Number 19 Year 2017

Journal of New Theory

ISSN: 2149-1402

Editor-in-Chief
Naim Çağman

www.dergipark.org.tr/en/pub/jnt

Journal of New Theory (abbreviated by J. New Theory or JNT) is a mathematical journal focusing on new mathematical theories or the applications of a mathematical theory to science.

JNT founded on 18 November 2014 and its first issue published on 27 January 2015.

ISSN: 2149-1402

Editor-in-Chief: Naim Çağman

Email: journalofnewtheory@gmail.com

Language: English only.

Article Processing Charges: It has no processing charges.

Publication Frequency: Quarterly

Publication Ethics: The governance structure of J. New Theory and its acceptance procedures are transparent and designed to ensure the highest quality of published material. Journal of New Theory adheres to the international standards developed by the Committee on Publication Ethics (COPE).

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http://www.newtheory.org ISSN: 2149-1402

Journal of New Theory

Received: 25.11.2017
Published: 13.12.2017
Year: **2017**, *Number*: **19**, *Pages*: **1-19**
Original Article

A MATRIX REPRESENTATION OF A GENERALIZED FIBONACCI POLYNOMIAL

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Abstaract − The Fibonacci polynomial $F_n(x)$ defined recurrently by $F_{n+1}(x) = xF_n(x)+F_{n-1}(x)$, with $F_0(x) = 0$, $F_1(0) = 1$, for $n \geq 1$ is the topic of wide interest for many years. In this article, generalized Fibonacci polynomials $\widehat{F}_{n+1}(x)$ and $\widehat{L}_{n+1}(x)$ are introduced and defined by $\widehat{F}_{n+1}(x) = x\widehat{F}_n(x)+\widehat{F}_{n-1}(x)$ with $\widehat{F}_0(x) = 0$, $\widehat{F}_1(x) = x^2+4$, for $n \geq 1$ and $\widehat{L}_{n+1}(x) = x\widehat{L}_n(x)+\widehat{L}_{n-1}(x)$ with $\widehat{L}_0(x) = 2x^2 + 8$, $\widehat{L}_1(x) = x^3 + 4x$, for $n \geq 1$. Also some basic properties of these polynomials are obtained by matrix methods.

Keywords − *Fibonacci Sequence, Fibonacci Polynomial, Generalised Fibonacci Polynomial.*

1 Introduction

Horadam [9] introduced and studied the generalized Fibonacci sequence $W_n = W_n(a; b; p; q)$ defined by $W_n = pW_{n-1} - qW_{n-2}$ with $W_0 = a$, $W_1 = b$, for $n \geq 1$ where $a; b; p$ and q are arbitrary complex numbers with $q \neq 0$. These numbers were first studied by Horadam and they are called Horadam numbers. In [7] Silvester shows that a number of the properties of the Fibonacci sequence can be derived from a matrix representation.

In [27] Demirturk obtained summation formulae for the Fibonacci and Lucas sequences by matrix methods. In [28] the authors presented some important relationship between k-Jacobsthal matrix sequence and k-Jacobsthal-Lucas matrix sequence. In [22] Godase and Dhakne described some properties of k- Fibonacci and k-Lucas numbers by matrix terminology.

In [18] Catarino and Vasco introduced a 2×2 matrix for the k-Pell sequence with its nth power. The well-known Fibonacci polynomial is studied over several years. Many authors are dedicated to study this polynomial. The most research on Fibonacci polynomials are dedicated to study the generalizations of Fibonacci

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polynomials [4],[5], [20], [19], [21], [28]. The main aim of the present paper is to study other generalized Fibonacci polynomial by matrix methods.

Somnuk Srisawat and Wanna Sriprad [1] investigated generalization of Pell and Pell-Lucas numbers, which is called (s,t)-Pell and (s,t)-Pell-Lucas numbers, also they defined the 2×2 matrix

$$W = \begin{bmatrix} s & 2(s^2+t) \\ \frac{1}{2} & s \end{bmatrix}$$

using this matrix they established many identities of (s,t)-Pell and (s,t)-Pell-Lucas numbers. Hasan Huseyin Gulec, Necati Taskara [2] established a new generalizations for (s,t)-Pell (s,t)- Pell Lucas $\{q_n(s,t)\}n \in N$ sequences for Pell and Pell Lucas numbers. Considering these sequences, they defined the matrix sequences which have elements of $\{p_n(s,t)\}n \in N$ and $\{q_n(s,t)\}n \in N$.

Yuan, Yi, and Wenpeeg Zhang [3] introduced different methods to calculate the summations involving the Fibonacci polynomials. Fikri Koken and Durmus Bozkurt [6] defined the Jacobsthal Lucas E-matrix and R-matrix alike to the Fibonacci Q-matrix. Using this matrix representation they found some equalities and Binet-like formula for the Jacobsthal and Jacobsthal-Lucas numbers.

Falcon and Plaza[10] presented the derivatives of Fibonacci polynomials in the form of convolution of k-Fibonacci polynomials and many relations for the derivatives of these polynomials are proved.

We denote the Fibonacci and Lucas polynomial by $\widehat{F}_n(x)$ and $\widehat{L}_n(x)$ respectively. Most of the identities for Fibonacci and Lucas polynomials can be found in the articles [11], [12], [13], [14], [15], [16], [17], [23], [24], [25], [26] on Fibonacci and Lucas sequences and their applications. The ultimate aim of this paper is to introduce new generalization $\widehat{F}_n(x)$ and $\widehat{L}_n(x)$ of Fibonacci and Lucas polynomials and establish a collection of identities for the $\widehat{F}_n(x)$ and $\widehat{L}_n(x)$ using matrix method.

2 Generalized Fibonacci polynomials $\widehat{F}_n(x)$ and $\widehat{L}_n(x)$

In [8] the Fibonacci polynomial $F_n(x)$ defined recurrently by $F_{n+1}(x) = xF_n(x) + F_{n-1}(x)$, with $F_0(x) = 0$, $F_1(0) = 1$, for $n \geq 1$. In this paper we defined generalized Fibonacci polynomials $\widehat{F}_n(x)$ and $\widehat{L}_n(x)$.

Definition 2.1. The generalized Fibonacci polynomial $\widehat{F}_n(x)$ is defined by the recurrence relation

$$\widehat{F}_{n+1}(x) = x\widehat{F}_n(x) + \widehat{F}_{n-1}(x) \quad \text{with} \quad \widehat{F}_0(x) = 0, \quad \widehat{F}_1(x) = x^2+4, \text{ for } n \geq 1 \tag{1}$$

Definition 2.2. The generalized Lucas polynomial $\widehat{L}_n(x)$ is defined by the recurrence relation

$$\widehat{L}_{n+1}(x) = x\widehat{L}_n(x) + \widehat{L}_{n-1}(x) \quad \text{with} \quad \widehat{L}_0(x) = 2x^2+8, \quad \widehat{L}_1(x) = x^3+4x, \text{ for } n \geq 1 \tag{2}$$

Polynomial	Initial value $G_0(x) = a(x)$	Initial value $G_1(x) = b(x)$	Recursive Formula $G_{n+1}(x) = a(x)G_n(x) + b(x)G_{n-1}(x)$
Fibonacci	0	1	$F_{n+1}(x) = F_n(x) + F_{n-1}(x)$
Lucas	2	x	$L_{n+1}(x) = L_n(x) + L_{n-1}(x)$
Pell	0	1	$P_{n+1}(x) = 2xP_n(x) + P_{n-1}(x)$
Pell-Lucas	2	2x	$Q_{n+1}(x) = 2xQ_n(x) + Q_{n-1}(x)$
Jacobsthal	0	1	$J_{n+1}(x) = J_n(x) + 2xJ_{n-1}(x)$
Jacobsthal-Lucas	2	1	$j_{n+1}(x) = j_n(x) + 2xj_{n-1}(x)$
Generalized Fibonacci	0	$x^2 + 4$	$\widehat{F}_{n+1}(x) = x\widehat{F}_n(x) + \widehat{F}_{n-1}(x)$
Generalized Lucas	$2x^2 + 8$	$x^3 + 4x$	$\widehat{L}_{n+1}(x) = x\widehat{L}_n(x) + \widehat{L}_{n-1}(x)$

Table 1: **Recurrence relation of some GFP.**

Characteristic equation of the initial recurrence relation (1 and 2) is,

$$r^2 - xr - 1 = 0 \tag{3}$$

Characteristic roots of (3) are

$$r_1(x) = \frac{x + \sqrt{x^2 + 4}}{2}, \quad r_2(x) = \frac{x - \sqrt{x^2 + 4}}{2} \tag{4}$$

Characteristic roots (4) satisfy the properties

$$r_1(x) - r_2(x) = \sqrt{x^2 + 4} = \sqrt{\Delta(x)}, \quad r_1(x) + r_2(x) = x, \quad r_1(x)r_2(x) = -1 \tag{5}$$

Binet forms for both $\widehat{F}_n(x)$ and $\widehat{L}_n(x)$ are given by

$$\widehat{F}_n(x) = r_1(x)^n - r_2(x)^n \tag{6}$$

$$\widehat{L}_{k,n} = \left[r_1(x)^2 + r_2(x)^2 + 2\right]\left[r_1(x)^n + r_2(x)^n)\right] \tag{7}$$

The most commonly used matrix in relation to the recurrence relation(3) is

$$M(x) = \begin{bmatrix} x & 1 \\ 1 & 0 \end{bmatrix} \tag{8}$$

Using Principle of Mathematical induction the matrix M can be generalized to

$$M(x)^n = \begin{bmatrix} \dfrac{\widehat{F}_{n+1}(x)}{\Delta(x)} & \dfrac{\widehat{F}_n(x)}{\Delta(x)} \\ \dfrac{\widehat{F}_n(x)}{\Delta(x)} & \dfrac{\widehat{F}_{n-1}(x)}{\Delta(x)} \end{bmatrix} \quad \text{where, } n \text{ is an integer}$$

3 Auxiliary Results

Several identities for $\widehat{F}_n(x)$ and $\widehat{L}_n(x)$ can be established using (6), (7). Some of these are listed below

$$\widehat{L}_n(x) = \widehat{F}_{n+1}(x) + \widehat{F}_{n-1}(x) \tag{9}$$

$$\Delta(x)\widehat{F}_n(x) = \widehat{F}_{n+1}(x) + \widehat{L}_{n-1}(x) \tag{10}$$

$$\Delta(x)\widehat{F}_n^2(x) = \widehat{F}_{2n}(x) - 2\Delta(x)(-1)^n \tag{11}$$

$$\widehat{F}_m(x)\widehat{L}_n(x) = \Delta\widehat{F}_{m+n}(x) - \Delta(x)(-1)^m\widehat{L}_{n-m}(x) \tag{12}$$

$$\widehat{F}_m(x)\widehat{F}_n(x) = \widehat{L}_{m+n}(x) - (-1)^m\widehat{L}_{n-m}(x) \tag{13}$$

$$(-1)^{n-m+1}\widehat{F}_m(x)^2 = \widehat{F}_{m+n}(x)\widehat{F}_{n-m}(x) - \widehat{F}_n(x)^2 \tag{14}$$

$$(-1)^{n-m}\widehat{F}_m(x)^2 = \widehat{L}_{m+n}(x)\widehat{L}_{n-m}(x) - \widehat{L}_n(x)^2 \tag{15}$$

$$\widehat{F}_m(x)\widehat{F}_{n+r+m}(x) = \widehat{F}_{m+n}(x)\widehat{F}_{r+m}(x) - (-1)^m\widehat{F}_n(x)\widehat{F}_r(x) \tag{16}$$

$$2\Delta(x)\widehat{L}_{(m+1)n}(x) = \widehat{L}_{mn}(x)\widehat{L}_n(x) + \Delta(x)\widehat{F}_{mn}(x)\widehat{F}_n(x) \tag{17}$$

$$2\Delta(x)\widehat{F}_{(m+1)n}(x) = \widehat{F}_{mn}(x)\widehat{L}_n(x) + \widehat{L}_{mn}(x)\widehat{F}_n(x) \tag{18}$$

$$\Delta(x)\widehat{F}_{2n+m}(x)\widehat{F}_m(x) = \widehat{L}_{m+n}(x)^2 + (-1)^{m-1}\widehat{L}_n(x)^2 \tag{19}$$

$$\Delta(x)\widehat{F}_{2n}(x)\widehat{F}_m(x) = \widehat{L}_{m+n}(x)\widehat{L}_n(x) + (-1)^{m+1}\widehat{L}_{n-m}(x)\widehat{L}_n(x) \tag{20}$$

$$\widehat{F}_{2(r+1)n+m}(x)\widehat{F}_m(x) = \widehat{F}_{m+2rn}(x)\widehat{F}_{2n+m}(x) + (-1)^{m+1}\widehat{F}_{2rn}(x)\widehat{F}_{2n}(x) \tag{21}$$

$$\widehat{F}_{-n}(x) = (-1)^{n+1}\widehat{F}_n(x) \tag{22}$$

$$\widehat{F}_{-n-1}(x) = (-1)^n\widehat{F}_{n+1}(x) \tag{23}$$

4 Main Results

Lemma 4.1. If X is a square matrix with $\Delta(x)X^2 = xX + I$, then $\Delta(x)X^n = \widehat{F}_n(x)X + \widehat{F}_{n-1}(x)I$, for all $n \in Z$

Proof. For $n = 0$ the result is true
For $n = 1$

$$\begin{aligned}\Delta(x)(X)^1 &= \widehat{F}_1(x)X + \widehat{F}_0(x)I \\ &= \Delta(x)X + 0I \\ &= \Delta(x)X\end{aligned}$$

Hence result is true for $n = 1$.
Assume that, $\Delta(x)X^n = \widehat{F}_n(x)X+\widehat{F}_{n-1}(x)I$, and prove that, $\Delta(x)X^{n+1} = \widehat{F}_{n+1}(x)X+\widehat{F}_n(x)I$

Consider,

$$
\begin{aligned}
\widehat{F}_{n+1}(x)X + \widehat{F}_n(x)I &= (\widehat{F}_n(x)x + \widehat{F}_{n-1}(x)I)X + \widehat{F}_n(x)I \\
&= (xX + I)\widehat{F}_n(x) + X\widehat{F}_{n-1}(x) \\
&= X^2\widehat{F}_n(x) + X\widehat{F}_{n-1}(x) \\
&= X(X\widehat{F}_n(x) + \widehat{F}_{n-1}(x)) \\
&= X(\Delta(x)X^n) \\
&= \Delta(x)X^{n+1} \\
\text{Hence,} \quad \Delta(x)X^{n+1} &= \widehat{F}_{n+1}(x)X + \widehat{F}_n(x)I
\end{aligned}
$$

We now show that, $\Delta(x)X^{-n} = \widehat{F}_{-n}(x)X + \widehat{F}_{-n-1}(x)I$, for all $n \in Z^+$
Let, $Y = xI - X$, then

$$
\begin{aligned}
Y^2 &= (xI - X)^2 \\
&= x^2I - 2xX + X^2 \\
&= x^2I - 2xX + xX + I \\
&= x^2I - xX + I \\
&= x(xI - X) + X + I \\
&= xY + I \\
\text{This shows that} \quad \Delta Y^n &= \widehat{F}_n(x)Y + \widehat{F}_{n-1}(x)I \\
\Delta(x)(-X^{-1})^n &= \widehat{F}_n(x)(xI - X) + \widehat{F}_{n-1}(x)I \\
\Delta(x)(-1)^nX^{-n} &= -\widehat{F}_n(x)X + \widehat{F}_{n+1}(x)I \\
\Delta(x)X^{-n} &= (-1)^{n+1}\widehat{F}_n(x)X + (-1)^n\widehat{F}_{n+1}(x)I
\end{aligned}
$$

Using equations (22 and 23), it gives that $\Delta(x)X^{-n} = \widehat{F}_{-n}(x)X + \widehat{F}_{-n-1}(x)I$, for all $n \in Z^+$ □

Corollary 4.2. Let, $S(x) = \begin{bmatrix} \frac{x}{2} & \frac{\Delta(x)}{2} \\ \frac{1}{2} & \frac{x}{2} \end{bmatrix}$, then $\Delta(x)S(x)^n = \begin{bmatrix} \frac{\widehat{L}_n(x)}{2} & \frac{\Delta\widehat{F}_n(x)}{2} \\ \frac{\widehat{F}_n(x)}{2} & \frac{\widehat{L}_n(x)}{2} \end{bmatrix}$, for every $n \in Z$

Proof.

$$
\begin{aligned}
\text{Since} \quad S(x)^2 &= \begin{bmatrix} \frac{x^2+2}{2} & \frac{x\Delta(x)}{2} \\ \frac{x}{2} & \frac{x^2+2}{2} \end{bmatrix} \\
&= xS(x) + I
\end{aligned}
$$

Using Lemma (4.1), it is clear that

$$
\begin{aligned}
\Delta(x)S(x)^n &= \widehat{F}_n(x)S(x) + \widehat{F}_{n-1}(x)I \\
&= \begin{bmatrix} \frac{x\widehat{F}_n(x)}{2} & \frac{\Delta(x)\widehat{F}_n(x)}{2} \\ \frac{\widehat{F}_n(x)}{2} & \frac{x\widehat{F}_n(x)}{2} \end{bmatrix} + \begin{bmatrix} \widehat{F}_{n-1}(x) & 0 \\ 0 & \widehat{F}_{n-1}(x) \end{bmatrix} \\
&= \begin{bmatrix} \frac{\widehat{L}_n(x)}{2} & \frac{\Delta(x)\widehat{F}_n(x)}{2} \\ \frac{\widehat{F}_n(x)}{2} & \frac{\widehat{L}_n(x)}{2} \end{bmatrix}
\end{aligned}
$$

□

Lemma 4.3.

$$\widehat{L}_n^2(x) - \Delta(x)\widehat{F}_n^2(x) = 4\Delta(x)^2(-1)^n \qquad \text{for all}, \quad n \in Z \tag{24}$$

Proof.

$$\begin{aligned}
\text{Since,} \qquad \det(S(x)) &= -1 \\
\det(S(x)^n) &= (-1)^n \\
\text{Moreover since,} \quad \Delta(x)S(x)^n &= \begin{bmatrix} \frac{\widehat{L}_n(x)}{2} & \frac{\Delta\widehat{F}_n(x)}{2} \\ \frac{\widehat{F}_n(x)}{2} & \frac{\widehat{L}_n(x)}{2} \end{bmatrix} \\
\text{We get,} \quad \det(\Delta(x)S(x)^n) &= \frac{\widehat{L}_n^2(x)}{4} - \frac{\Delta(x)\widehat{F}_n^2(x)}{4} \\
\text{Thus it follows that,} \quad \widehat{L}_n^2(x) - \Delta(x)\widehat{F}_n^2(x) &= 4\Delta(x)^2(-1)^n \qquad \text{for all}, \quad n \in Z
\end{aligned}$$

□

Lemma 4.4.

$$2\Delta(x)\widehat{L}_{n+m}(x) = \widehat{L}_n(x)\widehat{L}_m(x) + \Delta(x)\widehat{F}_n(x)\widehat{F}_m(x) \quad \text{for all} \quad n, m \in Z \tag{25}$$

$$2\Delta(x)\widehat{F}_{n+m}(x) = \widehat{F}_n(x)\widehat{L}_m(x) + \widehat{L}_n(x)\widehat{F}_m(x) \quad \text{for all} \quad n, m \in Z \tag{26}$$

Proof.

$$\begin{aligned}
\text{Since,} \quad \Delta(x)^2S(x)^{n+m} &= \Delta(x)S(x)^n.\Delta(x)S(x)^m \\
&= \begin{bmatrix} \frac{\widehat{L}_n(x)}{2} & \frac{\Delta(x)\widehat{F}_n(x)}{2} \\ \frac{\widehat{F}_n(x)}{2} & \frac{\widehat{L}_n(x)}{2} \end{bmatrix} . \begin{bmatrix} \frac{\widehat{L}_m(x)}{2} & \frac{\Delta(x)\widehat{F}_m(x)}{2} \\ \frac{\widehat{F}_m(x)}{2} & \frac{\widehat{L}_m(x)}{2} \end{bmatrix} \\
&= \begin{bmatrix} \frac{\widehat{L}_n(x)\widehat{L}_m(x)+\Delta(x)\widehat{F}_n(x)\widehat{F}_m(x)}{4} & \frac{\Delta(x)[\widehat{L}_n(x)\widehat{F}_m(x)+\widehat{F}_n(x)\widehat{L}_m(x)}{4} \\ \frac{\widehat{L}_n(x)\widehat{F}_m(x)+\widehat{F}_n(x)\widehat{L}_m(x)}{4} & \frac{\widehat{L}_n(x)\widehat{L}_m(x)+\Delta(x)\widehat{F}_n(x)\widehat{F}_m(x)}{4} \end{bmatrix} \\
\text{But,} \quad \Delta(x)S(x)^{n+m} &= \begin{bmatrix} \frac{\widehat{L}_{n+m}(x)}{2} & \frac{\Delta(x)\widehat{F}_{n+m}(x)}{2} \\ \frac{\widehat{F}_{n+m}(x)}{2} & \frac{\widehat{L}_{n+m}(x)}{2} \end{bmatrix}
\end{aligned}$$

$$\begin{aligned}
\text{It gives that} \quad 2\Delta(x)\widehat{L}_{n+m}(x) &= \widehat{L}_n(x)\widehat{L}_m(x) + \Delta(x)\widehat{F}_n(x)\widehat{F}_m(x) \quad \text{for all} \quad n, m \in Z \\
2\Delta(x)\widehat{F}_{n+m}(x) &= \widehat{F}_n(x)\widehat{L}_m(x) + \widehat{L}_n(x)\widehat{F}_m(x) \quad \text{for all} \quad n, m \in Z
\end{aligned}$$

□

Lemma 4.5.

$$2(-1)^m\Delta(x)^2\widehat{L}_{n-m}(x) = \widehat{L}_n(x)\widehat{L}_m(x) - \Delta(x)\widehat{F}_n(x)\widehat{F}_m(x) \quad \text{for all} \quad n, m \in Z \tag{27}$$

$$2(-1)^m\Delta(x)^2\widehat{F}_{n-m}(x) = \widehat{F}_n(x)\widehat{L}_m(x) - \widehat{L}_n(x)\widehat{F}_m(x) \quad \text{for all} \quad n, m \in Z \tag{28}$$

Proof. Since

$$\begin{aligned}\Delta(x)^2S(x)^{n-m} &= \Delta(x)S(x)^n.\Delta(x)S(x)^{-m}\\ &= \Delta(x)S(x)^n.\Delta(x)[S(x)^m]^{-1}\\ &= \Delta(x)S(x)^n.(-1)^m\begin{bmatrix}\frac{\widehat{L}_m(x)}{2} & \frac{-\Delta(x)\widehat{F}_m(x)}{2}\\ \frac{-\widehat{F}_m(x)}{2} & \frac{\widehat{L}_m(x)}{2}\end{bmatrix}\\ &= (-1)^m\begin{bmatrix}\frac{\widehat{L}_n(x)}{2} & \frac{\Delta(x)\widehat{F}_n(x)}{2}\\ \frac{\widehat{F}_n(x)}{2} & \frac{\widehat{L}_n(x)}{2}\end{bmatrix}.\begin{bmatrix}\frac{\widehat{L}_m(x)}{2} & \frac{-\Delta(x)\widehat{F}_m(x)}{2}\\ \frac{-\widehat{F}_m(x)}{2} & \frac{\widehat{L}_m(x)}{2}\end{bmatrix}\\ &= \begin{bmatrix}\frac{\widehat{L}_n(x)\widehat{L}_m(x)-\Delta\widehat{F}_n(x)\widehat{F}_m(x)}{4} & \frac{\Delta(x)[\widehat{L}_n(x)\widehat{F}_m(x)-\widehat{F}_n(x)\widehat{L}_m(x)}{4}\\ \frac{\widehat{L}_n(x)\widehat{F}_m(x)-\widehat{F}_n(x)\widehat{L}_m(x)}{4} & \frac{\widehat{L}_n(x)\widehat{L}_m(x)-\Delta(x)\widehat{F}_n(x)\widehat{F}_m(x)}{4}\end{bmatrix}\end{aligned}$$

But

$$\Delta(x)^2S(x)^{n-m} = \begin{bmatrix}\frac{L_{n-m}(x)}{2} & \frac{\Delta(x)F_{n-m}(x)}{2}\\ \frac{F_{n-m}(x)}{2} & \frac{L_{n-m}(x)}{2}\end{bmatrix}$$

It gives

$$2(-1)^m\Delta(x)^2\widehat{L}_{n-m}(x) = \widehat{L}_n(x)\widehat{L}_m(x) - \Delta(x)\widehat{F}_n(x)\widehat{F}_m(x) \quad \text{for all} \quad n,m \in Z$$
$$2(-1)^m\Delta(x)^2\widehat{F}_{n-m}(x) = \widehat{F}_n(x)\widehat{L}_m(x) - \widehat{L}_n(x)\widehat{F}_m(x) \quad \text{for all} \quad n,m \in Z$$

□

Lemma 4.6.

$$(-1)^m\Delta(x)\widehat{L}_{n-m}(x) + \Delta(x)\widehat{L}_{n+m}(x) = \widehat{L}_n(x)\widehat{L}_m(x) \tag{29}$$
$$(-1)^m\Delta(x)\widehat{F}_{n-m}(x) + \Delta(x)\widehat{F}_{n+m}(x) = \widehat{F}_n(x)\widehat{L}_m(x) \tag{30}$$

Proof.

$$\Delta(x)^2S(x)^{n+m} + (-1)^m\Delta(x)^2S(x)^{n-m}$$
$$= \begin{bmatrix}\frac{\Delta(x)\widehat{L}_{n+m}(x)+(-1)^m\Delta(x)\widehat{L}_{n-m}(x)}{2} & \frac{\Delta(x)[\widehat{F}_{n+m}(x)+(-1)^m\widehat{F}_{n-m}(x)}{2}\\ \frac{\Delta(x)\widehat{F}_{n+m}(x)+(-1)^m\Delta(x)\widehat{F}_{n-m}(x)}{2} & \frac{\Delta(x)\widehat{L}_{n+m}(x)+(-1)^m\Delta(x)\widehat{L}_{n-m}(x)}{2}\end{bmatrix}$$

On the other hand

$$\begin{aligned}&\Delta(x)^2S(x)^{n+m} + (-1)^m\Delta(x)^2S(x)^{n-m}\\ &= \Delta(x)S(x)^n\Delta(x)S(x)^m + (-1)^m\Delta(x)S(x)^n\Delta(x)S(x)^{-m}\\ &= \Delta(x)S(x)^n[\Delta(x)S(x)^m + (-1)^m\Delta(x)S(x)^{-m}]\end{aligned}$$

$$\begin{aligned}&= \begin{bmatrix}\frac{\widehat{L}_n}{2} & \frac{\Delta(x)\widehat{F}_n(x)}{2}\\ \frac{\widehat{F}_n(x)}{2} & \frac{\widehat{L}_n(x)}{2}\end{bmatrix}\left\{\begin{bmatrix}\frac{\widehat{L}_m(x)}{2} & \frac{\Delta(x)\widehat{F}_m(x)}{2}\\ \frac{\widehat{F}_m(x)}{2} & \frac{\widehat{L}_m(x)}{2}\end{bmatrix} + \begin{bmatrix}\frac{\widehat{L}_m(x)}{2} & \frac{-\Delta(x)\widehat{F}_m(x)}{2}\\ \frac{-\widehat{F}_m}{2} & \frac{\widehat{L}_m(x)}{2}\end{bmatrix}\right\}\\ &= \begin{bmatrix}\frac{\widehat{L}_n(x)}{2} & \frac{\Delta(x)\widehat{F}_n(x)}{2}\\ \frac{\widehat{F}_n(x)}{2} & \frac{\widehat{L}_n(x)}{2}\end{bmatrix}.\begin{bmatrix}\widehat{L}_m(x) & 0\\ 0 & \widehat{L}_m(x)\end{bmatrix}\\ &= \begin{bmatrix}\frac{\widehat{L}_m(x)\widehat{L}_n(x)}{2} & \frac{\Delta(x)\widehat{F}_n(x)\widehat{L}_m(x)}{2}\\ \frac{\widehat{F}_n(x)\widehat{L}_m(x)}{2} & \frac{\widehat{L}_m(x)\widehat{L}_n(x)}{2}\end{bmatrix}\end{aligned}$$

It gives

$$(-1)^m\Delta(x)\widehat{L}_{n-m}(x)+\Delta(x)\widehat{L}_{n+m}(x)=\widehat{L}_n(x)\widehat{L}_m(x)$$
$$(-1)^m\Delta(x)\widehat{F}_{n-m}(x)+\Delta(x)\widehat{F}_{n+m}(x)=\widehat{F}_n(x)\widehat{L}_m(x)$$

□

Lemma 4.7.

$$8\Delta(x)^2\widehat{F}_{t+y+z}(x)=\widehat{L}_t(x)\widehat{L}_y(x)\widehat{F}_z(x)+\widehat{F}_t(x)\widehat{L}_y(x)\widehat{L}_z(x) \quad (31)$$
$$+\widehat{L}_t(x)\widehat{F}_y(x)\widehat{L}_z(x)+\Delta(x)\widehat{F}_t(x)\widehat{F}_y(x)\widehat{F}_z(x) \quad (32)$$

$$8\Delta(x)^2\widehat{L}_{t+y+z}(x)=\widehat{L}_t(x)\widehat{L}_y(x)\widehat{L}_z(x) \quad (33)$$
$$+\Delta(x)\left[\widehat{L}_t(x)\widehat{F}_y(x)\widehat{F}_z(x)+\widehat{F}_t(x)\widehat{L}_y(x)\widehat{F}_z(x)+\widehat{F}_t(x)\widehat{F}_y(x)\widehat{L}_z(x)\right] \quad (34)$$

Proof. Since

$$\Delta(x)^2S(x)^{t+y+z}=\begin{bmatrix}\frac{\Delta(x)L_{t+y+z}(x)}{2} & \frac{\Delta(x)^2F_{t+y+z}(x)}{2}\\ \frac{\Delta(x)F_{t+y+z}(x)}{2} & \frac{\Delta(x)L_{t+y+z}(x)}{2}\end{bmatrix}$$

On the other hand

$$\begin{aligned}\Delta(x)^2S(x)^{t+y+z}&=\Delta(x)S(x)^{t+y}\Delta(x)S(x)^z\\ &=\begin{bmatrix}\frac{\widehat{L}_{t+y}(x)}{2} & \frac{\Delta(x)\widehat{F}_{t+y}(x)}{2}\\ \frac{\widehat{F}_{t+y}(x)}{2} & \frac{\widehat{L}_{t+y}(x)}{2}\end{bmatrix}\cdot\begin{bmatrix}\frac{\widehat{L}_z(x)}{2} & \frac{\Delta(x)\widehat{F}_z(x)}{2}\\ \frac{\widehat{F}_z(x)}{2} & \frac{\widehat{L}_z(x)}{2}\end{bmatrix}\\ &=\begin{bmatrix}\frac{\widehat{L}_{t+y}(x)\widehat{L}_z(x)+\Delta(x)\widehat{F}_{t+y}(x)\widehat{F}_z(x)}{4} & \frac{\Delta(x)[\widehat{L}_{t+y}(x)\widehat{F}_z(x)+\widehat{F}_{t+y}(x)\widehat{L}_z(x)}{4}\\ \frac{\widehat{L}_z(x)\widehat{F}_{t+y}(x)+\widehat{F}_z(x)\widehat{L}_{t+y}(x)}{4} & \frac{\widehat{L}_{t+y}(x)\widehat{L}_z(x)+\Delta(x)\widehat{F}_{t+y}(x)\widehat{F}_z(x)}{4}\end{bmatrix}\end{aligned}$$

Using equations (25) and (26), it gives that

$$8\Delta(x)^2\widehat{F}_{t+y+z}(x)=\widehat{L}_t(x)\widehat{L}_y(x)\widehat{F}_z(x)+\widehat{F}_t(x)\widehat{L}_y(x)\widehat{L}_z(x)$$
$$+\widehat{L}_t(x)\widehat{F}_y(x)\widehat{L}_z(x)+\Delta(x)\widehat{F}_t(x)\widehat{F}_y(x)\widehat{F}_z(x)$$

$$8\Delta(x)^2\widehat{L}_{t+y+z}(x)=\widehat{L}_t(x)\widehat{L}_y(x)\widehat{L}_z(x)$$
$$+\Delta(x)\left[\widehat{L}_t(x)\widehat{F}_y(x)\widehat{F}_z(x)+\widehat{F}_t(x)\widehat{L}_y(x)\widehat{F}_z(x)+\widehat{F}_t(x)\widehat{F}_y(x)\widehat{L}_z(x)\right]$$

□

Theorem 4.8.

$$\widehat{L}^2_{t+y}(x)-(-1)^{t+y+1}\widehat{F}_{z-t}(x)\widehat{L}_{t+y}(x)\widehat{F}_{y+z}(x)-\Delta(x)(-1)^{t+z}(x)\widehat{F}^2_{y+z}(x)$$
$$=(-1)^{y+z}\widehat{L}^2_{z-t}(x) \quad (35)$$

$$\Delta(x)\widehat{L}_{t+y}^2(x) - (-1)^{x+z}\widehat{L}_{z-t}(x)\widehat{L}_{t+y}(x)\widehat{L}_{y+z}(x) + (-1)^{x+z}\Delta(x)\widehat{L}_{y+z}^2(x)$$
$$= (-1)^{y+z+1}\Delta(x)^2\widehat{F}_{z-t}^2(x) \quad (36)$$

$$\Delta(x)\widehat{F}_{t+y}^2(x) - \widehat{L}_{t-z}(x)\widehat{F}_{t+y}(x)\widehat{F}_{y+z}(x) + \Delta(x)(-1)^{x+z}\widehat{F}_{y+z}^2(x)$$
$$= (-1)^{y+z}\Delta(x)\widehat{F}_{z-t}^2, \quad (37)$$

for all $t,\ y,\ z \in Z,\ t \neq z$

Proof.

Consider the matrix multiplication

$$\begin{bmatrix} \frac{\widehat{L}_t(x)}{2} & \frac{\Delta(x)\widehat{F}_t(x)}{2} \\ \frac{\widehat{F}_z(x)}{2} & \frac{\widehat{L}_z(x)}{2} \end{bmatrix}\begin{bmatrix} \widehat{L}_y(x) \\ \widehat{F}_y(x) \end{bmatrix} = \begin{bmatrix} \Delta(x)\widehat{L}_{t+y}(x) \\ \Delta(x)\widehat{F}_{y+z}(x) \end{bmatrix}$$

$$\begin{aligned} \det\begin{bmatrix} \frac{\widehat{L}_t(x)}{2} & \frac{\Delta(x)\widehat{F}_t(x)}{2} \\ \frac{\widehat{F}_z(x)}{2} & \frac{\widehat{L}_z(x)}{2} \end{bmatrix} &= \frac{\widehat{L}_t(x)\widehat{L}_z(x) - \Delta(x)\widehat{F}_t\widehat{F}_z(x)}{4} \\ &= \frac{(-1)^t\Delta(x)\widehat{L}_{z-t}(x)}{2} \\ &= R \\ &\neq 0 \end{aligned}$$

Hence

$$\begin{aligned} \begin{bmatrix} \widehat{L}_y(x) \\ \widehat{F}_y(x) \end{bmatrix} &= \begin{bmatrix} \frac{\widehat{L}_t(x)}{2} & \frac{\Delta(x)\widehat{F}_t(x)}{2} \\ \frac{\widehat{F}_z(x)}{2} & \frac{\widehat{L}_z(x)}{2} \end{bmatrix}^{-1} . \begin{bmatrix} \Delta(x)\widehat{L}_{t+y}(x) \\ \Delta(x)\widehat{F}_{y+z}(x) \end{bmatrix} \\ &= \frac{1}{R}\begin{bmatrix} \frac{\widehat{L}_z(x)}{2} & \frac{-\Delta(x)\widehat{F}_t(x)}{2} \\ \frac{-\widehat{F}_z(x)}{2} & \frac{\widehat{L}_t(x)}{2} \end{bmatrix}\begin{bmatrix} \Delta(x)\widehat{L}_{t+y}(x) \\ \Delta(x)\widehat{F}_{y+z}(x) \end{bmatrix} \\ \widehat{L}_y(x) &= \frac{(-1)^t[\widehat{L}_z(x)\widehat{L}_{t+y}(x) - \Delta(x)\widehat{F}_t\widehat{F}_{y+z}(x)]}{\widehat{L}_{z-t}(x)} \\ \widehat{F}_y(x) &= \frac{(-1)^t[\widehat{L}_t(x)\widehat{F}_{z+y}(x) - \widehat{F}_z(x)\widehat{L}_{y+t}(x)]}{\widehat{L}_{z-t}(x)} \end{aligned}$$

Since

$$\widehat{L}_y^2(x) - \Delta(x)\widehat{F}_y^2(x) = 4\Delta(x)^2(-1)^y$$

We get

$$[\widehat{L}_z(x)\widehat{L}_{t+y}(x) - \Delta(x)\widehat{F}_t(x)\widehat{F}_{y+z}(x)]^2 - \Delta(x)[\widehat{L}_t(x)\widehat{F}_{z+y}(x) - \widehat{F}_z(x)\widehat{L}_{y+t}(x)]^2$$
$$= 4(-1)^y\Delta(x)^2\widehat{L}_{z-t}^2$$

Using equations (27), (28), (31) and (33)

$$(\widehat{L}_z^2(x)\widehat{L}_{t+y}^2(x) - 2\Delta(x)\widehat{L}_z(x)\widehat{F}_{t+y}(x)\widehat{F}_{y+z}(x) + \Delta(x)^2\widehat{F}_t^2(x)\widehat{F}_{y+z}^2(x)) - \Delta(x)(\widehat{L}_t^2(x)\widehat{F}_{y+z}^2(x) - 2\widehat{L}_t(x)\widehat{F}_z(x)\widehat{F}_{y+z}(x)\widehat{L}_{t+y}(x) + \widehat{F}_z^2(x)\widehat{L}_{t+y}^2(x)) = 4(-1)^y\Delta(x)^2L_{z-t}^2(x)$$

It gives that

$$\widehat{L}_{t+y}^2(x) - (-1)^{t+y+1}\widehat{F}_{z-t}(x)\widehat{L}_{t+y}(x)\widehat{F}_{y+z}(x) - \Delta(x)(-1)^{t+z}\widehat{F}_{y+z}^2(x) = (-1)^{y+z}\widehat{L}_{z-t}^2(x) \quad \text{for all} \quad t,y,z \in Z$$

□

Proof of (36) and (37) is similar to (35).

Theorem 4.9. For $n \in N$ and $m, t \in Z$ with $m \neq 0$

$$\sum_{j=0}^{j=n}\widehat{L}_{mj+t}(x) = \frac{\Delta(x)\widehat{L}_t(x) - \Delta(x)\widehat{L}_{mn+m+t}(x) + (-1)^m\Delta(x)(\widehat{L}_{mn+t}(x) - \widehat{L}_{t-m}(x))}{\Delta(x) + (-1)^m\Delta(x) - \widehat{L}_m(x)} \tag{38}$$

$$\sum_{j=0}^{j=n}\widehat{F}_{mj+t}(x) = \frac{\Delta(x)\widehat{F}_t(x) - \Delta(x)\widehat{F}_{mn+m+t}(x) + (-1)^m\Delta(x)(\widehat{F}_{mn+t}(x) - \widehat{F}_{t-m}(x))}{\Delta(x) + (-1)^m\Delta(x) - \widehat{L}_m(x)} \tag{39}$$

Proof.

Since $\quad \Delta(x)^2I - \Delta(x)^2(S(x)^m)^{n+1} = (\Delta(x)I - \Delta(x)S(x)^m)\sum_{j=0}^{j=n}\Delta(x)(S(x)^m)^j$

From Lemma (4.3) it is clear that $\det(\Delta(x)I - \Delta(x)S(x)^m) \neq 0$, therefore we get

$$\begin{aligned}(\Delta(x)I - \Delta(x)(S(x)^m))^{-1}(\Delta(x)I - \Delta(x)(S(x)^m)^{n+1})\Delta(x)S(x)^t \\ = \sum_{j=0}^{j=n}(\Delta(x)S(x)^{mj+t}) \\ = \begin{bmatrix} \frac{1}{2}\sum_{j=0}^{j=n}(\widehat{L}_{mj+t}(x)) & \frac{\Delta(x)}{2}\sum_{j=0}^{j=n}(\widehat{F}_{mj+t}(x)) \\ \frac{1}{2}\sum_{j=0}^{j=n}(\widehat{F}_{mj+t}(x)) & \frac{1}{2}\sum_{j=0}^{j=n}(\widehat{L}_{mj+t}(x)) \end{bmatrix}\end{aligned}$$

For $m \neq 0$, take $D = \Delta(x) + (-1)^m\Delta(x) - \widehat{L}_m(x)$, then we can write

$$\begin{aligned}(\Delta(x)I - \Delta(x)S(x)^m)^{-1} &= \frac{1}{D}\begin{bmatrix} \Delta(x) - \frac{(\widehat{L}_m(x))}{2} & \Delta(x)\frac{(\widehat{F}_m(x))}{2} \\ \frac{(\widehat{F}_m(x))}{2} & \Delta(x) - \frac{(\widehat{L}_m(x))}{2} \end{bmatrix} \\ &= \frac{1}{D}\left[(\Delta(x) - \frac{(\widehat{L}_m(x))}{2})I + \frac{(\widehat{F}_m(x))}{2}R\right]\end{aligned}$$

It gives that,

$$(\Delta(x)I - \Delta(x)S(x)^m)^{-1}(\Delta(x)^2S(x)^t - \Delta(x)^2S(x)^{mn+m+t})$$
$$= \frac{1}{D}\left[(\Delta(x) - \frac{(\widehat{L}_m(x))}{2})I + \frac{(\widehat{F}_m(x))}{2}R\right](\Delta(x)^2S(x)^t - \Delta(x)^2S(x)^{mn+m+t})$$

$$(\Delta(x)I - \Delta(x)S(x)^m)^{-1}(\Delta(x)^2S(x)^t - \Delta(x)^2S(x)^{mn+m+t})$$
$$= \frac{1}{D}\left[(\Delta(x) - \frac{(\widehat{L}_m(x))}{2})(\Delta(x)^2S(x)^t - \Delta(x)^2S(x)^{mn+m+t})\right]$$
$$+\frac{1}{D}\left[\frac{(\widehat{F}_m(x))}{2}R(\Delta(x)^2S(x)^t - \Delta(x)^2S(x)^{mn+m+t})\right]$$

From Corollary (4.2) and (4.3), we get

$$R(\Delta(x)^2S(x)^t - \Delta(x)^2S(x)^{mn+m+t}) =$$
$$\Delta(x)\begin{bmatrix} \Delta(x)\frac{(\widehat{F}_t(x)-\widehat{F}_{mn+m+t}(x))}{2} & \Delta(x)\frac{(\widehat{L}_t(x)-\widehat{L}_{mn+m+t}(x))}{2} \\ \frac{(\widehat{L}_t(x)-\widehat{L}_{mn+m+t}(x))}{2} & \Delta(x)\frac{(\widehat{F}_t(x)-\widehat{F}_{mn+m+t}(x))}{2} \end{bmatrix}$$

Hence, we get

$$(\Delta(x)I - \Delta(x)S(x)^m)^{-1}(\Delta(x)^2S(x)^t - \Delta(x)^2S(x)^{mn+m+t}) = \frac{\Delta(x)}{D}(\Delta(x) - \frac{\widehat{L}_m(x)}{2})$$
$$\begin{bmatrix} \Delta(x)\frac{(\widehat{L}_t(x)-\widehat{L}_{mn+m+t}(x))}{2} & \Delta(x)\frac{(\widehat{F}_t(x)-\widehat{F}_{mn+m+t}(x))}{2} \\ \frac{(\widehat{F}_t(x)-\widehat{F}_{mn+m+t}(x))}{2} & \frac{(\widehat{L}_t(x)-\widehat{L}_{mn+m+t}(x))}{2} \end{bmatrix} + \frac{\Delta(x)\widehat{F}_m(x)}{2D}$$
$$\begin{bmatrix} \frac{\Delta(x)(\widehat{F}_t(x)-\widehat{F}_{mn+m+t}(x))}{2} & \Delta(x)\frac{(\widehat{L}_t(x)-\widehat{L}_{mn+m+t}(x))}{2} \\ \frac{(\widehat{L}_t(x)-\widehat{L}_{mn+m+t}(x))}{2} & \Delta(x)\frac{(\widehat{F}_t(x)-\widehat{F}_{mn+m+t}(x))}{2} \end{bmatrix}$$

Hence, it gives that

$$\sum_{j=0}^{j=n}\widehat{L}_{mj+t}(x)$$
$$= \frac{\Delta(x)^2}{D}\left[(\Delta(x) - \frac{\widehat{L}_m(x)}{2})(\widehat{L}_t(x) - \widehat{L}_{mn+m+t}(x)) + \frac{\widehat{F}_m(x)}{2}(\widehat{F}_t(x) - \widehat{F}_{mn+m+t}(x)\right]$$
$$\sum_{j=0}^{j=n}\widehat{F}_{mj+t}(x)$$
$$= \frac{\Delta(x)^2}{D}\left[(\Delta(x) - \frac{\widehat{L}_m(x)}{2})(\widehat{F}_t(x) - \widehat{F}_{mn+m+t}(x)) + \frac{\widehat{F}_m(x)}{2}(\widehat{L}_t(x) - \widehat{L}_{mn+m+t}(x)\right]$$

Using (31) and (33), we get

$$\sum_{j=0}^{j=n}\widehat{L}_{mj+t}(x)=\frac{\Delta(x)\widehat{L}_t(x)-\Delta(x)\widehat{L}_{mn+m+t}(x)+(-1)^m\Delta(x)(\widehat{L}_{mn+t}(x)-\widehat{L}_{t-m}(x))}{\Delta(x)+(-1)^m\Delta(x)-\widehat{L}_m(x)}$$

$$\sum_{j=0}^{j=n}\widehat{F}_{mj+t}(x)=\frac{\Delta(x)\widehat{F}_t(x)-\Delta(x)\widehat{F}_{mn+m+t}(x)+(-1)^m\Delta(x)(\widehat{F}_{mn+t}(x)-\widehat{F}_{t-m}(x))}{\Delta(x)+(-1)^m\Delta(x)-\widehat{L}_m(x)}$$

□

Theorem 4.10. For $n \in N$ and $m,\, t \in Z$

$$\sum_{j=0}^{j=n}(-1)^j\widehat{L}_{mj+t}(x)=\frac{\Delta(x)\widehat{L}_t(x)-\Delta(x)\widehat{L}_{mn+m+t}(x)+(-1)^m\Delta(x)(\widehat{L}_{mn+t}(x)-\widehat{L}_{t-m}(x))}{\Delta(x)+(-1)^m\Delta(x)-\widehat{L}_m(x)} \tag{40}$$

$$\sum_{j=0}^{j=n}(-1)^j\widehat{F}_{mj+t}(x)=\frac{\Delta(x)\widehat{F}_t(x)-\Delta(x)\widehat{F}_{mn+m+t}(x)+(-1)^m\Delta(x)(\widehat{F}_{mn+t}(x)-\widehat{F}_{t-m}(x))}{\Delta(x)+(-1)^m\Delta(x)-\widehat{L}_m(x)} \tag{41}$$

Proof. **Case:** 1 If n is an even natural number then we have

$$\Delta(x)^2I+\Delta(x)^2(S(x)^m)^{n+1}=(\Delta(x)I+\Delta(x)S(x)^m)\sum_{j=0}^{j=n}(\Delta(x)S(x)^m)^j(-1)^j$$

From Lemma (4.3) it is clear that $\det(\Delta(x)I+\Delta(x)S(x)^m)\neq 0$, therefore we get

$$\begin{aligned}\Delta(x)I+\Delta(x)(S(x)^m)^{-1}(\Delta(x)I+\Delta(x)(S(x)^m)^{n+1})\Delta(x)S(x)^t\\ =\sum_{j=0}^{j=n}(-1)^j\Delta(x)(S(x)^{mj+t})\end{aligned}$$

$$=\begin{bmatrix}\frac{1}{2}\sum_{j=0}^{j=n}(-1)^j(\widehat{L}_{mj+t}(x)) & \frac{\Delta(x)}{2}\sum_{j=0}^{j=n}(-1)^j(\widehat{F}_{mj+t}(x))\\ \frac{1}{2}\sum_{j=0}^{j=n}(-1)^j(\widehat{F}_{mj+t}(x)) & \frac{1}{2}\sum_{j=0}^{j=n}(-1)^j(\widehat{L}_{mj+t}(x))\end{bmatrix}$$

For $m\neq 0$ take $d=\Delta(x)+\Delta(x)(-1)^m+\widehat{L}_m(x)$, then we get

$$\begin{aligned}(\Delta(x)I+\Delta(x)S(x)^m)^{-1}&=\frac{1}{d}\begin{bmatrix}\Delta(x)+\frac{(\widehat{L}_m(x))}{2} & -\Delta(x)\frac{(\widehat{F}_m(x))}{2}\\ \frac{(-\widehat{F}_m(x))}{2} & \Delta(x)+\frac{(\widehat{L}_m(x))}{2}\end{bmatrix}\\ &=\frac{1}{d}\left[(\Delta(x)+\frac{(\widehat{L}_m(x))}{2})I-\frac{(\widehat{F}_m(x))}{2}R\right]\end{aligned}$$

Thus, it is seen that

$$\begin{aligned}(\Delta(x)I+\Delta(x)S(x)^m)^{-1}(\Delta(x)^2S(x)^t+\Delta(x)^2S(x)^{mn+m+t})\\ =\frac{1}{d}\left[(\Delta(x)+\frac{(\widehat{L}_m(x))}{2})I-\frac{(\widehat{F}_m)}{2}R\right]\\ (\Delta(x)^2S(x)^t+\Delta(x)^2S(x)^{mn+m+t})\end{aligned}$$

By Corollary (4.2) and (4.3), we get

$$R(\Delta(x)^2S(x)^t+\Delta(x)^2S(x)^{mn+m+t}) = \Delta(x)\begin{bmatrix} \Delta(x)\frac{(\widehat{F}_t(x)+\widehat{F}_{mn+m+t}(x))}{2} & \Delta(x)\frac{(\widehat{L}_t(x)+\widehat{L}_{mn+m+t}(x))}{2} \\ \frac{(\widehat{L}_t(x)+\widehat{L}_{mn+m+t}(x))}{2} & \Delta(x)\frac{(\widehat{F}_t(x)+\widehat{F}_{mn+m+t}(x))}{2}\end{bmatrix}$$

Thus, it follows that

$$(\Delta(x)I+\Delta(x)S(x)^m)^{-1}(\Delta(x)^2S(x)^t+\Delta(x)^2S(x)^{mn+m+t}) = \frac{\Delta(x)}{d}(\Delta(x)+\frac{\widehat{L}_m(x)}{2}) \begin{bmatrix} \Delta(x)\frac{(\widehat{L}_t(x)+\widehat{L}_{mn+m+t}(x))}{2} & \Delta(x)\frac{(\widehat{F}_t(x)+\widehat{F}_{mn+m+t}(x))}{2} \\ \frac{(\widehat{F}_t(x)+\widehat{F}_{mn+m+t}(x))}{2} & \frac{(\widehat{L}_t(x)+\widehat{L}_{mn+m+t}(x))}{2}\end{bmatrix} - \frac{\Delta(x)\widehat{F}_m(x)}{2d}\begin{bmatrix} \Delta(x)\frac{\Delta(x)(\widehat{F}_t(x)+\widehat{F}_{mn+m+t}(x))}{2} & \Delta(x)\frac{(\widehat{L}_t(x)+\widehat{L}_{mn+m+t}(x))}{2} \\ \frac{(\widehat{L}_t(x)+\widehat{L}_{mn+m+t}(x))}{2} & \Delta(x)\frac{(\widehat{F}_t(x)+\widehat{F}_{mn+m+t}(x))}{2}\end{bmatrix}$$

Thus, it gives that

$$\sum_{j=0}^{j=n}(-1)^j\widehat{L}_{mj+t}(x) = \frac{\Delta(x)^2}{d}\left[(\Delta(x)+\frac{\widehat{L}_m(x)}{2})(\widehat{L}_t(x)+\widehat{L}_{mn+m+t}(x)) - \frac{\widehat{F}_m(x)\Delta(x)}{2}(\widehat{F}_t(x)+\widehat{F}_{mn+m+t}(x)\right]$$

$$\sum_{j=0}^{j=n}(-1)^j\widehat{F}_{mj+t}(x) = \frac{\Delta(x)^2}{d}\left[(\Delta(x)+\frac{\widehat{L}_m(x)}{2})(\widehat{F}_t(x)+\widehat{F}_{mn+m+t}(x)) - \frac{\widehat{F}_m(x)\Delta(x)}{2}(\widehat{L}_t(x)+\widehat{L}_{mn+m+t}(x)\right]$$

Using (31) and (33), we get

$$\sum_{j=0}^{j=n}(-1)^j\widehat{L}_{mj+t}(x) = \frac{\Delta(x)\widehat{L}_t(x)-\Delta(x)\widehat{L}_{mn+m+t}(x)+(-1)^m\Delta(x)(\widehat{L}_{mn+t}(x)-\widehat{L}_{t-m}(x))}{\Delta(x)+(-1)^m\Delta(x)-\widehat{L}_m(x)}$$

$$\sum_{j=0}^{j=n}(-1)^j\widehat{F}_{mj+t}(x) = \frac{\Delta(x)\widehat{F}_t(x)-\Delta(x)\widehat{F}_{mn+m+t}(x)+(-1)^m\Delta(x)(\widehat{F}_{mn+t}(x)-\widehat{F}_{t-m}(x))}{\Delta(x)+(-1)^m\Delta(x)-\widehat{L}_m(x)}$$

Case: 2 If n is an odd natural number, we get

$$\sum_{j=0}^{j=n}(-1)^j\widehat{L}_{mj+t}(x) = \sum_{j=0}^{j=n-1}(-1)^j\widehat{L}_{mj+t}(x)-\widehat{L}_{mn+t}(x)$$

Since n is an odd natural number then $(n-1)$ is an even number, hence using case-I, it follows that,

$$\sum_{j=0}^{j=n}(-1)^j\widehat{L}_{mj+t}(x) = \frac{\Delta(x)\widehat{L}_t(x)+\Delta(x)\widehat{L}_{mn+t}(x)+(-1)^m\Delta(x)(\widehat{L}_{mn-m+t}(x)+\widehat{L}_{t-m}(x))}{\Delta(x)+(-1)^m\Delta(x)+\widehat{L}_m(x)} - \widehat{L}_{mn+t}(x)$$
$$= \frac{\Delta(x)\widehat{L}_t(x)+(-1)^m\Delta(x)(\widehat{L}_{mn-m+t}(x)+\widehat{L}_{t-m}(x))-(-1)^m\Delta(x)\widehat{L}_{mn+t}(x)-\Delta(x)\widehat{L}_m(x)\widehat{L}_{mn+t}(x)}{\Delta(x)+(-1)^m\Delta(x)+\widehat{L}_m(x)}$$

Using equations (31) and (33), we get

$$\sum_{j=0}^{j=n}(-1)^j\widehat{L}_{mj+t}(x) = \frac{\Delta(x)\widehat{L}_t(x)+\Delta(x)\widehat{L}_{mn-m+t}(x)+(-1)^m\Delta(x)(\widehat{L}_{t-m}(x)-\widehat{L}_{mn+t}(x))}{\Delta(x)+(-1)^m\Delta(x)+\widehat{L}_m(x)}$$
$$\sum_{j=0}^{j=n}(-1)^j\widehat{F}_{mj+t}(x) = \frac{\Delta(x)\widehat{F}_t(x)-\Delta(x)\widehat{L}_{mn+m+t}(x)+(-1)^m\Delta(x)(\widehat{F}_{t-m}(x)-\widehat{F}_{mn+t}(x))}{\Delta(x)+(-1)^m\Delta(x)+\widehat{L}_m(x)}$$

□

5 Tables

n	$\widehat{F}_n(x)$	$\widehat{L}_n(x)$
1	$4+x^2$	$4x+x^3$
2	$4x+x^3$	$8+6x^2+x^4$
3	$4+5x^2+x^4$	$12x+7x^3+x^5$
4	$8x+6x^3+x^5$	$8+18x^2+8x^4+x^6$
5	$4+13x^2+7x^4+x^6$	$20x+25x^3+9x^5+x^7$
6	$12x+19x^3+8x^5+x^7$	$8+38x^2+33x^4+10x^6+x^8$
7	$4+25x^2+26x^4+9x^6+x^8$	$28x+63x^3+42x^5+11x^7+x^9$
8	$16x+44x^3+34x^5+10x^7+x^9$	$8+66x^2+96x^4+52x^6+12x^8+x^{10}$
9	$4+41x^2+70x^4+43x^6+11x^8+x^{10}$	$36x+129x^3+138x^5+63x^7+13x^9+x^{11}$
10	$20x+85x^3+104x^5+53x^7+12x^9+x^{11}$	$8+102x^2+225x^4+190x^6+75x^8+14x^{10}+x^{12}$
11	$4+61x^2+155x^4+147x^6+64x^8+13x^{12}+x^{12}$	$44x+231x^3+363x^5+253x^7+88x^9+15x^{11}+x^{13}$
12	$24x+146x^3+259x^5+200x^7+76x^9+14x^{11}+x^{13}$	$8+146x^2+456x^4+553x^6+328x^8+102x^{10}+16x^{12}+x^{14}$

Table 2: **First 12 terms of $\widehat{F}_n(x)$ and $\widehat{L}_n(x)$**

$\widehat{F}_n(x)$	$\widehat{F}_1(x)$	$\widehat{F}_2(x)$	$\widehat{F}_3(x)$	$\widehat{F}_4(x)$	$\widehat{F}_5(x)$	$\widehat{F}_6(x)$	$\widehat{F}_7(x)$	$\widehat{F}_8(x)$	$\widehat{F}_9(x)$	$\widehat{F}_{10}(x)$
$x = 1$	5	5	10	15	25	40	65	105	170	275
$x = 2$	8	16	40	96	232	560	1352	3264	7880	19024
$x = 3$	13	39	130	429	1417	4680	15457	51051	168610	556881
$x = 4$	20	80	340	1440	6100	25840	109460	463680	1964180	8320400
$x = 5$	29	145	754	3915	20329	105560	548129	2846205	14779154	76741975
$x = 6$	40	240	1480	9120	56200	346320	2134120	13151040	81040360	499393200
$x = 7$	53	371	2650	18921	135097	964600	6887297	49175679	351117050	2506995029
$x = 8$	68	544	4420	35904	291652	2369120	19244612	156326016	1269852740	10315147936
$x = 9$	85	765	6970	63495	578425	5269320	48002305	437290065	3983612890	36289806075
$x = 10$	104	1040	10504	106080	1071304	10819120	109262504	1103444160	11143704104	112540485200
$x = 11$	125	1375	15250	169125	1875625	20801000	230686625	2558353875	28372579250	314656725625
$x = 12$	148	1776	21460	259296	3133012	37855440	457398292	5526634944	66777017620	806850846384
$x = 13$	173	2249	29410	384579	5028937	65760760	859918817	11244705381	147041088770	1922778859391
$x = 14$	200	2800	39400	554400	7801000	109768400	1544558600	21733588800,	305814801800	4303140814000
$x = 15$	229	3435	51754	779745	11747929	176998680	2666728129	40177920615,	605335537354	9120210980925
$x = 16$	260	4160	66820	1073280	17239300	276902080	4447672580	71439663360	1147482286340	18431156244800
$x = 17$	293	4981	84970	1449471	24725977	421791080	7195174337	122739754809	2093771006090	35716846858339
$x = 18$	328	5904	106600	1924704	34751272	627447600	11328808072	204545992896	3693156680200	66681366236496
$x = 19$	365	6935	132130	2517405	47962825	913811080	17410373345	331710904635	6319917561410	120410144571425
$x = 20$	404	8080	162004	3248160	65125204	1305752240	26180170004	524909152320	10524363216404	211012173480400

Table 3: **First few terms of** $\widehat{F}_n$

$\widehat{L}_n(x)$	$\widehat{L}_1(x)$	$\widehat{L}_2(x)$	$\widehat{L}_3(x)$	$\widehat{L}_4(x)$	$\widehat{F}_5(x)$	$\widehat{L}_6(x)$	$\widehat{L}_7(x)$	$\widehat{L}_8(x)$	$\widehat{L}_9(x)$	$\widehat{L}_{10}(x)$
$x=1$	5	15	20	35	55	90	145	235	380	615
$x=2$	16	48	112	272	656	1584	3824	9232	22288	53808
$x=3$	39	143	468	1547	5109	16874	55731	184067	607932	2007863
$x=4$	80	360	1520	6440	27280	115560	489520	2073640	8784080	3720996
$x=5$	145	783	4060	21083	109475	568458	2951765	15327283	79588180	413268183
$x=6$	240	1520	9360	57680	355440	2190320	13497360	83174480	512544240	3158439920
$x=7$	371	2703	19292	137747	983521	7022394	50140279	358004347	2556170708	18251199303
$x=8$	544	4488	36448	296072	2405024	19536264	158695136	1289097352	10471473952	85060888968
$x=9$	765	7055	64260	585395	5332815	48580730	442559385	4031615195	36727096140	334575480455
$x=10$	1040	10608	107120	1081808	10925200	110333808	1114263280	11252966608	113643929360	1147692260208
$x=11$	1375	15375	170500	1890875	20970125	232562250	2579154875	28603265875	317215079500	3517969140375
$x=12$	1776	21608	261072	3154472	38114736	460531304	5564490384	67234415912	812377481328	9815764191848
$x=13$	2249	29583	386828	5058347	66145339	864947754	11310466141	147901007587	1934023564772	25290207349623
$x=14$	2800	39600	557200	7840400	110322800	1552359600	21843357200	307359360400	4324874402800	60855600999600
$x=15$	3435	51983	783180	11799683	177778425	2678476058	40354919295	608002265483	9160388901540	138013835788583
$x=16$	4160	67080	1077440	17306120	277975360	4464911880	71716565440	1151929958920	18502595908160	297193464489480
$x=17$	4981	85263	1454452	24810947	423240551	7219900314	123161545889	2100966180427	35839586613148	611373938603943
$x=18$	5904	106928	1930608	34857872	629372304	11363559344	205173440496	3704485488272	66885912229392	1207650905617328
$x=19$	6935	132495	2524340	48094955	916328485	17458336170	332624715715	6337327934755	120741855476060	2300432581979895
$x=20$	8080	162408	3256240	65287208	1309000400	26245295208	526214904560	10550543386408	211537082632720	4241292196040808

Table 4: **First few terms of** $\widehat{L}_n(x)$

References

[1] Srisawat, Somnuk, and Wanna Sriprad, On the (s,t)-Pell and (s,t)-Pell Lucas numbers by matrix methods, *Applied Mathematics LettersIn Annales Mathematicae et Informaticae*, Vol. 46, (2016), 195204.
http://ami.ektf.hu/uploads/papers/finalpdf/AMI_46_from195to204

[2] H.H. Gulec, N. Taskara, On the (s,t)-Pell and (s,t)-Pell Lucas sequences and their matrix representations, *Applied Mathematics Letters*, Vol. 25, (2012), 15541559.
https://doi.org/10.1016/j.aml.2012.01.014

[3] Yuan, Yi, and Wenpeeg Zhang, Some identities involving the Fibonacci polynomials. *Fibonacci Quarterly*, Vol. 40(02), (2002), 314–318.
http://www.fq.math.ca/Scanned/40-4/yuan

[4] Wu, Zhengang, and Wenpeng Zhang, Several identities involving the Fibonacci polynomials and Lucas polynomials, *Journal of Inequalities and Applications*, Vol. 2013(1), (2013), 205–218.
http://link.springer.com/article/10.1186/1029-242X-2013-205

[5] Wu, Zhengang, and Wenpeng Zhang, The sums of the reciprocals of Fibonacci polynomials and Lucas polynomials, *Journal of Inequalities and Applications*, Vol. 2012(1), (2012), 134–140.
http://link.springer.com/article/10.1186/1029-242X-2012-134

[6] Koken, F., Bozkurt, D. On the Jacobsthal numbers by matrix methods, *Int. J. Contemp. Math. Sciences*, Vol. 3(13), (2008), 605-614.
http://m-hikari.com/ijcms-password2008/33-36-2008/kokenIJCMS33-36-2008

[7] Silvester, J. R. Fibonacci properties by matrix methods, *The Mathematical Gazette*, Vol. 63(425), (1979), 188-191.
http://www.jstor.org/stable/3617892

[8] Nalli A, Haukkanen P. On generalized Fibonacci and Lucas polynomials, *Chaos, Solitons and Fractals*, Vol. 42(5), (2009), 3179-86.
https://doi.org/10.1016/j.chaos.2009.04.048

[9] A. F. Horadam, Basic Properties of a Certain Generalized Sequence of Numbers, *Fibonacci Quarterly*, Vol. 03(3), (1965), 61 – 76.
http://www.fq.math.ca/Scanned/3-3/horadam.

[10] S. Falcon, A. Plaza, On k -Fibonacci sequences and polynomials and their derivatives, *Chaos, Solitons and Fractals*, Vol. 39(03), (2009), 1005 – 1019.
https://doi.org/10.1016/j.chaos.2007.03.007.

[11] S. Falcon, A. Plaza, On the k Fibonacci numbers, *Chaos, Solitons and Fractals*, Vol. 32(05), (2007), 1615 – 1624.
https://doi.org/10.1016/j.chaos.2006.09.022.

[12] S. Falcon, A. Plaza, The k Fibonacci sequence and the Pascal 2 triangle, *Chaos, Solitons and Fractals*, Vol. 33(1), (2007), 38 - 49 .
https://doi.org/10.1016/j.chaos.2006.10.022.

[13] S. Falcon, A. Plaza, The k Fibonacci hyperbolic functions, *Chaos, Solitons and Fractals*, Vol. 38(2), (2008), 409 – 20.
https://doi.org/10.1016/j.chaos.2006.11.019.

[14] P. Filipponi, A. F. Horadam, A Matrix Approach to Certain Identities, *The Fibonacci Quarterly*, Vol. 26(2), (1988), 115 – 126.
http://www.fq.math.ca/Scanned/26-2/filipponi

[15] Waddill, Marcellus E., Louis Sacks , Another generalized Fibonacci sequence, *The Fibonacci Quarterly*, Vol. 5(03), (1967) 209 – 222.
http://www.fq.math.ca/Scanned/5-3/waddill

[16] D. Kalman, Generalized Fibonacci numbers by matrix methods, *Fibonacci Quarterly*, Vol. 20 (1),(1982), 73–76.
http://www.fq.math.ca/Scanned/20-1/kalman

[17] C. Bolat, Properties and applications of k -Fibonacci, k-Lucas numbers, *M. S. Thesis, Selcuk University*, (2008), Konya, Turkey.
http://www.m-hikari.com/ijcms-2010/21-24-2010/bolatIJCMS21-24-2010

[18] Paula Catarino, Vasco P., ASome basic properties and a two-by-two matrix involving the k-Pell numbers, *Int. Journal of Math. Analysis*, Vol. 7(45),(2013), 2209–2215.
https://doi.org/10.1016/j.chaos.2015.04.017

[19] S. Falcon, A. Plaza, On the 3 - dimensional k - Fibonacci spirals, *Chaos, Solitons and Fractals*, Vol. 38, (2008), 993–1003.
https://doi.org/10.1016/j.chaos.2007.02.009

[20] S. Falcon, A. Plaza, On k-Fibonacci numbers of arithmetic indexes, *Applied Mathematics and Computation*, Vol. 208 (1),(2009), 180–185.
https://doi.org/10.1016/j.amc.2008.11.031

[21] F. Sergio, Generalized Fibonacci sequences generated from a k-Fibonacci sequence, *Journal of Mathematics Research*, Vol. 4 (2),(2012), 97–100.
http://www.ccsenet.org/journal/index.php/jmr/article/download/14516/10822

[22] A. D. Godase, M. B. Dhakne, On the properties of k Fibonacci and k Lucas numbers, *International Journal of Advances in Applied Mathematics and Mechanics*, Vol. 02 (2014), 100 – 106.
http://www.ijaamm.com/uploads/2/1/4/8/21481830/v2n1p12

[23] A. D. Godase, M. B. Dhakne, Summation Identities for k Fibonacci and k Lucas Numbers using Matrix Methods, *International Journal of Mathematics and Scientific Computing* , Vol. 05(2), (2015), 75 – 78.

https://www.veltechuniv.edu.in/wp-content/uploads/2016/04/Paper-03-15

[24] A. D. Godase, M. B. Dhakne, Determinantal Identities for k Lucas Sequence, *Journal of New Theory*, Vol. 12(1), (2015), 01 – 07.
http://www.newtheory.org/jnt-papers/12-2016/JNT-12-2016-01

[25] R. S. Melham, Sums of certain products of Fbonacci and Lucas Numbers, *The Fibonacci Quarterly*, Vol. 37(3) (1999), 248–251.
https://opus.lib.uts.edu.au/bitstream/10453/17904/1/2011004786OK

[26] Y. Yazlik, N. Yilmaz and N. Taskara, On the Sums of Powers of k-Fibonacci and k-Lucas Sequences, *Seluk J. Appl. Math.*, Vol. Special Issue (2012), 47–50.
http://sjam.selcuk.edu.tr/sjam/article/view/327/249

[27] Demirturk B., Fibonacci and Lucas Sums by matrix Methods, *International Mathematical Forum*, Vol. 5(3) (2010), 99–107.
http://m-hikari.com/imf-2010/1-4-2010/demirturkIMF1-4-2010

[28] Ugyun, S. and Eldogan H., k-Jacobsthal and k-Jacobsthal Lucas matrix sequences, *International Mathematical Forum*, Vol. 11(3) (2016), 145–154.
http://m-hikari.com/imf-2010/1-4-2010/demirturkIMF1-4-2010

http://www.newtheory.org ISSN: 2149-1402

Journal of New Theory

Received: 21.11.2017 *Year*:**2017**, *Number*: **19**, *Pages*: **20-26**
Published: 13.12.2017 *Original Article*

ON NANO πgp-CLOSED SETS

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Abstaract − In this paper, new classes of sets called πgp-closed sets in nano topological spaces is introduced and its properties and studied of nano πgp-closed sets.

Keywords − *Nano π-closed set, nano πg-closed set, nano πgp-closed sets and nano gpr-closed set.*

1 Introduction

Thivagar et al. [4] introduced a nano topological space with respect to a subset X of an universe which is defined in terms of lower approximation and upper approximation and boundary region. The classical nano topological space is based on an equivalence relation on a set, but in some situation, equivalence relations are nor suitable for coping with granularity, instead the classical nano topology is extend to general binary relation based covering nano topological space

Bhuvaneswari et al. [3] introduced and investigated nano g-closed sets in nano topological spaces. Recently, Parvathy and Bhuvaneswari the notions of nano gpr-closed sets which are implied both that of nano rg-closed sets. In 2017, Rajasekaran et.al [7] introduced the notion of nano πgp-closed sets in nano topological spaces. new classes of sets called πgp-closed sets in nano topological spaces is introduced and its properties and studied of nano πgp-closed sets.

2 Preliminaries

Throughout this paper $(U, \tau_R(X))$ (or X) represent nano topological spaces on which no separation axioms are assumed unless otherwise mentioned. For a subset H of a

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space $(U, \tau_R(X))$, $Ncl(H)$ and $Nint(H)$ denote the nano closure of H and the nano interior of H respectively. We recall the following definitions which are useful in the sequel.

Definition 2.1. *[6] Let U be a non-empty finite set of objects called the universe and R be an equivalence relation on U named as the indiscernibility relation. Elements belonging to the same equivalence class are said to be indiscernible with one another. The pair (U, R) is said to be the approximation space. Let $X \subseteq U$.*

1. *The lower approximation of X with respect to R is the set of all objects, which can be for certain classified as X with respect to R and it is denoted by $L_R(X)$. That is, $L_R(X) = \bigcup_{x\in U}\{R(x) : R(x) \subseteq X\}$, where R(x) denotes the equivalence class determined by x.*

2. *The upper approximation of X with respect to R is the set of all objects, which can be possibly classified as X with respect to R and it is denoted by $U_R(X)$. That is, $U_R(X) = \bigcup_{x\in U}\{R(x) : R(x) \cap X \neq \phi\}$.*

3. *The boundary region of X with respect to R is the set of all objects, which can be classified neither as X nor as not - X with respect to R and it is denoted by $B_R(X)$. That is, $B_R(X) = U_R(X) - L_R(X)$.*

Property 2.2. *[4] If (U, R) is an approximation space and $X, Y \subseteq U$; then*

1. $L_R(X) \subseteq X \subseteq U_R(X)$;
2. $L_R(\phi) = U_R(\phi) = \phi$ *and* $L_R(U) = U_R(U) = U$;
3. $U_R(X \cup Y) = U_R(X) \cup U_R(Y)$;
4. $U_R(X \cap Y) \subseteq U_R(X) \cap U_R(Y)$;
5. $L_R(X \cup Y) \supseteq L_R(X) \cup L_R(Y)$;
6. $L_R(X \cap Y) \subseteq L_R(X) \cap L_R(Y)$;
7. $L_R(X) \subseteq L_R(Y)$ *and* $U_R(X) \subseteq U_R(Y)$ *whenever* $X \subseteq Y$;
8. $U_R(X^c) = [L_R(X)]^c$ *and* $L_R(X^c) = [U_R(X)]^c$;
9. $U_RU_R(X) = L_RU_R(X) = U_R(X)$;
10. $L_RL_R(X) = U_RL_R(X) = L_R(X)$.

Definition 2.3. *[4] Let U be the universe, R be an equivalence relation on U and $\tau_R(X) = \{U, \phi, L_R(X), U_R(X), B_R(X)\}$ where $X \subseteq U$. Then by the Property 2.2, R(X) satisfies the following axioms:*

1. *U and $\phi \in \tau_R(X)$,*
2. *The union of the elements of any sub collection of $\tau_R(X)$ is in $\tau_R(X)$,*
3. *The intersection of the elements of any finite subcollection of $\tau_R(X)$ is in $\tau_R(X)$.*

That is, $\tau_R(X)$ is a topology on U called the nano topology on U with respect to X. We call $(U, \tau_R(X))$ as the nano topological space. The elements of $\tau_R(X)$ are called as nano open sets and $[\tau_R(X)]^c$ is called as the dual nano topology of $[\tau_R(X)]$.

Remark 2.4. *[4] If $[\tau_R(X)]$ is the nano topology on U with respect to X, then the set $B = \{U, \phi, L_R(X), B_R(X)\}$ is the basis for $\tau_R(X)$.*

Definition 2.5. *[4] If $(U, \tau_R(X))$ is a nano topological space with respect to X and if $H \subseteq U$, then the nano interior of H is defined as the union of all nano open subsets of H and it is denoted by $Nint(H)$.*

That is, $Nint(H)$ is the largest nano open subset of H. The nano closure of H is defined as the intersection of all nano closed sets containing H and it is denoted by $Ncl(H)$.

That is, $Ncl(H)$ is the smallest nano closed set containing H.

Definition 2.6. *A subset H of a nano topological space $(U, \tau_R(X))$ is called*

1. *nano semi open [4] if $H \subseteq Ncl(Nint(H))$.*
2. *nano regular-open [4] if $H = Nint(Ncl(H))$.*
3. *nano π-open [1] if the finite union of nano regular-open sets.*
4. *nano pre-open [4] if $H \subseteq Nint(Ncl(H))$.*

The complements of the above mentioned sets is called their respective closed sets.

Definition 2.7. *A subset H of a nano topological space $(U, \tau_R(X))$ is called;*

1. *nano g-closed [2] if $Ncl(H) \subseteq G$, whenever $H \subseteq G$ and G is nano open.*
2. *nano rg-closed set [8] if $Ncl(H) \subseteq G$ whenever $H \subseteq G$ and G is nano regular-open.*
3. *nano πg-closed [7] if $Ncl(H) \subseteq G$, whenever $H \subseteq G$ and G is nano π-open.*
4. *nano gp-closed set [3] if $Npcl(H) \subseteq G$, whenever $H \subseteq G$ and G is nano open.*
5. *nano gpr-closed set [5] if $Npcl(H) \subseteq G$, whenever $H \subseteq G$ and G is nano regular open.*

3 On Nano πgp-closed Sets

Definition 3.1. *A subset H of a space $(U, \tau_R(X))$ is nano πgp-closed if $Npcl(H) \subseteq G$ whenever $H \subseteq G$ and G is nano π-open.*

The complement of nano πgp-open if $H^c = U - H$ is nano πgp-closed.

Example 3.2. *Let $U = \{a, b, c, d\}$ with $U/R = \{\{a, b\}, \{c\}, \{d\}\}$ and $X = \{a, d\}$. Then the nano topology $\tau_R(X) = \{\phi, \{d\}, \{a, b\}, \{a, b, d\}, U\}$.*

1. *then $\{a\}$ is nano πgp-closed set.*

2. *then $\{b\}$ is not nano πgp-closed set.*

Theorem 3.3. *In a space $(U, \tau_R(X))$,*

1. *If H is nano π-open and nano πgp-closed, then H is nano pre-closed and hence nano clopen.*

2. *If H is nano semi-open and nano πgp-closed, then H is nano πg-closed.*

Proof. 1. If H is nano π-open and nano πgp-closed, then $Npcl(H) \subseteq H$ and so H is nano pre-closed. Hence H is nano clopen, since nano π-open set is nano open and nano pre-closed open set is nano closed.

2. Let $H \subseteq G$ and G be nano π-open. Since H is nano πgp-closed, $Npcl(H) \subseteq G$. Since H is nano semi-open, $Npcl(H) = Ncl(H) \subseteq G$ and hence H is nano πg-closed.

Remark 3.4. *For a subset of a space $(U, \tau_R(X))$, we have the following implications:*

$$
\begin{array}{ccccc}
\textbf{\textit{nano closed}} & & & & \\
\Downarrow & & & & \\
\textbf{\textit{nano }} g\textbf{\textit{-closed}} & \Rightarrow & \textbf{\textit{nano }} \pi g\textbf{\textit{-closed}} & \Rightarrow & \textbf{\textit{nano }} rg\textbf{\textit{-closed}} \\
\Downarrow & & \Downarrow & & \Downarrow \\
\textbf{\textit{nano }} gp\textbf{\textit{-closed}} & \Rightarrow & \textbf{\textit{nano }} \pi gp\textbf{\textit{-closed}} & \Rightarrow & \textbf{\textit{nano }} gpr\textbf{\textit{-closed}}
\end{array}
$$

None of the above implications are reversible as shown by the following Examples.

Example 3.5. *Let $U = \{a, b, c\}$ with $U/R = \{\{a, c\}, \{b\}\}$ and $X = \{c\}$. Then the nano topology $\tau_R(X) = \{\phi, \{a, c\}, U\}$. Then $\{a, c\}$ is nano πgp-closed set but not nano gp-closed.*

Example 3.6. *In Example 3.2,*

1. *then $\{a\}$ is nano πgp-closed set but not nano πg-closed.*

2. *then $\{a, d\}$ is nano gpr-closed set but not nano πgp-closed.*

Lemma 3.7. *In a space $(U, \tau_R(X))$,*

1. *every nano open set is nano πgp-closed.*

2. *every nano closed set is nano πgp-closed.*

Remark 3.8. *The converses of statements in Lemma 3.7 are not necessarily true as seen from the following Examples.*

Example 3.9. *In Example 3.2,*

1. *then $\{a, c, d\}$ is nano πgp-closed set but not nano open.*

2. *then $\{b, c, d\}$ is nano πgp-closed set but not nano closed.*

Theorem 3.10. *Let H be nano πgp-closed. Then $Npcl(H) - H$ does not contain any non-empty nano π-closed set.*

Proof. Let K be a nano π-closed set such that $K \subseteq Npcl(H) - H$. Then $H \subseteq U - K$. Since H is nano πgp-closed and $U - K$ is nano π-open, $Npcl(H) \subseteq U - K$, i.e. $K \subseteq U - Npcl(H)$. Hence $K \subseteq Npcl(H) \cap (U - Npcl(H)) = \phi$. This shows that $K = \phi$.

Corollary 3.11. *Let H be nano πgp-closed. Then H is nano pre-closed $\iff$ $Npcl(H) - H$ is nano π-closed $\iff$ $H = Npcl(Nint(H))$.*

Theorem 3.12. *In a space $(U, \tau_R(X))$, the union of two nano πgp-closed sets is nano πgp-closed.*

Proof. Let $H \cup Q \subseteq G$, then $H \subseteq G$ and $Q \subseteq G$ where G is nano π-open. As H and Q are πgp-closed, $Ncl(H) \subseteq G$ and $Ncl(Q) \subseteq G$. Hence $Ncl(H \cup Q) = Ncl(H) \cup Ncl(Q) \subseteq G$.

Example 3.13. *In Example 3.2, then $H = \{a\}$ and $Q = \{c\}$ is nano πgp-closed. Clearly $H \cup Q = \{a, c\}$ is nano πgp-closed.*

Theorem 3.14. *In a space $(U, \tau_R(X))$, the intersection of two nano πgp-open sets are nano πgp-open.*

Proof. Obvious by Theorem 3.12.

Example 3.15. *In Example 3.2, then $H = \{a, d\}$ and $Q = \{b, d\}$ is nano πgp-open. Clearly $H \cap Q = \{d\}$ is nano πgp-open.*

Remark 3.16. *In a space $(U, \tau_R(X))$, the union of two nano πgp-closed sets but not nano πgp-closed.*

Example 3.17. *In Example 3.2, then $H = \{a\}$ and $Q = \{b\}$ is nano πgp-closed. Clearly $H \cup Q = \{a, b\}$ is but not nano πgp-closed.*

Remark 3.18. *In a space $(U, \tau_R(X))$, the intersection of two nano πgp-open sets but not nano πgp-open.*

Example 3.19. *In Example 3.2, then $H = \{a, c, d\}$ and $Q = \{b, c, d\}$ is nano πgp-open sets. Clearly $H \cup Q = \{c, d\}$ is but not nano πgp-open.*

Corollary 3.20. *If H ia nano πgp-closed and nano regular open and P is nano pre-closed in U, then $H \cap P$ is nano πgp-closed.*

Proof. Let $H \cap P \subseteq G$ and G is nano π-open in H. Since P is nano pre-closed in U, $H \cap P$ is nano pre-closed in H and so $Npcl_H(H \cap P) = H \cap P$. That is $Npcl_H(H \cap P) \subseteq G$. Then $H \cap P$ is nano πgp-closed in the nano πgp-closed and nano regular open set H and hence $H \cap P$ is nano πgp-closed in U.

Theorem 3.21. *If H is nano πgp-closed in U and $H \subseteq P \subseteq Npcl(H)$, then P is nano πgp-closed.*

Proof. Let $P \subseteq G$ and G be nano π-open in U. Since $H \subseteq G$ and H is nano πgp-closed, $Npcl(H) \subseteq G$ and then $Npcl(P) = Npcl(H) \subseteq G$. hence P is nano πgp-closed.

Theorem 3.22. *A set H of a space $(U, \tau_R(X))$ is called nano πgp-open $\iff$ if $K \subseteq Npint(H)$ whenever K is nano π-closed and $K \subseteq H$.*

Proof. Obvious.

Theorem 3.23. *A subset H of U is nano πgp-open $\iff$ $G = U$ whenever G is nano π-open and $Npint(H) \cup (U - H) \subseteq G$.*

Proof. Let G ba a nano π-open set and $Npint(H) \cup (U - H) \subseteq G$. Then $U - G \subseteq (U - Npint(H)) \cap H$, i.e., $(U - G) \subseteq Npcl(U - H) - (U - H)$. Since $U - H$ is nano πgp-closed, by Theorem 3.10, $U - G = \phi$ and hence $G = U$.

Conversely, let K be a nano π-open set of U and $K \subseteq H$. Since $Npint(H) \cup (U - H) = Nint(Ncl(H)) \cup (U - K)$ is nano π-open and $Npint(H) \cup (U - H) \subseteq Npint(H) \cup (U - K)$, by hypothesis, $Npint(H) \cup (U - K) = U$ and hence $K \subseteq Npint(H)$.

Theorem 3.24. *Let $H \subseteq V \subseteq U$ and V nano π-open and nano closed in U. If H is nano πgp-open in V, then H is nano πgp-open in U.*

Proof. Let K be any nano π-closed set and $K \subseteq H$. Since K is nano π-closed in V and H is nano πgp-open in V, $K \subseteq Npint_V(H)$ and then $K \subseteq Npint_U(H) \cap V$. Hence $K \subseteq Npint_U(H)$ and so H is nano πgp-open in U.

Theorem 3.25. *If H is nano πgp-open in U and $Npint(H) \subseteq P \subseteq H$, then P is nano πgp-open.*

Proof. Let $K \subseteq P$ and K be nano π-closed in U. Since H is nano πgp-open and $K \subseteq H$. $K \subseteq Npint(H)$ and then $K \subseteq Npint(P)$. Hence P is nano πgp-open.

Theorem 3.26. *A subset H of U is nano πgp-closed $\iff$ $Npcl(H) - H$ is nano πgp-open.*

Proof. Let $K \subseteq Npcl(H) - H$ and K be nano π-closed in U. Then by Theorem 3.10, $K = \phi$ and so $K \subseteq Npint(Npcl(H) - H)$. This shows that $Npcl(H) - H$ is nano πgp-open.

Conversely, let G be a nano π-open set of U and $H \subseteq G$. Then $Npcl(H) \cap (U - G) = Ncl(Nint(H)) \cap (U - G)$ is nano π-closed set contained in $Npcl(H) - H$. Since $Npcl(H) - H$ is nano πgp-open, by Theorem 3.22, $Npcl(H) \cap (U - G) \subseteq Npint(Npcl(H) - H = \phi)$ and hence $Npcl(H) \subseteq G$.

Acknowledgement

The authors thank the referees for their valuable comments and suggestions for improvement of this my article.

References

[1] Adinatha C. Upadhya, *On quasi nano p-normal spaces*, International Journal of Recent Scientific Research, 8(6) (2017) 17748-17751.

[2] K. Bhuvaneshwari and K. Mythili Gnanapriya, *Nano Generalizesd closed sets*, International Journal of Scientific and Research Publications, 4(5) (2014) 1-3.

[3] K. Bhuvaneswari and K. Mythili Gnanapriya, *On Nano Generalised Pre Closed Sets and Nano Pre Generalised Closed Sets in Nano Topological Spaces*, International Journal of Innovative Research in Science, Engineering and Technology, 3(10) (2014) 16825-16829.

[4] M. Lellis Thivagar and Carmel Richard, *On Nano forms of weakly open sets*, International Journal of Mathematics and Statistics Invention, 1(1) (2013) 31-37.

[5] C. R. Parvathy and , S. Praveena, *On Nano Generalized Pre Regular Closed Sets in Nano Topological Spaces*, IOSR Journal of Mathematics (IOSR-JM), 13(2) (2017) 56-60.

[6] Z. Pawlak, *Rough sets*, International journal of computer and Information Sciences, 11(5) (1982) 341-356.

[7] I. Rajasekaran and O. Nethaji, *On some new subsets of nano topological spaces*, Journal of New Theory, 16 (2017) 52-58.

[8] P. Sulochana Devi and K. Bhuvaneswari , *On Nano Regular Generalized and Nano Generalized Regular Closed Sets in Nano Topological Spaces*, International Journal of Engineering Trends and Technology (IJETT), 8(13) (2014) 386-390.

http://www.newtheory.org ISSN: 2149-1402

Journal of New Theory

Received: 02.12.2017
Published: 17.12.2017
Year: **2017**, *Number*: **19**, *Pages*: **27-47**
Original Article

TRAPEZOIDAL LINGUISTIC CUBIC HESITANT FUZZY TOPSIS METHOD AND APPLICATION TO GROUP DECISION MAKING PROGRAM

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Abstract – In this paper, we define a new idea of trapezoidal linguistic cubic hesitant fuzzy number. We discuss some basic operational laws of trapezoidal linguistic cubic hesitant fuzzy number and hamming distance of trapezoidal linguistic cubic hesitant fuzzy number. Furthermore, we develop Trapezoidal linguistic cubic hesitant fuzzy TOPSIS method to solve the MCDM method based on trapezoidal linguistic cubic hesitant fuzzy TOPSIS method. Finally, an illustrative example is given to verify and demonstrate the practicality and effectiveness of the proposed method.

Keywords *– Trapezoidal linguistic cubic hesitant fuzzy number, trapezoidal linguistic cubic hesitant fuzzy TOPSIS method, MCDM, Numerical Application*

1 Introduction

Selecting a proper supplier amongst different suppliers is a grave substance for highest organization. In skills that are worried with huge gage production the raw resources and unit parts can equal up to 70% creation cost. In such circumstances the procurement department can presentation a vital role in cost reduction, and supplier selection is one of the most energetic functions of getting management [7]. So, by means of an fitting method for this purpose is a vital issue, supplier selection has been revealed to be a multiple criteria decision making (MCDM) problem [13]. In a typical MCDM problem, multiple and usually incompatible criteria are instantaneously booked into description for making a decision. A wide-ranging review and organization of the MCDM approaches for vendor

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selection has been carried out in [5]. Torra [25] defined the Hesitant fuzzy sets. Xia et al. [26] defined hesitant fuzzy information aggregation in decision making.

(MCDM) is apprehensive with arranging and explaining decision and development problems relating multiple criteria. Typically, there does not exist an inimitable optimal solution for such problems and it is necessary to use decision maker's performance to differentiate between solutions. MCDM has been an active area of research since the 1970's. Different methods have been offered by many researchers, including the Analytic Hierarchy Process (AHP), Technique for Order Preference by Similarity to Ideal Solution (TOPSIS) and MCDM. Some of the most broadly used multi-criteria decision exploration methods is the TOPSIS method, which was proposed by Hwang and Yoon in 1981 [8], and extended by Yoon in 1987 [15], as well as by Hwang et al. in 1993 [16]. In the TOPSIS method, the optimal alternative is nearest to the positive ideal solution (PIS) and farthest from the negative ideal solution (NIS). When control inaccurate and imprecise date, mostly modelling social Judgments, it is more representative and instinctive to use linguistic assessments instead of numerical evaluations. Thus, in many former studies, the TOPSIS method was used in concurrence with fuzzy logic. Numerous fuzzy TOPSIS methods and applications have been established since the 1990s, e.g., for supplier selection [17] [18], finance [19] [20], power industry [21] [22], and negotiation problems [23]. In our study, we employ a fuzzy extension of the TOPSIS method presented by Chen [24].

Cubic sets introduced by Jun et al. [9], are the generalizations of fuzzy sets and intuitionistic fuzzy sets, in which there are two representations, one is used for the degree of membership and other is used for the degree of non-membership. The membership function is hold in the form of interval while non-membership is thought over the normal fuzzy set. Aliya et al. [2] defined the cubic TOPSIS method and cubic grey ananlysis set. Aliya et al. [1] defined the idea of triangular cubic fuzzy set and hamming distance. Due to the motivation and inspiration of the above discussion in this paper we generalized the concept of trapezoidal linguistic hesitant fuzzy sets, trapezoidal linguistic intutionistic hesitant fuzzy sets, interval-valued trapezoidal linguistic intuitionstic hesitant fuzzy number, trapezoidal linguistic hesitant fuzzy TOPSIS method, interval-valued trapezoidal linguistic hesitant fuzzy TOPSIS method, interval-valued trapezoidal linguistic intuitionstic hesitant fuzzy TOPSIS method and introudce the concept of trapezoidal linguistic cubic hesitant fuzzy sets. If we take only one element in the membership degree of the trapezoidal linguistic cubic hesitant fuzzy number, i.e.instead of interval we take a fuzzy number, than we get trapezoidal linguistic intuitionistic hesitant fuzzy numbers, similarly if we take memebrship degree as hesitant fuzzy number and nonmembership degree equal to zero, than we get trapezoidal linguistic hesitant fuzzy numbers.

In section 2, we firstly introduced some basic definitions of the fuzzy set and cubic set. In section 3, we develop trapezoidal linguistic cubic hesitant fuzzy number and hamming distance. In section 4, we develop trapezoidal linguistic cubic hesitant fuzzy TOPSIS method different steps in the proposed trapezoidal linguistic cubic hesitant fuzzy TOPSIS method are presented. A numerical example of the proposed model is presented in section 5. In section 6, we discuss comparison to different method. The paper is concluded in section 7.

2 **Preliminaries**

Definition [14] 2.1. *Let* H *be a universe of discourse. The idea of fuzzy set was presented by Zadeh and defined as following:* $J=\{h,\Gamma_J(h)\mid h\in H\}$. *A fuzzy set in a set* H *is defined* $\Gamma_J : H\to I$, *is a membership function,* $\Gamma_J(h)$ *denoted the degree of membership of the element* h *to the set* H *, where* $I=[0,1]$ *. The collection of all fuzzy subsets of* H *is denoted by* I^H *. Define a relation on* I^H *as follows:*

$$(\forall\Gamma,\eta\in I^H)(\Gamma\le\eta\Leftrightarrow(\forall h\in H)(\Gamma(h)\le\eta(h))).$$

Definition [4] 2.2. *An Atanassov intuitionistic fuzzy set on* H *is a set*

$$J=\{h,\Gamma(h),\eta(h):\ h\in H\}$$

where Γ_J *and* η_j *are membership and non-membership function, respectively*

$$\Gamma_J(h):\ h\to[0,1],h\in H\to\Gamma_J(h)\in[0,1];\eta_J(h):\ h\to[0,1],h\in H\to\eta_J(h)\in[0,1]$$

and

$$0\le\Gamma_J(h)+\eta_J(h)\le 1\ \textit{for all}\ h\in H\ .\ \pi_J(h)=1-\Gamma_J(h)-\eta_J(h).$$

Definition [9] 2.3. *Let* H *be a nonempty set. By a cubic set in* H *we mean a structure* $F=\{h,\alpha(h),\beta(h):\ h\in H\}$ *in which* α *is an IVF set in* H *and* β *is a fuzzy set in* H. *A cubic set* $F=\{h,\alpha(h),\beta(h):\ h\in H\}$ *is simply denoted by* $F=\langle\alpha,\beta\rangle$. *Denote by* C^H *the collection of all cubic sets in* H. *A cubic set* $F=\langle\alpha,\beta\rangle$ *in which* $\alpha(h)=0$ *and* $\beta(h)=1$ *(resp.* $\alpha(h)=1$ *and* $\beta(h)=0$ *for all* $h\in H$ *is denoted by 0 (resp. 1). A cubic set* $D=\langle\lambda,\xi\rangle$ *in which* $\lambda(h)=0$ *and* $\xi(h)=0$ *(resp.* $\lambda(h)=1$ *and* $\xi(h)=1$ *) for all* $h\in H$ *is denoted by 0 (resp. 1).*

Definition [9] 2.4. *Let* H *be a non-empty set. A cubic set* $F=(C,\lambda)$ *in* H *is said to be an internal cubic set if* $C^-(h)\le\lambda(h)\le C^+(h)$ *for all* $h\in H$.

Definition [9] 2.5. *Let* H *be a non-empty set. A cubic set* $F=(C,\lambda)$ *in* H *is said to be an external cubic set if* $\lambda(h)\notin(C^-(h),C^+(h))$ *for all* $h\in H$.

3. Trapezoidal Linguistic Cubic Hesitant Fuzzy Number

Definition 3.1. *Let* $\tilde{b}$ *be the trapezoidal linguistic cubic hesitant fuzzy number on the set of real numbers, its interval value trapezoidal linguistic hesitant fuzzy set is defined as:*

$$\lambda_{\tilde{b}}(h)=\begin{cases} s_{\theta},\frac{(h-r)}{(s-r)}[\omega_{b,}^{-}\omega_{b}^{+}] & r\le h<s \\ [\omega_{b}^{-},\omega_{b}^{+}] & s\le h<t \\ s_{\theta},\frac{(t-h)}{(d^{-}-s^{-})}[\omega_{b}^{-},\omega_{b}^{+}] & t\le h<u \\ 0 & otherwise \end{cases}$$

and its trapezoidal linguistic hesitant fuzzy set

$$\Gamma_{\tilde{b}}(h)=\begin{cases} s_{\theta},\frac{s-h+(h-r)\eta_{\tilde{b}}}{(r-s)}, & r\le h<s \\ \eta_{\tilde{b}} & s\le h\le t \\ s_{\theta},\frac{h-t+(u-h)\eta_{\tilde{b}}}{(u-t)} & t<h\le u \\ 0 & \text{otherwise} \end{cases}$$

where $0\le\lambda_{\tilde{b}}(h)\le 1, 0\le\Gamma_{\tilde{b}}(h)\le 1$ and $r,s,t,u,$ are real numbers. The values of $\omega_{b^{-}},\omega_{b^{+}}$ consequently the maximum values of interval value hesitant fuzzy set and $\eta_{\tilde{b}}$ minimum value of hesitant fuzzy set. Then the TrLCHFN $\tilde{b}$ basically denoted by

$$\tilde{b}=s_{\theta},[(r,s,t,u)];\langle[\omega_{b}^{-},\omega_{b}^{+}\quad],\eta_{\tilde{b}}\rangle.$$

Further, the TrLCHFN reduced to a TLCHFN. Moreover, if $\omega_{b}^{-}=1,\omega_{b}^{+}=1$ and $\eta_{\tilde{b}}=0$, if the TrLCHFN $\tilde{b}$ is called a normal TLCHFN denoted as

$$\tilde{b}=s_{\theta},[(r,s,t,u)];[\langle(1,1)],(0)]\rangle.$$

Therefore, the TrLCHFN considered now can be regarded as generalized TrLCHFN. Such numbers remand the doubt information in a more flexible approach than normal fuzzy numbers as the values $\omega_{b}^{-},\omega_{b}^{+},\eta_{\tilde{b}}\in[0,1]$ can be interpreted as the degree of confidence in the quantity characterized by $r,s,t,u.$ Then $\tilde{b}$ is called trapezoidal linguistic cubic hesitant fuzzy number (TrLCHFN).

Definition 3.2. Let

$$h=\left\{\begin{matrix} s_{\theta}, \\ [r, \\ s, \\ t, \\ u], \\ \langle[\omega_{b}^{-}, \\ \omega_{b}^{+}], \\ \eta_{\tilde{b}}\rangle \end{matrix}\right\}, h_1=\left\{\begin{matrix} s_{\theta_1}, \\ [r_1, \\ s_1, \\ t_1, \\ u_1], \\ \langle[\omega_{b_1}^{-}, \\ \omega_{b_1}^{+}], \\ \eta_{\tilde{b}_1}\rangle \end{matrix}\right\} \quad and \quad h_2=\left\{\begin{matrix} s_{\theta_2}, \\ [r_2, \\ s_2, \\ t_2, \\ u_2], \\ \langle[\omega_{b_2}^{-}, \\ \omega_{b_2}^{+}], \\ \eta_{\tilde{b}_2}\rangle \end{matrix}\right\}$$

are three TrCLHFNs, which can be described as follows:

$$\ddot{h}^c = \{\alpha^c \mid \alpha \in \ddot{h}\} = \left\{\begin{matrix} s_\theta, [r,s,t,u], \\ \langle [\omega_b^-, \omega_b^+], \eta_{\tilde{b}} \rangle \end{matrix}\right\};$$

$$\ddot{h}_1 \cup \ddot{h}_2 = \left\{\begin{matrix} s_{\theta(h_1)\cup\,\theta(h_2)}, [r_1 \cup r_2, s_1 \cup s_2, t_1 \cup t_2, u_1 \cup u_2], \\ \langle [\omega_{b_1}^- \cup \omega_{b_2}^-, \omega_{b_1}^+ \cup \omega_{b_2}^+], \eta_{\tilde{b}_1} \cap \eta_{\tilde{b}_2} \rangle \end{matrix}\right\};$$

$$\ddot{h}_1 \cap \ddot{h}_2 = \left\{\begin{matrix} s_{\theta(h_1)\cap\theta(h_2)}, [r_1 \cap r_2, s_1 \cap s_2, t_1 \cap t_2, u_1 \cap u_2], \\ \langle [\omega_{b_1}^- \cap \omega_{b_2}^-, \omega_{b_1}^+ \cap \omega_{b_2}^+], \eta_{\tilde{b}_1} \cup \eta_{\tilde{b}_2} \rangle \end{matrix}\right\};$$

$$\ddot{h}_1 \oplus \ddot{h}_2 = \left\{\begin{matrix} s_{\theta(h_1)+\theta(h_2)}, [r_1 + r_2 - r_1 r_2, s_1 + s_2 - s_1 s_2, \\ t_1 + t_2 - t_1 t_2, u_1 + u_2 - u_1 u_2], \\ \langle [\omega_{b_1}^- + \omega_{b_2}^- - \omega_{b_1}^- \omega_{b_2}^-, \omega_{b_1}^+ + \omega_{b_2}^+ - \omega_{b_1}^+ \omega_{b_2}^+], \\ \eta_{\tilde{b}_1} \eta_{\tilde{b}_2} \rangle \end{matrix}\right\};$$

$$\lambda\ddot{h} = \left\{\begin{matrix} \{\lambda h \mid a \in \ddot{h}\} = s_{\theta^\lambda(h)}, [1-(1-r)^\lambda], [1-(1-s)^\lambda], \\ [1-(1-t)^\lambda], [1-(1-u)^\lambda]; \\ \langle [1-(1-\omega_b^-)^\lambda], [1-(1-\omega_b^+)^\lambda], \\ (\eta_{\tilde{b}})^\lambda \rangle \end{matrix}\right\};$$

$$\ddot{h}^\lambda = \left\{\begin{matrix} \{\alpha^\lambda \mid a \in \ddot{h}\} = [s_{\lambda\times\theta(h)}], [(r)^\lambda, (s)^\lambda, (t)^\lambda, (u)^\lambda], \\ \langle [(\omega_b^-)^\lambda, (\omega_b^+)^\lambda], 1-(1-\eta_{\tilde{b}})^\lambda \rangle \end{matrix}\right\}.$$

Example 3.3. Let

$$\alpha = \left\{\begin{matrix} s_3, [0.4, 0.6, \\ 0.8, 0.10], \\ \langle [0.2, 0.4], \\ 0.3 \rangle \end{matrix}\right\}, \quad \alpha_1 = \left\{\begin{matrix} s_4, [0.2, 0.4, \\ 0.6, 0.8], \\ \langle [0.4, 0.6], \\ 0.5 \rangle \end{matrix}\right\} \text{ and } \alpha_2 = \left\{\begin{matrix} s_5, [0.9, 0.11, \\ 0.13, 0.15], \\ \langle [0.11, 0.13], \\ 0.10 \rangle \end{matrix}\right\}$$

be three TrCLHFNs. Then,

$$\alpha^c = \left\{\begin{matrix} s_3, [0.4, 0.6, 0.8, 0.10], \\ \langle [0.2, 0.4], 0.3 \rangle \end{matrix}\right\},$$

$$\alpha_1 \cup \alpha_2 = \left\{ \begin{matrix} s_{4\cup 5}, [0.2 \cup 0.9, \\ 0.4 \cup 0.11, 0.6 \cup 0.13, \\ 0.8 \cup 0.15], \\ \langle [0.4 \cup 0.11, 0.6 \cup 0.13], \\ 0.5 \cap 0.10 \rangle \end{matrix} \right\};$$

$$a_1 \cap a_2 = \left\{ \begin{matrix} s_{4\cap 5}, [0.2 \cap 0.9, \\ 0.4 \cap 0.11, 0.6 \cap 0.13, \\ 0.8 \cap 0.15], \\ \langle [0.4 \cap 0.11, 0.6 \cap 0.13], \\ 0.5 \cup 0.10 \rangle \end{matrix} \right\};$$

$$\alpha_1 \oplus \alpha_2 = \left\{ \begin{matrix} s_{4+5}, [0.2 + 0.9 - (0.2)(0.9), \\ 0.4 + 0.11 - (0.4)(0.11), \\ 0.6 + 0.13 - (0.6)(0.13), \\ 0.8 + 0.15 - (0.8)(0.15)], \\ \langle [0.4 + 0.11 - (0.4)(0.11), \\ 0.6 + 0.13 - (0.6)(0.13)], \\ (0.5)(0.10) \rangle \end{matrix} \right\};$$

$$\alpha_1 \otimes \alpha_2 = \left\{ \begin{matrix} s_{4\times 5}, [(0.2)(0.9), (0.4)(0.11), (0.6)(0.13), \\ (0.8)(0.15)], [(0.4)(0.11), (0.6)(0.13)], \\ 0.5 + 0.10 - (0.5)(0.10) \rangle \end{matrix} \right\};$$

$$\lambda = 0.25, 0.25, 0.25, 0.25$$

$$\lambda\alpha = \left\{ \begin{matrix} s_{3\times 0.25}, [1-(1-0.4)^{0.25}, 1-(1-0.6)^{0.25}, \\ 1-(1-0.8)^{0.25}, 1-(1-0.10)^{0.25}], \\ \langle [1-(1-0.2)^{0.25}, 1-(1-0.4)^{0.25}], \\ (0.3)^{0.25} \rangle \end{matrix} \right\};$$

$$\alpha^{\lambda} = \left\{ \begin{matrix} s_{3^{0.25}}, [(0.4)^{0.25}, (0.6)^{0.25}, \\ (0.8)^{0.25}, (0.10)^{0.25}], \\ \langle [(0.2)^{0.25}, (0.4)^{0.25}], \\ 1-(1-0.3)^{0.25} \rangle \end{matrix} \right\}.$$

Definition 3.4. Let

$$b_1 = \left\{ \begin{matrix} s_{\theta_1}, [r_1, s_1, t_1, u_1]; \\ \langle [\omega_{b_1}^-, \omega_{b_1}^+], \eta_{b_1} \rangle \end{matrix} \right\} \quad and \quad b_2 = \left\{ \begin{matrix} s_{\theta_2}, [r_2, s_2, t_2, u_2]; \\ \langle [\omega_{b_2}^-, \omega_{b_2}^+], \eta_{b_2} \rangle \end{matrix} \right\}$$

be two TrLCHFNs. The hammimg distance between b_1 and b_2 is defined as follows:

$$\underset{H}{d}(\tilde{b}_1, \tilde{b}_2) = \left\{ \begin{matrix} \frac{1}{12}[| s_{\theta_1-\theta_2} |, \langle [| r_1 - r_2 | + | s_1 - s_2 | + | t_1 - t_2 | + \\ | u_1 - u_2 |] + \max[\{| \omega_{b_1}^- - \omega_{b_2}^- |, | \omega_{b_1}^+ - \omega_{b_2}^+ |\}, | \eta_{b_1} - \eta_{b_2} |]] \end{matrix} \right\}$$

the TrLCHFN

$$b_1 = \left\{ \begin{matrix} s_{\theta_1}, [r_1, s_1, \\ t_1, u_1]; \\ \langle [\omega_{b_1^-}, \omega_{b_1^+}], \\ \eta_{b_1} \rangle \end{matrix} \right\} \quad and \quad b_2 = \left\{ \begin{matrix} s_{\theta_2}, [r_2, s_2, \\ t_2, u_2]; \\ \langle [\omega_{b_2^-}, \omega_{b_2^+}], \\ \eta_{b_2}]\rangle \end{matrix} \right\}$$

reduces to a TrLCHF

$$b_1 = \left\{ \begin{matrix} s_{\theta_1}, [r_1, s_1, \\ t_1, u_1]; \\ \langle [1,1], \\ 0 \rangle \end{matrix} \right\} \quad and \quad b_2 = \left\{ \begin{matrix} s_{\theta_2}, [r_2, s_2, \\ t_2, u_2]; \\ \langle [1,1], \\ 0 \rangle \end{matrix} \right\}.$$

If $\omega_{b_1^-} = 1, \omega_{b_2^-} = 1$ and $\omega_{b_1^+} = 1, \omega_{b_2^+} = 1,$ if $\eta_{b_1} = 0$ and $\eta_{b_2} = 0$.

Example 3.5. Let

$$b_1 = \left\{ \begin{matrix} s_4, [0.6, 0.8, \\ 0.10, 0.12]; \\ \langle [0.22, 0.24], \\ 0.23 \rangle \end{matrix} \right\} \quad and \quad b_2 = \left\{ \begin{matrix} s_2, [0.2, 0.4, \\ 0.6, 0.8]; \\ \langle [0.11, 0.13], \\ 0.12 \rangle \end{matrix} \right\}$$

be two TrLCHFNs. The hammimg distance between $\tilde{b}_1$ and $\tilde{b}_2$ is defined as follows:

$$\underset{H}{d}(\tilde{b}_1,\tilde{b}_2)=\left\{\begin{array}{c}\frac{1}{12}[|\,s_{4-2}\,|,\langle[|\,0.6-0.2\,|+|\,0.8-0.4\,| \\ +|\,0.10-0.6\,|+ \\ |\,0.12-0.8\,|]+ \\ \max[\left\{\begin{array}{l}|\,0.22-0.11\,|, \\ |\,0.24-0.13\,|\end{array}\right\}, \\ |\,0.23-0.12\,|]] \\ \frac{1}{12}[|\,s_2\,|,\langle[|\,0.4\,|+ \\ |\,0.4\,|+|-0.5\,|+ \\ |-0.68\,|]+\max[\left\{\begin{array}{l}|\,0.11\,|, \\ |\,0.11\,|\end{array}\right\}, \\ |\,0.11\,|]]=s_{0.3483}\end{array}\right\}.$$

4 Trapezoidal Linguistic Cubic Hesitant Fuzzy TOPSIS Method

In this section we apply trapezoidal cubic fuzzy set to linguistic hesitant TOPSIS method. We define a new extension of trapezoidal linguistic cubic hesitant fuzzy TOPSIS method by using trapezoidal cubic fuzzy set.

Step 1: Suppose that a trapezoidal linguistic cubic hesitant fuzzy TOPSIS method decision-making problem under multiple attributes has m students and n decision attributes. The framework of trapezoidal linguistic cubic hesitant decision matric can be exhibit as follows:

$$\beta=\left\{\begin{array}{c} s_{\theta_{11}}[r_{11},s_{11},t_{11},u_{11}]\langle[B_{11}^-,B_{11}^+],\eta_{11}\rangle \\ s_{\theta_{12}}[r_{12},s_{12},t_{12},u_{12}]\langle[B_{12}^-,B_{12}^+],\eta_{12}^S\rangle \ldots\ldots. \\ s_{\theta_{n1}}[r_{n1},s_{n1},t_{n1},u_{n1}]\langle[B_{n1}^-,B_{n1}^+],\eta_{1n}\rangle \\ s_{\theta_{12}}[r_{21},s_{21},t_{21},u_{21}]\langle[B_{21}^-,B_{21}^+],\eta_{21}\rangle \\ s_{\theta_{22}}[r_{22},s_{22},t_{22},u_{22}]\langle[B_{22}^-,B_{22}^+],\eta_{22}\rangle \ldots\ldots. \\ s_{\theta_{n2}}[r_{n2},s_{n2},t_{n2},u_{n2}]\langle[B_{n2}^-,B_{n2}^+],\eta_{2n}\rangle \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ s_{\theta_{m1}}[r_{m1},s_{m1},t_{m1},u_{m1}]\langle[B_{m1}^-,B_{m1}^+],\eta_{m1}\rangle \\ s_{\theta_{m2}}[r_{m2},s_{m2},t_{m2},u_{m2}]\langle[B_{m2}^-,B_{m2}^+],\eta_{m2}\rangle \ldots\ldots. \\ s_{\theta_{mn}}[r_{mn},s_{mn},t_{mn},u_{mn}]\langle[B_{mn}^-,B_{mn}^+],\eta_{mn}\rangle \end{array}\right\}$$

Step 2: Construct normalized trapezoidal linguistic cubic hesitant fuzzy TOPSIS decision matrix $R=[\beta_{ij}]$. The normalized value r_{ij} is calculated as:

$$\beta=\left\{\begin{array}{c}\dfrac{s_\theta}{\sqrt{\sum_{i=1}^{n}(s_{\theta_{ij}})^2}},\left[\dfrac{r}{\sqrt{\sum_{i=1}^{n}(r_{ij})^2}},\dfrac{s}{\sqrt{\sum_{i=1}^{n}(s_{ij})^2}},\dfrac{t}{\sqrt{\sum_{i=1}^{n}(t_{ij})^2}},\dfrac{u}{\sqrt{\sum_{i=1}^{n}(u_{ij})^2}}\right]; \\ \left\langle\left[\dfrac{B^-}{\sqrt{\sum_{i=1}^{n}(B_{ij}^-)^2}},\dfrac{B^+}{\sqrt{\sum_{i=1}^{n}(B_{ij}^+)^2}}\right],\dfrac{\eta}{\sqrt{\sum_{i=1}^{n}(\eta)^2}}\right\rangle\end{array}\right\}$$

Step 3: Make the weighted normalized trapezoidal linguistic cubic hesitant fuzzy TOPSIS decision matrix by multiplying the normalized trapezoidal linguistic cubic hesitant decision matrix by its associated weights. The weight vector $W=(w_1,w_2,...w_n)$ collected of the isolated weights $w_j(j=1,2,3,...n)$ for each attribute C_j satisfying $\sum_{j=1}^{n}W_j=1$. The weighted normalized value is deliberate by $B_j=v_{ij}w_j$ where $0\leq B_j\leq 1$ $i=1,2,3....,m$ and $j=1,2,3...,n.$

Step 4: Identify positive ideal solution (α^*) and negative ideal solution (α^-). The trapezoidal linguistic cubic hesitant fuzzy TOPSIS method positive-ideal solution (TrLCHFPIS, α^*) and the trapezoidal linguistic cubic hesitant fuzzy negative-ideal solution (TrLCHFNIS, α^-) is shown as

$$\alpha_i^+=\left\{\begin{array}{c}s_{\theta_1},[(r_1,s_1,t_1,u_1),\\(r_2,s_2,t_2,u_2).....\\s_{\theta_n}(r_n,s_n,t_n,u_n)]\\\{(B_1^+,\eta_1)(B_2^+,\eta_2)\\.......(B_n^+,\eta_n)\end{array}\right\}=\max_i s_{\theta_{ij}},\max_i(r_{ij},s_{ij},t_{ij},u_{ij})\{\max_i(B_{ij}^+)\}\{\min_i(\eta_i)\},$$

$$\alpha_i^-=\left\{\begin{array}{c}s_{\theta_1},[(r_1,s_1,t_1,u_1),\\(r_2,s_2,t_2,u_2).....\\s_{\theta_n}(r_n,s_n,t_n,u_n)],\\\{(B_1^-,\eta_1)(B_2^-,\eta_2)\\.......(B_n^-,\eta_n)\end{array}\right\}=\min_i s_{\theta_{ij}},\min_i(r_{ij},s_{ij},t_{ij},u_{ij}),\{\min_i(B_1)\}\{\max_i(\eta_i)\}.$$

Step 5: Estimated separation measures, using the n-dimensional euclidean distance. The separation of each candidate from the TrLCHPIS $q_i^*\langle[B^-,B^+],\eta\rangle$ is given as

$$q_i^+\langle[B^-,B^+],\eta\rangle=\langle\tfrac{1}{12n}\left(\begin{array}{c}\left|s_{\theta_{ij}-\theta_j}\right|,\left|r_{ij}-r_j\right|+\left|s_{ij}-s_j\right|+\\\left|t_{ij}-t_j\right|+\left|u_{ij}-u_j\right|,\\\max\{\left|B_{ij}-B_j^-\right|,\\\left|B_{ij}-B_j^+\right|,\left|\eta_{ij}-\eta_j\right|\}\end{array}\right)\rangle.$$

The separation of each candidate from the TrLCHNIS $q_i^-\langle[B^-,B^+],\eta\rangle$ is given as

$$q_i^-\langle[B^-,B^+],\eta\rangle=\langle\tfrac{1}{12n}\begin{pmatrix}\left|s_{\theta_{ij}-\theta_j}\right|,\left|r_{ij}-r_j\right|+\\ \left|s_{ij}-s_j\right|+\left|t_{ij}-t_j\right|,\\ \max\{\left|B_{ij}-B_j^-\right|,\\ \left|B_{ij}-B_j^+\right|,\left|\eta_{ij}-\eta_j\right|\}\end{pmatrix}\rangle.$$

Step 6: Calculate similarities to ideal solution. This progression comprehends the similitudes to an ideal solution by Eqs. $Z_i=\frac{q_i^-\langle[B^-,B^+],\eta\rangle}{q_i^-\langle[B^-,B^+],\eta\rangle+q_i^+\langle[B^-,B^+],\eta\rangle}$.

Figure 1

Flow chart of the extended trapezoidal linguistic cubic hesitant fuzzy TOPSIS method

5 Numerical Example

Assume that an automotive company is in a decision-making situation for purchasing one of the main items for their newly introduced automobile. A committee of three decision-makers $\{D_1, D_2, D_3\}$ want to select the most promising vendor for supplying the item. After a preliminary screening, three alternatives $B_1, B_2, ..., B_n$ remain for further evaluations.

Linguistic variables of ratings of alternatives by decision-maker 1

	C_1	C_2	C_3
B_1	$\left\{ s_2, \begin{bmatrix} 0.2, 0.3, \\ 0.4, 0.5 \end{bmatrix}; \langle [0.20, 0.24], 0.22 \rangle \right\}$	$\left\{ s_4, \begin{bmatrix} 0.4, 0.5, \\ 0.6, 0.7 \end{bmatrix}; \langle [0.10, 0.12], 0.11 \rangle \right\}$	$\left\{ s_2, \begin{bmatrix} 0.1, 0.2, \\ 0.3, 0.4 \end{bmatrix}; \langle [0.5, 0.7], 0.6 \rangle \right\}$
B_2	$\left\{ s_6, \begin{bmatrix} 0.14, 0.15, \\ 0.16, 0.17 \end{bmatrix}; \langle [0.1, 0.3], 0.2 \rangle \right\}$	$\left\{ s_6, \begin{bmatrix} 0.11, 0.12, \\ 0.13, 0.14 \end{bmatrix}; \langle [0.4, 0.6], 0.5 \rangle \right\}$	$\left\{ s_4, \begin{bmatrix} 0.4, 0.5, \\ 0.6, 0.7 \end{bmatrix}; \langle [0.10, 0.12], 0.11 \rangle \right\}$
B_3	$\left\{ s_2, \begin{bmatrix} 0.2, 0.3, \\ 0.4, 0.5 \end{bmatrix}; \langle [0.20, 0.24], 0.22 \rangle \right\}$	$\left\{ s_2, \begin{bmatrix} 0.1, 0.2, \\ 0.3, 0.4 \end{bmatrix}; \langle [0.5, 0.7], 0.6 \rangle \right\}$	$\left\{ s_6, \begin{bmatrix} 0.11, 0.12, \\ 0.13, 0.14 \end{bmatrix}; \langle [0.4, 0.6], 0.5 \rangle \right\}$

Linguistic variables of ratings of alternatives by decision-maker 2

	C_1	C_2	C_3
B_1	$\left\langle s_4, \begin{bmatrix} 0.4, 0.5, \\ 0.6, 0.7 \end{bmatrix}; [0.10, 0.12], 0.11 \right\rangle$	$\left\langle s_2, \begin{bmatrix} 0.2, 0.3, \\ 0.4, 0.5 \end{bmatrix}; [0.20, 0.24], 0.22 \right\rangle$	$\left\langle s_6, \begin{bmatrix} 0.14, 0.15, \\ 0.16, 0.17 \end{bmatrix}; [0.1, 0.3], 0.2 \right\rangle$
B_2	$\left\langle s_6, \begin{bmatrix} 0.11, 0.12, \\ 0.13, 0.14 \end{bmatrix}; [0.4, 0.6], 0.5 \right\rangle$	$\left\langle s_2, \begin{bmatrix} 0.1, 0.2, \\ 0.3, 0.4 \end{bmatrix}; [0.5, 0.7], 0.6 \right\rangle$	$\left\langle s_6, \begin{bmatrix} 0.11, 0.12, \\ 0.13, 0.14 \end{bmatrix}; [0.4, 0.6], 0.5 \right\rangle$
B_3	$\left\langle s_2, \begin{bmatrix} 0.2, 0.3, \\ 0.4, 0.5 \end{bmatrix}; [0.20, 0.24], 0.22 \right\rangle$	$\left\langle s_6, \begin{bmatrix} 0.14, 0.15, \\ 0.16, 0.17 \end{bmatrix}; [0.1, 0.3], 0.2 \right\rangle$	$\left\langle s_2, \begin{bmatrix} 0.1, 0.2, \\ 0.3, 0.4 \end{bmatrix}; [0.5, 0.7], 0.6 \right\rangle$

$v_1 = 0.2, v_2 = 0.3, v_3 = 0.5$

Step 1: In this step we aggregated TrLCHF-decision matrix D_1, D_2, D_3 based on opinions of the experts after weights values for the experts are obtained, the evaluating values provided by different experts can be aggregated based on the TrLCHFWG operator as below: The aggregated TrLCHF-decision matrix can be defined as follows:

Table 3. The aggregated TrLCHF-decision matrix

	C_1	C_2	C_3
B_1	$\left\langle s_{1.5157}, \begin{bmatrix} 0.1365, \\ 0.1893, \\ 0.2483, \\ 0.3157 \end{bmatrix}; \begin{bmatrix} 0.0635, \\ 0.0773 \end{bmatrix}, 0.4751 \right\rangle$	$\left\langle s_{1.8661}, \begin{bmatrix} 0.1976, \\ 0.2701, \\ 0.3482, \\ 0.4339 \end{bmatrix}; \begin{bmatrix} 0.0938, \\ 0.1136 \end{bmatrix}, 0.3274 \right\rangle$	$\left\langle s_{3.4641}, \begin{bmatrix} 0.1202, \\ 0.1753, \\ 0.2331, \\ 0.2943 \end{bmatrix}; \begin{bmatrix} 0.3291, \\ 0.5417 \end{bmatrix}, 0.3464 \right\rangle$
B_2	$\left\langle s_{2.0476}, \begin{bmatrix} 0.0521, \\ 0.0564, \\ 0.0607, \\ 0.0652 \end{bmatrix}; \begin{bmatrix} 0.1159, \\ 0.2247 \end{bmatrix}, 0.6309 \right\rangle$	$\left\langle s_{2.1074}, \begin{bmatrix} 0.0644, \\ 0.0999, \\ 0.1382, \\ 0.1800 \end{bmatrix}; \begin{bmatrix} 0.3031, \\ 0.4706 \end{bmatrix}, 0.6968 \right\rangle$	$\left\langle s_{4.8989}, \begin{bmatrix} 0.2692, \\ 0.3366, \\ 0.4100, \\ 0.4921 \end{bmatrix}; \begin{bmatrix} 0.2651, \\ 0.4067 \end{bmatrix}, 0.2345 \right\rangle$
B_3	$\left\langle s_{1.3195}, \begin{bmatrix} 0.0853, \\ 0.1329, \\ 0.1848, \\ 0.2421 \end{bmatrix}; \begin{bmatrix} 0.0854, \\ 0.1039 \end{bmatrix}, 0.5457 \right\rangle$	$\left\langle s_{2.1074}, \begin{bmatrix} 0.0739, \\ 0.1092, \\ 0.1472, \\ 0.1887 \end{bmatrix}; \begin{bmatrix} 0.2131, \\ 0.3738 \end{bmatrix}, 0.5293 \right\rangle$	$\left\langle s_{3.4641}, \begin{bmatrix} 0.1051, \\ 0.1609, \\ 0.2196, \\ 0.2816 \end{bmatrix}; \begin{bmatrix} 0.4522, \\ 0.6535 \end{bmatrix}, 0.5477 \right\rangle$

Step 2: Construct normalized trapezoidal linguistic cubic hesitant fuzzy TOPSIS decision matrix $R = [\beta_{ij}]$. The normalized value r_{ij} is calculated as:

Table 4. The normalized trapezoidal linguistic cubic hesitant fuzzy TOPSIS decision matrix

	C_1	C_2	C_3
B_1	$\left\{ s_{0.9142}, \begin{bmatrix} 0.0823, \\ 0.1141, \\ 0.1497, \\ 0.1904 \end{bmatrix}; \left\langle \begin{bmatrix} 0.0383, \\ 0.0466 \end{bmatrix}, 0.2865 \right\rangle \right\}$	$\left\{ s_{0.9292}, \begin{bmatrix} 0.0984, \\ 0.1345, \\ 0.1733, \\ 0.2161 \end{bmatrix}; \left\langle \begin{bmatrix} 0.0467, \\ 0.0565 \end{bmatrix}, 0.1630 \right\rangle \right\}$	$\left\{ s_{0.9717}, \begin{bmatrix} 0.0337, \\ 0.0491, \\ 0.0653, \\ 0.0825 \end{bmatrix}; \left\langle \begin{bmatrix} 0.0923, \\ 0.1519 \end{bmatrix}, 0.0971 \right\rangle \right\}$
B_2	$\left\{ s_{0.9476}, \begin{bmatrix} 0.0241, \\ 0.0261, \\ 0.0281, \\ 0.0301 \end{bmatrix}; \left\langle \begin{bmatrix} 0.0536, \\ 0.1039 \end{bmatrix}, 0.2921 \right\rangle \right\}$	$\left\{ s_{0.9149}, \begin{bmatrix} 0.0279, \\ 0.0433, \\ 0.0600, \\ 0.0781 \end{bmatrix}; \left\langle \begin{bmatrix} 0.1315, \\ 0.2043 \end{bmatrix}, 0.3025 \right\rangle \right\}$	$\left\{ s_{0.9821}, \begin{bmatrix} 0.0539, \\ 0.0674, \\ 0.0821, \\ 0.0986 \end{bmatrix}; \left\langle \begin{bmatrix} 0.0531, \\ 0.0815 \end{bmatrix}, 0.0471 \right\rangle \right\}$
B_3	$\left\{ s_{0.8948}, \begin{bmatrix} 0.0578, \\ 0.0901, \\ 0.1253, \\ 0.1641 \end{bmatrix}; \left\langle \begin{bmatrix} 0.0579, \\ 0.0704 \end{bmatrix}, 0.3700 \right\rangle \right\}$	$\left\{ s_{0.9442}, \begin{bmatrix} 0.0331, \\ 0.0489, \\ 0.0659, \\ 0.0845 \end{bmatrix}; \left\langle \begin{bmatrix} 0.0954, \\ 0.1674 \end{bmatrix}, 0.2371 \right\rangle \right\}$	$\left\{ s_{0.9572}, \begin{bmatrix} 0.0291, \\ 0.0444, \\ 0.0606, \\ 0.0778 \end{bmatrix}; \left\langle \begin{bmatrix} 0.1249, \\ 0.1805 \end{bmatrix}, 0.1513 \right\rangle \right\}$

Step 3: Make the weighted normalized trapezoidal linguistic cubic hesitant fuzzy TOPSIS decision matrix by multiplying the normalized trapezoidal linguistic cubic hesitant decision matrix by its associated weights. $w_1 = 0.3427, w_2 = 0.3492, w_3 = 0.3079$.

Table 5. The weighted normalized trapezoidal linguistic cubic hesitant fuzzy TOPSIS decision matrix

	C_1	C_2	C_3
B_1	$\left\{ s_{0.3132}, [0.0282, 0.0391, 0.0513, 0.0652]; \left\langle [0.131, 0.0159], 0.0981 \right\rangle \right\}$	$\left\{ s_{0.3244}, [0.0343, 0.0469, 0.0605, 0.0754]; \left\langle [0.0153, 0.0197], 0.0569 \right\rangle \right\}$	$\left\{ s_{0.2991}, [0.0103, 0.0151, 0.0201, 0.0254]; \left\langle [0.0284, 0.0467], 0.0298 \right\rangle \right\}$
B_2	$\left\{ s_{0.3247}, [0.0082, 0.0089, 0.0096, 0.0103]; \left\langle [0.0183, 0.0356], 0.1001 \right\rangle \right\}$	$\left\{ s_{0.3194}, [0.0097, 0.0151, 0.0209, 0.0272]; \left\langle [0.0459, 0.0713], 0.1056 \right\rangle \right\}$	$\left\{ s_{0.3023}, [0.0165, 0.0207, 0.0252, 0.0303]; \left\langle [0.0163, 0.0251], 0.0145 \right\rangle \right\}$
B_3	$\left\{ s_{0.3066}, [0.0198, 0.0308, 0.0429, 0.0562]; \left\langle [0.0198, 0.0241], 0.1267 \right\rangle \right\}$	$\left\{ s_{0.3297}, [0.0115, 0.0171, 0.0231, 0.0295]; \left\langle [0.0333, 0.0584], 0.0827 \right\rangle \right\}$	$\left\{ s_{0.2947}, [0.0089, 0.0136, 0.0186, 0.0239]; \left\langle [0.0384, 0.0555], 0.0465 \right\rangle \right\}$

Step 4: Identify positive ideal solution (α^*) and negative ideal solution (α^-). The trapezoidal linguistic cubic hesitant fuzzy TOPSIS method positive-ideal solution (TrLCHPIS, α^*) and the trapezoidal linguistic cubic hesitant fuzzy negative-ideal solution (TrLCHNIS, α^-) is shown as

$$\alpha_i^{+}=\left\{\begin{Bmatrix} s_{0.3244}, \\ \begin{bmatrix}0.0343, 0.0469, \\ 0.0605, 0.0754\end{bmatrix}; \\ \left\langle\begin{bmatrix}0.0284, \\ 0.0467\end{bmatrix}, \\ 0.0298\right\rangle\end{Bmatrix}, \begin{Bmatrix} s_{0.3247}, \\ \begin{bmatrix}0.0165, 0.0207, \\ 0.0252, 0.0303\end{bmatrix}; \\ \left\langle\begin{bmatrix}0.0459, \\ 0.0713\end{bmatrix}, \\ 0.0145\right\rangle\end{Bmatrix}, \begin{Bmatrix} s_{0.3297}, \\ \begin{bmatrix}0.0198, 0.0308, \\ 0.0429, 0.0562\end{bmatrix}; \\ \left\langle\begin{bmatrix}0.0384, \\ 0.0584\end{bmatrix}, \\ 0.0465\right\rangle\end{Bmatrix}\right\}$$

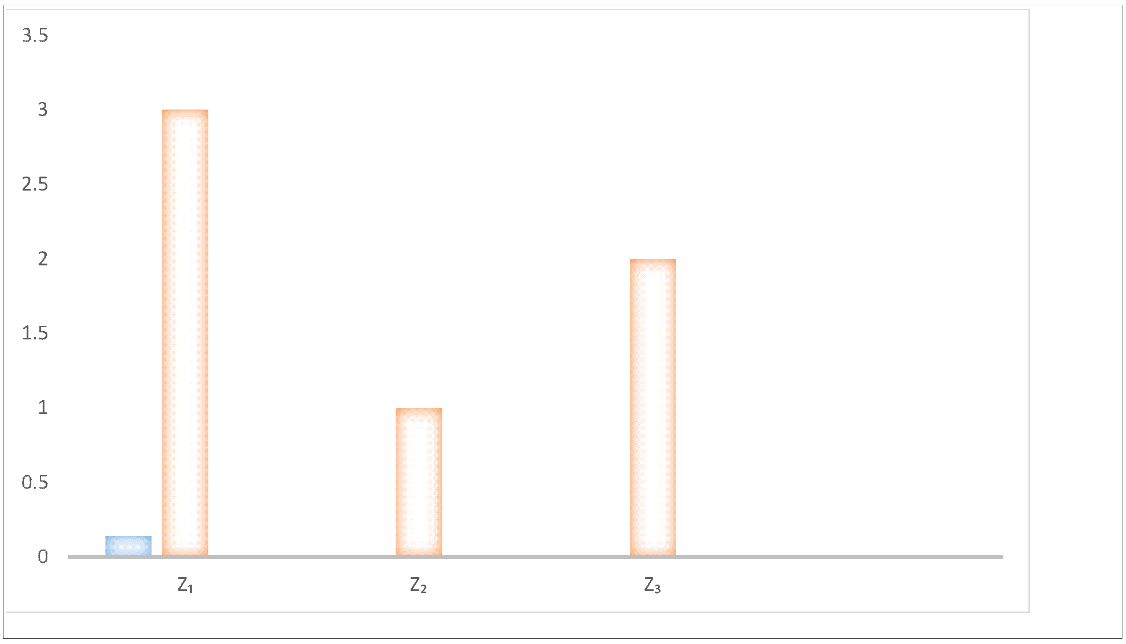
$$\alpha_i^{-}=\left\{\begin{Bmatrix} s_{0.2991}, \\ \begin{bmatrix}0.0103, 0.0151, \\ 0.0201, 0.0254\end{bmatrix}; \\ \left\langle\begin{bmatrix}0.131, \\ 0.0159\end{bmatrix}, \\ 0.0981\right\rangle\end{Bmatrix}, \begin{Bmatrix} s_{0.3023}, \\ \begin{bmatrix}0.0082, 0.0089, \\ 0.0096, 0.0103\end{bmatrix}; \\ \left\langle\begin{bmatrix}0.0163, \\ 0.0251\end{bmatrix}, \\ 0.0145\right\rangle\end{Bmatrix}, \begin{Bmatrix} s_{0.2947}, \\ \begin{bmatrix}0.0089, 0.0136, \\ 0.0186, 0.0239\end{bmatrix}; \\ \left\langle\begin{bmatrix}0.0198, \\ 0.0241\end{bmatrix}, \\ 0.1267\right\rangle\end{Bmatrix}\right\}$$

Step 5: Estimated separation measures, using the n-dimensional euclidean distance. The separation of each candidate from the TrLCHPIS $q_i^{*}\langle[B^-,B^+],\eta\rangle$ is given as $q_1^{+}\langle[B^-,B^+],\eta\rangle = 0.2104$, $q_2^{+}\langle[B^-,B^+],\eta\rangle = 0.3288, q_3^{+}\langle[B^-,B^+],\eta\rangle = 0.6732$.

The separation of each candidate from the TrLCHNIS $q_i^{-}\langle[B^-,B^+],\eta\rangle$ is given as the separation of each candidate from the TrLCNIS $q_i^{-}\langle[B^-,B^+],\eta\rangle$ is given as $q_1^{-}\langle[B^-,B^+],\eta\rangle = 0.0345$. $q_2^{-}\langle[B^-,B^+],\eta\rangle = 0.2344, q_3^{-}\langle[B^-,B^+],\eta\rangle = 0.1234$.

Step 6: Calculate similarities to ideal solution. This progression comprehends the similitudes to an ideal solution by Eqs.

$$Z_1 = \tfrac{0.0054}{0.2449} = 0.1408, Z_2 = \tfrac{0.0127}{0.5632} = 0.4161, Z_3 = \tfrac{0.1234}{0.7966} = 0.1549.$$



6 Comparison Analyses

In direction to verify the rationality and efficiency of the proposed approach, a comparative study is steered consuming the methods of cubic TOPSIS method [2] , which is special case of TrLCHTFNs, to the similar expressive example.

6.1. A Comparison Analysis With The Existing MCDM Method Cubic TOPSIS Method

[2] TOPSIS (Technique for Order Preference by Similarity to an Ideal Solution) method is a popular approach to multi-attribute decision making problems. Assuming that there are N alternatives and M attributes, the procedure of TOPSIS starts from the construction of the scores matrix $X = [x_{ij}]$ where x_{ij} denotes score of the ith alternative with respect to the jth attribute and can be summarized as follows: The proposed method which is applied to solve this problem and the computational procedure are summarized as follows:
Step 1: Set a number of alternatives and some attributes or criteria. There are 3 criteria used as a basis for decision making in academic scholarship. The criteria include:

Table 6 Cubic TOPSIS method

	C_1	C_2	C_3
B_1	$\langle[0.20, 0.24], 0.22\rangle$	$\langle[0.10, 0.12], 0.11\rangle$	$\langle[0.5, 0.7], 0.6\rangle$
B_2	$\langle[0.1, 0.3], 0.2\rangle$	$\langle[0.4, 0.6], 0.5\rangle$	$\langle[0.10, 0.12], 0.11\rangle$
B_3	$\langle[0.20, 0.24], 0.22\rangle$	$\langle[0.5, 0.7], 0.6\rangle$	$\langle[0.4, 0.6], 0.5\rangle$

Step 2: Calculation of normalized decision matrix

Table 7 Normalized decision matrix

	C_1	C_2	C_3
B_1	$\langle [0.5234, 0.6281], 0.5757 \rangle$	$\langle [0.5232, 0.6279], 0.5756 \rangle$	$\langle [0.4767, 0.6674], 0.5721 \rangle$
B_2	$\langle [0.2673, 0.8019], 0.5346 \rangle$	$\langle [0.4558, 0.6838], 0.5698 \rangle$	$\langle [0.5232, 0.6279], 0.5756 \rangle$
B_3	$\langle [0.5234, 0.6281], 0.5757 \rangle$	$\langle [0.4767, 0.6674], 0.5721 \rangle$	$\langle [0.4558, 0.6838], 0.5698 \rangle$

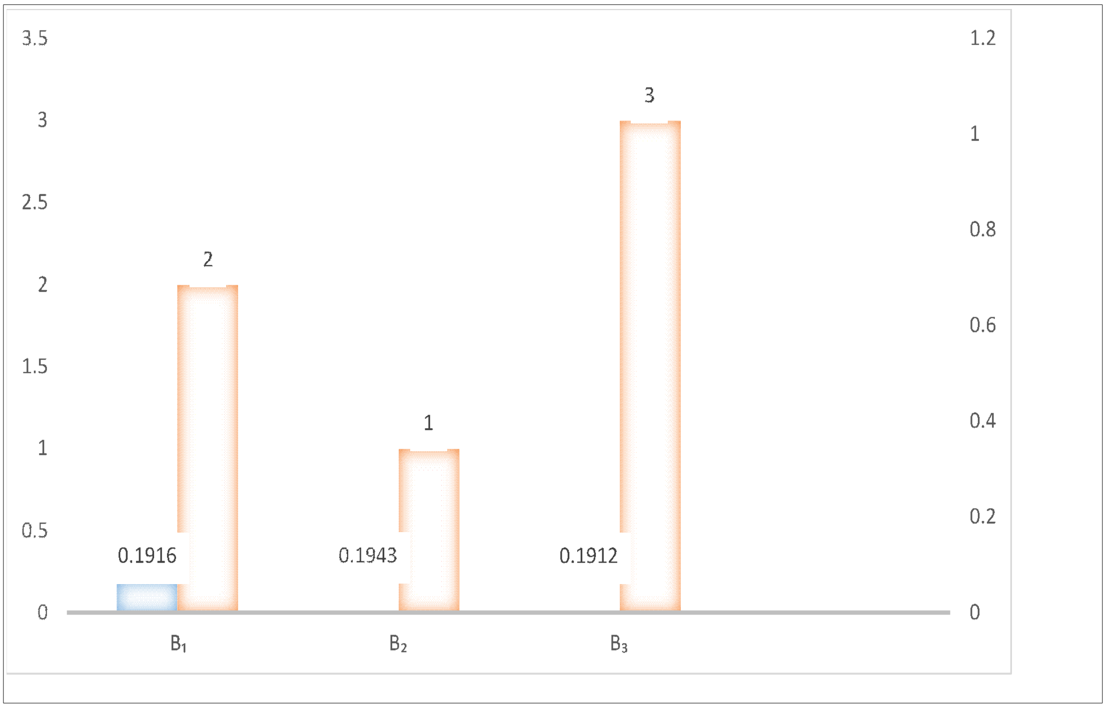
Step 3: Calculation of the weighted aggregated CF-decision matrix

$$v_1 = 0.3, v_2 = 0.2, v_3 = 0.5,$$

Table 8 weighted aggregated CF-decision matrix	
B_1	$\langle [0.5005, 0.6482], 0.5738 \rangle$
B_2	$\langle [0.4431, 0.7018], 0.5618 \rangle$
B_3	$\langle [0.4811, 0.6646], 0.5721 \rangle$

Step 4: Determination of the score value

$$Z_1 = 0.1916, Z_2 = 0.1943, Z_3 = 0.1912.$$



Comparison analysis with existing methods Table 9

Method	Ranking
Trapezoidal linguistic cubic hesitant fuzzy TOPSIS method	$B_2 > B_3 > B_1$
Cubic TOPSIS method [2]	$B_2 > B_1 > B_3$

7 Conclusion

In this paper, we define a new idea of trapezoidal linguistic cubic hesitant fuzzy number. We discuss some basic operational laws of trapezoidal linguistic cubic hesitant fuzzy number and hamming distance of trapezoidal linguistic cubic hesitant fuzzy number. Furthermore, we develop Trapezoidal linguistic cubic hesitant fuzzy TOPSIS method to solve the MCDM method based on trapezoidal linguistic cubic hesitant fuzzy TOPSIS method. Finally, an illustrative example is given to verify and demonstrate the practicality and effectiveness of the proposed method. We compared the proposed method to the existing methods, which shows the trapezoidal linguistic Cubic hesitant TOPSIS method are more fexible to deal uncertainties and fuzziness. In fact, this method is very simple and flexible. Hence, it is expected that proposed in this study may have more potential management applications.

Graphical Abstract

References

[1] A. Fahmi, S. Abdullah, F. Amin ,N. Siddque and A Ali. AGGREGATION OPERATORS ON TRIANGULAR CUBIC FUZZY NUMBERS AND ITS APPLICATION TO MULTI-CRITERIA DECISION MAKING PROBLEMS, Journal of Intelligent and Fuzzy System, 33 (2017),3323-3337..

[2] A. Fahmi, S. Abdullah, F. Amin and A. Ali, Precursor Selection for Sol--Gel Synthesis of Titanium Carbide Nanopowders by a New Cubic Fuzzy Multi-Attribute Group Decision-Making Model. Journal of Intelligent Systems. 0(0): -. Retrieved 6 Oct. 2017, from doi:10.1515/jisys-2017-0083.

[3] A. Fahmi, S. Abdullah, F. Amin and A. Ali, Weighted Average Rating (War) Method for Solving Group Decision Making Problem Using Triangular Cubic Fuzzy Hybrid Aggregation (Tcfha), Punjab University Journal of Mathematics, Vol. 50(1)(2018) pp. 23-34.

[4] K. T. Atanassov, Intuitionistic fuzzy sets, Fuzzy Sets Syst. 20 (1) (1986) 87--96.

[5] De Boer, L., E. Labro, and P. Morlacchi, A review of methods supporting supplier selection, European Journal of Purchasing and Supply Management, vol.7, pp.75-89, 200.
[6] B. Gülçin, and G. Çifçi. "A novel hybrid MCDM approach based on fuzzy DEMATEL, fuzzy ANP and fuzzy TOPSIS to evaluate green suppliers." Expert Systems with Applications 39.3 (2012): 3000-3011.
[7] S. H. Ghodsypour., and C. O'Brien, A decision support system for supplier selection using an integrated analytic hierarchy process and linear programming, International Journal of Production Economics, vol.56-57, pp.199-212, 1998.
[8] C. L. Hwang and Yoon K.: Multiple Attribute Decision Making: Methods and Application, Springer, New York (1981).
[9] Y. B. Jun, C. S. Kim and Ki. O. Yang, Cubic sets ,Annals of Fuzzy Mathematics and Informatics, 4(1), 83- 98: (2011).
[10] Yang, T. & Chou P.: Solving a multi-response simulation--optimization problem with discrete variables using a multipleattribute decision making method. Math Comput Simulat 68, 9--21, (2005).
[11] K. Yoon,. and C. Hwang. Multiple attribute decision making. Thousand Oaks, CA, Sage Publication, (1995).
[12] T. Yang and C. Hung Multiple-attribute decision making methods for plant layout design problem. Robotics and ComputerIntegrated Manufacturing 23, 126--137, (2005).
[13] C. A. Weber., J.R. Current, and W.C. Benton, Vendor selection criteria and methods, European Journal of Operational Research, vol.50, no.1, pp.2-18, 1991.
[14] L.A. Zadeh, Fuzzy sets, Inf. Control 18 (1965) 338--353.
[15] K. Yoon, A Reconciliation Among Discrete Compromise Solutions, Journal of The Operational Research Society 38 (1987) 277--286.
[16] C. lai Hwang, Y. jou Lai, T. yun Liu, A new approach for multiple objective decision making, Computers and Operations Research 20 (1993) 889--899.
[17] X. lin Wang, Z. bo Qing, Y. Zhang, Decision making of selecting manufacturing partner based on the supply chain: study on topsis method application, in: Cybernetics and Intelligent Systems, 2004 IEEE Conference on, Vol. 2, 2004, pp. 1118--1122. doi:10.1109/ICCIS.2004.1460746.
[18] Y. Yang, X. Li, D. Chen, T. Yu, W. Wang, Application of gc-topsis method in the process of supplier evaluation, in: Management and Service Science, 2009. MASS '09. International Conference on, 2009, pp. 1--4.doi:10.1109/ICMSS.2009.5301059.
[19] A. Bilbao-Terol, M. Arenas-Parra, V. Cañal-Fernández, J. Antomil-Ibias, Using {TOPSIS} for assessing the sustainability of government bond funds, Omega 49 (2014) 1 -- 17. doi:http://dx.doi.org/10.1016/j.omega.2014.04.005.
[20] M. Tavana, M. Keramatpour, F. J. Santos-Arteaga, E. Ghorbaniane, A fuzzy hybrid project portfolio selection method using data envelopment analysis, {TOPSIS} and integer programming, Expert Systems with Applications 42 (22) (2015) 8432 -- 8444. doi:http://dx.doi.org/10.1016/j.eswa.2015.06.057.
[21] L. Zhao, X. Gao, Y. Zhang, Compare and contrast analysis to the development pattern of energy producing provinces with the aim of carbon emission reducing, in: Management and Service Science (MASS), 2010 International Conference on, 2010, pp. 1--6. doi:10.1109/ICMSS.2010.5577654.
[22] L. yun Wu, Y. zhong Yang, Topsis method for green vendor selection in coal industry group, in: Machine Learning and Cybernetics, 2008 International Conference on, Vol. 3, 2008, pp. 1721--1725. doi:10.1109/ICMLC.2008.4620683.
[23] E. Roszkowska, T. Wachowicz, Application of fuzzy {TOPSIS} to scoring the negotiation offers in ill-structured negotiation problems, European Journal of

Operational Research 242 (3) (2015) 920 -- 932. doi:http://dx.doi.org/10.1016/j.ejor.2014.10.050.

[24] C. T. Chen, Extensions of the {TOPSIS} for group decision-making under fuzzy environment, Fuzzy Sets and Systems 114 (1) (2000) 1 -- 9.

[25] V. Torra, Hesitant fuzzy sets, International Journal of Intelligent Systems, vol. 25, no. 6, pp. 529--539, 2010.

[26] M. Xia and Z. Xu, Hesitant fuzzy information aggregation in decision making, International Journal of Approximate Reasoning, vol. 52, no. 3, pp. 395--407, 2011.

http://www.newtheory.org ISSN: 2149-1402

Journal of New Theory

Received: 02.12.2017 *Year*: **2017**, *Number*: **19**, *Pages*: **48-55**
Published: 19.12.2017 *Original Article*

GENERATING FUNCTIONS OF q-ANALOGUE OF I-FUNCTION SATISFYING TRUESDELL'S DESCENDING F_q-EQUATION

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Abstract – In this paper, the authors have obtained various forms of q-analogue of I-function satisfying Truesdell's descending F_q-equation. These forms have been employed to arrive at certain generating functions for q-analogue of I-function. Some particular cases of these results in terms of q-analogue of H-and G-functions which appear to be new have also been obtained.

Keywords – *F_q-equation, Generating function, q-analogue of I-Function, q-analogue of H-Function, q-analogue of G-Function.*

1 Introduction

Recent developments in the theory of Basic hypergeometric functions have gained much interest due to its introduction of certain new generalized forms of Basic hypergeometric functions. These functions are Mac-Roberts's E-Function, Meijer's G-Function, Fox's H-Function, Saxena's I-Function and their q-analogues. The q-analogue of I-Function have been introduced by Saxena et al. [1] in terms of Mellin-Barnes type basic contour integral as

$$I(z)=I^{m,n}_{A_i,B_i;r}\left[z;q\left|\begin{matrix}(a_j,\alpha_j)_{1,n};(a_{ji},\alpha_{ji})_{n+1,A_i}\\(b_j,\beta_j)_{1,m};(b_{ji},\beta_{ji})_{m+1,B_i}\end{matrix}\right.\right]=$$

$$\frac{1}{2\pi i}\int_L\frac{\prod_{j=1}^{m}G(q^{b_j-\beta_j s})\prod_{j=1}^{n}G(q^{1-a_j+\alpha_j s})}{\sum_{i=1}^{r}\left[\prod_{j=m+1}^{B_i}G(q^{1-b_{ji}+\beta_{ji}s})\prod_{j=n+1}^{A_i}G(q^{a_{ji}-\alpha_{ji}s})G(q^s)G(q^{1-s})\sin\pi s\right]}\pi z^s ds \tag{1}$$

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where $\alpha_j, \beta_j, \alpha_{ji}, \beta_{ji}$ are real and positive, a_j, b_j, a_{ji}, b_{ji} are complex numbers and

$$G(q^\alpha) = \prod_{n=0}^{\infty} (1 - q^{\alpha+n})^{-1} = \frac{1}{(q^\alpha; q)_\infty}$$

where L is contour of integration running from - i∞ to i∞ in such a manner so that all poles of $G(q^{b_j - \beta_j s}); 1 \le j \le m$ are to right of the path and those of $G(q^{1-a_j+\alpha_j s}); 1 \le j \le n$ are to left. The integral converges if *Re [s log (x) – log sin πs} < 0,* for large values of |s| on the contour L.

Setting $r = 1, A_i = A, B_i = B$ in equation (1) we get q-analogue of H-Function defined by Saxena et.al. [1] as follows:

$$H_{A,B}^{m,n}\left[z;q\middle|\begin{matrix}(a_j,\alpha_j)_{1,A}\\(b_j,\beta_j)_{1,B}\end{matrix}\right] = \frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^{m} G(q^{b_j-\beta_j s})\prod_{j=1}^{n} G(q^{1-a_j+\alpha_j s})}{\prod_{j=m+1}^{B} G(q^{1-b_j+\beta_j s})\prod_{j=n+1}^{A} G(q^{a_j-\alpha_j s})G(q^s)G(q^{1-s})\sin\pi s}\pi z^s ds \quad (2)$$

Further if we put $\alpha_j = \beta_j = 1$, equation (2) reduces to the basic analogue of Meijer's G-Function given by Saxena et. al. [1].

$$G_{A,B}^{m,n}\left[z;q\middle|\begin{matrix}a_1,a_2,...,a_A\\b_1,b_2,...,b_B\end{matrix}\right] = \frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^{m} G(q^{b_j-s})\prod_{j=1}^{n} G(q^{1-a_j+s})}{\prod_{j=m+1}^{B} G(q^{1-b_j+s})\prod_{j=n+1}^{A} G(q^{a_j-s})G(q^s)G(q^{1-s})\sin\pi s}\pi z^s ds \quad (3)$$

Farooq et. al. [2] defined the basic analogue of I-function in terms of Gamma function as follows

$$I_{q,A_i,B_i;R}^{m,n}\left[z;q\middle|\begin{matrix}(a_j,\alpha_j)_{1,n}\ (a_{ji},\alpha_{ji})_{n+1,A_i}\\(b_j,\beta_j)_{1,m}\ (b_{ji},\beta_{ji})_{m+1,B_i}\end{matrix}\right] = \frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^{m}\Gamma_q(b_j-\beta_j s)\prod_{j=1}^{n}\Gamma_q(1-a_j+\alpha_j s)}{\sum_{i=1}^{R}\left\{\prod_{j=m+1}^{B_i}\Gamma_q(1-b_{ji}+\beta_{ji}s)\prod_{j=n+1}^{A_i}\Gamma_q(a_{ji}-\alpha_{ji}s)\right\}\Gamma_q(s)\Gamma_q(1-s)\sin\pi s}\pi z^s ds \quad (4)$$

In his effort towards achieving unification of special functions, Truesdell [3] has put forward a theory which yielded a number of results for special functions satisfying the so called Truesdell's F_q-equation. Agrawal B. M. [4] extended this theory further and derived results for descending F_q-equation. He obtained various properties like orthogonality, Rodrigue's and Schalafli's formulae for F_q-equation, which turn out to be special functions.

Renu Jain et. al. [7] derived some generating functions of q-analogue of Mittag-Leffler function and Hermite polynomial satisfying Truesdell's ascending and descending Fq-equation. The function $F(z,\alpha)$ is said to satisfy the descending F-equation if

$$D_z^r F(z,\alpha) = F(z,\alpha - r) \tag{5}$$

For $F(z,\alpha)$ satisfying descending F-equation, Agrawal B. M. [6] has obtained following generating functions:

$$F(z+y,\alpha) = \sum_{n=0}^{\infty} y^n \frac{F(z,\alpha-n)}{n!} \tag{6}$$

The q-derivative of equation (5) can be written in the following manner:

$$D_{q,z}^r F(z,\alpha) = F(z,\alpha + r) \tag{7}$$

In order to obtain main result of this paper we will make use of the following results which we have obtained on multiplication formulae for q-analogue of Gamma functions:

$$\prod_{k=0}^{m-1} \Gamma_q\left(\frac{\alpha - r + k}{m}\right) = \frac{(1-q)^r q^{\frac{r}{m}(r-2\alpha+1)}}{(-1)^r (q^{1-\alpha};q)_r} \prod_{k=0}^{m-1} \Gamma_q\left(\frac{\alpha + k}{m}\right) \tag{8}$$

$$\prod_{k=0}^{m-1} \Gamma_q\left(1 - \frac{\alpha - r + k}{m}\right) = \frac{(q^{1-\alpha};q)_r}{(1-q)^r} \prod_{k=0}^{m-1} \Gamma_q\left(1 - \frac{\alpha + k}{m}\right) \tag{9}$$

In the results that follow, by $\Delta(\mu,\alpha)$ we shall mean the array of μ parameters

$$\frac{\alpha}{\mu}, \frac{\alpha+1}{\mu}, \ldots, \frac{\alpha+\mu-1}{\mu}; \qquad (\mu = 1,2,3,\ldots) \tag{10}$$

and $(\Delta(\mu,\alpha),\beta)$ stands for $\left(\frac{\alpha}{\mu},\beta\right), \left(\frac{\alpha+1}{\mu},\beta\right), \ldots, \left(\frac{\alpha+\mu-1}{\mu},\beta\right)$ (11)

2 Generating Functions for q-analogue of I-Function

(A): Renu Jain et. al.[5] obtained various forms of I-Function which satisfy Truesdell's descending F-equation and hence in this connection, we established in this section the different forms of q-analogue of I-Function which satisfy Truesdell's descending F_q-equation:

$$(I) \left(q^{\frac{\alpha}{2}\left[\frac{1-\rho}{\rho}\right]} z \right)^{\alpha-1} I_{p_i,q_i,l}^{m,n} \left[\frac{q^{\alpha h(\lambda+1)}}{z^{h\lambda}}; q \,\middle|\, \begin{matrix} \{\Delta(\lambda,\alpha),h\}, \{(a_j,\alpha_j)_{\lambda+1,n}\}, \{(a_{ji},\alpha_{ji})_{n+1,p_i-\rho}\}, \{\Delta(\rho,\alpha),h\} \\ \{(b_j,\beta_j)_{1,m}\}, \{(b_{ji},\beta_{ji})_{m+1,q_i-\rho}\}, \{\Delta(\rho,\alpha),h\} \end{matrix} \right] \tag{12}$$

$$(II)\ \frac{\left(q^{\frac{-\alpha}{2}\left[\frac{1+\lambda}{\lambda}\right]}z\right)^{\alpha-1}}{(1-q)^{\alpha}} I^{m,n}_{p_i,q_i,l}\left[\frac{q^{\alpha h(\lambda-1)}}{z^{h\lambda}};q\left|\begin{matrix}\{\Delta(2\lambda,2\alpha),h\},\{(a_j,\alpha_j)_{2\lambda+1,n}\},\{(a_{ji},\alpha_{ji})_{n+1,p_i}\}\\ \left\{\Delta(\lambda,\alpha+\frac{1}{2}),h\right\},\{(b_j,\beta_j)_{\lambda+1,m}\},\{(b_{ji},\beta_{ji})_{m+1,q_i}\}\end{matrix}\right.\right] \tag{13}$$

$$(III)\ \frac{\left(q^{\frac{-\alpha}{2(\alpha-1)}\left[\frac{3\alpha+1}{3\lambda}+\alpha-1\right]}z\right)^{\alpha-1}}{(1-q)^{\alpha}} I^{m,n}_{p_i,q_i,l}\left[\frac{q^{\alpha h(\lambda+1)}}{z^{h\lambda}};q\left|\begin{matrix}\{\Delta(3\lambda,3\alpha),h\},\{(a_j,\alpha_j)_{3\lambda+1,n}\},\{(a_{ji},\alpha_{ji})_{n+1,p_i}\}\\ \left\{\Delta(\lambda,\alpha+\frac{2}{3}),h\right\},\{(b_j,\beta_j)_{\lambda+1,m}\},\{(b_{ji},\beta_{ji})_{m+1,q_i-\lambda}\},\left\{\Delta(\lambda,\alpha+\frac{1}{3}),h\right\}\end{matrix}\right.\right] \tag{14}$$

$$(IV)\ \left(q^{\frac{-\alpha}{2}}z\right)^{\alpha-1} e^{\pi i\alpha} I^{m,n}_{p_i,q_i,l}\left[\frac{q^{\alpha h\lambda}}{z^{h\lambda}};q\left|\begin{matrix}\{\Delta(\lambda,\alpha),h\},\{(a_j,\alpha_j)_{\lambda+1,n}\},\{(a_{ji},\alpha_{ji})_{n+1,p_i}\}\\ \{(b_j,\beta_j)_{1,m}\},\{(b_{ji},\beta_{ji})_{m+1,q_i}\}\end{matrix}\right.\right] \tag{15}$$

$$(V)\ \left(q^{\frac{\alpha}{2}(\frac{1-\lambda}{\lambda})}z\right)^{\alpha-1} I^{m,n}_{p_i,q_i,l}\left[\frac{q^{h\alpha(\lambda-1)}}{z^{h\lambda}};q\left|\begin{matrix}\{(a_j,\alpha_j)_{1,n}\},\{(a_{ji},\alpha_{ji})_{n+1,p_i-\lambda}\},\{\Delta(\lambda,\alpha),h\}\\ \{(b_j,\beta_j)_{1,m}\},\{(b_{ji},\beta_{ji})_{m+1,q_i}\}\end{matrix}\right.\right] \tag{16}$$

$$(VI)\ \left(q^{\frac{\alpha}{2}\left(\frac{1-\lambda}{\lambda}-\frac{1}{\rho}\right)}z\right)^{\alpha-1} e^{\pi i\alpha} I^{m,n}_{p_i,q_i,l}\left[\frac{q^{\alpha h\lambda}}{z^{h\lambda}};q\left|\begin{matrix}\{\Delta(\rho,\alpha),h\}\{(a_j,\alpha_j)_{\rho+1,n}\},\{(a_{ji},\alpha_{ji})_{n+1,p_i-\lambda}\},\{\Delta(\lambda,\alpha),h\}\\ \{\Delta(\rho,\alpha),h\}\{(b_j,\beta_j)_{\rho+1,m}\},\{(b_{ji},\beta_{ji})_{m+1,q_i}\}\end{matrix}\right.\right] \tag{17}$$

Assuming that form (12) is $A(z,\alpha)$, replacing q-analogue of I-Function by its definition (4) and then interchanging order of integration and differentiation, which is justified under the conditions of convergence [2], we observe

$$D^r_{q,z}A(z,\alpha)$$
$$=\frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^{m}\Gamma_q(b_j-\beta_j s)\prod_{k=0}^{\lambda-1}\Gamma_q\left(1-\frac{\alpha+k}{\lambda}-hs\right)\prod_{j=\lambda+1}^{n}\Gamma_q(1-a_j+\alpha_j s)\, q^{\frac{\alpha(\alpha-1)(1-\rho)}{2\rho}} q^{\alpha h(\lambda+1)s} D^r_{q,z}\left\{z^{\alpha-1-h\lambda s}\right\}\pi ds}{\sum_{i=1}^{l}\left\{\prod_{j=m+1}^{q_i-\rho}\Gamma_q(1-b_{ji}+\beta_{ji}s)\prod_{k=0}^{\rho-1}\Gamma_q\left(1-\frac{\alpha+k}{\rho}+hs\right)\prod_{j=n+1}^{p_i-\rho}\Gamma_q(a_{ji}-\alpha_{ji}s)\prod_{k=0}^{\rho-1}\Gamma_q\left(\frac{\alpha+k}{\rho}-hs\right)\right\}\Gamma_q(s)\Gamma_q(1-s)\sin\pi s} \tag{18}$$

Now results (8) and (9) lead to two very important identities

$$\prod_{k=0}^{\lambda-1}\Gamma_q\left(\frac{\alpha+k}{\lambda}-hs\right)=\frac{(-1)^r(q^{1-(\alpha-h\lambda s)};q)_r}{(1-q)^r q^{\frac{r}{2m}(r-2(\alpha-h\lambda s)+1)}}\prod_{k=0}^{\lambda-1}\Gamma_q\left(\frac{\alpha-r+k}{\lambda}-hs\right) \tag{19}$$

$$\prod_{k=0}^{\lambda-1}\Gamma_q\left(1-\frac{\alpha+k}{\lambda}+hs\right)=\frac{(1-q)^r}{(q^{1-(\alpha-h\lambda s)};q)_r}\prod_{k=0}^{\lambda-1}\Gamma_q\left(1-\frac{\alpha-r+k}{\lambda}+hs\right) \tag{20}$$

Using these identities (19) and (20) we see that (18) takes the form

$$D_{q,z}^{r} A(z,\alpha) = \left[A(z,\alpha - r)\right] \tag{21}$$

This is the Truesdell's form of descending F-equation.
Similarly forms (13) to (17) can be shown to satisfy Truesdell's descending F-equation.

(B): In this section we employ forms (12) to (17), to establish the following generating functions for q-analogue of I-Functions using Truesdell's descending F-equation technique:

(I)

$$(1+q^{\alpha})^{\alpha-1} I_{p_i,q_i,l}^{m,n}\left[(1+q^{\alpha})^{-h\lambda}x;q\left|\begin{matrix}\{\Delta(\lambda,\alpha),h\},\{(a_j,\alpha_j)_{\lambda+1,n}\},\{(a_{ji},\alpha_{ji})_{n+1,p_i-\rho}\},\{\Delta(\rho,\alpha),h\}\\ \{(b_j,\beta_j)_{1,m}\},\{(b_{ji},\beta_{ji})_{m+1,q_i-\rho}\},\{\Delta(\rho,\alpha),h\}\end{matrix}\right.\right]$$

$$=\sum_{r=0}^{\infty}\frac{\left(q^{(-\alpha+r/2+1/2)(\frac{1-\rho}{\rho})+\alpha}\right)^{r}}{r!} I_{p_i,q_i,l}^{m,n}\left[q^{-rh(\lambda+1)}x;q\left|\begin{matrix}\{\Delta(\lambda,\alpha-r),h\},\{(a_j,\alpha_j)_{\lambda+1,n}\},\{(a_{ji},\alpha_{ji})_{n+1,p_i-\rho}\},\{\Delta(\rho,\alpha-r),h\}\\ \{(b_j,\beta_j)_{1,m}\},\{(b_{ji},\beta_{ji})_{m+1,q_i-\rho}\},\{\Delta(\rho,\alpha-r),h\}\end{matrix}\right.\right] \tag{22}$$

(II)

$$(1+q^{\alpha})^{\alpha-1} I_{p_i,q_i,l}^{m,n}\left[(1+q^{\alpha})^{-h\lambda}x;q\left|\begin{matrix}\{\Delta(2\lambda,2\alpha),h\},\{(a_j,\alpha_j)_{2\lambda+1,n}\},\{(a_{ji},\alpha_{ji})_{n+1,p_i}\}\\ \left\{\Delta(\lambda,\alpha+\frac{1}{2}),h\right\},\{(b_j,\beta_j)_{\lambda+1,m}\},\{(b_{ji},\beta_{ji})_{m+1,q_i}\}\end{matrix}\right.\right]$$

$$=\sum_{r=0}^{\infty}\frac{\left(q^{\left(\alpha-\frac{1}{2}-\frac{r}{2}\right)\left(\frac{1+\lambda}{\lambda}\right)+\alpha}(1-q)\right)^{r}}{r!} I_{p_i,q_i,l}^{m,n}\left[q^{rh(1-\lambda)}x;q\left|\begin{matrix}\{\Delta(2\lambda,2\alpha-2r),h\},\{(a_j,\alpha_j)_{2\lambda+1,n}\},\{(a_{ji},\alpha_{ji})_{n+1,p_i}\}\\ \left\{\Delta(\lambda,\alpha-r+\frac{1}{2}),h\right\},\{(b_j,\beta_j)_{\lambda+1,m}\},\{(b_{ji},\beta_{ji})_{m+1,q_i}\}\end{matrix}\right.\right] \tag{23}$$

(III)

$$(1+q^{\alpha})^{\alpha} I_{p_i,q_i,l}^{m,n}\left[(1+q^{\alpha})^{-h\lambda}x;q\left|\begin{matrix}\{\Delta(3\lambda,3\alpha),h\},\{(a_j,\alpha_j)_{3\lambda+1,n}\},\{(a_{ji},\alpha_{ji})_{n+1,p_i}\}\\ \left\{\Delta(\lambda,\alpha+\frac{2}{3}),h\right\},\{(b_j,\beta_j)_{\lambda+1,m}\},\{(b_{ji},\beta_{ji})_{m+1,q_i-\lambda}\},\left\{\Delta(\lambda,\alpha+\frac{1}{3}),h\right\}\end{matrix}\right.\right]$$

$$=\sum_{r=0}^{\infty}\frac{\left(q^{\left(2\alpha+\frac{\alpha}{\lambda}-\frac{r}{2\lambda}-\frac{r}{2}+\frac{1}{6\lambda}+\frac{1}{2}\right)}(1-q)\right)^{r}}{r!} I_{p_i,q_i,l}^{m,n}\left[q^{-rh(1+\lambda)}x;q\left|\begin{matrix}\{\Delta(3\lambda,3\alpha-3r),h\},\{(a_j,\alpha_j)_{3\lambda+1,n}\},\{(a_{ji},\alpha_{ji})_{n+1,p_i}\}\\ \left\{\Delta(\lambda,\alpha-r+\frac{2}{3}),h\right\},\{(b_j,\beta_j)_{\lambda+1,m}\},\{(b_{ji},\beta_{ji})_{m+1,q_i-\lambda}\},\left\{\Delta(\lambda,\alpha-r+\frac{1}{3}),h\right\}\end{matrix}\right.\right] \tag{24}$$

(IV)

$$(1+q^{\alpha})^{\alpha-1} I_{p_i,q_i,l}^{m,n}\left[(1+q^{\alpha})^{-h\lambda}x;q\left|\begin{matrix}\{\Delta(\lambda,\alpha),h\},\{(a_j,\alpha_j)_{\lambda+1,n}\},\{(a_{ji},\alpha_{ji})_{n+1,p_i}\}\\ \{(b_j,\beta_j)_{1,m}\},\{(b_{ji},\beta_{ji})_{m+1,q_i}\}\end{matrix}\right.\right]$$

$$=\sum_{r=0}^{\infty}\frac{(-1)^{r}\left(q^{2\alpha+\frac{r}{2}-\frac{1}{2}}\right)^{r}}{r!} I_{p_i,q_i,l}^{m,n}\left[q^{-rh\lambda}x;q\left|\begin{matrix}\{\Delta(\lambda,\alpha-r),h\},\{(a_j,\alpha_j)_{\lambda+1,n}\},\{(a_{ji},\alpha_{ji})_{n+1,p_i}\}\\ \{(b_j,\beta_j)_{1,m}\},\{(b_{ji},\beta_{ji})_{m+1,q_i}\}\end{matrix}\right.\right] \tag{25}$$

(V)

$$(1+q^{\alpha})^{\alpha-1} I_{p_i,q_i,l}^{m,n}\left[(1+q^{\alpha})^{-h\lambda}x;q\left|\begin{matrix}\{(a_j,\alpha_j)_{1,n}\},\{(a_{ji},\alpha_{ji})_{n+1,p_i-\lambda}\},\{\Delta(\lambda,\alpha),h\}\\ \{(b_j,\beta_j)_{1,m}\},\{(b_{ji},\beta_{ji})_{m+1,q_i}\}\end{matrix}\right.\right]$$

$$=\sum_{r=0}^{\infty}\frac{\left(q^{\left(\frac{1-\lambda}{\lambda}\right)\left[-\alpha+\frac{r}{2}+\frac{1}{2}\right]+\alpha}\right)^r}{r!} I_{p_i,q_i,l}^{m,n}\left[q^{rh(1-\lambda)}x;q\left|\begin{matrix}\{(a_j,\alpha_j)_{1,n}\},\{(a_{ji},\alpha_{ji})_{n+1,p_i-\lambda}\},\{\Delta(\lambda,\alpha-r),h\}\\ \{(b_j,\beta_j)_{1,m}\},\{(b_{ji},\beta_{ji})_{m+1,q_i}\}\end{matrix}\right.\right] \quad (26)$$

(VI)

$$(1+q^{\alpha})^{\alpha-1} I_{p_i,q_i,l}^{m,n}\left[(1+q^{\alpha})^{-h\lambda}x;q\left|\begin{matrix}\{\Delta(\rho,\alpha),h\}\{(a_j,\alpha_j)_{\rho+1,n}\},\{(a_{ji},\alpha_{ji})_{n+1,p_i-\lambda}\},\{\Delta(\lambda,\alpha),h\}\\ \{\Delta(\rho,\alpha),h\}\{(b_j,\beta_j)_{\rho+1,m}\},\{(b_{ji},\beta_{ji})_{m+1,q_i}\}\end{matrix}\right.\right]$$

$$=\sum_{r=0}^{\infty}\frac{(-1)^r\left(q^{\left(\frac{1}{\lambda}-\frac{1}{\rho}-1\right)\left[-\alpha+\frac{r}{2}+\frac{1}{2}\right]+\alpha}\right)^r}{r!} I_{p_i,q_i,l}^{m,n}\left[q^{-rh\lambda}x;q\left|\begin{matrix}\{\Delta(\rho,\alpha-r),h\}\{(a_j,\alpha_j)_{\rho+1,n}\},\{(a_{ji},\alpha_{ji})_{n+1,p_i-\lambda}\},\{\Delta(\lambda,\alpha-r),h\}\\ \{\Delta(\rho,\alpha-r),h\}\{(b_j,\beta_j)_{\rho+1,m}\},\{(b_{ji},\beta_{ji})_{m+1,q_i}\}\end{matrix}\right.\right] \quad (27)$$

Proof: To establish (22) we substitute the form (12) in Truesdell's descending F-equation (6) and replace z by $\frac{y}{q^{\alpha}}$ and $\frac{q^{\alpha h+2\alpha h\lambda}}{y^{h\lambda}}$ by x in succession to get the required result. Similarly, result (24) can be proved by substituting the form (14) in Truesdell's descending F-equation (6) and using same replacement.
To establish (23) we substitute the form (13) in Truesdell's descending F-equation (6) and replace z by $\frac{y}{q^{\alpha}}$ and $\frac{q^{2\alpha h\lambda-\alpha h}}{y^{h\lambda}}$ by x in succession to get the required result. Similarly, result (26) can be proved by substituting the form (16) in Truesdell's descending F-equation (6) and using same replacement.
To establish (25) we substitute the form (15) in Truesdell's descending F-equation (6) and replace z by $\frac{y}{q^{\alpha}}$ and $\frac{q^{2\alpha h\lambda}}{y^{h\lambda}}$ by x in succession to get the required result. Similarly, result (27) can be proved by substituting the form (17) in Truesdell's descending F-equation (6) and using same replacement.

3 Special Cases

These results yield as special cases of certain generating function for q-analogue of Fox's H-Function [1] and q-analogue of Meijer's G-Function [1].

(i): If we take $l=1$ then the series (22) reduces to generating function of q-analogue of Fox's H-Function.

$$(1+q^{\alpha})^{\alpha-1}H^{m,n}_{P,Q}\left[(1+q^{\alpha})^{-h\lambda}x;q\left|\begin{matrix}\{\Delta(\lambda,\alpha),h\},(a_j,\alpha_j),\{\Delta(\rho,\alpha),h\}\\(b_j,\beta_j),\{\Delta(\rho,\alpha),h\}\end{matrix}\right.\right]$$

$$=\sum_{r=0}^{\infty}\frac{\left(q^{(-\alpha+r/2+1/2)(\frac{1-\rho}{\rho})+\alpha}\right)^r}{r!}H^{m,n}_{P,Q}\left[q^{-rh(\lambda+1)}x;q\left|\begin{matrix}\{\Delta(\lambda,\alpha-r),h\},(a_j,\alpha_j),\{\Delta(\rho,\alpha-r),h\}\\(b_j,\beta_j),\{\Delta(\rho,\alpha-r),h\}\end{matrix}\right.\right] \quad (28)$$

Again taking $\alpha_j=\beta_j=1\, and\, h=1$ in (28), it gives Meijer's G-Function as:

$$(1+q^{\alpha})^{\alpha-1}H^{m,n}_{P,Q}\left[(1+q^{\alpha})^{-\lambda}x;q\left|\begin{matrix}\Delta(\lambda,\alpha),a_j,\Delta(\rho,\alpha)\\b_j,\Delta(\rho,\alpha)\end{matrix}\right.\right]$$

$$=\sum_{r=0}^{\infty}\frac{\left(q^{(-\alpha+r/2+1/2)(\frac{1-\rho}{\rho})+\alpha}\right)^r}{r!}H^{m,n}_{P,Q}\left[q^{-r(\lambda+1)}x;q\left|\begin{matrix}\Delta(\lambda,\alpha-r),a_j,\Delta(\rho,\alpha-r)\\b_j,\Delta(\rho,\alpha-r)\end{matrix}\right.\right] \quad (29)$$

Similarly, (23) to (27) can be employed to yield apparently new and interesting results for q-analogue of Fox's H-Function and Meijer's G-Function [1].

4 Conclusions

The results proved in this paper give some contributions to the theory of Truesdell's F_q-equation and are believed to be new to the theory of q-calculus and are likely to find certain applications in the theory of q-calculus.

Acknowledgements

The authors are grateful to the anonymous referees for a careful checking of the details and for helpful comments that improved this paper.

References

[1] Saxena, R. K., Modi, G. C. and Kalla, S. L., *A basic analogue of Fox's H-function*, Rev. Technology Univ, Zulin, 6, (1983) 139-143.

[2] Farooq, A., Jain, R. and Jain, D.K. (2012), *q-integral and Basic Analogue of I-function*. Asian Journal of Mathematics & Statistics, 5: 99-103.

[3] Truesdell, C., *An essay toward the unified theory of special functions*, Princeton, (1948).

[4] Agrawal, B. M., *A unified theory of special functions*, A Ph.D. thesis, Vikram University, Ujjain (1966).

[5] Jain, R., Chaturvedi, P. K. and Jain, D. K., *Some Generating Functions of I-Functions Involving Truesdell's F-Equation Technique,* Rajasthan Ganit Parishad, Vol. 21, No.2, (2008).

[6] Agrawal, B. M.: *Rodrigue's and Schlafli's formulae for the solution of F- equations*, J. Ind. Math. Soc., Vol. XXXXI, Nos 3-4. (1968), 173-177.

[7] Jain R., Ahmad A. and Jain D.K., *Some generating functions of q-analogue of Mittag-Leffler Function and Hermite Polynomial Satisfying Truesdell's Ascending and Descending Fq-equation,* GAMS journal of Mathematics and Mathematical Biosciences, Vol. 6 (Dec. 2017) (accepted).

http://www.newtheory.org ISSN: 2149-1402

Journal of New Theory

Received: 27.11.2017 *Year*: **2017**, *Number*: **19**, *Pages*: **56-62**
Published: 20.12.2017 *Original Article*

ON NANO πgs-CLOSED SETS

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Abstaract — In this paper, a new class of sets called πgs-closed sets in nano topological spaces is introduced and its properties are studied and studied of nano πgs-closed sets which is implied by that of nano gs-closed sets.

Keywords — *Nano π-closed set, nano πg-closed set, nano πgp-closed sets and nano πgs-closed set*

1 Introduction

Lellis Thivagar et al [5] introduced a nano topological space with respect to a subset X of an universe which is defined in terms of lower approximation and upper approximation and boundary region. The classical nano topological space is based on an equivalence relation on a set, but in some situation, equivalence relations are nor suitable for coping with granularity, instead the classical nano topology is extend to general binary relation based covering nano topological space

Bhuvaneswari et.al [4] introduced and investigated nano g-closed sets in nano topological spaces. Recently, Rajasekaran et.al [8, 9] initiated the study nano πg-closed sets and new classes of sets called πgp-closed sets in nano topological spaces is introduced and its properties and studied of nano πgp-closed sets.

In this paper, a new class of sets called πgs-closed sets in nano topological spaces is introduced and its properties are studied and studied of nano πgs-closed sets which is implied by that of nano gs-closed sets.

2 Preliminaries

Throughout this paper $(U, \tau_R(X))$ (or X) represent nano topological spaces on which no separation axioms are assumed unless otherwise mentioned. For a subset H of a

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space $(U, \tau_R(X))$, $Ncl(H)$ and $Nint(H)$ denote the nano closure of H and the nano interior of H respectively. We recall the following definitions which are useful in the sequel.

Definition 2.1. *[7] Let U be a non-empty finite set of objects called the universe and R be an equivalence relation on U named as the indiscernibility relation. Elements belonging to the same equivalence class are said to be indiscernible with one another. The pair (U, R) is said to be the approximation space. Let* $X \subseteq U$.

1. *The lower approximation of X with respect to R is the set of all objects, which can be for certain classified as X with respect to R and it is denoted by* $L_R(X)$. *That is,* $L_R(X) = \bigcup_{x \in U}\{R(x) : R(x) \subseteq X\}$, *where R(x) denotes the equivalence class determined by x.*

2. *The upper approximation of X with respect to R is the set of all objects, which can be possibly classified as X with respect to R and it is denoted by* $U_R(X)$. *That is,* $U_R(X) = \bigcup_{x \in U}\{R(x) : R(x) \cap X \neq \phi\}$.

3. *The boundary region of X with respect to R is the set of all objects, which can be classified neither as X nor as not - X with respect to R and it is denoted by* $B_R(X)$. *That is,* $B_R(X) = U_R(X) - L_R(X)$.

Property 2.2. *[5] If* (U, R) *is an approximation space and* $X, Y \subseteq U$; *then*

1. $L_R(X) \subseteq X \subseteq U_R(X)$;
2. $L_R(\phi) = U_R(\phi) = \phi$ *and* $L_R(U) = U_R(U) = U$;
3. $U_R(X \cup Y) = U_R(X) \cup U_R(Y)$;
4. $U_R(X \cap Y) \subseteq U_R(X) \cap U_R(Y)$;
5. $L_R(X \cup Y) \supseteq L_R(X) \cup L_R(Y)$;
6. $L_R(X \cap Y) \subseteq L_R(X) \cap L_R(Y)$;
7. $L_R(X) \subseteq L_R(Y)$ *and* $U_R(X) \subseteq U_R(Y)$ *whenever* $X \subseteq Y$;
8. $U_R(X^c) = [L_R(X)]^c$ *and* $L_R(X^c) = [U_R(X)]^c$;
9. $U_R U_R(X) = L_R U_R(X) = U_R(X)$;
10. $L_R L_R(X) = U_R L_R(X) = L_R(X)$.

Definition 2.3. *[5] Let U be the universe, R be an equivalence relation on U and* $\tau_R(X) = \{U, \phi, L_R(X), U_R(X), B_R(X)\}$ *where* $X \subseteq U$. *Then by the Property 2.2, R(X) satisfies the following axioms:*

1. *U and* $\phi \in \tau_R(X)$,
2. *The union of the elements of any sub collection of* $\tau_R(X)$ *is in* $\tau_R(X)$,
3. *The intersection of the elements of any finite subcollection of* $\tau_R(X)$ *is in* $\tau_R(X)$.

That is, $\tau_R(X)$ is a topology on U called the nano topology on U with respect to X. We call $(U, \tau_R(X))$ as the nano topological space. The elements of $\tau_R(X)$ are called as nano open sets and $[\tau_R(X)]^c$ is called as the dual nano topology of $[\tau_R(X)]$.

Remark 2.4. *[5] If $[\tau_R(X)]$ is the nano topology on U with respect to X, then the set $B = \{U, \phi, L_R(X), B_R(X)\}$ is the basis for $\tau_R(X)$.*

Definition 2.5. *[5] If $(U, \tau_R(X))$ is a nano topological space with respect to X and if $H \subseteq U$, then the nano interior of H is defined as the union of all nano open subsets of H and it is denoted by $Nint(H)$.*

That is, $Nint(H)$ is the largest nano open subset of H. The nano closure of H is defined as the intersection of all nano closed sets containing H and it is denoted by $Ncl(H)$.

That is, $Ncl(H)$ is the smallest nano closed set containing H.

Definition 2.6. *A subset H of a nano topological space $(U, \tau_R(X))$ is called*

1. *nano semi open [5] if $H \subseteq Ncl(Nint(H))$.*
2. *nano regular-open [5] if $H = Nint(Ncl(H))$.*
3. *nano π-open [1] if the finite union of nano regular-open sets.*
4. *nano pre-open [5] if $H \subseteq Nint(Ncl(H))$.*

The complements of the above mentioned sets is called their respective closed sets.

Definition 2.7. *A subset H of a nano topological space $(U, \tau_R(X))$ is called;*

1. *nano g-closed [3] if $Ncl(H) \subseteq G$, whenever $H \subseteq G$ and G is nano open.*
2. *nano gs-closed [2] if $Nscl(H) \subseteq G$, whenever $H \subseteq G$ and G is nano open.*
3. *nano πg-closed [8] if $Ncl(H) \subseteq G$, whenever $H \subseteq G$ and G is nano π-open.*
4. *nano gp-closed set [4] if $Npcl(H) \subseteq G$, whenever $H \subseteq G$ and G is nano open.*
5. *nano πgp-closed set [9] if $Npcl(H) \subseteq G$, whenever $H \subseteq G$ and G is nano π-open.*

The complements of the above mentioned sets is called their respective open sets.

3 On Nano πgs-closed Sets

Definition 3.1. *A subset H of a space $(U, \tau_R(X))$ is nano πgs-closed if $Nscl(H) \subseteq G$ whenever $H \subseteq G$ and G is nano π-open.*

The complement of nano πgs-open if $H^c = U - H$ is nano πgs-closed.

Example 3.2. *Let $U = \{1, 2, 3\}$ with $U/R = \{\{1\}, \{2, 3\}\}$ and $X = \{1\}$. Then the nano topology $\tau_R(X) = \{\phi, \{1\}, U\}$.*

1. *then $\{2, 3\}$ is nano πgs-closed set.*

2. *then* $\{1\}$ *is nano* πgs*-open set.*

Definition 3.3. *A subset* H *of a space* $(U, \tau_R(X))$ *is called a nano strong* $\mathcal{B}_Q$*-set if* $Nint(Ncl(H)) = Ncl(Nint(H))$.

Example 3.4. *In Example 3.2, then* $\{1, 2\}$ *is nano strong* $\mathcal{B}_Q$*-set.*

Theorem 3.5. *In a space* $(U, \tau_R(X))$*, the following properties are equivalent:*

1. *If* H *is nano* gs*-closed, then* H *is nano* πgs*-closed.*
2. *If* H *is nano* πg*-closed, then* H *is nano* πgs*-closed.*

Proof. Obvious.

Remark 3.6. *For a subset of a space* $(U, \tau_R(X))$*, we have the following implications:*

$$\begin{array}{ccccc}
 & & \textbf{nano } \pi\textbf{-closed} & & \\
 & & \Downarrow & & \\
\textbf{nano semi-closed} & \Leftarrow & \textbf{nano closed} & \Rightarrow & \textbf{nano pre-closed} \\
\Downarrow & & \Downarrow & & \Downarrow \\
\textbf{nano } gs\textbf{-closed} & \Leftarrow & \textbf{nano } g\textbf{-closed} & \Rightarrow & \textbf{nano } gp\textbf{-closed} \\
\Downarrow & & \Downarrow & & \Downarrow \\
\textbf{nano } \pi gs\textbf{-closed} & \Leftarrow & \textbf{nano } \pi g\textbf{-closed} & \Rightarrow & \textbf{nano } \pi gp\textbf{-closed}
\end{array}$$

None of the above implications are reversible as shown by the following Examples.

Example 3.7.
1. *Let* $U = \{1, 2, 3, 4\}$ *with* $U/R = \{\{1, 2\}, \{3\}, \{4\}\}$ *and* $X = \{1, 4\}$*. Then the nano topology* $\tau_R(X) = \{\phi, \{4\}, \{1, 2\}, \{1, 2, 4\}, U\}$*. Then* $\{1\}$ *is nano* πgs*-closed set but not nano* πg*-closed.*
2. *Let* $U = \{1, 2, 3\}$ *with* $U/R = \{\{1, 3\}, \{2\}\}$ *and* $X = \{3\}$*. Then the nano topology* $\tau_R(X) = \{\phi, \{1, 3\}, U\}$*. Clearly* $\{1\}$ *is nano* πgs*-closed set but not nano* gs*-closed.*

Lemma 3.8. *In a space* $(U, \tau_R(X))$*,*

1. *every nano open set is nano* πgs*-closed.*
2. *every nano closed set is nano* πgs*-closed.*

Remark 3.9. *The converses of statements in Lemma 3.8 are not necessarily true as seen from the following Examples.*

Example 3.10. *In Example 3.2,*

1. *then* $\{2, 3\}$ *is nano* πgs*-closed set but not nano open.*
2. *then* $\{1, 2\}$ *is nano* πgs*-closed set but not nano closed.*

Theorem 3.11. *In a space* $(U, \tau_R(X))$*, the following properties are equivalent:*

1. H *is* H *is nano* π*-open and nano* πgs*-closed.*

2. *H is nano regular-open.*

Proof. (1) ⇒ (2). By (1) $Nscl(H) \subseteq H$, since H is nano π-open and nano πgs-closed. Thus $Nint(Ncl(H)) \subseteq H$. Since H is nano open, then H is clearly nano pre-open and thus $H \subseteq Nint(Ncl(H))$. Therefore $Nint(Ncl(H)) \subseteq H \subseteq Nint(Ncl(H))$ or equivalently $H = Nint(Ncl(H))$, which shows that H is nano regular-open.

(2) ⇒ (1). Every nano regular-open set is nano π-open. For the second claim note that nano regular-open sets are even nano semi-closed.

Corollary 3.12. *If H is nano π-open and nano πgs-closed, then H is nano semi-closed and hence nano gs-closed.*

Proof. By assumption and Theorem 3.11, H is nano regular-open. Thus H is nano semi-closed. Since every nano semi-closed set is nano gs-closed, H is nano gs-closed.

Remark 3.13. *In a space $(U, \tau_R(X))$, $Nscl(U - H) = U - sint(H)$, for any subset H of a space U.*

Theorem 3.14. *In a space $(U, \tau_R(X))$, $H \subseteq U$ is nano πgs-open $\iff$ $F \subseteq Nsint(H)$ whenever K is nano π-closed and $K \subseteq H$.*

Proof. Necessity. Let H be nano πgs-open. Let K be nano π-closed and $K \subseteq H$. Then $U - H \subseteq U - K$ where $U - K$ is nano π-open. nano πgs-closedness of $U - H$ implies $Nscl(U - H) \subseteq U - K$. By Remark 3.13, $Nscl(U - H) = U - Nsint(H)$. So $K \subseteq Nsint(H)$.

Sufciency. Suppose K is nano π-closed and $K \subseteq H$ imply $K \subseteq Nsint(H)$. Let $U - H \subseteq G$ where G is nano π-open. Then $U - G \subseteq H$ where $U - G$ is nano π-closed. By hypothesis $U - G \subseteq Nsint(H)$. That is $U - Nsint(H) \subseteq G$. By Remark 3.13, $Nscl(U - H) \subseteq G$. So, $U - H$ is nano πgs-closed and H is nano πgs-open.

Theorem 3.15. *In a space $(U, \tau_R(X))$, the following properties are equivalent:*

1. *H is nano π-clopen.*

2. *H is nano π-open, a nano strong $\mathcal{B}_Q$-set and nano πgs-closed.*

Proof. (1) ⇒ (2) is Obvious.

(1) ⇒ (2). By Theorem 3.11, H is nano regular-open. Since H is a nano strong $\mathcal{B}_Q$-set, $H = Nint(Ncl(H)) = Ncl(Nint(H))$. So H is nano regular-closed. This shows that H is nano π-closed and hence H is nano π-clopen.

Theorem 3.16. *In a space $(U, \tau_R(X))$, the union of two nano πgs-closed sets is nano πgs-closed.*

Proof. Let $H \cup Q \subseteq G$, then $H \subseteq G$ and $Q \subseteq G$ where G is nano π-open. As H and Q are πgs-closed, $Ncl(H) \subseteq G$ and $Ncl(Q) \subseteq G$. Hence $Ncl(H \cup Q) = Ncl(H) \cup Ncl(Q) \subseteq G$.

Example 3.17. *In Example 3.2, then $H = \{2\}$ and $Q = \{3\}$ is nano πgs-closed sets. Clearly $H \cup Q = \{2, 3\}$ is nano πgs-closed.*

Theorem 3.18. *In a space $(U, \tau_R(X))$, the intersection of two nano πgs-open sets are nano πgs-open.*

Proof. Obvious by Theorem 3.16.

Example 3.19. *In Example 3.2, then* $H = \{1, 3\}$ *and* $Q = \{1, 2\}$ *is nano* πgs*-open. Clearly* $H \cap Q = \{1\}$ *is nano* πgs*-open.*

Remark 3.20. *In a space* $(U, \tau_R(X))$*, the union of two nano* πgs*-closed sets but not nano* πgs*-closed.*

Example 3.21. *In Example 3.7(1), then* $H = \{1\}$ *and* $Q = \{4\}$ *is nano* πgs*-closed sets. Clearly* $H \cup Q = \{1, 4\}$ *is but not nano* πgs*-closed.*

Remark 3.22. *In a space* $(U, \tau_R(X))$*, the intersection of two nano* πgs*-open sets but not nano* πgs*-open.*

Example 3.23. *In Example 3.7(1), then* $H = \{3, 4\}$ *and* $Q = \{1, 2, 3\}$ *is nano* πgs*-open sets. Clearly* $H \cap Q = \{3\}$ *is but not nano* πgs*-open.*

Theorem 3.24. *Let* H *be nano* πgs*-closed. Then* $Nscl(H) - H$ *does not contain any non-empty nano* π*-closed set.*

Proof. Let K be a nano π-closed set such that $K \subseteq Nscl(H) - H$. Then $K \subseteq U - H$ implies $H \subseteq U - K$. Therefore $Nscl(H) \subseteq U - K$. That is $K \subseteq U - Nscl(H)$. Hence $K \subseteq Nscl(H) \cap (U - Nscl(H)) = \phi$. This shows $K = \phi$.

Theorem 3.25. *If* H *is nano* πgs*-closed and* $H \subseteq P \subseteq Nscl(H)$*, then* P *is nano* πgs*-closed.*

Proof. Let H be nano πgs-closed and $P \subseteq G$, where G is nano π-open. Then $H \subseteq P$ implies $H \subseteq G$. Since H is nano πgs-closed, $Nscl(H) \subseteq G$. $P \subseteq Nscl(H)$ implies $Nscl(P) \subseteq Nscl(H)$. Therefore $Nscl(P) \subseteq G$ and hence P is nano πgs-closed.

Acknowledgement

The authors thank the referees for their valuable comments and suggestions for improvement of this my article.

References

[1] Adinatha C. Upadhya, *On quasi nano p-normal spaces*, International Journal of Recent Scientific Research, 8/6 (2017) 17748-17751.

[2] K. Bhuvaneshwari and K. Ezhilarasi, *On Nano semi generalized and Nano generalized semi closed sets*, IJMCAR, 4/3 (2014) 117-124.

[3] K. Bhuvaneshwari and K. Mythili Gnanapriya, *Nano Generalizesd closed sets*, International Journal of Scientific and Research Publications, 4/5 (2014) 1-3.

[4] K. Bhuvaneswari and K. Mythili Gnanapriya, *On Nano Generalised Pre Closed Sets and Nano Pre Generalised Closed Sets in Nano Topological Spaces*, International Journal of Innovative Research in Science, Engineering and Technology, 3/10 (2014) 16825-16829.

[5] M. Lellis Thivagar and Carmel Richard, *On Nano forms of weakly open sets*, International Journal of Mathematics and Statistics Invention, 1/1 (2013) 31-37.

[6] C. R. Parvathy and , S. Praveena, *On Nano Generalized Pre Regular Closed Sets in Nano Topological Spaces*, IOSR Journal of Mathematics (IOSR-JM), 13/2 (2017) 56-60.

[7] Z. Pawlak, *Rough sets*, International journal of computer and Information Sciences, 11/5 (1982) 341-356.

[8] I. Rajasekaran and O. Nethaji, *On some new subsets of nano topological spaces*, Journal of New Theory, 16 (2017) 52-58.

[9] I. Rajasekaran, T. Siva Subramania Raja and O. Nethaji, *On nano πgp-closed sets*, Journal of New Theory, 19 (2017) 20-26.

http://www.newtheory.org ISSN: 2149-1402

Journal of New Theory

Received: 27.11.2017
Published: 20.12.2017

Year: **2017**, *Number*: **19**, *Pages*: **63-81**
Original Article

CHARACTERIZATION OF SOFT α-SEPARATION AXIOMS AND SOFT β-SEPARATION AXIOMS IN SOFT SINGLE POINT SPACES AND IN SOFT ORDINARY SPACES

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Abstract - The main aim of this article is to introduce soft α and soft β separations axioms, soft α-separations axioms and soft β separations axioms in soft single point topology. We discuss soft $\alpha n - T_4$ separation axioms and soft $\beta n - T_4$ separation axioms in soft topological spaces with respect to ordinary points and soft points. Further study the hereditary properties at different angles with respect to ordinary points as well as with respect to soft points. Some of their fundamental properties in soft single point topological spaces are also studied.

Keywords - *Soft sets, soft points, soft α open set, soft α closed set, soft β open set, soft β closed in soft topological space, soft single point topology, soft α and soft β separation axioms.*

1. Introduction

In real life condition the problems in economics, engineering, social sciences, medical science etc. We cannot beautifully use the traditional classical methods because of different types of uncertainties presented in these problems. To overcome these difficulties, some kinds of theories were put forwarded like theory of Fuzzy set, intuitionistic fuzzy set, rough set and bi polar fuzzy sets, inwhich we can safely use a mathematical techniques for businessing with uncertainties. But, all these theories have their inherent difficulties. To overcome these difficulties in the year 1999, Russian scientist Molodtsov [4], initiated the notion of soft set as a new mathematical technique for uncertainties. Which is free from the above complications. In [4,5], Mololdtsov successfully applied the soft set theory in different directions, such as smoothness of functions, game theory, operation research, Riemann integration, perron integration, probability ,theory of measurement and so on. After presentation of the operations of soft sets [6], the properties and applications of the soft set theory have been studied increasingly [7,8,6]. Xiao et al. [9] and Pei and Maio [10] discussed the linkage between soft sets and information systems. They showed that soft sets are class of special information system. In the recent year, many interesting applications of soft sets theory have been extended by embedding the ideas of fuzzy sets [11,12,13,14,15,16,17,18,20,21,22] industrialized soft set theory, the operations of the soft sets are redefined and in indecision making method was constructed by using their new operations [23].

Recently, in 20011, Shabir and Naz [24] launched the study of soft Topological spaces, they beautiful defined soft Topology as a collection of τ of soft sets over X. They also defined the basic conception of soft topological spaces such as open set and closed soft sets, soft nbd of a point, soft separation axiom, soft regular and soft normal spaces and published their several behaviors. Min in [25] scrutinized some belongings of this soft separation axiom. In [26] Kandil et al. introduced some soft operations such as semi open soft, pre-open soft, α-open soft and β-open soft and examined their properties in detail. Kandil et al. [27] introduced the concept of soft semi–separation axioms, in particular soft semi-regular spaces. The concept of soft ideal was discussed for the first time by Kandil et al. [28]. They also introduced the concept of soft local function; these concepts are discussed with a view to find new soft topological from the original one, called soft topological spaces with soft ideal (X, τ, E, I).

Applications to different zone were further discussed by Kandil et al. [28,29,30,32, 33,34,35]. The notion of super soft topological spaces was initiated for the first time by El-Sheikh and Abd-e-Latif [36]. They also introduced new different types of sub-sets of supra soft topological spaces and study the dealings between them in great detail. Bin Chen [41] introduced the concept of semi open soft sets and studied their related properties, Hussain [42] discussed soft separation axioms. Mahanta [39] introduced semi open and semi closed soft sets. Arokialancy in [43] generalized soft g β closed and soft gs β closed sets in soft topology are exposed. Mukharjee [44] introduced some new bi topological notion with respect to ordinary points. Gocur and Kopuzlu [45] discussed some new properties on soft separation axioms in soft single point space over El-Sheikh and Abd-e-Latif [46] discussed Characterization of soft b-open sets in soft topological spaces and defined pre-open, semi-open, α-open and β-open soft sets in soft topological spaces with respect to ordinary points. Yumak and Kaymaker [47] discussed Soft β-open sets and their applications.

In this present paper the concept of soft α-T_i spaces (i=1, 2, 3) and soft β_i spaces (i=1, 2, 3) are introduced in soft single point space with respect to ordinary and soft points of a topological space. Soft $\alpha n - T_4$ space and Soft $\beta n - S_4$ are introduced in soft topological space with respect to ordinary and soft points. Many mathematicians discussed soft separation axioms in soft topological spaces at full length with respect to soft open set, soft b-open set, soft semi-open set. They also worked over the hereditary properties of different soft topological structures in soft topology. In this present work hand is tried and work is encouraged over the gap that exists in soft topology. Related to Soft spaces, some theorems in soft single topological spaces are discussed with respect to ordinary points as well as with respect to soft points. Focus is laid upon the characters of soft $\alpha n - T_4$ and soft $\beta n - S_4$ space and their sub spaces in soft topological structures. When we talk about the distances between the points in soft topology then the concept of soft separation axioms will automatically come in play. That is why these structures are catching our attentions. We hope that these results will be valuable for the future study on soft single point topological spaces to accomplish general framework for the practical applications and to solve the most intricate problems containing scruple in economics, engineering, medical, environment and in general mechanic systems of various kinds

2. Preliminaries

The following Definitions which are pre-requisites for present study.

Definition 1 [4]. Let X be an initial universe of discourse and E be a set of parameters. Let P(X) denotes the power set of X and A be a non-empty sub-set of E. A pair (F, A) is called a soft set over U, where F is a mapping given by $F: A \rightarrow P(X)$

In other words, a set over X is a parameterized family of sub set of universe of discourse X. For $e \in A, F(e)$ may be considered as the set of e-approximate elements of the soft set (F, A) and if $e \notin A$ then$F(e) = \phi$, that is $F_A = \{F(e): e \in A \subseteq E, F: A \rightarrow P(X)\}$ the family of all these soft sets over X denoted by$SS(X)_A$.

Definition 2 [4]. Let $F_A, G_B \in SS(X)_E$ then F_A is a soft subset of G_B denoted by $F_A \sqsubseteq G_B$, if

1. $A \subseteq B$ and
2. $F(e) \subseteq G(e), \forall \in A$

In this case F_A is said to be a soft subset of G_B and G_B is said to be a soft super setF_A, $G_B \sqsupseteq F_A$.

Definition 3 [6]. Two soft subsets F_A and G_B over a common universe of discourse set X are said to be equal if F_A is a soft subset of G_B and G_B is a soft subset of F_A.

Definition 4 [6]. The complement of soft subset(F, A) denoted by $(F, A)^C$ is defined by $(F, A)^C = (F^C, A)F^C \rightarrow P(X)$ is a mapping given by $F^C(e) = U - F(e) \forall e \in A$ and $F^C$is called the soft complement function of F. Clearly $(F^C)^C$ is the same as F and $((F, A)^C)^C = (F, A)$.

Definition 5 [7]. The difference between two soft subset (G, E) and (G, E) over common of universe discourse X denoted by $(F, E) - (G, E)$ is the soft set (H, E) where for all $e \in E. \overline{\emptyset} or\ \emptyset_A$ if $\forall e \in A, F(e) = \emptyset$.

Definition 6 [7]. Let (G, E) be a soft set over$X$ and $x \in X$ We say that $x \in (F, E)$ and read as x belong to the soft set (F, E) whenever $x \in F(e) \forall e \in E$ The soft set (F, E) over X such that $F(e) = \{x\} \forall e \in E$ is called singleton soft point and denoted by $x_E, or\ (x, E)$.

Definition 7 [6]. A soft set (F, A) over X is said to be Null soft set denoted by $\overline{\emptyset}\ or\ \emptyset_A$ if $\forall e \in A, F(e) = \emptyset$.

Definition 8 [6]. A soft set (F, A) over X is said to be an absolute soft denoted by $\bar{A}$ or X_A if $\forall e \in A, F(e) = X$.

Clearly, we have. $X_A^c = \emptyset_A$ and $\emptyset_A^c = X_A$.

Definition 9 [7]. Let (G, E) be a soft set over X and $e_G \in X_A$, we say that $e_G \in (F, E)$ and read as e_G belong to the soft set(F, E)whenever $e_G \in F(e) \forall e \in E$. the soft set (F, E) over X such that $F(e) = \{e_G\}$, $\forall e \in E$ is called singleton soft point and denoted bye_G, or (e_G, E).

Definition 10 [42]. The soft set $(F, A) \in SSX_A$ is called a soft point in X_A, denoted by e_F, if for the element $e \in A, F(e) \neq \emptyset$ and $F(e^/) = \phi$ if for all $e^/ \in A - \{e\}$

Definition 11 [42]. The soft point e_F is said to be in the soft set (G, A), denoted by $e_F \in (G, A)$ if for the element $e \in A, F(e) \subseteq G(e)$.

Definition 12 [42]. Two soft sets $(G, A), (H, A)$ in SSX_A are said to be soft disjoint, written

$$(G, A) \cap (H, A) = \emptyset_A \text{ If } G(e) \cap H(e) = \emptyset \forall\, e \in A.$$

Definition 13 [42]. The soft point $e_G, e_H \in X_A$ are disjoint, written $e_G \neq e_{H,}$ if their corresponding soft sets (G, A) and (H, A) are disjoint.

Definition 14[6]. The union of two soft sets (F, A) and (G, B) over the common universe of discourse X is the soft set (H, C), where, $C = AUB \forall e \in C$

$$H(e) = \left\{ \begin{array}{ll} F(e) & if\ e \in A - B \\ G(e) & if\ e \in (B - A) \\ F(e), & if\ e \in A \cap B \end{array} \right\}$$

Written as $(F, A) \cup (G, B) = (H, C)$

Definition 15 [6]. The intersection $(H. C)$ of two soft sets (F, A) and (G, B) over common universe X, denoted $(F, A) \bar{\sqcap} (G,B)$ is defined as

$$C = A \cap B \text{ and } H(e) = F(e) \cap G(e), \forall e \in C.$$

Definition 16 [2]. Let (F, E) be a soft set over X and Y be a non-empty sub set of X. Then the sub soft set of (F, E) over Y denoted by (Y_F, E), is defined as follow $Y_{F(\alpha)} = Y \cap F(\alpha), \forall \in E$ in other words
$(Y_F, E) = Y \cap (F, E)$.

Definition 17 [2]. Let τ be the collection of soft sets over X , then $\tau$is said to be a soft topology on X, if

1. $\emptyset$, X belong to τ
2. The union of any number of soft sets in τ belongs to τ
3. The intersection of any two soft sets in τ belong to τ

The triplet (X, F, E) is called a soft topological space.

Definition 18 [1]. Let (X, τ, E) be a soft topological space over X then the member of τ are said to be soft open sets in X.

Definition 19 [1]. Let (X, τ, E) be a soft topological space over X. A soft set (F, A) over X is said to be a soft closed set in X if its relative complement $(F, E)^C$ belong to τ.

Definition 20 [46]. Let (X, τ, E) be a soft topological space and $(F, E) \subseteq SS(X)_E$ then (F, E) is said to be α-open soft set if $((F, E) \subseteq int(cl(int(F, E)))$.

The set of all α- open soft set is denoted $\alpha OS(X, \tau, E)$ or $BOS(X)$ and the set of all α-closed soft set is denoted by $\alpha CS(X, \tau, E)$ or $\alpha CS(X)$.

Definition 21 [46]. Let (X, τ, E) be a soft topological space and $(F, E) \subseteq SS(X)_E$ then (F, E) is called β open soft set $((F, E) \subseteq Cl(int(Cl(F, E))))$.

The set of all β *open soft* set is denoted by $S\beta O(X, \tau, E)$ or $S\beta O(X)$ and the set of all β *closed soft* set is denoted by $\beta CS(X, \tau, E)$ or $\beta CS(X)$.

Definition22[45]. Let X be an initial universe set, E be the set of parameters, x ∈ X and A be a subset of X. Let (A, E) be defined as A(e) = A, for all e ∈ E.

Then τ = {(A, E)|∀A ⊂ X} is a soft topology over X. In this case, τ is called soft Single point topology over X and (X, τ, E) is said to be a soft single point space over X.

Theorem*[48]. A sub space (Y, τ_Y, E) of a soft β T_1 space is soft βT_1.

3. Separation Axioms of Soft Topological Spaces With Respect to Ordinary Points as Well as Soft Points

Definition 23 [23]. Let (X, τ, A) be a soft Topological space over X and $x, y \in X$ such that $x \neq y$ if there exist at least one soft open set (F_1, A) OR (F_2, A) such that $x \in (F_1, A), y \notin (F_1, A)$ or $y \in (F_2, A), x \notin ((F_2, A)$ then (X, τ, A) is called a soft T_0 *space*.

Definition 24 [23]. Let (X, τ, A) be a soft Topological spaces over X and $x, y \in X$ such that $x \neq y$ if there exist soft open sets $(F_1, A)$and (F_2, A) such that $x \in (F_1, A), y \notin (F_1, A)$ and $y \in (F_2, A), x \notin ((F_2, A)$ then (X, τ, A) is called a soft T_1 *space*.

Definition 25 [23]. Let (X, τ, A) b e a soft Topological space over X and $x, y \in X$ such that $x \neq y$ if there exist soft open set (F_1, A) and (F_2, A) such that $x \in (F_1, A)$, and $y \in (F_2, A)$ and $F_1 \cap F_2 = \varphi$

Then (X, τ, A) is called soft T_2 spaces.

Definition 26 [42]. Let (X, τ, A) be a soft Topological space over X and $e_G, e_H \in X_A$ such that $e_G \neq e_H$ if we can search at least one soft open set (F_1, A) or (F_2, A) such that

$e_G \in (F_1, A), e_H \notin (F_1, A)$ or $e_H \in (F_2, A), e_G \notin ((F_2, A)$ then (X, τ, A) is called a soft T_0 space.

Definition 27 [42]. Let (X, τ, A) be a soft Topological spaces over X and $e_G, e_H \in X_A$ such that $e_G \neq e_H$ if we can search soft open sets $(F_1, A)$and (F_2, A) such that $e_G \in (F_1, A), e_H \notin (F_1, A)$ and $e_H \in (F_2, A), e_G \notin ((F_2, A)$ then (X, τ, A) is called a soft T_1 $space$.

Definition 28 [42]. Let (X, τ, A) b e a soft Topological space over X and $e_G, e_H \in X_A$ such that $e_G \neq e_H$ if we can search soft open set $(F_1, A)$and (F_2, A) such that $e_G \in (F_1, A)$, and $e_H \in (F_2, A)$

$(F_1, A) \cap (F_2, A) = \phi_A$ Then (X, τ, A) is called soft T_2 space.

Definition 29 [23]. Let (X, τ, E) be a soft topological space (G, E) be closed soft set in X and $e_G \in X_A$ such that $e_G \notin (G, E)$. If there occurs soft open sets (F_1, E) and (F_2, E) such that $e_G \in (F_1, E), (G, E) \subseteq (F_2, E)$ and $(F_1, E) \cap (F_2, E) = \varphi$. Then (X, τ, E) is called soft regular spaces. A soft regular T_1 $Space$ is called soft T_3 space

Definition 30 [23]. Let (X, τ, E) be a soft topological space $(F_1, E), (G, E)$ be closed soft sets in X such that $(F, E) \cap (G, E) = \varphi$ if there exists open soft sets (F_1, E) and (F_2, E) such that $(F, E) \subseteq (F_1, E), (G, E) \subseteq (F_2, E)$ and $(F_1, E) \cap (F_2, E) = \varphi$ then (X, τ, E) is called soft normal space. A soft normal T_1 $Space$ is called soft T_4 $Space$.

Definition 32 [45]. Let (X, τ, E) be a soft topological space X and $x, y \in X$ such that $x \neq y$. Let (F, E) and (G, E) be soft closed sets such that that $x \in (F, E)$ and $(F, E) \cap (G, E) = \phi$. If there exist soft open sets (F_1, E) and (F_2, E) such that $y \in (F_2, E)$, $(F, E) \subseteq (F_1, E)$, $(G, E) \subseteq (F_2, E)$ and $(F_1, E) \cap (F_2, E) = \phi$, then (X, τ, E) is called soft n-normal space.

Definition 33 [45]. Let (X, τ, E) be a soft topological space X. If (X, τ, E) is a soft n-normal space and T_1 space, then (X, τ, E) is a soft n-T_4 space.

4. Soft α Separation Axioms of Soft Single Point Topological Spaces

In this section we introduced the concept of soft αT_i spaces (i=1, 2, 3) in soft single point space with respect to ordinary and soft points of a soft single point topological space and some of its basic properties are studied and applied to different results in this section.

4.1 Soft α Separation Axioms of Soft Single Point Topological Spaces With Respect to Ordinary Points

In this section we introduced soft separation axioms in soft single point topological space with respect to ordinary points and discussed some results with respect to these points in detail.

Definition 34. Let (X, τ, E) be a soft topological space and $(x, E) \subseteq SS(X)_E$ then (x, E) is said to be α-open soft set if $((x, E) \subseteq int(cl(int(x, E)))$.

The set of all α- open soft set is denoted $\alpha OS(X,\tau,E)$ or $BOS(X)$ and the set of all α-closed soft set is denoted by $\alpha CS(X,\tau,E)$or $\alpha CS(X)$.

Definition 35. Let (X,τ,A) be a soft Topological space over X and $x,y \in X$ such that $x \neq y$ if there exist at least one soft α open set (F_1,A) OR (F_2,A) such that $x \in (F_1,A), y \notin (F_1,A)$ or $y \in (F_2,A), x \notin ((F_2,A)$then (X,τ,A) is called a soft $\alpha\ T_0\ space$.

Definition 36. Let (X,τ,A) be a soft Topological spaces over X and $x,y \in X$ such that $x \neq y$ if there exist soft α open sets (F_1,A) and (F_2,A) such that $x \in (F_1,A), y \notin (F_1,A)$ and $y \in (F_2,A), x \notin ((F_2,A)$ then (X,τ,A) is called a soft $\alpha\ T_1\ space$.

Definition 37. Let (X,τ,A) b e a soft topological space over X and $x,y \in X$ such that $x \neq y$ if there exist soft α open set $(F_1,A)$and (F_2,A) such that $y \notin (F_1,A)$, and $y \in (F_2,A)$ and $F_1 \cap F_2 = \varphi$

Then (X,τ,A) is called soft $\alpha\ T_2$ spaces.

Definition 38. Let (X,τ,E) be a soft topological space X and $x,y \in X$ such that$x \neq y$. Let (F,E) and (G,E) be soft α closed sets such that that $x \in (F,E)$ and $(F,E) \cap (G,E) = \phi$. If there exist soft α open sets (F_1,E) and (F_2,E) such that $y \in (F_2,E)$,$(F,E) \subseteq (F_1,E)$, $(G,E) \subseteq (F_2,E)$ and $(F_1,E) \cap (F_2,E) = \phi$, then (X,τ,E) is called soft α n-normal space.

Definition 39. Let (X,τ,E) be a soft topological space X. If (X,τ,E) is a soft α n-normal space and αT_1 space, then (X,τ,E) is a soft α n-T_4 space.

Theorem**[**]** A sub (Y,τ_Y,E) of a soft α T_1 space is soft αT_1.

Proof. Let $x,y \in Y$ such that$x \neq y$. Then $x,y \in X$ such that $x \neq y$. Hence there exists soft α open sets (F,E) and (G,E) such that $x \in (F,E), y \notin (F,E)$ and $y \in (G,E), x \notin ((G,E)$. Since$x \in Y$. Hence $x \in Y \cap (F,E) = (F_Y,E)$, (F,E) is soft α open set. Consider $, y \notin (F,A)$, This implies that, $y \notin F(e)\ for\ some\ e \in E$. Therefore $y \notin Y \cap (F,E) = (F_Y,E)$. Similarly, if $y \in (G,E)$ and $x \notin ((G,A)$, Then $y \in (G_Y,E)$ and $x \notin (G_Y,E)$, Then $y \in (G_Y,E)$ and $x \notin (G_Y,E)$.Thus, (Y,τ_Y,E) of a soft α T_1 space is soft αT_1.

Theorem 1. Let X be an initial universe set, E be the set of parameters. If (X,τ,E) is a soft single point space, then each soft element of (X,τ,E) is both soft α open and soft α closed set.

Proof. Let X be an initial universe set, E be the set of parameters and(A,E) is a soft single point space. Let (A,E) be defined as$A(e) = A, \forall e \in E$. $\tau = \{(A,E): \forall A \subseteq X\}$ From Definition 22[45], since$\tau^{/} = \{(A,E)^{/}: \forall A^{/} \subseteq X\}, (A,E)^{/}$is a soft α open set$\forall A^{/} \subseteq X$. Thus (A,E) is soft α open and soft α closed set in $\forall A \subseteq X$.

Theorem 2. Let X be an initial universe set, E be the set of parameters. **If** (X,τ,E) is a soft single point space, then (X,τ_e) is a discrete space$\forall e \in E$.

Proof. Let X be an initial universe set, E be the set of parameters and (X, τ, E) is a soft single point space, (A, E) is defined as $A(e) = A, \forall\ e \in E$. Then $\tau = \{(A, E): \forall\ A \subseteq X\}$ is a soft topology over X from Definition 22[45]. Here A is soft α open set in $(X, \tau_e) \forall A \subseteq X \forall\ e \in E$. Thus (X, τ_e) is a discrete space for all $e \in E$.

Theorem 3. Let (X, τ, E) be a soft single point space X. Then (X, τ, E) is soft αT_1 space.

Proof. Let (X, τ, E) be a soft single point space X and $x, y \in X$ such that $x \neq y$.Then there exist soft α open sets (x, E), (y, E) such that $x \in (x, E) \in \tau$, $y \notin (x, E)$ and $y \in (y, E) \in \tau$, $x \notin (y, E)$. Hence (X, τ, E) is soft αT_1 space.

Theorem 4. Let (X, τ, E) be a soft single point space over X then (X, τ, E) is soft αT_2 space.

Proof. Let (X, τ, E) be a soft single point space over X and $x, y \in X$ such that $x \neq y$. Then there exist soft α open sets (x, E) and (y, E) such that $x \in (x, E) \in \tau$, $y \in (y, E) \in \tau$ and $(x, E) \cap (y, E) = \phi$. Hence (X, τ, E) is soft αT_2 space.

Theorem 5. Let (X, τ, E) be a soft single point space over X. Then, (X, τ, E) is a soft αT_3 space.

Proof. Let (X, τ, E) be a soft single point space over X, (G, E) be a soft α closed set in X and $x \in X$ such that , $x \notin (G, E)$. From Theorem 1, there exists soft α open sets (x, E) and (G, E) such that $x \in (x, E)$, $(G, E) \subseteq (G, E)$and$(x, E) \cap (G, E) = \phi$. Also, from Theorem 3, (X, τ, E) is a soft αT_1 point space, so (X, τ, E) is soft αT_3 space.

Theorem 6. Let (X, τ, E) be a soft single point space over X then (X, τ, E) is a soft αT_4 space

Proof. Let (X, τ, E) be a soft single point space over $X$and let (F, E) and (G, E) be soft α closed sets in X such that $(F, E) \cap (G, E) = \phi$. From Theorem1, there exists soft α open sets (F, E) and (G, E) such that $F, E) \subseteq (F, E)$, $(G, E) \subseteq (G, E)$. Since $(F, E) \cap (G, E) = \phi$. (X, τ, E) Is called a soft α normal space. Also Theorem 3, (X, τ, E) is a soft αT_1 space, so (X, τ, E) is a soft αT_4.

Theorem 7. Let (X, τ, E) be a soft single point space over X and $x, y \in X$. Then (X, τ, E) is a soft α n- T_4 space.

Proof. Let (X, τ, E) be a soft single point space over X and$x, y \in X$, let (F, E) and (G, E) be soft α closed sets such that $x \in (F, E)$ and$(F, E) \cap (G, E) = \phi$. Then there exist soft α open sets (F, E) and (G, E) such that $y \in (G, E)$, $(F, E) \subseteq (F, E)$, $(G, E) \subseteq (G, E) and$ $(F, E) \cap (G, E) = \phi$ from Theorem 1, Thus (X, τ, E) is a soft α normal space. Also from Theorem 3, (X, τ, E) is soft αT_1 space so (X, τ, E) is a soft α n-T_4 space.

4.2 Soft α-Separation Axioms of Soft Single Point Topological Spaces With Respect to Soft Points

In this section we introduced soft α separation axiom in soft single point topological space with respect to soft points and discussed some results with respect to these points in detail.

Definition 40. Let (X, τ, E) be a soft topological space and $(e_G, E) \subseteq SS(X)_E$ then (e_G, E) is said to be α-open soft set if $((e_G, E) \subseteq int(cl(int(e_G, E)))$.

The set of all α- open soft set is denoted $\alpha OS(X, \tau, E)$ or $BOS(X)$ and the set of all α-closed soft set is denoted by $\alpha CS(X, \tau, E)$or $\alpha CS(X)$.

Definition 41. Let (X, τ, E) be a soft topological space X and $e_G, e_H \in X_E$ such that $e_G \neq e_H$. Let $(e_G, E)$and(e_H, E) be soft α closed sets such that that $e_G \in (e_G, E)$ and $(F, E) \cap (\mathrm{G}, \mathrm{E}) = \phi$. If there exist soft α open sets (F_1, E) and (F_2, E) such that $e_H \in (F_2, E)$,$(F, E) \subseteq (F_1, E)$, $(G, E) \subseteq (F_2, E)$ and $(F_1, E) \cap (F_2, E) = \phi$, then (X, τ, E) is called soft α n-normal space.

Definition 42. Let (X, τ, E) be a soft topological space X. If (X, τ, E) is a soft α n-normal space and α T_1 space, then (X, τ, E) is a soft α n-T_4 space.

Theorem 8. Let (X, τ, E) be a soft single point space$X$. Then (X, τ, E) is soft αT_1 space.

Proof. Let (X, τ, E) be a soft single point space X and $e_G, e_H \in X_E$such that $e_G \neq e_H$.Then there exist soft α open sets (e_G, E), (e_H, E) such that$e_G \in (e_G, E) \in \tau, e_H \notin (e_G, E)$ and $e_H \in (e_H, E) \in \tau$, $e_G \notin (e_H, E)$. Hence (X, τ, E) is soft αT_1 space.

Theorem 9. Let (X, τ, E) be a soft single point space over X then (X, τ, E) is soft αT_2 space

Proof. Let (X, τ, E) be a soft single point space over X_E and $e_G, e_H \in X_E$ such that $e_G \neq e_H$. Then there exist soft α open sets (e_G, E) and (e_H, E) such that $e_G \in (e_G, E) \in \tau, e_H \in (e_H, E) \in \tau$ and $(e_G, E) \cap (e_H, \mathrm{E}) = \phi$. Hence (X, τ, E) is soft αT_2 space

Theorem 10. Let (X, τ, E) be a soft single point space over X .Then, (X, τ, E) is a soft αT_3 space.

Proof. Let (X, τ, E) be a soft single point space over X_E , (G, E) be a soft α closed set in X and $e_G \in X$ such that,$e_G \notin (G, E)$ From Theorem 1, there exists soft α open sets (e_G, E) and (G, E) such that$e_G \in (e_G, E)$, $(G, E) \subseteq (G, E)$and$(e_G, E) \cap (\mathrm{G}, \mathrm{E}) = \phi$. Also, from Theorem 8, (X, τ, E) is a soft αT_1 point space, so (X, τ, E) is soft αT_3 space. Theorem 11. Let (X, τ, E) be a soft single point space over X and $x, y \in X$.then (X, τ, E) is a soft α n- T_4 space.

Proof. Let (X, τ, E) be a soft single point space over X and $e_G, e_H \in X_E$, let (F, E) and (G, E) Be soft α closed sets such that $e_G \in (F, E)$and$(F, E) \cap (\mathrm{G}, \mathrm{E}) = \phi$. Then there exist soft α open sets (F, E) and (G, E) such that $e_H \in (G, E)$, $(F, E) \subseteq (F, E)$, $(G, E) \subseteq (G, E) and$ $(F, E) \cap (\mathrm{G}, \mathrm{E}) = \phi$ from Theorem 1, Thus (X, τ, E) is a soft α normal space. Also from Theorem 8, (X,τ,E) is soft αT_1 space so (X,τ,E) is a soft α n-T_4 space.

4.3 Soft β-Separation Axioms of Soft Single Point Space With Respect to Ordinary Points

In this section we introduced soft β-separation axioms in soft single point topological space with respect to ordinary points and discussed some results with respect to these points in detail.

Definition 43 [46]. Let (X,τ,E)be a soft topological space and $(x,E) \subseteq SS(X)_E$ then (x,E) is called β open soft set $((x,E) \subseteq Cl(int(Cl(x,E)))$.

The set of all β open soft set is denoted by $S\beta O(X,\tau,E)$ or $S\beta O(X)$ and the set of all β closed soft set is denoted by $\beta CS(X,\tau,E)$ or $\beta CS(X)$.

Theorem 12. Let X be an initial universe set, E be the set of parameters. If(X,τ,E) is a soft single point space, then each soft element of (X,τ,E) is both soft β open and soft β closed set.

Proof. Let X be an initial universe set, E be the set of parameters and(A,E) is a soft single point space. Let (A,E) be defined as$A(e) = A, \forall\ e \in E$. $\tau = \{(A,E): \forall\ A \subseteq X\}$ From Definition 22[45], since$\tau^{/} = \{(A,E)^{/}: \forall\ A^{/} \subseteq X\}, (A,E)^{/}$is a soft β open set$\forall\ A^{/} \subseteq X$. Thus (A,E) is soft β open and soft β closed set in $\forall\ A \subseteq X$.

Theorem 13. Let X be an initial universe set, E be the set of parameters. If (X,τ,E) is a soft single point space, then (X,τ_e) is a discrete space $\forall\ e \in E$.

Proof. Let X be an initial universe set, E be the set of parameters and(X,τ,E) is a soft single point space, (A,E) is defined as $A(e) = A, \forall\ e \in E$. Then $\tau = \{(A,E): \forall A \subseteq X\}$ is a soft topology over X Definition 22[45]. Here A is soft β open set in $(X,\tau_e) \forall A \subseteq X \forall\ e \in E$. Thus (X,τ_e) is a discrete space for all $e \in E$.

Theorem 14. Let (X,τ,E) be a soft single point space$X$. Then (X,τ,E) is soft βT_1 space.

Proof. Let (X,τ,E) be a soft single point space $X$and $x, y \in X$ such that $x \neq y$.Then there exist soft β open sets (x,E), (y,E) such that$x \in (x,E) \in \tau$, $y \notin (x,E)$ and $y \in (y,E) \in \tau$, $x \notin (y,E)$. Hence (X,τ,E) is soft βT_1 space.

Theorem 15. Let (X,τ,E) be a soft single point space over X then (X,τ,E) is soft βT_2 space

Proof. Let (X,τ,E) be a soft single point space over $X$and$x, y \in X$ such that $x \neq y$. Then there exist soft β open sets (x,E) and (y,E) such that $x \in (x,E) \in \tau$, $y \in (y,E) \in \tau$ and $(x,E) \cap (y,E) = \phi$. Hence (X,τ,E) is soft βT_2 space

Theorem 16. Let (X,τ,E) be a soft single point space over X . Then, (X,τ,E) is a soft βT_3 space.

Proof. Let (X,τ,E) be a soft single point space over , (G,E) be a soft β closed set in X and $x \in X$ such that , $x \notin (G,E)$. From Theorem 12, there exists soft β open sets (x,E) and (G,E) such that $x \in (x,E)$, $(G,E) \subseteq (G,E)$and$(x,E) \cap (G,E) = \phi$. Also, from Theorem 12, there exists soft β open sets (F,E) and (G,E) such that $F,E) \subseteq (F,E)$, $(G,E) \subseteq (G,E)$. Since $(F,E) \cap (G,E) = \phi$.(X,τ,E) Is called a soft β normal space. Also Theorem 14, (X,τ,E) is a soft βT_1 space, so (X,τ,E) is a soft βT_4. (X,τ,E) is a soft βT_1 point space, so (X,τ,E) is soft βT_3 space.

Theorem 17. Let (X,τ,E) be a soft single point space over X then (X,τ,E) is a soft βT_4 space

Proof. Let (X,τ,E) be a soft single point space over $X$and let (F,E) and (G,E) be soft β closed sets in X such that$(F,E) \subseteq (F,E)$, $(G,E) \subseteq (G,E)$. Since $(F,E) \cap (G,E) = \phi$. From Theorem 12. (X,τ,E) is called a soft β normal space. Also Theorem 14, (X,τ,E) is a soft βT_1 space, so (X,τ,E) is a soft βT_4.

Theorem 18. Let (X,τ,E) be a soft single point space over X and $x,y \in X$.then (X,τ,E) is a soft β n- T_4 space.

Proof. Let (X,τ,E) be a soft single point space over X and,$y \in Xx,y \in X$, let (F,E) and (G,E) Be soft β closed sets such that $x \in (F,E)$ and $(F,E) \cap (G,E) = \phi$. Then there exist soft β open sets (F,E) and (G,E) such that $y \in (G,E)$, $(F,E) \subseteq (F,E)$, $(G,E) \subseteq (G,E) and$ $(F,E) \cap (G,E) = \phi$ from Theorem 12. Thus (X,τ,E) is a soft β normal space. Also from Theorem14, (X,τ,E) is soft βT_1 space so (X,τ,E) is a softβ n-T_4 space.

4.4 Soft β-Separation Axioms of Soft Single Point Topological Spaces With Respect to Soft Points

In this section we introduced soft β-separation axioms in soft single point topological space with respect to soft points and discussed some results with respect to these points in detail.

Theorem 19. Let (X,τ,E) be a soft single point space X. Then (X,τ,E) is soft βT_1 space.

Proof. Let (X,τ,E) be a soft single point space X and $e_G,e_H \in X_E$ such that $e_G \neq e_H$. Then there exist soft β open sets (e_G,E), (e_H,E) such that$e_G \in (e_G,E) \in \tau$, $e_H \notin (e_G,E)$ and $e_H \in (e_H,E) \in \tau$, $e_G \notin (e_H,E)$. Hence (X,τ,E) is soft βT_1 space.

Theorem 20. Let (X,τ,E) be a soft single point space over X then (X,τ,E) is soft βT_2 space

Proof. Let (X,τ,E) be a soft single point space over X_E and $e_G,e_H \in X_E$ such that $e_G \neq e_H$. Then there exist soft β open sets (e_G,E) and (e_H,E) such that $e_G \in (e_G,E) \in \tau$, $e_H \in (e_H,E) \in \tau$ and$(e_G,E) \cap (e_H,E) = \phi$. Hence (X,τ,E) is soft βT_2 space

Theorem 21. Let (X,τ,E) be a soft single point space over X . Then, (X,τ,E) is a soft βT_3 space.

Proof. Let (X, τ, E) be a soft single point space over X_E , (G, E) be a soft β closed set in X and $e_G \in X$ such that, $e_G \notin (G, E)$ From Theorem 1, there exists soft β open sets (e_G, E) and (G, E) such that $e_G \in (e_G, E)$, $(G, E) \subseteq (G, E)$ and $(e_G, E) \cap (G, E) = \phi$. Also, from Theorem 8, (X, τ, E) is a soft βT_1 point space, so (X, τ, E) is soft βT_3 space.

Proof. Let (X, τ, E) be a soft single point space over X_E , (G, E) be a soft β closed set in X and $e_G \in X$ such that,$e_G \notin (G, E)$ From Theorem 12, there exists soft β open sets (e_G, E) and (G, E) such that$e_G \in (e_G, E)$, $(G, E) \subseteq (G, E)$ and $(e_G, E) \cap (G, E) = \phi$. Also, from Theorem 19, (X, τ, E) is a soft βT_1 point space, so (X, τ, E) is soft βT_3 space.

Theorem 22. Let (X, τ, E) be a soft single point space over X and $x, y \in X$.then (X, τ, E) is a soft β n- T_4 space.

Proof. Let (X, τ, E) be a soft single point space over X and $e_G, e_H \in X_E$, let (F, E) and (G, E) be soft β closed sets such that $e_G \in (F, E)$ and $(F, E) \cap (G, E) = \phi$. Then there exist soft β open sets (F, E) and (G, E) such that $e_H \in (G, E)$, $(F, E) \subseteq (F, E)$, $(G, E) \subseteq (G, E) and$ $(F, E) \cap (G, E) = \phi$ from Theorem 12, Thus (X, τ, E) is a soft β normal space. Also from Theorem 19, (X, τ, E) is soft βT_1 space so (X, τ, E) is a soft β n- T_4 space.

4.5 Soft α-Separation Axioms of Soft Topological Spaces With Respect to Ordinary Points

In this section we introduced soft α separation axioms in soft topological space with respect to ordinary points and discussed some results with respect to these points in detail. Soft αT_4 may not be a soft αT_3 space and soft αT_2 space. But breaking news is that we launched a new soft α separation axioms which is both soft αT_3 space and soft αT_2 space. It enjoys all the properties of both the soft αT_3 space and soft αT_2 space.

Theorem 23. Let (Y, τ_Y, E) be a soft sub space of a soft topological space (X, τ, E) and $(F, E) \in SS(X)$ then

1) If (F, E) is α *open soft* set in Y and $Y \in \tau$, then $(F, E) \in \tau$.
2) (F, E) is α *open soft* set in Y if and only if $(F, E) = Y \cap (G, E)$ for some $(G, E) \in \tau$.
3) (F, E) is α *closed soft* set in Y if and only if $(F, E) = Y \cap (H, E)$ for some $(H, E)\ is\ \tau$ *α close soft* set.

Proof. 1) Let (F, E) be a soft α set in Y then there does exists a soft α open set (G, E) in X such that $(F, E) = Y \cap (G, E)$. Now, if $Y \in \tau$ then $Y \cap (G, E) \in \tau$ by the third condition of the definition of a soft topological space and hence$(F, E) \in \tau$.

2) Fallows from the definition of a soft subspace.

3) If (F, E) is soft α closed in Y then we have $(F, E) = Y(G, E)$, for some $(G, E) \in \tau_Y$. Now, $(G, E) = Y \cap (H, E)$ for some soft α open set $(H, E) \in \tau$. for any $\gamma \in E$. $F(\gamma) = Y(\gamma) \backslash G(\gamma)$ $= Y \backslash G(\gamma)$ $=$ $Y(Y(\gamma) \cap H(\gamma))$ $= Y \backslash (Y \cap H(\gamma)) = Y \backslash H(\gamma) = Y \cap (X \backslash H(\gamma))$ $=$ $Y \cap (H(\gamma))^c = Y(\gamma) \cap (H(\gamma))^c$. Thus $(F, E) = Y \cap (H, E)$ is soft α closed in X as$(H, E) \in \tau$. Conversely, suppose that $(F, E) = Y \cap (G, E)$ for some soft α close set (G, E) in X. This qualifies us to say

that$(G,E)^{-} \in \tau$. Now, if $(G,E) = (X,E)(H,E)$ where (H,E) is soft α open set τ then for any $\gamma \in EF(\gamma) = Y(\gamma) \cap G(\gamma) = Y \cap G(\gamma)$ =Y$\cap (X(\gamma)\backslash H(\gamma))$ $=Y \cap (X(\gamma)\backslash H(\gamma))$ $=Y \cap (X\backslash H(\gamma))Y\backslash H(\gamma)$ $=Y\backslash(Y \cap H(\gamma))$ $=Y(\gamma)\backslash(Y(\gamma) \cap H(\gamma))$. Thus $(F,E) = Y\backslash(Y \cap (H,E))$. Since $(H,E) \in \tau$, So$(Y \cap (H,E) \in \tau$. So $(Y \cap (H,E) \in \tau_Y$ and hence (F,E) is soft α closed in Y.

Theorem 24. Let (X,τ,E) be a soft topological space over$X$. And let (Y,E) be a soft α closed set in X and let $(G,E) \subseteq (Y,E)$. Then, (G,E) is a soft α closed set in sub space Y iff, (G,E) Is a soft α closed set in X.

Proof. This implies since (G,E) is a soft α closed set in soft sub space Y, there exists a soft α closed set (F,E) in X such that $(G,E) = Y \cap (F,E)$ from Theorem 23, Because (Y,E) and (F,E) are soft α closed set in X. Is implied by Since (G,E) is a soft α closed set in X and $(G,E) = Y \cap (G,E)$, (G,E) is a soft α closed set in sub space Y from Theorem 23.

Theorem 25. Let (X,τ,E) be a soft topological space over X and Y be a non-empty soft set of X. If (X,τ,E) is a soft $\alpha n = T_4$ space and Y be a soft α closed set, (Y,τ_Y,E) is a soft $\alpha n = T_4$ space.

Proof. Let (X,τ,E) is a soft $\alpha n = T_4$ space and Y be a soft α closed set in X. Because (X,τ,E) is soft αT_1 space, (Y,τ_Y,E) is soft αT_1 from ***Theorem****. Let,(F,E) and (G,E) be soft α closed set in Y such that that$x \in (F,E)$. Then (F,E) and (G,E) are soft α closed sets in X from Theorem 1, Because (X,τ,E) is a soft $\alpha n = T_4$ space, $(F,E) \cap (\mathrm{G,E}) = \phi$. Since (X,τ,E) is a soft α n-normal space, there exists soft α open sets and (F_1,E) and (F_2,E) such that and$y \in (F_2,E)$, $(F,E) \subseteq (F_1,E)$, $(\mathrm{G,E}) \subseteq (F_2,E)$ from **Theorem 1**, In this case,$y \in Y \cap (F_2,E)$, $(F,E) \subseteq Y \cap (F_1,E)$, $(G,E) \subseteq Y \cap (F_2,E)$ and $(Y \cap (F_1,E)) \cap (Y \cap (F_2,E)) = \phi$. Hence (Y,τ_Y,E) is a soft α n-normal space, so (Y,τ_Y,E) is a soft $\alpha n = T_4$ space.

Theorem 26. Soft $\alpha n = T_4$ space is soft αT_3 space.

Proof. Let (X,τ,E) be a soft $\boldsymbol{\alpha n = T_4}$ space over$X$ and let $x \in X$. And let (F,E) and let (G,E) be soft α closed sets such that let $x \in (F,E)$ and $(F,E) \cap (\mathrm{G,E}) = \phi$. Then there exists soft α open sets (F_1,E) and (F_2,E) such that $y \in (F_2,E)$,$(F,E) \subseteq (F_1,E)$, $(G,E) \subseteq (F_2,E)$ and $(F_1,E) \cap (F_2,\mathrm{E}) = \phi$. Because of $x \in (F,E)$ and $(F,E) \cap (\mathrm{G,E}) = \phi$, $x \notin (G,E)$ (for all α∈E, $x \notin (G(\alpha)$. Then $(\mathrm{G,E}) \subseteq (F_2,E) \in \tau$, $x \in (F,E)(F_1,E) \in \tau$ and $(F_1,E) \cap (F_2,E) = \phi$. And then, (X,τ,E) is soft α regular space. Also (X,τ,E) is soft αT_1 space, so (X,τ,E) is soft αT_3 space.

4.6 Soft α-Separation Axioms of Soft Topological Spaces With Respect to Soft Points

In this section we introduced soft **α** separation axioms in soft topological space with respect to soft points and discussed some results with respect to these points in detail. Soft $\boldsymbol{\alpha T_4}$ may not be a soft $\boldsymbol{\alpha T_3}$ space soft α T_2 space

But breaking news is that we launched a new soft α separation axiom which is both soft αT_3 space and soft αT_2 space. It enjoys all the properties of both the soft αT_3 space and soft αT_2 space.

Theorem 27. Let (X,τ,E) be a soft topological space over$X$ and Y be a non-empty soft set of X . If (X,τ,E) is a soft $\alpha n-T_4$ space and Y be a soft α closed set, (Y,τ_Y,E) is a soft $\alpha n-T_4$ space.

Proof. Let (X,τ,E) is a soft $\alpha n=T_4$ space and Y be a soft α closed set in$X$. Because (X,τ,E) is soft αT_1 space, (Y,τ_Y,E) is soft αT_1 from $\boldsymbol{Theorem}$**. Let, (F,E) and (G,E) be soft α closed set in Y such that that $e_G\in(F,E)$. Then (F,E) and (G,E) are soft α closed sets in X from Theorem 24. Because (X,τ,E) is a soft $\alpha n-T_4$ space $(F,E)\cap(\mathrm{G,E})=\phi$. Since (X,τ,E) is a soft α n-normal space, there exists soft α open sets and $(F_1,E)$and (F_2,E) such that and $e_H\in(F_2,E)$, $(F,E)\subseteq(F_1,E)$, $(\mathrm{G,E})\subseteq(F_2,E)$ and $(F_1,E)\cap(F_2,E)=\phi$. $Y\cap(F_1,E)$ and $Y\cap(F_2,E)$ are soft α open sets in Y.

Theorem 24. In this case,$e_H\in Y\cap(F_2,E)$, $(F,E)\subseteq Y\cap(F_1,E)$, $(G,E)\subseteq Y\cap(F_2,E)$ and $\big(Y\cap(F_1,E)\big)\cap\big(Y\cap(F_2,E)\big)=\phi$. Hence (Y,τ_Y,E) is a soft α n-normal space, so (Y,τ_Y,E) is a soft $\alpha n-T_4$ space.

Theorem 28. Soft $\alpha n-T_4$ space is soft αT_3 space.

Proof. Let(X,τ,E) be a soft $\boldsymbol{\alpha}n-T_4$ space over$X$ and let $x\in X$. And let (F,E) and let (G,E) be soft α closed sets such that let $e_G\in(F,E)$ and $(F,E)\cap(\mathrm{G,E})=\phi$. Then there exists soft α open sets (F_1,E) and (F_2,E) such that $e_H\in(F_2,E)$,$(F,E)\subseteq(F_1,E)$, $(G,E)\subseteq(F_2,E)$ and $(F_1,E)\cap(F_2,\mathrm{E})=\phi$. Because of $e_G\in(F,E)$ and $(F,E)\cap(\mathrm{G,E})=\phi$, $e_G\notin(G,E)$ (for all α∈E, $e_G\notin(G(\alpha)$. Then $(\mathrm{G,E})\subseteq(F_2,E)\in\tau$, $e_G\in(F,E)(F_1,E)\in\tau$ and $(F_1,E)\cap(F_2,E)=\phi$. And then, (X,τ,E) is soft α regular space. Also (X,τ,E) is soft αT_1 space, so (X,τ,E) is soft $\alpha n-T_4$ space.

4.7 Soft β-Separation Axioms of Soft Topological Spaces With Respect to Ordinary Points

In this section we introduced soft β-separation axioms in soft topological space with respect to ordinary points and discussed some results with respect to these points in detail. Soft βT_4 may not be a soft βT_3 space and soft βT_2 space. But breaking news is that we launched a new soft β-separation axiom which is both soft βT_3 space and soft βT_2 space. It enjoys all the properties of both the soft βT_3 space and soft βT_2 space.

Theorem 29. Let (Y,τ_Y,E) be a soft sub space of a soft topological space (X,τ,E) and $(F,E)\in SS(X)$ then

1) If(F,E) is β *open soft* set in Y and$Y\in\tau$, then$(F,E)\in\tau$.
2) (F,E) is β *open soft* set in Y if and only if $(F,E)=Y\cap(G,E)$ for some $(G,E)\in\tau$.
3) (F,E) is β *closed soft* set in Y if and only if $(F,E)=Y\cap(H,E)$ for some $(H,E)\ is\ \tau\beta$ *close soft* set.

Proof. 1) Let (F,E) be a soft β open set in Y then there does exists a soft β open set (G,E) in X such that $(F,E)=Y\cap(G,E)$. Now, if $Y\in\tau$ then $Y\cap(G,E)\in\tau$ by the third condition of the definition of a soft topological space and hence$(F,E)\in\tau$.

2) Fallows from the definition of a soft subspace.

3) If (F,E) is soft β closed in Y then we have $(F,E) = Y(G,E)$, for some $(G,E) \in \tau_Y$. Now, $(G,E) = Y \cap (H,E)$ for some soft β open set $(H,E) \in \tau$. for any $\gamma \in E$. $F(\gamma) = Y(\gamma)\backslash G(\gamma)$ $= Y\backslash G(\gamma)$ $=$ $Y\backslash(Y(\gamma) \cap H(\gamma))$ $=$ $Y\backslash(Y \cap H(\gamma)) = Y\backslash H(\gamma) = Y \cap (X\backslash H(\gamma))$ $=$ $Y \cap (H(\gamma))^C = Y(\gamma) \cap (H(\gamma))^C$. Thus $(F,E) = Y \cap (H,E)'$ is soft β closed in X as $(H,E) \in \tau$. Conversely, suppose that $(F,E) = Y \cap (G,E)$ for some soft β close set (G,E) in X. This qualifies us to say that $(G,E)' \in \tau$. Now, if $(G,E) = (X,E)\backslash(H,E)$ where (H,E) is soft βopen set τ then for any $\gamma \in E F(\gamma) = Y(\gamma) \cap G(\gamma) = Y \cap G(\gamma)$ $= Y \cap (X(\gamma)\backslash H(\gamma)$ $= Y \cap (X(\gamma)\backslash H(\gamma))$ $= Y \cap (X\backslash H(\gamma)) Y\backslash H(\gamma)$ $= Y\backslash(Y \cap H(\gamma))$ $= Y(\gamma)\backslash(Y(\gamma) \cap H(\gamma))$. Thus $(F,E) = Y\backslash(Y \cap (H,E))$. Since $(H,E) \in \tau$, So $(Y \cap (H,E) \in \tau$. So $(Y \cap (H,E) \in \tau_Y$ and hence (F,E) is soft β closed in Y.

Theorem 30. Let (X,τ,E) be a soft topological space over X. And let (Y,E) be a soft β closed set in X and let $(G,E) \subseteq (Y,E)$. Then, (G,E) is a soft β closed set in sub space Y iff, (G,E) is a soft β closed set in X.

Proof. This implies since (G,E) is a soft β closed set in soft sub space Y, there exists a soft β closed set (F,E) in X such that $(G,E) = Y \cap (F,E)$ from Theorem 29. Because (Y,E) and (F,E) are soft β closed set in X.Is implied by Since (G,E) is a soft β closed set in X and, $(G,E) = Y \cap (G,E)$, (G,E) is a soft β closed set in sub space Y from Theorem 29.

Theorem 31. Let (X,τ,E) be a soft topological space over X and Y be a non-empty soft set of X. If (X,τ,E) is a soft β$n - T_4$ space and Y be a soft β closed set, (Y,τ_Y,E) is a soft β$n - T_4$ space.

Proof. Let (X,τ,E) is a soft β$n = T_4$ space and Y be a β soft closed set in X. Because (X,τ,E) is soft βT_1 space, (Y,τ_Y,E) is soft βT_1 from ***Theorem**** [48]. Let, (F,E) and (G,E) be soft β closed set in Y such that that $x \in (F,E)$. Then (F,E) and (G,E) are soft β closed sets in X from Theorem 30. Because (X,τ,E) is a soft β$n - T_4$ space, $(F,E) \cap (G,E) = \phi$. Since (X,τ,E) is a soft β n-normal space, there exists soft β open sets and (F_1,E) and (F_2,E)such that and $y \in (F_2,E)$, $(F,E) \subseteq (F_1,E)$, $(G,E) \subseteq (F_2,E)$ and $(F_1,E) \cap (F_2,E) = \phi$. $Y \cap (F_1,E)$ and $Y \cap (F_2,E)$ are soft β open sets in Y from Theorem 29. In this case, $y \in Y \cap (F_2,E)$, $(F,E) \subseteq Y \cap (F_1,E)$, $(G,E) \subseteq Y \cap (F_2,E)$ and $(Y \cap (F_1,E)) \cap (Y \cap (F_2,E)) = \phi$. Hence $(Y,\tau_Y S,E)$ is a soft β n-normal space, so (Y,τ_Y,E) is a soft β$n - T_4$ space.

Theorem 32. Soft β$n - T_4$ space is soft βT_3 space.

Proof. Let (X,τ,E) be a soft **β**$n - T_4$ space over X and let $x \in X$. And let (F,E) and let (G,E) be soft β closed sets such that let $x \in (F,E)$ and $(F,E) \cap (G,E) = \phi$. Then there exists soft β open sets (F_1,E) and (F_2,E) such that $y \in (F_2,E)$, $(F,E) \subseteq (F_1,E)$, $(G,E) \subseteq (F_2,E)$ and $(F_1,E) \cap (F_2,E) = \phi$. Because of $x \in (F,E)$ and $(F,E) \cap (G,E) = \phi$, $x \notin (G,E)$ (for all γ∈ E, $x \notin (G(\gamma)$. Then $(G,E) \subseteq (F_2,E) \in \tau$, $x \in (F,E)(F_1,E) \in \tau$ and $(F_1,E) \cap (F_2,E) = \phi$. And then, (X,τ,E) is soft β regular space. Also (X,τ,E) is soft βT_1 space, so (X,τ,E) is soft βT_3 space.

4.8 Soft β-Separation Axioms of Soft Topological Spaces With Respect to Soft Points

In this section we introduced soft β-separation axioms in soft topological space with respect to soft points and discussed some results with respect to these points in detail. Soft βT_4 may not be a soft βT_3 space

But breaking news is that we launched a new soft β-separation axiom which is both soft βT_3 space and soft βT_2 space. It enjoys all the properties of both the soft βT_3 space and soft βT_2 space.

Theorem 33. Let (X, τ, E) be a soft topological space over$X$ and Y be a non-empty soft set of X .If (X, τ, E) is a soft β$n - T_4$ space and Y be a soft β closed set, (Y, τ_Y, E) is a soft β$n = T_4$ space.

Proof. Let (X, τ, E) is a soft β$n - T_4$ space and Y be a β soft closed set in X. Because (X, τ, E) is soft βT_1 space, (Y, τ_Y, E) is soft βT_1 from $\boldsymbol{Theorem}^*$ [48]. Let, (F, E) and (G, E) be soft β closed set in Y such that that$e_G \in (F, E)$. Then (F, E) and (G, E) are soft β closed sets in X from Theorem 30. . Because (X, τ, E) is a soft β$n = T_4$ space$(F, E) \cap (\mathrm{G}, \mathrm{E}) = \phi$. Since $(X, \tau, E)$is a soft β n-normal space, there exists soft β open setsand (F_1, E) and (F_2, E)such that and$e_H \in (F_2, E)$, $(F, E) \subseteq (F_1, E)$, $(\mathrm{G}, \mathrm{E}) \subseteq (F_2, E)$ and $(F_1, E) \cap (F_2, E) = \phi$. $Y \cap (F_1, E)$ and $Y \cap (F_2, E)$ are soft β open sets in Y from Theorem 29. In this case,$e_H \in Y \cap (F_2, E)$, $(F, E) \subseteq Y \cap (F_1, E)$, $(G, E) \subseteq Y \cap (F_2, E)$ and $\left(Y \cap (F_1, E)\right) \cap \left(Y \cap (F_2, E)\right) = \phi$. Hence (Y, τ_Y, E) is a soft β n-normal space, so (Y, τ_Y, E) is a soft β$n - T_4$ space.

Theorem 34. Soft β$n - T_4$ space is soft βT_3 space.

Proof. Let(X, τ, E) be a soft β$n - T_4$ space over$X$ and let $x \in X$. And let (F, E) and let (G, E) be soft β closed sets such that let $e_G \in (F, E)$ and $(F, E) \cap (\mathrm{G}, \mathrm{E}) = \phi$. Then there exists soft β open sets (F_1, E) and (F_2, E) such that $e_H \in (F_2, E)$,$(F, E) \subseteq (F_1, E)$, $(G, E) \subseteq (F_2, E)$ and $(F_1, E) \cap (F_2, E) = \phi$. Because of$e_G \in (F, E)$ and $(F, E) \cap (\mathrm{G}, \mathrm{E}) = \phi$, $e_G \notin (G, E)$ (for all $\gamma \in$E, $e_G \notin (G(\gamma)$.Then $(\mathrm{G}, \mathrm{E}) \subseteq (F_2, E) \in \tau$, $e_G \in (F, E)(F_1, E) \in \tau$ and $(F_1, E) \cap (F_2, E) = \phi$. And then, (X, τ, E) is soft β regular space. Also (X, τ, E) is soft βT_1 space, so (X, τ, E) is soft βT_3 space.

5. Conclusion

Topology is the most important branch of mathematics which deals with mathematical structures. Recently, many researchers have studied the soft set theory which is initiated by Molodtsov [4] and safely applied to many problems which contain uncertainties in our social life. Shabir and Naz in [23] introduced and deeply studied the concept of soft topological spaces. They also studied topological structures and exhibited their several properties with respect to ordinary points. In this present paper the concept of soft αT_i spaces (i=1, 2, 3) and soft β_i spaces (i=1, 2, 3) are introduced in soft single point space with respect to ordinary and soft points of a topological space. Soft α$n - T_4$ space and Soft β$n - S_4$ are introduced in soft topological space with respect to ordinary and soft points.

Many mathematicians discussed soft separation axioms in soft topological spaces at full length with respect to soft open set, soft b-open set, soft semi-open set and soft set. They also worked over the hereditary properties of different soft topological structures in soft topology. In this present work hand is tried and work is encouraged over the gap that exists in soft topology. Related to Soft spaces, some theorems in soft single topological spaces are discussed with respect to ordinary points as well as with respect to soft points. Focus is laid upon the characters of soft $\alpha n - T_4$ and soft $\beta n - S_4$ space and their sub spaces in soft topological structures. We also beautifully discussed some soft transmissible properties with respect to ordinary as well as soft points. We hope that these results in this paper will help the researchers for strengthening the toolbox of soft topology. In the next study, we extend the concept of semi open, Pre-open and b^{**}open soft sets in soft bi topological spaces with respect to ordinary as well as soft points. We also extended these axioms to different results. These soft separation axioms would be useful for the growth of the theory of soft topology to solve complex problems, comprising doubts in economics, engineering, medical etc.

References

[1] Basavaraj M. Ittanagi, Soft Bi topological spaces international journal of Computer application (0975 8887), v107-No. 7. December 2014.

[2] S.A El-Sheikh, Rodyna A. Hosny and A.M Abd-e-Latif, Characterization of b soft separation Axioms in soft topological spaces in Inf. Sci. Lett.4 (3) (2015) 125-133.

[3] M. Naz, M. Shabir and M. I. Ali, Separation axiom in bi soft topological spaces, Arxiv. 1850 of .00866 vl [Math, Gn] 29 August 2015.

[4] D. A. Molodtsov, Soft set theory first results, Comp. Math. Appl., 37, (1999), 19-31.

[5] B. Ahmad and A. Kharal, Mapping of soft classes, New Math. Nat. Comp., 7 (3) (2011) 471-481.

[6] P. K. Maji, R, Biswas and A. R. Roy, Soft set theory, Comp. Math. Appl., 45, (2003), 555-562.

[7] M. I. Ali, F. Feng, X. Liu, W. K. Min and M. Shabir, On some new operations in soft set theory, Comp. Math. Appl., 57, (2009), 1547-1553.

[8]. D.V. Kovkov, V. M. Kolbanov and .D.A Moldtsov, Soft set theory based optimization, Journal of Computer and System, 46 (6) (2007), 872-880.

[9] Z.Xiao, L.Chen, B.Zhong and S.Ye, Recognition for soft information based on the theory of soft sets, J. Chen (Ed) Proceeding of ICSSM – 05, 2, IEEE, 2005,Appl. 1404 -1106.

[10] D. Pei and D. Miao, From soft sets to information systems , in :X, Hu, Q liu, A skowron , T,Y, Lin, R.R Yager, B.Zhang (Eds), Proceedings of Granular computing in IEEE, Vol , 21,2005,pp,617-621.

[11] B. Ahmad and A. Kharal. On fuzzy soft sets, Adv, Fuzzy syst, 2009, Avt, ID 586507, 6PP.

[12] H. Aktas and N. Cagman, Soft sets and soft groups, Information sciences, 1(77) (2007) 2726-2735.

[13] N. Cagman, F. Citak and S. Enginoglu, Fuzzy parameterized fuzzy soft sets theory and its application, Turkish, Journal of Fuzzy systems, 1 (1) (2010)21-35.

[14] A. Kandil, O A. E. Tantawy , S. A. El-Sheikh and A. M. Abd El-e-Latif , Fuzzy soft semi connection properties in fuzzy soft topological spaces math, Sci, Lett, 4(2015)171-179.

[15] N. Levine, Semi open sets and semi continuity in topological spaces, Amer, Math. Monthly, 70, 1963, 36-41.

[16] P. K. Majk, R. Biswas and A. R. Roy, Fuzzy soft sets, Journal of Fuzzy Mathematics, 9(3) (2001)589- 602.
[17] P. K. Maji, R. Biswas, and A. R. Roy, Intuitionistic fuzzy soft sets, Journal of fuzzy Mathematics, 9(3) (2001) 677 – 691.
[18] P. Majumdar and S. K. Samanta, Gernalized fuzzy soft sets, Comp, Math, Appl. 59, (2010), 1425 – 1432.
[19] T. T. Buhaescu, Some observations Intuitionistic fuzzy relations, Itimerat Seminar on Functional Equations, 111-118.
[20] D. Andrijevic, On b-open sets, Mat.Vesnik, 48(1996), 24-32.
[21] Y. Zou and Z. Xiao, Data analysis approaches of soft sets under incomplete information, Knowledge Bond systems, 21, (2008), 941-945.
[22] C. L. Chang, Fuzzy topological spaces, J. Math, Anal, Appl, 24(1968), 182-190.
[23] M. Shabir and M. Naz, On soft topological spaces, Compt, Math, Appl, 61(2011)1786–1799.
[24] W. K. Min, A note on soft topological spaces, Compt, Math, Appl., 62, (2011), 3524 – 3528.
[25] A. Kandil, O. A.E. Tantawy , S.A. El-Sheikh and A. M. Abd-e- Latif, R- Operation and decomposition of some forms of soft continuity in soft topological spaces, FMI, 7(2014)181-196.
[26] A.Kandil,O. A.E .Tantawy , S.A EI-Sheikh and A.M Abd-e-Latif, Soft semi separation axioms and irresolute softfunction, FMI, 8(2) (2014) 305- 318.
[27] A. Kandil, O. A. E. Tantawy, S. A. El-Sheikh and A.M Abd-e-Latif, Soft ideal theory, soft local function and generated soft topological spaces, Appl. Math. Inf. Sci., 8, (4) (2014), 1595-1603.
[28] A. Kandil, O, A. E. Tantawy, S. A. El- Sheikh and A. M. Abd-e-Latif, Operation and decomposition of some forms of soft continuity of soft topological spaces via soft ideal, FMI , (3)(2015) 385- 420.
[29] A. Kandil, O. A. E. Tantawy , S. A. El-Sheikh and A. M. Abd- e-Latif, Soft connected ness via soft ideals, Journal of New results in science, 4 (2014) 90-108.
[30] A. Kandil, O, A. E. Tantawy, S. A. El-Sheikh and A. M. Abd-e-Latif, Soft regularity and normally based on semi – open soft sets and soft ideal, Appl. Math. Inf. Sci., lett, 3(2015) 47-55.
[31] A. Kandil, O. A. E. Tantawy, S. A. El-Sheikh and A. M. Abd-e-Latif, Soft semi compactness via soft ideal Appl. Math. Inf. Sci., 8 (5) (2014), 2297- 2306.
[32] A. Kandkil, O. A. E. Tantawy, S. A. El-Sheikh and A. M. abd-e- Latif, Soft semi soft ideal south Asian j. Math, 4 (6) (2014) 265- 284.
[33] A. Kandil, O. A. E. Tantawy, S. A. El- Sheikh and A. M. Abd El-e- Latif , Supra generalized closed soft sets with respect to an soft ideal in supra soft topological spaces Appl. Math.inf . Sci., (4) (2014) 1731- 1740.
[34] S. A. El-Sheikh and A. M. Abd-e-Latif, Decomposition of some type of supra soft sets and soft continuity, international journal of mathematics trends and technology, 9 (1) (2014) 37 -56.
[35] I. Zorlutuna, N. Akdog, W.K Min, S, Atmaca, Remarks on soft topological spaces, FMI , 3 (2) (2012) 171-185.
[36] L. A. Zadeh, Fuzzy sets, Information and control 8(1965) 338-353.
[37] Q. M. Sun, Z. L. Zhang, J. Liu. Soft sets and soft modules , Proceeding of rough sets and Knowledge technology , third international conference , RSKT 2008 ,17-19 May , Chengdu , China , PP. 403- 409, 2008.
[38] C. H. Park, Y. B. Jun, M. A. Ozturk. Soft W.S- Algebras, Communications of the Korean Mathematical society 23(3) (2008) 313- 324.

[39] J. Mahanta, P. K. Das. On soft topological spaces via semi open and semi close soft sets are XIV: 1203, 1203.4133 VI P [Math, GN] 16 Mar 2012.

[40] A. Lancy and Arockiarani, On Soft β-Separation Axioms, IJMRS, 1 (5), 2013.

[41] Bin Chen, Soft Semi open sets and related properties in soft topological spaces, App, Math. Inf. Sci., 7.No.1.287-294 (2013).

[42] S. Hussain, B. Ahmad, Soft separation axioms in soft topological space submitted to elsvier 28 september 2015.

[43] I. Arockiarani, A. Arokialancy, Generalized soft g β closed sets and soft gs β closed sets in soft topological spaces, International Journal of Mathematical Archive, 4(2), (2013), 1-7.

[44] A. Mukharjee, Some new bi topological notions, De L, institute Mathematique, 93 (107) (2013), 165-172.

[45] O. Gocur and A. Kopuzlu, Some new properties on soft separation axioms, Ann. Fuzzy. Math, 9(3) (2015) 421-429).

[46] S. A. El-Sheikh and A. M. Abd-e-Latif, Characterization of soft b-open sets in soft topological spaces, New theory, 2 (2015), 8-18.

[47] Y. Yumak and A. K. Kaymaker, Soft β-open sets and their applications, Journal of New Theory, 4 (2015), 80-89.

[48] A. M. Abd-e-Latif, Rodyna. A. Hosny, On soft separation axioms via β-open soft sets, SAJM, 5(1), (2015), 252-265.

http //www. newtheory. org **ISSN: 2149-1402**

Journal of New Theory

Received: 06.12.2017 *Year*: **2017**, *Number:* **19**, *Page:* **82-107**
Published: 24.12.2017 *Original Article*

SOME CONTRIBUTION OF SOFT RUGULAR OPEN SETS TO SOFT SEPARATION AXIOMS IN SOFT QUAD TOPOLOGICAL SPACES

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Abstract-Our main interest in this study is to look for soft regular separations axioms in soft quad topological spaces. We talk over and focus our attention on soft regular separation axioms in soft quad topological spaces with respect to ordinary points and soft points. Moreover study the inherited characteristics at different angles with respect to ordinary points and soft points. Some of their central properties in soft quad topological spaces are also brought under examination.

Keywords- *Soft sets, soft topology, soft regular open set, soft regular closed set, soft quad topological space, soft R-qT_0 structure, R-soft qT_1structure, soft R-qT_2 structure, soft R-qT_3 structure and soft R-qT_4 structure.*

1 Introduction

In real life condition the complications in economics, engineering, social sciences, medical science etc. We cannot handsomely use the old-fashioned classical methods because of different types of uncertainties existing in these problems. To finish out these complications, some types of theories were put forwarded like theory of Fuzzy set, intuitionistic fuzzy set, rough set and bi polar fuzzy sets, inwhich we can safely use a mathematical methods for dealing with uncertainties. But, all these theories have their inherent worries. To overcome these difficulties in the year 1999, Russian scholar Molodtsov [4]introduced the idea of soft set as a new mathematical methods to deal with uncertainties. This is free from the above difficulties. Kelly [5] studied Bi topological spaces and discussed different results. Tapi et al. beautifully discussed separation axioms in quad topological spaces. Hameed and Abid discussed separation axioms in Tri-topological spaces.

Recently, in 2011, Shabir and Naz [7] initiated the idea of soft topological space and discussed different results with respect to ordinary points, they beautifully defined soft topology as a collection of τ of soft sets over X. they also defined the basic concept of soft topological spaces such as open set and closed soft sets, soft nbd of a point, soft separation axioms, soft regular and soft normal spaces and published their several performances. soft separation axioms are also discussed at detail. Aktas and Cagman [9] discussed soft sets and soft groups. Chen [10] discovered the parameterization reduction of soft sets and its applications. Feng et al. [11] studied soft semi rings and its applications. In the recent years, many interesting applications of soft sets theory and soft topology have been discussed at great depth [12,13,14,15,16,17,18,19,20,21,22] Kandil at al. [25] explained soft connectedness via soft ideal developed soft set theory. Kandil et al. [27] launched soft regular ity and normality based on semi open soft sets and soft ideals.

In [28,29,30,31,32,33,34,35,36] discussion is launched soft semi Hausdorff spaces via soft ideals, semi open and semi closed sets, separation axioms,decomposition of some type supra soft sets and soft continuity are discussed. Hussain and Ahmad [51] defined soft points, soft separation axioms in soft topological spaces with respect to soft points and used it in different results. Kandil et al. [52] studied soft semi separation axioms and some types of soft functions and their characteristics.

In this present paper, concept of soft regular separation axioms in soft quad topological spaces is broadcasted with respect to ordinary and soft points.

Many mathematicians made discussion over soft separation axioms in soft topological spaces at full length with respect to soft open set, soft b-open set, soft semi-open, soft α-open set and soft β-open set. They also worked over the hereditary properties of different soft topological structures in soft topology. In this present article h and is tried and work is encouraged over the gap that exists in soft quad-topology related to soft regular R- qT_0, soft regular R-qT_1 soft regular R-qT_2, soft regular R-qT_3 and soft regular-qT_4structures. Some propositions in soft quid topological spaces are discussed with respect to ordinary points and soft points. When we talk about distance between the points in soft topology then the concept of soft separation axioms will auto medically come in force. that is why these structures are catching our attentions. We hope that these results will be valuable for the future study on soft quad topological spaces to accomplish general framework for the practical applications and to solve the most complicated problems containing doubts in economics, engineering, medical, environment and in general mechanic systems of various varieties. In upcoming these beautiful soft topological structures may be extended in to soft n-topological spaces provided n is even.

2. Preliminaries

The following Definition s which are pre-requisites for present study

Definition 1 [4] Let X be an initial universe of discourse and E be a set of parameters. Let P (X) denotes the power set of X and A be a non-empty sub-set of E. A pair (F, A) is called a soft set over U, where F is a mapping given by $F\ A \rightarrow P\ (X)$

In other words, a set over X is a parameterized family of sub set of universe of discourse X. For $e \in A, F\ (e)$ may be considered as the set of e-approximate elements of the soft set (F, A) and if $e \notin A$ then $F(e) = \phi$ that is$F_{A=}\{F(e)\ e \in A \subseteq E, F\ A \rightarrow P(X)\}$ the family of all these soft sets over X denoted by $SS\ (X)\ _A$

Definition 2 [4] Let $F_{A,} G_B \in SS\ (X)\ _E$ then F_A, is a soft subset of G_B denoted by $F_A \sqsubseteq G_B$, if

1. $A \subseteq B$ and
2. $F(e) \subseteq G(e), \forall \in A$

In this case F_A is said to be a soft subset of G_B and G_B is said to be a soft super set $F_A, G_B \sqsupseteq F_A$

Definition 3 [6] Two soft subsets F_A and G_B over a common universe of discourse set X are said to be equal if F_A is a soft subset of G_B and G_B is a soft subset of F_A

Definition4 [6] The complement of soft subset (F, A) denoted by $(F, A)\ ^C$ is defined by $(F, A)\ ^C = (F^C, A)\ F^C \rightarrow P\ (X)$ is a mapping given by $F^C(e) = U - F\ (e)\ \forall e \in A$ and $F^C$is called the soft complement function of F. Clearly $(F^C)\ ^C$ is the same as F and $(\ (F, A)\ ^C)\ ^C = (F, A)$

Definition 5 [7] The difference between two soft subset (G, E) and (G, E) over common of universe discourse X denoted by $(F, E) \backslash\ (G, E)$ is defined as $F\ (e) \backslash G\ (e)$ for all $e \in E$

Definition 6 [7] Let (G, E) be a soft set over X and $x \in X$ We say that $x \in\ (F, E)$ and read as x belong to the soft set (F, E) whenever $x \in F\ (e)\ \forall e \in E$. The soft set (F, E) over X such that $F(e) = \{x\} \forall e \in E$ is called sing Let on soft point and denoted by $x, or\ \ (x, E)$

Definition 7 [6] A soft set (F, A) over X is said to be Null soft set denoted by $\overline{\emptyset}\ or\ \emptyset_A$ if $\forall e \in A, F(e) = \emptyset$

Definition 8 [6] A soft set (F, A) over X is said to be an absolute soft denoted by $\bar{A}\ or\ X_A$ if $\forall e \in A, F(e) = X$.

Clearly, we have $X_A^C = \emptyset_A\ and\ \emptyset_A^C = X_A$

Definition 9 [38] The soft set $(F, A)\ \in SS(X)A$ is called a soft point inX_A, denoted by e_F, if for the element $e \in A, F(e) \neq \{x\}$ and $F\ (e^/)\ = \phi$ if for all $e^/ \in A - \{e\}$.

A soft point is an element of a soft set F_A. The class of all soft sets over U is denoted by $S\ (U)$.

For Example $U = \{u_1, u_2, u_3\}, E = \{x_1, x_2, x_3\}, A = \{x_1, x_2\}$ and $F_A = \{ (x_1, \{u_1, u_2\}) \}, \{ (x_2, \{u_2, u_3\}) \}$. Then $F_{A_1} = \{(x_1, \{u_1\})\}, F_{A_2} = \{(x_1, \{u_2\})\}, F_{A_3} = \{(x_1, \{u_2, u_2\})\}$, $F_{A_4} = \{(x_2, \{u_2\})\}, F_{A_5} = \{(x_2, \{u_3\})\}, F_{A_6} = \{(x_2, \{u_1, u_3\})\}, F_{A_7} = \{(x_1, \{u_2\}, (x_{2,}\{u_2\})\}$, $F_{A_8} = \{(x_1, \{u_1\}, (x_2, \{u_3\})\}$,
$F_{A_9} = \{(x_1, \{u_1\})\}, (x_2, \{u_2, u_3\}) \}, F_{A_{10}} = \{(x_1, \{u_2\})\}, (x_2, \{u_2\}) \}, F_{A_{11}} = \{(x_1, \{u_2\}), (x_{2,}\{u_3\}) \}, F_{A_{12}} = \{(x_1, \{u_2\}), (x_{2,}\{u_2, u_3\}, F_{A_{13}} = \{(x_1, \{u_1, u_2\}), (x_2, \{u_2\}) \}, F_{A_{14}} = \{(x_1, \{u_1, u_2\}), (x_2, u_3\}, F_{A_{15}} = F_A, F_{A_{16}} = F_\emptyset$, are all soft sub sets of F_A.

Definition 10 [38] The soft point e_F is said to be in the soft set (G, A), denoted by $e_F \in (G, A)$ if for the element$e \in A, F(e) \subseteq G(e)$.

Definition 11 [6] The union of two soft sets (F, A) and (G, B) over the common universe of discourse X is the soft set (H, C), where, $C = AUB$ For all $e \in C$

$$H(e) = \begin{cases} F(e) & if\ e \in A - B \\ G(e) & if\ e \in (B - A) \\ F(e)UG(e), & if\ e \in A \cap B \end{cases}$$

Written as $(F, A) \cup (G, B) = (H, C)$

Definition 12 [6] The intersection $(H.C)$ of two soft sets (F, A) and (G, B) over common universe X, denoted $(F, A)\ \bar{\sqcap}\ (G, B)$ is defined as $C = A \cap B$ and $H(e) = F(e) \cap G(e), \forall e \in C$

Definition 13 [41] Two soft sets $(G, A), (H, A)$ in $SS\ (X)\ _A$ are said to be soft disjoint, written $(G, A) \cap (H, A) = \emptyset_A$ if $G(e) \cap H(e) = \emptyset$ for all$e \in A$.

Definition 14 [38] The soft point e_G, e_H in X_A are disjoint, written $e_G \neq e_H$ if their corresponding soft sets (G, A) and (H, A) are disjoint.

Definition 15 [2] Let (F, E) be a soft set over X and Y be a non-empty sub set of X. Then the sub soft set of (F, E) over Y denoted by (Y_F, E), is defined as follow $Y_{F\ (\alpha)} = Y \cap F(\alpha), \forall \in E$ in other words

$$(Y_F, E) = Y \cap (F, E).$$

Definition 16 [3] Let τ be the collection of soft sets over X, then $\tau$is said to be a soft topology on X, if

1. $\emptyset, X$ belong to τ
2. The union of any number of soft sets in τ belongs to τ
3. The intersection of any two soft sets in τ belong to τ

The trip Let (X, F, E) is called a soft topological space.

Definition 17 [1] Let (X, F, E) be a soft topological space over X, then the member of τ are said to be soft open sets in X.

Definition 18 [1] Let (X, F, E) be a soft topological space over X. A soft set (F, A) over X is said to be a soft closed set in X if its relative complement $(F, E)^{C}$ belong to τ

Definition 19 [42] A soft set (A, E) in a soft topological space (X, τ, E) will be termed soft regular open set denoted as $S, R, O\ (X)$ if and only if there exists a soft open set $(F, E) = int\ (cl\ (F, E)$ and soft regular closed set if set $(F, E) = cl\ (in\ (F, E)$ denoted by as $S, R, C\ (X)$ in short h .

3. Soft Regular Separation Axioms of Soft Quad Topological Spaces

In this section we introduced soft regular Separation Axioms in soft Quad topological space with respect to ordinary points and discussed some results with respect to these points in detail.

Definition 20 Let $\left(\breve{X}, \overset{1}{\tau}, \breve{E}\right)$, $\left(\breve{X}, \overset{2}{\tau}, E\right)$, $\left(\breve{X}, \overset{3}{\tau}, \breve{E}\right)$ and $\left(\breve{X}, \overset{4}{\tau}, \breve{E}\right)$ be four different soft topologies on $\breve{X}$. Then $\left(\breve{X}, \overset{1}{\tau}, \overset{2}{\tau}, \overset{3}{\tau}, \overset{4}{\tau}, \breve{E}\right)$is called a *soft quad topological space.* The soft four topologies $\left(\breve{X}, \overset{1}{\tau}, \breve{E}\right)$, $\left(\breve{X}, \overset{2}{\tau}, \breve{E}\right)$, $\left(\breve{X}, \overset{3}{\tau}, E\right)$ and $\left(\breve{X}, \overset{4}{\tau}, \breve{E}\right)$ are independently satisfying the axioms of soft topology. The members of $\overset{1}{\tau}$ are called $\overset{1}{\tau}$ *soft open* set. and complement of $\overset{1}{\tau}$ soft open set is called $\overset{1}{\tau}$ *soft closed* set. Similarly, the member of $\overset{2}{\tau}$ are called $\overset{2}{\tau}$ *soft open* sets and the complement of $\overset{2}{\tau}$ soft open sets are called $\overset{2}{\tau}$ *soft closed* set. The members of $\overset{3}{\tau}$ are called $\overset{3}{\tau}$ *soft open* set. and complement of $\overset{3}{\tau}$ soft open set is called $\overset{3}{\tau}$ *soft closed* set and the members of $\overset{4}{\tau}$ are called $\overset{4}{\tau}$ *soft open* set. and complement of $\overset{4}{\tau}$ soft open set is called $\overset{4}{\tau}$ *soft closed* set.

Definition 21 Let $\left(\breve{X}, \overset{1}{\tau}, \overset{2}{\tau}, \overset{3}{\tau}, \overset{4}{\tau}, \breve{E}\right)$ be a soft quad topological space over $\breve{X}$ and $\breve{Y}$be a non-empty subset of $\breve{X}$. Then $\overset{1}{\tau}_{\breve{Y}} = \{(Y_F, E)\ (F, \breve{E}) \in \overset{1}{\tau})\}$, $\overset{2}{\tau}_{\breve{Y}} = \{(Y_G, E)\ (G, \breve{E}) \in \overset{2}{\tau})\}$, $\overset{3}{\tau}_{\breve{Y}} = \{(\breve{Y}_H, \breve{E})\ (H, \breve{E}) \in \overset{3}{\tau})\}$ and $\overset{4}{\tau}_{\breve{Y}} = \{(I_E, \breve{E})\ (I, \breve{E}) \in \overset{4}{\tau})\}$are said to be the relative topological on Y. Then $\left(\breve{Y}, \overset{1}{\tau}, \overset{2}{\tau}, \overset{3}{\tau}, \overset{4}{\tau}, \breve{E}\right)$ is called relative soft quad-topological space $\left(X, \overset{1}{\tau}, \overset{2}{\tau}, \overset{3}{\tau}, \overset{4}{\tau}, \breve{E}\right)$.

Let $\left(\breve{X},\overset{1}{\breve{\tau}},\overset{2}{\breve{\tau}},\overset{3}{\breve{\tau}},\overset{4}{\breve{\tau}},E\right)$ be a soft quad topological space over X, where $\left(\breve{X},\overset{1}{\breve{\tau}},\breve{E}\right)$, $\left(\breve{X},\overset{2}{\breve{\tau}},\breve{E}\right)$,$\left(\breve{X},\overset{3}{\breve{\tau}},\breve{E}\right)$ and $\left(\breve{X},\overset{4}{\breve{\tau}},\breve{E}\right)$ be four different soft topologies on $\breve{X}$. Then a sub set $(F,\breve{E})$ is said to be quad-open (in short h and q-open) if $(F,\breve{E}) \subseteq \overset{1}{\breve{\tau}} \cup \overset{2}{\breve{\tau}} \cup \overset{3}{\breve{\tau}} \cup \overset{4}{\breve{\tau}}$ and its complement is said to be soft q-closed.

3. 1 Soft Regular Separation Axioms of Soft Quad Topological Spaces with Respect to Ordinary Points

In this section we introduced soft semi separation axioms in soft quad topological space with respect to ordinary points and discussed some attractive results with respect to these points in detail.

Definition 22 Let $\left(\breve{X},\overset{1}{\breve{\tau}},\overset{2}{\breve{\tau}},\overset{3}{\breve{\tau}},\overset{4}{\breve{\tau}},\breve{E}\right)$ be a soft quad topological space over $\breve{X}$ and $x,y \in \breve{X}$ such that $x \neq y$. if we can find soft q-open sets $(F,\breve{E})$ and (G,E) such that $x \in (F,\breve{E})$ and $y \notin (F,\breve{E})$ or $y \in (G,\breve{E})$ and $x \notin (G,\breve{E})$ $then$ $\left(\breve{X},\overset{1}{\breve{\tau}},\overset{2}{\breve{\tau}},\overset{3}{\breve{\tau}},\overset{4}{\breve{\tau}},\breve{E}\right)$ is called soft qT_0 space.

Definition 23 Let $\left(\breve{X},\overset{1}{\breve{\tau}},\overset{2}{\breve{\tau}},\overset{3}{\breve{\tau}},\overset{4}{\breve{\tau}},\breve{E}\right)$ be a soft quad topological space over X and $x,y \in X$ such that $x \neq y$ if we can find two soft q-open sets $(F,\breve{E})$ and $(G,\breve{E})$ such that $x \in (F,\breve{E})$ and $y \notin (F,\breve{E})$ and $y \in (G,\breve{E})$ and $x \notin (G,\breve{E})$ then $\left(\breve{X},\overset{1}{\breve{\tau}},\overset{2}{\breve{\tau}},\overset{3}{\breve{\tau}},\overset{4}{\breve{\tau}},\breve{E}\right)$ is called soft qT_1 space.

Definition 22 Let $\left(\breve{X},\overset{1}{\breve{\tau}},\overset{2}{\breve{\tau}},\overset{3}{\breve{\tau}},\overset{4}{\breve{\tau}},\breve{E}\right)$be a soft quad topological space over X and $x,y \in X$ such that $x \neq y$. If we can find two q-open soft sets such that $x \in (F,\breve{E})$ and $y \in (G,\breve{E})$ moreover $(F,\breve{E}) \cap (G,\breve{E}) = \phi$. Then $(X,\tau_1,\tau_2,\tau_3,\tau_4,E)$ is called a soft qT_2 space.

Definition 25 Let $\left(\breve{X},\overset{1}{\breve{\tau}},\overset{2}{\breve{\tau}},\overset{3}{\breve{\tau}},\overset{4}{\breve{\tau}},\breve{E}\right)$ be a soft topological space $(G,\breve{E})$ be q-closed soft set in X and $x \in X_A$ such that $x \notin (G,\breve{E})$. If there occurs soft q-open sets $(F_1,\breve{E})$ and $(F_2,\breve{E})$ such that $x \in (F_1,\breve{E})$, $(G,\breve{E}) \subseteq (F_2,\breve{E})$ and $(F_1,\breve{E}) \cap (F_2,\breve{E}) = \varphi$. Then $\left(\breve{X},\overset{1}{\breve{\tau}},\overset{2}{\breve{\tau}},\overset{3}{\breve{\tau}},\overset{4}{\breve{\tau}},\breve{E}\right)$is called soft q-regular spaces. A soft q-regular qT_1 $Space$ is called soft qT_3 space.

Then $\left(\breve{X},\overset{1}{\breve{\tau}},\overset{2}{\breve{\tau}},\overset{3}{\breve{\tau}},\overset{4}{\breve{\tau}},\breve{E}\right)$ is called a soft q-regular spaces. A soft q-regular $T_1\,Space.$ is called soft qT_3 space.

Definition 26 $\left(\breve{X},\overset{1}{\breve{\tau}},\overset{2}{\breve{\tau}},\overset{3}{\breve{\tau}},\overset{4}{\breve{\tau}},\breve{E}\right)$ be a *soft quad topological* space $(F_1,\breve{E})$, $(G,\breve{E})$ be closed soft sets in X such that $(F,\breve{E})\cap(G,\breve{E})=\varphi$ if there exists q-open soft sets $(F_1,\breve{E})$ and $(F_2,\breve{E})$ such that $(F,\breve{E})\subseteq(F_1,\breve{E}),(G,\breve{E})\subseteq(F_2,\breve{E})$ and $(F_1,\breve{E})\cap(F_2,\breve{E})=\varphi$. Then $\left(\breve{X},\overset{1}{\breve{\tau}},\overset{2}{\breve{\tau}},\overset{3}{\breve{\tau}},\overset{4}{\breve{\tau}},\breve{E}\right)$ is called a q-soft normal space. A soft q-normal $qT_1\,Space$ is called soft $qT_4\,Space.$

Definition 27 Let $\left(\breve{X},\overset{1}{\breve{\tau}},\overset{2}{\breve{\tau}},\overset{3}{\breve{\tau}},\overset{4}{\breve{\tau}},\breve{E}\right)$ be a soft topological space over X and $e_G,e_H\in X_A$ such that $e_G\neq e_H$ if there can happen at least one soft q-open set $(F_1,\breve{A})$ or $(F_2,\breve{A})$ such that $e_G\in(F_1,\breve{A}),e_H\notin(F_1,\breve{A})$ or $e_H\in(F_2,\breve{A}),e_G\notin((F_2,\breve{A})$ then $\left(\breve{X},\overset{1}{\breve{\tau}},\overset{2}{\breve{\tau}},\overset{3}{\breve{\tau}},\overset{4}{\breve{\tau}},\breve{E}\right)$ is called a *soft* qT_0 *space.*

Definition 28 Let $\left(\breve{X},\overset{1}{\breve{\tau}},\overset{2}{\breve{\tau}},\overset{3}{\breve{\tau}},\overset{4}{\breve{\tau}},\breve{E}\right)$ be a soft topological spaces over X and $e_G,e_H\in X_{\breve{A}}$ such that $e_G\neq e_H$ if there can happen *soft q-open sets* $(F_1,\breve{A})$ and $(F_2,\breve{A})$ such that $e_G\in(F_1,\breve{A}),e_H\notin(F_1,\breve{A})$ and $e_H\in(F_2,\breve{A}),e_G\notin((F_2,\breve{A})$ then $\left(\breve{X},\overset{1}{\breve{\tau}},\overset{2}{\breve{\tau}},\overset{3}{\breve{\tau}},\overset{4}{\breve{\tau}},\breve{E}\right)$ is called soft $qT_1\,space.$

Definition 29 Let $\left(\breve{X},\overset{1}{\breve{\tau}},\overset{2}{\breve{\tau}},\overset{3}{\breve{\tau}},\overset{4}{\breve{\tau}},\breve{E}\right)$ be a *soft topological* space over X and $e_G,e_H\in X_A$ such that $e_G\neq e_H$ if there can happen *soft q-open sets* $(F_1,\breve{A})$ and $(F_2,\breve{A})$ such that $e_G\in(F_1,\breve{A})$, and $e_H\in(F_2,\breve{A})$, $(F_1,\breve{A})\cap(F_2,\breve{A})=\phi_A$. Then $\left(\breve{X},\overset{1}{\breve{\tau}},\overset{2}{\breve{\tau}},\overset{3}{\breve{\tau}},\overset{4}{\breve{\tau}},\breve{E}\right)$ is called soft $qT_2\,space.$

Definition 30 Let $\left(\breve{X},\overset{1}{\breve{\tau}},\overset{2}{\breve{\tau}},\overset{3}{\breve{\tau}},\overset{4}{\breve{\tau}},\breve{E}\right)$ be a soft topological space $(G,\breve{E})$ be q-closed soft set in X and $e_G\in X_{\breve{A}}$ such that $e_G\notin(G,\breve{E})$. if there occurs soft q-open sets $(F_1,\breve{E})$ and $(F_2,\breve{E})$ such that $e_G\in(F_1,\breve{E})$, $(G,\breve{E})\subseteq(F_2,\breve{E})$ and $(F_1,\breve{E})\cap(F_2,E)=\breve{\emptyset}$. Then $\left(\breve{X},\overset{1}{\breve{\tau}},\overset{2}{\breve{\tau}},\overset{3}{\breve{\tau}},\overset{4}{\breve{\tau}},\breve{E}\right)$ is called soft q-regular spaces. A soft q-regular $qT_1\,Space$ is called soft qT_3 space.

Definition 31 In a *soft quad topological* space $\left(\breve{X}, \overset{1}{\breve{\tau}}, \overset{2}{\breve{\tau}}, \overset{3}{\breve{\tau}}, \overset{4}{\breve{\tau}}, \breve{E}\right)$

1) $\overset{1}{\breve{\tau}} \cup \overset{2}{\breve{\tau}}$ is said to be *soft regular* T_0 space with respect to $\overset{3}{\breve{\tau}} \cup \overset{4}{\breve{\tau}}$ if for each pair of points $x, y \in \breve{X}$ such that $x \neq y$ there exists $\overset{1}{\breve{\tau}} \cup \overset{2}{\breve{\tau}}$ soft *regular* open set $(F, \breve{E})$ and a to $\overset{3}{\breve{\tau}} \cup \overset{4}{\breve{\tau}}$ soft *regular* open set $(G, \breve{E})$ such that $x \in (F, \breve{E})$ and $y \notin (G, \breve{E})$ or $y \in (G, \breve{E})$ and $x \notin (G, \breve{E})$ similarly, to $\overset{3}{\breve{\tau}} \cup \overset{4}{\breve{\tau}}$is said to be *soft regular* T_0 space with respect to $\overset{1}{\breve{\tau}} \cup \overset{2}{\breve{\tau}}$ if for each pair of points $x, y \in \breve{X}$such that $x \neq y$ there exists $\overset{3}{\breve{\tau}} \cup \overset{4}{\breve{\tau}}$ *regular open set* $(F, \breve{E})$ and $\overset{1}{\breve{\tau}} \cup \overset{2}{\breve{\tau}}$ soft *regular open* set $(G, \breve{E})$ such that $x \in (F, \breve{E})$ and $y \notin (F, \breve{E})$ or $y \in (G, \breve{E})$ and $x \notin (G, \breve{E})$. *soft quad topological spaces* $\left(\breve{X}, \overset{1}{\breve{\tau}}, \overset{2}{\breve{\tau}}, \overset{3}{\breve{\tau}}, \overset{4}{\breve{\tau}}, \breve{E}\right)$is said to be *pair wise soft regular* T_0 space if $\overset{1}{\breve{\tau}} \cup \overset{2}{\breve{\tau}}$ is *soft regular* T_0 *space* with respect to $\overset{3}{\breve{\tau}} \cup \overset{4}{\breve{\tau}}$ and $\overset{3}{\breve{\tau}} \cup \overset{4}{\breve{\tau}}$ and is *soft regular* T_0 space with respect to $\overset{1}{\breve{\tau}} \cup \overset{2}{\breve{\tau}}$.

2) $\overset{1}{\breve{\tau}} \cup \overset{2}{\breve{\tau}}$ is said to be soft regular T_1 space with respect to $\overset{3}{\breve{\tau}} \cup \overset{4}{\breve{\tau}}$ if for each pair of points $x, y \in X$ such that $x \neq y$ there exists $\overset{1}{\breve{\tau}} \cup \overset{2}{\breve{\tau}}$ *soft regular open set* $(F, \breve{E})$ and to $\overset{3}{\breve{\tau}} \cup \overset{4}{\breve{\tau}}$ soft *regular open* set $(G, \breve{E})$ such that $x \in (F, \breve{E})$ and $y \notin (G, \breve{E})$ and $y \in (G, \breve{E})$ and $x \notin (G, \breve{E})$. Similarly, $\overset{3}{\breve{\tau}} \cup \overset{4}{\breve{\tau}}$is said to be soft regular T_1 space with respect to $\overset{1}{\breve{\tau}} \cup \overset{2}{\breve{\tau}}$ if for each pair of distinct points $x, y \in X$ such that $x \neq y$ there exists$\overset{3}{\breve{\tau}} \cup \overset{4}{\breve{\tau}}$ soft *regular open* set $(F, \breve{E})$ and a to $\overset{1}{\breve{\tau}} \cup \overset{2}{\breve{\tau}}$ soft *regular open* set $(G, \breve{E})$ such that $x \in (F, \breve{E})$ and $y \notin (F, \breve{E})$ and $y \in (G, \breve{E})$ and $x \notin (G, \breve{E})$. *soft quad topological spaces* ($\left(\breve{X}, \overset{1}{\breve{\tau}}, \overset{2}{\breve{\tau}}, \overset{3}{\breve{\tau}}, \overset{4}{\breve{\tau}}, \breve{E}\right)$is said to be *pair wise soft regular* T_1 space if $\overset{1}{\breve{\tau}} \cup \overset{2}{\breve{\tau}}$is *soft regular* T_1 *space* with respect to $\overset{3}{\breve{\tau}} \cup \overset{4}{\breve{\tau}}$ and to $\overset{3}{\breve{\tau}} \cup \overset{4}{\breve{\tau}}$is *soft regular* T_1 space with respect to $\overset{3}{\breve{\tau}} \cup \overset{4}{\breve{\tau}}$.

3) $\overset{1}{\breve{\tau}} \cup \overset{2}{\breve{\tau}}$is said to be soft regular T_2 space with respect to $\overset{1}{\breve{\tau}} \cup \overset{2}{\breve{\tau}}$ if for each pair of points $x, y \in \breve{X}$ such that $x \neq y$ there exists a $\overset{1}{\breve{\tau}} \cup \overset{2}{\breve{\tau}}$ soft *regular* open set $(F, \breve{E})$ and a $\overset{3}{\breve{\tau}} \cup \overset{4}{\breve{\tau}}$ soft *regular open* set $(G, \breve{E})$ such that $x \in (F, \breve{E})$ and $y \in (G, \breve{E})$, $\left(F, \breve{E}\right) \cap \left(G, \breve{E}\right) = \phi$. Similarly, $\overset{3}{\breve{\tau}} \cup \overset{4}{\breve{\tau}}$is said to be *soft regular* T_2 space with respect to $\overset{1}{\breve{\tau}} \cup \overset{2}{\breve{\tau}}$ if for each pair of points $x, y \in \breve{X}$ such that $x \neq y$there exists a $\overset{3}{\breve{\tau}} \cup \overset{4}{\breve{\tau}}$ *soft regular open* set (F, E) and a $\overset{1}{\breve{\tau}} \cup \overset{2}{\breve{\tau}}$ *soft regular open* set $(G, \breve{E})$ such that $x \in \left(F, \breve{E}\right), y \in (G, \breve{E})$

and $\left(F,\breve{E}\right)\cap\left(G,\breve{E}\right)=\phi$. The soft quad totopological space $\left(\breve{X},\overset{1}{\breve{\tau}},\overset{2}{\breve{\tau}},\overset{3}{\breve{\tau}},\overset{4}{\breve{\tau}},\breve{E}\right)$is said to be pair wise soft regular T_2 space if $\overset{1}{\breve{\tau}}\cup\overset{2}{\breve{\tau}}$is soft regular T_2 space with respect to $\overset{3}{\breve{\tau}}\cup\overset{4}{\breve{\tau}}$ and $\overset{3}{\breve{\tau}}\cup\overset{4}{\breve{\tau}}$ is soft regular T_2 space with respect to $\overset{1}{\breve{\tau}}\cup\overset{2}{\breve{\tau}}$

Definition 32 In a soft quad topological space $\left(\breve{X},\overset{1}{\breve{\tau}},\overset{2}{\breve{\tau}},\overset{3}{\breve{\tau}},\overset{4}{\breve{\tau}},\breve{E}\right)$

1) $\overset{1}{\breve{\tau}}\cup\overset{2}{\breve{\tau}}$ is said to be soft *regular* qT_3 space with respect to a $\tau_3\cup\tau_4$ if $\overset{1}{\breve{\tau}}\cup\overset{2}{\breve{\tau}}$is soft *regular* T_1 space with respect to $\overset{3}{\breve{\tau}}\cup\overset{4}{\breve{\tau}}$ and for each pair of points $x,y\in\breve{X}$such that $x\neq y$ there exists $\overset{1}{\breve{\tau}}\cup\overset{2}{\breve{\tau}}$. soft *regular* closed set $(G,\breve{E})$ such that $x\notin(G,\breve{E})$, a$\overset{1}{\breve{\tau}}\cup\overset{2}{\breve{\tau}}$ soft *regular open* set $(F_1,\breve{E})$ and $\overset{3}{\breve{\tau}}\cup\overset{4}{\breve{\tau}}$ soft *regular* open set $(F_2,\breve{E})$ such that $x\in\left(F_1,\breve{E}\right),(G,\breve{E})\subseteq(F_2,\breve{E})$ and $\left(F_1,\breve{E}\right)\cap\left(F_2,\breve{E}\right)=\emptyset$. Similarly,$\overset{3}{\breve{\tau}}\cup\overset{4}{\breve{\tau}}$is said to be soft regular T_3 space with respect to $\overset{1}{\breve{\tau}}\cup\overset{2}{\breve{\tau}}$ if $\overset{3}{\breve{\tau}}\cup\overset{4}{\breve{\tau}}$ is soft regular T_1 space with respect to $\tau_1\cup\tau_2$ and for each pair of points $x,y\in\breve{X}$such that $x\neq y$ there exists a $\tau_3\cup\tau_4$ soft *regular* closed set $(G,\breve{E})$ such that $x\notin(G,\breve{E})$, $\tau_3\cup\tau_4$ soft regular open set $(F_1,\breve{E})$ and $\overset{1}{\breve{\tau}}\cup\overset{2}{\breve{\tau}}$ soft *regular open* set $(F_2,\breve{E})$ such that $x\in\left(F_1,\breve{E}\right),(G,\breve{E})\subseteq(F_2,\breve{E})$ and $\left(F_1,\breve{E}\right)\cap\left(F_2,\breve{E}\right)=\phi$. $\left(\breve{X},\overset{1}{\breve{\tau}},\overset{2}{\breve{\tau}},\overset{3}{\breve{\tau}},\overset{4}{\breve{\tau}},\breve{E}\right)$is said to be pair wise soft regular T_3 space if $\overset{1}{\breve{\tau}}\cup\overset{2}{\breve{\tau}}$ is soft regular T_3 space with respect to $\overset{3}{\breve{\tau}}\cup\overset{4}{\breve{\tau}}$ and $\overset{3}{\breve{\tau}}\cup\overset{4}{\breve{\tau}}$is soft s *regular* T_3 space with respect to $\overset{1}{\breve{\tau}}\cup\overset{2}{\breve{\tau}}$.

2) $\overset{1}{\breve{\tau}}\cup\overset{2}{\breve{\tau}}$is said to be soft regular T_4 space with respect to $\overset{3}{\breve{\tau}}\cup\overset{4}{\breve{\tau}}$ if $\overset{1}{\breve{\tau}}\cup\overset{2}{\breve{\tau}}$is soft regular T_1 space with respect to $\overset{3}{\breve{\tau}}\cup\overset{4}{\breve{\tau}}$, there exists a $\overset{1}{\breve{\tau}}\cup\overset{2}{\breve{\tau}}$ soft *regular* closed set $(F_1,\breve{E})$ and $\overset{3}{\breve{\tau}}\cup\overset{4}{\breve{\tau}}$ soft *regular* closed set $(F_2,\breve{E})$ such that $\left(F_1,\breve{E}\right)\cap\left(F_2,\breve{E}\right)=\emptyset$. Also there exists $(F_3,\breve{E})$ and $(G_1,\breve{E})$ such that $(F_3,\breve{E})$ is soft $\overset{1}{\breve{\tau}}\cup\overset{2}{\breve{\tau}}$ *regular open* set, $(G_1,\breve{E})$ is soft $\overset{3}{\breve{\tau}}\cup\overset{4}{\breve{\tau}}$ *regular open* set such that $\left(F_1,\breve{E}\right)\subseteq\left(F_3,\breve{E}\right)$, $\left(F_2,\breve{E}\right)\subseteq\left(G_1,\breve{E}\right)$. Similarly,$\overset{3}{\breve{\tau}}\cup\overset{4}{\breve{\tau}}$is said to be soft regular T_4 space with respect to $\overset{1}{\breve{\tau}}\cup\overset{2}{\breve{\tau}}$ if $\overset{3}{\breve{\tau}}\cup\overset{4}{\breve{\tau}}$is soft regular T_1 space with respect to $\overset{1}{\breve{\tau}}\cup\overset{2}{\breve{\tau}}$, there exists $\overset{3}{\breve{\tau}}\cup\overset{4}{\breve{\tau}}$ soft *regular* closed set $(F_1,\breve{E})$ and $\overset{1}{\breve{\tau}}\cup\overset{2}{\breve{\tau}}$ soft *regular* closed set $(F_2,\breve{E})$ such that $\left(F_1,\breve{E}\right)\cap\left(F_2,\breve{E}\right)=\phi$. Also there exist $(F_3,\breve{E})$ and $(G_1,\breve{E})$ such that $(F_3,\breve{E})$ is soft $\overset{3}{\breve{\tau}}\cup\overset{4}{\breve{\tau}}$ *regular* i open set, $(G_1,\breve{E})$ is soft $\overset{1}{\breve{\tau}}\cup\overset{2}{\breve{\tau}}$ *regular open* set such that $\left(F_1,\breve{E}\right)\subseteq$

$(F_3,\breve{E}),(F_2,\breve{E}) \subseteq (G_1,\breve{E})$ and $(F_3,\breve{E}) \cap (G_1,\breve{E}) = \phi$. Thus, $(\breve{X},\overset{1}{\tau},\overset{2}{\tau},\breve{E})$ is said to be pair wise soft regular T_4 space if $\overset{1}{\tau} \cup \overset{2}{\tau}$ is soft regular T_4 space with respect to $\overset{3}{\tau} \cup \overset{4}{\tau}$ and $\tau_3 \cup \tau_4$ is soft regular T_4 space with respect to $\overset{1}{\tau} \cup \overset{2}{\tau}$.

Proposition 1. Let $(\breve{X},\tau,\breve{E})$ be a soft topological space over X. if $(\breve{X},\tau,\breve{E})$ is soft-R_3-space , then for all $x \in \breve{X}, x_E = (x,\breve{E})$ is regular-closed soft set.

Proof. We want to prove that $x_{\breve{E}}$is regular-closed soft set, which is sufficient to prove that $x_{\breve{E}}^c$ is regular soft-open set for all $y \in \{x\}^c$. Since $(X,\tau,\breve{E})$ is soft R_3-space , then there exists soft regular set sets $(F,\breve{E})_Y$ and (G,E) such that $y_{\breve{E}} \subseteq (F,\breve{E})_Y$ and $x_{\breve{E}} \cap (F,\breve{E})_Y = \phi$ and $x_{\breve{E}} \subseteq (G,\breve{E})$ and $y_{\breve{E}} \cap (G,\breve{E}) = \phi$. It follows that , $\cup_{y \in (x)^c}(F,\breve{E})_y \subseteq x_{\breve{E}}^c$. Now, we want to prove that $x_{\breve{E}}^c \subseteq \cup_{y \in (x)^c} (F,\breve{E})_y$. Let $\cup_{y \in (x)^c} (F,E)_y = (H,E)$. where $H(e) = \cup_{y \in (x)^c} (F(e))_y$for all $e \in \breve{E}$. Since, $x_{\breve{E}}^c(e) = \{x\}^c$ for all $e \in \breve{E}$from Definition 6, so, for all $y \in \{x\}^c$ and $e \in \breve{E} x_{\breve{E}}^c(e) = \{x\}^c = \cup_{y \in (x)^c} \{y\} = \cup_{y \in (x)^c} F(e)_y = H(e)$. Thus, $x_{\breve{E}}^c \subseteq \cup_{y \in (x)^c} (F,\breve{E})_y$ from Definition 2, and so$x_{\breve{E}}^c = \cup_{y \in (x)^c} (F,\breve{E})_y$. This means that , $x_{\breve{E}}^c$ is soft regular-open set for all$y \in \{x\}^c$. Hence $x_{\breve{E}}$ is soft regular-closed set.

Proposition 2. Let $(\breve{Y},\overset{Y}{\tau},\breve{E})$ be a soft sub space of a soft topological space $(\breve{X},\tau,\breve{E})$ and $(F,\breve{E}) \in SS(\breve{X})$ then,

1. if $(F,\breve{E})$ is soft regular open *soft* set in $\breve{Y}$ and $\breve{Y} \in \tau$, then $(F,\breve{E}) \in \tau$.

2. $(F,\breve{E})$ is soft regular *open soft* set in $\breve{Y}$ if and only if $(F,\breve{E}) = Y \cap (G,\breve{E})$ for some $(G,\breve{E}) \in \tau$.

3. $(F,\breve{E})$ is soft regular *closed soft* set in$\breve{Y}$ if and only if $(F,\breve{E}) = \breve{Y} \cap (H,\breve{E})$ for some $(H,\breve{E})$ *is* τ *soft* regular *close* set.

Proof. 1) Let $(F,\breve{E})$ be a soft regular open set in $\breve{Y}$, then there does exists a soft regular open set (G,E) in $\breve{X}$ such that $(F,\breve{E}) = \breve{Y} \cap (G,\breve{E})$. Now, if $\breve{Y} \in \tau$ then $\breve{Y} \cap (G,\breve{E}) \in \tau$ by the third condition of the definition of a soft topological space and hence $(F,\breve{E}) \in \tau$.

2) Fallows from the definition of a soft subspace.

3) if $(F,\breve{E})$ is soft regular closed in Y then we have $(F,\breve{E}) = \breve{Y} \setminus (G,\breve{E})$, for some $(G,\breve{E}) \in \tau_{\breve{Y}}$. Now,$(G,\breve{E}) = \breve{Y} \cap (H,\breve{E})$ for some soft regular open set $(H,\breve{E}) \in \tau$. for any$\alpha \in \breve{E}$. $F(\alpha) = \breve{Y}(\alpha) \setminus G(\alpha) = \breve{Y} \setminus G(\alpha) = Y \setminus (Y(\alpha) \cap H(\alpha)) = \breve{Y} \setminus (\breve{Y} \cap H(\alpha)) = Y \setminus H(\alpha) = \breve{Y} \cap (X \setminus H(\alpha)) = \breve{Y} \cap (H(\alpha))^C = \breve{Y}(\alpha) \cap (H(\alpha))^C$. Thus $(F,\breve{E}) = \breve{Y} \cap (H,\breve{E})^{/}$is soft regular closed in $\breve{X}$ as$(H,\breve{E}) \in \tau$. Conversely, suppose that $(F,\breve{E}) = \breve{Y} \cap$

$(G,\breve{E})$ for some soft regular closed set (G,E) in $\breve{X}$. This qualifies us to say that $(G,\breve{E})^{/} \in \tau$. Now, if $(G,\breve{E}) = (X,\breve{E}) \setminus (H,\breve{E})$ where $(H,\breve{E})$ is soft regular open.

Proposition 3. Let $\left(\breve{X},\overset{1}{\breve{\tau}},\overset{2}{\breve{\tau}},\overset{3}{\breve{\tau}},\overset{4}{\breve{\tau}},\breve{E}\right)$ be a soft quad topological space over $\breve{X}$. Then, if $\left(\breve{X},\overset{1}{\breve{\tau}},\overset{2}{\breve{\tau}},\breve{E}\right)$ and $\left(\breve{X},\overset{3}{\breve{\tau}},\overset{4}{\breve{\tau}},\breve{E}\right)$ are soft regular T_3 space then $\left(\breve{X},\overset{1}{\breve{\tau}},\overset{2}{\breve{\tau}},\overset{3}{\breve{\tau}},\overset{4}{\breve{\tau}},\breve{E}\right)$ is a pair wise soft regular T_2 space.

Proof. Suppose $\left(\breve{X},\overset{1}{\breve{\tau}},\overset{2}{\breve{\tau}},\breve{E}\right)$ is a soft regular T_3 space with respect to $\left(\breve{X},\overset{3}{\breve{\tau}},\overset{4}{\breve{\tau}},\breve{E}\right)$ then according to definition for $x,y \in \breve{X}$, which distinct, by using Proposition 1, $(\breve{Y},\breve{E})$ is soft *regular* closed set in $\overset{3}{\breve{\tau}} \cup \overset{4}{\breve{\tau}}$ and $x \notin (\breve{Y},\breve{E})$ there exists a$\overset{1}{\breve{\tau}} \cup \overset{2}{\breve{\tau}}$ soft *regular* open set $(F,\breve{E})$ and a $\overset{3}{\breve{\tau}} \cup \overset{4}{\breve{\tau}}$ soft *regular* open set $(G,\breve{E})$ such that $x \in (F,\breve{E}), y \in (Y,\breve{E}) \subseteq (G,\breve{E})$ and $(F_1,\breve{E}) \cap (F_2,\breve{E}) = \phi$. Hence $\overset{1}{\breve{\tau}} \cup \overset{2}{\breve{\tau}}$is soft regular T_2 space with respect to $\overset{3}{\breve{\tau}} \cup \overset{4}{\breve{\tau}}$. Similarly, if $\left(\breve{X},\overset{3}{\breve{\tau}},\overset{4}{\breve{\tau}},\breve{E}\right)$ is a soft regular T_3 space with respect to $\left(\breve{X},\overset{1}{\breve{\tau}},\overset{2}{\breve{\tau}},\breve{E}\right)$ then according to definition for $x,y \in \breve{X}, x \neq y$, by using Theorem 2, (x, $\breve{E}$) is *regular* closed soft set in $\overset{1}{\breve{\tau}} \cup \overset{2}{\breve{\tau}}$ and $y \notin (x,E)$ there exists a $\overset{3}{\breve{\tau}} \cup \overset{4}{\breve{\tau}}$ soft *regular open* set $(F,\breve{E})$ and a $\overset{1}{\breve{\tau}} \cup \overset{2}{\breve{\tau}}$ soft *regular open* set $(G,\breve{E})$ such that $y \in (F,\breve{E})$, $x \in (x,\breve{E}) \subseteq (G,\breve{E})$ and $(F_1,\breve{E}) \cap (F_2,\breve{E}) = \phi$. Hence $\overset{3}{\breve{\tau}} \cup \overset{4}{\breve{\tau}}$is soft regular T_2 space. This implies that $\left(\breve{X},\overset{1}{\breve{\tau}},\overset{2}{\breve{\tau}},\overset{3}{\breve{\tau}},\overset{4}{\breve{\tau}},\breve{E}\right)$is a pair wise soft regular T_2 space.

Proposition 4. Let $\left(\breve{X},\overset{1}{\breve{\tau}},\overset{2}{\breve{\tau}},\overset{3}{\breve{\tau}},\overset{4}{\breve{\tau}},\breve{E}\right)$be a soft quad topological space over $\breve{X}$. $\left(\breve{X},\overset{1}{\breve{\tau}},\overset{2}{\breve{\tau}},\breve{E}\right)$ and $\left(\breve{X},\overset{3}{\breve{\tau}},\overset{4}{\breve{\tau}},\breve{E}\right)$ are soft regular T_3 space then $\left(\breve{X},\overset{1}{\breve{\tau}},\overset{2}{\breve{\tau}},\overset{3}{\breve{\tau}},\overset{4}{\breve{\tau}},\breve{E}\right)$is a pair wise soft regular T_3 space.

Proof. Suppose $\left(\breve{X},\overset{1}{\breve{\tau}},\overset{2}{\breve{\tau}},\breve{E}\right)$ is a soft regular T_3 space with respect to $\left(\breve{X},\overset{1}{\breve{\tau}},\overset{2}{\breve{\tau}},\breve{E}\right)$ then according to definition for $x,y \in X, x \neq y$ there exists a $\left(\breve{X},\overset{1}{\breve{\tau}},\overset{2}{\breve{\tau}},\breve{E}\right)$ soft *regular open* set $(F,\breve{E})$ and a $\left(\breve{X},\overset{3}{\breve{\tau}},\overset{4}{\breve{\tau}},\breve{E}\right)$ soft *regular open* set $(G,\breve{E})$ such that $x \in (F,\breve{E})$ and $y \notin (F,\breve{E})$ or $y \in (G,\breve{E})$ and $x \notin (G,\breve{E})$ and for each point $x \in X$ and each $\left(\breve{X},\overset{1}{\breve{\tau}},\overset{2}{\breve{\tau}},\breve{E}\right)$ *regular* closed soft set $(G_1,\breve{E})$ such that $x \notin (G_1,\breve{E})$ there exists

$\left(\breve{X},\overset{1}{\breve{\tau}},\overset{2}{\breve{\tau}},\breve{E}\right)$ soft *regular* open set $(F_1,\breve{E})$ and $\left(\breve{X},\overset{3}{\breve{\tau}},\overset{4}{\breve{\tau}},\breve{E}\right)$ soft *regular* open set (F_2,E) such that $x\in\left(F_1,\breve{E}\right),(G_1,\breve{E})\subseteq(F_2,\breve{E})$ and $\left(F_1,\breve{E}\right)\cap\left(F_2,\breve{E}\right)=\phi$. Similarly, to $\left(\breve{X},\overset{3}{\breve{\tau}},\overset{4}{\breve{\tau}},\breve{E}\right)$ is a soft regular T_3 space with respect to $\left(\breve{X},\overset{1}{\breve{\tau}},\overset{2}{\breve{\tau}},\breve{E}\right)$ So according to definition for $x,y\in X,x\neq y$ there exists a $\left(\breve{X},\overset{3}{\breve{\tau}},\overset{4}{\breve{\tau}},\breve{E}\right)$ soft *regular* open set $(F,\breve{E})$ and a $\left(\breve{X},\overset{1}{\breve{\tau}},\overset{2}{\breve{\tau}},\breve{E}\right)$ soft *regular open* set (G,E) such that $x\in(F,\breve{E})$ and $y\notin(F,\breve{E})$ or $y\in(G,\breve{E})$ and $x\notin(G,\breve{E})$ and for each point $x\in\breve{X}$ and each $\left(\breve{X},\overset{3}{\breve{\tau}},\overset{4}{\breve{\tau}},\breve{E}\right)$ *regular* closed soft set $(G_1,\breve{E})$ such that $x\notin(G_1,\breve{E})$ there exists $\left(\breve{X},\overset{3}{\breve{\tau}},\overset{4}{\breve{\tau}},\breve{E}\right)$ soft *regular open* set $\left(F_1,\breve{E}\right)$ and a $\left(\breve{X},\overset{1}{\breve{\tau}},\overset{2}{\breve{\tau}},\breve{E}\right)$ soft semi open set $\left(F_2,\breve{E}\right)$ such that $x\in\left(F_1,\breve{E}\right),(G_1,\breve{E})\subseteq(F_2,\breve{E})$ and $\left(F_1,\breve{E}\right)\cap\left(F_2,\breve{E}\right)=\phi$. Hence $\left(\breve{X},\overset{1}{\breve{\tau}},\overset{2}{\breve{\tau}},\overset{3}{\breve{\tau}},\overset{4}{\breve{\tau}},\breve{E}\right)$is pair wise soft regular T_3 space.

Proposition 5. If $\left(\breve{X},\overset{1}{\breve{\tau}},\overset{2}{\breve{\tau}},\overset{3}{\breve{\tau}},\overset{4}{\breve{\tau}},\breve{E}\right)$ be a soft quad topological space over X. $\left(\breve{X},\overset{1}{\breve{\tau}},\overset{2}{\breve{\tau}},\breve{E}\right)$ and $\left(\breve{X},\overset{3}{\breve{\tau}},\overset{4}{\breve{\tau}},\breve{E}\right)$ are soft regular T_4 space then $\left(\breve{X},\overset{1}{\breve{\tau}},\overset{2}{\breve{\tau}},\overset{3}{\breve{\tau}},\overset{4}{\breve{\tau}},\breve{E}\right)$ is pair wise soft regular T_4 space.

Proof. Suppose $\left(\breve{X},\overset{1}{\breve{\tau}},\overset{2}{\breve{\tau}},\breve{E}\right)$ is soft regular T_4 space with respect to $\left(\breve{X},\overset{3}{\breve{\tau}},\overset{4}{\breve{\tau}},\breve{E}\right)$ So according to definition for $x,y\in\breve{X},x\neq y$ there exist a $\left(\breve{X},\overset{1}{\breve{\tau}},\overset{2}{\breve{\tau}},\breve{E}\right)$ soft *regular open* set $(F,\breve{E})$ and a $\left(\breve{X},\overset{3}{\breve{\tau}},\overset{4}{\breve{\tau}},\breve{E}\right)$ soft *regular* open set $(G,\breve{E})$ such that $x\in(F,\breve{E})$ and $y\notin(F,\breve{E})$ or $y\in(G,\breve{E})$ and $x\notin(G,\breve{E})$ each $\left(\breve{X},\overset{1}{\breve{\tau}},\overset{2}{\breve{\tau}},\breve{E}\right)$ soft *regular* closed set $(F_1,\breve{E})$ and a $\left(\breve{X},\overset{3}{\breve{\tau}},\overset{4}{\breve{\tau}},\breve{E}\right)$ soft *regular* closed set $(F_2,\breve{E})$ such that $\left(F_1,\breve{E}\right)\cap\left(F_2,\breve{E}\right)=\phi$. There exist (F_3,E) and $\left(G_1,\breve{E}\right)$ such that $\left(F_3,\breve{E}\right)$ is soft $\left(\breve{X},\overset{3}{\breve{\tau}},\overset{4}{\breve{\tau}},\breve{E}\right)$ and soft regular open set $(G_1,\breve{E})$ is soft $\left(\breve{X},\overset{1}{\breve{\tau}},\overset{2}{\breve{\tau}},\breve{E}\right)$ *regular open* set $\left(F_1,\breve{E}\right)\subseteq\left(F_3,\breve{E}\right),(F_2,\breve{E})\subseteq(G_1,\breve{E})$ and $\left(F_3,\breve{E}\right)\cap\left(G_1,\breve{E}\right)=\phi$. Similarly, $\left(\breve{X},\overset{3}{\breve{\tau}},\overset{4}{\breve{\tau}},\breve{E}\right)$ is soft regular T_4 space with respect to τ_1 so according to definition for $x,y\in\breve{X},x\neq y$ there exists a $\left(X,\tau_3,\tau_4,\breve{E}\right)$ soft *regular* open set $(F,\breve{E})$ and a $\left(\breve{X},\overset{1}{\breve{\tau}},\overset{2}{\breve{\tau}},\breve{E}\right)$ soft *regular open* set $(G,\breve{E})$ such that $x\in(F,\breve{E})$ and $y\notin(F,\breve{E})$ or $y\in(G,\breve{E})$ and $x\notin(G,\breve{E})$ and for each $\left(\breve{X},\overset{3}{\breve{\tau}},\overset{4}{\breve{\tau}},\breve{E}\right)$ soft *regular* closed set (F_1,E) and $\left(\breve{X},\overset{1}{\breve{\tau}},\overset{2}{\breve{\tau}},\breve{E}\right)$ soft *regular* closed set $(F_2,\breve{E})$ such that $(F_1,E)\cap(F_2,E)=\phi$. there exists soft *regular open* sets$\left(F_3,\breve{E}\right)$ and $(G_1,\breve{E})$ such that $\left(F_3,\breve{E}\right)$ is soft $\left(\breve{X},\overset{1}{\breve{\tau}},\overset{2}{\breve{\tau}},\breve{E}\right)$ *regular* open set $(G_1,\breve{E})$ is soft $\left(\breve{X},\overset{3}{\breve{\tau}},\overset{4}{\breve{\tau}},\breve{E}\right)$ *regular open* set such that $\left(F_1,\breve{E}\right)\subseteq\left(F_3,\breve{E}\right),(F_2,\breve{E})\subseteq(G_1,\breve{E})$ and $\left(F_3,\breve{E}\right)\cap\left(G_1,\breve{E}\right)=\phi$. Hence $\left(\breve{X},\overset{1}{\breve{\tau}},\overset{2}{\breve{\tau}},\overset{3}{\breve{\tau}},\overset{4}{\breve{\tau}},\breve{E}\right)$ is pair wise soft regular T_4 space.

Proposition 6. Let $\left(\breve{X},\overset{1}{\breve{\tau}},\overset{2}{\breve{\tau}},\overset{3}{\breve{\tau}},\overset{4}{\breve{\tau}},\breve{E}\right)$ be a soft quad topological space over X and Y be a non-empty subset of X. if $(X,\overset{1^Y}{\breve{\tau}}\ \overset{2^Y}{\breve{\tau}},\overset{3^Y}{\breve{\tau}},\overset{4^Y}{\breve{\tau}},\breve{E})$ is pair wise soft regular T_3 space. Then $(\breve{Y},\overset{1^Y}{\breve{\tau}}\ \overset{2^Y}{\breve{\tau}},\overset{3^Y}{\breve{\tau}},\overset{4^Y}{\breve{\tau}},\breve{E})$ is pair wise soft regular T_3 space. `

Proof. First we prove that $(\breve{Y},\overset{1^Y}{\breve{\tau}},\overset{2^Y}{\breve{\tau}},\overset{3^Y}{\breve{\tau}},\overset{4^Y}{\breve{\tau}},\breve{E})$ is pair wise soft regular T_1 space.

Let $x,y\in X,x\neq y$ if $\left(\breve{X},\overset{1}{\breve{\tau}},\overset{2}{\breve{\tau}},\overset{3}{\breve{\tau}},\overset{4}{\breve{\tau}},\breve{E}\right)$ is pair wise space then this implies that $\left(\breve{X},\overset{1}{\breve{\tau}},\overset{2}{\breve{\tau}},\overset{3}{\breve{\tau}},\overset{4}{\breve{\tau}},\breve{E}\right)$ is pair wise soft space. So there exists $\left(\breve{X},\overset{1}{\breve{\tau}},\overset{2}{\breve{\tau}},\breve{E}\right)$ soft *regular open* $(F,\breve{E})$ and $\left(\breve{X},\overset{3}{\breve{\tau}},\overset{4}{\breve{\tau}},\breve{E}\right)$ soft *regular* open set $(G,\breve{E})$ such that $x\in(F,\breve{E})$ and $y\notin(F,\breve{E})$ or $y\in(G,\breve{E})$ and $x\notin(G,\breve{E})$ now $x\in\breve{Y}\ and\ x\notin(G,\breve{E})$. Hence $x\in\breve{Y}\cap\left(F,\breve{E}\right)=(Y_F,E)$ then $y\notin Y\cap(\alpha)$ for some $\alpha\in\breve{E}$. this means that $\alpha\in\breve{E}$ then $y\notin Y\cap F(\alpha)$ for some $\alpha\in\breve{E}$.

Therefore, $y\notin Y\cap\left(F,\breve{E}\right)=\left(Y_F,\breve{E}\right)$. Now $y\in\breve{Y}$ and $y\in\left(G,\breve{E}\right)$. Hence $y\in Y\cap\left(G,\breve{E}\right)=\left(G_Y,\breve{E}\right)$ where $\left(G,\breve{E}\right)\in(X,\tau_3,\tau_4,\breve{E})$. Consider $x\notin(G,E)$ this means that $\alpha\in\breve{E}$ then $x\notin Y\cap G(\alpha)$ for some α$\in\breve{E}$. Therefore $x\notin\breve{Y}\cap\left(G,\breve{E}\right)=(G_Y,\breve{E})$ thus $(\breve{Y},\tau_{1\breve{Y}},\tau_{2\breve{Y}},\tau_{3\breve{Y}},\tau_{4\breve{Y}},\breve{E})$ is pair wise soft regular T_1 space.

Now we prove that $(\breve{Y},\overset{1^Y}{\breve{\tau}}\ \overset{2^Y}{\breve{\tau}},\overset{3^Y}{\breve{\tau}},\overset{4^Y}{\breve{\tau}},\breve{E})$ is pair wise soft regular T_3 space then $(\breve{Y},\overset{1^Y}{\breve{\tau}}\ \overset{2^Y}{\breve{\tau}},\overset{3^Y}{\breve{\tau}},\overset{4^Y}{\breve{\tau}},\breve{E})$ is pair wise soft R-regular space.

Let $y \in \breve{Y}$ and $(G, \breve{E})$ be a soft *regular* closed set in Y such that $y \notin (G, E)$ where $(G, E) \in \left(\breve{X}, \overset{1}{\breve{\tau}}, \overset{2}{\breve{\tau}}, \overset{3}{\breve{\tau}}, \overset{4}{\breve{\tau}}, \breve{E}\right)$ then $(G, \breve{E}) = (Y, \breve{E}) \cap (F, \breve{E})$ for some soft *regular* closed set in $(\breve{X}, \overset{1}{\breve{\tau}}{}^{Y}\ \overset{2}{\breve{\tau}}{}^{Y}, \overset{3}{\breve{\tau}}{}^{Y}, \overset{4}{\breve{\tau}}{}^{Y}, \breve{E})$ Hence $y \notin (Y, \breve{E}) \cap (F, \breve{E})$ but $y \in (Y, \breve{E})$, so $y \notin (F, \breve{E})$ since $\left(\breve{X}, \overset{1}{\breve{\tau}}, \overset{2}{\breve{\tau}}, \overset{3}{\breve{\tau}}, \overset{4}{\breve{\tau}}, \breve{E}\right)$ is soft regular T_3 space $(X, \tau_1, \tau_2, \tau_3, \tau_4, \breve{E})$ is soft *regular* space so there exists $\left(\breve{X}, \overset{1}{\breve{\tau}}, \overset{2}{\breve{\tau}}, \breve{E}\right)$ soft *regular open* set $(F_1, \breve{E})$ and $\left(\breve{X}, \overset{3}{\breve{\tau}}, \overset{4}{\breve{\tau}}, \breve{E}\right)$ soft *regular* open set $(F_2, \breve{E})$ such that

$$y \in (F_1, \breve{E}), (G, E) \subseteq (F_2, \breve{E})$$
$$(F_1, E)(F_2, E) = \phi$$

Take $(G_1, \breve{E}) = (Y, \breve{E}) \cap (F_2, \breve{E})$ then $(G_1, \breve{E}), (G_2, \breve{E})$ are soft *regular* open set in $\breve{Y}$ such that

$$y \in (G_1, \breve{E}), (G, \breve{E}) \subseteq (Y, \breve{E}) \cap (F_2, \breve{E}) = (G_2, \breve{E})$$
$$(G_1, \breve{E}) \cap (G_2, \breve{E}) \subseteq (F_1, \breve{E}) \cap (F_2, \breve{E}) = \phi$$
$$(G_1, \breve{E}) \cap (G_2, \breve{E}) = \phi$$

There fore $\overset{1}{\breve{\tau}}{}^{Y} \cup \overset{2}{\breve{\tau}}{}^{Y}$ is soft R-regular space with respect to $\overset{3}{\breve{\tau}}{}^{Y} \cup \overset{4}{\breve{\tau}}{}^{Y}$. Similarly, Let $y \in \breve{Y}$ and $(G, \breve{E})$ be a soft *regular* closed sub set in $\breve{Y}$ such that $y \notin (G, \breve{E})$, where $(G, \breve{E}) \in$ ($\left(\breve{X}, \overset{3}{\breve{\tau}}, \overset{4}{\breve{\tau}}, \breve{E}\right)$ then $(G, \breve{E}) = (Y, \breve{E}) \cap (F, \breve{E})$ where $(F, \breve{E})$ is some soft *regular* closed set in $\left(\breve{X}, \overset{1}{\breve{\tau}}, \overset{2}{\breve{\tau}}, \breve{E}\right)$ $y \notin (Y, \breve{E}) \cap (F, \breve{E})$ But $y \in (Y, \breve{E})$ so $y \notin (F, \breve{E})$ since $\left(\breve{X}, \overset{1}{\breve{\tau}}, \overset{2}{\breve{\tau}}, \breve{E}\right)$ is soft R-regular space so there exists $\left(\breve{X}, \overset{3}{\breve{\tau}}, \overset{4}{\breve{\tau}}, \breve{E}\right)$ soft *regular* open set $(F_1, \breve{E})$ and $\left(\breve{X}, \overset{1}{\breve{\tau}}, \overset{2}{\breve{\tau}}, \breve{E}\right)$ soft *regular* open set (F_2, E). Such that

$$y \in (F_1, \breve{E}), (G, \breve{E}) \subseteq (F_2, \breve{E})$$
$$(F_1, \breve{E}) \cap (F_2, \breve{E}) = \phi$$

Take

$$(G_1, \breve{E}) = (Y, \breve{E}) \cap (F_1, \breve{E})$$
$$(G_1, \breve{E}) = (Y, \breve{E}) \cap (F_1, \breve{E})$$

Then (G_1, E) and $(G_2, \breve{E})$ are soft *regular* open set in Y such that

$$y \in (G_1, \breve{E}), (G, \breve{E}) \subseteq (Y, \breve{E}) \cap (F_2, \breve{E}) = (G_2, \breve{E})$$
$$(G_1, \breve{E}) \cap (G_2, \breve{E}) \subseteq (F_1, \breve{E}) \cap (F_2, \breve{E}) = \phi$$

Therefore $\overset{3}{\tau}^{Y} \cup \overset{4}{\tau}^{Y}$ is soft *regular* space with respect $\overset{1}{\tau}^{Y} \cup \overset{2}{\tau}^{Y} \Rightarrow (\breve{Y}, \overset{1}{\tau}^{Y} \overset{2}{\tau}^{Y}, \overset{3}{\tau}^{Y}, \overset{4}{\tau}^{Y}, \breve{E})$ is pair wise soft regular T_3 space.

Proposition 7. Let $\left(\breve{X}, \overset{1}{\tau}, \overset{2}{\tau}, \overset{3}{\tau}, \overset{4}{\tau}, \breve{E}\right)$ be a soft quad topological space over $\breve{X}$ and $\breve{Y}$ be a soft *regular* closed sub space of X. If $\left(\breve{X}, \overset{1}{\tau}, \overset{2}{\tau}, \overset{3}{\tau}, \overset{4}{\tau}, \breve{E}\right)$ is pair wise soft regular T_4 space then $(\breve{Y}, \overset{1}{\tau}^{Y} \overset{2}{\tau}^{Y}, \overset{3}{\tau}^{Y}, \overset{4}{\tau}^{Y}, \breve{E})$ is pair wise soft regular T_4 space.

Proof. Since $\left(\breve{X}, \overset{1}{\tau}, \overset{2}{\tau}, \overset{3}{\tau}, \overset{4}{\tau}, \breve{E}\right)$ is pair wise soft regular T_4 space. So this implies that $\left(\breve{X}, \overset{1}{\tau}, \overset{2}{\tau}, \overset{3}{\tau}, \overset{4}{\tau}, \breve{E}\right)$ is pair wise soft regular T_1 space as proved above.

We prove $\left(\breve{X}, \overset{1}{\tau}, \overset{2}{\tau}, \overset{3}{\tau}, \overset{4}{\tau}, \breve{E}\right)$ is pair wise soft *regular* normal space. Let $(G_1, \breve{E}), (G_2, \breve{E})$ be soft *regular* closed sets in $\breve{Y}$ such that

$$(G_1, \breve{E}) \cap (G_2, \breve{E}) = \phi$$

Then

$$(G_1, \breve{E}) = (Y, \breve{E}) \cap (F_1, \breve{E})$$

and

$$(G_2, \breve{E}) = (\breve{Y}, \breve{E}) \cap (F_2, \breve{E})$$

For some soft *regular* closed sets such that $(F_1, \breve{E})$ is soft *regular* closed set in $\overset{1}{\tau} \cup \overset{2}{\tau}$ soft *regular* closed set $(F_2, \breve{E})$ in $\overset{3}{\tau}^{Y} \cup \overset{4}{\tau}^{Y}$ and $(F_1, \breve{E}) \cap (F_2, \breve{E}) = \phi$ From Proposition 2**.** Since, $\breve{Y}$ is soft *regular* closed sub set of X then $(G_1, \breve{E}), (G_2, \breve{E})$ are soft *regular* closed sets in $\breve{X}$ such that

$$(G_1, \breve{E}) \cap (G_2, \breve{E}) = \phi$$

Since $\left(\breve{X}, \overset{1}{\tau}, \overset{2}{\tau}, \overset{3}{\tau}, \overset{4}{\tau}, \breve{E}\right)$ is pair wise softs *regular* normal space. So there exists soft *regular open* sets $(H_1, \breve{E})$ and (H_2, E) such that $(H_1, \breve{E})$ is soft *regular open* set in $\tau_1 \cup \tau_2$ and $(H_2, \breve{E})$ is soft *regular* open set in $\overset{3}{\tau}^{Y} \cup \overset{4}{\tau}^{Y}$ such that

$$(G_1, \breve{E}) \subseteq (H_1, \breve{E})$$
$$(G_2, \breve{E}) \subseteq (H_2, \breve{E})$$
$$(H_1, \breve{E}) \cap (H_2, \breve{E}) = \phi$$

Since

$$\left(G_1,\breve{E}\right),\left(G_2,\breve{E}\right) \subseteq \ (Y,\breve{E})$$

Then

$$\left(G_1,\breve{E}\right) \subseteq \left(Y,\breve{E}\right) \cap \ (H_1,\breve{E})$$
$$\left(G_2,\breve{E}\right) \subseteq \left(Y,\breve{E}\right) \cap \ (H_2,\breve{E})$$

and

$$\left[\left(Y,\breve{E}\right) \cap \left(H_1,\breve{E}\right)\right] \cap \left[\left(Y,\breve{E}\right) \cap \left(H_2,\breve{E}\right)\right] = \phi$$

Where $\left(Y,\breve{E}\right) \cap \left(H_1,\breve{E}\right)$ and $\left(Y,\breve{E}\right) \cap \left(H_2,\breve{E}\right)$ are soft *regular* open sets in Y there fore $\overset{1}{\breve{\tau}}{}^{Y} \cup \overset{2}{\breve{\tau}}{}^{Y}$ is soft *regular* normal space with respect to $\overset{3}{\breve{\tau}}{}^{Y} \cup \overset{4}{\breve{\tau}}{}^{Y}$. Similarly, Let $\left(G_1,\breve{E}\right),\left(G_2,\breve{E}\right)$ be soft *regular* closed sub set in Y such that

$$\left(G_1,\breve{E}\right) \cap \left(G_2,\breve{E}\right) = \phi$$

Then

$$\left(G_1,\breve{E}\right) = \ (Y,\breve{E}) \ \cap \left(F_1,\breve{E}\right)$$

and

$$\left(G_2,\breve{E}\right) = \ (Y,\breve{E}) \ \cap \left(F_2,\breve{E}\right)$$

For some soft *regular* closed sets such that $(F_1,\breve{E})$ is soft *regular* closed set in $\overset{3}{\breve{\tau}} \cup \overset{4}{\breve{\tau}}$ *and* $(F_2,\breve{E})$ soft *regular* closed set in $\overset{1}{\breve{\tau}}{}^{Y} \cup \overset{2}{\breve{\tau}}{}^{Y}$ and $\left(F_1,\breve{E}\right)\left(F_2,\breve{E}\right) = \phi$ from Proposition 2. Since, $\breve{Y}$ is soft *regular* closed sub set in $\breve{X}$ then $\left(G_1,\breve{E}\right),\left(G_2,\breve{E}\right)$ are soft *regular* closed sets in $\breve{X}$ such that

$$\left(G_1,\breve{E}\right) \cap \left(G_2,\breve{E}\right) = \phi$$

Since $\left(\breve{X},\overset{1}{\breve{\tau}},\overset{2}{\breve{\tau}},\overset{3}{\breve{\tau}},\overset{4}{\breve{\tau}},\breve{E}\right)$ is pair wise soft *regular* normal space so there exists soft *regular* open sets $\left(H_1,\breve{E}\right)$ and $\left(H_2,\breve{E}\right)$. Such that $\left(H_1,\breve{E}\right)$ is soft *regular open* set is $\overset{3}{\breve{\tau}} \cup \overset{4}{\breve{\tau}}$ and (H_2,E) is soft *regular open* set in $\overset{1}{\breve{\tau}}{}^{Y} \cup \overset{2}{\breve{\tau}}{}^{Y}$ such that

$$\left(G_1,\breve{E}\right) \subseteq \left(H_1,\breve{E}\right)$$
$$\left(G_2,\breve{E}\right) \subseteq \left(H_2,\breve{E}\right)$$
$$\left(H_1,\breve{E}\right) \cap \left(H_2,\breve{E}\right) = \phi$$

Since

$$\left(G_1,\breve{E}\right),\left(G_2,\breve{E}\right) \subseteq \ (Y,\breve{E})$$

Then

$$\left(G_1,\breve{E}\right) \subseteq \left(Y,\breve{E}\right) \cap \ (H_1,\breve{E})$$
$$\left(G_2,\breve{E}\right) \subseteq \left(Y,\breve{E}\right) \cap \ (H_2,\breve{E})$$

and

$$\left[\left(Y,\breve{E}\right) \cap \left(H_1,\breve{E}\right)\right] \cap \left[\left(Y,\breve{E}\right) \cap \left(H_2,\breve{E}\right)\right] = \phi$$

Where $(Y,\breve{E})\cap(H_1,\breve{E})$ and $(Y,\breve{E})\cap(H_2,\breve{E})$ are soft *regular open* sets in $\breve{Y}$ there fore $\overset{3}{\breve{\tau}}^{Y}\cup\overset{4}{\breve{\tau}}^{Y}$ is soft *regular* normal space with respect to $\overset{1}{\breve{\tau}}^{Y}\cup\overset{2}{\breve{\tau}}^{Y} \Rightarrow (\breve{Y},\overset{1}{\breve{\tau}}^{Y}\ \overset{2}{\breve{\tau}}^{Y},\overset{3}{\breve{\tau}}^{Y},\overset{4}{\breve{\tau}}^{Y},\breve{E})$ is pair wise soft regular T_4 space.

3. 2 Soft Regular Separation Axioms in Soft Quad Topological Spaces with Respect to Soft Points

In this section, we introduced soft topological structures known as soft regular separation axioms in soft quad topology with respect to soft points. With the applications of these soft *regular* separation axioms different result are brought under examination.

Definition 33 In a *soft quad topological* space $\left(\breve{X},\overset{1}{\breve{\tau}},\overset{2}{\breve{\tau}},\overset{3}{\breve{\tau}},\overset{4}{\breve{\tau}},\breve{E}\right)$

1) $\overset{1}{\breve{\tau}}\cup\overset{2}{\breve{\tau}}$ said to be *soft regular* T_0 space with respect to $\overset{3}{\breve{\tau}}\cup\overset{4}{\breve{\tau}}$ if for each pair of distinct points $e_G,e_H\in X_A$there happens $\overset{1}{\breve{\tau}}\cup\overset{2}{\breve{\tau}}$ soft *regular* open set $(F,\breve{E})$ and a $\overset{3}{\breve{\tau}}\cup\overset{4}{\breve{\tau}}$ soft *regular* open set $(G,\breve{E})$ such that $e_G\in(F,\breve{E})$ and $e_H\notin(G,\breve{E})$, Similarly, $\overset{3}{\breve{\tau}}\cup\overset{4}{\breve{\tau}}$is said to be *soft regular* T_0 space with respect to $\overset{1}{\breve{\tau}}\cup\overset{2}{\breve{\tau}}$ if for each pair of distinct points $e_G,e_H\in X_A$there happens $\overset{3}{\breve{\tau}}\cup\overset{4}{\breve{\tau}}$ *soft regular open set* (F,E) and a $\overset{1}{\breve{\tau}}\cup\overset{2}{\breve{\tau}}$ *regular* soft open set (G,E) such that $e_G\in(F,\breve{E})$ and $e_H\notin(F,\breve{E})$ or $e_H\in(G,\breve{E})$ and $e_G\notin(G,\breve{E})$. *Soft quad topological spaces* $\left(\breve{X},\overset{1}{\breve{\tau}},\overset{2}{\breve{\tau}},\overset{3}{\breve{\tau}},\overset{4}{\breve{\tau}},\breve{E}\right)$ is said to be *pair wise soft regular* T_0 space if $\overset{1}{\breve{\tau}}\cup\overset{2}{\breve{\tau}}$ is *soft regular* T_0 *space* with respect to $\overset{3}{\breve{\tau}}\cup\overset{4}{\breve{\tau}}$ and $\overset{3}{\breve{\tau}}\cup\overset{4}{\breve{\tau}}$ is soft regular T_0 spaces with respect to $\overset{1}{\breve{\tau}}\cup\overset{2}{\breve{\tau}}$

2) $\overset{1}{\breve{\tau}}\cup\overset{2}{\breve{\tau}}$is said to be soft *regular* T_1 space with respect to $\overset{3}{\breve{\tau}}\cup\overset{4}{\breve{\tau}}$ if for each pair of distinct points $e_G,e_H\in X_A$ there happens $\overset{1}{\breve{\tau}}\cup\overset{2}{\breve{\tau}}$ *soft regular open set* $(F,\breve{E})$ and $\overset{3}{\breve{\tau}}\cup\overset{4}{\breve{\tau}}$ soft *regular open* set $(G,\breve{E})$ such that $e_G\in(F,\breve{E})$ and $e_H\notin(G,\breve{E})$ and $e_H\in(G,\breve{E})$ and $e_G\notin(G,E)$. Similarly, $\overset{3}{\breve{\tau}}\cup\overset{4}{\breve{\tau}}$is said to be soft regular T_1 space with respect to $\overset{1}{\breve{\tau}}\cup\overset{2}{\breve{\tau}}$ if for each pair of distinct points $e_G,e_H\in X_A$there exist $\overset{3}{\breve{\tau}}\cup\overset{4}{\breve{\tau}}$ soft *regular open* set $(F,\breve{E})$ and a $\overset{1}{\breve{\tau}}\cup\overset{2}{\breve{\tau}}$ soft *regular* open set $(G,\breve{E})$ such that $e_G\in(F,\breve{E})$ and $e_H\notin(G,\breve{E})$ and $e_H\in(G,\breve{E})$ and $e_G\notin(G,\breve{E})$. *Soft quad topological space* $\left(\breve{X},\overset{1}{\breve{\tau}},\overset{2}{\breve{\tau}},\overset{3}{\breve{\tau}},\overset{4}{\breve{\tau}},\breve{E}\right)$ is said to be *pair wise soft regular* T_1 space if $\overset{1}{\breve{\tau}}\cup\overset{2}{\breve{\tau}}$ is *soft regular* T_1 *space* with respect to $\overset{3}{\breve{\tau}}\cup\overset{3}{\breve{\tau}}$ and $\overset{3}{\breve{\tau}}\cup\overset{4}{\breve{\tau}}$is soft regular T_1 spaces with respect to $\overset{1}{\breve{\tau}}\cup\overset{2}{\breve{\tau}}$.

3) $\overset{1}{\tau} \cup \overset{2}{\tau}$ is said to be soft regular T_2 space with respect to $\overset{3}{\tau} \cup \overset{4}{\tau}$, if for each pair of distinct points $e_G, e_H \in X_A$ there happens a $\overset{1}{\tau} \cup \overset{2}{\tau}$ soft *regular* open set $(F, \breve{E})$ and a $\overset{3}{\tau} \cup \overset{4}{\tau}$ soft *regular* open set $(G, \breve{E})$ such that $e_G \in (F, \breve{E})$ and $e_H \notin (G, \breve{E})$ and $e_H \in (G, \breve{E})$ and $e_G \notin (G, \breve{E})$ and $(F, \breve{E}) \cap (G, \breve{E}) = \phi$. Similarly, $\tau_3 \cup \tau_4$ is aid to be *soft regular* T_2 space with respect to $\overset{1}{\tau} \cup \overset{2}{\tau}$ if for each pair of distinct points $e_G, e_G \in X_A$ there happens $\overset{3}{\tau} \cup \overset{4}{\tau}$ *soft regular open* set (F, E) and $\overset{1}{\tau} \cup \overset{2}{\tau}$ *soft regular open* set $(G, \breve{E})$ such that $e_G \in (F, \breve{E})$ and $e_G \in (G, \breve{E})$ and $(F, \breve{E}) \cap (G, \breve{E}) = \phi$. The soft quad topological space $\left(\breve{X}, \overset{1}{\tau}, \overset{2}{\tau}, \overset{3}{\tau}, \overset{4}{\tau}, \breve{E}\right)$ is said to be pair wise soft regular T_2 space if $\overset{1}{\tau} \cup \overset{2}{\tau}$ is soft regular T_2 space with respect to $\overset{3}{\tau} \cup \overset{4}{\tau}$ and $\overset{3}{\tau} \cup \overset{4}{\tau}$ is soft regular T_2 space with respect to $\overset{1}{\tau} \cup \overset{2}{\tau}$.

Definition 34 In a soft quad topological space $\left(\breve{X}, \overset{1}{\tau}, \overset{2}{\tau}, \overset{3}{\tau}, \overset{4}{\tau}, \breve{E}\right)$

1) $\overset{1}{\tau} \cup \overset{2}{\tau}$ is said to be soft regular T_3 space with respect to $\overset{3}{\tau} \cup \overset{4}{\tau}$ if $\overset{1}{\tau} \cup \overset{2}{\tau}$ is soft regular T_1 space with respect to $\tau_3 \cup \tau_4$ and for each pair of distinct points $e_G, e_H \in X_A$, there exists $\overset{1}{\tau} \cup \overset{2}{\tau}$.

regular closed soft set $(G, \breve{E})$ such that $e_G \notin (G, \breve{E})$, $\overset{1}{\tau} \cup \overset{2}{\tau}$ soft *regular open* set $(F_1, \breve{E})$ and $\overset{3}{\tau} \cup \overset{4}{\tau}$ soft *regular open* set $(F_2, \breve{E})$ such that $e_G \in (F_1, \breve{E}), (G, \breve{E}) \subseteq (F_2, \breve{E})$ and $(F_1, \breve{E}) \cap (F_2, \breve{E}) = \emptyset$. Similarly, $\tau_3 \cup \tau_4$is said to be soft regular T_3 space with respect to $\overset{1}{\tau} \cup \overset{2}{\tau}$ if $\overset{3}{\tau} \cup \overset{4}{\tau}$ is soft regular T_1 space with respect to $\overset{1}{\tau} \cup \overset{2}{\tau}$ and for each pair of distinct points $e_G, e_H \in X_A$there exists a $\overset{3}{\tau} \cup \overset{4}{\tau}$ soft *regular* closed set (G, E) such that $e_G \notin (G, E)$, $\overset{3}{\tau} \cup \overset{4}{\tau}$ soft *regular* open set (F_1, E) amd $\overset{1}{\tau} \cup \overset{2}{\tau}$ soft *regular* open set (F_2, E) such that $e_G \in (F_1, E), (G, E) \subseteq (F_2, \breve{E}$ and $(F_1, \breve{E}) \cap (F_2, \breve{E}) = \phi$. $\left(\breve{X}, \overset{1}{\tau}, \overset{2}{\tau}, \overset{3}{\tau}, \overset{4}{\tau}, \breve{E}\right)$ is said to be pair wise soft regular T_3 space if $\overset{1}{\tau} \cup \overset{2}{\tau}$is soft regular T_3 space with respect to $\overset{3}{\tau} \cup \overset{4}{\tau}$ and $\overset{3}{\tau} \cup \overset{4}{\tau}$is soft regular T_3 space with respect to $\overset{1}{\tau} \cup \overset{2}{\tau}$.

2) $\overset{1}{\tau} \cup \overset{2}{\tau}$is said to be soft regular T_4 space with respect to $\overset{3}{\tau} \cup \overset{4}{\tau}$ if $\overset{1}{\tau} \cup \overset{2}{\tau}$is soft regular T_1 space with respect to $\overset{3}{\tau} \cup \overset{4}{\tau}$, there exists a $\overset{1}{\tau} \cup \overset{2}{\tau}$ soft *regular* closed set (F_1, E) and $\overset{3}{\tau} \cup \overset{4}{\tau}$ soft *regular* closed set $(F_2, \breve{E})$ such that $(F_1, \breve{E}) \cap (F_2, \breve{E}) = \emptyset$,also,open there exists $(F_3, \breve{E})$ and $(G_1, \breve{E})$ such that $(F_3, \breve{E})$ is soft $\overset{1}{\tau} \cup \overset{2}{\tau}$ *regular*

open set, (G_1, E) is soft $\overset{3}{\breve{\tau}} \cup \overset{4}{\breve{\tau}}$ *regular* set such that $(F_1, \breve{E}) \subseteq (F_3, \breve{E})$, $(F_2, \breve{E}) \subseteq (G_1, \breve{E})$. Similarly, $\overset{3}{\breve{\tau}} \cup \overset{4}{\breve{\tau}}$is said to be soft regular T_4 space with respect to $\overset{1}{\breve{\tau}} \cup \overset{2}{\breve{\tau}}$ if $\overset{3}{\breve{\tau}} \cup \overset{4}{\breve{\tau}}$is soft regular T_1 space with respect to $\overset{1}{\breve{\tau}} \cup \overset{2}{\breve{\tau}}$there exists $\overset{3}{\breve{\tau}} \cup \overset{4}{\breve{\tau}}$ soft *regular* closed set $(F_1, \breve{E})$ and $\overset{1}{\breve{\tau}} \cup \overset{2}{\breve{\tau}}$ soft *regular* closed set $(F_2, \breve{E})$ such that $(F_1, \breve{E}) \cap (F_2, \breve{E}) = \phi$. Also there exists $(F_3, \breve{E})$ and $(G_1, \breve{E})$ such that $(F_3, \breve{E})$ is soft $\overset{3}{\breve{\tau}} \cup \overset{4}{\breve{\tau}}$ *regular* open set, $(G_1, \breve{E})$ is soft $\overset{1}{\breve{\tau}} \cup \overset{2}{\breve{\tau}}$ *regular* soft set such that $(F_1, E) \subseteq (F_3, E), (F_2, E) \subseteq (G_1, E)$ and $(F_3, E) \cap (G_1, E) = \phi$. Thus, $(\left(\breve{X}, \overset{1}{\breve{\tau}}, \overset{2}{\breve{\tau}}, \overset{3}{\breve{\tau}}, \overset{4}{\breve{\tau}}, \breve{E}\right)$is said to be pair wise soft regular T_4 space if $\overset{1}{\breve{\tau}} \cup \overset{2}{\breve{\tau}}$is soft regular T_4 space with respect to $\overset{3}{\breve{\tau}} \cup \overset{4}{\breve{\tau}}$ and to $\overset{3}{\breve{\tau}} \cup \overset{4}{\breve{\tau}}$is soft regular T_4 space with respect to $\overset{1}{\breve{\tau}} \cup \overset{2}{\breve{\tau}}$.

Proposition 8. Let $\left(\breve{X}, \overset{1}{\breve{\tau}}, \overset{2}{\breve{\tau}}, \overset{3}{\breve{\tau}}, \overset{4}{\breve{\tau}}, \breve{E}\right)$ be a soft topological space over X. $\left(\breve{X}, \overset{1}{\breve{\tau}}, \overset{2}{\breve{\tau}}, \overset{3}{\breve{\tau}}, \overset{4}{\breve{\tau}}, \breve{E}\right)$ is soft *regular* T_3 space , then for all $e_G \in X_E e_G = (e_G, \breve{E})$ is soft *regular*-closed set.

Proof. We want to prove that $e_G$is *regular* closed soft set, which is sufficient to prove that e_G^c is *regular open* soft set for all $e_H \in \{e_G\}^c$. Since $(X, \tau_1, \tau_2, \tau_3, \tau_4, \breve{E})$ is soft regular T_3 space , then there exists soft *regular* sets ($F, E)_{e_H}$ and (G, E) such that $e_{H_E} \subseteq (F, \breve{E})_{e_H}$ and $e_{G_E} \cap (F, \breve{E})_{e_H} = \phi$ and $e_{G_E} \subseteq (G, \breve{E})$ and $e_{H_E} \cap (G, \breve{E}) = \phi$. It follows that, $\cup_{e_H \in (e_G)^c} (F, \breve{E})_{e_H} \subseteq e_{G_E}^c$ Now, we want to prove that $e_{G_{\breve{E}}}^c \subseteq \cup_{e_H \in (e_G)^c} (F, \breve{E})_{e_H}$. Let $\cup_{e_H \in (e_G)^c} (F, \breve{E})_{e_H} = (H, \breve{E})$. Where $H(e) = \cup_{e_H \in (e_G)^c} (F(e)_{e_H}$ for all $e \in \breve{E}$. Since $e_{G_E}^c(e) = (e_G)^c$for all $e \in \breve{E}$ from Definition 9, so, for all $e_H \in \{e_G\}^c$ and $e \in \breve{E} e_{G_E}^c(e) = \{e_G\}^c = \cup_{e_H \in (e_G)^c} \{e_H\} = \cup_{e_H \in (e_G)^c} F(e)_{e_H} = H(e)$. Thus, $e_{G_E}^c \subseteq \cup_{e_H \in (e_G)^c} (F, E)_{e_H}$ from Definition 2, and so, $e_{G_E}^c = \cup_{e_H \in (e_G)^c} (F, E)_{e_H}$. This means that , $e_{G_E}^c$ is soft *regular open* set for all $e_H \in \{e_G\}^c$. Therefore, $e_{G_E}$is *regular* closed soft set.

Proposition 9. Let $\left(\breve{X}, \overset{1}{\breve{\tau}}, \overset{2}{\breve{\tau}}, \overset{3}{\breve{\tau}}, \overset{4}{\breve{\tau}}, \breve{E}\right)$be a soft quad topological space over X. Then, if $\left(\breve{X}, \overset{1}{\breve{\tau}}, \overset{2}{\breve{\tau}}, \breve{E}\right)$ and $\left(\breve{X}, \overset{3}{\breve{\tau}}, \overset{4}{\breve{\tau}}, \breve{E}\right)$are soft regular qT_3 space , then $\left(\breve{X}, \overset{1}{\breve{\tau}}, \overset{2}{\breve{\tau}}, \overset{3}{\breve{\tau}}, \overset{4}{\breve{\tau}}, \breve{E}\right)$is a pair wise soft regular T_2 space.

Proof. Suppose if $\left(\breve{X}, \overset{1}{\breve{\tau}}, \overset{2}{\breve{\tau}}, \breve{E}\right)$ is a soft regular T_3 space with respect to $\left(\breve{X}, \overset{3}{\breve{\tau}}, \overset{4}{\breve{\tau}}, \breve{E}\right)$, then according to definition for, $e_G \neq e_{H,} e_G, e_H \in X_A$, by using Theorem 8, $(e_H, \breve{E})$ is soft *regular* closed set in $\left(\breve{X},\overset{3}{\tau},\overset{4}{\tau},\breve{E}\right)$ and $e_G \notin (e_H,\breve{E})$ there exist a $\left(\breve{X},\overset{1}{\tau},\overset{2}{\tau},\breve{E}\right)$ soft *regular* open set (F,E) and a $\left(\breve{X},\overset{3}{\tau},\overset{4}{\tau},\breve{E}\right)$ soft *regular* open set (G,E) such that $e_G \in (F,\breve{E})$, $e_H \in (y,\breve{E}) \subseteq (G,\breve{E})$ and $(F_1,\breve{E}) \cap (F_2,\breve{E}) = \phi$. Hence$\left(\breve{X},\overset{1}{\tau},\overset{2}{\tau},\breve{E}\right)$ is soft regular T_2 space with respect to $\left(\breve{X},\overset{3}{\tau},\overset{4}{\tau},\breve{E}\right)$ Similarly, if $\left(\breve{X},\overset{3}{\tau},\overset{4}{\tau},\breve{E}\right)$ is a soft regular T_3 space with respect to $\left(\breve{X},\overset{1}{\tau},\overset{2}{\tau},\breve{E}\right)$, then according to definition for, $e_G \neq e_{H,}e_G, e_H \in X_A$, by using Theorem 8, $(e_G,\breve{E})$ is *regular* closed soft set in $\left(\breve{X},\overset{1}{\tau},\overset{2}{\tau},\breve{E}\right)$ is and $y \notin (x,\breve{E})$ there exists a $\left(\breve{X},\overset{3}{\tau},\overset{4}{\tau},\breve{E}\right)$ soft *regular open* set $(F,\breve{E})$ and a $\left(\breve{X},\overset{1}{\tau},\overset{2}{\tau},\breve{E}\right)$ soft *regular* open set $(G,\breve{E})$ such that $e_H \in (F,\breve{E})$, $e_G \in (x,\breve{E}) \subseteq (G,\breve{E})$ and $(F_1,\breve{E}) \cap (F_2,\breve{E}) = \phi$. Hence $\left(\breve{X},\overset{3}{\tau},\overset{4}{\tau},\breve{E}\right)$ is a soft regular T_2 space. Thus $\left(\breve{X},\overset{1}{\tau},\overset{2}{\tau},\overset{3}{\tau},\overset{4}{\tau},\breve{E}\right)$ is a pair wise soft s *regular* T_2 space.

Proposition 10. Let $\left(\breve{X},\overset{1}{\tau},\overset{2}{\tau},\overset{3}{\tau},\overset{4}{\tau},\breve{E}\right)$ be a soft quad topological space over $\breve{X}$. if $\left(\breve{X},\overset{1}{\tau},\overset{2}{\tau},\breve{E}\right)$ and $\left(\breve{X},\overset{3}{\tau},\overset{4}{\tau},\breve{E}\right)$ are soft regular T_3 space then $\left(\breve{X},\overset{1}{\tau},\overset{2}{\tau},\overset{3}{\tau},\overset{4}{\tau},\breve{E}\right)$ is a pair wise soft regular T_3 space.

Proof. Suppose $\left(\breve{X},\overset{1}{\tau},\overset{2}{\tau},\breve{E}\right)$ is a soft regular T_3 space with respect to $\left(\breve{X},\overset{3}{\tau},\overset{4}{\tau},\breve{E}\right)$ then according to definition for $e_G, e_H \in X_A e_G \neq e_H$there happens $\tau_1 \cup \tau_2$ soft *regular open* set (F,E) and a $\overset{3}{\tau} \cup \overset{4}{\tau}$ soft *regular* open set (G,E) such that $e_G \in (F,\breve{E})$ and $e_H \notin (F,\breve{E})$ or $e_H \in (G,\breve{E})$ and $e_G \notin (G,\breve{E})$ and for each point $e_G \in X_A$ and each $\overset{1}{\tau} \cup \overset{2}{\tau}$ *regular* closed soft set $(G_1,\breve{E})$ such that $e_G \notin (G_1,\breve{E})$ there happens a $\overset{1}{\tau} \cup \overset{2}{\tau}$ soft *regular* open set $(F_1,\breve{E})$ and $\overset{3}{\tau} \cup \overset{4}{\tau}$ soft *regular open* set $(F_2,\breve{E})$ such that $e_G \in (F_1,\breve{E}), (G_1,\breve{E}) \subseteq (F_2,\breve{E})$ and $(F_1,\breve{E}) \cap (F_2,\breve{E}) = \phi$. Similarly $\left(\breve{X},\overset{3}{\tau},\overset{4}{\tau},\breve{E}\right)$ is a soft regular T_3 space with respect to $\left(\breve{X},\overset{1}{\tau},\overset{2}{\tau},\breve{E}\right)$ So according to definition for $e_G, e_H \in X_A$, $e_G \neq e_H$ there exists a $\overset{3}{\tau} \cup \overset{4}{\tau}$ soft *regular open* set $(F,\breve{E})$ and $\overset{1}{\tau} \cup \overset{2}{\tau}$ soft *regular open* set $(G,\breve{E})$ such that $e_H \in (F,\breve{E})$ and $e_H \notin (F,\breve{E})$ or $e_H \in (G,\breve{E})$ and

$e_G \notin (G, \breve{E})$ and for each point $e_G \in X_A$ and each $\overset{3}{\breve{\tau}} \cup \overset{4}{\breve{\tau}}$s *regular* closed soft set $(G_1, \breve{E})$ such that $e_G \notin (G_1, \breve{E})$ there exists $\overset{3}{\breve{\tau}} \cup \overset{4}{\breve{\tau}}$ soft *regular* open set $(F_1, \breve{E})$ and $\overset{1}{\breve{\tau}} \cup \overset{2}{\breve{\tau}}$ soft *regular open* set $(F_2, \breve{E})$ such that $e_G \in (F_1, \breve{E}), (G_1, \breve{E}) \subseteq (F_2, \breve{E})$ and $(F_1, \breve{E}) \cap (F_2, \breve{E}) = \phi$. Hence $\left(\breve{X}, \overset{1}{\breve{\tau}}, \overset{2}{\breve{\tau}}, \overset{3}{\breve{\tau}}, \overset{4}{\breve{\tau}}, \breve{E}\right)$ is pair wise soft *regular* T_3 space.

Proposition 11. if $\left(\breve{X}, \overset{1}{\breve{\tau}}, \overset{2}{\breve{\tau}}, \overset{3}{\breve{\tau}}, \overset{4}{\breve{\tau}}, \breve{E}\right)$ be a soft quad topological space over $\breve{X}$. $\left(\breve{X}, \overset{1}{\breve{\tau}}, \overset{2}{\breve{\tau}}, \breve{E}\right)$ and $\left(\breve{X}, \overset{3}{\breve{\tau}}, \overset{4}{\breve{\tau}}, \breve{E}\right)$ are soft regular T_4 space then $\left(\breve{X}, \overset{1}{\breve{\tau}}, \overset{2}{\breve{\tau}}, \overset{3}{\breve{\tau}}, \overset{4}{\breve{\tau}}, \breve{E}\right)$ is pair wise soft regular T_4 space.

Proof. Suppose $\left(\breve{X}, \overset{1}{\breve{\tau}}, \overset{2}{\breve{\tau}}, \breve{E}\right)$ is soft regular T_4 space with respect to $\left(\breve{X}, \overset{3}{\breve{\tau}}, \overset{4}{\breve{\tau}}, \breve{E}\right)$ So according to definition for $e_G, e_H \in \breve{X}$, $e_G \neq e_H$ there happens a $\overset{1}{\breve{\tau}} \cup \overset{2}{\breve{\tau}}$ soft *regular* open set $(F, \breve{E})$ and a $\overset{3}{\breve{\tau}} \cup \overset{4}{\breve{\tau}}$ soft *regular* open set $(G, \breve{E})$ such that $e_G \in (F, \breve{E})$ and $e_H \notin (F, \breve{E})$ or $e_H \in (G, E)$ and $e_G \notin (G, \breve{E})$ each $\overset{1}{\breve{\tau}} \cup \overset{2}{\breve{\tau}}$ soft *regular* closed set (F_1, E) and a $\overset{3}{\breve{\tau}} \cup \overset{4}{\breve{\tau}}$ soft *regular* closed set (F_2, E) such that $(F_1, E) \cap (F_2, E) = \phi$. There occurs (F_3, E) and (G_1, E) such that (F_3, E) is soft $\overset{3}{\breve{\tau}} \cup \overset{4}{\breve{\tau}}$ *regular open* set (G_1, E) is soft a $\overset{1}{\breve{\tau}} \cup \overset{2}{\breve{\tau}}$ *regular* open set $(F_1, E) \subseteq (F_3, E), (F_2, E) \subseteq (G_1, E)$ and $(F_3, E) \cap (G_1, E) = \phi$. Similarly, $\overset{3}{\breve{\tau}} \cup \overset{4}{\breve{\tau}}$is soft regular T_4 space with respect to $\overset{1}{\breve{\tau}} \cup \overset{2}{\breve{\tau}}$so according to definition for $e_G, e_H \in X_A, e_G \neq e_H$there happens a $\overset{3}{\breve{\tau}} \cup \overset{4}{\breve{\tau}}$ soft *regular* open set $(F, \breve{E})$ and a $\overset{1}{\breve{\tau}} \cup \overset{2}{\breve{\tau}}$ soft *regular open* set $(G, \breve{E})$ such that $e_G \in (F, \breve{E})$ and $e_H \notin (F, \breve{E})$ or $e_H \in (G, \breve{E})$ and $e_G \notin (G, \breve{E})$ and for each $\overset{3}{\breve{\tau}} \cup \overset{4}{\breve{\tau}}$ soft *regular* closed set $(F_1, \breve{E})$ and $\overset{1}{\breve{\tau}} \cup \overset{2}{\breve{\tau}}$ soft *regular* closed set $(F_2, \breve{E})$ such that $(F_1, \breve{E}) \cap (F_2, \breve{E}) = \phi$. there occurs $(F_3, \breve{E})$ and $(G_1, \breve{E})$ such that $(F_3, \breve{E})$ is soft $\overset{3}{\breve{\tau}} \cup \overset{4}{\breve{\tau}}$ *regular open* set $(G_1, \breve{E})$ is soft $\overset{1}{\breve{\tau}} \cup \overset{2}{\breve{\tau}}$ $regular$semi open set such that $(F_1, \breve{E}) \subseteq (F_3, \breve{E}), (F_2, \breve{E}) \subseteq (G_1, \breve{E})$ and $(F_3, \breve{E}) \cap (G_1, \breve{E}) = \phi$ hence $\left(\breve{X}, \overset{1}{\breve{\tau}}, \overset{2}{\breve{\tau}}, \overset{3}{\breve{\tau}}, \overset{4}{\breve{\tau}}, \breve{E}\right)$ is pair wise soft regular T_4 space.

Proposition 12. Let $\left(\breve{X},\overset{1}{\breve{\tau}},\overset{2}{\breve{\tau}},\overset{3}{\breve{\tau}},\overset{4}{\breve{\tau}},\breve{E}\right)$ be a soft quad topological space over $\breve{X}$ and $\breve{Y}$ be a non-empty subset of $\breve{X}$. if $(\breve{Y},\overset{1^Y}{\breve{\tau}}\ \overset{2^Y}{\breve{\tau}}\ ,\overset{3^Y}{\breve{\tau}}\ ,\overset{4^Y}{\breve{\tau}}\ ,\breve{E})$ is pair wise soft regular T_3 space. Then $(\breve{Y},\overset{1^Y}{\breve{\tau}}\ \overset{2^Y}{\breve{\tau}}\ ,\overset{3^Y}{\breve{\tau}}\ ,\overset{4^Y}{\breve{\tau}}\ ,\breve{E})$ is pair wise soft regular T_3 space.

Proof. First we prove that $(\breve{Y},\overset{1^Y}{\breve{\tau}}\ \overset{2^Y}{\breve{\tau}}\ ,\overset{3^Y}{\breve{\tau}}\ ,\overset{4^Y}{\breve{\tau}}\ ,\breve{E})$ is pair wise soft *regular* T_1 space.

Let $e_G, e_H \in X_A, e_G \neq e_H$ if $\left(\breve{X},\overset{1}{\breve{\tau}},\overset{2}{\breve{\tau}},\overset{3}{\breve{\tau}},\overset{4}{\breve{\tau}},\breve{E}\right)$ is pair wise soft space then this implies that $\left(\breve{X},\overset{1}{\breve{\tau}},\overset{2}{\breve{\tau}},\overset{3}{\breve{\tau}},\overset{4}{\breve{\tau}},\breve{E}\right)$ is pair wise soft $\overset{1}{\breve{\tau}} \cup \overset{2}{\breve{\tau}}$ space. So there exists $\overset{1}{\breve{\tau}} \cup \overset{2}{\breve{\tau}}$ soft *regular* open set $(G,\breve{E})$ such that $e_G \in (F,\breve{E})$ and $e_H \notin (F,\breve{E})$ or $e_H \in (G,E)$ and $e_G \notin (G,E)$ now$e_G \in Y\ and\ e_G \notin (G,\breve{E})$. Hence $e_G \in Y \cap \left(F,\breve{E}\right) = (Y_F,\breve{E})$ then $e_H \notin Y \cap F\ (\alpha)$ for some $\alpha \in \breve{E}$, this means that $\alpha \in \breve{E}$ then $e_H \notin \breve{Y} \cap F\ (\alpha)$ for some $\alpha \in \breve{E}$.

There fore, $e_H \notin Y \cap \left(F,\breve{E}\right) = \left(Y_F,\breve{E}\right)$. Now $e_H \in \breve{Y}$ and $e_H \in \left(G,\breve{E}\right)$. Hence, $e_H \in \breve{Y} \cap \left(G,\breve{E}\right) = \left(G_Y,\breve{E}\right)$ where$\left(G,\breve{E}\right) \in \overset{3}{\breve{\tau}} \cup \overset{4}{\breve{\tau}}$. Consider $x \notin \left(G,\breve{E}\right)$. this means that $\alpha \in E$ then $x \notin \breve{Y} \cap G\ (\alpha)$ for some$\alpha \in \breve{E}$. There fore $e_G \notin \breve{Y} \cap \left(G,\breve{E}\right) = (G_Y,E)$ thus $((\breve{Y},\overset{1^Y}{\breve{\tau}}\ \overset{2^Y}{\breve{\tau}}\ ,\overset{3^Y}{\breve{\tau}}\ ,\overset{4^Y}{\breve{\tau}}\ ,\breve{E})$ is pair wise soft regular T_1 space.

Now, we prove that $(\breve{Y},\overset{1^Y}{\breve{\tau}}\ \overset{2^Y}{\breve{\tau}}\ ,\overset{3^Y}{\breve{\tau}}\ ,\overset{4^Y}{\breve{\tau}}\ ,\breve{E})$ is pair wise soft regular T_3 space.

Let $e_H \in \breve{Y}$ and $\left(G,\breve{E}\right)$ be soft *regular* closed set in $\breve{Y}$ such that $e_H \notin (G,\breve{E})$ where $\left(G,\breve{E}\right) \in \overset{1}{\breve{\tau}} \cup \overset{2}{\breve{\tau}}$then $\left(G,\breve{E}\right) = (\breve{Y},\breve{E}) \cap (F,\breve{E})$ for some soft *regular* closed set in $\overset{1}{\breve{\tau}} \cup \overset{2}{\breve{\tau}}$hence$e_H \notin (Y,\breve{E}) \cap (F,\breve{E})$ but $e_H \in (Y,\breve{E})$, so $e_H \notin (F,\breve{E})$ since $\left(\breve{X},\overset{1}{\breve{\tau}},\overset{2}{\breve{\tau}},\overset{3}{\breve{\tau}},\overset{4}{\breve{\tau}},\breve{E}\right)$ is soft regular T_3 space $\left(\breve{X},\overset{1}{\breve{\tau}},\overset{2}{\breve{\tau}},\overset{3}{\breve{\tau}},\overset{4}{\breve{\tau}},\breve{E}\right)$is soft R-regular space so there happens $\tau_1 \cup \tau_2$ soft *regular open* set $(F_1,\breve{E})$ and $\overset{3}{\breve{\tau}} \cup \overset{4}{\breve{\tau}}$ soft *regular* open set $(F_2,\breve{E})$ such that

$$e_H \in \left(F_1,\breve{E}\right), (G,\breve{E}) \subseteq (F_2,\breve{E})$$
$$\left(F_1,\breve{E}\right)\left(F_2,\breve{E}\right) = \breve{\emptyset}$$

Take $\left(G_1,\breve{E}\right) = (Y,\breve{E}) \cap (F_2,\breve{E})$ then $\left(G_1,\breve{E}\right),\left(G_2,\breve{E}\right)$ are soft *regular open* sets in $\breve{Y}$ such that

$$e_H \in \left(G_1,\breve{E}\right), \left(G,\breve{E}\right) \subseteq \left(Y,\breve{E}\right) \cap \left(F_2,\breve{E}\right) = \left(G_2,\breve{E}\right)$$

$$\left(G_1,\breve{E}\right)\cap\left(G_2,\breve{E}\right)\subseteq\left(F_1,\breve{E}\right)\cap\left(F_2,\breve{E}\right)=\breve{\emptyset}$$
$$\left(G_1,\breve{E}\right)\cap\left(G_2,\breve{E}\right)=\breve{\emptyset}$$

Therefore, $\left(\breve{X},\overset{1}{\breve{\tau}},\overset{2}{\breve{\tau}},\breve{E}\right)$ is soft R-regular space with respect to $\overset{3}{\breve{\tau}}^{Y}\cup\overset{4}{\breve{\tau}}^{Y}$ Similarly, Let $e_H\in\breve{Y}$ and $\left(G,\breve{E}\right)$ be a soft *regular* closed sub set in $\breve{Y}$ such that $e_H\notin\left(G,\breve{E}\right)$, Where $(G,E)\in\overset{3}{\breve{\tau}}\cup\overset{4}{\breve{\tau}}$then $\left(G,\breve{E}\right)=\left(Y,\breve{E}\right)\cap\left(F,\breve{E}\right)$ where $\left(F,\breve{E}\right)$ is some soft *regular* closed set in$\overset{3}{\breve{\tau}}\cup\overset{4}{\breve{\tau}}$ $e_H\notin(Y,E)\cap(F,E)$ but $e_H\in(Y,E)$ so $e_H\notin(F,E)$ since $\left(\breve{X},\overset{1}{\breve{\tau}},\overset{2}{\breve{\tau}},\overset{3}{\breve{\tau}},\overset{4}{\breve{\tau}},\breve{E}\right)$ is soft *R*-regular space so there happens $\overset{3}{\breve{\tau}}\cup\overset{4}{\breve{\tau}}$ soft *regular* open set $\left(F_1,\breve{E}\right)$ and $\overset{1}{\breve{\tau}}\cup\overset{2}{\breve{\tau}}$ soft *regular open* set $\left(F_{2,},\breve{E}\right)$. Such that

$$e_H\in\left(F_1,\breve{E}\right),\left(G,\breve{E}\right)\subseteq\left(F_2,\breve{E}\right)$$
$$\left(F_1,\breve{E}\right)\cap\left(F_{2,},\breve{E}\right)=\phi$$

Take

$$\left(G_1,\breve{E}\right)=\left(Y_,,\breve{E}\right)\cap\left(F_1,\breve{E}\right)$$
$$\left(G_1,\breve{E}\right)=\left(Y_,,\breve{E}\right)\cap\left(F_1,\breve{E}\right)$$

Then $\left(G_{1,},\breve{E}\right)$ and $\left(G_{2,},\breve{E}\right)$ are soft *regular open* set in,$\breve{Y}$ such that

$$e_H\in\left(G_1,\breve{E}\right),\left(G,\breve{E}\right)\subseteq\left(Y,\breve{E}\right)\cap\left(F_2,\breve{E}\right)=\left(G_2,\breve{E}\right)$$
$$\left(G_1,\breve{E}\right)\cap\left(G_2,\breve{E}\right)\subseteq\left(F_1,\breve{E}\right)\cap\left(F_2,\breve{E}\right)=\breve{\emptyset}.$$

Therefore$\overset{3}{\breve{\tau}}^{Y}\cup\overset{4}{\breve{\tau}}^{Y}$ is soft R-regular space.

4 Conclusions

A soft set with single specific topological structure is unable to shoulder up the responsibility to construct the whole theory. So to make the theory healthy, some additional structures on soft set has to be introduced. It makes, it more springy to develop the soft topological spaces with its infinite applications. In this regards we introduced soft topological structure known as soft quad topological structure with respect to soft regular open sets. In this article, topology is the supreme branch of pure mathematics which deals with mathematical structures. Freshly, many scholars have studied the soft set theory which is coined by Molodtsov [4] and carefully applied to many difficulties which contain uncertainties in our social life. Shabir and Naz [7] familiarized and deeply studied the origin of soft topological spaces. They also studied topological structures and exhibited their several properties with respect to ordinary points.

In the present work, we constantly study the behavior of soft regular separation axioms in soft quad topological spaces with respect to soft points as well as ordinary points of a soft topological space. We introduce soft regular qT_0structure, soft regular qT_1 structure, soft regular qT_2 structure, soft regular qT_3 and soft regular qT_4structure with respect to soft and ordinary points. In future we will plant these structures in different results. More over defined soft regular T_0structure w. r. t. soft regular T_1structure and vice versa, soft regular T_1structure w. r. t soft regular T_2structure and vice versa and soft regular T_3 space w. r. t soft regular T_4 and vice versa with respect to ordinary and soft points in soft quad topological spaces and studied their activities in different results with respect to ordinary and soft points. We also planted these axioms to different results. These soft semi separation axioms in quad structure would be valuable for the development of the theory of soft topology to solve complicated problems, comprising doubts in economics, engineering, medical etc. We also attractively discussed some soft transmissible properties with respect to ordinary as well as soft points. I have fastidiously studied numerous homes on the behalf of soft topology, and lastly I determined that soft topology is totally linked or in other sense we can correctly say that soft topology (Separation Axioms) are connected with structure. Provided if it is related with structures then it gives the idea of non-linearity beautifully. In other ways we can rightly say soft topology is somewhat directly proportional to non-linearity. Although we use non-linearity in Applied Math. So it is not wrong to say that soft topology is applied Math in itself. It means that soft topology has the taste of both of pure and applied math. In future I will discuss Separation Axioms in soft topology With respect to soft points. We expect that these results in this article will do help the researchers for strengthening the tool box of soft topological structures. In the forthcoming, we spread the idea of soft α-open, and soft b^{**}open sets in soft quad topological structure with respect to ordinary and soft points.

References

[1] B. M. Ittanagi, 2014, soft bi-topological spaces, International Journal of Computer Application (0975 8887), v107-No. 7.

[2] S. A El-Sheikh, Rodyna A. Hosny and A. M. Abd-e-Latif, 2015, Characterization of b soft separation Axioms in soft topological spaces in Inf. Sci. Let t. 4 (3) 125-133.

[3] M. Naz, M. Shabir and M. I. Ali, 2015, Separation axiom in bi soft topological spaces, Arxiv. 1850 of . 00866 vl [Math, GN]

[4] D. A Molodtsov, 1999, soft set theory first results, Comp. Math. Appl., 37, 19-31.

[5] J. C. Kelly, 1963, Bitopological spaces proc London math-soc, 13, 71-81.

[6] P. K. Maji, R, Biswas and A. R. Roy, 2003, soft set theory, Comp. Math. Appl., 45, 555-562.

[7] M. Shabir and M. Naz, on soft topological space , 2011, Comp. Math. Appl., 61, 1786-1799.

[8] H. Aktas and N. Cagman, 2007, soft sets and soft groups, Inform. Sci 1 (77), 2726-2735.

[9] Chen, 2005, the paramatrization reduction of soft sets and application,Comp. Math, Appl., 49,757-763.

[10] F. Feng, Y. B. Jun, X. Z. Zhao, 2008, soft semi rings, Comp. Math. Appl., 56, 2621-2628

[11] Z. Kong, L. Gao, L. Wang, S. Li, 2008, The normal parameter reduction of soft set and its algorithm. Comp., Math. Appl., 56, 3029-3027.

[12] Y. B. Jun, C. H. Park, 2008, Application of soft sets and ideal theory of BCK/BCI-algebra, Inf, Sci 178, 2466-2475.
[13] Pie and D. Miao, 2005, Form soft sets to information system,in X. Hu, Q. Liu,A. Skowron, T. Y . Lin, R. R. Yager, B. Zhang (Eds), Proceedings of Granular Computing,Vol. 2,IEEE,pp. 617-621.
[14] Majumdar and S. K Samanta, 2010, on soft mapping, Comp. Math. Appl., 60, 2666-2672.
[15] B. Jun, 2008, soft BSK/BCI-algehra, Comp. Math. Appl., 56, 1408-1413.
[16] Shahir and M. I. Ali, 2009) soft ideals and generalized fuzzy ideals in semi-groups, New Math. Wat. Comp. 5,599-615
[17] Y. Zou and Z. Xiao, 2008, Data analysis approach of soft sets under incomp Let e information,
Knowledge-Based System, 21, 941-945.
[18] A. Aygunoglu and H. Aygun, Some notes on soft topological space , Neural Comp and Applic DoI 10. 1007/s. 00521-011-0722-3.
[19] Husain, B Ahmad, 2011, Some properties of soft topological spaces, Comp. Math. App., 62, 4058-4067.
[20] D. N Geogiou, A. C Megart is, V. I. 2013, petropounlos On soft topological spaces, Appl, Math. Inf, Sci 7 (2), 1889-190
[21] Hen B, 2013, soft semi open sets and related properties in soft topological spaces. Appl, math, Inf, sci 7 (1), 287-294.
[22] A. Kandil, O. A . E. Tantawy, S. A,EI Sheikh and A. M, Abd-e-Lat if, 2015, γ operation and decomposition of some form of soft continuity of soft topological spaces via soft ideal, Ann Fuzzy Math, inf, 9 (3),385-402.

[23] A. Kandil, O. A . E. Tantawy, S. A, EI Sheikh and A. M, Abd-e-Latif, 2014, soft connectedness via soft ideal, Journal of New result in science, 4,90-108.
[24] A. Kandil, O. A . E. Tantawy, S. A, EI Sheikh and A. M, Abd-e-Latif, 2014, soft ideal theory, soft local function and generalize soft topological spaces, Appl. Math. Inf. Sci, 8 (4),1595-1603.
[25] A. Kandil. O. A . E. Tantawy, S. A,EI Sheikh and A. M, Abd-e-Latif, 2015, soft regular ity and normality based on semi open soft sets and soft ideals,App. Math. Inf. Sci. Let t, 3,47-55.
[26] A. Kandil, O. A . E. Tantawy, S. A,EI Sheikh and A. M, Abd-e-Latif, 2014) soft semi compactness, via soft ideals, Appl. Math . Inf. Sci, 8 (5),2297-2306
[27] A. Kandil, O. A . E. Tantawy, S. A,EI Sheikh and A. M, Abd-e-Latif, 2014) soft semi (Quasi) husdorff spaces via soft ideals, South Asian, J. Math, 4 (6),265-284.

[28] I. Zorlutana,N. Akdog,W. K. Min, S. Atmaca, 2012, Remarks on soft topological spaces, Annals, of Fuzzy Mathematics and informatics 3 (2),171-185
[29] J. Mahanta. P. K. Das, 2012, On soft topological spaces Via semi open and semi closed soft sets, ar XIV 1203. 1433VI [math, G. N]
[30] A. Kandil'O. A. E. Tantawy, S. A El-Sheikh and A. M Abd-Latif, 2014, Sopra generalized closed soft sets with respect to an soft ideal in supra soft topological spaces Appl. Math . Inf. Sci, (4), 1731-1740.
[31] S. A El-Sheikh and A. M Abd-e-Latif, 2014, Decomposition of some type of supra soft sets and soft continuity, international journal of mathematics trends and technology, 9 (1), 37-56.
[32] L. A. Zadeh, 1965, Fuzzy sets, information and control 8,338-353.

[33] Q. M. Sun, Z. L Zhang, J. Liu soft sets and soft modules, proceeding of rough sets and Knowledge technology, third international conference, RSKT 2008,17-19 May, Chengdu, Chaina, PP 403-409, 2008.

[34] C. H. Park, Y. B. Jun, M. A. Ozturk, 2008, soft WS Algebras, Communications of the Korean Mathematical Society 23 (3), 313-324.

[35] I. Arockiarani and A. A. Lancy, 2013, Generalized soft gβ-closed sets and soft gsβ-closed sets in soft topological spaces, IJMRS, 4 (2), 1-7.

[36] B. Chen, 2013, soft semi-open sets and related properties in soft topological spaces. Appl. Math. Inf. Sci, 7 (1), 287-294.

[37] A. S. Mashhour, M. E. Abd El-Monsef and S. N. El-Deeb, 1982, A note on semi-continuity and pre-continuity. Indian J. Pure Appl. Math. 13 (10), 1119-1123.

[38] S. Hussain, B. Ahmad, 2015, soft separation axioms in soft topological space submitted to elsvier.

[39] A. Kandil, O. A . E. Tantawy, S. A,EI Sheikh and A. M, Abd-e-Latif, 2014, soft semi separation axioms and some

[40] U. D. Tapi,Ranu Sharma,Bhagyashri A. Deole, 2014, Separation Axioms in Quad topological spaces, Bul Let in of Mathematical Science and Applications İS SN 2278-9634, Vol. 10, pp 16-18.

[41] N. F. Hameed and Yahya Abid, 2011,Certain types of separation axioms in Tri-topological spaces,Iraqi Journal of science,Vol 52,No. 2,PP212-217.

[42] Y. Yumak and A. K. Kaymaci, 2015, some β-Open sets and their Applications, journal of New Theory, 4 (2015) 80-89.

http://www.newtheory.org ISSN: 2149-1402

Journal of New Theory

Year: **2017**, *Number*: **19**, *Pages*: **108**

EDITORIAL

First of all, on behalf of the team of Journal of New Theory (JNT), I am writing to wish you a Happy Christmas and a peaceful and prosperous New Year in 2018. JNT had another good year. I am extremely grateful for your contribution to the journal and your hard work over the past year. I am looking forward to work with you in the coming year.

We are happy to inform you that, final number of 2017, Number 19 of the JNT, is completed with 7 articles.

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We would like to express our deepest thanks to all of the members of the editorial board and reviewers of the papers in this issue who are H. Günal, F. Smarandache, M. A. Noor, J. Zhan, S. Pramanik, M. I. Ali, P. K. Maji, S. Broumi, O. Muhtaroğlu, A. A. Ramadan, S. Enginoğlu, S. J. John, M. Ali, A. S. Sezer, A. A. El-latif, J. Ye,D. Mohamad, B. Mehmetoğlu, İ. Zorlutuna, B. H. Çadırcı, C. Kaya, Ç. Çekiç, H. M. Doğan, H. Kızılaslan,İ. Gökce, İ. Türkekul, R. Yayar, A. Yıldırım, Y. Budak, N. Sağlam, N. Yeşilayer, N. Kizilaslan, S. Karaman, S. Demiriz, S. Öztürk, S. Eğri, Ş. Sözen, E. H. Hamouda, K. Mondal, T. Muhammad.

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Happy reading!

25 December 2017

Prof. Dr. Naim Çağman
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