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CONTENT

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PROPERTIES AND APPLICATIONS OF $\theta g^*\alpha$-CLOSED SETS IN TOPOLOGICAL SPACES

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Abstaract − In this paper we discussed properties and applications of $\theta g^*\alpha$-closed sets. $\theta g^*\alpha$-closed sets is introduced by Chandrasekar et al. Moreover we analyze some basic properties and applications of neighbourhoods, limit points, border, frontier and exterior of $\theta g^*\alpha$-closed sets.

Keywords − *$\theta g^*\alpha$-closed sets, $\theta g^*\alpha$-neighbourhoods, $\theta g^*\alpha$-limit points, $\theta g^*\alpha$-border, $\theta g^*\alpha$-frontier, $\theta g^*\alpha$-exterior.*

1 Introduction

The concepts of θ-closed set, δ-closed set, first introduced by Velicko [16]. θ-closed set have been studied intensively by many authors. Since the advent of these notions, several researches have been done which produced interesting results. θg-closed set introduced by Dontchev and Maki [7] in 1999. In 1965 Njastad [9] introduced α-open sets. In [3], $\theta g^*\alpha$ closed set introduced by Chandrasekar et al. In this paper, we discussed properies and applications of $\theta g^*\alpha$-neighbourhoods, $\theta g^*\alpha$-limit points, $\theta g^*\alpha$- border, $\theta g^*\alpha$-frontier and $\theta g^*\alpha$-exterior.

2 Preliminary

Let us recall the following definition, which are useful in the sequel.

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Definition 2.1. A subset A of a space (X, τ) is called:

1. α- closed set [9] if $cl(int(cl(A))) \subseteq A$
2. θ-closed [16] if $A = cl_\theta(A)$, where $cl_\theta(A) = \{x \in X : cl(G) \cap A \neq \phi, G \in \tau$ and $x \in G\}$
3. a generalized closed (briefly, g-closed) set [8] if $cl(A) \subseteq G$ whenever $A \subseteq G$and G is open in (X, τ).
4. a θ-generalized closed (briefly, θg-closed) set [6] if $cl_\theta(A) \subseteq G$ whenever A $\subseteq G$ and G is open in (X, τ).
5. $\theta g^*\alpha$-closed set if $\alpha cl(A) \subseteq G$ whenever $A \subseteq G$ and G is θg-open in (X, τ).

The complements of the above mentioned closed sets are their respective open sets.

Definition 2.2. The union (respectively intersection) of all $\theta g^*\alpha$-open (respectively $\theta g^*\alpha$-closed) sets, each contained in (respectively containing) a set A of X is called the $\theta g^*\alpha$-interior (respectively $\theta g^*\alpha$-closure) of A, which is denoted by $\theta g^*\alpha$-int(A) (respectively $\theta g^*\alpha$-cl(A)).

Proposition 2.3. If A and B are subsets of X, then

1. A is $\theta g^*\alpha$-open if and only if $\theta g^*\alpha$-int(A)=A.
2. $\theta g^*\alpha$-int(A) is $\theta g^*\alpha$-open.
3. A is $\theta g^*\alpha$-closed if and only if $\theta g^*\alpha$-cl(A)=A.
4. $\theta g^*\alpha$-cl(A) is $\theta g^*\alpha$-closed.
5. $\theta g^*\alpha$-cl(X∩ A)=X∩ $\theta g^*\alpha$-int(A).
6. $\theta g^*\alpha$-int(X∩ A)=X∩ $\theta g^*\alpha$-cl(A).
7. If A is $\theta g^*\alpha$-open in X and B is open in X, then A B is $\theta g^*\alpha$-open in X.
8. A point x ∈ $\theta g^*\alpha$-cl(A) if and only if every $\theta g^*\alpha$-open set in X containing x intersects A.
9. Arbitrary intersection of $\theta g^*\alpha$-closed sets in X is also $\theta g^*\alpha$-closed in X.

3 $\theta g^*\alpha$-Neighbourhoods

In this section, we define and study about $\theta g^*\alpha$-neighbourhood, and some of their properties are analogous to those for open sets.

Definition 3.1. Let (X, τ) be a topological space and let x∈ X. A subset N of X is said to be $\theta g^*\alpha$-neighbourhood of a point x∈ X if there exists a $\theta g^*\alpha$-open set G such that x∈G⊂ N

Definition 3.2. Let(X, τ) be a topological space and A be a subset of X .A subset N of X is said to be $\theta g^{*}\alpha$-neighbourhood of A if there exists a $\theta g^{*}\alpha$-open set G such that A∈G⊂ N.

The collection of all $\theta g^{*}\alpha$-neighbourhood of x∈ X is called the $\theta g^{*}\alpha$-neighbourhood system at x and shall be denoted by $\theta g^{*}\alpha$-N(x).

It is evident from the above definition that a $\theta g^{*}\alpha$-open set is a $\theta g^{*}\alpha$-neighbourhood of each of its points. But a $\theta g^{*}\alpha$-neighbourhood of a point need not be a $\theta g^{*}\alpha$-open set. Also every $\theta g^{*}\alpha$-open set containing x is a $\theta g^{*}\alpha$-neighbourhood of x.

Theorem 3.3. A subset of a topological space is $\theta g^{*}\alpha$-open if it is a $\theta g^{*}\alpha$-neighbourhood of each of its points.

Proof: Let a subset G of a topological space be $\theta g^{*}\alpha$-open. Then for every x∈G,x∈G⊂ N and therefore G is a $\theta g^{*}\alpha$-neighbourhood of each of its points. analogous to those for open sets. The converse of the above Theorem need not be true as seen from the following example.

Example 3.4. Let (X, τ) be topological space and X={a,b,c,d} with topology

τ ={X,ϕ ,{a},{b},{a, b},{b,c},{a,b,c}}.
$\theta g^{*}\alpha$-Cl(X) = {X, ϕ ,{c},{d},{a, d},{b, d},{c, d},{a, c, d},{a ,b ,d},{b, c, d}}
$\theta g^{*}\alpha$-O(X) = {X,ϕ ,{a},{b},{c},{a, b},{b, c},{a ,c},{a, b ,c},{a, b, d}}

the set{a, c, d} is the neighbourhood of {a, c}, since a, c ∈{a, c}⊂ {a, c, d}, and {a, c, d} is the $\theta g^{*}\alpha$-neighbourhood of each of its points.

Theorem 3.5. Let (X, τ) be a topological space. If A is a $\theta g^{*}\alpha$-closed subset of X and x∈ X\ A, then there exists a $\theta g^{*}\alpha$-neighbourhood N of x such that N ∩A=ϕ

Proof: Since A is $\theta g^{*}\alpha$-closed, then X\ A is $\theta g^{*}\alpha$-open set in (X, τ) . By the above Theorem 3.3, X\ A contains a $\theta g^{*}\alpha$-neighbourhood of each of its points. Hence there exists a $\theta g^{*}\alpha$-neighbourhood N of x, such that N⊂ X\ A.That is, no point of N belongs to A and hence N∩A=ϕ .

Theorem 3.6. Let (X, τ) be a topological space and A⊆X. Then x∈ $\theta g^{*}\alpha$-cl(A) if and only if for any $\theta g^{*}\alpha$-neighbourhood N of x in (X, τ),A∩ N≠ϕ.

Proof: Suppose x∈ $\theta g^{*}\alpha$-cl(A). Let us assume that there is a$\theta g^{*}\alpha$-neighbourhood N of the point x in (X, τ) such that N ∩ A=ϕ . Since N is a $\theta g^{*}\alpha$-neighbourhood of x in (X, τ) by definition of $\theta g^{*}\alpha$-neighbourhood there exists an $\theta g^{*}\alpha$-open set G of x such that x∈G⊂ N. Therefore we have G ∩ A=ϕ and so A ⊆ H^{c}. Since X\G is an $\theta g^{*}\alpha$-closed set containing A.We have by definition of $\theta g^{*}\alpha$-closure, $\theta g^{*}\alpha$-cl(A)⊆ X\G and therefore x∉$\theta g^{*}\alpha$-cl(A), which is a contradiction to hypothesis x∈ $\theta g^{*}\alpha$-cl(A). Therefore A∩ N≠ϕ . Conversely, Suppose for each $\theta g^{*}\alpha$-neighbourhood N of x in (X, τ) .A ∩ N≠ϕ . Suppose that x ∈ $\theta g^{*}\alpha$-cl(A). Then by definition of $\theta g^{*}\alpha$-cl(A), there exists a $\theta g^{*}\alpha$-closed set G of (X, τ) such that A⊆ G and x∉G. Thus x∈ X\G and X\ G is $\theta g^{*}\alpha$-open in (X, τ) and hence X\G is a $\theta g^{*}\alpha$-neighbourhood of x in (X, τ) .But A∩ (X\ G)= ϕ which a contradiction.Hence x∈ $\theta g^{*}\alpha$-cl(A).

Theorem 3.7. Let (X, τ) be a topological space and p∈ X. Let $\theta g^*\alpha$-N(p) be the collection of all $\theta g^*\alpha$-neighbourhoods of p. Then

1. $\theta g^*\alpha$-N(p)$\neq\phi$ and p each member of $\theta g^*\alpha$-N(p).

2. The intersection of any two members of $\theta g^*\alpha$-N(p) is again a member of $\theta g^*\alpha$-N(p).

3. If N $\in \theta g^*\alpha$-N(p) and M⊆N, then M $\in\theta g^*\alpha$-N(p).

4. Each member N∈ $\theta g^*\alpha$ -N(p) is a superset of a member G∈ $\theta g^*\alpha$-N(p) where G is a $\theta g^*\alpha$-open set.

Proof:

1. Since X is a $\theta g^*\alpha$-open set containing p, it is a $\theta g^*\alpha$-neighbourhood of every p∈ X. Hence there exists at least one $\theta g^*\alpha$-neighbourhood namely X for each p ∈X. Here $\theta g^*\alpha$-N(p)$\neq\phi$. Let N $\in\theta g^*\alpha$-N(p), N is a $\theta g^*\alpha$-neighbourhood of p.Then there exists a $\theta g^*\alpha$-open set G such that p∈ G ⊆N.So p∈ N.Therefore p ∈ every member N of $\theta g^*\alpha$-N(p).

2. Let N ∈ $\theta g^*\alpha$-N(p) and M∈$\theta g^*\alpha$-N(p). Then by definition of $\theta g^*\alpha$-neighbourhood, there exists $\theta g^*\alpha$-open sets G and F such that p∈ N and p∈ F ⊆ M. Hence p∈ G∩F⊆ M ∩N, Note that G ∩ F is a $\theta g^*\alpha$-open set since intersection of $\theta g^*\alpha$-open sets is $\theta g^*\alpha$-open.Therefore it follows that N ∩ M is a $\theta g^*\alpha$ - neighbourhood of p. Hence N∩ M $\theta g^*\alpha$-N(p).

3. If N∈ $\theta g^*\alpha$-N(p) then there is an $\theta g^*\alpha$-open set G such that p∈ G⊆ N.Since M ∩ N, M is a $\theta g^*\alpha$- neighbourhood of p. Hence M ∈ $\theta g^*\alpha$-N(p).

4. Let N ∈ $\theta g^*\alpha$-N(p).Then there exist an $\theta g^*\alpha$-open set G such that p∈ G⊆ N.Since G is $\theta g^*\alpha$-open and p ∈ G, G is $\theta g^*\alpha$-neighbourhood of p. Therefore G ∈ $\theta g^*\alpha$-N(p) and also G⊆ N.

4 $\theta g^*\alpha$-limit Points

In this section, we define and study about $\theta g^*\alpha$-limit point and $\theta g^*\alpha$-derived set of a set and show that some of their properties.

Definition 4.1. Let(X, τ) be a topological space and A be a subset of X.Then a point x∈ X is called a $\theta g^*\alpha$-limit point of A if and only if every $\theta g^*\alpha$-neighbourhood of x contains a point of A distinct from x. That is [N\x]∩ A$\neq\phi$ for each $\theta g^*\alpha$-neighbourhood N of x.

Also equivalently if and only if every $\theta g^*\alpha$-open set G containing x contains a point of A other than x.

In a topological space (X, τ) the set of all $\theta g^*\alpha$-limit points of a given subset A of X is called a $\theta g^*\alpha$-derived set of A and it is denoted by $\theta g^*\alpha$-d(A).

Theorem 4.2. Let A and B be subsets of a topological space(X, τ). Then

1. $\theta g^*\alpha$-D(ϕ) = ϕ
2. $\theta g^*\alpha$-D(A) $\subset$ D(A) where D(A) is the derived set of A.
3. If A\B, then a $\theta g^*\alpha$-D (A)$\subseteq$ $\theta g^*\alpha$-D(B),
4. If x$\in\theta g^*\alpha$-D(A), then x$\in$ $\theta g^*\alpha$-D[A\{x}],
5. $\theta g^*\alpha$-D(A$\cup$ B)$\supset$ $\theta g^*\alpha$-D(A)$\cup$ $\theta g^*\alpha$-D(B),
6. $\theta g^*\alpha$-D(A$\cap$ B)$\subseteq\theta g^*\alpha$-D(A) $\cap$ $\theta g^*\alpha$-D(B).

Proof:

1. For all $\theta g^*\alpha$ open set U and for all x$\in$X, U$\cap$ {ϕ\x} =ϕ . Hence $\theta g^*\alpha$-D(ϕ) = ϕ.
2. Since every open set is $\theta g^*\alpha$-open, the proof follows.
3. If X$\in\theta g^*\alpha$-D(A), that is if x is $\theta g^*\alpha$-limit point of A, then by Definition 4.1[G\{x}]A$\cap$ ϕ . for every $\theta g^*\alpha$-open set G containing x. Since A$\subseteq$ B implies [G\{x}]$\cap$ A $\subseteq$ [G\{x}]$\cap$ B. Thus if x is a $\theta g^*\alpha$-limit point of A it is also a $\theta g^*\alpha$-limit point of B, that is x $\in$ $\theta g^*\alpha$-D(B). Hence $\theta g^*\alpha$-D(A)$\subseteq$ $\theta g^*\alpha$-D(B).
4. If x$\in$ $\theta g^*\alpha$-D(A), that is x is a $\theta g^*\alpha$-limit point of A. Then by Definition 4.1 every $\theta g^*\alpha$- open set G containing x contains at least one point other than x of A \ {x}. That is G$\cap$(A \{x})$\neq$ ϕ Hence x is a $\theta g^*\alpha$-limit point of A\{x} and as such it belongs to $\theta g^*\alpha$-D[A\{x}].Therefore x$\in$ $\theta g^*\alpha$-D(A) $\Rightarrow$ x$\in$ $\theta g^*\alpha$-D[A\ {x}].
5. Since A$\subseteq$ A$\cup$B and B $\subseteq$ A$\cup$ B,it follows from (2)$\theta g^*\alpha$-D(A)$\subseteq$ $\theta g^*\alpha$-D(A$\cup$ B) and $\theta g^*\alpha$-D(B)$\subseteq$ $\theta g^*\alpha$-D(A$\cup$ B) and hence $\theta g^*\alpha$-D(A) $\cup$ $\theta g^*\alpha$-D(B)$\subseteq$ $\theta g^*\alpha$-(A$\cup$ B).To prove the other way that is $\theta g^*\alpha$-D(A$\cup$B)$\subseteq$ $\theta g^*\alpha$-d(A) $\cup$ $\theta g^*\alpha$-d(B).If x$\in$ $\theta g^*\alpha$-d(A)$\cup$ $\theta g^*\alpha$-d(B),then x$\in$ $\theta g^*\alpha$-d(A) and x$\in$ $\theta g^*\alpha$-d(B),that is x is neither a $\theta g^*\alpha$-limit point of A nor a $\theta g^*\alpha$-limit point of B. Hence there exist $\theta g^*\alpha$-neighbourhoods G_1 and G_2 of x such that G_1 $\cap$ (A\{x}) = ϕ and $G_2$$\cap$ (B\x)=ϕ Since $G_1$$\cap$ G_2 is a $\theta g^*\alpha$-neighbourhood of x,we have ($G_1$$\cap$ G_2)$\cap$ [(A$\cup$B)\{x}] =ϕ . Therefore x$\in$ $\theta g^*\alpha$-D(A$\cup$ B).Thus $\theta g^*\alpha$-D(A$\cup$ B) $\subseteq$ $\theta g^*\alpha$-D(A) $\theta g^*\alpha$-(B). Hence $\theta g^*\alpha$-D(A$\cup$ B) = $\theta g^*\alpha$-D(A)$\cup$ $\theta g^*\alpha$-D(B).
6. Since A $\subseteq$ B$\cup$ A and A $\subseteq$ B $\cup$ A, by (2)$\theta g^*\alpha$-D(A$\cap$ B) $\subseteq$ $\theta g^*\alpha$-D(A) and $\theta g^*\alpha$-D(A$\cap$ B) $\subseteq$ $\theta g^*\alpha$-D(B). Consequently $\theta g^*\alpha$-D(A$\cap$ B) $\subseteq$ $\theta g^*\alpha$-D(A)$\cap$ $\theta g^*\alpha$-D(B).

Example 4.3. Let X = {a, b, c,d} and τ ={ϕ, {a, b}, X}. Then β *O(τ)=(P(X) \ {a}, {b}, {a, b}). Let A={a, b, d} and B ={c}. Then Dβ * (A$\cup$ B) = {a, b} and Dβ^*(A)=ϕ, Dβ *(B)=ϕ.

Theorem 4.4. Let (X, τ) be a topological space and A be subset of X. If A is $\theta g^*\alpha$-closed, then $\theta g^*\alpha$-D(A)$\subseteq$A.

Proof: Let A be to $\theta g^*\alpha$-closed, Now we will show that $\theta g^*\alpha$-D(A) $\subseteq$ A. Since A is $\theta g^*\alpha$-closed, X \A is $\theta g^*\alpha$-open. To each x$\in$ X\ A there exists $\theta g^*\alpha$-neighbourhood G of x such that G$\subset$ X\A. Since A $\cap$ (X\A) =ϕ , the $\theta g^*\alpha$-neighbourhood G contains no point of A and so x is not a $\theta g^*\alpha$-limit point of A. Thus no point of X \ A can be $\theta g^*\alpha$-limit point of A that is, A contains all its $\theta g^*\alpha$-limit points. That is $\theta g^*\alpha$-D(A)$\subseteq$ A.

5 $\theta g^*\alpha$-Border of a Set

Definition 5.1. For any subset A of X,The border of A is defined by Bd(A)=A \int(A).

Definition 5.2. For any subset A of X, $\theta g^*\alpha$-border of A is defined by $\theta g^*\alpha$-Bd(A)=A \ $\theta g^*\alpha$-int(A).

Theorem 5.3. In a topological space (X, τ), for any subset A of X, the following statements hold.

1. $\theta g^*\alpha$-Bd(ϕ) = $\theta g^*\alpha$-Bd(X) =ϕ .
2. $\theta g^*\alpha$-Bd(A) $\subseteq$ Bd(A).
3. A=$\theta g^*\alpha$-int(A)$\cup$ $\theta g^*\alpha$-Bd(A).
4. $\theta g^*\alpha$-int(A)$\cap\theta g^*\alpha$-Bd(A) = ϕ .
5. $\theta g^*\alpha$-int(A) = A \$\theta g^*\alpha$-Bd(A).
6. $\theta g^*\alpha$-int($\theta g^*\alpha$-Bd(A)) =ϕ .
7. A is $\theta g^*\alpha$-open if and only if $\theta g^*\alpha$-Bd(A) =ϕ .
8. $\theta g^*\alpha$-Bd($\theta g^*\alpha$-int(A)) = ϕ .
9. $\theta g^*\alpha$-Bd($\theta g^*\alpha$-Bd(A)) = $\theta g^*\alpha$-Bd(A).
10. $\theta g^*\alpha$-Bd(A) = A$\cap$ $\theta g^*\alpha$-cl(X\A).

Proof: (1) , straight forward. (2) , (3) , (4) and (5) follows From the Definition 5.2 To prove(6) if possible let x$\in\theta g^*\alpha$-int($\theta g^*\alpha$-Bd(A)).Then x$\in$ $\theta g^*\alpha$-Bd(A),since $\theta g^*\alpha$-Bd(A)$\subseteq$A, x$\in\theta g^*\alpha$-int($\theta g^*\alpha$-Bd(A)) $\subseteq$ $\theta g^*\alpha$-int(A). Therefore x$\in$ $\theta g^*\alpha$-int(A)$\cap$ $\theta g^*\alpha$-Bd(A) which is a contradiction to (4). Thus (6)is proved. A is $\theta g^*\alpha$-open if and only if $\theta g^*\alpha$-int(A)=A . But $\theta g^*\alpha$-Bd(A)= A\ $\theta g^*\alpha$-int(A) implies $\theta g^*\alpha$-Bd(A)= ϕ . This proves (7) and (8).When A = $\theta g^*\alpha$-Bd(A),Definition 5.2 becomes $\theta g^*\alpha$-Bd($\theta g^*\alpha$-Bd(A))= $\theta g^*\alpha$-Bd(A)\$\theta g^*\alpha$-int($\theta g^*\alpha$-Bd(A)). Using (8), we get (9). To prove (10).$\theta g^*\alpha$-Bd(A)=A\$\theta g^*\alpha$-int(A)= A$\cap$ (X\ $\theta g^*\alpha$-int(A))= A$\cap$ $\theta g^*\alpha$-cl(X\A). Hence (10)is proved.

Example 5.4. Let X={a,b,c,d} with topology τ ={X,ϕ ,{a},{b},{c},{a,b},{a,c},{b,c}, {a,b,c}}. In this topological space(X, τ), $\theta g^*\alpha$-O(X) = {X, ϕ ,{a},{b},{c},{a, b},{b, c},{a ,c}{a, b ,c}} Let A = {a, c}, then $\theta g^*\alpha$-Bd(A) = {a, c} -{a, c} = ϕ and Bd-(A) = {a, c}\{a} = {c}. Therefore Bd-(A) $\nsubseteq$ $\theta g^*\alpha$-Bd(A)

6 $\theta g^*\alpha$- Frontier of a Set

Definition 6.1. For any subset A of X,The frontier of A is defined by Fr(A)=cl(A) $\setminus$ int(A).

Definition 6.2. For any subset A of X, its $\theta g^*\alpha$-Frontier is defined by $\theta g^*\alpha$-Fr(A)= $\theta g^*\alpha$-cl(A) $\setminus$ $\theta g^*\alpha$-int(A).

Theorem 6.3. For any subset A of X, in a topological space(X, τ),the following statements hold.

(1). $\theta g^*\alpha$-Fr(ϕ)= $\theta g^*\alpha$-Fr(X)= ϕ .

(2). $\theta g^*\alpha$-cl(A)= $\theta g^*\alpha$-int(A) $\cap$ $\theta g^*\alpha$-Fr(A).

(3). $\theta g^*\alpha$-int(A) $\cap$ $\theta g^*\alpha$-Fr(A)=ϕ .

(4). $\theta g^*\alpha$-bd(A) $\subseteq$ $\theta g^*\alpha$-Fr(A) $\subseteq$ $\theta g^*\alpha$-cl(A).

(5). If A is $\theta g^*\alpha$-closed, then A=$\theta g^*\alpha$-int(A) $\cup$ $\theta g^*\alpha$-Fr(A).

(6). $\theta g^*\alpha$-Fr(A)= $\theta g^*\alpha$-cl(A)$\cap\theta g^*\alpha$-cl(X$\setminus$A).

(7). A point x$\in\theta g^*\alpha$-Fr(A), if and only if every $\theta g^*\alpha$-open set containing x intersects both A and its complement X$\setminus$A.

(8). $\theta g^*\alpha$-cl($\theta g^*\alpha$-Fr(A))= $\theta g^*\alpha$-Fr(A), i.e, $\theta g^*\alpha$-Fr(A) is $\theta g^*\alpha$-closed.

(9). $\theta g^*\alpha$-Fr(A) = $\theta g^*\alpha$-Fr(X$\setminus$A).

(10). A is $\theta g^*\alpha$-closed if and only if $\theta g^*\alpha$-Fr(A) = $\theta g^*\alpha$-bd(A), i.e, A is $\theta g^*\alpha$-closed if and only if A contains its $\theta g^*\alpha$-frontier.

(11). $\theta g^*\alpha$-Fr($\theta g^*\alpha$-int(A))$\subseteq$ $\theta g^*\alpha$-Fr(A).

(12). $\theta g^*\alpha$-Fr($\theta g^*\alpha$-cl(A))$\subseteq$ $\theta g^*\alpha$-Fr(A).

(13). $\theta g^*\alpha$-Fr($\theta g^*\alpha$-Fr(A))$\subseteq$ $\theta g^*\alpha$-Fr(A).

(14). X=$\theta g^*\alpha$-int(A)$\cup$ $\theta g^*\alpha$-int(X$\setminus$A)$\cup$ $\theta g^*\alpha$-Fr(A).

(15). $\theta g^*\alpha$-int(A) = A$\setminus\theta g^*\alpha$-Fr(A).

(16). If A is $\theta g^*\alpha$-open, then A $\cap$ $\theta g^*\alpha$-Fr(A) =ϕ , i.e, $\theta g^*\alpha$-Fr(A) $\subseteq$ X$\setminus$A.

Proof: (1), (2), (3) and (4) follows from Definition 6.2 (5) follows from (2) and Proposition2.3 (2). (6) follows from Proposition2.3 (5). (7) can be proved using (6)and Proposition2.3 (8). From (6), we can prove (8) by applying the results of Proposition2.3 (3) and (9). Proof of (9)is similar. To prove (10): If A is $\theta g^*\alpha$-closed, then A = $\theta g^*\alpha$-cl(A).Hence Definition 6.2 reduces to $\theta g^*\alpha$-Fr(A)= A$\setminus$ $\theta g^*\alpha$-int(A) = $\theta g^*\alpha$-bd(A). Conversely, suppose that $\theta g^*\alpha$-Fr(A) = $\theta g^*\alpha$-bd(A),using Definition 6.2 and 6.1, we get $\theta g^*\alpha$-cl(A) = A, which proves the sufficient part.

(11) and (12) Since $\theta g^*\alpha$-int(A) is $\theta g^*\alpha$-open, (11) holds. Similarly (12) can also be proved. Since $\theta g^*\alpha$-Fr(A) is $\theta g^*\alpha$-closed, invoking (10), (13) can be proved. since X=$\theta g^*\alpha$-cl(A) (X\$\theta g^*\alpha$-cl(A)), but from (2) $\theta g^*\alpha$-cl(A) = $\theta g^*\alpha$-int(A) $\theta g^*\alpha$-Fr(A). Also X\$\theta g^*\alpha$-cl(A)=$\theta g^*\alpha$-int(X\A). Hence X = $\theta g^*\alpha$-int(A) $\theta g^*\alpha$-Fr(A) $\theta g^*\alpha$-int(X\ A). Thus (14) is proved. Proof of (15) is obvious. If A is $\theta g^*\alpha$-open, A= $\theta g^*\alpha$-int(A). (16) follows from (3).

Theorem 6.4. If a subset A of X is $\theta g^*\alpha$-open or $\theta g^*\alpha$-closed in (X, τ), then $\theta g^*\alpha$-Fr($\theta g^*\alpha$-Fr(A))=$\theta g^*\alpha$-Fr(A).

Proof: By Theorem 6.3 (6), we have $\theta g^*\alpha$-Fr($\theta g^*\alpha$-Fr(A))= $\theta g^*\alpha$-cl($\theta g^*\alpha$-Fr(A)) $\cap$ $\theta g^*\alpha$-cl(X $\setminus$ $\theta g^*\alpha$-Fr(A))= $\theta g^*\alpha$-Fr(A)$\cap\theta g^*\alpha$-cl(X\$\theta g^*\alpha$-Fr(A))= $\theta g^*\alpha$-cl(A)\$\theta g^*\alpha$-cl(X$\cap$ A)$\cap$ $\theta g^*\alpha$-cl(X$\cap$ $\theta g^*\alpha$-Fr(A)).If A is $\theta g^*\alpha$-open in X, by Theorem 6.3 (16), we have $\theta g^*\alpha$-Fr(A) $\cap$A =ϕ . Therefore A$\subseteq$ X $\cap$ $\theta g^*\alpha$-Fr(A). Hence $\theta g^*\alpha$-cl(A) $\theta g^*\alpha$-cl(X$\cap$ $\theta g^*\alpha$-Fr(A)).i.e,$\theta g^*\alpha$-cl(A)\ $\theta g^*\alpha$-cl(X$\cap$ $\theta g^*\alpha$-Fr(A))=$\theta g^*\alpha$-cl(A). If A is $\theta g^*\alpha$-closed in X,then X $\cap$ A is $\theta g^*\alpha$-open and hence From the above case, we have $\theta g^*\alpha$-cl(X\A)\ $\theta g^*\alpha$-cl(X$\cap\theta g^*\alpha$-Fr(X\A))= $\theta g^*\alpha$-cl(X\A). In both the cases using Theorem 6.3(6), we get $\theta g^*\alpha$-Fr($\theta g^*\alpha$-Fr(A))=$\theta g^*\alpha$-cl(A)$\notin\theta g^*\alpha$-cl(X \A)= $\theta g^*\alpha$-Fr(A).

Theorem 6.5. If A is any subset of X, then $\theta g^*\alpha$-Fr($\theta g^*\alpha$-Fr($\theta g^*\alpha$-Fr(A)))= $\theta g^*\alpha$-Fr($\theta g^*\alpha$-Fr(A)).

Proof: It follows From Theorem 6.3 (8) and Theorem 6.4.

Theorem 6.6. If A and B are subsets of X such that A$\cap$B=ϕ, where A is $\theta g^*\alpha$-open in X, then A$\cap\theta g^*\alpha$ cl(B)=ϕ.

Proof: If possible, let x$\in$A$\cap\theta g^*\alpha$-cl(B). Then A is a $\theta g^*\alpha$-open set containing x and also x$\in$ $\theta g^*\alpha$-cl(B).By Proposition2.3 (8)A$\cap$B =ϕ, which is a contradiction. Thus A$\cap\theta g^*\alpha$-cl(B)=ϕ.

Theorem 6.7. If A and B are subsets of X such that A B and B is $\theta g^*\alpha$-closed in X, then $\theta g^*\alpha$-Fr(A)$\subseteq$ B.

Proof: $\theta g^*\alpha$-Fr(A)= $\theta g^*\alpha$-cl(A) $\notin$ $\theta g^*\alpha$-int(A) $\theta g^*\alpha$-cl(B)\$\theta g^*\alpha$-int(A)= B \$\theta g^*\alpha$-int(A) $\subseteq$ B.

Theorem 6.8. If A and B are subsets of X such that A $\cap$ B=ϕ, where A is $\theta g^*\alpha$-open in X, then A $\notin\theta g^*\alpha$-Fr(B)=ϕ.

Proof: Since $\theta g^*\alpha$-Fr(B) $\subseteq$ $\theta g^*\alpha$-cl(B), proof is obvious From Theorem 6.6.

Theorem 6.9. If A , B$\subseteq$ X such that $\theta g^*\alpha$-Fr(A)$\cap$Fr(B) =ϕ and Fr(A)$\cap\theta g^*\alpha$-Fr(B) =ϕ, then $\theta g^*\alpha$-int(A $\cup$ B)=$\theta g^*\alpha$-int(A) $\cup\theta g^*\alpha$-int(B).

Proof: We know that $\theta g^*\alpha$-int(A) $\cup\theta g^*\alpha$-int(B) $\subseteq$ $\theta g^*\alpha$-(A $\cup$ B). Let x$\in\theta g^*\alpha$-int(A $\cup$ B). i.e, x$\in$ U$\subseteq$ A $\cup$ B, U is a $\theta g^*\alpha$-open set. Thus either x$\in\theta g^*\alpha$-Fr(A) $\notin$Fr(B), since $\theta g^*\alpha$-Fr(A) $\cap$ Fr(B) = ϕ. Hence x$\in$int(B). i.e, x$\notin$cl(B). Since x$\in$ int(B) $\subseteq$ $\theta g^*\alpha$-int(B), x$\subseteq$ $\theta g^*\alpha$-int(B). Moreover since x $\notin$cl(B), there exists an open set V containing x which is disjoint From B, i.e, V $\subseteq$ X \B. So x$\in$ U$\cap$V$\subseteq$ A. Hence

U∩V is a $\theta g^*\alpha$-open subset of A containing x.(By Proposition2.3 (7)). i.e, x∈$\theta g^*\alpha$-int(A). Thus x∈$\theta g^*\alpha$-int(A) ∪$\theta g^*\alpha$-int(B). If x∉ $\theta g^*\alpha$-Fr(A), x∈$\theta g^*\alpha$-int(A) or x ∉ $\theta g^*\alpha$-cl(A). If x ∉ $\theta g^*\alpha$-cl(A), there exists a $\theta g^*\alpha$-open set W containing x which is disjoint From A, i.e, W⊆ X\A. i.e, x∈ U∩W⊆ B $\theta g^*\alpha$-cl(B). i.e, x∈$\theta g^*\alpha$-Fr(B).Hence From the above case, we get x∈$\theta g^*\alpha$-int(A) ∪$\theta g^*\alpha$-int(B). So $\theta g^*\alpha$-int(A∪B)⊆ $\theta g^*\alpha$-int(A) ∪$\theta g^*\alpha$-int(B). Thus $\theta g^*\alpha$-int(A ∪ B)=$\theta g^*\alpha$-int(A)∪$\theta g^*\alpha$-int(B).

7 $\theta g^*\alpha$ -Exterior of a Set

Definition 7.1. For any subset A of X,The exterior of A is defined by Ext(A)=int(X\A).

Definition 7.2. For any subset A of X, its $\theta g^*\alpha$-Exterior is defined by $\theta g^*\alpha$-Ext(A)= $\theta g^*\alpha$-int(X\A).

Theorem 7.3. For any subset A of X, in a topological space(X, τ),the following statements hold.

(1). $\theta g^*\alpha$-Extϕ) = $\theta g^*\alpha$-Ext(X)=ϕ.

(2). Ext(A) ⊂ $\theta g^*\alpha$-Ext(A) where Ext(A) denote the exterior of A.

(3). If A ⊆ B, then $\theta g^*\alpha$-Ext(B)⊆ $\theta g^*\alpha$-Ext(A).

(4). $\theta g^*\alpha$-Ext(A) is $\theta g^*\alpha$-open.

(5). A is $\theta g^*\alpha$-closed if and only if $\theta g^*\alpha$-Ext(A)= X\ A.

(6). $\theta g^*\alpha$-Ext(A) = X\$\theta g^*\alpha$-cl(A).

(7). $\theta g^*\alpha$-Ext($\theta g^*\alpha$-Ext(A))=$\theta g^*\alpha$-int($\theta g^*\alpha$-cl(A)).

(8). Ext(A) ⊆ $\theta g^*\alpha$-Ext(A) where Ext(A) denote the exterior of A.

(9). $\theta g^*\alpha$-Ext(A)= $\theta g^*\alpha$-Ext(X\$\theta g^*\alpha$-Ext(A)).

(10). $\theta g^*\alpha$-int(A) $\theta g^*\alpha$-Ext($\theta g^*\alpha$-Ext(A)).

(11). X=$\theta g^*\alpha$-int(A) ∪$\theta g^*\alpha$-Ext(A) ∪ $\theta g^*\alpha$-Fr(A).

(12). $\theta g^*\alpha$-Ext(A ∪B)⊆ $\theta g^*\alpha$-Ext(A)∩$\theta g^*\alpha$-Ext(B)

(13). $\theta g^*\alpha$-Ext(A ∪B) ⊆ $\theta g^*\alpha$-Ext(A) ∪$\theta g^*\alpha$-Ext(B)r(A) ⊆X\A.

Proof: (1) ,(2)and (3)can be proved From Definition 7.2. Since $\theta g^*\alpha$-int(A) is $\theta g^*\alpha$-open, Proof of (4) is obvious. Since $\theta g^*\alpha$-int(X\A)= X\$\theta g^*\alpha$-cl(A),(5) follows From Definition 7.2.Similarly (6) and (7)can be proved.

To Prove (8), $\theta g^*\alpha$-Ext(X\ $\theta g^*\alpha$-Ext(A))=$\theta g^*\alpha$-Ext(X\ $\theta g^*\alpha$-int(X\A))= $\theta g^*\alpha$-int(X\(X\ $\theta g^*\alpha$- int(X\A)))=$\theta g^*\alpha$-int($\theta g^*\alpha$-int(X\A))=$\theta g^*\alpha$- int(X\A)= $\theta g^*\alpha$-Ext(A). Hence (8)is proved. Since A ⊆ $\theta g^*\alpha$-cl(A), using (6), (9)can be proved. (10).follows From Theorem 6.3 (14) and Definition 7.2. Proof of (11),(12)and (13)are obvious.

Example 7.4. LetX={a,b,c,d}with topology τ ={X,ϕ ,{a},{b},{a,b}, {a,b,c}}. In this topological space(X, τ), $\theta g^*\alpha$-O(X) ={X,ϕ ,{a},{b},{c},{a, b},{b, c}, {a ,c},{a, b ,c},{a,b,d}} Let A = {a,b.d}, then $\theta g^*\alpha$-Ext (A) = {c} and Ext-(A)=ϕ. Therefore $\theta g^*\alpha$-Ext-(A) $\nsubseteq$ Ext (A)

8 Conclusion

Every year many topologist introduced diffrent type of closed sets.we introduced $\theta g^*\alpha$-closed sets in topological spaces. In this paper, we discussed properies and applications of $\theta g^*\alpha$-neighbourhoods,$\theta g^*\alpha$-limit points, $\theta g^*\alpha$- border, $\theta g^*\alpha$-frontier and $\theta g^*\alpha$-exterior.This shall be extended in the future Research with some applications

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FOUR NEW OPERATORS OVER THE GENERALIZED INTUITIONISTIC FUZZY SETS

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Abstract – In this paper, newly defined four level operators over generalized intuitionistic fuzzy sets ($GIFS_B$s) are proposed. Some of the basic properties of the new operators are discussed. Geometric interpretation of operators over generalized intuitionistic fuzzy sets is given.

Keywords – *Generalized intuitionistic fuzzy sets, intuitionistic fuzzy sets, level operators.*

1 Introduction

The theory of intuitionistic fuzzy sets (IFSs), proposed by Atanassov [1], and has earned successful applications in various fields. Modal operators, topological operators, level operators, negation operators and aggregation operators are different groups of operators over the IFSs due to Atanassov[1]. Atanassov [2] defined level operators $P_{\alpha,\beta}$ and $Q_{\alpha,\beta}$ over the IFSs. Lupianez [3] show relations between topological operators and intuitionistic fuzzy topology. In 2008, Atanassov [4] studied some relations between intuitionistic fuzzy negations and intuitionistic fuzzy level operators $P_{\alpha,\beta}$ and $Q_{\alpha,\beta}$. Atanassov [5] introduced extended level operators over intuitionistic fuzzy sets. In 2009, Parvathi and Geetha [6] defined some level operators, max-min implication operators and $P_{\alpha,\beta}$ and $Q_{\alpha,\beta}$ operators on temporal intuitionistic fuzzy sets. Atanassov [7] introduced two new operators that partially extend the intuitionistic fuzzy operators from modal type. Yilmaz and Cuvalcioglu [8] introduced level operators of temporal intuitionistic fuzzy sets. Sheik Dhavudh and Srinivasan [9] proposed level operators on intuitionistic L-fuzzy sets and establish some of their properties. The intuitionistic fuzzy operators are important from the point of view

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application. The intuitionistic fuzzy operators applied in contracting a classifier recognizing imbalanced classes, image recognition, image processing, multi-criteria decision making, deriving the similarity measure, sales analysis, new product marketing, medical diagnosis, financial services, solving optimization problems etc.. Baloui Jamkhaneh and Nadarajah [10] considered a new generalized intuitionistic fuzzy sets ($GIFS_B s$) and introduced some operators over $GIFS_B$. Baloui Jamkhaneh and Garg [11] considered some new operations over the generalized intuitionistic fuzzy sets and their application to decision making process. In 2017 Baloui Jamkhnaeh [12] defined level operators $P_{\alpha,\beta}$ and $Q_{\alpha,\beta}$ over the $GIFS_B$s. In this paper we shall introduce the sum of level operators over $GIFS_B$ and we will discuss their properties.

2 Preliminaries

In this section we will briefly remind some of the basic definition and notions of IFS which will be helpful in further study of the paper. Let X be a non-empty set.

Definition 2.1. [1] An intuitionistic fuzzy setsA in X is defined as an object of the form $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\}$ where the functions $\mu_A : X \to [0,1]$ and $\nu_A : X \to [0,1]$ denote the degree of membership and non-membership functions of A respectively and $0 \le \mu_A(x) + \nu_A(x) \le 1$for each $x \in X$.

Definition 2.2. [10] A generalized intuitionistic fuzzy set ($GIFS_B$) A in X, is defined as an object of the form $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\}$ where the functions $\mu_A : X \to [0,1]$ and $\nu_A : X \to [0,1]$, denote the degree of membership and degree of non-membership functions of A respectively, and $0 \le \mu_A(x)^\delta + \nu_A(x)^\delta \le 1$ for each $x \in X$ where $\delta = n$ or $\frac{1}{n}, n = 1,2,\dots,N$. The collection of all generalized intuitionistic fuzzy sets is denoted by $GIFS_B(\delta, X)$. Let X is a universal set and F is a subset in the Euclidean plane with the Cartesian coordinates. For a $GIFS_B$ A, a function f_A from X to F can be constructed, such that if $x \in X$ then

$$(\nu_A(x), \mu_A(x)) = f_A(x) \in F, \quad 0 \le \nu_A(x),\ \mu_A(x) \le 1$$

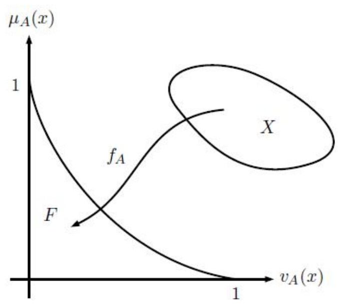

Figure 1.A geometrical interpretation of $GIFS_B$ with $\delta = 0.5$

Definition2.3. Let A and B be two $GIFS_B$s such that

$$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\}\ ,\ B = \{\langle x, \mu_B(x), \nu_B(x)\rangle : x \in X\},$$

define the following relations and operations on A and B

i. A⊂B if and only if μ_A(x)≤ μ_B(x) andν_A(x)≥ ν_A(x) , ∀xϵX,
ii. A=B if and only if μ_A(x) =μ_B(x) and ν_A(x)=ν_B(x) , ∀xϵX,
iii. A∪B $=\{\langle x, \max(\mu_A(x), \mu_B(x)), \min(\nu_A(x), \nu_B(x))\rangle : x\epsilon X\}$,
iv. A∩B $=\{\langle x, \min(\mu_A(x), \mu_B(x)), \max(\nu_A(x), \nu_B(x))\rangle : x\epsilon X\}$,
v. $\bar{A} = \{\langle x, \nu_A(x), \mu_A(x)\rangle : x \in X\}$.

Let X is a non-empty finite set, and $A \in GIFS_B$, as $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\}$. Baloui Jamkhaneh and Nadarajah [10] and Baloui Jamkhaneh [12] introduced following operators of $GIFS_B$ and investigated some their properties.

i. $\Box A = \left\{\langle x, \mu_A(x), (1 - \mu_A(x)^{\delta})^{\frac{1}{\delta}}\rangle : x\epsilon X\right\}$,(modal logic: the necessity measure),
ii. $\Diamond A = \left\{\langle x, (1 - \nu_A(x)^{\delta})^{\frac{1}{\delta}}, \nu_A(x)\rangle : x\epsilon X\right\}$,(modal logic: the possibility measure),
iii. $P_{\alpha,\beta}(A) = \left\{\langle x, \max(\alpha^{\frac{1}{\delta}}, \mu_A(x)), \min(\beta^{\frac{1}{\delta}}, \nu_A(x))\rangle : x\epsilon X\right\}$, where $\alpha + \beta \leq 1$,
iv. $Q_{\alpha,\beta}(A) = \left\{\langle x, \min(\alpha^{\frac{1}{\delta}}, \mu_A(x)), \max(\beta^{\frac{1}{\delta}}, \nu_A(x))\rangle : x\epsilon X\right\}$, where $\alpha + \beta \leq 1$.

The geometrical interpretations of operators of $GIFS_B$ are shown on Fig. 2-4.

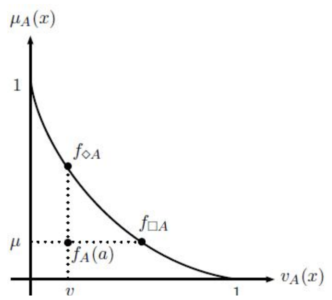

Figure 2.A geometrical interpretation of $\Diamond A$ and $\Box A$ with $\delta = 0.5$

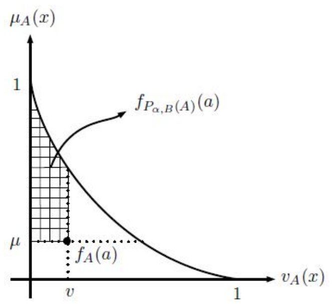

Figure 3.A geometrical interpretation of $P_{\alpha,\beta}(A)$ with $\delta = 0.5$

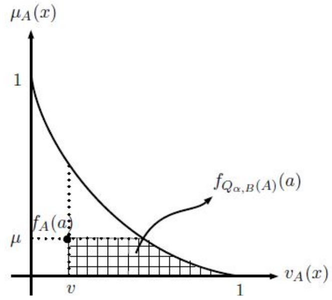

Figure 4. A geometrical interpretation of $Q_{\alpha,\beta}(A)$ with $\delta = 0.5$

3 Main Results

Here, we will introduce new operators over the $\mathrm{GIFS_B}$, which extend some operators in the research literature related to IFSs. Let X is a non-empty finite set.

Definition 3.1. Letting $\alpha, \beta \in [0,1]$, where $\alpha + \beta \leq 1$. For every $\mathrm{GIFS_B}$ as $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\}$, we define the level operators as follows:

i. $P^{(1)}_{\alpha,\beta}(A) = \left\{\langle x, \min\left(\max\left(\alpha^{\frac{1}{\delta}}, \mu_A(x)\right), \left(1 - \nu_A(x)^{\delta}\right)^{\frac{1}{\delta}}\right), \min\left(\beta^{\frac{1}{\delta}}, \nu_A(x)\right)\rangle : x \epsilon\, X\right\}$,

ii. $P^{(2)}_{\alpha,\beta}(A) = \left\{\langle x, \max\left(\max\left(\alpha^{\frac{1}{\delta}}, \mu_A(x)\right), \left(1 - \nu_A(x)^{\delta}\right)^{\frac{1}{\delta}}\right), \min\left(\beta^{\frac{1}{\delta}}, \nu_A(x)\right)\rangle : x \epsilon\, X\right\}$,

iii. $Q^{(1)}_{\alpha,\beta}(A) = \left\{\langle x, \min\left(\alpha^{\frac{1}{\delta}}, \mu_A(x)\right), \min\left(\max\left(\beta^{\frac{1}{\delta}}, \nu_A(x)\right), \left(1 - \mu_A(x)^{\delta}\right)^{\frac{1}{\delta}}\right), \rangle : x \epsilon\, X\right\}$,

iv. $Q^{(2)}_{\alpha,\beta}(A) = \left\{\langle x, \min\left(\alpha^{\frac{1}{\delta}}, \mu_A(x)\right), \max\left(\max\left(\beta^{\frac{1}{\delta}}, \nu_A(x)\right), \left(1 - \mu_A(x)^{\delta}\right)^{\frac{1}{\delta}}\right), \rangle : x \epsilon\, X\right\}$.

The geometrical interpretations of new operators of $\mathrm{GIFS_B}$ are shown on Fig. 5-8.

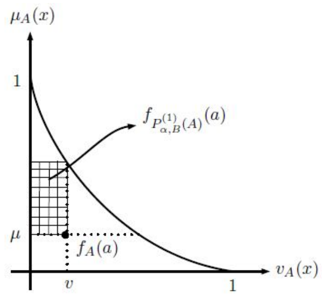

Figure 5. A geometrical interpretation of $P^{(1)}_{\alpha,\beta}(A)$ with $\delta = 0.5$

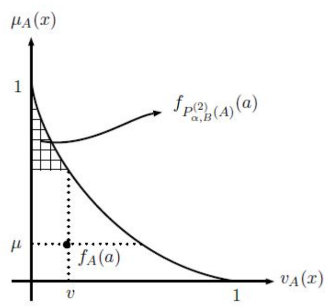

Figure 6. A geometrical interpretation of $P^{(2)}_{\alpha,\beta}(A)$ with $\delta = 0.5$

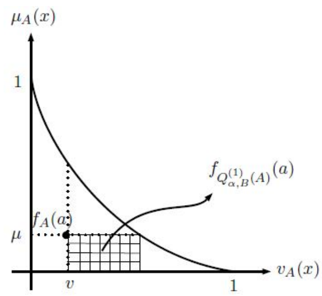

Figure 7.A geometrical interpretation of $Q^{(1)}_{\alpha,\beta}(A)$ with $\delta = 0.5$

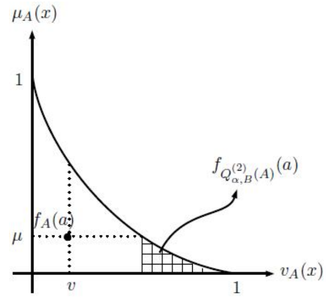

Figure 8.A geometrical interpretation of $Q^{(2)}_{\alpha,\beta}(A)$ with $\delta = 0.5$

Corollary 3.2. According to definition of new operators and geometrical interpretations of them, we have

i. If $A = \{\langle x, 1, 0\rangle : x \in X\}$, then$P_{\alpha,\beta}(A) = P^{(1)}_{\alpha,\beta}(A) = P^{(2)}_{\alpha,\beta}(A) = \{\langle x, 1, 0\rangle : x \in X\}$,

ii. If $A = \{\langle x, 0, 1\rangle : x \in X\}$, then $Q_{\alpha,\beta}(A) = Q^{(1)}_{\alpha,\beta}(A) = Q^{(2)}_{\alpha,\beta}(A) = \{\langle x, 0, 1\rangle : x \in X\}$.

Remark 3.3. If $\mu_A(x) \geq \alpha^{\frac{1}{\delta}}$ and $\nu_A(x) \leq \beta^{\frac{1}{\delta}}$ then $P^{(1)}_{\alpha,\beta}(A) = A$, $P^{(2)}_{\alpha,\beta}(A) = \lozenge A$ and if $\mu_A(x) \leq \alpha^{\frac{1}{\delta}}$ and $\nu_A(x) \geq \beta^{\frac{1}{\delta}}$ then $Q^{(1)}_{\alpha,\beta}(A) = A$, $Q^{(2)}_{\alpha,\beta}(A) = \square A$.

Remark 3.4. If $\alpha_1 \leq \alpha_2$ then$P^{(i)}_{\alpha_1,\beta}(A) \subseteq P^{(i)}_{\alpha_2,\beta}(A)$and $Q^{(i)}_{\alpha_1,\beta}(A) \subseteq Q^{(i)}_{\alpha_2,\beta}(A)$, also if $\beta_1 \leq \beta_2$then $P^{(i)}_{\alpha,\beta_2}(A) \subseteq P^{(i)}_{\alpha,\beta_1}(A)$ and $Q^{(i)}_{\alpha,\beta_2}(A) \subseteq Q^{(i)}_{\alpha,\beta_1}(A)$, i=1,2.

Theorem 3.5. For every $A \in GIFS_B$ and $\alpha, \beta \in [0,1]$, where $\alpha + \beta \leq 1$, we have

i. $P^{(i)}_{\alpha,\beta}(A) \in GIFS_B$, i=1,2,

ii. $Q^{(i)}_{\alpha,\beta}(A) \in GIFS_B$, i=1,2,

iii. $\overline{P^{(i)}_{\alpha,\beta}(\bar{A})} = Q^{(i)}_{\beta,\alpha}(A)$, i=1,2.

Proof. (i) Let i=1

$\mu_{P^{(i)}_{\alpha,\beta}(A)}(x)^\delta + \nu_{P^{(i)}_{\alpha,\beta}(A)}(x)^\delta =$

$(\min(\max(\alpha^{\frac{1}{\delta}}, \mu_A(x)), (1 - \nu_A(x)^\delta)^{\frac{1}{\delta}}))^\delta + (\min(\beta^{\frac{1}{\delta}}, \nu_A(x)))^\delta =$
$\min(\max(\alpha, \mu_A(x)^\delta), (1 - \nu_A(x)^\delta)) + (\min(\beta, \nu_A(x)^\delta) = I.$
If $\max(\alpha, \mu_A(x)^\delta) \leq (1 - \nu_A(x)^\delta)$ then

(1) If $\max(\alpha, \mu_A(x)^\delta) = \alpha$ and $\min(\beta, \nu_A(x)^\delta) = \beta$ then

$$I = \alpha + \beta \leq 1.$$

(2) If $\max(\alpha, \mu_A(x)^\delta) = \alpha$ and $\min(\beta, \nu_A(x)^\delta) = \nu_A(x)^\delta$ then

$$I = \alpha + \nu_A(x)^\delta \leq \alpha + \beta \leq 1,$$

(3) If $\max(\alpha, \mu_A(x)^\delta) = \mu_A(x)^\delta$ and $\min(\beta, \nu_A(x)^\delta) = \beta$ then

$$I = \mu_A(x)^\delta + \beta \leq \mu_A(x)^\delta + \nu_A(x)^\delta \leq 1,$$

(4) If $\max(\alpha, \mu_A(x)^\delta) = \mu_A(x)^\delta$ and $\min(\beta, \nu_A(x)^\delta) = \nu_A(x)^\delta$ then

$$I = \mu_A(x)^\delta + \nu_A(x)^\delta \leq 1.$$

If $\max(\alpha, \mu_A(x)^\delta) \geq (1 - \nu_A(x)^\delta)$ then

(1) If $\min(\beta, \nu_A(x)^\delta) = \beta$ then $I = 1 - \nu_A(x)^\delta + \beta \leq 1 - \nu_A(x)^\delta + \nu_A(x)^\delta = 1$,

(2) If $\min(\beta, \nu_A(x)^\delta) = \nu_A(x)^\delta$ then $I = 1 - \nu_A(x)^\delta + \nu_A(x)^\delta = 1$.

The proof is completed. Proof of (ii) is similar to that of (i).

(iii) Let i=1

$\overline{A} = \{\langle x, \nu_A(x), \mu_A(x)\rangle : x \in X\}$.

$$P_{\alpha,\beta}^{(1)}(\overline{A}) = \left\{\langle x, \min(\max(\alpha^{\frac{1}{\delta}}, \nu_A(x)), (1 - \mu_A(x)^\delta)^{\frac{1}{\delta}}), \min(\beta^{\frac{1}{\delta}}, \mu_A(x))\rangle : x \in X\right\},$$

$$\overline{P_{\alpha,\beta}^{(1)}(\overline{A})} = \left\{\langle x, \min(\beta^{\frac{1}{\delta}}, \mu_A(x)), \min(\max(\alpha^{\frac{1}{\delta}}, \nu_A(x)), (1 - \mu_A(x)^\delta)^{\frac{1}{\delta}})\rangle : x \in X\right\} = Q_{\beta,\alpha}^{(1)}(A).$$

The proof is completed.

Theorem 3.6. For every $A \in GIFS_B$, we have

i. $P_{0,1}^{(1)}(A) = A$,
ii. $P_{0,1}^{(2)}(A) = P_{1,1}^{(1)}(A) = \Diamond A$,
iii. $Q_{1,0}^{(1)}(A) = A$,
iv. $Q_{1,0}^{(2)}(A) = Q_{1,1}^{(1)}(A) = \Box A$.

Proof. Proofs are obvious.

Theorem 3.7. For every $A \in GIFS_B$ and $\alpha, \beta \in [0,1]$, where $\alpha + \beta \leq 1$, we have

i. $P_{\alpha,\beta}^{(1)}(A) \subset P_{\alpha,\beta}(A) \subset P_{\alpha,\beta}^{(2)}(A)$,
ii. $Q_{\alpha,\beta}^{(2)}(A) \subset Q_{\alpha,\beta}(A) \subset Q_{\alpha,\beta}^{(1)}(A)$,

iii. $Q^{(1)}_{\alpha,\beta}(A) \subset P^{(1)}_{\alpha,\beta}(A)$,
iv. $Q^{(i)}_{\alpha,\beta}(A) \subset A$, i=1,2,
v. $A \subset P^{(i)}_{\alpha,\beta}(A)$, i=1,2.

Proof. (i) Since

$\min\left(\max\left(\alpha^{\frac{1}{\delta}}, \mu_A(x)\right), \left(1-\nu_A(x)^{\delta}\right)^{\frac{1}{\delta}}\right) \leq \max\left(\alpha^{\frac{1}{\delta}}, \mu_A(x)\right)$ then $P^{(1)}_{\alpha,\beta}(A) \subset P_{\alpha,\beta}(A)$,

and

$\max\left(\max\left(\alpha^{\frac{1}{\delta}}, \mu_A(x)\right), \left(1-\nu_A(x)^{\delta}\right)^{\frac{1}{\delta}}\right) \geq \max\left(\alpha^{\frac{1}{\delta}}, \mu_A(x)\right)$ then $P_{\alpha,\beta}(A) \subset P^{(2)}_{\alpha,\beta}(A)$,

therefore, we have $P^{(1)}_{\alpha,\beta}(A) \subset P_{\alpha,\beta}(A) \subset P^{(2)}_{\alpha,\beta}(A)$.

The proof is completed. (ii)-(v) are proved analogically.

Corollary 3.8. According to definition of new operators and Theorem3.7 (iv)-(v), the operators of $\mathrm{P}^{(\mathrm{i})}_{\alpha,\beta}(\mathrm{A}), \mathrm{i}=1,2,$ increases the membership degree of A and reduces non-membership degree of A, the operators of $\mathrm{Q}^{(\mathrm{i})}_{\alpha,\beta}(\mathrm{A}), \mathrm{i}=1,2,$ reduces the membership degree of A and increases non-membership degree of A.

Theorem 3.9. For every $\mathrm{A}, \mathrm{B} \in \mathrm{GIFS}_{\mathrm{B}}$, we have
i. $P^{(i)}_{\alpha,\beta}(A \cup B) = P^{(i)}_{\alpha,\beta}(A) \cup P^{(i)}_{\alpha,\beta}(B)$, i=1,2,
ii. $P^{(i)}_{\alpha,\beta}(A \cap B) = P^{(i)}_{\alpha,\beta}(A) \cap P^{(i)}_{\alpha,\beta}(B)$, i=1,2,
iii. $Q^{(i)}_{\alpha,\beta}(A \cup B) = Q^{(i)}_{\alpha,\beta}(A) \cup Q^{(i)}_{\alpha,\beta}(B)$, i=1,2,
iv. $Q^{(i)}_{\alpha,\beta}(A \cap B) = Q^{(i)}_{\alpha,\beta}(A) \cap Q^{(i)}_{\alpha,\beta}(B)$. i=1,2.

Proof. (i) Let i=1

$$P^{(1)}_{\alpha,\beta}(\mathrm{A} \cup \mathrm{B}) = \left\{\langle x, \min\left(\max\left(\alpha^{\frac{1}{\delta}}, \max(\mu_A(x), \mu_B(x)\right), (1 - \min(\nu_A(x), \nu_B(x))^{\delta})^{\frac{1}{\delta}}, \min\left(\beta^{\frac{1}{\delta}}, \min(\nu_A(x), \nu_B(x)\right)\right\rangle : x\epsilon\, X\right\},$$

$$= \left\{\langle x, \min(\max(\max\left(\alpha^{\frac{1}{\delta}}, \mu_A(x)\right), \max\left(\alpha^{\frac{1}{\delta}}, \mu_B(x)\right)), max\left(\left(1-\nu_A(x)^{\delta}\right)^{\frac{1}{\delta}}, (1-\nu_B(x)^{\delta})^{\frac{1}{\delta}}\right)\ ), \min(\min\left(\beta^{\frac{1}{\delta}}, \nu_A(x)\right), \min\left(\beta^{\frac{1}{\delta}}, \nu_B(x)\right)\rangle : x\epsilon\, X\right\},$$

$$= \left\{\langle x, \min(\max\left(\alpha^{\frac{1}{\delta}}, \mu_A(x)\right), \left(1-\nu_A(x)^{\delta}\right)^{\frac{1}{\delta}}), \min\left(\beta^{\frac{1}{\delta}}, \nu_A(x)\right), \rangle : x\epsilon\, X\right\} \cup$$
$$\left\{\langle x, \min(\max\left(\alpha^{\frac{1}{\delta}}, \mu_B(x)\right), \left(1-\nu_B(x)^{\delta}\right)^{\frac{1}{\delta}}), \min\left(\beta^{\frac{1}{\delta}}, \nu_B(x)\right), \rangle : x\epsilon\, X\right\},$$
$$= P^{(1)}_{\alpha,\beta}(A) \cup P^{(1)}_{\alpha,\beta}(B).$$

(ii) Let i=1

$P_{\alpha,\beta}^{(1)}(\mathrm{A} \cap \mathrm{B}) =$
$\Big\{\langle x, \min\,(\max\,(\alpha^{\frac{1}{\delta}}, min(\mu_A(x), \mu_B(x)), (1 -$
$\max(\nu_A(x), \nu_B(x))^{\delta})^{\frac{1}{\delta}}, \min\,(\beta^{\frac{1}{\delta}}, \max(\nu_A(x), \nu_B(x))\rangle : x\epsilon\, X\Big\},$

$= \Big\{\langle x, \min(\min(\max\Big(\alpha^{\frac{1}{\delta}}, \mu_A(x)\Big), \max\Big(\alpha^{\frac{1}{\delta}}, \mu_B(x)\Big)), min\Big(\big(1 - \nu_A(x)^{\delta}\big)^{\frac{1}{\delta}}, \big(1 -$
$\nu_B(x)^{\delta}\big)^{\frac{1}{\delta}}\Big) \;), \max(\min\,(\beta^{\frac{1}{\delta}}, \nu_A(x)), \min\,(\beta^{\frac{1}{\delta}}, \nu_B(x))\rangle : x\epsilon\, X\Big\},$
$= \Big\{\langle x, \min(\max\Big(\alpha^{\frac{1}{\delta}}, \mu_A(x)\Big), \big(1 - \nu_A(x)^{\delta}\big)^{\frac{1}{\delta}}), \min\,(\beta^{\frac{1}{\delta}}, \nu_A(x)),\rangle : x\epsilon\, X\Big\} \cap$
$\Big\{\langle x, \min(\max\Big(\alpha^{\frac{1}{\delta}}, \mu_B(x)\Big), \min\,(\beta^{\frac{1}{\delta}}, \nu_B(x))), \min\,(\beta^{\frac{1}{\delta}}, \nu_B(x)),\rangle : x\epsilon\, X\Big\},$

$= P_{\alpha,\beta}^{(1)}(A) \cap P_{\alpha,\beta}^{(1)}(B),$

The proof is completed. Proofs of (iii) and (iv) are similar to that of (i) and (ii).

Corollary 3.10. For every $\mathrm{A_j} \in \mathrm{GIFS_B}$, $\mathrm{j} = 1, \ldots \mathrm{n}$, we have

i. $P_{\alpha,\beta}^{(i)}\big(\bigcup_{j=1}^{n} A_j\big) = \bigcup_{j=1}^{n} P_{\alpha,\beta}^{(i)}(A_j)$, i=1,2,
ii. $P_{\alpha,\beta}^{(i)}\big(\bigcap_{j=1}^{n} A_j\big) = \bigcap_{j=1}^{n} P_{\alpha,\beta}^{(i)}(A_j)$, i=1,2,
iii. $Q_{\alpha,\beta}^{(i)}\big(\bigcup_{j=1}^{n} A_j\big) = \bigcup_{j=1}^{n} Q_{\alpha,\beta}^{(i)}(A_j)$, i=1,2,
iv. $Q_{\alpha,\beta}^{(i)}\big(\bigcap_{j=1}^{n} A_j\big) = \bigcap_{j=1}^{n} Q_{\alpha,\beta}^{(i)}(A_j)$, i=1,2.

Theorem 3.11. For every $\mathrm{A}, \mathrm{B} \in \mathrm{GIFS_B}$, where $\mathrm{A} \subseteq \mathrm{B}$ we have

i. $P_{\alpha,\beta}^{(i)}(A) \subseteq P_{\alpha,\beta}^{(i)}(B)$,
ii. $Q_{\alpha,\beta}^{(i)}(A) \subseteq Q_{\alpha,\beta}^{(i)}(B)$.

Proof. (i) Let i=1, since $\mathrm{A} \subseteq \mathrm{B}$ then $\mu_A(x) \leq \mu_B(x)$ and $\nu_A(x) \geq \nu_B(x)$ therefore

$\max\,(\alpha^{\frac{1}{\delta}}, \mu_A(x) \leq \max\,(\alpha^{\frac{1}{\delta}}, \mu_B(x))\,,\ \big(1 - \nu_B(x)^{\delta}\big)^{\frac{1}{\delta}} \geq \big(1 - \nu_A(x)^{\delta}\big)^{\frac{1}{\delta}}$

and

$\min\,(\beta^{\frac{1}{\delta}}, \nu_A(x)) \geq \min\,(\beta^{\frac{1}{\delta}}, \nu_B(x)).$

Finally

$\mu_{P_{\alpha,\beta}^{(1)}(A)}(x) = \min(\max\Big(\alpha^{\frac{1}{\delta}}, \mu_A(x)\Big), \big(1 - \nu_A(x)^{\delta}\big)^{\frac{1}{\delta}}),$
$\leq \mu_{P_{\alpha,\beta}^{(1)}(B)}(x) = \min(\max\Big(\alpha^{\frac{1}{\delta}}, \mu_B(x)\Big), \min\,(\beta^{\frac{1}{\delta}}, \nu_B(x))),$

and similarly, we have$\nu_{P^{(1)}_{\alpha,\beta}(B)}(x) \leq \nu_{P^{(1)}_{\alpha,\beta}(A)}(x)$.

Proof is complete. Proof of (ii) is similar to that of (i).

Example 3.12. Let $A = \{\langle x, 0.36, 0.09\rangle\}, \delta = 0.5$, then

$$\mu_{P_{\alpha,\beta}(A)}(x) = \begin{cases} 0.36, & \alpha \leq 0.6 \\ \alpha^2, & \alpha > 0.6 \end{cases}, \upsilon_{P_{\alpha,\beta}(A)}(x) = \begin{cases} 0.09, & \beta \geq 0.3 \\ \beta^2, & \beta < 0.3 \end{cases},$$

$$\mu_{Q_{\alpha,\beta}(A)}(x) = \begin{cases} 0.36, & \alpha > 0.6 \\ \alpha^2, & \alpha \leq 0.6 \end{cases}, \upsilon_{Q_{\alpha,\beta}(A)}(x) = \begin{cases} 0.09, & \beta < 0.3 \\ \beta^2, & \beta \geq 0.3 \end{cases},$$

$$\mu_{P^{(1)}_{\alpha,\beta}(A)}(x) = \begin{cases} 0.36, & \alpha \leq 0.6 \\ \alpha^2, & 0.6 < \alpha < 0.7 \\ 0.49, & 0.7 \leq \alpha \end{cases}, \upsilon_{P^{(1)}_{\alpha,\beta}(A)}(x) = \begin{cases} 0.09, & \beta \geq 0.3 \\ \beta^2, & \beta < 0.3 \end{cases},$$

$$\mu_{P^{(2)}_{\alpha,\beta}(A)}(x) = \begin{cases} 0.49, & \alpha < 0.7 \\ \alpha^2, & \alpha \geq 0.7 \end{cases}, \upsilon_{P^{(2)}_{\alpha,\beta}(A)}(x) = \begin{cases} 0.09, & \beta \geq 0.3 \\ \beta^2, & \beta < 0.3 \end{cases},$$

$$\mu_{Q^{(1)}_{\alpha,\beta}(A)}(x) = \begin{cases} 0.36, & \alpha > 0.6 \\ \alpha^2, & \alpha \leq 0.6 \end{cases}, \upsilon_{Q^{(1)}_{\alpha,\beta}(A)}(x) = \begin{cases} 0.09, & \beta < 0.3 \\ \beta^2, & 0.3 < \beta \leq 0.4, \\ 0.16, & 0.4 < \beta \end{cases}$$

$$\mu_{Q^{(2)}_{\alpha,\beta}(A)}(x) = \begin{cases} 0.36, & \alpha > 0.6 \\ \alpha^2, & \alpha \leq 0.6 \end{cases}, \upsilon_{Q^{(2)}_{\alpha,\beta}(A)}(x) = \begin{cases} 0.16, & \beta < 0.4 \\ \beta^2, & \beta \geq 0.4 \end{cases},$$

where $\alpha + \beta \leq 1$.

4 Conclusions

We have introduced four level operators over $GIFS_B$s and showed geometrical interpretation of new operators in the generalized intuitionistic fuzzy sets. Also we proved their relationships and showed that these operators are $GIFS_B$. These operators are well defined since, if $\delta = 1$, the results agree with IFS.

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CHARACTERIZATIONOF SOFT STRONG SEPARATION AXIOMS IN SOFT BI TOPOLOGICAL SPACES

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Abstract – The main aim of this article is to introduce Soft strong separation axioms in soft bi topological spaces. We discuss soft strong separation axioms in soft bi topological spaces with respect to ordinary point and soft points. Further study the behavior of soft strong T_0 space, soft strong $T_1$and soft strong T_2spaces at different angles with respect to ordinary points as well as with respect to soft points. Hereditary properties are also talked over.

Keywords – *Soft sets, soft open sets, soft points, soft bi-topological space, soft strong* T_i *spaces* $(i = 0,1,2)$ *spaces in soft bi topology.*

1. Introduction.

In real life condition the problems in economics, engineering, social sciences, medical science etc we cannot beautifully use the traditional classical methods because of different types of uncertainties presented in these problems. To overcome these difficulties, some kinds of theories were put forwarded like theory of Fuzzy set, intuitionistic fuzzy set, rough set and bi polar fuzzy sets, inwhich we can safely use a mathematical technique for businessing with uncertainties. But, all these theories have their inherent difficulties. To overcome these difficulties in the year 1999, Russian scientist Molodtsov [4], originated the notion of soft set as a new mathematical technique for uncertainties, which is free from the above complications. Mololdtsov [4] and Ahmad [5], successfully applied the soft set theory in different directions, such as smoothness of functions, game theory, operation research, Riemann integration, perron integration, probability, theory of measurement and so on.

After presentation of the operations of soft sets [6], the properties and applications of the soft set theory have been studied increasingly [7,8,6]. Xiao et al. [9] and Pei and Maio [10] discussed the linkage between soft sets and information systems. They showed that soft sets are class of special information system. In the recent year, many attention-grabbing applications of soft sets theory have been extended by embedding the ideas of fuzzy sets [11,12,13,14,15,16,17,18,20,21,22] industrialized soft set theory, the operations of the soft sets are redefined and indecision making method was constructed by using their new operations [23].

Recently, in 2011, Shabir and Naz [23] launched the study of soft Topological spaces; they beautiful defined soft Topology as a collection of τ of soft sets over X. They also defined the basic conceptionof soft topological spaces such as open set and closed soft sets, soft nbd of a point, soft separation axiom, soft regular and soft normal spaces and published their several behaviors. Min in [25] scrutinized some belongings of these soft separation axioms. Kandil et al. [26] introduced some soft operations such as semi open soft, pre-open soft, α-open soft and β-open soft and examined their properties in detail. Kandil et al. [27] introduced the concept of soft semi – separation axioms, in particular soft semi- regular spaces. The concept of soft ideal was discussed for the first time by Kandil et al. [28] .They also introduced the concept of soft local function. These ideas are discussed with a view to find new soft topological from the original one, called soft topological spaces with soft ideal(X, τ, E, I).

Applications of different zone were further discussed by Kandil et al. [28,29,30,32,33, 34,35]. The notion of super soft topological spaces was initiated for the first time by El-Sheikh and Abd-e-Latif [36]. They also introduced new different types of sub-sets of supra soft topological spaces and study the dealings between them in great detail. Bin Chen [41] introduced the concept of semi open soft sets and studied their related properties, Hussain [42] discussed soft separation axioms. Mahanta [39] introduced semi open and semi closed soft sets. Lancy and Arockiarani [40], On Soft β-Separation Axioms, Arockiarani and Arokialancy in [43] generalized soft g β closed and soft gs β closed sets in soft topology are exposed. Analogous to [44] and based on the concept of b-open sets in topological spaces, the notion of quasi-b-open set in bi topological spaces is introduced and discovered. [43] discussedbi topological strong separation axioms, pair wise intersection property and pair wise strong ly regular property is also studied.

In this present paper we introduce the soft strong separation axioms, soft strong pair wise regularity and soft strong pair wise normal. Concept of soft strong T_0, soft strong T_1 and soft strong T_2 spaces in Soft bi topological spaces is introduced with respect to soft points. Many mathematicians discussed soft separation axioms in soft topological spaces at full length with respect to soft open set, soft b-open set, soft semi-open set, soft α-open set and soft β-open set. They also worked over the hereditary properties of different soft topological structures in soft topology. In this present work hand is tried and work is encouraged over the gap that exists in soft bi-topology. Related to soft strong T_0, soft strong T_1 and soft strong T_2 spaces, some Proposition in soft bi topological spaces are discussed with respect to ordinary points as well as with respect to soft points. When we talk about the distance between the points in soft topology then the concept of soft separation axioms will automatically come in play. That is why these structures are catching our attention. We hope that these results will be valuable for the future study on soft bi topological spaces to accomplish general framework for the practical applications and to solve the most intricate problems containing scruple in economics, engineering, medical, atmosphere and in general mechanic systems of various kinds.

2. Preliminaries

The following Definitions which are pre-requisites for present study.

Definition 1 [4]. Let X be an initial universe of discourse and E be a set of parameters. Let P(X) denotes the power set of X and A be a non-empty sub-set of E. A pair (F, A) is called a soft set over U, where F is a mapping given by $F: A \to P(X)$

In other words, a set over X is a parameterized family of sub set of universe of discourse X For $e \in A, F(e)$ may be considered as the set of e-approximate elements of the soft set $(F, A)$and if$e \notin A$ then$F(e) = \phi$, that is$F_{A=}\{F(e): e \in A \subseteq E, F: A \to P(X)\}$ the family of all these soft sets over X denoted by$SS(X)_A$.

Definition 2 [4]. Let$F_{A,}G_B \in SS(X)_E$ then $F_A$is a soft subset of G_B denoted by$F_A \,\overline{\sqsubseteq}\, G_B$, if

1. $A \subseteq B$ and
2. $F(e) \subseteq G(e), \forall \in A$

In this case F_A is said to be a soft subset of G_B and G_B is said to be a soft super setF_A, $G_B \,\overline{\sqsupseteq}\, F_A$.

Definition 3 [6].Two soft subsets F_A and G_B over a common universe of discourse set X are said to be equal if F_A is a soft subset of G_B and G_B is a soft subset of F_A.

Definition 4 [6].The complement of soft subset(F, A)ddenoted by $(F, A)^C$ is defined by $(F, A)^C = (F^C, A)F^C \to P(X)$is a mapping given by $F^C(e) = U - F(e) \forall e \in A$ and $F^C$is called the soft complement function of$F$. Clearly $(F^C)^C$ is the same as F and $((F, A)^C)^C = (F, A)$.

Definition 5 [7]. The difference between two soft subset (G, E) and (G, E) over common of universe discourse X denoted by $(F, E) - (G, E)$ is the soft set(H, E) where for all $e \in E. \overline{\emptyset} or\ \emptyset_A$if $\forall e \in A, F(e) = \emptyset$.

Definition 6 [7]. Let (G, E) be a soft set over$X$ and $x \in X$ We say that $x \in (F, E)$ and read as x belong to the soft set(F, E) whenever $x \in F(e) \forall e \in E$ The soft set(F, E) over$X$such that $F(e) = \{x\} \forall e \in E$ is called singleton soft point and denoted by $x_E, or\ (x, E)$.

Definition 7 [6]. A soft set (F, A) over X is said to be Null soft set denoted by $\overline{\emptyset}\ or\ \emptyset_A$ if $\forall e \in A, F(e) = \emptyset$.

Definition 8 [6]. A soft set(F, A) over X is said to be an absolute soft denoted by $\bar{A}\ or\ X_A$ if $\forall e \in A, F(e) = X$.

Clearly, we have. $X_A^C = \emptyset_A\ and\ \emptyset_A^C = X_A$.

Definition 9 [42]. Let(G, E) be a soft set over$X$and$e_G \in X_A$, we say that $e_G \in (F, E)$ and read as e_G belong to the soft set(F, E)whenever $e_G \in F(e) \forall e \in E$. the soft set$(F, E)$ over$X$such that $F(e) = \{e_G\}$, $\forall e \in E$ is called singleton soft point and denoted bye_G, $or\ (e_G, E)$.

A soft point is an element of a soft set F_A. the class of all soft sets over U is denoted by$S(U)$.

For example, $U = \{u_1, u_2, u_3\}, E = \{x_1, x_2, x_3\}$, $A = \{x_1, x_2\}$ and $F_A = \{(x_1, \{u_1, u_2\})\}$, $\{(x_2, \{u_2, u_3\})\}$.

Then

$F_{A_1} = \{(x_1, \{u_1\})\}, F_{A_2} = \{(x_1, \{u_2\})\}, F_{A_3} = \{(x_1, \{u_2, u_2\})\}$,
$F_{A_4} = \{(x_2, \{u_2\})\}$,$F_{A_5} = \{(x_2, \{u_3\})\}$,
$F_{A_6} = \{(x_2, \{u_1, u_3\})\}, F_{A_7} = \{(x_1, \{u_2\}, (x_{2,}\{u_2\})\}$,
$F_{A_8} = \{(x_1, \{u_1\}, (x_2, \{u_3\})\}$,
$F_{A_9} = \{(x_1, \{u_1\})\}, (x_2, \{u_2, u_3\})\}, F_{A_{10}} = \{(x_1, \{u_2\})\}, (x_2, \{u_2\})\}$,
$F_{A_{11}} = \{(x_1, \{u_2\}), (x_{2,}\{u_3\})\}, F_{A_{12}} = \{(x_1, \{u_2\}), (x_{2,}\{u_2, u_3\}, F_{A_{13}} =$
$\{(x_1, \{u_1, u_2\}), (x_2, \{u_2\})\}, F_{A_{14}} = \{(x_1, \{u_1, u_2\}), (x_2, u_3\}, F_{A_{15}} = F_A, F_{A_{16}} = F_\emptyset$,

are all soft sub sets of F_A

Definition 10 [42].The soft set $(F, A) \in SSX_A$is called a soft point inX_A, denoted by e_F, if for the element $e \in A, F(e) \neq \{x\}$ and $F(e^/) = \phi$ if for all $e^/ \in A - \{e\}$

Definition 11 [42]. The soft point e_F is said to be in the soft set(G, A), denoted by $e_F \in (G, A)$ if for the element $e \in A, F(e) \subseteq G(e)$.

Definition 12 [42].Two soft sets $(G, A), (H, A)$ in SSX_A are said to be soft disjoint, written $(G, A) \cap (H, A) = \emptyset_A$If $G(e) \cap H(e) = \emptyset \forall\ e \in A$.

Definition 13 [42]. The soft point $e_G, e_H \in X_A$ are disjoint, written $e_G \neq e_{H,}$ if their corresponding soft sets (G, A) and (H, A) are disjoint.

Definition 14 [6]. The union of two soft sets(F, A) and (G, B) over the common universe of discourse X is the soft set(H, C), where, $C = AUB \forall e \in C$

$$H(e) = \begin{cases} F(e) & if\ e\ \in A - B \\ G(e) & if\ e\ \in (B - A) \\ F(e), & if\ e\ \in A \cap B \end{cases}$$

Written as $(F, A) \cup (G, B) = (H, C)$

Definition 15 [6]. The intersection$(H.C)$of two soft sets$(F,A)$and (G,B)over common universe X, denoted $(F,A)\,\overline{\cap}\,(\mathrm{G,B})$ is defined as $C=A\cap B$and $H(e)=F(e)\cap G(e), \forall e\in C$.

Definition 16 [2]. Let(F,E) be a soft set over X and Y be a non-empty sub set of X. Then the sub soft set of (F,E)over Y denoted by (Y_F,E),is defined as follow$Y_{F(\alpha)}=Y\cap F(\alpha), \forall\in E$ in other words

$$(Y_F,E)=Y\cap(F,E).$$

Definition 17 [2]. Let τ be the collection of soft sets over X , then τ is said to be a soft topology on X, if

1. $\emptyset,\ X\,\epsilon\tau$
2. The union of any number of soft sets in τ belongs to τ
3. The intersection of any two soft sets in τ belong to τ

The triplet (X,F,E) is called a soft topological space.

Definition 18 [1]. Let (X,τ,E) be a soft topological space over X then the member of τ are said to be soft open sets in X.

Definition 19 [1]. Let(X,τ,E) be a soft topological space over$X$.A soft set(F,A) over X is said to be a soft closed set in X , if its relative complement $(F,E)^C$ belong to τ.

Definition 20 [3]. Let(X,τ_1,E) and (X,τ_2,E) be two different soft topologies on X. Let $\tau_1 v\tau_2$be the smallest soft topology on X that contains$\tau_1\cup\tau_2$.

Example: Suppose there are three Houses in the universe $U=\{h_1,h_2,h_3\}$ under observation, and that $E=\{e_1,e_2\}$ is a decision parameters which stands for "beautiful", and "in green surrounding". In this case to define a soft set means to point out beautiful house and in green surrounding house.

Let$\tau_1=\{\emptyset,\widetilde{X,}(F_1,E),(F_2,E)\}$ and
$\{\tau_1=$
$\emptyset,\widetilde{X,}(G_1,E),(G_2,E),(G_3,E),(G_4,E)\}$ where
$(F_1,E),(F_1,E),(G_1,E),(G_2,E),(G_3,E),(G_4,E)$ are soft sets over$X$, defined as follow:

$F_1(e_1)=\{h_1\},F_1(e_2)=\{h_1,h_2\},F_2(e_1)=\{h_1,h_3\},F_2(e_2)=X,G_1(e_1)=\{h_2\},G_1(e_2)=\{h_2\},\ G_2(e_1)=\{h_1,h_2\},G_2(e_2)=\{h_2\},G_3(e_1)=\{h_2\},G_3(e_2)=\{h_2\},G_4(e_1)=\{\quad\},G_4(e_2)=\{h_2\}$.

Then τ_1 and τ_2 are soft topology on X.Now,

$$\tau_1 v\tau_2=\{\emptyset,X,(F_1,E),(F_2,E),(G_1,E),(G_2,E),(G_3,E),(G_4,E),(H_1,E)\}.$$

Where, $H_1(e_1)=\{h_1,h_2\},H_1(e_2)=\{h_1,h_2\}$. Thus $(X,\tau_1 v\tau_2,E)$ is the smallest soft topological space over X that contains $\tau_1\cup\tau_2$.

3. Separation Axioms of Soft Topological Spaces With Respect to Ordinary Points as Well as Soft Points

Definition 21 [1]. Let (X, τ, A) be a soft topological space over X and $x, y \in X$ such that $x \neq y$ if there exist at least one soft open set (F_1, A) OR (F_2, A) such that $x \in (F_1, A), y \notin (F_1, A)ory \in (F_2, A), x \notin ((F_2, A)$then (X, τ, A) is called a soft T_0 $space$.

Definition 22 [1]. Let (X, τ, A) be a soft topological spaces over X and $x, y \in X$ such that $x \neq y$ if there exist soft open sets (F_1, A) and (F_2, A) such that $x \in (F_1, A), y \notin (F_1, A)$ and $y \in (F_2, A), x \notin ((F_2, A)$ then (X, τ, A) is called a soft T_1 $space$.

Definition 23 [1]. Let (X, τ, A) b e a soft Topological space over X and $x, y \in X$ such that $x \neq y$ if there exist soft open set $(F_1, A)$and (F_2, A) such that $x \in (F_1, A)$, and $y \in (F_2, A)$ and $F_1 \cap F_2 = \varphi$

Then (X, τ, A) is called soft T_2 spaces.

Definition 24 [42].Let (X, τ, A) be a soft topological space over X and $e_G, e_H \in X_A$ such that $e_G \neq e_H$ if we can search at least one soft open set (F_1, A) or (F_2, A) such that $e_G \in (F_1, A), e_H \notin (F_1, A)$or $e_H \in (F_2, A), e_G \notin ((F_2, A)$then (X, τ, A) is called a soft T_0 space.

Definition 25 [42]. Let (X, τ, A) be a soft topological spaces over X and $e_G, e_H \in X_A$ such that $e_G \neq e_H$ if we can search soft open sets $(F_1, A)$and (F_2, A) such that $e_G \in (F_1, A), e_H \notin (F_1, A)$ and $e_H \in (F_2, A), e_G \notin ((F_2, A)$ then (X, τ, A) is calleda soft T_1 $space$.

Definition 26 [42]. Let (X, τ, A) b e a soft topological space over X and $e_G, e_H \in X_A$ such that $e_G \neq e_H$ if we can search soft open set $(F_1, A)$and (F_2, A) such that $e_G \in (F_1, A)$, and $e_H \in (F_2, A)$

$(F_1, A) \cap (F_2, A) = \phi_A$ Then (X, τ, A) is called soft T_2 space.

4. Soft Strong Separation Axioms of Soft Bi Topological Spaces

Let X is an initial set and E be the non-empty set of parameter. In [1] soft bi topological space over the soft set X is introduced. Soft separation axioms in soft bi topological spaces were introduced by Basavaraj and Ittanagi [1]. In this section we introduced the concept of soft b T_3 and bT_4 spaces in soft bi topological spaces with respect to ordinary as well as soft points and some of its basic properties are studied and applied to different results in this section.

Definition 27 [1]. Let(X, τ_1, E) and (X, τ_2, E) be two different soft topologies on X. Then (X, τ_1, τ_2, E) is called a *soft bi topological space.* The two soft topologies (X, τ_1, E) and (X, τ_2, E) are independently satisfy the axioms of soft topology. The members ofτ_1 are called τ_1*soft open*. And complement ofτ_1. Soft open set is called τ_1*soft closed* set.

Similarly, the member of τ_2 are called τ_2 *soft open* sets and the complement of τ_2 soft open sets are called τ_2*soft closed* set.

Definition 28 [1]**.** Let (X, τ_1, τ_2, E)be a soft topological space over X and Y be a non-empty subset of X. Then$\tau_{1Y} = \{(Y_E, E): (F, E) \in \tau_1)\}$ and $\tau_{2Y} = \{(G_E, E): (G, E) \in \tau_2)\}$ are said to be the relative topological on Y. Then $(Y, \tau_{1Y}, \tau_{2Y}, E)$ is called relative soft bi-topological space of (X, τ_1, τ_2, E).

4.1 Soft Strong Separation Axioms of Soft Bi Topological Spaces with Respect to Ordinary Points.

In this section we introduced soft strong separation axioms in soft bi topological space with respect to ordinary points and discussed some results with respect to these points in detail.

Definition 29. In a *soft bi topological* space (X, τ_1, τ_2, E)

1) τ_1 said to be *soft strong* T_0 space with respect to τ_2 if for each pair of distinct points $x, y \in X$ there exists τ_1 soft open set (F, E) and a τ_2 soft open set (G, E) such that $x \in \tau_2 int(F, E)$ and $y \notin \tau_2 int(F, E)$ or $y \in \tau_1 int(G, E)$ and $x \notin \tau_1 int(G, E)$ similarly τ_2 is said to be *soft strong* T_0 space with respect to τ_1 if for each pair of distinct points $x, y \in X$ there exists τ_2*soft open set*(F, E) and a $\tau_1$soft open set (G, E) such that $x \in \tau_1\, int(F, E)$ and $y \notin \tau_1 int(F, E)$ or $y \in \tau_2 int(G, E)$ and $x \notin \tau_2 int(G, E)$. *Soft bi topological spaces*(X, τ_1, τ_2, E) is said to be *pair wise soft strong* T_0 space if τ_1 is *soft strong* T_0 *space* with respect to τ_2 and τ_2 is *soft strong* T_0 space with respect to τ_1.

2) τ_1 is said to be soft strong T_1 space with respect to τ_2 if for each pair of distinct points $x, y \in X$ there exists τ_1 *soft open set* (F, E) and τ_2 soft open set (G, E) such that $x \in \tau_2\text{int}\,(F, E)$ and $y \notin \tau_2\text{int}\,(F, E)$ and $y \in \tau_1 int(G, E)$ and $x \notin \tau_1 int(G, E)$. Similarly,$\tau_2$ is said to be soft strong T_1 space with respect to τ_1 if for each pair of distinct points $x, y \in X$there exist a τ_2 soft open set (F, E) and a τ_1 soft open set (G, E) such that $x \in \tau_1 int(F, E)$ and $y \notin \tau_1 int(F, E)$ and $y \in \tau_2 int(G, E)$ and $x \notin \tau_2 int(G, E)$. *Soft bi topological spaces* (X, τ_1, τ_2, E) is said to be *pair wise soft strong* T_1 space if τ_1 is *soft strong* T_1 *space*with respect to $\tau_2$and τ_2 is *soft strong* T_1space with respect to τ_1.

3) τ_1 said to be soft strong T_2 space with respect to $\tau_2$if for each pair of distinct points $x, y \in X$ there exists a τ_1 soft open set (F, E) and a τ_2 soft open set (G, E) such that $x \in \tau_2 int(F, E)$and $y \in \tau_1\text{int}\,(G, E)$ and$(F, E) \cap (G, E) = \phi$. Similarly,$\tau_2$ is said to be *soft strong* T_2space with respect to τ_1 if for each pair of distinct points $x, y \in X$ there exists a τ_2*soft open* set (F, E) and a τ_1*soft open*set(G, E) such that $x \in \tau_1 int(F, E)$ and $y \in \tau_2 int(G, E)$ and $(F, E) \cap (G, E) = \phi$. The soft bi topological space (X, τ_1, τ_2, E) is said to be pair wise soft strong T_2 space if τ_1 is soft strong T_2 space with respect to τ_2 and τ_2 is soft strong T_2 space with respect toτ_1.

Proposition 1. Let (X, τ_1, τ_2, E) be a *bi soft topological* space over X . If (X, τ_1, τ_2, E) is a *pair wise soft strong* T_0 space then $(X, \tau_1 v \tau_2, E)$ is a *soft strong* T_0 space.

Proof. A *soft bi topological* space (X, τ_1, τ_2, E) is called *pair wise soft strong* T_0 space if τ_1 is a *soft strong* T_0 space with respect to τ_2 and τ_2 is *soft strong* T_0 space with respect toτ_1. If $x, y \in X$, $x \neq y$. Then, since $\tau_1$is *soft strong* T_0 space with respect to τ_2 so there exists τ_1 *soft open* set (F, E) and τ_2 *soft open* set (G, E) such that $x \in \tau_2\text{int}(F, E)$ and $y \notin \tau_2 int(F, E)$ or $y \in \tau_1 int(G, E)$ and $x \notin \tau_1 int(G, E)$ and since $\tau_2$is *soft* T_0 space with respect to τ_1 so there existsτ_2 *soft open* set$(G,E)$and $\tau_1$soft openset(F,E) such that and $x \in \tau_2 int(F,E)$ and $y \notin \tau_2 int(F,E)$ or $y \in \tau_1 int(G,E)$ and $x \notin \tau_1 int(G,E)$. In either case $(F,E),(G,E) \in (X,\tau_1 v \tau_2,E)$. Hence $(X,\tau_1 v \tau_2,E)$ is a soft strong T_0 *space.*

Proposition 2. Let (X,τ_1,τ_2,E) be a *soft bi topological* space over X and Y be a non-empty subset of X. if (X,τ_1,τ_2,E) is *pair wise soft strong* T_0 space. Then$(X,\tau_{1Y},\tau_{2Y},E)$ is *pair wise soft strong* T_0 space.

Proof. Let (X,τ_1,τ_2,E) be a *soft bi topological* space over X $x,y \in X$ such that$x \neq y$. If(X,τ_1,τ_2,E) is *pair wise soft strong* T_0 *space*. Then there exist τ_1 *soft open* set (F,E) and τ_2 s*oft open* set (G,E). Such that $x \in \tau_2\text{int}(F,E)$ and $y \notin \tau_2\text{int}(F,E)$ or $y \in \tau_1\text{int}\,(G,E)$ and$x \notin \tau_1 int(G,E)$. Now, $x \in Y$ and $x \in \tau_2\text{int}\,(F,E)$. Hence, where $x \in Y \cap (F,E) = (Y_F,E)$ where $(F,E) \in \tau_1$. Consider $y \notin \tau_2\text{int}(F,E)$, this means that $\alpha \in E$ for some $\alpha \in E. y \notin Y \cap (F,E) = (Y_E,E)$ There fore τ_{1Y} is *soft strong* T_0 *space* with respect τ_{2Y}. Similarly, can proved thatτ_{2Y} is *soft strong* T_0 *space* with respect to τ_{1Y}, that is $y \in \tau_1\text{int}(G,E)$ and $x \notin \tau_1\text{int}(G,E)$ then $y \in (Y_{G,}E)$ and $x \notin \left(Y_{G,}E\right)$. Thus $(Y,\tau_{1Y},\tau_{2Y},E)$ is *pair wise soft strong* T_0 *space.*

Proposition 3. Let(X,τ_1,τ_2,E) be a *soft bi topological* space over X . Then (X,τ_1,τ_2,E) is a *pair wise soft strong* T_1 space $\Leftrightarrow$ then(X,τ_1,E) and (X,τ_2,E) are *soft strong* T_1 space.

Proof. Suppose (X,τ_1,E) and (X,τ_2,E) are *soft strong* T_1 spaces. Let $x,y \in X, x \neq y$ then

1). $\tau_1$is a soft strong T_1 space with respect to τ_2. So there exists τ_1 s*oft open* set (F,E) and τ_2 *soft open* set (G,E) Such that $x \in \tau_2\text{int}(F,E)$, $y \notin \tau_2\text{int}(F,E)$ and $y \in \tau_1\text{int}(G,E)$, $x \notin \tau_1\text{int}(G,E)$.

2). $\tau_2$is a *soft strong* T_1 space with respect to τ_1So there exists $\tau_2$soft open set (G,E) and τ_1*soft open* set (F,E) such that $x \in \tau_1\text{int}(G,E)$ and $y \notin \tau_1\text{int}(G,E)$ and $y \in \tau_2\text{int}(F,E)$ and $x \notin \tau_2\text{int}(F,E)$. In either case we obtained the requirement and so (X,τ_1,τ_2,E) is a *pair wise soft strong* T_1space. Conversely, we suppose that (X,τ_1,τ_2,E) is a pair *wise soft strong* T_1 space. Then

1). There exists some *soft open set* $(F,E) \in \tau_1$ with respect to *soft open* set $(G,E) \in \tau_2$ such $x \in \tau_2\text{int}(F,E)$, $y \notin \tau_2\text{int}(F,E)$ and $y \in \tau_1\text{int}(G,E)$, $x \notin \tau_1\text{int}(G,E)$.

2). There exists some *soft open* set$(G,E) \in \tau_2$with respect to *soft open* set $(F,E) \in \tau_1$such that $x \in \tau_1\text{int}(G,E)$ and $y \notin \tau_1\text{int}(G,E)$ and $y \in \tau_2\text{int}(F,E)$ and $x \notin \tau_2\ \text{int}(F,E)$ Thus(X,τ_1,E) and (X,τ_2,E) are *soft strong* T_1 Spaces.

Proposition 4. Let (X,τ_1,τ_2,E) be a *soft bi topological* space over X. if (X,τ_1,τ_2,E) is *pair wise soft strong* T_1 space then $(X,\tau_1 v\,\tau_2,E)$ is also *soft strong* T_1 space.

Proof. Let$x,y \in X$ such that $x \neq y$.then exists *soft open* set $(F,E) \in \tau_1$with respect to soft open $(G,E) \in \tau_2$ such that $x \in \tau_2\text{int}(F,E)$,$y \notin \tau_2\text{int}(F,E)$ and $y \in \tau_1\text{int}(G,E)$ and $x \notin \tau_1\text{int}(G,E)$.Similarly ,There exists *soft open* set$(G,E) \in \tau_2$ with respect to τ_1 *soft open* set$(F,E) \in \tau_1$ such that $x \in \tau_1\text{int}(G,E)$ and $y \notin \tau_1\text{int}(G,E)$ and $y \in \tau_2\text{int}(F,E)$ and$x \notin \tau_2\text{int}(F,E)$ So $(F,E),(G,E) \in \tau_1 v \tau_2$ and thus $(X,\tau_1 v\,\tau_2,E)$ is *soft strong* T_1 space.

Proposition 5. Let(X,τ_1,τ_2,E) be a *soft bi topological* space over X and Y be non-empty sub set of X . If (X,τ_1,τ_2,E) is *pair wise soft strong* T_1 space then $(X,\tau_{1Y},\tau_{2Y},E)$ is *pair wise soft strong* T_1 space.

Proof. Let (X,τ_1,τ_2,E) be a *soft bi topological space* over X and $x,y\in Y, x\neq y$. If (X,τ_1,τ_2,E) is *pair wise soft* T_1 then there exists τ_1 *soft open set*(F,E) and τ_2 *soft open* set (G,E) such that $x\in\tau_2\text{int}(F,E)$ and $y\notin\tau_2\text{int}(F,E)$ and $y\in\tau_1\text{int}(G,E)$ and $x\notin\tau_1\text{int}(G,E)$. Now $x\in Y$and $x\in(F,E)$. Hence$x\in Y\cap(F,E)=(Y_F,E)$. Then $y\notin Y\cap F(\alpha)$ for some $\alpha\in E$. This means that $\alpha\in E$ then $y\notin Y\cap F(\alpha)$ for some $\alpha\in E$. Therefore $y\notin Y\cap(F,E)=(Y_F,E)$ Now $y\in Y$ and $y\in(G,E)$.

Hence $y\in Y\cap(G,E)=(Y_G,E)$ where $(G,E)\in\tau_2$. Consider$x\notin(G,E)$ this means that $\alpha\in E$ then$x\notin Y\cap G(\alpha)$ for some $\alpha\in E$. Therefore. $x\notin Y\cap(G,E)=(Y_G,E)$ hence τ_{1Y} is bT_1 space with respect (Y_G,E) to τ_{2Y}. Similarly it can be provide that τ_{2_Y} is *soft strong* T_1 space with respect to τ_{1Y}, that is $y\in\tau_1\text{int}(G,E)$ and $x\notin\tau_1\text{int}(G,E)$. Then $y\in(Y_G,E)$ and $x\notin(Y_G,E)$ .Thus $(X,\tau_{1Y},\tau_{2Y},E)$ is *pair wise soft strong* T_1space.

Proposition 6. Every *pair wise soft strong* T_1 space is *pair wise sot strong* T_0 space.

Proof. Let (X,τ_1,τ_2,E) be a *soft bi topological* space over X and $x,y\in X$ such that $x\neq y$ If (X,τ_1,τ_2,E) is *pair wise soft strong* T_1 space. That is, (X,τ_1,τ_2,E) is *pair wise soft strong* T_1 space with respect toτ_2 and τ_2 is s*oft strong* T_1 space with respect to τ_1 .If τ_1 is *soft strong* T_1 space with respect to τ_2 then there exists a τ_1*soft open* set (F,E) and a τ_2 *soft open* set (G,E) such that $x\in\tau_2\text{int}(F,E)$ and $y\notin\tau_2\text{int}(F,E)$ and $y\in\tau_1\text{int}(G,E)$ and $x\notin\tau_1\text{int}(G,E)$.

Obviously $x\in\tau_2\text{int}(F,E)$ and $y\notin\tau_2\text{int}(F,E)$ or $y\in\tau_1\text{int}(G,E)$ and $x\notin\tau_1\text{int}(G,E)$. Therefore τ_1 is *soft strong* T_0 space with respect to τ_2. Similarly, ifτ_2 is a *soft strong* T_1 space with respect to τ_1 then, there exists τ_2*soft open* set $(F,E)$and τ_1*soft open* set (G,E) such that $x\in\tau_2\text{int}(F,E)$ and $y\notin\tau_2\text{int}(F,E)$ and $y\in\tau_1\text{int}(G,E)$ and$x\notin\tau_1\text{int}(G,E)$. Obviously $x\in\tau_2$ $\text{int}(F,E)$ and $y\notin\tau_2\text{int}(F,E)$ or $y\in(G,E)$ and $x\notin\tau_1\text{int}(G,E)$. Therefore$\tau_2$ soft strong T_0 space with respect to τ_1. Thus (X,τ_1,τ_2,E) is a *pair wise* soft strong T_0 space.

Proposition 7. Let (X,τ_1,τ_2,E) be a *soft bi topological* space over X. if (X,τ_1,τ_2,E) is a pair wise soft strong T_2 space over X then $(X,\tau_{1e},\tau_{2e},E)$ is a *pair wise soft strong* T_2 space for each$e\in E$.

Proof. Suppose that (X,τ_1,τ_2,E) is a *pair wise soft strong* T_2 space over X. For any $e\in E$.

$$\tau_{1e}=\{F(e),(F,E)\in\tau_1$$
$$\tau_{2e}=\{G(e),(G,E)\in\tau_2$$

Let $x,y\in X$ such that $x\neq y$ then there *exists soft open set* $(F,E)\in\tau_1$ and *soft open* set $(G,E)\in\tau_2$ such that $x\in\tau_2\text{int}(F,E), y\in\tau_1\text{int}(G,E)$ and $(F,E)\cap(G,E)=\emptyset$. This implies that $x\in F(e)\in\tau_{1e}, y\in G(e)\in\tau_{2e}$for each $e\in E$. Similarly, for the other case. Thus $(X,\tau_{1e},\tau_{2e},E)$ is *a pair wise strong* T_2 space for each $e\in E$.

Proposition 8. Let (X,τ_1,τ_2,E) be a *soft bi topological* space over$X$. If (X,τ_1,τ_2,E) is *Pair wise soft strong* T_2 space. Then $(X,\tau_1 v\tau_2,E)$ also a *soft strong* T_2 space.

Proof. Let $x,y\in X$ such that$x\neq y$. Since (X,τ_1,τ_2,E) be a soft *bi topological* space over $X(X,\tau_1,\tau_2,E)$ is a *pair wise soft strong* T_2space if τ_1 is *soft strong* T_2space with respect τ_2 and τ_2 is *soft strong* T_2 space with respect to τ_1. If τ_1 be *soft strong* T_2 space with respect to τ_2 then there exists a τ_1 *soft open* set $(F,E)$and τ_2*soft open* set(G,E) such that $x\in\tau_2\mathrm{int}(F,E), y\in\tau_2\mathrm{int}(G,E)$ and $(F,E)\cap(G,E)=\emptyset$. Obviously $x\in\tau_2\mathrm{int}(F,E)$ and $y\notin\tau_2\mathrm{int}(F,E)$ and $y\in\tau_1\mathrm{int}(G,E)$ and $x\notin\tau_1\mathrm{int}(G,E)$. Therefore τ_1 is *soft strong* T_1 space with respect to τ_2.

Similarly, τ_2 is soft strong T_2 space with respect to τ_1 then there exists a τ_2 *soft open* set (F,E) and τ_1

Soft open set (G,E) such that $x\in\tau_1\mathrm{int}(F,E), y\in\tau_2\mathrm{int}(G,E)$ and$(F,E)\cap(G,E)=\emptyset$.

Obviously $x\in\tau_1\mathrm{int}(F,E)$ and $y\notin\tau_1\mathrm{int}(F,E)$ and $y\in\tau_2\mathrm{int}(G,E)$ and $x\notin\tau_2\mathrm{int}(G,E)$. Therefore τ_2 soft strong T_2 space with respect to τ_1 thus in either case $(F,E),(G,E)\in\tau_1 v\tau_2$. Hence $(X,\tau_1 v\tau_2,E)$ is *soft strong* T_2 space over X.

Proposition 9. Let (X,τ_1,τ_2,E) be a *soft bi topological* space over X and Y be a non-empty sub set of X. if (X,τ_1,τ_2,E) *is pairwise soft strong* T_2 space then $(Y,\tau_{1Y},\tau_{2Y},E)$ is *pair wise soft strong* T_2 space.

Proof. Let (X,τ_1,τ_2,E) be *soft bi topological* space over X $x,y\in X$ such that $x\neq y$ such that (X,τ_1,τ_2,E) is *pair wise soft strong* T_2 space. Then there exists a τ_1 *soft open* set(F,E) and a τ_2 *soft open* (G,E) such that $x\in\tau_2\,\mathrm{int}(F,E), y\in\tau_1\mathrm{int}(G,E)$ and $(F,E)\cap(G,E)=\emptyset$. So for each $\alpha\in E$, $x\in F(\alpha), y\in G(\alpha)$ and $F(\alpha)\cap G(\alpha)=\emptyset$. This implies that $x\in Y\cap F(\alpha), y\in Y\cap G(\alpha)$ and $F(\alpha)\cap G(\alpha)=\emptyset$. Hence $x\in(Y_F,E), y\in(Y_G,E)$ $(Y_F,E)\cap(Y_G,E)=\emptyset$. Where (Y_F,E) is *soft open* set in τ_{1Y}, and (Y_G,E) is *soft open* set in τ_{2Y} therefore τ_{1Y} is *soft* P_2 space with respect to τ_{2Y}. Similarly, it can be proved that τ_{2Y} is *soft* T_2 space with respect to τ_{1Y}. Thus $(Y,\tau_{1Y},\tau_{2Y},E)$ is *pair wise soft strong* T_2 space.

Proposition 10. Every *pair wise soft strong* T_1 space is *pair wise soft strong* T_1 space.

Proof. Let (X,τ_1,τ_2,E) be a *soft bi topological* space over X and $x,y\in X$ such that $x\neq y$ If (X,τ_1,τ_2,E) is *pair wise soft strong* T_2 space. That is (X,τ_1,τ_2,E) is *pair wise soft strong* T_2 space if τ_1 is *soft* T_2 space with respect τ_2 then there exists a τ_1 *soft open* set (F,E) and τ_2 *soft open* set(G,E) such that $x\in\tau_2\mathrm{int}(F,E), y\in\tau_1\mathrm{int}(G,E)$ and $(F,E)\cap(G,E)=\emptyset$. Obviously, $x\in\tau_2\mathrm{in}(F,E)$ and $y\notin\tau_2\mathrm{in}(F,E)$ and $y\in\tau_1\mathrm{in}(G,E)$ and $x\notin\tau_1\mathrm{in}((G,E)$. Therefore τ_1 is *soft* $strong$ T_1 space with respect to τ_2. Similarly, if τ_2 is *soft strong* T_2 space with respect to τ_1 then there exists a τ_2 *soft open* set (F,E) and τ_1 *soft open* set(G,E) such that $x\in\tau_2\mathrm{int}(F,E), y\in\tau_1\mathrm{int}(G,E)$ and $(F,E)\cap(G,E)=\emptyset$. Obviously, $x\in\tau_2\mathrm{int}(F,E)$ and $y\notin\tau_1\mathrm{int}(F,E)$ and $y\in\tau_1\mathrm{int}(G,E)$ and $x\notin\tau_1\mathrm{int}(G,E)$. Therefore τ_2 is *soft strong* T_1space with respect to τ_1 thus $(X,\tau_1,\tau_2,E)$is *pair wise soft strong* T_1 space.

4.2 Soft Strong Separation Axioms of Soft Bi Topological Spaces with Respect to Soft Points

In this section, we introduced soft strong separation axioms in soft topology and in soft bi topology with respect to soft points. With the application of these soft strong separation axioms different results are discussed. Soft point is beautifully defined in Definition 9 [42].

Definition 30. In a *soft bi topological* space (X, τ_1, τ_2, E)

1) τ_1 said to be *soft strong* T_0 space with respect to τ_2 if for each pair of distinct points $e_G, e_H \in X_A$ there happens τ_1 soft open set (F, E) and a τ_2 soft open set (G, E) such that $e_G \in \tau_2\text{int}(F, E)$ and $e_H \notin \tau_2\text{int}(F, E)$or $e_H \in \tau_1\text{int}(G, E)$ and $e_G \notin \tau_1\text{int}(G, E)$, Similarly, τ_2 is said to be *soft strong* T_0 space with respect to τ_1 if for each pair of distinct points $e_G, e_H \in X_A$ there happens τ_2 *soft open set*(F, E) and a $\tau_1$soft open set (G, E) such that $e_G \in \tau_1\text{int}(F, E)$ and $e_H \notin \tau_1\text{int}(F, E)$ or $e_H \in \tau_2\text{int}(G, E)$ and $e_G \notin \tau_2\text{int}(G, E)$. *Soft bi topological spaces* (X, τ_1, τ_2, E) is said to be *pair wise soft strong* T_0 space if τ_1 is *soft strong* T_0 *space* with respect to τ_2 andτ_2 is soft strong T_0 space with respect to τ_1.

2) τ_1 is said to be soft strong T_1 space with respect to τ_2 if for each pair of distinct points $e_G, e_H \in X_A$ there happens a τ_1*soft open set* (F, E) and τ_2 soft open set (G, E) such that $e_G \in \tau_2\text{int}(F, E)$ and $e_H \notin \tau_2\text{int}(F, E)$ and $e_H \in \tau_1\text{int}(G, E)$ and $e_G \notin \tau_2\text{int}(G, E)$. Similarly, τ_2 is said to be soft strong T_1 space with respect to τ_1 if for each pair of distinct points $e_G, e_H \in X_A$ there exist a τ_2 soft open set (F, E) and a τ_1 soft open set (G, E) such that $e_G \in \tau_1\text{int}(F, E)$ and $e_H \notin \tau_1\text{int}(F, E)$ and $e_H \in \tau_2\text{int}(G, E)$ and $e_G \notin \tau_2\text{int}(G, E)$. *Soft bi topological spaces* (X, τ_1, τ_2, E) is said to be *pair wise soft strong* T_1 space if τ_1 is *soft strong* T_1 *space* with respect to τ_2 and τ_2 is soft strong T_1 spaces with respect to τ_1.

3) τ_1 is said to be soft strong T_2 space with respect to τ_2, if for each pair of distinct points $e_G, e_H \in X_A$ there happens a τ_1 softopen set (F, E) and a τ_2 soft open set (G, E) such that $e_G \in \tau_2\text{int}(F, E)$ and $e_H \in \tau_1\text{int}(G, E)$ and $(F, E) \cap (G, E) = \phi$. Similarly, τ_2 is aid to be *soft strong* T_2 space with respect to τ_1 if for each pair of distinct points $e_G, e_G \in X_A$there happens a τ_2 *soft open* set (F, E) and a τ_1 *soft open* set(G, E) such that $e_G \in \tau_1\text{int}(F, E)$ and $e_H \in \tau_2\text{int}(G, E)$ and $(F, E) \cap (G, E) = \phi$. The soft bi topological space (X, τ_1, τ_2, E) is said to be pairwise soft strong T_2 space if τ_1 is soft strong T_2 space with respect to τ_2 and τ_2 is soft strong T_2 space with respect to τ_1.

Proposition 11. Let (X, τ_1, τ_2, E) be a *soft bi topological* space over X . If (X, τ_1, E) and (X, τ_2, E) is a *Soft strong* T_0space. Then (X, τ_1, τ_2, E) is a *pair wise soft strong* T_0 space.

Proof. Let$e_G, e_H \in X$, $e_G \neq e_H$and suppose that $(X, \tau_1, E)$is a *soft strong* T_0 *space* with respect to (X, τ_2, E). Then, according to definition, there exists τ_1 *softopen* set $(F, E)$and τ_2 *soft open* set(G, E)such that $e_G \in \tau_2\text{int}(F, E)$ and $e_H \notin \tau_2\text{int}(F, E)$ or$e_H \in \tau_1\text{int}(G, E)$ and $e_G \notin \tau_1\text{int}(G, E)$. Similarly, let $e_G, e_H \in X$, $e_G \neq e_H$ and suppose that (X, τ_2, E) is a *soft strong* T_0 *space* with respect to (X, τ_1, E)then, according to definition there exists τ_2 *soft open* set (G, E) and τ_1 *soft open* set (F, E) such that $e_G \in \tau_2\text{int}(F, E)$ and $e_H \notin \tau_2\text{int}(F, E)$ or $e_H \in \tau_1\text{int}(G, E)$ and $e_G \notin \tau_1\text{int}(G, E)$. Hence (X, τ_1, τ_2, E) is a *pair wise soft strong* T_0 space.

Proposition 12. Let (X, τ_1, τ_2, E) be a *soft bi topological* space over X . If (X, τ_1, τ_2, E) is a *pair wise soft strong* T_0 space then $(X, \tau_1 v, \tau_2, E)$ is a *soft strong* T_0 space.

Proof. A *soft bi topological* space (X,τ_1,τ_2,E) is called *pair wise soft strong* T_0 space if τ_1 is a *soft strong* T_0 space with respect to τ_2 and τ_2 is *soft strong* T_0 space with respect toτ_1. If$e_G,e_H \in X$, $e_G \neq e_H$ then since $\tau_1$is *soft strong* T_0 space with respect to τ_2 so there exists τ_1 *soft open* set (F,E) and τ_2 *soft open* set (G,E) such that $e_G \in \tau_2\text{int}(F,E)$ and $e_H \notin \tau_2\text{int}(F,E)$ or$e_H \in \tau_1\text{int}(G,E)$ and $e_G \notin \tau_1\text{int}(G,E)$ and since $\tau_2$is *soft strong* T_0 space with respect to τ_1 so there existsτ_2 *soft open* set$(G,E)$and $\tau_1$soft open set(F,E) such that and $e_G \in \tau_2\text{int}(F,E)$ and $e_H \notin \tau_2\text{int}(F,E)$ or $e_H \in \tau_1\text{int}(G,E)$ and $e_G \notin \tau_1\text{int}(G,E)$. In either case $(F,E),(G,E) \in (X,\tau_1 v,\tau_2,E)$. Hence $(X,\tau_1 v,\tau_2,E)$ is a *soft strong* T_0 *space.*

Proposition 13. Let (X,τ_1,τ_2,E) be a *soft bi topological* space over X and Y be a non-empty subset of X. If (X,τ_1,τ_2,E) is *pair wise soft strong* T_0 space. Then $(Y,\tau_{1Y},\tau_{2Y},E)$ is *pair wise soft strong* T_0 space.

Proof. Let (X,τ_1,τ_2,E) be a *soft bi topological* space over X, $e_G,e_H \in X$ such that$e_G \neq e_H$. If (X,τ_1,τ_2,E) is *pair wise soft strong* T_0 *space*. Then there exist τ_1 *soft open* set (F,E) and τ_2 *Soft open* set(G,E) such that $e_G \in \tau_2\text{int}(F,E)$ and $e_H \notin \tau_2\text{int}(F,E)$ or $e_H \in \tau_1\text{int}(G,E)$ and$e_G \notin \tau_1\text{int}(G,E)$. Now, $e_G \in Y$ and $e_G \in \tau_2\text{int}(F,E)$. Hence where $e_G \in Y \cap \tau_2\text{int}((F,E) = (Y_F,E)$ where $(F,E) \in \tau_1$. Consider$e_H \notin \tau_2\text{int}(F,E)$ this means that $\alpha \in E$ for some $\alpha \in E. y \notin Y \cap \tau_2\text{int}(F,E) = (Y_E,E)$ There fore τ_{1Y} is *soft strong* T_0 *space* with respect τ_{2Y}. Similarly, can proved that τ_{2Y} is *soft strong* T_0 *space* with respect to τ_{1Y}, that is $e_H \in \tau_1\text{int}(G,E)$ and $e_G \notin \tau_1\text{int}(G,E)$ then $e_H \in (Y_{G,}E)$ and $e_G \notin (Y_{G,}E)$. Thus $(X,\tau_{1Y},\tau_{2Y},E)$ is *pair wise soft strong* T_0 *space.*

Proposition 14. Let (X,τ_1,τ_2,E) be a *soft bi topological* space over X . Then (X,τ_1,τ_2,E) is a *pair wise soft strong* T_1 space if and only if (X,τ_1,E) and (X,τ_2,E) are *soft strong* T_1 space.

Proof. Suppose (X,τ_1,E) and (X,τ_2,E) are *soft strong* T_1 spaces. Let $e_G,e_H \in X, e_G \neq e_H$ then

1) $\tau_1$is a *soft strong* T_1 space with respect to τ_2. So there exists τ_1 *soft open* set (F,E) and τ_2 *soft open* set (G,E) Such that $e_G \in \tau_2\text{int}(F,E)$ and $e_H \notin \tau_2\text{int}(F,E)$ and $e_H \in \tau_1\text{int}(G,E)$ and $e_G \notin \tau_1\text{int}(G,E)$.

2) $\tau_2$is a *soft strong* T_1 space with respect to τ_1So there exists $\tau_2$softopen set (G,E) and τ_1 *soft open* set (F,E) such that $e_G \in \tau_1\text{int}(G,E)$ and $e_H \notin \tau_1\text{int}(G,E)$ and $e_H \in \tau_2\text{int}(F,E)$ and $e_G \notin \tau_2\text{int}(F,E)$. In either case we obtained the requirement and so (X,τ_1,τ_2,E) is a *pair wise soft strong* T_1 space. Conversely, we suppose that (X,τ_1,τ_2,E) is a *pair wise soft strong* T_1 space. Then

1) There exists some *soft open set* $(F,E) \in \tau_1$ with respect to *soft open* set $(G,E) \in \tau_2$such $e_G \in \tau_2\text{int}(F,E)$ and $e_H \notin \tau_2\text{int}(F,E)$ and $e_H \in \tau_1\text{int}(G,E)$ and $e_G \notin \tau_1\text{int}(G,E)$.

2) There exists some *soft open* set$(G,E) \in \tau_2$with respect to *soft open* set $(F,E) \in \tau_1$such that $e_G \in \tau_1\text{int}(G,E)$ and $e_H \notin \tau_1\text{int}(G,E)$ and $e_H \in \tau_2\text{int}(F,E)$ and $e_G \notin \tau_2\text{int}(F,E)$ Thus(X,τ_1,E) and (X,τ_2,E) are *soft strong* T_1 Space.

Proposition 15. Let (X, τ_1, τ_2, E) be a *soft bi topological* space over X. If (X, τ_1, τ_2, E) is *pair wise soft strong* T_1 space then $(X, \tau_1 v\, \tau_2, E)$ is also *a soft strong* T_1 space.

Proof. Let$e_G, e_H \in X$, $e_G \neq e_H$. Then exists *soft open* set$(F, E) \in \tau_1$with respect to soft open $(G, E) \in \tau_2$ such that $e_G \in \tau_2\text{int}(F, E)$ and $e_H \notin \tau_2\text{int}(F, E)$ and $e_H \in \tau_1\text{int}(G, E)$ and $e_G \notin \tau_1\text{int}(G, E)$. Similarly, There exists *soft open* set$(G, E) \in \tau_2$ with respect to τ_1 *soft open* set$(F, E) \in \tau_1$ such that $e_G \in \tau_1\text{int}(G, E)$ and $e_H \notin \tau_1\text{int}(G, E)$ and $e_H \in \tau_2\text{int}(F, E)$ and $e_G \notin \tau_2\text{int}(F, E)$ so $(F, E), (G, E) \in \tau_1 v \tau_2$ and thus $(X, \tau_1 v\, \tau_2, E)$ is *soft strong* T_1 space.

Proposition 16. Let (X, τ_1, τ_2, E) be a *soft bi topological* space over X and Y be non-empty sub set of X . If(X, τ_1, τ_2, E) is *pair wise soft strong* T_1 space then $(X, \tau_{1Y}, \tau_{2Y}, E)$ is *pair wise soft strong* T_1 space.

Proof. Let (X, τ_1, τ_2, E) be a *soft bi topological space over* X , $e_G, e_H \in Y, e_G \neq e_H$, If (X, τ_1, τ_2, E) is *pair wise soft strong* T_1 then there exists τ_1 *soft open set*(F, E) andτ_2*soft open* set (G, E) such that $e_G \in \tau_2\text{int}(F, E)$ and $e_H \notin \tau_2\text{int}(F, E)ande_H \in \tau_1\text{int}(G, E)$ and $e_G \notin \tau_1\text{int}(G, E)$. Now$e_G \in Y$and $e_G \in \tau_2\text{int}(F, E)$. Hence $e_G \in Y \cap \tau_2\text{int}(F, E) = (Y_F, E)$ Then $e_H \notin Y \cap F(\alpha)$ for some $\alpha \in E$. This means that $\alpha \in E$ then $e_H \notin Y \cap F(\alpha)$ for some $\alpha \in E$. Therefore, $e_H \notin Y \cap \tau_2\text{int}(F, E) = (Y_F, E)$. Now $e_H \in Y$ and $e_H \in (G, E)$. Hence $e_H \in Y \cap \tau_1\text{int}(G, E) = (Y_G, E)$ where $(G, E) \in \tau_2$. Consider $e_G \notin (G, E)$ this means that $\alpha \in E$ then $e_G \notin YG(\alpha)$ for some $\alpha \in E$. Therefore, $e_G \notin Y \cap \tau_1\text{int}(G, E) = (Y_G, E)$ hence τ_{1Y} is *soft strong* T_1 space with respect (Y_G, E) to τ_{2Y}. Similarly, it can be provide that τ_{2Y} is *soft strong* T_1 space with respect to τ_{1Y}, that is $e_H \in \tau_1\text{int}(G, E)$ and $e_G \notin \tau_1\text{int}(G, E)$. Then $e_H \in (Y_G, E)$ and $e_G \notin (Y_G, E)$ thus $(X, \tau_{1Y}, \tau_{2Y}, E)$ is *pair wise soft strong* T_1 space.

Proposition 17. Every *pair wise soft strong* T_1 space is *pair wise soft strong* T_0 space.

Proof. Let (X, τ_1, τ_2, E) be a *soft bi topological* space over X and $e_G, e_H \in X$ such that $e_G \neq e_H$. If (X, τ_1, τ_2, E) is *pair wise soft strong* T_1 space, that is (X, τ_1, τ_2, E) is *pair wise soft strong* T_1 space with respect toτ_2 and τ_2 is *soft strong* T_1 space with respect to τ_1 . If τ_1 is *soft strong* T_1 space with respect to τ_2 then there exists a τ_1 *soft open* set (F, E) and a τ_2 *soft open* set(G, E) such that $e_G \in \tau_2\text{int}(F, E)$ and $e_H \notin \tau_2\text{int}(F, E)$ and $e_H \in \tau_1\text{int}(G, E)$ and $e_G \notin \tau_1\text{int}(G, E)$ Obviously $e_G \in \tau_2\text{int}(F, E)$ and $e_H \notin \tau_2\text{int}(F, E)$ or $e_H \in \tau_1\text{int}(G, E)$ and $e_G \notin \tau_1\text{int}(G, E)$. Therefore τ_1 is *soft strong* T_0 space with respect to τ_2. Similarly if τ_2 is a *soft strong* T_1 space with respect to τ_1 then their exists τ_2 *soft open* set (F, E) and τ_1*soft open* set(G, E) such that such that $e_G \in \tau_1\text{int}(F, E)$ and $e_H \notin \tau_1\text{int}(F, E)$ and $e_H \in \tau_2\text{int}(G, E)$ and $e_G \notin \tau_2\text{int}(G, E)$. Obviously $e_G \in \tau_1\text{int}(F, E)$ and $e_H \notin \tau_1\text{int}(F, E)$ or $e_H \in \tau_2\text{int}(G, E)$ and $e_G \notin \tau_2\text{int}(G, E)$. Therefore τ_2 soft strong T_0 space with respect to τ_1. Thus (X, τ_1, τ_2, E) is a *pair wise soft strong* T_0.

Proposition 18. Let (X, τ_1, τ_2, E) *be soft bi topological* space over X. If (X, τ_1, τ_2, E) is a pair wise soft strong T_2 space over X then $(X, \tau_{1e}, \tau_{2e}, E)$ is a *pair wise soft strong* T_2 space for each $e \in E$

Proof. Suppose that (X, τ_1, τ_2, E) is a *pair wise soft strong* T_2 space over X. For any$e \in E$

$$\tau_{1e} = \{F(e), (F,E) \in \tau_1$$
$$\tau_{2e} = \{G(e), (G,E) \in \tau_2$$

Let$e_G, e_H \in X$ such that $e_G \neq e_H$

Case 1) then there *exists open set* $(F,E) \in \tau_1$ and *soft open* set $(G,E) \in \tau_2$ such that $e_G \in \tau_1\text{int}(F,E), e_H \in \tau_2\text{int}(G,E)$ and $(F,E) \cap (G,E) = \emptyset$. This implies that $e_G \in F(e) \in \tau_{1e}, e_H \in G(e) \in \tau_{2e}$ for each $\in E$ and $F(e) \cap G(e) = \emptyset$. Similarly,

Case 2) then there *exists open set* $(F,E) \in \tau_2$ and *soft open* set $(G,E) \in \tau_1$ such that $e_G \in \tau_2\text{int}(F,E), e_H \in \tau_1\text{int}(G,E)and(F,E) \cap (G,E) = \emptyset$. This implies that $e_G \in F(e) \in \tau_{2e}, e_H \in G(e) \in \tau_{1e}$ for each $\in E$and$F(e) \cap G(e) = \emptyset$. This implies that $e_G \in F(e) \in \tau_{2e}, e_H \in G(e) \in \tau_{1e}$ for each $\in E$ and $F(e) \cap G(e) = \emptyset$ thus $(X, \tau_{1e}, \tau_{2e}, E)$ is *a pair wise soft strong* T_2 space for each$e \in E$.

Proposition 19. Let (X, τ_1, τ_2, E) be a *soft bi topological* space over$X$. If (X, τ_1, τ_2, E) is p*air wise soft strong* T_2space. Then$(X, \tau_1 v \tau_2, E)$ also a *soft strong* T_2 space.

Proof. Let $e_G, e_H \in X$ such that$e_G \neq e_H$. Since (X, τ_1, τ_2, E) be a soft *bi topological* space over X. (X, τ_1, τ_2, E) is a *pair wise soft strong* T_2 space if τ_1 is *soft strong* T_2 space with respect τ_2 and τ_2 is *soft strong* T_2 space with respect to τ_1. If τ_1 be *soft strong* T_2 space with respect to τ_2, then there exists a τ_1*soft open* set (F,E) and τ_2 *soft open* set (G,E) such that $e_G \in \tau_2\text{int}(F,E), e_H \in \tau_1\text{int}(G,E)$ and $(F,E) \cap (G,E) = \emptyset$. Obviously $e_G \in \tau_2\text{int}(F,E)$ and $e_H \notin \tau_2\text{int}((F,E)$ and $e_H \in \tau_1\text{int}(G,E)$ and $e_G \notin \tau_1\text{int}(G,E)$. Therefore τ_1 is *soft strong* T_1 space with respect to τ_2.

Similarly, τ_2 is soft strong T_2 space with respect to τ_1 then there exists a τ_2 *soft open* set (F,E) and τ_1 s*oft open* set (G,E) such that

$$e_G \in \tau_2\text{int}(F,E), e_H \in \tau_1\text{int}(G,E) \; and \; (F,E) \cap (G,E) = \emptyset.$$

Obviously $e_G \in \tau_2\text{int}(F,E)$ and $e_H \notin \tau_2\text{int}(F,E)$, $e_H \in \tau_1\text{int}(G,E)$ and $e_G \notin \tau_1\text{int}(G,E)$. Therefore τ_2 soft strong T_1 space with respect to τ_1. Thus in either case $(F,E), (G,E) \in \tau_1 v \tau_2$. Hence $(X, \tau_1 v \tau_2, E)$ is *soft strong* T_2 space over X.

Proposition 20. Let (X, τ_1, τ_2, E) be a *soft bi topological* space over X and Y be a non-empty sub set of X. if (X, τ_1, τ_2, E)*is pair wise soft strong* T_2 space. Then$(Y, \tau_{1Y}, \tau_{2Y}, E)$ is *pair wise soft strong* T_2 space $e_G, e_H \in X$ and $e_G \neq e_H$.

Proof. Let (X, τ_1, τ_2, E) be *soft bi topological* space over$X$ $e_G, e_H \in X$ such that $e_G \neq e_H$ is *pair wise soft* strong T_2 space then there exist a τ_1 *soft open* set (F,E) and a τ_2 *soft open*(G,E) such that $e_G \in \tau_2\text{int}(F,E), e_H \in \tau_1\text{int}(G,E)$, $(F,E) \cap (G,E) = \emptyset$ so for each $\alpha \in E e_G \in F(\alpha), e_H \in G(\alpha)$ and $F(\alpha) \cap (G(\alpha) = \emptyset$. This implies that $e_G \in Y \cap F(\alpha), e_H \in Y \cap G(\alpha)$ and $F \in (\alpha) \cap G(\alpha) = \emptyset$. Hence $(Y_F, E) \cap (Y_G, E) = \emptyset$. Where (Y_F, E) is *soft open* set in τ_{1Y}, and (Y_G, E) is *soft open* set in τ_{2Y}. Therefore, τ_{1Y} is *soft strong* T_2 space with respect to τ_{2Y}. Similarly, it can be proved that $\tau_{2Y}$is soft strong T_2 space with respect to τ_{1Y}. Thus $(Y, \tau_{1Y}, \tau_{2Y}, E)$ is *pair wise soft strong* T_2 space.

Proposition 21. Every *pair wise soft strong* T_2 space is *pair wise soft strong* T_1 space.

Proof. Let (X, τ_1, τ_2, E) be a *soft bi topological* space over X and $e_G, e_H \in X$ such that $e_G \neq e_H$. If (X, τ_1, τ_2, E) is *pair wise soft strong* T_2 space. That is (X, τ_1, τ_2, E) is *pair wise soft strong* T_2 space. If τ_1 is *soft strong* T_2 space with respect τ_2 then there exists a τ_1*soft open* set (F, E) and τ_2*soft open* set(G, E)such that $e_G \in \tau_2\text{int}(F, E), e_H \in \tau_1\text{int}(G, E)$ and $(F, E) \cap (G, E) = \emptyset$. Obviously, $e_G \in \tau_2\text{int}(F, E)$ and $e_H \notin \tau_2\text{int}(F, E)$ and $e_H \in \tau_1\text{int}(G, E)$ and $e_G \notin \tau_1\text{int}(G, E)$. Therefore τ_1 is *soft strong* T_1 space with respect to τ_2. Similarly, if τ_2 is *soft strong* T_2 space with respect to τ_1 then there exists a τ_2*soft open* set $(F, E)$and τ_1*soft open* set (G, E)such that$e_G \in \tau_1\text{int}(F, E), e_H \in \tau_2\text{int}(G, E)$ and $(F, E) \cap (G, E) = \emptyset$. Obviously $e_G \in \tau_1\text{int}(F, E)$ and $e_H \notin \tau_1\text{int}(F, E)$ and $e_H \in \tau_2\text{int}(G, E)$ and $e_G \notin \tau_2\text{int}(G, E)$. Therefore τ_2 is *soft strong* T_1 space with respect to τ_1 thus (X, τ_1, τ_2, E) is *pair wise soft strong* T_1 space.

5. Conclusions

Topology is the most important and major area of mathematics and it can make a marriage between other scientific area and mathematical structures beautifully. Recently, many researchers have studied the soft set theory which is initiated by Molodtsov [4] and safely applied to many problems which contain uncertainties in our social life. Shabir and Naz in [23] introduced and deeply studied the concept of soft topological spaces. They also studied topological structures and exhibited their several properties with respect to ordinary points.

In the present work, we have continued to study the properties of soft separation axioms in soft bi topological spaces with respect to soft points as well as ordinary points of a soft topological space. We defined soft strong T_0, T_1, T_2 and spaces with respect to soft points and studied their behaviors in soft bi topological spaces. We also extended these axioms to different results. These soft separation axioms would be useful for the growth of the theory of soft topology to solve complex problems, comprising doubts in economics, engineering, medical etc. We also beautifully discussed some soft transmissible properties with respect to ordinary as well as soft points. We hope that these results in this paper will help the researchers for strengthening the toolbox of soft topology. In the next study, we extend the concept of soft semi open, α- open, Pre-open and b^{**}open soft sets in soft bi topological spaces with respect to ordinary as well as soft points.

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CHARACTERIZATIONS OF INTUITIONISTIC FUZZY SUBSEMIRINGS OF SEMIRINGS AND THEIR HOMOMORPHISMS BY NORMS

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Abstaract − In this paper, we introduce the notion of intuitionistic fuzzy subsemirings, level subsets of of intuitionistic fuzzy subsemirings, intersection and direct sum of intuitionistic fuzzy subsemirings under norms and investigate many properties of them. We also made an attempt to study the characterizations of them under homomorphism and anti-homomorphism.

Keywords − *Ring theory, norms, fuzzy set theory, intuitionistic fuzzy subsemirings, homomorphisms, anti-homomorphisms, direct sum.*

1 Introduction

In abstract algebra, a semiring is an algebraic structure similar to a ring, but without the requirement that each element must have an additive inverse. After the introduction of fuzzy sets by Zadeh [26], a number of generalizations of this fundamental concept have come up. Algebraic structures play a vital role in Mathematics and numerous applications of these structures are seen in many disciplines such as computersciences, information sciences, theoretical physics, control engineering and so on. This inspires researchers to study and carry out research in various concepts of abstract algebra in fuzzy setting. There are natural ways to fuzzify various algebraic structures and it has been done successfully by many mathematicians. For instance, Rosenfeld [23] is the father of fuzzy abstract algebra and the reader may consult the papers [12] or [13] about fuzzy semigroups; [11], [10], [15], [24] or [27] about fuzzy ideals and fuzzy rings; [14] or [17] about fuzzy modules; [16] about fuzyy vector spaces; [7] about fuzzy coalgebras over a field; [25] about Lie algebras, and so on. In 1993, Ahsan et al. [1] introduced the notion of fuzzy semirings. In1994, Dutta and Biswas [8] characterized fuzzy prime ideals of a semiring. Recently, many results of semiring theory are investigated by many researchers in fuzzy context. The notion of intuitionistic fuzzy sets introduced by Atanassov [3] (also see [4], [5]) is one among them. Biswas [6] applied the concept of intuitionistic fuzzy sets to the theory of groups and studied intuitionistic fuzzy subgroups of a group. Norms originated from the studies of probabilistic metric spaces in which triangular inequalities were extended using the theory of norms. Later, Hohle [9], Alsina et al. [2] introduced the norms into fuzzy set theory and suggested that norms be used for the intersection of fuzzy sets. The author by using norms, investigated some properties of fuzzy submodules, fuzzy subrings, fuzzy ideals of subtraction semigroups, intuitionistic fuzzy subrings and ideals of a ring, fuzzy Lie algebra (See [18, 19, 20, 21, 22]).

In this paper, we introduce the notions of intuitionistic fuzzy subsemirings of a semiring with respect to norms and establish necessary and sufficient conditions for them. We also investigate the algebraic nature of such type of them under intersection, direct some, homomorphism and anti-homomorphism.

2 Preliminary

Definition 2.1. A semiring is a set R equipped with two binary operations " $+$ " and "." called addition and multiplication, such that:

(1) $(R, +)$ is a commutative monoid with identity element 0:
(a) $(a + b) + c = a + (b + c)$,
(b) $0 + a = a + 0 = a$,
(c) $a + b = b + a$.
(2) $(R, .)$ is a monoid with identity element 1:
(a) $(a.b).c = a.(b.c)$,
(b) $1.a = a.1 = a$.
(3) Multiplication left and right distributes over addition:
(a) $a.(b + c) = (a.b) + (a.c)$,
(b) $(a + b).c = (a.c) + (b.c)$.
(4) Multiplication by 0 annihilates R: $0.a = a.0 = 0$.

This last axiom is omitted from the definition of a ring: it follows from the other ring axioms. Here it does not, and it is necessary to state it in the definition. The difference between rings and semirings, then, is that addition yields only a commutative monoid, not necessarily a commutative group. Specifically, elements in semirings do not necessarily have an inverse for the addition. The symbol . is usually omitted from the notation; that is, $a.b$ is just written ab. Similarly, an order of operations is accepted, according to which . is applied before +; that is, $a+bc$ is $a+(bc)$.

A commutative semiring is one whose multiplication is commutative. An idempotent semiring is one whose addition is idempotent: $a + a = a$, that is, $(R, +, 0)$ is a join-semilattice with zero.

Example 2.2. (1) By definition, any ring is also a semiring. A motivating example of a semiring is the set of natural numbers $\mathbb{N}$ (including zero) under ordinary addition and multiplication. Likewise, the non-negative rational numbers and the non-negative real numbers form semirings. All these semirings are commutative.

(2) The set of all ideals of a given ring form a semiring under addition and multiplication of ideals.

(3) Any unital quantale is an idempotent semiring.

(4) Any bounded, distributive lattice is a commutative, idempotent semiring under join and meet.

Definition 2.3. Let R be a semiring. A nonempty subset S of R is a subsemiring of R if and only if $x + y \in S$ and $xy \in S$ for all $x, y \in S$.

Definition 2.4. A t-norm T is a function $T : [0,1] \times [0,1] \to [0,1]$ having the following four properties: For all $x, y, z \in [0,1]$;

(T1) $T(x,1) = x$ (neutral element),
(T2) $T(x,y) \leq T(x,z)$ if $y \leq z$ (monotonicity),
(T3) $T(x,y) = T(y,x)$ (commutativity),
(T4) $T(x,T(y,z)) = T(T(x,y),z)$ (associativity),

It is clear that if $x_1 \geq x_2$ and $y_1 \geq y_2$, then $T(x_1,y_1) \geq T(x_2,y_2)$.

Example 2.5. (1) Standard intersection T-norm $T_m(x,y) = \min\{x,y\}$.

(2) Bounded sum T-norm $T_b(x,y) = \max\{0, x+y-1\}$.

(3) algebraic product T-norm $T_p(x,y) = xy$.

(4) Drastic T-norm

$$T_D(x,y) = \begin{cases} y & \text{if } x = 1 \\ x & \text{if } y = 1 \\ 0 & \text{otherwise.} \end{cases}$$

(5) Nilpotent minimum T-norm

$$T_{nM}(x,y) = \begin{cases} \min\{x,y\} & \text{if } x + y > 1 \\ 0 & \text{otherwise.} \end{cases}$$

(6) Hamacher product T-norm

$$T_{H_0}(x,y) = \begin{cases} 0 & \text{if } x = y = 0 \\ \frac{xy}{x+y-xy} & \text{otherwise.} \end{cases}$$

The drastic t-norm is the pointwise smallest t-norm and the minimum is the pointwise largest t-norm: $T_D(x,y) \leq T(x,y) \leq T_{\min}(x,y)$ for all $x, y \in [0,1]$.

Definition 2.6. A t-conorm C is a function $C : [0,1] \times [0,1] \to [0,1]$ having the following four properties: For all $x, y, z \in [0,1]$;

(C1) $C(x,0) = x$,
(C2) $C(x,y) \leq C(x,z)$ if $y \leq z$,
(C3) $C(x,y) = C(y,x)$,
(C4) $C(x, C(y,z)) = C(C(x,y), z)$,

Example 2.7. (1) Standard union t-conorm $C_m(x,y) = \max\{x,y\}$.
(2) Bounded sum t-conorm $C_b(x,y) = \min\{1, x+y\}$.
(3) Algebraic sum t-conorm $C_p(x,y) = x + y - xy$.
(4) Drastic T-conorm

$$C_D(x,y) = \begin{cases} y & \text{if } x = 0 \\ x & \text{if } y = 0 \\ 1 & \text{otherwise,} \end{cases}$$

dual to the drastic T-norm.
(5) Nilpotent maximum T-conorm , dual to the nilpotent minimum T-norm:

$$C_{nM}(x,y) = \begin{cases} \max\{x,y\} & \text{if } x + y < 1 \\ 1 & \text{otherwise.} \end{cases}$$

(6) Einstein sum (compare the velocity-addition formula under special relativity) $C_{H_2}(x,y) = \dfrac{x+y}{1+xy}$ is a dual to one of the Hamacher t-norms. Note that all t-conorms are bounded by the maximum and the drastic t-conorm: $C_{\max}(x,y) \leq C(x,y) \leq C_D(x,y)$ for any t-conorm C and all $x, y \in [0,1]$.

Recall that t-norm T(t-conorm C) is idempotent if for all $x \in [0,1]$, $T(x,x) = x(C(x,x) = x)$.

Definition 2.8. For sets X, Y and Z, $f = (f_1, f_2) : X \to Y \times Z$ is called a complex mapping if $f_1 : X \to Y$ and $f_2 : X \to Z$ are mappings.

Definition 2.9. Let Let φ be a function from set X into set Y such that $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ be two intuitionistic fuzzy sets in X and Y respectively.
For all $x \in X, y \in Y$, we define

$$\varphi(A)(y) = (\varphi(\mu_A)(y), \varphi(\nu_A)(y)) =$$

$$= \begin{cases} (\sup\{\mu_A(x) \mid x \in R, \varphi(x) = y\}, \inf\{\nu_A(x) \mid x \in R, \varphi(x) = y\}), & \text{if } \varphi^{-1}(y) \neq \emptyset \\ (0,1), & \text{if } \varphi^{-1}(y) = \emptyset \end{cases}$$

Also $\varphi^{-1}(B)(x) = (\varphi^{-1}(\mu_B)(x), \varphi^{-1}(\nu_B)(x)) = (\mu_B(\varphi(x)), \nu_B(\varphi(x)))$.

Lemma 2.10. Let T be a t-norm. Then

$$T(T(x,y), T(w,z)) = T(T(x,w), T(y,z)),$$

for all $x, y, w, z \in [0,1]$.

Lemma 2.11. Let C be a t-conorm. Then

$$C(C(x,y),C(w,z)) = C(C(x,w),C(y,z))$$

for all $x, y, w, z \in [0,1]$

Definition 2.12. Let X be a nonempty set. A complex mapping $A = (\mu_A, \nu_A) : X \to [0,1] \times [0,1]$ is called an intuitionistic fuzzy set (in short, IFS) in X if $\mu_A + \nu_A \leq 1$ where the mappings $\mu_A : X \to [0,1]$ and $\nu_A : X \to [0,1]$ denote the degree of membership (namely $\mu_A(x)$) and the degree of non-membership (namely $\nu_A(x)$) for each $x \in X$ to A, respectively. In particular $0_\sim$ and $1_\sim$ denote the intuitionistic fuzzy empty set and intuitionistic fuzzy whole set in X defined by $0_\sim(x) = (0,1)$ and $1_\sim(x) = (1,0)$, respectively.

We will denote the set of all $IFSs$ in X as $IFS(X)$.

Definition 2.13. Let X be a nonempty set and let $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ be $IFSs$ in X. Then

(1) $A \subset B$ iff $\mu_A \leq \mu_B$ and $\nu_A \geq \nu_B$.
(2) $A = B$ iff $A \subset B$ and $B \subset A$.

Definition 2.14. If A is intuitionistic fuzzy subset of R, then the sets $\{< x, \mu_A(x) > | x \in R\}$ and $\{< x, \nu_A(x) > | x \in R\}$, are called fuzzy subset and anti-fuzzy subset of R with respect to intuitionistic fuzzy set A. For $\alpha, \beta \in [0,1]$, we define the following sets

(1) $U_1(A, \alpha) = \{x \in R \mid \mu_A(x) \geq \alpha\}$,
(2) $U_2(A, \alpha) = \{x \in R \mid \nu_A(x) \geq \alpha\}$,
(3) $L_1(A, \beta) = \{x \in R \mid \mu_A(x) \leq \beta\}$,
(4) $L_2(A, \beta) = \{x \in R \mid \nu_A(x) \leq \beta\}$
(5) $C_{\alpha,\beta} = \{x \in R \mid \mu_A(x) \geq \alpha, \nu_A(x) \leq \beta\}$.

The sets $U_1(A, \alpha)$ and $L_1(A, \beta)$ are respectively called the upper α-level cut and lower β-level cut of the fuzzy subset of R w.r.t. $IFSA$ and the sets $U_2(A, \alpha)$ and $L_2(A, \beta)$ are respectively called the upper α-level cut and lower β-level cut of the anti-fuzzy subset of R w.r.t. $IFSA$.

Definition 2.15. Let R and S be any two semirings and $f : R \to S$ be a function:

(1) f is called a homomorphism if $f(x + y) = f(x) + f(y)$ and $f(xy) = f(x)f(y)$ for all $x, y \in R$.

(2) f is called an anti-homomorphism if $f(x+y) = f(x)+f(y)$ and $f(xy) = f(y)f(x)$ for all $x, y \in R$.

3 Level Subsets of Intuitionistic Fuzzy Subsemiring af a Semiring with Respect to Norms

Definition 3.1. Let R be a semiring. An $A = (\mu_A, \nu_A)$ is said to be intuitionistic fuzzy subsemiring with respect to norms(a t-norm T and a t-conorm C) (in short, $IFSN(R)$) of R if

(1) $\mu_A(x + y) \geq T(\mu_A(x), \mu_A(y))$
(2) $\mu_A(xy) \geq T(\mu_A(x), \mu_A(y))$
(3) $\nu_A(x + y) \leq C(\nu_A(x), \nu_A(y))$
(4) $\nu_A(xy) \leq C(\nu_A(x), \nu_A(y))$,

for all $x, y \in R$.

Example 3.2. Let $R = (\mathbb{Z}, +, .)$ be a semiring of integer. For all $x \in R$ we define a fuzzy subset μ_A and ν_A of R as

$$\mu_A(x) = \begin{cases} 0.75 & \text{if } x \in \{0, \pm 2, \pm 4, ...\} \\ 0.60 & \text{if } x \in \{\pm 1, \pm 3, ...\} \end{cases}$$

$$\nu_A(x) = \begin{cases} 0.35 & \text{if } x \in \{0, \pm 2, \pm 4, ...\} \\ 0.55 & \text{if } x \in \{\pm 1, \pm 3, ...\} \end{cases}$$

Let $T(x, y) = T_p(x, y) = xy$ and $C(x, y) = C_p(x, y) = x + y - xy$ for all $x, y \in R$, then $A = (\mu_A, \nu_A) \in IFSN(R)$.

Proposition 3.3. Let $A \in IFSN(R)$ and T, C be idempotent. If $\alpha, \beta \in [0, 1]$, then $C_{\alpha,\beta}$ is a subsemiring of R.

Proof. If $x, y \in C_{\alpha,\beta}$, then $\mu_A(x), \mu_A(y) \geq \alpha$ and $\nu_A(x), \nu_A(y) \leq \beta$. Now

(1) $\mu_A(x + y) \geq T(\mu_A(x), \mu_A(y)) \geq T(\alpha, \alpha) = \alpha$.
(2) $\mu_A(xy) \geq T(\mu_A(x), \mu_A(y)) \geq T(\alpha, \alpha) = \alpha$.
(3) $\nu_A(x + y) \leq C(\nu_A(x), \nu_A(y)) \leq C(\beta, \beta) = \beta$.
(4) $\nu_A(xy) \leq C(\nu_A(x), \nu_A(y)) \leq C(\beta, \beta) = \beta$.

Thus $x + y, xy \in C_{\alpha,\beta}$ and therefore $C_{\alpha,\beta}$ is a subsemiring of R.

Proposition 3.4. Let R be a semiring and $A \in IFS(R)$. If T, C be idempotent and $C_{\alpha,\beta}$ be a subsemiring of R for all $\alpha, \beta \in [0, 1]$, then $A \in IFSN(R)$.

Proof. Let $x, y \in R$ and for C_{α_i,β_i} with $i = 1, 2$ we have $\mu_A(x) = \alpha_1$, $\mu_A(y) = \alpha_2$, $\nu_A(x) = \beta_1$ and $\nu_A(y) = \beta_2$ such that $\alpha_1, \alpha_2, \beta_1, \beta_2 \in [0, 1]$. Since $x, y \in C_{\alpha_i,\beta_i}$ and C_{α_i,β_i} is a subsemiring of R so $x + y, xy \in C_{\alpha_i,\beta_i}$. Now we prove that $A \in IFSN(R)$ in the following conditions.

(a) Let $\alpha_1 > \alpha_2$ and $\beta_1 < \beta_2$ such that $x, y \in C_{\alpha_1,\beta_1}$. Then

(1) $\mu_A(x + y) \geq \alpha_1 = T(\alpha_1, \alpha_1) \geq T(\alpha_1, \alpha_2) = T(\mu_A(x), \mu_A(y))$.

(2) $\mu_A(xy) \geq \alpha_1 = T(\alpha_1, \alpha_1) \geq T(\alpha_1, \alpha_2) = T(\mu_A(x), \mu_A(y))$.

(3) $\nu_A(x+y) \leq \beta_1 = C(\beta_1, \beta_1) \leq C(\beta_1, \beta_2) = C(\nu_A(x), \nu_A(y))$.

(4) $\nu_A(xy) \leq \beta_1 = C(\beta_1, \beta_1) \leq C(\beta_1, \beta_2) = C(\nu_A(x), \nu_A(y))$.

(b) Let $\alpha_1 < \alpha_2$ and $\beta_1 < \beta_2$ such that $x, y \in C_{\alpha_2,\beta_1}$. Then

(1) $\mu_A(x+y) \geq \alpha_2 = T(\alpha_2, \alpha_2) \geq T(\alpha_1, \alpha_2) = T(\mu_A(x), \mu_A(y))$.

(2) $\mu_A(xy) \geq \alpha_2 = T(\alpha_2, \alpha_2) \geq T(\alpha_1, \alpha_2) = T(\mu_A(x), \mu_A(y))$.

(3) $\nu_A(x+y) \leq \beta_1 = C(\beta_1, \beta_1) \leq C(\beta_1, \beta_2) = C(\nu_A(x), \nu_A(y))$.

(4) $\nu_A(xy) \leq \beta_1 = C(\beta_1, \beta_1) \leq C(\beta_1, \beta_2) = C(\nu_A(x), \nu_A(y))$.

(c) Let $\alpha_1 > \alpha_2$ and $\beta_1 > \beta_2$ such that $x, y \in C_{\alpha_1,\beta_2}$. Then

(1) $\mu_A(x+y) \geq \alpha_1 = T(\alpha_1, \alpha_1) \geq T(\alpha_1, \alpha_2) = T(\mu_A(x), \mu_A(y))$.

(2) $\mu_A(xy) \geq \alpha_1 = T(\alpha_1, \alpha_1) \geq T(\alpha_1, \alpha_2) = T(\mu_A(x), \mu_A(y))$.

(3) $\nu_A(x+y) \leq \beta_2 = C(\beta_2, \beta_2) \leq C(\beta_1, \beta_2) = C(\nu_A(x), \nu_A(y))$.

(4) $\nu_A(xy) \leq \beta_2 = C(\beta_2, \beta_2) \leq C(\beta_1, \beta_2) = C(\nu_A(x), \nu_A(y))$.

(d) Let $\alpha_1 < \alpha_2$ and $\beta_1 > \beta_2$ such that $x, y \in C_{\alpha_2,\beta_2}$. Then

(1) $\mu_A(x+y) \geq \alpha_2 = T(\alpha_2, \alpha_2) \geq T(\alpha_1, \alpha_2) = T(\mu_A(x), \mu_A(y))$.

(2) $\mu_A(xy) \geq \alpha_2 = T(\alpha_2, \alpha_2) \geq T(\alpha_1, \alpha_2) = T(\mu_A(x), \mu_A(y))$.

(3) $\nu_A(x+y) \leq \beta_2 = C(\beta_2, \beta_2) \leq C(\beta_1, \beta_2) = C(\nu_A(x), \nu_A(y))$.

(4) $\nu_A(xy) \leq \beta_2 = C(\beta_2, \beta_2) \leq C(\beta_1, \beta_2) = C(\nu_A(x), \nu_A(y))$.

Thus from (a) to (d) we get that $A \in IFSN(R)$.

Proposition 3.5. Let R be a semiring and $A \in IFS(R)$ defined by

$$\mu_A(x) = \begin{cases} 1 & \text{if } x \in H \\ 0 & \text{if } x \notin H \end{cases}$$

and

$$\nu_A(x) = \begin{cases} 0 & \text{if } x \in H \\ 1 & \text{if } x \notin H. \end{cases}$$

If H is a subsemiring of R and T, C be idempotent, then $A \in IFSN(R)$.

Proof. Let $x, y \in R$ and H is a subsemiring of R. Then

(a) If $x, y \in H$, then $x + y, xy \in H$ and we have:

(1) $\mu_A(x + y) = 1 \geq 1 = T(1, 1) = T(\mu_A(x), \mu_A(y))$.

(2) $\mu_A(xy) = 1 \geq 1 = T(1, 1) = T(\mu_A(x), \mu_A(y))$.

(3) $\nu_A(x + y) = 0 \leq= 0 = C(0, 0) = C(\nu_A(x), \nu_A(y))$.

(4) $\nu_A(xy) = 0 \leq 0 = C(0, 0) = C(\nu_A(x), \nu_A(y))$.

(b) If $x \in H$ and $y \notin H$, then $x + y, xy \notin H$ and then:

(1) $\mu_A(x + y) = 0 \geq 0 = T(1, 0) = T(\mu_A(x), \mu_A(y))$.

(2) $\mu_A(xy) = 0 \geq 0 = T(1, 0) = T(\mu_A(x), \mu_A(y))$.

(3) $\nu_A(x + y) = 1 \leq 1 = C(0, 1) = C(\nu_A(x), \nu_A(y))$.

(4) $\nu_A(xy) = 1 \leq 1 = C(0, 1) = C(\nu_A(x), \nu_A(y))$.

(c) If $x, y \notin H$, then $x + y, xy \notin H$ and so:

(1) $\mu_A(x + y) = 0 \geq 0 = T(0, 0) = T(\mu_A(x), \mu_A(y))$.

(2) $\mu_A(xy) = 0 \geq 0 = T(0, 0) = T(\mu_A(x), \mu_A(y))$.

(3) $\nu_A(x + y) = 1 \leq 1 = C(1, 1) = C(\nu_A(x), \nu_A(y))$.

(4) $\nu_A(xy) = 1 \leq 1 = C(1, 1) = C(\nu_A(x), \nu_A(y))$.

Now fram (a) to (c) we obtain that $A \in IFSN(R)$.

Definition 3.6. Let $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ be two intuitionistic fuzzy sets in semiring R. Define $A \cap B = (\mu_{A\cap B}, \nu_{A\cap B})$ as $\mu_{A\cap B}(x) = T(\mu_A(x), \mu_B(x))$ and $\nu_{A\cap B}(x) = C(\nu_A(x), \nu_B(y))$ for all $x \in R$.

Proposition 3.7. Let $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ be two intuitionistic fuzzy sets in semiring R. If $A, B \in IFSN(R)$, then $(A \cap B) \in IFSN(R)$.

Proof. Let $x, y \in R$. Then

$$
\begin{aligned}
(1)\ \mu_{A\cap B}(x + y) &= T(\mu_A(x + y), \mu_B(x + y)) \\
&\geq T(T(\mu_A(x), \mu_A(y)), T(\mu_B(x), \mu_B(y))) \\
&= T(T(\mu_A(x), \mu_B(x)), T(\mu_A(y), \mu_B(y))) \\
&= T(\mu_{A\cap B}(x), \mu_{A\cap B}(y))
\end{aligned}
$$

$$\begin{aligned}(2)\ \mu_{A\cap B}(xy) &= T(\mu_A(xy),\mu_B(xy))\\ &\geq T(T(\mu_A(x),\mu_A(y)),T(\mu_B(x),\mu_B(y)))\\ &= T(T(\mu_A(x),\mu_B(x)),T(\mu_A(y),\mu_B(y)))\\ &= T(\mu_{A\cap B}(x),\mu_{A\cap B}(y))\end{aligned}$$

$$\begin{aligned}(3)\ \nu_{A\cap B}(x+y) &= C(\nu_A(x+y),\nu_B(x+y))\\ &\leq T(T(\mu_A(x),\mu_A(y)),T(\mu_B(x),\mu_B(y)))\\ &= C(C(\nu_A(x),\nu_A(y)),C(\nu_B(x),\nu_B(y)))\\ &= C(\nu_{A\cap B}(x),\nu_{A\cap B}(y))\end{aligned}$$

$$\begin{aligned}(4)\ \nu_{A\cap B}(xy) &= C(\nu_A(xy),\nu_B(xy))\\ &\leq T(T(\mu_A(x),\mu_A(y)),T(\mu_B(x),\mu_B(y)))\\ &= C(C(\nu_A(x),\nu_A(y)),C(\nu_B(x),\nu_B(y)))\\ &= C(\nu_{A\cap B}(x),\nu_{A\cap B}(y))\end{aligned}$$

Therefore $(A\cap B)\in IFSN(R)$.

Corollary 3.8. Let $\{A_i=(\mu_{A_i},\nu_{A_i})\mid i=1,2,3,...,n\}\subseteq IFSN(R)$. Then so does $\cap_{A_i}=(\mu_{\cap A_i},\nu_{\cap A_i})$.

Proposition 3.9. Let $A\in IFSN(R)$ and T,C be idempotent.

(1) For all $\alpha\in[0,1]$, the μ-level α-cut $U(\mu_A,\alpha)=\{x\in R\mid \mu_A\geq\alpha\}$ is a subsemiring of R.

(2) For all $\beta\in[0,1]$, the ν-level β-cut $L(\nu_A,\beta)=\{x\in R\mid \nu_A\leq\beta\}$ is a subsemiring of R.

Proof. (1) Let $x,y\in U(\mu_A,\alpha)$. Since $A\in IFSN(R)$ so $\mu_A(x+y)\geq T(\mu_A(x),\mu_A(y))\geq T(\alpha,\alpha)=\alpha$ and $\mu_A(xy)\geq T(\mu_A(x),\mu_A(y))\geq T(\alpha,\alpha)=\alpha$. Thus $x+y,xy\in U(\mu_A,\alpha)$ and then $U(\mu_A,\alpha)$ is a subsemiring of R.

(2) Let $x,y\in L(\nu_A,\beta)$. As $A\in IFSN(R)$ then $\nu_A(x+y)\leq C(\nu_A(x),\nu_A(y))\leq C(\beta,\beta)=\beta$ and $\nu_A(xy)\leq C(\nu_A(x),\nu_A(y))\leq C(\beta,\beta)=\beta$. Therefore $x+y,xy\in L(\nu_A,\beta)$ and $L(\nu_A,\beta)$ is a subsemiring of R.

Definition 3.10. Let $A=(\mu_A,\nu_A)$ and $B=(\mu_B,\nu_B)$ be two intuitionistic fuzzy sets in R and S, respectively. The direct som of A and B, denoted by $A\oplus B=(\mu_A\oplus\mu_B,\nu_A\oplus\nu_B)$, is an intuitionistic fuzzy set in $R\oplus S$ such that for all x in R and y in S,$(\mu_A\oplus\mu_B)(x,y)=T(\mu_A(x),\mu_B(y))$ and $(\nu_A\oplus\nu_B)(x,y)=C(\nu_A(x),\nu_B(y))$

Proposition 3.11. If $A_i=(\mu_{A_i},\nu_{A_i})\in IFSN(R_i)$ for $i=1,2$, then $A_1\oplus A_2\in IFSN(R_1\oplus R_2)$.

Proof. Let $(x_1,y_1),(x_2,y_2)\in R_1\oplus R_2$. Then

$$
\begin{aligned}
(1)\ \ (\mu_{A_1} \oplus \mu_{A_2})((x_1,y_1)+(x_2,y_2)) &= (\mu_{A_1} \oplus \mu_{A_2})(x_1+x_2, y_1+y_2) \\
&= T(\mu_{A_1}(x_1+x_2), \mu_{A_2}(y_1+y_2)) \\
&\geq T(T(\mu_{A_1}(x_1), \mu_{A_1}(x_2)), T(\mu_{A_2}(y_1), \mu_{A_2}(y_2))) \\
&= T(T(\mu_{A_1}(x_1), \mu_{A_2}(y_1)), T(\mu_{A_1}(x_2), \mu_{A_2}(y_2))) \\
&= T((\mu_{A_1} \oplus \mu_{A_2})(x_1,y_1), (\mu_{A_1} \oplus \mu_{A_2})(x_2,y_2))
\end{aligned}
$$

$$
\begin{aligned}
(2)\ \ (\mu_{A_1} \oplus \mu_{A_2})((x_1,y_1)(x_2,y_2)) &= (\mu_{A_1} \oplus \mu_{A_2})(x_1x_2, y_1y_2) \\
&= T(\mu_{A_1}(x_1x_2), \mu_{A_2}(y_1y_2)) \\
&\geq T(T(\mu_{A_1}(x_1), \mu_{A_1}(x_2)), T(\mu_{A_2}(y_1), \mu_{A_2}(y_2))) \\
&= T(T(\mu_{A_1}(x_1), \mu_{A_2}(y_1)), T(\mu_{A_1}(x_2), \mu_{A_2}(y_2))) \\
&= T((\mu_{A_1} \oplus \mu_{A_2})(x_1,y_1), (\mu_{A_1} \oplus \mu_{A_2})(x_2,y_2))
\end{aligned}
$$

$$
\begin{aligned}
(3)\ \ (\nu_{A_1} \oplus \mu_{A_2})((x_1,y_1)+(x_2,y_2)) &= (\nu_{A_1} \oplus \nu_{A_2})(x_1+x_2, y_1+y_2) \\
&= C(\nu_{A_1}(x_1+x_2), \nu_{A_2}(y_1+y_2)) \\
&\leq C(C(\nu_{A_1}(x_1), \nu_{A_1}(x_2)), C(\nu_{A_2}(y_1), \nu_{A_2}(y_2))) \\
&= C(C(\nu_{A_1}(x_1), \nu_{A_2}(y_1)), C(\nu_{A_1}(x_2), \nu_{A_2}(y_2))) \\
&= C((\nu_{A_1} \oplus \nu_{A_2})(x_1,y_1), (\nu_{A_1} \oplus \nu_{A_2})(x_2,y_2))
\end{aligned}
$$

$$
\begin{aligned}
(4)\ \ (\nu_{A_1} \oplus \mu_{A_2})((x_1,y_1)(x_2,y_2)) &= (\nu_{A_1} \oplus \nu_{A_2})(x_1x_2, y_1y_2) \\
&= C(\nu_{A_1}(x_1x_2), \nu_{A_2}(y_1y_2)) \\
&\leq C(C(\nu_{A_1}(x_1), \nu_{A_1}(x_2)), C(\nu_{A_2}(y_1), \nu_{A_2}(y_2))) \\
&= C(C(\nu_{A_1}(x_1), \nu_{A_2}(y_1)), C(\nu_{A_1}(x_2), \nu_{A_2}(y_2))) \\
&= C((\nu_{A_1} \oplus \nu_{A_2})(x_1,y_1), (\nu_{A_1} \oplus \nu_{A_2})(x_2,y_2))
\end{aligned}
$$

Corollary 3.12. Let $A_i = (\mu_{A_i}, \nu_{A_i}) \in IFSN(R_i)$ for $i = 1, 2, ..., n$. Then

$$A_1 \oplus A_2 \oplus ... \oplus A_n \in IFSN(R_1 \oplus R_2 \oplus ... \oplus R_n).$$

4 Homomorphisms and Anti-Homomorphisms of Intuitionistic Fuzzy Subsemirings of Semirings Under Norms

Proposition 4.1. Let φ be an epihomomorphism from semiring R into semiring S. If $A = (\mu_A, \nu_A) \in IFSN(R)$, then $\varphi(A) = (\varphi(\mu_A), \varphi(\nu_A)) \in IFSN(S)$.

Proof. Let $y_1, y_2 \in S$. Then

$$
\begin{aligned}
(1)\quad & \varphi(\mu_A)(y_1+y_2) \\
= \ & \sup\{\mu_A(x_1+x_2) \mid x_1, x_2 \in R, \varphi(x_1)y_1, \varphi(x_2) = y_2\} \\
\geq \ & \sup\{T(\mu_A(x_1), \mu_A(x_2)) \mid x_1, x_2 \in R, \varphi(x_1) = y_1, \varphi(x_2) = y_2\} \\
= \ & T(\sup\{\mu_A(x_1) \mid x_1 \in R, \varphi(x_1) = y_1\}, \sup\{\mu_A(x_2) \mid x_2 \in R, \varphi(x_2))y_2\}) \\
= \ & T(\varphi(\mu_A)(y_1), \varphi(\mu_A)(y_2))
\end{aligned}
$$

$$
\begin{array}{rl}
(2) & \varphi(\mu_A)(y_1y_2) \\
= & \sup\{\mu_A(x_1x_2) \mid x_1, x_2 \in R, \varphi(x_1) = y_1, \varphi(x_2) = y_2\} \\
\geq & \sup\{T(\mu_A(x_1), \mu_A(x_2)) \mid x_1, x_2 \in R, \varphi(x_1) = y_1, \varphi(x_2) = y_2\} \\
= & T(\sup\{\mu_A(x_1) \mid x_1 \in R, \varphi(x_1) = y_1\}, \sup\{\mu_A(x_2) \mid x_2 \in R, \varphi(x_2) = y_2\}) \\
= & T(\varphi(\mu_A)(y_1), \varphi(\mu_A)(y_2))
\end{array}
$$

$$
\begin{array}{rl}
(3) & \varphi(\nu_A)(y_1 + y_2) \\
= & \inf\{\nu_A(x_1 + x_2) \mid x_1, x_2 \in R, \varphi(x_1) = y_1, \varphi(x_2) = y_2\} \\
\leq & \inf\{C(\nu_A(x_1), \nu_A(x_2)) \mid x_1, x_2 \in R, \varphi(x_1) = y_1, \varphi(x_2) = y_2\} \\
= & C(\inf\{\nu_A(x_1) \mid x_1 \in R, \varphi(x_1) = y_1\}, \inf\{\nu_A(x_2) \mid x_2 \in R, \varphi(x_2) = y_2\}) \\
= & C(\varphi(\nu_A)(y_1), \varphi(\nu_A)(y_2))
\end{array}
$$

$$
\begin{array}{rl}
(4) & \varphi(\nu_A)(y_1y_2) \\
= & \inf\{\nu_A(x_1x_2) \mid x_1, x_2 \in R, \varphi(x_1) = y_1, \varphi(x_2) = y_2\} \\
\leq & \inf\{C(\nu_A(x_1), \nu_A(x_2)) \mid x_1, x_2 \in R, \varphi(x_1) = y_1, \varphi(x_2) = y_2\} \\
= & C(\inf\{\nu_A(x_1) \mid x_1 \in R, \varphi(x_1) = y_1\}, \inf\{\nu_A(x_2) \mid x_2 \in R, \varphi(x_2) = y_2\}) \\
= & C(\varphi(\nu_A)(y_1), \varphi(\nu_A)(y_2))
\end{array}
$$

Hence $\varphi(A) \in IFSN(S)$.

Corollary 4.2. Let φ be an anti-epihomomorphism from semiring R into semiring S. If $A = (\mu_A, \nu_A) \in IFSN(R)$, then $\varphi(A) \in IFSN(S)$.

Proposition 4.3. Let Let φ be a homorphism from semiring R into semiring S. If $B = (\mu_B, \nu_B) \in IFSN(S)$, then $\varphi^{-1}(B) = (\varphi^{-1}(\mu_B), \varphi^{-1}(\nu_B)) \in IFSN(R)$.

Proof. Let $x_1, x_2 \in R$.

$$
\begin{array}{rcl}
(1)\ \varphi^{-1}(\mu_B)(x_1 + x_2) & = & \mu_B(\varphi(x_1 + x_2)) \\
& = & \mu_B(\varphi(x_1) + \varphi(x_2)) \\
& \geq & T(\mu_B(\varphi(x_1)), \mu_B(\varphi(x_2))) \\
& = & T(\varphi^{-1}(\mu_B)(x_1), \varphi^{-1}(\mu_B)(x_2))
\end{array}
$$

$$
\begin{array}{rcl}
(2)\ \varphi^{-1}(\mu_B)(x_1x_2) & = & \mu_B(\varphi(x_1x_2)) \\
& = & \mu_B(\varphi(x_1)\varphi(x_2)) \\
& \geq & T(\mu_B(\varphi(x_1)), \mu_B(\varphi(x_2))) \\
& = & T(\varphi^{-1}(\mu_B)(x_1), \varphi^{-1}(\mu_B)(x_2))
\end{array}
$$

$$
\begin{array}{rcl}
(3)\ \varphi^{-1}(\nu_B)(x_1 + x_2) & = & \nu_B(\varphi(x_1 + x_2)) \\
& = & \nu_B(\varphi(x_1) + \varphi(x_2)) \\
& \leq & C(\nu_B(\varphi(x_1)), \nu_B(\varphi(x_2))) \\
& = & C(\varphi^{-1}(\nu_B)(x_1), \varphi^{-1}(\nu_B)(x_2))
\end{array}
$$

$$
\begin{array}{rcl}
(4)\ \varphi^{-1}(\nu_B)(x_1x_2) & = & \nu_B(\varphi(x_1x_2)) \\
& = & \nu_B(\varphi(x_1)\varphi(x_2)) \\
& \leq & C(\nu_B(\varphi(x_1)), \nu_B(\varphi(x_2))) \\
& = & C(\varphi^{-1}(\nu_B)(x_1), \varphi^{-1}(\nu_B)(x_2))
\end{array}
$$

Then $\varphi^{-1}(B) \in IFSN(R)$.

Proposition 4.4. Let Let φ be a anti-homorphism from semiring R into semiring S. If $B = (\mu_B, \nu_B) \in IFSN(S)$, then $\varphi^{-1}(B) \in IFSN(R)$.

Proof. Let $x_1, x_2 \in R$.

$$
\begin{aligned}
(1)\ \varphi^{-1}(\mu_B)(x_1+x_2) &= \mu_B(\varphi(x_1+x_2)) \\
&= \mu_B(\varphi(x_1)+\varphi(x_2)) \\
&\geq T(\mu_B(\varphi(x_1)), \mu_B(\varphi(x_2))) \\
&= T(\varphi^{-1}(\mu_B)(x_1), \varphi^{-1}(\mu_B)(x_2))
\end{aligned}
$$

$$
\begin{aligned}
(2)\ \varphi^{-1}(\mu_B)(x_1x_2) &= \mu_B(\varphi(x_1x_2)) \\
&= \mu_B(\varphi(x_2)\varphi(x_1)) \\
&\geq T(\mu_B(\varphi(x_2)), \mu_B(\varphi(x_1))) \\
&= T(\mu_B(\varphi(x_1)), \mu_B(\varphi(x_2))) \\
&= T(\varphi^{-1}(\mu_B)(x_1), \varphi^{-1}(\mu_B)(x_2))
\end{aligned}
$$

$$
\begin{aligned}
(3)\ \varphi^{-1}(\nu_B)(x_1+x_2) &= \nu_B(\varphi(x_1+x_2)) \\
&= \nu_B(\varphi(x_1)+\varphi(x_2)) \\
&\leq C(\nu_B(\varphi(x_1)), \nu_B(\varphi(x_2))) \\
&= C(\varphi^{-1}(\nu_B)(x_1), \varphi^{-1}(\nu_B)(x_2))
\end{aligned}
$$

$$
\begin{aligned}
(4)\ \varphi^{-1}(\nu_B)(x_1x_2) &= \nu_B(\varphi(x_1x_2)) \\
&= \nu_B(\varphi(x_2)\varphi(x_1)) \\
&\leq C(\nu_B(\varphi(x_2)), \nu_B(\varphi(x_1))) \\
&= C(\nu_B(\varphi(x_1)), \nu_B(\varphi(x_2))) \\
&= C(\varphi^{-1}(\nu_B)(x_1), \varphi^{-1}(\nu_B)(x_2))
\end{aligned}
$$

Therefore $\varphi^{-1}(B) \in IFSN(R)$.

Proposition 4.5. Let φ be an epihomomorphism from semiring R into semiring S and T, C be idempotent. If $A = (\mu_A, \nu_A) \in IFSN(R)$ and $C_{\alpha,\beta} = \{x \in R \mid \mu_A(x) \geq \alpha, \nu_A(x) \leq \beta\}$ be subsemiring of A, then $\varphi(C_{\alpha,\beta}) = C_{\acute{\alpha},\acute{\beta}} = \{\varphi(x) = y \in S \mid \mu_{\varphi(A)}(y) \geq \acute{\alpha}, \nu_{\varphi(A)}(y) \leq \acute{\beta}\}$ will be a subsemiring of $\varphi(A)$.

Proof. Since $A = (\mu_A, \nu_A) \in IFSN(R)$ so from Proposition 4.1 $\varphi(A) = (\mu_{\varphi(A)}, \nu_{\varphi(A)}) \in IFSN(S)$. Let $y_1, y_2 \in C_{\acute{\alpha},\acute{\beta}}$. Then

(1) $\mu_{\varphi(A)}(y_1+y_2) \geq T(\mu_{\varphi(A)}(y_1), \mu_{\varphi(A)}(y_2)) \geq T(\acute{\alpha}, \acute{\alpha}) = \acute{\alpha}$.
(2) $\mu_{\varphi(A)}(y_1y_2) \geq T(\mu_{\varphi(A)}(y_1), \mu_{\varphi(A)}(y_2)) \geq T(\acute{\alpha}, \acute{\alpha}) = \acute{\alpha}$.
(3) $\nu_{\varphi(A)}(y_1+y_2) \leq C(\nu_{\varphi(A)}(y_1), \nu_{\varphi(A)}(y_2)) \leq C(\acute{\beta}, \acute{\beta}) = \acute{\beta}$.
(4) $\nu_{\varphi(A)}(y_1y_2) \leq C(\nu_{\varphi(A)}(y_1), \nu_{\varphi(A)}(y_2)) \leq C(\acute{\beta}, \acute{\beta}) = \acute{\beta}$.

Then $y_1+y_2, y_1y_2 \in C_{\acute{\alpha},\acute{\beta}}$ and $\varphi(C_{\alpha,\beta}) = C_{\acute{\alpha},\acute{\beta}}$ is a subsemiring of $\varphi(A)$.

Proposition 4.6. Let φ be a homorphism from semiring R into semiring S and T, C be idempotent. If $B = (\mu_B, \nu_B) \in IFSN(S)$ and $C_{\alpha,\beta} = \{y \in S \mid \mu_B(y) \geq \alpha, \nu_B(y) \leq \beta\}$ be a subsemiring of B, then $\varphi^{-1}(C_{\alpha,\beta}) = C_{\acute{\alpha},\acute{\beta}} = \{\varphi^{-1}(y) = x \in R \mid \mu_{\varphi^{-1}(B)}(x) \geq \acute{\alpha}, \nu_{\varphi^{-1}(B)}(x) \leq \acute{\beta}\}$ be a subsemiring of $\varphi^{-1}(B)$.

Proof. Let $x_1, x_2 \in C_{\acute{\alpha},\acute{\beta}}$. As Proposition 4.3 $\varphi^{-1}(B) \in IFSN(R)$ and then

(1) $\mu_{\varphi^{-1}(B)}(x_1 + x_2) \geq T(\mu_{\varphi^{-1}(B)}(x_1), \mu_{\varphi^{-1}(B)}(x_2)) \geq T(\acute{\alpha}, \acute{\alpha}) = \acute{\alpha}$.
(2) $\mu_{\varphi^{-1}(B)}(x_1 x_2) \geq T(\mu_{\varphi^{-1}(B)}(x_1), \mu_{\varphi^{-1}(B)}(x_2)) \geq T(\acute{\alpha}, \acute{\alpha}) = \acute{\alpha}$.
(3) $\nu_{\varphi^{-1}(B)}(x_1 + x_2) \leq C(\nu_{\varphi^{-1}(B)}(x_1), \nu_{\varphi^{-1}(B)}(x_2)) \leq C(\acute{\beta}, \acute{\beta}) = \acute{\beta}$.
(4) $\nu_{\varphi^{-1}(B)}(x_1 x_2) \leq C(\nu_{\varphi^{-1}(B)}(x_1), \nu_{\varphi^{-1}(B)}(x_2)) \leq C(\acute{\beta}, \acute{\beta}) = \acute{\beta}$.

Thus $x_1 + x_2, x_1 x_2 \in C_{\acute{\alpha},\acute{\beta}}$ and so $\varphi^{-1}(C_{\alpha,\beta}) = C_{\acute{\alpha},\acute{\beta}}$ is a subsemiring of $\varphi^{-1}(B)$.

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GREEN SUPPLY CHAIN MANAGEMENT IN PRODUCTION SECTORS AND ITS IMPACT ON FIRM REPUTATION

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Abstract – The purpose of this paper is to investigate the effect of Green Supply Chain Management practice on firm reputation. To investigate this, data were collected from executives and managers of production companies of reputed industries. A descriptive, correlational methodology was adopted and data were analyzed using structural equation modeling by using exploratory analysis and linear multiple regression analysis. The results revealed that the green purchasing, green manufacturing/material management, green distribution/marketing of production companies have a positive and significant impact on firm reputation. Finally, the results suggest that strengthening green supply chain management practice in production sectors improves firm reputation, which in turn increases firm revenue.

***Keywords* –** Green Supply Chain Management, Green Purchasing, Green Manufacturing/Material Management, Green Distribution/Marketing, Firm Reputation.

1 Introduction

In the last couple of decades, researchers have shown great interest on to investigate the effect of implementation of green supply chain management in firms. But the implementation of green supply chain in firms plays a significant role impact in the environmental and financial performances of any firm. Many firms are now showing more willingness to implement environmental-friendly practices in their businesses. This is mainly because of two factors. One factor is customer's pressure to implement green manufacturing and green distribution or marketing. Other factor is pressure from government using strict laws regarding environment to force enterprises to adopt green practices. Also there are some motivating factors to some firms to implement green practice to go towards sustainable development. Some firms believe that implementing green practice they can increase financial gains and reduce cost using recycling, reuse, and remanufacturing. On the other hand, some firms believe that green practices have negatively impact on overall firm performance and execution of green practices can be a waste of their resources. The proposal of Bansal (2005) supported the claim of these firms. A similar result was studied by Zhang and Yang (2016), where they found that show that

environmental-friendly practices have no positive effects on business' economic performance. Also in a survey with hundreds CEOs from around the world exposed that about more than half of them feel that implementing green sustainability programs may be critical to their businesses. But several researchers found that adapting green or ecological practices in the business would lead to overall improvement of company's performance and execution of green practices not only improves the environment but also creates competitive advantages(Christmann (2000) & Porter and Van der Linde (1995)). Freeman(1984) explained the stakeholder approach and according to him pressure on a company to implement to force something has a negative effect on manufacturing and business activities. Several researchers found that pressurization to implement green practices may have negative impact on the overall firm performance and heavy investment in green technology may reduce the overall profit. According to Christman(2000), company's lack of basic green capabilities may lead a financial burden on the firm. Hart(1995) suggested that incorporating green practice in an enterprise's strategic planning will improve its ability to overcome uncertainties and firm operations and help the enterprise to develop firm competitive advantage and consequently, increase its financial performance. Klassen and McLaughlin(1996) and Jacobs et al.(2010) also suggested that proper implementation of green practices in supply chain can enhance operational, environmental and financial performances. They also suggested that green practices will reduce the cost and create good image and reputation in the market. Many researchers conducted similar studies to investigate the relationship between enterprise green practice with economic performance of the firms and firm reputation. Although, their investigations found a mixed results on firm economic performance but they all agreed that implementation of green practice have a positive and significant impact on firm reputation.
Our investigation in the proposed paper is to find the impacts of implementation of green supply chain management in firm on firm reputation. This study will find the impacts (positive or negative) of green purchasing, green manufacturing and green distribution/marketing on firm reputation.

The pressure to implement green practice in firms by the customers and the governmental laws are the driving forces of green purchasing, green manufacturing, green distribution and marketing. International companies always try to satisfy their customers by providing proper service and better quality products through the innovation and research carried out by the research and development(R&D) section of the companies. Sometimes this takes the form of improving green performance by observing environmental laws and standards, increasing customer knowledge in this area, and reducing the negative environmental effects of their products and services (Koplin, Seuring, & Mesterharm, 2007). Green performance involves assessing the relationship between trade and the environment (Olsthoorn, Tyteca, Wehrmeyer, & Wagner, 2001). Sustainable development is key to ensuring a company's survival and requires the commitment and participation of all employees and managers. Many industries are facing competitive pressure to coordinate and cooperate through the supply chain management practice to improve agility, flexibility and proper functioning of their product. On the other hand implementation of green supply chain management practice has a positive impact on firm reputation. Sigala (2008) suggested that concern about environmental issues and governmental policies drive the industries to adopt green supply chain management practice to maintain competitiveness.

Many researchers found several studies that the green practice of the organizations for environment-friendly business operations have a significant and positive impact on firm reputation. From a macro perspective, attention to green issues is important in relation to both the design of new green products and the creation of markets for products that are compatible with the environment. The creation of a green supply chain requires the development of opportunities for companies to invest in the design and manufacture of greener products and to meet the requirements of sustainability. It involves not only the production of green consumer goods, but also the involvement of suppliers in the creation of green markets (Sheu, Chou, & Hu, 2005).

This study sought to investigate the role of green purchasing, green manufacturing, green distribution/marketing of the firm on its reputation. Internal green practice of the company recognizes that different administrative areas within the company need to be integrated for optimum performance (Flynn, Huo, & Zhao, 2010). External green collaboration to use green distribution and marketing involves mutual understanding of environmental responsibilities and risks and shared decision-making to solve environmental problems and allocate resources, skills and knowledge between suppliers, partners and customers in the supply chain to achieve common environmental goals (Vachon, S., & Klassen, R. D. 2008). Our investigation is significantly different from the existing investigations. No statistical investigation has been carried out to investigate the impact of green supply chain management practices i.e. green purchasing, green manufacturing, green distribution/marketing on firm reputation. The rest of the paper is structured as follows. Section 2 covers hypothesis, Section 3 describes the methodology, Section 4 descries analysis and results and section 5 describes conclusion, managerial implications and future research directives.

2 Hypothesis

The study of the proposed conceptual model is shown in Figure 1, in which green supply chain management is understood as comprising green purchasing, green manufacturing/material management, green distribution/marketing and firm reputation. In the present study, green purchasing is identified as green raw materials, green shipping practices and green accumulation. Green manufacturing/material management is comprised

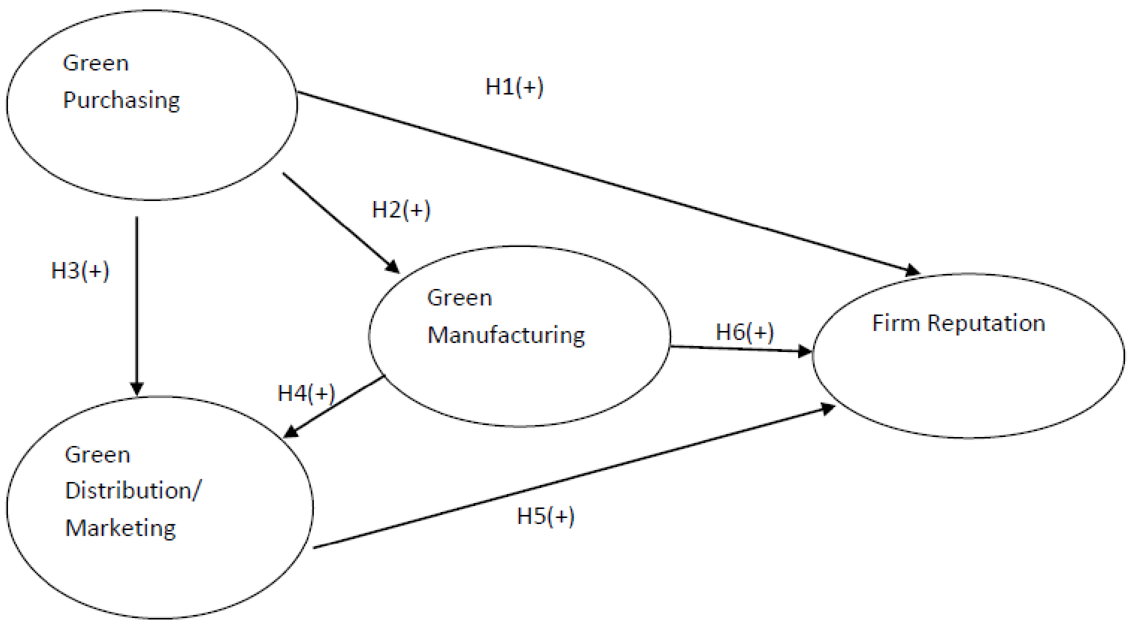

Figure 1: Research conceptual model.

as green processing, green packaging. Green distribution/marketing comprises of reduction in cost of transportation, reduction in pollutants. Firm reputation is identified by quality of services, productivity, and corporate Profit. Firm reputation may be assessed by the accelerated sales of goods which in turns increased profits.

The Figure 1 shows six testable hypotheses in which all of the direct associations indicated are hypothesised as positive. The theoretical structural model incorporates green manufacturing as the focal construct with green purchasing and green distribution/marketing as antecedents and firm reputation as a consequence. The above model is designed to assess the impact of green supply chain indicators on firm reputation. Our proposed investigation claims that the combination of green purchasing, green manufacturing and green distribution/marketing will enhance firm reputation which ultimately leads to increased revenues of the firm.

Some researchers found that green purchasing (GP) create significant improvement in the overall firm reputation. In studying the relationship between green practices in supply chain and firm reputation, it is found that there is a positive and significant impact of green purchasing and firm reputation. According to Allenby(1991) and Zailani et al.(2012) implementation of green purchasing practices improved the firm reputation and brand image. According to Mitra and Datta (2014) and Zhu and Sarkis(2004) also found that implementing green purchasing of the firm have a significant impact on firm reputation. According to Min and Galle(2001), green purchasing practices reduce the sources of waste and encourage recycling and reuse activities without any hindrance to the firm performance which in turns increases firm reputation. Zhu et al. (2008) found that there is a relationship between green procurement and company's financial performance. The result also shows that purchasing activities are connected with firm performance. In addition, they suggest that firms need to implement green purchasing to maximize resource utilization and reduce the harmful effect of manufacturing activities. Therefore, we suggest the following hypotheses for testing:

H1: *Green purchasing of raw materials have a positive and significant impact on firm reputations.*

It is expected that a firm can manufacture green product only if it purchases green raw materials. Many researchers investigated this phenomenon. Therefore the firm should be careful to buy only green raw materials to manufacture green products. Green procurement activities have viable environmental properties such as reusability and recyclability. Rao and Holt (2005) found green purchasing practices has a positive and significant impact on firm performance in the context of firm internal management and supplier selection which ultimately leads to firm reputation. These results also indicate that adoption of green purchasing not only reduces pollution and waste but also improves overall firm brand image and reputation. Therefore we propose the following hypothesis for testing:

H2: Green purchases of raw materials have a positive and significant impact in green manufacturing.

Min and Galle(2001) found that green purchasing practices reduce the sources of waste and encourage recycling and reuse activities without any hindrance to the firm distribution and marketing. Zhu et al. (2008) found that there is a relationship between green procurement and company's financial performance. The result also shows that green purchasing activities are connected with firm internal and external distribution policies. The green procurement or green purchase of raw products also influences green marketing of the firm. In addition, researchers also suggest that firms need to implement green purchasing to maximize resource utilization and reduce the harmful effect of manufacturing activities. Therefore, we suggest the following hypotheses for testing:

H3: Green purchases of raw materials have a positive and significant impact in green distribution/marketing.

Porter and Van der Linde (1995) proposed that green manufacturing (GM) and green process can reduce the resource wastage and play a vital role in energy reduction, optimizing manufacturing steps, and to improve overall firm green distribution/marketing. Therefore, Green manufacturing has a positive and significant impact on green distribution and marketing. Therefore, we suggest the following hypotheses for testing:
H4: Green manufacturing has a positive and significant impact on a green distribution/ marketing.

Several researchers like Droge, Jayaram, & Vickery (2004), O'Leary-Kelly & Flores (2002), Rosenzweig, Roth, & Dean Jr, (2003), Swink & Nair (2007), Zailani & Rajagopal (2005) investigated the impact green distribution and green marketing on firm reputation. They also derived that there is a positive relationship between internal performance and operational performance. Stank, Keller & Daugherty (2001) and Ellinger et al. (2007) investigated that collaboration between marketing and logistics had a positive effect on distribution services performance. Zhu and Sarkis (2004) proposed that companies with high levels of adaptation of green activity achieve improved environmental performance. Hence there is considerable evidence to support the hypothesis that the implementation of i green distribution/marketing practices will lead to improved firm reputation. Based on these investigations we hypothesize that:

H5: Green distribution/marketing has a positive and significant impact on firm reputation.

The Increasing environmental concern from customers, buyers, communities, and government regulations force companies to implement Green Supply Chain Management (GSCM) and green innovation. Zhu, Sarkis, & Lai, 2008 suggested that GSCM and green innovation have strategic interconnection in developing new green product and this ultimately has a positive impact on firm reputation. Vachon and Klassen (2008) proposed that green cooperation between the organization and the members of its green supply chain enables the company to implement GSCM which ultimately have a very high impact on firm reputation. Rao & Holt (2005) claimed that green supply chain management using green manufacturing practice ultimately improves firm reputation and consequently, the following hypothesis follows:

H6: Green manufacturing has a positive and significant impact on firm reputation.

3 Methodology

Our aim is to investigate the impact of green supply chain management practice on firm reputation. This research paper is based on quantitative research approach since the objective of this paper is to compute the research variables namely green purchasing (GP), green manufacturing(GM), green distribution/marketing and firm reputation(FR). After analyzing the content validity of the questionnaire by industrial experts, the final questionnaire was sent to 100 manufacturing firms. There were totally 30 questions in the questionnaire related to three variables such as GP, GM, GD/M and FR. The questionnaire is based on five point Likert scale (1: strongly disagree to 5: strong agree). A total of 78 questionnaire responses were received, in which 11 responses were excluded due to not being properly filled or unusable. Thus, the total sample size included for analysis is 67. We use the hierarchical multiple regression method on SPSS (version 22) to test the hypotheses. The research model is shown in Fig. 1.

4 Analysis and Results

We study the reliability analysis, exploratory factor analysis, and multiple regressions for dataset. The exploratory analysis was conducted to determine the underlying structure for the 30 items of the enterprise performance questionnaire. Based on the hypotheses and model shown before, three factors were requested. This is due to the fact that the items were designed to index four constructs: firm reputation is a dependent variable, while green purchasing, green manufacturing and green distribution/marketing were predictors. The value of KMO measure of sampling adequacy (0.743) indicates that the sample has fulfilled the requirement to run factor analysis. Moreover, a significant result of Bartlett's test (p < 0:05) shows that the matrix is not an identify matrix. In other words, these four components do relate enough to one another to further run a substantial factor analysis. Table 2 illustrates the results of KMO and Bartlett's test.

Also, the initial solution of exploratory factor analysis was rotated by using the orthogonal (varimax) rotation approach with Kaiser normalization which extracted the six required uncorrelated factors. We accounted for the variances of 19.725%, 18.276%, 17.753% and 27.348% respectively. However, these four components explained the cumulative 83.102% of the total variance. For the internal consistency of measuring scale, Cronbach's alpha of every variable also was calculated. The overall reliability of scale of the 30 items was 0.824 because, to enhance clarity, values less than 0.30 were omitted. Table 3 indicates the items and factor loading for the rotated factors. Since all of these 24 items were loaded onto their own components in the rotated solution, there were no cross-loadings as well, while both the discriminant and construct validity were ensured already.

4.1. Hypotheses testing

After satisfying the basic parametric assumptions, linear multiple regression was used to determine: the size of the association among variables of green purchasing (independent) and firm reputation (dependent variable); and to what extent every independent variable (i.e. green manufacturing and green design/marketing) individually contributed to predicting firm reputation. Table 3 illustrates the values of mean, standard deviation of enterprise performance, and its predictors. Table 4 illustrates the results of hypotheses testing through simultaneous multiple regression for predicting enterprise performance. The combination of variances significantly predicted 33.6% of the total variance in predicting enterprise performance.

Table 1. Results of KMO and Bartlett's test.

Kaiser–Meyer–Olkin measure of sampling adequacy	**0.743**
Bartlett's test of sphericity Approx. chi-square	**4321.621**
Degrees of freedom	**263**
Significance	**0.000**

Table 2. Rotated components matrix (extraction method: principal component analysis and rotation method: varimax with Kaiser normalization).

		Component			
	Alpha	**1**	**2**	**3**	**4**
Green Purchasing	0.811	0.782			
Green manufacturing	0.930		0.837		
Green Distribution/Marketing	0.814			0.852	
Firm Reputation	0.924				0.872
Eigenvalues		3.531	3.051	2.553	1.978
% of variance explained		13.221	12. 217	10.753	8.348
Cumulative % of variance explained		21.72	41.89	57.65	78.05

Table 3. Descriptive statistics

	N	Mean	Std. deviation
Firm Reputation	67	29.47	3.93
Green Purchasing	67	17.42	4.13
Green Manufacturing	67	21.33	4.21
Green Distribution/Marketing	67	18.91	3.42

The results of hypotheses testing through simultaneous multiple regression for predicting enterprise performance is shown in Table 4. . The combination of variances significantly predicted 32.7% of the total variance in predicting firm reputation ($F = 23.7$, $p < 0:001$), with three independent variables that significantly predicted firm reputation. Moreover, the issue of multicollinearity is not found among independent variables because the variance \VIF" value for each independent variable is less than 10. The coefficients of parameter estimates suggest the green purchasing (0.217, $p < 0.05$), green manufacturing (0.237, $p < 0.05$) and green distribution/marketing (0.232, $p < 0.05$) reflect a statistically significant and positive impact on firm reputation.

Table 4. Hypotheses testing for firm reputation through standard regression analysis. Unstandardized coefficients

Hypothesis	B	Standard Error	VIT	T-Statistics	Significance	Accept or Reject the Hypothesis
Constant	8.273	1.432		5.231	0.00	
Green Purchasing → Firm Reputation (H1)	0.217	0.053	1.432	2.33	Sig<0.05	Not rejected
Green Purchasing → Green Manufacturing (H2)	0.227	0.042	1.263	2.19	Sig<0.05	Not rejected
Green Purchasing → Green Distribution/Marketing (H3)	0.206	0.039	1.124	2.11	Sig<0.05	Not rejected
Green Manufacturing → Green Distribution/Marketing (H4)	0.219	0.032	1.137	2.14	Sig<0.05	Not rejected
Green Distribution/Marketing → Firm Reputation (H5)	0.232	0.051	1.411	2.28	Sig<0.05	Not rejected
Green Manufacturing → Firm Reputation (H6)	0.237	0.054	1.672	2.44	Sig<0.05	Not rejected

When dependent variable: Firm Reputation ($F = 23.7$, $p < 0:001$, and adjusted $R^2 = 32.7\%$).

5. Conclusions

The simultaneous multiple regression analysis prove that green purchasing, green manufacturing and green distribution/marketing have a significant and positive impact with firm reputation. The statistical results found that green manufacturing is the most important predictor of firm reputation. This result is also supported by previous empirical studies. Then comes the green distribution/marketing next important predictor and then comes to green purchasing. All have a positive and significant impact on firm reputation. Green purchasing, green manufacturing and green distribution/marketing all reduces the resources and cost in terms of recycling, reuse, and remanufacturing and also improves the firm reputation. The following equation shows the regression equation to predict firm reputation M= 8.273+0.217 X Green Purchasing +0.232 X Green Distribution/Marketing +0.237 X Green Manufacturing.

This research examined the impact of green purchasing, green manufacturing and green distribution/marketing on firm reputation. Three dimensions of green purchasing, green manufacturing, green distribution/marketing were assessed and the result suggest that all these have a positive and significant impact on firm reputation. Therefore, it is recommended that the senior management of manufacturing firms should implement green practices to improve the overall environmental performance and also enhance the operational and reputational performance which in turns leads to overall economic performances along with firm positive brand image. The main objective of senior management of having an optimum enterprise performance can also be reinforced by a \green awareness and training program" to their employees and distributors.

5.1. Managerial Implications

In terms of managerial implications, this research has verified the significance of green purchasing, green manufacturing, green distribution/marketing to firm reputation. Hence, when it comes to managing their respective manufacturing firms, the management should pay attention to these three aspects of green supply chain management practices to enhance firm reputation. In addition, this study has put forward some valuable insights to senior management of manufacturing firms in practice, to identify problematic areas in their own firms and devise corrective actions.

Another valuable result is that although green purchasing, green manufacturing green distribution/marketing has a significant and positive impact on firm reputation, this requires a long-term infrastructural requirements. There is no doubt that green manufacturing processes are long-term investments of firms in favor of the environment, firm's brand image, and firm's financial performance. The green manufacturing is very crucial for continuous improvement (CI) of enterprise in the long run. The green manufacturing processes will also help firms to achieve their financial targets through reduction of cost. The research results suggested that senior management should review firm's green practices, initiatives, and relevant policies, and conduct them in a way that supports high level of enterprise performance. Therefore, senior management should not ignore the importance of green purchasing, green manufacturing and green distribution/marketing for making a brand image and reputation of the enterprise.

Future research should include the additional measure of performance, such as the operational performance of the firm and the overall performance of the green supply chain. This study is limited to only the manufacturing firms, however, future researches may concentrate on comparative studies between manufacturing industry and other industries. Future researches can be conducted with other predictors, including green logistics, co-operation with customers and suppliers, and internal environmental management.

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C^{∞} SOFT MANIFOLDS

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Abstaract − In this paper, we briefly recall several basic notions of soft sets and soft topological spaces and we continue investigating the properties of soft mappings, soft continuous mappings and soft homeomorphisms. We introduce and discuss the properties of the soft topological manifolds of dimension n and define C^{∞} soft manifolds which will strengthen the foundations of the theory of soft geometry. We study restriction of a soft mapping and then define submanifolds.

Keywords − *Soft set, soft mapping, soft continuity, soft topological space, C^{∞} soft manifold.*

1 Introduction

The concept of soft sets is introduced by Molodtsov [7] which is a completely new mathematical tool to deal with uncertainties while modeling problems in engineering physics, computer science, economics, social sciences and medical sciences.

Soft sets are convenient to be applied in practice and this theory has potential application in many different fields such as smoothness of functions, game theory, Riemann integration, Perron integration, probability theory and measure theory.

Shabir and Naz [10], defined soft topology and studied many properties. Zorlutuna et al. [14] studied some concepts in soft topological spaces such as interior point, interior, neighborhood, continuity, and compactness. In [4], Maji et al. combined fuzzy sets and soft sets and introduced fuzzy soft sets. They described an application of soft set theory to a decision-making problem [5]. Tanay et al. [11] introduced the fuzzy soft topology. Later, Roy et al. [9] and Varol et al. [12] independently modified the definition of fuzzy soft sets and redefined fuzzy soft topology. Research on the soft set theory has been accelerated [1, 2, 3, 6]. In this paper, we introduce the soft topological manifolds of dimension n. Also we define C^{∞} soft manifolds and C^{∞} soft submanifolds.

2 Preliminary

Definition 2.1. [7] Let X be an initial universe and E be a set of parameters. Let A be a non-empty subset of E. A pair (F, A) is called a soft set over X, where F is a mapping given by $F : A \to P(X)$.

The set of all of soft sets over X, is denoted by $S(X, E)$.

Example 2.2. [4] Zadeh's fuzzy sets [13] may be concidered as a special case of the soft set.

Let $D : X \to [0, 1]$ be a fuzzy set. Let us concider the family of α-level sets for D given by:

$$F(\alpha) = \{x \in X : \ \ D(x) \geqq \alpha\}, \quad \alpha \in [0, 1].$$

Then we can write

$$D(x) = sup\{\alpha : \alpha \in [0, 1], \ \ x \in F(\alpha)\}.$$

Thus the fuzzy set D may be considered as a soft set $(F, [0, 1])$.

Definition 2.3. [5] Let $(F, A) \in S(X, E)$.

i. The complement of a soft set (F, A) is denoted by $(F, A)^c$ and is defined by $(F, A)^c = (F^c, \ A)$ where, $F^c : A \to P(U)$ is a mapping given by $F^c(a) = U - F(a)$, for all $a \in A$.

ii. Let $E = \{e_1, e_2, ..., e_n\}$ be a set of parameters. The NOT set of E denoted by $\rceil E$ is defined by $\rceil E = \{\rceil e_1, \rceil e_2, ..., \rceil e_n\}$ where, $\rceil e_i = note_i$ for all i.

Definition 2.4. [5] Let $(F, A), (G, B) \in S(X, E)$.

i. (F,A) is a soft subset of (G,B), denoted by $(F, A)\tilde{\subset}(G, B)$, if $F(e) \subset G(e)$ for each $e \in E$.

ii. (F,A) and (G,B) are said to be soft equal, denoted by $(F, A) = (G, B)$ if $(F, A)\tilde{\subset}(G, B)$ and $(G, B)\tilde{\subset}(F, A)$.

iii. Union of (F, A) and (G, B) is a soft set (H, C), where $C = A \cup B$ and $H(e) = F(e) \cup G(e)$ for each $e \in E$. This relationship is written as $(F, A)\tilde{\cup}(G, B) = (H, C)$.

iv. Intersection of (F, A) and (G, B) is a soft set (H, C), where $C = A \cap B$ and $H(e) = F(e) \cap G(e)$ for each $e \in E$. This relationship is written as:
$(F, A)\tilde{\cap}(G, B) = (H, C)$.

v. The difference (H, E) of (F, E) and (G, E), denoted by $(F, E)\backslash(G, E)$, is defined as $H(e) = F(e)\backslash G(e)$ for all $e \in E$.

Definition 2.5. [10]

i. Let (F, E) be a soft set over X and $x \in X$. We say that $x \in (F, E)$ and read as x belongs to the soft set (F, E) whenever $x \in F(a)$ for all $a \in E$.

ii. Let $x \in X$, then (x, E) denotes the soft set over X for which $x(a) = x$, for all $a \in E$.

Definition 2.6. [10] Let (F,E) be a soft set over$X$ and Y be a non-empty subset of X. Then the sub soft set of (F,E) over Y denoted by $({}^{Y}\mathrm{F},E)$, is defined as follows: ${}^{Y}\mathrm{F}(a) = Y \cap F(a)$, for all $a \in E$. In other words $({}^{Y}\mathrm{F},E) = \tilde{Y} \tilde{\cap} (F,E)$.

Definition 2.7. [3] Let $S(X,E)$ and $S(Y,K)$ be families of soft sets. Let $u : X \to Y$ and $p : E \to K$ be mappings. Then $f_{pu} : S(X,E) \to S(Y,K)$ is defined as:

i. Let $(F,A) \in S(X,E)$. The image of (F,A) under f_{pu} written as $f_{pu}(F,A) = (f_{pu}(F), p(A))$ is a soft set in $S(Y,K)$ such that:

$$f_{pu}(F)(k) = \begin{cases} \bigcup_{e \in p^{-1}(k) \cap A} u(F(e)) & p^{-1}(k) \cap A \neq \phi, \\ \phi & otherwise \end{cases}$$

ii. Let $(G,B) \in S(Y,K)$. The invers image of (G,B) under f_{pu} written as $f_{pu}^{-1}(G,B) = \left(f_{pu}^{-1}(G), p^{-1}(B)\right)$ is a soft set in $S(X,E)$ such that:

$$f_{pu}^{-1}(G)(e) = \begin{cases} u^{-1}(G(p(e))) & p(e) \in B, \\ \phi & otherwise \end{cases}$$

The soft function f_{pu} is called surjective if p and u are surjective, also is said to be injective if p and u are injective.

Theorem 2.8. [1] Let $S(X,E)$ and $S(Y,K)$ be families of soft sets. For the soft function
$f_{pu} : S(X,E) \to S(Y,K)$, the following statements hold:

i. $f_{pu}^{-1}((G,B)^c) = \left(f_{pu}^{-1}(G,B)\right)^c$, for all $(G,B) \in S(Y,K)$.

ii. $f_{pu}\left(f_{pu}^{-1}(G,B)\right) \tilde{\subset} (G,B)$ for all $(G,B) \in S(Y,K)$. If f_{pu} is surjective, then the equality holds.

iii. $(F,A) \tilde{\subset} f_{pu}^{-1}(f_{pu}(F,A))$ for all $(F,A) \in S(X,E)$. If f_{pu} is injective, then the equality holds.

Theorem 2.9. [3] Let $\{(F_i,E)\}_{i \in I} \subseteq S(X,E)$ and $\{(G_i,k)\}_{i \in I} \subseteq S(Y,K)$. Then for a soft mapping $f_{pu} : S(X,E) \to S(Y,K)$, the following are true.

i. If $(F_1,E) \tilde{\subseteq} (F_2,E)$, then $f_{pu}(F_1,E) \tilde{\subseteq} f_{pu}(F_2,E)$

ii. $(G_1,K) \tilde{\subseteq} (G_2,K)$ then $f_{pu}^{-1}(G_1,K) \tilde{\subseteq} f_{pu}^{-1}(G_2,K)$

iii. $f_{pu}(\tilde{\cup}_i (F_i,E)) = \tilde{\cup}_i f_{pu}(F_i,E)$

iv. $f_{pu}^{-1}((G_1,K) \tilde{\cap} (G_2,K)) = f_{pu}^{-1}(G_1,K) \tilde{\cap} f_{pu}^{-1}(G_2,K)$.

v. $f_{pu}^{-1}((G_1,K) \tilde{\cup} (G_2,K)) = f_{pu}^{-1}(G_1,K) \tilde{\cup} f_{pu}^{-1}(G_2,K)$.

Definition 2.10. [10] Let τ be the collection of soft sets over X, then τ is said to be a soft topology on X if

i. ϕ, X belong to τ

ii. the union of any number of soft sets in τ belongs to τ

iii. the intersection of any two soft sets in τ belongs to τ .

The triplet (X, τ, E) is called a soft topological space over X.

The members of τ are said to be soft open sets in X.

Definition 2.11. [14]

i. The soft set $(F, A) \in (X, \tau, E)$ is called a soft point in X, denoted by e_F, if for the element $e \in A, F(e) \neq \phi$ and $F(e') = \phi$ for all $e \in A\{e\}$.

ii. The soft point e_F is said to be in the soft set (G, A), denoted by $e_F \tilde{\in} (G, A)$, if for the element $e \in A,\ F(e) \subseteq G(e)$.

Definition 2.12. Let (X, τ, E) be a soft topological space and $(F, E) \in S(X, E)$.

i. [10] The soft closure of (F, E), denoted by $\overline{(F, E)}$ is the intersection of all closed soft supersets of (F, E) is the smallest closed soft set over X which contains (F, E), i.e.

$$\overline{(F, E)} = \tilde{\cap}\{(H, E) : (H, E) \in \tau', (F, E) \tilde{\subseteq} (H, E)\}.$$

ii. [14] The soft interior of (F, E), denoted by $(F, E)^{\circ}$ is the union of all open soft subsets of (F,E). Clearly (F,E) is the largest open soft set over X which contained in (F,E), i.e.

$$(F, E)^{\circ} = \tilde{\cup}\{(H, E) : (H, E) \in \tau \ and \ (H, E) \tilde{\subset} (F, E)\}.$$

iii. [2] The soft boundary of (F, E) is the soft set

$$\partial (F, E) = \overline{(F, E)} \tilde{\cap} \overline{(F, E)^{c}}$$

Theorem 2.13. [14] Let $f_{pu} : (X, \tau, E) \to (Y, \upsilon, K)$ be a soft mapping. Then the following statements are equivalent:

i. f_{pu} is soft continuous;

ii. $f_{pu}^{-1} (G, K) \in \tau', \forall (G, K) \in \upsilon'$;

iii. $\overline{f_{pu}^{-1} (G, K)} \tilde{\subseteq} f_{pu}^{-1} \overline{(G, K)}, \forall (G, K) \in S (Y, K)$;

iv. $\partial \left(f_{pu}^{-1} (G, K)\right) \tilde{\subseteq} f_{pu}^{-1} (\partial (G, K)), \forall (G, K) \in S (Y, K)$;

v. $f_{pu} (\partial (F, E)) \tilde{\subseteq} \partial (f_{pu} (F, E)), \forall (F, E) \in S (X, E)$

vi. $f_{pu} \left(\overline{(F, E)}\right) \tilde{\subseteq} \overline{f_{pu} (F, E)}, \forall (F, E) \in S (X, E)$

vii. $f_{pu}^{-1} ((G, K)^{\circ}) \tilde{\subseteq} \left(f_{pu}^{-1} (G, K)\right)^{\circ}, \forall (G, K) \in S (Y, K)$

Definition 2.14. A family β of members of τ is called a basis of soft topological space (X, τ), if each element of τ is a union of members of β .

Example 2.15. Let $(\mathbb{R}^n, \tau_{\epsilon n})$ be the fuzzy topology induced by $\beta = \{B(p, \epsilon, r) \mid p \in \mathbb{R}^n,\ \epsilon \in \mathbb{R}^+,\ r \in [0, 1]\}$ which $B(p, \epsilon, r)$ is a fuzzy subset that equals to zero outside the sphere $B(p, \epsilon)$ and equals to r inside $B(p, \epsilon)$. Since by 1.2 each fuzzy subset $B(p, \epsilon, r)$ can be considered as a soft set over $\mathbb{R}^n$ then β can be considered as a basis of a soft topology called soft Euqleadian space denoted by $(\mathbb{R}^n, \tau_{\epsilon n}, [0, 1])$.

Example 2.16. Let $\beta = \{B(p, q, r) \mid p \in \mathbb{R}^n,\ q \in \mathbb{Q}^+,\ r \in \mathbb{Q}^+ \cap [0, 1]\}$ which $B(p, q, r)$ is a fuzzy subset that equals to zero outside the sphere $B(p, q)$ and equals to r inside $B(p, q)$. Since $\mathbb{Q}$ is dence in $\mathbb{R}$, we can easily prove that the soft topology induced by β equels to $(\mathbb{R}^n, \tau_{\epsilon n}, [0, 1])$.

Example 2.17. Let E be a set of parameters and

$$\beta = \{A \mid A : E \to \prod_{i=1}^{n} (a_i, b_i) \ is \ \ surjective \ and \ \forall i \in I,\ a_i, b_i \in \mathbb{R}\}.$$

We call the soft topology τ induced by β, the natural soft topology over $\mathbb{R}^n$. So we have the natural soft topological space $(\mathbb{R}^n, \tau, E)$.

Example 2.18. Let $E = \{1\}$. Then $\{1\}$ is the single nonempty subset of E. We can consider

$$\beta = \{(F, \{1\}) \mid F : \{1\} \to B(p, \epsilon) \ is \ surjective \ and \ p \in \mathbb{R}^n,\ \epsilon \in \mathbb{R}^+\}$$

as a basis of soft Euqleadian space denoted by $(\mathbb{R}^n, \tau_\epsilon, \{1\})$. As in ordinary topology, this space is equel to the natural soft topological space.

3 Soft Topological Manifolds

Definition 3.1. A soft topological space (X, τ, E) is a soft topological space of dimention n if for any $x \in X$ there exists a soft open set (F, A) over X containing x and soft homeomorphic to a soft open set (G, B) of natural soft topology over $(\mathbb{R}^n,\ \tau_\epsilon,\ K)$.

Remark 3.2. When we write $f_{pu} : (F, A) \to (G, B)$, is a soft homeomorphism, it means that there is a soft homeomorphism $f_{pu} : (Z, \tau_Z, E) \to (Y,\ \tau_{\epsilon_Y}, K)$ where $Z = F(A)$, $Y = G(B)$. So there is a homeomorphism $u : Z \to Y$ and a bijective map $p : E \to K$.
The triple (F, A, f_{up}) is called a soft local coordinate neighborhood of each $q \in (F, A)$ and we assign to q the n soft coordinates $x_1(q), ..., x_n(q)$, of its image $u(q)$ in $\mathbb{R}^n$.

Proposition 3.3. With the above notations and soft local coordinate neighborhood (F, A, f_{up}), we have $f_{up}(A)(k) = u(A(e))$ where $p^{-1}(k) = \{e\}$.

Proof. Let $(H, D) \in \tau_Z$, then by definition 1.6 we have $f_{pu}(H, D) = (f_{pu}(H), p(D))$.

$$f_{pu}(H)(k) = \begin{cases} \bigcup_{e \in p^{-1}(k) \cap H} u(H(e)) & p^{-1}(k) \cap H \neq \phi \\ \phi & otherwise \end{cases}$$

for each $k \in D$. Since p is bijective, there is exactly one element of A such that $p^{-1}(k) = \{e\}$. So we have $f_{up}(H)(k) = u(H(e))$. If we set $H = A$ then we have $f_{up}(A)(k) = u(A(e))$ where $p^{-1}(k) = \{e\}$. □

Definition 3.4. A soft topological space (X, τ, E) is called a soft topologica manifold of dimension n if satisfies the following axoims:

i. X is a a soft topological space of dimention n,

ii. X is a T_2-space,

iii. X has a countable soft basis of soft open sets.

Definition 3.5. Let $\mathfrak{B} = \{(F_i,\ A_i,\ f_{u_ip_i}) :\ i \in I\}$ be a countable collection of soft local coordinate neighborhoods such that $\tilde{X} = \tilde{\bigcup}_{i\in I}(F_i,\ A_i)$. Since $f_{u_ip_i}$ is a soft homeomorphism for all $i \in I$, then

$$f_{u_ip_i}\, f^{-1}_{u_jp_j} :\ f_{u_jp_j}((F_i,\ A_i)\tilde{\cap}(F_j,\ A_j)) \to f_{u_ip_i}((F_i,\ A_i)\tilde{\cap}(F_j,\ A_j))$$

is a soft homeomorphism for all $i, j \in I$ whenever $(F_i,\ A_i)\tilde{\cap}(F_j,\ A_j) \neq \phi$ that is called a soft transition function.

Let $f_{u_ip_i} :\ (F_i,\ A_i) \to (G_i,\ B_i)$, be a soft homeomorphism for each $i \in I$, then

$$f_{u_i,p_i} : (Z_i, \tau_{Z_i}, E)\ \to\ (Y_i, \sigma_{Y_i}, K)$$

is a soft homeomorphism where $Y_i = G_i(B_i),\ \ Z_i = F_i(A_i)$.

Now for each $(H, D) \in \tau_{Z_i\cap\, Z_j}$ and $k \in D$, we have $f_{u_jp_j}\,(H)\,(k) = u_j\,(H(e))$ where $p_j^{-1}(k) = e$. Therefore

$$f_{u_ip_i}\, f^{-1}_{u_jp_j}\left(f_{u_jp_j}(H)(k)\right) = f_{u_ip_i}\, f^{-1}_{u_jp_j}\,(u_j(H(e))) = u_i(H(e))$$

Since $u_i : Z_i \to Y_i$ is a homeomorphism for each $i \in I$, hence for all $q \in H(e)$,

$$u_i\ u_j^{-1}(u_j(q)) = u_i\ u_j^{-1}(x_1^j, x_2^j,\ ...,\ x_n^j) = (x_1^i, x_2^i,\ ...,\ x_n^i) = u_i(q)$$

Proposition 3.6. With the above notations and soft local coordinate neighborhood (F, A, f_{up}), we have $f_{up}\,(A)\,(k) = u\,(A(e))$ where $p^{-1}(k) = \{e\}$.

Proof. Let $(H, D) \in \tau_Z$, then by definition 1.6 we have $f_{pu}(H, D) = (f_{pu}(H), p(D))$.

$$f_{pu}\,(H)\,(k) = \begin{cases} \bigcup_{e\in p^{-1}(k)\cap H}\ u(H(e)) & p^{-1}(k)\cap H \neq \phi \\ \phi & otherwise. \end{cases}$$

for each $k \in D$. Since p is bijective, there is exactly one element of A such that $p^{-1}(k) = \{e\}$. So we have $f_{up}\,(H)\,(k) = u\,(H(e))$. If we set $H = A$ then we have $f_{up}\,(A)\,(k) = u\,(A(e))$ where $p^{-1}(k) = \{e\}$. □

Definition 3.7. A soft topological space (X, τ, E) is called a soft topologica manifold of dimension n if satisfies the following axoims:

i. X is a a soft topological space of dimention n,

ii. X is a T_2-space,

iii. X has a countable soft basis of soft open sets.

Definition 3.8. Let $\mathfrak{B} = \{(F_i,\ A_i,\ f_{u_ip_i}) :\ i \in I\}$ be a countable collection of soft local coordinate neighborhoods such that $\tilde{X} = \tilde{\bigcup}_{i\in I}(F_i,\ A_i)$. Since $f_{u_ip_i}$ is a soft homeomorphism for all $i \in I$, then

$$f_{u_ip_i}\ f^{-1}_{u_jp_j} :\ f_{u_jp_j}((F_i,\ A_i)\tilde{\cap}(F_j,\ A_j)) \to f_{u_ip_i}((F_i,\ A_i)\tilde{\cap}(F_j,\ A_j))$$

is a soft homeomorphism for all $i, j \in I$ whenever $(F_i,\ A_i)\tilde{\cap}(F_j,\ A_j) \neq \phi$ that is called a soft transition function.

Let $f_{u_ip_i} :\ (F_i,\ A_i) \to (G_i,\ B_i)$, be a soft homeomorphism for each $i \in I$, then

$$f_{u_i,p_i} : (Z_i, \tau_{Z_i}, E)\ \to\ (Y_i, \sigma_{Y_i}, K)$$

is a soft homeomorphism where $Y_i = G_i(B_i),\ \ Z_i = F_i(A_i)$.

Now for each $(H, D) \in \tau_{Z_i\cap Z_j}$ and $k \in D$, we have $f_{u_jp_j}(H)(k) = u_j(H(e))$ where $p_j^{-1}(k) = e$. Hence

$$f_{u_ip_i}\ f^{-1}_{u_jp_j}\left(f_{u_jp_j}(H)(k)\right) = f_{u_ip_i}\ f^{-1}_{u_jp_j}\left(u_j(H(e))\right) = u_i(H(e))$$

Since $u_i : Z_i \to Y_i$ is a homeomorphism for each $i \in I$, hence for all $q \in H(e)$,

$$u_i\ u_j^{-1}(u_j(q)) = u_i\ u_j^{-1}(x_1^j, x_2^j,\ ...,\ x_n^j) = (x_1^i, x_2^i,\ ...,\ x_n^i) = u_i(q)$$

4 C^∞ Soft Manifolds

Definition 4.1. With the above notations, we shall say that $((F_i,\ A_i),\ f_{u_ip_i})$ is C^∞-compatible with $((F_j,\ A_j),\ f_{u_jp_j})$ when $(F_i,\ A_i)\tilde{\cap}(F_j,\ A_j) \neq \phi$ if $u_i\ u_j^{-1}$ and $u_j\ u_i^{-1}$ changing the soft coordinates are C^∞ functions or we say that $f_{u_ip_i}\ f^{-1}_{u_jp_j}$ is a soft deffeomorphism of soft open subsets (G_j, B_j) and (G_i, B_i) of $\mathbb{R}^n$.

Definition 4.2. A soft differetiable or C^∞ structure on a soft topological manifold (X, τ, E) is a family $\mathfrak{A} = \{(F_i, A_i, f_{u_ip_i}),\ i \in I\}$ of soft coordinate neighborhoods s.t.

i. $\tilde{X} = \tilde{\bigcup}_{i\in I}(F_i,\ A_i)$;

ii. Each triple $((F_i,\ A_i,\ f_{u_ip_i})$ and $((F_j,\ A_j,\ f_{u_jp_j})$ are C^∞ soft compatible for all $i, j \in I$.

iii. Any soft coordinate neighborhood $((H,\ D,\ h_{up})$ that is fuzzy compatible with every $(F_i,\ A_i,\ f_{u_ip_i}), ; i \in I$, is itself in $\mathfrak{A}$.

A C^∞ soft manifold is a soft topological manifold with a soft C^∞ structure on it.

Example 4.3. Let $E = \{1\}$. Then $\{1\}$ is the single nonempty subset of E. We can consider the soft Euqleadian space $(\mathbb{R}^n, \tau_\epsilon, \{1\})$. Now let $X = \mathcal{M}_{m\times n}(\mathbb{R})$. Since there is a bijection $\psi : X \to \mathbb{R}^{mn}$:

$$\psi(a_{ij}) = (a_{11},\ ...,\ a_{1n};\ ...;\ a_{m1},\ ...,\ a_{mn}),$$

we can define a natural soft topology τ on X as follows:

$$\tau = \{\phi, X\} \cup \{(F, \{1\}) : F(\{1\})\ is\ an\ ordinary\ open\ set\ of\ X\}.$$

Also we can cover $(X, \tau, \{1\})$ by a single soft coodinate neighborhood $(F, \{1\}, f_{up})$ where$F(\{1\}) = X,\ \ u = \psi,\ \ p = id$. Hence $(X, \tau, \{1\})$ is a C^∞ soft manifold of dimention mn.

Definition 4.4. (Soft open submanifolds) Let (X, τ, E) be an C^∞ soft manifold and Z be a soft open subset of X. If $\mathfrak{A} = \{(F_i,\ A_i,\ f_{u_i p_i}),\ i \in I\}$ is a C^∞ structure on X, then $(Z,\ \tau_Z, E)$ is a C^∞ soft manifold with soft differentiable structure consisting of the soft coordinate neighborhoods $(F_i|_{A_i \cap Z},\ A_i \cap Z,\ f_{(u_i|_{F(A_i \cap Z)})p_i})$.

Example 4.5. Since $Z = Gl(n, \mathbb{R})$ is an open subset of $X = \mathcal{M}_{n\times n}(\mathbb{R})$, then $(Z,\ \tau_Z, E)$ is an C^∞ soft submanifold of (X, τ, E).

Example 4.6. Let $X = S^2$, the unit sphere and $E = 1$. As in example 3.3 we have a natural soft topology τ on X. We prove that $(X, \tau, \{1\})$ is a C^∞ soft manifold of dimention 2. Let $I = \{1, 2, 3\}$. We define six soft open subsets covering X, $F_i^\pm : \{1\} \to \mathbb{R}^3,\ i \in I$ by:

$$F_i^\pm(\{1\}) = \{(x_1, x_2, x_3)|\ \|x\| = 1,\ \pm x_i > 1\}.$$

Then we show that all soft sets $(F_i^\pm, \{1\})$ are homeomorphic to the soft open subset $(G, \{1\})$ which $G\{1\} = \{(y_1, y_2)\ |\|y\| < 1\}$, with six soft homeomorphisms $f_{u_i \pm p}$ where $p = id$, $u_i^\pm\ (F_i^\pm(\{1\})) \to G(\{1\}),\ \forall i \in I$ is defined by:

$$u_1^\pm(x_1, x_2, x_3) = (x_2, x_3)\ , \quad (u_1^\pm)^{-1}(y_1, y_2) = (\pm\sqrt{1 - y_1^2 - y_2^2}\ , y_1, y_2)$$

$$u_2^\pm(x_1, x_2, x_3) = (x_1, x_3)\ , \quad (u_2^\pm)^{-1}(y_1, y_2) = (y_1, \pm\sqrt{1 - y_1^2 - y_2^2}, y_2)$$

$$u_3^\pm(x_1, x_2, x_3) = (x_1, x_2)\ , \quad (u_3^\pm)^{-1}(y_1, y_2) = (y_1, y_2,\ \pm\sqrt{1 - y_1^2 - y_2^2})$$

Also it is seen that $u_j^\pm \circ (u_i^\pm)^{-1}$ is infinitly differentiable for all $i,\ j \in I$. For example:

$$u_2^\pm \circ (u_1^\pm)^{-1}(y_1, y_2) = u_2^\pm(\pm\sqrt{1 - y_1^2 - y_2^2}, y_1, y_2) = (\pm\sqrt{1 - y_1^2 - y_2^2}, y_2)$$

Therefor each triple $(F_i^\pm, \{1\}, f_{u_i^\pm p})$ and $(F_j^\pm, \{1\}, f_{u_j^\pm p})$ are C^∞ soft compatible for all $i,\ j \in I$.

Example 4.7. Let $E = \mathbb{R}^n$ and $X = T\mathbb{R}^n = \bigcup_{p\in\mathbb{R}^n} T_p\mathbb{R}^n$. We define the soft topology τ as follows:

$$\tau = \{(F, U) \mid F(U) = \bigcup_{p\in U} T_p\mathbb{R}^n,\ U\ is\ an\ open\ subset\ of\ \mathbb{R}^n\}.$$

We define soft homeomorphism $f_{up} : (F, E) \to (G, E)$, where $F(E) = X,\ G(E) = \mathbb{R}^{2n}$, $p = id$ and $u(v_p) = (x_1,\ ...,\ x_n,\ v_1,\ ...,\ v_n)$, $\forall p = (x_1,\ ...,\ x_n) \in \mathbb{R}^n$, $v = (v_1,\ ...,\ v_n) \in T_p\mathbb{R}^n$. Thus (X, τ, E) is an C^∞ soft manifold of dimension $2n$ with single soft coordinate neighborhood (F, E, f_{up})

Theorem 4.8. Let (M, σ) be an C^∞ topological manifold of dimension n and $E = \mathbb{R}^n$. Let $X = TM = \bigcup_{p\in M} T_pM$ and $\mathfrak{B} = \{(U_i,\ \psi_i,),\ i \in I\}$ be an C^∞ structure on M. We define the soft topology $\tau = \{(F, U) \mid F(U) = \bigcup_{p\in U} T_pM,\ U \in \sigma\}$. Then (X, τ, E) is an C^∞ soft manifold of dimension $2n$.

Proof. One can easily prove that τ is an soft topology on X. If $\psi_i(p) = (x_1^i, x_2^i, \ ..., \ x_n^i)$, $\forall p \in U_i$, then we define soft homeomorphisms:

$$f_{u_i p} : (F_i, U_i) \to (G, U_i), \quad F_i(U_i) = \bigcup_{p \in U_i} T_p M, \quad G(U_i) = U_i \times \mathbb{R}^n, \ p = id$$

$$u_i(v_p) = (x_1^i, \ ..., \ x_n^i, \ v_1^i, \ ..., \ v_n^i), \quad \forall p \in U_i, \quad \forall v_p \in T_p M$$

Note that $v_p = \sum_{k=1}^n v_k^i \ E_{ip}$ where $E_{ip} = \psi_{i*}^{-1}(\dfrac{\partial}{\partial x_k^i})$. Since $\psi_i \circ \psi_j^{-1}$ is C^∞ for all $i, j \in I$, then $u_i u_j^{-1}$ and then $f_{u_i p} \circ f_{u_j p}^{-1}$ are C^∞ for all $i, j \in I$. Therefore we have C^∞ soft structure:

$$\mathfrak{A} = \{(F_i, \ U_i, \ f_{u_i p}), \ i \in I\}.$$

□

5 Conclusion

In the present study, we have continued to study the properties of soft continuous, soft open and soft closed mappings between soft topological spaces.We obtain new characterizations of these mappings and investigate preservation properties. We expect that results in this paper will be basis for further applications of soft manifolds in soft sets theory.

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$(\tilde{T}, S)$ - CUBİC HYPER *KU*-İDEALS

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Abstract – It is known that, the concept of hyper KU–algebras is a generalization of KU–algebras. In this paper, the concepts $(\tilde{T}, S)$ – cubic theory of the (s-weak – strong) hyper KU-ideals in hyper KU–algebras are applied and the relations among them are obtained.

Keywords – *KU–algebra, hyper KU–algebra, cubic hyper KU–ideal.*

1. Introduction

Prabpayak and Leerawat [18,19] introduced a new algebraic structure which is called KU–algebras. They studied ideals and congruences in KU-algebras. Also, they introduced the concept of homomorphism of KU–algebra and investigated some related properties. Moreover, they derived some straightforward consequences of the relations between quotient KU–algebras and isomorphism. Mostafa et al. [15] introduced the notion of fuzzy KU–ideals of KU-algebras and then they investigated several basic properties which are related to fuzzy KU-ideals. The hyper structure theory (called also multi–algebras) is introduced in 1934 by Marty [14] at the 8th congress of Scandinvian Mathematiciens. Around the 40's, several authors worked on hyper groups, especially in France and in the United States, but also in Italy, Russia and Japan. Hyper structures have many applications to several sectors of both pure and applied sciences. Since then numerous mathematical papers [2,3,4,5,6,14,16,17,21] have been written investigating the algebraic properties of the hyper BCK/BCI–KU–algebras. Jun and Xin [6] considered the fuzzification of the notion of a (weak, strong, reflexive) hyper BCK–ideal, and investigated the relations among them. Mostafa et al. [16], applied the hyper structures to KU–algebras and introduced the concept of a hyper KU–algebra which is a generalization of a KU–algebra, and investigated some related properties. They also introduced the notion of a hyper KU–ideal, a weak hyper KU–ideal and gave relations between hyper KU–ideals and weak hyper KU–ideals.

On the other hand, triangular norm is a powerful tool in the theory research and application development of fuzzy sets. On the basis of the definition of the intuitionistic fuzzy groups, Li et al . [13], generalized the operators “ ^” and “ ∨ ” to T-norm and S-norm and defined the intuitionistic fuzzy groups of (T,S)–norms. as a generalization of the notion of fuzzy set Kim [12] Using t–norm T and s–norm S, they introduced the notion of intuitionistic (T,S) – normed fuzzy subalgebra in BCK/BCI-algebra, and some related properties are investigated. Based on the (interval-valued) fuzzy sets, Jun et al. [7-11] introduced the notion of cubic sub–algebras/ideals in BCK/BCI–algebras, and then they investigated several properties. There are several authors who applied the theory of cubic sets to different algebraic structures for instance, Jun et al. [7-11], Yaqoob et al. [20].

In this paper, we studied the idea of $(\tilde{T}, S)$ –cubic set theory to the (s–weak–strong) hyper KU–ideals in hyper KU–algebras and investigated some of related properties .

2. Preliminaries

Let H be a nonempty set and $P^*(H) = P(H) \setminus \{\phi\}$ the family of the nonempty subsets of H. A multi valued operation (said also hyper operation) "$\circ$" on H is a function, which associates with every pair $(x, y) \in H \times H = H^2$ a non empty subset of H denoted $x \circ y$ .An algebraic hyper structure or simply a hyper structure is a non empty set H endowed with one or more hyper operations.

We shall use the $x \circ y$ instead of $x \circ \{y\}$, $\{x\} \circ y$ or $\{x\} \circ \{y\}$.

Definition 2.1 [16] Let H be a nonempty set and "$\circ$" a hyper operation on H, such that $\circ : H \times H \to P^*(H)$. Then H is called a hyper KU-algebra if it contains a constant "0" and satisfies the following axioms: for all $x, y, z \in H$

(HKU_1) $[(y \circ z) \circ (x \circ z)] << x \circ y$
(HKU_2) $x \circ 0 = \{0\}$
(HKU_3) $0 \circ x = \{x\}$
(HKU_4) $if\ \ x << y,\ \ y << x\ implies\ \ \ x = y$.

where x << y is defined by $0 \in y \circ x$ and for every $A, B \subseteq H$, $A << B$ is defined by $\forall a \in A, \exists b \in B\ \ such\ that\ \ a << b$. In such case, we call “<<” the hyper order in H.

Note that if $A, B \subseteq H$, then by $A \circ B$ we mean the subset $\bigcup_{a \in A, b \in B} a \circ b\ \ of\ \ H$.

Example 2.2. [16] Let $H = \{0,1,2,3\}$ be a set. Define hyper operation $\circ$ on H as follows:

$\circ$	0	1	2	3
0	{0}	{1}	{2}	{3}
1	{0}	{0}	{1}	{3}
2	{0}	{0}	{0}	{0,3}
3	{0}	{0}	{1}	{0,3}

Then $(H, \circ, 0)$ is a hyper KU-algebra.

Proposition 2.3. [16] Let H be a hyper KU-algebra. Then for all $x, y, z \in H$, the following statements hold:

(P_1) $A \subseteq B$ implies $A << B$, for all nonempty subsets *A*, *B* of *H*.
(P_2) $0 \circ 0 = \{0\}$.
(P_3) $0 << x$.
(P_4) $z << z$.
(P_5) $x \circ z << z$
(P_6) $A \circ 0 = \{0\}$.
(P_7) $0 \circ A = A$.
(P_8) $(0 \circ 0) \circ x = \{x\}$ and $(x \circ (0 \circ x)) = \{0\}$.
(P_9) $x \circ x = \{x\} \Leftrightarrow x = 0$

Lemma 2.4. [16] In hyper KU-algebra $(X, \circ, 0)$, the following hold:

$$x << y \quad \text{imply} \quad y \circ z << x \circ z \text{ for all } x, y, z \in X$$

Lemma 2.5. [16] In hyper KU-algebra $(X, \circ, 0)$, we have

$$z \circ (y \circ x) = y \circ (z \circ x) \text{ for all } x, y, z \in X .$$

Lemma 2.6. [16] For all $x, y, z \in H$, the following statements hold:

(i) $x \circ y << z \Leftrightarrow z \circ y << x$,
(ii) $0 << A \Rightarrow 0 \in A$,
(iii) $y \in (0 \circ x) \Rightarrow y << x$.

Definition 2.7. [16] Let A be a non-empty subset of a hyper KU-algebra X. Then A is said to be a hyper ideal of X if

(HI_1) $0 \in A$,
(HI_2) $y \circ x << A$ and $y \in A$ imply $x \in A$ for all $x, y \in X$.

Definition 2.8. [16] Let I be a non-empty subset of a hyper KU-algebra H and $0 \in I$. Then,

(1) I is called a weak hyper ideal of H if $y \circ x \subseteq I$ and $y \in I$ imply that $x \in I$, for all $x, y \in H$.
(2) I is called a strong hyper ideal of H if $(y \circ x) \cap I \neq \phi$ and $y \in I$ imply that $x \in I$, for all $x, y \in H$.

Definition 2.9. [16] For a hyper KU-algebras H, a non-empty subset $I \subseteq H$, containing 0 are :

1- A weak hyper KU-ideal of H if $a \circ (b \circ c) \subseteq I$ and $b \in I$ imply $a \circ c \in I$.

2- A hyper KU-ideal of H if $a\circ(b\circ c)<<I$ and $b\in I$ imply $a\circ c\in I$.
3- A strong hyper KU-ideal of H if $(\forall x,y\in H)((a\circ(b\circ c)\bigcap I\neq\phi)$ and $b\in I$ imply $a\circ c\in I$.

Example 2.10. [16] Let $H=\{0,a,b,c\}$ be a set with the following Cayley table

∘	0	a	b	c
0	{0}	{a}	{b}	{c}
a	{0}	{0,a}	{0,b}	{b,c}
b	{0}	{0,b}	{0 }	{a}
c	{0}	{0,b}	{0 }	{0,a}

Then H is a hyper KU-algebra. Take $I=\{0,b\}$, then I is a weak hyper ideal, however, not a weak hyper KU-ideal of H as $b\circ(b\circ c)\subseteq I,\ b\in I,\ but\ b\circ c=a\notin I$.

Example 2.11.[16]. Let $H=\{0,a,b\}$ be a set with the following Cayley table:

∘	**0**	**a**	**b**
0	$\{0\}$	$\{a\}$	$\{b\}$
a	$\{0\}$	$\{0,a\}$	$\{b\}$
b	$\{0\}$	$\{b\}$	$\{0,b\}$

Then H is a hyper KU-algebra. Take $I=\{0,b\}$. Then I is a hyper ideal, but not a hyper KU-ideal, since $0\circ(b\circ a)<<I$ and $b\in I$ but $a\notin I$

Here $I=\{0,b\}$ is also a strong hyper ideal but it is not a strong hyper KU-ideal of H, since $0\circ(b\circ a)=\{b\}\cap I\neq\phi$ *and* $b\in I$ *but* $a\notin I$.

Definition 3.12. [12,13] An interval number is $\tilde{a}=[a_L,a_U]$, where $0\leq a_L\leq a_U\leq 1$. Let D[0, 1] denote the family of all closed subintervals of [0, 1], i.e.,

$$D[0,1]=\{\tilde{a}=[a_L,a_U]:a_L\leq a_U\quad for\ a_L,a_U\in I\}.$$

We define the operations ≤ , ≥ , =,rmin and rmax in case of two elements in D[0, 1]. We consider two elements $\tilde{a}=[a_L,a_U]$ and $\tilde{b}=[b_L,b_U]$ in D[0, 1]. Then

1- $\tilde{a}\leq\tilde{b}$ *iff* $a_L\leq b_L, a_U\leq b_U$;
2- $\tilde{a}\geq\tilde{b}$ *iff* $a_L\geq b_L, a_U\geq b_U$;
3- $\tilde{a}=\tilde{b}$ *iff* $a_L=b_L, a_U=b_U$;
4- $rmim\{\tilde{a},\tilde{b}\}=[\min\{a_L,b_L\},\min\{a_U,b_U\}]$;
5- $r\max\{\tilde{a},\tilde{b}\}=[\max\{a_L,b_L\},\max\{a_U,b_U\}]$

Here we consider that $\tilde{0}=[0,0]$ as least element and $\tilde{1}=[1,1]$ as greatest element. Let $\tilde{a}_i\in D[0,1]$, where $i\in\Lambda$. We define

$$r\inf_{i\in\Lambda} \tilde{a}_i = \left[\inf_{i\in\Lambda}(a_i)_L, \inf_{i\in\Lambda}(a_i)_U \right] \text{ and } r\sup_{i\in\Lambda} \tilde{a}_i = \left[\sup_{i\in\Lambda}(a_i)_L, \sup_{i\in\Lambda}(a_i)_U \right]$$

An interval valued fuzzy set (briefly, i-v-f-set) $\tilde{\mu}$ on a set X is defined as

$$\tilde{\mu} = \left\{ \left\langle x, \left[\mu^L(x), \mu^U(x) \right], x \in X \right\rangle \right\}$$

where $\tilde{\mu} : X \to D[0,1]$ and $\mu^L(x) \le \mu^U(x)$, for all $x \in X$. Jun et al. [7-11], introduced the concept of cubic sets defined on a non-empty set *X* as objects having the form:

$$A = \left\{ \left\langle x, \tilde{\mu}_A(x), \lambda_A(x) \right\rangle : x \in X \right\},$$

which is briefly denoted by $A = \langle \tilde{\mu}_A, \lambda_A \rangle$, where the functions $\tilde{\mu}_A : X \to D[0,1]$ and $\lambda_A : X \to [0,1]$.

Definition 2.12. [12,13] A triangular norm (t-norm) is a function $T : [0,1] \times [0,1] \to [0,1]$ that satisfies following conditions:

(T_1) boundary condition :T(x, 1) = x,
(T_2) commutativity condition: T(x, y) = T(y, x),
(T_3) associativity condition :T(x, T(y, z)) = T(T(x, y), z),
(T_4) monotonicity:T(x, y) ≤ T(x, z) ,whenever y ≤ z for all x, y, z ∈ [0, 1].

A simple example of such defined *t*-norm is a function $T(\alpha, \beta) = \min\{\alpha, \beta\}$. In the general case $T(\alpha, \beta) \le \min\{\alpha, \beta\}$ and $T(\alpha, 0) = 0$ for all $\alpha, \beta \in [0, 1]$.

Definition 2.13 [12,13] A triangular conorm (t-conorm S) is a mapping $S : [0,1] \times [0,1] \to [0,1]$ that satisfies following conditions:

(S1) S(x, 0) = x,
(S2) S(x, y) = S(y, x),
(S3) S(x, S(y, z)) = S(S(x, y), z),
(S4) S(x, y) ≤ S(x, z) , whenever y ≤ z for all x, y, z ∈ [0, 1].

A simple example of such definition s-norm S is a function S(x, y) = max{x, y}. Every S-conorm S has a useful property: max{α, β}≤ S (α, β) for all α, β ∈ [0, 1].

Definition 2.14. [1] An interval valued triangular norm (interval valued t-norm) is a function $\tilde{T} : D[0,1] \times D[0,1] \to D[0,1]$ that satisfies following conditions:

(T_1) interval valued boundary condition : $\tilde{T}(\tilde{x}, \tilde{1}) = \tilde{x}$,

(T_2)) interval valued commutativity condition: $\tilde{T}(\tilde{x}, \tilde{y}) = \tilde{T}(\tilde{y}, \tilde{x})$,

(T_3) interval valued associativity condition : $\tilde{T}(\tilde{x},\tilde{T}(\tilde{y},\tilde{z}))=\tilde{T}(\tilde{T}(\tilde{x},\tilde{y}),\tilde{z})$,
(T_4) interval valued monotonicity: $\tilde{T}(\tilde{x},\tilde{y})\le\tilde{T}(\tilde{y},\tilde{z})$,whenever $\tilde{y}\le\tilde{z}$ for all $\tilde{x},\tilde{y},\tilde{z}\in D[0,1]$.

A simple example of such defined interval valued *t*-norm is a function $\tilde{T}(\tilde{\alpha},\tilde{\beta})=r\min\{\tilde{\alpha},\tilde{\beta}\}$.

In the general case $\tilde{T}(\tilde{\alpha},\tilde{\beta})\le r\min\{\tilde{\alpha},\tilde{\beta}\}$ and $\tilde{T}(\tilde{\alpha},\tilde{0})=\tilde{0},\forall\tilde{\alpha},\tilde{\beta}\in D[0,1]$.

Definition 2.15.3 [1] An interval valued triangular conorm (interval valued t-conorm $\tilde{S}$) is a mapping $\tilde{S}$: $D[0,1]\times D[0,1]\to D[0,1]$ that satisfies following conditions:

(S1) $\tilde{S}(\tilde{x},\tilde{0})=\tilde{x}$,
(S2) $\tilde{S}(\tilde{x},\tilde{y})=\tilde{S}(\tilde{y},\tilde{x})$,
(S3) $\tilde{S}(\tilde{x},\tilde{S}(\tilde{y},\tilde{z}))=\tilde{S}(\tilde{S}(\tilde{x},\tilde{y}),\tilde{z})$
(S4) interval valued monotonicity : $\tilde{S}(\tilde{x},\tilde{y})\le\tilde{S}(\tilde{y},\tilde{z})$,whenever $\tilde{y}\le\tilde{z}$ for all $\tilde{x},\tilde{y},\tilde{z}\in D[0,1]$.

A simple example of such definition interval valued s-norm S is a function $\tilde{S}(\tilde{\alpha},\tilde{\beta})=r\max\{\tilde{\alpha},\tilde{\beta}\}$.

In the general case $\tilde{S}(\tilde{\alpha},\tilde{\beta})\le r\max\{\tilde{\alpha},\tilde{\beta}\}$ and $\tilde{S}(\tilde{\alpha},\tilde{1})=\tilde{1},\forall\tilde{\alpha},\tilde{\beta}\in D[0,1]$

Example.2.16. Here the set H = {0, 1, 2, 3,4} in which $\circ$ is defined by the following table

$\circ$	0	1	2	3	4
0	{0}	{1}	{2}	{3}	{4}
1	{0}	{0}	{2}	{3}	{4}
2	{0}	{1}	{0}	{3}	{3}
3	{0}	{0}	{2}	{0}	{2}
4	{0}	{0}	{0}	{0}	{0}

Then $(H,*,0)$ is a hyper KU-algebra . The function $\tilde{T}_m$ defined by

$$\tilde{T}_m(\tilde{\alpha},\tilde{\beta})=r\max\{\tilde{\alpha}+\tilde{\beta}-\tilde{1},\tilde{0}\},\forall\tilde{\alpha},\tilde{\beta}\in D[0,1]$$

By routine calculations, we known that a fuzzy set $\tilde{\mu}$ in H defined by $\tilde{\mu}$ (1) =[0.3,0.5] and $\tilde{\mu}$(0) = $\tilde{\mu}$ (2) = $\tilde{\mu}$ (3) = $\tilde{\mu}$ (4) = [0.3 0.9] is interval valued $\tilde{T}_m$ -fuzzy KU-ideal of H, which is interval valued $\tilde{T}_m$ -fuzzy KU-ideal because $\tilde{\mu}_A(x)\ge\tilde{T}\{\tilde{\mu}_A(z*(y*x)),\tilde{\mu}_A(y*z)\}$.

3. $(\tilde{T}, S)$ - Cubic hyper *KU*–ideals

Now some fuzzy logic concepts are reviewed .A fuzzy set μ in a set H is a function $\mu : H \to [0,1]$. A fuzzy set μ in a set H is said to satisfy the inf (resp. sup) property if for any subset T of H there exists $x_0 \in T$ such that $\mu(x_0) = \inf_{x \in T} \mu(x)$ (resp. $\mu(x_0) = \sup_{x \in T} \mu(x)$). For a fuzzy set μ in X and a ∈ [0, 1] the set U(μ ; a):={x ∈ H, μ (x) ≥ a}, which is called a level set of μ.

Definition 3.1. A fuzzy set μ in H is said to be a fuzzy hyper KU-subalgebra of H if it satisfies the inequality:

$$\inf_{z \in x \circ y} \mu(z) \geq \min\{\mu(x), \mu(y)\} \ \forall x, y \in H .$$

Proposition 3.2. Let μ be a fuzzy hyper KU-sub-algebra of H. Then $\mu(0) \geq \mu(x)$ for all x ∈ H.

Proof. Using Proposition **2.3** (P_9), we see that 0 ∈ x ∘ x for all x ∈ H. Hence

$$\inf_{0 \in x \circ x} \mu(0) \geq \min\{\mu(x), \mu(x)\} = \mu(x) \text{ for all } x \in H .$$

Example 3.3. Let $H = \{0, a, b\}$ be a set. Define hyper operation ∘ on H as follows:

∘	0	a	b
0	$\{0\}$	$\{a\}$	$\{b\}$
a	$\{0\}$	$\{0,a\}$	$\{a,b\}$
b	$\{0\}$	$\{0,a\}$	$\{0,a,b\}$

Then $(H, \circ, 0)$ is a hyper KU-algebra. Define a fuzzy set μ : $H \to [0, 1]$ by μ(0) = μ(a)=α_1>α_2= μ(b). Then μ is a fuzzy hyper sub-algebra of H .

A fuzzy set ν : $H \to [0, 1]$ defined by ν(0) = 0.7, ν(a)=0.5 and ν(b)=0.2 is also a fuzzy Hyper sub-algebra of H.

Definition3.4[17] Let H be nonempty set .A cubic set A in H is Structure $A = \{\langle x, \tilde{\mu}_A(x), \lambda_A(x) \rangle, x \in X\}$ which is briefly denoted by $A = \langle \tilde{\mu}_A(x), \lambda_A(x) \rangle$, where $\tilde{\mu}_A(x) = [\mu_A^L, \mu_A^U]$ is an interval value fuzzy set in H and λ_A is an fuzzy set in X.

Definition3.5 . A cubic set $A = \langle \tilde{\mu}_A(x), \lambda_A(x) \rangle$ in H is called a $(\tilde{T}, S)$ -cubic hyper ideal of H, if it satisfies the following conditions:

K_1 : $x << y$ *implies* $\tilde{\mu}(x) \geq \tilde{\mu}(y)$ and $\lambda_A(x) \leq \lambda_A(y)$

$K_2 : \tilde{\mu}(z) \geq \tilde{T}\left\{ \inf_{u\in((y\circ z)} \tilde{\mu}(u), \tilde{\mu}(y) \right\}, \quad \lambda_A(x) \leq S\left\{ \sup_{a\in(y\circ x)} \mu(a), \mu(y) \right\}$

Definition3.6. For a hyper KU-algebra H , a cubic $A = \langle \tilde{\mu}_A(x), \lambda_A(x) \rangle$" in H is called:

(I) $(\tilde{T}, S)$ -Cubic hyper KU-ideal of H , if

$K_1 :\ x << y\ \ implies\ \ \tilde{\mu}(x) \geq \tilde{\mu}(y), \lambda_A(x) \leq \lambda_A(y)$

$$\tilde{\mu}_A(x\circ z) \geq \tilde{T}\left\{ \inf_{u\in(x\circ(y\circ z)} \tilde{\mu}_A(u)\ ,\ \tilde{\mu}_A(y) \right\} \text{and } \lambda_A(x\circ z) \leq S\left\{ \sup_{a\in x\circ(y\circ z)} \mu(a), \mu(y) \right\}$$

(II) $(\tilde{T}, S)$ - Cubic weak hyper KU-ideal of H" if, for any x; y; z $\in$ H

$$\tilde{\mu}_A(0) \geq \tilde{\mu}_A(x\circ z) \geq \tilde{T}\left\{ \inf_{u\in(x\circ(y\circ z)} \tilde{\mu}_A(u), \tilde{\mu}_A(y) \right\}$$
$$\lambda_A(0) \leq \lambda_A(x\circ z) \leq S\left\{ \sup_{a\in x\circ(y\circ z)} \lambda_A(a), \lambda_A(y) \right\}$$

(III) $(\tilde{T}, S)$ -Cubic hyper KU-ideal of H" if, x << y implies

$K_1 :\ x << y\ \ implies\ \ \tilde{\mu}(x) \geq \tilde{\mu}(y)$ and $\lambda_A(x) \leq \lambda_A(y)$ for any x; y; z $\in$ H

$$\tilde{\mu}_A(x\circ z) \geq \tilde{T}\left\{ \inf_{u\in(x\circ(y\circ z))} \tilde{\mu}_A(u), \tilde{\mu}_A(y) \right\}, \ \lambda_A(0) \leq \lambda_A(x\circ z) \leq S\left\{ \sup_{a\in x\circ(y\circ z)} \mu(a), \mu(y) \right\}$$

(IV) $(\tilde{T}, S)$ -Cubic strong hyper KU-ideal of H" if, for any x; y; z $\in$ H

$$\inf_{u\in x\circ(y\circ z)} \tilde{\mu}_A(u) \geq \tilde{\mu}_A(z) \geq \tilde{T}\left\{ \inf_{u\in x\circ(y\circ z)} \tilde{\mu}_A(u), \tilde{\mu}_A(y) \right\},$$

$$\sup_{u\in x\circ(y\circ z)} \lambda_A(u) \leq \lambda_A(z) \leq S\left\{ \sup_{u\in x\circ(y\circ z)} \lambda_A(u), \lambda_A(y) \right\}.$$

Ending the proof.

Example 3.7. Let $H = \{0,1,2,3\}$ be a set. The hyper operations $\circ$ on H are defined as follows.

$\circ_1$	0	1	2	3
0	{0}	{1}	{2}	{3}
1	{0}	{0}	{1}	{3}
2	{0}	{0}	{0}	{0,3}
3	{0}	{0}	{1}	{0,3}

Then $(H,\circ,0)$ is hyper KU-algebras.Define $\tilde{\mu}_A(x)$, as follows:

$$\tilde{\mu}_A(x) = \begin{cases} [0.2,0.9] & if\ x = \{0,1\} \\ [0.1,0.4] & otherwise \end{cases}$$

H	0	1	2	3
$\lambda_A(x)$	0.2	0.2	0.6	0.7

The function $\tilde{T}_m$ defined by $\tilde{T}_m(\tilde{\alpha},\tilde{\beta}) = r\max\{\tilde{\alpha}+\tilde{\beta}-\tilde{1},\tilde{0}\}\forall\tilde{\alpha},\tilde{\beta}\in D[0,1]$, $S_m : [0,1]\times[0,1]\to[0,1]$ be a function defined by $S_m(\alpha,\beta) = \min\{1-(\alpha+\beta),1\}$. It is easy to check that

$$\tilde{\mu}_A(x\circ z) \geq \tilde{T}\left\{ \inf_{u\in(x\circ(y\circ z))} \tilde{\mu}_A(u)\ ,\ \tilde{\mu}_A(y) \right\} \text{ and } \lambda_A(x\circ z) \leq S\left\{ \sup_{a\in x\circ(y\circ z)} \mu(a), \mu(y) \right\}$$

Then $A = \langle \tilde{\mu}_A(x), \lambda_A(x) \rangle$ is $(\tilde{T}, S)$ -cubic hyper KU-ideal of H.

Theorem 3.8. Any $(\tilde{T}, S)$ - cubic (weak, strong) hyper KU-ideal is $(\tilde{T}, S)$ -cubic (weak, strong) hyper ideal.

Proof. Let $A = \langle \tilde{\mu}_A(x), \lambda_A(x) \rangle$ be $(\tilde{T}, S)$ - cubic hyper KU-ideal of H, we get for any

x; y; z $\in H$, $\tilde{\mu}_A(x\circ z) \geq \tilde{T}\left\{ \inf_{u\in x\circ(y\circ z)} \tilde{\mu}_A(u), \tilde{\mu}_A(y) \right\}$ Put x = 0 , we get

$$\tilde{\mu}_A(0\circ z) \geq \tilde{T}\left\{ \inf_{u\in 0\circ(y\circ z)} \tilde{\mu}_A(u)\ ,\ \tilde{\mu}_A(y) \right\}$$

which gives,

$$\tilde{\mu}_A(z) \geq \tilde{T}\left\{ \inf_{u\in(y\circ z)} \tilde{\mu}_A(u)\ ,\ \tilde{\mu}_A(y) \right\} \text{and } \lambda_A(x\circ z) \leq S\left\{ \sup_{u\in x\circ(y\circ z)} \lambda_A(u), \lambda_A(y) \right\}$$

Take x = 0 , we get $\lambda_A(0\circ z) \leq S\left\{ \sup_{u\in 0\circ(y\circ z)} \lambda_A(u)\ ,\ \lambda_A(y) \right\}$, which gives,

$\lambda_A(z) \leq S\left\{ \sup_{u\in(y\circ z)} \lambda_A(u)\ ,\ \lambda_A(y) \right\}$. Ending the proof.

Definition 3.9. A cubic set $A = \langle \tilde{\mu}_A(x), \lambda_A(x) \rangle$ in H is called a $(\tilde{T}, S)$ - cubic s-weak hyper KU-ideal of H if

(i) $\tilde{\mu}_A(0) \geq \tilde{\mu}_A(x)$, $\lambda_A(0) \leq \lambda_a(x)$ $\forall x\in H$,
(ii) for every $x, y, z\in H$ there exists $a\in x\circ(y\circ z)$ such that $\tilde{\mu}_A(x\circ z) \geq \tilde{T}\{\tilde{\mu}_A(a), \tilde{\mu}_A(y)\}$.

(iii) $\lambda_A(x \circ z) \le S\left\{ \sup_{a \in x\circ(y\circ z)} \lambda_A(a), \lambda_A(y) \right\}$.

Proposition 3.10. Let $A = \langle \tilde{\mu}_A(x), \lambda_A(x) \rangle$ be a $(\tilde{T}, S)$-cubic weak hyper KU-ideal of *H*. If $A = \langle \tilde{\mu}_A(x), \lambda_A(x) \rangle$ satisfies the inf-sup property, then $A = \langle \tilde{\mu}_A(x), \lambda_A(x) \rangle$ is a $(\tilde{T}, S)$-cubic fuzzy s-weak hyper KU -ideal of *H*.

Proof. Since $A = \langle \tilde{\mu}_A(x), \lambda_A(x) \rangle$ satisfies the inf property, there exists $a_0 \in x\circ(y\circ z)$, such that $\tilde{\mu}_A(a_0) = \inf_{a_0 \in x\circ(y\circ z)} \tilde{\mu}_A(a_0)$. It follows that

$$\tilde{\mu}_A(x \circ z) \ge \tilde{T}\left\{ \inf_{a \in x\circ(y\circ z)} \tilde{\mu}_A(a) \ , \ \tilde{\mu}_A(y) \right\}$$

And since $A = \langle \tilde{\mu}_A(x), \lambda_A(x) \rangle$ satisfies the sup property, there exists $b_0 \in x\circ(y\circ z)$, such that $\lambda_A(b_0) = \sup_{b_0 \in x\circ(y\circ z)} \lambda_A(a_0)$ It follows that

$$\lambda_A(x \circ z) \le S\left\{ \sup_{b \in x\circ(y\circ z)} \lambda_A(b) \ , \ \lambda_A(y) \right\}.$$

Ending the proof.

Note that, in a finite hyper KU-algebra, every $(\tilde{T}, S)$ - cubic set satisfies (inf -sup) property. Hence the concept of $(\tilde{T}, S)$-cubic weak hyper KU -ideals and $(\tilde{T}, S)$-cubic s-weak hyper KU-ideals coincide in a finite hyper KU -algebra.

Proposition 3.11. Let $A = \langle \tilde{\mu}_A(x), \lambda_A(x) \rangle$ be a $(\tilde{T}, S)$-cubic strong hyper KU-ideal of *H* and let x; y; z ∈ *H*. Then

(i) $\tilde{\mu}_A(0) \ge \tilde{\mu}_A(x)$, $\lambda_A(0) \le \lambda_A(x)$
(ii) x << y implies $\tilde{\mu}_A(x) \ge \tilde{\mu}_A(y)$.
(iii) $\tilde{\mu}_A(x \circ z) \ge \tilde{T}\{\tilde{\mu}_A(a), \tilde{\mu}_A(y)\}, \forall a \in x\circ(y\circ z)$
(v) x << y implies $\lambda_A(x) \le \lambda_A(y)$
(iv) $\lambda_A(x \circ z) \le S\left\{ \sup_{b \in x\circ(y\circ z)} \lambda_A(b) \ , \ \lambda_A(y) \right\}$

Proof. (i) Since $0 \in x \circ x \ \forall x \in H$,we have $\mu(0) \ge \inf_{a \in x\circ x} \mu(a) \ge \mu(x)$, $\lambda(0) \le \sup_{b \in x\circ x} \lambda(b) \le \lambda(x)$, which proves (i).

(ii) Let x; y ∈ *H* be such that x << y. Then $0 \in y \circ x \ \forall x, y \in H$ and so $\inf_{b \in (y\circ x)} \tilde{\mu}_A(b) \le \tilde{\mu}_A(0)$, it follows from (i) that ,

$$\tilde{\mu}_A(x) \geq \tilde{T}\left\{\inf_{a\in(y\circ x)} \tilde{\mu}_A(a), \tilde{\mu}_A(y)\right\} \geq \tilde{T}\{\tilde{\mu}_A(0), \tilde{\mu}_A(y)\} = \tilde{\mu}_A(y),$$

$$\lambda_A(x) \leq S\left\{\sup_{b\in(y\circ x)} \lambda_A(b), \lambda_A(y)\right\} \leq S\{\lambda_A(0), \lambda_A(y)\} = \lambda_A(y).$$

(iii) $\tilde{\mu}_A(x\circ z) \geq \tilde{T}\left\{\inf_{a\in x\circ(y\circ z)} \tilde{\mu}_A(a), \tilde{\mu}_A(y)\right\} \geq \tilde{T}\{\tilde{\mu}_A(a), \tilde{\mu}_A(y)\}, \forall a\in x\circ(y\circ z),$

$$\lambda(x\circ z) \leq S\left\{\sup_{b\in x\circ(y\circ z)} \lambda(b), \lambda(y)\right\} \leq S\{\mu(b), \mu(y)\}, \forall b\in x\circ(y\circ z)$$

we conclude that (iii), (v), (iv) are true. Ending the proof.

Corollary 3.12. Every $(\tilde{T}, S)$-cubic strong hyper KU-ideal is both a $(\tilde{T}, S)$-cubic s-weak hyper KU-ideal and a $(\tilde{T}, S)$-cubic hyper KU -ideal.

Proof. Straight forward.

Proposition 3.13. Let $A = \langle \tilde{\mu}_A(x), \lambda_A(x) \rangle$ be $(\tilde{T}, S)$- cubic hyper KU -ideal of H and let $x, y, z \in H$. Then, (i) $\tilde{\mu}_A(0) \geq \tilde{\mu}_A(x)$, $\lambda_A(0) \leq \lambda_A(x)$

(ii) if $A = \langle \tilde{\mu}_A(x), \lambda_A(x) \rangle$ satisfies the inf -sup property, then

$$\tilde{\mu}_A(x\circ z) \geq \tilde{T}\{\tilde{\mu}_A(a), \tilde{\mu}_A(y)\}, \forall a\in x\circ(y\circ z), \lambda_A(x\circ z) \leq S\left\{\sup_{b\in x\circ(y\circ z)} \lambda_A(b), \lambda_A(y)\right\}$$

Proof. (i) Since 0 << x for each $x \in H$; we have $\tilde{\mu}_A(0) \geq \tilde{\mu}_A(x)$, $\lambda_A(0) \leq \lambda_A(x)$ by Definition 3.6(I) and hence (i) holds.

(ii) Since $A = \langle \tilde{\mu}_A(x), \lambda_A(x) \rangle$ satisfies the inf property, there is $a_0 \in x\circ(y\circ z)$,such that $\tilde{\mu}(a_0) = \inf_{a\in x\circ(y\circ z)} \tilde{\mu}_A(a)$ .Hence $\tilde{\mu}_A(x\circ z) \geq \tilde{T}\left\{\inf_{a\in x\circ(y\circ z)} \tilde{\mu}_A(a), \tilde{\mu}_A(y)\right\} = \tilde{T}\{\tilde{\mu}_A(a_0), \tilde{\mu}_A(y)\}$
Since $A = \langle \tilde{\mu}_A(x), \lambda_A(x) \rangle$ satisfies the sup- property, there is $b_0 \in x\circ(y\circ z)$,such that $\lambda(b_0) = \sup_{b\in x\circ(y\circ z)} \lambda_A(b)$,Hence $\lambda_A(x\circ z) \leq S\left\{\sup_{b\in x\circ(y\circ z)} \lambda_A(b), \lambda_A(y)\right\} = S\{\lambda_A(b_0), \lambda_A(y)\}$

which implies that (ii) is true. The proof is complete.

Corollary 3.13. (i) Every $(\tilde{T}, S)$ - cubic hyper KU -ideal of *H* is a $(\tilde{T}, S)$ -cubic weak hyper KU –ideal of *H*.
(ii) If $A = \langle \tilde{\mu}_A(x), \lambda_A(x) \rangle$ is a $(\tilde{T}, S)$ -cubic hyper KU -ideal of H satisfying inf-sup property, then $A = \langle \tilde{\mu}_A(x), \lambda_A(x) \rangle$ is a $(\tilde{T}, S)$ -cubic s-weak Hyper KU -ideal of *H*.

Proof. Straightforward.

Theorem 3.15 . If $A=\langle\tilde{\mu}_A(x),\lambda_A(x)\rangle$ is a $(\tilde{T},S)$ -cubic strong hyper KU-ideal of H , then the set $\mu_{t,s}=\{x\in H,\tilde{\mu}_A(x)\geq\tilde{t},\lambda_A(x)\leq s\}$ is a $(\tilde{T},S)$ -strong hyper KU-ideal of H ,when $\mu_{t,s}\neq\Phi,\ for\ \tilde{t}\in D[0,1],s\in[0,1]$.

Proof. Let $A=\langle\tilde{\mu}_A(x),\lambda_A(x)\rangle$ be a $(\tilde{T},S)$ -cubic strong hyper KU-ideal of H and $\mu_{t,s}\neq\Phi,\ for\ \tilde{t}\in D[0,1],s\in[0,1]$ .Then there $a\in\mu_{t,s}$ and so $\tilde{\mu}_A(a)\geq\tilde{t},\lambda_A(a)\leq s$. By Proposition 3.13 (i), $\tilde{\mu}_A(0)\geq\tilde{\mu}_A(a)\geq\tilde{t},\lambda(0)\geq\lambda(a)\leq s$ and so $0\in\mu_{t,s}$. Let $x,y,z\in H$ such that $x\circ(y\circ z)\cap\mu_{t,s}\neq\Phi\ and\ y\in\mu_{t,s}$, Then there exist $a_0\in x\circ(y\circ z)\cap\mu_{t,s}\ and\ hence\ \tilde{\mu}_A(a_0)\geq\tilde{t},\lambda_A(a_0)\leq s$. By Definition 3.6 (iv), we have $\tilde{\mu}_A(x\circ z)\geq\tilde{T}\{\inf_{a\in x\circ(y\circ z)}\tilde{\mu}_A(a),\tilde{\mu}_A(y)\}\geq\tilde{T}\{\tilde{\mu}_A(a),\tilde{\mu}_A(y)\}\geq\tilde{T}\{\tilde{t},\tilde{t}\}=\tilde{t}$, and

$$\lambda_A(x\circ z)\leq S\left\{\sup_{a\in x\circ(y\circ z)}\lambda_A(a)\ ,\ \lambda_A(y)\right\}=S\{\ \lambda_A(a_0)\ ,\ \lambda_A(y)\ \}=S\{\ s\ ,\ s)\ \}=s\,.$$

So $(x\circ z)\in\mu_{t,s}$. It follows that $\mu_{t,s}$ is a strong hyper KU -ideal of H .

Ending the proof.

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Conflicts of Interest

State any potential conflicts of interest here or “The author declare no conflict of interest”.

Conclusion

In this paper concepts $(\tilde{T},S)$ -cubic theory of the (s-weak – strong) hyper KU-ideals in hyper KU-algebras are applied and the relations among them are obtained .Example is given in support of the definition of $(\tilde{T},S)$ -cubic fuzzy ideals. Some theorems are proved. And also introduced $(\tilde{T},S)$ -cubic fuzzy ideal extension. The main purpose of our future work is to investigate the following:

- Folding theory applied to some types of $(\tilde{T},S)$ -cubic positive implicative hyper KU-ideals in hyper KU-algebras

- Homomorphism and quotient of $(\tilde{T},S)$ -cubic fuzzy KU-hyper-ideals.
- On implicative $(\tilde{T},S)$ -cubic hyper ku-ideals of hyper KU-algebras.
- Intuitionistic fuzziness of $(\tilde{T},S)$ -cubic strong hyperKU-ideals.
- Filter theory on $(\tilde{T},S)$ -cubic hyper KU-algebras.

- On $(\tilde{T}, S)$-cubic intuitionistic Fuzzy Implicative Hyper KU-Ideals of Hyper KU-algebras.
- On $(\tilde{T}, S)$-cubic intuitionistic fuzzy commutative hyper KU-ideals.
- On $(\tilde{T}, S)$- interval-valued intuitionistic fuzzy Hyper KU-ideals of hyper KU-algebras.
- (T, S) Bipolar fuzzy implicative hyper KU-ideals in hyper KU-algebras.

Algorithm for hyper KU-algebras

Input (X : set, $\circ$ hyper operation)
Output (" X is a hyper KU-algebra or not")
Begin
If $X = \phi$ then go to (1.);
End If
If $0 \notin X$ then go to (1.);
End If
Stop: =false;
$i := 1$;
While $i \leq |X|$ and not (Stop) do
If $0 \notin x_i \circ x_i$ then
Stop: = true;
End If
$j := 1$
While $j \leq |X|$ and not (Stop) do
If $0 \notin x_i \circ (y_j \circ x_i)$ or $0 \in x_i \circ y_j$ *and* $0 \in (y_j \circ x_i)$ and $x_i \neq y_j$,then
Stop: = true;
End If
End If
$k := 1$
While $k \leq |X|$ and not (Stop) do
If $0 \notin (x_i * y_j) \circ ((y_j * z_k) \circ (x_i * z_k))$ then
Stop: = true;
  End If
 End While
 End While
End While
If Stop then
(1.) Output (" X is not hyper KU-algebra")
Else
 Output (" X is hyper KU-algebra")
  End If
End.

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ON NANO $\alpha^{\star}$-SETS AND NANO $\mathcal{R}_{\alpha^{\star}}$-SETS

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Abstaract − In this paper, nano $\alpha^{\star}$-set and nano $\mathcal{R}_{\alpha^{\star}}$-set are introduced and investigated.

Keywords − *nano α-open sets, nano $\alpha^{\star}$-set, nano $\mathcal{R}_{\alpha^{\star}}$-set.*

1 Introduction

Lellis Thivagar et al [2] introduced a nano topological space with respect to a subset X of an universe which is defined in terms of lower approximation and upper approximation and boundary region. The classical nano topological space is based on an equivalence relation on a set, but in some situation, equivalence relations are nor suitable for coping with granularity, instead the classical nano topology is extend to general binary relation based covering nano topological space.

In this paper, nano $\alpha^{\star}$-set and nano $\mathcal{R}_{\alpha^{\star}}$-set in nano topological spaces and investigate some of their properties.

2 Preliminaries

Throughout this paper $(U, \tau_R(X))$ (or X) represent nano topological spaces on which no separation axioms are assumed unless otherwise mentioned. For a subset H of a space $(U, \tau_R(X))$, $Ncl(H)$ and $Nint(H)$ denote the nano closure of H and the nano interior of H respectively. We recall the following definitions which are useful in the sequel.

Definition 2.1. *[3] Let U be a non-empty finite set of objects called the universe and R be an equivalence relation on U named as the indiscernibility relation. Elements belonging to the same equivalence class are said to be indiscernible with one another. The pair (U, R) is said to be the approximation space. Let $X \subseteq U$.*

1. *The lower approximation of X with respect to R is the set of all objects, which can be for certain classified as X with respect to R and it is denoted by* $L_R(X)$. *That is,* $L_R(X) = \bigcup_{x \in U}\{R(x) : R(x) \subseteq X\}$, *where R(x) denotes the equivalence class determined by x.*

2. *The upper approximation of X with respect to R is the set of all objects, which can be possibly classified as X with respect to R and it is denoted by* $U_R(X)$. *That is,* $U_R(X) = \bigcup_{x \in U}\{R(x) : R(x) \cap X \neq \phi\}$.

3. *The boundary region of X with respect to R is the set of all objects, which can be classified neither as X nor as not - X with respect to R and it is denoted by* $B_R(X)$. *That is,* $B_R(X) = U_R(X) - L_R(X)$.

Property 2.2. *[2] If* (U, R) *is an approximation space and* $X, Y \subseteq U$; *then*

1. $L_R(X) \subseteq X \subseteq U_R(X)$;
2. $L_R(\phi) = U_R(\phi) = \phi$ *and* $L_R(U) = U_R(U) = U$;
3. $U_R(X \cup Y) = U_R(X) \cup U_R(Y)$;
4. $U_R(X \cap Y) \subseteq U_R(X) \cap U_R(Y)$;
5. $L_R(X \cup Y) \supseteq L_R(X) \cup L_R(Y)$;
6. $L_R(X \cap Y) \subseteq L_R(X) \cap L_R(Y)$;
7. $L_R(X) \subseteq L_R(Y)$ *and* $U_R(X) \subseteq U_R(Y)$ *whenever* $X \subseteq Y$;
8. $U_R(X^c) = [L_R(X)]^c$ *and* $L_R(X^c) = [U_R(X)]^c$;
9. $U_R U_R(X) = L_R U_R(X) = U_R(X)$;
10. $L_R L_R(X) = U_R L_R(X) = L_R(X)$.

Definition 2.3. *[2] Let U be the universe, R be an equivalence relation on U and* $\tau_R(X) = \{U, \phi, L_R(X), U_R(X), B_R(X)\}$ *where* $X \subseteq U$. *Then by the Property 2.2, R(X) satisfies the following axioms:*

1. *U and* $\phi \in \tau_R(X)$,
2. *The union of the elements of any sub collection of* $\tau_R(X)$ *is in* $\tau_R(X)$,
3. *The intersection of the elements of any finite subcollection of* $\tau_R(X)$ *is in* $\tau_R(X)$.

That is, $\tau_R(X)$ *is a topology on U called the nano topology on U with respect to X. We call* $(U, \tau_R(X))$ *as the nano topological space. The elements of* $\tau_R(X)$ *are called as nano open sets and* $[\tau_R(X)]^c$ *is called as the dual nano topology of* $[\tau_R(X)]$.

Remark 2.4. *[2] If* $[\tau_R(X)]$ *is the nano topology on U with respect to X, then the set* $B = \{U, \phi, L_R(X), B_R(X)\}$ *is the basis for* $\tau_R(X)$.

Definition 2.5. *[2] If $(U, \tau_R(X))$ is a nano topological space with respect to X and if $H \subseteq U$, then the nano interior of H is defined as the union of all nano open subsets of H and it is denoted by $Nint(H)$.*

That is, $Nint(H)$ is the largest nano open subset of H. The nano closure of H is defined as the intersection of all nano closed sets containing H and it is denoted by $Ncl(H)$.

That is, $Ncl(H)$ is the smallest nano closed set containing H.

Definition 2.6. *[2] A subset H of a nano topological space $(U, \tau_R(X))$ is called*

1. *nano semi-open if $H \subseteq Ncl(Nint(H))$.*
2. *nano regular-open if $H = Nint(Ncl(H))$.*
3. *nano α-open if $H \subseteq Nint(Ncl(Nint(H)))$.*
4. *nano semi pre-open if $H \subseteq Ncl(Nint(Ncl(H)))$.*

The complements of the above mentioned sets is called their respective closed sets.

Definition 2.7. *[1] A subset H of a space $(U, \tau_R(X))$ is called nano t-set if*

$$Nint(H) = Nint(Ncl(H))$$

3 Nano $\alpha^\star$-sets and Nano $\mathcal{R}_{\alpha^\star}$-sets

Definition 3.1. *A subset H of a space $(U, \tau_R(X))$ is called;*

1. *an nano $\alpha^\star$-set if $Nint(Ncl(Nint(H))) = Nint(H)$.*
2. *nano $\mathcal{R}_{\alpha^\star}$-set if $H = P \cap Q$, where P is nano open and Q is nano $\alpha^\star$-set.*

The family of all nano $\alpha^\star$-sets (resp. nano $\mathcal{R}_{\alpha^\star}$-sets) of a space $(U, \tau_R(X))$ will be denoted by $N\alpha^\star(U, \tau_R(X))$ (resp. $N\mathcal{R}_{\alpha^\star}(U, \tau_R(X))$).

Example 3.2. *Let $U = \{1, 2, 3\}$ with $U/R = \{\{1\}, \{2, 3\}\}$ and $X = \{1\}$. Then the nano topology $\tau_R(X) = \{\phi, \{1\}, U\}$.*

1. *then $\{2\}$ is nano $\alpha^\star$-set.*
2. *then $\{1\}$ is nano $\mathcal{R}_{\alpha^\star}$-set.*

Proposition 3.3. *For a subset H of a space $(U, \tau_R(X))$, the following are equivalent:*

1. *$H \in N\alpha^\star(U, \tau_R(X))$.*
2. *H is nano semi-pre closed.*
3. *$Nint(H)$ is nano regular open.*

Proof. The proof is obvious.

Proposition 3.4. *In a space $(U, \tau_R(X))$.*

1. *If H is a nano t-set, then $H \in N\alpha^\star(U, \tau_R(X))$.*
2. *nano semi-open set H is a nano t-set $\Longleftrightarrow$ $H \in N\alpha^\star(U, \tau_R(X))$.*
3. *H is an nano α-open set and $H \in N\alpha^\star(U, \tau_R(X))$ $\Longleftrightarrow$ H is nano regular-open.*

Proof. (1) Let H be a nano t-set, then

$$Nint(H) = Nint(Ncl(H))$$

and

$$Nint(Ncl(Nint(H))) = Nint(Ncl(H)) = Nint(H)$$

Therefor H is an nano $\alpha^\star$-set.

(2). Let H be nano semi-open and $H \in N\alpha^\star(U, \tau_R(X))$. Since H is nano semi-open,

$$Ncl(Nint(H)) = Ncl(H)$$

and hence

$$Nint(Ncl(H)) = Nint(Ncl(Nint(H))) = Nint(H)$$

Therefore, H is a nano t-set.

(3). Let H be an nano α-open set and $H \in N\alpha^\star(U, \tau_R(X))$. By Proposition 3.3 and the Definition of an nano α-open set, we have

$$Nint(Ncl(Nint(H))) = H$$

and hence

$$Nint(Ncl(H)) = Nint(Ncl(Nint(H))) = H$$

The converse is obvious.

Remark 3.5. *In a space $(U, \tau_R(X))$, the union of two nano $\alpha^\star$-sets but not nano $\alpha^\star$-set.*

Example 3.6. *Let $U = \{a, b, c, d\}$ with $U/R = \{\{a\}, \{c\}, \{b, d\}\}$ and $X = \{a, b\}$. Then the nano topology $\tau_R(X) = \{\phi, \{a\}, \{b, d\}, \{a, b, d\}, U\}$. Then $H = \{a, b\}$ and $Q = \{d\}$ is nano $\alpha^\star$-sets. Clearly $H \cup Q = \{a, b, d\}$ is but not nano $\alpha^\star$-set.*

Remark 3.7. *In a space $(U, \tau_R(X))$, the union of two nano $\alpha^\star$-sets is nano $\alpha^\star$-set.*

Example 3.8. *In Example 3.6, then $H = \{a\}$ and $Q = \{b\}$ is nano $\alpha^\star$-sets. Clearly $H \cup Q = \{a, b\}$ is nano $\alpha^\star$-set.*

Remark 3.9. *In a space $(U, \tau_R(X))$, the intersection of two nano $\alpha^\star$-sets are nano $\alpha^\star$-set.*

Example 3.10. *In Example 3.6, then $H = \{a, b\}$ and $Q = \{b, c\}$ is nano $\alpha^\star$-sets. Clearly $H \cap Q = \{b\}$ is nano $\alpha^\star$-set.*

Proposition 3.11. *In a space* $(U, \tau_R(X))$, *then* $N\alpha^\star(U, \tau_R(X)) \subseteq N\mathcal{R}_{\alpha^\star}(U, \tau_R(X))$ *and* $\tau_R(X) \subseteq N\mathcal{R}_{\alpha^\star}(U, \tau_R(X))$.

Proof. Since $U \in \tau_R(X) \cap N\alpha^\star(U, \tau_R(X))$, the inclusions are obvious.

Example 3.12. *In Example 3.2, then* $\{1\}$ *is a nano* $\mathcal{R}_{\alpha^\star}$*-set but not a nano* $\alpha^\star$*-set and* $\{2, 3\}$ *is a nano* $\mathcal{R}_{\alpha^\star}$*-set but not nano open.*

Lemma 3.13. *In a space* $(U, \tau_R(X))$. *If either* H, P *is nano semi-open, then*

$$Nint(Ncl(H \cap P)) = Nint(Ncl(H)) \cap Nint(Ncl(P))$$

Proof. For any subset $H, P \subseteq U$, we generally have

$$Nint(Ncl(H \cap P)) \subseteq Nint(Ncl(H)) \cap Nint(Ncl(P))$$

Assume that H is nano semi-open. Then we have $Ncl(H) = Ncl(Nint(H))$. Therefore

$$\begin{aligned} Nint(Ncl(H)) \cap Nint(Ncl(P)) &= Nint(Ncl(Nint(Ncl(H)) \cap Nint(Ncl(P)))) \\ &\subseteq Nint(Ncl(Ncl(H) \cap Nint(Ncl(P)))) \\ &= Nint(Ncl(Ncl(Nint(H)) \cap Nint(Ncl(P)))) \\ &\subseteq Nint(Ncl(Nint(H) \cap Ncl(P))) \\ &\subseteq Nint(Ncl(Nint(H) \cap P)) \\ &\subseteq Nint(Ncl(H \cap P) \end{aligned}$$

Proposition 3.14. *A subset* H *is nano open in a space* $(U, \tau_R(X)) \iff$ *it is a nano* α*-open set and a nano* $\mathcal{R}_{\alpha^\star}$*-set.*

Proof. It is obvious that every nano open set is a nano α-open set and a nano $\mathcal{R}_{\alpha^\star}$-set. let H be a nano α-open set and a nano $\mathcal{R}_{\alpha^\star}$-set. Since H is a nano $\mathcal{R}_{\alpha^\star}$-set, there exist $G \in \tau_R(X)$ and $Q \in N\alpha^\star(U, \tau_R(X))$ such that $H = G \cap Q$. Since H is a nano α-open set, by using Lemma 3.13, we have

$$\begin{aligned} H \subseteq Nint(Ncl(Nint(H))) &= Nint(Ncl(Nint(G \cap Q))) \\ &= Nint(Ncl(G)) \cap Nint(Ncl(Nint(Q))) \\ &= Nint(Ncl(G)) \cap Nint(Q) \end{aligned}$$

and hence

$$\begin{aligned} H &= G \cap H \\ &\subseteq G \cap (Nint(Ncl(G)) \cap Nint(Q)) \\ &= G \cap Nint(Q) \\ &\subseteq H \end{aligned}$$

Consequently, we obtain $H = G \cap Nint(H)$ and H is nano open.

Remark 3.15. *The following example shows that the concepts of nano* α*-open set and nano* $\mathcal{R}_{\alpha^\star}$*-set are independent of each other.*

Example 3.16. *In a Example 3.2,*

1. *then* $\{1, 2\}$ *is nano* α*-open set but not nano* $\mathcal{R}_{\alpha^\star}$*-set.*
2. *then* $\{2, 3\}$ *is nano* $\mathcal{R}_{\alpha^\star}$*-set but not nano* α*-open set.*

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SOFT IDEALS OVER A SEMIGROUP GENERATED BY A SOFT SET

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Abstract – In this paper, the concept of soft singletons is defined. Consequently, we introduce the soft principal left (right) ideals over a semigroup S. The smallest soft right (left) ideals over S generated by a soft set over S are studied. Some illustrative examples are given**.**

Keywords *– Soft sets, soft semigroups, soft ideals, soft singleton.*

1. Introduction and Preliminaries

The concept of a soft set was first introduced by Molodtsov in [6]. Aktas and Cagman [1] adapted this concept to define soft groups. In [2], the authors introduced the concept of soft semigroups as a collection of subsemigroups of a semigroupand defined soft (left, right, quasi, bi) ideals of a semigroup. Shabir and Ahmad applied soft sets theory of ternary semigroups [7]. Jun and et al introduced concepts of soft ideals over ordered semigroups [5]. Properties of soft Γ-semigroups and soft ideals over a Γ-semigroup were studied in [3]. In Section 2 we introduce the definition of soft singletons andsome basic propositions. In Section 3 we define the soft left (right) ideal generated by a soft set over a semigroup and the soft ideal generated by a softset over a semigroup, and find, as special cases, those soft ideals generated by soft sets over monoids.

Let S be a semigroup. A nonempty subset A of S is called a subsemigroup of Sif $A^2 \subseteq A$, a left (right) ideal of S if $SA \subseteq A$($AS \subseteq A$) and a two-sided ideal (or simply ideal) of S if it is both a left and a right ideal of S.

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Definition 2.1 [1]. Let U be a universal set and let E be a set of parameters. Let $P(U)$ denote the power set of U and let $A \subseteq E$. A pair (F, A) is called a soft set over U if F is a mapping $F: A \rightarrow P(U)$.

Definition 2.2 [5]. Let (F, A) and (G, B) be soft sets over U, then (G, B) is called a soft subset of (F, A), denoted by $(G, B) \sqsubseteq (F, A)$If $B \subseteq A$ and $G(b) \subseteq F(b)$ for all $b \in B$.

Definition 2.3 [2]. Let U be an initial universe set, Ebe the universe set of parameters and $A \subseteq E$.

a) $(F, A)$is called a relative null soft set (with respect to the parameter set$A$), denoted by$(\mathcal{N}, A)$if $F(e) = \emptyset$, for all $e \in A$.
b) We shall denote by $\emptyset_U$ the unique soft set over $U$with an empty parameter set which is called the empty soft set over U.

Definition 2.4 [2]. Let U be an initial universe set, E be the universe set of parameters and $A \subseteq E$. Then (U, A) is said to be an absolute soft set over Uif $U(e) = U$, for all $e \in A$.

Definition 2.3 [2]. Let (F, A) and (G, B) be two soft sets over a common universe U, then

1) The extended intersection of (F, A) and (G, B) denoted by $(F, A) \cap_\varepsilon (G, B)$, is defined as soft set (H, C) where $C = A \cap B, \forall c \in C,$

$$H(c) = \begin{cases} F(c) & \text{if } c \in A \setminus B \\ G(c) & \text{if } c \in B \setminus A \\ F(c) \cap G(c) & \text{if } c \in A \cap B \end{cases}$$

2) The restricted intersection of $(F, A)$and (G, B), denoted by $(F, A) \sqcap (G, B)$, is defined as soft set (H, C) where $C = A \cap B$and $H(c) = F(c) \cap G(c)$ for all $c \in C$.

Definition 2.4 [2]. Let (F, A) and (G, B) be two soft sets over a common universe U, then

1) The extended union of (F, A) and (G, B) denoted by $(F, A) \cup_\varepsilon (G, B)$, is defined as soft set (H, C) where $C = A \cup B, \forall c \in C,$

$$H(c) = \begin{cases} F(c) & \text{if } c \in A \setminus B \\ G(c) & \text{if } c \in B \setminus A \\ F(c) \cup G(c) & \text{if } c \in A \cap B \end{cases}$$

2) The restricted union of (F, A) and (G, B), denoted by $(F, A) \sqcup (G, B)$, is defined as soft set (H, C) where $C = A \cap B$ and $H(c) = F(c) \cup G(c)$ for all $c \in C$.

2. Principle Soft Ideals

In the rest of this paper, S is a semigroup and S^1 denotes the monoid generated by S.

Definition 2.1. [2]. Let (F,A) and (G,B) be two soft sets over a semigroup S. The restricted product of (F,A) and (G,B) denoted by $(F,A) \tilde{\circ} (G,B)$ is defined as the soft set $(H;C)$ where $C = A \cap B$ and $H(c) = F(c)G(c)$ for all $c \in C$.

Definition 2.2. [2]. A soft set (F,A) over a semigroup S is called a soft semigroup if by $(\mathcal{N},A) \neq (F,A) \neq \emptyset_S$ and $(F,A) \tilde{\circ} (F,A) \subseteq (F,A)$.

It is shown that (F,A) is a soft semigroup over S if and only if $\forall x \in A,\ F(x) \neq \emptyset$is a subsemigroup of S [2].

Definition 3.3. [2]. A soft set $(\mathcal{N},A) \neq (F,A) \neq \emptyset_S$ over a semigroup S is called a soft left (right) ideal over S, if $(S,A) \tilde{\circ} (F,A) \subseteq (F,A)$ $((F,A) \tilde{\circ} (S,A) \subseteq (F,A))$ Where (S,A) is an absolute soft set over S. A soft set over S is a soft ideal if it is both a soft left and a soft right ideal over S.

It is shown that a soft set (F,A) over S is a soft ideal over S ifand only if $F(a) \neq \emptyset$ is an ideal of S [2].

Definition 2.3. Let $x \in S$. A soft set (x,A) over a semigroup S is called a soft singleton if $x(a) = \{x\}$ for all $a \in A$.

Definition 2.3. For a soft singleton (x,A) and a soft set (F,A) over S, we say(x,A) belongs to (F,A), denoted by $(x,A) \Subset (F,A)$, if $x \in F(a)$, for all $a \in A$.

Example 2.4. Let $S = (N,+)$ be the semigroup of natural numbers. Define $F : A = \{1,2,3\} \rightarrow P(N)$ by $F(1) = \{2,3,4,\ldots\}$, $F(2) = \{3,4,5,\ldots\}$ and $F(3) = \{4,5,6,\ldots\}$. It is obvious that $(4,A) \Subset (F,A)$ because $4 \in F(a)$ for all $a \in A$ while (x,A) does not belong to (F,A) for all $x \in A$.

Proposition 2.5. Let (F,A) be a soft set over a semigroup S. If (F,A) is a soft semigroup, then $(x,A) \tilde{\circ} (y,A) \Subset (F,A)$ for any $(x,A),(y,A) \Subset (F,A)$.

Proof. Assume that(F,A) is a soft semigroup, then for all $a \in A, F(a)$is a subsemigroup of S. Let$(x,A),(y,A) \Subset (F,A) \Rightarrow (x,A) \tilde{\circ} (y,A) = (xy,A) \Subset (F,A)$ because $xy \in F(a)$ for all $a \in A$. □

Proposition 2.6. If(F,A) is a soft left (right) ideal over a semigroup S, then $(S,A) \tilde{\circ} (x,A) \sqsubseteq (F,A)$, $((x,A) \tilde{\circ} (S,A) \sqsubseteq (F,A))$ for all $(x,A) \Subset (F,A)$.

Proof. Suppose that (F,A) is a soft left (right) ideal over, then for all $a \in A, F(a)$ is a left (right) ideal of S. Let $x \in F(a)$ for all $a \in A$, then $Sx \subseteq F(a)$ $(xS \subseteq F(a))$ for all $a \in A$. Thus $(S,A) \tilde{\circ} (x,A) \sqsubseteq (F,A)$,$((x,A) \tilde{\circ} (S,A) \sqsubseteq (F,A))$ for all $(x,A) \Subset (F,A)$. □

Generally, the opposite direction of the above proposition is not true. Also, it is not necessary that a soft set (F,A) equals union of all soft singletons belonging to it. This fact is depicted in the following example.

Example 2.7. Let $S = \{1,2,3,4,5\}$ be a semigroup defined by the following table

.	1	2	3	4	5
1	1	2	3	4	5
2	2	2	2	2	2
3	3	2	3	3	2
4	4	2	4	4	2
5	5	5	5	5	5

For $A = \{1,2\} \subset S$, define the soft set $(F,A)$by $F(1) = \{4,5\}$ and $F(2) = \{4\}$. Clearly, $(4,A)$ is the only soft singleton belonging to (F,A). Moreover, $(4,A) \tilde{\circ} (4,A) \tilde{\in} (F,A)$ but (F,A) is not a soft semigroup over S because $F(1) = \{4,5\}$ is not subsemigroup of S. It is obvious that (F,A) is not the union of its soft singletons. Let (G,A) be a soft set over S defined as $G(1) = \{1,2,4\}$ and $G(2) = \{2,4\}$. The soft singletons belonging to (G,A) are $(2,A)$ and $(4,A)$. Easily, one can show that $(x,A) \tilde{\circ} (S,A) \sqsubseteq (G,A)$ for all $(x,A) \tilde{\in} (G,A)$ but (G,A) is not a soft right ideal over S because $G(1) = \{1,2,4\}$ is not an ideal of S.

Definition 2.8. The smallest soft right (left) ideal over S containing (x,A) is called the principal soft right (left) ideal generated by (x,A). The smallest soft ideal over S containing (x,A) is called the principal soft ideal generated by (x,A).

By definition, $(x,A) \tilde{\circ} (S^1,A) = (H,A)$ such that $H(a) = xS^1 = \{x\} \cup xS$. That is, $(x,A) \tilde{\circ} (S^1,A)$ is a soft set over S with a constant value equals the principal right ideal of S generated by $\{x\}$.

Lemma2.9. $(x,A) \tilde{\circ} (S^1,A)$ is the principal soft right ideal over Sgenerated by (x,A).

Proof. Clearly, $(x,A) \tilde{\circ} (S^1,A)$ is a soft right ideal over S and $(x,A) \tilde{\in} (x,A) \tilde{\circ} (S^1,A)$. Let (G,A) be a soft right ideal over S containing (x,A), then

$$xS^1 \subseteq G(a)S^1 = G(a) \cup G(a)S \subseteq G(a)$$

hence $(x,A) \tilde{\circ} (S^1,A) \sqsubseteq (G,A)$. Then $(x,A) \tilde{\circ} (S^1,A)$ is the principal soft right ideal over Sgenerated by (x,A). □

Similarly, we get the dual result.

Lemma2.9. $(S^1,A) \tilde{\circ} (x,A)$ is the principal soft left ideal over S generated by (x,A).

Lemma2.10. $(S^1,A) \tilde{\circ} (x,A) \tilde{\circ} (S^1,A)$ is the principal soft ideal over S generated by (x,A).

Proof. Since $x = 1x1 \subseteq S^1xS^1$, then $(x,A) \Subset (S^1,A) \tilde{\circ} (x,A) \tilde{\circ} (S^1,A)$. Obviously, $(S^1,A) \tilde{\circ} (x,A) \tilde{\circ} (S^1,A)$ is a soft ideal over S. Suppose that (G,A) be a soft ideal over S containing (x,A), then

$$S^1xS^1 \subseteq S^1G(a)S^1 = G(a) \cup G(a)S \cup SG(a) \cup SG(a)S \subseteq G(a)$$

thus $(S^1,A) \tilde{\circ} (x,A) \tilde{\circ} (S^1,A) \sqsubseteq (G,A)$. Then $(S^1,A) \tilde{\circ} (x,A) \tilde{\circ} (S^1,A)$ is the principal soft ideal over S generated by (x,A). □

Lemma 2.12. (Principle soft left Ideal Lemma). Let $x, y \in S$, then the following statements are equivalent;

1) $(S^1,A) \tilde{\circ} (x,A) \sqsubseteq (S^1,A) \tilde{\circ} (y,A)$,
2) $(x,A) \Subset (S^1,A) \tilde{\circ} (y,A)$,
3) $x = y$ or $x = sy$ for some $s \in S$.

Proof. Straightforward.

Lemma 2.13. (Principle Soft Right Ideal Lemma). Let $x, y \in S$, then the following statements are equivalent;

1) $(x,A) \tilde{\circ} (S^1,A) \sqsubseteq (y,A) \tilde{\circ} (S^1,A)$,
2) $(x,A) \Subset (y,A) \tilde{\circ} (S^1,A)$,
3) $x = y$ or $x = ys$ for some $s \in S$.

Proof. Straightforward.

Theorem 2.14. Let $\mathcal{L}$,$\mathcal{R}$ be relations on a semigroup S defined by

1) $x\mathcal{L}y$ if and only if $(S^1,A) \tilde{\circ} (x,A) = (S^1,A) \tilde{\circ} (y,A)$,
2) $x\mathcal{R}y$ if and only if $(x,A) \tilde{\circ} (S^1,A) = (y,A) \tilde{\circ} (S^1,A)$.

Then $\mathcal{L}[\mathcal{R}]$ is aright [left] congruence relation.

Proof. $x\mathcal{L}x$ $(x\mathcal{R}x)$ because $S^1x = S^1x$ $(xS^1 = xS^1)$. It is clear that$\mathcal{L}$ and are symmetric and transitive relations. Then $\mathcal{L}$ and $\mathcal{R}$ are equivalence relations. To show that $\mathcal{L}$ $[\mathcal{R}]$ is a right [left] congruence, assume $x\mathcal{L}y$ $[x\mathcal{R}y]$ and $s \in S$ then

$$(S^1,A) \tilde{\circ} (x,A) = (S^1,A) \tilde{\circ} (y,A) \; [(x,A) \tilde{\circ} (S^1,A) = (y,A) \tilde{\circ} (S^1,A)]$$

that is,

$$S^1x = S^1y \Rightarrow S^1xs = S^1ys \; [xS^1 = yS^1 \Rightarrow sxS^1 = syS^1].$$

Hence

$$\begin{aligned}(S^1,A) \tilde{\circ} (x,A) \tilde{\circ} (s,A) = (S^1,A) \tilde{\circ} (y,A) \tilde{\circ} (s,A)[(s,A) \tilde{\circ} (x,A) \tilde{\circ} (S^1,A) \\ = (s,A) \tilde{\circ} (y,A) \tilde{\circ} (S^1,A)],\end{aligned}$$

This implies that $xs\mathcal{L}ys$ [$sx\mathcal{R}sy$]. Thus $\mathcal{L}$ [$\mathcal{R}$] is a right [left] congruence. □

Corollary 2.15. For $x, y \in S$, we have

- $x\mathcal{L}y \Leftrightarrow \exists\, s, t \in S^1$ such that $(s, A) \tilde{\circ} (y, A) = (x, A)$ and $(t, A) \tilde{\circ} (x, A) = (y, A)$.
- $x\mathcal{R}y \Leftrightarrow \exists\, s, t \in S^1$ such that $(y, A) \tilde{\circ} (s, A) = (x, A)$ and $(x, A) \tilde{\circ} (t, A) = (y, A)$.

Proof. Let $x\mathcal{L}y \Leftrightarrow$ if $(S^1,A) \tilde{\circ} (x,A) = (S^1,A) \tilde{\circ} (y,A) \Leftrightarrow (S^1,A) \tilde{\circ} (x,A) \sqsubseteq (S^1,A) \tilde{\circ} (y,A)$ and $(S^1,A) \tilde{\circ} (y,A) \sqsubseteq (S^1,A) \tilde{\circ} (x,A) \Leftrightarrow x = sy$ and $y = tx$ for some $t, s \in S \Leftrightarrow (s,A) \tilde{\circ} (y,A) = (x,A)$ and $(t,A) \tilde{\circ} (x,A) = (y,A)$, by lemma 3.7. For $x\mathcal{R}y$, the result comes directly by a similar argument.□

Definition 2.16. We define the equivalence relation $\mathcal{H} = \mathcal{L} \cap \mathcal{R}$. For $x \in S$, we define L_x to be the $\mathcal{L}$-class of x; R_x to be the $\mathcal{R}$-class of x and H_x is the $\mathcal{H}$-class of x.

Example 2.17. Let $x, y \in S = (N, +)$, then

$$x\mathcal{L}y \Leftrightarrow (N^1, A) \tilde{\circ} (y, A) = (N^1, A) \tilde{\circ} (y, A) \Leftrightarrow N^1 + x = N^1 + y \Leftrightarrow x = y.$$

Thus $\mathcal{L} = \mathcal{R} = \mathcal{H} = \{(x,x): \forall x \in N\}$ and then $L_x = R_x = H_x = \{x\}$, for all $x \in N$.

3. Soft Ideals Generated by Soft Sets

Authors in [2], showed that $(F, A) \cap_{\mathcal{R}} (G, B)$ for any soft ideals (F, A) and (G, B) over S is a soft ideal. Hence the restricted intersection of all soft ideals over S containing the soft set (H, A) is the soft ideal over S generated by (H, A).

Definition 3.1. The smallest soft right (left) ideal over S containing (F, A) is called the soft right (left) ideal generated by (F, A), denoted by $([F], A)$ $((\langle F \rangle, A))$. The smallest soft ideal over S containing (F, A) is called the soft ideal generated by (F, A), denoted by $((F), A)$.

Theorem 3.2. Let (F, A) be a soft set over S, then

$$(\langle F \rangle, A) = (F, A) \sqcup (S, A) \tilde{\circ} (F, A).$$

Proof. Let $\{(F_i, A): i \in I\}$ the family of all soft left ideals over S containing (F, A), then $F_i(a)$ is a left ideal of S for all $i \in I, a \in A$. Since $SF(a) \subseteq SF_i(a) \subseteq F_i(a)$ for each $i \in I$, $a \in A$, then

$$(S, A) \tilde{\circ} (F, A) \sqsubseteq \sqcap_{i \in I} \{(F_i, A)\}.$$

As a result, $(F, A) \sqcup (S, A) \tilde{\circ} (F, A) \sqsubseteq \sqcap_{i \in I} \{(F_i, A)\}$. We notice that $(\langle F \rangle, A)$ is a soft left ideal over S^1 because $\langle F \rangle(a)$ is the left ideal of S generated by $F(a)$ for all $a \in A$. This follows that we have $\sqcap_{i \in I} \{(F_i, A)\} \sqsubseteq (\langle F \rangle, A)$. By definition, we get

$$\sqcap_{i \in I} \{(F_i, A)\} = (\langle F \rangle, A).$$

Similarly, we prove the following result.

Theorem 3.3. Let (F,A) be a soft set over S, then

$$([F],A) = (F,A) \sqcup (F,A) \tilde{\circ} (S,A).$$

Theorem 3.4. Let (F,A) be a soft set over S^1, then

$$\langle F\rangle(a) = \bigcup_{x\in S^1} xF(a)$$

Proof. Since for all $a \in A, F(a) = 1F(a) \subseteq \langle F\rangle(a)$, then $(F,A) \sqsubseteq (\langle F\rangle,A)$. The soft set $(\langle F\rangle,A)$ is a soft left ideal over S^1. Indeed, by definition $(S^1,A) \tilde{\circ} (\langle F\rangle,A) = (H,A)$ where

$$H(a) = S^1\langle F\rangle(a) = S^1\left(\bigcup_{x\in S^1} xF(a)\right) = \bigcup_{x\in S^1} S^1xF(a) \subseteq \bigcup_{x\in S^1} xF(a) = \langle F\rangle(a)$$

Thus $H(a) \subseteq \langle F\rangle(a)$ for all $a \in A$. As a result, $(\langle F\rangle,A)$ is a soft left ideal over S^1. Let (G,A) be a soft left ideal over S^1 containing (F,A), then

$$\langle F\rangle(a) = \bigcup_{x\in S^1} xF(a) \subseteq \bigcup_{x\in S^1} xG(a) \subseteq G(a).$$

Hence $(\langle F\rangle,A) \sqsubseteq (G,A)$. By definition, we conclude that $(\langle F\rangle,A) = (G,A)$. This ends the proof. □

Similarly, we prove the following result.

Theorem 3.5. Let (F,A) be a soft set over S^1, then

$$\langle F\rangle(a) = \bigcup_{x\in S^1} F(a)x.$$

Example 3.6. Consider the non-commutative semigroup $S = \{1,a,b,c\}$

.	1	a	b	c
1	1	a	b	c
a	a	a	a	a
b	b	b	b	b
c	c	b	a	c

For $A = \{1\} \subset S$, define a soft set (F,A) over S by $F(1) = \{b\}$. By definition, $(S,A) \tilde{\circ} (F,A) = (H,A)$ such that $H(1) = SF(1) = S\{b\} = \{a,b\}$. Then

$$\langle F\rangle(1) = F(1) \cup SF(1) = \{a, b\}.$$

That is, $(\langle F\rangle, A) = (F, A) \sqcup (S, A) \tilde{\circ} (F, A)$ is a soft left ideal over S containing (F, A). Let (G, A)be a soft left ideal over S containing (F, A). Then $\{b\} = F(1) \subseteq G(1) = \{a, b, c\}$ or $G(1) = \{a, b\}$. For all cases, $(\langle F\rangle, A) \sqsubseteq (G, A)$. Therefore, $(\langle F\rangle, A)$ is the soft left ideal over S containing (F, A).

Let (F, A)be a soft set over Sdefined by $F(1) = \{c\}$. By definition, $(F, A) \tilde{\circ} (S, A) = (H, A)$ such that $H(1) = F(1)S = \{c\}S = \{a, b, c\}$. Then

$$[F](1) = F(1) \cup F(1)S = \{a, b, c\}.$$

That is, $([F], A) = (F, A) \sqcup (F, A) \tilde{\circ} (S, A)$ is a soft right ideal over S containing (F, A). Let (G, A) be a soft right ideal over Scontaining (F, A). Then $\{c\} = F(1) \subseteq G(1) = \{a, b, c\}$ is the only right ideal of S that contains $F(1)$. Thus $([F], A) \sqsubseteq (G, A)$. Therefore, $([F], A)$ is the soft right ideal over$S$containing (F, A). □

Theorem 3.7. Let (F, A) be a soft set over S, then

$$((F), A) = (F, A) \sqcup (F, A) \tilde{\circ} (S, A) \sqcup (F, A) \tilde{\circ} (S, A) \sqcup (S, A) \tilde{\circ} (F, A) \tilde{\circ} (S, A).$$

Proof. Let $\{(F_i, A): i \in I\}$ the family of all soft ideals over S containing (F, A), then $F_i(a)$ is an ideal of S for all $i \in I, a \in A$. By the same way as in theorem, we show that

$$(S, A) \tilde{\circ} (F, A) \sqsubseteq \sqcap_{i \in I} \{(F_i, A)\},$$
$$(F, A) \tilde{\circ} (S, A) \sqsubseteq \sqcap_{i \in I} \{(F_i, A)\}$$

and

$$(S, A) \tilde{\circ} (F, A) \tilde{\circ} (S, A) \sqsubseteq \sqcap_{i \in I} \{(F_i, A)\}$$

for each $i \in I, a \in A$. Hence $((F), A) \sqsubseteq \sqcap_{i \in I} \{(F_i, A)\}$. Because

$$(F)(a) = F(a) \cup SF(a) \cup F(a)S \cup SF(a)S$$

is the ideal of S generated by $F(a)$ for all $a \in A$. Thus we have $\sqcap_{i \in I} \{(F_i, A)\} \sqsubseteq ((F), A)$. By definition, we get $\sqcap_{i \in I} \{(F_i, A)\} = ((F), A)$.□

Theorem 3.8. Let (F, A) be a soft set over S^1, then

$$([\langle F\rangle], A) = ((F), A) = (\langle [F]\rangle, A).$$

Proof. By definition, $([\langle F\rangle], A)$ is a soft right ideal over S^1. Also $([\langle F\rangle], A)$ is a soft left ideal over S^1. Indeed, we have

$$S^1[\langle F\rangle](a) = S^1(\bigcup_{x \in S^1} \langle F\rangle(a)x) = \bigcup_{x \in S^1} S^1\langle F\rangle(a)x \subseteq \bigcup_{x \in S^1} \langle F\rangle(a)x = [\langle F\rangle](a).$$

So $([\langle F\rangle],A)$ is a soft ideal over S^1 containing (F,A). Let (G,A) be a soft ideal over S^1 containing (F,A), then $(\langle F\rangle,A) \sqsubseteq (G,A)$ and $([\langle F\rangle],A) \sqsubseteq (G,A)$. This means $([\langle F\rangle],A)$ is a soft ideal over S^1 generated by (F,A), hence $([\langle F\rangle],A) = \big((F),A\big)$. Similarly, we can show that $\big((F),A\big) = (\langle[F]\rangle,A)$. This completes the proof. □

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