q-1}}\right)^{\frac{q-1}{q}}\Bigg],$$

where

$$S_5 = \int_0^{\frac{x^p-a^p}{(a+\eta(b,a))^p-a^p}} t[(1-t)a^p+t(a+\eta(b,a))^p]^{\frac{q(1-p)}{p(q-1)}}dt$$

$$S_6 = \int_0^{\frac{x^p-a^p}{(a+\eta(b,a))^p-a^p}} (1-t)[(1-t)a^p+t(a+\eta(b,a))^p]^{\frac{q(1-p)}{p(q-1)}}dt$$

$$S_7 = \int_{\frac{x^p-a^p}{(a+\eta(b,a))^p-a^p}}^1 t[(1-t)a^p+t(a+\eta(b,a))^p]^{\frac{q(1-p)}{p(q-1)}}dt$$

$$S_8 = \int_{\frac{x^p-a^p}{(a+\eta(b,a))^p-a^p}}^{1} (1-t)[(1-t)a^p + t(a+\eta(b,a))^p]^{\frac{q(1-p)}{p(q-1)}} dt$$

Proof. Using Lemma 2.1, Holder's inequality and p-preinvexity of $|f'|^{\frac{q}{q-1}}$ on S, we have

$$\left| f(x) - \frac{p}{[(a+\eta(b,a))^p - a^p]} \int_a^{a+\eta(b,a)} \frac{f(u)}{u^{1-p}} du \right|$$

$$\leq \frac{(a+\eta(b,a))^p - a^p}{p} \Bigg[\int_0^{\frac{x^p-a^p}{(a+\eta(b,a))^p-a^p}} t[(1-t)a^p + t(a+\eta(b,a))^p]^{\frac{1-p}{p}} \times \left| f'\left([(1-t)a^p + t(a+\eta(b,a))^p]^{\frac{1}{p}} \right) \right| dt$$

$$+ \int_{\frac{x^p-a^p}{(a+\eta(b,a))^p-a^p}}^{1} (1-t)[(1-t)a^p+t(a+\eta(b,a))^p]^{\frac{1-p}{p}} \left| f'\left([(1-t)a^p+t(a+\eta(b,a))^p]^{\frac{1}{p}} \right) \right| dt \Bigg]$$

$$\leq \frac{(a+\eta(b,a))^p - a^p}{p} \Bigg[\left(\int_0^{\frac{x^p-a^p}{(a+\eta(b,a))^p-a^p}} t^q dt \right)^{\frac{1}{q}} \left(\int_0^{\frac{x^p-a^p}{(a+\eta(b,a))^p-a^p}} [(1-t)a^p+t(a+\eta(b,a))^p]^{\frac{q(1-p)}{p(q-1)}} \times \left| f'\left([(1-t)a^p + t(a+\eta(b,a))^p]^{\frac{1}{p}} \right) \right|^{\frac{q}{q-1}} dt \right)^{\frac{q-1}{q}}$$

$$+ \left(\int_{\frac{x^p-a^p}{(a+\eta(b,a))^p-a^p}}^{1} (1-t)^q dt \right)^{\frac{1}{q}} \left(\int_{\frac{x^p-a^p}{(a+\eta(b,a))^p-a^p}}^{1} [(1-t)a^p + t(a+\eta(b,a))^p]^{\frac{q(1-p)}{p(q-1)}} \times \left| f'\left([(1-t)a^p + t(a+\eta(b,a))^p]^{\frac{1}{p}} \right) \right|^{\frac{q}{q-1}} dt \right)^{\frac{q-1}{q}} \Bigg]$$

$$\leq \frac{(a+\eta(b,a))^p - a^p}{p} \Bigg[\left(\int_0^{\frac{x^p-a^p}{(a+\eta(b,a))^p-a^p}} t^q dt \right)^{\frac{1}{q}} \left(\int_0^{\frac{x^p-a^p}{(a+\eta(b,a))^p-a^p}} [(1-t)a^p+t(a+\eta(b,a))^p]^{\frac{q(1-p)}{p(q-1)}} \times (t|f(a)|^{\frac{q}{q-1}} + (1-t)|f(b)|^{\frac{q}{q-1}}) dt \right)^{\frac{q-1}{q}}$$

$$+ \left(\int_{\frac{x^p-a^p}{(a+\eta(b,a))^p-a^p}}^{1} (1-t)^q dt \right)^{\frac{1}{q}}$$

$$\times \left(\int_{\frac{x^p-a^p}{(a+\eta(b,a))^p-a^p}}^{1} [(1-t)a^p + t(a+\eta(b,a))^p]^{\frac{q(1-p)}{p(q-1)}} (t|f(a)|^{\frac{q}{q-1}} + (1-t)|f(b)|^{\frac{q}{q-1}}) dt \right)^{\frac{q-1}{q}} \Bigg]$$

$$= \frac{[(a+\eta(b,a))^p - a^p]}{2^{1+\frac{1}{q}}(q+1)^{\frac{1}{q}}} \Bigg[\left(S_5|f(a)|^{\frac{q}{q-1}} + S_6|f(b)|^{\frac{q}{q-1}} \right)^{\frac{q-1}{q}} + \left(S_7|f(a)|^{\frac{q}{q-1}} + S_8|f(b)|^{\frac{q}{q-1}} \right)^{\frac{q-1}{q}} \Bigg]$$

The proof is completed. □

Theorem 2.4. Let $S \subseteq \mathbb{R}$ be an open invex set with respect to $\eta : S \times S \to \mathbb{R}$ and $a, b \in S$ with $a < a + \eta(b,a)$. Suppose that $f : S \to \mathbb{R}$ is a differentiable function such that $|f'|^q$ is preinvex on $[a, a+\eta(b,a)]$ with $p = 2k+1$ or $p = \frac{n}{m}$, $n = 2r+1$, $m = 2t+1$ where $k, r, t \in N$, for some fixed $q \geq 1$. If f' is integral on $[a, a+\eta(b,a)]$ and η satisfies condition C, then for each $x \in [a, a+\eta(b,a)]$, the following inequality holds

$$\left| f(x) - \frac{p}{[(a+\eta(b,a))^p - a^p]} \int_a^{a+\eta(b,a)} \frac{f(u)}{u^{1-p}} du \right|$$

$$\leq \frac{[(a+\eta(b,a))^p - a^p]}{2^{1+\frac{1}{r}}(r+1)^{\frac{1}{r}}} \left[\left(S_9|f(a)|^q + S_{10}|f(b)|^q \right)^{\frac{1}{q}} + \left(S_{11}|f(a)|^q + S_{12}|f(b)|^q \right)^{\frac{1}{q}} \right],$$

where

$$S_9 = \int_0^{\frac{x^p - a^p}{(a+\eta(b,a))^p - a^p}} t[(1-t)a^p + t(a+\eta(b,a))^p]^{\frac{q(1-p)}{p}} dt$$

$$S_{10} = \int_0^{\frac{x^p - a^p}{(a+\eta(b,a))^p - a^p}} (1-t)[(1-t)a^p + t(a+\eta(b,a))^p]^{\frac{q(1-p)}{p}} dt$$

$$S_{11} = \int_{\frac{x^p - a^p}{(a+\eta(b,a))^p - a^p}}^1 t[(1-t)a^p + t(a+\eta(b,a))^p]^{\frac{q(1-p)}{p}} dt$$

$$S_{12} = \int_{\frac{x^p - a^p}{(a+\eta(b,a))^p - a^p}}^1 (1-t)[(1-t)a^p + t(a+\eta(b,a))^p]^{\frac{q(1-p)}{p}} dt$$

Proof. Using Lemma 2.1, Holder's inequality and p-preinvexity of $|f'|^q$ on S, we have

$$\left| f(x) - \frac{p}{[(a+\eta(b,a))^p - a^p]} \int_a^{a+\eta(b,a)} \frac{f(u)}{u^{1-p}} du \right|$$

$$\leq \frac{(a+\eta(b,a))^p - a^p}{p} \Bigg[\int_0^{\frac{x^p - a^p}{(a+\eta(b,a))^p - a^p}} t[(1-t)a^p + t(a+\eta(b,a))^p]^{\frac{1-p}{p}}$$

$$\times \left| f'\left([(1-t)a^p + t(a+\eta(b,a))^p]^{\frac{1}{p}} \right) \right| dt$$

$$+ \int_{\frac{x^p - a^p}{(a+\eta(b,a))^p - a^p}}^1 (1-t)[(1-t)a^p + t(a+\eta(b,a))^p]^{\frac{1-p}{p}} \left| f'\left([(1-t)a^p + t(a+\eta(b,a))^p]^{\frac{1}{p}} \right) \right| dt \Bigg]$$

$$\leq \frac{(a+\eta(b,a))^p - a^p}{p} \Bigg[\left(\int_0^{\frac{x^p - a^p}{(a+\eta(b,a))^p - a^p}} t^r dt \right)^{\frac{1}{r}} \Bigg(\int_0^{\frac{x^p - a^p}{(a+\eta(b,a))^p - a^p}} [(1-t)a^p + t(a+\eta(b,a))^p]^{\frac{q(1-p)}{p}}$$

$$\times \left| f'\left([(1-t)a^p + t(a+\eta(b,a))^p]^{\frac{1}{p}} \right) \right|^q dt \Bigg)^{\frac{1}{q}}$$

$$+\left(\int_{\frac{x^p-a^p}{(a+\eta(b,a))^p-a^p}}^{1}(1-t)^r dt\right)^{\frac{1}{r}}\left(\int_{\frac{x^p-a^p}{(a+\eta(b,a))^p-a^p}}^{1}[(1-t)a^p+t(a+\eta(b,a))^p]^{\frac{q(1-p)}{p}}\right.$$

$$\left.\times\left|f'\left([(1-t)a^p+t(a+\eta(b,a))^p]^{\frac{1}{p}}\right)\right|^q dt\right)^{\frac{1}{q}}\Bigg]$$

$$\leq \frac{(a+\eta(b,a))^p-a^p}{p}\Bigg[\left(\int_0^{\frac{x^p-a^p}{(a+\eta(b,a))^p-a^p}}t^r dt\right)^{\frac{1}{r}}\left(\int_0^{\frac{x^p-a^p}{(a+\eta(b,a))^p-a^p}}[(1-t)a^p+t(a+\eta(b,a))^p]^{\frac{q(1-p)}{p}}\right.$$

$$\left.\times(t|f(a)|^q+(1-t)|f(b)|^q)dt\right)^{\frac{1}{q}}$$

$$+\left(\int_{\frac{x^p-a^p}{(a+\eta(b,a))^p-a^p}}^{1}(1-t)^r dt\right)^{\frac{1}{r}}$$

$$\times\left(\int_{\frac{x^p-a^p}{(a+\eta(b,a))^p-a^p}}^{1}[(1-t)a^p+t(a+\eta(b,a))^p]^{\frac{q(1-p)}{p}}(t|f(a)|^q+(1-t)|f(b)|^q)dt\right)^{\frac{1}{q}}\Bigg]$$

$$=\frac{[(a+\eta(b,a))^p-a^p]}{2^{1+\frac{1}{r}}(r+1)^{\frac{1}{r}}}\left[\left(S_9|f(a)|^q+S_{10}|f(b)|^q\right)^{\frac{1}{q}}+\left(S_{11}|f(a)|^q+S_{12}|f(b)|^q\right)^{\frac{1}{q}}\right]$$

The proof is completed. □

Acknowledgement

The authors are grateful for financial support from Higher Education Commission of Pakistan.

References

[1] I.A. Baloch, İ. İşcan, Some Hermite-Hadamard Type Inequalities For p-preinvex functions, available on RGMIA Res. Rep. Coll. 19(2016), Art. 84.

[2] W.W. Breckner, Stetigkeitsaussagen für eine klasse verallgemeinerter konvexer funktonen in topologischen linearen Räumen, Publ. Inst. Math. (Beograd), 23, (1978), 13-20.

[3] İ. İşcan, Ostrowski type inequalities for functions whose derivatives are preinvex, Availabe on arXive:1204.2010v2[math.CA]9 Nov 2013.

[4] İ. İşcan, Hermite-Hadamard type inequalities for harmonically convex functions, Hacettepe Journal of Mathematics and statistics, vol 43 (6) (2014),935-942.

[5] İ. İşcan, Ostrowski type inequalities for harmonically s-convex functions, Konuralp journal of Mathematics, 3(1) (2015), 63-74 .

[6] İ. İşcan, Hermite-Hadamard type inequalities for harmonically (α, m) convex functions, Hacettepe Journal of Mathematics and statistics. Accepted for publication "arxiv:1307.5402v2[math.CA]".

[7] J. Park, New Ostrowski-Like type inequalities for differentiable (s, m)-convex mappings, International journal of pure and applied mathematics,vol.78 No.8 2012,1077-1089.

[8] A. Ostrowski, ¨Über die Absolutabweichung einer differentiebaren funktion von ihren integralmittelwert, Comment. Math. Helv. 10(1938) 226-227.

[9] Z. Liu, A note on Ostrowski type inequalities related to some s-convex functions in the second sense, Bull. Korean Math. Soc. 49 (4) (2012), 775-785. Available online at http://dx.doi.org/10.4134/BKMS.2012.49.4.775.

[10] M. Alomari, M. Darus, S.S Dragomir, and P. Cerone, Ostrowski type inequalities for functions whose derivatives are s-convex in the second sense, Appl. Math. Lett. 23 (1) (2010), 1071-1076.

[11] M. Abramowitz and I. A Stegun (Eds), Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, New Yark, 1995.

[12] S.S. Dragomir and R.P. Agarwal, Two inequalities for differentiable mappings and applica- tions to special means of real numbers and trapezoidal formula, Appl. Math. Lett., 11(5) (1998), 91.95.

[13] S.S. Dragomir and C. E. M. Pearce, Selected Topics on Hermite-Hadamard Inequalities and Applications, RGMIA Monographs, Victoria University, 2000.

[14] A. Barani, A.G. Ghazanfari, S.S. Dragomir, Hermite-Hadamard inequality through prequsi- invex functions, (submitted)

[15] X.M. Yang and D. Li, On properties of preinvex functions, J. Math. Anal. Appl. 256 (2001) 229-241.

[16] M. Aslam Noor, Hadamard integral inequalities for product of two preinvex function, Nonl. Anal. Forum, 14 (2009), 167-173.

[17] M. Aslam Noor, Some new classes of nonconvex functions, Nonl. Funct. Anal. Appl., 11 (2006) 165-171

[18] M. Aslam Noor, On Hadamard integral inequalities involving two log-preinvex functions, J. Inequal. Pure Appl. Math., 8(2007), No. 3, 1-6, Article 75.

[19] U.S. Kırmacı, Inequalities for differentiable mappings and applications to special means of real numbers and to midpoint formula, Appl. Math. Comp., 147 (2004), 137-146.

[20] U.S. Kırmacı and M.E. Özdemir, On some inequalities for differentiable mappings and ap- plications to special means of real numbers and to midpoint formula, Appl. Math. Comp., 153 (2004), 361-368.

[21] U.S. Kırmacı, Improvement and further generalization of inequalities for differentiable map- pings and applications, Computers and Math. with Appl., 55 (2008), 485-493.

[22] C.E.M. Pearce and J. Pecaric, Inequalities for differentiable mappings with application to special means and quadrature formulae, Appl. Math. Lett., 13(2) (2000), 51.55.

[23] M.A. Hanson, On sufficiency of the Kuhn-Tucker conditions, J. Math. Anal. Appl. 80 (1981) 545-550.

[24] S.R. Mohan and S. K. Neogy, On invex sets and preinvex functions, J. Math. Anal. Appl. 189 (1995), 901.908.

[25] A. Ben-Israel and B. Mond, What is invexity?, J. Austral. Math. Soc., Ser. B, 28(1986), No. 1, 1-9.

[26] R. Pini, Invexity and generalized Convexity, Optimization 22 (1991) 513-525.

[27] M.Z. Sarıkaya, H. Bozkurt and N. Alp, On Hermite-Hadamard type inequalities for preinvex and log-preinvex functions,arXiv:1203.4759v1[math.FA]21 Mar 2012.

[28] M.Z. Sarıkaya, A. Saglam and H. Yıldırım, New inequalities of Hermite-Hadamard type for functions whose second derivatives absolute values are convex and quasi-convex, International Journal of Open Problems in Computer Science and Mathematics (IJOPCM), 5(3), 2012.

[29] M.Z. Sarıkaya, A. Saglam and H. Yıldırım, On some Hadamard-type inequalities for h-convex functions, Journal of Mathematical Inequalities, Volume 2, Number 3 (2008), 335-341.

[30] M.Z. Sarıkaya, M. Avci and H. Kavurmacı, On some inequalities of Hermite-Hadamard type for convex functions, ICMS Iternational Conference on Mathematical Science. AIP Conference Proceedings 1309, 852 (2010).

[31] Y. Wang, B.Y. Xi and F. Qi, Hermite Hadamard type integral inequalities when the power of the absolute value of the first derivative of the integrand is preinvex, Le Matematche, Vol. LXIX (2014)-Fasc.I, pp.89-96. doi:10.4418/2014.69.1.6

http://www.newtheory.org ISSN: 2149-1402

Journal of New Theory

Received: 05.10.2017 *Year*: **2017,** *Number*: **16**, *Pages*: **80-91**
Published: 21.10.2017 *Original Article*

CUBIC HYPER *KU*-IDEALS

Samy Mohammed Mostafa[1,*] <samymostafa@yahoo.com>
Fatema Faysal Kareem[2] <fa_sa20072000@yahoo.com >
Reham Abd Allah Ghanem[3] <ghanemreham@yahoo.co>

[1]Department of Mathematics, Faculty of Education, Ain Shams University, Roxy, Cairo, Egypt.
[2]Department of Mathematics, Ibn-Al-Haitham college of Education, University of Baghdad, Iraq.
[3]Department of Basic science, Deanship of Preparatory Year and Supporting studies Imam Abdulrahman Bin Faisal UniversityDammam, Saudi Arabia

Abstract – It is known that, the concept of hyper KU-algebras is a generalization of KU-algebras. In this paper, we define cubic (strong, weak,s-weak) hyper KU-ideals of hyper KU-algebras and related properties are investigated.

Keywords – *KU-algebra, hyper KU-algebra, cubic* (strong, weak,s-weak) *hyper KU-ideal.*

1. Introduction

Prabpayak and Leerawat [10,11] introduced a new algebraic structure which is called KU-algebras. They studied ideals and congruences in KU-algebras. Also, they introduced the concept of homomorphism of KU-algebra and investigated some related properties. Moreover, they derived some straightforward consequences of the relations between quotient KU-algebras and isomorphism. Mostafa et al. [7]introduced the notion of fuzzy KU-ideals of KU-algebras and then they investigated several basic properties which are related to fuzzy KU-ideals .The hyper structure theory (called also multi-algebras) is introduced in 1934 by Marty [6] at the 8th congress of Scandinvian Mathematiciens. Around the 40's, several authors worked on hyper groups, especially in France and in the United States, but also in Italy, Russia and Japan. Hyper structures have many applications to several sectors of both pure and applied sciences, since then numerous mathematical papers [2,3,4,8] have been written investigating the algebraic properties of the hyper BCK / BCI- KU algebras. Jun and Xin [3] considered the fuzzification of the notion of a (weak, strong, reflexive) hyper BCK-ideal, and investigated the relations among them. In [8], Mostafa et al. applied the hyper structures to KU- algebras and introduced the concept of a

*Corresponding Author.

hyper KU-algebra which is a generalization of a KU-algebra, and investigated some related properties. They also introduced the notion of a hyper KU-ideal, a weak hyper KU-ideal and gave relations between hyper KU-ideals and weak hyper KU-ideals. Mostafa et al [9] the bipolar fuzzy set theory to the (s-weak-strong) hyper KU-ideals in hyper KU-algebras are applied and discussed. In this paper, we define cubic (strong, weak, s-weak) hyper KU-ideals of hyper KU-algebras and related properties are investigated.

2. Preliminaries

Let H be a nonempty set and $P^*(H) = P(H) \setminus \{\phi\}$ the family of the nonempty subsets of H. A multi valued operation (said also hyper operation) "$\circ$" on H is a function, which associates with every pair $(x, y) \in H \times H = H^2$ a non empty subset of H denoted $x \circ y$ .An algebraic hyper structure or simply a hyper structure is a non empty set H endowed with one or more hyper operations.

We shall use the $x \circ y$ instead of $x \circ \{y\}$, $\{x\} \circ y$ or $\{x\} \circ \{y\}$.

Definition2.1[8]. Let H be a nonempty set and "$\circ$" a hyper operation on H, such that $\circ : H \times H \to P^*(H)$. Then H is called a hyper KU-algebra if it contains a constant "0" and satisfies the following axioms: for all $x, y, z \in H$

(HKU_1) $[(y \circ z) \circ (x \circ z)] << x \circ y$
(HKU_2) $x \circ 0 = \{0\}$
(HKU_3) $0 \circ x = \{x\}$
(HKU_4) *if* $x << y,\ y << x$ *implies* $x = y$.

where x << y is defined by $0 \in y \circ x$ and for every $A, B \subseteq H$, $A << B$ is defined by $\forall a \in A, \exists b \in B$ *such that* $a << b$. In such case, we call "<<" the hyper order in H.
Note that if $A, B \subseteq H$, then by $A \circ B$ we mean the subset $\bigcup_{a \in A, b \in B} a \circ b$ *of* H.

Example 2.2. [8] Let $H = \{0,1,2,3\}$ be a set. Define hyper operation $\circ$ on H as follows:

$\circ$	0	1	2	3
0	{0}	{1}	{2}	{3}
1	{0}	{0}	{1}	{3}
2	{0}	{0}	{0}	{0,3}
3	{0}	{0}	{1}	{0,3}

Then $(H, \circ, 0)$ is a hyper KU-algebra.

Proposition 2.3. [8]Let H be a hyper KU-algebra. Then for all $x, y, z \in H$, the following statements hold:
(P_1) $A \subseteq B$ implies $A << B$, for all nonempty subsets A, B of H.

(P_2) $0 \circ 0 = \{0\}$.
(P_3) $0 << x$.
(P_4) $z << z$.
(P_5) $x \circ z << z$
(P_6) $A \circ 0 = \{0\}$.
(P_7) $0 \circ A = A$.
(P_8) $(0 \circ 0) \circ x = \{x\}$ and $(x \circ (0 \circ x)) = \{0\}$.
(P_9) $x \circ x = \{x\} \Leftrightarrow x = 0$

Lemma 2.4. [8] In hyper KU-algebra $(X, \circ, 0)$, the following hold:

$$x << y \quad \text{imply} \quad y \circ z << x \circ z \text{ for all } x, y, z \in X.$$

Lemma 2.5. [8] In hyper KU-algebra $(X, \circ, 0)$, we have

$$z \circ (y \circ x) = y \circ (z \circ x) \text{ for all } x, y, z \in X.$$

Lemma 2.6. [8] For all $x, y, z \in H$, the following statements hold:

(i) $x \circ y << z \Leftrightarrow z \circ y << x$,
(ii) $0 << A \Rightarrow 0 \in A$,
(iii) $y \in (0 \circ x) \Rightarrow y << x$.

Definition 2.7. [8] For a hyper KU-algebras H, a non-empty subsets $I \subseteq H$, containing 0 are called :

1- A weak hyper KU-ideal of H if $a \circ (b \circ c) \subseteq I$ and $b \in I$ imply $a \circ c \in I$.
2- A hyper KU-ideal of H if $a \circ (b \circ c) << I$ and $b \in I$ imply $a \circ c \in I$.
3- A strong hyper KU-ideal of H if $(\forall x, y \in H)((a \circ (b \circ c) \cap I \neq \phi)$ and $b \in I$ imply $a \circ c \in I$.

Example 2.8. [8] Let $H = \{0, a, b, c\}$ be a set with the following Cayley table

∘	0	a	b	c
0	{0}	{a}	{b}	{c}
a	{0}	{0,a}	{0,b}	{b,c}
b	{0}	{0,b}	{0 }	{a}
c	{0}	{0,b}	{0 }	{0,a}

Then H is a hyper KU-algebra. Take $I = \{0, b\}$, then I is a weak hyper ideal, however, not a weak hyper KU-ideal of H as $b \circ (b \circ c) \subseteq I,\ b \in I,\ but\ b \circ c = a \notin I$.

Example 2.9. [8] Let $H = \{0, a, b\}$ be a set with the following Cayley table:

$\circ$	**0**	**a**	**b**
0	$\{0\}$	$\{a\}$	$\{b\}$
a	$\{0\}$	$\{0,a\}$	$\{b\}$
b	$\{0\}$	$\{b\}$	$\{0,b\}$

Then H is a hyper KU-algebra. Take $I=\{0,b\}$. Then I is a hyper ideal, but not a hyper KU-ideal, since $0\circ(b\circ a) << I$ and $b\in I$ but $a\notin I$

Here $I=\{0,b\}$ is also a strong hyper ideal but it is not a strong hyper KU-ideal of H, since $0\circ(b\circ a)=\{b\}\cap I\neq\phi$ *and* $b\in I$ *but* $a\notin I$.

Definition 3.10. [1] An interval number is $\tilde{a}=[a_L,a_U]$, where $0\le a_L\le a_U\le 1$.

Let D[0, 1] denote the family of all closed subintervals of [0, 1], i.e.,

$$D[0,1]=\{\tilde{a}=[a_L,a_U]: a_L\le a_U \quad for\ a_L,a_U\in I\}.$$

We define the operations $\le$, $\ge$, $=$,rmin and rmax in case of two elements in D[0, 1]. We consider two elements $\tilde{a}=[a_L,a_U]$ and $\tilde{b}=[b_L,b_U]$ in D[0, 1]. Then

1- $\tilde{a}\le\tilde{b}$ *iff* $a_L\le b_L, a_U\le b_U$;
2- $\tilde{a}\ge\tilde{b}$ *iff* $a_L\ge b_L, a_U\ge b_U$;
3- $\tilde{a}=\tilde{b}$ *iff* $a_L=b_L, a_U=b_U$;
4- $rmim\{\tilde{a},\tilde{b}\}=[\min\{a_L,b_L\},\min\{a_U,b_U\}]$;
5- $r\max\{\tilde{a},\tilde{b}\}=[\max\{a_L,b_L\},\max\{a_U,b_U\}]$

Here we consider that $\tilde{0}=[0,0]$ as least element and $\tilde{1}=[1,1]$ as greatest element.
Let $\tilde{a}_i\in D[0,1]$,where $i\in\Lambda$. We define

$$r\inf_{i\in\Lambda}\tilde{a}_i=\left[\inf_{i\in\Lambda}(a_i)_L,\inf_{i\in\Lambda}(a_i)_U\right] \text{ and } r\sup_{i\in\Lambda}\tilde{a}_i=\left[\sup_{i\in\Lambda}(a_i)_L,\sup_{i\in\Lambda}(a_i)_U\right]$$

An interval valued fuzzy set (briefly, i-v-f-set) $\tilde{\mu}$ on a set X is defined as

$$\tilde{\mu}=\left\{\left\langle x,\ [\mu^L(x),\mu^U(x)],\ x\in X\right\rangle\right\}$$

where $\tilde{\mu}: X\to D[0,1]$ and $\mu^L(x)\le\mu^U(x)$, for all $x\in X$. A cubic fuzzy set A over a set X (see [5]) is an object having the form $A=\{(x,\tilde{\mu}_A(x),\lambda_A(x))\mid x\in X\}$, where $\tilde{\mu}_A(x)\subseteq D[0,1]$ and $\lambda_A(x)\in[0,1]$ Jun et al. [5], introduced the concept of cubic sets defined on a non-empty set X as objects having the form: $A=\{\langle x,\tilde{\mu}_A(x),\lambda_A(x)\rangle : x\in X\}$, which is briefly denoted by $A=\langle\tilde{\alpha}_A,\lambda_A\rangle$, where the functions $\tilde{\alpha}_A : X \to D[0,1]$ and $\lambda_A : X \to [0,1]$.

3. Cubic Hyper KU–ideals

Now some fuzzy logic concepts are reviewed .A fuzzy set μ in a set H is a function $\mu : H \to [0,1]$. A fuzzy set μ in a set H is said to satisfy the inf (resp. sup) property if for any subset T of H there exists $x_0 \in T$ such that $\mu(x_0)=\inf_{x\in T}\mu(x)$ (resp. $\mu(x_0)=\sup_{x\in T}\mu(x)$). For a fuzzy set μ in X and a ∈ [0, 1] the set U(μ ; a):={x ∈ H, μ (x) ≥ a}, which is called a level set of μ.

Definition 3.1. A fuzzy set μ in H is said to be a fuzzy hyper KU-subalgebra of H if it satisfies the inequality: $\inf_{z\in x\circ y}\mu(z) \geq \min\{\mu(x),\mu(y)\}\ \forall x, y \in H$.

Proposition 3.2. Let μ be a fuzzy hyper KU-sub-algebra of H. Then $\mu(0) \geq \mu(x)$ for all x ∈ H.

Proof. Using Proposition **2.3** (P_9), we see that 0 ∈ x ∘ x for all x ∈ H. Hence

$$\inf_{0\in x\circ x}\mu(0) \geq \min\{\mu(x),\mu(x)\} = \mu(x) \text{ for all } x \in H.$$

Example 3.3. Let $H=\{0,a,b\}$ be a set. Define hyper operation ∘ on H as follows:

∘	0	a	b
0	$\{0\}$	$\{a\}$	$\{b\}$
a	$\{0\}$	$\{0,a\}$	$\{a,b\}$
b	$\{0\}$	$\{0,a\}$	$\{0,a,b\}$

Then $(H,\circ,0)$ is a hyper KU-algebra. Define a fuzzy set μ : H →[0, 1] by

$$\mu(0) = \mu(a)=\alpha_1 > \alpha_2 = \mu(b)$$

Then μ is a fuzzy hyper sub-algebra of H. A fuzzy set ν : H →[0, 1] defined by

$$\nu(0) = 0.7,\ \nu(a)=0.5 \text{ and } \nu(b)=0.2$$

is also a fuzzy Hyper sub-algebra of H.

Definition3.4. Let X be nonempty set .A cubic set A in X is structure

$$A=\{\langle x,\tilde{\mu}_A(x),\lambda_A(x)\rangle, x\in X\}$$

which is briefly denoted by $A=\langle\tilde{\mu}_A(x),\lambda_A(x)\rangle$, where $\tilde{\mu}_A(x)=[\mu_A^L\ ,\mu_A^U\ ]$ is an interval value fuzzy set in X and λ_A is an fuzzy set in X.

Definition3.5. For a hyper KU-algebra H , a cubic $A=\langle\tilde{\mu}_A(x),\lambda_A(x)\rangle$" in H is called:

(I) Cubic hyper ideal of H ,if $K_1:\ x<<y\ \textit{implies}\ \tilde{\mu}(x)\geq\tilde{\mu}(y)$,

$$\lambda_A(x)\leq\lambda_A(y)\,,\tilde{\mu}_A(z)\geq r\min\left\{\inf_{u\in(y\circ z)}\tilde{\mu}_A(u)\ ,\ \tilde{\mu}_A(y)\right\}$$

and

$$\lambda_A(z)\leq\max\left\{\sup_{a\in(y\circ z)}\mu(a),\mu(y)\right\}$$

(II) Cubic weak hyper ideal of H" if, for any x; y; z $\in H$

$$\tilde{\mu}_A(0)\geq\tilde{\mu}_A(z)\geq r\min\left\{\inf_{u\in(y\circ z)}\tilde{\mu}_A(u),\tilde{\mu}_A(y)\right\}$$

$$\lambda_A(0)\leq\lambda_A(z)\leq\max\left\{\sup_{a\in(y\circ z)}\lambda_A(a),\lambda_A(y)\right\}$$

(III) Cubic strong hyper ideal of H" if, for any x; y; z $\in H$

$$\inf_{u\in(y\circ z)}\tilde{\mu}_A(u)\geq\tilde{\mu}_A(z)\geq r\min\left\{\inf_{u\in(y\circ z)}\tilde{\mu}_A(u),\tilde{\mu}_A(y)\right\}$$

$$\sup_{u\in(y\circ z)}\lambda_A(u)\leq\lambda_A(z)\leq\max\left\{\sup_{u\in(y\circ z)}\lambda_A(u),\lambda_A(y)\right\}$$

Definition3.6. For a hyper KU-algebra H , a cubic $A=\langle\tilde{\mu}_A(x),\lambda_A(x)\rangle$" in H is called:

(I) Cubic hyper KU-ideal of H, if $K_1:\ x<<y\ \textit{implies}\ \tilde{\mu}(x)\geq\tilde{\mu}(y)$,

$$\lambda_A(x)\leq\lambda_A(y)\,,\tilde{\mu}_A(x\circ z)\geq r\min\left\{\inf_{u\in(x\circ(y\circ z)}\tilde{\mu}_A(u)\ ,\ \tilde{\mu}_A(y)\right\}$$

and

$$\lambda_A(x\circ z)\leq\max\left\{\sup_{a\in x\circ(y\circ z)}\mu(a),\mu(y)\right\}$$

(II) Cubic weak hyper KU-ideal of H" if, for any x; y; z $\in H$

$$\tilde{\mu}_A(0)\geq\tilde{\mu}_A(x\circ z)\geq r\min\left\{\inf_{u\in(x\circ(y\circ z)}\tilde{\mu}_A(u),\tilde{\mu}_A(y)\right\}$$

$$\lambda_A(0) \le \lambda_A(x \circ z) \le \max\left\{ \sup_{a \in x \circ (y \circ z)} \lambda_A(a), \lambda_A(y) \right\}$$

(III) Cubic strong hyper KU-ideal of H" if, for any x; y; z $\in H$

$$\inf_{u \in x \circ (y \circ z)} \tilde{\mu}_A(u) \ge \tilde{\mu}_A(z) \ge r\min\left\{ \inf_{u \in x \circ (y \circ z)} \tilde{\mu}_A(u), \tilde{\mu}_A(y) \right\},$$

$$\sup_{u \in x \circ (y \circ z)} \lambda_A(u) \le \lambda_A(z) \le \max\left\{ \sup_{u \in x \circ (y \circ z)} \lambda_A(u), \lambda_A(y) \right\}$$

Example 3.7. Let $H = \{0, a, b\}$ be a set with a binary operation $\circ$ as Example 3.3. Then $(H, \circ, 0)$ is a hyper KU-algebra. . Define $\tilde{\mu}_A(x)$, as follows:

$$\tilde{\mu}_A(x) = \begin{cases} [0.2, 0.9] & if\ x = \{0,1\} \\ [0.1, 0.4] & otherwise \end{cases}$$

H	0	1	2	3
$\lambda_A(x)$	0.2	0.2	0.6	0.7

It is easy to check that $A = \langle \tilde{\mu}_A(x), \lambda_A(x) \rangle$ is cubic hyper KU-ideal of H.

Example 3.8. Let $H = \{0, a, b\}$ be a set. Define hyper operation $\circ$ on H as follows:

$\circ$	0	a	b
0	$\{0\}$	$\{a\}$	$\{b\}$
a	$\{0\}$	$\{0\}$	$\{b\}$
b	$\{0\}$	$\{a\}$	$\{0, b\}$

Then (H, $\circ$) is a Hyper KU-algebra. Define a cubic set $A = \langle \tilde{\mu}_A(x), \lambda_A(x) \rangle$ in H by

$$\tilde{\mu}_A(0) = [0.4, 0.9],\ \tilde{\mu}_A(a) = [0.5, 0.7], \tilde{\mu}_A(b) = [0.2, 0.3]$$

and

H	0	1	2	3
$\lambda_A(x)$	0.2	0.3	0.5	0.7

It is easy to check that $A = \langle \tilde{\mu}_A(x), \lambda_A(x) \rangle$ is a cubic strong hyper KU-ideal of H

Definition 3.9. A cubic set $A=\langle\tilde{\mu}_A(x),\lambda_A(x)\rangle$ in H is called a cubic s-weak hyper KU-ideal of H if

(i) $\tilde{\mu}_A(0)\geq\tilde{\mu}_A(x)$, $\lambda_A(0)\leq\lambda_a(x)$ $\forall x\in H$,

(ii) for every $x,y,z\in H$ there exists $a\in x\circ(y\circ z)$ such that

$$\tilde{\mu}_A(x\circ z)\geq r\min\left\{\inf_{a\in x\circ(y\circ z)}\tilde{\mu}_A(a),\tilde{\mu}_A(y)\right\}$$

(iii) $\lambda_A(x\circ z)\leq\max\left\{\sup_{a\in x\circ(y\circ z)}\lambda_A(a),\lambda_A(y)\right\}$.

Theorem 3.10. Any cubic (weak, strong) hyper KU-ideal is a cubic (weak, strong) hyper ideal.

Proof. Let $A=\langle\tilde{\mu}_A(x),\lambda_A(x)\rangle$ be cubic a hyper KU-ideal of *H*, we get for any x; y; z $\in H$,

$$\tilde{\mu}_A(x\circ z)\geq r\min\left\{\inf_{u\in x\circ(y\circ z)}\tilde{\mu}_A(u),\tilde{\mu}_A(y)\right\} \text{ Put } x=0$$

we get

$$\tilde{\mu}_A(0\circ z)\geq r\min\left\{\inf_{u\in 0\circ(y\circ z)}\tilde{\mu}_A(u)\ ,\ \tilde{\mu}_A(y)\right\}$$

which gives,

$$\tilde{\mu}_A(z)\geq r\min\left\{\inf_{u\in(y\circ z)}\tilde{\mu}_A(u)\ ,\ \tilde{\mu}_A(y)\right\}$$

And

$$\lambda_A(x\circ z)\leq\max\left\{\sup_{u\in x\circ(y\circ z)}\lambda_A(u),\lambda_A(y)\right\}$$

Take x = 0 , we get

$$\lambda_A(0\circ z)\leq\max\left\{\sup_{u\in 0\circ(y\circ z)}\lambda_A(u)\ ,\lambda_A(y)\right\}$$

which gives,

$$\lambda_A(z)\leq\max\left\{\sup_{u\in(y\circ z)}\lambda_A(u)\ ,\ \lambda_A(y)\right\}$$

Ending the proof.

Theorem 3.11. Every cubic s-weak hyper KU-ideal of H is a cubic weak hyper KU-ideal.

Proof. Let $A=\langle\tilde{\mu}_A(x),\lambda_A(x)\rangle$ be a cubic s- weak hyper KU-ideal of *H*, then there exists $a,b\in x\circ(y\circ z)$ such that

$$\tilde{\mu}_A(x \circ z) \geq r\min\left\{ \inf_{a \in x \circ (y \circ z)} \tilde{\mu}_A(a), \tilde{\mu}_A(y) \right\}$$

$$\lambda_A(x \circ z) \leq \max\left\{ \sup_{b \in x \circ (y \circ z)} \lambda_A(b), \lambda_A(y) \right\}$$

Since $\tilde{\mu}_A(a) \geq \inf_{c \in x \circ (y \circ z)} \tilde{\mu}_A(c)$ and $\lambda_A(b) \leq \sup_{d \in x \circ (y \circ z)} \lambda_A(d)$, it follows that

$$\tilde{\mu}_A(x \circ z) \geq r\min\left\{ \inf_{c \in x \circ (y \circ z)} \tilde{\mu}_A(c), \tilde{\mu}_A(y) \right\}$$

$$\lambda_A(x \circ z) \leq \max\left\{ \sup_{d \in x \circ (y \circ z)} \lambda_A(d), \lambda_A(y) \right\}$$

Proposition 3.12. Let $A = \langle \tilde{\mu}_A(x), \lambda_A(x) \rangle$ be a cubic weak hyper KU-ideal of *H*. If $A = \langle \tilde{\mu}_A(x), \lambda_A(x) \rangle$ satisfies the inf-sup property, then $A = \langle \tilde{\mu}_A(x), \lambda_A(x) \rangle$ is a cubic s-weak hyper KU -ideal of *H*.

Proof. Since $A = \langle \tilde{\mu}_A(x), \lambda_A(x) \rangle$ satisfies the inf property, there exists $a_0 \in x \circ (y \circ z)$,such that $\tilde{\mu}_A(a_0) = \inf_{a_0 \in x \circ (y \circ z)} \tilde{\mu}_A(a_0)$. It follows that

$$\tilde{\mu}_A(x \circ z) \geq r\min\left\{ \inf_{a \in x \circ (y \circ z)} \tilde{\mu}_A(a) , \tilde{\mu}_A(y) \right\}$$

And since $A = \langle \tilde{\mu}_A(x), \lambda_A(x) \rangle$ satisfies the sup property, there exists $b_0 \in x \circ (y \circ z)$,such that $\lambda_A(b_0) = \sup_{b_0 \in x \circ (y \circ z)} \lambda_A(a_0)$ It follows that

$$\lambda_A(x \circ z) \leq \max\left\{ \sup_{b \in x \circ (y \circ z)} \lambda_A(b) , \lambda_A(y) \right\}$$

Proposition 3.13. Let $A = \langle \tilde{\mu}_A(x), \lambda_A(x) \rangle$ be a cubic strong hyper KU-ideal of *H* and let x; y; z $\in$ *H*. Then

(i) $\tilde{\mu}_A(0) \geq \tilde{\mu}_A(x)$, $\lambda_A(0) \leq \lambda_A(x)$
(ii) x << y implies $\tilde{\mu}_A(x) \geq \tilde{\mu}_A(y)$.
(iii) $\tilde{\mu}_A(x \circ z) \geq r\min\{\tilde{\mu}_A(a), \tilde{\mu}_A(y)\}, \forall a \in x \circ (y \circ z)$
(v) x << y implies $\lambda_A(x) \leq \lambda_A(y)$
(iv) $\lambda_A(x \circ z) \leq \max\left\{ \sup_{b \in x \circ (y \circ z)} \lambda_A(b) , \lambda_A(y) \right\}$

Proof. (i) Since $0 \in x \circ x \ \forall x \in H$, we have

$$\mu(0) \geq \inf_{a \in x \circ x} \mu(a) \geq \mu(x), \ \lambda(0) \leq \sup_{b \in x \circ x} \lambda(b) \leq \lambda(x)$$

which proves (i).

(ii) Let x; y ∈ H be such that x << y. Then $0 \in y \circ x \ \forall x, y \in H$ and so $\inf_{b \in (y \circ x)} \tilde{\mu}_A(b) \le \tilde{\mu}_A(0)$, it follows from (i) that,

$$\tilde{\mu}_A(x) \ge r\min\left\{ \inf_{a \in (y \circ x)} \tilde{\mu}_A(a), \tilde{\mu}_A(y) \right\} \ge r\min\{\tilde{\mu}_A(0), \tilde{\mu}_A(y)\} = \tilde{\mu}_A(y)$$

$$\lambda_A(x) \le \max\left\{ \sup_{b \in (y \circ x)} \lambda_A(b), \lambda_A(y) \right\} \le \max\{\lambda_A(0), \lambda_A(y)\} = \lambda_A(y)$$

(iii) $\tilde{\mu}_A(x \circ z) \ge r\min\left\{ \inf_{a \in x \circ (y \circ z)} \tilde{\mu}_A(a), \tilde{\mu}_A(y) \right\} \ge r\min\{\tilde{\mu}_A(a), \tilde{\mu}_A(y)\}, \forall a \in x \circ (y \circ z)$,

$$\lambda(x \circ z) \le \max\left\{ \sup_{b \in x \circ (y \circ z)} \lambda(b), \lambda(y) \right\} \le \max\{\mu(b), \mu(y)\}, \forall b \in x \circ (y \circ z)$$

we conclude that (iii), (v), (iv) are true. Ending the proof.

Proposition 3.14. Every cubic strong hyper KU-ideal is both a cubic s-weak hyper KU-ideal and a cubic hyper KU-ideal.

Proof. Straight forward.

Proposition 3.15. Let $A = \langle \tilde{\mu}_A(x), \lambda_A(x) \rangle$ be a cubic hyper KU -ideal of H and let $x, y, z \in H$. Then,

(i) $\tilde{\mu}_A(0) \ge \tilde{\mu}_A(x)$, $\lambda_A(0) \le \lambda_A(x)$
(ii) if $A = \langle \tilde{\mu}_A(x), \lambda_A(x) \rangle$ satisfies the inf -sup property, then

$$\tilde{\mu}_A(x \circ z) \ge r\min\{\tilde{\mu}_A(a), \tilde{\mu}_A(y)\}, \forall a \in x \circ (y \circ z), \lambda_A(x \circ z) \le \max\left\{ \sup_{b \in x \circ (y \circ z)} \lambda_A(b), \lambda_A(y) \right\}$$

Proof. (i) Since 0 << x for each $x \in H$; we have $\tilde{\mu}_A(0) \ge \tilde{\mu}_A(x)$, $\lambda_A(0) \le \lambda_A(x)$ by Definition 3.6(I) and hence (i) holds.

(ii) Since $A = \langle \tilde{\mu}_A(x), \lambda_A(x) \rangle$ satisfies the inf property, there is $a_0 \in x \circ (y \circ z)$, such that $\tilde{\mu}(a_0) = \inf_{a \in x \circ (y \circ z)} \tilde{\mu}_A(a)$. Hence

$$\tilde{\mu}_A(x \circ z) \ge r\min\left\{ \inf_{a \in x \circ (y \circ z)} \tilde{\mu}_A(a), \tilde{\mu}_A(y) \right\} = r\min\{ \tilde{\mu}_A(a_0), \tilde{\mu}_A(y) \}$$

Since $A = \langle \tilde{\mu}_A(x), \lambda_A(x) \rangle$ satisfies the sup- property, there is $b_0 \in x \circ (y \circ z)$, such that $\lambda(b_0) = \sup_{b \in x \circ (y \circ z)} \lambda_A(b)$, Hence

$$\lambda_A(x \circ z) \leq \max\left\{ \sup_{b \in x \circ (y \circ z)} \lambda_A(b) \; , \; \lambda_A(y) \right\} = \max\{ \; \lambda_A(b_0) \; , \; \lambda_A(y) \; \}$$

which implies that (ii) is true. The proof is complete.

Proposition 3.16. Let $A = \langle \tilde{\mu}_A(x), \lambda_A(x) \rangle$ be a cubic strong hyper KU -ideal of H, then $\tilde{\mu}_A(x \circ z) \geq r\min\left\{ \inf_{a \in x \circ (y \circ z)} \tilde{\mu}_A(a) \; , \; \tilde{\mu}_A(y) \right\}$ and $\lambda_A(x \circ z) \leq \max\left\{ \sup_{b \in x \circ (y \circ z)} \lambda_A(b) \; , \; \lambda_A(y) \right\}$. $\forall x, y, z \in H$.

Proof. For any $x, y, z \in H$, we have

$$\sup_{a \in x \circ (y \circ z)} \tilde{\mu}_A(a) \geq \inf_{a \in x \circ (y \circ z)} \tilde{\mu}_A(a) \text{ and } \inf_{b \in x \circ (y \circ z)} \lambda_A(b) \leq \sup_{b \in x \circ (y \circ z)} \lambda_A(b)$$

It follows from the definition, we get

$$\tilde{\mu}_A(x \circ z) \geq r\min\left\{ \sup_{a \in x \circ (y \circ z)} \tilde{\mu}_A(a) \; , \; \tilde{\mu}_A(y) \right\} \geq r\min\left\{ \inf_{a \in x \circ (y \circ z)} \tilde{\mu}_A(a) \; , \; \tilde{\mu}_A(y) \right\}$$

and

$$\lambda_A(x \circ z) \leq \max\left\{ \inf_{b \in x \circ (y \circ z)} \lambda_A(b) \; , \; \lambda_A(y) \right\} \leq \max\left\{ \sup_{b \in x \circ (y \circ z)} \lambda_A(b) \; , \; \lambda_A(y) \right\}.$$

Corollary 3.17. (i) Every cubic hyper KU-ideal of *H* is a cubic weak hyper KU-ideal of *H*.
(ii) If $A = \langle \tilde{\mu}_A(x), \lambda_A(x) \rangle$ is a cubic hyper KU -ideal of H satisfying inf-sup property, then $A = \langle \tilde{\mu}_A(x), \lambda_A(x) \rangle$ is a cubic s-weak Hyper KU -ideal of *H*.

Proof. Straightforward.

Theorem 3.18 . If $A = \langle \tilde{\mu}_A(x), \lambda_A(x) \rangle$ is a cubic strong hyper KU-ideal of H , then the set

$$\mu_{t,s} = \{ x \in H, \tilde{\mu}_A(x) \geq \tilde{t}, \lambda_A(x) \leq s \}$$

is a strong hyper KU-ideal of H , when $\mu_{t,s} \neq \Phi, \; for \; \tilde{t} \in D[0,1], s \in [0,1]$.

Proof. Let $A = \langle \tilde{\mu}_A(x), \lambda_A(x) \rangle$ be a cubic strong hyper KU-ideal of H and $\mu_{t,s} \neq \Phi, \; for \; \tilde{t} \in D[0,1], s \in [0,1]$ .Then there $a \in \mu_{t,s}$ and so $\tilde{\mu}_A(a) \geq \tilde{t}, \lambda_A(a) \leq s$.

By Proposition 3.13 (i), $\tilde{\mu}_A(0) \geq \tilde{\mu}_A(a) \geq \tilde{t}, \lambda(0) \geq \lambda(a) \leq s$ and so $0 \in \mu_{t,s}$. Let $x, y, z \in H$ such that $x \circ (y \circ z) \cap \mu_{t,s} \neq \Phi \; and \; y \in \mu_{t,s}$, Then there exist

$a_0 \in x \circ (y \circ z) \cap \mu_{t,s}$ *and hence* $\tilde{\mu}_A(a_0) \geq \tilde{t}, \lambda_A(a_0) \leq s$ By Definition 3.6(B) (III), we have

$$\tilde{\mu}_A(x \circ z) \geq r\min\left\{\inf_{a \in x \circ (y \circ z)} \tilde{\mu}_A(a), \tilde{\mu}_A(y)\right\} \geq r\min\{\tilde{\mu}_A(a), \tilde{\mu}_A(y)\} \geq r\min\{\tilde{t}, \tilde{t}\} = \tilde{t}$$

and

$$\lambda_A(x \circ z) \leq \max\left\{\sup_{a \in x \circ (y \circ z)} \lambda_A(a), \lambda_A(y)\right\} = \max\{\lambda_A(a_0), \lambda_A(y)\} = \max\{s, s)\} = s$$

So $(x \circ z) \in \mu_{t,s}$. It follows that $\mu_{t,s}$ is a strong hyper KU-ideal of H.

Acknowledgment

The author is greatly appreciate the referees for their valuable comments and suggestions for improving the paper.

Conflicts of Interest

State any potential conflicts of interest here or "The author declare no conflict of interest".

References

[1] K. Atanassov, G. Gargov, Interval valued intuitionistic fuzzy sets, Fuzzy Sets and Systems31 (1989) 343{349.[3] Y. Huang, *BCI-algebra,* Science Press, Beijing, 2006.

[2] R. A. Borzooei and M. Bakhshi, Some results on hyper BCK-algebras, Quasigroups and Related Systems 11 (2004), 9-24.

[3] Y. B. Jun, X. L. Xin, E. H. Roh, M. M. Zahedi, Strong hyper BCK-ideals of hyper BCK-algebra, Math. Jap. 51(3) (2000), 493-498.

[4] Y. B. Jun, M. M. Zahedi, X. L. Xin, R. A. Borzooei, On hyper BCK-algebra, Italian J. Pure App. Math., Oxford Ser. 10 (2000), 127-136.

[5] Y. B. Jun, C. S. Kim, K.O. Yang, "Cubic sets," *Annals of Fuzzy Mathematics and Informatics*, vol. 4, no. 1, pp. 83–98, 2012

[6] F. Marty, Sur une generalization de la notion de groupe*,* 8th Congress Math. Scandinaves, Stockholm (1934), 45-49.

[7] S. M. Mostafa, M.A. Abd-Elnaby, M.M.M. Yousef, Fuzzy ideals of KU-Algebras, Int. Math. Forum, 63(6) (2011) 3139-3149.

[8] S. M. Mostafa , F.Kareem, B. Davvaz , Hyper structure theory applied to KU-algebras. Accepted for publication in the Journal of Hyper structures .

[9] S. M. Mostafa, R. Ghanem, Bipolar fuzzy hyper ku–ideals (sub algebras), *Journal of New Theory* Year: 2017, Number: 15, Pages: 81-98.

[10] C. Prabpayak and U.Leerawat, On ideals and congruence in KU-algebras, scientia Magna Journal, 5(1) (2009), 54-57.

[11] C. Prabpayak, U. Leerawat, On isomorphisms of KU-algebras, scientiamagna journal, 5(3) (2009) 25-31.

http://www.newtheory.org ISSN: 2149-1402

Journal of New Theory

Received: 04.10.2017
Published: 21.10.2017

Year: **2017**, *Number*: **16**, *Pages*: **92-101**
Original Article

COMPACTNESS IN INTUITIONISTIC FUZZY MULTISET TOPOLOGY

Shinoj Thekke Kunnambath[1,*] <shinojthrissur@gmail.com>
Sunil Jacob John[1] <sunil@nitc.ac.in>

[1]*Department of Mathematics, NIT Calicut, INDIA*

Abstract – In this paper, we discuss the compactness properties of Intuitionistic Fuzzy Multiset Topological spaces. Various properties of Compact and Homeomorphic Intuitionistic Fuzzy Multiset Topological spaces are discussed.

Keywords -- *Intuitionistic Fuzzy Multiset, Intuitionistic Fuzzy Multiset Topology, Compact spaces, Homeomorphism.*

1. INTRODUCTION

The theory of setsconsidered to have begun with Cantor (1845-1918). For considering the uncertainty factor, Zadeh [1] introduced Fuzzy sets in 1965, in which a membership function assigns to each element of the universe of discourse, a number from the unit interval [0,1] to indicate the degree of belongingness to the set under consideration.

If repeated occurrences of any object are allowed in a set, then the mathematical structure is called as multiset [11,12]. As a generalization of this concept, Yager [2] introduced fuzzy multisets. An element of a Fuzzy Multiset can occur more than once with possibly the sameor different membership values.

In 1983, Atanassov [3,10] introduced the concept of Intuitionistic Fuzzy sets. An Intuitionistic Fuzzy set is characterized by two functions expressing the degree of membership and the degree of nonmembership of elements of the universe to the Intuitionistic Fuzzy set. Among the various notions of higher-order Fuzzy sets, Intuitionistic Fuzzy sets proposed by Atanassov provide a flexible framework to explain uncertainty and vagueness.

[*]*Corresponding Author.*

The concept of Intuitionistic Fuzzy Multiset is introduced in [4] by combining the all the above concepts. Intuitionistic Fuzzy Multiset has applications in medical diagnosis and robotics [13,14]. In [5] Shinoj et al. introduced algebraic structures on Intuitionistic Fuzzy Multiset.

In 1968, Chang [9] introduced Fuzzy topological spaces. And as a continuation of this, in 1997, Coker [6] introduced the concept of Intuitionistic fuzzy topological spaces. In [15], Shinoj and John generalized this concept into Intuitionistic Fuzzy Multiset by introducing Intuitionistic Fuzzy Multiset Topology. In the present work we introduced the concept of Compactness, which is considered as a "global" property in general topology. The advantage of this concept is that, one can study the whole space by studying a finite number of open subsets. Also we introduced the concept of Homeomorphism which will help to compare two spaces and corresponding properties.

2. Preliminaries

Definition 2.1. [1] Let *X* be a nonempty set. A *Fuzzy setA* drawn from *X* is defined as

$$A = \{<x : \mu_A(x) > : x \in X\}.$$

Where : $X \rightarrow [0,1]$ is the membership function of the Fuzzy Set *A*.

Definition 2.2. [2] Let X be a nonempty set. A *Fuzzy Multiset* (FMS) A drawn from X is characterized by a function, 'count membership' of A denoted by CM_A such that $CM_A : X \rightarrow Q$ where Q is the set of all crisp multisets drawn from the unit interval [0,1]. Then for any $x \in X$, the value CM_A (x) is a crisp multiset drawn from [0,1]. For each $x \in X$, the membership sequence is defined as the decreasingly ordered sequence of elements in $CM_A(x)$. It is denoted by $(\mu^1_A(x), \mu^2_A(x),...,\mu^P_A(x))$ where $\mu^1_A(x) \geq \mu^2_A(x) \geq,..., \geq \mu^P_A(x)$.

A complete account of the applications of Fuzzy Multisets in various fields can be seen in [9].

Definition 2.3. [3] Let X be a nonempty set. An *Intuitionistic Fuzzy Set* (IFS) A is an object having the form $A = \{< x: \mu_A(x), \nu_A(x) >: x \in X\}$, where the functions $\mu_A: X \rightarrow [0,1]$ and $\nu_A: X \rightarrow [0,1]$ define respectively the degree of membership and the degree of non membership of the element $x \in X$ to the set A with $0 \leq \mu_A(x) + \nu_A(x) \leq 1$ for each $x \in X$.

Remark 2.4. Every Fuzzy set *A* on a nonempty set *X* is obviously an IFS having the form

$$A = \{<x : \mu_A(x), 1 - \mu_A(x) > : x \in X\}$$

Using the definition of FMS and IFS, a new generalized concept can be defined as follows:

Definition 2.5. [4] Let *X* be a nonempty set. An *Intuitionistic Fuzzy Multiset* A denoted by IFMS drawn from *X* is characterized by two functions : 'count membership' of *A* (CM_A) and 'count non membership' of A (CN_A) given respectively by $CM_A : X \rightarrow Q$ and $CN_A : X \rightarrow Q$

where Q is the set of all crisp multisets drawn from the unit interval [0, 1] such that for each x ϵ X, the membership sequence is defined as a decreasingly ordered sequence of elements in CMA(x) which is denoted by $(\mu^1{}_A(x), \mu^2{}_A(x),...,\mu^P{}_A(x))$ where $(\mu^1{}_A(x) \geq \mu^2{}_A(x) \geq,... \geq,\mu^P{}_A(x)$ and the corresponding non membership sequence will be denoted by $(v^1{}_A(x), v^2{}_A(x),...,v^P{}_A(x))$ such that $0 \leq \mu^i{}_A(x) + v^i{}_A(x) \leq 1$ for every x ϵ X and i = 1,2,...,p.

An IFMS A is denoted by

$$A = \{<x : (\mu^1{}_A(x), \mu^2{}_A(x),...,\mu^P{}_A(x)), (v^1{}_A(x), v^2{}_A(x), ... ,v^P{}_A(x)) >: x \in X\}$$

Remark 2.6. We arrange the membership sequence in decreasing order but the corresponding non membership sequence may not be in decreasing or increasing order.

Definition 2.7. [15] Let *X* and *Y* be two nonempty sets and f : X →Y be a mapping. Then

a) The image of an IFMS A in X under the mapping f is denoted by f(A) is defined as

$$CM_{f[A]}(y) = \begin{cases} \vee_{f(x)=y} CM_A(x) \ ; & f^{-1}(y) \neq \emptyset \\ 0 & \text{otherwise} \end{cases}$$
$$CN_{f[A]}(y) = \begin{cases} \wedge_{f(x)=y} CN_A(x) \ ; & f^{-1}(y) \neq \emptyset \\ 1 & \text{otherwise} \end{cases}$$

b) The inverse image of the IFMS B in Y under the mapping f is denoted by $f^{-1}(B)$ where

$$CM_{f^{-1}[B]}(x) = CM_B f[x], CN_{f^{-1}[B]}(x) = CN_B f[x]$$

2.1. Intuitionistic FuzzyMultiset Topological spaces

In this section we introduced the concept of Intuitionistic Fuzzy multiset Topology (IFMT). Here we extend the concept of Intuitionistic fuzzy topological spaces introduced by Dogan Coker in [6] to the case of Intuitionistic fuzzy multisets.

For this first we introduced $\neg 0$ and $\neg 1$ in a nonempty set X as follows.

Definition 2.8. [15] Let

$$\neg 0 = \{< x: (0,0,........,0), (1,1,.........,1): x \in X \}$$
$$\neg 1 = \{< x: (1,1,.......1), (0,0,..........,0) : x \in X \}$$

Definition 2.9. [15] Anintuitionistic Fuzzy multiset topology (IFMT) on X is a family ꚍ of intuitionistic fuzzy multisets (IFMSs) such that

1. $\neg 0, \neg 1 \in$ ꚍ
2. $G_1 \cap G_2 \in$ ꚍ for any $G_1, G_2 \in$ ꚍ
3. $\cup G_i \in$ ꚍ for any arbitrary family $\{G_i : i \in I\}$ in ꚍ

Then the pair (X, ｒ) is called **IntuitionisticFuzzy multiset topological space** (IFMT for short) and any IFMS in ｒ is known as an open intuitionistic fuzzy multiset (OIFMS in short) in X.

Remark 2.10. [15] The complement of an OIFMS is called closed intuitionistic Fuzzy multiset (CIFMS in short)

2.2. Construction of IFMTs [15]

Here we construct Intuitionistic fuzzy multiset topology from a given IFMT.Consider a nonempty set X. LetA = $\{<x : (\mu^1{}_A(x), \mu^2{}_A(x),...,\mu^P{}_A(x)), (v^1{}_A(x), v^2{}_A(x),...,v^P{}_A(x)) > : x \in X\}$ be an IFMS. Define

$$[]A = \{< x : (\mu^1{}_A(x), \mu^2{}_A(x),...,\mu^P{}_A(x)),(1-\mu^1{}_A(x),1-\mu^2{}_A(x),...,1-\mu^P{}_A(x)) > : x \in X\}$$

Proposition 2.11. Let (X,ｒ) be an IFMT on X. Then $ｒ_{0,1}$= {[]A: A$\in$ ｒ} is an IFMS.

2.3. Closure and Interior

Definition 2.12. [15] Let (X, ｒ) be an IFMT and A be an IFMS in X. Then **closure** of A denoted by cl(A) is defined as cl(A) = ∩{M: M is closed in X and A ⊆ M}.

Definition 2.13. [15] Let (X, ｒ) be an IFMT and B be an IFMS in X. Then **interior** of B is denoted by

int(B) is defined as int(B) = U{N: N is open in X and N ⊆ B}.

Proposition 2.14. [15] Let (X, ｒ) be an IFMT and A be an IFMS in X. Then cl(A) is a CIFMS.

Proposition 2.15. [15] Let (X, ｒ) be an IFMT and A be an IFMS in X. Then int(A) is an OIFMS.

Proposition 2.16. [15] Let (X, ｒ) be an IFMT and A be an IFMS. Then cl(∇A)= ∇(int(A))

Proposition 2.17. [15] Let (X, ｒ) be an IFMT and A be an IFMS in X. Then A is a CIFMS if and only if cl(A) = A.

Proposition 2.18.[15] Let (X, ｒ) be an IFMT and A be an IFMS in X. Then A is an OIFMS if and only if int(A) = A.

2.4. Continuous Functions

Definition 2.19. [15] Let (X, ｒ) and (Y, ɸ) be two IFMTs. A function f : X →Y is said to be **Continuous** if and only if inverse image of each OIFMS in ɸ is an OIFMS in ｒ.

Theorem 2.20. [15] Let (X, ґ) and (Y, ϕ) be two IFMTs. Then the function $f : X \rightarrow Y$ is Continuous if and only if inverse image of each CIFMS in ϕ is a CIFMS in ґ.

Theorem 2.21. [15] Let (X, ґ) and (Y, ϕ) be two IFMTs. Then the function $f : X \rightarrow Y$ is Continuous if and only if for each IFMT A in X, $f[cl(A)] \subseteq cl[f(A)]$

Theorem 2.22. [15] Let (X, ґ) and (Y, ϕ) be two IFMTs. Then the function $f: X \rightarrow Y$ is Continuous if and only if for each IFMT B inY, $cl[f^{-1}(B)] \subseteq f^{-1}[cl(B)]$

Theorem 2.23. [15] Let (X, ґ) and (Y, ϕ) be two IFMTs. Then the function $f: X \rightarrow Y$ is Continuous if and only if for each IFMT A in X, $int[f(A)] \subseteq f[int(A)]$.

Theorem 2.24.[15] Let (X, ґ) and (Y, ϕ) be two IFMTs. Then the function $f: X \rightarrow Y$ is Continuous if and only if for each IFMT B in Y, $f^{-1}[int(B)] \subseteq int[f^{-1}(B)]$

2.5. Subspace Topology

Definition2.25. [15] Let (X, ґ) and (Y, ϕ) be two IFMTs. The topological space Y is called a **subspace**of the topological space X if $Y \subseteq X$ and if the open subsets of Y are precisely the subsets $O'$of the form

$$O' = O \cap Y$$

for some open subsets O of X. Here we may say that each open subset O' of Y is the *restriction* to Y of an open subset O of X. O' is also called **relative open** in Y.

3. Compactness on Intuitionistic Fuzzy Multisets

Definition 3.1. Let(X, ґ) be an IFMT. Let $\{G_i : i \in I\}$ be a family of OIFMSs in X such that $\cup\{G_i : i \in I\} = \neg 1$, then it is called an **open cover**of X. A finite subfamily of $\{G_i : i \in I\}$ is an open cover of X, then it is called a *finite subcover* of X.

Definition 3.2. A family $\{H_i : i \in I\}$ of CIFMSs in X satisfies the **finite intersection property**iff every finite subfamily $\{H_i : i=1,2,\ldots,n\}$ of the family satisfies the condition

$$\cap^{n}_{i=1} H_i \neq \neg 0.$$

Definition3.3. Let(X, ґ) be an IFMT. Then X is **compact**iff every open cover of X has a finite subcover.

Example 3.4. Let X = {1, 2} and define the IFMSs in X as follows. For $n \in N^+$, $p \in N$

$$G_n = \{<1: (n/n+1, n+1/n+2,\ldots,n+p/n+p+1), (1/n+2, 1/n+3,\ldots,1/n+p+2)>,$$
$$<2: (n+1/n+2, n+2/n+3,\ldots,n+p+1/n+p+2), (1/n+3, 1/n+4,\ldots,1/n+p+3)> \}$$

Let $\Gamma = \{\neg 0, \neg 1\} \cup \{G_n\}$. Then (X, Γ) forms an IFMT.

The above example is not compact, since $\{G_n: n \in N^+\}$ has no finite subcover.

Theorem 3.5. Let(X, Γ) be an IFMT. Then (X, Γ) is compactiff $(X,\Gamma_{0,1})$ is compact**.**

Proof. Let (X, Γ) is compact. Let $\{[]A_i;\ i\in I,\ A_i \in \Gamma\}$ be an open cover of X in $(X,\Gamma_{0,1})$.

$$\Rightarrow \cup([]A_i) = \neg 1 = \{< x: (1,1,\ldots\ldots.1), (0,0,\ldots\ldots\ldots,0) : x \in X \} \quad (1)$$

Where $[]A_i = \{< x : (\mu^1_{Ai}(x), \mu^2_{Ai}(x),\ldots,\mu^p_{Ai}(x)),(1-\mu^1_{Ai}(x),1-\mu^2_{Ai}(x),\ldots,1-\mu^p_{Ai}(x)) > : x \in X\}$,

$A_i = \{<x : (\mu^1_{Ai}(x), \mu^2_{Ai}(x),\ldots,\mu^p_{Ai}(x)),((v^1_{Ai} (x), v^2_{Ai} (x),\ldots,v^P_{Ai} (x)) >) > : x \in X\}$

Now (1) $\Rightarrow$

$$\begin{cases} \mu^1{}_{A_1}(x)\vee\mu^1{}_{A_2}(x)\vee\ldots\ldots = 1\text{and } 1-\mu^1{}_{A_1}(x) \wedge 1-\mu^1{}_{A_2}(x) \ \wedge\ldots\ldots = 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \mu^p{}_{A_1}(x) \ \vee\mu^p{}_{A_2}(x) \vee\ldots\ldots = 1 \text{ and } 1-\mu^p{}_{A_1}(x) \wedge 1-\mu^p{}_{A_2}(x)) \wedge\ldots\ldots = 0 \end{cases} \quad (2)$$

Now for $j = 1,\ldots..,p$

$$\begin{aligned} v^1_{A1}(x) \wedge v^2_{A2}(x) \wedge \ldots\ldots.. &\leq (1-\mu^j_{A1}(x)) \wedge (1-\mu^j_{A2}(x)) \wedge\ldots\ldots.. \\ &= 1- (\mu^j_{A1}(x)\vee\mu^j_{A2}(x)\vee\ldots\ldots) \\ &= 1-1 = 0 \end{aligned} \quad (3)$$

(1) And (3) $\Rightarrow \cup A_i = \neg 1$

$\Rightarrow \{A_i; i\in I,\ A_i\in \Gamma\}$ is an open cover of X in (X,Γ).

Since (X,Γ) is compact, there exist a finite subcover $\{A_i;\ A_i\in \Gamma, i = 1,2,\ldots.,n\}$ such that

$$\cup^n_{i=1} A_i = \neg 1 \quad (4)$$

From (4) for $j = 1,\ldots..,p$

$$\mu^j_{A1}(x)\vee\ldots..\vee\mu^j_{An}(x) = 1$$

and

$$\begin{aligned} 1-\mu^j_{A1}(x)\wedge\ldots..\wedge 1-\mu^j_{An}(x) &= 1 - (\mu^j_{A1}(x)\vee\ldots..\vee\mu^j_{An}(x)) \\ &= 1-1 = 0 \end{aligned}$$

$\Rightarrow \{[]A_i;\ A_i\in \Gamma, I = 1,\ldots,n\} \in \Gamma_{0,1}$ is a finite subcover of $(X,\Gamma_{0,1})$.
$\Rightarrow (X,\Gamma_{0,1})$ is compact.

Similarly we can prove the converse part.

The well known theorems in the modern Topology are also holds good for IFMTs. Some of them are given below.

Theorem 3.6. Any closed subspace of a compact IFMT is compact.

Proof. Let (X, г) be an IFMT on X. Also assume that (X, г) is compact. Let (Y, ф) be a closed subspace of X. Let $\{A_i : i \in I\}$ be an open cover of Y, where

$$A_i = \{<x : (\mu^1_{Ai}(x), \mu^2_{Ai}(x),...,\mu^p_{Ai}(x)),((v^1_{Ai}(x),v^2_{Ai}(x),...,v^P_{Ai}(x)) >) > : x \in X\}$$

ie,

$$UA_i = \neg 1 \quad (1)$$

By Definition 4.16, ∃ open sets B_i in X ∍

$$A_i = B_i \cap Y \quad (2)$$

Since Y is closed, ∇Y U $\{B_i\}$ forms an open cover of X.

Since X is compact, this open cover has a finite subcover. Discard ∇Y if it occurs in this subcover. Let $\{B_1,B_2,\ldots,B_n\}$ be the finite subcover of X. Then from (2), the corresponding $\{A_1,A_2,\ldots,A_n\}$ forms a finite subcover of Y. ie.

$$U^n_{i=1}A_i = \neg 1 \quad (3)$$

Then by definition 4.20, Y is compact. Hence the proof.

Theorem 3.7. Continuous image of a compact IFMT is compact.

Proof. Let (X, г) be an IFMT on X and assume that (X, г) is compact. Let f : X →Y be continuous. To prove f(X) is a compact subspace of X.

Let $\{A_i : i \in I\}$ be an open cover of f(X), where

$$A_i = \{<x : (\mu^1_{Ai}(x), \mu^2_{Ai}(x),...,\mu^p_{Ai}(x)),((v^1_{Ai} (x), v^2_{Ai} (x),...,v^P_{Ai} (x)) >) > : x \in X\}$$

ie,

$$UA_i = \neg 1 \quad (1)$$

Since f is continuous, $\{f^{-1}(A_i)\}_{i\in I}$ is an open cover of X. Since X is compact ∃ a finite subcover $\{f^{-1}(A_i): I = 1,2,\ldots,n\}$ which covers X.

$\Rightarrow U^{n}_{i=1}A_i = \neg 1$ (2)

$\Rightarrow$ f(X) is compact. Hence the proof.

Theorem 3.8. An IFMT is compact if and only if every class of CIFMSs with empty intersection has a finite subclass with empty intersection.

Proof: Let (X, г) be a compact IFMT.

Let $\{C_i : i \in I\}$ be a family of closed sets ∍

$\cap C_i = \neg 0$ (1)

where $C_i = \{< x : (\mu^1_{Ci}(x), \mu^2_{Ci}(x),...,\mu^p_{Ci}(x)),((v^1_{Ci}(x), v^2_{Ci}(x),...,v^P_{Ci}(x)) >) > : x \in X\}$

(1) $\Rightarrow U(\nabla C_i) = \neg 1$

$\Rightarrow \{\nabla C_i : i \in I\}$ be an open cover of X.

Since X is compact, $\exists\{\nabla C_1,\nabla C_2,\ldots,\nabla C_n\}$ ∍ $U^{n}_{i=1}(\nabla C_i) = \neg 1$

$\Rightarrow \cap^{n}_{i=1}C_i = \neg 0$

Conversely assume that every class of CIFMSs with empty intersection has a finite subclass with empty intersection.

To prove X is compact. Let $\{A_i : i \in I\}$ be an open cover of X.

$\Rightarrow UA_i = \neg 1$

$\Rightarrow \cap(\nabla A_i) = \neg 0$

Hence by assumption $\exists\ \{\nabla A_1,\nabla A_2,\ldots,\nabla A_n\}$ ∍ $\cap^{n}_{i=1}(\nabla A_i) = \neg 0$

$\Rightarrow U^{n}_{i=1}A_i = \neg 1$

Hence the proof.

3.1. Homeomorphism on Intuitionistic Fuzzy Multisets

Definition 3.9. A homeomorphism is a one-to-one continuous mapping of one topological space onto another. The IFMTs (X, г) and (Y, ф) are said to be **homeomorphic** if there exist functions $f : X \rightarrow Y$ and $g : Y \rightarrow X$ such that f and g are continuous. If X and Y are homeomorphic, then their points can be put into one-to-one correspondence in such a way that their open sets also correspond to one another. The two spaces differ only in the nature of their points, so it can be considered that they are identical.

Theorem 3.10. Let (X, г) and (Y, ф) are homeomorphic. Then X is compact if and only if Y is compact.

Proof. Let f : X →Y be a homeomorphism. Let (X, г) be a compact IFMT. To prove Y is compact. Let {A_i :i∊ I}be an open cover of Y. ie

$$UA_i = \neg 1 \text{ in } Y.$$

Then $\{f^{-1}(A_i)\}_{i\in I}$be an open cover of X. Since X is compact there exist $\{f^{-1}(A_i)$: i=1,2,…,n} ∍

$$U^n_{i=1}f^{-1}(A_i)=\neg 1 \text{ in } X.$$

Since f is onto

$$U^n_{i=1}A_i = \neg 1 \text{ in } Y.$$

Hence Y is compact. Similarly we can prove the converse.

4. Conclusion

In this work we extended the concept topological structures of Intuitionistic Fuzzy Multisets.We introduced the concept of Intuitionistic Fuzzy Multiset Topology in our previous work. In the current work weintroduced the concept of compactnesson Intuitionistic Fuzzy Multisets.The homeomorphism between two Intuitionistic Fuzzy Multisets are defined. Characterization of compactness is discussed in detail.

References

[1] L. A. Zadeh, *Fuzzy sets*, Inform. and Control 8 (1965) 338-353.

[2] R.R Yager, *On the theory of bags*, Int.J. of General Systems 13(1986) 23-37.

[3] K.T.Atanassov, *Intuitionistic Fuzzy sets*, Fuzzy sets and Systems 20 (1986) 87-96.

[4] T. K. Shinoj, S. J. John, *Intutionistic fuzzy multisets*, International Journal of Engineering Science and Innovative Technology, Volume 2, Issue 6, November 2013.

[5] T. K, Shinoj, A. Baby, S. J. John, *On Some Algebraic Structures of Fuzzy Multisets*, Annals of Fuzzy Mathematics and Informatics, Volume 9, No 1, January 2015, 77-90.

[6] D. Çoker, *An Introduction to intuitionistic fuzzy topological spaces*, Fuzzy sets and systems, 88 (1997), 81-89.

[7] G. F. Simmons, *Introduction to Topology and Modern Analysis*, McGraw-Hill International Student Edition.

[8] B. Mendelson, *Introduction to Topology*, 3rd Edition

[9] C. Chang, *Fuzzy Topological spaces*, J. Math. Anal. Appl. 24 (1968), 142-145.

[10] K. T. Atanassov, *Intuitionistic Fuzzy sets,*VII ITKRs Session, Sofia (deposed in Central Science-Technical Library of Bulgarian Academy of Science, 1697/84) (in Bulgarian) (1983)
[11] W. D. Blizard, *Dedekind multisets and function shells*, Theoretical ComputerScience, 110,(1993) 79-98.
[12] W. D. Blizard, *Multiset Theory,* Notre Dame Journal of Formal Logic, Vol.30,No.1, (1989)36-66.
[13] T. K. Shinoj, S. J. John, *Intuitionistic Fuzzy Multiset and its application in Medical Diagnosis*, International Journal of Mathematical and Computational Sciences, World Academy of Science, Engineering and Technology(WASET), Volume 6 (2012) 34-38
[14] T. K. Shinoj, S. J. John, *Accuracy in Collaborative Robotics: An Intuitionistic FuzzyMultiset Approach*, Global Journal of Science Frontier Research (GJSFR), Volume 13 (2013) Issue 10.
[15] T. K. Shinoj, S. J. John, *Topological Structures on Intuitionistic Fuzzy Multisets*, International Journal of Scientific & Engineering Research, volume 6, issue 1, March-2015.

http://www.newtheory.org ISSN: 2149-1402

Journal of New Theory

Year: **2017**, *Number*: **16**, *Pages*: **102**

EDITORIAL

We are happy to inform you that Number 16 of the Journal of New Theory (JNT) is completed with 10 articles.

JNT publishes original research articles, reports, reviews and commentaries that are based on a theory of mathematics. However, the topics are not limited to only mathematics, but also include statistics, computer science, physics, engineering, chemistry, biology, economics or social sciences that use a theory of mathematics.

We would like to express our deepest thanks to all of the members of the editorial board and reviewers of the papers in this issue who are H. Günal, A. Yıldırım, A. S. Sezer, B. Mehmetoğlu, B. H. Çadırcı, C. Kaya, Ç. Çekiç, D. Mohamad, E. Turgut, F. Smarandache, H. M. Doğan, H. Kızılaslan, H. Şimşek, İ. Zorlutuna, İ. Gökce, İ. Türkekul, J. Zhan, J. Ye, H. Kızılaslan, M. A. Noor, M. I. Ali, M. Ali, N. Sağlam, N. Yeşilayer, N. Kizilaslan, O. Muhtaroğlu, P. K. Maji, R. Yayar, S. Broumi, S. Karaman, S. Enginoğlu, S. Demiriz, S. Öztürk, S. Eğri, Ş. Sözen, Y. Budak, K. Aydemir.

JNT is a refereed, electronic, open access and international journal.

Papers in JNT are published free of charge.

Pleases, write any original idea. If it is true, it gives an opportunity to use. If it is incomplete, it gives an opportunity to complete. If it is incorrect, it gives an opportunity to correct.

You can reach us from journal homepage at http://www.newtheory.org. To receive further information and to send your recommendations and remarks, or to submit articles for consideration, please e-mail us at jnt@newtheory.org

We hope you will enjoy this issue of JNT. We are looking forward to hearing your feedback and receiving your contributions.

Happy reading!

21 October 2017

Prof. Dr. Naim Çağman
Editor-in-Chief
Journal of New Theory
http://www.newtheory.org