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DOUBLE CONNECTED SPACES

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Abstaract − In this paper, we introduce new types of double connected topological spaces. The first one depends on the separated double sets and the other one depends on the quasi-coincident separated double sets. The properties and the relation between them have investigated. Also, we defined and study the component of each type and the properties of these types of component have obtained.

Keywords − *Separated double sets, q-separated double sets, double connected, q-double connected, DC_1-connected, strongly double connected, strongly q-double connected, double components, q-double components and q-hyperconnected in double topological spaces.*

1 Introduction

Atanassov [1, 2, 3, 4] introduced the concept of intuitionistic fuzzy sets as a generalization of fuzzy sets. Coker [5] generalized topological structures in intuitionistic fuzzy case. The concept of intuitionistic sets and the topology on intuitionistic sets was first given by Coker [7, 6].

In 2005, the suggestion of J. G. Garcia et al. [8] that double set is a more appropriate name than flou (intuitionistic) set, and double topology for the flou (intuitionistic) topology. In 2007, Kandil et al. [10] proved the $1-1$ correspondence mapping f between the set of all double sets and the set of all intuitionistic sets defined as: $f(A_1, A_2) = (A_1, A_2^c), A_2^c$ is the complement of A_2. Kandil et al. [9, 10] introduced the concept of double sets, (D-set, for short), double topological spaces,

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(DTS, for short), continuous functions between these spaces and the concept of double point, (D-point, for short).

In this paper, we define (q-)separated double sets, (q)double connected, DC_1-connected, strongly double connected, (q)double components and quasi-hyperconnected in double topological spaces. Moreover, we give some related results to these notions.

2 Preliminary

In this section, we collect some definitions and theorems which will be needed in the sequel. For more details see [9, 10].

Definition 2.1. [10] Let X be a non-empty set.

1. A D-set $\underline{A}$ is an ordered pair $(A_1, A_2) \in P(X)\hat{\times}P(X)$ such that $A_1 \subseteq A_2$.
2. $D(X) = \{(A_1, A_2) \in P(X)\hat{\times}P(X), A_1 \subseteq A_2\}$ is the family of all D-sets on X.
3. Let $\eta_1, \eta_2 \subseteq P(X)$. The product of η_1 and η_2, denoted by $\eta_1 \times \eta_2$, defined by: $\eta_1\hat{\times}\eta_2 = \{(A_1, A_2) : A_1 \in \eta_1, A_1 \in \eta_2, A_1 \subseteq A_2\}$.
4. The D-set $\underline{X} = (X, X)$ is called the universal D-set.
5. The D-set $\underline{\emptyset} = (\emptyset, \emptyset)$ is called the empty D-set.

Definition 2.2. [10] Let $\underline{A} = (A_1, A_2), \underline{B} = (B_1, B_2)$ and $\underline{C} = (C_1, C_2) \in D(X)$.

1. $\underline{A} = \underline{B} \Leftrightarrow A_1 = B_1, A_2 = B_2$.
2. $\underline{A} \subseteq \underline{B} \Leftrightarrow A_1 \subseteq B_1, A_2 \subseteq B_2$.
3. $\underline{A} \cup \underline{B} = (A_1 \cup B_1, A_2 \cup B_2)$.
4. $\underline{A} \cap \underline{B} = (A_1 \cap B_1, A_2 \cap B_2)$.
5. $\underline{A}^c = (A_2^c, A_1^c)$, where $\underline{A}^c$ is the complement of $\underline{A}$.
6. $\underline{A} \setminus \underline{B} = (A_1 \setminus B_2, A_2 \setminus B_1)$.
7. $\underline{A} \cap (\underline{B} \cup \underline{C}) = (\underline{A} \cap \underline{B}) \cup (\underline{A} \cup \underline{C})$.
8. Let $x \in X$. Then, the D-sets $\underline{x}_1 = (\{x\}, \{x\})$ and $\underline{x}_{\frac{1}{2}} = (\emptyset, \{x\})$ are said to be D-points in X. The family of all D-points, denoted by $DP(X)$, i.e., $DP(X) = \{\underline{x}_t : x \in X, t \in \{\frac{1}{2}, 1\}\}$.
9. $\underline{x}_1 \underline{\in} \underline{A} \Leftrightarrow x \in A_1$ and $\underline{x}_{\frac{1}{2}} \underline{\in} \underline{A} \Leftrightarrow x \in A_2$.

Definition 2.3. [9] Two D-sets $\underline{A}$ and $\underline{B}$ are said to be a quasi-coincident, denoted by $\underline{A}q\underline{B}$, if $A_1 \cap B_2 \neq \emptyset$ or $A_2 \cap B_1 \neq \emptyset$. $\underline{A}$ is not quasi-coincident with $\underline{B}$, denoted by $\underline{A} \not{q} \underline{B}$, if $A_1 \cap B_2 = \emptyset$ and $A_2 \cap B_1 = \emptyset$.

Theorem 2.4. [9] Let $\underline{A}, \underline{B}, \underline{C} \in D(X)$ and $\underline{x}_t \in DP(X)$. Then,

1. $\underline{A} \not q \underline{B} \Leftrightarrow \underline{A} \subseteq \underline{B}^c$.
2. $\underline{A} \not q \underline{B}, \underline{C} \subseteq \underline{B} \Rightarrow \underline{A} \not q \underline{C}$.

Definition 2.5. [10] Let X be a non-empty set. The family η of D-sets in X is called a double topology on X if it satisfies the following axioms:

1. $\underline{\emptyset}, \underline{X} \in \eta$,
2. If $\underline{A}, \underline{B} \in \eta$, then $\underline{A} \sqcap \underline{B} \in \eta$,
3. If $\{\underline{A}_s : s \in S\} \subseteq \eta$, then $\sqcup_{s\in S}\underline{A}_s \in \eta$.

The pair (X, η) is called a DTS. Each element of η is called an open D-set in X. The complement of open D-set is called closed D-set.

Definition 2.6. [10] Let (X, η) be a DTS and $\underline{A} \in D(X)$. The double closure of $\underline{A}$, denoted by $cl_\eta(\underline{A})$ or $\overline{\underline{A}}$, defined by: $cl_\eta(\underline{A}) = \sqcap\{\underline{B} : \underline{B} \in \eta^c \; and \; \underline{A} \subseteq \underline{B}\}$.

Theorem 2.7. [10] Let (X, η) be a DTS and let $\underline{A}$, $\underline{B}$, $\underline{C} \in D(X)$. Then,

1. $cl_\eta(\underline{A})$ is the smallest closed D-set containing $\underline{A}$.
2. $cl_\eta(\underline{A} \sqcap \underline{B}) \subseteq cl_\eta(\underline{A}) \sqcap cl_\eta(\underline{B})$.
3. $\underline{A} \; q \; \underline{C} \Leftrightarrow cl_\eta(\underline{A}) \; q \; cl_\eta(\underline{C})$, $\underline{C} \in \eta$.

Definition 2.8. [10] Let X be a non-empty set. The family τ of D-sets in X is called a stratified double topology on X if it satisfies the following axioms:

1. $\underline{\emptyset} \in \tau, \underline{X} \in \tau \; and \; (\emptyset, X) \in \tau$,
2. If $\underline{A}, \underline{B} \in \tau$, then $\underline{A} \sqcap \underline{B} \in \tau$,
3. If $\{\underline{A}_s : s \in S\} \subseteq \tau$, then $\sqcup_{s\in S}\underline{A}_s \in \tau$.

The pair (X, τ) is called a stratified DTS.

Definition 2.9. [10] Let X be a non-empty set.

1. $I(X) = \{\underline{\emptyset}, \underline{X}\}$ is a DTS, which is called indiscrete DTS.
2. $i(X)\hat{\times}i(X) = \{\underline{\emptyset}, \underline{X}, (\emptyset, X)\}$ is a DTS, which is called indiscrete stratified DTS, $i(X)$ is the indiscrete topology on X.
3. $D(X) = P(X)\hat{\times}P(X)$ is a DTS, which is called discrete DTS.

Theorem 2.10. [10] Let η be a double topology on X. Then, the following collections are ordinary topologies on X :

1. $\pi_1 = \{A_1 : \underline{A} \in \eta\}$.

2. $\pi_2 = \{A_2 : \underline{A} \in \eta\}$.

Definition 2.11. [10] Let (X, η) be a DTS and Y be a non-empty subset of X. Then, $\eta_Y = \{\underline{A} \bigcap \underline{Y} : \underline{A} \in \eta \ and \ \underline{Y} = (Y, Y)\}$ is a double topology on Y. The DTS (Y, η_Y) is called a double topological subspace of (X, η) ($DT-$subspace, for short).

Definition 2.12. [10] Let (X, η) be a DTS, $\underline{F} \in D(X)$ and Y be a non-empty subset of X. Then, the D-subset over Y, denoted by F^Y, defined by: $F^Y = \underline{F} \bigcap \underline{Y}$.

Definition 2.13. [10] Consider two ordinary sets X and Y. Let f be a mapping from X into Y. The image of a D-set $\underline{A} \ in \ D(X)$ defined by: $f(\underline{A}) = (f(A_1), f(A_2))$. Also the inverse image of a D-set $\underline{B} \in D(Y)$ defined by: $f^{-1}(\underline{B}) = (f^{-1}(B_1), f^{-1}(B_2))$.

Definition 2.14. [10] Let $f : X \to Y$ be a mapping and let (X, η) and (Y, η^*) be DTS. Then, f is called a D-continuous if $f^{-1}(\underline{B}) \in \eta$, whenever $\underline{B} \in \eta^*$.

Theorem 2.15. [10] Let (X, η) and (Y, η^*) be two DTS and let $f : X \to Y$ be a mapping, $\underline{A} \in D(X)$ and $\underline{B} \in D(Y)$. Then, the following conditions are equivalent:

1. f is a D-continuous,

2. $f^{-1}(\underline{B}) \in \eta^c$, $\forall \underline{B} \in \eta^{*c}$,

3. $f(cl_\eta(\underline{A})) \subseteq cl_{\eta^*}(f(\underline{A}))$, $\forall \underline{A} \in D(X)$,

4. $cl_\eta(f^{-1}(\underline{B})) \subseteq f^{-1}(cl_{\eta^*}(\underline{B}))$, $\forall \underline{B} \in D(Y)$,

5. $f^{-1}(int_{\eta^*}(\underline{B})) \subseteq int_\eta(f^{-1}(\underline{B}))$, $\forall \underline{B} \in D(Y)$.

Definition 2.16. [10] Let (X, η) and (Y, η^*) be two DTS and let $f : X \to Y$ be a mapping and $\underline{A} \in D(X)$.

1. f is called D-open if $f(\underline{A}) \in \eta^*$, $\forall \underline{A} \in \eta$.

2. f is called D-closed if $f(\underline{A}) \in \eta^{*c}$, $\forall \underline{A} \in \eta^c$

Theorem 2.17. [10] Let (X, η) and (Y, η^*) be two DTS and let $f : X \to Y$ be a mapping and $\underline{A} \in D(X)$. f is D-closed iff $cl_{\eta^*}(f(\underline{A})) \subseteq f(cl_\eta(\underline{A}))$, $\forall \underline{A} \in D(X)$.

Definition 2.18. [9] Let (X, η) be a DTS. and let $\underline{A} \in D(X)$. $\underline{A}$ is said to be:

1. D-dense if $cl_\eta(\underline{A}) = \underline{X}$.

2. D-nowhere dense if $int_\eta(cl_\eta(\underline{A})) = \underline{\emptyset}$.

3 Connectedness in DTS

Definition 3.1. Let (X, η) be a DTS and let $\underline{A}, \underline{B} \in D(X)$.

1. $\underline{A}, \underline{B}$ are said to be separated double sets (separated D-sets, for short) if $cl_\eta(\underline{A}) \sqcap \underline{B} = \underline{\emptyset}$ and $\underline{A} \sqcap cl_\eta(\underline{B}) = \underline{\emptyset}$.

2. $\underline{A}, \underline{B}$ are said to be quasi-coincident separated double sets (q-separated D-sets, for short) if $cl_\eta(\underline{A}) \not{q} \underline{B}$ and $\underline{A} \not{q}\, cl_\eta(\underline{B})$.

Proposition 3.2. Let (X, η) be a DTS and let $\underline{A}, \underline{B} \in D(X)$. Then, if $\underline{A}, \underline{B}$ are separated D-sets, then $\underline{A} \sqcap \underline{B} = \underline{\emptyset}$.

Proof. Suppose $\underline{A}, \underline{B}$ are separated D-sets, then $cl_\eta(\underline{A}) \sqcap \underline{B} = \underline{\emptyset}$ and $\underline{A} \sqcap cl_\eta(\underline{B}) = \underline{\emptyset}$. But $\underline{A} \subseteq cl_\eta(\underline{A})$, then $\underline{A} \sqcap \underline{B} = \underline{\emptyset}$. Hence, the result.

The following example shows that the converse of Proposition 3.2 is not true in general.

Example 3.3. Let $X = \{a, b, c, d\}$, $\eta = \{\underline{\emptyset}, \underline{X}, (\{a\}, \{a, b\}), (\emptyset, \{a, b\}), (\{b, c, d\}, X)\}$ and $\eta^c = \{\underline{\emptyset}, \underline{X}, (\{c, d\}, \{b, c, d\}), (\{c, d\}, X), (\emptyset, \{a\})\}$. Then, (X, η) is a DTS. Now, $(\{a\}, \{a, b\}) \sqcap (\{c\}, \{c, d\}) = \underline{\emptyset}$, but $\underline{X} = cl_\eta(\{a\}, \{a, b\}) \sqcap (\{c\}, \{c, d\}) = (\{c\}, \{c, d\}) \neq \underline{\emptyset}$.

Proposition 3.4. Let (X, η) be a DTS and let $\underline{A}, \underline{B} \in D(X)$. Then, if $\underline{A}, \underline{B}$ are q-separated D-sets, then $\underline{A} \not{q} \underline{B}$.

Proof. Suppose $\underline{A}, \underline{B}$ are q-separated D-sets, then $cl_\eta(\underline{A}) \not{q}\, \underline{B}$ and $\underline{A} \not{q}\, cl_\eta(\underline{B})$. But $\underline{A} \subseteq cl_\eta(\underline{A})$, then $\underline{A} \not{q}\, \underline{B}$. Hence, the result.

The following example shows that the converse of Proposition 3.4 is not true in general.

Example 3.5. Let $X = \{a, b, c\}$ and $\eta = \{\underline{\emptyset}, \underline{X}, (\emptyset, \{a\}), (\emptyset, \{b\}), (\emptyset, \{a, b\})\}$. Then, $\exists (\{a\}, \{a\}), (\{b\}, \{b\}) \in D(X)$ such that $(\{a\}, \{a\}) \not{q}\, (\{b\}, \{b\})$. But, $cl_\eta(\{a\}, \{a\}) = (\{a, c\}, X)\ q\ (\{b\}, \{b\})$ and $cl_\eta(\{b\}, \{b\}) = (\{b, c\}, X)\ q\ (\{a\}, \{a\})$.

Proposition 3.6. Let (X, η) be a DTS and let $\underline{A}, \underline{B} \in D(X)$. Then, if $\underline{A} \sqcap \underline{B} = \underline{\emptyset}$, then $\underline{A} \not{q}\, \underline{B}$.

Proof. Suppose that $(A_1, A_2) = \underline{A} \sqcap \underline{B} = (B_1, B_2) = \underline{\emptyset}$, then $A_1 \cap B_1 = \emptyset$ and $A_2 \cap B_2 = \emptyset$, but $A_1 \subseteq A_2, B_1 \subseteq B_2$, then $A_1 \cap B_2 = \emptyset$ and $A_2 \cap B_1 = \emptyset$. Therefore, $\underline{A} \not{q}\, \underline{B}$.

The following example shows that the converse of Proposition 3.6 is not true in general.

Example 3.7. Let $X = \{a, b\}$ and $\eta = \{\underline{\emptyset}, \underline{X}, (\{a\}, \{a\}), (\emptyset, \{b\}), (\{a\}, X)\}$. Then, $(\emptyset, \{b\}) \not{q} (\{a\}, X)$, but $(\emptyset, \{b\}) \sqcap (\{a\}, X) = (\emptyset, \{b\}) \neq \underline{\emptyset}$.

Proposition 3.8. Let (X, η) be a DTS and let $\underline{A}, \underline{B} \in D(X)$. Then, if $\underline{A}, \underline{B}$ are separated D-sets, then $\underline{A}, \underline{B}$ are q-separated D-sets.

Proof. Straightforward.

The following example shows that the converse of Proposition 3.8 is not true in general.

Example 3.9. In Example 3.7, we see that: $(\emptyset, \{b\}) \not q\, cl_\eta(\{a\}, X) = (\{a\}, X)$ and $cl_\eta(\emptyset, \{b\}) = (\emptyset, \{b\}) \not q\, (\{a\}, X)$, but $(\emptyset, \{b\}) = cl_\eta(\emptyset, \{b\}) \sqcap (\{a\}, X) = (\emptyset, \{b\}) \neq \underline{\emptyset}$.

Remark 3.10. $\underline{A} \sqcap \underline{B} = \underline{\emptyset} \not\Leftrightarrow cl_\eta(\underline{A}) \not q\, \underline{B}$ and $\underline{A} \not q\, cl_\eta(\underline{B})$.

Example 3.11.

1. In Example 3.3, we see that: $(\{a\}, \{a, b\}) \sqcap (\{c\}, \{c, d\}) = \underline{\emptyset}$, but $\underline{X} = cl_\eta(\{a\}, \{a, b\})\ q\ (\{c\}, \{c, d\})$.
2. In Example 3.7, we see that: $(\emptyset, \{b\}) \not q\, cl_\eta(\{a\}, X) = (\{a\}, X)$ and $cl_\eta(\emptyset, \{b\}) = (\emptyset, \{b\}) \not q\, (\{a\}, X)$, but $(\emptyset, \{b\}) \sqcap (\{a\}, X) = (\emptyset, \{b\}) \neq \underline{\emptyset}$.

Theorem 3.12. Let (X, η) be a DTS and let $\underline{A}, \underline{B}, \underline{C}, \underline{D} \in D(X)$ such that $\underline{C} \sqsubseteq \underline{A}$ and $\underline{D} \sqsubseteq \underline{B}$. Then, if $\underline{A}, \underline{B}$ are separated D-sets, then $\underline{C}, \underline{D}$ are separated D-sets.

Proof. Suppose $\underline{A}, \underline{B}$ are separated D-sets, then $cl_\eta(\underline{A}) \sqcap \underline{B} = \underline{\emptyset}$ and $\underline{A} \sqcap cl_\eta(\underline{B}) = \underline{\emptyset}$. Since $\underline{C} \sqsubseteq \underline{A}$ and $\underline{D} \sqsubseteq \underline{B}$, then $cl_\eta(\underline{C}) \sqsubseteq cl_\eta(\underline{A})$ and $cl_\eta(\underline{D}) \sqsubseteq cl_\eta(\underline{B})$. Implies, $\underline{D} \sqcap cl_\eta(\underline{C}) = \underline{\emptyset}$ and $cl_\eta(\underline{D}) \sqcap \underline{C} = \underline{\emptyset}$. Hence, $\underline{C}, \underline{D}$ are separated D-sets.

Theorem 3.13. Let (X, η) be a DTS and let $\underline{A}, \underline{B}, \underline{C}, \underline{D} \in D(X)$ such that $\underline{C} \sqsubseteq \underline{A}$ and $\underline{D} \sqsubseteq \underline{B}$. Then, if $\underline{A}, \underline{B}$ are q-separated D-sets, then $\underline{C}, \underline{D}$ are q-separated D-sets.

Proof. Suppose $\underline{A}, \underline{B}$ are q-separated D-sets, then $cl_\eta(\underline{A}) \not q\, \underline{B}$ and $\underline{A} \not q\, cl_\eta(\underline{B})$. Since $\underline{C} \sqsubseteq \underline{A}$ and $\underline{D} \sqsubseteq \underline{B}$, then $cl_\eta(\underline{C}) \sqsubseteq cl_\eta(\underline{A})$ and $cl_\eta(\underline{D}) \sqsubseteq cl_\eta(\underline{B})$. Implies, $\underline{D} \not q\, cl_\eta(\underline{C})$ and $\underline{C} \not q\, cl_\eta(\underline{D})$ [by Theorem 2.4]. Hence, $\underline{C}, \underline{D}$ are q-separated D-sets.

Definition 3.14. Let (X, η) be a DTS, and let E be a nonempty subset of X.

1. If there exist two non-empty separated D-sets $\underline{A}, \underline{B} \in D(X)$ such that $\underline{A} \sqcup \underline{B} = \underline{E}$, then the D-sets $\underline{A}$ and $\underline{B}$ form a D-separation of E and it is said to be double disconnected set (D-disconnected set, for short). Otherwise, E is said to be double connected set (D-connected set, for short).
2. If there exist two non-empty q-separated D-sets $\underline{A}, \underline{B} \in D(X)$ such that $\underline{A} \sqcup \underline{B} = \underline{E}$, then the D-sets $\underline{A}$ and $\underline{B}$ form a $qD-$separation of E and it is said to be quasi-coincident double disconnected set ($qD-$disconnected set, for short). Otherwise, E is said to be quasi-coincident double connected set ($qD-$connected set, for short).

Remark 3.15. The D-point $\underline{x}_t$ in any DTS (X, η) is a $qD-$connected set, provided that $\underline{x}_{\frac{1}{2}}\ q\ cl_\eta(\underline{x}_{\frac{1}{2}})$.

Example 3.16. Let $X = \{a, b\},\ \eta = \{\underline{\emptyset}, \underline{X}, (\emptyset, \{a\}), (\{b\}, X)\}$ and $\eta^c = \{\underline{\emptyset}, \underline{X}, (\emptyset, \{a\}), (\{b\}, X)\}$. Then,

1. $(\emptyset, \{a\}) = cl(\emptyset, \{a\}) \not q\, (\emptyset, \{a\})$ and $(\emptyset, \{a\}) \sqcup (\emptyset, \{a\}) = (\emptyset, \{a\})$. Therefore, $(\emptyset, \{a\})$ is not $qD-$connected set.

2. $(\{b\}, X) = cl(\emptyset, \{b\})\ q\ (\emptyset, \{b\})$ and $(\emptyset, \{b\}) \underline{\cup} (\emptyset, \{b\}) = (\emptyset, \{b\})$. Therefore, $(\emptyset, \{b\})$ is $qD-$connected set.

Definition 3.17. Let (X, η) be a DTS.

1. If there exist two non-empty separated D-sets $\underline{A}, \underline{B} \in D(X)$ such that $\underline{A} \cup \underline{B} = \underline{X}$, then $\underline{A}$ and $\underline{B}$ are said to be double division (D-division, for short) for DTS (X, η). (X, η) is said to be double disconnected space (D-disconnected space, for short), if (X, η) has a D-division. Otherwise, (X, η) is said to be double connected space (D-connected space, for short).

2. If there exist two non-empty q-separated D-sets $\underline{A}, \underline{B} \in D(X)$ such that $\underline{A} \cup \underline{B} = \underline{X}$, then $\underline{A}$ and $\underline{B}$ are said to be a double quasi division ($qD-$division, for short) for DTS (X, η). (X, η) is said to be quasi-coincident double disconnected space ($qD-$disconnected space, for short), if (X, η) has a $qD-$division. Otherwise, (X, η) is said to be quasi-coincident double connected space ($qD-$connected space, for short).

Corollary 3.18. 1. Each indiscrete (stratified) DTS is D-connected ($qD-$connected).

2. Each discrete DTS is D-disconnected ($qD-$disconnected).

Proof. 1. It is obvious.

2. Suppose that (X, η) is a discrete DTS, then $\underline{x}_1 \in DP(X)$. Implies, $\underline{X} = \underline{x}_1 \cup \underline{x}_1^c$, $\underline{x}_1 = cl_\eta(\underline{x}_1) \sqcap \underline{x}_1^c = \emptyset$ and $\underline{x}_1^c = cl_\eta(\underline{x}_1^c) \sqcap \underline{x}_1 = \underline{\emptyset}$. Therefore, (X, η) is a D-disconnected. Similarly, (X, η) is a $qD-$disconnected.

Theorem 3.19. Let (X, η) be a DTS. Then, the following are equivalent:

1. (X, η) has a D-division,

2. There exist two disjoint closed D-sets $\underline{A}$ and $\underline{B}$ such that $\underline{A} \cup \underline{B} = \underline{X}$,

3. There exist two disjoint open D-sets $\underline{A}$ and $\underline{B}$ such that $\underline{A} \cup \underline{B} = \underline{X}$.

Proof. $(1 \to 2)$ Suppose that (X, η) has a D-division $\underline{A}$ and $\underline{B}$, then $\underline{A} \cup \underline{B} = \underline{X}$ and $cl_\eta(\underline{A}) \sqcap \underline{B} = \underline{\emptyset}$. Implies, $cl_\eta(\underline{A}) \subseteq \underline{B}^c = \underline{X} \setminus \underline{B} \subseteq \underline{A}$, but $\underline{A} \subseteq cl_\eta(\underline{A})$, so that $\underline{A} = cl_\eta(\underline{A})$. Therefore, $\underline{A}$ is a closed D-set. Similarly, we can see that $\underline{B}$ is also a closed D-set. Since $\underline{A} = cl_\eta(\underline{A}), cl_\eta(\underline{A}) \sqcap \underline{B} = \underline{\emptyset}$, then $\underline{A} \sqcap \underline{B} = \underline{\emptyset}$. Hence, the result.
$(2 \to 3)$ Suppose that (X, η) has a D-division $\underline{A}$ and $\underline{B}$ such that $\underline{A}$ and $\underline{B}$ are closed D-sets, then $\underline{A}^c$ and $\underline{B}^c$ are open D-sets, $\underline{A} = \underline{B}^c$ and $\underline{B} = \underline{A}^c$. Therefore, $\underline{A}^c \cup \underline{B}^c = \underline{X}$ and $\underline{A}^c \sqcap \underline{B}^c = \underline{\emptyset}$. Hence, the result.
$(3 \to 1)$ Since $\underline{X} = \underline{A} \cup \underline{B}$ such that $\underline{A} \sqcap \underline{B} = \underline{\emptyset}$ and $\underline{A}, \underline{B}$ are open D-sets, then $\underline{A}^c$ and $\underline{B}^c$ are closed D-sets, $\underline{A} = \underline{B}^c$ and $\underline{B} = \underline{A}^c$. This implies that, $\underline{A} = cl_\eta(\underline{A})$. Therefore, $cl_\eta(\underline{A}) \sqcap \underline{B} = \underline{\emptyset}$. Similarly, we have $\underline{A} \sqcap cl_\eta(\underline{B}) = \underline{\emptyset}$. Hence, (X, η) has a D-division.

Theorem 3.20. Let (X, η) be a DTS. Then, the following are equivalent:

1. (X, η) has a $qD-$division,

2. There exist two non quasi-coincident closed D-sets $\underline{A}$ and $\underline{B}$ such that $\underline{A} \cup \underline{B} = \underline{X}$,

3. There exist two non quasi-coincident open D-sets $\underline{A}$ and $\underline{B}$ such that $\underline{A} \cup \underline{B} = \underline{X}$.

Proof. $(1 \rightarrow 2)$ Suppose that (X, η) has a $qD-$division $\underline{A}$ and $\underline{B}$, then $\underline{A} \cup \underline{B} = \underline{X}$ and $cl_\eta(\underline{A}) \not{q} \underline{B}$, Implies, $cl_\eta(\underline{A}) \subseteq \underline{B}^c = \underline{X} \setminus \underline{B} \subseteq \underline{A}$. but $\underline{A} \subseteq cl_\eta(\underline{A})$, so that $\underline{A} = cl_\eta(\underline{A})$. Therefore, $\underline{A}$ is a closed D-set. Similarly, we can see that $\underline{B}$ is also a closed D-set and $\underline{A} \not{q} \underline{B}$ [by theorem 3.4]. Hence, the result.

$(2 \rightarrow 3)$ Suppose that (X, η) has a $qD-$division $\underline{A}$ and $\underline{B}$ such that $\underline{A}$ and $\underline{B}$ are closed D-sets, then $\underline{A}^c$ and $\underline{B}^c$ are open D-sets, $\underline{A} = \underline{B}^c$ and $\underline{B} = \underline{A}^c$. Therefore, $\underline{A}^c \cup \underline{B}^c = \underline{X}$ and $\underline{A}^c \not{q} \underline{B}^c$. Hence, the result.

$(3 \rightarrow 1)$ Since $\underline{X} = \underline{A} \cup \underline{B}$ such that $\underline{A} \not{q} \underline{B}$ and $\underline{A}, \underline{B}$ are open D-sets, then $\underline{A} \subseteq \underline{X} \setminus \underline{B}$ This implies that, $cl_\eta(\underline{A}) \subseteq \underline{X} \setminus \underline{B}$. Therefore, $cl_\eta(\underline{A}) \not{q} \underline{B}$. Similarly, we have $\underline{A} \not{q}\, cl_\eta(\underline{B})$. Hence, (X, η) has a $qD-$division.

Theorem 3.21. Let (X, η) be a DTS. Then, the following are equivalent:

1. (X, η) is D-connected,

2. $\underline{X}$ cannot be written as the union of two disjoint non-empty closed D-subsets,

3. $\underline{X}$ cannot be written as the union of two disjoint non-empty open D-subsets.

Proof. It follows from Theorem 3.19.

Theorem 3.22. Let (X, η) be a DTS. Then, the following are equivalent:

1. (X, η) is $qD-$connected,

2. $\underline{X}$ cannot be written as the union of two non quasi-coincident closed D-subsets,

3. $\underline{X}$ cannot be written as the union of two non quasi-coincident open D-subsets.

Proof. It follows from Theorem 3.20.

Theorem 3.23. Let (X, η) be a DTS and let Y be a non-empty subset of X. Then, if $\underline{A}$ and $\underline{B}$ are D-sets in Y, then $\underline{A}$ and $\underline{B}$ are separated D-sets in Y if and only if $\underline{A}$ and $\underline{B}$ are separated D-sets in X.

Proof. $cl_\eta(\underline{A}) \sqcap \underline{B} = \underline{Y} \sqcap cl_\eta(\underline{A}) \sqcap \underline{B},\ \underline{B} \subseteq \underline{Y}$
$= \underline{Y} \sqcap \underline{B} \sqcap cl_\eta(\underline{A})$
$= \underline{B} \sqcap \underline{Y} \sqcap cl_\eta(\underline{A})$
$= \underline{B} \sqcap cl_{\eta_Y}(\underline{A})$
$= \underline{\emptyset}$.

Similarly, we have: $cl_\eta(\underline{B}) \sqcap \underline{A} = \underline{\emptyset}$.

Conversely, $cl_{\eta_Y}(\underline{A}) \cap \underline{B} = \underline{Y} \cap cl_{\eta}(\underline{A}) \cap \underline{B} = \underline{Y} \cap (cl_{\eta}(\underline{A}) \cap \underline{B}) = \underline{Y} \cap \underline{\emptyset} = \underline{\emptyset}$.

Similarly, we have: $cl_{\eta_Y}(\underline{B}) \cap \underline{A} = \underline{\emptyset}$. Hence, the result.

Theorem 3.24. Let (X, η) be a DTS and let Y be a non-empty subset of X. Then, if $\underline{A}$ and $\underline{B}$ are D-sets in Y, then $\underline{A}$ and $\underline{B}$ are q-separated D-sets in Y if $\underline{A}$ and $\underline{B}$ are q-separated D-sets in X.

Proof. Suppose that $\underline{A}$ and $\underline{B}$ are $qSD-$sets in Y, then $cl_{\eta}(\underline{A}) \not q\, \underline{B}$ and $cl_{\eta}(\underline{B}) \not q\, \underline{A}$. So that, $cl_{\eta}(\underline{A}) \subseteq \underline{B}^c$. Thus, $\underline{Y} \cap cl_{\eta}(\underline{A}) \subseteq \underline{Y} \cap \underline{B}^c$. It follows that, $cl_{\eta_Y}(\underline{A}) \subseteq \underline{B}^c$.
Therefore, $cl_{\eta_Y}(\underline{A}) \not q\, \underline{B}$.
Similarly, we have: $cl_{\eta_Y}(\underline{B}) \not q\, \underline{A}$. Hence, the result.

Lemma 3.25. Let (Y, η_Y) be a $DT-$subspace of a DTS (X, η). Then, if (Y, η_Y) is a D-connected, then for every pair $\underline{A}$ and $\underline{B}$ of a separated D-subsets of $\underline{X}$ such that $\underline{Y} = \underline{A} \cup \underline{B}$, we have either $\underline{A} = \underline{\emptyset}$ or $\underline{B} = \underline{\emptyset}$.

Proof. Let $\underline{A} \neq \underline{\emptyset} \neq \underline{B}$ and $\underline{Y} = \underline{A} \cup \underline{B}$. Since $\underline{A}, \underline{B} \subseteq \underline{Y}$ and separated D-sets in X, then they are separated D-sets in Y [by Theorem 3.23]. This implies that, (Y, η_Y) is D-disconnected, which a contradiction. Hence, the result.

Lemma 3.26. Let (Y, η_Y) be a $DT-$subspace of a DTS (X, η). Then, if (Y, η_Y) is a $qD-$connected, then for every pair $\underline{A}$ and $\underline{B}$ of a q-separated D-subsets of $\underline{X}$ such that $\underline{Y} = \underline{A} \cup \underline{B}$, we have either $\underline{A} = \underline{\emptyset}$ or $\underline{B} = \underline{\emptyset}$.

Proof. Let $\underline{A} \neq \underline{\emptyset} \neq \underline{B}$ and $\underline{Y} = \underline{A} \cup \underline{B}$. Since $\underline{A}, \underline{B} \subseteq \underline{Y}$ and q-separated D-sets in X, then they are q-separated D-sets in Y [by Theorem 3.24] This implies that, (Y, η_Y) is $qD-$disconnected, which a contradiction. Hence, the result.

Theorem 3.27. Let (X, η) be a DTS and let Y be a non-empty subset of X such that (Y, η_Y) is D-connected. Then, if $\underline{A}$ and $\underline{B}$ are separated D-subsets of X such that $\underline{Y} \subseteq \underline{A} \cup \underline{B}$, then $\underline{Y} \subseteq \underline{A}$ or $\underline{Y} \subseteq \underline{B}$.

Proof. Since $\underline{Y} \subseteq \underline{A} \cup \underline{B}$, then $\underline{Y} = \underline{Y} \cap (\underline{A} \cup \underline{B}) = (\underline{Y} \cap \underline{A}) \cup (\underline{Y} \cap \underline{B})$. By Theorem 3.23, $\underline{Y} \cap \underline{A}$ and $\underline{Y} \cap \underline{B}$ are separated D-sets of Y. Since (Y, η_Y) is D-connected, then $\underline{Y} \cap \underline{A} = \underline{\emptyset}$ or $\underline{Y} \cap \underline{B} = \underline{\emptyset}$ [by Lemma 3.25]. Therefore, $\underline{Y} \subseteq \underline{A}$ or $\underline{Y} \subseteq \underline{B}$.

Theorem 3.28. Let (X, η) be a DTS and let Y be a non-empty subset of X such that (Y, η_Y) is $qD-$connected. Then, if $\underline{A}$ and $\underline{B}$ are q-separated D-subsets of X such that $\underline{Y} \subseteq \underline{A} \cup \underline{B}$, then $\underline{Y} \subseteq \underline{A}$ or $\underline{Y} \subseteq \underline{B}$.

Proof. Since $\underline{Y} \subseteq \underline{A} \cup \underline{B}$, then $\underline{Y} = \underline{Y} \cap (\underline{A} \cup \underline{B}) = (\underline{Y} \cap \underline{A}) \cup (\underline{Y} \cap \underline{B})$. By Theorem 3.24 and Theorem 2.4, $\underline{Y} \cap \underline{A}$ and $\underline{Y} \cap \underline{B}$ are q-separated D-sets of Y. Since (Y, η_Y) is $qD-$connected, then $\underline{Y} \cap \underline{A} = \underline{\emptyset}$ or $\underline{Y} \cap \underline{B} = \underline{\emptyset}$ [by Lemma 3.26]. Therefore, $\underline{Y} \subseteq \underline{A}$ or $\underline{Y} \subseteq \underline{B}$.

Theorem 3.29. Let $\{(X_\alpha, \eta_{X_\alpha}) : \alpha \in J\}$ be a family of non-empty D-connected subspaces of DTS (X, η). Then, if $\bigcap_{\alpha \in J} X_\alpha \neq \underline{\emptyset}$, then $(\bigcup_{\alpha \in J} X_\alpha, \eta_{\bigcup_{\alpha \in J} X_\alpha})$ is D-connected subspace of a (X, η).

Proof. Let $Y = \bigcup_{\alpha\in J} X_\alpha$. Choose a D-point $\underline{x_t}\underline{\in}\underline{Y}$. Let $\underline{A}$ and $\underline{B}$ be D-division of $(\bigcup_{\alpha\in J} X_\alpha, \eta_{\bigcup_{\alpha\in J} X_\alpha})$. Then, $\underline{x_t}\underline{\in}\underline{A}$ or $\underline{x_t}\underline{\in}\underline{B}$ without loss of generality, we may assume that $\underline{x_t}\underline{\in}\underline{A}$. For each $\alpha \in J$, since $(X_\alpha, \eta_{X_\alpha})$ is D-connected. It follows from Theorem 3.27 that, $\underline{X}_\alpha \subseteq \underline{A}$ or $\underline{X}_\alpha \subseteq \underline{B}$. Therefore, we have $\underline{Y} \subseteq \underline{A}$, since $\underline{x_t}\underline{\in}\underline{A}$, and then $\underline{B} = \underline{\emptyset}$, which a contradiction. Hence, $(\bigcup_{\alpha\in J} X_\alpha, \eta_{\bigcup_{\alpha\in J} X_\alpha})$ is a D-connected subspace of the DTS (X, η).

Theorem 3.30. Let $\{(X_\alpha, \eta_{X_\alpha}) : \alpha \in J\}$ be a family of non-empty $qD-$connected subspaces of DTS (X, η). Then, if $\bigcap_{\alpha\in J} X_\alpha \neq \underline{\emptyset}$, then $(\bigcup_{\alpha\in J} X_\alpha, \eta_{\bigcup_{\alpha\in J} X_\alpha})$ is qD-connected subspace of a (X, η).

Proof. Straightforward.

Theorem 3.31. Let $\{(X_\alpha, \eta_{X_\alpha}) : \alpha \in J\}$ be a family of non-empty D-connected subspaces of DTS (X, η). Then, if $X_\alpha \bigcap X_\beta \neq \emptyset$ for arbitrary $\alpha, \beta \in J$, then $(\bigcup_{\alpha\in J} X_\alpha, \eta_{\bigcup_{\alpha\in J} X_\alpha})$ is D-connected subspace of a (X, η).

Proof. Fix an $\alpha_o \in J$. For arbitrary $\beta \in J$, put $A_\beta = X_{\alpha_o} \bigcup X_\beta$. By Theorem 3.29, each $(A_\beta, \eta_{A_\beta})$ is D-connected. Then, $\{(A_\beta, \eta_{A_\beta}) : \beta \in J\}$ is a family non-empty D-connected subspaces of DTS (X, η) and $\bigcap_{\beta\in J} A_\beta = X_{\alpha_o} \neq \emptyset$. Obvious, we have $\bigcup_{\alpha\in J} X_\alpha = \bigcup_{\beta\in J} A_\beta$. It follows from Theorem 3.29 that, $(\bigcup_{\alpha\in J} X_\alpha, \eta_{\bigcup_{\alpha\in J} X_\alpha})$ is D-connected subspace of the DTS (X, η).

Theorem 3.32. Let $\{(X_\alpha, \eta_{X_\alpha}) : \alpha \in J\}$ be a family of non-empty $qD-$connected subspaces of DTS (X, η). Then, if $X_\alpha \bigcap X_\beta \neq \emptyset$ for arbitrary $\alpha, \beta \in J$, then $(\bigcup_{\alpha\in J} X_\alpha, \eta_{\bigcup_{\alpha\in J} X_\alpha})$ is $qD-$connected subspace of a (X, η).

Proof. Straightforward.

Theorem 3.33. Let (X, η) be a DTS and let Y be a non-empty subset of X such that (Y, η_Y) is D-connected. Then, if $\underline{Y} \subseteq \underline{A} \subseteq cl_\eta(\underline{Y})$, then (A, η_A) is a D-connected subspace of (X, η). In particular, $(cl_\eta(\underline{Y}), \eta_{cl_\eta(\underline{Y})})$ is a D-connected subspace of (X, η).

Proof. Suppose that (A, η_A) is a D-disconnected subspace of (X, η), then $\underline{A}$ has a D-separation $\underline{F}$ and $\underline{G}$ Implies, $\underline{Y} \subseteq \underline{F}$ or $\underline{Y} \subseteq \underline{G}$ [by Theorem 3.27]. Without loss of generality, we may assume that $\underline{Y} \subseteq \underline{F}$, so $cl_\eta(\underline{Y}) \subseteq cl_\eta(\underline{F}), cl_\eta(\underline{F}) \sqcap \underline{G} = \underline{\emptyset}$. Thus, $cl_\eta(\underline{Y}) \sqcap \underline{G} = \underline{\emptyset}$. Otherwise, $\underline{G} \subseteq \underline{A} \subseteq cl_\eta(\underline{Y})$. Therefore, $(cl_\eta(\underline{Y}))^c \sqcap \underline{G} = \underline{\emptyset}$, which a contradiction with $cl_\eta(\underline{Y}) \sqcap \underline{G} = \underline{\emptyset}$. Also, $\underline{Y} \subseteq cl_\eta(\underline{Y}) \subseteq cl_\eta(\underline{Y})$. This complete the proof.

Theorem 3.34. Let (X, η) be a DTS and let Y be a non-empty subset of X such that (Y, η_Y) is $qD-$connected. Then, if $\underline{Y} \subseteq \underline{A} \subseteq cl_\eta(\underline{Y})$, then (A, η_A) is a $qD-$connected subspace of (X, η). In particular, $(cl_\eta(\underline{Y}), \eta_{cl_\eta(\underline{Y})})$ is a $qD-$connected subspace of (X, η).

Proof. Suppose that (A, η_A) is a $qD-$disconnected subspace of (X, η), then $\underline{A}$ has a $qD-$separation $\underline{F}$ and $\underline{G}$. Implies, $\underline{Y} \subseteq \underline{F}$ or $\underline{Y} \subseteq \underline{G}$ [by Theorem 3.28]. Without loss of generality, we may assume that $\underline{Y} \subseteq \underline{F}$, so $cl_\eta(\underline{Y}) \subseteq cl_\eta(\underline{F}), cl_\eta(\underline{F}) \not{q}\, \underline{G}$. Thus $cl_\eta(\underline{Y}) \not{q}\, \underline{G}$. Otherwise, $\underline{G} \subseteq \underline{A} \subseteq cl_\eta(\underline{Y})$, Therefore, $(cl_\eta(\underline{Y}))^c \not{q}\, \underline{G}$, which a contradiction with $cl_\eta(\underline{Y}) \not{q}\, \underline{G}$. Also, $\underline{Y} \subseteq cl_\eta(\underline{Y}) \subseteq cl_\eta(\underline{Y})$. This complete the proof.

Theorem 3.35. The image of D-connected under a D-continuous map are *D*-connected.

Proof. Let (X, η) and (Y, τ) be two DTS, where (X, η) is a D-connected and let f be a D-continuous from X onto Y.
Suppose that (Y, τ) is a D-disconnected space, then $\exists\ \underline{A}, \underline{B} \in \tau$ such that $\underline{A} \sqcap \underline{B} = \underline{\emptyset}$ and $\underline{A} \sqcup \underline{B} = \underline{Y}$. Implies $\underline{A} \sqsubseteq \underline{B}^c$, thus $f^{-1}(\underline{A}) \sqsubseteq f^{-1}(\underline{B}^c) = (f^{-1}(\underline{B}))^c$. So that, $f^{-1}(\underline{A}) \sqcap f^{-1}(\underline{B}) = \underline{\emptyset}$ and $f^{-1}(\underline{A} \sqcup \underline{B}) = f^{-1}(\underline{Y})$. It follows that $f^{-1}(\underline{A}) \sqcup f^{-1}(\underline{B}) = \underline{X}$, i.e., (X, η) is a D-disconnected, which a contradiction. Hence, (Y, τ) is a D-connected.

Theorem 3.36. The image of $qD-$connected under a D-continuous map are qD-connected.

Proof. Let (X, η) and (Y, τ) be two DTS, where (X, η) is a $qD-$connected and let f be a D-continuous from X onto Y.

Suppose that (Y, τ) is a $qD-$disconnected space, then $\exists\ \underline{A}, \underline{B} \in \tau$ such that $\underline{A} \not{q} \underline{B}$ and $\underline{A} \sqcup \underline{B} = \underline{Y}$. Implies, $\underline{A} \sqsubseteq \underline{B}^c$, thus $f^{-1}(\underline{A}) \sqsubseteq f^{-1}(\underline{B}^c) = (f^{-1}(\underline{B}))^c$. So that, $f^{-1}(\underline{A}) \not{q} f^{-1}(\underline{B})$ and $f^{-1}(\underline{A} \sqcup \underline{B}) = f^{-1}(\underline{Y})$. It follows that $f^{-1}(\underline{A}) \sqcup f^{-1}(\underline{B}) = \underline{X}$, i.e., (X, η) is a $qD-$disconnected, which a contradiction. Hence, (Y, τ) is a $qD-$connected.

Theorem 3.37. Let (X, η) be a DTS. Then, if (X, η) is a D-disconnected space, then $(X, \pi_i), (i = 1, 2)$ are disconnected spaces.

Proof. Let (X, η) be a D-disconnected. Then, $\exists\ \underline{A}, \underline{B} \in \eta$ such that $(A_1, A_2) = \underline{A} \sqcap \underline{B} = (B_1, B_2) = \underline{\emptyset}$ and $\underline{A} \sqcup \underline{B} = \underline{X}$. Thus, $A_1 \cap B_1 = \emptyset, A_2 \cap B_2 = \emptyset, A_1 \cup B_1 = X$ and $A_2 \cup B_2 = X$. Therefore, $A_i \cap B_i = \emptyset, A_i \cup B_i = X$ and $A_i, B_i \in \pi_i$, $(i = 1, 2)$. Hence, (X, π_1) and (X, π_2) are disconnected spaces.

The following Example shows that the converse of Theorem 3.37 is not true in general.

Example 3.38. Let $X = \{a, b, c\}$ and $\pi_1 = \{\emptyset, X, \{a\}, \{b, c\}\}, \pi_2 = \{\emptyset, X, \{b\}, \{a, c\}\}$. Then, (X, π_1) and (X, π_2) are topological spaces and disconnected spaces. Since $\eta = (\pi_1, \pi_2) = \{\underline{\emptyset}, \underline{X}, (\emptyset, \{b\}), (\emptyset, X), (\emptyset, \{a, c\}), (\{a\}, \{a, c\}), (\{b, c\}, X), (\{a\}, X)\}$, then (X, η) is not D-disconnected.

Theorem 3.39. Let (X, η) be a DTS. Then, if (X, η) is a $qD-$disconnected space, then (X, π_1) is disconnected space.

Proof. Let (X, η) be a $qD-$disconnected space. Then, $\exists\ \underline{A}, \underline{B} \in \eta$ such that $(A_1, A_2) = \underline{A} \not{q} \underline{B} = (B_1, B_2)$ and $\underline{A} \sqcup \underline{B} = \underline{X}$. Thus, $A_1 \cap B_2 = \emptyset$ and $A_1 \cup B_1 = X, B_1 \subseteq B_2$. So that $A_1 \cap B_1 = \emptyset$ and $A_1 \cup B_1 = X,\ A_1, B_1 \in \pi_1$. Hence, (X, π_1) is a disconnected.

The following Example shows that the converse of Theorem 3.39 is not true in general.

Example 3.40. In Example 3.38, we see that: (X, π_1) is topological space and disconnected. But, (X, η) is not $qD-$disconnected.

Theorem 3.41. Let $(X, \eta), (X, \eta^*)$ be two DTS. Then, if (X, η) is D-connected and $\eta^* \leq \eta$, then (X, η^*) is also D-connected.

Proof. Suppose that (X, η^*) is a D-disconnected and $\eta^* \leq \eta$, then $\exists\ \underline{A}, \underline{B} \in \eta^*$ such that $\underline{A} \cap \underline{B} = \underline{\emptyset}$ and $\underline{A} \cup \underline{B} = \underline{X}$. Implies, $\underline{A}, \underline{B} \in \eta, \underline{A} \cap \underline{B} = \underline{\emptyset}$ and $\underline{A} \cup \underline{B} = \underline{X}$. Therefore, (X, η) is a D-disconnected, which a contradiction. Hence, (X, η^*) is D-connected.

Theorem 3.42. Let $(X, \eta), (X, \eta^*)$ be two DTS. Then, if (X, η) is $qD-$connected and $\eta^* \leq \eta$, then (X, η^*) is also $qD-$connected.

Proof. Suppose that (X, η^*) is a $qD-$disconnected and $\eta^* \leq \eta$, then $\exists\ \underline{A}, \underline{B} \in \eta^*$ such that $\underline{A}\ \not{q}\ \underline{B}$ and $\underline{A} \cup \underline{B} = \underline{X}$. Implies, $\underline{A}, \underline{B} \in \eta, \underline{A}\ \not{q}\ \underline{B}$ and $\underline{A} \cup \underline{B} = \underline{X}$. Therefore, (X, η) is a $qD-$disconnected, which a contradiction. Hence, (X, η^*) is $qD-$connected.

Theorem 3.43. Every $qD-$connected space is D-connected.

Proof. Suppose that (X, η) is a D-disconnected space, then $\exists\ \underline{A}, \underline{B} \in D(X)$ such that $cl_\eta(\underline{A}) \cap \underline{B} = \underline{\emptyset}$, $\underline{A} \cap cl_\eta(\underline{B}) = \underline{\emptyset}$ and $\underline{A} \cup \underline{B} = \underline{X}$. It follows from Proposition 3.6 that $cl_\eta(\underline{A})\ \not{q}\ \underline{B}$, $\underline{A}\ \not{q}\ cl_\eta(\underline{B})$ and $\underline{A} \cup \underline{B} = \underline{X}$. Therefore, (X, η) is a $qD-$disconnected space, which a contradiction. Hence, (X, η) is D-connected space.

Definition 3.44. The DTS (X, η) is said to be:

1. DC_1-disconnected, if (X, η) has a proper open and closed D-set in X.

2. DC_1-connected, if (X, η) is not DC_1-disconnected.

Corollary 3.45. Let (X, η) be a DTS or stratified DTS. Then, if the only open and closed D-sets are $\underline{\emptyset}, \underline{X}$ in (X, η) and $\underline{\emptyset}, \underline{X}$ and $(\emptyset, X)$ in stratified DTS, then (X, η) is $qD-$connected.

Proposition 3.46. Every DC_1-connected space is $qD-$connected.

Proof. Suppose that (X, η) is a $qD-$disconnected space, then $\exists\ \underline{A} \neq \underline{\emptyset}, \underline{B} \neq \underline{\emptyset} \in \eta$ such that $\underline{A}\ \not{q}\ \underline{B}$ and $\underline{A} \cup \underline{B} = \underline{X}$. Implies $\underline{A} \subseteq \underline{B}^c$ and $\underline{B}^c \subseteq \underline{A}$, so that $\underline{A} = \underline{B}^c$. Therefore, $\underline{A}$ is a proper open and closed D-set in X. Thus, (X, η) be a DC_1-disconnected, which a contradiction. Hence, (X, η) is DC_1-connected.

The converse of Proposition3.46 is not true in general.

Example 3.47. Let $X = \{a, b, c\}$ and $\eta = \{\underline{\emptyset}, \underline{X}, (\emptyset, \{b\}), (\{a\}, \{a\}), (\{a\}, \{a, b\}), (\{c\}, \{c\}), (\{c\}, \{b, c\}), (\{a, c\}, \{a, c\}), (\{a, c\}, X)\}$, $\eta^c = \{\underline{\emptyset}, \underline{X}, (\{a, c\}, X), (\{b, c\}, \{b, c\}), (\{c\}, \{b, c\}), (\{a, b\}, \{a, b\}), (\{a\}, \{a, b\}), (\{b\}, \{b\}), (\emptyset, \{b\})\}$. Then, (X, η) is a DTS and $qD-$connected space. But, (X, η) is not DC_1-connected because, $\exists\ (\emptyset, \{b\}), (\{a\}, \{a, b\}), (\{c\}, \{b, c\})$ and $(\{a, c\}, X)$ are open and closed D-sets.

Example 3.48. Let $X = \{a, b, c\}$ and $\eta = \{\underline{\emptyset}, \underline{X}, (\emptyset, \{a\}), (\emptyset, \{b\}), (\emptyset, \{c\}), (\emptyset, \{a, b\}), (\emptyset, \{a, c\}), (\emptyset, \{b, c\}), (\emptyset, X), (\{a\}, \{a\}), (\{a\}, \{a, b\}), (\{a\}, \{a, c\}), (\{a\}, X), (\{b\}, \{b\}), (\{b\}, \{a, b\}), (\{b\}, \{b, c\}), (\{b\}, X), (\{a, b\}, \{a, b\}), (\{a, b\}, X)\}$,

$\eta^c = \{\underline{\emptyset}, \underline{X}, (\{b, c\}, X), (\{a, c\}, X), (\{a, b\}, X), (\{c\}, X), (\{b\}, X), (\{a\}, X), (\emptyset, X), (\{b, c\}, \{b, c\}), (\{c\}, \{b, c\}), (\{b\}, \{b, c\}), (\emptyset, \{b, c\}), (\{a, c\}, \{a, c\}), (\{c\}, \{a, c\}), (\{a\}, \{a, c\}), (\emptyset, \{a, c\}), (\{c\}, \{c\}), (\emptyset, \{c\})\}$.

Then, (X, η) is stratified DTS and $qD-$connected space. But, (X, η) is not DC_1-connected because, $\exists\ (\emptyset, \{c\}), (\{a\}, X), (\{a\}, \{a, c\}), (\emptyset, \{a, c\}), (\emptyset, \{b, c\}), (\{b\}, \{b, c\}), (\{b\}, X)$, and $(\{a, b\}, X)$ are open and closed D-sets.

Example 3.49. From Example 3.48 (X, η) is stratified DTS and $qD-$connected, but $\exists\ (\emptyset, \{c\})$,
$(\{a\}, X), (\{a\}, \{a, c\}), (\emptyset, \{a, c\}), (\emptyset, \{b, c\}), (\{b\}, \{b, c\}), (\{b\}, X)$, and $(\{a, b\}, X)$, are open and closed D-sets.

Corollary 3.50. For a DTS (X, η) we have the following implication:
DC_1-connected $\rightarrow$ $qD-$connected $\rightarrow$ D-connected.

Definition 3.51. The DTS (X, η) is said to be:

1. Strongly double connected (strongly $SD-$connected, for short), if there exist no non-empty closed D-sets $\underline{A}, \underline{B} \in X$ such that $\underline{A} \cap \underline{B} = \underline{\emptyset}$.

2. Strongly $SD-$disconnected, if (X, η) is not strongly $SD-$connected.

Proposition 3.52. (X, η) is strongly $SD-$connected if and only if there exist no open D-sets $\underline{A}, \underline{B}$ in X such that $\underline{A} \neq \underline{X} \neq \underline{B}$ and $\underline{A} \cup \underline{B} = \underline{X}$.

Proof. Let $\underline{A}, \underline{B}$ be open D-sets in X such that $\underline{A} \neq \underline{X} \neq \underline{B}$. If we take $\underline{C} = \underline{A}^c$ and $\underline{D} = \underline{B}^c$, then $\underline{C}$ and $\underline{D}$ become closed D-sets in X and $\underline{C} \neq \underline{\emptyset} \neq \underline{D}$, $\underline{C} \cap \underline{D} = \underline{\emptyset}$, which a contradiction.
Conversely, it is obvious.

Proposition 3.53. Strongly $SD-$connectedness does not imply DC_1-connectedness, and DC_1- connectedness does not imply $StD-$connectedness.

Example 3.54. In Example3.7, we see that: (X, η) is strongly $SD-$connected, but it is not DC_1- connected, for $\exists\ (\emptyset, \{b\})$ is both open and closed D-set.

Example 3.55. In Example3.3, we see that: (X, η) is DC_1-connected, but it is not strongly $SD-$connected, for

$$\exists\ (\{a\}, \{a, b\}), (\{b, c, d\}, X) \in \eta \text{ and } (\{a\}, \{a, b\}) \,\underline{\cup}\, (\{b, c, d\}, X) = \underline{X}.$$

Definition 3.56. Let (X, η) be a DTS and $\underline{Y} \subseteq \underline{X}$ with $\underline{x_t} \underline{\in} DP(Y)$. The union of all D-connected subsets of $\underline{Y}$ containing the D-point $\underline{x_t}$ is called double component (D-component, for short) of Y with respect to $\underline{x_t}$, denoted by $\underline{C}(\underline{Y}, \underline{x_t})$, i.e.,
$\underline{C}(\underline{Y}, \underline{x_t}) = \underline{\cup}\{\underline{A} \subseteq \underline{Y} : \underline{x_t} \underline{\in} \underline{A}\ \textit{and}\ \underline{A}\ \textit{is a}\ D - \textit{connected set}\}$.

Remark 3.57. The D-component $\underline{C}(\underline{Y}, \underline{x}_t)$ is the largest D-connected subset of Y containing $\underline{x}_t$.

Definition 3.58. Let (X, η) be a DTS and $\underline{Y} \subseteq \underline{X}$ with $\underline{x}_t \underline{\in} DP(Y), \underline{x}_{\frac{1}{2}}\ q\ cl_\eta(\underline{x}_{\frac{1}{2}})$. Then, the union of all $qD-$connected subsets of $\underline{Y}$ containing the D-point $\underline{x}_t$ is called quasi-coincident double component ($qD-$component, for short) of Y with respect to $\underline{x}_t$, denoted by $\underline{Cq}(\underline{Y}, \underline{x}_t)$, i.e., $\underline{Cq}(\underline{Y}, \underline{x}_t) = \underline{\cup}\{\underline{A} \subseteq \underline{Y} : \underline{x}_t \underline{\in} \underline{A}$ *and* $\underline{A}$ *is a* $qD-$ *connected set*$\}$.

Remark 3.59. The $qD-$component $\underline{Cq}(\underline{Y}, \underline{x}_t)$ is the largest $qD-$connected subset of Y containing $\underline{x}_t$.

Theorem 3.60. Every D-component of a DTS is a closed D-set.

Proof. Let (X, η) be a DTS and let $\underline{C}(\underline{Y}, \underline{x}_t)$ be a D-component of the DTS (X, η) with respect to an arbitrary D-point $\underline{x}_t \in DP(X)$. Then, $\underline{C}(\underline{Y}, \underline{x}_t)$ is a D-connected subset of X [Theorem 3.29]. Also, by Theorem 3.33 $cl_\eta(\underline{C}(\underline{Y}, \underline{x}_t))$ is D-connected subset of $\underline{X}$ containing $\underline{x}_t$, then $cl_\eta(\underline{C}(\underline{Y}, \underline{x}_t)) \subseteq \underline{C}(\underline{Y}, \underline{x}_t)$. But, $\underline{C}(\underline{Y}, \underline{x}_t) \subseteq cl_\eta(\underline{C}(\underline{Y}, \underline{x}_t))$. Hence, $\underline{C}(\underline{Y}, \underline{x}_t) = cl_\eta(\underline{C}(\underline{Y}, \underline{x}_t))$, which shows that the D-component $\underline{C}(\underline{Y}, \underline{x}_t)$ is a closed D-set.

Theorem 3.61. Every $qD-$component of a DTS is a closed D-set.

Proof. Let (X, η) be a DTS and let $\underline{Cq}(\underline{Y}, \underline{x}_t)$ be a $qD-$component of the DTS (X, η) with respect to an arbitrary D-point $\underline{x}_t \in DP(X)$. Then, $\underline{Cq}(\underline{Y}, \underline{x}_t)$ is a $qD-$connected subset of X [Theorem 3.30]. Also, by Theorem 3.34 $cl_\eta(\underline{Cq}(\underline{Y}, \underline{x}_t))$ is $qD-$connected subset of $\underline{X}$ containing $\underline{x}_t$, then $cl_\eta(\underline{Cq}(\underline{Y}, \underline{x}_t)) \subseteq \underline{Cq}(\underline{Y}, \underline{x}_t)$. But, $\underline{Cq}(\underline{Y}, \underline{x}_t) \subseteq cl_\eta(\underline{Cq}(\underline{Y}, \underline{x}_t))$. Hence, $\underline{Cq}(\underline{Y}, \underline{x}_t) = cl_\eta(\underline{Cq}(\underline{Y}, \underline{x}_t))$, which shows that the qD-component $\underline{Cq}(\underline{Y}, \underline{x}_t)$ is a closed D-set.

Theorem 3.62. Let (X, η) be a DTS. Then, each D-point in X is contained in exactly one D-component of X.

Proof. Let $\underline{x}_t \underline{\in} \underline{X}$ and consider the collection:
$\underline{C} = \{\underline{Y} \subseteq \underline{X} : \underline{x}_t \underline{\in} \underline{Y}$ *and* $\underline{Y}$ *is a* $D -$ *connected set*$\}$.
Then, we have:

1. $\underline{C} \neq \underline{\emptyset}$, for the D-point $\underline{x}_t$ is a D-connected subset of X. Then, $\underline{x}_t \underline{\in} \underline{C}$.
2. $\underline{\bigcap}\{\underline{Y} \subseteq \underline{X} : \underline{x}_t \underline{\in} \underline{Y}$ *and* $\underline{Y}$ *is a* $D-$*connected set*$\} \neq \underline{\emptyset}$. Since $\underline{x}_t \underline{\in} \underline{Y}$, $\forall\ \underline{Y} \underline{\in} \underline{C}$.
3. $\underline{\bigcup}\{\underline{Y} \subseteq \underline{X} : \underline{x}_t \underline{\in} \underline{Y}$ *and* $\underline{Y}$ *is a* $D -$ *connected set*$\}$, having non null double intersection, is D-connected subset of X containing $\underline{x}_t$.
4. $\underline{\bigcup}\{\underline{Y} \subseteq \underline{X} : \underline{x}_t \underline{\in} \underline{Y}$ *and* $\underline{Y}$ *is a* $D -$ *connected set*$\}$ is the largest D-connected subset of $\underline{X}$ containing $\underline{x}_t$, which is the D-component $\underline{C}(\underline{X}, \underline{x}_t)$ of X with respect to $\underline{x}_t$ and containing $\underline{x}_t$ from Definition 3.56.

Now, suppose that $\underline{C}^*(\underline{X}, \underline{x}_t)$ be another D-component containing $\underline{x}_t$, then $\underline{C}^*(\underline{X}, \underline{x}_t)$ is D-connected subset of X containing $\underline{x}_t$. Since $\underline{C}(\underline{X}, \underline{x}_t)$ is D-component containing $\underline{x}_t$, then $\underline{C}^*(\underline{X}, \underline{x}_t) \subseteq \underline{C}(\underline{X}, \underline{x}_t)$. Again, since $\underline{C}^*(\underline{X}, \underline{x}_t)$ is D-component containing $\underline{x}_t$, then $\underline{C}(\underline{X}, \underline{x}_t) \subseteq \underline{C}^*(\underline{X}, \underline{x}_t)$. Therefore, $\underline{C}(\underline{X}, \underline{x}_t) = \underline{C}^*(\underline{X}, \underline{x}_t)$. Hence, $\underline{x}_t$ is contained in exactly one D-component of X.

Theorem 3.63. Let (X, η) be a DTS. Then, each D-point in X, $\underline{x}_{\frac{1}{2}}\ q\ cl_\eta(\underline{x}_{\frac{1}{2}})$, is contained in exactly one $qD-$component of X.

Proof. Let $\underline{x}_t \underline{\in} \underline{X}$ and consider the collection:
$\underline{Cq} = \{\underline{Y} \subseteq \underline{X} : \underline{x}_t \underline{\in} \underline{Y}\ and\ \underline{Y}\ is\ a\ qD-connected\ set\}$.
Then, we have:

1. $\underline{Cq} \neq \underline{\emptyset}$, for the D-point $\underline{x}_t$ is a $qD-$connected subset of X, $\underline{x}_{\frac{1}{2}}\ q\ cl_\eta(\underline{x}_{\frac{1}{2}})$. Then, $\underline{x}_t \underline{\in} \underline{Cq}$.
2. $\underline{\bigcap}\{\underline{Y} \subseteq \underline{X} : \underline{x}_t \underline{\in} \underline{Y}\ and\ \underline{Y}\ is\ a\ qD-connected\ set\} \neq \underline{\emptyset}$. Since $\underline{x}_t \underline{\in} \underline{Y}$, $\forall\ \underline{Y} \underline{\in} \underline{C}$.
3. $\underline{\bigcup}\{\underline{Y} \subseteq \underline{X} : \underline{x}_t \underline{\in} \underline{Y}\ and\ \underline{Y}\ is\ a\ qD-connected\ set\}$, having non null double intersection, is $qD-$connected subset of X containing $\underline{x}_t$.
4. $\underline{\bigcup}\{\underline{Y} \subseteq \underline{X} : \underline{x}_t \underline{\in} \underline{Y}\ and\ \underline{Y}\ is\ a\ qD-connected\ set\}$ is the largest $qD-$connected subset of $\underline{X}$ containing $\underline{x}_t$, which is the $qD-$component $\underline{Cq}(\underline{X}, \underline{x}_t)$ of X with respect to $\underline{x}_t$ and containing $\underline{x}_t$ from Definition 3.58.

Now, suppose that $\underline{Cq}^*(\underline{X}, \underline{x}_t)$ be another $qD-$component containing $\underline{x}_t$, thus $\underline{Cq}^*(\underline{X}, \underline{x}_t)$ is $qD-$connected subset of X containing $\underline{x}_t$. Since $\underline{Cq}(\underline{X}, \underline{x}_t)$ is $qD-$component containing $\underline{x}_t$, then $\underline{Cq}^*(\underline{X}, \underline{x}_t) \subseteq \underline{Cq}(\underline{X}, \underline{x}_t)$. Again, since $\underline{Cq}^*(\underline{X}, \underline{x}_t)$ is $qD-$component containing $\underline{x}_t$, then $\underline{Cq}(\underline{X}, \underline{x}_t) \subseteq \underline{Cq}^*(\underline{X}, \underline{x}_t)$. Therefore, $\underline{Cq}(\underline{X}, \underline{x}_t) = \underline{Cq}^*(\underline{X}, \underline{x}_t)$. Hence, $\underline{x}_t$ is contained in exactly one $qD-$component of X.

Theorem 3.64. Let (X, η) be a DTS. Then, any two D-components with respect to two disjoint D-points in X are either disjoint or identical.

Proof. Let $\underline{C}(\underline{X}, \underline{x}_t)$ and $\underline{C}(\underline{X}, \underline{y}_r)$ be two D-components of the DTS (X, η) with respect to the D-points $\underline{x}_t, \underline{y}_r$ in X. and $\underline{x}_t \underline{\cap} \underline{y}_r = \underline{\emptyset}$. Then, if $\underline{C}(\underline{X}, \underline{x}_t) \sqcap \underline{C}(\underline{X}, \underline{y}_r) = \underline{\emptyset}$, then we are done. So let $\underline{C}(\underline{X}, \underline{x}_t) \sqcap \underline{C}(\underline{X}, \underline{y}_r) \neq \underline{\emptyset}$. We may choose $\underline{z}_s \underline{\in} \underline{C}(\underline{X}, \underline{x}_t) \sqcap \underline{C}(\underline{X}, \underline{y}_r)$. Clearly, $\underline{z}_s \underline{\in} \underline{C}(\underline{X}, \underline{x}_t)$ and $\underline{z}_s \underline{\in} \underline{C}(\underline{X}, \underline{y}_r)$. Thus, $\underline{C}(\underline{X}, \underline{x}_t) = \underline{C}(\underline{X}, \underline{y}_r)$ [from Theorem 3.62]. Therefore, $\underline{C}(\underline{X}, \underline{x}_t)$ and $\underline{C}(\underline{X}, \underline{y}_r)$ are identical. This completes the proof.

Theorem 3.65. Let (X, η) be a DTS. Then, any two $qD-$components with respect to two disjoint D-points in X are either not quasi-coincident or identical.

Proof. Let $\underline{Cq}(\underline{X}, \underline{x}_t)$ and $\underline{Cq}(\underline{X}, \underline{y}_r)$ be two $qD-$components of the DTS (X, η) with respect to the D-points $\underline{x}_t, \underline{y}_r$ in X, $\underline{x}_{\frac{1}{2}}\ q\ cl_\eta(\underline{x}_{\frac{1}{2}})$, $\underline{y}_{\frac{1}{2}}\ q\ cl_\eta(\underline{y}_{\frac{1}{2}})$ and $\underline{x}_t \underline{\cap} \underline{y}_r = \underline{\emptyset}$. Then, if
$\underline{Cq}(\underline{X}, \underline{x}_t)\ \not{q}\ \underline{Cq}(\underline{X}, \underline{y}_r)$, then we are done. So let $\underline{Cq}(\underline{X}, \underline{x}_t)\ q\ \underline{Cq}(\underline{X}, \underline{y}_r)$. Then, $\underline{Cq}(\underline{X}, \underline{x}_t) \sqcap$
$\underline{Cq}(\underline{X}, \underline{y}_r) \neq \underline{\emptyset}$. We may choose $\underline{z}_s \underline{\in} \underline{Cq}(\underline{X}, \underline{x}_t) \sqcap \underline{Cq}(\underline{X}, \underline{y}_r)$. Clearly, $\underline{z}_s \underline{\in} \underline{Cq}(\underline{X}, \underline{x}_t)$ and $\underline{z}_s \underline{\in} \underline{Cq}(\underline{X}, \underline{y}_r)$, thus $\underline{Cq}(\underline{X}, \underline{x}_t) = \underline{Cq}(\underline{X}, \underline{y}_r)$ [from Theorem 3.63]. Therefore, $\underline{Cq}(\underline{X}, \underline{x}_t)$ and $\underline{Cq}(\underline{X}, \underline{y}_r)$ are identical. This completes the proof.

Definition 3.66. A DTS (X, η) is said to be a quasi-coincident hyperconnected (q-hyperconnected, for short) if every pair of non null proper open D-sets $\underline{A}$ and $\underline{B}$ are quasi-coincident, i. e.,
(X, η) is said to be q-hyperconnected if $\forall\ \underline{A}, \underline{B} \in \eta$, we have $\underline{A}\ q\ \underline{B}$.

Theorem 3.67. Every q-hyperconnected DTS is $qD-$connected.

Proof. Suppose that (X, η) is is $qD-$disconnected, then there exist two open D-sets $\underline{A}$ and $\underline{B}$ such that $\underline{A}\ \not{q}\ \underline{B}$ [Theorem 3.20]. Hence, (X, η) is not q-hyperconnected DTS, which a contradiction. Thus, (X, η) is $qD-$connected.

Remark 3.68. The converse of Theorem 3.67 is not true in general, as shown in the following Examples.

Example 3.69. In Example3.7, we see that: The DTS (X, η) is a $qD-$connected, but it is not a q-hyperconnected, for $(\{a\}, \{a\})\ \not{q}\ (\emptyset, \{b\})$.

Example 3.70. Let $X = \{a, b\}$ and $\eta = \{\underline{\emptyset}, \underline{X}, (\emptyset, \{a\}), (\emptyset, \{b\}), (\emptyset, X)\}$. Then, η defines a stratified double topology on X. Hence, the DTS (X, η) is $qD-$connected, but it is not q-hyperconnected, for $(\emptyset, \{a\})\ \not{q}\ (\emptyset, \{b\})$.

Theorem 3.71. Let (X, η) be a DTS. Then, the following are equivalent:

1. (X, η) is q-hyperconnected,
2. $\underline{A}$ is D-dense, $\forall\ \underline{A} \in \eta$,
3. $\underline{A}$ is D-dense or D-nowhere dense, $\forall\ \underline{A} \in D(X)$.

Proof. (1) $\rightarrow$ (2) Suppose that $\exists\ \underline{B} \in \eta$ such that $\underline{B}$ is not D-dense in X, thus $cl_\eta(\underline{B}) \neq \underline{X}$. Hence, $\underline{X} \setminus cl_\eta(\underline{B})$ and $\underline{B}$ are not quasi-coincident [by Proposition 3.3], which a contradiction with q-hyperconnected of (X, η).
(2) $\rightarrow$ (3) Suppose that $\underline{B}$ is not D-nowhere dense, then $int_\eta(cl_\eta(\underline{B})) \neq \underline{\emptyset}$. So by (2), $cl_\eta(int_\eta(cl_\eta(\underline{B}))) = \underline{X}$. Since $cl_\eta(int_\eta(cl_\eta(\underline{B}))) \subseteq cl_\eta(\underline{B})$, then $cl_\eta(\underline{B}) = \underline{X}$. Hence, $\underline{B}$ is D-dense.
(3) $\rightarrow$ (1) Suppose that $\underline{A}\ \not{q}\ \underline{B}$, for some non-empty open D-subsets $\underline{A}, \underline{B}$ of X, then $cl_\eta(\underline{A})\ \not{q}\ \underline{B}$ [by Theorem 2.7], and $\underline{A}$ is not D-dense. Since $\underline{A} \in \eta$, then $\underline{\emptyset} \neq \underline{A} \subseteq int_\eta(cl_\eta(\underline{A}))$, which a contradiction with (3). Hence, the result.

Remark 3.72. If the DTS (X, η) is a stratified, then the Theorems 3.19, 3.20, ..., 3.71 and Propositions 3.2, ..., 3.53 are satisfied.

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ON CLASS OF FUNCTION VIA GRILL

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Abstaract − If we have (X, τ, G) be a topological space with a grill, our aim in this paper is to provide a study of new class of functions via grill called slightly G-precontinuous functions. From this new definition, we are enables some of the important characterizations and principle properties of this class of functions are obtained.

Keywords − *Grill topological spaces, G-preopen sets, Slightly G-precontinuity.*

1 Introduction

The idea of grill topological space depended on the two influential are Φ and Ψ. Choquet [4] in 1947 is the first exposure to the concept of grill and explain its conditions. There is a similarity between the notion of ideals, nets, filters and grill. Also, the theory of compactifications is similar to the investigation of many topological notions in terms of grill, proximity spaces was used (see [5], [6], [12] for details). In [10] Roy and Mukherjee introduced grill toplogical space τ_G, some of characterizations and the relationship between τ and τ_G. In 2012, Mandal and Mukherjee [9] have defined new classes of sets in grill topological space and obtained a new decoposition of continuity in terms of grills. Quite recently, Baskar and Rajesh [3] introduced new class of functions called slightly G-semi-continuous functions in grill topological spaces. Our purpose in this paper, slightly G-precontinuity is introduced and studied. Moreover, basic properties and preservation theorems of slightly G-precontinuous functions are investigated and relationships between this class of functions and graphs are investigated.

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2 Preliminaries

In what follows, by a space X we shall mean a topological space (X, τ). For a subset A of the space X, we shall adopt the usual notations $Cl(A)$ and $Int(A)$ to respectively denote the closure and interior of A in X. Also, the power set of X will be written as $P(X)$. A collection G of nonempty subsets of a space X is called a grill ([1]) if (i) $\phi \notin G$, (ii) $A \in G$ and $A \subset B \subset X$ implies that $B \in G$, (iii) $A, B \subset X$ and $A \cup B \in G$ implies that $A \in G$ or $B \in G$.

Definition 2.1 (2)**.** Let (X, τ, G) be a grill topological space. An operator $\Phi : P(X) \to P(X)$ is defined as follows: $\Phi(A) = \Phi_G(A, \tau) = \{x \in X : A \cap U \in G$ for every open set U containing x $\}$ for each $A \in P(X)$. The mapping Φ is called the operator associated with the grill G and the topology τ.

Definition 2.2 (2)**.** Let G be a grill on a topological space (X, τ). Then we define a map $\Psi : P(X) \to P(X)$ by $\Psi(A) = A \cup \Phi(A)$ for all $A \in P(X)$. The map Ψ is a Kuratowski closure axioms. Corresponding to a grill G on a topological space (X, τ), there exists a unique topology τ_G on X given by $\tau_G = \{U \subset X : \Psi(X \setminus U) = X \setminus U\}$, where for any $A \subset X$, $\Psi(A) = A \cup \Phi(A) = \tau_G Cl(A)$. For any grill G on a topological space (X, τ), $\tau \subset \tau_G$. If (X, τ) is a topological space with a grill G on X, then we call it a grill topological space and denote it by (X, τ, G).

Definition 2.3 (5)**.** Let (X, τ, G) be a grill topological space. A subset A in X is said to be G-preopen if $A \subseteq Int(\Psi(A))$. The complement of a G-preopen set is called a G-preclosed set.

Definition 2.4. The intersection of all G-preclosed sets containing $A \subset X$ is called the G-preclosure of A and is denoted by $pCl_G(A)$. The family of all G-preopen (resp. G-preclosed) sets of (X, τ, G) is denoted by $GPO(X)$ (resp. $GPC(X)$). The family of all G-preopen (resp. G-preclosed) sets of (X, τ, G) containing a point $x \in X$ is denoted by $GPO(X, x)$ (resp. $GPC(X, x)$).

Definition 2.5. A function $f : (X, \tau) \to (Y, \sigma)$ is said to be slightly continuous [8] (resp. slightly precontinuous [2]) if $f^{-1}(V)$ is open (resp. preopen) in X for every clopen set V of Y.

Remark 2.6. For several sets, we have implications as in the Figure 1. Where converses of the above implications need not be true as shown by the following examples.

Example 2.7. *Let $X = \{a, b, c\}$ and $\tau = \{\phi, X, \{a\}, \{b, c\}\}$. If $G = \{X, \{a\}, \{b\}, \{a, b\}, \{b, c\}, \{a, c\}\}$, then G is a grill on X. If $A = \{a, b\}$. Then*
1- A is $G-\gamma-$ open but niether G-semi-open nor $G-\alpha-$open.
2- A is γ-open but not semi-open.
3- A is preopen but not $\alpha-$open.

Example 2.8. *Let $X = \{a, b, c, d\}$ and $\tau = \{\phi, X, \{d\}, \{a, c\}, \{a, c, d\}\}$. If $G = \{X, \{a\}, \{a, c\}, \{a, c, d\}, \{a, b\}, \{a, bc\}, \{a, d\}, \{a, b, d\}\}$, then G is a grill on X. Then,*
1- A is G-preopen and G-semi-open but neither Φ-open nor α-open at $A = \{d\}$.
2- A is β-open, but not $G-\beta-$open at $A = \{c\}$.

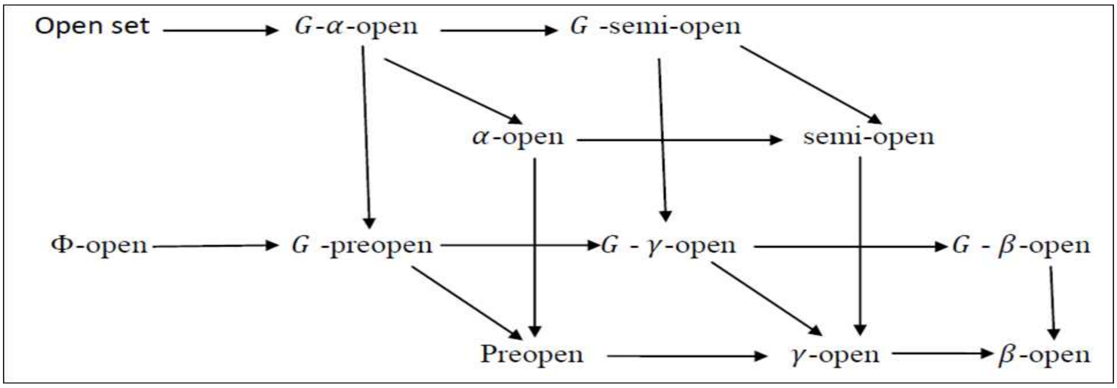

Figure 1: Relationships between some classes of open sets

3 Slightly G-precontinuous Functions

In this section, the notion of slightly G-precontinuous functions is introduced and characterizations and some relationships of precontinuous functions and basic properties of slightly G-precontinuous functions are investigated and obtained.

Definition 3.1. A function $f : (X, \tau, G) \to (Y, \sigma)$ is said to be:
1- Slightly G-precontinuous at $x \in X$ if for each clopen subset V of Y containing $f(x)$, there exists $U \in GPO(X, x)$ such that $f(U) \subset V$;
2- Slightly G-precontinuous if it is G-precontinuous at each point of X.

Theorem 3.2. *For a function $f : (X, \tau, G) \to (Y, \sigma)$, The following statements are equivalent :*
1- f is slightly G-precontinuous;
2- $f^{-1}(V)$ is G-preopen in X for each clopen subset V of Y;
3- $f^{-1}(V)$ is G-preclosed in X for each clopen subset V of Y;
4- $f^{-1}(V)$ is G-preclopen in X for each clopen subset V of Y.

Proof. The proof is clear. □

Proposition 3.3. *Every slightly G-precontinuous function is slightly precontinuous.*

Proof. It follows from proposition 3.3 of [1]. □

The converse of Proposition 3.3 is need not be true as shown by the following example.

Example 3.4. *Let $X = \{a, b, c\}, \tau = \{\phi, \{a\}, \{b, c\}, X\}$ and $G = P(X) \setminus \{\phi, \{b\}\}$. Then the function $f : (X, \tau, G) \to (X, \tau)$ is defined by $f(a) = c, f(b) = a$ and $f(c) = b$ is slightly precontinuous but not slightly G-precontinuous*

Proposition 3.5. *Every slightly continuous function is slightly G-precontinuous.*

Proof. It follows from Remark 1 of [1]. □

The converse of Proposition 3.5 is need not be true as shown by the following example.

Example 3.6. *Let* $X = \{a, b, c\}, \tau = \{\phi, \{a\}, \{b, c\}, X\}$ *and* $G = P(X)\backslash\{\phi\}$. *Then the function* $f : (X, \tau, G) \rightarrow (Y, \sigma)$ *is defined by* $f(a) = c, f(b) = a$ *and* $f(c) = b$ *is slightly* G*-precontinuous but not slightly continuous.*
i. e. Slightly continuiy $\Rightarrow$ *slightly* G*-precontinuity* $\Rightarrow$ *slightly precontinuity.*

Definition 3.7. A function $f : (X, \tau, G) \rightarrow (Y, \sigma, G)$ is said to be G-preirresolute if $f^{-1}(V) \in GPO(X)$ for every $V \in GPO(Y)$.

Theorem 3.8. *Let* $f : (X, \tau, G_1) \rightarrow (Y, \sigma, G_2)$ *and* $g : (Y, \sigma, G_2) \rightarrow (Z, \theta)$ *be functions, then the following properties holds:*
1- If f *is slightly* G_1*-precontinuous and* g *is slightly continuous, then* $g \circ f : (X, \tau, G_1) \rightarrow (Z, \theta)$ *is slightly* G_1*-precontinuous.*
2- If f *is* G_1*-preirresolute and* g *is slightly* G_2*-precontinuous, then* $g \circ f$ *is slightly* G_1*-precontinuous.*
3- If f *is* G_1*-preirresolute and* g *is slightly continuous, then* $g \circ f$ *is slightly* G_1*-precontinuous.*

Proof. The proof is clear. □

Definition 3.9. A function $f : (X, \tau, G_1) \rightarrow (Y, \sigma, G_2)$ is said to be strongly G-preopen if $f(U) \in G_2PO(Y)$ for every $U \in G_1PO(X)$.

Theorem 3.10. *Let* $f : (X, \tau, G_1) \rightarrow (Y, \sigma, G_2)$ *and* $g : (Y, \sigma, G_2) \rightarrow (Z, \eta)$ *be functions. Then the following properties hold:*
1- If f *is strongly* G_1*-preopen surjection and* $g \circ f$ *is slightly* G_1*-precontinuous, then* g *is slightly* G_2*-precontinuous.*
2- Let f *be strongly* G_1*-preopen and* G_1*-preirresolute surjection. then* g *is slightly* G_2*-precontinuous if and only if* $g \circ f$ *is slightly* G_1*-precontinuous*

Proof. The proof is clear. □

Remark 3.11. A subset A of a grill topological space (X, τ, G) is G-preopen if and only if for all $x \in A$, there exists $H_x \in GPO(X)$ such that $x \in H_x \subset A$.

Theorem 3.12. *A function* $f : (X, \tau, G) \rightarrow (Y, \sigma)$ *is slightly* G*-precontinuous if and only if the graph function* $g : X \rightarrow X \times Y$, *defined by* $g(x) = (x, f(x))$ *for each* $x \in X$ *is slightly* G*-precontinuous.*

Proof. Let $x \in X$ and let W be a clopen subset of $X \times Y$ containing $g(x)$. Then $W \cap (\{x\} \times Y)$ is clopen in $\{x\} \times Y$ containing $g(x)$. Also $\{x\} \times Y$ is homeomorphic to Y. Hence $\{y \in Y \mid (x, y) \in W\}$ is a clopen subset of Y. Since f is slightly G-precontinuous, $\cup\{f^{-1}(y) \mid (x, y) \in W\}$ is a G-preopen subset of (X, τ, G). Further, $x \in \cup\{f^{-1}(y) \mid (x, y) \in W\} \subset G^{-1}(W)$. Hence $g^{-1}(W)$ is G-preopen. Then g is slightly G-precontinuous. Conversely, Let F be a clopen subset of Y. Then $X \times F$ is a clopen subset of $X \times Y$. Since g is slightly G-precontinuous, $g^{-1}(X \times F)$ is a G-preopen subset of X. Also, $g^{-1}(X \times F) = f^{-1}(F)$. Hence f is slightly G-precontinuous. □

Definition 3.13. A grill topological space (X, τ, G) is said to be G-preconnected if X is not the union of two disjoint nonempty G-preopen sets of X.

Theorem 3.14. *If $f : (X, \tau, G) \to (Y, \sigma)$ is slightly G-precontinuous surjection and (X, τ, G) is G-preconnected, then Y is connected.*

Proof. Follows from the Theorem 3.2 and Definition 3.12. □

Theorem 3.15. *If f is slightly G-precontinuous function from a G-preconnected space (X, τ, G) onto space (Y, σ), then Y is not a discrete space.*

Proof. Since f is slightly G-precontinuous function, then for each clopen V of Y there exist $f^{-1}(V)$ is G-preopen in X and scince $X \neq f^{-1}(V_1) \cup f^{-1}(V_2)$, $V_1 \neq V_2$. Then Y is not discrate space. □

Theorem 3.16. *A grill topological space (X, τ, G) is G-preconnected if for every slightly G-precontinuous function from a space (X, τ, G) into any τ_0-space Y is constant.*

Proof. From Theorem 3.2 and Definition 3.12 □

Let $\{X_\alpha : \alpha \in I\}$ and $\{Y_\alpha : \alpha \in I\}$ be two families of topological spaces with the same index set I. The product space of $\{X_\alpha : \alpha \in I\}$ is denoted by $\Pi\{X_\alpha : \alpha \in I\}$(or simply ΠX_α). Let $f_\alpha : X_\alpha \to Y_\alpha$ be a function for each $\alpha \in I$. The product function $f : \Pi X_\alpha \to \Pi Y_\alpha$ is defined by $f(\{x_\alpha\}) = \{f_\alpha(x_\alpha)\}$ for each $\{x_\alpha\} \in \Pi X_\alpha$.

Theorem 3.17. *If a function $f : (X, \tau, G) \to \Pi Y_\alpha$ is slightly G-precontinuous, then $P_\alpha \circ f : (X, \tau, G) \to Y_\alpha$ is slightly G-precontinuous for each $\alpha \in I$, where P_α is the projection of ΠY_α into Y_α.*

Proof. Let V_α be any clopen set of Y_α. Then, $P_\alpha^{-1}(V_\alpha)$ is clopen in ΠY_α and hence $(P_\alpha \circ f)^{-1}(V_\alpha) = f^{-1}(P_\alpha^{-1}(V_\alpha))$ is G-preopen in X. Therefore, $P_\alpha \circ f$ is slightly G-precontinuous. □

4 Separation Axioms

In this section we begin by introducing a generalized classes of normal spaces, regular spaces, τ_1 and τ_2 in terms of grills.

Definition 4.1. A grill topological space (X, τ, G) is said to be
1- G-prei-τ_1 if for each pair of distinct points x and y of X, there exist G-preopen sets U and V of X such that $x \in U$ and $y \notin U$, and $y \in V$ and $x \notin V$.
2- G-prei-τ_2 if for each pair of distinct points x and y of X, there exist disjoint G-preopen sets U and V of X such that $x \in U$ and $y \in V$.
3- Clopen-τ_1 [11] if for each pair of distinct points x and y of X, there exist clopen sets U and V of X such that $x \in U$ and $y \notin U$, and $y \in V$ and $x \notin V$.
4- Clopen-τ_2 [11] if for each pair of distinct points x and y of X, there exist disjoint clopen sets U and V of X such that $x \in U$ and $y \in V$.

Theorem 4.2. *If $f : (X, \tau, G) \to (Y, \sigma)$ is slightly G-precontinuous injection and Y is a clopen-τ_1 space, then(X, τ, G) is a G-pre-τ_1 space.*

Proof. Suppose that Y is clopen-τ_1. For any two distinct points x and y of X, there exists clopen sets V and W of Y such that $f(x) \in V, f(y) \notin V, f(x) \notin W$ and $f(y) \in W$. Since f is slightly G-precontinuous, $f^{-1}(V)$ and $f^{-1}(W)$ are G-preopen subsets of (X, τ, G) such that $x \in f^{-1}(V), y \notin f^{-1}(V), x \notin f^{-1}(W)$ and $y \in f^{-1}(W)$. This shows that (X, τ, G) is a G-pre-τ_1 space.

□

Theorem 4.3. *If $f : (X, \tau, G) \to (Y, \sigma)$ is slightly G-precontinuous injection and Y is a clopen-τ_2 space, then (X, τ, G) is a G-pre-τ_2 space.*

Proof. For any pair of distinct points x and y of X, there exists clopen sets V and U of Y such that $f(x) \in U, f(y) \in V$ Since f is slightly G-precontinuous, $f^{-1}(U)$ and $f^{-1}(V)$ are G-preopen sets in (X, τ, G) containing x and y, respectively. Therefore, $f^{-1}(U) \cap f^{-1}(V) = \phi$ because $U \cap V = \phi$. This shows that the space (X, τ, G) is a G-pre-τ_2 space. □

Definition 4.4. A grill topological space (X, τ, G) is said to be G-preregular if for each closed set F and each point $x \notin F$, there exist disjoint G-preopen sets U and V of X such that $F \subset U$ and $x \in V$.

Example 4.5. *Let $X = \{a, b, c\}$, $\tau = \{X, \phi, \{a\}, \{b, c\}\}$ and $G = \{X, \{a\}, \{a, b\}\}$. Then a grill topological space (X, τ, G) is G-preregular space.*

Definition 4.6. A grill topological space (X, τ, G) is said to be G-prenormal if for any pair of disjoint closed subsets F_1 and F_2 of X, there exist disjoint G-preopen sets U and V of X such that $F_1 \subset U$ and $F_2 \subset V$.

Definition 4.7. A topological space (X, τ) is said to be:
1-Ultra Hausdroff [11] if every two distinct points of X can be separated by disjoint clopoen sets.
2- Ultra regular [11] if each pair of a point and a closed set not containing the point can be separated by disjoint clopen sets.
3- Ultra normal [11] if every two disjoint closed sets of X can be separated by clopen sets.

Theorem 4.8. *Let $f : (X, \tau, G) \to (Y, \sigma)$ be a slightly G-precontinuous injection then the following properties are hold:*
1-If (Y, σ) is ultra Hausdroff, then (X, τ, G) is G-pre-τ_2,
2- If (Y, σ) is ultra regular and f is open or closed, then (X, τ, G) is G-preregular,
3- If (Y, σ) is ultra normal and f is closed, then (X, τ, G) is G-prenormal.

Proof. (1) Let x_1, x_2 be two distinct points of X. Then since f is injective and Y is ultra Hausdroff, there exists clopen sets V_1 and V_2 of Y such that $f(x_1) \in V_1$, $f(x_2) \in V_2$, and $V_1 \cap V_2 = \phi$. By Theorem 3.2, $x_i \in f^{-1}(V_i) \in GPO(X)$ for $i = 1, 2$ and $f^{-1}(V_1) \cap f^{-1}(V_2) = \phi$. Thus, (X, τ, G) is G-pre-τ_2.

(2) Case-1: Suppose that f is open. Let $x \in X$ and U be an open set containing x. Then $f(U)$ is an open set of Y containing $f(x)$. Since Y is ultra regular, there exists a clopen set V such that $f(x) \in V \subset f(U)$. Since f is a slightly G-precontinuous injection, by Definition 3.1 $x \in f^{-1}(V) \subset U$ and $f^{-1}(V)$ is G-preclopen in X. Therefore, (X, τ, G) is G-preregular. Case-2: Suppose that f is closed. Let $x \in X$ and F be any closed se of X not containing x. Since f is injective and closed, $f(x) \notin f(F)$ and $f(F)$ is closed in Y. By the ultra regularity of Y, there exists a clopen set V such that $f(x) \in V \subset Y \setminus f(F)$. Therefore, $x \in f^{-1}(V)$ and $F \subset X \setminus f^{-1}(V)$. By Theorem 3.2, $f^{-1}(V)$ is an G preclopen set in (X, τ, G). Thus, (X, τ, G) is G preregular.
3- Similar to the proof of Statment(2). □

5 Conclusion

The study of grill topological spaces generalized most of near open sets, near continuous function, a new classes of functions, slightly G- semi-continuous functions and obtained a new decomposition of continuity in terms of grills. So we introduced a new class of function called slightly G- precontinuous functions and some basic properties, beside theorems of slightly G- precontinuous functions are investigated and relationships between this classes of functions.

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SOLVING ASSIGNMENT PROBLEM WITH LINGUISTIC COSTS

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Abstract – The purpose of this paper is to solve an assignment problem with linguistic costs/ times. In this paper, an assignment problem has been considered whose costs/times relating to the problem are linguistic variables and its solution methodology has been proposed. The solution method is based on fuzzy representation of linguistic costs/times and defuzzification of fuzzy costs/times. Here, a new defuzzification method based on statistical beta distribution has been used. Then the assignment problem has been converted into an assignment problem whose costs/times are crisp valued and solved by existing linear programming and/or Hungarian method. Finally, a numerical example has been considered and solved to illustrate the solution procedure.

Keywords – *Assignment problem, linguistic variables, fuzzy number, Beta distribution, Defuzzification*

1 Introduction

Assignment problem (AP) is the effective tool in solving real world decision making problem. AP plays a vital role in industrial management system. It is connected with the problem of production planning, telecommunication system, VLSI design problem etc. The main objective of the AP is how to assign/select several tasks/jobs in an effective way for which an optimal assignment can be performed in the best possible way. In an AP n resources (workers) and n activities (jobs) and effectiveness (which are generally represented in terms of cost, profit, time etc.) of each resource for each activity, the problem involves in assigning each resource to one and only one activity so that the total measure of effectiveness is optimized in an efficient way. In the existing literature, it has been noticed that several solution methods and algorithms, viz. linear programming method (Balinski [3], Barr et al. [4], Hung and Rom [10]), Hungarian algorithm (Khun [13]), Neural network method (Eberhardt et al. [9]), genetic algorithm (Avis et al. [2]) etc. were used to solve the assignment problem. But in reality the precise assumptions about

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cost/time related to assignment problem is not realistic in nature and it is observed that the decision makers may be not able to evaluate the same exactly due to lack of proper information. As a result, costs/times of the assignment problem are not measured precisely. To overcome these types of situation, the problem can be formulated using the concept of uncertainty and costs/times are treated as imprecise /uncertain in nature. In such cases fuzzy set theory plays an important role to handle such situation. In this paper, we have treated imprecise costs/times considering linguistic variables. A linguistic variable [22] is a variable whose values are represented using linguistic terms. To tackle the linguistic variables fuzzy representation is the best representation as qualitative data is converted into quantitative data using the same. Therefore, the fuzzy assignment problem provides an efficient framework which solves real-life problems with linguistic information. In the last few years, several attempts have been made in the existing literature for solving fuzzy assignment problem. Sakawa et al. [18] handled the problems on production work force assignment in a firm using interactive fuzzy programming for two level linear and linear functional programming models. Fuzzy assignment model that considers all persons to have same skill proposed by Chen [8].Based on restriction on qualification Huang and Zhang [11] presented a mathematical model for the fuzzy assignment problem and they transformed the model as certain assignment problem. Chen Liang-Hsuan and Lu Hai-Wen [12] developed a method for resolving assignments problem with multiple inadequate inputs and outputs in crisp form for each possible assignment using linear programming model. Majumdar and Bhunia [14] developed an efficient genetic algorithm to solve a generalized assignment problem with imprecise costs/times. X Ye and Jiuping Xu [20] developed a priority based genetic algorithm to a fuzzy vehicle routing assignment problem with connection network. Chen and Chen [7] well discussed fuzzy optimal solution of the assignment problem based on the ranking of generalized fuzzy numbers. Fuzzy risk analysis based on ranking fuzzy numbers using α-cuts proposed by Chen and Wang. Rommelfanger [17] proposed a ranking of fuzzy cost present in the fuzzy assignment problem, which takes more advantages over the existing fuzzy ranking methods. Kar et al/ [12] solved fuzzy generalized assignment problem with restriction on available cost. Solution of fuzzy assignment problem with ranking of generalized trapezoidal fuzzy numbers proposed by Thangavelu et al. [19]. Andal et al. [1] well discussed novel approach to minimum cost flow of fuzzy assignment problem with fuzzy membership functions. Pramanik and Biswas [15] developed multi-objective assignment problem with generalized trapezoidal fuzzy numbers.

In this paper, we have considered assignment problem with linguistic costs/times which is more realistic and has not been generally used in the existing literature. To solve the assignment problem, we have used fuzzy representation of linguistic costs/times and defuzzification of fuzzy costs/times. Here, a new defuzzification technique based on statistical beta distribution [16] has been used for defuzzification of fuzzy number. Then the assignment problem has been transformed into an assignment problem whose costs/times are crisp valued and solved by Linear programming method and/or Hungarian method [13]. Finally, a numerical example has been considered and solved in support of the solution method. The rest of this paper is organized as follows. Section 2 gives some basic definitions used in this paper. Section 3 presents the mathematical formulation of the assignment problem. Section 4 provides the solution methodology. Numerical example is presented in Section. Section 6 presents the results and discussion of a numerical example. Finally, conclusions are made in Section 7.

2 Basic Definitions

In this section, some basic concepts and methods used in this paper are briefly described.

2.1 Fuzzy Set Theory

The fuzzy set theory [21] is developed to deal with the extraction of the primary possible outcome from a set of multiple information that is presented in vague/imprecise terms. Fuzzy set theory handles imprecise data as probability distributions in terms of membership function.

Let X be a non empty set. A fuzzy set $\tilde{A}$ is defined by a membership function $\mu_{\tilde{A}}(x)$, which maps each element x in X to a real number in the interval $[0,1]$. The function $\mu_{\tilde{A}}(x)$ is called the membership function value of $x \in X$ in the fuzzy set $\tilde{A}$.

Definition 2.1 (Fuzzy number). A fuzzy number $\tilde{A}$ is a fuzzy set on the real line R, must satisfy the following conditions.

(i) There exists at least one $x_0 \in R$ for which $\mu_{\tilde{A}}(x_0) = 1$.

(ii) $\mu_{\tilde{A}}(x)$ is pair wise continuous.

(iii) $\tilde{A}$ must be convex and normal.

Definition 2.2 (Triangular fuzzy number). The triangular fuzzy number (TFN) is a normal fuzzy number denoted as $\tilde{A} = (a_1, a_2, a_3)$ where $a_1 \le a_2 \le a_3$ and its membership function $\mu_{\tilde{A}}(x): X \to [0,1]$ is defined by

$$\mu_{\tilde{A}}(x) = \begin{cases} \dfrac{x - a_1}{a_2 - a_1} & \text{if } a_1 \le x \le a_2 \\ 1 & \text{if } x = a_2 \\ \dfrac{a_3 - x}{a_3 - a_2} & \text{if } a_2 \le x \le a_3 \end{cases}$$

2.2 Linguistic Variables

Linguistic variable is an important concept in fuzzy set theory and plays a vital role in its applications, especially in the fuzzy expert system. Linguistic variable is a variable whose values are descriptive word in natural languages. For example "cost" is a linguistic variable, which can take the values as "Extremely Low", "Very Low", "Low", "Fairly Low" "Medium", "Fairly High", "High", "Very High" and "Extremely High".

2.3 Beta distribution:

A random variable X is said to have Beta distribution if its probability density function is as follows:

$$f(x) = \begin{cases} \dfrac{x^{p-1}(1-x)^{q-1}}{\beta(p,q)} & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

where $p > 0$ and $q > 0$ are parameters of Beta distribution.

The expected value/mean value $E(X)$ of Beta distribution is $\frac{p}{p+q}$ and is denoted by μ.

If, we have assumed that the density curve of Beta distribution is unimodal and $x^* \in (0,1)$ be the point at which the density function $f(x)$ has the maximum value then we have a relation as follows:

$$q-1=(p-1)\left(\frac{1-x^*}{x^*}\right)$$

Therefore, given both the values of p and x^*, the value of q can be obtained and the expected value of Beta distribution can be calculated by $\mu = \frac{p}{p+q}$.

Theorem 2.1. Let $[a,b]$ be a real interval on $\mathbb{R}$. Then every $\mu \in (a,b)$ there exists a unique $\mu' \in (a,b)$ and vice versa.

Proof. The uniqueness of μ' is proven by contradiction. Let $\mu \in (a,b)$ and $\mu' = \frac{\mu - a}{b-a}$, then $\mu' \in (0,1)$. Assuming that $\mu' \in (0,1)$ and $\mu'' \in (0,1)$ are two distinct numbers ($\mu' \neq \mu''$) corresponding to $\mu \in (a,b)$, then we have $\mu = \mu'(b-a)+a$ and $\mu = \mu''(b-a)+a$ and consequently we get $\mu' = \mu''$ which contradicts the assumption $\mu' \neq \mu''$. So every $\mu \in (a,b)$ there exists a unique $\mu' \in (a,b)$ and vice versa.

2.4 Defuzzification of Triangular Fuzzy Number Based on Statistical Beta Distribution

Let us consider a triangular fuzzy number $\tilde{A} = (a_1, a_2, a_3)$. Now, the projection of

$\tilde{A} = (a_1, a_2, a_3)$ on the interval $(0,1)$ is $\tilde{A} = (\frac{a_1 - a_1}{a_3 - a_1}, \frac{a_2 - a_1}{a_3 - a_1}, \frac{a_3 - a_1}{a_3 - a_1}) = (0, \frac{a_2 - a_1}{a_3 - a_1}, 1)$.

Now, we define the parameter p corresponding to the Beta distribution as follows:

$$p = \frac{a_2 - a_1}{a_3 - a_1} + 1 \tag{1}$$

From (1) it is clear that $m \geq 1$ and if $a_2 \neq a_1$, then the distribution curve is unimodal. If $a_3 - a_2 = -a_1$, which gives a symmetrical triangular fuzzy number, then $m = n = \frac{3}{2}$ and the distribution curve will be symmetric.

In the Beta distribution corresponding to the projection of fuzzy number $\tilde{A}=(a_1,a_2,a_3)$, we have $x^* = \dfrac{a_2-a_1}{a_3-a_1}$ and using the equation (1) we get $q=\dfrac{a_3-a_2}{a_3-a_1}+1$

Now we calculate the mean value of Beta distribution corresponding to the triangular fuzzy number $\tilde{A}=(a_1,a_2,a_3)$ is as follows:

$$\mu = \frac{p}{p+q} = \frac{a_3+a_2-2a_1}{3(a_3-a_1)} \tag{2}$$

The real number $\mu_{\tilde{A}}$ is the real number by transforming μ from the interval $(0,1)$ to the interval (a_1,a_3) is considered as the real number corresponding to the fuzzy number $\tilde{A}=(a_1,a_2,a_3)$ is as follows:

$$\mu_{\tilde{A}} = \mu(a_3-a_1)+a_1 = \frac{a_3+a_2-2a_1}{3(a_3-a_1)}(a_3-a_1)+a_1 = \frac{a_3+a_2+a_1}{3} \tag{3}$$

Remark 1. Using statistical beta distribution, the crisp real number $\mu_{\tilde{A}}$ corresponding to the triangular fuzzy number $\tilde{A}=(a_1,a_2,a_3)$ is $\mu_{\tilde{A}} = \dfrac{a_1+a_2+a_3}{3}$.

2.5 Ranking of Fuzzy Numbers

Let $\mu_{\tilde{A}}$ and $\mu_{\tilde{B}}$ are the crisp real numbers corresponding to two fuzzy numbers $\tilde{A}$ and $\tilde{B}$ respectively. Now, we have defined the ranking or order relations of two fuzzy numbers as follows:

(i) If $\mu_{\tilde{A}} > \mu_{\tilde{B}}$ then $\tilde{A} > \tilde{B}$

(ii) If $(\mu_{\tilde{A}} = \mu_{\tilde{B}}) \wedge (a_3 > \mathrm{b}_3)$ then $\tilde{A} > \tilde{B}$

(iii) If $(\mu_{\tilde{A}} = \mu_{\tilde{B}}, a_3 = \mathrm{b}_3) \wedge (a_2 > \mathrm{b}_2)$ then $\tilde{A} > \tilde{B}$

(iv) If $(\mu_{\tilde{A}} = \mu_{\tilde{B}}, a_3 = \mathrm{b}_3) \wedge (a_2 = \mathrm{b}_2)$ then $\tilde{A} = \tilde{B}$

Theorem 2.2. If a number is added to or subtracted from all of the entries of any row or column of the cost matrix, then an optimal assignment for the resulting cost matrix is also an optimal assignment for the original cost matrix.

2.6 The Hungarian Method

The following algorithm applies the Theorem 2 to a given $n\times n$ cost matrix to find an optimal assignment.

Step-1 Subtract the smallest entry in each row from all the entries of its row.

Step-2 Subtract the smallest entry in each column from all the entries of its column.

Step-3 Draw lines through appropriate rows and columns so that all the zero entries of the cost matrix are covered and the minimum number of such lines is used.

Step-4 Test for optimality: (i) If the minimum number of covering lines is n, an optimal assignment of zeros is possible and we are finished. (ii) If the minimum number of covering lines is less than n, an optimal assignment of zero is not yet possible. In that case proceed to next step.

Step-5 Determine the smallest entry not covered by any line, subtract this entry from each uncovered row, and then add it to each covered column and return to **Step-3**.

3 Mathematical Formulation of Assignment Problem

Let us consider a problem of assignment of n resources (workers) to n activities (jobs) so as to minimize the total cost or time in such a way that each resource can associate with one and only one job. The assignment problem is a special case of transportation problem. The cost or time of the assignment problem is same as that of a transportation problem except that availability at each of the resource and the requirement at each of the destination is unity. The cost matrix $(c_{ij})_{n\times n}$ is given as follows:

		Activities				
		J_1	J_2	$\ldots$	J_n	Availability
	W_1	c_{11}	c_{12}	$\ldots$	c_{1n}	1
Resource	W_2	c_{21}	c_{22}	$\ldots$	c_{2n}	1
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	W_n	c_{n1}	c_{n2}	$\ldots$	c_{nn}	1
	Requirement	1	1	$\ldots$	1	

Here c_{ij} denotes the cost associated with assigning i-th resource to j-th activity.

Let x_{ij} denotes the assignment of i-th resource to j-th activity such that

$$x_{ij} = \begin{cases} 1, \text{ if resource } i \text{ is assigned to activity } j \\ 0, \text{otherwise} \end{cases}$$

Mathematically assignment problem can be formulated as follows:

$$\text{Minimize } z = \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}x_{ij}$$

$$\text{subject to } \sum_{i=1}^{n} x_{ij} = 1,\ j = 1,2,\ldots,n$$

$$\sum_{j=1}^{n} x_{ij} = 1,\ i = 1,2,...,n$$
$$x_{ij} \in \{0,1\}$$

In most of the cases, it is seen that the decision maker decision is not a precise type due to lack of information between workers and assigned costs of the job. For this reason, let us consider that $\tilde{c}_{ij}$ be the imprecise cost. Here, we have considered imprecise cost $\tilde{c}_{ij}$ as a linguistic variable and it is represented by descriptive word in natural language. When the costs $\tilde{c}_{ij}$ are linguistic variables, then the assignment problem becomes

$$\text{Minimize } z = \sum_{i=1}^{n}\sum_{j=1}^{n} \tilde{c}_{ij} x_{ij}$$

subject to $\sum_{i=1}^{n} x_{ij} = 1,\ j = 1,2,...,n$

$$\sum_{j=1}^{n} x_{ij} = 1,\ i = 1,2,...,n$$
$$x_{ij} \in \{0,1\}$$

4 Solution Methodology

Proposed approach of solution methodology of linguistic assignment problem is as follows:

Step-1 In linguistic cost matrix, replace all the cost with linguistic variables by triangular fuzzy numbers. The resulting cost matrix is fuzzy cost matrix.

Step-2 Find the crisp number $\mu_{\tilde{c}_{ij}}$ of each $\tilde{c}_{ij}$ using statistical beta distribution and re-constructed crisp cost matrix.

Step-3 Test whether the given assignment problem is balanced or not.

(i) If it is a balanced then go to **Step-5.**
(ii) If it is unbalanced then go to **Step-4**.

Step-4 Introduce dummy rows and/or columns with zero cost so as to form a balanced assignment problem.

Step-5 Solved by linear programming problem method and/or by the Hungarian method to obtain the optimal assignments.

(a) In Case of linear programming method, solve

$$\text{Minimize } z = \sum_{i=1}^{n}\sum_{j=1}^{n} \mu_{\tilde{c}_{ij}} x_{ij}$$

subject to $\sum_{i=1}^{n} x_{ij} = 1,\ j = 1,2,...,n$

$$\sum_{j=1}^{n} x_{ij} = 1,\ i = 1,2,...,n$$

$$x_{ij} \in \{0,1\}$$

(b) In case of using Hungarian method to a given $n \times n$ cost matrix to find an optimal assignment follow the Hungarian Algorithm.

Step-6 Print the total minimum cost/time and optimal assignment for the assignment problem.

5 Numerical Example

Consider an assignment problem with four resources W_1, W_2, W_3, W_4 and fours jobs J_1, J_2, J_3, J_4 with assignment cost varying between 0$ to 50$. The cost matrix $(c_{ij})_{4\times4}$ is given whose elements are linguistic variables which are replaced by triangular fuzzy numbers. The linguistic cost matrix is given below:

Activities

		J_1	J_2	J_3	J_4
	W_1	Extremely Low	Low	Fairly High	Extremely High
Resource	W_2	Low	Very Low	High	Very High
	W_3	Medium	Extremely High	Very Low	Extremely Low
	W_4	Very High	Low	Fairly High	Fairly Low

Table 1. Linguistic term and their corresponding fuzzy data

Linguistic variable (qualitative data)	**Fuzzy data (quantitative data)**
Extremely Low	(0,2,5)
Very Low	(1,2,4)
Low	(4,8,12)
Fairly Low	(15,18,20)
Medium	(23,25,27)
Fairly High	(28,30,32)
High	(33,36,38)
Very High	(37,40,42)
Extremely High	(44,48,50)

Solution. The linguistic variables showing the qualitative data is converted into quantitative data using the Table 1. As the assignment cost varies between Rs.0 to Rs. 50.00 the minimum possible value is taken as 0 and the maximum possible value is taken as 50. Using Table 1, the fuzzy cost matrix is given below:

Activities

		J_1	J_2	J_3	J_4
	W_1	(0,2,5)	(4,8,12)	(28,30,32)	(44,48,50)
Resource	W_2	(4,8,12)	(1,2,4)	(33,36,38)	(37,40,42)
	W_3	(23,25,27)	(44,48,50)	(1,2,4)	(0,2,5)
	W_4	(37,40,42)	(4,8,12)	(28,30,32)	(15,18,20)

Here it is seen that the fuzzy assignment problem is balanced one. Now, we have obtained crisp data $\mu_{\tilde{c}_{ij}}$ of each $\tilde{c}_{ij}$ using statistical beta distribution and presented in Table 2.

Table 2. Crisp data $\mu_{\tilde{c}_{ij}}$ of each $\tilde{c}_{ij}$ using statistical beta distribution

Fuzzy data $\tilde{c}_{ij}$ (quantitative data)	**Crisp Data $\mu_{\tilde{c}_{ij}}$**
(0,2,5)	2.33
(1,2,4)	2.33
(4,8,12)	8.00
(15,18,20)	17.67
(23,25,27)	25.00
(28,30,32)	30.00
(33,36,38)	35.67
(37,40,42)	39.67
(44,48,50)	47.33

Now we calculate crisp real number $\mu_{\tilde{A}}$ corresponding to the fuzzy costs $\tilde{c}_{ij}$ and the reduced crisp cost matrix is as follows:

Activities

		J_1	J_2	J_3	J_4
	W_1	$\mu_{(0,2,5)}$	$\mu_{(4,8,12)}$	$\mu_{(28,30,32)}$	$\mu_{(44,48,50)}$
Resource	W_2	$\mu_{(4,8,12)}$	$\mu_{(1,2,4)}$	$\mu_{(33,36,38)}$	$\mu_{(37,40,42)}$
	W_3	$\mu_{(23,25,27)}$	$\mu_{(44,48,50)}$	$\mu_{(1,2,4)}$	$\mu_{(0,2,5)}$
	W_4	$\mu_{(37,40,42)}$	$\mu_{(4,8,12)}$	$\mu_{(15,18,20)}$	$\mu_{(15,18,20)}$

Then the assignment problem can be formulated in the following mathematical programming:

Minimize $z = z_1 + z_2 + z_3 + z_4$ (3)

Subject to

$x_{11} + x_{21} + x_{31} + x_{41} = 1$; $x_{12} + x_{22} + x_{32} + x_{42} = 1$; $x_{13} + x_{23} + x_{33} + x_{43} = 1$; $x_{14} + x_{24} + x_{34} + x_{44} = 1$

$x_{11} + x_{12} + x_{13} + x_{14} = 1$; $x_{21} + x_{22} + x_{23} + x_{24} = 1$; $x_{31} + x_{32} + x_{33} + x_{34} = 1$; $x_{41} + x_{42} + x_{43} + x_{44} = 1$

where

$z_1 = \mu_{(0,2,5)}x_{11} + \mu_{(4,8,12)}x_{21} + \mu_{(23,25,27)}x_{31} + \mu_{(37,40,42)}x_{41}$

$z_2 = \mu_{(4,8,12)}x_{12} + \mu_{(1,2,4)}x_{22} + \mu_{(44,48,50)}x_{32} + \mu_{(4,8,12)}x_{42}$

$z_3 = \mu_{(28,30,32)}x_{13} + \mu_{(33,36,38)}x_{23} + \mu_{(1,2,4)}x_{33} + \mu_{(15,18,20)}x_{43}$

$z_4 = \mu_{(44,48,50)}x_{14} + \mu_{(37,40,42)}x_{24} + \mu_{(0,2,5)}x_{34} + \mu_{(15,18,20)}x_{44}$

$x_{ij} \in \{0,1\}, i = 1,2,3,4$ and $j = 1,2,3,4$

Solving the problem (3) we get the solution as $x_{11} = 1$, $x_{22} = 1$, $x_{34} = 1$ and $x_{43} = 1$

Therefore the optimal assignment is $W_1 \to J_1, W_2 \to J_2, W_3 \to J_4, W_4 \to J_3$

Total optimal cost is $(0,2,5) + (1,2,4) + (0,2,5) + (15,18,20) = (16,24,34)$ and corresponding $\mu_{(16,24,34)} = 24.66$.

Also proceeding by Hungarian method, the optimal assignments are $W_1 \to J_1, W_2 \to J_2, W_3 \to J_4, W_4 \to J_3$ and $W_1 \to J_1, W_2 \to J_2, W_3 \to J_3, W_4 \to J_4$. For the optimal assignment $W_1 \to J_1, W_2 \to J_2, W_3 \to J_4, W_4 \to J_3$ the minimum cost is $(0,2,5) + (1,2,4) + (1,2,4) + (15,18,20) = (17,24,33)$ and corresponding $\mu_{(17,24,33)} = 24.66$, where as for optimal assignment $W_1 \to J_1, W_2 \to J_2, W_3 \to J_4, W_4 \to J_3$ the minimum cost is $(0,2,5) + (1,2,4) + (0,2,5) + (15,18,20) = (16,24,34)$ and corresponding $\mu_{(16,24,34)} = 24.66$.

6 Results and Discussion

Generally, in real life situation the assignment cost cannot be taken as precise one. So, in this paper we have solved an assignment problem considering each cost of the assignment is linguistic variable. For this purpose, to solve the assignment problem with linguistic cost we have used fuzzy representation of linguistic variables and defuzzification of fuzzy cost. Here we have considered triangular fuzzy number to represent linguistic variable and for defuzzification we have used newly defuzzification technique based on statistical beta distribution. For optimal solution we have used linear programming method and Hungarian method. By using linear programming method we have found optimal assignment as $x_{11} = 1$, $x_{22} = 1$, $x_{34} = 1$ and $x_{43} = 1$, where as using Hungarian method we have found multiple assignment either as $x_{11} = 1$, $x_{22} = 1$, $x_{34} = 1$ and $x_{43} = 1$ or $x_{11} = 1$, $x_{22} = 1$, $x_{33} = 1$ and $x_{44} = 1$. But it is seen that optimal assignment cost is 24.66 for all solution sets. Hence, the optimal assignments are $W_1 \to J_1, W_2 \to J_2, W_3 \to J_4, W_4 \to J_3$ and $W_1 \to J_1, W_2 \to J_2, W_3 \to J_3, W_4 \to J_4$. For the fuzzy optimal costs (17, 24, 33) and (16, 24, 34), we have obtained $\mu_{(17,24,33)} = 24.66$ and $\mu_{(16,24,34)} = 24.66$. Here, we have seen that $(\mu_{(17,24,33)} = \mu_{(16,24,34)}) \wedge (34 > 33)$ so (16, 24, 34) is greater than (17, 24, 33). Also from Figure 1, it is seen that intersection of (16, 24, 34) and (17, 24, 33) is (17, 24, 33). Therefore we have taken (17, 24, 33) is the optimal fuzzy assignment cost and its linguistic value has considered as Medium cost.

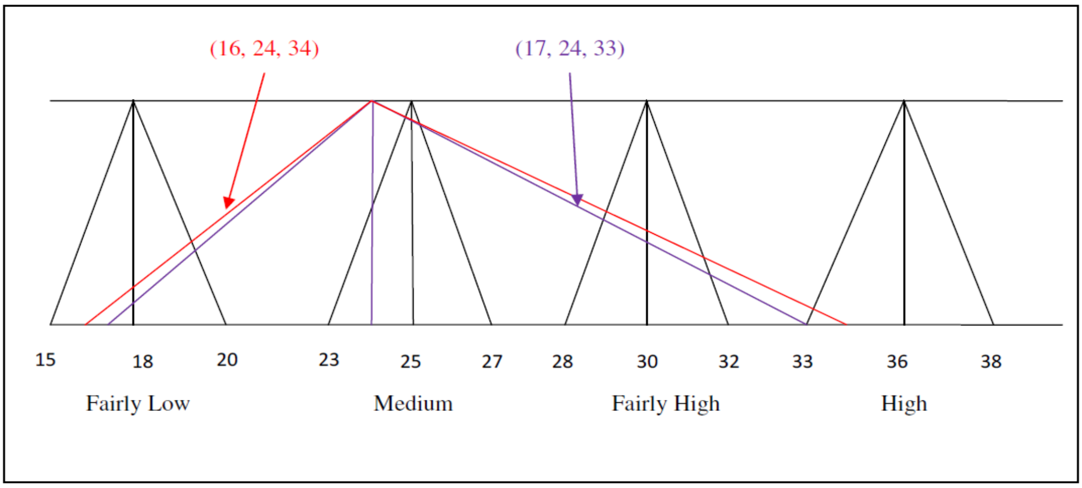

Figure 1. Pictorial representation of linguistic variables and fuzzy data

7 Conclusions

Here assignment problem with linguistic cost/time is presented and its solution methodology has been discussed. Assignment problem is one of the most important problems in decision making. In many real life situations, costs/times of the assignment problem are not precise. The assignment problem with linguistic costs is more realistic than the assignment problem with precise costs because most of the real life situations are uncertain. The proposed method described here is very simple and easy to implement. This method is based on fuzzy representation of linguistic costs/times and defuzzification of fuzzy costs/times. Here, a new defuzzification technique based on the statistical beta distribution has been used for defuzzification of fuzzy number. Then the assignment problem has been converted into an assignment problem whose costs/times are crisp /precise valued and solved by existing Hungarian method/ linear programming method. Finally, an assignment problem with linguistic cost has been solved and computed results has been presented and compared. It may be claimed that the proposed method attempted in this paper can be applied to solve realistic decision making problems in the near future involving linguistic parameters.

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ON NANO $\wedge_g$-CLOSED SETS

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Abstaract − In this paper, we introduce nano $\wedge_g$-closed sets in nano topological spaces. Some properties of nano $\wedge_g$-closed sets and nano $\wedge_g$-open sets are weaker forms of nano closed sets and nano open sets.

Keywords − *Nano $\wedge$-set, nano λ-closed set, nano $\wedge_g$-closed set.*

1 Introduction

In 2017, Rajasekaran et.al [5] introduced the notion of nano $\wedge$-sets in nano topological spaces and nano $\wedge$-set is a set H which is equal to its nano kernel and we introduced the notion of nano λ-closed set and nano λ-open sets. In this paper to introduce new classes of sets called nano $\wedge_g$-closed sets and nano $\wedge_g$-open sets in nano topological spaces. We also some properties of such sets and nano $\wedge_g$-closed sets and nano $\wedge_g$-open sets are weaker forms of nano closed sets and nano open sets.

2 Preliminaries

Throughout this paper $(U, \tau_R(X))$ (or X) represent nano topological spaces on which no separation axioms are assumed unless otherwise mentioned. For a subset H of a space $(U, \tau_R(X))$, $Ncl(H)$ and $Nint(H)$ denote the nano closure of H and the nano interior of H respectively. We recall the following definitions which are useful in the sequel.

Definition 2.1. *[4] Let U be a non-empty finite set of objects called the universe and R be an equivalence relation on U named as the indiscernibility relation. Elements*

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*Corresponding Author.

belonging to the same equivalence class are said to be indiscernible with one another. The pair (U, R) is said to be the approximation space. Let $X \subseteq U$.

1. *The lower approximation of X with respect to R is the set of all objects, which can be for certain classified as X with respect to R and it is denoted by $L_R(X)$. That is, $L_R(X) = \bigcup_{x \in U}\{R(x) : R(x) \subseteq X\}$, where R(x) denotes the equivalence class determined by x.*
2. *The upper approximation of X with respect to R is the set of all objects, which can be possibly classified as X with respect to R and it is denoted by $U_R(X)$. That is, $U_R(X) = \bigcup_{x \in U}\{R(x) : R(x) \cap X \neq \phi\}$.*
3. *The boundary region of X with respect to R is the set of all objects, which can be classified neither as X nor as not - X with respect to R and it is denoted by $B_R(X)$. That is, $B_R(X) = U_R(X) - L_R(X)$.*

Proposition 2.2. *[2] If (U, R) is an approximation space and $X, Y \subseteq U$; then*

1. $L_R(X) \subseteq X \subseteq U_R(X)$;
2. $L_R(\phi) = U_R(\phi) = \phi$ *and* $L_R(U) = U_R(U) = U$;
3. $U_R(X \cup Y) = U_R(X) \cup U_R(Y)$;
4. $U_R(X \cap Y) \subseteq U_R(X) \cap U_R(Y)$;
5. $L_R(X \cup Y) \supseteq L_R(X) \cup L_R(Y)$;
6. $L_R(X \cap Y) \subseteq L_R(X) \cap L_R(Y)$;
7. $L_R(X) \subseteq L_R(Y)$ *and* $U_R(X) \subseteq U_R(Y)$ *whenever* $X \subseteq Y$;
8. $U_R(X^c) = [L_R(X)]^c$ *and* $L_R(X^c) = [U_R(X)]^c$;
9. $U_R U_R(X) = L_R U_R(X) = U_R(X)$;
10. $L_R L_R(X) = U_R L_R(X) = L_R(X)$.

Definition 2.3. *[2] Let U be the universe, R be an equivalence relation on U and $\tau_R(X) = \{U, \phi, L_R(X), U_R(X), B_R(X)\}$ where $X \subseteq U$. Then by the Property 2.2, R(X) satisfies the following axioms:*

1. *U and $\phi \in \tau_R(X)$,*
2. *The union of the elements of any sub collection of $\tau_R(X)$ is in $\tau_R(X)$,*
3. *The intersection of the elements of any finite subcollection of $\tau_R(X)$ is in $\tau_R(X)$.*

That is, $\tau_R(X)$ is a topology on U called the nano topology on U with respect to X. We call $(U, \tau_R(X))$ as the nano topological space. The elements of $\tau_R(X)$ are called as nano open sets and $[\tau_R(X)]^c$ is called as the dual nano topology of $[\tau_R(X)]$.

Remark 2.4. *[2] If $[\tau_R(X)]$ is the nano topology on U with respect to X, then the set $B = \{U, \phi, L_R(X), B_R(X)\}$ is the basis for $\tau_R(X)$.*

Definition 2.5. *[2] If $(U, \tau_R(X))$ is a nano topological space with respect to X and if $H \subseteq U$, then the nano interior of H is defined as the union of all nano open subsets of H and it is denoted by $Nint(H)$.*

That is, $Nint(H)$ is the largest nano open subset of H. The nano closure of H is defined as the intersection of all nano closed sets containing H and it is denoted by $Ncl(H)$.

That is, $Ncl(H)$ is the smallest nano closed set containing H.

Definition 2.6. *[3] Let $(U, \tau_R(X))$ be a nano topological spaces and $H \subseteq U$. The nano $Ker(H) = \bigcap\{U : H \subseteq U, U \in \tau_R(X)\}$ is called the nano kernal of H and is denoted by $\mathcal{N}Ker(H)$.*

Definition 2.7. *[5] A subset H of a space $(U, \tau_R(X))$ is called*

1. *a nano $\wedge$-set if $H = NKer(H)$.*
2. *nano λ-closed if $H = L \cap F$ where L is a nano $\wedge$-set and F is nano closed.*

Definition 2.8. *A subset H of a nano topological space $(U, \tau_R(X))$ is called nano g-closed [1] if $Ncl(H) \subseteq G$, whenever $H \subseteq G$ and G is nano open.*

Remark 2.9. *[5]In a nano topological space, the concepts of nano g-closed sets and nano λ-closed sets are independent.*

3 Nano $\wedge_g$-closed Sets

Definition 3.1. *A subset H of a space $(U, \tau_R(X))$ is called nano λ-open if $H^c = U - H$ is nano λ-closed.*

Example 3.2. *Let $U = \{a, b, c, d\}$ with $U/R = \{\{a\}, \{c\}, \{b, d\}\}$ and $X = \{a, b\}$. Then the nano topology $\tau_R(X) = \{\phi, \{a\}, \{b, d\}, \{a, b, d\}, U\}$. Then $\{a\}$ is nano λ-open.*

Definition 3.3. *A subset H of a space $(U, \tau_R(X))$ is called a nano $\wedge_g$-closed set if $Ncl(H) \subseteq G$, whenever $H \subseteq G$ and G is nano λ-open.*

The complement of nano $\wedge_g$-open if $H^c = U - H$ is nano $\wedge_g$-closed.

Example 3.4. *In Example 3.2, then $\{a, c\}$ is nano $\wedge_g$-closed set.*

Lemma 3.5. *In a space $(U, \tau_R(X))$, every nano open set is nano $\wedge_g$-open but not conversely*

Remark 3.6. *The converse of statements in Lemma 3.5 are not necessarily true as seen from the following Example.*

Example 3.7. *In Example 3.2, then $\{b\}$ is nano $\wedge_g$-open but not nano open.*

Remark 3.8. *The following example shows that the concepts of nano $\wedge_g$-closed sets and nano λ-closed are independent for each other.*

Example 3.9. *In Example 3.2,*

1. *then $\{b, c\}$ is nano $\wedge_g$-closed but not nano λ-closed.*
2. *then $\{a\}$ is nano λ-closed but not nano $\wedge_g$-closed.*

Theorem 3.10. *In a space $(U, \tau_R(X))$, the union of two nano $\wedge_g$-closed sets is nano $\wedge_g$-closed.*

Proof. Let $H \cup Q \subseteq G$, then $H \subseteq G$ and $Q \subseteq G$ where G is nano λ-open. As H and Q are $\wedge_g$-closed, $Ncl(H) \subseteq G$ and $Ncl(Q) \subseteq G$. Hence $Ncl(H \cup Q) = Ncl(H) \cup Ncl(Q) \subseteq G$.

Example 3.11. *In Example 3.2, then $H = \{a, c\}$ and $Q = \{b, c\}$ is nano $\wedge_g$-closed. Clearly $H \cup Q = \{a, b, c\}$ is nano $\wedge_g$-closed.*

Theorem 3.12. *In a space $(U, \tau_R(X))$, the intersection of two nano $\wedge_g$-open sets is nano $\wedge_g$-open.*

Proof. Obvious by Theorem 3.10.

Example 3.13. *Let $U = \{a, b, c, d\}$ with $U/R = \{\{a\}, \{b\}, \{c, d\}\}$ and $X = \{b, d\}$. Then the nano topology $\tau_R(X) = \{\phi, \{b\}, \{c, d\}, \{b, c, d\}, U\}$ Then $H = \{b, c\}$ and $Q = \{b, d\}$ is nano $\wedge_g$-open. Clearly $H \cap Q = \{b\}$ is nano $\wedge_g$-open.*

Remark 3.14. *In a space $(U, \tau_R(X))$, the intersection of two nano $\wedge_g$-closed sets but not nano $\wedge_g$-closed.*

Example 3.15. *Let $U = \{1, 2, 3\}$ with $U/R = \{\{1\}, \{2, 3\}\}$ and $X = \{1\}$. Then the nano topology $\tau_R(X) = \{\phi, \{1\}, U\}$, Then $H = \{1, 21\}$ and $Q = \{1, 3\}$ is nano $\wedge_g$-closed. Clearly $H \cap Q = \{1\}$ is but not nano $\wedge_g$-closed.*

Theorem 3.16. *In a space $(U, \tau_R(X))$ is nano $\wedge_g$-closed, then $Ncl(H) - H$ contains no nonempty nano closed.*

Proof. Let P be a nano closed subset contains in $Ncl(H) - H$. Clearly $H \subseteq P^c$ where H is nano $\wedge_g$-closed and P^c is an nano open set of U. Thus $Ncl(H) \subseteq P^c$ (or) $P \subseteq (Ncl(H))^c$. Then $P \subseteq (Ncl(H))^c \cap (Ncl(H) - H) \subseteq (Ncl(H))^c \cap Ncl(H) = \phi$. This is show that $P = \phi$.

Theorem 3.17. *A subset H of a space $(U, \tau_R(X))$ is nano $\wedge_g$-closed $\Longleftrightarrow$ $Ncl(H) - H$ contains no nonempty nano λ-closed.*

Proof. Necessity. Assume that H is nano $\wedge_g$-closed. Let K be a nano λ-closed subset of $Ncl(H) - H$. Then $H \subseteq K^c$. Since H is nano $\wedge_g$-closed, we have $Ncl(H) \subseteq K^c$. Consequently $K \subseteq (Ncl(H))^c$. Hence $K \subseteq Ncl(H) \cap (Ncl(H))^c = \phi$. Therefore K is empty.

Sufficiency. Assume that $Ncl(H) - H$ contains no nonempty nano λ-closed sets. Let $H \subseteq C$ and C be a nano λ-open. If $Ncl(H) \nsubseteq C$, then $Ncl(H) \cap C^c$ is a nonempty nano λ-closed subset of $Ncl(H) - H$. Therefore H is nano $\wedge_g$-closed.

Theorem 3.18. *In a space $(U, \tau_R(X))$, if H is a nano $\wedge_g$-closed and $H \subseteq Q \subseteq Ncl(H)$, then Q is a nano $\wedge_g$-closed.*

Proof. Let $H \subseteq Q$ and $Ncl(Q) \subseteq Ncl(H)$. Hence $(Ncl(Q) - Q) \subseteq (Ncl(H) - H)$. But by Theorem 3.17, $Ncl(H) - H$ contains no nonempty nano λ-closed subset of U and hence neither does $Ncl(B) - B$. By Theorem 3.17, Q is nano $\wedge_g$-closed.

Theorem 3.19. *In a space $(U, \tau_R(X))$, if H is nano λ-open and nano $\wedge_g$-closed, then hence H is nano closed.*

Proof. Since H is nano λ-open and nano λ-closed, $Ncl(H) \subseteq H$ and hence H is nano closed.

Theorem 3.20. *For each $x \in U$, either $\{x\}$ is nano λ-closed (or) $\{x\}^c$ is nano $\wedge_g$-closed.*

Proof. Assume $\{x\}$ is not nano λ-closed. Then $\{x\}^c$ is not nano λ-open and the only nano λ-open set containing $\{x\}^c$ is the space of U itself. Therefore $Ncl(\{x\}^c) \subseteq U$ and so $\{x\}^c$ is nano $\wedge_g$-closed.

Theorem 3.21. *In a space $U, \tau_R(X)$, H is nano $\wedge_g$-open $\iff$ $P \subseteq Nint(H)$ whenever P is nano λ-closed and $P \subseteq H$.*

Proof. Assume that $P \subseteq Nint(H)$ whenever P is nano λ-closed and $P \subseteq H$. Let $H^c \subseteq C$, where C is nano λ-open. Hence $C^c \subseteq H$. By assumption $C^c \subseteq Nint(H)$ which implies that $(Nint(H))^c \subseteq C$, so $Ncl(H^c) \subseteq C$. Hence H^c is nano $\wedge_g$-closed that is, H is nano $\wedge_g$-open.

Conversely, let H be nano $\wedge_g$-open. Then H^c is nano $\wedge_g$-closed. Also let P be a nano λ-closed set contained in H. Then P^c is nano λ-open. Therefore whenever $H^c \subseteq P^c$, $Ncl(H^c) \subseteq P^c$. This implies that $P \subseteq (Ncl(H^c))^c = Nint(H)$. Thus $H \subseteq Nint(H)$.

Theorem 3.22. *In a space $(U, \tau_R(X))$, H is $\wedge_g$-open $\iff$ $C = U$ whenever C is nano λ-open and $Nint(H) \cup H^c \subseteq C$.*

Proof. Let H be a nano $\wedge_g$-open, C be a nano λ-open and $Nint(H) \cup H^c \subseteq C$. Then $C^c \subseteq (Nint(H))^c \cap (H^c)^c = (Nint(H))^c - H^c) = Ncl(H^c) - H^c$. Since H^c is nano $\wedge_g$-closed and C^c is nano λ-closed, by Theorem 3.17 it follows that $C^c = \phi$. Therefore $U = C$. Conversely, suppose that P is nano λ-closed and $P \subseteq H$. Then $Nint(H) \cup H^c \subseteq Nint(H) \cup P^c$. It follows that $Nint(H) \cup P^c = U$ and hence $P \subseteq Nint(H)$. Therefore H is nano $\wedge_g$-open.

Theorem 3.23. *In a space $(U, \tau_R(X))$, if $Nint(H) \subseteq Q \subseteq H$ and H is nano $\wedge_g$-open, then Q is nano $\wedge_g$-open.*

Proof. Assume $Nint(H) \subseteq Q \subseteq H$ and H is nano $\wedge_g$-open. Then $H^c \subseteq Q^c \subseteq Ncl(H^c)$ and H^c is nano $\wedge_g$-closed. By Theorem 3.18, Q is nano $\wedge_g$-open.

Theorem 3.24. *In a space $(U, \tau_R(X))$, H is nano $\wedge_g$-closed $\iff$ $Ncl(H) - H$ is nano $\wedge_g$-open.*

Proof. Necessity. Assume that H is nano $\wedge_g$-closed. Let $P \subseteq Ncl(H) - H$, where P is nano λ-closed. By Theorem 3.17, $P = \phi$, Therfore $P \subseteq Nint(Ncl(H) - H)$ and by Theorem 3.21, $Ncl(H) - H$ is nano $\wedge_g$-open.

Sufficiency. Let $H \subseteq C$ where C is a nano λ-open set. Then $Ncl(H) \cap C^c \subseteq Ncl(H) \cap H^c = Ncl(H) - H$. Since $Ncl(H) \cap C^c$ is nano λ-closed and $Ncl(H) - H$ is nano $\wedge_g$-open, by Theorem 3.21, we have $Ncl(H) \cap C^c \subseteq Nint(Ncl(H) - H) = \phi$. Hence H is nano $\wedge_g$-closed.

Theorem 3.25. *In a nano topological space $(U, \tau_R(X))$, the following properties are equivalent:*

1. *H is nano $\wedge_g$-closed.*
2. *$Ncl(H) - H$ contains no nonempty nano λ-closed set.*
3. *$Ncl(H) - H$ is nano $\wedge_g$-open.*

Proof. This follows from by Theorems 3.17 and 3.24.

Definition 3.26. *A subset H of a space $(U, \tau_R(X))$ is called*

1. *a nano ${}_g\wedge$-closed set if $N\lambda cl(H) \subseteq G$, whenever $H \subseteq G$ and G is nano open.*
2. *a nano $\wedge$-g-closed set if $N\lambda cl(H) \subseteq G$, whenever $H \subseteq G$ and G is nano λ-open.*

The complement of the above mentioned sets are called their respective open sets.

Example 3.27. *In Example 3.2, then $\wp(U)$ is nano ${}_g\wedge$-closed and nano $\wedge$-g-closed.*

Remark 3.28. *For a subset of a space $(U, \tau_R(X))$, we have the following implications:*

$$\begin{array}{ccc}
\textbf{\textit{nano closed}} & \rightarrow & \textbf{\textit{nano }}\lambda\textbf{\textit{-closed}} \\
\downarrow & & \downarrow \\
\textbf{\textit{nano }}\wedge_g\textbf{\textit{-closed}} & \rightarrow & \textbf{\textit{nano }}\wedge\textbf{\textit{-}}g\textbf{\textit{-closed}} \\
\downarrow & & \downarrow \\
\textbf{\textit{nano }}g\textbf{\textit{-closed}} & \rightarrow & \textbf{\textit{nano }}{}_g\wedge\textbf{\textit{-closed}}
\end{array}$$

None of the above implications is reversible.

Theorem 3.29. *In a space $(U, \tau_R(X))$, H is nano $\wedge_g$-closed $\Longleftrightarrow$ $N\lambda cl(\{x\}) \cap H \neq \phi$ for every $x \in Ncl(H)$.*

Proof. Necessity. Suppose that $N\lambda cl(\{x\}) \cap H = \phi$ for some $x \in Ncl(H)$. Then $U - N\lambda cl(\{x\})$ is a nano λ-open set containing H. Furthermore, $x \in Ncl(H) - (U - N\lambda cl(\{x\}))$ and hence $Ncl(H) \nsubseteq U - N\lambda cl(\{x\})$. This shows that H is not nano $\wedge_g$-closed.

Sufficiency. Suppose that H is not nano $\wedge_g$-closed. There exist a nano λ-open set G containing H such that $Ncl(H) - G \neq \phi$. There exist $x \in Ncl(H)$ such that $x \notin G$, hence $N\lambda cl(\{x\}) \cap G = \phi$. Therefore, $N\lambda cl(\{x\}) \cap H = \phi$ for some $x \in Ncl(H)$.

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VECTOR APPROACH TO A NEW GENERALIZATION OF FIBONACCI POLYNOMIAL

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Abstaract − In this paper we introduce a new generalization of Fibonacci polynomial and vectors of length d are defined for these Polynomials. Using these vectors some properties for Generalized Fibonacci Polynomial (GFP) are established.

Keywords − *Fibonacci polynomial, Lucas polynomial, Pell polynomial, Pell-Lucas polynomial, Jacobsthal polynomial, Jacobsthal-Lucas polynomial.*

1 Introduction

The Generalized Fibonacci Polynomial (GFP) is a natural generalization of the Fibonacci polynoial $F_n(x)$ which defined recurrently by $F_{n+1}(x) = xF_n(x) + F_{n-1}(x)$, with $F_0(x) = 0$, $F_1(0) = 1$, for $n \geq 1$. Fibonacci Polynomial is the topic of wide interest, for the literature of these Polynomials one can refer to articles by many authors like [1, 5, 6, 7, 11, 13, 14, 15, 16, 17, 18, 19, 20, 21, 25, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 2, 3, 4, 10, 9]. The study of properties for Fibonacci polynomials and Lucas polynomials have received less attention than numerical sequences. The identities involving Fibonacci and Lucas sequences extend naturally to the GFP that satisfy closed formulas similar to the Binet formulas satisfied by Fibonacci and Lucas sequences. We denote the GFP of Lucas type and Fibonacci type by $\widehat{F}_n(x)$ and $\widehat{L}_n(x)$ respectively. We adapt some known identities given for Fibonacci or Lucas polynomials to the $\widehat{F}_n(x)$ and $\widehat{L}_n(x)$. Most of the identities for Fibonacci and Lucas polynomials that we extend to GFP can be found in the books, articles on Fibonacci and Lucas sequences and their applications, [2, 3, 4, 8, 9, 10, 22, 23, 24, 26, 12] The ultimate aim of this paper is to introduce new generalization $\widehat{F}_n(x)$ and $\widehat{L}_n(x)$ of Fibonacci and Lucas polynomials and establish a collection of identities for the $\widehat{F}_n(x)$ and $\widehat{L}_n(x)$ using vector method.

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2 Generalized Fibonacci Polynomial (GFP)

In this section we introduce the new generalized Fibonacci polynomials $\widehat{F}_n(x)$ and $\widehat{L}_n(x)$.

Definition 2.1. The generalized Fibonacci polynomial $\widehat{F}_n(x)$ is defined by the recurrence relation

$$\widehat{F}_{n+1}(x) = x\widehat{F}_n(x) + \widehat{F}_{n-1}(x) \quad \text{with} \quad \widehat{F}_0(x) = 0, \quad \widehat{F}_1(x) = x^2 + 4, \text{ for } n \geq 1 \tag{1}$$

Definition 2.2. The generalized Lucas polynomial $\widehat{L}_n(x)$ is defined by the recurrence relation

$$\widehat{L}_{n+1}(x) = x\widehat{L}_n(x) + \widehat{L}_{n-1}(x) \quad \text{with} \quad \widehat{L}_0(x) = 2x^2 + 8, \quad \widehat{L}_1(x) = x^3 + 4x, \text{ for } n \geq 1 \tag{2}$$

Polynomial	Initial value $G_0(x) = a(x)$	Initial value $G_1(x) = b(x)$	Recursive Formula $G_{n+1}(x) = a(x)G_n(x) + b(x)G_{n-1}(x)$
Fibonacci	0	1	$F_{n+1}(x) = F_n(x) + F_{n-1}(x)$
Lucas	2	x	$L_{n+1}(x) = L_n(x) + L_{n-1}(x)$
Pell	0	1	$P_{n+1}(x) = 2xP_n(x) + P_{n-1}(x)$
Pell-Lucas	2	2x	$Q_{n+1}(x) = 2xQ_n(x) + Q_{n-1}(x)$
Jacobsthal	0	1	$J_{n+1}(x) = J_n(x) + 2xJ_{n-1}(x)$
Jacobsthal-Lucas	2	1	$j_{n+1}(x) = j_n(x) + 2xj_{n-1}(x)$
Generalized Fibonacci	0	$x^2 + 4$	$\widehat{F}_{n+1}(x) = x\widehat{F}_n(x) + \widehat{F}_{n-1}(x)$
Generalized Lucas	$2x^2 + 8$	$x^3 + 4x$	$\widehat{L}_{n+1}(x) = x\widehat{L}_n(x) + \widehat{L}_{n-1}(x)$

Table 1: **Recurrence relation of some GFP.**

Characteristic equation of the initial recurrence relation (1 and 2) is,

$$r^2 - xr - 1 = 0 \tag{3}$$

Characteristic roots of (3) are

$$r_1(x) = \frac{x + \sqrt{x^2 + 4}}{2}, \quad r_2(x) = \frac{x - \sqrt{x^2 + 4}}{2} \tag{4}$$

Characteristic roots (4) satisfy the properties

$$r_1(x) - r_2(x) = \sqrt{x^2 + 4} = \sqrt{\Delta(x)}, \quad r_1(x) + r_2(x) = x, \quad r_1(x)r_2(x) = -1 \tag{5}$$

Binet forms for both $\widehat{F}_n(x)$ and $\widehat{L}_n(x)$ are given by

$$\widehat{F}_n(x) = r_1(x)^n - r_2(x)^n \tag{6}$$

$$\widehat{L}_n(x) = \left[r_1(x)^2 + r_2(x)^2 + 2\right] \left[r_1(x)^n + r_2(x)^n)\right] \tag{7}$$

Definition 2.3. (Generalized Fibonacci polynomial vector of length d)
For all integers n, Generalized Fibonacci polynomial vector $\overrightarrow{F_n^d}(x)$ and Generalized Lucas polynomial vector $\overrightarrow{L_n^d}(x)$ of length d are defined as follows

$$\overrightarrow{F_n^d}(x)=\begin{bmatrix}\widehat{F}_n(x)\\ \widehat{F}_{n+1}(x)\\ \widehat{F}_{n+2}(x)\\ \vdots\\ \widehat{F}_{n+d-1}(x)\end{bmatrix}\text{ and }\overrightarrow{L_n^d}(x)=\begin{bmatrix}\widehat{L}_n(x)\\ \widehat{L}_{n+1}(x)\\ \widehat{L}_{n+2}(x)\\ \vdots\\ \widehat{L}_{n+d-1}(x)\end{bmatrix}$$

Definition 2.4. (Reverse Generalized Fibonacci polynomial vector of length d)
For all integers n, Reverse Generalized Fibonacci polynomial vector $\overrightarrow{f_n^d}(x)$ and Reverse Generalized Lucas polynomial vector $\overrightarrow{l_n^d}(x)$ of length d are defined as follows

$$\overrightarrow{f_n^d}(x)=\begin{bmatrix}\widehat{L}_{n+d-1}(x)\\ \widehat{L}_{n+d-2}(x)\\ \widehat{L}_{n+d-3}(x)\\ \vdots\\ \widehat{L}_n(x)\end{bmatrix}\text{ and }\overrightarrow{l_n^d}(x)=\begin{bmatrix}\widehat{L}_{n+d-1}(x)\\ \widehat{L}_{n+d-2}(x)\\ \widehat{L}_{n+d-3}(x)\\ \vdots\\ \widehat{L}_n(x)\end{bmatrix}$$

Definition 2.5. (Vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ and $\vec{d}$)
For all integers n, the vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ and $\vec{d}$ of length d are defined as follows

$$\vec{a}=\begin{bmatrix}1\\ r_1(x)\\ r_1^2(x)\\ \vdots\\ r_1^{d-2}(x)\\ r_1^{d-1}(x)\end{bmatrix},\ \vec{b}=\begin{bmatrix}1\\ r_2(x)\\ r_2^2(x)\\ \vdots\\ r_2^{d-2}(x)\\ r_2^{d-1}(x)\end{bmatrix},\ \vec{c}=\begin{bmatrix}r_1^{d-1}(x)\\ r_1^{d-2}(x)\\ r_1^{d-3}(x)\\ \vdots\\ r_1(x)\\ 1\end{bmatrix}\text{ and }\vec{d}=\begin{bmatrix}r_2^{d-1}(x)\\ r_2^{d-2}(x)\\ r_2^{d-3}(x)\\ \vdots\\ r_2(x)\\ 1\end{bmatrix}$$

Definition 2.6. Define $d \times d$ matrix T by

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & 0 & 0 & 0 & 1 \\ 0 & \cdots & 0 & 1 & 0 & 1 & x \end{bmatrix}_{d \times d}$$

Definition 2.7. Define $d \times d$ matrix S by

$$\begin{bmatrix} x & 1 & 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & 0 & 1 & 0 & 0 \\ 0 & \cdots & 0 & 0 & 0 & 1 & 0 \end{bmatrix}_{d \times d}$$

The matrices T and S have characteristic polynomial $r^{(m-2)}(r^2 - xr - 1)$. The non zero eigenvalues of both are $r_1(x)$ and $r_2(x)$ and eigenspace associated with $r_1(x)$ and $r_2(x)$ for T are spanned respectively by $\overrightarrow{a}$, $\overrightarrow{b}$ and for S are spanned respectively by $\overrightarrow{c}$ and $\overrightarrow{d}$.

3 Auxiliary Results

Theorem 3.1. (Binet form for $\overrightarrow{F_n^d}(x)$, $\overrightarrow{L_n^d}(x)$, $\overrightarrow{f_n^d}(x)$ and $\overrightarrow{l_n^d}(x)$) For all integers n,

$$\overrightarrow{F_n^d}(x) = [r_1(x) - r_2(x)][r_1^n(x)\overrightarrow{a} - r_2^n(x)\overrightarrow{b}] \tag{8}$$

$$\overrightarrow{L_n^d}(x) = [r_1(x) - r_2(x)]^2[r_1^n(x)\overrightarrow{a} + r_2^n(x)\overrightarrow{b}] \tag{9}$$

$$\overrightarrow{f_n^d}(x) = [r_1(x) - r_2(x)][r_1^n(x)\overrightarrow{c} - r_2^n(x)\overrightarrow{d}] \tag{10}$$

$$\overrightarrow{l_n^d}(x) = [r_1(x) - r_2(x)]^2[r_1^n(x)\overrightarrow{c} + r_2^n(x)\overrightarrow{d}] \tag{11}$$

Theorem 3.2. For all integers n

$$\overrightarrow{F_{n+1}^d}(x) = T\,\overrightarrow{F_n^d}(x)$$

$$\overrightarrow{L_{n+1}^d}(x) = T\,\overrightarrow{L_n^d}(x)$$

$$\overrightarrow{f_{n+1}^d}(x) = S\,\overrightarrow{F_n^d}(x)$$

$$\overrightarrow{l_{n+1}^d}(x) = S\,\overrightarrow{F_n^d}(x)$$

Theorem 3.3. For all integers $t \geq n+1$

$$\overrightarrow{F_{n+1}^{d}}(x) = T^{(n-t+1)}\,\overrightarrow{F_{n}^{d}}(x)$$
$$\overrightarrow{L_{n+1}^{d}}(x) = T^{(n-t+1)}\,\overrightarrow{L_{n}^{d}}(x)$$
$$\overrightarrow{f_{n+1}^{d}}(x) = S^{(n-t+1)}\,\overrightarrow{F_{n}^{d}}(x)$$
$$\overrightarrow{l_{n+1}^{d}}(x) = S^{(n-t+1)}\,\overrightarrow{F_{n}^{d}}(x)$$

Theorem 3.4. For $k > 0$

$$\vec{a}\cdot\vec{a}=\vec{c}\cdot\vec{c}=\|\vec{a}\|^2=\|\vec{c}\|^2=\begin{cases}\dfrac{\widehat{F}_n(x)r_1^{d-1}(x)}{(r_1(x)-r_2(x))}, & \text{If } d \text{ is even;}\\ \dfrac{\widehat{L}_n(x)r_1^{d-1}(x)}{(r_1(x)-r_2(x))^2}, & \text{If } d \text{ is odd}\end{cases}$$

$$\vec{b}\cdot\vec{b}=\vec{d}\cdot\vec{d}=\|\vec{b}\|^2=\|\vec{d}\|^2=\begin{cases}\dfrac{-\widehat{F}_n(x)r_2^{d-1}(x)}{(r_1(x)-r_2(x))}, & \text{If } d \text{ is even;}\\ \dfrac{\widehat{L}_n(x)r_2^{d-1}(x)}{(r_1(x)-r_2(x))^2}, & \text{If } d \text{ is odd}\end{cases}$$

$$\vec{a}\cdot\vec{b}=\vec{c}\cdot\vec{d}=\begin{cases}0, & \text{If } d \text{ is even;}\\ 1, & \text{If } d \text{ is odd}\end{cases}$$

Theorem 3.5. For all positive integer d and for all integers n_1 and n_2

$$\begin{aligned}\overrightarrow{F_{n_1}^{d}}(x)\cdot\overrightarrow{F_{n_2}^{d}}(x) &= \overrightarrow{f_{n_1}^{d}}(x)\cdot\overrightarrow{f_{n_2}^{d}}(x)\\ &= \begin{cases}\dfrac{\widehat{F}_d(x)\widehat{F}_{d+n_1+n_2-1}(x)}{(r_1(x)-r_2(x))^2}, & \text{If } d \text{ is even;}\\ \widehat{L}_d(x)\widehat{L}_{d+n_1+n_2-1}(x)-\dfrac{(-1)^{n_1}\widehat{L}_{n_2-n_1}(x)}{(r_1(x)-r_2(x))}, & \text{If } d \text{ is odd}\end{cases}\end{aligned}$$

$$\begin{aligned}\overrightarrow{L_{n_1}^{d}}(x)\cdot\overrightarrow{L_{n_2}^{d}}(x) &= \overrightarrow{l_{n_1}^{d}}(x)\cdot\overrightarrow{l_{n_2}^{d}}(x)\\ &= \begin{cases}\widehat{F}_d(x)\widehat{F}_{d+n_1+n_2-1}(x), & \text{If } d \text{ is even;}\\ \dfrac{\widehat{L}_d(x)\widehat{L}_{d+n_1+n_2-1}(x)}{(r_1(x)-r_2(x))^2}+\dfrac{(-1)^{n_1}\widehat{L}_{n_2-n_1}(x)}{(r_1(x)-r_2(x))}, & \text{If } d \text{ is odd}\end{cases}\end{aligned}$$

$$\begin{aligned}\overrightarrow{F_{n_1}^{d}}(x)\cdot\overrightarrow{L_{n_2}^{d}}(x) &= \overrightarrow{f_{n_1}^{d}}(x)\cdot\overrightarrow{l_{n_2}^{d}}(x)\\ &= \begin{cases}\widehat{F}_d(x)\widehat{L}_{d+n_1+n_2-1}(x), & \text{If } d \text{ is even;}\\ \dfrac{\widehat{L}_d(x)\widehat{F}_{d+n_1+n_2-1}(x)}{(r_1(x)-r_2(x))^2}+\dfrac{(-1)^{n_1}\widehat{F}_{n_2-n_1}(x)}{(r_1(x)-r_2(x))}, & \text{If } d \text{ is odd}\end{cases}\end{aligned}$$

Theorem 3.6. For all positive integer d and for all integer n

$$\| \overrightarrow{F_{n_1}^{d}}(x)\|^2 = \| \overrightarrow{f_{n_1}^{d}}(x)\|^2 = \begin{cases} \widehat{F}_d(x)\widehat{F}_{d+2n-1}(x), & \text{If } d \text{ is even;} \\ \widehat{L}_d(x)\widehat{L}_{d+2n-1}(x) - \dfrac{2(-1)^n}{(r_1(x)-r_2(x))^2}, & \text{If } d \text{ is odd} \end{cases}$$

$$\| \overrightarrow{L_{n_1}^{d}}(x)\|^2 = \| \overrightarrow{l_{n_1}^{d}}(x)\|^2 = \begin{cases} \widehat{F}_d(x)\widehat{F}_{d+2n-1}(x), & \text{If } d \text{ is even;} \\ \widehat{L}_d(x)\widehat{L}_{d+2n-1}(x) + \dfrac{2(-1)^n}{(r_1(x)-r_2(x))^2}, & \text{If } d \text{ is odd} \end{cases}$$

4 Main Results

Theorem 4.1. For all positive integer d and for all integers n_1 and n_2

$$\sum_{i=0}^{i=d-1} \widehat{F}_{n_1+i}(x)\widehat{F}_{n_2+i}(x) = \begin{cases} \widehat{F}_d(x)\widehat{F}_{d+n_1+n_2-1}(x), & \text{If } d \text{ is even;} \\ \widehat{L}_d(x)\widehat{L}_{d+n_1+n_2-1}(x) - \dfrac{(-1)^{n_1}\widehat{L}_{n_2-n_1}(x)}{(r_1(x)-r_2(x))}, & \text{If } d \text{ is odd} \end{cases}$$

$$\sum_{i=0}^{i=d-1} \widehat{L}_{n_1+i}(x)\widehat{L}_{n_2+i}(x) = \begin{cases} (r_1(x)-r_2(x))^2\widehat{F}_d(x)\widehat{F}_{d+n_1+n_2-1}(x), & \text{If } d \text{ is even;} \\ \widehat{L}_d(x)\widehat{L}_{d+n_1+n_2-1}(x) + \dfrac{(-1)^{n_1}\widehat{L}_{n_2-n_1}(x)}{(r_1(x)-r_2(x))}, & \text{If } d \text{ is odd} \end{cases}$$

$$\sum_{i=0}^{i=d-1} \widehat{F}_{n_1+i}(x)\widehat{L}_{n_2+i}(x) = \begin{cases} (r_1(x)-r_2(x))^2\widehat{F}_d(x)\widehat{L}_{d+n_1+n_2-1}(x), & \text{If } d \text{ is even;} \\ \widehat{L}_d(x)\widehat{F}_{d+n_1+n_2-1}(x) + \dfrac{(-1)^{n_1}\widehat{F}_{n_2-n_1}(x)}{(r_1(x)-r_2(x))}, & \text{If } d \text{ is odd} \end{cases}$$

Corollary 4.2. For all positive integer d and for all integer n

$$\sum_{i=0}^{i=d-1} \widehat{F}_n(x)^2 = \begin{cases} \widehat{F}_d(x)\widehat{F}_{d+2n-1}(x), & \text{If } d \text{ is even;} \\ \dfrac{\widehat{L}_d(x)\widehat{L}_{d+2n-1}(x)}{(r_1(x)-r_2(x))^2} - 2(-1)^n, & \text{If } d \text{ is odd} \end{cases}$$

$$\sum_{i=0}^{i=d-1} \widehat{L}_n(x)^2 = \begin{cases} \widehat{F}_d(x)\widehat{F}_{d+2n-1}(x), & \text{If } d \text{ is even;} \\ \widehat{L}_d\widehat{L}_{d+2n-1}(x) + 2(-1)^{n_1})(r_1(x)-r_2(x))^2, & \text{If } d \text{ is odd} \end{cases}$$

Corollary 4.3. For all positive integer d and for all integer t and n

$$\widehat{F}_{n+d-2}(x)\widehat{F}_{n+d-t-1}(x) + \widehat{F}_{n-1}(x)\widehat{F}_{n-t}(x) =$$

$$\begin{cases} (r_1(x)+r_2(x))\widehat{F}_{d-1}(x)\widehat{F}_{d+2n-t-2}(x), & \text{If } d \text{ is even;} \\ \dfrac{(r_1(x)+r_2(x))^2}{(r_1(x)^2-r_2(x)^2)}\widehat{L}_{d-1}(x)\widehat{L}_{d+2n-t}(x) - \dfrac{2(-1)^{n-t}\widehat{L}_{t-1}(x)}{(r_1(x)-r_2(x))}, & \text{If } d \text{ is odd} \end{cases}$$

$$\widehat{L}_{n+d-2}(x)\widehat{L}_{n+d-t-1}(x) + \widehat{L}_{n-1}(x)\widehat{L}_{n-t}(x) =$$
$$\begin{cases} (r_1(x)+r_2(x))(r_1(x)-r_2(x))^2\widehat{L}_{d-1}(x)\widehat{L}_{d+2n-t-2}(x), & \text{If } d \text{ is even;} \\ (r_1(x)+r_2(x))\widehat{L}_{d-1}(x)\widehat{L}_{d+2n-t}(x) + 2(-1)^{n-1}\widehat{L}_{t+1}(x)(r_1(x)-r_2(x)), & \text{If } d \text{ is odd} \end{cases}$$

$$\widehat{F}_{n+d-2}(x)\widehat{L}_{n+d-t-1}(x) + \widehat{F}_{n-1}(x)\widehat{L}_{n-t}(x) =$$
$$\begin{cases} (r_1(x)+r_2(x))(r_1(x)-r_2(x))^2\widehat{F}_{d-1}(x)\widehat{L}_{d+2n-t-2}(x), & \text{If } d \text{ is even;} \\ (r_1(x)+r_2(x))\widehat{L}_{d-1}(x)\widehat{F}_{d+2n-t}(x) + 2(-1)^{n-1}\widehat{F}_{t+1}(x)(r_1(x)-r_2(x)), & \text{If } d \text{ is odd} \end{cases}$$

5 Concluding Remarks

In this paper, we have generalized and derived some identities of $\widehat{F}_n(x)$ and $\widehat{L}_n(x)$ using matrix method. In the future, we would like to continue working on the more generalizations of these type of the polynomials.

6 Tables

n	$\widehat{F}_n(x)$	$\widehat{L}_n(x)$
1	$4+x^2$	$4x+x^3$
2	$4x+x^3$	$8+6x^2+x^4$
3	$4+5x^2+x^4$	$12x+7x^3+x^5$
4	$8x+6x^3+x^5$	$8+18x^2+8x^4+x^6$
5	$4+13x^2+7x^4+x^6$	$20x+25x^3+9x^5+x^7$
6	$12x+19x^3+8x^5+x^7$	$8+38x^2+33x^4+10x^6+x^8$
7	$4+25x^2+26x^4+9x^6+x^8$	$28x+63x^3+42x^5+11x^7+x^9$
8	$16x+44x^3+34x^5+10x^7+x^9$	$8+66x^2+96x^4+52x^6+12x^8+x^{10}$
9	$4+41x^2+70x^4+43x^6+11x^8+x^{10}$	$36x+129x^3+138x^5+63x^7+13x^9+x^{11}$
10	$20x+85x^3+104x^5+53x^7+12x^9+x^{11}$	$8+102x^2+225x^4+190x^6+75x^8+14x^{10}+x^{12}$
11	$4+61x^2+155x^4+147x^6+64x^8+13x^{12}+x^{12}$	$44x+231x^3+363x^5+253x^7+88x^9+15x^{11}+x^{13}$
12	$24x+146x^3+259x^5+200x^7+76x^9+14x^{11}+x^{13}$	$8+146x^2+456x^4+553x^6+328x^8+102x^{10}+16x^{12}+x^{14}$

Table 2: **First** 12 **terms of** $\widehat{F}_n(x)$ **and** $\widehat{L}_n(x)$

$\widehat{F}_n(x)$	$\widehat{F}_1(x)$	$\widehat{F}_2(x)$	$\widehat{F}_3(x)$	$\widehat{F}_4(x)$	$\widehat{F}_5(x)$	$\widehat{F}_6(x)$	$\widehat{F}_7(x)$	$\widehat{F}_8(x)$	$\widehat{F}_9(x)$	$\widehat{F}_{10}(x)$
$x = 1$	5	5	10	15	25	40	65	105	170	275
$x = 2$	8	16	40	96	232	560	1352	3264	7880	19024
$x = 3$	13	39	130	429	1417	4680	15457	51051	168610	556881
$x = 4$	20	80	340	1440	6100	25840	109460	463680	1964180	8320400
$x = 5$	29	145	754	3915	20329	105560	548129	2846205	14779154	76741975
$x = 6$	40	240	1480	9120	56200	346320	2134120	13151040	81040360	499393200
$x = 7$	53	371	2650	18921	135097	964600	6887297	49175679	351117050	2506995029
$x = 8$	68	544	4420	35904	291652	2369120	19244612	156326016	1269852740	10315147936
$x = 9$	85	765	6970	63495	578425	5269320	48002305	437290065	3983612890	36289806075
$x = 10$	104	1040	10504	106080	1071304	10819120	109262504	1103444160	11143704104	112540485200
$x = 11$	125	1375	15250	169125	1875625	20801000	230686625	2558353875	28372579250	314656725625
$x = 12$	148	1776	21460	259296	3133012	37855440	457398292	5526634944	66777017620	806850846384
$x = 13$	173	2249	29410	384579	5028937	65760760	859918817	11244705381	147041088770	1922778859391
$x = 14$	200	2800	39400	554400	7801000	109768400	1544558600	21733588800,	305814801800	4303140814000
$x = 15$	229	3435	51754	779745	11747929	176998680	2666728129	40177920615,	605335537354	9120210980925
$x = 16$	260	4160	66820	1073280	17239300	276902080	4447672580	71439663360	1147482286340	18431156244800
$x = 17$	293	4981	84970	1449471	24725977	421791080	7195174337	122739754809	2093771006090	35716846858339
$x = 18$	328	5904	106600	1924704	34751272	627447600	11328808072	204545992896	3693156680200	66681366236496
$x = 19$	365	6935	132130	2517405	47962825	913811080	17410373345	331710904635	6319917561410	120410144571425
$x = 20$	404	8080	162004	3248160	65125204	1305752240	26180170004	524909152320	10524363216404	211012173480400

Table 3: **First few terms of** $\widehat{F}_n(x)$

$\widehat{L}_n(x)$	$\widehat{L}_1(x)$	$\widehat{L}_2(x)$	$\widehat{L}_3(x)$	$\widehat{L}_4(x)$	$\widehat{F}_5(x)$	$\widehat{L}_6(x)$	$\widehat{L}_7(x)$	$\widehat{L}_8(x)$	$\widehat{L}_9(x)$	$\widehat{L}_{10}(x)$
$x=1$	5	15	20	35	55	90	145	235	380	615
$x=2$	16	48	112	272	656	1584	3824	9232	22288	53808
$x=3$	39	143	468	1547	5109	16874	55731	184067	607932	2007863
$x=4$	80	360	1520	6440	27280	115560	489520	2073640	8784080	3720996
$x=5$	145	783	4060	21083	109475	568458	2951765	15327283	79588180	413268183
$x=6$	240	1520	9360	57680	355440	2190320	13497360	83174480	512544240	3158439920
$x=7$	371	2703	19292	137747	983521	7022394	50140279	358004347	2556170708	18251199303
$x=8$	544	4488	36448	296072	2405024	19536264	158695136	1289097352	10471473952	85060888968
$x=9$	765	7055	64260	585395	5332815	48580730	442559385	4031615195	36727096140	334575480455
$x=10$	1040	10608	107120	1081808	10925200	110333808	1114263280	11252966608	113643929360	1147692260208
$x=11$	1375	15375	170500	1890875	20970125	232562250	2579154875	28603265875	317215079500	3517969140375
$x=12$	1776	21608	261072	3154472	38114736	460531304	5564490384	67234415912	812377481328	9815764191848
$x=13$	2249	29583	386828	5058347	66145339	864947754	11310466141	147901007587	1934023564772	25290207349623
$x=14$	2800	39600	557200	7840400	110322800	1552359600	21843357200	307359360400	4324874402800	60855600999600
$x=15$	3435	51983	783180	11799683	177778425	2678476058	40354919295	608002265483	9160388901540	138013835788583
$x=16$	4160	67080	1077440	17306120	277975360	4464911880	71716565440	1151929958920	18502595908160	297193464489480
$x=17$	4981	85263	1454452	24810947	423240551	7219900314	123161545889	2100966180427	35839586613148	611373938603943
$x=18$	5904	106928	1930608	34857872	629372304	11363559344	205173440496	3704485488272	66885912229392	1207650905617328
$x=19$	6935	132495	2524340	48094955	916328485	17458336170	332624715715	6337327934755	120741855476060	2300432581979895
$x=20$	8080	162408	3256240	65287208	1309000400	26245295208	526214904560	10550543386408	211537082632720	4241292196040808

Table 4: **First few terms of** $\widehat{L}_n(x)$

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ON L-FUZZY (K, E)-SOFT NEIGHBORHOOD SPACES

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Abstaract − In this paper, we define L-fuzzy (K, E)-soft neighborhood spaces and investigated the topological properties of L-fuzzy (K, E)-soft uniformity in stsc-quantales. We obtain L-fuzzy (K, E)-soft topology and L-fuzzy (K, E)-soft neighborhood spaces induced by L-fuzzy (K, E)-soft uniformity. Moreover, we study the relations among L-fuzzy (K, E)-soft topology, L-fuzzy (K, E)-soft neighborhood system and L-fuzzy (K, E)-soft uniformity.

Keywords − *Stsc-quantales, L-fuzzy (K, E)-soft neighborhood space, L-fuzzy (K, E)-soft uniform space, L-fuzzy (K, E)-soft topologies.*

1 Introduction

In 1999 Molodtsov [15] initiated the theory of soft sets as a new mathematical tool to deal with uncertainties while modeling problems in engineering physics, computer science, economics, social sciences and medical sciences. In [15], Molodtsov applied successfully in directions such as, smoothness of functions, game theory, operations research, Riemann integration, Perron integration, probability and theory of measurement. Maji et al. [13,14] gave the first practical application of soft sets in decision making problems. In 2003, Maji et al. [13] defined and studied several basic notions of soft set theory. Many researchers have contributed towards the algebraic structure of soft set theory [1,3,7]. In 2011, Shabir and Naz [23] initiated the study of soft topological spaces. They defined soft topology on the collection of soft sets over X and established their several properties. Aygünoğlu et.al [4] introduced the concept of soft topology in the sense of Šostak [24].

Hájek [8] introduced a complete residuated lattice which is an algebraic structure for many valued logic and decision rules in complete residuated lattices. Höhle

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[9,10] introduced L-fuzzy topologies with algebraic structure L(cqm, quantales, MV-algebra). It has developed in many directions [11,17,18]. Lowen [12] introduced the notion of fuzzy uniformities as a view of the enourage approach. Many reachers studied the different approach as powerset or the uniform covering. Kim [11] introduced the notion of fuzzy uniformities as an extension of Lowen in a stsc-quantale L. Ramadan et al. [19,20] define the the concept , L- fuzzy (K, E)-soft uniform spaces, L-fuzzy (K, E)-soft topological spaces in strictly two sided commutative quantales and investigated the relation between them.

In this paper, we define L-fuzzy (K, E)-soft neighborhood spaces and investigated the topological properties of L-fuzzy (K, E)-soft uniformity in stsc-quantales. We obtain L-fuzzy (K, E)-soft topology and L-fuzzy (K, E)-soft neighborhood spaces induced by L-fuzzy (K, E)-soft uniformity. Moreover, we study the relations among L-fuzzy (K, E)-soft topology, L-fuzzy (K, E)-soft neighborhood system and L-fuzzy (K, E)-soft uniformity.

2 Preliminaries

Let $L = (L, \leq, \vee, \wedge, 0, 1)$ be a completely distributive lattice with the least element 0 and the greatest element 1 in L.

Definition 2.1. [8,10,22] A complete lattice $(L, \leq, \odot)$ is called a strictly two-sided commutative quantale (stsc-quantale, for short) iff it satisfies the following properties.

(L1) $(L, \odot)$ is a commutative semigroup,
(L2) $x = x \odot 1$, for each $x \in L$ and 1 is the universal upper bound,
(L3) $\odot$ is distributive over arbitrary joins, i.e. $(\bigvee_i x_i) \odot y = \bigvee_i (x_i \odot y)$.

There exists a further binary operation $\rightarrow$ (called the implication operator or residuated) satisfying the following condition

$$x \rightarrow y = \bigvee \{z \in L | x \odot z \leq y\}.$$

Then it satisfies Galois correspondence; i.e, $(x \odot z) \leq y$ iff $z \leq (x \rightarrow y)$.

In this paper, we always assume that $(L, \leq, \odot, \rightarrow, ^*)$ is a stsc-quantales with an order reversing involution * which is defined by $x^* = x \rightarrow 0$ unless otherwise specified.

Remark 2.2. Every completely distributive lattice $(L, \leq, \wedge, ^*)$ with order reversing involution * is a stsc-quantale $(L, \leq, \odot, \oplus, ^*)$ with a strong negation * where $\odot = \wedge$.

Lemma 2.3. [8,22] For each $x, y, z, x_i, y_i, w \in L$, we have the following properties.

(1) $1 \to x = x$, $0 \odot x = 0$.
(2) If $y \le z$, then $x \odot y \le x \odot z$, $x \to y \le x \to z$ and $z \to x \le y \to x$.
(3) $x \le y$ iff $x \to y = 1$.
(4) $(\bigwedge_i y_i)^* = \bigvee_i y_i^*$, $(\bigvee_i y_i)^* = \bigwedge_i y_i^*$.
(5) $x \to (\bigwedge_i y_i) = \bigwedge_i (x \to y_i)$.
(6) $(\bigvee_i x_i) \to y = \bigwedge_i (x_i \to y)$.
(7) $x \to (\bigvee_i y_i) \ge \bigvee_i (x \to y_i)$.
(8) $(\bigwedge_i x_i) \to y \ge \bigvee_i (x_i \to y)$.
(9) $(x \odot y) \to z = x \to (y \to z) = y \to (x \to z)$.
(10) $x \odot y = (x \to y^*)^*$,
(11) $(x \to y) \odot (z \to w) \le (x \odot z) \to (y \odot w)$ and $x \le y \to x \odot y$.
(12) $x \to y \le (x \odot z) \to (y \odot z)$ and $(x \to y) \odot (y \to z) \le x \to z$.
(13) $y \to z \le (x \to y) \to (x \to z)$.
(14) $x \to y = y^* \to x^*$.
(15) $\bigvee_{i\in\Gamma} x_i \to \bigvee_{i\in\Gamma} y_i \ge \bigwedge_{i\in\Gamma}(x_i \to y_i)$ and $\bigwedge_{i\in\Gamma} x_i \to \bigwedge_{i\in\Gamma} y_i \ge \bigwedge_{i\in\Gamma}(x_i \to y_i)$.

Throughout this paper, X refers to an initial universe, E and K are the sets of all parameters for X, and L^X is the set of all L-fuzzy sets on X.

Definition 2.4. [4,7] A map f is called an L- fuzzy soft set on X, where f is a mapping from E into L^X, i.e., $f_e := f(e)$ is an L- fuzzy set on X, for each $e \in E$. The family of all L- fuzzy soft sets on X is denoted by $(L^X)^E$. Let f and g be two L- fuzzy soft sets on X.

(1) f is an L-fuzzy soft subset of g and we write $f \sqsubseteq g$ if $f_e \le g_e$, for each $e \in E$. f and g are equal if $f \sqsubseteq g$ and $g \sqsubseteq f$.

(2) The intersection of f and g is an L- fuzzy soft set $h = f \sqcap g$, where $h_e = f_e \wedge g_e$, for each $e \in E$.

(3) The union of f and g is an L- fuzzy soft set $h = f \sqcup g$, where $h_e = f_e \vee g_e$, for each $e \in E$.

(4) An L- fuzzy soft set $h = f \odot g$ is defined as $h_e = f_e \odot g_e$, for each $e \in E$.

(5) The complement of an L- fuzzy soft sets on X is denoted by f^*, where $f^* : E \to L^X$ is a mapping given by $f_e^* = (f_e)^*$, for each $e \in E$.

(6) 0_X (resp. 1_X) is an L-fuzzy soft set if $(0_X)_e(x) = 0$ (resp. $(1_X)_e(x) = 1$), for each $e \in E$, $x \in X$.

Definition 2.5. [4] Let $\varphi : X \to Y$ and $\psi : E \to F$ be two mappings, where E and K are parameters sets for the crisp sets X and Y, respectively. Then $\varphi_\psi : (X, E) \to (Y, F)$ is called a fuzzy soft mapping. Let f and g be two fuzzy soft sets over X and Y, respectively and let φ_ψ be a fuzzy soft mapping from (X, E) into (Y, F).

(1) The image of f under the fuzzy soft mapping φ_ψ, denoted by $\varphi_\psi(f)$ is the fuzzy soft set on Y defined by ,$\forall e_1 \in F, \forall y \in Y$,

$$\varphi(f)_{e_1}(y) = \begin{cases} \bigvee_{\varphi(x)=y} \Big(\bigvee_{\psi(e)=e_1} f_e(x) \Big), & \text{if } x \in \varphi^{-1}(\{y\}), e \in \psi^{-1}(\{e_1\}) \\ 0, & \text{otherwise,} \end{cases}$$

(2) The pre-image of g under the fuzzy soft mapping φ_ψ, denoted by $\varphi_\psi^{-1}(g)$ is the fuzzy soft set on X defined by

$$\varphi_\psi^{-1}(g)_e(x) = g_{\psi(e)}(\varphi(x)), \forall e \in E, \forall x \in X.$$

Definition 2.6. [4,19] A mapping $\mathcal{T} : K \to L^{(L^X)^E}$ (where $\mathcal{T}_k := \mathcal{T}(k) : (L^X)^E \to L$ is a mapping for each $k \in K$) is called an L-fuzzy (K, E)-soft topology on X if it satisfies the following conditions for each $k \in K$.

(SO1) $\mathcal{T}_k(0_X) = \mathcal{T}_k(1_X) = 1$,
(SO2) $\mathcal{T}_k(f \odot g) \geq \mathcal{T}_k(f) \odot \mathcal{T}_k(g) \;\; \forall\, f, g \in (L^X)^E$,
(SO3) $\mathcal{T}_k(\bigsqcup_i f_i) \geq \bigwedge_{i\in I} \mathcal{T}_k(f_i) \; \forall\, f_i \in (L^X)^E,\, i \in I$.

The pair $(X, \mathcal{T})$ is called an L-fuzzy (K, E)-soft topological space.

An L-fuzzy (K, E)-soft topology is called enriched if

(SR) $\mathcal{T}_k(\alpha \odot f) \geq \mathcal{T}_k(f)$ for all $f \in (L^X)^E$ and $\alpha \in L$.

Let $(X, \mathcal{T}^1)$ be an L-fuzzy (K_1, E_1)-soft topological space and $(Y, \mathcal{T}^2)$ be an L-fuzzy (K_2, E_2)-soft topological space. Let $\varphi : X \to Y$, $\psi : E_1 \to E_2$ and $\eta : K_1 \to K_2$ be mappings. Then $\varphi_{\psi,\eta}$ from $(X, \mathcal{T}^1)$ into $(Y, \mathcal{T}^2)$ is called L-fuzzy soft continuous if

$$\mathcal{T}^2_{\eta(k)}(f) \leq \mathcal{T}^1_k(\varphi_\psi^{-1}(f)) \;\; \forall\, f \in (L^Y)^{E_2}, k \in K_1.$$

Lemma 2.7. [18] Define a binary mapping $S : (L^X)^E \times (L^X)^E \to L$ by

$$S(f, g) = \bigwedge_{x\in X} \bigwedge_{e\in E} (f_e(x) \to g_e(x)) \;\; \forall\, f, g \in (L^X)^E, \;\; \forall\, e \in E.$$

Then $\forall f, g, h, m, n \in (L^X)^E$ the following statements hold.

(1) $f \sqsubseteq g$ iff $S(f, g) = 1$.
(2) If $f \sqsubseteq g$, then $S(h, f) \leq S(h, g)$ and $S(f, h) \geq S(g, h)$.
(3) $S(f, h) \odot S(h, g) \leq S(f, g)$. Moreover, $\bigvee_{h\in(L^X)^E}(S(f, h) \odot S(h, g)) = S(f, g)$
(4) $S(f, g) \odot S(m, n) \leq S(f \odot m, g \odot n)$.
(5) If $\varphi_\psi : (X, E) \to (Y, F)$ is a fuzzy soft mapping, then $S(p, q) \leq S(\varphi_\psi^{-1}(p), \varphi_\psi^{-1}(q))$, for each $p, q \in (L^Y)^F$.

Definition 2.8. [18] An L- fuzzy (K, E)-soft quasi- uniformity is a mapping $\mathcal{U} : K \to L^{(L^{X\times X})^E}$ which satisfies the following conditions .

(SU1) There exists $u \in (L^{X\times X})^E$ such that $\mathcal{U}_k(u) = 1$.
(SU2) If $v \sqsubseteq u$, then $\mathcal{U}_k(v) \le \mathcal{U}_k(u)$.
(SU3) For every $u, v \in (L^{X\times X})^E, \mathcal{U}_k(u \odot v) \ge \mathcal{U}_k(u) \odot \mathcal{U}_k(v)$.
(SU4) If $\mathcal{U}_k(u) \ne 0$ then $\top_\triangle \sqsubseteq u$ where, for each $e \in E$,

$$(\top_\triangle)_e(x,y) = \begin{cases} 1, & \text{if } x = y, \\ 0, & \text{if } x \ne y. \end{cases}$$

(SU5) $\mathcal{U}_k(u) \le \bigvee\{\mathcal{U}_k(v) \mid v \circ v \sqsubseteq u\}$, where

$$v_e \circ w_e(x,z) = \bigvee_{y\in X} v_e(x,y) \odot w_e(y,z),$$

The pair $(X,\mathcal{U})$ is called an L-fuzzy (K,E)-soft quasi-uniform space.

An L-fuzzy (K,E)-soft quasi-uniform space $(X,\mathcal{U})$ is said to be an L-fuzzy (K,E)-soft uniform space if

(U) $\mathcal{U}_k(u) \le \mathcal{U}_k(u^{-1})$, where $(u^{-1})_e(x,y) = u_e(y,x)$ for each $k \in K$ and $u \in (L^{X\times X})^E$.

An L-fuzzy (K,E)-soft quasi-uniformity $\mathcal{U}$ on X is said to be stratified if

(SR) $\mathcal{U}_k(\alpha \odot u) \ge \alpha \odot \mathcal{U}_k(u), \ \ \forall\, u \in (L^{X\times X})^E, \alpha \in L$.

Let $(X,\mathcal{U}^1)$ be an L-fuzzy (K_1,E_1)-soft quasi-uniform space and $(Y,\mathcal{U}^2)$ be an L-fuzzy (K_2,E_2)-soft quasi-uniform space. Let $\varphi : X \to Y$, $\psi : E_1 \to E_2$ and $\eta : K_1 \to K_2$ be mappings. Then $\varphi_{\psi,\eta}$ from $(X,\mathcal{U}^1)$ into $(Y,\mathcal{U}^2)$ is called L-fuzzy soft uniformly continuous if

$$\mathcal{U}^2_{\eta(k)}(v) \le \mathcal{U}^1_k((\varphi\times\varphi)^{-1}_\psi(v)) \ \ \forall\, v \in (L^{Y\times Y})^{E_2}, k \in K_1.$$

Remark 2.9. Let $(X,\mathcal{U})$ be an L-fuzzy (K,E)-soft uniform space.

(1) By (SU1) and (SU2), we have $\mathcal{U}_k(1_{X\times X}) = 1$ because $u \sqsubseteq 1_{X\times X}$ for all $u \in (L^{X\times X})^E$.
(2) Since $\mathcal{U}_k(u) \le \mathcal{U}_k(u^{-1}) \le \mathcal{U}_k((u^{-1})^{-1}) = \mathcal{U}_k(u)$, then $\mathcal{U}_k(u) = \mathcal{U}_k(u^{-1})$.

3 Topological Properties of L-fuzzy (K,E)-soft Uniform Spaces

Definition 3.1. An L-fuzzy (K,E)-soft neighborhood system on X is a set $N = \{N^x \mid x \in X\}$ of mappings $N^x : K \to L^{(L^X)^E}$ such that for each $k \in K$:

(SN1) $N^x_k(\top_X) = \top$ and $N^x_k(\bot_X) = \bot$,
(SN2) $N^x_k(f \odot g) \ge N^x_k(f) \odot N_x(g)$ for each $f, g \in (L^X)^E$,

(SN3) If $f \sqsubseteq g$, then $N_k^x(f) \leq N_k^x(g)$,
(SN4) $N_k^x(f) \leq \bigwedge_{e\in E} f_e(x)$ for all $f \in (L^X)^E$. (Here $N^x(k) =: N_k^x : (L^X)^E \to L$)

An L-fuzzy (K,E)-soft neighborhood system is called stratified if

(SR) $N_k^x(\alpha \odot f) \geq \alpha \odot N_k^x(f)$ for all $f \in (L^X)^E$ and $\alpha \in L$.

The pair (X,N) is called an L-fuzzy (K,E)-soft neighborhood space.

Let (X,N) be an L-fuzzy (K_1,E_1)-soft neighborhood space and (Y,M) be an L-fuzzy (K_2,E_2)-soft neighborhood space. Let $\varphi : X \to Y$, $\psi : E_1 \to E_2$ and $\eta : K_1 \to K_2$ be mappings. Then $\varphi_{\psi,\eta}$ from (X,N) into (Y,M) is called L-fuzzy soft continuous at every $x \in X$ if $M_{\eta(k)}^{\phi(x)}(f) \leq N_k^x(\varphi_\psi^{-1}(f))$ $\forall\, f \in (L^Y)^{E_2}, k \in K_1$.

Theorem 3.2. Let $(X,\mathcal{T})$ be an L-fuzzy (K,E)- soft topological space. Define a map $(N^{\mathcal{T}})^x : K \to L^{(L^X)^E}$ by

$$(N^{\mathcal{T}})_k^x(f) = \bigvee\{\bigwedge_{e\in E}(\mathcal{T}_k(g) \odot g_e(x)) \mid g \sqsubseteq f\}.$$

Then,

(1) $N^{\mathcal{T}}$ is an L-fuzzy (K,E)-soft neighborhood on X,
(2) If $\mathcal{T}$ is enriched, then $N^{\mathcal{T}}$ is stratified.

Proof. (SN1)

$$(N^{\mathcal{T}})_k^x(\top_X) \geq \bigwedge_{e\in E}(\mathcal{T}_k(\top_X) \odot (\top_X)_e(x)) = \top,$$

$$(N^{\mathcal{T}})_k^x(\bot_X) = \bigwedge_{e\in E}(\mathcal{T}_k(\bot_X) \odot (\bot_X)_e(x)) = \bot.$$

(SN2)

$$\begin{aligned}
&\bigwedge_{e\in E}(\mathcal{T}_k(f \odot g) \odot (f_e \odot g_e)(x)) \\
&\geq \bigwedge_{e\in E}(\mathcal{T}_k(f) \odot \mathcal{T}_k(g) \odot (f_e \odot g_e)(x)) \\
&\geq \bigwedge_{e\in E}(\mathcal{T}_k(f) \odot f_e(x)) \odot \bigwedge_{e\in E}(\mathcal{T}_k(g) \odot g_e(x))
\end{aligned}$$

$$\begin{aligned}
&(N^{\mathcal{T}})_k^x(f_1) \odot (N^{\mathcal{T}})_k^x(f_2) \\
&= \bigvee\{\bigwedge_{e\in E}(\mathcal{T}_k(g_1) \odot (g_1)_e(x)) \mid g_1 \sqsubseteq f_1\} \odot \bigvee\{\bigwedge_{e\in E}(\mathcal{T}_k(g_2) \odot (g_2)_e(x)) \mid g_2 \sqsubseteq f_2\}
\end{aligned}$$

$$
\begin{aligned}
&\leq \bigvee\{\bigwedge_{e\in E}(\mathcal{T}_k(g_1\odot g_2)\odot((g_2)_e\odot(g_2)_e)(x)) \mid g_1\odot g_2\sqsubseteq f_1\odot f_2\}\\
&\leq (N^{\mathcal{T}})^x_k(f\odot g).
\end{aligned}
$$

(SN3) and (SN4) are easily proved.

(2) Let $\mathcal{T}$ be enriched. Then

$$
\begin{aligned}
&\alpha\odot(N^{\mathcal{T}})^x_k(g)=\alpha\odot\bigvee\{\bigwedge_{e\in E}(\mathcal{T}_k(f)\odot f_e(x)) \mid f\sqsubseteq g\}\\
&\leq \bigvee\{\bigwedge_{e\in E}(\alpha\odot\mathcal{T}_k(f)\odot f_e(x)) \mid f\sqsubseteq g\}\\
&\leq \bigvee\{\bigwedge_{e\in E}(\mathcal{T}_k(\alpha\odot f)\odot(\alpha\odot f_e)(x)) \mid \alpha\odot f\sqsubseteq \alpha\odot g\}\\
&\leq (N^{\mathcal{T}})^x_k(\alpha\odot f).
\end{aligned}
$$

Theorem 3.3. Let $(X,\mathcal{T}^1)$ be an L-fuzzy (K_1,E_1)-soft topological space and $(Y,\mathcal{T}^2)$ be an L-fuzzy (K_2,E_2)-soft topological space. Let $\varphi:X\to Y$, $\psi:E_1\to E_2$ and $\eta:K_1\to K_2$ be mappings. If $\varphi_{\psi,\eta}:(X,\mathcal{T}^1)\to(Y,\mathcal{T}^2)$ is L-fuzzy soft continuous, then $\varphi_{\psi,\eta}:(X,N^{\mathcal{T}^1})\to(Y,N^{\mathcal{T}^2})$ is L-fuzzy soft continuous.

Proof.

$$
\begin{aligned}
&(N^{\mathcal{T}^2})^{\varphi(x)}_{\eta(k)}(g)\to(N^{\mathcal{T}^1})^x_k(\varphi^{-1}_\psi(g))\\
&=\bigvee\{\bigwedge_{e_2\in E_2}(\mathcal{T}^2_{\eta(k)}(f)\odot f_{e_2}(\varphi(x))) \mid g\sqsubseteq f\}\to\bigvee\{\bigwedge_{e_1\in E_1}(\mathcal{T}^1_k(h)\odot h_{e_1}(x)) \mid h\sqsubseteq\varphi^{-1}_\psi(g)\}\\
&\geq\bigvee\{\bigwedge_{e_2\in E_2}(\mathcal{T}^2_{\eta(k)}(f)\odot f_{e_2}(\varphi(x))) \mid g\sqsubseteq f\}\to\\
&\bigvee\{\bigwedge_{e_1\in E_1}(\mathcal{T}^1_k(\varphi^{-1}_\psi(f))\odot\varphi^{-1}_\psi(f)_{e_1}(x)) \mid \varphi^{-1}_\psi(f)\sqsubseteq\varphi^{-1}_\psi(g)\}\\
&\geq\bigwedge\{\bigwedge_{e_2\in E_2}(\mathcal{T}^2_{\eta(k)}(f)\odot f_{e_2}(\varphi(x)))\to\bigwedge_{e_1\in E_1}(\mathcal{T}^1_k(\varphi^{-1}_\psi(f))\odot\varphi^{-1}_\psi(f)_{e_1}(x)) \mid f\sqsubseteq g\}\\
&\geq\bigwedge\{\bigwedge_{e_1\in E_1}(\mathcal{T}^2_{\eta(k)}(f)\odot f_{\psi(e_1)}(\varphi(x)))\to\bigwedge_{e_1\in E_1}(\mathcal{T}^1_k(\varphi^{-1}_\psi(f))\odot f_{\psi(e_1)}(\varphi(x))) \mid f\sqsubseteq g\}\\
&\geq\bigwedge\bigwedge_{e_1\in E_1}\{(\mathcal{T}^2_{\eta(k)}(f)\odot f_{\psi(e_1)}(\varphi(x)))\to(\mathcal{T}^1_k(\varphi^{-1}_\psi(f))\odot f_{\psi(e_1)}(\varphi(x))) \mid f\sqsubseteq g\}\\
&\geq\bigwedge\{\mathcal{T}^2_{\eta(k)}(f)\to\mathcal{T}^1_k(\varphi^{-1}_\psi(f)) \mid f\sqsubseteq g\}
\end{aligned}
$$

Thus, if $\mathcal{T}_{\eta(k)}(f)\leq\mathcal{T}^{\mathcal{U}}_k(\varphi^{-1}_\psi(f))$, then $(N^{\mathcal{T}^2})^{\varphi(x)}_{\eta(k)}(g)\leq(N^{\mathcal{T}^1})^x_k(\varphi^{-1}_\psi(g))$. So, $\varphi_{\psi,\eta}$

is L-fuzzy soft continuous.

Theorem 3.4. Let (X, N) be an L-fuzzy (K, E)-soft neighborhood space. Define a map $\mathcal{T}^N : K \to L^{(L^X)^E}$ by

$$\mathcal{T}_k^N(f) = \bigwedge_{x\in X}\bigwedge_{e\in E}(f_e(x) \to (N_k^x)(f)).$$

Then we have the following properties.

(1) $\mathcal{T}^N$ is an L-fuzzy (K, E)-soft topology on X.
(2) If N is stratified, then $\mathcal{T}^N$ is an enriched L-fuzzy (K, E)-soft topology.
(3) $N^{\mathcal{T}^N} \le N$.
(4) If $(X, \mathcal{T})$ is an L-fuzzy (K, E)-soft topological space with $E = \{e\}$, then $\mathcal{T}^{N^{\mathcal{T}}} \ge \mathcal{T}$.

Proof. (1) (SO1)

$$\mathcal{T}_k^N(\top_X) = \bigwedge_{x\in X}\bigwedge_{e\in E}((\top_X)_e(x) \to N_k^x(\top_X) = \top \to \top = \top,$$

$$\mathcal{T}_k^N(\perp_X) = \bigwedge_{x\in X}\bigwedge_{e\in E}((\perp_X)_e(x) \to N_k^x(\perp_X)) = \perp \to \perp = \perp.$$

(SO2)

$$\begin{aligned}\mathcal{T}_k^N(f \odot g) &= \bigwedge_{x\in X}\bigwedge_{e\in E}((f_e \odot g_e)(x) \to N_k^x(f \odot g))\\ &\ge \bigwedge_{x\in X}\bigwedge_{e\in E}((f_e \odot g_e)(x) \to (N_k^x(f) \odot N_k^x(g)))\\ &\quad \text{(by Lemma 2.3 (11))}\\ &\ge \bigwedge_{x\in X}\bigwedge_{e\in E}(f_e(x) \to N_k^x(f)) \odot \bigwedge_{x\in X}\bigwedge_{e\in E}(g_e(x) \to N_k^x(g))\\ &= \mathcal{T}_k^N(f) \odot \mathcal{T}_k^N(g).\end{aligned}$$

(SO3)

$$\begin{aligned}\mathcal{T}_k^N(\bigsqcup_i f_i) &= \bigwedge_{x\in X}\bigwedge_{e\in E}((\bigsqcup_i (f_i)_e)(x) \to N_k^x(\bigsqcup_i f_i))\\ &\ge \bigwedge_{x\in X}\bigwedge_{e\in E}((\bigsqcup_i (f_i)_e)(x) \to \bigvee_i N_k^x(f_i))(x))\\ &\quad \text{(by Lemma 2.3 (15))}\\ &\ge \bigwedge_i\bigwedge_{x\in X}\bigwedge_{e\in E}((f_i)_e)(x) \to N_k^x(f_i)) = \bigwedge_i \mathcal{T}_k^N(f_i).\end{aligned}$$

(2) (SR) By Lemma 2.3 (12), we have

$$\begin{aligned}\mathcal{T}^N_k(\alpha \odot f) &= \bigwedge_{x\in X}\bigwedge_{e\in E}((\alpha \odot f_e)(x) \to N^x_k(\alpha \odot f))\\ &\geq \bigwedge_{x\in X}\bigwedge_{e\in E}((\alpha \odot f_e)(x) \to (\alpha \odot N^x_k(f)))\\ &\geq \bigwedge_{x\in X}\bigwedge_{e\in E}(f_e(x) \to N^x_k(f)) = \mathcal{T}^N_k(f).\end{aligned}$$

(3) We have $N^{\mathcal{T}^N} \leq N$ from:

$$\begin{aligned}(N^{\mathcal{T}^N})^x_k(f) &= \bigwedge_{e\in E}(\mathcal{T}^N_k(f) \odot f_e(x))\\ &= \bigwedge_{e\in E}((\bigwedge_{z\in X}\bigwedge_{e_1\in E}(f_{e_1}(z) \to N^z_k(f)) \odot f_e(x))\\ &\leq \bigwedge_{e\in E}((f_e(x) \to N^x_k(f)) \odot f_e(x)) \leq N^x_k(f).\end{aligned}$$

(4) Let $E = \{e\}$ be given. We have $\mathcal{T}^{N^{\mathcal{T}}} \geq \mathcal{T}$ from:

$$\begin{aligned}\mathcal{T}^{N^{\mathcal{T}}}_k(f) &= \bigwedge_{x\in X}(f_e(x) \to N^{\mathcal{T}}_k(f))\\ &= \bigwedge_{x\in X}(f_e(x) \to \bigvee\{\mathcal{T}_k(g) \odot g_e(x)) \mid g \sqsubseteq f\}\\ &\geq \bigwedge_{x\in X}(f_e(x) \to \mathcal{T}_k(f) \odot f_e(x))\\ &\geq \mathcal{T}_k(f) \text{ (by Lemma 2.3 (11))}.\end{aligned}$$

Theorem 3.6. Let (X, N^1) be an L-fuzzy (K_1, E_1)-soft neighborhood space and (Y, N^2) be an L-fuzzy (K_2, E_2)-soft neighborhood space. Let $\varphi : X \to Y$, $\psi : E_1 \to E_2$ and $\eta : K_1 \to K_2$ be mappings. If $\varphi_{\psi,\eta} : (X, N^1) \to (Y, N^2)$ is L-fuzzy soft continuous, then $\varphi_{\psi,\eta} : (X, \mathcal{T}^{N^1}) \to (Y, \mathcal{T}^{N^2})$ is L-fuzzy soft continuous.

Proof.

$$\begin{aligned}&\mathcal{T}^{N^2}_{\eta(k)}(f) \to \mathcal{T}^{N^1}_k(\varphi^{-1}_\psi(f))\\ &= \bigwedge_{y\in Y}\bigwedge_{e_2\in E_2}(f_{e_2}(y) \to (N^2)^y_{\eta(k)}(f)) \to \bigwedge_{x\in X}\bigwedge_{e_1\in E_1}(\varphi^{-1}_\psi(f)_{e_1}(x) \to (N^1)^x_k(\varphi^{-1}_\psi(f)))\\ &\geq \bigwedge_{x\in X}\bigwedge_{e_1\in E_1}(\varphi^{-1}_\psi(f)_{e_1}(x) \to (N^2)^{\varphi_\psi(x)}_{\eta(k)}(f)) \to \bigwedge_{x\in X}\bigwedge_{e_1\in E_1}(\varphi^{-1}_\psi(f)_{e_1}(x) \to (N^1)^x_k(\varphi^{-1}_\psi(f)))\end{aligned}$$

$$\geq \bigwedge_{x\in X}\bigwedge_{e_1\in E_1}\Big((\varphi_\psi^{-1}(f)_{e_1}(x)\to (N^2)^{\varphi_\psi(x)}_{\eta(k)}(f))\to(\varphi_\psi^{-1}(f)_{e_1}(x)\to (N^1)^x_k(\varphi_\psi^{-1}(f)))\Big)$$

(by Lemma 2.3 (13))

$$\geq \bigwedge_{x\in X}\Big((N^2)^{\varphi_\psi(x)}_{\eta(k)}(f)\to (N^1)^x_k(\varphi_\psi^{-1}(f))\Big).$$

Thus, if $(N^2)^{\varphi_\psi(x)}_{\eta(k)})(f)\leq (N^1)^x_k(\varphi_\psi^{-1}(f))$, then $\mathcal{T}^2_{\eta(k)}(f)\leq \mathcal{T}^1_k(\varphi_\psi^{-1}(f))$. So, $\varphi_{\psi,\eta}$ is L-fuzzy soft continuous.

Theorem 3.7. Let $(X,\mathcal{U})$ be an L-fuzzy (K,E)-soft uniform space. Define a map $N^{\mathcal{U}}: K\to L^{(L^X)^E}$ by

$$(N^{\mathcal{U}})^x_k(f)=\bigvee_u \mathcal{U}_k(u)\odot S(u[x],f),\quad \forall\, f\in (L^X)^E,\ x\in X,$$

where $u_e[x](y)=u_e(y,x)$. Then the following properties hold.

(1) $(X,N^{\mathcal{U}})$ is an L-fuzzy (K,E)-soft neighborhood space.
(2) If $\mathcal{U}$ is stratified, then $N^{\mathcal{U}}$ is also stratified.

Proof. (1) (SN1) For $\mathcal{U}_k(u)\neq\perp$, $\top_\triangle\sqsubseteq u$. Then

$$\begin{aligned}(N^{\mathcal{U}})^x_k(\perp_X)&=\bigvee_u \mathcal{U}_k(u)\odot S(u[x],\perp_X)\\ &\leq \bigvee_u \big(\mathcal{U}_k(u)\odot(u_e(x,x)\to\perp)\big)\\ &\leq \bigvee_u \big(\mathcal{U}_k(u)\odot((\top_\triangle)_e(x,x)\to\perp)\big)=\perp.\end{aligned}$$

Hence $(N^{\mathcal{U}})^x_k(\perp_X)=\perp$. Also, $(N^{\mathcal{U}})^x_k(\top_X)=\top$, because

$$(N^{\mathcal{U}})^x_k(\top_X)\geq \mathcal{U}_k(\top_{X\times X})\odot\bigwedge_{e\in E}\bigwedge_{y\in X}((\top_{X\times X})_e(x,y)\to(\top_X)_e(y))=\top.$$

(SN2) By Lemma 2.7 (4) , we have

$$\begin{aligned}&(N^{\mathcal{U}})^x_k(f)\odot (N^{\mathcal{U}})^x_k(g)\\ &=\big(\bigvee_u \mathcal{U}_k(u)\odot S(u[x],f)\big)\odot\big(\bigvee_u \mathcal{U}_k(v)\odot S(v[x],g)\big)\\ &=\bigvee_{u,v}\mathcal{U}_k(u)\odot\mathcal{U}_k(v)\odot S(u[x],f)\odot S(v[x],g)\\ &\leq \bigvee_{u,v}\mathcal{U}_k(u\odot v)\odot S((u\odot v)[x],f\odot g)\\ &\leq \bigvee_w \mathcal{U}_k(w)\odot S(w[x],f\odot g)=(N^{\mathcal{U}})^x_k(f\odot g).\end{aligned}$$

(SN3) By Lemma 2.7 (2), we have

$$(N^{\mathcal{U}})^x_k(f) = \bigvee_u \mathcal{U}_k(u) \odot S(u[x], f)$$
$$\leq \bigvee_u \mathcal{U}_k(u) \odot S(u[x], g) = (N^{\mathcal{U}})^x_k(g).$$

(SN4) For $\mathcal{U}_k(u) \neq \perp$, $\top_\triangle \sqsubseteq u$.

$$(N^{\mathcal{U}})^x_k(f) = \bigvee_u \mathcal{U}_k(u) \odot \bigwedge_{e\in E} \bigwedge_{y\in X} (u_e(y, x) \to f_e(y))$$
$$\leq \bigvee_u \big\{\mathcal{U}_k(u) \odot (u_e(x, x) \to f_e(x))\big\} \leq f_e(x).$$

This implies that $(X, N^{\mathcal{U}}_x)$ is an L-fuzzy soft neighborhood space.

(2)

$$\alpha \odot (N^{\mathcal{U}})^x_k(f) = \alpha \odot \bigvee_u \mathcal{U}_k(u) \odot S(u[x], f)$$
$$= \bigvee_u \alpha \odot \mathcal{U}_k(u) \odot S(\alpha, \alpha) \odot S(u[x], f)$$
$$\leq \bigvee_u \mathcal{U}_k(\alpha \odot u) \odot S(\alpha \odot u[x], \alpha \odot f)$$
$$\leq (N^{\mathcal{U}})^x_k(\alpha \odot f).$$

Theorem 3.8. Let $(X, \mathcal{U})$ be an L-fuzzy (K_1, E_1)-soft uniform space and $(Y, \mathcal{V})$ be an L-fuzzy (K_2, E_2)-soft uniform space. Let $\varphi : X \to Y$, $\psi : E_1 \to E_2$ and $\eta : K_1 \to K_2$ be mappings. If $\varphi_{\psi,\eta} : (X, \mathcal{U}) \to (Y, \mathcal{V})$ is L-fuzzy soft uniformly continuous, then $\varphi_{\psi,\eta} : (X, N^{\mathcal{U}}) \to (Y, N^{\mathcal{V}})$ is L-fuzzy soft continuous.

Proof. First we show that $\varphi_\psi^{-1}(v_e[\varphi_\psi(x)]) = (\varphi_\psi \times \varphi_\psi)^{-1}(v_e)[x]$ from

$$\varphi_\psi^{-1}(v_e[\varphi_\psi(x)])(z) = v_e[\varphi_\psi(x)](\varphi_\psi(z)) = v_e(\varphi_\psi(z), \varphi_\psi(x))$$
$$= (\varphi_\psi \times \varphi_\psi)^{-1}(v_e)(z, x) = (\varphi_\psi \times \varphi_\psi)^{-1}(v_e)[x](z).$$

Thus, by Lemma 2.7(4), we have

$$S(v[\varphi_\psi(x)], f) \leq S(\varphi_\psi^{-1}(v[\varphi_\psi(x)]), \varphi_\psi^{-1}(f))$$
$$= S((\varphi_\psi \times \varphi_\psi)^{-1}(v)[x], \varphi_\psi^{-1}(f)).$$

$$
\begin{aligned}
&(N^{\mathcal{V}})^{\varphi_\psi(x)}_{\eta(k)}(f) = \bigvee_v \mathcal{V}_{\eta(k)}(v) \odot S(v[\varphi_\psi(x)], f) \\
&\leq \bigvee_v \mathcal{V}_{\eta(k)}(v) \odot S((\varphi_\psi \times \varphi_\psi)^{-1}(v)[x], \varphi_\psi^{-1}(f)) \\
&\leq \bigvee_u \mathcal{U}_k((\varphi_\psi \times \varphi_\psi)^{-1}(v)) \odot S((\varphi_\psi \times \varphi_\psi)^{-1}(v)[x], \varphi_\psi^{-1}(f)) \\
&\leq (N^{\mathcal{U}})^x_k(\varphi_\psi^{-1}(f)).
\end{aligned}
$$

From Theorems 3.4 and 3.7, we obtain the following corollary.

Corollary 3.9. Let $(X,\mathcal{U})$ be an L-fuzzy (K,E) soft uniform space and $N^{\mathcal{U}} = \{N^{\mathcal{U}}_x \mid x \in X\}$ be an L-fuzzy (K,E) soft neighborhood system on X. Define a map $\mathcal{T}^{\mathcal{U}} : K \to L^{(L^X)^E}$ by

$$\mathcal{T}^{\mathcal{U}}(f) = \bigwedge_{x\in X} (f_e(x) \to (N^{\mathcal{U}})^x_k(f)).$$

Then,

(1) $\mathcal{T}^{\mathcal{U}}$ is an L-fuzzy (K,E) soft topology on X,
(2) If $N^{\mathcal{U}}$ is stratified, then $\mathcal{T}^{\mathcal{U}}$ is an enriched L-fuzzy topology.

From Theorems 3.6 and 3.8, we obtain the following corollary.

Corollary 3.10. Let $(X,\mathcal{U})$ be an L-fuzzy (K_1,E_1)-soft uniform space and $(Y,\mathcal{V})$ be an L-fuzzy (K_2,E_2)-soft uniform space. Let $\varphi : X \to Y$, $\psi : E_1 \to E_2$ and $\eta : K_1 \to K_2$ be mappings. If $\varphi_{\psi,\eta} : (X,\mathcal{U}) \to (Y,\mathcal{V})$ is L-fuzzy soft uniformly continuous, then $\varphi_{\psi,\eta} : (X,\mathcal{T}^{\mathcal{U}}) \to (Y,\mathcal{T}^{\mathcal{V}})$ is L-fuzzy soft continuous.

Example 3.11. Let $X = \{h_i \mid i = \{1,2,3\}\}$ with h_i=house and $E = \{e,b\}$ with e=expensive, b= beautiful. Define a binary operation $\odot$ on $[0,1]$ by

$$x \odot y = \max\{0, x+y-1\},\ x \to y = \min\{1-x+y, 1\}$$

Then $([0,1],\wedge,\to,0,1)$ is a stsc-quantle (ref [8-10]). Put $f,g \in (L^X)^E$ such that

$$
\begin{gathered}
f_e(h_1) = 0.5, f_e(h_2) = 0.5, f_e(h_3) = 0.6 \\
f_b(h_1) = 0.6, f_b(h_2) = 0.3, f_b(h_3) = 0.6 \\
g_e(h_1) = 0.8, g_e(h_2) = 0.7, g_e(h_3) = 0.5 \\
g_b(h_1) = 0.9, g_b(h_2) = 0.6, g_b(h_3) = 0.6
\end{gathered}
$$

(1) Put $K = \{k_1, K_2\}$. We define $\mathcal{T} : K \to [0,1]^{([0,1]^X)^E}$ as follows:

$$\mathcal{T}_{k_1}(h) = \begin{cases} 1, & \text{if } h = 1_X \text{or } h = 0_X \\ 0.6, & \text{if } h = f, \\ 0.3, & \text{if } h = f \odot f, \\ 0, & \text{otherwise.} \end{cases}$$

$$\mathcal{T}_{k_2}(h) = \begin{cases} 1, & \text{if } h = 1_X \text{or } h = 0_X \\ 0.5, & \text{if } h = g, \\ 0, & \text{otherwise.} \end{cases}$$

We obtain a $[0,1]$-fuzzy (K,E)-soft neighborhood system on X as:

$$(N^{\mathcal{T}})_{k_1}^{h_1}(h) = \begin{cases} 1, & \text{if } h = 1_X \\ 0.1, & \text{if } f \sqsubseteq h \neq 1_X, \\ 0, & \text{otherwise.} \end{cases} \quad (N^{\mathcal{T}})_{k_2}^{h_1}(h) = \begin{cases} 1, & \text{if } h = 1_X \\ 0.2, & \text{if } g \sqsubseteq h \neq 1_X, \\ 0, & \text{otherwise.} \end{cases}$$

$$(N^{\mathcal{T}})_{k_1}^{h_2}(h) = \begin{cases} 1, & \text{if } h = 1_X \\ 0, & \text{otherwise.} \end{cases} \quad (N^{\mathcal{T}})_{k_2}^{h_2}(h) = \begin{cases} 1, & \text{if } h = 1_X \\ 0.1, & \text{if } g \sqsubseteq h \neq 1_X, \\ 0, & \text{otherwise.} \end{cases}$$

$$(N^{\mathcal{T}})_{k_1}^{h_3}(h) = \begin{cases} 1, & \text{if } h = 1_X \\ 0.2, & \text{if } f \sqsubseteq h \neq 1_X, \\ 0, & \text{otherwise.} \end{cases} \quad (N^{\mathcal{T}})_{k_2}^{h_3}(h) = \begin{cases} 1, & \text{if } h = 1_X \\ 0.1, & \text{if } g \sqsubseteq h \neq 1_X, \\ 0, & \text{otherwise.} \end{cases}$$

Then

$$\mathcal{T}_{k_1}^{N^{\mathcal{T}}}(f) = \bigwedge_{x\in X}(\bigvee_{e\in E} f_e(x) \to N_{k_1}^x(f)) = (0.6 \to 0.1) \wedge (0.5 \to 0) \wedge (0.6 \to 0.2) = 0.5$$

$$\mathcal{T}_{k_1}^{N^{\mathcal{T}}}(f \odot f) = (0.2 \to 0) \wedge (0 \to 0) \wedge (0.2 \to 0) = 0.8,$$
$$\mathcal{T}_{k_2}^{N^{\mathcal{T}}}(g) = (0.9 \to 0.2) \wedge (0.7 \to 0.1) \wedge (0.6 \to 0.1) = 0.3.$$

Since $E = \{e, b\}$, by Theorem 3.4(4), in general, $\mathcal{T}^{N^{\mathcal{T}}} \not\geq \mathcal{T}$.

(2) Put $v, v \odot v, w \in ([0,1]^{X\times X})^E$ as

$$v_e = \begin{pmatrix} 1 & 0.6 & 0.5 \\ 0.3 & 1 & 0.5 \\ 0.4 & 0.6 & 1 \end{pmatrix} v_b = \begin{pmatrix} 1 & 0.5 & 0.3 \\ 0.7 & 1 & 0.5 \\ 0.6 & 0.6 & 1 \end{pmatrix}$$

$$(v \odot v)_e = \begin{pmatrix} 1 & 0.2 & 0 \\ 0 & 1 & 0 \\ 0 & 0.2 & 1 \end{pmatrix} (v \odot v)_b = \begin{pmatrix} 1 & 0 & 0 \\ 0.4 & 1 & 0 \\ 0.2 & 0.2 & 1 \end{pmatrix}$$

$$w_e = \begin{pmatrix} 1 & 0.4 & 0.5 \\ 0.4 & 1 & 0.5 \\ 0.4 & 0.6 & 1 \end{pmatrix} v_b = \begin{pmatrix} 1 & 0.5 & 0.3 \\ 0.3 & 1 & 0.5 \\ 0.2 & 0.3 & 1 \end{pmatrix}$$

We define $\mathcal{U} : K \to [0,1]^{([0,1]^{X\times X})^E}$ as follows:

$$\mathcal{U}_{k_1}(u) = \begin{cases} 1, & \text{if } u = 1_{Y\times Y} \\ 0.6, & \text{if } v \sqsubseteq u \neq 1_{Y\times Y}, \\ 0.3, & \text{if } v \odot v \sqsubseteq u \not\sqsupseteq v, \\ 0, & \text{otherwise.} \end{cases}$$

$$\mathcal{U}_{k_2}(u) = \begin{cases} 1, & \text{if } u = 1_{Y\times Y} \\ 0.5, & \text{if } w \sqsubseteq u \neq 1_{Y\times Y}, \\ 0, & \text{otherwise.} \end{cases}$$

Since $(N^{\mathcal{U}})^x_k(f) = \bigvee_u \mathcal{U}_k(u) \odot S(u[x], f)$, we have

$$(N^{\mathcal{U}})^{h_1}_{k_1}(f) = 0.1, (N^{\mathcal{U}})^{h_2}_{k_1}(f) = 0.1, (N^{\mathcal{U}})^{h_3}_{k_1}(f) = 0.2$$
$$(N^{\mathcal{U}})^{h_1}_{k_2}(f) = 0, (N^{\mathcal{U}})^{h_2}_{k_2}(f) = 0, (N^{\mathcal{U}})^{h_3}_{k_2}(f) = 0.1$$

Since $\mathcal{T}^{\mathcal{U}}_k(f) = \bigwedge_{x\in X}(\bigvee_{e\in E} f_e(x) \to (N^{\mathcal{U}})^x_k(f))$,

$$\mathcal{T}^{\mathcal{U}}_{k_1}(f) = (0.6 \to 0.1) \wedge (0.5 \to 0.1) \wedge (0.6 \to 0.2) = 0.5,$$

$$\mathcal{T}^{\mathcal{U}}_{k_2}(f) = (0.6 \to 0) \wedge (0.5 \to 0) \wedge (0.6 \to 0.1) = 0.4.$$

$$(N^{\mathcal{U}})^{h_1}_{k_1}(g) = 0.4, (N^{\mathcal{U}})^{h_2}_{k_1}(g) = 0.2, (N^{\mathcal{U}})^{h_3}_{k_1}(g) = 0.1,$$
$$(N^{\mathcal{U}})^{h_1}_{k_2}(g) = 0.3, (N^{\mathcal{U}})^{h_2}_{k_2}(g) = 0.1, (N^{\mathcal{U}})^{h_3}_{k_2}(g) = 0.1,$$
$$\mathcal{T}^{\mathcal{U}}_{k_1}(g) = (0.9 \to 0.4) \wedge (0.7 \to 0.2) \wedge (0.6 \to 0.1) = 0.5,$$
$$\mathcal{T}^{\mathcal{U}}_{k_2}(g) = (0.9 \to 0.3) \wedge (0.7 \to 0.1) \wedge (0.6 \to 0.1) = 0.4.$$

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CONTRA-CONTINUITY BETWEEN GRILL-TOPOLOGICAL SPACES

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Abstaract − Recently noted the importance of the concept of grill between topological space. It helps us to measure the things that was difficult to measure and it is also used in many applications such as computer and information systems. Our purpose is to introduce the notation of $e-G-$open, $r-G-$open sets and discuss new class of function named contra $e-G-$continuous function with their various assets, depiction and relationships. Relevance between there new class and other proportion of functions are obtained and several depictions of a new class of functions are discussed.

Keywords − *r-G-open sets, e-G-open sets, e-G-continuous, Contra-e-G-continuous.*

1 Introduction

The concept of grill topological spaces, which is grounded on two operators, is Φ and Ψ. Choquet [1] was the first introduced this concept in 1947. A number of theories and features has been handled in [2, 3, 4, 5, 6, 7]. It helps to expand the topological structure which is used to measure the description rather than quantity, such as love, intelligence, beauty, quality of education and etc. In 1996, Dontchev [8] pass the notation of contra continuous functions. Ekici [9] debate the concepts of contra-e-continuous functions and new class named as e-open sets. Jafari and Noiri [10, 11] exhibited contra-α-continuous and contra-pre-continuous functions. We are working to provide the previous concepts and definitions by using the concept of grill. Some important characteristics and special relations of these concepts are obtained.

2 Preliminary

Definition 2.1. A nonempty subcollection G of a space L which carries topology τ is named grill [1] on this space if the following conditions are true:
(1) $\phi \notin G$,
(2) $A \in G$ and $A \subseteq B \subseteq L \Rightarrow B \in G$,
(3) if $A \cup B \in G$ for $A, B \subseteq L$, then $A \in G$ or $B \in G$.

Since the grill depends on the two mappings Φ and Ψ which is generated a unique grill topological space finer than τ on space L denoted by τ_G on L have been discussion in [3, 5].

A subset B of a space (L, τ, G) is named regular open (resp. regular closed) if $B = Int(Cl(B))$ (resp. $B = Cl(Int(B))$. B is named δ-open [12] if for $x \in B$, there exist a regular open set D such that $x \in D \subset B$. The complement of δ-open set is called δ-closed. A point $x \in L$ is called a δ-derived point of B if $Int(Cl(U)) \cap B \neq \varphi$ for each open set U containing x. The set of all δ-derived points of B is called the δ-closure of B and is denoted by $\delta Cl(B)$ [12]. The set δ-interior of B [12] is the union of all regular open sets of L contained in B and its denoted by $\delta Int(B)$. B is δ-open if $\delta Int(B) = B$. δ-open sets forms a topology τ^δ. The collection of all δ-open sets in L is denoted by $\delta O(L)$. A subset B of a space (L, τ) is called e-open [13] (resp. α-open [14], β-open [15]) if $B \subset Cl(\delta Int(B)) \cup Int(\delta Cl(B))$ (resp. $B \subset Int(Cl(Int(B)), B \subset Cl(Int(Cl(B))$. The complement of an e-open set is called is an e-closed set. The intersection of all e-closed sets containing a set B in a topological space (L, τ) is called the e-closure [13]. The union of all e-open sets contained in a set B in a topological space (L, τ) is called the e-interior of B and it is denoted by $e - Int(B)$. A subset B of a topological L is e-regular [16] if it is e-open and e-closed. The family of all e-open (resp. e-closed, e-regular) sets in L will be denoted by $EO(L)$ (resp. $EC(L)$, $ER(L)$. The family of all e-open (resp. e-closed, e-regular) sets which contain x in L will be denoted by $EO(L, x)$ (resp. $EC(L, x)$, $ER(L, x)$).

A function $z : (L, \tau, I) \to (M, \sigma)$ is called contra continuous [8] (resp. contra-e-continuous [17], contra-$e - I-$continuous [18]) if the inverse image of each open set of M is closed (resp.$z^{-1}(V)$ is e-closed in L for every open set V of M, $z^{-1}(V)$ is $e - I$-closed in (L, τ, I) for every open set V of (M, σ))".

3 Essential Results

The third section, we define and study some new definitions called $r - G-$open, $e - G-$open and contra $e - G-$continuous. Depiction and basic assets of $e - G-$continuous function are studied.

Definition 3.1. For $B \subseteq L$ which carries topology τ with grill G is called $r-G-$open (resp. $r - G-$closed) if $B = Int(\Psi(B))$ (resp. $B = \Psi(Int(B))$. A point $x \in L$ is called a $\delta-G-$derived point of B if $Int(\Psi(U)) \cap B \neq \varphi$ for each open set U containing x. The family of all $\delta - G-$derived points of B is named the $\delta - G-$closure of B and is denoted by $\delta\Psi(B)$. The set $\delta - G-$interior of B is the union of all $r - G-$open sets of X contained in B and is denoted by $\delta Int_G(B)$. B is said to be $\delta - G-$closed if $\delta\Psi(B) = B$.

Definition 3.2. Let the space L which carries topology τ with grill G then, A subset B of L named $e - G-$open if $B \subset Cl(\delta Int_G(B)) \cup Int(\delta\Psi(B))$ and $e - G-$closed if

$Cl(\delta Int_G(B)) \cap Int(\delta\Psi(B)) \subset B$.

The class of all $e-G-$open sets of an grill with a space L which carries topology τ is denoted by $EGO(L)$.

Theorem 3.3. (1) The union of any family of $e-G-$open sets is an $e-G-$open set;
(2) The intersection of even two $e-G-$open sets need not to be $e-G-$open.

Proof. (1) Let $\{S_\alpha : \alpha \in \triangle\}$ be a family of $e-G-$open set, $S_\alpha \subset Cl(\delta Int_G(S_\alpha)) \cup Int(\delta\Psi(S_\alpha))$. Hence, $\cup_\alpha S_\alpha \subset \cup_\alpha[Cl(\delta Int_G(S_\alpha)) \cup Int(\delta\Psi(S_\alpha))]$
$\subset \cup_\alpha[Cl(\delta Int_G(S_\alpha))] \cup \cup_\alpha[Int(\delta\Psi(S_\alpha))]$
$\subset [Cl(\cup_\alpha(\delta Int_G(S_\alpha))] \cup [Int(\cup_\alpha(\delta\Psi(S_\alpha))]$
$\subset [Cl(\delta Int_G(\cup_\alpha S_\alpha))] \cup [Int(\delta\Psi(\cup_\alpha S_\alpha))]$. Then, $\cup_\alpha S_\alpha$ is $e-G-$open. $\square$

Example 3.4. Let $L = \{1, 2, 3\}$ with $\tau = \{L, \varphi, \{1\}, \{2\}, \{1, 2\}\}$ and $G = P(L) \setminus \{\varphi\}$. Then the set $A = \{1, 3\}$ and $B = \{2, 3\}$ are $e-G-$open sets, but $A \cap B = \{3\}$ is not $e-G-$open.

Definition 3.5. A mapping $z : (L, \tau, G) \to (M, \sigma)$ is said to be contra $e-G-$continuous functions if $z^{-1}(V)$ is $e-G-$closed in (L, τ, G) for every open set V in (M, σ).

Example 3.6. Let $L = M = \{1, 2, 3\}$ and $\tau = \sigma = \{L, \varphi, \{1\}, \{2, 3\}\}$ with $G = \{\{2\}, \{1, 2\}, \{2, 3\}, L\}$. If $z : (L, \tau, G) \to (Y, \sigma)$ is identity function then, we notice that z is contra-e-continuous but not contra-$e-G-$continuous. Since $z^{-1}\{2, 3\} = \{2, 3\}$ is not $e-G-$closed.

Definition 3.7. For the space L which carries topology τ with grill G and $B \subset L$,
(1) Intersection of all open set U containing B is named kernel of B and denoted by $Ker(B)$.
(2) Intersection of all $e-G-$closed in (L, τ, G) containing B is named the $e-G-$closure of B and its denoted by $\Psi_e(B)$.
(3) The $e-G-$interior of B, denoted by $Int_{eG}(B)$, is defined by the union of all $e-G-$open sets contained in B.

Lemma 3.8. For a space L which carries topology τ with grill G and $C, B \subset L$,
(1) $x \in Ker(B)$ if and only if $B \cap F \neq \varphi$, where F closed subset of L containing x.
(2) $B \subset Ker(B)$ and $B = Ker(B)$ if A is open in L.
(3) if $C \subset B$, then $Ker(C) \subset Ker(B)$.

Lemma 3.9. For a subset B of a a space L which carries topology τ with grill G, the following properties are holds,
(1) $Int_{eG}(B) = L \setminus \Psi_e(L \setminus B)$.
(2) $x \in \Psi_e(B)$ if and only if $B \cup U \neq \varphi$ for each $U \in EGO(L, x)$.
(3) B is $e-G-$open if and only if $Int_{eG}(B) = B$.
(4) B is $e-G-$closed if and only if $\Psi_e(B) = B$.

Theorem 3.10. Let $z : (L, \tau, G) \to (M, \sigma)$ be a given function then, the next are equivalent:

(1) z is contra $e-G-$continuous,
(2) for each $x \in L$ and each closed set F in M with $z(x) \in F$, there exist $e-G-$open set U containing x such that $z(U) \subset F$,
(3) for each $x \in L$ and each closed set F in Y with $z(x) \in F, z^{-1}(F)$ is $e-G-$open in L ,
(4) $z(\Psi_e(B)) \subset Ker(z(B))$ for every $B \subset L$,
(5) $z(\Psi_e(B)) \subset z^{-1}(Ker(B))$.

Proof. (1) $\Rightarrow$ (2): By the assumption, suppose that $z(x) \in F$ such that $x \in L$ and F be any closed set in M, we have $z^{-1}(M \setminus F) = L \setminus z^{-1}(F)$ is $e-G-$closed in L and so $z^{-1}(F)$ is $e-G-$open. By butting $U = z^{-1}(F)$ containing x, we have$z(U) \subset F$.

(2) $\Rightarrow$ (3): Let $F \subset M$ be any closed set and $x \in z(x)$. Then $z(x) \in F$ and there exists $e-G-$open subset U_x containing x such that $z(U_x) \subset F$. Therefore, we obtain $z^{-1}(F)$ is $e-G-$open in L.
(3) $\Rightarrow$ (1): If U is any open set of Y, then $z^{-1}(Y \setminus U) = L \setminus z^{-1}(U)$ is $e-G-$open in L. Therefore $z^{-1}(U)$ is $e-G-$closed in L.

(3) $\Rightarrow$ (4): Let $B \subset L$ and $y \notin Ker(B)$. Then, by Lemma 3.7, there exists a closed set F of M such that $M \in F$ and $z(B) \cap F = \varphi$. This implies that $B \cap z^{-1}(F)) = \varphi$ and $\Psi_e(B) \cap z^{-1}(F) = \varphi$. Therefore, we obtain $z(\Psi_e(B)) \cap F = \varphi$ and $y \notin z(\Psi_e(B))$. That $z(\Psi_e(B)) \subset Ker(z(B))$.

(4) $\Rightarrow$ (5): Let C be any subset of M. By Lemma 3.7, we have $z(\Psi_e(z^{-1}(C)) \subset Ker(z(z^{-1}(C))) \subset Ker(C)$ and $z(\Psi_e(z^{-1}(C)) \subset z^{-1}(Ker(C))$.

(5) $\Rightarrow$ (1): Let N be any subset of M. By Lemma 3.7, we have $\Psi_e(z^{-1}(N)) \subset z^{-1}(Ker(N)) = z^{-1}(N)$ and $\Psi_e(z^{-1}(N)) = z^{-1}(N)$. This shows that $z^{-1}(N)$ is $e-G-$closed in L. □

Definition 3.11. A mapping $z : (L, \tau, G) \to (M, \sigma)$ is named $e-G-$continuous if $z^{-1}(D)$ is $e-G-$-open in L for every $D \in \sigma$.

Lemma 3.12. The next declarations are equivalent for a mapping $z : (L, \tau, G) \to (M, \sigma)$
(1) z is $e-G-$continuous,
(2) for each open set V of Y and each $x \in X$ with $z(x) \in V$, there exist $U \in EGO(L, x)$ such that $z(U) \subset V$.

Theorem 3.13. Let $z : (L, \tau, G) \to (M, \sigma)$ be a contra $e-G-$continuous functions such that M is regular set, then z is $e-G-$continuous functions.

Proof. Let N be an open set of M containing $z(x)$ for each $x \in L$. Since M is regular, there exists an open set C in Y containing $z(x)$ such that $Cl(C) \subset N$. Since z is contra $e-G-$continuous, then there exists $U \in EGO(L)$ containing x such that $z(U) \subset Cl(C)$. Then $f(U) \subset Cl(C) \subset N$. that, z is $e-G-$continuous. □

Definition 3.14. An grill with the space Y which carries topology τ is named $e-G-$connected if Y not equal the union of two disjoint non-null $e-G-$open subsets of Y.

Theorem 3.15. Let $z:(L,\tau,G)\to(M,\sigma)$ be a contra $e-G-$continuous functions from a $e-G-$connected onto any space M, then M is not a discrete space.

Proof. Let M be a discrete. Suppose that A be an appropriate non-null clopen set in M. Then $z^{-1}(A)$ is appropriate non-null $e-G-$clopen subset of L, which conflict with L is $e-G-$connected. □

Theorem 3.16. If $z:(L,\tau,G)\to(M,\sigma)$ is a contra $e-G-$continuous surjection functions and X is $e-G-$connected, then Y is connected.

Proof. Let $z:(L,\tau,G)\to(M,\sigma)$ be a contra $e-G-$continuous functions from a $e-G-$connected L onto any space M. Suppose that M is disconnected. Then $M=A\cup B$, where A and B are non-null clopen sets in M with $A\cap B=\varphi$. Because of z is contra $e-G-$continuous, that $z^{-1}(A)$ and $z^{-1}(B)$ are $e-G-$open non-null sets in L with $z^{-1}(A)\cup z^{-1}(B)=f^{-1}(A\cup B)=f^{-1}(M)=L$ and $z^{-1}(A)\cap z^{-1}(B)=z^{-1}(A\cap B)=z^{-1}(\varphi)=\varphi$. This leads to L is not $e-G-$connected, this is a contradiction. Then M is connected. □

Definition 3.17. A function $z:(L,\tau,G)\to(M,\sigma)$ is called almost- $e-G-$continuous if $z^{-1}(V)\in EGO(X)$ for every regular open set V of M.

Definition 3.18. A function $z:(L,\tau,G)\to(M,\sigma)$ is called pre- $e-G-$open if $z^{-1}(V)$ is $e-G-$open for every $e-$open set V of M.

Definition 3.19. The $e-G-$boundary of a subset A of a space L denoted by $e-G-B(A)$, is defined as $e-G-B(A)=\Psi_e(A)\cap\Psi_e(L\setminus A)$.

Theorem 3.20. For each $x\in L$ at $z:(L,\tau,G)\to(M,\sigma)$ is not contra-$e-G-$continuous is identical with the union of $e-G-$boundary of the inverse images of closed sets of M containing $z(x)$.

Proof. Firstly, let z be not satisfy the condition of contra-$e-G-$continuous at a point $x\in L$. Then there exists a closed set F of M containing $z(x)$ such that $z(U)\cap(M\setminus F)\neq\varphi,\forall U\in EGO(L,x)$, which leads to $U\cap z^{-1}(M\setminus F)\neq\varphi$, that by Theorem 3.9. Therefore, $x\in\Psi_e(z^{-1}(M\setminus z))=\Psi_e(L\setminus z^{-1}(F))$. Also, since $x\in z^{-1}(F)$, we get $x\in\Psi_e(z^{-1}(F))$ and so, $x\in e-G-B(z^{-1}(F))$. Secondly, let $x\in e-G-B(z^{-1}(F))$ for some closed set F of M containing $z(x)$ and z is contra-$e-G-$continuous at a point x. Then there exists $U\in EGO(L,x)$ such that $z(U)\subset F$. So, $x\in U\subset z^{-1}(F)$ and then $x\in Int_{eG}(z^{-1}(F))\subset L\backslash e-G-B(z^{-1}(F))$, which is a contradiction. So z is not contra-$e-G-$continuous at x.

□

Theorem 3.21. If $z:(L,\tau,G)\to(M,\sigma)$ is pre-$e-G-$open, contra-$e-G-$continuous. Then, its almost-$e-G-$continuous.

Proof. Let $x \in L$ and V be an open set containing $z(x)$. Because of z is contra-$e-G-$continuous, then by Theorem 3.9, there exists $U \in EGO(L, x)$ such that $z(U) \subset Cl(V)$. Also, since z is pre-$e-G-$open, $z(U)$ is $e-\sigma-$open in M. Therefore, $z(U) = Int_{eG}(z(U))$ and hence $z(U) \subset Int_{eG}(Cl(z(U))) \subset Int_{eG}(Cl(V))$. So z is almost-$e-G-$continuous. □

Definition 3.22. A mapping $z : (L, \tau, G) \rightarrow (M, \sigma)$ is named almost weakly-$e-G-$continuous if for each $x \in L$ and for each open set V of M containing $z(x)$, there exist $U \in EGO(L, x)$ such that $z(U) \subset Cl(V)$.

Theorem 3.23. If $z : (L, \tau, G) \rightarrow (M, \sigma)$ is contra $e-G-$continuous then, z is almost weakly-$e-G-$continuous.

Proof. For any open set V of M, $Cl(V)$ is closed in M. Since z is contra $e-G-$continuous, $z^{-1}(Cl(V))$ is $e-G-$open set in L. We take $U = z^{-1}(Cl(V))$, then $z(U) \subset Cl(V)$. Hence z is almost weakly-$e-G-$continuous. □

Example 3.24. Let $L = \{1, 2, 3\}$ with a topology $\tau = \{L, \varphi, \{1\}, \{2\}, \{1, 2\}\}$ and a grill $G = \{\{1, 2\}, L\}$. If $z : (L, \tau, G) \rightarrow (L, \tau)$ is the identity function. We notice that z is almost weakly-$e-G-$continuous. But z is not contra $e-G-$continuous since $z^{-1}(V)$ is not $e-G-$closed at $V = \{1, 2\}$.

Definition 3.25. An a space L which carries topology τ with grill G is called $e-G-T_1$ if for each $x, y \in L$ such that $x \neq y$, there exist $e-G-$open sets U and V containing x and y, respectively, such that $y \notin U$ and $x \notin V$.

Definition 3.26. An a space L which carries topology τ with grill G is called $e-G-T_2$ if for each $x, y \in L$ such that $x \neq y$, there exist $e-G-$open sets U and V containing x and y, respectively, such that $U \cap V = \varphi$.

Theorem 3.27. Let $z : (L, \tau, G) \rightarrow (M, \sigma)$ be a contra $e-G-$continuous injection. If M is a Urysohn space, then L is $e-G-T_2$.

Proof. If $x, y \in L$ such that $x \neq y$, then $z(x) \neq z(y)$. Since M is a Urysohn space, there exist open sets U and V of m such that $z(x) \in U, z(y) \in V$ and $Cl(U) \cap Cl(V) = \varphi$. Since z is contra $e-G-$continuous at x and y, there exist $e-G-$open sets A and B in L such that $x \in A, y \in B$ and $z(A) \subset Cl(U), z(B) \subset Cl(V)$. Then, $z(A) \cap z(B) = \varphi$, so $A \cap B = \varphi$ and L is $e-G-T_2$.

□

Definition 3.28. A mapping $z : (L, \tau, G) \rightarrow (L, \sigma, \jmath)$ is named be $e-G-$irresolute if $z^{-1}(B) \in EGO(L)$ for each $B \in EGO(M)$.

Theorem 3.29. If $z : (L, \tau, G) \rightarrow (M, \sigma, \jmath)$ and $g : (M, \sigma, \jmath) \rightarrow (N, \zeta)$ is two maps then,the following are holds:
(1) If z is contra $e-G-$continuous function and g is a continuous function, then $g \circ z$ is contra $e-G-$continuous.
(2) If z is $e-G-$irresolute function and g is a contra $e-G-$continuous function, then $g \circ z$ is contra $e-G-$continuous.

Proof. (1) For $x \in L$, let W be any closed set of N containing $(g \circ z)(x)$. Since g is continuous, $V = g^{-1}(W)$ is closed in M. Also, since z is contra $e - G-$continuous, there exists $U \in EGO(L, x)$ such that $z(U) \subset V$. Therefore $(g \circ z(U)) \subset W$. Hence, $g \circ z$ is is contra $e - G-$continuous.

(2) For $x \in L$, let W be any closed set of N containing $(g \circ z)(x)$. Since g is a contra $e - G-$continuous, there exists $V \in E\jmath O(M, z(x))$ such that $g(V) \subset W$. Also, since z is $e - G-$irresolute there exist $U \in EGO(L, x)$ such that $z(U) \subset V$. This shows that $(g \circ z(U)) \subset W$. Hence, $g \circ z$ is contra $e - G-$continuous. □

Definition 3.30. A space (L, τ, G) is named $e - G-$normal if each pair of nonempty disjoint closed sets can be separated by disjoint $e - G-$open sets. Also, it is called ultra normal if each pair of non-null disjoint closed sets can be separated by disjoint clopen sets.

Theorem 3.31. If $z : (L, \tau, G) \to (M, \sigma, \jmath)$ is a contra $e - G-$continuous, closed injection and M is ultra normal, then L is $e - G-$normal.

Proof. Let C_1 and C_2 be disjoint closed subsets of L. Since z is closed and injective, $z(C_1)$ and $z(C_2)$ are separated by disjoint clopen sets V_1 and V_2, respectively. Since z is contra $e - G-$continuous, $z^{-1}(V_1)$ and $z^{-1}(V_2)$ are $e - G-$open, with $C_1 \subset z^{-1}(V_1), C_2 \subset z^{-1}(V_2)$ and $z^{-1}(V_1) \cap z^{-1}(V_2) = \varphi$. Hence, L is $e - G-$normal. □

Definition 3.32. A topological space (L, τ) is ultra Hausdorff [19] if for each pair of distinct points x and y of L there exist closed sets U and V such that $x \in U$ and $y \in V$ and $U \cap V = \varphi$. A topological space (L, τ) is weakly Hausdorff [20]if each element of L is the intersection of regular closed sets of L.

Theorem 3.33. If $z : (L, \tau, G) \to (M, \sigma, \jmath)$ is a contra $e - G-$continuous injection and M is ultra Hausdorff, then L is $e - G - T_2$.

Proof. Let $x, y \in L$ where $x \neq y$. Then, since z is an injection and M is ultra Hausdorff, $z(x) \neq z(y)$ and there exist disjoint closed sets U and V containing $z(x)$ and $z(y)$ respectively. Also, since f is contra $e - G-$continuous, $z^{-1}(U) \in EGO(L, x)$ and $z^{-1}(V) \in EGO(L, y)$ with $z^{-1}(U) \cap z^{-1}(V) = \varphi$. This shows that L is $e - G - T_2$. □

4 Conclusion

The concept of grill with topological space generating finer topological space τ_G.vIt helps us to measure the things that was difficult to measure and it is also used in many applications such as computer and information systems. Also, $e-G$-continuous and contra $e-G-$continuous functions is very important so, we have introduced the definition of $e - G-$continuous and contra $e - G-$continuous functions. Characterizations and basic properties of contra $e - G-$continuous functions are discussed.

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A TWO ECHELON SUPPLY CHAIN INVENTORY MODEL FOR A DETERIORATING ITEM WHEN DEMAND IS TIME AND CREDIT SENSITIVE

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Abstaract − While formulating inventory models with trade credit facilities, most of the inventory modelers so far assumed that the up-stream suppliers offer the down-stream retailers a fixed credit period. However, some times the retailers also provide a credit period to the end customers to promote the market competition. In the present paper, a supply chain inventory model of a deteriorating item has been formulated under two levels of trade credit policy, namely by the suppliers to the retailers and also by the retailers to the end customers with default risk consideration. In the proposed model, it has been assumed the demand is dependent linearly on time as well as on credit facility offered by the retailers. The model is solved analytically to find the retailer's optimal credit period and optimal cycle time which maximizes the total average profit. The solution of the model is illustrated with the help of a numerical example. The sensitivity of the optimal solution is also carried out with respect to the changes to the different system parameters.

Keywords − *Inventory, Deterioration, Time and trade credit dependent demand, Supply chain management.*

1 Introduction

In formulating inventory models, two factors are of growing interest among the researchers. One is the deterioration rate of items and another is the variation of demand function over time. Now it is well accepted that deterioration of items should not be neglected for all the cases. Items like chemicals, foodstuff, pharmaceuticals and also electronic components deteriorate significantly. In the first inventory model developed by Harris(1915), the deterioration of item was neglected simplicity. Many inventory models have been developed without considering the deterioration of items. Researchers like Whitin (1957), Shah and Jaiswal (1977), Dave and patel (1981), Dave (1986), Aggarwal(1978), RoyChaudhuri and Chaudhuri(1983), Dave(1979), Bahari-Kashani(1989), Wee (1995), Hariga (1996a, 1996b), Benkherouf (1995, 1997, 1998), Hariga and Al-Alyan (1997), Chang and Dye (1999a, 1999b) assumed that items deteriorate at a constant rate, for simplicity.The assumption of constant deterioration rate was relaxed by Covert and Philip (1973), Philip (1974), Misra (1975), Deb and Chaudhuri (1986), Goswami and Chaudhuri (1992), Giri, Goswami and Chaudhuri (1996), Ghosh and Chaudhuri (2006), Sana (2010), Sarkar(2012), Sarkar and Sarkar (2013a), Sarkar and Sarkar (2013b), Sarkar et al. (2013) etc. They all assumed time-dependent deterioration in their model. On the other hand a production inventory model with probabilistic deterioration in supply chain management was discussed by Sarkar(2013).

In the first EOQ model developed by F.W. Harris in 1915, the demand of items was assumed to be constant. But demand of an item is always a dynamic state. can not be constant throughout the inventory cycle. This unrealistic assumption of constant demand was relaxed by Silver and Meal (1969). They were the first to consider time varying demand. Later, Silver and Meal (1973) developed an approximate solution procedure, known as *Silver-Meal-Heuristic*, for the case of general deterministic time-varying demand pattern. Donaldson (1977) was the first to derive an analytical solution procedure of obtaining the optimal number of replenishments and optimal replenishment times of an EOQ model with linearly time-dependent demand pattern over a finite time horizon. Later, Silver (1979) takes a special case of a positive linear trend and applies his heuristic to the problem of Donaldson (1977). To determine the optimal reorder times, much computational effort was needed in that model. Rather, simpler and less complicated techniques were used to solve the same model by the researchers like Mitra, Cox and Jesse (1984), Henry (1979), Richie (1980, 1984, 1985), etc. Several researchers like Dave and Patel(1981), McDonald(1979), Goyal (1985), Sachan (1984), etc., contributed in this direction.

Some researchers also devoted their attention to inventory replenishment problems with exponentially time-varying demand patterns. Some of the contributions in this direction came from the researchers like Hollier and Mak (1983), Aggarwal and Bahari-Kashani (1991), Hariga and Benkherouf (1994), Wee (1995), etc. Time-dependent quadratic demand function was first used in the model of Khanra and Chaudhuri (2003). Later on Ghosh and Chaudhuri (2006), Manna, Chaudhuri and Chiang (2007), etc. contributed in this direction. The distinct advantage of taking time-dependent quadratic demand function is that it accommodates all the three types of increasing, decreasing or constant demand function depending upon the parameters of the demand function.

During the last few decades, many inventory models have been developed considering trade credit facility in their models. A single-item inventory model under permissible delay in payments was discussed by Goyel(1985). This model was further extended by Aggarwal and Jaggi(1995) considering deterioration of item. The model of Aggarwal and Jaggi(1995) was further extended by Jamal et al.(1997) considering inventory shortages. Later, Teng (2002) suggested that it make economic sense for a well-established retailer to order less quantity and take the benefits of payment delay more frequently considering the difference between selling price and purchasing cost as in the the model of Goyel (1985). Chang et al. (2003) developed an EOQ model for deteriorating items under supplier credits linked to ordering quantity. The up-stream and down-stream trade credits in which the length of down-stream trade credit period is less than or equal to the length of the up-stream trade credit period was discussed by Huang (2003). A more generalized result of the above model was proposed by Teng and Goyal (2007). Khanra et al (2011) developed an EOQ model for a deteriorating item under permissible delay in payment. An EPQ model with time proportional deterioration and permissible delay in payment was discussed by Sarkar et al.(2013). A two-level trade credit policy in the fuzzy sense was discussed by Mahata and Goswami (2007). The model of Huang (2003) was further extended by Huang (2007) model to investigate the two-level trade credit policy in the EPQ frame work. Inventory model under partial delay in payment and partial backordering was discussed by Taleizadeh et al. (2013). Retailer's partial trade policy was discussed by Chen et al. (2014). Taleizadeh and Nematollahi (2014) investigated the effects of time value of money and inflation on the optimal ordering policy for a perishable item over the finite time horizon under backordering and delay in payment payment. Ouyang et al. (2015) discussed an integrated inventory model with the order-size dependent trade credit and a constant demand. A partial up-stream advanced payment and partial downstream delayed payment in a three-level supply chain model was proposed by Lashgari et al. (2015). Inventory model model with backordering under hybrid linked-to-order multiple advance payments and delayed payment was discussed by Zia and Taleizadeh (2015). Several contributions came from the researchers Chang and Teng (2004), Chung (2008), Chung and Liao (2004), Goyal et al. (2007), Huang and Hsu (2008).

The dynamic behavior of demand function was not considered by the above researchers in their model while considering trade credit facility. The dynamic behavior of the demand function as time dependent demand or stock dependent demand were discussed by some researchers in case of trade credit facility. Time dependent demand function was considered in the model of Sarkar (2012) and Teng et al. (2012) in case case of trade credit financing. Soni (2013) developed the optimal replenishment policies under trade credit financing by considering price and stock-sensitive demand. The inventory model with the credit-linked demand were discussed by Jaggi et al. (2008) and Jaggi et al. (2012). Lou and Wang (2012) discussed an inventory model with considering trade credit with a positive correlation of market sales, but with negatively correlated with credit risk. Giri and Maiti(2013) discuss the supply chain model with price and trade credit sensitive demand by considering a retailer's shares as a fraction of the profit earned during the credit period. Wu et al.(2014) studied optimal credit period and lot size model by considering demand dependent delay in payment with default risk for deteriorating items. A deteriorating inventory model with stock-dependent demand and shortages under two-level trade credit was discussed by Annadurai and Uthayakumar (2015). A sustainable trade credit policy with credit -linked demand and credit risk considering the carbon emission constraints was discussed by Dye and Yang (2015).

In case of any supplier-retailer-buyer supply chain models, sometimes the supplier offers the retailer a trade credit of M periods, and the retailer in turn provides a trade credit of N periods to the buyer. This stimulates sales and consequently earns revenue for the supplier as well as the retailer. On the other hand, the reduction in inventory also reduces holding cost for both supplier and retailer. But, when the retailer offers the customers a trade credit period, it also increases opportunity cost (the capital opportunity loss during credit period) and default risk. In a product life cycle, at the maturity stage demand comes to more or less constant and during the growth stage of the product life cycle (especially for newly launched high-tech products), the demand function increases with time. Therefore, at the end of the product life cycle, it is necessary to stimulate the demand of the product and consequently, the proposed result of this paper is suitable for both the growth and maturity stages of a product life cycle. In the proposed paper, an economic order quantity model for the retailer has been derived when the supplier provides an up-stream trade credit and the retailer also offers a down-stream trade credit. The retailer's down-stream trade credit to the buyer increases sales and revenue but at the same time it also increases opportunity cost and default risk. The demand is dependent linearly on time as well as on the trade credit period offered by the retailer. Also it has been assumed that a constant fraction on the on hand inventory deteriorates with time. The retailer's annual profit is maximized to obtain the the optimal replenishment period and the optimal trade credit period offered to the customers by the retailer. The optimal solution of the model is solved analytically and then it is illustrated with the help of a numerical example. The sensitivity of the optimal solution with respect to the changes of different system parameters is also studied.

2 Notations and Assumptions

The inventory model is developed considering the following *notations* which includes *decision variables* and *Parameters* :

Decision Variables

T^* : the optimal replenishment period for the retailer

N^* : the optimal trade credit period offered to the customers by the retailer

Parameters

A: the ordering cost per order for the retailer.

h: the holding cost per unit per year for the retailer, excluding interest charges.

c: the purchasing cost per unit of the retailer.

c_1 the cost of deterioration per unit item.

s: the selling price per unit of the retailer ($s > c$).

I_e: the interest earned per \$ per year by the retailer.

I_p: the interest charged per \$ in stocks per year to the retailer.

M: the retailer's trade credit period offered by supplier in years.

N: the customer's trade credit period offered by retailer in years.

T: the replenishment time period for the retailer.

Q: the order quantity fr the retailer.

$I(t)$: the inventory level at any time t.

θ: the deterioration rate of items.

$D(t, N)$: the annual demand rate, which is a function of time t and trade credit period N.

$\pi(N, T)$: the total profit per year for the retailer.

The following *assumptions* have been used in developing the model:

1. Initially the inventory level is zero.
2. Lead time is zero.
3. The deterioration rate of items is constant over time.
4. The holding cost, ordering cost and shortage cost remain constant over time.
5. The planning horizon is finite.
6. The rate of replenishment is infinite.
7. The supplier provides the trade credit period M to the retailer, and the retailer offers its customers trade credit period N.
8. Demand rate $D(t, N)$ is a function of time t and the credit period N and is given by $D(t, N) = a + bt + \lambda(N), a, b > 0$. Since in real practice, the demand increases with with the increase of trade credit period, we must have $\lambda^{'}(N) > 0$.

7. There is no repair or replacement of the deteriorated items.

8. Longer is credit period offered by the retailer to the customer, higher is the default risk to the retailer. It is assumed that the rate of the default risk given the credit period offered by the retailer is $F(N) = 1 - e^{-kN}$, where $k > 0$. This default risk pattern was also used by Lou and Wang (2012) and Zhang et al. (2014).

3 Formulation and Solution of the Model

The inventory starts with Q units at time $t = 0$, when the retailer orders and receives the order quantity Q units from the supplier. The procurement made at time $t = 0$ is partly used to satisfy the demand and the rest of the procurement accounts for the deterioration in $[0, T]$. The inventory level gradually falls to zero at T. The cycle repeats itself for every cycle.

The instantaneous inventory level $I(t)$ is given by the following differential equation

$$\frac{dI(t)}{dt} + \theta I(t) = -D(t, N) = -a - bt - \lambda(N), \quad 0 \le t \le T \tag{1}$$

with the boundary condition $I(T) = 0$.

The solution of the differential equation (1) is given by

$$I(t) = \frac{b}{\theta^2} - \frac{(a + bt + \lambda(N))}{\theta} + \frac{\theta}{b}(a + bT + \lambda(N))e^{\theta(T-t)}, \qquad 0 \le t \le T \tag{2}$$

Therefore, the retailers order quantity Q is given by

$$Q = I(0) = \frac{b}{\theta^2} - \frac{(a + \lambda(N))}{\theta} + \frac{\theta}{b}(a + bT + \lambda(N))e^{\theta T} \tag{3}$$

To find the retailer's annual profit per cycle, we need to calculate the revenue, ordering cost, purchasing cost, cost od deterioration, holding cost, interest charged and interest earned.

1. Average annual ordering cost is $\frac{A}{T}$.

2. Average annual purchasing cost per cycle is

$$\frac{cQ}{T} = \frac{c}{T}[\frac{b}{\theta^2} - \frac{(a + \lambda(N))}{\theta} + \frac{\theta}{b}(a + bT + \lambda(N))e^{\theta T}]$$

3. Average annual holding cost excluding interest charges per cycle is

$$\begin{aligned}\frac{h}{T}\int_0^T I(t)dt \quad &= \frac{h}{T}\int_0^T \frac{b}{\theta^2} - \frac{(a + bt + \lambda(N))}{\theta} + \frac{\theta}{b}(a + bT + \lambda(N))e^{\theta(T-t)} \\ &= \frac{h}{T}[\{(\frac{b}{\theta^2} - \frac{a}{\theta} - \frac{\lambda(N)}{\theta}) + (\frac{a\theta}{b} + \theta T + \frac{N(\lambda))}{b}e^{\theta T}\}T - \frac{bT^2}{2\theta} \\ &-\frac{(a + bT + \lambda(N))}{b}(e^{-\theta T} - 1)]\end{aligned}$$

4. The sales revenue considering default risk is

$$\frac{se^{-kN}}{T}\int_0^T D(t,N)dt$$
$$= se^{-kN}(a+\frac{b}{2}T+\lambda(N))$$

5. The cost of deteriorated items is

$$c_1\{I(0)-\int_0^T I(t)dt\}$$
$$= c_1[(-\frac{b}{\theta^2}+\frac{a}{\theta}+\frac{\lambda(N)}{\theta})(1-T)+(\frac{a\theta}{b}+\theta T+\frac{\theta\lambda(N)}{b})(e^{\theta T}-T)$$
$$+\frac{bT^2}{2\theta}+\frac{a+bT+\lambda(N)}{b}(e^{-\theta T}-1)]$$

6. The interest earned and interest charged is calculated considering two cases namely $M \leq N$ and $M \geq N$ separately.

Case 1: $M \leq N$

In this case, we have two possible sub cases subcases:(1) $T+N \leq M$ and (2) $T+N \geq M$. Now we discuss two subcases separately
Sub-case 1.1: $T+N \leq M$. In this case the retailer receives the sales revenue at time T + N and is able to pay off the total purchasing cost at time M. Therefore, there is no interest charged. During the period $[N, T+N]$, the retailer can obtain the interest earned on the sales revenues received and on the full sales revenue during the period $[T+N, M]$. Therefore, the annual interest earned is

$$\frac{sI_e}{T}[\int_0^{T+N}\int_N^{t+T} D(u-N,N)dudt+(M-T-N)\int_0^T D(u,N)du]$$
$$= \frac{sI_e}{T}[\frac{(a+\lambda(N))T^2}{2}+\frac{bT^3}{6}+(M-T-N)\{(a+\lambda(N))T+\frac{bT^2}{2}\}] \quad (4)$$

Therefore, the retailer's annual total profit $\pi_1(N,T)$ is given by $\Pi_1(N,T)$= Net revenue after default risk-annual ordering cost - annual purchasing cost - annual holding cost + annual interest earned - interest charged- cost of deterioration

Therefore

$$\Pi_1(N,T) = se^{-kN}(a+\frac{b}{2}T+\lambda(N)) + \frac{sI_e}{T}[\frac{(a+\lambda(N))T^2}{2}+\frac{bT^3}{6}$$
$$+(M-T-N)\{(a+\lambda(N))T+\frac{bT^2}{2}\}] - \frac{A}{T} - \frac{c}{T}[\frac{b}{\theta^2} - \frac{(a+\lambda(N))}{\theta}$$
$$+\frac{\theta}{b}(a+bT+\lambda(N))e^{\theta T}] - \frac{h}{T}[\{(\frac{b}{\theta^2}-\frac{a}{\theta}-\frac{\lambda(N)}{\theta})+(\frac{a\theta}{b}+\theta T+\frac{N(\lambda))}{b}e^{\theta T}\}T - \frac{bT^2}{2\theta}$$
$$-\frac{(a+bT+\lambda(N))}{b}(e^{-\theta T}-1)] - c_1[(-\frac{b}{\theta^2}+\frac{a}{\theta}+\frac{\lambda(N)}{\theta})(1-T)$$
$$+(\frac{a\theta}{b}+\theta T+\frac{\theta\lambda(N)}{b})(e^{\theta T}-T)+\frac{bT^2}{2\theta}+\frac{a+bT+\lambda(N)}{b}(e^{-\theta T}-1)] \tag{5}$$

Sub-case 1.2: $M \leq T+N$. During the interval $[M, T+N]$, the retailer must pay the interest for the items unsold. Therefore, have interest charged per \$ in stocks per year to the retailer as

$$\frac{cI_p}{T}\int_M^{T+N} I(t-N)dt = \frac{cI_p}{T}\int_{M-N}^{T} I(t,N)dt$$
$$= \frac{cI_p}{T}[\frac{a+\lambda(N)}{2}\{T-(M-N)\}^2 + \frac{bT^2}{2}\{T-(M-N)\} - \frac{b}{6}\{T^3-(M-N)^3\}] \tag{6}$$

Also the retailer can earn interest from the delayed payment during period [N, M]. Therefore, we have the interest earned by the retailer is

$$\frac{sI_e}{T}\int_N^M\int_N^{t+N} D(u-N,N)dudt = \frac{sI_e}{T}[\frac{a+\lambda(N)}{2}(M-N)^2+\frac{b}{6}(M-N)^3] \tag{7}$$

As in *subcase 1.1*, the retailer's annual total profit is

$$\Pi_2(N,T) = se^{-kN}(a+\frac{b}{2}T+\lambda(N)) + \frac{sI_e}{T}[\frac{a+\lambda(N)}{2}(M-N)^2+\frac{b}{6}(M-N)^3]$$
$$-\frac{cI_p}{T}[\frac{a+\lambda(N)}{2}\{T-(M-N)\}^2+\frac{bT^2}{2}\{T-(M-N)\}-\frac{b}{6}\{T^3-(M-N)^3\}]$$
$$-\frac{A}{T}-\frac{c}{T}[\frac{b}{\theta^2}-\frac{(a+\lambda(N))}{\theta}+\frac{\theta}{b}(a+bT+\lambda(N))e^{\theta T}]$$
$$-\frac{h}{T}[\{(\frac{b}{\theta^2}-\frac{a}{\theta}-\frac{\lambda(N)}{\theta})+(\frac{a\theta}{b}+\theta T+\frac{N(\lambda))}{b}e^{\theta T}\}T-\frac{bT^2}{2\theta}$$
$$-\frac{(a+bT+\lambda(N))}{b}(e^{-\theta T}-1)] - c_1[(-\frac{b}{\theta^2}+\frac{a}{\theta}+\frac{\lambda(N)}{\theta})(1-T)$$
$$+(\frac{a\theta}{b}+\theta T+\frac{\theta\lambda(N)}{b})(e^{\theta T}-T)+\frac{bT^2}{2\theta}+\frac{a+bT+\lambda(N)}{b}(e^{-\theta T}-1)] \tag{8}$$

Case 2: $N \geq M$. Since $M \leq N$, there is no interest earned. The retailer must finance all the purchasing cost from $[M, N]$ and pay off the loan from $[N, T+N]$. Therefore, the interest charged per cycle is

$$\begin{aligned}
&\frac{cI_p}{T}[(N-M)Q + \int_N^{T+N} I(t-N)dt] \\
&\frac{cI_p}{T}[(N-M)(\frac{b}{\theta^2} - \frac{(a+\lambda(N))}{\theta} + \frac{\theta}{b}(a+bT+\lambda(N))e^{\theta T}) \qquad (9)
\end{aligned}$$

Therefore, the retailer's average total profit is given by

$$\begin{aligned}
\Pi_3(N,T) &= se^{-kN}(a+\frac{b}{2}T+\lambda(N)) - \frac{A}{T} - \frac{c}{T}[\frac{b}{\theta^2} - \frac{(a+\lambda(N))}{\theta} + \frac{\theta}{b}(a+bT+\lambda(N))e^{\theta T}] \\
&-\frac{h}{T}[\{(\frac{b}{\theta^2} - \frac{a}{\theta} - \frac{\lambda(N)}{\theta}) + (\frac{a\theta}{b} + \theta T + \frac{N(\lambda))}{b}e^{\theta T}\}T - \frac{bT^2}{2\theta} - \frac{(a+bT+\lambda(N))}{b}(e^{-\theta T}-1)] \\
&-c_1[(-\frac{b}{\theta^2} + \frac{a}{\theta} + \frac{\lambda(N)}{\theta})(1-T+(\frac{a\theta}{b} + \theta T + \frac{\theta\lambda(N)}{b})(e^{\theta T}-T) + \frac{bT^2}{2\theta} \\
&+\frac{a+bT+\lambda(N)}{b}(e^{-\theta T}-1)] - \frac{cI_p}{T}[(N-M)(\frac{b}{\theta^2} - \frac{(a+\lambda(N))}{\theta} + \frac{\theta}{b}(a+bT+\lambda(N))e^{\theta T})
\end{aligned} \qquad (10)$$

Therefore, the objective is to determine optimal customer's trade credit period N^* offered by retailer in years and the optimal replenishment time period T^* for the retailer such that the annual total profit $\Pi_i(N,T)$ for $i = 1,2,3$ is maximized.

Treating N and T as decision variables, the necessary conditions for the minimization of the average system cost are

$$\frac{\partial \Pi_i}{\partial N} = \frac{\partial \Pi_i}{\partial T} = 0, i = 1,2,3. \qquad (11)$$

The optimal values N^* of N and T^* of T are obtained by solving equations (11). The sufficient conditions that these values minimize $\Pi_i(N,T)$ are

$$\frac{\partial^2(\Pi_i)}{\partial N^{*2}}\frac{\partial^2(\Pi_i)}{\partial T^{*2}} - \left(\frac{\partial^2(\Pi_i)}{\partial N^* \partial T^*}\right)^2 > 0 \qquad (12)$$

$$\frac{\partial^2(\Pi_i)}{\partial N^{*2}} > 0, \quad \frac{\partial^2(\Pi_i)}{\partial T^{*2}} > 0, i = 1,2,3. \qquad (13)$$

Equations (11) can only be solved with the help of a computer oriented numerical technique for a given set of parameter values. Once N^* and T^* are obtained, we get ${\Pi_i}^*, i = 1,2,3$ from (5), (8) and (10) respectively.

4 Numerical Examples

Example 1: Let $a = 100$ units/year, $b = 0.2$/year, $d = 1$/year, $\theta = 0.01$, $u = 0.1$/year, $s = \$20$/unit, $k = 0.2$/year, $A = \$10$/order, $M = 0.5$/year, $h = \$5$/year, $c =\$10$/unit, $I_e = 0.09$, $I_p = 0.14$. Using software Mathematica 7.0, we have the optimal solution as follow:

${N_1}^* = 0.0$ years, ${T_1}^* = 0.1487$ years ; and ${\Pi_1}^*(N.T) = \$994.33$

${N_2}^* = 0.0$ years, ${T_2}^* = 0.2332$ years ; and ${\Pi_2}^*(N.T) = \$973.54$

${N_1}^* = 0.5$ years, ${T_1}^* = 0.3177$ years ; and ${\Pi_1}^*(N.T) = \$813.87$

Therefore, the retailer's optimal solution is ${N_1}^* = 0.0$ years, ${T_1}^* = 0.1487$ years ; and ${\Pi_1}^*(N.T) = \$994.33$.

Example 2 Taking the same parameter values as in Example 1,except $d = 5$/year, $u = 5$/year, $s = \$30$/unit, $k = 0.5$/year, the optimal solution can be obtained as follows.

${N_1}^* = 0.4327$ years, ${T_1}^* = 0.1883$ years ; and ${\Pi_1}^*(N.T) = \$2444.71$

${N_2}^* = 0.0$ years, ${T_2}^* = 0.5$ years ; and ${\Pi_2}^*(N.T) = \$2573.53$

${N_1}^* = 1.4421$ years, ${T_1}^* = 0.0678$ years ; and ${\Pi_1}^*(N.T) = \$27793.53$

Therefore, the retailer's optimal solution is ${N_1}^* = 1.4421$ years, ${T_1}^* = 0.0678$ years; and ${\Pi_1}^*(N.T) = \$27793.53$

5 Conclusion

From the first development of the inventory model by Harris in 1915, Inventory modellers and researchers are still very much curious about the nature of demand function. In the inventory model developed by Harris, the demand of item was assumed to be constant. This was taken only because to make the model simple. But in reality, there is hardly available product in market whose demand remains constant over time. When it was realized that this inventory model has very little real applications, inventory modellers tried to investigate which are the parameters on which the demand function is dependent. Numerous inventory models have been developed considering linear, exponential, quadratic and general nonlinear time dependent demand functions. Some researchers also understood that a display of large quantity of goods may increase its sales. This means that in this case the demand is dependent on the stock of goods. This dynamic behavior of demand was discussed by several researchers. Now most of the inventory management researches believe that demand is affected by several factors such as price, time, inventory level, and delay in payment period etc. In today's high-tech products demand rate increases significantly during the growth stage. Moreover, the marginal influence of the credit period on sales is associated with the unrealized potential market demand. In this paper, we have developed an EOQ model of an deteriorating item with under two-level trade credit financing involving default risk by considering demand to a credit-sensitive and linear non-decreasing function of time. Our objective is to find optimal customer's trade credit period offered by retailer in years and optimal replenishment time period for the retailer. The model is solved analytically and the optimal solution is illustrated with the help of numerical examples.

This model can be extended by several ways, for example, we may extend the model to allow for shortages and partially backlogging. Also the effect of inflation rates on the economic order quantity can also be considered.

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EFFECTS OF LONG-TERM TILLAGE SYSTEMS ON SOIL WATER CONTENT AND WHEAT YIELD UNDER MEDITERRANEAN CONDITIONS

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Abstract – Agricultural practices conserving soil water are needed to sustain agricultural production under changing climate. Long-term (2006-2014) effects of six different tillage systems on soil water content (SWC) and wheat yield were investigated in a clayey soil of the Çukurova region, Turkey. The tillage treatments were; conventional tillage with stubble (moldboard plowing) (CT1), conventional tillage with stubbles burned (CT2), heavy disc harrow reduced tillage (RT1), rototiller reduced tillage (RT2), heavy disc harrow zero soil tillage (RNT) and no till or zero tillage (NT). Soil moisture content was measured at different times of the rainfed wheat production season in 2014-2015. Tillage practices had statistically important effects on SWC on 6 February, 9 March, 17 April and 8 May 2015. Although moisture values measured on February, 6 and March, 9 were optimal for plant growth, SWC under conservation tillage practices were higher compared to conventional tillage practices. However, tillage practices had no significant effect on the wheat yield. These results showed that reduced and no-tillage practices can be alternative to conventional tillage practices under Mediterranean conditions.

***Keywords* –** *Soil tillage, Soil water content, Wheat yield, Mediterranean*

1 Introduction

The shortage of water resources worldwide is one of the major limiting factors of agricultural development, which severely threats the global food security [15]. Rainfed crops such as durum wheat are highly sensitive to the dry conditions. Due to the strong influence of soil moisture on crop yield [7], management practices cause to increase soil

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water storage potential are important to be adopted for sustaining crop production in arid and semi-arid regions of the world.

Tillage treatments directly or indirectly influence soil hydraulic properties such as water infiltration, hydraulic conductivity, and water retention which determine the ability of the soil to capture and store water through precipitation or irrigation [2]. Tillage changes flow path and rate of water by rearranging aggregate size distribution [9]. Therefore, tillage systems conserving water in soil are important for plant growth under arid and semiarid conditions. Conservation tillage systems increase soil water by reducing evaporation [17, 18], increasing water infiltration [13] because of crop residues remained on soil surface. In contrast, Blanco-Canqui et al. [2] reported that disrupting compacted layers and loosening the soil by tillage may increase infiltration of water relative to no-till management.

Management practices improve soils' ability to drain when wet and hold more water under dry season are preferred to support rainfed crop growth. The extent of tillage effect on the ability of the soil to adsorb and retain water depends on the level of soil disturbance [2]. Studies from different regions of the world have shown that conservation tillage systems are important for crop production because of increasing soil water content [10, 11, 12, 19, 21]. Fernandez-Garcia et al. [11] reported that more water was stored at sowing depth under no-tillage compared to conventional tillage which improved chickpea grain yield. Copec et al. [8] reported that the highest average SWC was measured under the no-tillage compared to conventional system which reduces water infiltration by weakening soil aggregate stability and decreasing macro-porosity and increasing surface crusting [33]. In another study, SWC was improved by no-till system during whole agricultural season [19].

Understanding the changes in soil hydraulic properties under long term tillage systems is important to manage soil water under different soil types, management options, and climates [2]. However, lack of data on water retention of soils obtained in long-term experiments limits the understanding of the efficiency of tillage systems on soil water management. Therefore, this information is particularly necessary in the regions where water is limited for crop production or crop production is performed under rainfed conditions. The long-term effects of conservation tillage systems on SWC have scarcely been studied in Turkey. The objective of this study was to determine the long-term (8-year) effects of conventional, reduced and no-tillage practices on SWC and wheat yield in a heavy clay soil under Mediterranean climate.

2 Materials and Methods

2.1 Experimental Site

A long-term field experiment established in 2006 at the Experimental Farm (37^o 00′ 54″ N, 35^o 21′ 27″ E; 32 m above sea level) of the Çukurova University in Adana, Turkey under wheat, corn and soybean rotation (Figure 1). The soils of study were clayey Arık soils and classified as fine, smectitic, active, mesic Typic Haploxererts [29] with a pH of 7.82, $CaCO_3$ of 244 g kg^{-1}, electrical conductivity of 0.15 dS m^{-1} and particle size distribution of 50% clay, 32% silt and 18% sand at the surface horizon (0-30 cm) [5]. Arık soils are deep formed over old terraces of Seyhan River and well drained with almost zero slope.

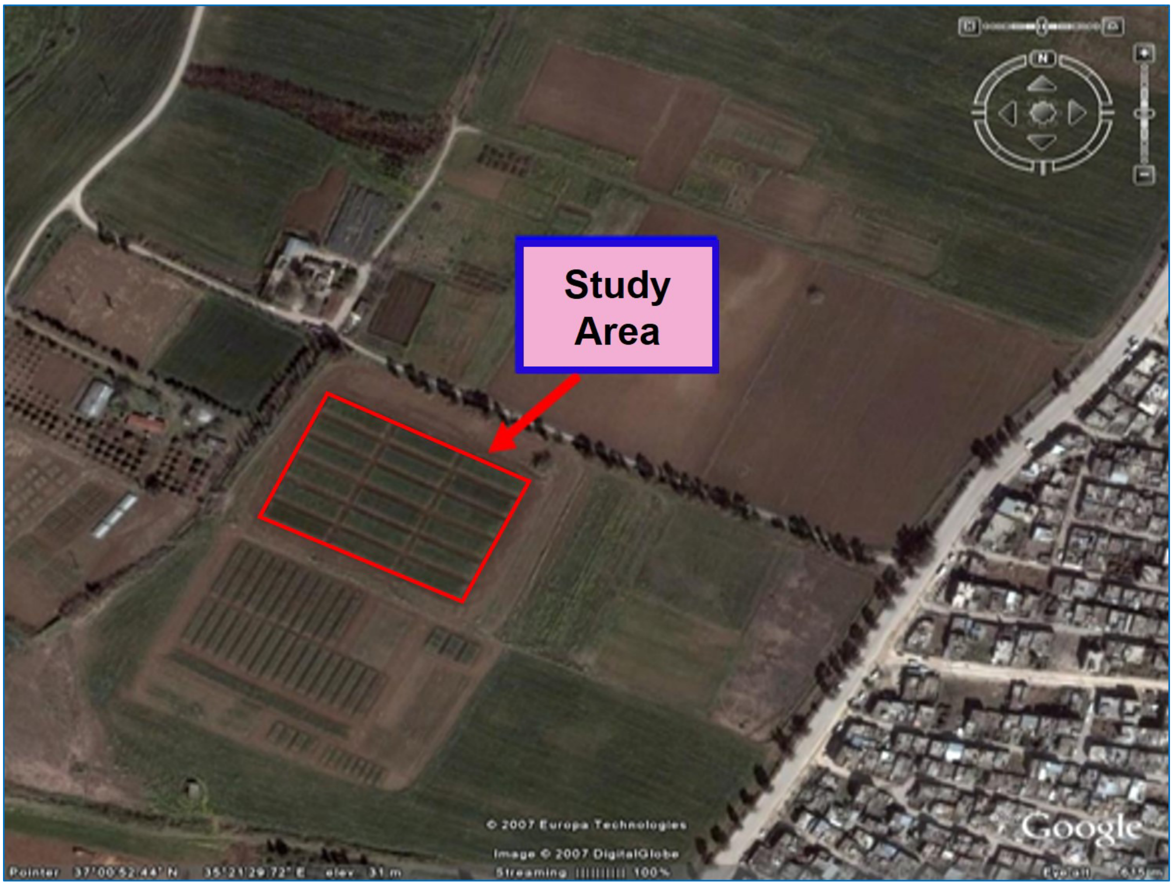

Figure 1. Study area

Table 1. Long-term (1985-2014) mean climate data of Adana province

	Months												
	1.	2.	3.	4.	5.	6.	7.	8.	9.	10.	11.	12.	Annual
Maximum Temperature (^oC)	15.2	16.3	19.7	24.1	28.2	31.6	33.8	34.7	33.1	29.0	22.2	16.7	25.4
Minimum Temperature (^oC)	5.5	6.0	8.5	12.4	16.3	20.5	24.0	24.4	21.0	16.4	10.7	7.1	14.4
Annual Temperature (^oC)	9.6	10.4	13.5	17.8	22.0	25.9	28.5	29.0	26.4	21.7	15.2	11.0	19.2
Evaporation (mm)	48.4	55.4	87.2	118.7	165.2	214.3	239.7	226.9	175.5	120.9	66.5	48.1	130.6 (Total:1567)
Annual Precipitation (mm)	96.1	81.7	62.0	46.7	46.5	17.9	9.0	6.8	17.6	48.1	81.6	125.1	53.3 (Total: 639)

The prevailing climate of the study area is Mediterranean with a long-term (30 years) mean annual precipitation from 1985 to 2014 was 639 mm, about 75% of which falls during the winter and spring (from November to May) and the long-term mean annual potential evapotranspiration is 1567 mm. The summers are hot and dry, and winters are wet and mild.

Table 2. Daily rainfall in wheat production season (November 2014-June 2015)

Days	November 2014	December 2014	January 2015	February 2015	March 2015	April 2015	May 2015	June 2015
	(mm)							
1	2.1	-	3.1	3.5	-	0.4	-	-
2	-	-	18.0	-	8.9	-	-	-
3	-	-	6.7	-	1.5	1.7	-	-
4	-	-	-	1.2	-	1.7	-	4.0
5	-	-	1.5	-	-	4.4	-	0.1
6	-	-	58.5	-	-	-	-	-
7	-	-	3.2	-	-	-	-	-
8	-	-	0.9	-	-	-	-	-
9	-	6.7	-	1.8	-	0.3	-	-
10	-	34.6	-	3.3	-	-	0.5	-
11	-	-	-	26.5	0.5	8.5	-	-
12	-	-	-	47.0	32.2	-	32.4	-
13	-	36.0	-	10.8	28.8	-	0.7	-
14	-	-	-	4.9	0.4	-	4.3	-
15	-	-	-	0.7	-	-	0.4	-
16	-	-	-	4.2	18.2	-	-	-
17	0.1	-	-	0.6	-	-	-	-
18	-	-	-	1.5	-	-	-	-
19	-	2.9	-	1.2	1.2	-	-	-
20	-	0.3	-	10.0	8.2	-	-	-
21	6.2	-	-	-	18.9	-	-	-
22	15.4	0.2	-	-	3.8	1.0	-	-
23	-	-	-	-	-	3.5	-	-
24	-	-	-	-	3.7	-	-	-
25	-	-	-	4.8	-	-	-	-
26	39.5	-	-	-	-	-	-	-
27	3.2	-	-	-	-	-	-	0.7
28	-	4.9	-	-	-	-	-	-
29	-	5.5	2.3	-	5.3	-	-	-
30	-	0.2	4.9	-	3.5	-	10.4	-
31	-	15.1	8.4	-	-	-	17.0	-
Total	66.5	106.4	107.5	122.0	135.1	21.5	65.7	4.8
General total (November 2014-June 2015): 630 mm								

Long term total mean precipitation between November and June, wheat production season for Çukurova region was recorded as 558 mm. Total mean rainfall from November 2014 through June 2015 wheat production season was 630 mm which corresponds to %13 more rainfall compared to the last 30 years' average. During the wheat production season, the rainfalls recorded for the months of January, February, March and May were above the average of the last 30 years. In these months, %12, %49, %118 and %41 more rain was obtained, respectively.

2.2 Experimental Design and Tillage Systems

The experiment was designed in a randomized complete block where similar experimental units were grouped into the blocks or replicates. The treatments were conventional tillage with residue incorporated in the soil (CT1), conventional tillage with residue burned (CT2), reduced tillage with heavy tandem disc-harrow (RT1), reduced tillage with rotary tiller (RT2), reduced tillage with heavy tandem disc harrow followed by no-tillage (RNT) for the second crop, and no tillage (NT). The tillage plots were 12 m wide and 40 m long (480 m^2). A buffer-zone of 4 m was reserved around each plot for tractor and, tillage equipment operations. The detailed information on treatments within each practice and sowing methods were given in detail by Celik et al. [6].

The rotation of winter wheat (*Triticum aestivum* L.)-corn (*Zea Mays* L.), winter wheat (*Triticum aestivum* L.)-soybean (*Glycine max.* L.) were applied in all tillage treatments from 2006 to 2014. In each growing season, the first crop was winter wheat and the second crop was either corn or soybean. Two weeks prior to sowing, the total herbicide (500 g ha^{-1} Glyphosate) was used to control weeds in the NT and RNT treatments. Compound NP-fertilizers were applied in the seedbed at the rates of 172 kg N ha^{-1} and 55 kg P ha^{-1} for wheat, 250 kg N ha^{-1} and 60 kg P ha^{-1} for corn, and 120 kg N ha^{-1} and 40 kg P ha^{-1} for soybean. Winter wheat was sown in the first week of November from 2006 to 2013 at a seeding rate of 240 kg ha^{-1}, and harvested in the first week of June 2007 to 2014. The second crop (corn or soybean) were sown in the third week of June, and harvested in the second week of October. Corn and soybean seeding rates were 8.4 and 23.6 plants per m^2, respectively. Soybean and corn were irrigated nine times by sprinklers in 13-day intervals. The amount of water applied for each irrigation was identical for all treatments and no irrigation water was applied to the wheat.

2.3 Soil Moisture Measurements

After the wheat planting, with the aim of monitoring and determining the effects of tillage practices on SWC, three time-domain reflectometry (TDR) probes were placed in each research plot and moisture measurements were performed from spring rains till wheat harvest (Figure 2). The moisture measurements were conducted on February 6, March 9, April 17, April 28, May 8, May 29 and June 8, 2015. Since the deepest tillage equipment had an impact on 30-32 cm depth, the TDR probes were placed in 0-30 cm depth.

Figure 2. Measuring with time domain reflectometer (TDR) in experimental plots

For the calibration of the moisture measurements made with TDR device, the gravimetric moisture content of the soil was measured on each measurement period. In order to convert gravimetric moisture measurement to volumetric moisture, bulk density was determined in the undisturbed soil samples taken from parcels and converted to the volumetric moisture (%) as shown on equation 1 [34].

$$\theta = \rho_b . P_w \tag{1}$$

θ: Volumetric moisture content (%),
ρ_b: Bulk density (g cm^{-3}),
P_w: Gravimetric moisture content (%).

In order to calibrate the volumetric moisture values measured with the TDR device, a calibration curve was created. The equation for the created curve was given on Equation 2. With the help of this equation, each moisture value measured in 0-30 cm soil depth with TDR device was calibrated.

$$y = 0{,}68x + 13{,}47$$
$$R^2 = 0{,}7914 \tag{2}$$

y: Calibrated volumetric moisture values (%).
x: Volumetric moisture value measured with TDR device.

2.4 Statistical Analyses

The effects of tillage practices on SWC was assessed by one-way analyses of variance test (ANOVA) using JMP statistical program. The least-significant difference (LSD) method was used for mean comparisons among different treatments. Differences among means of tillage treatments were reported at the 0.05 probability level.

3 Results and Discussion

3.1 Soil Water Content

Water contents of soils measured at different periods of wheat growing season were presented in Table 3. Tillage practices had significant effect on soil water retention measured in February 6, March 9, April 17 and May 8. The SWC decreased by time as precipitation decreased. The highest soil water contents in February 6 and March 9 were obtained under RT2 whereas the lowest soil water contents at the same measurements were obtained in CT2 treatment (Table 3). The differences in SWC for February 6 and March 9 measurements were distinct between CT that moldboard plow used and the rest of tillage systems. Higher moisture content retention of conservation tillage practices is a result of crop residues left on the soil surface which increased the water infiltration [13, 17, 23, 25, 30, 31, 32] and water storage capacity of soils [3, 4, 16, 25, 27]. The highest SWC in April 17 and May 8 were measured under RNT and CT2 systems, respectively. However, the lowest values were obtained in CT1 and RT2 practices in April 17 and May 8. Inconsistent results have been published on comparing soil hydraulic properties among different tillage systems. Lyon et al. [24] reported a significant increase in soil water retention under no-till practice, whereas McVay et al. [26] and Blanco-Canqui et al. [2] could not observe a meaningful change in soil water retention under no-till compared to reduced and conventional practices.

Table 3. Effects of different tillage treatments on soil water content

Tillage treatments	Measure times							
	6 February	9 March	17 April	28 April	8 May	20 May	29 May	8 June
	Soil water content (Volumetric, %)							
CT1	37.6 ± 0.6^{d}	34.7 ± 1.3^{c}	30.9 ± 2.7^{c}	31.7 ± 1.8^{a}	31.1 ± 0.9^{abc}	32.3 ± 0.6^{a}	31.6 ± 0.6^{a}	32.5 ± 0.8^{a}
CT2	37.8 ± 0.6^{d}	35.7 ± 0.7^{b}	33.4 ± 1.8^{ab}	33.1 ± 0.9^{a}	32.1 ± 0.9^{a}	33.3 ± 0.7^{a}	32.2 ± 0.5^{a}	34.2 ± 0.8^{a}
RT1	38.7 ± 0.4^{ab}	37.9 ± 0.4^{a}	32.2 ± 1.5^{abc}	31.9 ± 1.4^{a}	30.3 ± 0.3^{c}	32.2 ± 2.1^{a}	30.7 ± 0.8^{a}	33.2 ± 0.9^{a}
RT2	39.1 ± 0.3^{a}	38.1 ± 0.7^{a}	31.9 ± 0.5^{bc}	31.0 ± 1.3^{a}	29.9 ± 1.8^{c}	32.0 ± 1.4^{a}	31.9 ± 1.7^{a}	33.4 ± 1.3^{a}
RNT	38.4 ± 0.6^{bc}	37.7 ± 0.3^{a}	34.0 ± 0.7^{a}	32.5 ± 0.6^{a}	31.5 ± 0.4^{ab}	32.3 ± 0.7^{a}	31.5 ± 0.2^{a}	33.0 ± 1.0^{a}
NT	39.0 ± 0.5^{ab}	37.8 ± 0.5^{a}	31.8 ± 0.7^{bc}	30.9 ± 0.3^{a}	30.7 ± 0.2^{bc}	31.7 ± 0.6^{a}	31.3 ± 0.9^{a}	34.2 ± 0.8^{a}
LSD_{till}	0.16**	0.98**	2.00*	ns	1.22*	ns	ns	ns

*Mean values ± standard deviation. Values followed by the same letters in a column are not significantly different (P < 0.05). *: Difference is significant at P ≤ 0.05 level, **: Difference is significant at P ≤ 0.01 level, ns: Difference is not significant. CT1: Conventional tillage with residue incorporated, CT2: Conventional tillage with residues burned, RT1: Reduced tillage with heavy tandem disc harrow, RT2: Reduced tillage with rotary tiller, RNT: Reduced tillage with heavy tandem disc harrow fallowed by no tillage for the second crop, NT: No tillage*

The moisture contents measured in February 6 and March 9 at all experimental plots were optimal for plant growth, and SWC under conservation tillage practices were higher compared to conventional tillage practices. The results showed that especially for the high moisture contents, conservation tillage practices retained higher moisture compared to conventional tillage practices. The absence of crop residue on soil surface under conventional practices may increase the formation of surface crust and consequently decreases the infiltration of water to soil. The SWC under conservational tillage systems decreased from April 17 to April 28, whereas SWC under CT1 and CT2 either not changed or slightly increased. The moisture contents under different tillage practices seemed was similar and non-significant in April 28 measurement.

The highest moisture content in April 17 was obtained under RNT treatment (34.0%) and this value was 10% higher than the CT1 (%30.9) that had the lowest value in April 17. As the SWC decreased with the evaporation and plant use, soil water content in May 8 under CT2 system was 6% and 7% higher than that obtained in RT1 and RT2 systems, respectively. The effects of tillage practices on soil water retention was not significant in May 20, May 29 and June 8 measurements. The precipitation (32.4 mm) occurred eight days prior to the May 20 measurement increased the soil moisture content compared to May 8 water content. Celik et al. [6] reported higher organic matter content under no-till (2.56%) compared to CT2 (1.48%) and CT1 (1.51%) systems at 0-15 cm depth. Indeed, increased organic matter content of soils is expected to increase ability of soils water retention. We have observed higher SWC under no-till system than CT system in rainy months of February and March, but the SWC was not significantly higher under no-till for the rest of the moths. Raws et al. [28] reported that soil water retention may or may not change with the change in soil organic carbon. Because soil water retention depends on clay content, initial organic matter content and site-specific interactions of clay and organic matter. High clay content of soil which has inherently high water holding capacity, may mask the effects of increased organic matter on water retention under no-till system. Blanco-Canqui et al. [2] recommended additional conservative practices such as addition of cover crops to improve hydraulic properties of soils under no-till management

3.2 Winter Wheat Yield

The tillage practices had no significant effect on wheat yield. The highest wheat yield was obtained under RT1 (6.29 t ha^{-1}), and the lowest (5.66 t ha^{-1}) yield was obtained under CT2 (Table 4).

Table 4. Effects of different tillage treatments on wheat yield

Tillage Treatments	Wheat yield (t ha^{-1})
CT1	6.10 ± 0.50^a
CT2	5.66 ± 0.79^a
RT1	5.96 ± 0.40^a
RT2	6.29 ± 0.67^a
RNT	6.11 ± 0.10^a
NT	6.24 ± 0.55^a

Mean values ± standard deviation. Values followed by the same letters in a column are not significantly different (P < 0.05). CT1: Conventional tillage with residue incorporated, CT2: Conventional tillage with residues burned, RT1: Reduced tillage with heavy tandem disc harrow, RT2: Reduced tillage with rotary tiller, RNT: Reduced tillage with heavy tandem disc harrow fallowed by no tillage for the second crop, NT: No tillage

Total rainfall during 2014-2015 wheat growing season was %13 higher compared to the last 30 years average. High rainfall probably masked the effects of tillage practices on water retention and wheat yield. Our results are in accordance with the Anken et al. [1] and Kosutic et al. [20] who also no significant effect of tillage practices on wheat crop yield. On the contrary, some researchers reported lower wheat yield under conservation tillage practices compared to conventional tillage practices [14, 22]. The results showed that as an alternative to conventional tillage, reduced and no-tillage practices provided successful wheat crop production in a clay soil under a semi-arid Mediterranean climate.

4 Conclusions

The long-term (8-years) effects of six different tillage practices on SWC and wheat yield were evaluated. The results indicated that SWC under conservation tillage practices at rainy months was higher than that obtained under conventional tillage practices. Significant differences on SWC were obtained among tillage practices evaluated, whereas the effect of tillage practices on wheat yield was not statistically significant. The results of water content indicated that six tillage systems were similar in their ability to capture precipitation during wheat growing period. In spite of non-significant effect, wheat yield under no-till and one of the reduced till systems was higher compared to CT2 where moldboard plow is used and crop residue is burned. The results revealed that reduced and no-tillage practices can be an alternative to conventional tillage practices under Mediterranean climate. It is important to note that the data presented in this paper is belonged to a single season which had more rain than the long-term average. The differences among tillage systems can be better defined by monitoring hydraulic properties on a regular basis.

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