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Generalized Cubic Aggregation Operators with Application in Decision Making Problem

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Abstract - There are many aggregation operators and their applications have been developed up to date, but in this paper we introduced the idea of generalized aggregation operator. The main idea of this paper is to study the generalized aggregation operators with cubic numbers. In this paper, we introduced three types of cubic aggregation operators called generalized cubic weighted averaging $(GCWA)$ operator, generalized cubic ordered weighted averaging $(GCOWA)$ operator and generalized cubic hybrid averaging $(GCHA)$ operator. We extend the theory of cubic numbers to generalized ordered weighted averaging operators that are characterized by interval membership and exact membership. In last section we provide an application of these aggregation operators to multiple attribute group decision making problem.

***Keywords* -** Cubic sets, GCWA Operator, GCOWA Operator, GCHA operator.

1. Introduction

In 1965, Zadeh generalized the classical set theory to fuzzy set theory. Fuzzy set (Fs) has been studied in many fields such that decision making theory, information science, medical diagnosis, pattern recognition, fuzzy algebra and fuzzy topology. Fuzzy set has not explained every concept due to not available of non- membership. In [2] , Atanassov introduced the concept of intuitionistic fuzzy set (IFs) , intuitionistic fuzzy set is generalized structure of fuzzy set. Intuitionistic fuzzy set characterized by membership and non-membership of an element in a set. The application of intuitionistic fuzzy set has been studied in many fields, logic program, algebra, topology, medical diagnosis and decision making theory. IFs aggregation operator has been studied [3,4,5,6,7] i.e., intuitionistic fuzzy ordered weighted $(IFOW)$ operator, intuitionistic fuzzy ordered weighted

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geometric $(IFOWG)$ operator, intuitionistic fuzzy hybrid averaging $(IFHA)$ operator. The intuitionistic fuzzy set does not explain the problem when arise uncertainty. Therefore Jun et al defined the new concept so called cubic set (CS). In [8] Jun introduced a new theory which is called cubic (CS) set theory. They introduced many concepts of cubic set cubic to deal with uncertainty problem. Cubic set explain all the satisfied, unsatisfied and uncertain information, while fuzzy and intuitionistic fuzzy set fail to explain these terms. In classical fuzzy set, to explain i.e., the experts degree of certainty in various statement, the value of interval $[0,1]$ is used. It is often more difficult for a decision maker's to exactly quantify his certainty. Therefore instead of real number, it is more adequate to explain this degree of certainty by an interval or even by a fuzzy set. In case of cubic set (CS) the membership is represented by interval-valued fuzzy set and non-membership in fuzzy set. Interval - valued fuzzy set has applied to real life application i.e., Sambuc applied it to medical diagnosis in thyroidian pathology. Kohout also applied it to medical, in a system CLINAID [9]. Turlesen [10,11] used interval-valued logic to preference modeling. Cubic set theory applied many areas in BCK/BCI algebra and other structures [12,13,14].

The weighted aggregation (WA) operator and ordered weighted aggregation (OWA) operator are rich area for research and the generalized aggregation operators are new class of aggregation operator. Thus an advantage of the above mentioned aggregation operators. In this paper, we introduced three types of cubic aggregation operators so-called generalized cubic weighted averaging $(GCWA)$ operator, generalized cubic ordered weighted averaging $(GCOWA)$ operator and generalized cubic hybrid averaging $(GCHA)$ operator.

This paper is organized as follows: In section 2, we give some basic definitions and laws of cubic numbers which will be used in our later sections. In section 3, we develop the concept of generalized cubic weighted averaging $(GCWA)$ operator, generalized cubic ordered weighted averaging $(GCOWA)$ operator and generalized cubic hybrid averaging $(GCHA)$ operator. In section 4, we provide an applications of these aggregation operators to multi-criteria decision making. For this purpose we develop a general algorithm for these cubic aggregation operators. In section 5, numerical an application to decision making problems. In section 6, we discuss and compare the proposed operator with $GIFA$ operator. Concluding remarks are made in section 7.

2. Preliminaries

Atanassov generalized the concept of fuzzy set (FS) and defined the concept of IFS as follows [2] .Let X be a fixed set. An IFS A in X is an object having the form:

$$A=\{\langle x,\mu_A(x),\nu_A(x)\rangle \mid x\in X\}, \qquad (1)$$

where the functions $\mu_A : X\to[0,1]$ and $\nu_A : X\to[0,1]$ define the degree of membership and the degree of non-membership of the element $x\in X$ to A, respectively, and for every $x\in X$,

$$0\le \mu_A(x)+\nu_A(x)\le 1. \qquad (2)$$

For each IFS A in X, if

$$\pi_A(x) = 1 - \mu_A(x) - \nu_A(x), \text{ for all } x \in X. \qquad (3)$$

Then, $\pi_A(x)$ is called the degree of indeterminacy of x to A.

Definition 2.1 [8] Let X be a fixed non empty set. A cubic set is an object of the form:

$$\widetilde{C} = \{\langle a, A(a), \lambda(a)\rangle : a \in X\},$$

where A is an interval-valued fuzzy set $(IVFS)$ and λ is a fuzzy set in X. A cubic set $\widetilde{C} = \langle a, A(a), \lambda(a)\rangle$ is simply denoted by $\widetilde{C} = \langle \widetilde{A}, \lambda\rangle$, the collection of cubic sets is denoted by $\widetilde{C}(X)$.

(1) if $\lambda \in \widetilde{A}(x)$ for all $x \in X$ so it is called interval cubic set，

(2) If $\lambda \notin \widetilde{A}(x)$ for all $x \in X$ so it is called external cubic set，

(3) If $\lambda \in \widetilde{A}(x)$ or $\lambda \notin \widetilde{A}(x)$ its called cubic set for all $x \in X$.

Definition 2.2 [8] Let $\widetilde{A} = \langle A, \lambda\rangle$ and $\widetilde{B} = \langle B, \mu\rangle$ are two cubic sets in X , such that,

(1) （Equality） $A = B \Leftrightarrow A = B$ and $\lambda = \mu,$

(2) $(P-$ order） $A \subseteq_A B \Leftrightarrow A \subseteq B$ and $\lambda \leq \mu,$

(3) $(R-$ order） $A \subseteq_R B \Leftrightarrow A \subseteq B$ and $\lambda \geq \mu.$

Definition 2.3 [8] The complement of $\widetilde{A} = \langle A, \lambda\rangle$ is defined to be the cubic set as follows:

$$A^c = \{\langle x, A^c(x), 1 - \lambda(x)\rangle \mid x \in X\}.$$

Cubic Numbers Score and Accuracy Function

In this section, we define some operational laws of cubic numbers. We define score function and accuracy function of a cubic set which will be used in our later sections.

Definition 2.4 Let $\widetilde{C} = \langle A_c, \eta_c\rangle$, $\widetilde{C}_1 = \left\langle \bar{A}_{c_1}, \eta_{c_1}\right\rangle$, and $\widetilde{C}_2 = \left\langle \bar{A}_{c_2}, \eta_{c_2}\right\rangle$ be any three cubic values (CV) . Then, the following operational laws hold:

(1)

$$\widetilde{C}_1 \oplus \widetilde{C}_2 = \left\langle \begin{bmatrix} \bar{a}_{c_1} + \bar{a}_{c_2} - \bar{a}_{c_1}\bar{a}_{c_2}, \\ a^+_{c_1} + a^+_{c_2} - a^+_{c_1}a^+_{c_2} \end{bmatrix}, \eta_{c_1}\eta_{c_2} \right\rangle,$$

(2)

$$\widetilde{C}_1 \otimes \widetilde{C}_2 = \left\langle \left[\bar{a}_{c_1}\bar{a}_{c_2}, \ a^+_{c_1}a^+_{c_2}\right], \eta_{c_1} + \eta_{c_2} - \eta_{c_1}\eta_{c_2} \right\rangle,$$

(3)

$$\delta\tilde{C}=\left\langle\left[1-(1-\bar{a}_c)^{\delta},(1-(1-a_c^{+})^{\delta}\right],\ \eta_c^{\delta}\right\rangle,\quad \delta\geq 0,$$

(4)

$$\tilde{C}^{\delta}=\left\langle\left[(\bar{a}_c)^{\delta},\ (a_c^{+})^{\delta}\right],1-(1-\eta_c)^{\delta}\right\rangle,\quad \delta\geq 0.$$

Example 2.5 Let $\tilde{C}_1=\langle[0.5,0.6],0.4\rangle$, $\tilde{C}_2=\langle[0.4,0.5],0.7\rangle$, $\tilde{C}_3=\langle[0.6,0.8],0.3\rangle$, be any three cubic numbers, and let $\delta=\ 2$. Then, we verify the above results as follows:

(1) $\tilde{C}_1\oplus\tilde{C}_2$

$$=\left\langle\begin{bmatrix}0.4+0.6-0.4\times0.6,\\ 0.5+0.8-0.5\times0.8\end{bmatrix},0.3\times0.7\right\rangle,$$
$$=\langle[1.0-0.24,1.3-0.40],0.21\rangle,$$
$$=\langle[0.76-0.90],0.21\rangle.$$

(2) $\tilde{C}_1\otimes\tilde{C}_2$

$$=\langle[0.4\times0.6,0.5\times0.8],0.7+0.3-0.7\times0.3\rangle$$
$$=\langle[0.24,0.40],1.0-0.21\rangle$$
$$=\langle[0.24-0.40],0.79\rangle.$$

(3) $\delta\tilde{C}$

$$=\left\langle\left[1-(1-0.5)^2,1-(1-0.6)^2\right],(0.4)^2\right\rangle$$
$$=\left\langle\left[1-(0.5)^2,1-(0.4)^2\right],0.16\right\rangle$$
$$=\langle[0.75-0.84],0.16\rangle.$$

(4) $\tilde{C}^{\delta}$

$$=\left\langle\left[(0.5)^2,(0.6)^2\right],1-(1-0.4)^2\right\rangle$$
$$=\left\langle[0.25,0.36],1-(0.6)^2\right\rangle$$
$$=\langle[0.25-0.36],0.64\rangle.$$

Theorem 2.6 Let $\tilde{C}=\left\langle\bar{A}_c,\eta_c\right\rangle$, $\tilde{C}_1=\left\langle\bar{A}_{c_1},\eta_{c_1}\right\rangle$, and $\tilde{C}_2=\left\langle\bar{A}_{c_2},\eta_{c_2}\right\rangle$, be any three cubic values. Then, the following operational laws hold:

$$\tilde{C}_1^{\cdot} = \tilde{C}_1 \oplus \tilde{C}_2, \tilde{C}_2^{\cdot} = \tilde{C}_1 \otimes \tilde{C}_2, \tilde{C}_3^{\cdot} = \delta\tilde{C}, \tilde{C}_4^{\cdot} = \tilde{C}^{\delta}, \delta > 0$$

then all $\tilde{C}_i^{\cdot}$ $(i = 1,2,3,4)$ are cubic values.

Theorem 2.7 Let $\tilde{C} = \left\langle \bar{A}_c, \eta_c \right\rangle$, $\tilde{C}_1 = \left\langle \bar{A}_{c_1}, \eta_{c_1} \right\rangle$, $\tilde{C}_2 = \left\langle \bar{A}_{c_2}, \eta_{c_2} \right\rangle$ and $\tilde{C}_3 = \left\langle \bar{A}_{c_3}, \eta_{c_3} \right\rangle$ be any four (CVs), and δ, δ_1, δ_2 are any sclar numbers grater then zero such that,

(1)

$$\delta_1\tilde{C} \oplus \delta_2\tilde{C} = (\delta_1 + \delta_2)\tilde{C},$$

(2)

$$(\tilde{C}_1 \oplus \tilde{C}_2) \oplus \tilde{C}_3 = \tilde{C}_1 \oplus (\tilde{C}_2 \oplus \tilde{C}_3),$$

(3)

$$((\tilde{C})^{\delta_1})^{\delta_2} = (\tilde{C})^{\delta_1 \ \delta_2}.$$

Example 2.8 Let

$$\tilde{C} = \langle [0.3, 0.4], 0.5 \rangle, \tilde{C}_1 = \langle [0.4, 0.6], 0.3 \rangle,$$
$$\tilde{C}_2 = \langle [0.5, 0.7], 0.8 \rangle, \tilde{C}_3 = \langle [0.6, 0.3], 0.4 \rangle$$

be any four cubic numbers, and let $\delta_1 =$ 2 and $\delta_2 =$ 3. Then, we verify the above results as follows;

(1) $\delta_1\tilde{C} \oplus \delta_2\tilde{C} = (\delta_1 + \delta_2)\tilde{C}$. In this case first we take $\delta_1\tilde{C} \oplus \delta_2\tilde{C}$ and then we take $(\delta_1 + \delta_2)\tilde{C}$. We apply cubic laws to verify the result such that,

$$\begin{aligned}
\delta_1\tilde{C} &= \left\langle \left[1 - (1 - 0.3)^2, 1 - (1 - 0.4)^2\right], (0.5)^2 \right\rangle \\
&= \langle [1 - 0.49, 1 - 0.84], 0.25 \rangle \\
&= \langle [0.51, 0.64], 0.25 \rangle, \text{ and} \\
\delta_2\tilde{C} &= \left\langle \left[1 - (1 - 0.3)^3, 1 - (1 - 0.4)^3\right], (0.5)^3 \right\rangle \\
&= \langle [1 - 0.343, 1 - 0.216], 0.125 \rangle \\
&= \langle [0.675, 0.784], 0.125 \rangle.
\end{aligned}$$

By using $\delta_1\tilde{C}$ and $\delta_2\tilde{C}$ such that,

$$(\delta_1+\delta_2)\tilde{C}=\left\langle\begin{bmatrix}0.51+0.657-0.51\times 0.657,\\ 0.64+0.784-0.64\times 0.784\end{bmatrix}, \atop 0.25\times 0.125\right\rangle$$
$$=\langle[0.8319-0.9224],0.0312\rangle.$$

Similarly we can find $(\delta_1+\delta_2)\tilde{C}$ if we use $\delta=5$ such that,

$$\delta\tilde{C}=\left\langle\left[1-(1-0.3)^5,1-(1-0.4)^5\right],(0.5)^5\right\rangle$$
$$=\langle[1-0.1680,1-0.0776],0.0312\rangle$$
$$=\langle[0.8319,0.9224],0.0312\rangle.$$

(2) $(\tilde{C}_1\oplus\tilde{C}_2)\oplus\tilde{C}_3=\tilde{C}_1\oplus(\tilde{C}_2\oplus\tilde{C}_3)$. In this case first we take $(\tilde{C}_1\oplus\tilde{C}_2)\oplus\tilde{C}_3$ and then we take $\tilde{C}_1\oplus(\tilde{C}_2\oplus\tilde{C}_3)$, we apply cubic laws to verify the result such that,

Let

$$\tilde{C}=\langle[0.4,0.6],0.5\rangle,\tilde{C}_1=\langle[0.4,0.6],0.3\rangle,$$
$$\tilde{C}_2=\langle[0.6,0.7],0.4\rangle,\tilde{C}_3=\langle[0.5,0.7],0.9\rangle.$$

Then,

$$(\tilde{C}_1\oplus\tilde{C}_2)=\left\langle\begin{bmatrix}0.4+0.6-0.4\times 0.6,\\ 0.5+0.8-0.5\times 0.8\end{bmatrix},0.3\times 0.7\right\rangle,$$
$$=\langle[1.0-0.24,1.3-0.40],0.21\rangle,$$
$$=\langle[0.76,0.90],0.21\rangle$$
$$(\tilde{C}_1\oplus\tilde{C}_2)\oplus\tilde{C}_3=\langle[0.76,0.90],0.21\rangle\oplus\langle[0.5,0.7],0.9\rangle$$
$$=\left\langle\begin{bmatrix}0.76+0.5-0.76\times 0.5,\\ 0.90+0.7-0.90\times 0.7\end{bmatrix},0.21\times 0.9\right\rangle$$
$$=\langle[1.26-0.38,1.6-0.63],0.18\rangle$$
$$=\langle[0.88,0.97],0.18\rangle.$$

Similarly we find $\tilde{C}_1\oplus(\tilde{C}_2\oplus\tilde{C}_3)$

$$
\begin{aligned}
(\tilde{C}_2 \oplus \tilde{C}_3) &= \left\langle \begin{bmatrix} 0.6+0.5-0.6\times 0.5, \\ 0.7+0.7-0.7\times 0.7 \end{bmatrix}, 0.4\times 0.9 \right\rangle, \\
&= \langle [1.1-0.3, 1.4-0.49], 0.36 \rangle, \\
&= \langle [0.80, 0.91], 0.36 \rangle \\
\tilde{C}_1 \oplus (\tilde{C}_2 \oplus \tilde{C}_3) &= \langle [0.4, 0.6], 0.3 \rangle \oplus \langle [0.80, 0.91], 0.36 \rangle \\
&= \left\langle \begin{bmatrix} 0.4+0.80-0.4\times 0.80, \\ 0.6+0.91-0.6\times 0.91 \end{bmatrix}, 0.3\times 0.36 \right\rangle \\
&= \langle [1.2-0.32, 1.51-0.54], 0.18 \rangle \\
&= \langle [0.88, 0.97], 0.18 \rangle .
\end{aligned}
$$

(3) $((\tilde{C})^{\delta_1})^{\delta_2} = (\tilde{C})^{\delta_1 \ \delta_2}$. Let $\tilde{C} = \langle [0.3, 0.4], 0.6 \rangle$ be any cubic number and let $\delta_1 =$ 0.3 and $\delta_2 =$ 0.2 in this case first we find $((\tilde{C})^{\delta_1})^{\delta_2}$ then we find $\tilde{C}^{\delta_1 \ \delta_2}$ such that,

$$
\begin{aligned}
C^{\delta_1} &= \left\langle \left[(0.3)^{0.3}, (0.4)^{0.3}\right], 1-(1-0.6)^{0.3} \right\rangle \\
&= \langle [0.69, 0.83], 0.24 \rangle \\
\left(C^{\delta_1}\right)^{\delta_2} &= \left\langle \left[(0.69)^{0.2}, (0.83)^{0.2}\right], 1-(1-0.24)^{0.2} \right\rangle \\
&= \langle [0.93, 0.94], 0.0.05 \rangle \text{ and} \\
C^{\delta_1 \delta_2} &= \left\langle \left[(0.3)^{0.06}, (0.4)^{0.06}\right], 1-(1-0.6)^{0.06} \right\rangle \\
&= \langle [0.93, 0.94], 0.0.05 \rangle .
\end{aligned}
$$

Based on the cubic value (CVs) sets · We introduced a score function $s(\tilde{C})$ such that,

Let $\tilde{C} =$ $\left\langle \bar{A}_c, \eta_c \right\rangle$ be an cubic value, where

$$\bar{A}_c \in [0,1], \ \eta_c \in [0,1]. \qquad (4)$$

The score of $\tilde{C}$ can be evaluated by the score function s shown as follows:

$$s(\tilde{C}) = \frac{\bar{A}_c - \eta_c}{3} = \frac{\bar{a} + \ a^+ - \eta}{3}, \qquad (5)$$

where $s(\tilde{C}) \in [-1,1].$ The function s is used to measure the score of a (CV) . Now an accuracy function to evaluate the degree of accuracy of the cubic value $\tilde{C} = \left\langle \bar{A}_c, \eta_c \right\rangle$ as follows;

$$h(\tilde{C}) = \frac{1 + \bar{A}_c - \eta_c}{3} = \frac{1 + \bar{a} + \ a^+ - \eta}{3}, \qquad (6)$$

where $h(\tilde{C}) \in [0,1].$

Definition 2.9 Let $\tilde{C} = \left\langle \bar{A}_c, \eta_c \right\rangle$ and $\tilde{D} = \left\langle \bar{A}_D, \eta_D \right\rangle$ be any two cubic set such that,

$$s(\tilde{C}) = \frac{\bar{A}_c - \eta_c}{3} = \frac{\bar{a} + 2a^{+} - \eta}{3}$$

$$s(\tilde{D}) = \frac{\bar{A}_D - \eta_D}{3} = \frac{\bar{a} + 2a^{+} - \eta}{3}$$

be the scores function of $\tilde{C}$ and $\tilde{D}$, respectively, and be the accuracy degrees of $\tilde{C}$ and $\tilde{D}$, respectively, then

Remarks:

1. If $s(\tilde{C}) < s(\tilde{D})$, then $\tilde{C} < \tilde{D}$,
2. If $s(\tilde{C}) = s(\tilde{D})$, then,

i. If $h(\tilde{C}) = h(\tilde{D})$, then $\tilde{C} = \tilde{D}$,

ii. If $h(\tilde{C}) < h(\tilde{D})$, then $\tilde{C}$ is smaller than $\tilde{D}$, denoted by $\tilde{C} < \tilde{D}$.

3. The GCWA, GCOWA, And GCHA Operators

Definition 3.1 [15] A generalized weighted averaging (GWA) operator of dimension n is a mapping $GWA: \ (R^{+})^{n} \rightarrow R^{+}$ $\left(R \text{ denotes the set of real number}\right)$ which has the following form:

$$GWA(a_1, a_2, ..., a_n) = \left(\sum_{j=1}^{n} w_j a_j^{\delta} \right)^{\frac{1}{\delta}} \qquad (7)$$

where $\delta > 0$, and $w = (w_1, w_2, ..., w_n)^T$ be the weighting vector of the arguments $a_j (j = 1,2,...,n)$ with $j \geq 0$, $j = 1,2,...,n$ and $\sum_{j=1}^{n} w_j = 1$, R^{+} is the set of all non-negative real numbers. Another aggregation operator called the $GOWA$ operators is the generalization of the OWA operator.

Definition 3.2 [15] A generalized ordered weighted averaging $(GOWA)$ operator of dimension n is a mapping $GOWA: R^n \rightarrow R$ which has the following form:

$$GOWA(a_1, a_2, ..., a_n) = \left(\sum_{j=1}^{n} w_j b_j^{\delta} \right)^{\frac{1}{\delta}} \qquad (8)$$

where $\delta > 0$, and $w = (w_1, w_2, ..., w_n)^T$ be the weighting vector of $(a_1, a_2, ..., a_n)$, $w_j \geq 0$, $j = 1,2,...,n$, and $\sum_{j=1}^{n} w_j = 1$, b_j is jth largest of a_i, $I = [0,1]$.

The GCWA Operator

In this section, we define $GCWA$ operator and study different results relevant to $GCWA$ operator. For our convenience, let $\tilde{C}$ denotes all of cubic set.

Definition 3.3 Let $\tilde{C}_j = \left\langle \bar{A}_{c_j}, \eta_{c_j} \right\rangle$ $(j = 1, 2, ..., n)$ be a collection of cubic value set and $GCWA : \tilde{C}^n \to \tilde{C}$, if

$$GCWA_w(\tilde{c}_1, \tilde{c}_2, ..., \tilde{c}_n) = (w_1 c_1^\delta \oplus w_2 c_2^\delta \oplus ... \oplus w_n c_n^\delta)^{\frac{1}{\delta}}, \qquad (9)$$

then the function $GCWA$ is called a $GCWA$ operator, where $\delta > 0$, and $w = (w_1, w_2, ..., w_n)^T$ is the weighting vector associated with the $GCWA$ operator, with $w_j \geq 0$, $j = 1, 2, ..., n$, and $\sum_{j=1}^n w_j = 1$. By using the operation laws of cubic numbers we will prove the following theorems.

Theorem 3.4 Let $\tilde{C}_j = \left\langle \bar{A}_{c_j}, \eta_{c_j} \right\rangle$ $(j = 1, 2, ..., n)$ be a collection of cubic value set. Then, their aggregated value by using the $GCWA$ operator is also cubic value such that,

$$GCWA_w(c_1, c_2, ..., c_n) = \left\langle \begin{array}{c} \left[\left(1 - \prod_{j=1}^n (1 - \bar{a}_{c_j}^{\delta})^{w_j}\right)^{\frac{1}{\delta}}, \left(1 - \prod_{j=1}^n (1 - a_{c_j}^{+\delta})^{w_j}\right)^{\frac{1}{\delta}} \right], \\ 1 - \left(1 - \prod_{j=1}^n (1 - (1 - \eta_{c_j})^\delta)^{w_j}\right)^{\frac{1}{\delta}} \end{array} \right\rangle \qquad (10)$$

where $\delta > 0$, and $w = (w_1, w_2, ..., w_n)^T$ be the weighting vector associated with the $GCWA$ operator, with $w_j \geq 0$, $j = 1, 2, \ldots, n$, and $\sum_{j=1}^n w_j = 1$.

Proof The first result follows quickly from Definition 6 and Theorem 1 . In the following, we first prove

$$w_1 c_1^\delta \oplus w_2 c_2^\delta \oplus ... \oplus w_n c_n^\delta = \left\langle \begin{array}{c} \left[\left(1 - \prod_{j=1}^n (1 - \bar{a}_{c_j}^{\delta})^{w_j}\right), \left(1 - \prod_{j=1}^n (1 - a_{c_j}^{+\delta})^{w_j}\right) \right], \\ \left(\prod_{j=1}^k (1 - (1 - \eta_{c_j})^\delta)^{w_j}\right) \end{array} \right\rangle \qquad (11)$$

By using mathematically induction on n,

1. For $n = 2$. As we know that

$$\tilde{C}^\delta = \left\langle \left[(\bar{a}_{c_j})^\delta, (a_{c_j}^+)^\delta \right], 1 - (1 - \eta_{c_j})^\delta \right\rangle.$$

Then

$$\tilde{C}_1^{\delta} = \left\langle \left[(\bar{a}_{c_1})^{\delta}, (a_{c_1}^{+})^{\delta} \right], 1-(1-\eta_{c_1})^{\delta} \right\rangle,$$

$$\tilde{C}_2^{\delta} = \left\langle \left[(\bar{a}_{c_2})^{\delta}, (a_{c_2}^{+})^{\delta} \right], 1-(1-\eta_{c_2})^{\delta} \right\rangle.$$

Therefore

$$w_1 c_1^{\delta} \oplus w_2 c_2^{\delta} = \left\langle \begin{array}{c} \left[\left(1-\prod_{j=1}^{2}(1-\bar{a}_{c_j}^{\delta})^{w_j}\right), \left(1-\prod_{j=1}^{2}(1-a_{c_j}^{+\delta})^{w_j}\right) \right], \\ \left(\prod_{j=1}^{2}(1-(1-\eta_{c_j})^{\delta})^{w_j}\right) \end{array} \right\rangle$$

2. If Eq. 11 holds for $n = k$, then

$$w_1 c_1^{\delta} \oplus w_2 c_2^{\delta} \oplus \ldots \oplus w_k c_k^{\delta} = \left\langle \begin{array}{c} \left[\left(1-\prod_{j=1}^{k}(1-\bar{a}_{c_j}^{\delta})^{w_j}\right), \left(1-\prod_{j=1}^{k}(1-a_{c_j}^{+\delta})^{w_j}\right) \right], \\ \left(\prod_{j=1}^{k}(1-(1-\eta_{c_j})^{\delta})^{w_j}\right) \end{array} \right\rangle.$$

when $n = k+1$, by the operational laws 1, 2 and 4 such that,

$$w_1 c_1^{\delta} \oplus w_2 c_2^{\delta} \oplus \ldots \oplus w_{k+1} c_{k+1}^{\delta} = \left\langle \begin{array}{c} \left[\left(1-\prod_{j=1}^{k}(1-\bar{a}_{c_j}^{\delta})^{w_j}\right), \left(1-\prod_{j=1}^{k}(1-a_{c_j}^{+\delta})^{w_j}\right) \right], \\ \left(\prod_{j=1}^{k}(1-(1-\eta_{c_j})^{\delta})^{w_j}\right) \end{array} \right\rangle$$

$$\oplus \left\langle \begin{array}{c} \left[\begin{array}{c} \left(1-(1-\bar{a}_{c_{k+1}}^{\delta})^{w_{k+1}}\right), \\ \left(1-(1-a_{c_{k+1}}^{+\delta})^{w_{k+1}}\right) \end{array} \right], \\ (1-(1-\eta_{c_{k+1}})^{\delta})^{w_{k+1}} \end{array} \right\rangle$$

$$= \left\langle \begin{array}{c} \left[\left(1-\prod_{j=1}^{k+1}(1-\bar{a}_{c_j}^{\delta})^{w_j}\right), \left(1-\prod_{j=1}^{k+1}(1-a_{c_j}^{+\delta})^{w_j}\right) \right], \\ \left(\prod_{j=1}^{k+1}(1-(1-\eta_{c_j})^{\delta})^{w_j}\right) \end{array} \right\rangle$$

i.e. Eq. 11 holds for $n = k+1$. Thus, Eq. 11 holds for all n such that,

Example 3.5 Let

$$\tilde{C}_1 = \langle [0.3, 0.4], 0.5 \rangle, \tilde{C}_2 = \langle [0.2, 0.5], 0.3 \rangle, \tilde{C}_3 = \langle [0.4, 0.6], 0.3 \rangle$$

be any three cubic numbers, and $w = (0.2, 0.3, 0.5)^T$ be weighting vector of $\tilde{C}_j\ (j = 1, 2, 3),$ and $\delta = 2.$ Then we have calculated the $GCWA$ by applying E.q 10 such that,

$$GCWA_w(\tilde{C}_1, \tilde{C}_2, \tilde{C}_3) = \langle [0.3342, 0.5260], 0.3297 \rangle.$$

On the basis of Theorem 2, we have the following properties of the $GCWA$ operators.

Theorem 3.6 Let $\tilde{C}_j = \left\langle \bar{A}_{c_j}, \eta_{c_j} \right\rangle$ $(j = 1, 2, ..., n)$ be a collection of cubic value set where $\delta > 0,$ and $w = (w_1, w_2, ..., w_n)^T$ is the weighting vector associated with the $GCWA$ operator, with $w_j \geq 0,\ j = 1, 2, ..., n,$ and $\sum_{j=1}^{n} w_j = 1.$ If all $C_j\ (j = 1, 2, ..., n)$ are equal such that $\tilde{C}_j = \tilde{C},$ for all $j,$ then $GCWA_w(\tilde{C}_1, \tilde{C}_2, ..., \tilde{C}_n) = \tilde{C}.$

Proof By Theorem 2, we have

$$\begin{aligned} GCWA_w(\tilde{C}_1, \tilde{C}_2, ..., \tilde{C}_n) &= (w_1 C_1^{\delta} \oplus w_2 C_2^{\delta} \oplus ... \oplus w_n C_n^{\delta})^{\frac{1}{\delta}} \\ &= (w_1 C^{\delta} \oplus w_2 C^{\delta} \oplus ... \oplus w_n C^{\delta})^{\frac{1}{\delta}} \\ &= ((w_1 + w_2 + ... + w_n) C^{\delta})^{\frac{1}{\delta}} \\ &= (C^{\delta})^{\frac{1}{\delta}} = \tilde{C}. \end{aligned}$$

Theorem 3.7 Let $\tilde{C}_j = \left\langle \bar{A}_{c_j}, \eta_{c_j} \right\rangle$ $(j = 1, 2, ..., n)$ be a collection of cubic value set where $\delta > 0,$ and $w = (w_1, w_2, ..., w_n)^T$ be the weighting vector associated with the $GCWA$ operator with $w_j \geq 0,\ j = 1, 2, ..., n,$ and $\sum_{j=1}^{n} w_j = 1,$

Let

$$\bar{C} = \left(\min_j (\bar{A}_{c_j}), \max_j (\eta_{c_j}) \right),\ C^{+} = \left(\max_j (\bar{A}_{c_j}), \min_j (\eta_{c_j}) \right).$$

Then,

$$\bar{C} \leq GCWA_w(C_1, C_2, ..., C_n) \leq C^{+}.$$

Proof Since

$$\min_j(\bar{A}_{c_j}) \le (\bar{A}_{c_j}) \le \max_j(\bar{A}_{c_j}) \text{ and}$$

$$\min_j(\eta_{c_j}) \le \eta_{c_j} \le \max_j(\eta_{c_j}), \forall\ j,.\text{Then,}$$

$$\Rightarrow \min_j(\bar{a}_{c_j}) \le \bar{a}_{c_j} \le \max_j(\bar{a}_{c_j}) \text{ and}$$

$$\min_j(a^{+}_{c_j}) \le a^{+}_{c_j} \le \max_j(a^{+}_{c_j})$$

$$\prod_{j=1}^{n}(1-\bar{a}^{\delta}_{c_j})^{w_j} \ge \prod_{j=1}^{n}\left(1-(\max_j(\bar{a}_{c_j})\right)^{w_j} = 1-(\max_j(\bar{a}_{c_j})^{\delta}$$

$$\text{and} \prod_{j=1}^{n}(1-a^{+\delta}_{c_j})^{w_j} \ge \prod_{j=1}^{n}\left(1-(\max_j(a^{+\delta}_{c_j})\right)^{w_j} = 1-(\max_j(a^{+}_{c_j})^{\delta}$$

$$\left(1-\prod_{j=1}^{n}(1-\bar{a}^{\delta}_{c_j})^{w_j}\right)^{\frac{1}{\delta}} \le \max_j(\bar{a}^{\delta}_{c_j}) \text{ and}$$

$$\left(1-\prod_{j=1}^{n}(1-a^{+\delta}_{c_j})^{w_j}\right)^{\frac{1}{\delta}} \le \max_j(a^{+\delta}_{c_j})$$

$$= \begin{bmatrix} \left(1-\prod_{j=1}^{n}(1-\bar{a}^{\delta}_{c_j})^{w_j}\right)^{\frac{1}{\delta}}, \\ \left(1-\prod_{j=1}^{n}(1-a^{+\delta}_{c_j})^{w_j}\right)^{\frac{1}{\delta}} \end{bmatrix}$$

$$\le \max_j(\bar{a}^{\delta}_{c_j}, a^{+\delta}_{c_j}) \qquad (13)$$

Similarly

$$\left(1-\prod_{j=1}^{n}(1-\bar{a}^{\delta}_{c_j})^{w_j}\right)^{\frac{1}{\delta}} \ge \min_j(\bar{a}^{\delta}_{c_j})$$

$$\text{and} \left(1-\prod_{j=1}^{n}(1-a^{+\delta}_{c_j})^{w_j}\right)^{\frac{1}{\delta}} \ge \max_j(a^{+\delta}_{c_j})$$

$$= \begin{bmatrix} \left(1-\prod_{j=1}^{n}(1-\bar{a}^{\delta}_{c_j})^{w_j}\right)^{\frac{1}{\delta}}, \\ \left(1-\prod_{j=1}^{n}(1-a^{+\delta}_{c_j})^{w_j}\right)^{\frac{1}{\delta}} \end{bmatrix}$$

$$\ge \max_j(\bar{a}^{\delta}_{c_j}, a^{+\delta}_{c_j}) \qquad (14)$$

$$\prod_{j=1}^{n}\left(1-(1-\eta_{c_j})^{\delta}\right)^{w_j} \le \prod_{j=1}^{n}\left(1-(1-\max_j(\eta_{c_j}))^{\delta}\right)^{w_j}$$

$$= 1-\left(1-\max_j(\eta_{c_j})\right)^{\delta}$$

$$\left(1-\prod_{j=1}^{n}(1-(1-\eta_{c_j})^{\delta})^{w_j}\right) \ge \left(1-\max_j(\eta_{c_j})\right)^{\delta}$$

$$\left(1-\prod_{j=1}^{n}(1-(1-\eta_{c_j})^{\delta})^{w_j}\right)^{\frac{1}{\delta}} \ge 1-\max_j(\eta_{c_j})$$

$$1-\left(1-\prod_{j=1}^{n}(1-(1-\eta_{c_j})^{\delta})^{w_j}\right)^{\frac{1}{\delta}} \le \max_j(\eta_{c_j}) \qquad (15)$$

Similarly

$$1-\left(1-\prod_{j=1}^{n}(1-(1-\eta_{c_j})^{\delta})^{w_j}\right)^{\frac{1}{\delta}} \ge \min_j(\eta_{c_j}). \qquad (16)$$

Let $GCWAw \quad (c_1,c_2,,...,c_n)=\tilde{C}=\left\langle \bar{A}_{c_j},\eta_{c_j}\right\rangle$, where $\left\langle \bar{A}_c,\eta_c\right\rangle=\left\langle\left[\bar{a}_c,a_c^{+}\right],\eta_c\right\rangle$. Then,

$$S(\tilde{C})=\frac{A_c-\eta_c}{3}=\frac{\bar{a}+2a^{+}-\eta}{3}\le \max_j(A_{c_j})-\min_j(\eta_{c_j})=S(C^{+})$$

$$S(\tilde{C})=\frac{A_c-\eta_c}{3}\ge \min_j(A_{c_j})-\max_j(\eta_{c_j})=S(\bar{C}).$$

If $S(\tilde{C})\le S(\tilde{C}^{+})$ and $S(\tilde{C})\ge S(\bar{C})$, then by Definition 5, we have

$$\bar{C}<GCWA_w(C_1,C_2,...,C_n)<C^{+}. \qquad (17)$$

If

$$S(\tilde{C})=S(\tilde{C}^{+}), i.e. A_c-\eta_c=\max_j(A_{c_j})-\min_j(\eta_{c_j}).$$

Then, by Eq. 13 and Eq. 16 such that,

$$A_c=\max_j(A_{c_j}),\ \eta_c=\min_j(\eta_{c_j}).$$

Hence,

$$h(\tilde{C})=\frac{1+A_c-\eta_c}{4}=\frac{1+\bar{a}+2a^{+}-\eta}{4}$$

$$=\max_j(A_{c_j})+\min_j(\eta_{c_j})=h(\tilde{C}^{+}).$$

Then, by Definition 5, we have

$$GCWA_w(\tilde{C}_1,\tilde{C}_2,...,\tilde{C}_n)=\tilde{C}^{+}. \qquad (18)$$

If $S(\tilde{C}) = S(\tilde{\bar{C}})$ such that,

$$A_c - \eta_c = \min_j (A_{c_j}) - \max_j (\eta_{c_j}).$$

Then by Eq. 14 and Eq. 15 we have

$$A_c = \min_j (A_{c_j}),\ \eta_c = \max_j (\eta_{c_j}).$$

Therefore,

$$h(\tilde{C}) = \frac{1 + A_c - \eta_c}{4} = \frac{1 + \bar{a} + 2a^{+} - \eta}{4}$$

$$h(\tilde{C}) = \min_j (A_{c_j}) + \max_j (\eta_{c_j}) = h(\bar{C}).$$

Thus, from Definition 5, we have

$$GCWA_w(\tilde{C}_1, \tilde{C}_2, ..., \tilde{C}_n) = \bar{C} \qquad (19)$$

From Eqs. 17–19, Eq. 12, always hold.

Theorem 3.8 Let $\tilde{C}_j = \left\langle \bar{A}_{c_j}, \eta_{c_j} \right\rangle$ $(j = 1, 2, ..., n)$ and $\tilde{C}^*_j = \left\langle \bar{A}_{c^*_j}, \eta_{c^*_j} \right\rangle$ $(j = 1, 2, ..., n)$ be a collection of any two cubic value set and $w = (w_1, w_2, ..., w_n)^T$ be the weighting vector related to the $GCWA$ operator, with $w_j \geq 0$, $j = 1, 2, ..., n$, and $\sum_{j=1}^{n} w_j = 1$, and $\delta > 0$. If $\bar{A}_{c_j} \leq \bar{A}_{c^*_j}$ and $\eta_{c_j} \geq \eta_{c^*_j}$, for all j, such that,

$$GCWA_w(c_1, c_2, ,..., c_n) \leq GCWA_w(c^*_1, c^*_2, ,..., c^*_n). \qquad (20)$$

Proof As we know that

$$\tilde{C}_j = \left\langle \bar{A}_{c_j}, \eta_{c_j} \right\rangle = \left\langle \left[\bar{a}_{c_j}, a^{+}_{c_j} \right], \eta_{c_j} \right\rangle \text{ and}$$

$$\tilde{C}^*_j = \left\langle \bar{A}_{c^*_j}, \eta_{c^*_j} \right\rangle = \left\langle \left[\bar{a}_{c^*_j}, a^{+}_{c^*_j} \right], \eta_{c^*_j} \right\rangle$$

Therefor $\bar{A}_{c_j} \leq \bar{A}_{c^*_j}$ and $\eta_{c_j} \geq \eta_{c^*_j}$, for all j, such that,

$$\left[\prod_{j=1}^{n}(1-a^{-\delta}_{c_j})^{w_j},\prod_{j=1}^{n}(1-a^{+\delta}_{c_j})^{w_j}\right]$$
$$\geq\left[\prod_{j=1}^{n}(1-a^{-\delta}_{c_j^*})^{w_j},\prod_{j=1}^{n}(1-a^{+\delta}_{c_j^*})^{w_j}\right]$$
$$\left[1-\prod_{j=1}^{n}(1-a^{-\delta}_{c_j})^{w_j},1-\prod_{j=1}^{n}(1-a^{+\delta}_{c_j})^{w_j}\right]$$
$$\leq\left[1-\prod_{j=1}^{n}(1-a^{-\delta}_{c_j^*})^{w_j},1-\prod_{j=1}^{n}(1-a^{+\delta}_{c_j^*})^{w_j}\right]$$
$$\left[1-\prod_{j=1}^{n}(1-a^{-\delta}_{c_j})^{w_j},1-\prod_{j=1}^{n}(1-a^{+\delta}_{c_j})^{w_j}\right]^{\frac{1}{\delta}}$$
$$\leq\left[1-\prod_{j=1}^{n}(1-a^{-\delta}_{c_j^*})^{w_j},1-\prod_{j=1}^{n}(1-a^{+\delta}_{c_j^*})^{w_j}\right]^{\frac{1}{\delta}}$$

and

$$\prod_{j=1}^{n}(1-(1-\eta_{c_j})^{\delta})^{w_j}\geq\prod_{j=1}^{n}(1-(1-\eta_{c_j^*})^{\delta})^{w_j}$$
$$\left(1-\prod_{j=1}^{n}(1-(1-\eta_{c_j})^{\delta})^{w_j}\right)\leq\left(1-\prod_{j=1}^{n}(1-(1-\eta_{c_j^*})^{\delta})^{w_j}\right)$$
$$\left(1-\prod_{j=1}^{n}(1-(1-\eta_{c_j})^{\delta})^{w_j}\right)^{\frac{1}{\delta}}\leq\left(1-\prod_{j=1}^{n}(1-(1-\eta_{c_j^*})^{\delta})^{w_j}\right)^{\frac{1}{\delta}}$$
$$1-\left(1-\prod_{j=1}^{n}(1-(1-\eta_{c_j})^{\delta})^{w_j}\right)^{\frac{1}{\delta}}\geq 1-\left(1-\prod_{j=1}^{n}(1-(1-\eta_{c_j^*})^{\delta})^{w_j}\right)^{\frac{1}{\delta}}$$

Hence

$$\frac{\left(1-\prod_{j=1}^{n}(1-a^{-\delta}_{c_j})^{w_j}\right)^{\frac{1}{\delta}}+\left(1-\prod_{j=1}^{n}(1-a^{+\delta}_{c_j})^{w_j}\right)^{\frac{1}{\delta}}-\left(1-\left(1-\prod_{j=1}^{n}(1-(1-\eta_{c_j})^{\delta})^{w_j}\right)^{\frac{1}{\delta}}\right)}{3}$$
$$\leq\frac{\left(1-\prod_{j=1}^{n}(1-a^{-\delta}_{c_j^*})^{w_j}\right)^{\frac{1}{\delta}}+\left(1-\prod_{j=1}^{n}(1-a^{+\delta}_{c_j^*})^{w_j}\right)^{\frac{1}{\delta}}-\left(1-\left(1-\prod_{j=1}^{n}(1-(1-\eta_{c_j^*})^{\delta})^{w_j}\right)^{\frac{1}{\delta}}\right)}{3}\quad(21)$$

Let

$\widetilde{C} = GCWA_w(c_1, c_2, ,..., c_n), \widetilde{C}^* = GCWA_w(c_1^*, c_2^*, ,..., c_n^*).$

Then by Eq. 21, we have $S(\widetilde{C}) \le S(\widetilde{C}^*)$. If $S(\widetilde{C}) < S(\widetilde{C}^*)$, then by Definition 5, we have

$GCWA_w(\widetilde{c}_1, \widetilde{c}_2, ,..., \widetilde{c}_n), < GCWA_w(\widetilde{c}_1^*, \widetilde{c}_2^*, ,..., \widetilde{c}_n^*)$ (22) If $S(\widetilde{C}) = S(\widetilde{C}^*)$, such that,

$$\frac{\left(1-\prod_{j=1}^{n}(1-\bar{a}_{c_j}^{\delta})^{w_j}\right)^{\frac{1}{\delta}} + \left(1-\prod_{j=1}^{n}(1-a_{c_j}^{+\delta})^{w_j}\right)^{\frac{1}{\delta}} - \left(1-\left(1-\prod_{j=1}^{n}(1-(1-\eta_{c_j})^{\delta})^{w_j}\right)^{\frac{1}{\delta}}\right)}{3}$$

$$= \frac{\left(1-\prod_{j=1}^{n}(1-\bar{a}_{c_j^*}^{\delta})^{w_j}\right)^{\frac{1}{\delta}} + \left(1-\prod_{j=1}^{n}(1-a_{c_j^*}^{+\delta})^{w_j}\right)^{\frac{1}{\delta}} - \left(1-\left(1-\prod_{j=1}^{n}(1-(1-\eta_{c_j^*})^{\delta})^{w_j}\right)^{\frac{1}{\delta}}\right)}{3}$$

Since $\bar{A}_{c_j} \le \bar{A}_{c_j^*}$ and $\eta_{c_j} \ge \eta_{c_j^*}$, for all j , such that,

$$\left[\left(1-\prod_{j=1}^{n}(1-\bar{a}_{c_j}^{\delta})^{w_j}\right)^{\frac{1}{\delta}}, \left(1-\prod_{j=1}^{n}(1-a_{c_j}^{+\delta})^{w_j}\right)^{\frac{1}{\delta}}\right]$$

$$= \left[\left(1-\prod_{j=1}^{n}(1-\bar{a}_{c_j^*}^{\delta})^{w_j}\right)^{\frac{1}{\delta}}, \left(1-\prod_{j=1}^{n}(1-a_{c_j^*}^{+\delta})^{w_j}\right)^{\frac{1}{\delta}}\right]$$

and

$$1-\left(1-\prod_{j=1}^{n}(1-(1-\eta_{c_j})^{\delta})^{w_j}\right)^{\frac{1}{\delta}}$$

$$= 1-\left(1-\prod_{j=1}^{n}(1-(1-\eta_{c_j^*})^{\delta})^{w_j}\right)^{\frac{1}{\delta}}.$$

Hence

$$h(C)=\frac{1+\left(1-\prod_{j=1}^{n}(1-\bar{a}_{c_j}^{\,\delta})^{w_j}\right)^{\frac{1}{\delta}}+\left(1-\prod_{j=1}^{n}(1-a_{c_j}^{+\delta})^{w_j}\right)^{\frac{1}{\delta}}-\left(1-\left(1-\prod_{j=1}^{n}(1-(1-\eta_{c_j})^{\delta})^{w_j}\right)^{\frac{1}{\delta}}\right)}{4}$$

$$h(C^*)=\frac{1+\left(1-\prod_{j=1}^{n}(1-\bar{a}_{c_j^*}^{\,\delta})^{wj}\right)^{\frac{1}{\delta}}+\left(1-\prod_{j=1}^{n}(1-a_{c_j^*}^{+\delta})^{wj}\right)^{\frac{1}{\delta}}-\left(1-\left(1-\prod_{j=1}^{n}(1-(1-\eta_{c_j^*})^{\delta})^{wj}\right)^{\frac{1}{\delta}}\right)}{4}$$

By Definition 5, such that,

$$GCWA_w(\tilde{c}_1,\tilde{c}_2,,...,\tilde{c}_n)=GCWA_w(\tilde{c}_1^*,\tilde{c}_2^*,,...,\tilde{c}_n^*). \qquad (23)$$

From Eq. 22 and Eq. 23, we know that Eq. 20 always holds.

Now we have some special cases which obtained by using choices of the parameters w and δ.

Theorem 3.9 Let $\tilde{C}_j=\left\langle \bar{A}_{c_j},\eta_{c_j}\right\rangle$ $(j=1,2,...,n)$ be a collection of cubic value set $\delta>0$, and $w=(w_1,w_2,...,w_n)^T$ be the weighting vector related to the $GCWA$ operator, with $w_j\geq 0$ $(j=1,2,...,n)$, and $\sum_{j=1}^{n}w_j=1$.

1. If $\delta=1$, then the $GCWA$ operator (9) is reduced to the following:

$$CWA_w(\tilde{c}_1,\tilde{c}_2,,...,\tilde{c}_n)=w_1c_1\oplus w_2c_2\oplus...\oplus w_nc_n,$$

which is called cubic weighted average operator.

2. $\delta\to 0$, then the $GCWA$ operator (9) is reduced to the following:

$$CWG_w(\tilde{c}_1,\tilde{c}_2,,...,\tilde{c}_n)=c_1^{w_1}\otimes c_2^{w_2}\otimes...\otimes c_n^{w_n},$$

which is called cubic weighted geometric operator.

3. $\delta\to+\infty$, then the $GCWA$ operator (9) is reduced to the following:

$$CMAX_w(\tilde{c}_1,\tilde{c}_2,,...,\tilde{c}_n)=\max_j(\tilde{C}_j),$$

which is called cubic maximum operator.

4. If $w=(\frac{1}{n},\frac{1}{n},...,\frac{1}{n})^T$ and $\delta=1$, then the $GCWA$ operator (9) is reduced to the following:

$$CA_w(\tilde{c}_1,\tilde{c}_2,,\ldots,\tilde{c}_n)=\frac{1}{n}(c_1\oplus c_2\oplus\ldots\oplus c_n),$$

which is called cubic average operator.

5. If $w=(\frac{1}{n},\frac{1}{n},...,\frac{1}{n})^T$ and $\delta\to 0,$ then the $GCWA$ operator (9) is reduced to the following:

$$CG_w(\tilde{c}_1,\tilde{c}_2,,...,\tilde{c}_n)=(c_1\otimes c_2\otimes...\otimes c_n),$$

which is called cubic geometric operator.

The GCOWA Operator

In this section we shall define $GCOWA$ operator and study different results relevant to $GCOWA$ operator.

Definition 3.10 Let $\tilde{C}_j=\left\langle \bar{A}_{c_j},\eta_{c_j}\right\rangle$ $(j=1,2,...,n)$ be a collection of cubic value set and $GCOWA: C^n\to C,$ if

$$GCOWA_w(\tilde{c}_1,\tilde{c}_2,,...,\tilde{c}_n)=(w_1c^{\delta}_{\sigma(1)}\oplus w_2c^{\delta}_{\sigma(2)}\oplus...\oplus w_nc^{\delta}_{\sigma(n)})^{\frac{1}{\delta}}, \tag{24}$$

where $\delta>0,$ and $w=(w_1,w_2,...,w_n)^T$ be the weighting vector such that $w_j\geq 0,$ $j=1,2,...,n,$ and $\sum_{j=1}^{n}w_j=1,$ $\tilde{C}$ is the jth largest of $\tilde{C}_j$, then the function $GCOWA$ is called a $GCOWA$ operator.

The $GCOWA$ operator has some properties similar to those of the $GCWA$ operator.

Theorem 3.11 Let $\tilde{C}_j=\left\langle \bar{A}_{c_j},\eta_{c_j}\right\rangle$ $(j=1,2,...,n)$ be a collection of cubic value set then their aggregated value by using the $GCOWA$ operator is also a cubic value such that,

$$GCOWA_w(\tilde{c}_1,\tilde{c}_2,,...,\tilde{c}_n)$$
$$=\left\langle \begin{array}{c}\left[\left(1-\prod_{j=1}^{n}(1-\bar{a}^{-\delta}_{c_{\sigma(j)}})^{w_j}\right)^{\frac{1}{\delta}},\left(1-\prod_{j=1}^{n}(1-a^{+\delta}_{c_{\sigma(j)}})^{w_j}\right)^{\frac{1}{\delta}}\right],\\ 1-\left(1-\prod_{j=1}^{n}(1-(1-\eta_{c_{\sigma(j)}})^{\delta})^{w_j}\right)^{\frac{1}{\delta}}\end{array}\right\rangle. \tag{25}$$

where $\delta>0,$ and $w=(w_1,w_2,...,w_n)^T$ be weighting vector associated with the $GCOWA$ operator, with $w_j\geq 0,$ $j=1,2,...,n$ and $\sum_{j=1}^{n}w_j=1,$ $\tilde{c}_{\sigma(j)}$ is the jth largest of C_j.

Example 3.12 Let

$$\tilde{C}_1=\langle[0.2,0.4],0.3\rangle,\ \tilde{C}_2=\langle[0.3,0.4],0.5\rangle,\ \tilde{C}_3=\langle[0.3,0.6],0.2\rangle,$$
$$\tilde{C}_4=\langle[0.4,0.6],0.6\rangle,\text{ and }\tilde{C}_5=\langle[0.6,0.8],0.4\rangle,$$

be any five cubic numbers and $w=(0.2,0.3,0.12,0.16,0.22)^T$ be the weighting vector of $\tilde{C}j$ $(j=1,2,3,4,5)$. Let $\delta=2$. We calculate the scores of $\tilde{C}_j$ $(j=1,2,3,4,5)$.

$$S(\tilde{C}_1)=0.2333,\ S(\tilde{C}_2)=0.20,\ S(\tilde{C}_3)=0.4333,$$
$$S(\tilde{C}_4)=0.3333,\ \text{and}\ S(\tilde{C}_5)=0.60$$

Since

$$S(\tilde{C}_5)>S(\tilde{C}_3)>S(\tilde{C}_4)>S(\tilde{C}_1)>S(\tilde{C}_2).$$

then

$$\tilde{C}_{\sigma(1)}=\langle[0.6,0.8],0.4\rangle,\ \tilde{C}_{\sigma(2)}=\langle[0.3,0.6],0.2\rangle,$$
$$\tilde{C}_{\sigma(3)}=\langle[0.4,0.6],0.6\rangle,\ \tilde{C}_{\sigma(4)}=\langle[0.2,0.4],0.3\rangle,$$
$$\tilde{C}_{\sigma(5)}=\langle[0.3,0.4],0.5\rangle.$$

and thus, by Eq. 25, we have

$$GCOWA_w(\tilde{C}_1,\tilde{C}_2,\tilde{C}_3,\tilde{C}_4,\tilde{C}_5)=\langle[0.3910,0.6063],0.3332\rangle.$$

Theorem 3.13 Let $\tilde{C}_j=\left\langle \bar{A}_{c_j},\eta_{c_j}\right\rangle$ $(j=1,2,...,n)$ be a collection of cubic value set and $w=(w_1,w_2,...,w_n)^T$ be the weighting vector related to the $GCOW$ operator, with $w_j\geq 0,$ $j=1,2,...,n,$ and $\sum_{j=1}^{n}w_j=1.$ If all $\tilde{C}_j\,(j=1,2,...,n)$ are equal, i.e. $\tilde{C}_j=\tilde{C},$ for all j, then $GCOWA_w(\tilde{C}_1,\tilde{C}_2,,...,\tilde{C}_n)=\tilde{C}.$

Theorem 3.14 Let $\tilde{C}_j=\left\langle \bar{A}_{c_j},\eta_{c_j}\right\rangle$ $(j=1,2,...,n)$ be a collection of cubic value set and $w=(w_1,w_2,...,w_n)^T$ be the weighting vector related to the $GCOWA$ operator, with $w_j\geq 0,$ $j=1,2,...,n,$ and $\sum_{j=1}^{n}w_j=1.$

Let

$$\bar{C}=(\min_j(\bar{A}_{c_j}),\ \max_j(\eta_{c_j})),\ C^{+}=(\max_j(\bar{A}_{c_j}),\ \min_j(\eta_{c_j})).$$

Then,

$$\bar{C}\leq GCOWA_w(\tilde{c}_1,\tilde{c}_2,,...,\tilde{c}_n)\leq C^{+}.$$

Theorem 3.15 Let $\tilde{C}_j=\left\langle \bar{A}_{c_j},\eta_{c_j}\right\rangle$ $(j=1,2,...,n)$ and $\tilde{C}_j^{*}=\left\langle \bar{A}_{c_j^{*}},\eta_{c_j^{*}}\right\rangle$ $(j=1,2,...,n)$ be a collection of two cubic value set and $w=(w_1,w_2,...,w_n)^T$ be the weighting vector related to the $GCWA$ operator, with $w_j\geq 0,$ $j=1,2,...,n,$ and $\sum_{j=1}^{n}w_j=1,$ if $\bar{A}_{c_j}\leq \bar{A}_{c_j^{*}}$ and $\eta_{c_j}\geq\eta_{c_j^{*}},$ for all j, such that,

$$GCOWA_w(\tilde{c}_1,\tilde{c}_2,,...,\tilde{c}_n)\leq GCOWA_w(\tilde{c}_1^{*},\tilde{c}_2^{*},,...,\tilde{c}_n^{*}).$$

Theorem 3.16 Let $\widetilde{C}_j = \left\langle \bar{A}_{c_j}, \eta_{c_j} \right\rangle$ $(j = 1, 2, ..., n)$ and $\widetilde{C}_j^{/} = \left\langle \bar{A}_{c_j^{/}}, \eta_{c_j^{/}} \right\rangle$ $(j = 1, 2, ..., n)$ be a collection of two cubic value set， $\delta > 0$ and $w = (w_1, w_2, ..., w_n)^T$ be the weighting vector related to the $GCOWA$ operator, with $w_j \geq 0$, $j = 1, 2, ..., n$, and $\sum_{j=1}^{n} w_j = 1$ such that,

$$GCOWA_w(\widetilde{c}_1, \widetilde{c}_2, ,..., \widetilde{c}_n) = GCOWA_w(\widetilde{c}_1^{/}, \widetilde{c}_2^{/}, ,..., \widetilde{c}_n^{/}). \quad (26)$$

where $(c_1^{/}, c_2^{/}, ,..., c_n^{/})^T$ is any permutation of $(\widetilde{c}_1, \widetilde{c}_2, ,..., \widetilde{c}_n)^T$.

$$GCOWA_w(\widetilde{c}_1, \widetilde{c}_2, ,..., \widetilde{c}_n) = \left(w_1 c_{\alpha(1)}^{\delta} \oplus w_2 c_{\alpha(2)}^{\delta} \oplus ... \oplus w_n c_{\alpha(n)}^{\delta} \right)^{\frac{1}{\delta}}$$

$$GCOWA_w(c_1^{/}, c_2^{/}, ,..., c_n^{/}) = \left(w_1 (c_{\alpha(1)}^{/})^{\delta} \oplus w_2 (c_{\alpha(2)}^{/})^{\delta} \oplus ... \oplus w_n (c_{\alpha(n)}^{/})^{\delta} \right)^{\frac{1}{\delta}}.$$

Since $(c_1^{/}, c_2^{/}, ,..., c_n^{/})^T$ is any permutation of $(\widetilde{c}_1, \widetilde{c}_2, ,..., \widetilde{c}_n)^T$. Then, $c_{\alpha(j)} = c_{\alpha(j)}^{/}$, $j = 1, 2, ..., n$. From E.q 26, we now take a look at some the $GCOWA$ operator has commutativity property that we desire. It is worth noting that the $GCWA$ operator does not have this property. We now take a look at some special cases obtained by using different choices of the parameter w and δ.

Theorem 3.17 Let $\widetilde{C}_j = \left\langle \bar{A}_{c_j}, \eta_{c_j} \right\rangle$ $(j = 1, 2, ..., n)$ be a collection of cubic value set， $\delta > 0$ and $w = (w_1, w_2, ..., w_n)^T$ be the weighted vector related to the $GCOWA$ operator, with $w_j \geq 0$ $(j = 1, 2, ..., n)$, $\sum_{j=1}^{n} w_j = 1$, then

1. If $\delta = 1$, then the $GCOWA$ operator (24) is reduced to the following:

$$COWA_w(\widetilde{c}_1, \widetilde{c}_2, ,..., \widetilde{c}_n) = w_1 c_{\sigma(1)} \oplus w_2 c_{\sigma(2))} \oplus ... \oplus w_n c_{\sigma(n)},$$

which is called cubic ordered weighted average operator.

2. $\delta \to 0$, then the $GCOWA$ operator (24) is reduced to the following:

$$COWG_w(\widetilde{c}_1, \widetilde{c}_2, ,..., \widetilde{c}_n) = c_{\sigma(1)}^{w_1} \otimes c_{\sigma(2)}^{w_2} \otimes ... \otimes c_{\sigma(n)}^{w_n},$$

which is called cubic ordered weighted geometric operator.

3. $\delta \to +\infty$, then the $GCOWA$ operator (24) is reduced to the following:

$$CMAX_w(\widetilde{c}_1, \widetilde{c}_2, ,..., \widetilde{c}_n) = \max_j (C_j),$$

which is called cubic maximum operator.

4. If $w = (\frac{1}{n}, \frac{1}{n}, ..., \frac{1}{n})^T$ and $\delta = 1$, then the $GCOWA$ operator (24) is reduced to the following:

$$CA_w(\widetilde{c}_1, \widetilde{c}_2, ,..., \widetilde{c}_n) = \frac{1}{n}(c_1 \oplus c_2 \oplus ... \oplus c_n),$$

which is calld cubic averaging operator.

5. If $w=(\frac{1}{n},\frac{1}{n},...,\frac{1}{n})^T$ and $\delta \to 0$, then the $GCOWA$ operator (24) is reduced to the following:

$$CG_w(\tilde{c}_1,\tilde{c}_2,,...,\tilde{c}_n)=(c_1\otimes c_2\otimes ...\otimes c_n)^{\frac{1}{n}},$$

which is called cubic geometric operator.

The GCHA Operator

Consider that the $GCWA$ operator weights only the cubic value set whereas the $GCOWA$ operator weights only the ordered positions of the CVs instead of the weighting the cubic value set themselves. To overcome this limitation, motivated by the idea of combining the WA and OWA operators, in what follows, we developed a generalized cubic hybrid aggregation $(GCHA)$ operator, which weights both the given cubic value and its ordered position.

Definition 3.18 $GCHA$ operator of dimension n is a mapping $GCHA: C^n \to C$, which has an associated vector $w=(w_1,w_2,...,w_n)^T$, with $w_j \geq 0$, $j=1,2,...,n$, and $\sum_{j=1}^n w_j=1$, such that,

$$GCHA_{w,w}(\tilde{c}_1,\tilde{c}_2,,...,\tilde{c}_n)=\left(w_1(c^{\cdot}_{\sigma(1)})^{\delta}\oplus w_2(c^{\cdot}_{\sigma(2)})^{\delta}\oplus ...\oplus w_n(c^{\cdot}_{\sigma(n)})^{\delta}\right)^{\frac{1}{\delta}}. \quad (27)$$

where $\delta>0$, $c^{\cdot}_{\sigma(j)}$ is the jth largest of the weighted CVs $c^{\cdot}_j(c^{\cdot}_j=nw_jc_j$, $j=(1,2,...,n)$, and $w=(w_1,w_2,...,w_n)^T$ is the weighting vector of $c_j($ $j=1,2,...,n)$ with $w_j\geq 0$, and $\sum_{j=1}^n w_j=1$, and n is balancing coefficient, which plays a role of balance if the vector $w=(w_1,w_2,...,w_n)^T$ approaches $(\frac{1}{n},\frac{1}{n},...,\frac{1}{n})^T$. Then, the vector $(nw_1c_1,nw_2c_2,...,nw_nc_n)^T$ approaches $(c_1,c_2,...,c_n)^T$. Let $C^{\cdot}_{\alpha(j)}=\left\langle \bar{A}_{c^{\cdot}_{\sigma(j)}},\eta_{c_{\sigma(j)}}\right\rangle$, then, similar to Theorem 3, such that,

$$GCHA_{w,w}(c_1,c_2,...,c_n)$$
$$=\left\langle \begin{array}{c}\left[\left(1-\prod_{j=1}^n(1-\bar{a}^{\delta}_{c^{\cdot}_{\sigma(j)}})^{w_j}\right)^{\frac{1}{\delta}},\left(1-\prod_{j=1}^n(1-a^{+\delta}_{c^{\cdot}_{\sigma(j)}})^{w_j}\right)^{\frac{1}{\delta}}\right],\\ 1-\left(1-\prod_{j=1}^n(1-(1-\eta_{c^{\cdot}_{\sigma(j)}})^{\delta})^{w_j}\right)^{\frac{1}{\delta}}\end{array}\right\rangle \quad (28)$$

and the aggregated value derived by using the $GCHA$ operator is also CVs . Especially, if $\delta=1$, then (28) is reduced to the following form:

$$CHA_{w,w}(c_1,c_2,...,c_n) = \left\langle \begin{array}{c} \left[\left(1-\prod_{j=1}^{n}(1-\bar{a}^{\delta}_{c^{\cdot}_{\alpha(j)}})^{w_j}\right),\left(1-\prod_{j=1}^{n}(1-\overset{+}{a}^{\delta}_{c^{\cdot}_{\alpha(j)}})^{w_j}\right)\right], \\ \left(\prod_{j=1}^{n}(\eta_{c^{\cdot}_{\alpha(j)}})^{w_j}\right) \end{array} \right\rangle,$$

which is called cubic hybrid averaging (CHA) operator.

Theorem 3.19 The $GCOWA$ operator is a special case of the $GCHA$ operator.

Proof Let $w=(\frac{1}{n},\frac{1}{n},...,\frac{1}{n})^T$, then $\dot{C}_j = C_j\ (j=1,2,...,n)$, so we have

$$\begin{aligned} GCHA_{w,w}(c_1,c_2,,...,c_n) &= \left(w_1(c^{\cdot}_{\sigma(1)})^{\delta} \oplus w_2(c^{\cdot}_{\sigma(2)})^{\delta} \oplus ... \oplus w_n(c^{\cdot}_{\sigma(n)})^{\delta}\right)^{\frac{1}{\delta}} \\ &= (w_1(c^{\delta}_{\sigma(1)}) \oplus w_2(c^{\delta}_{\sigma(2)}) \oplus ... \oplus w_n(c^{\delta}_{\sigma(n)}))^{\frac{1}{\delta}} \\ &= GCOWA_w(c_1,c_2,...,c_n). \end{aligned}$$

which completes the proof of Theorem.

Example 3.20 Let $\tilde{C}_1 = \langle[0.2,0.3],0.5\rangle$, $\tilde{C}_2 = \langle[0.4,0.6],0.2\rangle$, $\tilde{C}_3 = \langle[0.5,0.7],0.3\rangle$, and $\tilde{C}_4 = \langle[0.6,0.7],0.1\rangle$, be any four cubic numbers and let $w=(0.1,0.3,0.2,0.4)^T$ be the weighting vector of $\tilde{C}_j$ $(j=1,2,3,4)$, and $\delta=2$, then by applying operational law 3, and Definition 4 we get

$$\tilde{C}^{\cdot}_1 = \langle[0.0853,0.1329],0.7578\rangle,\ \tilde{C}^{\cdot}_2 = \langle[0.4582,0.6669],0.1449\rangle,$$
$$\tilde{C}^{\cdot}_3 = \langle[0.4256,0.6183],0.3816\rangle,\ \tilde{C}^{\cdot}_4 = \langle[0.7691,0.8543],0.0251\rangle.$$

By using Eq. 5, we calculate the scores of $\tilde{C}_j\ (j=1,2,3,4)$

$$S(\tilde{C}^{\cdot}_1) = -0.1355, S(\tilde{C}^{\cdot}_2) = 0.5490, S(\tilde{C}^{\cdot}_3) = 0.4268, S(\tilde{C}^{\cdot}_4) = 0.8275,$$

$S(\tilde{C}^{\cdot}_4) > S(\tilde{C}^{\cdot}_2) > S(\tilde{C}^{\cdot}_3) > S(\tilde{C}^{\cdot}_1).$

Then,

$$\tilde{C}^{\cdot}_{\sigma(1)} = \langle[0.7691,0.8543],0.0251\rangle,\ \tilde{C}^{\cdot}_{\sigma(2)} = \langle[0.4582,0.6669],0.1449\rangle,$$
$$\tilde{C}^{\cdot}_{\sigma(3)} = \langle[0.4256,0.6183],0.3816\rangle,\ \tilde{C}^{\cdot}_{\sigma(4)} = \langle[0.0853,0.1329],0.7578\rangle.$$

Now we find the weighting vector of $GCHA$ operator by means of the normal distribution based method such that, $w=(0.1550,0.3450,0.3450,0.1550)^T$. Then, by Eq. 28 it follows that,

$$GCHA_{w,w}(\tilde{C}_1,\tilde{C}_2,\tilde{C}_3,\tilde{C}_4) = \langle[0.5020,0.6612],\ 0.1842\rangle.$$

4. Applications in Decision Making Problem

In this section, we provide an application of proposed score function, accuracy function and aggregation operators. We develop a general algorithm frame work of proposed aggregation operators and their application. In decision support system (DSS) the group decision making problem under consideration is explained as follows;

Algorithm 1. Let $X=\{A_1, A_2,..., A_n\}$ be the set of n alternatives, and $C=\{C_1, C_2,..., C_m\}$ be set of criteria of the each alternative with weighting vector of m criteria $w=(w, w,...w_m)^T$ such that $w_j \in [0,1]$ and $\sum_{j=1}^{m} w_j =1$. Let $D_{ij}=\left\langle \bar{A}_{ij}, \eta_{ij} \right\rangle$ be cubic matrices, where $\langle A_{ij}, \eta_{ij} \rangle$ is an evaluation in term of cubic sets provided by decision maker related to the alternative $A_i \in A$ based on the criterion $C_j \in C$. The main goal of decision maker is finding the best alternative or ranking the alternative given information. In decision making process it depends on the weights of criteria of the alternatives. In this method we proposed an algorithm to rank the alternative or find out the best one of alternatives. our method is based on more knowledge about the criteria of each alternative. Then decision making method consists of the following steps.

Step 1. The decision makers give their opinions related to each alternative with respect to each criterion. The evaluation of each alternative with respect to each given criterion is listed in decision matrices .

Cubic Decision Matrix

$$(D_{ij})_{nm} = \begin{array}{c|cccc} & C_1 & C_2 & \cdots & C_m \\ \hline A_1 & \left\langle \bar{A}_{11}, \eta_{11} \right\rangle & \left\langle \bar{A}_{12}, \eta_{12} \right\rangle & \cdots & \left\langle \bar{A}_{1m}, \eta_{1m} \right\rangle \\ A_2 & \left\langle \bar{A}_{21}, \eta_{21} \right\rangle & \left\langle \bar{A}_{22}, \eta_{22} \right\rangle & \cdots & \left\langle \bar{A}_{2m}, \eta_{2m} \right\rangle \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ A_n & \left\langle \bar{A}_{n1}, \eta_{n1} \right\rangle & \left\langle \bar{A}_{n2}, \eta_{n2} \right\rangle & \cdots & \left\langle \bar{A}_{nm}, \eta_{nm} \right\rangle \end{array}$$

Step 2. Applying generalized cubic weighted aggregation $(GCWA)$ operator to cubic decision matrices, the aggregated information of each alternative with respect the criteria.

Step 3. In this step, we calculate the scores to aggregate the value of each alternative. If there is no difference between two or more than two scores then we have must to find out the accuracy degrees of the aggregated values of each alternative.

Step 4. In this step, we arrange all the score values of the alternatives in the form of descending order and select the best alternative which has the highest degree of the score value.

Algorithm 2. Let $X=\{A_1, A_2,..., A_n\}$ be the set of n alternatives and $C=\{C_1, C_2,..., C_m\}$ be set of criteria of the each alternative with weighting vector of m criteria

$w = (w, w, \ldots w_m)^T$ such that $w_j \in [0,1]$ and $\sum_{j=1}^{m} w_j = 1$. Let $D_{ij} = \left\langle \bar{A}_{ij}, \eta_{ij} \right\rangle$ be cubic matrices, where $\left\langle \bar{A}_{ij}, \eta_{ij} \right\rangle$ is an evaluation in term of cubic sets provided by decision maker related to the alternative $A_i \in A$ based on the criterion $C_j \in C.$ The main goal of decision maker is finding the best alternative or ranking the alternative given information. In decision making process it depends on the weights of criteria of the alternative. In this method we proposed an algorithm to rank the alternative or find out the best one of alternatives. Our method is based on more knowledge about the criteria of each alternative. Then decision making method consists of the following steps.

Step 1. The decision makers give their opinions related to each alternative with respect to each criterion. The evaluation of each alternative with respect to each given criterion is listed in decision matrices

Cubic Decision Matrix

$$(D_{ij})_{nm} = \begin{array}{c|cccc} & C_1 & C_2 & \cdots & C_m \\ \hline A_1 & \left\langle \bar{A}_{11}, \eta_{11} \right\rangle & \left\langle \bar{A}_{12}, \eta_{12} \right\rangle & \cdots & \left\langle \bar{A}_{1m}, \eta_{1m} \right\rangle \\ A_2 & \left\langle \bar{A}_{21}, \eta_{21} \right\rangle & \left\langle \bar{A}_{22}, \eta_{22} \right\rangle & \cdots & \left\langle \bar{A}_{2m}, \eta_{2m} \right\rangle \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ A_n & \left\langle \bar{A}_{n1}, \eta_{n1} \right\rangle & \left\langle \bar{A}_{n2}, \eta_{n2} \right\rangle & \cdots & \left\langle \bar{A}_{nm}, \eta_{nm} \right\rangle \end{array}$$

Step 2. In this step, we apply the known weightted vector by using operational law 3 in Definition 4, and score function to order the cubic values in cubic decision matrix.

Step 3. Applying generalized cubic hybrid aggregation ($GCHA$) operator to cubic decision matrix the aggregated information of each alternative with respect the criteria.

Step 4. In this step, we calculate the scores of the aggregated values of each alternative. If there is no difference between two or more than two scores then we have to find out the accuracy degrees of the aggregated values of each alternative.

Step 5. In this step, we arrange all the score values of the alternatives in the form of descending order and select the best alternative which has the highest degree of the score value.

Step 6. End

5. Illustrative Example

In this section, we are going to present an illustrative example of the new approach in a decision-making problem. We analyze a company that operates in Europe and North America that wants to invest some money in a new market. They consider four possible alternatives

- A_1 = Invest in the Asian market.
- A_2 = Invest in the South American market.
- A_3 = Invest in the African market.
- A_4 = Invest in all three markets.

To evaluate these alternatives, the investor has brought together a group of three alternatives. After analyzing the information, this group considers that the key factor is the economic situation of the world economy for the next period. They consider five main possible states of nature that could happen in the future:

Let C_1, C_2, C_3, C_4, C_5 be criteria for these four markets, In the process of choosing one of the market, five factor are considered;

- $C_1 =$ Very bad economic situation.
- $C_2 =$ Bad economic situation.
- $C_3 =$ Regular economic situation.
- $C_4 =$ Good economic situation.
- $C_5 =$ Very good economic situation.

Suppose that the weighting vector of $C_j\,(j=1,2,...,5)$ is and $w=(0.2,0.3,0.13,0.17,0.20,)^T$, $\delta=2$, the cubic values of the alternatives $A_i\,(i=1,2,3,4)$ are represented by the cubic decision matrix $a_{ij}\,(i=1,2,3,4,;$ $j=1,2,3,4,5)$ listed in Table 1.

Step 1. The decision makers give their opinions in Table 1.

Table 1. Cubic decision matrix

$D_{ij} =$

	C_1	C_2	C_3	C_4	C_5
A_1	$\langle[0.6,0.7],0.9\rangle$	$\langle[0.4,0.5],0.2\rangle$	$\langle[0.4,0.6],0.7\rangle$	$\langle[0.6,0.7],0.3\rangle$	$\langle[0.5,0.6],0.6\rangle$
A_2	$\langle[0.5,0.8],0.7\rangle$	$\langle[0.6,0.7],0.8\rangle$	$\langle[0.4,0.7,0.5\rangle$	$\langle[0.6,0.8],0.2\rangle$	$\langle[0.2,0.4],0.1\rangle$
A_3	$\langle[0.3,0.4],0.5\rangle$	$\langle[0.6,0.8],0.3\rangle$	$\langle[0.7,0.8],0.9\rangle$	$\langle[0.4,0.6],0.5\rangle$	$\langle[0.3,0.7],0.4\rangle$
A_4	$\langle[0.7,0.9],0.6\rangle$	$\langle[0.4,0.6],0.2\rangle$	$\langle[0.5,0.6],0.8\rangle$	$\langle[0.4,0.6],0.3\rangle$	$\langle[0.4,0.8],0.6\rangle$

Step 2. Now we normalized the decision making matrices by using normalized procedure.

Table 2. Normalized cubic decision matrix $R_{ij} =$

	C_1	C_2	C_3	C_4	C_5
A_1	$\langle[0.3,0.4],0.1\rangle$	$\langle[0.5,0.6],0.8\rangle$	$\langle[0.4,0.6],0.3\rangle$	$\langle[0.3,0.4],0.7\rangle$	$\langle[0.4,0.5],0.4\rangle$
A_2	$\langle[0.2,0.5],0.3\rangle$	$\langle[0.3,0.4],0.2\rangle$	$\langle[0.3,0.6],0.5\rangle$	$\langle[0.2,0.4],0.8\rangle$	$\langle[0.6,0.8],0.9\rangle$
A_3	$\langle[0.6,0.7],0.5\rangle$	$\langle[0.2,0.4],0.7\rangle$	$\langle[0.2,0.3],0.1\rangle$	$\langle[0.4,0.6],0.5\rangle$	$\langle[0.3,0.7],0.6\rangle$
A_4	$\langle[0.1,0.3],0.4\rangle$	$\langle[0.4,0.6],0.8\rangle$	$\langle[0.4,0.5],0.2\rangle$	$\langle[0.4,0.6],0.7\rangle$	$\langle[0.2,0.6],0.4\rangle$

Step 2. Now using generalized cubic weighted aggregation operator by using Eq. 10, we have the aggregated values of the normalized cubic decision matrix is given in Table 3.

Table 3. Aggregated values

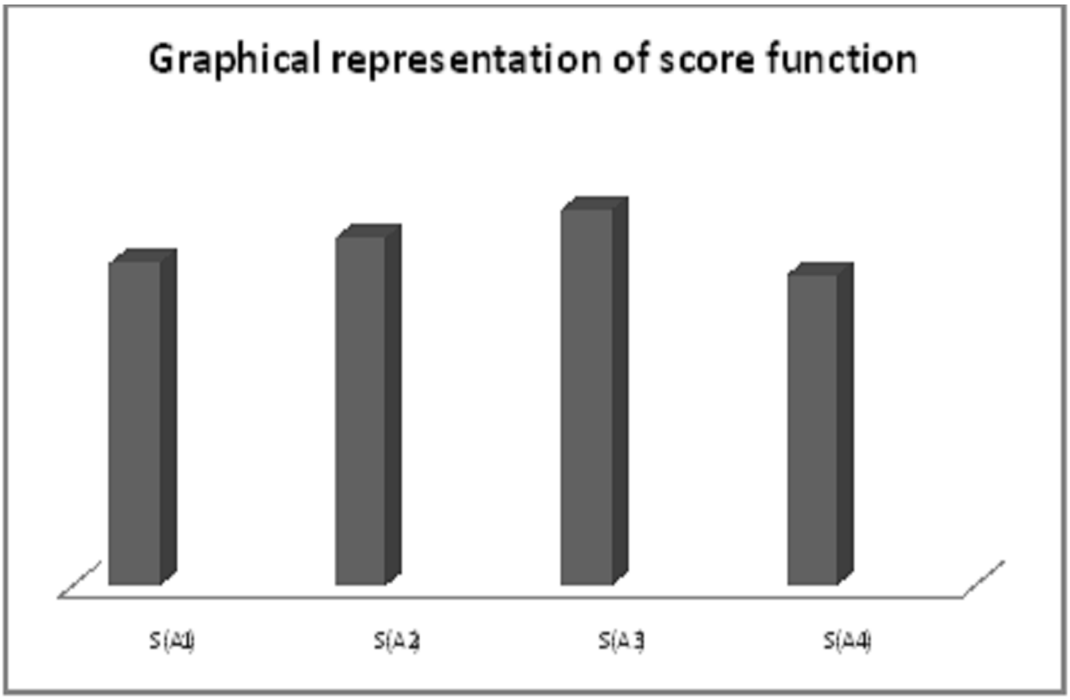
A_1	$\langle[0.4511, 0.5641], 0.2361\rangle$
A_2	$\langle[0.4608, 0.6398], 0.2938\rangle$
A_3	$\langle[0.4919, 0.6965], 0.3227\rangle$
A_4	$\langle[0.3637, 0.6127], 0.2983\rangle$

Step 3. In this step, we calculate the scores to aggregate the value of each alternative. If there is no difference between two or more than two scores then we have to find out the accuracy degrees of the aggregated values of each alternative.

$S(A_1)$	$S(A_2)$	$S(A_3)$	$S(A_4)$
0.4477	0.4822	0.5207	0.4302

Step 4. In this step we arrange all the score values of the alternatives in the form of descending order and select the best alternative which has the highest degree of the score value. Here $A_3 > A_2 > A_1 > A_4$. Thus most wanted alternative is A_3.

Figure 1



a. Illustrative Example

A computer center in a university desires to select an information system to improve the product, for this purpose suppose A_1, A_2, A_3, A_4 are four alternatives $A_i(i = 1, 2, 3, 4)$ have remained the list of candidate. There are four experts from a committee to act the decision makers having weighting vector $\lambda =$ $(0.3, 0.2, 0.4, 0.1)^T$. Consider there are four attributes C_1, C_2, C_3, C_4 such that $C_j(j = 1, 2, 3, 4)$,

(i) C_1 is cost for software investment,

(ii) C_2 is contribution for organization performance,

(iii) C_3 is effort to transformation current system,

(iv) C_4 is for out sourcing software reliability.

Consider that the weighting vector of $C_j\,(j=1,2,...,4)$ is $w=(0.1,0.3,0.2,0.4,)^T$, $\delta=2,$ and the cubic values of the alternatives $A_i\,(i=1,2,3,4)$ are represented by the cubic decision matrix $a_{ij}\,(i=1,2,3,4$; $j=1,2,3,4)$ listed in Table 1. (Cubic decision matrix), to rank the given four projects, we first weight all the (CVs) $a_{ij}\,(i=1,2,3,4$; $j=1,2,3,4)$ by the weighting vector $w=(0.1,0.3,0.2,0.4)^T$ of the attribute $C_j\,(j=1,2,...,4)$ and multiply these values by the balancing coefficient $n=4,$ and we get (CVs) $4w_j a_{ij}$, listed in Table 2. Then, we utilize the *GCHA* operator $w=(0.1550,0.3450,0.3450,0.1550)^T$ be the weighting vector derived by the normal distribution based method to get the overall values.

Step 1. The decision makers give their opinions in table 1.

Table 1. Cubic Decision Matrix

$D_{ij}=$

	C_1	C_2	C_3	C_4
A_1	$\langle[0.2,0.3],0.5\rangle$	$\langle[0.4,0.6],0.2\rangle$	$\langle[0.5,0.7],0.3\rangle$	$\langle[0.6,0.7],0.1\rangle$
A_2	$\langle[0.1,0.2],0.3\rangle$	$\langle[0.3,0.5],0.4\rangle$	$\langle[0.6,0.8],0.6\rangle$	$\langle[0.3,0.5],0.3\rangle$
A_3	$\langle[0.4,0.5],0.9\rangle$	$\langle[0.8,0.9],0.3\rangle$	$\langle[0.5,0.6],0.3\rangle$	$\langle[0.5,0.7],0.2\rangle$
A_4	$\langle[0.3,0.8],0.2\rangle$	$\langle[0.6,0.7],0.5\rangle$	$\langle[0.6,0.8],0.2\rangle$	$\langle[0.3,0.4],0.3\rangle$

Step 2. Using known weighting vector by applying Definition 4 and operational law 3 in Table 2.

Table 2. Order cubic decision matrix $R_{ij}=$

	C_1	C_2	C_3	C_4
A_1	$\langle[0.76,0.85],0.02\rangle$	$\langle[0.45,0.66],0.14\rangle$	$\langle[0.42,0.61],0.38\rangle$	$\langle[0.08,0.13],0.75\rangle$
A_2	$\langle[0.66,0.85],0.54\rangle$	$\langle[0.34,0.56],0.23\rangle$	$\langle[0.34,0.56],0.33\rangle$	$\langle[0.11,0.23],0.23\rangle$
A_3	$\langle[0.72,0.84],0.38\rangle$	$\langle[0.42,0.61],0.27\rangle$	$\langle[0.42,0.51],0.38\rangle$	$\langle[0.33,0.42],0.94\rangle$
A_4	$\langle[0.76,0.92],0.07\rangle$	$\langle[0.76,0.85],0.32\rangle$	$\langle[0.43,0.92],0.07\rangle$	$\langle[0.43,0.55],0.14\rangle$

Step 3. Now using generalized cubic hybrid aggregation $(GCHA)$ operator by using Eq. 28, we have the aggregated values of the cubic decision matrix is given in Table 3.

Table 3. Aggregated values

A_1	$\langle [0.5020, 0.6612], 0.1842 \rangle$
A_2	$\langle [0.4084, 0.6161], 0.2982 \rangle$
A_3	$\langle [0.4878, 0.6256], 0.3736 \rangle$
A_4	$\langle [0.6516, 0.8774], 0.2812 \rangle$

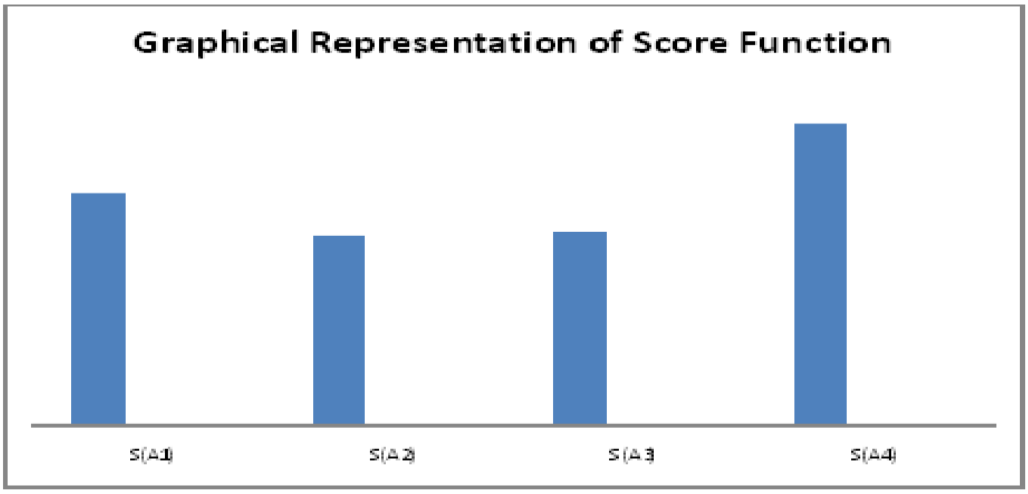
Step 4. In this step, we calculate the scores of the aggregated values of each alternative. If there is no difference between two or more than two scores then we have to find out the accuracy degrees of the aggregated values of each alternative.

$S(A_1)$	$S(A_2)$	$S(A_3)$	$S(A_4)$
0.5467	0.4462	0.4551	0.7084

Step 5. In this step, we arrange all the score values of the alternatives in the form of descending order and select the best alternative which has the highest degree of the score value. Here $S(A_4) > S(A_1) > S(A_3) > S(A_2)$. Thus most wanted alternative is (A_4).

Step 6. End,

Figure 2



6. Further Discussion

In order to show the validity of the proposed methods, we utilize intuitionistic fuzzy (IFs) sets to solve the same problem described above. We apply the proposed aggregation operators developed in this paper. After simplification we obtained the ranking result as $A_4 > A_1 > A_3 > A_2$, and we find that A_4 is best alternative. In the above example, if we use IFs sets to express the decision maker's evaluations then the decision matrix D_{ij} can be written as decision matrix $D_{ij}^{(1)}$ by applying intuitionistic fuzzy numbers. In [16] the proposed $GIFW$ operators to deal with multiple attribute decision making with intuitionistic fuzzy information respectively;

Table 1. Cubic Decision Matrix

$D_{ij}^{(1)} =$

	C_1	C_2	C_3	C_4
A_1	$\langle 0.2, 0.5\rangle$	$\langle 0.4, 0.2\rangle$	$\langle 0.5, 0.3\rangle$	$\langle 0.6, 0.1\rangle$
A_2	$\langle 0.1, 0.3\rangle$	$\langle 0.3, 0.4\rangle$	$\langle 0.6, 0.6\rangle$	$\langle 0.3, 0.3\rangle$
A_3	$\langle 0.4, 0.9\rangle$	$\langle 0.8, 0.3\rangle$	$\langle 0.5, 0.3\rangle$	$\langle 0.5, 0.2\rangle$
A_4	$\langle 0.3, 0.2\rangle$	$\langle 0.6, 0.5\rangle$	$\langle 0.6, 0.2\rangle$	$\langle 0.3, 0.3\rangle$

We further explain to find the best alternative of *IFs*, after the computation process of the aggregated values of each alternative $D_{ij}^{(1)}$ as follows. By applying score function of such that,

$S(A_1)$	$S(A_2)$	$S(A_3)$	$S(A_4)$
0.3178	0.1102	0.1142	0.3704

Now we find the ranking as $A_4 > A_1 > A_3 > A_2$. In this case A_4 is the best alternative.

It is noted that the ranking orders obtained by this paper and by [16] are very different. Therefore, *CFNs* may better reflect the decision information than *IFNs*, hence our proposed approach is more better than *IFNs*

7. Conclusion

In this paper, we constructed new kinds of aggregation operators, consists of the *GCWA* operator, the *GCOWA* operator and the *GCHA* operator which extend the *GOWA* operator. We also discussed some basic properties of these operators, the weighting vector of *GCOWA* operator and *GCHA* operator can be determined by the normal distribution based method. At the end of this paper we have developed two numerical example by applying these operators to multiple attribute group decision making $(MAGD)$ problem based on cubic sets. We can extend this to various field.

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Some Issues on Properties of the Extended IOWA Operators in Cubic Group Decision Making

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Abstract – The concept of this paper to study some IOWA operator to aggregating the individual cubic preference relations (CPR). This paper deal further the study of their properties of group decision problems with the help of CPR, we have proved that the collective preference relation obtained by IOWA operator, then we applied the aggregation operator of individual judgment by using IOWA operators as aggregation procedure by (RAMM) method. Additionally, the result of group Consistency IOWA (C-IOWA) operator is greater than the arithmetic mean of all the individual consistency degree. The numerical application verified the result of this paper.

Keywords – *Cubic preference relation (CPR), induced ordered weighted averaging (IOWA), group decision making*

1. Introduction

The theory of fuzzy sets is developed in 1965 [15] which has been generally used in many area of our present society. Atanassov [1] generalized fuzzy set to intuitionistic fuzzy set (IFS) [2] The IFS categorized by membership and as a non-membership. Atanassov and Gargov further extend the concept of IFS to interval value intuitionistic fuzzy set. IFS the membership and non-membership are the fuzzy number while $IVIFS$ are interval valued intuitionistic fuzzy numbers.

The IFS does not explain the problem when there is some uncertainty. Therefore Jun, defined the new concept so called cubic set [3] In 2012, Jun introduced a new theory which is called cubic set theory. They introduced many concept of cubic set. Cubic deal with uncertainty problem. Jun cubic set explain all the satisfied, unsatisfied and uncertain information, while fuzzy and intuitionistic fuzzy set fail to explain these term. Szmidt and

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Kacprzyk [4] proposed the concept of intuitionistic preference relation (PR) and Xu [5] defined the consistency of intuitionistic fuzzy relation by extending the notion of consistent reciprocal preference relation. Since it is often more difficult for a decision maker to exactly quantify his certainty properties of these $IOWA$ operators.

The application of PR applied to DM [6,7,8,9,10]. Therefore the verification of such preference relation (PR) is some significant to construct worthy DM method. Where the consistency property is most benefit property, in these properties the non existence of consistency in DM must be inconsistent in the conclusions. Therefore this show the important conditions. Its plays a vital role to study the conditions under which consistency is satisfied [11.10]. The obtaing of perfect consistency practice is challenging mostly, when calculting the preference on a classical set with big numbers of choices. There are two problems of consistency

(1) The individually consideration of an expert is called consistent.
(2) when the consideration of consistent in the group.

We define the method of computing consistency in CPR. By using this consistency measure, we verified that if different judgement matrix $(C-IOW)$ have a adequate, then combined judgement matrix $(C-IOWACJM)$ also is of acceptable consistency. Moreover, our result guarantees that the consistency of $(C-IOWACJM)$ is smaller than the arithmetic mean of all the individual consistency. The $(I-IOWA)$ operator also has similar properties.

The paper is consists of the following sections, such that. In Section 2 we review some fundamental concepts such that the $IOWA, (C-IOWA)$ and $(I-IOWA)$ operators. We also defines the concept of consistency degree of (CPR) in Section 3 . In Section 4 , we study the preferred properties of these $(IOWA)$ operators in cubic (GDM) . In Section 5 we provides illustrative examples. This paper is concluded in Section 6 .

2. Preliminaries

$(IOWA), (C-IOWA)$ and $(I-IOWA)$ Operators

In this section we generalized the concept of induced ordered weighted average $(IOWA)$, consistency $IOWA$ $(C-IOWA)$ and individual $(I-IOWA)$ operators, which will be used throughout this paper. [15] Yager and Filev defined an induced OWA $(IOWA)$ operator in which the ordering of the $a_i (i \in n)$ is induced by other variables $u_i (i \in n)$ called the order inducing variables, where a_i and u_i are the factor of OWA set $\langle u_i, a_i \rangle (i \in n)$.

Definition 2.1 [15] An $(IOWA)$ operator of dimension n is a mapping, $\varphi_w^G : R^{+^n} \to R^+$ to which a set of weights or a weighting vector is related,

$$W=(w_1,w_2,...,w_n)^T, w_j\in[0,1] and \sum_{j=1}^{n} w_j=1,$$

and it is defined to aggregate the set of $2nd$ arguments of list of two pairs $\left\{\begin{matrix}\langle u_1,a_1\rangle,...,\\ \langle u_n,a_n\rangle\end{matrix}\right\}$, given on the basis of a positive ratio scale, define as following:

$$f_w^G=(\langle u_1,a_1\rangle,...,\langle u_n,a_n\rangle)=\sum_{j=1}^{n} w_j b_{j,} \qquad (1)$$

where $w=(w_1,w_2,...,w_n)^T$ is a weighting vector, i.e. $\sum_{j=1}^{n} w_j=1, w_j\in[0,1], b_j$ is the a_i value of the $IOWA$ pair having the jth largest u_i, and u_i in $\langle u_i,a_i\rangle$ is referred to as the order inducing variable and a_i as the argument variable.

Definition 2.2 [12] If a set of (DMs) $D=\{d_1,d_2,...,d_m\}$ provides preference about a set of alternatives $X=\{x_1,x_2,...,x_n\}$ by means of (CPR) $\{M^{(1)},...,M^{(l)},...,M^{(m)}\}$, and each have an importance degree $\mu(d_k)\in[0,1]$, related to him or her, then an $(I-IOWA)$ operator is an $(IOWA)$ operator in which its order-inducing values is the set of importance degree.

Definition 2.3 If a set of (DMs) $D=\{d_1,d_2,...,d_m\}$ provides preference about a set of alternatives $X=\{x_1,x_2,...,x_n\}$ by means of (CPR), $\{M^{(1)},...,M^{(l)},...,M^{(m)}\}$, $M^{(l)}\in M$, then a $(C-IOWA)$ operator is an $(IOWA)$ operator in which its order-inducing values is the set of consistency index values such that,

$$\{CI(M^{(1)}),...,CI(M^{(l)}),...,CI(M^{(m)})\}$$

Definition 2.4 [3] Let X be a fixed non empty set. A cubic set is an object of the form:

$$C=\{\langle a,A(a),\lambda(a)\rangle : a\in X\},$$

where A is an $(IVFS)$ and λ is a fuzzy set in X. A cubic set $\widetilde{C}=\langle a,A(a),\lambda(a)\rangle$ is simply denoted by $\widetilde{C}=\langle\widetilde{A},\lambda\rangle$. The collection of all cubic set is denoted by $C(X)$.

(a) if $\lambda\in\widetilde{A}(x)$ $\forall$ $x\in X$ so it is called interval cubic set.

(b) If $\lambda\notin\widetilde{A}(x)$ $\forall$ $x\in X$ so it is called external cubic set.

(c) If $\lambda\in\widetilde{A}(x)$ or $\lambda\notin\widetilde{A}(x)$ its called cubic set for all $x\in X$.

Definition 2.5 [3] Let $A=\langle A,\lambda\rangle$ and $B=\langle B,\mu\rangle$ be cubic set in X, then we define

(a) (Equality) $A=B$ if and only if $A=B$ and $\lambda=\mu$.

(b) $(P-$ order) $A\subseteq_A B$ if and only if $A\subseteq B$ and $\lambda\le\mu$.

(c) $(R-$ order) $A\subseteq_R B$ if and only if $A\subseteq B$ and $\lambda\ge\mu$.

Definition 2.6 [3] The complement of $A=\langle A,\lambda\rangle$ is defined to be the cubic set

$$A^c=\{\langle x,A^c(x),1-\lambda(x)\rangle \mid x\in X\}.$$

3. The Measure of Consistency Index of CPR

In GD atmosphere, the problem of consistency itself consist of two problems

(1) The individually consideration of an expert is called consistent.
(2) when the consideration of consistent in the group.

First problem is emphasis in this section. First of all we define the idea's of the additive transitive CPR . Then we define the CI of CPR . In the following section, we will emphasis on the 2nd problem.

Definition 3.1 Suppose $X=\{x_1,x_2,...,x_n\}$ be a finite set of alternatives. If the DM gives his/her PR information on X by means of a preference relation $M=(C_{ij})_{n\times n}$, where $\widetilde{C}_{ij}=\left\langle \widetilde{A}_{ij},\lambda_{ij}\right\rangle$ and we have,

$$\widetilde{A}_{ij}+\widetilde{A}_{ji}=1,\ \widetilde{A}_{ii}=0.5 \text{ and } \lambda_{ij}+\lambda_{ji}=1,\ \lambda_{ii}=0.5\ \forall\, i,j\in N.$$

Where C_{ij} denotes the preference degree or intensity of the alternative X_i over X_j , then M is called a CPR.

Definition 3.2 Suppose $M=(C_{ij})_{n\times n}$ where $\widetilde{C}_{ij}=\left\langle \widetilde{A}_{ij},\lambda_{ij}\right\rangle$ be a CPR , then M is called an additive transitive CPR , if the following additive transitivity is satisfied:

$$\widetilde{A}_{ij}=\widetilde{A}_{ik}-\widetilde{A}_{jk}+0.5, \text{ and } \lambda_{ij}=\lambda_{ik}-\lambda_{jk}+0.5\ \forall\, i,j,k\in N.$$

Definition 3.3 If we utilize the row arithmetic mean method $(RAMM)$, then can get the priority vector $w^{(l)}=(w_1^{(l)},w_2^{(l)},...,w_n^{(l)})^T$ of the CPR, $M^{(l)}$, where

$$w_i^{(l)}=\frac{1}{n}\sum_{j=1}^{n}C_{ij}^{(l)},\quad i=1,2,...n;l=1,2,...,m.$$

Definition 3.4 Suppose $A=(a_{ij})_{n\times n}\in M$ and $b=(b_{ij})_{n\times n}\in$ M , then the distance between A and B define as follows:

$$d(A,B)=\frac{1}{3n}\sum_{i=1}^{n}\sum_{j=1}^{n}[\left|\overline{a}_{ij}-\overline{b}_{ij}\right|+|\,a_{ij}^{+}-b_{ij}^{+}\,|+|\,\lambda_{ij}-\lambda_{ij}\,|] \qquad (2)$$

Clearly, the smaller the value of distance degree $d(A,B)$, the nearer of the CPR , A and B .

Theorem 3.5 Let $A=(a_{ij})_{n\times n}\in M$ and $b=(b_{ij})_{n\times n}\in$ M , then

(1) $d(A,B)\geq 0$;
(2) .$d(A,B)=0$ $\Leftrightarrow A$ and B are perfectly consistent.

Proof. (1)

$$d(A,B)=\frac{1}{3n}\sum_{i=1}^{n}\sum_{j=1}^{n}[\left|\overline{a}_{ij}-\overline{b}_{ij}\right|+|\,a_{ij}^{+}-b_{ij}^{+}\,|+|\,\lambda_{ij}-\lambda_{ij}\,|]\geq 0 \qquad (3)$$

(2) Necessity. If $d(A,B)=0$, then $a_{ij}=b_{ij}$ for all $i,j\in N$. Hence, A and B are perfectly consistent.
(3) Sufficiency. If A and B are perfectly consistent, then $a_{ij}=b_{ij}$ $\forall$ $i,j\in N$. Thus, we have $a_{ij}-b_{ij} \;\; =0$ $\forall$ $i,j\in N$. Therefore, $d(A,B)=0$.
In (GD) problems based on (CPR) , the study of consistency is related to the transitivity property. And gave a categorization of the consistency property defined by the additive transitivity property of a cubic preference relation

$$M^K=(C_{ij}^k):\; C_{ij}^k+C_{jl}^k+C_{il}^k=\left\langle \widetilde{\frac{3}{2}},\frac{3}{2}\right\rangle, \forall i,j,l\in\{1,...,n\}.$$

Applying this categorization technique, a method to construct a consistent reciprocal (CPR) M on $X=\{x_1,x_2,...,x_n,\quad n\geq 2\}$ from $n-1$ preference values $\{C_{12}$ $C_{23},...,C_{n-1n}$ $\}$ define as followes:

(1) $M=(C_{ij})$ i.e.

$$C_{ij}=\begin{cases} C_{ij} & \text{if } i\leq j\leq i+1,\\ (C_{ii+1}+C_{i+1i+2}+C_{i+2i+3},...,C_{j-1j})-\frac{j-(i+1)}{2} & \text{if } i+1<j,\\ 1-C_{ij} & \text{if } j<i. \end{cases}$$

But the matrix M could have entries not in the interval $[0,1]$, but in an interval $[-x,1+x]$, being $x=\lfloor\min\{C_{ij};C_{ij}\in M\}\rfloor$ For this case. [13] the alteration function which reserves reciprocity and additive consistency, that is a function $[-x,1+x]\rightarrow[0,1]$ satisfying

$(i) f(-x)=0.$
$(ii) f(1+x)=1.$
$(iii) f(a)+f(1-a)=1,\quad \forall \quad a\in[-x,1+x].$
$(iv)\quad f(a)+f(b)+f(c)=\frac{3}{2},\quad \forall \quad a,b,c\in[-x,1+x].$ i.e. $a+b+c=\frac{3}{2}.$

(**2**) The consistent (CPR) , N is obtained as $N=f(M)$. This (CI) has a certain physical consequence and reflects the deviance degree b/w the (CPR) $M^{(l)}$ and its equivalent consistent matrix $N^{(l)}$. The distance b/w $M^{(l)}$ and its equivalent consistent matrix $N^{(l)}$ define as follows.

Definition 3.6 Let $M^{(1)},...,M^{(l)},...,M^{(m)}$ be the (CPR) provided by m decision maker's and $N^{(1)},...,N^{(l)},...,N^{(m)}$ be their equivalent consistent matrix, then we define a measure of (CI) of the (CPR) $M^{(l)}$ as follows:

$$CI(M^{(l)})=1-d(M^{(l)},N^{(l)}). \qquad (4)$$

Clearly, the nearer $CI(M^{(l)})$ is to 1 the ultimate consistent the information provided by the (DM) $d^{(l)}$, and thus more importance should be placed on that information. By using this (CI) , we obtain some preferred properties of $(C-IOWA)$ operator.

4. The Properties Of $IOWA$ Operators In Cubic Group Decision Making

We appliance the $(C-IOWA)$ operator and the ($I-IOWA)$ operator to aggregate individual (CPR) in group decision making problems, and then study their desired properties. in this section.

The Consistency $IOWA$ $(C-IOWA)$ **Operator**

In a standardized group decision making problem, the decision maker's have identical importance. Therefore, every decision maker's continuously can have a (CI) value related with them, which measures the level of consent b/w group preferences and individual preference. Therefore, the (DM) provided further consistency information, the greater weighting value should be placed on that information. We discuss the reciprocity and consistency properties of the $(C-IOWACJM)$, which is found by applying $(C-IOWA)$ operator, in this section.

Definition 4.1 If $M^{(1)},...,M^{(l)},...,M^{(m)}$ are the (CPR) provided by m (DMs) , then the $(C-IOWACJM)$ $M=(C_{ij})_{n\times n}$ is difined as follows:

$$\bar{M} = C-IOWA\begin{pmatrix}\langle CI(M^{(1)}),M^{(1)}\rangle,\langle CI(M^{(2)}),M^{(2)}\rangle,\\ ...,\langle CI(M^{(m)}),M^{(m)}\rangle\end{pmatrix}$$

$$= C-IOWA\begin{pmatrix}\langle CI(M^{(\alpha(1))}),M^{(\alpha(1))}\rangle,\langle CI(M^{(\alpha(2))}),M^{(\alpha(2))}\rangle,\\ ...,\langle CI(M^{(\alpha(m))}),M^{(\alpha(m))}\rangle\end{pmatrix}$$

$$= \begin{pmatrix}M^{(\alpha(1))}\times\delta_{(\alpha(1))})+(M^{(\alpha(2))}\times\delta_{(\alpha(2))})+\\ ...+(M^{(\alpha(m))}\times\delta_{(\alpha(m))}\end{pmatrix} \quad (5)$$

$$C_{ij} = \begin{pmatrix}C_{ij}^{(\alpha(1))}\times\delta_{(\alpha(1))})+(C_{ij}^{(\alpha(2))}\times\delta_{(\alpha(2))})+\\ ...+(C_{ij}^{(\alpha(m))}\times\delta_{(\alpha(m))}\end{pmatrix}$$

$$= \prod_{l=1}^{m}(a_{ij}^{(\alpha(l))}\times\delta_{(\alpha(l))}), \quad (6)$$

where $(\alpha(1),\alpha(2),...,$ $\alpha(n)$ is a permutation of $(1,2,...,n)$ such that

$$CI(M^{(\alpha(l-1))})\geq CI(M^{(\alpha(l))}) \text{ and } \delta_{\alpha(l-1)}\geq\delta_{\alpha(l)}\ \forall\, l=2,...,m;$$

$$\langle CI(M^{(\alpha(l))}),M^{(\alpha(l))}\rangle \text{ is two tuple with } CI(M^{(\alpha(l))})$$

the lth largest value in the set $\{CI(M^{(1)}),...,CI(M^{(m)})\}$;

$$\delta=(\delta_{\alpha(1)},\delta_{\alpha(2)},...,\delta_{\alpha(m)})^{T} \text{ is a weighting vector i.e.}$$

$$\sum_{l=1}^{m}\delta_{\alpha(l)}=1 \text{ and } \delta_{\alpha(l)}\in[0,1].$$

Yager [14] provided a method to define the weighting vector related to an $(IOWA)$ operator. In this case, each remark in the aggregation contains of a triple $(p_{ij}^{(l)}, u_l, v_l)$: $p_{ij}^{(l)}$ is the argument value to aggregate, u_l is the significance weight value related to $p_{ij}^{(l)}$, and v_l is the order inducing value. Therefore, the aggregation is

$$IOWA_Q(p_{ij}^{(1)},...,p_{ij}^{(m)}) = \sum_{l=1}^{m} w_l p_{ij}^{\alpha(l)}, \text{ with}$$

$$w_l = Q\left(\frac{S(l))}{S(n)}\right) - Q\left(\frac{S(l-1)}{S(n)}\right), \qquad (7)$$

where $S(l) = \sum_{k=1}^{l} u_{\alpha(k)}$, and α is permutation i.e. $u_{\alpha(l)}$ in $(p_{ij}^{\alpha(l)}, u_{\alpha(l)}, v_{\alpha(l)})$ is the lth largest value in the set of $\{v_1,...,v_n\}$. Q is a function $:[0,1] \to [0,1]$ i.e. $Q(0)=0$, $Q(1)=1$ and if $x > y$ then $Q(x) \geq Q(y)$. In this case, we suggest to use the consistency values related to each one of the (DM) both as a weight related to the argument and as the order inducing values $u_i \ = v_i \ = CI(M^{(i)})$. Therefore the ordering of the preference values is first induced by the ordering of the (DMs) from greatest to smallest consistency one, and the weights of the $(C-IOWA)$ operator is obtained by using the above, E.q. (7) , with decreases to

$$\delta_{\alpha(l)} = Q\left(\frac{S(\alpha(l))}{S(\alpha(n))}\right) - Q\left(\frac{S(\alpha(l-1))}{S(\alpha(n))}\right), \qquad (8)$$

where $S(\alpha(l)) = \ \sum_{k=1}^{l} CI(M^{(\alpha(k))})$, and α is the permutation such that

$$CI(M^{(\alpha(l))}) \text{ in } (C_{ij}^{(\alpha(l))}, CI(M^{(\alpha(l))}), CI(M^{(\alpha(l))}))$$

is the lth largest value in the set $\{CI(M^{(\alpha(l))}),..., CI(M^{(\alpha(n))})\}$. In an aggregation process, we consider that the weighting value of (DMs) should be implemented in such a way that the effect from those (DMs) who are less consistency is reduced, and therefore the above is obtained if the linguistic quantifier Q verifiers that the most the consistency of an (DM) the higher the weighting value of that (DM) in the aggregation, i.e.:

$$CI(M^{(\alpha(1))}) \geq CI(M^{(\alpha(2))}) \geq,..., CI(M^{(\alpha(n))}) \geq 0$$
$$\Rightarrow \delta_{\alpha(1)} \geq \delta_{\alpha(2)},..., \geq \delta_{\alpha(n)} \geq 0.$$

Theorem 4.2 Let the parameterized family of RIM quantifiers $Q(\lambda) = \lambda^{\alpha}, \alpha \geq 0,$ if $a \in [0,1]$ and

$$S(\alpha(l)) = \sum_{k=1}^{l} CI(M^{(\alpha(k))}), then\, \delta_{\alpha(l)} \geq \delta_{\alpha(l+1)}, \forall l = 1,2,...,m$$

Proof If $\alpha \in [0,1]$, then the function $Q(\lambda)=\lambda^{\alpha}$ is concave and, we have $Q(T_l)-Q(T_{l-1}) \geq Q(T_{l+1})-Q(T_l)$. Suppose

$$T_l = \frac{S(\alpha(l))}{S(\alpha(n))} \text{ and } S(\alpha(l)) = \sum_{k=1}^{l} CI(M^{(\alpha(k))}), \text{ then}$$

$$\delta_{\alpha(l)} = Q\left(\frac{S(\alpha(l))}{S(\alpha(n))}\right) - Q\left(\frac{S(\alpha(l-1))}{S(\alpha(n))}\right) = Q(T_l)-Q(T_{l-1}) \text{ and}$$

$$\delta_{\alpha(l+1)} = Q\left(\frac{S(\alpha(l+1))}{S(\alpha(n))}\right) - Q\left(\frac{S(\alpha(l))}{S(\alpha(n))}\right) = Q(T_{l+1})-Q(T_l)$$

Thus, we can obtain $\delta_{\alpha(l)} \geq \delta_{\alpha(l+1)}$.

In group decision making models with (CP) assessments, it is frequently supposed that the (CPR) , to express the judgments are reciprocal. The $(C-IOWA)$ operator is able to maintain both the reciprocity and the consistency properties in the collective (CPR) . In order to study these properties, we construct the next theorem.

Theorem 4.3 Let $M^{(1)}, M^{(2)}, ..., M^{(m)}$ be (CPR) provided by m decision maker's where $M^{(l)} = (C_{ij}^{(l)})_{n\times n}$, $l=1,2,...,m;$ $i,j=1,2,...,n,$ then their $(CI-IOWACJM)$ $\bar{M} = (C_{ij}^{(l)})_{n\times n}$ is also a (CPR), where

$$\begin{aligned} C_{ij} &= CI-IOWA\begin{pmatrix} \langle CI(M^{(1)}, C_{ij}^{(1)})\rangle, \langle CI(M^{(2)}), C_{ij}^{(2)}\rangle \\ ,..., \langle CI(M^{(m)}), C_{ij}^{(m)}\rangle \end{pmatrix} \\ &= CI-IOWA\begin{pmatrix} \langle CI(M^{(\alpha(1))}), C_{ij}^{(\alpha(1))}\rangle, \langle CI(M^{(\alpha(2))}), C_{ij}^{(\alpha(2))},\rangle \\ ,..., \langle CI(M^{(\alpha(m))}), C_{ij}^{(\alpha(m))}\rangle \end{pmatrix} \\ &= (C_{ij}^{(\alpha(1))} \times \delta_{(\alpha(1))}) + (C_{ij}^{(\alpha(2))} \times \delta_{(\alpha(2))}) + ... + (C_{ij}^{(\alpha(m))} \times \delta_{(\alpha(m))}) \end{aligned}$$

and

$$C_{ij} \geq 0,\ \widetilde{A}_{ij} + \widetilde{A}_{ji} = 1,\ A_{ii} = 0.5 \text{ and}$$
$$\lambda_{ij} + \lambda_{ji} = 1,\ \lambda_{ii} = 0.5\ \forall\, i,j \in N.$$

Also $\bar{M}$ is also consistent, subject to $\{M^{(1)}, M^{(2)}, ..., M^{(m)}\}$ are consistent.

Proof Since $M^{(1)}, M^{(2)}, ..., M^{(m)}$ are (CPR) , we have then

$$C_{ij}=\begin{pmatrix}(C_{ij}^{(\alpha(1))}\times\delta_{(\alpha(1))})+(C_{ij}^{(\alpha(2))}\times\delta_{(\alpha(2))})\\ +\ldots+(C_{ij}^{(\alpha(m))}\times\delta_{(\alpha(m))})\end{pmatrix}$$
$$\geq(0\times\delta_{(\alpha(1))})+(0\times\delta_{(\alpha(2))})+\ldots+(0\times\delta_{(\alpha(m))})=0.$$
$$C_{ij}+C_{ji}=\begin{pmatrix}(C_{ij}^{(\alpha(1))}+C_{ji}^{(\alpha(1))})\delta_{(\alpha(1))}+(C_{ij}^{(\alpha(2))}+C_{ji}^{(\alpha(2))})\delta_{(\alpha(2))}\\ +\ldots+(C_{ij}^{(\alpha(m))}+C_{ji}^{(\alpha(m))})\delta_{(\alpha(m))}\end{pmatrix}$$
$$=\delta_{\alpha(1)}+\delta_{\alpha(2)}+\ldots+\delta_{\alpha(m)}=1,$$
$$C_{ij}=(C_{ii}^{(\alpha(1))}\times\gamma_{\alpha(1)})+(C_{ii}^{(\alpha(2))}\times\gamma_{\alpha(2)})+\ldots+(C_{ii}^{(\alpha(m))}\times\gamma_{\alpha(m)})$$
$$=(\frac{1}{2}\delta_{\alpha(1)})+(\frac{1}{2}\delta_{\alpha(2)})+\ldots+(\frac{1}{2}\delta_{\alpha(m)})=\frac{1}{2}.$$

Thus, $M=(C_{ij})_{n\times n}$ is also a (CPR).

(ii) Since all the $M^{(1)},M^{(2)},\ldots,M^{(m)}$ are consistent, *i.e.*, then

$$\widetilde{A}_{ij}^{l}=\widetilde{A}_{ik}^{l}+\widetilde{A}_{kj}^{l},-0.5 \text{ and}$$
$$\lambda_{ij}=\lambda_{ik}^{l}+\lambda_{kj}^{l}-0.5\;\forall\, l=1,2,\ldots,m\;\; i,j\in N.$$

Thus

$$C_{ik}+C_{kj}=\sum_{l=1}^{m}C_{ik}^{(\alpha(l))}\delta_{(\alpha(l))}+\sum_{l=1}^{m}C_{kj}^{(\alpha(l))}\delta_{(\alpha(l))}$$
$$=\sum_{l=1}^{m}(C_{ik}^{(\alpha(l))}+C_{kj}^{(\alpha(l))})\delta_{(\alpha(l))}$$
$$=\sum_{l=1}^{m}(C_{ij}^{(\alpha(l))}+\langle 0.\widetilde{5},0.5\rangle)\delta_{(\alpha(l))}$$
$$=C_{ij}+\langle 0.\widetilde{5},0.5\rangle$$

and thus, $\bar{M}$ is also consistent.

Definition 4.4 Denote $M^{(l)}\in M$ be the cubic judgement matrix provided by the lth (DM) when comparing n alternatives, $w^{(l)}=(w_1^{(l)},w_2^{(l)},\ldots,w_n^{(l)})^T$ as its priority vector, $P^{(l)}=(P_{ij}^{(l)})_{n\times n}$ as the equivalent consistent matrix; $\bar{w}=(\bar{w}_1,\bar{w}_2,\ldots,\bar{w}_n)^T$ as the priority vector of $(C-IOWACJM)$ $\bar{M}$, and $\bar{N}=(p_{ij})_{n\times n}$ as the equivalent consistent matrix of $\bar{M}$.

Theorem 4.5 Applying the $(C-IOWACJM)$ as the aggregation method, the weighting vector

$$\delta=(\delta_{\alpha(1)},\delta_{\alpha(2)}+\ldots+\delta_{\alpha(m)})^T,\;\delta_{\alpha(l-1)}\geq\delta_{\alpha(l)},\;\sum_{l=1}^{m}\delta_{\alpha(l)}=1,$$

and the $(RAMM)$ as the prioritization method, such that the (AIJ) and the (AIP) offers the same priorities of alternatives.

Proof Let $w^{(l)} = (w_1^{(l)}, w_2^{(l)}, ..., w_n^{(l)})^T$ be the priority of the individual judgement matrix $M^{(l)}$ and $\bar{w} = (w_1, w_2, ..., w_n)^T$ be the group priorities, so we define as following,

$$w_i(AIP) = C - IOWA\begin{pmatrix} \langle CI(M^{(1)}), w^{(1)} \rangle, \langle CI(M^{(2)}), w^{(2)} \rangle \\ ,..., \langle CI(M^{(m)}), w^{(m)} \rangle \end{pmatrix}$$

$$= C - IOWA\begin{pmatrix} \langle CI(M^{(\alpha(1))}), w^{(\alpha(1))} \rangle, \langle CI(M^{(\alpha(2))}), w^{(\alpha(2))} \rangle \\ ,..., \langle CI(M^{(\alpha(m))}), w^{(\alpha(m))} \rangle \end{pmatrix}$$

$$= (w^{(\alpha(1))}\delta_{\alpha(1)}) + (w^{(\alpha(2))}\delta_{\alpha(2)}) + ... + (w^{(\alpha(m))}\delta_{\alpha(m)})$$

$$w_i(AIP) = \sum_{l=1}^{m} w_i^{\alpha(l)} \times \delta_{\alpha(l)}$$

$$w_i(AIJ) = \frac{1}{n}\sum_{j=1}^{n} C_{ij} = \frac{1}{n}\sum_{j=1}^{n}\sum_{l=1}^{m} C_{ij}^{(\alpha(l))} \times \delta_{\alpha(l)}$$

$$= \sum_{l=1}^{m} \delta_{\alpha(l)}\left(\sum_{j=1}^{n}\frac{1}{n}C_{ij}^{(\alpha(l))}\right) = \sum_{l=1}^{m} w_i^{\alpha(l)} \times \delta_{\alpha(l)}.$$

Thus $w_i(AIP) = w_i(AIJ)$.

Definition 4.6 Let $CI(\bar{M})$ be a measure of the consistency of the collective matrix $\bar{M}$, and $CI(M^{(l)})$ be a measure of the consistency of matrix $M^{(l)}$.

Theorem 4.7 Suppose $M^{(1)}, M^{(2)}, ..., M^{(m)}$ be the (CPR) provided by m decision maker's when comparing n alternatives with the corresponding weighting vector

$$\delta = (\delta_{\alpha(1)}, \delta_{\alpha(2)} + ... + \delta_{\alpha(m)})^T,\ \delta_{\alpha(l-1)} \geq \delta_{\alpha(l)},\ \sum_{l=1}^{n}\delta_{\alpha(l)} = 1.$$

Using the $(C - IOWACJM)$ as the aggregation procedure and the row arithmetic mean method as the prioritization method i.e.

$$CI(\bar{M}) \geq \frac{1}{m}\sum_{l=1}^{m} CI(M^{(l)}) \qquad (9)$$

Proof By Definition 16 and E.g. (4), we have

$$CI(\bar{M}) = 1 - d(\bar{M}, \bar{N}) = 1 - \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n}\left|C_{ij} - \bar{p}_{ij}\right|$$

$$= 1 - \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n}\left|\begin{array}{l}\sum_{l=1}^{m} C_{ij}^{(\alpha(l))} \times \delta_{\alpha(l)} \\ -\sum_{l=1}^{q} p_{ij}^{(\alpha(l))} \times \delta_{\alpha(l)}\end{array}\right|$$

$$= 1 - \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n}\left|\sum_{l=1}^{m}(C_{ij}^{(\alpha(l))} - p_{ij}^{(\alpha(l))})\delta_{\alpha(l)})\right|.$$

Since $\left|\sum_{l=1}^{m}(C_{ij}^{(\alpha(l))} - p_{ij}^{(\alpha(l))})\delta_{\alpha(l)}\right| \le \sum_{l=1}^{m}\left|(C_{ij}^{(\alpha(l))} - p_{ij}^{(\alpha(l))})\delta_{\alpha(l)}\right|.$

Then $CI(\bar{M}) \ge 1 - \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{l=1}^{m}\left|(C_{ij}^{(\alpha(l))} - p_{ij}^{(\alpha(l))})\delta_{\alpha(l)}\right|$

$$= 1 - \sum_{l=1}^{m}\delta_{\alpha(l)}(\frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n}\left|(C_{ij}^{(\alpha(l))} - p_{ij}^{(\alpha(l))})\right|$$

$$= \sum_{l=1}^{m}\delta_{\alpha(l)}\left(1 - \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n}\left|(\lambda_{ij}^{(\alpha(l))} - p_{ij}^{(\alpha(l))})\right|\right)$$

$$= \sum_{l=1}^{m}\delta_{\alpha(l)}CI(M^{(l)})$$

$$CI(M^{(\alpha(l))}) \ge CI(M^{(\alpha(l+1))}) \text{ and } \delta_{\alpha(1)} \ge \delta_{\alpha(2)} \dots \ge \delta_{\alpha(m)}$$

Then we have,

$$\sum_{l=1}^{m}\delta_{\alpha(l)}CI(M^{(\alpha(l)}) \ge \frac{1}{m}\sum_{l=1}^{n}CI(M^{(\alpha(l)}) = \frac{1}{m}\sum_{l=1}^{n}CI(M^{(l)}).$$

Thus $CI(\bar{M}) \ge \frac{1}{m}\sum_{l=1}^{m}CI(M^{(l)}).$

The importance $IOWA$ $(I - IOWA)$ **operator**

In a heterogeneous group decision making problem every expert has an importance degree related with the $(I - IOWA)$ operator, which used this importance degree variable as the order-inducing variable to induce the ordering of the argument values before their aggregation. In this section, we study the reciprocity and consistency properties of the $(I - IOWACJM)$, which is obtained by using $(I - IOWA)$ operator.

Definition 4.8 If a set of (DMs) $D = \{d_1, d_2, ..., d_m\}$ provides preference about a set of alternatives $X = \{x_1, x_2, ..., x_n\}$ by means of (CPR) $\{M^{(1)}, M^{(2)}, ..., M^{(m)}\}$, whose associated importance degree $\mu = (\mu_1, \mu_2, ..., \mu_m)$, $\sum_{l=1}^{m}\mu_1 = 1, 0 \le \mu_1 \le 1,$ then the $(I - IOWACJM)$ $\bar{M} = (C_{ij})_{n\times n}$ is defined as follows:

$$\bar{M} = I - IOWA(\langle \mu_1, M^{(1)} \rangle, \langle \mu_2, M^{(2)} \rangle, ..., \langle \mu_m, M^{(m)} \rangle)$$
$$= (M^{(1)} \times \mu_1) + (M^{(2)} \times \mu_2) + ... + (M^{(m)} + \mu_m) \quad (10)$$
$$C_{ij} = \sum_{l=1}^{m} M_{ij}^{(l)} \times \mu_1 \quad (11)$$

In group decision making models with (CP) calculations, it usually is supposed that the (CPR) to express the judgments are reciprocal. The $(I - ILOWA)$ operator also is able to maintain both the reciprocity and consistency properties in the collective (CPR). therefore we define the following theorems.

Theorem 4.9 Consider $\{M^{(1)}, M^{(2)}, ..., M^{(m)}\}$ be (CPR) provided by m (DMs), where $M^{(l)} = (C_{ij}^{(l)})_{n \times n}$ $(l = 1,2,...,m;$ $i,$ $J = 1,2,...,n)$, then their $(I - IOWACJM)$ $\bar{M} = (C_{ij})_{n \times n}$ is also a (CPR),

$$C_{ij} = \sum_{l=1}^{m} ((C_{ij}^{(l)}) \times \mu_1)$$

where $C_{ij} \geq 0, \widetilde{A}_{ij} + \widetilde{A}_{ji} = 1, A_{ii} = 0.5$ and $\lambda_{ij} + \lambda_{ji} = 1, \lambda_{ii} = 0.5$

Also $\bar{M}$ is also consistent, subject to $\{M^{(1)}, M^{(2)}, ..., M^{(m)}\}$ are consistent.

Proof (i). Since $\{M^{(1)}, M^{(2)}, ..., M^{(m)}\}$ are (CPR), we have

$$C_{ij} \geq 0,\ \widetilde{A}_{ij} + \widetilde{A}_{ji} = 1,\ A_{ii} = 0.5 \text{ and } \lambda_{ij} + \lambda_{ji} = 1,\ \lambda_{ii} = 0.5\, \forall\, i, j \in N.$$

$$C_{ij} = \sum_{l=1}^{m} ((C_{ij}^{(l)}) \times \mu_1) \geq \sum_{l=1}^{m} (0 \times \mu_1) = 0,$$

$$C_{ij} + C_{ji} = \sum_{l=1}^{m} (C_{ij}^{(l)}) \times \mu_1 + \sum_{l=1}^{m} (C_{ji}^{(l)}) \times \mu_1$$
$$= \sum_{l=1}^{m} (C_{ij}^{(l)} + C_{ji}^{(l)}) \times \mu_1 = \sum_{l=1}^{m} \mu_1 = 1.$$
$$C_{ii} = \sum_{l=1}^{m} C_{ii}^{(l)} \times \mu_1 = \sum_{l=1}^{m} \frac{1}{2} \times \mu_1 = \frac{1}{2}.$$

Thus, $M = (C_{ij})_{n \times n}$ is also a (CPR).

(**ii**) Since the $\{M^{(1)}, M^{(2)}, ..., M^{(m)}\}$ are consistent such that,

$$\widetilde{A}_{ij}^{l} = \widetilde{A}_{ik}^{l} + \widetilde{A}_{kj}^{l}, -0.5 \text{ and}$$
$$\lambda_{ij} = \lambda_{ik}^{l} + \lambda_{kj}^{l} - 0.5, \forall\, l = 1,2,...,m\ \ i, j \in N.$$

Then

$$\bar{C}_{ik} + \bar{C}_{kj} = \sum_{l=1}^{m} (C_{ik}^{(l)} + C_{kj}^{(l)}) \times \mu_1 = \sum_{l=1}^{m} (C_{ij}^{(l)} + \langle 0.\widetilde{5}, 0.5 \rangle) \mu_1 = \bar{C}_{ij} + \langle 0.\widetilde{5}, 0.5 \rangle$$

Hence, $\bar{M}$ is also consistent.

Definition 4.10 Denote $M^{(l)} \in M$ be the cubic judgement matrix provided by the $l-th$ DM when comparing n alternatives, $w^{(l)} = (w_1^{(l)}, w_2^{(l)}, ..., w_n^{(l)})^T$ as its priority vector, $P^{(l)} =$ $(P_{ij}^{(l)})_{n\times n}$ as equivalent consistent matrix; $\bar{w} = (\bar{w}_1, \bar{w}_2, ..., \bar{w}_n)^T$ as the priority vector of $(I-IOWACJM)$ M, and $\bar{N} = (\bar{p}_{ij})_{n\times n}$ as the equavelent consistent matrix of M.

Theorem 4.11 Applying the $(I-IOWACJM)$ as the aggregation technique, the weighting vector

$$\lambda = (\mu_1,\ \mu_2, ..., \mu_n)^T,\ \sum_{l=1}^{m} \mu_1 = 1,$$

The row arithmetic mean method as the prioritization method, such that the (AIP) and the (AIJ) provides the same priorities of alternatives.

Proof. Let $w^{(l)} = (w_1^{(l)}, w_2^{(l)}, ..., w_n^{(l)})^T$ be the priority of the individual judgement matrix $M^{(l)}$ and $\bar{w} = (\bar{w}_1, \bar{w}_2, ..., \bar{w}_n)^T$ be the group priorities, then we get, i.e.

$$w_i(AIP) = I - IOWA(\langle \mu_1, w^{(1)} \rangle, \langle \mu_2, w^{(2)} \rangle, ..., \langle \mu_m, w^{(m)} \rangle)$$

$$= \sum_{l=1}^{m} w_i^{(l)} \mu_1$$

$$w_i(AIP) = \sum_{l=1}^{m} w_i^{(l)} \times \mu_1 \text{ and}$$

$$w_i(AIJ) = \frac{1}{n} \sum_{j=1}^{n} C_{ij} = \frac{1}{n} \sum_{j=1}^{n} \sum_{l=1}^{m} C_{ij}^{l} \times \mu_1$$

$$= \sum_{l=1}^{m} \mu_1 (\sum_{j=1}^{n} \frac{1}{n} C_{ij}^{(l)}) = \sum_{l=1}^{m} w_i^{(l)} \mu_1$$

Thus $w_i(AIP) = w_i(AIJ)$.

Theorem 4.12 Suppose $M^{(1)}, M^{(2)}, ..., M^{(m)}$ be the (CPR) provided by m decision maker's when comparing n alternatives with the corresponding weighting vector

$$\delta = (\delta_{\alpha(1)}, \delta_{\alpha(2)}, ..., \delta_{\alpha(m)})^T, \delta_{\alpha(l-1)} \geq \delta_{\alpha(l)},\ \sum_{l=1}^{m} \delta_{\alpha(l)} = 1.$$

Applying the $(I-IOWACJM)$ as the aggregation procedure and the $(RAMM)$ as the prioritization procedure, it holds that:

$$CI(\bar{M}) \geq \sum_{l=1}^{m} \mu_1 CI(M^{(l)}) \qquad (11)$$

Proof. Definition 22 and Eq. (4), we have

$$CI(\bar{M}) = 1 - d(\bar{M}, \bar{N}) = 1 - \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n}\left|C_{ij} - \bar{p}_{ij}\right|$$

$$= 1 - \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n}\left|\begin{array}{l}\sum_{l=1}^{m} C_{ij}^{(l)} \times \mu_1 \\ -\sum_{l=1}^{m} p_{ij}^{(l)} \times \mu_1\end{array}\right|$$

$$= 1 - \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n}\left|\sum_{l=1}^{m}(C_{ij}^{(l)} - p_{ij}^{(l)})\mu_1\right|.$$

Since $\left|\sum_{l=1}^{m}(C_{ij}^{(l)} - p_{ij}^{(l)})\mu_1\right| \le \sum_{l=1}^{m}\left|(C_{ij}^{(l)} - p_{ij}^{(l)})\mu_1\right|.$

Then $CI(\bar{M}) \ge 1 - \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{l=1}^{m}\left|(C_{ij}^{(l)} - p_{ij}^{(l)})\mu_1\right|$

$$= 1 - \frac{1}{n}\sum_{l=1}^{m}\mu_1\left(\frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n}\left|(C_{ij}^{(l)} - p_{ij}^{(l)})\right|\right.$$

$$= \sum_{l=1}^{m}\mu_1\left(1 - \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n}\left|(C_{ij}^{(l)} - p_{ij}^{(l)})\right|\right)$$

$$= \sum_{l=1}^{m}\mu_1 CI(M^{(l)}).$$

Corollary If the individual cubic judegements $\{M^{(1)}, M^{(2)}, ..., M^{(m)}\}$ are of acceptable consistency, then the $(I-IOWACJM)$ M is also acceptable consistency, that is to say,

$$CI(M^{(l)}) \ge \tau, \text{ for all } l = 1, ..., m \Rightarrow CI(\bar{M}) \ge \tau, \quad (12)$$

where τ is for acceptable consistency.

Corollary The consistency degree of $\overline{M}$ is more than the minimum of the consistency degree between $M^{(l)}$, i.e.

$$CI(\bar{M}) \ge Min_{l=1,...,m}\{CI(M^{(l)})\} \quad (13)$$

4. Numerical Example

Consider there are the set of four alternatives $X = \{x_1, x_2, x_3, x_4\}$, and four (DMs) , $D = \{d_1, d_2, d_3, d_4\}$. Suppose that these decision maker's provide the following (CPR) on the set of alternative.

$$M^{(1)} = \begin{bmatrix} \langle[0.5,0.5],0.5\rangle & \langle[0.3,0.4],0.6\rangle & \langle[0.6,0.7],0.3\rangle & \langle[0.7,0.8],0.3\rangle \\ \langle[0.6,0.7],0.4\rangle & \langle[0.5,0.5],0.5\rangle & \langle[0.6,0.7],0.8\rangle & \langle[0.3,0.5],0.4\rangle \\ \langle[0.3,0.4],0.7\rangle & \langle[0.3,0.4],0.2\rangle & \langle[0.5,0.5],0.5\rangle & \langle[0.6,0.7],0.5\rangle \\ \langle[0.2,0.3],0.7\rangle & \langle[0.5,0.7],0.6\rangle & \langle[0.3,0.4],0.5\rangle & \langle[0.5,0.5],0.5\rangle \end{bmatrix}$$

$$M^{(2)}=\begin{bmatrix}\langle[0.5,0.5],0.5\rangle\langle[0.3,0.4],0.2\rangle\langle[0.4,0.5],0.4\rangle\langle[0.2,0.3],0.6\rangle\\ \langle[0.6,0.7],0.8\rangle\langle[0.5,0.5],0.5\rangle\langle[0.4,0.5],0.6\rangle\langle[0.5,0.6],0.4\rangle\\ \langle[0.5,0.6],0.6\rangle\langle[0.5,0.6],0.4\rangle\langle[0.5,0.5],0.5\rangle\langle[0.3,0.5],0.6\rangle\\ \langle[0.7,0.8],0.4\rangle\langle[0.4,0.5],0.6\rangle\langle[0.5,0.7],0.4\rangle\langle[0.5,0.5],0.5\rangle\end{bmatrix}$$

$$M^{(3)}=\begin{bmatrix}\langle[0.5,0.5],0.5\rangle\langle[0.3,0.4],0.6\rangle\langle[0.6,0.7],0.2\rangle\langle[0.4,0.5],0.3\rangle\\ \langle[0.6,0.7],0.4\rangle\langle[0.5,0.5],0.5\rangle\langle[0.5,0.6],0.1\rangle\langle[0.7,0.8],0.2\rangle\\ \langle[0.3,0.4],0.8\rangle\langle[0.4,0.5],0.9\rangle\langle[0.5,0.5],0.5\rangle\langle[0.3,0.6],0.5\rangle\\ \langle[0.5,0.6],0.7\rangle\langle[0.2,0.3],0.8\rangle\langle[0.4,0.7],0.5\rangle\langle[0.5,0.5],0.5\rangle\end{bmatrix}$$

$$M^{(4)}=\begin{bmatrix}\langle[0.5,0.5],0.5\rangle\langle[0.4,0.5],0.6\rangle\langle[0.6,0.7],0.2\rangle\langle[0.6,0.7],0.3\rangle\\ \langle[0.5,0.6],0.4\rangle\langle[0.5,0.5],0.5\rangle\langle[0.3,0.4],0.5\rangle\langle[0.4,0.5],0.4\rangle\\ \langle[0.3,0.4],0.8\rangle\langle[0.6,0.7],0.5\rangle\langle[0.5,0.5],0.5\rangle\langle[0.3,0.4],0.8\rangle\\ \langle[0.3,0.4],0.7\rangle\langle[0.5,0.6],0.6\rangle\langle[0.6,0.7],0.2\rangle\langle[0.5,0.5],0.5\rangle\end{bmatrix}$$

By using the above procedure, we can obtain four consistent matrices as follows:

$$N^{(1)}=\left[\begin{array}{|c|c|}\hline \langle[0.5000,0.5000],0.5000\rangle & \langle[0.3753,0.4444],0.5357\rangle\\ \hline \langle[0.5556,0.6251],0.4643\rangle & \langle[0.5000,0.5000],0.5000\rangle\\ \hline \langle[0.4445,0.5630],0.3581\rangle & \langle[0.3889,0.4375],0.3929\rangle\\ \hline \langle[0.3334,0.5000],0.3929\rangle & \langle[0.2778,0.3751],0.3929\rangle\\ \hline \langle[0.4371,0.5505],0.4221\rangle & \langle[0.5000,0.6666],0.6071\rangle\\ \hline \langle[0.5625,0.6111],0.6071\rangle & \langle[0.6253,0.7222],0.6071\rangle\\ \hline \langle[0.5000,0.5000],0.5000\rangle & \langle[0.5625,0.6111],0.5000\rangle\\ \hline \langle[0.3889,0.4375],0.5000\rangle & \langle[0.5000,0.5000],0.5000\rangle\\ \hline\end{array}\right]$$

$$N^{(2)}=\left[\begin{array}{|c|c|}\hline \langle[0.5000,0.5000],0.5000\rangle & \langle[0.3571,0.4375],0.3750\rangle\\ \hline \langle[0.5625,0.6429],0.6250\rangle & \langle[0.5000,0.5000],0.5000\rangle\\ \hline \langle[0.5625,0.7143],0.5417\rangle & \langle[0.4375,0.5000],0.5417\rangle\\ \hline \langle[0.5625,0.5872],0.5417\rangle & \langle[0.5625,0.7143],0.4167\rangle\\ \hline \langle[0.2857,0.4375],0.6166\rangle & \langle[0.4128,0.4375],0.4583\rangle\\ \hline \langle[0.5000,0.5625],0.4583\rangle & \langle[0.2857,0.4375],0.5833\rangle\\ \hline \langle[0.5000,0.5000],0.5000\rangle & \langle[0.3571,0.5000],0.5416\rangle\\ \hline \langle[0.5000,0.6429],0.5484\rangle & \langle[0.5000,0.5000],0.5000\rangle\\ \hline\end{array}\right]$$

$$N^{(3)}=\begin{bmatrix}\langle[0.5000,0.5000],0.5000\rangle & \langle[0.3333,0.4444],0.5416\rangle\\ \langle[0.5555,0.6667],0.4584\rangle & \langle[0.5000,0.5000],0.5000\rangle\\ \langle[0.5000,0.6667],0.6253\rangle & \langle[0.4445,0.5000],0.6667\rangle\\ \langle[0.4445,0.3344],0.4167\rangle & \langle[0.3889,0.6667],0.6667\rangle\\ \langle[0.3333,0.5000],0.3750\rangle & \langle[0.1666,0.5555],0.5833\rangle\\ \langle[0.5000,0.5555],0.3333\rangle & \langle[0.3333,0.6111],0.3330\rangle\\ \langle[0.5000,0.5000],0.5000\rangle & \langle[0.3333,0.5555],0.5000\rangle\\ \langle[0.4445,0.6667],0.5000\rangle & \langle[0.5000,0.5000],0.5000\rangle\end{bmatrix}$$

$$N^{(4)}=\begin{bmatrix}\langle[0.5000,0.5000],0.5000\rangle & \langle[0.4164,0.5000],0.5384\rangle\\ \langle[0.5000,0.5834],0.4616\rangle & \langle[0.5000,0.5000],0.5000\rangle\\ \langle[0.5625,0.7510],0.4616\rangle & \langle[0.6251,0.7500],0.5000\rangle\\ \langle[0.6235,0.8334],0.3834\rangle & \langle[0.6253,0.3847],0.3847\rangle\\ \langle[0.2502,0.7375],0.5384\rangle & \langle[0.1666,0.3750],0.6153\rangle\\ \langle[0.2500,0.3750],0.5000\rangle & \langle[0.1666,0.3759],0.6153\rangle\\ \langle[0.5000,0.5000],0.5000\rangle & \langle[0.3333,0.4375],0.6135\rangle\\ \langle[0.5625,0.6667],0.6667\rangle & \langle[0.5000,0.5000],0.5000\rangle\end{bmatrix}$$

According to E.q.(4), we can calculate the consistency degree $CI(M^{l}),\quad l=1,2,3,4$:

$$CI(M^1)=0.5481,\ CI(M^2)=0.6701,\ CI(M^3)=0.5984,\ CI(M^4)=0.499$$

and the judgment matrices $M^{(1)},M^{(2)},M^{(3)},M^{(4)}$ and having equivalent consistent matrices $N^{(1)},N^{(2)},N^{(3)},N^{(4)}$ are reordered as follows respectively:

$$M^{(\alpha(1))}=M^{(2)};\ M^{(\alpha(2))}=M^{(3)};\ M^{(\alpha(3))}=M^{(1)};\ M^{(\alpha(4))}=M^{(4)};$$
$$N^{(\alpha(1))}=N^{(2)};\ N^{(\alpha(2))}=N^{(3)};\ N^{(\alpha(3))}=N^{(1)};\ N^{(\alpha(4))}=N^{(4)};$$

Using E.q. (8) with $Q(r)=r^{\frac{1}{2}}$, we obtain the weight as followes:

$$\delta_{\alpha(1)}=0.51;\ \delta_{\alpha(2)}=0.19;\ \delta_{\alpha(3)}=0.23;\ \delta_{\alpha(4)}=0.07.$$

Then, the $(C-IOWACJM)$ $\overline{M}_1$ and its equivalent consistent matrix $\overline{P}_1$ are calculated as;

$$\overline{M}_1 = \begin{bmatrix} \langle [0.5000, 0.5000], 0.5000 \rangle & \langle [0.3075, 0.4077], 0.3426 \rangle \\ \langle [0.3937, 0.6939], 0.5696 \rangle & \langle [0.5000, 0.5000], 0.5000 \rangle \\ \langle [0.4014, 0.5121], 0.6421 \rangle & \langle [0.4494, 0.5510], 0.4041 \rangle \\ \langle [0.5604, 0.6713], 0.5262 \rangle & \langle [0.4001, 0.5334], 0.6336 \rangle \\ \langle [0.5081, 0.6107], 0.3126 \rangle & \langle [0.4242, 0.5361], 0.4227 \rangle \\ \langle [0.4664, 0.5684], 0.4503 \rangle & \langle [0.5035, 0.6225], 0.3507 \rangle \\ \langle [0.5000, 0.5000], 0.5000 \rangle & \langle [0.3845, 0.5684], 0.5670 \rangle \\ \langle [0.4494, 0.6481], 0.5000 \rangle & \langle [0.5000, 0.5000], 0.5000 \rangle \end{bmatrix}$$

$$\overline{P}_1 = \begin{bmatrix} \langle [0.5000, 0.5000], 0.5000 \rangle & \langle [0.3675, 0.4449], 0.4479 \rangle \\ \langle [0.5554, 0.6369], 0.5388 \rangle & \langle [0.5000, 0.5000], 0.5000 \rangle \\ \langle [0.5259, 0.6786], 0.5197 \rangle & \langle [0.4441, 0.5106], 0.5204 \rangle \\ \langle [0.5011, 0.5567], 0.4673 \rangle & \langle [0.4825, 0.6608], 0.4470 \rangle \\ \langle [0.3257, 0.4764], 0.3126 \rangle & \langle [0.4874, 0.5137], 0.5216 \rangle \\ \langle [0.5012, 0.5021], 0.4630 \rangle & \langle [0.3855, 0.5509], 0.5312 \rangle \\ \langle [0.5000, 0.5000], 0.5000 \rangle & \langle [0.4059, 0.5347], 0.5284 \rangle \\ \langle [0.4708, 0.8024], 0.5146 \rangle & \langle [0.5000, 0.5000], 0.5000 \rangle \end{bmatrix}$$

A/to Definition 16 and, E.q. (4) we get such that

$$CI(\overline{M}_1) = 0.7487 > \frac{1}{4}\sum_{l=1}^{4} CI(M^l) = \frac{0.5481 + 0.6701 + 0.5984 + 0.499}{4} = 0.5789.$$

This result is in accordance with Theorem 5.

5. Conclusion

We have discussed the properties of $IOWA$ operators in the aggregation of CPR in group decision making problems in this paper. We have also defined that the collective preference get by these cases of $IOWA$ operators which shown the reciprocity and consistency conditions. Then, it is verified that the aggregation of individual judgments and the aggregation of individual properties define the same properties of the alternatives by applying $RAMM$ as prioritization technique and $IOWA$ operators as aggregation technique. By using the distance between $M^{(l)}$ and its corresponding consistent matrix N $^{(l)}$, we present the consistency index of CPR. Using this consistency measure, we proved that the $C-IOWA$ and the $I-IOWA$ operator can improve consistency degree in the collective CPR. In a future we plan that we will extend this work.

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Upper and Lower δ_{ij}-Continuous Multifunctions

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Abstaract − In this paper we introduce and study the notions of upper and lower δ_{ij}-continuous multifunctions. Several characterizations and properties concerning upper and lower δ_{ij}-continuous multifunctions and other known forms of multifunctions introduced previously are investigated.

Keywords − *Upper(lower)δ_{ij}-continuous multifunction.*

1 Introduction

A multifunction or a multivalued function is set valued function. In last thirty years the theory of multifunctions has advanced in variety of ways. Applications of this theory can be found in economic theory, viability theory, noncooperative games, decision theory, artificial intelligence, medicine and existence of solutions for differential equations. In topology there has been recently significant interest in characterizing and investigating the properties of several weak and strong forms of continuity of multifuctions. The development of such a theory is in fact very well motivated in [1, 4, 5, 6, 7, 12, 14, 15, 17]. Kucuk [10] and Cao and Reilly [3] independently defined and investigated upper(lower)δ_{ij}-continuous multifunction. The invariance of some separating properties of the bitoplogical spaces by multifunctions was studied by Smithson [18]. The notions of continuous (resp. upper semicontinuous, lower semicontinuous) multifunctions between bitopological spaces wear defined and studied by Popa [15] and Ganguly [13] introduced and studied the concept of upper (lower) almost multifunction between bitopological spaces. Several characterizations of these

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concepts were given by Kucuk and Kucuk in [11]. In this paper we introduce and study the notions of upper and lower δ_{ij}-continuous multifunctions between bitopological spaces. As a consequence, some characterizations and several proprties concerning upper (lower) δ_{ij}-continuous multifunctions are obtained. The relationship between upper (lower) δ_{ij}-continuous multifunctions and with other known forms of multifuctions introduced previously are established.

2 Preliminary

Let (X, τ_1, τ_2) be a bitopological space. The closure and interior of a subset A of X with respect to τ_i are denoted by $\tau_i.cl(A)$ and $\tau_i.int(A)$, respectively. The set $N(A, \tau_i)$ denotes the family of all τ_i-open set containing A. In particular, $N(x, \tau)$ is the family of all τ_i-open neighborhood (τ_i-nbds, for short) of x. The set of all τ_i-closed sets will be denoted by $\acute{\tau_i}$. A subset A of a bts (X, τ_1, τ_2) is called $ij-$regular closed (resp. $ij-$regular open) if $A = \tau_i.cl(\tau_j.int(A))$(resp.$A = \tau_i.int(\tau_j.cl(A))$). The set of all $ij-$regular closed (resp. $ij-$regular open) sets of (X, τ_1, τ_2) is denoted by $ijRC(X)$ (resp. $ijRO(X)$). By a multifunction $F : X \to Y$, we mean a point-to-set correspondence from X into Y, and we always assume that $F(x) \neq \phi$ for all $x \in X$. For a multifunction $F : X \to Y$, we shall denote the upper and lower inverse of a set B of Y by $F^-(B)$ and $F_-(B)$ [2], respectively, that is $F^-(B) = \{x \in X : F(X) \subseteq B\}$ and $F_- = \{x \in X : F(x) \cap B \neq \phi\}$. In particular, $F^-(y) = \{x \in X : y \in F(x)\}$, for each $y \in Y$. For $A \subseteq X, F(A) = \cup_{x \in A} F(x)$. Then F is said to be a surjection if $F(x) = Y$, or equivalently if for each $y \in Y$, there exists an $x \in X$ such that $y \in F(x)$. Also, F is said to be injective if for any $x_1, x_2 \in X, x_1 \notin x_2$, we have $F(x_1) \cap F(x_2) = \phi$. The reader can find undefined notions of some generalizing continuities for multifunctions from the references.

Definition 2.1. Let (X, τ_1, τ_2) be a bts.[8, 13, 16]. A point x in X will be called an δ_{ij}-adherent (resp. θ_{ij}-adherent) point of a subset A of X if and only if $A \cap \tau_i.int(\tau_j.cl(U)) \neq \phi$ (resp. $A \cap \tau_j..cl(U)) \neq \phi$ for each τ_i-open nbd U of x. The set of all δ_{ij}-adherent (resp. θ_{ij}-adherent) points of A is called δ_{ij}-closure (resp. θ_{ij}-closure) of A and it is denoted by $\delta_{ij}.cl(A)$ (resp. $\theta_{ij}.cl(A)$). If $A = \delta_{ij}.cl(A)$ (resp. $A = \theta_{ij}.cl(A)$), then A is called δ_{ij}-closed (resp. θ_{ij}-closed). The complement of a δ_{ij}-closed (resp. θ_{ij}-closed) set is called a δ_{ij}-open(resp. θ_{ij}-open) set. The family of all δ_{ij}-closed (resp. δ_{ij}-open , θ_{ij}-closed, θ_{ij}-open) sets of X is denoted by $\delta_{ij}.C(X)$ (resp. $\delta_{ij}.O(X), \theta_{ij}.C(X), \theta_{ij}.O(X)$). It is clear that in any bts (X, τ_1, τ_2), we have $\theta_{ij}.O(X) \subseteq \delta_{ij}.O(X) \subseteq \tau_i$ and $ijRC(X) \subseteq \delta_{ij}.C(X)$.

Definition 2.2. Let (X, τ_1, τ_2) be a bts.[8, 13]. A point x in X will be called an δ_{ij}-interior (resp. θ_{ij}-interior) point of a subset A of X if and only if there exists τ_i-open nbd U of x such that $\tau_i.int\tau_i.cl(U)) \subseteq A$ (resp. $\tau_i.cl(U)) \subseteq A$) equivalently, if there exists $ij-$regular open (resp. $ij-$regular closed) nbd U of x such that $U \subseteq A$. The family of all δ_{ij}-interior (resp. θ_{ij}-interior) points of A will be denoted by $\delta_{ij} - int(A)$ (resp. $\theta_{ij} - int(A)$). A subset A of a bts (X, τ_1, τ_2) is δ_{ij}-open (resp. θ_{ij}-open) if and only if $\delta_{ij} - int(A) = A$ (resp. $\theta_{ij} - int(A) = A$).

Definition 2.3. A bts (X, τ_1, τ_2) [8, 9, 15] is called:
(a) PR_2 if and only if $\forall x \in X, F \in \acute{\tau_i} s.t. x \notin F \; \exists U \in N(x, \tau_i), V \in N(F, \tau_j) s.t. U \cap$

$V = \phi$.
(b)PSR_2 if and only if $\forall x \in X, U \in N(x, \tau_i)\ \exists V \in N(x, \tau_i), \tau_i - int(\tau_i.cl(V)) \subseteq U$.
(c)PAR_2 if and only if $\forall x \in X, U \in N(x, ijRO(X))\ \exists V \in N(x, \tau_i), \tau_i.cl(V) \subseteq U$.

Theorem 2.4. Let (X, τ_1, τ_2) be a bts.[8, 15].
(a) For each $A \subseteq X$, then $\tau_i.cl(A) \subseteq \delta_{ij}.cl(A) \subseteq \theta_{ij}.cl(A)$.
(b) If $A \in \tau_j$, then $\tau_i.cl(A) = \delta_{ij}.cl(A)$.
(c) If (X, τ_1, τ_2) is PSR_2-space, then $\tau_i.cl(A) = \delta_{ij}.cl(A)$.
(d) If (X, τ_1, τ_2) is PAR_2-space, then $\delta_{ij}.cl(A) = \theta_{ij}.cl(A)$.

3 Upper and Lower δ_{ij}-Continuous Multifunctions

In this section we define and study the concept of upper and lower δ_{ij}-continuous multifunctions.some of their properties are obtained.

Definition 3.1. A multifunction $F : (X, \tau_1, \tau_2) \to (Y, \triangle_1, \triangle_2)$ is called:
(a) Lower δ_{ij}-continuous at a point x in X if and only if for every increasing $\triangle_i$-open set V in Y with $F(x) \cap V \neq \phi$, there exists increasing $\triangle_i$-open nbd U of x such that $F(x_0) \cap \triangle_i.int(\triangle_j.cl(V)) \neq \phi$, for each $x_0 \in \tau_i.int(\tau_j.cl(U))$.
(b) Upper δ_{ij}-continuous at a point x in X if and only if for every decreasing $\triangle_i$-open set V in Y with $F(x) \subseteq V$, there exists decreasing τ_i-open nbd U of x such that $F(\tau_i.int(\tau_j.cl(U)) \subseteq \triangle_i.int(\triangle_j.cl(V))$.
(c) Lower (resp. upper) δ_{ij}-continuous if it has this property at each point $x \in X$.
The following theorem give us some characterizations of lower δ_{ij}-continuity of F.

Theorem 3.2. For a multifunction $F : (X, \tau_1, \tau_2) \to (Y, \triangle_1, \triangle_2)$ the following statements are equivalent:
(a) F is lower δ_{ij}-continuous,
(b) For every increasing $ij-$regular open set $V \subseteq Y$ and for each $x \in X$ with $F(x) \cap V \neq \phi$, there exists increasing $ij-$regular open nbd U of x such that $F(x_0) \cap V \neq \phi$, for each $x_0 \in U$.
(c) For every increasing $ij-$regular open set $V \subseteq Y, F_-(V)$ is δ_{ij}-open set in X.
(d) For every increasing δ_{ij}-open set $V \subseteq Y, F_-(V)$ is δ_{ij}-open set in X.
(e) For every increasing δ_{ij}-closed set $K \subseteq Y, F^-(K)$ is δ_{ij}-closed set in X.
(f) For every increasing $ij-$regular closed set $K \subseteq Y, F^-(K)$ is δ_{ij}-closed set in X.
(g) For each $B \subseteq Y, F^-(\delta_{ij}.int(B)) \subseteq \delta_{ij}.int(F^-(B))$.
(h) For each $A \subseteq X, F(\delta_{ij}.cl(A)) \subseteq \delta_{ij}.cl(F(A))$.

Proof. (a)→ (b): Let x in X and let V by an $ij-$regular open set in Y with $F(x) \cap V \neq \phi$. Then V is $\triangle_i-$open set in Y. By (a), there exists $W \in N(x, \tau_i)$ such that $F(x_0) \cap \triangle_i.int(\triangle_i.cl(V)) \neq \phi$, for each $x_0 \in \tau_i.int(\tau_j.cl(W))$. But V is ij-regular open set, so $F(x_0) \cap V \neq \phi$, for each $x_0 \in \tau_i.int(\tau_j.cl(W))$. Put $U = \tau_i.int(\tau_j.cl(W))$. Then U is ij-regular open set in X. So $F(x_0) \cap V \neq \phi$ for $x_0 \in U$.
(b)→ (c): Let $V \subseteq Y$ be an ij-regular open set and let x in X with $x \in F^-(V)$. Then $F(x) \cap V \neq \phi$. By (b), there exists ij-regular open nbd U of x such that $F(x_0) \cap V \neq \phi$, for each $x_0 \in U$. Which implies that $U \subseteq F_-(V)$. Consequently $F_-(V)$ is δ_{ij}-open set in X.
(c)→ (d): Let $V \subseteq Y$ be a δ_{ij}-open set and let x in X with $x \in F_-(V)$. So,

$F(x) \cap V \neq \phi$ and so there exists $y \in Y$ such that $y \in F(x) \cap V$. Hence, $y \in F(x)$ and $y \in V$. Since V is δ_{ij}-open set, then there exist ij-regular open set $W \subseteq Y$ such that $y \in W \subseteq V$. Thus $F(x) \cap W \neq \phi$ and so $x \in F_{-}(W)$. Since W is ij-regular open set, by (c), $F_{-}(W)$ is a δ_{ij}-open set of X and from $x \in F_{-}(W)$, there exists an ij-regular open set $U \subseteq X$ such that $x \in U \subseteq F_{-}(W) \subseteq F_{-}(V)$. Thus $F_{-}(V)$ is a δ_{ij}-open set in X.
(d)$\rightarrow$ (e): Let $K \subseteq Y$ be any δ_{ij}-closed set. Then $T\backslash K$ is a δ_{ij}-open set. By (d), $F_{-}(Y\backslash K)$ is a δ_{ij}-open set. As we can write $F^{-}(K) = X\backslash F_{-}(Y\backslash K)$ so $F^{-}(K)$ is a δ_{ij}-closed set in X.
(e)$\rightarrow$ (f): Let $K \subseteq Y$ be any δ_{ij}-regular closed set. Then K is a δ_{ij}-closed set. By (e), $F^{-}(K)$ is a δ_{ij}-closed set in X.
(f)$\rightarrow$ (c): Let $V \subseteq Y$ be an ij-regular open set. Then $Y\backslash V$ is an ij-regular closed set of Y. By (f), $F^{-}(Y\backslash V)$ is δ_{ij}-closed set in X. Thus $F_{-}(V)$ is δ_{ij}-open set in X.
(c)$\rightarrow$ (a): Let x in X and let $V \subseteq Y$ be any $\triangle_i$-open set with $F(x)\cap V \neq \phi$. Since $V \subseteq \triangle_i.int(\triangle_j.cl(V))$, then $F(x)\cap\triangle_i.int(\triangle_j.cl(V)) \neq \phi$. So, x is $F^{-}(\triangle_i.int(\triangle_j.cl(V)))$. By (c), there exists $ij-$regular open nbd U of x such that $U \subseteq F^{-}(\triangle_i.int(\triangle_j.cl(V)))$. Thus $F(x_0)\cap\triangle_i.int(\triangle_j.cl(V)) \neq \phi$ for each x_0 in U. Thus F is lower δ_{ij}-continuous.
(d)$\rightarrow$ (g): Let $B \subseteq Y$. Since $\delta_{ij}.int(B) \subseteq B$, then $F_{-}(\delta_{ij}.int(B)) \subseteq F_{-}(B)$. Since $\delta_{ij}.int(B)$ is δ_{ij}-open set of Y, then by (d), $F_{-}(\delta_{ij}.int(B)) = \delta_{ij}int(F_{-}(\delta_{ij}.int(B))) \subseteq \delta_{ij}.int(F_{-}(B))$. Thus $F_{-}(\delta_{ij}.int(B))) \subseteq \delta_{ij}.int(F_{-}(B))$.
(g)$\rightarrow$ (d): Let V be δ_{ij}-open set of Y. By (g), we have $F_{-}(V) = F_{-}(\delta_{ij}.int(V)) \subseteq \delta_{ij}.int(F_{-}(V))$. Thus $F_{-}(V)$ is δ_{ij}-open set of X.
(d)$\rightarrow$ (h): Under the assumption (e) suppose that (h) is not true, i.e. for some $A \subseteq X$, we have $F(\delta_{ij}.cl(A)) \nsubseteq \delta_{ij}.cl(F(A))$. Then there exists y in Y such that $y \in F(\delta_{ij}.cl(A))$, but $y \notin \delta_{ij}.cl(F(A))$. So, $Y\backslash(\delta_{ij}.cl(F(A)))$ is δ_{ij}-open set containing y. By (d), we have $F_{-}(Y\backslash(\delta_{ij}.cl(F(A))))$ is δ_{ij}-open set in X and $F_{-}(Y) \subseteq F_{-}(Y\backslash(\delta_{ij}.cl(F(A))))$. Since $Y\backslash(\delta_{ij}.cl(F(A))) \cap F(A) = \phi$ and $A \subseteq F^{-}(F(A))$ we have $F_{-}(Y\backslash(\delta_{ij}.cl(F(A))))\cap F^{-}(F(A)) = \phi$ and $F_{-}(Y\backslash(\delta_{ij}.cl(F(A))))\cap A = \phi$. Since $F_{-}(Y\backslash(\delta_{ij}.cl(F(A))))$ is δ_{ij}-open set in X, then $F_{-}(Y\backslash(\delta_{ij}.cl(F(A))))\cap\delta_{ij}.cl(A) = \phi$. On the other hand, because of $y \in F(\delta_{ij}.cl(A))$, we have $F_{-}(Y)\cap\delta_{ij}.cl(A) \neq \phi$, which is contradiction with $F_{-}(Y\backslash(\delta_{ij}.cl(F(A)))) \cap \delta_{ij}.cl(A) = \phi$. Thus $y \in F(\delta_{ij}.cl(A))$ implies $y \in \delta_{ij}.cl(F(A))$. Consequently, $F(\delta_{ij}.cl(A)) \subseteq \delta_{ij}.cl(F(A))$.
(h)$\rightarrow$ (e): Let $K \subseteq Y$ be any δ_{ij}-closed set. Since we have always $FF^{-}(K) \subseteq K$, then we obtain $\delta_{ij}.cl(FF^{-}(K)) \subseteq \delta_{ij}.cl(K) = K$. By (h), $F(\delta_{ij}.cl(F^{-}(K))) \subseteq \delta_{ij}.cl(FF^{-}(K))$. Thus $F(\delta_{ij}.cl(F^{-}(K))) \subseteq K$ and so

$\delta_{ij}.cl(F^{-}(K)) \subseteq F^{-}F(\delta_{ij}.cl(F^{-}(K))) \subseteq F^{-}(K)$. Hence $F^{-}(K)$ is δ_{ij}-closed set in X. □

Theorem 3.3. For multifunction $F : (X, \tau_1, \tau_2) \rightarrow (Y, \triangle_1, \triangle_2)$ the following statements are equivalent:
(a) F is upper δ_{ij}-continuous,
(b) For every ij-regular open set $V \subseteq Y$ for each $x \in X$ with $F(x) \subseteq V$, there exists ij-regular open nbd U of x such that $F(U) \subseteq V$.
(c) For each ij-regular open set $V \subseteq Y, F^{-}(V)$ is δ_{ij}-open set in X.
(d) For each ij-open set $V \subseteq Y, F^{-}(\triangle_i.int(\triangle_j.cl(V)))$ is δ_{ij}-closed set in X.
(e) For each δ_{ij}-closed set $K \subseteq Y, F_{-}(\triangle_j.cl(\triangle_i.int(K)))$ is δ_{ij}-closed set in X.
(f) For each δ_{ij}-regular closed set $K \subseteq Y, F_{-}(K)$ is δ_{ij}-open set in X.

Proof. It is quite similar to that of Theorem 3.2 and so it is omitted. □

Definition 3.4. A multifunction $F : (X, \tau_1, \tau_2) \to (Y, \triangle_1, \triangle_2)$ is called pairwise point compact if the induced multifunctions $F : (X, \tau_i) \to (Y, \triangle_i), i = 1, 2$ are point compact.

Theorem 3.5. Let $F : (X, \tau_1, \tau_2) \to (Y, \triangle_1, \triangle_2)$ be a pairwise point compact multifunction and $(Y, \triangle_1, \triangle_2)$ be PAR_2-space. Then the following statements are equivalent:
(a) F is upper δ_{ij}-continuous,
(b) For each δ_{ij}-open set $V \subseteq Y, F^-(V)$ is δ_{ij}-open set in X.
(c) For each δ_{ij}-closed set $K \subseteq Y, F_-(K)$ is δ_{ij}-closed set in X.
(d) For each $B \subseteq Y, \delta_{ij}.cl(F_-(B)) \subseteq F_-(\delta_{ij}.cl(B))$.

Proof. (a)→ (b): Let V be a δ_{ij}-open set in Y and let x in X with $x \in F^-(V)$. Then $F(x) \subseteq V$. Since V is δ_{ij}-open, then for each $y \in F(x)$, there exists ij-regular open set W_y such that $y \in W_y \subseteq V$. Since $(Y, \triangle_1, \triangle_2)$ is PAR_2-space. Then there exists an $\triangle_i$-open set τ_y such that $y \in \tau_y \subseteq \triangle_j.cl(\tau_y) \subseteq \triangle_i.int(\triangle_j cl(W_y)) = W_y$. Hence we have $F(x) \subseteq \cup\{T_y : y \in F(x)\} \subseteq \cup\{\triangle_j.cl(\tau_y) : y \in F(x)\} \subseteq \cup\{W_y : y \in F(x)\} \subseteq V$. Since $F(x)$ is a $\triangle_i$-compact set, there exists points $y_1, y_2, ..., y_n \in F(x)$ such that $F(x) \subseteq \cup\{\tau_{y_s} : y_s \in F(x), s = 1, 2, ..., n\} \subseteq U\{\triangle_j.cl(\tau_{y_s}) : y_s \in F(x), s = 1, 2, ..., n\} \subseteq \cup\{W_{y_s} : y_s \in F(x), s = 1, 2, ..., n\} \subseteq V$. Therefore, we obtain $F(x) \subseteq \triangle_i.int(\cup\{\tau_{y_s} : y_s \in F(x), s = 1, 2, ..., n\} = \cup\{\tau_{y_s} : y_s \in F(x), s = 1, 2, ..., n\} \subseteq \triangle_i.int(\triangle_j.cl(\cup\{\tau_{y_s}) : y_s \in F(x), s = 1, 2, ..., n\}) \subseteq V$. Put $H = \triangle_i.int((\cup\{\triangle_j.cl\{\tau_{y_s}) : y_s \in F(x), s = 1, 2, ..., n\}))$. Then H is ij-regular open set of Y with $F(x) \subseteq H$. By (a), there exists ij-regular open nbd U of x such that $U \subseteq F^-(H) \subseteq F^-(V)$. Therefore, $x \subseteq U \subseteq F^-(V)$ and this mean that $F^-(V)$ is δ_{ij}-open set in X.
(b)→ (c): Let $K \subseteq Y$ be δ_{ij}-closed set. Then $Y \backslash K$ is δ_{ij}-open set in Y. By (b) we conclude that $F^-(Y \backslash K)$ is a δ_{ij}-open set in X, so $F^-(K)$ is δ_{ij}-closed set in X.
(c)→ (a): Let x in X and let $V \subseteq Y$ be ij-regular open set of Y such that $F(x) \subseteq V$. So, $Y \backslash V$ is a δ_{ij}-closed set in Y. By (c) $F^-(Y \backslash V)$ is a δ_{ij}-closed set in X. Thus $F^-(V) = X \backslash F_-(Y \backslash V)$ is δ_{ij}-open set in X. Since $x \in F^-(V)$, there exists ij-regular open nbd U of x such that $x \in U \in F^-(V)$. Thus F is upper δ_{ij}-continuous.
(c)→ (d): Let $B \subseteq Y$. Since $B \subseteq \delta_{ij}.cl(B)$, then $F_-(B) \subseteq F_-(\delta_{ij}.cl(B))$. Since $\delta_{ij}.cl(B)$ is a δ_{ij}-closed set of Y, then by (c), $F_-(\delta_{ij}.cl(B))$ is δ_{ij}-closed set of X. Hence, we have $\delta_{ij}.cl(F_-(B)) \subseteq \delta_{ij}.cl(F_-(\delta_{ij}.cl(B))) = F_-(\delta_{ij}.cl(B))$ and so $\delta_{ij}.cl(F_-(B)) \subseteq F_-(\delta_{ij}.cl(B))$.
(d)→ (c): Let B a δ_{ij}-closed set in Y. Then $F_-(B) = F_-(\delta_{ij}.cl(B))$. By (d), we have $\delta_{ij}.cl(F_-(B)) \subseteq F_-(\delta_{ij}.cl(B)) = F_-(B)$ and $F_-(B)$ is δ_{ij}-closed set in X. □

Theorem 3.6. Let $F_1 : (X, \tau_1, \tau_2) \to (Y, \triangle_1, \triangle_2)$ and $F_2 : (Y, \triangle_1, \triangle_2) \to (Z, \Gamma_1, \Gamma_2)$ are lower δ_{ij}-continuous function then $F_2 \circ F_1 : (X, \tau_1, \tau_2) \to (Z, \Gamma_1, \Gamma_2)$ is lower δ_{ij}-continuous function.

Proof. Let K be δ_{ij}-closed set in Z. From lower δ_{ij}-continuity of F_2, we have $F_2^-(K)$ is δ_{ij}-closed set in Y. Since F_1 is lower δ_{ij}-continuous, then $F_1^-(F_2^-(K))$ is δ_{ij}-closed set in Y. But $(F_2 \circ F_1)^-(K) = F_1^-(F_2^-(K))$. Therefore $F_2 \circ F_1$ is lower δ_{ij}-continuous function.

□

Proposition 3.7. Let (X, τ_1, τ_2) be a bts, $A \subseteq X$ be τ_i-open set and $U \subseteq X$ be ij-regular open set. Then $W = A \cap U$ is ij-regular open set in $(A, \tau_{1A}, \tau_{2A})$.

Proof. It is very similar to that of Proposition 2.6 in[10].

□

Theorem 3.8. For a multifunction $F_1 : (X, \tau_1, \tau_2) \to (Y, \triangle_1, \triangle_2)$, the following statement are true:
(a) If F is lower(resp. upper) δ_{ij}-continuous and A is an τ_i-open set in X, then $F|_A$: $(A, \tau_{1|A}, \tau_{2|A}) \to (Y, \triangle_1, \triangle_2)$ is lower (resp. upper) δ_{ij}-continuous.
(b) Let $U = \{U_\alpha : \alpha \in \Omega\}$ be ij-regular open cover of X. Then a $p-$multifunction $F : (X, \tau_1, \tau_2) \to (Y, \triangle_1, \triangle_2)$ is lower (resp. upper) δ_{ij}-continuous if and only if the restrictions $F_\alpha = F \mid U_\alpha : (U_\alpha, \tau_{1|U_\alpha}, \tau_{2|U_\alpha}) \to (Y, \triangle_1, \triangle_2)$ are lower (resp. upper) δ_{ij}-continuous, for each $\alpha \in \Omega$.

Proof. (a): Let $x \in A$ and V be any ij-regular open set in Y with $F|_A (x) \cap V \neq \phi$. Hence $F(x) \cap V \neq \phi$. Since F is lower δ_{ij}-continuous, then there exists $U \in N(x, ijRO(x))$ such that $F(x_0) \cap V \neq \phi$, for each $x_0 \in U$. Then $U \subseteq F_-$. Put $W = U \cap A$. Then W is ij-regular open set in A with $W \subseteq A \cap F_- = F|_A (V)$. Hence $F|_A (x_0) \cap V \neq \phi$, for each $x_0 \in W$. Thus $F|_A$ is lower δ_{ij}-continuous. The proof is the upper δ_{ij}-continuous of F is similar.
(b): Let F be lower δ_{ij}-continuous and $\alpha \in \Omega$ be such that $x \in U_\alpha$ and let V be any ij-regular open set in Y such that $F_\alpha(x) \cap V \neq \phi$. Since $F(x) = F_\alpha(x)$ and F is lower δ_{ij}-continuous, then there exists an ij-regular open nbd U_0 of x such that $F(x_0) \cap V \neq \phi$, for each $x_0 \in U_0$. Hence $U_0 \in V_0$. Put $U = U_\alpha \cap U_0$, thus U is ij-regular open subset of U_α and $x \in U$. Therefore $U = U_\alpha \cap U_0 \subseteq U_\alpha \cap F_-(V) = F_{-\alpha}(V)$. Thus F_α is lower δ_{ij}-continuous at x. Conversely, suppose that F_α is lower δ_{ij}-continuous, for each $\alpha \in \Omega$. Let $x \in X$ and V be an ij-regular open set in Y such that $F(x) \cap V \neq \phi$. Then there exists $\alpha \in \Omega$ such that $x \in U_\alpha$. Hence $F(x) = F_\alpha(x)$ and so $F_\alpha(x) \cap V \neq \phi$. Since F_α is lower δ_{ij}-continuous, there exists ij-regular open set U in U_α with $x \in U$ such that $F_\alpha(x_0) \cap V \neq \phi$, for each $x_0 \in U$. Then $U \subseteq F_\alpha(V) = F_-(V) \cap U_\alpha \subseteq F_-(V)$. Thus $F_\alpha(U) \cap V \neq \phi$ implies $U \subseteq F_{-\alpha}$, but $F_{-\alpha}(V) = F_-(V) \cap U_\alpha$. Take ij-regular open set W in X such that $U = U_\alpha \cap W$. Thus U is ij-regular open set W in X. Hence F is lower δ_{ij}-continuous.
The proof of the upper δ_{ij}-continuous of F is similar. □

4 Mutual Relationships

This section explain some of types of multifunction with some examples.

Definition 4.1. A multifunction $F : (X, \tau_1, \tau_2) \to (Y, \triangle_1, \triangle_2)$ is called [15]:
(a) pairwise lower semicontinuous (p. l. s. c, for short) at a point $x \in X$ if the induced multifuctions $F : (X, \tau_i) \to (Y, \triangle_i), i = 1, 2$ are lower semicontinuous at a point $x \in X$.
(b) pairwise upper semicontinuous (p. u. s. c, for short) at a point $x \in X$ if the induced multifuctions $F : (X, \tau_i) \to (Y, \triangle_i), i = 1, 2$ are upper semicontinuous at a point $x \in X$.
(c) pairwise lower (resp. pairwise upper) semicontinuous if it has this property at

each point $x \in X$.
Now we give two examples in order to show that the concepts of upper (resp. lower) δ_{ij}-continuity and pairwise upper (resp. pairwise lower) semicontinuous are independent.

Example 4.2. Let $X = \{a, b, c\}, \tau_1 = \{X, \phi, \{a, b\}\}, \tau_2 = \{X, \phi, \{b, c\}\}, Y = \{1, 2, 3\}$, $\triangle_1 = \{Y, \phi, \{2\}\}$ and $\triangle_2 = \{Y, \phi, \{3\}\}$. Define a multifunction $F : (X, \tau_1, \tau_2) \rightarrow (Y, \triangle_1, \triangle_2)$ as follows: $F(a) = \{1, 2\}, F(b) = \{2, 3\}$ and $F(c) = \{1, 3\}$. Then F is pairwise lower semicontinuous multifunction but it is not lower δ_{ij}-continuous multifunction, since $\{2\} \in 12RO(Y)$ and $\{3\} \in 21RO(Y)$, but $F_-(\{2\}) = \{a, b\} \notin \delta_{12}O(X)$ and $F_-(\{3\}) = \{a, b\} \notin \delta_{21}O(X)$.

Example 4.3. Let $X = \{a, b, c\}, \tau_1 = \{X, \phi, \{a, b\}\}, \tau_2 = \{X, \phi, \{b, c\}\}, Y = \{1, 2, 3\}$, $\triangle_1 = \{Y, \phi, \{2\}\}$ and $\triangle_2 = \{Y, \phi, \{3\}\}$. Define a multifunction $F : (X, \tau_1, \tau_2) \rightarrow (Y, \triangle_1, \triangle_2)$ as follows: $F(a) = \{2\}, F(b) = \{3\}$ and $F(c) = \{1, 2\}$. Then F is pairwise upper semicontinuous multifunction but it is not upper δ_{ij}-continuous multifunction. Indeed, $\{2\} \in 12RO(Y)$ and $\{3\} \in 21RO(Y)$, but $F^-(\{2\}) = \{a\} \notin \delta_{12}O(X)$ and $F^-(\{3\}) = \{b\} \notin \delta_{21}O(X)$.

Theorem 4.4. Ever upper (resp. lower) δ_{ij}-continuous multifunction from any bts to a PSR_2-space is p-upper (resp. p-lower) semicontinuous.

Proof. Let $F : (X, \tau_1, \tau_2) \rightarrow (Y, \triangle_1, \triangle_2)$ be upper (resp. lower) δ_{ij}-continuous multifunction and $(Y, \triangle_1, \triangle_2)$ is PSR_2-space. Let $V \subseteq Y$ be $\triangle_i$-open set. Since $(Y, \triangle_1, \triangle_2)$ is PSR_2-space, then V is ij-regular open. By upper (resp. lower) δ_{ij}-continuity of F, $F^-(V)$ (resp. $F_-(V)$ is δ_{ij}-open set in X, then $F^-(V)$ (resp. $F_-(V)$) is τ_i-open set in X. So F is p-upper (resp. p-lower) semicontinuous. □

Theorem 4.5. Ever p-upper (resp. p-lower) semicontinuous multifunction from a PSR_2-space to any bts-space is upper (resp. lower) δ_{ij}-continuous.

Proof. Let $F : (X, \tau_1, \tau_2) \rightarrow (Y, \triangle_1, \triangle_2)$ be p-upper (resp. p-lower) continuous multifunction and (X, τ_1, τ_2) is PSR_2-space. Let $V \subseteq Y$ be ij-regular open, then V is $\triangle_i$-open set. By p-upper (resp. p-lower) continuity of F, $F^-(V)$ (resp. $F_-(V)$ is τ_i-open set in X. Since (X, τ_1, τ_2) is PSR_2-space, then $F^-(V)$ (resp. $F_-(V)$) is ij-regular open set in X. So F is upper (resp. lower) δ_{ij}-continuous. □

Definition 4.6. A p-multifunction $F : (X, \tau_1, \tau_2) \rightarrow (Y, \triangle_1, \triangle_2)$ is called:
(a) lower strongly θ_{ij}-continuous at a point x in X if and only if for every $\triangle_i$-open set V in Y with $F(x) \cap V \neq \phi$, there exists τ_i-open nbd U of x such that $F(x_0) \cap V \neq \phi$ for each $x_0 \in \tau_i.cl(U)$.
(b) upper strongly θ_{ij}-continuous at a point x in X if and only if for every $\triangle_i$-open set V in Y with $F(x) \subseteq Y$, there exists τ_i-open nbd U of x such that $F(\tau_i.cl(U)) \subseteq V$.
(c) lower (resp. upper) strongly θ_{ij}-continuous if it has this property at each point $x \in X$.

Theorem 4.7. Every upper (resp. lower) strongly θ_{ij}-continuous multifunction is upper (resp. lower) δ_{ij}-continuous.

Proof. Let $F : (X, \tau_1, \tau_2) \to (Y, \triangle_1, \triangle_2)$ be upper (resp. lower) strongly θ_{ij}-continuous multifunction and $V \subseteq Y$ be ij-regular open set, then V is $\triangle_i$-open. By upper (resp. lower) strongly θ_{ij}-continuity of F, $F^-(V)$ (resp. $F_-(V)$) is θ_{ij}-open set in X. Hence $F^-(V)$ (resp. $F_-(V)$) is δ_{ij}-open set in X. So F is upper (resp. lower) δ_{ij}-continuous. The following example shows the converse of Theorem 4.7 is not true in general. □

Example 4.8. Let $X = \{a, b, c\}$ with $\tau_1 = \{\phi, X, \{a\}\}$, $\tau_2 = \{\phi, X, \{b, c\}\}, Y = \{1, 2, 3\}$, $\triangle_1 = \{Y, \phi, \{1\}\}$ and $\triangle_2 = \{Y, \phi\}$. Define a multifunction $F : (X, \tau_1, \tau_2) \to (Y, \triangle_1, \triangle_2)$ as follows: $F(a) = \{1\}, F(b) = \{2\}$ and $F(c) = \{2, 3\}$. Then F is upper (resp. lower) δ_{ij}-continuous multifunction but it is not upper (resp. lower) strongly θ_{ij}-continuous multifunction. Indeed, $\{1\} \in \triangle_1$ but $F_-(\{1\}) = \{a\} \notin \theta_{12}O(X)$ and $F^-(\{1\}) = \{a\} \notin \theta_{12}O(X)$.
The following theorem give us the condition for converse.

Theorem 4.9. Every upper (resp.lower) δ_{ij}-continuous multifunction from a PAR_2-space is upper (resp. lower) strongly δ_{ij}-continuous.

Proof. Let $F : (X, \tau_1, \tau_2) \to (Y, \triangle_1, \triangle_2)$ be upper (resp. lower) θ_{ij}-continuous multifunction, (X, τ_1, τ_2) be a PAR_2-space and $(Y, \triangle_1, \triangle_2)$ be a PR_2-space. Let $V \subseteq Y$ be $\triangle_i$-open set. Since $(Y, \triangle_1, \triangle_2)$ is PR_2-space, then V is ij-regular open set. By upper (resp. lower) δ_{ij}-continuity of F, $F^-(V)$ (resp. $F_-(V)$) is δ_{ij}-open set in X. Since (X, τ_1, τ_2) is a PAR_2-space. Then $F^-(V)$ (resp. $F_-(V)$)is θ_{ij}-open set in X. Thus F is upper (resp. lower) strongly θ_{ij}-continuous. □

Definition 4.10. A multifunction $F : (X, \tau_1, \tau_2) \to (Y, \triangle_1, \triangle_2)$ is called:
(a) pairwise lower almost continuous at a point x in X if and only if for every $\triangle_i$-open set V in Y with $F(x) \cap V \neq \phi$, there exists τ_i-open nbd U of x such that $F(x_0) \cap \triangle_i.int(\triangle_j.cl(v)) \neq \phi$, for each $x_0 \in \tau_i.int(\tau_j.cl(U))$.
(b) Pairwise upper almost at a point x in X if and only if for every $\triangle_i$-open set V in Y with $F(x) \subseteq V$, there exists $\triangle_i$-open nbd U of x such that $F(\tau_i.int(\tau_j.cl(U)) \subseteq \triangle_i.int(\triangle_i.cl(V)))$.
(c) pairwise lower(resp. pairwise upper) continuous if it has this property at each point $x \in X$.

Theorem 4.11. Every upper (resp.lower) δ_{ij}-continuous multifunction is P- upper (resp. P-lower) almost continuous.

Proof. Let $F : (X, \tau_1, \tau_2) \to (Y, \triangle_1, \triangle_2)$ be upper (resp. lower) δ_{ij}-continuous multifunction and let $V \subseteq Y$ be ij-regular open set. By upper (resp. lower) δ_{ij}-continuity of F, $F^-(V)$ (resp. $F_-(V)$) is δ_{ij}-open set in X. Thus $F^-(V)$ (resp. $F_-(V)$)is τ_i-open set in X. So F is $P-$upper (resp. $P-$lower) almost continuous.
The following examples show the converse of Theorem 4.11 is not true in general. □

Example 4.12. Let $X = \{a, b, c\}$, $\tau_1 = \{\phi, X, \{a, b\}\}$, $\tau_2 = \{\phi, X, \{b\}, \{a, b\}\}, Y = \{1, 2, 3\}$, $\triangle_1 = \{Y, \phi, \{1\}, \{2, 3\}\}$ and $\triangle_2 = 2^Y$,. Define a multifunction $F : (X, \tau_1, \tau_2) \to (Y, \triangle_1, \triangle_2)$ as follows: $F(a) = \{1, 2\}, F(b) = \{1, 3\}$ and $F(c) = \{2, 3\}$. Then F is P-lower almost continuous multifunction but it is not lower δ_{ij}-continuous multifunction. Indeed, $\{1\} \in ijRO(Y)$ but $F_-(\{1\}) = \{a, b\} \notin \delta_{ij}O(X)$.

Example 4.13. Let $F : (X, \tau_1, \tau_2) \to (Y, \triangle_1, \triangle_2)$ as in Example 4.12. Define a multifunction $F : (X, \tau_1, \tau_2) \to (Y, \triangle_1, \triangle_2)$ as follows: $F(a) = F(b) = \{1\}$ and $F(c) = Y$. Then F is P-upper almost continuous multifunction but it is not upper δ_{ij}-continuous multifunction. Indeed, $\{1\} \in ijRO(Y)$, but $F_-(\{1\}) = \{a, b\} \notin \delta_{ij}O(X)$. The following theorem gives us the condition for converse.

Theorem 4.14. Every $P-$upper (resp.$P-$lower) almost continuous multifunction from a PSR_2- space to any *bts*-space is $P-$upper (resp. P-lower) δ_{ij}-continuous.

Proof. Let $F : (X, \tau_1, \tau_2) \to (Y, \triangle_1, \triangle_2)$ be $P-$upper (resp. $P-$lower) almost continuous multifunction and (X, τ_1, τ_2) is PSR_2- space . Let $V \subseteq Y$ be ij-regular open set. By $P-$upper (resp. $P-$lower) almost continuity of F, $F^-(V)$ (resp. $F_-(V)$) is τ_i-open set in X. Since (X, τ_1, τ_2) is PSR_2- space, then $F^-(V)$ (resp. $F_-(V)$)is ij-regular open set in X. So F is upper (resp. lower) δ_{ij}-continuous.
The applications of multifunctions with closed graphs, cluster (inverse cluster) set of functions, separation axioms and weak and strong forms of compactness in bitopological spaces are now under consideration and will e the subject of the next paper. □

5 Conclusion

The filed of mathematical science which goes under the name of topology is concerned with all questions directly or indirectly related to continuity. Therefore, generalization of continuity is one of the most important subject in topology. On the other hand, topology plays a significant role in quantum physics, high energy physics and supersting theory [5, 6]. Thus we studies upper and lower δ_{ij}-continuous multifunctions which are some generalized continuity may have possible applications in quantum physics, high energy physics and supersting theory.

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On Topology of Fuzzy Strong $b-$Metric Spaces

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Abstaract − In this study, we introduce and investigate the concept of fuzzy strong b-metric space such that is a fuzzy analogy of strong b-metric spaces. By using the open balls, we define a topology on these spaces which is Hausdorff and first countable. Later we show that open balls are open and closed balls are closed. After defining the standard fuzzy strong b-metric space induced by a strong b-metric, we show that these spaces have same topology. We also note that every separable fuzzy strong b-metric space is second countable. Moreover, we give the uniform convergence theorem for these spaces.

Keywords − *Fuzzy strong b-metric space, strong b-metric space, b-metric spaces, uniform convergence.*

1 Introduction and Preliminaries

The concept of b-metric space obtained by modifying the triangle inequality has been introduced by many authors.

Definition 1.1 ([3, 14, 8, 4, 13]). An ordered triple (X, D, K) is called b-metric (metric type) space and D is called b-metric on X if X is a nonempty set, $K \geq 1$ is a given real number and D:$X \times X \rightarrow [0, \infty)$ satisfies the following conditions for all $x, y, z \in X$
1) $D(x, y) = 0$ if and only if $x = y$,
2) $D(x, y) = D(y, x)$,
3) $D(x, z) \leq K[D(x, y) + D(y, z)]$.

For a b-metric space (X, D, K), the b-metric D need not be continuous, an open ball is not necessarily open and a closed ball is not necessarily closed where $B(x, r) = \{y : D(x, y) < r\}$ is an open ball, $B[x, r] = \{y : D(x, y) \leq r\}$ is a closed ball and A is an open set if for any $x \in A$ there exists an open ball $B(x, r)$ such $B(x, r) \subset A$ [15, 16, 11].

This fact suggests a strengthening of the notion of b-metric spaces.

Definition 1.2 ([16]). An ordered triple (X, D, K) is called strong b-metric space and D is called strong b-metric on X if X is a nonempty set, $K \geq 1$ is a given real number and $D{:}X \times X \to [0, \infty)$ satisfies the following conditions for all $x, y, z \in X$
1) $D(x, y) = 0$ if and only if $x = y$,
2) $D(x, y) = D(y, x)$,
3) $D(x, z) \leq D(x, y) + KD(y, z)$.

Remark 1.3 ([16]). Let (X, D, K) be a strong b-metric space.
(1) The strong b-metric D is continuous.
(2) Every open ball $B(x, r)$ is open.

After Zadeh [6] introduced the theory of fuzzy sets, many authors have introduced and studied several notions of metric fuzziness [1, 9, 17, 7, 10] from different points of view.

Fuzzy metric type spaces, which is a generalization of fuzzy metric space in sense of George and Veeramani [1] have been introduced and studied in [12] as a fuzzy analogy of b-metric spaces.

Definition 1.4 ([2]). A binary operation $* : [0, 1] \times [0, 1] \longrightarrow [0, 1]$ is a continuous t-norm if $*$ satisfies the following conditions;
1) $*$ is associative and commutative,
2) $*$ is continuous,
3) $a * 1 = a$ for all $a \in [0, 1]$,
4) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$, $a, b, c, d \in [0, 1]$.

Definition 1.5 ([12]). A 4-tuple $(X, M, *, K)$ is called a fuzzy metric type (fuzzy b-metric) space and M is called fuzzy metric type (fuzzy b-metric) on X if X is an arbitrary (non-empty) set, $*$ is a continuous t-norm, and M is a fuzzy set on $X \times X \times (0, \infty)$, satisfying the following conditions for each $x, y, z \in X$ and $t, s > 0$,
1) $M(x, y, t) > 0$,
2) $M(x, y, t) = 1$ if and only if $x = y$,
3) $M(x, y, t) = M(y, x, t)$,
4) $M(x, y, t) * M(y, z, s) \leq M(x, z, K(t + s))$ for some constant $K \geq 1$,
5) $M(x, y, .) : (0, \infty) \to [0, 1]$ is continuous.

In a similar manner, in this study, we introduce a new concept, fuzzy strong b-metric space, as a fuzzy analogy of strong b-metric spaces and present some elementary results.

Remark 1.6 ([1]). For any $r_1 > r_2$, we can find a r_3 such that $r_1 * r_3 \geq r_2$ and for any r_4 we can find a r_5 such that $r_5 * r_5 \geq r_4$ $(r_1,r_2,r_3,r_4,r_5 \in (0, 1))$.

2 Fuzzy strong b-metric space

Definition 2.1. Let X be a non-empty set, $K > 1$, $*$ is a continuous t-norm and M be a fuzzy set on $X \times X \times (0, \infty)$ such that for all $x, y, z \in X$ and $t, s > 0$,
1) $M(x, y, t) > 0$,
2) $M(x, y, t) = 1$ if and only if $x = y$,

3) $M(x,y,t) = M(y,x,t)$,
4) $M(x,y,t) * M(y,z,s) \leq M(x,z,t+Ks)$,
5) $M(x,y,.) : (0,\infty) \to [0,1]$ is continuous.
Then M is called a fuzzy strong b-metric on X and $(X,M,*,K)$ is called a fuzzy strong b-metric space.

Example 2.2. Let (X,D,K) be a strong b-metric space. Define

$$M_D(x,y,t) = \frac{t}{t+D(x,y)}$$

for $t>0$ and $x,y \in X$. Then $(X,M_D,\cdot,K)$ is a fuzzy strong b-metric space and is called standard fuzzy strong b-metric space induced by D. Here (1)–(3) and (5) are obvious and we show (4).

$$\begin{aligned} M_D(x,z,t)\cdot M_D(z,y,s) &= \frac{t}{t+D(x,z)}\cdot\frac{s}{s+D(z,y)} \\ &= \frac{1}{1+\frac{D(x,z)}{t}}\cdot\frac{1}{1+\frac{D(z,y)}{s}} \\ &\leq \frac{1}{1+\frac{D(x,z)}{t+Ks}}\cdot\frac{1}{1+\frac{KD(z,y)}{t+Ks}} \\ &\leq \frac{1}{1+\frac{D(x,z)+KD(z,y)}{t+Ks}} \\ &\leq \frac{1}{1+\frac{D(x,z)}{t+Ks}} \\ &= \frac{t+Ks}{t+Ks+D(x,z)} \\ &= M_D(x,y,t+Ks) \end{aligned}$$

Proposition 2.3. Let $(X,M,*,K)$ be a fuzzy strong b-metric space. Then $M(x,y,\_) : (0,\infty) \longrightarrow [0,1]$ is nondecreasing for all $x,y \in X$.

Proof. Assume that $M(x,y,t) > M(x,y,s)$, for $s>t>0$. We have $M(x,y,t) * M\left(y,y,\frac{s-t}{K}\right) \leq M(x,y,s) < M(x,y,t)$. Since $M(y,y,s-t) = 1$, we have $M(x,y,t) < M(x,y,t)$ that is a contradiction. □

Definition 2.4. Let $(X,M,*,K)$ be a fuzzy strong b-metric space. For $t>0$, the open ball $B(x,r,t)$ with center $x \in X$ and radius $0<r<1$ is defined by

$$B(x,r,t) = \{y \in X : M(x,y,t) > 1-r\}.$$

A subset $A \subset X$ is called open if for any $x \in A$, there exist $r \in (0,1)$ and $t>0$ such that $B(x,r,t) \subset A$.

Proposition 2.5. Let $(X,M,*,K)$ be a fuzzy strong b-metric space and τ_M be the family of all open sets in X. Then τ_M is a topology on X.

Proof. 1. Clearly $\emptyset, X \in \tau_M$.

2. Let $A, B \in \tau_M$ and $x \in A \cap B$. Then $x \in A$ and $x \in B$, so there exist $t_1, t_2 > 0$ and $r_1, r_2 \in (0,1)$ such that $B(x, r_1, t_1) \subset A$ and $B(x, r_2, t_2) \subset B$. Let $t = \min\{t_1, t_2\}$ and $r = \min\{r_1, r_2\}$. Then $B(x, r, t) \subset B(x, r_1, t_1) \cap B(x, r_2, t_2) \subset A \cap B$. Thus $A \cap B \in \tau_M$.

3. Let $A_i \in \tau_M$ for each $i \in I$ and $x \in \bigcup_{i \in I} A_i$. Then there exists $i_0 \in I$ such that $x \in A_{i_0}$. So, there exist $t > 0$ and $r \in (0,1)$ such that $B(x, t, r) \subset A_{i_0}$. Since $A_{i_0} \subset \bigcup_{i \in I} A_i$, $B(x, r, t) \subset \bigcup_{i \in I} A_i$. Thus $\bigcup_{i \in I} A_i \in \tau_M$. Hence, τ_M is a topology on X. □

Proposition 2.6. Let $(X, M, *, K)$ be a fuzzy strong b-metric space. Then an open ball is an open set.

Proof. We will show that an open ball $B(x, r, t)$ is an open set. Let $y \in B(x, r, t)$. Then we have $M(x, y, t) > 1 - r$. Since $M(x, y, \_)$ is nondecreasing and continuous, there exists $t_0 \in (0, t)$ such that $M(x, y, t_0) > 1 - r$. Let $r_0 = M(x, y, t_0)$. Therefore $r_0 > 1 - r$ and we can find a $s, 0 < s < 1$ such that $r_0 > 1 - s > 1 - r$. For r_0 and s such that $r_0 > 1 - s$ we can find $r_1, 0 < r_1 < 1$ such that $r_0 * r_1 \geq 1 - s$. Now we will show that $B(y, 1 - r_1, \frac{t - t_0}{K}) \subset B(x, r, t)$. $z \in B(y, 1 - r_1, \frac{t - t_0}{K})$ implies that $M(y, z, \frac{t - t_0}{K}) > r_1$. Hence we have

$$\begin{aligned} M(x, z, t) &\geq M(x, y, t_0) * M(y, z, \frac{t - t_0}{K}) \\ &\geq r_0 * r_1 \geq 1 - s > 1 - r. \end{aligned}$$

Therefore $z \in B(x, r, t)$ and $B(y, 1 - r_1, \frac{t - t_0}{K}) \subset B(x, r, t)$. □

Proposition 2.7. Let $(X, M, *, K)$ be a fuzzy strong b-metric space. Then (X, τ_M) is Hausdorff.

Proof. Let $x, y \in X$ such that $x \neq y$. From the definition of fuzzy strong b-metric space, $1 > M(x, y, t) > 0$ say $M(x, y, t) = r$. For all r_0 such that $1 > r_0 > r$ we can find $r_1 \in (0,1)$ such that $r_1 * r_1 > r_0$. Now consider, the sets $B(x, 1 - r_1, \frac{t}{2})$ and $B(y, 1 - r_1, \frac{t}{2K})$. Clearly $B(x, 1 - r_1, \frac{t}{2}) \cap B(y, 1 - r_1, \frac{t}{2K}) = \varnothing$.
Otherwise, if there exists $z \in B(x, 1 - r_1, \frac{t}{2}) \cap B(y, 1 - r_1, \frac{t}{2K})$. Then

$$\begin{aligned} r &= M(x, y, t) \geq M(x, z, \frac{t}{2}) * M(z, y, \frac{t}{2K}) \\ &\geq r_1 * r_1 \geq r_0 > r \end{aligned}$$

which is a contradiction. □

Proposition 2.8. Let $(X, M, *, K)$ be a fuzzy strong b-metric space. Then (X, τ_M) is first countable.

Proof. Let $x \in X$. We need to show that $\mathcal{B}_x = \{B(x, \frac{1}{n}, \frac{1}{n}) : n \in \mathbb{N}\}$ is a local basis for $x \in X$. Let $U \in \tau_M$ such that $x \in U$. Since U is open, then there exists $r \in (0,1)$ and $t > 0$ such that $B(x, r, t) \subset U$. Choose $n \in \mathbb{N}$ such that $\frac{1}{n} < r$ and $\frac{1}{n} < t$. Now we need to show $B(x, \frac{1}{n}, \frac{1}{n}) \subset B(x, r, t)$. Let $z \in B(x, \frac{1}{n}, \frac{1}{n})$. Then

$M(x,z,\frac{1}{n}) > 1-\frac{1}{n} > 1-r$. Since $\frac{1}{n} < t$, we have $1-r < M(x,z,\frac{1}{n}) \leq M(x,z,t)$. Hence $z \in B(x,r,t)$ which implies $B(x,\frac{1}{n},\frac{1}{n}) \subset B(x,r,t) \subset U$. Consequently, $\mathcal{B}_x$ is countable local basis for x. Hence (X,τ_M) is first countable topological space. □

Definition 2.9. Let $(X,M,*,K)$ be a fuzzy strong b-metric space, $x \in X$ and $\{x_n\}$ be a sequence in X. Then
i) $\{x_n\}$ is said to converge to x if for any $t > 0$ and any $r \in (0,1)$ there exists a natural number n_0 such that $M(x_n,x,t) > 1-r$ for all $n \geq n_0$. We denote this by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$ as $n \to \infty$.
ii) $\{x_n\}$ is said to be a Cauchy sequence if for any $r \in (0,1)$ and any $t > 0$ there exists a natural number n_0 such that $M(x_n,x_m,t) > 1-r$ for all $n,m \geq n_0$.
iii) $(X,M,*,K)$ is said to be a complete fuzzy strong b-metric space if every Cauchy sequence is convergent.

Theorem 2.10. Let $(X,M,*,K)$ be a fuzzy strong b-metric space, $x \in X$ and $\{x_n\}$ be a sequence in X. $\{x_n\}$ converges to x if and only if $M(x_n,x,t) \to 1$ as $n \to \infty$, for each $t > 0$.

Proof. ($\Rightarrow$:) Suppose that, $x_n \to x$. Then, for each $t > 0$ and $r \in (0,1)$, there exists a natural number n_0 such that $M(x_n,x,t) > 1-r$ for all $n \geq n_0$. We have $1-M(x_n,x,t) < r$. Hence $M(x_n,x,t) \to 1$ as $n \to \infty$.

($\Leftarrow$:) Now, suppose that $M(x_n,x,t) \to 1$ as $n \to \infty$. Then, for each $t > 0$ and $r \in (0,1)$, there exists a natural number n_0 such that $1-M(x_n,x,t) < r$ for all $n \geq n_0$. In that case, $M(x_n,x,t) > 1-r$. Hence $x_n \to x$ as $n \to \infty$. □

Let X be a first countable space. Then X is Hausdorff if and only if sequential limits in X are unique [5]. Then the following is obvious.

Proposition 2.11. Let $(X,M,*,K)$ be a fuzzy strong b-metric space and $\{x_n\} \subset X$. If $\{x_n\}$ is convergent, then the limit point of $\{x_n\}$ is unique.

Proposition 2.12. Let $(X,M,*,K)$ be a fuzzy strong b-metric space and $\{x_n\} \subset X$. If $\{x_n\}$ is convergent, then $\{x_n\}$ is Cauchy.

Proof. Let r and t be arbitrary real number such that $r \in (0,1)$, $t > 0$ and $\lim_{n\to\infty} x_n = x$ for $x \in X$. Since $r \in (0,1)$, there exists $r_0 \in (0,1)$ such that

$$(1-r_0)*(1-r_0) > 1-r.$$

Since $\lim_{n\to\infty} x_n = x$, for $\frac{t}{2K} > 0$ and $r_0 \in (0,1)$ there exists $n_0 \in \mathbb{N}$ such that

$$n \geq n_0 \Longrightarrow M(x_n,x,\frac{t}{2K}) > 1-r_0.$$

Therefore we have

$$\begin{aligned} M(x_n,x_m,t) &\geq M(x_n,x,\frac{t}{2}) * M(x,x_m,\frac{t}{2K}) \\ &\geq M(x_n,x,\frac{t}{2K}) * M(x,x_m,\frac{t}{2K}) \\ &> (1-r_0)*(1-r_0) > 1-r \end{aligned}$$

for $m, n \geq n_0$ which means $\{x_n\}$ is Cauchy. □

Definition 2.13. Let $(X, M, *)$ be a fuzzy strong b-metric space. For $t > 0$, the closed ball $B[x, r, t]$ with center x and radius $r \in (0, 1)$ is defined by $B[x, r, t] = \{y \in X : M(x, y, t) \geq 1 - r\}$.

Proposition 2.14. Let $(X, M, *, K)$ be a fuzzy strong b-metric space. Then a closed ball is a closed set.

Proof. Let $y \in \overline{B[x, r, t]}$. We need to show that $y \in B[x, r, t]$. Since X is first countable space, there exists a sequence $\{y_n\}$ in $B[x, r, t]$ such that $y_n \to y$. Hence $M(y_n, y, t) \to 1$ for all $t > 0$. For a given $\epsilon > 0$

$$M(x, y, t + \epsilon) \geq M(x, y_n, t) * M(y_n, y, \tfrac{\epsilon}{K}).$$

Hence

$$\begin{aligned} M(x, y, t + \epsilon) &\geq \lim_{n \to \infty} M(x, y_n, t) * \lim_{n \to \infty} M(y_n, y, \tfrac{\epsilon}{K}) \\ &\geq (1 - r) * 1 = 1 - r. \end{aligned}$$

(If $M(x, y_n, t)$ is bounded, the sequence $\{y_n\}$ has a subsequence, which we again denote by $\{y_n\}$ for which $\lim_{n \to \infty} M(x, y_n, t)$ exists.) In particular for $n \in \mathbb{N}$, take $\epsilon = \dfrac{t}{n}$. Then we have

$$M(x, y, t + \frac{t}{n}) \geq (1 - r)$$

and

$$M(x, y, t) \geq \lim_{n \to \infty} M(x, y, t + \frac{t}{n}) \geq 1 - r.$$

Therefore $y \in B[x, r, t]$. □

Proposition 2.15. Let (X, D, K) be a strong b-metric space and $(X, M_D, \cdot, K)$ be the standard fuzzy strong b-metric space induced by D. Then the topology τ_D induced by D and the topology τ_{M_D} induced by M_D are the same.

Proof. ($\Rightarrow$) Let $A \in \tau_D$. For every $x \in A$, there exists $\epsilon > 0$ such that $B(x, \epsilon) \subset A$. For a fixed $t > 0$, we have

$$M_D(x, y, t) = \frac{t}{t + D(x, y)} > \frac{t}{t + \epsilon}.$$

If we write $1 - r = \frac{t}{t+\epsilon}$, then we have $M_D(x, y, t) > 1 - r$ which means $B(x, r, t) \subset A$ and $A \in \tau_{M_D}$.
($\Leftarrow$). Let $A \in \tau_{M_D}$. For every $x \in A$, there exists $0 < r < 1$ and $t > 0$ such that $B(x, r, t) \subset A$. We have

$$\begin{aligned} M_D(x, y, t) &= \frac{t}{t + D(x, y)} > 1 - r \\ t &> (1 - r)t + (1 - r)D(x, y) \\ D(x, y) &< \frac{rt}{1 - r} \end{aligned}$$

If we write $\epsilon = \frac{rt}{1-r}$ where $0 < \epsilon < 1$, then we have $D(x, y) < \epsilon$ which means $B(x, \epsilon) \subset A$ and $A \in \tau_D$. Therefore $\tau = \tau_D$. □

Theorem 2.16. Let $(X, M, *, K)$ be a fuzzy strong b-metric space. If (X, τ_M) is separable then (X, τ_M) is second countable.

Proof. Let $A = \{a_n : n \in \mathbb{N}\}$ be a countable dense subset of X. Consider

$$\mathcal{B} = \{B(a_j, \frac{1}{k}, \frac{1}{k}) : j, k \in \mathbb{N}\}.$$

We will show that B is a countable base for τ_M. Clearly B is countable. Let U be an open set in X. For any $x \in U$, there exists $r \in (0,1)$ and $t > 0$ such that $B(x, r, t) \subset U$. For $r \in (0,1)$, we can find an $s \in (0,1)$ such that $(1-s) * (1-s) > (1-r)$. Let $m \in \mathbb{N}$ such that $\frac{1}{m} < s$ and $\frac{1}{m} < \frac{t}{2K}$. Since A is dense in X, there exists $a_j \in A$ such that $a_j \in B(x, \frac{1}{m}, \frac{1}{m})$. If $y \in B(a_j, \frac{1}{m}, \frac{1}{m})$ then,

$$\begin{aligned} M(x, y, t) &\geq M(x, a_j, \frac{t}{2}) * M(y, a_j, \frac{t}{2K}) \\ &\geq M(x, a_j, \frac{1}{m}) * M(y, a_j, \frac{1}{m}) \\ &\geq (1 - \frac{1}{m}) * (1 - \frac{1}{m}) \\ &\geq (1-s) * (1-s) \\ &> (1-r). \end{aligned}$$

Hence $y \in B(x, r, t)$ and $\mathcal{B}$ is a basis. □

Definition 2.17. Let X be a topological space, $(Y, M, *, K)$ be a fuzzy strong b-metric space and $f_n : X \to Y$ be a sequence of functions. Then $\{f_n\}$ is said to converge uniformly to a function f from X to Y if for given $r \in (0,1)$ and $t > 0$, there exists $n_0 \in \mathbb{N}$ such that $M(f_n(x), f(x), t) > 1 - r$ for all $n \geq n_0$ and for all $x \in X$.

Theorem 2.18. Let X be a topological space, $(Y, M, *, K)$ be a fuzzy strong b-metric space and $f_n : X \to Y$ be a sequence of continuous functions. If $\{f_n\}$ converges uniformly to f then f is continuous.

Proof. Let V be an open set in Y, $x_0 \in f^{-1}(V)$ and let $y_0 = f(x_0)$. Then there exist $r \in (0,1)$ and $t > 0$ such that $B(y_0, r, t) \subset V$. For $r \in (0,1)$, we can find an $s \in (0,1)$ such that $(1-s) * (1-s) * (1-s) > 1-r$. Since $\{f_n\}$ converges uniformly to f, for given $s \in (0,1)$ and $t > 0$, there exists $n_0 \in N$ such that $M(f_n(x), f(x), \frac{t}{4K^2}) > 1-s$ for all $n \geq n_0$ which also implies $M(f_n(x), f(x), \frac{t}{2}) > 1-s$. Since f_n is continuous for all $n \in \mathbb{N}$, we can find a neighborhood U of x_0, for a fixed $n \geq n_0$, such that $f_n(U) \subset B(f_n(x_0), s, \frac{t}{4K})$. Therefore $M(f_n(x), f_n(x_0), \frac{t}{4K}) > 1-s$ for all x in U an we have

$$\begin{aligned} M(f(x), f(x_0), t) &\geq M(f(x), f_n(x), \tfrac{t}{2}) * M(f_n(x), f_n(x_0), \tfrac{t}{4K}) \\ &\quad * M(f_n(x_0), f(x_0), \tfrac{t}{4K^2}) \\ &\geq (1-s) * (1-s) * (1-s) \\ &\geq 1-r. \end{aligned}$$

Hence, $f(x) \in B(f(x_0), r, t) \subset V$ for all $x \in U$ which means $f(U) \subset V$ and f is continuous. □

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On αrω-Homeomorphisms in Topological Spaces

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Abstract - A bijection $f:(X, \tau)\to(Y, \sigma)$ is called αrω-homeomorphism if f and f^{-1} are αrω-continuous. Also we introduce new class of maps, namely αrω*c*-homeomorphisms which form a subclass of αrω-homeomorphisms. This class of maps is closed under composition of maps. We prove that the set of all αrω*c*-homeomorphisms forms a group under the operation composition of maps.

Keywords *- αrω-closed maps, αrω*-closed maps and αrω-open maps, αrω*-open maps, αrω-homeomorphism, αrωc-homeomorphism.*

1 introduction

Mappings plays an important role in the study of modern mathematics, especially in Topology and Functional Analysis. Closed and open mappings are one such mappings which are studied for different types of closed sets by various mathematicians for the past many years. Levine [16] introduced the notion of generalized closed sets. After him different mathematicians worked and studied on different versions of generalized closed sets and related topological properties.

Generalized Homeomorphisms, wgrα-homeomorphisms, rgα-homeomorphisms, rps-homeomorphisms and gs and sg homeomorphisms have been introduced and studied by Maki et al. [19], Sakthivel and Uma [25], Vadivel and Vairamanickam [30], Mary and Thangavelu [27], Devi et al. [9] respectively.

We give the definitions of some of them which are used in our present study. In this paper, we introduce the concept of αrω-homeomorphism and study the relationship between homeomorphisms, wgrα-homeomorphisms, rgα-homeomorphisms, rps-homeomorphisms, w-homeomorphisms, g-homeomorphisms and rwg-homeomorphisms. Also we introduce

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new class of maps αωc-homeomorphisms which form a subclass of αrω-homeomorphisms. This class of maps is closed under composition of maps. We prove that the set of all αrω*c*-homeomorphisms forms a group under the operation composition of maps.

2. Preliminaries

Throughout this paper (X,τ) and (Y,σ) (or simply X and Y) always denote topological spaces on which no separation axioms are assumed unless otherwise mentioned. For a subset A of a space X, cl(A) and int(A) denote the closure of A and the interior of A respectively. X\A or A^c denotes the complement of A in X.

We recall the following definitions and results.

Definition 2.1 A subset A of a topological space (X, τ) is called

(i) semi-open set [17] if A ⊆ cl(int(A)) and semi-closed set if int(cl(A)) ⊆ A.
(ii) pre-open set [21] if A ⊆ int(cl(A)) and pre-closed set if cl(int(A)) ⊆ A.
(iii) α-open set [13] if A ⊆ int(cl(int(A))) and α -closed set if cl(int(cl(A)))⊆ A.
(iv) semi-pre open set [2] (=β-open[1] if A⊆cl(int(cl(A)))) and a semi-pre closed set (=β-closed) if int(cl(int(A)))⊆A.
(v) regular open set [28] if A = int(clA)) and a regular closed set if A = cl(int(A)).
(vi) Regular semi open set [8] if there is a regular open set U such that U ⊆ A ⊆ cl(U).
(vii) Regular α-open set[31] (briefly,rα-open) if there is a regular open set U such that U ⊆A ⊆αcl(U).

Definition 2.2 A subset A of a topological space (X, τ) is called

(i) regular generalized α-closed set (briefly, rgα-closed)[31] if αcl (A)⊆U whenever A⊆ U and U is regular α-open in X.
(ii) generalized closed set(briefly g-closed) [16] if cl(A)⊆U whenever A⊆U and U is open in X.
(iii) generalized semi-closed set(briefly gs-closed)[4] if scl(A) ⊆ U whenever A⊆U and U is open in X.
(iv) generalized semi pre regular closed (briefly gspr-closed) set[24] if spcl(A)⊆U whenever A⊆U and U is regular open in X.
(v) strongly generalized closed set [24]](briefly,g*-closed) if cl(A)⊆U whenever A⊆U and U is g-open in X.
(vi) α-generalized closed set(briefly αg-closed)[20] if αcl(A)⊆U whenever A⊆U and U is open in X.
(vii) ω-closed set[29] if cl(A)⊆U whenever A⊆U and U is semi-open in X.
(viii) weakely generalized closed set(briefly, wg-closed)[23] if cl(int(A))⊆U whenever A⊆U and U is open in X.
(ix) regular weakly generalized closed set (briefly, rwg-closed)[23] if cl(int(A))⊆U whenever A⊆U and U is regular open in X.
(x) semi weakly generalized closed set (briefly, swg-closed)[23] if cl(int(A))⊆U whenever A⊆U and U is semi open in X.
(xi) generalized pre closed (briefly gp-closed) set [18] if pcl(A) ⊆U whenever A⊆U and U is open in X.

(xii) regular ω-closed (briefly rω -closed) set [5] if cl(A) ⊆ U whenever A⊆U and U is regular semi-open in X.
(xiii) g*-pre closed (briefly g*p-closed) [32] if pcl(A) ⊆ U whenever A⊆ U and U is g-open in X
(xiv) generalized regular closed (briefly gr–closed)set[7] if rcl(A)⊆ U whenever A⊆ U and U is open in X.
(xv) regular generalized weak (briefly rgw-closed) set[22] if cl(int(A))⊆U whenever A⊆U and U is regular semi open in X.
(xvi) weak generalized regular–α closed (briefly wgrα-closed) set[14]if cl(int(A)⊆U whenever A⊆ U and U is regular α-open in X.
(xvii) regular pre semi–closed (briefly rps-closed) set [26]if spcl(A)⊆U whenever A⊆U and U is rg- open in X.
(xviii) generalized pre regular weakly closed (briefly gprw-closed) set [15] if pcl(A)⊆U whenever A⊆ U and U is regular semi- open in X.
(xix) α-generalized regular closed (briefly αgr-closed) set [33] if αcl(A)⊆U whenever A⊆U and U is regular open in X.
(xx) R*-closed set [12] if rcl(A)⊆U whenever A⊆U and U is regular semi- open in X.
(xxi) generalized pre regular closed set(briefly gpr-closed)[11] if pcl(A)⊆U whenever A⊆U and U is regular open in X.
(xxii) ωα- closed set [6] if αcl(A)⊆U whenever A⊆ U and U is ω-open in X.
(xxiii) α regular ω- closed (briefly αrω -closed) set[37] if αcl(A) ⊆ U whenever A⊆U and U is rw-open in X.

The compliment of the above mentioned closed sets are their open sets respectively.

Definition 2.3 A map f: (X,τ)→(Y,σ) is said to be
(i) regular-continuous(r-continuous) [3] if $f^{-1}(V)$ is r-closed in X for every closed subset V of Y.
(ii) Completely-continuous [3]if $f^{-1}(V)$ is regular closed in X for every closed subset V of Y.
(iii) strongly α-continuous [38] if $f^{-1}(V)$ is α-closed in X for every semi-closed subset V of Y.
(iv) αrω-continuous [35] if $f^{-1}(V)$ is αrω –closed in X for every closed subset V of Y
(v) Strongly-continuous[28] if $f^{-1}(V)$ is Clopen (both open and closed) in X for every subset V of Y.
(vi) α-continuous[13] if $f^{-1}(V)$ is α–closed in X for every closed subset V of Y.
(vii) αg-continuous[20] if $f^{-1}(V)$ is αg–closed in X for every closed subset V of Y.
(viii) wg-continuous[23] if $f^{-1}(V)$ is wg–closed in X for every closed subset V of Y.
(ix) rwg-continuous[23] if $f^{-1}(V)$ is rwg–closed in X for every closed subset V of Y.
(x) gs-continuous [4] if $f^{-1}(V)$ is gs–closed in X for every closed subset V of Y.
(xi) gp-continuous [18] if $f^{-1}(V)$ is gp–closed in X for every closed subset V of Y.
(xii) gpr-continuous [11] if $f^{-1}(V)$ is gpr–closed in X for every closed subset V of Y.
(xiii) αgr-continuous [33] if $f^{-1}(V)$ is αgr–closed in X for every closed subset V of Y.
(xiv) ωα-continuous [6] if $f^{-1}(V)$ is ωα–closed in X for every closed subset V of Y.
(xv) gspr-continuous [24] if $f^{-1}(V)$ is gspr–closed in X for every closed subset V of Y.
(xvi) g-continuous [6] if $f^{-1}(V)$ is g–closed in X for every closed subset V of Y
(xvii) ω-continuous [29] if $f^{-1}(V)$ is ω–closed in X for every closed subset V of Y
(xviii) rgα-continuous [31] if $f^{-1}(V)$ is rgα–closed in X for every closed subset V of Y.
(xix) gr-continuous [7] if $f^{-1}(V)$ is gr–closed in X for every closed subset V of Y.

(xx) g*p-continuous [32] if $f^{-1}(V)$ is g*p–closed in X for every closed subset V of Y.
(xxi) rps-continuous [26] if $f^{-1}(V)$ is rps–closed in X for every closed subset V of Y.
(xxii) R*-continuous [12] if $f^{-1}(V)$ is R*–closed in X for every closed subset V of Y.
(xxiii) gprw-continuous [15] if $f^{-1}(V)$ is gprw–closed in X for every closed subset V of Y.
(xxiv) wgrα-continuous [14] if $f^{-1}(V)$ is wgrα–closed in X for every closed subset V of Y.
(xxv) swg-continuous [23] if $f^{-1}(V)$ is swg–closed in X for every closed subset V of Y.
(xxvi) rw-continuous [5] if $f^{-1}(V)$ is rw–closed in X for every closed subset V of Y.
(xxvii) rgw-continuous [22] if $f^{-1}(V)$ is rgw–closed in X for every closed subset V of Y.

Definition 2.4 A map f: (X,τ)→(Y,σ) is said to be
(i) α-irresolute [13] if $f^{-1}(V)$ is α-closed in X for every α-closed subset V of Y.
(ii) irresolute [6] if $f^{-1}(V)$ is semi-closed in X for every semi-closed subset V of Y.
(iii) contra ω-irresolute [29] if $f^{-1}(V)$ is ω-open in X for every ω-closed subset V of Y.
(iv) contra irresolute [13] if $f^{-1}(V)$ is semi-open in X for every semi-closed subset V of Y.
(v) contra r-irresolute [3] if $f^{-1}(V)$ is regular-open in X for every regular-closed subset V of Y
(vi) rw*-open(resp rw*-closed) [5] map if f(U) is rw-open (resp. rw-closed) in Y for every rw-open (resp. rw-closed) subset U of X.
(vii) contra continuous [10] if $f^{-1}(V)$ is open in X for every closed subset V of Y .

Lemma 2.5 [37]
i) Every closed (resp. regular-closed, α-closed) set is αrω-closed set in X.
ii) Every αrω-closed set is αg-closed set
iii) Every αrω-closed set is αgr-closed (resp. ωα-closed, gs-closed, gspr-closed, wg-closed, rwg-closed, gp-closed, gpr-closed) set in X

Lemma 2.6 [37] If a subset A of a topological space X and
i) If A is regular open and αrω-closed then A is α-closed set in X
ii) If A is open and αg-closed then A is αrω-closed set in X
iii) If A is open and gp-closed then A is αrω-closed set in X
iv) If A is regular open and gpr-closed then A is αrω-closed set in X
v) If A is open and wg-closed then A is αrω-closed set in X
vi) If A is regular open and rwg-closed then A is αrω-closed set in X
vii) If A is regular open and αgr-closed then A is αrω-closed set in X
viii) If A is ω-open and ωα-closed then A is αrω-closed set in X

Lemma 2.7 [37] If a subset A of a topological space X and
i) If A is semi-open and sg-closed then it is αrω-closed.
ii) If A is semi-open and ω-closed then it is αrω-closed.
iii) A is αrω-open iff U⊆αint(A), whenever U is rw-closed and U ⊆ A.

Definition 2.8 A topological space (X,τ) is called an α-space if every α-closed subset of X is closed in X.

Definition 2.9 A map f : (X,τ)→(Y,σ) is said to be
(i) **g-closed** [29] if f(F) is g-closed in (Y, σ) for every closed set F of (X, τ),
(ii) **w-closed** [22] if f(F) is w-closed in (Y, σ) for every closed set F of (X, τ),
(iii) **wg-closed** [23] if f(F) is wg-closed in (Y, σ) for every closed set F of (X, τ),
(iv) **rwg-closed** [23] if f(F) is rwg-closed in (Y, σ) for every closed set F of (X, τ),
(v) **rg-closed** [19] if f(F) is rg-closed in (Y, σ) for every closed set F of (X, τ) ,

(vi) **gpr-closed** [11] if f(F) is gpr-closed in (Y, σ) for every closed set F of (X, τ),
(vii) **regular closed** [31] if f(F) is closed in (Y, σ) for every regular closed set F of (X, τ).

Definition 2.10 A map f : (X, τ) → (Y, σ) is said to be
(i) **g-open** [15] if f(U) is g-open in (Y, σ) for every open set U of (X, τ),
(ii) **w-open** [22] if f(U) is w-open in (Y, σ) for every open set U of (X, τ),
(iii) **wg-open** [23] if f(U) is wg-open in (Y, σ) for every open set U of (X, τ),
(iv) rwg-**open** [23] if f(U) is rwg-open in (Y, σ) for every open set U of (X, τ),
(v) rg-**open** [19] if f(U) is rg-open in (Y, σ) for every open set U of (X, τ),
(vi) gpr-**open** [11] if f(U) is gpr-open in (Y, σ) for every open set U of (X, τ),
(vii) **regular open** [31] if f(U) is open in (Y, σ) for every regular open set U of (X, τ).

Definition 2.11 A bijective function f : (X, τ) → (Y, σ) is called
(i) **generalized homeomorphism** (g-homeomorphism) [2]if both f and f^{-1} are g-continuous,
(ii) **gc-homeomorphism** [2] if both f and f^{-1} are gc-irresolute,
(iii) **rwg-homeomorphism** [16] if both f and f^{-1} are rwg-continuous,
(iv) **w∗-homeomorphism** [20] if both f and f^{-1} are w-irresolute,
(v) **w-homeomorphism** [20] if both f and f^{-1} are w-contiuous.
(vi) **rps-homeomorphism** [27] if both f and f^{-1} are rps-contiuous.
(vii) **rgα-homeomorphism** [30] if both f and f^{-1} are rgα-contiuous.
(viii) **wgrα-homeomorphism** [25] if both f and f^{-1} are wgrα-contiuous

3 αrω-Homeomorphisms in Topological Spaces

We introduce the following definition.

Definition 3.1 A bijection f : (X, τ) → (Y, σ) is called α **regular ω-homeomorphism** (briefly, αrω-homeomorphism) if f and f^{-1} are αrω-continuous.

We denote the family of all αrω-homeomorphisms of a topological space (X, τ) onto itself by αrω-h(X, τ).

Example 3.2 Consider X = Y = {a, b, c, d} with topologies τ = σ ={X, ϕ, {a},{b}, {a, b}, {a, b, c}} Let f:(X,τ)→(Y,σ) by f(a)=c, f(b)=a, f(c)= b, f(d)=d. Then f is αrω-continuous and f^{-1} is αrω-continuous. Therefore f is αrω-homeomorphisms.

Theorem 3.3 Every homeomorphism is an αrω-homeomorphism, but not conversely.
Proof: Let f :(X,τ)→(Y,σ) be a homeomorphism. Then f and f^{-1} are continuous and f is bijection. As every continuous function is αrω-continuous, we have f and f^{-1} are αrω-continuous. Therefore f is αrω-homeomorphism.

The converse of the above theorem need not be true, as seen from the following example.

Example 3.4 Consider X = Y ={a, b, c, d} with topologies τ = σ ={X, ϕ, {a},{b},{a, b}, {a, b, c}} and Define a map f :(X, τ)→(Y, σ) by f(a)=c, f(b)=a, f(c)=b, f(d)=d. Then f is αrω-homeomorphism but it is not homeomorphism, since the inverse image of closed F = {c, d} in Y then f^{-1}(F)= {a, d} which is not closed set in X .

Theorem 3.5 Every α-homeomorphism is an αrω-homeomorphism but not conversely.

Proof: Let f : (X, τ) → (Y, σ) be a α-homeomorphism. Then f and f^{-1} are α-continuous and f is bijection. As every α-continuous function is αrω-continuous, we have f and f^{-1} are αrω-continuous. Therefore f is αrω-homeomorphism.

The converse of the above theorem is not true in general as seen from the following example.

Example 3.6 Consider X = Y = {a, b, c, d} with topologies τ = σ = {X, ϕ, {a}, {b},{a, b}, {a, b, c}} and define a map f : (X, τ) → (Y, σ) by f(a)=c , f(b)=a ,f(c)=b , f(d)=d. Then f is αrω-homeomorphism but it is not α-homeomorphism, since the inverse image of closed F={c, d} in Y then f^{-1}(F)= {a, d} which is not α-closed set in X .

Theorem 3.7 i) Every αrω-homeomorphism is an αg-homeomorphism.
ii) Every αrω-homeomorphism is an wg-homeomorphism (resp. gs-homeomorphism, rwg-homeomorphism, gp-homeomorphism, gspr-homeomorphism, gpr-homeomorphism, ωα-homeomorphism, αgr-homeomorphism)

Proof: i) Let f:(X,τ)→(Y,σ) be a αrω-homeomorphism. Then f and f^{-1} are αrω-continuous and f is bijection. As every αrω-continuous function is αg-continuous, we have f and f^{-1} are αg-continuous. Therefore f is αg-homeomorphism.

Similarly we can prove **ii)**

The converse of the above theorem is not true in general as seen from the following example.

Example 3.8 Consider X = Y = {a, b, c} with topologies τ = {X, ϕ, {a} ,{b ,c}} and σ = {Y, ϕ ,{a}}. Let f :(X,τ)→(Y,σ) by f(a)=b, f(b)=a, f(c)=c. Then this function is αg-homeomorphism, (resp. wg-homeomorphism, gs-homeomorphism, rwg-homeomorphism, gp-homeomorphism, gspr-homeomorphism, gpr-homeomorphism, ωα-homeomorphism, αgr–homeomorphism) but it is not αrω-homeomorphism, since the closed set F={b, c} in Y, f^{-1}(F)= {a, c} which is not αrω-closed set in X .

Remark 3.9 The following examples shows that αrω-homeomorphism are independent of pre-homeomorphism, β-homeomorphism, g-homeomorphism, ω-homeomorphism, rw-homeomorphism, swg-homeomorphism, rgw-homeomorphism, wgrα-homeomorphism, rgα-homeomorphism, gprw-homeomorphism, g*p-homeomorphism, gr-homeomorphism, R*- homeomorphism, rps-homeomorphism, semi-homeomorphism .

Example 3.10 Let X=Y={a, b, c}, τ = {X, ϕ, {a} , {b,c}} σ ={Y, ϕ, {a}, {b,c}}, Let map f: X→Y defined by f(a)=b f(b)=a f(c)=c. Then pre-homeomorphism, β-homeomorphism, g-homeomorphism, ω-homeomorphism, rw-homeomorphism, swg-homeomorphism, rgw-homeomorphism, wgrα-homeomorphism, rgα-homeomorphism, gprw-homeomorphism, g*p-homeomorphism, gr-homeomorphism, R*-homeomorphism , rps-homeomorphism but it is not αrω-homeomorphism, since the inverse image of the closed set {b,c} in Y is {a, c} which is not αrω-closed set in X .

Example 3.11 X=Y={a,b,c,d}, τ = {X, ϕ ,{a},{b},{a,b},{a,b,c}} σ ={Y, ϕ, {a},{b},{a,b},{a,b,c}} Let map f: X→Y defined by f(a)=b, f(b)=a , f(c)=d, f(d)=c then αrω-homeomorphism but not, g-homeomorphism, ω-homeomorphism, rw-homeomorphism, gprw-homeomorphism, g*p-homeomorphism, gr-homeomorphism, R*-homeomorphism as closed set F={d} in X, then f(F)={c} in Y, which is not gr-closed (resp. g-closed, g*p-closed, ω-closed, rw-closed, gprw-closed, gr-closed, R*-closed) set in Y.

Theorem 3.12 Let f:(X,τ)→(Y,σ) be a bijective αrω-continuous map. Then the following are equivalent.
(i) f is a αrω-open map,
(ii) f is αrω-homeomorphism,
(iii) f is a αrω-closed map.

Proof: Proof follows from theorem 3.39 in [36].

Remark 3.13 The composition of two αrω-homeomorphism need not be a αrω-homeomorphism in general as seen from the following example.

Example 3.14 Consider X = Y = Z= {a, b, c, d} with topologies τ = σ = μ = {X, ϕ, {a}, {b}, {a, b}, {a, b, c}} and define a map f : (X, τ)→(Y, σ) by f(a)=c, f(b)=a, f(c)=b, f(d)=d. and g :Y→Z defined by g(a) = b, g(b) = a, g(c) = d, g(d) = c then both f and g are αrω-homeomorphisms but their composition g ∘ f : (X, τ)→(Z, μ) is not αrω-homeomorphism because for the open set {a, b} of (X,τ), g ∘ f ({a,b}) =g(f({a, b}) = g(a,,c}) = {a, d}, which is not αrω-open in (z, μ). Therefore g ∘ f is not αrω-open and so g ∘ f is not αrω-homeomorphism.

Definition 3.15 A bijection f :(X,τ)→(Y,σ) is said to be **αrωc-homeomorphism** if both f and f^{-1} are αrω-irresolute. We say that spaces (X, τ) and (Y, σ) are αrωc-homeomorphic if there exists a αrωc-homeomorphism from (X, τ) onto (Y, σ).

We denote the family of all αrωc-homeomorphisms of a topological space (X,τ) onto itself by αrωc-h(X, τ).

Theorem 3.16: Every αrωc-homeomorphism is an αrω-homeomorphism but not conversely.

Proof: Let f :(X,τ)→(Y,σ) be an αrωc-homeomorphism. Then f and f^{-1} are αrω-irresolute and f is bijection. By theorem 3.20 in [35] f and f^{-1} are αrω-continuous. Therefore f is αrω-homeomorphism.

The converse of the above Theorem is not true in general as seen from the following example.

Example 3.17 Consider X = Y = Z= {a, b, c, d} with topologies τ = σ = μ = {X, ϕ, {a}, {b}, {a,b}, {a,b,c}} and define a map f : (X, τ) → (Y, σ) by f(a)=c, f(b)=a, f(c)=b, f(d)=d. Then f is αrω-homeomorphism but it is not αrωc-homeomorphism, since f is not αrω-irresolute.

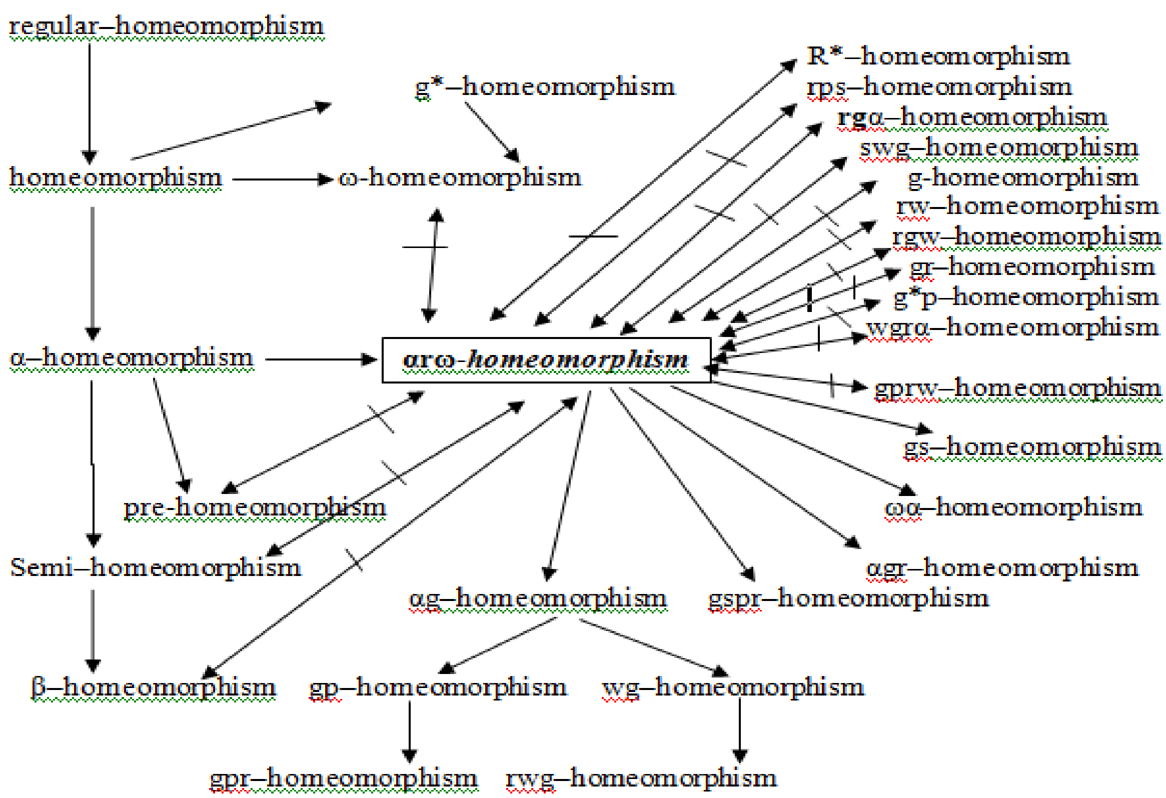
Theorem 3.18 Every αrωc-homeomorphism is wg-homeomorphism (resp. αg-homeomorphism, gs-homeomorphism, rwg-homeomorphism, gp-homeomorphism, gspr-homeomorphism, gpr-homeomorphism, ωα-homeomorphism, αgr-homeomorphism) but not conversely.

Proof: Proof follows from lemma 2.5 and 2.6.

Remark 3.19 From the above discussions and known results we have the following implications.

A ⟷ B means A & B are independent of each other

A ─┼→ B means A implies B but not conversely



Theorem 3.20 Let f :(X,τ)→(Y,σ) and g :(Y,σ)→(Z,η) are αrωc-homeomorphisms, then their composition $g \circ f : (X, \tau) \to (Z, \eta)$ is also αrωc-homeomorphism.

Proof: Let U be a αrω-open set in (Z, η). Since g is αrω-irresolute, $g^{-1}(U)$ is αrω-open in (Y,σ). Since f is αrω-irresolute, $f^{-1}(g^{-1}(U)) = (g \circ f)^{-1}(U)$ is αrω open set in (X, τ). Therefore $g \circ f$ is αrω-irresolute. Also for a αrω-open set G in (X,τ), we have $(g \circ f)(G) = g(f(G)) = g(W)$, where W =f(G). By hypothesis, f(G) is αrω-open in (Y, σ) and so again by hypothesis, g(f(G)) is a αrω-open set in (Z, η). That is $(g \circ f)(G)$ is a αrω-open set in (Z, η) and therefore $(g \circ f)^{-1}$ is αrω-irresolute. Also $g \circ f$ is a bijection. Hence $g \circ f$ is αrωc-homeomorphism.

Theorem 3.21: The set αrωc-h(X, τ) is a group under the composition of maps.

Proof: Define a binary operation $*$: αrωc-h(X,τ) × αrωc-h(X,τ)→αrωc-h(X,τ) by $f * g = g \circ f$ for all f,g∈ αrωc-h(X,τ) and $\circ$ is the usual operation of composition of maps. Then by Lemma 2.8, $g \circ f$∈ αrωc-h(X, τ). We know that the composition of maps is associative and the identity map I:(X,τ)→(X,τ) belonging to αrωc-h(X,τ) serves as the identity element. If f∈ αrωc-h(X,τ), then f^{-1}∈ αrωc-h(X,τ) such that $f \circ f^{-1} = f^{-1} \circ f = I$ and so inverse exists for each element of αrωc-h(X,τ). Therefore (αrωc-h(X,τ), ∘) is a group under the operation of composition of maps.

Theorem 3.22 Let f : (X, τ)→(Y, σ) be a αrωc-homeomorphism. Then f induces an isomorphism from the group αrωc-h(X, τ) onto the group αrωc-h(Y, σ).

Proof: Using the map f, we define a map Ψf : αrωc-h(X,τ)→αrωc-h(Y, σ) by $\Psi f(h) = f \circ h \circ f^{-1}$ for every h ∈ αrωc-h(X,τ). Then Ψf is a bijection. Further, for all h_1 , h_2 ∈ αrωc-h(X,τ), $\Psi f(h_1 \circ h_2) = f \circ (h_1 \circ h_2) \circ f^{-1} = (f \circ h_1 \circ f^{-1}) \circ (f \circ h_2 \circ f^{-1}) = \Psi f(h_1) \circ \Psi f(h_2)$. Therefore Ψf is a homeomorphism and so it is an isomorphism induced by f.

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Topological Mappings via $B\delta g$-Closed Sets

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Abstaract − In this paper we introduce a new class of functions called $B\delta g$-continuous functions. We obtain several characterizations and some their properties. Also we investigate its relationship with other types of functions. Further we introduce and study a new class of functions namely $B\delta g$-irresolute.

Keywords − *$B\delta g$-closed set, δ-continuous function, $B\delta g$-continuous function, $B\delta g$-irresolute function.*

1 Introduction

Levine [6], Noiri [10], Balachandran et al [2] and Dontchev and Ganster [3] introduced generalized closed sets, δ-continuity, generalized continuity and δ-generalized continuity (beiefly δg - continuity) & δ-generalized irresolute functions respectively. Devi et al [2] and Veerakumar [12] introduced semi-generalized continuity and $\hat{g}$-continuity in topological spaces. The purpose of this present paper is to define a new class of generalized continuous functions called $B\delta g$-continuous functions and investigate their relationships to other generalized continuous functions. We further introduce and study a new class of functions namely $B\delta g$-irresolute.

2 Preliminaries

Throughout this paper (X, τ) and, (Y, σ) and (Z, η) represent non-empty topological spaces on which no separation axioms are assumed unless or otherwise mentioned.

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For a subset A of X, $cl(A)$, $int(A)$ and A^c denote the closure of A, the interior of A and the complement of A respectively.

Let us recall the following definitions, which are useful in the sequel.

Definition 2.1. A subset A of a space (X,τ) is called a

(i) semi-open set [5] if A $\subseteq$ cl(int(A)).

(ii) pre-open set [7] if A$\subseteq$ int(cl(A)).

(iii) α-open set [9] if A$\subseteq$ int(cl(int(A))).

The complement of a semi-open (resp. pre-open, α-open) set is called semi-closed (resp. semi-closed, α-closed).

Definition 2.2. The δ-interior [11] of a subset A of X is the union of all regular open sets of X contained in A and is denoted by $int_\delta(A)$. The subset A is called δ-open [11] if $A = int_\delta(A)$, i.e. a set is δ-open if it is the union of regular open sets. The complement of a δ-open set is called δ-closed. Alternatively, a set A$\subseteq$ (X,τ) is called δ-closed [11] if $A = cl_\delta(A)$, where $cl_\delta(A) = \{x \in X : int(cl(U)) \cap A \neq \phi, U \in \tau$ and $x \in U\}$.

Definition 2.3. [11] A subset A of a space (X, τ) is called a

(i) t-set if $int(A) = int(cl(A))$.

(ii) B-set if $A = G \cap F$ where G is open and F is a t-set in X .

Definition 2.4. A subset A of (X,τ) is called

(i) generalized closed (briefly g-closed) set [6] if cl(A)$\subseteq$U whenever A $\subseteq$U and U is open in (X,τ).

(ii) generalized semi-closed (briefly gs-closed) set [1] if scl(A)$\subseteq$U whenever A $\subseteq$U and U is open in (X,τ).

(iii) α- generalized closed (briefly αg-closed) set [2] if αcl(A)$\subseteq$U whenever A $\subseteq$U and U is open in (X,τ).

(iv) δ-generalized closed (briefly δg-closed) set [3] if $cl_\delta(A)$ $\subseteq$U whenever A $\subseteq$U and U is open in (X,τ).

(v) $\hat{g}$-closed set [12] if cl(A)$\subseteq$U whenever A $\subseteq$U and U is semi-open in (X, τ).

(vi) δ-$\hat{g}$-closed (briefly $\delta\hat{g}$-closed) set [4] if $cl_\delta(A) \subseteq U$ whenever A $\subseteq$U and U is $\hat{g}$-open in (X,τ).

(vii) $B\delta$-generalized closed (briefly $B\delta g$-closed) set [8] if $cl_\delta(A) \subseteq U$ whenever A $\subseteq$U and U is B-set in (X,τ).

The complement of a g-closed (resp. gs-closed, αg-closed, δg-closed, $\hat{g}$-closed, $\delta\hat{g}$-closed, $B\delta g$-closed) set is called g-open (resp. gs-open, αg-open, δg-open, $\hat{g}$-open, $\delta\hat{g}$-open, $B\delta g$-open).

Definition 2.5. A function $f : (X, \tau) \to (Y, \sigma)$ is called

(i) semi-continuous [5] if $f^{-1}(V)$ is semi-closed in (X, τ) for every closed set V of (Y, σ).

(ii) g-continuous [2] if $f^{-1}(V)$ is g-closed in (X, τ) for every closed set V of (Y, σ).

(iii) gs-continuous [2] if $f^{-1}(V)$ is gs-closed in (X, τ) for every closed set V of (Y, σ).

(iv) αg-continuous [2] if $f^{-1}(V)$ is αg-closed in (X, τ) for every closed set V of (Y, σ).

(v) super continuous [10] if $f^{-1}(V)$ is δ-open in (X, τ) for every open set V of (Y, σ).

(vi) $\hat{g}$-continuous [12] if $f^{-1}(V)$ is $\hat{g}$-closed in (X, τ) for every $\hat{g}$-closed set V of (Y, σ).

(vii) δ-continuous [10] if $f^{-1}(V)$ is δ-open in (X, τ) for every δ-open set V of (Y, σ).

(viii) δ-closed [10] if $f(V)$ is δ-closed in (Y, σ) for every δ-closed set V of (X, τ).

(ix) δg-continuous [3] if $f^{-1}(V)$ is δg-closed in (X, τ) for every closed set V of (Y, σ).

(x) $\delta\hat{g}$-continuous [4] if $f^{-1}(V)$ is $\delta\hat{g}$-closed in (X, τ) for every closed set V of (Y, σ).

Proposition 2.6. [8] If A and B are $B\delta g$-closed sets, then $A \cup B$ is $B\delta g$-closed.

3 $B\delta g$-Continuous and $B\delta g$-Irresolute Functions

In this section we introduce the following definitions.

Definition 3.1. A function $f : (X, \tau) \to (Y, \sigma)$ is called $B\delta g$-continuous if $f^{-1}(V)$ is $B\delta g$-closed in (X, τ) for every closed set V of (Y, σ).

Example 3.2. Let $X = \{a, b, c\}$and $Y = \{p, q, r\}$ with the topologies $\tau = \{\phi, \{a\}, \{b\}, \{a, b\}, X\}$ and $\sigma = \{\phi, \{q\}, \{p, q\}, Y\}$. Define $f : (X, \tau) \to (Y, \sigma)$ by $f(a) = p$, $f(b) = q$ and $f(c) = r$. Clearly f is $B\delta g$-continuous.

Definition 3.3. A function $f : (X, \tau) \to (Y, \sigma)$ is called $B\delta g$-irresolute if $f^{-1}(V)$ is $B\delta g$-closed in (X, τ) for every $B\delta g$-closed set V of (Y, σ).

Example 3.4. Let $X = \{a, b, c\} = Y = \{p, q, r\}$ with the topologies $\tau = \{\phi, \{a\}, X\}$ and $\sigma = \{\phi, \{q\}, \{q, r\}, Y\}$. Define $f : (X, \tau) \to (Y, \sigma)$ by $f(a) = p$, $f(b) = r$ and $f(c) = q$. Clearly f is $B\delta g$-irresolute.

Proposition 3.5. If $f : (X, \tau) \to (Y, \sigma)$ is $B\delta g$-continuous then f is g-continuous, αg-continuous, gs-continuous and δg-continuous maps.

Proof. It is true that every $B\delta g$-closed set is g-closed, αg-closed, gs-closed and δg-closed. □

Remark 3.6. The converses of the above proposition are not true in general as seen from the following examples.

Example 3.7. Let $X = \{a, b, c\}$and $Y = \{p, q, r\}$ with the topologies $\tau = \{\phi, \{a\}, X\}$ and $\sigma = \{\phi, \{p\}, \{p, r\}, Y\}$. Define the map $f : (X, \tau) \to (Y, \sigma)$ by $f(a) = p$, $f(b) = q$ and $f(c) = r$. Clearly f is not $B\delta g$-continuous because $\{q, r\}$ is closed in (Y, σ) but $f^{-1}(\{q, r\}) = \{b, c\}$ is not $B\delta g$-closed in (X, τ). However f is g-continuous.

Example 3.8. Let $X = \{a, b, c\}$and $Y = \{p, q, r\}$ with the topologies $\tau = \{\phi, \{c\}, X\}$ and $\sigma = \{\phi, \{p\}, \{p, q\}, \{p, r\}, Y\}$. Let $f : (X, \tau) \to (Y, \sigma)$ be a function defined by $f(a) = r$, $f(b) = q$ and $f(c) = p$. Then f is αg-continuous and sg-continuous. But f is not $B\delta g$-continuous, for the closed set $\{q\}$ of (Y, σ), $f^{-1}(\{q\}) = \{b\}$ is not $B\delta g$-closed in (X, τ).

Example 3.9. Let $X = \{a, b, c\}$and $Y = \{p, q, r\}$ with the topologies $\tau = \{\phi, \{a\}, \{a, c\}, X\}$ and $\sigma = \{\phi, \{p\}, \{q\}, \{p, q\}, Y\}$. Define $f : (X, \tau) \to (Y, \sigma)$ by $f(a) = q$, $f(b) = r$ and $f(c) = p$. Then f is not $B\delta g$-continuous, for $\{r\}$ is closed in (Y, σ), $f^{-1}(\{r\}) = \{b\}$ is not $B\delta g$-closed in (X, τ). However f is δg-continuous function.

Theorem 3.10. Every super continuous function is $B\delta g$-continuous.

Proof. It is true that every δ-closed set is $B\delta g$-closed. □

Remark 3.11. The converse of Theorem 3.10 need not be true as shown in the following example.

Example 3.12. Let $X = \{a, b, c\}$ and $Y = \{p, q, r\}$ with the topologies $\tau = \{\phi, \{a\}, \{a, b\}, X\}$ and $\sigma = \{\phi, \{r\}, Y\}$. Let $f : (X, \tau) \to (Y, \sigma)$ be a function defined by $f(a) = p$, $f(b) = r$ and $f(c) = q$. Then f is $B\delta g$-continuous. But f is not super continuous, for $\{r\}$ is open in (Y, σ), $f^{-1}(\{r\}) = \{b\}$ is not δ-open in (X, τ).

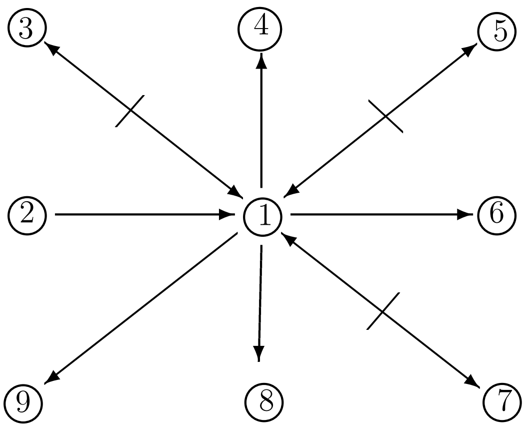
Remark 3.13. The following examples show that $B\delta g$-continuity is independent of semi-continuity, $\hat{g}$-continuity and $\delta\hat{g}$-continuity.

Example 3.14. Let $X = \{a, b, c\}$and $Y = \{p, q, r\}$ with the topologies $\tau = \{\phi, \{a\}, \{a, b\}, \{a, c\}, X\}$ and $\sigma = \{\phi, \{p, q\}, Y\}$. Let $f : (X, \tau) \to (Y, \sigma)$ be a function defined by $f(a) = q$, $f(b) = r$ and $f(c) = p$. Then f is semi-continuous but not $B\delta g$-continuous.

Example 3.15. Let $X = \{a, b, c\}$and $Y = \{p, q, r\}$ with the topologies $\tau = \{\phi, \{a\}, X\}$ and $\sigma = \{\phi, \{q\}, Y\}$. Define a function $f : (X, \tau) \to (Y, \sigma)$ by $f(a) = q$, $f(b) = p$ and $f(c) = r$. Then f is $\hat{g}$-continuous and $\delta\hat{g}$-continuous but not $B\delta g$-continuous.

Example 3.16. Let $X = \{a, b, c\}$and $Y = \{p, q, r\}$ with the topologies $\tau = \{\phi, \{a\}, \{a, c\}, X\}$ and $\sigma = \{\phi, \{p\}, Y\}$. Define a function $f : (X, \tau) \to (Y, \sigma)$ by $f(a) = q$, $f(b) = r$ and $f(c) = p$. Then f is neither semi-continuous nor $\hat{g}$-continuous. Moreover it is not $\delta\hat{g}$-continuous. However f is $B\delta g$-continuous function.

Remark 3.17. All the above discussions of this section can be represented by the following diagram. $A \to B(A \nleftrightarrow B)$ represents A implies B but not conversely (A and B are independent of each other).



1.$B\delta g$-continuous 2.δ-continuous 3.*semi*-continuous 4.αg-continuous
5.$\hat{g}$-continuous 6.δg-continuous 7.$\delta\hat{g}$-continuous 8.gs-continuous 9.g-continuous

4 Characterizations

Theorem 4.1. A function $f : (X, \tau) \to (Y, \sigma)$ is $B\delta g$-continuous iff $f^{-1}(U)$ is $B\delta g$-open in (X, τ) for every open set U in (Y, σ).

Proof. Let $f : (X, \tau) \to (Y, \sigma)$ be an $B\delta g$-continuous function and U be an open set in (Y, σ). Then $f^{-1}(U^c)$ is $B\delta g$-closed set in (X, τ). But $f^{-1}(U^c) = [f^{-1}(U)]^c$ and hence $f^{-1}(U)$ is $B\delta g$-open in (X, τ). Conversely $f^{-1}(U)$ is $B\delta g$-open in (X, τ) for every open set U in (Y, σ). Then U^c is closed set in (Y, σ) and $[f^{-1}(U)]^c$ is $B\delta g$-closed in (X, τ). But $[f^{-1}(U)]^c = f^{-1}(U^c)$, so $f^{-1}(U^c)$ is $B\delta g$-closed set in (X, τ). Hence f is $B\delta g$-continuous. □

Theorem 4.2. Let $f : (X, \tau) \to (Y, \sigma)$ be a $B\delta g$-irresolute and $g : (Y, \sigma) \to (Z, \eta)$ a $B\delta g$-irresolute. Then their composition is $g \circ f : (X, \tau) \to (Z, \eta)$ is $B\delta g$-irresolute.

Proof. Let F be $B\delta g$-closed set in (Z, η). Then $g^{-1}(F)$ is $B\delta g$-closed in (Y, σ). Since f is $B\delta g$-irresolute, $(g \circ f)^{-1}(F) = f^{-1}(g^{-1}(F))$ is $B\delta g$-closed set of (X, τ) and so $g \circ f$ is $B\delta g$-irresolute function. □

Remark 4.3. The composition of two $B\delta g$-continuous functions need not be $B\delta g$-continuous as the following examples shows.

Example 4.4. Let $X = \{a, b, c\} = Y = Z$ with the topologies $\tau = \{\phi, \{b\}, \{c\}, \{a, b\}, \{b, c\}, X\}$ and $\sigma = \{\phi, \{b\}, \{a, c\}, Y\}$ and $\eta = \{\phi, \{a\}, \{b\}, \{a, b\}, Z\}$. Define a function $f : (X, \tau) \to (Y, \sigma)$ by $f(a) = a$, $f(b) = c$ and $f(c) = b$ and let $g : (Y, \sigma) \to (Z, \eta)$ be the identity function. Clearly f and g are $B\delta g$-continuous. But $g \circ f : (X, \tau) \to (Z, \eta)$ is not an $B\delta g$-continuous function because $(g \circ f)^{-1}(\{c\}) = f^{-1}(g^{-1}(\{c\})) = f^{-1}(\{c\}) = \{b\}$ is not an $B\delta g$-closed in (X, τ) where as $\{c\}$ is a closed set of (Z, η).

Theorem 4.5. Let $f : (X, \tau) \to (Y, \sigma)$ and $g : (Y, \sigma) \to (Z, \eta)$ be two functions. Then

(i) $g \circ f : (X, \tau) \to (Z, \eta)$ is $B\delta g$-continuous, if g is continuous and f is $B\delta g$-continuous.

(ii) $g \circ f : (X, \tau) \to (Z, \eta)$ is $B\delta g$-continuous, if g is $B\delta g$-continuous and f is $B\delta g$-irresolute.

Proof. (i) Let F be any closed set in (Z, η). Since g is continuous, $g^{-1}(F)$ is closed in (Y, σ). But f is $B\delta g$-continuous, $(g \circ f)^{-1}(F) = f^{-1}(g^{-1}(F))$ is $B\delta g$-closed of (X, τ) and hence $g \circ f$ is $B\delta g$-continuous function.
(ii) Let G be any closed set in (Z, η). Then $g^{-1}(G)$ is $B\delta g$-closed in (Y, σ). Since f is $B\delta g$-irresolute, $(g \circ f)^{-1}(G) = f^{-1}(g^{-1}(G))$ is $B\delta g$-closed of (X, τ) and so $g \circ f$ is $B\delta g$-continuous functions. □

Theorem 4.6. Let $f : (X, \tau) \to (Y, \sigma)$ be continuous and δ-closed map. Then for every $B\delta g$-closed subset A of (X, τ), $f(A)$ is $B\delta g$-closed in (Y, σ).

Proof. Let A be $B\delta g$-closed in (X, τ). Let $f(A) \subseteq$ O where O is open in (Y, σ). Since A$\subseteq f^{-1}(O)$ is open in (X, τ), $f^{-1}(O)$ is B-set in (X, τ). Since A is $B\delta g$-closed and since $f^{-1}(O)$ is B-set in (X, τ), then $cl_\delta(A) \subseteq f^{-1}(O)$. Thus $f(cl_\delta(A)) \subseteq O$. Hence $cl_\delta(f(A)) \subseteq cl_\delta(f(cl_\delta(A)) = f(cl_\delta(A)) \subseteq O$, since f is δ-closed. Hence $f(A)$ is $B\delta g$-closed in (Y, σ). □

Remark 4.7. $B\delta g$-continuity and $B\delta g$-irresoluteness are independent notions as seen in the following examples.

Example 4.8. Let $X = \{a, b, c\}, Y = \{p, q, r\}$ with the topologies $\tau = \{\phi, \{a\}, \{a, c\}, X\}$ and $\sigma = \{\phi, \{r\}, Y\}$. Define a function $f : (X, \tau) \to (Y, \sigma)$ by $f(a) = p$, $f(b) = q$ and $f(c) = r$. Then f is $B\delta g$-continuous but it is not $B\delta g$-irresolute function because $f^{-1}(\{q, r\}) = \{b, c\}$ is not $B\delta g$-closed in (X, τ), where $\{q, r\}$ is $B\delta g$-closed in (Y, σ).

Example 4.9. Let $X = \{a, b, c\}$, $Y = \{p, q, r\}$ with the topologies $\tau = \{\phi, \{a\}, \{b\}, \{a, b\}, \{a, c\}, X\}$ and $\sigma = \{\phi, \{r\}, \{q, r\}, Y\}$. Define a function $f : (X, \tau) \to (Y, \sigma)$ by $f(a) = p$, $f(b) = q$ and $f(c) = r$. Then f is $B\delta g$-irresolute but it is not $B\delta g$-continuity function because $f^{-1}(\{p\}) = \{a\}$ is not $B\delta g$-closed in (X, τ), when $\{p\}$ is closed in (Y, σ).

Proposition 4.10. The product of two $B\delta g$-open sets of two spaces is $B\delta g$-open set in the product space.

Proof. Let A and B be two $B\delta g$-open sets of two spaces (X, τ) and (Y, σ) respectively and $V = A \times B \subseteq X \times Y$. Let F$\subseteq V$ be a complement of B-set in $X \times Y$, then there exists two complement of B-sets $F_1 \subseteq A$ and $F_2 \subseteq B$. So, $F_1 \subseteq int_\delta(A)$ and $F_2 \subseteq int_\delta(B)$. Hence $F_1 \times F_2 \subseteq A \times B$ and $F_1 \times F_2 \subseteq int_\delta(A) \times int_\delta(B) = int_\delta(A \times B)$. Therefore $A \times B$ is $B\delta g$-open subset of the space $X \times Y$. □

Theorem 4.11. Let $f_i : X_i \to Y_i$ be $B\delta g$-continuous functions for each $i \in \{1, 2\}$ and let $f : X_1 \times X_2 \to Y_1 \times Y_2$ be defined by $f((x_1, x_2)) = (f(x_1), f(x_2))$. Then f is $B\delta g$-continuous.

Proof. Let V_1 and V_2 be two open sets in Y_1 and Y_2 respectively. Since $f_i : X_i \to Y_i$ are $B\delta g$-continuous functions, for each i$\in$ {1,2}, $f_1^{-1}(V_1)$ and $f_2^{-1}(V_2)$ are $B\delta g$-open sets in X_1 and X_2 respectively. By the Proposition 4.10, $f_1^{-1}(V_1) \times f_2^{-1}(V_2)$ is $B\delta g$-open set in $X_1 \times X_2$. Therefore f is $B\delta g$-continuous. □

Theorem 4.12. Let $f : (X, \tau) \to (Y, \sigma)$ be a function. Then the following statements are equivalent.

(i) f is $B\delta g$-irresolute.

(ii) For $x \in (X, \tau)$ and any $B\delta g$-closed set V of (Y, σ) containing $f(x)$ there exists an $B\delta g$-closed set U such that $x \in U$ and $f(U) \subset V$.

(iii) Inverse image of every $B\delta g$-open set of (Y, σ) is $B\delta g$-open in (X, τ).

Proof. $(i) \Rightarrow (ii)$. Let V be an $B\delta g$-closed set of (Y, σ) and $f(x) \in V$. Since f is $B\delta g$-irresolute, $f^{-1}(V)$ is $B\delta g$-closed in (X, τ) and $x \in f^{-1}(V)$. Put $U = f^{-1}(V)$. Then $x \in U$ and $f(U) \subset V$.
(ii) $\Rightarrow (i)$. Let V be an $B\delta g$-closed set of (Y, σ) and $x \in f^{-1}(V)$. Then $f(x) \in V$. Therefore by (ii) there exists an $B\delta g$-closed set U_x such that $x \in U_x$ and $f(U_x) \subset V$. Hence $x \in U_x \subset f^{-1}(V)$. This implies that $f^{-1}(V)$ is a union of $B\delta g$-closed sets of (X, τ). By Proposition 2.6, $f^{-1}(V)$ is $B\delta g$-closed set. This shows that f is $B\delta g$-irresolute.
$(i) \Leftrightarrow (iii)$. It is true that $f^{-1}(Y - V) =$ X$-f^{-1}(V)$. □

5 Applications

Definition 5.1. [8] A space (X, τ) is called a ${}_BT_{\delta g}$-space if every $B\delta g$-closed set in it is δ-closed.

Theorem 5.2. Let $f : (X, \tau) \to (Y, \sigma)$ be $B\delta g$-irresolute. Then f is δ-continuous if (X, τ) is ${}_BT_{\delta g}$-space.

Proof. Let V be a δ-closed subset of (Y, σ). Every δ-closed is $B\delta g$-closed and hence V is $B\delta g$-closed in (Y, σ). Since f is $B\delta g$-irresolute, $f^{-1}(V)$ is $B\delta g$-closed in (X, σ). Since X is ${}_BT_{\delta g}$-space, $f^{-1}(V)$ is δ-closed in (X, τ). Thus f is δ-continuous. □

Theorem 5.3. Let $f : (X, \tau) \to (Y, \sigma)$ and $g : (Y, \sigma) \to (Z, \eta)$ be two functions. Let (Y, σ) be ${}_BT_{\delta g}$-space. Then $g \circ f$ is $B\delta g$-continuous if g is $B\delta g$-continuous and f is $B\delta g$-continuous.

Proof. Let G be any closed set in (Z, η). Then $g^{-1}(G)$ is $B\delta g$-closed in (Y, σ). Since (Y, σ) is ${}_BT_{\delta g}$-space, $g^{-1}(G)$ is closed in (Y, σ). Since f is $B\delta g$-continuous, $(g \circ f)^{-1}(G) = f^{-1}(g^{-1}(G))$ is $B\delta g$-closed in (X, τ). Hence $g \circ f$ is $B\delta g$-continuous function. □

Theorem 5.4. Let $f : (X, \tau) \to (Y, \sigma)$ be onto, $B\delta g$-irresolute and δ-closed. If (X, τ) is a ${}_BT_{\delta g}$-space, then (Y, σ) is also a ${}_BT_{\delta g}$-space.

Proof. Let B be a $B\delta g$-closed subset of (Y, σ). Since f is $B\delta g$-irresolute, then $f^{-1}(B)$ is $B\delta g$-closed set in (X, τ). Since (X, τ) is ${}_BT_{\delta g}$-space, $f^{-1}(B)$ is δ-closed in (X, τ). Also since f is surjective, B is δ-closed in (Y, σ). Hence (Y, σ) is ${}_BT_{\delta g}$-space. □

Theorem 5.5. If $f : (X, \tau) \to (Y, \sigma)$ is bijection, open and $B\delta g$-continuous, then f is $B\delta g$-irresolute.

Proof. Let V be $B\delta g$-closed in (Y,σ) and let $f^{-1}(V) \subseteq U$ where U is open in (X,τ). Since f is open, $f(U)$ is open in (Y,σ). Every open set is B-set and hence $f(U)$ is B-set. Clearly $V \subseteq f(U)$. Then $cl_\delta(V) \subseteq f(U)$ and hence $f^{-1}(cl_\delta(V)) \subseteq U$. Since f is $B\delta g$-continuous and since $cl_\delta(V)$ is a closed subset of (Y,σ), then $cl_\delta(f^{-1}(V)) \subseteq cl_\delta(f^{-1}(cl_\delta(V)) = f^{-1}(cl_\delta(V)) \subseteq U$. U is open and hence B-set in (X,τ). Thus we have $cl_\delta(f^{-1}(V)) \subseteq U$ whenever $f^{-1}(V) \subseteq U$ and U is B-set in (X,τ). This shows that $f^{-1}(V)$ is $B\delta g$-closed in (X,τ). Hence f is $B\delta g$-irresolute. □

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On Nano $\wedge_{g^\star}$-Closed Sets

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Abstaract − In this paper, we introduce and study nano topological properties of nano $\wedge_{g^\star}$-closed sets and nano $\wedge_{g^\star}$-open sets and its relationships with other nano generalized closed sets are investigated.

Keywords − *Nano $\wedge_{g^\star}$-open sets, nano λ-closed set, nano $\wedge$-set*

1 Introduction

Thivagar et al. [4] introduced a nano topological space with respect to a subset X of an universe which is defined in terms of lower approximation and upper approximation and boundary region. The classical nano topological space is based on an equivalence relation on a set, but in some situation, equivalence relations are nor suitable for coping with granularity, instead the classical nano topology is extend to general binary relation based covering nano topological space.

In 2017, Rajasekaran et al. [7, 8] introduced nano $\wedge$-sets and nano $\wedge_g$-sets in nano topological spaces and we introduced the notion of nano λ-closed set and nano λ-open sets. In this paper, we introduce and study nano topological properties of nano $\wedge_{g^\star}$-closed sets and nano $\wedge_{g^\star}$-open sets and its relationships with other nano generalized closed sets are investigated.

2 Preliminaries

Throughout this paper $(U, \tau_R(X))$ (or X) represent nano topological spaces on which no separation axioms are assumed unless otherwise mentioned. For a subset H of a

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space $(U, \tau_R(X))$, $Ncl(H)$ and $Nint(H)$ denote the nano closure of H and the nano interior of H respectively. We recall the following definitions which are useful in the sequel.

Definition 2.1. *[6] Let U be a non-empty finite set of objects called the universe and R be an equivalence relation on U named as the indiscernibility relation. Elements belonging to the same equivalence class are said to be indiscernible with one another. The pair (U, R) is said to be the approximation space. Let $X \subseteq U$.*

1. *The lower approximation of X with respect to R is the set of all objects, which can be for certain classified as X with respect to R and it is denoted by $L_R(X)$. That is, $L_R(X) = \bigcup_{x \in U}\{R(x) : R(x) \subseteq X\}$, where R(x) denotes the equivalence class determined by x.*

2. *The upper approximation of X with respect to R is the set of all objects, which can be possibly classified as X with respect to R and it is denoted by $U_R(X)$. That is, $U_R(X) = \bigcup_{x \in U}\{R(x) : R(x) \cap X \neq \phi\}$.*

3. *The boundary region of X with respect to R is the set of all objects, which can be classified neither as X nor as not - X with respect to R and it is denoted by $B_R(X)$. That is, $B_R(X) = U_R(X) - L_R(X)$.*

Property 2.2. *[4] If (U, R) is an approximation space and $X, Y \subseteq U$; then*

1. $L_R(X) \subseteq X \subseteq U_R(X)$;
2. $L_R(\phi) = U_R(\phi) = \phi$ *and* $L_R(U) = U_R(U) = U$;
3. $U_R(X \cup Y) = U_R(X) \cup U_R(Y)$;
4. $U_R(X \cap Y) \subseteq U_R(X) \cap U_R(Y)$;
5. $L_R(X \cup Y) \supseteq L_R(X) \cup L_R(Y)$;
6. $L_R(X \cap Y) \subseteq L_R(X) \cap L_R(Y)$;
7. $L_R(X) \subseteq L_R(Y)$ *and* $U_R(X) \subseteq U_R(Y)$ *whenever* $X \subseteq Y$;
8. $U_R(X^c) = [L_R(X)]^c$ *and* $L_R(X^c) = [U_R(X)]^c$;
9. $U_R U_R(X) = L_R U_R(X) = U_R(X)$;
10. $L_R L_R(X) = U_R L_R(X) = L_R(X)$.

Definition 2.3. *[4] Let U be the universe, R be an equivalence relation on U and $\tau_R(X) = \{U, \phi, L_R(X), U_R(X), B_R(X)\}$ where $X \subseteq U$. Then by the Property 2.2, R(X) satisfies the following axioms:*

1. *U and $\phi \in \tau_R(X)$,*
2. *The union of the elements of any sub collection of $\tau_R(X)$ is in $\tau_R(X)$,*

3. *The intersection of the elements of any finite subcollection of* $\tau_R(X)$ *is in* $\tau_R(X)$.

That is, $\tau_R(X)$ *is a topology on* U *called the nano topology on* U *with respect to* X. *We call* $(U, \tau_R(X))$ *as the nano topological space. The elements of* $\tau_R(X)$ *are called as nano open sets and* $[\tau_R(X)]^c$ *is called as the dual nano topology of* $[\tau_R(X)]$.

Remark 2.4. *[4] If* $[\tau_R(X)]$ *is the nano topology on* U *with respect to* X, *then the set* $B = \{U, \phi, L_R(X), B_R(X)\}$ *is the basis for* $\tau_R(X)$.

Definition 2.5. *[4] If* $(U, \tau_R(X))$ *is a nano topological space with respect to* X *and if* $H \subseteq U$, *then the nano interior of* H *is defined as the union of all nano open subsets of* H *and it is denoted by* $Nint(H)$.

That is, $Nint(H)$ *is the largest nano open subset of* H. *The nano closure of* H *is defined as the intersection of all nano closed sets containing* H *and it is denoted by* $Ncl(H)$.

That is, $Ncl(H)$ *is the smallest nano closed set containing* H.

Definition 2.6. *[4] A subset* H *of a nano topological space* $(U, \tau_R(X))$ *is called;*

1. *nano pre-open set if* $H \subseteq Nint(Ncl(H))$.
2. *nano semi-open set if* $H \subseteq Ncl(Nint(H))$.

The complements of the above mentioned sets are called their respective closed sets.

Definition 2.7. *A subset* H *of a nano topological space* $(U, \tau_R(X))$ *is called;*

1. *nano g-closed [1] if* $Ncl(H) \subseteq G$, *whenever* $H \subseteq G$ *and* G *is nano open.*
2. *nano gs-closed set [2] if* $Nscl(H) \subseteq G$ *whenever* $H \subseteq G$, G *is nano open.*
3. *nano gp-closed set [3] if* $Npcl(H) \subseteq G$, *whenever* $H \subseteq G$ *and* G *is nano open.*

Definition 2.8. *[5] Let* $(U, \tau_R(X))$ *be a nano topological spaces and* $H \subseteq U$. *The nano* $Ker(H) = \bigcap\{U : H \subseteq U, U \in \tau_R(X)\}$ *is called the nano kernal of* H *and is denoted by* $\mathcal{N}Ker(H)$.

Definition 2.9. *[7] A subset* H *of a space* $(U, \tau_R(X))$ *is called;*

1. *nano* $\wedge$*-set if* $H = \mathcal{N}Ker(H)$.
2. *nano* λ*-closed if* $H = L \cap F$ *where* L *is a nano* $\wedge$*-set and* F *is nano closed.*

Definition 2.10. *[8] A subset* H *of a space* $(U, \tau_R(X))$ *is called;*

1. *nano* $\wedge_g$*-closed set if* $Ncl(H) \subseteq G$, *whenever* $H \subseteq G$ *and* G *is nano* λ*-open.*
2. *a nano* $_g\wedge$*-closed set if* $N\lambda cl(H) \subseteq G$, *whenever* $H \subseteq G$ *and* G *is nano open.*
3. *a nano* $\wedge$*-g-closed set if* $N\lambda cl(H) \subseteq G$, *whenever* $H \subseteq G$ *and* G *is nano* λ*-open.*

The complement of the above mentioned sets are called their respective open sets.

Lemma 2.11. *[7] For a subset H of a space $(U, \tau_R(X))$, the following conditions are equivalent.*

1. *H is nano λ-closed.*
2. *$H = L \cap Ncl(H)$ where L is a nano $\wedge$-set.*
3. *$H = \mathcal{N}Ker(H) \cap Ncl(H)$.*

Lemma 2.12. *[7]*

1. *Every nano $\wedge$-set is nano λ-closed.*
2. *Every nano open set is nano λ-closed.*
3. *Every nano closed set is nano λ-closed.*

3 Nano $\wedge_{g^\star}$-Closed Sets

Definition 3.1. *A subset H of a space $(U, \tau_R(X))$ is called a nano $\wedge_{g^\star}$-closed if $N\lambda cl(H) \subseteq G$, whenever $H \subseteq G$ and G is nano g-open.*

The complement of nano $\wedge_{g^\star}$-open if $H^c = U - H$ is nano $\wedge_{g^\star}$-closed.

Example 3.2. *Let $U = \{a, b, c, d\}$ with $U/R = \{\{a, b\}, \{c\}, \{d\}\}$ and $X = \{a, d\}$. Then the nano topology $\tau_R(X) = \{\phi, \{d\}, \{a, b\}, \{a, b, d\}, U\}$.*

1. *Then $\{a, b\}$ is nano $\wedge_{g^\star}$-closed.*
2. *Then $\{a, d\}$ is not nano $\wedge_{g^\star}$-closed.*

Theorem 3.3. *In a space $(U, \tau_R(X))$, every nano λ-closed is nano $\wedge_{g^\star}$-closed.*

Proof. Let $H \subseteq G$, where G is nano g-open. Since H is nano λ-closed, we have $\lambda cl(H) = H \subseteq G$. Hence H is nano $\wedge_{g^\star}$-closed.

Remark 3.4. *The converse of statements in Theorem 3.3 are not necessarily true as seen from the following Example.*

Example 3.5. *In Example 3.2, then $\{a, c, d\}$ is nano $\wedge_{g^\star}$-closed but not nano λ-closed.*

Theorem 3.6. *In a space $(U, \tau_R(X))$, every nano closed is nano $\wedge_{g^\star}$-closed.*

Proof. Proof follows from Lemma 2.12 and Theorem 3.3.

Remark 3.7. *The converse of statements in Theorem 3.6 are not necessarily true as seen from the following Example.*

Example 3.8. *In Example 3.2, then $\{d\}$ is nano $\wedge_{g^\star}$-closed but not nano closed.*

Theorem 3.9. *In a space $(U, \tau_R(X))$, every nano open is nano $\wedge_{g^\star}$-closed.*

Proof. Obvious by the Definitions.

Remark 3.10. *The converse of statements in Theorem 3.9 are not necessarily true as seen from the following Example.*

Example 3.11. *In Example 3.2, then $\{c\}$ is nano $\wedge_{g^\star}$-closed but not nano open.*

Theorem 3.12. *Let H be a nano g-open. Then H is nano λ-closed if H is nano $\wedge_{g^\star}$-closed.*

Proof. Let H is nano $\wedge_{g^\star}$-closed and nano g-open. Since as $H \subseteq H$, $N\lambda cl(H) \subseteq H$. Hence H is nano λ-closed.

Theorem 3.13. *In a space $(U, \tau_R(X))$, every nano $\wedge_{g^\star}$-closed is nano ${}_g\wedge$-closed.*

Proof. Let H is nano $\wedge_{g^\star}$-closed and $H \subseteq G$, with G is nano open. Since every nano open is nano g-open and H is nano $\wedge_{g^\star}$-closed, we have $\lambda cl(H) \subseteq G$. Hence H is nano ${}_g\wedge$-closed.

Remark 3.14. *The converse of statements in Theorem 3.13 are not necessarily true as seen from the following Example.*

Example 3.15. *In Example 3.2, then $\{a\}$ is nano ${}_g\wedge$-closed but not nano $\wedge_{g^\star}$-closed.*

Theorem 3.16. *In a space $(U, \tau_R(X))$, every nano $\wedge_{g^\star}$-closed is nano $\wedge$-g-closed.*

Proof. Obvious.

Remark 3.17. *The converse of statements in Theorem 3.16 are not necessarily true as seen from the following Example.*

Example 3.18. *In Example 3.2, then $\{a\}$ is nano $\wedge$-g-closed but not nano $\wedge_{g^\star}$-closed.*

Theorem 3.19. *In a space $(U, \tau_R(X))$, every nano g-closed is nano $\wedge_{g^\star}$-closed.*

Proof. Obvious.

Remark 3.20. *The converse of statements in Theorem 3.19 are not necessarily true as seen from the following Example.*

Example 3.21. *In Example 3.2, then $\{a, b, d\}$ is nano $\wedge_{g^\star}$-closed but not nano g-closed.*

Remark 3.22. *In a space $(U, \tau_R(X))$, every nano $\wedge_g$-closed is nano $\wedge_{g^\star}$-closed.*

Example 3.23. *In Example 3.2, then $\{a, b, d\}$ is nano $\wedge_{g^\star}$-closed but not nano $\wedge_g$-closed.*

Remark 3.24. *The concepts of nano $\wedge_{g^\star}$-closed and being nano gs-closed, nano gp-closed are independent.*

Example 3.25.

1. *Let $U = \{a, b, c\}$ with $U/R = \{\{b\}, \{a, c\}\}$ and $X = \{c\}$. Then the nano topology $\tau_R(X) = \{\phi, \{a, c\}, U\}$. Then $\{a, c\}$ is nano $\wedge_{g^\star}$-closed but not nano gp-closed.*

2. *In Example 3.2, then $\{a\}$ is nano gp-closed but not nano $\wedge_{g^\star}$-closed.*

Example 3.26. *In Example 3.2,*

1. *then* $\{a, b, d\}$ *is nano* $\wedge_{g^\star}$*-closed but not nano gs-closed.*

2. *then* $\{a\}$ *is nano gs-closed but not nano* $\wedge_{g^\star}$*-closed.*

Remark 3.27. *In a space* $(U, \tau_R(X))$*,*

1. *the intersection of two nano* $\wedge_{g^\star}$*-open sets but not nano* $\wedge_{g^\star}$*-open.*

2. *the union of two nano* $\wedge_{g^\star}$*-closed sets but not nano* $\wedge_{g^\star}$*-closed.*

Example 3.28. *In Example 3.2,*

1. *then* $P = \{b\}$ *and* $Q = \{c\}$ *is nano* $\wedge_{g^\star}$*-open sets. Hence* $P \cup Q = \{b, c\}$ *is not nano* $\wedge_{g^\star}$*-open.*

2. *then* $P = \{a, b\}$ *and* $Q = \{a, c\}$ *is nano* $\wedge_{g^\star}$*-closed sets. Hence* $P \cap Q = \{a\}$ *is not nano* $\wedge_{g^\star}$*-closed.*

4 Properties of $\wedge_{g^\star}$-Closed Sets

Theorem 4.1. *If a subset* H *is nano* $\wedge_{g^\star}$*- closed set, then nano* $N\lambda cl(H) - H$ *does not contain any non empty nano closed in* U*.*

Proof. Let H be nano $\wedge_{g^\star}$-closed, suppose K is a non empty nano closed contained in $N\lambda cl(H) - H$, which clearly implies $H \subseteq K^c$, where K^c is nano open. Since H is nano $\wedge_{g^\star}$-closed and as every nano open is nano g-open, we have $N\lambda cl(H) \subseteq K^c$. Hence $K \subseteq U - N\lambda cl(H)$. Also we have $K \subseteq N\lambda cl(H)$. Therefore $K \subseteq (U - N\lambda cl(H)) \cap N\lambda cl(H) = \phi$. Hence $N\lambda cl(H) - H$ does not contain any non empty nano closed.

Theorem 4.2. *If a subset* H *is nano* $\wedge_{g^\star}$*- closed, then* $N\lambda cl(H) - H$ *does not contain any non empty nano g-closed.*

Proof. Let H be nano $\wedge_{g^\star}$-closed. Suppose K is a nano g-closed contained in $N\lambda cl(H) - H$, which clearly implies $H \subseteq K^c$, where K^c is nano g-open. Since H is nano $\wedge_{g^\star}$-closed and $N\lambda cl(H) \subseteq K^c$. Hence $K \subseteq U - N\lambda cl(H)$. Also we have $K \subseteq N\lambda cl(H)$. Therefore $K \subseteq (U - N\lambda cl(H)) \cap N\lambda cl(H) = \phi$. Hence $N\lambda cl(H) - H$ does not contain a non empty nano g-closed.

Theorem 4.3. *In a space* $(U, \tau_R(X))$*, for each* $x \in U$*,* $\{x\}$ *is nano g-closed or nano* $\wedge_{g^\star}$*-open.*

Proof. Suppose $\{x\}$ is not nano g-closed then $U - \{x\}$ is not nano g-open, then the only nano g-open containing $U - \{x\}$ is U. That is $U - \{x\} \subseteq U$. So $N\lambda cl(U - \{x\}) \subseteq U$. Hence $U - \{x\}$ is nano $\wedge_{g^\star}$-closed set. Hence $\{x\}$ is nano $\wedge_{g^\star}$-open.

Theorem 4.4. *Let* H *be nano* $\wedge_{g^\star}$*-closed. Then* H *is nano* λ*-closed* $\Longleftrightarrow$ $N\lambda cl(H) - H$ *is nano closed.*

Proof. Necessity : Suppose H be nano $\wedge_{g^\star}$-closed and nano λ-closed. H is nano λ-closed implies $N\lambda cl(H) = H$. Hence $N\lambda cl(H) - H = \phi$ is nano closed.

Sufficiency : Suppose H is nano $\wedge_{g^\star}$-closed and $N\lambda cl(H) - H$ is nano closed. Then by Theorem 4.1. $N\lambda cl(H) - H$ contains no non empty nano closed. Hence we should have $N\lambda cl(H) - H = \phi$, which in turn implies $N\lambda cl(H) = H$. Therefore H is nano λ-closed.

Theorem 4.5. *If every nano $\wedge_{g^\star}$-closed is nano λ-closed then $\{x\}$ is nano g-closed or nano λ-open.*

Proof. Suppose $\{x\}$ is not a nano g-closed, then $U-\{x\}$ is not a nano g-open. Hence we have U is the only nano g-open containing $U-\{x\}$. Obviously $N\lambda cl(U-\{x\}) \subseteq U$. Therefore $U - \{x\}$ is nano $\wedge_{g^\star}$-closed. By hypothesis $U - \{x\}$ is nano λ-closed set. Hence $\{x\}$ is nano λ-open.

Theorem 4.6. *Let H is nano g-open set and nano $\wedge_{g^\star}$-closed. If K is nano λ-closed then $H \cap K$ is nano $\wedge_{g^\star}$-closed.*

Proof. By Theorem 3.12 if a set H is both nano g-open and nano $\wedge_{g^\star}$-closed then H is nano λ-closed. Hence if K is nano λ-closed then $H \cap K$ is nano λ-closed as the intersection of nano λ-closed sets is a nano λ-closed. Hence by Theorem 3.3 $H \cap K$ is nano $\wedge_{g^\star}$-closed set.

Theorem 4.7. *For a subset H of a space $(U, \tau_R(X))$, the following are equivalent:*

1. *every nano g-open set is nano λ-closed.*

2. *every subset is a nano $\wedge_{g^\star}$- closed.*

Proof. (1) $\Rightarrow$ (2). Let H be any subset of $(U, \tau_R(X))$ such that $H \subseteq G$ where G is nano g-open. Hence we get $N\lambda cl(H) \subseteq N\lambda cl(G)$. By hypothesis G is nano λ-closed set. Then we get $N\lambda cl(H) \subseteq N\lambda cl(G) = G$. Hence H is nano $\wedge_{g^\star}$-closed.

(1) $\Rightarrow$ (2). Let H be a nano g-open. By hypothesis H is nano $\wedge_{g^\star}$-closed. Then we have $N\lambda cl(H) \subseteq H$. Therefore H is nano λ-closed. Hence every nano g-open is nano λ-closed.

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Mathematical Model of Tuberculosis with Drug Resistance to the First and Second Line of Treatment

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Abstract - This study proposed a mathematical model of tuberculosis with drug resistance to a first and second line of treatment. The basic reproduction number for the model using next generation method is obtained. The equilibrium point of the model was investigated and also found the global stability of the disease free equilibrium and endemic equilibrium for the model. This study shows the effect of resistance rate of the first and second line of treatment to the infected and resistant population. If basic reproduction number is less than one, the disease free equilibrium is globally asymptotically stable and if basic reproduction number is greater than one, then the endemic equilibrium is a globally asymptotically stable.

Keywords *- Tuberculosis, Mycobacterium tuberculosis bacteria [Mtb], developed multi-drug resistant [MDR], Basic reproduction number, Stability.*

1. Introduction

Tuberculosis is an airborne disease caused by Mycobacterium tuberculosis bacteria (Mtb). Ullah et al. [8] discuss a general SIR epidemic model which represents the direct transmission of infectious disease. It is an ancient disease with evidence of its existence being found in relics from ancient Egypt, India and China [1]. Today, this disease ranks as the second leading cause of morbidity and mortality in the world from a single infectious agent, after the human immunodeficiency virus (HIV) according to Daniel. [10] Interestingly, about one third of the world's population is infected with Mycobacterium tuberculosis bacteria with approximately nine million people developing active tuberculosis and up to nearly two million people worldwide die from the disease every year. Approximately 480,000 people

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developed multidrug resistant (MDR) tuberculosis globally with 210,000 of those who developed MDR tuberculosis succumbing to it. In addition to posing a, major health concern to low and middle income countries, tuberculosis affect economic growth negatively. [3] Psycho-social distress that communities go through is enormous. This involves thinking about the loss of their loved ones and the economic impact of taking care of sick ones especially among the low income individuals. This impacts not only the individuals, but also the economic progress of the country. Zaman [7] gives, another category of people largely at risk of contracting tuberculosis are those who work closely or live close to a person with active tuberculosis and they could include health care workers, people living in crowded living spaces or confined places such as schools or prisons. According to Semenza et al. [5] over the last twenty five years, the mortality rate of tuberculosis has greatly decreased by 45% since and this is largely due to effective diagnosis and treatment. However, the world is still far from defeating the disease. About 8 billion US dollars per year is needed for a full response to the global tuberculosis epidemic in low and middle income countries by the year 2015 with a funding gap of 2 billion US dollars per year. This amount excluded resources required for research and development, which was estimated to be about 2 billion US dollars yearly. Clearly, this reveals that the current investment in tuberculosis falls below the low and middle income country's needs.

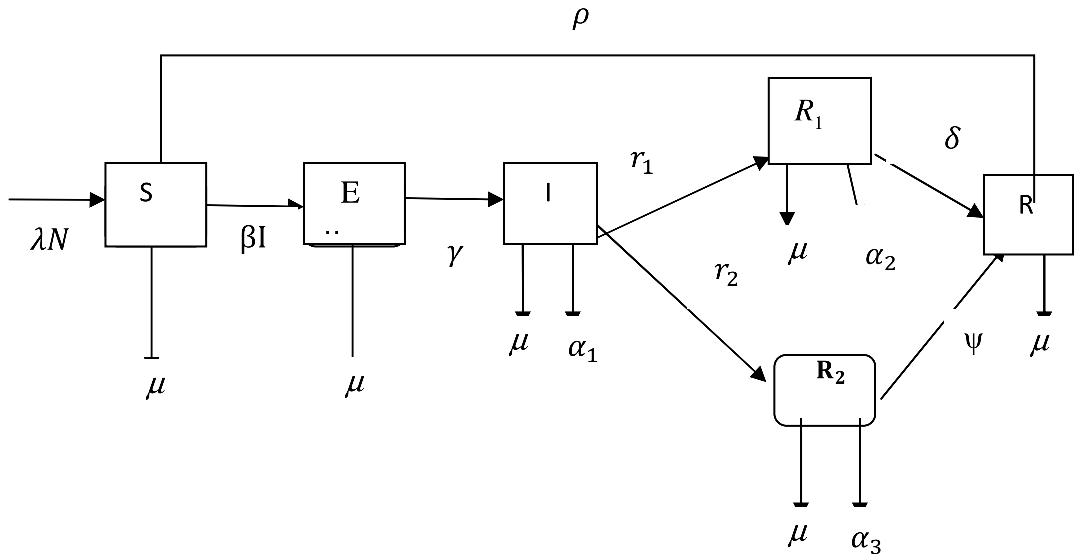
Tuberculosis is responsible for more deaths worldwide than any other infectious agent. Waaler and Anderson [4] developed a first tuberculosis model for the transmission dynamics of tuberculosis. The enormous progress in prevention and treatment, tuberculosis disease remains a leading cause of death worldwide and one of the major sources of concern is the drug resistant strain, MDR-TB (multidrug resistant tuberculosis) and XDR-TB (extensively drug resistant tuberculosis). Young et al. [2] studies, tuberculosis is curable provided an early diagnosis is made and one follows the proper treatment regimen which would take six months upto two years for the active tuberculosis to clear. Sharma et al. [9] given that the infected population is similar on the sociological and psychological effect rate. Cohen and Murray [11] modelled epidemics of multi-drug resistant tuberculosis of heterogeneous fitness.

2. Model Analysis

This study will first extend the standard SEIRS mathematical model for the transmission of tuberculosis which will demonstrate the transmission of the Mycobacterium tuberculosis in human hosts taking into account the multidrug resistant (MDR) tuberculosis.

2.1. The Model Equations

This study presents a simple model that can easily be analysed so as to properly understand the dynamics of this disease. Humans can contract MTB tuberculosis through contact with individuals who are infected with the disease after which they enter the exposed phase where a proportion of this class develop active tuberculosis thus moving into the infectious class. If treatment is administered promptly, those who recover from the disease will move to the recovered class and those who delay treatment and develop MDR tuberculosis will move to the resistant class. Those who recover from MDR tuberculosis will move to the recovered class. Given that there is no permanent immunity to tuberculosis, the recovered can lose their immunity and become susceptible again. Figure represent the flow of individuals into the different compartments and it has been constructed with these assumptions: recruitment is by birth only, a variable population, a constant mortality rate, no permanent immunity to tuberculosis, no immediate infectively.



The human population is categorized into six cl… …ich that at time $t \geq 0$ there are S, susceptible humans, E, exposed humans to tuberculosis, I, infected humans with active tuberculosis, R_1, resistant humans to the first line of treatment , R_2, resistant humans to the second line of treatment, R, recovered humans. Thus the size of the human population is given as $N = S + E + I + R_{ES} + R$. In our model, the recruitment into the susceptible human population is by birth λ. The size of the human population is further increased by the partial immune humans in R after they lose their immunity at the rate ρ. The size of human population is decreased by natural deaths (μ) and exposure to Mtb. The exposed susceptible to Mtb move to the exposed classes E with the force of infection being β resulting in an increase in the exposed class. The exposed class is further decreased by natural death(μ) and the proportion who move to the infected class I after developing active tuberculosis. The infected class I is also reduced by natural deaths (μ), disease induced death (α_1), those who recover (δ) and also by those resistance rate to the first and second line of treatment $r_1$and r_2 respectively. Thus the infected class (I), and the resistant classes ($R_1 and\ R_2$) gain partial immunity at the rates (δ) and (ψ) respectively thus moving to the recovered class R thus reducing their respective classes and also increasing the recovered class. The resistant classes R_1, R_2 also reduced by natural deaths (μ) and disease induced deaths while the recovered class is reduced by natural deaths (μ) and those who lose their partial immunity at the rate ρ.

Following Table (1) and (2) gives the description of variables and parameters

Table 1

Description of variables
$S(t)$ = Susceptible humans
$E(t)$ = exposed humans
$I(t)$ = infected humans
$R_1(t)$ = resistant to the first line of treatment
$R_2(t)$ = resistant to the second line of treatment
R(t) = Recovered humans

Table 2

Description of Parameters
$\beta =$ Rate at which the susceptible become exposed to Mtb
$\gamma =$ Infection rate
$\alpha_1 =$ Disease induced death rate
$\mu =$ Rate of natural death
$r_1 =$ Resistance rate of first line treatment
$r_2 =$ Resistance rate of second line treatment
$\delta =$ Recovery after first line of treatment
$\psi =$ Recovery after second line of treatment
$\rho =$ Rate at which recovered loss their immunity
$\alpha_2, \alpha_3 =$ Disease induced death rate after first and second resistance respectively

2.2. Differential Equations

From the above discussion, we get the following system of ordinary differential equations

$$\left.\begin{aligned} \frac{dS}{dt} &= \lambda N - \mu S - \beta SI + \rho R, \\ \frac{dE}{dt} &= \beta SI - (\mu + \gamma)E, \\ \frac{dI}{dt} &= \gamma E - (\mu + \alpha_1 + r_1 + r_2)I, \\ \frac{dR_1}{dt} &= r_1 I - (\mu + \alpha_2 + \delta)R_1, \\ \frac{dR_2}{dt} &= r_2 I - (\mu + \alpha_3 + \psi)R_2, \\ \frac{dR}{dt} &= \delta R_1 + \pi R_2 - (\mu + \rho)R \end{aligned}\right\} \quad (1)$$

2.3. Equilibrium Points

To obtain the equilibrium points for the system of differential equation (1) by equating each of the equations to 0 as shown in below

$$\left.\begin{aligned} \frac{dS}{dt} &= \lambda N - \mu S - \beta SI + \rho R = 0, \\ \frac{dE}{dt} &= \beta SI - (\mu + \gamma)E = 0, \\ \frac{dI}{dt} &= \gamma E - (\mu + \alpha_1 + r_1 + r_2)I = 0, \\ \frac{dR_1}{dt} &= r_1 I - (\mu + \alpha_2 + \delta)R_1 = 0, \\ \frac{dR_2}{dt} &= r_2 I - (\mu + \alpha_3 + \psi)R_2 = 0, \\ \frac{dR}{dt} &= \delta R_1 + \psi R_2 - (\mu + \rho)R = 0, \end{aligned}\right\} \quad (2)$$

Solving system (2), to get two equilibrium points, one being the diseases free equilibrium while the other being the endemic equilibrium. Disease free equilibrium Point $(S, E, I, R_1, R_{2,} R)$ is expressed as follows: $X_0 = (S, E, I, R_1, R_2, R) = \left(\frac{\lambda N}{\mu}, 0,0,0,0,0\right)$ and endemic equilibrium point $(S^*, E^*, I^*, R_1^*, R_2^*\ R^*)$ is expressed as follows:

$$S^* = \frac{(\mu+\gamma)(\mu+\alpha_1+r_1+r_2)}{\beta\gamma}, \qquad E^* = \frac{\beta x(\mu+\rho)(\lambda N-\mu x)}{(\mu+\gamma)(\beta x(\mu+\rho)-p)},$$

$$I^* = \frac{(\mu+\rho)(\lambda N-\mu x)}{\beta x(\mu+\rho)-p}, \qquad R_1^* = \frac{r_1(\mu+\rho)(\lambda N-\mu x)}{(\mu+\alpha_2+\delta)(\beta x(\mu+\rho)-p)} \tag{3}$$
$$R_2^* = \frac{r_2(\mu+\rho)(\lambda N-\mu x)}{(\mu+\alpha_3+\psi)(\beta x(\mu+\rho)-p)}, \qquad R^* = \frac{(\lambda N-\mu x)p}{(\beta x(\mu+\rho)-p)\rho}$$

where $x = S^*$ and $p = \rho\left(\frac{\delta r_1}{\mu+\alpha_2+\delta} + \frac{\psi r_2}{\mu+\alpha_3+\psi}\right)$.

2.4. Condition of Existence/Positivity of Equilibrium

The system will remain positive provided this condition holds:

$$\frac{\lambda N - \mu x}{\beta x(\mu+\rho) - p} > 0$$
$$\Leftrightarrow \lambda N - \mu x > 0$$
$$\Leftrightarrow \lambda N > \mu x$$

Substituting for x

$$\Leftrightarrow \lambda N > \mu \frac{(\mu+\gamma)(\mu+\alpha_1+r_1+r_2)}{\beta\gamma}$$

$$\Leftrightarrow \lambda N\beta\gamma > \mu(\mu+\gamma)(\mu+\alpha_1+r_1+r_2)$$

$$\frac{\lambda N\beta\gamma}{\mu(\mu+\gamma)(\mu+\alpha_1+r_1+r_2)} > 1$$

This expression is the condition of existence.

2.5. The Basic Reproduction Number R_0

Let us look at the following system of differential equations.

$$\frac{dE}{dt} = \beta SI - (\mu+\gamma)E,$$

$$\frac{dI}{dt} = \gamma E - (\mu+\alpha_1+r_1+r_2)I,$$

$$\frac{dR_1}{dt} = r_1 I - (\mu+\alpha_2+\delta)R_1,$$

$$\frac{dR_2}{dt} = r_2 I - (\mu+\alpha_3+\psi)R_2,$$

Let $X = (E, I, R_1, R_2)^T$ then above system can be represented in matrix form as shown below:

$$\frac{dX}{dt} = F(X) - V(X)$$

where

$$F(X)=\begin{pmatrix} \beta SI \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad V(X)=\begin{pmatrix} -\gamma E+(\mu+\alpha_1+r_1+r_2)I \\ (\mu+\gamma)E \\ -r_1+(\mu+\alpha_2+\delta)R_1 \\ r_2-(\mu+\alpha_3+\pi)R_2 \end{pmatrix}$$

The Jacobian matrix of $F(X)$ and $V(X)$ at the disease free equilibrium X_0 are,

$$DF(X_o)=\begin{pmatrix} F_1 & 0 \\ 0 & 0 \end{pmatrix}, \; DV(X_o)=\begin{pmatrix} V_1 & 0 \\ 0 & 0 \end{pmatrix} \text{ respectively,}$$

where

$$F_1=\begin{pmatrix} 0 & \frac{\beta\lambda N}{\mu} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and

$$V_1=\begin{pmatrix} \mu+\gamma & 0 & 0 & 0 \\ -\gamma & \mu+\alpha_1+r_1+r_2 & 0 & 0 \\ 0 & -r_1 & \mu+\alpha_2+\delta & 0 \\ 0 & r_2 & 0 & -(\mu+\alpha_3+\pi) \end{pmatrix}.$$

Now

$$V_1^{-1}=\begin{pmatrix} \frac{1}{\mu+\gamma} & 0 & 0 & 0 \\ \frac{\gamma}{(\mu+\gamma)(\mu+\alpha_1+r_1+r_2)} & \frac{1}{(\mu+\alpha_1+r_1+r_2)} & 0 & 0 \\ \frac{\gamma r_1}{(\mu+\gamma)(\mu+\alpha_1+r_1+r_2)(\mu+\alpha_2+\delta)} & \frac{r_1}{(\mu+\alpha_1+r_1+r_2)(\mu+\alpha_2+\delta)} & \frac{1}{\mu+\alpha_2+\delta} & 0 \\ \frac{\gamma r_2}{(\mu+\gamma)(\mu+\alpha_1+r_1+r_2)(\mu+\alpha_3+\pi)} & \frac{r_2}{(\mu+\alpha_1+r_1+r_2)(\mu+\alpha_3+\pi)} & 0 & -\frac{1}{\mu+\alpha_3+\pi} \end{pmatrix}.$$

The next generation matrix of the system is given by

$$F_1V_1^{-1}=\begin{pmatrix} \frac{\beta\gamma\lambda N}{\mu(\mu+\gamma)(\mu+\alpha_1+r_1+r_2)} & \frac{\beta\lambda N}{\mu(\mu+\alpha_1+r_1+r_2)} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Now, to obtain the spectral radius of $F_1V_1^{-1}$, which is defined as the largest eigen value of $F_1V_1^{-1}$ and the spectral radius for the above system is the basic reproduction number and its expression is given by

$$R_0 = \frac{\beta\gamma\lambda N}{\mu(\mu+\gamma)(\mu+\alpha_1+r_1+r_2)}$$

2.6. Stability Analysis

In this section this study will determine the stability of the diseases free equilibrium point. This study can easily establish the local stability of the equilibriums by Routh Hurwitz criteria, so left it. This study will discuss only on the global stability of the disease free and endemic equilibrium.

Global Stability of the Disease Free Equilibrium

The local dynamics of a general SEIRS model is determined by the reproduction number R_0. If $R_0 \leq 1$, then each infected individual in its entire period of infectiousness will produce less than one infected individual on average. This means that the disease will be wiped out of the population. If $R_0 > 1$, then each infected individual in its entire infectious period having contact with susceptible individuals will produce more than one infected individual implying that the disease persist in the population. If $R_0 = 1$ and this is defined as the disease threshold, then one individual infects one more individual. For $R_0 \leq 1$, the disease free equilibrium is locally asymptotically stable while for $R_0 > 1$ the disease free equilibrium becomes unstable. The disease free equilibrium point is $(S, E, I, R_1, R_2, R) = \left(\frac{\lambda N}{\mu}, 0,0,0,0\right)$.

Theorem 1. If $R_0 \leq 1$, then the disease free equilibrium is of the system $(S, E, I, R_1, R_2, R) = \left(\frac{\lambda N}{\mu}, 0,0,0,0,0\right)$ of the system is globally asymptotically stable on Ω.

Proof. Construct the following Lasalle-Lyapunov function $V(S, E, I, R_1, R_2, R)$ on the positively invariant compact set Ω.

Define

$$V(S, E, I, R_1, R_2, R) = \gamma E + (\mu+\gamma)I. \tag{4}$$

Differentiate (4) and using the second and third equations of the system (1), we get

$$\frac{dV}{dt} = \gamma\frac{dE}{dt} + (\mu+\gamma)\frac{dI}{dt}$$

$$\frac{dV}{dt} = [\beta\gamma S - (\mu+\gamma)(\mu+\alpha_1+r_1+r_2)]I.$$

$$\frac{dV}{dt} = (\mu+\gamma)(\mu+\alpha_1+r_1+r_2)(R_o-1)I,$$

which is strictly decreasing when $R_0 < 1$.

Define the set Define the set $E = \{(E, I, R_1, R_2) \in \Omega \,/\, (E, I, R_1, R_2 = 0)$.The largest invariant set is contained in the set E for which $E = 0$ or $I = 0$ or $R_1 = 0, R_2 = 0$ Thus by Lasalle invariant principal the disease free equilibrium is globally asymptotically stable on Ω.

Global Stability of The Endemic Equilibrium Theorem 2. The endemic equilibrium $\emptyset = (E^*, I^*, {R_1}^*, {R_2}^*)$ given by equation (3) is globally asymptotically stable on Ω.

Proof. To establish the global stability of the endemic equilibrium $\emptyset$, so construct the Lyapunov function $V_1: \Omega \to R$ where $\Omega = \{(E(t), I(t), R_1(t), R_2(t)/E(t) > 0, I(t) > 0, R_1 > 0, R_2 > 0\}$ as described by Ullah, Zaman and Islam[10] and it is given as

$$V_1(E, I, R_1, R_2) = L_1\left[E - E^*\ln\left(\frac{E}{E^*}\right)\right] + L_2\left[I - I^*\ln\left(\frac{I}{I^*}\right)\right] + L_3\left[R_1 - {R_1}^*\ln\left(\frac{R_1}{{R^*}_1}\right)\right] + L_4[R_2 - {R^*}_2\ln\left(\frac{R_2}{{R^*}_2}\right)] \quad (5)$$

Where $L_1, L_2, L_{3,} L_4$ are positive constant to be later considered.

Taking the derivative of the Lyapunov function V_1as given in equation (5) yields

$$\frac{dV_1}{dt} = L_1\left[E - E^*\left(\frac{\beta SI}{E} - (\mu+\gamma)\right)\right] + L_2\left[I - I^*\left(\frac{\gamma E}{I} - (\mu+\alpha_1+r_1+r_2)\right)\right] + L_3\left[R_1 - {R_1}^*\left(\frac{r_1 I}{R_1} - (\mu+\alpha_2+\delta)\right)\right] + L_4[R_2 - {R_2}^*(\frac{r_2 I}{R_2} - (\mu+\alpha_3+\psi))] \quad (6)$$

Choosing $L_1 = L_2 = L_3 = L_4 = 1$, equation (6) becomes

$$= (E-E^*)(\mu+\gamma)(W_1R_0-1) + (I-I^*)(\mu+\alpha_1+r_1+r_2)(W_2R_0-1) + r1(R_1-{R^*}_1)\left(\frac{{R^*}_1 I - R_1 I^*}{R_1{R^*}_1}\right) + r_2(R_2-{R^*}_2)(\frac{{R^*}_2 I - R_2 I^*}{R_2{R^*}_2})$$

Thus $\frac{dV_1}{dt} \le 0\ iff\ R_0 < 1\ and\ {R^*}_1 I < R_1 I^*$ and ${R^*}_2 I < R_2 I^*$

To have that $\frac{dV_1}{dt} = 0\ iff\ E = E^*, I = I^*$

$$R_1 < R^*{}_1$$

$$R_1 = R^*{}_1$$

$$R_2 = R^*{}_2$$

Or when $R_0 = 1$ *and* $R^*{}_1 I = R_1 I^*$

$$R^*{}_2 I = R_2 I^*$$

Define the set $\emptyset = \{E^*, I^*, R^*{}_1, R^*{}_2\} \in \Omega / \frac{dV_1}{dt} = 0\}$

Therefore the largest compact invariant set is singletone set Φ which is the endemic equilibrium. By Lasalle invariant principle Φ is globally asymptotically stable on Ω.

3. Numerical Simulation

Explain this result through graphically. Consider the parameters as: $\lambda = 0.001, N = 1{,}000, \beta = 0.398, \gamma = 1, r_1 = 0.4, r_2 = 0.5, \mu = 0.7,\ \alpha_1 = 0.8, \alpha_2 = 0.4, \alpha_3 = 0.3,$ $\delta = 1, \pi = 1.2, \rho = 0.4.$ Then this study give $R_0 = 0.1395 < 1$ and if the initial values of susceptible, exposed, infected, resistant of first and second line treatment population are 1, 2, 1, 1, 1 and 1 respectively. The susceptible population goes to its steady state value while exposed, infected, resistant of first and second line treatment population approach to zero as time increase as shown in Figure 1. So that the disease free equilibrium is globally asymptotically stable.

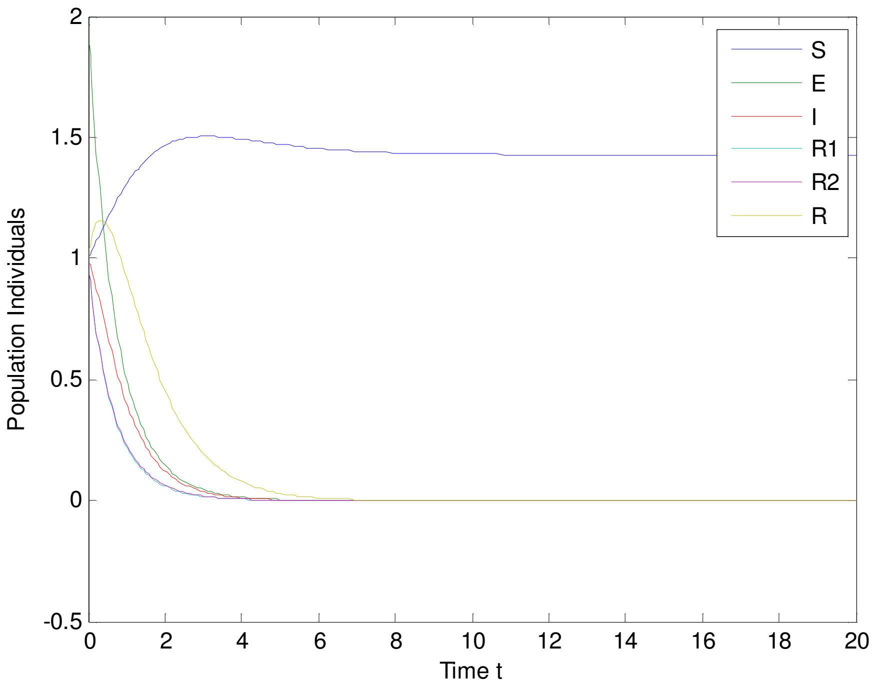


Figure 1. When $R_0 = 0.1395 < 1$.

Again if, we take the parameters of the system as: $\lambda = 0.015, N = 1,000, \beta = 0.398, \gamma = 1,$ $r_1 = 0.4,\ r_2 = 0.5, \mu = 0.7, \alpha_1 = 0.8, \alpha_2 = 0.4, \alpha_3 = 0.3,\ \delta = 1, \pi = 1.2, \rho = 0.4.$ Then $E^*(S^*, E^*, I^*, R_1^*, R_2^*, R^*) = (10.25, 4.8, 2, .38, .45, .84)$ and $R_0 = 2.091 > 1$. If the initial values of susceptible, exposed, infected, resistant of first and second line treatment population are 1, 2, 1, 1, 1 and 1 respectively. Therefore by theorem (2), the endemic equilibrium is a global asymptotically stable as shown in Figure 2.

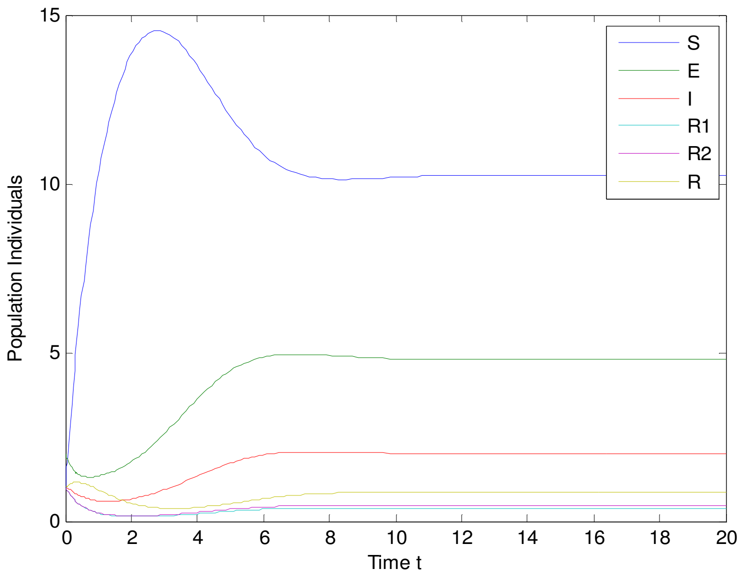


Figure 2. When $R_0 = 2.091 > 1$.

Let us take all the parameters are fixed except the resistance rate of the first and second line of treatments, found that the infected population decreases as the resistance rate of the first and second line of treatment increases which is shown in figure 3(a) and (b). Therefore infected population moves to resistant population of the first line of treatment and to the resistant population of the second line of treatment, as resistant rate increases respectively.

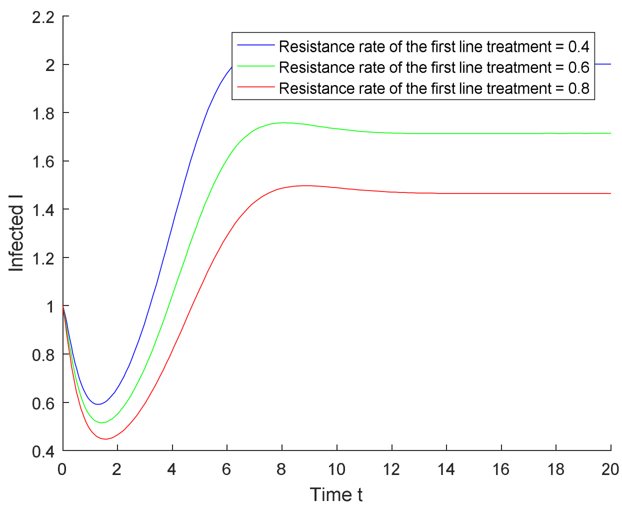


Figure.3(a) Changes in the infected population with respect to resistance rate of the first line treatment, keeping all other parameters are fixed.

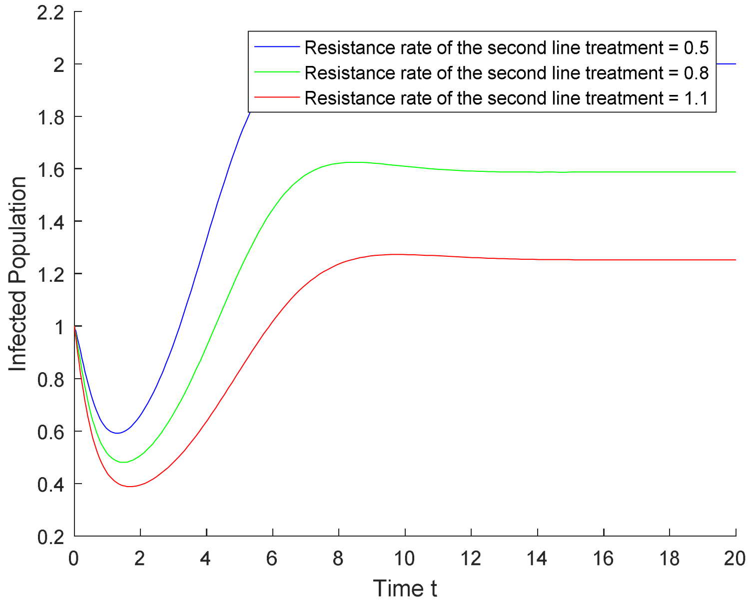


Figure.3(b) Changes in the infected population with respect to resistance rate of the second line treatment, keeping all other parameters are fixed.

Similarly again we take all parameters are fixed except the resistance rate of the first line and the second line of treatment, found that the resistant population of the first line treatment decreases when resistance rate of the first line treatment increases i.e. resistant population □*1* moves to recovered population while the resistant population of the second line treatment increases when the resistance rate of the second line of treatment increases i.e. after the second line treatment, the infected population comes into resistant population which shown in figure 4(a) and 4(b) respectively.

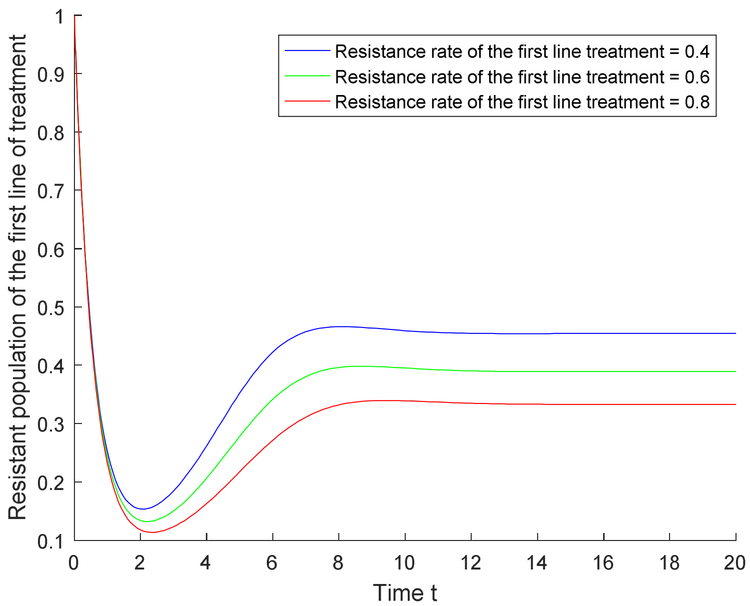


Figure. 4(a) Changes in the resistant population with respect to resistance rate of the first line of treatment, keeping all the other parameters are fixed.

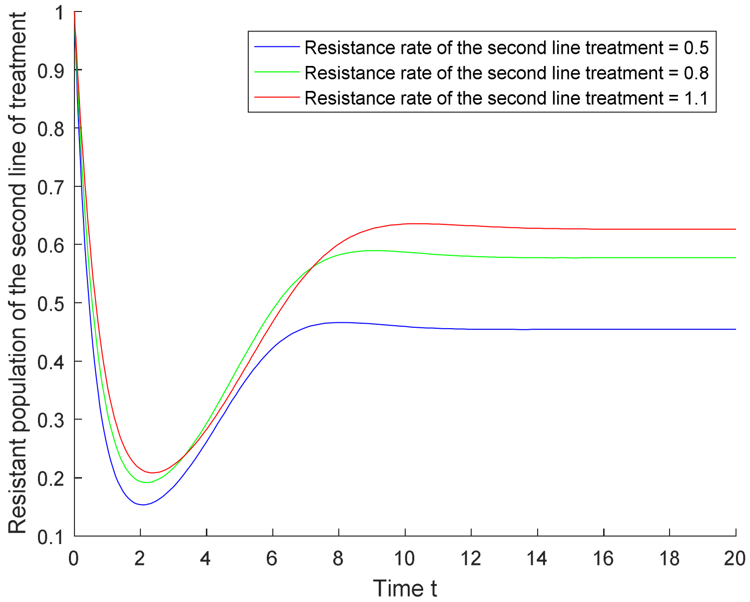


Figure. 4(b) Changes in the resistant population with respect to resistance rate of the second line of treatment, keeping all the other parameters are fixed.

4. Conclusion

This study analyzed the local and global stability of the equilibrium points, found that when the basic reproduction number $R_o < 1$, then disease dies out and when the basic reproduction number $R_o > 1$, then disease persists.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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