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Original Article

A Note on the Integral Representation of Some Relative Growth Indicators of Entire Algebroidal Functions

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Abstaract — Let p and q be any two positive integers. In this paper the concept of two relative growth indicators namely *relative* (p, q)*-th type* and *relative* (p, q)*-th weak type* of entire functions with respect to entire algebroidal functions have been introduced from the view point of their integral representations. Here we also investigate the equivalence of the computational definitions with their respective integral representations.

Keywords — *Entire function, entire algebroidal function, growth, order (lower order), relative order (relative lower order), growth indicator.*

1 Introduction

The *order* and *lower order* of an entire function f which is generally used in computational purposes are classical in complex analysis. Bernal [1] and [2], introduced the *relative order* (respectively *relative lower order)* between two entire functions to avoid comparing growth just with $\exp z$. Extending the notion of *relative order* (respectively *relative lower order)* Ruiz et al. [8] introduced the *relative* (p, q)*-th order* (respectively *relative lower* (p, q)*-th order)* where p and q are any two positive integers. Now to compare the growth of entire functions having the same *relative* (p, q)*-th order* or *relative lower* (p, q)*-th order*, we would like to introduce the definition of *relative* (p, q)*-th type* and *relative* (p, q)*-th weak type* of entire functions with respect to entire algebroidal functions and establish their respective integral representations. We also investigate the equivalence of the computational definitions and their corresponding integral representations of the relative growth indicators as stated above in case of entire algebroidal functions.

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Let F and G be two k-valued function defined by the following irreducible equation

$$f_kF^k + f_{k-1}F^{k-1} + f_{k-2}F^{k-2} + ... + f_0 = 0$$

$$g_kG^k + g_{k-1}G^{k-1} + g_{k-2}G^{k-2} + ... + g_0 = 0$$

where $f_k \neq 0$, $g_k \neq 0$ where f_i $(i = 0, 1, 2, \ldots, k-1)$ and g_i $(i = 0, 1, 2, \ldots, k-1)$ are entire functions having no common zeros. If at least one of the f_i $(i = 0, 1, 2, \ldots, k)$ is transcendental then F is called a k-valued algebroidal function. Further, if $f_k \equiv 1$ then F is called a k-valued entire algebroidal function and similar for G.

Let us consider the definition of *relative* (p, q)*-th order* $\rho_G^{(p,q)}(f_i)$ (respectively *relative* (p, q)*-th lower order* $\lambda_G^{(p,q)}(f_i)$) of an entire functions f_i with respect to an entire algebroidal function G, in the light of *index-pair* which is as follows:

Definition 1.1. [8] Let G be any entire algebroidal function as defined above with index-pair (m, p) .Also let f_i's$(i = 0, 1, 2, \ldots, k-1)$ be entire functions with index-pair (m, q) where p, q, m are positive integers such that $m \geq \max(p, q)$. Then the relative (p, q)-th order of f_i with respect to G is defined as

$$\rho_G^{(p,q)}(f_i) = \limsup_{r\to\infty} \frac{\log^{[p]} M_G^{-1} M_{f_i}(r)}{\log^{[q]} r}.$$

Analogously, the relative (p, q)-th lower order of f_i with respect to $G$is defined by:

$$\lambda_G^{(p,q)}(f_i) = \liminf_{r\to\infty} \frac{\log^{[p]} M_G^{-1} M_{f_i}(r)}{\log^{[q]} r}.$$

In order to refine the above growth scale, now we intend to introduce the definition of another growth indicator, called *relative* (p, q) *-th type* of entire algebroidal function with respect to another entire algebroidal function in the light of their *index-pair* which is as follows:

Definition 1.2. Let $f_i's$ $(0 \leq i \leq k-1)$ be entire functions with index-pair (m_1, q) and G be any entire algebroidal function with index-pair (m_2, p) where $m_1 = m_2 = m$ and p, q, m are all positive integers such that $m \geq \max\{p, q\}$. The relative (p, q) th type of entire functions f_i with respect to the entire algebroidal function G having finite positive relative (p, q) th order $\rho_G^{(p,q)}(f_i)$ $\left(0 < \rho_G^{(p,q)}(f_i) < \infty\right)$ is defined as :

$$\sigma_G^{(p,q)}(f_i) = \inf \left\{ \begin{array}{c} \phi > 0 : M_{f_i}(r) < M_G\left[\exp^{[p-1]}\left(\phi(\log^{[q-1]} r)^{\rho_G^{(p,q)}(F)}\right)\right] \\ \text{for all } r > r_0(\phi) > 0 \end{array} \right\}$$
$$= \limsup_{r\to\infty} \frac{\log^{[p-1]} M_G^{-1} M_{f_i}(r)}{\left(\log^{[q-1]} r\right)^{\rho_G^{(p,q)}(f_i)}} .$$

The above definition can alternatively defined in the following manner:

Definition 1.3. Let $f_i's$ $(0 \leq i \leq k-1)$ be entire functions having finite positive *relative* (p,q) *-th order* $\rho_G^{(p,q)}(f_i)$ $\left(0 < \rho_G^{(p,q)}(F) < \infty\right)$with respect to an entire algebroidal function G defined as earlier where p and q are any two positive integers. Then the *relative* (p,q) *-th type* $\sigma_G^{(p,q)}(f_i)$ of entire functions f_i with respect to the entire algebroidal function G is defined as: The integral

$$\int_{r_0}^{\infty} \frac{\log^{[p-2]} M_G^{-1} M_{f_i}(r)}{\left[\exp\left(\left(\log^{[q-1]} r\right)^{\rho_G^{(p,q)}(f_i)}\right)\right]^{t+1}} dr \ (r_0 > 0)$$

converges for $t > \sigma_G^{(p,q)}(f_i)$and diverges for $t < \sigma_G^{(p,q)}(f_i)$.

Analogously, to determine the relative growth of two entire functions having same non zero finite *relative* (p,q) *-th lower order* with respect to an entire algebroidal function, one can introduce the definition of *relative* (p,q) *-th weak type* of entire functionf_i with respect to an entire algebroidal function G of finite positive *relative* (p,q) *-th lower order* $\lambda_G^{(p,q)}(f_i)$ in the following way:

Definition 1.4. Let $f_i's$ $(i = 0, 1, 2, \ldots, k-1)$ be entire functions having finite positive relative (p,q) th lower order $\lambda_G^{(p,q)}(f_i)$ $\left(a < \lambda_G^{(p,q)}(f_i) < \infty\right)$ with respect to an entire algebroidal function G where p and q are any two positive integers. Then the *relative* (p,q) *-th weak type* of entire functions f_i with respect to the entire algebroidal function G is defined as :

$$\tau_G^{(p,q)}(f_i) = \liminf_{r\to\infty} \frac{\log^{[p-1]} M_G^{-1} M_{f_i}(r)}{\left(\log^{[q-1]} r\right)^{\lambda_G^{(p,q)}(f_i)}}.$$

The above definition can also be alternatively defined as:

Definition 1.5. Let $f_i's$ $(i = 0, 1, 2, \ldots, k-1)$ be entire functions having finite positive *relative* (p,q) *-th lower order* $\lambda_G^{(p,q)}(f_i)$ $\left(a < \lambda_G^{(p,q)}(f_i) < \infty\right)$ where p and q are any two positive integers. Then the *relative* (p,q) *-th weak type* $\tau_G^{(p,q)}(f_i)$ of entire functions f_i with respect to the entire algebroidal function G is defined as: The integral

$$\int_{r_0}^{\infty} \frac{\log^{[p-2]} M_G^{-1} M_{f_i}(r)}{\left[\exp\left(\left(\log^{[q-1]} r\right)^{\lambda_G^{(p,q)}(f_i)}\right)\right]^{t+1}} dr \ (r_0 > 0)$$

converges for $t > \tau_G^{(p,q)}(f_i)$ and diverges for $t < \tau_G^{(p,q)}(f_i)$.

Next we introduce the following two relative growth indicators which will also enable us for subsequent study.

Definition 1.6. Let f_i's be entire functions having finite positive relative (p,q) th order $\rho_G^{(p,q)}(f_i)$ $\left(a < \rho_G^{(p,q)}(f_i) < \infty\right)$ with respect to an entire algebroidal function

G where p and q are any two positive integers. Then the *relative* (p, q)*-th lower type* of entire functions f_i with respect to an entire algebroidal function G is defined as :

$$\overline{\sigma}_G^{(p,q)}(f_i) = \liminf_{r\to\infty} \frac{\log^{[p-1]} M_G^{-1} M_{f_i}(r)}{\left(\log^{[q-1]} r\right)^{\rho_G^{(p,q)}(f_i)}} .$$

The above definition can alternatively be defined in the following manner:

Definition 1.7. Let f_i's be entire functions having finite positive relative $(p, q) -$ th order $\rho_G^{(p,q)}(f_i)$ $\left(a < \rho_G^{(p,q)}(f_i) < \infty\right)$ with respect to an entire algebroidal function G where p and q are any two positive integers. Then the *relative* (p, q) *-th lower type* $\overline{\sigma}_G^{(p,q)}(f_i)$ of entire function f_i with respect to tan entire algebroidal function G is defined as: The integral

$$\int_{r_0}^{\infty} \frac{\log^{[p-2]} M_G^{-1} M_{f_i}(r)}{\left[\exp\left(\left(\log^{[q-1]} r\right)^{\rho_G^{(p,q)}(f_i)}\right)\right]^{t+1}} dr \; (r_0 > 0)$$

converges for $t > \overline{\sigma}_G^{(p,q)}(f_i)$ and diverges for $t < \overline{\sigma}_G^{(p,q)}(f_i)$.

Definition 1.8. Let f_i's be entire functions having finite positive relative (p, q)*-th lower order* $\lambda_G^{(p,q)}(f_i)$ $\left(a < \lambda_G^{(p,q)}(f_i) < \infty\right)$ and G be an entire algebroidal function . Then the growth indicator $\overline{\tau}_G^{(p,q)}(f_i)$ of an entire function $f_i$with respect to the entire algebroidal function G is defined as :

$$\overline{\tau}_G^{(p,q)}(f_i) = \limsup_{r\to\infty} \frac{\log^{[p-1]} M_G^{-1} M_{f_i}(r)}{\left(\log^{[q-1]} r\right)^{\lambda_G^{(p,q)}(f_i)}} .$$

The above definition can also be alternatively defined as:

Definition 1.9. Let f_i's be entire functions having finite positive *relative* (p, q) *-th lower order* $\lambda_G^{(p,q)}(f_i)$ $\left(a < \lambda_G^{(p,q)}(f_i) < \infty\right)$with respect to the entire algebroidal function G where p and q are any two positive integers. Then the growth indicator $\overline{\tau}_G^{(p,q)}(f_i)$ of entire function f_i with respect to the entire algebroidal function G is defined as: The integral

$$\int_{r_0}^{\infty} \frac{\log^{[p-2]} M_G^{-1} M_{f_i}(r)}{\left[\exp\left(\left(\log^{[q-1]} r\right)^{\lambda_G^{(p,q)}(f_i)}\right)\right]^{t+1}} dr \; (r_0 > 0)$$

converges for $t > \overline{\tau}_G^{(p,q)}(f_i)$ and diverges for $t < \overline{\tau}_G^{(p,q)}(f_i)$.

Now a question may arise about the equivalence of the definitions of *relative* (p, q) *-th type* and *relative* (p, q) *-th weak type* with their integral representations. In the present paper we would like to establish such equivalence of Definition 1.2 with Definition 1.3 and Definition 1.4 with Definition 1.5 and also investigate some growth properties related to *relative* (p, q) *-th type* and *relative* (p, q) *-th weak type* of entire function with respect to an entire algebroidal function.

2 Lemma

In this section we present a lemma which will be needed in the sequel.

Lemma 2.1. Let the integral $\int_{r_0}^{\infty} \frac{\log^{[p-2]} M_G^{-1} M_{f_i}(r)}{\left[\exp\left(\left(\log^{[q-1]} r\right)^A\right)\right]^{t+1}} dr \; (r_0 > 0)$ converges where $0 < A < \infty$. Then

$$\lim_{r\to\infty} \frac{\log^{[p-2]} M_G^{-1} M_{f_i}(r)}{\left[\exp\left(\left(\log^{[q-1]} r\right)^A\right)\right]^{t}} = 0 \ .$$

Proof. Since the integral $\int_{r_0}^{\infty} \frac{\log^{[p-2]} M_G^{-1} M_{f_i}(r)}{\left[\exp\left(\left(\log^{[q-1]} r\right)^A\right)\right]^{t+1}} dr \; (r_0 > 0)$ converges, then

$$\int_{r_0}^{\infty} \frac{\log^{[p-2]} M_G^{-1} M_{f_i}(r)}{\left[\exp\left(\left(\log^{[q-1]} r\right)^A\right)\right]^{t+1}} dr < \varepsilon, \text{ if } r_0 > R(\varepsilon) \ .$$

Therefore,

$$\int_{r_0}^{\exp\left(\log^{[q-1]} r_0\right)^A + r_0} \frac{\log^{[p-2]} M_G^{-1} M_{f_i}(r)}{\left[\exp\left(\left(\log^{[q-1]} r\right)^A\right)\right]^{t+1}} dr < \varepsilon \ .$$

Since $\log^{[p-2]} M_G^{-1} M_{f_i}(r)$ increases with r, so

$$\int_{r_0}^{\exp\left(\log^{[q-1]} r_0\right)^A + r_0} \frac{\log^{[p-2]} M_G^{-1} M_{f_i}(r)}{\left[\exp\left(\left(\log^{[q-1]} r\right)^A\right)\right]^{t+1}} dr \geq$$

$$\frac{\log^{[p-2]} M_G^{-1} M_{f_i}(r_0)}{\left[\exp\left(\left(\log^{[q-1]} r_0\right)^A\right)\right]^{t+1}} \cdot \left[\exp\left(\left(\log^{[q-1]} r_0\right)^A\right)\right] \ .$$

i.e., for all sufficiently large values of r,

$$\int_{r_0}^{\exp\left(\log^{[q-1]} r_0\right)^A + r_0} \frac{\log^{[p-2]} M_G^{-1} M_{f_i}(r)}{\left[\exp\left(\left(\log^{[q-1]} r\right)^A\right)\right]^{t+1}} dr \geq$$

$$\frac{\log^{[p-2]} M_G^{-1} M_{f_i}(r_0)}{\left[\exp\left(\left(\log^{[q-1]} r_0\right)^A\right)\right]^{t}} \ ,$$

so that

$$\frac{\log^{[p-2]} M_G^{-1} M_{f_i}(r_0)}{\left[\exp\left(\left(\log^{[q-1]} r_0\right)^A\right)\right]^{t}} < \varepsilon \text{ if } r_0 > R(\varepsilon) \, .$$

$$i.e., \lim_{r\to\infty} \frac{\log^{[p-2]} M_G^{-1} M_{f_i}(r)}{\left[\exp\left(\left(\log^{[q-1]} r\right)^A\right)\right]^t} = 0.$$

This proves the lemma. □

3 Theorems

In this section we state the main results of this paper.

Theorem 3.1. Let f_i's $(i = 0, 1, 2, ...k-1)$ be entire functions having finite positive *relative* (p, q) *-th order* $\rho_G^{(p,q)}(f_i)$ $\left(0 < \rho_g^{(p,q)}(f) < \infty\right)$ and *relative* (p, q) *-th type* $\sigma_G^{(p,q)}(f_i)$with respect to an entire algebroidal function G as defined in the introductory section where p and q are any two positive integers. Then Definition 1.2 and Definition 1.3 are equivalent.

Proof. Let us considerf_i's $(i = 0, 1, 2, ...k-1)$ be entire functions and G be an entire algebroidal function such that $\rho_G^{(p,q)}(f_i)$ $\left(0 < \rho_G^{(p,q)}(f_i) < \infty\right)$ exists for any two positive integers p and q.

Case I. $\sigma_G^{(p,q)}(f_i) = \infty$.

Definition 1.2 ⇒ Definition 1.3.

As $\sigma_G^{(p,q)}(f_i) = \infty$, from Definition 1.2 we have for arbitrary positive C and for a sequence of values of r tending to infinity that

$$\log^{[p-1]} M_G^{-1} M_{f_i}(r) > C \cdot \left(\log^{[q-1]} r\right)^{\rho_G^{(p,q)}(f_i)}$$

$$i.e., \ \log^{[p-2]} M_G^{-1} M_{f_i}(r) > \left[\exp\left(\left(\log^{[q-1]} r\right)^{\rho_G^{(p,q)}(f_i)}\right)\right]^C . \tag{1}$$

If possible, let the integral $\int_{r_0}^{\infty} \frac{\log^{[p-2]} M_G^{-1} M_{f_i}(r)}{\left[\exp\left(\left(\log^{[q-1]} r\right)^{\rho_G^{(p,q)}(f_i)}\right)\right]^{C+1}} dr$ $(r_0 > 0)$ be converge.

Then by Lemma 2.1,

$$\limsup_{r\to\infty} \frac{\log^{[p-2]} M_G^{-1} M_{f_i}(r)}{\left[\exp\left(\left(\log^{[q-1]} r\right)^{\rho_G^{(p,q)}(f_i)}\right)\right]^C} = 0 .$$

So for all sufficiently large values of r,

$$\log^{[p-2]} M_G^{-1} M_{f_i}(r) < \left[\exp\left(\left(\log^{[q-1]} r\right)^{\rho_G^{(p,q)}(f_i)}\right)\right]^C . \tag{2}$$

Therefore from (1) and (2) we arrive at a contradiction.

Hence $\int\limits_{r_0}^{\infty} \frac{\log^{[p-2]} M_G^{-1} M_{f_i}(r)}{\left[\exp\left(\left(\log^{[q-1]} r\right)^{\rho_G^{(p,q)}(f_i)}\right)\right]^{C+1}} dr$ $(r_0 > 0)$ diverges whenever C is finite, which is the Definition 1.3.

Definition 1.3 ⇒ Definition 1.2.

Let C be any positive number. Since $\sigma_G^{(p,q)}(f_i) = \infty$, from Definition 1.3, the divergence of the integral $\int\limits_{r_0}^{\infty} \frac{\log^{[p-2]} M_G^{-1} M_{f_i}(r)}{\left[\exp\left(\left(\log^{[q-1]} r\right)^{\rho_G^{(p,q)}(f_i)}\right)\right]^{C+1}} dr$ $(r_0 > 0)$ gives for arbitrary positive ε and for a sequence of values of r tending to infinity

$$\log^{[p-2]} M_G^{-1} M_{f_i}(r) > \left[\exp\left(\left(\log^{[q-1]} r\right)^{\rho_G^{(p,q)}(f_i)}\right)\right]^{C-\varepsilon}$$

$$i.e., \ \log^{[p-1]} M_G^{-1} M_{f_i}(r) > (C-\varepsilon)\left(\log^{[q-1]} r\right)^{\rho_G^{(p,q)}(f_i)},$$

which implies that

$$\limsup_{r\to\infty} \frac{\log^{[p-1]} M_G^{-1} M_{f_i}(r)}{\left(\log^{[q-1]} r\right)^{\rho_G^{(p,q)}(f_i)}} \geq G - \varepsilon .$$

Since $C > 0$ is arbitrary, it follows that

$$\limsup_{r\to\infty} \frac{\log^{[p-1]} M_G^{-1} M_{f_i}(r)}{\left(\log^{[q-1]} r\right)^{\rho_G^{(p,q)}(f_i)}} = \infty .$$

□

Thus Definition 1.2 follows.

Case II. $0 \leq \sigma_G^{(p,q)}(f_i) < \infty$.

Definition 1.2 ⇒ Definition 1.3.

Subcase (A). $0 < \sigma_G^{(p,q)}(f_i) < \infty$.

Let f_i 's$(i = 0, 1, 2, ...k-1)$ be entire functions and G be an entire algebroidal function such that $0 < \sigma_G^{(p,q)}(f_i) < \infty$ exists for any two positive integers p and q. Then according to the Definition 1.2, for arbitrary positive ε and for all sufficiently large values of r, we obtain that

$$\log^{[p-1]} M_G^{-1} M_{f_i}(r) < \left(\sigma_G^{(p,q)}(f_i) + \varepsilon\right)\left(\log^{[q-1]} r\right)^{\rho_G^{(p,q)}(f_i)}$$

$$i.e., \ \log^{[p-2]} M_G^{-1} M_{f_i}(r) < \left[\exp\left(\left(\log^{[q-1]} r\right)^{\rho_G^{(p,q)}(f_i)}\right)\right]^{\sigma_G^{(p,q)}(f_i)+\varepsilon}$$

$$i.e., \ \frac{\log^{[p-2]} M_G^{-1} M_{f_i}(r)}{\left[\exp\left(\left(\log^{[q-1]} r\right)^{\rho_G^{(p,q)}(f_i)}\right)\right]^{t}} < \frac{\left[\exp\left(\left(\log^{[q-1]} r\right)^{\rho_G^{(p,q)}(f_i)}\right)\right]^{\sigma_G^{(p,q)}(f_i)+\varepsilon}}{\left[\exp\left(\left(\log^{[q-1]} r\right)^{\rho_G^{(p,q)}(f_i)}\right)\right]^{t}}$$

$$i.e., \ \frac{\log^{[p-2]} M_G^{-1} M_{f_i}(r)}{\left[\exp\left(\left(\log^{[q-1]} r\right)^{\rho_G^{(p,q)}(f_i)}\right)\right]^{t}} < \frac{1}{\left[\exp\left(\left(\log^{[q-1]} r\right)^{\rho_G^{(p,q)}(f_i)}\right)\right]^{t-\sigma_G^{(p,q)}(f_i)+\varepsilon}}.$$

Therefore $\int_{r_0}^{\infty} \frac{\log^{[p-2]} M_G^{-1} M_{f_i}(r)}{\left[\exp\left(\left(\log^{[q-1]} r\right)^{\rho_G^{(p,q)}(f_i)}\right)\right]^{t+1}} dr \ (r_0 > 0)$ converges for $t > \sigma_G^{(p,q)}(f_i)$.

Again by Definition 1.2, we obtain for a sequence values of r tending to infinity that

$$\log^{[p-1]} M_G^{-1} M_{f_i}(r) > \left(\sigma_G^{(p,q)}(f_i) - \varepsilon\right)\left(\log^{[q-1]} r\right)^{\rho_G^{(p,q)}(f_i)}$$

$$i.e., \ \log^{[p-2]} M_G^{-1} M_{f_i}(r) > \left[\exp\left(\left(\log^{[q-1]} r\right)^{\rho_G^{(p,q)}(f_i)}\right)\right]^{\sigma_G^{(p,q)}(f_i)-\varepsilon}. \quad (3)$$

So for $t < \sigma_G^{(p,q)}(f_i)$, we get from (3) that

$$\frac{\log^{[p-2]} M_G^{-1} M_{f_i}(r)}{\left[\exp\left(\left(\log^{[q-1]} r\right)^{\rho_G^{(p,q)}(f_i)}\right)\right]^{t}} > \frac{1}{\left[\exp\left(\left(\log^{[q-1]} r\right)^{\rho_G^{(p,q)}(f_i)}\right)\right]^{t- \ \sigma_G^{(p,q)}(f_i)-\varepsilon}}.$$

Therefore $\int_{r_0}^{\infty} \frac{\log^{[p-2]} M_G^{-1} M_{f_i}(r)}{\left[\exp\left(\left(\log^{[q-1]} r\right)^{\rho_G^{(p,q)}(f_i)}\right)\right]^{t+1}} dr \ (r_0 > 0)$ diverges for $t < \sigma_G^{(p,q)}(f_i)$.

Hence $\int_{r_0}^{\infty} \frac{\log^{[p-2]} M_G^{-1} M_{f_i}(r)}{\left[\exp\left(\left(\log^{[q-1]} r\right)^{\rho_G^{(p,q)}(f_i)}\right)\right]^{t+1}} dr \ (r_0 > 0)$ converges for $t > \sigma_G^{(p,q)}(f_i)$ and diverges for $t < \sigma_G^{(p,q)}(f_i)$.

Subcase (B). $\sigma_G^{(p,q)}(f_i) = 0.$

When $\sigma_G^{(p,q)}(f_i) = 0$ for any two positive integers p and q , Definition 1.2 gives for all sufficiently large values of r that

$$\frac{\log^{[p-1]} M_G^{-1} M_{f_i}(r)}{\left(\log^{[q-1]} r\right)^{\rho_G^{(p,q)}(f_i)}} < \varepsilon \ .$$

Then as before we obtain that $\int_{r_0}^{\infty} \frac{\log^{[p-2]} M_G^{-1} M_{f_i}(r)}{\left[\exp\ \left(\log^{[q-1]} r\right)^{\rho_G^{(p,q)}(f_i)}\right)\right]^{t+1}} dr\ (r_0 > 0)$ converges for $t > 0$ and diverges for $t < 0$.

Thus combining Subcase (A) and Subcase (B), Definition 1.3 follows.

Definition 1.3 $\Rightarrow$ Definition 1.2.

From Definition 3 and for arbitrary positive ε,the integral

$$\int_{r_0}^{\infty} \frac{\log^{[p-2]} M_G^{-1} M_{f_i}(r)}{\left[\exp\left(\left(\log^{[q-1]} r\right)^{\rho_G^{(p,q)}(f_i)}\right)\right]^{\sigma_G^{(p,q)}(f_i)+\varepsilon+1}} dr\ (r_0 > 0)$$

converges. Then by Lemma 2.1, we get that

$$\limsup_{r\to\infty} \frac{\log^{[p-2]} M_G^{-1} M_{f_i}(r)}{\left[\exp\left(\left(\log^{[q-1]} r\right)^{\rho_G^{(p,q)}(f_i)}\right)\right]^{\sigma_G^{(p,q)}(f_i)+\varepsilon}} = 0 \ .$$

So we obtain all sufficiently large values of r that

$$\frac{\log^{[p-2]} M_G^{-1} M_{f_i}(r)}{\left[\exp\left(\left(\log^{[q-1]} r\right)^{\rho_G^{(p,q)}(f_i)}\right)\right]^{\sigma_G^{(p,q)}(f_i)+\varepsilon}} < \varepsilon$$

$$i.e.,\ \log^{[p-2]} M_G^{-1} M_{f_i}(r) < \varepsilon \cdot \left[\exp\left(\left(\log^{[q-1]} r\right)^{\rho_G^{(p,q)}(f_i)}\right)\right]^{\sigma_G^{(p,q)}(f_i)+\varepsilon}$$

$$i.e.,\ \log^{[p-1]} M_G^{-1} M_{f_i}(r) < \log\varepsilon + \left(\sigma_G^{(p,q)}(f_i) + \varepsilon\right)\left(\log^{[q-1]} r\right)^{\rho_G^{(p,q)}(f_i)}$$

$$i.e.,\ \limsup_{r\to\infty} \frac{\log^{[p-1]} M_G^{-1} M_{f_i}(r)}{\left(\log^{[q-1]} r\right)^{\rho_G^{(p,q)}(f_i)}} \leq \sigma_G^{(p,q)}(f_i) + \varepsilon \ .$$

Since $\varepsilon\,(> 0)$ is arbitrary, it follows from above that

$$\limsup_{r\to\infty} \frac{\log^{[p-1]} M_G^{-1} M_{f_i}(r)}{\left(\log^{[q-1]} r\right)^{\rho_G^{(p,q)}(f_i)}} \leq \sigma_G^{(p,q)}(f_i) \,. \tag{4}$$

On the other hand, the divergence of the integral

$$\int_{r_0}^{\infty} \frac{\log^{[p-2]} M_G^{-1} M_{f_i}(r)}{\left[\exp\left(\left(\log^{[q-1]} r\right)^{\rho_G^{(p,q)}(f_i)}\right)\right]^{\sigma_G^{(p,q)}(f_i)-\varepsilon+1}} dr \; (r_0 > 0)$$

implies that there exists a sequence of values of r tending to infinity such that

$$\frac{\log^{[p-2]} M_G^{-1} M_{f_i}(r)}{\left[\exp\left(\left(\log^{[q-1]} r\right)^{\rho_G^{(p,q)}(f_i)}\right)\right]^{\sigma_G^{(p,q)}(f_i)-\varepsilon+1}} > \frac{1}{\left[\exp\left(\left(\log^{[q-1]} r\right)^{\rho_G^{(p,q)}(f_i)}\right)\right]^{1+\varepsilon}}$$

$$\begin{aligned}
&i.e., \;\; \log^{[p-2]} M_G^{-1} M_{f_i}(r) > \left[\exp\left(\left(\log^{[q-1]} r\right)^{\rho_G^{(p,q)}(f_i)}\right)\right]^{\sigma_G^{(p,q)}(f_i)-2\varepsilon} \\
&i.e., \;\; \log^{[p-1]} M_G^{-1} M_{f_i}(r) > \left(\sigma_G^{(p,q)}(f_i) - 2\varepsilon\right)\left(\left(\log^{[q-1]} r\right)^{\rho_G^{(p,q)}(f_i)}\right) \\
&i.e., \;\; \frac{\log^{[p-1]} M_G^{-1} M_{f_i}(r)}{\left(\log^{[q-1]} r\right)^{\rho_G^{(p,q)}(f_i)}} > \left(\sigma_G^{(p,q)}(f_i) - 2\varepsilon\right) .
\end{aligned}$$

As $\varepsilon\,(> 0)$ is arbitrary, it follows from above that

$$\limsup_{r\to\infty} \frac{\log^{[p-1]} M_G^{-1} M_{f_i}(r)}{\left(\log^{[q-1]} r\right)^{\rho_G^{(p,q)}(f_i)}} \geq \sigma_G^{(p,q)}(f_i) \; . \tag{5}$$

So from (4) and (5) , we obtain that

$$\limsup_{r\to\infty} \frac{\log^{[p-1]} M_G^{-1} M_{f_i}(r)}{\left(\log^{[q-1]} r\right)^{\rho_G^{(p,q)}(f_i)}} = \sigma_G^{(p,q)}(f_i) .$$

This proves the theorem.

Remark 3.2. The similar results follows if we consider an entire algebroidal function F and the entire functions g_i $(i = 0, 1, 2, ..., k-1)$ instead of G and f_i respectively in Definition 1.2 and Definition 1.3 **.**

Theorem 3.3. Let $f_i's$ $(i = 0, 1, 2, ..., k-1)$ be entire functions having finite positive *relative* (p, q) *-th lower order* $\lambda_G^{(p,q)}(f_i)$ $\left(0 < \lambda_G^{(p,q)}(f_i) < \infty\right)$ and *relative* (p, q) *-th weak type* $\tau_G^{(p,q)}(f_i)$ with respect to an algebroidal functions G where p and q are any two positive integers. Then Definition 1.4 and Definition 1.5 are equivalent.

Proof. Let us consider $f_i's$ be entire function and G be an entire algebroidal function such that $\lambda_G^{(p,q)}(f_i)\left(0<\lambda_G^{(p,q)}(f_i)<\infty\right)$ exists for any two positive integers p and q.

Case I. $\tau_G^{(p,q)}(f_i)=\infty$.

Definition 1.4 $\Rightarrow$ Definition 1.5.

As $\tau_G^{(p,q)}(f_i)=\infty$, from Definition 1.4 we obtain for arbitrary positive C and for all sufficiently large values of r that

$$\log^{[p-1]} M_G^{-1}M_{f_i}(r) > C\cdot\left(\log^{[q-1]} r\right)^{\lambda_G^{(p,q)}(f_i)}$$

$$i.e.,\ \log^{[p-2]} M_G^{-1}M_{f_i}(r) > \left[\exp\left(\left(\log^{[q-1]} r\right)^{\lambda_G^{(p,q)}(f_i)}\right)\right]^C. \tag{6}$$

Now if possible let the integral $\int_{r_0}^{\infty}\frac{\log^{[p-2]} M_G^{-1}M_{f_i}(r)}{\left[\exp\left(\log^{[q-1]} r\right)^{\lambda_G^{(p,q)}(f_i)}\right]^{C+1}}dr\ (r_0>0)$ be converge.

Then by Lemma 2.1,

$$\liminf_{r\to\infty}\frac{\log^{[p-2]} M_G^{-1}M_{f_i}(r)}{\left[\exp\left(\left(\log^{[q-1]} r\right)^{\lambda_G^{(p,q)}(f_i)}\right)\right]^C}=0\ .$$

So for a sequence of values of r tending to infinity we get that

$$\log^{[p-2]} M_G^{-1}M_{f_i}(r) < \left[\exp\left(\left(\log^{[q-1]} r\right)^{\lambda_G^{(p,q)}(f_i)}\right)\right]^C. \tag{7}$$

Therefore from (6) and (7), we arrive at a contradiction.
Hence $\int_{r_0}^{\infty}\frac{\log^{[p-2]} M_G^{-1}M_{f_i}(r)}{\left[\exp\left(\log^{[q-1]} r\right)^{\lambda_G^{(p,q)}(f_i)}\right]^{C+1}}dr\ (r_0>0)$ diverges whenever G is finite, which is Definition 1.5.

Definition 1.5 $\Rightarrow$ Definition 1.4.

Let C be any positive number. Since $\tau_G^{(p,q)}(f_i)=\infty$, from Definition 1.5, the divergence of the integral $\int_{r_0}^{\infty}\frac{\log^{[p-2]} M_G^{-1}M_{f_i}(r)}{\left[\exp\left(\log^{[q-1]} r\right)^{\lambda_G^{(p,q)}(f_i)}\right]^{C+1}}dr\ (r_0>0)$ gives for arbitrary positive ε and for all sufficiently large values of r that

$$\log^{[p-2]} M_G^{-1}M_{f_i}(r) > \left[\exp\left(\left(\log^{[q-1]} r\right)^{\lambda_G^{(p,q)}(f_i)}\right)\right]^{C-\varepsilon}$$

$$i.e.,\ \log^{[p-1]} M_G^{-1}M_{f_i}(r) > (C-\varepsilon)\left(\log^{[q-1]} r\right)^{\lambda_G^{(p,q)}(f_i)},$$

which implies that

$$\liminf_{r\to\infty}\frac{\log^{[p-1]} M_G^{-1}M_{f_i}(r)}{\left(\log^{[q-1]} r\right)^{\lambda_G^{(p,q)}(f_i)}} \geq C-\varepsilon\ .$$

Since $C>0$ is arbitrary, it follows that

$$\liminf_{r\to\infty}\frac{\log^{[p-1]} M_G^{-1}M_{f_i}(r)}{\left(\log^{[q-1]} r\right)^{\lambda_G^{(p,q)}(f_i)}} = \infty\ .$$

Thus Definition 1.4 follows.

Case II. $0 \leq \tau_G^{(p,q)}(f_i) < \infty$.

Definition 1.4 $\Rightarrow$ Definition 1.5.

Subcase (C). $0 < \tau_G^{(p,q)}(f_i) < \infty$.

Let f_i's$(i=0,1,2,....,k-1)$ be entire functions and G be an entire algebroidal function such that $0<\tau_G^{(p,q)}(f_i)<\infty$ exists for any two positive integers p and q. Then according to Definition 1.4, for a sequence of values of r tending to infinity, we get that

$$\log^{[p-1]} M_G^{-1}M_{f_i}(r) < \left(\tau_G^{(p,q)}(f_i)+\varepsilon\right)\left(\log^{[q-1]} r\right)^{\lambda_G^{(p,q)}(f_i)}$$

$$i.e.,\ \log^{[p-2]} M_G^{-1}M_{f_i}(r) < \left[\exp\left(\left(\log^{[q-1]} r\right)^{\lambda_G^{(p,q)}(f_i)}\right)\right]^{\tau_G^{(p,q)}(f_i)+\varepsilon}$$

$$i.e.,\ \frac{\log^{[p-2]} M_G^{-1}M_{f_i}(r)}{\left[\exp\left(\left(\log^{[q-1]} r\right)^{\lambda_G^{(p,q)}(f_i)}\right)\right]^t} < \frac{\left[\exp\left(\left(\log^{[q-1]} r\right)^{\lambda_G^{(p,q)}(f_i)}\right)\right]^{\tau_G^{(p,q)}(f_i)+\varepsilon}}{\left[\exp\left(\left(\log^{[q-1]} r\right)^{\lambda_G^{(p,q)}(f_i)}\right)\right]^t}$$

$$i.e.,\ \frac{\log^{[p-2]} M_G^{-1}M_{f_i}(r)}{\left[\exp\left(\left(\log^{[q-1]} r\right)^{\lambda_G^{(p,q)}(f_i)}\right)\right]^t} < \frac{1}{\left[\exp\left(\left(\log^{[q-1]} r\right)^{\lambda_G^{(p,q)}(f_i)}\right)\right]^{t-\ \tau_G^{(p,q)}(f_i)+\varepsilon}}\ .$$

Therefore $\int\limits_{r_0}^{\infty}\frac{\log^{[p-2]} M_G^{-1}M_{f_i}(r)}{\left[\exp\ \left(\log^{[q-1]} r\right)^{\lambda_G^{(p,q)}(f_i)}\right)\right]^{t+1}}dr\ (r_0>0)$ converges for $k>\tau_G^{(p,q)}(f_i)$. □

Again by Definition 1.4, we obtain for all sufficiently large values of r that

$$\log^{[p-1]} M_G^{-1} M_{f_i}(r) > \left(\tau_G^{(p,q)}(f_i) - \varepsilon\right)\left(\log^{[q-1]} r\right)^{\lambda_G^{(p,q)}(f_i)}$$

$$i.e., \ \log^{[p-2]} M_G^{-1} M_{f_i}(r) > \left[\exp\left(\left(\log^{[q-1]} r\right)^{\lambda_G^{(p,q)}(f_i)}\right)\right]^{\tau_G^{(p,q)}(f_i)-\varepsilon} . \tag{8}$$

So for $k < \tau_G^{(p,q)}(f_i)$, we get from (8) that

$$\frac{\log^{[p-2]} M_G^{-1} M_{f_i}(r)}{\left[\exp\left(\left(\log^{[q-1]} r\right)^{\lambda_G^{(p,q)}(f_i)}\right)\right]^t} > \frac{1}{\left[\exp\left(\left(\log^{[q-1]} r\right)^{\lambda_G^{(p,q)}(f_i)}\right)\right]^{t- \ \tau_G^{(p,q)}(f_i)-\varepsilon}} .$$

Therefore $\int\limits_{r_0}^{\infty} \frac{\log^{[p-2]} M_G^{-1} M_{f_i}(r)}{\left[\exp\left(\left(\log^{[q-1]} r\right)^{\lambda_G^{(p,q)}(f_i)}\right)\right]^{t+1}} dr \ (r_0 > 0)$ diverges for $t < \tau_G^{(p,q)}(f_i)$.

Hence $\int\limits_{r_0}^{\infty} \frac{\log^{[p-2]} M_G^{-1} M_{f_i}(r)}{\left[\exp\left(\left(\log^{[q-1]} r\right)^{\lambda_G^{(p,q)}(f_i)}\right)\right]^{t+1}} dr \ (r_0 > 0)$ converges for $t > \tau_G^{(p,q)}(f_i)$ and diverges for $t < \tau_G^{(p,q)}(f_i)$.

Subcase (D). $\tau_G^{(p,q)}(f_i) = 0$.

When $\tau_G^{(p,q)}(f_i) = 0$ for any two positive integers p and q , Definition 1.4 gives for a sequence of values of r tending to infinity that

$$\frac{\log^{[p-1]} M_G^{-1} M_{f_i}(r)}{\left(\log^{[q-1]} r\right)^{\lambda_G^{(p,q)}(f_i)}} < \varepsilon .$$

Then as before we obtain that $\int\limits_{r_0}^{\infty} \frac{\log^{[p-2]} M_G^{-1} M_{f_i}(r)}{\left[\exp\left(\left(\log^{[q-1]} r\right)^{\lambda_G^{(p,q)}(f_i)}\right)\right]^{t+1}} dr \ (r_0 > 0)$ converges for $t > 0$ and diverges for $t < 0$.

Thus combining Subcase(C) and Subcase(D) , Definition 1.5 follows.

Definition 1.5 ⇒ Definition 1.4.

From Definition 1.5 and for arbitrary positive ε, the integral

$$\int\limits_{r_0}^{\infty} \frac{\log^{[p-2]} M_G^{-1} M_{f_i}(r)}{\left[\exp\left(\left(\log^{[q-1]} r\right)^{\lambda_G^{(p,q)}(f_i)}\right)\right]^{\tau_G^{(p,q)}(f_i)+\varepsilon+1}} dr \ (r_0 > 0)$$

converges. Then by Lemma 2.1, we get that

$$\liminf_{r\to\infty}\frac{\log^{[p-2]} M_G^{-1}M_{f_i}(r)}{\left[\exp\left(\left(\log^{[q-1]} r\right)^{\lambda_G^{(p,q)}(f_i)}\right)\right]^{\tau_G^{(p,q)}(f_i)+\varepsilon}}=0\ .$$

So we get for a sequence of values of r tending to infinity that

$$\frac{\log^{[p-2]} M_G^{-1}M_{f_i}(r)}{\left[\exp\left(\left(\log^{[q-1]} r\right)^{\lambda_G^{(p,q)}(f_i)}\right)\right]^{\tau_G^{(p,q)}(f_i)+\varepsilon}}<\varepsilon$$

$$i.e.,\ \log^{[p-2]} M_G^{-1}M_{f_i}(r)<\varepsilon\cdot\left[\exp\left(\left(\log^{[q-1]} r\right)^{\lambda_G^{(p,q)}(f_i)}\right)\right]^{\tau_G^{(p,q)}(f_i)+\varepsilon}$$

$$i.e.,\ \log^{[p-1]} M_G^{-1}M_{f_i}(r)<\log\varepsilon+\left(\tau_G^{(p,q)}(f_i)+\varepsilon\right)\left(\log^{[q-1]} r\right)^{\lambda_G^{(p,q)}(f_i)}$$

$$i.e.,\ \liminf_{r\to\infty}\frac{\log^{[p-1]} M_G^{-1}M_{f_i}(r)}{\left(\log^{[q-1]} r\right)^{\lambda_G^{(p,q)}(f_i)}}\le\tau_G^{(p,q)}(f_i)+\varepsilon\ .$$

Since $\varepsilon\,(>0)$ is arbitrary, it follows from above that

$$\liminf_{r\to\infty}\frac{\log^{[p-1]} M_G^{-1}M_{f_i}(r)}{\left(\log^{[q-1]} r\right)^{\lambda_G^{(p,q)}(f_i)}}\le\tau_G^{(p,q)}(f_i)\ . \tag{9}$$

On the other hand, the divergence of the integral

$$\int_{r_0}^{\infty}\frac{\log^{[p-2]} M_G^{-1}M_{f_i}(r)}{\left[\exp\left(\left(\log^{[q-1]} r\right)^{\lambda_G^{(p,q)}(f_i)}\right)\right]^{\tau_G^{(p,q)}(f_i)-\varepsilon+1}}dr\ (r_0>0)$$

implies for all sufficiently large values of r that

$$\frac{\log^{[p-2]} M_G^{-1}M_{f_i}(r)}{\left[\exp\left(\left(\log^{[q-1]} r\right)^{\lambda_G^{(p,q)}(f_i)}\right)\right]^{\tau_G^{(p,q)}(f_i)-\varepsilon+1}}>\frac{1}{\left[\exp\left(\left(\log^{[q-1]} r\right)^{\lambda_G^{(p,q)}(f_i)}\right)\right]^{1+\varepsilon}}$$

$$i.e.,\ \log^{[p-2]} M_G^{-1}M_{f_i}(r)>\left[\exp\left(\left(\log^{[q-1]} r\right)^{\lambda_G^{(p,q)}(f_i)}\right)\right]^{\tau_G^{(p,q)}(f_i)-2\varepsilon}$$

$$i.e.,\ \log^{[p-1]} M_G^{-1}M_{f_i}(r)>\left(\tau_G^{(p,q)}(f_i)-2\varepsilon\right)\left(\left(\log^{[q-1]} r\right)^{\lambda_G^{(p,q)}(f_i)}\right)$$

$$i.e.,\ \frac{\log^{[p-1]} M_G^{-1}M_{f_i}(r)}{\left(\log^{[q-1]} r\right)^{\lambda_G^{(p,q)}(f_i)}}>\left(\tau_G^{(p,q)}(f_i)-2\varepsilon\right)\ .$$

As $\varepsilon\,(>0)$ is arbitrary, it follows from above that

$$\liminf_{r\to\infty}\frac{\log^{[p-1]}M_G^{-1}M_{f_i}(r)}{\left(\log^{[q-1]}r\right)^{\lambda_G^{(p,q)}(f_i)}}\geq\tau_G^{(p,q)}(f_i)\ . \tag{10}$$

So from (9) and (10) , we obtain that

$$\liminf_{r\to\infty}\frac{\log^{[p-1]}M_G^{-1}M_{f_i}(r)}{\left(\log^{[q-1]}r\right)^{\lambda_G^{(p,q)}(f_i)}}=\tau_G^{(p,q)}(f_i)\ .$$

This proves the theorem.

Now we state the following two theorems without their proofs as those can easily be carried out with help of Lemma 2.1 and in the line of Theorem 3.1 and Theorem 3.3 respectively.

Theorem 3.4. Let f_i 's be entire functions having finite positive *relative* (p,q)-*th order* $\rho_G^{(p,q)}(f_i)$ $\left(0<\rho_G^{(p,q)}(f_i)<\infty\right)$ and *relative* (p,q) *-th lower type* $\overline{\sigma}_G^{(p,q)}(f_i)$ with respect to an entire algebroidal function G where p and q are any two positive integers. Then Definition 1.6 and Definition 1.7 are equivalent.

Theorem 3.5. Let f_i's be entire functions having finite positive relative (p,q)-*th lower order* $\lambda_G^{(p,q)}(f_i)$ $\left(a<\lambda_G^{(p,q)}(f_i)<\infty\right)$ and the growth indicator $\overline{\tau}_G^{(p,q)}(f_i)$ with respect to an entire algebroidal function G where p and q are any two positive integers. Then Definition 1.8 and Definition 1.9 are equivalent.

Theorem 3.6. Let f_i's be entire functions and G be an entire algebroidal function with $0<\lambda_G^{(p,q)}(f_i)\leq\rho_G^{(p,q)}(f_i)<\infty$ where p and q are any two positive integers. Then

$$(i)\ \ \sigma_G^{(p,q)}(f_i)=\limsup_{r\to\infty}\frac{\log^{[p-1]}M_G^{-1}(r)}{\left[\log^{[q-1]}M_{f_i}^{-1}(r)\right]^{\rho_G^{(p,q)}(f_i)}},$$

$$(ii)\ \ \overline{\sigma}_G^{(p,q)}(f_i)=\liminf_{r\to\infty}\frac{\log^{[p-1]}M_G^{-1}(r)}{\left[\log^{[q-1]}M_{f_i}^{-1}(r)\right]^{\rho_G^{(p,q)}(f_i)}},$$

$$(iii)\ \ \tau_G^{(p,q)}(f_i)=\liminf_{r\to\infty}\frac{\log^{[p-1]}M_G^{-1}(r)}{\left[\log^{[q-1]}M_{f_i}^{-1}(r)\right]^{\lambda_G^{(p,q)}(f_i)}}\mathit{and}$$

$$(iv)\ \ \overline{\tau}_G^{(p,q)}(f_i)=\limsup_{r\to\infty}\frac{\log^{[p-1]}M_G^{-1}(r)}{\left[\log^{[q-1]}M_{f_i}^{-1}(r)\right]^{\lambda_G^{(p,q)}(f_i)}}\ .$$

Proof. Taking $M_{f_i}(r)=R$, the theorem follows from the definitions of $\sigma_G^{(p,q)}(f_i)$, $\overline{\sigma}_G^{(p,q)}(f_i)$, $\tau_G^{(p,q)}(f_i)$and $\overline{\tau}_g^{(p,q)}(f)$ respectively.

In the following theorem we obtain a relationship among $\sigma_G^{(p,q)}(f_i)$, $\overline{\sigma}_G^{(p,q)}(f_i)$, $\overline{\tau}_G^{(p,q)}(f_i)$ and $\tau_G^{(p,q)}(f_i)$. □

Theorem 3.7. Let f_i's be entire functions such that f_i is of regular relative $(p,q)-$growth with respect to an entire algebroidal function G i.e., $\rho_G^{(p,q)}(f_i) = \lambda_G^{(p,q)}(f_i)$ $\left(0 < \lambda_G^{(p,q)}(f_i) = \rho_G^{(p,q)}(f_i) < \infty\right)$ where p and q are any two positive integers, then the following quantities

$$(i)\ \sigma_G^{(p,q)}(f_i),\ \ (ii)\ \tau_G^{(p,q)}(f_i),\ \ (iii)\ \overline{\sigma}_G^{(p,q)}(f_i)\ \text{ and }\ \ (iv)\ \overline{\tau}_G^{(p,q)}(f_i)$$

are all equivalent.

From Definition 1.5, it follows that the integral

$$\int_{r_0}^{\infty} \frac{\log^{[p-2]} M_G^{-1} M_{f_i}(r)}{\left[\exp\left(\left(\log^{[q-1]} r\right)^{\lambda_G^{(p,q)}(f_i)}\right)\right]^{t+1}} dr\ (r_0 > 0)$$

converges for $t > \tau_G^{(p,q)}(f_i)$and diverges for $t < \tau_G^{(p,q)}(f_i)$.

On the other hand, Definition 1.3 implies that the integral

$$\int_{r_0}^{\infty} \frac{\log^{[p-2]} M_G^{-1} M_{f_i}(r)}{\left[\exp\left(\left(\log^{[q-1]} r\right)^{\rho_G^{(p,q)}(f_i)}\right)\right]^{t+1}} dr\ (r_0 > 0)$$

converges for $t > \sigma_G^{(p,q)}(f_i)$ and diverges for $t < \sigma_G^{(p,q)}(f_i)$.

(i)⇒(ii).

Now it is obvious that all the quantities in the expression

$$\left[\frac{\log^{[p-2]} M_G^{-1} M_{f_i}(r)}{\left[\exp\left(\left(\log^{[q-1]} r\right)^{\lambda_G^{(p,q)}(f_i)}\right)\right]^{t+1}} - \frac{\log^{[p-2]} M_G^{-1} M_{f_i}(r)}{\left[\exp\left(\left(\log^{[q-1]} r\right)^{\rho_G^{(p,q)}(f_i)}\right)\right]^{t+1}}\right]$$

are of non negative type. So

$$\int_{r_0}^{\infty} \frac{\log^{[p-2]} M_G^{-1} M_{f_i}(r)}{\left[\exp\left(\left(\log^{[q-1]} r\right)^{\lambda_G^{(p,q)}(f_i)}\right)\right]^{t+1}} - \frac{\log^{[p-2]} M_G^{-1} M_{f_i}(r)}{\left[\exp\left(\left(\log^{[q-1]} r\right)^{\rho_G^{(p,q)}(f_i)}\right)\right]^{t+1}} dr\ (r_0 > 0) \geq 0$$

$$i.e.,\ \int_{r_0}^{\infty} \frac{\log^{[p-2]} M_G^{-1} M_{f_i}(r)}{\left[\exp\left(\left(\log^{[q-1]} r\right)^{\lambda_G^{(p,q)}(f_i)}\right)\right]^{t+1}} dr \geq$$

$$\int_{r_0}^{\infty} \frac{\log^{[p-2]} M_G^{-1} M_{f_i}(r)}{\left[\exp\left(\left(\log^{[q-1]} r\right)^{\rho_G^{(p,q)}(f_i)}\right)\right]^{t+1}} dr \text{ for } r_0 > 0 \ .$$

$$i.e., \ \tau_G^{(p,q)}(f_i) \geq \sigma_G^{(p,q)}(f_i) \ . \tag{11}$$

Further f_i 's are of regular relative (p,q) growth with respect to G. Therefore we get that

$$\sigma_G^{(p,q)}(f_i) = \limsup_{r\to\infty} \frac{\log^{[p-1]} M_G^{-1} M_{f_i}(r)}{\left(\log^{[q-1]} r\right)^{\rho_G^{(p,q)}(f_i)}}$$
$$\geq \liminf_{r\to\infty} \frac{\log^{[p-1]} M_G^{-1} M_{f_i}(r)}{\left(\log^{[q-1]} r\right)^{\rho_G^{(p,q)}(f_i)}} = \liminf_{r\to\infty} \frac{\log^{[p-1]} M_G^{-1} M_{f_i}(r)}{\left(\log^{[q-1]} r\right)^{\lambda_G^{(p,q)}(f_i)}} = \tau_G^{(p,q)}(f_i) \ . \tag{12}$$

Hence from (11) and (12), we obtain that

$$\sigma_G^{(p,q)}(f_i) = \tau_G^{(p,q)}(f_i) \ . \tag{13}$$

(ii)⇒(iii).

Since f_i 's are of regular relative (p,q) growth with respect to G i.e., $\rho_G^{(p,q)}(f_i) = \lambda_G^{(p,q)}(f_i)$ we get that

$$\tau_G^{(p,q)}(f_i) = \liminf_{r\to\infty} \frac{\log^{[p-1]} M_G^{-1} M_{f_i}(r)}{\left(\log^{[q-1]} r\right)^{\lambda_G^{(p,q)}(f_i)}} = \liminf_{r\to\infty} \frac{\log^{[p-1]} M_G^{-1} M_{f_i}(r)}{\left(\log^{[q-1]} r\right)^{\rho_G^{(p,q)}(f_i)}} = \overline{\sigma}_G^{(p,q)}(f_i) \ .$$

(iii)⇒(iv).

In view of (13) and the condition $\rho_G^{(p,q)}(f_i) = \lambda_G^{(p,q)}(f_i)$, it follows that

$$\overline{\sigma}_G^{(p,q)}(f_i) = \liminf_{r\to\infty} \frac{\log^{[p-1]} M_G^{-1} M_{f_i}(r)}{\left(\log^{[q-1]} r\right)^{\rho_G^{(p,q)}(f_i)}}$$
$$i.e., \overline{\sigma}_G^{(p,q)}(f_i) = \liminf_{r\to\infty} \frac{\log^{[p-1]} M_G^{-1} M_{f_i}(r)}{\left(\log^{[q-1]} r\right)^{\lambda_G^{(p,q)}(f_i)}}$$

$$i.e., \ \overline{\sigma}_G^{(p,q)}(f_i) = \tau_G^{(p,q)}(f_i)$$
$$i.e., \ \overline{\sigma}_G^{(p,q)}(f_i) = \sigma_G^{(p,q)}(f_i)$$

$$i.e., \ \overline{\sigma}_G^{(p,q)}(f_i) = \limsup_{r\to\infty} \frac{\log^{[p-1]} M_G^{-1} M_{f_i}(r)}{\left(\log^{[q-1]} r\right)^{\rho_G^{(p,q)}(f_i)}}$$

$$i.e., \ \overline{\sigma}_G^{(p,q)}(f_i) = \limsup_{r\to\infty} \frac{\log^{[p-1]} M_G^{-1} M_{f_i}(r)}{\left(\log^{[q-1]} r\right)^{\lambda_G^{(p,q)}(f_i)}}$$

$$i.e., \ \overline{\sigma}_G^{(p,q)}(f_i) = \overline{\tau}_G^{(p,q)}(f_i) \ .$$

(iv)⇒(i).

As f_i 's are of regular relative (p,q) growth with respect to G i.e., $\rho_G^{(p,q)}(f_i) = \lambda_G^{(p,q)}(f_i)$, we obtain that

$$\limsup_{r\to\infty} \frac{\log^{[p-1]} M_G^{-1} M_{f_i}(r)}{\left(\log^{[q-1]} r\right)^{\lambda_G^{(p,q)}(f_i)}} = \limsup_{r\to\infty} \frac{\log^{[p-1]} M_G^{-1} M_{f_i}(r)}{\left(\log^{[q-1]} r\right)^{\rho_G^{(p,q)}(f_i)}} = \sigma_G^{(p,q)}(f_i).$$

Thus the theorem follows.

4 Conclusion

The results carried out in this present paper may be viewed from the angle of slowly changing functions as well as for the functions analytic in the unit disc and ploydisc.

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References

[1] L. Bernal, Crecimiento relativo de funciones enteras. Contribución al estudio de lasfunciones enteras con índice exponencial finito, Doctoral Dissertation, University of Seville, Spain, 1984.

[2] L. Bernal, Orden relative de crecimiento de funciones enteras , Collect. Math., 39 (1988), 209-229.

[3] J. Clunie, The composition of entire and meromorphic functions, Mathematical essays dedicated to A.J.Macintyre, Ohio University Press, (1970), 75 - 92.

[4] A. S. B. Holland, Introduction to the theory of entire functions, Academic Press, New York and London (1973).

[5] O. P. Juneja, G. P. Kapoor and S. K. Bajpai, On the (p,q)-order and lower (p,q)-order of an entire function, J. Reine Angew. Math., 282**,** (1976), 53-67.

[6] O. P. Juneja, G. P. Kapoor and S. K. Bajpai, On the (p, q)-type and lower (p, q)-type of an entire function, J. Reine Agew. Math., 290 (1977), 180-189.

[7] C. Roy, Some properties of entire functions in one and several complex vaiables, Ph.D. Thesis (2010), University of Calcutta.

[8] L. M. S. Ruiz, S. K. Datta, T. Biswas and G. K. Mondal, On the (p,q)-th relative order oriented growth properties of entire functions,Abstract and Applied Analysis,Hindawi Publishing Corporation, Volume 2014, Article ID 826137, 8 pages.

[9] E. C. Titchmarsh, The theory of functions, 2nd ed. Oxford University Press, Oxford, 1939.

[10] G. Valiron, Lectures on the general theory of integral functions, Chelsea Publishing Company, 1949.

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Certain Classes of Analytic Functions Associated with Conic Domains

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Abstract – In this paper, we define new subclasses of k-uniformly Janowski starlike and k-uniformly Janowski convex functions associated with m-symmetric points. The integral representations, convolution properties and sufficient conditions for the functions belong to this class are investigated.

Keywords – *Subordination, convolution, m-symmetric points.*

1. Introduction

Let A denote the class of functions f (z) of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the open unit disk $E = \{z : z \in C \text{ and } |z| < 1\}$. Furthermore S, represents class of all functions in A which are univalent in E. Sakaguchi [6] introduced a class S_s^* of functions starlike with respect to symmetric points, it consists of functions $f(z) \in S$ satisfying the inequality

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z) - f(-z)}\right) > 0, \quad (z \in E). \tag{1.2}$$

Following him, many authors studied this class and its subclasses see [7, 8, 9].

Das and Singh [16] in 1977 extend the results of Sakaguchi to other class in E, namely convex functions with respect to symmetric points. Let C_s denote the class of convex functions with respect to symmetric points and satisfying the condition

$$\operatorname{Re}\left(\frac{(zf'(z))'}{f'(z) - f'(-z)}\right) > 0, \quad (z \in E).$$

It is also well known [16] that $f \in C_s$ if and only if $zf'(z) \in S_s^*$.

Chand and Singh [1] introduced a class S_s^m of functions starlike with respect to m-symmetric points, which consists of functions $f(z) \in S$, satisfying the inequality

$$\mathrm{Re}\left(\frac{zf'(z)}{f_m(z)}\right) > 0, \quad (z \in E), \tag{1.3}$$

where

$$f_m(z) = \frac{1}{m}\sum_{\mu=0}^{m-1} \varepsilon^{-\mu} f(\varepsilon^{\mu} z), \quad (\varepsilon^{\mu} = 1, m \in N). \tag{1.4}$$

From equation (1.4) we can write

$$f_m(z) = \frac{1}{m}\sum_{\mu=0}^{m-1} \varepsilon^{-\mu} f(\varepsilon^{\mu} z) = \frac{1}{m}\sum_{\mu=0}^{m-1} \varepsilon^{-\mu}\left(\varepsilon^{\mu} z + \sum_{n=2}^{\infty} a_n (\varepsilon^{\mu} z)^n\right)$$

$$= z + \sum_{n=2}^{\infty} b_n a_n z^n \tag{1.5}$$

where

$$b_n = \frac{1}{m}\sum_{\mu=0}^{m-1} \varepsilon^{(n-1)\mu} = \begin{cases} 1, & n = lm+1, \\ 0, & n \neq lm+1, \end{cases} \tag{1.6}$$

where $l, m \in N$; $n \geq 2, \varepsilon^{m} = 1$.

Note that the accompanying characters follow directly from the above definition [10].

$$f_m(\varepsilon^{\mu} z) = \varepsilon^{\mu} f_m(z), \tag{1.7}$$

$$f_m'(\varepsilon^{\mu} z) = f_m(z) = \frac{1}{m}\sum_{\mu=0}^{m-1} f'(\varepsilon^{\mu} z), \quad (z \in E). \tag{1.8}$$

Definition 1. For $f(z) \in A$ given by (1.1) and $g(z) \in A$ of the form

$$g(z) = z + \sum_{n=2}^{\infty} c_n z^n$$

the Hadamard product (or convolution) of $f(z)$ and $g(z)$ is given by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n c_n z^n = (g * f)(z), \quad (z \in E).$$

For two functions F(z) and G(z) analytic in E, we say that F(z) is subordinate to G(z) denoted by $F \prec G$ or $F(z) \prec G(z)$, if there exists an analytic function $w(z)$ with $|w(z)| < 1$ such that $F(z) = G(w(z))$. Furthermore if the function G(z) is univalent in E then we have the following equivalence [13,14,15]

$$F(z) \prec G(z) \Leftrightarrow F(0) = G(0) \text{ and } F(E) \subseteq G(E).$$

Definition 2. A function $p(z)$ is said to be in the class P[A,B], if it is analytic in E with $p(0) = 1$ and

$$p(z) \prec \frac{1+Az}{1+Bz}, \qquad -1 \le B < A \le 1.$$

Geometrically, if a function p belongs to P [A,B], then it maps the open unit disc E onto the disk characterized by the domain

$$\Omega[A,B] = \left\{ w : \left| w - \frac{1-AB}{1-B^2} \right| < \frac{A-B}{1-B^2} \right\}.$$

The class P [A,B], is connected with the class P of functions with positive real part by the relation

$$p(z) \in P, \text{ if and only if } \frac{(A+1)p(z)-(A-1)}{(B+1)p(z)-(B-1)} \in P[A,B].$$

This class was presented by Janowski [2] and explored by a few creators. Kanas and Wisniowska [4,3] presented and examined the class k-ST of k-starlike functions and the relating class k-UCV of k-uniformly convex functions. These were characterized subject to the conic region k, Ω_k, $k \ge 0$, as

$$\Omega_k = \left\{ u + iv : u > k\sqrt{(u-1)^2 + v^2} \right\}.$$

This domain represents the right half plane, a parabola, a hyperbola and an ellipse for k = 0, k = 1, 0 < k < 1 and k > 1 respectively. The external functions for these conic regions are

$$p_k(z) = \begin{cases} \dfrac{1+z}{1-z}, & k = 0, \\ 1 + \dfrac{2}{\pi^2}\left(\log\left(\dfrac{1+\sqrt{z}}{1-\sqrt{z}} \right) \right)^2, & k = 1, \\ 1 + \dfrac{2}{1-k^2}\sinh^2\left\{ \left(\dfrac{2}{\pi}\arccos k \right) \arctan h\sqrt{z} \right\}, & 0 < k < 1, \\ 1 + \dfrac{2}{k^2-1}\sin\left(\dfrac{\pi}{2R(t)} \displaystyle\int_0^{\frac{u(z)}{\sqrt{t}}} \dfrac{d(x)}{\sqrt{1-x^2}\sqrt{1-(tx)^2}} \right) + \dfrac{1}{k^2-1}, & k > 1, \end{cases} \tag{1.9}$$

where

$$u(z)=\frac{z-\sqrt{t}}{1-\sqrt{tx}}, \qquad (z\in E),$$

and $t\in(0,1)$ and z is chosen such that $k=\cosh\left(\frac{\pi R'(t)}{4R(t)}\right)$. Here $R(t)$ is Legendre's complete elliptic integral of first kind and $R'(t)$ is the complementary integral of $R(t)$.

Following are the definitions of classes k - ST and k - UCV.

Definition 3. A function $f(z)\in A$ is said to be in the class k - ST, if and only if

$$\frac{zf'(z)}{f(z)}\prec p_k(z), \quad (z\in E, k\geq 0).$$

Definition 4. A function $f(z)\in A$ is said to be in the class k - UCV, if and only if

$$\frac{(zf'(z))'}{f'(z)}\prec p_k(z), \quad (z\in E, k\geq 0).$$

The classes k - ST and k - UCV were further generalized by Shams et al, [11], to the $KD(k,\beta)$ and $SD(k,\beta)$, respectively subject to the conic domain $G(k,\beta)$, $k\geq 0$ and $0\leq\beta<1$ which is

$$G(k,\beta)=\{w:\operatorname{Re}w>k|w-1|+\beta\}.$$

Now using the concepts of Janowski functions and the conic regions, we defne the following

$$p(z)\prec\frac{(A+1)p_k(z)-(A-1)}{(B+1)p_k(z)-(B-1)}, \qquad k\geq 0,$$

where $p_k(z)$ is defined by (1.9) and $-1\leq B<A\leq 1$.

Geometrically, the function $p(z)\in k-[A,B]$, takes all values from the domain $\Omega_k[A,B]$, $-1\leq B<A\leq 1, k\geq 0$ which is define as

$$\Omega_k[A,B]=\left\{w:\ \operatorname{Re}\left(\frac{(B-1)w(z)-(A-1)}{(B+1)w(z)-(A+1)}\right)>k\left|\frac{(B-1)w(z)-(A-1)}{(B+1)w(z)-(A+1)}-1\right|\right\},$$

or equivalently

$$\begin{aligned}\Omega_k[A,B]=\{u+iv:&\left[(B^2-1)(u^2+v^2)-2(AB-1)u+(A^2-1)\right]^2\\&>k^2\left[\left(-2(B+1)(u^2+v^2)+2(A+B+2)u-2(A+1)\right)^2+4(A-B)^2v^2\right]\}.\end{aligned}$$

The domain $\Omega_k[A,B]$ retains the conic domain Ω_k inside the circular region defined by $\Omega[A,B]$. The impact of $\Omega[A,B]$ on the conic domain Ω_k changes the original shape of the conic regions. The ends of hyperbola and parabola gets closer to one another but never meet anywhere and the ellipse gets the oval shape. When $A \to 1, B \to -1$ the radiuses of the circular disk define by $\Omega[A,B]$ tends to infinity, consequently the arm of the hyperbola and parabolas expand to the oval terns into ellipse. We see that $\Omega_k[1,-1] = \Omega_k$, the conic domain define by Kanas and Wisniowska [3].

Definition 4. A function $f(z) \in \mathrm{A}$ is said to be in the class $k\text{-}\mathrm{ST}_s^{(m)}[\mathrm{A,B}]$, $-1 \leq B < A \leq 1$, $k \geq 0$, if and only if

$$\operatorname{Re}\left(\frac{(\mathrm{B}-1)\frac{zf'(z)}{f_m(z)} - (A-1)}{(\mathrm{B}+1)\frac{zf'(z)}{f_m(z)} - (A+1)}\right) > k\left|\frac{(B-1)\frac{zf'(z)}{f_m(z)} - (A-1)}{(B+1)\frac{zf'(z)}{f_m(z)} - (A+1)} - 1\right|,$$

or equivalently

$$\frac{\mathrm{z}f'(\mathrm{z})}{f_m(z)} \in k - \mathrm{P}[A,B], \tag{1.10}$$

where $f_m(z)$ is defined by (1.4).

Special Cases:

i). $k\text{-}\mathrm{ST}_s^{(1)}[\mathrm{A,B}] = k\text{-}\mathrm{ST}[\mathrm{A,B}]$, we have the well known class presented and studied in [5].
ii). $0\text{-}\mathrm{ST}_s^{(m)}[\mathrm{A,B}] = \mathrm{S}_s^{(m)}[\mathrm{A,B}]$, see [10].
iii). $k\text{-}\mathrm{ST}_s^{(1)}[1,-1] = k\text{-}\mathrm{ST}$. For this we refer to [4].
iv). $k\text{-}\mathrm{ST}_s^{(1)}[1-2\beta,-1] = \mathrm{SD}[\mathrm{k},\beta,]$, we have the well known class presented and studied in [11].
v). $0\text{-}\mathrm{ST}_1^{(1)}[\mathrm{A,B}] = \mathrm{S}^*[\mathrm{A,B}]$, we have the well known class presented and studied in [2].

Definition 4. A function $f(z) \in \mathrm{A}$ is said to be in the class $k\text{-}\mathrm{UCV}_s^{(m)}[\mathrm{A,B}]$, $k \geq 0$, $-1 \leq B < A \leq 1$, if and only if

$$\operatorname{Re}\left(\frac{(\mathrm{B}-1)\frac{(zf'(z))'}{f_m'(z)} - (A-1)}{(\mathrm{B}+1)\frac{(zf'(z))'}{f_m'(z)} - (A+1)}\right) > k\left|\frac{(B-1)\frac{(zf'(z))'}{f_m'(z)} - (A-1)}{(B+1)\frac{(zf'(z))'}{f_m'(z)} - (A+1)} - 1\right|,$$

or equivalently

$$\frac{(zf'(z))'}{f_m'(z)} \in k-\mathrm{P}[A,B], \tag{1.11}$$

where $f_m(z)$ is defined by (1.4).

Special Cases:

i). $k\text{-}\mathrm{UCV}_s^{(1)}[\mathrm{A,B}] = k\text{-}\mathrm{UCV}[\mathrm{A,B}]$, we have the class introduced and studied in [5].
ii). $k\text{-}\mathrm{UCV}_s^{(1)}[1,-1] = k\text{-}\mathrm{UCV}$, and this is well known class introduced and studied in [3].
iv). $k\text{-}\mathrm{UCV}_s^{(1)}[1-2\beta,-1] = K\mathrm{D}[\mathrm{k},\beta,]$, see [11].
v). $0\text{-}\mathrm{UCV}_s^{(1)}[\mathrm{A,B}] = C[\mathrm{A,B}]$, we have the well known class introduced and studied in [2].

It is easy to see that:

$$\mathrm{f} \in k\text{-}\mathrm{UCV}_s^{(m)}[\mathrm{A,B}] \Leftrightarrow zf' \in k\text{-}\mathrm{ST}_s^{(m)}[\mathrm{A,B}].$$

2. Main Results

Integral representation. First we give two meaningful conclusions about the classes $k\text{-}\mathrm{ST}_s^{(m)}[\mathrm{A,B}]$ and $k\text{-}\mathrm{UCV}_s^{(m)}[\mathrm{A,B}]$.

Theorem 1. Let $f(z) \in k\text{-}\mathrm{ST}_s^{(m)}[\mathrm{A,B}]$. Then $f_m(z) \in k\text{-}\mathrm{ST}[\mathrm{A,B}] \subseteq k\text{-}\mathrm{ST} \subseteq \mathrm{S}$.

Proof. For $f(z) \in k\text{-}\mathrm{ST}_s^{(m)}[\mathrm{A,B}]$, we can obtain

$$\frac{\mathrm{z}f'(\mathrm{z})}{f_m(z)} \prec \frac{(A+1)p_k(z)-(A-1)}{(B+1)p_k(z)-(B-1)}, \quad (z \in \mathrm{E}). \tag{2.1}$$

Substituting z by $\varepsilon^{\mu} z$ respectively $(\mu = 0,1,2,...m-1)$, we have

$$\frac{\varepsilon^{\mu} z\mathrm{f}'(\varepsilon^{\mu} z)}{f_m(\varepsilon^{\mu} z)} \prec \frac{(A+1)p_k(\varepsilon^{\mu} z)-(A-1)}{(B+1)p_k(\varepsilon^{\mu} z)-(B-1)} \prec \frac{(A+1)p_k(z)-(A-1)}{(B+1)p_k(z)-(B-1)}, \quad (z \in \mathrm{E}). \tag{2.2}$$

By definition of $f_m(z)$ and $\varepsilon = \exp\left(\frac{2\pi i}{m}\right)$, we know that $\varepsilon^{-\mu} f_m(\varepsilon^{\mu} z) = f_m(z)$. Then equation (2.2), becomes

$$\frac{z\mathrm{f}'(\varepsilon^{\mu} z)}{f_m(z)} \prec \frac{(A+1)p_k(z)-(A-1)}{(B+1)p_k(z)-(B-1)}, \quad (z \in \mathrm{E}). \tag{2.3}$$

Let $(\mu = 0,1,2,...m-1)$ in (2.3), respectively and sum them to get

$$\frac{z\mathrm{f}_{\mathrm{m}}'(z)}{f_m(z)} \prec \frac{1}{m}\sum_{\mu=0}^{m-1}\frac{z\mathrm{f}'(\varepsilon^{\mu} z)}{f_m(z)} \prec \frac{(A+1)p_k(z)-(A-1)}{(B+1)p_k(z)-(B-1)}, \quad (z \in \mathrm{E}).$$

Thus $f_m(z) \in k\text{-ST}[A,B] \subseteq S$.

Putting k = 0 in Theorem 1, we can obtain Corollary 1, below which is comparable to the result obtained by Kwon and Sim [10].

Corollary 1. Let $f(z) \in \mathrm{ST}_s^{(m)}[A,B]$. Then $f_m(z) \in \mathrm{ST}[A,B] \subseteq k\text{-ST} \subseteq S$.

Theorem 2. Let $f(z) \in k\text{-UCV}_s^{(m)}[A,B]$. Then $f_m(z) \in k\text{-UCV}[A,B] \subseteq S$.

Proof. The proof of Theorem 2 is similar to that of Theorem 1 so the details are omitted.

Now we give the integral representations of the functions belonging to the classes $k\text{-ST}_s^{(m)}[A,B]$ and $k\text{-UCV}_s^{(m)}[A,B]$.

Theorem 3. Let $f(z) \in k\text{-ST}_s^{(m)}[A,B]$. Then

$$f_m(z) = z \cdot \left\{ \exp(A-B) \frac{1}{m} \sum_{\mu=0}^{m-1} \int_0^{\varepsilon^\mu z} \frac{(p_k(w(t))-1)}{t(B+1)p_k(w(t))-(B-1)} dt \right\}, \tag{2.4}$$

where $w(z)$ analytic function E, with $w(0)=0$ and $|w(z)|<1$.

Proof. Let $f(z) \in k\text{-ST}_s^{(m)}[A,B]$, from definition of the subordination, we can have

$$\frac{zf'(z)}{f_m(z)} \prec \frac{(A+1)p_k(w(z))-(A-1)}{(B+1)p_k(w(z))-(B-1)}, \quad (z \in E), \tag{2.5}$$

where $w(z)$ analytic function E, with $w(0)=0$ and $|w(z)|<1$. Substituting z by $\varepsilon^\mu z$ respectivelty $(\mu = 0,1,2,...m-1)$, we have

$$\frac{zf'(\varepsilon^\mu z)}{\varepsilon^{-\mu} f_m(\varepsilon^\mu z)} = \frac{(A+1)p_k(w(\varepsilon^\mu z))-(A-1)}{(B+1)p_k(w(\varepsilon^\mu z))-(B-1)}, \quad (z \in E). \tag{2.6}$$

For $(\mu = 0,1,2,...m-1)$, $z \in E$. Using the equalities (1.7) and (1.8) we have

$$\frac{zf'_m(z)}{f_m(z)} = \frac{1}{m} \sum_{\mu=0}^{m-1} \frac{(A+1)p_k(w(\varepsilon^\mu z))-(A-1)}{(B+1)p_k(w(\varepsilon^\mu z))-(B-1)}, \quad (z \in E). \tag{2.7}$$

or equivalently

$$\frac{f'_m(z)}{f_m(z)} - \frac{1}{z} = \frac{1}{m} \sum_{\mu=0}^{m-1} \frac{(A-B)(p_k(w(\varepsilon^\mu z))-1)}{z((B+1)p_k(w(\varepsilon^\mu z))-(B-1))}, \quad (z \in E). \tag{2.8}$$

Integrating equality (2.8) , we have

$$\log\frac{f_m(z)}{z}=(A-B)\frac{1}{m}\sum_{\mu=0}^{m-1}\int_0^z\frac{\left(p_k\left(w\left(\varepsilon^\mu\zeta\right)\right)-1\right)}{\zeta\left((B+1)p_k\left(w\left(\varepsilon^\mu\zeta\right)\right)-(B-1)\right)}d\zeta$$
$$=(A-B)\frac{1}{m}\sum_{\mu=0}^{m-1}\int_0^{\varepsilon^\mu z}\frac{\left(p_k(w(t))-1\right)}{t\left((B+1)p_k(w(t))-(B-1)\right)}dt.$$

Therefore arranging equality (2.9) for $f_m(z)$ we can obtain

$$f_m(z)=z\cdot\left\{\exp(A-B)\frac{1}{m}\sum_{\mu=0}^{m-1}\int_0^{\varepsilon^\mu z}\frac{\left(p_k(w(t))-1\right)}{t\left((B+1)p_k(w(t))-(B-1)\right)}dt\right\},$$

and so the proof of Theorem 3 is complete.

Putting m = 1, in Theorem 3, we can obtain Corollary 2.

Corollary 2. Let $f(z)\in k\text{-ST}[A,B]$. Then

$$f(z)=z\cdot\left\{\exp(A-B)\int_0^z\frac{\left(p_k(w(t))-1\right)}{t(B+1)p_k(w(t))-(B-1)}dt\right\}, \tag{2.11}$$

where $w(z)$ analytic function E, with $w(0)=0$ and $|w(z)|<1$.

Putting k = 0, in Theorem 3, we can obtain Corollary 3, below which is comparable to the result obtained by Kwon and Sim [10].

Corollary 3. Let $f(z)\in ST_s^{(m)}[A,B]$. Then

$$f_m(z)=z\cdot\left\{\exp(A-B)\frac{1}{m}\sum_{\mu=0}^{m-1}\int_0^{\varepsilon^\mu z}\frac{w(t)}{t(1+Bw(t))}dt\right\}, \tag{2.12}$$

where $w(z)$ analytic function E, with $w(0)=0$ and $|w(z)|<1$.

Putting m = 1, A = 1 and B = -1 in Theorem 3, we can obtain Corollary 4.

Corollary 4. Let $f(z)\in k\text{-ST}$. Then

$$f(z)=z\cdot\left\{\exp\int_0^z\left(p_k(w(t))-1\right)dt\right\}, \tag{2.13}$$

where $w(z)$ analytic function E, with $w(0)=0$ and $|w(z)|<1$.

Theorem 4. Let $f(z)\in k\text{-}\mathrm{UCV}_s^{(m)}[\mathrm{A},\mathrm{B}]$. Then

$$f_m(z)=\int_0^z \exp\left((A-B)\frac{1}{m}\sum_{\mu=0}^{m-1}\int_0^{\varepsilon^\mu z}\frac{(p_k(w(t))-1)}{t(B+1)p_k(w(t))-(B-1)}dt\right)d\zeta. \quad (2.14)$$

where $\mathrm{w}(\mathrm{z})$ analytic function E, with $\mathrm{w}(0)=0$ and $|w(\mathrm{z})|<1$.

Proof. The proof of Theorem 4 is similar to that of Theorem 3 so the details are omitted. ▪

Theorem 4. Let $f(z)\in k\text{-}\mathrm{ST}_s^{(m)}[\mathrm{A},\mathrm{B}]$. Then

$$f(z)=\int_0^z \exp\left((A-B)\frac{1}{m}\sum_{\mu=0}^{m-1}\int_0^{\varepsilon^\mu z}\frac{(p_k(w(t))-1)}{t(B+1)p_k(w(t))-(B-1)}dt\right)$$
$$\times\left(\frac{((A+1)p_k(w(\zeta))-(A-1))}{(B+1)p_k(w(\zeta))-(B-1)}\right)d\zeta. \quad (2.15)$$

where $\mathrm{w}(\mathrm{z})$ analytic function E, with $\mathrm{w}(0)=0$ and $|w(\mathrm{z})|<1$.

Proof. Let $f(z)\in k\text{-}\mathrm{ST}_s^{(m)}[\mathrm{A},\mathrm{B}]$. Then from equalities (2.4) and (2.5) we have

$$f'(z)=\left(\frac{f_m(z)}{z}\right)\left(\frac{((A+1)p_k(w(\zeta))-(A-1))}{(B+1)p_k(w(\zeta))-(B-1)}\right)$$
$$=\exp\left((A-B)\frac{1}{m}\sum_{\mu=0}^{m-1}\int_0^{\varepsilon^\mu z}\frac{(p_k(w(t))-1)}{t(B+1)p_k(w(t))-(B-1)}dt\right)$$
$$\times\left(\frac{((A+1)p_k(w(\zeta))-(A-1))}{(B+1)p_k(w(\zeta))-(B-1)}\right)d\zeta, \quad (2.16)$$

Integrating the equality (2.16) , we have

$$f(z)=\int_0^z \exp\left((A-B)\frac{1}{m}\sum_{\mu=0}^{m-1}\int_0^{\varepsilon^\mu z}\frac{(p_k(w(t))-1)}{t(B+1)p_k(w(t))-(B-1)}dt\right)$$
$$\times\left(\frac{((A+1)p_k(w(\zeta))-(A-1))}{(B+1)p_k(w(\zeta))-(B-1)}\right)d\zeta.$$

and so the proof of Theorem 5 is completed.

Putting k = 0, in Theorem 5, we can obtain Corollary 5, below which is comparable to the result obtained by Kwon and Sim [10].

Corollary 5. Let $f(z) \in k\text{-}\mathrm{ST}_s^{(m)}[\mathrm{A,B}]$. Then

$$f(z)=\int_0^z \exp\left((A-B)\frac{1}{m}\sum_{\mu=0}^{m-1}\int_0^{\varepsilon^\mu z}\frac{w(t)}{t(1+Bw(t))}dt\right)\left(\frac{1+Aw(\zeta)}{1+Bw(\zeta)}\right)d\zeta, \qquad (2.17)$$

where $\mathrm{w}(\mathrm{z})$ analytic function E, with $\mathrm{w}(0)=0$ and $|w(\mathrm{z})|<1$.

Theorem 4. Let $f(z) \in k\text{-}\mathrm{USV}_s^{(m)}[\mathrm{A,B}]$. Then

$$f_m(z)=\int_0^z\left\{\frac{1}{\zeta}\int_0^\zeta \exp\left((A-B)\frac{1}{m}\sum_{\mu=0}^{m-1}\int_0^{\varepsilon^\mu \zeta}\frac{(p_k(w(t))-1)}{t(B+1)p_k(w(t))-(B-1)}dt\right)\right.$$
$$\times\left(\frac{((A+1)p_k(w(\zeta))-(A-1))}{(B+1)p_k(w(\zeta))-(B-1)}\right)d\zeta d\xi.$$

where $\mathrm{w}(\mathrm{z})$ analytic function E, with $\mathrm{w}(0)=0$ and $|w(\mathrm{z})|<1$.

Convolution conditions: In this section, we provide the convolutions conditions for the classes $k\text{-}\mathrm{ST}_s^{(m)}[\mathrm{A,B}]$ and $k\text{-}\mathrm{UCV}_s^{(m)}[\mathrm{A,B}]$.

Theorem 5. A function $f(z) \in k\text{-}\mathrm{ST}_s^{(m)}[\mathrm{A,B}]$, if and only if

$$\frac{1}{z}\left\{f(z)*\left(\frac{z}{(1-z)^2}((B+1)p_k(e^{i\vartheta})-(B-1))-((A+1)p_k(e^{i\vartheta})-(A-1))h(z)\right)\right\}\neq 0, \quad (2.18)$$

for all $z \in \mathrm{E}$ and $0 \le \vartheta < 2\pi$, where h(z) is given by (2.24).

Proof. Assume that $f(z) \in k\text{-}\mathrm{ST}_s^{(m)}[\mathrm{A,B}]$, then we have

$$\frac{\mathrm{z}f'(\mathrm{z})}{f_m(z)} \prec \frac{(A+1)p_k(z)-(A-1)}{(B+1)p_k(z)-(B-1)}, \quad (z\in \mathrm{E}), \qquad (2.19)$$

if and only if

$$\frac{\mathrm{z}f'(\mathrm{z})}{f_m(z)} \neq \frac{(A+1)p_k(e^{i\vartheta})-(A-1)}{(B+1)p_k(e^{i\vartheta})-(B-1)}, \quad (z\in \mathrm{E}). \qquad (2.20)$$

for all $z \in \mathrm{E}$ and $0 \le \vartheta < 2\pi$. The condition (2.20), can be written as

$$\frac{1}{z}\left\{zf'(z)(((B+1)p_k(e^{i\vartheta})-(B-1))-f_m(z)((A+1)p_k(e^{i\vartheta})-(A-1))h(z))\right\}\neq 0, \quad (2.21)$$

On the other hand it is well known that

$$zf'(z) = f(z) * \frac{z}{(1-z)^2}, \quad (z \in \mathrm{E}). \tag{2.22}$$

And from the definition of $f_m(z)$ we have

$$f_m(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n = (f * h)(z), \tag{2.23}$$

where

$$h(z) = z + \sum_{n=2}^{\infty} b_n z^n \tag{2.24}$$

where b_n is given by (1.6). Substituting (2.22) and (2.23) in (2.21), we can get (2.18). This completes the proof of the Theorem 7.

Putting m = 1, in Theorem 7, we can obtain Corollary 6.

Corollary 6. A function $f(z) \in k\text{-}\mathrm{ST}[\mathrm{A,B}]$, if and only if

$$\frac{1}{z}\left\{ f(z) * \left(\frac{z}{(1-z)^2}\left(1 + Be^{i\vartheta}\right) - \frac{z}{(1-z)}\left(1 + Ae^{i\vartheta}\right) \right) \right\} \neq 0, \tag{2.25}$$

for all $z \in \mathrm{E}$.

Putting k = 0, in Theorem 7, we can obtain Corollary 7.

Corollary 7. A function $f(z) \in \mathrm{ST}_s^{(m)}[\mathrm{A,B}]$, if and only if

$$\frac{1}{z}\left\{ f(z) * \left(\frac{z}{(1-z)^2}\left(1 + Be^{i\vartheta}\right) - h(z)\left(1 + Ae^{i\vartheta}\right) \right) \right\} \neq 0, \tag{2.26}$$

for all $z \in \mathrm{E}$ and $0 \leq \vartheta < 2\pi$, where h(z) is given by (2.24).

Theorem 8. A function $f(z) \in k\text{-}\mathrm{UCV}_s^{(m)}[\mathrm{A,B}]$, if and only if

$$\frac{1}{z}\left\{ f(z) * \left(\frac{z(B+1)p_k\left(e^{i\vartheta}\right) - (B-1)}{(1-z)^2} - \left((A+1)p_k\left(e^{i\vartheta}\right) - (A-1)\right)h(z) \right) \right\} \neq 0, \tag{2.27}$$

for all $z \in \mathrm{E}$ and $0 \leq \vartheta < 2\pi$, where h(z) is given by (2.24).

Proof. The proof of Theorem 8, is similar to that of Theorem 7, so the details are omitted.

Coefficient inequalities: Finally, we provided the sufficient conditions for the functions belonging to classes $k\text{-}\mathrm{ST}_s^{(m)}[\mathrm{A,B}]$ and $k\text{-}\mathrm{UCV}_s^{(m)}[\mathrm{A,B}]$.

Theorem 9. A function $f(z) \in \mathrm{A}$ is said to be in the class $k\text{-}\mathrm{ST}_s^{(m)}[\mathrm{A,B}]$, if it satisfies the condition

$$\sum_{n=1}^{\infty} 2(k+1)mn + |(mn(B+1)+(B-A))||a_{mn+1}| + \sum_{n=2,\, n\neq lm+1}^{\infty} 2(k+1)n + |n(B+1)||a_n| < |B-A|, \tag{2.28}$$

where $f_m(z)$ is given by (1.5) with $k \geq 0$, $-1 \leq B < A \leq 1$.

Proof. Assume that (2.28) holds, then it suffices to show that

$$k\left|\frac{(B-1)\frac{zf'(z)}{f_m(z)}-(A-1)}{(B+1)\frac{zf'(z)}{f_m(z)}-(A+1)}-1\right| - \mathrm{Re}\left(\frac{(\mathrm{B}-1)\frac{zf'(z)}{f_m(z)}-(A-1)}{(\mathrm{B}+1)\frac{zf'(z)}{f_m(z)}-(A+1)}-1\right) < 1 \tag{2.29}$$

we have

$$k\left|\frac{(B-1)\frac{zf'(z)}{f_m(z)}-(A-1)}{(B+1)\frac{zf'(z)}{f_m(z)}-(A+1)}-1\right| - \mathrm{Re}\left(\frac{(\mathrm{B}-1)\frac{zf'(z)}{f_m(z)}-(A-1)}{(\mathrm{B}+1)\frac{zf'(z)}{f_m(z)}-(A+1)}-1\right)$$

$$< (k+1)\left|\frac{(B-1)zf'(z)-(A-1)f_m(z)}{(B+1)zf'(z)-(A+1)f_m(z)}-1\right|$$

$$= 2(k+1)\left|\frac{f_m(z)-zf'(z)}{(B+1)zf'(z)-(A+1)f_m(z)}-1\right|$$

$$\leq 2(k+1)\frac{\sum_{n=2}^{\infty}|b_n-n||a_n|}{|B-A|-\sum_{n=2}^{\infty}|n(B+1)-(A+1)b_n||a_n|}.$$

The last expression is bounded by 1, if

$$\sum_{n=2}^{\infty} 2(k+1)(n-b_n) + |(n(B+1)-(A+1)b_n)||a_n| < |B-A|. \tag{2.30}$$

Using (1.6) in (2.30) we have

$$\sum_{n=1}^{\infty} 2(k+1)mn + |(mn(B+1)+(B-A))||a_{mn+1}| + \sum_{n=2,\, n\neq lm+1}^{\infty} 2(k+1)n + |n(B+1)||a_n| < |B-A|,$$

and this completes the proof of Theorem 9.

Putting m = 1, in Theorem 9, we can obtain Corollary 8, below which is comparable to the result obtained by Noor and Malik [5].

Corollary 8. A function $f(z) \in \mathrm{A}$ is said to be in the class $k\text{-ST[A,B]}$, if it satisfies the condition

$$\sum_{\mathrm{n}=2}^{\infty}\{2(k+1)(n-1)+|(n(B+1)+(A+1))|\}|a_n|<|B-A|,$$

where $k \geq 0$, $-1 \leq B < A \leq 1$.

Putting k = 0, in Theorem 9, we can obtain Corollary 9, below which is comparable to the result obtained by Kwon and Sim [10].

Corollary 9. A function $f(z) \in \mathrm{A}$ is said to be in the class $\mathrm{ST}_{\mathrm{s}}^{(\mathrm{m})}[\mathrm{A,B}]$, if it satisfies the condition

$$\sum_{\mathrm{n}=1}^{\infty} mn+(mn+1)(B-A)|a_{mn+1}|+\sum_{\mathrm{n}=2,\, \mathrm{n}\neq \mathrm{lm}+1}^{\infty}|n(B+1)||a_n|<|B-A|,$$

where $f_m(z)$ is given by (1.5) with $-1 \leq B < A \leq 1$.

Putting m = 1, A = 1 and B = -1 in Theorem 9, we can obtain Corollary 10, below which is comparable to the result obtained by Kanas and Wisniowska [3].

Corollary 10. A function $f(z) \in \mathrm{A}$ is said to be in the class $\mathrm{k\text{-}ST}$, if it satisfies the condition

$$\sum_{\mathrm{n}=2}^{\infty}\{n+k(n-1)\}|a_n|<1, \quad \mathrm{k} \geq 0.$$

Putting m = 1, $\mathrm{A} = 1 - 2\beta, \mathrm{B} = -1$, with $0 \leq \beta < 1$ in Theorem 9, we can obtain Corollary 11, below which is comparable to the result obtained by Shams et-al [11].

Corollary 11. A function $f(z) \in \mathrm{A}$ is said to be in the class $\mathrm{SD}(\mathrm{k}, \beta)$, if it satisfies the condition

$$\sum_{\mathrm{n}=2}^{\infty}\{n(k+1)-(k+\beta)\}|a_n|<1-\beta,$$

with $0 \leq \beta < 1$, with $\mathrm{k} \geq 0$.

Putting m = 1, $\mathrm{A} = 1 - 2\beta, \mathrm{B} = -1$, with $0 \leq \beta < 1$ and $\mathrm{k} = 0$ in Theorem 9, we can obtain Corollary 12, below which is comparable to the result obtained by Shams et-al [11].

Corollary 12. A function $f(z) \in \mathrm{A}$ is said to be in the class $\mathrm{S}^*(\beta)$, if it satisfies the condition

$$\sum_{\mathrm{n}=1}^{\infty}\{n-\beta\}|a_n|<1-\beta,$$

with $0 \leq \beta < 1$.

Theorem 10. A function $f(z) \in A$ is said to be in the class k - $\mathrm{UCV}_s^{(m)}[A,B]$, if it satisfies the condition

$$\sum_{n=2}^{\infty}\left[2(k+1)mn+\left|(mn(B+1)+(B-A))\right|\right](mn+1)\left|a_{mn+1}\right|$$
$$+\sum_{n=2,\, n\neq lm+1}^{\infty}(2(k+1)n+n(B+1))\left|na_n\right|<\left|B-A\right|, \tag{2.28}$$

where $f_m(z)$ is given by (1.5) with $k \geq 0$, $-1 \leq B < A \leq 1$.

Proof. The proof of Theorem 10, is similar to that of Theorem 9, so the details are omitted.

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References

[1] R. Chand and P. Singh, *On certain schlicht mapping*, Ind. J. Pure App. Math. 10 (1979), 1167-1174.
[2] W. Janowski, *Some external problem for certain families of analytic functions I,* Ann. Polon. Math., 28 (1973), 298-326.
[3] S. Kanas and A. Wisniowska, *Conic regions and k-uniform convexity*, J. Comput. Appl. Math. 105(1999), 327-336.
[4] S. Kanas and A. Wisniowska, *Conic domains and starlike functions*, Rev. Roumaine Math. Pure. Appl. 45 (2000), 647-657.
[5] K. I. Noor and S. N. Malik, *On coefficient inequalities of functions associated with conic domains*, Comput. Math. Appl. 62 (2011), 2209-2217.
[6] K. Sakaguchi, *On a certain univalent mapping*, J. Math. Soc. Japan. 11 (1959), 72-75.
[7] M. Arif, K. I. Noor and R. Khan, *On subclasses of analytic functions with respect to symmetrical points*, Abst. Appl. Anal. (2012), ID 790689, 11 pages.
[8] K. I. Noor and S. Mustafa, *Some classes of analytic functions related with functions of bounded radius rotation with respect to symmetrical points*, J. Math. Ineq. 3 (2) (2009), 267-276.
[9] T. N. Shanmugam, C. Ramachandran, and V. Ravichandran, *Fekete-Szego problem for subclass of starlike functions with respect to symmetric points*, Bull. Korean Math. Soc. 43 (3) (2006), 589-598.
[10] O. Kwon and Y. Sim, *A certain subclass of Janowski type functions associated with k-symmetric points*, Commun. Korean Math. Soc. 28 (2013), 143-154.
[11] S. Shams, S. R. Kulkarni, and J. M. Jahangiri, *Classes of uniformly starlike and convex functions*, Int. J. Math. Math. Sci. 55 (2004), 2959–2961.
[12] H. Silverman, *Univalent functions with negative coefficients*, Proc. Amer. Math. Soc. 51 (1975), 109–116.
[13] S. S. Miller and P. T Mocanu, *Subordinates of differential superordinations*, Complex Variables, 48 (10) (2003), 815–826.

[14] T. Bulboac a, *Differential Subordinations and Superordinations, Recent Results*, House of Scientic Book Publ, Cluj-Napoca, 2005.
[15] A. Aral and V. Gupta, *On q-Baskakov type operators*, Demonstratio Math., 42 (1) (2009), 109–122.
[16] R. N. Das and P. Singh, *On subclasses of Schlicht mappings*. Ind. J. Pure. App. Math. 8(1977), 864-872.

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Simple Forms of Nano Open Sets in an Ideal Nano Topological Spaces

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Abstaract — In this paper, we introduce and study the new concepts called α-nI-open sets, semi-nI-open sets, pre-nI-open sets, b-nI-open sets and β-nI-open sets, which are simple forms of nano open sets in an ideal nano topological spaces. Also we characterize the relations between them and the related properties. And the independence of n-open sets and nI-open sets is established.

Keywords — *n-open set, semi-nI-open set, α-nI-open set, pre-nI-open set.*

1 Introduction

An ideal I [10] on a topological space (X, τ) is a non-empty collection of subsets of X which satisfies the following conditions.

1. $A \in I$ and $B \subset A$ imply $B \in I$ and
2. $A \in I$ and $B \in I$ imply $A \cup B \in I$.

Given a topological space (X, τ) with an ideal I on X. If $\wp(X)$ is the family of all subsets of X, a set operator $(.)^\star : \wp(X) \rightarrow \wp(X)$, called a local function of A with respect to τ and I is defined as follows: for $A \subset X$, $A^\star(I, \tau) = \{x \in X : U \cap A \notin I$ for every $U \in \tau(x)\}$ where $\tau(x) = \{U \in \tau : x \in U\}$ [2]. The closure operator defined by $cl^\star(A) = A \cup A^\star(I, \tau)$ [9] is a Kuratowski closure operator which generates a topology $\tau^\star(I, \tau)$ called the $\star$-topology finer than τ. The topological space together with an ideal on X is called an ideal topological space or an ideal space denoted by (X, τ, I). We will simply write $A^\star$ for $A^\star(I, \tau)$ and $\tau^\star$ for $\tau^\star(I, \tau)$.

Some new notions in the concept of ideal nano topological spaces were introduced by Parimala et al. [4, 5].

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In this paper, we introduce and study the new concepts called α-nI-open sets, semi-nI-open sets, pre-nI-open sets, b-nI-open sets and β-nI-open sets, which are simple forms of nano open sets in an ideal nano topological spaces. Also we characterize the relations between them and the related properties. And the independence of n-open sets and nI-open sets is established.

2 Preliminaries

Definition 2.1. *[7] Let U be a non-empty finite set of objects called the universe and R be an equivalence relation on U named as the indiscernibility relation. Elements belonging to the same equivalence class are said to be indiscernible with one another. The pair (U, R) is said to be the approximation space. Let $X \subseteq U$.*

1. *The lower approximation of X with respect to R is the set of all objects, which can be for certain classified as X with respect to R and it is denoted by $L_R(X)$. That is, $L_R(X) = \bigcup_{x\in U}\{R(x) : R(x) \subseteq X\}$, where R(x) denotes the equivalence class determined by x.*

2. *The upper approximation of X with respect to R is the set of all objects, which can be possibly classified as X with respect to R and it is denoted by $U_R(X)$. That is, $U_R(X) = \bigcup_{x\in U}\{R(x) : R(x) \cap X \neq \phi\}$.*

3. *The boundary region of X with respect to R is the set of all objects, which can be classified neither as X nor as not - X with respect to R and it is denoted by $B_R(X)$. That is, $B_R(X) = U_R(X) - L_R(X)$.*

Definition 2.2. *[3] Let U be the universe, R be an equivalence relation on U and $\tau_R(X) = \{U, \phi, L_R(X), U_R(X), B_R(X)\}$ where $X \subseteq U$. Then $\tau_R(X)$ satisfies the following axioms:*

1. *U and $\phi \in \tau_R(X)$,*

2. *The union of the elements of any sub collection of $\tau_R(X)$ is in $\tau_R(X)$,*

3. *The intersection of the elements of any finite subcollection of $\tau_R(X)$ is in $\tau_R(X)$.*

Thus $\tau_R(X)$ is a topology on U called the nano topology with respect to X and $(U, \tau_R(X))$ is called the nano topological space. The elements of $\tau_R(X)$ are called nano-open sets (briefly n-open sets). The complement of a n-open set is called n-closed.

In the rest of the paper, we denote a nano topological space by $(U, \mathcal{N})$, where $\mathcal{N} = \tau_R(X)$. The nano-interior and nano-closure of a subset A of U are denoted by n-$int(A)$ and n-$cl(A)$, respectively.

Definition 2.3. *A subset A of a space $(U, \mathcal{N})$ is called*

1. *nano α-open [3] if $A \subseteq n\text{-}int(n\text{-}cl(n\text{-}int(A)))$.*

2. *nano semi-open [3] if $A \subseteq n\text{-}cl(n\text{-}int(A))$.*

3. *nano pre-open [3] if* $A \subseteq n\text{-}int(n\text{-}cl(A))$.

4. *nano b-open [6] if* $A \subseteq n\text{-}int(n\text{-}cl(A)) \cup n\text{-}cl(n\text{-}int(A))$.

5. *nano* β*-open [8] if* $A \subseteq n\text{-}cl(n\text{-}int(n\text{-}cl(A)))$.

The complements of the above mentioned sets are called their respective closed sets.

A nano topological space $(U, \mathcal{N})$ with an ideal I on U is called [4] an ideal nano topological space and is denoted by $(U, \mathcal{N}, I)$. $G_n(x) = \{G_n \mid x \in G_n, G_n \in \mathcal{N}\}$, denotes [4] the family of nano open sets containing x.

In future an ideal nano topological spaces $(U, \mathcal{N}, I)$ is referred as a space.

Definition 2.4. *[4] Let* $(U, \mathcal{N}, I)$ *be a space with an ideal* I *on* U. *Let* $(.)^\star_n$ *be a set operator from* $\wp(U)$ *to* $\wp(U)$ *(*$\wp(U)$ *is the set of all subsets of* U*). For a subset* $A \subseteq U$, $A^\star_n(I, \mathcal{N}) = \{x \in U : G_n \cap A \notin I,$ *for every* $G_n \in G_n(x)\}$ *is called the nano local function (briefly, n-local function) of* A *with respect to* I *and* $\mathcal{N}$. *We will simply write* $A^\star_n$ *for* $A^\star_n(I, \mathcal{N})$.

Theorem 2.5. *[4] Let* $(U, \mathcal{N}, I)$ *be a space and* A *and* B *be subsets of* U. *Then*

1. $A \subseteq B \Rightarrow A^\star_n \subseteq B^\star_n$,

2. $A^\star_n = n\text{-}cl(A^\star_n) \subseteq n\text{-}cl(A)$ *(*$A^\star_n$ *is a n-closed subset of* $n\text{-}cl(A)$*)*,

3. $(A^\star_n)^\star_n \subseteq A^\star_n$,

4. $(A \cup B)^\star_n = A^\star_n \cup B^\star_n$,

5. $V \in \mathcal{N} \Rightarrow V \cap A^\star_n = V \cap (V \cap A)^\star_n \subseteq (V \cap A)^\star_n$,

6. $J \in I \Rightarrow (A \cup J)^\star_n = A^\star_n = (A - J)^\star_n$.

Theorem 2.6. *[4] Let* $(U, \mathcal{N}, I)$ *be a space with an ideal* I *and* $A \subseteq A^\star_n$, *then* $A^\star_n = n\text{-}cl(A^\star_n) = n\text{-}cl(A)$.

Definition 2.7. *[4] Let* $(U, \mathcal{N}, I)$ *be a space. The set operator* $n\text{-}cl^\star$ *called a nano* $\star$*-closure is defined by* $n\text{-}cl^\star(A) = A \cup A^\star_n$ *for* $A \subseteq X$.

It can be easily observed that $n\text{-}cl^\star(A) \subseteq n\text{-}cl(A)$.

Theorem 2.8. *[5] In a space* $(U, \mathcal{N}, I)$, *if* A *and* B *are subsets of* U, *then the following results are true for the set operator* $n\text{-}cl^\star$.

1. $A \subseteq n\text{-}cl^\star(A)$,

2. $n\text{-}cl^\star(\phi) = \phi$ *and* $n\text{-}cl^\star(U) = U$,

3. *If* $A \subset B$, *then* $n\text{-}cl^\star(A) \subseteq n\text{-}cl^\star(B)$,

4. $n\text{-}cl^\star(A) \cup n\text{-}cl^\star(B) = n\text{-}cl^\star(A \cup B)$,

5. $n\text{-}cl^\star(n\text{-}cl^\star(A)) = n\text{-}cl^\star(A)$.

Definition 2.9. *[5]*

A subset A *of a space* $(U, \mathcal{N}, I)$ *is said to be nano-I-open (briefly,* nI*-open) if* $A \subseteq n\text{-}int(A^\star_n)$.

3 Simple Forms of n-open Sets in $(U, \mathcal{N}, I)$

Definition 3.1. *A subset A of space $(U, \mathcal{N}, I)$ is said to be*

1. *nano α-I-open (briefly α-nI-open) if $A \subset n\text{-}int(n\text{-}cl^{\star}(n\text{-}int(A)))$,*
2. *nano semi-I-open (briefly semi-nI-open) if $A \subset n\text{-}cl^{\star}(n\text{-}int(A))$,*
3. *nano pre-I-open (briefly pre-nI-open) if $A \subset n\text{-}int(n\text{-}cl^{\star}(A))$,*
4. *nano b-I-open (briefly b-nI-open) if $A \subset n\text{-}int(n\text{-}cl^{\star}(A)) \cup n\text{-}cl^{\star}(n\text{-}int(A))$,*
5. *nano β-I-open (briefly β-nI-open) if $A \subset n\text{-}cl^{\star}(n\text{-}int(n\text{-}cl^{\star}(A)))$.*

The complements of the above mentioned sets are called their respective closed sets.

Theorem 3.2. *In a space $(U, \mathcal{N}, I)$, for a subset A, the following relations hold.*

1. A is n-open $\Rightarrow A$ is α-nI-open.
2. A is α-nI-open $\Rightarrow A$ is semi-nI-open.
3. A is α-nI-open $\Rightarrow A$ is pre-nI-open.
4. A is semi-nI-open $\Rightarrow A$ is b-nI-open.
5. A is pre-nI-open $\Rightarrow A$ is b-nI-open.
6. A is b-nI-open $\Rightarrow A$ is β-nI-open.

Proof.

1. A is n-open $\Rightarrow A = n\text{-}int(A)$. But $A \subseteq n\text{-}cl^{\star}(A) = n\text{-}cl^{\star}(n\text{-}int(A)) \subseteq n\text{-}cl^{\star}(n\text{-}int(n\text{-}cl^{\star}(A)))$ which proves that A is α-nI-open.
2. A is α-nI-open $\Rightarrow A \subseteq n\text{-}int(n\text{-}cl^{\star}(n\text{-}int(A))) \subseteq n\text{-}cl^{\star}(n\text{-}int(A))$ which proves that A is semi-nI-open.
3. A is α-nI-open $\Rightarrow A \subseteq n\text{-}int(n\text{-}cl^{\star}(n\text{-}int(A))) \subseteq n\text{-}int(n\text{-}cl^{\star}(A))$ which proves that A is pre-nI-open.
4. A is semi-nI-open $\Rightarrow A \subseteq n\text{-}cl^{\star}(n\text{-}int(A)) \subseteq n\text{-}cl^{\star}(n\text{-}int(A)) \cup n\text{-}int(n\text{-}cl^{\star}(A))$ which proves that A is b-nI-open.
5. A is pre-nI-open $\Rightarrow A \subseteq n\text{-}int(n\text{-}cl^{\star}(A)) \subseteq n\text{-}int(n\text{-}cl^{\star}(A)) \cup n\text{-}cl^{\star}(n\text{-}int(A))$ which proves that A is b-nI-open.
6. A is b-nI-open $\Rightarrow A \subseteq n\text{-}int(n\text{-}cl^{\star}(A)) \cup n\text{-}cl^{\star}(n\text{-}int(A)) \subseteq n\text{-}cl^{\star}(n\text{-}int(n\text{-}cl^{\star}(A))) \cup n\text{-}cl^{\star}(n\text{-}int(n\text{-}cl^{\star}(A))) = n\text{-}cl^{\star}(n\text{-}int(n\text{-}cl^{\star}(A)))$ which proves that A is β-nI-open.

Remark 3.3. *These relations are shown in the diagram.*

$$\begin{array}{ccccc} n\text{-}\textbf{open} & \longrightarrow & \alpha\text{-}nI\text{-}\textbf{open} & \longrightarrow & \textbf{pre-}nI\text{-}\textbf{open} \\ & & \downarrow & & \downarrow \\ & & \textbf{semi-}nI\text{-}\textbf{open} & \longrightarrow & \textbf{b-}nI\text{-}\textbf{open} \\ & & & & \downarrow \\ & & & & \beta\text{-}nI\text{-}\textbf{open} \end{array}$$

The converses of each statement in Theorem 3.2 are not true as shown in the following Example.

Example 3.4. *1. Let $U = \{e_1, e_2, e_3, e_4, e_5\}$ with $U/R = \{\{e_1\}, \{e_2, e_3\}, \{e_4, e_5\}\}$ and $X = \{e_1, e_2\}$. Then $\mathcal{N} = \{\phi, U, \{e_1\}, \{e_2, e_3\}, \{e_1, e_2, e_3\}\}$. Let the ideal be $I = \{\phi, \{e_2\}\}$.*

(a) *Then $A = \{e_1, e_2, e_3, e_4\}$ is α-nI-open but not n-open.*
n-int$(A) = \{e_1, e_2, e_3\}$ and $\{e_1, e_2, e_3\}^\star_n = \{e_1, e_2, e_3, e_4, e_5\} = U$. Therefore n-cl*(n-int$(A)) = U$ and n-int(n-cl*(n-int$(A))) = U \supseteq A$. Thus A is α-nI-open. But A is not n-open.

(b) *$B = \{e_2, e_3, e_4\}$ is semi-nI-open but not α-nI-open.*

(c) *$F = \{e_3, e_4\}$ is β-nI-open but not b-nI-open.*

2. Let $U = \{e_1, e_2, e_3, e_4\}$ with $U/R = \{\{e_1\}, \{e_3\}, \{e_2, e_4\}\}$ and $X = \{e_1, e_2\}$. Then $\mathcal{N} = \{\phi, U, \{e_1\}, \{e_2, e_4\}, \{e_1, e_2, e_4\}\}$. Let the ideal be $I = \{\phi, \{e_1\}\}$.

(a) *$C = \{e_2\}$ is pre-nI-open but not α-nI-open.*

(b) *$D = \{e_1, e_4\}$ is b-nI-open but not semi-nI-open.*

(c) *$E = \{e_2, e_3, e_4\}$ is b-nI-open but not pre-nI-open.*

Remark 3.5. *In a space the family of n-open sets and the family of nI-open sets are independent.*

Example 3.6. *In Example 3.4(2), $A = \{e_2\}$ is nI-open but not n-open and $B = \{e_1, e_2, e_4\}$ is n-open but not nI-open.*

Theorem 3.7. *A subset A of a space $(U, \mathcal{N}, I)$ is α-nI-open $\Longleftrightarrow$ A is semi-nI-open and pre-nI-open.*

Proof. $\Rightarrow$ Part follows from (2) and (3) of Theorem 3.2.

$\Leftarrow$ If A is semi-nI-open and pre-nI-open then $A \subseteq$ *n-int*(*n-cl*$^\star(A)$) and $A \subseteq$ *n-cl**(*n-int*(A)).
Thus $A \subseteq$ *n-int*(*n-cl*$^\star(A)$) $\subseteq$ *n-int*(*n-cl**(*n-cl**(*n-int*(A)))) $=$ *n-int*(*n-cl**(*n-int*(A))) which proves that A is α-nI-open.

Remark 3.8. *In a space $(U, \mathcal{N}, I)$, the family of semi-nI-open sets and the family of pre-nI-open sets are independent of each other as shown in the following Example.*

Example 3.9. *Let $U = \{p, q, r, s\}$ with $U/R = \{\{p\}, \{s\}, \{q, r\}\}$ and $X = \{p, r\}$. Then $\mathcal{N} = \{\phi, U, \{p\}, \{q, r\}, \{p, q, r\}\}$. Let the ideal be $I = \{\phi, \{r\}\}$. Then the subset*

1. *$\{p, s\}$ is semi-nI-open but not pre-nI-open.*

2. *$\{q\}$ is pre-nI-open but not semi-nI-open.*

Theorem 3.10. *If a subset A of a space $(U,\mathcal{N},I)$ is both $n\star$-closed and β-nI-open, then A is semi-nI-open.*

Proof. Since A is β-nI-open, $A \subset n\text{-}cl^\star(n\text{-}int(n\text{-}cl^\star(A))) = n\text{-}cl^\star(n\text{-}int(A))$, A being $n\star$-closed. Therefore A is semi-nI-open.

Theorem 3.11. *A subset A of a space $(U,\mathcal{N},I)$ is semi-nI-open if and only if $n\text{-}cl^\star(A) = n\text{-}cl^\star(n\text{-}int(A))$.*

Proof. Let A be semi-nI-open. Then $A \subset n\text{-}cl^\star(n\text{-}int(A))$ and $n\text{-}cl^\star(A) \subset n\text{-}cl^\star(n\text{-}int(A))$. But $n\text{-}cl^\star(n\text{-}int(A)) \subset n\text{-}cl^\star(A)$. Thus $n\text{-}cl^\star(A) = n\text{-}cl^\star(n\text{-}int(A))$.

Conversely, let the condition hold. We have $A \subset n\text{-}cl^\star(A) = n\text{-}cl^\star(n\text{-}int(A))$, by assumption. Thus $A \subset n\text{-}cl^\star(n\text{-}int(A))$ and hence A is semi-nI-open.

Proposition 3.12. *In (U, $\mathcal{N}$, I) if A is a b-nI-open set such that $n\text{-}cl^\star(A) = \phi$, then A is semi-nI-open.*

Theorem 3.13. *A subset A of a space $(U,\mathcal{N},I)$ is semi-nI-open if and only if there exists a n-open set G such that $G \subset A \subset n\text{-}cl^\star(G)$.*

Proof. Let A be semi-nI-open. Then $A \subset n\text{-}cl^\star(n\text{-}int(A))$. Take $n\text{-}int(A) = G$. Then $G \subset A \subset n\text{-}cl^\star(G)$, where G is n-open.

Conversely, let $G \subset A \subset n\text{-}cl^\star(G)$ for some n-open set G. Since $G \subset A$, $G \subset n\text{-}int(A)$ and $A \subset n\text{-}cl^\star(G) \subset n\text{-}cl^\star(n\text{-}int(A))$ which implies A is semi-nI-open.

Theorem 3.14. *If A is a semi-nI-open set in a space $(U,\mathcal{N},I)$ and $A \subset B \subset n\text{-}cl^\star(A)$, then B is semi-nI-open.*

Proof. By assumption $B \subset n\text{-}cl^\star(A) \subset n\text{-}cl^\star(n\text{-}cl^\star(n\text{-}int(A)))$ (for A is semi-nI-open) $= n\text{-}cl^\star(n\text{-}int(A)) \subset n\text{-}cl^\star(n\text{-}int(B))$ by assumption. This implies B is semi-nI-open.

Theorem 3.15. *In a space $(U,\mathcal{N},I)$, for a subset A, the following results hold.*

1. A is nI-open $\Rightarrow A$ is pre-nI-open.
2. A is nI-open $\Rightarrow A$ is β-nI-open.
3. A is nI-open $\Rightarrow A$ is b-nI-open.

Proof.
1. A is nI-open $\Rightarrow A \subseteq n\text{-}int(A_n^\star) \subseteq n\text{-}int(n\text{-}cl^\star(A))$ which proves that A is pre-nI-open.
2. A is nI-open $\Rightarrow A \subseteq n\text{-}int(A_n^\star) \subseteq n\text{-}cl^\star(n\text{-}int(n\text{-}cl^\star(A)))$ which proves that A is β-nI-open.
3. A is nI-open $\Rightarrow A \subseteq n\text{-}int(A_n^\star) \subseteq n\text{-}int(n\text{-}cl^\star(A)) \subseteq n\text{-}int(n\text{-}cl^\star(A)) \cup n\text{-}cl^\star(n\text{-}int(A))$ which proves that A is b-nI-open.

Remark 3.16. *The converses of (1), (2) and (3) in Theorem 3.15 are not true as shown in the following Example.*

Example 3.17. *In Example 3.4 (2),*

1. *$A = \{e_1\}$ is pre-nI-open but not nI-open and b-nI-open but not nI-open.*

2. $A = \{e_3, e_4\}$ *is* β-nI-*open but not* nI-*open.*

Remark 3.18. *1. In a space* $(U, \mathcal{N}, I)$, *the family of* nI-*open sets and the family of* α-nI-*open sets are independent of each other.*

2. *In a space* $(U, \mathcal{N}, I)$, *the family of* nI-*open sets and the family of semi-*nI-*open sets are independent of each other.*

Example 3.19. *In Example 3.4(2),*

1. $A = \{e_2\}$ is nI-open but not α-nI-open.

2. $B = \{e_1\}$ is α-nI-open but not nI-open.

   Examples (1) and (2) verify (1) of Remark 3.18.

3. $C = \{e_2\}$ is nI-open but not semi-nI-open.

4. $D = \{e_1\}$ is semi-nI-open but not nI-open.

   Examples (3) and (4) verify (2) of Remark 3.18.

Proposition 3.20. *For a subset of* A *a space* $(U, \mathcal{N}, I)$, *the following properties hold:*

1. A *is* α-nI-*open* $\Rightarrow$ A *is nano* α-*open.*

2. A *is pre-*nI-*open* $\Rightarrow$ A *is nano pre-open.*

3. A *is b-*nI-*open* $\Rightarrow$ A *is nano b-open.*

4. A *is* β-nI-*open* $\Rightarrow$ A *is nano* β-*open.*

Proof. 1. Let A be a α-nI-open set. Then $A \subset n\text{-}int(n\text{-}cl^\star(n\text{-}int(A))) \subset n\text{-}int(n\text{-}cl(n\text{-}int(A)))$. This shows that A is nano α-open.

2. Let A be a pre-nI-open set. Then $A \subset n\text{-}int(n\text{-}cl^\star(A)) \subset n\text{-}int(n\text{-}cl(A))$. This shows that A is nano pre-open.

3. Let A be a b-nI-open set. Then $H \subset n\text{-}int(n\text{-}cl^\star(A)) \cup n\text{-}cl^\star(n\text{-}int(A)) \subset n\text{-}int(n\text{-}cl(A)) \cup n\text{-}cl(n\text{-}int(A))$. This shows that A is nano b-open.

4. Let A be a β-nI-open set. Then $A \subset n\text{-}cl^\star(n\text{-}int(n\text{-}cl^\star(A))) \subset n\text{-}cl(n\text{-}int(n\text{-}cl(A)))$. This shows that A is nano β-open.

Remark 3.21. *The converses of Proposition 3.20 are not true in general as shown in the following Example.*

Example 3.22. *1. Let* $U = \{e_1, e_2, e_3, e_4, e_5\}$ *with* $U/R = \{\{e_1\}, \{e_2, e_3\}, \{e_4, e_5\}\}$, $X = \{e_1, e_2\}$ *and* $I = \wp(U)$. *Then in the space* $(U, \mathcal{N}, I)$, $\mathcal{N} = \{\phi, U, \{e_1\}, \{e_2, e_3\}, \{e_1, e_2, e_3\}\}$. $A = \{e_1, e_2, e_3, e_4\}$ *is nano* α-*open but not* α-nI-*open since* $n\text{-}cl^\star(A) = A$.

2. *Let* $U = \{k_1, k_2, k_3\}$ *with* $U/R = \{\{k_1\}, \{k_2, k_3\}\}$ *and* $X = \{k_1, k_2\}$. *Then* $\mathcal{N} = \{\phi, U, \{k_1\}, \{k_2, k_3\}\}$. *Let the ideal be* $I = \{\phi, k_2\}$. *Then in* $(U, \mathcal{N}, I)$, $B = \{k_1, k_2\}$ *is nano pre-open but pre-*nI-*open.*

3. *In Example 3.4(2),*

   (a) $C = \{e_1, e_3\}$ *is nano b-open but not b-nI-open.*

   (b) $D = \{e_1, e_3\}$ *is nano β-open but not β-nI-open.*

Lemma 3.23. *Let $(U, \mathcal{N}, I)$ be a space and A a subset of U. If H is n-open in $(U, \mathcal{N}, I)$, then $H \cap$ n-cl$^\star(A) \subseteq$ n-cl$^\star(H \cap A)$.*

Proof. $H \cap$ *n-cl*$^\star(A) = H \cap (A^\star_n \cup A) = (H \cap A^\star_n) \cup (H \cap A) \subseteq (H \cap A)^\star_n \cup (H \cap A)$ by (5) of Theorem 2.5. Thus $H \cap$ *n-cl*$^\star(A) \subseteq (H \cap A)^\star_n \cup (H \cap A) =$ *n-cl*$^\star(H \cap A)$.

Proposition 3.24. *The intersection of a pre-nI-open set and n-open set is pre-nI-open.*

Proof. Let A be pre-nI-open and G be n-open. Then $A \subset$ *n-int*(*n-cl*$^\star(A)$) and $G \cap A \subset$ *n-int*$(G) \cap$ *n-int*(*n-cl*$^\star(A)$) $=$ *n-int*($G \cap$ *n-cl*$^\star(A)$) $\subset$ *n-int*(*n-cl*$^\star(G \cap A)$) by Lemma 3.23. This shows that $G \cap A$ is pre-nI-open.

Proposition 3.25. *The intersection of a semi-nI-open set and n-open set is semi-nI-open.*

Proof. Let A be semi-nI-open and G be n-open in U. Then $A \subset$ *n-cl**(*n-int*(A)) and *n-int*$(G) = G$. $G \cap A \subset G \cap$ *n-cl**(*n-int*(A)) $\subseteq$ *n-cl**($G \cap$ *n-int*(A)) $=$ *n-cl**(*n-int*$(G) \cap$ *n-int*(A)) $=$ *n-cl**(*n-int*$(G \cap A)$) by Lemma 3.23. Hence A is semi-nI-open.

Proposition 3.26. *The intersection of a α-nI-open set and n-open set is α-nI-open.*

Proof. Let G be a n-open and A be an α-nI-open in a space $(U, \mathcal{N}, I)$. Then A is both pre-nI-open and semi-nI-open by (2) and (3) of Theorem 3.2. $A \cap G$ is both pre-nI-open and semi-nI-open by Proposition 3.24 and 3.25. Hence by Theorem 3.7, $A \cap G$ is α-nI-open.

Proposition 3.27. *The intersection of a b-nI-open set and n-open set is b-nI-open.*

Proof. Let A be b-nI-open and G be n-open. Then $A \subset$ *n-int*(*n-cl*$^\star(A)$)$\cup$*n-cl**(*n-int*(A)) and $G \cap A \subset G \cap [$*n-int*(*n-cl*$^\star(A)$) $\cup$ *n-cl**(*n-int*(A))$] = [G \cap$ *n-int*(*n-cl*$^\star(A)$)$] \cup [G \cap$ *n-cl**(*n-int*(A))$] = [$*n-int*$(G) \cap$ *n-int*(*n-cl*$^\star(A)$)$] \cup [G \cap$ *n-cl**(*n-int*(A))$] \subset [$*n-int*($G \cap$ *n-cl*$^\star(A)$)$] \cup [$*n-cl**($G \cap$ *n-int*(A))$]$ by Lemma 3.23. Thus $G \cap A \subset [$*n-int*(*n-cl*$^\star(G \cap A)$)$] \cup [$*n-cl**(*n-int*$(G \cap A)$)$]$. This shows that $G \cap A$ is b-nI-open.

Proposition 3.28. *The intersection of a β-nI-open set and n-open set is β-nI-open.*

Proof. Let A be β-nI-open and G be n-open. Then $A \subset$ *n-cl**(*n-int*(*n-cl*$^\star(A)$)) and $G \cap A \subset G \cap$ *n-cl**(*n-int*(*n-cl*$^\star(A)$)) $\subset$ *n-cl**($G \cap$ *n-int*(*n-cl*$^\star(A)$)) $\subset$ *n-cl**(*n-int*$(G) \cap$ *n-int*(*n-cl*$^\star(A)$)) $=$ *n-cl**(*n-int*($G \cap$ *n-cl*$^\star(A)$)) $\subset$ *n-cl**(*n-int*(*n-cl*$^\star(G \cap A)$)) by Lemma 3.23. This shows that $G \cap A$ is β-nI-open.

Remark 3.29. *The intersection of two semi-nI-open (resp. pre-nI-open, b-nI-open, β-nI-open) sets need not be semi-nI-open (resp. pre-nI-open, b-nI-open, β-nI-open) as shown in the following Example.*

Example 3.30. 1. *In Example 3.9, $H = \{p, s\}$ and $K = \{q, r, s\}$ are semi-nI-open. But $H \cap K = \{s\}$ is not semi-nI-open.*

2. *In Example 3.4(2),*

   (a) $H = \{e_1, e_2, e_3\}$ *and* $K = \{e_1, e_3, e_4\}$ *are pre-nI-open. But* $H \cap K = \{e_1, e_3\}$ *is not pre-nI-open.*

   (b) $H = \{e_1, e_2, e_3\}$ *and* $K = \{e_2, e_3, e_4\}$ *are b-nI-open. But* $H \cap K = \{e_2, e_3\}$ *is not b-nI-open.*

   (c) $H = \{e_2, e_3\}$ *and* $K = \{e_3, e_4\}$ *are* β*-nI-open. But* $H \cap K = \{e_3\}$ *is not* β*-nI-open.*

References

[1] D. Jankovi'c and T. R. Hamlett, *Compatible extensions of ideals*, Boll. Un. Mat. Ital., 7 (6-B) (1992), 453-465.

[2] K. Kuratowski, *Topology*, Vol I. Academic Press (New York) 1966.

[3] M. L. Thivagar and C. Richard, *On nano forms of weakly open sets*, International Journal of Mathematics and Statistics Invention, 1 (1) (2013), 31-37.

[4] M. Parimala, T. Noiri and S. Jafari, *New types of nano topological spaces via nano ideals* (to appear).

[5] M. Parimala and S. Jafari, *On some new notions in nano ideal topological spaces*, International Balkan Journal of Mathematics (IBJM), 1(3)(2018), 85-92.

[6] M. Parimala, C. Indirani and S. Jafari, *On nano b-open sets in nano topological spaces*, Jordan Journal of Mathematics and Statistics 9 (3) (2016), 173-184.

[7] Z. Pawlak, *Rough sets*, International journal of computer and Information Sciences, 11 (5) (1982), 341-356.

[8] A. Revathy and G. Ilango, *On nano* β*-open sets*, Int. Jr. of Engineering, Contemporary Mathematics and Sciences, 1 (2) (2015), 1-6.

[9] R. Vaidyanathaswamy, *The localization theory in set topology*, Proc. Indian Acad. Sci., 20 (1945), 51-61.

[10] R. Vaidyanathaswamy, *Set topology*, Chelsea Publishing Company, New York, 1946.

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A Geometric Solution to the Jacobian Problem

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Annotation – In this article given a geometric solution to the well-known Jacobian problem. The two-dimensional polynomial Keller map is considered in four-dimensional Euclidean space R^4. Used the concept of parallel. A well-known example of Vitushkin is also considered. Earlier it was known that Vitushkin's map has a nonzero constant Jacobian and it is not injective. We will show that the Vitushkin map is not surjective and moreover it has two inverse maps in the domain of its definition.

1. Introduction

In works [1], [2], [5], [6] the Jacobian problem is reduced to the injectivity problem of polynomial mapping. And in papers [3], [4] the Jacobian problem is reduced to the reversibility of a polynomial map with a non-constant nilpotent Jacobi matrix.

2. Properties of Tangent Spaces

Consider the polynomial mapping

$$F(x,y) = (u,v)$$

where $u = f(\mathrm{x},\mathrm{y}), \mathrm{v} = \mathrm{g}(\mathrm{x},\mathrm{y})$ are polynomials from two variables and their Jacobians

$$f_x(\mathrm{x},\mathrm{y})\cdot \mathrm{g}_y(\mathrm{x},\mathrm{y}) - \mathrm{f}_y(\mathrm{x},\mathrm{y})\cdot \mathrm{g}_x(\mathrm{x},\mathrm{y}) = 1.$$

Such polynomial maps are called kellerovas. The main result of this paper reads as follows:

Theorem 1. Any Keller polynomial map is injective over a field of real numbers *R*.

The proof of the theorem relies on methods of analytic geometry in the four-dimensional space R^4, where R^4 is the field of real numbers. We define the surface π in space R^4 as a graph of Keller mapping $F: R^4 \to R^4$

$$\pi = \{(x, y, u, v) \in R^4 \mid u = f(x, y), v = g(x, y)\} \qquad (1)$$

Tangent plane K of the surface π at the point $(x_0, y_0, u_0, v_0) \in \pi$ is determined by the following equations:

$$K: \begin{cases} u = u_0 + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0), \\ v = v_0 + g_x(x_0, y_0(x - x_0) + g_y(x_0, y_0)(y - y_0). \end{cases}$$

Let's write parametric equations of tangent plane K:

$$K: \begin{cases} x = t, \\ y = s, \\ u = f_x(x_0, y_0)t + f_y(x_0, y_0)s + u_0', \\ v = g_x(x_0, y_0)t + g_y(x_0, y_0)s + v_0'. \end{cases} \qquad (2)$$

where

$$u_0' = u_0 - f_x(x_0, y_0)x_0 - f_y(x_0, y_0)y_0,$$
$$v_0' = v_0 - g_x(x_0, y_0)x_0 - g_y(x_0, y_0)y_0.$$

Then

$$K = \langle M, \vec{a}, \vec{b} \rangle = \{X \in R^4 \mid \overrightarrow{MX} = x\vec{a} + y\vec{b}, x, y \in R\},$$

where

$M(0, 0, u_0', v_0')$ is the starting point in K,

$$\vec{a} = (1, 0, f_x(x_0, y_0), g_x(x_0, y_0)),$$
$$\vec{b} = (0, 1, f_y(x_0, y_0), g_y(x_0, y_0)), -$$

the guiding vectors of the plane K. As we seen from the parametric equations of the tangent plane K, the surface $\pi(1)$ at the every point has a two - dimensional tangent plane. Therefore, the surface $\pi(1)$ has a dimension equal to two.

The following Lemma plays a key role in the proof of the theorem.

Lemma 1. Any tangential plane $K(2)$ of the surface $\pi(1)$ in space $\mathbb{R}^4$ does not contain a line parallel to coordinate planes Oxy and Ouv.

Proof. Let $M_1 = (x_1, y_1, u_1, v_1)$ and $M_2 = (x_2, y_2, u_2, v_2)$ two different points of the tangent plane K. If the line (M_1, M_2) is parallel to the Oxy plane, then vector $\overrightarrow{M_1M_2} = (x_2 - x_1, y_2 - y_1, u_2 - u_1, v_2 - v_1)$ is expressed linearly via vectors $\vec{e_1} = (1, 0, 0, 0)$ and $\vec{e_2} = (0, 1, 0, 0)$ .Then from (2) we get

$$u_2 - u_1 = f_x(x_0, y_0)(x_2 - x_1) + f_y(x_0, y_0)(y_2 - y_1) = 0$$
$$v_2 - v_1 = g_x(x_0, y_0)(x_2 - x_1) + g_y(x_0, y_0)(y_2 - y_1) = 0$$

Since the Jacobian equal to 1, then $x_2 - x_1 = 0, y_2 - y_1 = 0$. Hence $u_2 - u_1 = 0, v_2 - v_1 = 0$, that is $M_1 \equiv M_2$. Contradiction.

If the line (M_1M_2) is parallel to the Ouv plane, then the vector $\overrightarrow{M_1M_2}$ is expressed linearly via vectors $\vec{e_3} = (0,0,1,0)$ and $\vec{e_4} = (0,0,0,1)$. Then $x_2 - x_1 = 0,$ and $y_2 - y_1 = 0$. Hence $u_2 = u_1$ and $v_2 = v_1$, that is, the points M_1 and M_2 coincide again. Contradiction.

Consequence. Any tangent plane $K(2)$ of the surface $\pi(1)$ in space R^4 is not parallel to the coordinate planes Oxy and Ouv.

3. Proof of Theorem

Let $F(x_1, y_1) = (u_1, v_1) = F(x_2, y_2)$. Then a nonzero vector $\overrightarrow{M_1M_2} = (x_2 - x_1, y_2 - y_1, 0, 0)$, where $M_1 = (x_1, y_1, u_1, v_1)$ and $M_2 = (x_2, y_2, u_2, v_2)$, is parallel to the coordinate plane Oxy. Replacing the mapping $F(x, y)$ with the mapping $F(x + x_1, y + y_1) - F(x_1, y_1)$, we can assume that $M_1 = (x_1, y_1, u_1, v_1)$ coincides with the origin $O(0,0,0,0)$ and the point $M_2 = (x_2, y_2, u_2, v_2)$ coincides with the point $M(a, b, 0, 0)$, where $a = x_2 - x_1, b = y_2 - y_1$. Let

$$\prod = \left\langle O, \overrightarrow{OM}, \vec{e_3} = (0,0,1,0), \vec{e_4} = (0,0,0,1) \right\rangle = \left\{ X \in R^4 \mid \overrightarrow{OX} = x\overrightarrow{OM} + y\vec{e_3} + x\vec{e_4}, x, y, z \in \mathbb{R} \right\}$$

three-dimensional hyperplane in R^4. Parametric equations of a plane $\prod$ have the form:

$$\prod : \begin{cases} x = a\tau, \\ y = b\tau, \\ u = p, \\ v = q, \end{cases}$$

where $\tau, p, q \in R$. Parametric equations of a plane π have the form:

$$\pi : \begin{cases} x = t, \\ y = s, \\ u = f(t,s), \\ v = g(t,s). \end{cases}$$

where $t,s \in R$. We find the intersection of $\pi \cap \prod$. Have,

$$\pi \cap \prod : \begin{cases} x = at, \\ y = bt, \\ u = f(\text{at}, \text{bt}), \\ v = g(\text{at}, \text{bt}). \end{cases}$$

As you can see, the curve $\pi \cap \prod$ has the following radius-vector

$$r(t) = (at, bt, f(at, bt), g(at, bt))$$

Then the tangent vector to the curve $\pi \cap \prod$ looks like:

$$r'(t) = (a, b, \text{c}(t), \text{d}(t)),$$

where

$$c(t) = f_x(\text{at}, \text{bt})\,\text{a} + f_y(\text{at}, \text{bt})\,\text{b}, \text{d}(t) = g_x(\text{at}, \text{bt})\,\text{a} + \text{g}_y(\text{at}, \text{bt})\,\text{b}.$$

Have $r'(t) = \overrightarrow{OM} + c(t)\overrightarrow{e_3} + d(t)\overrightarrow{e_4}$, where the vector $\overrightarrow{OM}$ is perpendicular to the vector $c(t)\overrightarrow{e_3} + d(t)\overrightarrow{e_4}$. We find the outer product of vectors $\overrightarrow{OM}$ and $r'(t)$. Have

$$\overrightarrow{OM} \wedge r'(t) = 0 \cdot \overrightarrow{e_1} \wedge \overrightarrow{e_2} + a \cdot c(t)\overrightarrow{e_1} \wedge \overrightarrow{e_3} + a \cdot d(t)\overrightarrow{e_1} \wedge \overrightarrow{e_4} + b \cdot c(t)\overrightarrow{e_2} \wedge \overrightarrow{e_3} + b \cdot d(t)\overrightarrow{e_2} \wedge \overrightarrow{e_4} + 0 \cdot \overrightarrow{e_3} \wedge \overrightarrow{e_4}.$$

Have $\left| r'(t) \wedge \overrightarrow{OM} \right|^2 = (a^2 + b^2)\left(c(t)^2 + d(t)^2\right).$

On the other hand $\left| r'(t) \wedge \overrightarrow{OM} \right| = \left| r'(t) \right| \cdot \left| \overrightarrow{OM} \right| \cdot \sin(\alpha(t))$, where $\alpha(t)$ is the angle between the vectors $r'(t)$ and $\overrightarrow{OM}$. Here the area of a parallelogram is understood as a focused area. In the vicinity of the point $O(0,0,0,0)$ $\sin \alpha(t)$ is positive , and in vicinity of the points $M(a, \text{b}, 0, 0)$ $\sin \alpha(t)$ is negative or vice versa. Here we assume that map F between the points O and M has no zeros. Then at some $\text{t} \in [0,1] \sin(\alpha(t))$ has zero value. Then, $a^2 + b^2 = 0$ or $c(t)^2 + d(t)^2 = 0$. Contradiction. The theorem is proved.

3. Vitushkin Example

Is considered the following well-known example of Vitushkin:

$$u(x, y) = x^2 y^6 + 2xy^2$$

$$v(x, y) = xy^3 + \frac{1}{y}, y \neq 0$$

The map $F: R^2 \rightarrow R^2$ is defined as

$$F(x,y)=(u,v),\, y\neq 0 .$$

Vitushkin's map is not injective, namely $F(-3,-1)=F(1,1)=(3,2)$ and has a constant nonzero Jacobian: $J(F)=-2$. Since $\lim_{(x,y)\to(x,+0)} v(x,y)=+\infty$ and $\lim_{(x,y)\to(x,-0)} v(x,y)=-\infty$, domain of the Vitushkin map is divided into two parts, with the points $(-3,-1,3,2)$ and $(1,1,3,2)$ lying in different parts of the domin. Namely, these points lie in different sides of the hyperplane $y=0$ of dimension three.

Theorem 2. *Vitushkin's map not surjective and has two reverse-mapping.*

Proof. $v^2-u=\frac{1}{y^2}>0$, that is, $u<v^2$. Hence, the upper part of the three-dimensional paraboloid $u=v^2$ has no inverse image. Consider the following maps:

$$G_+(x,y)=\left(\frac{x\left(y^2-x\right)^{\frac{3}{2}}}{\sqrt{y^2-x}+y},\frac{1}{\sqrt{y^2-x}}\right), y>0, y^2>x,$$

$$G_-(x,y)=\left(\frac{x\left(y^2-x\right)^{\frac{3}{2}}}{\sqrt{y^2-x}-y},-\frac{1}{\sqrt{y^2-x}}\right), y<0, y^2>x,$$

An immediate check indicates that

$$F\circ G_+=E=G_+\circ F \text{, for } y>0, y^2>x$$

and

$$F\circ G_-=E=G_-\circ F\, , y>0, y^2>x .$$

Thus, the Vitushkin's map has the following four properties:

1. The Vitushkin's map has a nonzero constant Jacobian,

$$J(F)=-2;$$

2. The Vitushkin's map is not injective,

$$F(-3,-2)=(3,2)=F(1,1);$$

3. The Vitushkin's map not surjective,

$$v^2-u=\frac{1}{y^2}>0\, ,(u,v)=F(x,y);$$

4. The Vitushkin map has two inverse mappings,

$$F^{-1} = G_{+}, y > 0, y^2 > x,$$
$$F^{-1} = G_{-}, y < 0, y^2 > x.$$

Bibliography

[1] Newman D. T., One-one polynomial maps, Proc. Amer. Math. Soc., **11**, 1960, 867–870.
[2] Bialynicki-Birula A., Rosenlicht V., Injective morphisms of real algebraic varieties, Proc.of the AMS., **13**, 1962 , 200–203.
[3] Yagzhev A. V., On Keller's problem, Siberian Math.J., **21**, 1980, 747–754.
[4] Bass H., Connel E., Wright D., The Jacobian Conjecture: Reduction of Degree and Formal Expansion of the Inverse, Bulletin of the AMS, №7 (1982), 287–330.
[5] S. Cynk and K. Rusek, Injective endomorphisms of algebraic and analytic sets, Annales Polonici Mathematici, **56** № 1 (1991), 29–35 .
[6] Aro van den Essen, Polynomial Automorphisms and the Jacobian Conjecture, Progress in Mathematics, 2000, 77–79 .

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The R_0 and R_1 Properties in Fuzzy Soft Topological Spaces

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Abstaract − The purpose of this paper is to introduce and study some new properties so-called fuzzy soft R_i (for short, FSR_i, $i = 0, 1$) on fuzzy soft spaces by using quasi-coincident relation for fuzzy soft points, we get some characterizations and properties of them. Also, the relationships of these properties in fuzzy soft topologies which are constructed from crisp topology and soft topology over X and vice versa are studied with some illustrative examples.

Keywords − *Fuzzy soft set, Fuzzy soft point, Fuzzy soft quasi-coincident, Fuzzy soft topology.*

1 Introduction

In 1999, Molodtsov [8] introduced the concept of soft set as one of mathematical tools for dealing with uncertainties. The works on the soft set theory have been applied in several directions. Maji et al.[7] introduced the concept of fuzzy soft set with some its properties. Then fuzzy soft theory and its applications have been studied by many authors. Chang [2] introduced the concept of fuzzy topology. Tanay et al.[12] introduced the definition of fuzzy soft topology over a subset of the initial universe set while Roy and Samanta [9] gave the definition of fuzzy soft topology over the initial universe set. In recent time, many of notions and results in fuzzy soft topology have been studied as in [1, 3, 4, 5, 10].

In this paper, we define and study some new properties and results related to fuzzy soft spaces. The main aim of our work is to introduce and study the R_0 and R_1 properties in fuzzy soft topological spaces by using quasi-coincidence for fuzzy soft points. Some characterizations and basic properties of them are studied. Also we, investigate the relationships of these properties in fuzzy soft topologies which are derived from crisp topology and vice versa with some necessary examples.

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2 Definitions and Notions

Throughout this work, X refers to a universe set, E be the set of all parameters for X, $P(X)$ is the power set of X and I^X be the set of all fuzzy subsets of X, where $I = [0, 1]$.

Definition 2.1. [2, 13] A fuzzy set A on X is a set characterized by a membership function $\mu_A : X \longrightarrow I$ whose value $\mu_A(x)$ represents the degree of membership of x in A for $x \in X$. A fuzzy point x_λ $(0 < \lambda \leq 1)$ is a fuzzy set in X given by $x_\lambda(y) = \lambda$ at $x = y$ and $x_\lambda(y) = 0$ otherwise for all $y \in X$. Here x and λ are called support and the value of x_λ, respectively. The set of all fuzzy point in X denoted by $FP(X)$. For $\alpha \in I$, $\underline{\alpha} \in I^X$ refers to the fuzzy constant function where, $\underline{\alpha}(x) = \alpha$ $\forall x \in X$ and for $x_\lambda \in FP(X)$, O_{x_λ} refers to a fuzzy open set contains x_λ and called fuzzy open neighborhood of x_λ. For $A,\ B \in I^X$, the basic operations for fuzzy sets are given by Zadah [13].

Definition 2.2. [6, 8] A soft set $F_E = (F, E)$ over X with the set E of parameters is a mapping $F : E \longrightarrow P(X)$ the value $F(e)$ is a set called e-element of the soft set for all $e \in E$. Thus a soft set over X can be represented by the set of ordered pairs $F_E = \{(e, F(e)) :\ e \in E,\ F(e) \in P(X)\ \}$, we denote the family of all soft sets over X by $SS(X, E)$.

Definition 2.3. [6, 11] Let $F_E \in SS(X, E)$ be a soft set over X. Then:

i. F_E is called a null soft set, denoted by $\emptyset_E$, if $F(e) = \emptyset$ for every $e \in E$. And if $F(e) = X$ for all $e \in E$, then F_E is called an universal soft set, denoted by X_E.

ii. If $F(e) = \{x\}$ and $F(e^{'}) = \emptyset$ for all $e^{'} \in E - \{e\}$, then F_E is called a soft point and denoted by x^e. The complement of a soft point x^e is a soft set over X dented by $(x^e)^c$ and given by $(x^e)^c(e) = X - \{x\}$, $(x^e)^c(e^{'}) = X$ for all $e^{'} \in E - \{e\}$. The set of all soft points over X is denoted by $SP(X, E)$.

Definition 2.4. [7, 9] A fuzzy soft set $f_E = (f, E)$ over X with the set E of parameters is defined by the set of ordered pairs $f_E = \{(e, f(e)) : e \in E\ ,\ f(e) \in I^X\}$. Here f is a mapping given by $f : E \longrightarrow I^X$ and the value $f(e)$ is a fuzzy set called e-element of the fuzzy soft set for all $e \in E$. The family of all fuzzy soft sets over X is denoted by $FSS(X, E)$.

Definition 2.5. [7, 9] Let f_E, g_E are two fuzzy soft sets over X. Then:

i. f_E is called a null fuzzy soft set, denoted by $\tilde{0}_E$ if $f(e) = \underline{0}$ for all $e \in E$. And if $f(e) = \underline{1}$ for all $e \in E$, then f_E is called universal fuzzy soft, denoted by $\tilde{1}_E$.

ii. A fuzzy soft f_E is subset of g_E if $f(e) \leq g(e)$ for all $e \in E$, denoted by $f \sqsubseteq g$.

iii. f_E and g_E are equal if $f_E \sqsubseteq g_E$ and $g_E \sqsubseteq f_E$. It is denoted by $f_E = g_E$.

iv. The complement of f_E is denoted by f^c_E, where $f^c\ : E \longrightarrow I^X$ is a mapping defined by $f(e)^C = \underline{1} - f(e)$ for all $e \in E$. Clearly, $f^c_E)^c = f_E$.

v. The union of f_E , g_E is a fuzzy soft set h_E defined by $h(e) = f(e) \cup g(e)$ for all $e \in E$. h_E is denoted by $f_E \sqcup g_E$.

vi. The intersection of f_E and g_E is a fuzzy soft set l_E defined by, $l(e) = f(e) \cap g(e)$ for all $e \in E$. l_E is denoted by $f_E \sqcap g_E$.

Definition 2.6. [1] A fuzzy soft point x_α^e over X is a fuzzy soft set over X defined as follows:
$x_\alpha^e(e') = \begin{cases} x_\alpha & if \quad e' = e \\ \underline{0} & if \quad e' \in E - \{e\} \end{cases}$ where,
x_α is the fuzzy point in X with support x and value α, $\alpha \in (0,1]$. The set of all fuzzy soft points in X is denoted by $FSP(X,E)$. The fuzzy soft point x_α^e is called belongs to a fuzzy soft set f_E , denoted by $x_\alpha^e \widetilde{\in} f_E$ iff $\alpha \leq f(e)(x)$. Every non-null fuzzy soft set f_E can be expressed as the union of all the fuzzy soft points belonging to f_E. The complement of a fuzzy soft point x_α^e is a fuzzy soft set over X.

Definition 2.7. [1, 9] Let X be a universe set, E be a fixed set of parameters and δ be the family of fuzzy soft sets over X, then δ is said to be a fuzzy soft topology on X iff:

i. $\tilde{0}_E$, $\tilde{1}_E$ belong to δ,

ii. The union of any number of fuzzy soft sets in δ is in δ,

iii. The intersection of any two fuzzy soft sets in δ is in δ.

In this case, (X,δ,E) is called a fuzzy soft topological space. The members of δ are called fuzzy soft open sets in X, denoted by $FSO(X,\delta,E)$. A fuzzy soft set f_E over X is called fuzzy soft closed in X iff $f_E^c \in \delta$, the set of all fuzzy soft closed sets over X, denoted by $FSC(X,\delta,E)$.

Notation.[10] Let (X,δ,E) be a fuzzy soft topological space. For $x_\alpha^e \in FSP(X,E)$ the fuzzy soft set $O_{x_\alpha^e}$ refers to a fuzzy soft open set contains x_α^e and $O_{x_\alpha^e}$ is called a fuzzy soft open neighborhood of x_α^e. The fuzzy soft open neighborhood system of x_α^e denoted by, $N_E(x_\alpha^e)$ is the family of all its fuzzy soft open neighborhoods.
In general for, $f_E \in FSS(X,E)$ the notation O_{f_E} refers to a fuzzy soft open set contains f_E and is called a fuzzy soft open neighborhood of f_E.

Definition 2.8. [1, 9] Let (X,δ,E) be a fuzzy soft topological space and $f_E \in FSS(X,E)$. Then:

i. The fuzzy soft interior of f_E is the fuzzy soft set denoted by f_E° and given by $f_E^\circ = \sqcup\{g_E : \ g_E \in \delta \text{ and } \ g_E \sqsubseteq f_E\}$, that is f_E° is a fuzzy soft open set. Indeed it is the largest fuzzy soft open set contained in f_E.

ii. The fuzzy soft closure of f_E is the fuzzy soft set denoted by $\overline{f_E}$ and given by $\overline{f_E} = \sqcap\{g_E : g_E \in \delta^c \text{ and } f_E \sqsubseteq g_E\}$, that is $\overline{f_E}$ is a fuzzy soft closed set. Clearly, $\overline{f_E}$ is the smallest fuzzy soft closed set over X which contains f_E.

Definition 2.9. [4] Let (X,δ,E) be a fuzzy soft topological space and $Y \subseteq X$. Let h_E^Y be a fuzzy soft set over (Y,E) such that $h_E^Y : E \longrightarrow I^Y$ such that $h_E^Y(e) \in I^Y$,
$h_E^Y(e)(x) = \begin{cases} 1 & \text{if} \quad x \in Y \\ 0 & \text{if} \quad x \notin Y \end{cases}$.
Let $\delta_Y = \{h_E^Y \sqcap g_E : g_E \in \delta\}$, then the fuzzy soft topology δ_Y on (Y,E) is called

fuzzy soft subspace topology for (Y, E) and (Y, δ_Y, E) is called a fuzzy soft subspace of (X, δ, E). If $h_E^Y \in \delta$ (resp. $h_E^Y \in \delta^c$), then (Y, δ_Y, E) is called fuzzy open (resp. closed) soft subspace of (X, δ, E).

Definition 2.10. [10] For $A \subseteq X$. The soft characteristic of A, denoted by $\widetilde{\chi}_A$ is a fuzzy soft set $\widetilde{\chi}_A : E \longrightarrow I^X$ defined by, $\widetilde{\chi}_A(e) = \chi_A\ \forall e \in E$, where χ_A is the characteristic of A. *i.e.* $\widetilde{\chi}_A = \{(e, \chi_A) : e \in E\}$, where $\chi_A : X \longrightarrow \{0, 1\}$.

Definition 2.11. [10] Let $f_E \in FSS(X, E)$. Then the soft support of f_E, denoted by $Ssup(f_E)$ is a soft set given by, $Ssup(f_E) = \{(e, S(f(e)) : e \in E\}$, where $S(f(e))$ is the support of fuzzy set $f(e)$, which is given by the set $S(f(e)) = \{x \in X : f(e)(x) > 0\} \subseteq X$.

Definition 2.12. [1] The fuzzy soft sets f_E and g_E in (X, E) are called fuzzy soft quasi-coincident, denoted by $f_E q g_E$ iff there exist $e \in E$, $x \in X$ such that $f(e)(x) + g(e)(x) > 1$. If f_E is not fuzzy soft quasi-coincident with g_E, then we write $f_E \tilde{q} g_E$, that is $f_E \tilde{q} g_E$ iff $f(e)(x) + g(e)(x) \leq 1$, *i.e.* $f(e)(x) \leq g^c(e)(x)$ for all $x \in X$ and $e \in E$.
A fuzzy soft point x_α^e is said to be soft quasi-coincident with f_E, denoted by $x_\alpha^e q f_E$ iff there exists $e \in E$ such that $\alpha + f_E(e)(x) > 1$.

Proposition 2.13. [1, 10] Let x_α^e, $y_\beta^e \in FSP(X, E)$, f_E, g_E, $h_E \in FSS(X, E)$ and $\{f_{iE} : i \in J\} \subseteq FSS(X, E)$. Then we have:

1. $f_E \tilde{q} g_E \Longleftrightarrow f_E \sqsubseteq g_E^c$,
2. $f_E \sqcap g_E = \widetilde{0}_E \Longrightarrow f_E \tilde{q} g_E$,
3. $f_E \tilde{q} g_E$, $h_E \sqsubseteq g_E \Longrightarrow f_E \tilde{q} h_E$,
4. $f_E q g_E \Longleftrightarrow x_\alpha^e q g_E,\ for\ some\ x_\alpha^e \widetilde{\in} f_E$,
5. $x_\alpha^e \tilde{q} f_E \Longleftrightarrow x_\alpha^e \widetilde{\in} f_E^c$,
6. $f_E \sqsubseteq g_E \Longleftrightarrow (x_\alpha^e q f_E \Longrightarrow x_\alpha^e q g_E$ for all $x_\alpha^e)$,
7. $f_E \tilde{q} f_E^c$,
8. If $x_\alpha^e q(\sqcap_{i \in J} f_{iE})$, then $x_\alpha^e q f_{iE}$ for all $i \in j$,
9. $x \neq y \Longrightarrow x_\alpha^e \tilde{q} y_\beta^e$ for all $\alpha, \beta \in I$,
10. $x_\alpha^e \tilde{q} y_\beta^e \Longleftrightarrow x \neq y\ \ or\ \ (x = y\ and\ \alpha + \beta \leq 1)$.

Lemma 2.14. [10] Let (X, δ, E) be a fuzzy soft topological space and $x_\alpha^e \in FSP(X, E)$. Then:

i. $g_E q f_E \Longleftrightarrow g_E q \overline{f_E}$ for all $g_E \in FSO(X, \delta, E)$,

ii. $x_\alpha^e q \overline{f_E} \Longleftrightarrow O_{x_\alpha^e} q f_E$ for all $O_{x_\alpha^e} \in N_E(x_\alpha^e)$.

Theorem 2.15. [10]

i. Let (X,τ) be a crisp topological space, then the family $\delta_\tau = \{\ \widetilde{\chi}_A : A \in \tau\}$ forms a fuzzy soft topology on X induced by τ,

ii. Every fuzzy soft topological space (X,δ,E) defines a crisp topology on X in the form $\tau_\delta = \{A \subseteq X : \widetilde{\chi}_A \in \delta\}$ which is induced by δ.

Theorem 2.16. [10]

i. Let (X,τ^*,E) be a soft topological space, then the collection $\delta_{\tau^*} = \{f_E \in FSS(X,E) : Ssup(f_E) \in \tau^*\}$ defines the fuzzy soft topology on X which is induced by τ^*.

ii. Let (X,δ,E) be a fuzzy soft topological space, then the family $\tau^*_\delta = \{Ssup(f_E) : f_E \in \delta\}$ defines the soft topology on X which is induced by δ.

Proposition 2.17. [10] Let (X,τ) be a topological space, (X,τ^*,E) be a soft topological space and (X,δ,E) be a fuzzy soft topological space. Then:

i. $\underline{\alpha}_E \in \delta_{\tau^*}$ for all $\alpha \in I$,

ii. $F_E \in \tau^* \Longrightarrow \widetilde{\chi}_{F_E} \in \delta_{\tau^*}$, in particular $\delta_\Delta \subseteq \delta_{\tau^*}$.

3 Fuzzy Soft R_i -Spaces, $i = 0$,1.

Definition 3.1. A fuzzy soft topological space (X,δ,E) is said to be:

i. Fuzzy soft R_0 (FSR_0, for short) iff for every x^e_α , $y^e_\beta \in FSP\,(X,E)$ with $x^e_\alpha \tilde{q} \overline{y^e_\beta}$ implies $\overline{x^e_\alpha}\, \tilde{q} y^e_\beta$.

ii. Fuzzy soft R_1 (FSR_1, for short) iff for every x^e_α , $y^e_\beta \in FSP(X,E)$ with $x^e_\alpha \tilde{q} \overline{y^e_\beta}$ implies there exist $O_{x^e_\alpha}$ and $O_{y^e_\beta} \in \delta$ such that $O_{x^e_\alpha} \tilde{q} O_{y^e_\beta}$.

In the following we get some characteristics of $FSR_i -$ spaces, $i = 0, 1$.

Theorem 3.2. Let (X,δ,E) be a fuzzy soft topological space. Then the following items are equivalent:

i. (X,δ,E) is FSR_0.

ii. $\overline{x^e_\alpha} \sqsubseteq O_{x^e_\alpha}$ for all $O_{x^e_\alpha} \in \delta$.

iii. $\overline{x^e_\alpha} \sqsubseteq \sqcap\{O_{x^e_\alpha} : O_{x^e_\alpha} \in \delta\}$ for all $x^e_\alpha \in FSP(X,E)$.

Proof. i $\Longrightarrow$ ii) Let (X,δ,E) be FSR_0 and $y^e_\beta q \overline{x^e_\alpha}$, then $x^e_\alpha q \overline{y^e_\beta}$ implies $y^e_\beta q O_{x^e_\alpha}\ \ \forall\ O_{x^e_\alpha}$. Hence $\overline{x^e_\alpha} \sqsubseteq O_{x^e_\alpha}\ \ \forall\ O_{x^e_\alpha}$ (by 6) of Proposition 2.13).
ii $\Longrightarrow$ iii) Obvious.
iii $\Longrightarrow$ i) Let $\overline{x^e_\alpha} \sqsubseteq \sqcap\{O_{x^e_\alpha} : O_{x^e_\alpha} \in N_E(x^e_\alpha)\} \sqsubseteq O_{x^e_\alpha}\ \forall\ O_{x^e_\alpha}$. Now let x^e_α , $y^e_\beta \in FSP(X,E)$ with $x^e_\alpha \tilde{q} \overline{y^e_\beta}$, then $x^e_\alpha \in \overline{y^e_\beta}^c = O_{x^e_\alpha}$ and so, by hypothesis $\overline{x^e_\alpha} \sqsubseteq O_{x^e_\alpha} = \overline{y^e_\beta}^c = (y^e_\beta)^{co} \sqsubseteq (y^e_\beta)^c \Longrightarrow \overline{x^e_\alpha} \tilde{q} y^e_\beta$. Hence (X,δ,E) is FSR_0.

Theorem 3.3. Let (X,δ,E) be a fuzzy soft topological space and $f_E \in FSC(X,\delta,E)$. Then the following items are equivalent:

i. (X,δ,E) is FSR_0.

ii. $x^e_\alpha \tilde{q} f_E$ implies there exists $O_{f_E} \in \delta$ contains f_E such that $x^e_\alpha \tilde{q} O_{f_E}$.

iii. $x^e_\alpha \tilde{q} f_E \Longrightarrow \overline{x^e_\alpha} \tilde{q} f_E$.

iv. $x^e_\alpha \tilde{q} \overline{y^e_\beta} \Longrightarrow \overline{x^e_\alpha} \tilde{q} \overline{y^e_\beta}$.

Proof. i $\Longrightarrow$ ii) Let (X,δ,E) be FSR_0 ,$f_E \in FSC(X,\delta,E)$ and $x^e_\alpha \tilde{q} f_E$, then $x^e_\alpha \in f^c_E = O_{x^e_\alpha} \Longrightarrow \overline{x^e_\alpha} \sqsubseteq f^c_E = O_{x^e_\alpha}$ (by Theorem 3.2) $\Longrightarrow f_E \sqsubseteq \overline{x^e_\alpha}^c = O_{f_E}$. Since $x^e_\alpha \sqsubseteq \overline{x^e_\alpha}$, then $\overline{x^e_\alpha}^c \sqsubseteq (x^e_\alpha)^c$. Hence $x^e_\alpha \tilde{q} \overline{x^e_\alpha}^c = O_{\ f_E}$.
ii) $\Longrightarrow$ iii) Let $x^e_\alpha \tilde{q} f_E$, then by hypothesis there exists O_{f_E} such that $x^e_\alpha \tilde{q} O_{f_E} \Longrightarrow \overline{x^e_\alpha} \tilde{q} f_E$ (by ii. of Lemma 2.14).
iii $\Longrightarrow$ iv) it is clear.
iv $\Longrightarrow$ i) Let x^e_α , $y^e_\beta \in FSP(X,E)$ with $x^e_\alpha \tilde{q} \overline{y^e_\beta} \Longrightarrow \overline{x^e_\alpha} \tilde{q} \overline{y^e_\beta}$ (by given). Since $y^e_\beta \sqsubseteq \overline{y^e_\beta}$, then $\overline{x^e_\alpha} \tilde{q} y^e_\beta$. Hence (X,δ,E) is FSR_0.

Theorem 3.4. Every $FSR_1 - space$ is a $FSR_0 - space$.

Proof. Obvious.

Corollary 3.5. Let (X,δ,E) be a fuzzy soft topological space. Then (X,δ,E) is FSR_1 if and only if for all x^e_α , $y^e_\beta \in FSP(X,E)$ with $x^e_\alpha \tilde{q} \overline{y^e_\beta}$ implies there exist $O_{\overline{x^e_\alpha}}$, $O_{\overline{y^e_\beta}} \in \delta$ such that $O_{\overline{x^e_\alpha}} \tilde{q} O_{\overline{y^e_\beta}}$.

Proof. Follows from the above theorem and from ii. of Theorem 3.2 .

Theorem 3.6. Every subspace (Y,δ_Y,E) of a $FSR_i - space$ (X,δ,E) is a $FSR_i - space$, $i = 0,1$.

Proof. As a sample we prove the case $i = 1$.
Let x^e_α, y^e_β are fuzzy soft points in (Y,E) with $x^e_\alpha \tilde{q} \overline{y^e_\beta}$. Then x^e_α, y^e_β also in (X,E) with $x^e_\alpha \tilde{q} \overline{y^e_\beta}$. Since (X,δ,E) is FSR_1, then there exist $O_{x^e_\alpha}$, $O_{y^e_\beta} \in \delta$ such that $O_{x^e_\alpha} \tilde{q} O_{y^e_\beta}$ and so, there exist $O^*_{x^e_\alpha} = O_{x^e_\alpha} \sqcap h^Y_E \in \delta_Y$, $O^*_{y^e_\beta} = O_{y^e_\beta} \sqcap h^Y_E \in \delta_Y$ such that $O^*_{x^e_\alpha} \tilde{q} O^*_{y^e_\beta}$. Hence (Y,δ_Y,E) is FSR_1

Lemma 3.7. Let (X,τ) and (X,τ^*,E) be a topological space and a soft topological space respectively, then we have:

i. $\overline{\widetilde{\chi}}^{\delta_\tau}_{\{x\}} = \widetilde{\chi}_{\overline{\{x\}}^\tau}$ for all $x \in X$.

ii. $\overline{\widetilde{\chi}}^{\delta_{\tau^*}}_{\{x^e\}} = \widetilde{\chi}_{\overline{\{x^e\}}^{\tau^*}}$ for all $x^e \in SP(X,E)$.

Proof. Straightforward.

In the following, we introduce some relationships for FSR_i-axioms, $i = 0,1$ in fuzzy soft topologies and that on crisp and soft topologies.

Theorem 3.8. Let (X,τ) be a topological space. Then (X,δ_τ,E) is a FSR_i-space if and only if (X,τ) is an $R_i -$ space, $i = 0,1$.

Proof. 1) For the case $i = 0$. Let (X, δ_τ, E) be FSR_0 and $x \in \overline{y}$. Then $x_1^e \widetilde{\in} \overline{y_1^e}$ and $x_1^e q \overline{y_1^e}$ (by i. of the above lemma). Since (X, δ_τ, E) is FSR_0, then $x_1^e q O_{y_1^e} \Longrightarrow y_1^e q \overline{x_1^e}$ (by ii. of Lemma 2.14). Thus $y_1^e \widetilde{\in} \overline{x_1^e}$ and so, $y \in \overline{x}$ (by i. of the above lemma). Hence (X, τ) is an $R_0 -$ space.
Conversely, let (X, τ) be R_0 and x_α^e, $y_\beta^e \in FSP(X, E)$ with $x_\alpha^e q \overline{y_\beta^e}$, in particular $x_1^e q \overline{y_1^e} \Longrightarrow x_1^e \not\sqsubseteq \overline{y_1^e}^c \Longrightarrow x_1^e \sqsubseteq \overline{y_1^e} \Longrightarrow x \in \overline{y}$(by i. of the above lemma). Since (X, τ) is R_0, then $y \in \overline{x} \Longrightarrow y_1^e \widetilde{\in} \overline{x_1^e} = \overline{x_\alpha^e} \Longrightarrow y_\beta^e \sqsubseteq y_1^e \widetilde{\notin} \overline{x_1^e}^c = \overline{x_\alpha^e}^c$(by i. of the above lemma)$\Longrightarrow y_\beta^e q \overline{x_\alpha^e}$. Hence we obtain the result.

2) For the case $i = 1$. Let (X, τ) be R_1 and x_α^e, $y_\beta^e \in FSP(X, E)$ with $x_\alpha^e \tilde{q} \overline{y_\beta^e}$, in particular $x_1^e \tilde{q} \overline{y_1^e} \Longrightarrow x_1^e \sqsubseteq \overline{y_1^e}^c \Longrightarrow x_1^e \not\sqsubseteq \overline{y_1^e} \Longrightarrow x \notin \overline{y} \Longrightarrow \overline{x} \neq \overline{y}$, then there exist O_x, $O_y \in \tau$ such that $O_x \cap O_y = \emptyset$. Take $O_{x_\alpha^e} = \widetilde{\chi}_{O_x} \in \delta_\tau$ and $O_{y_\beta^e} = \widetilde{\chi}_{O_y} \in \delta_\tau$, then $O_{x_\alpha^e} \tilde{q} O_{y_\beta^e}$. Hence (X, δ_τ, E) is FSR_1.
Conversely, let (X, δ_τ, E) is FSR_1 and $\overline{x} \neq \overline{y} \Longrightarrow$ there exists $x \in X$ such that $x \in \overline{x}$ and $x \notin \overline{y} \Longrightarrow x_1^e \not\sqsubseteq \overline{y_1^e} \Longrightarrow x_1^e \tilde{q} \overline{y_1^e}$, then there exist $O_{x_1^e}$, $O_{y_1^e} \in \delta_\tau$ such that $O_{x_1^e} \tilde{q} O_{y_1^e}$ and so, there exist O_x, $O_y \in \tau$ such that $O_{x_1^e} = \widetilde{\chi}_{O_x}$ and $O_{y_1^e} = \widetilde{\chi}_{O_y}$, then $\widetilde{\chi}_{O_x} \sqsubseteq \widetilde{\chi}_{O_y}^c \Longrightarrow O_x \subseteq O_y^c \Longrightarrow O_x \cap O_y = \emptyset$. Hence the result holds.

Theorem 3.9. Let (X, δ, E) be a fuzzy soft topological space. If (X, δ, E) is a FSR_0-space, then (X, τ_δ) is a R_0-space.

Proof. It is similar to that of the necessity part of the above theorem.

Note. An R_i-space (X, τ_δ) need not imply (X, δ, E) FSR_i-space, $i = 0, 1$, this fact can be shown by the following examples.

Examples 3.10. 1) Let $X = \{x, y, z\}$ and $E = \{e_1, e_2\}$, then the family $\delta = \{\tilde{0}_E, \tilde{1}_E, f_E = \{(e_1, (x_1, y_{0.5})), (e_2, \underline{1})\}, g_E = \{(e_1, x_{0.5})\}\}$ is a fuzzy soft topology on X and $_\delta = \{\emptyset, X\}$ is a topology on X which is induced by δ. It is easy to check that $(X,_\delta)$ *is* R_0, but the fuzzy soft topological space (X, δ, E) is not FSR_0 . Indeed, for $x_{0.5}^{e_1} \in FSP(X, E)$, $\overline{x_{0.5}^{e_1}} = \{(e_1, (x_{0.5}, y_1, z_1)), (e_2, \underline{1})\}$, but there exists $O_{x_{0.5}^{e_1}} = \{(e_1, (x_1, y_{0.5})), (e_2, \underline{1})\}$ such that $\overline{x_{0.5}^{e_1}} \not\sqsubseteq O_{x_{0.5}^{e_1}}$.

2) Let $X = \{a, b, c\}$ and $E = \{e_1, e_2\}$. Then the family $\delta = \{\tilde{0}_E, \tilde{1}_E, f_E\}$, where $f_E = \{(e_1, (a_{0.3}, c_{0.5})), (e_2, (a_{0.3}, c_{0.5}))\}$ is a fuzzy soft topology on X and $\tau_\delta = \{\emptyset, X\}$ is a topology on X which is induced by δ. It is clear that (X, τ_δ) is R_1, but (X, δ, E) is not FSR_1, because for $a_{0.3}^{e_2}$, $b_1^{e_2} \in FSP(X, E)$ with $a_{0.3}^{e_2} \tilde{q} \overline{b_1^{e_2}} \Longrightarrow O_{a_{0.3}^{e_2}} q O_{b_1^{e_2}}$ for all $O_{a_{0.3}^{e_2}}$, $O_{b_1^{e_2}} \in \delta$.

Definition 3.11. A soft topological space (X, τ^*, E) is said to be:

i. Soft R_0(for short, SR_0) iff for every pair of soft points x^e, $y^e(x \neq y) \in SP(X, E)$ with $x^e \in \overline{y^e}$ implies $y^e \in \overline{x^e}$.

ii. Soft R_1(for short, SR_1) iff for every pair of soft points x^e, $y^e(x \neq y) \in SP(X, E)$ with $\overline{x^e} \neq \overline{y^e}$ implies there exist two soft open sets F_E, G_E contains x^e and y^e respectively, such that $F_E \widetilde{\cap} G_E = \emptyset_E$.

Theorem 3.12. Let (X, τ^*, E) be a soft topological space, then we have:

i. (X, δ_τ^*, E) is FSR_0 if and only if (X, τ^*, E) is SR_0.

ii. If (X, τ^*, E) is SR_1, then $(X, \delta_\tau{}^*, E)$ is FSR_1.

Proof. i.) Let (X, δ_{τ^*}, E) be FSR_0 and $x^e \in \overline{y^e}$, then $x_1^e q \overline{y_1^e}$ (by Lemma 3.7). Since (X, δ_{τ^*}, E) is FSR_0, then $x_1^e q O_{y_1^e} \Longrightarrow y_1^e q \overline{x_1^e}$ (by ii. of Lemma 2.14)$\Longrightarrow y_1^e \not\sqsubseteq \overline{x_1^e}^c \Longrightarrow y_1^e \sqsubseteq \overline{x_1^e}$. Thus $y^e \in \overline{x^e}$(by Lemma 3.7). Hence (X, τ^*, E) is SR_0.
Conversely, let (X, τ^*, E) be SR_0 and $x_\alpha^e \in FSP(X, E)$. Since $x_\alpha^e \in \delta_{\tau^*}^c$ $\forall \alpha \in I - \{0, 1\}$, then $\overline{x_\alpha^e} = x_\alpha^e \sqsubseteq O_{x_\alpha^e}$ $\forall O_{x_\alpha^e}$. When $\alpha = 1$, then clearly $\overline{x_1^e} = O_{x_1^e}$. Hence we obtain the result.

ii.) Let (X, τ^*, E) be SR_1 and x_α^e, $y_\beta^e \in FSP(X, E)$ with $x_\alpha^e \tilde{q} \overline{y_\beta^e} \Longrightarrow x_\alpha^e \tilde{q} y_\beta^e$. Then we have, either $x \neq y$ or ($x = y$ and $\alpha + \beta \leq 1$) (by 10. of Proposition 3.13).
Case I. If $x \neq y$, then $x^e \neq y^e \Longrightarrow (\overline{x^e} \neq \overline{y^e}$ *or* $\overline{x^e} = \overline{y^e})$. Now we have:

a. If $\overline{x^e} \neq \overline{y^e}$, then there exsit $O_{x^e}, O_{y^e} \in \tau^*$ such that $O_{x^e} \tilde{\sqcap} O_{y^e} = \emptyset_E$. Take $O_{x_\alpha^e} = \widetilde{\chi}_{O_{x^e}} \in \delta_{\tau^*}$ and $O_{y_\beta^e} = \widetilde{\chi}_{O_{y^e}} \in \delta_{\tau^*}$, then $O_{x_\alpha^e} \tilde{q} O_{y_\beta^e}$. Hence (X, δ_{τ^*}, E) is a FSR_1-space.

b. If $\overline{x^e} = \overline{y^e}$, then this case is excluded (since (X, τ^*, E) is SR_1).

Case II. If ($x = y$ and $\alpha + \beta \leq 1$). Take $O_{x_\alpha^e} = \underline{\alpha_E} \in \delta_{\tau^*}$, $O_{y_\beta^e} = \underline{\beta_E} \in \delta_{\tau^*}$, then $O_{x_\alpha^e} = \underline{\alpha_E} \tilde{q} O_{y_\beta^e} = \underline{\beta_E}$. Hence (X, δ_{τ^*}, E) is a FSR_1-space.

Note. A soft R_i-space (X, τ_δ^*, E) need not imply (X, δ, E) FSR_i , $i = 0, 1$, this fact can be shown by the following example.

Example 3.13. Let $X = \{a, b\}$ and $E = \{e_1, e_2\}$. The family $\delta = \{\tilde{0}_E, \tilde{1}_E, f_E, g_E, h_E\}$, where $f_E = \{(e_1, a_{0.6}), (e_2, a_{0.6})\}$, $g_E = \{(e_1, b_{0.9}), (e_2, b_{0.9})\}$, $h_E = \{(e_1, (a_{0.6}, b_{0.9})), (e_2, (a_{0.6}, b_{0.9}))\}$ is a fuzzy soft topology on X and $\tau_\delta^* = \{\emptyset_E, X_E, F_E = \{(e_1, \{a\}), (e_2, \{a\})\}, G_E = \{(e_1, \{b\}), (e_2, \{b\})\}\}$ is a soft topology on X which is induced by δ. It is clear that (X, τ_δ^*, E) is soft R_1, but (X, δ, E) is not FSR_0, because for $a_{0.3}^{e_1}$, $b_1^{e_1} \in FSP(X, E)$ with $a_{0.3}^{e_1} \tilde{q} \overline{b_1^{e_1}}$, but $b_1^{e_1} q \overline{a_{0.3}^{e_1}}$. Also, it is not FSR_1, because for $a_{0.3}^{e_1}$, $b_1^{e_1} \in FSP(X, E)$ with $a_{0.3}^{e_1} \tilde{q} \overline{b_1^{e_1}} \Longrightarrow O_{a_{0.3}^{e_1}} q O_{b_1^{e_1}}$ for all $O_{a_{0.3}^{e_1}}$, $O_{b_1^{e_1}} \in \delta$.

4 Conclusion

In this paper, we defined and studied some new axioms are called the R_0 and R_1 properties in fuzzy soft topological spaces and some of its properties. Also, the relationships of these properties are studied. We hope these basic results will help the researchers to enhance and promote the research on fuzzy soft theory and its applications. In the next work, by the same manner, we defined and study a new set of separation axioms on fuzzy soft spaces.

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References

[1] S. Atmaca and I. Zorlutuna, *On fuzzy soft topological spaces*, Ann. Fuzzy Math. Inform. 5(2) (2013) 377-386.

[2] C. L. Chang, *Fuzzy topological spaces*, J. of Math. analysis and Appl. 24(1986) 182-190.

[3] İ. Demir and O. Özbakır, *Some properties of fuzzy proximity spaces*, The Scientific World Journal, (2015), ID 752634, 10 PP.

[4] A. Kandil, O. A. Tantawy, S. A. El-Sheikh and A. M. Abd El-latif, *Some fuzzy toplogical properties based on fuzzy semi-open soft sets*, South Asian J. of Math. 4(2014) 154-169.

[5] A. Kharal and B. Ahmad, *Mappings on fuzzy soft classes*, Adv. Fuzzy Syst. 2009, Artical ID 407890, 6 pp.

[6] P. K. Maji, R. Biswas and A. R. Roy, *Soft set theory*, Comp. Math. Appl. 45 (2003) 555-562.

[7] P. K. Maji, R. Biswas and A. R. Roy, *Fuzzy soft sets*, J. Fuzzy Math. 9(3) (2001) 589-602.

[8] D. Molodtsov, *Soft set theory-first results*, Comp. Math. Appl. 37 (1999) 19-31.

[9] S. Roy and T. K. Samanta, *A note on fuzzy soft topological spaces*, Ann. Fuzzy Math. Inform. 3(2) (2011) 305-311.

[10] S. Saleh and Amani Al-Salemi, *Some new results on fuzzy soft spaces*, Accepted (2018).

[11] M. Shabir and M. Naz, *On soft topological spaces*, Comp. Math. Appl. 61 (2011) 1786-1799.

[12] B. Tanay and M. B. Kandemir, *Topological structures of fuzzy soft sets*, Comput. Math. Appl. 61 (2011) 412-418.

[13] L. A. Zadeh, *Fuzzy Sets*, Information and Control 8 (1965) 338-353.