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Meromorphic Function of Fuzzy Complex Variables

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Abstaract — The fuzzy complex set is a fuzzy set whose values lies in the unit circle $|z| \leq 1$ in the complex plane. The Nevanlinna characteristic function playes an important role in the theory of entire and meromorphic function. In this paper we introduce the notion of fuzzy to the Nevanlinna theory and investigate some important properties of Nevanlinna characteristic function of fuzzy complex variables.

Keywords — *Fuzzy complex variavles, fuzzy complex numbers, Nevanlinna characteristic of fuzzy variables.*

1 Introduction.

In the year 1965 Zadeh [8] proposed the fuzzy sets. Many thousand of papers, articles have published in different journals. The idea of fuzzy sets based on real number system. Buckley [1] and [2] introduced the idea of fuzzy complex sets in the year 1987. In Buckley's definition, the representation of fuzzy complex number in the polar form is quite unstable. In the year 2003, Ramot et al. [4] and [5] proposed a new concept of defining a fuzzy complex set.

In this newly defined fuzzy complex number, as the phase and membership grade are present, the Fuzzy complex number(FCN) takes the well-known (wavelike) property of complex numbers. This wavelike property distinguishes the fuzzy complex sets with the traditional fuzzy sets. Many continuations of work of Ramot et al. [4] has been studied by various authors ([5], [9], etc.).

Now we are trying to establish the Nevanlinna Characteristic function in FCN and also investigate some useful properties.

2 Basic Concepts on Fuzzy Complex Numbers

Although this paper is in the line different as that of Buckley [1], we recall the definition of FCN introduced by Buckley.

Definition 2.1. [8] Let X be an universal set. Then the fuzzy subset A of X is defined by its membership function $\mu_A(x) : X \to [0,1]$ which will assign a real number $\mu_A(x)$ in the interval $[0,1]$ to each element $x \in X$, where the value of $\mu_A(x)$ shows the grade of membership of x in A.

We are not providing the basic definitions and notations such as $\alpha - cut$ or weak $\alpha - cut$ of fuzzy sets as they are available in [6] and [7].

We are now giving two basic definitions introduced by Buckley in [1].

Definition 2.2. Fuzzy complex set: Let $\mathbb{C}$ be a complex field. Then the fuzzy subset $\tilde{Z}$ of $\mathbb{C}$ is defined by the membership function $\mu_{\tilde{z}}(z) : \mathbb{C} \to [0,1]$.

Definition 2.3. $\tilde{z}$ is a fuzzy complex number if and only if

(i) $\alpha_{\tilde{z}}(z)$ is continuous ,

(ii) ${}^{\alpha-}\mu_{\tilde{z}}(z)$ is open, bounded, connected and simply connected for $0 \leq \alpha < 1$,

(iii) ${}^{1+}\mu_{\tilde{z}}(z)$ is non empty.

We now present the definition introduced by Ramot et al ([4]) .

Definition 2.4. A fuzzy complex set $\mathbb{C}_\mu$, defined on a universe or discourse U is characterized by a membership function $\mu_{\mathbb{C}}(z)$ that assigns any $z \in U$ a complex valued grade of membership in $\mathbb{C}_\mu$, i.e ,

$\mu_{\mathbb{C}}(z) = r_\mu(z) . \exp\left(i \arg_\mu(z)\right)$, where $r_\mu(z)$ and $\arg_\mu(z)$ are both real valued functions and $i = \sqrt{(-1)}$.

Here $\arg_\mu(z)$ is the principal argument and $0 \leq r_\mu(z) \leq 1$.

Also, $Arg_\mu(z) = \arg_\mu(z) + 2k\pi,\ k = 0,\ \pm 1,\ \pm 2, \cdots$

This $\arg_\mu(z)$ gives $\mu_{\mathbb{C}}(z)$ a wavelike property. Clearly $\mu_{\mathbb{C}}(z)$ lies on the unit circle centered at origin in the complex plane.

In this paper we use the notation $\phi_\mu(z)$ for the argument of the fuzzy complex numbers.

Definition 2.5. [9] Let A and B be two fuzzy complex sets on U such that $\mu_A(z) = r_A(z) . \exp(i\phi_A(z))$ and $\mu_B(z) = r_B(z) . \exp(i\phi_B(z))$, then

$$\begin{aligned} (i)\ \ \mu_{A\cup B}(z) &= r_{A\cup B}(z) . \exp i\left(\phi_{A\cup B}(z)\right) \\ &= \max\left(r_A(z), r_B(z)\right) . \exp\left(i \max\left(\phi_A(z), \phi_B(z)\right)\right) \\ (ii)\ \ \mu_{A\cap B}(z) &= r_{A\cap B}(z) . \exp i\left(\phi_{A\cap B}(z)\right) \\ &= \min\left(r_A(z), r_B(z)\right) . \exp\left(i \min\left(\phi_A(z), \phi_B(z)\right)\right) . \end{aligned} \tag{1}$$

Definition 2.6. [9] Let $F_{\mathbb{C}}(U)$ be the set of all fuzzy complex sets on U. Let $C_\alpha \in F_{\mathbb{C}}(U)$, $\alpha \in I$ and $\mu_{C_\alpha}(z) = r_{C_\alpha}(z) . \exp(i\phi_{C_\alpha}(z))$ then $\underset{\alpha\in I}{\cup} C_\alpha \in F_{\mathbb{C}}(U)$ and its membership function is

$$\mu_{\underset{\alpha\in I}{\cup} C_\alpha}(z) = \sup_{\alpha\in I}\left(r_{C_\alpha}(z)\right) . \exp\left(i \sup_{\alpha\in I}\left(\phi_{C_\alpha}(z)\right)\right) . \tag{2}$$

Definition 2.7. [9] Let C_μ be a fuzzy complex set on U and

$$\mu_C(z) = r_\mu(z) \,. \exp(i\phi_\mu(z))$$

then the fuzzy complex complement of C is denoted by $\bar{C}$ and is specified by the membership function

$$\begin{aligned} \mu_{\bar{C}}(z) &= r_{\bar{C}}(z) \,. \exp(i\phi_{\bar{\mu}}(z)) \\ &= (1 - r_C(z)) \,. \exp\{i(2\pi - \phi_\mu(z))\} \ . \end{aligned} \tag{3}$$

Definition 2.8. Logarithm of a fuzzy complex number: The logarithm of a fuzzy complex number is defined as $\log z_\mu = \log(r_\mu) + i\phi_\mu(z)\,,\ r_\mu \in (0,1]$ or equivalently $\log(\mu_C(z)) = \log(r_\mu(z)) + i\phi_\mu(z)\,,\ 0 < r_\mu(z) \le 1$.

Definition 2.9. Let A and B be two fuzzy complex sets in the universe U , and $\mu_A(z) = r_\mu(z) \,. \exp(i\phi_\mu(z))$ abd $\mu_B(z) = r_\mu(z) \,. \exp(i\phi_\mu(z))$ be the membership functions defined on it. Then the fuzzy product of A and B is defined by

$$\mu_{A\circ B}(z) = (r_{\mu_A}(z) \,. r_{\mu_B}(z)) \,. \exp\left[i\left\{2\pi \frac{\phi_{\mu_A}(z)}{\phi_{\mu_B}(z)}\right\}\right] \ .$$

Definition 2.10. Positive logarithm: The positive logarithm is defined as

$$\begin{aligned} \log^+(x) &= \log x,\ if\ x \ge 1 \\ &= 0,\ if\ 0 \le x < 1\ . \end{aligned}$$

Remark 2.1. With the above definition it can easily be verified that

$$\log x = \log^+ x - \log^+\left(\frac{1}{x}\right) \ .$$

Now we present some basic definitions of the theory of entire and meromorphic function.

Definition 2.11. A complex valued function which has no singularities other than poles in the finite complex plane is known as meromorphic function.

Definition 2.12. The proximity function: Let $f(z)$ be meromorphic on $|z| \le R$, $(0 < R < \infty)$. Then the proximity function is defined by

$$m(R,f) = \frac{1}{2\pi}\int_0^{2\pi} \log^+\left|f\left(Re^{i\phi}\right)\right| d\phi \ .$$

Definition 2.13. The counting function: Let $f(z)$ be a non constant meromorphic function on the complex plane. For any complex number $'a'$ we denote by $n(r,a) = n(r,a,f)$, the number of zeros of the equation $f(z) = a$ (counting multiplicities).

The function $N(r,a) = \int_0^r \frac{n(t,a)-n(0,a)}{t} dt + n(0,a)\log r$ is called the counting function. By $n(r,\infty,f)$ or simply $n(r,f)$ we mean the poles of the function $f(z)$ in $|z| \le r$.

Definition 2.14. Nevanlinna characteristic function: The sum of the proximity function and the counting function is denoted by $T(r,f)$. Rolf Nevanlinna defined the characteristic function as $T(r,f) = N(r,f) + m(r,f) + O(1)$.

We do not explain the basic definition and notation of the Nevanlinna theory as they are available in [3].

In the line of Nevanlinna it can easily be verifed that

$$T(r_\mu,f) = N(r_\mu,f) + m(r_\mu,f) + O(1) \ . \tag{4}$$

3 Known Results

We now present the well-known result of the Nevanlinna theory.

Theorem 3.1. $T(r,f)$ is an increasing function of r and convex function of $\log r$.

We are not providing the proof as it is available in [3].

4 Main Results

We now discuss our main results of this paper.

Theorem 4.1. $T(r_\mu,f)$ is an increasing function of r_μ and convex function of $\log r_\mu$.

Proof. The well known Jensen's formula is given by

$$\log|f(0)| = \frac{1}{2\pi}\int_0^{2\pi} \log^+\left|f\left(R.e^{i\phi_\mu}\right)\right| d\phi_\mu + \sum_{\alpha=1}^{M} \log\frac{|a_\alpha|}{R} - \sum_{\alpha=1}^{N}\log\frac{|a_\beta|}{R},$$

provided $f(0) \neq 0$ *or* ∞ , $a_\alpha\,(\alpha = 1, 2, \cdots, M)$ and $b_\beta\,(\beta = 1, 2, \cdots N)$ are the zeros and poles of $f(z)$ in $|z| < r$.

Now by Jensen's theorem with $R = 1$ and $f(z_\mu) = a_\mu - z_\mu$, we get

$$\frac{1}{2\pi}\int_0^{2\pi} \log^+\left|R.e^{i\phi_\mu} - a_\mu\right| d\phi_\mu = \log^+|a_\mu|;\ for\ all\ a_\mu \in \mathbb{C}_\mu\ . \tag{5}$$

Now applying Jensen's formula to the function $f(z_\mu) - e^{i\theta_\mu}$, we get

$$\begin{aligned}\log\left|f(0) - e^{i\theta_\mu}\right| &= \frac{1}{2\pi}\int_0^{2\pi} \log\left|f\left(r_\mu.e^{i\phi_\mu}\right) - e^{i\theta_\mu}\right| d\phi_\mu \\ &\quad -N\left(r_\mu, \frac{1}{f - e^{i\theta_\mu}}\right) + N\left(r_\mu, f - e^{i\theta_\mu}\right)\ .\end{aligned}$$

On integrating from 0 to 2π, one may get

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi} \log\left|f(0) - e^{i\theta_\mu}\right| d\theta_\mu &= \frac{1}{2\pi}\int_0^{2\pi} \log\left|f\left(r_\mu.e^{i\phi_\mu}\right) - e^{i\theta_\mu}\right| d\phi_\mu \\ &\quad - \frac{1}{2\pi}\int_0^{2\pi} N\left(r_\mu, \frac{1}{f - e^{i\theta_\mu}}\right) d\theta_\mu + \frac{1}{2\pi}\int_0^{2\pi} N\left(r_\mu, f - e^{i\theta_\mu}\right) d\theta_\mu\ .\end{aligned}$$

Now replacing a_μ by $f\left(r.e^{i\phi_\mu}\right)$, we get from 5 that

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi} \log\left|f(0) - e^{i\theta_\mu}\right| d\theta_\mu &= \frac{1}{2\pi}\int_0^{2\pi} \log^+\left|f\left(r_\mu.e^{i\phi_\mu}\right)\right| d\phi_\mu \\ &\quad -\frac{1}{2\pi}\int_0^{2\pi} N\left(r_\mu, \frac{1}{f - e^{i\theta_\mu}}\right) d\theta_\mu + N(r_\mu, f) \\ &= m(r_\mu, f) + N(r_\mu, f) \\ &\quad -\frac{1}{2\pi}\int_0^{2\pi} N\left(r_\mu, \frac{1}{f - e^{i\theta_\mu}}\right) d\theta_\mu \\ &= T(r_\mu, f) - \frac{1}{2\pi}\int_0^{2\pi} N\left(r_\mu, \frac{1}{f - e^{i\theta_\mu}}\right) d\theta_\mu\ .\end{aligned}$$

Thus we get

$$T(r_\mu, f) = \frac{1}{2\pi}\int_0^{2\pi} N\left(r_\mu, \frac{1}{f - e^{i\theta_\mu}}\right) d\theta_\mu + \log^+ |f(0)| \tag{6}$$

This is the fuzzy version of Cartan's identity. Now differentiating 6 with respect to r_μ we get

$$\begin{aligned}\frac{d(T(r_\mu, f))}{dr_\mu} &= \frac{d}{dr_\mu}\left\{\frac{1}{2\pi}\int_0^{2\pi} N\left(r_\mu, \frac{1}{f - e^{i\theta_\mu}}\right) d\theta_\mu + \log^+ |f(0)|\right\} \\ &= \frac{d}{dr_\mu}\left\{\frac{1}{2\pi}\int_0^{2\pi} N\left(r_\mu, \frac{1}{f - e^{i\theta_\mu}}\right) d\theta_\mu\right\}\end{aligned}$$

Hence

$$\frac{d(T(r_\mu, f))}{dr_\mu} = \frac{d}{dr_\mu}\{\frac{1}{2\pi}\int_0^{2\pi}(\frac{1}{2\pi}\int_0^{r_\mu}\frac{n\left(t, e^{i\theta_\mu}\right) - n\left(0, e^{i\theta_\mu}\right)}{t}dt \\ + n\left(0, e^{i\theta_\mu}\right).\log r_\mu) d\theta_\mu\}$$

That is

$$\begin{aligned}\frac{d(T(r_\mu, f))}{dr_\mu} &= \frac{d}{dr_\mu}\left[\frac{1}{2\pi}\int_0^{2\pi}\left\{\frac{1}{2\pi}\int_0^{r_\mu}\frac{n\left(t, e^{i\theta_\mu}\right)}{t}dt\right\}\right] \\ &= \frac{1}{2\pi}\int_0^{r_\mu}\frac{n\left(t, e^{i\theta_\mu}\right)}{t}dt\ .\end{aligned}$$

Similarly differentiating 6 with respect to $\log r_\mu$, we get

$$\frac{d(T(r_\mu, f))}{d(\log r_\mu)} = \frac{1}{2\pi}\int_0^{r_\mu} n\left(t, e^{i\theta_\mu}\right) dt\ .$$

Now $n\left(t, e^{i\theta_\mu}\right)$ is counting function and hence non negative and non decreasing. That is $\frac{d(T(r_\mu,f))}{d(\log r_\mu)} \geq 0$ and $\frac{d(T(r_\mu,f))}{dr_\mu} \geq 0$ for all $r_\mu \in \mathbb{C}_\mu$.

Therefore $T(r_\mu, f)$ is a convex function of $\log r_\mu$ and an increasing function of r_μ. □

Theorem 4.2. If A and B be two fuzzy complex sets on the universe U and $\mu_A(z) = r_A(z).e^{i\phi_{\mu_A}(z)}$ and $\mu_B(z) = r_B(z).e^{i\phi_{\mu_B}(z)}$ then $T(r_{\mu_{A\cup B}}, f) \geq T(r_{\mu_A}, f)$ and $T(r_{\mu_{A\cup B}}, f) \geq T(r_{\mu_B}, f)$.

Proof. By theorem 4.1, it is obvious that if $r_{\mu_A} \geq r_{\mu_B}$, then $T(r_{\mu_A}, f) \geq T(r_{\mu_B}, f)$. Also by definition 2.5 we have

$$\begin{aligned}\mu_{A\cup B}(z) &= r_{\mu_{A\cup B}}(z)\, e^{i\left(\phi_{\mu_{A\cup B}}(z)\right)} \\ &= \max\left(r_{\mu_A}(z), r_{\mu_B}(z)\right).e^{\left(i\max\left(\phi_{\mu_A}(z), \phi_{\mu_B}(z)\right)\right)}\ .\end{aligned}$$

Therefore

$$\begin{aligned} T\left(r_{\mu_{A\cup B}}, f\right) &= T\left(\max\left(r_{\mu_A}(z), r_{\mu_B}(z)\right), f\right) \\ &\geq T\left(r_{\mu_A}, f\right) . \end{aligned}$$

Similarly

$$T\left(r_{\mu_{A\cup B}}, f\right) \geq T\left(r_{\mu_B}, f\right)$$

□

Example 4.1. To give an example we use the popular convention notation for fuzzy sets. When the universe U is infinite and continuous, the fuzzy set A in the universe U can be expressed as

$$A = \int \frac{\mu_A(z)}{z},\ z \in U\ .$$

In this notation the integral sign is not the integral used in calculus or algebraic integral, but a set union notation for continuous variable.

When the discourse is finite or discrete, the fuzzy set can be expressed as

$$A = \frac{\mu_A(z_1)}{z_1} + \frac{\mu_A(z_2)}{z_2} + \frac{\mu_A(z_3)}{z_3} + \cdots + \frac{\mu_A(z_n)}{z_n}\ .$$

In both notation the fraction or horizontal bar is not a quotient but a delimiter. In both notation the numerator represents the membership value in the set A associated with the element of the universe indicated in the denominator.

Now consider

$$A = \int \frac{\frac{1}{z}e^{i(z\pi)}}{z},\ z \geq 1,\ B = \int \frac{\frac{1}{2z}e^{i(2z\pi)}}{z},\ z \geq 1,$$

and $f(z) = \exp z$. Now

$$A \cup B = \int \frac{\frac{1}{z}e^{i(z\pi)}}{z}\ .$$

Then

$$\begin{aligned} T\left(r_{\mu_A}, f\right) &= \frac{r_\mu}{\pi} = T\left(r_{\mu_{A\cup B}}, f\right), \\ T\left(r_{\mu_B}, f\right) &= \frac{r_\mu}{2\pi},\ \left(0 \leq r_\mu \leq 1\right)\ . \end{aligned}$$

Thus

$$\begin{aligned} T\left(r_{\mu_{A\cup B}}, f\right) &\geq T\left(r_{\mu_A}, f\right)\ \textit{and} \\ T\left(r_{\mu_{A\cup B}}, f\right) &\geq T\left(r_{\mu_B}, f\right)\ . \end{aligned}$$

and the result follows .

Theorem 4.3. Let A and B be two fuzzy complex sets on the universe U and $\mu_A(z) = r_A(z).e^{i\phi_{\mu_A}(z)}$ and $\mu_B(z) = r_B(z).e^{i\phi_{\mu_B}(z)}$. Then $T\left(r_{\mu_{A\cap B}}, f\right) \leq T\left(r_{\mu_A}, f\right)$ and $T\left(r_{\mu_{A\cap B}}, f\right) \leq T\left(r_{\mu_B}, f\right)$.

Proof. By theorem 4.1,it is obvious that if $r_{\mu_A} \geq r_{\mu_B}$ then $T(r_{\mu_A}, f) \geq T(r_{\mu_B}, f)$. Also by definition 2.5 we have

$$\mu_{A\cap B}(z) = r_{\mu_{A\cap B}}(z) \, .e^{i\left(\phi_{\mu_{A\cap B}}(z)\right)}$$
$$= \min\left(r_{\mu_A}(z), r_{\mu_B}(z)\right) .e^{\left(i\min\left(\phi_{\mu_A}(z),\phi_{\mu_B}(z)\right)\right)} .$$

Therefore

$$\begin{aligned} T\left(r_{\mu_{A\cap B}}, f\right) &= T\left(\min\left(r_{\mu_A}(z), r_{\mu_B}(z)\right), f\right) \\ &\leq T\left(r_{\mu_A}, f\right) . \end{aligned}$$

Similarly

$$T\left(r_{\mu_{A\cap B}}, f\right) \leq T\left(r_{\mu_B}, f\right) .$$

□

Example 4.2. We go with the same sets as in example 4.1 .

Let,

$$A = \int \frac{\frac{1}{z}e^{i(z\pi)}}{z}, \; z \geq 1, \; B = \int \frac{\frac{1}{2z}e^{i(2z\pi)}}{z}, \; z \geq 1,$$

and $f(z) = \exp z$. Now,

$$A \cap B = \int \frac{\frac{1}{2z}e^{i(2z\pi)}}{z} .$$

Then

$$T\left(r_{\mu_B}, f\right) = \frac{r_\mu}{2\pi} = T\left(r_{\mu_{A\cap B}}, f\right), \; T\left(r_{\mu_A}, f\right) = \frac{r_\mu}{\pi}, \; \left(0 \leq r_\mu \leq 1\right) .$$

Thus

$$T\left(r_{\mu_{A\cap B}}, f\right) \leq T\left(r_{\mu_B}, f\right) \; and \; T\left(r_{\mu_{A\cap B}}, f\right) \leq T\left(r_{\mu_A}, f\right) ,$$

and the result follows .

Theorem 4.4. For any three fuzzy complex sets A, B and C on the universe U , $T(r_\mu, f)$ follows the associativity property with respect to the union of fuzzy sets, *i.e,*

$$T\left(r_{\mu_{(A\cup B)\cup C}}, f\right) = T\left(r_{\mu_{A\cup(B\cup C)}}, f\right) .$$

Proof. As

$$\begin{aligned} \mu_{(A\cup B)\cup C}(z) &= \max\left\{r_{\mu_{(A\cup B)}}(z), r_{\mu_C}(z)\right\} .e^{i\left\{\max\left(\phi_{\mu_{A\cup B}}(z),\phi_{\mu_C}(z)\right)\right\}} \\ &= \max\left\{r_{\mu_A}(z), r_{\mu_B}(z), r_{\mu_C}(z)\right\} .e^{i\left\{\max\left(\phi_{\mu_A}(z),\phi_{\mu_B}(z),\phi_{\mu_C}(z)\right)\right\}} \end{aligned}$$

and $T(r_\mu, f)$ is an increasing function, therefore

$$\begin{aligned} T\left(r_{\mu_{(A\cup B)\cup C}}, f\right) &= T\left(\max\left\{r_{\mu_{(A\cup B)}}(z), r_{\mu_C}(z)\right\}, f\right) \\ &= T\left(\max\left\{r_{\mu_A}(z), r_{\mu_B}(z), r_{\mu_C}(z)\right\}, f\right) \\ &= T\left(\max\left\{r_{\mu_A}(z), \left(r_{\mu_B}(z), r_{\mu_C}(z)\right)\right\}, f\right) \\ &= T\left(\max\left\{r_{\mu_A}(z), r_{\mu_{B\cup C}}(z)\right\}, f\right) \\ &= T\left(r_{\mu_{A\cup(B\cup C)}}, f\right) . \end{aligned}$$

□

Example 4.3. Let,

$$A=\int\frac{\frac{1}{z}e^{i(z\pi)}}{z},\ B=\int\frac{\frac{1}{2z}e^{i(2z\pi)}}{z},\ C=\int\frac{\frac{1}{3z}e^{i(3z\pi)}}{z},\ z\geq 1$$

and $f(z)=\exp z$,then

$$\begin{aligned} A\cup B &= \int\frac{\frac{1}{z}e^{i(z\pi)}}{z} \\ B\cup C &= \int\left(\frac{\frac{1}{2z}e^{i(2z\pi)}}{z}+\frac{\frac{1}{3z}e^{i(3z\pi)}}{z}\right) \\ A\cup B\cup C &= \int\frac{\frac{1}{e^{z}}e^{i(z\pi)}}{z}=(A\cup B)\cup C=A\cup(B\cup C), \end{aligned}$$

and the result follows trivially.

Theorem 4.5. For any three fuzzy complex sets A, B and C on the universe U , $T(r_\mu,f)$ follows the associativity property with respect to the intersection of fuzzy sets, *i.e,*

$$T\left(r_{\mu_{(A\cap B)\cap C}},f\right)=T\left(r_{\mu_{A\cap(B\cap C)}},f\right)\ .$$

Proof. The proof is similar as the proof of previous theorem. So we left the proof. □

Example 4.4. Let,

$$A=\int\frac{\frac{1}{2z}e^{i(2z\pi)}}{z},\ B=\int\frac{\frac{1}{3z}e^{i(3z\pi)}}{z},\ C=\int\frac{\frac{1}{4z}e^{i(4z\pi)}}{z},\ z\geq 1$$

and $f(z)=\exp z$, therefore

$$\begin{aligned} A\cap B &= \int\frac{\frac{1}{6z}e^{i(6z\pi)}}{z},\ B\cap C=\int\frac{\frac{1}{12z}e^{i(12z\pi)}}{z}, \\ A\cap B\cap C &= \int\frac{\frac{1}{12z}e^{i(12z\pi)}}{z}=(A\cap B)\cap C=A\cap(B\cap C), \end{aligned}$$

and the result follows trivially.

Theorem 4.6. Let $F(U)$ be the set of all fuzzy complex sets on the universe U, and $A_\alpha\in F(U)$, $\alpha\in I$. Also let $\mu_{A_\alpha}(z)=r_{\mu_{A_\alpha}}(z).e^{i\phi_{\mu_{A_\alpha}}(z)}$ be its membership function. Then

$$\begin{aligned} (i)\ T\left(r_{\mu_{A_1\cup A_2\cup\cdots\cup A_n}},f\right) &= T\left(\sup_{\alpha\in[1,n]}\left\{r_{\mu_{A_\alpha}}\right\},f\right)\geq\sup_{\alpha\in[1,n]}\left\{T\left(r_{\mu_{A_\alpha}},f\right)\right\} \\ (ii)\ T\left(r_{\mu_{A_1\cap A_2\cap\cdots\cap A_n}},f\right) &= T\left(\inf_{\alpha\in[1,n]}\left\{r_{\mu_{A_\alpha}}\right\},f\right)\leq\inf_{\alpha\in[1,n]}\left\{T\left(r_{\mu_{A_\alpha}},f\right)\right\} \end{aligned}$$

Proof. *(i)* Clearly, since

$$\mu_{A_1\cup A_2\cup\cdots\cup A_n}(z)=\left[\sup_{\alpha\in[1,n]}\left\{r_{\mu_{A_\alpha}}(z)\right\}\right].\exp\left\{i\left(\sup_{\alpha\in[1,n]}\left\{\phi_{\mu_{A_\alpha}}(z)\right\}\right)\right\}$$

therefore, we have

$$\begin{aligned} T\left(r_{\mu_{A_1\cup A_2\cup\cdots\cup A_n}}, f\right) &= T\left(\sup_{\alpha\in[1,n]}\left\{r_{\mu_{A_\alpha}}\right\}, f\right) \\ &\geq T\left(r_{\mu_{A_\alpha}}, f\right),\ for\ all\ \alpha\in I\ . \end{aligned}$$

Thus

$$T\left(r_{\mu_{A_1\cup A_2\cup\cdots\cup A_n}}, f\right) = T\left(\sup_{\alpha\in[1,n]}\left\{r_{\mu_{A_\alpha}}\right\}, f\right) \geq \sup_{\alpha\in[1,n]}\left\{T\left(r_{\mu_{A_\alpha}}, f\right)\right\}$$

(ii) Similarly like above. □

Example 4.5. Let

$$A_\alpha = \int \frac{\frac{1}{\alpha z}e^{i(\alpha z\pi)}}{z},\ z\geq 1,\ \alpha\in\mathbb{N}$$

and $f(z) = z$,then

$$\bigcup_{\alpha=1}^{n} A_\alpha = \int \frac{\frac{1}{z}e^{i(z\pi)}}{z}\ and \bigcap_{\alpha=1}^{n} = \int \frac{\frac{1}{\beta z}e^{i(z\beta\pi)}}{z},$$

where β is the $\operatorname{lcm}(1,2,\cdots n)$. Thus the results follows.

Corollary 4.1. Let $F(U)$ be the set of all fuzzy complex sets on the universe U, and $A_\alpha \in F(U)$, $\alpha\in I$. Also let $\mu_{A_\alpha}(z) = r_{A_\alpha}(z).e^{i\phi_{\mu_{A_\alpha}}(z)}$ be its membership function. Then

$$T\left(r_{\mu_{\cup A_\alpha}}, f\right) = T\left(\sup_{\alpha\in I}\left\{r_{\mu_{A_\alpha}}\right\}, f\right) \geq \sup_{\alpha\in I}\left\{T\left(r_{\mu_{A_\alpha}}, f\right)\right\}$$

Proof. Clearly, since $\mu_{\cup A_\alpha}(z) = \left[\sup_{\alpha\in I}\left\{r_{\mu_{A_\alpha}}(z)\right\}\right].e^{i\ \sup_{\alpha\in I}\left\{\phi_{\mu_{A_\alpha}}(z)\right\}}$, therefore, we have

$$\begin{aligned} T\left(r_{\mu_{\cup A_\alpha}}, f\right) &= T\left(\sup_{\alpha\in I}\left\{r_{\mu_{A_\alpha}}\right\}, f\right) \\ &\geq T\left(r_{\mu_{A_\alpha}}, f\right),\ for\ all\ \alpha\in I\ . \end{aligned}$$

Thus

$$T\left(r_{\mu_{\cup A_\alpha}}, f\right) = T\left(\sup_{\alpha\in I}\left\{r_{\mu_{A_\alpha}}\right\}, f\right) \geq \sup_{\alpha\in I}\left\{T\left(r_{\mu_{A_\alpha}}, f\right)\right\}$$

□

Example 4.6. Let

$$A_\alpha = \int \frac{\frac{1}{\alpha z}e^{i(\alpha z\pi)}}{z},\ z\geq 1,\ \alpha\in\mathbb{N}$$

and $f(z) = \exp z$,then

$$\cup A_\alpha = \int \frac{\frac{1}{z}e^{i(z\pi)}}{z},$$

and the result follows trivially.

The similar expression cannot be obtained for the intersection of fuzzy sets. which can be followed from the following corollary.

Corollary 4.2. Let $F(U)$ be the set of all fuzzy complex sets on the universe U, and $A_\alpha \in F(U)$, $\alpha \in I$. Also let $\mu_{A_\alpha}(z) = r_{A_\alpha}(z).e^{i\phi_{\mu_{A_\alpha}}(z)}$ be its membership function. Then

$$T\left(r_{\mu_{\cap A_\alpha}}, f\right) \neq T\left(\inf_{\alpha\in I}\left\{r_{\mu_{A_\alpha}}\right\}, f\right) \leq \inf_{\alpha\in I}\left\{T\left(r_{\mu_{A_\alpha}}, f\right)\right\} ,$$

i.e., no such relationship can be obtained.

Proof. We give a counterexample to establish the result.
for this consider the following fuzzy sets:

$$A_\alpha = \int \frac{\frac{1}{\alpha z} e^{i(\alpha z \pi)}}{z}, \ z \geq 1, \ \alpha \in \mathbb{N} .$$

Then $\bigcap_{\alpha=1}^{\infty} A_\alpha = \{0\}$, so no such $T(r, f)$ can be obtained.
Hence the result follows. □

Now we present a theorem with the complement of the fuzzy complex numbers.

Theorem 4.7. If A and B be two fuzzy complex sets in the universe U. Then

$$T\left(r_{\mu_{A\bar{\cap}B}}, f\right) = T\left(r_{\mu_{\bar{A}\cup\bar{B}}}, f\right) .$$

Proof. Clearly $\mu_{A\bar{\cap}B}(z) = r_{\mu_{A\bar{\cap}B}}(z).\exp i\left\{\phi_{\mu_{A\bar{\cap}B}}(z)\right\}$, we have

$$\begin{aligned}
\mu_{A\bar{\cap}B}(z) &= r_{\mu_{A\bar{\cap}B}}(z).\exp i\left\{\phi_{\mu_{A\bar{\cap}B}}\right\} \\
&= \left(1 - r_{\mu_{A\cap B}}(z)\right).\exp i\left\{2\pi - \phi_{\mu_{A\cap B}}(z)\right\} \\
&= \left\{1 - \min\left(r_{\mu_A}(z), r_{\mu_B}(z)\right)\right\}.\exp i\left\{2\pi - \min\left(\phi_{\mu_A}(z), \phi_{\mu_B}(z)\right)\right\} \\
&= \max\left\{\left(1 - r_{\mu_A}(z)\right), \left(1 - r_{\mu_B}(z)\right)\right\} \times \\
&\quad \exp i\left[\max\left\{\left(2\pi - \phi_{\mu_A}(z)\right), \left(2\pi - \phi_{\mu_B}(z)\right)\right\}\right] \\
&= \max\left(r_{\mu_{\bar{A}}}(z), r_{\mu_{\bar{B}}}(z)\right).\exp i\left[\max\left\{\phi_{\mu_{\bar{A}}}(z), \phi_{\mu_{\bar{B}}}(z)\right\}\right] \\
&= \mu_{\bar{A}\cup\bar{B}}(z) .
\end{aligned}$$

Therefore the result follows. □

Theorem 4.8. Let A be a fuzzy complex set in the universe U with the membership function $\mu_A(z) = r_A(z).e^{i\phi_{\mu_A}(z)}$. Also let, $f_1, f_2, \cdots, f_n$ be n fuzzy meromorphic functions. Then

$$\begin{aligned}
(i)\ T\left(r_{\mu_A}, \sum_{\alpha=1}^{n} f_\alpha\right) &\leq \sum_{\alpha=1}^{n} T\left(r_{\mu_A}, f_\alpha\right) \\
(ii)\ T(r_{\mu_A}, \prod_{\alpha=1}^{n} f_\alpha &\leq \sum_{\alpha=1}^{n} T\left(r_{\mu_A}, f_\alpha\right) .
\end{aligned}$$

Proof. We know by the construction of positive logarithm that (cf. [3])

$$\log^+\left|\prod_{\alpha=1}^{n} r_\alpha\right| \leq \sum_{\alpha=1}^{n}\log^+|r_\alpha| \quad and$$
$$\log^+\left|\sum_{\alpha=1}^{n} r_\alpha\right| \leq \sum_{\alpha=1}^{n}\log^+|r_\alpha|$$

therefore on applying it to the counting function and proximity function we get

$$N\left(r_{\mu_A}, \prod_{\alpha=1}^{n} f_\alpha\right) \leq \sum_{\alpha=1}^{n} N\left(r_{\mu_A}, f_\alpha\right) \quad and$$
$$N\left(r_{\mu_A}, \sum_{\alpha=1}^{n} f_\alpha\right) \leq \sum_{\alpha=1}^{n} N\left(r_{\mu_A}, f_\alpha\right),$$

also

$$m\left(r_{\mu_A}, \prod_{\alpha=1}^{n} f_\alpha\right) \leq \sum_{\alpha=1}^{n} m\left(r_{\mu_A}, f_\alpha\right) \quad and$$
$$m\left(r_{\mu_A}, \sum_{\alpha=1}^{n} f_\alpha\right) \leq \sum_{\alpha=1}^{n} m\left(r_{\mu_A}, f_\alpha\right) .$$

Thus $T\left(r_\mu, f\right) = N\left(r_\mu, f\right) + m\left(r_\mu, f\right)$ gives

$$(i)\ T\left(r_{\mu_A}, \sum_{\alpha=1}^{n} f_\alpha\right) \leq \sum_{\alpha=1}^{n} T\left(r_{\mu_A}, f_\alpha\right)$$
$$(ii)\ T\left(r_{\mu_A}, \prod_{\alpha=1}^{n} f_\alpha\right) \leq \sum_{\alpha=1}^{n} T\left(r_{\mu_A}, f_\alpha\right) .$$

□

Theorem 4.9. Let A, B and C be three fuzzy sets in the universe U. Then $T\left(r_{\mu_{(A\circ B)\circ C}}, f\right) = T\left(r_{\mu_{A\circ(B\circ C)}}, f\right)$. That is Nevanlinna Characteristic of the fuzzy complex product is associative.

Proof. Let the membership functions are $\mu_A(z) = r_A(z).e^{i\phi_{\mu_A}(z)}$, $\mu_B(z) = r_B(z).e^{i\phi_{\mu_B}(z)}$ and $\mu_C(z) = r_C(z).e^{i\phi_{\mu_C}(z)}$ for the fuzzy complex sets A, B and C respectively. Then

$$\begin{aligned}
\mu_{(A\circ B)\circ C}(z) &= r_{(A\circ B)\circ C}(z).\exp\left\{i.\phi_{\mu_{(A\circ B)\circ C}}(z)\right\} \\
&= \left(r_{A\circ B}(z).r_B(z)\right).\exp\left\{i.2\pi\left(\frac{\phi_{\mu_{A\circ B}}(z)}{2\pi}\cdot\frac{\phi_{\mu_B}(z)}{2\pi}\right)\right\} \\
&= \left(r_A(z).r_B(z).r_C(z)\right).\exp\left\{i.2\pi.\frac{2\pi\left(\frac{\phi_{\mu_A}(z)}{2\pi}\cdot\frac{\phi_{\mu_B}(z)}{2\pi}\right)}{2\pi}\cdot\frac{\phi_{\mu_C}(z)}{2\pi}\right\} \\
&= \mu_{A\circ(B\circ C)}(z),
\end{aligned}$$

and the result follows. □

Remark 4.1. From the above theorems it is obvious that, if $f(z) = z$, then

$$T\left(r_{\mu_{(A\circ B)\circ C}}, f\right) \leq T\left(r_{\mu_A}, f\right) + T\left(r_{\mu_B}, f\right) + T\left(r_{\mu_C}, f\right) \ .$$

Remark 4.2. In general, it can be proved that if $f(z) = z$, then

$$T\left(r_{\mu_{\prod_{\alpha=1}^{n} C_\alpha}}, f\right) \leq \sum_{\alpha=1}^{n} T\left(r_{\mu_{C_\alpha}}, f\right),$$

where $\{C_\alpha, \alpha \in I\}$ be the collection of fuzzy complex sets in the universe U and $\prod_{\alpha=1}^{n} C_\alpha = C_1 \circ C_2 \circ \cdots \circ C_n$.

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On Generalized Digital Topology and Root Images of Median Filters

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Abstract – In this paper, we extend the concepts of semi-open sets and λ-open sets in the digital topology. In addition, we introduce the concepts of regular semi-open and regular λ-open sets. A relationship between digital topology and image processing is established.

Keywords – *Digital topology, median filter, root image, regular open,* λ-open *and semi-open.*

1 Introduction

Over the last decades, digital topology has proved to be an important concept in image analysis and image processing. Rosenfeld [15] introduced the fundamentals of digital topology, which provides a sound mathematical basis for image processing operations such as image thinning, border following, contour filling, object counting, and signal processing. Whenever spatial relations are modeled on a computer, digital topology is needed.

Digital topology aims to transfer concepts from classical topology to digital spaces such as: connectivity, boundary, neighborhood, and continuity which are used to model computer images. The classes of semi-open and λ-open sets are finer than the class of the open sets in the 8-semi-topology which is studied in Alpers [3] proved that if $B \subseteq \mathbb{Z}^2$ is a regular open set in 8-semi-topology, then $(\mathbb{Z}^2, B)$is a root image of median filter Med_4. We found that the converse of this implication holds for the regular semi-open sets in *Marcus-Wyse* topology on $\mathbb{Z}^2$ and the regular λ-open sets in *Marcus-Wyse* or *Khalimsky* topologies on $\mathbb{Z}^2$.

In this paper, we extend the concepts of semi-open, λ-open, regular semi-open, and regular λ-open sets in the digital topology. We study the connections between these concepts and the root images of the median filters in the digital picture.

This paper is organized as following: In section 3 we study the notions of digital picture, median filter, root image, and some of its properties. In section 4 and 5 we study these notions in digital topology. Furthermore, we obtained a relationship between regular semi-open sets, regular λ-open sets, and root images of median filters.

2 Preliminaries

Definition 2. 1. [4, 11] Let (X, τ) be a topological space. A subset A of X is called:
(1) Semi-open if $A \subseteq \overline{A^\circ}$.
(2) Semi-closed if its complement is a semi-open.
(3) Λ-set if $A := \cap \{G \mid A \subseteq G, G \in \tau\}$.
(4) $\vee$-set if $A := \cup \{F \mid F \subseteq A, F^C \in \tau\}$.
(5) λ-closed if $A = G \cap H; G$ is a closed set and H is a Λ-set.
(6) λ-open if A^c is a λ-open.

Definition 2. 2. [5] Let (X, τ) be a topological space and $A \subseteq X$. The semi-interior (λ-interior) of A, denoted by $int_s(A)$ ($int_\lambda(A)$), is the union of all semi-open (λ-open) subsets of A.

Definition 2. 3 [5] Let (X, τ) be a topological space and $A \subseteq X$. The semi-closure (λ-closure) of A, denoted by $cl_s(A)$ ($cl_\lambda(A)$), is the intersection of all semi-closed (λ-closed) supersets of A.

In this paper, for any topological space (X, τ), let $N_{min}(p)$ (respectively, $N^\lambda_{min}(p) N^\lambda_{min}(p)$, $N^S_{min}(p)$) denotes the smallest open (respectively, λ-open and semi-open) set containing p. Let O^λ_p (respectively, O^S_p) be a λ- open (respectively, semi-open) set containing p. In addition, the collection of all open singletons of a subset A of X, i.e., one point open subset of A, is denoted by A_{op}.

Definition 2. 4. [14] Let (X, τ) be a topological space. A point $x \in X$ is called an open singleton if $\{x\}$ is an open set. The set of all open points of a subset A of X is denoted by A_{os}.

Definition 2. 4. [7] The digital n-space $\mathbb{Z}^n$ is the set of all n-tuples $p = (p_1, \ldots, p_n); p_i \in \mathbb{Z}, i \in \{1, \ldots, n\}$. A point $p = (p_1, \ldots, p_n)$ of the digital plane $\mathbb{Z}^n$ is called a pure vertex if its coordinates p_i are all even or odd, otherwise p is called mixed vertex. Every point p of the digital space $\mathbb{Z}^n$ has $2n$- and $(3^n - 1)$- neighbors. The $2n$-neighbors is the set $\mathcal{N}_{2n}(x) := (q_1, q_2, \ldots, q_n) \in \mathbb{Z}^n \mid \sum_{i=1}^n |q_i - p_i| = 1\}$ and the $(3^n - 1)$- neighbors is the set:

$$\mathcal{N}_{(3^n-1)}(x) := (q_1, q_2 \ldots, q_n) \in \mathbb{Z}^n \mid \max\{|q_1 - p_1|, |q_2 - p_2|\} = 1\}.$$

Definition 2. 4. [7] Two points p, q of the digital n-space $\mathbb{Z}^n$ are called k-adjacent if they are k-neighbors, i:e, one of them belongs to the $2n$ -neighbors or $(3^n - 1)$- neighbors of the other, for $k = n$ or $k = (3n - 1)$. Also for two points p, q of the digital n-space $\mathbb{Z}^n$; a k −path from p to q is a sequence of points $p = p_1, p_2, \ldots, p_j = q$ such that p_i and p_{i+1} are k −adjacent, $i = 1,2, \ldots, j - 1$.

Definition 2. 6 [7] Any subset X of a digital n-space $\mathbb{Z}^n$ is called k −connected, $k = 2n$ or $(3^n - 1)$; if for every pair of points p, q of X, there is a k-path contained in X from p to q.

The digital picture is a pair $(\mathbb{Z}^n, B)$, where $B \subseteq \mathbb{Z}^n$. The elements of $\mathbb{Z}^n$ are called the points of the digital picture, the points of B are called the black points of the picture, and the points of $\mathbb{Z}^n \setminus B$ are called the white points of the picture.

Median filters are quite popular tools in image processing. The median filters are firstly introduced by Tuckey [16], they are used to de-noise images. To deal with the median filters we need the following subsets:

Definition 2. 7 [9] For any subset p of a digital space $\mathbb{Z}^n$, consider the subsets

$$U_{2n}(p) = \{(q_1, q_2 \ldots, q_n) \in \mathbb{Z}^n \mid \textstyle\sum_{i=1}^{n} |q_i - p_i| \leq 1\}$$

and

$$U_{(3^n-1)}(p) = \{\left(q_1, q_2 \ldots, q_n\right) \in \mathbb{Z}^n \mid \max_i |q_i - p_i| \leq 1\}.$$

The median filter Med_k on a digital picture is a mapping which maps $(\mathbb{Z}^n, B)$ to $(\mathbb{Z}^n, B^*)$ with

$$B^* = \{p \in \mathbb{Z}^n : |U_k(p) \cap B| \geq \frac{|U_k| + 1}{2}\}$$

for $k = 4$ or 8 in $\mathbb{Z}^2$, and $k = 6$ or 26 in $\mathbb{Z}^3$. A root image of Med_k is a digital picture $(\mathbb{Z}^n, B)$ with $Med_k((\mathbb{Z}^n, B)) = (\mathbb{Z}^n, B)$.

It is clear that, if $B \subseteq \mathbb{Z}^n$ is a root image of the median filter Med_k , $x \in B$, then x has at least one of its k −neighbors in B.

An important property of the root images of any median filters will be given in the following proposition:

Proposition 2. 1. The root images of any median filter Med_{2n} in the digital n-space $\mathbb{Z}^n$ are $2n$-connected set for $n = 2, 3$.

Proof. Let $(\mathbb{Z}^n, B)$ be a root image of Med_{2n} and suppose that B is not $2n$ −connected. Then there exists at least $x \in B$ such that x has no $2n$ −neighbors in $B \setminus \{x\}$. Then $|U_{2n}(x) \cap B| = 1$, and so $x \notin Med_{2n}(\mathbb{Z}^n, B)$. Which contradicts that $(\mathbb{Z}^n, B)$ is a root image of Med_{2n}.

The converse of this Proposition is not true. Figure 1 shows a 4-connected set, but it is not root image of Med_4.

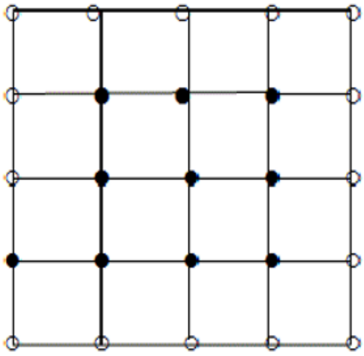

Figure 1. 4-connected set which is neither root image of Med_4 nor root image of Med_8

To study with the concept of connectedness in the digital spaces from the topological point of view, many topological structures are introduced. In this article, we need the following topologies on the digital spaces.

Definition 2. 8 [10] The Kalimsky line is the set of all integers $\mathbb{Z}$ with the topology generated by the following subbase: $\eta = \{\{2n, 2n \pm 1\}, n \in \mathbb{Z}\}$. The Kalimsky n-space is $\mathbb{Z}^n$ with the product space of n-Kalimsky line.

Definition 2. 8 [13] The Marcus-Wyse topological structure on $\mathbb{Z}^n$; $n = 2,3$ is the topology generated by the following base: $\beta = \{N_{min}(p); p \in \mathbb{Z}^n \; n = 2,3\}$ where

$$N_{min}(p = (p_1, p_2, \dots, p_n)) = \left\{ \begin{matrix} \{p\} & ; \sum_{i=1}^{n} p_i \, odd \, number \\ U_{2n}(p) & ; othewise \end{matrix} \right\}$$

Theorem 2. 1 [7] The two topologies Khalimsky and Marcus-Wyse on the digital n −space $\mathbb{Z}^n$ satisfies the following two conditions:
If $S \subseteq \mathbb{Z}^n$ is $2n$-connected set, then S is topologically connected.
If $S \subseteq \mathbb{Z}^n$ is not $(3^n - 1)$-connected set, then S is not topologically connected.

Corollary 2. 1 In any digital n −space $\mathbb{Z}^n$ with the Khalimsky topology or Marcus-Wyse topologies, any rot image of any median filter Med_{2n} is topologically connected.

Note that, neither of the following sets is in general a root image, the union of two root images, the intersection of two root images or difference between any two root images.
Figure 2 shows that the intersection and the difference between two root images is not necessarily to be root image. Let $A = \{a, b, f, e\}$ and $B = \{b, c, e, d\}$. Then, $A \cap B = \{b, e\}$ and $A - B = \{\{a, f\}$ which is not root image of Med_4.

Figure 3 shows that the union of two root images is not necessarily to be root image. Let $A = \{a, b, \; c, d, e, f, g, h\}$ and $B = \{1, 2, 3, 4\}$. It is clear that, A and B are root images of Med_4 but $A \cup B$ is not since $(A \cup B)^* \neq A \cup B$.

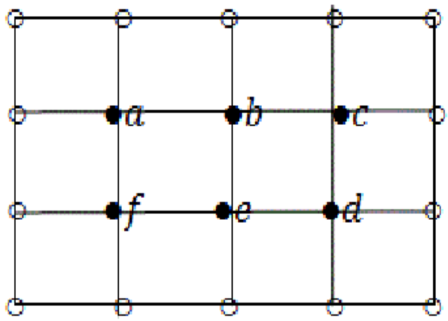

Figure 2. shows that the intersection of two root images is not root image

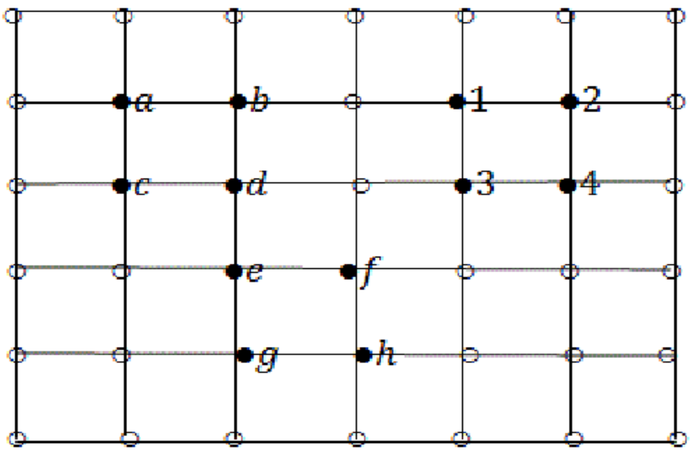

Figure 3. shows that the union of two root images is not root image

3 Semi-openness and Root Images

Theorem 3. 1 [7] A subset A of a digital n-space $\mathbb{Z}^n$ is a semi-open set if and only if $N_{min}(x) \cap A_{op} \neq \emptyset$ for all $x \in A$.

Proposition 3. 1

(1) The collection of the smallest semi-open neighborhoods of a point p in the Marcus-Wyse topology on $\mathbb{Z}^2$ can be given as follows: for every $p = (p_1, p_2) \in \mathbb{Z}^2$,

$$N^S_{min}(p) = \begin{cases} \{p = (p_1, p_2)\} & ; p_1 + p_2 \text{ is odd number, otherwise} \\ \left.\begin{matrix} \{(p_1, p_2), (p_1 + i, p_2 + j)\} \\ \{(p_1, p_2), (p_1 - i, p_2 - j)\} \end{matrix}\right\} & ; p_1 + p_2 \text{ is even number and} \\ & \text{either } i = 0, j = 1 \text{ or } i = 1, j = 0 \end{cases}$$

(2) The collection of the smallest semi-open neighborhoods of a point p in the Marcus-Wyse topology on $\mathbb{Z}^3$ can be given as follows: for every $p = (p_1, p_2, p_3) \in \mathbb{Z}^3$,

$$N^S_{min}(p) = \left\{ \begin{array}{ll} \{p\} & ;p_1 + p_2 + p_3 \text{ is odd number, otherwise} \\ \left.\begin{array}{l}\{p,(p_1+i,p_2+j,p_3+k)\} \\ \{p,(p_1-i,p_2-j,p_3-k)\}\end{array}\right\} & ;p_1 + p_2 + p_3 \text{ is even number and} \\ & \text{either } i = 1, j = k = 0 \text{ or } j = 1, \\ & i = k = 0 \text{ or } k = 1, i = j = 0 \end{array} \right\}$$

(3) The collection of the smallest semi-open neighborhoods of a point p in the Khalimsky topology on $\mathbb{Z}^2$ can be given as follows: for every $p = (p_1, p_2) \in \mathbb{Z}^2$,

$$N^S_{min}(p) = \left\{ \begin{array}{ll} \{p\} & ;p_1 \text{ and } p_2 \text{ are both odd numbers} \\ \{p,(p_1{}^*,p_2{}^*)\} & ;p_i{}^* = p_i \text{ if } p_i \text{ odd number either} \\ & p_i{}^* = p_i + 1 \text{ or } p_i - 1 \text{ if } pi \text{ even for } i = 1,2 \end{array} \right\}$$

(4) The collection of the smallest semi-open neighborhoods of a point p in the Khalimsky topology on $\mathbb{Z}^3$ can be given as follows: for every $p = (p_1, p_2, p_3) \in \mathbb{Z}^3$,

$$N^S_{min}(p) = \left\{ \begin{array}{l} \{p\} \quad ;p_i \text{ are odd numbers, otherwise} \\ \{p,(p_1{}^*,p_2{}^*,p_3{}^*)\} \text{ where } p_i{}^* = \left\{ \begin{array}{ll} p_i & \text{if } p_i \text{ is an odd number for } i = 1,2,3 \\ p_i + 1 \text{ or } p_i - 1 & \text{if } p_i \text{ is an odd number for } i = 1,2,3 \end{array} \right\} \end{array} \right\}$$

Proof. (1) Let τ be Marcus-Wyse topology on $\mathbb{Z}^2$. Since $\{p = (p_1, p_2)\}$ is an open set if $p_1 + p_2$ is an odd number, then $\{p\}$ is a semi-open set. Let $A = \{(p_1, p_2), (p_1 + 1, p_2)\} \subseteq \mathbb{Z}^2$ and $p_1 + p_2$ is an even number. Since $N_{min}(p) \cap A_{op} \neq \emptyset$ for all $p \in A$, then A is a semi-open set by Theorem 3.1. The same results will be given if $A = \{(p_1, p_2), (p_1 - 1, p_2)\}$ or $A = \{(p_1, p_2), (p_1, p_2 + 1)\}$ or $A = \{(p_1, p_2), (p_1, p_2 - 1)\}$. The rest of the proof is of the same argument.

Theorem 3. 2 If $B \subseteq \mathbb{Z}^2$ is a root image of the median filter Med_4, then B is a regular semi-open set in Marcus-Wyse topology on $\mathbb{Z}^2$.

Proof. Let A be a root image of the median filter Med_4, and $x \in A$. Then at least two of the 4-neighbors of this x are in A. Then A is a semi-open subset by Proposition 3.1 case (1). Let $x \in Cl_s(A)$. Then, $N^S_{min}(x) \cap A \neq \emptyset$. Suppose that $x \notin A$. Then, $x \in A^*$. Which has a contradiction with A is a root image of the cross median filter Med_4.Then A is a semi-closed set, hence A is a regular semi-open set.

The converse of the previous theorem is not true. The following example shows a regular semi-open set in *Marcus-Wyse* topology on $\mathbb{Z}^2$ which is not a root image of the cross median filter Med_4.

Example 3. 1 Let $B \subseteq \mathbb{Z}^2$as shown in Figure 4. Then, B is a regular semi-open set in *Marcus-Wyse* topology on $\mathbb{Z}^2$, but it is not a root image of the cross median filter Med_4.

Solution. Since $|U_4(x) \cap B| = 3$, then $x \in B^* \setminus B$ and so B is not a root image of the cross median filter Med_4. Since $|U_8(x) \cap B| = 5$, then $x \in B^* \setminus B$ and so B is not a root image of the median filter Med_8. Since int_s(B) = B and $Cl_s(B) = B$, then B is a regular semi-open set in *Marcus-Wyse* topology on $\mathbb{Z}^2$.

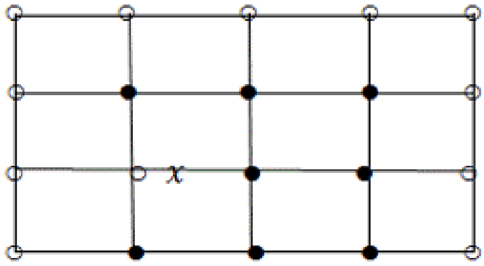

Figure 4. Regular semi-open set in Marcus-Wyse topology on $\mathbb{Z}^2$ which is not root image of Med_4.

Lemma 3. 1 If $B \subseteq \mathbb{Z}^2$ is a root image of the median filter Med_8, τ be *Marcus-Wyse* topology on $\mathbb{Z}^2$ and $x \in Cl_s(B)$, then $x \in B$.

Proof. Let $B \subseteq \mathbb{Z}^2$ be a root image of the median filter Med_8, τ be Marcus-Wyse topology on $\mathbb{Z}^2$, and $x \in Cl_s(B)$. Then, $N^S_{min}(x) \cap B \neq \emptyset$, and all the 4-neighbors of x are in B. Suppose that $x \notin B$ as shown in Figure 5. Then the points $a, c \notin B^*$ which is a contradiction with B is a root image of the median filter Med_8.

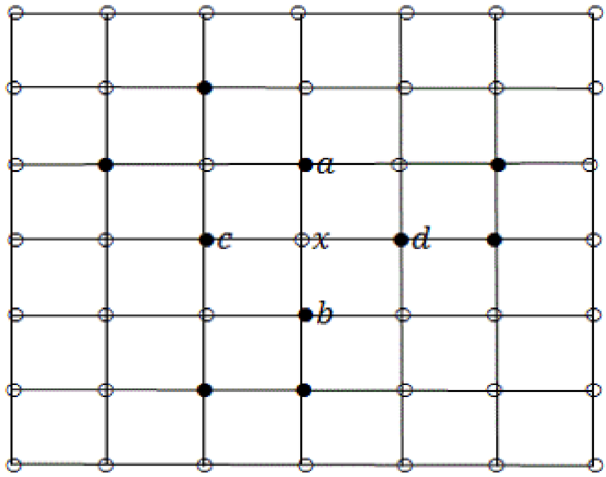

Figure 5. A contradiction with root image of Med_8 if $x \in Cl_s(B)$ and $x \notin B$.

Theorem 3. 3 If $B \subseteq \mathbb{Z}^2$ is a root image of the median filter Med_8, then B is a regular semi-open set in *Marcus-Wyse* topology on $\mathbb{Z}^2$.

Proof. Let B be a root image of the median filter Med_8 and $x \in B$. Then x has at least 4 of its 8-neighbors in B and at least one the 4-neighbors of x is in B. Then, B is a semi-open set in *Marcus-Wyse* topology on $\mathbb{Z}^2$. Let $x \in Cl_s(B)$. Then according to Lemma 3.2, $x \in B$.

The converse of the previous theorem is not true in general. Example 3.1 shows a regular semi-open set in *Marcus-Wyse* topology on $\mathbb{Z}^2$, but it is not a root image of the median filter Med_8.

Example 3. 2 Let $B \subseteq \mathbb{Z}^2$ as shown in Figure 6 and let $x = (x_1, x_2) \in B$ such that x_1,x_2 are even numbers. Then, B is a root image of the median filter Med_4, but it is semi-open in *Khalimsky* topology on $\mathbb{Z}^2$.

Solution. Since there is no $O^S(x)$ such that $O^S(x) \subseteq B$, then $x \notin int_S(B)$ and so B is not semi-open set in *Khalimsky* topology on $\mathbb{Z}^2$. Since $|U_4(y) \cap B| \geq 3$ for all $y \in B$ and $|U_4(y) \cap B| < 3$ for all $y \notin B$, then B is a root image of the median filter Med_4.

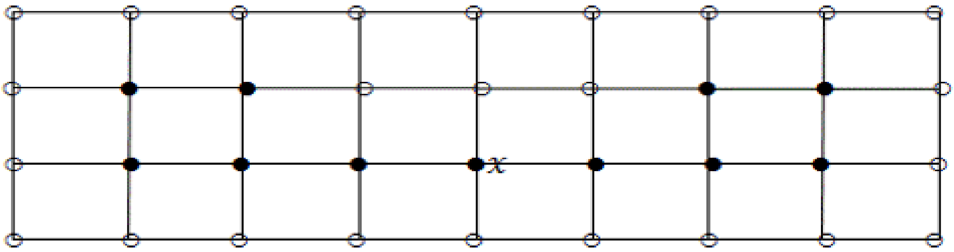

Figure 6. A root image of Med_4 which is not semi-open in *Khalimsky* topology on $\mathbb{Z}^2$.

Example 3. 3 Let $B \subseteq \mathbb{Z}^2$ as shown in Figure 7 and $x = (x_1, x_2) \in B$ such that x_1,x_2 are even numbers. Then B is a regular semi-open set in *Khalimsky* topology on $\mathbb{Z}^2$, but it is neither a root image of the cross median filter Med_4 nor a root image of the median filter Med_8.

Solution. Since int_s(B) = B and $Cl_s(B) = B$, then B is a regular semi-open set in *Khalimsky* topology on $\mathbb{Z}^2$. Since $Med_4(\mathbb{Z}^2, B) = \{y, z, w, t\}$ and $Med_8(\mathbb{Z}^2, B) = \{z, y\}$, then B neither a root image of the cross median filter Med_4 nor a root image of the median filter Med_8.

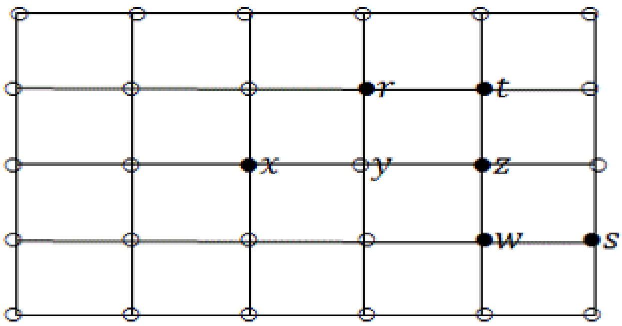

Figure 7. A regular semi-open subset in *Khalimsky* topology on $\mathbb{Z}^2$ which is neither root image of Med_4 nor root image of Med_8.

Example 3. 4 Let $B \subseteq \mathbb{Z}^2$ as shown in Figure 8. Let $x = (x_1, x_2) \in B$ such that x_1,x_2 are even numbers. Then, B is a root image of the median filter Med_8, but it is not a semi-open set in the *Khalimsky* topology on $\mathbb{Z}^2$.

Solution. Since x has no $O^S(x) \subseteq BO^S(x) \subseteq B$, then B is not a semi-open set in the *Khalimsky* topology on $\mathbb{Z}^2$. Since $|U_8(y) \cap B| \geq 5$ for all $y \in B$ and $|U_8(y) \cap B| < 5$ for all $y \notin B$, then B is a root image of the median filter Med_8.

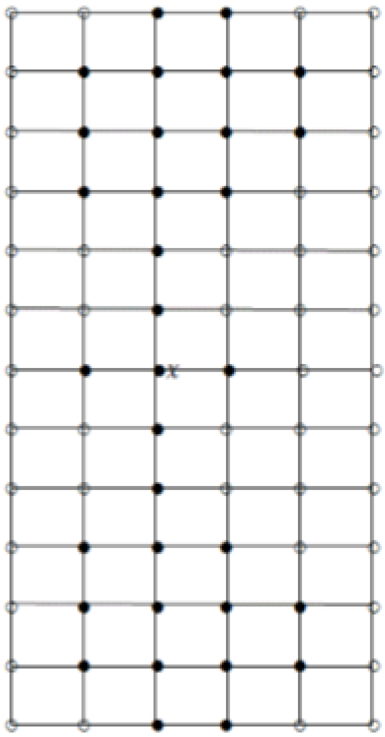

Figure 8. A root image of Med_8 which is not a semi-open subset in the *Khalimsky* topology on $\mathbb{Z}^2$

Theorem 3. 4 If $B \subseteq \mathbb{Z}^2$ is a root image of the median filter Med_6, then B is a regular semi-open set in the *Marcus-Wyse* topology on $\mathbb{Z}^3$.

Proof. Let B be a root image of the median filter Med_6 and $x \in B$. Then x has at least three of its 6-neighbors in B and so B is a semi-open set in Marcus-Wyse topology on $\mathbb{Z}^3$ by Proposition 3. 1 (2). Let $x \in Cl_S(B)$. Then $N^S_{min}(x) \cap B \neq \emptyset$ and $x \in B^* = B$. Then B is a semi-closed, hence B is a regular semi-open set in the *Marcus-Wyse* topology on $\mathbb{Z}^3$.

The converse of the previous theorem is not true in general as shown in the following example:

Example 3. 5 Let $B \subseteq \mathbb{Z}^3$ such that

$$B = \{(0,1,1), (2,1,1), (1,0,1), (1,1,0), (0,2,1), (2,2,1), (1,0,2), (1,2,0)\}.$$

Then, B is a regular semi-open set in the *Marcus-Wyse* topology on $\mathbb{Z}^3$, but B is not a root image of the median filter Med_6.

Solution. Since $|U_6(x) \cap B| < 4$ for all $x \in B$, then B is not a root image of the median filter Med_6. Since int_s(B) = B and $Cl_s(B) = B$, then B is a regular semi-open set in the *Marcus-Wyse* topology on $\mathbb{Z}^3$.

The previous example shows also that the regular semi-open set in the *Marcus-Wyse* topology on $\mathbb{Z}^3$ is not necessary to be a root image of the median filter Med_6 or a root image of the median filter Med_{26}.

Example 3. 6 Let $B \subseteq \mathbb{Z}^3$ such that:

$B = \{(0,0,0), (0,1,0), (1,0,0), (0,0,1), (1,1,0), (0,1,1), (1,0,1), (1,0,2), (1,2,0), (0,1,2), (0,0,2), (1,1,1), (1,1,2), (0,2,0), (1,2,1), (1,2,2), (0,2,1), (0,2,2)\}$

Then B is a root image of the median filter Med_6, but it is not semi-open set in the *Khalimsky* topology on $\mathbb{Z}^3$.

Solution. Since there is no $O^S(x)$ such that $O^S(x) \subseteq B$, then B is not semi-open set in the *Khalimsky* topology on $\mathbb{Z}^3$ while B is a root image of the median filter Med_6.

Example 3. 7 Let $B \subseteq \mathbb{Z}^3$ such that:

$$B = \{(1,1,1), (2,1,1), (0,0,0,), (0,1,1), (1,0,1), (1,1,0)\}.$$

Then, B is a regular semi-open set in the *Khalimsky* topology on $\mathbb{Z}^3$, but it is neither a root image of the median filter Med_6 nor a root image of the median filter Med_{26}.

Solution. Since int_s(B) = B and $Cl_s(B) = B$, then, B is a regular semi-open set in *Khalimsky* topology on $\mathbb{Z}^3$. Since $Med_6(\mathbb{Z}^3, B) = \{(1,1,1)\}$ and $Med_{26}(\mathbb{Z}^3, B) = \emptyset$, then B is neither a root image of the median filter Med_6 nor a root image of the median filter Med_{26}.

4. λ-open and Root Image

A digital topology is an *Alexandroff* space [2]. So, a subset A of a digital topology is called λ-open set if it can be written as a union of an open and a closed set. Then, every open set is also a λ-open set, and every closed set is also a λ-open set. Since *Marcus-Wyse* topology is $T_{1/2}$-space, then every singleton in *Marcus-Wyse* topology is a λ-open set. Consequently,

Corollary 4. 1 If $B \subseteq \mathbb{Z}^2$ is a root image of the median filters Med_4 or Med_8, then B is a regular λ-open set in the *Marcus-Wyse* topology on $\mathbb{Z}^2$.

Corollary 4. 2 If $B \subseteq \mathbb{Z}^3$ is a root image of the median filters Med_6 or Med_{26}, then B is a regular λ-open set in *Marcus-Wyse* topology on $\mathbb{Z}^3$.

Example 4. 1 Let $B \subseteq \mathbb{Z}^2$ as shown in Figure 9. Then B is a regular λ-open in *Marcus-Wyse* topology on $\mathbb{Z}^2$, but it is neither a root image of the median filter Med_4 nor a root image of the median filter Med_8.

Solution. Since $x \in B^* \setminus B$, then B is neither root image of the median filter Med_4 nor a root image of the median filter Med_8. It is clear that B is both λ-open set and λ-closed set, hence B is a regular λ-open in *Marcus-Wyse* topology on $\mathbb{Z}^2$.

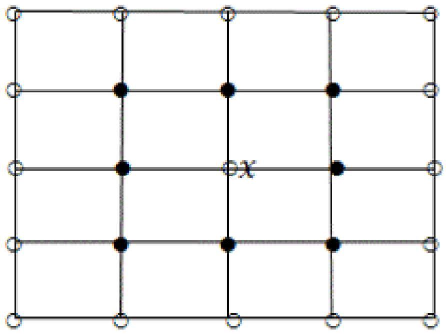

Figure 9. A regular λ-open in *Marcus-Wyse* topology on $\mathbb{Z}^2$ which is neither a root image Med_4 nor a root image of Med_8.

Different results are found with the *Khalimsky* topology:

(1) The collection of the smallest λ-open neighborhoods of a point p in *Khalimsky* topology on $\mathbb{Z}^2$ can be given as follows: for every $p = (p_1, p_2) \in \mathbb{Z}^2$,

$$N^{\lambda}_{\min}(p) = \begin{cases} \{p = (p_1, p_2)\} & if\ p\ is\ pure\ vertex \\ \begin{Bmatrix} \{(p_1, p_2), (p_1 \pm 1, p_2)\} \\ \{(p_1, p_2), (p_1, p_2 \pm 1)\} \end{Bmatrix} & if\ p\ is\ mixed\ vertex \end{cases}$$

(2) The collection of the smallest λ-open neighborhoods of a point p in the *Khalimsky* topology on $\mathbb{Z}^3$ can be given as follows: for every $p = (l, m, n) \in \mathbb{Z}^3$,

$$N^{\lambda}_{\min}(p) = \begin{cases} \{p = (l, m, n\} & if\ p\ is\ pure\ vertex \\ \{(l, m, n), (l \pm 1, m, n)\} & if \begin{Bmatrix} l\ even, m, n\ odd, or \\ l\ odd\ m, n, even \end{Bmatrix} \\ \{(l, m, n), (l, m \pm 1, n)\} & if \begin{Bmatrix} m\ even, l, n\ odd, or \\ m\ odd\ l, n, even \end{Bmatrix} \\ \{(l, m, n), (l, m, n \pm 1)\} & if \begin{Bmatrix} n\ even, l, m\ odd, or \\ n\ odd\ l, m\ even \end{Bmatrix} \end{cases}$$

Theorem 4. 1 If $B \subseteq \mathbb{Z}^2$ is a root image of the median filter Med_4 , then B is a λ-closed in the *Khalimsky* topology on $\mathbb{Z}^2$.

Proof. Let B be a root image of the median filter Med_4. Let $x \in Cl_{\lambda}(B)$. Then $N^{\lambda}_{min}(x) \cap B \neq \phi$ and all the 4-neighbors of x are in B. Then, $x \in B^* = B$.

Example 4. 2 Let $B \subseteq \mathbb{Z}^2$ as shown in Figure 10. Let $x \in B$ is pure vertex. Then, B is a regular λ-open set in the *Khalimsky* topology on $\mathbb{Z}^2$, but B is neither a root image of the cross median filter Med_4 nor a root image of the median filter Med_8.

Solution. Since int_{λ}(B) = B and $Cl_{\lambda}(B) = B$, then B is a regular λ-open in the *Khalimsky* topology on $\mathbb{Z}^2$. Since $Med_4(\mathbb{Z}^2, B) = \{y, z, s\}$ and $Med_8(\mathbb{Z}^2, B) = \{w\}$, then B is neither a root image of the cross median filter Med_4 nor a root image of the median filter Med_8.

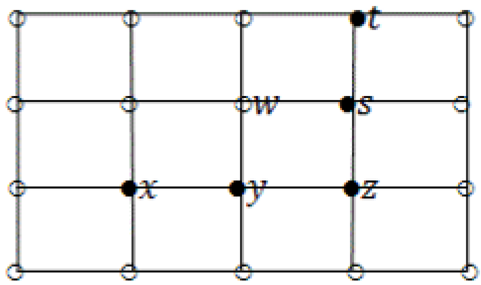

Figure 10. A regular λ-open in *Khalimsky* topology on $\mathbb{Z}^2$ which is neither root image of Med_4 nor a root image of Med_8.

Example 4. 3 Let $B \subseteq \mathbb{Z}^2$ be as shown in Figure 11 and let $x \in B$ be mixed vertex. Then, B is a root image of the median filter Med_8, but it is not λ-open set in the *Khalimsky* topology on $\mathbb{Z}^2$.

Solution. Since there is no $O^\lambda(x)$ such that $O^\lambda(x) \subseteq B$, then B is not λ-open in the *Khalimsky* topology on $\mathbb{Z}^2$. Since $Med_8(\mathbb{Z}^2, B) = B$, then B is root image of the median filter Med_8.

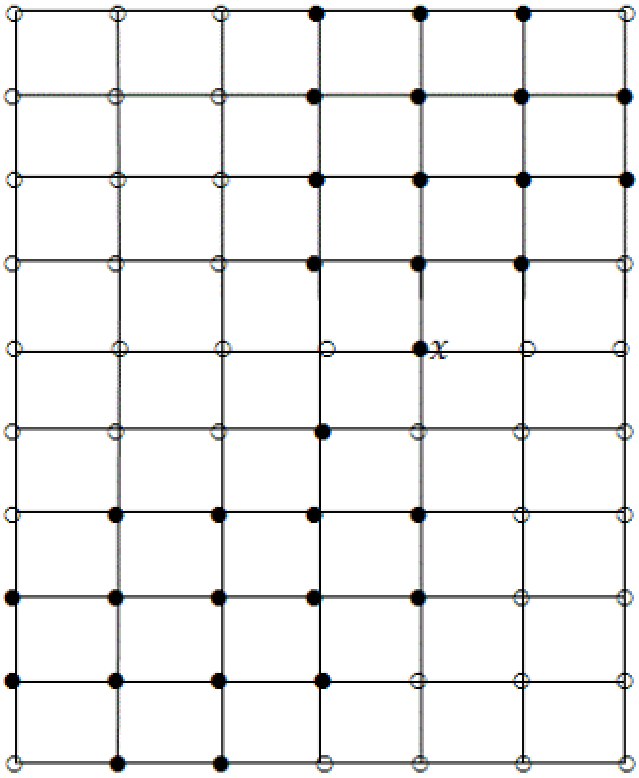

Figure 11. A root image of Med_8 which is not λ-open in *Khalimsky* topology on $\mathbb{Z}^2$.

Example 4. 4 Let $B \subseteq \mathbb{Z}^3$ such that:

$B = \{(0,0,0), (0,1,0),\ (1,0,0), (0,0,1), (1,1,0), (0,1,1), (1,0,1), (1,0,2),\ (1,2,0), (0,1,2),$ $(0,0,2),\ (1,1,1), (1,1,2),\ (0,2,0), (1,2,1), (1,2,2),\ (0,2,1), (0,2,2)\}$

Then, B is a root image of the median filter Med_6, but it is not λ-open set in the *Khalimsky* topology on $\mathbb{Z}^3$.

Solution. Since there is no $O^{\lambda}((1,0,0))$ such that $O^{\lambda}((1,0,0)) \subseteq B$, then B is not λ-open set in the *Khalimsky* topology on $\mathbb{Z}^3$. But B is a root image of the median filter Med_6 as illustrated in Example 3.6.

Example 4. 5 Let $B \subseteq \mathbb{Z}^3$ such that $B = \{(0,0,0),\ (1,1,1),\ (2,2,2,),\ (3,5,7)\}$. Then, B is a regular λ-open set in the *Khalimsky* topology on $\mathbb{Z}^3$, but it is neither a root image of the median filter Med_6 nor a root image of the median filter Med_{26}.

Solution. Since x is a pure vertex for all $x \in B$, then $\{x\}$ is a λ-open set for all $x \in B$ and B is a λ-open set. Since there is $O^{\lambda}(x)$ such that $O^{\lambda}(x)\ \cap B = \phi$ for all $x \notin B$, then B is a λ-closed set, hence B is a regular λ-open set in the *Khalimsky* topology on $\mathbb{Z}^3$. Since $Med_k(\mathbb{Z}^3, B) = \phi$ for k=6 or 26, then B is neither a root image of the median filter Med_6 nor a root image of the median filter Med_{26}.

6. Conclusion

In this paper, we show how the topological concepts such as: λ-open, semi-open, regular λ-open set, regular semi-open set, and topologically connected can be transferred to the digital topology. In addition, we explain how we can apply these concepts in median filter. The results may be summarized as following:

1- Every root image of median filter Med_4 and Med_8 is a regular semi-open set in Marcus-Wyse topology on $\mathbb{Z}^2$.
2- Every root image of median filter Med_4 and Med_8 are regular λ-open set in Marcus-Wyse topology on $\mathbb{Z}^2$.
3- Every root image of median filter Med_4 is a every root image of λ-closed set in Khalimsky topology on $\mathbb{Z}^2$ which are the converse of the implication does Alpers have.
4- Every root image of Med_6 is regular semi-open set in Marcus-Wyse topology on $\mathbb{Z}^3$.
5- Every root image of median filter Med_6 and Med_{26} are regular λ-open set in Marcus-Wyse topology on $\mathbb{Z}^3$.
6- Every root image of Med_4 and Med_6 is topologically connected set in the digital topology.

We aim to have a generalization of the digital topology which provide the implication or find another filter that make the root image of this filter is regular semi-open set (regular λ-open) and vice versa.

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On Distances and Similarity Measures between Two Interval Neutrosophic Sets

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Abstract – An Interval Neutrosophic set (INS) is an instance of a Neutrosophic set and also an emerging tool for uncertain data processing in real scientific and engineering applications. In this paper, several distance and similarity measures between two Interval Neutrosophic sets have been discussed. Distances and similarities are very useful techniques to determine interacting segments in a data set. Here we have also shown an application of our similarity measures in solving a multicriteria decision making method based on INS's. Finally, we take an illustrative example from [14] to apply the proposed decision making method. We use the distance as well as the similarity measures between each alternative and ideal alternative to form a ranking order and also to find the best alternative. We compare the obtained results with the existing result in [14] and also reveal the best distance and similarity measure to find the best alternative and also point out the best alternative.

Keywords – *Interval Neutrosophic Set, Distance, Similarity Measure, Multicriteria Decision Making.*

1. Introduction

"As far as the laws of Mathematics refer to reality, they are not certain; and as far they are certain, they do not refer to reality." – Albert Einstein. Uncertainty is a common phenomenon in our daily life; because in our real or daily life we have to take account a lot of uncertainties. From centuries, numerous theories have been developed in both Science and Philosophy to understand and represent the features of uncertainty. Probability theory and stochastic techniques are such theories, which were developed in early eighteenth century and probability was the sole technique to handle a certain type of uncertainty called Randomness. But there are several other kinds of uncertainties, such as vagueness, imprecision, cloudiness, haziness, ambiguity, variety etc. It is generally agreed that the most important invention in the evolution of the concept of uncertainty was made by Zadeh in 1965, when he coined the theory of Fuzzy sets [17], which was a remarkable step to deal with such types of uncertainties, though some ideas presented by him, were borrowed from the envisions of American philosopher Max Black (1937). In his theory, Zadeh introduced the fuzzy sets, which have imprecise boundaries. When A is a fuzzy set and x is an object of A, then the statement ‘ x is a member of A’ is not only either true or false as in crisp sets, but also it is true only to some degree to which x is actually a member of A. The membership degrees are within the closed interval [0,1]. Later, this theory leads to a highly commendable theory of Fuzzy logic, which was applied to engineering such as washing machine or shifting gears of cars with great efficiency. After Zadeh’s invention of Fuzzy sets, many other concepts began to develop. In 1986, K. Atanassov [1], introduced the idea of Intuitionistic fuzzy sets (IFS), which is a generalization of Fuzzy sets. The IFS is a set with each member having a degree of belongingness and a degree of non-belongingness as well. There is a restriction that sum of the membership grade and non-membership grade of an element is less or equal to 1. IFS is quite useful to deal with applications like expert systems, information fusion etc., where ‘degree of non belongingness’ of an object is equally important as the ‘degree of belongingness’. Besides IFS, there are other generalizations of Fuzzy sets and intuitionistic fuzzy sets like L-Fuzzy sets, interval valued fuzzy sets, intuitionistic L-Fuzzy sets, interval valued intuitionistic fuzzy sets [11,2] etc.

In 1995, Smarandache [9, 10], introduced a more generalized tool to handle Uncertainty, called as Neutrosophic logic and sets. It is a logic, in which each proposition has a degree of truth (T), a degree of indeterminacy (I), and a degree of falsity (F). Also an element x in a Neutrosophic set (NS) X has a truth membership, an indeterminacy membership and a falsity membership, which are independent and which lies between [0, 1], and sum of them is less or equal to 3. Thus Neutrosophic set is a generalization of fuzzy set [17], interval valued fuzzy set [11], intuitionistic fuzzy set [1], interval valued intuitionistic fuzzy set [2], paraconsistent set [9], dialetheist set [9], paradoxist set [9] and tautological set [9]. Though the NS generalized the above mentioned sets, but the generalization was only from philosophical point of view. For application in engineering and other areas of science, NS needed to be more specific. Further Wang et. al., in 2005, developed an instance of NS, called as single valued Neutrosophic sets (SVNS) [13]. Later they have also introduced the notion of Interval valued neutrosophic sets (INS) [12]. The INS is more capable to handle the uncertain, imprecise, incomplete and inconsistent information that exist in real world. In INS, the degree of truth, indeterminacy and falsity membership of an object are expressed in closed subintervals of [0, 1].

In many problems, it is often needed to compare two sets, which may be fuzzy, intuitionistic fuzzy, vague etc. We are often interested to reveal the similarity or the least degree of similarity of two images or patterns. Distance and similarity measures are the efficient tools to do this. Many authors have done extensive research regarding distance and similarity of fuzzy and intuitionistic fuzzy sets and their interval valued versions [7, 8, 15, 16]. Similarity measures are also a very good tool for solving many decision making problems. The notion of distance and similarity was first introduced in [5,6]. Later Broumi et. al. [3] has defined several other similarity measures on Single valued neutrosophic sets. The notion of similarity of INS is introduced in [4, 14].This paper also deals with distance and similarity of Interval neutrosophic sets. However, in this article, our motive is to establish the best suitable distance and similarity measures by comparing the numerical value of various distances and similarities between two INSs. We are to also point out the best alternative, similar to the ideal alternative in the decision making problem stated and solved by Jun Ye [14], by comparing numerical values of distances and similarities of each alternative with the ideal alternative and also comparing with the existing results [14].

The organization of the rest of this paper is as follows: In section 2, definitions of Fuzzy set, Intuitionistic Fuzzy set, Neutrosophic Set (NS) and Interval valued Neutrosophic set (INS) are given and some operations on NS and INS have been defined and also Set theoretic properties on INS are also given. Several distances and Similarities on INSs are defined in section 3 and 4. A decision making method is established in Interval Neutrosophic setting by means of distance and similarity measures between each alternative and ideal alternative in section 5. In section 6, an illustrative example is adapted from [14], to illustrate the proposed method. Finally a comparative study has been made with the existing results in section 7 and at last section 8 concludes the article.

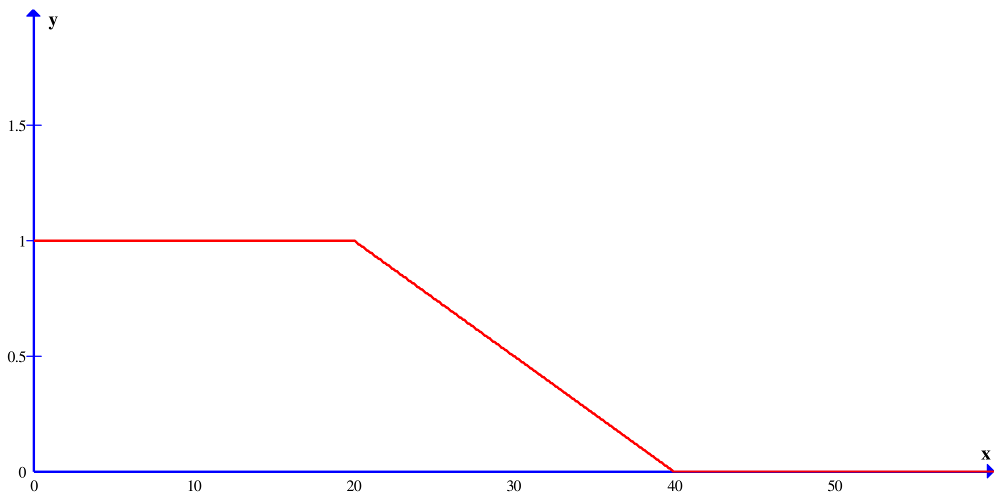
2. Preliminaries

In this section, we give some useful definitions, examples and results which will be used in the rest of this paper.

Definition 2.1 (*Type I Fuzzy set*) If *X* is a collection of objects denoted by *x*, then a fuzzy set (or type I fuzzy set) *A* in *X is* a set of ordered pairs: $A = \{(x, \mu_A(x)) \mid x \in X\}$ where $\mu_A(x)$ is called the membership function or grade of membership (also degree of compatibility or degree of truth) of x in *A* that maps X to the membership space, i.e. $\mu_A : X \to M = [0,1]$. A becomes a crisp set when M contains only two points 0 and 1and μ_A is the characteristic function χ_A of *A*.

Example 2.2 As an illustration, consider the following example. Let, the set 'P' is the set of people. To each person in 'P' we have to assign a degree of membership in the fuzzy subset YOUTH, which is defined as follows:

$$\text{Youth } (x) = \{\ 1,\ if\ age(x) \le 20,$$
$$(40 - age(x))/20,\ if\ 20 < age(x) \le 40,$$
$$0,\ if\ age(x) > 40\ \}$$



Then the set YOUTH is a fuzzy set of type I or an ordinary fuzzy set.

Definition 2.3 (*Intuitionistic fuzzy set)* Intuitionistic fuzzy sets generalize fuzzy sets, since with membership function μ, a non-membership function ν is also introduced for each object in it.

Let us have a fixed universe ***X***. Let $\boldsymbol{A} \subseteq \boldsymbol{X}$. Let us construct the set:

$$A^* = \{(x, \mu_A(x), \nu_A(x)) \mid x \in X \;\&\; 0 \le \mu_A(x) + \nu_A(x) \le 1\}$$

where $\mu_A : X \to [0,1], \nu_A : X \to [0,1]$ and $\forall x \in X$ .We call the set ***A**** intuitionistic fuzzy set (IFS).

Example 2.4 Let us illustrate the concept of IFS by an example as follows: Let *X* be the set of all Secondary schools in a district. We assume that, for every school $x \in X$, the number of students qualified in the final exam is known and say it is *P*(*x*). Let,

$$\mu_X(x) = P(x) \Big/ (total\ number\ of\ students)$$

Take $\nu_X(x) = 1 - \mu_X(x)$, which indicates the part of students couldn't qualify the exam. By Fuzzy set theory, we cannot obtain that how many students have not given the exam. But, if we take $\nu_X(x)$ as the number of students failed to qualify the exam, then we can easily obtain the part of the students, have not given the exam at all and the value will be $1 - \mu_X(x) - \nu_X(x)$. Thus we construct the IFS, $\{(x, \mu_X(x), \nu_X(x)) : x \in X\}$ and obviously $0 \le \mu_X(x) + \nu_X(x) \le 1$

Definition 2.5 (*Neutrosophic set)* Neutrosophic sets (NS) further generalizes the IFS. As in NS, the indeterminacy is explicitly defined and also the truth membership, falsity membership and indeterminacy membership are beyond any restriction. Let *X* be a collection of objects denoted by ***x***. A Neutrosophic set ***A*** in ***X*** is characterized by a truth membership function $\boldsymbol{T_A}$, an Indeterminacy membership function $\boldsymbol{I_A}$ and a falsity membership function $\boldsymbol{F_A}$, where,

$$T_A(x),\ I_A(x) \text{ and } F_A(x) : X \to [0,1] \text{ and } 0 \le \sup T_A(x) + \sup I_A(x) + \sup F_A(x) \le 3.$$

The NS ***A*** in ***X*** can be denoted as $A = \{x, T_A(x), I_A(x), F_A(x) : x \in X\}$

Example 2.6 If x_1 be an element of a set A and if we take the probability of x_1 in A is 60%, probability of x_1 not in A is 20% and probability of x_1 in A is undetermined is 10%, then the NS can be denoted as x_1(0.6,0.1,0.2). Also to generalize the example, Take X be the set of 'rainy days'. Consider A be the set "today it will rain heavily." Let according to an observer x_1, probability of heavy raining is 80%, that of not raining is 10%, and also the indeterminacy is 10%. According to another observer x_2 , those probabilities are 40%, 50% and 10% respectively. Then NS A in X can be denoted as follows:

$$A = \langle 0.8, 0.1, 0.1 \rangle / x_1 + \langle 0.4, 0.1, 0.5 \rangle / x_2$$

Definition 2.7(*Interval Neutrosophic set*) Let ***X be*** a space of objects, whose elements are denoted by *x*. An INS ***A*** in ***X*** is characterized by a truth-membership function. $T_A(x)$, an indeterminacy-membership function $I_A(x)$ and a falsity-membership function $F_A(x)$. For each point ***x*** in ***X***, we have:

$$T_A(x) = [\inf T_A(x), \sup T_A(x)] \subseteq [0,1],$$
$$I_A(x) = [\inf I_A(x), \sup I_A(x)] \subseteq [0,1],$$
$$F_A(x) = [\inf F_A(x), \sup F_A(x)] \subseteq [0,1]$$

and

$$0 \le \sup T_A(x) + \sup I_A(x) + \sup F_A(x) \le 3, \forall x \in X.$$

When *X* is continuous, an INS *A* can be written as :

$$A = \int_X \langle T(x), I(x), F(x) \rangle / x, \quad x \in X$$

When *X* is discrete, an INS *A* can be written as :

$$A = \sum_{i=1}^{n} \langle T(x_i), I(x_i), F(x_i) \rangle / x_i, \quad x_i \in X$$

Example 2.8 For example, Assume that x_1 is quality, x_2 is trustworthiness and x_3 is price of a book. The values of x_1, $x_2$and x_3 are in [0, 1]. They are obtained from some questionnaires, having options as 'degree of good', 'degree of indeterminacy' and 'degree of bad'. Take *A* and *B* are interval neutrosophic sets of X defined as:

$$A = \langle [0.1,0.3],[0,0.2],[0.5,0.7] \rangle / x_1 + \langle [0.4,0.5],[0.1,0.2],[0.6,0.7] \rangle / x_2 + \langle [0.7,0.8],[0,0.3],[0.1,0.2] \rangle / x_3$$

$$B = \langle [0.2,0.4],[0.1,0.3],[0.6,0.8] \rangle / x_1 + \langle [0.7,0.9],[0.4,0.6],[0.2,0.4] \rangle / x_2 + \langle [0.3,0.5],[0.2,0.4],[0.1,0.3] \rangle / x_3$$

Some operations on Neutrosophic sets

Definition 2.9

(i) **Complement:** Let *A* be a Neutrosophic set. Then *complement* of *A* is denoted by A^c or $\bar{A}$ and is defined by

$$T_{\bar{A}}(x) = F_A(x), I_{\bar{A}}(x) = 1 - I_A(x), F_{\bar{A}}(x) = T_A(x), \forall x \in X$$

(ii) **Containment:** A NS *A* is *contained* in the other NS *B*, denoted as $A \subseteq B$, if and only if:

$$T_A(x) \leq T_B(x)\,;\, I_A(x) \geq I_B(x)\,;\, F_A(x) \geq F_B(x)\,;\, x \in X$$

(iii) **Union:** The *union* of two NS *A* and *B* is a NS *C,* written as $C = A \cup B$, whose truth-membership, indeterminacy-membership and falsity membership functions are related to those of *A* and *B* by:

$$\begin{aligned} T_C(x) &= T_A(x) \vee T_B(x), \\ I_C(x) &= I_A(x) \wedge I_B(x), \\ F_C(x) &= F_A(x) \wedge F_B(x), \forall x \in X \end{aligned}$$

(iv) **Intersection:** The *intersection* of two NS *A* and *B* is a NS *C*, denoted as $C = A \cap B$, whose truth-membership, indeterminacy-membership and falsity membership functions are related to those of *A* and *B* by:

$$\begin{aligned} T_C(x) &= T_A(x) \wedge T_B(x), \\ I_C(x) &= I_A(x) \vee I_B(x), \\ F_C(x) &= F_A(x) \vee F_B(x), \forall x \in X \end{aligned}$$

Some operations on Interval Neutrosophic set

The notion of IVNS was defined by Wang et. al. [13]. Here we give some definitions and examples of IVNS

Definition 2.10 (Complement): Let *A* be an Interval Neutrosophic set. Then *complement* of A is denoted by A^c or $\bar{A}$ and is defined by:

$$\begin{aligned} &T_{\bar{A}}(x) = F_A(x), \\ &\inf I_{\bar{A}}(x) = 1 - \sup I_A(x), \\ &\sup I_{\bar{A}}(x) = 1 - \inf I_A(x), \\ &F_{\bar{A}}(x) = T_A(x) \end{aligned}$$

Example 2.11 Let A be the interval valued Neutrosophic set defined in *example 2.8*. Then

$$\begin{aligned} \bar{A} = &\langle [0.5, 0.7], [0.8, 1.0], [0.1, 0.3] \rangle / x_1 + \\ &\langle [0.6, 0.7], [0.8, 0.9], [0.4, 0.5] \rangle / x_2 + \\ &\langle [0.1, 0.2], [0.7, 1.0], [0.7, 0.8] \rangle / x_3 \end{aligned}$$

Definition 2.12 (Containment) A INS *A* is *contained* in the other INS *B*, denoted as $A \subseteq B$, if and only if:

$$\inf T_A(x) \le \inf T_B(x),\ \sup T_A(x) \le \sup T_B(x);$$
$$\inf I_A(x) \ge \inf I_B(x),\ \sup I_A(x) \ge \sup I_B(x);$$
$$\inf F_A(x) \ge \inf F_B(x),\ \sup F_A(x) \ge \sup F_B(x); \forall\ x \in X$$

Two interval neutrosophic sets A and B are ***equal***, written as A = B, if and only if $A \subseteq B$ and $B \subseteq A$

Example 2.13 Let A and B be two INS defined in *example 3.1.4*, then it can be easily observed that those INSs do not satisfy all the required properties for containment of A in B. So here $A \not\subset B$.

Definition 2.14 (Union): The *union* of two INS *A* and *B* is a INS *C,* written as $C = A \cup B$, whose truth-membership, indeterminacy-membership and falsity membership functions are related to those of *A* and *B* by:

$$\inf T_C(x) = \max(\inf T_A(x), \inf T_B(x)),$$
$$\sup T_C(x) = \max(\sup T_A(x), \sup T_B(x)),$$
$$\inf I_C(x) = \min(\inf I_A(x), \inf I_B(x)),$$
$$\sup I_C(x) = \min(\sup I_A(x), \sup I_B(x)),$$
$$\inf F_C(x) = \min(\inf F_A(x), \inf F_B(x)),$$
$$\sup F_C(x) = \min(\sup F_A(x), \sup F_B(x)), \forall x \in X$$

Example 2.15: Consider two INS A and B defined in example 2.8. Then their union C =A∪B is

$$C = \langle [0.2,0.4],[0,0.2],[0.5,0.7]\rangle / x_1 + \langle [0.7,0.9],[0.1,0.2],[0.2,0.4]\rangle / x_2 + \langle [0.7,0.8],[0,0.3],[0.1,0.2]\rangle / x_3$$

Definition 2.16 (Intersection) The *intersection* of two INS *A* and *B* is a INS *C*, denoted as *C*=*A*∩*B*, whose truth-membership, indeterminacy-membership and falsity membership functions are related to those of *A* and *B* by:

$$\inf T_C(x) = \min(\inf T_A(x), \inf T_B(x)),$$
$$\sup T_C(x) = \min(\sup T_A(x), \sup T_B(x)),$$
$$\inf I_C(x) = \max(\inf I_A(x), \inf I_B(x)),$$
$$\sup I_C(x) = \max(\sup I_A(x), \sup I_B(x)),$$
$$\inf F_C(x) = \max(\inf F_A(x), \inf F_B(x)),$$
$$\sup F_C(x) = \max(\sup F_A(x), \sup F_B(x)), \forall x \in X$$

Example 2.17 Take A and B be two INS defined in *example 2.8*. Then their intersection C=A∩B is as follows:

$$C = \langle [0.1,0.3],[0.1,0.3],[0.6,0.8] \rangle / x_1 + \langle [0.4,0.5],[0.4,0.6],[0.6,0.7] \rangle / x_2 + \langle [0.3,0.5],[0.2,0.4],[0.1,0.3] \rangle / x_3$$

Set theoretical properties

Here we will give some properties of set-theoretic operators defined on interval neutrosophic sets.

Let, A, B and C be three INSs. Then the properties satisfied by A, B and C are as follows:

Property 1 ***(Commutativity)***

$$A \cup B = B \cup A$$
$$A \cap B = B \cap A$$

Property 2 ***(Associativity)***

$$A \cup (B \cup C) = (A \cup B) \cup C$$
$$A \cap (B \cap C) = (A \cap B) \cap C$$

Property 3 ***(Distributivity)***

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$
$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

Property 4 ***(Idempotency)***

$$A \cup A = A, A \cap A = A.$$

Property 5 $A \cap \Phi = \Phi, A \cup X = X$, Where Φ and X are respectively Null set and absolute INS defined below:

$$\inf T_\Phi = \sup T_\Phi = 0,$$
$$\inf I_\Phi = \sup I_\Phi = \inf F_\Phi = \sup F_\Phi = 1,$$
$$\inf T_X = \sup T_X = 1,$$
$$\inf I_X = \sup I_X = \inf F_X = \sup F_X = 0$$

Property 6

$A \cup \Phi = A, A \cap X = A$, Where Φ and X are defined above.

Property 7 ***(Absorption)***

$$A \cup (A \cap B) = A, A \cap (A \cup B) = A$$

Property8 (Involution)

$$\overline{\overline{A}} = A$$

Here, we notice that by the definitions of complement, union and intersection of interval neutrosophic set as defined previously, INS satisfies the most properties of crisp set, fuzzy set and intuitionistic fuzzy set. Also, it does not satisfy the principle of excluded middle, same as fuzzy set and intuitionistic fuzzy set.

3. Distance Measure

In this section, we investigate several distance measures for two INS's A and B. Also, we take the weights of the element x_i (i= 1, 2,….,n) into account. In the following, we consider some weighted distance measures between INSs. For this we take $w=\{w_1,w_2,\dots,w_n\}$ as the weight vector of the element x_i (i =1,2,….,n) and also $w_i \in [0,1], \forall i = 1,2,.....,n$ .We adopt some distance and similarity measures from [15] and extend those in INS setting as follows:

a. *Hamming Distance :*

$$d_1(A,B) = \frac{1}{6}\sum_{i=1}^{n}[|\inf T_A(x_i) - \inf T_B(x_i)| + |\sup T_A(x_i) - \sup T_B(x_i)| + \\ |\inf I_A(x_i) - \inf I_B(x_i)| + |\sup I_A(x_i) - \sup I_B(x_i)| + \\ |\inf F_A(x_i) - \inf F_B(x_i)| + |\sup F_A(x_i) - \sup F_B(x_i)|]$$

b. *Normalized Hamming Distance :*

$$d_2(A,B) = \frac{1}{6n}\sum_{i=1}^{n}[|\inf T_A(x_i) - \inf T_B(x_i)| + |\sup T_A(x_i) - \sup T_B(x_i)| + \\ |\inf I_A(x_i) - \inf I_B(x_i)| + |\sup I_A(x_i) - \sup I_B(x_i)| + \\ |\inf F_A(x_i) - \inf F_B(x_i)| + |\sup F_A(x_i) - \sup F_B(x_i)|]$$

c. *Euclidean distance :*

$$d_3(A,B) = \{\frac{1}{6}\sum_{i=1}^{n}[|\inf T_A(x_i) - \inf T_B(x_i)|^2 + |\sup T_A(x_i) - \sup T_B(x_i)|^2 + \\ |\inf I_A(x_i) - \inf I_B(x_i)|^2 + |\sup I_A(x_i) - \sup I_B(x_i)|^2 + \\ |\inf F_A(x_i) - \inf F_B(x_i)|^2 + |\sup F_A(x_i) - \sup F_B(x_i)|^2]\}^{1/2}$$

d. Normalized Euclidean distance :

$$d_4(A,B) = \{\frac{1}{6n}\sum_{i=1}^{n}[|\inf T_A(x_i) - \inf T_B(x_i)|^2 + |\sup T_A(x_i) - \sup T_B(x_i)|^2 + |\inf I_A(x_i) - \inf I_B(x_i)|^2 + |\sup I_A(x_i) - \sup I_B(x_i)|^2 + |\inf F_A(x_i) - \inf F_B(x_i)|^2 + |\sup F_A(x_i) - \sup F_B(x_i)|^2]\}^{1/2}$$

e. Hausdroff distance :

$$d_5(A,B) = \sum_{i=1}^{n}\max[|\inf T_A(x_i) - \inf T_B(x_i)|, |\sup T_A(x_i) - \sup T_B(x_i)|, |\inf I_A(x_i) - \inf I_B(x_i)|, |\sup I_A(x_i) - \sup I_B(x_i)|, |\inf F_A(x_i) - \inf F_B(x_i)|, |\sup F_A(x_i) - \sup F_B(x_i)|]$$

f. Normalized Hausdroff distance :

$$d_6(A,B) = \frac{1}{n}\sum_{i=1}^{n}\max[|\inf T_A(x_i) - \inf T_B(x_i)|, |\sup T_A(x_i) - \sup T_B(x_i)|, |\inf I_A(x_i) - \inf I_B(x_i)|, |\sup I_A(x_i) - \sup I_B(x_i)|, |\inf F_A(x_i) - \inf F_B(x_i)|, |\sup F_A(x_i) - \sup F_B(x_i)|]$$

g. Weighted Hamming Distance :

$$d_7(A,B) = \frac{1}{6}\sum_{i=1}^{n} w_i[|\inf T_A(x_i) - \inf T_B(x_i)| + |\sup T_A(x_i) - \sup T_B(x_i)| + |\inf I_A(x_i) - \inf I_B(x_i)| + |\sup I_A(x_i) - \sup I_B(x_i)| + |\inf F_A(x_i) - \inf F_B(x_i)| + |\sup F_A(x_i) - \sup F_B(x_i)|]$$

h. Weighted normalized Hamming distance :

$$d_8(A,B) = \frac{1}{6n}\sum_{i=1}^{n} w_i[|\inf T_A(x_i) - \inf T_B(x_i)| + |\sup T_A(x_i) - \sup T_B(x_i)| + |\inf I_A(x_i) - \inf I_B(x_i)| + |\sup I_A(x_i) - \sup I_B(x_i)| + |\inf F_A(x_i) - \inf F_B(x_i)| + |\sup F_A(x_i) - \sup F_B(x_i)|]$$

i. Weighted Euclidean distance :

$$d_9(A,B) = \{\frac{1}{6}\sum_{i=1}^{n} w_i[|\inf T_A(x_i) - \inf T_B(x_i)|^2 + |\sup T_A(x_i) - \sup T_B(x_i)|^2 + |\inf I_A(x_i) - \inf I_B(x_i)|^2 + |\sup I_A(x_i) - \sup I_B(x_i)|^2 + |\inf F_A(x_i) - \inf F_B(x_i)|^2 + |\sup F_A(x_i) - \sup F_B(x_i)|^2]\}^{1/2}$$

j. Weighted normalized Euclidean distance

$$d_{10}(A,B) = \{\frac{1}{6n}\sum_{i=1}^{n} w_i[|\inf T_A(x_i) - \inf T_B(x_i)|^2 + |\sup T_A(x_i) - \sup T_B(x_i)|^2 + |\inf I_A(x_i) - \inf I_B(x_i)|^2 + |\sup I_A(x_i) - \sup I_B(x_i)|^2 + |\inf F_A(x_i) - \inf F_B(x_i)|^2 + |\sup F_A(x_i) - \sup F_B(x_i)|^2]\}^{1/2}$$

k. Weighted Hausdroff distance :

$$d_{11}(A,B) = \sum_{i=1}^{n} w_i \max[|\inf T_A(x_i) - \inf T_B(x_i)|, |\sup T_A(x_i) - \sup T_B(x_i)|, |\inf I_A(x_i) - \inf I_B(x_i)|, |\sup I_A(x_i) - \sup I_B(x_i)|, |\inf F_A(x_i) - \inf F_B(x_i)|, |\sup F_A(x_i) - \sup F_B(x_i)|]$$

l. Weighted normalized Hausdroff distance:

$$d_{12}(A,B) = \frac{1}{n}\sum_{i=1}^{n} w_i \max[|\inf T_A(x_i) - \inf T_B(x_i)|, |\sup T_A(x_i) - \sup T_B(x_i)|, |\inf I_A(x_i) - \inf I_B(x_i)|, |\sup I_A(x_i) - \sup I_B(x_i)|, |\inf F_A(x_i) - \inf F_B(x_i)|, |\sup F_A(x_i) - \sup F_B(x_i)|]$$

m. Euclidean Hausdroff distance :

$$d_{13}(A,B) = \{\sum_{i=1}^{n} \max[|\inf T_A(x_i) - \inf T_B(x_i)|^2, |\sup T_A(x_i) - \sup T_B(x_i)|^2, |\inf I_A(x_i) - \inf I_B(x_i)|^2, |\sup I_A(x_i) - \sup I_B(x_i)|^2, |\inf F_A(x_i) - \inf F_B(x_i)|^2, |\sup F_A(x_i) - \sup F_B(x_i)|^2]\}^{1/2}$$

n. Weighted Euclidean Hausdroff distance :

$$d_{14}(A,B) = \{\sum_{i=1}^{n} w_i \max[|\inf T_A(x_i) - \inf T_B(x_i)|^2, |\sup T_A(x_i) - \sup T_B(x_i)|^2, |\inf I_A(x_i) - \inf I_B(x_i)|^2, |\sup I_A(x_i) - \sup I_B(x_i)|^2, |\inf F_A(x_i) - \inf F_B(x_i)|^2, |\sup F_A(x_i) - \sup F_B(x_i)|^2]\}^{1/2}$$

o. Normalized Euclidean Hausdroff Distance :

$$d_{15}(A,B) = \{\frac{1}{n}\sum_{i=1}^{n} \max[|\inf T_A(x_i) - \inf T_B(x_i)|^2, |\sup T_A(x_i) - \sup T_B(x_i)|^2, |\inf I_A(x_i) - \inf I_B(x_i)|^2, |\sup I_A(x_i) - \sup I_B(x_i)|^2, |\inf F_A(x_i) - \inf F_B(x_i)|^2, |\sup F_A(x_i) - \sup F_B(x_i)|^2]\}^{1/2}$$

p. Normalized Weighted Euclidean Hausdroff Distance :

$$d_{16}(A,B)=\{\frac{1}{n}\sum_{i=1}^{n} w_i \max[|\inf T_A(x_i)-\inf T_B(x_i)|^2,|\sup T_A(x_i)-\sup T_B(x_i)|^2,$$
$$|\inf I_A(x_i)-\inf I_B(x_i)|^2,|\sup I_A(x_i)-\sup I_B(x_i)|^2,$$
$$|\inf F_A(x_i)-\inf F_B(x_i)|^2,|\sup F_A(x_i)-\sup F_B(x_i)|^2]\}^{1/2}$$

Some other distances between two INS's are given as follows

We consider 'p' as a positive integer in the following.

q. $$d_{17}(A,B)=\{\frac{1}{6}\sum_{i=1}^{n}[|\inf T_A(x_i)-\inf T_B(x_i)|^p+|\sup T_A(x_i)-\sup T_B(x_i)|^p+$$
$$|\inf I_A(x_i)-\inf I_B(x_i)|^p+|\sup I_A(x_i)-\sup I_B(x_i)|^p+$$
$$|\inf F_A(x_i)-\inf F_B(x_i)|^p+|\sup F_A(x_i)-\sup F_B(x_i)|^p]\}^{1/p},\quad \forall p>0$$

r. $$d_{18}(A,B)=\{\frac{1}{6}\sum_{i=1}^{n}w_i[|\inf T_A(x_i)-\inf T_B(x_i)|^p+|\sup T_A(x_i)-\sup T_B(x_i)|^p+$$
$$|\inf I_A(x_i)-\inf I_B(x_i)|^p+|\sup I_A(x_i)-\sup I_B(x_i)|^p+$$
$$|\inf F_A(x_i)-\inf F_B(x_i)|^p+|\sup F_A(x_i)-\sup F_B(x_i)|^p]\}^{1/p},\quad \forall p>0$$

s. $$d_{19}(A,B)=\{\frac{1}{6n}\sum_{i=1}^{n}[|\inf T_A(x_i)-\inf T_B(x_i)|^p+|\sup T_A(x_i)-\sup T_B(x_i)|^p+$$
$$|\inf I_A(x_i)-\inf I_B(x_i)|^p+|\sup I_A(x_i)-\sup I_B(x_i)|^p+$$
$$|\inf F_A(x_i)-\inf F_B(x_i)|^p+|\sup F_A(x_i)-\sup F_B(x_i)|^p]\}^{1/p},\quad \forall p>0$$

t. $$d_{20}(A,B)=\{\frac{1}{6n}\sum_{i=1}^{n}w_i[|\inf T_A(x_i)-\inf T_B(x_i)|^p+|\sup T_A(x_i)-\sup T_B(x_i)|^p+$$
$$|\inf I_A(x_i)-\inf I_B(x_i)|^p+|\sup I_A(x_i)-\sup I_B(x_i)|^p+$$
$$|\inf F_A(x_i)-\inf F_B(x_i)|^p+|\sup F_A(x_i)-\sup F_B(x_i)|^p]\}^{1/p},\quad \forall p>0$$

u. $$d_{21}(A,B)=\{\sum_{i=1}^{n}\max[|\inf T_A(x_i)-\inf T_B(x_i)|^p,|\sup T_A(x_i)-\sup T_B(x_i)|^p,$$
$$|\inf I_A(x_i)-\inf I_B(x_i)|^p,|\sup I_A(x_i)-\sup I_B(x_i)|^p,$$
$$|\inf F_A(x_i)-\inf F_B(x_i)|^p,|\sup F_A(x_i)-\sup F_B(x_i)|^p]\}^{1/p},\quad \forall p>0$$

v. $$d_{22}(A,B)=\{\sum_{i=1}^{n} w_i \max[|\inf T_A(x_i)-\inf T_B(x_i)|^p, |\sup T_A(x_i)-\sup T_B(x_i)|^p,$$
$$|\inf I_A(x_i)-\inf I_B(x_i)|^p, |\sup I_A(x_i)-\sup I_B(x_i)|^p,$$
$$|\inf F_A(x_i)-\inf F_B(x_i)|^p, |\sup F_A(x_i)-\sup F_B(x_i)|^p]\}^{1/p}, \quad \forall p>0$$

w. $$d_{23}(A,B)=\{\frac{1}{n}\sum_{i=1}^{n} \max[|\inf T_A(x_i)-\inf T_B(x_i)|^p, |\sup T_A(x_i)-\sup T_B(x_i)|^p,$$
$$|\inf I_A(x_i)-\inf I_B(x_i)|^p, |\sup I_A(x_i)-\sup I_B(x_i)|^p,$$
$$|\inf F_A(x_i)-\inf F_B(x_i)|^p, |\sup F_A(x_i)-\sup F_B(x_i)|^p]\}^{1/p}, \quad \forall p>0$$

x. $$d_{24}(A,B)=\{\frac{1}{n}\sum_{i=1}^{n} w_i \max[|\inf T_A(x_i)-\inf T_B(x_i)|^p, |\sup T_A(x_i)-\sup T_B(x_i)|^p,$$
$$|\inf I_A(x_i)-\inf I_B(x_i)|^p, |\sup I_A(x_i)-\sup I_B(x_i)|^p,$$
$$|\inf F_A(x_i)-\inf F_B(x_i)|^p, |\sup F_A(x_i)-\sup F_B(x_i)|^p]\}^{1/p}, \quad \forall p>0$$

Properties of Distance Measure

The above defined distance $d_k(A,B)$ (k=1, 2, 3, …)between INSs A and B satisfies the following properties (D1–D3) :

D1: $d_k(A,B) \ge 0$;
D2: $d_k(A,B)=0$ if and only if $A=B$
D3: $d_k(A,B)=d_k(B,A)$;

It can be easily shown that the distances as defined above satisfy the said properties.

4. Algorithm

Now we present an algorithm to solve a decision making problem in Interval Neutrosophic Sets by means of distance and similarity measures in INSs.
Let $\{A_i : i=1,2,...,m\}$ be a set of alternatives and $\{C_j : j=1,2,...,n\}$ be a set of criteria. Assume that the weight of the criterion C_j is $w_j \in [0,1]$ and $\sum_{j=1}^{n} w_j = 1$. In this case the INS A_i can be denoted as follows:

$$A_i = \{\langle C_j, (T_{A_i}(C_j), I_{A_i}(C_j), F_{A_i}(C_j))\rangle : C_j \in C\},$$

where

$$T_{A_i}(C_j) = [\inf T_{A_i}(C_j), \sup T_{A_i}(C_j)] \in [0,1],$$
$$I_{A_i}(C_j) = [\inf I_{A_i}(C_j), \sup I_{A_i}(C_j)] \in [0,1],$$
$$F_{A_i}(C_j) = [\inf F_{A_i}(C_j), \sup F_{A_i}(C_j)] \in [0,1],$$

and $0 \le \sup T_{A_i}(C_j) + \sup I_{A_i}(C_j) + \sup F_{A_i}(C_j) \le 3$, i=1,2,…,m and j=1,2,…,n.
Now let us consider an INS denoted as:

$$\alpha_{ij} = ([a_{ij}, b_{ij}], [c_{ij}, d_{ij}], [e_{ij}, f_{ij}])$$

where

$$[a_{ij}, b_{ij}] = [\inf T_{A_i}(C_j), \sup T_{A_i}(C_j)],$$
$$[c_{ij}, d_{ij}] = [\inf I_{A_i}(C_j), \sup I_{A_i}(C_j)],$$
$$[e_{ij}, f_{ij}] = [\inf F_{A_i}(C_j), \sup F_{A_i}(C_j)]$$

Now, an INS is derived from the evaluation of an alternative A_i with respect to a criterion C_j, by means of score law and data processing. Therefore, we can introduce an interval neutrosophic decision matrix $D = (\alpha_{ij})_{m\times n}$.

The evaluation criteria are generally taken of two kinds, benefit criteria and cost criteria. Let ***B*** be a collection of benefit criteria and ***P*** be a collection of cost criteria. Then we define an ideal INS for a benefit criterion in the ideal alternative A^* as:

$$\alpha^*_j = ([a^*_j, b^*_j], [c^*_j, d^*_j], [e^*_j, f^*_j]) = ([1,1],[0,0],[0,0]) \text{ for } j \in B$$

and for a cost criterion, we define the ideal alternative A^{**} as:

$$\alpha^{**}_j = ([a^{**}_j, b^{**}_j], [c^{**}_j, d^{**}_j], [e^{**}_j, f^{**}_j]) = ([0,0],[1,1],[1,1]) \text{ for } j \in P.$$

Although, the ideal alternative doesn't exist in real world, it is only used to identify the best alternative in decision set.

Now if we denote the ideal alternative as the INS E, then by the distance measures $d_k(E, A_i)$, (i =1,2,….,m), (k=1,2,…,24) and the similarity measures $s_k(E, A_i)$, (i=1,2,….,m), (k=1,2,…,21) (as defined in previous section), between each alternative A_i and the ideal alternative E *(For benefit criteria* $E = A^*$ *and for cost criteria* $E = A^{**}$*)*, the ranking order of all alternatives can be determined and the best one can be easily identified as well.

5. Problem

To illustrate the above algorithm we take a multi-criteria decision making problem of alternatives to apply the proposed decision making method.

We adapt the required problem from the article by Jun Ye [14], stated as follows:

There is an investment company, which wants to invest a sum of money in the best option. There is a panel with four possible alternatives to invest the money:

(1) A_1 is a car company; (2) A_2 is a food company; (3) A_3 is a computer company; (4) A_4 is an arms company.

The investment company must take a decision according to the following three criteria:

(1) C_1 is the risk analysis; (2) C_2 is the growth analysis; (3) C_3 is the environmental impact analysis, where C_1 and C_2 are benefit criteria and C_3 is a cost criterion. The weight vector of the criteria is given by :w = (0.35, 0.25, 0.40). The four possible alternatives are to be evaluated under the above three criteria by corresponding to the INSs, as shown in the following interval neutrosophic decision matrix *D:*

$$D=\begin{bmatrix} \langle[0.4,0.5],[0.2,0.3],[0.3,0.4]\rangle & \langle[0.4,0.6],[0.1,0.3],[0.2,0.4]\rangle & \langle[0.7,0.9],[0.2,0.3],[0.4,0.5]\rangle \\ \langle[0.6,0.7],[0.1,0.2],[0.2,0.3]\rangle & \langle[0.6,0.7],[0.1,0.2],[0.2,0.3]\rangle & \langle[0.3,0.6],[0.3,0.5],[0.8,0.9]\rangle \\ \langle[0.3,0.6],[0.2,0.3],[0.3,0.4]\rangle & \langle[0.5,0.6],[0.2,0.3],[0.3,0.4]\rangle & \langle[0.4,0.5],[0.2,0.4],[0.7,0.9]\rangle \\ \langle[0.7,0.8],[0.0,0.1],[0.1,0.2]\rangle & \langle[0.6,0.7],[0.1,0.2],[0.1,0.3]\rangle & \langle[0.6,0.7],[0.3,0.4],[0.8,0.9]\rangle \end{bmatrix}$$

Now we measure the distances and also the similarities between each alternative A_i and the ideal alternatives E, as defined earlier.

To calculate the Hamming distance between E and A_1 we take :

$$d_1(E,A_1)=\frac{1}{6}\sum_{i=1}^{n}\begin{matrix}[|\inf T_E(x_i)-\inf T_{A_1}(x_i)|+|\sup T_E(x_i)-\sup T_{A_1}(x_i)|+ \\ |\inf I_E(x_i)-\inf I_{A_1}(x_i)|+|\sup I_E(x_i)-\sup I_{A_1}(x_i)|+ \\ |\inf F_E(x_i)-\inf F_{A_1}(x_i)|+|\sup F_E(x_i)-\sup F_{A_1}(x_i)|]\end{matrix}$$

=1/6[|1-0.4|+|1-0.5|+|0-0.2|+|0-0.3|+|0-0.3|+|0-0.4|+|1-0.4|+|1-0.6|+|0-0.1|+|0-0.3|+|0-0.2|+|0-0.4|+|0-0.7|+|0-0.9|+|1-0.2|+|1-0.3|+|1-0.4|+|1-0.5|] =1.4167

Similarly, $d_1(E,A_2)=0.9$, $d_1(E,A_3)=1.25$ and $d_1(E,A_4)=0.86$.

In this way, the obtained results are presented in tabular form as follows:

For Distance measurement

Distance	Obtained Results	Rank of Alternatives (descending order)	Best alternative obtained
$d_1(E,A_i)$	$A_1 = 1.4167$ $A_2 = 0.9$ $A_3 = 1.25$ $A_4 = 0.86$	$A_1 > A_3 > A_2 > A_4$	A_4
$d_2(E,A_i)$	$A_1 = 0.4722$ $A_2 = 0.3$ $A_3 = 0.4167$ $A_4 = 0.2867$	$A_1 > A_3 > A_2 > A_4$	A_4
$d_3(E,A_i)$	$A_1 = 0.8990$ $A_2 = 0.5916$ $A_3 = 0.7450$ $A_4 = 0.6245$	$A_1 > A_3 > A_4 > A_2$	A_2
$d_4(E,A_i)$	$A_1 = 0.5190$ $A_2 = 0.3416$ $A_3 = 0.4301$ $A_4 = 0.3606$	$A_1 > A_3 > A_4 > A_2$	A_2
$d_5(E,A_i)$	$A_1 = 2.1$ $A_2 = 1.5$ $A_3 = 2.0$ $A_4 = 1.4$	$A_1 > A_3 > A_2 > A_4$	A_4
$d_6(E,A_i)$	$A_1 = 0.7000$ $A_2 = 0.5000$ $A_3 = 0.6667$ $A_4 = 0.4667$	$A_1 > A_3 > A_2 > A_4$	A_4
$d_7(E,A_i)$	$A_1 = 0.4975$ $A_2 = 0.3100$ $A_3 = 0.4233$ $A_4 = 0.3042$	$A_1 > A_3 > A_2 > A_4$	A_4
$d_8(E,A_i)$	$A_1 = 0.1658$ $A_2 = 0.1033$ $A_3 = 0.1411$ $A_4 = 0.1014$	$A_1 > A_3 > A_2 > A_4$	A_4
$d_9(E,A_i)$	$A_1 = 0.5428$ $A_2 = 0.3545$ $A_3 = 0.4401$ $A_4 = 0.3800$	$A_1 > A_3 > A_4 > A_2$	A_2
$d_{10}(E,A_i)$	$A_1 = 0.3134$ $A_2 = 0.2047$ $A_3 = 0.2541$ $A_4 = 0.2194$	$A_1 > A_3 > A_4 > A_2$	A_2
$d_{11}(E,A_i)$	$A_1 = 0.7200$ $A_2 = 0.5200$ $A_3 = 0.6900$ $A_4 = 0.4850$	$A_1 > A_3 > A_2 > A_4$	A_4
$d_{12}(E,A_i)$	$A_1 = 0.2400$ $A_2 = 0.1733$ $A_3 = 0.2300$ $A_4 = 0.1617$	$A_1 > A_3 > A_2 > A_4$	A_4
$d_{13}(E,A_i)$	$A_1 = 1.2369$ $A_2 = 0.9000$ $A_3 = 1.1747$ $A_4 = 0.8602$	$A_1 > A_3 > A_2 > A_4$	A_4
$d_{14}(E,A_i)$	$A_1 = 0.7348$ $A_2 = 0.5404$ $A_3 = 0.7000$ $A_4 = 0.5172$	$A_1 > A_3 > A_2 > A_4$	A_4
$d_{15}(E,A_i)$	$A_1 = 0.7141$ $A_2 = 0.5196$ $A_3 = 0.6782$ $A_4 = 0.4966$	$A_1 > A_3 > A_2 > A_4$	A_4
$d_{16}(E,A_i)$	$A_1 = 0.4242$ $A_2 = 0.3120$ $A_3 = 0.4041$ $A_4 = 0.2986$	$A_1 > A_3 > A_2 > A_4$	A_4
$d_{17}(E,A_i)$	For p = 6 $A_1 = 0.033700$ $A_2 = 0.005336$ $A_3 = 0.013387$ $A_4 = 0.009309$	$A_1 > A_3 > A_4 > A_2$	A_2
	For p = 10 $A_1 = 0.00888288$ $A_2 = 0.00059184$ $A_3 = 0.00240292$ $A_4 = 0.00114518$	$A_1 > A_3 > A_4 > A_2$	A_2

Distance	Obtained Results	Rank of Alternatives (descending order)	Best alternative obtained
$d_{18}(E,A_i)$	For p = 6 $A_1 = 0.01317057$ $A_2 = 0.00210276$ $A_3 = 0.00524945$ $A_4 = 0.00624732$	$A_1 > A_4 > A_3 > A_2$	A_2
	For p = 10 $A_1 = 0.00353154$ $A_2 = 0.00023634$ $A_3 = 0.00093445$ $A_4 = 0.00045777$	$A_1 > A_3 > A_4 > A_2$	A_2
$d_{19}(E, A_i)$	For p = 6 $A_1 = 0.06740000$ $A_2 = 0.01067160$ $A_3 = 0.02677516$ $A_4 = 0.01861966$	$A_1 > A_3 > A_4 > A_2$	A_2
	For p = 10 $A_1 = 0.02960961$ $A_2 = 0.00197280$ $A_3 = 0.00800974$ $A_4 = 0.00381729$	$A_1 > A_3 > A_4 > A_2$	A_2
$d_{20}(E, A_i)$	For p =6 $A_1 = 0.02634115$ $A_2 = 0.00420552$ $A_3 = 0.01049890$ $A_4 = 0.01249464$	$A_1 > A_4 > A_3 > A_2$	A_2
	For p = 10 $A_1 = 0.01177182$ $A_2 = 0.00078782$ $A_3 = 0.00311483 A_4 = 0.00152592$	$A_1 > A_3 > A_4 > A_2$	A_2
$d_{21}(E, A_i)$	For p = 6 $A_1 = 0.1041255$ $A_2 = 0.0209735$ $A_3 = 0.0659030$ $A_4 = 0.0204123$	$A_1 > A_3 > A_2 > A_4$	A_4
	For p = 10 A_1=0.03607716 $A_2 = 0.00284572$ A_3 =0.01365982 $A_4 = 0.00283582$	$A_1 > A_3 > A > A_4$	A_4
$d_{22}(E, A_i)$	For p = 6 $A_1 = 0.0400950$ $A_2 = 0.0082528$ $A_3 = 0.0249901$ $A_4 = 0.0080200$	$A_1 > A_3 > A_2 > A_4$	A_4
	For p =10 $A_1 = 0.4975$ $A_2 = 0.3100$ $A_3 = 0.4233$ $A_4 = 0.3042$	$A_1 > A_2 > A_3 > A_4$	A_4
$d_{23}(E, A_i)$	For p = 6 $A_1 = 0.208251$ $A_2 = 0.041947$ $A_3 = 0.131806$ $A_4 = 0.040824$	$A_1 > A_3 > A_2 > A_4$	A_4
	For p = 10 $A_1 = 0.208251$ $A_2 = 0.041947$ $A_3 = 0.131806$ $A_4 = 0.040824$	$A_1 > A_3 > A_2 > A_4$	A_4
$d_{24}(E, A_i)$	For p = 6 $A_1 = 0.0801900$ $A_2 = 0.0165057$ $A_3 = 0.0499803$ $A_4 = 0.0160401$	$A_1 > A_3 > A_2 > A_4$	A_4
	For p = 10 $A_1 = 0.0475990$ $A_2 = 0.0037873$ $A_3 = 0.0126310$ $A_4 = 0.0037757$	$A_1 > A_3 > A_2 > A_4$	A_4

For similarity measurement

Similarity	Obtained Results	Rank of Alternatives (descending order)	Best alternative obtained
$s_1(E,A_i)$	$A_1 = 0.6678$ $A_2 = 0.7634$ $A_3 = 0.7026$ $A_4 = 0.7668$	$A_4 > A_2 > A_3 > A_1$	A_4
$s_2(E,A_i)$	$A_1 = 0.8342$ $A_2 = 0.8967$ $A_3 = 0.8589$ $A_4 = 0.8986$	$A_4 > A_2 > A_3 > A_1$	A_4
$s_3(E,A_i)$	$A_1 = 0.6481$ $A_2 = 0.7383$ $A_3 = 0.6944$ $A_4 = 0.7246$	$A_2 > A_4 > A_3 > A_1$	A_2
$s_4(E,A_i)$	$A_1 = 0.6866$ $A_2 = 0.7953$ $A_3 = 0.7459$ $A_4 = 0.7806$	$A_2 > A_4 > A_3 > A_1$	A_2
$s_5(E,A_i)$	$A_1 = 0.5814$ $A_2 = 0.6579$ $A_3 = 0.5917$ $A_4 = 0.6734$	$A_4 > A_2 > A_3 > A_1$	A_4
$s_6(E,A_i)$	$A_1 = 0.7600$ $A_2 = 0.8267$ $A_3 = 0.7700$ $A_4 = 0.8383$	$A_4 > A_2 > A_3 > A_1$	A_4
$s_7(E,A_i)$	$A_1 = 0.52778$ $A_2 = 0.70000$ $A_3 = 0.60555$ $A_4 = 0.71111$	$A_4 > A_2 > A_3 > A_1$	A_4
$s_8(E,A_i)$	For p = 6 $A_1 = 0.99321146$ $A_2 = 0.99974531$ $A_3 = 0.99926028$ $A_4 = 0.99928211$	$A_2 > A_4 > A_3 > A_1$	A_2
	For p =10 $A_1 = 0.99905556$ $A_2 = 0.99999644$ $A_3 = 0.99998544$ $A_4 = 0.99997679$	$A_2 > A_3 > A_4 > A_1$	A_2
$s_9(E,A_i)$	For p = 6 $A_1 = 0.99191449$ $A_2 = 0.99970251$ $A_3 = 0.99918473$ $A_4 = 0.99914266$	$A_2 > A_3 > A_4 > A_1$	A_2
	For p = 10 $A_1 = 0.99886727$ $A_2 = 0.99999574$ $A_3 = 0.99998329$ $A_4 = 0.99997215$	$A_2 > A_3 > A_4 > A_1$	A_2
$s_{10}(E,A_i)$	For p = 6 $A_1 = 0.92097654$ $A_2 = 0.98738343$ $A_3 = 0.96956463$ $A_4 = 0.97780405$	$A_2 > A_4 > A_3 > A_1$	A_2
	For p = 10 $A_1 = 0.97881070$ $A_2 = 0.99858191$ $A_3 = 0.99439330$ $A_4 = 0.99725333$	$A_2 > A_4 > A_3 > A_1$	A_2
$s_{11}(E,A_i)$	$A_1 = 0.66245024$ $A_2 = 0.74828426$ $A_3 = 0.67495694$ $A_4 = 0.76380514$	$A_4 > A_2 > A_3 > A_1$	A_4
$s_{12}(E,A_i)$	$A_1 = 0.61290322$ $A_2 = 0.70459388$ $A_3 = 0.62601626$ $A_4 = 0.98138033$	$A_4 > A_2 > A_3 > A_1$	A_4
$s_{13}(E,A_i)$	For p = 6 $A_1 = 0.9326000$ $A_2 = 0.9893284$ $A_3 = 0.9732248$ $A_4 = 0.9813803$	$A_2 > A_4 > A_3 > A_1$	A_2
	For p = 10 $A_1 = 0.97039038$ $A_2 = 0.99802719$ $A_3 = 0.99199025$ $A_4 = 0.99618270$	$A_2 > A_4 > A_3 > A_1$	A_2

Similarity	Obtained Results	Rank of Alternatives (descending order)	Best alternative obtained
$s_{14}(E,A_i)$	For p = 6 $A_1 = 0.97365884$ $A_2 = 0.99579447$ $A_3 = 0.98950109$ $A_4 = 0.98750536$	$A_2 > A_3 > A_4 > A_1$	A_2
	For p = 10 $A_1 = 0.98822817$ $A_2 = 0.99921217$ $A_3 = 0.99688516$ $A_4 = 0.99847407$	$A_2 > A_4 > A_3 > A_1$	A_2
$s_{15}(E,A_i)$	$A_1 = 0.29661016$ $A_2 = 0.35238095$ $A_3 = 0.37168141$ $A_4 = 0.50000000$	$A_4 > A_3 > A_2 > A_1$	A_4
$s_{16}(E,A_i)$	$A_1 = 0.300$ $A_2 = 0.550$ $A_3 = 0.450$ $A_4 = 0.483$	$A_2 > A_4 > A_3 > A_1$	A_2
$s_{17}(E,A_i)$	$A_1 = 0.43708609$ $A_2 = 0.65384615$ $A_3 = 0.54193548$ $A_4 = 0.66666666$	$A_4 > A_2 > A_3 > A_1$	A_4
$s_{18}(E,A_i)$	$A_1 = 0.20283243$ $A_2 = 0.37547646$ $A_3 = 0.26990699$ $A_4 = 0.38997923$	$A_4 > A_2 > A_3 > A_1$	A_4
$s_{19}(E,A_i)$	$A_1 = 0.18945738$ $A_2 = 0.30270010$ $A_3 = 0.22405482$ $A_4 = 0.33782415$	$A_4 > A_2 > A_3 > A_1$	A_4
$s_{20}(E,A_i)$	$A_1 = 0.4125$ $A_2 = 0.6375$ $A_3 = 0.5250$ $A_4 = 0.6500$	$A_4 > A_2 > A_3 > A_1$	A_4
$s_{21}(E,A_i)$	$A_1 = 0.140625$ $A_2 = 0.222500$ $A_3 = 0.183750$ $A_4 = 0.226250$	$A_4 > A_2 > A_3 > A_1$	A_4

7. Comparative study with existing work

Hence we compare the results given in [14] and the results obtained in previous section (section 6). In the article [14], the authors have used the similarity measures $\boldsymbol{s_1(A,B)}$and $\boldsymbol{s_3(A,B)}$ (as stated in the section 4, where $\boldsymbol{A}$ is the ideal alternative $\boldsymbol{E}$ and $\boldsymbol{B}$ is the alternative to be measured), to obtain the best alternatives. Using $\boldsymbol{s_1(A,B)}$ the best alternative obtained is $\boldsymbol{A_4}$ and using $\boldsymbol{s_3(A,B)}$ the best alternative is $\boldsymbol{A_2}$. Also the similarity measure of $\boldsymbol{A_4}$ with ideal alternative is ***0.9600*** and the same of $\boldsymbol{A_2}$ is ***0.9323***. However, we have measured using various numbers of similarities and distances as well, between the alternatives and the ideal alternative, to obtain the best alternative. According to the results, $\boldsymbol{A_4}$ is the best alternative (in both distances and similarity measures) when the distance or the similarity is in linear form i.e. Hamming distance, Hausdroff distance and their related distance and similarity measures, etc. (except $\boldsymbol{d_{21}(A,B)}$ and its related distance measures, where though they are not linear, the best alternative obtained is $\boldsymbol{A_4}$). Otherwise the best alternative is $\boldsymbol{A_2}$ (except $\boldsymbol{s_{16}(A,B)}$, where being linear similarity measure, the best alternative given is $\boldsymbol{A_2}$). Now, one can decide the best alternative considering the alternative obtained as best alternative according to numerical value in most number of cases in both distance and similarity measures and also this decision can be made considering more distance and similarities besides those defined in this paper. So, we suggest that, according to the number of cases, $\boldsymbol{A_4}$ can be taken as the best alternative.

8. Conclusion

In this article, at first we have defined various distances $d_k(A,B)$, (k =1,2,…,24) and similarity measures $s_k(A,B)$, (k =1,2,…,21) , between two Interval Neutrosophic sets. Then we have shown an application of these distances and similarities in solving a multicriteria decision-making problem. A method, for the solution of this type of problems, has been established by means of distance and similarity measures between each alternative and the respective ideal alternative. Then, as an illustrative example, a problem from [14] has been reconsidered and applying our distance and similarity measures, the ranking order of all alternatives has been calculated and stated in tabular form and the best alternative has also been identified as well. Finally we have made a comparison between the existing result in [14] and the results obtained in this article and finally conclude that the result obtained in this paper is more precise and more specific. The proposed similarity measures are also useful in real life applications of science and engineering such as medical diagnosis, pattern recognitions etc. Furthermore, the proposed techniques, based on distance and similarity measures, can be more useful for decision makers as it extend the existing decision making methods.

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Generalized Pre $\alpha-$Closed Sets in Topology

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Abstaract − In this paper, a new class of sets called generalized pre α-closed sets are introduced and studied in topological spaces, which are properly placed between the class of pre closed and the class of generalized star pre closed (g*p-closed) sets.

Keywords − *Closed sets, gpα-closed sets, gpα-open sets.*

1 Introduction

The concept of stronger forms of open sets and closed sets were introduced by Stone[17], which were called as regular open and regular closed sets respectively. Levine[10]introduced the generalized closed sets in topology as generalization of closed sets. The concept of Levine[10] opened the flood gates of research in weaker forms of closed sets in general topology. Many researchers like [1], [2], [4], [7], [12], [13], [14], [16], [18], [19] and others have studied many weaker forms of closed sets in topological spaces. Recently, Benchalli et al.[3] and Jafari et al.[8] studied $\omega\alpha$-closed and pre g*-closed sets. The aim of this paper is to continue the study of generalization of closed sets namely generalized pre α-closed(briefly gpα-closed) set using α-open [16] in topological spaces. Also, we introduce the concept of gpα-closure, gpα-interior and gpα-neighborhood in topological spaces.

2 Preliminaries

Throughout this paper, spaces X and Y(or (X, τ) and (Y, σ)) denote topological spaces, in which no separation axioms are assumed unless explicitly stated. The

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following definitions are useful in the sequel.

Definition 2.1. A subset A of a topological space X is called a

1. semi-open [9] if $A \subseteq cl(int(A))$ and semi-closed set if $int(cl(A)) \subseteq A$.
2. pre-open set [14] if $A \subseteq int(cl(A))$ and pre-closed set if $cl(int(A)) \subseteq A$.
3. α-open set [16] if $A \subseteq int(cl(int(A)))$ and α-closed set if $cl(int(cl(A))) \subseteq A$.
4. semi-preopen set [1] if $A \subseteq cl(int(cl(A)))$ and semi-preclosed set if $int(cl(int(A))) \subseteq A$.

Definition 2.2. A subset A of a topological space X is called a

1. generalized closed (briefly g-closed)[10](briefly ω-closed[18], pre $g^{\star}$-closed[8]) if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is open(resp. semi-open, $\omega\alpha$-open) in X.
2. generalized preclosed (briefly gp-closed)[13],(resp. generalized pre regular closed (briefly gpr-closed[7])), if $pcl(A) \subseteq U$ whenever $A \subseteq U$ and U is open(resp. regular open) in X.
3. generalized semi-pre closed(briefly gsp-closed)[5], if $spcl(A) \subseteq U$ whenever $A \subseteq U$ and U is open in X.
4. semi generalized closed(briefly sg-closed)[4] (resp.generalized semi-closed[2]), if $scl(A) \subseteq U$ whenever $A \subseteq U$ and U is semi-open (resp. open) in X.
5. $\omega\alpha$-closed [3], if $\alpha cl(A) \subseteq U$ whenever $A \subseteq U$ and U is ω-open in X.

3 Generalized Pre α Closed Sets

In this section, the concept of generalized pre α closed set is introduced and studied some of its properties in topological spaces.

Definition 3.1. In a topological space X, a subset A of X is called generalized pre α-closed (briefly gpα-closed) if $pcl(A) \subseteq U$ whenever $A \subseteq U$ and U is α-open in X.

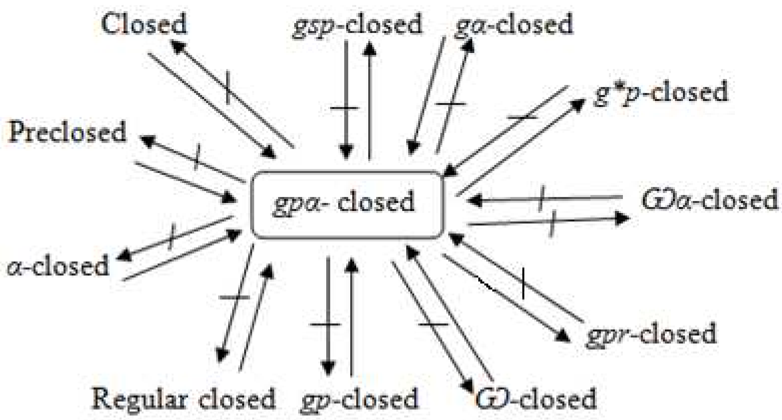
The compliment of gpα-closed is gpα-open in X. The family of all gpα-closed sets in X is denoted by Gp$\alpha C\,(X)$.

Example 3.2. Let $X = \{a, b, c\}$ and $\tau = \{X, \phi, \{a\}, \{a, b\}\}$. Then the family of gp$\alpha$-closed sets in X is given by Gp$\alpha C\,(X) = \{X, \phi, \{b\}, \{c\}\,\{b, c\}\}$.

Remark 3.3. From the definition 3.1, it is clear that every pre closed set is gpα-closed but not conversely.

Example 3.4. Let $X = \{a, b, c, d, e\}$ and $\tau = \{X, \phi, \{a\}, \{c, d\}, \{a, c, d\}\}$. Then the subset A=$\{a, e\}$ of X is gpα-closed but not pre closed in X.

Remark 3.5. From the definition 3.1 and from [1,3,4,5,7,12,18,19,20],we have the following implications. However converse implications are not true in general.



Example 3.6. Let $X = \{a, b, c\}$ and $\tau = \{X, \phi, \{a\}, \{b, c\}\}$. Then the subset A=$\{a, b\}$ is gpα-closed but not closed, regular closed and $\omega\alpha$-closed in X and B=$\{c\}$ is gpα-closed but not α-closed.

Example 3.7. Let $X = \{a, b, c\}$ and $\tau = \{X, \phi, \{a, b\}\}$. Then the subset A=$\{a\}$ is gpα-closed but not gα-closed in X and the subset B=$\{a, b\}$ is gpr-closed but not gpα-closed set in X.

Example 3.8. Let $X = \{a, b, c\}$ and $\tau = \{X, \phi, \{a\}, \{a, b\}\}$. Then the subset $A = \{b\}$ is gpα-closed but not ω-closed in X and $B = \{a, c\}$ is $\omega\alpha$-closed, gp-closed,gsp-closed and g*p-closed but not gpα-closed in X.

From the above observations, the class of gpα-closed sets are properly placed between the class of preclosed and g*p-closed sets.

Remark 3.9. The following examples show that semi-closed(resp. semi-preclosed) and gpα-closed sets are independent of each other.

Example 3.10. Let $X = \{a, b, c\}$ and $\tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}\}$. Then the subset $A = \{b\}$ is semi-closed (resp.semi-pre closed) but not gpα-closed.

Example 3.11. In Example 3.7, the subset $A = \{a\}$ is gpα-closed but not semi-closed and semi-pre closed in X.

Remark 3.12. From the following examples it is clear that gpα-closed and sg-closed (resp. g-closed, gs-closed) sets are independent of each other.

Example 3.13. Let $X = \{a, b, c\}$ and $\tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}\}$. Then the subset A=$\{a\}$ is sg-closed, gs-closed but not gpα-closed in X.

Example 3.14. In Example 3.7, the subset A={b} is gpα-closed but not gs-closed, sg-closed and g-closed in X.

Example 3.15. Let $X = \{a, b, c\}$ and $\tau = \{X, \phi, \{a\}\}$. Then the subset A={a, c} is g-closed but not gpα-closed in X.

Remark 3.16. From the examples 3.6 and 3.7, the $\omega\alpha$-closed and gpα-closed sets are independent of each other.

Theorem 3.17. If A is gpα-closed in X, then $pcl(A) - A$ does not contain any non-empty α-closed set in X.

Proof. Let F be a α-closed set in X contained in $pcl(A) - A$. Then $F \subseteq X - A$ and $A \subseteq X - A$. A subset A is α-closed and $X - F$ is α-open in X, then $pcl(A) \subseteq X - F$. So, $F \subseteq X - pcl(A)$. Therefore $F \subseteq (pcl(A) - A) \cap (X - cl(A)) = \phi$. Hence, $pcl(A) - A$ does not contain any non-empty α-closed set in X.

Theorem 3.18. Let A and B are gpα-closed sets, then $A \cup B$ is gpα-closed.

Proof. Let U be an α-open set in X such that $A \subseteq U$ and $B \subseteq U$. Since A and B are gpα-closed sets, then $pcl(A) \subseteq U$ and $pcl(B) \subseteq U$. But $pcl(A \cup B) = pcl(A) \cup pcl(B) \subseteq U$, so $pcl(A \cup B) \subseteq U$. Hence $A \cup B$ is gpα-closed.

Theorem 3.19. If A is gpα-closed set and $A \subseteq B \subseteq pcl(A)$, then B is gpα-closed.

Proof. Let U be an α-open in X such that $B \subseteq U$. Then $A \subseteq B$ implies that $A \subseteq U$. Since A is gpα-closed, then $pcl(A) \subseteq U$. But $B \subseteq pcl(A)$, so $pcl(B) \subseteq pcl(A)$. Then $pcl(B) \subseteq U$. Hence B is gpα-closed.

Theorem 3.20. If A is α-open and gpα-closed set of X, then A is preclosed.

Proof. Let $A \subseteq A$, where A is α-open. Then $pcl(A) \subseteq A$ as A is gpα-closed. But $A \subseteq pcl(A)$ is always true. Therefore $A = pcl(A)$. Hence A is preclosed.

Theorem 3.21. Let $A \subseteq Y \subseteq X$ and suppose that A is gpα-closed in X, then A is gpα-closed relative to Y.

Proof. Consider $A \subseteq Y \cap G$, where G is open and so α-open in X. Since$A$ is gpα-closed in X, $A \subseteq G$ which implies $pcl(A) \subseteq G$. That is $Y \cap pcl(A) \subseteq Y \cap G$, where $Y \cap pcl(A)$ is the pre-closure of A. Thus A is gpα-closed relative to Y.

Definition 3.22. [11] For a topological space X, the kernel of a subset A of X is defined as the intersection of all open supersets of A and denoted by ker(A) or $A^\wedge$.

Definition 3.23. A subset A of X is called p star-closed (briefly $p^\star$-closed), if $A = pcl(A) \cap A^\wedge$ and its compliment is $p^\star$-open.

Theorem 3.24. For a subset A of X, the following are equivalent:

(i) A is preclosed.

(ii) A is gpα-closed and $A = pcl(A) \cap U$, for some open set U.

(iii) A is gpα-closed and p^*-closed.

Proof. $(i) \rightarrow (ii)$ Every preclosed set is gpα-closed and $A = pcl(A)$ and X is open. Then $A = X \cap A$, implies that $A = pcl(A) \cap X$.
$(ii) \rightarrow (iii)$ Let $A = pcl(A) \cap U$, where U is some open set. Then $A \subseteq pcl(A)$ and $A \subseteq U$. But $A \subseteq ker(\text{A}) \subseteq U$. So,$A \subseteq ker(\text{A}) \subseteq pcl(A)$ implies $A \subseteq pcl(A) \cap U = A$. Then we have $A = ker(\text{A}) \cap pcl(A)$. Hence A is p^*-closed.
$(iii) \rightarrow (i)$ Let A be gpα-closed, by definition, $pcl(A) \subseteq A$, wherever $A \subseteq U$ and U is α-open. Then $pcl(A) \subseteq ker(\text{A}) \subseteq U$, therefore $A = ker(\text{A}) \cap pcl(A)$ and hence A is preclosed.

Theorem 3.25. For each $x \in X$, $\{x\}$ is α-closed or $X - \{x\}$ is gpα-closed in X.

Proof. Let $\{x\}$ be α-closed, then the proof is completed. Suppose $\{x\}$ is not α-closed in X, then $X - \{x\}$ is not α-open and only α-open set containing $X - \{x\}$ is space X itself. Therefore $pcl(X - \{x\}) \subseteq X$ and hence $X - \{x\}$ is gpα-closed in X.

4 gpα-Closure and gpα-Interior

In this section we introduce gpα-closure and gpα-interior of a subset A of X by using the gpα-closed and gpα-open sets also studied their properties.

Definition 4.1. A subset A of X, the intersection of all gpα-closed sets containing A is called the gpα-closure of A and is denoted by gp$\alpha - cl(A)$.

That is gp$\alpha - cl(A) = \cap\{G : A \subseteq G$, G is gp$\alpha$ -closed in X $\}$.

Definition 4.2. A subset A of X, gpα-interior of A and denoted by gp$\alpha - int(A)$, defined as gp$\alpha - int(A) = \cup\{G : G \subseteq A$, G is gp$\alpha$-open in X $\}$.

Remark 4.3. If $A \subseteq X$, then

(i) $A \subseteq$ gp$\alpha - cl(A) \subseteq cl(A)$

(ii) $int(A) \subseteq$ gp$\alpha - int(A) \subseteq A$.

Theorem 4.4. If A and B are subsets of X, then

(i) gp$\alpha - cl(X) = X$ and gp$\alpha - cl(\phi) = \phi$.

(ii) $A \subseteq$ gp$\alpha - cl(A)$

(iii) If B is any gpα-closed set containing A, then gp$\alpha - cl(A) \subseteq B$

(iv) If $A \subseteq B$, then gp$\alpha - cl(A) \subseteq$ gp$\alpha - cl(B)$

(v) gp$\alpha - cl(A)$=gpα-cl(gp$\alpha - cl(A)$)

(vi) gp$\alpha - cl(A \cup B)$=gp$\alpha - cl(A)\cup$ gp$\alpha - cl(B)$

Proof. (i),(ii), (iii) and (iv) follows from the definition 4.1.
(v) Let E be gpα-closed set containing A. Then by definition 4.1, gp$\alpha - cl(A) \subseteq E$. Since, E is gpα-closed and contains gp$\alpha - cl(A)$ and is contained in every gpα-closed set containing A, it follows gp$\alpha - cl($gp$\alpha - cl(A)) \subseteq$ gp$\alpha - cl(A)$. Therefore gp$\alpha - cl($gp$\alpha - cl(A)$)=gp$\alpha - cl(A)$.

(vi) Since gp$\alpha - cl(A) \subseteq$ gp$\alpha - cl(A \cup B)$ and gp$\alpha - cl(B) \subseteq$ gp$\alpha - cl(A \cup B)$ implies that gp$\alpha - cl(A)\cup$ gp$\alpha - cl(A \cup B)$. Let x be any point in X such that $x \notin$ gp$\alpha - cl(A)\cup$ gp$\alpha - cl(B)$, then there exist gpα-closed sets E and$F$, such that $A \subseteq E$ and $B \subseteq F$, $x \notin E$ and $x \notin F$ implies that $x \notin E \cup F$, $A \cup B \subseteq E \cup F$ and $E \cup F$ is gpα-closed. Thus $x \notin$ gp$\alpha - cl(A \cup B)$, gp$\alpha - cl(A \cup B) \subseteq$ gp$\alpha - cl(A)\cup$ gp$\alpha(B)$. Hence, we conclude that gp$\alpha - cl(A \cup B)$=gp$\alpha - cl(A)\cup$ gp$\alpha - cl(B)$.

Theorem 4.5. Let A and B be subsets of X, then

$$gp\alpha - cl(A \cap B) \subseteq gp\alpha - cl(A) \cap gp\alpha - cl(B)$$

Remark 4.6. The equality of Theorem 4.5 does not hold in general as seen from the following example.

Example 4.7. Let $X = \{a, b, c\}$ and $\tau = \{X, \phi, \{a\}, \{a, b\}\}$ be a topology on X. For subsets of X, $A = \{a\}$ and $B = \{b\}$. The gp$\alpha - cl(A) = \{a, c\}$ and gp$\alpha - cl(B) = \{b, c\}$, then gp$\alpha - cl(\{A \cap B\} = \phi$. Hence

$$gp\alpha - cl(A) \cap gp\alpha - cl(B) \nsubseteq gp\alpha(A \cap B)$$

Remark 4.8. If $A \subseteq X$ and A is gpα-closed, then gp$\alpha - cl(A)$ is smallest gpα-closed subset of X containing A.

Theorem 4.9. For any $x \in X$, $x \in$ gp$\alpha - cl(A)$ if and only if $A \cap V \neq \phi$ for every gpα-open set V containing x.

Proof. Let $x \in$ gp$\alpha - cl(A)$. Suppose there exists gpα-open set V containing x, such that $A \cap V \neq \phi$, then $A \subseteq X - V$, where $X - V$ is gpα-closed set. So, that gp$\alpha - cl(A) \subseteq X - V$. This implies that $x \notin$ gp$\alpha - cl(A)$, which contradicts to the fact that $x \in$ gp$\alpha - cl(A)$. Hence $A \cap V \neq \phi$ for every open set containing x.
Conversely, let $x \notin$ gp$\alpha - cl(A)$, then there exists gpα-closed set G containing A, such that, $x \notin G$. Then $x \in X - F$ is gpα-open. Also $(X - F) \cap A = \phi$, which is contradiction. Hence, $x \in gp\alpha - cl(A)$.

Theorem 4.10. Let A be subset of X, then gp$\alpha - int(A)$ is the largest gpα-open subset of X contained in$A$, if A is gpα-open.

The converse of the above theorem need not be true as seen from following example.

Example 4.11. In the example 3.7, the subset $A = \{b, c\}$ of X, then gpα-int(A)=$\{b\}$ is gpα-open in (X, τ), but A is not gpα-open in X.

Theorem 4.12. Let A and B be subsets of X, then

(i) gp$\alpha - int(X) = X$ and gp$\alpha - int(\phi) = \phi$.

(ii) gp$\alpha - int(A) \subseteq A$.

(iii) If B is any gpα-open set contained in A, then $B \subseteq$ gp$\alpha - int(A)$.

Proof. (i) and (ii) follows from the definition 4.2.
(iii) Suppose B is any gpα-open set contained in A. Let $x \in B$,since B is gpα-open set contained in A. Then $x \in$ gpα-int(A). Hence, $B \subseteq$ gpα-int(A).

Remark 4.13. For any subset of X, $int(A) \subseteq$ gp$\alpha - int(A)$

5 gpα-Neighborhoods and gpα-Limit points

In this section we define the gpα-neighborhood, gpα-limit points and gpα-derived set of a set and study some of their basic properties.

Definition 5.1. A subset N of X is said to be gpα-neighborhood of a point $x \in X$, if there exists an gpα-open set G containing x, such that $x \in G \subseteq N$.

Definition 5.2. Let (X, τ) be a topological space and A be a subset of X. A subset N of X is said to be gpα-neighborhood of A if there exists an gpα-open set G such that $A \in G \subseteq N$.

The collection of all gpα-neighborhood of $x \in X$ is called the gpα-neighborhood system and denoted by gp$\alpha N(x)$.

Theorem 5.3. If $N \subseteq X$ is gpα-open if it is a gpα-neighborhood of each of its points.

Proof. Let $x \in N$. Since N is gpα-open such that $x \in N \subseteq N$. Also x is an arbitrary point of N, it follows that N is a gpα-neighborhood of each of its points.

Remark 5.4. The converse of the above theorem need not to be true as seen from following example.

Example 5.5. Let $X = \{a, b, c\}$ and $\tau = \{X, \phi, \{b\}\}$. A subset $A = \{b, c\}$ is gpα-neighborhood of each of its points b and c but A is not gpα-open.

Theorem 5.6. If A be subset of X and $x \in$ gp$\alpha - cl(A)$ if and only if any gpα-neighborhood N of x in X, $N \cap A \neq \phi$.

Proof. Suppose there is a gpα-neighborhood of N of x in X, such that $N \cap A = \phi$. Then there exist an gpα-open set G of X, such that $x \in G \subseteq N$. So, $G \cap A = \phi$ and $x \in X - G$. This implies gp$\alpha - cl(A) \in X - G$ and therefore $x \notin$ gp$\alpha - cl(A)$, which contradicts to the fact that $x \in$ gp$\alpha - cl(A)$. Hence, $N \cap A \neq \phi$.
Conversely, let us assume that $x \notin$ gp$\alpha - cl(A)$, there exists a gpα-closed set G of X, such that $A \subseteq G$ and $x \notin G$. So, $x \in X - G$ and $X - G$ is gpα-open in X. It becomes a gpα-neighborhood of x in X. Since $A \cap (X - G) = \phi$, which leads to a contradiction. Hence, $x \in$ gp$\alpha - cl(A)$.

Definition 5.7. A point $x \in X$ is called a gpα-limit point of a subset A of X, if and only if every gpα-neighborhood of x contains a point of A distinct from x. That is $[N - \{x\}] \cap A \neq \phi$ for each gpα-neighborhood of N of x.

Equivalently, if and only if every gpα-open set G containing x contains a point of A other than x.
In topological space (X, τ), the set of all gpα-limit points of A is called a gpα-derived set of A and is denoted by gp$\alpha - d(A)$.

Theorem 5.8. Let A and B be subsets of X,then

(i) gp$\alpha - d(\phi) = \phi$.

(ii) If $A \subseteq B$, then gp$\alpha - d(A) \subseteq$ gp$\alpha - d(B)$.

(iii) If $x \in$ gp$\alpha - d(A)$, then $x \in$ gp$\alpha - d[A - \{x\}]$.

(iv) gp$\alpha - d(A \cap B)$=gp$\alpha - d(A)\cap$ gp$\alpha - d(B)$.

(v) gp$\alpha - d(A \cap B) \subseteq$ gp$\alpha d(A) \cap -d(B)$.

Proof. (i) and (ii) follows from the definition 5.7.
(iii)Let $x \in gp\alpha - d(A)$. By definition 5.7, every gpα-open set G containing x contains at least one point other than x. Hence, $x \in$ gp$\alpha - d\,[A - \{x\}]$, that is x is gpα-limit point of $[A - \{x\}]$. Thus $x \in$ gp$\alpha - d\,[A - \{x\}]$.
(iv) We know that $A \subseteq A \cup B$ and $B \subseteq A \cup B$. From (ii) gp$\alpha - d(A)\cup$ gp$\alpha - d(B) \subseteq$ gp$\alpha - d(A \cup B)$. In other way, suppose $x \notin$ (gp$\alpha - d(A)\cup$ gp$\alpha - d(B)$), then $x \notin$ gp$\alpha - d(A)$ and $x \notin$ gp$\alpha - d(B)$, hence there exists gpα-open sets U and V each containing x, such that $U \cap (A - \{x\}) = \phi$ and $V \cap (B - \{x\}) = \phi$. Then $(U \cap V) \cap (A - \{x\}) = \phi$ and $U \cap V) \cap (B - \{x\}) = \phi$. On combining $(U \cap V) \cup ((A \cup B) - \{x\}) = \phi$. Therefore $x \notin$ gp$\alpha - d(A \cup B)$. Hence, gp$\alpha - d(A \cup B)$=gp$\alpha - d(A)\cup$ gp$\alpha - d(B)$.
(v) Since $A \cap B \subseteq A$ and $A \cap B \subseteq B$, from (ii) gp$\alpha - d(A \cap B) \subseteq$ gp$\alpha - d(A)$ and gp$\alpha - d(A \cap B) \subseteq$ gp$\alpha - d(B)$. Consequently, gp$\alpha - d(A \cap B) \subseteq$ gp$\alpha - d(A)\cap$ gp$\alpha - d(B)$.

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On Nano πgb-Closed Sets

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Abstaract − In this paper we introduce a new class of sets called nano πgb-closed sets and nano πgb-open sets. We study some of its basic properties.

Keywords − *Nano π-closed set, nano πg-closed set, nano πgp-closed set, nano πgs-closed set and nano πgb-closed set*

1 Introduction

Lellis Thivagar et al. [3] introduced a nano topological space with respect to a subset X of an universe which is defined in terms of lower approximation and upper approximation and boundary region. The classical nano topological space is based on an equivalence relation on a set, but in some situation, equivalence relations are nor suitable for coping with granularity, instead the classical nano topology is extend to general binary relation based covering nano topological space

Recently, Rajasekaran et al. [6, 7, 8] initiated the study nano πg-closed sets and new classes of sets called nano πgp-closed sets and nano πgs-closed sets in nano topological spaces is introduced and its properties and studied.

In this paper, a new class of sets called nano πgb-closed sets in nano topological spaces is introduced and its properties are studied and studied of nano πgb-closed sets.

2 Preliminaries

Throughout this paper $(U, \tau_R(X))$ (or X) represent nano topological spaces on which no separation axioms are assumed unless otherwise mentioned. For a subset H of a space $(U, \tau_R(X))$, $Ncl(H)$ and $Nint(H)$ denote the nano closure of H and the nano

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interior of H respectively. We recall the following definitions which are useful in the sequel.

Definition 2.1. *[5] Let U be a non-empty finite set of objects called the universe and R be an equivalence relation on U named as the indiscernibility relation. Elements belonging to the same equivalence class are said to be indiscernible with one another. The pair (U, R) is said to be the approximation space. Let* $X \subseteq U$*.*

1. *The lower approximation of X with respect to R is the set of all objects, which can be for certain classified as X with respect to R and it is denoted by* $L_R(X)$*. That is,* $L_R(X) = \bigcup_{x\in U}\{R(x) : R(x) \subseteq X\}$*, where R(x) denotes the equivalence class determined by x.*

2. *The upper approximation of X with respect to R is the set of all objects, which can be possibly classified as X with respect to R and it is denoted by* $U_R(X)$*. That is,* $U_R(X) = \bigcup_{x\in U}\{R(x) : R(x) \cap X \neq \phi\}$*.*

3. *The boundary region of X with respect to R is the set of all objects, which can be classified neither as X nor as not - X with respect to R and it is denoted by* $B_R(X)$*. That is,* $B_R(X) = U_R(X) - L_R(X)$*.*

Property 2.2. *[3] If* (U, R) *is an approximation space and* $X, Y \subseteq U$*; then*

1. $L_R(X) \subseteq X \subseteq U_R(X)$;

2. $L_R(\phi) = U_R(\phi) = \phi$ *and* $L_R(U) = U_R(U) = U$;

3. $U_R(X \cup Y) = U_R(X) \cup U_R(Y)$;

4. $U_R(X \cap Y) \subseteq U_R(X) \cap U_R(Y)$;

5. $L_R(X \cup Y) \supseteq L_R(X) \cup L_R(Y)$;

6. $L_R(X \cap Y) \subseteq L_R(X) \cap L_R(Y)$;

7. $L_R(X) \subseteq L_R(Y)$ *and* $U_R(X) \subseteq U_R(Y)$ *whenever* $X \subseteq Y$;

8. $U_R(X^c) = [L_R(X)]^c$ *and* $L_R(X^c) = [U_R(X)]^c$;

9. $U_RU_R(X) = L_RU_R(X) = U_R(X)$;

10. $L_RL_R(X) = U_RL_R(X) = L_R(X)$.

Definition 2.3. *[3] Let U be the universe, R be an equivalence relation on U and* $\tau_R(X) = \{U, \phi, L_R(X), U_R(X), B_R(X)\}$ *where* $X \subseteq U$*. Then by the Property 2.2, R(X) satisfies the following axioms:*

1. *U and* $\phi \in \tau_R(X)$*,*

2. *The union of the elements of any sub collection of* $\tau_R(X)$ *is in* $\tau_R(X)$*,*

3. *The intersection of the elements of any finite subcollection of* $\tau_R(X)$ *is in* $\tau_R(X)$*.*

That is, $\tau_R(X)$ is a topology on U called the nano topology on U with respect to X. We call $(U, \tau_R(X))$ as the nano topological space. The elements of $\tau_R(X)$ are called as nano open sets and $[\tau_R(X)]^c$ is called as the dual nano topology of $[\tau_R(X)]$.

Remark 2.4. *[3] If $[\tau_R(X)]$ is the nano topology on U with respect to X, then the set $B = \{U, \phi, L_R(X), B_R(X)\}$ is the basis for $\tau_R(X)$.*

Definition 2.5. *[3] If $(U, \tau_R(X))$ is a nano topological space with respect to X and if $H \subseteq U$, then the nano interior of H is defined as the union of all nano open subsets of H and it is denoted by $Nint(H)$.*

That is, $Nint(H)$ is the largest nano open subset of H. The nano closure of H is defined as the intersection of all nano closed sets containing H and it is denoted by $Ncl(H)$.

That is, $Ncl(H)$ is the smallest nano closed set containing H.

Definition 2.6. *A subset H of a nano topological space $(U, \tau_R(X))$ is called*

1. *nano semi-open [3] if $H \subseteq Ncl(Nint(H))$.*
2. *nano pre-open [3] if $H \subseteq Nint(Ncl(H))$.*
3. *nano regular-open [3] if $H = Nint(Ncl(H))$.*
4. *nano π-open [1] if the finite union of nano regular-open sets.*
5. *nano b-open [4] if $H \subseteq Nint(Ncl(H)) \cup Ncl(Nint(H))$.*

The complements of the above mentioned sets is called their respective closed sets.

Definition 2.7. *[6] The nano π-Kernel of the set H, denoted by $\mathcal{N}\pi$-Ker(H), is the intersection of all nano π-open supersets of H.*

Definition 2.8. *[8] A subset H of a space $(U, \tau_R(X))$ is called a nano strong $\mathcal{B}_Q$-set if $Nint(Ncl(H)) = Ncl(Nint(H))$.*

Definition 2.9. *A subset H of a nano topological space $(U, \tau_R(X))$ is called;*

1. *nano gb-closed set [2] if $Nbcl(H) \subseteq G$ whenever $H \subseteq G$ and G is nano open.*
2. *nano πg-closed [6] if $Ncl(H) \subseteq G$, whenever $H \subseteq G$ and G is nano π-open.*
3. *nano πgp-closed set [7] if $Npcl(H) \subseteq G$, whenever $H \subseteq G$ and G is nano π-open.*
4. *nano πgs-closed set [8] if $Nscl(H) \subseteq G$, whenever $H \subseteq G$ and G is nano π-open.*

The complements of the above mentioned sets is called their respective open sets.

3 On Nano πgb-Closed Sets

Definition 3.1. *A subset H of a space $(U, \tau_R(X))$ is nano πgb-closed if $Nbcl(H) \subseteq G$ whenever $H \subseteq G$ and G is nano π-open.*

The complement of nano πgb-open if $H^c = U - H$ is nano πgb-closed.

Example 3.2. *Let $U = \{a, b, c, d\}$ with $U/R = \{\{a\}, \{b, c\}, \{d\}\}$ and $X = \{b, d\}$. Then the nano topology $\tau_R(X) = \{\phi, \{d\}, \{b, c\}, \{b, c, d\}, U\}$.*

1. *then $\{a\}$ is nano πgb-closed set.*
2. *then $\{b, c\}$ is nano πgb-open set.*

Remark 3.3. *For a subset of a space $(U, \tau_R(X))$, we have the following implications:*

$$\begin{array}{ccccc} & & \textbf{nano } \pi gp\textbf{-closed} & \Leftarrow & \textbf{nano } \pi g\textbf{-closed} \\ & & \Downarrow & & \Downarrow \\ \textbf{nano } b\textbf{-closed} & \Rightarrow & \textbf{nano } \pi gb\textbf{-closed} & \Leftarrow & \textbf{nano } \pi gs\textbf{-closed} \end{array}$$

None of the above implications are reversible as shown by the following Examples.

Remark 3.4. *A subset H of a space U is nano πgb-closed $\iff$ $Nbcl(H) \subseteq \mathcal{N}\pi$-$Ker(H)$.*

Remark 3.5. *In a space $(U, \tau_R(X))$, every nano b-closed set is nano πgb-closed.*

Example 3.6. *Let $U = \{a, b, c\}$ with $U/R = \{\{a, c\}, \{b\}\}$ and $X = \{c\}$. Then the nano topology $\tau_R(X) = \{\phi, \{a, c\}, U\}$. Then $\{a, c\}$ is nano πgb-closed but not nano b-closed.*

Proposition 3.7. *In a space $(U, \tau_R(X))$, every nano πgp-closed set is nano πgb-closed.*

Proof. Let H be nano πgp-closed subset of U and G be nano π-open such that $H \subseteq G$. Then $Npcl(H) \subseteq G$. Since every nano pre-closed set is nano b-closed. Therefore $Nbcl(H) \subseteq Npcl(H)$. Hence H is nano πgb-closed.

Example 3.8. *In Example 3.2, then $\{b, c\}$ is nano πgb-closed but not nano πgp-closed.*

Proposition 3.9. *If H is nano π-open and nano πgb-closed, then H is nano b-closed and hence nano gb-closed.*

Proof. Since H is nano π-open and nano πgb-closed. So $Nbcl(H) \subseteq H$. But $H \subseteq Nbcl(H)$. So $H = Nbcl(H)$. Hence H is nano b-closed. Hence nano gb-closed.

Theorem 3.10. *Let H be a nano πgb-closed. Then $Nbcl(H) - H$ does not contain any nonempty nano π-closed set.*

Proof. Let K be a nano π-closed set such that $K \subseteq Nbcl(H) - H$, so $K \subseteq U - H$. Hence $H \subseteq U - K$. Since H is nano πgb-closed and $U - K$ is nano π-open. So $Nbcl(H) \subseteq U - K$. That is $K \subseteq U - Nbcl(H)$. Therefore $K \subseteq Nbcl(H) \cap (U - Nbcl(H)) = \phi$. Thus $K = \phi$.

Corollary 3.11. *Let H be nano πgb-closed set. Then H is nano b-closed $\Longleftrightarrow$ $Nbcl(H) - H$ is nano π-closed.*

Proof. Let H be nano πgb-closed. By hypothesis $Nbcl(H) = H$ and so $Nbcl(H) - H = \phi$, which is nano π-closed.

Conversely, suppose that $Nbcl(H) - H$ is nano π-closed. Then by Theorem 3.10, $Nbcl(H) - H = \phi$, that is $Nbcl(H) = H$. Hence H is nano b-closed.

Theorem 3.12. *If H is nano πgb-closed and $H \subseteq P \subseteq Nbcl(H)$. Then P is nano πgb-closed.*

Proof. Let $P \subseteq G$, where G is nano π-open. Then $H \subseteq P$ implies $H \subseteq G$. Since H is nano πgb-closed, so $Nbcl(H) \subseteq G$ and since $P \subseteq Nbcl(H)$, then $Nbcl(P) \subseteq Nbcl(H) = Nbcl(H)$. Therefore $Nbcl(P) \subseteq G$. Hence P is nano πgb-closed.

Remark 3.13. *In a space $(U, \tau_R(X))$, every nano πgs-closed set is nano πgb-closed.*

Example 3.14. *In Example 3.2, then $\{c, d\}$ is nano πgb-closed set but not nano πgs-closed.*

Theorem 3.15. *For a subset H of U, the following statements are equivalent:*

1. *H is nano π-open and nano πgb-closed.*
2. *H is nano regular-open.*

Proof. (1) $\Rightarrow$ (2) Let H be a nano π-open and nano πgb-closed subset of U. Then $Nbcl(H) \subseteq H$ and so $Nint(Ncl(H)) \subseteq H$ holds. Since H is nano open then H is nano pre-open and thus $H \subseteq Nint(Ncl(H))$. Therefore, we have $Nint(Ncl(H)) = H$, which shows that H is nano regular-open.

(2) $\Rightarrow$ (1) Since every nano regular-open set is nano π-open then $Nbcl(H) = H$ and $Nbcl(H) \subseteq H$. Hence H is nano πgb-closed.

Theorem 3.16. *For a subset H of U, the following statements are equivalent:*

1. *H is nano π-clopen.*
2. *H is nano π-open, nano strong $\mathcal{B}_Q$-set and nano πgb-closed.*

Proof. (1) $\Rightarrow$ (2) Let H be a nano π-clopen subset of U. Then H is nano π-closed and nano π-open. Thus H is nano closed and nano open.

Therefore, H is nano strong $\mathcal{B}_Q$-set. Since every nano π-closed is nano πgb-closed then H is nano πgb-closed.

(2) $\Rightarrow$ (1) By Theorem 3.15, H is nano regular-open. Since H is nano strong $\mathcal{B}_Q$-set, $H = Nint(Ncl(H)) = Ncl(Nint(H))$.Therefore, H is nano regular-closed. Then H is nano π-closed. Hence H is nano π-clopen.

Theorem 3.17. *Let H be a nano πgb-closed set such that $Ncl(H) = U$. Then H is nano πgp-closed.*

Proof. Suppose that H be nano πgb-closed set such that $Ncl(H) = U$. Let G be an nano π-open set containing H. Since $Nbcl(H) = H \cup (Nint(Ncl(H)) \cap Ncl(Nint(H)))$ and $Ncl(H) = U$, we obtain $Nbcl(H) = H \cup Ncl(Nint(H)) = Npcl(H) \subseteq G$. Therefore, H is nano πgp-closed.

Lemma 3.18. *In a space* $(U, \tau_R(X))$,

1. *every nano open set is nano* πgb*-closed.*
2. *every nano closed set is nano* πgb*-closed.*

Remark 3.19. *The converses of statements in Lemma 3.18 are not necessarily true as seen from the following Examples.*

Example 3.20. *In Example 3.2,*

1. *then* $\{a, b\}$ *is nano* πgb*-closed set but not nano open.*
2. *then* $\{a, c\}$ *is nano* πgb*-closed set but not nano closed.*

Theorem 3.21. *In a space* $(U, \tau_R(X))$*, the union of two nano* πgb*-closed sets is nano* πgb*-closed.*

Proof. Let $H \cup Q \subseteq G$, then $H \subseteq G$ and $Q \subseteq G$ where G is nano π-open. As H and Q are nano πgb-closed, $Ncl(H) \subseteq G$ and $Ncl(Q) \subseteq G$. Hence $Ncl(H \cup Q) = Ncl(H) \cup Ncl(Q) \subseteq G$.

Example 3.22. *In Example 3.2, then* $H = \{a\}$ *and* $Q = \{b, c\}$ *is nano* πgb*-closed sets. Clearly* $H \cup Q = \{a, b, c\}$ *is nano* πgb*-closed.*

Theorem 3.23. *In a space* $(U, \tau_R(X))$*, the intersection of two nano* πgb*-open sets are nano* πgb*-open.*

Proof. Obvious by Theorem 3.21.

Example 3.24. *In Example 3.2, then* $H = \{a, c\}$ *and* $Q = \{b, c\}$ *is nano* πgb*-open. Clearly* $H \cap Q = \{c\}$ *is nano* πgb*-open.*

Remark 3.25. *In a space* $(U, \tau_R(X))$*, the union of two nano* πgb*-closed sets but not nano* πgb*-closed.*

Example 3.26. *In Example 3.2, then* $H = \{b\}$ *and* $Q = \{d\}$ *is nano* πgb*-closed sets. Clearly* $H \cup Q = \{b, d\}$ *is but not nano* πgb*-closed.*

Remark 3.27. *In a space* $(U, \tau_R(X))$*, the intersection of two nano* πgb*-open sets but not nano* πgb*-open.*

Example 3.28. *In Example 3.2, then* $H = \{a, b\}$ *and* $Q = \{a, d\}$ *is nano* πgb*-open sets. Clearly* $H \cap Q = \{a\}$ *is but not nano* πgb*-open.*

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On Bipolar Soft Topological Spaces

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Abstract — In this present study, some properties of bipolar soft closed sets are introduced and the concept of closure, interior, basis and subspaces which are the building blocks of classical topology are defined on bipolar soft topological spaces. In addition, examples have been presented so that the subject can be better understood.

Keywords — *Bipolar soft set, bipolar soft topology, bipolar soft topological spaces, bipolar soft open(close), bipolar soft interior, bipolar soft basis.*

1 Introduction

Introducing fuzzy sets [11], intiutionistic fuzzy sets [1], soft sets [6] and etc. theories which contribute to solution of problems such as decision making and uncertainity. A lot of researcher has been done on these theories [2, 3, 7, 10].

In the past years, Shabir & Naz [9] and Karaaslan & Karatas [4] differently defined bipolar soft set. Obviously, bipolar soft sets satisfied more sharp results than soft sets. Therefore the concept of bipolar soft topology has a great importance.

In this study, we define a short notation for writing simplicity in the application of bipolar soft sets and investigate the relationship between the soft topological spaces and the bipolar soft topological spaces. Moreover, we define the notion of bipolar soft closure, bipolar soft interior, bipolar soft basis, bipolar soft subspace. The basis theorems of these notations are provided and supported with examples.

2 Preliminary

In this section, we will give some preliminary information about bipolar soft sets and bipolar soft topological spaces. Let X be an initial universe set and E be a set of parameters. Let $P(X)$ denotes the power set of X and $A, B, C \subseteq E$.

Definition 2.1. [5] Let $E = \{e_1,\ e_2,\ e_3, ..., e_n\}$ be a set of parameters. The not set of E denoted by $\neg E$ is defined by $\neg E = \{\neg e_1,\ \neg e_2,\ \neg e_3, ..., \neg e_n\}$ where for all i, $\neg e_i = not\ e_i$.

Definition 2.2. [9] A triplet (F, G, A) is called a bipolar soft set over X, where F and G are mappings, $F : A \to P(X)$ and $G : \neg A \to P(X)$ such that $F(e) \cap G(\neg e) = \emptyset$ for all $e \in A$ and $\neg e \in \neg A$.

Definition 2.3. [9] For two bipolar soft sets (F_1, G_1, A) and (F_2, G_2, B) over X, (F_1, G_1, A) is called a bipolar soft subset of (F_2, G_2, B) if

1. $A \subseteq B$ and
2. $F_1(e) \subseteq F_2(e)$ and $G_2(\neg e) \subseteq G_1(\neg e)$ for all $e \in A$.

   This relationship is denoted by $(F_1, G_1, A) \,\widetilde{\subseteq}\, (F_2, G_2, B)$. (F_1, G_1, A) and (F_2, G_2, B) are said to be equal if (F_1, G_1, A) is a bipolar soft subset of (F_2, G_2, B) and (F_2, G_2, A) is a bipolar soft subset of (F_1, G_1, B).

Definition 2.4. [9] Bipolar soft complement of a bipolar soft set (F, G, A) over X is denoted by $(F, G, A)^c$ and is defined by $(F, G, A)^c = (F^c, G^c, A)$ where $F^c : A \to P(X)$ and $G^c : \neg A \to P(X)$ are given by $F^c(e) = G(\neg e)$ and $G^c(\neg e) = F(e)$ for all $e \in A$ and $\neg e \in \neg A$.

Definition 2.5. [9] Bipolar soft union of two bipolar soft sets (F_1, G_1, A) and (F_2, G_2, B) over X is the bipolar soft set (H, I, C) over X where $C = A \cup B$ and for all $e \in C$,

$$H(e) = \begin{cases} F_1(e), & \text{if } e \in A - B, \\ F_2(e), & \text{if } e \in B - A, \\ F_1(e) \cup F_2(e), & \text{if } e \in A \cap B. \end{cases}$$

$$I(\neg e) = \begin{cases} G_1(\neg e), & \text{if } \neg e \in (\neg A) - (\neg B), \\ G_2(\neg e), & \text{if } \neg e \in (\neg B) - (\neg A), \\ G_1(\neg e) \cap G_2(\neg e), & \text{if } \neg e \in (\neg A) \cap (\neg B). \end{cases}$$

It is denoted by $(F_1, G_1, A) \,\widetilde{\cup}\, (F_2, G_2, B) = (H, I, C)$.

Definition 2.6. [9] Bipolar soft intersection of two bipolar soft sets (F_1, G_1, A) and (F_2, G_2, B) over X is the bipolar soft set (H, I, C) over X where $C = A \cup B$ is non-empty and for all $e \in C$,

$$H(e) = F_1(e) \cap F_2(e) \text{ and } I(\neg e) = G_1(\neg e) \cup G_2(\neg e).$$

It is denoted by $(F_1, G_1, A) \,\widetilde{\cap}\, (F_2, G_2, B) = (H, I, C)$.

Definition 2.7. [9] Let (F_1, G_1, A) and (F_2, G_2, B) be two bipolar soft sets over X. Then,

1. $\left((F_1, G_1, A) \,\widetilde{\cup}\, (F_2, G_2, B)\right)^c = (F_1, G_1, A)^c \,\widetilde{\cap}\, (F_2, G_2, B)^c$,

2. $\left((F_1, G_1, A)\,\widetilde{\cap}\,(F_2, G_2, B)\right)^c = (F_1, G_1, A)^c\,\widetilde{\cup}\,(F_2, G_2, B)^c$.

Definition 2.8. [9] A bipolar soft set (F, G, A) over X is said to be relative null bipolar soft set, denoted by $\left(\Phi, \widetilde{X}, A\right)$, if for all $e \in A$, $F(e) = \emptyset$ and for all $\neg e \in \neg A$, $G(\neg e) = X$.

The relative null bipolar soft set with respect to the universe set of parameters E is called a NULL bipolar soft set over X and is denoted by $\left(\Phi, \widetilde{X}, E\right)$.

Definition 2.9. [9] A bipolar soft set (F, G, A) over X is said to be relative absolute bipolar soft set, denoted by $\left(\widetilde{X}, \Phi, A\right)$, if for all $e \in A$, $F(e) = X$ and for all $\neg e \in \neg A$, $G(\neg e) = \emptyset$.

The relative absolute bipolar soft set with respect to the universe set of parameters E is called a ABSOLUTE bipolar soft set over X and is denoted by $\left(\widetilde{X}, \Phi, E\right)$.

Definition 2.10. [8] Let $\widetilde{\widetilde{\tau}}$ be the collection of bipolar soft sets over X with E as the set of parameters. Then $\widetilde{\widetilde{\tau}}$ is said to be a bipolar soft topology over X if

1. $\left(\Phi, \widetilde{X}, E\right)$ and $\left(\widetilde{X}, \Phi, E\right)$ belong to $\widetilde{\widetilde{\tau}}$
2. the bipolar soft union of any number of bipolar soft sets in $\widetilde{\widetilde{\tau}}$ belongs to $\widetilde{\widetilde{\tau}}$
3. the bipolar soft intersection of finite number of bipolar soft sets in $\widetilde{\widetilde{\tau}}$ belongs to $\widetilde{\widetilde{\tau}}$.

   Then $\left(X, \widetilde{\widetilde{\tau}}, E, \neg E\right)$ is called a bipolar soft topological space over X .

Definition 2.11. [8] Let $\left(X, \widetilde{\widetilde{\tau}}, E, \neg E\right)$ be a bipolar soft topological space over X, then the members of $\widetilde{\widetilde{\tau}}$ are said to be bipolar soft open sets in X.

Definition 2.12. [8] Let $\left(X, \widetilde{\widetilde{\tau}}, E, \neg E\right)$ be a bipolar soft topological space over X. A bipolar soft set (F, G, E) over X is said to be a bipolar soft closed set in X, if its bipolar soft complemet $(F, G, E)^c$ belongs to $\widetilde{\widetilde{\tau}}$.

Definition 2.13. [8] Let $\left(X, \widetilde{\widetilde{\tau}}, E, \neg E\right)$ be a bipolar soft topological space over X. A bipolar soft set (F, G, E) over X is said to be a bipolar soft clopen set in X, if it is both a bipolar soft closed set and a bipolar soft open set over X.

3 The Main Results

Definition 3.1. Let (F, G, A) be a bipolar soft set over X. The presentation of

$$(F, G, A) = \{(e,\ F(e),\ G(\neg e)) : e \in A \subseteq E,\ \neg e \in \neg A \subseteq \neg E \text{ and } F(e),\ G(\neg e) \in P(X)\}$$

is said to be a short expansion of bipolar soft set (F, G, A).

From now on, $BSS\left(X\right)_{E,\neg E}$ denotes the family of all bipolar soft sets over X with E as the set of parameters and $BSTS$ denotes a bipolar soft topolological space.

Example 3.2. Let $X = \{x_1, x_{2,}x_3, x_4\}$ be an universe set, $E = \{e_1, e_{2,}e_3\}$ be the set of parameters and $A = \{e_1, e_3\} \subseteq E$ be a subset of parameters. Then $\neg E = \{\neg e_1, \neg e_{2,}\neg e_3\}$ and $\neg A = \{\neg e_1, \neg e_3\}$. Suppose that a bipolar soft set (F, G, A) is given as follows.

$$F(e_1) = \{x_1, x_3\}, \; F(e_3) = \{x_4\}$$
$$G(\neg e_1) = \{x_2\}, \; G(\neg e_3) = \{x_1, x_2\}.$$

Then the short expansion of bipolar soft set (F, G, A) is denoted by $(F, G, A) = \{(e_1, \{x_1, x_3\}, \{x_2\}), \; (e_3, \{x_4\}, \{x_1, x_2\})\}$.

Proposition 3.3. [8] Let $\left(X, \widetilde{\widetilde{\tau}}, E, \neg E\right)$ be a $BSTS$ over X. Then the collection $\tau_e = \{F(e) : (F, G, E) \in \widetilde{\widetilde{\tau}}\}$ for each $e \in E$, defines a topology on X.

Theorem 3.4. [8] Let $(X, \widetilde{\tau}, E)$ be a soft topological space over X. Then the collection $\widetilde{\widetilde{\tau}}$ consisting of bipolar soft sets (F, G, E) such that $(F, E) \in \widetilde{\tau}$ and $G(\neg e) = F^{'}(e) = U \backslash F(e)$ for all $\neg e \in \neg E$, defines a $BSTS$ over X.

Proposition 3.5. Let $\left(X, \widetilde{\widetilde{\tau}}, E, \neg E\right)$ be a $BSTS$ over X. Then the collection $\widetilde{\tau} = \left\{(F, E) : (F, G, E) \in \widetilde{\widetilde{\tau}}\right\}$ defines a soft topology and $(X, \widetilde{\tau}, E)$ is a soft topological space over X.

Proof. Suppose that $\left(X, \widetilde{\widetilde{\tau}}, E, \neg E\right)$ is a $BSTS$ over X. Let us show that the collection $\widetilde{\tau} = \left\{(F, E) : (F, G, E) \in \widetilde{\widetilde{\tau}}\right\}$ provides the conditions of soft topological spaces.

1. $\left(\Phi, \widetilde{X}, E\right)$, $\left(\widetilde{X}, \Phi, E\right) \in \widetilde{\widetilde{\tau}}$ implies that (Φ, E), $\left(\widetilde{X}, E\right) \in \widetilde{\tau}$.

2. Let $\{(F_i, E)\}_{i \in I}$ be a collection of sets in $\widetilde{\tau}$. Since $(F_i, G_i, E) \in \widetilde{\widetilde{\tau}}$, for all $i \in I$ so that $\underset{i \in I}{\widetilde{\cup}} (F_i, G_i, E) \in \widetilde{\widetilde{\tau}}$ thus $\underset{i \in I}{\widetilde{\cup}} (F_i, E) \in \widetilde{\tau}$.

3. Let $\{(F_i, E)\}_{i = \overline{1,n}}$ be a collection of finite sets in $\widetilde{\tau}$. Then $\overset{n}{\underset{i=1}{\widetilde{\cap}}} (F_i, G_i, E) \in \widetilde{\widetilde{\tau}}$ so $\overset{n}{\underset{i=1}{\widetilde{\cap}}} (F_i, E) \in \widetilde{\tau}$.

Hence $\widetilde{\tau}$ defines a soft topology over X. □

Remark 3.6. Let $\left(X, \widetilde{\widetilde{\tau}}, E, \neg E\right)$ be a $BSTS$ over X. It can be easily shown that, if the collection $\widetilde{\widetilde{\tau}}$ is finite then $\widetilde{\neg \tau} = \left\{(G, \neg E) : (F, G, E) \in \widetilde{\widetilde{\tau}}\right\}$ defines a soft topology and $(X, \widetilde{\neg \tau}, \neg E)$ is a soft topological space over X.

Similarly, if the collection $\widetilde{\widetilde{\tau}}$ is finite then $\widetilde{\tau}_{\neg e} = \left\{G\left(\neg e\right) : (F, G, E) \in \widetilde{\widetilde{\tau}}, \text{ for all } \neg e \in \neg E\right\}$ defines a topology and $(X, \widetilde{\tau}_{\neg e})$ is a topological space over X.

Definition 3.7. Let X be an initial universe set, E be the set of parameters and $\widetilde{\widetilde{\tau}} = \left\{ \left(\Phi, \widetilde{X}, E \right), \left(\widetilde{X}, \Phi, E \right) \right\}$. Then $\widetilde{\widetilde{\tau}}$ is called the bipolar soft indiscrete topology over X and $\left(X, \widetilde{\widetilde{\tau}}, E, \neg E \right)$ is said to be a bipolar soft indiscrete topological space over X.

Definition 3.8. Let X be an initial universe set, E be the set of parameters and $\widetilde{\widetilde{\tau}}$ be the collection of all bipolar soft sets that can be defined over X. Then $\widetilde{\widetilde{\tau}}$ is called the bipolar soft discrete topology over X and $\left(X, \widetilde{\widetilde{\tau}}, E, \neg E \right)$ is said to be a bipolar soft discrete topological space over X.

Definition 3.9. Let $\left(X, \widetilde{\widetilde{\tau_1}}, E, \neg E \right)$ and $\left(X, \widetilde{\widetilde{\tau_2}}, E, \neg E \right)$be two $BSTS'$s over the same initial universe set X. Then $\widetilde{\widetilde{\tau_2}}$ is said to be bipolar soft finer than $\widetilde{\widetilde{\tau_1}}$, or $\widetilde{\widetilde{\tau_1}}$ is said to be bipolar soft coarser than $\widetilde{\widetilde{\tau_2}}$ if $\widetilde{\widetilde{\tau_2}} \supseteq \widetilde{\widetilde{\tau_1}}$.

Example 3.10. Let X be an initial universe set and E be the set of parameters. The bipolar soft indiscrete topology is the coarsest bipolar soft topology and the bipolar discrete topology is the finest bipolar soft topology over X.

Proposition 3.11. Let $\left(X, \widetilde{\widetilde{\tau_1}}, E, \neg E \right)$ and $\left(X, \widetilde{\widetilde{\tau_2}}, E, \neg E \right)$be two $BSTS'$s over the same initial universe set X, then $\left(X, \widetilde{\widetilde{\tau_1}} \cap \widetilde{\widetilde{\tau_2}}, E, \neg E \right)$ is a $BSTS$ over X.

Proof. 1) $\left(\Phi, \widetilde{X}, E \right), \left(\widetilde{X}, \Phi, E \right) \in \widetilde{\widetilde{\tau_1}} \cap \widetilde{\widetilde{\tau_2}}$.

2) Let $\{(F_i, G_i, E)\}_{i \in I}$ be a family of bipolar soft sets in $\widetilde{\widetilde{\tau_1}} \cap \widetilde{\widetilde{\tau_2}}$. Then $(F_i, G_i, E) \in \widetilde{\widetilde{\tau_1}}$ and $(F_i, G_i, E) \in \widetilde{\widetilde{\tau_2}}$, for all $i \in I$, so $\widetilde{\bigcup_{i \in I}} (F_i, G_i, E) \in \widetilde{\widetilde{\tau_1}}$ and $\widetilde{\bigcup_{i \in I}} (F_i, G_i, E) \in \widetilde{\widetilde{\tau_2}}$. Therefore $\widetilde{\bigcup_{i \in I}} (F_i, G_i, E) \in \widetilde{\widetilde{\tau_1}} \cap \widetilde{\widetilde{\tau_2}}$.

3) Let $\{(F_i, G_i, E)\}_{i = \overline{1,n}}$ be a finite family of bipolar soft sets in $\widetilde{\widetilde{\tau_1}} \cap \widetilde{\widetilde{\tau_2}}$. Then $(F_i, G_i, E) \in \widetilde{\widetilde{\tau_1}}$ and $(F_i, G_i, E) \in \widetilde{\widetilde{\tau_2}}$ for $i = \overline{1,n}$. Since $\widetilde{\bigcap_{i=1}^{n}} (F_i, G_i, E) \in \widetilde{\widetilde{\tau_1}}$ and $\widetilde{\bigcap_{i=1}^{n}} (F_i, G_i, E) \in \widetilde{\widetilde{\tau_2}}$, then $\widetilde{\bigcap_{i=1}^{n}} (F_i, G_i, E) \in \widetilde{\widetilde{\tau_1}} \cap \widetilde{\widetilde{\tau_2}}$. □

Remark 3.12. The union of two bipolar soft topologies over the same initial universe set X may not be a bipolar soft topology over X.

Example 3.13. Let $X = \{x_1, x_2, x_3, x_4\}$, $E = \{e_1, e_2, e_3\}$. Then $\neg E = \{\neg e_1, \neg e_2, \neg e_3\}$. Suppose that $\widetilde{\widetilde{\tau_1}} = \left\{ \left(\Phi, \widetilde{X}, E \right), \left(\widetilde{X}, \Phi, E \right), (F_1, G_1, E), (F_2, G_2, E), (F_3, G_3, E) \right\}$, $\widetilde{\widetilde{\tau_2}} = \left\{ \left(\Phi, \widetilde{X}, E \right), \left(\widetilde{X}, \Phi, E \right), (H_1, K_1, E), (H_2, K_2, E), (H_3, K_3, E) \right\}$ are two bipolar soft topologies defined over X where (F_1, G_1, E), (F_2, G_2, E), (F_3, G_3, E), (H_1, K_1, E), (H_2, K_2, E), (H_3, K_3, E) are bipolar soft sets over X, defined as follows:

$$
\begin{aligned}
(F_1, G_1, E) &= \{(e_1, \{x_1, x_3, x_4\}, \{x_2\}), (e_2, \{x_2, x_3\}, \{x_4\}), (e_3, \{x_3, x_4\}, \{x_1\})\}, \\
(F_2, G_2, E) &= \{(e_1, \{x_2, x_4\}, \{x_1\}), (e_2, \{x_1, x_4\}, \{x_2\}), (e_3, \{x_1, x_2\}, \{x_3\})\}, \\
(F_3, G_3, E) &= \{(e_1, \{x_4\}, \{x_1, x_2\}), (e_2, \emptyset, \{x_2, x_4\}), (e_3, \emptyset, \{x_1, x_3\})\},
\end{aligned}
$$

and

$$\begin{aligned}(H_1,K_1,E) &= \{(e_1,\{x_2,x_3\},\{x_1\}),(e_2,\{x_1,x_2\},\{x_3,x_4\}),(e_3,\{x_3,x_4\},\{x_2\})\},\\(H_2,K_2,E) &= \{(e_1,\{x_1,x_4\},\{x_2,x_3\}),(e_2,X,\emptyset),(e_3,\{x_1,x_2,x_3\},\emptyset)\},\\(H_3,K_3,E) &= \{(e_1,\emptyset,\{x_1,x_2,x_3\}),(e_2,\{x_1,x_2\},\{x_3,x_4\}),(e_3,\{x_3\},\{x_2\})\}.\end{aligned}$$

Then

$$\widetilde{\widetilde{\tau}}_1\cup\widetilde{\widetilde{\tau}}_2=\left\{\begin{array}{c}\left(\Phi,\widetilde{X},E\right),\left(\widetilde{X},\Phi,E\right),(F_1,G_1,E),(F_2,G_2,E),\\(F_3,G_3,E),(H_1,K_1,E),(H_2,K_2,E),(H_3,K_3,E)\end{array}\right\}.$$

For example, we take
$(F_1,G_1,E)\widetilde{\cup}(H_1,K_1,E)=(S,T,E)=\{(e_1,X,\emptyset),(e_2,\{x_1,x_2,x_3\},\{x_4\})\ \ (e_3,\{x_3,x_4\},\emptyset)\}$, but $(S,T,E)\notin\widetilde{\widetilde{\tau}}_1\cup\widetilde{\widetilde{\tau}}_2$. Therefore $\widetilde{\widetilde{\tau}}_1\cup\widetilde{\widetilde{\tau}}_2$ is not a bipolar soft topology over X.

Theorem 3.14. Let $\left(X,\widetilde{\widetilde{\tau}},E,\neg E\right)$ be a $BSTS$ over X. Then

1. $\left(\Phi,\widetilde{X},E\right),\left(\widetilde{X},\Phi,E\right)$ are bipolar soft closed sets over X,
2. Arbitrary bipolar soft interscetions of the bipolar soft closed sets are bipolar soft closed set over X,
3. Finite bipolar soft unions of the bipolar soft closed sets are bipolar soft closed set over X.

Proof. 1. Since $\left(\Phi,\widetilde{X},E\right)^c=\left(\widetilde{X},\Phi,E\right)\in\widetilde{\widetilde{\tau}}$ and $\left(\widetilde{X},\Phi,E\right)^c=\left(\Phi,\widetilde{X},E\right)\in\widetilde{\widetilde{\tau}}$, then $\left(\Phi,\widetilde{X},E\right),\left(\widetilde{X},\Phi,E\right)$ are bipolar soft closed sets over X.

2. Let $\{(F_i,G_i,E)\}_{i\in I}$ be a family of bipolar soft closed sets over X. Then

$$\left(\underset{i\in I}{\widetilde{\cap}}(F_i,G_i,E)\right)^c=\underset{i\in I}{\widetilde{\cup}}(F_i,G_i,E)^c\in\widetilde{\widetilde{\tau}}.$$

Therefore, $\underset{i\in I}{\widetilde{\cap}}(F_i,G_i,E)$ is a bipolar soft closed set over X.

3. Let $\{(F_i,G_i,E)\}_{i=\overline{1,n}}$ be a finite family of bipolar soft closed sets over X. Then

$$\left(\overset{n}{\underset{i=1}{\widetilde{\cup}}}(F_i,G_i,E)\right)^c=\overset{n}{\underset{i=1}{\widetilde{\cap}}}(F_i,G_i,E)^c\in\widetilde{\widetilde{\tau}}.$$

Thus, $\overset{n}{\underset{i=1}{\widetilde{\cup}}}(F_i,G_i,E)$ is a bipolar soft closed set over X. □

Definition 3.15. Let $\left(X,\widetilde{\widetilde{\tau}},E,\neg E\right)$ be a $BSTS$ over X and (F,G,E) be a bipolar soft set over X. Then the bipolar soft closure of (F,G,E), denoted by $\overline{(F,G,E)}$, is the bipolar soft intersection of all bipolar soft closed super sets of (F,G,E).

Obviously, $\overline{(F,G,E)}$ is the smallest bipolar soft closed set over X that containing (F,G,E).

Theorem 3.16. Let $\left(X, \widetilde{\widetilde{\tau}}, E, \neg E\right)$ be a $BSTS$ over X, (F, G, E) and (F_1, G_1, E) be two bipolar soft sets over X. Then

1. $\overline{\left(\Phi, \widetilde{X}, E\right)} = \left(\Phi, \widetilde{X}, E\right)$, $\overline{\left(\widetilde{X}, \Phi, E\right)} = \left(\widetilde{X}, \Phi, E\right)$,
2. $(F, G, E)\widetilde{\subseteq}\overline{(F, G, E)}$,
3. (F, G, E) is a bipolar soft closed set iff $(F, G, E) = \overline{(F, G, E)}$,
4. $\overline{\overline{(F, G, E)}} = \overline{(F, G, E)}$,
5. $(F, G, E)\widetilde{\subseteq}(F_1, G_1, E) \Rightarrow \overline{(F, G, E)}\widetilde{\subseteq}\overline{(F_1, G_1, E)}$
6. $\overline{(F, G, E)\widetilde{\cup}(F_1, G_1, E)} = \overline{(F, G, E)}\widetilde{\cup}\overline{(F_1, G_1, E)}$
7. $\overline{(F, G, E)\widetilde{\cap}(F_1, G_1, E)}\widetilde{\subseteq}\overline{(F, G, E)}\widetilde{\cap}\overline{(F_1, G_1, E)}$.

Proof. 1. and 2. are obvious.

3. Suppose that (F, G, E) is a bipolar soft closed. Then $\overline{(F, G, E)}$ is the smallest bipolar soft closed set containing (F, G, E) and $(F, G, E) = \overline{(F, G, E)}$.

Conversely, let $(F, G, E) = \overline{(F, G, E)}$. Since $\overline{(F, G, E)}$ is a bipolar soft closed set, then (F, G, E) is a bipolar soft closed set over X.

4. Since $\overline{(F, G, E)}$ is a bipolar soft closed then we have $\overline{\overline{(F, G, E)}} = \overline{(F, G, E)}$ from the part (3.).

5. Let $(F, G, E)\widetilde{\subseteq}(F_1, G_1, E)$. From the part (2.), $(F, G, E)\widetilde{\subseteq}\overline{(F, G, E)}$ and $(F_1, G_1, E)\widetilde{\subseteq}\overline{(F_1, G_1, E)}$. $\overline{(F, G, E)}$ is the smallest bipolar soft closed set that containing (F, G, E). Then $\overline{(F, G, E)}\widetilde{\subseteq}\overline{(F_1, G_1, E)}$.

6. Since $(F, G, E)\widetilde{\subseteq}(F, G, E)\widetilde{\cup}(F_1, G_1, E)$ and $(F_1, G_1, E)\widetilde{\subseteq}(F, G, E)\widetilde{\cup}(F_1, G_1, E)$ then $\overline{(F, G, E)}\widetilde{\subseteq}\overline{(F, G, E)\widetilde{\cup}(F_1, G_1, E)}$ and $\overline{(F_1, G_1, E)}\widetilde{\subseteq}\overline{(F, G, E)\widetilde{\cup}(F_1, G_1, E)}$ from the part (5.). Therefore, $\overline{(F, G, E)}\widetilde{\cup}\overline{(F_1, G_1, E)}\widetilde{\subseteq}\overline{(F, G, E)\widetilde{\cup}(F_1, G_1, E)}$.
Conversely, since $(F, G, E)\widetilde{\subseteq}\overline{(F, G, E)}$ and $(F_1, G_1, E)\widetilde{\subseteq}\overline{(F_1, G_1, E)}$. Then $(F, G, E)\widetilde{\cup}(F_1, G_1, E)\widetilde{\subseteq}\overline{(F, G, E)}\widetilde{\cup}\overline{(F_1, G_1, E)}$. Since $\overline{(F, G, E)}\widetilde{\cup}\overline{(F_1, G_1, E)}$ is a bipolar soft closed set and $\overline{(F, G, E)\widetilde{\cup}(F_1, G_1, E)}$ is the smallest bipolar soft closed set that containing $(F, G, E)\widetilde{\cup}(F_1, G_1, E)$. Then $\overline{(F, G, E)\widetilde{\cup}(F_1, G_1, E)}\widetilde{\subseteq}\overline{(F, G, E)}\widetilde{\cup}\overline{(F_1, G_1, E)}$.
Hence, $\overline{(F, G, E)\widetilde{\cup}(F_1, G_1, E)} = \overline{(F, G, E)}\widetilde{\cup}\overline{(F_1, G_1, E)}$.

7. Since $(F, G, E)\widetilde{\cap}(F_1, G_1, E)\widetilde{\subseteq}(F, G, E)$ and $(F, G, E)\widetilde{\cap}(F_1, G_1, E)\widetilde{\subseteq}(F_1, G_1, E)$ then $\overline{(F, G, E)\widetilde{\cap}(F_1, G_1, E)}\widetilde{\subseteq}\overline{(F, G, E)}$ and $\overline{(F, G, E)\widetilde{\cap}(F_1, G_1, E)}\widetilde{\subseteq}\overline{(F_1, G_1, E)}$. Therefore, $\overline{(F, G, E)\widetilde{\cap}(F_1, G_1, E)}\widetilde{\subseteq}\overline{(F, G, E)}\widetilde{\cap}\overline{(F_1, G_1, E)}$. □

Example 3.17. Let $X = \{x_1, x_2, x_3, x_4, x_5\}$, $E = \{e_1, e_2, e_3\}$. Then $\neg E = \{\neg e_1, \neg e_2, \neg e_3\}$. Suppose that $\widetilde{\widetilde{\tau}} = \left\{\left(\Phi, \widetilde{X}, E\right), \left(\widetilde{X}, \Phi, E\right), (F_1, G_1, E), (F_2, G_2, E), (F_3, G_3, E), (F_4, G_4, E)\right\}$, is a bipolar soft topology defined over X where (F_1, G_1, E), (F_2, G_2, E), (F_3, G_3, E), (F_4, G_4, E) are bipolar soft sets over X, defined as follows:

$$\begin{aligned}
(F_1, G_1, E) &= \{(e_1, \{x_2, x_3, x_4\}, \{x_5\}), (e_2, \{x_1, x_2, x_3\}, \{x_4, x_5\}), (e_3, \{x_3, x_4, x_5\}, \{x_2\})\}, \\
(F_2, G_2, E) &= \{(e_1, \{x_1, x_2, x_5\}, \{x_3, x_4\}), (e_2, \{x_2, x_4\}, \{x_3, x_5\}), (e_3, \{x_1, x_5\}, \{x_2, x_3\})\}, \\
(F_3, G_3, E) &= \{(e_1, \{x_2\}, \{x_3, x_4, x_5\}), (e_2, \{x_2\}, \{x_3, x_4, x_5\}), (e_3, \{x_5\}, \{x_2\})\}, \\
(F_4, G_4, E) &= \{(e_1, X, \emptyset), (e_2, \{x_1, x_2, x_3, x_4\}, \{x_5\}), (e_3, \{x_1, x_3, x_4, x_5\}, \{x_2\})\}.
\end{aligned}$$

According to the bipolar soft topological space $\left(X, \widetilde{\widetilde{\tau}}, E, \neg E\right)$;

$$\left(\widetilde{\widetilde{\tau}}\right)^c = \left\{\left(\Phi, \widetilde{X}, E\right)^c, \left(\widetilde{X}, \Phi, E\right)^c, (F_1, G_1, E)^c, (F_2, G_2, E)^c, (F_3, G_3, E)^c, (F_4, G_4, E)^c\right\}$$

is the family of all bipolar soft closed sets such that

$$\begin{aligned}
(F_1, G_1, E)^c &= \{(e_1, \{x_5\}, \{x_2, x_3, x_4\}), (e_2, \{x_4, x_5\}, \{x_1, x_2, x_3\}) \ (e_3, \{x_2\}, \{x_3, x_4, x_5\})\}, \\
(F_2, G_2, E)^c &= \{(e_1, \{x_3, x_4\}, \{x_1, x_2, x_5\}), (e_2, \{x_3, x_5\}, \{x_2, x_4\}) \ (e_3, \{x_2, x_3\}, \{x_1, x_5\})\}, \\
(F_3, G_3, E)^c &= \{(e_1, \{x_3, x_4, x_5\}, \{x_2\}), (e_2, \{x_3, x_4, x_5\}, \{x_2\}) \ (e_3, \{x_2\}, \{x_5\})\}, \\
(F_4, G_4, E)^c &= \{(e_1, \emptyset, X), (e_2, \{x_5\}, \{x_1, x_2, x_3, x_4\}), (e_3, \{x_2\}, \{x_1, x_3, x_4, x_5\})\}.
\end{aligned}$$

Let $(K, S, E) = \{(e_1, \{x_3\}, \{x_1, x_2, x_4, x_5\}), (e_2, \{x_5\}, \{x_1, x_2, x_4\}), (e_3, \{x_2\}, \{x_1, x_3, x_5\})\}$ be a bipolar soft set over X. Then the bipolar soft closure of (K, S, E),

$$\overline{(K, S, E)} = (F_2, G_2, E)^c \widetilde{\cap} (F_3, G_3, E)^c \widetilde{\cap} \left(\widetilde{X}, \Phi, E\right) = (F_2, G_2, E)^c.$$

Corollary 3.18. It is clear that whereas only intersection operation on soft closed sets containing (F, E) depending on an appropriate parameter is performed for the soft closure operation of (F, E) in the studies [3, 10], in the bipolar soft set theory, an intersection operation according to an appropriate parameter on the bipolar soft closed sets containing the set and union operation according to not element of parameter on the bipolar soft closed sets containing the set are performed.

Definition 3.19. Let $\left(X, \widetilde{\widetilde{\tau}}, E, \neg E\right)$ be a $BSTS$ over X and (F, G, E) be a bipolar soft set over X. Then the bipolar soft interior of (F, G, E), denoted by $(F, G, E)^\circ$, is the bipolar soft union of all bipolar soft open subsets of (F, G, E).

Obviously, $(F, G, E)^\circ$ is the biggest bipolar soft open set over X that is contained by (F, G, E).

Theorem 3.20. Let $\left(X, \widetilde{\widetilde{\tau}}, E, \neg E\right)$ be a $BSTS$ over X, (F, G, E) and (F_1, G_1, E) be two bipolar soft sets over X. Then

1. $\left(\Phi, \widetilde{X}, E\right)^\circ = \left(\Phi, \widetilde{X}, E\right)$, $\left(\widetilde{X}, \Phi, E\right)^\circ = \left(\widetilde{X}, \Phi, E\right)$,
2. $(F, G, E)^\circ \widetilde{\subseteq} (F, G, E)$,
3. (F, G, E) is a bipolar soft open set iff $(F, G, E) = (F, G, E)^\circ$,
4. $((F, G, E)^\circ)^\circ = (F, G, E)^\circ$,
5. $(F, G, E) \widetilde{\subseteq} (F_1, G_1, E) \Rightarrow (F, G, E)^\circ \widetilde{\subseteq} (F_1, G_1, E)^\circ$,
6. $(F, G, E)^\circ \widetilde{\cap} (F_1, G_1, E)^\circ = \left[(F, G, E) \widetilde{\cap} (F_1, G_1, E)\right]^\circ$,
7. $(F, G, E)^\circ \widetilde{\cup} (F_1, G_1, E)^\circ \widetilde{\subseteq} \left[(F, G, E) \widetilde{\cup} (F_1, G_1, E)\right]^\circ$.

Proof. 1. and 2. are obvious.

3. Suppose that (F,G,E) is a bipolar soft open set. Then (F,G,E) is the biggest bipolar soft open set that is contained by (F,G,E) and $(F,G,E)=(F,G,E)^\circ$.

Conversely, let $(F,G,E)=(F,G,E)^\circ$. Since $(F,G,E)^\circ$ is a bipolar soft open set, then (F,G,E) is a bipolar soft open set over X.

4. Let $(F,G,E)^\circ=(K,S,E)$. Then (K,S,E) is a bipolar soft open set iff $(K,S,E)=(K,S,E)^\circ$. Therefore, $((F,G,E)^\circ)^\circ=(F,G,E)^\circ$.

5. Suppose that $(F,G,E)\widetilde{\subseteq}(F_1,G_1,E)$. From the part (2.), $(F,G,E)^\circ\widetilde{\subseteq}(F,G,E)$ and $(F_1,G_1,E)^\circ\widetilde{\subseteq}(F_1,G_1,E)$. $(F_1,G_1,E)^\circ$ is the biggest bipolar soft open set that is contained by (F_1,G_1,E). So, $(F,G,E)^\circ\widetilde{\subseteq}(F_1,G_1,E)^\circ$.

6. Since $(F,G,E)^\circ\widetilde{\subseteq}(F,G,E)$ and $(F_1,G_1,E)^\circ\widetilde{\subseteq}(F_1,G_1,E)$, then $(F,G,E)^\circ\widetilde{\cap}(F_1,G_1,E)^\circ\widetilde{\subseteq}(F,G,E)\widetilde{\cap}(F_1,G_1,E)$. $\left[(F,G,E)\widetilde{\cap}(F_1,G_1,E)\right]^\circ$ is the biggest bipolar soft open set that is contained by $(F,G,E)\widetilde{\cap}(F_1,G_1,E)$. Therefore, $(F,G,E)^\circ\widetilde{\cap}(F_1,G_1,E)^\circ\widetilde{\subseteq}\left[(F,G,E)\widetilde{\cap}(F_1,G_1,E)\right]^\circ$.

Conversely, since $(F,G,E)\widetilde{\cap}(F_1,G_1,E)\widetilde{\subseteq}(F,G,E)$ and $(F,G,E)\widetilde{\cap}(F_1,G_1,E)\widetilde{\subseteq}(F_1,G_1,E)$, then $\left[(F,G,E)\widetilde{\cap}(F_1,G_1,E)\right]^\circ\widetilde{\subseteq}(F,G,E)^\circ$ and $\left[(F,G,E)\widetilde{\cap}(F_1,G_1,E)\right]^\circ\widetilde{\subseteq}(F_1,G_1,E)^\circ$. Hence, $\left[(F,G,E)\widetilde{\cap}(F_1,G_1,E)\right]^\circ\widetilde{\subseteq}(F,G,E)^\circ\widetilde{\cap}(F_1,G_1,E)^\circ$.

7. Since $(F,G,E)^\circ\widetilde{\subseteq}(F,G,E)$ and $(F_1,G_1,E)^\circ\widetilde{\subseteq}(F_1,G_1,E)$, then $(F,G,E)^\circ\widetilde{\cup}(F_1,G_1,E)^\circ\widetilde{\subseteq}(F,G,E)\widetilde{\cup}(F_1,G_1,E)$. $\left[(F,G,E)\widetilde{\cup}(F_1,G_1,E)\right]^\circ$ is the biggest bipolar soft open set that is contained by $(F,G,E)\widetilde{\cup}(F_1,G_1,E)$. So, $(F,G,E)^\circ\widetilde{\cup}(F_1,G_1,E)^\circ\widetilde{\subseteq}\left[(F,G,E)\widetilde{\cup}(F_1,G_1,E)\right]^\circ$. □

Example 3.21. Let us consider the bipolar soft topology over X that is given in Example 3.17. Suppose that

$$(K,S,E)=\{(e_1,\{x_2,x_3,x_4\},\{x_5\}),(e_2,X,\emptyset)\ \ (e_3,X,\emptyset)\}$$

is a bipolar soft set over X. Then the bipolar soft interior of (K,S,E),

$$(K,S,E)^\circ=(F_1,G_1,E)\widetilde{\cup}(F_3,G_3,E)\widetilde{\cup}\left(\Phi,\widetilde{X},E\right)=(F_1,G_1,E).$$

Corollary 3.22. It is clear that whereas only union operation on soft open sets contained in (F,E) depending on an appropriate parameter is performed for the soft interior operation of (F,E) in the studies [3, 10], in the bipolar soft set theory, a union operation according to an appropriate parameter on the bipolar soft open sets contained in the set and an intersection operation according to not element of parameter on the bipolar soft open sets contained in the set are performed.

Theorem 3.23. Let $\left(X,\widetilde{\widetilde{\tau}},E,\neg E\right)$ be a $BSTS$ over X, (F,G,E) be a bipolar soft sets over X. Then $\left[\overline{(F,G,E)}\right]^c=[(F,G,E)^c]^\circ$.

Proof. From the definitions of a bipolar soft closure and a bipolar soft interior, we have

$$\left[\overline{(F,G,E)}\right]^c=\left(\widetilde{\bigcap_{\substack{(F_i,G_i,E)\widetilde{\supseteq}(F,G,E)\\(F_i,G_i,E)^c\in\widetilde{\widetilde{\tau}}}}}(F_i,G_i,E)\right)^c=\widetilde{\bigcup}(F_i,G_i,E)^c=[(F,G,E)^c]^\circ.$$

□

Definition 3.24. Let $\left(X, \widetilde{\widetilde{\tau}}, E, \neg E\right)$ be a $BSTS$ over X and $\widetilde{\widetilde{B}} \subseteq \widetilde{\widetilde{\tau}}$. $\widetilde{\widetilde{B}}$ is said to be a bipolar soft basis for the bipolar soft topology $\widetilde{\widetilde{\tau}}$ if every element of $\widetilde{\widetilde{\tau}}$ can be written as the bipolar soft union of elements of $\widetilde{\widetilde{B}}$.

Theorem 3.25. Let $\left(X, \widetilde{\widetilde{\tau}}, E, \neg E\right)$ be a $BSTS$ over X and $\widetilde{\widetilde{B}}$ be a bipolar soft basis for $\widetilde{\widetilde{\tau}}$. Then, $\widetilde{\widetilde{\tau}}$ equals the collection of all bipolar soft unions of elements of $\widetilde{\widetilde{B}}$.

Proof. This is eaisly seen from the definition of bipolar soft basis. □

Example 3.26. Let us consider the bipolar soft topology over X that is given in Example 3.17. Then $\widetilde{\widetilde{B}} = \left\{\left(\Phi, \widetilde{X}, E\right), (F_1, G_1, E), (F_2, G_2, E), (F_3, G_3, E)\right\}$ is a bipolar soft basis for the bipolar soft topology $\widetilde{\widetilde{\tau}}$.

Definition 3.27. [8] Let (F, G, E) be a bipolar soft set over X and Y be a non-empty subset of X. Then the bipolar sub soft set of (F, G, E) over Y denoted by $({}^Y F, {}^Y G, E)$, is defined as follows

$$ {}^Y F(e) = Y \cap F(e) \text{ and } {}^Y G(\neg e) = Y \cap G(\neg e), \text{ for each } e \in E. $$

Proposition 3.28. [8] Let $\left(X, \widetilde{\widetilde{\tau}}, E, \neg E\right)$ be a $BSTS$ over X and Y be a non-empty subset of X. Then $\widetilde{\widetilde{\tau}}_Y = \left\{({}^Y F, {}^Y G, E) : (F, G, E) \in \widetilde{\widetilde{\tau}}\right\}$ is a bipolar soft topology on Y.

The collection $\widetilde{\widetilde{\tau}}_Y$ is called a bipolar soft subspace topology.

In the above Definition 3.25., Shabir and Bakhtawar have defined bipolar soft subspace according to universal subset $Y \subseteq X$. However, the following definition defines a bipolar soft subspace according to a bipolar soft set.

Theorem 3.29. Let $\left(X, \widetilde{\widetilde{\tau}}, E, \neg E\right)$ be a $BSTS$ over X and $(F, G, E) \widetilde{\subseteq} \left(\widetilde{X}, \Phi, E\right)$. Then the collection

$$ \widetilde{\widetilde{\tau}}_{(F,G,E)} = \left\{(F, G, E) \widetilde{\cap} (F_i, G_i, E) : (F_i, G_i, E) \in \widetilde{\widetilde{\tau}} \text{ for } i \in I\right\} $$

is a bipolar soft topology on (F, G, E) and $\left(X_{(F,G,E)}, \widetilde{\widetilde{\tau}}_{(F,G,E)}, E, \neg E\right)$ is a bipolar soft topological space.

Proof. Since $\left(\Phi, \widetilde{X}, E\right) \widetilde{\cap} (F, G, E) = \left(\Phi, \widetilde{X}, E\right)$ and $\left(\widetilde{X}, \Phi, E\right) \widetilde{\cap} (F, G, E) = (F, G, E)$, then $\left(\Phi, \widetilde{X}, E\right)$, $(F, G, E) \in \widetilde{\widetilde{\tau}}_{(F,G,E)}$.

Moreover,

$$ \widetilde{\bigcap_{i=1}^{n}} \left((F_i, G_i, E) \widetilde{\cap} (F, G, E)\right) = \left(\widetilde{\bigcap_{i=1}^{n}} (F_i, G_i, E)\right) \widetilde{\cap} (F, G, E) $$

and

$$\widetilde{\bigcup_{i\in I}}\left((F_i,G_i,E)\widetilde{\cap}(F,G,E)\right)=\left(\widetilde{\bigcup_{i\in I}}(F_i,G_i,E)\right)\widetilde{\cap}(F,G,E)$$

for $\widetilde{\widetilde{\tau}}=\{(F_i,G_i,E):i\in I\}$. Therefore, the bipolar soft union of any number of bipolar soft sets in $\widetilde{\widetilde{\tau}}_{(F,G,E)}$ belongs to $\widetilde{\widetilde{\tau}}_{(F,G,E)}$ and the finite bipolar soft intersection of bipolar soft sets in $\widetilde{\widetilde{\tau}}_{(F,G,E)}$ belongs to $\widetilde{\widetilde{\tau}}_{(F,G,E)}$. Hence, $\widetilde{\widetilde{\tau}}_{(F,G,E)}$ is a bipolar soft topology on (F,G,E). □

Definition 3.30. Let $\left(X,\widetilde{\widetilde{\tau}},E,\neg E\right)$ be a $BSTS$ over X and $(F,G,E)\widetilde{\subseteq}\left(\widetilde{X},\Phi,E\right)$. Then the collection

$$\widetilde{\widetilde{\tau}}_{(F,G,E)}=\left\{(F,G,E)\widetilde{\cap}(F_i,G_i,E):(F_i,G_i,E)\in\widetilde{\widetilde{\tau}}\text{ for }i\in I\right\}$$

is called a bipolar soft subspace topology on (F,G,E) and $\left(X_{(F,G,E)},\widetilde{\widetilde{\tau}}_{(F,G,E)},E,\neg E\right)$ is called a bipolar soft topological subspace of $\left(X,\widetilde{\widetilde{\tau}},E,\neg E\right)$.

Example 3.31. Let us consider the bipolar soft topology over X that is given in Example 3.17. and $(F,G,E)\widetilde{\subseteq}\left(\widetilde{X},\Phi,E\right)$ such that

$$(F,G,E)=\{(e_1,\{x_1,x_2,x_3\},\{x_4,x_5\}),(e_2,\{x_3,x_5\},\{x_2,x_4\})\ \ (e_3,\{x_2,x_4,x_5\},\{x_1\})\}.$$

Then the collection

$$\widetilde{\widetilde{\tau}}_{(F,G,E)}=\left\{\begin{array}{l}\left(\Phi,\widetilde{X},E\right)\widetilde{\cap}(F,G,E),\left(\widetilde{X},\Phi,E\right)\widetilde{\cap}(F,G,E),(F_1,G_1,E)\widetilde{\cap}(F,G,E),\\(F_2,G_2,E)\widetilde{\cap}(F,G,E),(F_3,G_3,E)\widetilde{\cap}(F,G,E),(F_4,G_4,E)\widetilde{\cap}(F,G,E)\end{array}\right\}$$

is a bipolar soft subspace topology on (F,G,E) and $\left(X_{(F,G,E)},\widetilde{\widetilde{\tau}}_{(F,G,E)},E,\neg E\right)$ is a bipolar soft topological subspace of $\left(X,\widetilde{\widetilde{\tau}},E,\neg E\right)$.

4 Conclusion

In this paper, we introduced some properties of bipolar soft topological spaces and the relationships between soft topological spaces and bipolar soft topological spaces. We hope that, the results of this study may help to next studies for many researchers.

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On Some Maps in Supra Topological Ordered Spaces

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Abstaract: In [6] the notion of supra semi open sets was presented and some of its properties were discussed. In this study, we introduce and investigate four main concepts namely supra continuous (supra open, supra closed, supra homeomorphism) maps via supra topological ordered spaces. Our findings in this work generalize some previous results in ([1], [13]). Many examples are considered to show the concepts introduced and main results obtained herein.

Keywords: *I(D,B)-supra semi continuous map, I(D,B)-supra semi open map, I(D,B)-supra semi homeomorphism map, Ordered supra semi separation axioms.*

2010 Mathematics Subject Classification: *54F05, 54F15.*

1 Introduction

Nachbin [18] in 1965, initiated the concept of topological ordered spaces and studied its main features. He also investigated the main properties of increasing and decreasing sets. Then McCartan [17], in 1968, carried out a detailed study on ordered separation axioms by utilizing the notions of increasing and decreasing neighborhoods. Mashhour et al. [16] generalized a topology notion to a supra topology and discussed some supra topological notions such as supra continuity and supra separation axioms. In 1991, Arya and Gupta [8] utilized semi open sets [15] to introduce semi separation axioms in topological ordered spaces. In 2002, Kumar [14] introduced and studied the concepts of continuity, openness, closedness and homeomorphism between topological ordered spaces. In 2004, Das [9] introduced and studied ordered separation axioms via some ordered spaces. In 2016, Abo-elhamayel and Al-shami [1] formulated the concepts of x-supra continuous, x-supra open, x-supra closed and x-supra homeomorphism maps in supra topological ordered spaces, for x= {I, D, B} and studied their properties. El-Shafei et al. [11] utilized the monotone open sets instead of monotone neighborhoods to present and investigate strong ordered separation axioms. They also used a notion of supra R-open sets [10] to define several kinds of maps [12] in topological ordered spaces. It is worth noting that the supra R-open sets except for the non-empty set were studied in topological spaces under the name of somewhere dense sets [4]. Recently, some studies on ordered maps via supra topological ordered spaces were done (see for example, [5], [7]).

The aim of the present paper is to establish some types of x-supra semi continuous, x-supra semi open, x-supra semi closed and x-supra semi homeomorphism maps in supra topological spaces, for x= {I, D, B}. Also, we give necessary and sufficient conditions for these maps and investigate under what conditions these maps preserve some separation axioms. Many of the findings that raised at are generalizations of those findings in supra topological ordered spaces which introduced in [1].

2 Preliminary

Hereinafter, several concepts and results of supra topological ordered spaces are recalled.

Definition 2.1. ([18], [1]) A triple $(X, \tau, \preceq)$ is called a topological ordered space, where (X, τ) is a topological space and $\preceq$ is a partial order relation on X. If we replace a topology τ by a supra topology μ, then a triple $(X, \mu, \preceq)$ is called a supra topological ordered space.

Remark 2.2. Throughout this paper, $(X, \tau, \preceq_1)$ and $(Y, \tau, \preceq_2)$ stand for topological ordered spaces and $(X, \mu, \preceq_1)$ and $(Y, \mu, \preceq_2)$ stand for supra topological ordered spaces. A diagonal relation is denoted by $\triangle$.

Definition 2.3. [18] Let $(X, \preceq)$ be a partially ordered set. Then:

(i) $i(b) = \{a \in X : b \preceq a\}$ and $d(b) = \{a \in X : a \preceq b\}$.

(ii) $i(B) = \bigcup\{i(b) : b \in B\}$ and $d(B) = \bigcup\{d(b) : b \in B\}$.

(iii) A set B is called increasing (resp. decreasing), if $A = i(A)(resp. A = d(A))$.

Definition 2.4. [14] A subset B of a partially ordered set $(X, \preceq)$ is called balancing if $B = i(B) = d(B)$.

Definition 2.5. [16] Let E be a subset of a supra topological space (X, μ). Then:

(i) Supra interior of E, denoted by $sint(E)$, is the union of all supra open sets contained in E.

(ii) Supra closure of E, denoted by $scl(E)$, is the intersection of all supra closed sets containing E.

Definition 2.6. [16]

(i) A map $g : (X, \tau) \to (Y, \theta)$ is said to be supra continuous if the inverse image of each open subset of Y is a supra open subset of X.

(ii) Let (X, τ) be a topological space and μ be a supra topology on X. We say that μ is associated supra topology with τ if $\tau \subseteq \mu$.

Definition 2.7. [6] A subset E of a supra topological space (X, μ) is called supra semi open if $E \subseteq scl(sint(E))$ and its complement is called supra semi closed.

Definition 2.8. [6] Let E be a subset of a supra topological space (X, μ). Then:

(i) Supra semi interior of E, denoted by $ssint(E)$, is the union of all supra semi open sets contained in E.

(ii) Supra semi closure of E, denoted by $sscl(E)$, is the intersection of all supra semi closed sets containing E.

Definition 2.9. [6] A map $g : (X, \tau) \to (Y, \theta)$ is said to be:

(i) Supra semi continuous if the inverse image of each open subset of Y is a supra semi open subset of X.

(ii) Supra semi open (resp. supra semi closed) if the image of each open (resp. closed) subset of X is a supra semi open (resp. supra semi closed) subset of Y.

(iii) Supra semi homeomorphism if it is bijective, supra semi continuous and supra semi open.

Definition 2.10. [1] A map $g : (X, \tau) \to (Y, \theta)$ is said to be supra open (resp. supra closed) if the image of any open (resp. closed) subset of X is a supra open (resp. supra closed) subset of Y.

Definition 2.11. A map $f : (X, \preceq_1) \to (Y, \preceq_2)$ is called:

(i) Order preserving (or increasing) if $a \preceq_1 b$, then $f(a) \preceq_2 f(b)$.

(ii) Order embedding if $a \preceq_1 b$ if and only if $f(a) \preceq_2 f(b)$.

Definition 2.12. [17] A topological ordered space $(X, \tau, \preceq)$ is called:

(i) Lower (Upper) strong T_1-ordered if for each $a, b \in X$ such that $a \npreceq b$, there exists an increasing (a decreasing) open set G containing $a(b)$ such that $b(a)$ belongs to G^c.

(ii) Strong T_1-ordered if it is strong lower T_1-ordered and strong upper T_1-ordered.

(iii) Strong T_0-ordered if it is strong lower T_1-ordered or strong upper T_1-ordered.

(iv) Strong T_2-ordered if for every $a, b \in X$ such that $a \npreceq b$, there exist disjoint open sets W_1 and W_2 containing a and b, respectively, such that W_1 is increasing and W_2 is decreasing.

Remark 2.13. In definition above, McCartan [17] named the above axioms, T_i-ordered spaces instead of strong T_i-ordered spaces if it is replaced the words open set by neighborhood.

Definition 2.14. [11] A supra topological ordered space $(X, \mu, \preceq)$ is called:

(i) Lower (Upper) SST_1-ordered if for each $a, b \in X$ such that $a \not\preceq b$, there exists an increasing (a decreasing) supra open set G containing $a(b)$ such that $b(a)$ belongs to G^c.

(ii) SST_1-ordered space if it is both lower SST_1-ordered and upper T_1-ordered space.

(iii) SST_0-ordered space if it is lower SST_1-ordered or upper SST_1-ordered space.

(iv) SST_2-ordered if for every $a, b \in X$ such that $a \not\preceq b$, there exist disjoint supra open sets W_1 and W_2 containing a and b, respectively, such that W_1 is increasing and W_2 is decreasing.

3 Supra Semi Continuous Maps in Supra Topological Ordered Spaces

The concepts of I-supra semi continuous, D-supra semi continuous and B-supra semi continuous maps in supra topological ordered spaces are presented and their main properties are investigated. The relationships among them are illustrated with the help of examples. The enough conditions for these three types of supra semi continuous maps to preserve some of ordered supra semi separation axioms are given.

Definition 3.1. A subset E of $(X, \mu, \preceq_1)$ is said to be:

(i) I-supra (resp. D-supra, B-supra) semi open if it is supra semi open and increasing (resp. decreasing, balancing).

(ii) I-supra (resp. D-supra, B-supra) semi closed if it is supra semi closed and increasing (resp. decreasing, balancing).

Definition 3.2. A map $f : (X, \mu, \preceq_1) \rightarrow (Y, \tau, \preceq_2)$ is called I-supra (resp. D-supra, B-supra) semi continuous at $p \in X$ if for each open set H containing $f(p)$, there exists an I-supra (resp. a D-supra, a B-supra) semi open set G containing p such that $f(G) \subseteq H$.
Also, the map is called I-supra (resp. D-supra, B-supra) semi continuous if it is I-supra (resp. D-supra, B-supra) semi continuous at each point $p \in X$.

Theorem 3.3. A map $f : (X, \mu, \preceq_1) \rightarrow (Y, \tau, \preceq_2)$ is I-supra (resp. D-supra, B-supra) semi continuous if and only if the inverse image of each open subset of Y is an I-supra (resp. a D-supra, a B-supra) semi open subset of X.

Proof. We only prove the theorem in case of f is an I-supra semi continuous map and the other follow similar lines.
To prove the necessary part, let G be an open subset of Y, Then we have the following two cases:

(i) $f^{-1}(G) = \emptyset$ which is an I-supra semi open subset of X.

(ii) $f^{-1}(G) \neq \emptyset$. By choosing $p \in X$ such that $p \in f^{-1}(G)$, we obtain that $f(p) \in G$. So there exists an I-supra semi open set H_p containing p such that $f(H_p) \subseteq G$. Since p is chosen arbitrary, then $f^{-1}(G) = \bigcup_{p \in f^{-1}(G))} H_p$. Thus $f^{-1}(G)$ is an I-supra semi open subset of X.

To prove the sufficient part, let G be an open subset of Y containing $f(p)$. Then $p \in f^{-1}(G)$. By hypothesis, $f^{-1}(G)$ is an I-supra semi open set. Since $f(f^{-1}(G)) \subseteq G$, then f is an I-supra semi continuous at $p \in X$ and since p is chosen arbitrary, then f is an I-supra semi continuous. □

Remark 3.4. **(i)** Every I-supra (D-supra, B-supra) semi continuous map is supra semi continuous.

(ii) Every B-supra semi continuous map is I-supra semi continuous and D-supra semi continuous.

The following two examples illustrate that a supra semi continuous (resp. an I-supra semi continuous) map need not be I-supra semi continuous or D-supra semi continuous or B-supra semi continuous (resp. B-supra semi continuous).

Example 3.5. Let the supra topology $\mu = \{\emptyset, X, \{a\}, \{b\}, \{a,b\}\}$ and the topology $\theta = \{\emptyset, Y, \{x\}\}$ on $X = \{a,b,c,d\}$ and $Y = \{x,y,z\}$, respectively. Let the partial order relation $\preceq_1 = \triangle \bigcup \{(a,b),(b,c),(a,c)\}$ on X and let the map $f : X \to Y$ be defined as follows $f(a) = f(c) = f(d) = x$, $f(b) = y$. Obviously, f is supra semi continuous. Now, $\{x\}$ is an open subset of Y, whereas $f^{-1}(\{x\}) = \{a,c,d\}$ is neither a decreasing nor an increasing supra semi open subset of X. Then f is not I-supra (D-supra, B-supra) semi continuous.

Example 3.6. We replace only the partial order relation in Example 3.5 by $\preceq = \triangle \bigcup \{(b,c)\}$. Then the map f is I-supra semi continuous, but not B-supra semi continuous.

The relationships among the introduced types of supra continuous maps are illustrated in the following figure.

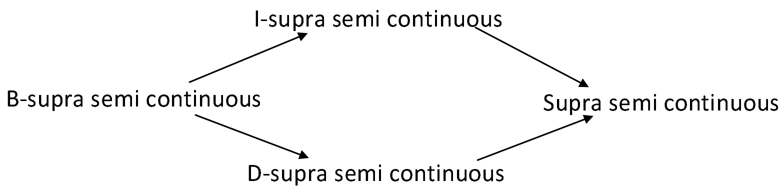

Figure 1: The relationships among types of supra continuous maps

Definition 3.7. Let E be a subset of $(X, \mu, \preceq)$. Then:

(i) $E^{isso} = \bigcup\{G : G$ is an I-supra semi open set included in $E\}$.

(ii) $E^{dsso} = \bigcup\{G : G$ is a D-supra semi open set included in $E\}$.

(iii) $E^{bsso} = \bigcup\{G : G$ is a B-supra semi open set included in $E\}$.

(iv) $E^{isscl} = \bigcap\{H : H$ is an I-supra semi closed set including $E\}$.

(v) $E^{dsscl} = \bigcap\{H : H$ is a D-supra semi closed set including $E\}$.

(vi) $E^{bsscl} = \bigcap\{H : H$ is a B-supra semi closed set including $E\}$.

Lemma 3.8. Let E be a subset of $(X, \mu, \preceq)$. Then:

(i) $((E)^{dsscl})^c = ((E)^c)^{isso}$.

(ii) $((E)^{isscl})^c = ((E)^c)^{dsso}$.

(iii) $((E)^{bsscl})^c = ((E)^c)^{bsso}$.

Proof. **(i)** $((E)^{dsscl})^c = \{\bigcup F : F$ is a D-supra semi closed set including $E\}^c$
$= \bigcap\{F^c : F^c$ is an I-supra semi open set included in $E^c\}$
$= ((E)^c)^{isso}$.

The proof of **(ii)** and **(iii)** is similar to that of **(i)**. □

Theorem 3.9. Let $g : (X, \mu, \preceq_1) \to (Y, \tau, \preceq_2)$ be a map. Then the following five statements are equivalent:

(i) g is I-supra semi continuous;

(ii) The inverse image of each closed subset of Y is a D-supra semi closed subset of X;

(iii) $(g^{-1}(H))^{dsscl} \subseteq g^{-1}(cl(H))$, for every $H \subseteq Y$;

(iv) $g(A^{dsscl}) \subseteq cl(g(A))$, for every $A \subseteq X$;

(v) $g^{-1}(int(H)) \subseteq (g^{-1}(H))^{isso}$, for every $H \subseteq Y$.

Proof. **(i)** ⇒ **(ii)** Consider H is a closed subset of Y. Then H^c is open. Therefore $g^{-1}(H^c) = (g^{-1}(H))^c$ is an I-supra semi open subset of X. So $g^{-1}(H)$ is D-supra semi closed.
(ii) ⇒ **(iii)** For any subset H of Y, we have that $cl(H)$ is closed. Since $g^{-1}(cl(H))$ is a D-supra semi closed subset of X, then $(g^{-1}(H))^{dsscl} \subseteq (g^{-1}(cl(H))^{dsscl} = g^{-1}(cl(H))$.
(iii) ⇒ **(iv)**: Consider A is a subset of X. Then $A^{dsscl} \subseteq (g^{-1}(g(A))^{dsscl} \subseteq g^{-1}(cl(g(A))$. Therefore $g(A^{dsscl}) \subseteq g(g^{-1}(cl(g(A))) \subseteq cl(g(A))$.
(iv) ⇒ **(v)**: Let H be a subset of Y. By Lemma (3.8), we obtain that $g(X - (g^{-1}(H))^{isso}) = g(((g^{-1}(H))^c)^{dsscl})$. By **(iv)** $g(((g^{-1}(H))^c)^{dsscl}) \subseteq cl(g(g^{-1}(H))^c) = cl(g(g^{-1}(H^c))) \subseteq cl(Y - H) = Y - int(H)$. Therefore $X - (g^{-1}(H))^{isso} \subseteq g^{-1}(Y - int(H)) = X - g^{-1}(int(H))$. Thus $g^{-1}(int(H)) \subseteq (g^{-1}(H))^{isso}$.
(v) ⇒ **(i)**: Consider H is an open subset of Y. Then $g^{-1}(H) = g^{-1}(int(H)) \subseteq (g^{-1}(H))^{isso}$. Since $g^{-1}(H)$ is I-supra semi open, then $(g^{-1}(H))^{isso} \subseteq g^{-1}(H)$. Therefore $g^{-1}(H)$ is an I-supra semi open subset of X. Thus g is I-supra semi continuous. □

Theorem 3.10. Let $g : (X, \mu, \preceq_1) \to (Y, \tau, \preceq_2)$ be a map. Then the following five statements are equivalent:

(i) g is D-supra semi continuous;

(ii) The inverse image of each closed subset of Y is an I-supra semi closed subset of X;

(iii) $(g^{-1}(H))^{isscl} \subseteq g^{-1}(cl(H))$, for every $H \subseteq Y$;

(iv) $g(A^{isscl}) \subseteq cl(g(A))$, for every $A \subseteq X$;

(v) $g^{-1}(int(H)) \subseteq (g^{-1}(H))^{dsso}$, for every $H \subseteq Y$.

Proof. The proof is similar to that of Theorem (3.9). □

Theorem 3.11. Let $g : (X, \mu, \preceq_1) \to (Y, \tau, \preceq_2)$ be a map. Then the following five statements are equivalent:

(i) g is B-supra semi continuous;

(ii) The inverse image of each closed subset of Y is a B-supra semi closed subset of X;

(iii) $(g^{-1}(H))^{bsscl} \subseteq g^{-1}(cl(H))$, for every $H \subseteq Y$;

(iv) $g(A^{bsscl}) \subseteq cl(g(A))$, for every $A \subseteq X$;

(v) $g^{-1}(int(H)) \subseteq (g^{-1}(H))^{bsso}$, for every $H \subseteq Y$.

Proof. The proof is similar to that of Theorem (3.9). □

Definition 3.12. A supra topological ordered space $(X, \mu, \preceq)$ is called:

(i) Lower (Upper) strong supra semi T_1-ordered (briefly, Lower (Upper) $SSST_1$-ordered) if for each $a, b \in X$ such that $a \not\preceq b$, there exists an increasing (a decreasing) supra semi open set G containing $a(b)$ such that $b(a)$ belongs to G^c.

(ii) $SSST_0$-ordered space if it is lower $SSST_1$-ordered or upper $SSST_1$-ordered.

(iii) $SSST_1$-ordered space if it is both lower $SSST_1$-ordered and upper $SSST_1$-ordered.

(iv) $SSST_2-$ordered if for every $a, b \in X$ such that $a \not\preceq b$, there exist disjoint supra semi open sets W_1 and W_2 containing a and b, respectively, such that W_1 is increasing and W_2 is decreasing.

Theorem 3.13. Let a bijective map $f : (X, \mu, \preceq_1) \to (Y, \tau, \preceq_2)$ be I-supra semi continuous and f^{-1} be an order preserving map. If $(Y, \tau, \preceq_2)$ is a lower T_1-ordered space, then $(X, \mu, \preceq_1)$ is a lower $SSST_1$-ordered space.

Proof. Let $a, b \in X$ such that $a \not\preceq_1 b$. Then there exist $x, y \in Y$ such that $x = f(a), y = f(b)$. Since f^{-1} is an order preserving map, then $x \not\preceq_2 y$. Since $(Y, \tau, \preceq_2)$ is a lower T_1-ordered space, then there exists an increasing neighborhood W of x in Y such that $x \in W$ and $y \notin W$. Therefore there exists an open set G such that $x \in G \subseteq W$. Since f is bijective I-supra semi continuous, then $a \in f^{-1}(G)$ which is I-supra semi open and $b \notin f^{-1}(G)$. Thus $(X, \mu, \preceq_1)$ is a lower $SSST_1$-ordered space. □

Theorem 3.14. Let a bijective map $f : (X, \mu, \preceq_1) \to (Y, \tau, \preceq_2)$ be D-supra semi continuous and f^{-1} be an order preserving map. If $(Y, \tau, \preceq_2)$ is an upper T_1-ordered space, then $(X, \mu, \preceq_1)$ is an upper $SSST_1$-ordered space.

Proof. The proof is similar to that of Theorem (3.16). □

Theorem 3.15. Let a bijective map $f : (X, \mu, \preceq_1) \to (Y, \tau, \preceq_2)$ be B-supra semi continuous and f^{-1} be an order preserving map. If $(Y, \tau, \preceq_2)$ is a T_i-ordered space, then $(X, \mu, \preceq_1)$ is an $SSST_1$-ordered space for $i = 0, 1, 2$.

Proof. We prove the theorem in case of $i = 2$. Let $a, b \in X$ such that $a \not\preceq_1 b$. Then there exist $x, y \in Y$ such that $x = f(a)$ and $y = f(b)$. Since f^{-1} is an order preserving map, then $x \not\preceq_2 y$. Since $(Y, \tau, \preceq_2)$ is a T_2-ordered space, then there exist disjoint balancing neighborhoods W_1 and W_2 of x and y, respectively. Therefore there are disjoint open sets G and H containing x and y, respectively. Since f is bijective B-supra semi continuous, then $a \in f^{-1}(G)$ which is an I-supra semi open subset of X, $b \in f^{-1}(H)$ which is a D-supra semi open subset of X and $f^{-1}(G) \bigcap f^{-1}(H) = \emptyset$. Thus $(X, \mu, \preceq_1)$ is a $SSST_2$-ordered space.

In a similar way, we can prove the theorem in case of $i = 0, 1$. □

Theorem 3.16. Consider $f : (X, \mu, \preceq_1) \to (Y, \tau, \preceq_2)$ is a bijective supra semi continuous map such that f is ordered embedding. If $(Y, \tau, \preceq_2)$ is strong T_i-ordered, then $(X, \mu, \preceq_1)$ is $SSST_i$-ordered, for $i = 0, 1, 2$.

Proof. We prove the theorem in case of $i = 2$. Let $a, b \in X$ such that $a \not\preceq_1 b$. Then there exist $x, y \in Y$ such that $x = f(a)$ and $y = f(b)$. Since f is ordered embedding, then $x \not\preceq_2 y$. Since $(Y, \tau, \preceq_2)$ is strong T_2-ordered, then there exist disjoint open sets W_1 and W_2 containing x and y, respectively, such that W_1 is increasing and W_2 is decreasing. Since f is bijective supra semi continuous and order preserving, then $f^{-1}(W_1)$ is an I-supra semi open set containing a, $f^{-1}(W_2)$ is a D-supra semi open set containing b and $f^{-1}(W_1) \bigcap f^{-1}(W_2) = \emptyset$. Thus $(X, \mu, \preceq_1)$ is $SSST_2$-ordered.
Similarly, one can prove theorem in case of $i = 0, 1$. □

Theorem 3.17. Consider $f : (X, \mu, \preceq_1) \to (Y, \tau, \preceq_2)$ is an injective B-supra semi continuous map. If $(Y, \tau, \preceq_2)$ is a T_i-space, then $(X, \mu, \preceq_1)$ is an $SSST_i$-ordered space, for $i = 1, 2$.

Proof. We prove the theorem in case of $i = 2$ and the other case is similar. Let $a, b \in X$ such that $a \not\preceq_1 b$. Then there exist $x, y \in Y$ such that $f(a) = x, f(b) = y$ and $x \neq y$. Since $(Y, \tau, \preceq_2)$ is a T_2-space, then there exist disjoint open sets G and H such that $x \in G$ and $y \in H$. Therefore $a \in f^{-1}(G)$ which is an I-supra semi open subset of X, $b \in f^{-1}(H)$ which is a D-supra semi open subset of X and $f^{-1}(G) \bigcap f^{-1}(H) = \emptyset$. Thus $(X, \mu, \preceq_1)$ is an $SSST_2$-ordered space.
Similarly, one can prove the theorem in case of $i = 1$. □

4 Supra Semi Open (Supra Semi Closed) Maps in Supra Topological Ordered Spaces

In this section, we introduce the concepts of I-supra semi open (I-supra semi closed), D-supra semi open (D-supra semi closed) and B-supra semi open (B-supra semi closed) maps in supra topological ordered spaces. We demonstrate their main properties and illustrate the relationships among them with the help of examples. Finally, some results concerning the image and per image of some separation axioms under these maps are presented.

Definition 4.1. A map $g : (X, \tau, \preceq_1) \to (Y, \mu, \preceq_2)$ is said to be:

(i) I-supra (resp. D-supra, B-supra) semi open if the image of any open subset of X is an I-supra (resp. a D-supra, a B-supra) semi open subset of Y.

(ii) I-supra (resp. D-supra, B-supra) semi closed if the image of any closed subset of X is an I-supra (resp. a D-supra, a B-supra) semi closed subset of Y.

Remark 4.2. (i) Every I-supra (D-supra, B-supra) semi open map is supra semi open.

(ii) Every I-supra (D-supra, B-supra) semi closed map is supra semi closed.

(iii) Every B-supra semi open (resp. B-supra semi closed) map is I-supra semi open and D-supra semi open (resp. I-supra semi closed and D-supra semi closed).

The following two examples illustrate that a supra semi open (resp. D-supra semi open) map need not be I-supra semi open or D-supra semi open or B-supra semi open (resp. B-supra semi open).

Example 4.3. Let the topology $\tau = \{\emptyset, X, \{1, 2\}\}$ and the partial order relation $\preceq_2 = \triangle \bigcup \{(1, 3), (3, 2), (1, 2)\}$ on $X = \{1, 2, 3\}$. Let the supra topology associated with τ be $\{\emptyset, X, \{1, 2\}, \{1, 3\}\}$ on X. The identity map $f : (X, \tau) \to (X, \mu, \preceq_2)$ is a supra semi open map. Now, $\{1, 2\}$ is an open subset of X. Since $f(\{1, 2\}) = \{1, 2\}$ is neither an increasing nor a decreasing supra semi open subset of Y, then f is not x-supra semi open map, for x={I, D, B}.

Example 4.4. We replace only the partial order relation in Example (4.3) by $\preceq = \triangle \bigcup \{(1, 3),\ (2, 3)\}$. Then the map f is D-supra semi open, but is not B-supra semi open.

The following two examples illustrate that a supra semi closed (resp. an I-supra semi closed) map need not be I-supra semi closed or D-supra semi closed or B-supra semi closed (resp. B-supra semi closed).

Example 4.5. Let the topology $\tau = \{\emptyset, X, \{a, b\}\}$ on $X = \{a, b, c\}$, the supra topology associated with τ be $\{\emptyset, X, \{c\}, \{a, b\}\}$ and the partial order relation $\preceq_2 = \triangle \bigcup \{(a, c), (c, b), (a, b)\}$ on X. The map $f : (X, \tau) \to (X, \mu, \preceq_2)$ is defined as follows $f(a) = f(c) = c$ and $f(b) = b$. Obviously, f is supra semi closed. Now, $\{c\}$ is a closed subset of X, but $f(\{c\}) = \{c\}$ is neither a decreasing nor an increasing supra semi closed subset of Y. Then f is not x-supra semi closed map, for x={I, D, B}..

Example 4.6. We replace only the partial order relation in Example (4.5) by $\preceq = \triangle \bigcup \{(b, c)\}$. Then the map f is I-supra semi closed, but is not B-supra semi closed.

The relationships among the introduced types of supra semi open (supra semi closed) maps are illustrated in the following figure.

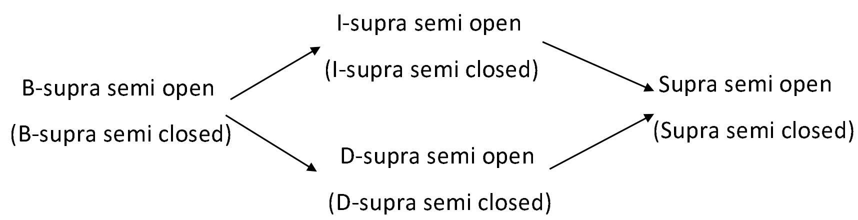

Figure 2: The relationships among types of supra open (supra closed) maps

Theorem 4.7. The following statements are equivalent, for a map $f : (X, \tau, \preceq_1) \to (Y, \mu, \preceq_2)$:

(i) f is I-supra semi open;

(ii) $int(f^{-1}(H)) \subseteq f^{-1}(H^{isso})$, for every $H \subseteq Y$;

(iii) $f(int(G)) \subseteq (f(G))^{isso}$, for every $G \subseteq X$.

Proof. **(i)** $\Rightarrow$ **(ii)**: Since $int(f^{-1}(H))$ is an open subset of X, then $f(int(f^{-1}(H)))$ is an I-supra semi open subset of Y. Since $f(int(f^{-1}(H))) \subseteq f(f^{-1}(H)) \subseteq H$, then $int(f^{-1}(H)) \subseteq f^{-1}(H^{isso})$.
(ii) $\Rightarrow$ **(iii)**: By replacing H by $f(G)$ in **(ii)**, we obtain that $int(f^{-1}(f(G))) \subseteq f^{-1}((f(G))^{isso})$. Since $int(G) \subseteq f^{-1}(f(int(f^{-1}(f(G))))) \subseteq f^{-1}((f(G))^{isso})$, then $f(int(G)) \subseteq (f(G))^{isso}$.
(iii) $\Rightarrow$ **(i)**: Let G be an open subset of X. Then $f(int(G)) = f(G) \subseteq (f(G))^{isso}$. So f is an I-supra semi open map. □

In a similar way, one can prove the following two theorems.

Theorem 4.8. The following statements are equivalent, for a map $f : (X, \tau, \preceq_1) \to (Y, \mu, \preceq_2)$:

(i) f is D-supra semi open;

(ii) $int(f^{-1}(H)) \subseteq f^{-1}(H^{dsso})$, for every $H \subseteq Y$;

(iii) $f(int(G)) \subseteq (f(G))^{dsso}$, for every $G \subseteq X$.

Theorem 4.9. The following statements are equivalent, for a map $f : (X, \tau, \preceq_1) \to (Y, \mu, \preceq_2)$:

(i) f is B-supra semi open;

(ii) $int(f^{-1}(H)) \subseteq f^{-1}(H^{bsso})$, for every $H \subseteq Y$;

(iii) $f(int(G)) \subseteq (f(G))^{bsso}$, for every $G \subseteq X$.

Theorem 4.10. Let $f : (X, \tau, \preceq_1) \to (Y, \mu, \preceq_2)$ be a map. Then we have the following results.

(i) f is I-supra semi closed if and only if $(f(G))^{isscl} \subseteq f(cl(G))$, for any $G \subseteq X$.

(ii) f is D-supra semi closed if and only if $(f(G))^{dsscl} \subseteq f(cl(G))$, for any $G \subseteq X$.

(iii) f is B-supra semi closed if and only if $(f(G))^{bsscl} \subseteq f(cl(G))$, for any $G \subseteq X$.

Proof. **(i)** Necessity: Consider f is an I-supra semi closed map. Then $f(cl(G))$ is an I-supra semi closed subset of Y. Since $f(G) \subseteq f(cl(G))$, then $(f(G))^{isscl} \subseteq f(cl(G))$. Sufficiency: Consider B is a closed subset of X. Then $f(B) \subseteq (f(B))^{isscl} \subseteq f(cl(B)) = f(B)$. Therefore $f(B) = (f(B))^{isscl}$ is an I-supra semi closed set. Thus f is an I-supra semi closed map.
The proof of **(ii)** and **(iii)** is similar to that of **(i)**. □

Theorem 4.11. Let $f : (X, \tau, \preceq_1) \to (Y, \mu, \preceq_2)$ be a bijective map. Then we have the following results.

(i) f is I-supra semi open if and only if f is D-supra semi closed.

(ii) f is D-supra semi open if and only if f is I-supra semi closed.

(iii) f is B-supra semi open if and only if f is B-supra semi closed.

Proof. **(i)** Necessity: Let f be an I-supra semi open map and let G be a closed subset of X. Then G^c is open. Since f is bijective, then $f(G^c) = (f(G))^c$ is I-supra semi open. Therefore $f(G)$ is a D-supra semi closed subset of Y. Thus f is D-supra semi closed.
Sufficiency: Let f be a D-supra semi closed map and let B be an open subset of X. Then B^c is closed. Since f is bijective, then $f(B^c) = (f(B))^c$ is D-supra semi closed. Therefore $f(B)$ is I-supra semi open. Thus f is I-supra semi closed.
The proof of **(ii)** and **(iii)** is similar to that of **(i)**. □

Theorem 4.12. The following two statements hold.

(i) If the maps $f : (X, \tau, \preceq_1) \to (Y, \theta, \preceq_2)$ is open and $g : (Y, \theta, \preceq_2) \to (Z, \nu, \preceq_3)$ is I-supra (resp. D-supra, B-supra) semi open, then a map $g \circ f$ is I-supra (resp. D-supra, B-supra) semi open.

(ii) If the maps $f : (X, \tau, \preceq_1) \to (Y, \theta, \preceq_2)$ is closed and $g : (Y, \theta, \preceq_2) \to (Z, \nu, \preceq_3)$ is I-supra (resp. D-supra, B-supra) semi closed, then a map $g \circ f$ is I-supra (resp. D-supra, B-supra) semi closed.

Proof. It is clear. □

Theorem 4.13. If the maps $g \circ f$ is I-supra (resp. D-supra, B-supra) semi open and $f : (X, \tau, \preceq_1) \to (Y, \theta, \preceq_2)$ is surjective continuous, then a map $g : (Y, \theta, \preceq_2) \to (Z, \nu, \preceq_3)$ is I-supra (resp. D-supra, B-supra) semi open.

Proof. Consider $g \circ f$ is I-supra semi open and let G be an open subset of Y. Then $f^{-1}(G)$ is an open subset of X. Since $g \circ f$ is I-supra semi open and f is surjective, then $(g \circ f)(f^{-1}(G)) = g(G)$ is an I-supra semi open subset of Z. Therefore g is I-supra semi open.

A similar proof can be given for the cases between parentheses. □

Theorem 4.14. If the maps $g \circ f : (X, \tau, \preceq_1) \to (Z, \mu, \preceq_3)$ is closed and $g : (Y, \theta, \preceq_2) \to (Z, \mu, \preceq_3)$ is I-supra (resp. D-supra, B-supra) semi continuous injective, then a map $f : (X, \tau, \preceq_1) \to (Y, \theta, \preceq_2)$ is D-supra (resp. I-supra, B-supra) semi closed.

Proof. Consider g is I-supra semi continuous. Let G be a closed subset of X. Then $(g \circ f)(G)$ is a closed subset of Z. Since g is injective and I-supra semi continuous, then $g^{-1}(g \circ f)(G) = f(G)$ is a D-supra semi closed subset of Y. Therefore f is D-supra semi closed.

A similar proof can be given for the cases between parentheses. □

Theorem 4.15. We have the following results for a bijective map $f : (X, \tau, \preceq_1) \to (Y, \theta, \preceq_2)$.

(i) f is I-supra (resp. D-supra, B-supra) semi open if and only if f^{-1} is I-supra (resp. D-supra, B-supra) semi continuous.

(ii) f is D-supra (resp. I-supra, B-supra) semi closed if and only if f^{-1} is I-supra (resp. D-supra, B-supra) semi continuous.

Proof. **(i)** We prove **(i)** when f is B-supra semi open, and the other cases follow similar lines.

$' \Rightarrow '$ Let f be a B-supra semi open map and let G be an open subset of X. Then $(f^{-1})^{-1}(G) = f(G)$ is a B-supra semi open subset of Y. Therefore f^{-1} is a B-supra semi continuous.

$' \Leftarrow '$ let G be an open subset of X and f^{-1} be a B-supra semi continuous. Then $f(G) = (f^{-1})^{-1}(G)$ is a B-supra semi open subset of Y. Therefore f is B-supra semi open.

(ii) Similarly, one can prove **(ii)**.

□

Theorem 4.16. Let a bijective map $f : (X, \tau, \preceq_1) \to (Y, \mu, \preceq_2)$ be I-supra semi open (D-supra semi closed) and order preserving. If $(X, \tau, \preceq_1)$ is a lower T_1-ordered space, then $(Y, \mu, \preceq_2)$ is a lower $SSST_1$-ordered space.

Proof. We prove the theorem when a map f be I-supra semi open.
Let $x, y \in Y$ such that $x \npreceq_2 y$. Since f is bijective, then there exist $a, b \in X$ such that $a = f^{-1}(x)$ and $b = f^{-1}(y)$ and since f is an order preserving map, then $a \npreceq_1 b$. By hypotheses $(X, \tau, \preceq_1)$ is a lower T_1-ordered space, then there exists an increasing neighborhood W in X such that $a \in W$ and $b \notin W$. Therefore there exists an open set G such that $a \in G \subseteq W$. Thus $x \in f(G)$ which is an I-supra semi open and $y \notin f(G)$. Hence $(Y, \mu, \preceq_2)$ is a lower $SSST_1$-ordered space.
The proof for a D-supra semi closed map is achieved similarly. □

Theorem 4.17. Let a bijective map $f : (X, \tau, \preceq_1) \rightarrow (Y, \mu, \preceq_2)$ be D-supra semi open (I-supra semi closed) and order preserving. If $(X, \tau, \preceq_1)$ is an upper T_1-ordered space, then $(Y, \mu, \preceq_2)$ is an upper $SSST_1$-ordered space.

Proof. The proof is similar to that of Theorem (4.16). □

Theorem 4.18. Let a bijective map $f : (X, \tau, \preceq_1) \rightarrow (Y, \mu, \preceq_2)$ be B-supra semi open (B-supra semi closed) and order preserving. If $(X, \tau, \preceq_1)$ is a T_i-ordered space, then $(Y, \mu, \preceq_2)$ is an $SSST_i$-ordered space for $i = 0, 1, 2$.

Proof. When a map f is B-supra semi open and $i = 2$.
For all $x, y \in Y$ such that $x \npreceq_2 y$, there are $a, b \in X$ such that $a = f^{-1}(x), b = f^{-1}(y)$. Since f is an order preserving, then $a \npreceq_1 b$. Since $(X, \tau, \preceq_1)$ is a T_2-ordered space, then there exist disjoint neighborhoods W_1 and W_2 of a and b, respectively, such that W_1 is increasing and W_2 is decreasing. Therefore there are disjoint open sets G and H such that $a \in G \subseteq W_1$ and $b \in H \subseteq W_2$. Thus $x \in f(G)$ which is a balancing supra semi open, $y \in f(H)$ which is a balancing supra semi open and $f(G) \bigcap f(H) = \emptyset$. Thus $(Y, \mu, \preceq_2)$ is an $SSST_2$-ordered space.
In a similar way, we can prove the theorem in case of $i = 0, 1$.
The proof for a B-supra semi closed map is achieved similarly. □

Theorem 4.19. Consider a bijective map $f : (X, \tau, \preceq_1) \rightarrow (Y, \mu, \preceq_2)$ is supra semi open such that f and f^{-1} are order preserving. If $(X, \tau, \preceq_1)$ is strong T_i-ordered, then $(Y, \mu, \preceq_2)$ is $SSST_i$-ordered, for $i = 0, 1, 2$.

Proof. We prove the theorem in case of $i = 2$. Let $x, y \in Y$ such that $x \npreceq_2 y$. Then there exist $a, b \in X$ such that $a = f^{-1}(x)$ and $b = f^{-1}(y)$. Since f is an order preserving, then $a \npreceq_1 b$. Since $(X, \tau, \preceq_1)$ is strong T_2-ordered space, then there exist disjoint an increasing open set W_1 containing a and a decreasing open set W_2 containing b such that $a \in W_1$ and $b \in W_2$. Since f is a bijective supra semi open and f^{-1} is an order preserving, then $f(W_1)$ is an I-supra semi open set containing x, $f(W_2)$ is a D-supra semi open set containing y and $f(W_1) \bigcap f(W_2) = \emptyset$. Therefore $(Y, \mu, \preceq_2)$ is $SSST_2$-ordered.

Similarly, one can prove theorem in case of $i = 0, 1$. □

Theorem 4.20. Let $f : (X, \tau, \preceq_1) \rightarrow (Y, \mu, \preceq_2)$ be a bijective supra open map such that f and f^{-1} are order preserving. If $(X, \tau, \preceq_1)$ is strong T_i-ordered, then $(Y, \mu, \preceq_2)$ is $SSST_i$-ordered, for $i = 0, 1, 2$.

Proof. The proof is similar to that of Theorem (4.19). □

5 Supra Semi Homeomorphism Maps in Supra Topological Ordered Spaces

The concepts of I-supra semi homeomorphism, D-supra semi homeomorphism and B-supra semi homeomorphism maps are introduced and many of their properties are established. Some illustrative examples are provided.

Definition 5.1. Let $\tau^\star$ and $\theta^\star$ be associated supra topologies with τ and θ, respectively. A bijective map $g : (X, \tau, \preceq_1) \to (Y, \theta, \preceq_2)$ is called I-supra (resp. D-supra, B-supra) semi homeomorphism if it is I-supra semi continuous and I-supra semi open (resp. D-supra semi continuous and D-supra semi open, B-supra semi continuous and B-supra semi open).

Remark 5.2. (i) Every I-supra (D-supra, B-supra) semi homeomorphism map is supra semi homeomorphism.

(ii) Every B-supra semi homeomorphism map is I-supra semi homeomorphism and D-supra semi homeomorphism.

The following two examples illustrate that a supra semi homeomorphism (resp. D-supra semi homeomorphism) map need not be I-supra semi homeomorphism or D-supra semi homeomorphism or B-supra semi homeomorphism (resp. B-supra semi homeomorphism).

Example 5.3. Let the topology $\tau = \{\emptyset, X, \{a, c\}\}$ on $X = \{a, b, c\}$, the supra topology associated with τ be $\{\emptyset, X, \{a\}, \{a, c\}\}$ and the partial order relation $\preceq_1 = \triangle \bigcup \{(c, a), (c, b)\}$. Let the topology $\theta = \{\emptyset, Y, \{y, z\}\}$ on $Y = \{x, y, z\}$, the supra topology associated with θ be $\{\emptyset, Y, \{y\}, \{y, z\}\}$ and the partial order relation $\preceq_2 = \triangle \bigcup \{(y, z)\}$ on Y. The map $f : (X, \tau, \preceq_1) \to (Y, \theta, \preceq_2)$ is defined as $f(a) = y,\ f(b) = z$ and $f(c) = x$. Now, f is supra semi homeomorphism, but is not x-supra semi homeomorphism, for $x = \{\text{I,D,B}\}$.

Example 5.4. We replace only the partial order relation $\preceq_1$ in Example (5.3) by $\preceq = \triangle \bigcup \{(a, c)\}$. Then the map f is D-supra semi homeomorphism, but not B-supra semi homeomorphism.

The relationships among the presented types of supra semi homeomorphism maps are illustrated in the following figure.

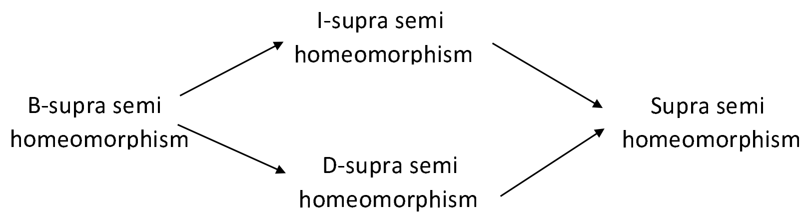

Figure 3: The relationships among types of supra homeomorphism maps

Theorem 5.5. Let a map $f : X \to Y$ be bijective and I-supra semi continuous. Then the following statements are equivalent:

(i) f is I-supra semi homeomorphism;

(ii) f^{-1} is I-supra semi continuous;

(iii) f is D-supra semi closed.

Proof. **(i) ⇒ (ii)** Let G be an open subset of X. Then $(f^{-1})^{-1}(G) = f(G)$ is an I-supra semi open set in Y. Therefore f^{-1} is I-supra semi continuous.
(ii) ⇒ (iii) Let G be a closed subset of X. Then G^c is an open subset of X and $(f^{-1})^{-1}(G^c) = f(G^c) = (f(G))^c$ is an I-supra semi open set in Y. Therefore $f(G)$ is a D-supra semi closed subset of Y. Thus f is D-supra semi closed.
(iii) ⇒ (i) Let G be an open subset of X. Then G^c is a closed set and $f(G^c) = (f(G))^c$ is D-supra semi closed. Therefore $f(G)$ is an I-supra semi open subset of Y. Thus f is I-supra semi open. Hence f is an I-supra semi homeomorphism map. □

In a similar way one can prove the following two theorems.

Theorem 5.6. Let a map $f : X \to Y$ be bijective and D-supra semi continuous. Then the following statements are equivalent:

(i) f is D-supra semi homeomorphism;

(ii) f^{-1} is D-supra semi continuous;

(iii) f is I-supra semi closed.

Theorem 5.7. Let a map $f : X \to Y$ be bijective and B-supra semi continuous. Then the following statements are equivalent:

(i) f is B-supra semi homeomorphism;

(ii) f^{-1} is B-supra semi continuous;

(iii) f is B-supra semi closed.

Theorem 5.8. Consider $(X, \tau, \preceq_1)$ and $(Y, \theta, \preceq_2)$ are two topological ordered spaces, and $\tau^\star$ and $\theta^\star$ are associated supra topologies with τ and θ, respectively. Let $f : X \to Y$ be a supra semi homeomorphism map such that f and f^{-1} are order preserving. If $X(resp.\ Y)$ is strong T_i-ordered, then $Y(resp.\ X)$ is $SSST_i$-ordered, for $i = 0, 1, 2$.

Proof. **(i)** Let $(X, \tau, \preceq_1)$ be a strong T_i-ordered space, then by Theorem (4.19), $(Y, \theta, \preceq_2)$ is an $SSST_i$-ordered space, for $i = 0, 1, 2$.

(ii) Let $(Y, \theta, \preceq_2)$ be a strong T_i-ordered space, then by Theorem (3.16), $(X, \tau, \preceq_1)$ is an $SSST_i$-ordered space, for $i = 0, 1, 2$.

□

Conclusion

In the present paper, the concepts of I-supra (D-supra, B-supra) semi continuous, I-supra (D-supra, B-supra) semi open, I-supra (D-supra, B-supra) closed and I-supra (D-supra, B-supra) semi homeomorphism maps are given and studied. The sufficient conditions for maps to preserve some separation axioms (which introduced in [9], [11] and [17]) are determined. In particular, we investigate the equivalent conditions for each concept and present their properties. Apart from that, we point out the relationships among them with the help of illustrative examples. In the end, the presented concepts in this paper are fundamental background for studying several topics in supra topological ordered spaces.

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On Path Laplacian Eigenvalues and Path Laplacian Energy of Graphs

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Abstaract − We introduce the concept of Path Laplacian Matrix for a graph and explore the eigenvalues of this matrix. The eigenvalues of this matrix are called the path Laplacian eigenvalues of the graph. We investigate path Laplacian eigenvalues of some classes of graph. Several results concerning path Laplacian eigenvalues of graphs have been obtained.

Keywords − *Path, Real symmetric matrix, Laplacian matrix.*

1 Introduction

For a graph G the eigenvalues of G are the eigenvalues of its adjacency matrix. The spectrum of of a graph G is the set of its eigenvalues. Several properties and applications of eigenvalues of graph are useful. For undefined terminology and notations we refer to Lowel W. Beineke [1] and West [2]. For an extensive survey on graph spectra we refer to R. B. Bapat [3], Brouwer A. E. [4] and Verga R. S. [5].

We have defined the path matrix [6, 7] of the graph G as follows. Let G be a graph without loops and let $V(G) = \{v_1, v_2, ..., v_n\}$ be the vertex set of G.
Define the matrix $P = (p_{ij})$ of size $n \times n$ such that

$$p_{ij} = \begin{cases} \text{maximum number of vertex disjoint paths from } v_i \text{ to } v_j & \text{if } i \neq j \\ 0 & \text{if } i = j \end{cases}$$

We call P as Path Matrix of G. The matrix P is real symmetric matrix. Therefore, its eigenvalues are real. We call eigenvalues of P as path eigenvalues of G.

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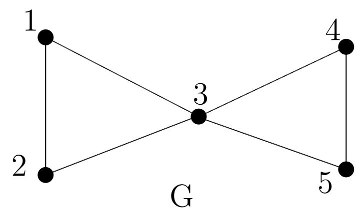
2 Preliminary

We define the path Laplacian matrix of G, $PL(G)$ as follows.

Definition 2.1. The rows and columns of $PL(G)$ are indexed by $V(G)$. If $i \neq j$ then the (i, j)- entry of $PL(G)$ is 0 if there is no path between i and j, and it is $-k$ if the maximum number of vertex disjoint paths between i and j is k. The (i, i) entry of $PL(G)$ is d_i, the degree of the vertex i, $i = 1, 2, 3, ..., n$.

Thus $PL(G)$ is an $n \times n$ matrix. The path Laplacian matrix of G can be defined in an alternative way. Let $D(G)$ be the diagonal matrix of vertex degrees. If $P(G)$ is the path matrix of G, then $PL(G) = D(G) - P(G)$. We call the path eigenvalues of $PL(G)$ as path Laplacian eigenvalues of G.

Example 2.2. Consider the graph G as shown in the following figure.



Then the path Laplacian matrix of G is

$$\mathbf{PL}(G) = \begin{bmatrix} 2 & -2 & -2 & -1 & -1 \\ -2 & 2 & -2 & -1 & -1 \\ -2 & -2 & 4 & -2 & -2 \\ -1 & -1 & -2 & 2 & -2 \\ -1 & -1 & -2 & -2 & 2 \end{bmatrix}.$$

The characteristic polynomial of the matrix $PL(G)$ is
$C_{PL(G)}(x) = |PL - xI| = (x+4)(x-2)(x-4)^2(x-6)$. The path Laplacian eigenvalues of G are $-4, 2, 4, 4$ and 6. The ordinary Laplacian eigenvalues of G are $0, 1, 3, 3$ and 5.

The ordinary Laplacian spectrum of the graph G, consisting of the numbers $\mu_1, \mu_2, ..., \mu_n$ is the spectrum of its Laplacian matrix [8, 9, 10, 11]. In analogy, the path Laplacian spectrum of a graph G is defined as the spectrum of the corresponding path Laplacian matrix.

3 Path Laplacian Eigenvalues of Graphs

In this section, we investigate path Laplacian eigenvalues of some special classes of graphs. In this paper, we define path Laplacian matrix of a graph and investigate the eigenvalues (called path Laplacian eigenvalues) of this matrix. We obtain several properties concerning the path Laplacian eigenvalues. A notion of path Laplacian energy has been introduced and some of its basic properties have been obtained.

Proposition 3.1. Let S_n be a star with n vertices. Then the path Laplacian eigenvalues of S_n are 2 with multiplicity $n-2$, $1+\sqrt{n^2-3n+3}$ with multiplicity 1 and $1-\sqrt{n^2-3n+3}$ with multiplicity 1.

Proof. We can write the path Laplacian matrix of S_n as

$$\mathbf{PL(S_n)} = \begin{bmatrix} n-1 & -1 & -1 & ... & -1 & -1 \\ -1 & 1 & -1 & ... & -1 & -1 \\ -1 & -1 & 1 & ... & -1 & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & -1 & -1 & ... & 1 & -1 \\ -1 & -1 & -1 & ... & -1 & 1 \end{bmatrix}$$

The characteristic polynomial of $PL(S_n)$ is

$$C_{PL(S_n)}(x) = (x-2)^{n-2}(x-1-\sqrt{n^2-3n+3})(x-1+\sqrt{n^2-3n+3}).$$

Consequently the path Laplacian eigenvalues of S_n are 2 with multiplicity $n-2$, $1+\sqrt{n^2-3n+3}$ with multiplicity 1 and $1-\sqrt{n^2-3n+3}$ with multiplicity 1. □

Proposition 3.2. Let P_n be a path graph with n vertices. Then the path Laplacian eigenvalues of P_n are 2 with multiplicity 1, 3 with multiplicity $n-3$, $\frac{(-n+5)+\sqrt{n^2-2n+9}}{2}$ with multiplicity 1 and $\frac{(-n+5)-\sqrt{n^2-2n+9}}{2}$ with multiplicity 1.

Proof. The path Laplacian matrix of P_n is

$$\mathbf{PL(P_n)} = \begin{bmatrix} 1 & -1 & -1 & ... & -1 & -1 \\ -1 & 2 & -1 & ... & -1 & -1 \\ -1 & -1 & 2 & ... & -1 & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & -1 & -1 & ... & 2 & -1 \\ -1 & -1 & -1 & ... & -1 & 1 \end{bmatrix}$$

The characteristic polynomial of $PL(P_n)$ is $C_{PL(P_n)}(x) =$

$$(x-2)(x-3)^{n-3}(x-\frac{(-n+5)+\sqrt{n^2-2n+9}}{2})(x-\frac{(-n+5)-\sqrt{n^2-2n+9}}{2}).$$

Consequently the path Laplacian eigenvalues of P_n are 2 with multiplicity 1, 3 with multiplicity $n-3$, $\frac{(-n+5)+\sqrt{n^2-2n+9}}{2}$ with multiplicity 1 and $\frac{(-n+5)-\sqrt{n^2-2n+9}}{2}$ with multiplicity 1. □

Proposition 3.3. Let W_n be a wheel graph with n vertices. Then the path Laplacian eigenvalues of W_n are 6 with multiplicity $n-2$, $-(n-4)+\sqrt{4n^2-11n+16}$ with multiplicity 1 and
$-(n-4)-\sqrt{4n^2-11n+16}$ with multiplicity 1.

Proof. The path Laplacian matrix of W_n is

$$\mathbf{PL(W_n)} = \begin{bmatrix} n-1 & -3 & -3 & ... & -3 & -3 \\ -3 & 3 & -3 & ... & -3 & -3 \\ -3 & -3 & 3 & ... & -3 & -3 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -3 & -3 & -3 & ... & 3 & -3 \\ -3 & -3 & -3 & ... & -3 & 3 \end{bmatrix}$$

The characteristic polynomial of $PL(W_n)$ is $C_{PL(W_n)}(x) = (x-6)^{n-2}(x+(n-4)-\sqrt{4n^2-11n+16})(x+(n-4)+\sqrt{4n^2-11n+16})$. Consequently the path Laplacian eigenvalues of W_n are 6 with multiplicity $n-2$, $-(n-4)+\sqrt{4n^2-11n+16}$ with multiplicity 1 and
$-(n-4)-\sqrt{4n^2-11n+16}$ with multiplicity 1. □

Proposition 3.4. The path Laplacian eigenvalues of the complete bipartite graph $K_{m,n}$ $(1 < m \leq n)$ are m with multiplicity $n-1$, n with multiplicity $m-1$, $(m+n-mn)+\sqrt{[m+n-mn]^2+mn[1+3(m-1)]}$ with multiplicity 1 and $(m+n-mn)-\sqrt{[m+n-mn]^2+mn[1+3(m-1)]}$ with multiplicity 1.

Proof. The path Laplacian matrix of $K_{m,n}$ is

$$\mathbf{PL(K_{m,n})} = \begin{bmatrix} n & -n & ... & -n & -m & -m & ... & -m \\ -n & n & ... & -n & -m & -m & ... & -m \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -n & -n & ... & n & -m & -m & ... & -m \\ -m & -m & ... & -m & m & -m & ... & -m \\ -m & -m & ... & -m & -m & m & ... & -m \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -m & -m & ... & -m & -m & -m & ... & m \end{bmatrix}$$

$$= \begin{bmatrix} 2nI_m - nJ_m & B \\ B' & 2mI_n - mJ_n \end{bmatrix}.$$

where B is $m \times n$ matrix with all entries $-m$ and B' is the transpose of the matrix B. Therefore the path Laplacian eigenvalues of $K_{m,n}$ are $2m$ with multiplicity $n-1$, $2n$ with multiplicity $m-1$, $(m+n-mn)+\sqrt{[m+n-mn]^2+mn[1+3(m-1)]}$ with multiplicity 1 and $(m+n-mn)-\sqrt{[m+n-mn]^2+mn[1+3(m-1)]}$ with multiplicity 1. □

Remark: Let G be a graph on n vertices with m edges. Then the sum of the path Laplacian eigenvalues of G is $2m$. For instance, let G be a graph with vertex degrees $d_1, d_2, ..., d_n$ and with path Laplacian eigenvalues $\mu_1, \mu_2, ..., \mu_n$. Then $tracePL(G) = \sum_{i=1}^{n} d_i = 2m$, also $tracePL(G) = \sum_{i=1}^{n} \mu_i$. Thus $\sum_{i=1}^{n} \mu_i = 2m$.

The following theorem gives path Laplacian eigenvalues of r-regular, r-connected graph.

Theorem 3.5. Let G be a r- regular, r-connected graph with n vertices. Then the path Laplacian matrix $PL(G)$ of G is of the form $2rI_n - rJ_n$ and the path Laplacian eigenvalues of G are of the form $2r - nr$ with multiplicity 1 and $2r$ with multiplicity $n-1$.

Proof. We can write $PL(G)$ as

$$\mathbf{PL(G)} = \begin{bmatrix} r & -r & ... & -r \\ -r & r & ... & -r \\ \vdots & \vdots & \ddots & \vdots \\ -r & -r & ... & r \end{bmatrix}$$
$$= 2rI_n - rJ_n.$$

Consequently the path Laplacian eigenvalues of a graph G are $r(2-n)$ with multiplicity 1 and $2r$ with multiplicity $n-1$.

□

Corollary 3.6. Let G_1 be a r_1-regular, r_1-connected graph with n_1 vertices and G_2 be a r_2-regular, r_2-connected graph with n_2 vertices. Then the path Laplacian eigenvalues of their cartesian product are $(r_1+r_2)(2-n)$ with multiplicity 1 and $2(r_1+r_2)$ with multiplicity $n-1$, where $n = n_1.n_2$.

Proof. Let G denote the cartesian product of G_1 and G_2. Then G is r_1+r_2-regular, r_1+r_2-connected with n vertices. By Theorem 3.5, the path Laplacian eigenvalues of G are $(r_1+r_2)(2-n)$ with multiplicity 1 and $2(r_1+r_2)$ with multiplicity $n-1$. □

Remark: Let G be an r-regular, r-connected graph with n vertices. Then $PL(G) + P(G) = rI_n$.

Proposition 3.7. Let G be a r-regular, r-connected graph with n vertices and m edges. Let $\mu_1, ..., \mu_n$ and $d_1, ..., d_n$ be the path Laplacian eigenvalues and degrees of vertices of G, respectively. Then

$$\sum_{i=1}^{n} \mu_i^2 = \sum_{i=1}^{n} d_i^2 + n(n-1)r^2 = \sum_{i=1}^{n} d_i^2 + \frac{4m^2(n-1)}{n}.$$

Proof. Let $PL(G)$ be the path Laplacian matrix of G. Then

$$PL(G)^2 = \begin{bmatrix} nr^2 & (n-4)r^2 & ... & (n-4)r^2 \\ (n-4)r^2 & nr^2 & ... & (n-4)r^2 \\ \vdots & \vdots & \ddots & \vdots \\ (n-4)r^2 & (n-4)r^2 & ... & nr^2 \end{bmatrix}$$

Since G is r-regular, $d_i = r = \frac{2m}{n}$, $\quad i = 1, 2, ..., n$ and $\sum_{i=1}^{n} d_i^2 = nr^2$.
$\sum_{i=1}^{n} \mu_i^2 = trPL(G)^2 = n^2r^2 = nr^2 + n^2r^2 - nr^2 = \sum_{i=1}^{n} d_i^2 + n(n-1)r^2 = \sum_{i=1}^{n} d_i^2 + \frac{4m^2(n-1)}{n}$. □

In the following Proposition, we give the relation between path Laplacian eigenvalues and maximum vertex degree Δ.

Proposition 3.8. Let G be a graph on n vertices with degrees d_i and $PL(G)$ be its path Laplacian matrix. Let $\Delta = \max_i d_i$ and $\mu_1, \mu_2, ..., \mu_n$ be the path Laplacian eigenvalues of PL(G). Then $\sum_i \mu_i \leq n\Delta$.

Proof. We know that $\sum_i \mu_i = \sum_i d_i$ and $\sum_i d_i \leq n\Delta$. Therefore we conclude that $\sum_i \mu_i \leq n\Delta$. □

Proposition 3.9. (Bounds for μ_1 and μ_n:) Let G be a graph on n vertices, m edges with degrees of vertices d_i and $PL(G)$ be its path Laplacian matrix. Let $\mu_1 \geq \mu_2 \geq ... \geq \mu_n$ be the path Laplacian eigenvalues of $PL(G)$. Then $\mu_n \leq \frac{2m}{n} \leq \mu_1$.

Proof. We know, $\sum_i \mu_i = 2m$ and $n\mu_n \leq \sum_i \mu_i \leq n\mu_1$. This implies that $\mu_n \leq \frac{2m}{n}$ and $\mu_1 \geq \frac{2m}{n}$. Thus $\mu_n \leq \frac{2m}{n} \leq \mu_1$. □

4 Path Laplacian Energy of Graphs

In this section, we find path Laplacian energy of some graphs.
Definition: Let G be a graph with n vertices and m edges. Let μ_1, $\mu_2, ...,$ μ_n be the path Laplacian eigenvalues of G. We define the path Laplacian energy as

$$PLE(G) = \sum_{i=1}^{n} |\mu_i - 2m/n|.$$

In the following table, we explore the path Laplacian energy of some classes of graphs which have just two distinct path Laplacian eigenvalues denoted by μ_1 and μ_2.

Graphs	μ_1	μ_2	Path Laplacian Energy
K_n	$(n-1)(2-n)$	$2(n-1)$	$3(n-1)^2$
C_n	$2(2-n)$	4	$3(n-1)$
Q_n	$n(2-2^n)$	$2n$	$2n(2^n-1)$
Petersen Graph	6	-24	54

From Propositions 3.1-3.4, we get the path Laplacian energies of S_n, P_n, W_n and $K_{m,n}$ as follows.
The path Laplacian energy of the star graph S_n is $\frac{2(n-2)}{n} + 2\sqrt{n^2-3n+3}$.
The path Laplacian energy of the path graph P_n is $\frac{n^2-n-4}{n} + \sqrt{n^2-2n+9}$.
The path Laplacian energy of the wheel graph W_n is $\frac{2(n^2-4)}{n} + 2\sqrt{4n^2-11n+16}$.
The path Laplacian energy of the complete bipartite graph $K_{m,n}$ $(1 < m \leq n)$ is $\frac{2mn(n-m)}{m+n} + (m-n) + \sqrt{[m+n-mn]^2 + mn[1+3(m-1)]}$.

The following result follows from the definitions of the path energy and path Laplacian energy.

Proposition 4.1. Let G be a r-regular, r-connected graph on n vertices $(1 \leq r \leq n-1)$ and m edges. Then $PE(G) = PLE(G) = \frac{4(n-1)}{n}m$.

Proof. By [6], the path eigenvalues of G are $r(n-1)$ with multiplicity 1 and $-r$ with multiplicity $n-1$. Since G is r-regular, $r = \frac{2m}{n}$, this implies that

$$PE(G) = |r(n-1)| + (n-1)|-r| = 2r(n-1) = \frac{4(n-1)}{n}m.$$

By Theorem 3.5, the path Laplacian eigenvalues of G are $2r - nr$ with multiplicity 1 and $2r$ with multiplicity $n-1$. Thus

$$PLE(G) = |r(2-n)-r|+(n-1)|2r-r| = |r-nr|+(n-1)|r| = 2r(n-1) = \frac{4(n-1)}{n}m.$$

□

Let G be a disconnected graph with two components G_1 and G_2, then $PLE(G)$ need not be equal to $PLE(G_1) + PLE(G_2)$. Consider the following example.

Example 4.2. Consider the graph G with two connected components P_4 and C_3, then $PLE(G) \neq PLE(P_4) + PLE(C_3)$ as the value of LHS is 13.982 and the value of RHS is 12.123. We observe that average vertex degree of $P_4 = 1.5 \neq 2=$ average vertex degree of C_3.

In the following Proposition, we give a sufficient condition so that $PLE(G) = PLE(G_1) + PLE(G_2)$.

Proposition 4.3. If the graph G consists of disconnected components G_1 and G_2, and if G_1 and G_2 have equal average vertex degrees, then $PLE(G) = PLE(G_1) + PLE(G_2)$.

Proof. Let G, G_1, and G_2 be $(n,\ m)$, $(n_1,\ m_1)$, and $(n_2,\ m_2)$-graphs, respectively. Then from $2m_1/n_1 = 2m_2/n_2$ it follows $2m/n = 2m_i/n_i,\ i = 1, 2$. Therefore

$$PLE(G) = \sum_{i=1}^{n_1+n_2} |\mu_i - \frac{2m}{n}| = \sum_{i=1}^{n_1} |\mu_i - \frac{2m_1}{n_1}| + \sum_{i=n_1+1}^{n_1+n_2} |\mu_i - \frac{2m_2}{n_2}|$$
$$= PLE(G_1) + PLE(G_2).$$

□

Let G_1 and G_2 be two graphs with disjoint vertex sets. Let V_i and E_i be the vertex and edge sets of G_i $(i = 1, 2)$, respectively. The union of G_1 and G_2 is the graph $G_1 \cup G_2$ with vertex set $V_1 \cup V_2$ and the edge set $E_1 \cup E_2$. If G_1 is an $(n_1,\ m_1)$-graph and G_2 is an $(n_2,\ m_2)$-graph then $G_1 \cup G_2$ has $n_1 + n_2$ vertices and $m_1 + m_2$ edges.

In the following Theorem, we obtain bound for the path Laplacian energy of the union of two graphs.

Theorem 4.4. If G_1 be an $(n_1,\ m_1)$-graph and G_2 be an $(n_2,\ m_2)$-graph, such that $\frac{2m_1}{n_1} > \frac{2m_2}{n_2}$. Then

$$PLE(G_1)+PLE(G_2)-\frac{4(n_2m_1 - n_1m_2)}{n_1 + n_2} \leq PLE(G_1 \cup G_2) \leq PLE(G_1)+PLE(G_2)+\frac{4(n_2m_1 - n_1m_2)}{n_1 + n_2}.$$

Proof. Let $G = G_1 \cup G_2$. Then G is an $(n_1 + n_2,\ m_1 + m_2)$-graph. By the definition of path Laplacian energy,

$$PLE(G_1 \cup G_2) = \sum_{i=1}^{n_1+n_2} |\mu_i(G) - \frac{2(m_1+m_2)}{n_1+n_2}|$$

$$= \sum_{i=1}^{n_1} |\mu_i(G) - \frac{2(m_1+m_2)}{n_1+n_2}| + \sum_{i=n_1+1}^{n_1+n_2} |\mu_i(G) - \frac{2(m_1+m_2)}{n_1+n_2}|$$

$$= \sum_{i=1}^{n_1} |\mu_i(G_1) - \frac{2(m_1+m_2)}{n_1+n_2}| + \sum_{i=1}^{n_2} |\mu_i(G_2) - \frac{2(m_1+m_2)}{n_1+n_2}|$$

$$= \sum_{i=1}^{n_1} |\mu_i(G_1) - \frac{2m_1}{n_1} + \frac{2m_1}{n_1} - \frac{2(m_1+m_2)}{n_1+n_2}| + \sum_{i=1}^{n_2} |\mu_i(G_2) - \frac{2m_2}{n_2} + \frac{2m_2}{n_2} - \frac{2(m_1+m_2)}{n_1+n_2}|$$

$$\leq \sum_{i=1}^{n_1} |\mu_i(G_1) - \frac{2m_1}{n_1}| + n_1|\frac{2m_1}{n_1} - \frac{2(m_1+m_2)}{n_1+n_2}| + \sum_{i=1}^{n_2} |\mu_i(G_2) - \frac{2m_2}{n_2}| + n_2|\frac{2m_2}{n_2} - \frac{2(m_1+m_2)}{n_1+n_2}|.$$

Since $n_2m_1 > n_1m_2$, above inequality becomes $PLE(G_1 \cup G_2) \leq PLE(G_1) + n_1(\frac{2m_1}{n_1} - \frac{2(m_1+m_2)}{n_1+n_2}) + PLE(G_2) + n_2(-\frac{2m_2}{n_2} + \frac{2(m_1+m_2)}{n_1+n_2}) = PLE(G_1) + PLE(G_2) + \frac{4(n_2m_1 - n_1m_2)}{n_1+n_2}$ which is an upper bound for path Laplacian energy of $G_1 \cup G_2$.

To get the lower bound, we just have to note that in full analogy to the above arguments,

$PLE(G_1 \cup G_2) \geq \sum_{i=1}^{n_1} |\mu_i(G_1) - \frac{2m_1}{n_1}| - n_1|\frac{2m_1}{n_1} - \frac{2(m_1+m_2)}{n_1+n_2}| + \sum_{i=1}^{n_2} |\mu_i(G_2) - \frac{2m_2}{n_2}| - n_2|\frac{2m_2}{n_2} - \frac{2(m_1+m_2)}{n_1+n_2}|$.

Since $n_2m_1 > n_1m_2$, above inequality becomes

$PLE(G_1 \cup G_2) \geq PLE(G_1) - n_1(\frac{2m_1}{n_1} - \frac{2(m_1+m_2)}{n_1+n_2}) + PLE(G_2) - n_2(-\frac{2m_2}{n_2} + \frac{2(m_1+m_2)}{n_1+n_2}) = PLE(G_1) + PLE(G_2) - \frac{4(n_2m_1 - n_1m_2)}{n_1+n_2}$

which is a lower bound for path Laplacian energy of $G_1 \cup G_2$. □

Corollary 4.5. Let G_1 be an r_1 regular graph on n_1 vertices and G_2 be an r_2 regular graph on n_2 vertices, such that $r_1 > r_2$. Then

$PLE(G_1) + PLE(G_2) - \frac{2n_1n_2(r_1 - r_2)}{n_1+n_2} \leq PLE(G_1 \cup G_2) \leq PLE(G_1) + PLE(G_2) + \frac{2n_1n_2(r_1 - r_2)}{n_1+n_2}$.

Proof. Since G_1 is r_1 regular, the number of edges in G_1 is $m_1 = \frac{n_1r_1}{2}$ and since G_2 is r_2 regular, the number of edges in G_2 is $m_2 = \frac{n_2r_2}{2}$. Now $\frac{2m_1}{n_1} = r_1 > r_2 = \frac{2m_2}{n_2}$. By Theorem 4.4, we get the required inequality. □

Corollary 4.6. Let G_1 be an $(n,\ m)$-graph and G_2 be the graph obtained from G_1 by removing k edges, $0 \leq k \leq m$. Then
$PLE(G_1) + PLE(G_2) - 2k \leq PLE(G_1 \cup G_2) \leq PLE(G_1) + PLE(G_2) + 2k$.

Proof. The number of vertices of G_2 is n and the number of edges in G_2 is $m - k$. By Theorem 4.4, the result follows. □

5 Conclusion

In the present paper, the concepts of path Laplacian matrix, path Laplacian eigenvalues and path Laplacian energy of a graph are given and studied. Also, some bounds on Path Laplacian Energy of graphs are given and studied.

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