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Original Article

Stability of Waste Paper Recycling through Graph Theory

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Abstract - The process in which waste paper is collected and then reprocessed for reuse is called paper recycling. Paper recycling is very often routine of human life. Paper recycling is very important to reduce deforestation and pollution. It has been analyzed that scrap dealer plays a vital role in this cycle that collects waste paper from distributor and customer and then sent it to the paper industry, so the effect of scrap dealers is observed here. This waste paper recycling model is divided into four compartments namely paper industry, distributor, customer and scrap dealer. The model is proposed as a system of non-linear differential equation. The basic reproduction number is computed to see the impact of scrap dealer.

Keywords *- Dynamical model, System of non-linear ordinary differential equation, Basic reproduction number, Local stability, Global stability, Weighted graphs, Paper.*

1 Introduction

Recycling is one of the best ways for us to have an optimistic impact on the world in which we want to live healthily. Paper is one of the best materials that we can recycle easily. Recycled paper made from paper and paper products that has already been used. The paper recycling starts with us. It can be learnt at schools, colleges, home, offices, and local communities and even at drop off centers. Recycling of paper helps us in many ways. As the pulp of tree is an only source for producing paper, the recycled fibres from waste paper provides a better alternative. Creating recycled paper pulp, compare to manufacturing pulp from trees to make paper products, and devours less energy and water. By recycling one ton of paper, we save 17 trees, 7000 gallons of water, 463 gallons of oil and more than 3.3 cubic yards of landfill space.

In the case of recycling, many researchers have studied the process of recycling wastepaper. Kleineidam *et al.* [14] obtained optimizing product recycling chains by control theory in 2000. Clement and Marie [4] in 1988 considered method for producing pulp from printed unselected waste paper. In 1994, Nadeau and Allan [18] have studied integrated

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waste paper treatment process. In 1996, production of soft paper products from old newspaper was deliberated by Back and Sangho [1]. Denem and Dennis [5] learned waste paper minimizing paper dispenser in 2004. Kinney and Roland [15] in 1994 wrote book on technology of paper recycling. Bleakley *et al.* [2] in 2000 examined waste paper treatment process. Some of the researchers have analyzed the impact of recycling waste paper on the environment. In 1997, Bystrom *et al.* [3] observed paper recycling: environmental and economic impact. Miranda [17] *et al.* investigated environmental awareness and paper recycling in 2010. Environment impacts of waste paper recycling were deliberated by Virtanen *et al.* [23] in 2013. Different model for the awareness of recycling for waste paper was developed by many researchers. Kara *et al.* [13] studied a stochastic optimization approach for paper recycling reverse logistics network design under uncertainty in 2010. Merrild *et al.* [16] generated life cycle assessment of waste paper management: the importance of technology data and system boundaries in assessing recycling and incineration in 2008. A different model 'A goal programming model for paper recycling system' was formulated in 2008 by Pati *et al.* [19]. Some environment related models like forest model [22] and green belt model [21] was also developed by some researchers to revive the natural resources.

Mathematical modeling of paper recycling is developed in Section 2. Using weighted graph, the stability analysis is carried out in Section 3. In Section 4, sensitivity analysis is analyzed. Numerical analysis is calculated in Section 5 and validated data is given.

2 Mathematical Modeling

We live in the society where deforestation and pollution has taken place. The process of recycled paper from paper industries can reduce it. Paper industries (P) are those which manufacture the paper and even recycle the waste paper. Distributors (D) are those who collect produced paper sold by paper industry. Customer (C) are those who buy paper from distributors. Scrap dealers are those who gather water paper from distributor and customer and give it to the paper industry. Thus, it becomes a cycle. And this process of recycling of paper helps us to save energy, water and resources. Here, scrap dealer works as a control for waste paper recycling model.

Parameters and their notations along with parametric values used to formulate waste paper recycling model are as given in the Table 1.

Table 1. Notation and parametric values

Notation		Parametric value
B	Recruitment rate from wood	0.7
δ	The rate at which distributor buys paper	0.9
η	The rate at which customer buys paper	0.75
α	The rate at which distributor gives paper to scrap dealers	0.02
β	The rate at which customer gives paper to scrap dealers	0.7
γ	The rate at which scrap dealer transports waste paper to paper industry	0.6
μ	Natural waste of paper from each compartment	0.28

To formulate a mathematical model of waste water recycling we have used above notation and necessary assumptions whose transmission diagram is as given in Figure 1.

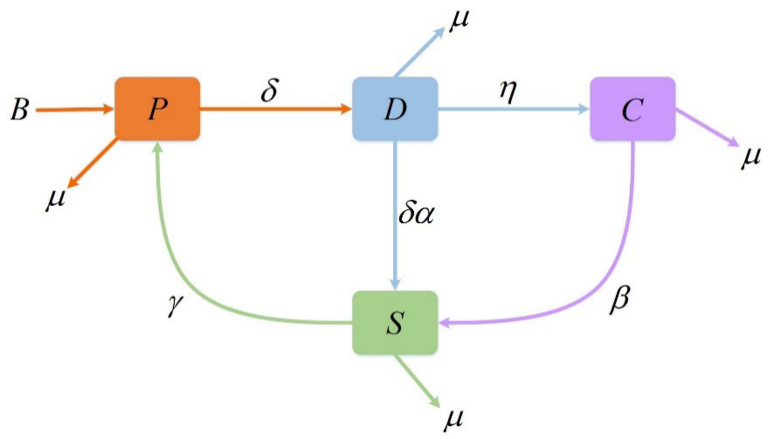

Figure 1. Transmission diagram of waste paper recycling model

The system of non-linear differential equation of transmission of waste paper is as given below:

$$\begin{aligned}
\frac{dP}{dt} &= B - \delta PD + \gamma S - \mu P \\
\frac{dD}{dt} &= \delta PD - \eta D - \delta\alpha DS - \mu D \\
\frac{dC}{dt} &= \eta D - \beta CS - \mu C \\
\frac{dS}{dt} &= \delta\alpha DS + \beta CS - \gamma S - \mu S
\end{aligned} \tag{1}$$

where $P + D + C + S = N$. Also, $P > 0; D, C, S \geq 0$.

Adding above system of differential equations, we get

$$\begin{aligned}
\frac{d}{dt}(P + D + C + S) &= B - \delta PD + \gamma S - \mu P + \delta PD - \eta D - \delta\alpha DS - \mu D + \eta D - \beta CS - \mu C \\
&\quad + \delta\alpha DS + \beta CS - \gamma S - \mu S \\
&= B - \mu(P + D + C + S) \geq 0
\end{aligned}$$

which implies that $\limsup_{t\to\infty}(P + D + C + S) \leq \frac{B}{\mu}$.

Therefore, the feasible region of the waste paper recycling model is

$$\Lambda = \left\{ (P + D + C + S) \Big/ P + D + C + S \le \frac{B}{\mu}, P > 0; D, C, S \ge 0 \right\}.$$

Now, solving system of differential equations, we get four equilibrium points:

1) $E_P^* = \left(\frac{B}{\mu}, 0, 0, 0 \right)$

2) $E_{PCS}^* = \left(\frac{-\mu\gamma + B\beta}{\mu\beta}, 0, \frac{\gamma + \mu}{\beta}, \frac{-\mu}{\beta} \right)$

3) $E_{PDC}^* = \left(\frac{\eta + \mu}{\delta}, \frac{B\delta - \mu(\eta + \mu)}{\delta(\eta + \mu)}, \frac{\eta(B\delta - \mu(\eta + \mu))}{\delta\mu(\eta + \mu)}, 0 \right)$

4) $E_{PDCS}^* = \left(\frac{-\delta\alpha Y\mu + \gamma\eta + \mu\eta + \mu\beta Y}{\delta Y\beta}, \frac{-\beta Y + \gamma + \mu}{\delta\alpha}, X, \frac{-\delta\alpha Y\mu - \beta Y\eta + \gamma\eta + \mu\eta}{\delta\alpha Y\beta} \right)$

where

$$\begin{aligned} Y = RootOf\,(&\left(-\beta^2\mu + \beta\delta\alpha\mu\right)x^2 + \left(\mu\beta\gamma - \mu^2\alpha^2\delta - \mu\eta\beta + \mu^2\beta - \beta\delta\alpha B - \delta\alpha\mu^2 + \mu^2\alpha\beta\right)x \\ &+ \mu\alpha\gamma\eta + \mu^2\eta + \gamma\eta\mu + \mu^2\alpha\eta) \end{aligned} \tag{2}$$

Next, we compute basic reproduction number R_0 for each equilibrium point E^*, using next generation matrix method.

Let us consider $X' = (P, D, C, S)'$, where derivative is denoted by dash. So,

$$X' = \frac{dX}{dt} = F(X) - V(X)$$

where $F(X)$ is the rate of presence of new individual in compartment and $V(X)$ is the rate of transfer of culture. They are given by

$$F = \begin{bmatrix} \delta PD \\ \delta\alpha DS + \beta CS \\ 0 \\ 0 \end{bmatrix} \text{ and } V = \begin{bmatrix} \eta D + \delta\alpha DS + \mu D \\ \gamma S + \mu S \\ -B + \delta PD - \gamma S + \mu P \\ -\eta D + \beta CS + \mu C \end{bmatrix}$$

Now, $DF(E^*) = \begin{bmatrix} f & 0 \\ 0 & 0 \end{bmatrix}$ and $DV(E^*) = \begin{bmatrix} v & 0 \\ J_1 & J_2 \end{bmatrix}$

where f and v are 4×4 matrices defined as

$$f=\left[\frac{\partial F_i\left(E^*\right)}{\partial X_j}\right] \text{ and } v=\left[\frac{\partial V_i\left(E^*\right)}{\partial X_j}\right].$$

1) Finding f and v for the equilibrium $E_P^*\left(\frac{B}{\mu},0,0,0\right)$, we get

$$f_P=\begin{bmatrix}\frac{\delta B}{\mu} & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix} \text{ and } v_P=\begin{bmatrix}\eta+\mu & 0 & 0 & 0\\ 0 & \gamma+\mu & 0 & 0\\ \frac{\delta B}{\mu} & -\gamma & 0 & 0\\ -\eta & 0 & 0 & \mu\end{bmatrix}$$

Here, v_P is non-singular.

Therefore, the expression of basic reproduction number R_{0_P} is as below:

$R_{0_P} =$ spectral radius of $f_P {v_P}^{-1}$

$$\Rightarrow R_{0_P}=\frac{\delta B}{\mu(\eta+\mu)}$$

After putting parametric values given in the Table 1, we get $R_{0_{PCS}}=2.1840$ which is grater that 1 that makes model unstable.

2) Finding f and v for the equilibrium $E_{PCS}^*=\left(\frac{-\mu\gamma+B\beta}{\mu\beta},0,\frac{\gamma+\mu}{\beta},\frac{-\mu}{\beta}\right)$, we get

$$f_{PCS}=\begin{bmatrix}\frac{\delta(-\mu\gamma+B\beta)}{\mu\beta} & 0 & 0 & 0\\ -\frac{\delta\alpha\mu}{\beta} & \frac{\beta(\gamma+\mu)}{\mu} & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix} \text{ and } v_{PCS}=\begin{bmatrix}\eta-\frac{\delta\alpha\mu}{\beta}+\mu & 0 & 0 & 0\\ 0 & \gamma+\mu & 0 & 0\\ \frac{\delta(-\mu\gamma+B\beta)}{\mu\beta} & -\gamma & \mu & 0\\ -\eta & 0 & 0 & 0\end{bmatrix}$$

Here, v_{PCS} is non-singular.

Therefore, the expression of basic reproduction number $R_{0_{PCS}}$ is as below:

$R_{0_{PCS}} =$ spectral radius of $f_{PCS} {v_{PCS}}^{-1}$

$$\Rightarrow R_{0_{PCS}}=\frac{\delta(-\mu\gamma+B\beta)}{\mu(\eta\beta-\delta\alpha\mu+\mu\beta)}$$

After putting parametric values given in the Table 1, we get $R_{0_{PCS}} = 2.5000$ which is grater that 1 that makes model unstable.

3) Finding f and v for the equilibrium

$$E^{*}_{PDC} = \left(\frac{\eta+\mu}{\delta}, \frac{B\delta-\mu(\eta+\mu)}{\delta(\eta+\mu)}, \frac{\eta\left(B\delta-\mu(\eta+\mu)\right)}{\delta\mu(\eta+\mu)}, 0 \right)$$

we get

$$f_{PDC} = \begin{bmatrix} \eta+\mu & 0 & \frac{B\delta-\mu\eta-\mu^2}{\eta+\mu} & 0 \\ 0 & \frac{\beta\left(B\delta-\mu\eta-\mu^2\right)}{\eta+\mu} + \frac{\beta\eta\left(B\delta-\mu\eta-\mu^2\right)}{\delta\mu(\eta+\mu)} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and

$$v_{PDC} = \begin{bmatrix} \eta+\mu & \frac{\alpha\left(B\delta-\mu\eta-\mu^2\right)}{\eta+\mu} & 0 & 0 \\ 0 & \gamma+\mu & 0 & 0 \\ \eta+\mu & -\gamma & \frac{B\delta-\mu\eta-\mu^2}{\eta+\mu}+\mu & 0 \\ -\eta & \frac{\beta\eta\left(B\delta-\mu\eta-\mu^2\right)}{\delta\mu(\eta+\mu)} & 0 & \mu \end{bmatrix}$$

Here, v_{PDC} is non-singular.

Therefore, the expression of basic reproduction number $R_{0_{PDC}}$ is as below:

$$R_{0_{PDC}} = \text{ spectral radius of } f_{PDC}{v_{PDC}}^{-1}$$
$$\Rightarrow R_{0_{PDC}} = \frac{\left(B\delta-\mu(\eta+\mu)\right)(\alpha\delta\mu+\beta\eta)}{\delta\mu(\eta+\mu)(\gamma+\mu)}$$

After putting parametric values given in the Table 1, we get $R_{0_{PDC}} = 0.6456$.

4) Finding f and v for the equilibrium

$$E^*_{PDCS} = \left(\frac{-\delta\alpha Y\mu + \gamma\eta + \mu\eta + \mu\beta Y}{\delta Y\beta}, \frac{-\beta Y + \gamma + \mu}{\delta\alpha}, Y, \frac{-\delta\alpha Y\mu - \beta Y\eta + \gamma\eta + \mu\eta}{\delta\alpha Y\beta} \right)$$

where Y is as equation (2) and taking

$$\frac{-\delta\alpha Y\mu + \gamma\eta + \mu\eta + \mu\beta Y}{\delta Y\beta} = P^*, \frac{-\beta Y + \gamma + \mu}{\delta\alpha} = D^*, Y = C^*$$

and

$$\frac{-\delta\alpha Y\mu - \beta Y\eta + \gamma\eta + \mu\eta}{\delta\alpha Y\beta} = S^*$$

we get

$$f_{PDCS} = \begin{bmatrix} \delta P^* & 0 & \delta D^* & 0 \\ \delta\alpha S^* & \delta\alpha D^* + \beta C^* & 0 & \beta S^* \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and

$$v_{PDCS} = \begin{bmatrix} \eta + \delta\alpha S^* + \mu & \delta\alpha D^* & 0 & 0 \\ 0 & \gamma + \mu & 0 & 0 \\ \delta P^* & -\gamma & \delta D^* + \mu & 0 \\ -\eta & \beta C^* & 0 & \beta S^* + \mu \end{bmatrix}$$

Here, v_{PDCS} is non-singular.

Therefore, the expression of basic reproduction number $R_{0_{PDCS}}$ is as below:

$$R_{0_{PDCS}} = \text{ spectral radius of } f_{PDCS}{v_{PDCS}}^{-1}$$

$$\Rightarrow R_{0_{PDCS}} = \frac{\delta^2\alpha^2 S^* D^*}{(\gamma+\mu)(\eta + \delta\alpha S^* + \mu)} + \frac{\delta\alpha D^* + \beta C^*}{\gamma + \mu} - \frac{\beta S^*\left(\beta\eta C^* + \beta\delta\alpha C^* S^* + \beta\mu C^* + \eta\delta\alpha D^*\right)}{(\eta + \delta\alpha S^* + \mu)(\gamma+\mu)(\beta S^* + \mu)}$$

After putting parametric values given in the Table 1, we get $R_{0_{PDCS}} = 0.7346$.

Above four basic reproduction numbers for distinct equilibrium point of waste paper recycling model shows that equilibrium point E^*_{PDCS} is more appropriate than E^*_{PDC} which indicate that scrap dealers are the most significant factor for the process of waste paper recycling. Therefore, in the further section, we will discuss stability only at two equilibrium point E^*_{PDC} and E^*_{PDCS}.

3 Equilibrium

In the current section, we will establish local stability and global stability for the waste water recycling model for two equilibrium points.

3.1 Local Stability

The local stability for waste paper recycling model will be discovered here.

First, we will begin with the equilibrium point

$$E^*_{PDC} = \left(\frac{\eta+\mu}{\delta}, \frac{B\delta-\mu(\eta+\mu)}{\delta(\eta+\mu)}, \frac{\eta\left(B\delta-\mu(\eta+\mu)\right)}{\delta\mu(\eta+\mu)}, 0 \right)$$

using Jacobian matrix J_{PDC}. The Jacobian matrix J_{PDC} of the waste paper recycling model is as follows:

$$J_{PDC} = \begin{bmatrix} -\frac{B\delta-\mu\eta-\mu^2}{\eta+\mu}-\mu & -\eta-\mu & 0 & \gamma \\ \frac{B\delta-\mu\eta-\mu^2}{\eta+\mu} & 0 & 0 & -\frac{\alpha\left(B\delta-\mu\eta-\mu^2\right)}{\eta+\mu} \\ 0 & \eta & -\mu & -\frac{B\eta\left(B\delta-\mu\eta-\mu^2\right)}{\delta\mu(\eta+\mu)} \\ 0 & 0 & 0 & \frac{\alpha\left(B\delta-\mu\eta-\mu^2\right)}{\eta+\mu}+\frac{B\eta\left(B\delta-\mu\eta-\mu^2\right)}{\delta\mu(\eta+\mu)}-\gamma-\mu \end{bmatrix}$$

Above Jacobian matrix J_{PDC} has four distinct Eigenvalues:

$$\omega_1 = -\mu,$$

$$\omega_2 = -\frac{1}{2}\left(\frac{B\delta+\sqrt{B^2\delta^2-4\eta^2B\delta+4\eta^3\mu+12\eta^2\mu^2-8\mu\eta B\delta+12\mu^3\eta-4\eta^2B\delta+4\mu^2}}{\eta+\mu} \right),$$

$$\omega_3 = -\frac{1}{2}\left(\frac{B\delta+\sqrt{B^2\delta^2-4\eta^2B\delta+4\eta^3\mu+12\eta^2\mu^2-8\mu\eta B\delta+12\mu^3\eta-4\eta^2B\delta+4\mu^2}}{\eta+\mu} \right),$$

$$\omega_4 = \frac{\alpha\delta^2\mu B-\alpha\delta\mu^2\eta-\alpha\delta\mu^3+\beta\eta\left(B\delta-\mu\eta-\mu^2\right)-\gamma\delta\mu\eta-\gamma\delta\mu^2-\delta\mu^2\eta-\delta\mu^3}{\delta\mu(\eta+\mu)}$$

One can easily see that ω_1, ω_2 and ω_3 have negative value. And if $\omega_4 < 0$ then one can write

$$\frac{\alpha\delta^2\mu B - \alpha\delta\mu^2\eta - \alpha\delta\mu^3 + \beta\eta\left(B\delta - \mu\eta - \mu^2\right) - \gamma\delta\mu\eta - \gamma\delta\mu^2 - \delta\mu^2\eta - \delta\mu^3}{\delta\mu(\eta+\mu)} < 0$$
$$\Rightarrow \frac{\alpha\delta\mu\left(B\delta - \mu\eta - \mu^2\right) + \beta\eta\left(B\delta - \mu\eta - \mu^2\right) - \delta\mu\left(\gamma\eta + \gamma\mu + \mu\eta + \mu^2\right)}{\delta\mu(\eta+\mu)} < 0$$
$$\Rightarrow \left(B\delta - \mu\eta - \mu^2\right)(\alpha\delta\mu + \beta\eta) - \delta\mu\left(\gamma(\eta+\mu) + \mu(\eta+\mu)\right) < 0$$
$$\Rightarrow \left(B\delta - \mu\eta - \mu^2\right)(\alpha\delta\mu + \beta\eta) - \delta\mu(\eta+\mu)(\gamma+\mu) < 0$$
$$\Rightarrow \frac{\left(B\delta - \mu\eta - \mu^2\right)(\alpha\delta\mu + \beta\eta)}{\delta\mu(\eta+\mu)(\gamma+\mu)} - 1 < 0$$
$$\Rightarrow R_0 - 1 < 0$$
$$\Rightarrow R_0 < 1$$

So, if $R_0 < 1$ then ω_4 has negative value.

Further, we will also discuss the behaviour of the equilibrium point

$$E^*_{PDCS} = \left(\frac{-\delta\alpha Y\mu + \gamma\eta + \mu\eta + \mu\beta Y}{\delta Y\beta}, \frac{-\beta Y + \gamma + \mu}{\delta\alpha}, Y, \frac{-\delta\alpha Y\mu - \beta Y\eta + \gamma\eta + \mu\eta}{\delta\alpha Y\beta} \right)$$

using Jacobian matrix J_{PDCS}. The Jacobian matrix J_{PDCS} of the waste paper recycling model is as follows:

$$J_{PDCS} = \begin{bmatrix} -\delta D^* - \mu & -\delta P^* & 0 & \gamma \\ \delta D^* & \delta P^* - \eta - \delta\alpha S^* - \mu & 0 & -\delta\alpha D^* \\ 0 & \eta & -\beta S^* - \mu & -\beta C^* \\ 0 & \delta\alpha S^* & \beta S^* & \delta\alpha D^* + \beta C^* - \gamma - \mu \end{bmatrix}$$

where

$$P^* = \frac{-\delta\alpha Y\mu + \gamma\eta + \mu\eta + \mu\beta Y}{\delta Y\beta},\ D^* = \frac{-\beta Y + \gamma + \mu}{\delta\alpha},$$
$$C^* = Y,\ S^* = \frac{-\delta\alpha Y\mu - \beta Y\eta + \gamma\eta + \mu\eta}{\delta\alpha Y\beta}$$

and the value of Y is as in the Equation 2. The characteristics equation of the Jacobian matrix J_{PDCS} about the equilibrium point E^*_{PDCS} is as given below

$$A_1\lambda^3 + A_2\lambda^2 + A_3\lambda + A_4 = 0$$

where

$$
\begin{aligned}
A_1 &= -\delta\alpha D^* - \beta C^* + \gamma + 4\mu + \beta S^* - \delta P^* + \eta + \delta\alpha S^* + \delta D^* \\
&= \delta\alpha\left(S^* - D^*\right) + \beta\left(S^* - C^*\right) + \delta\left(D^* - P^*\right) + \gamma + 4\mu > 0
\end{aligned}
$$

$$
\begin{aligned}
A_2 &= \delta^2\alpha SD^* + 3\delta\alpha S^*\mu - \beta S^*\delta P^* + \beta S^{*2}\delta\alpha + \beta S^*\delta D^* - \delta\alpha D^*\eta - 3\alpha\delta D^*\mu + \delta^2 P\alpha D^* \\
&\quad - 3\delta P^*\mu + \eta\delta D^* + 3\mu\delta D^* + \beta S^*\eta + 3\beta S^*\mu + \gamma\delta D^* - 3\beta C^*\mu - \delta^2\alpha D^{*2} - \beta C^*\eta \\
&\quad + \gamma\beta S^* - \gamma\delta P^* \\
&> \delta^2\alpha^2 S^* D^*\left(\beta S^* + \mu\right) - \left(\delta\alpha D^* + \beta C^*\right)\left(\eta + \delta\alpha S^* + \mu\right)\left(\beta S^* + \mu\right) \\
&\quad + \beta S^*\left(\beta C^*\eta + \beta C^*\delta\alpha S^* + \beta C^*\mu + \eta\delta\alpha D^*\right) + \left(\eta + \delta\alpha S^* + \mu\right)(\gamma + \mu)\left(\beta S^* + \mu\right) \\
&> 1 + \frac{\delta^2\alpha^2 S^* D^*}{\left(\eta + \delta\alpha S^* + \mu\right)(\gamma + \mu)} - \frac{\delta\alpha D^* + \beta C^*}{(\gamma + \mu)} + \frac{\beta S^*\left(\beta C^*\eta + \beta C^*\delta\alpha S^* + \beta C^*\mu + \eta\delta\alpha D^*\right)}{\left(\eta + \delta\alpha S^* + \mu\right)(\gamma + \mu)\left(\beta S^* + \mu\right)} \\
&> 1 - R_0 > 0 \\
&\Rightarrow R_0 < 1
\end{aligned}
$$

$$
\begin{aligned}
A_3 &= 2\delta D^*\gamma\mu + \delta^2\alpha D^*\beta S^* P^* - 2\delta\alpha D^*\beta S^*\mu - 2\beta C^*\delta\alpha S^*\mu - \beta C^*\delta^2\alpha S^* D^* + 2\mu\delta^2 P^*\alpha D^* \\
&\quad - 2\eta\delta\alpha D^*\mu + 2\gamma\eta\mu - 3\beta C^*\mu^2 + 3\gamma\mu^2 + 3\eta\mu^2 - 3\delta\alpha D^*\mu^2 + \eta\gamma\delta D^* - 2\delta^2\alpha D^{*2}\mu \\
&\quad - \delta^2\alpha D^{*2}\eta - 2\beta C^*\eta\mu + 2\mu\beta S^*\eta + 3\delta\alpha S^*\mu^2 + 2\mu\eta\delta D^* + \gamma\beta S^*\eta + 2\gamma\beta S^*\mu - 2\gamma\delta P^*\mu \\
&\quad - \delta^2\alpha D^{*2}\beta S^* + 2\beta C^*\delta P^*\mu - 2\beta C^*\mu\delta D^* - \beta C^*\eta\delta D^* - 2\mu\beta S^*\delta P^* + 2\mu\beta S^{*2}\delta\alpha \\
&\quad + 2\mu\beta S^*\delta D^* + 2\mu\delta^2\alpha S^* D^* + \beta S^*\eta\delta D^* + \beta S^{*2}\delta^2\alpha D^* - \gamma\beta S^*\delta P^* + \gamma\beta S^{*2}\delta\alpha \\
&\quad + \gamma\beta S^*\delta D^* + 2\gamma\delta\alpha S^*\mu + 4\mu^3 + 3\beta S^*\mu - 3\delta P^*\mu^2 + 3\mu^2\delta D^* \\
&> \mu(\delta^2\alpha^2 S^* D^*(\beta S^* + \mu) - (\delta\alpha D^* + \beta C^*)(\eta + \delta\alpha S^* + \mu)(\beta S^* + \mu) \\
&\quad + \beta S^*(\beta C^*\eta + \beta C^*\delta\alpha S^* + \beta C^*\mu + \eta\delta\alpha D^*)) + (\eta + \delta\alpha S^* + \mu)(\gamma + \mu)(\beta S^* + \mu) \\
&> 1 + \frac{\delta^2\alpha^2 S^* D^*}{\left(\eta + \delta\alpha S^* + \mu\right)(\gamma + \mu)} - \frac{\delta\alpha D^* + \beta C^*}{(\gamma + \mu)} + \frac{\beta S^*\left(\beta C^*\eta + \beta C^*\delta\alpha S^* + \beta C^*\mu + \eta\delta\alpha D^*\right)}{\left(\eta + \delta\alpha S^* + \mu\right)(\gamma + \mu)\left(\beta S^* + \mu\right)} \\
&> 1 - R_0 > 0 \\
&\Rightarrow R_0 < 1
\end{aligned}
$$

Similarly, one can prove,

$$
\begin{aligned}
A_4 &= \delta^2\alpha D^*\beta S^* P^*\mu - \beta C^*\mu\delta^2\alpha S^* D^* + \mu^3\eta + \gamma\mu^3 - \delta^2\alpha D^{*2}\mu^2 - \delta\alpha D^*\mu^3 - \beta C^*\mu^2\eta \\
&\quad + \beta S^*\mu^2\eta + \mu^2\eta\delta D^* + \mu^3\delta\alpha S^* + \gamma\beta S^*\mu^2 - \gamma\mu^2\delta P^* + \gamma\mu^2\delta D^* + \delta^2\alpha D^*\mu^2 P^* \\
&\quad - \delta^2\alpha D^{*2}\mu\eta - \delta\alpha D^*\mu^2\eta + \beta C^*\mu^2\delta P^* - \beta C^*\mu^2\delta D^* - \beta S^*\mu^2\delta P^* + \beta S^{*2}\mu^2\delta\alpha \\
&\quad + \beta S^*\mu^2\delta D^* + \mu^2\delta^2\alpha S^* D^* + \gamma\beta S^*\eta\mu + \gamma\mu\eta\delta D^* + \gamma\mu^2\delta\alpha S^* - \beta C^*\mu^3 + \beta S^*\mu^3 \\
&\quad - \mu^3\delta P^* + \mu^3\delta D^* + \gamma\mu^2\eta^* \\
&> 1 - R_0 > 0 \\
&\Rightarrow R_0 < 1
\end{aligned}
$$

It follows that $A_1 > 0, A_2 > 0, A_3 > 0, A_4 > 0$ and also $A_1A_2A_3 > A_3^2 + A_1^2A_4$.

Theorem 1. Using Routh-Hurwitz criterion [20], the equilibrium points

$$E^*_{PDC} = \left(\frac{\eta+\mu}{\delta}, \frac{B\delta-\mu(\eta+\mu)}{\delta(\eta+\mu)}, \frac{\eta\left(B\delta-\mu(\eta+\mu)\right)}{\delta\mu(\eta+\mu)}, 0 \right)$$

and

$$E^*_{PDCS} = \left(\frac{-\delta\alpha Y\mu+\gamma\eta+\mu\eta+\mu\beta Y}{\delta Y\beta}, \frac{-\beta Y+\gamma+\mu}{\delta\alpha}, Y, \frac{-\delta\alpha Y\mu-\beta Y\eta+\gamma\eta+\mu\eta}{\delta\alpha Y\beta} \right)$$

of the waste water recycling model are locally asymptotically stable with the condition $R_0 < 1$.

3.2 Global Stability

The global stability will be conversed in this section using some graph theoretical results [6, 24] given as below:

- Any graph will consist of the set of vertices and the set of edges.
- (i, j) is called *an edge* from initial vertex i to terminal vertex j.
- *A directed graph* G is the set of vertices and the set of edges where all the edges are directed from one vertex to another [7].
- *The out-degree of a vertex* i is the number of edges whose initial vertex is i denoted as $d^+(i)$.
- *The in-degree of a vertex* i is the number of edges whose terminal vertex is i denoted as $d^-(i)$.
- A directed graph G is called *a weighted directed graph* if each edge is assigned a positive weight.
- *The weight* $w(H)$ of sub-directed graph H is the product of weights on all its edges.
- *A path* in a graph is a finite or infinite sequence of edged which connect the sequence of vertices where all are distinct from others [10].
- *A directed path* in a directed graph is a sequence of edges which connect a sequence of edges which connect a sequence of vertices where all the edges should be directed in the same direction.
- *A cycle graph* is a graph where some number of vertices connected in a closed chain [8].
- *A directed cycle graph* is a directed version of a cycle graph with all the edges being oriented in the same direction.
- *A loop* (or *buckle*) is an edge that connects a vertex i to itself [9].
- *A tree* is any acyclic connected graph [12].
- If tree is directed then it is called *directed tree*.
- *A spanning tree* is a subgraph of a graph G which includes all the vertices of G with minimum number of edges [11].

- If G is a weighted directed graph with n vertices then the weight matrix has order $n \times n$ denoted as $A = \left[a_{ij} \right]$ with entries $a_{ij} > 0$ which is equal to the weight of edge if it exists otherwise it is 0. This kind of weighted directed graph is noted by (G, A).
- G is called *strongly connected directed graph* if for any pair of discrete vertices there exists a directed path.
- (G, A) is called *a strongly connected weighted directed graph* if and only if the weighted matrix A is irreducible.
- The Laplacian matrix $L = \left[l_{ij} \right]$ of (G, A) is defined as $l_{ij} = \begin{cases} -a_{ij} & ; i \neq j \\ \sum\limits_{k \neq i} a_{ik} & ; i = j \end{cases}$.

Proposition 2 (Kirchhoff's matrix tree theorem). Assume $n \geq 2$ and let c_i be the cofactor of l_{ii} in L. Then $c_i = \sum\limits_{\tau \in \mathrm{T}_i} w(\tau), i = 1, 2, ..., n$ where T_i is the set of all spanning trees τ of weighted directed graph (G, A) which makes tree at vertex i and $w(\tau)$ is the weight of τ. If the weighted graph (G, A) is strongly connected then $c_i > 0$ for $1 \leq i \leq n$.

Theorem 3. Let c_i be as given in the Kirchhoff's matrix tree theorem. If $a_{ij} > 0$ and $d^{+}(j) = 1$ for some i, j then $c_i a_{ij} = \sum\limits_{k=1}^{n} c_j a_{jk}$.

Theorem 4. Let c_i be as given in the Kirchhoff's matrix tree theorem. If $a_{ij} > 0$ and $d^{-}(i) = 1$ for some i, j then $c_i a_{ij} = \sum\limits_{k=1}^{n} c_k a_{ki}$.

Theorem 5. Suppose that the following assumptions are satisfied:

(1) There exists function $V_i : U \to \square, G_{ij} : U \to \square$ and constants $a_{ij} \geq 0$ such that for every $1 \leq i \leq n, V_i' \leq \sum\limits_{j=1}^{n} G_{ij}(z)$ for $z \in U$.

(2) For $A = \left[a_{ij} \right]$, each directed cycle C of (G, A) has $\sum\limits_{(s,r) \in \varepsilon(C)} G_{rs}(z) \leq 0$ for $z \in U$, where $\varepsilon(C)$ denotes the arc set of the directed cycle C.

Then, the function $V(z) = \sum\limits_{i=1}^{n} c_i V_i(z)$, with constant $c_i \geq 0$ as given in the proposition of Kirchhoff's matrix tree theorem, satisfies $V' \leq 0$ then V is a Lyapunov function for the system. First, we will discuss about the global stability for the equilibrium point

$$E_{PDC}^{*} = \left(\frac{\eta + \mu}{\delta}, \frac{B\delta - \mu(\eta + \mu)}{\delta(\eta + \mu)}, \frac{\eta\left(B\delta - \mu(\eta + \mu)\right)}{\delta\mu(\eta + \mu)}, 0 \right)$$

using graph theory.

Let $V_{1_{PDC}} = \frac{1}{2}\left(P - P^* + D - D^* + C - C^*\right)^2, V_{2_{PDC}} = D - D^* - D^* \ln\frac{D}{D^*}, V_{3_{PDC}} = \frac{1}{2}\left(C - C^*\right)^2$

Differentiation of $V_{1_{PDC}}, V_{2_{PDC}}, V_{3_{PDC}}$, gives us

$$\begin{aligned} V'_{1_{PDC}} &= \left(\left(P - P^* + C - C^*\right) + \left(D - D^*\right)\right)\left(\mu\left(P - P^* + C - C^*\right) + \mu\left(D - D^*\right)\right) \\ &\le 2\mu\left(P - P^*\right)\left(D - D^*\right) + 2\mu\left(C - C^*\right)\left(D - D^*\right) \\ &= a_{12}G_{12} + a_{23}G_{23} \end{aligned}$$

$$\begin{aligned} V'_{2_{PDC}} &= \beta\left(P - P^*\right)\left(D - D^*\right) \\ &= a_{21}G_{21} \end{aligned}$$

$$\begin{aligned} V'_{3_{PDC}} &= \left(C - C^*\right)\left(\eta D - \eta D^* + \mu C^* - \mu C\right) \\ &\le \eta\left(D - D^*\right)\left(C - C^*\right) \\ &= a_{31}G_{31} \end{aligned}$$

Using above results and the set of three vertices, the weighted graph is created as shown in Figure 2.

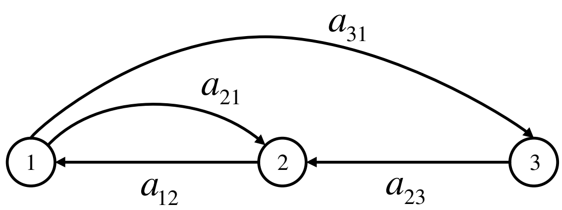

Figure 2. The weighted graph of waste paper model without scrap dealer

With $a_{12} = a_{23} = 2\mu, a_{21} = \beta, a_{31} = \eta$ and others $a_{ij} = 0$.

The related weighted graph has three vertices and two cycles where each cycle $G_{12} + G_{21} = 0$ and $G_{12} + G_{31} + G_{23} = 0$. Then, as assumptions taken in theorem 5, there exists $c_i, 1 \le i \le 3$ such that $V_{PDC} = \sum_{i=1}^{3} c_i V_{i_{PDC}}$ is Lyapunov function. Using theorem 3,

$$d^+(3) = 1 \Rightarrow c_2 a_{23} = c_3 a_{31}$$

Now, taking $c_2 = k$ and putting the values of a_{23} and a_{31}, we get

$$c_3 = \frac{2\mu k}{\eta}$$

Also, $d^+(2) = 1 \Rightarrow c_1 a_{12} = c_2 a_{21} + c_2 a_{23}$

$$\Rightarrow c_1 = \frac{(\beta + 2\mu)k}{2\mu}$$

Therefore,

$$\begin{aligned} V_{PDC} &= \sum_{i=1}^{3} c_i V_{i_{PDC}} \\ &= c_1 V_{1_{PDC}} + c_2 V_{2_{PDC}} + c_3 V_{3_{PDC}} \\ &= \frac{(\beta + 2\mu)k}{2\mu} V_{1_{PDC}} + k V_{2_{PDC}} + \frac{2\mu k}{\eta} V_{3_{PDC}} \end{aligned}$$

where t is an arbitrary constant.

This verifies that E^*_{PDC} is the only invariant set in $\operatorname{int}(\Lambda)$, where $V'_{PDC} = 0.$ Hence, E^*_{PDC} is globally asymptotically stable in $\operatorname{int}(\Lambda)$.

Next, we will deliberate the global stability for the equilibrium point

$$E^*_{PDCS} = \left(\frac{-\delta\alpha Y\mu + \gamma\eta + \mu\eta + \mu\beta Y}{\delta Y \beta}, \frac{-\beta Y + \gamma + \mu}{\delta\alpha}, Y, \frac{-\delta\alpha Y\mu - \beta Y\eta + \gamma\eta + \mu\eta}{\delta\alpha Y \beta} \right)$$

using graph theory. Consider, $V_{1_{PDCS}} = P - P^* \ln \frac{P}{P^*}$, $V_{2_{PDCS}} = D - D^* \ln \frac{D}{D^*}$, $V_{3_{PDCS}} = C - C^* \ln \frac{C}{C^*}$, $V_{4_{PDCS}} = S - S^* \ln \frac{S}{S^*}$

Differentiate $V_{1_{PDCS}}$ and we get,

$$\begin{aligned} V'_{1_{PDCS}} &= \left(1 - \frac{P^*}{P}\right) P' \\ &= \left(1 - \frac{P^*}{P}\right)(B - \delta PD + \gamma S - \mu P) \\ &= \left(1 - \frac{P^*}{P}\right)\left(\delta P^* D^* - \delta PD - \gamma S^* + \gamma S + \mu P^* - \mu P\right) \\ &= \delta\left(1 - \frac{P^*}{P}\right)\left(P^* D^* - PD\right) + \gamma S^* \left(1 - \frac{P^*}{P}\right)\left(1 - \frac{S}{S^*}\right) + \mu\left(1 - \frac{P^*}{P}\right)\left(P^* - P\right) \\ &= -\delta P^* D^* \left(1 - \frac{P^*}{P}\right)\left(1 - \frac{PD}{P^* D^*}\right) + \gamma S^* \left(1 - \frac{P^*}{P}\right)\left(1 - \frac{S}{S^*}\right) + \mu \frac{\left(P - P^*\right)^2}{P} \\ &\le \gamma S^* \left(1 - \frac{P^*}{P}\right)\left(1 - \frac{S}{S^*}\right) \\ &= a_{14} G_{14} \end{aligned}$$

Similarly, differentiating $V_{2_{PDCS}}, V_{3_{PDCS}}$ and $V_{4_{PDCS}}$, we get

$$\begin{aligned}
V'_{2_{PDCS}} &= \left(1-\frac{D^*}{D}\right)D' \\
&= \left(1-\frac{D^*}{D}\right)\left(\delta PD - \eta D - \delta\alpha DS - \mu D\right) \\
&= \delta P^* D^* \left(1-\frac{D^*}{D}\right)\left(1-\frac{PD}{P^*D^*}\right) - \eta D^*\left(1-\frac{D^*}{D}\right)\left(1-\frac{D}{D^*}\right) - \delta\alpha D^* S^*\left(1-\frac{D^*}{D}\right)\left(1-\frac{DS}{D^*S^*}\right) \\
&\quad - \mu D^*\left(1-\frac{D^*}{D}\right)\left(1-\frac{D}{D^*}\right) \\
&\le \delta P^* D^* \left(1-\frac{D^*}{D}\right)\left(1-\frac{PD}{P^*D^*}\right) \\
&= a_{21}G_{21}
\end{aligned}$$

$$\begin{aligned}
V'_{3_{PDCS}} &= \left(1-\frac{C^*}{C}\right)C' \\
&= \left(1-\frac{C^*}{C}\right)\left(\eta D - \beta CS - \mu C\right) \\
&= \eta D^*\left(1-\frac{C^*}{C}\right)\left(1-\frac{D}{D^*}\right) - \beta C^* S^*\left(1-\frac{C^*}{C}\right)\left(1-\frac{CS}{C^*S^*}\right) - \mu C^*\left(1-\frac{C^*}{C}\right)\left(1-\frac{C}{C^*}\right) \\
&\le \eta D^*\left(1-\frac{C^*}{C}\right)\left(1-\frac{D}{D^*}\right) \\
&= a_{32}G_{32}
\end{aligned}$$

$$\begin{aligned}
V'_{4_{PDCS}} &= \left(1-\frac{S^*}{S}\right)S' \\
&= \left(1-\frac{S^*}{S}\right)\left(\delta\alpha DS + \beta CS - \gamma S - \mu S\right) \\
&= \delta\alpha D^* S^*\left(1-\frac{S^*}{S}\right)\left(1-\frac{DS}{D^*S^*}\right) + \beta C^* S^*\left(1-\frac{S^*}{S}\right)\left(1-\frac{CS}{C^*S^*}\right) - \gamma S^*\left(1-\frac{S^*}{S}\right)\left(1-\frac{S}{S^*}\right) \\
&\quad - \mu S^*\left(1-\frac{S^*}{S}\right)\left(1-\frac{S}{S^*}\right) \\
&\le \delta\alpha D^* S^*\left(1-\frac{S^*}{S}\right)\left(1-\frac{DS}{D^*S^*}\right) + \beta C^* S^*\left(1-\frac{S^*}{S}\right)\left(1-\frac{CS}{C^*S^*}\right) \\
&= a_{42}G_{42} + a_{43}G_{43}
\end{aligned}$$

Using above results and the set of four vertices, the weighted graph is created as shown in Figure 3.

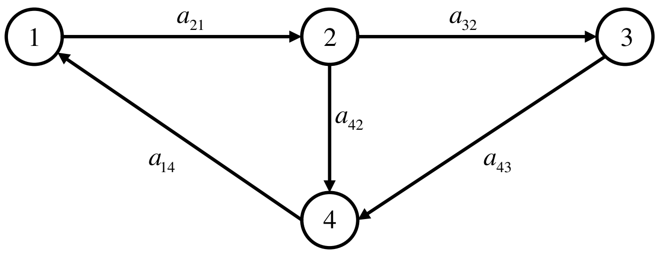

Figure 3. The weighted graph of waste paper model

with $a_{14} = \gamma S^*, a_{21} = \delta P^* D^*, a_{32} = \eta D^*, a_{42} = \delta\alpha D^* S^*, a_{43} = \beta C^* S^*$ and others $a_{ij} = 0$.

The related weighted graph has three vertices and two cycles where each cycle $G_{14} + G_{21} + G_{42} = 0$ and $G_{14} + G_{21} + G_{32} + G_{43} = 0$. Then, as assumptions taken in theorem 5, there exists $c_i, 1 \le i \le 4$ such that $V_{PDCS} = \sum_{i=1}^{4} c_i V_{i_{PDCS}}$ is Lyapunov function. Using theorem 3,

$$d^+(1) = 1 \Rightarrow c_2 a_{21} = c_1 a_{14}$$

Now, taking $c_1 = t$ and putting the values of a_{21} and a_{14}, we get

$$c_2 = \frac{t\gamma S^*}{\delta P^* D^*}$$

Also, $d^+(4) = 1 \Rightarrow c_1 a_{14} = c_4 a_{42} + c_4 a_{43}$

$$\Rightarrow c_4 = \frac{t\gamma S^*}{\delta\alpha D^* S^* + \beta C^* S^*}$$

and

$$d^+(3) = 1 \Rightarrow c_4 a_{43} = c_3 a_{32}$$

$$\Rightarrow \frac{t\gamma S^*}{\delta\alpha D^* S^* + \beta C^* S^*} \beta C^* S^* = c_3 \eta D^*$$

$$\Rightarrow c_3 = \frac{t\beta\gamma C^* S^{*2}}{\left(\delta\alpha D^* S^* + \beta C^* S^*\right)\eta D^*}$$

Therefore,

$$V_{PDCS} = \sum_{i=1}^{4} c_i V_{i_{PDCS}}$$
$$= c_1 V_{1_{PDCS}} + c_2 V_{2_{PDCS}} + c_3 V_{3_{PDCS}} + c_4 V_{4_{PDCS}}$$
$$= tV_{1_{PDCS}} + \frac{t\gamma S^*}{\delta P^* D^*} V_{2_{PDCS}} + \frac{t\beta\gamma C^* S^{*^2}}{\left(\delta\alpha D^* S^* + \beta C^* S^*\right)\eta D^*} V_{3_{PDCS}} + \frac{t\gamma S^*}{\delta\alpha D^* S^* + \beta C^* S^*} V_{4_{PDCS}}$$

where t is an arbitrary constant.

This confirms that E^*_{PDCS} is the only invariant set in $\mathrm{int}(\Lambda)$, where $V'_{PDCS} = 0$. Hence, E^*_{PDCS} is globally asymptotically stable in $\mathrm{int}(\Lambda)$.

Theorem 6. The positive equilibrium points

$$E^*_{PDC} = \left(\frac{\eta+\mu}{\delta}, \frac{B\delta - \mu(\eta+\mu)}{\delta(\eta+\mu)}, \frac{\eta\left(B\delta - \mu(\eta+\mu)\right)}{\delta\mu(\eta+\mu)}, 0\right)$$

and

$$E^*_{PDCS} = \left(\frac{-\delta\alpha Y\mu + \gamma\eta + \mu\eta + \mu\beta Y}{\delta Y\beta}, \frac{-\beta Y + \gamma + \mu}{\delta\alpha}, X, \frac{-\delta\alpha Y\mu - \beta Y\eta + \gamma\eta + \mu\eta}{\delta\alpha Y\beta}\right)$$

of the waste water recycling model are globally asymptotically stable in $\mathrm{int}(\Lambda)$.

4 Sensitivity Analyses

In this section, we will confer about the sensitivity for each parameter used to formulate waste paper recycling model shown in Table 2.

The normalised sensitivity index of the parameters is calculated by using the following formula: $\Upsilon_\alpha^{R_0} = \frac{\partial R_0}{\partial \alpha} \cdot \frac{\alpha}{R_0}$ where α is the model parameter.

The rate at which distributor buys paper, the rate at which distributor gives paper to scrap dealers and the rate at which customer gives paper to scrap dealers have positive impact on R_0 which means they are factors which help us to recycle more waste paper. And others have negative effect for the waste paper recycling model and we should not increase them.

Table 2. Sensitivity analysis

Parameter	Value
δ	+
η	-
α	+
β	+
γ	-
μ	-

5 Numerical Simulations

In the current section, some numerical simulation has been done using the parametric values given in the Table 1.

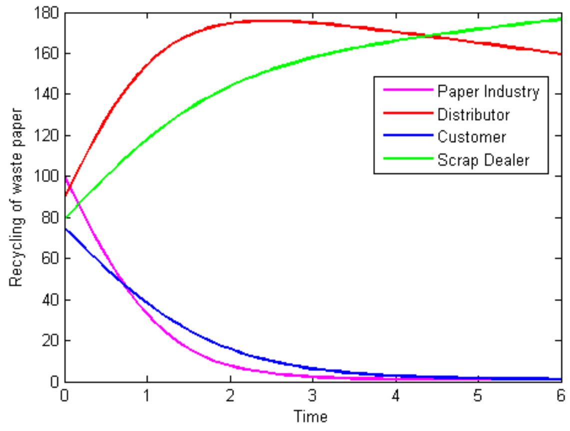

Figure 4. Transmission of waste paper recycling model

Figure 4 depicts that initially paper industry supplies to distributors and which gets maximum at 2 weeks. After buying it from the distributor, the customer runs away with the sticks in almost 6 weeks. When the used papers from the distributor, paper industry and customer increases pile up for scrap dealer and this result recycling of paper again to the paper industry.

Figure 5-7 shows the effect of recycling of waste paper due to scarp dealers. From all figures we can state that as the rate increases the recycling of waste paper is also increases. Which also mean that all the rates mentioned in following three figures have helpful influence on the model?

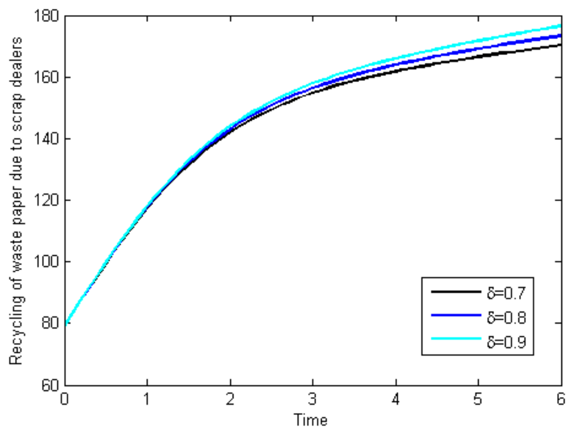

Figure 5. Effect of the rate at which distributor buys paper on the scrap dealer

From Figure 5, it can be determined that as the rate at which distributor buys paper (δ) is varied from 70% to 90%, the recycling of waste paper is increased from 170 to 178. This expressed that we can recycle more waste paper by 4.70% when δ is increased by 20%.

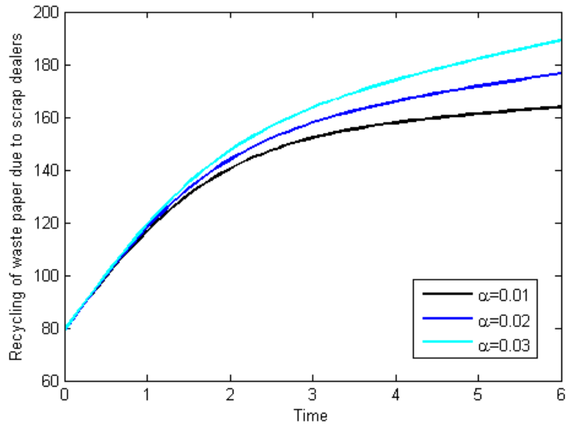

Figure 6. Effect of the rate at which distributor gives paper to scrap dealers on the scrap dealer

From Figure 6 as the rate at which distributor gives paper to scrap dealers (α) is varied from 1% to 3%, the recycling of waste paper is increased from 164 to 189. This shows that when α is increased by 2% there will be an increment of 15.24% in the recycled waste paper.

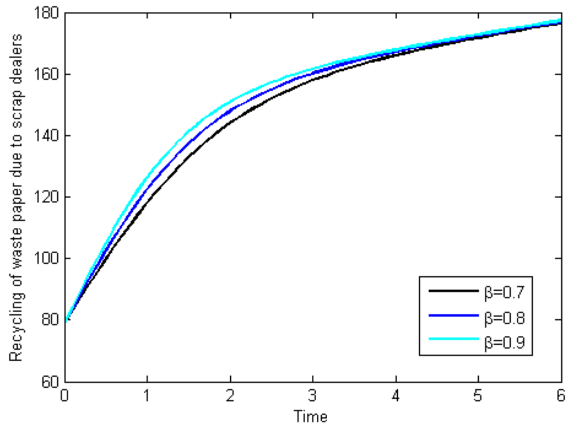

Figure 7. Effect of the rate at which customer gives paper to scrap dealers on the scrap dealer

In Figure 7, the rate at which customer gives paper to scrap dealers (β) is varied from 70% to 90% then the recycling of waste paper is increased from 188 to 189. It indicates that as 20% β is increased only 1% more waste paper is recycled.

Hence, from above three figures it can be concluded that α is more effective factor than δ and β in the recycling of waste paper which directs that distributors should have better awareness than customer and they should not waste leftover paper inappropriately. In fact, they should give as much as waste paper to scrap dealers.

6 Conclusions

In the proposed paper, a mathematical model of the waste paper recycling is formulated to examine the importance of scrap dealers in the process of waste paper recycling. Recycling paper reduces the need for raw material, it also requires much less energy, so it could preserve natural resources like trees, forest, water, fuel, etc. for the future and also condenses greenhouse gases. For that we all need to show our interest in recycling to make it successful. We need to take our time and save the paper products so that it can be recycled. We can minimize our use of paper, too. We can use electronic storage rather than paper storage. If we really need to buy paper, we just could buy a recycled one. Also, we can go digital for the further step in a right direction to protect the environment.

In the section 2, we have found the basic reproduction numbers for all the equilibrium points. From that it can be determined that when only paper industry exists and when distributor doesn't exist, the system will not physically stable. Also, we can conclude that if scrap dealers are not there, the paper recycling increases only by 64.56% but with the existence of scrap dealers, paper recycling will be increased by 73.46%. So, scrap dealers are noteworthy for the waste paper recycling.

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Soft β-Separation Axioms in Soft Quad Topological Spaces

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Abstract - The major purpose of this article is to publicize soft β-separations axioms in soft quad topological spaces. We talk over and focus our attention on soft β-separation axioms in soft quad topological spaces with respect to ordinary points and soft points. Moreover study the inherited characteristics at different respects with respect to ordinary points and soft points. Some of their `principal properties in soft quad topological spaces are also brought under consideration.

Key words *- Soft sets, soft topology, soft β open set, soft β closed set, soft quad topological space, soft βqT_0structure, soft bqT_1structure, soft βqT_2structure, soft βqT_3structure and soft βqT_4structure.*

1 Introduction

In actual life situation the complexity in economics, engineering, social sciences, medical science etc. we cannot attractively use the outdated classical methods because of different kinds of uncertainties existing in these problems. To finish out these complexity, some types of theories were put forwarded like theory of Fuzzy set, intuitionistic fuzzy set, rough set and bi polar fuzzy sets, inwhich we can safely use mathematical methods for dealing with uncertainties. But, all these theories have their inherent worries. To bury these difficulties in the year 1999, Russian scholar Molodtsov [4] introduced the idea of soft set as a new mathematical methods to deal with uncertainties. Which is free from the above difficulties? Kelly [5] studied bi-topological spaces and discussed different results. Tapi et al. [53] beautifully discussed separation axioms in quad topological spaces . Hameed and Abid [54] discussed separation axioms in Tri-topological spaces .

Recently, in 2011, Shabir and Naz [7]opened the idea of soft topological space and discussed different results with respect to ordinary points. They beautifully defined soft topology as a collection of τ of soft sets over X. they also defined the basic idea of soft topological spaces such as open set and closed soft sets, soft nbd of a point, soft separation axioms, soft regular and soft normal spaces and published their several performances. Soft separation axioms are also discussed at detail. Aktas and Cagman [9] discussed Soft sets and soft groups. Chen [10] discovered the parameterization reduction of soft sets and its applications. Feng et al. [11]. Studied Soft semi rings and its applications. In the recent years, many interesting applications of soft sets theory and soft topology have been discussed at great depth [12,13,14,,15,16,17,18,19,20,21,22]. Kandil at al. [25] explained Soft connectedness via soft ideal developed soft set theory. Kandil et al. [27] launched Soft regularity and normality based on semi open soft sets and soft ideals.

In [28,29,30,31,32,33,34,35,36] discussion is launched soft semi hausdorff spaces via soft ideals, semi open and semi closed sets, separation axioms ,decomposition of some type supra soft sets and soft continuity are discussed. Hussain and Ahmad [51] defined soft points, soft separation axioms in soft topological spaces with respect to soft points and used it in different results, Kandil et al. [52] studied Soft semi separation axioms and some types of soft functions and their characteristics.

In this present paper, concept of soft β-separation axioms in Soft quad topological spaces is introduced with respect to ordinary and soft points.

Many mathematicians threw light on soft separation axioms in soft topological spaces at full length with respect to soft open set, soft b-open set, soft semi-open, soft α-open set and soft β-open set. They also worked over the hereditary properties of different soft topological structures in soft topology. In this present article bridge is built over the gap that exists in soft quad-topology related to soft $\beta\, qT_0$, soft βqT_1, soft βqT_2, soft βqT_3 and soft βqT_4 structures. Some propositions in soft quid topological spaces are discussed with respect to ordinary points and soft points. When we talk about distances between the points in soft topology then the concept of soft separation axioms is automatically taking birth. That is why these structures are catching our attentions. It is hoped that these results will be the driving force for the future study on soft quad topological spaces to achieve general framework for the practical applications and to solve the most complicated problems containing doubts in economics, engineering, medical, environment and in general mechanic systems of various varieties. In future these beautiful soft topological structures may be extended to soft n-topological spaces provided n is even.

2. Preliminaries

The following Definition s which are pre-requisites for present study

Definition 1 [4]. Let X be an initial universe of discourse and E be a set of parameters. Let P(X) denotes the power set of X and A be a non-empty sub-set of E. A pair (F, A) is called a soft set over U, where F is a mapping given by $F: A \to P(X)$

In other words, a set over X is a parameterized family of sub set of universe of discourse X. For $e \in A, F(e)$ may be considered as the set of e-approximate elements of the soft set (F, A)

and if $e \notin A$ then $F(e) = \phi$ that is $F_{A=}\{F(e): e \in A \subseteq E, F: A \to P(X)\}$ the family of all these soft sets over X denoted by $SS(X)_A$

Definition 2 [4]. Let$F_{A,}G_B \in SS(X)_E$ then F_A, is a soft subset of G_B denoted by $F_A \sqsubseteq G_B$, if
1. $A \subseteq B$ and
2.$F(e) \subseteq G(e), \forall \in A$

In this case F_A is said to be a soft subset of G_B and G_B is said to be a soft super set $F_A, G_B \sqsupseteq F_A$

Definition 3 [6]. Two soft subsets F_A and G_B over a common universe of discourse set X are said to be equal if F_A is a soft subset of G_B and G_B is a soft subset of F_A

Definition 4 [6]. The complement of soft subset (F, A) denoted by $(F, A)^C$ is defined by $(F, A)^C = (F^C, A) F^C \to P(X)$ is a mapping given by $F^C(e) = U - F(e) \forall e \in A$ and F^C is called the soft complement function of F . Clearly $(F^C)^C$ is the same as F and $((F, A)^C)^C = (F, A)$

Definition 5 [7]. The difference between two soft subset (G, E) and (G, E) over common of universe discourse X denoted by $(F, E)\backslash(G, E)$ is defined as $F(e)\backslash \mathrm{G}(e)$ for all $e \in E$

Definition 6 [7]. Let(G, E) be a soft set over X and $x \in X$ We say that $x \in (F, E)$ and read as x belong to the soft set(F, E) whenever $x \in F(e) \forall e \in E$The soft set (F, E) over X such that $F(e) = \{x\}, \forall e \in E$ is called singleton soft point and denoted by $x, or\ (x, E)$

Definition 7 [6]. A soft set (F, A) over X is said to be Null soft set denoted by $\overline{\emptyset}\ or\ \emptyset_A$if $\forall e \in A, F(e) = \emptyset$

Definition 8 [6]. A soft set (F, A) over X is said to be an absolute soft denoted by $\bar{A}\ or\ X_A$ if $\forall e \in A, F(e) = X$

Clearly, we have, $X_A^C = \emptyset_A\ and\ \emptyset_A^C = X_A$

Definition 9 [51]. The soft point e_F is said to be in the soft set(G, A) , denoted by $e_F \in (G, A)$ if for the element$e \in A, F(e) \subseteq G(e)$.

Definition 10 [44]. Two soft sets $(G, A), (H, A)$ in $SS(X)_A$ are said to be soft disjoint, written $(G, A) \cap (H, A) = \emptyset_A$ If $G(e) \cap H(e) = \emptyset$ for all $e \in A$.

Definition 11 [51]. The soft point e_G, e_H in X_A are disjoint, written $e_G \neq e_H$ if their corresponding soft sets (G, A) and (H, A) are disjoint.

Definition 12 [6]. The union of two soft sets (F, A) and (G, B) over the common universe of discourse X is the soft set (H, C), where, $C = AUB, \forall e \in C$

$$H(e) = \left\{ \begin{array}{cc} F(e) & if\ e\ \in A - B \\ G(e) & if\ e\ \in (B - A\ ) \\ F(e) U G(e), & if\ e\ \in A \cap B \end{array} \right\}$$

Written as $(F,A) \cup (G,B) = (H,C)$

Definition 13 [6]. The intersection $(H.C)$ of two soft sets(F,A) and (G,B)over common universe X, denoted $(F,A) \,\overline{\cap}\, (\mathrm{G},\mathrm{B})$ is defined as

$$C = A \cap B \text{ and } H(e) = F(e) \cap G(e), \forall e \in C$$

Definition 14 [2]. Let (F,E) be a soft set over X and Y be a non-empty sub set of X. Then the sub soft set of (F,E)over Y denoted by (Y_F,E), is defined as follow $Y_{F(\alpha)} = Y \cap F(\alpha), \forall \in E$ in other words

$$(Y_F, E) = Y \cap (F,E).$$

Definition 16 [3]. Let τbe the collection of soft sets over $\tilde{X}$ then τ is said to be a soft topology on$\tilde{X}$ if

1. $\emptyset, \tilde{X} \epsilon \tau$
2. The union of any number of soft sets in τbelongs to τ
3. The intersection of any two soft sets in τ belong to τ

   The triplet $(\tilde{X}, \tau, E)$ is called a soft topological space.

Definition 17 [1]. Let $(\tilde{X}, F, E)$ be a soft topological space over X, then the member of $\imath$ are said to be soft open sets in X.

Definition 18 [1]. Let (X,τ,E) be a soft topological space over X. A soft set (F,A) over $\tilde{X}$ is said to be a soft closed set in X if its relative complement $(F,E)^C$ belong to $\imath$.

Definition 20 [51]. Let $(\tilde{X},\tau,E)$ be a soft topological space and $(F,E) \subseteq SS(\tilde{X})_A$ then (F,E) is called β-open soft set if $((F,E) \subseteq Cl(int(Cl(F,E))))$. The set of all β-*open soft* set is denoted by $S\beta O(X,\tau,E)$ or $S\beta O(\tilde{X})$ and the set of all β-*closed soft* set is denoted by $S\beta C(\tilde{X},\tau,E)$ or $S\beta C(\tilde{X})$.

Proposition 1. Let (X,τ,E) be a soft topological space over X.If (X,τ,E) is soft βT_3-space, then for all $x \in X$, $x_E = (x,E)$ is β-closed soft set.

Proposition 2. Let $(Y,\tau_{Y,}E)$ be a soft sub space of a soft topological space (X,τ,E) and $(F,E) \in SS(X)$ then

1. If (F,E) is soft β *open* set in Y and $Y \in \tau$, then $(F,E) \in \tau$
2. (F,E) is soft β *open soft* set in Y if and only if $(F,E) = Y \cap (G,E)$ for some $(G,E) \in \tau$.
3. (F,E) is soft β *closed soft* set in Y if and only if $(F,E) = Y \cap (H,E)$ for some (H,E) *is* τ *soft β closed.*

4. Soft β-Separation Axioms in Soft Quad Topological Spaces

In this section we inaugurated soft β separation axioms in soft quad topological space with respect to ordinary points and discussed some results with respect to these points in detail.

Definition 27. Let $(X,\tau_1,E),(X,\tau_2,E)$,$(X,\tau_3,E)$ and (X,τ_4,E) be four different soft topologies on X. Then $(X,\tau_1,\tau_2,\tau_3,\tau_4,E)$ is called a *soft quad topological space.* The soft four topologies (X,τ_1,E), (X,τ_2,E), (X,τ_3,E) and (X,τ_4,E) are independently satisfying the axioms of soft topology. The members ofτ_1 are called τ_1*soft open* set. And complement ofτ_1Soft open set is called τ_1 *soft closed* set. Similarly, the member of τ_2 are called τ_2 *soft open* sets and the complement of τ_2 soft open sets are called τ_2 *soft closed* set. The members of τ_3 are called τ_3 *soft open* set. And complement of τ_3 Soft open set is called τ_3 *soft closed* set and the members of τ_4 are called τ_4 *soft open* set. And complement of τ_4 Soft open set is called τ_4 *soft closed* set.

Definition 28. Let $(X,\tau_1,\tau_2,\tau_3,\tau_4,E)$ be a soft quad topological space over X and Y be a non-empty subset of X. Then $\tau_{1Y}=\{(Y_F,E):(F,E)\in\tau_1)\}$,$\tau_{2Y}=\{(Y_G,E):(G,E)\in\tau_2)\}$,$\tau_{3Y}=\{(Y_H,E):(H,E)\in\tau_1)\}$ and $\tau_{4Y}=\{(I_E,E):(I,E)\in\tau_2)\}$ are said to be the relative topological on Y. Then $(Y,\tau_{1Y},\tau_{2Y},\tau_{3Y},\tau_{4Y},E)$ is called relative soft quad-topological space of $(X,\tau_1,\tau_2,\tau_3,\tau_4,E)$.

Let $(X,\tau_1,\tau_2,\tau_3,\tau_4,E)$ be a soft quad topological space over X, where $(X,\tau_1,E),(X,\tau_2,E)$,$(X,\tau_3,E)$ and (X,τ_4,E) be four different soft topologies on X. Then a sub set $(F,E)$is said to be quad-open (in short hand q-open) if $(F,E)\subseteq\tau_1\cup\tau_2\cup\tau_3\cup\tau_4$ and its complement is said to be soft q-closed.

4.1 Soft β-Separation Axioms of Soft Quad Topological Spaces with Respect to Ordinary Points.

In this section inauguration of soft β separation axioms in soft quad topological space with respect to ordinary points is launched and discussed some eye-catching results with respect to these points in detail.

Definition 29. Let $(X,\tau_1,\tau_2,\tau_3,\tau_4,E)$ be a soft quad topological space over X and $x,y\in X$ such that $x\neq y$. If we can find soft q-open sets (F,E) and (G,E) such that $x\in(F,E)$ and$y\notin(F,E)$ or $y\in(G,E)$ and $x\notin(G,E)\ then\ (X,\tau_1,\tau_2,\tau_3,\tau_4,E)$ is called soft qT_0 space.

Definition 30. Let $(X,\tau_1,\tau_2,\tau_3,\tau_4,E)$be a soft quad topological space over X and $x,y\in X$ such that $x\neq y$ If we can find two soft q-open sets (F,E) and (G,E) such that $x\in(F,E)$ and $y\notin(F,E)$ and $y\in(G,E)$ and $x\notin(G,E)$ then $(X,\tau_1,\tau_2,\tau_3,\tau_4,E)$ is called soft qT_1 space.

Definition 31. Let $(X,\tau_1,\tau_2,\tau_3,\tau_4,E)$ be a soft quad topological space over X and $x,y\in X$ such that $x\neq y$. If we can find two q- open soft sets such that $x\in(F,E)$ and $y\in(G,E)$ moreover $(F,E)\cap(G,E)=\phi$. Then $(X,\tau_1,\tau_2,\tau_3,\tau_4,E)$is called a soft qT_2 space.

Definition 32. Let $(X,\tau_1,\tau_2,\tau_3,\tau_4,E)$ be a soft topological space (G,E) be q-closed soft set in X and $x\in X_A$ such that $x\notin(G,E)$. If there occurs soft q-open sets (F_1,E) and (F_2,E) such that $x\in(F_1,E),(G,E)\subseteq(F_2,E)$ and $(F_1,E)\cap(F_2,E)=\varphi$. Then $(X,\tau_1,\tau_2,\tau_3,\tau_4,E)$ is called soft q-regular spaces . A soft q-regular $qT_1\ Space$ is called soft qT_3space.

Then $(X,\tau_1,\tau_2,\tau_3,\tau_4,E)$is called a soft q-regular spaces . A soft q-regular T_1 $Space$ is called soft qT_3space.

Definition 33. $(X,\tau_1,\tau_2,\tau_3,\tau_4,E)$ be a *soft quad topological* space $(F_1,E),(G,E)$ be closed soft sets in X such that $(F,E)\cap(G,E)=\varphi$ If there exists q- open soft sets (F_1,E) and (F_2,E) such that $(F,E)\subseteq(F_1,E),(G,E)\subseteq(F_2,E)$ and $(F_1,E)\cap(F_2,E)=\varphi$. Then $(X,\tau_1,\tau_2,\tau_3,\tau_4,E)$ is called a q-soft normal space. A soft q-normal qT_1 $Space$ is called soft qT_4 $Space$.

Definition 34. Let (X,τ,A) be a soft Topological space over X and $e_G,e_H\in X_A$ such that $e_G\neq e_H$ if there can happen at least one soft semi open set (F_1,A) or (F_2,A) such that $e_G\in(F_1,A),e_H\notin(F_1,A)$ or $e_H\in(F_2,A),e_G\notin((F_2,A)$ then $(X,\tau_1,\tau_2,\tau_3,\tau_4,E)$ is called a *soft* qT_0 *space.*

Definition 35. Let $(X,\tau_1,\tau_2,\tau_3,\tau_4,E)$be a soft Topological spaces over X and $e_G,e_H\in X_A$ such that $e_G\neq e_H$ if there can happen *soft q-open sets* (F_1,A) and (F_2,A) such that $e_G\in(F_1,A),e_H\notin(F_1,A)$ and $e_H\in(F_2,A),e_G\notin((F_2,A)$ then $(X,\tau_1,\tau_2,\tau_3,\tau_4,E)$ is called soft qT_1 $space$.

Definition 36. Let $(X,\tau_1,\tau_2,\tau_3,\tau_4,E)$ be a *soft Topological* space over X and $e_G,e_H\in X_A$ such that $e_G\neq e_H$ if there can happen *soft q-open sets* $(F_1,A)$and (F_2,A) such that $e_G\in(F_1,A)$, and $e_H\in(F_2,A)$, $(F_1,A)\cap(F_2,A)=\phi_A$. Then $(X,\tau_1,\tau_2,\tau_3,\tau_4,E)$ is called soft qT_2 $space$

Definition 37. Let $(X,\tau_1,\tau_2,\tau_3,\tau_4,E)$ be a soft topological space (G,E) be q-closed soft set in X and $e_G\in X_A$ such that$e_G\notin(G,E)$. If there occurs soft q-open sets (F_1,E) and (F_2,E) such that $e_G\in(F_1,E),(G,E)\subseteq(F_2,E)$ and $(F_1,E)\cap(F_2,E)=\varphi$. Then $(X,\tau_1,\tau_2,\tau_3,\tau_4,E)$ is called soft q-regular spaces . A soft q- regular qT_1 $Space$ is called soft qT_3space.

Definition 38. In a *soft quad topological* space $(X,\tau_1,\tau_2,\tau_3,\tau_4,E)$

1) $\tau_1\cup\tau_2$is said to be *soft* βT_0 space with respect to $\tau_3\cup\tau_4$ if for each pair of points $x,y\in X$ such that $x\neq y$ there exists $\tau_1\cup\tau_2$ soft β-open set (F,E) and a $\tau_3\cup\tau_4$ soft β-open set (G,E) such that $x\in(F,E)$ and $y\notin(G,E)$ or $y\in(G,E)$ and $x\notin(G,E)$. Similarly, $\tau_3\cup\tau_4$is said to be *soft* βT_0 space with respect to $\tau_1\cup\tau_2$ if for each pair of points $x,y\in X$such that $x\neq y$ there exists $\tau_3\cup\tau_4$*soft βopen set*(F,E) and $\tau_1\cup\tau_2$soft β open set (G,E) such that $x\in(F,E)$ and $y\notin(F,E)$ or $y\in(G,E)$ and $x\notin(G,E)$. *Soft quad topological spaces* $(X,\tau_1,\tau_2,\tau_3,\tau_4,E)$ is said to be *pair wise soft* βT_0 space if $\tau_1\cup\tau_2$ is *soft* βT_0 *space* with respect to $\tau_3\cup\tau_4$ and to $\tau_3\cup\tau_4$ and is *soft* βT_0space with respect to$\tau_1\cup\tau_2$.

2)$\tau_1\cup\tau_2$ is said to be soft βT_1 space with respect to $\tau_3\cup\tau_4$ if for each pair of points $x,y\in X$such that $x\neq y$ there exists $\tau_1\cup\tau_2$ *soft β open set* (F,E) and to $\tau_3\cup\tau_4$ soft β open set (G,E) such that $x\in(F,E)$ and $y\notin(G,E)$ and $y\in(G,E)$ and $x\notin(G,E)$. Similarly, $\tau_3\cup\tau_4$is said to be soft βT_1 space with respect to $\tau_1\cup\tau_2$ if for each pair of distinct points $x,y\in X$ such that $x\neq y$ there exists $\tau_3\cup\tau_4$ soft β open set (F,E) and a to $\tau_1\cup\tau_2$ soft β open set (G,E) such that $x\in(F,E)$ and $y\notin(F,E)$ and $y\in(G,E)$ and $x\notin(G,E)$. *Soft quad topological spaces* $(X,\tau_1,\tau_2,\tau_3,\tau_4,E)$ is said to be *pair wise soft*

βT_1 space if $\tau_1 \cup \tau_2$ is *soft* βT_1 *space*with respect to $\tau_3 \cup \tau_4$and to $\tau_3 \cup \tau_4$is *soft* βT_1space with respect to $\tau_1 \cup \tau_2$.

3)$\tau_1 \cup \tau_2$is said to be soft βT_2 space with respect to $\tau_1 \cup \tau_2$ if for each pair of points $x, y \in X$ such that $x \neq y$ there exists a $\tau_1 \cup \tau_2$ soft β open set (F, E) and a $\tau_3 \cup \tau_4$ soft β open set (G, E) such that $x \in (F, E)$ and $y \in (G, E)$, $(F, E) \cap (G, E) = \phi$. Similarly, $\tau_3 \cup \tau_4$is said to be *soft* βT_2space with respect to $\tau_1 \cup \tau_2$ if for each pair of points $x, y \in X$ such that $x \neq y$there exists a $\tau_3 \cup \tau_4$*soft* β *open* set (F, E) and a $\tau_1 \cup \tau_2$*soft* β *open* set (G, E) such that $x \in (F, E)$,$y \in (G, E)$ and $(F, E) \cap (G, E) = \phi$. The soft quad topological space $(X, \tau_1, \tau_2, \tau_3, \tau_4, E)$ is said to be pair wise soft βT_2 space if $\tau_1 \cup \tau_2$ is soft βT_2 space with respect to $\tau_3 \cup \tau_4$ and $\tau_3 \cup \tau_4$ is soft βT_2 space with respect to$\tau_1 \cup \tau_2$.

Definition 39. In a soft quad topological space $(X, \tau_1, \tau_2, \tau_3, \tau_4, E)$

1) $\tau_1 \cup \tau_2$is said to be soft βT_3 space with respect to a $\tau_3 \cup \tau_4$if $\tau_1 \cup \tau_2$ is soft βT_1 space with respect to $\tau_3 \cup \tau_4$ and for each pair of points $x, y \in X$ such that $x \neq y$ there exists $\tau_1 \cup \tau_2$ Soft β closed set (G, E) such that $x \notin (G, E)$, a$\tau_1 \cup \tau_2$soft β open set (F_1, E) and $\tau_3 \cup \tau_4$ soft β open set (F_2, E) such that $x \in (F_1, E), (G, E) \subseteq (F_2, E)and(F_1, E) \cap (F_2, E) = \emptyset$. Similarly, $\tau_3 \cup \tau_4$is said to be soft βT_3 space with respect to $\tau_1 \cup \tau_2$ if $\tau_3 \cup \tau_4$ is soft βT_1 space with respect to $\tau_1 \cup \tau_2$ and for each pair of points $x, y \in X$ such that $x \neq y$ there exists a $\tau_3 \cup \tau_4$ soft β closed set (G, E) such that $x \notin (G, E)$, $\tau_3 \cup \tau_4$ soft β open set (F_1, E) and$\tau_1 \cup \tau_2$ soft β open set (F_2, E) such that $x \in (F_1, E), (G, E) \subseteq (F_2, E)$ and $(F_1, E) \cap (F_2, E) = \phi$. $(X, \tau_1, \tau_2, \tau_3, \tau_4, E)$is said to be pair wise soft βT_3 space if $\tau_1 \cup \tau_2$ is soft βT_3 space with respect to $\tau_3 \cup \tau_4$ and $\tau_3 \cup \tau_4$is soft βT_3 space with respect to$\tau_1 \cup \tau_2$.

2) $\tau_1 \cup \tau_2$is said to be soft βT_4 space with respect to $\tau_3 \cup \tau_4$if $\tau_1 \cup \tau_2$ is soft βT_1 space with respect to $\tau_3 \cup \tau_4$, there exists a $\tau_1 \cup \tau_2$ soft β closed set (F_1, E) and $\tau_3 \cup \tau_4$ soft β closed set (F_2, E) such that $(F_1, E) \cap (F_2, E) = \emptyset$. Also there exists (F_3, E) and (G_1, E) such that (F_3, E) is soft $\tau_1 \cup \tau_2$ β open set, (G_1, E) is soft $\tau_3 \cup \tau_4$ β open set such that $(F_1, E) \subseteq (F_3, E)$, $(F_2, E) \subseteq (G_1, E)$. Similarly,$\tau_3 \cup \tau_4$is said to be soft βT_4 space with respect to $\tau_1 \cup \tau_2$ if $\tau_3 \cup \tau_4$ is soft βT_1 space with respect to$\tau_1 \cup \tau_2$, there exists $\tau_3 \cup \tau_4$ soft β closed set (F_1, E) and $\tau_1 \cup \tau_2$ soft β closed set (F_2, E) such that $(F_1, E) \cap (F_2, E) = \phi$. Also there exist$(F_3, E)$ and (G_1, E) such that (F_3, E) is soft $\tau_3 \cup \tau_4$ β open set, (G_1, E) is soft $\tau_1 \cup \tau_2$ β open set such that $(F_1, E) \subseteq (F_3, E), (F_2, E) \subseteq (G_1, E)$ and $(F_3, E) \cap (G_1, E) = \phi$. Thus, (X, τ_1, τ_2, E) is said to be pair wise soft βT_4 space if $\tau_1 \cup \tau_2$is soft βT_4 space with respect to$\tau_3 \cup \tau_4$ and $\tau_3 \cup \tau_4$ is soft βT_4 space with respect to$\tau_1 \cup \tau_2$.

Proposition 3. Let $(X, \tau_1, \tau_2, \tau_3, \tau_4, E)$ be a soft quad topological space over X. Then, if (X, τ_1, τ_2, E) and (X, τ_3, τ_4, E) are soft βT_3 space then $(X, \tau_1, \tau_2, \tau_3, \tau_4, E)$ is a pair wise soft βT_2 space.

Proof: Suppose $(X, \tau_1, \tau_2, E)$is a soft βT_3 space with respect to (X, τ_3, τ_4, E) then according to definition for $x, y \in X$, which distinct, by using Proposition 1, (Y, E) is soft β closed set in $\tau_3 \cup \tau_4$ and $x \notin (Y, E)$ there exists a $\tau_1 \cup \tau_2$ soft β open set (F, E) and a $\tau_3 \cup \tau_4$ soft β open set (G, E) such that $x \in (F, E)$, $y \in (Y, E) \subseteq (G, E)$ and $(F_1, E) \cap (F_2, E) = \phi$. Hence $\tau_1 \cup \tau_2$ is soft βT_2 space with respect to$\tau_3 \cup \tau_4$. Similarly, if (X, τ_3, τ_4, E) is a soft βT_3 space with respect to (X, τ_1, τ_2, E) then according to definition for $x, y \in X, x \neq y$, by using Theorem 2, (x, E) is β closed soft set in $\tau_1 \cup \tau_2$and $y \notin (x, E)$ there exists a $\tau_3 \cup \tau_4$ soft β open set (F, E) and a $\tau_1 \cup \tau_2$ soft β open set (G, E) such that $y \in (F, E)$, $x \in (x, E) \subseteq$

(G,E) and $(F_1,E) \cap (F_2,E) = \phi$ .Hence $\tau_3 \cup \tau_4$is soft βT_2space. This implies that $(X,\tau_1,\tau_2,\tau_3,\tau_4,E)$ is a pair wise soft βT_2 space.

Proposition 4. Let $(X,\tau_1,\tau_2,\tau_3,\tau_4,E)$ be a soft quad topological space over X. if (X,τ_1,τ_2,E) and (X,τ_3,τ_4,E) are soft βT_3 space then $(X,\tau_1,\tau_2,\tau_3,\tau_4,E)$ is a pair wise soft βT_3 space.

Proof: Suppose (X,τ_1,τ_2,E) is a soft βT_3 space with respect to (X,τ_3,τ_4,E) then according to definition for $x,y \in X, x \neq y$ there exists a (X,τ_1,τ_2,E) soft β open set (F,E) and a (X,τ_3,τ_4,E) soft β open set (G,E) such that $x \in (F,E)$ and $y \notin (F,E)$ or $y \in (G,E)$ and $x \notin (G,E)$ and for each point $x \in X$ and each (X,τ_1,τ_2,E) β closed soft set (G_1,E) such that $x \notin (G_1,E)$ there exists a (X,τ_1,τ_2,E) soft β open set (F_1,E) and (X,τ_3,τ_4,E) soft β open set (F_2,E) such that $x \in (F_1,E), (G_1,E) \subseteq (F_2,E)$ and $(F_1,E) \cap (F_2,E) = \phi$. Similarly, to (X,τ_3,τ_4,E) is a soft βT_3 space with respect to (X,τ_1,τ_2,E). So according to definition for $x,y \in X, x \neq y$ there exists a (X,τ_3,τ_4,E) soft β open set (F,E) and a (X,τ_1,τ_2,E) soft β open set (G,E) such that $x \in (F,E)$ and $y \notin (F,E)$ or $y \in (G,E)$ and $x \notin (G,E)$ and for each point $x \in X$ and each (X,τ_3,τ_4,E) β closed soft set (G_1,E) such that $x \notin (G_1,E)$ there exists (X,τ_3,τ_4,E) soft β open set $(F_1,E)$and (X,τ_1,τ_2,E) soft β open set (F_2,E) such that $x \in (F_1,E), (G_1,E) \subseteq (F_2,E)$ and $(F_1,E) \cap (F_2,E) = \phi$. Hence $(X,\tau_1,\tau_2,\tau_3,\tau_4,E)$ is pair wise soft βT_3 space.

Proposition 5. If $(X,\tau_1,\tau_2,\tau_3,\tau_4,E)$ be a soft quad topological space over X. if(X,τ_1,τ_2,E) and $(X,\tau_3,\tau_4,E)$are soft βT_4 space then $(X,\tau_1,\tau_2,\tau_3,\tau_4,E)$ is pair wise soft βT_4 space.

Proof: Suppose $(X,\tau_1,\tau_2,E)$is soft βT_4 space with respect to (X,τ_3,τ_4,E). So according to definition for $x,y \in X, x \neq y$ there exist a (X,τ_1,τ_2,E) soft βopen set (F,E) and a (X,τ_3,τ_4,E) soft β open set (G,E) such that $x \in (F,E)$ and $y \notin (F,E)$ or $y \in (G,E)$ and $x \notin (G,E)$ each (X,τ_1,τ_2,E) soft β closed set (F_1,E) and a (X,τ_3,τ_4,E) soft β closed set (F_2,E) such that $(F_1,E) \cap (F_2,E) = \phi$. There exist (F_3,E) and (G_1,E) such that (F_3,E) is soft (X,τ_3,τ_4,E) and soft β open set (G_1,E) is soft (X,τ_1,τ_2,E) β open set $(F_1,E) \subseteq (F_3,E), (F_2,E) \subseteq (G_1,E)$ and $(F_3,E) \cap (G_1,E) = \phi$. Similarly, $(X,\tau_3,\tau_4,E)$is soft βT_4 space with respect to (X,τ_1,τ_2,E) is so according to definition for $x,y \in X, x \neq y$ there exists a (X,τ_3,τ_4,E) soft semi open set (F,E) and a (X,τ_1,τ_2,E)soft β open set (G,E) such that $x \in (F,E)$ and $y \notin (F,E)$ or $y \in (G,E)$ and $x \notin (G,E)$ and for each (X,τ_3,τ_4,E)soft β closed set (F_1,E) and (X,τ_1,τ_2,E) soft β closed set (F_2,E) such that $(F_1,E) \cap (F_2,E) = \phi$.there exists soft β open sets (F_3,E) and (G_1,E) such that (F_3,E) is soft (X,τ_3,τ_4,E) β open set (G_1,E) is soft (X,τ_1,τ_2,E) β open set such that $(F_1,E) \subseteq (F_3,E), (F_2,E) \subseteq (G_1,E)$ and $(F_3,E) \cap (G_1,E) = \phi$. Hence $(X,\tau_1,\tau_2,\tau_3,\tau_4,E)$is pair wise soft βT_4 space.

Proposition 6. Let $(X,\tau_1,\tau_2,\tau_3,\tau_4,E)$ be a soft quad topological space over X and Y be a non-empty subset of X. if $(X,\tau_{1Y},\tau_{2Y},\tau_{3Y},\tau_{4Y},E)$ is pair wise soft βT_3 space. Then $(Y,\tau_{1Y},\tau_{2Y},\tau_{3Y},\tau_{4Y},E)$is pair wise soft βT_3 space.

Proof: First we prove that $(X,\tau_{1Y},\tau_{2Y},\tau_{3Y},\tau_{4Y},E)$ is pair wise soft βT_1 space. Let $x,y \in X, x \neq y$ if $(X,\tau_1,\tau_2,\tau_3,\tau_4,E)$is pair wise space then this implies that $(X,\tau_1,\tau_2,\tau_3,\tau_4,E)$is pair wise soft space. So there exists (X,τ_1,τ_2,E) soft β open (F,E) and (X,τ_3,τ_4,E) soft β open set (G,E) such that $x \in (F,E)$ and $y \notin (F,E)$ or $y \in (G,E)$ and $x \notin (G,E)$ now

$x \in Y$ *and* $x \notin (G,E)$. Hence $x \in Y \cap (F,E) = (Y_F,E)$ then $y \notin Y \cap (\alpha)$ for some $\alpha \in E$. this means that $\alpha \in E$ then $y \notin Y \cap F(\alpha)$ for some $\alpha \in E$.

Therefore, $y \notin Y \cap (F,E) = (Y_F,E)$. Now $y \in Y$ and $y \in (G,E)$. Hence $y \in Y \cap (G,E) = (G_Y,E)$ where $(G,E) \in (X,\tau_3,\tau_4,E)$. Consider $x \notin (G,E)$ this means that $\alpha \in E$ then $x \notin Y \cap G(\alpha)$ for some $\alpha \in E$. There fore $x \notin Y \cap (G,E) = (G_Y,E)$ thus $(Y,\tau_{1Y},\tau_{2Y},\tau_{3Y},\tau_{4Y},E)$ is pair wise soft βT_1 space.

Now we prove that $(Y,\tau_{1Y},\tau_{2Y},\tau_{3Y},\tau_{4Y},E)$ is pair wise soft βT_3 space then $(Y,\tau_{1Y},\tau_{2Y},\tau_{3Y},\tau_{4Y},E)$ is pair wise soft β regular space.

Let $y \in Y$ and (G,E) be a soft β closed set in Y such that $y \notin (G,E)$ where $(G,E) \in (X,\tau_1,\tau_2,\tau_3,\tau_4,E)$ then $(G,E) = (Y,E) \cap (F,E)$ for some soft β closed set in $(X,\tau_{1Y},\tau_{2Y},\tau_{3Y},\tau_{4Y},E)$. Hence $y \notin (Y,E) \cap (F,E)$ but $y \in (Y,E)$, so $y \notin (F,E)$ since $(X,\tau_1,\tau_2,\tau_3,\tau_4,E)$ is soft βT_3 space $(X,\tau_1,\tau_2,\tau_3,\tau_4,E)$ is soft β regular space so there exists (X,τ_1,τ_2,E) soft β open set (F_1,E) and (X,τ_3,τ_4,E) soft β open set (F_2,E) such that $y \in (F_1,E), (G,E) \subseteq (F_2,E),\ (F_1,E)(F_2,E) = \phi$

Take $(G_1,E) = (Y,E) \cap (F_2,E)$ then $(G_1,E),(G_2,E)$ are soft β open set in Y such that
$y \in (G_1,E), (G,E) \subseteq (Y,E) \cap (F_2,E) = (G_2,E)$
$(G_1,E) \cap (G_2,E) \subseteq (F_1,E) \cap (F_2,E) = \phi$
$(G_1,E) \cap (G_2,E) = \phi$

There fore $\tau_{1Y} \cup \tau_{2Y}$ is soft β regular space with respect to $\tau_{3Y} \cup \tau_{4Y}$. Similarly, Let $y \in Y$ and (G,E) be a soft β closed sub set in Y such that $y \notin (G,E)$, where $(G,E) \in (X,\tau_3,\tau_4,E)$ then $(G,E) = (Y,E) \cap (F,E)$ where (F,E) is some soft β closed set in (X,τ_3,τ_4,E). $y \notin (Y,E) \cap (F,E)$ But $y \in (Y,E)$ so $y \notin (F,E)$ since (X,τ_1,τ_2,E) is soft β regular space so there exists (X,τ_3,τ_4,E) soft β open set (F_1,E) and (X,τ_1,τ_2,E) soft β open set (F_2,E). Such that
$y \in (F_1,E), (G,E) \subseteq (F_2,E)$
$(F_1,E) \cap (F_2,E) = \phi$

Take $$(G_1,E) = (Y,E) \cap (F_1,E)$$
$$(G_1,E) = (Y,E) \cap (F_1,E)$$

Then (G_1,E) and (G_2,E) are soft β open set in Y such that

$$y \in (G_1,E), (G,E) \subseteq (Y,E) \cap (F_2,E) = (G_2,E)$$
$$(G_1,E) \cap (G_2,E) \subseteq (F_1,E) \cap (F_2,E) = \phi$$

There for e$\tau_{3Y} \cup \tau_{4Y}$ is soft β regular space with respect $\tau_{1Y} \cup \tau_{2Y} \Rightarrow (Y,\tau_{1Y},\tau_{2Y},\tau_{3Y},\tau_{4Y},E)$ is pair wise soft βT_3 space.

Proposition 7. Let $(X,\tau_1,\tau_2,\tau_3,\tau_4,E)$ be a soft quad topological space over X and Y be a soft β closed sub space of X. if $(X,\tau_1,\tau_2,\tau_3,\tau_4,E)$ is pair wise soft βT_4 space then $(Y,\tau_{1Y},\tau_{2Y},\tau_{3Y},\tau_{4Y},E)$ is pair wise soft βT_4 space.

Proof: Since $(X,\tau_1,\tau_2,\tau_3,\tau_4,E)$ is pair wise soft βT_4 space so this implies that $(X,\tau_1,\tau_2,\tau_3,\tau_4,E)$ is pair wise soft βT_1 space as proved above.

We prove $(X,\tau_1,\tau_2,\tau_3,\tau_4,E)$is pair wise soft β normal space.

Let $(G_1,E),(G_2,E)$ be soft β closed sets in Y such that

$$(G_1,E)\cap(G_2,E)=\phi$$

Then $$(G_1,E)=(Y,E)\cap(F_1,E)$$

And $$(G_2,E)=(Y,E)\cap(F_2,E)$$

For some soft β closed sets such that (F_1,E) is soft β closed set in $\tau_1\cup\tau_2$ soft β closed set(F_2,E) in $\tau_3\cup\tau_4$.

And $$(F_1,E)\cap(F_2,E)=\phi$$

From Proposition 2. Since, Y is soft β closed sub set of X then $(G_1,E),(G_2,E)$ are soft β closed sets in X such that

$$(G_1,E)\cap(G_2,E)=\phi$$

Since $(X,\tau_1,\tau_2,\tau_3,\tau_4,E)$ is pair wise soft β normal space. So there exists soft β open sets (H_1,E) and (H_2,E) such that

$(H_1,E)$is soft β open set in $\tau_1\cup\tau_2$and (H_2,E) is soft β open set in $\tau_3\cup\tau_4$such that

$$(G_1,E)\subseteq(H_1,E)$$
$$(G_2,E)\subseteq(H_2,E)$$
$$(H_1,E)\cap(H_2,E)=\phi$$

Since $$(G_1,E),(G_2,E)\subseteq(Y,E)$$

Then $$(G_1,E)\subseteq(Y,E)\cap(H_1,E)$$
$$(G_2,E)\subseteq(Y,E)\cap(H_2,E)$$

And $$[(Y,E)\cap(H_1,E)]\cap[(Y,E)\cap(H_2,E)]=\phi$$

Where $(Y,E)\cap(H_1,E)$ and $(Y,E)\cap(H_2,E)$ are soft β open sets in Y there fore $\tau_{1Y}\cup\tau_{2Y}$ is soft β normal space with respect to$\tau_{3Y}\cup\tau_{4Y}$. Similarly, let $(G_1,E),(G_2,E)$ be soft β closed sub set in Y such that

$$(G_1,E)\cap(G_2,E)=\phi$$

Then $$(G_1,E)=(Y,E)\cap(F_1,E)$$

And $$(G_2,E)=(Y,E)\cap(F_2,E)$$

For some soft β closed sets such that (F_1,E) is soft βclosed set in $\tau_3\cup\tau_4 and\ (F_2,E)$ soft β closed set in $\tau_1\cup\tau_2$and

$$(F_1,E)(F_2,E)=\phi$$

From Proposition 2. Since, Y is soft β closed sub set in X then $(G_1,E),(G_2,E)$ are soft β closed sets in X such that

$$(G_1,E)\cap(G_2,E)=\phi$$

Since $(X,\tau_1,\tau_2,\tau_1,\tau_2,E)$ is pair wise soft β normal space so there exists soft β open sets (H_1,E) and (H_2,E)

Such that (H_1,E) is soft β open set is $\tau_3\cup\tau_4$ and (H_2,E) is soft β open set in $\tau_1\cup\tau_2$ such that

$$(G_1,E)\subseteq(H_1,E)$$
$$(G_2,E)\subseteq(H_2,E)$$
$$(H_1,E)\cap(H_2,E)=\phi$$

Since $$(G_1,E),(G_2,E)\subseteq(Y,E)$$

Then $$(G_1,E)\subseteq(Y,E)\cap(H_1,E)$$
$$(G_2,E)\subseteq(Y,E)\cap(H_2,E)$$

And $$[(Y,E)\cap(H_1,E)]\cap[(Y,E)\cap(H_2,E)]=\phi$$

Where $(Y,E)\cap(H_1,E)$ and $(Y,E)\cap(H_2,E)$ are soft β open sets in Y there fore $\tau_{3Y}\cup\tau_{4Y}$ is soft β normal space with respect to$\tau_{1Y}\cup\tau_{2Y}$

$\Rightarrow(Y,\tau_{1Y},\tau_{2Y},\tau_{3Y},\tau_{4Y}E)$is pair wise soft βT_4 space.

4.2 Soft β-Separation Axioms in Soft Quad Topological Spaces with Respect to Soft Points.

In this section, we brought out soft topological structures known as β separation axioms in soft quad topology with respect to soft points. With the applications of this soft β separation axioms different result are brought under examination.

Definition 40. In a *soft quad topological* space $(X, \tau_1, \tau_2, \tau_1, \tau_2, E)$

1) $\tau_1 \cup \tau_2$ said to be *soft* βT_0 space with respect to $\tau_3 \cup \tau_4$if for each pair of distinct points $e_G, e_H \in X_A$there happens $\tau_1 \cup \tau_2$ soft β open set (F, E) and a $\tau_3 \cup \tau_4$soft β open set (G, E) such that $e_G \in (F, E)$ and $e_H \notin (G, E)$, Similarly, $\tau_3 \cup \tau_4$is said to be *soft* βT_0 space with respect to $\tau_1 \cup \tau_2$if for each pair of distinct points $e_G, e_H \in X_A$there happens $\tau_3 \cup \tau_4$*soft βopen set*(F, E) and a $\tau_1 \cup \tau_2$β soft open set (G, E) such that $e_G \in (F, E)$ and $e_H \notin (F, E)$ or $e_H \in (G, E)$ and $e_G \notin (G, E)$.*Soft quad topological spaces* $(X, \tau_1, \tau_2, \tau_1, \tau_2, E)$is said to be *pair wise soft* βT_0 space if $\tau_1 \cup \tau_2$ is *soft* βT_0 *space* with respect to $\tau_3 \cup \tau_4$ and$\tau_3 \cup \tau_4$ is soft βT_0 spaces with respect to$\tau_1 \cup \tau_2$.

2) $\tau_1 \cup \tau_2$is said to be soft βT_1 space with respect to $\tau_3 \cup \tau_4$ if for each pair of distinct points $e_G, e_H \in X_A$there happens a$\tau_1 \cup \tau_2$*soft β open set* $(F. E)$and $\tau_3 \cup \tau_4$ soft β open set (G, E) such that $e_G \in (F, E)$ and $e_H \notin (G, E)$ and $e_H \in (G, E)$ and $e_G \notin (G, E)$.Similarly, $\tau_3 \cup \tau_4$is said to be soft βT_1 space with respect to $\tau_1 \cup \tau_2$ if for each pair of distinct points $e_G, e_H \in X_A$there exist a $\tau_3 \cup \tau_4$soft β open set (F, E) and a $\tau_1 \cup \tau_2$ soft β open set (G, E) such that $e_G \in (F, E)$ and$e_H \notin (G, E)$ and $e_H \in (G, E)$ and $e_G \notin (G, E)$. *Soft quad topological space*$(X, \tau_1, \tau_2, \tau_1, \tau_2, E)$is said to be *pair wise soft* βT_1 space if $\tau_1 \cup \tau_2$ is *soft* βT_1 *space* with respect to $\tau_3 \cup \tau_4$ and$\tau_3 \cup \tau_4$is soft βT_1 spaces with respect to$\tau_1 \cup \tau_2$.

3) $\tau_1 \cup \tau_2$ is said to be soft βT_2 space with respect to$\tau_3 \cup \tau_4$, if for each pair of distinct points $e_G, e_H \in X_A$there happens a $\tau_1 \cup \tau_2$ soft β open set (F, E) and a $\tau_3 \cup \tau_4$ soft β open set (G, E) such that $e_G \in (F, E)$ and $e_H \notin (G, E)$ and $e_H \in (G, E)$ and $e_G \notin (G, E)$ and $(F, E) \cap (G, E) = \phi$. Similarly, $\tau_3 \cup \tau_4$is aid to be*soft* βT_2space with respect to $\tau_1 \cup \tau_2$ if for each pair of distinct points$e_G, e_G \in X_A$there happens a $\tau_3 \cup \tau_4$*soft βopen* set (F, E) and a $\tau_1 \cup \tau_2$*soft βopen* set (G, E) such that $e_G \in (F, E)$ and $e_G \in (G, E)$ and $(F, E) \cap (G, E) = \phi$. The soft quad topological space $(X, \tau_1, \tau_2, \tau_1, \tau_2, E)$is said to be pair wise soft βT_2 space if $\tau_1 \cup \tau_2$ is soft βT_2 space with respect to $\tau_3 \cup \tau_4$ and $\tau_3 \cup \tau_4$ is soft βT_2 space with respect to$\tau_1 \cup \tau_2$.

Definition 41. In a soft quad topological space $(X, \tau_1, \tau_2, \tau_1, \tau_2, E)$

1) $\tau_1 \cup \tau_2$ is said to be soft βT_3 space with respect to $\tau_3 \cup \tau_4$ if $\tau_1 \cup \tau_2$ is soft βT_1 space with respect to $\tau_3 \cup \tau_4$ and for each pair of distinct points$e_G, e_H \in X_A$, there exists a $\tau_1 \cup \tau_2$ β closed soft set (G, E) such that$e_G \notin (G, E)$, $\tau_1 \cup \tau_2$soft β open set (F_1, E) and $\tau_3 \cup \tau_4$ soft β open set (F_2, E) such that $e_G \in (F_1, E), (G, E) \subseteq (F_2, E)$ and $(F_1, E) \cap (F_2, E) = \emptyset$. Similarly, $\tau_3 \cup \tau_4$is said to be soft βT_3 space with respect to $\tau_1 \cup \tau_2$ if $\tau_3 \cup \tau_4$ is soft βT_1 space with respect to $\tau_1 \cup \tau_2$ and for each pair of distinct points $e_G, e_H \in X_A$ there exists a $\tau_3 \cup \tau_4$ soft β closed set (G, E) such that $e_G \notin (G, E)$, $\tau_3 \cup \tau_4$soft β open set (F_1, E) amd $\tau_1 \cup \tau_2$soft β open set (F_2, E) such that $e_G \in (F_1, E), (G, E) \subseteq (F_2, E)$ and $(F_1, E) \cap (F_2, E) = \phi$. $(X, \tau_1, \tau_2, \tau_1, \tau_2, E)$is said to be pair wise soft βT_3 space if $\tau_1 \cup \tau_2$ is soft βT_3 space with respect to $\tau_3 \cup \tau_4$ and $\tau_3 \cup \tau_4$ is soft βT_3 space with respect to$\tau_1 \cup \tau_2$.

2) $\tau_1 \cup \tau_2$is said to be soft βT_4 space with respect to $\tau_3 \cup \tau_4$ if $\tau_1 \cup \tau_2$ is soft βT_1 space with respect to $\tau_3 \cup \tau_4$, there exists a $\tau_1 \cup \tau_2$ soft β closed set (F_1, E) and $\tau_3 \cup \tau_4$ soft β closed set (F_2, E) such that $(F_1, E) \cap (F_2, E) = \emptyset$,also, there exists (F_3, E) and (G_1, E) such that (F_3, E) is soft $\tau_1 \cup \tau_2$ βopen set, (G_1, E) is soft $\tau_3 \cup \tau_4$ βopen set such that $(F_1, E) \subseteq (F_3, E)$, $(F_2, E) \subseteq (G_1, E)$. Similarly, $\tau_3 \cup \tau_4$is said to be soft βT_4 space with respect to $\tau_1 \cup \tau_2$ if $\tau_3 \cup \tau_4$ is soft βT_1 space with respect to$\tau_1 \cup \tau_2$there exists $\tau_3 \cup \tau_4$ soft β closed set (F_1, E) and $\tau_1 \cup \tau_2$ soft β closed set (F_2, E) such that $(F_1, E) \cap (F_2, E) = \phi$. Also there exists (F_3, E) and (G_1, E) such that (F_3, E) is soft $\tau_3 \cup \tau_4$ β open set, (G_1, E) is soft $\tau_1 \cup \tau_2$ β soft set such that $(F_1, E) \subseteq (F_3, E), (F_2, E) \subseteq (G_1, E)$ and $(F_3, E) \cap (G_1, E) = \phi$. Thus, $(X, \tau_1, \tau_2, \tau_1, \tau_2, E)$ is said to be pair wise soft βT_4 space if $\tau_1 \cup \tau_2$ is soft $\beta\, T_4$ space with respect to $\tau_3 \cup \tau_4$and $\tau_3 \cup \tau_4$ is soft βT_4 space with respect to $\tau_1 \cup \tau_2$.

Proposition 8. Let $(X, \tau_1, \tau_2, \tau_3, \tau_4, E)$ be a soft topological space over X. $(X, \tau_1, \tau_2, \tau_3, \tau_4, E)$ is soft βT_3 space, then for all $e_G \in X_E e_G = (e_G, E)$ is soft β-closed set.

Proof: We want to prove that $e_G$is βclosed soft set, which is sufficient to prove that e_G^c is βopen soft set for all $e_H \in \{e_G\}^c$. Since $(X, \tau_1, \tau_2, \tau_3, \tau_4, E)$is soft βT_3 space, then there exists soft β sets$(F, E)_{e_H}$ and (G, E) such that $e_{H_E} \subseteq (F, E)_{e_H}$ and $e_{G_E} \cap (F, E)_{e_H} = \phi$ and $e_{G_E} \subseteq (G, E)$ and $e_{H_E} \cap (G, E) = \phi$. It follows that, $\cup_{e_H \in (e_G)^c (F,E)_{e_H} \subseteq e_{G_E}^c}$Now, we want to prove that $e_{G_E}^c \subseteq \cup_{e_H \in (e_G)^c} (F, E)_{e_H}$. Let $\cup_{e_H \in (e_G)^c} (F, E)_{e_H} = (H, E)$. Where $H(e) = \cup_{e_H \in (e_G)^c (F(e)_{e_H}}$for all $e \in E$. Since $e_{G_E}^c(e) = (e_G)^c$for all $e \in E$ from Definition 9, so, for all $e_H \in \{e_G\}^c$ and $e \in E e_{G_E}^c(e) = \{e_G\}^c = \cup_{e_H \in (e_G)^c} \{e_H\} = \cup_{e_H \in (e_G)^c F(e)_{e_H} = H(e)}$. Thus, $e_{G_E}^c \subseteq \cup_{e_H \in (e_G)^c} (F, E)_{e_H}$ from Definition 2, and so, $e_{G_E}^c = \cup_{e_H \in (e_G)^c} (F, E)_{e_H}$.

This means that, $e_{G_E}^c$ is soft β open set for all $e_H \in \{e_G\}^c$. Therefore, $e_{G_E}$is βclosed soft set.

Proposition 9. Let $(X, \tau_1, \tau_{2,} \tau_3, \tau_4, E)$ be a soft quad topological space over X. Then, if (X, τ_1, τ_2, E) and (X, τ_3, τ_4, E) are soft βT_3 space, then $(X, \tau_1, \tau_{2,} \tau_3, \tau_4, E)$ is a pair wise soft βT_2 space.

Proof: Suppose if (X, τ_1, τ_2, E) is a soft βT_3 space with respect to (X, τ_3, τ_4, E), then according to definition for, $e_G \neq e_{H\,,} e_G, e_H \in X_A$, by using Theorem 8, ($e_H$, E) is soft β closed set in (X, τ_3, τ_4, E) and $e_G \notin (e_H, E)$ there exist a (X, τ_1, τ_2, E)soft β open set (F, E) and a (X, τ_3, τ_4, E) soft β open set (G, E) such that $e_G \in (F, E)$, $e_H \in (y, E) \subseteq (G, E)$ and $(F_1, E) \cap (F_2, E) = \phi$. Hence, $(X, \tau_1, \tau_2, E)$is soft βT_2 space with respect to (X, τ_3, τ_4, E) Similarly, if $(X, \tau_3, \tau_4, E)$is a soft βT_3 space with respect to (X, τ_1, τ_2, E), then according to definition for , $e_G \neq e_{H\,,} e_G, e_H \in X_A$, by using Theorem 8, ($e_G$, E) is β closed soft set in $(X, \tau_1, \tau_2, E)$is and $y \notin (x, E)$ there exists a (X, τ_3, τ_4, E)soft β open set (F, E) and a (X, τ_1, τ_2, E)soft β open set (G, E) such that $e_H \in (F, E)$, $e_G \in (x, E) \subseteq (G, E)$ and $(F_1, E) \cap (F_2, E) = \phi$. Hence, $(X, \tau_3, \tau_4, E)$is a soft βT_2 space. Thus $(X, \tau_1, \tau_{2,} \tau_3, \tau_4, E)$is a pair wise soft βT_2 space.

Proposition 10. Let $(X, \tau_1, \tau_{2,} \tau_3, \tau_4, E)$be a soft quad topological space over X. If if (X, τ_1, τ_2, E) and (X, τ_3, τ_4, E) are soft βT_3 space then $(X, \tau_1, \tau_{2,} \tau_3, \tau_4, E)$ is a pair wise soft βT_3 space.

Proof: Suppose $(X,\tau_1,\tau_2,E)$is a soft βT_3 space with respect to (X,τ_3,τ_4,E) then according to definition for $e_G,e_H \in X_A e_G \neq e_H$ there happens $\tau_1 \cup \tau_2$soft β open set (F,E) and a $\tau_3 \cup \tau_4$soft β open set (G,E) such that $e_G \in (F,E)$ and $e_H \notin (F,E)$ or $e_H \in (G,E)$ and $e_G \notin (G,E)$ and for each point $e_G \in X_A$and each $\tau_1 \cup \tau_2$ βclosed soft set (G_1,E) such that $e_G \notin (G_1,E)$ there happens a $\tau_1 \cup \tau_2$soft β open set (F_1,E) and $\tau_3 \cup \tau_4$soft β open set (F_2,E) such that $e_G \in (F_1,E),(G_1,E) \subseteq (F_2,E)$ and $(F_1,E) \cap (F_2,E) = \phi$. Similarly $(X,\tau_3,\tau_4,E)$is a soft βT_3 space with respect to (X,τ_1,E). So according to definition for$e_G,e_H \in X_A$, $e_G \neq e_H$ there exists a $\tau_3 \cup \tau_4$soft β open set (F,E) and $\tau_1 \cup \tau_2$ soft β open set (G,E) such that $e_H \in (F,E)$ and $e_H \notin (F,E)$ or $e_H \in (G,E)$ and $e_G \notin (G,E)$ and for each point $e_G \in X_A$ and each $\tau_3 \cup \tau_4$βclosed soft set (G_1,E) such that $e_G \notin (G_1,E)$ there exists $\tau_3 \cup \tau_4$ soft β open set (F_1,E) and $\tau_1 \cup \tau_2$soft β open set (F_2,E) such that $e_G \in (F_1,E),(G_1,E) \subseteq (F_2,E)$ and $(F_1,E) \cap (F_2,E) = \phi$. Hence $(X,\tau_1,\tau_{2,}\tau_3,\tau_4,E)$is pair wise soft βT_3 space.

Proposition 11. If $(X,\tau_1,\tau_{2,}\tau_3,\tau_4,E)$ be a soft quad topological space over X. (X,τ_1,τ_2,E) and (X,τ_3,τ_4,E) are soft βT_4 space then $(X,\tau_1,\tau_{2,}\tau_3,\tau_4,E)$is pair wise soft βT_4 space.

Proof: Suppose $((X,\tau_1,\tau_2,E))$ is soft βT_4 space with respect to(X,τ_3,τ_4,E). So according to definition for $e_G,e_H \in X_A$, $e_G \neq e_H$ there happens a $\tau_1 \cup \tau_2$soft β open set (F,E) and a $\tau_3 \cup \tau_4$soft β open set (G,E) such that$e_G \in (F,E)$ and $e_H \notin (F,E)$ or $e_H \in (G,E)$ and $e_G \notin (G,E)$ each $\tau_1 \cup \tau_2$soft β closed set (F_1,E) and a $\tau_3 \cup \tau_4$soft β closed set (F_2,E) such that $(F_1,E) \cap (F_2,E) = \phi$. There occurs (F_3,E) and (G_1,E) such that (F_3,E) is soft $\tau_3 \cup \tau_4$β open set (G_1,E) is soft a $\tau_1 \cup \tau_2$β open set $(F_1,E) \subseteq (F_3,E),(F_2,E) \subseteq (G_1,E)$ and$(F_3,E) \cap (G_1,E) = \phi$. Similarly, $\tau_3 \cup \tau_4$ is soft βT_4 space with respect to $\tau_1 \cup \tau_2$so according to definition for $e_G,e_H \in X_A,e_G \neq e_H$there happens a $\tau_3 \cup \tau_4$soft β open set (F,E) and a $\tau_1 \cup \tau_2$soft β open set (G,E) such that $e_G \in (F,E)$ and $e_H \notin (F,E)$ or $e_H \in (G,E)$ and $e_G \notin (G,E)$ and for each $\tau_3 \cup \tau_4$soft β closed set (F_1,E) and $\tau_1 \cup \tau_2$soft β closed set (F_2,E) such that $(F_1,E) \cap (F_2,E) = \phi$. there occurs (F_3,E) and (G_1,E) such that (F_3,E) is soft $\tau_3 \cup \tau_4$ β open set (G_1,E) is soft $\tau_1 \cup \tau_2$ βopen set such that $(F_1,E) \subseteq (F_3,E),(F_2,E) \subseteq (G_1,E)$ and $(F_3,E) \cap (G_1,E) = \phi$ hence $(X,\tau_1,\tau_{2,}\tau_3,\tau_4,E)$ is pair wise soft βT_4 space.

Proposition 12. Let $(X,\tau_1,\tau_{2,}\tau_3,\tau_4,E)$ be a soft quad topological space over X and Y be a non-empty subset of X. if $(Y,\tau_{1Y},\tau_{2Y},\tau_{3Y},\tau_{4Y},E)$ is pair wise soft βT_3 space. Then $(Y,\tau_{1Y},\tau_{2Y},\tau_{3Y},\tau_{4Y},E)$ is pair wise soft βT_3 space.

Proof: First we prove that $(Y,\tau_{1Y},\tau_{2Y},\tau_{3Y},\tau_{4Y},E)$is pair wise soft βT_1 space.

Let $e_G,e_H \in X_A,e_G \neq e_H$if $(X,\tau_1,\tau_{2,}\tau_3,\tau_4,E)$ is pair wise soft space then this implies that $(X,\tau_1,\tau_{2,}\tau_3,\tau_4,E)$is pair wise soft $\tau_1 \cup \tau_2$space. So there exists $\tau_1 \cup \tau_2$soft β open set (G,E) such that $e_G \in (F,E)$ and $e_H \notin (F,E)$ or $e_H \in (G,E)$ and $e_G \notin (G,E)$ now$e_G \in Y\, and\, e_G \notin (G,E)$. Hence $e_G \in Y \cap (F,E) = (Y_F,E)$ then $e_H \notin Y \cap F(\alpha)$ for some $\alpha \in E$. this means that $\alpha \in E$ then $e_H \notin Y \cap F(\alpha)$ for some $\alpha \in E$.

There fore, $e_H \notin Y \cap (F,E) = (Y_F,E)$. Now $e_H \in Y$ and $e_H \in (G,E)$. Hence, $e_H \in Y \cap (G,E) = (G_Y,E)$ where $(G,E) \in \tau_3 \cup \tau_4$. Consider $x \notin (G,E)$. this means that $\alpha \in E$ then $x \notin Y \cap G(\alpha)$ for some $\alpha \in E$. There fore $e_G \notin Y \cap (G,E) = (G_Y,E)$ thus $(Y,\tau_{1Y},\tau_{2Y},\tau_{3Y},\tau_{4Y},E)$ is pair wise soft βT_1 space.

Now, we prove that $(Y, \tau_{1Y}, \tau_{2Y}, \tau_{3Y}, \tau_{4Y}, E)$ is pair wise soft βT_3 space.

Let $e_H \in Y$ and (G, E) be soft β closed set in Y such that $e_H \notin (G, E)$ where $(G, E) \in \tau_1 \cup \tau_2$ then $(G, E) = (Y, E) \cap (F, E)$ for some soft β closed set in $\tau_1 \cup \tau_2$hence $e_H \notin (Y, E) \cap (F, E)$ but $e_H \in (Y, E)$, so $e_H \notin (F, E)$ since $(X, \tau_1, \tau_{2,} \tau_3, \tau_4, E)$ is soft βT_3 space $(X, \tau_1, \tau_{2,} \tau_3, \tau_4, E)$ is soft β regular space so there happens $\tau_1 \cup \tau_2$soft β open set (F_1, E) and $\tau_3 \cup \tau_4$ soft β open set (F_2, E) such that $e_H \in (F_1, E), (G, E) \subseteq (F_2, E)$, $(F_1, E)(F_2, E) = \phi$

Take $(G_1, E) = (Y, E) \cap (F_2, E)$ then $(G_1, E), (G_2, E)$ are soft β open sets in Y such that
$e_H \in (G_1, E), (G, E) \subseteq (Y, E) \cap (F_2, E) = (G_2, E)$
$(G_1, E) \cap (G_2, E) \subseteq (F_1, E) \cap (F_2, E) = \phi$
$(G_1, E) \cap (G_2, E) = \phi$

Therefore, $(\tau_{1Y}, \tau_{2Y})$is soft β regular space with respect to (τ_{3Y}, τ_{4Y})Similarly, Let $e_H \in Y$ and (G, E) be a soft β closed sub set in Y such that $e_H \notin (G, E)$, Where $(G, E) \in \tau_3 \cup \tau_4$ then $(G, E) = (Y, E) \cap (F, E)$ where (F, E) is some soft β closed set in$\tau_3 \cup \tau_4$. $e_H \notin (Y, E) \cap (F, E)Bute_H \in (Y, E)$ so $e_H \notin (F, E)$ since $(X, \tau_1, \tau_{2,} \tau_3, \tau_4, E)$ is soft β regular space so there happens $\tau_3 \cup \tau_4$ soft β open set (F_1, E) and $\tau_1 \cup \tau_2$ soft β open set (F_2, E). Such that
$e_H \in (F_1, E), (G, E) \subseteq (F_2, E)$
$(F_1, E) \cap (F_2, E) = \phi$
Take $(G_1, E) = (Y, E) \cap (F_1, E)$
$(G_1, E) = (Y, E) \cap (F_1, E)$
Then $(G_1, E)$and (G_2, E) are soft β open set in Y such that
$e_H \in (G_1, E), (G, E) \subseteq (Y, E) \cap (F_2, E) = (G_2, E)$
$(G_1, E) \cap (G_2, E) \subseteq (F_1, E) \cap (F_2, E) = \phi$.
Therefore$(\tau_{3Y}, \tau_{4Y})$is soft β regular space.

4. Conclusion

A soft set with single specific topological structure is unable to shoulder up the responsibility to construct the whole theory. So to make the theory healthy, some additional structures on soft set has to be introduced. It makes, it more springy to develop the soft topological spaces with its infinite applications. In this regard we introduce strong topological structure known as soft quad topological structure in this paper.

Topology is the supreme branch of pure mathematics which deals with mathematical structures. Freshly, many scholars have studied the soft set theory which is coined by Molodtsov [4] and carefully applied to many difficulties which contain uncertainties in our social life. Shabir and Naz [7] familiarized and deeply studied the origin of soft topological spaces. They also studied topological structures and exhibited their several properties with respect to ordinary points.

In the present work, we constantly study the behavior of soft semi separation axioms in soft quad topological spaces with respect to soft points as well as ordinary points of a soft topological space. We introduce soft β qT_0 structure, soft βqT_1 structure, soft βqT_2 structure, Soft βqT_3 and soft βqT_4 structure with respect to soft and ordinary points. In future we will plant these structures in different results. More over defined soft βT_0 structure w.r.t. soft βT_1structure and vice versa, soft βT_1structure w.r.t soft βT_2structure and vice versa and soft βT_3space w.r.t soft βT_4 and vice versa with respect to ordinary and soft points in soft quad topological spaces and studied their activities in different results with respect to ordinary and soft points . We also planted these axioms to different results. These soft semi separation axioms in quad structure would be valuable for the development of the theory of soft topology to solve complicated problems, comprising doubts in economics, engineering, medical etc. We also attractively discussed some soft transmissible properties with respect to ordinary as well as soft points. We expect that these results in this article will do help the researchers for strengthening the toolbox of soft topological structures. In the forthcoming, we spread the idea of soft α- open, and soft b^{**}open sets in soft quad topological structure with respect to ordinary and soft points.

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Partial Constant Hesitant Fuzzy Sets on UP-Algebras

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Abstaract − In this paper, partial constant hesitant fuzzy sets on UP-algebras are introduced and proved some results. Further, we discuss the relation between partial constant hesitant fuzzy sets and UP-subalgebras (resp. UP-filters, UP-ideals and strongly UP-ideals).

Keywords − *UP-algebra, hesitant fuzzy set, partial constant hesitant fuzzy set.*

1 Introduction and Preliminaries

Among many algebraic structures, algebras of logic form important class of algebras. Examples of these are BCK-algebras [5], BCI-algebras [6], BCH-algebras [3], KU-algebras [15], SU-algebras [12], UP-algebras [4] and others. They are strongly connected with logic. For example, BCI-algebras introduced by Iséki [6] in 1966 have connections with BCI-logic being the BCI-system in combinatory logic which has application in the language of functional programming. BCK and BCI-algebras are two classes of logical algebras. They were introduced by Imai and Iséki [5, 6] in 1966 and have been extensively investigated by many researchers. It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras.

A fuzzy subset f of a set S is a function from S to a closed interval $[0, 1]$. The concept of a fuzzy subset of a set was first considered by Zadeh [21] in 1965. The fuzzy set theories developed by Zadeh and others have found many applications in the domain of mathematics and elsewhere.

In 2009 - 2010, Torra and Narukawa [20, 19] introduced the notion of hesitant fuzzy sets, that is a function from a reference set to a power set of the unit interval. The notion of hesitant fuzzy sets is the other generalization of the notion fuzzy sets.

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The hesitant fuzzy set theories developed by Torra and others have found many applications in the domain of mathematics and elsewhere.

After the introduction of the notion of hesitant fuzzy sets by Torra and Narukawa [20, 19], several researches were conducted on the generalizations of the notion of hesitant fuzzy sets and application to many logical algebras such as: In 2012, Rodríguez et al. [16] introduced the notion of hesitant fuzzy linguistic term sets and several basic properties and operations to carry out the processes of computing with words. Zhu et al. [22] introduced the notion of dual hesitant fuzzy sets, which is a new extension of fuzzy sets. In 2014, Jun et al. [8] introduced the notions of hesitant fuzzy soft subalgebras and (closed) hesitant fuzzy soft ideals in BCK/BCI-algebras, and investigated related properties. Jun and Song [10] introduced the notions of (Boolean, prime, ultra, good) hesitant fuzzy filters and hesitant fuzzy MV-filters of MTL-algebras, and investigated their relations. In 2015, Ali et al. [1] introduced the notions of hesitant fuzzy products, characteristic hesitant fuzzy sets, hesitant fuzzy $\mathcal{AG}$-groupoids, hesitant fuzzy left (resp. right, twosided) ideals, hesitant fuzzy bi-ideals, hesitant fuzzy interior ideals and hesitant fuzzy quasi-ideals on $\mathcal{AG}$-groupoids, and investigated several properties. They also characterized regular, completely regular, weakly regular and quasi-regular $\mathcal{AG}$-groupoids in term of hesitant fuzzy ideals. Jun and Song [11] introduced the notions of hesitant fuzzy prefilters (resp. filters) and positive implicative hesitant fuzzy prefilters (resp. filters) of EQ-algebras, and investigated several properties. Jun et al. [9] introduced the notions of hesitant fuzzy (generalized) bi-ideals, and investigated related properties. In 2016, Jun and Ahn [7] introduced the notions of hesitant fuzzy subalgebras and hesitant fuzzy ideals of BCK/BCI-algebras, and investigated their relations and properties. Muhiuddin [14] introduced the notion of hesitant fuzzy filters of residuated lattices. In 2017, Mosrijai et al. [13] introduced the notion of hesitant fuzzy sets which is a new extension of fuzzy sets on UP-algebras and the notions of hesitant fuzzy UP-subalgebras, hesitant fuzzy UP-filters, hesitant fuzzy UP-ideals and hesitant fuzzy strongly UP-ideals of UP-algebras and investigated some of its essential properties. Satirad et al. [17] characterized the relationships among (prime, weakly prime) hesitant fuzzy UP-subalgebras (resp. hesitant fuzzy UP-filters, hesitant fuzzy UP-ideals and hesitant fuzzy strongly UP-ideals) and some level subsets of a hesitant fuzzy set on UP-algebras.

The notions of hesitant fuzzy subalgebras, hesitant fuzzy filters and hesitant fuzzy ideals play an important role in studying the many logical algebras. In this paper, partial constant hesitant fuzzy sets are introduced and proved some results. Further, we discuss the relation between partial constant hesitant fuzzy sets and UP-subalgebras (resp. UP-filters, UP-ideals and strongly UP-ideals), and study the concept of prime and weakly prime of subsets and of hesitant fuzzy sets of a UP-algebra.

Before we begin our study, we will introduce the definition of a UP-algebra.

Definition 1.1. [4] An algebra $A = (A, \cdot, 0)$ of type $(2, 0)$ is called a *UP-algebra*, where A is a nonempty set, $\cdot$ is a binary operation on A, and 0 is a fixed element of A (i.e., a nullary operation) if it satisfies the following axioms: for any $x, y, z \in A$,

(UP-1) $(y \cdot z) \cdot ((x \cdot y) \cdot (x \cdot z)) = 0,$

(UP-2) $0 \cdot x = x$,

(UP-3) $x \cdot 0 = 0$, and

(UP-4) $x \cdot y = y \cdot x = 0$ implies $x = y$.

From [4], we know that the notion of UP-algebras is a generalization of KU-algebras.

Example 1.2. [4] Let X be a universal set. Define a binary operation $\cdot$ on the power set of X by putting $A \cdot B = A' \cap B$ for all $A, B \in \mathcal{P}(X)$. Then $(\mathcal{P}(X), \cdot, \emptyset)$ is a UP-algebra and we shall call it the *power UP-algebra of type 1*.

Example 1.3. [4] Let X be a universal set. Define a binary operation $*$ on the power set of X by putting $A * B = A' \cup B$ for all $A, B \in \mathcal{P}(X)$. Then $(\mathcal{P}(X), *, X)$ is a UP-algebra and we shall call it the *power UP-algebra of type 2*.

Example 1.4. [4] Let $A = \{0, 1, 2, 3\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:

$\cdot$	0	1	2	3
0	0	1	2	3
1	0	0	0	0
2	0	1	0	3
3	0	1	2	0

Then $(A, \cdot, 0)$ is a UP-algebra.

In what follows, let A and B denote UP-algebras unless otherwise specified. The following proposition is very important for the study of UP-algebras.

Proposition 1.5. [4] In a UP-algebra A, the following properties hold: for any $x, y, z \in A$,

(1) $x \cdot x = 0$,

(2) $x \cdot y = 0$ and $y \cdot z = 0$ imply $x \cdot z = 0$,

(3) $x \cdot y = 0$ implies $(z \cdot x) \cdot (z \cdot y) = 0$,

(4) $x \cdot y = 0$ implies $(y \cdot z) \cdot (x \cdot z) = 0$,

(5) $x \cdot (y \cdot x) = 0$,

(6) $(y \cdot x) \cdot x = 0$ if and only if $x = y \cdot x$, and

(7) $x \cdot (y \cdot y) = 0$.

Definition 1.6. [4] A subset S of A is called a *UP-subalgebra* of A if the constant 0 of A is in S, and $(S, \cdot, 0)$ itself forms a UP-algebra.

Proposition 1.7. [4] A nonempty subset S of a UP-algebra $A = (A, \cdot, 0)$ is a UP-subalgebra of A if and only if S is closed under the $\cdot$ multiplication on A.

Definition 1.8. [4] A subset B of A is called a *UP-ideal* of A if it satisfies the following properties:

(1) the constant 0 of A is in B, and

(2) for any $x, y, z \in A, x \cdot (y \cdot z) \in B$ and $y \in B$ imply $x \cdot z \in B$.

Definition 1.9. [18] A subset F of A is called a *UP-filter* of A if it satisfies the following properties:

(1) the constant 0 of A is in F, and

(2) for any $x, y \in A, x \cdot y \in F$ and $x \in F$ imply $y \in F$.

Definition 1.10. [2] A subset C of A is called a *strongly UP-ideal* of A if it satisfies the following properties:

(1) the constant 0 of A is in C, and

(2) for any $x, y, z \in A, (z \cdot y) \cdot (z \cdot x) \in C$ and $y \in C$ imply $x \in C$.

From [2], we know that the notion of fuzzy UP-subalgebras is a generalization of fuzzy UP-filters, the notion of fuzzy UP-filters is a generalization of fuzzy UP-ideals, and the notion of fuzzy UP-ideals is a generalization of fuzzy strongly UP-ideals.

Definition 1.11. [18] A nonempty subset B of A is called a *prime subset* of A if it satisfies the following property: for any $x, y \in A$,

$$x \cdot y \in B \text{ implies } x \in B \text{ or } y \in B.$$

Definition 1.12. [18] A UP-subalgebra (resp. UP-filter, UP-ideal, strongly UP-ideal) B of A is called a *prime UP-subalgebra* (resp. *prime UP-filter*, *prime UP-ideal*, *prime strongly UP-ideal*) of A if B is a prime subset of A.

Theorem 1.13. [2] Let S be a subset of A. Then the following statements are equivalent:

(1) S is a prime UP-subalgebra (resp. prime UP-filter, prime UP-ideal, prime strongly UP-ideal) of A,

(2) $S = A$, and

(3) S is a strongly UP-ideal of A.

Definition 1.14. [2] A nonempty subset B of A is called a *weakly prime subset* of A if it satisfies the following property: for any $x, y \in A$ and $x \neq y$,

$$x \cdot y \in B \text{ implies } x \in B \text{ or } y \in B.$$

Definition 1.15. [2] A UP-subalgebra (resp. UP-filter, UP-ideal, strongly UP-ideal) B of A is called a *weakly prime UP-subalgebra* (resp. *weakly prime UP-filter*, *weakly prime UP-ideal*, *weakly prime strongly UP-ideal*) of A if B is a weakly prime subset of A.

Definition 1.16. [19] Let X be a reference set. A *hesitant fuzzy set* on X is defined in term of a function h that when applied to X return a subset of $[0,1]$, that is, $\mathrm{h}\colon X \to \mathcal{P}([0,1])$.

If $Y \subseteq X$, the *characteristic hesitant fuzzy set* h_Y on X is a function of X into $\mathcal{P}([0,1])$ defined as follows:

$$\mathrm{h}_Y(x) = \begin{cases} [0,1] & \text{if } x \in Y, \\ \emptyset & \text{if } x \notin Y. \end{cases}$$

By the definition of characteristic hesitant fuzzy sets, h_Y is a function of X into $\{\emptyset, [0,1]\} \subset \mathcal{P}([0,1])$. Hence, h_Y is a hesitant fuzzy set on X.

If $Y \subseteq X$ and $\varepsilon \in \mathcal{P}([0,1])$, the *partial constant hesitant fuzzy set* $\mathrm{P}_{Y,\varepsilon}$ on X is a function of X into $\mathcal{P}([0,1])$ defined as follows:

$$\mathrm{P}_{Y,\varepsilon}(x) = \begin{cases} [0,1] & \text{if } x \in Y, \\ \varepsilon & \text{if } x \notin Y. \end{cases}$$

By the definition of partial constant hesitant fuzzy sets, $\mathrm{P}_{Y,\varepsilon}$ is a function of X into $\{\varepsilon, [0,1]\} \subset \mathcal{P}([0,1])$. Hence, $\mathrm{P}_{Y,\varepsilon}$ is a hesitant fuzzy set on X. We note that $\mathrm{P}_{Y,\emptyset} = \mathrm{h}_Y$.

Definition 1.17. [13] Let h be a hesitant fuzzy set on A. The hesitant fuzzy set $\overline{\mathrm{h}}$ defined by $\overline{\mathrm{h}}(x) = [0,1] - \mathrm{h}(x)$ for all $x \in A$ is said to be the *complement* of h on A.

Remark 1.18. For all hesitant fuzzy set h on A, we have $\mathrm{h} = \overline{\overline{\mathrm{h}}}$.

Definition 1.19. [13] A hesitant fuzzy set h on A is called a *hesitant fuzzy UP-subalgebra* (HFUPS) of A if it satisfies the following property: for any $x, y \in A$,

$$\mathrm{h}(x \cdot y) \supseteq \mathrm{h}(x) \cap \mathrm{h}(y).$$

By Proposition 1.5 (1), we have $\mathrm{h}(0) = \mathrm{h}(x \cdot x) \supseteq \mathrm{h}(x) \cap \mathrm{h}(x) = \mathrm{h}(x)$ for all $x \in A$.

Definition 1.20. [13] A hesitant fuzzy set h on A is called a *hesitant fuzzy UP-filter* (HFUPF) of A if it satisfies the following properties: for any $x, y \in A$,

(1) $\mathrm{h}(0) \supseteq \mathrm{h}(x)$, and

(2) $\mathrm{h}(y) \supseteq \mathrm{h}(x \cdot y) \cap \mathrm{h}(x)$.

Definition 1.21. [13] A hesitant fuzzy set h on A is called a *hesitant fuzzy UP-ideal* (HFUPI) of A if it satisfies the following properties: for any $x, y, z \in A$,

(1) $\mathrm{h}(0) \supseteq \mathrm{h}(x)$, and

(2) $\mathrm{h}(x \cdot z) \supseteq \mathrm{h}(x \cdot (y \cdot z)) \cap \mathrm{h}(y)$.

Definition 1.22. [13] A hesitant fuzzy set h on A is called a *hesitant fuzzy strongly UP-ideal* (HFSUPS) of A if it satisfies the following properties: for any $x, y, z \in A$,

(1) $\mathrm{h}(0) \supseteq \mathrm{h}(x)$, and

(2) $\mathrm{h}(x) \supseteq \mathrm{h}((z \cdot y) \cdot (z \cdot x)) \cap \mathrm{h}(y)$.

From [13], we know that the notion of hesitant fuzzy UP-ideals of UP-algebras is the generalization of the notion of hesitant fuzzy strongly UP-ideals, the notion of hesitant fuzzy UP-filters of UP-algebras is the generalization of the notion of hesitant fuzzy UP-ideals, and the notion of hesitant fuzzy UP-subalgebras of UP-algebras is the generalization of the notion of hesitant fuzzy UP-filters.

2 Main Results

In this section, we discuss the relation between partial constant hesitant fuzzy sets and UP-subalgebras (resp. UP-filters, UP-ideals and strongly UP-ideals), and study the concept of prime and weakly prime of subsets and of hesitant fuzzy sets of a UP-algebra.

Theorem 2.1. Let S be a nonempty subset of A. Then the following statements hold:

(1) if S is a UP-subalgebra of A, then the partial constant hesitant fuzzy set $\mathrm{P}_{S,\varepsilon}$ is a hesitant fuzzy UP-subalgebra of A for all $\varepsilon \in \mathcal{P}([0,1])$, and

(2) if there exists $\varepsilon \in \mathcal{P}([0,1])$ such that the partial constant hesitant fuzzy set $\mathrm{P}_{S,\varepsilon}$ is a hesitant fuzzy UP-subalgebra of A, then S is a UP-subalgebra of A.

Proof. (1) Assume that S is a UP-subalgebra of A. For any $\varepsilon \in \mathcal{P}([0,1])$ and let $x, y \in A$.

Case 1: $x \in S$ and $y \in S$. Then $\mathrm{P}_{S,\varepsilon}(x) = [0,1]$ and $\mathrm{P}_{S,\varepsilon}(y) = [0,1]$. Thus $\mathrm{P}_{S,\varepsilon}(x) \cap \mathrm{P}_{S,\varepsilon}(y) = [0,1]$. Since S is a UP-subalgebra of A, we have $x \cdot y \in S$ and so $\mathrm{P}_{S,\varepsilon}(x \cdot y) = [0,1]$. Therefore, $\mathrm{P}_{S,\varepsilon}(x \cdot y) = [0,1] \supseteq [0,1] = \mathrm{P}_{S,\varepsilon}(x) \cap \mathrm{P}_{S,\varepsilon}(y)$.

Case 2: $x \in S$ and $y \notin S$. Then $\mathrm{P}_{S,\varepsilon}(x) = [0,1]$ and $\mathrm{P}_{S,\varepsilon}(y) = \varepsilon$. Thus $\mathrm{P}_{S,\varepsilon}(x) \cap \mathrm{P}_{S,\varepsilon}(y) = \varepsilon$. If $x \cdot y \in S$, then $\mathrm{P}_{S,\varepsilon}(x \cdot y) = [0,1]$ and so $\mathrm{P}_{S,\varepsilon}(x \cdot y) = [0,1] \supseteq \varepsilon = \mathrm{P}_{S,\varepsilon}(x) \cap \mathrm{P}_{S,\varepsilon}(y)$. If $x \cdot y \notin S$, then $\mathrm{P}_{S,\varepsilon}(x \cdot y) = \varepsilon$ and so $\mathrm{P}_{S,\varepsilon}(x \cdot y) = \varepsilon \supseteq \varepsilon = \mathrm{P}_{S,\varepsilon}(x) \cap \mathrm{P}_{S,\varepsilon}(y)$. Therefore, $\mathrm{P}_{S,\varepsilon}(x \cdot y) \supseteq \mathrm{P}_{S,\varepsilon}(x) \cap \mathrm{P}_{S,\varepsilon}(y)$.

Case 3: $x \notin S$ and $y \in S$. Then $\mathrm{P}_{S,\varepsilon}(x) = \varepsilon$ and $\mathrm{P}_{S,\varepsilon}(y) = [0,1]$. Thus $\mathrm{P}_{S,\varepsilon}(x) \cap \mathrm{P}_{S,\varepsilon}(y) = \varepsilon$. If $x \cdot y \in S$, then $\mathrm{P}_{S,\varepsilon}(x \cdot y) = [0,1]$ and so $\mathrm{P}_{S,\varepsilon}(x \cdot y) = [0,1] \supseteq \varepsilon = \mathrm{P}_{S,\varepsilon}(x) \cap \mathrm{P}_{S,\varepsilon}(y)$. If $x \cdot y \notin S$, then $\mathrm{P}_{S,\varepsilon}(x \cdot y) = \varepsilon$ and so $\mathrm{P}_{S,\varepsilon}(x \cdot y) = \varepsilon \supseteq \varepsilon = \mathrm{P}_{S,\varepsilon}(x) \cap \mathrm{P}_{S,\varepsilon}(y)$. Therefore, $\mathrm{P}_{S,\varepsilon}(x \cdot y) \supseteq \mathrm{P}_{S,\varepsilon}(x) \cap \mathrm{P}_{S,\varepsilon}(y)$.

Case 4: $x \notin S$ and $y \notin S$. Then $\mathrm{P}_{S,\varepsilon}(x) = \varepsilon$ and $\mathrm{P}_{S,\varepsilon}(y) = \varepsilon$. Thus $\mathrm{P}_{S,\varepsilon}(x) \cap \mathrm{P}_{S,\varepsilon}(y) = \varepsilon$. If $x \cdot y \in S$, then $\mathrm{P}_{S,\varepsilon}(x \cdot y) = [0,1]$ and so $\mathrm{P}_{S,\varepsilon}(x \cdot y) = [0,1] \supseteq \varepsilon = \mathrm{P}_{S,\varepsilon}(x) \cap \mathrm{P}_{S,\varepsilon}(y)$. If $x \cdot y \notin S$, then $\mathrm{P}_{S,\varepsilon}(x \cdot y) = \varepsilon$ and so $\mathrm{P}_{S,\varepsilon}(x \cdot y) = \varepsilon \supseteq \varepsilon = \mathrm{P}_{S,\varepsilon}(x) \cap \mathrm{P}_{S,\varepsilon}(y)$. Therefore, $\mathrm{h}_S(x \cdot y) \supseteq \mathrm{h}_S(x) \cap \mathrm{h}_S(y)$.
Hence, $\mathrm{P}_{S,\varepsilon}$ is a hesitant fuzzy UP-subalgebra of A.

(2) Assume that $\mathrm{P}_{S,\varepsilon}$ is a hesitant fuzzy UP-subalgebra of A for some $\varepsilon \in \mathcal{P}([0,1])$. Let $x, y \in S$. Then $\mathrm{P}_{S,\varepsilon}(x) = [0,1]$ and $\mathrm{P}_{S,\varepsilon}(y) = [0,1]$. Thus $\mathrm{P}_{S,\varepsilon}(x \cdot y) \supseteq \mathrm{P}_{S,\varepsilon}(x) \cap \mathrm{P}_{S,\varepsilon}(y) = [0,1]$, so $\mathrm{P}_{S,\varepsilon}(x \cdot y) = [0,1]$. Since $\varepsilon \neq [0,1]$, we have $x \cdot y \in S$. Hence, S is a UP-subalgebra of A. □

Lemma 2.2. Let B be a nonempty subset of A. Then the following statements hold:

(1) if the constant 0 of A is in B, then $\mathrm{P}_{B,\varepsilon}(0) \supseteq \mathrm{P}_{B,\varepsilon}(x)$ for all $\varepsilon \in \mathcal{P}([0,1])$ and for all $x \in A$, and

(2) if there exists $\varepsilon \in \mathcal{P}([0,1])$ and $\varepsilon \neq [0,1]$ such that $\mathrm{P}_{B,\varepsilon}(0) \supseteq \mathrm{P}_{B,\varepsilon}(x)$ for all $x \in A$, then the constant 0 of A is in B.

Proof. (1) If $0 \in B$, then $\mathrm{P}_{B,\varepsilon}(0) = [0,1]$ for all $\varepsilon \in \mathcal{P}([0,1])$. Thus $\mathrm{P}_{B,\varepsilon}(0) = [0,1] \supseteq \mathrm{P}_{B,\varepsilon}(x)$ for all $\varepsilon \in \mathcal{P}([0,1])$ and for all $x \in A$.

(2) Assume that there exists $\varepsilon \in \mathcal{P}([0,1])$ and $\varepsilon \neq [0,1]$ such that $\mathrm{P}_{B,\varepsilon}(0) \supseteq \mathrm{P}_{B,\varepsilon}(x)$ for all $x \in A$. Since B is a nonempty subset of A, we have $a \in B$ for some $a \in A$. Then $\mathrm{P}_{B,\varepsilon}(0) \supseteq \mathrm{P}_{B,\varepsilon}(a) = [0,1]$, so $\mathrm{P}_{B,\varepsilon}(0) = [0,1]$. Since $\varepsilon \neq [0,1]$, we have $0 \in B$. □

Theorem 2.3. Let F be a nonempty subset of A. Then the following statements hold:

(1) if F is a UP-filter of A, then the partial constant hesitant fuzzy set $\mathrm{P}_{F,\varepsilon}$ is a hesitant fuzzy UP-filter of A for all $\varepsilon \in \mathcal{P}([0,1])$, and

(2) if there exists $\varepsilon \in \mathcal{P}([0,1])$ and $\varepsilon \neq [0,1]$ such that the partial constant hesitant fuzzy set $\mathrm{P}_{F,\varepsilon}$ is a hesitant fuzzy UP-filter of A, then F is a UP-filter of A.

Proof. (1) Assume that F is a UP-filter of A. Let $\varepsilon \in \mathcal{P}([0,1])$. Since $0 \in F$, it follows from Lemma 2.2 (1) that $\mathrm{P}_{F,\varepsilon}(0) \supseteq \mathrm{P}_{F,\varepsilon}(x)$ for all $x \in A$. Next, let $x, y \in A$.

Case 1: $x \in F$ and $y \in F$. Then $\mathrm{P}_{F,\varepsilon}(x) = [0,1]$ and $\mathrm{P}_{F,\varepsilon}(y) = [0,1]$. Therefore, $\mathrm{P}_{F,\varepsilon}(y) = [0,1] \supseteq \mathrm{P}_{F,\varepsilon}(x \cdot y) = \mathrm{P}_{F,\varepsilon}(x) \cap \mathrm{P}_{F,\varepsilon}(x \cdot y)$.

Case 2: $x \notin F$ and $y \in F$. Then $\mathrm{P}_{F,\varepsilon}(x) = \varepsilon$ and $\mathrm{P}_{F,\varepsilon}(y) = [0,1]$. Thus $\mathrm{P}_{F,\varepsilon}(y) = [0,1] \supseteq \varepsilon = \mathrm{P}_{F,\varepsilon}(x) \cap \mathrm{P}_{F,\varepsilon}(x \cdot y)$.

Case 3: $x \in F$ and $y \notin F$. Then $\mathrm{P}_{F,\varepsilon}(x) = [0,1]$ and $\mathrm{P}_{F,\varepsilon}(y) = \varepsilon$. Since F is a UP-filter of A, we have $x \cdot y \notin F$ or $x \notin F$. But $x \in F$, so $x \cdot y \notin F$. Then $\mathrm{P}_{F,\varepsilon}(x \cdot y) = \varepsilon$. Thus $\mathrm{P}_{F,\varepsilon}(y) = \varepsilon \supseteq \varepsilon = \mathrm{P}_{F,\varepsilon}(x) \cap \mathrm{P}_{F,\varepsilon}(x \cdot y)$.

Case 4: $x \notin F$ and $y \notin F$. Then $\mathrm{P}_{F,\varepsilon}(x) = \varepsilon$ and $\mathrm{P}_{F,\varepsilon}(y) = \varepsilon$. Thus $\mathrm{P}_{F,\varepsilon}(y) = \varepsilon \supseteq \varepsilon = \mathrm{P}_{F,\varepsilon}(x) \cap \mathrm{P}_{F,\varepsilon}(x \cdot y)$.

Hence, $\mathrm{P}_{F,\varepsilon}$ is a hesitant fuzzy UP-filter of A.

(2) Assume that $\mathrm{P}_{F,\varepsilon}$ is a hesitant fuzzy UP-filter of A for some $\varepsilon \in \mathcal{P}([0,1])$ and $\varepsilon \neq [0,1]$. Since $\mathrm{P}_{F,\varepsilon}(0) \supseteq \mathrm{P}_{F,\varepsilon}(x)$ for all $x \in A$, it follows from Lemma 2.2 (2) that $0 \in F$. Next, let $x, y \in A$ be such that $x \cdot y \in F$ and $x \in F$. Then $\mathrm{P}_{F,\varepsilon}(x \cdot y) = [0,1]$ and $\mathrm{P}_{F,\varepsilon}(x) = [0,1]$. Thus $\mathrm{P}_{F,\varepsilon}(y) \supseteq \mathrm{P}_{F,\varepsilon}(x) \cap \mathrm{P}_{F,\varepsilon}(x \cdot y) = [0,1]$, so $\mathrm{P}_{F,\varepsilon}(y) = [0,1]$. Since $\varepsilon \neq [0,1]$, we have $y \in F$ and so F is a UP-filter of A. □

Theorem 2.4. Let B be a nonempty subset of A. Then the following statements hold:

(1) if B is a UP-ideal of A, then the partial constant hesitant fuzzy set $\mathrm{P}_{B,\varepsilon}$ is a hesitant fuzzy UP-ideal of A for all $\varepsilon \in \mathcal{P}([0,1])$, and

(2) if there exists $\varepsilon \in \mathcal{P}([0,1])$ and $\varepsilon \neq [0,1]$ such that the partial constant hesitant fuzzy set $\mathrm{P}_{B,\varepsilon}$ is a hesitant fuzzy UP-ideal of A, then B is a UP-ideal of A.

Proof. (1) Assume that B is a UP-ideal of A. Let $\varepsilon \in \mathcal{P}([0,1])$. Since $0 \in B$, it follows from Lemma 2.2 (1) that $\mathrm{P}_{B,\varepsilon}(0) \supseteq \mathrm{P}_{B,\varepsilon}(x)$ for all $x \in A$. Next, let $x, y, z \in A$.

Case 1: $x\cdot(y\cdot z) \in B$ and $y \in B$. Then $\mathrm{P}_{B,\varepsilon}(x\cdot(y\cdot z)) = [0,1]$ and $\mathrm{P}_{B,\varepsilon}(y) = [0,1]$. Thus $\mathrm{P}_{B,\varepsilon}(x\cdot(y\cdot z))\cap\mathrm{P}_{B,\varepsilon}(y) = [0,1]$. Since B is a UP-ideal of A, we have $x\cdot z \in B$ and so $\mathrm{P}_{B,\varepsilon}(x\cdot z) = [0,1]$. Therefore, $\mathrm{P}_{B,\varepsilon}(x\cdot z) = [0,1] \supseteq [0,1] = \mathrm{P}_{B,\varepsilon}(x\cdot(y\cdot z))\cap\mathrm{P}_{B,\varepsilon}(y)$.

Case 2: $x\cdot(y\cdot z) \in B$ and $y \notin B$. Then $\mathrm{P}_{B,\varepsilon}(x\cdot(y\cdot z)) = [0,1]$ and $\mathrm{P}_{B,\varepsilon}(y) = \varepsilon$. Thus $\mathrm{P}_{B,\varepsilon}(x\cdot(y\cdot z))\cap\mathrm{P}_{B,\varepsilon}(y) = \varepsilon$. Therefore, $\mathrm{P}_{B,\varepsilon}(x\cdot z) \supseteq \varepsilon = \mathrm{P}_{B,\varepsilon}(x\cdot(y\cdot z))\cap\mathrm{P}_{B,\varepsilon}(y)$.

Case 3: $x\cdot(y\cdot z) \notin B$ and $y \in B$. Then $\mathrm{P}_{B,\varepsilon}(x\cdot(y\cdot z)) = \varepsilon$ and $\mathrm{P}_{B,\varepsilon}(y) = [0,1]$. Thus $\mathrm{P}_{B,\varepsilon}(x\cdot(y\cdot z))\cap\mathrm{P}_{B,\varepsilon}(y) = \varepsilon$. Therefore, $\mathrm{P}_{B,\varepsilon}(x\cdot z) \supseteq \varepsilon = \mathrm{P}_{B,\varepsilon}(x\cdot(y\cdot z))\cap\mathrm{P}_{B,\varepsilon}(y)$.

Case 4: $x\cdot(y\cdot z) \notin B$ and $y \notin B$. Then $\mathrm{P}_{B,\varepsilon}(x\cdot(y\cdot z)) = \varepsilon$ and $\mathrm{P}_{B,\varepsilon}(y) = \varepsilon$. Thus $\mathrm{P}_{B,\varepsilon}(x\cdot(y\cdot z))\cap\mathrm{P}_{B,\varepsilon}(y) = \varepsilon$. Therefore, $\mathrm{P}_{B,\varepsilon}(x\cdot z) \supseteq \varepsilon = \mathrm{P}_{B,\varepsilon}(x\cdot(y\cdot z))\cap\mathrm{P}_{B,\varepsilon}(y)$.
Hence, $\mathrm{P}_{B,\varepsilon}$ is a hesitant fuzzy UP-ideal of A.

(2) Assume that $\mathrm{P}_{B,\varepsilon}$ is a hesitant fuzzy UP-ideal of A for some $\varepsilon \in \mathcal{P}([0,1])$ and $\varepsilon \neq [0,1]$. Since $\mathrm{P}_{B,\varepsilon}(0) \supseteq \mathrm{P}_{B,\varepsilon}(x)$ for all $x \in A$, it follows from Lemma 2.2 (2) that $0 \in B$. Next, let $x, y, z \in A$ be such that $x\cdot(y\cdot z) \in B$ and $y \in B$. Then $\mathrm{P}_{B,\varepsilon}(x\cdot(y\cdot z)) = [0,1]$ and $\mathrm{P}_{B,\varepsilon}(y) = [0,1]$. Thus $\mathrm{P}_{B,\varepsilon}(x\cdot z) \supseteq \mathrm{P}_{B,\varepsilon}(x\cdot(y\cdot z))\cap\mathrm{P}_{B,\varepsilon}(y) = [0,1]$, so $\mathrm{P}_{B,\varepsilon}(x\cdot z) = [0,1]$. Since $\varepsilon \neq [0,1]$, we have $x\cdot z \in B$ and so B is a UP-ideal of A. □

Theorem 2.5. Let C be a nonempty subset of A. Then the following statements hold:

(1) if C is a strongly UP-ideal of A, then the partial constant hesitant fuzzy set $\mathrm{P}_{C,\varepsilon}$ is a hesitant fuzzy strongly UP-ideal of A for all $\varepsilon \in \mathcal{P}([0,1])$, and

(2) if there exists $\varepsilon \in \mathcal{P}([0,1])$ and $\varepsilon \neq [0,1]$ such that the partial constant hesitant fuzzy set $\mathrm{P}_{C,\varepsilon}$ is a hesitant fuzzy strongly UP-ideal of A, then C is a strongly UP-ideal of A.

Proof. (1) Assume that C is a strongly UP-ideal of A. Let $\varepsilon \in \mathcal{P}([0,1])$. Since $0 \in C$, it follows form Lemma 2.2 (1) that $\mathrm{P}_{C,\varepsilon}(0) \supseteq \mathrm{P}_{C,\varepsilon}(x)$ for all $x \in A$. Next, let $x, y, z \in A$.

Case 1: $(z\cdot y)\cdot(z\cdot x) \in C$ and $y \in C$. Then $\mathrm{P}_{C,\varepsilon}((z\cdot y)\cdot(z\cdot x)) = [0,1]$ and $\mathrm{P}_{C,\varepsilon}(y) = [0,1]$. Thus $\mathrm{P}_{C,\varepsilon}((z\cdot y)\cdot(z\cdot x))\cap\mathrm{P}_{C,\varepsilon}(y) = [0,1]$. Since C is a strongly UP-ideal of A, we have $x \in C$ and so $\mathrm{P}_{C,\varepsilon}(x) = [0,1]$. Therefore, $\mathrm{P}_{C,\varepsilon}(x) = [0,1] \supseteq [0,1] = \mathrm{P}_{C,\varepsilon}((z\cdot y)\cdot(z\cdot x))\cap\mathrm{P}_{C,\varepsilon}(y)$.

Case 2: $(z\cdot y)\cdot(z\cdot x) \in C$ and $y \notin C$. Then $\mathrm{P}_{C,\varepsilon}((z\cdot y)\cdot(z\cdot x)) = [0,1]$ and $\mathrm{P}_{C,\varepsilon}(y) = \varepsilon$. Thus $\mathrm{P}_{C,\varepsilon}((z\cdot y)\cdot(z\cdot x))\cap\mathrm{P}_{C,\varepsilon}(y) = \varepsilon$. Therefore, $\mathrm{P}_{C,\varepsilon}(x) \supseteq \varepsilon = \mathrm{P}_{C,\varepsilon}((z\cdot y)\cdot(z\cdot x))\cap\mathrm{P}_{C,\varepsilon}(y)$.

Case 3: $(z\cdot y)\cdot(z\cdot x) \notin C$ and $y \in C$. Then $\mathrm{P}_{C,\varepsilon}((z\cdot y)\cdot(z\cdot x)) = \varepsilon$ and $\mathrm{P}_{C,\varepsilon}(y) = [0,1]$. Thus $\mathrm{P}_{C,\varepsilon}((z\cdot y)\cdot(z\cdot x))\cap\mathrm{P}_{C,\varepsilon}(y) = \varepsilon$. Therefore, $\mathrm{P}_{C,\varepsilon}(x) \supseteq \varepsilon = \mathrm{P}_{C,\varepsilon}((z\cdot y)\cdot(z\cdot x))\cap\mathrm{P}_{C,\varepsilon}(y)$.

Case 4: $(z\cdot y)\cdot(z\cdot x) \notin C$ and $y \notin C$. Then $\mathrm{P}_{C,\varepsilon}((z\cdot y)\cdot(z\cdot x)) = \varepsilon$ and $\mathrm{P}_{C,\varepsilon}(y) = \varepsilon$. Thus $\mathrm{P}_{C,\varepsilon}((z\cdot y)\cdot(z\cdot x))\cap\mathrm{P}_{C,\varepsilon}(y) = \varepsilon$. Therefore, $\mathrm{P}_{C,\varepsilon}(x) \supseteq \varepsilon = \mathrm{P}_{C,\varepsilon}((z\cdot y)\cdot(z\cdot x))\cap\mathrm{P}_{C,\varepsilon}(y)$.
Hence, $\mathrm{P}_{C,\varepsilon}$ is a hesitant fuzzy strongly UP-ideal of A.

(2) Assume that $\mathrm{P}_{C,\varepsilon}$ is a hesitant fuzzy strongly UP-ideal of A for some $\varepsilon \in \mathcal{P}([0,1])$ and $\varepsilon \neq [0,1]$. Since $\mathrm{P}_{C,\varepsilon}(0) \supseteq \mathrm{P}_{C,\varepsilon}(x)$ for all $x \in A$, it follows from Lemma

2.2 (2) that $0 \in C$. Next, let $x, y, z \in A$ be such that $(z \cdot y) \cdot (z \cdot x) \in C$ and $y \in C$. Then $\mathrm{P}_{C,\varepsilon}((z \cdot y) \cdot (z \cdot x)) = [0, 1]$ and $\mathrm{P}_{C,\varepsilon}(y) = [0, 1]$. Thus $\mathrm{P}_{C,\varepsilon}(x) \supseteq \mathrm{P}_{C,\varepsilon}((z \cdot y) \cdot (z \cdot x)) \cap \mathrm{P}_{C,\varepsilon}(y) = [0, 1]$, so $\mathrm{P}_{C,\varepsilon}(x) = [0, 1]$. Since $\varepsilon \neq [0, 1]$, we have $x \in C$ and so C is a strongly UP-ideal of A. □

Definition 2.6. [13] A hesitant fuzzy set h on A is called a *prime hesitant fuzzy set* on A if it satisfies the following property: for any $x, y \in A$,

$$\mathrm{h}(x \cdot y) \subseteq \mathrm{h}(x) \cup \mathrm{h}(y).$$

Theorem 2.7. Let B be a nonempty subset of A. Then the following statements hold:

(1) if B is a prime subset of A, then the partial constant hesitant fuzzy set $\mathrm{P}_{B,\varepsilon}$ is a prime hesitant fuzzy set on A for all $\varepsilon \in \mathcal{P}([0, 1])$, and

(2) if there exists $\varepsilon \in \mathcal{P}([0, 1])$ and $\varepsilon \neq [0, 1]$ such that the partial constant hesitant fuzzy set $\mathrm{P}_{B,\varepsilon}$ is a prime hesitant fuzzy set on A, then B is a prime subset of A.

Proof. (1) Assume that B is a prime subset of A. For any $\varepsilon \in \mathcal{P}([0, 1])$ and let $x, y \in A$.

Case 1: $x \cdot y \in B$. Then $\mathrm{P}_{B,\varepsilon}(x \cdot y) = [0, 1]$. Since B is a prime subset of A, we have $x \in B$ or $y \in B$. Then $\mathrm{P}_{B,\varepsilon}(x) = [0, 1]$ or $\mathrm{P}_{B,\varepsilon}(y) = [0, 1]$, so $\mathrm{P}_{B,\varepsilon}(x) \cup \mathrm{P}_{B,\varepsilon}(y) = [0, 1]$. Therefore, $\mathrm{P}_{B,\varepsilon}(x \cdot y) = [0, 1] \subseteq [0, 1] = \mathrm{P}_{B,\varepsilon}(x) \cup \mathrm{P}_{B,\varepsilon}(y)$.

Case 2: $x \cdot y \notin B$. Then $\mathrm{P}_{B,\varepsilon}(x \cdot y) = \varepsilon \subseteq \mathrm{P}_{B,\varepsilon}(x) \cup \mathrm{P}_{B,\varepsilon}(y)$.

Therefore, $\mathrm{P}_{B,\varepsilon}$ is a prime hesitant fuzzy set on A.

(2) Assume that $\mathrm{P}_{B,\varepsilon}$ is a prime hesitant fuzzy set on A for some $\varepsilon \in \mathcal{P}([0, 1])$ and $\varepsilon \neq [0, 1]$. Let $x, y \in A$ be such that $x \cdot y \in B$. Then $\mathrm{P}_{B,\varepsilon}(x \cdot y) = [0, 1]$, so $[0, 1] = \mathrm{P}_{B,\varepsilon}(x \cdot y) \subseteq \mathrm{P}_{B,\varepsilon}(x) \cup \mathrm{P}_{B,\varepsilon}(y)$. Thus $\mathrm{P}_{B,\varepsilon}(x) \cup \mathrm{P}_{B,\varepsilon}(y) = [0, 1]$, so $\mathrm{P}_{B,\varepsilon}(x) = [0, 1]$ or $\mathrm{P}_{B,\varepsilon}(y) = [0, 1]$. Since $\varepsilon \neq [0, 1]$, we have $x \in B$ or $y \in B$ and so B is a prime subset of A. □

Definition 2.8. [13] A hesitant fuzzy UP-subalgebra (resp. hesitant fuzzy UP-filter, hesitant fuzzy UP-ideal, hesitant fuzzy strongly UP-ideal) h of A is called a *prime hesitant fuzzy UP-subalgebra* (resp. *prime hesitant fuzzy UP-filter, prime hesitant fuzzy UP-ideal, prime hesitant fuzzy strongly UP-ideal*) if h is a prime hesitant fuzzy set on A.

Definition 2.9. [13] A hesitant fuzzy set h on A is called a *weakly prime hesitant fuzzy set* on A if it satisfies the following property: for any $x, y \in A$ and $x \neq y$,

$$\mathrm{h}(x \cdot y) \subseteq \mathrm{h}(x) \cup \mathrm{h}(y).$$

Definition 2.10. [13] A hesitant fuzzy UP-subalgebra (resp. hesitant fuzzy UP-filter, hesitant fuzzy UP-ideal, hesitant fuzzy strongly UP-ideal) h of A is called a *weakly prime hesitant fuzzy UP-subalgebra* (resp. *weakly prime hesitant fuzzy UP-filter, weakly prime hesitant fuzzy UP-ideal, weakly prime hesitant fuzzy strongly UP-ideal*) if h is a weakly prime hesitant fuzzy set on A.

From [13], we know that the notion of weakly prime hesitant fuzzy UP-subalgebras (resp. weakly prime hesitant fuzzy UP-filters, weakly hesitant fuzzy UP-ideals) is a generalization of prime hesitant fuzzy UP-subalgebras (resp. prime hesitant fuzzy UP-filters, prime hesitant fuzzy UP-ideals), and the notions of weakly prime hesitant fuzzy strongly UP-ideals and prime hesitant fuzzy strongly UP-ideals coincide.

Theorem 2.11. Let B be a nonempty subset of A. Then the following statements hold:

(1) if B is a weakly prime subset of A, then the partial constant hesitant fuzzy set $\mathrm{P}_{B,\varepsilon}$ is a weakly prime hesitant fuzzy set on A for all $\varepsilon \in \mathcal{P}([0,1])$, and

(2) if there exists $\varepsilon \in \mathcal{P}([0,1])$ and $\varepsilon \neq [0,1]$ such that the partial constant hesitant fuzzy set $\mathrm{P}_{B,\varepsilon}$ is a weakly prime hesitant fuzzy set on A, then B is a weakly prime subset of A.

Proof. (1) Assume that B is a weakly prime subset of A and let $x, y \in A$ be such that $x \neq y$ and $\varepsilon \in \mathcal{P}([0,1])$.

Case 1: $x \cdot y \in B$. Then $\mathrm{P}_{B,\varepsilon}(x \cdot y) = [0,1]$. Since B is a weakly prime subset of A, we have $x \in B$ or $y \in B$. Then $\mathrm{P}_{B,\varepsilon}(x) = [0,1]$ or $\mathrm{P}_{B,\varepsilon}(y) = [0,1]$, so $\mathrm{P}_{B,\varepsilon}(x) \cup \mathrm{P}_{B,\varepsilon}(y) = [0,1]$. Therefore, $\mathrm{P}_{B,\varepsilon}(x \cdot y) = [0,1] \subseteq [0,1] = \mathrm{P}_{B,\varepsilon}(x) \cup \mathrm{P}_{B,\varepsilon}(y)$.

Case 2: $x \cdot y \notin B$. Therefore, $\mathrm{P}_{B,\varepsilon}(x \cdot y) = \varepsilon \subseteq \mathrm{P}_{B,\varepsilon}(x) \cup \mathrm{P}_{B,\varepsilon}(y)$.

Hence, $\mathrm{P}_{B,\varepsilon}$ is a weakly prime hesitant fuzzy set on A.

(2) Assume that $\mathrm{P}_{B,\varepsilon}$ is a weakly prime hesitant fuzzy set on A for some $\varepsilon \in \mathcal{P}([0,1])$ and $\varepsilon \neq [0,1]$. Let $x, y \in A$ be such that $x \neq y$ and $x \cdot y \in B$. Then $\mathrm{P}_{B,\varepsilon}(x \cdot y) = [0,1]$, so $[0,1] = \mathrm{P}_{B,\varepsilon}(x \cdot y) \subseteq \mathrm{P}_{B,\varepsilon}(x) \cup \mathrm{P}_{B,\varepsilon}(y)$. Thus $\mathrm{P}_{B,\varepsilon}(x) \cup \mathrm{P}_{B,\varepsilon}(y) = [0,1]$, so $\mathrm{P}_{B,\varepsilon}(x) = [0,1]$ or $\mathrm{P}_{B,\varepsilon}(y) = [0,1]$. Since $\varepsilon \neq [0,1]$, we have $x \in B$ or $y \in B$ and so B is a weakly prime subset of A. □

Theorem 2.12. Let S be a nonempty subset of A. Then the following statements hold:

(1) if S is a weakly prime UP-subalgebra of A, then the partial constant hesitant fuzzy set $\mathrm{P}_{S,\varepsilon}$ is a weakly prime hesitant fuzzy UP-subalgebra of A for all $\varepsilon \in \mathcal{P}([0,1])$, and

(2) if there exists $\varepsilon \in \mathcal{P}([0,1])$ and $\varepsilon \neq [0,1]$ such that the partial constant hesitant fuzzy set $\mathrm{P}_{S,\varepsilon}$ is a weakly prime hesitant fuzzy UP-subalgebra of A, then S is a weakly prime UP-subalgebra of A.

Proof. (1) It is straightforward by Theorem 2.1 (1) and 2.11 (1).

(2) It is straightforward by Theorem 2.1 (2) and 2.11 (2). □

Theorem 2.13. Let F be a nonempty subset of A. Then the following statements hold:

(1) if F is a weakly prime UP-filter of A, then the partial constant hesitant fuzzy set $\mathrm{P}_{F,\varepsilon}$ is a weakly prime hesitant fuzzy UP-filter of A for all $\varepsilon \in \mathcal{P}([0,1])$, and

(2) if there exists $\varepsilon \in \mathcal{P}([0,1])$ and $\varepsilon \neq [0,1]$ such that the partial constant hesitant fuzzy set $\mathrm{P}_{F,\varepsilon}$ is a weakly prime hesitant fuzzy UP-filter of A, then F is a weakly prime UP-filter of A.

Proof. (1) It is straightforward by Theorem 2.3 (1) and 2.11 (1).
(2) It is straightforward by Theorem 2.3 (2) and 2.11 (2). □

Theorem 2.14. Let B be a nonempty subset of A. Then the following statements hold:

(1) if B is a weakly prime UP-ideal of A, then the partial constant hesitant fuzzy set $\mathrm{P}_{B,\varepsilon}$ is a weakly prime hesitant fuzzy UP-ideal of A for all $\varepsilon \in \mathcal{P}([0,1])$, and

(2) if there exists $\varepsilon \in \mathcal{P}([0,1])$ and $\varepsilon \neq [0,1]$ such that the partial constant hesitant fuzzy set $\mathrm{P}_{B,\varepsilon}$ is a weakly prime hesitant fuzzy UP-ideal of A, then B is a weakly prime UP-ideal of A.

Proof. (1) It is straightforward by Theorem 2.4 (1) and 2.11 (1).
(2) It is straightforward by Theorem 2.4 (2) and 2.11 (2). □

Theorem 2.15. Let C be a nonempty subset of A. Then the following statements hold:

(1) if C is a weakly prime strongly UP-ideal of A, then the partial constant hesitant fuzzy set $\mathrm{P}_{C,\varepsilon}$ is a weakly prime hesitant fuzzy strongly UP-ideal of A for all $\varepsilon \in \mathcal{P}([0,1])$, and

(2) if there exists $\varepsilon \in \mathcal{P}([0,1])$ and $\varepsilon \neq [0,1]$ such that the partial constant hesitant fuzzy set $\mathrm{P}_{C,\varepsilon}$ is a weakly prime hesitant fuzzy strongly UP-ideal of A, then C is a weakly prime strongly UP-ideal of A.

Proof. (1) It is straightforward by Theorem 2.5 (1) and 2.11 (1).
(2) It is straightforward by Theorem 2.5 (2) and 2.11 (2). □

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Expected Values of Aggregation Operators on Cubic Trapezoidal Fuzzy Number and its Application to Multi-Criteria Decision Making Problems

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Abstract – In this paper, we define trapezoidal cubic fuzzy numbers and their operational laws. Started on these operational laws, each collection operators, with trapezoidal cubic fuzzy weighted arithmetic averaging operator and weighted geometric averaging operator are purposed. Expected values, score function, and accuracy function of trapezoidal cubic fuzzy numbers are defined. Overcoming on these, mindful of trapezoidal cubic fuzzy multi-criteria decision making program is proposed. A delineation illustration example is given to exhibit the sound judgment and openness of the procedure.

Keywords – Trapezoidal cubic fuzzy number, aggregation operators, multi-criteria decision making

1 Introduction

Their get at a considerable lot of multi-criteria decision-making (MCDM) issues in indicating sociology. Different a period past the point of no return the fuzzy set institutionalization was offered and passed down to illuminate MCDM issues by Zadeh [14]. Therefor in [1], Atanassov presented the concept of intuitionistic fuzzy set (IFS) and discussed the degree of membership as well as the degree of non-membership function. Li reachable by theories and uses of fuzzy multi-criteria decision-making [9]. Wang displayed reading on multi-criteria decision-making drawing near with divided undoubting data [11]. There are differentiating preparing on the instrument of multi-criteria decision-making issues, in which the measures' weight coefficients are obvious and the criteria's principles are changed or are fuzzy numbers in [5,7,12], and here are likewise efficient readings on multi-criteria decision making or multi-criteria group decision making in [10,13], in which the weight sizes are tight and the standards' morals are fuzzy numbers.

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Cubic set appeared by Jun in [8]. Cubic sets are the speculations of fuzzy sets and intuitionistic fuzzy sets, in which there are two portrayals, one is utilized for the degree of membership and other is utilized for the degree of non-membership. The membership function is hold as interim while non-membership is inside and out seen as the constant fuzzy set.

Aliya et al., [4] defined the triangular cubic fuzzy number and operational laws. We developed the triangular cubic fuzzy hybrid aggregation (TCFHA) administrator to total all individual fuzzy choice structure provide by the decision makers into the aggregate cubic fuzzy decision matrix. Aliya et al., [3] proposed the cubic TOPSIS method and cubic gray relation analysis (GRA) method. Finally, the proposed method is used for selection in sol--gel synthesis of titanium carbide Nano powders. Aliya et al., [2] defined weighted average operator of triangular cubic fuzzy numbers and hamming distance of the TCFN. We develop an MCDM method approach based on an extended VIKOR method using triangular cubic fuzzy numbers (TCFNS) and multi-criteria decision-making (MCDM) method using triangular cubic fuzzy numbers (TCFNs) are developed. Aliya et al., [5] defined the generalized triangular cubic linguistic hesitant fuzzy weighted geometric (GTCHFWG) operator, generalized triangular cubic linguistic hesitant fuzzy ordered weighted average (GTCLHFOWA) operator, generalized triangular cubic linguistic hesitant fuzzy ordered weighted geometric (GTCLHFOWG) operator, generalized triangular cubic linguistic hesitant fuzzy hybrid averag-ing (GTCLHFHA) operator and generalized triangular cubic linguistic hesitant fuzzy hybrid geometric (GTCLHFHG) operator. Aliya et al., [6] developed Trapezoidal linguistic cubic hesitant fuzzy TOPSIS method to solve the MCDM method based on trapezoidal linguistic cubic hesitant fuzzy TOPSIS method.

Thus, it is very necessary to introduce a new extension of cubic set to address this issue. The aim of this paper is to present the notion of Trapezoidal cubic fuzzy set, which extends the cubic set to Trapezoidal cubic fuzzy environments and permits the membership of an element to be a set of several possible Trapezoidal cubic fuzzy numbers. Thus, Trapezoidal cubic fuzzy set is a very useful tool to deal with the situations in which the experts hesitate between several possible Trapezoidal cubic fuzzy numbers to assess the degree to which an alternative satisfies an attribute. In the current example, the degree to which the alternative satisfies the attribute can be represented by the Trapezoidal cubic fuzzy set. Moreover, in many multiple attribute group decision-making (MAGDM) problems, considering that the estimations of the attribute values are Trapezoidal cubic fuzzy sets, it therefore is very necessary to give some aggregation techniques to aggregate the Trapezoidal cubic fuzzy information. However, we are aware that the present aggregation techniques have difficulty in coping with group decision-making problems with Trapezoidal cubic fuzzy information. Therefore, we in the current paper propose a series of aggregation operators for aggregating the Trapezoidal cubic fuzzy information and investigate some properties of these operators. Then, based on these aggregation operators, we develop an approach to MAGDM with Trapezoidal cubic fuzzy information. Moreover, we use a numerical example to show the application of the developed approach.

The rest parts of this paper are organized as follows: Section 2, we define the definotion of fuzzy set and cubic set. Section 3, we exhibit trapezoidal cubic fuzzy set and operational laws. Section 4, we exhibit Aggregation operators on trapezoidal cubic fuzzy numbers. Section 5, we define Expected values of trapezoidal cubic fuzzy numbers and comparison between them. Section 6, we define Multi-criteria decision making method based on trapezoidal cubic fuzzy numbers. Section 7, the application of the developed approach in group decision-making problems is shown by an illustrative example. Results and discussion are given in section 8. Finally, we give the conclusions in Section 9.

2. Preliminaries

Definition.2.1. [14] *Give p a chance to be a nature of talk. The possibility of fuzzy set was speak to by Zadeh, and characterized as taking after: $j = \{p, \Upsilon_j(p) \mid \quad p \in P\}$. A fuzzy set in a set $\mathring{p}$ is defined $\Upsilon_j : p \to I$, is a membership function, $\Upsilon_j(p)$ denoted the degree of membership of the element p to the set P , where $I = [0,1]$. The accumulation of every single fuzzy subset of p is meant by I^p . Characterize a connection on I^p as takes after:* $(\forall \Upsilon, \eta \in I^p)(\Upsilon \leq \eta \Leftrightarrow (\forall p \in P)(\Upsilon(p) \leq \eta(p)))$.

Definition.2.2. [8] *Give P a chance to be a nonempty set. By a cubic set in P we mean a structure $F = \{q, \Upsilon(q), \chi(q) : q \in P\}$ in which Υ is an IVF set in q and χ is a fuzzy set in P. A cubic set $\tilde{F} = \{q, \Upsilon(q), \chi(q) : q \in P\}$ is simply denoted by $\tilde{F} = \langle \Upsilon, \chi \rangle$. Denote by C^P the collection of all cubic sets in p. A cubic set $\tilde{F} = \langle \Upsilon, \chi \rangle$ in which $\Upsilon(q) = 0$ and $\chi(q) = 1$ (resp. $\Upsilon(q) = 1$ And $\chi(q) = 0$ for all $q \in P$ is denoted by 0 (resp. 1). A cubic set $\tilde{D} = \langle \tilde{\lambda}, \tilde{\xi} \rangle$ in which $\tilde{\lambda}(q) = 0$ and $\tilde{\xi}(q) = 0$ (resp. $\tilde{\lambda}(q) = 1$ and $\tilde{\xi}(q) = 1$) for all $q \in P$ is denoted by 0 (resp. 1).*

Definition.2.3.[8] *Let P be a non empty set. A cubic set $F = (\varsigma, \tilde{\lambda})$ in P is said to be an internal cubic set if $\varsigma^-(p) \leq \tilde{\lambda}(p) \leq \varsigma^+(p)$ for all $p \in P$.*

Definition.2.4.[8] *Let P be a non empty set. A cubic set $F = (\varsigma, \tilde{\lambda})$ on P is said to be an external cubic set if $\tilde{\lambda}(p) \notin (\varsigma^-(p), \varsigma^+(p))$ for all $p \in P$.*

3. Trapezoidal Cubic Fuzzy Numbers

Definition.3.1. *Let Υ be trapezoidal cubic fuzzy number in the set of real numbers, its membership function is defined as* $\mu_\Upsilon(q) = \begin{cases} f_\Upsilon^L(q) & \Gamma \leq q < \varphi \\ \mu_\Upsilon & \varphi \leq q < \Theta \\ f_\Upsilon^R(q) & \Theta \leq q < \tau \\ 0, & \text{otherwise} \end{cases}$

Its non-membership function is defined as $\upsilon_{\Upsilon}(q) = \begin{cases} g_{\Upsilon}^{L}(q) & \Gamma_1 \le q < \varphi_1 \\ \upsilon_{\Upsilon} & \varphi_1 \le q < \Theta_1 \\ g_{\Upsilon}^{R}(q) & \Theta_1 \le q < \tau_1 \\ 0, & \text{otherwise} \end{cases}$

The trapezoidal cubic fuzzy number is denoted as $\Upsilon = \left\{ \begin{matrix} \left\langle \begin{matrix} [\Gamma, \varphi, \\ \Theta, \tau]; \\ [\mu^{-}, \\ \mu^{+}], \upsilon \end{matrix} \right\rangle, \\ \left\langle \begin{matrix} [\Gamma_1, \varphi_1, \\ \Theta_1, \tau_1]; \\ [\mu_1^{-}, \mu_1^{+}], \\ \upsilon_1 \end{matrix} \right\rangle \end{matrix} \right\}$. Generally from fuzzy numbers, trapezoidal cubic fuzzy numbers have another parameter: non-membership function, which is utilized to unequivocal the admeasurements to which the decision making that the component does not have a place with $((\Gamma, \varphi, \Theta, \tau); \upsilon_{\Gamma})$. When $\mu_{\Upsilon}^{-}(q) = 1, \quad \mu_{\Upsilon}^{+}(q) = 1, \quad \upsilon_{\Upsilon} = 0$, a is called normal trapezoidal cubic fuzzy number, by method for detail, conventional fuzzy number.

$$f_{\Upsilon}^{L}(q) = \tfrac{x-\Gamma}{\Upsilon-\Gamma}[\mu_{\Upsilon}^{-}, \mu_{\Upsilon}^{+}],\ f_{\Upsilon}^{R}(q) = \tfrac{\tau-x}{\tau-\Theta}[\mu_{\Upsilon}^{-}, \mu_{\Upsilon}^{+}], g_{\Upsilon}^{L}(q) = \tfrac{\varphi_1 - x + \upsilon_{\Gamma}(x-\Gamma_1)}{\varphi_1 - \Gamma_1}, g_{\Upsilon}^{R}(q) = \tfrac{x-\Theta_1+\upsilon_{\Gamma}(\tau_1 - x)}{\tau_1 - \Theta_1},$$

the cubic fuzzy number is called trapezoidal cubic fuzzy number.

Definition.3.2. *Let* $h_1 = \left\langle \begin{matrix} [\chi_1, \varpi_1, \alpha_1, \tau_1]; \\ [\mu_1^{-}, \mu_1^{+})], \upsilon_1 \end{matrix} \right\rangle$ *and* $h_2 = \left\langle \begin{matrix} [\chi_2, \varpi_2, \alpha_2, \tau_2]; \\ [\mu_2^{-}, \mu_2^{+})], \upsilon_2 \end{matrix} \right\rangle$ *be two trapezoidal cubic fuzzy numbers; then,*

(1): $h_1 + h_2 = \left\langle \begin{matrix} [\chi_1 + \chi_2, \varpi_1 + \varpi_2, \alpha_1 + \alpha_2, \tau_1 + \tau_2], \\ [\mu_1^{-} + \mu_2^{-} - \mu_1^{-}\mu_2^{-}, \mu_1^{+} + \mu_2^{+} - \mu_1^{+}\mu_2^{+}], \upsilon_1\upsilon_2 \end{matrix} \right\rangle,$

(2): $h_1 - h_2 = \left\langle \begin{matrix} [\chi_1 - \chi_2, \varpi_1 - \varpi_2, \alpha_1 - \alpha_2, \tau_1 - \tau_2], \\ [\mu_1^{-} - \mu_2^{-} + \mu_1^{-}\mu_2^{-}, \mu_1^{+} - \mu_2^{+} + \mu_1^{+}\mu_2^{+}], \upsilon_1\upsilon_2 \end{matrix} \right\rangle,$

(3): $\lambda h_1 = \left\langle [\lambda\chi_1, \lambda\varpi_1, \lambda\alpha_1, \lambda\tau_1]; [1-(1-\mu_1^{-})^{\lambda}), 1-(1-\mu_1^{+})^{\lambda}], (\upsilon_1^{\lambda})^{\lambda} \right\rangle$,

(4): $h_1^{\lambda} = \left\langle [(\chi_1)^{\lambda}, (\varpi_1)^{\lambda}, (\alpha_1)^{\lambda}, (\tau_1)^{\lambda}]; [(\mu_{\chi_1}^{-})^{\lambda}, (\mu_{\chi_1}^{+})^{\lambda}], 1-(1-\upsilon_{\chi_1})^{\lambda} \right\rangle$

Example.3.3. *Let* $h_1 = \left\langle \begin{matrix} [0.4, 0.8, 0.12, 0.16]; \\ [0.7, 0.9], 0.8 \end{matrix} \right\rangle$ *and* $h_2 = \left\langle \begin{matrix} [0.3, 0.5, 0.7, 0.11]; \\ [0.1, 0.5], 0.3 \end{matrix} \right\rangle$ *be two trapezoidal cubic fuzzy numbers; then,*

$$(1):\ h_1 + h_2 = \left\langle \begin{matrix} [0.4+0.3, 0.8+0.5, 0.12+0.7, 0.16+0.11] \\ [0.7+0.1-(0.7)(0.1)), (0.9+0.5-(0.9)(0.5))], \\ ((0.8)(0.3)) = [0.7, 1.3, 0.82, 0.27][0.73, 0.95], 0.24 \end{matrix} \right\rangle,$$

$$(2):\ h_1 - h_2 = \left\langle \begin{matrix} [0.4-0.3, 0.8-0.5, 0.12-0.7, 0.16-0.11], \\ [0.7-0.1+(0.7)(0.1)), \\ (0.9-0.5+(0.9)(0.5))], ((0.8)(0.3)) \\ = [0.1, 0.3, 0.58, 0.05][0.67, 0.85], 0.24 \end{matrix} \right\rangle.$$

4. Aggregation Operators on Trapezoidal Cubic Fuzzy Numbers

Definition.4.1. *Let* $\ddot{\alpha}_j\ (j = 1, ..., n)$ *be a set of trapezoidal cubic fuzzy numbers, and TrC - WAA* : $\Omega_n \to \Omega$ *; if TrC -WAA* $_\omega(\ddot{\alpha}_1, \ddot{\alpha}_2, ..., \ddot{\alpha}_n) = \sum_{j=1}^{n} w_j \ddot{\alpha}_j$ *where* Ω *is the set of all trapezoidal cubic fuzzy numbers, and* $\omega = (\omega_1, \omega_2, ..., \omega_n)^T$ *is the weight vector of* $\ddot{\alpha}_j\ (j = 1, ..., n), \omega_j \in [0,1], \sum_{j=1}^{n} w_j = 1,$ *then, TrC-WAA is called the weighted arithmetic average operator on trapezoidal cubic fuzzy numbers.*

Specially, if $\omega = (\frac{1}{n}, \frac{1}{n}, \ldots \frac{1}{n})^T$ *. TrC-WAA is the arithmetic average operator (TrC-WA) on trapezoidal cubic fuzzy numbers.*

Theorem.4.2. *Let* $h = \langle [((\alpha, \Upsilon, \Theta, \tau); \mu^-, \mu^+)], \upsilon) \rangle$ *be a set of trapezoidal cubic fuzzy numbers; then, the results aggregated from Definition 4.1 are still trapezoidal cubic fuzzy numbers, and even*

$$TrC-WAA_\omega(h_1, h_2, ..., h_n) = \sum_{j=1}^{n} \omega_j h_j = \left\langle \begin{matrix} [\prod_{j=1}^{n} (\alpha)^{\omega_j}, \prod_{j=1}^{n} (\Upsilon)^{\omega_j}, \prod_{j=1}^{n} (\Theta)^{\omega_j}, \prod_{j=1}^{n} (\tau)^{\omega_j}]; \\ [1-\prod_{j=1}^{n} (1-\mu^-)^{\omega_j}, 1-\prod_{j=1}^{n} (1-\mu^+)^{\omega_j}], \prod_{j=1}^{n} (\upsilon)^{\omega_j} \end{matrix} \right\rangle$$

where $\omega = (\omega_1, \omega_2, ..., \omega_n)^T$ *is the weight vector of* $h_j\ (j = 1, ..., n), \omega_j \in [0,1], \sum_{j=1}^{n} \omega_j = 1.$

Example.4.3. *Let*

$[0.5, 0.6, 0.7, 0.8]\langle[0.9, 0.15], 0.13\rangle$
$[0.1, 0.2, 0.3, 0.4]\langle[0.23, 0.27], 0.25\rangle$
$[0.4, 0.6, 0.8, 0.10]\langle[0.6, 0.10], 0.8\rangle$
$[0.5, 0.6, 0.7, 0.8]\langle[0.9, 0.15], 0.13\rangle$
$[0.8, 0.10, 0.12, 0.14]\langle[0.22, 0.28], 0.24\rangle$
$[0.25, 0.26, 0.27, 0.28]\langle[0.39, 0.45], 0.40\rangle$

$$= [0.002, 0.0012, 0.0039, 0.0010], \langle[0.9985, 0.8121], 0.0003\rangle$$

Definition.4.4. *Let* $h_j\,(j = 1, ..., n)$ *be a set of trapezoidal cubic fuzzy numbers, and TrC - WGA :* $\Omega_n \to \Omega$ *; if TrC -WGA* $_\omega(h_1, h_2, ..., h_n) = \sum_{j=1}^{n} h_j^{\omega_j}$ *where* Ω *is the set of all trapezoidal cubic fuzzy numbers, and* $\omega = (\omega_1, \omega_2, ..., \omega_n)^T$ *is the weight vector of* $h_j\,(j = 1, ..., n), \omega_j \in [0,1], \sum_{j=1}^{n} \omega_j = 1,$ *then, TrC-WAA is called the weighted arithmetic average operator on trapezoidal cubic fuzzy numbers. Specially, if* $\omega = (\frac{1}{n}, \frac{1}{n}, \ldots \frac{1}{n})^T$ *. TrC-WAA is the arithmetic average operator (TrC-WA) on trapezoidal cubic fuzzy numbers.*

Theorem.4.5. *Let* $h = \langle[(\alpha, \kappa, \varpi, \tau); [\mu^-, \mu^+)], \upsilon)\rangle$ *be a set of trapezoidal cubic fuzzy numbers; then, the results aggregated from Definition 4.4 are still trapezoidal cubic fuzzy numbers, and even*

$$TrC - WGA_\omega(h_1, h_2, ..., h_n) = \sum_{j=1}^{n} h_j \omega_j = \left\langle \begin{array}{c} [\prod_{j=1}^{n} (\alpha)^{\omega_j}, \prod_{j=1}^{n} (\kappa)^{\omega_j}, \prod_{j=1}^{n} (\varpi)^{\omega_j}, \prod_{j=1}^{n} (\tau)^{\omega_j}]; \\ [\prod_{j=1}^{n} (\mu_{\ddot{\alpha}}^-)^{\omega_j}, \prod_{j=1}^{n} (\mu_{\ddot{\alpha}}^+)^{\omega_j}], 1 - \prod_{j=1}^{n} (1 - \upsilon)^{\omega_j} \end{array} \right\rangle$$

where $\omega = (\omega_1, \omega_2, ..., \omega_n)^T$ *is the weight vector of* $h_j\,(j = 1, ..., n), \omega_j \in [0,1], \sum_{j=1}^{n} \omega_j = 1.$

Example.4.6. *Let*

	table
h_1	$[0.3, 0.5, 0.7, 0.9]\langle[0.29, 0.35], 0.33\rangle$
h_2	$[0.21, 0.22, 0.23, 0.24]\langle[0.2, 0.7], 0.5\rangle$
h_3	$[0.44, 0.46, 0.48, 0.50]\langle[0.12, 0.18], 0.16\rangle$
h_4	$[0.1, 0.3, 0.7, 0.9]\langle[0.21, 0.26], 0.23\rangle$
h_5	$[0.3, 0.4, 0.5, 0.6]\langle[0.2, 0.8], 0.4\rangle$
h_6	$[0.5, 0.6, 0.7, 0.8]\langle[0.9, 0.15], 0.13\rangle$

$$= [0.0004, 0.0036, 0.0189, 0.0466], \langle[0.0002, 0.0005], 0.8868\rangle$$

5. Expected Values of Trapezoidal Cubic Fuzzy Numbers and Comparison between them

For trapezoidal cubic fuzzy numbers, $f_L(p)$, are strictly linear increasing function, and $f_R(p)$ is strictly linear decreasing function in Definition 3.1. There in lay functions are respectively,

$$p_{\varpi}^{L}(p) = \varpi + \tfrac{y}{[\mu_{\varpi}^{-},\mu_{\varpi}^{+}]} \times (\Upsilon - \varpi),\ p_{\varpi}^{R}(\Upsilon) = \tau + \tfrac{y}{[\mu_{\varpi}^{-},\mu_{\varpi}^{+}]} \times (\Theta - \tau),$$

The assurance degree of trapezoidal cubic fuzzy number $\tilde{a}$ is between $\langle [\mu_{\varpi}^{-}, \mu_{\varpi}^{+}], 1-\upsilon_{\varpi})\rangle$.

Definition.5.1.

$$I_{\lambda}(\varpi) = \frac{1}{3}\int_{0}^{[\mu_{\varpi}^{-},\mu_{\varpi}^{+}]} (1-\lambda)\times g_{\varpi}^{L}(p) + \lambda \times g_{\varpi}^{R}(p)]\}dy + \int_{0}^{1-\upsilon_{\varpi}} (1-\lambda)\times g_{\varpi}^{L}(p) + \lambda \times g_{\varpi}^{R}(p)]\}dy$$

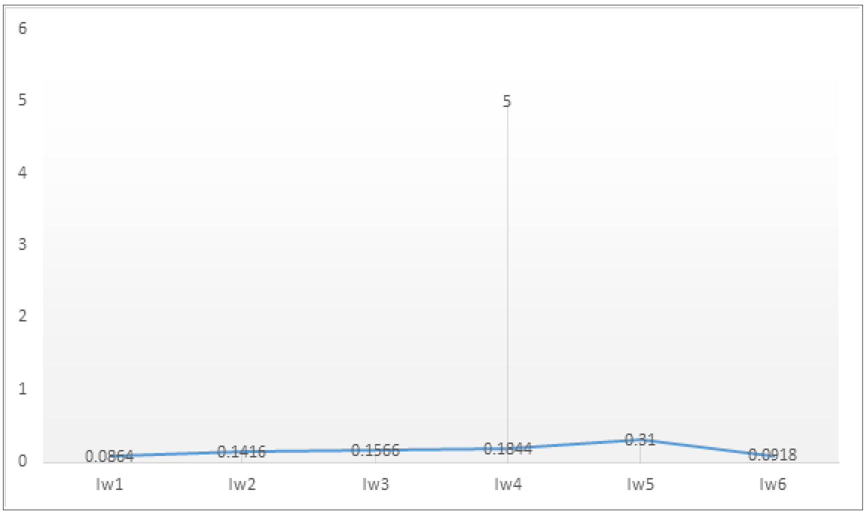
is called the expected value of trapezoidal cubic fuzzy number $\tilde{a}$.

Theorem.5.2. *The trapezoidal cubic fuzzy number* $\tilde{a} = \langle [((\varpi, \lambda, \zeta, \tau); \mu^{-}, \mu^{+})], \upsilon)\rangle$,
$I(\varpi) = \frac{1}{12}\langle [((\varpi + \lambda + \zeta + \tau) \times [1 + \mu^{-} - \upsilon) \times (1 + \mu^{+} - \upsilon)\rangle$

Example.5.3.

	table
h_1	$[0.25, 0.26, 0.27, 0.28]\langle [0.29, 0.35], 0.33\rangle$
h_2	$[0.41, 0.42, 0.43, 0.44]\langle [0.53, 0.57], 0.55\rangle$
h_3	$[0.44, 0.46, 0.48, 0.50]\langle [0.56, 0.60], 0.58\rangle$
h_4	$[0.55, 0.56, 0.57, 0.58]\langle [0.59, 0.65], 0.63\rangle$
h_5	$[0.88, 0.91, 0.92, 0.94]\langle [0.92, 0.98], 0.94\rangle$
h_6	$[0.25, 0.26, 0.27, 0.28]\langle [0.39, 0.45], 0.40\rangle$

$$\begin{aligned}
I_1(\varpi) &= \frac{1}{12}[1.06] \times \langle 0.96 \times 1.02\rangle \\
&= 0.0864, \\
I_2(\varpi) &= \frac{1}{12}[1.7] \times \langle [0.98 \times 1.02]\rangle \\
&= 0.1416, \\
I_3(\varpi) &= \frac{1}{12}[1.88] \times \langle [0.98 \times 1.02]\rangle \\
&= 0.1566, \\
I_4(\varpi) &= \frac{1}{12}[2.26] \times \langle [0.96 \times 1.02]\rangle \\
&= 0.1844, \\
I_5(\varpi) &= \frac{1}{12}[3.65] \times \langle [0.98 \times 1.04]\rangle \\
&= 0.3100, \\
I_6(\varpi) &= \frac{1}{12}[1.06] \times \langle [0.99 \times 1.05]\rangle \\
&= 0.0918.
\end{aligned}$$



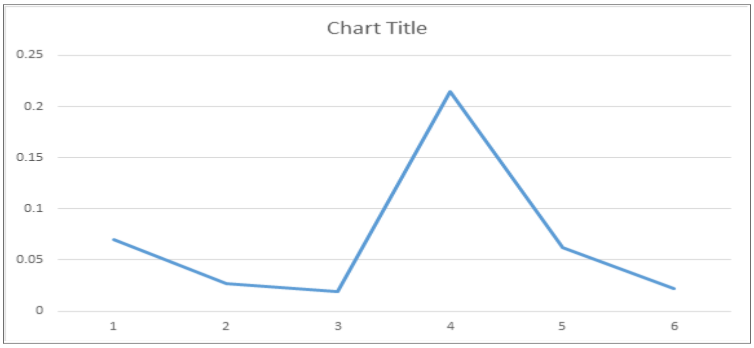
The score function and accuracy function of trapezoidal cubic fuzzy numbers.

Definitio.5.4. *Let* $\tilde{a} = \langle [((\Gamma, \Upsilon, \Theta, \tau); \mu^-, \mu^+)], \upsilon)\rangle$ *be the trapezoidal cubic fuzzy number; then,* $S(h) = I_h \times [(\mu_h^-, \mu_h^+) - (1 - \upsilon_h)]$ *is called the score function of* $\tilde{a}$ *, where* I_h *is the expected value of trapezoidal cubic fuzzy number* h *.*

Example.5.5. *Let*

	table
h_1	$[0.5, 0.6, 0.7, 0.8]\langle[0.9, 0.15], 0.13\rangle$
h_2	$[0.1, 0.2, 0.3, 0.4]\langle[0.9, 0.15], 0.13\rangle$
h_3	$[0.4, 0.6, 0.8, 0.10]\langle[0.6, 0.10], 0.8\rangle$
h_4	$[0.15, 0.16, 0.17, 0.18]\langle[0.9, 0.15], 0.13\rangle$
h_5	$[0.8, 0.10, 0.12, 0.14]\langle[0.9, 0.28], 0.14\rangle$
h_6	$[0.25, 0.26, 0.27, 0.28]\langle[0.39, 0.45], 0.40\rangle$

$$S_1(h) = \frac{1}{12}[2.6 * 1.77 * 1.02] * [1.05 - 0.87) = 0.3911 * 0.18$$
$$= 0.0703,$$
$$S_2(h) = \frac{1}{12}[1 * 1.77 * 1.02] * [1.05 - 0.87) = 0.1504 * 0.18$$
$$= 0.0270,$$
$$S_3(h) = \frac{1}{12}[1.9 * 0.8 * 0.3] * [0.7 - 0.2) = 0.038 * 0.5$$
$$= 0.019,$$
$$S_4(h) = \frac{1}{12}[0.66 * 1.77 * 1.02] * [1.05 - 0.87) = 1.1915 * 0.18$$
$$= 0.2144,$$
$$S_5(h) = \frac{1}{12}[1.16 * 1.76 * 1.14][1.18 - 0.86) = 0.1939 * 0.32$$
$$= 0.0620,$$
$$S_6(h) = \frac{1}{12}[1.06 * 0.99 * 1.05][0.84 - 0.6) = 0.0918 * 0.24$$
$$= 0.0220$$



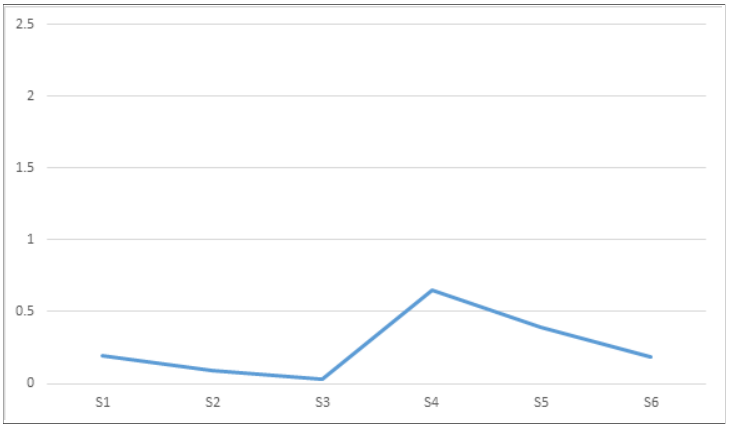
Definition.5.6. *Let* $\tilde{a} = \langle[(\Gamma, \Upsilon, \Theta, \tau);[\mu^-, \mu^+)], \upsilon)\rangle$ *be the trapezoidal cubic fuzzy number; then,* $p(h) = I \times [(\mu^-, \mu^+) + (1-\upsilon)]$ *is called the accuracy function of* $\tilde{a}$ *, where* I_Γ *is the expected value of trapezoidal cubic fuzzy number* Γ *.*

Example.5.7. *Let* $I(\varpi) = \frac{1}{12}\langle[((\varpi + \lambda + \zeta + \tau) \times [1 + \mu_\varpi^- - \upsilon_\varpi) \times (1 + \mu_\varpi^+ - \upsilon_\varpi))\rangle$

Table

h_1	$[0.15, 0.16, 0.17, 0.18]\langle[0.9, 0.15], 0.13\rangle$
h_2	$[0.21, 0.22, 0.23, 0.24]\langle[0.23, 0.27], 0.25\rangle$
h_3	$[0.44, 0.46, 0.48, 0.50]\langle[0.6, 0.10], 0.8\rangle$
h_4	$[0.55, 0.56, 0.57, 0.58]\langle[0.9, 0.15], 0.13\rangle$
h_5	$[0.88, 0.90, 0.92, 0.94]\langle[0.22, 0.28], 0.24\rangle$
h_6	$[0.35, 0.36, 0.37, 0.38]\langle[0.39, 0.45], 0.40\rangle$

$$p(h_1) = \frac{1}{12}[0.66 * 1.77 * 1.02] * [1.05 + 0.87]$$
$$= [0.0992] * 1.92 = 0.1904,$$
$$p(h_2) = \frac{1}{12}[0.9 * 0.98 * 1.02] * [0.5 + 0.75]$$
$$= [0.0749] * [1.25] = 0.0936,$$
$$p(h_3) = \frac{1}{12}[1.88 * 0.8 * 0.3] * [0.7 + 0.2]$$
$$= [0.0376] * 0.9 = 0.0338,$$
$$p(h_4) = \frac{1}{12}[2.26 * 1.77 * 1.02] * [1.05 + 0.87]$$
$$= [0.3400] * 1.92 = 0.6528,$$
$$p(h_5) = \frac{1}{12}[3.64 * 0.98 * 1.04] * [0.5 + 0.76]$$
$$= [0.3091] * 1.26 = 0.3894,$$
$$p(h_6) = \frac{1}{12}[1.46 * 0.99 * 1.05] * [0.84 + 0.6]$$
$$= 0.1264 * 1.44 = 0.1820.$$



6. Multi-Criteria Decision Making Method Based on Trapezoidal Cubic Fuzzy Numbers

For a brief fuzzy multi-criteria decision making issue, guess that there are m choices $A=\{h_1,h_2,...,h_n\},l$ decision criteria $C=\{\Theta_1,\Theta_2,...,\Theta_n\}$, and the relating weight coefficients are $\omega=\{\omega_1,\omega_2,...,\omega_l\},\omega_j\in[0,1],\omega_1+\omega_2+...+\omega_l=1$. The value of alternative h_i on the criteria Θ_j is trapezoidal cubic fuzzy number

$$h_{ij}=\langle([m_{1j}(h_i),m_{2j}(h_i),m_{3j}(h_i),m_{4j}(h_i)];[\mu_j^-(h_i),\mu_j^+(h_i)],\upsilon_j(h_i)\rangle$$

The local sorts of criteria are benefit and cost in multi criteria decision making problems. To dispense with the impact from various physical measurements to choice outcomes, the matrix

$$T=(t_{ij})_{m\times n},\quad (t_{ij})_{m\times n}=\langle([m_{1j}(h_i),m_{2j}(h_i),m_{3j}(h_i),m_{4j}(h_i)];[\mu_j^-(h_i),\mu_j^+(h_i)],\upsilon_j(h_i)\rangle$$

created by trapezoidal fuzzy numbers of fuzzy decision matrix $D=(h_{ij})_{n\times l}$ is revamp into homogenized matrix $R=(r_{ij})_{n\times l},r_{ij}=\langle[r_{ij}^1,r_{ij}^2,r_{ij}^3,r_{ij}^4]\rangle$ using formulas to homogenize the fuzzy decision matrix.

For cost criteria:

Decision steps:

(1) homogenize decision matrix

(2) Using weighted arithmetic average operator
$h_i=TrC-WAA_\omega(\Theta_1(h_i),\Theta_2(h_i),...,\Theta_l(h_i)),i=1,2,...,n$
or using weighted geometric average operator
$h_i=TrC-WGA_\omega(\Theta_1(h_i),\Theta_2(h_i),...,\Theta_l(h_i)),i=1,2,...,n$
Aggregate criteria's weights and qualities to land at the mixed trapezoidal cubic fuzzy numbers h_i , $i=1,2,...,n$ of alternative h_i .

(3) Enumerate the score value and the accuracy, respectively.

(4) Reeking the alternatives by Definition 5.4.

7. Example. There are 4 options $h_1, h_2, ..., h_4$ and 4 criteria $\Theta_1, \Theta_2, ..., \Theta_4$ in a multi-criteria decision making problem; the weight vector of criteria is $\omega = (0.20, 0.30, 0.40, 0.10)$, and and the choice data is given as Table 1 by chiefs, extreme to impact positioning of the 4 options.

Steps applying the ways and means in this unit are as continue from

(1) homogenize material in Table 1;

	Θ_1	Θ_2	Θ_3	Θ_4
A_1	$\left\{ [0.2, 0.4, 0.6, 0.8], \langle [0.2, 0.6], 0.4 \rangle \right\}$	$\left\{ [0.4, 0.8, 0.10, 0.18], \langle [0.8, 12], 0.10 \rangle \right\}$	$\left\{ [0.12, 0.14, 0.16, 0.18], \langle [0.1, 9], 0.6 \rangle \right\}$	$\left\{ [0.12, 0.14, 0.16, 0.18] \langle [0.52, 60], 0.56 \rangle \right\}$
A_2	$\left\{ [0.4, 0.6, 0.8, 0.10], \langle [0.14, 0.22], 0.19 \rangle \right\}$	$\left\{ [0.21, 0.41, 0.61, 0.81], \langle [0.22, 28], 0.25 \rangle \right\}$	$\left\{ [0.11, 0.14, 0.19, 0.28], \langle [0.2, 0.7], 0.5 \rangle \right\}$	$\left\{ [0.9, 0.14, 0.19, 0.28], \langle [0.42, 0.50], 0.45 \rangle \right\}$
A_3	$\left\{ [0.12, 0.14, 0.16, 0.18], \langle [0.2, 0.6], 0.4 \rangle \right\}$	$\left\{ [0.14, 0.17, 0.20, 0.28], \langle [0.8, 0.12], 0.10 \rangle \right\}$	$\left\{ [0.21, 0.31, 0.41, 0.51], \langle [0.1, 0.9], 0.6 \rangle \right\}$	$\left\{ [0.4, 0.8, 0.16, 0.18], \langle [0.52, 0.60], 0.56 \rangle \right\}$
A_4	$\left\{ [0.22, 0.24, 0.26, 0.28], \langle [0.14, 0.22], 0.19 \rangle \right\}$	$\left\{ [0.42, 0.44, 0.46, 0.48], \langle [0.22, 0.28], 0.25 \rangle \right\}$	$\left\{ [0.10, 0.14, 0.16, 0.18], \langle [0.2, 0.7], 0.5 \rangle \right\}$	$\left\{ [0.20, 0.22, 0.26, 0.29], \langle [0.42, 0.50], 0.45 \rangle \right\}$

(2) Total every one of the components a_{ij} $(j = 1, ..., 4)$ in the ith row of decision matrix D using TrC-WAA; then, the coordinated trapezoidal cubic fuzzy numbers

$h_i, i = 1, 2, ..., 4$ of alternative Γ_i are achieved.

$h_1 = \langle [(0.22, 0.39, 0.23, 0.304), [0.4742, 0.7089], 0.3209 \rangle$

$h_2 = \langle [0.324, 0.447, 0.716, 0.147)[0.2199, 0.5029], 0.3311 \rangle$

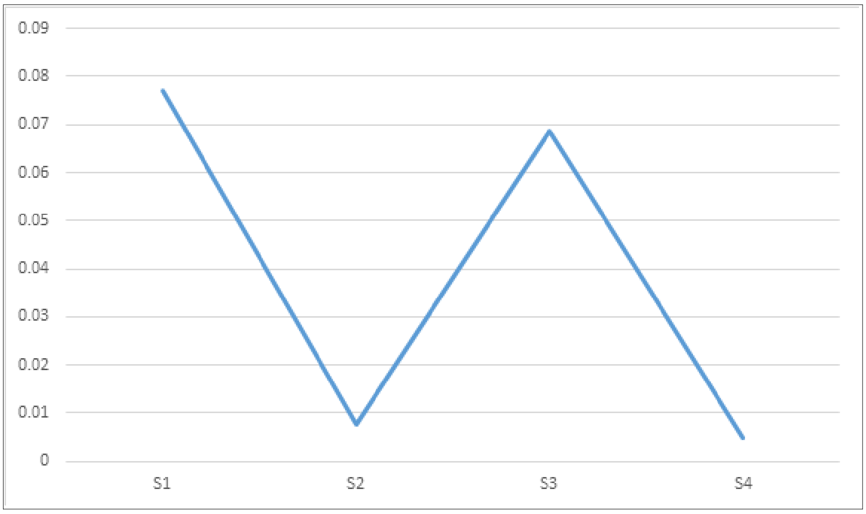
$h_3 = \langle [0.19, 0.219, 0.272, 0.342]), [0.4742, 0.7089], 0.3209 \rangle$

$h_4 = \langle [(0.23, 0.258, 0.28, 0.298), [0.2199, 0.5029], 0.3311 \rangle$

(3) Calculate the score values $S(h_i)$ of h_i

$S_1 = 0.0769, S_2 = 0.0076,$
$S_3 = 0.0687, S_4 = 0.0049.$

Step 4 : Rank the score value $S_1 > S_3 > S_2 > S_4$



$H_1 = 0.2841, H_2 = 0.1974, \ H_3 = 0.5975, H_4 = 0.1287.$

	Θ_1	Θ_2	Θ_3	Θ_4
A_1	$\left\{ [0.2, 0.4, 0.6, 0.8], \langle [0.2, 0.6], 0.4 \rangle \right\}$	$\left\{ [0.4, 0.8, 0.10, 0.18], \langle [0.8, 12], 0.10 \rangle \right\}$	$\left\{ [0.12, 0.14, 0.16, 0.18], \langle [0.1, 9], 0.6 \rangle \right\}$	$\left\{ [0.12, 0.14, 0.16, 0.18] \langle [0.52, 60], 0.56 \rangle \right\}$
A_2	$\left\{ [0.4, 0.6, 0.8, 0.10], \langle [0.2, 0.6], 0.4 \rangle \right\}$	$\left\{ [0.21, 0.41, 0.61, 0.81], \langle [0.8, 12], 0.10 \rangle \right\}$	$\left\{ [0.11, 0.14, 0.19, 0.28], \langle [0.1, 0.9], 0.6 \rangle \right\}$	$\left\{ [0.9, 0.14, 0.19, 0.28] \langle [0.52, 0.60], 0.56 \rangle \right\}$
A_3	$\left\{ [0.12, 0.14, 0.16, 0.18], \langle [0.2, 0.6], 0.4 \rangle \right\}$	$\left\{ [0.14, 0.17, 0.20, 0.28], \langle [0.8, 0.12], 0.10 \rangle \right\}$	$\left\{ [0.21, 0.31, 0.41, 0.51], \langle [0.1, 0.9], 0.6 \rangle \right\}$	$\left\{ [0.4, 0.8, 0.16, 0.18], \langle [0.52, 0.60], 0.56 \rangle \right\}$
A_4	$\left\{ [0.22, 0.24, 0.26, 0.28], \langle [0.2, 0.6], 0.4 \rangle \right\}$	$\left\{ [0.42, 0.44, 0.46, 0.48], \langle [0.8, 0.12], 0.10 \rangle \right\}$	$\left\{ [0.10, 0.14, 0.16, 0.18] \langle [0.1, 0.9], 0.6 \rangle \right\}$	$\left\{ [0.20, 0.22, 0.26, 0.29], \langle [0.52, 0.60], 0.56 \rangle \right\}$

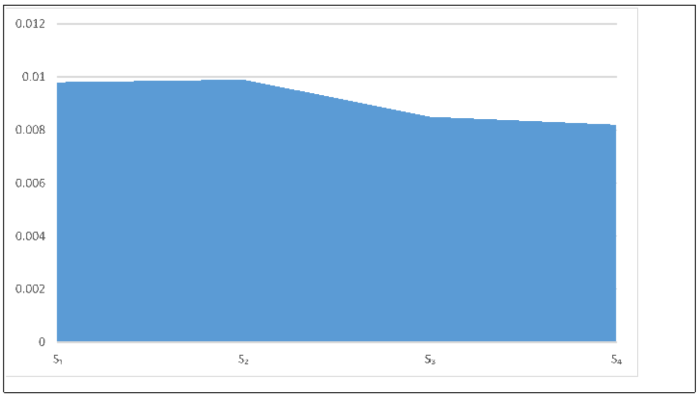
In the event that all components h_{ij} $(j=1,\ldots,4)$ in the i th row of decision matrix D are aggregated using $TC-WGA$, the coordinated trapezoidal cubic fuzzy numbers Γ_i, $i=1,2,\ldots,4$ of alternative h_i are as per the following

(2) Total every one of the components h_{ij} $(j=1,\ldots,4)$ in the ith row of decision matrix D using TrC-WGA; then, the coordinated trapezoidal cubic fuzzy numbers $h_i, i=1,2,\ldots,4$ of alternative h_i are accomplished.

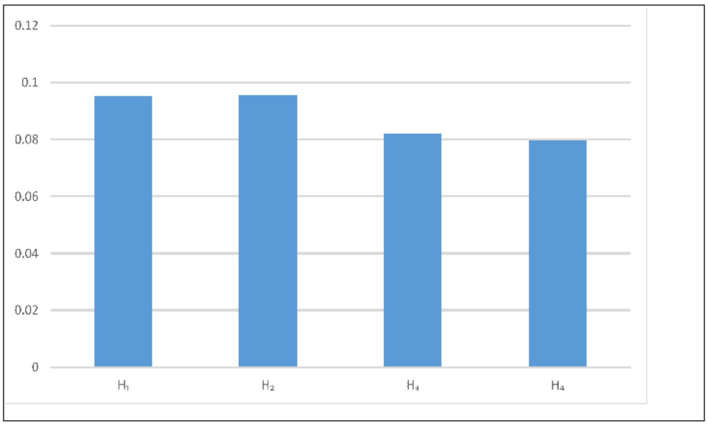
$$h_1 = \langle[(0.2583, 0.2183, 0.0749, 0.5846),[0.2527, 0.4354], 0.4414\rangle$$
$$h_2 = \langle[0.2134, 0.2585, 0.3594, 0.3134),[0.2527, 0.4354], 0.4414\rangle$$
$$h_3 = \langle[0.1773, 0.2427, 0.2492, 0.3117]),[0.2527, 0.4354], 0.4414\rangle$$
$$h_4 = \langle[(0.1930, 0.2300, 0.2540, 0.2767),[0.2527, 0.4354], 0.4414\rangle$$

(3) Calculate the score values $S(\Gamma_i)$ of
$S_1 = 0.0098, S_2 = 0.0099,$
$S_3 = 0.0085, S_4 = 0.0082.$

Step 4: Rank the alternatives by $S_2 > S_1 > S_3 > S_4$



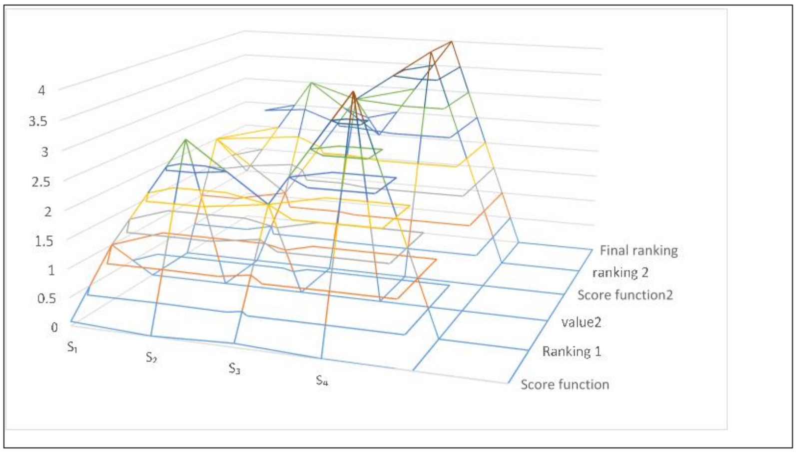
$H_1 = 0.0951, H_2 = 0.0958,$ $H_3 = 0.0821, H_4 = 0.0797.$



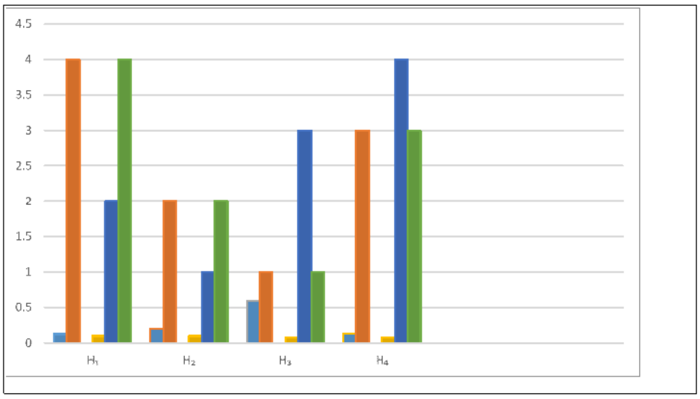
8. Comparison Analyses

The result of the score value 1 and score vaue 2 of the numerical examples are tabulated below

	Score function	Ranking 1		Score function	ranking 2	Final ranking
S_1	0.0769	1	S_1	0.0098	2	1
S_2	0.0076	3	S_2	0.0099	1	3
S_3	0.0687	2	S_3	0.0085	3	2
S_4	0.0049	4	S_4	0.0082	4	4



	Accuracy function	Ranking 1		Accuracy function	ranking 2	Final ranking
H_1	0.1241	4	H_1	0.0951	2	4
H_2	0.1974	2	H_2	0.0958	1	2
H_3	0.5975	1	H_3	0.0821	3	1
H_4	0.1287	3	H_4	0.0797	4	3



9. Conclusion

In this paper, we define trapezoidal cubic fuzzy numbers and their operational laws. Started on these operational laws, each collection operators, with trapezoidal cubic fuzzyy weighted arithmetic averaging operator and weighted geometric averaging operator are purposed. Expected values, score function, and accuracy function of trapezoidal cubic fuzzy numbers are defined.

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On Nano $\pi g\alpha$-Closed Sets

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Abstaract − In this paper, we define and study the properties of a nano $\pi g\alpha$-closed set which is a weaker form of a nano πg-closed set but strong than a nano πgp-closed sets and we define a new class of sets called nano $\pi g\alpha$-closed sets and some of their properties.

Keywords − *Nano π-closed set, nano πg-closed set, nano αg-closed set, nano πgp-closed set, nano gpr-closed set and nano $\pi g\alpha$-closed set*

1 Introduction

Thivagar et al. [4] introduced a nano topological space with respect to a subset X of an universe which is defined in terms of lower approximation and upper approximation and boundary region. The classical nano topological space is based on an equivalence relation on a set, but in some situation, equivalence relations are nor suitable for coping with granularity, instead the classical nano topology is extend to general binary relation based covering nano topological space

Bhuvaneswari et al. [3] introduced and investigated nano g-closed sets in nano topological spaces. Recently, Parvathy and Bhuvaneswari the notions of nano gpr-closed sets which are implied both that of nano rg-closed sets. In 2017, Rajasekaran et al. [7] introduced the notion of nano πgp-closed sets in nano topological spaces. In this paper, we define and study the properties of a nano $\pi g\alpha$-closed set which is a weaker form of a nano πg-closed set but strong than a nano πgp-closed sets and we define a new class of sets called nano $\pi g\alpha$-closed sets and some of their properties.

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2 Preliminaries

Throughout this paper $(U, \tau_R(X))$ (or X) represent nano topological spaces on which no separation axioms are assumed unless otherwise mentioned. For a subset H of a space $(U, \tau_R(X))$, $n\text{-}cl(H)$ and $n\text{-}int(H)$ denote the nano closure of H and the nano interior of H respectively. We recall the following definitions which are useful in the sequel.

Definition 2.1. *[6] Let U be a non-empty finite set of objects called the universe and R be an equivalence relation on U named as the indiscernibility relation. Elements belonging to the same equivalence class are said to be indiscernible with one another. The pair (U, R) is said to be the approximation space. Let $X \subseteq U$.*

1. *The lower approximation of X with respect to R is the set of all objects, which can be for certain classified as X with respect to R and it is denoted by $L_R(X)$. That is, $L_R(X) = \bigcup_{x \in U}\{R(x) : R(x) \subseteq X\}$, where R(x) denotes the equivalence class determined by x.*

2. *The upper approximation of X with respect to R is the set of all objects, which can be possibly classified as X with respect to R and it is denoted by $U_R(X)$. That is, $U_R(X) = \bigcup_{x \in U}\{R(x) : R(x) \cap X \neq \phi\}$.*

3. *The boundary region of X with respect to R is the set of all objects, which can be classified neither as X nor as not - X with respect to R and it is denoted by $B_R(X)$. That is, $B_R(X) = U_R(X) - L_R(X)$.*

Property 2.2. *[4] If (U, R) is an approximation space and $X, Y \subseteq U$; then*

1. $L_R(X) \subseteq X \subseteq U_R(X)$;
2. $L_R(\phi) = U_R(\phi) = \phi$ *and* $L_R(U) = U_R(U) = U$;
3. $U_R(X \cup Y) = U_R(X) \cup U_R(Y)$;
4. $U_R(X \cap Y) \subseteq U_R(X) \cap U_R(Y)$;
5. $L_R(X \cup Y) \supseteq L_R(X) \cup L_R(Y)$;
6. $L_R(X \cap Y) \subseteq L_R(X) \cap L_R(Y)$;
7. $L_R(X) \subseteq L_R(Y)$ *and* $U_R(X) \subseteq U_R(Y)$ *whenever* $X \subseteq Y$;
8. $U_R(X^c) = [L_R(X)]^c$ *and* $L_R(X^c) = [U_R(X)]^c$;
9. $U_R U_R(X) = L_R U_R(X) = U_R(X)$;
10. $L_R L_R(X) = U_R L_R(X) = L_R(X)$.

Definition 2.3. *[4] Let U be the universe, R be an equivalence relation on U and $\tau_R(X) = \{U, \phi, L_R(X), U_R(X), B_R(X)\}$ where $X \subseteq U$. Then by the Property 2.2, R(X) satisfies the following axioms:*

1. *U and* $\phi \in \tau_R(X)$,

2. *The union of the elements of any sub collection of $\tau_R(X)$ is in $\tau_R(X)$,*

3. *The intersection of the elements of any finite subcollection of $\tau_R(X)$ is in $\tau_R(X)$.*

This means that $\tau_R(X)$ is a topology on U called the nano topology on U with respect to X and $(U, \tau_R(X))$ as a nano topological space. The elements of $\tau_R(X)$ are called nano open sets (briefly n-open sets).

In the rest of the paper, we denote a nano topological space by $(U, \mathcal{N})$, where $\mathcal{N} = \tau_R(X)$. The nano-interior and nano-closure of a subset A of U are denoted by n--$int(A)$ and n--$cl(A)$, respectively.

Remark 2.4. *[4] If $[\tau_R(X)]$ is the nano topology on U with respect to X, then the set $B = \{U, \phi, L_R(X), B_R(X)\}$ is the basis for $\tau_R(X)$.*

Definition 2.5. *A subset H of a space $(U, \mathcal{N})$ is called*

1. *nano regular-open [4] if $H = n\text{-}int(n\text{-}cl(H))$.*

2. *nano pre-open [4] if $H \subseteq n\text{-}int(n\text{-}cl(H))$.*

3. *nano α-open [4] if $H \subseteq n\text{-}int(n\text{-}cl(n\text{-}int(H)))$.*

4. *nano π-open [1] if the finite union of nano regular-open sets.*

The complements of the above mentioned sets is called their respective closed sets.

Definition 2.6. *A subset H of a space $(U, \mathcal{N})$ is called;*

1. *nano g-closed [2] if $n\text{-}cl(H) \subseteq G$, whenever $H \subseteq G$ and G is n-open.*

2. *nano gα-closed [9] if $n\text{-}\alpha cl(H) \subseteq G$ whenever $H \subseteq G$ and G is nano α-open.*

3. *nano αg-closed set [9] if $n\text{-}\alpha cl(H) \subseteq G$ whenever $H \subseteq G$ and G is n-open.*

4. *nano πg-closed [7] if $n\text{-}cl(H) \subseteq G$, whenever $H \subseteq G$ and G is nano π-open.*

5. *nano gp-closed [3] if $n\text{-}pcl(H) \subseteq G$, whenever $H \subseteq G$ and G is n-open.*

6. *nano gpr-closed [5] if $n\text{-}pcl(H) \subseteq G$, whenever $H \subseteq G$ and G is nano regular open.*

7. *nano πgp-closed [8] if $n\text{-}pcl(H) \subseteq G$, whenever $H \subseteq G$ and G is nano π-open.*

3 On Nano $\pi g\alpha$-Closed Sets

Definition 3.1. *A subset H of a space $(U, \mathcal{N})$ is nano $\pi g\alpha$-closed if $n\text{-}\alpha cl(H) \subseteq G$ whenever $H \subseteq G$ and G is nano π-open.*

The complement of nano $\pi g\alpha$-open if $H^c = U - H$ is nano $\pi g\alpha$-closed.

Example 3.2. *Let $U = \{a, b, c, d\}$ with $U/R = \{\{a\}, \{c\}, \{b, d\}\}$ and $X = \{c, d\}$. Then the nano topology $\mathcal{N} = \{\phi, \{c\}, \{b, d\}, \{b, c, d\}, U\}$.*

1. *then $\{a\}$ is nano $\pi g\alpha$-closed set.*

2. *then $\{b\}$ is not nano $\pi g\alpha$-closed set.*

Remark 3.3. *For a subset of a space $(U, \mathcal{N})$, we have the following implications:*

$$\begin{array}{ccc} n\text{-}\textbf{closed} & \Rightarrow & \textbf{nano } g\text{-}\textbf{closed} \\ \Downarrow & & \Downarrow \\ \textbf{nano } \pi\text{-}\textbf{closed} & \Rightarrow & \textbf{nano } \pi g\text{-}\textbf{closed} \\ \Downarrow & & \\ \textbf{nano regular-closed} & & \end{array}$$

None of the above implications are reversible.

Theorem 3.4. *In a space $(U, \mathcal{N})$, every n-closed, every nano g-closed, every nano πg-closed, every nano αg-closed and every nano $g\alpha$-closed is nano $\pi g\alpha$-closed.*

Proof. Let $H \subseteq G$ where G is nano π-open. By hypothesis. $n\text{-}cl(H) = H \subseteq G$. Since every n-closed set is nano α-closed, $n\text{-}\alpha cl(H) \subseteq n\text{-}cl(H) \subseteq G$. Therefore H is nano $\pi g\alpha$-closed.

Let H be nano g-closed and $H \subseteq G$ where G is nano π-open. Since every nano π-open set is n-open and H is nano g-closed, $n\text{-}cl(H) \subseteq G$. Hence $n\text{-}\alpha cl(H) \subseteq n\text{-}cl(H) \subseteq G$ implies H is nano $\pi g\alpha$-closed.

Let H be a nano πg-closed set and $H \subseteq G$ where G is nano π-open. By assumption, $n\text{-}cl(H) \subseteq G$. Hence $n\text{-}\alpha cl(H) \subseteq n\text{-}cl(H) \subseteq G$ implies H is nano $\pi g\alpha$-closed.

Let H be a nano αg-closed set and $H \subseteq G$ where G is nano π-open. By Remark 3.3 and by assumption, it follows that $n\text{-}\alpha cl(H) \subseteq G$ and hence H is nano $\pi g\alpha$-closed.

Obvious every nano π-open is nano α-open.

Remark 3.5. *The converses of statements in Theorem 3.4 are not necessarily true as seen from the following Examples.*

Example 3.6. *In Example 3.3, then $\{a, b\}$ is nano $\pi g\alpha$-closed set but not n-closed.*

Example 3.7.

Let $U = \{a, b, c\}$ with $U/R = \{\{c\}, \{a, b\}\}$ and $X = \{c\}$. Then the nano topology $\mathcal{N} = \{\phi, \{c\}, U\}$.

1. *then $\{c\}$ is nano $\pi g\alpha$-closed set but not nano g-closed.*

2. *then $\{c\}$ is nano $\pi g\alpha$-closed set but not nano αg-closed.*

3. *then $\{a, c\}$ is nano $\pi g\alpha$-closed set but not nano $g\alpha$-closed.*

Theorem 3.8. *In a space $(U, \mathcal{N})$, every nano $\pi g\alpha$-closed is nano gpr-closed and nano πgp-closed.*

Proof. Let H be a nano $\pi g\alpha$-closed set and $H \subseteq G$ where G is nano regular open. By Remark 3.3 and since H is nano $\pi g\alpha$-closed set, we have $n\text{-}\alpha cl(H) \subseteq G$. Every nano α-closed set is nano pre-closed implies $n\text{-}pcl(H) \subseteq G$ and hence H is nano gpr-closed.

Let H be a nano $\pi g\alpha$-closed set and $H \subseteq G$ where G is nano π-open. By hypothesis, $n\text{-}\alpha cl(H) \subseteq G$. Now $n\text{-}pcl(H) \subseteq n\text{-}\alpha cl(H) \subseteq G$ implies that H is nano πgp-closed.

Theorem 3.9. *In a space $(U,\mathcal{N})$, every nano gp-closed set is nano $\pi g\alpha$-closed.*

Proof. Obvious.

Remark 3.10. *The converses of statements in Theorem 3.9 are not necessarily true as seen from the following Examples.*

Example 3.11. *In Example 3.7, then $\{c\}$ is nano $\pi g\alpha$-closed but not nano gp-closed.*

Theorem 3.12. *In a space $(U,\mathcal{N})$, if H is nano regular open and nano $\pi g\alpha$-closed, then H is nano α-closed and hence n-clopen.*

Proof. If H is nano regular open and nano $\pi g\alpha$-closed, then $n\text{-}\alpha cl(H) \subseteq H$. This implies H is a nano α-closed. Since every nano α-closed and nano regular open set is n-closed, H is n-clopen .

Theorem 3.13. *In a space $(U,\mathcal{N})$, for $x \in U$, its complement $U - \{x\}$ is nano $\pi g\alpha$-closed or nano π-open.*

Proof. Suppose $U - \{x\}$ is not nano π-open. Then U is the only nano π-open set containing $U - \{x\}$. This implies $n\text{-}\alpha cl(U - \{x\}) \subseteq U$. Hence $U - \{x\}$ is nano $\pi g\alpha$-closed.

Theorem 3.14. *In a space $(U,\mathcal{N})$, if H is nano $\pi g\alpha$-closed and $H \subseteq K \subseteq n\text{-}\alpha cl(H)$, then K is nano $\pi g\alpha$-closed.*

Proof. Let $K \subseteq G$ where G is nano π-open. Then $H \subseteq K$ implies $H \subseteq G$. Since H is nano $\pi g\alpha$-closed we have $n\text{-}\alpha cl(H) \subseteq G$. Also $K \subseteq n\text{-}\alpha cl(H)$ implies $n\text{-}\alpha cl(K) \subseteq n\text{-}\alpha cl(H)$. Thus $n\text{-}\alpha cl(K) \subseteq G$ and so K is nano $\pi g\alpha$-closed.

Theorem 3.15. *In a space $(U,\mathcal{N})$, let H be a nano $\pi g\alpha$-closed set in U. Then $n\text{-}\alpha cl(H) - H$ does not contain any non-empty nano π-closed set.*

Proof. Let P be a non-empty nano π-closed set such that $P \subseteq n\text{-}\alpha cl(H) - H$. Then $P \subseteq n\text{-}\alpha cl(H) \cap (U - H) \subseteq U - H$ implies $H \subseteq U - P$. H is nano $\pi g\alpha$-closed and $U - P$ is nano π-open implies that nano $n\text{-}\alpha cl(H) \subseteq U - P$. That is $P \subseteq (n\text{-}\alpha cl(H))^c$. Now $P \subseteq n\text{-}\alpha cl(H) \cap (n\text{-}\alpha cl(H))^c$ implies P is empty.

Theorem 3.16. *In a space $(U,\mathcal{N})$, if H is a nano $\pi g\alpha$-closed set, then $n\text{-}\pi cl(x) \cap H \neq \phi$ holds for each $x \in n\text{-}\alpha cl(H)$.*

Proof. Let H be a nano $\pi g\alpha$-closed set. Suppose $n\text{-}\pi cl(x) \cap H = \phi$, for some $x \in n\text{-}\alpha cl(H)$. We have $H \subseteq U - n\text{-}\pi cl(x)$. Since H is nano $\pi g\alpha$-closed set, $n\text{-}\alpha cl(H) \subseteq U - n\text{-}\pi cl(x)$ implies $x \notin n\text{-}\alpha cl(H)$ which is a contradiction. Hence $n\text{-}\pi cl(x) \cap H \neq \phi$ holds for each $x \in n\text{-}\alpha cl(H)$.

Corollary 3.17. *Let H be nano $\pi g\alpha$- closed in $(U,\mathcal{N})$. Then H is nano α-closed $\Longleftrightarrow n\text{-}\alpha cl(H) - H$ is nano π-closed.*

Lemma 3.18. *Let $(U,\mathcal{N})$ be a space and H is subset of U. Then the following properties are equivalent.*

1. *H is n-clopen.*

2. *H is nano regular open and nano $\pi g\alpha$-closed.*

3. *H is nano π-open and nano $\pi g\alpha$-closed.*

Proof. Follows from Theorem 3.12 and Remark 3.3.

Proposition 3.19. *In a space $(U, \mathcal{N})$, the union of two nano $\pi g\alpha$-closed sets is nano $\pi g\alpha$-closed.*

Proof. Let $H \cup K \subseteq G$ where G is nano π-open. Since H and K are nano $\pi g\alpha$-closed sets, $n\text{-}\alpha cl(H) \subseteq G$ and $n\text{-}\alpha cl(K) \subseteq G$. Now $n\text{-}\alpha cl(H \cup K) = n\text{-}\alpha cl(H) \cup n\text{-}\alpha cl(K) \subseteq G$. Hence $H \cup K$ is nano $\pi g\alpha$-closed.

Example 3.20. *In Example 3.7, then $H = \{a\}$ and $K = \{b\}$ is nano $\pi g\alpha$-closed sets. Clearly $H \cup K = \{a, b\}$ is nano $\pi g\alpha$-closed.*

Remark 3.21. *In sa space $(U, \mathcal{N})$,*

1. *$n\text{-}\alpha cl(U - H) = U - n\text{-}int(H)$*

2. *for any $H \subseteq U$, $n\text{-}\alpha int(n\text{-}\alpha cl(H) - H) = \phi$.*

Theorem 3.22. *A subset H of a space $(U, \mathcal{N})$ is nano $\pi g\alpha$-open $\Longleftrightarrow P \subseteq n\text{-}\alpha int(H)$ whenever P is nano π-closed and $P \subseteq H$.*

Proof. Necessity. Let H be nano $\pi g\alpha$-open. Let P be a nano π-closed set such that $P \subseteq H$. Then $U - H \subseteq U - P$ where $U - P$ is nano π-open. Then $U - H$ is nano $\pi g\alpha$-closed implies $n\text{-}\alpha cl(U - H) \subseteq U - P$. By Remark 3.21. $U - n\text{-}\alpha int(H) \subseteq U - P$. That is $P \subseteq n\text{-}\alpha int(H)$.

Sufficiency. Suppose P is a nano π-closed set and $P \subseteq H$ implies $P \subseteq n\text{-}\alpha int(H)$. Let $U - H \subseteq G$ where G is nano π-open. Then $U - G \subseteq H$ and $U - G$ is nano π-closed. By hypothesis, $U - G \subseteq n\text{-}\alpha int(H)$. That is $U - n\text{-}\alpha int(H) \subseteq G$ implies $n\text{-}\alpha cl(U - H) \subseteq G$. This implies $U - H$ is nano $\pi g\alpha$-closed and H is nano $\pi g\alpha$-open.

Remark 3.23. *From the above Propositions, Examples and Remarks, we obtain the following diagram, where $A \longrightarrow B$ represents A implies B but not conversely.*

$$\begin{tikzcd}
 & \textbf{nano } \pi\textbf{-closed} \arrow[d] & \\
\textbf{nano } g\textbf{-closed} \arrow[d] & \textbf{nano closed} \arrow[l] \arrow[r] \arrow[d] & \textbf{nano } \alpha\textbf{-closed} \arrow[d] \\
\textbf{nano } gp\textbf{-closed} \arrow[dd] & \textbf{nano } g\alpha\textbf{-closed} \arrow[l] \arrow[r] & \textbf{nano } \alpha g\textbf{-closed} \arrow[dd] \\
 & \textbf{nano } gpr\textbf{-closed} & \\
\textbf{nano } \pi gp\textbf{-closed} \arrow[ur] & & \textbf{nano } \pi g\alpha\textbf{-closed} \arrow[ul] \arrow[ll] \\
 & \textbf{nano } \pi g\textbf{-closed} \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

None of the above implications are reversible

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Space Time Fractional Telegraph Equation and its Application by Using Adomian Decomposition Method

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Abstract – The telegraph equations are a pair of linear differential equations which describe the voltage and current on an electrical transmission line with distance and time. In this paper the authors give a brief overview of fractional calculus and extend its application to space-time fractional telegraph equation by using Adomian decomposition method. The time- space derivates are considered as Caputo fractional derivate. The solutions are obtained in the series form.

Keywords – *Adomian decomposition method, time-space fractional telegraph equation.*

1 Introduction

Fractional Calculus is a field of mathematical study that grows out of the traditional definition of the calculus of integral and derivative operators in much the same way as fractional exponents grew from exponents with integer value. It was originated from the L-hospital and Leibnitz's inquisition about considering the result if n was taken as half in the n^{th} derivative of a function. Fractional calculus is of great importance in the field of Science and Technology as it is the generalization of ordinary differentiation and integration to arbitrary order [1]. Telegraph equations are a pair of linear differential equations that are very important due to their vast applications in high frequency transmission lines, optimization of guided communication system, propagation of electrical signals and many other physical and chemical phenomena. The theoretical background on transmission and transmission lines including open wire lines was given by Tomasi [2]. The fractional Telegraph equation has been studied extensively in literature. Cascaval [3] studied the time fractional Telegraph equation with applications to suspension flows using the Riemann-Liouville approach and presented asymptotic concepts. Orsingher and Beghin [4] obtained the fundamental solutions of time-fractional Telegraph equations of order 2α. Chen [5]

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discussed and derived the solution of the time-fractional telegraph equation with three kinds of non-homogenous boundary conditions making use of the separation of variables method. Momani [6] discussed the analytic and approximate solutions of the space and time fractional Telegraph differential equations by means of the Adomian decomposition method.

The Adomian decomposition method is a semi-analytical method for solving ordinary and partial non-linear differential equations. This method has been introduced and developed by Adomian [7,8]. This method has been used to obtain approximate solutions of a large class of linear and non-linear differential equations [8,9]. This method provides solutions in the form of power series with easily computed terms. It has many advantages over some classical techniques. After Adomian, this method has been further modified by Wazwaz [10] and more recently by Luo [11] and zhang and Luo [12]. Recently a lot of work has been done to apply this method to a large number of linear and non-linear ordinary differential equations, partial differential equations and integro-differential equations.

2 Mathematical Preliminaries

The Caputo fractional derivative of order $\alpha > 0$ is defined as [13]

$$C\,D_t^\alpha f(t) = \begin{cases} J^{n-\alpha} D^n f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} f^{(n)}(\tau) d\tau, & n-1 < \alpha \leq n, n \in N \\ = \frac{\partial^n f}{dt^n}, & if\ \alpha = n \in N \end{cases} \tag{1}$$

Here $D^n = \frac{d^m}{dx^m}$ and J_x^α is called the Riemann-Liouville integral operator of order $\alpha > 0$. According to this definition,

$$CD_t^\alpha A = 0, \qquad f(t) = A$$
$$\Rightarrow f(t) = \text{Constant}$$

That is Caputo's fractional derivative of a constant is zero.

Furthermore the relation between Riemann-Liouville fractional integral and Caputo fractional derivative is given by the following relation,

$$J^\alpha CD_t^\alpha f(t) = J^\alpha J^{n-\alpha} CD^n f(t) = J^n CD^n f(t) = f(t) - \sum_{k=0}^{n-1} f^{(k)}(0) \frac{t^k}{k!}$$
$$= CD_t^\alpha f(t) - \sum_{k=0}^{n-1} \frac{t^{k-\alpha}}{\Gamma k-\alpha+1} f^{(k)}(0) \tag{2}$$

The Laplace transform of Caputo's fractional derivative gives an interesting formula

$$\mathcal{L}\{CD_t^\alpha f(t)\} = s^\alpha \bar{f}(s) - \sum_{k=0}^{n-1} f^{(k)}(0) s^{\alpha-k-1} \tag{3}$$

The Weyl fractional integral and the Mellin transform

The Weyl fractional integral ${}_tW_\infty^{-\alpha} f(t)$ can be regarded as the convolution of $\varphi_\alpha(-t)$ with $f(t)$ [14] so that,

$$ {}_tW_\infty^{-\alpha} f(t) = \frac{1}{\Gamma\alpha}\int_t^\infty (x-t)^{\alpha-1} f(x)dx \tag{4}$$

$$= \varphi_\alpha(-t) * f(t) \tag{5}$$

We next calculate the Mellin transform of the Riemann-Liouville fractional integrals and derivatives.

$$ {}_0J_x^\alpha f(x) = \frac{1}{\Gamma\alpha}\int_0^x (x-t)^{\alpha-1} f(t)dt \tag{6}$$

$$= \frac{1}{\Gamma\alpha}\int_0^x x^{\alpha-1}\left(1-\frac{t}{x}\right)^{\alpha-1} f(t)dt \tag{7}$$

Let

$$\frac{t}{x} = \eta \Rightarrow t = x\eta \Rightarrow dt = xd\eta$$

$$\Rightarrow \qquad {}_0J_x^\alpha f(x) = \frac{x^\alpha}{\Gamma\alpha}\int_0^\infty \varphi(\eta) f(\eta t)\,d\eta \tag{8}$$

Where

$$\varphi(\eta) = (1-\eta)^{\alpha-1}.H(1-\eta),\ H(t)$$

is the Heaviside unit step function.

$$H(t) = \begin{cases} 0, & t < 0 \\ 1, & t \geq 0 \end{cases}$$

Using the properties of the Mellin Transform of $f(t)$, we obtain

$$M\{\varphi(t)\} = \hat{\varphi}(p) = \int_0^\infty t^{p-1}\varphi(t)dt = \frac{\Gamma(\alpha)\Gamma(p)}{\Gamma(\alpha+p)} \tag{9}$$

Now Mellin transform of ${}_0J_t^\alpha f(t)$ is given by

$$M\{{}_0J_t^\alpha f(t)\} = \frac{\Gamma 1-\alpha-p}{\Gamma 1-p}.\hat{f}(p+\alpha) \tag{10}$$

The Mellin transform of ${}_0D_t^\alpha f(t)$ is given by

$$M[{}_0D_t^\alpha f(t)] = \frac{\Gamma 1-p+\alpha}{\Gamma 1-p}.\hat{f}(p-\alpha) \tag{11}$$

We next find the Mellin transform of the Weyl fractional integral

$$ {}_xW_\infty^{-\alpha} f(x) = \frac{1}{\Gamma\alpha}\int_x^\infty (x-t)^{\alpha-1} f(t)dt \tag{12}$$

$$M\{{}_xW_\infty^{-\alpha} f(x)\} = \frac{\Gamma p}{\Gamma p+\alpha}.\hat{f}(p+\alpha) \tag{13}$$

or

$$\{{}_xW_\infty^{-\alpha} f(x)\} = M^{-1}\left\{\frac{\Gamma p}{\Gamma p+\alpha}.\hat{f}(p+\alpha)\right\} \tag{14}$$

Similarly the Mellin transform of Weyl fractional derivative is given by

$$M\{W^v f(x)\} = \frac{\Gamma p}{\Gamma p - v}.\hat{f}(p-v)$$

$$W^v f(x) = M^{-1}\left[\frac{\Gamma p}{\Gamma p-v}.\hat{f}(p-v)\right] \tag{15}$$

Adomian decomposition method

To illustrate the method consider the following differential equation of the form

$$y = f(x,y),\ y(a) = y_0 \tag{16}$$

In order to solve the problem, we put the highest degree differential operator $\mathcal{L}$ on the left side in the following way,

[15] $$\mathcal{L}(y) = f(x,y) \tag{17}$$

Where the differential operator $\mathcal{L}$ is given as

$$\mathcal{L} = \frac{d}{dx}\ and\ \mathcal{L}^{-1} = \int_0^t (\,.\,)$$

Operate $\mathcal{L}^{-1}$ on both sides of equation (17) and use the initial condition $y(a) = y_0$, we get,

$$y(x) = y_0 + \mathcal{L}^{-1}f(x,y) \tag{18}$$

The solution through Adomian decomposition method is obtained in an infinite series form as

$$y(x) = \sum_{n=0}^{\infty} y_n(x) \tag{19}$$

where the components $y_n(x)$ are determined recursively. Moreover the non linear function $f(x,y)$ is defined by the infinite series of the form

$$f(x,y) = \sum_{n=0}^{\infty} A_n \tag{20}$$

by using equations (19), (20) in (18) we get

$$\sum_{n=0}^{\infty} y_n(x) = y_0 + \mathcal{L}^{-1}\sum_{n=0}^{\infty} A_n \tag{21}$$

To determine the component $y_n(x)$, the zeroth component $y_0(x)$ is identified by the term that arises from the initial condition. The remaining components are obtained by using the preceding component.

3 Solution of the Space Time Fractional Telegraph Equation by Using Adomian Decomposition Method

In this section, we have obtained a solution of following time and space fractional telegraph equation using Adomian decomposition method. The space time telegraph equation is given by

$$D_t^{2\alpha}u(x,t) + \lambda\, D_t^{\alpha}u(x,t) = \mu\, D_x^{\beta}u(x,t), \quad t \geq 0, \quad 0 < \alpha \leq 1 \tag{22}$$

subject to the boundary and initial conditions

$$\begin{cases} u(x,0) = h_1(x) \\ u_t(x,0) = h_2(x) \\ u(0,t) = s(t) \end{cases} \tag{23}$$

we write (22) in an operator form as,

$$D_t^{2\alpha}\, u = \mu\, L_x^{\beta}\, u - \lambda\, D_t^{\alpha}\, u \tag{24}$$

where $L_x^{\beta} = \frac{\partial^{\beta}}{\partial t^{\beta}}$ and the fractional differential operational $D_t^{\alpha} = \frac{\partial^{\alpha}}{\partial t^{\alpha}}$ is defined in the Caputo's sense as follows,

$$u(x,t) = \frac{\partial^{\alpha} u(x,t)}{\partial t^{\alpha}} = \begin{cases} \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} f^{(n)}(\tau)d\tau, & n-1 < \alpha \leq n, n \in N \\ = \frac{\partial^n f}{dt^n}, & if\ \alpha = n \in N \end{cases} \tag{25}$$

Operating with $J_t^{2\alpha} = J_0^{2\alpha}$ on both sides of equation (24) and using initial conditions, equation (23) yields

$$u(x,t) = u(x,0) + tu_t(x,0) + J_t^{2\alpha}\left[\mu\, L_x^{\beta}\, u - \lambda\, D_t^{\alpha}\, u\right] \tag{26}$$

The Adomian's decomposition method assumes a series solution for $u(x,t)$ given by

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t) \tag{27}$$

Following Adomian method analysis, equation (26) is transformed into a set of recursive relations given by

$$u_0 = h_1(x) + th_2(x) \tag{28}$$

$$u_{n+1} = J_t^{2\alpha}\left[\mu\, L_x^{\beta}\, u_n - \lambda\, D_t^{\alpha}\, u_n\right] \tag{29}$$

Using the above recursive relationship and mathematics the first three terms of the decomposition series are given by

$$u_0 = u(x,0) + tu_t(x,0) \tag{30}$$

$$u_0 = h_1(x) + th_2(x) \tag{31}$$

$$u_1 = J_t^{2\alpha}\left[\mu\, L_x^{\beta}\, u_0 - \lambda\, D_t^{\alpha}\, u_0\right] \tag{32}$$

$$u_1 = J_t^{2\alpha}\left[\mu \frac{\partial^{\beta}}{\partial x^{\beta}}\, u_0 - \lambda \frac{\partial^{\alpha}}{\partial t^{\alpha}}\, u_0\right] \tag{33}$$

$$u_1 = J_t^{2\alpha}\left[\mu\{h_1{}^{\beta}(x) + th_2{}^{\beta}(x)\} - \lambda\, D_t^{\alpha}\, \{h_1(x)t^0 + h_2(x)t^1\}\right] \tag{34}$$

$$u_1 = J_t^{2\alpha}\left[\mu\{h_1{}^{\beta}(x) + th_2{}^{\beta}(x)\} - \lambda\, \left\{h_1(x)\frac{\Gamma(1+0)}{\Gamma(1+0-\alpha)}t^{0-\alpha} + h_2(x)\frac{\Gamma(1+1)}{\Gamma(1+1-\alpha)}t^{1-\alpha}\right\}\right] \tag{35}$$

$$u_1 = J_t^{2\alpha}\left[\mu\{h_1{}^{\beta}(x) + th_2{}^{\beta}(x)\} - \lambda\, \left\{h_1(x)\frac{t^{-\alpha}}{\Gamma(1-\alpha)} + h_2(x)\frac{t^{1-\alpha}}{\Gamma(2-\alpha)}\right\}\right] \tag{36}$$

$$u_1 = \left[\mu\{h_1{}^{\beta}(x)(J_t^{2\alpha}\, t^0) + h_2{}^{\beta}(x)(J_t^{2\alpha} t^1)\} - \lambda\left\{\frac{h_1(x)}{\Gamma(1-\alpha)}(J_t^{2\alpha} t^{-\alpha}) + \frac{h_2(x)}{\Gamma(2-\alpha)}(J_t^{2\alpha} t^{1-\alpha})\right\}\right] \tag{37}$$

$$u_1 = \left[\mu\{h_1{}^{\beta}(x)(J_t^{2\alpha}\, t^0) + h_2{}^{\beta}(x)(J_t^{2\alpha} t^1)\} - \lambda\left\{\frac{h_1(x)}{\Gamma(1-\alpha)}(J_t^{2\alpha} t^{-\alpha}) + \frac{h_2(x)}{\Gamma(2-\alpha)}(J_t^{2\alpha} t^{1-\alpha})\right\}\right] \tag{38}$$

$$u_1 = \left[\mu\left\{h_1{}^{\beta}(x)\frac{\Gamma(1+0)}{\Gamma(1+0+2\alpha)}t^{0+2\alpha} + h_2{}^{\beta}(x)\frac{\Gamma(1+1)}{\Gamma(1+1+2\alpha)}t^{1+2\alpha}\right\} - \lambda\left\{\frac{h_1(x)}{\Gamma(1-\alpha)}\frac{\Gamma(1-\alpha)}{\Gamma(1-\alpha+2\alpha)}t^{-\alpha+2\alpha} + \frac{h_2(x)}{\Gamma(2-\alpha)}\frac{\Gamma(1+1-\alpha)}{\Gamma(1+1-\alpha+2\alpha)}t^{1-\alpha+2\alpha}\right\}\right] \tag{39}$$

$$u_1 = \mu\left\{h_1{}^{\beta}(x)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + h_2{}^{\beta}(x)\frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)}\right\} - \lambda\left\{h_1(x)\frac{t^{\alpha}}{\Gamma(\alpha+1)} + h_2(x)\frac{t^{\alpha+1}}{\Gamma(\alpha+2)}\right\} \tag{40}$$

$$u_2 = J_t^{2\alpha}\left[\mu\, L_x^{\beta}\, u_1 - \lambda\, D_t^{\alpha}\, u_1\right] \tag{41}$$

$$u_2 = J_t^{2\alpha}\left[\mu \frac{\partial^{\beta}}{\partial x^{\beta}}\, u_1 - \lambda \frac{\partial^{\alpha}}{\partial t^{\alpha}}\, u_1\right] \tag{42}$$

$$u_2 = J_t^{2\alpha}\left[\mu \frac{\partial^{\beta}}{\partial x^{\beta}}\left\{\mu\left(h_1{}^{\beta}(x)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + h_2{}^{\beta}(x)\frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)}\right) - \lambda\left(h_1(x)\frac{t^{\alpha}}{\Gamma(\alpha+1)} + h_2(x)\frac{t^{\alpha+1}}{\Gamma(\alpha+2)}\right)\right\} - \lambda\frac{\partial^{\alpha}}{\partial t^{\alpha}}\left\{\mu\left(h_1{}^{\beta}(x)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + h_2{}^{\beta}(x)\frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)}\right) - \lambda\left(h_1(x)\frac{t^{\alpha}}{\Gamma(\alpha+1)} + h_2(x)\frac{t^{\alpha+1}}{\Gamma(\alpha+2)}\right)\right\}\right] \tag{43}$$

$$u_2 = J_t^{2\alpha}\left[\left\{\frac{\mu^2 h_1{}^{2\beta}(x)}{\Gamma(2\alpha+1)}t^{2\alpha} + \frac{\mu^2 h_2{}^{2\beta}(x)}{\Gamma(2\alpha+2)}t^{2\alpha+1} - \frac{\mu\lambda h_1{}^{\beta}(x)}{\Gamma(\alpha+1)}t^{\alpha} - \frac{\mu\lambda h_2{}^{\beta}(x)}{\Gamma(\alpha+2)}t^{\alpha+1}\right\} - \left\{\frac{\mu\lambda h_1{}^{\beta}(x)}{\Gamma(2\alpha+1)}(D_t^{\alpha}t^{2\alpha}) + \frac{\mu\lambda h_2{}^{\beta}(x)}{\Gamma(2\alpha+2)}(D_t^{\alpha}t^{2\alpha+1}) - \frac{\lambda^2 h_1(x)}{\Gamma(\alpha+1)}(D_t^{\alpha}t^{\alpha}) - \frac{\lambda^2 h_2(x)}{\Gamma(\alpha+2)}(D_t^{\alpha}t^{\alpha+1})\right\}\right] \quad (44)$$

Which on simplification gives

$$u_2 = J_t^{2\alpha}\left[\left\{\frac{\mu^2 h_1{}^{2\beta}(x)}{\Gamma(2\alpha+1)}t^{2\alpha} + \frac{\mu^2 h_2{}^{2\beta}(x)}{\Gamma(2\alpha+2)}t^{2\alpha+1} - \frac{\mu\lambda h_1{}^{\beta}(x)}{\Gamma(\alpha+1)}t^{\alpha} - \frac{\mu\lambda h_2{}^{\beta}(x)}{\Gamma(\alpha+2)}t^{\alpha+1}\right\} - \left\{\frac{\lambda\mu h_1{}^{\beta}(x)}{\Gamma(\alpha+1)}t^{\alpha} + \frac{\lambda\mu h_2{}^{\beta}(x)}{\Gamma(\alpha+2)}t^{\alpha+1} - \lambda^2 h_1(x)t^0 - \lambda^2 h_2(x)t^1\right\}\right] \quad (45)$$

$$u_2 = \left[\frac{\mu^2 h_1{}^{2\beta}(x)}{\Gamma(2\alpha+1)}(J_t^{2\alpha}t^{2\alpha}) + \frac{\mu^2 h_2{}^{2\beta}(x)}{\Gamma(2\alpha+2)}(J_t^{2\alpha}t^{2\alpha+1}) - \frac{\mu\lambda h_1{}^{\beta}(x)}{\Gamma(\alpha+1)}(J_t^{2\alpha}t^{\alpha}) - \frac{\mu\lambda h_2{}^{\beta}(x)}{\Gamma(\alpha+2)}(J_t^{2\alpha}t^{\alpha+1}) - \frac{\lambda\mu h_1{}^{\beta}(x)}{\Gamma(\alpha+1)}(J_t^{2\alpha}t^{\alpha}) - \frac{\lambda\mu h_2{}^{\beta}(x)}{\Gamma(\alpha+2)}(J_t^{2\alpha}t^{\alpha+1}) + \lambda^2 h_1(x)(J_t^{2\alpha}t^0) + \lambda^2 h_2(x)(J_t^{2\alpha}t^1)\right] \quad (46)$$

$$u_2 = \left[\frac{\mu^2 h_1{}^{2\beta}(x)}{\Gamma(2\alpha+1)}\frac{\Gamma(2\alpha+1)}{\Gamma(1+2\alpha+2\alpha)}t^{2\alpha+2\alpha} + \frac{\mu^2 h_2{}^{2\beta}(x)}{\Gamma(2\alpha+2)}\frac{\Gamma(1+2\alpha+1)}{\Gamma(1+2\alpha+1+2\alpha)}t^{2\alpha+1+2\alpha} - \frac{\mu\lambda h_1{}^{\beta}(x)}{\Gamma(\alpha+1)}\frac{\Gamma(\alpha+1)}{\Gamma(1+\alpha+2\alpha)}t^{\alpha+2\alpha} - \frac{\mu\lambda h_2{}^{\beta}(x)}{\Gamma(\alpha+2)}\frac{\Gamma(1+\alpha+1)}{\Gamma(1+\alpha+1+2\alpha)}t^{\alpha+1+2\alpha} - \frac{\lambda\mu h_1{}^{\beta}(x)}{\Gamma(\alpha+1)}\frac{\Gamma(\alpha+1)}{\Gamma(1+\alpha+2\alpha)}t^{\alpha+2\alpha} - \frac{\lambda\mu h_2{}^{\beta}(x)}{\Gamma(\alpha+2)}\frac{\Gamma(1+\alpha+1)}{\Gamma(1+\alpha+1+2\alpha)}t^{\alpha+1+2\alpha} + \lambda^2 h_1(x)\frac{\Gamma(1+0)}{\Gamma(1+0+2\alpha)}t^{0+2\alpha} + \lambda^2 h_2(x)\frac{\Gamma(1+1)}{\Gamma(1+1+2\alpha)}t^{1+2\alpha}\right] \quad (47)$$

$$u_2 = \mu^2\left(h_1{}^{2\beta}(x)\frac{t^{4\alpha}}{\Gamma(2\alpha+1)} + h_2{}^{2\beta}(x)\frac{t^{4\alpha+1}}{\Gamma(4\alpha+2)}\right) - 2\mu\lambda\left(h_1{}^{\beta}(x)\frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + h_2{}^{\beta}(x)\frac{t^{3\alpha+1}}{\Gamma(3\alpha+2)}\right) + \lambda^2\left(h_1(x)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + h_2(x)\frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)}\right) \quad (48)$$

and so on. In this manner the rest of components of the decomposition series can be obtained. the solution in series form is given by

$$u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + \cdots \quad (49)$$

$$u(x,t) = h_1(x) + th_2(x) + \mu\left(h_1{}^{\beta}(x)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + h_2{}^{\beta}(x)\frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)}\right) - \lambda\left(h_1(x)\frac{t^{\alpha}}{\Gamma(\alpha+1)} + h_2(x)\frac{t^{\alpha+1}}{\Gamma(\alpha+2)}\right) + \mu^2\left(h_1{}^{2\beta}(x)\frac{t^{4\alpha}}{\Gamma(2\alpha+1)} + h_2{}^{2\beta}(x)\frac{t^{4\alpha+1}}{\Gamma(4\alpha+2)}\right) - 2\mu\lambda\left(h_1{}^{\beta}(x)\frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + h_2{}^{\beta}(x)\frac{t^{3\alpha+1}}{\Gamma(3\alpha+2)}\right) + \lambda^2\left(h_1(x)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + h_2(x)\frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)}\right) \quad (50)$$

Special case

Setting $\alpha = 1$ and $\beta = 2$ in equation (50) we obtain the solution of classical telegraph equation by

$$u(x,t) = h_1(x) + th_2(x) + \mu\left(h_1{}^{\beta}(x)\frac{t^2}{2!} + h_2{}^{\beta}(x)\frac{t^3}{3!}\right) - \lambda\left(h_1(x)t + h_2(x)\frac{t^2}{2!}\right) +$$
$$\mu^2\left(h_1{}^{2\beta}(x)\frac{t^4}{4!} + h_2{}^{2\beta}(x)\frac{t^5}{5!}\right) - 2\mu\lambda\left(h_1{}^{\beta}(x)\frac{t^3}{3!} + h_2{}^{\beta}(x)\frac{t^4}{4!}\right) + \lambda^2\left(h_1(x)\frac{t^2}{2!} + h_2(x)\frac{t^4}{4!}\right) \quad (51)$$

4 Conclusions

Clear conclusion can be drawn from the analytical results in equation (50) and equation (51) that the Adomian method provides highly accurate numerical solutions without spatial discretization for the problem. It is evident that the overall errors can be made smaller by adding new terms of the decomposition series.

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Fuzzy Implicative Ideals in KU-algebras

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Abstract – In this paper, we consider KU-implicative ideal (briefly implicative ideal) in KU-algebras. The notion of fuzzy implicative ideals in KU-algebras are introduced, several appropriate examples are provided and their some properties are investigated. The image and the inverse image of fuzzy implicative ideals in KU-algebras are defined and how the image and the inverse image of fuzzy implicative ideals in KU-algebras become fuzzy implicative ideals are studied. Moreover, the Cartesian product of fuzzy implicative ideals in Cartesian product of KU-algebras are given.

***Keywords* –** Fuzzy implicative ideal ,image (inverse image) of fuzzy implicative ideals, Cartesian product of fuzzy implicative ideals.

1. Introduction

BCK-algebras form an important class of logical algebras introduced by Iseki [11,12,13] and was extensively investigated by several researchers. It is an important way to research the algebras by its ideals. The notions of ideals in BCK-algebras and positive implicative ideals in BCK-algebras (i.e. Iseki's implicative ideals) were introduced by Iseki [11,12,13]. The notions of commutative ideals in BCK-algebras and implicative ideals in BCK-algebras were introduced by [18-24]. Zadeh [33] introduced the notion of fuzzy sets. At present this concept has been applied to many mathematical branches, such as group, functional analysis, probability theory, topology, and so on. In 1991, Xi [32] applied this concept to BCK-algebras, and he introduced the notion of fuzzy sub-algebras (ideals) of the BCK-algebras with respect to minimum, and since then Y.B. Jun et al studied fuzzy ideals (cf.[10], [14], [15]), and moreover several fuzzy structures in BCC-algebras are considered (cf [2-9]). Prabpayak and Leerawat [29,30] introduced a new algebraic structure which is called KU-algebra. They gave the concept of homomorphisms of KU-algebras and investigated some related properties. Mostafa et al. [25 - 28] introduced the notion of fuzzy KU-ideals of KU-algebras and then they investigated several basic properties which are related to fuzzy KU-ideals. The idea of implicative ideal was introduced by Meng et al.

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[22,23], they established the concepts of implicative ideals and commutative ideals in BCI-algebras and investigated some of their properties. Mostafa et al. [26,27] introduced the notion of implicative ideals and commutative ideals of KU-algebras and investigated of their properties.

In this paper, the notion of fuzzy implicative ideals of KU - algebras is introduced and then the several basic properties are investigated. How the image and the pre-image of fuzzy implicative ideal under homomorphism of KU-algebras become fuzzy of implicative ideal are studied. Moreover, the product of fuzzy implicative ideal to product fuzzy implicative ideal is established.

2. Preliminaries

Definition 2.1. [29,30] Algebra (X,*,0) of type (2,0) is said to be a KU-algebra, if it satisfies the following conditions:
(ku_1) $(x*y)*[(y*z))*(x*z)]=0$,
(ku_2) $x*0=0$,
(ku_3) $0*x=x$,
(ku_4) $x*y=0$ and $y*x=0$ implies $x=y$,
(ku_5) $x*x=0$, For all $x,y,z\in X$.

On a KU-algebra $(X,*,0)$ we can define a binary relation $\leq$ on X by putting

$$x\leq y \Leftrightarrow y*x=0$$

Thus a KU-algebra X satisfies the conditions:

$(ku_{1^\backslash})$ $(y*z)*(x*z)\leq(x*y)$
$(ku_{2^\backslash})$ $0\leq x$
$(ku_{3^\backslash})$ $x\leq y, y\leq x$ implies $x=y$,
$(ku_{4^\backslash})$ $y*x\leq x$.

Theorem 2.2. [25] In a KU-algebra X , the following axioms are satisfied:
For all $x,y,z\in X$,
(1) $x\leq y$ imply $y*z\leq x*z$,
(2) $x*(y*z)=y*(x*z)$,for all $x,y,z\in X$,
(3) $((y*x)*x)\leq y$.
(4) $((y*x)*x)*x))=(y*x)$

Proof. No. (4) Since $\overbrace{(y*z)*(x*z)\leq(x*y)}^{KU_1'}$ implies $x*((y*z)*z)\leq(x*y)$, put x=0, we have

$$0*((y*z)*z)\leq(0*y)\Rightarrow(y*x)*x\leq y, we\ have\ \overbrace{(y*x)\leq((y*x)*x)*x)}^{by(1)}-\tilde{1}$$

But, $(y*x)*[((y*x)*x)*x)]=[(y*x)*x)]*[(y*x)*x)]=0$
$i.e((y*x)*x)*x)\leq(y*x)----------\tilde{2}$

From $\tilde{1}$, $\tilde{2}$,we have $((y*x)*x)*x))=(y*x)$.

We will refer to X is a KU-algebra unless otherwise indicated.

Definition 2.3. [29,30] Let I be a non empty subset of a KU-algebra X. Then I is said to be an ideal of X, if
(I_0) $0\in I$
(I_1) $\forall y,z\in X$, if $(y*z)\in I$ and $y\in I$, imply $z\in I$.

Definition 2.4. [25] Let I be a non empty subset of a KU-algebra X. Then I is said to be an KU- ideal of X, if
(F_0) $0\in I$
(F_{KU}) $\forall x,y,z\in X$, if $x*(y*z)\in I$ and $y\in I$, imply $x*z\in I$.

Definition 2.5 [27] A KU-algebra X is said to be implicative if it satisfies the identity

$$x = (x * y) * x \text{ for all } x, y \in X.$$

For the properties of KU-algebras, we refer the reader to [12 - 16].

3. Fuzzy Implicative Ideals

We now review some fuzzy logic concepts

Definition 3.1. [33] Let X be a non-empty set, a fuzzy subset μ in X is a function

$$f : X \to [0,1].$$

Definition 2.11. [25] Let X be a KU-algebra, a fuzzy set μ in X is called a fuzzy ideal of X if it satisfies the following conditions:
(F_0) $\mu(0)\geq\mu(x)$ for all $x\in X$.
(FI) $\forall x,y\in X$, $\mu(y)\geq\min\{\mu(x*y),\mu(x)\}$.

Definition 3.2. [25] Let μ be a fuzzy set in a set X. For t $\in$ [0, 1], the set

$$\mu_t = \{x \in X \mid \mu(x) \geq t\}$$

is called upper level cut (level subset) of μ.

Definition 3.3. A non empty subset μ of a KU-algebra X is called a fuzzy implicative ideal of X, if $\forall x,y,z\in X$,

(F_0) $\mu(0) \geq \mu(x)$
(F_1) $\mu((x * y) * x) \geq \min\{\mu(z * ((x * y) * x)), \mu(z)\}$.

Example 3.4. Let $X = \{0,1,2,3,4\}$ in which the operation $*$ is given by the table

*	0	1	2	3	4
0	0	1	2	3	4
1	0	0	1	3	4
2	0	0	0	3	4
3	0	0	0	0	4
4	0	0	0	0	0

Then $(X, *, 0)$ is a KU-Algebra. Define a fuzzy set $\mu : X \to [0,1]$ by

$$\mu(0) = t_0,\ \mu(1) = \mu(2) = t_1,\ \mu(3) = t_2,\ \mu(4) = t_3$$

where $t_0, t_1, t_2, t_3 \in [0,1]$ with $t_0 > t_1 > t_2 > t_3$.

Routine calculation gives that μ is a fuzzy implicative ideal of KU-algebra X.

Lemma 3.5. If μ is a fuzzy implicative ideal of KU - algebra X and if $x \leq z$, then $\mu(x) \geq \mu(z)$.

Proof. if $x \leq z$, then $z * x = 0$, this together with $0 * x = x$, $x * x = x * 0 = 0$ and $\mu(0) \geq \mu(z)$. Put $y = 0$ in (F_1), we get

$$\mu((x * 0) * x) \geq \min\{\mu(z * ((x * 0) * x)), \mu(z)\}$$
$$\mu(0 * x) \geq \min\{\mu(z * (0 * x)), \mu(z)\} = \min\{\mu(z * x), \mu(z)\}$$
$$\mu(x) \geq \min\{\mu(0), \mu(z)\} = \mu(z)$$

Lemma 3.6. Let μ be a fuzzy implicative ideal of KU-algebra X, if the inequality

$z * x \leq y$ hold in X, Then $\mu(x) \geq \min\{\mu(y), \mu(z)\}$.

Proof. Assume that the inequality $y * x \leq z$ holds in X, then

$$\mu(z * x) \geq \mu(y)$$

by (**Lemma 3.5**). Put x =y in (F_2), we have $\mu((x * x) * x) \geq \min\{\mu(z * ((x * x) * x)), \mu(z)\}$ i.e. $\mu(x) \geq \min\{\mu(z * x), \mu(z)\}$, but $\mu(z * x) \geq \mu(y)$, then $\mu(x) \geq \min\{\mu(y), \mu(z)\}$, this completes the proof.

Proposition 3.7. The intersection of any set of fuzzy implicative ideals of KU-algebra X is also fuzzy implicative ideal.

Proof. Let $\{\mu_i\}$ be a family of fuzzy implicative-ideals of KU-algebra X, then for any $x, y, z \in X$,

$$(\cap\mu_i)(0) = \inf(\mu_i(0)) \geq \inf(\mu_i(x)) = (\cap\mu i)(x)$$

and

$$\begin{aligned}(\cap\mu_i)((x * y)*x) &= \inf(\mu_i((x * y)*x)) \geq \inf(\min\{\mu_i(z*((x * y)*x)), \mu_i(z)\}) \\ &= \min\{\inf(\mu_i(z*((x * y)*x)), \inf(\mu_i(z)\} \\ &= \min\{(\cap\mu_i)(z*((x * y)*x)), (\cap\mu_i(z)\}.\end{aligned}$$

This completes the proof.

Theorem 3.8. A fuzzy subset μ of KU - algebra X is a fuzzy implicative-ideal of X if and only if, for every $t \in [0,1]$, μ_t is either empty or an implicative ideal of X.

Proof. Assume that μ is a fuzzy implicative - ideal of X, by (F1), we have $\mu(0) \geq \mu(x)$ for all $x \in X$ therefore $\mu(0) \geq \mu(x) \geq t$ for $x \in \mu t$ and so $0 \in \mu_t$.

Let $z*((x * y) * x) \in \mu t$ and $z \in \mu t$, then $\mu(z*((x * y) * x)) \geq t$ and $\mu(z) \geq t$, since μ is a fuzzy implicative - ideal it follows that $\mu((x * y)*x) \geq \min\{\mu(z*((x * y) * x)), \mu(z)\} \geq t$ and therefore $(x * y)*x \in \mu_t$. Hence μ_t is an KU-ideal of X.

Conversely, we only need to show that (F_1) and (F_2) are true. If (F_1) is false then there exist $x` \in X$ such that $\mu(0) < \mu(x`)$. If we take $t` = (\mu(x`)\ \mu(0))/2$, then $\mu(0) < t`$ and $0 \leq t` < \mu(x`) \leq 1$, thus $x` \in \mu$ and $\mu \neq \phi$ As μ is an KU–implicative ideal of X, we have $0 \in \mu_{t`}$ and so $\mu(0) \geq t`$. This is a contradiction.

Now, assume (F_2) is not true,then there exist x`, y` and z` such that,

$$\mu((x`* y`)* x`) < \min\{\mu(z`*(x`* y`)* x`), \mu(z`)\}$$

Putting $t` = \{\mu((x`* y`)* x`)+\min\{\mu(z`*(x`* y`)* x`), \mu(z`)\}\}/2$, then $\mu((x`* y`)* x`) < t`$ and $0 \leq t` < \min\{\mu(z`*(x`* y`)* x`), \mu(z`)\}\}/2 \leq 1$, hence $\mu(z`*(x`* y`)* x`) > t`$ and $\mu(z`) > t`$,which imply that $(x`* y`)* x`) \in \mu(t`)$ and $z` \in \mu_{t`}$, since μ_t is an implicative-ideal, it follows that $(x`* y`)* x`) \in \mu_{t`}$ and that $\mu(x`* y`)* x`) \geq t`$, this is also a contradiction. Hence μ is a fuzzy implicative ideal of X.

Corollary 3.9. If a fuzzy subset μ of KU-algebra X is a fuzzy implicative-ideal, then for every $t \in \mathrm{Im}(\mu)$, μ_t is an implicative-ideal of X.

Definition 3. 10. [32] Let *f* be a mapping from the set *X* to a set *Y*. If μ is a fuzzy subset of *X*, then the fuzzy subset *B* of *Y* defined by

$$f(\mu)(y) = B(y) = \begin{cases} \sup\limits_{x \in f^{-1}(y)} \mu(x), & \text{if } f^{-1}(y) = \{x \in X, f(x) = y\} \neq \phi \\ 0 & \text{otherwise} \end{cases}$$

is said to be the image of μ under *f*.

Similarly if β is a fuzzy subset of Y, then the fuzzy subset μ = β o f in X (i.e the fuzzy subset defined by μ (x) = β (f (x)) for all x ∈ X) is called the primage of β under f.

Theorem 3.11. An onto homomorphic preimage of a fuzzy implicative-ideal is also a fuzzy implicative-ideal.

Proof. Let f : X → X` be an into homomorphism of KU-algebras, β a fuzzy implicative-ideal of X` and μ the preimage of β under f, then β (f (x)) = μ (x), for all x ∈ X.

Let x ∈ X, then μ (0) = β (f (0)) ≥ β (f (x)) = μ (x). Now let x, y, z ∈ X then

$$\begin{aligned}\mu((x * y)*x) &= \beta(f(x * y)*x)) = \beta((f(x) *` f(y))* f(x)) \\ &\geq \min\{\beta(f(z) *(f(x) *` f(y))* f(x))), \beta(f(z))\} \\ &= \min\{\beta(f(z* ((x * y) * x)), \beta(f(z))\} \\ &= \min\{\mu(z* ((x * y) * x)), \mu(z)\}.\end{aligned}$$

The proof is completed.

Definition 3.12. [31] A fuzzy subset μ of X has sup property if for any subset T of X, there exist $t_0 \in T$ such that $\mu(t_0) = \sup_{t\in T} \mu(t)$.

Theorem 3.13. Let X → Y be a homomorphism between KU-algebras X and Y. For every fuzzy implicative ideal μ in X, f (μ) is a fuzzy implicative-ideal of Y.

Proof. By definition

$$B(y') = f(\mu)(y') = \sup_{x\in f^{-1}(y')} \mu(x) \text{ for all } y' \in Y \text{ and } \sup\phi = 0$$

We have to prove that $B((x' * y') * x') \geq \min\{B(z' * (x' * y') * x'), B(z')\}$, ∀ x`, y`, z`∈ Y.

Let f : X → Y be an onto a homomorphism of KU - algebras, μ a fuzzy implicative - ideal of X with sup property and β the image of μ under f, since μ is a fuzzy implicative - ideal of X, we have μ(0) ≥ μ(x) for all x∈ X. Note that 0 ∈ f^{-1}(0`), where 0, 0` are the zero of X and Y respectively. Thus, $B(0') = \sup_{t\in f^{-1}(0')} \mu(t) = \mu(0) \geq \mu(x)$, for all $x \in X$, which implies that $B(0') \geq \sup_{t\in f^{-1}(x')} \mu(t) = B(x')$, for any $x' \in Y$. For any $x', y', z' \in Y$, let

$$x_0 \in f^{-1}(x') \quad , \quad y_0 \in f^{-1}(y') \quad , \quad z_0 \in f^{-1}(z')$$

be such that

$$\mu(z_0 * ((x_0 * y_0) * x_0)) = \sup_{t\in f^{-1}(z_0*((x_0*y_0)*x_0))} \mu(t) \quad , \quad \mu(z_0) = \sup_{t\in f^{-1}(z')} \mu(t)$$

and

$$\mu(z_0 * ((x_0 * y_0) * x_0)) = B\{f(z_0 * ((x_0 * y_0) * x_0)\} = B(z' * ((x' * y') * x'))$$

$$= \sup_{z_0*((x_0*y_0)*x_0)\in f^{-1}(z'*(x'*y')*x')} \mu(z_0*((x_0*y_0)*x_0)) = \sup_{t\in f^{-1}((z'*(x'*y')*x')} \mu(t).$$

Then

$$B((x'*y)*x') = \sup_{t\in f^{-1}(x'*y)*x')} \mu(t) = \mu((x_0*y_0)*x_0) \geq \min\{\mu(z_0*(x_0*y_0)*x_0)), \mu(z_0)\} =$$

$$\min\left\{\sup_{t\in f^{-1}(z'*((x'*y')*x'))} \mu(t), \quad \sup_{t\in f^{-1}(z')} \mu(t)\right\} = \min\{B(z'*((x'*y')*x')), \ B(z')\}.$$

Hence B is a fuzzy implicative -ideal of Y.

4. Cartesian Product of Fuzzy Implicative-ideal

Definition 4.1. [1] A fuzzy μ is called a fuzzy relation on any set S, if μ is a fuzzy subset

$$\mu : S \times S \rightarrow [0,1]$$

Definition 4.2. [1] If μ is a fuzzy relation a set S and β is a fuzzy subset of S, then μ is fuzzy relation on β if $\mu(x, y) \leq \min\{\beta(x), \beta(y)\}$, $\forall$ x, y ∈ S.

Definition 4.3. [1] Let μ and β be fuzzy subset of a set S, the Cartesian product of μ and β is define by $(\mu \times \beta)(x, y) = \min\{\mu(x), \beta(y)\}$, $\forall$ x, y ∈ S.

Lemma 4.4. [1] let μ and β be fuzzy subset of a set S then,
(i) $\mu \times \beta$ is a fuzzy relation on S.
(ii) $(\mu \times \beta)_t = \mu_t \times \beta_t$ for all t ∈ [0,1].

Definition 4.5. [1] If β is a fuzzy subset of a set S, the strongest fuzzy relation on S, that is, a fuzzy relation on β is μ_β given by $\mu_\beta(x, y) = \min\{\beta(x), \beta(y)\}$, $\forall$ x, y ∈ S.

Lemma 4.6. [1] For a given fuzzy subset S, let μ_β be the strongest fuzzy relation on S the for t ∈ [0,1], we have $(\mu_\beta)_t = \beta_t \times \beta_t$.

Proposition 4.7. For a given fuzzy subset β of KU- algebra X, let μ_β be the strongest fuzzy relation on X. If μ_β is a fuzzy implicative ideal of X × X, then $\beta(x) \leq \beta(0)$ for all x ∈ X.

Proof. Since μ_β is a fuzzy implicative ideal of X × X, it follows from (F_1) that

$$\mu_\beta(x, x) = \min\{\beta(x), \beta(x)\} \leq (0, 0) = \min\{\beta(0), \beta(0)\}$$

where (0, 0) ∈ X × X, then $\beta(x) \leq \beta(0)$.

Remark 4.8. Let X and Y be KU-algebras, we define $*$ on X × Y for every (x, y), (u, v)∈ X x Y, $(x, y) * (u, v) = (x * u, y * v)$, then clearly $(x * y, *, (0, 0))$ is a KU-algebra.

Theorem 4.9. let μ and β be a fuzzy implicative-ideals of KU - algebra X, $\mu \times \beta$ is a fuzzy implicative-ideal of X × X.

Proof. for any $(x, y) \in X \times X$, we have,

$(\mu \times \beta)(0, 0) = \min\{\mu(0), \beta(0)\} \geq \min\{\mu(x), \beta(x)\} = (\mu \text{ x } \beta)(x, y)$.

Now let $(x_1, x_2), (y_1, y_2), (z_1, z_2) \in X \times X$, then,

$$
\begin{aligned}
&(\mu \text{ x } \beta)((x_1 * y_1) * x_1), ((x_2 * y_2) * x_2)) \\
&= \min\{\mu((x_1 * y_1) * x_1)), \beta((x_2 * y_2) * x_2))\} \\
&\geq \min\{\min\{\mu(z_1 * (x_1 * y_1) * x_1))), \mu(z_1)\}, \min\{\beta((x_2 * y_2) * x_2)), \beta(z_2)\}\} \\
&= \min\{\min\{\mu(z_1 * (x_1 * y_1) * x_1)), \beta(z_2 * (x_2 * y_2) * x_2))\}, \min\{\mu(z_1), \beta(z_2)\}\} \\
&= \min\{(\mu \times \beta)(z_1 * (x_1 * y_1) * x_1), z_2 * (x_2 * y_2) * x_2)), (\mu \times \beta)(z_1, z_2)\}.
\end{aligned}
$$

Hence $\mu \times \beta$ is a fuzzy implicative ideal of $X \times X$.

Analogous to [28], we have a similar results for implicative-ideal, which can be proved in similar manner, we state the results without proof.

Theorem 4.10. let μ and β be a fuzzy subset of KU-algebra X,such that $\mu \times \beta$ is fuzzy implicative -ideal of $X \times X$, then
(i) either $\mu(x) \leq \mu(0)$ or $\beta(x) \leq \beta(0)$ for all $x \in X$,
(ii) if $\mu(x) \leq \mu(0)$ for all $x \in X$, then either $\mu(x) \leq \beta(0)$ or $\beta(x) \leq \beta(0)$,
(iii) if $\beta(x) \leq \beta(0)$ for all $x \in X$, then either $\mu(x) \leq \mu(0)$ or $\beta(x) \leq \mu(0)$,
(v) either μ or β is a fuzzy implicative - ideal of X.

Theorem 4.11. let β be a fuzzy subset of KU-algebra X and let μ_β be the strongest fuzzy relation on X, then β is a fuzzy implicative ideal of X if and only if μ_β is a fuzzy implicative-ideal of $X \times X$.

Proof. Assume that β is a fuzzy implicative-ideal X, we note from (F_1) that

$$\mu_\beta(0, 0) = \min\{\beta(0), \beta(0)\} \geq \min\{\beta(x), \beta(y)\} = \mu_\beta(x, y)$$

Now, for any $(x_1,x_2), (y_1,y_2),(z_1,z_2) \in X \text{ x } X$, we have from ($F_2$)

$$
\begin{aligned}
&\mu_\beta((x_1 * y_1) * x_1, (x_2 * y_2) * x_2) \\
&= \min\{\beta((x_1 * y_1) * x_1), \beta((x_2 * y_2) * x_2)\} \\
&\geq \min\{\min\{\beta(z_1 * ((x_1 * y_1) * x_1)), \beta(z_1)\}, \min\{\beta(z_2 * ((x_2 * y_2) * x_2)), \beta(z_2)\}\} \\
&= \min\{\min\{\beta(z_1 * ((x_1 * y_1) * x_1)), \beta(z_2 * ((x_2 * y_2) * x_2))\}, \min\{\beta(z_1), \beta(z_2)\}\} \\
&= \min\{\mu_\beta(z_1 * ((x_1 * y_1) * x_1), z_2 * ((x_2 * y_2) * x_2)), \mu_\beta(z_1, z_2)\}.
\end{aligned}
$$

Hence μ_β is a fuzzy implicative-ideal of $X \times X$.

Conversely: for all $(x, y) \in X \times X$, we haveMin $\{\beta(0), \beta(0)\} = \mu_\beta(x, y) = \min\{\beta(x), \beta(y)\}$ It follows that $\beta(0) \geq \beta(x)$ for all $x \in X$, which prove (F_1).

Now, let $(x_1, x_2), (y_1, y_2), (z_1, z_2) \in X \times X$, then

$$
\begin{aligned}
&\min\{\beta(((x_1 * y_1) * x_1), \beta((x_2 * y_2) * x_2)\} = \mu_\beta((x_1 * y_1) * x_1), (x_2 * y_2) * x_2)) \\
&\geq \min\{\mu_\beta((z_1, z_2) * ((x_1, x_2) * (y_1, y_2)) * (x_1, x_2)), \mu_\beta(z_1,z_2))\}
\end{aligned}
$$

$= \min \{\mu_\beta (z_1 * ((x_1 * y_1) * x_1), z_2 * ((x_2 * y_2) * x_2)), \mu_\beta (z_1, z_2)\}$
$= \min \{\min \{\beta (z_1 * ((x_1 * y_1) * x_1)), \beta (z_2 * ((x_2 * y_2) * x_2))\}, \min \{\beta (z_1), \beta (z_2)\}\}$
$= \min \{\min \{\beta (z_1 * ((x_1 * y_1) * x_1)), \beta (z_1)\}, \min \{\beta(z_2 * ((x_2 * y_2) * x_2)), \beta (z_2)\}\}$

In particular, if we take $x_2 = y_2 = z_2 = 0$, then, $\beta ((x_1 * y_1) * x_1) \geq \min \{ \beta (z_1 * ((x_1 * y_1) * x_1)), \beta (z_1)\}$.

This prove (F_1) and completes the proof.

5. Conclusion

we have studied the fuzzy of implicative ideal in KU-algebras. Also we discussed few results of fuzzy of implicative ideal in KU-algebras under homomorphism, the image and the pre- image of fuzzy implicative ideal under homomorphism of KU-algebras are defined. How the image and the pre-image of fuzzy implicative ideal under homomorphism of KU-algebras become fuzzy of implicative ideal are studied. Moreover, the product of fuzzy implicative ideal to product fuzzy implicative ideal is established. Furthermore, the main purpose of our future work is to investigate the foldedness of other types of fuzzy ideals with special properties such as a bipolar intuitionistic (interval value) fuzzy n-fold of implicative ideals in some algebras.

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Conflicts of Interest

State any potential conflicts of interest here or "The author declare no conflict of interest".

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$T-$Fuzzy Submodules of $R \times M$

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Abstaract — In this paper, we introduce the concept of T-fuzzy submodule of $R \times M$ and give new results on this subject. Next we study the concept of the extension of T-fuzzy submodule of $R \times M$ and prove some results on these. Also we investigate T-fuzzy submodule of $R \times M$ under homomorphisms or R-modules.

Keywords — *Theory of modules, extensions, homomorphism, fuzzy set theory, norms.*

1 Introduction

In algebra, ring theory is the study of rings algebraic structures in which addition and multiplication are defined and have similar properties to those operations defined for the integers. Ring theory studies the structure of rings, their representations, or, in different language, modules, special classes of rings (group rings, division rings, universal enveloping algebras), as well as an array of properties that proved to be of interest both within the theory itself and for its applications, such as homological properties and polynomial identities. In mathematics, a module is one of the fundamental algebraic structures used in abstract algebra. A module over a ring is a generalization of the notion of vector space over a field, wherein the corresponding scalars are the elements of an arbitrary given ring (with identity) and a multiplication (on the left and/or on the right) is defined between elements of the ring and elements of the module. Thus, a module, like a vector space, is an additive abelian group; a product is defined between elements of the ring and elements of the module that is distributive over the addition operation of each parameter and is compatible with the ring multiplication. Modules are very closely related to the representation theory of groups. They are also one of the central notions of commutative algebra and homological algebra, and are used widely in algebraic geometry and algebraic topology. In 1965, Zadeh [17] introduced the notion of a fuzzy subset of a set as a method for representing uncertainty. It provoked, at first (and as expected), a strong negative reaction from some influential scientists and mathematicians many of whom turned openly hostile. However, despite the controversy, the subject also attracted the attention of other mathematicians and in the following years, the field grew enormously, finding applications in areas as diverse as washing machines to handwriting recognition. In its trajectory of stupendous growth, it has also come to include the theory of fuzzy algebra and for the past five decades, several researchers have been working on concepts like fuzzy semigroup, fuzzy groups, fuzzy rings, fuzzy ideals, fuzzy semirings, fuzzy near-rings and so on. Solairaju and Nagarajan [5,6] have introduced and defined a new algebraic structure called Q-fuzzy subgroups. The triangular norm, T-norm, originated from the studies of probabilistic metric spaces in which triangular inequalities were extended using the theory of T-norm. Later, Hohle [4], Alsina et al. [1] introduced the T-norm into fuzzy set theory and suggested that the T-norm be used for the intersection of fuzzy sets. Since then, many other researchers have presented various types of T-norms for particular purposes [3, 16]. Anthony and Sherwood [2] gave the definition of fuzzy subgroup based on t-norm. The author by using norms, investigated some properties of fuzzy submodules, fuzzy subrings, fuzzy ideals of subtraction semigroups, intuitionistic fuzzy subrings and ideals of a ring, fuzzy Lie algebra, fuzzy subgroups on direct product of groups, characterizations of intuitionistic fuzzy subsemirings of semirings and their homomorphisms, characterization of Q-fuzzy subrings (anti Q-fuzzy subrings) ([7, 8, 9, 10, 11, 12, 13, 14, 15]). In this work, by using a t-norm T, we introduce the notion of T-fuzzy submodule of $R \times M$, and investigate some of their properties. Also we use a t-norm to construct the concept of the extension of T-fuzzy submodule of $R \times M$ and prove some results on these. Finally we obtain some new results of T-fuzzy submodule of $R \times M$ with respect to t-norm T under homomorphisms of R-modules

2 Preliminary

Definition 2.1. A ring $< R, +, . >$ consists of a nonempty set R and two binary operations $+$ and . that satisfy the axioms:
(1) $< R, +, . >$ is an abelian group;
(2) $(ab)c = a(bc)$ (associative multiplication) for all $a, b, c \in R$;
(3) $a(b + c) = ab + ac, (b + c)a = ba + ca$ (distributive laws) for all $a, b, c \in R$

Moreover, the ring R is a commutative ring if $ab = ba$ and ring with identity if R contains an element 1_R such that $1_R a = a 1_R = a$ for all $a \in R$.

Example 2.2. (1) The ring $\mathbb{Z}$ of integers is a commutative ring with identity. So are $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$, $\mathbb{Z}_n$, $\mathbb{R}[x]$, etc.
(2) $3\mathbb{Z}$ is a commutative ring with no identity.
(3) The ring $\mathbb{Z}^{2\times 2}$ of 2×2 matrices with integer coe?cients is anoncommutative ring with identity.
(4) $(3\mathbb{Z})^{2\times 2}$ is a noncommutative ring with no identity.

Definition 2.3. Let R be a ring. A commutative group $(M, +)$ is called a left R-module or a left module over R with respect to a mapping

$$. : R \times M \to M$$

if for all $r, s \in R$ and $m, n \in M$,
(1) $r.(m+n) = r.m + r.n$,
(2) $r.(s.m) = (rs).m$,
(3) $(r+s).m = r.m + s.m$.

If R has an identity 1 and if $1.m = m$ for all $m \in M$, then M is called a unitary or unital left R-module.

A right R-module can be defined in a similar fashion.

Definition 2.4. Let X a non-empty sets. A fuzzy subset μ of X is a function $\mu : X \to [0,1]$. Denote by $[0,1]^X$, the set of all fuzzy subset of X.

Definition 2.5. A t-norm T is a function $T : [0,1] \times [0,1] \to [0,1]$ having the following four properties:
(T1) $T(x,1) = x$ (neutral element),
(T2) $T(x,y) \leq T(x,z)$ if $y \leq z$ (monotonicity),
(T3) $T(x,y) = T(y,x)$ (commutativity),
(T4) $T(x, T(y,z)) = T(T(x,y), z)$ (associativity),
for all $x, y, z \in [0,1]$.

We say that T be idempotent if $T(x,x) = x$ for all $x \in [0,1]$.

It is clear that if $x_1 \geq x_2$ and $y_1 \geq y_2$, then $T(x_1, y_1) \geq T(x_2, y_2)$.

Example 2.6. (1) Standard intersection T-norm $T_m(x,y) = \min\{x,y\}$.
(2) Bounded sum T-norm $T_b(x,y) = \max\{0, x+y-1\}$.
(3) algebraic product T-norm $T_p(x,y) = xy$.
(4) Drastic T-norm

$$T_D(x,y) = \begin{cases} y & \text{if } x = 1 \\ x & \text{if } y = 1 \\ 0 & \text{otherwise.} \end{cases}$$

(5) Nilpotent minimum T-norm

$$T_{nM}(x,y) = \begin{cases} \min\{x,y\} & \text{if } x+y > 1 \\ 0 & \text{otherwise.} \end{cases}$$

(6) Hamacher product T-norm

$$T_{H_0}(x,y) = \begin{cases} 0 & \text{if } x = y = 0 \\ \frac{xy}{x+y-xy} & \text{otherwise.} \end{cases}$$

The drastic t-norm is the pointwise smallest t-norm and the minimum is the pointwise largest t-norm: $T_D(x,y) \leq T(x,y) \leq T_{\min}(x,y)$ for all $x, y \in [0,1]$.

Lemma 2.7. Let T be a t-norm. Then

$$T(T(x,y), T(w,z)) = T(T(x,w), T(y,z)),$$

for all $x, y, w, z \in [0,1]$.

3 $T-$Fuzzy Submodules of $R \times M$

Definition 3.1. Let M be an R-module. A M-fuzzy subset μ of R is a function $\mu : R \times M \to [0,1]$. Denote by $[0,1]^{R\times M}$, the set of all M-fuzzy subset of R.

Definition 3.2. Let $S \subseteq R$ and $a \in [0,1]$. Define $a_{\{S\times M\}} \in [0,1]^{R\times M}$ as follows;

$$a_{\{S\times M\}}(r,m) = \begin{cases} a & \text{if } r \in S, m \in M \\ 0 & \text{if } r \in R - S, m \in M \end{cases}$$

Definition 3.3. Let $\mu \in [0,1]^{R\times M}$ and T be a t-norm. We say that μ is a T-fuzzy submodule of $R \times M$ if for all $r, s \in R$ and $x, y \in M$
(1) $\mu(r, 0_M) = 1$,
(2) $\mu(r, sx) \geq \mu(r, x)$,
(3) $\mu(r, x + y) \geq T(\mu(r, x), \mu(r, y))$.

We denote the set of all fuzzy submodules of $R \times M$ by $TF(R \times M)$.
Since $-1x = -x$, condition (2) implies that $\mu(r, -x) \geq \mu(r, x)$.

Example 3.4. Let $R = (\mathbb{Z}, +, .)$ be a ring of integer. If $M = \mathbb{Z}$, then M is an R-module. For all $x \in R$ we define a fuzzy subset μ of $\mathbb{Z} \times \mathbb{Z}$ as

$$\mu(r,x) = \begin{cases} 1 & \text{if } (r,x) \in \mathbb{Z} \times \{0_{\mathbb{Z}}\} \\ 0.90 & \text{if } (r,x) \in \mathbb{Z} \times (2\mathbb{Z} - \{0_{\mathbb{Z}}\}) \\ 0.80 & \text{if } (r,x) \in \mathbb{Z} \times (2\mathbb{Z} + 1) \end{cases}$$

Let $T(x,y) = T_p(x,y) = xy$ for all $x, y \in \mathbb{Z}$, then $\mu \in TF(\mathbb{Z} \times \mathbb{Z})$.

Definition 3.5. Let $\mu, \nu \in TF(R\times M)$ and $r \in R$ and $x \in M$. Define $\mu + \nu$, $\mu \cup \nu$, $\mu \cap \nu$, and $-\mu$ as follows:
$(\mu + \nu)(r, x) = \sup\{T(\mu(r, y), \nu(r, z)) \mid y, z \in M, y + z = x\}$,
$(\mu \cup \nu)(r, x) = \sup\{\mu(r, x), \nu(r, x)\}$,
$(\mu \cap \nu)(r, x) = T(\mu(r, x), \nu(r, x))$,
$(-\mu)(r, x) = \mu(r, -x)$.

Then $\mu + \nu, \mu \cup \nu, \mu \cap \nu$ are called the sum, union, intersection of μ and ν respectively, and $-\mu$ the negative of μ.

Let $\mu_i \in TF(R \times M)$. The least upper bound $\cup_{i\in I}\mu_i$ of the $x_i^{,}s$ is given by $(\cup_{i\in I}\mu_i)(r, x) = \sup\{\mu_i(r, x) \mid i \in I\}$ for all $i \in I, r \in R, x \in M$.

Definition 3.6. Let $\mu_i \in TF(R \times M), 1 \leq i \leq n$ and $n \in \mathbb{N}$. Since $+$ is associative and commutative, we can consider $\mu_1 + \mu_2 + ... + \mu_n$ and write it as $\Sigma_{i=1}^n \mu_i$.

If $\mu_i \in TF(R \times M)$ for each $i \in I$, then $\Sigma_{i\in I}\mu_i$ is defined by

$$(\Sigma_{i\in I}\mu_i)(r, x) = \sup\{T_{i\in I}(\mu_i(r, x_i)) \mid x_i \in M, i \in I, \Sigma x_i = x\}$$

such that $\Sigma x_i = \Sigma_{i\in I} x_i$ and there are at most finitely many $x_i^{,}s$ not equal to 0_M.

Definition 3.7. Let $r, s \in R, x \in M$ and $\mu \in TF(R \times M)$. Define $s\mu$ as follow: $s\mu(r,x) = \sup\{\mu(r,y) \mid y \in M, sy = x\}$ which is called the product of s and μ.

Proposition 3.8. Let $r, s, t \in R$ and $\mu, \nu, \xi, \mu_i \in TF(R \times M), i \in I$. Then for all $x, y \in M$
(1) $1\mu = \mu, (-1)\mu = (-\mu)$.
(2) $s1_{\{R\times 0_M\}} = 1_{\{R\times 0_M\}}$.
(3) If $\mu \leq \nu$, then $s\mu \leq s\nu$.
(4) $(ts)\mu = t(s\mu)$.
(5) $s(\mu + \nu) = s\mu + s\nu$.
(6) $s(\cup_{i\in I}\mu_i) = \cup_{i\in I} s\mu_i$.
(7) $(s\mu)(r, sx) \geq \mu(r,x)$.
(8) $\xi(r, sx) \geq \mu(r,x)$ if and only if $s\mu \leq \xi$.
(9) $(s\mu + t\nu)(r, sx + ty) \geq T(\mu(r,x), \nu(r,y))$.
(10) $\xi(r, sx + ty) \geq T(\mu(r,x), \nu(r,y))$ if and only if $s\mu + t\nu \leq \xi$.

Proof. Let $r, s, t \in R$ and $x, y, z \in M$. Then

(1) $1\mu(r,x) = \sup\{\mu(r,y) \mid y \in M, 1y = x\} = \mu(r,x)$. Also
$(-1)\mu(r,x) = \sup\{\mu(r,y) \mid y \in M, -1y = x\} = \mu(r,-x) = (-\mu)(r,x)$.

(2) It is clear.

(3) $s\mu(r,x) = \sup\{\mu(r,y) \mid y \in M, sy = x\} \leq \sup\{\nu(r,y) \mid y \in M, sy = x\}$
$= s\nu(r,x)$.

(4) $(ts)\mu(r,x) = \sup\{\mu(r,y) \mid y \in M, (ts)y = x\} = \sup\{\mu(r,y) \mid y \in M, t(sy) = x\} = t(s\mu)(r,x)$.

(5) $(s\mu + s\nu)(r,x) = \sup\{T(s\mu(r,y), s\nu(r,z)) \mid y, z \in M, y + z = x\}$
$= \sup\{T(\sup\{\mu(r,y_1) \mid y_1 \in M, sy_1 = y\}, \sup\{\nu(r,z_1) \mid z_1 \in M, sz_1 = z\})\}$
$\mid y, z \in M, s(y_1 + z_1) = sy_1 + sz_1 = x = s(\mu + \nu)(r,x)\}$

(6) $s(\cup_{i\in I}\mu_i)(r,x) = \sup\{(\cup_{i\in I}\mu_i)(r,y) \mid y \in M, sy = x\}$
$= \sup\{\sup_{i\in I}\mu_i(r,y) \mid y \in M, sy = x\} = \sup_{i\in I}\{\sup \mu_i(r,y) \mid y \in M, sy = x\}$
$= \cup_{i\in I} s\mu_i(r,x)$.

(7) $s\mu(r, sx) = \sup\{\mu(r,y) \mid y \in M, sy = sx\} \geq \mu(r,x)$.

(8) Let $\xi(r, sx) \geq \mu(r,x)$. Then $s\mu(r,x) = \sup\{\mu(r,y) \mid y \in M, sy = x\}$
$\leq \sup\{\xi(r, sy) \mid y \in M, sy = x\} = \xi(r,x)$.

(9) By Definition 3.5 and (part 7) we obtain that
$(s\mu + t\nu)(r, sx + ty) = \sup\{T((s\mu)(r,sx), (t\nu)(r,ty))\} \geq \sup\{T\mu(r,x), \nu(r,y))\}$
$\geq T(\mu(r,x), \nu(r,y))$.

(10) Let $\xi(r, sx + ty) \geq T(\mu(r,x), \nu(r,y))$. Then $(s\mu + t\nu)(r,z)$
$= \sup\{T(s\mu(r,z_1), t\nu(r,z_2) \mid z_1, z_2 \in M, z_1 + z_2 = z\}$

$= \sup\{T(\sup\{\mu(r,x) \mid x \in M, sx = z_1\}, \sup\{\nu(r,y) \mid y \in M, ty = z_2\}) \mid z_1, z_2 \in M, sx + ty = z\}$
$= \sup\{T(\mu(r,x), \nu(r,y)) \mid x, y \in M, sx + ty = z\} \leq \sup\{\xi(r, sx + ty) \mid x, y \in M, sx + ty = z\} = \xi(r,z)$.

Conversely, suppose that $s\mu + t\nu \leq \xi$. Then $\xi(r, sx+ty) \geq (s\mu + t\nu)(r, sx+ty) \geq T(s\mu(r,sx), t\nu(r,ty)) \geq T(\mu(r,x), \nu(r,y))$ $(by(7))$. □

Corollary 3.9. Let $r, s, t \in R$ and $\mu \in TF(R \times M)$. Then
(1) $s\mu \leq \mu$ if and only if $\mu(r,sx) \geq \mu(r,x)$,
(2) $s\mu + t\mu \leq \mu$ if and only if $\mu(r, sx+ty) \geq T(\mu(r,x), \mu(r,y))$.

Proof. (1) In Proposision 3.8 part(8), put $\mu = \xi$.

(2) In Proposision 3.8 part(10), put $\mu = \nu = \xi$. □

Corollary 3.10. Let $s \in R$ and $\mu \in [0,1]^{R\times M}$. Then $\mu \in TF(R \times M)$ if and only if μ satisfies the following conditions:
(1) $1_{\{R\times 0_M\}} \leq \mu$,
(2) $s\mu \leq \mu$,
(3) $\mu + \mu \leq \mu$.

Proof. Let $\mu \in TF(R \times M)$. Then

(1) $\mu(r, 0_M) = 1 \geq 1 = 1_{\{R\times 0_M\}(r,0_M)}$ and so $1_{\{R\times 0_M\}} \leq \mu$.

(2) For all $r, s \in R$ and $x \in M$ we have that $\mu(r,sx) \geq \mu(r,x)$, and by Corollary 3.9 (part 1) we get $s\mu \leq \mu$.

(3) Let $r, s, t \in R$ and $x, y \in M$. Then from $\mu(r, x+y) \geq T(\mu(r,x), \mu(r,y))$ and Corollary 3.9 (part 2 with $s = 1 = t$) we obtain that $\mu + \mu \leq \mu$.

Conversely, we prove that $\mu \in TF(R \times M)$.

From condition (1) we have $\mu(r, 0_M) \geq 1_{\{R\times 0_M\}(r,0_M)}$ and so $\mu(r, 0_M) = 1$.

By condition (2) and Corollary 3.9 (part 1) we get $\mu(r,sx) \geq \mu(r,x)$.

Also as condition (3) and Corollary 3.9 (part 2) we have $\mu(r, x+y) \geq T(\mu(r,x), \mu(r,y))$. Therefore $\mu \in TF(R \times M)$. □

Proposition 3.11. Let $r, s, t \in R$ and $x, y \in M$. If $\mu \in [0,1]^{R\times M}$, then $\mu \in TF(R \times M)$ if and only if μ satisfies condition (1) from Definition 3.3 and the following condition:
(4) $\mu(r, sx+ty) \geq T(\mu(r,x), \mu(r,y))$.

Proof. Suppose $\mu \in TF(R \times M)$. By Definition 3.3, μ satisfies condition (1). Since μ also satisfies conditions (2) and (3), it follows that $\mu(r, sx+ty) \geq T(\mu(r,sx), \mu(r,ty)) \geq$

$T(\mu(r,x),\mu(r,y))$.

Conversely, assume that μ satisfies conditions (1) and (4). Then $\mu(r,sx) = \mu(r,sx+s0_M) \geq T(\mu(r,x),\mu(r,0_M)) = T(\mu(r,x),1) = \mu(r,x)$.

Also $\mu(r,x+y) = \mu(r,1x+1y) \geq T(\mu(r,x),\mu(r,y))$. Hence μ satisfies conditions (2) and (3) and so $\mu \in TF(R\times M)$. □

Corollary 3.12. Let $r,s \in R$ and $\mu \in [0,1]^{R\times M}$. Then $\mu \in TF(R\times M)$ if and only if μ satisfies the following conditions:
(1) $1_{\{R\times 0_M\}} \leq \mu$,
(2) $r\mu + s\mu \leq \mu$.

Proof. Let $\mu \in TF(R\times M)$. Then from Corollary 3.10 we get that $1_{\{R\times 0_M\}} \leq \mu$ and $r\mu + s\mu \leq \mu$. Conversely, we show that $\mu \in TF(R\times M)$. As $1_{\{R\times 0_M\}} \leq \mu$ so $\mu(r,0_M) = 1$. By $r\mu + s\mu \leq \mu$ and Proposision 3.8(part 10) and Proposion 3.11 we obtain that $\mu \in TF(R\times M)$. □

Proposition 3.13. Let $\mu,\nu \in TF(R\times M)$. Then $\mu\cap\nu \in TF(R\times M)$.

Proof. Let $r,s \in R$ and $x,y \in M$. If $\mu,\nu \in TF(R\times M)$, then

(1) $(\mu\cap\nu)(r,0_M) = T(\mu(r,0_M),\nu(r,0_M)) = T(1,1) = 1$.

(2) $(\mu\cap\nu)(r,sx) = T(\mu(r,sx),\nu(r,sx)) \geq T(\mu(r,x),\nu(r,x)) = (\mu\cap\nu)(r,x)$.

(3) $(\mu\cap\nu)(r,x+y) = T(\mu(r,x+y),\nu(r,x+y)) \geq T(T(\mu(r,x),\mu(r,y)),T(\nu(r,x),\nu(r,y))) = T(T(\mu(r,x),\nu(r,x)),T(\mu(r,y),\nu(r,y)))(by\ Lemma\ 2.7) = T((\mu\cap\nu)(r,x),(\mu\cap\nu)(r,y))$.

Thus $\mu\cap\nu \in TF(R\times M)$. □

Corollary 3.14. If $\{\mu_i \mid i = 1,2,...\} \subseteq TF(R\times M)$, then $\cap_i\mu_i \in TF(R\times M)$.

Proposition 3.15. Let $\mu,\nu \in TF(R\times M)$ abd T be idempotent. Then $\mu+\nu \in TF(R\times M)$.

Proof. Let $\mu,\nu \in TF(R\times M)$.

(1) Let $r \in R, x \in M$. Then
$(\mu+\nu)(r,x) = \sup\{T(\mu(r,x_1),\nu(r,x_2)) \mid x_1,x_2 \in M, x_1+x_2 = x\}$
$\geq \sup\{T(1_{\{R\times 0_M\}}(r,x_1),1_{\{R\times 0_M\}}(r,x_2)) \mid x_1,x_2 \in M, x_1+x_2 = x\}$
$= \sup\{T(1,1) \mid x_1,x_2 \in M, x_1+x_2 = x\} = 1 = 1_{\{R\times 0_M\}}(r,x)$.

(2) Let $s \in R$. Then $s(\mu+\nu) = s\mu + s\nu \subseteq \mu+\nu$.

(3) $(\mu+\nu)+(\mu+\nu) = (\mu+\mu)+(\nu+\nu) \subseteq (\mu+\nu)$.

Hence from Corollary 3.10 we have $\mu+\nu \in TF(R\times M)$. □

Corollary 3.16. If $\{\mu_i \mid i = 1,2,...\} \subseteq TF(R\times M)$, then $\Sigma_i\mu_i \in TF(R\times M)$.

Definition 3.17. Let $\mu \in [0,1]^{R\times M}$ and $s \in R$. For all $(r, y) \in R\times M$ the fuzzy subset $< s, \mu > \in [0,1]^{R\times M}$ defined by $< s, \mu > (r, y) = \mu(r, sy)$ is called the extension of μ by s.
Also we define $Supp\mu = \{(r, x) \in R \times M \mid \mu(r, x) > 0\}$.

Proposition 3.18. Let $\mu \in TF(R \times M)$ and $s \in R$. Then $< s, \mu > \in TF(R \times M)$.

Proof. Let $r, s, t \in R$ and $x, y \in M$. If $\mu \in TF(R \times M)$, then
(1) $< s, \mu > (r, 0_M) = \mu(r, 0_M) = 1$.
(2) $< s, \mu > (r, tx) = \mu(r, stx) = \mu(r, tsx) \geq \mu(r, sx) =< s, \mu > (r, x)$.
(3) $< s, \mu > (r, x + y) = \mu(r, s(x + y)) = \mu(r, sx + sy) \geq T(\mu(r, sx), \mu(r, sy)) = T(< s, \mu > (r, x), < s, \mu > (r, y))$.
Hence $< s, \mu > \in TF(R \times M)$. □

Corollary 3.19. If $s \in R$ and $\{\mu_i \mid i = 1, 2, ...\} \subseteq TF(R \times M)$, then $< s, \cap_i \mu_i > \in TF(R \times M)$.

Proposition 3.20. Let $\mu \in TF(R \times M)$ and $s \in R$. Then we have the following:
(1) $\mu \subseteq < s, \mu >$,
(2) $< s^n, \mu > \subseteq < s^{n+1}, \mu >$ for every $n \in N$,
(3) If $x \in M$ and $\mu(r, x) > 0$, then $Supp < s, \mu > = R \times M$.

Proof. (1) If $(r, x) \in R \times M$, then $< s, \mu > (r, x) = \mu(r, sx) \geq \mu(r, x)$.

(2) From every $n \in N$ and $(r, x) \in R \times M$ we have that $< s^{n+1}, \mu > (r, x) = \mu(r, s^{n+1}x) = \mu(r, ss^n x) \geq \mu(r, s^n x) =< s^n, \mu > (r, x)$.

(3) By Definition 3.17, $Supp < s, \mu > \subseteq R \times M$. Now if $(r, x) \in R \times M$, then $< s, \mu > (r, x) = \mu(r, sx) \geq \mu(r, x) > 0$ and so $Supp < s, \mu > = R \times M$. □

4 Homomorphisms Over T-Fuzzy Submodules of $R \times M$

Definition 4.1. Let f be a mapping from R-module M into R-module N. Let $\mu \in TF(R\times M)$ and $\nu \in TF(R\times N)$. Define $f(\mu) \in [0,1]^{R\times N}$ and $f^{-1}(\nu) \in [0,1]^{R\times M}$ as $\forall y \in N$, $\forall r \in R$, $f(\mu)(r, y) = \sup\{\mu(r, x) \mid x \in M, f(x) = y\}$ if $f^{-1}(y) \neq \emptyset$ and $f(\mu)(r, y) = 0$ if $f^{-1}(y) = \emptyset$. Also $\forall x \in M$, $\forall r \in R$ $f^{-1}(\nu)(r, x) = \nu(r, f(x))$.

Proposition 4.2. Let f be a mapping from R-module M into R-module N. Let $\mu, \mu_1, \mu_2 \in TF(R \times M)$ and $\nu, \nu_1, \nu_2 \in TF(R \times N)$.
(1) Let $\mu_1 \leq \mu_2$. Then $f(\mu_1) \leq f(\mu_2)$.
(2) Let $\nu_1 \leq \nu_2$. Then $f^{-1}(\nu_1) \leq f^{-2}(\nu_2)$.
(3) $\mu \leq f^{-1}(f(\mu))$. Inparticular, if f is an injection, then $\mu = f^{-1}(f(\mu))$.
(4) $\nu \geq f(f^{-1}(\nu))$. Inparticular, if f is a surjection,, then $\nu = f(f^{-1}(\nu))$.
(5) $f(\mu) \leq \nu$ if and only if $\mu \leq f^{-1}(\nu)$.

Proof. Clearly, assertions (1) and (2) hold.

(3) $f^{-1}(f(\mu))(r,x) = f(\mu)(r,f(x)) = \sup\{\mu(r,z) \mid z \in M, f(z) = f(x)\} \geq \mu(r,x)$.
If f is an injection, then $f^{-1}(f(\mu))(r,x) = \sup\{\mu(r,z) \mid z \in M, f(z) = f(x)\} = \mu(r,x)$.

(4) $f(f^{-1}(\nu))(r,y) = \sup\{f^{-1}(\nu)(r,x) \mid x \in M, f(x) = y\} = \sup\{\nu(r,f(x)) \mid x \in M, f(x) = y\} = \{\nu(r,y) \mid y \in f(M)\} \leq \nu(y)$.

Assertion (5) is an immediate consequence of the four preceding assertions. □

Proposition 4.3. Suppose that f be an epiomorphism from R-module M into R-module N. Let $r,s,t \in R$ and $\mu,\nu \in TF(R \times M)$. Then
(1) $f(\mu+\nu) = f(\mu) + f(\nu)$,
(2) $f(s\mu) = sf(\mu)$,
(3) $f(s\mu + t\nu) = sf(\mu) + tf(\nu)$.

Proof. (1) If $y_1, y_2 \in N$, then we have $x_1, x_2 \in M$ such that $y_1 = f(x_1)$ and $y_2 = f(x_2)$. Now $f(\mu+\nu)(r,y) = \sup\{(\mu+\nu)(r,x) \mid x \in M, f(x) = y\}$
$= \sup\{\sup\{T(\mu(r,x_1),\nu(r,x_2)) \mid x_1,x_2 \in M, x_1+x_2 = x\} \mid y = f(x) = f(x_1) + f(x_2) = y_1 + y_2\}$
$= \sup\{T(\sup\{\mu(r,x_1) \mid x_1 \in M, f(x_1) = y_1\}, \sup\{\mu(r,x_2) \mid x_2 \in M, f(x_2) = y_2\}) \mid y = y_1 + y_2\} = (f(\mu) + f(\nu))(r,y)$.

(2) $f(s\mu)(r,y) = \sup\{(s\mu)(r,x_1) \mid x_1 \in M, f(x_1) = y\}$
$= \sup\{\sup\{\mu(r,x_2) \mid x_2 \in M, x_1 = sx_2\} \mid x_1 \in M, f(x_1) = y\}$
$= \sup\{\sup\{\mu(r,x_2) \mid x_2 \in M, x_1 = sx_2\} \mid x_1 \in M, sf(x_2) = y\} = sf(\mu)(r,y)$.

(3) This assertion follows from (1) and (2). □

Proposition 4.4. Let $\mu \in TF(R \times M)$ and N be an R-module. Suppose that f is an isomorphism of M onto N. Then $f(\mu) \in TF(R \times N)$.

Proof. (1) $f(\mu)(r,0_N) = \sup\{\mu(r,x) \mid f(x) = 0_N\} = \sup\{\mu(r,x) \mid x \in ker f = 0\} = \sup\{\mu(r,0_M)\} = 1$.

(2) $f(\mu)(r,sy) = \sup\{\mu(r,z) \mid f(z) = sy = sf(x) = f(sx)\} = \sup\{\mu(r,sx) \mid f(x) = y\} \geq \sup\{\mu(r,x) \mid f(x) = y\} = f(\mu)(r,y)$.

(3) $f(\mu)(r,y_1+y_2) = \sup\{\mu(r,z) \mid f(z) = y_1 + y_2 = f(x_1) + f(x_2) = f(x_1+x_2)\}$
$= \sup\{\mu(r,x_1+x_2) \mid y_1 = f(x_1), y_2 = f(x_2)\}$
$\geq \sup\{T(\mu(r,x_1),\mu(r,x_2)) \mid y_1 = f(x_1), y_2 = f(x_2)\}$
$\geq T(\sup\{\mu(r,x_1) \mid f(x_1) = y_1\}, \sup\{\mu(r,x_2) \mid f(x_2) = y_2\})$
$= T(f(\mu)(r,y_1), f(\mu)(r,y_2))$. □

Proposition 4.5. Let $\nu \in TF(R \times N)$ and M be an R-module. Suppose that f is a homomorphism of M onto N. Then $f^{-1}(\nu) \in TF(R \times M)$.

Proof. Let $r,s \in R$ and $x_1, x_2 \in M$. Then

(1) $f^{-1}(\nu)(r, 0_M) = \nu(r, f(0_M)) = \nu(r, 0_N) = 1.$

(2) $f^{-1}(\nu)(r, sx) = \nu(r, f(sx)) = \nu(r, sf(x)) \geq \nu(r, f(x)) = f^{-1}(\nu)(r, x).$

(3) $f^{-1}(\nu)(r, x_1 + x_2) = \nu(r, f(x_1 + x_2)) = \nu(r, f(x_1) + f(x_2))$
$\geq T(\nu(r, f(x_1)), \nu(r, f(x_2))) = T(f^{-1}(\nu)(r, x_1), f^{-1}(\nu)(r, x_2)).$ □

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