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Fuzzy Sub Implicative Ideals of KU-Algebras

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Abstract – We consider the fuzzification of sub-implicative (sub-commutative) ideals in KU-algebras, and investigate some related properties. We give conditions for a fuzzy ideal to be a fuzzy sub-implicative (sub-commutative) ideal. We show that any fuzzy sub-implicative (sub-commutative) ideal is a fuzzy ideal, but the converse is not true. Using a level set of a fuzzy set in a KU-algebra; we give a characterization of a fuzzy sub-implicative (sub-commutative) ideal.

***Keywords* –** KU-algebras - fuzzy sub implicative ideals- fuzzy sub-commutative

1. Introduction

BCK-algebras form an important class of logical algebras introduced by Iseki [2] and was extensively investigated by several researchers. It is an important way to research the algebras by its ideals. The notions of ideals in BCK-algebras and positive implicative ideals in BCK-algebras (i.e Isekis implicative ideals) were introduced by Iseki [2]. The notions of commutative (sub-commutative) ideals in BCK-algebras, positive implicative and implicative (Sub-implmicative), ideals in BCK-algebras were introduced by [4,5]. Zadeh [15] introduced the notion of fuzzy sets. At present this concept has been applied to many mathematical branches, such as group, functional analysis, probability theory, topology, and so on. In 1991, Xi [14] applied this concept to BCK-algebras, and he introduced the notion of fuzzy sub - algebras (ideals) of the BCK-algebras. Prabpayak and Leerawat [12,13] introduced a new algebraic structure which is called KU-algebra. They gave the concept of homomorphisms of KU-algebras and investigated some related properties. Mostafa et al. [8] introduced the notion of fuzzy KU-ideals of KU-algebras and then they investigated several basic properties which are related to fuzzy KU-ideals. Senapati et al. [6,7] introduced the notion of fuzzy KU-subalgebras (fuzzy *KU*-ideals) of KU-algebras with respect to a given *t*-norm, intuitionistic fuzzy bi-normed KU-ideals of a KU-algebra and obtained some of their properties. Mostafa et al. [10] introduced the notion of sub implicative (sub-commutative) ideals of KU-algebras and investigated of their properties.

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In this paper, the notion of fuzzy sub implicative (sub commutative) ideals of KU-algebras are introduced and then the several basic properties are investigated.

2. Preliminaries

Now we will recall some known concepts related to KU-algebra from the literature which will be helpful in further study of this article.

Definition 2.1. [12,13] Algebra(X, $*$, 0) of type (2, 0) is said to be a KU -algebra, if it satisfies the following axioms:

(ku_1) $(x*y)*[(y*z))*(x*z)]=0$,
(ku_2) $x*0=0$,
(ku_3) $0*x=x$,
(ku_4) $x*y=0$ and $y*x=0$ implies $x=y$,
(ku_5) $x*x=0$, for all $x,y,z\in X$.

On a KU-algebra $(X,*,0)$ we can define a binary relation $\leq$ on X by putting:

$x\leq y\Leftrightarrow y*x=0$.

Thus a KU - algebra X satisfies the conditions:

$(ku_{1'})$: $(y*z)*(x*z)\leq(x*y)$
$(ku_{2'})$: $0\leq x$
$(ku_{3'})$: $x\leq y, y\leq x$ implies $x=y$,
$(ku_{4'})$: $y*x\leq x$.

Remark 2.2. Substituting $z*x$ for x and $z*y$ for y in ku_1 ,we get

$[(z*x)*(z*y)]*[(z*y)*z))*[(z*x)*z)]\leq[(z*x)*(z*y)]*[(z*x)*(z*y)]=0$ by (ku_1) ,hence $(x*y)*[(z*x)*(z*y)]=0$ that mean the condition (ku_1) and $(x*y)*[(z*x)*(z*y)]=0$ are equivalent.

For any elements x and y of a KU-algebra, $y*x^n$ denotes by $\overbrace{(y*x)*x)......*x}^{n\,times}$

Theorem 2.3. [8] In a KU-algebra X , the following axioms are satisfied:
For all $x,y,z\in X$,

(1) $x\leq y$ imply $y*z\leq x*z$,
(2) $x*(y*z)=y*(x*z)$, for all $x,y,z\in X$,
(3) $((y*x)*x)\leq y$.
(4) $(y*x^3)=(y*x)$

We will refer to X is a KU-algebra unless otherwise indicated.

Definition 2.4. [12,13**]** Let I be a non empty subset of a KU-algebra X. Then I is said to be an ideal of X, if

(I_1) $0 \in I$
(I_2) $\forall y, z \in X$, if $(y * z) \in I$ and $y \in I$, imply $z \in I$.

Definition 2.5. [8] Let I be a non empty subset of a KU-algebra X. Then I is said to be an KU- ideal of X, if
(I_1) $0 \in I$
(I_3) $\forall x, y, z \in X$, if $x * (y * z) \in I$ and $y \in I$, imply $x * z \in I$.

Definition 2.6. [11] KU- algebra X is said to be implicative if it satisfies
$(x * y^2) = (x * y) * (y * x^2)$

Definition 2.7. [11] KU- algebra X is said to be commutative
if it satisfies $x \leq y$ *implies* $(x * y^2) = x$

Lemma 2.8. [10] Let X be a KU-algebra. X is KU-implicative iff X is KU-positive implicative and KU-commutative.

Definition 2.9. [10] A non empty subset A of a KU-algebra X is called a ***sub*** implicative ideal of X, if $\forall x, y, z \in X$,

(1) $\mathbf{0} \in A$
(2) $z * ((x * y) * ((y * x^2)) \in A$ and $z \in A$, imply $(x * y^2) \in A$.

Definition 2.10. [10] Let $(X, *, 0)$ be a KU-algebra, a nonempty subset A of X is said to be a ***ku* -** positive implicative ideal if it satisfies, for all x, y, z in X,

(1) $0 \in A$,
(2) $z * (x * y) \in A$ and $z * x \in A$ imply $z * y \in A$.

Definition 2.11. [10] A non empty subset A of a KU-algebra X is called a ***ku* –** sub commutative ideal of X, if

(1) $\mathbf{0} \in A$
(2) $z * \{((y * x^2)) * y^2)\} \in A$ and $z \in A$, imply $(y * x^2) \in A$.

Definition 2.12. [10] A nonempty subset A of a KU-algebra X is called a kp-ideal of X if it satisfies

(1) $\mathbf{0} \in A$,
(2) $(z * y) * (z * x) \in A$, $y \in A \Rightarrow x \in A$

Definition 2.13. [8] A fuzzy set μ in a KU-algebra X is called a fuzzy sub -algebra of X if $\mu(x*y) \geq \min\{\mu(x), \mu(y)\}$ $\forall x, y \in X$.

Definition 2.14. [8] Let X be a KU-algebra, a fuzzy set μ in X is called a fuzzy ideal of X if it satisfies the following conditions:

(F_1) $\mu(0) \geq \mu(x)$ for all $x \in X$.
(F_2) $\forall x, y \in X$, $\mu(y) \geq \min\{\mu(x*y), \mu(x)\}$.

3. Fuzzy Sub-Implicative Ideals

Definition 3.1. [15] Let X be a non-empty set, a fuzzy subset μ in X is a function $f : X \rightarrow [0,1]$.

Definition 3.2. [1.15] Let μ be a fuzzy set in a set X . For t ∈ [0, 1], the set

$$\mu_t = \{x \in X \mid \mu(x) \geq t\}$$

is called upper level cut (level subset) of μ and the set L(μ, t) = {x ∈ X | μ(x) ≤ t} is called lower level cut of μ.

Definition 3.3. A non empty subset μ of a KU-algebra X is called a fuzzy ***sub*** implicative ideal (briefly FSI - ideal) of X , if $\forall x, y, z \in X$,

(F_1) $\mu(0) \geq \mu(x)$
(FSI_1) $\mu(x*y^2) \geq \min\{\mu(z*((x*y)*((y*x^2)), \mu(z)\}$

Example 3.4. Let $X = \{0,1,2,3,4\}$ in which the operation $*$ is given by the table

*	0	1	2	3	4
0	0	1	2	3	4
1	0	0	1	3	4
2	0	0	0	3	4
3	0	0	0	0	4
4	0	0	0	0	0

Then $(X,*,0)$ is a KU-Algebra. Define a fuzzy set μ : X→ [0,1] by $\mu(0) = t_0$, $\mu(1) = \mu(2) = t_1$, $\mu(3) = \mu(4) = t_2$, where $t_0, t_1, t_2 \in [0,1]$ with $t_0 > t_1 > t_2$. Routine calculation gives that μ is FSI- ideal of KU- algebra X.

Proposition 3.5. Every FSI- ideal of a KU-algebra X is order reversing.

Proof. Let μ be FSI -ideal of X and let x, y, z ∈ X be such that x ≤ z, then z* x = 0 and by (F_1) $\mu(x*y^2) \geq \min\{\mu(z*((x*y)*((y*x^2)),\mu(z)\}$. Let y = x,

$$\mu(x*x^2) \geq \min\{\mu(z*((x*x)*((x*x^2)),\mu(z)\}$$
$$\mu(x) \geq \min\{\mu(z*x),\mu(z)\} = \min\{\mu(0),\mu(z)\} = \mu(z)$$

Lemma 3.6. Let μ be a fuzzy FSI - ideal of KU - algebra X , if the inequality y * x ≤ z hold in X , Then μ (x) ≥ min {μ (y) , μ (z)} .

Proof. Let μ be FSI -ideal of X and let x, y, z ∈ X be such that y*x ≤ z, then z* (y*x)= 0 or y* (z*x)= 0 i.e $z*x \leq y$ we get

$$\mu(z*x) \geq \mu(y) \tag{a}$$

By (FSI_1): $\mu(x*y^2) \geq \min\{\mu(z*((x*y)*((y*x^2)),\mu(z)\}$. Let y = x

$$\mu(x*x^2) \geq \min\left\{\mu(z*(\overbrace{(x*x)}^{0}*\overbrace{((x*x^2)}^{x}),\mu(z)\right\} = \min\{\mu(z*x),\mu(z)\}, \text{i.e}$$

$$\mu(x) \geq \min\{\mu(z*x),\mu(z)\} \geq \min\{\mu(y),\mu(z)\} \text{ by (a)}.$$

Definition 2.7. [9,10] KU- algebra X is said to be implicative if it satisfies

$$(x*y^2) = (x*y)*(y*x^2)$$

Lemma 3.8. If X is implicative KU-algebra, then every fuzzy ideal of X is an FSI-ideal of X.

Proof. Let μ be an fuzzy ideal of X, then by (F_2)

$$\forall y,z \in X,\ \mu(y) \geq \min\{\mu(z*y),\mu(z)\}.$$

Substituting $x*y^2$ for y in (F_2) $\mu(x*y^2) \geq \min\{\mu(z*(x*y^2)),\mu(z)\}$, but KU- algebra is implicative i.e $(x*y^2) = (x*y)*(y*x^2)$, hence

$$\mu(x*y^2) \geq \min\{\mu(z*(x*y^2)),\mu(z)\} = \min\{\mu(z*(x*y)*(y*x^2)),\mu(z)\}$$

Which shows that μ is FSI-ideal of X.

Theorem 3.9. Let μ be a fuzzy set in X satisfying the condition (FSI_1) , then μ satisfies the following inequality:

$$\mu(x*y^2) \geq \mu((x*y)*(y*x^2)) \tag{FSI_2}$$

Proof. Let μ satisfying (FSI_1) i.e $\mu(x*y^2) \geq \min\{\mu(z*((x*y)*(y*x^2))), \mu(z)\}$, then by taking $z = 0$ in (FSI_1) and using (F_1) and (ku_3) we get

$$\mu(x*y^2) \geq \min\{\mu(0*((x*y)*((y*x^2)), \mu(0)\} = \mu((x*y)*(y*x^2))$$

Theorem 3.10. Every FSI-ideal is a fuzzy ideal, but the converse does not hold.

Proof. Let μ be FSI-ideal FSI-ideal of X; put x=y in (FSI_1), we get

$$\mu(\overbrace{x*x^2}^{x}) \geq \min\{\mu(z*((x*x)*((x*x^2)), \mu(z)\} \textit{ then}$$

$$\mu(x) \geq \min\left\{\mu(z*((\overbrace{x*x}^{0})*((\overbrace{x*x^2}^{x})), \mu(z)\right\} = \min\{\mu(z*x), \mu(z)\}$$

Hence μ is a fuzzy ideal of X .

The following example shows that the converse of Theorem 3.10 may not be true.

Example 3.11. Let $X = \{0,1,2,3,4\}$ in which the operation $*$ is given by the table

*	0	1	2	3	4
0	0	1	2	3	4
1	0	0	1	3	4
2	0	0	0	3	4
3	0	0	0	0	4
4	0	0	0	0	0

Then $(X,*,0)$ is a KU-Algebra. Define a fuzzy set μ : X→ [0,1] by μ (0) = 0.7 , μ (1) = μ (2) = μ (3) = μ (4) = 0.2 , we get for z=0 , x=1 and y=2. L.H.S of (FI_1)

$$\mu((1*2)*2) = \mu(1) = 0.2 \;\; R.H.S \; of \; (FI_1) \; \min\left\{\mu(0*((\overbrace{1*2}^{1})*((\overbrace{2*1}^{0})*1), \mu(0)\right\} = \mu(0) = 0.7$$

i.e in this case $\mu(x*y^2) \ngeq \min\{\mu(z*((x*y)*((y*x^2)), \mu(z)\}$.

We now give a condition for a fuzzy ideal to be a FSI-ideal.

Theorem 3.12. Every fuzzy ideal μ of X satisfying the condition (FSI_2) is a FSI-ideal ideal of X .

Proof. Let μ be fuzzy ideal of X satisfying the condition (FSI_2). We get

$$\mu(x*y^2) \geq \{\mu(((x*y)*((y*x^2))\} \text{ and } \mu(x*y^2) \geq \min\{\mu(z*((x*y)*((y*x^2)), \mu(z)\}$$

by (Definition of fuzzy ideal (F_2)), hence

$$\mu(x*y^2) \geq \ \mu(((x*y)*((y*x^2)) \ \geq \min\{\mu(z*((x*y)*((y*x^2)),\mu(z)\}$$

(Definition of fuzzy ideal (F_2)) ,which proves the condition (FSI_1) . This completes the proof.

Theorem 3.13. Let μ be a fuzzy ideal of X. Then the following are equivalent

(i) μ is an FSI-ideal of X,
(ii) $\mu(x*y^2) \geq \mu(z*((x*y)*((y*x^2))$
(iii) $\mu(x*y^2) = \mu(z*((x*y)*((y*x^2))$.

Proof. (i)) $\Rightarrow$ (ii) Suppose that μ is an FSI-ideal of X. By (FSI_1) and (F_1) we have
$\mu(x*y^2) \geq \min\{\mu(0*((x*y)*((y*x^2)),\mu(0)\} = \mu(0*((x*y)*((y*x^2)$ *i.e*
$\mu(x*y^2) \geq \mu(((x*y)*((y*x^2))$

(ii) $\Rightarrow$ (iii) Since $(x*y)*((y*x^2) \leq x*y^2$, by Lemma 3.5 we obtain ,
$\mu(x*y^2) \geq \mu((x*y)*((y*x^2))$ Combining (ii) we have $\mu(x*y^2) = \mu((x*y)*((y*x^2))$.

(iii) $\Rightarrow$ (i) Since

$$\begin{aligned}[(z*((x*y)*((y*x^2))]*[(x*y)*((y*x^2)] &= [(x*y)*(z*((y*x^2))]*[(x*y)*((y*x^2)] \\ &\leq [(z*((y*x^2))]*[(y*x^2)] \\ &= [(z*((y*x^2))]*[0*(y*x^2)] \\ &\leq 0*z = z ,\end{aligned}$$

by Lemma 3.6. we obtain $\mu((x*y)*((y*x^2) \geq \min\{ \ \mu,((x*y)*((y*x^2),\mu(z)\}$. From (iii), we have $\mu(x*y^2) \geq \min\{\mu(z*((x*y)*((y*x^2)),\mu(z)\}$. Hence μ is an FSI-ideal of X The proof is complete.

Theorem 3.14. A fuzzy set μ of a KU-algebra X is a sub-implicative fuzzy ideal of X if and only if $\mu_t \neq \Phi$ is a sub-implicative ideal of X.

Proof: Suppose that μ is a fuzzy sub-implicative ideal of X and $\mu_t \neq \Phi$ for any $t \in (0,1]$, there exists $x \in \mu_t$ so that $\mu(x) \geq t$. It follows from (F_1) that $\mu(0) \geq \mu(x) \geq t$ so that $0 \in \mu_t$. Let $x, y, z \in X$ be such that $z*((x*y)*((y*x^2) \in \mu_t$ and $z \in \mu_t$. Using (FI_1) , we know that

$$\mu(x*y^2) \geq \min\{\mu(z*((x*y)*((y*x^2)),\mu(z)\} = \min\{t,t\} = t$$

thus $x*y^2 \in \mu_t$. Hence μ_t is a sub-implicative ideal of X.

Conversely, suppose that $\mu_t \neq \Phi$ is a sub-implicative ideal of X ,for every $t \in (0,1]$. and any $x \in X$, let $\mu(x) = t$. Then $x \in \mu_t$. Since $0 \in \mu_t$, it follows that $\mu(0) \geq t = \mu(x)$ so that

$\mu(0) \geq \mu(x)$ for all $x \in X$. Now, we need to show that μ satisfies (FI_1) . If not, then there exist $a, b, c \in X$ such that

$$\mu(a * b^2) \leq \min\{\mu(c * ((a * b) * ((b * a^2)), \mu(c)\}$$

Taking

$$t_0 = \frac{1}{2}(\mu(a * b^2) + \{\mu(c * ((a * b) * ((b * a^2)), \mu(c)\})$$

then we have

$$\mu(a * b^2) < t_0 < \{\mu(c * ((a * b) * ((b * a^2)), \mu(c)\}$$

Hence $c * ((a * b) * (b * a^2)) \in \mu_t$ and $c \in \mu_t$, but $a * b^2 \notin \mu_t$ which means that μ_t is not a sub-implicative ideal of X.this is contradiction. Therefore μ is a fuzzy sub-implicative ideal of X .

4. Fuzzy Sub-Commutative Ideals

Definition4.1. A non empty subset A of a KU-algebra X is called a sub commutative ideal of X , if

(1) $0 \in A$

(2) $z * \{((y * x^2)) * y^2)\} \in A$ and $z \in A$, imply $(y * x^2) \in A$.

Lemma4 .2. Every fuzzy FSC ideal of a KU-algebra X is order reversing.

Proof. Let μ be FSC -ideal of X and let x, y, z ∈ X be such that x ≤ z, then z* x = 0 and by $(FSCI_1)$ $\mu(y * x^2) \geq \min\{\mu(z * ((y * x^2)) * y^2), \mu(z)\}$. Let y = x ,then

$$\mu(x) \geq \min\{\mu(z * ((x * x^2)) * x^2), \mu(z)\} = \min\{\mu[(z * x)], \mu(z)\} = \min\{\mu(0), \mu(z)\} = \mu(z)$$

Lemma 4.3. let μ be a fuzzy FSCk - ideal of KU - algebra X , if the inequality y * x ≤ z hold in X , Then μ (x) ≥ min {μ (y) , μ (z)} .

Proof. Let μ be FSC -ideal of X and let x, y, z ∈ X be such that z*x ≤ y, then z* (y*x)= 0 or y* (z*x)= 0 i.e $z * x \leq y$ [$\mu(z * x) \geq \mu(y)$]. By $(FSCI_1)$:

$$\mu(y * x^2) \geq \min\{\mu(z * ((y * x^2)) * y^2), \mu(z)\}$$

Put x =y

$$\mu(x) \geq \min\{\mu(z * ((x * x^2)) * x^2), \mu(z)\} = \min\{\mu[(z * x)], \mu(z)\} \geq \min\{\mu(y), \mu(z)\}$$

Lemma 4.4. If X is commutative KU-algebra, then every fuzzy ideal of X is an FSC-ideal of X.

Proof. Let μ be an fuzzy ideal of X, then by (F_2) $\forall y, z \in X$,

$$\mu(y) \geq \min\{\mu(z*y), \mu(z)\}.$$

Substituting $y*x^2$ for y in (F_2)

$$\mu(y*x^2) \geq \min\{\mu(z*(y*x^2)), \mu(z)\},$$

but KU- algebra is commutative i.e $(y*x)*x = (x*y)*y$, hence

$$\mu(y*x^2) \geq \min\{\mu(z*(y*x^2)), \mu(z)\} = \min\{\mu(z*(y*x^2)), \mu(z)\}$$

since

$$(y*x^2)*y^2 = ((y*x)*x)*y)*y = ((y*x)*x)*y)*(0*y) \leq (y*x)*x)$$

Then $z*[(y*x^2)*y^2] \geq z*(y*x)*x)$ by i.e $\mu[z*((y*x^2)*y^2)] \leq \mu\{z*(y*x^2)\}$ by theorem 4.2. Therefore

$$\mu(y*x^2) \geq \min\{\mu(z*(y*x^2)), \mu(z)\} \geq \min\{\mu(z*((y*x^2)*y^2)), \mu(z)\}.$$

Which shows that μ is FSIk-ideal of X.

Theorem 4.5. Let μ be a fuzzy set in X satisfying the condition $(FSCI_1)$, then μ satisfies the following inequality

$$\mu(y*x^2) \geq \mu((y*x^2))*y^2) \qquad (FSCI_2)$$

Proof. Let μ satisfying $(FSCI_1)$ i.e $\mu(y*x^2) \geq \min\{\mu(z*((y*x^2))*y^2), \mu(z)\}$, then by taking z = 0 in (FI_1) and using (F_1) and (ku_3) we get

$$\mu(y*x^2) \geq \min\{\mu(0*((y*x^2))*y^2), \mu(0)\}.$$

Hence $\mu(y*x^2) \geq \mu((y*x^2))*y^2)$

Theorem 4.6. Every fuzzy SCI is a fuzzy ideal, but the converse does not hold.

Proof . Let μ be fuzzy fuzzy SCI of X; put x=y in $(FSCI_1)$, we get

$$\mu(x*x^2) \geq \min\{\mu(z*((x*x^2))*x^2), \mu(z)\} = \min\{\mu(z*x), \mu(z)\}$$

for all x, z □ X . Hence μ is a fuzzy ideal of X .

The following example shows that the converse of Theorem 4.6 may not be true.

Example 4.7. Let $X = \{0,1,2,3,4\}$ in which the operation $*$ is given by the table

*	0	1	2	3	4
0	0	1	2	3	4
1	0	0	1	3	4
2	0	0	0	3	4
3	0	0	0	0	4
4	0	0	0	0	0

Then $(X,*,0)$ is aKU-Algebra. . Define a fuzzy set μ : X→ [0,1] by μ (0) = 0.7 , μ (1) = μ (2) = μ (3) = μ (4) = 0.2 , we get for z=0 , x=1 and y=3, L.H.S of $(FSCI_1)$
$\mu((3*1)*1) = \mu(1) = 0.2$

$R.H.S\ of\ (FSCI_1)\ \min\left\{\mu(0*(((\overbrace{3*1}^{0})*1)*3)*3), \mu(0)\right\} = \mu(0) = 0.7$, i.e in this case

$\mu(y*x^2) \ngeq \min\{\mu(z*((y*x^2)*y^2), \mu(z)\}$

We now give a condition for a fuzzy ideal to be a fuzzy sub- commutative ideal.

Theorem 4.8. Every fuzzy ideal μ of X satisfying the condition $(FSCI_2)$ is a fuzzy FSC of X .

Proof. Let μ be fuzzy ideal of X satisfying the condition $(FSCI_2)$. We get

$$\mu(y*x^2) \geq \mu((y*x^2))*y^2)$$

and by (Definition (F_2) fuzzy ideal), hence

$$\mu(y*x^2) \geq \mu((y*x^2)*y^2) \geq \min\{\mu(z*((y*x^2)*y^2), \mu(z)\}$$

by F_2 which proves the condition $(FSCI_1)$. This completes the proof.

Theorem 4.9. Let μ be a fuzzy ideal of X. Then the following are equivalent

(i) μ is an FSC-ideal of X,
(ii) $\mu(y*x^2) \geq \mu((y*x^2)*y^2)$
(iii) $\mu(y*x^2) = \mu((y*x^2)*y^2)$.

Proof. (i)$\Rightarrow$(ii) Suppose that μ is an FSC-ideal of X. By $(FSCI_1)$ and (F_1) we have
$\mu(y*x^2) \geq \min\{\mu(z*((y*x^2))*y^2), \mu(z)\} = \min\{\mu(0*((y*x^2))*y^2), \mu(0)\} =$
$\mu((y*x^2))*y^2)$

(ii) $\Rightarrow$ (iii) Since $(y*x^2)*y^2 \leq y*x^2$, we have $\mu(y*x^2) \geq \mu((y*x^2)*y^2)$
Combining (ii) we have $\mu(y*x^2) = \mu((y*x^2)*y^2)$.

(iii) $\Rightarrow$ (i) Since $[(z*((y*x^2)*y^2))]*[0*((y*x^2)*y^2] \leq 0*z = z$, by Lemma 4.3 we obtain $\mu((y*x^2)*y^2) \geq \min\{\mu(z*((y*x^2)*y^2), \mu(z)\}$

Hence μ is an FSC-ideal of X The proof is complete.

Theorem 4.10. A fuzzy set μ of a KU-algebra X is a fuzzy sub- commutative ideal of X if and only if $\mu_t \neq \Phi$ is a sub- commutative ideal of X.

Proof: Suppose that μ is a fuzzy sub- commutative ideal of X and $\mu_t \neq \Phi$ for any $t \in (0,1]$, there exists $x \in \mu_t$ so that $\mu(x) \geq t$. It follows from (F_1) that $\mu(\mathbf{0}) \geq \mu(x) \geq t$ so that $0 \in \mu_t$. Let $x, y, z \in X$ be such that $z*((y*x^2)*y^2) \in \mu_t$ and $z \in \mu_t$. Using $(FSCI_1)$, we know that $\mu(y*x^2) \geq \min\{\mu(z*((y*x^2)*y^2)), \mu(z)\} = \min\{t,t\} = t$, thus $y*x^2 \in \mu_t$. Hence μ_t is a sub- commutative ideal of X.

Conversely, suppose that $\mu_t \neq \Phi$ is a sub- commutative ideal of X ,for every $t \in (0,1]$. and any $x \in X$, let $\mu(x) = t$. Then $x \in \mu_t$. Since $0 \in \mu_t$, it follows that $\mu(\mathbf{0}) \geq t = \mu(x)$ so that $\mu(\mathbf{0}) \geq \mu(x)$ for all $x \in X$. Now, we need to show that μ satisfies $(FSCI_1)$. If not, then there exist $a, b, c \in X$ such that $\mu(b*a^2) \leq \min\{\mu(c*((b*a^2)*b^2), \mu(c)\}$. Taking

$$t_0 = \frac{1}{2}(\mu(b*a^2) + \{\mu(c*((b*a^2)*b^2), \mu(c)\})$$

then we have $\mu(b*a^2) < t_0 < \{\mu(c*((b*a^2)*b^2), \mu(c)\}$. Hence $c*((b*a^2)*b^2) \in \mu_t$ and $c \in \mu_t$, but $b*x^2 \notin \mu_t$ which means that μ_t is not a sub- commutative ideal of X, this is contradiction. Therefore μ is a fuzzy sub- commutative ideal of X .

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Conflicts of Interest

State any potential conflicts of interest here or “The author declare no conflict of interest”.

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Neutrosophic Crisp Tri-Topological Spaces

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Abstract – In this paper we will introduce neutrosophic crisp Tri-topological spaces, and we will introduce four new types of open and closed sets in neutrosophic crisp Tri-topological spaces. Then, the closure and interior neutrosophic crisp set will be defined via this new concept of open and closed sets. Finally, we will introduce the basic properties of these types of open and closed sets and the properties of new concept of closure and the interior.

Keywords – Neutrosophic crisp Tri-topological spaces, neutrosophic crisp Tri-open set, neutrosophic crisp Tri-closed set, neutrosophic crisp S-open sets and neutrosophic crisp S-closed.

1 Introduction

Smarandache introduces neutrosophy. He has laid the foundation of new mathematical theories generalizing their fuzzy counterparts, [8,9,10]. Many introduced the introduction of the Neutrosophic set concepts in many of their works [11,12,13,14,15,16, 5, 6,7]. In [12, 17] provides a natural foundation for treating mathematically the neutrosophic phenomena which exist pervasively in our real world and for building new branches of neutrosophic mathematics. Smarandache introduces the concept of neutrosophic sete as generalization of the concept of fuzzy sets [1] and intuitionistic fuzzy sets [2,3]. Lupianez has developed and modified many of papers about neutrosophic in his papers in [21, 22,23,24,25]. Hamido introduces neutrosophic crisp Bi-topological space [1].

In this paper we will introduce the concept of neutrosophic crisp Tri-topological as generalization of the concept of neutrosophic crisp Bi-topological [1]. Then, we will introduce new types of open and closed sets as neutrosophic crisp Tri-open sets, neutrosophic crisp Tri-closed sets, neutrosophic crisp TriS-open sets and neutrosophic crisp TriS-closed sets. We investigated the properties of these new four types of neutrosophic crisp sets.

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2 Preliminaries

In this section, we recollect some basic preliminaries, and in particular, the work of Smarandache in [8,9,10], and Salama in [11, 12,13,14, 15,16, 5, 4,7]. Smarandache in his work introduced the neutrosophic components T, I, F which represent the membership, indeterminacy, and non-membership values respectively, where $\rfloor^{-}0,1^{+}\lfloor$ is a non-standard unit interval. Hanafy and Salama et al. [7,15] considered some possible definitions for basic concepts of the neutrosophic crisp set and its operations.

Definition 2.1. [19] Let X be a non-empty fixed set. A neutrosophic crisp set (NCS) A is an object having the form A $=\{A_1, A_2, A_3\}$, where A_1, A_2, and A_3 are subsets of X satisfying $A_1 \cap A_2 = \phi$, $A_1 \cap A_3 = \phi$, and $A_2 \cap A_1 = \phi$.

Definition 2.2. [19] Types of $NCSs$ ϕ_N and X_N [20] in X as follows:

1- ϕ_N may be defined in many ways as a NCS, as follows

1. $\phi_N = (\phi,\phi, X)$ or
2. $\varphi_N = (\phi, X, X)$ or
3. $\phi_N = (\phi, X, \phi)$ or
4. $\phi_N = (\phi,\phi,\phi)$

2- X_N may be defined in many ways as a NCS, as follows

1. $X_N = (X, \phi, \phi)$ or
2. $X_N = (X, X, \phi)$ or
3. $X_N = (X, X, X)$.

Definition 2.3. [19] Let X is a non-empty set, and the NCSs A and B in the form $A=\{A_1, A_2, A_3\}$, $B = \{B_1, B_2, B_3\}$. Then we may consider two possible definitions for subsets $A\subseteq B$, may defined in two ways:

1. $A\subseteq B \Leftrightarrow A_1\subseteq B_1, A_2\subseteq B_2$, and $A_3\supseteq B_3$ or
2. $A\subseteq B\Leftrightarrow A_1\subseteq B_1, A_2\supseteq B_2$, and $A_3 \supseteq B_3$

Definition 2.4. [19] Let X is a non-empty set, and the NCSs A and B in the form $A = \{A_1, A_2, A_3\}$, $B = \{B_1, B_2, B_3\}$. Then:

1. $A \cap B$ may be defined in two ways as a NCS, as follows:

i) $A \cap B = (A_1 \cap B_1, A_2 \cap B_2, A_3 \cup B_3)$
ii) $A \cap B = (A_1 \cap B_1, A_2 \cup B_2, A_3 \cup B_3)$

2. $A \cup B$ may be defined in two ways as a NCS, as follows:

i) $A \cup B = (A_1 \cup B_1, A_2 \cap B_2, A_3 \cap B_3)$
ii) $A \cup B = (A_1 \cup B_1, A_2 \cup B_2, A_3 \cap B_3)$

Definition 2.5. [19] A neutrosophic crisp topology (NCT) on a non-empty set X is a family Γ of neutrosophic crisp subsets in X satisfying the following axioms.

1. $\phi_N, X_N \in \Gamma$.
2. $A_1 \cap A_2 \in \Gamma$, for any A_1 and $A_2 \in \Gamma$.
3. $\cup A_j \in \Gamma$, $\forall \{A_j : j \in J\} \subseteq \Gamma$.

The pair (X,Γ) is said to be a neutrosophic crisp topological space (NCTS) in X. Moreover, the elements in Γ are said to be neutrosophic crisp open sets (NCOS), A neutrosophic crisp set F is closed (NCCS) if and only if its complement F^c is an open neutrosophic crisp set.

Definition 2.6. [19] Let X is a non-empty set, and the NCSs A in the form $A=\{A_1,A_2,A_3\}$. Then A^c may be defined in three ways as a NCS, as follows:

$i)\ A^c = < A_1^c, A_2^c, A_3^c >$ or
$ii)\ A^c = < A_3, A_2, A_1 >$ or
$iii)\ A^c = < A_3, A_2^c, A_1 >$.

Definition 2.7. *[1]* Let Γ_1, Γ_2 be two neutrosophic crisp topology (NCT) on a nonempty set X then (X,Γ_1,Γ_2) neutrosophic crisp Bi-topological space (Bi-NCTS for short).In this case:

- The elements in $\Gamma_1 \cup \Gamma_2$ are said to be neutrosophic crisp Bi-open sets (Bi-NCOS for short). A neutrosophic crisp set F is closed (Bi-NCCS for short) if and only if its complement F^c is an neutrosophic crisp Bi-open set.

- the family of all neutrosophic crisp Bi-open sets is denoted by (Bi-NCOS(X)).

- the family of all neutrosophic crisp Bi-closed sets is denoted by (Bi-NCCS(X)).

3 Neutrosophic Crisp Tri-Topological Spaces

In this section, We will introduce Neutrosophic Tri-topological crisp Spaces .

Moreover we will introduce new types of open and closed sets in Neutrosophic Tri-topological crisp Spaces.

Definition 3.1. Let Γ_1,Γ_2 and Γ_3 be three neutrosophic crisp topology (NCT) on a nonempty set X then $(X,\Gamma_1,\Gamma_2,\Gamma_3)$ neutrosophic crisp Tri-topological space (Tri-NCTS for short).

Example 3.2. Let X={1,2,3,4}, $\Gamma_1=\{\Phi_N, X_N, D, C\}$, $\Gamma_2=\{\Phi_N, X_N, A\}$, $\Gamma_3=\{\Phi_N, X_N, B\}$, $A=<\{1\},\{2,4\},\{3\}>=C$, $B=<\{1\},\{2\},\{3,4\}>$, $D=<\{1\},\{2\},\{3\}>$. Then (X,Γ_1), (X,Γ_2) and (X,Γ_3) are neutrosophic crisp spaces therefore $(X,\Gamma_1,\Gamma_2,\Gamma_3)$ is neutrosophic crisp Tri-topological space (Tri-NCTS).

Definition 3.3. Let $(X,\Gamma_1,\Gamma_2,\Gamma_3)$ be neutrosophic crisp Tri-topological space (Tri-NCTS) then:

-The elements in $\Gamma_1\cup\Gamma_2\cup\Gamma_3$ are said to be neutrosophic crisp Tri-open sets (Tri-NCOS for short). A neutrosophic crisp set F is closed (Tri-NCCS for short) if and only if its complement F^c is an neutrosophic crisp Tri-open set.

- the family of all neutrosophic crisp Tri-open sets is denoted by (Tri-NCOS(X)).

- the family of all neutrosophic crisp Tri-closed sets is denoted by (Tri-NCCS(X)).

Example 3.4. In Example 2 the neutrosophic crisp Tri-open sets (Tri-NCOS) are: Tri-NCOS(X) = $\Gamma_1\cup\Gamma_2\cup\Gamma_3$={A,B,C,D} the neutrosophic crisp Tri-closed sets (Tri-NCCS) are : Tri-NCCS(X) = $\Gamma_1\cup\Gamma_2\cup\Gamma_3=\{\phi_N, X_N, A_1, B_1, C_1, D_1\}$, where:

$A_1=<\{2,3,4\},\{1,3\},\{1,2,4\}>=C_1$, $B_1=<\{2,3,4\},\{1,3,4\},\{1,2\}>$,
$D_1=<\{2,3,4\},\{1,3,4\},\{1,2,4\}>$.

Remark 3.5.
1) Every neutrosophic crisp open sets in (X,Γ_1) or (X,Γ_2) or (X,Γ_3) is neutrosophic crisp Tri-open set.

2) Every neutrosophic crisp closed sets in (X,Γ_1) or (X,Γ_2) or (X,Γ_3) is neutrosophic crisp Tri-closed set.

Remark 3.6. Every neutrosophic crisp Tri-topological space $(X,\Gamma_1,\Gamma_2,\Gamma_3)$ induces three neutrosophic crisp topological spaces as (X,Γ_1), (X,Γ_2) and (X,Γ_3).

Remark 3.7. If (X,Γ) neutrosophic crisp topological space then (X,Γ,Γ,Γ) neutrosophic crisp Tri-topological space.

Theorem 3.8. Let $(X,\Gamma_1,\Gamma_2,\Gamma_3)$ be neutrosophic crisp Tri-topological space (Tri-NCTS) then: The union of two neutrosophic crisp Tri-open (Tri-closed) sets is not neutrosophic crisp Tri-open (Tri-closed) set as the following example:

Example 3.9. X={1,2,3,4}, $\Gamma_1=\{\Phi_N, X_N, A\}$, $\Gamma_2=\{\Phi_N, X_N, D\}$, $\Gamma_3=\{\Phi_N, X_N, C\}$. It is clear that (X,Γ_1), (X,Γ_2) and (X,Γ_3) are neutrosophic crisp topological spaces therefore is $(X,\Gamma_1,\Gamma_2,\Gamma_3)$ neutrosophic crisp Tri-topological space A, D are two neutrosophic crisp Tri-open sets but $A\cup D=<\{1,3\},\{2,4\},\varnothing>$ is not neutrosophic crisp Tri-open set. $A^c=<\{1,2,4\},\{1,3\},\{2,3,4\}>, D^c=<\{2,3,4\},\{1,3,4\},\{1,2,4\}>$ are two neutrosophic crisp Tri-closed sets but $A^c\cup D^c=<X,\{1,3\},\{2,4\}>$ is not neutrosophic crisp Tri-closed set.

Theorem 3.10. Let $(X,\Gamma_1,\Gamma_2,\Gamma_3)$ be neutrosophic crisp Tri-topological space (Tri-NCTS) then: The intersection of two neutrosophic crisp Tri-open (Tri-closed) sets is neutrosophic crisp Tri-open (Tri-closed) set as the following example:

Example 3.11. In example 3.9 A, D are two neutrosophic crisp Tri-open sets but $A \cap D = <\emptyset,\{2\},\{1,3\}>$ is not neutrosophic crisp Tri-open set.

$$A^c = <\{1,2,4\},\{1,3\},\{2,3,4\}>, D^c = <\{2,3,4\},\{1,3,4\},\{1,2,4\}>$$

are two neutrosophic crisp Tri-closed sets but $A^c \cap D^c = <\{2,4\},\{1,3\},X>$ is not neutrosophic crisp Tri-closed set.

4 The Closure and the Interior via Neutrosophic Crisp Tri-Open Sets (Tri-NCOS) and Neutrosophic Crisp Tri-closed (Tri-NCCS)

In this section we use this new concept of open and closed sets in the definition of closure and interior Neutrosophic crisp set, where we defined the closure and interior Neutrosophic crisp set based on these new varieties of open and closed Neutrosophic crisp sets.Also we introduced the basic properties of closure and the interior.

Definition 4.1. Let $(X,\Gamma_1,\Gamma_2,\Gamma_3)$ be neutrosophic crisp Tri-topological space (Tri-NCTS), and A is neutrosophic crisp set then: The union of any neutrosophic crisp Tri-open sets ,contain in A is called neutrosophic crisp Tri-interior of A ($NC^{Tri}Int(A)$ for short). $NC^{Tri}Int(A) = \cup\{B : B\subseteq A$; B is neutrosophic crisp tri-open set$\}$.

Theorem 4.2. Let $(X,\Gamma_1,\Gamma_2,\Gamma_3)$be neutrosophic crisp Tri-topological space (Tri-NCTS), A is neutrosophic crisp set then:

1. $NC^{Tri}Int(A) \subseteq A$.
2. $NC^{Tri}Int(A)$ is not neutrosophic crisp Tri-open set .

Proof:
1. Follow from the defintion of $NC^{Tri}Int(A)$ as a union of any *neutrosophic crisp Tri-open set*s ,contains in A.

2. Follow from Theorem 8 in section 3.

Theorem 4.3. Let $(X,\Gamma_1,\Gamma_2,\Gamma_3)$ be neutrosophic crisp Tri-topological space (Tri-NCTS), A, B are neutrosophic crisp sets then:

$$A\subset B \Rightarrow NC^{Tri}Int(A) \subset NC^{Tri}Int(B).$$

Proof: Obvious.

Definition 4.4. Let $(X,\Gamma_1,\Gamma_2,\Gamma_3)$ be neutrosophic crisp Tri-topological space (Tri-NCTS), A is neutrosophic crisp set then: The intersection of any neutrosophic crisp Tri-open sets ,contained A is called neutrosophic crisp Tri-closure of A (NC^{Tri}-Cl(A) for short). *NC^{Tri}-Cl(A)*$= \cap\{B : B\supseteq A$; B is an neutrosophic Tri-closed set$\}$.

Theorem 4.5. Let $(X,\Gamma_1,\Gamma_2,\Gamma_3)$ be neutrosophic crisp Tri-topological space (Tri-NCTS), A is neutrosophic crisp set then:

1. $A \subseteq NC^{Tri}$-Cl(A) .
2. NC^{Tri}-Cl(A) is not neutrosophic crisp Tri-closed set.

Proof:
1. Follow from the defintion of NC^{Tri}-Cl(A) as a intersection of any neutrosophic crisp Tri-closed set,contained in A.

2. Follow from Theorem 3.10.

5 The Neutrosophic crisp TriS-open Sets (TriS-NCOS) and Neutrosophic Crisp TriS-closed sets (TriS-NCOS)

We introduced new concept of open and closed sets in neutrosophic crisp Tri-topological space in this section, as neutrosophic crisp TriS-open sets (TriS-NCOS) and neutrosophic crisp TriS-closed sets (S-NCCS). Also we introduced the basic properties of this new concept of open and closed sets in Tri-NCTS , and their relationship with neutrosophic crisp Tri-open sets and neutrosophic crisp Tri-closed sets.

Definition 5.1. Let $(X,\Gamma_1,\Gamma_2,\Gamma_3)$be neutrosophic crisp Tri-topological space (Tri-NCTS) then: The neutrosophic crisp open set only in one of the three neutrosophic crisp topological space (X,Γ_1), (X,Γ_2) and (X,Γ_3) are called neutrosophic crisp TriS-open set (TriS-NCOS for short).

- The complement of neutrosophic crisp S-open set is called neutrosophic crisp TriS-closed set (Tri-NCCS for short).

- the family of all neutrosophic crisp triS-open sets is denoted by (TriS-NCOS(X)).

- the family of all neutrosophic crisp TriS-closed sets is denoted by (TriS-NCCS(X)).

Example 5.2. In example 3.2: B, D are two neutrosophic crisp S-open sets.

Theorem 5.3. Let $(X,\Gamma_1,\Gamma_2,\Gamma_3)$ be neutrosophic crisp Tri-topological space (Tri-NCTS) then:

1. Every neutrosophic crisp TriS-open sets (TriS-NCOS) is neutrosophic crisp Tri-open set (Tri-NCOS).

2. Every neutrosophic crisp TriS-closed sets (TriS-NCCS) is neutrosophic crisp Tri-closed set (Tri-NCCS).

Proof:
1. Let A neutrosophic crisp TriS-open set therefore A neutrosophic crisp open set in one of the three neutrosophic crisp topological spaces (X,Γ_1), (X,Γ_2) and (X,Γ_3) therefore A neutrosophic crisp Tri-open set.

2. Let A neutrosophic crisp TriS-closed set therefore A neutrosophic crisp closed set in one of the three neutrosophic crisp topological spaces (X,Γ_1), (X,Γ_2) and (X,Γ_3) therefore A neutrosophic crisp Tri- closed set.

Remark 5.4. The converse of Theorem 3 is not true , as the following example.

Example 5.5. In any neutrosophic crisp Tri-topological space, Φ_N, X_N are two neutrosophic crisp Tri-open sets, but Φ_N, X_N are not neutrosophic crisp TriS-open sets .

Also Φ_N, X_N are two neutrosophic crisp Tri-closed sets, but Φ_N, X_N are not neutrosophic crisp TriS-closed sets.

Theorem 5.6. Let $(X,\Gamma_1,\Gamma_2,\Gamma_3)$ be neutrosophic crisp Tri-topological space (Tri-NCTS) then: The union of two neutrosophic crisp TriS-open (TriS-closed) sets is neutrosophic crisp TriS-open (TriS-closed) set as the following example.

Example 5.7. In example 3.9. It is clear that (X,Γ_1), (X,Γ_2) and (X,Γ_3) are neutrosophic crisp topological spaces therefore $(X,\Gamma_1,\Gamma_2,\Gamma_3)$ is neutrosophic crisp Tri-topological space. A,D are two neutrosophic crisp TriS-open sets but $A \cup D = <\{1,3\},\{2,4\},\varnothing>$ is not neutrosophic crisp TriS-open set.

$$A^c = <\{1,2,4\},\{1,3\},\{2,3,4\}>, D^c = <\{2,3,4\},\{1,3,4\},\{1,2,4\}>$$

are two neutrosophic crisp TriS-closed sets but $A^c \cup D^c = <X,\{1,3\},\{2,4\}>$ is not neutrosophic crisp TriS-closed set.

Theorem 5.8. Let $(X,\Gamma_1,\Gamma_2,\Gamma_3)$ be neutrosophic crisp Tri-topological space (Tri-NCTS) then: The intersection of two neutrosophic crisp TriS-open (TriS-closed) sets is neutrosophic crisp TriS-open (TriS-closed) set as the following example.

Example 5.9. In example 3.9. A, D are two neutrosophic crisp TriS-open sets but $A \cap D = <\varnothing,\{2\},\{1,3\}>$ is not neutrosophic crisp TriS-open set.

$$A^c = <\{1,2,4\},\{1,3\},\{2,3,4\}>, D^c = <\{2,3,4\},\{1,3,4\},\{1,2,4\}>$$

are two neutrosophic crisp TriS-closed sets but $A^c \cap D^c = <\{2,4\},\{1,3\},X>$ is not neutrosophic crisp TriS-closed set.

Conclusions

In this paper we have introduced neutrosophic crisp *Tri*-Topological space. Then we have introduced neutrosophic crisp *Tri*-open, neutrosophic crisp *Tri*-closed, neutrosophic crisp TriS-open, neutrosophic crisp TriS-open set's. Also we studied some of their basic properties and their relationship with each other. Finally, these new concepts are going to pave the way for new types of open and closed sets as neutrosophic Crisp Tri-α-open sets, neutrosophic crisp Tri-β-open sets, neutrosophic crisp Tri-pre-open sets, neutrosophic crisp Tri-semi-open sets.

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On Finite Extension and Conditions on Infinite Subsets of Finitely Generated FC and FN_k-groups

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Abstract – Let k>0 an integer. F, τ, N, N_k, $N_k^{(2)}$ and A denote, respectively, the classes of finite, torsion, nilpotent, nilpotent of class at most k, group in which every two generator subgroup is in N_k and abelian groups. The main results of this paper is, firstly, to prove that in the class of finitely generated FN-group, the property FC is closed under finite extension. Secondly, we prove that a finitely generated τN-group in the class $((\tau N_k)\tau,\infty)$ (respectively $((\tau N_k)\tau,\infty)^*$) is a $\tau N_k^{(2)}$-group (respectively τN_c for certain integer c=c(k)) and deduce that a finitely generated FN-group in the class $((FN_k)F,\infty)$ (respectively $((FN_k)F,\infty)^*$) is $FN_k^{(2)}$-group (respectively FN_c for certain integer c=c(k)). Thirdly we prove that a finitely generated NF-group in the class $((FN_k)F,\infty)$ (respectively $((FN_k)F,\infty)^*$) is $N_k^{(2)}F$-group (respectively N_cF for certain integer c=c(k)). Finally and particularly, we deduce that a finitely generated FN-group in the class $((FA)F,\infty)$ (respectively $((FC)F,\infty)^*$, $((FN_2)F,\infty)^*$) is in the class FA (respectively FN_2, $FN_3^{(2)}$).

Keywords *–* *FC-group, (FC)F-group, $(\tau N_k)\tau$-group, $(FN_k)F$-group, $((FN_k)F,\infty)$-group, $((FN_k)F,\infty)^*$-group, finitely generated group.*

1 Introduction

Definition 1.1. A group G is said to be with finite contumacy classes (or shortly FC-group) if and only if every element of G has a finite contumacy class in G.

It is known that FIZ⊆FA⊆FC, where FIZ denotes the class of center-by-finite groups, and that for finitely generated equalities FIZ=FA=FC hold. These results and other have been studied and developed by Baer, Neumann, Erdos and Tomkinson and others in [5, 8, 13, 15, 22]. FC-groups have many similar properties with abelian groups and finite groups.

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On the one hand, several authors have studied the class of (χ,∞)-groups, where χ is a given property of groups, with some conditions on these groups. The question that interests mathematicians is the following: If G is a group in the class (χ,∞), where χ is a given property, and then does G have a property in relation to the property χ? For example is that G has the property $\chi\gamma$ or $\gamma\chi$, where γ is another group property, or in particular is it in the same class χ. For example, in 1976, B.H Neumann in [14], has shown that a group is in the class (A, ∞), if and only if, it is FIZ-group, where A is the class of abelian groups. In 1981, Lennox and Wiegold in [12] proved that a finitely generated solvable group is in the class (N, ∞) (resp. (P, ∞), (Co, ∞)) if and only if, it is FN, (resp. P, Co), where P, N, Co and F designates respectively polycyclic, nilpotent, coherent and finite class of groups. Other results of this type of this class can be found in section 2.

On the other hand, some authors give another extension of the problem of Paul Erdos and noted it $(\chi, \infty)^*$. For example in 2005, Trabelsi in [21] proved that a finitely generated soluble group in the class $(CN, \infty)^*$, where C is the class of cernikov group. Other results these types are given in section 2.

2 Preliminary

Before giving proof to the results in next section, we need some definitions and basic known facts from the theory of isolators in nilpotent groups, which has been developed in [11] (see also [6]).

Definition 2.1. A group G is said to be with finite contumacy classes (or shortly FC-group) if and only if every element of G has a finite contumacy class in G.

Nishigoryin [15] showed that every extension of a finite group by an FC-group is likewise an FC-group; in other words F(FC)=FC. As we mentioned in introduction the property FC is not closed under finite extension that means (FC)F is not always FC.

Therefore, we add some conditions on these groups so that it is. We prove in Theorem 1. that, in the class of finitely generated finite-by-nilpotent-group, the property FC is closed under taking finite extension.

Definition 2.2. If H is a subgroup of a group G. The isolator of H in G noted $I_G(H)$ is the set of elements $x \in G$ such that, for some integer $r > 0$, we have $x^r \in H$.

To prove the Theorem1 below in the next section, we begin by giving the next Lemma.

Lemma 2.1. Let G be a group and H a subgroup of G.
(i) If G is aτN-group, then the set of elements of finite order is a characteristic subgroup $\tau(G)$ of G and the group quotient $G/\tau(G)$ is torsion-free.
(ii) If G is a finitely generated FN-group, then $\tau(G)$ is finite.
(iii) If G is locally nilpotent torsion-free group, then the isolator $I_G(H) = \{x \in G / \exists n \in \mathbb{N} : x^n \in H\}$ is a subgroup of G containing H . If H is nilpotent of class k then $I_G(H)$ is nilpotent of class k as well. In particular, if H is abelian then $I_G(H)$ is abelian as well.

Definition 2.3. Let χ is a given property of groups. A group G is said to be in the class (χ,∞) (respectively $(\chi, \infty)^*$) if and only if every infinite subset X of G contains two distinct elements x, y such that the subgroup $<x, y>$ (respectively $<x, x^y>$) is a χ-group.

Note that if χ is a subgroup closed class, then $\chi \subset (\chi, \infty) \subset (\chi, \infty)^*$.

In addition to the first results mentioned in the introduction concerning category (χ, ∞), we recall other results. In 2000, 2002 and 2005, Abdollahi and Trabelsi, proved in [1, 19, 21] that a finitely generated solvable group is in the class (FN_k, ∞) (resp. (FN, ∞), (NF, ∞), $(\tau N, \infty)$) if and only if it is $FN_k^{(2)}$, (resp. FN, NF, τN). Other results of this type have been obtained, for example in [3, 4, 7, 9, 10, 20].

In this note we prove that a finitely generated τN-group G is in the class $((\tau N_k)\tau, \infty)$ is in the class $\tau N_k^{(2)}$ and deduce that a finitely generated FN-group (respectively NF-group) G in the class of $((FN_k)F, \infty)$-groups, is in the class of $FN_k^{(2)}$-groups (respectively in the class of $N_k^{(2)}F$-groups) and In particular a finitely generated FN-group G is in the class $((FC)F, \infty)$, if and only if, it is FA-group.

About other results on the class $(\chi, \infty)^*$. In 2007, Rouabehi and Trabelsi proved in [18] that a finitely generated soluble group in the class $(CN, \infty)^*$ where C is the class of cernikov group (respectively in the class $(\tau N, \infty)^*$) is FN-group (respectively τN-group).In 2007 too, Guerbi and Rouabhi proved in [9] that a finitely generated Hyper (abelian-by-finite) group in the class $(\Omega, \infty)^*$ is FN-group, where Ω the class of groups of finite depth, i.e. $G \in \Omega$, if and only if, there exists $k \in \mathbb{N}: \gamma_{k+1}(G) = \gamma_k(G)$ where $(\gamma_i(G))$ is the lower central series of G.
In this paper we prove that a finitely generated τN-group in the class $((\tau N_k)\tau, \infty)^*$ is in the class $(\tau N_c)\tau$ for certain integer c=c(k) and deduce that a finitely generated FN-group (respectively NF-group) G in the class $((FN_k)F, \infty)^*$) is in the class FN_c (respectively N_cF). Finally, if G is a finitely generated FN-group in the class $((FC)F, \infty)^*$ (respectively $((FN_2)F, \infty)^*$) then G is in the class of FN_2-groups (respectively in the class of $FN_3^{(2)}$-groups) .

3 Main results

3.1. Stability by finite extension

As we know, the property FC is not closed under the formation of extension. The following example shows that even, a finite extension of a FC-group is not always a FC-group.

Example 3.1. Let $G = D_\infty =< a; b/ a^2 = 1$ and $aba = b^{-1}>$ the infinite dihedral group, which is a finitely generated soluble group, generated by the involutions a, b. We have $K=C_\infty =<b>$ which is a infinite cyclic group isomorphic to Z therefore it is a FC-group and the quotient group G/K is isomorphic to $C_2 =< a >$ which is finite of order 2, thus G is a finite extension of a FC-group, but as the center of the infinite dihedral group is trivial then it is not a FC-group.

This example shows also that, in the class of finitely generated soluble groups, the property FC is not closed under the formation of finite extension. So we consider the class of finitely generated finite-by-nilpotent groups. We prove that, in this class, the property FC is closed under taking finite extension. Precisely we prove the following Theorem.

Theorem 3.1. Let G a finitely generated finite-by-nilpotent group. G is FC-by-finite group, if and only if, G is FC-group.

Proof. If G is FC-group, it is clear that, G is FC-by-finite. Conversely, since G is finitely generated finite-by-nilpotent group, there exists a finite normal subgroup F of G such that the quotient group G/F is nilpotent group. As the property FC-by-finite is closed under quotient, it is enough to show that G/F is a FC-group. For this it is sufficient to show that every FC-by-finite group G in the class of finitely generated nilpotent groups is a FC-group too. Assume that G is (FC)F, so there exists a normal FC-subgroup N with of finite index in the group G. Since G is finitely generated and nilpotent, it checks the maximal condition on subgroups. So N is finitely generated FC-subgroup. According to ([5], Theorem 6.2) N is center-by-finite which means that Z(N) is of finite index in N. Or N is of finite index in G. It follows that Z(N) is of finite index in G. Let T = τ(G), the torsion subgroup of G, by Lemma 2.1, (ii) above T is finite. Note that, since F(FC) = FC as pointed out above in [15], it is enough to prove the statement for G=T, that is we may assume T = 1, that is G is nilpotent torsion-free group. Since Z(N) is of finite index in G then $I_G(H)$ = G. So by using Lemma 2.1, (iii) with H = Z(N) we deduce that G is abelian group. This completes the proof.

Remark 3.1. The example below shows that Theorem 1. is falls when the condition "finitely generated" is omitted.

Example 3.2 Let A = $F_2[X]$ algebra of polynoms on the field F_2 and the isomorphism $\varphi: A \times A \to A \times A$, $(P,Q) \to (P + Q,Q)$. We put $H = A \times A$ and $K = <\varphi>$ such that $\varphi^2 = Id_{A\times A}$ the identity application on A×A. Since H is an abelian group, it is a FC-group. K is a finite group of order 2 and so it is FC too. We consider G = H ©K, the semi-direct product of H by K. G is a non finitely generated nilpotent group, which is a finite extension of the FC-group H. But G is not a FC-group.

3.2 τN_k and FN_k-groups and conditions on infinite subsets

Our first elementary propositions below follows from lemmas below.

Lemma 3.1. ([1], Corollary 1.8. (i)) If G a finitely generated soluble group in the class (FN_k,∞), then G is in the class of $FN_k^{(2)}$ -groups and there exists an integer t, depending only on k, such G=Z_t(G) is finite.

Lemma 3.2. ([4], Theorem) Let G be a finitely generated soluble group. Then G has the property (N_k,∞) if and only if G is a $FN_k^{(2)}$-group.

Proposition 3.1. If G is a finitely generated finite-by-soluble group in the class (FN_k,∞); then G is in the class of $FN_k^{(2)}$-groups.

Proof. Suppose that G is finite-by-soluble, there exists finite normal subgroup N such that G/N is soluble. As the class of (FN_k,∞)-group, is closed under taking quotient, then the quotient group G/N is a finitely generated soluble group in the class of (FN_k,∞)-group. By

Lemma 3.2 above, the quotient group G/N is in the class of $FN_k^{(2)}$-groups. Therefore G is finite-by- $FN_k^{(2)}$-groupand this gives that G is $FN_k^{(2)}$-group. This completes the proof.

Proposition 3.2. If G is a finitely generated torsion-by-soluble group in the class $(\tau N_k,\infty)$; then G is in the class of $\tau N_k^{(2)}$-groups.

Proof. Suppose that G is torsion-by-soluble, there exists a torsion and normal subgroup N such that G/N is soluble. As the class of $(\tau N_k,\infty)$-group, is closed under taking quotient, then the quotient group G/N is a finitely generated soluble group in the class of $(\tau N_k,\infty)$which is included in $(\tau N, \infty)$.By a result in [21], G/N is in the class of τN-groups. Using Lemma 2.1, (i), G/N admits a torsion group $\tau(G/N) = T/N$ such that the quotient G/T is torsion-free in the class $(\tau N_k,\infty)$. So G/T is a finitely generated soluble group in the class $(\tau N_k,\infty)$. It results by Lemma 3.2 above that G/T is in the class $FN_k^{(2)}$, therefore G is torsion-by-$FN_k^{(2)}$, and this gives that G is $\tau N_k^{(2)}$-group. This completes the proof.

Theorem 3.2 Let G a finitely generated τN-group. If G is in the class $((\tau N_k)\tau,\infty)$, then, G is $\tau N_k^{(2)}$-group.

Proof. Assume that G is finitely generated τN- group in the class $((\tau N_k)\tau,\infty)$. There exists a normal and torsion subgroup H of G such that G/H is nilpotent quotient group. Since G/H is finitely generated nilpotent group, it has a torsion subgroup T/H of finite order and as H is torsion group then T is torsion group too. So G/T is torsion-free nilpotent group in the class $((\tau N_k)\tau,\infty)$ which gives that G/T is in the class $(N_k\tau,\infty)$. We deduce by ([16], Lemma 6.33) that G/T is in the class $(N_k\tau,\infty)$ and so G/T is a finitely generated soluble group in the class (N_k,∞). It follows by ([4] Theorem) that G/T belongs in the class of $FN_k^{(2)}$-groups and as T is torsion, it gives that G is in the class of $\tau N_k^{(2)}$-groups. This completes the proof.

If we replace the property τN by the property FN, we obtain the result in the lemma below.

Lemma 3.3. Let G a finitely generated FN-group.
(i) If G is in the class $((FN_k)F,\infty)$, then G is in the class of $FN_k^{(2)}$-groups.
(ii) G is in the class $((FC)F,\infty)$, if and only if, G is FC-group.

Proof. (i) Assume that G is finitely generated FN-group in the class $((FN_k)F,\infty)$which is in the class$((\tau N_k)\tau,\infty)$. As G is FN-group, there exists a normal and finite subgroup H of G such that G/H is nilpotent. As in the above theorem, we found that the torsion subgroup T/Hof G/H is finite and so T is finite too. As the property $((\tau N_k)\tau,\infty)$ is closed under quotient then the quotient group G/T a torsion-free nilpotent group which verifies the conditions of the above theorem. It follows that G/T belongs in the class of $FN_k^{(2)}$-groups,which gives that G/Tis in the class$N_k^{(2)}$ and hence G is in the class $FN_k^{(2)}$.

(ii) As finitely generated FN-group verifies maximal condition on subgroups, then, $FC = FA = FN_1 = FN_1^{(2)}$ and $((FC)F,\infty) = ((FN_1)F,\infty)$. This completes the proof.

The Example 1 above shows that nilpotency is necessary for the results of the above theorem to remain true.

Remark 3.2. (i) As $(FN_k)F$ is a subgroup closed class, then$(FN_k)F\subset((FN_k)F,\infty)$, we deduce that a finitely generated FN-group in the class $(FN_k)F$, is in the class $FN_k^{(2)}$.

(ii) Theorem 1 can be proved by using (ii) in the lemma above and by seeing that (FC)F is a subgroup closed class so $(FC)F\subset((FC)F, \infty)$.

(iii) In (i) of the above lemma, as G is in the class $FN_k^{(2)}$ and as nilpotent groups of class at most k are k-Engel then G is finite-by-(k-Engel, torsion-free and soluble of derived length an integer d). So by a result of Gruenberg [16, Theorem 7.36 (i)] G is in the class of$FN_{k^{d-1}}$and by P. Hall [10] there exists an integer c=c(k, l) depending on k, d such that $G/Z_c(G)$.

Recall that FN-groups are NF-groups (see[9]).

Theorem 3.3. Let G a finitely generated NF-group.
(i) If G is in the class $((FN_k)F,\infty)$, then G is in the class of $N_k^{(2)}F$-groups.
(ii) In particular, if G is in the class $((FC)F,\infty)$, then G is in the class of AF-group.

Proof. (i) Assume that G is finitely generated NF-group in the class $((FN_k)F,\infty)$. As the group G is NF- group, and then it contains a normal nilpotent subgroup N such that G/N is finite. As the subgroup N is finitely generated and nilpotent of finite index then N is polycyclic so by ([14], Theorem 5.4.15) there exists a subgroup M normal in N and poly-infinite cyclic hence torsion-free and of finite index in N. Let $K=M_G$ the core of the subgroup M, so K is nilpotent torsion-free of finite index in G. Since the class $((FN_k)F,\infty)$ is closed under taking subgroups, then K is nilpotent subgroup in the class $((FN_k)F,\infty)$ and according to (i) in the above lemma we deduce that K is torsion-free subgroup in the class of $FN_k^{(2)}$-groups which gives that K is $N_k^{(2)}$-group and so G is $N_k^{(2)}F$-group. In particular, for k=1 $(FC)F=(FA)F=(FN_1)F$ and $N_1^{(2)}F=AF$.This completes the proof.

If we replace the property $((\tau N_k)\tau, \infty)$ by the property $((\tau N_k)\tau, \infty)^*$ in the above Theorem, we obtain the next result.

Theorem 3.4. Let G a finitely generated τN-group. G is in the class $((\tau N_k)\tau, \infty)^*$, then there exists an integer c=c(k) such that G is in the class of τN_c-group.

Proof. Assume that G is finitely generated τN- group in the class $((\tau N_k)\tau, \infty)^*$. Let $T=\tau(G)$ the torsion group of G. So by Lemma 2.1. (i) G/T is torsion-free nilpotent group and as $((\tau N_k)\tau, \infty)^*$is quotient closed class then G/T belongs in $((\tau N_k)\tau, \infty)^*$ and hence G/T is in the class $(N_k\tau, \infty)^*$. We deduce by ([16], Lemma 6.33) that G/T is in the class $(N_k, \infty)^*$. Note that the class $(N_k, \infty)^*$is included in the class $\varepsilon_{k+1}(\infty)$, where $\varepsilon_{k+1}(\infty)$ is the class of groups whose every infinite subset X contain two distinct elements x, y such that $[x,_{k+1}y]=1$. We deduce that G/T belongs in $\varepsilon_{k+1}(\infty)$. Since G/T is nilpotent so soluble then by ([2], Theorem 3) there exists an integer c=c(k) depending only on k such that $(G/T)/Z_c(G/T)$ is finite. By a result in ([10], Theorem 1) $\gamma_{c+1}(G/T)= \gamma_{c+1}(G)T/T$ is finite and so is torsion, and since T is torsion group , we deduce that $\gamma_{c+1}(G)$ is torsion group too. Therefore G is in the class of τN_c-group. This completes the proof.

Lemma 3.4. Let G a finitely generated FN-group.
(i) If G is in the class $((FN_k)F,\infty)^*$, then there exists an integer c=c(k) depending only on k such that G is in the class ofFN_c-group.
(ii) G is in the class $((FC)F,\infty)^*$, then, $G/Z_2(G)$ is finite and G is in the class of FN_2-groups.
(iii) If G is in the class $((FN_2)F, \infty)^*$, then, G is in the class of $FN_3^{(2)}$-groups.

Proof. (i) Assume that G is finitely generated FN- group in the class $((FN_k)F,\infty)^*$. Let $T=\tau(G)$ the torsion subgroup of G. So by Lemma 2.1. (ii) T is a characteristic (so normal) and finite subgroup in G and as the same way in the above theorem, we deduce by ([16], Lemma 6.33) that G/T is in the class $(N_k, \infty)^*$which is included in the class $\varepsilon_{k+1}(\infty)$and according to ([2], Theorem 3) we found that there exists an integer c=c(k) depending only on k such that $(G/T)/Z_c(G/T)$ is finite. By a result in ([10], Theorem 1) $\gamma_{c+1}(G/T)= \gamma_{c+1}(G)T/T$ is finite and since T is finite, $\gamma_{c+1}(G)$ is finite too. Therefore G is in the class of FN_c-groups.

(ii) As the same way in (i) and the above Theorem we found that G/T is in the class $(A,\infty)^*$which is included in the class $\varepsilon_2(\infty)$, where $\varepsilon_2(\infty)$ is the class of groups whose every infinite subset X contain two distinct elements x, y such that $[x,_2y]=1$. we deduce by ([7], Theorem) that $(G/T)/Z_2(G/T)$ is finite and as T is finite then $G/Z_2(G)$ is finite equivalently $\gamma_3(G)$ is finite. It follows that G is in the class of FN_2-groups.

(iii) For k=2, as the same way in the above theorem we found that G/T is in the class $(N_2,\infty)^*$which is included in the class $\varepsilon_3(\infty)$, where $\varepsilon_3(\infty)$ is the class of groups whose every infinite subset X contain two distinct elements x, y such that $[x,_3y]=1$.we deduce by ([2], Theorem 1) that G/T is in the class $FN_3^{(2)}$ and as the torsion subgroup T is finite, then G is $F(FN_3^{(2)})$-group. It follows that G is $FN_3^{(2)}$-group. This completes the proof.

Theorem 3.5. Let G a finitely generated NF-group.
(i) If G is in the class $((FN_k)F,\infty)^*$, then there exists an integer c=c(k) depending only on such that G is in the class of N_cF-groups.
(ii) If G is in the class of $((FC)F, \infty)^*$-groups, then, G is in the class of N_2F-group.
(iii) If G is in the class $((FN_2)F, \infty)^*$, then, G is in the class of $N_3^{(2)}F$-groups.

Proof. As the group G is NF- group, and then it contains a normal nilpotent subgroup N such that G/N is finite. As the subgroup N is finitely generated and nilpotent of finite index then N is polycyclic so by ([14], Theorem 5.4.15) there exists a normal subgroup M in N and poly-infinite cyclic hence torsion-free and of finite index in N. Let $K=M_G$ the core of the subgroup M, so K is nilpotent torsion-free of finite index in G. Since the class $((FN_k)F,\infty)^*$is closed under taking subgroups, then K is in this class too, so by (i) in the above lemma, we obtains that there exists an integer c=c(k) depending only on k such that K is FN_c-group and as K is torsion-free, it is N_c-group and so G is N_cF-group

(ii) Particulary for k=1, we have $((FC)F, \infty)^*=((FN_1)F, \infty)^*$, in this case the subgroup K is a finitely generated torsion-free nilpotent group in the class $((FN_1)F, \infty)^*$ and according to

(ii) in the above lemma, we deduce that K is in the class FN_2-groups and as K is torsion-free, it is N_2-group of finite index in G, this gives that G is N_2F-group.

(iii) In particular for k=2, we have the subgroup K in (i) is a finitely generated torsion-free nilpotent group in the class $((FN_2)F, \infty)^*$ and according to (iii) in the above lemma, we deduce that K is in the class $FN_3^{(2)}$-groups and as K is torsion-free it is the class $N_3^{(2)}$-group and as G/K if finite this gives that G is in the class of $N_3^{(2)}F$-groups.

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(M,N)-Int-Soft Generalized Bi-Hyperideals of Ordered Semihypergroups

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Abstaract − Molodtsov introduced the concept of soft set as a new mathematical tool for dealing with uncertainties that is free from the difficulties that have troubled the usual theoretical approaches. In this paper, we apply the notion of soft sets to the ordered semihypergroups and introduce the notion of (M, N)-int-soft generalized bi-hyperideals of ordered semihypergroups. Moreover their related properties are investigated. We prove that every int-soft generalized bi-hyperideal is an (M, N)-int-soft generalized bi-hyperideals of S over U but the converse is not true which is shown with help of an example. We present new characterization of ordered semihypergroups in terms of (M, N)-int-soft generalized bi-hyperideals.

Keywords − *Ordered semihypergroup, int-soft hyperideal, int-soft generalized bi-hyperideal, (M, N)-int-soft hyperideal, (M, N)-int-soft generalized bi-hyperideal.*

1 Introduction

The real world is too complex for our immediate and direct understanding. We create models of reality that are simplifications of aspects of the real word. Unfortunately these mathematical models are too complicated and we cannot find the exact solutions. The uncertainty of data while modeling the problems in engineering, physics, computer sciences, economics, social sciences, medical sciences and many other diverse fields makes it unsuccessful to use the traditional classical methods, such as fuzzy set theory [21], intuitionistic set theory [22], and probability theory are useful approaches to describe uncertainty, but each of these theories has its inherent difficulties. To overcome these problems, Molodtsov [7], introduced the concept of

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soft set that is free from the difficulties that have troubled the usual theoretical approaches. Molodtsov pointed out several directions for the applications of soft sets. Maji et al. [23], gave the operations of soft sets and their properties; furthermore, in [24], they introduced fuzzy soft sets which combine the strengths of both soft sets and fuzzy sets. As a generalization of the soft set theory, the fuzzy soft set theory makes description of the objective world more realistic, practical, and precise in some cases, making it very promising. Since its introduction, the concept of soft sets has gained considerable attention in many directions and has found applications in a wide variety of fields such as the theory of soft sets [3, 4] and soft decision making [25, 26]. Since the notion of soft groups was proposed by Aktas and Cagman [1], then the soft set theory is used as a new tool to discuss algebraic structures Feng et al. soft semirings [2], Jun et al. [5] ordered semigroups. Soft sets were also applied to structure of hemirings [6, 8]. Song et al. [10], introduced the notions of int-soft semigroups and int-soft left (resp. right) ideals. Khan et al. [19], applied soft set theory to ordered semihypergroups and introduced the notions of (M, N)-int-soft hyperideals and (M, N)-int-soft interior hyperideals.

Algebraic hyperstructures represent a natural extension of classical algebraic structures and they were originally proposed in 1934 by a French mathematician Marty [9], at the 8^{th} Congress of Scandinavian Mathematicians. One of the main reason which attracts researches towards hyperstructures is its unique property that in hyperstructures composition of two elements is a set, while in classical algebraic structures the composition of two elements is an element. Thus algebraic hyperstructures are natural extension of classical algebraic structures. Since then, hyperstructures are widely investigated from the theoretical point of view and for their applications to many branches of pure and applied mathematics. Especially, semihypergroups are the simplest algebraic hyperstructures which possess the properties of closure and associativity. Nowadays many researchers have studied different aspects of semihypergroups (see [12, 13, 14, 15, 16, 17, 18]).

In this paper, we study the notion of (M, N)-int-soft generalized bi-hyperideals of ordered semihypergroups and give some related examples of this notion. We show that every int-soft generalized bi-hyperideals is an (M, N)-int-soft generalized bi-hyperideals but the converse is not true in general. We characterize ordered semihypergroups in terms of (M, N)-int-soft generalized bi-hyperideals.

2 Preliminaries

By an ordered semihypergroup we mean a structure $(S, \circ, \leq)$ in which the following conditions are satisfied:

(i) $(S, \circ)$ is a semihypergroup.

(ii) $(S, \leq)$ is a poset.

(iii) $(\forall a, b, x \in S)$ $a \leq b$ implies $x \circ a \leq x \circ b$ and $a \circ x \leq b \circ x$.

For $A \subseteq S$,we denote $(A] := \{t \in S : t \leq h \text{ for some } h \in A\}$.

For $A, B \subseteq S$, we have $A \circ B := \bigcup\{a \circ b : a \in A,\, b \in B\}$.

A nonempty subset A of an ordered semihypergroup S is called a subsemihypergroup of S if $A \circ A \subseteq A$.

A nonempty subset A of S is called a left (resp. right) hyperideal of S if it satisfies the following conditions:

(i) $S \circ A \subseteq A$ (resp. $A \circ S \subseteq A$).

(ii) If $a \in A, b \in S$ and $b \leq a$, implying $b \in A$.

By a two sided hyperideal or simply a hyperideal of S we mean a nonempty subset of S which is both a left hyperideal and a right hyperideal of S.

A nonempty B of S is called a generalized bi-hyperideal of S if it satisfies the following conditions:

(i) $B \circ S \circ B \subseteq B$.

(ii) If $a \in B$, $b \in S$ and $b \leq a$, implying $b \in B$.

For $x \in S$, we define $A_x = \{(y, z) \in S \times S \mid x \leq y \circ z\}$.

3 Soft Sets

In what follows, we take $E = S$ as the set of parameters, which is an ordered semihypergroup, unless otherwise specified.

From now on, U is an initial universe set, E is a set of parameters, $P(U)$ is the power set of U and $A, B, C... \subseteq E$.

Definition 3.1. (see [7, 20]). A *soft set* f_A over U is defined as

$$f_A : E \longrightarrow P(U) \text{ such that } f_A(x) = \emptyset \text{ if } x \notin A.$$

Hence f_A is also called an *approximation function.*

A soft set f_A over U can be represented by the set of ordered pairs

$$f_A = \{(x, f_A(x)) | x \in E, f_A(x) \in P(U)\}.$$

It is clear that a soft set is a *parameterized family* of subsets of U. Note that the set of all soft sets over U will be denoted by $S(U)$.

Definition 3.2. (see [20]). Let $f_A, f_B \in S(U)$. Then f_A is called a *soft subset* of f_B, denoted by $f_A \widetilde{\subseteq} f_B$ if $f_A(x) \subseteq f_B(x)$ for all $x \in E$.

Definition 3.3. (see [20]). Two soft sets f_A and f_B are said to be equal soft sets if $f_A \widetilde{\subseteq} f_B$ and $f_B \widetilde{\subseteq} f_A$ and is denoted by $f_A \widetilde{=} f_B$.

Definition 3.4. (see [20]). Let $f_A, f_B \in S(U)$. Then the *soft union* of f_A and f_B, denoted by $f_A \widetilde{\cup} f_B = f_{A \cup B}$, is defined by $\left(f_A \widetilde{\cup} f_B\right)(x) = f_A(x) \cup f_B(x)$ for all $x \in E$.

Definition 3.5. (see [20]). Let $f_A, f_B \in S(U)$. Then the *soft intersection* of f_A and f_B, denoted by $f_A \widetilde{\cap} f_B = f_{A \cap B}$, is defined by $\left(f_A \widetilde{\cap} f_B\right)(x) = f_A(x) \cap f_B(x)$ for all $x \in E$.

Definition 3.6. (see [11]). Let f_A and g_B be two soft sets of an ordered semihypergroup S over U. Then, the intersectional soft product, denoted by $f_A \widetilde{\odot} g_B$, is defined by $f_A \widetilde{\odot} g_B : S \longrightarrow P(U), x \longmapsto \left(f_A \widetilde{\odot} g_B\right)(x) = \begin{cases} \bigcup\limits_{(y,z) \in A_x} \{f_A(y) \cap g_B(z)\}, & \text{if } A_x \neq \emptyset, \\ \emptyset, & \text{if } A_x = \emptyset, \end{cases}$

for all $x \in S$.

Definition 3.7. (see [11]). For a nonempty subset A of S the characteristic soft set is defined to be the soft set $\mathcal{S}_{\mathcal{A}}$ of A over U in which $\mathcal{S}_{\mathcal{A}}$ is given by

$$\mathcal{S}_{\mathcal{A}} : S \longmapsto P(U). \quad x \longmapsto \begin{cases} U, & \text{if} \quad x \in A \\ \emptyset, & \text{otherwise} \end{cases}$$

For an ordered semihypergroup S, the soft set $\mathcal{S}_{\mathcal{S}}$ of S over U is defined as follows:

$$\mathcal{S}_{\mathcal{S}} : S \longrightarrow P(U), x \longmapsto \mathcal{S}_{\mathcal{S}}(x) = U \text{ for all } x \in S.$$

The soft set $\mathcal{S}_{\mathcal{S}}$ of an ordered semihypergroup S over U is called the whole soft set of S over U.

Definition 3.8. (see [11]). Let f_A be a soft set of an ordered semihypergroup S over U a subset δ such that $\delta \in P(U)$. The δ-inclusive set of f_A is denoted by $i_A(f_A, \delta)$ and defined to be the set

$$i_A(f_A, \delta) = \{x \in S \mid \delta \subseteq f_A(x)\}.$$

Definition 3.9. (see [11]). A soft set f_A of an ordered semihypergroup S over U is called *an int-soft subsemihypergroup of* S over U if:

$(\forall x, y \in S)$ $\bigcap\limits_{\alpha \in x \circ y} f_A(\alpha) \supseteq f_A(x) \cap f_A(y)$.

Definition 3.10. (see [11]). Let f_A be a soft set of an ordered semihypergroup S over U. Then f_A is called an int-soft left (resp. right) hyperideal of S over U if it satisfies the following conditions:

(1) $(\forall x, y \in S)$ $\bigcap\limits_{\alpha \in x \circ y} f_A(\alpha) \supseteq f_A(y)$ (resp. $\bigcap\limits_{\alpha \in x \circ y} f_A(\alpha) \supseteq f_A(x)$).

(2) $(\forall x, y \in S)$ $x \leq y \Longrightarrow f_A(x) \supseteq f_A(y)$.

A soft set f_A of an ordered semihypergroup S over U is called an *int-soft hyperideal (* or int-soft two-sided hyperideal) of S over U if it is both an int-soft left hyperideal and an int-soft right hyperideal of S over U.

Definition 3.11. (see [17]). A soft set f_A of an ordered semihypergroup S over U is called an int-soft generalized bi-*hyperideal* of S over U if it satisfies the following conditions:

(1) $(\forall x, y, z \in S)$ $\bigcap\limits_{\alpha \in x \circ y \circ z} f_A(\alpha) \supseteq f_A(x) \cap f_A(z)$.

(2) $(\forall x, y \in S)$ $x \leq y \Longrightarrow f_A(x) \supseteq f_A(y)$.

4 *(M,N)*-Int-Soft Generalized Bi-Hyperideals

In this section, we introduce the notion of (M, N)-int-soft generalized bi-hyperideals of ordered semihypergroups and investigate some related properties. From now on, $\emptyset \subseteq M \subset N \subseteq U$.

Definition 4.1. (see [19]). A soft set f_A of an ordered semihypergroup S over U is called an (M, N)*-int-soft subsemihypergroup of* S over U if:

$(\forall x, y \in S)$ $(\bigcap\limits_{\alpha \in x \circ y} f_A(\alpha)) \cup M \supseteq f_A(x) \cap f_A(y) \cap N$.

Definition 4.2. (see [19]). A soft set f_A of an ordered semihypergroup S over U is called an (M,N)-int-soft left (resp. right) hyperideal of S over U if it satisfies the following conditions:

(1) $(\forall x,y\in S)$ $(\bigcap_{\alpha\in x\circ y} f_A(\alpha))\cup M\supseteq f_A(y)\cap N$

(resp. $(\bigcap_{\alpha\in x\circ y} f_A(\alpha))\cup M\supseteq f_A(x)\cap N$).

(2) $(\forall x,y\in S)$ $x\leq y\Longrightarrow f_A(x)\cup M\supseteq f_A(y)\cap N$.

A soft set f_A of an ordered semihypergroup S over U is called an (M,N)*-int-soft hyperideal* of S over U, if it is both an (M,N)-int-soft left hyperideal and an (M,N)-int-soft right hyperideal of S over U.

Definition 4.3. A soft set f_A of an ordered semihypergroup S over U is called an (M,N)*-int-soft generalized bi-hyperideal of* S over U if it satisfies the following conditions:

(1) $(\forall x,y,z\in S)$ $(\bigcap_{\alpha\in x\circ y\circ z} f_A(\alpha))\cup M\supseteq f_A(x)\cap f_A(z)\cap N$.

(2) $(\forall x,y\in S)$ $x\leq y\Longrightarrow f_A(x)\cup M\supseteq f_A(y)\cap N$.

Example 4.4. Let $(S,\circ,\leq)$ be an ordered semihypergroup where the hyperoperation and the order relation are defined by:

$\circ$	a	b	c	d
a	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$
b	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$
c	$\{a\}$	$\{a\}$	$\{a,b\}$	$\{a\}$
d	$\{a\}$	$\{a\}$	$\{a,b\}$	$\{a,b\}$

$$\leq:=\{(a,a),(b,b),(c,c),(d,d),(a,b)\}.$$

Suppose $U=\{p,q,r,s\}$, $A=\{a,c,d\}$, $M=\{p,q\}$ and $N=\{p,q,s\}$. Let us define $f_A(a)=\{p,q,r,s\}$, $f_A(b)=\emptyset$, $f_A(c)=\{q,r,s\}$ and $f_A(d)=\{p,s\}$. Then f_A is an (M,N)-int-soft generalized bi-hyperideal of S over U.

Remark 4.5. Every int-soft generalized bi-hyperideal is an (M,N)-int-soft generalized bi-hyperideal of S over U. But the converse is not true. We can illustrate it by the following example.

Example 4.6. Let $(S,\circ,\leq)$ be an ordered semihypergroup where the hyperoperation and the order relation are defined by:

$\circ$	e_1	e_2	e_3	e_4	e_5
e_1	$\{e_1\}$	$\{e_1,e_2,e_4\}$	$\{e_1\}$	$\{e_1,e_2,e_4\}$	$\{e_1,e_2,e_4\}$
e_2	$\{e_1\}$	$\{e_2\}$	$\{e_1\}$	$\{e_1,e_2,e_4\}$	$\{e_1,e_2,e_4\}$
e_3	$\{e_1\}$	$\{e_1,e_2,e_4\}$	$\{e_1,e_3\}$	$\{e_1,e_2,e_4\}$	$\{e_1,e_2,e_3,e_4,e_5\}$
e_4	$\{e_1\}$	$\{e_1,e_2,e_4\}$	$\{e_1\}$	$\{e_1,e_2,e_4\}$	$\{e_1,e_2,e_4\}$
e_5	$\{e_1\}$	$\{e_1,e_2,e_4\}$	$\{e_1,e_3\}$	$\{e_1,e_2,e_4\}$	$\{e_1,e_2,e_3,e_4,e_5\}$

$\leq:=\{(e_1,e_1),(e_2,e_2),(e_3,e_3),(e_4,e_4),(e_5,e_5),(e_1,e_3),(e_1,e_4),(e_1,e_5),(e_2,e_4),(e_2,e_5),$ $(e_3,e_5),(e_4,e_5)\}$.

Suppose $U = \{1,2,3\}$, $A = \{e_1, e_2, e_4\}$, $M = \{2\}$ and $N = \{2,3\}$. Let us define $f_A(e_1) = \{1,2,3\}$, $f_A(e_2) = \{1,2\}$, $f_A(e_3) = \emptyset$, $f_A(e_4) = \{2\}$ and $f_A(e_5) = \emptyset$. Then f_A is an (M,N)-int-soft generalized bi-hyperideal of S over U. This is not int-soft generalized bi-hyperideal of S over U, as $\bigcap_{\alpha \in e_1 \circ e_1 \circ e_2 = \{e_1, e_2, e_4\}} f_A(\alpha) = f_A(e_1) \cap f_A(e_2) \cap f_A(e_4) = \{2\} \not\supseteq \{1,2\} = f_A(e_1) \cap f_A(e_2)$.

Theorem 4.7. A non-empty subset A of an ordered semihypergroup $(S, \circ, \leq)$ is a generalized bi-hyperideal of S if and only if the soft set f_A is defined by

$$f_A(x) = \begin{cases} N \text{ if } x \in A \\ M \text{ if } x \notin A \end{cases}$$

is an (M,N)-int-soft generalized bi-hyperideal of S over U.

Proof. Suppose A is a generalized bi-hyperideal of S. If there exist $x, y \in S$ such that $x \leq y$. If $y \in A$, then $x \in A$. Hence $f_A(x) = N$. Therefore $f_A(x) \cup M \supseteq N = f_A(y) \cap N$. If $y \notin A$, then $f_A(y) \cap N = M$. Thus $f_A(x) \cup M \supseteq M = f_A(y) \cap N$. Let $x, y, z \in S$, such that $x, z \in A$. Then $f_A(x) = N$ and $f_A(z) = N$. Hence for any $\alpha \in x \circ y \circ z$, $(\bigcap_{\alpha x \circ y \circ z} f_A(\alpha)) \cup M \supseteq N = f_A(x) \cap f_A(z) \cap N$. If $x \notin A$ or $z \notin A$ then $f_A(x) \cap f_A(z) \cap N = M$. Thus $(\bigcap_{\alpha x \circ y \circ z} f_A(\alpha)) \cup M \supseteq M = f_A(x) \cap f_A(z) \cap N$. Hence $(\bigcap_{\alpha x \circ y \circ z} f_A(\alpha)) \cup M \supseteq f_A(x) \cap f_A(z) \cap N$. Consequently, f_A is an (M,N)-int-soft generalized bi-hyperideal of S over U.

Theorem 4.8. If $\{f_{A_i} \mid i \in I\}$ is a family of (M,N)-int-soft generalized bi-hyperideal of an ordered semihypergroup S over U. Then $f_A = \bigcap_{i \in I} f_{A_i}$ is an (M,N)-int-soft generalized bi-hyperideal of S over U.

Proof. Let $\{f_{A_i} \mid i \in I\}$ be a family of (M,N)-int-soft generalized bi-hyperideal of S over U. Let $x, y, z \in S$ and $(\bigcap_{\beta \in x \circ y \circ z} f_{A_i}(\beta)) \cup M \supseteq f_{A_i}(x) \cap f_{A_i}(z) \cap N$. Since each f_{A_i} $(i \in I)$ is an (M,N)-int-soft generalized bi-hyperideal of S over U. Thus for any $\beta \in x \circ y \circ z$, $f_{A_i}(\beta) \cup M \supseteq f_{A_i}(x) \cap f_{A_i}(z) \cap N$. Then $f_A(\beta) \cup M = \left(\bigcap_{i \in I} f_{A_i}\right)(\beta) \cup M = (\bigcap_{i \in I} f_{A_i}(\beta)) \cup M \supseteq \bigcap_{i \in I} (f_{A_i}(x) \cap f_{A_i}(z) \cap N) = \left(\bigcap_{i \in I} f_{A_i}\right)(x) \cap \left(\bigcap_{i \in I} f_{A_i}\right)(z) \cap N = f_A(x) \cap f_A(z) \cap N$. Thus $(\bigcap_{\beta \in x \circ y \circ z} f_A(\beta)) \cup M \supseteq f_A(x) \cap f_A(y) \cap N$. Furthermore, if $x \leq y$, then $f_A(x) \cup M \supseteq f_A(y) \cap N$. Indeed: Since every f_{A_i} $(i \in I)$ is an (M,N)-int-soft generalized bi-hyperideal of S over U, it can be obtained that $f_{A_i}(x) \cup M \supseteq f_{A_i}(y) \cap N$ for all $i \in I$. Thus $f_A(x) \cup M = \left(\bigcap_{i \in I} f_{A_i}\right)(x) \cup M = (\bigcap_{i \in I} (f_{A_i}(x))) \cup M \supseteq (\bigcap_{i \in I} (f_{A_i}(y))) \cap N = \left(\bigcap_{i \in I} f_{A_i}\right)(y) \cap N = f_A(y) \cap N$. Thus f_A is is an (M,N)-int-soft generalized bi-hyperideal of S over U.

Theorem 4.9. Let $(S, \circ, \leq)$ be an ordered semihypergroup and A be a nonempty subset of S. Then A is a generalized bi-hyperideal of S if and only if the characteristic function $\mathcal{S}_{\mathcal{A}}$ of A is an (M, N)-int-soft generalized bi-hyperideal of S over U.

Proof. Suppose that A is a generalized bi-hyperideal of S. Let x, y and z be any elements of S. Then $(\bigcap_{\alpha \in x \circ y \circ z} \mathcal{S}_{\mathcal{A}}(\alpha)) \cup M \supseteq \mathcal{S}_{\mathcal{A}}(x) \cap \mathcal{S}_{\mathcal{A}}(z) \cap N$. Indeed, If $x, z \in A$, then $\mathcal{S}_{\mathcal{A}}(x) = U$ and $\mathcal{S}_{\mathcal{A}}(z) = U$. Since A is a generalized bi-hyperideal of S, we have $\alpha \in x \circ y \circ z \subseteq A \circ S \circ A \subseteq A$ we have $\mathcal{S}_{\mathcal{A}}(\alpha) = U$ and $\emptyset \subseteq M \subset N \subseteq U$. Thus $(\bigcap_{\alpha \in x \circ y \circ z} \mathcal{S}_{\mathcal{A}}(\alpha)) \cup M = U \supseteq \mathcal{S}_{\mathcal{A}}(x) \cap \mathcal{S}_{\mathcal{A}}(z) \cap N$. If $x \notin A$ or $z \notin A$ then $\mathcal{S}_{\mathcal{A}}(x) = \emptyset$ or $\mathcal{S}_{\mathcal{A}}(z) = \emptyset$. Since $\mathcal{S}_{\mathcal{A}}(p) \supseteq \emptyset$ for all $p \in S$. Thus $(\bigcap_{\alpha \in x \circ y \circ z} \mathcal{S}_{\mathcal{A}}(\alpha)) \cup M \supseteq \emptyset = \mathcal{S}_{\mathcal{A}}(x) \cap \mathcal{S}_{\mathcal{A}}(z) \cap N$. Let $x, y \in S$ with $x \leq y$. Then $\mathcal{S}_{\mathcal{A}}(x) \cup M \supseteq \mathcal{S}_{\mathcal{A}}(y) \cap N$. Indeed, if $y \notin A$ then $\mathcal{S}_{\mathcal{A}}(y) = \emptyset$ and $\emptyset \subseteq M \subset N \subseteq U$ so $\mathcal{S}_{\mathcal{A}}(x) \cup M \supseteq \emptyset = \mathcal{S}_{\mathcal{A}}(y) \cap N$. If $y \in A$ then $\mathcal{S}_{\mathcal{A}}(y) = U$. Since $x \leq y$ and A is a generalized bi-hyperideal of S, we have $x \in A$ and thus $\mathcal{S}_{\mathcal{A}}(x) \cup M = U \supseteq \mathcal{S}_{\mathcal{A}}(y) \cap N$.

Conversely, let $\emptyset \neq A \subseteq S$ such that $\mathcal{S}_{\mathcal{A}}$ is an (M, N)-int-soft generalized hyperideal of S over U. Let $\alpha \in A \circ S \circ A$, then there exist $x, z \in A$ and $y \in S$ such that $\alpha \in x \circ y \circ z$. Since $(\bigcap_{\alpha \in x \circ y \circ z} \mathcal{S}_{\mathcal{A}}(\alpha)) \cup M \supseteq \mathcal{S}_{\mathcal{A}}(x) \cap \mathcal{S}_{\mathcal{A}}(z) \cap N$, and $x, z \in A$ we have $\mathcal{S}_{\mathcal{A}}(x) = U$ and $\mathcal{S}_{\mathcal{A}}(z) = U$. Hence for each $\alpha \in A \circ S \circ A$, we have $(\bigcap_{\alpha \in x \circ y \circ z} \mathcal{S}_{\mathcal{A}}(\alpha)) \cup M \supseteq U \cap U \cap N = N$. Thus by $\emptyset \subseteq M \subset N \subseteq U$, $\bigcap_{\alpha \in x \circ y \circ z} \mathcal{S}_{\mathcal{A}}(\alpha) \supseteq N \supset \emptyset$. On the other hand $\mathcal{S}_{\mathcal{A}}(x) \subseteq U$ for all $x \in S$. Thus for any $\alpha \in x \circ y \circ z$, $\mathcal{S}_{\mathcal{A}}(\alpha) = U$ implies that $\alpha \in A$. Thus $A \circ S \circ A \subseteq A$. Furthermore, let $x \in A$, $S \ni y \leq x$. Then $y \in A$. Indeed, it is enough to prove that $\mathcal{S}_{A}(y) = U$. By $x \in A$ we have $\mathcal{S}_{A}(x) = U$. Since $\mathcal{S}_{A}$ is an (M, N)-int-soft generalized-hyperideal of S over U and $y \leq x$, we have $\mathcal{S}_{A}(y) \cup M \supseteq \mathcal{S}_{A}(x) \cap N = U \cap N = N$. Notice that $\emptyset \subseteq M \subset N \subseteq U$, we conclude that $\mathcal{S}_{A}(y) \supseteq \emptyset$. Thus $\mathcal{S}_{A}(y) = U$. Therefore A is a generalized bi-hyperideal of S.

Theorem 4.10. Let f_A be a soft set of an ordered semihypergroup S over U and $\delta \in P(U)$. Then f_A is an (M, N)-int-soft generalized bi-hyperideal of S over U if and only if each nonempty δ-inclusive set $i_A(f_A, \delta)$ of f_A is a generalized bi-hyperideal of S where $M \subset \delta \subseteq N$.

Proof. Assume that f_A is an (M, N)-int-soft generalized bi-hyperideal of S over U, and $i_A(f_A, \delta) \neq \emptyset$. Let $x, y, z \in S$ and $x, z \in i_A(f_A, \delta)$ where $M \subset \delta \subseteq N$. Then $f_A(x) \supseteq \delta$ and $f_A(z) \supseteq \delta$. Since f_A is an (M, N)-int-soft generalized bi-hyperideal of S over U. Thus $(\bigcap_{w \in x \circ y \circ z} f_A(w)) \cup M \supseteq f_A(x) \cap f_A(z) \cap N \supseteq \delta \cap \delta \cap N = \delta$. Since $\emptyset \subseteq M \subset \delta \subseteq N \subseteq U$, we can write as $\bigcap_{w \in x \circ y \circ z} f_A(w) \supseteq \delta$. Hence $f_A(w) \supseteq \delta$ for any $w \in x \circ y \circ z$ implies that $w \in i_A(f_A, \delta)$. Thus $i_A(f_A, \delta) \circ S \circ i_A(f_A, \delta) \subseteq i_A(f_A, \delta)$. Furthermore, let $x \in i_A(f_A, \delta)$, $S \ni y \leq x$. Then $y \in i_A(f_A, \delta)$. Indeed, since $x \in i_A(f_A, \delta)$, $f_A(x) \supseteq \delta$ and f_A is an (M, N)-int-soft generalized bi-hyperideal of S over

U, we have $f_A(y) \cup M \supseteq f_A(x) \cap N \supseteq \delta \cap N = \delta$. By $M \subset \delta$, we have $f_A(y) \supseteq \delta$, i.e., $y \in e_A(f_A, \delta)$. Therefore $i_A(f_A, \delta)$ is a generalized bi-hyperideal of S.

Conversely, suppose that $i_A(f_A, \delta) \neq \emptyset$ is a generalized bi-hyperideal of S for all $M \subset \delta \subseteq N$. Now let $x, y, z \in S$. We will prove that $(\bigcap_{\alpha \in x \circ y \circ z} f_A(\alpha)) \cup M \supseteq f_A(x) \cap f_A(z) \cap N$ for all $x, y, z \in S$. If there exist x_1, y_1, z_1 such that $(\bigcap_{\alpha \in x_1 \circ y_1 \circ z_1} f_A(\alpha)) \cup M \subset f_A(x_1) \cap f_A(z_1) \cap N$, and $M \subset \delta \subseteq N$ such that $(\bigcap_{\alpha \in x_1 \circ y_1 \circ z_1} f_A(\alpha)) \cup M \subset \delta \subseteq f_A(x_1) \cap f_A(z_1) \cap N$, so $f_A(x_1) \supseteq \delta$, $f_A(z_1) \supseteq \delta$ and $\bigcap_{\alpha \in x_1 \circ y_1 \circ z_1} f_A(\alpha) \subset \delta$ then $x_1, z_1 \in i_A(f_A, \delta)$ and $x_1 \circ y_1 \circ z_1 \not\subseteq i_A(f_A, \delta)$. This is a contradiction that $i_A(f_A, \delta)$ is a generalized bi-hyperideal of S. Moreover if $x \leq y$ then $f_A(x) \cup M \supseteq f_A(y) \cap N$. Indeed, if there exist $x_1, y_1 \in S$ such that $x_1 \leq y_1$ and $f_A(x_1) \cup M \subset f_A(y_1) \cap N$, $M \subset \delta \subseteq N$ such that $f_A(x_1) \cup M \subset \delta \subseteq f_A(y_1) \cap N$ and we have $f_A(y_1) \supseteq \delta$ and $f_A(x_1) \subset \delta$. Then $y_1 \in i_A(f_A, \delta)$ and $x_1 \notin i_A(f_A, \delta)$. This is a contradiction that $i_A(f_A, \delta)$ is a generalized bi-hyperideal of S. Thus if $x \leq y$ then $f_A(x) \cup M \supseteq f_A(y) \cap N$.

Theorem 4.11. Every (M, N)-int-soft right (resp. left) hyperideal of S over U is an (M, N)-int-soft generalized bi-hyperideal of S over U.

Proof. Let f_A is an (M, N)-int-soft right hyperideal of S over U. Let $x, y, z \in S$. Then $(\bigcap_{\alpha \in x \circ y \circ z} f_A(\alpha)) \cup M = (\bigcap_{\substack{\alpha \in x \circ \beta \\ \beta \in y \circ z}} f_A(\alpha)) \cup M \supseteq f_A(x) \cap N \supseteq f_A(x) \cap f_A(z) \cap N$. Thus f_A is an (M, N)-int-soft generalized bi-hyperideal of S over U.

Let g_B is an (M, N)-int-soft left hyperideal of S over U. Then $(\bigcap_{\alpha \in x \circ y \circ z} g_B(\alpha)) \cup M = (\bigcap_{\substack{\alpha \in \gamma \circ z \\ \gamma \in x \circ y}} g_B(\alpha)) \cup M \supseteq g_B(z) \cap N \supseteq g_B(x) \cap g_B(z) \cap N$. Thus g_B is an (M, N)-int-soft generalized bi-hyperideal of S over U.

Definition 4.12. Let $(S, \circ, \leq)$ be an ordered semihypergroup. Let f_A be a soft set of S over U. We define the the soft set f_A^* of S as follows:

$$f_A^*(x) = f_A(x) \cap N \cup M$$

for all $x \in S$.

Definition 4.13. Let $(S, \circ, \leq)$ be an ordered semihypergroup. Let f_A and g_B be soft set of S over U. We define $f_A \widetilde{\cap^*} g_B$, $f_A \widetilde{\cup^*} g_B$ and $f_A \widetilde{\odot^*} g_B$ of S as follows:

$$\left(f_A \widetilde{\cap^*} g_B\right)(x) = \left(\left(f_A \widetilde{\cap} g_B\right)(x) \cap N\right) \cup M$$
$$\left(f_A \widetilde{\cup^*} g_B\right)(x) = \left(\left(f_A \widetilde{\cup} g_B\right)(x) \cap N\right) \cup M$$
$$\left(f_A \widetilde{\odot^*} g_B\right)(x) = \left(\left(f_A \widetilde{\odot} g_B\right)(x) \cap N\right) \cup M$$

for all $x \in S$.

Lemma 4.14. Let f_A and g_B be soft sets of an ordered semihypergroup S over U. Then the following conditions hold:

(1) $f_A \widetilde{\cap^*} g_B = f_A^* \widetilde{\cap} g_B^*$.
(2) $f_A \widetilde{\cup^*} g_B = f_A^* \widetilde{\cup} g_B^*$.
(3) $f_A \widetilde{\odot^*} g_B \widetilde{\supseteq} f_A^* \widetilde{\odot} g_B^*$.

Proof. (1) Let $x \in S$. Then

$$\begin{aligned}\left(f_A \widetilde{\cap^*} g_B\right)(x) &= \left(\left(f_A \widetilde{\cap} g_B\right)(x) \cap N\right) \cup M \\ &= \left(\left(f_A(x) \widetilde{\cap} g_B(x)\right) \cap N\right) \cup M \\ &= \left((f_A(x) \cap N) \widetilde{\cap} (g_B(x) \cap N)\right) \cup M \\ &= (((f_A(x) \cap N)) \cup M) \widetilde{\cap} (((g_B(x) \cap N)) \cup M) \\ &= f_A^* \widetilde{\cap} g_B^*.\end{aligned}$$

(2) Proof is similar to the proof of (1).
(3) If $A_x = \emptyset$. Then $\left(f_A \widetilde{\odot} g_B\right)(x) = \emptyset$. Thus

$$\begin{aligned}\left(f_A \widetilde{\odot^*} g_B\right)(x) &= \left(\left(f_A \widetilde{\odot} g_B\right)(x) \cap N\right) \cup M \\ &= (\emptyset \cap N) \cup M \\ &= M = N \cap M \\ \left(f_A \widetilde{\odot^*} g_B\right)(x) &\supseteq M = f_A^* \widetilde{\odot} g_B^*.\end{aligned}$$

If $A_x \neq \emptyset$. So there exist $y, z \in S$ such that $x \leq y \circ z$. Then $(y, z) \in A_x$. Thus

$$\begin{aligned}\left(f_A \widetilde{\odot^*} g_B\right)(x) &= \left(\left(f_A \widetilde{\odot} g_B\right)(x) \cap N\right) \cup M \\ &= \left(\left(\bigcup_{(y,z)\in A_x} \{f_A(y) \cap g_B(z)\}\right) \cap N\right) \cup M \\ &= \left(\bigcup_{(y,z)\in A_x} \{(f_A(y) \cap N) \cap (g_B(z) \cap N)\}\right) \cup M \\ &= \bigcup_{(y,z)\in A_x} \{((f_A(y) \cap N) \cup M) \cap (g_B(z) \cap N) \cup M\} \\ &= \bigcup_{(y,z)\in A_x} \{f_A^*(y) \cap g_B^*(z)\} \\ &= (f_A^* \odot g_B^*)(x).\end{aligned}$$

Thus $f_A \widetilde{\odot^*} g_B \widetilde{\supseteq} f_A^* \widetilde{\odot} g_B^*$.

Definition 4.15. If $\mathcal{S}_{\mathcal{A}}$ is the characteristic soft function of A. Then $\mathcal{S}_{\mathcal{A}}^*$ is defined over U in which $\mathcal{S}_{\mathcal{A}}^*$ is given by

$$\mathcal{S}_{\mathcal{A}}^*(x) = \begin{cases} N \text{ if } x \in A \\ M \text{ if } x \notin A \end{cases}$$

Lemma 4.16. Let A and B be the nonempty subsets of an ordered semihypergroup S. Then the following holds:
(1) $\mathcal{S}_{\mathcal{A}} \widetilde{\cap^*} \mathcal{S}_{\mathcal{B}} = \mathcal{S}^*_{\mathcal{A}\cap\mathcal{B}}$.
(2) $\mathcal{S}_{\mathcal{A}} \widetilde{\cup^*} \mathcal{S}_{\mathcal{B}} = \mathcal{S}^*_{\mathcal{A}\cup\mathcal{B}}$.
(3) $\mathcal{S}_{\mathcal{A}} \widetilde{\odot^*} \mathcal{S}_{\mathcal{B}} = \mathcal{S}^*_{(\mathcal{A}\circ\mathcal{B}]}$.

Proof. (1) and (2) are obvious.

(3) Let $x \in (A \circ B]$. Then $\mathcal{S}_{(A\circ B]}(x) = U$. Hence $\left(\mathcal{S}_{(A\circ B]} \cap N\right) \cup M = (U \cap N) \cup M = N \cup M = N$. Thus $\mathcal{S}^*_{(\mathcal{A}\circ\mathcal{B}]}(x) = N$. Since $x \in (A \circ B]$, we have $x \leq a \circ b$ for some $a \in A$ and $b \in B$. Then $(a, b) \in A_x$ and $A_x \neq \emptyset$. Thus

$$\begin{aligned}\left(\mathcal{S}_{\mathcal{A}} \widetilde{\odot^*} \mathcal{S}_{\mathcal{B}}\right)(x) &= \left(\left(\mathcal{S}_{\mathcal{A}} \widetilde{\odot} \mathcal{S}_{\mathcal{B}}\right)(x) \cap N\right) \cup M \\ &= \left[\left\{\bigcup_{(y,z)\in A_x} \left(\mathcal{S}_{\mathcal{A}}(y) \cap \mathcal{S}_{\mathcal{B}}(z)\right)\right\} \cap N\right] \cup M \\ &\supseteq \left[\left\{\mathcal{S}_{\mathcal{A}}(a) \cap \mathcal{S}_{\mathcal{B}}(b)\right\} \cap N\right] \cup M.\end{aligned}$$

Since $a \in A$ and $b \in B$, we have $\mathcal{S}_{\mathcal{A}}(a) = U$ and $\mathcal{S}_{\mathcal{B}}(b) = U$ and so

$$\begin{aligned}\left(\mathcal{S}_{\mathcal{A}} \widetilde{\odot^*} \mathcal{S}_{\mathcal{B}}\right)(x) &\supseteq \left[\left\{\mathcal{S}_{\mathcal{A}}(a) \cap \mathcal{S}_{\mathcal{B}}(b)\right\} \cap N\right] \cup M \\ &= \left[\{U \cap U\} \cap N\right] \cup M \\ &= N \cup M = N.\end{aligned}$$

Thus,

$$\left(\mathcal{S}_{\mathcal{A}} \widetilde{\odot^*} \mathcal{S}_{\mathcal{B}}\right)(x) = \mathcal{S}^*_{(\mathcal{A}\circ\mathcal{B}]}(x).$$

Let $x \notin (A \circ B]$, then $\mathcal{S}_{(\mathcal{A}\circ\mathcal{B}]}(x) = \emptyset$ and hence,

$$\left\{\mathcal{S}_{(\mathcal{A}\circ\mathcal{B}]}(x) \cap N\right\} \cup M = \{\emptyset \cap N\} \cup M = M.$$

So $\mathcal{S}^*_{(\mathcal{A}\circ\mathcal{B}]}(x) = M$. Let $(y, z) \in A_x$. Then

$$\begin{aligned}\left(\mathcal{S}_{\mathcal{A}} \widetilde{\odot^*} \mathcal{S}_{\mathcal{B}}\right)(x) &= \left(\left(\mathcal{S}_{\mathcal{A}} \widetilde{\odot} \mathcal{S}_{\mathcal{B}}\right)(x) \cap N\right) \cup M \\ &= \left[\left\{\bigcup_{(y,z)\in A_x} \left(\mathcal{S}_{\mathcal{A}}(y) \cap \mathcal{S}_{\mathcal{B}}(z)\right)\right\} \cap N\right] \cup M.\end{aligned}$$

Since $(y, z) \in A_x$, then $x \leq y \circ z$. If $y \in A$ and $z \in B$, then $y \circ z \subseteq A \circ B$ and so $x \in (A \circ B]$. This is a contradiction. If $y \notin A$ and $z \in B$, then

$$\left[\left\{\bigcup_{(y,z)\in A_x} \left(\mathcal{S}_{\mathcal{A}}(y) \cap \mathcal{S}_{\mathcal{B}}(z)\right)\right\} \cap N\right] \cup M = \left[\left\{\bigcup_{(y,z)\in A_x} (\emptyset \cap U)\right\} \cap N\right] \cup M = M.$$

Hence $\left(\mathcal{S}_{\mathcal{A}} \widetilde{\odot^*} \mathcal{S}_{\mathcal{B}}\right)(x) = M = \mathcal{S}^*_{(\mathcal{A}\circ\mathcal{B}]}(x)$. Similarly, for $y \in A$ and $z \notin B$, we have $\left(\mathcal{S}_{\mathcal{A}} \widetilde{\odot^*} \mathcal{S}_{\mathcal{B}}\right)(x) = M = \mathcal{S}^*_{(\mathcal{A}\circ\mathcal{B}]}(x)$.

Theorem 4.17. If f_A is an (M,N)-int-soft subsemihypergroup of S over U. Then f_A^* is an (M,N)-int-soft subsemihypergroup of S over U.

Proof. Suppose that f_A is an (M,N)-int-soft subsemihypergroup of S over U. Let $x,y \in S$. Then

$$\begin{aligned}
\bigcap_{\alpha\in x\circ y} f_A^*(\alpha)\cup M &= \left[\bigcap_{\alpha\in x\circ y}\{(f_A(\alpha)\cap N)\cup M\}\right]\cup M\\
&= \left[\bigcap_{\alpha\in x\circ y}(f_A(\alpha)\cup M)\cap(N\cup M)\right]\cup M\\
&= \left[\bigcap_{\alpha\in x\circ y}(f_A(\alpha)\cup M)\cap N\right]\cup M\\
&\supseteq \{(f_A(x)\cap f_A(y)\cap N)\cap N\}\cup M\\
&= \{(f_A(x)\cap N)\cap(f_A(y)\cap N)\cap N\}\cup M\\
&= \{(f_A(x)\cap N)\cup M\}\cap\{(f_A(y)\cap N)\cup M\}\cap(N\cup M)\\
&= f_A^*(x)\cap f_A^*(y)\cap N.
\end{aligned}$$

Thus f_A^* is an (M,N)-int-soft subsemihypergroup of S over U.

Theorem 4.18. A soft set f_A is an (M,N)-int-soft subsemihypergroup of S over U if and only if $f_A \widetilde{\odot^*} f_A \widetilde{\subseteq} f_A^*$.

Proof. Assume that f_A is an (M,N)-int-soft subsemihypergroup of S over U. Let $x \in S$. If $A_x = \emptyset$. Then $\left(f_A \widetilde{\odot} f_A\right)(x) = \emptyset$. Thus

$$\begin{aligned}
\left(f_A \widetilde{\odot^*} f_A\right)(x) &= \left\{\left(f_A \widetilde{\odot} f_A\right)(x)\cap N\right\}\cup M\\
&= (\emptyset\cap N)\cup M\\
&= M
\end{aligned}$$

$\left(f_A \widetilde{\odot^*} f_A\right)(x) \supseteq M = f_A^*(x)$.

If $A_x \neq \emptyset$. Then

$$\begin{aligned}
\left(f_A \widetilde{\odot^*} f_A\right)(x) &= \left\{\left(f_A \widetilde{\odot} f_A\right)(x)\cap N\right\}\cup M\\
&= \left\{\left(\bigcup_{(a,b)\in A_x}\{f_A(a)\cap f_A(b)\}\right)\cap N\right\}\cup M\\
&= \left\{\bigcup_{(a,b)\in A_x}(f_A(a)\cap f_A(b)\cup M)\cap N\right\}\cup M\\
&\subseteq \left\{\bigcup_{(a,b)\in A_x}(f_A(x)\cap N)\cup M\right\}\cup M\\
&= (f_A(x)\cap N)\cup M\\
&= f_A^*(x).
\end{aligned}$$

Thus $f_A \widetilde{\odot^*} f_A \widetilde{\subseteq} f_A^*$.

Conversely, assume that $f_A \widetilde{\odot^*} f_A \widetilde{\subseteq} f_A^*$. Let $x, y \in S$. Then for each $\alpha \in x \circ y$, we have,

$$\begin{aligned}
(f_A(\alpha) \cap N) \cup M = f_A^*(\alpha) &\supseteq \left(f_A \widetilde{\odot^*} f_A\right)(\alpha) \\
&= \left\{\left(f_A \widetilde{\odot} f_A\right)(\alpha) \cap N\right\} \cup M \\
&= \left[\left\{\bigcup_{(a,b) \in A_\alpha} (f_A(a) \cap f_A(b))\right\} \cap N\right] \cup M \\
&\supseteq \{(f_A(x) \cap f_A(y)) \cap N\} \cup M \\
&\supseteq \{(f_A(x) \cap f_A(y)) \cap N\}.
\end{aligned}$$

Thus $\bigcap_{\alpha \in x \circ y} f_A(\alpha) \cup M \supseteq f_A(x) \cap f_A(y) \cap N$. Hence f_A is an (M, N)-int-soft subsemihypergroup of S over U.

Theorem 4.19. The characteristic function $\mathcal{S}_A^*$ of A is an (M, N)-int-soft generalized bi-hyperideal of S over U, if and only if A is a generalized bi-hyperideal of S.

Proof. Suppose that A is a generalized bi-hyperideal of S. Then by Theorem 4.9, $\mathcal{S}_A^*$ is an (M, N)-int-soft generalized bi-hyperideal of S over U.

Conversely, assume that $\mathcal{S}_A^*$ is an (M, N)-int-soft generalized bi-hyperideal of S over U. Let $x, y \in S$, $x \leq y$ be such that $y \in A$. It implies that $\mathcal{S}_A^*(y) = N$. Since $\mathcal{S}_A^*$ is an (M, N)-int-soft generalized bi-hyperideal of S over U. Therefore $\mathcal{S}_A^*(x) \cup M \supseteq \mathcal{S}_A^*(y) \cap N = N \cap N = N$. Since $M \subset N$. Hence $\mathcal{S}_A^*(y) = N$. Implies that $x \in A$. Now if there exist $x, y, z \in S$ such that $x, z \in A$. Then $\mathcal{S}_A^*(x) = N$ and $\mathcal{S}_A^*(z) = N$. Since $\mathcal{S}_A^*$ is an (M, N)-int-soft generalized bi-hyperideal of S over U. We have

$$\begin{aligned}
\bigcap_{\alpha \in x \circ y \circ z} \mathcal{S}_A^*(\alpha) \cup M &\supseteq \mathcal{S}_A^*(x) \cap \mathcal{S}_A^*(z) \cap N \\
&= N \cap N \cap N \\
&= N.
\end{aligned}$$

Since $M \subset N$. Hence $\mathcal{S}_A^*(\alpha) = N$. Thus $\alpha \in x \circ y \circ z \subseteq A$. Consequently, A is a generalized bi-hyperideal of S.

Proposition 4.20. If f_A is an (M, N)-int-soft generalized bi-hyperideal of S over U. Then f_A^* is an (M, N)-int-soft generalized bi-hyperideal of S over U.

Proof. Assume that f_A is an (M, N)-int-soft generalized bi-hyperideal of S over U.

Let $x, y, z \in S$, then

$$\begin{aligned}
\bigcap_{\alpha\in x\circ y\circ z} f_A^*(\alpha)\cup M &= \left\{\left(\bigcap_{\alpha\in x\circ y\circ z} f_A(\alpha)\cap N\right)\cup M\right\}\cup M \\
&= \left(\bigcap_{\alpha\in x\circ y\circ z} f_A(\alpha)\cap N\right)\cup M \\
&= \left(\bigcap_{\alpha\in x\circ y\circ z} f_A(\alpha)\cup M\right)\cap (N\cup M) \\
&= \left(\bigcap_{\alpha\in x\circ y\circ z} f_A(\alpha)\cup M\right)\cap N \\
&= \left\{\left(\bigcap_{\alpha\in x\circ y\circ z} f_A(\alpha)\cup M\right)\cup M\right\}\cap N \\
&\supseteq \{(f_A(x)\cap f_A(z)\cap N)\cup M\}\cap N \\
&= \{(f_A(x)\cap f_A(z)\cap N\cap N)\cup M\cup M\}\cap N \\
&= [\{(f_A(x)\cap N)\cup M\}\cap\{(f_A(z)\cap N)\cup M\}]\cap N \\
&= [f_A^*(x)\cap f_A^*(z)]\cap N \\
&= f_A^*(x)\cap f_A^*(z)\cap N.
\end{aligned}$$

Let $x, y \in S$ such that $x \le y$. Then $f_A^*(x)\cup M \supseteq f_A^*(y)\cap N$. Indeed. Thus

$$\begin{aligned}
f_A^*(x)\cup M &= \{(f_A(x)\cap N)\cup M\}\cup M \\
&= \{(f_A(x)\cap N)\cup M\} \\
&= \{(f_A(x)\cup M)\cap (N\cup M)\} \\
&= \{(f_A(x)\cup M)\cap N\} \\
&= \{(f_A(x)\cup M)\cup M\}\cap N \\
&\supseteq \{(f_A(y)\cap N)\cup M\}\cap N \\
&= f_A^*(y)\cap N.
\end{aligned}$$

Hence f_A^* is an (M, N)-int-soft generalized bi-hyperideal of S over U.

Corollary 4.21. If $\{f_{A_i} \mid i \in I\}$ is a family of (M, N)-int-soft generalized bi-hyperideal of an ordered semihypergroup S over U. Then $f_A^* = \bigcap_{i\in I} f_{A_i}^*$ is an (M, N)-int-soft generalized bi-hyperideal of S over U.

Theorem 4.22. A soft set f_A satisfies condition (2) of Definition 4.3 is an (M, N)-int-soft generalized bi-hyperideal of S over U if and only if $f_A \widetilde{\odot}^* \mathcal{S}_\mathcal{S} \widetilde{\odot}^* f_A \widetilde{\subseteq} f_A^*$.

Proof. Suppose that f_A is an (M, N)-int-soft generalized bi-hyperideal of S over U. Let $x \in S$. If $A_x = \emptyset$. Then $\left(f_A \widetilde{\odot}^* \mathcal{S}_\mathcal{S} \widetilde{\odot}^* f_A\right)(x) \widetilde{\subseteq} f_A^*(x)$. Let $A_x \neq \emptyset$, then there exist $a, b \in S$ such that $x \leq a \circ b$. So $(a, b) \in A_x$. Thus

$$
\begin{aligned}
&\left(f_A \widetilde{\odot^*} \mathcal{S}_{\mathcal{S}} \widetilde{\odot^*} f_A\right)(x) \\
&= \left\{\left(f_A \widetilde{\odot}\left(\mathcal{S}_{\mathcal{S}} \widetilde{\odot^*} f_A\right)\right)(x) \cap N\right\} \cup M \\
&= \left(\left(\bigcup_{(a,b)\in A_x}\left\{f_A(a) \cap \left(\mathcal{S}_{\mathcal{S}} \widetilde{\odot^*} f_A\right)(b)\right\}\right) \cap N\right) \cup M \\
&= \left(\bigcup_{(a,b)\in A_x}\left\{f_A(a) \cap \left[\left(\left(\bigcup_{(c,d)\in A_b}\left\{\mathcal{S}_{\mathcal{S}}(c) \cap f_A(d)\right\}\right) \cap N\right) \cup M\right]\right\} \cap N\right) \cup M \\
&= \left(\bigcup_{(a,b)\in A_x}\left\{f_A(a) \cap \left[\left(\left(\bigcup_{(c,d)\in A_b} f_A(d)\right) \cap N\right) \cup M\right]\right\} \cap N\right) \cup M \\
&= \left(\left(\bigcup_{(a,b)\in A_x}\left\{\bigcup_{(c,d)\in A_b}\left[f_A(a) \cap f_A(d)\right] \cap N\right\} \cup M\right) \cap N\right) \cup M \\
&= \left(\left(\bigcup_{(a,b)\in A_x}\left\{\bigcup_{(c,d)\in A_b}\left[f_A(a) \cap f_A(d) \cup M\right] \cap N\right\}\right) \cap N\right) \cup M \\
&= \left(\bigcup_{(a,b)\in A_x}\left\{\bigcup_{(c,d)\in A_b}\left[f_A(a) \cap f_A(d) \cup M\right] \cap N\right\}\right) \\
&\subseteq \left(\bigcup_{x\leq a\circ b\leq a\circ c\circ d}\left\{f_A(x) \cap N\right\} \cup M\right) \\
&= \left(f_A(x) \cap N\right) \cup M \\
&= f_A^*(x).
\end{aligned}
$$

Thus $f_A \widetilde{\odot^*} \mathcal{S}_{\mathcal{S}} \widetilde{\odot^*} f_A \widetilde{\subseteq} f_A^*$.

Conversely, assume that $f_A^* \widetilde{\supseteq} f_A \widetilde{\odot^*} \mathcal{S}_{\mathcal{S}} \widetilde{\odot^*} f_A$ and $x, y, z \in S$. Then for every $\beta \in x \circ y \circ z$, we have

$$
\begin{aligned}
\left(f_A(\beta) \cap N\right) \cup M = f_A^*(\beta) &\\
&\widetilde{\supseteq} \left(f_A \widetilde{\odot^*} \mathcal{S}_{\mathcal{S}} \widetilde{\odot^*} f_A\right)(\beta) \\
&= \left(\left(\bigcup_{(x,p)\in A_\beta}\left\{f_A(x) \cap \left(\mathcal{S}_{\mathcal{S}} \widetilde{\odot^*} f_A\right)(p)\right\}\right) \cap N\right) \cup M
\end{aligned}
$$

(because there exist $p \in y \circ z$ such that $\beta \leq x \circ p$)

$$\supseteq \left(\left(f_A(x) \cap \left(\mathcal{S}_\mathcal{S} \widetilde{\odot^*} f_A \right)(p) \right) \cap N \right) \cup M$$
$$\supseteq \left(\left(f_A(x) \cap \left[\left(\bigcup_{(y,z) \in A_p} \{ \mathcal{S}_\mathcal{S}(y) \cap f_A(z) \} \cap N \right) \right] \cup M \right) \cap N \right) \cup M$$
$$\supseteq ((f_A(x) \cap ([f_A(z) \cap N] \cup M)) \cap N) \cup M$$
$$\supseteq (((f_A(x) \cap f_A(z)) \cup M) \cap N) \cup M$$
$$\supseteq ((f_A(x) \cap f_A(z) \cap N) \cap N)$$
$$= f_A(x) \cap f_A(z) \cap N.$$

Thus $\bigcap_{\beta \in x \circ y \circ z} f_A(\beta) \cup M \supseteq f_A(x) \cap f_A(z) \cap N$. Thus f_A is an (M, N)-int-soft generalized bi-hyperideal of S over U.

5 Conclusion

In this paper, we have presented a detail theoretical study of intersectional soft sets. We introduced the notion of (M, N)-int-soft generalized bi-hyperideals of ordered semihypergroups and studied them. When $M = \emptyset$ and $N = U$, we meet intersectional soft generalized bi-hyperideals. From this analysis, we say that (M, N)-int-soft generalized bi-hyperideals are more general concept than usual intersectional soft ones. We characterized ordered semihypergroups in the framework of (M, N)-int-soft generalized bi-hyperideals. Hopefully that the obtained new characterizations will be very useful for future study of ordered semihypergroups. In future we will define other (M, N)-int-soft hyperideals of ordered semihypergroups and study their applications.

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Soft Sub Spaces and Soft b-Separation Axioms in Binary Soft Topological Spaces

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Abstract – In this article, we introduce binary soft pre-separation axioms in binary soft topological space along with several properties of binary soft pre$\tau_{\triangle_i}$, i = 0; 1; 2, binary soft pre regular, binary soft pre$\tau_{\triangle_3}$, binary soft pre normal and binary soft $\tau_{\triangle_4}$ axiom using binary soft points. We also mention some binary soft invariance properties namely binary soft topological property and binary soft hereditary property. We hope that these results will be useful for the future study on binary soft topology to carry out general background for the practical applications and to solve the thorny problems containing doubts in different grounds.

Keywords – *Binary soft topology, binary soft pre-open sets, binary soft pre closed sets, binary soft pre separation axioms.*

1 Introduction

The concept of soft sets was first introduced by Molodtsov [3] in 1999 as a general mathematical technique for dealing with uncertain substances. In [3,4] Molodtsov magnificently applied the soft theory in numerous ways, such as smoothness of functions, game theory, operations research, Riemann integration, Perron integration, probability, theory of measurement, and so on. Point soft set topology deals with a non-empty set X

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together with a collection τ of sub set X under some set of parameters satisfying certain conditions. Such a collection τ is called a soft topological structure on X. General soft topology studied the characteristics of sub set of Xby using the members of. Therefore the study of point soft topology can be thought of the study of information. But in the real world situation there may be two or more universal sets. Our attempt is to introduce a single structure which carries the sub sets of X and Y for studying the information about ordered pair of sub sets of X and Y. Such a structure is called a binary soft structure from X to Y.

In 2016 Açıkgöz andTas [1] introduced the notion of binary soft set theory on two master sets and studied some basic characteristics. In prolongation, Benchalli et al. [2] planned the idea of binary soft topology and linked fundamental properties which are defined over two master sets with appropriate parameters. Benchalli et al. [6] threw his detailed discussion on Binary Soft Topological. Kalaichelvi and Malini [7] beautifully discussed Application of Fuzzy Soft Sets to Investment Decision and also discussed some more results related to this particular field. Özgür and Taş, [8] studied some more Application of Fuzzy Soft Sets to Investment Decision Making Problem. Taş et al. [9] worked over An Application of Soft Set and Fuzzy Soft Set Theories to Stock Management Alcantud et al. [10] carefully discussed Valuation Fuzzy Soft Sets: A Flexible Fuzzy Soft Set Based Decision Making Procedure for the Valuation of Assets. Çağman and Enginoğlu [11] attractively explored Soft Matrix Theory and some very basic results related to it and its Decision Making.

In continuation, in the present paper we have defined and explored several properties of binary soft b-$\tau_{\triangle_i}, i = 0; 1; 2$ binary soft b-regular, binary soft b-$\tau_{\triangle_3}$, binary soft b-normal and binary soft b-$\tau_{\triangle_4}$ axioms using binary soft points. Also, we have talked over some binary soft invariance properties i.e. binary soft topological property and binary soft hereditary property in binary soft topological spaces.

The arrangement of this paper is as follows: Section 1 briefly reviews some basic concepts about soft sets, binary soft sets and their related properties; Section 2 some hereditary properties are discussed in a beautiful way. Section 3 is devoted to Binary Soft b-Separation Axioms. Section 4 is devoted to Binary Soft b-Regular, Binary Soft b-Normal and Binary b-Soft $\boldsymbol{\tau}_{\Delta_i}$ (i=4, 3) Spaces.

2. Preliminaries

Definition 2.1. [5] Let X be an initial universe and let E be a set of parameters. Let P(X) denote the power set of X and let A be a nonempty subset ofE. A pair (F, A) iscalled a soft set over X, where F is a mapping given by $F: A \rightarrow P(X)$. In other words, a soft set over X is a parameterized family of subsets of the universe X. For $\in A$,F (ε) may be considered as the set of ε-approximate elements of the soft set (F, A). Clearly, a soft set is not a set.

Let U_1, U_2be two initial universe sets and E be a set of parameters.

Let$P(U_1)$, $P(U_2)$ denote the power set of U_1, U_2 respectively. Also, let $A, B, C \subseteq E$.

Definition 2.2. [1] A pair (F, A) is said to be a binary soft set over $U_1, U_2$where F is defined as below:

$F: A \to P(U_1) \times P(U_2)$, $F(e) = (X, Y)$ for each $e \in A$ such that $X \subseteq U_1, Y \subseteq U_2$.

Definition 2.3. [1] A binary soft set (G, A) over U_1, U_2 is called a binary absolute soft set, denoted by$\tilde{\tilde{A}}$if F (e) =(U_1, U_2) for each$e \in A$.

Definition 2.4. [1] The intersection of two binary soft sets of (F, A) and (G, B) over the common U_1, U_2 is the binary soft set (H, C), where $C = A \cap B$ and for all $e \in C$

$$H(e) = \begin{cases} (X_1, Y_1) \text{ if } e \in A - B \\ (X_2, Y_2) \text{ if } e \in B - A \\ (X_1 \cup X_2, Y_1 \cup Y_2) \text{ if } e \in A \cap B \end{cases}$$

Such that $F(e) = (X_1, Y_1)$ for each $e \in A$ and $G(e) = (X_2, Y_2)$ for each$e \in B$. We denote it $(F, A) \tilde{\tilde{\cup}} (G, A) = (H, C)$

Definition 2.5. [1] The intersection of two binary soft sets (F, A) and (G, B) over a common U_1, U_2 is the binary soft set (H, C), where

$$C = A \cap B, \text{ and } H(e) = (X_1 \cap X_2, Y_1 \cap Y_2)$$

for each $e \in C$ such that $F(e) = (X_1, Y_1)$ for each $e \in A$ and $G(e) = (X_2, Y_2)$ for each $e \in B$. We denote it as $(F, A) \tilde{\tilde{\cap}} (G, B) = (H, C)$

Definition 2.6. [1] Let (F, A) and (G, B) be two binary soft sets over a commonU_1, U_2. (F, A) is called a binary soft subset of (G, B) if

(i) $A \subseteq B$,
(ii) $X_1 \subseteq X_2$ and $Y_1 \subseteq Y_2$ Such that $F(e) = (X_1, Y_1)$, $G(e) = X_2, Y_2$ for each$e \in A$.

We denote it as $(F, A) \tilde{\tilde{\subseteq}} (G, B)$.

Definition 2.7. [1] A binary soft set (F, A) over U_1, U_2 is called a binary null soft set, denoted by if $F(e) = (\varphi, \varphi)$ for each $e \in A$.

Definition 2.8. [1] The difference of two binary soft sets (F, A) and (G, A) over the Common U_1, U_2 is the binary soft set (H, A), where $H(e)$ $(X_1 - X_2, Y_1 - Y_2)$ for each $e \in A$ such that $(F, A) = (X_1, Y_1)$ and $(G, A) = (X_2, Y_2)$.

Definition 2.9. [2] Let $\tau_\triangle$ be the collection of binary soft sets over U_1, U_2 then $\tau_\triangle$ issaid to be a binary soft topology on U_1, U_2 if

(i) $\tilde{\tilde{\varphi}}, \tilde{\tilde{X}} \in \tau_\triangle$
(ii) The union of any member of binary soft sets in $\tau_\triangle$ belongs to $\tau_\triangle$
(iii) The intersection of any two binary soft sets in $\tau_\triangle$ belongs to$\tau_\triangle$

Then $(U_1, U_2, \tau_\triangle, E)$ is called a binary soft topological space over U_1, U_2.

Definition 2.10. [2] Let $(U_1, U_2, \tau_\triangle, E)$ be a binary soft topological spaces on $\widetilde{X}$ over $U_1 \times U_2$ and $\widetilde{Y}$ be non empty binary soft subset of $\widetilde{\widetilde{X}}$. Then $\tau_{\triangle_Y} = \{^Y (F, E)/(F, E) \in \Delta$ is said to be the binary soft relative topology on $\widetilde{\widetilde{Y}}$and $(\widetilde{\widetilde{Y}}, \tau_{\triangle_Y}, E)$ is called a binary soft subspace of $(U_1, U_2, \tau_\Delta, E)$. We can easily verify that $\tau_{\triangle_Y}$ is a binary soft topology on $\widetilde{\widetilde{Y}}$.

Example 2.1. [2] Any binary soft subspace of a binary soft indiscrete topological space is binary soft indiscrete topological space.

Definition 2.11. Let (F, A) be any binary soft sub set of a binary soft topological space $(U_1, U_2, \tau_\triangle, E)$then (F, A) is called

1) Binary soft b-open set of $(U_1, U_2, \tau_\triangle, E)$ if$(F, A) \subseteq cl(int((F, A) \cup in(cl((F, A)$
2) Binary soft b-closed set of $(U_1, U_2, \tau_\triangle, E)$ if$(F, A) \supseteq cl(int(F, A)))in(cl(F, A))))$)

The set of all binary *b-open soft* sets is denoted by BSBO(U) and the set of all binary b-closed sets is denoted by BSBO(U).

Proposition 2.1. Let $(U_1, U_2, \tau_\Delta, E)$ be a binary soft topological spaces on $\widetilde{X}$ over $U_1 \times U_2$ and $\widetilde{Y}$ be a non-empty binary soft subset of $\widetilde{\widetilde{X}}$. Then $(U_1, U_2, \tau_{\triangle_Y}, \alpha)$ is subspace of $(U_1, U_2, \tau_{\triangle_Y}, E)$ for each $\alpha \widetilde{\widetilde{\in}} E$.

Proof. Let $(U_1, U_2, \tau_{\triangle_Y}, \alpha)$ is a binary soft topological space for each$\alpha \in E$. Now by definition for any $\alpha \in E$

$$\begin{aligned} \tau_{\triangle_Y} &= \{^Y F(\alpha)/(F, E) \text{is binary soft b} - \text{open set}\} \\ &= \{\widetilde{\widetilde{Y}} \cap F(\alpha)/(F, E) \text{ isbinary soft b} - \text{open set}\} \\ &= \{\widetilde{\widetilde{Y}} \cap F(\alpha)/(F, E)\text{isbinary soft b} - \text{open set} \\ &= \{\widetilde{\widetilde{Y}} \cap F(\alpha)/F(\alpha) \in \tau_{\triangle_\alpha}\} \end{aligned}$$

Thus $(U_1, U_2, \tau_{\triangle_Y}, \alpha)$ is a subspace of$(U_1, U_2, \tau_\Delta, \alpha)$.

Proposition 2.2. Let $(U_1, U_2, \tau_{\triangle_Y}, E)$ be a binary soft subspace of a binary softTopological space $(U_1, U_2, \tau_\Delta, E)$ and (G, E) be a binary soft b-open in $\widetilde{Y}$. If $\widetilde{\widetilde{Y}} \in \tau_\Delta$, Then (G, E) $\in \tau_\Delta$.

Proof. Let (G, E) be a binary soft b-open set in $\widetilde{\widetilde{Y}}$, then there exists a binary soft b-open set (H, E) in $\widetilde{\widetilde{X}}$ over $U_1 \times U_2$ such that (G,E) = $\widetilde{\widetilde{Y}} \cap (H, E)$. Now, if $\widetilde{\widetilde{Y}} \in \tau_\Delta$, then $\widetilde{\widetilde{Y}} \cap (H, E) \in \tau_\Delta$ by the third axiom of the definition of binary soft topological space and hence (G, E) $\in \tau_\Delta$.

Proposition 2.3. Let $(U_1, U_2, \tau_{\triangle_Y}, E)$ be a binary soft subspace of a binary soft topological space $(U_1, U_2, \tau_\Delta, E)$ and (G, E) be a binary soft b-open set of $\widetilde{\widetilde{X}}$ over $U_1 \times U_2$, then

(i) (G, E) is binary soft b-open in $\widetilde{\widetilde{Y}}$ if and only if $(G, E) = \widetilde{\widetilde{Y}} \mathbin{\widetilde{\widetilde{\cap}}} (H, E)$ for some $(H, E) \in \tau_\Delta$.
(ii) (G, E) is binary soft b-closed in $\widetilde{\widetilde{Y}}$ if and only if $(G, E) = \widetilde{\widetilde{Y}} \mathbin{\widetilde{\widetilde{\cap}}} (H, E)$ for some binary soft b-closed set in $(H, E) \in \widetilde{\widetilde{X}}$ over $U_1 \times U_2$.

Proof. (i) Follows from the definition of binary soft subspace.

(ii) If (G, E) is binary soft b-closed in $\tilde{\tilde{\tilde{Y}}}$ then we have $(G, E) = \tilde{\tilde{\tilde{Y}}}$, then we have $(G, E) = \tilde{\tilde{\tilde{Y}}} - (H, E)$, for some binary soft b-open $(H, E) \in \tau_{\Delta_Y}$, now $(H, E) = \tilde{\tilde{\tilde{Y}}} \tilde{\tilde{\cap}} (H, E)$ for some binary soft b-open $(K, E) \in \tau_\Delta$ for any $\beta \in E$,

$$
\begin{aligned}
G(\beta) &= \tilde{\tilde{\tilde{Y}}}(\beta) - H(\beta) \\
&= \tilde{\tilde{\tilde{Y}}} - H(\beta) \\
&= \tilde{\tilde{\tilde{Y}}} - [\, \tilde{\tilde{\tilde{Y}}}(\beta) \cap K(\beta)] \\
&= \tilde{\tilde{\tilde{Y}}} - [\, \tilde{\tilde{\tilde{Y}}} \cap K(\beta)] \\
&= \tilde{\tilde{\tilde{Y}}} - K(\beta) \\
&= \tilde{\tilde{\tilde{Y}}} \tilde{\tilde{\cap}} (\tilde{\tilde{\tilde{X}}} - K(\beta))] \\
&= \tilde{\tilde{\tilde{Y}}} \tilde{\tilde{\cap}} [K(\beta)] \\
&= \tilde{\tilde{\tilde{Y}}}(\beta) \tilde{\tilde{\cap}} [K(\beta)]^C
\end{aligned}
$$

Thus $(G, E) = \tilde{\tilde{\tilde{Y}}}(\beta) \tilde{\tilde{\cap}} [K(\beta)]^C$ Where $(K, E)^C$ is binary soft b-closed set in $\tilde{\tilde{\tilde{X}}}$ over $U_1 \times U_2$ as $(K, E) \in \tau_\Delta$.

Conversely, assume that $(G, E) = \tilde{\tilde{\tilde{Y}}} \cap (H, E)$ for some binary soft b-closed set (H, E) in $\tilde{\tilde{\tilde{X}}}$ over $U_1 \times U_2$ which means that$(H, E) \in \tau_\Delta$. Now if $(H, E) = \tilde{\tilde{\tilde{X}}} - (K, E)$ where $(K, E) \tilde{\tilde{\in}} \tau_\Delta$ then for any $\beta \tilde{\tilde{\in}} E$,

$$
\begin{aligned}
G(\beta) &= \tilde{\tilde{\tilde{Y}}}(\beta) \tilde{\tilde{\cap}} H(\beta) \\
&= \tilde{\tilde{\tilde{Y}}} \tilde{\tilde{\cap}} H(\beta) \\
&= \tilde{\tilde{\tilde{Y}}} \tilde{\tilde{\cap}} [\tilde{\tilde{\tilde{X}}} - K(\beta)] \\
&= \tilde{\tilde{\tilde{Y}}} - [\tilde{\tilde{\tilde{Y}}} \tilde{\tilde{\cap}} K(\beta)] \\
&= \tilde{\tilde{\tilde{Y}}}(\beta) - [\tilde{\tilde{\tilde{Y}}}(\beta) \tilde{\tilde{\cap}} K(\beta)] \\
&= \tilde{\tilde{\tilde{Y}}} - [\tilde{\tilde{\tilde{Y}}} \tilde{\tilde{\cap}} (K, E)].
\end{aligned}
$$

Since$(K, E) \in \tau_\Delta$, so $[\tilde{\tilde{\tilde{Y}}} \tilde{\tilde{\cap}} (K, E)] \in \tau_{\Delta_Y}$ and hence (G, E) is binary soft b- closed set in $\tilde{\tilde{\tilde{Y}}}$. This finishes the proof.

Let $(U_1, U_2, \tau_\Delta, E)$ be a binary soft topological space. Let $(U_1, U_2, \tau_{\Delta_Y}, E)$ be a binary soft subspace of $(U_1, U_2, \tau_\Delta, E)$. Let $(F, E) \tilde{\tilde{\subseteq}} \tilde{\tilde{\tilde{Y}}}$ be a binary soft subset of $\tilde{\tilde{\tilde{Y}}}$. Thenwe can find the binary soft b-closure of (F, E) in the space $(U_1, U_2, \tau_{\Delta_Y}, E)$. The binary soft b-closure of (F, E) in $(U_1, U_2, \tau_{\Delta_Y}, E)$ is denoted by $\overline{\overline{(F, E)}}^{y}$.

Proposition 2.4. Let $(U_1, U_2, \tau_{\Delta_Y}, E)$ be a binary soft subspace of binary soft topological space $(U_1, U_2, \tau_\Delta, E)$. Let $(F, E) \tilde{\tilde{\subseteq}} \tilde{\tilde{\tilde{Y}}}$ be a binary soft subset of $\tilde{\tilde{\tilde{Y}}}$. Thenwe have the following results as follows.

(i) $\overline{\overline{(F, E)}}^{y} = \tilde{\tilde{\tilde{Y}}} \tilde{\tilde{\cap}} \overline{\overline{(F, E)}}$.

(ii) $(F, E)^{*y} = \tilde{\tilde{\tilde{Y}}} \tilde{\tilde{\cap}} (F, E)^*$

(iii) $\underline{\underline{(F,E)}}_{y} \tilde{\tilde{\subseteq}} \tilde{\tilde{Y}} \tilde{\tilde{\cap}} \underline{\underline{(F,E)}}$

Proof. (i) To prove, let $\overline{\overline{(F,E)}}^{y} = \tilde{\tilde{Y}} \tilde{\tilde{\cap}} \overline{\overline{(F,E)}}$. We have $\overline{\overline{(F,E)}}^{y}$ =the binary soft intersection of all thebinary soft b-closed sets containing (F, E) = $\tilde{\tilde{\cap}}${(G, E) $_{y}$: (G, E) $_{y}$ is τ_{Δ_Y}-binary soft b-closed set and (G, E) $_{y}\tilde{\tilde{\supseteq}}$(F, E)} = $\tilde{\tilde{\cap}}${$\tilde{\tilde{Y}} \tilde{\tilde{\cap}}$ (G, E): (G, E) is -binary soft b-closed set and $\tilde{\tilde{Y}} \tilde{\tilde{\cap}}$ (G, E)$\tilde{\tilde{\supseteq}}$ (F, E)} = $\tilde{\tilde{\cap}}${$\tilde{\tilde{Y}} \tilde{\tilde{\cap}}$ (G, E): (G, E) is τ_{Δ_Y}-binary soft b-closed set and (G, E)$\tilde{\tilde{\supseteq}}$ (F, E)}= $\tilde{\tilde{Y}} \tilde{\tilde{\cap}}${$\tilde{\tilde{\cap}}$ (G, E): (G, E) is τ_{Δ} -binary soft b-closed set and (G, E)$\tilde{\tilde{\supseteq}}$ (F, E)} =$\tilde{\tilde{Y}} \tilde{\tilde{\cap}} \overline{\overline{(F,E)}}$. Thus $\overline{\overline{(F,E)}}^{y} = \tilde{\tilde{Y}} \tilde{\tilde{\cap}} \overline{\overline{(F,E)}}$.

(ii) To prove that (F, E) ^{y}= $\tilde{\tilde{Y}} \tilde{\tilde{\cap}} (F,E)^{*}$,we know that, (F,E) The binary soft union of all the τ_{Δ_Y}-binary soft b-open Sets contained in (F,E)= $\tilde{\tilde{\cup}}$ {(H, E): (H, E) isτ_{Δ_Y}-binary soft b-open and (H, E)$\tilde{\tilde{\supseteq}}$ (F, E)}= $\tilde{\tilde{\cup}}$ {(H, E) =$\tilde{\tilde{Y}} \tilde{\tilde{\cap}}$ (K, E): (K, E) is τ_{Δ}-binary soft b-open set and $\tilde{\tilde{Y}} \tilde{\tilde{\cap}}$ (K, E) $\tilde{\tilde{\supseteq}}$ (F, E)}. Also we know that (F, E) ^{e} = $\tilde{\tilde{Y}} \tilde{\tilde{\cap}}$ [$\tilde{\tilde{\cup}}$(L, E) γ]: (L, E) is τ_{Δ}-binary soft b-open set and (L, E)γ $\tilde{\tilde{\subseteq}}$ (F, E)}. Now let $(M,E) \tilde{\tilde{\in}} \tilde{\tilde{Y}} \tilde{\tilde{\cap}} (F,E)^{*}$ which implies $(M,E) \tilde{\tilde{\in}} \tilde{\tilde{Y}}$ and $(M,E) \tilde{\tilde{\in}} (F,E)^{*}(M,E) \tilde{\tilde{\in}} \tilde{\tilde{Y}}$and $(M,E) \tilde{\tilde{\in}}\tilde{\tilde{\cup}}$(L,E) γ : (L, E) γ is τ_{Δ} -binary soft b-open set and $(L,E)\,\gamma \tilde{\tilde{\subseteq}} (F,E) \Rightarrow (M,E) \tilde{\tilde{\in}} \tilde{\tilde{Y}}and(M,E) \tilde{\tilde{\in}} (L,E)\,\gamma$, where $(L,E)_{\gamma_i}$, is τ_{Δ} -b-open and $(L,E)_{\gamma_i} \tilde{\tilde{\subseteq}} (F,E) \Rightarrow (M,E) \tilde{\tilde{\in}} \tilde{\tilde{Y}} \tilde{\tilde{\cap}} (L,E)_{\gamma_i}$. Where $(L,E)_{\gamma_i}$ isτ_{Δ}-b-open and $(L,E)_{\gamma_i} \tilde{\tilde{\subseteq}} (F,E)$ that is $\tilde{\tilde{Y}} \tilde{\tilde{\cap}} (L,E)_{\gamma_i} \tilde{\tilde{\subseteq}} \tilde{\tilde{Y}} \tilde{\tilde{\cap}} (F,E) \Rightarrow (M,E) \tilde{\tilde{\in}}\tilde{\tilde{\cup}}$ {$\tilde{\tilde{Y}} \tilde{\tilde{\cap}} (K,E)$: (K, E) is τ_{Δ}- b-open $\tilde{\tilde{Y}} \tilde{\tilde{\cap}} (K,E) \tilde{\tilde{\subseteq}} (F,E)$} $\Rightarrow (M,E) \tilde{\tilde{\in}} (F,E)^{*y}$. Thus $(M,E) \tilde{\tilde{\in}} \tilde{\tilde{Y}} \tilde{\tilde{\cap}} (F,E)^{*}$ which implies $(M,E) \tilde{\tilde{\in}} (F,E)^{*}$. Therefore $\tilde{\tilde{Y}} \tilde{\tilde{\cap}} (F,E)^{*} \tilde{\tilde{\subseteq}} (F,E)^{*}$.

(iii) To prove, $\underline{\underline{(F,E)}}_{y} \tilde{\tilde{\subseteq}} \tilde{\tilde{Y}} \tilde{\tilde{\cap}} \underline{\underline{(F,E)}}$. Now consider $\underline{\underline{(F,E)}}_{y} = \overline{\overline{(F,E)}}^{y} \tilde{\tilde{\cap}} \overline{\overline{\tilde{\tilde{Y}} - (F,E)}}^{y} \Rightarrow$ $[\tilde{\tilde{Y}} \tilde{\tilde{\cap}} \overline{\overline{(F,E)}}] \tilde{\tilde{\cap}} \tilde{\tilde{Y}} \tilde{\tilde{\cap}} [\overline{\overline{\tilde{\tilde{Y}} - (F,E)}}]$. Since using the result (i) $[\tilde{\tilde{Y}} \tilde{\tilde{\cap}} \overline{\overline{(F,E)}}] \tilde{\tilde{\cap}} \tilde{\tilde{Y}} \tilde{\tilde{\cap}} [\overline{\overline{\tilde{\tilde{Y}} - (F,E)}}]$. (since $\tilde{\tilde{Y}} \tilde{\tilde{\subseteq}} \tilde{\tilde{X}}$). $\tilde{\tilde{\subseteq}} \tilde{\tilde{Y}} \tilde{\tilde{\cap}} \overline{\overline{(F,E)}}] \tilde{\tilde{\cap}} [\overline{\overline{\tilde{\tilde{X}} - (F,E)}}]$=$\tilde{\tilde{Y}} \tilde{\tilde{\cap}} \underline{\underline{(F,E)}}$. Thus $\underline{\underline{(F,E)}}_{y} \tilde{\tilde{\subseteq}} \tilde{\tilde{Y}} \tilde{\tilde{\cap}} \underline{\underline{(F,E)}}$

This finishes the proof.

3. Binary Soft b-Separation Axioms

In this section binary soft b-separation axioms in Binary Soft Topological Spaces are reflected**.**

Definition 3.1. Let $(U_1, U_2, \tau_{\Delta}, A)$ be a binary soft topological space of $\tilde{X}$ over $(U_1 \times U_2)$ and $F_e, G_e \tilde{\tilde{\in}} \tilde{\tilde{X}}_A$ such that $F_e \tilde{\tilde{\neq}} G_e$. Then the binary soft topological space is said to be a binary soft b-τ_o space denoted as b-T_{Δ_o}.If there exists at least one binary soft b-open set (F_1, A) or (F_2, A) such that $F_e \tilde{\tilde{\in}} (F_1, A)$, $G_e \tilde{\tilde{\notin}} (F_1, A)$ or $F_e \tilde{\tilde{\notin}} (F_2, A)$, $G_e \tilde{\tilde{\in}} (F_2, A)$.

Definition 3.2. Let $(U_1, U_2, \tau_{\Delta}, A)$ be a binary soft topological space of $\tilde{X}$ over $(U_1 \times U_2)$ and $F_e, G_e \tilde{\tilde{\in}} \tilde{\tilde{X}}_A$ such that $F_e \tilde{\tilde{\neq}} G_e$. Then the binary soft topological space is said to be a binary soft b-τ_1 space denoted as b-T_{Δ_1}. If there exists at least one binary soft b-open set (F_1, A) or (F_2, A) such that $F_e \,\tilde{\tilde{\in}}\, (F_1, A)$, $G_e \,\tilde{\tilde{\notin}}\, (F_1, A)$ or $F_e \,\tilde{\tilde{\in}}\, (F_2, A)$, $G_e \,\tilde{\tilde{\notin}}\, (F_2, A)$.

Definition 3.3. Let $(U_1, U_2, \tau_\Delta, A)$ be a binary soft topological space of $\tilde{\tilde{X}}$ over $(U_1 \times U_2)$ and $F_e, G_e \,\tilde{\tilde{\in}}\, \tilde{\tilde{X}}_A$ such that $F_e \,\tilde{\tilde{\neq}}\, G_e$. Then the binary soft topological space is said to be a binary soft b-τ_2 space denoted as b-T_{Δ_2}. If there exists at least one binary soft b-open set (F_1, A) or (F_2, A) such that $F_e \,\tilde{\tilde{\in}}\, (F_1, A)$, $H_e \,\tilde{\tilde{\in}}\, (F_2, A)$ and$(F_1, E) \,\tilde{\tilde{\cap}}\, (F_2, E) = \tilde{\tilde{\varphi}}_A$.

Proposition 3.1. (i) Every b-T_{Δ_1}-space is b-T_{Δ_o}space.
(ii) Every b-T_{Δ_2}-space is b-T_{Δ_1}-space.

Proof. (i) is obvious. (ii) If $(U_1, U_2, \tau_\Delta, A)$ is a T_{Δ_2}-space then by definition for $F_e, G_e \,\tilde{\tilde{\in}}\, \tilde{\tilde{X}}_A$, $F_e \,\tilde{\tilde{\neq}}\, G_e$ there exists at least one binary soft b-open set (F_1, A) and (F_2, A) such that $F_e \,\tilde{\tilde{\in}}\, (F_1, A)$, $H_e \,\tilde{\tilde{\in}}\, (F_2, A)$ and $(F_1, E) \,\tilde{\tilde{\cap}}\, (F_2, E) = \tilde{\tilde{\varphi}}_A$. Since $(F_1, E) \,\tilde{\tilde{\cap}}\, (F_2, E) = \tilde{\tilde{\varphi}}_A$; $F_e \,\tilde{\tilde{\notin}}\, (F_2, A)$ and $G_e \,\tilde{\tilde{\notin}}\, (F_1, A)$. Thus it follows that $(U_1, U_2, \tau_\Delta, A)$ is b-T_{Δ_1}space.

Note that every b-T_{Δ_1}space is b-T_{Δ_o}space. Every b-T_{Δ_2}space is b-T_{Δ_1}space.

Proposition 3.2. Let $(U_1, U_2, \tau_\Delta, E)$ be a binary soft topological space of $\tilde{\tilde{X}}$ over $(U_1 \times U_2)$ and $\tilde{\tilde{Y}}$ be a non-empty subset of $\tilde{\tilde{X}}$. If $(U_1, U_2, \tau_\Delta, E)$ is a binary soft b-T_{Δ_o}Space then $(U_1, U_2, \tau_{\Delta_y}, E)$ is a binary soft b-T_{Δ_o}space.

Proof. Let $(U_1, U_2, \tau_\Delta, E)$ be a binary soft topological space of $\tilde{\tilde{X}}$ over $(U_1 \times U_2)$ Now let $F_e, G_e \,\tilde{\tilde{\in}}\, \tilde{\tilde{Y}}$ such that $F_e \,\tilde{\tilde{\neq}}\, G_e$. If there exist a binary soft b-open set (F_1, E) in $\tilde{\tilde{X}}$ such that $F_e \,\tilde{\tilde{\in}}\, (F_1, E)andG_e \,\tilde{\tilde{\notin}}\, (F_1, E)$. Now if $F_e \,\tilde{\tilde{\in}}\, \tilde{\tilde{Y}}$implies that$F_e \,\tilde{\tilde{\in}}\, \tilde{\tilde{Y}}$. So $F_e \,\tilde{\tilde{\in}}\, \tilde{\tilde{Y}}andF_e \,\tilde{\tilde{\in}}\, (F_1, E)$. Hence $F_e \,\tilde{\tilde{\in}}\, \tilde{\tilde{Y}} \,\tilde{\tilde{\cap}}\, (F_1, E) = [^Y(F_1, E)]$, where,$(F_1, E)$ is binary soft b-open set. That is$(F_1, E) \in \tau_\Delta$. Let$G_e \,\tilde{\tilde{\notin}}\, (F_1, E)$, this means that $G_e \,\tilde{\tilde{\notin}}\, F(\beta)$ for some $\beta \,\tilde{\tilde{\in}}\, E$. $G_e \,\tilde{\tilde{\notin}}\, \tilde{\tilde{Y}} \,\tilde{\tilde{\cap}}\, (F_1, E)_\alpha = Y_\beta(F_1, E)_\alpha$. Therefore, $G_e \,\tilde{\tilde{\notin}}\, \tilde{\tilde{Y}} \,\tilde{\tilde{\cap}}\, (F_1, E) = [^Y(F_1, E)]$. Similarly, it can prove that if $G_e \,\tilde{\tilde{\in}}\, (F_2, E)$ and $F_e \,\tilde{\tilde{\notin}}\, (F_2, E)$ then $G_e \,\tilde{\tilde{\in}}\, [^Y(F_2, E)]$ and $F_e \,\tilde{\tilde{\in}}\, [^Y(F_2, E)]$. Thus $(U_1, U_2, \tau_{\Delta_y}, E)$ is a binary soft b-T_{Δ_o}space.

Example 3.1. Let $U_1 = \{c_1, c_2, c_3\}, U_2 = \{m_1, m_2\}$ $E = \{e_1, e_2\}$ and

$$\tau_\Delta = \{\tilde{\tilde{X}}, \tilde{\tilde{\varphi}}, \{(e_1(\{c_2\}\{m_2\})),(e_2(\{c_1\}\{m_1\}))\},$$
$$\{(e_1(\{c_1\}\{m_1\})), (e_2(\{c_2\}\{m_2\}))\}, \{(e_1(\{c_1\}\{m_1\}))\}, \{\left(e_1\left(\{\tilde{\tilde{X}}\}\{\tilde{\tilde{X}}\}\right)\right), (e_2(\{c_1\}\{m_1\}))\}\}$$

where

$$(F_1, E) = \{(e_1(\{c_2\}\{m_2\}), (e_2(\{c_1\}\{m_1\}))\},$$
$$(F_2, E) = (e_1(\{c_1\}\{m_1\}), (e_2(\{c_2\}\{m_1\}))\},$$
$$(F_3, E) = \{(e_1(\{c_1\}\{m_1\}))\}$$
$$(F_4, E) = \{\left(e_1\left(\{\tilde{\tilde{X}}\}\{\tilde{\tilde{X}}\}\right)\right), (e_2(\{c_1\}\{m_1\}))\}$$

Clearly $(U_1, U_2, \tau_\Delta, E)$ is binary sot topological space of $\tilde{\tilde{\tilde{X}}}$ over $(U_1 \times U_2)$.

Note that

$$\tau_{\Delta_1} = \{\tilde{\tilde{\tilde{X}}}, \tilde{\tilde{\varphi}}, \{(e_1(\{c_1\}\{m_1\}))\}, (e_1(\{c_2\}\{m_2\}))\}$$
$$T_{\Delta_2} = \{\tilde{\tilde{\tilde{X}}}, \tilde{\tilde{\varphi}}, \{(e_2(\{c_1\}\{m_1\}))\}, (e_2(\{c_2\}\{m_2\}))\}$$

are binary soft topological spaces on $\tilde{\tilde{\tilde{X}}}$ over $(U_1 \times U_2)$. There are two pairs of distinct binary soft points namely

$F_{e_1} = \{(e_1(\{c_2\}\{m_2\})\}, G_{e_1} = \{(e_1(\{c_1\}\{m_1\})\}$ and
$F_{e_2} = \{(e_2(\{c_1\}\{m_1\})\}, G_{e_2} = \{(e_2(\{c_2\}\{m_2\})\}$.

Then for binary soft pair $F_{e_1} \neq G_{e_1}$ of points there are binary soft open sets (F_1, E) and (F_2, E) such that$F_{e_1} \tilde{\tilde{\in}} (F_1, E)$, $G_{e_1} \tilde{\tilde{\notin}} (F_1, E)$ and $G_{e_1} \tilde{\tilde{\in}} (F_1, E)$, $F_{e_1} \tilde{\tilde{\notin}} (F_2, E)$. Similarly for the pair $F_{e_2} \neq G_{e_2}$, there are binary soft b-open sets (F_1, E) and (F_2, E) such that $F_{e_2} \tilde{\tilde{\notin}} (F_2, E)$, $G_{e_2} \tilde{\tilde{\in}} (F_2, E)$ and $G_{e_2} \tilde{\tilde{\notin}} (F_1, E)$, $F_{e_2} \tilde{\tilde{\in}} (F_1, E)$. This shows that $(U_1, U_2, \tau_\Delta, E)$ is binary soft space b-T_{Δ_1}-space and hence a binary soft b-T_{Δ_o}space. Note that $(U_1, U_2, \tau_\Delta, E)$ is binary soft b-T_{Δ_2}space.

Proposition 3.3. Let $(U_1, U_2, \tau_\Delta, E)$ is binary soft topological space on $\tilde{\tilde{\tilde{X}}}$ over $(U_1 \times U_2)$. Then each binary soft point is binary soft b-closed if and only if $(U_1, U_2, \tau_\Delta, E)$ is binary soft b-T_{Δ_1}space.

Proof. Let $(U_1, U_2, \tau_\Delta, E)$ is binary soft topological space on $\tilde{\tilde{\tilde{X}}}$ over $(U_1 \times U_2)$. Now to prove let$(U_1, U_2, \tau_\Delta, E)$ is binary soft b-T_{Δ_1}space, suppose binary soft points$F_{e_1} \tilde{\tilde{\equiv}} (F, E)$, $G_{e_1} \tilde{\tilde{\equiv}} (G, E)$ are binary soft b-closed and$F_{e_1} \neq G_{e_1}$. Then $(F, E)^c$ and $(G, E)^c$ are binary soft b-open in$(U_1, U_2, \tau_\Delta, E)$. Then by definition $(F, E)^c = (F^c, E)$ where $F^c(e_1) = \tilde{\tilde{\tilde{X}}} - F(e_1)$ and$(G, E)^c = (G^c, E)$, where${G^c}_{e_1} = \tilde{\tilde{\tilde{X}}} - G(e_1)$. Since $F(e_1) \tilde{\tilde{\cap}} G(e_1) = \tilde{\tilde{\varphi}}$. This implies $F(e_1) = \tilde{\tilde{\tilde{X}}} - G(e_1) = {G^c}_{e_1} \forall$ e. This implies $F(e_1) = (F, E) \tilde{\tilde{\in}} (G, E)^c$. Similarly $G(e_1) = (G, E) \tilde{\tilde{\in}} (F, E)^c$. Thus we have$(e_1) \tilde{\tilde{\in}} (G, E)^c$, $G(e_1) \tilde{\tilde{\notin}} (G, E)^c$ and $F(e_1) \tilde{\tilde{\notin}} (F, E)^c$, $G\ (e_1) \tilde{\tilde{\in}} (F, E)^c$. This proves that $(U_1, U_2, \tau_\Delta, E)$ is binary soft b-T_1space.

Conversely, let $(U_1, U_2, \tau_\Delta, E)$ is binary soft b-T_{Δ_1}space, to prove that $F(e_1) = (F, E) \tilde{\tilde{\in}} \tilde{\tilde{\tilde{X}}}$ is binary soft pre-closed, we show that $(F, E)^c$ is binary soft b-open in$(U_1, U_2, \tau_\Delta, E)$. Let $G_{e_1} = (G, E) \tilde{\tilde{\in}} (F, E)^c$ is binary soft b-closed. Then$F_{e_1} \neq G_{e_1}$, since $(U_1, U_2, \tau_\Delta, E)$ is binary soft b-T_{Δ_1}space, there exists binary soft b-open set (L, E) such that $G(e_1) \tilde{\tilde{\in}} (L, E) \tilde{\tilde{\subseteq}} (F, E)^c$ and hence $\tilde{\tilde{U}}_{G_{e_1}} \{(L, E),\ G_{e_1} \tilde{\tilde{\in}} (F, E)^c\}$. This proves that $(F, E)^c$ is binary soft b-open in $(U_1, U_2, \tau_\Delta, E)$ that is $F_{e_1} = (F, E)$is binary soft b-closed in$(U_1, U_2, \tau_\Delta, E)$. Which completes the proof.

Proposition 3.4. Let $(U_1, U_2, \tau_\Delta, E)$ be a binary soft topological space of $\tilde{\tilde{X}}$ over $(U_1 \times U_2)$ and $F_e, G_e \tilde{\tilde{\in}} \tilde{\tilde{X}}$ such that $F_e \neq G_e$. If there exist binary soft b-open sets (F_1, E), (F_2, E) such that $F_e \tilde{\tilde{\in}} (F_1, E)$ and $G_e \tilde{\tilde{\in}} (F_2, E)^c$, then $(U_1, U_2, \tau_\Delta, E)$ is a binary soft b-T_{Δ_o}space and $(U_1, U_2, \tau_\Delta, E)$ is a binary soft b-T_{Δ_o}space for each $e \tilde{\tilde{\in}} E$.

Proof. Clearly $G_{e_1} \tilde{\tilde{\in}} (F_1, E)^c = (F_1{}^c, E)$ implies $G_{e_1} \tilde{\tilde{\notin}} (F_2, E)$similarly $F_e \tilde{\tilde{\in}} (F_2, E)^c = (F_2{}^c, E)$ implies $F_e \tilde{\tilde{\notin}} (F_2, E)$. Thus we have $F_e \tilde{\tilde{\in}} (F_1, E)$, $G_e \tilde{\tilde{\notin}} (F_1, E)$ or $G_e \tilde{\tilde{\in}} (F_2, E)$, $F_e \tilde{\tilde{\notin}} (F_2, E)$. This proves $(U_1, U_2, \tau_\Delta, E)$ is a binary soft b-T_{Δ_o}space. Now for any$\tilde{\tilde{\in}}$ E, $(U_1, U_2, \tau_\Delta, E)$ is a binary soft topological space and $F_e \tilde{\tilde{\in}} (F_1, E)$ and $G_e \tilde{\tilde{\in}} (F_1, E)^c$or $G_e \tilde{\tilde{\in}} (F_2, E)$ and $F_e \tilde{\tilde{\notin}} (F_2, E)^c$so that $F_e \tilde{\tilde{\in}} F_1(e)$, $G_e \tilde{\tilde{\notin}} F_1(e)$, $G_e \tilde{\tilde{\in}} F_2(e)$, $G_e \tilde{\tilde{\notin}} F_2(e)$. Thus $(U_1, U_2, \tau_\Delta, E)$ is a binary soft b-T_{Δ_o}space.

Proposition 3.5. Let $(U_1, U_2, \tau_\Delta, E)$ be a binary soft topological space of $\tilde{\tilde{X}}$ over $(U_1 \times U_2)$ and $F_e, G_e \tilde{\tilde{\in}} \tilde{\tilde{X}}$ such that $F_e \neq G_e$. If there exist binary soft b-open sets (F_1, E), (F_2, E) such that $F_e \tilde{\tilde{\in}} (F_1, E)$ and $G_e \tilde{\tilde{\in}} (F_1, E)^c$or $F_e \tilde{\tilde{\in}} (F_2, E)$ and $G_e \tilde{\tilde{\in}} (F_2, E)$, then $(U_1, U_2, \tau_\Delta, E)$ is a binary soft b-T_{Δ_o}space and $(U_1, U_2, \tau_\Delta, E)$ is a binary soft b-T_{Δ_o}space for each $e \tilde{\tilde{\in}} E$.

Proof. Clearly $G_e \tilde{\tilde{\in}} (F_1, E)^c = (F_1{}^c, E)$ implies $G_e \tilde{\tilde{\notin}} (F_2, E)$ similarly $F_e \tilde{\tilde{\in}} (F_2, E)^c = (F_2{}^c, E)$ implies $F_e \tilde{\tilde{\notin}} (F_2, E)$. Thus we have$F_e \tilde{\tilde{\in}} (F_1, E)$, $G_e \tilde{\tilde{\notin}} (F_1, E)$ or $G_e \tilde{\tilde{\in}} (F_2, E)$, $F_e \tilde{\tilde{\notin}} (F_2, E)$. This proves $(U_1, U_2, \tau_\Delta, E)$ is a binary soft b-T_{Δ_o}space. Now, for any $\tilde{\tilde{\in}}$ E, $(U_1, U_2, \tau_\Delta, E)$ is a binary soft topological space and $F_e \tilde{\tilde{\in}} (F_1, E)$ and $G_e \tilde{\tilde{\in}} (F_1, E)^c$ or $G_e \tilde{\tilde{\in}} (F_2, E)$ and $F_e \tilde{\tilde{\notin}} (F_2, E)^c$. So that $F_e \tilde{\tilde{\in}} F_1(e)$, $G_e \tilde{\tilde{\notin}} F_1(e)$ or $G_e \tilde{\tilde{\in}} F_2(e)$, $F_e \tilde{\tilde{\notin}} F_1(e)$. Thus $(U_1, U_2, \tau_\Delta, E)$ is a binary soft b-T_{Δ_o}space.

Proposition 3.6. Let $(U_1, U_2, \tau_\Delta, E)$ be a binary soft topological space of $\tilde{\tilde{X}}$ over $(U_1 \times U_2)$ and $F_e, G_e \tilde{\tilde{\in}} \tilde{\tilde{X}}$ such that $F_e \neq G_e$. If there exist binary soft b-open sets (F_1, E), (F_2, E) such that $F_e \tilde{\tilde{\in}} (F_1, E)$ and $G_e \tilde{\tilde{\in}} (F_1, E)^c$or $G_e \tilde{\tilde{\in}} (F_2, E)$ and $F_e \tilde{\tilde{\in}} (F_2, E)^c$, then $(U_1, U_2, \tau_\Delta, E)$ is a binary soft b-T_{Δ_o}space and $(U_1, U_2, \tau_\Delta, E)$ is a binary soft b-T_{Δ_1}space and $(U_1, U_2, \tau_\Delta, E)$ is a binary soft b-T_{Δ_1}space for each $e \tilde{\tilde{\in}} E$.

Proof. The proof is similar to the proof **9**.

Now we shill discuss some of the binary soft hereditary properties of b-T_{Δ_i} $(i = 0, 1)$ spaces.

Proposition 3.7. Let $(U_1, U_2, \tau_\Delta, E)$ be a binary soft topological space of $\tilde{\tilde{X}}$ over $(U_1 \times U_2)$ and $\tilde{\tilde{Y}} \tilde{\tilde{\subseteq}} \tilde{\tilde{X}}$. Then if $(U_1, U_2, \tau_\Delta, E)$ is a binary soft b-T_{Δ_o}space then $(U_1, U_2, \tau_{\Delta_y}, E)$ is binary soft b-T_{Δ_o}space.

Proof. $F_e, G_e \tilde{\tilde{\in}} \tilde{\tilde{Y}}$ such that $F_e \neq G_e$,Then$F_e, G_e \tilde{\tilde{\in}} \tilde{\tilde{X}}$. Since $(U_1, U_2, \tau_\Delta, E)$ is a binary soft b-T_{Δ_o}space, thus there exists binary soft b-open sets (F, E) and (G, E) in $(U_1, U_2, \tau_\Delta, E)$ such that $F_e \tilde{\tilde{\in}} (F,E)$ and $G_e \tilde{\tilde{\notin}} (F,E)$ or $G_e \tilde{\tilde{\in}} (G,E)$ and $F_e \tilde{\tilde{\notin}} (G,E)$. Therefore $F_e \tilde{\tilde{\in}} \tilde{\tilde{Y}} \tilde{\tilde{\cap}} (F,E) =^Y (F,E)$. Similarly I can be shown that if $G_e \tilde{\tilde{\in}} (G,E)$ and $F_e \tilde{\tilde{\notin}} (G,E)$, then $G_e \tilde{\tilde{\in}}^Y (G,E)$ and $F_e \tilde{\tilde{\in}}^Y (G,E)$ and $F_e \tilde{\tilde{\notin}}^Y (G,E)$. Thus $(U_1, U_2, \tau_{\Delta_y}, E)$ is binary soft b-T_{Δ_o}space.

Proposition 3.8. Let $(U_1, U_2, \tau_\Delta, E)$ be a binary soft topological space of $\tilde{\tilde{X}}$ over $(U_1 \times U_2)$ and $\tilde{\tilde{Y}} \tilde{\tilde{\subseteq}} \tilde{\tilde{X}}$. Then if $(U_1, U_2, \tau_\Delta, E)$ is a binary soft b-T_{Δ_1}space then $(U_1, U_2, \tau_{\Delta_y}, E)$ is binary soft b-T_{Δ_1}space.

Proof. The proof is similar to the proof 11.

Proposition 3.9. Let $(U_1, U_2, \tau_\Delta, E)$ be a binary soft topological space on $\tilde{\tilde{X}}$ over $(U_1 \times U_2)$. If $(U_1, U_2, \tau_\Delta, E)$ is a binary soft bτ_{Δ_2}space on $\tilde{\tilde{X}}$over $(U_1 \times U_2)$ then $(U_1, U_2, \tau_{\Delta_e}, E)$ is binary soft b-T_{Δ_2}space for each $e \tilde{\tilde{\in}} E$.

Proof. Let $(U_1, U_2, \tau_\Delta, E)$ be a binary soft topological space on $\tilde{\tilde{X}}$ over $(U_1 \times U_2)$. For any$e \tilde{\tilde{\in}} E$, $\tau_{\Delta_e} = \{F(e): (F,E) \tilde{\tilde{\in}} \tau_\Delta\}$ is a binary soft topology on $\tilde{\tilde{X}}$ over $(U_1 \times U_2)$. Let $x, y \tilde{\tilde{\in}} \tilde{\tilde{X}}$ such that $x \neq y$, since $(U_1, U_2, \tau_\Delta, E)$ is a binary soft b-T_{Δ_2}space, therefore binary soft points $F_e, G_e \tilde{\tilde{\in}} \tilde{\tilde{X}}$ such that $F_e \neq G_e$ and $x \tilde{\tilde{\in}} F(e)$, $y \tilde{\tilde{\in}} G(e)$, there exists binary soft b-open sets (F_1, E) , (F_2, E) such that $F_e \tilde{\tilde{\in}} (F_1, E)$,$G_e \tilde{\tilde{\in}} (F_2, E)$ and $(F_1, E) \tilde{\tilde{\cap}} (F_2, E) = \tilde{\tilde{\varphi}}$. Which implies that $\tilde{\tilde{\in}} F(e) \tilde{\tilde{\subseteq}} F_1(e)$, $y \tilde{\tilde{\in}} G(e) \tilde{\tilde{\subseteq}} F_2(e)$ and $F_1(e) \tilde{\tilde{\cap}} F_2(e) = \tilde{\tilde{\varphi}}$. This proves that $(U_1, U_2, \tau_{\Delta_e}, E)$ is binary soft b-T_{Δ_2}space.

Proposition 3.10. Let $(U_1, U_2, \tau_\Delta, E)$ be a binary soft topological space on $\tilde{\tilde{X}}$ over $(U_1 \times U_2)$ and $\tilde{\tilde{Y}} \tilde{\tilde{\subseteq}} \tilde{\tilde{X}}$. Then if $(U_1, U_2, \tau_\Delta, E)$ is a binary soft b-τ_{Δ_2}space then $\left(U_1, U_2, \tau_{\Delta_y}, E\right)$ is binary soft b-T_{Δ_2}space and $\left(U_1, U_2, \tau_{\Delta_e}, E\right)$ is binary soft b-T_{Δ_2}space for each $e \tilde{\tilde{\in}} E$.

Proof. Let $F_e, G_e \tilde{\tilde{\in}} \tilde{\tilde{Y}}$such that $F_e \neq G_e$. Then$F_e, G_e \tilde{\tilde{\in}} \tilde{\tilde{X}}$. Since $(U_1, U_2, \tau_\Delta, E)$ is a binary soft b-T_{Δ_2}space, thus there exists binary soft b-open sets (F_1, E) and (F_2, E) such that $F_e \tilde{\tilde{\in}} (F_1, E)$ and $G_e \tilde{\tilde{\in}} (,E)$ and $(F_1, E) \tilde{\tilde{\cap}} (F_2, E) = \tilde{\tilde{\varphi}}$. Therefore $F_e \tilde{\tilde{\in}} \tilde{\tilde{Y}} \tilde{\tilde{\cap}} (F_2, E) =^Y (F_2, E)$ and ${}^Y(F_2, E) \tilde{\tilde{\cap}}^Y (F_2, E) = \tilde{\tilde{\varphi}}$. Thus it proves that $(U_1, U_2, \tau_{\Delta_y}, E)$ is binary soft b-T_{Δ_2}space.

Proposition 3.11. Let $(U_1, U_2, \tau_\Delta, E)$ be a binary soft topological space of $\tilde{\tilde{X}}$ over $(U_1 \times U_2)$. If $(U_1, U_2, \tau_\Delta, E)$ is a binary soft b-T_{Δ_2}space and for any two binary soft points $F_e, G_e \tilde{\tilde{\in}} \tilde{\tilde{X}}$ such that $F_e \neq G_e$. Then there exist binary soft b-closed sets (F_1, E) and (F_2, E) such that $F_e \tilde{\tilde{\in}} (F_1, E)$ and $G_e \tilde{\tilde{\notin}} (F_1, E)$or $G_e \tilde{\tilde{\in}} (F_2, E)$ and $(F_1, E) \tilde{\tilde{\cup}} (F_2, E) = \tilde{\tilde{X}}$.

Proof. Let $(U_1, U_2, \tau_\Delta, E)$ be a binary soft topological space of $\tilde{\tilde{X}}$ over $(U_1 \times U_2)$. Since $(U_1, U_2, \tau_\Delta, E)$ is a binary soft b-τ_{Δ_2}space and $F_e, G_e \tilde{\tilde{\in}} \tilde{\tilde{X}}$ such that $F_e \neq G_e$ there exists binary soft b-open sets (H,E) and (L,E) such that $F_e \tilde{\tilde{\in}} (H,E)$ and $G_e \tilde{\tilde{\in}} (L,E)$ and

$(H,E)\ \tilde{\tilde{\cap}}\ (L,E) = \tilde{\tilde{\varphi}}$. Clearly $(H,E)\ \tilde{\tilde{\subseteq}}\ (L,E)^c$ and $(L,E)\ \tilde{\tilde{\subseteq}}\ (H,E)^c$. Hence $F_e\ \tilde{\tilde{\in}}\ (L,E)^c$, put $(L,E)^c = (F_1,E)$ which gives $F_e\ \tilde{\tilde{\in}}\ (F_1,E)$ and $G_e \notin (F_1,E)$. Also $G_e\ \tilde{\tilde{\in}}\ (F_1,E)^c$, then put $(H,E)^c = (F_2,E)$. Therefore $F_e\ \tilde{\tilde{\in}}\ (F_1,E)$ and $G_e\ \tilde{\tilde{\in}}\ (F_2,E)$. Moreover, $(F_1,E)\ \tilde{\tilde{\cup}}\ (F_2,E) = (L,E)^c\ \tilde{\tilde{\cup}}\ (H,E)^c = \tilde{\tilde{X}}$. Which completes the proof.

4. Binary Soft b-T_{Δ_i} (i=4,3) Spaces

In this section binary soft b-separation axioms in Binary Soft Topological Spaces are discussed**.**

In this section, we define binary soft b-regular and binary soft b-T_{Δ_i}- spaces using binary soft points. We also characterize binary soft b-regular and binary soft b-normal spaces. Moreover, we prove that binary soft b-regular and binary soft b-T_{Δ_3} properties are binary soft hereditary, whereas binary soft b- normal and binary soft b-T_{Δ_4} are binary soft b-closed hereditary properties.

Now we define binary soft b-regular space as follows:

Definition 4.1. Let (U_1,U_2,τ_Δ,E) be a binary soft topological space of $\tilde{\tilde{X}}$ over $(U_1 \times U_2)$. Let (F,E) be a binary soft b-closed set in $(U_1,U_2,\tau_\Delta,E)$and$F_e\ \tilde{\tilde{\notin}}\ (F,E)$. If there exists binary soft b-open sets (G,E) and (H,E) such that $F_e\ \tilde{\tilde{\in}}\ (G,E)$, $(F,E)\ \tilde{\tilde{\subseteq}}\ (H,E)$ and $(F,E)\ \tilde{\tilde{\cap}}\ (H,E) = \tilde{\tilde{\varphi}}$, then (U_1,U_2,τ_Δ,E) is called a binary soft b- regular space.

Proposition 4.1. Let (U_1,U_2,τ_Δ,E) be a binary soft topological space of $\tilde{\tilde{X}}$ over $(U_1 \times U_2)$. Then the following statements are equivalent:

(i) (U_1,U_2,τ_Δ,E) is binary soft b-regular.
(ii) For any binary soft b- open set (F,E) in $(U_1,U_2,\tau_\Delta,E)$and$G_e\ \tilde{\tilde{\in}}\ (F,E)$, there is binary soft b-open set (G,E) containing G_e such that$G_e\ \tilde{\tilde{\in}}\ \overline{\overline{(G,E)}}\ \tilde{\tilde{\subseteq}}\ (F,E)$.
(iii) Each binary soft point in (U_1,U_2,τ_Δ,E) has a binary soft neighborhood base consisting of binary soft b-closed sets.

Proof. (i)⇒ (ii)

Let (F,E) be a binary soft b-open set in $(U_1,U_2,\tau_\Delta,E)$and$G_e\ \tilde{\tilde{\in}}\ (F,E)$. Then $(F,E)^c$ is binary soft b-closed set such that $G_e\ \tilde{\tilde{\notin}}\ (F,E)^c$. By he binary soft regularity of (U_1,U_2,τ_Δ,E) there are binary soft b-open sets $(F_1,E),(F_2,E)$ such that$G_e\ \tilde{\tilde{\notin}}\ (F_1,E)$,$(F,E)^c\ \tilde{\tilde{\subseteq}}\ (F_2,E)$ and $(F_1,E)\ \tilde{\tilde{\cap}}\ (F_2,E) = \tilde{\tilde{\varphi}}$. Clearly $(F_2,E)^c$ is a binary soft set contained in (F,E). Thus $(F_1,E)\ \tilde{\tilde{\subseteq}}\ (F_2,E)^c\ \tilde{\tilde{\subseteq}}\ (F,E)$. This gives $\overline{\overline{(F_1,E)}}\ \tilde{\tilde{\subseteq}}\ (F_2,E)^c\ \tilde{\tilde{\subseteq}}\ (F,E)$, put $(F_1,E) = (G,E)$. Consequently $G_e\ \tilde{\tilde{\in}}\ (G,E)$ and $\overline{\overline{(G,E)}}\ \tilde{\tilde{\subseteq}}\ (F,E)$. This proves (ii).

(ii)⇒ (iii)

Let $G_e \stackrel{\approx}{\in} \widetilde{\widetilde{\widetilde{X}}}$, for binary soft b-open set (F, E) in $(U_1, U_2, \tau_\Delta, E)$ there is a binary soft b-open set (G, E) containing G_e such that $G_e \stackrel{\approx}{\in} (G, E)$, $\overline{\overline{(G, E)}} \stackrel{\approx}{\subseteq} (F, E)$. Thus for each $G_e \stackrel{\approx}{\in} \widetilde{\widetilde{\widetilde{X}}}$, the sets $\overline{\overline{(G, E)}}$ from a binary soft neighborhood base consisting of binary soft b-closed sets of $(U_1, U_2, \tau_\Delta, E)$ which proves (iii).

(iii) ⇒ (i)

Let (F, E) be a binary soft b-closed set such that $G_e \stackrel{\approx}{\notin} (F, E)$. Then $(F, E)^c$is a binary soft b-open neighborhood of G_e. By (iii) there is a binary soft b-closed set (F_1, E) which contains G_e and is a binary soft neighborhood of G_e with $(F_1, E) \stackrel{\approx}{\subseteq} (F_1, E)^c$. Then$G_e \stackrel{\approx}{\notin} (F, E)^c$, $(F, E) \stackrel{\approx}{\subseteq} (F_1, E)^c = (F_2, E)$ and $(F_1, E) \stackrel{\approx}{\cap} (F_2, E) = \widetilde{\widetilde{\varphi}}$. Therefore $(U_1, U_2, \tau_\Delta, E)$ is binary soft b-regular.

Proposition 4.2. Let $(U_1, U_2, \tau_\Delta, E)$ be a binary soft b-regular space on $\widetilde{\widetilde{\widetilde{X}}}$ over $(U_1 \times U_2)$. Then every binary soft subspace of $(U_1, U_2, \tau_\Delta, E)$ is binary soft b- regular.

Proof. Let $(U_1, U_2, \tau_\Delta, E)$ be a binary soft subspace of a binary soft pre-regular space $(U_1, U_2, \tau_\Delta, E)$. Suppose (F, E) is a binary soft b-closed set in $(U_1, U_2, \tau_{\Delta_y}, E)$ and $F_e \stackrel{\approx}{\in} \widetilde{\widetilde{\widetilde{Y}}}$ such that $F_e \stackrel{\approx}{\notin} (F, E)$.Then$(F, E) = (G, E) \stackrel{\approx}{\cap} \widetilde{\widetilde{\widetilde{Y}}}$; Where$(G, E)$ is binary soft b-closed set in $(U_1, U_2, \tau_\Delta, E)$. Then $F_e \stackrel{\approx}{\notin} (F, E)$, since $(U_1, U_2, \tau_\Delta, E)$ be a binary soft subspace of a binary soft b-regular, there exists soft disjoint binary b-open sets (F_1, E), (F_2, E) in $(U_1, U_2, \tau_\Delta, E)$. Then $F_e \stackrel{\approx}{\notin} (G, E)$, Since $(U_1, U_2, \tau_\Delta, E)$ is binary soft pre- regular, there exist binary soft disjoint binary b-open sets (F_1, E) , (F_2, E) in $(U_1, U_2, \tau_\Delta, E)$ such that $F_e \stackrel{\approx}{\in} (F_1, E)$, $(G, E) \stackrel{\approx}{\in} (F_2, E)$. Clearly $F_e \stackrel{\approx}{\in} (F_1, E) \stackrel{\approx}{\cap} \widetilde{\widetilde{\widetilde{Y}}} =^Y (F_2, E)$ and $(F, E) \stackrel{\approx}{\subseteq} (F_2, E) \stackrel{\approx}{\cap} \widetilde{\widetilde{\widetilde{Y}}} =^Y (F_2, E)$ such that $^Y(F_1, E) \stackrel{\approx}{\cap}{}^Y(F_2, E) = \widetilde{\widetilde{\varphi}}$. Therefore it proves that $(U_1, U_2, \tau_{\Delta_y}, E)$ is a binary soft b-regular subspace of $(U_1, U_2, \tau_\Delta, E)$.

Proposition 4.3. Let $(U_1, U_2, \tau_\Delta, E)$ be a binary soft regular space on $\widetilde{\widetilde{\widetilde{X}}}$ over $(U_1 \times U_2)$. A binary space $(U_1, U_2, \tau_\Delta, E)$ is binary soft b-regular if and only if for each $F_e \stackrel{\approx}{\in} \widetilde{\widetilde{\widetilde{X}}}$and a binary soft b-closed set (F, E)in $(U_1, U_2, \tau_\Delta, E)$such that $F_e \stackrel{\approx}{\notin} (F, E)$ there exist binary soft b-open sets (F_1, E) , (F_2, E)in $(U_1, U_2, \tau_\Delta, E)$ such that $F_e \stackrel{\approx}{\in} (F_1, E)$, $(F_1, E) \stackrel{\approx}{\subseteq} (F_2, E)$ and $\overline{\overline{(F_1, E)}} \stackrel{\approx}{\cap} \overline{\overline{(F_2, E)}} = \widetilde{\widetilde{\varphi}}$.

Proof. For each $F_e \stackrel{\approx}{\in} \widetilde{\widetilde{\widetilde{X}}}$ and a binary soft b-closed set (G, E) such that $F_e \stackrel{\approx}{\notin} (F, E)$ by theorem 16there is a binary soft b-open set (G, E) such that $F_e \stackrel{\approx}{\in} (G, E)$, $\overline{\overline{(G, E)}} \stackrel{\approx}{\subseteq} (F_1, E)^c$. Again by theorem16 there is a binary soft b-open (F_1, E) containing F_e such that $\overline{\overline{(F_1, E)}} \stackrel{\approx}{\subseteq} (G, E)$. Let$(F_2, E) = (\overline{\overline{(G, E)}})^c$, then $\overline{\overline{(F_1, E)}} \stackrel{\approx}{\subseteq} (G, E) \stackrel{\approx}{\subseteq} \overline{\overline{(G, E)}} \stackrel{\approx}{\subseteq} (F, E)^c$ Implies $\overline{\overline{(F_1, E)}} \stackrel{\approx}{\subseteq} (\overline{\overline{(G, E)}})^c = (F_2, E)$ or$(F, E) \stackrel{\approx}{\subseteq} (F_2, E)$. Also

$$\overline{\overline{(F_1, E)}} \stackrel{\approx}{\cap} \overline{\overline{(F_2, E)}} = \overline{\overline{(F_1, E)}} \stackrel{\approx}{\cap} \left(\overline{\overline{(G, E)}}\right)^c \stackrel{\approx}{\subseteq} (G, E) \stackrel{\approx}{\cap} \left(\overline{\overline{(G, E)}}\right)^c \stackrel{\approx}{\subseteq} \overline{\overline{(G, E) \stackrel{\approx}{\cap} \left(\overline{\overline{(G, E)}}\right)^c}} = \overline{\overline{\widetilde{\widetilde{\varphi}}}}$$
$$= \varphi .$$

Thus (F_1, E) , (F_2, E) are the required binary soft b-open sets in $(U_1, U_2, \tau_\Delta, E)$. This proves the necessity. The sufficiency is immediate.

Definition 4.2. Let $(U_1, U_2, \tau_\Delta, E)$ be a binary soft regular space on $\widetilde{\widetilde{X}}$ over $(U_1 \times U_2)$. (F, E) , (G, E) are binary soft b-closed sets over $(U_1 \times U_2)$ such that $(F, E) \,\widetilde{\widetilde{\cap}}\, (G, E) = \widetilde{\widetilde{\varphi}}$. If there exist binary soft b-open sets (F_1, E), and (F_2, E) such that $(F, E) \,\widetilde{\widetilde{\subseteq}}\, (F_1, E)$, $(G, E) \,\widetilde{\widetilde{\subseteq}}\, (F_2, E)$ and $(F_1, E) \,\widetilde{\widetilde{\cap}}\, (F_2, E) = \varphi$, then $(U_1, U_2, \tau_\Delta, E)$ is called a binary soft b-normal space.

Definition 4.3. Let $(U_1, U_2, \tau_\Delta, E)$ be a binary soft regular space on $\widetilde{\widetilde{X}}$ over $(U_1 \times U_2)$. Then $(U_1, U_2, \tau_\Delta, E)$ is said to be a binary soft b-τ_{Δ_3}space if it is binary soft b-regular and a binary soft b-τ_{Δ_1} space.

Proposition 4.4. Let $(U_1, U_2, \tau_\Delta, E)$ be a binary soft regular space on $\widetilde{\widetilde{X}}$ over $(U_1 \times U_2)$ and $\widetilde{\widetilde{Y}} \,\widetilde{\widetilde{\subseteq}}\, \widetilde{\widetilde{X}}$. If $(U_1, U_2, \tau_\Delta, E)$ is a binary softb-τ_{Δ_3}space then $(U_1, U_2, \tau_{\Delta_Y}, E)$ is a binary soft b-τ_{Δ_3}space.

Proof. Straightforward

Definition 4.4. A binary soft topological space $(U_1, U_2, \tau_\Delta, E)$ on $\widetilde{\widetilde{X}}$ over $(U_1 \times U_2)$ is said to be a binary soft b-τ_{Δ_4}space if it is binary soft b- normal and binary soft b-τ_{Δ_1}space.

Proposition 4.5. A binary soft topological space $(U_1, U_2, \tau_\Delta, E)$ is binary soft b-normal if and only if for soft b-closed set (F, E) and a binary soft b-open set (G, E), such that $(F, E) \,\widetilde{\widetilde{\subseteq}}\, (G, E)$ these exist at least one binary soft b-open set (H, E) containing (F, E) such that $(F, E) \,\widetilde{\widetilde{\subseteq}}\, (H, E) \,\widetilde{\widetilde{\subseteq}}\, \overline{\overline{(H, E)}} \,\widetilde{\widetilde{\subseteq}}\, (G, E)$.

Proof. Let us suppose that $(U_1, U_2, \tau_\Delta, E)$ is a binary soft normal space and (F, E) is any binary soft b- closed subset of $(U_1, U_2, \tau_\Delta, E)$ and $(G, E)$is a binary soft b-open set such that $(F, E) \,\widetilde{\widetilde{\subseteq}}\, (G, E)$. Then $(G, E)^c$ is binary soft b-closed and $(F, E) \,\widetilde{\widetilde{\cap}}\, (G, E)^c = \varphi$. So by supposition, there are binary soft b-open sets (H, E) and (K, E) such that $(F, E) \,\widetilde{\widetilde{\subseteq}}\, (H, E)$, $(G, E)^c \,\widetilde{\widetilde{\subseteq}}\, (K, E)$ and $\widetilde{\widetilde{\cap}}\, (K, E) = \widetilde{\widetilde{\varphi}}$.

Since $(H, E) \,\widetilde{\widetilde{\cap}}\, (K, E) = \widetilde{\widetilde{\varphi}}$, $(H, E) \,\widetilde{\widetilde{\subseteq}}\, (K, E)^c$. But $(K, E)^c$is binary soft b-closed, so that

$$(F, E) \,\widetilde{\widetilde{\subseteq}}\, (H, E) \,\widetilde{\widetilde{\subseteq}}\, \overline{\overline{(H, E)}} \,\widetilde{\widetilde{\subseteq}}\, (K, E)^c \,\widetilde{\widetilde{\subseteq}}\, (G, E).$$

Hence

$$(F, E) \,\widetilde{\widetilde{\subseteq}}\, (H, E) \,\widetilde{\widetilde{\subseteq}}\, \overline{\overline{(H, E)}} \,\widetilde{\widetilde{\subseteq}}\, (K, E)^c \,\widetilde{\widetilde{\subseteq}}\, (G, E).$$

Conversely, suppose that for every binary soft b-closed set (F, E) and a binary soft b-open set (G, E) such that $(F, E) \,\widetilde{\widetilde{\subseteq}}\, (H, E)$, there is a binary soft b-open set (H, E) such that $(F, E) \,\widetilde{\widetilde{\subseteq}}\, (H, E) \,\widetilde{\widetilde{\subseteq}}\, \overline{\overline{(H, E)}} \,\widetilde{\widetilde{\subseteq}}\, (G, E)$. Let (F_1, E) , (F_2, E) be any two soft disjoint b-closed sets, then $(F_1, E) \tilde{\tilde{\subseteq}} (F_2, E)^c$ where $(F_2, E)^c$binary soft b-open. Hence there is a binary soft b-open set (H, E) such that $(F, E) \tilde{\tilde{\subseteq}} (H, E) \tilde{\tilde{\subseteq}} \overline{\overline{(H, E)}} \tilde{\tilde{\subseteq}} (F_2, E)^c$. But then $(F_2, E) \tilde{\tilde{\subseteq}} (\overline{\overline{(H, E)}})^c$ and $(H, E) \tilde{\tilde{\cap}} (\overline{\overline{(H, E)}})^c \neq \varphi$.

Hence $(F_1, E) \tilde{\tilde{\subseteq}} (H, E)$ and $(F_2, E) \tilde{\tilde{\subseteq}} (\overline{\overline{(H, E)}})^c$ with $(H, E) \tilde{\tilde{\cap}} (\overline{\overline{(H, E)}})^c = \varphi$.

Hence $(U_1, U_2, \tau_\Delta, E)$ is binary soft b- normal space.

5. Conclusion

A soft topology between two sets other than the product soft topology has been touched through proper channel. A soft set with single specific topological structure is unable to shoulder up the responsibility to build the whole theory. So to make the theory strong, some additional structures on soft set has to be introduced. It makes, it more bouncy to grow the soft topological spaces with its infinite applications. In this regards we familiarized soft topological structure known as binary soft b-separation axioms in binary soft topological structure with respect to soft b-open sets.

Topology is the most important branch of pure mathematics which deals with mathematical structures by one way or the others. Recently, many scholars have studied the soft set theory which is coined by Molodtsov [3] and carefully applied to many difficulties which contain uncertainties in our social life. Shabir and Naz familiarized and profoundly studied the foundation of soft topological spaces. They also studied topological structures and displayed their several properties with respect to ordinary points.

In the present work, we constantly study the behavior of binary soft b-separation axioms in binary soft topological spaces with respect to soft points as well as ordinary points. We introduce (b- τ_{Δ_o}, pre- τ_{Δ_1}, b- τ_{Δ_2},b- τ_{Δ_3} and b- τ_{Δ_4}) structures with respect to soft points. In future we will plant these structures in different results. We also planted these axioms to different results. These binary soft b-separation structure would be valuable for the development of the theory of soft in binary soft topology to solve complicated problems, comprising doubts in economics, engineering, medical etc. We also attractively discussed some soft transmissible properties with respect to ordinary as well as soft points. I have fastidiously studied numerous homes on the behalf of Soft Topology. And lastly I determined that soft Topology is totally linked or in other sense we can correctly say that Soft Topology (Separation Axioms) are connected with structure. Provided if it is related with structures then it gives the idea of non-linearity beautifully. In other ways we can rightly say Soft Topology is somewhat directly proportional to non-linearity. Although we use non-linearity in Applied Math. So it is not wrong to say that Soft Topology is applied Math in itself. It means that Soft Topology has the taste of both of pure and applied math. In future I will discuss Separation Axioms in Soft Topology with respect to soft points. We expect that these results in this article will do help the researchers for strengthening the toolbox of soft topological structures. Soft topology provides less information on the behalf of a few choices. The reason for this is that we use a single set in soft topology and in binary soft topology we use double sets .It means that binary soft topology exceeds soft topology in all respect. In the light of above mentioned discussion I can literary say that number of sets is directly proportional to choices. Therefore all mathematicians are kindly informed to emphasize upon it.

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New Types of Some Nano $\mathcal{R}$-Sets

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Abstaract — In this paper, we introduce the notions of nano $\mathcal{R}$-set, nano $\mathcal{R}_r$-set and $\mathcal{R}_r^\star$-set in nano topological spaces and study some of their properties.

Keywords — *Nano $\mathcal{B}$-set, nano t-set, nano $\alpha^\star$-set, nano $\mathcal{R}$-set, nano $\mathcal{R}_r$-set, nano $\mathcal{R}_r^\star$-set.*

1 Introduction

Thivagar and Richard [3] introduced a nano topological space with respect to a subset X of an universe which is defined in terms of lower approximation and upper approximation and boundary region. The classical nano topological space is based on an equivalence relation on a set, but in some situation, equivalence relations are nor suitable for coping with granularity, instead the classical nano topology is extend to general binary relation based covering nano topological space

Bhuvaneshwari and Gnanapriya [1] introduced and investigated nano g-closed sets in nano topological spaces. Recently, Devi and Bhuvaneswari [6] introduced the notions of nano rg-closed sets. In this paper we introduce the notions of nano $\mathcal{R}$-set, nano $\mathcal{R}_r$-set, nano $\mathcal{R}_r^\star$-set and study some of their properties.

2 Preliminaries

Definition 2.1. [4] Let U be a non-empty finite set of objects called the universe and R be an equivalence relation on U named as the indiscernibility relation. Elements belonging to the same equivalence class are said to be indiscernible with one another. The pair (U, R) is said to be the approximation space. Let $X \subseteq U$.

1. The lower approximation of X with respect to R is the set of all objects, which can be for certain classified as X with respect to R and it is denoted by

$L_R(X)$. That is, $L_R(X) = \bigcup_{x\in U}\{R(x) : R(x) \subseteq X\}$, where R(x) denotes the equivalence class determined by x.

2. The upper approximation of X with respect to R is the set of all objects, which can be possibly classified as X with respect to R and it is denoted by $U_R(X)$. That is, $U_R(X) = \bigcup_{x\in U}\{R(x) : R(x) \cap X \neq \phi\}$.

3. The boundary region of X with respect to R is the set of all objects, which can be classified neither as X nor as not - X with respect to R and it is denoted by $B_R(X)$. That is, $B_R(X) = U_R(X) - L_R(X)$.

Definition 2.2. [3] Let U be the universe, R be an equivalence relation on U and $\tau_R(X) = \{U, \phi, L_R(X), U_R(X), B_R(X)\}$ where $X \subseteq U$. Then R(X) satisfies the following axioms:

1. U and $\phi \in \tau_R(X)$,

2. The union of the elements of any sub collection of $\tau_R(X)$ is in $\tau_R(X)$,

3. The intersection of the elements of any finite subcollection of $\tau_R(X)$ is in $\tau_R(X)$.

Thus $\tau_R(X)$ is a topology on U called the nano topology with respect to X and $(U, \tau_R(X))$ is called the nano topological space. The elements of $\tau_R(X)$ are called nano-open sets (briefly n-open sets). The complement of a n-open set is called n-closed.

In the rest of the paper, we denote a nano topological space by $(U, \mathcal{N})$, where $\mathcal{N} = \tau_R(X)$. The nano-interior and nano-closure of a subset A of U are denoted by $n\text{-}int(A)$ and $n\text{-}cl(A)$, respectively.

Definition 2.3. A subset H of a space $(U, \mathcal{N})$ is called nano regular-pen set [3] if $H = n\text{-}int(n\text{-}cl(H))$.

The complement of the above mentioned set is called their respective closed set.

Definition 2.4. [2] A subset H of a space $(U, \mathcal{N})$ is called:

1. nano t-set (briefly, nt-set) if $n\text{-}int(H) = n\text{-}int(n\text{-}cl(H))$.

2. nano $\mathcal{B}$-set (briefly, $n\mathcal{B}$-set) if $H = P \cap Q$, where P is n-open and Q is nt-set.

Definition 2.5. [5] A subset H of a space $(U, \mathcal{N})$ is called a nano $\alpha^\star$-set (briefly, $n\alpha^\star$-set) if $n\text{-}int(n\text{-}cl(n\text{-}int(H))) = n\text{-}int(H)$.

Definition 2.6. A subset H of a space $(U, \mathcal{N})$ is called;

1. nano g-closed (briefly, ng-closed) [1] if $n\text{-}cl(H) \subseteq G$, whenever $H \subseteq G$ and G is n-open.

2. nano rg-closed set (briefly, nrg-closed) [6] if $n\text{-}cl(H) \subseteq G$ whenever $H \subseteq G$ and G is nano regular-open.

3 Properties of Some Nano $\mathcal{R}$-Sets

Definition 3.1. A subset H of a space $(U, \mathcal{N})$ is called;

1. nano $\mathcal{R}$-set (briefly, $n\mathcal{R}$-set) if $H = P \cap K$ where P is ng-open and K is nt-set.
2. nano $\mathcal{R}_r$-set (briefly, $n\mathcal{R}_r$-set) if $H = P \cap K$ where P is nrg-open and K is nt-set.
3. nano $\mathcal{R}_r^\star$-set (briefly, $n\mathcal{R}_r^\star$-set) if $H = P \cap K$ where P is nrg-open and K is $n\alpha^\star$-set.

Example 3.2. Let $U = \{a, b, c, d\}$ with $U/R = \{\{a\}, \{b, c\}, \{d\}\}$ and $X = \{b, d\}$. Then the nano topology $\mathcal{N} = \{\phi, \{d\}, \{b, c\}, \{b, c, d\}, U\}$.

1. then $\{a\}$ is $n\mathcal{R}$-set.
2. then $\{a, b\}$ is $n\mathcal{R}_r$-set.
3. then $\{a, b, c\}$ is $n\mathcal{R}_r^\star$-set.

Theorem 3.3. In a space $(U, \mathcal{N})$, for a subset H, the following relations hold.

1. H is $n\mathcal{B}$-set $\Rightarrow$ H is $n\mathcal{R}$-set.
2. H is nt-set $\Rightarrow$ H is $n\mathcal{R}_r$-set.
3. H is nrg-open set $\Rightarrow$ H is $n\mathcal{R}_r$-set.
4. H is $n\alpha^\star$-set $\Rightarrow$ H is $n\mathcal{R}_r^\star$-set
5. H is nrg-open set $\Rightarrow$ H is $n\mathcal{R}_r^\star$-set.
6. H is nt-set $\Rightarrow$ H is $n\mathcal{R}_r^\star$-set.
7. H is $n\mathcal{R}_r$-set $\Rightarrow$ H is $n\mathcal{R}_r^\star$-set.

Proof.
1. Since every n-open set is ng-open, every $n\mathcal{B}$-set is a $n\mathcal{R}$-set.
2. Let H be a nt-set in U. Then $H = U \cap H$ where U is clearly nrg-open in U. Therefore, H is $n\mathcal{R}_r$-set in U.
3. Let H be a nrg-open set in U. Then $H = H \cap U$ where U is clearly a nt-set in U. Therefore, H is $n\mathcal{R}_r$-set in U.
4. Let H be a $n\alpha^\star$-set in U. Then $H = U \cap H$ where U is clearly nrg-open in U. Therefore, H is a $n\mathcal{R}_r^\star$-set in U.
5. Let H be a nrg-open set in U. Then $H = H \cap U$ where U is clearly a $n\alpha^\star$-set in U. Therefore, H is a $n\mathcal{R}_r^\star$-set in U.
6. Let H be a nt-set in U. Since every nt-set is $n\alpha^\star$-set in U. So, H is $n\alpha^\star$-set in U. By (4), U is a $n\mathcal{R}_r^\star$-set in U.

7. Let H be a $n\mathcal{R}_r$-set in U. Then $H = P \cap Q$ where P is nrg-open in U and Q is a nt-set in U. Since every nt-set in U is a $n\alpha^\star$-set in U, Q is a $n\alpha^\star$-set in U. Therefore, H is a $n\mathcal{R}_r^\star$-set in U.

Remark 3.4. These relations are shown in the diagram.

$$\begin{array}{ccccccc}
 & & & & n\mathcal{R}\textbf{-set} & \longleftarrow & n\mathcal{B}\textbf{-set} \\
 & & & & \downarrow & & \\
 & & nt\textbf{-set} & \longrightarrow & n\mathcal{R}_r\textbf{-set} & & \\
 & & \downarrow & \swarrow & \uparrow & & \\
n\alpha^\star\textbf{-set} & \longrightarrow & n\mathcal{R}_r^\star\textbf{-set} & \longleftarrow & nrg\textbf{-open} & &
\end{array}$$

The converses of each statement in Theorem 3.3 are not true as shown in the following Example.

Example 3.5. Let $U = \{p, q, r\}$ with $U/R = \{\{p, q\}, \{r\}\}$ and $X = \{p\}$. Then the nano topology $\mathcal{N} = \{\phi, \{p, q\}, U\}$. Then $H = \{p\}$ is $n\mathcal{R}$-set but not $n\mathcal{B}$-set.

Example 3.6. In Example 3.2,

1. $\{b\}$ is $n\mathcal{R}_r$-set but not nt-set.
2. $\{a, d\}$ is $n\mathcal{R}_r$-set but not nrg-open.
3. $\{b, c, d\}$ is $n\mathcal{R}_r^\star$-set but not $n\alpha^\star$-set.
4. $\{a, d\}$ is $n\alpha^\star$-set, so $n\mathcal{R}_r^\star$-set. But {a,d} is not nrg-open.
5. $\{a, b\}$ is $n\mathcal{R}_r^\star$-set but not nt-set.
6. $\{a, b, d\}$ is $n\mathcal{R}_r^\star$-set but not $n\mathcal{R}_r$-set.

Remark 3.7. In a space $(U, \mathcal{N})$,

1. the intersection of two $n\mathcal{R}$-sets are $n\mathcal{R}$-set.
2. the intersection of two $n\mathcal{R}_r$-sets are $n\mathcal{R}_r$-set.
3. the intersection of two $n\mathcal{R}_r^\star$-sets are $n\mathcal{R}_r^\star$-set.
4. the union of two $n\mathcal{R}$-sets but not $n\mathcal{R}$-set.
5. the union of two $n\mathcal{R}_r$-sets but not $n\mathcal{R}_r$-set.

Example 3.8. In Example 3.2,

1. then $H = \{a, d\}$ and $Q = \{b, d\}$ is $n\mathcal{R}$-sets, $n\mathcal{R}_r$-sets and $n\mathcal{R}_r^\star$-sets. But $H \cap Q = \{d\}$ is $n\mathcal{R}$-set, $n\mathcal{R}_r$-set and $n\mathcal{R}_r^\star$-set.
2. then $H = \{a\}$ and $Q = \{b\}$ is $n\mathcal{R}$-sets. But $H \cup Q = \{a, b\}$ is not $n\mathcal{R}$-set.
3. then $H = \{a, b\}$ and $Q = \{d\}$ is $n\mathcal{R}_r$-sets. But $H \cup Q = \{a, b, d\}$ is not $n\mathcal{R}_r$-sets.

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On Some Identities and Symmetric Functions for Balancing Numbers

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Abstract - In this paper, we derive new generating functions of the product of balancing numbers, Lucas balancing numbers and the Chebychev polynomials of the second kind by making use of useful properties of the symmetric functions mentioned in the paper.

Keywords *- Balancing numbers, Lucas balancing number, Chebychev polynomials*.

1 Introduction and Preliminaries

Recently, Behera and Panda [1] introduced balancing numbers $n \in \mathbb{Z}_+$ as solutions of the Diophantine equation

$$1+2+...+(n-1)=(n+1)+(n+2)+...+(n+r). \qquad (1.1)$$

for some positive integer r which is called balancer or cobalancing number. For example $6;35;204;1189$ and 6930 are balancing numbers with balancers $2;14;84;492$ and 2870, respectively. If n is a balancing number with balancer r, then from (1.1) one has

$$\frac{n(n+1)}{2}=rn+\frac{r(r+1)}{2},$$

and so

$$r=\frac{-(2n+1)+\sqrt{8n^2+1}}{2} \text{ and } n=\frac{2r+1+\sqrt{8r^2+8r+1}}{2}.$$

Let B_n denote the n^{th} balancing number and let b_n denote the n^{th} cobalancing number. Then

$$\begin{cases} B_{n+1}=6B_n-B_{n-1},\ n\geq 1 \\ B_0=0,\ B_1=1 \end{cases},$$

and

$$\begin{cases} b_{n+1} = 6b_n - b_{n-1} + 2, \ n \geq 2 \\ b_1 = 0, \ b_2 = 2 \end{cases}.$$

Definition 1.1. [14] The Lucas-balancing $\{C_n\}_{n\in\mathbb{N}^*}$ is defined recurrently by

$$\begin{cases} C_{n+1} = 6C_n - C_{n-1}, \ n \geq 1 \\ C_0 = 1, \ C_1 = 3 \end{cases}.$$

The main purpose of this paper is to present some results involving the balancing number and Lucas-balancing number using define a new useful operator denoted by $\delta_{p_1p_2}$ for which we can formulate, extend and prove new results based on our previous ones [3, 4, 5]. In order to determine generating functions of the product of balancing number, Lucas-balancing number and Chebychev polynomials of first and second kind, we combine between our indicated past techniques and these presented polishing approaches.

In order to render the work self-contained we give the necessary preliminaries tools; we recall some definitions and results.

Definition 1.2. [5] Let k and n be tow positive integer and $\{a_1, a_2, ..., a_n\}$ are set of given variables the k-th elementary symmetric function $e_k(a_1, a_{2,} ..., a_n)$ is defined by

$$e_k(a_1, a_{2,} ..., a_n) = \sum_{i_1+i_2+...+i_n=k} a_1^{i_1} a_2^{i_2} ... a_n^{i_n}, \quad 0 \leq k \leq n,$$

with $i_1, i_2, ..., i_n = 0$ or 1.

Definition 1.3. [5] Let k and n be tow positive integer and $\{a_1, a_2, ..., a_n\}$ are set of given variables the k-th complete homogeneous symmetric function $h_k(a_1, a, ..., a_n)$ is defined by

$$h_k(a_1, a_{2,} ..., a_n) = \sum_{i_1+i_2+...+i_n=k} a_1^{i_1} a_2^{i_2} ... a_n^{i_n}, \quad 0 \leq k \leq n,$$

with $i_1, i_2, ..., i_n \geq 0.$

Remark 1.1. We set $e_0(a_1, a_{2,} ..., a_n) = 1$ and $h_k(a_1, a_{2,} ..., a_n) = 1$ (by convention). For $k > n$ or $k < 0,$ we set $e_0(a_1, a_{2,} ..., a_n) = 0$ and $h_k(a_1, a_{2,} ..., a_n) = 0.$

Definition 1.4. [7] Let B and P be any two alphabets. We define $S_n(B-P)$ by the following form

$$E(-z)H(z) = \sum_{n=0}^{\infty} S_n(B-P)z^n,$$

with $H(z) = \Pi_{b\varepsilon B}(1-bz)^{-1}, \ E(-z) = \Pi_{p\varepsilon P}(1-pz).$

Remark 1.2. $S_n(B-P)=0$ for $n<0$.

Definition 1.5. [5] Given a function f on $\mathbb{R}^n$, the divided difference operator is defined as follows

$$\partial_{p_i p_{i+1}}(f)=\frac{f(p_1,\cdots,p_i,p_{i+1},\cdots p_n)-f(p_1,\cdots p_{i-1},p_{i+1,}p_i,p_{i+2}\cdots p_n)}{p_i-p_{i+1}}.$$

Definition 1.6. The symmetrizing operator $\delta^k_{p_1p_2}$ is defined by

$$\delta^k_{p_1p_2}(g)=\frac{p_1^k g(p_1)-p_2^k g(p_2)}{p_1-p_2} \text{ for all } k\in\mathbb{N}.$$

Proposition 1.1. [6] Let $P=\{p_1,p_2\}$ an alphabet, we define the operator $\delta^k_{p_1p_2}$ as follows

$$\delta^k_{p_1p_2}g(p_1)=S_{k-1}(p_1+p_2)g(p_1)+p_2^k\partial_{p_1p_2}g\,(p_1),\ \textit{for all } k\in\mathbb{N}.$$

2 Main Results

In our main results, we will combine all these results in a unified way such that they can be considered as a special case of the following Theorem.

Theorem 2.1. Let A and P be two alphabets, respectively, $\{a_1,a_2\}$ and $\{b_1,b_2\}$, then we have

$$\sum_{n=0}^{\infty}h_n(a_1,a_2)h_{n+k-1}(p_1,p_2)z^n=\frac{h_{k-1}(p_1,p_2)+p_1p_2(a_1+a_2)h_{k-2}(p_1,p_2)z-a_1a_2p_1p_2\delta_{p_1p_2}(p_2^{k-1})z^2}{\left(\sum_{n=0}^{\infty}e_n(a_1,a_2)(-p_1z)^n\right)\left(\sum_{n=0}^{\infty}e_n(a_1,a_2)(-p_2z)^n\right)}, \tag{2.1}$$

for all $k\in\mathbb{N}$.

Proof. By applying the operator $\partial_{p_1p_2}$ to the series $f(p_1z)=\sum_{n=0}^{\infty}h_n(a_1,a_2)\,p_1^{n+k}z^n$, we have

$$\begin{aligned}\partial_{p_1p_2}\left(\sum_{n=0}^{\infty}h_n(a_1,a_2)\,p_1^{n+k}z^n\right)&=\frac{\sum_{n=0}^{\infty}h_n(a_1,a_2)\,p_1^{n+k}z^n-\sum_{n=0}^{\infty}h_n(a_1,a_2)\,p_2^{n+k}z^n}{p_1-p_2}\\&=\sum_{n=0}^{\infty}h_n(a_1,a_2)\left(\frac{p_1^{n+k}-p_2^{n+k}}{p_1-p_2}\right)z^n\\&=\sum_{n=0}^{\infty}h_n(a_1,a_2)\,h_{n+k-1}(p_1,p_2)\,z^n.\end{aligned}$$

On the other hand,

$$\partial_{p_1p_2}\left(\frac{p_1^k}{\left(\sum_{n=0}^{\infty}e_n(a_1,a_2)(-p_1z)^n\right)}\right)=\frac{\frac{p_1^k}{\left(\sum_{n=0}^{\infty}e_n(a_1,a_2)(-p_1z)^n\right)}-\frac{p_2^k}{\left(\sum_{n=0}^{\infty}e_n(a_1,a_2)(-p_2z)^n\right)}}{p_1-p_2}$$

$$=\frac{p_1^k\left(\sum_{n=0}^{\infty}e_n(a_1,a_2)(-p_2z)^n\right)-p_2^k\left(\sum_{n=0}^{\infty}e_n(a_1,a_2)(-p_1z)^n\right)}{(p_1-p_2)\left(\sum_{n=0}^{\infty}e_n(a_1,a_2)(-p_1z)^n\right)\left(\sum_{n=0}^{\infty}e_n(a_1,a_2)(-p_2z)^n\right)}$$

$$=\frac{p_1^k-p_2^k-(a_1+a_2)\left(p_1^kp_2-p_2^kp_1\right)z-a_1a_2\left(p_2^kp_1^2-p_1^kp_2^2\right)z^2}{(p_1-p_2)\left(\sum_{n=0}^{\infty}e_n(a_1,a_2)(-p_1z)^n\right)\left(\sum_{n=0}^{\infty}e_n(a_1,a_2)(-p_2z)^n\right)}$$

$$=\frac{\frac{p_1^k-p_2^k}{p_1-p_2}-(a_1+a_2)p_1p_2\left(\frac{p_1^{k-1}-p_2^{k-1}}{p_1-p_2}\right)z-a_1a_2p_1p_2\left(\frac{p_1p_2^{k-1}-p_2p_1^{k-1}}{p_1-p_2}\right)z^2}{\left(\sum_{n=0}^{\infty}e_n(a_1,a_2)(-p_1z)^n\right)\left(\sum_{n=0}^{\infty}e_n(a_1,a_2)(-p_2z)^n\right)}$$

$$=\frac{h_{k-1}(p_1,p_2)+p_1p_2(a_1+a_2)h_{k-2}(p_1,p_2)z-a_1a_2p_1p_2\delta_{p_1p_2}(p_2^{k-1})z^2}{\left(\sum_{n=0}^{\infty}e_n(a_1,a_2)(-p_1z)^n\right)\left(\sum_{n=0}^{\infty}e_n(a_1,a_2)(-p_2z)^n\right)}.$$

Thus, this completes the proof.

3 Generating Functions of Some Well-known Numbers

We now derive new generating functions of the products of some well-known numbers. Indeed, we consider Theorem 2.1 in order to derive balancing numbers, Lucas balancing numbers and Tchebychev polynomials of second kind and the symmetric functions.

If $k=0,1$ and $A=\{1,0\}$, we deduce the following lemmas

Lemma 3.1. [2] Given an alphabet $P=\{p_1,p_2\}$, we have

$$\sum_{n=0}^{\infty}h_n(p_1,p_2)z^n=\frac{1}{(1-p_1z)(1-p_2z)}. \tag{3.1}$$

Lemma 3.2. [3] Given an alphabet $P=\{p_1,p_2\}$, we have

$$\sum_{n=0}^{\infty}h_{n-1}(p_1,p_2)z^n=\frac{z}{(1-p_1z)(1-p_2z)}. \tag{3.2}$$

Replacing p_2 by $(-p_2)$ in (3.1) and (3.2), we obtain

$$\sum_{n=0}^{\infty} h_n(p_1,[-p_2])z^n = \frac{1}{(1-p_1z)(1+p_2z)}, \quad (3.3)$$

$$\sum_{n=0}^{\infty} h_{n-1}(p_1,[-p_2])z^n = \frac{z}{(1-p_1z)(1+p_2z)}. \quad (3.4)$$

Choosing p_1 and p_2 such that

$$\begin{cases} p_1p_2 = -1, \\ p_1 - p_2 = 6, \end{cases}$$

and substituting in (3.3) and (3.4) we end up with

$$\sum_{n=0}^{\infty} h_n(p_1,[-p_2])z^n = \frac{1}{1-6z+z^2}, \quad (3.5)$$

$$\sum_{n=0}^{\infty} h_{n-1}(p_1,[-p_2])z^n = \frac{z}{1-6z+z^2} \text{ with } p_{1,2} = 3 \pm 2\sqrt{2}, \quad (3.6)$$

Which represents a generating function for balancing numbers, such that $B_n = S_{n-1}(p_1+[-p_2])$.

Multiplying the equation (3.6) by (**-3**) and added to (3.5), we obtain

$$\sum_{n=0}^{\infty} (h_n(p_1,[-p_2]) - 3h_{n-1}(p_1,[-p_2]))z^n = \frac{1-3z}{1-6z+z^2},$$

which represents a generating function for Lucas-balancing numbers.

Corollary 3.1. For all $n \in \mathbb{N}$, we have

$$C_n = h_n(p_1,[-p_2]) - 3h_{n-1}(p_1,[-p_2]), \text{with } p_{1,2} = 3 \pm 2\sqrt{2}.$$

Theorem 3.1. For $n \in \mathbb{N}$, the generating function of the cobalancing numbers numbers is given by

$$\sum_{n=0}^{\infty} b_n z^n = \frac{2z^2}{(1-6z+z^2)(1-z)}.$$

Proof. The ordinary generating function associated is defined by $G(b_n, z) = \sum_{n=1}^{\infty} b_n z^n$.
Using the initial conditions, we get

$$\sum_{n=1}^{\infty} b_n z^n = b_1 z + b_2 z^2 + \sum_{n=3}^{\infty} b_n z^n$$
$$= 2z^2 + \sum_{n=3}^{\infty} (6b_n - b_{n-1} + 2) z^n.$$

Consider that $j = n-2$ and $p = n-1$. Then can be written by

$$\sum_{n=1}^{\infty} b_n z^n = 2z^2 + 6z \sum_{n=1}^{\infty} b_n z^n - z^2 \sum_{n=3}^{\infty} b_n z^n + 2z^3 \sum_{n=0}^{\infty} z^n,$$

which is equivalent to

$$(1 - 6z + z^2) \sum_{n=1}^{\infty} b_n z^n = 2z^2 + \frac{2z^3}{1-z},$$

Therefore

$$\sum_{n=0}^{\infty} b_n z^n = \frac{2z^2}{(1 - 6z + z^2)(1-z)}.$$

This completes the proof.

If $k = 0,\ k = 1$ and $A = \{a_1, a_2\}$, we deduce the following theorems

Theorem 3.2. [8] Given two alphabets $A = \{a_1, a_2\}$ and $P = \{p_1, p_2\}$ we have

$$\sum_{n=0}^{\infty} h_n(a_1, a_2) h_{n-1}(p_1, p_2) z^n = \frac{(a_1 + a_2) z - a_1 a_2 (p_1 + p_2) z^2}{\left(\sum_{n=0}^{\infty} e_n(a_1, a_2)(-p_1 z)^n\right)\left(\sum_{n=0}^{\infty} e_n(a_1, a_2)(-p_2 z)^n\right)}. \tag{3.7}$$

Theorem 3.3. [9] Given two alphabets $A = \{a_1, a_2\}$ and $P = \{p_1, p_2\}$ we have

$$\sum_{n=0}^{\infty} h_n(a_1, a_2) h_n(p_1, p_2) z^n = \frac{1 - a_1 a_2 p_1 p_2 z^2}{\left(\sum_{n=0}^{\infty} e_n(a_1, a_2)(-p_1 z)^n\right)\left(\sum_{n=0}^{\infty} e_n(a_1, a_2)(-p_2 z)^n\right)}. \tag{3.8}$$

From (3.8) we can deduce

$$\sum_{n=0}^{\infty} h_{n-1}(a_1, a_2) h_{n-1}(p_1, p_2) z^n = \frac{z - a_1 a_2 p_1 p_2 z^3}{\left(\sum_{n=0}^{\infty} e_n(a_1, a_2)(-p_1 z)^n\right)\left(\sum_{n=0}^{\infty} e_n(a_1, a_2)(-p_2 z)^n\right)}. \tag{3.9}$$

Case 1: Replacing p_2 by $(-p_2)$ and a_2 by $(-a_2)$ in (3.7) and (3.9) yields

$$\sum_{n=0}^{\infty} h_n(a_1,[-a_2])h_{n-1}(p_1,[-p_2])z^n = \frac{(a_1-a_2)z + a_1a_2(p_1-p_2)z^2}{(1-a_1p_1z)(1+a_2p_1z)(1+a_1p_2z)(1-a_2p_2z)}, \qquad (3.10)$$

$$\sum_{n=0}^{\infty} h_{n-1}(a_1,[-a_2])h_{n-1}(p_1,[-p_2])z^n = \frac{z - a_1a_2p_1p_2z^3}{(1-a_1p_1z)(1+a_2p_1z)(1+a_1p_2z)(1-a_2p_2z)}. \qquad (3.11)$$

This case consists of four related parts.

Firstly, the substitutions of

$$\begin{cases} a_1 - a_2 = 1, \\ a_1a_2 = 1, \end{cases} \text{ and } \begin{cases} p_1 - p_2 = 6, \\ p_1p_2 = -1, \end{cases}$$

in (3.10) give

$$\sum_{n=0}^{\infty} h_n(a_1,[-a_2])h_{n-1}(p_1,[-p_2])z^n = \frac{z+6z^2}{1-6z+35z^2+6z^3+z^4},$$

which represents a new generating function for product of Fibonacci numbers with balancing numbers, such that $F_nB_n = h_n(a_1,[-a_2])h_{n-1}(p_1,[-p_2])$ with $a_{1,2} = \frac{1\pm\sqrt{5}}{2}$, $p_{1,2} = 3\pm 2\sqrt{2}$.

Secondly, the substitution of

$$\begin{cases} a_1 - a_2 = 6, \\ a_1a_2 = -1, \end{cases} \text{ and } \begin{cases} p_1 - p_2 = 6, \\ p_1p_2 = -1, \end{cases}$$

in (3.11) give

$$\sum_{n=0}^{\infty} h_{n-1}(a_1,[-a_2])h_{n-1}(p_1,[-p_2])z^n = \frac{z-z^3}{1-36z+2z^2-36z^3+z^4},$$

which represents a new generating function for balancing numbers of second order, such that $B_n^2 = h_{n-1}(a_1,[-a_2])h_{n-1}(p_1,[-p_2])$ with $a_{1,2} = p_{1,2} = 3\pm 2\sqrt{2}$.

Thirdly, the substitution of

$$\begin{cases} a_1 - a_2 = 1, \\ a_1a_2 = 2, \end{cases} \text{ and } \begin{cases} p_1 - p_2 = 6, \\ p_1p_2 = -1, \end{cases}$$

in (3.11) give

$$\sum_{n=0}^{\infty} h_{n-1}(a_1,[-a_2])h_{n-1}(p_1,[-p_2])z^n = \frac{z+2z^3}{1-6z-13z^2+12z^3+4z^4},$$

which represents a new generating function for product of Jacobsthal numbers with balancing numbers, such that $J_nB_n = h_{n-1}(a_1,[-a_2])h_{n-1}(p_1,[-p_2])$ with $a_1 = 2,\ a_2 = -1$ and $p_{1,2} = 3\pm 2\sqrt{2}$.

Finally, the substitution of

$$\begin{cases} a_1 - a_2 = 2, \\ a_1 a_2 = 1, \end{cases} \text{ and } \begin{cases} p_1 - p_2 = 6, \\ p_1 p_2 = -1, \end{cases}$$

in (3.11) give

$$\sum_{n=0}^{\infty} h_{n-1}(a_1,[-a_2])h_{n-1}(p_1,[-p_2])z^n = \frac{z+z^3}{1-12z+38z^2+12z^3+z^4}.$$

which represents a new generating function for product of Pell numbers with balancing numbers, such that $P_n B_n = h_{n-1}(a_1,[-a_2])h_{n-1}(p_1,[-p_2])$ with $a_1 = 1 \pm \sqrt{2}$ and $p_{1,2} = 3 \pm 2\sqrt{2}.$

Case 2: Replacing p_2 by $(-p_2)$ and a_1 by $2a_1$ and a_2 by $(-2a_2)$ in (3.10) yields

$$\sum_{n=0}^{\infty} h_n(2a_1,[-2a_2])h_{n-1}(p_1,[-p_2])z^n = \frac{2(a_1-a_2)z + 4a_1a_2(p_1-p_2)z^2}{(1-2a_1p_1z)(1+2a_2p_1z)(1+2a_1p_2z)(1-2a_2p_2z)}, \quad (3.12)$$

The substitution of

$$\begin{cases} p_1 - p_2 = 6, \\ p_1 p_2 = -1, \\ a_1 a_2 = \frac{-1}{4}, \end{cases}$$

in (3.12) and set for ease on notations $x = a_1 - a_2,$ we reach

$$\sum_{n=0}^{\infty} h_n(2a_1,[-2a_2])h_{n-1}(p_1,[-p_2])z^n = \sum_{n=0}^{\infty} B_n U_n(x) z^n$$

$$= \frac{2xz - 6z^2}{1-12xz+2(17+2x)z^2-12xz^3+z^4},$$

which corresponds to a new generating function for the combined balancing numbers and Tchebychev polynomials of the second kind.

Theorem 3.4. For $n \in \mathbb{N},$ the new generating function of the product of balancing numbers B_n and Tchebychev polynomials of first kind is given by

$$\sum_{n=0}^{\infty} B_n T_n(x) z^n = \frac{xz^3 - 6z^2 + 2xz - x}{1-12xz+2(17+2x)z^2-12xz^3+z^4}.$$

Proof . We see that

$$\begin{aligned}\sum_{n=0}^{\infty} B_n T_n(x) z^n &= \sum_{n=0}^{\infty} h_{n-1}(p_1,[-p_2])\left(h_n(2a_1,[-2a_2]) - x h_{n-1}(2a_1,[-2a_2])\right) z^n \\ &= \sum_{n=0}^{\infty} h_{n-1}(p_1,[-p_2]) h_n(2a_1,[-2a_2]) z^n - x\sum_{n=0}^{\infty} h_{n-1}(p_1,[-p_2]) h_{n-1}(2a_1,[-2a_2]) z^n \\ &= \sum_{n=0}^{\infty} B_n U_n(x) z^n - \frac{x}{2(a_1+a_2)} \sum_{n=0}^{\infty} S_{n-1}(p_1+[-p_2])((2a_1)^n - (-2a_2)^n) z^n \\ &= \sum_{n=0}^{\infty} B_n U_n(x) z^n - \frac{x}{2(a_1+a_2)} \left(\begin{array}{l} \sum_{n=0}^{\infty} h_{n-1}(p_1,[-p_2])(2a_1 z)^n - \\ \sum_{n=0}^{\infty} h_{n-1}(p_1+[-p_2])(-2a_2 z)^n \end{array} \right).\end{aligned}$$

On the other hand, we know that

$$\sum_{n=0}^{\infty} h_{n-1}(p_1,[-p_2]) z^n = \sum_{n=0}^{\infty} B_n z^n = \frac{z}{1-6z+z^2},$$

from which it follows

$$\begin{aligned}\sum_{n=0}^{\infty} B_n T_n(x) z^n = &\frac{2xz-6z^2}{1-12xz+2(17+2x)z^2-12xz^3+z^4} \\ &- \frac{x}{2(a_1+a_2)}\left(\frac{2a_1 z}{1-12a_1 z+4a_1^2 z^2} + \frac{2a_2 z}{1+12a_2 z+4a_2^2 z^2} \right),\end{aligned}$$

therefore

$$\sum_{n=0}^{\infty} B_n T_n(x) z^n = \frac{xz^3-6z^2+2xz-x}{1-12xz+2(17+2x)z^2-12xz^3+z^4}.$$

This completes the proof.

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Perceptions of Several Sets in Ideal Nano Topological Spaces

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Abstaract − In this paper, we introduce the concepts of t-nI-set and $\mathcal{R}$-nI-set are investigate and deal with an ideal nano topological spaces.

Keywords − *t-nI-open set, $\mathcal{R}$-nI-open set, and t_α-nI-open set α-nI-open set, pre-nI-open set and $\mathcal{R}_\alpha$-nI-open set.*

1 Introduction

An ideal I [8] on a topological space (X, τ) is a non-empty collection of subsets of X which satisfies the following conditions.

1. $A \in I$ and $B \subset A$ imply $B \in I$ and
2. $A \in I$ and $B \in I$ imply $A \cup B \in I$.

Given a topological space (X, τ) with an ideal I on X. If $\wp(X)$ is the family of all subsets of X, a set operator $(.)^\star : \wp(X) \to \wp(X)$, called a local function of A with respect to τ and I is defined as follows: for $A \subset X$, $A^\star(I, \tau) = \{x \in X : U \cap A \notin I$ for every $U \in \tau(x)\}$ where $\tau(x) = \{U \in \tau : x \in U\}$ [1]. The closure operator defined by $cl^\star(A) = A \cup A^\star(I, \tau)$ [7] is a Kuratowski closure operator which generates a topology $\tau^\star(I, \tau)$ called the $\star$-topology finer than τ. The topological space together with an ideal on X is called an ideal topological space or an ideal space denoted by (X, τ, I). We will simply write $A^\star$ for $A^\star(I, \tau)$ and $\tau^\star$ for $\tau^\star(I, \tau)$.

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Some new notions in the concept of ideal nano topological spaces were introduced by Parimala et al. [3, 4].

In this paper, we introduce the notions of t-nI-set, $\mathcal{R}$-nI-set, t_α-nI-set and $\mathcal{R}_\alpha$-nI-set are investigate and deal with an ideal nano topological spaces.

2 Preliminaries

Definition 2.1. *[5] Let U be a non-empty finite set of objects called the universe and R be an equivalence relation on U named as the indiscernibility relation. Elements belonging to the same equivalence class are said to be indiscernible with one another. The pair (U, R) is said to be the approximation space. Let $X \subseteq U$.*

1. *The lower approximation of X with respect to R is the set of all objects, which can be for certain classified as X with respect to R and it is denoted by $L_R(X)$. That is, $L_R(X) = \bigcup_{x\in U}\{R(x) : R(x) \subseteq X\}$, where R(x) denotes the equivalence class determined by x.*

2. *The upper approximation of X with respect to R is the set of all objects, which can be possibly classified as X with respect to R and it is denoted by $U_R(X)$. That is, $U_R(X) = \bigcup_{x\in U}\{R(x) : R(x) \cap X \neq \phi\}$.*

3. *The boundary region of X with respect to R is the set of all objects, which can be classified neither as X nor as not - X with respect to R and it is denoted by $B_R(X)$. That is, $B_R(X) = U_R(X) - L_R(X)$.*

Definition 2.2. *[2] Let U be the universe, R be an equivalence relation on U and $\tau_R(X) = \{U, \phi, L_R(X), U_R(X), B_R(X)\}$ where $X \subseteq U$. Then R(X) satisfies the following axioms:*

1. *U and $\phi \in \tau_R(X)$,*

2. *The union of the elements of any sub collection of $\tau_R(X)$ is in $\tau_R(X)$,*

3. *The intersection of the elements of any finite subcollection of $\tau_R(X)$ is in $\tau_R(X)$.*

Thus $\tau_R(X)$ is a topology on U called the nano topology with respect to X and $(U, \tau_R(X))$ is called the nano topological space. The elements of $\tau_R(X)$ are called nano-open sets (briefly n-open sets). The complement of a n-open set is called n-closed.

In the rest of the paper, we denote a nano topological space by $(U, \mathcal{N})$, where $\mathcal{N} = \tau_R(X)$. The nano-interior and nano-closure of a subset A of U are denoted by n-int(A) and n-cl(A), respectively.

A nano topological space $(U, \mathcal{N})$ with an ideal I on U is called [3] an ideal nano topological space and is denoted by $(U, \mathcal{N}, I)$. $G_n(x) = \{G_n \mid x \in G_n, G_n \in \mathcal{N}\}$, denotes [3] the family of nano open sets containing x.

In future an ideal nano topological spaces $(U, \mathcal{N}, I)$ is referred as a space.

Definition 2.3. *[3] Let $(U, \mathcal{N}, I)$ be a space with an ideal I on U. Let $(.)^\star_n$ be a set operator from $\wp(U)$ to $\wp(U)$ ($\wp(U)$ is the set of all subsets of U). For a subset $A \subseteq U$, $A^\star_n(I, \mathcal{N}) = \{x \in U : G_n \cap A \notin I$, for every $G_n \in G_n(x)\}$ is called the nano local function (briefly, n-local function) of A with respect to I and $\mathcal{N}$. We will simply write $A^\star_n$ for $A^\star_n(I, \mathcal{N})$.*

Theorem 2.4. *[3] Let $(U, \mathcal{N}, I)$ be a space and A and B be subsets of U. Then*

1. $A \subseteq B \Rightarrow A^\star_n \subseteq B^\star_n$,
2. $A^\star_n = n\text{-}cl(A^\star_n) \subseteq n\text{-}cl(A)$ ($A^\star_n$ *is a n-closed subset of* $n\text{-}cl(A)$),
3. $(A^\star_n)^\star_n \subseteq A^\star_n$,
4. $(A \cup B)^\star_n = A^\star_n \cup B^\star_n$,
5. $V \in \mathcal{N} \Rightarrow V \cap A^\star_n = V \cap (V \cap A)^\star_n \subseteq (V \cap A)^\star_n$,
6. $J \in I \Rightarrow (A \cup J)^\star_n = A^\star_n = (A - J)^\star_n$.

Theorem 2.5. *[3] Let $(U, \mathcal{N}, I)$ be a space with an ideal I and $A \subseteq A^\star_n$, then $A^\star_n = n\text{-}cl(A^\star_n) = n\text{-}cl(A)$.*

Definition 2.6. *[3] Let $(U, \mathcal{N}, I)$ be a space. The set operator $n\text{-}cl^\star$ called a nano $\star$-closure is defined by $n\text{-}cl^\star(A) = A \cup A^\star_n$ for $A \subseteq X$.*

It can be easily observed that $n\text{-}cl^\star(A) \subseteq n\text{-}cl(A)$.

Theorem 2.7. *[4] In a space $(U, \mathcal{N}, I)$, if A and B are subsets of U, then the following results are true for the set operator $n\text{-}cl^\star$.*

1. $A \subseteq n\text{-}cl^\star(A)$,
2. $n\text{-}cl^\star(\phi) = \phi$ *and* $n\text{-}cl^\star(U) = U$,
3. *If* $A \subset B$, *then* $n\text{-}cl^\star(A) \subseteq n\text{-}cl^\star(B)$,
4. $n\text{-}cl^\star(A) \cup n\text{-}cl^\star(B) = n\text{-}cl^\star(A \cup B)$.
5. $n\text{-}cl^\star(n\text{-}cl^\star(A)) = n\text{-}cl^\star(A)$.

Definition 2.8. *[6] A subset A of space $(U, \mathcal{N}, I)$ is said to be*

1. *nano α-I-open (briefly, α-nI-open) if* $A \subset n\text{-}int(n\text{-}cl^\star(n\text{-}int(A)))$,
2. *nano pre-I-open (briefly, pre-nI-open) if* $A \subset n\text{-}int(n\text{-}cl^\star(A))$.

3 On nano t-I-set, nano t_α-I-set, nano $\mathcal{R}$-I-set and nano $\mathcal{R}_\alpha$-I-sets

Definition 3.1. *A subset A of a space $(U, \mathcal{N}, I)$ is called*

1. *nano t-I-set (briefly, t-nI-set) if* $n\text{-}int(A) = n\text{-}int(n\text{-}cl^\star(A))$,
2. *nano t_α-I-set (briefly, t_α-nI-set) if* $n\text{-}int(A) = n\text{-}int(n\text{-}cl^\star(n\text{-}int(A)))$,
3. *nano $\mathcal{R}$-I-set (briefly, $\mathcal{R}$-nI-set) if $A = P \cap Q$, where P is n-open and Q is t-nI-set,*
4. *nano $\mathcal{R}_\alpha$-I-set (briefly, $\mathcal{R}_\alpha$-nI-set) if $A = P \cap Q$, where P is n-open and Q is t_α-nI-set.*

Example 3.2. *Let $U = \{a_1, a_2, a_3, a_4\}$ with $U/R = \{\{a_2\}, \{a_4\}, \{a_1, a_3\}\}$ and $X = \{a_3, a_4\}$. Then $\mathcal{N} = \{\phi, \{a_4\}, \{a_1, a_3\}, \{a_1, a_3, a_4\}, U\}$. Let the ideal be $I = \{\phi, \{a_3\}\}$.*

1. *$A = \{a_2\}$ is t-nI-set.*
2. *$B = \{a_4\}$ is t_α-nI-set.*
3. *$C = \{a_2, a_3\}$ is $\mathcal{R}$-nI-set*
4. *$D = \{a_1, a_3\}$ is $\mathcal{R}_\alpha$-nI-set*

Remark 3.3. *In a space $(U, \mathcal{N}, I)$,*

1. *each n-open set is $\mathcal{R}$-nI-set.*
2. *each t-nI-set is $\mathcal{R}$-nI-set.*

Proposition 3.4. *Let A and B be subsets of a space $(U, \mathcal{N}, I)$. If A and B are t-nI-sets, then $A \cap B$ is t-nI-set.*

Proof. Let A and B be t-nI-sets. Then we have

$$\begin{aligned} n\text{-}int(A \cap B) &\subset n\text{-}int(n\text{-}cl^\star(A \cap B)) \\ &\subset n\text{-}int(n\text{-}cl^\star(A) \cap n\text{-}cl^\star(B)) \\ &= n\text{-}int(n\text{-}cl^\star(A)) \cap n\text{-}int(n\text{-}cl^\star(B)) \\ &= n\text{-}int(A) \cap n\text{-}int(B) \\ &= n\text{-}int(A \cap B). \end{aligned}$$

Then $n\text{-}int(A \cap B) = n\text{-}int(n\text{-}cl^\star(A \cap B))$ and hence $A \cap B$ is a t-nI-set.

Example 3.5. *In Example 3.2, $H = \{a_1, a_3\}$ and $K = \{a_3, a_4\}$ is t-nI-set. But $H \cap K = \{a_3\}$ is t-nI-set.*

Proposition 3.6. *For a subset A of a space $(U, \mathcal{N}, I)$, the following properties are equivalent:*

1. *A is n-open,*

2. *A is pre-nI-open and $\mathcal{R}$-nI-set.*

Proof. (1) ⇒ (2): Let A be n-open. Then

$$A = n\text{-}int(A) \subset n\text{-}int(n\text{-}cl^{\star}(A))$$

and A is pre-nI-open. Also by Remark 3.3 A is $\mathcal{R}$-nI-set.

(2) ⇒ (1): Given A is $\mathcal{R}$-nI-set. So $A = P \cap Q$ where P is n-open and

$$n\text{-}int(Q) = n\text{-}int(n\text{-}cl(Q))$$

Then $A \subset P = n\text{-}int(P)$. Also, A is pre-nI-open implies

$$A \subset n\text{-}int(n\text{-}cl(A)) \subset n\text{-}int(n\text{-}cl^{\star}(Q)) = n\text{-}int(Q)$$

by assumption. Thus

$$A \subset n\text{-}int(P) \cap n\text{-}int(Q) = n\text{-}int(P \cap Q) = n\text{-}int(A)$$

and hence A is n-open.

Remark 3.7. *In a space the family of pre-nI-open sets and the family of $\mathcal{R}$-nI-sets are independent.*

Example 3.8. *In Example 3.2,*

1. *$A = \{a_1, a_4\}$ is pre-nI-open but not $\mathcal{R}$-nI-set.*
2. *$B = \{a_2\}$ is $\mathcal{R}$-nI-set but not pre-nI-open.*

Remark 3.9. *In a space $(U, \mathcal{N}, I)$,*

1. *each n-open set is $\mathcal{R}_\alpha$-nI-set.*
2. *each t_α-nI-set is $\mathcal{R}_\alpha$-nI-set.*

These relations are shown in the diagram.

$$\begin{array}{ccccc} t\text{-}nI\text{-}\mathbf{set} & & & & t_\alpha\text{-}nI\text{-}\mathbf{set} \\ \downarrow & & & & \downarrow \\ \mathcal{R}\text{-}nI\text{-}\mathbf{set} & \longleftarrow & n\text{-}\mathbf{open} & \longrightarrow & \mathcal{R}_\alpha\text{-}nI\text{-}\mathbf{set} \end{array}$$

The converses of diagram is not true as shown in the following Example.

Example 3.10. *In Example 3.2,*

1. *$A = \{a_2\}$ is $\mathcal{R}$-nI-set but not n-open set.*
2. *$B = \{a_1, a_3, a_4\}$ is $\mathcal{R}$-nI-set but not t-nI-set.*
3. *$A = \{a_1\}$ is $\mathcal{R}_\alpha$-nI-set but not n-open set.*
4. *$B = \{a_1, a_3, a_4\}$ is $\mathcal{R}_\alpha$-nI-set but not t_α-nI-set.*

Proposition 3.11. *If A and B are t_α-nI-sets of a space $(U, \mathcal{N}, I)$, then $A \cap B$ is a t_α-nI-set.*

Proof. Let A and B be t_α-nI-sets. Then we have

$$\begin{aligned} n\text{-}int(A \cap B) &\subset n\text{-}int(n\text{-}cl^\star(n\text{-}int(A \cap B))) \\ &\subset n\text{-}int[n\text{-}cl^\star(n\text{-}int(A)) \cap n\text{-}cl^\star(n\text{-}int(B))] \\ &= n\text{-}int(n\text{-}cl^\star(n\text{-}int(A))) \cap n\text{-}int(n\text{-}cl^\star(n\text{-}int(B))) \\ &= n\text{-}int(A) \cap n\text{-}int(B) \\ &= n\text{-}int(A \cap B). \end{aligned}$$

Then $n\text{-}int(A \cap B) = n\text{-}int(n\text{-}cl^\star(n\text{-}int(A \cap B)))$ and hence $A \cap B$ is a t_α-nI-set.

Example 3.12. *In Example 3.2, $H = \{a_2, a_3\}$ and $K = \{a_1, a_2\}$ is t_α-nI-set. But $H \cap K = \{a_2\}$ is t_α-nI-set.*

Proposition 3.13. *For a subset A of a space $(U, \mathcal{N}, I)$, the following properties are equivalent:*

1. *A is n-open;*

2. *A is α-nI-open and a $\mathcal{R}_\alpha$-nI-set.*

Proof. (1) ⇒ (2): Let A be n-open. Then

$$A = n\text{-}int(A) \subset n\text{-}cl^\star(n\text{-}int(A))$$

and

$$A = n\text{-}int(A) \subset n\text{-}int(n\text{-}cl^\star(n\text{-}int(A)))$$

Therefore A is α-nI-open. Also by (1) of Remark 3.9, A is a $\mathcal{R}_\alpha$-nI-set.

(2) ⇒ (1): Given A is a $\mathcal{R}_\alpha$-nI-set. So $A = P \cap Q$ where P is n-open and

$$n\text{-}int(Q) = n\text{-}int(n\text{-}cl^\star(n\text{-}int(Q)))$$

Then $A \subset P = n\text{-}int(P)$. Also A is α-nI-open implies

$$A \subset n\text{-}int(n\text{-}cl^\star(n\text{-}int(H))) \subset n\text{-}int(n\text{-}cl^\star(n\text{-}int(Q))) = n\text{-}int(Q)$$

by assumption. Thus

$$A \subset n\text{-}int(P) \cap n\text{-}int(Q) = n\text{-}int(P \cap Q) = n\text{-}int(A)$$

and A is n-open.

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On Grill S_p-Open Set in Grill Topological Spaces

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Abstract - In this paper, we introduce a new type of grill set namely; Gs_p-open sets, which is analogous to the G-semiopen sets in a grill topological space (X, τ, G). Further, we define Gs_p-continuous and Gs_p-open functions by using a Gs_p-open set and we investigate some of their important properties.

Keywords - *Gs_p-open set, G$s_p O(X)$, Gs_p-continuous function, Gs_p-open function.*

1. Introduction and Preliminaries

Choquet [2] introduced the concept of grill on a topological space and the idea of grills has shown to be a essential tool for studying some topological concepts. A collection G of nonempty subsets of a topological space (X, τ) is called a grill on X if (i) $A \in$ G and $A \subseteq B$ implies that $B \in$ G, and (ii) A, $B \subseteq X$ and $A \cup B \in$ G implies that $A \in$ G or $B \in$ G. A triple (X, τ, G) is called a grill topological space.

Roy and Mukherjee [17] defined a unique topology by a grill and they studied topological concepts. For any point x of a topological space (X, τ), $\tau(x)$ denotes the collection of all open neighborhoods of x. A mapping $\varphi : P(X) \to P(X)$ is defined as $\varphi(A) = \{x \in X : A \cap U \in$ G for all $U \in \tau(x)\}$ for each $A \in P(X)$. A mapping $\psi : P(X) \to P(X)$ is defined as $\psi(A) = A \cup \varphi(A)$ for all $A \in P(X)$. The map ψ satisfies Kuratowski closure axioms:

(i) $\psi(\emptyset) = \emptyset$,
(ii) if $A \subseteq B$, then $\psi(A) \subseteq \psi(B)$,
(iii) if $A \subseteq X$, then $\psi(\psi(A)) = \psi(A)$, and

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(iv) if A, $B \subseteq X$, then $\psi(A \cup B) = \psi(A) \cup \psi(B)$.

Corresponding to a grill G on a topological space (X, τ), there exists a unique topology τ_G (say) on X given by $\tau_G = \{U \subseteq X : \psi(X - U) = X - U\}$, where for any $A \subseteq X$, $\psi(A) = A \cup \varphi(A) = \tau_G$-cl$(A)$ and $\tau \subseteq \tau_G$.

The concept of decompositions of continuity on a grill topological space and some classes of sets were defined with respect to grill (see [3, 7, 10] for details). A subset A in X is said to be

(i) φ-open if $A \subseteq \text{int}(\varphi(A))$,
(ii) G-α.open if $A \subseteq \text{int}(\psi(\text{int}(A)))$,
(iii) G-preopen if $A \subseteq \text{int}(\psi(A))$,
(iv) G-semiopen if $A \subseteq \psi(\text{int}(A))$,
(v) G-β.open if $A \subseteq \text{cl}(\text{int}(\psi(A)))$.

The family of all G-α.open (resp. G-preopen, G-semiopen, G-β.open) sets in a grill topological space (X,τ,G) is denoted by $G\alpha O(X)$ (rep. $GPO(X)$, $GSO(X)$, $G\beta O(X)$). A function $f: (X,\tau,G) \to (Y, \sigma)$ is said to be G-semicontinuous if $f^{-1}(V) \in GSO(X)$ for each $V \in \sigma$.

Mashhour et al. [14] introduced a class of preopen sets and he defined pre interior and pre closure in a topological space. A subset A in X is said to be preopen if $A \subseteq \text{int}(\text{cl}(A))$ and $PO(X)$ denotes the family of preopen sets. For any subset A of X, (i) pint$(A) = \cup\{U : U \in PO(X)$ and $U \subseteq A\}$; (ii) pcl$(A) = \cap\{F : X - F \in PO(X)$ and $A \subseteq F\}$.

In this paper, we define a Gs_p-open set in a grill topological space (X, τ, G) and we study some of its basic properties. Moreover, we define Gs_p-continuous, Gs_p-open, Gs_p-closed and Gs_p^*-continuous functions on a grill topological space (X, τ, G) and we discuss some of their essential properties.

Proposition 1.1. [17] Let (X, τ, G) be a grill topological space. Then for all A, $B \subseteq X$:
(i) $A \subseteq B$ implies that $\varphi(A) \subseteq \varphi(B)$;
(ii) $\varphi(A \cup B) = \varphi(A) \cup \varphi(B)$;
(iii) $\varphi(\varphi(A)) \subseteq \varphi(A) = \text{cl}(\varphi(A)) \subseteq \text{cl}(A)$.

2. Gs_p-Open Sets

Definition 2.1. Let (X,τ,G) be a grill topological space and let A be a subset A of X. Then A is said to be Gs_p-open if and only if there exist a $U \in PO(X)$ such that $U \subseteq A \subseteq \psi(U)$. A set A of X is Gs_p-closed if its complement $X - A$ is Gs_p-open. The family of all Gs_p-open (resp. Gs_p-closed) sets is denoted by G$s_pO(X)$ (resp. G$s_pC(X)$).

Example 2.1. Let $X = \{a, b, c, d\}$, $\tau = \{\emptyset, X, \{a, b\}, \{c, d\}, \{a,b,c\}, \{a,b,d\}\}$ and $G = \{\{d\}, \{a, d\}, \{b, d\}, \{c, d\}, \{a,b,d\}, \{a,c,d\}, \{b,c,d\}, X\}$. Then G$s_pO(X) = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}, \{a,b,c\}, \{a,b,d\}, \{a,c,d\}, \{b,c,d\}\}$.

Theorem 2.1. Let (X,τ,G) be a grill topological space and let $A \subseteq X$. Then $A \in \mathrm{G}s_pO(X)$ if and only if $A \subseteq \psi(\mathrm{pint}(A))$.

Proof. If $A \in \mathrm{G}s_pO(X)$, then there exist a $U \in PO(X)$ such that $U \subseteq A \subseteq \psi(U)$. But $U \subseteq A$ implies that $U \subseteq \mathrm{pint}(A)$. Hence $\psi(U) \subseteq \psi(\mathrm{pint}(A))$. Therefore $A \subseteq \psi(\mathrm{pint}(A))$. Conversely, let $A \subseteq \psi(\mathrm{pint}(A))$. To prove that $A \in \mathrm{G}s_pO(X)$, take $U = \mathrm{pint}(A)$, then $U \subseteq A \subseteq \psi(U)$. Hence $A \in \mathrm{G}s_pO(X)$.

Corollary 2.1. If $A \subseteq X$, then $A \in \mathrm{G}s_pO(X)$ if and only if $\psi(A) = \psi(\mathrm{pint}(A))$.

Proof. Let $\mathrm{A} \in \mathrm{G}s_pO(X)$. Then as ψ is monotonic and idempotent, $\psi(A) \subseteq \psi(\psi(\mathrm{pint}(A))) = \psi(\mathrm{pint}(A)) \subseteq \psi(A)$ implies that $\psi(A) = \psi(\mathrm{pint}(A))$. The converse is obvious.

Theorem 2.2. Let (X,τ,G) be a grill topological space. If $\mathrm{A} \in \mathrm{G}s_pO(X)$ and $B \subseteq X$ such that $A \subseteq B \subseteq \psi(\mathrm{pint}(A))$, then $B \in \mathrm{G}s_pO(X)$.

Proof. Given $A \in \mathrm{G}s_pO(X)$. Then by Theorem 2.1, $A \subseteq \psi(\mathrm{pint}(A))$. But $A \subseteq B$ implies that $\mathrm{pint}(A) \subseteq \mathrm{pint}(B)$ and hence by Theorem 2.4[17], $\psi(\mathrm{pint}(A)) \subseteq \psi(\mathrm{pint}(B))$. Therefore $B \subseteq \psi(\mathrm{pint}(A)) \subseteq \psi(\mathrm{pint}(B))$. Hence $B \in \mathrm{G}s_pO(X)$.

Corollary 2.2. If $\mathrm{A} \in \mathrm{G}s_pO(X)$ and $B \subseteq X$ such that $A \subseteq B \subseteq \psi(A)$, then $B \in \mathrm{G}s_\alpha O(X)$.

Proof. Follows from the Theorem 2.2 and Corollary 2.1.

Proposition 2.1. If $U \in PO(X)$, then $U \in \mathrm{G}s_pO(X)$.

Proof. Let $U \in PO(X)$, it implies that $U = \mathrm{pint}(U) \subseteq \psi(\mathrm{pint}(U))$. Hence $U \in \mathrm{G}s_pO(X)$.

Note that the converse of the above proposition need not be true. Let $X = \{a, b, c, d\}$, $\tau = \{\emptyset, X, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, b, c\}\}$ and $\mathrm{G} = \{\{a\}, \{b\}, \{d\}, \{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}, X\}$. Then $PO(X) = \{\emptyset, X, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, b, c\}, \{b, c, d\}\}$ and $\mathrm{G}s_pO(X) = \{\emptyset, X, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{b, d\}, \{a, b, c\}, \{a, b, d\}, \{b, c, d\}\}$. Here $\{b, d\}$ and $\{a, b, d\}$ are $\mathrm{G}s_p$-open sets but not preopen.

Theorem 2.3. Let (X,τ,G) be a grill topological space. If $A \in \mathrm{G}SO(X)$, then $A \in \mathrm{G}s_pO(X)$.

Proof. Given $A \in \mathrm{G}SO(X)$. Then $A \subseteq \psi(\mathrm{int}(A))$. Since $\mathrm{int}(A) \subseteq \mathrm{pint}(A)$, we have that $\psi(\mathrm{int}(A)) \subseteq \psi(\mathrm{pint}(A))$ (by Theorem 2.4[17]). Hence $A \subseteq \psi(\mathrm{pint}(A))$ and thus $A \in \mathrm{G}s_pO(X)$.

Note that the converse of the above theorem need not be true. By Example 2.1, we have that $\mathrm{G}SO(X) = \{\emptyset, X, \{a, b\}, \{c, d\}, \{a,b,c\}, \{a,b,d\}\}$. Therefore $\{a\}$, $\{b\}$, $\{a, c\}$, $\{a, d\}$, $\{b, c\}$, $\{b, d\}$, $\{a,c,d\}$ and $\{b,c,d\}$ are $\mathrm{G}s_p$-open sets but not G-semiopen.

Proposition 2.2. If $PO(X) = \tau$, then $\mathrm{G}s_pO(X) = \mathrm{G}SO(X)$.

Proof. By Theorem 2.3, $\mathrm{G}SO(X) \subseteq \mathrm{G}s_pO(X)$. Let $A \in \mathrm{G}s_pO(X)$. Then by Theorem 2.1, $A \subseteq \psi(\mathrm{pint}(A))$. Since $PO(X) = \tau$, we have that $\mathrm{pint}(A) = \mathrm{int}(A)$ implies that $A \subseteq \psi(\mathrm{pint}(A)) = \psi(\mathrm{int}(A))$ and hence $A \in \mathrm{G}SO(X)$. Thus $\mathrm{G}s_pO(X) \subseteq \mathrm{G}SO(X)$.

Theorem 2.4. Let (X,τ,G) be a grill topological space.
(i) If $A_i \in \mathrm{G}s_pO(X)$ for each $i \in J$, then $\cup_{i\in J}A_i \in \mathrm{G}s_pO(X)$;
(ii) If $A \in \mathrm{G}s_pO(X)$ and $U \in PO(X)$, then $A \cap U \in \mathrm{G}s_pO(X)$.

Proof. (i) Since $A_i \in \mathrm{G}s_pO(X)$, we have that $A_i \subseteq \psi(\mathrm{pint}(A_i))$ for each $i \in J$. Thus, we obtain $A_i \subseteq \psi(\mathrm{pint}(A_i)) \subseteq \psi(\mathrm{pint}(\cup_{i\in J}A_i))$ and hence $\cup_{i\in J}A_i \subseteq \psi(\mathrm{pint}(\cup_{i\in J}A_i))$. This shows that $\cup_{i\in J}A_i \in \mathrm{G}s_pO(X)$.

(ii) Let $A \in \mathrm{G}s_pO(X)$ and $U \in PO(X)$. Then $A \subseteq \psi(\mathrm{pint}(A))$ and $\mathrm{pint}(U) = U$. Now, $A \cap U \subseteq \psi(\mathrm{pint}(A)) \cap U = (\mathrm{pint}(A) \cup \varphi(\mathrm{pint}(A))) \cap U = (\mathrm{pint}(A) \cap U) \cup (\varphi(\mathrm{pint}(A)) \cap U) \subseteq \mathrm{pint}(A \cap U) \cup \varphi(\mathrm{pint}(A) \cap U)$ (by Theorem 2.10[17]) $= \mathrm{pint}(A \cap U) \cup \varphi(\mathrm{pint}(A \cap U)) = \psi(\mathrm{pint}(A \cap U))$. Therefore $A \cap U \in \mathrm{G}s_pO(X)$.

Remark 2.1. The following example shows that if $A, B \in \mathrm{G}s_pO(X)$, then $A \cap B \notin \mathrm{G}s_pO(X)$.

From Example 2.1, take $A = \{b, c\}$ and $B = \{c, d\}$, then $A, B \in \mathrm{G}s_pO(X)$ but $A \cap B = \{c\} \notin \mathrm{G}s_pO(X)$.

Theorem 2.5. Let (X,τ,G) be a grill topological space and $A \subseteq X$. If $A \in \mathrm{G}s_pC(X)$, then $\mathrm{pint}(\psi(A)) \subseteq A$.

Proof. Suppose $A \in \mathrm{G}s_pC(X)$. Then $X - A \in \mathrm{G}s_pO(X)$ and hence $X - A \subseteq \psi(\mathrm{pint}(X - A)) \subseteq \mathrm{pcl}(\mathrm{pint}(X - A)) = X - \mathrm{pint}(\mathrm{pcl}(A)) \subseteq X - \mathrm{pint}(\psi(A))$, implies that $\mathrm{pint}(\psi(A)) \subseteq A$.

Theorem 2.6. Let (X,τ,G) be a grill topological space and $A \subseteq X$ such that $X - \mathrm{pint}(\psi(A)) = \psi(\mathrm{pint}(X - A))$. Then $A \in \mathrm{G}s_pC(X)$ if and only if $\mathrm{pint}(\psi(A)) \subseteq A$.

Proof. Necessary part is proved by Theorem 2.5. Conversely, suppose that $\mathrm{pint}(\psi(A)) \subseteq A$. Then $X - A \subseteq X - \mathrm{pint}(\psi(A)) = \psi(\mathrm{pint}(X - A))$, implies that $X - A \in \mathrm{G}s_pO(X)$. Hence $A \in \mathrm{G}s_pC(X)$.

Definition 2.2. Let (X,τ,G) be a grill topological space and $A \subseteq X$. Then
(i) $\mathrm{G}s_p$-interior of A is defined as union of all $\mathrm{G}s_p$-open sets contained in A.
Thus $\mathrm{G}s_p\mathrm{int}(A) = \cup\{U : U \in \mathrm{G}s_pO(X) \text{ and } U \subseteq A\}$;
(ii) $\mathrm{G}s_p$-closure of A is defined as intersection of all $\mathrm{G}s_p$-closed sets containing A.
Thus $\mathrm{G}s_p\mathrm{cl}(A) = \cap\{F : X - F \in \mathrm{G}s_pO(X) \text{ and } A \subseteq F\}$.

Theorem 2.7. Let (X,τ,G) be a grill topological space and $A \subseteq X$. Then
(i) $\mathrm{G}s_p\mathrm{int}(A)$ is a $\mathrm{G}s_p$-open set contained in A;
(ii) $\mathrm{G}s_p\mathrm{cl}(A)$ is a $\mathrm{G}s_p$-closed set containing A;
(iii) A is $\mathrm{G}s_p$-closed if and only if $\mathrm{G}s_p\mathrm{cl}(A) = A$;
(iv) A is $\mathrm{G}s_p$-open if and only if $\mathrm{G}s_p\mathrm{int}(A) = A$;
(v) $\mathrm{G}s_p\mathrm{int}(A) = X - \mathrm{G}s_p\mathrm{cl}(X - A)$;
(vi) $\mathrm{G}s_p\mathrm{cl}(A) = X - \mathrm{G}s_p\mathrm{int}(X - A)$.

Proof. Follows form the Definition 2.15 and Theorem 2.4(i).

Theorem 2.8. Let (X,τ,G) be a grill topological space and A, $B \subseteq X$. Then the following are hold: (i) If $A \subseteq B$, then $\mathrm{G}s_p\mathrm{int}(A) \subseteq \mathrm{G}s_p\mathrm{int}(B)$;
(ii) $\mathrm{G}s_p\mathrm{int}(A \cup B) \supseteq \mathrm{G}s_p\mathrm{int}(A) \cup \mathrm{G}s_p\mathrm{int}(B)$;
(iii) $\mathrm{G}s_p\mathrm{int}(A \cap B) = \mathrm{G}s_p\mathrm{int}(A) \cap \mathrm{G}s_p\mathrm{int}(B)$.

Proof. Follows from the Theorem 2.8.

Definition 2.3. A function f: $(X,\tau,\mathrm{G}) \to (Y, \sigma)$ is said to be $\mathrm{G}s_p$-continuous if $f^{-1}(V) \in \mathrm{G}s_pO(X)$ for each $V \in PO(Y)$.

Example 2.2. Let $X = \{a, b, c, d\}$, $Y = \{1, 2, 3, 4\}$, $\tau = \{\emptyset, X, \{a,b\}, \{c,d\}\}$, $\sigma = \{\emptyset, Y, \{1, 2\}, \{3, 4\}, \{1, 2, 3\}, \{1, 2, 4\}\}$and $\mathrm{G} = \{\{a,b, c\}, X\}$. Then $\mathrm{G}s_pO(X) = P(X)$ and $PO(Y) = \{\emptyset, Y, \{1\}, \{2\}, \{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{3, 4\}, \{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\}\}$. Define f: $(X, \tau,\mathrm{G}) \to (Y, \sigma)$ by $f(a) = 2$, $f(b) = 1$, $f(c) = 4$ and $f(d) = 3$. Then inverse image of every preopen sets in Y is $\mathrm{G}s_p$-open in X. Hence f is $\mathrm{G}s_p$-continuous.

Remark 2.2. The concepts of G-semicontinuous and $\mathrm{G}s_p$-continuous are independent.

(i) From Example 2.2, we have that $\mathrm{G}SO(X) = \{\emptyset, X, \{a,b\}, \{c,d\}\}$ and the function f is $\mathrm{G}s_p$-continuous. Also $f^{-1}(\{1, 2, 3\}) = \{a, b, d\}$ is not G-semiopen in X for the open set $\{1, 2, 3\}$ of Y. Hence f is not G-semicontinuous.

(ii) Let $X = \{a, b, c, d\}$, $Y = \{1, 2, 3, 4\}$, $\tau = \{\emptyset, X, \{a\}, \{a,b\}, \{c,d\}, \{a,c,d\}, \{b,c,d\}\}$, $\sigma = \{\emptyset, Y, \{1, 2\}, \{3, 4\}\}$ and $\mathrm{G} = \{\{b\}, \{a, b\}, \{b, c\}, \{b, d\}, \{c, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}, X\}$. Then $\mathrm{G}SO(X) = \tau$, $\mathrm{G}s_pO(X) = \{\emptyset, X, \{a\}, \{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}\}$ and $PO(Y) = P(Y)$. Define f: $(X, \tau,\mathrm{G}) \to (Y, \sigma)$ by $f(a) = 4$, $f(b) = 3$, $f(c) = 2$ and $f(d) = 1$. Then the function f is G-semicontinuous. Also the inverse image $f^{-1}(\{3\}) = \{b\}$ is not $\mathrm{G}s_p$-open in X for the preopen set $\{3\}$ of Y. Hence f is not $\mathrm{G}s_p$-continuous.

From (i) and (ii), we got the concepts of G-semicontinuous and $\mathrm{G}s_p$-continuous are independent.

Theorem 2.9. For a function f: $(X,\tau,G) \to (Y, \sigma)$, the following are equivalent:
(i) f is Gs_p-continuous;
(ii) For each F $\in PC(Y)$, $f^{-1}(F) \in$ G$s_pC(X)$;
(iii) For each $x \in X$ and each $V \in PO(Y)$ containing $f(x)$, there exists a $U \in$ G$s_pO(X)$ containing x such that $f(U) \subseteq V$.

Proof. (i) $\Leftrightarrow$ (ii): It is obvious.

(i) $\Rightarrow$ (iii): Let $V \in PO(Y)$ and $f(x) \in V(x \in X)$. Then by (i), $f^{-1}(V) \in$ G$s_pO(X)$ containing x. Taking $f^{-1}(V) = U$, we have that $x \in U$ and $f(U) \subseteq V$.

(iii) $\Rightarrow$ (i): Let $V \in PO(Y)$ and $x \in f^{-1}(V)$. Then $f(x) \in V \in PO(Y)$ and hence by (iii), there exists a $U \in$ G$s_pO(X)$ containing x such that $f(U) \subseteq V$. Thus, we obtain $x \in U \subseteq \psi$ (pint(U)) $\subseteq$ ψ(pint($f^{-1}(V)$)). This shows that $f^{-1}(V) \subseteq \psi$(pint($f^{-1}(V)$)). Hence f is Gs_p-continuous.

Theorem 2.10. A function f: $(X,\tau,G) \to (Y, \sigma)$ is Gs_p-continuous if and only if the graph function $g : X \to X \times Y$, defined by $g(x) = (x, f(x))$ for each $x \in X$, is Gs_p-continuous.

Proof. Suppose that f is Gs_p-continuous. Let $x \in X$ and $W \in PO(X \times Y)$ containing $g(x)$. Then there exist a $U \in PO(X)$ and $V \in PO(Y)$ such that $g(x) = (x, f(x)) \in U \times V \subseteq W$. Since f is Gs_p-continuous, there exists a $G \in$ G$s_pO(X)$ containing x such that $f(G) \subseteq V$. By Theorem 2.4(b), $G \cap U \in$ G$s_pO(X)$ and $g(G \cap U) \subseteq U \times V \subseteq W$. This shows that g is Gs_p-continuous. Conversely, suppose that g is Gs_p-continuous. Let $x \in X$ and $V \in \alpha(Y)$ containing $f(x)$. Then $X \times V \in PO(X \times Y)$ and by Gs_p-continuity of g, there exists a $U \in$ G$s_pO(X)$ containing x such that $g(U) \subseteq X \times V$. Thus we have that $f(U) \subseteq V$ and hence f is G-s_p.continuous.

Definition 2.3. Let (X, τ) be a topological space and (Y,σ,G) a grill topological space. A function f: $(X, \tau) \to (Y,\sigma,G)$ is said to be Gs_p-open (resp. Gs_p-closed) if for each $U \in PO(X)$ (resp. for each $U \in PC(X)$), $f(U)$ is Gs_p-open (resp. Gs_p-closed) in (Y, σ,G).

Theorem 2.11. A function f: $(X, \tau) \to (Y,\sigma,G)$ is Gs_p-open if and only if for each $x \in X$ and each pre-neighbourhood U of x, there exists a $V \in$ G$s_pO(Y)$ such that $f(x) \in V \subseteq f(U)$.

Proof. Suppose that f is a G-s_p.open function and let $x \in X$. Also let U be any pre-neighbourhood of x. Then there exists $G \in PO(X)$ such that $x \in G \subseteq U$. Since f is Gs_p-open, $f(G) = V$ (say) $\in$ G$s_pO(Y)$ and $f(x) \in V \subseteq f(U)$. Conversely, suppose that $U \in PO(X)$. Then for each $x \in U$, there exists a $V_x \in$ G$s_pO(X)$ such that $f(x) \in V_x \subseteq f(U)$. Thus $f(U) = \cup\{V_x : x \in U\}$ and hence by Theorem 2.4(a), $f(U) \in$ G$s_pO(Y)$. This shows that f is Gs_p-open.

Theorem 2.12. Let f: $(X, \tau) \to (Y,\sigma,G)$ be a G-s_p.open function. If $V \subseteq Y$ and $F \in PC(X)$ containing $f^{-1}(V)$, then there exists a $H \in$ G$s_pO(Y)$ containing V such that $f^{-1}(H) \subseteq F$.

Proof. Suppose that f is G-s_p.open. Let $V \subseteq Y$ and $F \in PC(X)$ containing $f^{-1}(V)$. Then $X - F \in PO(X)$ and by Gs_p-openness of f, $f(X - F) \in$ G$s_pO(X)$. Thus H = $Y - f(X - F) \in$ G$s_pC(Y)$ consequently $f^{-1}(V) \subseteq F$ implies that $V \subseteq H$. Further, we obtain that $f^{-1}(H) \subseteq F$.

Theorem 2.13. For any bijection f: $(X, \tau) \rightarrow (Y,\sigma$,G), the following are equivalent:
(i) f^{-1}: $(Y,\sigma$,G) $\rightarrow (X, \tau)$ is Gs_p-continuous;
(ii) f is Gs_p-open;
(iii) f is Gs_p-closed.

Proof. It is obvious.

Definition 2.4. Let $(X,\tau$,G) be a grill topological space. A subset A of X is said to be a Gs_p^*-set if $A = U \cap V$, where $U \in PO(X)$ and ψ(pint(V)) = pint(V).

Theorem 2.14. Let $(X,\tau$,G) be a grill topological space and let $A \subseteq X$. Then $A \in PO(X)$ if and only if $A \in$ G$s_pO(X)$ and A is Gs_p^*-set in $(X,\tau$,G).

Proof. Let $A \in PO(X)$. Then $A \in$ G$s_pO(X)$, implies that $A \subseteq \psi$(pint(A)). Also A can be expressed as $A = A \cap X$, where $A \in PO(X)$ and ψ(pint(X)) = pint(X). Thus A is a Gs_p^*-set. Conversely, Let $A \in$ G$s_pO(X)$ and A be a Gs_p^*-set. Thus $A \subseteq \psi$(pint(A)) = ψ(pint($U \cap V$)), where $U \in PO(X)$ and ψ(pint(V)) = pint(V). Now $A \subseteq U \cap A \subseteq U \cap \psi$(pint($U \cap V$)) = $U \cap \psi(U \cap$ pint(V)) $\subseteq U \cap \psi(U) \cap \psi$(pint($V$)) = $U \cap$ pint(V) = pint(A). Hence $A \in PO(X)$.

Definition 2.5. A function f: $(X,\tau$,G) $\rightarrow (Y, \sigma)$ is Gs_p^*-continuous if for each $V \in PO(Y)$, $f^{-1}(V)$ is a Gs_p^*-set in $(X, \tau$,G).

Theorem 2.15. Let $(X,\tau$,G) be a grill topological space. Then for a function f: $(X,\tau$,G) $\rightarrow (Y, \sigma)$, the following are equivalent:
(i) f is precontinuous;
(ii) f is Gs_p-continuous and Gs_p^*-continuous.

Proof. Straightforward.

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Soft Almost b-Continuous Mappings

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Abstaract — In the present paper, we introduce one class of soft mappings,namely soft almost b-continuous mappings and investigate several properties of these mappings.This notion is stronger than soft almost β-continuous mappings and is weaker than both soft almost pre-continuous mappings and soft almost semi-continuous mappings.The diagrams of implications among these soft classes of mappings and some known classes of mappings have been established.

Keywords — *Soft regular open set, Soft b-open set, Soft δ-open set, Soft almost continuous mappings, Soft b-continuous mappings.*

1 Introduction

In 1999, Molodtsov [14] introduced the concept of soft sets to deal with uncertainties while modelling the problems with incomplete information. In 2011 Shabir and Naz [15] initiated the study of soft topological spaces.Theoretical study of soft sets and soft topological spaces have been by some authors in [6, 8, 9, 10, 11, 14, 15, 23, 25, 26]. Soft regular-open sets[5], soft semi-open sets[12],soft preopen sets [2], soft α-open sets [4],soft β-open sets [3], soft b-open sets [1] play an important part in the researches of generalizations of continuity in soft topological spaces. The aim of this paper is to introduce one class of soft mappings, namely soft almost b-continuous mappings by utilizing the notions of soft b-open sets due to [1]. We investigate several properties of this class. The class of soft almost b-continuous mappings is a generalization of soft almost pre-continuous mappings and soft almost semi-continuous mappings. At the same time, the class of soft almost β-continuous mappings is a generalization of soft almost b-continuous mappings.

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2 Preliminary

Let U is an initial universe set, E be a set of parameters, P(U) be the power set of U and A $\subseteq$ E.

Definition 2.1. [14] A pair (F, A) is called a soft set over U, where F is a mapping given by F: A $\to$ P(U). In other words, a soft set over U is a parameterized family of subsets of the universe U. For all e $\in$ A, F(e) may be considered as the set of e-approximate elements of the soft set (F, A).

Let X and Y be an initial universe sets and E and K be the non empty sets of parameters, S(X, E) denotes the family of all soft sets over X and S(Y, K) denotes the family of all soft sets over Y.

Definition 2.2. [15] A subfamily τ of S(X , E) is called a soft topology on X if:

1. $\widetilde{\phi}$, $\widetilde{X}$ belong to τ.
2. The union of any number of soft sets in τ belongs to τ.
3. The intersection of any two soft sets in τ belongs to τ.

The triplet (X, τ, E) is called a soft topological space over X. The members of τ are called soft open sets in X and their complements called soft closed sets in X.

Definition 2.3. [25] The soft set (F, E)$\in$ S(X, E) is called a soft point if there exist x $\in$ X and e $\in$ E such that F(e) = {x} and F(e') = ϕ for each e' $\in$ E – {e}, and the soft point (F, E) is denoted by $(x_e)_E$.

Definition 2.4. [23, 4, 7, 5, 1, 3] A soft set (F,E) in a soft topological space (X,τ,E) is said to be :
(a) Soft regular open if (F,E)= Int(Cl(F,E)).
(b) Soft α-open if (F,E) $\subseteq$ Int(Cl(Int(F,E))).
(c) Soft semi-open if (F,E) $\subseteq$ Cl(Int(F,E)).
(d) Soft pre-open if (F,E) $\subseteq$ Int(Cl(F,E)).
(e) Soft b-open if (A, E) $\subset$ Int(Cl(A, E)) $\cup$ Cl(Int (A, E)).
(f) Soft β-open if (A, E) $\subset$ Cl(Int(Cl(A, E))).
The complement of soft α-open set (resp. soft semi-open set, soft pre-open,soft b-open, soft β-open) set is called Soft α-closed(resp. soft semi-closed ,soft pre-closed,soft b-closed, soft β-closed) set.

Definition 2.5. [17]A soft point $(x_e)_E$ in a soft topological space (X,τ,E) is called δ-cluster point of a soft set (A,E) of X if Int(Cl(V,E)) $\cap$ (A,E) $\neq \phi$ for each soft open set (V,E) containing $(x_e)_E$.The union of all δ-cluster points of (A, E) is called δ-closure of (A,E) and is denoted by δCl(A, E).

Remark 2.6. [4, 23, 1]
(a) Every soft regular open (resp. soft regular closed) set is soft open (resp. closed), every soft open (resp. soft closed) set is soft α-open (resp. soft α-closed), every soft α-open (resp. soft α-closed) set is soft pre-open (resp. pre-closed) and soft semi-open (resp. semi-closed) but the converses may not be true.

(b) The concepts of soft semi-open (resp. soft semi-closed) and soft pre-open (resp.soft pre-closed)sets are independent to each other.

(c) Every soft pre-open (resp. pre-closed) and soft semi-open (resp. semi-closed) is soft b-open(resp. soft b-closed) set and every soft b-open(resp. soft b-closed) set is soft β-open((resp. soft β-closed) set but the converses may not be true.

Definition 2.7. [1] Let (F,E) be a soft set in a soft topological space (X ,τ,E).

(a) The soft b-closure of (F, E) is defined as the smallest soft b-closed set over which contains (F, E) and it is denoted by bCl(F,E).

(b) The soft b-interior of (F, E) is defined as the largest soft b-open set over which is contained in (F, E) and is denoted by bInt(F,E).

Definition 2.8. [23, 4, 7, 5, 1, 3] Let (X,τ,E) and (Y,υ,K) be a soft topological spaces. A soft mapping f_{pu} : (X,τ,E)$\rightarrow$ (Y,υ,K) is said to be soft continuous (resp. soft α-continuous,soft semi-continuous ,soft pre-continuous,soft b-continuous,soft β-continuous) mapping if f_{pu}^{-1}(G, K) is soft open(resp. soft α-open ,soft semi-open,soft pre-open,soft b-open,soft β-open) over X , for all soft open set(G,K) over Y.

Definition 2.9. [1] A soft mapping f_{pu} : (X,τ,E)$\rightarrow$ (Y,υ,K) is said to be soft b-irresolute if f_{pu}^{-1}(G, K) is soft b-open over X , for all soft b-open set (G, K) over Y.

Definition 2.10. [23, 4, 7, 5, 1, 3] Let (X,τ,E) and (Y,υ,K) be a soft topological spaces. A soft mapping f_{pu} : (X,τ,E)$\rightarrow$ (Y,υ,K) is said to be soft open(resp. soft α-open,soft semi-open,soft pre-open,soft b-open,soft β-open) mapping if f_{pu} (F, E) is soft open(resp. soft α-open ,soft semi-open,soft pre-open,soft b-open,soft β-open) over Y, for all soft open set (F, E) over X.

Remark 2.11. [4, 3, 1]

(a) Every soft continuous (resp. soft open) mapping is soft α-continuous(resp. soft α-open) mapping ,every soft α-continuous (resp. soft α-open) mapping is soft pre-continuous (resp. soft pre-open) and soft semi-continuous (resp. soft semi-open) mapping but the converse may not be true.

(b) The concepts of soft semi-continuous and soft pre-continuous (resp. soft semi-open and soft pre-open) mappings are independent.

(c) Every soft pre-continuous (resp. soft pre-open) and soft semi-continuous (resp. soft semi-open) mappings are soft b-continuous and every soft b-continuous mapping is soft β-continuous mapping but the converse may not be true.

Definition 2.12. [18, 19, 20, 21, 22] A soft mapping f_{pu} : (X ,τ,E) $\rightarrow$ (Y,ϑ,K) is said to be soft almost(resp. α-continuous,semi-continuous,pre-continuous,β-continuous) mapping if the inverse image of every soft regular open set over Y is soft open(soft α-open,soft semi-open,soft pre-open,soft β-open) over X.

In this paper ,we use the abbreviations of soft almost continuous mapping , soft almost α-continuous mapping, soft almost semi-continuous mapping, soft almost pre-continuous mapping, soft almost β-continuous mapping by s.a.c., s.a.α.c., s.a.s.c., s.a.p.c., s.a.β.c. respectively.

Remark 2.13. [18, 19, 20, 21, 22]

(a) Every soft continuous mapping is soft almost continuous but the converse may not be true.

(b) Every soft α-continuous mapping is soft almost α-continuous but the converse may not be true.

(c) Every soft almost continuous(resp. soft almost-open) mapping is soft almost α-continuous(resp. soft almost α-open) but the converse may not be true.

(d) Every soft almost α-continuous(resp. soft almost α-open) mapping is almost pre-continuous(resp. soft almost pre-open) and almost semi-continuous(resp. soft almost semi-open) but the converse may not be true.

(e) Every soft semi-continuous mapping (resp. soft semi-open) is soft almost semi-continuous(resp. soft almost semi-open) but the converse may not be true.

(f) Every soft pre-continuous(resp. soft pre-open) mapping is soft almost pre-continuous(resp. soft almost pre-open) but the converse may not be true.

(g) The concepts of soft almost semi-continuous and soft almost pre-continuous (resp. soft almost semi-open and soft almost pre-open) mappings are independent.

(h) Every soft β-continuous(resp. soft β-open) mapping is soft almost β-continuous(resp. soft almost β-open) but the converse may not be true.

(i) Every soft almost pre-continuous(resp. soft almost pre-open) and soft almost semi-continuous(resp. soft almost semi-open) mapping is soft almost β-continuous(resp. soft almost β-open) but the converse may not be true.

Definition 2.14. [20] A soft topological space (X ,τ,E) is said to be soft semiregular if for each soft open set (F,E) and each soft point $(x_e)_E \in$ (F,E), there exists a soft open set (G,E) such that $(x_e)_E \in$ (G,E) and (G,E) $\subset$ Int(Cl(G,E)) $\subset$ (F,E).

Definition 2.15. [16] Let f_{pu} : (X ,τ , E) $\rightarrow$ (Y , ϑ , K) be a soft mapping. Then a soft mapping $G_{g_p g_u}$: (X ,τ,E) $\rightarrow$ (X $\times$ Y, $\tau \times \vartheta$, E $\times$ K) is said to be soft graph mapping of f_{pu} where g_u and g_p are respectively defined by g_u(x) = (x , u(x)) for all x $\in$ X and g_p(e) = (e , p(e)) for all e $\in$ E.

3 Soft Almost b-Continuous Mappings

Definition 3.1. A soft mapping f_{pu} : (X ,τ,E) $\rightarrow$ (Y,ϑ,K) soft almost b-continuous (brie?y s.a.b.c.) for each soft point $(x_e)_E$ over X and each soft regular open set (V,K) over Y containing $f_{pu}((x_e)_E)$, there exists soft b-open set(U,E) over X containing $(x_e)_E$ such that f_{pu}(U,E) $\subset$ (V,K).

Theorem 3.2. Let f_{pu} : (X ,τ,E) $\rightarrow$ (Y,ϑ,K) be a soft mapping. Then the following conditions are equivalent:

(a) f_{pu} soft almost b-continuous.

(b) For each soft point $(x_e)_E$ over X and each soft open set (V,K) over Y containing $f_{pu}((x_e)_E)$, there exists soft b-open set (U,E) over X containing $(x_e)_E$ such that f_{pu}(U,E) $\subset$ Int(Cl(V,K)).

(c) f_{pu}^{-1}(V,K) be a soft b-open set over X, for every soft regular open set (V,K) over Y.

Proof: It is obvious.

Remark 3.3. Every soft b-continuous mapping is soft almost b-continuous but the converse may not be true.

Example 3.4. Let X = $\{x_1, x_2\}$, E = $\{e_1, e_2\}$ and Y = $\{y_1, y_2\}$, K = $\{k_1, k_2\}$. The soft sets (F,E), (G,K) are defined as follows :
F(e_1) = $\{x_2\}$, F(e_2) = $\{x_1\}$
G(k_1) = $\{y_1\}$, G(k_2) = $\{y_2\}$
Let $\tau = \{\phi$,(F,E), $\tilde{X}\}$, and $\upsilon = \{\phi$,(G,K), $\tilde{Y}\}$ are topologies on X and Y respectively. Then soft mapping f_{pu} : (X,τ,E) $\rightarrow$ (Y,υ,K) defined by u(x_1) = y_1 , u(x_2) = y_2 and p(e_1)= k_1, p(e_2) = k_2 is soft almost b-continuous but not soft b-continuous.

Remark 3.5. Every soft almost semi-continuous is soft almost b-continuous but the converse may not be true.

Example 3.6. Let X = $\{x_1, x_2\}$, E = $\{e_1, e_2\}$ and Y = $\{y_1, y_2\}$, K = $\{k_1, k_2\}$. The soft sets (F_1,E), (F_2,E), (G,K) are defined as follows :
$G_1(k_1) = \{y_1\}$, $G_1(k_2) = \{y_2\}$,
$G_2(k_1) = \{y_2\}$, $G_2(k_2) = \{y_1\}$,
Let $\tau = \{\phi, \tilde{X}\}$, and $\upsilon = \{\phi$, (G_1,K), (G_2,K), $\tilde{Y}\}$ are topologies on X and Y respectively. Then soft mapping f_{pu} : (X,τ,E) $\rightarrow$ (Y,υ,K) defined by u(x_1) = y_1, u(x_2)= y_2 and p(e_1)= k_1, p(e_2) = k_2 is soft almost b-continuous mapping but not soft almost semi-continuous.

Remark 3.7. Every soft almost pre-continuous is soft almost b-continuous but the converse may not be true.

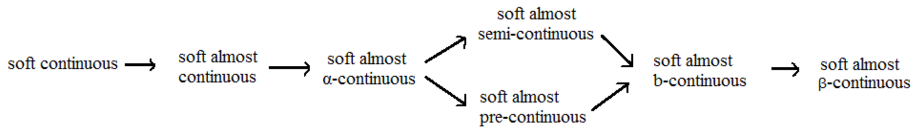
Example 3.8. Let X = $\{x_1, x_2\}$, E = $\{e_1, e_2\}$ and Y = $\{y_1, y_2\}$, K = $\{k_1, k_2\}$. The soft sets (F_1,E), (F_2,E), (G,K) are defined as follows :
$F_1(e_1) = \phi$, $F_1(e_2) = \{x_1\}$,
$F_2(e_1) = \{x_1\}$, $F_2(e_2) = \phi$,
$F_3(e_1) = \{x_1\}$, $F_3(e_2) = \{x_1\}$,
$G_1(k_1) = \{y_1\}$, $G_1(k_2) = \{y_2\}$,
$G_2(k_1) = \{y_2\}$, $G_2(k_2) = \{y_1\}$.
Let $\tau = \{\phi$, (F_1,E) ,(F_2,E),(F_3,E), $\tilde{X}\}$, and $\upsilon = \{\phi$,(G_1,K),(G_2,K), $\tilde{Y}\}$ are topologies on X and Y respectively. Then soft mapping f_{pu} : (X,τ,E) $\rightarrow$ (Y,υ,K) defined by u(x_1) = y_1 , u(x_2)= y_2 and p(e_1)= k_1 , p(e_2) = k_2 is soft almost b-continuous mapping but not soft almost pre-continuous.

Remark 3.9. Every soft almost b-continuous mapping is soft almost β-continuous but the converse may not be true.

Example 3.10. Let X = $\{x_1, x_2, x_3, x_4\}$, E = $\{e_1, e_2\}$ and Y = $\{y_1, y_2, y_3, y_4\}$, K = $\{k_1, k_2\}$. The soft sets (F_1,E) ,(F_2,E),(F_3,E),(G_1,K),(G_2,K) and (G_3,K) are defined as follows :
$F_1(e_1) = \{x_3\}$, $F_1(e_2) = \phi$,
$F_2(e_1) = \{x_1, x_4\}$, $F_2(e_2) = \phi$,
$F_3(e_1) = \{x_1, x_3, x_4\}$, $F_3(e_2) = \phi$,
$G_1(k_1) = \{y_3\}$, $G_1(k_2) = \phi$,

$G_2(k_1) = \{y_1,y_4\}$, $G_2(k_2) = \phi$,
$G_3(k_1) = \{y_1,y_3,y_4\}$, $G_3(k_2) = \phi$,
Let
$\tau = \{\phi$, $(F_1$,E), $(F_2$,E),$(F_3$,E), $\tilde{X}$ } and $\upsilon = \{\phi$, $(G_1$,K), $(G_2$,K), $(G_3$,K), $\tilde{Y}$ } are topologies on X and Y respectively. Then soft mapping f_{pu} : (X,τ,E) $\rightarrow$ (Y,υ,K) defined by u(x_1) = u(x_2)= y_1, u(x_3) = y_3, u(x_4) = y_4 and p(e_1)= k_1, p(e_2) = k_2 is soft almost β-continuous mapping but not soft almost b-continuous.

Thus we reach at the following diagram of implications.



Theorem 3.11. Let f_{pu} : (X ,τ,E) $\rightarrow$ (Y,ϑ,K) be a soft mapping. Then the following conditions are equivalent:

(a) f_{pu} is soft almost b-continuous.

(b) f_{pu}^{-1} (G,K) is soft b-closed set in X for every soft regular closed set (G,K) over Y.

(c) f_{pu}^{-1} (A,K) $\subset$ bInt(f_{pu}^{-1}(Int(Cl(A,K)))) for every soft open set (A,K) over Y.

(d) bCl(f_{pu}^{-1}(Cl(Int(G,K)))) $\subset$ f_{pu}^{-1}(G,K) for every soft closed set (G,K) over Y.

(e) For each soft point $(x_e)_E$ over X and each soft regular open set (G,K) over Y containing f_{pu} ($(x_e)_E$),there exists a soft b-open set (F,E) over X such that $(x_e)_E \in$ (F,E) and (F,E) $\subset$ f_{pu}^{-1}(G,K).

(f) For each soft point $(x_e)_E$ over X and each soft regular open set (G,K) over Y containing $f_{pu}((x_e)_E)$,there exists a soft b-open set (F,E) over X such that $(x_e)_E \in$ (F,E) and f_{pu}(F,E) $\subset$ (G,K).

Proof: (a)$\Leftrightarrow$(b) Since f_{pu}^{-1} ($(G,K)^C$) = $(f_{pu}^{-1}(G,K))^C$ for every soft set (G,K) over Y.

(a) $\Rightarrow$(c) Since (A,K) is soft open set over Y, (A,K) $\subset$ Int(Cl(A,K)) and hence, f_{pu}^{-1} (A,K) $\subset$ f_{pu}^{-1}(Int(Cl(A,K))).Now Int (Cl(A,K)) is a soft regular open set over Y. By (a), f_{pu}^{-1} (Int(Cl(A,K))) is soft b-open set over X. Thus, f_{pu}^{-1} (A,K) $\subset$ f_{pu}^{-1} (Int(Cl(A,K))) = bInt(f_{pu}^{-1}(Int(Cl(A,K)))).

(c) $\Rightarrow$(a) Let (A,K) be a soft regular open set over Y, then we have f_{pu}^{-1} (A,K) $\subset$ bInt(f_{pu}^{-1}(Int(Cl(A,K)))) = bInt(f_{pu}^{-1}(A,K)).Thus, f_{pu}^{-1} (A,K) = bInt(f_{pu}^{-1}(A,K)) shows that f_{pu}^{-1} (A,K) is a soft b-open set over X.

(b) $\Rightarrow$(d) Since (G,K) is soft closed set over Y, Cl(Int(G,K)) $\subset$ (G,K) and f_{pu}^{-1}(Cl(Int (G,K))) $\subset$ f_{pu}^{-1} (G,K). Cl(Int(G,K)) is soft regular closed set over Y. Hence, f_{pu}^{-1}(Cl(Int(G,K) is soft b-closed set over X.Thus, bCl(f_{pu}^{-1}(Cl(Int(G,K)))) = f_{pu}^{-1}(Cl(Int(G,K))) $\subset$ f_{pu}^{-1} (G,K).

(d) $\Rightarrow$(b) Let (G,K) be a soft regular closed set over Y,then we have bCl(f_{pu}^{-1}(G,K)) = βCl(f_{pu}^{-1}(Cl(Int(G,K)))) $\subset$ f_{pu}^{-1} (G,K) .Thus, bCl(f_{pu}^{-1}(G,K)) $\subset$ f_{pu}^{-1} (G,K), shows that f_{pu}^{-1} (G,K) is soft b-closed set over X.

(a) ⇒(e) Let $(x_e)_E$ be a soft point over X and (G,K) be a soft regular open set over Y such that $f_{pu}((x_e)_E) \in$ (G,K), Put (F,E) = f_{pu}^{-1}(G,K) .Then by (a), (F,E) is soft b-open set, $(x_e)_E \in$ (F,E) and (F,E) $\subset f_{pu}^{-1}$(G,K).

(e) ⇒(f) Let $(x_e)_E$ be a soft point over X and (G,K) be a soft regular open set over Y such that $f_{pu}((x_e)_E)$. By (e) there exists a soft b-open set (F,E) such that $(x_e)_E \in$ (F,E), (F,E) $\subset f_{pu}^{-1}$(G,K). And so , we have $(x_e)_E \in$ (F,E), f_{pu}(F,E) $\subset f_{pu}(f_{pu}^{-1}$(G,K)) $\subset$ (G,K).

(f) ⇒(a) Let (G,K) be a soft regular open set over Y and $(x_e)_E$ be a soft point over X such that $(x_e)_E \in f_{pu}^{-1}$(G,K). Then $f_{pu}((x_e)_E) \in f_{pu}(\ f_{pu}^{-1}$(G,K)) $\subset$(G,K). By (f) ,there exists a soft b-open set (F,E) such that $(x_e)_E \in$ (F,E) and f_{pu}(F,E) $\subset$(G,K) .This shows that $(x_e)_E \in$ (F,E) $\subset f_{pu}^{-1}$(G,K). it follows that f_{pu}^{-1}(G,K) is soft b-open set and hence f_{pu}^{-1} is soft almost b-continuous.

Definition 3.12. Let (X ,τ,E) be soft topological space and (A,E) be a soft set over X is called soft δ-open if for each soft point $(x_e)_E \in$ (A,E),there exists a soft regular open set (F,E) such that $(x_e)_E \in$ (F,E) $\subset$ (A,E) and its complement is called soft δ-closed.

Definition 3.13. Let (X ,τ,E) be soft topological space and (A,E) be a soft set over X,

The intersection of all soft δ-closed sets containing a soft set (A,E) is called the δ-closure of (A,E) and is denoted by δCl(A,E).

Theorem 3.14. Let f_{pu} : (X ,τ,E) $\rightarrow$ (Y,ϑ,K) be a soft mapping. Then the following conditions are equivalent:

(a) f_{pu} is soft almost b-continuous.

(b) f_{pu}(bCl(A,E)) $\subset \delta$Cl(f_{pu}(A,E)), for every soft set (A,E) over X.

(c) bCl(f_{pu}^{-1}(B,K)) $\subset f_{pu}^{-1}(\delta$Cl(B,K)), for every soft set (B,K) over Y.

(d) f_{pu}^{-1}(F,K) is soft b-closed set over X, for every soft δ-closed set (F,K) over Y.

(e) f_{pu}^{-1}(V,K) is soft b-open set over X, for every soft δ-open set (V,K) over Y.

Proof: (a) $\rightarrow$ (b)Let (A,E) be a soft set over X.Since, δCl(f_{pu}(A,E)) is a soft δ-closed set over Y.By theorem 3.11,we have (A,E) $\subset f_{pu}^{-1}(\ \delta$Cl($f_{pu}$(A,E))) which is soft b-closed set over X.Hence,bCl(A,E) $\subset f_{pu}^{-1}(\ \delta$Cl($f_{pu}$(A,E))).Hence,we obtain f_{pu}(bCl(A,E)) $\subset \delta$Cl(f_{pu}(A,E)).

(b) $\rightarrow$ (c) Let (B,K) be a soft set over Y . We have f_{pu}(bCl(f_{pu}^{-1}(B,K))) $\subset$ δCl($f_{pu}(f_{pu}^{-1}$(B,K)) $\subset \delta$Cl(B,K) and hence, bCl(f_{pu}^{-1}(B,K)) $\subset f_{pu}^{-1}(\delta$Cl(B,K)).

(c) $\rightarrow$ (d) Let (F,K) be a soft δ-closed set over Y. We have bCl(f_{pu}^{-1}(F,K)) $\subset$ $f_{pu}^{-1}(\delta$Cl(F,K)) = f_{pu}^{-1}(F,K) and f_{pu}^{-1}(F,K) is soft b-closed over X.

(d) $\rightarrow$ (e) Let (V,K) be a soft δ-open set over Y. By (d), we have $f_{pu}^{-1}(V,K)^c = (f_{pu}^{-1}(V,K))^c$, which is soft b-closed over X and so f_{pu}^{-1}(V,K) is soft b-open set in X.

(e)$\rightarrow$ (a) Let (V,K) be a soft regular open set over Y.Since (V,K) is soft δ-open set over Y, f_{pu}^{-1}(V,K) is soft b-open over X and hence by theorem 3.11,f_{pu} is soft almost b-continuous.

Theorem 3.15. Let f_{pu} : (X, τ, E) $\rightarrow$ (Y, ϑ, K) be a soft mapping and $G_{g_p g_u}$: (X ,τ,E) $\rightarrow$ (X $\times$ Y, $\tau \times \vartheta$, E $\times$ K) be the soft graph mapping of f_{pu}. Then g_{pu} is soft almost b-continuous mapping if and only if f_{pu} is soft almost b-continuous.

Proof: Necessity : Let $(x_e)_E \in$ be a soft point over X and (V,K) be a soft regular open set over Y containing $f_{pu}((x_e)_E)$. Then, we have $G_{g_p g_u} = ((x_e)_E, f_{pu}((x_e)_E)) \in$

(X $\times$ Y, τ $\times$ ϑ, E $\times$ K) which is soft regular open over (X $\times$ Y, τ $\times$ ϑ, E $\times$ K). Since $G_{g_pg_u}$ is soft almost b-continuous,there exists a soft b-open set (U,E) over X containing $(x_e)_E$ such that $G_{g_pg_u}$ (U,E) $\subset$ (X $\times$ Y, τ $\times$ ϑ , E $\times$ K). Therefore, we obtain f_{pu}(U,E) $\subset \tilde{Y}$ and hence, f_{pu} is soft almost b-continuous.

Sufficiency: Let $(x_e)_E$ be a soft point over X and (W, E $\times$ K) be a soft regular open set over (X $\times$ Y, τ $\times$ ϑ , E $\times$ K) containing $G_{g_pg_u}$ $((x_e)_E)$. There exist (U_1,E) be a soft regular open set Over X and (V,K) be a soft regular open set over Y such that $((x_e)_E$,$f_{pu}((x_e)_E))$ $\in$ (U_1,E) $\times$ (V,K) $\subset$ (W, E $\times$ K).Since f_{pu} is soft almost b-continuous ,there exists (U_2,E) be a soft b-open set Over X such that $(x_e)_E) \in$ (U_2,E) and f_{pu}(U_2,E) $\subset$ (V,K). Put (U,E) = (U_1,E) $\cap$ (U_2,E).we obtain $(x_e)_E) \in$ (U,E) which is soft b-open set over X and $G_{g_pg_u}$ (U,E) $\subset$ (U_1,E) $\times$ (V,K) $\subset$ (W, E $\times$ K).This shows that $G_{g_pg_u}$ is soft almost b-continuous.

Theorem 3.16. Let f_{pu} : (X ,τ,E) $\rightarrow$ (Y,ϑ,K) be a soft mapping from a soft topological space (X ,τ,E) to a soft semiregular space (Y,ϑ,K).Then f_{pu} is soft almost b-continuous if and only if f_{pu} is soft b-continuous.

Proof: Necessity: Let $(x_e)_E$ be a soft point over X and (F,K) be a soft open set over Y such that f_{pu}($(x_e)_E$) $\in$ (F,K) .Since (Y,ϑ,K) is soft semiregular there exists a soft open set (G,K) over Y such that $f_{pu}((x_e)_E)$ $\in$ (G,K) and (G,K) $\subset$ Int(Cl(G,K) $\subset$ (F,K). Since Int(Cl(G,K)) is soft regular open over Y and f_{pu} is soft almost b-continuous,by theorem 3.11 (f) there exists a soft b-open set (A,E) over X such that $(x_e)_E$ $\in$ (A,E) and f_{pu}(A,E) $\subset$ Int(Cl(G,K)).Thus, (A,E) is soft b-open set such that $(x_e)_E$ $\in$ (A,E) and f_{pu}(A,E) $\subset$ (F,K). Hence, f_{pu} is soft b-continuous.

Sufficiency : Obvious.

Lemma 3.17. If f_{pu} : (X ,τ,E) $\rightarrow$ (Y,ϑ,K) be a soft mapping and f_{pu} is a soft open and soft continuous mapping then f_{pu}^{-1}(G,K) is soft b-open over X for every (G,K) is soft b-open over Y.

Proof: Let (G,K) is soft b-open over Y. Then, (G,K) $\subseteq$ Int(Cl(Int(G,K))). Since f_{pu} is soft continuous we have,

f_{pu}^{-1}(G,K) $\subseteq$ f_{pu}^{-1}(Int(Cl(Int(G,K)))) $\subseteq$ Int(f_{pu}^{-1}(Cl(Int(G,K)))).

By the openness of f_{pu}, we have

f_{pu}^{-1}(Cl(Int(G,K))) $\subseteq$ Cl(f_{pu}^{-1}(Int(G,K))).

Again f_{pu} is soft continuous

f_{pu}^{-1}(Int(G,K)) $\subseteq$ Int(f_{pu}^{-1}(G,K)).

Thus,

f_{pu}^{-1}(G,K)$\subseteq$ Int(Cl(Int(f_{pu}^{-1}(G,K)))).

Consequently, f_{pu}^{-1}(G,K) is soft b-open over X.

Theorem 3.18. If soft mapping $f_{p_1u_1}$: (X ,τ,E) $\rightarrow$ (Y,ϑ,K) is soft open soft continuous and soft mapping $g_{p_2u_2}$:(Y,ϑ,K) $\rightarrow$ (Z,η,T) is soft almost b-continuous, then $g_{p_2u_2}$ o$f_{p_1u_1}$: (X ,τ,E) $\rightarrow$ (Z,η,T) is soft almost b-continuous.

Proof : Suppose (U,T) is a soft regular open set over Z. Then $g_{p_2u_2}^{-1}$(U,T) is a soft b-open set over Y because $g_{p_2u_2}$ is soft almost b-continuous. Since $f_{p_1u_1}$ being soft open and continuous.By lemma 3.17 ($f_{p_1u_1}^{-1}$ ($g_{p_2u_2}^{-1}$(U,T)) is soft b-open over X. Consequently, $g_{p_2u_2}$ o$f_{p_1u_1}$: (X ,τ,E) $\rightarrow$ (Z,η,T) is soft almost b-continuous.

Lemma 3.19. If (A,E) be a soft b-open set over X and (Y,E) is soft open in a soft topological space (X ,τ,E). Then (A,E) $\cap$ (Y,E) is soft b-open in (Y,E).

Proof: Obvious.

Theorem 3.20. Let f_{pu} : (X ,τ,E) $\rightarrow$ (Y,ϑ,K) be a soft almost b-continuous mapping and (A,E) is soft open set over X, Then f_{pu}/ (A,E) is soft almost b-continuous.

Proof : Let (G,K) be a soft regular open set in Y then f_{pu}^{-1} (G,K) is soft b-open in X. Since (A,E) is soft open in X, By lemma 3.19 (A,E) $\cap$ f_{pu}^{-1} (G,K) = $[f_{pu}/(A,E)]^{-1}$ (G,K) is soft b-open in (A,E). Therefore, f_{pu}/ (A,E) is soft almost b-continuous.

4 Soft Almost b-Open Mappings

Definition 4.1. A soft mapping f_{pu} : (X ,τ,E) $\rightarrow$ (Y,ϑ,K) is said to be soft almost b-open if for each soft regular open set (F,E) over X, f_{pu}(F,E) is soft b-open in Y.

Remark 4.2. Every soft b-open mapping is soft almost b-open but the converse may not be true.

Example 4.3. Let X = $\{x_1, x_2\}$, E = $\{e_1, e_2\}$ and Y = $\{y_1, y_2\}$, K = $\{k_1, k_2\}$. The soft sets (F,E), (G,K) are defined as follows :

F(e_1) = $\{x_1\}$, F(e_2) = $\{x_2\}$,

G(k_1) = $\{y_2\}$, G(k_2) = $\{y_1\}$.

Let τ = $\{\phi$,(F,E), $\tilde{X}$ $\}$, and υ = $\{\phi$,(G,K), $\tilde{Y}$ $\}$ are topologies on X and Y respectively. Then soft mapping f_{pu} : (X,τ,E) $\rightarrow$ (Y,υ,K) defined by u(x_1) = y_1 , u(x_2) = y_2 and p(e_1)= k_1, p(e_2) = k_2 is soft almost b-open but not soft b-open.

Remark 4.4. Every soft almost semi-open is soft almost b-open but the converse may not be true.

Example 4.5. Let X = $\{x_1, x_2\}$, E = $\{e_1, e_2\}$ and Y = $\{y_1, y_2\}$, K = $\{k_1, k_2\}$. The soft sets (F_1,E), (F_2,E) are defined as follows :

$F_1(e_1) = \{x_1\}$, $F_1(e_2) = \{x_2\}$,

$F_2(e_1) = \{x_2\}$, $F_2(e_2) = \{x_1\}$.

Let τ = $\{\phi$,(F_1,E) ,(F_2,E), $\tilde{X}$ $\}$, and υ = $\{\phi$, $\tilde{Y}$ $\}$ are topologies on X and Y respectively. Then soft mapping f_{pu} : (X,τ,E) $\rightarrow$ (Y,υ,K) defined by u(x_1) = y_1 , u(x_2)= y_2 and p(e_1)= k_1 , p(e_2) = k_2 is soft almost b-open mapping but not soft almost semi-open.

Remark 4.6. Every soft almost pre-open is soft almost b-open but the converse may not be true.

Example 4.7. Let X = $\{x_1, x_2\}$, E = $\{e_1, e_2\}$ and Y = $\{y_1 , y_2\}$, K = $\{k_1, k_2\}$. The soft sets (F_1,E), (F_2,E), (G_1,K), (G_2,K) and (G_3,K) are defined as follows :

$F_1(e_1) = \{x_1\}$, $F_1(e_2) = \{x_2\}$,

$F_2(e_1) = \{x_2\}$, $F_2(e_2) = \{x_1\}$,

$F_3(e_1) = \phi$, $F_3(e_2) = \{y_1\}$,

$G_1(k_1) = \{y_1\}$, $G_1(k_2) = \phi$,

$G_2(k_1) = \{y_1\}$, $G_2(k_2) = \{y_1\}$.

Let τ = $\{\phi$, (F_1,E), (F_2,E), $\tilde{X}$ $\}$, and υ = $\{\phi$, (G_1,K),(G_2,K), (G_3,K),$\tilde{Y}$ $\}$ are topologies on X and Y respectively. Then soft mapping f_{pu} : (X,τ,E) $\rightarrow$ (Y,υ,K) defined by u(x_1) = y_1 , u(x_2)= y_2 and p(e_1)= k_1 , p(e_2) = k_2 is soft almost b-open but not soft almost pre-open.

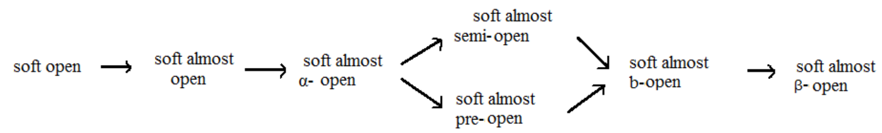
Remark 4.8. Every soft almost b-open mapping is soft almost β-open but the converse may not be true.

Example 4.9. Let X = $\{x_1, x_2, x_3, x_4\}$, E = $\{e_1, e_2\}$ and Y = $\{y_1, y_2, y_3, y_4\}$, K = $\{k_1, k_2\}$. The soft sets (F_1,E),(F_2,E),(F_3,E),(G_1,K),(G_2,K) and (G_3,K) are defined as follows :

$F_1(e_1) = \{x_3\}$, $F_1(e_2) = \phi$,
$F_2(e_1) = \{x_1, x_4\}$, $F_2(e_2) = \phi$,
$F_3(e_1) = \{x_1, x_3, x_4\}$, $F_3(e_2) = \phi$,
$G_1(k_1) = \{y_3\}$, $G_1(k_2) = \phi$,
$G_2(k_1) = \{y_1, y_4\}$, $G_2(k_2) = \phi$,
$G_3(k_1) = \{y_1, y_3, y_4\}$, $G_3(k_2) = \phi$.

Let $\tau = \{\phi$, (F_1,E), (F_2,E), (F_3,E), $\tilde{X}$ $\}$, and $\upsilon = \{\phi$,(G_1,K),(G_2,K),(G_3,K), $\tilde{Y}$ $\}$ are topologies on X and Y respectively. Then soft mapping f_{pu} : (X,τ,E) $\rightarrow$ (Y,υ,K) defined by u(x_1) = u(x_2)= y_1 ,u(x_3) = y_3,u(x_4) = y_4 and p(e_1)= k_1 , p(e_2) = k_2 is soft almost β-open mapping but not soft almost b-open.

Thus we reach at the following diagram of implications.



Theorem 4.10. Let $f_{p_1u_1}$: (X ,τ,E) $\rightarrow$ (Y,ϑ,K) and $g_{p_2u_2}$:(Y,ϑ,K) $\rightarrow$ (Z,η,T) be two soft mappings, If $f_{p_1u_1}$ is soft almost open and $g_{p_2u_2}$ is soft b-open. Then the soft mapping $g_{p_2u_2}$ o$f_{p_1u_1}$: (X ,τ,E) $\rightarrow$ (Z,η,T) is soft almost b-open.

Proof : Let (F,E) be soft regular open in X. Then $f_{p_1u_1}$(F,E) is soft open over Y because $f_{p_1u_1}$ is soft almost open.Therefore, $g_{p_2u_2}$ ($f_{p_1u_1}$)(F,E) is soft b-open over Z. Because $g_{p_2u_2}$ is soft b-open. Since ($g_{p_2u_2}$ o$f_{p_1u_1}$)(F,E) = ($g_{p_2u_2}$($f_{p_1u_1}$ (F,E)), it follows that the soft mapping ($g_{p_2u_2}$ o$f_{p_1u_1}$) is soft almost b-open.

Theorem 4.11. Let $f_{p_1u_1}$: (X ,τ,E) $\rightarrow$ (Y,ϑ,K) and $g_{p_2u_2}$:(Y,ϑ,K) $\rightarrow$ (Z,η,T) be two soft mappings, such that $g_{p_2u_2}$ o$f_{p_1u_1}$: (X ,τ,E) $\rightarrow$ (Z,η,T) is soft almost b-open and $g_{p_2u_2}$ is soft b-irresolute and injective then $f_{p_1u_1}$ is soft almost b-open.

Proof : Suppose (F,E) is soft regular open set over X. Then $g_{p_2u_2}$ o$f_{p_1u_1}$(F,E) is soft b-open over Z because $g_{p_2u_2}$ o$f_{p_1u_1}$ is soft almost b-open. Since $g_{p_2u_2}$ is injective, we have ($g^{-1}_{p_2u_2}$ ($g_{p_2u_2}$ o$f_{p_1u_1}$)(F,E)) = $f_{p_1u_1}$ (F,E). Therefore $f_{p_1u_1}$ (F, E) is soft b-open over Y, because $g_{p_2u_2}$ is soft b-irresolute. This implies $f_{p_1u_1}$ is soft almost b-open.

Theorem 4.12. Let soft mapping f_{pu} : (X ,τ,E) $\rightarrow$ (Y,ϑ,K) be soft almost b-open mapping. If (G,K) is soft set over Y and (F,E) is soft regular closed set of X containing f^{-1}_{pu} (G,K) then there is a soft b-closed set (A,K) over Y containing (G,K) such that f^{-1}_{pu} (A,K) $\subset$ (F,E).

Proof: Let (A, K) = $(f_{pu}(F,E)^C)^C$. Since f^{-1}_{pu} (G,K) $\subset$ (F,E) we have $f_{pu}(F,E)^C$ $\subset$ (G,K). Since f_{pu} is soft almost b-open then (A,K) is soft b-closed set of Y and f^{-1}_{pu} (A,K) = (f^{-1}_{pu} $(f_{pu}(F,E)^C)^C \subset ((F,E)^C)^C$ = (F,E). Thus, f^{-1}_{pu} (A,K) $\subset$ (F,E).

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