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Perception of Nano Generalized $t^{\#}$-Closed Sets

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Abstaract − In this paper, we introduce a new class of sets called $nt^{\#}_{g}$-closed sets, which is stronger than ng-closed sets and weaker than n-closed sets.

Keywords − *$n\pi$-closed set, $n\pi g$-closed set, $n\pi gp$-closed set, $n\pi gs$-closed set, $nt^{\#}$-set, $nt^{\#}_{g}$-closed set*

1 Introduction

Thivagar et al. [6] introduced a nano topological space with respect to a subset X of an universe which is defined in terms of lower approximation and upper approximation and boundary region. The classical nano topological space is based on an equivalence relation on a set, but in some situation, equivalence relations are nor suitable for coping with granularity, instead the classical nano topology is extend to general binary relation based covering nano topological space.

Bhuvaneswari et al. [4] introduced and investigated nano g-closed sets in nano topological spaces. Recently, Parvathy and Bhuvaneswari the notions of nano gpr-closed sets which are implied both that of nano rg-closed sets. In 2017, Rajasekaran et al. [9, 10] introduced the notion of nano πgp-closed sets and nano πgs-closed sets in nano topological spaces. In this paper, we introduce a new class of sets called nano $t^{\#}_{g}$-closed sets, which is stronger than nano g-closed sets and weaker than nano closed sets.

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2 Preliminaries

Definition 2.1. *[11] Let U be a non-empty finite set of objects called the universe and R be an equivalence relation on U named as the indiscernibility relation. Elements belonging to the same equivalence class are said to be indiscernible with one another. The pair (U, R) is said to be the approximation space. Let $X \subseteq U$.*

1. *The lower approximation of X with respect to R is the set of all objects, which can be for certain classified as X with respect to R and it is denoted by $L_R(X)$. That is, $L_R(X) = \bigcup_{x \in U}\{R(x) : R(x) \subseteq X\}$, where R(x) denotes the equivalence class determined by x.*

2. *The upper approximation of X with respect to R is the set of all objects, which can be possibly classified as X with respect to R and it is denoted by $U_R(X)$. That is, $U_R(X) = \bigcup_{x \in U}\{R(x) : R(x) \cap X \neq \phi\}$.*

3. *The boundary region of X with respect to R is the set of all objects, which can be classified neither as X nor as not - X with respect to R and it is denoted by $B_R(X)$. That is, $B_R(X) = U_R(X) - L_R(X)$.*

Property 2.2. *[6] If (U, R) is an approximation space and $X, Y \subseteq U$; then*

1. $L_R(X) \subseteq X \subseteq U_R(X)$;
2. $L_R(\phi) = U_R(\phi) = \phi$ *and* $L_R(U) = U_R(U) = U$;
3. $U_R(X \cup Y) = U_R(X) \cup U_R(Y)$;
4. $U_R(X \cap Y) \subseteq U_R(X) \cap U_R(Y)$;
5. $L_R(X \cup Y) \supseteq L_R(X) \cup L_R(Y)$;
6. $L_R(X \cap Y) \subseteq L_R(X) \cap L_R(Y)$;
7. $L_R(X) \subseteq L_R(Y)$ *and* $U_R(X) \subseteq U_R(Y)$ *whenever* $X \subseteq Y$;
8. $U_R(X^c) = [L_R(X)]^c$ *and* $L_R(X^c) = [U_R(X)]^c$;
9. $U_R U_R(X) = L_R U_R(X) = U_R(X)$;
10. $L_R L_R(X) = U_R L_R(X) = L_R(X)$.

Definition 2.3. *[6] Let U be the universe, R be an equivalence relation on U and $\tau_R(X) = \{U, \phi, L_R(X), U_R(X), B_R(X)\}$ where $X \subseteq U$. Then $\tau_R(X)$ satisfies the following axioms:*

1. *U and $\phi \in \tau_R(X)$,*
2. *The union of the elements of any sub collection of $\tau_R(X)$ is in $\tau_R(X)$,*
3. *The intersection of the elements of any finite subcollection of $\tau_R(X)$ is in $\tau_R(X)$.*

Thus $\tau_R(X)$ is a topology on U called the nano topology with respect to X and $(U, \tau_R(X))$ is called the nano topological space. The elements of $\tau_R(X)$ are called nano-open sets (briefly n-open sets). The complement of a n-open set is called n-closed.

In the rest of the paper, we denote a nano topological space by $(U, \mathcal{N})$, where $\mathcal{N} = \tau_R(X)$. The nano-interior and nano-closure of a subset A of U are denoted by n-int(A) and n-cl(A), respectively.

Definition 2.4. *A subset H of a space $(U, \mathcal{N})$ is called;*

1. *nano regular-open (briefly nr-open) set [6] if $H =$ n-int(n-cl(H)).*
2. *nano pre open (briefly np-open) set [6] if $H \subseteq$ n-int(n-cl(H)).*
3. *nano semi open (briefly ns-open) set [6] if $H \subseteq$ n-cl(n-int(H)).*
4. *nano π-open (briefly nπ-open) set [1] if the finite union of nr-open sets.*

The complements of the above mentioned sets is called their respective closed sets.

Definition 2.5. *[7] A subset H of a space $(U, \mathcal{N})$ is called a nt$^{\#}$-set if n-int$(H) =$ n-cl(n-int(H)).*

Definition 2.6. *A subset H of a space $(U, \mathcal{N})$ is called;*

1. *nano g-closed (briefly ng-closed) [2] if n-cl$(H) \subseteq G$, whenever $H \subseteq G$ and G is n-open.*
2. *nano πg-closed (briefly nπg-closed) [8] if n-cl$(H) \subseteq G$, whenever $H \subseteq G$ and G is nπ-open.*
3. *nano gp-closed set (briefly ngp-closed) [4] if n-pcl$(H) \subseteq G$, whenever $H \subseteq G$ and G is n-open.*
4. *nano gs-closed (briefly ngs-closed) [3] if n-scl$(H) \subseteq G$, whenever $H \subseteq G$ and G is n-open.*
5. *nano πgp-closed (briefly nπgp-closed) [9] if n-pcl$(H) \subseteq G$, whenever $H \subseteq G$ and G is nπ-open.*
6. *nano πgs-closed (briefly nπgs-closed) [10] if n-scl$(H) \subseteq G$, whenever $H \subseteq G$ and G is nπ-open.*
7. *nano g*-closed (briefly ng*-closed) [12] if n-cl$(H) \subseteq G$ whenever $H \subseteq G$ and G is ng-open.*
8. *nano LC-set (briefly nLC-set) [5] if $H = G \cap K$, where G is n-open and K is n-closed.*

3 On Nano $t_g^{\#}$-Closed Sets, Nano $t_{sg}^{\#}$-Closed Sets and Nano $t_{pg}^{\#}$-Closed Sets

Definition 3.1. *A subset H of a space $(U, \mathcal{N})$ is called;*

1. *nano $t_g^{\#}$-closed (briefly $nt_g^{\#}$-closed) if n-$cl(H) \subseteq G$ whenever $H \subseteq G$ and G is $nt^{\#}$-set.*

2. *nano $t_{sg}^{\#}$-closed (briefly $nt_{sg}^{\#}$-closed) if n-$scl(H) \subseteq G$ whenever $H \subseteq G$ and G is $nt^{\#}$-set.*

3. *nano $t_{pg}^{\#}$-closed (briefly $nt_{pg}^{\#}$-closed) if n-$pcl(H) \subseteq G$ whenever $H \subseteq G$ and G is $nt^{\#}$-set.*

The complements of the above mentioned sets are called their respective n-open sets.

Example 3.2. *Let $U = \{a, b, c, d\}$ with $U/R = \{\{a\}, \{b, c\}, \{d\}\}$ and $X = \{b, d\}$. Then the nano topology $\mathcal{N} = \{\phi, \{d\}, \{b, c\}, \{b, c, d\}, U\}$.*

1. *$H = \{a\}$ is $nt^{\#}$-closed set.*

2. *$H = \{d\}$ is $nt_{sg}^{\#}$-closed set.*

3. *$H = \{a, c\}$ is $nt_{pg}^{\#}$-closed set.*

Remark 3.3. *The family of nLC-set and the family of ng-closed sets are independent.*

Example 3.4. *In Example 3.2,*

1. *$H = \{b, c\}$ is nLC-set but not ng-closed set.*

2. *$H = \{a, b\}$ is ng-closed set but not nLC-set.*

Theorem 3.5. *For a subset H of a space $(U, \mathcal{N})$, the following are equivalent:*

1. *H is nLC-set.*

2. *$H = G \cap n$-$cl(H)$ for some G n-open set.*

Proof. (1) $\Rightarrow$ (2). Since H is a nLC-set, then $H = G \cap K$, where G is n-open and K is n-closed. So, $H \subseteq G$ and $H \subseteq K$. Hence, n-$cl(H) \subseteq n$-$cl(K)$. Therefore, $H \subseteq G \cap n$-$cl(H) \subseteq G \cap n$-$cl(K) = G \cap K = H$. Thus, $H = G \cap n$-$cl(H)$.

(2) $\Rightarrow$ (1). It is obvious because n-$cl(H)$ is n-closed.

Theorem 3.6. *For a subset H of a space $(U, \mathcal{N})$, the following are equivalent:*

1. *H is n-closed.*

2. *H is nLC-set and ng-closed.*

Proof. (1) $\Rightarrow$ (2). This is obvious.

(2) $\Rightarrow$ (1). Since H is nLC-set, then $H = G \cap n\text{-}cl(H)$, where G is n-open set in U. So, $H \subseteq G$ and since H is ng-closed, then $n\text{-}cl(H) \subseteq G$. Therefore, $n\text{-}cl(H) \subseteq G \cap n\text{-}cl(H) = H$. Hence, H is n-closed.

Theorem 3.7. *In a space* $(U, \mathcal{N})$,

1. *If H is n-closed set, then H is $nt^{\#}_{g}$-closed set.*
2. *If is $nt^{\#}_{g}$-closed set, then H is ng-closed set.*

Proof. (1). Obvious.

(2). Let H be a $nt^{\#}_{g}$-closed set and $H \subseteq G$ where $G \in \mathcal{N}$. Since each n-open set is $nt^{\#}$-set, so G is $nt^{\#}$-set. Since H is $nt^{\#}_{g}$-closed set, we obtain that $n\text{-}cl(H) \subseteq G$, hence H is ng-closed set

Remark 3.8. *The converses of Theorem 3.7 are not true as seen from the following Example.*

Example 3.9. *In Example 3.2,*

1. $H = \{d\}$ *is $nt^{\#}_{g}$-closed set but not n-closed.*
2. $H = \{a, b\}$ *is ng-closed set but not $nt^{\#}_{g}$-closed set.*

Theorem 3.10. *In a space* $(U, \mathcal{N})$,

1. *If H is $nt^{\#}_{g}$-closed set, then H is $nt^{\#}_{pg}$-closed set.*
2. *If H is $nt^{\#}_{g}$-closed set, then H is $nt^{\#}_{sg}$-closed set.*

Proof. Obvious.

Remark 3.11. *The converses of Theorem 3.10 are not true as seen from the following Example.*

Example 3.12.
1. *In Example 3.2,* $H = \{b\}$ *is $nt^{\#}_{pg}$-closed set but not $nt^{\#}_{g}$-closed set.*
2. *Let* $U = \{a, b, c\}$ *with* $U/R = \{\{a, b\}, \{c\}\}$ *and* $X = \{c\}$. *Then the nano topology* $\mathcal{N} = \{\phi, \{c\}, U\}$, $H = \{a\}$ *is $nt^{\#}_{sg}$-closed set but not $nt^{\#}_{g}$-closed set.*

Remark 3.13. *We obtain the following diagram, where $A \longrightarrow B$ represents A implies B but not conversely.*

		$ng^{\star}$-***closed***		
		↓		
nLC-***set***	←	n-***closed***	←	$n\pi$-***closed***
		↓		↓
$nt^{\#}_{sg}$-***closed***	←	$nt^{\#}_{g}$-***closed***	→	$nt^{\#}_{pg}$-***closed***
↓		↓		↓
ngs-***closed***	←	ng-***closed***	→	ngp-***closed***
↓		↓		↓
$n\pi gs$-***closed***	←	$n\pi g$-***closed***	→	$n\pi gp$-***closed***

None of the above implications are reversible

Example 3.14. *In Example 3.2,*

1. $H = \{b\}$ *is* $nt^{\#}_{pg}$*-closed but not* $n\pi$*-closed.*
2. $H = \{c\}$ *is ngs-closed but not* $nt^{\#}_{sg}$*-closed.*

Theorem 3.15. *In a space* $(U, \mathcal{N})$*, if* H *is* $ng^{\star}$*-closed set, then* H *is* $nt^{\#}_{g}$*-closed.*

Proof. Obvious.

Remark 3.16. *The converses of Theorem 3.15 are not true as seen from the following Example.*

Example 3.17. *In Example 3.2,* $H = \{a, b, d\}$ *is* $nt^{\#}_{g}$*-closed set but not* $ng^{\star}$*-closed set.*

Remark 3.18. *The family of* $nt^{\#}_{g}$*-closed sets and the family of* $nt^{\#}$*-sets are independent.*

Example 3.19. *In Example 3.2,*

1. $H = \{d\}$ *is* $nt^{\#}_{g}$*-closed set but not* $nt^{\#}$*-set.*
2. $H = \{b\}$ *is* $nt^{\#}$*-set but not* $nt^{\#}_{g}$*-closed set.*

Theorem 3.20. *In a space* $(U, \mathcal{N})$*, if* H *is both* $nt^{\#}$*-set and* $nt^{\#}_{g}$*-closed set, then* H *is* n*-closed.*

Proof. Let H be both $nt^{\#}$-set and $nt^{\#}_{g}$-closed set. Then $n\text{-}cl(H) \subseteq H$, whenever H is a $nt^{\#}$-set and $H \subseteq H$. So we obtain that $H = n\text{-}cl(H)$ and hence H is n-closed.

Proposition 3.21. *In a space* $(U, \mathcal{N}, I)$*, the union of two* $nt^{\#}_{g}$*-closed sets is* $nt^{\#}_{g}$*-closed.*

Proof. Let $H \cup K \subseteq G$, where G is a $nt^{\#}$-set. Since H, K are $nt^{\#}_{g}$-closed sets, $n\text{-}cl(H) \subseteq G$ and $n\text{-}cl(K) \subseteq G$, whenever $H \subseteq G$, $K \subseteq G$ and G is a $nt^{\#}$-set. Therefore, $n\text{-}cl(H \cup K) = n\text{-}cl(H) \cup n\text{-}cl(K) \subseteq G$. Hence we obtain that $H \cup K$ is a $nt^{\#}_{g}$-closed set.

Theorem 3.22. *In a space* $(U, \mathcal{N}, I)$*, the intersection of two* $nt^{\#}_{g}$*-closed sets need not be* $nt^{\#}_{g}$*-closed as illustrated in the following Example.*

Example 3.23. *In Example 3.2, then* $H = \{b, c\}$ *and* $Q = \{c, d\}$ *is* $nt^{\#}_{g}$*-closed sets. Clearly* $H \cap Q = \{c\}$ *is not* $nt^{\#}_{g}$*-closed set.*

Theorem 3.24. *In a space* $(U, \mathcal{N})$*, If* H *is* $nt^{\#}_{g}$*-closed set such that* $H \subseteq K \subseteq n\text{-}cl(H)$*, then* K *is also* $nt^{\#}_{g}$*-closed set.*

Proof. Let G be a $nt^{\#}$-set such that $K \subseteq G$. Then $H \subseteq G$. Since H is $nt^{\#}_{g}$-closed, we have $n\text{-}cl(H) \subseteq G$. Now $n\text{-}cl(K) \subseteq n\text{-}cl(n\text{-}cl(H)) = n\text{-}cl(H) \subseteq G$. Therefore, K is also $nt^{\#}_{g}$-closed set.

Theorem 3.25. *In a space* $(U, \mathcal{N})$*, let* H *be* $nt^{\#}_{g}$*-closed. Then* $n\text{-}cl(H) - H$ *does not contain any non-empty complement of* $nt^{\#}$*-set.*

Proof. Let H be $nt_g^{\#}$-closed set. Suppose that P is the complement of $nt^{\#}$-set and $P \subseteq n\text{-}cl(H) - H$. Since $P \subseteq n\text{-}cl(H) - H \subseteq U - H$, $H \subseteq U - P$ and $U - P$ is $nt^{\#}$-set. Therefore, $n\text{-}cl(H) \subseteq U - P$ and $P \subseteq U - n\text{-}cl(H)$. However, since $P \subseteq n\text{-}cl(H) - H$, $P = \phi$.

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N-Fuzzy BI-Topological Space and Separation Axioms

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Abstaract — In this article, we introduced N-fuzzy bi-topological space by using the concepts of fuzzy bi-topological space. We further define some basic properties of N-fuzzy bi-topological spaces, secondly we study the concepts of natural separation axioms of bi-topological in N-fuzzy bi-topological space which is pair wise separation Axioms mixed topology with the help of two N-fuzzy topologies of a N-fuzzy bi-topological space. Relation between such pairwise separation axioms and natural fuzzy separation axioms of the mixed fuzzy topological space are investigated.

Keywords — *N-fuzzy set, N-fuzzy bi-topological space, natural separation axioms, natural fuzzy separation axioms.*

1 Introduction

The concept of fuzzy sets was introduced by Zadeh in [1] and thereafter the paper of Chang [3] give the way for the farther growth of new direction. The notion of fuzzyness has been applied for studying different aspects of mathematics by Tripathy and Baruah [20], Tripathy and Borgoha in [21, 22], Tripathy and Tripathy and Sarma [23] and many research on sequence spaces in recent years. The notion of bi-topological spaces has been discuss from different aspects by Tripathy and Acharjee [24], Tripathy and Debnath [25] and others. Kandil introduced the concept of fuzzy bi-topological spaces. Later on several researcher were attracted by the notion of fuzzy bi-topological spaces. An N-fuzzy bi-topological space is a non-empty Set X together with two N-fuzzy topologies on it. We will apply the N-structure to fuzzy bi-topological space and pairwise, separation axioms. Different pair wise separation

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Axioms are defend which is the generalization of natural Separation axioms in the sense that such a notion reduces to the natural separation axioms of a N-fuzzy topological space when two topological spaces coincide. In this paper, pairwise, separation axioms are introduced and a mixed topology is introduced with the help of two N-fuzzy topologies of a N-fuzzy bi-topological space. Relation between such pairwise separation axioms and natural fuzzy separation axioms of the mixed fuzzy topological space are investigated. Finally, pairwise N-fuzzy normal bi-topological space, pairwise weakly and pairwise strongly separated space are introduced and investigated their properties with the mixed topology.

2 Preliminary

In this section we deals with basic concept of fuzzy set, fuzzy topological spaces N-fuzzy set and N-fuzzy topological and bi- topological spaces.

Definition 2.1. [1] Let **X** be a non-empty set. A fuzzy set **A** in **X** is characterized by its membership function $\mu_A : X \rightarrow [0, 1]$ and $\mu_A(x)$ is interpreted as the degree of membership of element x in fuzzy set A for each $x \epsilon X$. It is clear that A is completely determined by the set of tuples $A = \{(x, \mu_A(x)) : x \epsilon X\}$.

Definition 2.2. [2] A fuzzy topological space on a set **X** is a family τ of fuzzy sets in **X** which satisfies the following condition:
i For all $0, 1 \ \epsilon \ \tau$.
ii For all $A, B \epsilon \tau \Rightarrow A \wedge B \epsilon \tau$.
iii For all $(A_j)_{j \epsilon J} \epsilon \tau \Rightarrow \vee_{j \epsilon J} A_j \epsilon \tau$.
The pair (X, τ) is called fuzzy topological space and the members of τ are called fuzzy open sets.

Definition 2.3. [5] A fuzzy bi-topological space on a set **X** is a families (τ_1, τ_2) of fuzzy sets in **X** which satisfies the following condition:
i $0, 1 \ \epsilon \ \tau_1, \tau_2$.
ii For all $A, B \epsilon \tau_1, \tau_2 \Rightarrow A \wedge B \epsilon \tau_1 \tau_2$.
iii For all $(A_j)_{j \epsilon J} \epsilon \tau_1, \tau_2 \Rightarrow \vee_{j \epsilon J} A_j \epsilon \tau_1, \tau_2$.
The pair (X, τ_1, τ_2) is called fuzzy bi-topological space and the members of τ_1, τ_2 are called fuzzy open sets.

3 N-Fuzzy Set and its Properties

Definition 3.1. N-Structure:(Negative-valued function): A mapping from $f : X \rightarrow [-1, 0]$ is the collection of functions from a set X to [-1,0],(we say that an elements of $f : X \rightarrow [-1, 0]$) is a negative-valued function from X to [-1,0] (briefly,N-function on X). (By an N-structure we mean an ordered pair (X,f)) of X and an N-function f on X.

Note: Through out this paper we apply N-Structure to fuzzy bi-topological space and pairwise, separation axioms.

Definition 3.2. Let **X** be a set and **I** be the interval $\mathbf{I} = [-1, 0]$ a N-fuzzy set **A** is characterized by a membership function μ_A which associate with each point $x \epsilon X$ its grade of membership $\mu_A(x) \epsilon [-1, 0]$.

Definition 3.3. N-fuzzy topological space: A N-fuzzy topological space on a set **X** is a family τ of N-fuzzy sets in **X** which satisfies the following condition:
i $-1, 0 \epsilon \tau$.
ii if $\mathbf{A_j} \epsilon \tau$ then $\forall j \epsilon J, \bigcap_{j \epsilon J} \mathbf{A_j} \epsilon \tau$.
iii if **A** and $\mathbf{B} \epsilon \tau$ then $\mathbf{A} \bigcup \mathbf{B} \epsilon \tau$.
The pair (X, τ) is called N-fuzzy topological space and the members of τ are called N-fuzzy open sets.

Definition 3.4. A N-fuzzy set λ is called N-fuzzy closed set if $\lambda^c \epsilon \tau$. i.e the complement of N-fuzzy set λ of τ belongs to τ.

4 N-Fuzzy BI-Topological Space and Separation Axioms

Definition 4.1. A N-fuzzy bi-topological space on a set **X** is a families τ_1, τ_2 of N-fuzzy sets in **X** which satisfies the following properties:
i $-1, 0 \epsilon \tau_1, \tau_2$.
ii if $\mathbf{A_j} \epsilon \tau_1, \tau_2$ then $\forall j \epsilon J, \bigcap_{j \epsilon J} \mathbf{A_j} \epsilon \tau_1, \tau_2$.
iii if **A** and $\mathbf{B} \epsilon \tau_1, \tau_2$ then $\mathbf{A} \bigcup \mathbf{B} \epsilon \tau_1, \tau_2$.
The pair (X, τ_1, τ_2) is called N-fuzzy bi-topological space, and the members of τ_1, τ_2 are called N-fuzzy open sets.

Definition 4.2. A N-fuzzy set λ is called N-fuzzy closed set if $\lambda^c \epsilon (\tau_1, \tau_2)$, the complement belongs to (τ_1, τ_2).

Definition 4.3. A N-fuzzy bi-topological space $(\mathbf{X}, \tau_1, \tau_2)$ is said to be pairwise $\mathbf{T}_1$ if every pair of distinct N-fuzzy points x and y in X, there exits τ_1 open set R and a τ_2 open set S such that $R(x) = 0$, $y \notin R$ and $x \notin S$ then $S(y) = 0$.

Definition 4.4. A N-fuzzy bi-topological space $(\mathbf{X}, \tau_1, \tau_2)$ is said to be pairwise N-fuzzy Hausdorff space if for each pair of distinct points x and y, there are τ_1 open set R and a τ_2 open set S such that $R(x) = 0$, $S(y) = 0$, and $R \cup S = -1$.

Definition 4.5. A N-fuzzy bi-topological space $(\mathbf{X}, \tau_1, \tau_2)$ is said to be pairwise N-fuzzy regular w.r.t.τ_2 iff $\mu \epsilon (-1, 0)$, $R \epsilon \tau_1^c, x \epsilon X$ and $\lambda < -1 - R(x)$ imply that there exists $S \epsilon \tau_2$ and $T \epsilon \tau_2$ with $\lambda < S(x) \subseteq S$ and $S \subseteq -1 - T$. (X, τ_1, τ_2) is called pairwise N-fuzzy regular if it is N-fuzzy regular w.r.t.τ_1 and τ_2 N-fuzzy regular w.r.t.τ_1.
The following theorem play a fundamental role in this work. The relation between compactness and closeness of a subject of a pairwise Hausdorff bi-topological space. The ordinary subset Y is regarded as a N-fuzzy subsets.

Theorem 4.6. If (X, τ_1, τ_2) is a pairwise Hausdorff N-fuzzy bi-topological space and Y is an ordinary-1 compact N-fuzzy set in X, then Y is closed.

Proof. To show that x_λ is not in Y implies x_λ is not an accumulation point of Y. Means that $x_\lambda \notin Y$ implies that $-1 > \lambda > y(x)$ and so, $x \notin Y$. Then $x \neq Y$ for all $y \epsilon Y$, by the pairwise Hausdorff property of (X, τ_1, τ_2) there exists a τ_1 open set R and a τ_2 open set S such that $R(x) = S(y)$ and $R \cup S = -1$. Thus, for $x_\lambda \notin Y$ and $y \epsilon Y$. $R(x) = (R \cup S) = 0$ with $R \cup S = -1$. Varying y over all points belonging to Y, the collection $\{Y \cup S\}$ is a 1^* shading of Y, then so, it reduces to a finite 1^* subshading say, $\{Y \cup S_{y1,} Y \cup S_{y2,} .... Y \cup S_{yn}\}$. Simply write $S = S_{y1}$. Since $R_x^{y1}(x) = 0$ for all $0 \leq i \leq n$, we have $R(x) = 0$ and $x_\lambda \epsilon R$. Now $R \cup S = (R_x^{y1} \cup ..... R_x^{yn}) \cup (S_{y1} \cap S_{y2} \cap ..... \cap S_{yn}) = -1$ for $y \epsilon Y$, there exists $Y \cup S_{y1}$ such that $Y \cup S_{y1}(y) = 0$ implies $Y \cup S(y) = 0 = Y(y)$ so $Y \cup S = Y$. Also, $Y \cup R = Y \cup R = -1$ implies that $Y(x) = -1$ or $R(x) = -1$. Therefore, $Y(x) + U(x) > 0$. Hence Y and R are not quasi-coincident, and therefore x_λ is not a τ_1 accumulation point of Y. This proves that Y is τ_1 closed.

Definition 4.7. In a pairwise N-fuzzy Hausdorff space (X, τ_1, τ_2) is τ_1-1^* compact subset is τ_2 closed.

Definition 4.8. With the help of two N-fuzzy topologies of a N-fuzzy bi-topological space a third N-fuzzy topology is defined on it. This topology is named as mixed N-fuzzy topology. We then relate separation axioms and there property relative to the mixed topology with pairwise separation axioms of the N-fuzzy bi-topological space.

Theorem 4.9. Let (X, τ_1, τ_2) be a N-fuzzy bi-topological space, $\{Y_\mu\}$ be a collection of ordinary subsets of X which are τ_2 1^* compact as N-fuzzy subsets.

Proof. Let $\tau = \{i_{\mu,} : Y \rightarrow X\}$ and (τ_ν) be the collection of N-fuzzy topologies on X such that $E_\mu : (Y_\mu, \tau_1) \rightarrow (X, \tau_\nu)$ are continuous, where (Y_μ, τ_1) means subspace topology on X, then they are continuous. That is, $\tau_1(\tau_2)$ is topology such that

(a): $\tau_1(\tau_2) \supseteq \tau_\nu$ for all ν

$E_\mu : (Y_\mu, \tau_1) \rightarrow (X, \tau_1)(\tau_2)$ are continuous.

(b): If $(\tau_o) \supseteq \tau_\nu$ for all ν s. t.

$E_\mu : (Y_\mu, \tau_1) \rightarrow (X, \tau_o$ are continuous then $\tau_o \supseteq \tau_1(\tau_2)$ the N-fuzzy topology $\tau_1(\tau_2)$ is called a mixed N-fuzzy topology on X. Clearly, $\tau_1 \epsilon \{\tau_\nu\}$ and therefore, $\tau_1 \subseteq \tau_1(\tau_2)$. Although we have used the symbol $\tau_1(\tau_2)$ for the mixed topology arising out of N-fuzzy topologies τ_1 and τ_2. This theorem is applied to the rest of this paper. The following theorem shows the relation between Hausdorff property of the mixed topology and the pairwise Hausdorff property of the bi-topological space.

Theorem 4.10. If (X, τ_1, τ_2) is a pairwise N-fuzzy Hausdorff space then the mixed N-fuzzy topology $\tau_1(\tau_2)$ is a N-fuzzy Hausdorff topology.

Proof. The pair (X, τ_1, τ_2) is a pairwise N-fuzzy Hausdorff space, $x, y \epsilon X$ and $x \neq y$ there exist $R \epsilon \tau_1$ and $S \epsilon \tau_2$,$R(x) = S(x)$ such that $R \cup S = -1$ To show that $(X, \tau_1, (\tau_2))$ is a N-fuzzy Hausdorff space, we claim that both R and S are $\tau_1, (\tau_2)$ open. Let, Y_μ be ordinary subsets of X which are 1^* compact w.r.t. N-fuzzy topology

τ_2, let $A = \{i_\mu : Y \to X\}$ and $\{\tau_\mu$ be the collection of inclusion mappings and N-fuzzy topologies on X such that $i_\mu : (X, \tau_1, (\tau_2)) \to (X, \tau_\nu)$ are continuous. For each $z\epsilon Y_\mu$,$i_\mu^{-1}(R)(z) = R((i_\mu(z) = max\{Y_\mu(z), R(z)\}) = (Y_\mu \cup R)(z)$. Therefore, $i_\mu^{-1}(R)$ is $(Y_{\mu,}\tau_1)$ open since τ_1 is coarser then $\tau_1(\tau_2)$ $\tau_1 - open$ set R is $\tau_1(\tau_2)$ open. In the similar way, $B = i_\mu^{-1}(s) = (Y_\mu \cup S)$ is open in $(Y_\mu \cup \tau_2)$ and therefore its complement in Y_μ, Y_μ B is closes Y_μ, τ_2 from the result of 1.7.9 show that Y_μ-B is $(Y_{\mu,}\tau_2)$-1* compact. Also $(Y_{\mu,}\tau_1\tau_2)$ inherits pairwise Hausdorff property from (X, τ_1, τ_2). From the theorem 4.2, Y_μ B is $(Y_{\mu,}\tau_1)$ closed and $i_\mu^{-1}(s) = Y_\mu \cup s = B$ is $(Y_{\mu,}\tau_1)$ open for every Y_μ. We say that $s\epsilon\tau_1, (\tau_2)$. Now let $\tau_o = \{s|i_\mu^{-1}(s)\epsilon(Y_{\mu,}\tau_1)\}$ open for every Y_μ we have $s\epsilon\tau_1(\tau_2)$. Let $\tau_o = \{s|i_\mu^{-1}(s)\epsilon(Y_{\mu,}\tau_1)\}$ for all Y_μ so τ_o is a topology on X such that $i_\mu((X, Y_\mu, \tau_1)) \to (X, \tau_o)$ are continuous So, τ_o is one of the members of $\{\tau_\nu\}$ and hence τ_o is $\subseteq \tau_1(\tau_2)$. Now $B = Y_\mu \cup S = i_\mu^{-1}(s)\epsilon(Y_{\mu,}\tau_1)$ for all Y_μ then so $s\epsilon\tau_o \subseteq \tau_1(\tau_2)$, and $B = Y_\mu \cup S = i_\mu^{-1}(s)(Y_{\mu,}\tau_1)$ for all Y_μ so,$s\epsilon\tau_o \subseteq \tau_1(\tau_2)$. This prove that $S\epsilon\tau_1(\tau_2)$. Thus $x, y\epsilon X$ and $x \neq y$ implies that there exists $R\epsilon\tau_1(\tau_2)$ and $S\epsilon\tau_1(\tau_2)$ with the $R(x) = S(x) = 0$ and $r \cup s = -1$ then, $(X, \tau_1, (\tau_2))$ is a N-fuzzy Hausdorff space.
Note: Some authors have studied fuzzy regularity in different ways. Some of them are equivalent and others are independent as shown by Dewam M.ali. The following lemmas are related with the theorems of pairwise fuzzy regular bi-topological spaces.

Theorem 4.11. If R is a closed relative to subspace topology on y induced from τ then, $R = Y \cup R_o$ R_o is τ closed.

Proof. Here the subspace topology $\tau_{1y} = \{Y \cup H|G\epsilon\tau\}$. R is τ_{1y}-closed. Then $1 - R$ is τ_{1y}-open. $-1 - R = Y \cup S$ where S is τ-open. therefore $-1 - R(y) = max\{Y(y), S(y)\} = max\{-1, S(y)\} = S(y)$ implies that $R(y) = -1 - S(y) = max\{Y(y), -1 - S(y)\} = (Y \cup S^c)(y)$. Then $R = (Y \cup S^c) = Y \cup R_o$. Since S is τ open, $S^c = R_o$ is τ closed. Here is the representation of open subsets of the supremum topology $S\tau_\mu$.

Theorem 4.12. An the supremum topology $(S\tau_\mu)$ there open set are the intersections of finite unions of different τ_μ open set.

Proof. Let τ be the collection of intersection of finite unions of elements $R\tau_{\mu,}$ that is $\tau = \{\cap(U_{i=1}^n R_i), R_i\epsilon R\tau_\mu$ it clear that τ is a N-fuzzy topology and (i) $\tau \supseteq \tau_\mu$ for all μ. (ii) If τ_o is N-fuzzy in $\tau_o \supseteq \tau_\mu$ for all μ Then so, $\tau_o \supseteq \tau \supseteq \tau_\mu$. Now let $\cap(U_{i=1}^n R_i)\epsilon\tau$,Where $R_i\epsilon R\tau_\mu$ then $\cap_\mu(U_{i=1}^n R_i) \notin \tau$,$\tau_o$ for some $i = i_o$ implies that $R \notin \tau_\mu$ for all μ which contradicts that $R\epsilon R\tau_\mu$. Hence we have $\cap_n(U_{i=1}^n R_i)\epsilon\tau_o$ and so, $\tau \subseteq \tau_o$ and $\tau = l.u.b.\tau_o$.

Theorem 4.13. A function $f : (X, \tau) \to (Y, \tau_\mu)$ is continuous for all if $f : (X, \tau) \to (Y, S\tau_\mu)$ is continuous.

Proof. (i) Let the function $f : (X, \tau) \to (Y, \tau_\mu)$ is continuous for all μ and $R\epsilon S\tau_\mu$, then $R = \cap(U_{i=1}^n R_i)$. Now the inverse function is $f^{-1} : (\cap(U_{i=1}^n R_i)) = \cap(U_{i=1}^n f^{-1}R_i)\epsilon\tau$. Hence the function f is continuous from X, τ to (Y, τ_μ) for all μ.
(ii) Let the function $f : (X, \tau) \to (Y, S\tau_\mu)$ be continuous in every R in τ_μ is in $S\tau_\mu$ and we conclude that the function $f^{-1}(R)\epsilon\tau$. And hence the function $f : (X, \tau) \to (Y, \tau_\mu)$ is continuous.

Theorem 4.14. Let (X, τ_1, τ_2) be pairwise N-fuzzy Hausdorff and pairwise N-fuzzy regular space,Y_k be a τ_2-1^* compact ordinary subsets of X means N-fuzzy subsets and $\tau_1(\tau_2)_{1yk}$ is N-fuzzy regular for each Y_k.

Proof. Let us consider $x \epsilon Y_k$ and $R \epsilon (\tau_1(\tau_2)_{1yk})^c$ and $\mu \epsilon (-1, 0)$,such that $\mu < -1 - R(x)$. Then $R = Y_k \cup R_1$ where R_1 is $\tau_1(\tau_2)$ closed. From the continuity of $i_{yk} : (Yk, \tau_1) \rightarrow (X, \tau_1, \tau_2)$, and $i_{yk}^{-1}(R_1)\tau_{1\ 1yk}$ is closed, that is $R = Y_k \cup R_1$ is $\tau_{1\ 1yk}$-closed. Since Y_k is τ_1-1^* compact set in the pairwise Hausdorff space (X, τ_1, τ_2) and Y_k is τ_1 closed. So by lemma above, R is τ_1 closed. The pair (X, τ_1, τ_2) is pairwise N-fuzzy regular there exists $S \epsilon \tau_1$ and $U \epsilon \tau_2$ with the condition $\mu < S(x)$,$R \subseteq U$ and $S \subseteq -1 - U$ also $S \epsilon \tau_1 \subseteq \tau_1(\tau_2)$ implies that $S \epsilon \tau_1 \subseteq \tau_1(\tau_2)$. It can be shown that $U \epsilon \tau_1(\tau_2)$ that $(R \subseteq Y_k \cup U)$, $\mu < (Y_k \cup S)(x)$ and $Y_k \cup S \subseteq -1 - Y_k \cup U$. Therefore (Y_k, τ_1, τ_2), then i_{yk}^{-1} is N-fuzzy regular.

Theorem 4.15. Let (X, τ_1, τ_2) be a pairwise N-fuzzy Hausdorff space in which every τ_2-1^* compact sets are τ_2-1^* compact. Let $Y_k \tau_2$-1^* compact ordinary sets and $\tau_1(\tau_2)$ be the mixed topology on X. If (X, τ_1, τ_2) is a N-fuzzy regular space then $(Y_k, \tau_1 i_{yk}^{-1}, \tau_2 i_{yk}^{-1})$ is pairwise N-fuzzy regular for each Y_k.

Proof. Let us consider $x \epsilon Y_k$ and R is a $\tau_1 i_{yk}^{-1}$ closed set and $R = Y_k \cup R_o$ where R_o is τ_1 closed and Y_k is τ_1 closed. Then R is $\tau_1(\tau_2)$ closed. Since (X, τ_1, τ_2) is N-fuzzy regular, for $\mu \epsilon (-1, 0)$, $R \epsilon (\tau_1(\tau_2))^c$, $x \epsilon X$ and $\mu < -1 - R(x)$ then there exist $(S, U \epsilon \tau_1, \tau_2)$ with the condition $\mu < -1 - S(x)$, $R \subseteq U$ and $S \subseteq -1 - U$. Now $i_{yk}^{-1}(s) = Y_k \cup S$ is $\tau_{1\ 1yk}$-open and so $[-1 - Y_k \cup S]$ is $\tau_{1\ 1yk}$ closed. Since Y_k is τ_1-1^* compact, $[-1 - (Y_k \cup S)]$ is τ_{21yk} closed and hence $Y_k \cup S$ is τ_{21yk} open and $R \subseteq Y_k \cup U$. Now $i_{yk}^{-1}(U) = Y_k \cup U$ is $\tau_{2\ 1yk}$ open. Therefore $Y_k \cup S \epsilon \tau_{2\ 1yk}$ and $Y_k \cup U \epsilon \tau_{2\ 1yk}$, $Y_k \cup S \subseteq -1 - (Y_k \cup U)$. Hence, $(Y_k, \tau_{1\ 1yk}, \tau_{2\ 1yk})\tau_{2\ 1yk}$ is regular. So, $(Y_k, \tau_{1\ 1yk}, \tau_{2\ 1yk})$ is pairwise N-fuzzy regular for each Y_k. Which completes the theorem.

5 Conclusion

The results in this paper give us the structural properties of a N-Fuzzy bi-topological space and pairwise separation axioms which is the generalization of natural separation axioms. Many more results and its structural properties and applications can be expected.

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Generalized Topological Notions by Operators

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Abstract — In this paper, it is introduced the notion of r-fuzzy β-$T_i, i = 0, 1, 2$ separation axioms related to a fuzzy operator β on the initial set X which is a generalization of previous fuzzy separation axioms. An r-fuzzy α-connectedness related to a fuzzy operator α on the set X is introduced which is a generalization of many types of r-fuzzy connectedness. An r-fuzzy α-compactness related to a fuzzy operator α on the set X is introduced which is a generalization of many types of fuzzy compactness.

Keywords — *Fuzzy operators, fuzzy separation axioms, fuzzy compactness, fuzzy connectedness.*

1 Introduction

It is a way to use fuzzy operators α, β on the initial set X and to use fuzzy operators θ, δ on the set Y giving generalizations of many notions and results in fuzzy topological spaces. r-fuzzy β-$T_i,\ i = 0, 1, 2$ separation axioms of the set X is a new type of fuzzy separation axioms related with a fuzzy operator β on X. It is proved that the image of r-fuzzy β-$T_i,\ i = 0, 1, 2$ is r-fuzzy δ-$T_i,\ i = 0, 1, 2$, and also the preimage of r-fuzzy δ-$T_i,\ i = 0, 1, 2$ is r-fuzzy β-$T_i,\ i = 0, 1, 2$. r-fuzzy α-connectedness is introduced related with the fuzzy operator α on X giving a generalization of many of fuzzy connectedness notions. It is proved that the image of r-fuzzy α-connected is r-fuzzy θ-connected, and some particular cases are included. r-fuzzy α-compactness is introduced using the fuzzy operator α on X giving a generalization of many of fuzzy compactness notions. It is proved that the image of r-fuzzy r-fuzzy compact is r-fuzzy θ-compact, and many special cases are deduced.

2 Preliminaries

Throughout the paper, X refers to an initial universe, I^X is the set of all fuzzy sets on X (where $I = [0, 1], I_0 = (0, 1]$, $\lambda^c(x) = 1 - \lambda(x)\ \forall x \in X$ and for all $t \in I$, $\overline{t}(x) = t\ \forall x \in X$).

(X, τ) is a fuzzy topological space ([14]), if $\tau : I^X \to I$ satisfies the following conditions:

(O1) $\tau(\overline{0}) = \tau(\overline{1}) = 1$,

(O2) $\tau(\lambda_1 \wedge \lambda_2) \geq \tau(\lambda_1) \wedge \tau(\lambda_2)$ for all $\lambda_1, \lambda_2 \in I^X$,

(O3) $\tau(\bigvee_{j \in J} \lambda_j) \geq \bigwedge_{j \in J} \tau(\lambda_j)$ for all $\{\lambda_j\}_{j \in J} \subseteq I^X$.

By the concept of a fuzzy operator on a set X is meant a map $\gamma : I^X \times I_0 \to I^X$. Assume with respect to a fuzzy topology in Šostak sense defined on X, we have

$$\mathrm{int}_\tau(\mu, r) \leq \gamma(\mu, r) \leq \mathrm{cl}_\tau(\mu, r) \ \ \forall \mu \in I^X, \ \forall r \in I_0,$$

where $\mathrm{int}_\tau, \mathrm{cl}_\tau : I^X \times I_0 \to I^X$ are defined in Šostak sense for any $\mu \in I^X$ and each grade $r \in I_0$ as follows:

$$\mathrm{int}_\tau(\mu, r) = \bigvee \{\eta \in I^X : \eta \leq \mu, \ \tau(\eta) \geq r\}$$

and

$$\mathrm{cl}_\tau(\mu, r) = \bigwedge \{\eta \in I^X : \eta \geq \mu, \ \tau(\eta^c) \geq r\}$$

Let (X, τ_1) and (Y, τ_2) be two fuzzy topological spaces, α and β are fuzzy operators on X, θ and δ are fuzzy operators on Y, respectively. This type of maps α or β is called an expansion on X or a fuzzy operator on (X, τ_1), and the map θ or δ is called an expansion on Y or a fuzzy operator on (Y, τ_2) and let us fix that:

(1) β is a fuzzy operator on X such that $\beta(\mu, r) \leq \mu \ \ \forall \mu \in I^X, \ \forall r \in I_0$.

(2) α is a fuzzy operator on X such that $\alpha(\mu, r) \geq \mu \ \ \forall \mu \in I^X, \ \forall r \in I_0$.

As a special case of fuzzy operators, by the identity fuzzy operator id_X on a set X we mean that $id_X : I^X \times I_0 \to I^X$ so that $id_X(\nu, r) = \nu \ \ \forall \nu \in I^X, \forall r \in I_0$.

Recall that a fuzzy ideal $\mathcal{I}$ on X ([13]) is a map $\mathcal{I} : I^X \to I$ that satisfies the following conditions:

(1) $\lambda \leq \mu \ \Rightarrow \ \mathcal{I}(\lambda) \geq \mathcal{I}(\mu)$,

(2) $\mathcal{I}(\lambda \vee \mu) \geq \mathcal{I}(\lambda) \wedge \mathcal{I}(\mu)$.

Also, $\mathcal{I}$ is called proper if $\mathcal{I}(\overline{1}) = 0$ and there exists $\mu \in I^X$ such that $\mathcal{I}(\mu) > 0$. Define the fuzzy ideal $\mathcal{I}^\circ$ by

$$\mathcal{I}^\circ(\mu) = \begin{cases} 1 & \text{at } \mu = \overline{0}, \\ 0 & \text{otherwise} \end{cases}$$

Let us define the fuzzy difference between two fuzzy sets as follows:

$$(\lambda \,\overline{\wedge}\, \mu) = \begin{cases} \overline{0} & \text{if } \ \lambda \leq \mu, \\ \lambda \wedge \mu^c & \text{if } \ \text{otherwise.} \end{cases}$$

Definition 2.1. [4]

(1) A mapping $f : (X, \tau_1) \to (Y, \tau_2)$ is said to be fuzzy $(\alpha, \beta, \theta, \delta, \mathcal{I})$-continuous if for every $\mu \in I^Y$, any fuzzy ideal $\mathcal{I}$ on X,

$$\mathcal{I}[\alpha(f^{-1}(\delta(\mu, r)), r) \bar{\wedge} \beta(f^{-1}(\theta(\mu, r)), r)] \geq \tau_2(\mu); \ r \in I_0.$$

We can see that the above definition generalizes the concept of fuzzy continuity ([14]) when we choose $\alpha =$ identity operator, $\beta =$ interior operator, $\delta =$ identity operator, $\theta =$ identity operator and $\mathcal{I} = \mathcal{I}^\circ$.

(2) A mapping $f : (X, \tau_1) \to (Y, \tau_2)$ is said to be fuzzy $(\alpha, \beta, \theta, \delta, \mathcal{I}^*)$-open if for every $\lambda \in I^X$, any fuzzy ideal $\mathcal{I}^*$ on Y,

$$\mathcal{I}^*[\theta(f(\beta(\lambda, r)), r) \bar{\wedge} \delta(f(\alpha(\lambda, r)), r)] \geq \tau(\lambda); \ r \in I_0.$$

We can see that the above definition generalizes the concept of fuzzy openness ([14]) when we choose $\alpha =$ identity operator, $\beta =$ interior operator, $\delta =$ interior operator, $\theta =$ identity operator and $\mathcal{I}^* = \mathcal{I}^\circ$.

3 r-Fuzzy β-T_i Separation Axioms

Here, we introduce and study fuzzy separation axioms related with a fuzzy operator β on the initial set X.

Definition 3.1.

(1) A set X is called r-fuzzy β-T_0 if for all $x \neq y$ in X, there exists $\lambda \in I^X, r \in I_0$ with $t \leq \beta(\lambda, r)(x)$; $t \in I_0$ such that $t > \lambda(y)$ or there exists $\mu \in I^X, r \in I_0$ with $s \leq \beta(\mu, r)(y)$; $s \in I_0$ such that $s > \mu(x)$.

(2) A set X is called r-fuzzy β-T_1 if for all $x \neq y$ in X, there exist $\lambda, \mu \in I^X, r \in I_0$ with $t \leq \beta(\lambda, r)(x)$, $s \leq \beta(\mu, r)(y)$; $t, s \in I_0$ such that $t > \lambda(y)$, $s > \mu(x)$.

(3) A set X is called r-fuzzy β-T_2 if for all $x \neq y$ in X, there exist $\lambda, \mu \in I^X, r \in I_0$ with $t \leq \beta(\lambda, r)(x)$, $s \leq \beta(\mu, r)(y)$; $t, s \in I_0$ such that $(t \wedge s) > \sup(\lambda \wedge \mu)$.

Proposition 3.2. Every r-fuzzy β-T_i set X is an r-fuzzy β-T_{i-1}, $i = 1, 2$.

Proof. r-fuzzy β-$T_2 \Rightarrow r$-fuzzy β-T_1: Suppose that X is an r-fuzzy β-T_2 but it is not r-fuzzy β-T_1. Then, for all $x \neq y$ in X and for all $\lambda \in I^X$ with $t \leq \beta(\lambda, r)(x), r \in I_0$, suppose that $\lambda(y) \geq t; t \in I_0$. Now, for $\mu \in I^X$ with $s \leq \beta(\mu, r)(y) \leq \mu(y)$; $s \in I_0$, we get that

$$\sup(\lambda \wedge \mu) \geq (\lambda \wedge \mu)(y) \geq (t \wedge s),$$

which means a contradiction to X is r-fuzzy β-T_2. Hence, X is an r-fuzzy β-T_1.

r-fuzzy β-$T_1 \Rightarrow r$-fuzzy β-T_0: Direct.

Recall that: a fuzzy operator θ is finer than a fuzzy operator β on a set X, denoted by $\beta \sqsubseteq \theta$, if $\beta(\nu, r) \leq \theta(\nu, r)$ $\forall \nu \in I^X$, $\forall r \in I_0$.

Proposition 3.3. Let X be an r-fuzzy β-T_i, $i = 0, 1, 2$, and θ a fuzzy operator on X finer than β. Then X is also r-fuzzy θ-T_i space, $i = 0, 1, 2$.

Proof. For all the axioms r-fuzzy β-$T_i, i = 0, 1, 2$, the proof comes from that $\beta(\nu, r) \leq \theta(\nu, r)\ \forall \nu \in I^X,\ \forall r \in I_0$.

Example 3.4.

(1) Let $X = \{x, y\}$, $r \in I_0$ and

$$\beta(\nu, r) = \begin{cases} \nu & \text{at } \nu = \overline{0}, \overline{1} \\ x_1 & \text{at } x_1 \leq \nu < \overline{1}, \\ \overline{0} & \text{otherwise.} \end{cases}$$

Then, we get $\lambda = x_1 \in I^X, t = \frac{1}{4} \in I_0$ with $\beta(\lambda, r)\,(x) = x_1(x) = 1 \geq t$ and $\lambda(y) = x_1(y) = 0 < t$. Hence, the set X is an r-fuzzy β-T_0 set and it is neither r-fuzzy β-T_1 nor r-fuzzy β-T_2.

(2) Let $X = \{x, y\}$, $r \in I_0$ and

$$\beta(\nu, r) = \begin{cases} \nu & \text{at } \nu = \overline{0}, \overline{1} \\ x_1 & \text{at } x_1 \leq \nu < \overline{1}, \\ y_1 & \text{at } y_1 \leq \nu < \overline{1}, \\ \overline{0} & \text{otherwise.} \end{cases}$$

Then, we get $\lambda = y_1 \in I^X, t = \frac{1}{5} \in I_0$ with $\beta(\lambda, r)\,(y) = y_1(y) = 1 \geq t$ and $\lambda(x) = y_1(x) = 0 < t$. Similarly, we get $\mu = x_1 \in I^X, s = \frac{1}{3} \in I_0$ with $\beta(\mu, r)\,(x) = x_1(x) = 1 \geq s$ and $\mu(y) = x_1(y) = 0 < s$. Hence, the set X is an r-fuzzy β-T_1 set.

For $\lambda = x_1 \vee y_{\frac{1}{2}},\ \mu = y_1 \vee x_{\frac{1}{2}} \in I^X,\ t, s > \frac{1}{2} \in I_0$, we get that

$$\beta(\lambda, r)\,(x) = x_1(x) = 1 \geq t \quad \text{and} \quad \beta(\mu, r)(y) = y_1(y) = 1 \geq s$$

such that

$$(t \wedge s) > \frac{1}{2} = \sup(x_{\frac{1}{2}} \vee y_{\frac{1}{2}}) = \sup(\lambda \wedge \mu).$$

Hence, the set X is an r-fuzzy β-T_2 set.

(3) Let $X = \{x, y\}$, $r \in I_0$ and

$$\beta(\nu, r) = \begin{cases} \nu & \text{at} \quad \nu = \overline{0}, \overline{1} \\ \overline{0.2} & \text{at} \quad \overline{0.2} \leq \nu,\ \nu < x_1 \vee y_{0.2},\ \nu < x_{0.2} \vee y_1, \\ x_1 \vee y_{0.2} & \text{at} \quad x_1 \vee y_{0.2} \leq \nu < \overline{1}, \\ x_{0.2} \vee y_1 & \text{at} \quad x_{0.2} \vee y_1 \leq \nu < \overline{1} \\ \overline{0} & \text{otherwise.} \end{cases}$$

Then, there exist $\lambda = x_1 \vee y_{0.3},\ \mu = x_{0.3} \vee y_1$ such that $\beta(\lambda, r)\,(x) = 1 \geq t > 0.3 = \lambda(y)$ for $t \in I_0$ and $\beta(\mu, r)\,(y) = 1 \geq s > 0.3 = \mu(x)$ for $s \in I_0$, and then X is an r-fuzzy β-T_1 set.

Now, we study all possible fuzzy sets in I^X:

Then

(a) For any $\lambda = x_1 \vee y_p, \mu = x_1 \vee y_q, p, q \geq 0.2$, we get that: $\beta(\lambda, r)(x) = 1 \geq t,\ \beta(\mu, r)(y) = 0.2 \geq s;\ t, s \in I_0$ but $(t \wedge s) \leq 0.2 \leq \sup(\lambda \wedge \mu),\ p, q \geq 0.2$.

(b) For any $\lambda = x_p \vee y_1$ or $x_1 \vee y_p$, $\mu = x_q \vee y_1$ or $x_1 \vee y_q,\ p, q < 0.2$, we get that: $\beta(\lambda, r)(x) = \overline{0}(x) = 0 = \overline{0}(y) = \beta(\mu, r)(y)$.

(c) For any $\lambda = x_p, \mu = x_q$ or $\lambda = y_p, \mu = y_q$ or $\lambda = x_p, \mu = y_q,\ p, q \in I$, we get that: $\beta(\lambda, r)(x) = \overline{0}(x) = 0 = \overline{0}(y) = \beta(\mu, r)(y)$.

Hence, for every $\lambda, \mu \in I^X$ with $\beta(\lambda, r)(x) \geq t$ and $\beta(\mu, r)(y) \geq s;\ t, s \in I_0$, we have $(t \wedge s) \leq \sup(\lambda \wedge \mu)$, and thus X is not an r-fuzzy β-T_2 set.

Proposition 3.5. Let $f : X \to Y$ be an injective mapping. Assume that δ is a fuzzy operator on Y such that

$$f^{-1}(\delta(\lambda, r)) \leq \beta(f^{-1}(\lambda), r) \quad \forall \lambda \in I^Y,\ \forall r \in I_0.$$

Then, Y is an r-fuzzy δ-T_i implies that X is an r-fuzzy β-T_i, $i = 0, 1, 2$.

Proof. Since $x \neq y$ in X implies that $f(x) \neq f(y)$ in Y and Y is an r-fuzzy δ-T_1, then there exists $\lambda \in I^Y$ with $t \leq \delta(\lambda, r)(f(x)); t \in I_0$ so that $t > \lambda(f(y))$, that is,

$$t \leq [f^{-1}(\delta(\lambda, r))](x) \leq [\beta(f^{-1}(\lambda), r)](x) \text{ and } t > (f^{-1}(\lambda))(y),$$

which means that there exists $\mu = f^{-1}(\lambda) \in I^X$ with $t \leq \beta(\mu, r)(x);\ t \in I_0$ so that $t > \mu(y)$. Hence, X is an r-fuzzy β-T_1, and consequently X is an r-fuzzy β-T_0.

Now, for $x \neq y$ in X implies that $f(x) \neq f(y)$ in Y and Y is an r-fuzzy δ-T_2, then there exist $\lambda, \mu \in I^Y$ with $t \leq \delta(\lambda, r)(f(x)), s \leq \delta(\mu, r)(f(y));\ s, t \in I_0$ so that $(t \wedge s) > \sup(\lambda \wedge \mu)$.

Since $\sup(\lambda \wedge \mu) \geq \sup(f^{-1}(\lambda) \wedge f^{-1}(\mu))$, then $(t \wedge s) > \sup(f^{-1}(\lambda) \wedge f^{-1}(\mu))$. Also,

$$t \leq [f^{-1}(\delta(\lambda, r))](x) \leq [\beta(f^{-1}(\lambda), r)](x)$$

and

$$s \leq [f^{-1}(\delta(\mu, r))](y) \leq [\beta(f^{-1}(\mu), r)](y).$$

Hence, there exist $\nu = f^{-1}(\lambda), \rho = f^{-1}(\mu) \in I^X$ with $t \leq \beta(\nu, r)(x), s \leq \beta(\rho, r)(y);\ s, t \in I_0$ so that $(t \wedge s) > \sup(\nu \wedge \rho)$, and thus X is an r-fuzzy β-T_2.

Proposition 3.6. Let $f : X \to Y$ be a surjective mapping. Assume that δ is a fuzzy operator on Y such that

$$f(\beta(\lambda, r)) \leq \delta(f(\lambda), r) \quad \forall \lambda \in I^X,\ \forall r \in I_0.$$

Then, X is an r-fuzzy β-T_i implies that Y is an r-fuzzy δ-T_i, $i = 0, 1, 2$.

Proof. Since $p \neq q$ in Y implies that $x \neq y$ where $x = f^{-1}(p), y = f^{-1}(q)$ in X, and X is an r-fuzzy β-T_1, then there exists $\lambda \in I^X$ with $t \leq \beta(\lambda, r)(f^{-1}(p)); t \in I_0$ so that $t > \lambda(f^{-1}(q))$, that is,

$$t \leq [f(\beta(\lambda, r))](p) \leq [\delta(f(\lambda), r)](p) \text{ and } t > (f(\lambda))(q),$$

which means that there exists $\mu = f(\lambda) \in I^Y$ with $t \leq \delta(\mu, r)(p);\ t \in I_0$ so that $t > \mu(q)$. Hence, Y is an r-fuzzy δ-T_1, and consequently Y is an r-fuzzy δ-T_0.

Now, for $p \neq q$ in Y implies that $f^{-1}(p) \neq f^{-1}(q)$ in X and X is an r-fuzzy β-T_2, then there exist $\lambda, \mu \in I^X$ with $t \leq \beta(\lambda, r)(f^{-1}(p)), s \leq \beta(\mu, r)(f^{-1}(q));\ s, t \in I_0$ so that $(t \wedge s) > \sup(\lambda \wedge \mu)$.

Since $\sup(\lambda \wedge \mu)\ \geq \sup(f(\lambda) \wedge f(\mu))$, then $(t \wedge s) > \sup(f(\lambda) \wedge f(\mu))$. Also,

$$t \ \leq\ [f(\beta(\lambda, r))](p)\ \leq\ [\delta(f(\lambda), r)](p) \ \text{ and } \ s\ \leq\ [f(\beta(\mu, r))](q)\ \leq\ [\delta(f(\mu), r)](q).$$

Hence, there exist $\nu = f(\lambda), \rho = f(\mu) \in I^Y$ with $t \leq \delta(\nu, r)(p), s \leq \delta(\rho, r)(q);\ s, t \in I_0$ so that $(t \wedge s) > \sup(\nu \wedge \rho)$, and thus Y is an r-fuzzy δ-T_2.

Remark 3.7.

(1) For a fuzzy topological space (X, τ), by choosing $\beta =$ fuzzy interior operator, you can deduce the equivalence between the graded fuzzy separation axioms (t, s)-T_i, $i = 0, 1, 2;\ t, s \in I_0$ introduced in [5, 6] and the axioms r-fuzzy β-$T_i, i = 0, 1, 2$.

(2) For two fuzzy topological spaces (X, τ), (Y, σ), and $f : X \to Y$ a mapping, by choosing $\beta =$ fuzzy interior operator, we get that (X, τ) is (t, s)-T_i, $i = 0, 1, 2;\ t, s \in I_0$ whenever (Y, σ) is (t, s)-T_i, $i = 0, 1, 2;\ t, s \in I_0$ and f is injective fuzzy continuous (when $\delta =$ fuzzy interior operator in Proposition 3.5) as shown in [5]. This is equivalent to f is injective and $\alpha =$ identity operator, $\beta =$ interior operator, $\delta =$ interior operator, $\theta =$ identity operator and $\mathcal{I} = \mathcal{I}^\circ$ in Definition 2.1 (1).

(3) For two fuzzy topological spaces (X, τ), (Y, σ), and $f : X \to Y$ a mapping, by choosing $\delta =$ fuzzy interior operator, we get that (Y, σ) is (t, s)-T_i, $i = 0, 1, 2;\ t, s \in I_0$ whenever (X, τ) is (t, s)-T_i, $i = 0, 1, 2;\ t, s \in I_0$ and f is surjective fuzzy open (when $\beta =$ fuzzy interior operator in Proposition 3.6) as shown in [5]. This is equivalent to f is surjective and $\alpha =$ identity operator, $\beta =$ interior operator, $\delta =$ interior operator, $\theta =$ identity operator and $\mathcal{I} = \mathcal{I}^\circ$ in Definition 2.1 (2).

4 r-Fuzzy α-Connected Spaces

Here, we introduce the r-fuzzy connectedness of a space X relative to a fuzzy operator α. Assume (with respect to any fuzzy topology τ defined on X) that:

$$\lambda\ \leq\ \alpha(\lambda, r) \leq\ \mathrm{cl}_\tau(\lambda, r) \quad \forall \lambda \in I^X;\ r \in I_0.$$

Also, assume that α is a monotone operator, that is,

$$\mu \leq \nu \ \text{ implies } \ \alpha(\mu, r) \leq \alpha(\nu, r) \ \ \forall \mu, \nu \in I^X;\ r \in I_0.$$

Definition 4.1. Let X be a non-empty set. Then,

(1) the fuzzy sets $\lambda, \mu \in I^X$ are called r-fuzzy α-separated sets if

$$\alpha(\lambda, r) \wedge \mu\ =\ \lambda \wedge \alpha(\mu, r) = \overline{0};\ r \in I_0.$$

(2) X is called r-fuzzy α-connected space if it could not be found $\lambda, \mu \in I^X$, $\lambda \neq \overline{0}, \mu \neq \overline{0}$ such that λ, μ are r-fuzzy α-separated and $\lambda \vee \mu = \overline{1}$. That is, there are no r-fuzzy α-separated sets $\lambda, \mu \in I^X$ except $\lambda = \overline{0}$ or $\mu = \overline{0}$.

Definition 4.2. Let $\lambda, \mu \in I^X$, $\lambda \neq \overline{0}, \mu \neq \overline{0}$ such that:

(1) λ, μ are r-fuzzy α-separated and $\lambda \vee \mu = \overline{1}$. Then X is called r-fuzzy α-disconnected space.

(2) λ, μ are r-fuzzy α-separated and $\lambda \vee \mu = \nu$. Then ν is called r-fuzzy α-disconnected fuzzy set in I^X.

(3) λ, μ are r-fuzzy α-separated and $\lambda \vee \mu = \chi_A$, $A \subseteq X$. Then A is called r-fuzzy α-disconnected crisp set in I^X.

Remark 4.3. For a fuzzy topological space (X, τ)

(1) Taking α = fuzzy closure operator on (X, τ), then we have the r-fuzzy connectedness as given in [7].

(2) Taking α = fuzzy preclosure operator on (X, τ), then we have the r-fuzzy preconnectedness as given in [2].

(3) Taking α = fuzzy strongly semi-closure operator on (X, τ), then we have the r-fuzzy strongly connectedness as given in [10].

(4) Taking α = fuzzy semi-closure operator on (X, τ), then we have the 1-type of r-fuzzy strongly connectedness as given in [10].

(5) Taking α = fuzzy semi-preclosure operator on (X, τ), then we have the r-fuzzy semi-preconnectedness as given in [2].

(6) Taking α = fuzzy strongly preclosure operator on (X, τ), then we have the r-fuzzy strongly preconnectedness as given in [2].

Example 4.4. Let $X = \{x, y\}$, $r \in I_0$,

$$\alpha(\nu, r) = \begin{cases} \nu & \text{at } \nu = \overline{0}, \overline{1} \\ x_1 & \text{at } \overline{0} < \nu \leq x_1, \\ y_1 & \text{at } \overline{0} < \nu \leq y_1, \\ \overline{1} & \text{otherwise,} \end{cases}$$

Now, at $\lambda \neq \overline{0}, \lambda \leq x_1, \mu \neq \overline{0}, \mu \leq y_1, r \leq \frac{1}{4}$, then we have $\alpha(\lambda, r) \wedge \mu = x_1 \wedge \mu = \overline{0}$ and $\alpha(\mu, r) \wedge \lambda = y_1 \wedge \lambda = \overline{0}$, and thus λ, μ are r-fuzzy α-separated sets for $\lambda \neq \overline{0}, \lambda \leq x_1, \mu \neq \overline{0}, \mu \leq y_1$.

At $\lambda = x_1$ and $\mu = y_1$, we get r-fuzzy α-separated sets with $\overline{1} = \lambda \vee \mu$. Hence, X is an r-fuzzy α-disconnected space.

Proposition 4.5. Let (X, τ) be a fuzzy topological space. Then the following are equivalent.

(1) (X, τ) is r-fuzzy α-connected.

(2) $\lambda \wedge \mu = \overline{0}$, $\tau(\lambda) \geq r, \tau(\mu) \geq r$; $r \in I_0$, and $\overline{1} = \lambda \vee \mu$ imply $\lambda = \overline{0}$ or $\mu = \overline{0}$.

(3) $\lambda \wedge \mu = \overline{0}$, $\tau_c(\lambda) \geq r, \tau_c(\mu) \geq r$; $r \in I_0$, and $\overline{1} = \lambda \vee \mu$ imply $\lambda = \overline{0}$ or $\mu = \overline{0}$.

Proof. (1) $\Rightarrow$ (2): Let $\lambda, \mu \in I^X$ with $\tau(\lambda) \geq r, \tau(\mu) \geq r$; $r \in I_0$ such that $\lambda \wedge \mu = \overline{0}$ and $\overline{1} = \lambda \vee \mu$. Then, $\lambda = \mu^c$ and $\mu = \lambda^c$, and then

$$\overline{0} = \lambda \wedge \mu = \mu^c \wedge \lambda^c = \mathrm{cl}_\tau(\mu^c, r) \wedge \lambda^c \;\geq\; \alpha(\mu^c, r) \wedge \lambda^c \quad \text{and}$$
$$\overline{0} = \lambda \wedge \mu = \mu^c \wedge \lambda^c = \mu^c \wedge \mathrm{cl}_\tau(\lambda^c, r) \;\geq\; \mu^c \wedge \alpha(\lambda^c, r);\; r \in I_0,$$

which means that λ^c, μ^c are fuzzy α-separated so that $\lambda^c \vee \mu^c = \mu \vee \lambda = \overline{1}$. But (X, τ) is r-fuzzy α-connected implies that $\lambda^c = \overline{0}$ or $\mu^c = \overline{0}$, and thus $\lambda = \overline{0}$ or $\mu = \overline{0}$.

(2) $\Rightarrow$ (3): Clear.

(3) $\Rightarrow$ (1): Let $\lambda, \mu \in I^X$, $\lambda \neq \overline{0}$, $\mu \neq \overline{0}$ such that $\lambda \vee \mu = \overline{1}$. Taking $\nu = \mathrm{cl}_\tau(\lambda, r)$ and $\rho = \mathrm{cl}_\tau(\mu, r)$; $r \in I_0$, then $\nu \vee \rho = \overline{1}$ and $\tau_c(\nu) \geq r, \tau_c(\rho) \geq r$; $r \in I_0$.

Now, suppose that (3) is not satisfied. That is, $\nu \neq \overline{0}$, $\rho \neq \overline{0}$ and $\nu \wedge \rho = \overline{0}$. Then,

$$\alpha(\lambda, r) \wedge \mu \;\leq\; \mathrm{cl}_\tau(\lambda, r) \wedge \mathrm{cl}_\tau(\mu, r) \;=\; \nu \wedge \rho \;=\; \overline{0} \quad \text{and}$$
$$\alpha(\mu, r) \wedge \lambda \;\leq\; \mathrm{cl}_\tau(\lambda, r) \wedge \mathrm{cl}_\tau(\mu, r) \;=\; \nu \wedge \rho \;=\; \overline{0},$$

which means that λ, μ are r-fuzzy α-separated sets, $\lambda \neq \overline{0}$, $\mu \neq \overline{0}$ with $\lambda \vee \mu = \overline{1}$. Hence, (X, τ) is not r-fuzzy α-connected space.

Proposition 4.6. Let X be a non-empty set and $\lambda \in I^X$. Then the following are equivalent.

(1) λ is r-fuzzy α-connected.

(2) If μ, ρ are r-fuzzy α-separated sets with $\lambda \leq \mu \vee \rho$, then $\lambda \wedge \mu = \overline{0}$ or $\lambda \wedge \rho = \overline{0}$.

(3) If μ, ρ are r-fuzzy α-separated sets with $\lambda \leq \mu \vee \rho$, then $\lambda \leq \mu$ or $\lambda \leq \rho$.

Proof. (1) $\Rightarrow$ (2): Let μ, ρ be r-fuzzy α-separated with $\lambda \leq \mu \vee \rho$, that is, $\alpha(\mu, r) \wedge \rho = \alpha(\rho, r) \wedge \mu = \overline{0}$; $r \in I_0$ so that $\lambda \leq \mu \vee \rho$. Then, from that α is a monotone fuzzy operator, we get that

$$\alpha((\lambda \wedge \mu), r) \wedge (\lambda \wedge \rho) \leq \alpha(\lambda, r) \wedge \alpha((\mu, r) \wedge (\lambda \wedge \rho) = (\alpha(\lambda, r) \wedge \lambda) \wedge (\alpha((\mu, r) \wedge \rho) = \lambda \wedge \overline{0} = \overline{0}$$

and

$$\alpha((\lambda \wedge \rho), r) \wedge (\lambda \wedge \mu) \leq (\alpha(\lambda, r) \wedge \lambda) \wedge (\alpha(\rho, r) \wedge \mu) = \lambda \wedge \overline{0} = \overline{0};\; r \in I_0.$$

That is, $\lambda \wedge \mu$ and $\lambda \wedge \rho$ are r-fuzzy α-separated sets so that $\lambda = (\lambda \wedge \mu) \vee (\lambda \wedge \rho)$. But λ is r-fuzzy α-connected implies that $(\lambda \wedge \mu) = \overline{0}$ or $(\lambda \wedge \rho) = \overline{0}$.

(2) $\Rightarrow$ (3): If $\lambda \wedge \mu = \overline{0}$, then $\lambda = \lambda \wedge (\mu \vee \rho) = \lambda \wedge \rho$, and thus $\lambda \leq \rho$. Also, if $\lambda \wedge \rho = \overline{0}$, then $\lambda = \lambda \wedge \mu$, and then $\lambda \leq \mu$.

(3) $\Rightarrow$ (1): Let μ, ρ be r-fuzzy α-separated sets such that $\lambda = \mu \vee \rho$. Then, from (3), $\lambda \leq \mu$ or $\lambda \leq \rho$. If $\lambda \leq \mu$, then $\rho = \lambda \wedge \rho \leq \mu \wedge \rho \;\leq\; \alpha(\mu, r) \wedge \rho = \overline{0}$. Also, if $\lambda \leq \rho$, then $\mu = \lambda \wedge \mu \leq \rho \wedge \mu \;\leq\; \alpha(\rho, r) \wedge \mu = \overline{0}$. Hence, λ is r-fuzzy α-connected.

Theorem 4.7. Let $f : X \to Y$ be a mapping such that

$$\alpha(f^{-1}(\nu), r) \leq f^{-1}(\theta(\nu, r)) \ \ \forall \nu \in I^Y, \ r \in I_0,$$

where α is a fuzzy operator on X and θ is a fuzzy operator on Y. Then, $f(\lambda) \in I^Y$ is r-fuzzy θ-connected if $\lambda \in I^X$ is r-fuzzy α-connected.

Proof. Let $\mu, \rho \in I^Y$, $\mu \neq \overline{0}$, $\rho \neq \overline{0}$ be r-fuzzy θ-separated sets in I^Y with $f(\lambda) = \mu \vee \rho$. That is, $\theta(\mu, r) \wedge \rho = \theta(\rho, r) \wedge \mu = \overline{0}$; $r \in I_0$. Then, $\lambda \leq f^{-1}(\mu) \vee f^{-1}(\rho)$, and

$$\begin{aligned} \alpha(f^{-1}(\mu), r) \wedge f^{-1}(\rho) &\leq f^{-1}(\theta(\mu, r)) \wedge f^{-1}(\rho) \\ &= f^{-1}(\theta(\mu, r) \wedge \rho) \\ &= f^{-1}(\overline{0}) \ = \ \overline{0}, \end{aligned}$$

$$\begin{aligned} \alpha(f^{-1}(\rho), r) \wedge f^{-1}(\mu) &\leq f^{-1}(\theta(\rho, r)) \wedge f^{-1}(\mu) \\ &= f^{-1}(\theta(\rho, r) \wedge \mu) \\ &= f^{-1}(\overline{0}) \ = \ \overline{0}. \end{aligned}$$

Hence, $f^{-1}(\mu)$ and $f^{-1}(\rho)$ are r-fuzzy α-separated sets in X so that $\lambda \leq f^{-1}(\mu) \vee f^{-1}(\rho)$. But λ is r-fuzzy α-connected means, from (3) in Proposition 4.6, that $\lambda \leq f^{-1}(\mu)$ or $\lambda \leq f^{-1}(\rho)$, which means that $f(\lambda) \leq \mu$ or $f(\lambda) \leq \rho$. Thus, again from (3) in Proposition 4.6, we get that $f(\lambda)$ is r-fuzzy θ-connected.

Corollary 4.8. (Theorem 2.12 in [7]) Let $(X, \tau_1), (Y, \tau_2)$ be two fuzzy topological spaces. If $f : X \to Y$ is a fuzzy continuous mapping and $\lambda \in I^X$ is r-fuzzy connected in X, then $f(\lambda)$ is an r-fuzzy connected in Y.

Proof. Let $\alpha =$ fuzzy closure operator and $\theta =$ fuzzy closure operator. Then, the result follows from Theorem 4.7.

Corollary 4.9. (Theorems 2.12, 3.11 in [10]) Let $(X, \tau_1), (Y, \tau_2)$ be two fuzzy topological spaces. Let $f : (X, \tau_1) \to (Y, \tau_2)$ be S-irresolute (resp. irresolute). If $\lambda \in I^X$ is r-fuzzy strongly connected (resp. 1-type of r-fuzzy strongly connected) in X, then $f(\lambda)$ is r-fuzzy strongly connected (resp. 1-type of r-fuzzy strongly connected) in Y.

Proof. Let $\alpha =$ fuzzy strongly semi-closure (resp. semi-closure) operator and $\theta =$ fuzzy strongly semi-closure (resp. semi-closure) operator. Then, the result follows from Theorem 4.7.

Corollary 4.10. Let $(X, \tau_1), (Y, \tau_2)$ be two fuzzy topological spaces. Let $f : (X, \tau_1) \to (Y, \tau_2)$ be fuzzy semi-pre-irresolute. If $\lambda \in I^X$ is r-fuzzy semi-preconnected in X, then $f(\lambda)$ is r-fuzzy semi-preconnected in Y.

Proof. Let $\alpha =$ fuzzy semi-preclosure operator and $\theta =$ fuzzy semi-preclosure operator. Then, the result follows from Theorem 4.7.

Corollary 4.11. (Theorem 5.10 in [2]) Let $(X, \tau_1), (Y, \tau_2)$ be two fuzzy topological spaces. Let $f : (X, \tau_1) \to (Y, \tau_2)$ be fuzzy strongly pre-irresolute (resp. pre-irresolute). If $\lambda \in I^X$ is r-fuzzy s preconnected (resp. preconnected) in X, then $f(\lambda)$ is r-fuzzy s preconnected (preconnected) in Y.

Proof. Let α = fuzzy strongly preclosure (resp. preclosure) operator and θ = fuzzy strongly preclosure (resp. preclosure) operator. Then, the result follows from Theorem 4.7.

Corollary 4.12. Let $(X, \tau_1), (Y, \tau_2)$ be two fuzzy topological spaces. Let $f : (X, \tau_1) \to (Y, \tau_2)$ be fuzzy semi-continuous (resp. precontinuous, strongly semi-continuous, strongly precontinuous and semi-precontinuous) mapping. If $\lambda \in I^X$ is 1-type of r-fuzzy strongly connected (resp. r-fuzzy preconnected, r-fuzzy strongly connected, r-fuzzy strongly preconnected and r-fuzzy semi-preconnected) in X, then $f(\lambda)$ is r-fuzzy connected in Y.

Proof. Let α = fuzzy semi-closure (resp. preclosure, strongly semi-closure, strongly preclosure and semi-preclosure) operator and θ = fuzzy closure operator. Then, the result follows from Theorem 4.7.

Proposition 4.13. Any fuzzy point $x_t, t \in I_0$ is r-fuzzy α-connected, and consequently $x_1 \forall x \in X$ is r-fuzzy α-connected.

Proof. Clear.

Definition 4.14. Let X be a non-empty set and $\lambda \in I^X$. Then, λ is r-fuzzy α-component if λ is maximal r-fuzzy α-connected set in X, that is, if $\mu \geq \lambda$ and μ is r-fuzzy α-connected set, then $\lambda = \mu$.

Proposition 4.15. Let $\lambda \neq \overline{0}$ be r-fuzzy α-connected in X and $\lambda \leq \mu \leq \alpha(\lambda, r)$; $r \in I_0$. Then, μ is r-fuzzy α-connected.

Proof. Let ν, ρ be r-fuzzy α-separated sets such that $\mu = \nu \vee \rho$. That is, $\alpha(\nu, r) \wedge \rho = \alpha(\rho, r) \wedge \nu = \overline{0}$; $r \in I_0$. Since $\lambda \leq \mu$, then $\lambda \leq (\nu \vee \rho)$. From λ is r-fuzzy α-connected, and from (3) in Proposition 4.6, we have $\lambda \leq \nu$ or $\lambda \leq \rho$. If $\lambda \leq \nu$, then

$$\rho = \mu \wedge \rho \leq \alpha(\lambda, r) \wedge \rho \leq \alpha(\nu, r) \wedge \rho = \overline{0}.$$

If $\lambda \leq \rho$, then

$$\nu = \mu \wedge \nu \leq \alpha(\lambda, r) \wedge \nu \leq \alpha(\rho, r) \wedge \nu = \overline{0}.$$

Hence, μ is r-fuzzy α-connected.

5 Fuzzy α-Compact Spaces

This section is devoted to introduce the notion of r-fuzzy α-compact spaces.

Definition 5.1. Let (X, τ) be a fuzzy topological space, α a fuzzy operator on X, and $\mu \in I^X$, $r \in I_0$. Then, μ is called r-fuzzy α-compact if for each family $\{\lambda_j \in I^X : \tau(\lambda_j) \geq r,\ j \in J\}$ with $\mu \leq \bigvee_{j \in J} \lambda_j$, there exists a finite subset $J_0 \subseteq J$ such that $\mu \leq \bigvee_{j \in J_0} \alpha(\lambda_j, r)$.

Remark 5.2. For a fuzzy topological space (X, τ):

(1) if α = fuzzy identity operator, we get the r-fuzzy compactness as given in [1].

(2) if $\alpha =$ fuzzy closure operator, we get the r-fuzzy almost compactness as given in [1].

(3) if $\alpha =$ fuzzy interior closure operator, we get the r-fuzzy near compactness as given in [1].

(4) if $\alpha =$ fuzzy semi-closure (resp. preclosure, strongly semi-closure, strongly preclosure and semi-preclosure) operator, we get the r-fuzzy semi-compactness (resp. precompactness, strongly semi-compactness, strongly precompactness and semi-precompactness [11]).

Theorem 5.3. Let (X,τ) and (Y,σ) be two fuzzy topological spaces, α a fuzzy operator on X, θ is a fuzzy operators on Y. If $f : X \to Y$ is fuzzy $(\alpha, \mathrm{int}_\tau, \theta, id_Y, \mathcal{I}^\circ)$-continuous and $\mu \in I^X$ is r-fuzzy compact in X, then $f(\mu)$ is r-fuzzy θ-compact in Y.

Proof. Let $\{\lambda_j \in I^Y : \sigma(\lambda_j) \geq r,\ j \in J\}$ be a family with $f(\mu) \leq \bigvee_{j\in J} \lambda_j$. Since f is fuzzy $(\alpha, \mathrm{int}_\tau, \theta, id_Y, \mathcal{I}^\circ)$-continuous, we get that there exists $\mu_j = \mathrm{int}_\tau(f^{-1}(\theta(\lambda_j, r)), r) \in I^X$ with $\tau(\mu_j) \geq r\ \forall j \in J$ such that

$$\alpha(f^{-1}(\lambda_j), r) \leq \mu_j \leq f^{-1}(\theta(\lambda_j, r)).$$

Also, since $f^{-1}(\lambda_j) \leq \alpha(f^{-1}(\lambda_j), r)$, then

$$f^{-1}(\lambda_j) \leq \mu_j \leq f^{-1}(\theta(\lambda_j, r)),$$

which means that

$$\mu \leq \bigvee_{j\in J} f^{-1}(\lambda_j) \leq \bigvee_{j\in J} (\mu_j) \leq f^{-1}(\bigvee_{j\in J} \theta(\lambda_j, r)),$$

that is, $\mu \leq \bigvee_{j\in J} (\mu_j)$. By r-fuzzy compactness of μ, there exists a finite set $J_0 \subseteq J$ such that $\mu \leq \bigvee_{j\in J_0} (\mu_j)$, and thus

$$f(\mu) \leq \bigvee_{j\in J_0} f(\mu_j) \leq \bigvee_{j\in J_0} \theta(\lambda_j, r),$$

and therefore $f(\mu)$ is r-fuzzy θ-compact.

Corollary 5.4. ([11]) Let (X,τ) and (Y,σ) be two fuzzy topological spaces. Let $f : X \to Y$ be a fuzzy continuous mapping and $\mu \in I^X$ an r-fuzzy compact set in X, then $f(\mu)$ is r-fuzzy compact in Y.

Proof. Let $\alpha =$ fuzzy identity operator on X, $\theta =$ fuzzy identity operator and $\mathcal{I} = \mathcal{I}^\circ$, then the result follows from Theorem 5.3.

Corollary 5.5. ([11]) Let (X,τ) and (Y,σ) be two fuzzy topological spaces. Let $f : X \to Y$ be a fuzzy weakly continuous mapping ([8]) and $\mu \in I^X$ an r-fuzzy compact set in X, then $f(\mu)$ is r-fuzzy almost compact in Y.

Proof. Let $\alpha =$ fuzzy identity operator on X, $\theta =$ fuzzy closure operator and $\mathcal{I} = \mathcal{I}^\circ$, then the result follows from Theorem 5.3.

Corollary 5.6. ([11]) Let (X, τ) and (Y, σ) be two fuzzy topological spaces. Let $f : X \to Y$ be a fuzzy almost continuous mapping ([9]) and $\mu \in I^X$ an r-fuzzy compact set in X, then $f(\mu)$ is r-fuzzy nearly compact in Y.

Proof. Let $\alpha =$ fuzzy identity operator on X, $\theta =$ fuzzy interior closure operator and $\mathcal{I} = \mathcal{I}^\circ$, then the result follows from Theorem 5.3.

Corollary 5.7. Let (X, τ) and (Y, σ) be two fuzzy topological spaces. Let $f : X \to Y$ be a fuzzy semi-continuous [12] (resp. precontinuous [8], strongly semi-continuous [3], strongly precontinuous [2] and semi-precontinuous [8]) mapping, and $\mu \in I^X$ an r-fuzzy compact set in X, then $f(\mu)$ is r-fuzzy semi-compact (resp. precompact, strongly semi-compact, strongly precompact and semi-precompact) in Y.

Proof. Let $\alpha =$ fuzzy identity operator on X, $\theta =$ fuzzy semi-closure (resp. preclosure, strongly semi-closure, strongly preclosure and semi-preclosure) operator and $\mathcal{I} = \mathcal{I}^\circ$, then the result follows from Theorem 5.3.

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Separation Axioms Using Soft Turing Point of a Soft Ideal in Soft Topological Space

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Abstract – In this paper, we define new separation axioms in soft topological space via the concept of the Soft Truing Point and study the most important properties and results of it.

Keywords *– Soft Turing Point, Separation Axioms.*

1. **Introduction and Preliminaries**

In 1999, Molodtsov [2] begins whit the soft set theory as mathematical tool to solve many complicated problems in economics, engineering, and environment which we cannot successfully use classical methods because of various uncertainties typical of those problems. Maji et al. [13] studied a soft set theory and they introduced many of new concepts of this theory as an inclusion relation between the soft set, the formula of the empty set in this theory, the equality of two soft sets, the complement of a soft set also the soft intersection and the soft union with some of important results and properties. Cagman et al. [12] introduced a new type of a soft set and a new definition of a soft intersection and a soft union with many results and properties. Irfran et al. [11] discussed new operations in the soft set theory such as De Morgan laws in the soft set theory in addition to some anew views of the soft union and the soft intersection. S. Hussain and B. Ahmad [15] introduced the notion of soft topological spaces which are defined over an initial universe with a fixed set of parameters. The notion of soft ideal is initiated for the first time by Kandil et al. [1]. Al-Swidi, and Al-Amri [10] studied the soft sets theory as an analytical study and dividing the kinds to four families every family is different from other in

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some adjectives and properties and correspond to other. In the same year, [9] defined the separation axioms in soft topological space and practically in certain point of the parameters, and to study the most important properties and results of it. Recently, a new separation axiom in soft topological space is introduced by using the concept of the " ∗Soft turing point" as explained by [21]

Al-Swidi and Al-Fathly classified the soft points in soft topological spaces to three types. In this paper, we choose one of these types to be the focus of our work and which is defined in [19] as: let $F_A \in SS(X;A) = \{F_{i_A} : A \subseteq E_X \text{ and } F_i : A \longrightarrow IP(X)\}$, then F_A is called a soft point in $\tilde{X}_A$ denoted by F_a^x, where

$$F(P) = \begin{cases} \{x\} & \text{if } P = a \\ \varphi & P \in A \setminus \{a\} \end{cases}$$

So, a soft point F_a^x belong to the soft set F_A , denoted by $F_a^x \tilde{\in} F_A$ iff $x \in F(a)$. $F_a^x \tilde{\notin} F_A$ if $x \notin F(a)$

Definition 1.1 [2] A pair (F,A) denoted by F_A is called a soft set over X, where F is a mapping given by F: A → P(X). In other words, a soft set over X is a parameterized family of subsets of the universe X. For a particular a ∈ A, F(a) may be considered the set of a−approximate elements of the soft set (F,A) and if $a \notin A$, then $F(a)=\Phi$ i.e. $F_A=\{F(a): a \in A \subset E_X,\ F: A \longrightarrow P(X)\}$. We denote the family of all soft sets over X by SS(X;A).

A soft set F_A over X is said to be the null soft set , denoted by $\tilde{\Phi}_A$ if $\forall a \in A, F(a) = \varphi$. A soft set F_A over X is said to be the absolute soft set and denoted by $\tilde{X}_A$, if $\forall a \in A\ F(a) = X$.

Definition 1.2 [11] The complement of the soft set F_A is denoted by $(F_A)^C$ is defined by:

$$(F_A)^C = F^C{}_A = \tilde{X}_A \tilde{-} F_A ,$$

where $F^C: A \longrightarrow IP(X)$ is a mapping given by: $F^C(a) = X - F(a)\ \forall a \in A$. F^C is called the soft complement function of F

Definition 1.3 [15], [4] Let $\tilde{\tau}$ be the collection of soft sets over X, then $\tilde{\tau}$ is said to be a soft topology on X if

1- $\tilde{\Phi}_A$, $\tilde{X}_A$ belong to $\tilde{\tau}$
2- The union of any number of soft sets in $\tilde{\tau}$ belongs to $\tilde{\tau}$
3- The intersection of any two soft sets in $\tilde{\tau}$ belongs to $\tilde{\tau}$.

- The members of $\tilde{\tau}$ are called soft open sets.
- A soft set F_A is called soft closed set if the complement $(\tilde{\chi}_A - F_A)$ is soft open set (belong to $\tilde{\tau}$) .
- The family of all soft closed sets are denoted by $C(\tilde{\chi})$ and defined as follows: $C(\tilde{\chi}_A) = \{\tilde{\chi}_A - F_A , F_A \tilde{\in} \tilde{\tau}\}$.

Definition 1.4 [16] Let $(\tilde{X}_A, \tilde{\tau}, A)$ be a soft topological space , and let $Y \subseteq X$, the relative soft topology for $\tilde{Y}_A$ is the collection $\tilde{\tau}_Y$ given by: $\tilde{\tau}_Y = \{\tilde{Y}_A \tilde{\cap} F_A , F_A \tilde{\in} \tilde{\tau}\}$. Note that $\tilde{Y}_A$ means that $Y(a) = Y$, $\forall a \in A$. The soft topological space $(\tilde{Y}_A, \tilde{\tau}_Y, A)$ is called soft subspace of $(\tilde{X}_A, \tilde{\tau}, A)$. The soft topology $\tilde{\tau}_Y$ is called induced by $\tilde{\tau}$.

Theorem 1.5 [17] Let $(\tilde{Y}_A, \tilde{\tau}_Y, A)$ be a soft subspace of a soft topological space $(\tilde{X}_A, \tilde{\tau}, A)$ and F_A is a soft set over X .Then :F_A is soft open in $(\tilde{Y}_A, \tilde{\tau}_Y, A)$ iff $F_A = \tilde{Y}_A \tilde{\cap} G_A$, for some soft open set G_A in $(\tilde{X}_A, \tilde{\tau}, A)$ F_A is soft closed in $(\tilde{Y}_A, \tilde{\tau}_Y, A)$ iff $F_A = \tilde{Y}_A \tilde{\cap} K_A$, for some soft closed set K_A in $(\tilde{X}_A, \tilde{\tau}, A)$.

Definition 1.6 [20] Let χ and Y be two initial universal sets and A, B be sets of parameters, $u: \chi \longrightarrow Y$ and $p: A \longrightarrow B$, then the mapping $f: (\chi, A) \longrightarrow (Y, A)$ (i.e. $f: SS_{.}(\chi) \longrightarrow SS_{.}(Y)$) on A and B respectively is denoted by f_{pu} and can be shown as:

$$f_{pu} = \{\big(\big(f_{pu}(F_A), p(A)\big), p(A) \subseteq B\big)\} .$$

where

$$f_{pu}(F_A)(\beta) = \begin{cases} u\left(\bigcup\limits_{\alpha\in p^{-1}(\beta)\cap A\neq\varphi} \big(F(\alpha)\big)\right), & \text{if } p^{-1}(\beta) \neq \varphi \\ \varphi & \text{other wise} \end{cases}$$

For $\beta \in B$ $\exists$ $a \in p(A)$ such that $p(a) = \beta$ that is $p^{-1}(\beta) \neq \varphi$. Since $p^{-1}(\beta) \subseteq A$, hence $p^{-1}(\beta) \cap A \neq p^{-1}(\beta)$, hence we get that $f_{pu}(F_A)(\beta) = u\big(\bigcup_{\alpha\in p^{-1}(\beta)} F(\alpha)\big)$

Constructing: Since p is a mapping, so $p(A) \neq \varphi$, $\forall A \neq \varphi$, that is $\forall\beta \in p(A) \exists\, a \in A$ such that $p(a) = \beta$ and $p^{-1}(\beta) \neq \varphi$ since $a \in p^{-1}\big(p(a)\big)$ so:

$$f_{pu}(F_A)(\beta) = u\{\cup_{\alpha\in p^{-1}(\beta)}\big(F(\alpha)\big)\} \; \forall\, \beta \in p(A)$$

- If p is a one to one (1-1), then $p^{-1}\big(p(A)\big) = A$, that is $\forall\, \beta \in p(A) \exists$ a A such that $p(a) = \beta$ and $f_{pu}(F_A)(\beta) = u\big(F(a)\big)$.
- If $G_B \in SS(Y)$ then the inverse image of G_B under f_{pu} is denoted by ${f_{pu}}^{-1}(G_B)$ is a soft set $(F_A) \in SS_{.}(X)$ such that P(a)= $u^{-1}(G\big(p(\alpha)\big)$,for each $a \in A$.

Remark 1.7 [9, 20] For each $a \in A$ and $x \in X$,then we can define the soft mapping f_{pu} on soft point x_a ,as follows:

1- $\big(f_{pu}(x_a)\big)_{p(a)} = \{(p(a), \{u(x)\})\}$

2- Now, for $b \in B$ and $y \in Y$, ${f_{pu}}^{-1}(y_b)$(a)= $u_{.}^{-1}(y)$,for b=f(a).

Definition 1.8 [5] For a topological space (X,T), $x\in X$, $Y\subseteq X$, we define an ideal ${}^{Y}I_x$ respect to subspace (Y,T_Y), as follows

$$^{Y}I_x=\{G \subseteq Y: x \in (X-G)\}.$$

Definition 1.9 [1] Let $\tilde{I}_A$ be a non-null collection of soft sets over a universe X with the same set of parameters A. Then $\tilde{I}_A \in SS(X)$ is called a soft ideal on X with the same set A if

1- $F_A \tilde{\in} \tilde{I}_A$ and $G_A \tilde{\in} \tilde{I}_A$ then $F_A \tilde{\cup} G_A \tilde{\in} \tilde{I}_A$
2- $F_A \tilde{\in} \tilde{I}_A$ and $F_A \tilde{\subseteq} G_A$ then $G_A \tilde{\in} \tilde{I}_A$

Definition 1.10 [16] Let $(\tilde{X}_A, \tilde{\tau}, A)$, be a soft topological space, and let G_A be a soft set over the universe X, then the soft closure of G_A is a soft closed set defined as

$$\mathrm{Cl}G_A = \tilde{\cap}\{S_A, S_A \text{ is soft closed and } G_A \tilde{\subseteq} S_A\}$$

Proposition 1.11 [6] Let $(\tilde{X}_A, \tilde{\tau}, A)$, be a soft topological space, and let F_A, G_A be soft set over X , then G_A is soft closed iff $\mathrm{Cl}(G_A) = G_A$.

2. Separation Axioms Using Soft Turing Point

Definition 2.1 Let SI be a soft ideal in a soft topological space $(\tilde{X}_A, \tilde{\tau}, A)$, A soft point F_a^x in SS(X;A) is called "soft Turing point" of SI, if $(F_A)^C \tilde{\in}$ SI for each $F_A \tilde{\in} N_{\tilde{\tau}(F_a^x)}$ where $N_{\tilde{\tau}(F_a^x)}$ is collection of all soft open nhd of soft point F_a^x .

Example 1.2 Let $((\tilde{X}_A, \tilde{\tau}, A)$ be a soft topological space ,for some $a \in A$, $x \in X$, we define soft ideal $SI(F_a^x)$, as follows: $SI(F_a^x) = \{F_A \tilde{\subseteq} N_{\tilde{\tau}(F_a^x)} : F_a^x \tilde{\notin} (F_A)^C\}$. Then a soft point F_a^x is called "Soft Turing point" of $SI(F_a^x)$

Example 2.3 Let E_X be the set of all parameters and let X be the initial universe consisting of: $X = \{x, y\}$ and $A \tilde{\subseteq} E_X$ such that $A = \{a_1, a_2\}$.

$$\tilde{\tau} = \{\tilde{\varphi}_A, \tilde{\chi}_A, F_{1A}, F_{2A}, F_{3A}, F_{4A}, F_{5A}, F_{6A}, F_{7A}, F_{8A}, F_{9A}, F_{10A}, F_{11A}, F_{12A} F_{13A}, F_{14A}\},$$

where:

$$F_{1A}(p) = \begin{cases} \{x\} & \text{if } p = a_1 \\ \varphi & \text{if } p = a_2 \end{cases} \qquad F_{2A}(p) = \begin{cases} \{x\} & \text{if } p = a_2 \\ \varphi & \text{if } p = a_1 \end{cases}$$

$$F_{3A}(p) = \begin{cases} \{y\} & \text{if } p = a_2 \\ \varphi & \text{if } p = a_1 \end{cases} \qquad F_{4A}(p) = \begin{cases} \{y\} & \text{if } p = a_1 \\ \varphi & \text{if } p = a_2 \end{cases}$$

$$F_{5A}(p) = \begin{cases} \{x\} & \text{if } p = a_1 \\ \{y\} & \text{if } p = a_2 \end{cases} \qquad F_{6A}(p) = \begin{cases} \{x\} & \text{if } p = a_2 \\ \{y\} & \text{if } p = a_1 \end{cases}$$

$F_{7_A}(p) = \begin{cases} \{x\} & \text{if } p = a_1 \\ X & \text{if } p = a_2 \end{cases}$ $\quad F_{8_A}(p) = \begin{cases} \{x\} & \text{if } p = a_2 \\ X & \text{if } p = a_1 \end{cases}$

$F_{9_A}(p) = \begin{cases} X & \text{if } p = a_2 \\ \{y\} & \text{if } p = a_1 \end{cases}$ $\quad F_{10_A}(p) = \begin{cases} X & \text{if } p = a_1 \\ \varphi & \text{if } p = a_2 \end{cases}$

$F_{11_A}(p) = \begin{cases} X & \text{if } p = a_2 \\ \varphi & \text{if } p = a_1 \end{cases}$ $\quad F_{12_A}(p) = \begin{cases} X & \text{if } p = a_1 \\ \{y\} & \text{if } p = a_2 \end{cases}$

$F_{13_A}(p) = \begin{cases} \{x\} & \text{if } p = a_2 \\ \{x\} & \text{if } p = a_1 \end{cases}$ $\quad F_{14_A}(p) = \begin{cases} \{y\} & \text{if } p = a_1 \\ \{y\} & \text{if } p = a_2 \end{cases}$

Then

$SI(F_{a_1}^{x}) = \{\tilde{\varphi}_A, F_{2_A}(p), F_{3_A}(p), F_{4_A}(p), F_{6_A}(p), F_{9_A}(p), F_{11_A}(p), F_{14_A}(p)\}$.
$SI(F_{a_1}^{y}) = \{\tilde{\varphi}_A, F_{1_A}(p), F_{2_A}(p), F_{3_A}(p), F_{5_A}(p), F_{7_A}(p), F_{11_A}(p), F_{13_A}(p)\}$
$SI(F_{a_2}^{y}) = \{\tilde{\varphi}_A, F_{1_A}(p), F_{2_A}(p), F_{4_A}(p), F_{6_A}(p), F_{8_A}(p), F_{10_A}(p)\ F_{13_A}(p)\}$

$F_{a_1}^{x}$ is soft Turing point of $SI(F_{a_1}^{x})$,but $F_{a_1}^{y}$ is not soft Turing point of $SI(F_{a_1}^{x})$

Note: ((F_{a}^{y} is not soft Turing point of $SI(F_{a}^{x})$ i.e $F_{a}^{y}\ \tilde{\notin}\ SI(F_{a}^{x})$)).

Definition 2.4 Let $(\tilde{X}_A, \tilde{\tau}, A)$ be a soft topological space and $a \in A$, the space $\tilde{X}_A$ is called SI-T_0-space if and only if, for any pair of distinct points F_{a}^{x} and F_{a}^{y} of $\tilde{X}_A$, $F_{a}^{x}\ \tilde{\notin} SI(F_{a}^{y})$ or $F_{a}^{y}\ \tilde{\notin}$ $SI(F_{a}^{x})$.

Example 2.5 Consider [Example 2.3] Let $\tilde{\tau} = \{\tilde{\varphi}_A, \tilde{X}_A, F_{10_A}, F_{8_A}\}$ be a soft topology on $\tilde{X}_A$ .Then $(\tilde{X}_A, \tilde{\tau}, A)$ is SI-T_0-space because, for any pair of distinct points F_{a}^{x} and F_{a}^{y} of X, F_{a}^{x} $\tilde{\notin} SI(F_{a}^{y})$ or $F_{a}^{y}\ \tilde{\notin}\ SI(F_{a}^{x})$.

Lemma 2.6 Let $(\tilde{X}_A, \tilde{\tau}, A)$ be soft topological space ,for some $a \in A$ and $F_{a}^{x_1} \neq F_{a}^{x_2}$ in X, $F_{a}^{x_2}$ is soft closed set in $\tilde{X}_A$ if and only if $F_{a}^{x_1}\ \tilde{\notin}\ SI(F_{a}^{x_2})$.

Proof: Let $F_{a}^{x_1}, F_{a}^{x_2}$ in $\tilde{X}_A$ such that $F_{a}^{x_1} \neq F_{a}^{x_2}$. Assume that $F_{a}^{x_1}$ is a soft closed set in $\tilde{X}_A$, so that $F_{a}^{x_2} = cl(F_{a}^{x_2})$.But $F_{a}^{x_1} \neq F_{a}^{x_2}$, we get that $F_{a}^{x_1}\ \tilde{\notin} cl(F_{a}^{x_2})$.Therefore , there exists U $N_{\tilde{\tau}(F_{a}^{x_1})}$ such that, $F_{a}^{x_1}\ \tilde{\in}$ U, $U \cap F_{a}^{x_2} = \emptyset$ .So that $F_{a}^{x_1}\ \tilde{\in}$ U, $U^c \tilde{\notin} SI(F_{a}^{x_2})$,because if $U^c \tilde{\in}\ SI(F_{a}^{x_2})$, then $F_{a}^{x_2}\tilde{\in}$ U, that means $U \cap F_{a}^{x_2} \neq \emptyset$,this a contradiction. Hence $F_{a}^{x_1}\ \tilde{\notin}\ SI(F_{a}^{x_2})$.

Conversely, Let $F_{a}^{x_1}, F_{a}^{x_2}$ in $\tilde{X}_A$ such that $F_{a}^{x_1} \neq F_{a}^{x_2}$. Since $F_{a}^{x_1}\ \tilde{\notin}\ SI(F_{a}^{x_2})$, there exists $U\ \tilde{\in}\ N_{\tilde{\tau}(F_{a}^{x_1})}$such that, $F_{a}^{x_1}\ \tilde{\in}$ U, $U^c \tilde{\notin} SI(F_{a}^{x_2})$, so $F_{a}^{x_2}\ \tilde{\notin}$ U. Thus $F_{a}^{x_1}\ \tilde{\in}$ U, $U \cap F_{a}^{x_2} = \emptyset$ implies $F_{a}^{x_1}\ \tilde{\notin}\ cl(F_{a}^{x_2})$.Hence $F_{a}^{x_2} = cl(F_{a}^{x_2})$ .Thus, $F_{a}^{x_2}$ is a soft closed set in $\tilde{X}_A$.

Theorem 2.7 For every point F_a^x is a soft closed set, $a \in A$, then $(\tilde{X}_A, \tilde{\tau}, A)$ is SI-T_0-space.

Proof: Directly by Lemma 2.6.

Theorem 2.8 A soft topological space $(\tilde{X}_A, \tilde{\tau}, A)$ is SI-T_0-space, $a \in A$ iff for every distinct points $F_a^{x_1}, F_a^{x_2}$ in $\tilde{X}_A$ we have : $cl(F_a^{x_1}) \neq cl(F_a^{x_2})$

Proof: Suppose that $(\tilde{X}_A, \tilde{\tau}, A)$ is SI-T_0-space , $a \in A$ and $F_a^{x_1} \neq F_a^{x_2}$, then $F_a^{x_1} \tilde{\notin} SI(F_a^{x_2})$ or $F_a^{x_2} \tilde{\notin} SI(F_a^{x_1})$,so there exists $U \tilde{\in} N_{\tilde{\tau}(F_a^{x_1})}$ such that, $F_a^{x_1} \tilde{\in} U$, $U^c \tilde{\notin} SI(F_a^{x_2})$, or there exists $G \tilde{\in} N_{\tilde{\tau}(F_a^{x_2})}$ such that, $F_a^{x_2} \tilde{\in} G$, $G^c \tilde{\notin} SI(F_a^{x_1})$.Then by lemma 2.6, $cl(x_{1_a}) = x_{1_a}$ or $cl(x_{2_a}) = x_{2_a}$ That means $x_{1_a} \tilde{\in} cl(x_{1_a})$ and $x_{2_a} \tilde{\notin} cl(x_{1_a})$ or $x_{2_a} \tilde{\in} cl(x_{2_a})$ and $x_{1_a} \tilde{\notin} cl(x_{2_a})$.Thus, $x_{1_a} \tilde{\in} cl(x_{1_a})$ but $x_{1_a} \tilde{\notin} cl(x_{2_a})$.Hence,$cl(x_{1_a}) \neq cl(x_{2_a})$.

Conversely: Let $a \in A$ and $F_a^{x_1} \neq F_a^{x_2}$ in X, with $cl(F_a^{x_1}) \neq cl(F_a^{x_2})$, then there exist $F_a^Z \tilde{\in} cl(F_a^{x_1})$, but $F_a^Z \tilde{\notin} cl(F_a^{x_2})$, then $F_a^{x_1} \tilde{\notin} cl(F_a^{x_2})$ because, if $F_a^{x_1} \tilde{\in} cl(F_a^{x_2}$, then $cl(F_a^{x_1}) \tilde{\subseteq} cl(cl(F_a^{x_2}) = cl(F_a^{x_2})$, but $F_a^Z \tilde{\in} cl(F_a^{x_1}) \tilde{\subseteq} cl(F_a^{x_2})$ which is a contradiction, thus $F_a^{x_1} \tilde{\notin} cl(F_a^{x_2})$, that is, $F_a^{x_1} \tilde{\in} (\tilde{X}_A - cl(F_a^{x_2}))$ is a soft open nhf and $F_a^{x_2} \tilde{\notin} (\tilde{X}_A - cl(F_a^{x_2}))$, so $cl(x_{2_a}) \tilde{\notin} SI(x_{2_a})$. Hence $(\tilde{X}_A, \tilde{\tau}, A)$ is SI-T_0-space.

Theorem 2.9 Every soft subspace of SI-T_0-space is SI-T_0-space.

Proof: Suppose that $\tilde{Y}_A$ is a soft subspace of the of the SI-T_0-space $(\tilde{X}_A, \tilde{\tau}, A)$. Let $F_a^{y_1}$ and $F_a^{y_2}$ be two distinct points of $\tilde{Y}_A$. Again, since $\tilde{X}_A$ is SI-T_0 –space and $\tilde{Y}_A \subseteq \tilde{X}_A$, then $F_a^{y_1} \tilde{\notin} SI(F_a^{y_2})$ or $F_a^{y_2} \tilde{\notin} SI(F_a^{y_1})$, for some a. Suppose, $F_a^{y_1} \tilde{\notin} SI(F_a^{y_2})$, then there exists $U \tilde{\in} N_{\tilde{\tau}(F_a^{y_1})}$ such that, $F_a^{y_1} \tilde{\in} U$, $U^C \tilde{\notin} SI(F_a^{y_2})$. Then $U' = U \tilde{\cap} \tilde{Y}_A$ is $\tilde{\tau}_Y$- soft open contains $F_a^{y_1}$ but not $F_a^{y_2}$. So that $F_a^{y_1} \tilde{\in} U'$ and $(U')^c \tilde{\notin} SI(F_a^{y_2})$, hence $\tilde{Y}_A$ is SI-T_0-space.

Theorem 2.10 Let $(\tilde{X}_A, \tilde{\tau}, A)$ and $(\tilde{Y}_B, \tilde{\sigma}, B)$ be two soft topological spaces and let $\tilde{X}_A$ be SI-T_0-space, for some $a \in A$, if the map $f_{pu}: (\tilde{X}_A, \tilde{\tau}, A) \longrightarrow (\tilde{Y}_B, \tilde{\sigma}, B)$ is a soft open and u, p are onto maps, then $\tilde{Y}_B$ is SI-T_0-space.

Proof: Let $b \in B$ and $F_a^{y_1} \neq F_a^{y_1}$ in Y, then there exist $a \in A$ and $F_a^{x_1} \neq F_a^{x_1}$ in X such that $p(a) = b$, $u(F_a^{x_1}) = F_a^{y_1}$ and $u(F_a^{x_2}) = F_a^{y_2}$ because p and u are onto maps . Now by assumption ,then $F_a^{x_1} \tilde{\notin} SI(F_a^{x_2})$ or $F_a^{x_2} \tilde{\notin} SI(F_a^{x_1})$, so there exists U_A, G_A a soft open sets in $\tilde{X}_A$, such that, $F_a^{x_1} \tilde{\in} G_A$, $(G_A)^C \tilde{\notin} SI(F_a^{x_2})$ or $F_a^{x_2} \tilde{\in} U_A$,$(U_A)^C \tilde{\notin} SI(F_a^{x_1})$. Now: $f_{pu}(F_a^{x_1}) \tilde{\in} f_{pu}(G_A)$, $f_{pu}((G_A)^C) \tilde{\notin} SI(f_{pu}(F_a^{x_2}))$ or $f_{pu}(F_a^{x_2}) \tilde{\in} f_{pu}(U_A)$,$f_{pu}((U_A)^C) \tilde{\notin} SI(f_{pu}(F_a^{x_1}))$, but f_{pu} is soft open , so $f_{pu}(G_A), f_{pu}(U_A)$ are be a soft open sets in $\tilde{Y}_B$, and $F_b^{y_1} = f_{pu}(F_a^{x_1})$ and $F_b^{y_2} = f_{pu}(F_a^{x_2})$, i.e $F_b^{y_1} \tilde{\notin} SI(F_b^{y_1})$ or $F_b^{y_2} \tilde{\notin} SI(F_b^{y_1})$ Therefore , $\tilde{Y}_B$ is a SI-T_0-space

Theorem 2.11 Let $(\widetilde{Y}_B, \widetilde{\sigma}, B)$ be SI-T_0-spacefor $b \in B$ and let $(\widetilde{X}_A, \widetilde{\tau}, A)$ be any soft topological space such that the mapping $u: X \longrightarrow Y$ be a one to one and $p: A \longrightarrow B$ be an onto map, then there exist $a \in A$ with $p(a) = b$ and $\widetilde{X}_A$ is SI-T_0-space, if $f_{pu}: (\widetilde{X}_A, \widetilde{\tau}, A) \longrightarrow (\widetilde{Y}_B, \widetilde{\sigma}, B)$ is soft continuous map.

Definition 2.12 Let $(\widetilde{X}_A, \widetilde{\tau}, A)$ be a soft topological space and $a \in A$, the space $\widetilde{X}_A$ is called SI-T_1-space if and only if, for any pair of distinct points F_a^x and F_a^y of $\widetilde{X}_A$, $F_a^x \widetilde{\notin} SI(F_a^y)$ and $F_a^y \widetilde{\notin} SI(F_a^x)$.

Example 2.13 Consider [Example 2.3] Let $\widetilde{\tau} = \{\widetilde{\varphi}_A, \widetilde{X}_A, F_{10_A}, F_{8_A}, F_{12_A}\}$ be a soft topology on $\widetilde{X}_A$ .Then $(\widetilde{X}_A, \widetilde{\tau}, A)$ is SI-T_1-space because, for any pair of distinct points F_a^x , F_a^y of $\widetilde{X}_A$, $F_a^x \widetilde{\notin} SI(F_a^y)$ and $F_a^y \widetilde{\notin} SI(F_a^x)$.

Remark 2.14 Every SI-T_1-space is SI-T_0-space.

Proof: Direct from [Def].

Remark 2.15 The converse, need not be true, as seen in.

Example 2.16 Consider [Example 2.3] et $\widetilde{\tau} = \{\widetilde{\varphi}_A, \widetilde{X}_A, F_{10_A}, F_{8_A}\}$ be a soft topology on $\widetilde{X}_A$ .Then $(\widetilde{X}_A, \widetilde{\tau}, A)$ is SI-T_0-space ,but not SI-T_1-space,because,there exist distinct points $F_{a_1}^y$ and $F_{a_2}^y$ of $\widetilde{X}_A$, $F_{a_1}^y \widetilde{\notin} SI(F_{a_2}^y)$ and $F_{a_2}^y \widetilde{\in} SI(F_{a_1}^y)$.

Theorem 2.17 Let $(\widetilde{X}_A, \tilde{\tau}, A)$ be a soft topological space and $a \in A$,then the following properties are equivalent:

a) (X,T) is SI-T_1-space.
b) If the soft sets F_a^x and F_a^y such that $F_a^x \neq F_a^y$ are closed subsets in $\widetilde{X}_A, \forall\, a \in A$.

Proof: [Direct from definition and Lemma 2.6].

Theorem 2.18 Every soft subspace of SI-T_1-space is SI-T_1-space $\forall\, a \in A$

Proof: Suppose that $\widetilde{Y}_A$ is a soft subspace of the of the SI-T_1-space $(\widetilde{X}_A, \widetilde{\tau}, A)$. Let $F_a^{y_1}$ and $F_a^{y_2}$ be two distinct points of $\widetilde{Y}_A$. Again, since $\widetilde{X}_A$ is SI-T_0 –space and $\widetilde{Y}_A \subseteq \widetilde{X}_A$, then $F_a^{y_1} \widetilde{\notin} SI(F_a^{y_2})$ and $F_a^{y_2} \widetilde{\notin} SI(F_a^{y_1})$, for some a. So that, there exists $U \widetilde{\in} N_{\widetilde{\tau}(F_a^{y_1})}$ such that, $F_a^{y_1} \widetilde{\in} U$, $U^C \widetilde{\notin} SI(F_a^{y_2})$ and there exists $G \widetilde{\in} N_{\widetilde{\tau}(F_a^{y_2})}$ such that, $F_a^{y_2} \widetilde{\in} G$, $G^C \widetilde{\notin} SI(F_a^{y_1})$.Then, $U' = U \widetilde{\cap} \widetilde{Y}_A$ is $\widetilde{\tau}_{Y}$- soft open contains $F_a^{y_1}$ but not $F_a^{y_2}$ and $G' = G \widetilde{\cap} \widetilde{Y}_A$ is $\widetilde{\tau}_{Y}$- soft open contains $F_a^{y_2}$ but not $F_a^{y_1}$. So that $F_a^{y_1} \widetilde{\in} U'$ and $(U')^c \widetilde{\notin} SI(F_a^{y_2})$ and $F_a^{y_2} \widetilde{\in} G'$ and $(G')^c \widetilde{\notin} SI(F_a^{y_1})$, hence $\widetilde{Y}_A$ is SI-T_0-space.

Theorem 2.19 Let $(\widetilde{X}_A, \tilde{\tau}, A)$ and $(\tilde{Y}_B, \tilde{\sigma}, B)$ be two soft topological spaces and let $\widetilde{X}_A$ be SI-T_1-space, for some $a \in A$, if the map $f_{pu}: (\widetilde{X}_A, \widetilde{\tau}, A) \longrightarrow (\widetilde{Y}_B, \widetilde{\sigma}, B)$ is a soft open and u , p are onto maps, then $\tilde{Y}_B$ is SI-T_1-space .

Proof: By the same way of proof of Theorem 2.10.

Definition 2.20 Let $(\tilde{X}_A, \tilde{\tau}, A)$ be a soft topological space and $a \in A$, the space $\tilde{X}_A$ is called SI-T_2-space if and only if, for any pair of distinct points F_a^x and F_a^y of $\tilde{X}_A$, $F_a^x \tilde{\notin} SI(F_a^y)$ and $F_a^y \tilde{\notin} SI(F_a^x)$, $SI(F_a^x) \tilde{\cap} SI(F_a^y) = \emptyset$.

Example 2.21 Let E_X be the set of all parameters and let X be the initial universe consisting of: $X = \{x, y\}$ and $A \tilde{\subseteq} E_X$ such that $A = \{a\}$. Let $\tilde{\tau} = \{\tilde{\varphi}_A, \tilde{X}_A, F_{10_A}\}$ be a soft topology on $\tilde{X}_A$. where $F_{1_A} = \{(a, \{x\})\}$, $F_{2_A} = \{(a, \{y\})\}$. Then $(\tilde{X}_A, \tilde{\tau}, A)$ is SI-T_2-space.

Remark 2.22 Every SI-T_2-space is SI-T_1-space.

Proof: Direct from [Def].

Remark 2.23 The converse, need not be true, as seen in [Example 2.3].

Example 2.24 Consider [Example 2.3] Let $\tilde{\tau} = \{\tilde{\varphi}_A, \tilde{X}_A, F_{10_A}, F_{8_A}, F_{12_A}\}$ be a soft topology on $\tilde{X}_A$ .Then $(\tilde{X}_A, \tilde{\tau}, A)$ is SI-T_1-space because, for any pair of distinct points F_a^x , F_a^y of $\tilde{X}_A$, $F_a^x \tilde{\notin} SI(F_a^y)$ and $F_a^y \tilde{\notin} SI(F_a^x)$,but not SI-$T_2$-space,because,there exist distinct points $F_{a_2}^x$ and $F_{a_2}^y$ of $\tilde{X}_A$, $F_{a_2}^x \tilde{\notin} SI(F_{a_2}^y)$ and $F_{a_2}^y \tilde{\notin} SI(F_{a_2}^x)$. $SI(F_{a_2}^x) \tilde{\cap} SI(F_{a_2}^y) \neq \emptyset$.

Theorem 2.25 Every soft subspace of SI-T_2-space is SI-T_2-space, $\forall a \in A$.

Proof: Similar to theorem 2.9.

Theorem 2.26 Let $(\tilde{X}_A, \tilde{\tau}, A)$ and $(\tilde{Y}_B, \tilde{\sigma}, B)$ be two soft topological spaces and let $\tilde{X}_A$ be SI-T_2-space, for some $a \in A$, if the map $f_{pu}: (\tilde{X}_A, \tilde{\tau}, A) \longrightarrow (\tilde{Y}_B, \tilde{\sigma}, B)$ is a soft open and u , p are onto maps, then $\tilde{Y}_B$ is SI-T_2-space.

Proof: By the same way of proof of Theorem 2.10.

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Mathematical Model of Infertility Due to Obesity in Female

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Abstract - Now-a-days the issue of obesity is increasing worldwide due to food habits and has become a major problem. A primary reason for the exponential increase in obesity is due to consumption of high calorie food. Obesity has injurious effect on human health including the reproductive system in female. The risk of infertility, miscarriage rate and pregnancy complications are more in obese female. The impact of obesity is very risky in maturity phase of female. In order to study the impact of infertility in susceptible female due to obesity, the system of non-linear ordinary differential equations is formulated. The stability analysis is studied for infertility model. Numerical simulation is carried out using data to support the proposed analysis.

Keywords *- Mathematical Model, Obesity, Infertility, Basic Reproduction Number, Stability, Simulation*

1 Introduction

Healthy life and life-style of an individual are directly related with food habits. Food habits to maintain health is a challenge for an individual in fast era of life [2]. Unhealthy person may have many health problems like obesity, heart disease, diabetes, blood pressure etc.

In recent years, there has been an exponential rise in the number of obese individuals especially in developed countries like India, United States, Germany so that the World Health Organization (WHO) has declared obesity as an epidemic [3]. Obese individuals are at a much higher risk for serious medical conditions such as certain types of cancers, infertility, heart disease, diabetes, liver disease, brain stroke etc. The recent studies have proved that the rise of obesity among the world population because of high calorie intake coupled with less physical activity. Obesity occurs when someone regularly takes in more calories than it burns [4]. The average physically active man needs about 2,500 calories a day and woman need about 2,000 calories a day to maintain a Body Mass Index (BMI) [6]. Fast foods reduce the quality of diet and provide unhealthy choices especially among

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youngsters which are increasing since past four decades. Fast food contains calories, fat, sugar, carbohydrates, and salt that causes obesity [5].

Obesity has injurious effect for human health including reproductive system in female by causing hormonal imbalance and ovulation problem [7]. Obesity is considered as additional risk factor for infertility in female who have regular cycles. The female affected by obesity not only have problems with fertility, but are also at a greater risk for pregnancy complications.

In this paper, we will analyze how obesity causes infertility due to high calorie food in female using SEI model. Table 1 consisting of notations and its description along with its parametric values and mathematical model are described in Section 2. Local and global stability of the system is studied in Section 3. Section 4 validates proposed problem with numerical simulation.

2 Mathematical Model

Here, we formulate a mathematical model for the analysis of infertility due to obesity in female who is fond of high calorie food. The notations along with its parametric values are shown in Table 1.

Table 1. Notations and its Parametric Values

Notations		**Parametric Values**
$N(t)$	Sample size at any instant of time t	1000
$S(t)$	Number of susceptible female at any instant of time t	100
$C(t)$	Number of female taking calorie food at any instant of time t	35
$O_B(t)$	Number of female becoming obese at any instant of time t	25
$I(t)$	Number of Infertile female at any instant of time t	18
B	New recruitment rate	0.20
β	Rate at which susceptible female takes calorie food	0.20
δ	Rate at which susceptible female acquires obesity due to high calorie food	0.25
η	Rate at which creates infertility in female due to obesity	0.70
α	Rate at which susceptible female is naturally obese	0.05
ε	Rate at which infertile female starts consuming calorie food	0.40
μ	Natural escape rate	0.40

The transmission diagram of the female being infertile due to high calorie food is shown in Figure 1.

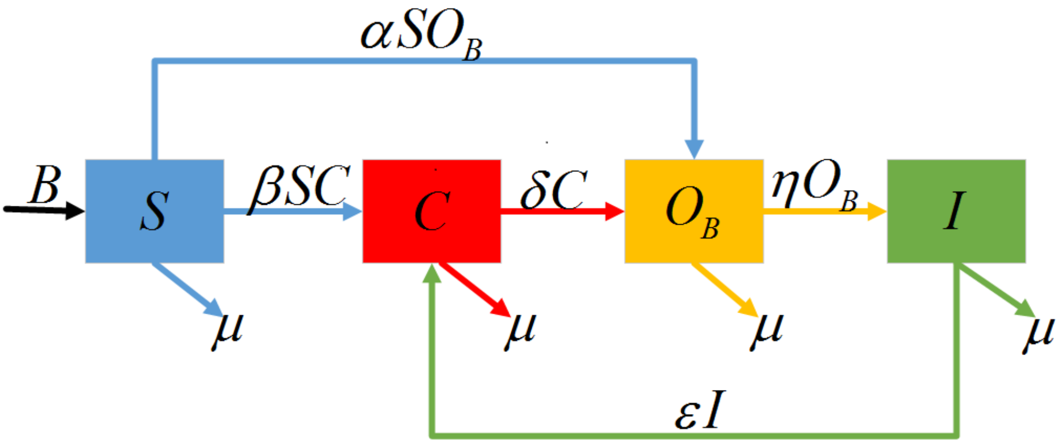

Figure 1. Transmission of infertility model

In this model, susceptible female (S) becomes obese in two ways. Firstly susceptible female becomes naturally obese at the rate α and secondly β is the rate which describes the female consuming high calorie food (C); and become obese at the rate δ. This obesity leads to infertility among female at the rate η. It may happen that this infertile may again starts to take calorie food in their diet at the rate ε. Here, B and μ describes new recruitment rate and natural escape rate respectively.

Figure 1 is described by the system of following non-linear ordinary differential equations.

$$\begin{aligned}
\frac{dS}{dt} &= B - \alpha S O_B - \beta S C - \mu S \\
\frac{dC}{dt} &= \beta S C - \delta C + \varepsilon I - \mu C \\
\frac{dO_B}{dt} &= \alpha S O_B + \delta C - \eta O_B - \mu O_B \\
\frac{dI}{dt} &= \eta O_B - \varepsilon I - \mu I
\end{aligned} \tag{1}$$

Here, $S + C + O_B + I \le N$ and $S > 0, C, O_B, I \ge 0$.

Adding the above set of systems of equations (1) we get,

$$\frac{d}{dt}\left(S + C + O_B + I\right) = B - \mu\left(S + C + O_B + I\right) \ge 0 \tag{2}$$

This gives, $\lim_{t\to\infty}\sup\left(S + C + O_B + I\right) \le \frac{B}{\mu}$. (3)

Thus the feasible region for (1) is,

$$\Lambda = \left\{ \left(S + C + O_B + I\right) \big/ S + C + O_B + I \le \frac{B}{\mu}, S > 0, C, O_B, I \ge 0 \right\}. \tag{4}$$

On solving these set of equation (1) by putting equal to zero we get the equilibrium point. Therefore, the equilibrium point of the model is $E_0 = \left(\frac{B}{\mu}, 0, 0, 0\right)$.

Now, we need to calculate the basic reproduction number to know the motion of susceptible female in the system using the next generation matrix method [1]. The next generation matrix method gives spectral radius of matrix fv^{-1} where f and v are the Jacobian matrices of F and V evaluated with respect to each compartment at an equilibrium state.

Let $X = \left(C, O_B, I, S\right)$

$$\therefore \frac{dX}{dt} = F\left(X\right) - V\left(X\right) \tag{5}$$

where $F\left(X\right)$ denotes the rate of new obese female in the compartment and $V\left(X\right)$ denotes the rate transmission of obese female from one compartment to other which is as follows

$$F\left(X\right) = \begin{bmatrix} \beta SC \\ \alpha SO_B \\ 0 \\ 0 \end{bmatrix} \text{ and } V\left(X\right) = \begin{bmatrix} \delta C - \varepsilon I + \mu C \\ -\delta C + \eta O_B + \mu O_B \\ -\eta O_B + \varepsilon I + \mu I \\ -B + \alpha SO_B + \beta SC + \mu S \end{bmatrix}$$

Now, the derivative of F and V calculated at equilibrium point $\left(E_0\right)$ gives matrices f and v of order 4×4 defined as,

$$f = \left[\frac{\partial F_i\left(E_0\right)}{\partial X_j}\right] \text{ and } v = \left[\frac{\partial V_i\left(E_0\right)}{\partial X_j}\right] \text{ for } i, j = 1, 2, 3, 4$$

$$\text{So, } f = \begin{bmatrix} \frac{\beta B}{\mu} & 0 & 0 & 0 \\ 0 & \frac{\alpha B}{\mu} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ and } v = \begin{bmatrix} \delta + \mu & 0 & -\varepsilon & 0 \\ -\delta & \eta + \mu & 0 & 0 \\ 0 & -\eta & \varepsilon + \mu & 0 \\ \frac{\beta B}{\mu} & \frac{\alpha B}{\mu} & 0 & \mu \end{bmatrix}$$

Thus, the basic reproduction number R_0 is calculated at an equilibrium point E_0 is

$$R_0 = \frac{\beta B\left(\varepsilon(\eta+\mu)+\mu(\eta+\mu)\right)}{\mu^2\left(\varepsilon(\eta+\delta+\mu)+\eta(\delta+\mu)+\mu(\delta+\mu)\right)} \tag{6}$$

On solving the above set of equation (1) we get the other equilibrium point which $E^* = \left(S^*, C^*, O_B^*, I^*\right)$

Here,

$S^* = A$

$$C^* = \frac{\left[\begin{array}{l} A\alpha\left(\beta B\left\{\mu(2\varepsilon+\mu+\eta)+\varepsilon(\eta+\varepsilon)\right\}+\mu\left(\begin{array}{l}\varepsilon\left\{\mu(-2\mu-2\delta-2\eta)-\varepsilon(\mu+\delta)-\eta(\delta+\varepsilon)\right\}\\ +\mu\left\{-\delta(\mu+\eta)-\mu(\mu+\eta)\right\}\end{array}\right)\right)\\ +\mu^2\left\{\eta\left(\eta(\mu+\delta+\varepsilon)+2\delta(\varepsilon+\mu)\right)+\mu\left(\mu(2\eta+\mu+\delta)+\varepsilon(3\eta+2\delta)\right)+\varepsilon^2(\eta+\delta+\mu)\right\}\\ +B\left\{\left(\beta\mu\left\{\mu(-2\varepsilon-2\eta-\mu)-\eta(3\varepsilon+\eta)\right\}+\varepsilon^2\eta(\alpha+\beta)\right)+\eta\varepsilon(\alpha\mu-\beta\eta)\right\} \end{array}\right]}{\left[\mu\left\{\mu(\mu+\delta+\eta+\varepsilon)+\delta\eta+\varepsilon(\delta+\eta)\right\}\left\{\mu(\beta-\alpha)+\beta(\varepsilon+\eta)-\alpha\varepsilon\right\}\right]}$$

$$O_B^* = \frac{\delta\left[A\mu(\mu+\varepsilon)+B(\varepsilon+\mu)\right]}{\mu\left[\mu(-\mu+A\alpha-\delta-\eta-\varepsilon)-\eta(\delta+\varepsilon)-\varepsilon(\delta-A\alpha)\right]}$$

$$I^* = \frac{-\left[(-A\mu+B)\delta\eta\right]}{\mu\left[\mu(-\mu+A\alpha-\delta-\eta-\varepsilon)-\eta(\delta+\varepsilon)-\varepsilon(\delta-A\alpha)\right]}$$

where,

$$A = \frac{1}{2\left(\beta\alpha(\varepsilon+\mu)\right)}\left[\begin{array}{l}(\mu+\varepsilon)\left(\alpha(\delta+\mu)\right)+\beta\left(\mu(\varepsilon+\mu+\eta)+\eta\varepsilon\right)\pm\\ \left(\begin{array}{l}\mu^2\left(\begin{array}{l}\beta\left[\begin{array}{l}\alpha\left\{2\left(\eta(-\delta-2\varepsilon-\mu)+\delta(-2\varepsilon-\mu)\right)-\varepsilon(2\varepsilon+4\mu)\right\}\\ +\beta\left\{\varepsilon^2+\mu\left(\mu+2(\varepsilon+\eta)\right)+\eta(\eta+4\varepsilon)\right\}\end{array}\right]\\ +\alpha\left[\left\{\mu\left(\mu(\alpha-2\beta)+2\alpha\varepsilon\right)\right\}+\alpha\left\{\delta\left(\delta+2\mu+\varepsilon(\varepsilon+4\delta)\right)\right\}\right]\end{array}\right)\\ +\varepsilon\left[\begin{array}{l}\varepsilon\left(\alpha\left\{\beta\left\{2\mu(-\delta-\eta)+2\delta\eta\right\}+\alpha\delta(\delta+2\mu)\right\}+\beta^2\eta(\eta+2\mu)\right)\\ +2\mu\left(\delta^2\alpha^2+\beta^2\eta^2\right)\end{array}\right]\end{array}\right)^{\frac{1}{2}}\end{array}\right]$$

3. Stability Analysis

The equilibrium for the local and global stability of the infertility model is discussed here.

3.1. Local Stability

First, we calculate the local stability behaviour of equilibrium $E_0 = \left(\frac{B}{\mu}, 0, 0\right)$ by using Jacobian matrix J_0 [10][11]. The Jacobian matrix of the given model is as given below

$$J_0 = \begin{bmatrix} -\mu & -\beta\frac{B}{\mu} & -\alpha\frac{B}{\mu} & 0 \\ 0 & \beta\frac{B}{\mu}-\delta-\mu & 0 & \varepsilon \\ 0 & \delta & \alpha\frac{B}{\mu}-\eta-\mu & 0 \\ 0 & 0 & \eta & -\varepsilon-\mu \end{bmatrix}$$

$$trace(J_0) < 0 \text{ provided } (\alpha+\beta)\frac{B}{\mu} < 4\mu+\delta+\eta+\varepsilon \tag{7}$$

Hence, E_0 is locally stable.

Next, we determine the local stability behavior of equilibrium $E^* = \left(S^*, C^*, O_B^*, I^*\right)$ by using the Ruth-Hurwitz criteria [8].

The Jacobian matrix J for equilibrium E^* of the model is as follows:

$$J = \begin{bmatrix} -\alpha O_B^* - \beta C^* - \mu & -\beta S^* & -\alpha S^* & 0 \\ \beta C^* & \beta S^* - \delta - \mu & 0 & \varepsilon \\ \alpha O_B^* & \delta & \alpha S^* - \eta - \mu & 0 \\ 0 & 0 & \eta & -\varepsilon-\mu \end{bmatrix}$$

Here, we take $\alpha O_B^* + \beta C^* + \mu = a_{11}$; $-\beta S^* + \delta + \mu = a_{22}$; $-\alpha S^* + \eta + \mu = a_{33}$; $\varepsilon + \mu = a_{44}$. Then the Jacobian J is reduced as

$$J = \begin{bmatrix} -a_{11} & -\beta S^* & -\alpha S^* & 0 \\ \beta C^* & -a_{22} & 0 & \varepsilon \\ \alpha O_B^* & \delta & -a_{33} & 0 \\ 0 & 0 & \eta & -a_{44} \end{bmatrix}$$

Now, the corresponding characteristic equation of Jacobian matrix J is

$$x^4 + A_{11}x^3 + A_{22}x^2 + A_{33}x + A_{44} = 0$$

where

$$\begin{aligned}
A_{11} &= a_{44} + a_{33} + a_{22} + a_{11} \\
A_{22} &= a_{44}\left(a_{33} + a_{22} + a_{11}\right) + a_{33}\left(a_{22} + a_{11}\right) + S^*\left(\alpha^2 O^* + \beta^2 C^*\right) \\
A_{33} &= a_{44}a_{33}\left(a_{22} + a_{11}\right) + a_{22}a_{11}\left(a_{44} + a_{33}\right) + a_{44}S^*\left(\alpha^2 O^* + \beta^2 C^*\right) \\
&\quad + \delta\left(\beta\alpha C^* S^* - \eta\varepsilon\right) + S^*\left(\alpha^2 O^* + \beta^2 C^*\right) \\
A_{44} &= a_{44}a_{33}\left(\beta^2 C^* S^* + a_{22}a_{11}\right) + a_{44}S^*\left(\alpha^2 O^* a_{22} + \alpha\beta\delta C^*\right) + \eta\varepsilon\left(\alpha\beta O^* S^* - \delta a_{11}\right)
\end{aligned}$$

Here, $A_{11} > 0, A_{22} > 0, A_{33} > 0 \, and \, A_{44} > 0$ and satisfy the condition of Routh-Hurwitz criterion (Routh E.J. 1877). Then the equilibrium E^* is locally stable**.**

3.2. Global Stability

The infertility model is globally stable if $\det\left(I - fv^{-1}\right) > 0$ [10][11].

$$\det\left(I - fv^{-1}\right) = 1 - R_0 = 1 - 0.1752 = 0.8248 > 0 \,. \tag{8}$$

Hence, E_0 is globally stable.

Now, we discuss the global stability behavior of E^* by Lyapunov function [9].

Consider the Lyapunov function

$$\begin{aligned}
L(t) &= \frac{1}{2}\left[\left(S - S^*\right) + \left(C - C^*\right) + \left(O_B - O_B^*\right) + \left(I - I^*\right)\right]^2 \\
L'(t) &= \left[\left(S - S^*\right) + \left(C - C^*\right) + \left(O_B - O_B^*\right) + \left(I - I^*\right)\right]\left[S' + C' + O_B' + I'\right] \\
&= \left[\left(S - S^*\right) + \left(C - C^*\right) + \left(O_B - O_B^*\right) + \left(I - I^*\right)\right] \\
&\qquad \left[\mu S^* + \mu C^* + \mu O_B^* + \mu I^* - \mu S - \mu C - \mu O_B - \mu I\right] \\
&= -\mu\left[\left(S - S^*\right) + \left(C - C^*\right) + \left(O_B - O_B^*\right) + \left(I - I^*\right)\right]^2 \le 0
\end{aligned}$$

Here we used, $B = \mu S^* + \mu C^* + \mu O_B^* + \mu I^*$.

Hence, E^* is globally stable.

4 Sensitivity Analysis

In this section, the sensitivity analysis for all model parameters is discussed in Table 2. The normalised sensitivity index of the parameters is computed by using the formula: $\gamma_{\theta}^{R_0} = \frac{\partial R_0}{\partial \theta} \cdot \frac{\theta}{R_0}$ where, θ denotes the model parameter [10].

Table 1. Notations and its Parametric Values

Parameter	Value	Parameter	Value
B	+	ε	+
β	+	δ	-
η	+	μ	-

New recruitment rate, rate at which susceptible female takes calorie food, rate at which creates infertility in female due to obesity and rate at which infertile female starts consuming calorie food have positive effect on R_0 which means that these parameters help to reduce infertility. The rate at which susceptible female is naturally obese have no impact on model. Other parameters have negative impact on model

5 Numerical Simulation

In this section, we will study the numerical results of all compartments.

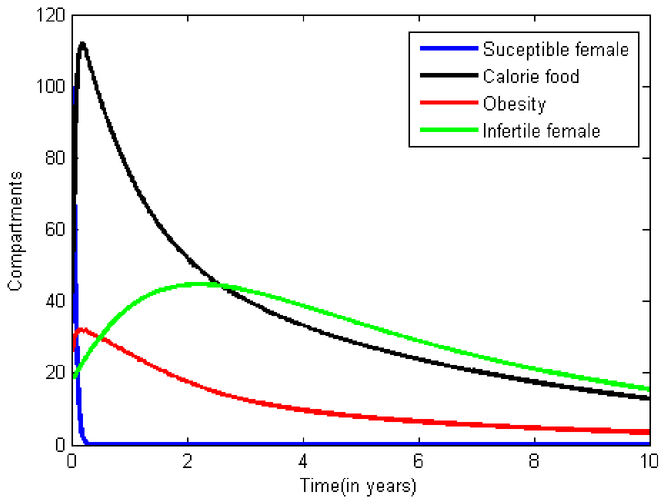

Figure 2. Movement of an individual in each compartment

Figure 2 shows that if the susceptible female who are likely to become obese starts taking more calorie food then the obesity due to it increases approximate in 1 year and this leads to increase in the infertility among female individuals.

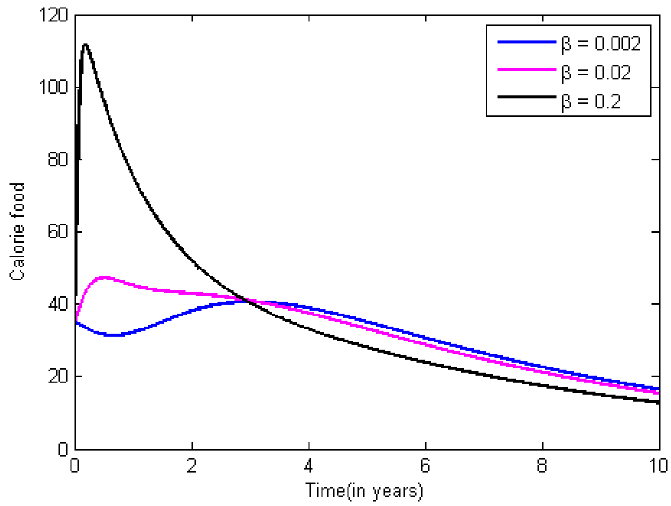

Figure 3. Effect of rate at which susceptible female takes calorie food

Figure 3 shows that if the rate at which susceptible female takes calorie food (β) is small then females are at low risk to be obese for almost 1 year and after that it increases but the opposite effect is observed for the large value of (β).

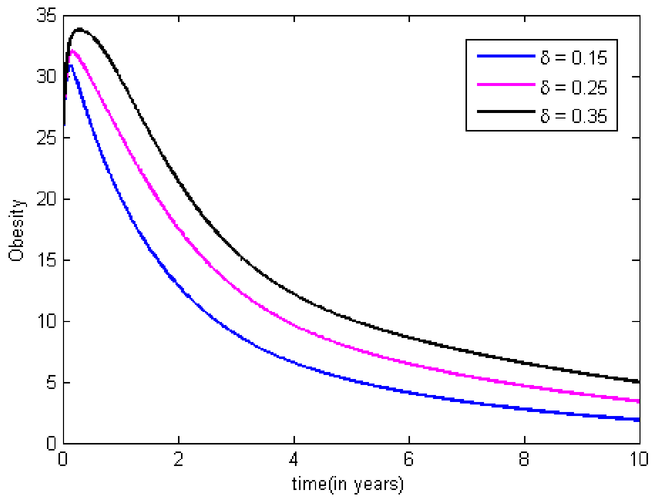

Figure 4. Effect of rate at which susceptible female acquires obesity due to high calorie food

Figure 4 indicates that if the rate of susceptible female become obese due to high calorie food (δ) is increased from 15% to 35% the, obesity in susceptible female is increased initially and there after it decreases.

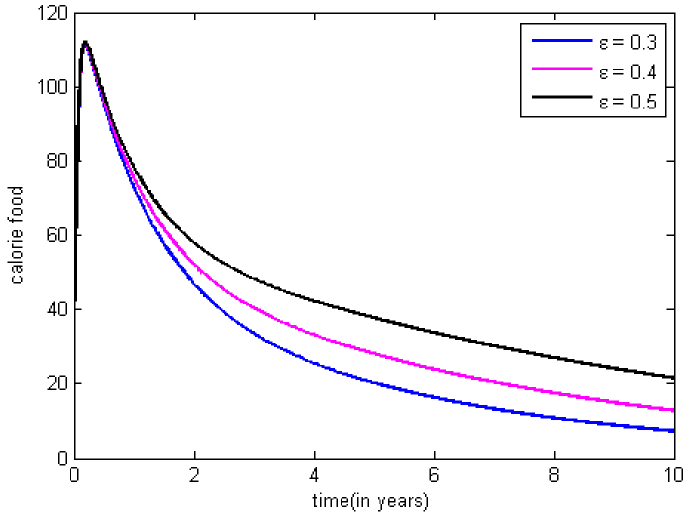

Figure 5. Effect of rate at which infertile female start consuming calorie food

Figure 5 shows that if the rate at which infertile female starts consuming calorie food (ε) is increased than number of females taking calorie food also increases almost equal in the one year after that it decreases which means individual female get conscious about their health.

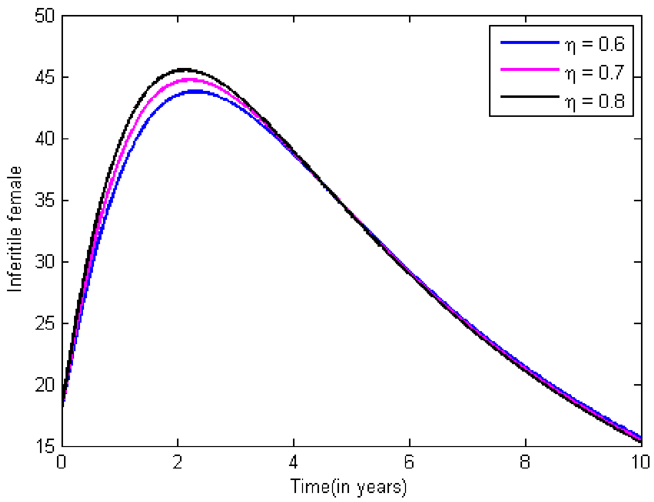

Figure 6. Effect of rate at which creates infertility in female due to obesity

Figure 6 indicates that if the rate which causes infertility in female due to obesity (η) is increased from 60% to 80% then the number of infertile female increase by approximate 43 to 46 in first two years after that it starts decreasing.

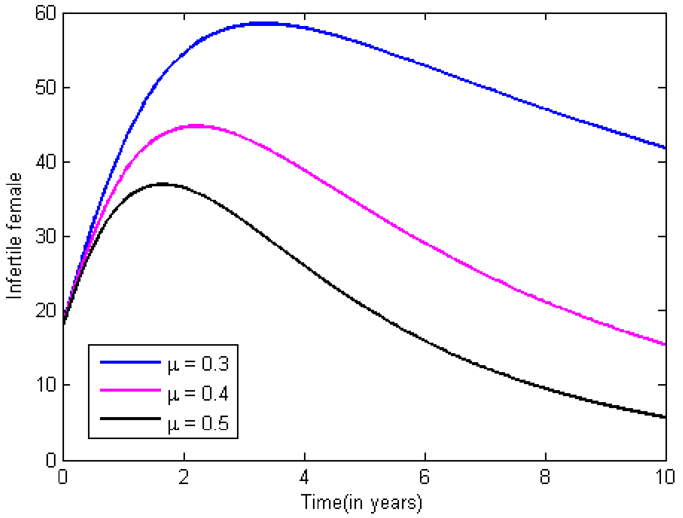

Figure 7. Effect of natural escape rate on infertility

Figure 7 shows that if one increases the natural escape rate from 30% to 50% the number of individuals increase in beginning for 3 years approximate by 36 to 60 and then after that it decreases which shows that after some years infertile female start taking healthy food and become less obese.

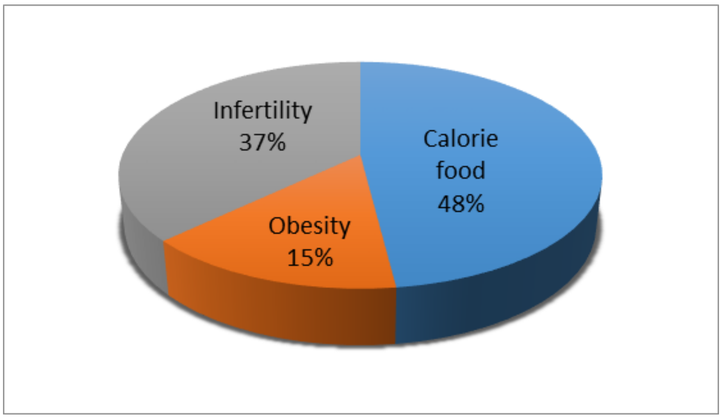

Figure 8. Percentage of calorie food, obesity and infertility in susceptible female

Figure 8 shows that if females start taking 48% calorie food in their diet, then there are 15% of chances for becoming obese, due to which 37% of infertility can prevail among these female individuals during their maturity stage.

5 Conclusion

In this paper, a mathematical model for the cause of infertility due to obesity is formulated. Also the system proves to be locally and globally stable at an equilibrium point by satisfying all its condition. The basic reproduction number is computed as 0.1752, which show that 17% of female affected by infertility due to high calorie food. The fact is infertility tends to go up in the proportion to the amount of access weight one is carrying. It is impossible to get read of this risk without losing weight, so it very important to maintain our health by regular physical activity or by any other means. Every individual should take care of their health so as to avoid risk in future. This model will help to educate society to become more health conscious.

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Bδg-Homeomorphisms

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Abstaract − In this paper we introduce two new classes of mappings called Bδg-homeomorphism and Bδgc-homeomorphism which are defined using Bδg-closed sets and study their basic properties. We also investigate its group structure of their subgroups. We also investigate its relationship with other types of mappings.

Keywords − *Semi-generalized set, semi-homeomorphism mapping, Bδg-homeomorphism mapping, Bδg-closed set.*

1 Introduction

Maki et al. [13] introduced the notions of generalized homeomorphism (briefly g-homeomorphism). Devi et al. [2] introduced two classes of mappings called generalized semi-homeomorphism (briefly gs-homeomorphism) and semi- generalized homeomorphism (briefly sg-homeomorphisms). In this present paper we introduce new class of generalization of homeomorphisms called Bδg-homeomorphisms using Bδg-closed sets. We further introduce generalization of homeomorphisms called Bδgc-homeomorphism. Basic properties of these two mappings are studied and the relation between these types and other existing ones are established.

2 Preliminary

Throughout this paper, a space (X,τ) (or simply X) represents a topological spaces on which no separation axioms are assumed unless otherwise mentioned. For a subset A of X, cl(A), int(A) and A^c denote the closure of A, the interior of A and

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the complement of A respectively. Let us recall the following definitions, which are useful in the sequel.

Definition 2.1. (i) semi-open set [11] if A ⊆ cl(int(A)).
(ii) preopen set [15] if A ⊆ int(cl(A)).
(iii) α-open set [19] if A ⊆ int(cl(int(A))).
(iv) regular open set [23] if A $\doteq$ int(cl(A)).

The complement of a semi-open (resp. preopen, -open) set is called semi-closed (resp. preclosed, -closed).

The α-closure (resp. semi-closure, preclosure) of A ⊆ X is the smallest -closed (resp. semi-closed, preclosed) set containing A. cl(A) (resp. scl(A), pcl(A)) is called the -closure (resp. semi-closure, preclosure) of A.

Definition 2.2. The -interior [27] of a subset A of X is the union of all regular open sets of X contained in A and is denoted by int_g (A). The subset A is called δ-open [27] if A = int_g (A), i.e. a set is δ -open if it is the finite union of regular open sets. The complement of a δ-open set is called δ-closed. Alternatively, a set A ⊆ (X, τ)is called δ-closed[27] if A = $cl_\delta(A)$, where

$$cl_\delta(A) = \{x \in X : int(cl(U) \bigcap A \neq \phi, U \in \tau \text{ and } x \in U\}$$

Definition 2.3. A subset A of a space (X, τ) is called a
(i) generalized closed (briefly g-closed) set [12] if cl(A)⊆ U whenever A ⊆ U and U is open in (X, τ).
(ii) generalized semi-closed (briefly gs-closed) set [1] if scl(A)⊆ U whenever A ⊆ U and U is open in (X,τ).
(iii) α-generalized closed (briefly αg-closed) set [13] if αcl(A)⊆ U whenever A ⊆ U and U is open in (X,τ).
(iv) δ-generalized closed (briefly δg-closed) set [3] if $cl_\delta(A)$ ⊆ U whenever A ⊆ U and U is open in (X,τ).
(v) $\widehat{g}$-closed set [26] if cl(A) ⊆ U whenever A ⊆ U and U is semi-open in (X, τ).
(vi) $\widehat{g}$-closed (briefly δ $\widehat{g}$-closed) set [8] if $cl_\delta(A)$ ⊆ U whenever A ⊆ U and U is $\widehat{g}$-open in (X, τ).

The complement of a g-closed (resp. gs-closed, αg-closed, δg-closed, $\widehat{g}$-closed and δ $\widehat{g}$ -closed)set is called g-open (resp. gs-open, αg-open, δg-open,$\widehat{g}$-open and δ $\widehat{g}$-open)

Definition 2.4. A subset A of a space (X, τ) is called a
(i) t-set infinite(A) = int(cl(A))
(ii) B-set if A = G ∩ F where G is open and F is a t-set in X.

Definition 2.5. A space (X, τ) is called a
(i) $T_{\frac{1}{2}}$-space [12] if every g-closed set in it is closed.
(ii) $T_{\frac{3}{4}}$ [3] if every δg-closed set in it is δ-closed.
(iii) $T_{\frac{3}{4}}$-space [8] if every δ $\widehat{g}$-closed set in it is δ - closed.

Definition 2.6. A map f:(X,τ) $\rightarrow$ (Y,σ)is called

(i) g-continuous [2] if f^{-1}(V) is g-closed in (X,τ) for every closed set V of (Y,σ).

(ii) gs-continuous [2] if f^{-1}(V) is gs-closed in (X,τ) for every closed set V of (Y,σ).

(iii) δ $\widehat{g}$-continuous [2] if f^{-1}(V) is δ $\widehat{g}$-closed in (X,τ) for every closed set V of (Y,σ).

Definition 2.7. A map f:(X,τ) $\rightarrow$ (Y,σ) is called

(i) generalized closed (briefly g-closed) (resp. g-open) [15] if the image of every closed (resp. open) set in (X,τ) is g-closed (resp. g-open) in (Y,σ).

(ii) generalized semi-closed (briefly gs-closed) (resp. gs-open) [2] if the image of every closed (resp. open) set in (X,τ) is gs-closed (resp. gs-open) in (Y,σ).

(iii) δ-closed [19] if f(V) is δ-closed in (Y,σ)for every δ-closed set V of (X,τ)

(iv) δ $\widehat{g}$-closed [8] if the image of every closed set in (X,τ) is δ $\widehat{g}$-closed in (Y,σ).

Definition 2.8. A map f:(X,τ) $\rightarrow$ (Y,σ)is called

(i) g-homeomorphism [13] if f is bijection, g-open and g-continuous.

(ii) gs-homeomorphism [2] if f is bijection, gs-open and gs-continuous.

(iii) δ-closed [19] if f(V) is δ-closed in (Y,σ)for every δ-closed set V of (X,τ)

(iv) δ $\widehat{g}$-homeomorphism [8] if f is bijection, δ $\widehat{g}$ - open and δ $\widehat{g}$ - continuous.

Proposition 2.9 (8)**.** Every δ -closed set is δ $\widehat{g}$ -closed set.

3 Properties Of Bδg-homeomorphisms

Definition 3.1. A subset A of (X,τ) is called Bδg- closed if cl_δ(A) $\subseteq$ U whenever A $\subseteq$ U and U is B-set

The Complement of Bδg - closed set is Bδg -open.

Definition 3.2. A bijection map f:(X,τ) $\rightarrow$ (Y,σ) is called Bδg - continuous if f is both Bδg -continuous and Bδg -open.

Definition 3.3. A bijection map f:(X,τ) $\rightarrow$ (Y,σ) is called Bδg - continuous if f^{-1}(V) is Bδg -closed in (X,τ) for every closed set V of (Y,σ)

Definition 3.4. A bijection map f:(X,τ) $\rightarrow$ (Y,σ) is called Bδg - irresolute if f^{-1}(V) is Bδg -closed in (X,τ) for every closed set V of (Y,σ)

Definition 3.5. A bijection map f:(X,τ) $\rightarrow$ (Y,σ) is called Bδg - closed if the image of closed set in (X,τ) is Bδg - closed in (Y,σ)

Definition 3.6. A bijection map f:(X,τ) $\rightarrow$ (Y,σ) is called Bδg - homeomorphism if f is both Bδg - continuous and Bδg - open.

Definition 3.7. A space X is called a $_BT_{\delta g}$- space if every Bδg ? closed set in it is δ closed.

Theorem 3.8. Every Bg-homeomorphism is gs-homeomorphism..

Proof. Let f:(X,τ) $\to$ (Y,σ) be Bδg-homeomorphism. Then f is bijective, Bδg-continuous and Bδg-open map. Let V be an closed set in (Y,σ). Then $f^{-1}(V)$ is Bδg-closed in (X,τ). Every Bδg-closed set is gs-closed and hence $f^{-1}(V)$ is gs-closed in (X,τ). This implies that f is gs-continuous. Let U be an open set in (X,τ). Then f(U) is Bδg-open in (Y,σ). This implies f is gs-open map. Hence f is gs-homeomorphism. □

Remark 3.9. The following example shows that the converse of the above theorem need not be true.

Example 3.10. Let X = $\{a, b, c\}$ and Y = $\{p, q, r\}$ with the topologies $\tau = \{\phi, \{a\}, \{b\}, \{a, b\}, X\}$ and $\sigma = \{\phi, \{p\}, \{q\}, \{p, q\}, Y\}$. Define f: (X,τ) $\to$ (Y,σ) by $f(a) = p$; $f(b) = q$ and $f(c) = r$. Clearly f is gs-homeomorphism but f is not Bδg-homeomorphism because f($\{b\}$) = f$\{q\}$ is not a Bδg-open in (Y,σ) where $\{b\}$ is open in (X,τ)

Theorem 3.11. Every Bδg-homeomorphism is g-homeomorphism.

Proof. Follows from the fact that every Bδg- continuous map is g- continuous map and every Bδg-open map is g-open map. □

Remark 3.12. The converse of the above theorem need not be true as it can be seen from the following example.

Example 3.13. Let X = $\{a, b, c\}$ and Y = $\{p, q, r\}$ with the topologies $\tau = \{\phi, \{a\}, \{b\}, \{a, b\}, X\}$ and $\sigma = \{\phi, \{p\}, \{p, r\}, Y\}$. Define f:(X,τ) $\to$ (Y,σ) by $f(a) = p$; $f(b) = r$ and $f(c) = q$. Clearly f is g-homeomorphism but f is not Bδg-homeomorphism because f($\{a, b\}$) = f$\{p, r\}$ is not a Bδg-open in (Y,σ) where (a,b) is open in (X,τ)

Remark 3.14. Homeomorphisms and Bδg-homeomorphisms are independent of each other as shown in the following examples.

Example 3.15. Let X = $\{a, b, c\}$ and Y = $\{p, q, r\}$ with the topologies $\tau = \{\phi, \{a\}, \{b, c\}, X\}$ and $\sigma = \{\phi, \{p\}, \{r\}, \{p, r\}, \{q, r\}, Y\}$. Define a map f:(X,τ) $\to$ (Y,σ) by $f(a) = p$; $f(b) = q$ and $f(c) = r$. Then f is Bδg - open and Bδg - continuous. Hence f is a Bδg homeomorphism. However $f^{-1}(\{p, q\}) = \{a, b\}$ is not closed in (X,τ) where $\{p, q\}$ is closed in (Y,σ) and hence f is not continuous. Therefore, f is not homeomorphism.

Example 3.16. Let X = $\{a, b, c\}$ and Y = $\{p, q, r\}$ with the topologies $\tau = \{\phi, \{a\}, \{a, b\}, X\}$ and $\sigma = \{\phi, \{q\}, \{p, q\}, Y\}$. Define a function f:(X,τ) $\to$ (Y,σ) by $f(a) = q$; $f(b) = p$ and $f(c) = r$. Then f is a homeomorphism. The set $\{a, b\}$ is open in (X,τ) but $f(\{a, b\}) = \{p, q\}$ is not Bδg - open in (Y,σ). This implies that f is not Bδg - open map. Hence f is not a Bδg - homeomorphism.

Remark 3.17. The concepts of Bδg-homeomorphism and $\delta\ \widehat{g}$-homeomorphism are independent of each other as shown in the following examples.

Example 3.18. Let X = $\{a, b, c\}$ and Y = $\{p, q, r\}$ with the topologies $\tau = \{\phi, \{a\}, X\}$ and $\sigma = \{\phi, \{q\}, Y\}$. Define a map f:(X,τ) $\rightarrow$ (Y,σ) by $f(a) = q$; $f(b) = p$ and $f(c) = r$. Then f is a δ $\widehat{g}$-homeomorphism. Here the set $\{a\}$ is open in (X,τ) but $f(\{a\}) = \{q\}$ is not Bδg - open in (Y,σ). This implies that f is not Bδg - open map. Hence f is not a Bδg - homeomorphism.

Example 3.19. Let X = $\{a, b, c\}$ and Y = $\{p, q, r\}$ with the topologies $\tau = \{\phi, \{a\}, \{b, c\}, X\}$ and $\sigma = \{\phi, \{r\}, \{p, q\}, Y\}$. Define a map f:(X,τ) $\rightarrow$ (Y,σ) by $f(a) = q$; $f(b) = p$ and $f(c) = r$. Then f is a Bδg-homeomorphism but f is not δ $\widehat{g}$-homeomorphism because $f(\{b, c\}) = \{p, r\}$ is not δ $\widehat{g}$- open in (Y,σ) where $\{b, c\}$ is open in (X,τ).

Proposition 3.20. For any bijective map f:(X,τ) $\rightarrow$ (Y,σ)the following statements are equivalent.
(i) f^{-1}:(Y,σ)$\rightarrow$(X,τ)is Bδg-continuous map.
(ii) f is a Bδg-open map.
(iii) f is a Bδg-closed map.

Proof. (i) $\Rightarrow$ (ii) Let U be an open set in (X,τ). Since, f^{-1} is Bδg continuous, $(f^{-1})^{-1}$(U) is Bδg- open in (Y,σ). Hence f is Bδg open map. (ii) $\Rightarrow$(iii). Let F be a closed set in (X,τ). Then F^c is open in (X,τ).Since f is Bδg open map, $f(F^c)$ is Bδg open set in (Y,σ). But $f(F^c) = f(F^c)$ is Bδg open set in (Y,σ). This implies that $f(F^c)$ is Bδg open set in (Y,σ). Hence f is Bδg closed map.(iii)?(i).Let V be a closed set of(X,τ).Since f is Bδg closed map, f(V) is Bδg closed in (Y,σ). That is $(f^{-1})^{-1}$(V) is Bδg closed set in (Y,σ). Hence f^{-1} is Bδg continuous. □

Theorem 3.21. Let f:(X,τ) $\rightarrow$ (Y,σ) be a bijective and Bδg - continuous map. Then the following statements are equivalent.
(i) f is a Bδg-open map.
(ii) f is a Bδg-homeomorphism.
(iii) f is an Bδg-closed map.

Proof. (i) $\Rightarrow$ (ii). Let f be Bδg-open map. By hypothesis, f is bijective and Bδg-continuous. Hence f is Bδg-homeomorphism. (ii) $\Rightarrow$ (iii). Let f be Bδg- homeomorphism. Then f is Bδg-open. By Proposition 3.19 f is Bδg-closed map.(iii) $\Rightarrow$ (i). It is obtained from Proposition 3.19. □

Remark 3.22. The composition of two Bδg-homeomorphisms need not be Bδg-homeomorphism as the following example shows.

Example 3.23. Let X = $\{a, b, c\}$ = Y = Z with the topologies $\tau = \{\phi, \{a\}, \{b, c\}, X\}$ and $\sigma = \{\phi, \{c\}, \{a, b\}, Y\}$ and $\eta = \{\phi, \{a\}, \{b\}, \{a, b\}, Z\}$. Let f:(X,τ) $\rightarrow$ (Y,σ) and g:(Y,σ) $\rightarrow$ (Z,η) be two identity maps. Then both f and g are Bδg-homeomorphism. The set $\{b, c\}$ is open in (X,τ) but $g \circ f$ $(\{b, c\}) = \{b, c\}$ is not Bδg - open in (Z,η). This implies that $g \circ f$ is not Bδg - open and hence $g \circ f$ is not Bδg - homeomorphism.

We introduce the following definition.

Definition 3.24. A bijection map f:(X,τ) $\rightarrow$ (Y,σ) is said to be Bδgc - homeomorphism if f is Bδg - irresolute and its inverse f^{-1} is Bδg - irresolute.

Remark 3.25. Bδgc - homeomorphisms and Bδg- homeomorphisms are independent notions as shown in the following examples.

Example 3.26. Let X $= \{a, b, c\}$ and Y $= \{p, q, r\}$ with the topologies $\tau = \{\phi, \{a\}, X\}$ and $\sigma = \{\phi, \{q\}, Y\}$. Define a map f:(X,τ) $\rightarrow$ (Y,σ) by $f(a) = p$; $f(b) = q$ and $f(c) = r$. Then f is a Bδg - homeomorphism. The set $\{q, r\}$ is Bδg - closed in (Y,σ) but $f^{-1}(\{q, r\}) = \{b, c\}$ is not Bδg - closed in (X,τ). Therfore f is not Bδg - irresolute and hence f is not a Bδgc - homeomorphism.

Example 3.27. Let X $= \{a, b, c\}$ and Y $= \{p, q, r\}$ with the topologies $\tau = \{\phi, \{b\}, \{b, c\}, X\}$ and $\sigma = \{\phi, \{r\}, \{p, r\}, Y\}$. Define a map f:(X,τ) $\rightarrow$ (Y,σ) by $f(a) = r$; $f(b) = q$ and $f(c) = p$. Then f is Bδgc - homeomorphism. But f is not Bδg - homeomorphism because $f(\{b, c\})$ is not Bδg - open in (Y,σ) where $\{b, c\}$ is open in (X,τ).

Remark 3.28. From the above discussion we get the following diagram. A $\rightarrow$ B represents A implies B. A $\nrightarrow$ B represents A does not implies B.

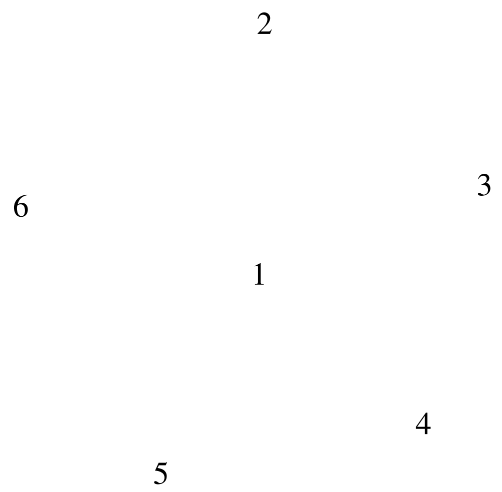

Figure 1: 1. Bδg-Homeomorphism 2. gs-Homeomorphism 3. g-Homeomorphism 4. $\delta\widehat{g}$-Homeomorphism 5. Bδgc-Homeomorphism 6.Homeomorphism

Theorem 3.29. The composition of two Bδgc-homeomorphisms is Bδgc-homeomorphism.

Proof. Let f:(X,τ) $\rightarrow$ (Y,σ) and g:(Y,σ) $\rightarrow$ (Z,η) be two Bδgc-homeomorphisms. Let F be a Bδg-closed set in (Z,η). Since g is Bδg-irresolute map, $g^{-1}(F)$ is Bδg-closed in (Y,σ). Since f is Bδg-irresolute, $f^{-1}(g^{-1}(F))$ is Bδg-closed in (X,τ). That is $(g \bigcirc f)^{-1}(F)$ is Bδg-closed in (X,τ). This implies $g \bigcirc f$ is Bδg-irresolute. Let G be Bδg-closed in (X,τ). Since f^{-1} is Bδg-irresolute, $(f^{-1})^{-1}(G) = f(G)$ is Bδg-closed in (Y,σ). Since g^{-1} is Bδg-irresolute, $(g^{-1})^{-1}(f(G))$ is Bδg-closed in (Z,η). That is $g(f(G))$ is Bδg-closed in (Z,η). Therefore $(g \bigcirc f)(G)$ is Bδg-closed in (Z;?)(Z,η). This implies that$((g \bigcirc f)^{-1})^{-1}(G)$ is Bδg-closed in (Z,η). This shows that $(g \bigcirc f)^{-1}$ is Bδg-irresolute. Hence $g \bigcirc f$ is Bδgc-homeomorphism. □

4 Application

Theorem 4.1. Every Bδg-homeomorphism from a ${}_BT_{\delta g}$-space into another ${}_BT_{\delta g}$-space is a homeomorphism.

Proof. Let f:(X,τ) $\rightarrow$ (Y,σ) be Bδg-homeomorphism. Then f is bijective, Bδg-open and Bδg-continuous maps. Let U be an open in (X,τ). Since f is Bδg-open and since (Y,σ) is ${}_BT_{\delta g}$ - space, f(U) is open set in (Y,σ). This implies f is open map. Let V be a closed set in (Y,σ). Since f is Bδg-continuous and since (X,τ) is ${}_BT_{\delta g}$-space, f^{-1}(V) is closed in (X,τ). Therefore f is continuous. Hence f is homeomorphism. □

Theorem 4.2. Let (Y,σ) be ${}_BT_{\delta g}$-space. If f:(X,τ) $\rightarrow$ (Y,σ) and g:(Y,σ) $\rightarrow$ (Z,η) are Bδg-homeomorphism then $g \bigcirc f$ is Bδg- homeomorphism.

Proof. Let f:(X,τ) $\rightarrow$ (Y,σ) and g:(Y,σ) $\rightarrow$ (Z,η) be two Bδg - homeomorphism. Let U be an open set in (X,τ). Since f is Bδg-open map, f(U) is Bδg-open in (Y,σ). Since (Y,σ) is ${}_BT_{\delta g}$-space, f(U) is open in (Y,σ). Also since g is Bδg-open map, $g(f$(U)) is Bδg-open in (Z,η). Hence $g \bigcirc f$ is Bδg-open map. Let V be a closed set in (Z,η). Since g is Bδg-continuous and since (Y,σ) is ${}_BT_{\delta g}$-space, g^{-1}(V) is closed in (Y,σ). Since f is Bδg-continuous, $f^{-1}(g^{-1}$(V)) = $(g \bigcirc f)^{-1}$(V) is Bδg-closed set in (X,τ). That is $g \bigcirc f$ is Bδg-continuous. Hence $g \bigcirc f$ is Bδg-homeomorphism. □

Theorem 4.3. Every Bδg-homeomorphism from a ${}_BT_{\delta g}$-space into another ${}_BT_{\delta g}$-space is $\delta\ \widehat{g}$-homeomorphism.

Proof. Let f:(X,τ) $\rightarrow$ (Y,σ) be Bδg-homeomorphism. Then f is bijective, Bδg-open and Bδg-continuous maps. Let U be an open set (X,τ). Since f is Bδg-open, and since (Y,σ) is ${}_BT_{\delta g}$-space, f(U) is δ-closed. By Proposition 2.8 every δ-closed set is $\delta\ \widehat{g}$-closed. Hence f(U) is $\delta\ \widehat{g}$-closed in (Y,σ). This implies f is $\delta\ \widehat{g}$-open. Let V be a closed set in (Y,σ). Since f is Bδg-continuous and since (X,τ) ${}_BT_{\delta g}$-space, f^{-1}(V) is $\delta\ \widehat{g}$-closed in (X,τ). Therefore f is $\delta\ \widehat{g}$-continuous. Thus f is $\delta\ \widehat{g}$-homeomorphism. □

Theorem 4.4. Every Bδg-homeomorphism from a ${}_BT_{\delta g}$-space into another ${}_BT_{\delta g}$-space is Bδgc-homeomorphism.

Proof. Let f :(X,τ) $\rightarrow$ (Y,σ) be Bδg-homeomorphism. Let U be Bδg-closed in (Y,σ). Since (Y,σ) is ${}_BT_{\delta g}$-space, U is closed in (Y,σ). Also Since f is Bδg-continuous, f^{-1}(U) is Bδg-closed in (X,τ). Hence f is Bδg-irresolute map. Let V be Bδg-open in (X,τ). Since (X,τ) is ${}_BT_{\delta g}$-space, V is open in (X,τ). Also since f is Bδg-open, f(V) is Bδg-open set in (Y,σ). That is $(f^{-1})^{-1}$(V) is Bδg-open in (Y,σ) and hence f^{-1} is Bδg-irresolute. Thus f is Bδgc- homeomorphism. □

We shall introduce the group structure of the set of all Bδgc- homeomorphisms from a topological space (X,τ) onto itself by Bδgc-h(X,τ).

Theorem 4.5. The set Bδgc-h(X,τ) is a group under composition of mappings.

Proof. By Theorem 3.28 $g \bigcirc f \in$ Bδgc-h(X,τ) for all f, $g \in$ Bδgc-h(X,τ). We know that the composition of mappings is associative. The identity map belonging to Bδgc-h(X,τ) acts as the identity element. If $f \in$ Bδgc-h(X,τ) then $f^{-1} \in$ Bδgc-h(X,τ). such that $f^{-1} \bigcirc f = f \bigcirc f^{-1} = I$ and so inverse exists for each element of Bδgc-h(X,τ). Hence Bδgc-h(X,τ) is a group under the composition of mappings. □

Theorem 4.6. Let f : Bδgc-h(X,τ)$\rightarrow$ Bδgc-h(Y,σ) be Bδgc-homeomorphism. Then f induces an isomorphism from the group Bδgc-h(X,τ) onto the group Bδgc-h(Y,σ).

Proof. We define a map f : Bδgc-h(X,τ)$\rightarrow$ Bδgc-h(Y,σ) by $f*(k) = f \bigcirc k \bigcirc f^{-1}$ for every $k \in$ Bδgc-h(X,τ).Then f is a bijection and also for all $k_1, k_2 \in$ Bδgc-h(X,τ), $f*(k_1 \bigcirc k_2) = f \bigcirc (k_1 \bigcirc k_2) \bigcirc f^{-1} = (f \bigcirc k_1 \bigcirc f^{-1}) \bigcirc (f \bigcirc k_2 \bigcirc f^{-1}) = f*(k_1) \bigcirc f*(k_2)$. Hence $f*$ is homeomorphism and so it is an isomorphism induced by f. □

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On Generalized (Ψ, φ)-Almost Weakly Contractive Maps in Generalized Fuzzy Metric Spaces

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Abstract – In this paper, we come out with the approach of generalized (Ψ, φ)-almost weakly contractive maps in the context of generalized fuzzy metric spaces. We prove theorem to show the existence of a fixed point and also provide an example in support to our result.

Keywords – *(Ψ, φ)-almost weakly contractive map, Fuzzy metric space, Generalized fuzzy metric spaces.*

1 Introduction

In Mathematics, the concept of fuzzy set was introduced by Zadeh [15]. It is a new way to represent vagueness in our daily life. In 1975 Kramosil and Michalek [3] introduced the concept of fuzzy metric spaces which opened a new way for further development of analysis in such spaces. George and Veeramani [2] modified the concept of fuzzy metric space. After that several fixed point theorems have been proved in fuzzy metric spaces. In 2008, Dutta and Choudary [8] introduced (Ψ, φ) – weakly contractive maps and showed the existence of fixed points in complete metric spaces. In 2009, Doric [7] unfolded it to a pair of maps by broadening the result that was proposed by Zhang and Song [14] Harjani and Sadarangani [9], Presented some fixed point results in a complete metric space bestowed with a partial order for weakly C-contractive mappings. Saha [12] established a weakened version of contraction mappings principle in fuzzy metric space with a partial ordering. In the present work, we insinuate the concept of (Ψ, φ)-almost weakly contractive maps in the panorama of fuzzy metric spaces and observe few results.

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2 Preliminaries

Definition 2.1. A 3 – tuple (X, $\mathcal{M}$, *) is called generalized fuzzy metric space if X is an arbitrary non – empty set, * is a continuous t – norm, and $\mathcal{M}$ is a fuzzy set on $X^3 \times (0,\infty)$ satisfying the following conditions; for each x, y, z, a ∈ X and t, s > 0

(GFM – 1) $\mathcal{M}(x, y, z, t) > 0$,
(GFM – 2) $\mathcal{M}(x, y, z, t) = 1$, if x = y= z,
(GFM – 3) $\mathcal{M}(x, y, z, t) = \mathcal{M}(p\{x, y, z\}, t)$, where p is a permutation function,
(GFM – 4) $\mathcal{M}(x, y, a, t) * \mathcal{M}(a, z, z, s) \leq \mathcal{M}(x, y, z, t + s)$,
(GFM – 5) $\mathcal{M}(x, y, z, .) : (0, \infty) \to [0, 1]$ is continuous,
(GFM – 6) $\lim_{t\to\infty} \mathcal{M}(x, y, z, t) = 1$.

Definition 2.2. If $\{x_n\}$ is a sequence in a generalized fuzzy metric spaces such that $\mathcal{M}(x_n, x, x, t) \to 1$ whenever $n \to \infty$, then $\{x_n\}$ is said to converges to $x \in X$.

(i) A sequence $\{x_n\}$ in X is said to be a converge to a point x in X if and only if for each $\varepsilon > 0$, t > 0 there exists $n_0 \in N$ such that $\mathcal{M}(x_n, x_m, x_m, t) > 1-\varepsilon$ for all $n \geq n_0$.
(ii) A generalized fuzzy metric space (X , $\mathcal{M}$,*) is said to be complete if every Cauchy sequence in it converges to a point in it.

Definition 2.3. Let (X , $\mathcal{M}$, *) be a complete generalized fuzzy metric space. Let C be a subset of X. Let T: C → C be a self mapping which satisfies the following inequality:

$\Psi(\mathcal{M}(Tx, Ty, Tz, t) \leq \Psi(\mathcal{M}(x, y, z, t)) - \varphi(\mathcal{M}(x, y, z, t)$ where x, y, z ∈ X, t > 0, Ψ and $\varphi : (0, 1] \to [0, \infty)$ are two functions such that,

(i) Ψ is continuous and monotone decreasing with $\Psi(t) = 0 \Leftrightarrow t = 1$
(ii) φ is continuous with $\varphi(s) = 0 \Leftrightarrow s = 1$

Then T is said to be a weak contraction on C.

Definition 2.4. Let (X , $\mathcal{M}$,*) be a generalized fuzzy metric space. Let there exists $\Psi, \varphi : (0,1] \to [0, \infty)$ such that

(i) Ψ is continuous and monotonically decreasing,
(ii) $\Psi(t) = 0 \Leftrightarrow t= 1$
(iii) φ is continuous with $\varphi(s) = 0 \Leftrightarrow s = 1$

Then T: X → X be a self map satisfying the inequality:

$\Psi(\mathcal{M}(Tx, Ty, Tz, t) \leq \Psi(\mathcal{M}(x, y, z, t)) - \varphi(\mathcal{M}(x, y, z, t) + L\{1-m(x,y, z)\}$ for all x,y, z ∈ X, t > 0, L ≥ 0, where m(x, y, z)= max $\{\mathcal{M}(x, Tx, z, t), \mathcal{M}(x, Ty, Tz, t), \mathcal{M}(y, Ty, Tz, t), \mathcal{M}(Tx, y, z, t)\}$. Then T is said to be a (Ψ, φ) - almost weakly contractive map on X.

3 Main Result

Theorem 3.1. Let $(X, \mathcal{M}, *)$ be a complete generalized fuzzy metric space. Let $T: X \to X$ be a (Ψ, φ)- almost weakly contractive map. Then, T has a fixed point in X which is unique.

Proof: Let $\{x_n\}$ be a sequence in X such that $Tx_n = x_{n+1}$. If $x_n = x_{n+1}$, then the theorem is obvious.If $x_n \neq x_{n+1}$, consider

$$\Psi(\mathcal{M}(x_n, x_{n+1}, x_{n+1}, t)) = \Psi(\mathcal{M}(Tx_{n-1}, Tx_n, Tx_n, t)) \leq \Psi\begin{pmatrix}\mathcal{M}(x_{n-1}, x_n, x_n, t)) - \varphi\ (\mathcal{M}\ (x_{n-1}, x_n, x_n, t))\ + \\ L\{1 - m(x_{n-1}, x_n, x_n)\end{pmatrix} \quad (3.1.1)$$

$$\begin{aligned} m(x_{n-1}, x_n, x_n) &= \max\begin{Bmatrix}\mathcal{M}(x_{n-1}, Tx_{n-1}, x_n, t), \mathcal{M}(Tx_{n-1}, Tx_n, Tx_n, t), \\ \mathcal{M}(x_n, Tx_n, Tx_n, t), \mathcal{M}(Tx_{n-1}, x_n, x_n, t),\end{Bmatrix} \\ &= \max\begin{Bmatrix}\mathcal{M}(x_{n-1}, x_n, x_n, t), \mathcal{M}(x_n, x_{n+1}, x_{n+1}, t), \\ \mathcal{M}(x_n, x_{n+1}, x_{n+1}, t), \mathcal{M}(x_{n+1}, x_n, x_n, t)\}\end{Bmatrix} \\ &= \max\{\mathcal{M}(x_{n-1}, x_n, x_n, t), 1, \mathcal{M}(x_{n-1}, x_{n+1}, x_{n+1}, t), \mathcal{M}(x_n, x_{n+1}, x_n, t)\} \\ &= 1 \end{aligned}$$

$$m(x_{n-1}, x_n, x_n) = 1 \quad (3.1.2)$$

from (3.1.1) and (3.1.2) , we get that

$$\Psi(\mathcal{M}(x_n, x_{n+1}, x_{n+1}, t)) \leq \Psi(\mathcal{M}(x_{n-1}, x_n, x_n, t)) - \varphi\ (\mathcal{M}(x_{n-1}, x_n, x_n, t)) \quad (3.1.3)$$

$$\Psi(\mathcal{M}(x_n, x_{n+1}, x_{n+1}, t)) < \Psi(\mathcal{M}(x_{n-1}, x_n, x_n, t)) \quad (3.1.4)$$

We know Ψ is monotonically decreasing $\mathcal{M}(x_n, x_{n+1}, x_{n+1}, t) > \mathcal{M}(x_{n-1}, x_n, x_n, t)$ (3.1.5)

$\{\mathcal{M}(x_n, x_{n+1}, x_{n+1}, t)\}$ is an increasing sequence of non-negative real numbers.

Let $\lim_{n\to\infty} \mathcal{M}(x_n, x_{n+1}, x_{n+1}, t) = r$ then taking limit as $n \to \infty$ in (3.1.3)

$\Longrightarrow \psi(r) \leq \psi(r) - \varphi(r)$
$\Longrightarrow \varphi(r) \leq 0 \Longrightarrow \varphi(r) = 0$.
$\Leftrightarrow r = 1$ (from definition (2.4)

Therefore $\lim_{n\to\infty} \mathcal{M}(x_n, x_{n+1}, x_{n+1}, t) = 1$. (3.1.6)

To prove that $\{x_n\}$is a Cauchy sequence.

Let $\{x_n\}$ is not Cauchy, then, for any given $\varepsilon > 0$, we can find subsequences $\{x_{n_k}\}$,$\{x_{m_k}\}$ of $\{x_n\}$ with $n_k > m_k$ such that

$$\mathcal{M}(x_{n_k}, x_{m_k}, x_{m_k}, t) \leq 1 - \varepsilon \quad (3.1.7)$$

then, we have

$\mathcal{M}(x_{n_{k-1}}, x_{m_k}, x_{m_k}, t) > 1-\varepsilon, \quad \mathcal{M}(x_{n_{k-1}}, x_{m_{k-1}}, x_{m_{k-1}}, t) > 1-\varepsilon.$ (3.1.8)

Consider

$1-\varepsilon \geq \mathcal{M}(x_{n_k}, x_{m_k}, x_{m_k}, t)$
$1-\varepsilon \geq \lim_{k\to\infty} \sup \mathcal{M}(x_{n_k}, x_{m_k}, x_{m_k}, t)$ (3.1.9)

$$\begin{aligned}\mathcal{M}(x_{n_k}, x_{m_k}, x_{m_k}, t) &\geq \mathcal{M}\left(x_{n_k}, x_{n_{k-1}}, x_{n_{k-1}}, \tfrac{t}{2}\right) * \mathcal{M}\left(x_{n_{k-1}}, x_{m_k}, x_{m_k}, \tfrac{t}{2}\right)\\ &> \mathcal{M}\left(x_{n_k}, x_{n_{k-1}}, x_{n_{k-1}}, \tfrac{t}{2}\right) * 1-\varepsilon \text{ (from (3.1.8))}\\ &> 1 * 1-\varepsilon \text{ as } k\to\infty \quad \text{(from (3.1.6))}\\ &\Rightarrow \lim_{k\to\infty}\mathcal{M}(x_{n_k}, x_{m_k}, x_{m_k}, t) > 1-\varepsilon \end{aligned} \quad (3.1.10)$$

Therefore

$\lim_{k\to\infty} \inf \mathcal{M}(x_{n_k}, x_{m_k}, x_{m_k}, t) > 1-\varepsilon,$ (3.1.11)

from (3.1.9) and (3..11) we see that

$1-\varepsilon < \lim_{k\to\infty} \inf \mathcal{M}(x_{n_k}, x_{m_k}, x_{m_k}, t) \leq \lim_{k\to\infty} \sup \mathcal{M}(x_{n_k}, x_{m_k}, x_{m_k}, t) < 1-\varepsilon$

$\lim_{k\to\infty} \mathcal{M}(x_{n_k}, x_{m_k}, x_{m_k}, t)$ exists and is equal to $1-\varepsilon$
$\lim_{k\to\infty} \mathcal{M}(x_{n_k}, x_{m_k}, x_{m_k}, t) = 1-\varepsilon.$ (3.1.12)

Consider

$$\begin{aligned}\Psi(\mathcal{M}(x_{n_k}, x_{m_k}, x_{m_k}, t)) &= \Psi(\mathcal{M}(Tx_{n_{k-1}}, Tx_{m_{k-1}}, Tx_{m_{k-1}}, t))\\ &\leq \Psi((\mathcal{M}(x_{n_{k-1}}, x_{m_{k-1}}, x_{m_{k-1}}, t) -\\ &\quad \varphi(\mathcal{M}(x_{n_{k-1}}, x_{m_{k-1}}, x_{m_{k-1}}, t) +\\ &\quad L\{1- m(x_{n_{k-1}}, x_{m_{k-1}}, x_{m_{k-1}}, t)\})\end{aligned} \quad (3.1.13)$$

from definition (2.4) , (3.1.8), and since we know that Ψ is a decreasing function, we have

$\mathcal{M}(x_{n_{k-1}}, x_{m_{k-1}}, x_{m_{k-1}}, t) > 1-\varepsilon \Rightarrow \Psi(\mathcal{M}(x_{n_{k-1}}, x_{m_{k-1}}, x_{m_{k-1}}, t) \leq \Psi(1-\varepsilon).$ (3.1.14)

Since φ is continuous, we have

$\mathcal{M}(x_{n_{k-1}}, x_{m_{k-1}}, x_{m_{k-1}}, t) > 1-\varepsilon \Rightarrow \varphi\, \mathcal{M}(x_{n_{k-1}}, x_{m_{k-1}}, x_{m_{k-1}}, t) \geq \varphi(1-\varepsilon)$ (3.1.15)

also,

$$m(x_{n_{k-1}}, x_{m_{k-1}}, x_{m_{k-1}}, t) = max\left\{\begin{array}{c}\mathcal{M}\left(x_{n_{k-1}}, Tx_{n_{k-1}}, x_{m_{k-1}}, t\right),\\ \mathcal{M}\left(x_{n_{k-1}}, Tx_{m_{k-1}}, Tx_{m_{k-1}}, t\right),\\ \mathcal{M}\left(x_{m_{k-1}}, Tx_{m_{k-1}}, Tx_{m_{k-1}}, t\right),\\ \mathcal{M}(Tx_{n_{k-1}}, x_{m_{k-1}}, x_{m_{k-1}}, t)\end{array}\right\} \quad (3.1.16)$$

$$= \max \begin{Bmatrix} \mathcal{M}(x_{n_{k-1}}, x_{n_k}, x_{m_{k-1}}, t), \\ \mathcal{M}(x_{n_{k-1}}, x_{m_k}, x_{m_k}, t), \\ \mathcal{M}(x_{m_{k-1}}, x_{m_k}, x_{m_k}, t), \\ \mathcal{M}(x_{n_k}, x_{m_{k-1}}, x_{m_k}, t) \end{Bmatrix}$$

Therefore

$$m(x_{n_{k-1}}, x_{m_{k-1}}, x_{m_{k-1}}) \to 1 \text{ as } k \to \infty. \qquad (3.1.17)$$

Using (3.1.12), (3.1.14), (3.1.15), and (3.1.17) , equation (3.1.13) becomes

$\Psi(\mathcal{M}(x_{n_k}, x_{m_k}, x_{m_k}, t) \leq \Psi(1-\varepsilon) - \varphi(1-\varepsilon) + L\{1- m(x_{n_{k-1}}, x_{m_{k-1}}, x_{m_{k-1}})\}$.

Since, X is complete, we can find a, $z \in X$ such that the sequence $\{x_n\}$ is convergent to z as $n \to \infty$. To prove z is a fixed point of T in X.

$$\begin{aligned} \Psi(\mathcal{M}(x_n, Tz, Tz, t) &= \Psi(\mathcal{M}(Tx_{n-1}, Tz, Tz, t) \\ &\leq \Psi(\mathcal{M}(x_{n-1}, z, z, t)) - \varphi(\mathcal{M}(x_{n-1}, z, z, t)) + \\ &\quad L\{1- m\{(x_{n-1}, z, z)\} \end{aligned} \qquad (3.1.18)$$

Where, $m(x_{n-1}, z, z) = \max \begin{Bmatrix} \mathcal{M}(x_{n-1}\ z, z, t), \mathcal{M}(Tx_{n-1}, x_{n-1}, x_{n-1}, t), \\ \mathcal{M}(Tx_{n-1}, z, z, t), \mathcal{M}(Tz, x_{n-1}, x_{n-1}, t), \mathcal{M}(Tz, z, z, t) \end{Bmatrix}$

as $n \to \infty$, (3.1.18) becomes

$\Psi(\mathcal{M}(z, Tz, Tz, t) \leq \Psi(\mathcal{M}(z, z, z, t) - \varphi(\mathcal{M}(z, z., z, t)) + L\{1 -1\} = \Psi(1) - \varphi(1) = 0.$

Therefore, $\Psi(\mathcal{M}(z, Tz, Tz, t) = 0 \Rightarrow \mathcal{M}(z, Tz, Tz, t) = 1$

Thus, $Tz = z \Rightarrow$ z is a fixed point of T in X.

To prove z is unique. If possible, let z, w be two fixed point of T in X, then

$$\begin{aligned} \Psi(\mathcal{M}(z, w, w, t) &\leq \Psi(\mathcal{M}(Tz, Tw, Tw, t)) \leq \Psi((\mathcal{M}(z, w, w, t) - \varphi(\mathcal{M}(z, w., w, t)) \\ &\quad + L\{1 - m(z, w, w)\}) \\ &= \Psi(\mathcal{M}(z, w, w, t) - \varphi(\mathcal{M}(z, w, w, t)) + L\{0\}. \quad (\text{since } m(z, w, w) = 1) \end{aligned}$$

Therefore $\mathcal{M}(z, w, w, t) = 1$ which implies z = w, That is fixed point is unique.

Example 3.2. Let X= [0, 1] and * be the continuous t-norm defined by

$a * b = ab$. $\mathcal{M}(x,y,z,t) = \begin{Bmatrix} 1, & if \quad either\ x = 0 \quad or \quad y = 0\ or\ z = 0 \\ \frac{\min\{x,y,z\}}{\max\{x,y,z\}} & if \quad x \neq 0 \quad , y \neq 0\ and\ z \neq 0 \end{Bmatrix}$

Then, clearly $(X, \mathcal{M}, *)$ is a complete generalized fuzzy metric space.

Let T: X→ X be defined by Tx = $\begin{cases} 0 & if\ x = \frac{1}{2} \\ 1 & if\ x \in \left[0, \frac{1}{2}\right) \cup \left(\frac{1}{2}, 1\right] \end{cases}$.

Let ψ and φ on (0, 1] be defined by ψ(s) = 1- s^2 and φ(s) = 1-s. Here, T satisfies the inequality (3.1.8) with any L ≥ 0. Therefore T is a (Ψ, φ) - almost weakly contractive map on X. Thus, T satisfies all the hypothesis of Theorem 3.1 and so, have a unique fixed point in X i.e., at x = 1.

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On (m, n)-bi-ideals in LA-semigroups

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Abstaract — In this paper we study define of an (m, n)-bi-ideals in LA-semigroup and study basic properties of it.

Keywords — *LA-semigroup, ideal, quasi-ideal,* (m, n)*-bi-ideals.*

1 Introduction

The concepts of on (m, n)-bi-Γ-ideals in Γ-semigroup of a semigroup was introduced by Ansari, M.A. and Khan, M.R. [2], in 1993. The left almost semigroup (LA-semigroup) was first introduced by Kazin and Naseerudin [3], in 1972. An LA-semigroup is a useful algebraic structure, midway between a groupoid and a commutative semigroup. An LA-semigroup is non-associative and non-commutative in general, however, there is a close relationship with semigroup as well as with commutative structures.

Later the concept of an (m, n)-ideal in LA-semigroup was first introduced and by M. Akram and N. Yaqood [3] in 2013 and study properties of (m, n)-ideal in LA-semigroup. In 2015 T. Gaketem [9] introduced concept of an (m, n)-quasi ideal in LA-semigroup and study properties of it.

In this paper, we discussed some properties of (m, n)-bi-ideal in LA-semigroup.

2 Preliminaries and basic definitions

Definition 2.1. [3, p.2188] A groupoid $(S, \cdot)$ is called an *LA-semigroup* or an *AG-groupoid*, if its satisfies left invertive law

$$(a \cdot b) \cdot c = (c \cdot b) \cdot a, \quad \text{for all} \quad a, b, c \in S.$$

Definition 2.2. [3, p.2188] An LA-semigroup S is called a *locally associative* LA-semigroup if its satisfies

$$(aa)a = a(aa), \quad \text{for all} \quad a \in S.$$

Lemma 2.3. [5, p.1] In an LA-semigroup S its satisfies the *medial law* if

$$(ab)(cd) = (ac)(bd), \quad \text{for all} \quad a, b, c, d \in S.$$

Definition 2.4. [7, p.1759] An element $e \in S$ is called *left identity* if $ea = a$ for all $a \in S$.

Lemma 2.5. [3, p.2188] If S is an LA-semigroup with left identity, then

$$a(bc) = b(ac), \quad \text{for all} \quad a, b, c \in S.$$

Lemma 2.6. [5, p.1] An LA-semigroup S with left identity its satisfies the *paramedial* if

$$(ab)(cd) = (dc)(ba), \quad \text{for all} \quad a, b, c, d \in S.$$

Definition 2.7. [4, p.2] Let S be an LA-semigroup. A non-empty subset A of S is called an LA-subsemigroup of S if $AA \subseteq A$.

Definition 2.8. [4, p.2] A non-empty subset A of an LA-semigroup S is called a *left (right) ideal* of S if $SA \subseteq A(AS \subseteq A)$. As usual A is called an *ideal* if it is both left and right ideal.

Definition 2.9. [4, p.2] Let S be an LA-semigroup. An LA-subsemigroup B of S is said to be *bi-ideal* of S if $(BS)B \subseteq B$.

Definition 2.10. [4, p.2] A non-empty subset A of an LA-semigroup S is called a *quasi-ideal* of S if $SA \cap AS \subseteq A$.

Definition 2.11. [3, p.107] A non-empty subset A of an LA-semigroup S is called an (m, n)*-ideal* of S if $(A^m S)A^n \subseteq A$ where m and n are positive integers.

Definition 2.12. [9, p.58] A non-empty subset Q of an LA-semigroup S is called an (m, n)*-quasi ideal* of S if $S^m Q \cap QS^n \subseteq Q$ where m and n are positive integers.

3 (m, n)-bi-ideal in LA-semigroups

In section we definition and study of (m, n)-bi-ideal in LA-semigroup is define the same as an (m, n)-bi-ideal in semigroup.

Definition 3.1. Let S be an LA-semigroup. An LA-subsemigroup B of S is called a (m, n)*-bi-ideal* of S if $(B^m S)B^n \subseteq B$, where m and n are arbitrary positive integers.

Note: The power B^m is canceled when $m = 0$ i.e. $(B^0 S) = S = (SB^0)$. Now we have the following definition:

Definition 3.2. Let S be an LA-semigroup. An LA-subsemigroup B of S is said to be $(m,0)$-*bi-ideal* of S if $(B^mS)B^0 \subseteq (B^mS) \subseteq B$ and $(0,n)$-*bi-ideal* of S if $(B^0S)B^n \subseteq (SB^n) \subseteq B$

In another words we can say that $(m,0)$-bi-ideal of S is exactly the m-left-ideal and $(0,n)$-bi-ideal of S is exactly the n-right-ideal.

Next following we will study basic properties of (m,n)-bi-ideal.

Theorem 3.3. Let S be an LA-semigroup and and B, C be an (m,n)-bi-ideal of S. then the intersection $B \cap C$ is an (m,n)-bi-ideal of S.

Proof. Since $B \cap C \subseteq B$ and $B \cap C \subseteq C$ we have $B \cap C \subseteq B \cap C$. Thus $B \cap C$ is an LA-subsemigroup. Next to show that $B \cap C$ is an (m,n)-bi-ideal of S. Consider

$$((B \cap C)^m S](B \cap C)^n \subseteq (B^m S]B^n \subseteq B,$$

since B is an (m,n)-bi-ideal of S. Secondly

$$((B \cap C)^m S](B \cap C)^n \subseteq (C^m S]C^n \subseteq (CC^n] \subseteq C.$$

Therefore from the above we get $((B\cap C)^m S](B\cap C)^n \subseteq B\cap C$. Thus the intersection $B \cap C$ is an (m,n)-bi-ideal of S. □

Theorem 3.4. Let S be an LA-semigroup and C is an LA-subsemigroup of S. Further let B be an (m,n)-bi-ideal of S. If $B \cap C \neq \emptyset$ then the intersection $B \cap C$ is an (m,n)-bi-ideal of C.

Proof. Assume that $B \cap C \neq \emptyset$ and $x, y \in B \cap C$. Then $x, y \in B$ and $x, y \in C$. Since B, C is an LA-subsemigroup of S we have $xy \in B \cap C$. Then $B \cap C$ is an LA-subsemigroup. Next to show that $B \cap C$ is an (m,n)-bi-ideal of C. Consider

$$((B \cap C)^m C)(B \cap C)^n \subseteq (B^m C)B^n \subseteq (B^m S)B^n \subseteq B,$$

since B is an (m,n)-bi-ideal of S. Secondly

$$((B \cap C)^m C)(B \cap C)^n \subseteq (C^m C)C^n \subseteq C.$$

Therefore from the above we get $((B\cap C)^m C)(B\cap C)^n \subseteq B\cap C$. Thus the intersection $B \cap C$ is an (m,n)-bi-ideal of C. □

In the Theorem 3.5 we can show that arbitrary intersection (m,n)-bi-ideal is an (m,n)-bi-ideal with can prove analogous [3, p.2190]

Theorem 3.5. Let $\{A_i : i \in I\}$ be a family of (m,n)-bi-ideal of an LA-semigroup S. Then $B = \bigcap_{i=1}^{k} A_i \neq \emptyset$ is an (m,n)-bi-ideal of S.

Proof. Since $\{A_i : i \in I\}$ be a family of (m,n)-bi-ideal of an LA-semigroup S we have the intersection of an LA-subsemigroup is an LA-subsemigroup. Next show that $B = \bigcap_{i=1}^{k} A_i$ is an (m,n)-bi-ideal of S. It suffice to prove that $(B^mS)B^n \subseteq B$. Let $x \in (B^mS)B^n$ then $x = (b_1^m s)b_2^n$ for some $b_1^m, b_2^n \in B$ and $s \in S$. Thus for any arbitrary $i \in I$ as $b_1^m, b_2^n \in B_i$ so $x \in (B_i^m S)B_i^n$. Since B_i is an (m,n)-bi-ideal of S we have $(B_i^m S)B_i^n \subseteq B_i$. Then $x \in B_i$. Since i was chosen arbitrarily so $x \in B_i$ for all $i \in I$ and hence $x \in B$. So $(B^mS)B^n \subseteq B$. Hence $B = \bigcap_{i=1}^{k} A_i$ is an (m,n)-bi-ideal of S. □

Theorem 3.6. Let A and B be LA-subsemigroups of a locally associative LA-semigroup S. If A is an $(m, 0)$-ideal and B is a $(0, n)$-ideal of S, then the product AB is an (m, n)-bi-ideal of S if $AB \subseteq A$.

Proof. By medial law we get

$$(AB)(AB) = (AA)(BB) \subseteq AB.$$

This shows that AB is an LA-subsemigroup. Now

$$((AB)^m S)(AB)^n \subseteq (A^m S)(A^n B^n) \subseteq A(SB^n) \subseteq AB$$

Hence the product AB is an (m, n)-ideal of S. □

Theorem 3.7. Let A and B be LA-subsemigroups of a locally associative LA-semigroup S with left identity . If A is an $(0, n)$-bi-ideal and B is a (m, n)-ideal of S, then the product BA is an (m, n)-bi-ideal of S.

Proof. By medial law we get

$$(BA)(BA) = (BB)(AA) \subseteq BA.$$

This shows that BA is an LA-subsemigroup. Now

$$\begin{aligned} ((BA)^m S)(BA)^n &= ((B^m A^m)S)(B^n A^n) \\ &= ((SA^m)B^m)(B^n A^n) \\ &= ((B^n A^n)B^m)(SA^n) \\ &= ((B^m A^n)B^n)(SA^n) \\ &= ((B^m S)B^n)(SA^n) \\ &\subseteq BA \end{aligned}$$

Hence BA is an (m, n)-ideal of S. □

Definition 3.8. [3, p.2190] An element a of an LA-semigroup S is called idempotent if $aa = a$. A subset I of an LA-semigroup S is called *idempotent* if all of its elements are idempotent.

Theorem 3.9. Suppose that S be a locally associative LA-semigroup, C be an (m, n)-bi-ideal of S and B be an (m, n)-bi-ideal of the LA-semigroup C such that B is an idempotent. Then B is an (m, n)-bi-ideal of S.

Proof. Since C be an (m, n)-bi-ideal of S and B be an (m, n)-bi-ideal of the LA-semigroup C we have B is an LA-subsemigroup of S.

Next show that $(B^m S)B^n$ is an (m, n)-bi-ideal of S. It is by media law. Thus

$$\begin{aligned} (B^m S)B^n &= ((B^2)^m (SS))(B^2)^n \\ &= ((B^m)^2 (SS))(B^n)^2 \\ &= ((B^m B^m)(SS))((B^n B^n) \\ &= ((B^m S)(B^m S))((B^n B^n) \\ &= ((B^m S)B^n)((B^m S)B^n) \\ &\subseteq BB \\ &\subseteq B. \end{aligned}$$

Then B is an (m, n)-bi-ideal of S. □

Theorem 3.10. Let S be a locally associative LA-semigroup. Let A and B are (m, n)-bi-ideal of S. Then the following assertions are true:

(1) AB is an (m, n)-bi-ideal of S.

(2) BA is an (m, n)-bi-ideal of S.

Proof. (1) Consider by media law

$$(AB)(AB) \subseteq (AB)(AB) \subseteq (AA)(BB) \subseteq AB.$$

This show that AB is a LA-subsemigroup S. Next we shows that AB is an (m, n)-bi-ideal of S. By medial law we have

$$\begin{aligned} ((AB)^m S)(AB)^n &= ((A^m B^m)S)(A^n B^n) \\ &= ((A^m B^m)(SS))(A^n B^n) \\ &= ((A^m S)(B^m S))(A^n B^n) \\ &= ((A^m S)A^n)((B^m S)B^n) \\ &\subseteq AB. \end{aligned}$$

Therefore AB is an (m, n) bi-ideal of S.

(2) Consider by media law

$$(BA)(BA) \subseteq (BB)(AA) \subseteq BA.$$

This show that BA is a sub LA-semigroup S. Next we shows that BA is an (m, n)- bi-ideal of S. By medial law we have

$$\begin{aligned} ((BA)^m S)(BA)^n &= ((B^m A^m)S)(B^n A^n) \\ &= ((B^m A^m)(SS))(B^n A^n) \\ &= ((B^m S)(A^m S))(B^n A^n) \\ &= ((B^m S)B^n)((A^m S)A^n) \\ &\subseteq BA. \end{aligned}$$

Therefore BA is an (m, n) bi-ideal of S.

□

Corollary 3.11. Suppose that S be an LA-semigroup and B is an (m, n)-bi-ideal and b be an element of S. Then the product Bb and bB are (m, n)-bi-ideal of S.

Proof. It followed by Theorem 3.10. □

Theorem 3.12. Let A be an ideal of an LA-subsemigroup S and Q an (m, n)-quasi-ideal of A, then Q is an (m, n)-bi-ideal of S.

Proof. Since $Q \subseteq A$ we have $Q^m S Q^n \subseteq Q^m SA \cap ASQ^n \subseteq Q^m A \cap AQ^n \subseteq Q$. Thus Q is an (m, n)-bi-ideal of S. □

Theorem 3.13. If B is an (m, n)-bi-ideal of an LA-semigroup S and A is an LA-subsemigroup of S such that $(B^m S)B^n \subseteq A \subseteq B$, then A is an (m, n)-bi-ideal of an LA-semigroup S.

Proof. Suppose that A is an LA-subsemigroup of S. We must show show that $(A^m S)A^n \subseteq A$. By assumption $(A^m S)A^n \subseteq (B^m S)B^n \subseteq A$. By the definition of bi-ideal of LA-semigroup S. Hence A is an (m, n) bi-ideal of LA-semigroup S. □

Theorem 3.14. Suppose that S be a locally associative LA-semigroup and B_1 be an m-left ideal and B_2 be an n-right ideal of S. Then the product B_1B_2 is an (m, n)-bi-ideal of S where m, n are arbitrary positive integers.

Proof. Consider by media law

$$(B_1B_2)(B_1B_2) = (B_1B_1)(B_2B_2) \subseteq B_1B_2.$$

This show that B_1B_2 is an LA-subsemigroup of S. Next to show that product B_1B_2 is an (m, n)-bi-ideal of S. By media law

$$\begin{aligned} ((B_1B_2)^m S)(B_1B_2)^n &= ((B_1B_2)^m (SS))(B_1B_2)^n \\ &= ((B_1^m B_2^m)(SS))(B_1B_2)^n \\ &= ((B_1^m S)(B_2^m S))(B_1B_2)^n \\ &\subseteq (B_1^m B_2^m)(B_1B_2)^n \\ &= (B_1B_2)^m (B_1B_2)^n. \end{aligned}$$

Similar

$$\begin{aligned} ((B_1B_2)^m S)(B_1B_2)^n &= ((B_1B_2)^m (SS))(B_1B_2)^n \\ &= ((B_1B_2)^m (SS))(B_1^n B_2^n) \\ &= (B_1B_2)^m (SB_1^n)(SB_2^n) \\ &\subseteq (B_1B_2)^m (B_1^n B_2^n) \\ &= (B_1B_2)^m (B_1B_2)^n. \end{aligned}$$

Then B_1B_2 is an (m, n)-bi-ideal of S. □

Theorem 3.15. Let I an (m, n)-ideal of an LA-semigroup S and Q be an (m, n)-quasi-ideal of I then Q is an (m, n)-bi-ideal of S.

Proof. Let Q be an (m, n)-quasi-ideal of I where I is (m,0) ideal and (0,n) ideal of LA-semigroup S. Then $Q \subseteq I$. Thus

$$(Q^m S)Q^n \subseteq (Q^m S)I^n \cap (I^m S)Q^n \subseteq Q.$$

By assumption we have $Q^m I \cap IQ^n \subseteq Q$, $SI^n \subseteq I$ and $I^m S \subseteq I$. Then

$$\begin{aligned} (Q^m S)Q^n &\subseteq (Q^m S)I^n \cap (I^m S)Q^n \\ &\subseteq Q^m I^n \cap I^m Q^n \\ &\subseteq Q^m I \cap IQ^m \subseteq Q \end{aligned}$$

This show that Q is an (m, n)-bi-ideal of S. □

Corollary 3.16. Let Q be an (m, n)-quasi-ideal of an LA-semigroup S. Then Q is an (m, n)-bi-ideal of S.

Proof. Let Q be an (m, n)-quasi-ideal of an LA-semigroup S. Then $Q^m S \cap SQ^n \subseteq Q$. Thus $Q^m S \subseteq Q$ and $SQ^n \subseteq Q$. Hence $(Q^m S)SQ^n \subseteq QQ$ implies that $(Q^m S)Q^n \subseteq Q$. Therefore Q is an (m, n)-bi-ideal of S. □

Definition 3.17. An LA-semigroup S is called (m, n)-*simple* if $SS \neq 0$ and S has no (m, n)-bi-ideal other than 0 and S. In other words S is said to be (m, n)-simple LA-semigroup if S is the unique (m, n)-bi-ideal of S.

Next we define (m, n)-simple and study relation of (m, n)-simple and (m, n)-bi-ideal.

Theorem 3.18. An LA-semigroup S is (m, n)-simple if $S = (A^m S)A^n$ for $A \subseteq S$.

Proof. Let S is an (m, n)-simple LA-semigroup. Further suppose that $B \subseteq S$. Then $(B^m S)B^n$ is an (m, n)-bi-ideal of S. Hence $S = (B^m S)B^n$. Further let $B \subseteq A$ be another (m, n)-bi-ideal of S. Then $S = (B^m S)B^n \subseteq B \subseteq A$. Hence $S = A$. Whence S is an (m, n)-simple LA-semigroup. □

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Single Valued Neutrosophic Sub Implicative Ideals of KU-Algebras

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Abstract – We consider the concepts single valued neutrosophic of sub-implicative ideals in KU-algebras, and investigate some related properties. We give conditions for a single valued neutrosophic ideal to be a single valued neutrosophic sub-implicative ideal. We show that any single valued neutrosophic sub-implicative ideal is a single valued neutrosophic ideal, but the converse is not true. Using a level set of a single valued neutrosophic set in a KU-algebra, we give a characterization of single valued neutrosophic sub-implicative ideal.

Keywords – *Single valued neutrosophic sub-algebra, Single valued neutrosophic ku-ideal of KU-ideals.*

1. Introduction

Prabpayak and Leerawat [10,11] introduced a new algebraic structure which is called KU-algebras. They studied ideals and congruences in KU-algebras. Also, they introduced the concept of homomorphism of KU-algebra and investigated some related properties. Moreover, they derived some straightforward consequences of the relations between quotient KU-algebras and isomorphism. Mostafa et al. [4,5,13] introduced the notion of fuzzy KU-ideals of KU-algebras and then they investigated several basic properties which are related to fuzzy KU-ideals. Mostafa et al. [6] introduced the notions of *ku-s*ub implicative / *ku*-positive implicative and *ku*-sub-commutative ideals in KU–algebras and investigated some their related properties. Fuzzy set theory was introduced by Zadeh since 1965 [14]. Immediately, it became a useful method to study the problems of imprecision and uncertainty. Since, a lot of new theories treating imprecision and uncertainty have been introduced. For instance, Intuitionistic fuzzy sets were introduced in1986, by Atanassov [2], which is a generalization of the notion of a fuzzy set. When fuzzy set give the degree of membership of an element in a given set, Intuitionistic fuzzy sets give a degree of membership and a degree of non-membership of an element in a given set. In 1998 [7,8], Smarandache gave the concept of neutrosophic set which generalized fuzzy set and

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intuitionistic fuzzy set. This new concept is difficult to apply in the real appliction. It is a set in which each proposition is estimated to have a degree of truth (T), adegree of indeterminacy (I) and a degree of falsity (F). Agboola and Davvaz introduced the concept of neutrosophic BCI/BCK-algebras in [1]. Davvaz et al. [3] introduce a neutrosophic KU-algebra and KU-ideal and investigate some related properties. Recently Wang et al. [12] introduced an instance of neutrosophic set known as single valued neutrosophic set which was motivated from the practical point of view and that can be used in real scientific and engineering applications. In this paper, we establish the concept of single valued neutrosophication sub-implicative ideals on KU-algebras, and investigate some of their properties.

2. Preliminaries

Now we will recall some known concepts related to KU-algebra from the literature which will be helpful in further study of this article.

Definition 2.1. [10,11] Algebra (X, $*$, 0) of type (2, 0) is said to be a KU-algebra, if it satisfies the following axioms:

(ku_1) $(x*y)*[(y*z))*(x*z)]=0$,
(ku_2) $x*0=0$,
(ku_3) $0*x=x$,
(ku_4) $x*y=0$ and $y*x=0$ implies $x=y$,
(ku_5) $x*x=0$, for all $x,y,z \in X$.

On a KU-algebra $(X,*,0)$ we can define a binary relation $\leq$ on X by putting:

$x \leq y \Leftrightarrow y*x=0$.

Thus a KU - algebra X satisfies the conditions:

$(ku_{1^\backslash})$: $(y*z)*(x*z) \leq (x*y)$
$(ku_{2^\backslash})$: $0 \leq x$
$(ku_{3^\backslash})$: $x \leq y, y \leq x$ implies $x=y$,
$(ku_{4^\backslash})$: $y*x \leq x$.

Remark 2.2. Substituting $z*x$ for x and $z*y$ for y in ku_1, we get

$$[(z*x)*(z*y)]*[(z*y)*z))*[(z*x)*z)] \leq [(z*x)*(z*y)]*[(z*x)*(z*y)]=0$$

by (ku_1), hence $(x*y)*[(z*x)*(z*y)]=0$ that mean the condition (ku_1) and $(x*y)*[(z*x)*(z*y)]=0$ are equivalent.

For any elements x and y of a KU-algebra, $y*x^n$ denotes by $(y*\overbrace{x)*x)......*x}^{n\,times}$

Theorem 2.3. [5] In a KU-algebra X, the following axioms are satisfied: For all $x, y, z \in X$,

(1) $x \leq y$ imply $y * z \leq x * z$,
(2) $x * (y * z) = y * (x * z)$,for all $x, y, z \in X$,
(3) $((y * x) * x) \leq y$.
(4) $(y * x^3) = (y * x)$

We will refer to X is a KU-algebra unless otherwise indicated.

Definition 2.4. [10,11] Let I be a non empty subset of a KU-algebra X. Then I is said to be an ideal of X, if

(I_1) $0 \in I$
(I_2) $\forall y, z \in X$, if $(y * z) \in I$ and $y \in I$, imply $z \in I$.

Definition 2.5. [5] Let I be a non empty subset of a KU-algebra X. Then I is said to be an KU- ideal of X, if

(I_1) $0 \in I$
(I_3) $\forall x, y, z \in X$, if $x * (y * z) \in I$ and $y \in I$, imply $x * z \in I$.

Definition 2.6. [6] KU- algebra X is said to be implicative if it satisfies

$$(x * y^2) = (x * y) * (y * x^2)$$

Definition 2.7. [6] KU- algebra X is said to be commutative if it satisfies $x \leq y$ *implies* $(x * y^2) = x$

Lemma 2.8. [6] Let X be a KU-algebra. X is *ku*-implicative iff X is *ku*-positive implicative and *ku*–commutative.

Definition 2.9 [6] A non empty subset A of a KU-algebra X is called a *ku-sub* implicative ideal of X, if $\forall x, y, z \in X$,

(1) $\mathbf{0} \in A$
(2) $z * ((x * y) * ((y * x^2)) \in A$ and $z \in A$, imply $(x * y^2) \in A$.

Definition 2.10. [6] Let $(X, *, 0)$ be a KU-algebra, a nonempty subset A of X is said to be a *ku*-positive implicative ideal if it satisfies, for all x, y, z in X,

(1) $0 \in A$,
(2) $z * (x * y) \in A$ and $z * x \in A$ imply $z * y \in A$.

Definition 2.11 [6] A non empty subset A of a KU-algebra X is called a *ku*–sub commutative ideal of X, if

(1) $\mathbf{0} \in A$
(2) $z * \{((y * x^2)) * y^2)\} \in A$ and $z \in A$, imply $(y * x^2) \in A$.

Definition 2.12.[6] A nonempty subset A of a KU-algebra X is called a kp-ideal of X if it satisfies

(1) $\mathbf{0} \in A$,

(2) $(z * y) * (z * x) \in A$, $y \in A \Rightarrow x \in A$.

3. Single Valued Neutrosophic Sub Implicative Ideals of KU-Algebras

Let X be a non-empty set. A neutrosophic set (NS) in X (see [8,9]) is a structure of the form: $A := \{\langle x, T_A(x), I_A(x), F_A(x) \rangle \mid x \in X\}$, where $T_A(x) : X \to [0,1]$ is a truth membership function, $I_A(x) : X \to [0,1]$ is an indeterminate membership function and $F_A(x) : X \to [0,1]$ is a false membership , we shall use the symbol $A := \{\langle x, T_A, I_A, F_A \rangle \mid x \in X\}$ for the neutrosophic set $A := \{\langle x, T_A(x), I_A(x), F_A(x) \rangle \mid x \in X\}$.

Definition 3.1. Let X be a KU-algebra, a neutrosophic set

$$A := \{\langle x, T_A(x), I_A(x), F_A(x) \rangle \mid x \in X\}$$

in X is called a single valued neutrosophic- ideal (briefly NF- ideal) of X if it satisfies the following conditions:

(F_1) $\mu(0) \geq \mu(x)$ $T_A(0) \geq T_A(x)$, $I_A(0) \geq I_A(x)$, $F_A(0) \leq F_A(x)$ for all $x \in X$.
(F_2) $\forall x, y \in X$, $T_A(y) \geq \min\{T_A(x * y), T_A(x)\}$.
(F_3) $\forall x, y \in X$, $I_A(y) \geq \min\{I_A(x * y), I_A(x)\}$
(F_4) $\forall x, y \in X$, $F_A(y) \leq \max\{F_A(x * y), F_A(x)\}$

Definition3.2. A non empty subset $A := \{\langle x, T_A(x), I_A(x), F_A(x) \rangle \mid x \in X\}$ of a KU-algebra X is called a single valued neutrosophic *sub* implicative ideal (briefly NFSI - ideal) of X , if $\forall x, y, z \in X$,

(F_1) $T_A(0) \geq T_A(x)$, $I_A(0) \geq I_A(x)$, $F_A(0) \leq F_A(x)$
$(NFSI_1)$ $T_A(x * y^2) \geq \min\{T_A(z * ((x * y) * ((y * x^2)), T_A(z)\}$
$(NFSI_2)$ $I_A(x * y^2) \geq \min\{I_A(z * ((x * y) * ((y * x^2)), I_A(z)\}$
$(NFSI_3)$ $F_A(x * y^2) \leq \max\{F_A(z * ((x * y) * ((y * x^2)), F_A(z)\}$

Example.3.3. Let $X = \{0,1,2,3\}$ be a set with a binary operation $*$ defined by the following table:

*	0	1	2	3
0	0	1	2	3
1	0	0	0	2
2	0	2	0	1
3	0	0	0	0

Let $A:=\{\langle x,T_A,I_A,F_A\rangle \mid x\in X\}$ be an neutrosophic Set in X defined by $T_A(0)=T_A(2)=0.7, T_A(3)=T_A(1)=0.2$, $I_A(0)=I_A(2)=0.6, I_A(3)=I_A(1)=0.2$ and $F_A(0)=F_A(2)=0, F_A(3)=F_A(1)=0.3$, By routine calculations we know that $A:=\{\langle x,T_A,I_A,F_A\rangle \mid x\in X\}$ is NFSI - ideal of algebra of *X*.

Proposition 3.4. Every NFSI- ideal of a KU-algebra X is order reversing.

Proof. Let $A:=\{\langle x,T_A,I_A,F_A\rangle \mid x\in X\}$be NFSI -ideal of X and let x, y, z $\in$ X be such that x $\leq$ z, then z$*$ x = 0 . Let y = x in $(NFSI_1)$, $(NFSI_2)$ and $(NFSI_3)$, we get

$$T_A(x)\geq \min\{T_A(z*x),T_A(z)\}=\min\{T_A(0),T_A(z)\}=T_A(z),$$
$$I_A(x)\geq \min\{I_A(z*x),I_A(z)\}=\min\{I_A(0),I_A(z)\}=I_A(z) \text{ and}$$
$$F_A(x)\leq \max\{F_A(z*x),F_A(z)\}=\max\{F_A(0),F_A(z)\}=F_A(z).$$

This completes the proof

Lemma 3.5. let $A:=\{\langle x,T_A,I_A,F_A\rangle \mid x\in X\}$ be a N FSI - ideal of KU - algebra X , if the inequality y $*$ x $\leq$ z hold in X , Then

$$T_A(y)\geq \min\{T_A(x)),T_A(z)\},\ I_A(y)\geq \min\{I_A(x)),I_A(z)\} \text{ and } F_A(y)\leq \max\{F_A(x)),F_A(z)\}.$$

Proof. Let $A:=\{\langle x,T_A,I_A,F_A\rangle \mid x\in X\}$be NFSI -ideal of X and let x, y, z $\in$ X be such that y*x $\leq$ z, then z$*$ (y*x)= 0 or y$*$ (z*x)= 0 i.e $z*x\leq y$, we get (by *Proposition 3.4),*

$$T_A(z*x)\geq T_A(y)\ ,\ I_A(z*x)\geq I_A(y) \text{ and } F_A(z*x)\leq F_A(y) \qquad \text{(a).}$$

Put in $(NFSI_1)$, $(NFSI_2)$ and $(NFSI_3)$, x=y, we get:

$$T_A(x*x^2)\geq \min\left\{T_A(z*(\overbrace{(x*x)}^{0}*\overbrace{((x*x^2)}^{x}),T_A(z)\right\} = \min\{T_A(z*x),T_A(z)\}, \text{ i.e}$$

$$T_A(x)\geq \min\{T_A(z*x),\mu(z)\}\geq \min\{T_A(y),T_A(z)\} \text{ by (a)}$$

$$I_A(x*x^2)\geq \min\left\{I_A(z*(\overbrace{(x*x)}^{0}*\overbrace{((x*x^2)}^{x}),I_A(z)\right\} = \min\{I_A(z*x),I_A(z)\}, \text{ i.e}$$

$I_A(x) \geq \min\{I_A(z*x), I(z)\} \geq \min\{I_A(y), I_A(z)\}$ by (a) . and

$$F_A(x*x^2) \leq \max\left\{F_A(z*(\overbrace{(x*x)}^{0}*\overbrace{((x*x^2)}^{x}), F_A(z)\right\} = \max\{F_A(z*x), F_A(z)\}, \text{i.e}$$

$F_A(x) \leq \max\{F_A(z*x), F(z)\} \leq \max\{F_A(y), F_A(z)\}$ by (a). This completes the proof.

Lemma 3.6. If X is implicative KU-algebra, then every NF ideal of X is an NFSI-ideal of X.

Proof. Let $A := \{\langle x, T_A, I_A, F_A\rangle \mid x \in X\}$ be NF ideal of X. Substituting $x*y^2$ for y in (F_2), (F_3) and (F_4), we get

$T_A(x*y^2) \geq \min\{T_A(z*(x*y^2)), T_A(z)\}$, but KU- algebra is implicative i.e
$I_A(x*y^2) \geq \min\{I_A(z*(x*y^2)), I_A(z)\}$ and
$F_A(x*y^2) \leq \max\{F_A(z*(x*y^2)), F_A(z)\}$, but KU- algebra is implicative i.e
$(x*y^2) = (x*y)*(y*x^2)$, hence
$T_A(x*y^2) \geq \min\{T_A(z*(x*y^2)), \mu(z)\} = \min\{T_A(z*(x*y)*(y*x^2)), T_A(z)\}$,
$I_A(x*y^2) \geq \min\{I_A(z*(x*y^2)), I(z)\} = \min\{I_A(z*(x*y)*(y*x^2)), I_A(z)\}$, and
$F_A(x*y^2) \leq \max\{F_A(z*(x*y^2)), F(z)\} = \max\{F_A(z*(x*y)*(y*x^2)), F_A(z)\}$,

which shows that $A := \{\langle x, T_A, I_A, F_A\rangle \mid x \in X\}$ is NFSI-ideal of X. This completes the proof.

Theorem 3.7. Let $A := \{\langle x, T_A, I_A, F_A\rangle \mid x \in X\}$ be N F set of KU - algebra X satisfying the conditions $(NFSI_1)$, $(NFSI_2)$ and $(NFSI_3)$ then $A := \{\langle x, T_A, I_A, F_A\rangle \mid x \in X\}$ satisfies the following inequalities ;

$(NFSI_4)$ $T_A(x*y^2) \geq T_A((x*y)*((y*x^2))$
$(NFSI_5)$ $I_A(x*y^2) \geq I_A((x*y)*(y*x^2))$
$(NFSI_6)$ $F_A(x*y^2) \leq F_A((x*y)*(y*x^2))$

Proof. Let $A := \{\langle x, T_A, I_A, F_A\rangle \mid x \in X\}$ satisfying conditions $(NFSI_1)$, $(NFSI_2)$ and $(NFSI_3)$ i.e.

$(NFSI_1)$ $T_A(x*y^2) \geq \min\{T_A(z*((x*y)*((y*x^2)), T_A(z)\}$
$(NFSI_2)$ $I_A(x*y^2) \geq \min\{I_A(z*((x*y)*((y*x^2)), I_A(z)\}$
$(NFSI_3)$ $F_A(x*y^2) \leq \max\{F_A(z*((x*y)*((y*x^2)), F_A(z)\}$

then by taking z = 0 in $(NFSI_1)$, $(NFSI_2)$ and $(NFSI_3)$ and using (F_1) $T_A(0) \geq T_A(x)$, $I_A(0) \geq I_A(x)$, $F_A(0) \leq F_A(x)$ and (ku_3) we get

$$T_A(x*y^2)\geq \min\{T_A(0*((x*y)*((y*x^2)),T_A(0)\}= T_A((x*y)*(y*x^2)) .$$
$$I_A(x*y^2)\geq \min\{I_A(0*((x*y)*((y*x^2)),I_A(0)\}= I_A((x*y)*(y*x^2)) .$$
$$F_A(x*y^2)\leq \max\{F_A(0*((x*y)*((y*x^2)),F_A(0)\}= F_A((x*y)*(y*x^2)).$$

This completes the proof

Theorem 3.8. Every NFSI- ideal of a KU-algebra X is a NF - ideal, but the converse does not hold.

Proof. Let $A:=\{\langle x,T_A,I_A,F_A\rangle \mid x\in X\}$be N FSI-ideal of X; put x=y in $(NFSI_1)$, $(NFSI_2)$ and $(NFSI_3)$, we get

$$T_A(\overbrace{x*x^2}^{x})\geq \min\{T_A(z*((x*x)*((x*x^2)),T_A(z)\}, \textit{ then}$$
$$T_A(x)\geq \min\left\{T_A(z*((\overbrace{x*x}^{0})*((\overbrace{x*x^2}^{x})),T_A(z)\right\}=\min\{T_A(z*x),T_A(z)\} \quad ,$$

$$I_A(\overbrace{x*x^2}^{x})\geq \min\{I_A(z*((x*x)*((x*x^2)),I_A(z)\}, \textit{ therefore}$$
$$I_A(x)\geq \min\left\{I_A(z*((\overbrace{x*x}^{0})*((\overbrace{x*x^2}^{x})),I_A(z)\right\}=\min\{I_A(z*x),I_A(z)\} \quad , \textit{and}$$

$$F_A(\overbrace{x*x^2}^{x})\geq \min\{F_A(z*((x*x)*((x*x^2)),F_A(z)\}, \textit{ we get}$$
$$F_A(x)\leq \max\left\{F_A(z*((\overbrace{x*x}^{0})*((\overbrace{x*x^2}^{x})),F_A(z)\right\}=\max\{F_A(z*x),F_A(z)\} .$$

Hence $A:=\{\langle x,T_A,I_A,F_A\rangle \mid x\in X\}$ is a a NF - ideal of X . This completes the proof

The following example shows that the converse of Theorem 3.8 may not be true.

Example 3.9. Let $X = \{0,1,2,3,4\}$ in which the operation $*$ is given by the table

*	0	1	2	3	4
0	0	1	2	3	4
1	0	0	1	3	4
2	0	0	0	3	4
3	0	0	0	0	4
4	0	0	0	0	0

Then $(X,*,0)$ is a KU-Algebra. Define a fuzzy set T_A: X→ [0,1] by T_A (0) = 0.7 , T_A (1) = T_A (2) = T_A (3) = T_A (4) = 0.2 , we get for z=0 , x=1 and y=2 L.H.S of $(NFSI_1)$ $T_A((1*2)*2)=T_A(1)=0.2$.

$R.H.S\ of\ (NFSI_1)\ \min\left\{T_A(0*((\overbrace{1*2}^{1})*((\overbrace{2*1}^{0})*1),T_A(0)\right\}=T_A(0)=0.7$, i.e in this case

$$T_A(x*y^2)\not\geq\ \min\{T_A(z*((x*y)*((y*x^2)),T_A(z)\}$$

We now give a condition for a NF- ideal to be a NFSI-ideal.

Theorem 3.10. Every a NF - ideal $A:=\{\langle x,T_A,I_A,F_A\rangle\mid x\in X\}$of X satisfying the condition $(FSI_4),(FSI_5),(FSI_6)$ is a NFSI-ideal ideal of X .

Proof. Let $A:=\{\langle x,T_A,I_A,F_A\rangle\mid x\in X\}$be NF ideal of X satisfying the conditions $(NFSI_4),(NFSI_5),(NFSI_6)$. we get $T_A(x*y^2)\geq\{T_A(((x*y)*((y*x^2))\}$,

$I_A(x*y^2)\geq\{I_A(((x*y)*((y*x^2))\}$ and $F_A(x*y^2)\leq\{F_A(((x*y)*((y*x^2))\}$

Therefore

$T_A(x*y^2)\geq\min\{T_A(z*((x*y)*((y*x^2)),T_A(z)\}$,
$I_A(x*y^2)\geq\min\{I_A(z*((x*y)*((y*x^2)),I_A(z)\}$, and
$F_A(x*y^2)\leq\max\{F_A(z*((x*y)*((y*x^2)),F_A(z)\}$

by (Definition of NF-ideal (F_2) , (F_3) , (F_4)), we get

$T_A(x*y^2)\geq\ T_A(((x*y)*((y*x^2))\ \geq\min\{T_A(z*((x*y)*((y*x^2)),T_A(z)\}$,
$I_A(x*y^2)\geq\ I_A(((x*y)*((y*x^2))\ \geq\min\{I_A(z*((x*y)*((y*x^2)),I_A(z)\}$, and
$F_A(x*y^2)\leq\ F_A(((x*y)*((y*x^2))\ \leq\max\{F_A(z*((x*y)*((y*x^2)),F_A(z)\}$,

which proves the condition $(NFSI_1),(NFSI_2),(NFSI_3)$. This completes the proof.

Theorem 3.11. Let $A:=\{\langle x,T_A,I_A,F_A\rangle\mid x\in X\}$ be NF ideal of X. Then the following are equivalent:

(i) $A:=\{\langle x,T_A,I_A,F_A\rangle\mid x\in X\}$is an NFSI-ideal of X,
(ii) $T_A(x*y^2)\geq T_A(z*((x*y)*((y*x^2))$, $I_A(x*y^2)\geq I_A(z*((x*y)*((y*x^2))$ and $F_A(x*y^2)\leq F_A(z*((x*y)*((y*x^2))$
(iii) $T_A(x*y^2)\geq T_A((x*y)*((y*x^2))$, $I_A(x*y^2)\geq I_A((x*y)*((y*x^2))$and $F_A(x*y^2)\leq F_A((x*y)*((y*x^2))$

Proof. (i) $\Rightarrow$(ii) Suppose that $A:=\{\langle x,T_A,I_A,F_A\rangle \mid x\in X\}$ be NFSI ideal of X. By $(NFSI_1),(NFSI_2),(NFSI_3)$ and (F_1) we have

$T_A(x*y^2)\geq \min\{T_A(0*((x*y)*((y*x^2)),T_A(0)\}=T_A(0*((x*y)*((y*x^2)$ *i.e*
$T_A(x*y^2)\geq T_A(((x*y)*((y*x^2))$,
$I_A(x*y^2)\geq \min\{I_A(0*((x*y)*((y*x^2)),I_A(0)\}=I_A(0*((x*y)*((y*x^2)$ *i.e*
$I_A(x*y^2)\geq I_A(((x*y)*((y*x^2))$ *and*
$F_A(x*y^2)\leq \max\{F_A(0*((x*y)*((y*x^2)),F_A(0)\}=F_A(0*((x*y)*((y*x^2)$ *i.e*
$F_A(x*y^2)\leq F_A(((x*y)*((y*x^2))$.

(ii) $\Rightarrow$ (iii) Since $(x*y)*(y*x^2)\leq x*y^2$, by Lemma 3.5 we obtain,

$T_A(x*y^2)\geq T_A((x*y)*((y*x^2))$, $I_A(x*y^2)\geq I_A((x*y)*((y*x^2))$ and
$F_A(x*y^2)\leq F_A((x*y)*((y*x^2))$

Combining (ii) we have

$T_A(x*y^2)\geq T_A((x*y)*((y*x^2))$, $I_A(x*y^2)\geq I_A((x*y)*((y*x^2))$ and
$F_A(x*y^2)\leq F_A((x*y)*((y*x^2))$

(iii) $\Rightarrow$ (i) Since $[(z*((x*y)*((y*x^2))]*[(x*y)*((y*x^2)]=$

$$=[(x*y)*(z*((y*x^2))]*[(x*y)*((y*x^2)]\leq$$
$$[(z*((y*x^2))]*[(y*x^2)]=[(z*((y*x^2))]*[0*(y*x^2)]$$
$$\leq 0*z=z.$$

By (Lemma 3.5) we obtain

$T_A((x*y)*((y*x^2)\geq \min\{\ T_A((x*y)*((y*x^2),T_A(z)\}$.
$I_A((x*y)*((y*x^2)\geq \min\{\ I_A((x*y)*((y*x^2),I_A(z)\}$, and
$F_A((x*y)*((y*x^2)\leq \max\{\ F_A((x*y)*((y*x^2),F_A(z)\}$.

From (iii), we have $T_A(x*y^2)\geq \min\{T_A(z*((x*y)*((y*x^2)),T_A(z)\}$,
$I_A(x*y^2)\geq \min\{I_A(z*((x*y)*((y*x^2)),I_A(z)\}$,and
$F_A(x*y^2)\leq \max\{F_A(z*((x*y)*((y*x^2)),F_A(z)\}$,

Hence $A:=\{\langle x,T_A,I_A,F_A\rangle \mid x\in X\}$ is an NFSI-ideal of X . The proof is complete.

Theorem 3.12. A single valued neutrosophic set $A:=\{\langle x,T_A,I_A,F_A\rangle \mid x\in X\}$ of a KU-algebra X is a NFSI-ideal of X if and only if $A_{t,s,m}=\{\langle x\in X \mid T_A\geq t, I_A\geq s, F_A\leq m\rangle\}\neq \Phi$, is a sub-implicative ideal of X.

Proof: Suppose that $A:=\{\langle x,T_A,I_A,F_A\rangle \mid x\in X\}$ is a A single valued neutrosophic sub-implicative ideal of X and $A_{t,s,m}\neq\Phi$ for any $t,s,m\in(0,1]$, there exists $x\in A_{t,s,m}$ so that $T_A\geq t, I_A\geq s, F_A\leq m$. It follows from (F_1) that $T_A(0)\geq T_A(x)\geq t$, $I_A(0)\geq I_A(x)\geq s$, $F_A(0)\leq F_A(x)\leq m$ so that $0\in A_{t,s,m}$. Let $x,y,z\in X$ be such that

$z*((x*y)*((y*x^2)\in A_{t,s,m}$ and $z\in A_{t,s,m}$. Using $(NFSI_1),(NFSI_2),(NFSI_3)$,

we know that

$$T_A(x*y^2)\geq\min\{T_A(z*((x*y)*((y*x^2)),T_A(z)\}=\min\{t,t\}=t\ ,$$
$$I_A(x*y^2)\geq\min\{I_A(z*((x*y)*((y*x^2)),I_A(z)\}=\min\{s,s\}=s\ , \text{ and}$$
$$F_A(x*y^2)\leq\max\{F_A(z*((x*y)*((y*x^2)),F_A(z)\}=\max\{m,m\}=m$$

thus $x*y^2\in A_{t,s,m}$. Hence $A_{t,s,m}$ is a sub-implicative ideal of X. Conversely, suppose that $A_{t,s,m}\neq\Phi$ is a sub-implicative ideal of X ,for every $t,s,m\in(0,1]$. and any $x\in X$, let $T_A(x)=t$, $I_A(x)=s$ and $F_A(x)=m$. Then $x\in A_{t,s,m}$. Since $0\in A_{t,s,m}$, it follows that $T_A(0)\geq t=T_A(x)$, $I_A(0)\geq s=I_A(x)$, $F_A(0)\leq m=F_A(x)$ so that $T_A(0)\geq T_A(x)$, $I_A(0)\geq I_A(x)$, $F_A(0)\leq F_A(x)$ for all $x\in X$. Now, we need to show that

$$A:=\{\langle x,T_A,I_A,F_A\rangle \mid x\in X\} \text{satisfies } (NFSI_1),(NFSI_2),(NFSI_3)$$

If not, then there exist $a,b,c\in X$ such that $T_A(a*b^2)\leq\min\{T_A(c*((a*b)*((b*a^2)),T_A(c)\}$

$$I_A(a*b^2)\leq\min\{I_A(c*((a*b)*((b*a^2)),I_A(c)\}\ , \text{ and}$$
$$F_A(a*b^2)\geq\max\{F_A(c*((a*b)*((b*a^2)),F_A(c)\}.$$

Taking

$$t_0=\frac{1}{2}(T_A(a*b^2)+\{T_A(c*((a*b)*((b*a^2)),T_A(c)\}),$$
$$s_0=\frac{1}{2}(I_A(a*b^2)+\{I_A(c*((a*b)*((b*a^2)),I_A(c)\}) \text{ and}$$
$$m_0=\frac{1}{2}(F_A(a*b^2)+\{F_A(c*((a*b)*((b*a^2)),F_A(c)\}),$$

then we have

$$T_A(a*b^2)<t_0<\{T_A(c*((a*b)*((b*a^2)),T_A(c)\}$$
$$I_A(a*b^2)<s_0<\{I_A(c*((a*b)*((b*a^2)),I_A(c)\}$$
$$F_A(a*b^2)>m_0>\{F_A(c*((a*b)*((b*a^2)),F_A(c)\}$$

Hence $c*((a*b)*(b*a^2)) \in A_{t,s,m}$ and $c \in A_{t,s,m}$, but $a*b^2 \notin A_{t,s,m}$ which means that $A_{t,s,m}$ is not a sub-implicative ideal of X, this is contradiction. Therefore $A := \{\langle x, T_A, I_A, F_A \rangle \mid x \in X\}$ is a A single valued neutrosophic sub-implicative ideal of X **.**

Conclusions

In the present paper, we have introduced the concept of single valued neutrosophic sub-implicative ideal KU-algebras and investigated some of their useful properties. In our opinion, these definitions and main results can be similarly extended to some other fuzzy algebraic systems such as hyper groups, hyper semigroups, hyper rings, hyper. It is our hope that this work would other foundations for further study of the theory of BC K/BC I- KU -algebras. Our obtained results can be perhaps applied in engineering, soft computing or even in medical diagnosis. In our future study of single valued neutrosophic sub commutative ideal structure of KU -algebras, may be the following topics should be considered:

(1) To establish single valued neutrosophic (s-weak–strong) hyper KU-ideals in hyper KU-algebras;

(2) To get more results in single valued neutrosophic ideals hyper KU-algebras and application.

(3) To consider the structure single valued neutrosophic dot (s-weak–strong) hyper KU-Ideals of hyper KU-Algebras.

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Comment (2) on Soft Set Theory and uni-int Decision Making [European Journal of Operational Research, (2010) 207, 848-855]

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Abstract − The uni-int decision-making method, which selects a set of optimum elements from the alternatives, was defined by Çağman and Enginoğlu [Soft set theory and uni-int decision making, European Journal of Operational Research 207 (2010) 848-855] via soft sets and their soft products. Lately, this method constructed by and-product/or-product has been configured by Enginoğlu and Memiş [A configuration of some soft decision-making algorithms via $fpfs$-matrices, Cumhuriyet Science Journal 39 (4) (2018) In Press] via fuzzy parameterized fuzzy soft matrices ($fpfs$-matrices), faithfully to the original, because a more general form is needed for the method in the event that the parameters or objects have uncertainties. In this study, we configure the method via $fpfs$-matrices and andnot-product/ornot-product, faithfully to the original. However, in the case that a large amount of data is processed, the method still has a disadvantage regarding time and complexity. To deal with this problem and to be able to use this configured method effectively denoted by CE10n, we suggest two new algorithms in this paper, i.e. EMA18an and EMA18on, and prove that CE10n constructed by andnot-product (CE10an) and constructed by ornot-product (CE10on) are special cases of EMA18an and EMA18on, respectively, if first rows of the $fpfs$-matrices are binary. We then compare the running times of these algorithms. The results show that EMA18an and EMA18on outperform CE10an and CE10on, respectively. Particularly in problems containing a large amount of parameters, EMA18an and EMA18on offer up to 99.9966% and 99.9964% of time advantage, respectively. Latterly, we apply EMA18on to a performance-based value assignment to the methods used in the noise removal, so that we can order them in terms of performance. Finally, we discuss the need for further research.

Keywords − *Fuzzy sets, Soft sets, Soft decision-making, Soft matrices, $fpfs$-matrices*

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1 Introduction

The concept of soft sets was produced by Molodtsov [1] to deal with uncertainties, and so far many theoretical and applied studies from algebra to decision-making problems [2–24] have been conducted on this concept.

Recently, some decision-making algorithms constructed by soft sets [3, 5, 25, 26], fuzzy soft sets [2, 8, 27–29], fuzzy parameterized soft sets [9, 30], fuzzy parameterized fuzzy soft sets ($fpfs$-sets) [7, 31], soft matrices [5, 32] and fuzzy soft matrices [10, 33] have been configured [34] via fuzzy parameterized fuzzy soft matrices ($fpfs$-matrices) [11], faithfully to the original, because a more general form is needed for the method in the event that the parameters or objects have uncertainties.

One of the configured methods above-mentioned is CE10 [5, 34] constructed by and-product (CE10a) or constructed by or-product (CE10o). Since the authors point to a configuration of these methods by using a different product such as andnot-product and ornot-product, in this study, we configure the uni-int decision-making method constructed by andnot-product/ornot-product via $fpfs$-matrices, faithfully to the original. However, in the case that a large amount of data is processed, this configured method denoted by CE10n has a disadvantage regarding time and complexity. It can be overcome this problem via simplification of the algorithms but in the event that first rows of the $fpfs$-matrices are binary, though there exist simplified versions of CE10n constructed by andnot-product (CE10an) and constructed by ornot-product (CE10on), no exist in the other cases. Therefore, in this study, we aim to develop two algorithms which have the ability of CE10an and CE10on and are also faster than them.

In Section 2 of the present study, we introduce the concept of $fpfs$-matrices. In Section 3, we configure the uni-int decision-making method constructed by andnot-product/ornot-product via $fpfs$-matrices. In Section 4, we suggest two new algorithms in this paper, i.e. EMA18an and EMA18on, and prove that CE10an and CE10on are special cases of EMA18an and EMA18on, respectively, if first rows of the $fpfs$-matrices are binary. A part of this section has been presented in [35]. In Section 5, we compare the running times of these algorithms. In Section 6, we apply EMA18on to the decision-making problem in image denoising. Finally, we discuss the need for further research.

2 Preliminary

In this section, we present the definition of $fpfs$-sets and $fpfs$-matrices. Throughout this paper, let E be a parameter set, $F(E)$ be the set of all fuzzy sets over E, and $\mu \in F(E)$. Here, $\mu := \{^{\mu(x)}x : x \in E\}$.

Definition 2.1. [7, 11] Let U be a universal set, $\mu \in F(E)$, and α be a function from μ to $F(U)$. Then the graphic of α, denoted by α, defined by

$$\alpha := \{(^{\mu(x)}x, \alpha(^{\mu(x)}x)) : x \in E\}$$

that is called fuzzy parameterized fuzzy soft set ($fpfs$-set) parameterized via E over U (or briefly over U).

In the present paper, the set of all $fpfs$-sets over U is denoted by $FPFS_E(U)$.

Example 2.2. Let $E = \{x_1, x_2, x_3, x_4\}$ and $U = \{u_1, u_2, u_3, u_4, u_5\}$. Then

$$\alpha = \{(^1x_1, \{^{0.3}u_1, ^{0.7}u_3\}), (^{0.8}x_2, \{^{0.2}u_1, ^{0.2}u_3, ^{0.9}u_5\}), (^{0.3}x_3, \{^{0.5}u_2, ^{0.7}u_4, ^{0.2}u_5\}), (^0x_4, \{^1u_2, ^{0.9}u_4\})\}$$

is a $fpfs$-set over U.

Definition 2.3. [11] Let $\alpha \in FPFS_E(U)$. Then $[a_{ij}]$ is called the matrix representation of α (or briefly $fpfs$-matrix of α) and defined by

$$[a_{ij}] = \begin{bmatrix} a_{01} & a_{02} & a_{03} & \dots & a_{0n} & \dots \\ a_{11} & a_{12} & a_{13} & \dots & a_{1n} & \dots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} & \dots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \end{bmatrix} \text{ for } i = \{0, 1, 2, \cdots\} \text{ and } j = \{1, 2, \cdots\}$$

such that

$$a_{ij} := \begin{cases} \mu(x_j), & i = 0 \\ \alpha(^{\mu(x_j)}x_j)(u_i), & i \neq 0 \end{cases}$$

Here, if $|U| = m - 1$ and $|E| = n$ then $[a_{ij}]$ has order $m \times n$.

From now on, the set of all $fpfs$-matrices parameterized via E over U is denoted by $FPFS_E[U]$.

Example 2.4. Let's consider the $fpfs$-set α provided in Example 2.2. Then the $fpfs$-matrix of α is as follows:

$$[a_{ij}] = \begin{bmatrix} 1 & 0.8 & 0.3 & 0 \\ 0.3 & 0.2 & 0 & 0 \\ 0 & 0 & 0.5 & 1 \\ 0.7 & 0.2 & 0 & 0 \\ 0 & 0 & 0.7 & 0.9 \\ 0 & 0.9 & 0.2 & 0 \end{bmatrix}$$

Definition 2.5. [11] Let $[a_{ij}], [b_{ik}] \in FPFS_E[U]$ and $[c_{ip}] \in FPFS_{E^2}[U]$ such that $p = n(j-1) + k$. For all i and p,

If $c_{ip} = \min\{a_{ij}, b_{ik}\}$, then $[c_{ip}]$ is called and-product of $[a_{ij}]$ and $[b_{ik}]$ and is denoted by $[a_{ij}] \wedge [b_{ik}]$.

If $c_{ip} = \max\{a_{ij}, b_{ik}\}$, then $[c_{ip}]$ is called or-product of $[a_{ij}]$ and $[b_{ik}]$ and is denoted by $[a_{ij}] \vee [b_{ik}]$.

If $c_{ip} = \min\{a_{ij}, 1 - b_{ik}\}$, then $[c_{ip}]$ is called andnot-product of $[a_{ij}]$ and $[b_{ik}]$ and is denoted by $[a_{ij}] \overline{\wedge} [b_{ik}]$.

If $c_{ip} = \max\{a_{ij}, 1 - b_{ik}\}$, then $[c_{ip}]$ is called ornot-product of $[a_{ij}]$ and $[b_{ik}]$ and is denoted by $[a_{ij}] \underline{\vee} [b_{ik}]$.

3 A Configuration of the uni-int Decision-Making Method

In this section, we configure the uni-int decision-making method [5] constructed by andnot-product/ornot-product via $fpfs$-matrices.

Algorithm Steps

Step 1. Construct two $fpfs$-matrices $[a_{ij}]$ and $[b_{ik}]$

Step 2. Find andnot-product/ornot-product $fpfs$-matrix $[c_{ip}]$ of $[a_{ij}]$ and $[b_{ik}]$

Step 3. Find andnot-product/ornot-product $fpfs$-matrix $[d_{it}]$ of $[b_{ik}]$ and $[a_{ij}]$

Step 4. Obtain $[s_{i1}]$ denoted by $\max\text{-}\min(c_{ip}, d_{it})$ defined by

$$s_{i1} := \max\{\max_j\min_k(c_{ip}), \max_k\min_j(d_{it})\}$$

such that $i \in \{1, 2, \ldots, m-1\}$, $I_a := \{j \,|\, a_{0j} \neq 0\}$, $I_b := \{k \,|\, b_{0k} \neq 0\}$, $I_a^* := \{j \,|\, 1 - a_{0j} \neq 0\}$, $I_b^* := \{k \,|\, 1 - b_{0k} \neq 0\}$, $p = n(j-1)+k$, $t = n(k-1)+j$, and

$$\max_j\min_k(c_{ip}) := \begin{cases} \max\limits_{j\in I_a}\left\{\min\limits_{k\in I_b^*} c_{0p}c_{ip}\right\}, & I_a \neq \emptyset \text{ and } I_b^* \neq \emptyset \\ 0, & \text{otherwise} \end{cases}$$

$$\max_k\min_j(d_{it}) := \begin{cases} \max\limits_{k\in I_b}\left\{\min\limits_{j\in I_a^*} d_{0t}d_{it}\right\}, & I_a^* \neq \emptyset \text{ and } I_b \neq \emptyset \\ 0, & \text{otherwise} \end{cases}$$

Step 5. Obtain the set $\{u_k \,|\, s_{k1} = \max\limits_i s_{i1}\}$

Preferably, the set $\{^{s_{i1}}u_i \,|\, u_i \in U\}$ or $\{^{\frac{s_{k1}}{\max s_{i1}}}u_k | u_k \in U\}$ can be attained.

4 The Soft Decision-Making Methods: EMA18an and EMA18on

In this section, firstly, we present a fast and simple algorithm denoted by EMA18an [35].

EMA18an's Algorithm Steps

Step 1. Construct two $fpfs$-matrices $[a_{ij}]$ and $[b_{ik}]$

Step 2. Obtain $[s_{i1}]$ denoted by $\max\text{-}\min(a_{ij}, b_{ik})$ defined by

$$s_{i1} := \max\{\max_j\min_k(a_{ij}, b_{ik}), \max_k\min_j(b_{ik}, a_{ij})\}$$

such that $i \in \{1, 2, \ldots, m-1\}$, $I_a := \{j \,|\, a_{0j} \neq 0\}$, $I_b := \{k \,|\, b_{0k} \neq 0\}$, $I_a^* := \{j \,|\, 1 - a_{0j} \neq 0\}$, $I_b^* := \{k \,|\, 1 - b_{0k} \neq 0\}$, and

$$\max_j\min_k(a_{ij}, b_{ik}) := \begin{cases} \min\left\{\max\limits_{j\in I_a}\{a_{0j}a_{ij}\}, \min\limits_{k\in I_b^*}\{(1-b_{0k})(1-b_{ik})\}\right\}, & I_a \neq \emptyset \text{ and } I_b^* \neq \emptyset \\ 0, & \text{otherwise} \end{cases}$$

$$\max_k\min_j(b_{ik}, a_{ij}) := \begin{cases} \min\left\{\max\limits_{k\in I_b}\{b_{0k}b_{ik}\}, \min\limits_{j\in I_a^*}\{(1-a_{0j})(1-a_{ij})\}\right\}, & I_a^* \neq \emptyset \text{ and } I_b \neq \emptyset \\ 0, & \text{otherwise} \end{cases}$$

Step 3. Obtain the set $\{u_k \,|\, s_{k1} = \max_i s_{i1}\}$

Preferably, the set $\{^{s_{i1}}u_i \,|\, u_i \in U\}$ or $\{^{\frac{s_{k1}}{\max s_{i1}}}u_k | u_k \in U\}$ can be attained.

Secondly, we propose a fast and simple algorithm denoted by EMA18on.

EMA18on's Algorithm Steps

Step 1. Construct two $fpfs$-matrices $[a_{ij}]$ and $[b_{ik}]$

Step 2. Obtain $[s_{i1}]$ denoted by $\max\text{-}\min(a_{ij}, b_{ik})$ defined by

$$s_{i1} := \max\{\text{max}_j\text{min}_k(a_{ij}, b_{ik}), \text{max}_k\text{min}_j(b_{ik}, a_{ij})\}$$

such that $i \in \{1, 2, \ldots, m-1\}$, $I_a := \{j \,|\, a_{0j} \neq 0\}$, $I_b := \{k \,|\, b_{0k} \neq 0\}$, $I_a^* := \{j \,|\, 1 - a_{0j} \neq 0\}$, $I_b^* := \{k \,|\, 1 - b_{0k} \neq 0\}$, and

$$\text{max}_j\text{min}_k(a_{ij}, b_{ik}) := \begin{cases} \max\left\{\max\limits_{j\in I_a}\{a_{0j}a_{ij}\}, \min\limits_{k\in I_b^*}\{(1-b_{0k})(1-b_{ik})\}\right\}, & I_a \neq \emptyset \text{ and } I_b^* \neq \emptyset \\ 0, & \text{otherwise} \end{cases}$$

$$\text{max}_k\text{min}_j(b_{ik}, a_{ij}) := \begin{cases} \max\left\{\max\limits_{k\in I_b}\{b_{0k}b_{ik}\}, \min\limits_{j\in I_a^*}\{(1-a_{0j})(1-a_{ij})\}\right\}, & I_a^* \neq \emptyset \text{ and } I_b \neq \emptyset \\ 0, & \text{otherwise} \end{cases}$$

Step 3. Obtain the set $\{u_k \,|\, s_{k1} = \max_i s_{i1}\}$

Preferably, the set $\{^{s_{i1}}u_i \,|\, u_i \in U\}$ or $\{^{\frac{s_{k1}}{\max s_{i1}}}u_k | u_k \in U\}$ can be attained.

Theorem 4.1. [35] CE10an is a special case of EMA18an provided that first rows of the $fpfs$-matrices are binary.

Proof. Suppose that first rows of the $fpfs$-matrices are binary. The functions s_{i1} provided in CE10an and EMA18an are equal in the event that $I_a = \emptyset$ or $I_b^* = \emptyset$. Assume that $I_a \neq \emptyset$ and $I_b^* \neq \emptyset$. Since $a_{0j} = 1$ and $b_{0k} = 0$, for all $j \in I_a := \{a_1, a_2, ..., a_s\}$ and $k \in I_b^* := \{b_1, b_2, ..., b_t\}$,

$$\begin{aligned} \text{max}_j\text{min}_k(c_{ip}) &= \max_{j\in I_a}\left\{\min_{k\in I_b^*} c_{0p}c_{ip}\right\} \\ &= \max_{j\in I_a}\left\{\min_{k\in I_b^*}\{\min\{a_{0j}, 1-b_{0k}\}.\min\{a_{ij}, 1-b_{ik}\}\}\right\} \\ &= \max_{j\in I_a}\left\{\min_{k\in I_b^*}\{\min\{a_{ij}, 1-b_{ik}\}\}\right\} \\ &= \max\{\min\{\min\{a_{ia_1}, 1-b_{ib_1}\}, \min\{a_{ia_1}, 1-b_{ib_2}\}, \ldots, \min\{a_{ia_1}, 1-b_{ib_t}\}\}, \\ &\qquad \min\{\min\{a_{ia_2}, 1-b_{ib_1}\}, \min\{a_{ia_2}, 1-b_{ib_2}\}, \ldots, \min\{a_{ia_2}, 1-b_{ib_t}\}\}, \\ &\qquad \ldots, \min\{\min\{a_{ia_s}, 1-b_{ib_1}\}, \min\{a_{ia_s}, 1-b_{ib_2}\}, \ldots, \min\{a_{ia_s}, 1-b_{ib_t}\}\}\} \end{aligned}$$

$$
\begin{aligned}
&= \max\{\min\{a_{ia_1}, \min\{1-b_{ib_1}, 1-b_{ib_2}, \dots, 1-b_{ib_t}\}\},\\
&\qquad \min\{a_{ia_2}, \min\{1-b_{ib_1}, 1-b_{ib_2}, \dots, 1-b_{ib_t}\}\}, \dots,\\
&\qquad \min\{a_{ia_s}, \min\{1-b_{ib_1}, 1-b_{ib_2}, \dots, 1-b_{ib_t}\}\}\}\\
&= \min\{\max\{a_{ia_1}, a_{ia_2}, \dots, a_{ia_s}\}, \min\{1-b_{ib_1}, 1-b_{ib_2}, \dots, 1-b_{ib_t}\}\}\\
&= \min\left\{\max_{j\in I_a}\{a_{ij}\}, \min_{k\in I_b^*}\{1-b_{ik}\}\right\}\\
&= \min\left\{\max_{j\in I_a}\{a_{0j}a_{ij}\}, \min_{k\in I_b^*}\{(1-b_{0k})(1-b_{ik})\}\right\}\\
&= \max_j \min_k (a_{ij}, b_{ik})
\end{aligned}
$$

In a similar way, $\max_k \min_j (d_{it}) = \max_k \min_j (b_{ik}, a_{ij})$. Consequently,

$$\max\text{-}\min(a_{ij}, b_{ik}) = \max\text{-}\min(c_{ip}, d_{it})$$

□

Theorem 4.2. CE10on is a special case of EMA18on provided that first rows of the $fpfs$-matrices are binary.

Proof. The proof is similar to that of Theorem 4.1. □

5 Simulation Results

In this section, we compare the running times of CE10an-EMA18an and CE10on-EMA18on by using MATLAB R2017b and a workstation with I(R) Xeon(R) CPU E5-1620 v4 @ 3.5 GHz and 64 GB RAM.

We, firstly, present the running times of CE10an and EMA18an in Table 1 and Fig. 1 for 10 objects and the parameters ranging from 10 to 100. We then give their running times in Table 2 and Fig. 2 for 10 objects and the parameters ranging from 1000 to 10000, in Table 3 and Fig. 3 for 10 parameters and the objects ranging from 10 to 100, in Table 4 and Fig. 4 for 10 parameters and the objects ranging from 1000 to 10000, in Table 5 and Fig. 5 for the parameters and the objects ranging from 10 to 100, and in Table 6 and Fig. 6 for the parameters and the objects ranging from 100 to 1000. The results show that EMA18an outperforms CE10an in any number of data under the specified condition.

Table 1. The results for 10 objects and the parameters ranging from 10 to 100

	10	**20**	**30**	**40**	**50**	**60**	**70**	**80**	**90**	**100**
CE10an	0.02798	0.01283	0.00623	0.00531	0.01103	0.00829	0.00966	0.01325	0.01637	0.01919
EMA18an	0.01249	0.00714	0.00090	0.00052	0.00244	0.00066	0.00039	0.00035	0.00048	0.00024
Difference	0.0155	0.0057	0.0053	0.0048	0.0086	0.0076	0.0093	0.0129	0.0159	0.0189
Advantage (%)	55.3709	44.3108	85.5050	90.1242	77.8817	92.0866	95.9250	97.3876	97.0461	98.7574

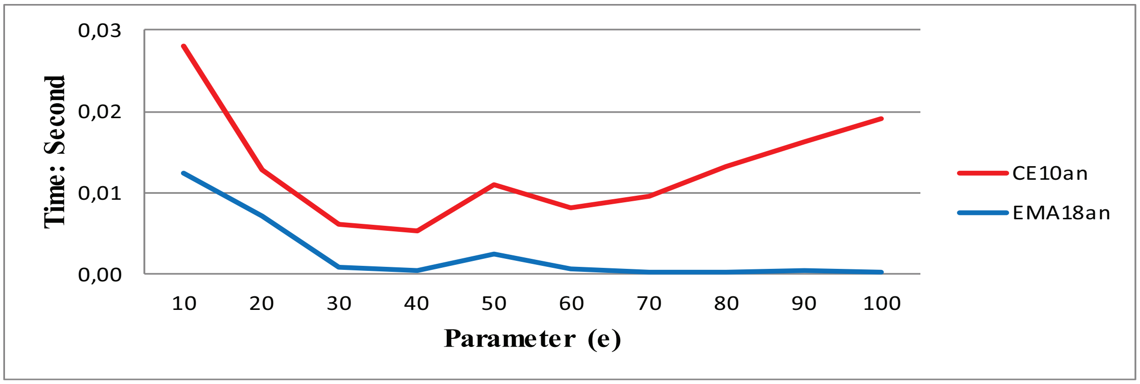


Fig. 1. The figure for Table 1

Table 2. The results for 10 objects and the parameters ranging from 1000 to 10000

	1000	2000	3000	4000	5000	6000	7000	8000	9000	10000
CE10an	1.7420	5.9795	12.4333	21.8006	34.2186	46.9271	66.0375	88.0452	110.2487	143.4280
EMA18an	0.0140	0.0050	0.0024	0.0027	0.0051	0.0053	0.0039	0.0044	0.0048	0.0049
Difference	1.7280	5.9745	12.4310	21.7979	34.2135	46.9218	66.0336	88.0408	110.2439	143.4230
Advantage (%)	99.1965	99.9163	99.9810	99.9875	99.9850	99.9887	99.9940	99.9950	99.9957	99.9966

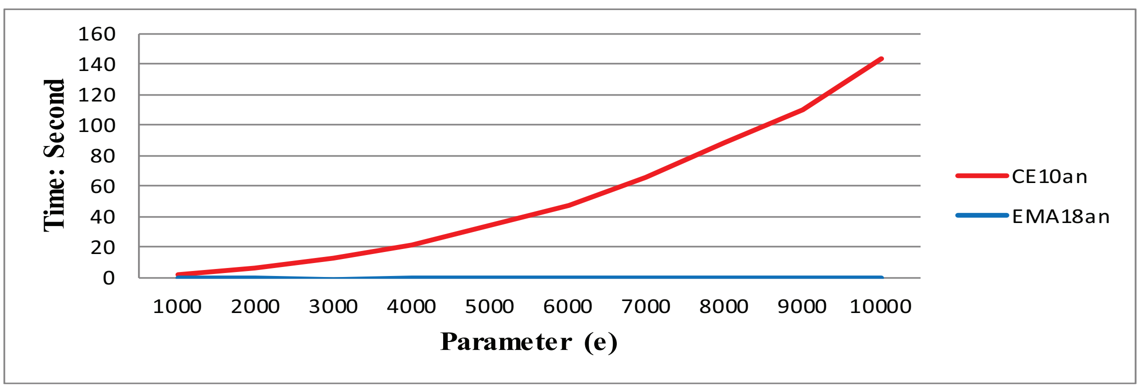


Fig. 2. The figure for Table 2

Table 3. The results for 10 parameters and the objects ranging from 10 to 100

	10	20	30	40	50	60	70	80	90	100
CE10an	0.0229	0.0087	0.0025	0.0023	0.0066	0.0095	0.0060	0.0060	0.0064	0.0072
EMA18an	0.0094	0.0040	0.0008	0.0009	0.0024	0.0023	0.0011	0.0012	0.0012	0.0018
Difference	0.0136	0.0048	0.0017	0.0015	0.0042	0.0072	0.0048	0.0048	0.0053	0.0054
Advantage (%)	59.1357	54.5995	67.8236	62.7065	63.9276	76.1559	81.2134	80.4675	81.6589	74.9437

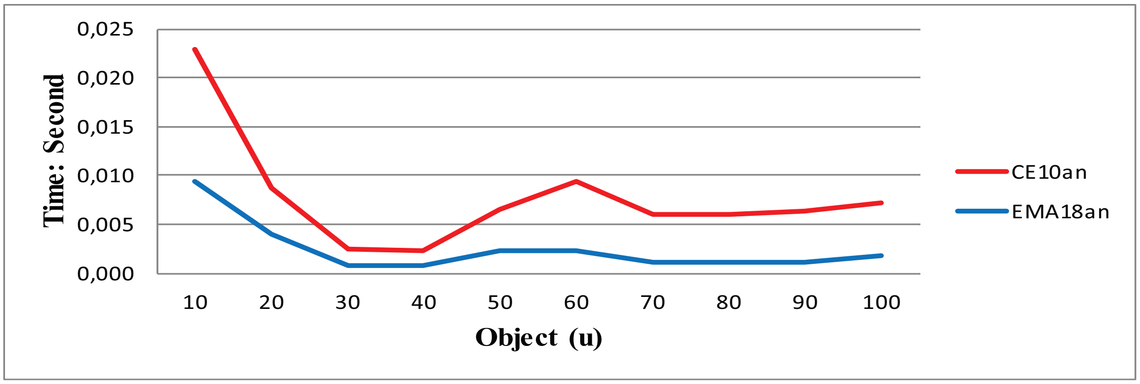


Fig. 3. The figure for Table 3

Table 4. The results for 10 parameters and the objects ranging from 1000 to 10000

	1000	**2000**	**3000**	**4000**	**5000**	**6000**	**7000**	**8000**	**9000**	**10000**
CE10an	0.1075	0.2303	0.4306	0.6850	1.0900	1.4666	1.9348	2.5576	3.1432	3.8415
EMA18an	0.0199	0.0250	0.0324	0.0447	0.0594	0.0736	0.0742	0.0993	0.1153	0.1313
Difference	0.0877	0.2053	0.3982	0.6404	1.0306	1.3930	1.8605	2.4583	3.0280	3.7102
Advantage (%)	81.5272	89.1420	92.4812	93.4776	94.5528	94.9825	96.1639	96.1160	96.3331	96.5811

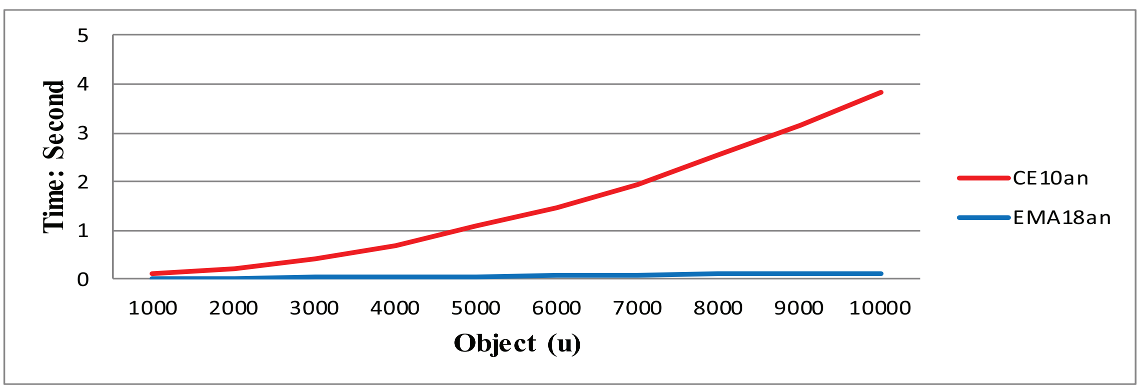


Fig. 4. The figure for Table 4

Table 5. The results for the parameters and the objects ranging from 10 to 100

	10	**20**	**30**	**40**	**50**	**60**	**70**	**80**	**90**	**100**
CE10an	0.0213	0.0109	0.0078	0.0166	0.0378	0.0645	0.0863	0.1156	0.1665	0.2299
EMA18an	0.0093	0.0041	0.0009	0.0009	0.0048	0.0023	0.0011	0.0014	0.0014	0.0014
Difference	0.0121	0.0069	0.0069	0.0157	0.0330	0.0622	0.0851	0.1142	0.1651	0.2285
Advantage (%)	56.4639	62.7094	88.6511	94.5591	87.3928	96.4563	98.6770	98.8164	99.1380	99.3720

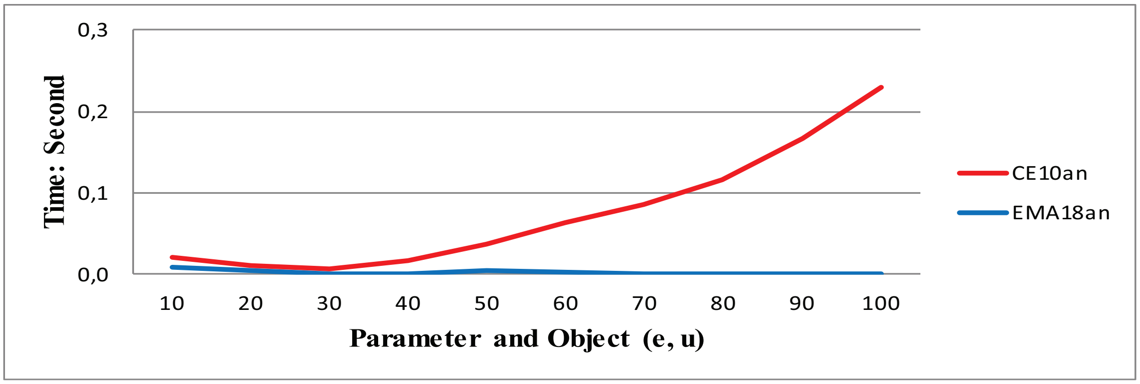


Fig. 5. The figure for Table 5

Table 6. The results for the parameters and the objects ranging from 100 to 1000

	100	200	300	400	500	600	700	800	900	1000
CE10an	0.2739	3.2532	14.0127	40.1959	93.9178	184.5333	335.5700	568.7381	914.9916	1412.0988
EMA18an	0.0113	0.0069	0.0068	0.0101	0.0162	0.0200	0.0244	0.0587	0.0396	0.0506
Difference	0.2626	3.2463	14.0060	40.1858	93.9015	184.5134	335.5456	568.6794	914.9520	1412.0482
Advantage (%)	95.8871	99.7870	99.9518	99.9748	99.9827	99.9892	99.9927	99.9897	99.9957	99.9964

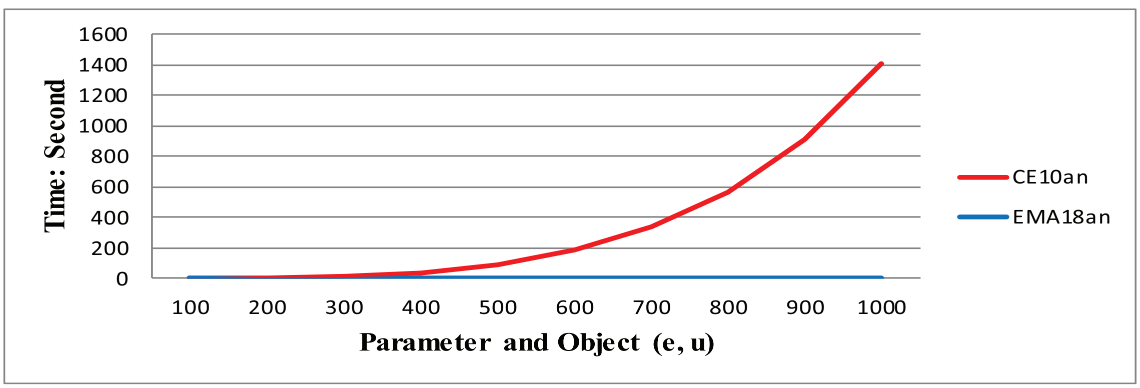


Fig. 6. The figure for Table 6

Secondly, we present the running times of CE10on and EMA18on in Table 7 and Fig. 7 for 10 objects and the parameters ranging from 10 to 100. We then give their running times in Table 8 and Fig. 8 for 10 objects and the parameters ranging from 1000 to 10000, in Table 9 and Fig. 9 for 10 parameters and the objects ranging from 10 to 100, in Table 10 and Fig. 10 for 10 parameters and the objects ranging from 1000 to 10000, in Table 11 and Fig. 11 for the parameters and the objects ranging from 10 to 100, and in Table 12 and Fig. 12 for the parameters and the objects ranging from 100 to 1000. The results show that EMA18on outperforms CE10on in any number of data under the specified condition.

Table 7. The results for 10 objects and the parameters ranging from 10 to 100

	10	20	30	40	50	60	70	80	90	100
CE10on	0, 0273	0, 0107	0, 0037	0, 0051	0, 0116	0, 0165	0, 0141	0, 0167	0, 0245	0, 0197
EMA18on	0, 0136	0, 0056	0, 0007	0, 0008	0, 0027	0, 0020	0, 0006	0, 0006	0, 0004	0, 0004
Difference	0, 0137	0, 0050	0, 0029	0, 0044	0, 0089	0, 0145	0, 0135	0, 0162	0, 0241	0, 0193
Advantage (%)	50, 1693	47, 3241	80, 2441	85, 2661	76, 7704	88, 1753	95, 9501	96, 4667	98, 3700	98, 0986

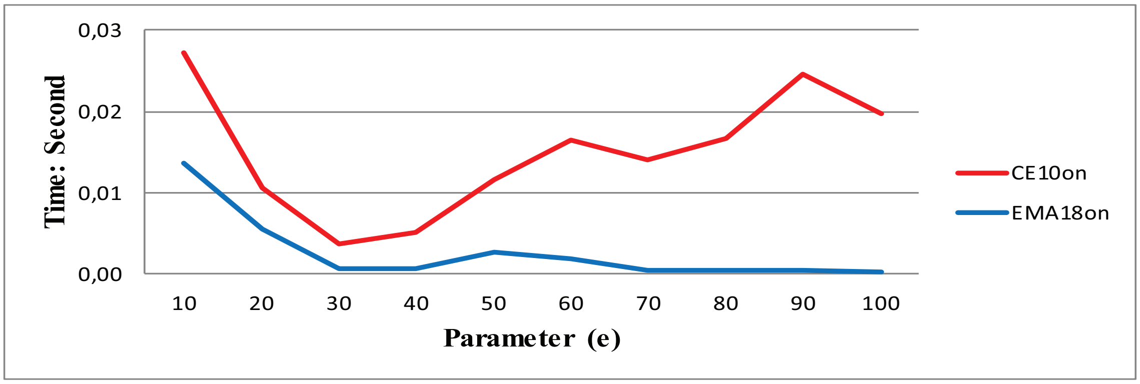


Fig. 7. The figure for Table 7

Table 8. The results for 10 objects and the parameters ranging from 1000 to 10000

	1000	2000	3000	4000	5000	6000	7000	8000	9000	10000
CE10on	1, 7399	6, 0070	12, 6272	21, 8605	34, 3464	47, 4500	68, 2726	90, 2301	112, 5468	145, 8467
EMA18on	0, 0111	0, 0061	0, 0024	0, 0028	0, 0053	0, 0052	0, 0040	0, 0042	0, 0051	0, 0053
Difference	1, 7287	6, 0009	12, 6249	21, 8577	34, 3411	47, 4448	68, 2687	90, 2259	112, 5417	145, 8414
Advantage (%)	99, 3597	99, 8982	99, 9812	99, 9872	99, 9845	99, 9890	99, 9942	99, 9954	99, 9954	99, 9964

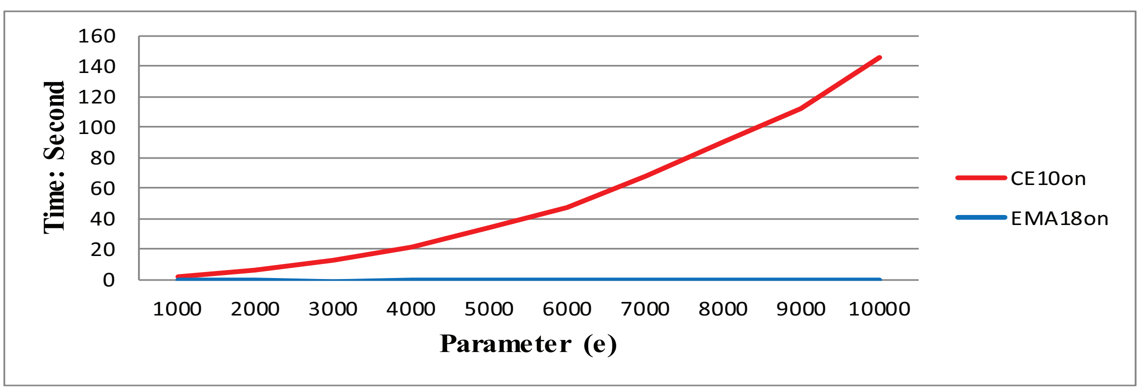


Fig. 8. The figure for Table 8

Table 9. The results for 10 parameters and the objects ranging from 10 to 100

	10	20	30	40	50	60	70	80	90	100
CE10on	0, 0225	0, 0087	0, 0022	0, 0023	0, 0066	0, 0084	0, 0044	0, 0036	0, 0043	0, 0054
EMA18on	0, 0101	0, 0048	0, 0007	0, 0008	0, 0030	0, 0023	0, 0010	0, 0009	0, 0012	0, 0013
Difference	0, 0124	0, 0039	0, 0014	0, 0015	0, 0036	0, 0062	0, 0034	0, 0027	0, 0031	0, 0041
Advantage (%)	55, 1341	44, 7449	66, 3800	66, 7695	54, 5929	73, 0843	76, 7851	74, 1328	72, 7223	75, 7520

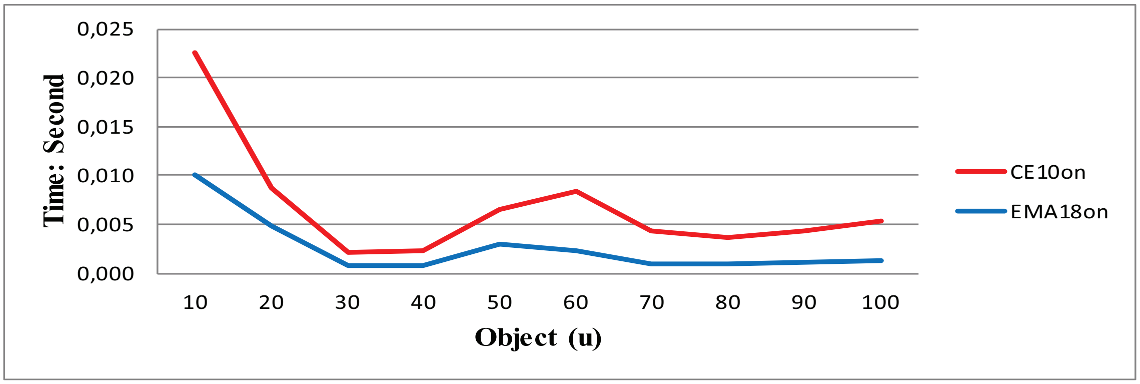


Fig. 9. The figure for Table 9

Table 10. The results for 10 parameters and the objects ranging from 1000 to 10000

	1000	2000	3000	4000	5000	6000	7000	8000	9000	10000
CE10on	0, 1106	0, 2317	0, 4371	0, 6954	1, 0671	1, 5367	1, 9939	2, 5698	3, 2157	4, 0413
EMA18on	0, 0220	0, 0241	0, 0299	0, 0409	0, 0550	0, 0676	0, 0779	0, 0908	0, 1062	0, 1203
Difference	0, 0886	0, 2076	0, 4072	0, 6545	1, 0121	1, 4691	1, 9160	2, 4790	3, 1094	3, 9210
Advantage (%)	80, 1058	89, 5835	93, 1572	94, 1181	94, 8425	95, 5995	96, 0947	96, 4649	96, 6968	97, 0232

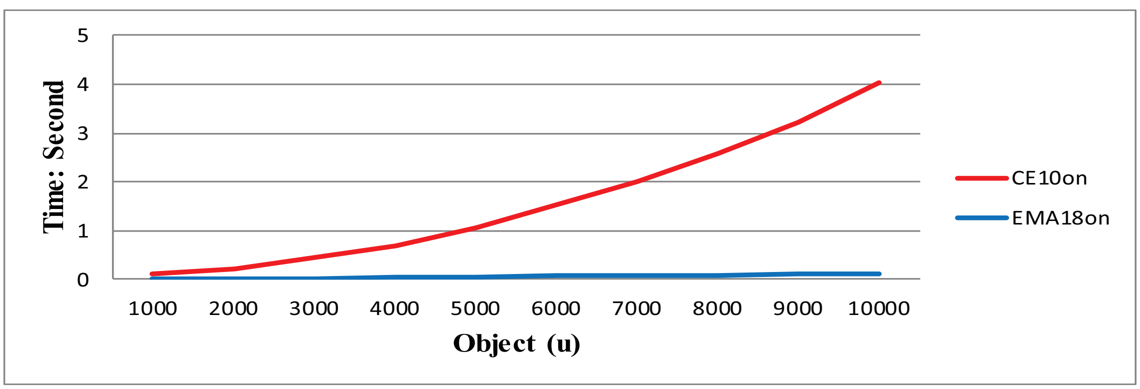


Fig. 10. The figure for Table 10

Table 11. The results for the parameters and the objects ranging from 10 to 100

	10	20	30	40	50	60	70	80	90	100
CE10on	0, 0207	0, 0108	0, 0084	0, 0170	0, 0343	0, 0629	0, 0872	0, 1145	0, 1688	0, 2283
EMA18on	0, 0108	0, 0045	0, 0016	0, 0011	0, 0031	0, 0024	0, 0012	0, 0013	0, 0017	0, 0016
Difference	0, 0099	0, 0063	0, 0068	0, 0159	0, 0312	0, 0605	0, 0860	0, 1132	0, 1670	0, 2267
Advantage (%)	47, 7823	58, 1857	81, 1154	93, 7711	90, 8651	96, 1574	98, 6538	98, 8966	98, 9715	99, 3208

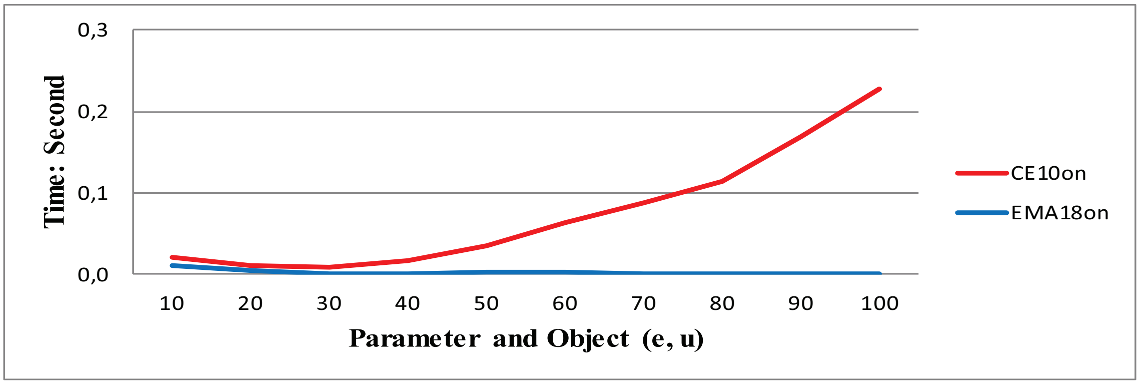


Fig. 11. The figure for Table 11

Table 12. The results for the parameters and the objects ranging from 100 to 1000

	100	200	300	400	500	600	700	800	900	1000
CE10on	0, 2714	3, 2456	14, 0665	40, 5834	93, 4571	182, 9835	325, 6018	545, 3415	870, 2537	1329, 9690
EMA18on	0, 0116	0, 0079	0, 0062	0, 0094	0, 0163	0, 0187	0, 0236	0, 0295	0, 0381	0, 0463
Difference	0, 2598	3, 2377	14, 0603	40, 5741	93, 4408	182, 9648	325, 5782	545, 3119	870, 2156	1329, 9228
Advantage (%)	95, 7199	99, 7562	99, 9556	99, 9769	99, 9826	99, 9898	99, 9927	99, 9946	99, 9956	99, 9965

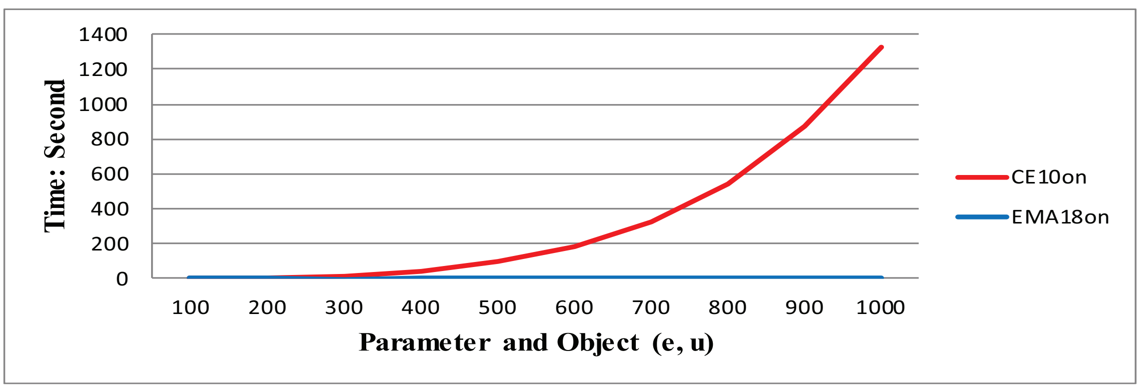


Fig. 12. The figure for Table 12

6 An Application of EMA18on

Being one of the most important topics in image processing, the noise removal directly affects the success rate of the procedures such as segmentation and classification. For this reason, the determining of the methods which perform better than the others is worthwhile to study.

In this section, in Table 13, we present the mean results of some well-known salt-and-pepper noise (SPN) removal methods Decision Based Algorithm (DBA) [36], Modified Decision Based Unsymmetrical Trimmed Median Filter (MDBUTMF) [37], Noise Adaptive Fuzzy Switching Median Filter (NAFSMF) [38]), Different Applied Median Filter (DAMF) [39], and Adaptive Weighted Mean Filter (AWMF) [40] by using 15 traditional images (Cameraman, Lena, Peppers, Baboon, Plane, Bridge, Pirate, Elaine, Boat, Lake, Flintstones, Living Room, House, Parrot, and Hill) with

512×512 pixels, ranging in noise densities from 10% to 90%, and an image quality metrics Structural Similarity (SSIM) [41], which is more preferred than the others. Secondly, in Table 14, we present the mean running times of these algorithms for the images. Finally, we then apply EMA18on to a performance-based value assignment to the methods used in the noise removal, so that we can order them in terms of performance.

Table 13. The mean-SSIM results of the algorithms for the 15 Traditional Images

Algorithm	**10%**	**20%**	**30%**	**40%**	**50%**	**60%**	**70%**	**80%**	**90%**
DBA	0.9655	0.9211	0.8605	0.7837	0.6915	0.5895	0.4846	0.3864	0.3138
MDBUTMF	0.9428	0.7961	0.8380	0.8391	0.7830	0.6322	0.3228	0.0969	0.0213
NAFSM	0.9753	0.9506	0.9244	0.8968	0.8660	0.8312	0.7888	0.7308	0.6094
DAMF	0.9865	0.9715	0.9538	0.9330	0.9083	0.8788	0.8412	0.7883	0.6975
AWMF	0.9738	0.9639	0.9507	0.9343	0.9133	0.8857	0.8481	0.7943	0.7044

Table 14. The mean running-time results of the algorithms for the 15 Traditional Images

Algorithm	**10%**	**20%**	**30%**	**40%**	**50%**	**60%**	**70%**	**80%**	**90%**
DBA	3.7528	3.7727	3.7827	3.7688	3.7734	3.7911	3.7954	3.7824	3.7866
MDBUTMF	2.4964	3.9101	5.5882	6.6506	7.2925	7.7194	7.9863	8.1317	8.1729
NAFSM	1.2528	2.4664	3.6903	4.8807	6.0873	7.3017	8.4870	9.6226	10.7410
DAMF	0.1567	0.3008	0.4478	0.5929	0.7399	0.8903	1.0464	1.2319	1.5205
AWMF	3.9340	3.2274	2.9008	2.7226	2.6228	2.5688	2.5946	2.7314	3.1366

Let's suppose that the success in low or high-noise density is more important than in the others. Furthermore, the long running time is a drawback. In that case, the values in Table 13 can be represented as an $fpfs$-matrices as follows:

$$[a_{ij}] := \begin{bmatrix} 0.9 & 0.7 & 0.5 & 0.3 & 0.1 & 0.3 & 0.5 & 0.7 & 0.9 \\ 0.9655 & 0.9211 & 0.8605 & 0.7837 & 0.6915 & 0.5895 & 0.4846 & 0.3864 & 0.3138 \\ 0.9428 & 0.7961 & 0.8380 & 0.8391 & 0.7830 & 0.6322 & 0.3228 & 0.0969 & 0.0213 \\ 0.9753 & 0.9506 & 0.9244 & 0.8968 & 0.8660 & 0.8312 & 0.7888 & 0.7308 & 0.6094 \\ 0.9865 & 0.9715 & 0.9538 & 0.9330 & 0.9083 & 0.8788 & 0.8412 & 0.7883 & 0.6975 \\ 0.9738 & 0.9639 & 0.9507 & 0.9343 & 0.9133 & 0.8857 & 0.8481 & 0.7943 & 0.7044 \end{bmatrix}$$

Similarly, the values given in Table 14 can be represented as an $fpfs$-matrices via the function $f : [0, 15] \to [0, 1]$ defined by $f(x) = 1 - x/15$, as follows:

$$[b_{ij}] := \begin{bmatrix} 0.1 & 0.3 & 0.5 & 0.7 & 0.9 & 0.7 & 0.5 & 0.3 & 0.1 \\ 0.7498 & 0.7485 & 0.7478 & 0.7487 & 0.7484 & 0.7473 & 0.7470 & 0.7478 & 0.7476 \\ 0.8336 & 0.7393 & 0.6275 & 0.5566 & 0.5138 & 0.4854 & 0.4676 & 0.4579 & 0.4551 \\ 0.9165 & 0.8356 & 0.7540 & 0.6746 & 0.5942 & 0.5132 & 0.4342 & 0.3585 & 0.2839 \\ 0.9896 & 0.9799 & 0.9701 & 0.9605 & 0.9507 & 0.9406 & 0.9302 & 0.9179 & 0.8986 \\ 0.7377 & 0.7848 & 0.8066 & 0.8185 & 0.8251 & 0.8287 & 0.8270 & 0.8179 & 0.7909 \end{bmatrix}$$

If we apply EMA18on to the $fpfs$-matrices $[a_{ij}]$ and $[b_{ij}]$, then the score matrix and the decision set are as follows:

$$[s_{i1}] = [0.8689 \ \ 0.8485 \ \ 0.8778 \ \ 0.8879 \ \ 0.8764]^T$$

and

$$\{^{0.9786}\text{DBA},\ ^{0.9556}\text{MDBUTMF},\ ^{0.9886}\text{NAFSM},\ ^{1}\text{DAMF},\ ^{0.9871}\text{AWMF}\}$$

The scores show that DAMF outperforms the other methods and the order DAMF, NAFSM, AWMF, DBA, and MDBUTMF is valid.

Let's suppose that the success in medium-noise density is more important than in the others. Furthermore, the long running time is a drawback. In that case, the values in Table 13 can be represented as an $fpfs$-matrices as follows:

$$[c_{ij}] := \begin{bmatrix} 0.1 & 0.3 & 0.5 & 0.7 & 0.9 & 0.7 & 0.5 & 0.3 & 0.1 \\ 0.9655 & 0.9211 & 0.8605 & 0.7837 & 0.6915 & 0.5895 & 0.4846 & 0.3864 & 0.3138 \\ 0.9428 & 0.7961 & 0.8380 & 0.8391 & 0.7830 & 0.6322 & 0.3228 & 0.0969 & 0.0213 \\ 0.9753 & 0.9506 & 0.9244 & 0.8968 & 0.8660 & 0.8312 & 0.7888 & 0.7308 & 0.6094 \\ 0.9865 & 0.9715 & 0.9538 & 0.9330 & 0.9083 & 0.8788 & 0.8412 & 0.7883 & 0.6975 \\ 0.9738 & 0.9639 & 0.9507 & 0.9343 & 0.9133 & 0.8857 & 0.8481 & 0.7943 & 0.7044 \end{bmatrix}$$

and

$$[d_{ij}] := \begin{bmatrix} 0.9 & 0.7 & 0.5 & 0.3 & 0.1 & 0.3 & 0.5 & 0.7 & 0.9 \\ 0.7498 & 0.7485 & 0.7478 & 0.7487 & 0.7484 & 0.7473 & 0.7470 & 0.7478 & 0.7476 \\ 0.8336 & 0.7393 & 0.6275 & 0.5566 & 0.5138 & 0.4854 & 0.4676 & 0.4579 & 0.4551 \\ 0.9165 & 0.8356 & 0.7540 & 0.6746 & 0.5942 & 0.5132 & 0.4342 & 0.3585 & 0.2839 \\ 0.9896 & 0.9799 & 0.9701 & 0.9605 & 0.9507 & 0.9406 & 0.9302 & 0.9179 & 0.8986 \\ 0.7377 & 0.7848 & 0.8066 & 0.8185 & 0.8251 & 0.8287 & 0.8270 & 0.8179 & 0.7909 \end{bmatrix}$$

If we apply EMA18on to the $fpfs$-matrices $[c_{ij}]$ and $[d_{ij}]$, then the score matrix and the decision set are as follows:

$$[s_{i1}] = [0.6748\ \ 0.7502\ \ 0.8248\ \ 0.8906\ \ 0.8220]^T$$

and

$$\{^{0.7577}\text{DBA},\ ^{0.8424}\text{MDBUTMF},\ ^{0.9262}\text{NAFSM},\ ^{1}\text{DAMF},\ ^{0.9229}\text{AWMF}\}$$

The scores show that DAMF performs better than the other methods and the order DAMF, NAFSM, AWMF, MDBUTMF, and DBA is valid.

7 Conclusion

The uni-int decision-making method was defined in 2010 [5]. Afterwards, this method has been configured [34] via $fpfs$-matrices [11]. However, the method suffers from a drawback, i.e. its incapability of processing a large amount of parameters on a standard computer, e.g. with 2.6 GHz i5 Dual Core CPU and 4GB RAM. For this reason, simplification of such methods is significant for a wide range of scientific and industrial processes. In this study, firstly, we have proposed two fast and simple soft decision-making methods EMA18an and EMA18on. Moreover, we have proved that these two methods accept CE10 as a special case, under the condition that the first rows of the $fpfs$-matrices are binary. It is also possible to investigate the simplifications of the other products such as andnot-product and ornot-product (see Definition 2.5).

We then have compared the running times of these algorithms. In addition to the results in Section 4, the results in Table 15 and 16 too show that EMA18an and EMA18on outperform CE10an and CE10on, respectively, in any number of data under the specified condition. Furthermore, other decision-making methods constructed by a different decision function such as minimum-maximum (min-max), max-max, and min-min can also be studied by the similar way.

Table 15. The mean/max advantage and max difference values of EMA18an over CE10an

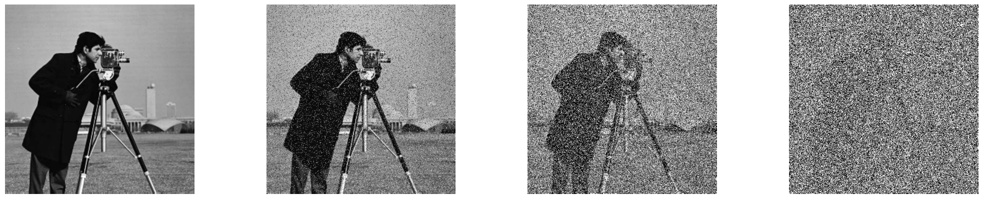
Location	Objects	Parameters	Mean Advantage%	Max Advantage%	Max Difference
Table 1	10	$10-100$	83.4395	98.7574	0.0189
Table 2	10	$1000-10000$	99.9036	99.9966	143.4230
Table 3	$10-100$	10	70.2632	81.6589	0.0136
Table 4	$1000-10000$	10	93.1357	96.5811	3.7102
Table 5	$10-100$	$10-100$	88.2236	99.3720	0.2285
Table 6	$100-1000$	$100-1000$	99.5547	99.9964	1412.0482

Table 16. The mean/max advantage and max difference values of EMA18on over CE10on

Location	Objects	Parameters	Mean Advantage%	Max Advantage%	Max Difference
Table 1	10	$10-100$	81.6835	98.3700	0.0241
Table 2	10	$1000-10000$	99.9181	99.9964	145.8414
Table 3	$10-100$	10	66.0098	76.7851	0.0124
Table 4	$1000-10000$	10	93.3686	97.0232	3.9210
Table 5	$10-100$	$10-100$	86.3720	99.3208	0.2267
Table 6	$100-1000$	$100-1000$	99.5361	99.9965	1329.9228

Finally, we have applied EMA18on to the determination of the performance of the known methods. It is clear that EMA18on, which is a fast and simple method, can be successfully applied to the decision-making problems in various areas such as machine learning and image enhancement.

Although we have no proof about the accuracy of the results of such methods, the results are in compliance with our observations. In order to help in checking the accuracy of the comparison made by a soft decision-making method, we give, in Fig. 13-16, the Cameraman image with different SPN ratios and show the denoised images via the above-mentioned filters. It must be noted that these images has no information of their running times. Whereas, the use of a filter in a software depends on its running time is short. In other words, the running time of a filter is so significant that it can not be ignored. As a result, it is understood that $fpfs$-matrices are an effective mathematical tool to deal with the situations in which more than one parameter or objects are used.



(a) **(b)** **(c)** **(d)**

Fig. 13. (a) Original image "Cameraman" (b) Noisy image with SPN ratio of 10%, (c) Noisy image with SPN ratio of 50%, and (d) Noisy image with SPN ratio of 90%

(a) DBA **(b)** MDBUTMF **(c)** NAFSM **(d)** DAMF **(e)** AWMF

Fig. 14. The images having with SPN ratio of 10% before denoising.

(a) DBA **(b)** MDBUTMF **(c)** NAFSM **(d)** DAMF **(e)** AWMF

Fig. 15. The images having with SPN ratio of 50% before denoising.

(a) DBA **(b)** MDBUTMF **(c)** NAFSM **(d)** DAMF **(e)** AWMF

Fig. 16. The images having with SPN ratio of 90% before denoising.

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EDITORIAL

First of all, on behalf of the team of Journal of New Theory (JNT), I am writing to wish you a Happy Christmas and a peaceful and prosperous New Year 2019. JNT had another good year. I am extremely grateful for your contribution to the journal and your hard work over the past year. I am looking forward to work with you in the coming year.

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