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Generalized Forms of Upper and Lower Continuous Fuzzy Multifunctions

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Abstract — In this paper, we introduce the concepts of upper and lower $(\alpha, \beta, \theta, \delta, \ell)$-continuous fuzzy multifunctions. It is in order to unify several characterizations and properties of some kinds of modifications of fuzzy upper and fuzzy lower semi-continuous fuzzy multifunctions, and to deduce a generalized form of these concepts, namely upper and lower $\eta\eta^*$-continuous fuzzy multifunctions.

Keywords — *General topology; fuzzy topology; multifunction; fuzzy multifunction.*

1 Introduction

Fuzzy multifunctions or multi-valued mappings have many applications in mathematical programming, probability, statistics, different inclusions, fixed point theorems and even in economics, and continuous fuzzy multifunctions have been generalized in manu ways. Many Mathematicians, see [1 - 6], devoted a great part of their research work on studying the generalized continuous fuzzy multifunctions, where their fuzzy fuzzy multifunction maps each point in a classical topological space into an arbitrary fuzzy set in a fuzzy topological space in the sense of Chang [7].

In this paper, we introduce the concepts of upper and lower $(\alpha, \beta, \theta, \delta, \ell)$-continuous fuzzy multifunctions and prove that if α, β are operators on the topological space (X, T) and θ, θ^*, δ are fuzzy operators on the fuzzy topological space (Y, τ) in Šostak sense [8], and ℓ is a proper ideal on X, then a fuzzy multifunction $F : X \multimap Y$ is upper (resp. lower) $(\alpha, \beta, \theta \sqcap \theta^*, \delta, \ell)$-continuous fuzzy multifunction iff F is both of upper (resp. lower) $(\alpha, \beta, \theta, \delta, \ell)$-continuous and upper (resp. lower) $(\alpha, \beta, \theta^*, \delta, \ell)$-continuous fuzzy multifunction. Also, we introduce new generalized notions that

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cover many of the generalized forms of upper (resp. lower) semi-continuous fuzzy multifunctions.

2 Preliminaries

Throughout this paper, X refers to an initial universe, 2^X denotes the power set of X, I^X denotes the set of all fuzzy sets of X, $\lambda^c(x) = 1 - \lambda(x)\ \forall x \in X$ (where $I = [0,1], I_0 = (0,1]$).

As applications, $\alpha, \beta, id_X : 2^X \to 2^X$ are operators on X and $\theta, \delta, id_Y : I^Y \times I_0 \to I^Y$ are fuzzy operators on Y. Recall that an ideal ℓ on X [9], is a collection $\ell \subseteq 2^X$ that satisfies the following conditions:

(1) $A \in \ell$ and $B \subseteq A$ implies that $B \in \ell$,

(2) $A \in \ell$ and $B \in \ell$ implies that $A \cup B \in \ell$.

ℓ is proper if $X \notin \ell$. Let by (X, T) and (Y, τ) be meant the classical and the fuzzy topological spaces due to Šostak [8], respectively. The closure and the interior of any set A in (X, T) will be denoted by T-cl(A) and T-int(A) while the fuzzy closure and the fuzzy interior of any fuzzy set $\mu \in I^Y$ will be denoted by $\mathrm{cl}_\tau(\mu, r)$ and $\mathrm{int}_\tau(\mu, r)$. The notion of quasi-coincidence is given for two fuzzy sets $\lambda, \mu \in I^Y$, denoted by $\lambda\ q\ \mu$, iff there exists a $y \in Y$ such that $\lambda(y) + \mu(y) > 1$. If they are not quasi-coincidence, it will be denoted by $\lambda\ \hat{q}\ \mu$. Any fuzzy set $\mu \in I^Y$ is called r-fuzzy semi-closed [10] (resp. r-fuzzy preclosed [11]) iff $\mu \geq \mathrm{int}_\tau(\mathrm{cl}_\tau(\mu, r), r)$ (resp. $\mu \geq \mathrm{cl}_\tau(\mathrm{int}_\tau(\mu, r), r)$),
while

$$\mathrm{s\,cl}_\tau(\lambda, r) = \bigwedge\{\mu : \lambda \leq \mu \ \text{ and } \quad \mu \text{ is } \ r - \text{fuzzy semi-closed}\}$$

and

$$\mathrm{pre\,cl}_\tau(\lambda, r) = \bigwedge\{\mu : \lambda \leq \mu \ \text{ and } \quad \mu \text{ is } \ r - \text{fuzzy preclosed}\}.$$

Also, $A \subseteq X$ is strongly semi-open [12] (resp. semi-preopen [12]) if

$$A \subseteq T - \mathrm{int}(T - \mathrm{cl}(T - \mathrm{int}(A))) \ \ (\text{ resp. } A \subseteq T - \mathrm{cl}(T - \mathrm{int}(T - \mathrm{cl}(A)))),$$

while

$$T - \mathrm{ss\,int}(A) = \bigcup\{B : B \subseteq A \ \text{ and } \quad B \text{ is } \ \text{strongly semi-open}\}$$

and

$$T - \mathrm{spre\,int}(A) = \bigcup\{B : B \subseteq A \ \text{ and } \quad B \text{ is } \ \text{semi-preopen}\}.$$

A mapping $F : X \multimap Y$ is called a fuzzy multifunction [1] if for each $x \in X$, $F(x)$ is a fuzzy set in Y. The upper inverse $F^+(\lambda)$ and the lower inverse $F^-(\lambda)$ of $\lambda \in I^Y$ are defined as follows:

$F^+(\lambda) = \{x \in X : F(x) \leq \lambda\}$ and $F^-(\lambda) = \{x \in X : F(x)\ q\ \lambda\}$.

For $A \subseteq X$, $F(A) = \bigvee\{F(x) : x \in A\}$. Also, $F^-(\lambda^c) = X - F^+(\lambda)$ for any $\lambda \in I^Y$.

3 Upper and Lower $(\alpha, \beta, \theta, \delta, \ell)$-continuous Fuzzy Multifunctions

Definition 3.1. *A mapping $F : (X, T) \multimap (Y, \tau)$ is said to be upper (resp. lower) $(\alpha, \beta, \theta, \delta, \ell)$-continuous fuzzy multifunction if for every $\mu \in I^Y$, $r \in I_0$, with $\tau(\mu) \geq r$,*

$$\alpha(F^+(\delta(\mu, r))) - \beta(F^+(\theta(\mu, r))) \in \ell \quad (\text{resp. } \alpha(F^-(\delta(\mu, r))) - \beta(F^-(\theta(\mu, r))) \in \ell).$$

We can see that the above definition generalizes the concept of upper (resp. lower) semi-continuous fuzzy multifunction [13] when we choose α = identity operator, β = interior operator, δ = r-fuzzy identity operator, θ = r-fuzzy identity operator and $\ell = \{\emptyset\}$.

Let us give a historical justification of the definition:

(1) In 2015, Ramadan and Abd El-Latif [13], defined the concept of upper (resp. lower) almost continuous fuzzy multifunction as: For every $\mu \in I^Y, r \in I_0$ with $\tau(\mu) \geq r$, then

$$F^+(\mu) \subseteq T\text{-int}(F^+(\text{int}_\tau(\text{cl}_\tau(\mu, r), r))) \quad (\text{resp. } F^-(\mu) \subseteq T\text{-int}(F^-(\text{int}_\tau(\text{cl}_\tau(\mu, r), r)))).$$

Here, α = identity operator, β = interior operator, δ = r-fuzzy identity operator, θ = r-fuzzy interior closure operator and $\ell = \{\emptyset\}$.

(2) In 2015, Ramadan and Abd El-Latif [13], defined the concept of upper (resp. lower) weakly continuous fuzzy multifunction as: For every $\mu \in I^Y, r \in I_0$ with $\tau(\mu) \geq r$, then

$$F^+(\mu) \subseteq T - \text{int}(F^+(\text{cl}_\tau(\mu, r))) \quad (\text{resp. } F^-(\mu) \subseteq T - \text{int}(F^-(\text{cl}_\tau(\mu, r)))).$$

Here, α = identity operator, β = interior operator, δ = r-fuzzy identity operator, θ = r-fuzzy closure operator and $\ell = \{\emptyset\}$.

(3) The concept of upper (resp. lower) almost weakly continuous fuzzy multifunction is defined as: For every $\mu \in I^Y, r \in I_0$ with $\tau(\mu) \geq r$, then

$$F^+(\mu) \subseteq T\text{-int}(T\text{-cl}(F^+(\text{cl}_\tau(\mu, r)))) \quad (\text{resp. } F^-(\mu) \subseteq T\text{-int}(T\text{-cl}(F^-(\text{cl}_\tau(\mu, r))))).$$

Here, α = identity operator, β = interior closure operator, δ = r-fuzzy identity operator, θ = r-fuzzy closure operator and $\ell = \{\emptyset\}$.

(4) The concept of upper (resp. lower) strongly semi-continuous fuzzy multifunction is defined as: For every $\mu \in I^Y, r \in I_0$ with $\tau(\mu) \geq r$, then

$$F^+(\mu) \subseteq T\text{-int}(T\text{-cl}(T\text{-int}(F^+(\mu)))) \quad (\text{resp. } F^-(\mu) \subseteq T\text{-int}(T\text{-cl}(T\text{-int}(F^-(\mu))))).$$

Here, α = identity operator, β = interior closure interior operator, δ = r-fuzzy identity operator, θ = r-fuzzy identity operator and $\ell = \{\emptyset\}$.

(5) The concept of upper (resp. lower) almost strongly semi-continuous fuzzy multifunction is defined as: For every $\mu \in I^Y, r \in I_0$ with $\tau(\mu) \geq r$, then
$F^+(\mu) \subseteq T\text{-ss}\,\text{int}(F^+(\text{s}\,\text{cl}_\tau(\mu, r)))$ (resp. $F^-(\mu) \subseteq T\text{-ss}\,\text{int}(F^-(\text{s}\,\text{cl}_\tau(\mu, r))))$.
Here, $\alpha =$ identity operator, $\beta =$ strongly semi-interior operator, $\delta = r$-fuzzy identity operator, $\theta = r$-fuzzy semi-closure operator and $\ell = \{\emptyset\}$.

(6) The concept of upper (resp. lower) weakly strongly semi-continuous fuzzy multifunction is defined as: For every $\mu \in I^Y, r \in I_0$ with $\tau(\mu) \geq r$, then

$$F^+(\mu) \subseteq T - \text{int}(T - \text{cl}(T - \text{int}(F^+(\text{cl}_\tau(\mu, r)))))$$

$$(\text{resp. } F^-(\mu) \subseteq T - \text{int}(T - \text{cl}(T - \text{int}(F^-(\text{cl}_\tau(\mu, r)))))).$$

Here, $\alpha =$ identity operator, $\beta =$ interior closure interior operator, $\delta = r$-fuzzy identity operator, $\theta = r$-fuzzy closure operator and $\ell = \{\emptyset\}$.

(7) The concept of upper (resp. lower) semi-precontinuous fuzzy multifunction is defined as: For every $\mu \in I^Y, r \in I_0$ with $\tau(\mu) \geq r$, then
$F^+(\mu) \subseteq T\text{-cl}(T\text{-int}(T\text{-cl}(F^+(\mu))))$ (resp. $F^-(\mu) \subseteq T\text{-cl}(T\text{-int}(T\text{-cl}(F^-(\mu)))))$.
Here, $\alpha =$ identity operator, $\beta =$ closure interior closure operator, $\delta = r$-fuzzy identity operator, $\theta = r$-fuzzy identity operator and $\ell = \{\emptyset\}$.

(8) The concept of upper (resp. lower) almost semi-precontinuous fuzzy multifunction is defined as: For every $\mu \in I^Y, r \in I_0$ with $\tau(\mu) \geq r$, then
$F^+(\mu) \subseteq T\text{-spre}\,\text{int}(F^+(\text{s}\,\text{cl}_\tau(\mu, r)))$ (resp. $F^-(\mu) \subseteq T\text{-spr}\,\text{int}(F^-(\text{s}\,\text{cl}_\tau(\mu, r))))$.
Here, $\alpha =$ identity operator, $\beta =$ semi-preinterior operator, $\delta = r$-fuzzy identity operator, $\theta = r$-fuzzy semi-closure operator and $\ell = \{\emptyset\}$.

(9) The concept of upper (resp. lower) weakly semi-precontinuous fuzzy multifunction is defined as: For every $\mu \in I^Y, r \in I_0$ with $\tau(\mu) \geq r$, then

$$F^+(\mu) \subseteq T - \text{cl}(T - \text{int}(T - \text{cl}(F^+(\text{cl}_\tau(\mu, r)))))$$

$$(\text{resp. } F^-(\mu) \subseteq T - \text{cl}(T - \text{int}(T - \text{cl}(F^-(\text{cl}_\tau(\mu, r)))))).$$

Here, $\alpha =$ identity operator, $\beta =$ closure interior closure operator, $\delta = r$-fuzzy identity operator, $\theta = r$-fuzzy closure operator and $\ell = \{\emptyset\}$.

(10) The concept of upper (resp. lower) precontinuous fuzzy multifunction is defined as: For every $\mu \in I^Y, r \in I_0$ with $\tau(\mu) \geq r$, then

$$F^+(\mu) \subseteq T - \text{int}(T - \text{cl}(F^+(\mu))) \quad (\text{resp. } F^-(\mu) \subseteq T - \text{int}(T - \text{cl}(F^-(\mu)))).$$

Here, $\alpha =$ identity operator, $\beta =$ interior closure operator, $\delta = r$-fuzzy identity operator, $\theta = r$-fuzzy identity operator and $\ell = \{\emptyset\}$.

(11) The concept of upper (resp. lower) strongly precontinuous fuzzy multifunction as: For every $\mu \in I^Y, r \in I_0$ with $\tau(\mu) \geq r$, then

$$F^+(\mu) \subseteq T\text{−int}(T\text{−pre cl}(F^+(\mu))) \ \ (\text{resp. } F^-(\mu) \subseteq T\text{−int}(T\text{−pre cl}(F^-(\mu)))).$$

Here, α = identity operator, β = interior preclosure operator, δ = r-fuzzy identity operator, θ = r-fuzzy identity operator and $\ell = \{\emptyset\}$.

Definition 3.2. *A mapping $F : (X, T) \multimap (Y, \tau)$ is called upper (resp. lower) P-continuous fuzzy multifunction* iff *$F^+(\mu) \in T$ (resp. $F^-(\mu) \in T$) for every $\mu \in I^Y$, $r \in I_0$, with $\tau(\mu) \geq r$, such that μ satisfies the property P.*

Let $\theta_P : I^Y \times I_0 \to I^Y$ be a fuzzy operator defined as:

$$\theta_P(\mu, r) \ = \ \begin{cases} \mu & \textit{if } \mu \in I^Y, r \in I_0 \textit{ with } \tau(\mu) \geq r \ \textit{ and } \ \mu \textit{ satisfies the property } P, \\ \overline{1} & \text{otherwise} \end{cases}$$

Theorem 3.3. A map $F : (X, T) \multimap (Y, \tau)$ is upper (resp. lower) P-continuous fuzzy multifunction iff it is upper (resp. lower) $(id_X, T\text{-int}, \theta_P, id_Y, \{\emptyset\})$-continuous fuzzy multifunction.

Proof. Suppose that F is an upper P-continuous fuzzy multifunction and let $\mu \in I^Y$, $r \in I_0$ with $\tau(\mu) \geq r$.

Case 1. If μ satisfies the property P, $\theta_P(\mu, r) = \mu$, and then by hypothesis $F^+(\mu) \in T$ and $F^+(\mu) \subseteq T\text{-int}(F^+(\mu)) = T\text{-int}(F^+(\theta_P(\mu, r)))$.

Case 2. μ does not satisfy the property P, then $\theta_P(\mu, r) = \overline{1}$, and thus $F^+(\mu) \subseteq X = T\text{-int}(F^+(\theta_P(\mu, r)))$. That is, F is upper $(id_X, T\text{-int}, \theta_P, id_Y, \{\emptyset\})$-continuous fuzzy multifunction.

Conversely, suppose that $F^+(\mu) \subseteq T\text{-int}(F^+(\theta_P(\mu, r)))$ for each $\mu \in I^Y$, $r \in I_0$ with $\tau(\mu) \geq r$. Take μ satisfying the property P, then $\theta_P(\mu, r) = \mu$, and thus $F^+(\mu) \subseteq T\text{-int}(F^+(\theta_P(\mu, r))) = T\text{-int}(F^+(\mu))$. We conclude that $F^+(\mu) \in T$ and thus F is an upper P-continuous fuzzy multifunction.

For lower P-continuous fuzzy multifunction, the proof is similar.

Definition 3.4. *If γ and γ^* are fuzzy operators on X, then the operator $\gamma \sqcap \gamma^*$ is defined as follows:*

$$(\gamma \sqcap \gamma^*)(\lambda, r) \ = \ \gamma(\lambda, r) \wedge \gamma^*(\lambda, r) \ \ \forall \lambda \in I^X, \ r \in I_0.$$

The fuzzy operators γ and γ^* are said to be mutually dual if $\gamma \sqcap \gamma^*$ is the identity operator.

Theorem 3.5. Let (X, T) be a topological space, (Y, τ) a fuzzy topological space and ℓ a proper ideal on X. Let α, β, β^* be operators on (X, T) and δ, θ, θ^* be fuzzy operators on (Y, τ). Then $F : X \multimap Y$ is:

(1) upper (resp. lower) $(\alpha, \beta, \theta \sqcap \theta^*, \delta, \ell)$-continuous fuzzy multifunction iff it is both upper (resp. lower) $(\alpha, \beta, \theta, \delta, \ell)$-continuous fuzzy multifunction and upper (resp. lower) $(\alpha, \beta, \theta^*, \delta, \ell)$-continuous fuzzy multifunction provided that for all $A, B \subseteq X$, we have $\beta(A \cap B) = \beta(A) \cap \beta(B)$.

(2) upper (resp. lower) $(\alpha, \beta \sqcap \beta^*, \theta, \delta, \ell)$-continuous fuzzy multifunction iff it is both upper (resp. lower) $(\alpha, \beta, \theta, \delta, \ell)$-continuous fuzzy multifunction and upper (resp. lower) $(\alpha, \beta^*, \theta, \delta, \ell)$-continuous fuzzy multifunction.

Proof. (1) If F is both upper $(\alpha, \beta, \theta, \delta, \ell)$-continuous fuzzy multifunction and upper $(\alpha, \beta, \theta^*, \delta, \ell)$-continuous fuzzy multifunction, then, for every $\mu \in I^Y$, $r \in I_0$, with $\tau(\mu) \geq r$, we have

$$\alpha(F^+(\delta(\mu, r))) - \beta(F^+(\theta(\mu, r))) \in \ell \text{ and } \alpha(F^+(\delta(\mu, r))) - \beta(F^+(\theta^*(\mu, r))) \in \ell$$

and then

$$(\alpha(F^+(\delta(\mu, r))) - \beta(F^+(\theta(\mu, r)))) \cup (\alpha(F^+(\delta(\mu, r))) - \beta(F^+(\theta^*(\mu, r)))) \in \ell.$$

But

$$(\alpha(F^+(\delta(\mu, r))) - \beta(F^+(\theta(\mu, r)))) \cup (\alpha(F^+(\delta(\mu, r))) - \beta(F^+(\theta^*(\mu, r))))$$
$$\begin{cases} = \alpha(F^+(\delta(\mu, r))) - (\beta(F^+(\theta(\mu, r))) \cap \beta(F^+(\theta^*(\mu, r)))) \\ = \alpha(F^+(\delta(\mu, r))) - \beta(F^+(\theta(\mu, r) \wedge \theta^*(\mu, r))) \\ = \alpha(F^+(\delta(\mu, r))) - \beta(F^+(\theta \sqcap \theta^*(\mu, r))). \end{cases}$$

That is, F is upper $(\alpha, \beta, \theta \sqcap \theta^*, \delta, \ell)$-continuous fuzzy multifunction.

Conversely; if F is upper $(\alpha, \beta, \theta \sqcap \theta^*, \delta, \ell)$-continuous fuzzy multifunction, then

$$\alpha(F^+(\delta(\mu, r))) - \beta(F^+(\theta \sqcap \theta^*(\mu, r))) \in \ell$$

Now, by the above equalities, we get that

$$(\alpha(F^+(\delta(\mu, r))) - \beta(F^+(\theta(\mu, r)))) \cup (\alpha(F^+(\delta(\mu, r))) - \beta(F^+(\theta^*(\mu, r)))) \in \ell,$$

which implies that

$$\alpha(F^+(\delta(\mu, r))) - \beta(F^+(\theta(\mu, r))) \in \ell \text{ and } \alpha(F^+(\delta(\mu, r))) - \beta(F^+(\theta^*(\mu, r))) \in \ell$$

which means that F is both upper $(\alpha, \beta, \theta, \delta, \ell)$-continuous fuzzy multifunction and upper $(\alpha, \beta, \theta^*, \delta, \ell)$-continuous fuzzy multifunction.

(2) Similar to the proof in (1).

The proof for lower continuity is typical.

Let Φ be the set of all operators on the topological space (X, T). Then a partial order could be defined by the relation:

$$\alpha \sqsubseteq \beta \quad \text{iff} \quad \alpha(A) \subseteq \beta(A) \text{ for all } A \in 2^X \text{ [14].}$$

Theorem 3.6. Let (X, T) be a topological space, (Y, τ) a fuzzy topological space and ℓ a proper ideal on X. Let $\alpha, \alpha^*, \beta, \beta^* : 2^X \to 2^X$ be operators on (X, T) and $\delta, \theta, \theta^* : I^Y \times I_0 \to I^Y$ are fuzzy operators on (Y, τ) and $F : X \multimap Y$ is a fuzzy multifunction. Then,

(1) If β is a monotone, $\theta \sqsubseteq \theta^*$ and F is upper (resp. lower) $(\alpha, \beta, \theta, \delta, \ell)$-continuous fuzzy multifunction, then F is upper (resp. lower) $(\alpha, \beta, \theta^*, \delta, \ell)$-continuous fuzzy multifunction,

(2) If $\alpha^* \sqsubseteq \alpha$ and F is upper (resp. lower) $(\alpha, \beta, \theta, \delta, \ell)$-continuous fuzzy multifunction, then F is upper (resp. lower) $(\alpha^*, \beta, \theta, \delta, \ell)$-continuous fuzzy multifunction,

(3) If $\beta \sqsubseteq \beta^*$ and F is upper (resp. lower) $(\alpha, \beta, \theta, \delta, \ell)$-continuous fuzzy multifunction, then F is upper (resp. lower) $(\alpha, \beta^*, \theta^*, \delta, \ell)$-continuous fuzzy multifunction.

Proof. (1) Since F is upper $(\alpha, \beta, \theta, \delta, \ell)$-continuous fuzzy multifunction, then for every $\mu \in I^Y$, $r \in I_0$ with $\tau(\mu) \geq r$, it happens that

$$\alpha(F^+(\delta(\mu, r))) - \beta(F^+(\theta(\mu, r))) \in \ell.$$

We know that $\theta \sqsubseteq \theta^*$, and then, for every $\mu \in I^Y$, $r \in I_0$, $\theta(\mu, r) \leq \theta^*(\mu, r)$, and thus $F^+(\theta(\mu, r)) \subseteq F^+(\theta^*(\mu, r))$ and $\beta(F^+(\theta(\mu, r))) \subseteq \beta(F^+(\theta^*(\mu, r)))$. Therefore,

$$\alpha(F^+(\delta(\mu, r))) - \beta(F^+(\theta^*(\mu, r))) \subseteq \alpha(F^+(\delta(\mu, r))) - \beta(F^+(\theta(\mu, r))) \in \ell,$$

which means that F is upper $(\alpha, \beta, \theta^*, \delta, \ell)$-continuous fuzzy multifunction.

(2) and (3) are similar.

The case of lower continuity is similar.

Definition 3.7. *A fuzzy operator γ on a fuzzy topological space (X, τ) induces another fuzzy operator* $(\mathrm{int}_\tau \gamma)$ *defined as follows:* $(\mathrm{int}_\tau \gamma)(\mu, r) = \mathrm{int}_\tau(\gamma(\mu, r), r)$. *Note that:* $\mathrm{int}_\tau \gamma \sqsubseteq \gamma$.

Theorem 3.8. Let $\alpha, \beta : 2^X \to 2^X$ be operators on (X, T) and $\delta, \theta : I^Y \times I_0 \to I^Y$ are fuzzy operators on (Y, τ) and ℓ a proper ideal on X. If $F : X \multimap Y$ is an upper (resp. lower) $(\alpha, \beta, \theta, \delta, \ell)$-continuous fuzzy multifunction and

$$\beta(F^+(\mu)) \subseteq \beta(F^+(\mathrm{int}_\tau(\mu, r))) \ \ (\text{ resp. } \ \beta(F^-(\mu)) \subseteq \beta(F^-(\mathrm{int}_\tau(\mu, r)))),$$

for every $\mu \in I^Y$, $r \in I_0$. Then F is upper (resp. lower) $(\alpha, \beta, \mathrm{int}_\tau \theta, \delta, \ell)$-continuous fuzzy multifunction.

Proof. Let $\mu \in I^Y$, $r \in I_0$ with $\tau(\mu) \geq r$, we have that $\alpha(F^+(\delta(\mu, r))) - \beta(F^+(\theta(\mu, r))) \in \ell$. Since $\beta(F^+(\mu)) \subseteq \beta(F^+(\mathrm{int}_\tau(\mu, r)))$, then $\beta(F^+(\theta(\mu, r))) \subseteq \beta(F^+(\mathrm{int}_\tau \theta(\mu, r)))$. Thus, $\alpha(F^+(\delta(\mu, r))) - \beta(F^+(\mathrm{int}_\tau \theta(\mu, r))) \subseteq \alpha(F^+(\delta(\mu, r))) - \beta(F^+(\theta(\mu, r))) \in \ell$, and it follows that F is upper $(\alpha, \beta, \mathrm{int}_\tau \theta, \delta, \ell)$-continuous fuzzy multifunction.

Definition 3.9. *Let (X, τ) be a fuzzy topological space, θ is a fuzzy operator on X and $\mu \in I^X$, $r \in I_0$. Then μ is called fuzzy θ-compact if for each family $\{\lambda_j \in I^X : \tau(\lambda_j) \geq r,\ j \in J\}$ with $\mu \leq \bigvee_{j \in J} (\lambda_j)$, there exists a finite subset $J_0 \subseteq J$ such that $\mu \leq \bigvee_{j \in J_0} (\theta(\lambda_j, r))$.*

An ordinary subset $A \in 2^X$ is called fuzzy θ-compact if for each family $\{\lambda_j \in I^X : \tau(\lambda_j) \geq r,\ j \in J\}$ with $\chi_A \leq \bigvee_{j \in J} (\lambda_j)$, there exists a finite subset $J_0 \subseteq J$ such that $\chi_A \leq \bigvee_{j \in J_0} (\theta(\lambda_j, r))$.

In crisp case (X, T); a fuzzy set $K \in 2^X$ is called θ-compact if for each family $\{B_j \in 2^X : B_j \in T\}$ with $K \subseteq \bigcup_{j \in J} (B_j)$, there exists a finite subset $J_0 \subseteq J$ such that $K \subseteq \bigcup_{j \in J_0} (\theta(B_j))$.

Theorem 3.10. Let (X, T) be a topological space, (Y, τ) a fuzzy topological space, $\alpha : 2^X \to 2^X$ an operator on (X, T) with $A \subseteq \alpha(A)\ \forall A \in 2^X$ and $\delta, \theta : I^Y \times I_0 \to I^Y$ with $\delta(\lambda, r) \geq \lambda\ \ \forall \lambda \in I^Y,\ r \in I_0$ are fuzzy operators on (Y, τ). If $F : X \multimap Y$ is upper (resp. lower) $(\alpha, T\text{-int}, \theta, \delta, \{\emptyset\})$-continuous fuzzy multifunction and K is a compact subset of X, then, $F(K)$ is fuzzy θ-compact in I^Y.

Proof. Suppose that each family $\{\mu_j : j \in J, r \in I_0$ with $\tau(\mu_j) \geq r\}$ satisfies that $F(K) \leq \bigvee_{j \in J} \mu_j$. By F is upper $(\alpha, T\text{-int}, \theta, \delta, \{\emptyset\})$-continuous fuzzy multifunction, then for each $j \in J$, we have $\alpha(F^+(\delta(\mu_j, r))) \subseteq T\text{-int}(F^+(\theta(\mu_j, r))) \subseteq F^+(\theta(\mu_j, r))$. Then there exists $G_j \in T$ such that $\alpha(F^+(\delta(\mu_j, r))) \subseteq G_j \subseteq F^+(\theta(\mu_j, r))$. Also, since $F^+(\delta(\mu_j, r)) \subseteq \alpha(F^+(\delta(\mu_j, r)))$ and $\mu_j \leq \delta(\mu_j, r)$, then

$$K \subseteq F^+(F(K)) \subseteq \bigcup_{j \in J} F^+(\mu_j) \subseteq \bigcup_{j \in J} G_j.$$

From the compactness of K, there exists a finite subset J_0 of J such that $K \subseteq \bigcup_{j \in J_0} G_j$. Then

$$F(K) \leq \bigvee_{j \in J_0} F(G_j) \leq \bigvee_{j \in J_0} F(F^+(\theta(\mu_j, r))) \leq \bigvee_{j \in J_0} \theta(\mu_j, r).$$

which means that $F(K)$ is fuzzy θ-compact.

Corollary 3.11. *Let (X, T) be a topological space and (Y, τ) a fuzzy topological space. Let $F : X \multimap Y$ be an upper (resp. lower) weakly continuous fuzzy multifunction and K a compact subset of X, then $F(K)$ is a fuzzy almost compact set in I^Y.*

Proof. Take $\alpha =$ identity operator on X, $\beta = T$-int, $\delta = r$-fuzzy identity operator, $\theta = r$-fuzzy closure operator on Y and $\ell = \{\emptyset\}$. Then the result is fulfilled directly from Theorem 2.5.

Corollary 3.12. *Let (X, T) be a topological space and (Y, τ) a fuzzy topological space. Let $F : X \multimap Y$ be an upper (resp. lower) almost continuous fuzzy multifunction and K a compact subset of X, then $F(K)$ is a fuzzy nearly compact set in I^Y.*

Proof. Take $\alpha =$ identity operator on X, $\beta = T$-int, $\delta = r$-fuzzy identity operator, $\theta = r$-fuzzy closure operator on Y and $\ell = \{\emptyset\}$. Then the result follows from Theorem 2.5.

4 Upper and Lower $\eta\eta^*$-continuous Fuzzy Multifunctions

Let X and Y be nonempty sets and $\eta \subseteq 2^X$ be any collection of subsets of X and $\eta^* : I^Y \to I$ any function.

Definition 4.1. *A function $F : X \multimap Y$ is said to be upper (resp. lower) $\eta\eta^*$-continuous fuzzy multifunction if $F^+(\mu) \in \eta$ (resp. $F^-(\mu) \in \eta$) whenever $\mu \in I^Y, r \in I_0$ with $\eta^*(\mu) \geq r$.*

Remark 4.2. *A generalized topology on a set X ([15]) is a collection η of subsets of X such that $\emptyset \in \eta$ and η is closed under arbitrary unions. Also, a generalized fuzzy topology on a set Y ([15]) is a function $\eta^* : I^Y \to I$ such that $\eta^*(\overline{0}) = 1$ and $\eta^*(\bigvee_{j\in J} \mu_j) \geq \bigwedge_{j\in J} (\eta^*(\mu_j))$ $\forall \mu_j \in I^Y$. Observe that if Definition 3.1, η and η^* are generalized topology and generalized fuzzy topology on X and Y respectively, then we just obtain the notion of upper (resp. lower) η, η^*-continuous fuzzy multifunctions. In [16], Maki et al., introduced the notion of minimal structure on a set X, as the collection m_X of subsets of X such that $\emptyset \in m_X$ and $X \in m_X$. Also, in [17], Yoo et al., introduced the notion of fuzzy minimal structure on a set Y, as $m_Y : I^Y \to I$ such that $m_Y(\overline{0}) = m_Y(\overline{1}) = 1$. Now, if in Definition 3.1, $\eta = m_X$ and $\eta^* = m_Y$, we obtain the notion of upper (resp. lower) m_X, m_Y-continuous fuzzy multifunctions.*

Any collection η of subsets of a set X and any function $\eta^* : I^Y \to I$ determine in a natural form an operator $\theta_\eta : 2^X \to 2^X$ and a fuzzy operator $\theta_{\eta^*} : I^Y \times I_0 \to I^Y$ respectively, so that

$$\theta_\eta(A) = \begin{cases} A & \text{if } A \in \eta \\ X & \text{otherwise} \end{cases}$$

and

$$\theta_{\eta^*}(\mu, r) = \begin{cases} \mu & \text{if } \mu \in I^Y, r \in I_0 \text{ with } \eta^*(\mu) \geq r \\ \overline{1} & \text{otherwise} \end{cases}$$

In the case that η is a generalized topology on X and η^* is a generalized fuzzy topology on Y, we obtain other operations (see [15]) that are important for its applications:

$$\eta - \mathrm{int}(A) = \bigcup\{B : B \subseteq A \text{ and } B \in \eta\},$$

$$\eta - \mathrm{cl}(A) = \bigcap\{B : A \subseteq B \text{ and } X - B \in \eta\},$$

$$\mathrm{int}_{\eta^*}(\lambda, r) = \bigvee\{\mu : \mu \leq \lambda \text{ and } \eta^*(\mu) \geq r\},$$

$$\mathrm{cl}_{\eta^*}(\lambda, r) = \bigwedge\{\mu : \lambda \leq \mu \text{ and } \eta^*(\overline{1} - \mu) \geq r\}.$$

Note that: η-int $\subseteq id_X \subseteq \theta_\eta$ and $\mathrm{int}_{\eta^*} \sqsubseteq id_Y \sqsubseteq \theta_{\eta^*}$. Similarly, in the case of a minimal structure m_X (see [18]) and a fuzzy minimal structure m_Y (see [17]), we have

$$m_X - \mathrm{int}(A) = \bigcup\{B : B \subseteq A \text{ and } B \in m_X\},$$

$$m_X - \mathrm{cl}(A) = \bigcap\{B : A \subseteq B \text{ and } X - B \in m_X\},$$

$$\mathrm{int}_{m_Y}(\mu, r) = \bigvee\{\nu : \nu \leq \mu \text{ and } m_Y(\nu) \geq r\},$$

$$\mathrm{cl}_{m_Y}(\mu, r) = \bigwedge\{\nu : \mu \leq \nu \text{ and } m_Y(\overline{1} - \nu) \geq r\}.$$

Note that: m_X-int $\subseteq id_X \subseteq \theta_\eta$ and $\mathrm{int}_{m_Y} \sqsubseteq id_Y \sqsubseteq \theta_{\eta^*}$. Also, m_X-int$(A) = A$ if $A \in m_X$ while m_X-int$(A) \in m_X$ whenever m_X is a minimal structure with the

Maki property [16]. $\text{int}_{m_Y}(\lambda, r) = \lambda$ if $m_Y(\lambda) \geq r$ while $m_Y(m_Y\text{-int}(\lambda, r)) \geq r$ whenever m_Y is a fuzzy minimal structure with the Yoo property [17].

The following results give the relationship between upper (resp. lower) $\eta\eta^*$-continuous fuzzy multifunctions and upper (resp. lower) $(\alpha, \beta, \theta, \delta, \ell)$-continuous fuzzy multifunctions. We obtain some interesting properties of upper (resp. lower) $\eta\eta^*$-continuous fuzzy multifunctions.

Theorem 4.3. Let X and Y be nonempty sets, $\eta \subseteq 2^X$, $\eta^* : I^Y \to I$. If $X \in \eta$, then $F : X \multimap Y$ is upper (resp. lower) $\eta\eta^*$-continuous fuzzy multifunction iff $F : X \multimap Y$ is upper (resp. lower) $(\theta_\eta, id_X, \theta_{\eta^*}, id_Y, \{\emptyset\})$-continuous fuzzy multifunction.

Proof. Suppose that $F : X \multimap Y$ is upper $\eta\eta^*$-continuous fuzzy multifunction. Let $\mu \in I^Y$, $r \in I_0$, we have two cases:
Case 1. If $\eta^*(\mu) \geq r$, then $\theta_{\eta^*}(\mu, r) = \mu$ and $\theta_\eta(F^+(\mu)) = F^+(\mu)$. This follows that $\theta_\eta(F^+(id_Y(\mu, r))) = F^+(\mu) = id_X(F^+(\theta_{\eta^*}(\mu, r)))$, and consequently

$$\theta_\eta(F^+(id_Y(\mu, r))) \subseteq id_X(F^+(\theta_{\eta^*}(\mu, r))).$$

Case 2. If $\eta^*(\mu) = 0$, $\theta_{\eta^*}(\mu, r) = \overline{1}$, then $\theta_\eta(F^+(id_Y(\mu, r))) \subseteq X = F^+(\overline{1}) = id_X(F^+(\theta_{\eta^*}(\mu, r)))$. Hence,

$$\theta_\eta(F^+(id_Y(\mu, r))) - id_X(F^+(\theta_{\eta^*}(\mu, r))) = \emptyset$$

for all $\mu \in I^Y, r \in I_0$. Thus, F is an upper $(\theta_\eta, id_X, \theta_{\eta^*}, id_Y, \{\emptyset\})$-continuous fuzzy multifunction.

Necessity; suppose that F is upper $(\theta_\eta, id_X, \theta_{\eta^*}, id_Y, \{\emptyset\})$-continuous fuzzy multifunction, then $\theta_\eta(F^+(id_Y(\mu, r))) - id_X(F^+(\theta_{\eta^*}(\mu, r))) = \emptyset$ for all $\mu \in I^Y, r \in I_0$ with $\eta^*(\mu) \geq r$. This implies that $\theta_\eta(F^+(\mu)) \subseteq F^+(\theta_{\eta^*}(\mu, r)))$. Assume that there is $\nu \in I^Y, r \in I_0$ such that $\eta^*(\nu) \geq r$ and $F^+(\nu)$ does not belong to η. Then we obtain $X \subseteq F^+(\nu)$. So, $F^+(\nu) = X$. Now, our hypothesis $X \in \eta$ implies that $F^+(\nu) \in \eta$, and a contradiction. Therefore, $F^+(\mu) \in \eta$ whenever $\mu \in I^Y, r \in I_0$ with $\eta^*(\mu) \geq r$, and thus $F : X \multimap Y$ is an upper $\eta\eta^*$-continuous fuzzy multifunction.

In the case that η is a generalized topology, then the following result is obtained.

Theorem 4.4. If η is a generalized topology such that $X \in \eta$ and $\eta^* : I^Y \to I$ is a function. Then $F : X \multimap Y$ is upper (resp. lower) $\eta\eta^*$-continuous fuzzy multifunction iff $F : X \multimap Y$ is upper (resp. lower) $(id_X, \eta\text{-int}, \theta_{\eta^*}, id_Y, \{\emptyset\})$-continuous fuzzy multifunction.

Proof. Suppose that $F : X \multimap Y$ is upper $\eta\eta^*$-continuous fuzzy multifunction. Let $\mu \in I^Y$, $r \in I_0$. Then consider two cases:
Case 1. If $\eta^*(\mu) \geq r$, then $\theta_{\eta^*}(\mu, r) = \mu$ and $id_X(F^+(\mu)) = F^+(\mu) = \eta\text{-int}(F^+(\mu))$. This follows that $id_X(F^+(id_Y(\mu, r))) = F^+(\mu) = \eta\text{-int}(F^+(\theta_{\eta^*}(\mu, r)))$, and consequently

$$id_X(F^+(id_Y(\mu, r))) \subseteq \eta - \text{int}(F^+(\theta_{\eta^*}(\mu, r))).$$

Case 2. If $\eta^*(\mu) = 0$, $\theta_{\eta^*}(\mu, r) = \overline{1}$, since $X \in \eta$, then

$$id_X(F^+(id_Y(\mu, r))) \subseteq X = F^+(\overline{1}) = \eta - \text{int}(F^+(\theta_{\eta^*}(\mu, r))).$$

So,

$$id_X(F^+(id_Y(\mu,r))) \; - \; \eta - \text{int}(F^+(\theta_{\eta^*}(\mu,r))) \; = \; \emptyset$$

for every $\mu \in I^Y, r \in I_0$. Thus, F is an upper $(id_X, \eta\text{-int}, \theta_{\eta^*}, id_Y, \{\emptyset\})$-continuous fuzzy multifunction.

Necessity; suppose that F is upper $(id_X, \eta\text{-int}, \theta_{\eta^*}, id_Y, \{\emptyset\})$-continuous fuzzy multifunction. Then

$$id_X(F^+(id_Y(\mu,r))) \; - \; \eta - \text{int}(F^+(\theta_{\eta^*}(\mu,r))) \; = \; \emptyset$$

for every $\mu \in I^Y, r \in I_0$ with $\eta^*(\mu) \geq r$. This implies that

$$F^+(\mu) \; \subseteq \; \eta - \text{int}(F^+(\theta_{\eta^*}(\mu,r))) = \eta - \text{int}(F^+(\mu)).$$

Assume that there is $\nu \in I^Y, r \in I_0$ such that $\eta^*(\nu) \geq r$ and $F^+(\nu)$ does not belong to η. Then we obtain $F^+(\nu) \subseteq \eta\text{-int}(F^+(\nu))$, and thus $F^+(\nu) = \eta\text{-int}(F^+(\nu))$, and $F^+(\nu) \in \eta$, and a contradiction. Therefore, $F^+(\mu) \in \eta$ whenever $\mu \in I^Y, r \in I_0$ with $\eta^*(\mu) \geq r$, that is, $F : X \multimap Y$ is an upper $\eta\eta^*$-continuous fuzzy multifunction.

The following corollaries are direct results.

Corollary 4.5. *Let $F : X \multimap Y$ be a fuzzy fuzzy multifunction. If F is upper (resp. lower) $m_X m_Y$-continuous fuzzy multifunction, then F is upper (resp. lower) $(id_X, m_X\text{-int}, \theta_{m_Y}, id_Y, \ell)$-continuous fuzzy multifunction whenever m_X has the Maki property.*

Corollary 4.6. *Let η be a generalized topology on X and η^* a generalized fuzzy topology on Y such that $X \in \eta$. Then, $F : X \multimap Y$ is upper (resp. lower) $\eta\eta^*$-continuous fuzzy multifunction* iff *F is upper (resp. lower) $(id_X, \eta\text{-int}, \text{int}_{\eta^*}, id_Y, \{\emptyset\})$-continuous fuzzy multifunction.*

5 Conclusions

In this article, we have introduced the notions of upper and lower continuous multifunctions from an ordinary topological space into a fuzzy topological space in Šostak sense.. We have investigated some of its properties. There are many other properties of the introduced notions, those could be investigated and applied for investigations in other branches of technology.

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Q-Soft Normal Subgroups

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Abstaract — This paper contains some definitions and results in Q-soft normal subgroup theory and cosets. Also some results are introduced which have been used by homomorphism and anti-homomorphism of Q-soft normal subgroups. Next we prove the analogue of the Lagrange's theorem.

Keywords — *Q-soft subsets, group theory, Q-soft subgroups, Q-soft normal subgroups, homomorphism.*

1 Introduction

In mathematics and abstract algebra, group theory studies the algebraic structures known as groups. The concept of a group is central to abstract algebra: other well-known algebraic structures, such as rings, fields, and vector spaces, can all be seen as groups endowed with additional operations and axioms. Groups recur throughout mathematics, and the methods of group theory have influenced many parts of algebra. Linear algebraic groups and Lie groups are two branches of group theory that have experienced advances and have become subject areas in their own right. Various physical systems, such as crystals and the hydrogen atom, may be modelled by symmetry groups. Thus group theory and the closely related representation theory have many important applications in physics, chemistry, and materials science. Group theory is also central to public key cryptography. Soft set theory is a generalization of fuzzy set theory, that was proposed by Molodtsov in 1999 to deal with uncertainty in a parametric manner [10]. A soft set is a parameterised family of sets - intuitively, this is "soft" because the boundary of the set depends on the parameters. Formally, a soft set, over a universal set X and set of parameters E is a pair (f, A) where A is a subset of E and f is a function from A to the power set of X. For each e in A, the set $f(e)$ is called the value set of e in (f, A). One of the most important steps for the new theory of soft sets was to define mappings on soft sets, which was achieved in 2009 by the mathematicians Athar Kharal and Bashir Ahmad, with the results published in 2011 [7]. Soft sets have also been applied to the problem of medical diagnosis for use in medical expert systems. In abstract algebra, a normal subgroup is a subgroup which is invariant under conjugation by members of the group of which it is a part. In other words, a subgroup H of a group G is normal in G if and only if $gH = Hg$ for all g in G. The definition of normal subgroup implies that the sets of left and right cosets coincide. In fact, a seemingly weaker condition that the sets of left and right cosets coincide also implies that the subgroup H of a group G is normal in G [6]. Normal subgroups (and only normal subgroups) can be used to construct quotient groups from a given group. Then Maji et al. [8] introduced several operations on soft sets. The works of the algebraic structure of soft sets was first started by Aktas and Cagman [1]. They presented the notion of the soft group and derived its some basic properties. For basic notions and the applications of soft sets, we incite to read [1, 2, 3, 4, 8, 9, 10, 11]. A. Solairaju and R. Nagarajan [14] introduced the new structures of Q-fuzzy groups. The author investigated soft Lie ideals and anti soft Lie ideals and extension of Q-soft ideals in semigroups [13, 12]. In [5] the author introduced the concept of Q-soft subgroups and discussed the characterisations Q-soft subgroups under homomorphism and anti-homomorphism. The purpose of this paper is to deal with the algebraic structure of Q-soft normal subgroups. The concept of Q-soft normal subgroups is introduced, their characterization and algebraic properties are investigated. The rest of this paper is organized as follows. In Section 2, we summarize some basic concepts which will be used throughout the paper. In Section 3, we introduce the concept of Q-soft normal subgroups and investigate some of their basic properties. Also we investigate Q-soft normal subgroups under homomorphism and anti-homomorphisms. Next we prove the analogue of the Lagrange's theorem.

2 Preliminary

In this section, we present basic definitions of soft sets and their operations. Throughout this work, Q is a non-empty set, U refers to an initial universe set, E is a set of parameters and $P(U)$ is the power set of U.

Definition 2.1. ([8, 10]) For any subset A of E, a Q-soft subset $f_{A\times Q}$ over U is a set, defined by a function $f_{A\times Q}$, representing a mapping $f_{A\times Q} : E\times Q \to P(U)$, such that $f_{A\times Q}(x, q) = \emptyset$ if $x \notin A$. A soft set over U can also be represented by the set of ordered pairs $f_{A\times Q} = \{((x, q), f_{A\times Q}(x, q)) \mid (x, q) \in E \times Q, f_{A\times Q}(x, q) \in P(U)\}$. Note that the set of all Q-soft subsets over U will be denoted by $QS(U)$. From here on, soft set will be used without over U.

Definition 2.2. ([8, 10]) Let $f_{A\times Q}, f_{B\times Q} \in QS(U)$. Then,

(1) $f_{A\times Q}$ is called an empty Q-soft subset, denoted by $\Phi_{A\times Q}$, if $f_{A\times Q}(x, q) = \emptyset$ for all $(x, q) \in E \times Q$,

(2) $f_{A\times Q}$ is called a $A \times Q$-universal soft set, denoted by $f_{A\tilde{\times}Q}$, if $f_{A\times Q}(x, q) = U$ for all $(x, q) \in A \times Q$,

(3) $f_{A\times Q}$ is called a universal Q-soft subset, denoted by $f_{E\tilde{\times}Q}$, if $f_{A\times Q}(x, q) = U$ for all $(x, q) \in E \times Q$,

(4) the set $Im(f_{A\times Q}) = \{f_{A\times Q}(x,q) : (x,q) \in A \times Q\}$ is called image of $f_{A\times Q}$ and if $A \times Q = E \times Q$, then $Im(f_{E\times Q})$ is called image of $E \times Q$ under $f_{A\times Q}$.

(5) $f_{A\times Q}$ is a Q-soft subset of $f_{B\times Q}$, denoted by $f_{A\times Q}\tilde{\subseteq} f_{B\times Q}$, if $f_{A\times Q}(x,q) \subseteq f_{B\times Q}(x,q)$ for all $(x,q) \in E \times Q$,

(6) $f_{A\times Q}$ and $f_{B\times Q}$ are soft equal, denoted by $f_{A\times Q} = f_{B\times Q}$, if and only if $f_{A\times Q}(x,q) = f_{B\times Q}(x,q)$ for all $(x,q) \in E \times Q$,

(7) the set $(f_{A\times Q}\tilde{\cup} f_{B\times Q})(x,q) = f_{A\times Q}(x,q) \cup f_{B\times Q}(x,q)$ for all $(x,q) \in E \times Q$ is called union of $f_{A\times Q}$ and $f_{B\times Q}$,

(8) the set $(f_{A\times Q}\tilde{\cap} f_{B\times Q})(x,q) = f_{A\times Q}(x,q) \cap f_{B\times Q}(x,q)$ for all $(x,q) \in E \times Q$ is called intersection of $f_{A\times Q}$ and $f_{B\times Q}$.

Example 2.3. Let $U = \{u_1, u_2, u_3, u_4, u_5\}$ be an initial universe set and $E = \{x_1, x_2, x_3, x_4, x_5\}$ be a set of parameters. Let $Q = \{q\}, A = \{x_1, x_2\}, B = \{x_2, x_3\}, C = \{x_4\}, D = \{x_5\}, F = \{x_1, x_2, x_3\}$. Define

$$f_{A\times Q}(x,q) = \begin{cases} \{u_1, u_2, u_3\} & \text{if } x = x_1 \\ \{u_1, u_5\} & \text{if } x = x_2 \end{cases}$$

$$f_{B\times Q}(x,q) = \begin{cases} \{u_1, u_2\} & \text{if } x = x_2 \\ \{u_2, u_4\} & \text{if } x = x_3 \end{cases}$$

$$f_{F\times Q}(x,q) = \begin{cases} \{u_1, u_2, u_3, u_4\} & \text{if } x = x_1 \\ \{u_1, u_2, u_5\} & \text{if } x = x_2 \\ \{u_2, u_4\} & \text{if } x = x_3 \end{cases}$$

$f_{C\times Q}(x_4, q) = U$ and $f_{D\times Q}(x_5, q) = \{\emptyset\}$. Then we will have

$$(f_{A\times Q}\tilde{\cup} f_{B\times Q})(x,q) = \begin{cases} \{u_1, u_2, u_3\} & \text{if } x = x_1 \\ \{u_1, u_2, u_5\} & \text{if } x = x_2 \\ \{u_2, u_4\} & \text{if } x = x_3 \end{cases}$$

$$(f_{A\times Q}\tilde{\cap} f_{B\times Q})(x,q) = \begin{cases} \{u_1\} & \text{if } x = x_2 \\ \{\} & \text{if } x \neq x_2 \end{cases}$$

Also $f_{C\times Q} = f_{\tilde{C}\times Q}$ and $f_{D\times Q} = \Phi_{D\times Q}$. Note that the difinition of classical subset is not valid for the soft subset. For example $f_{A\times Q}\tilde{\subseteq} f_{F\times Q}$ does not imply that every element of $f_{A\times Q}$ is an element of $f_{F\times Q}$. Thus $f_{A\times Q}\tilde{\subseteq} f_{F\times Q}$ but $f_{A\times Q} \nsubseteq f_{F\times Q}$ as classical subset.

Definition 2.4. ([5]) Let $\varphi : A \to B$ be a function and $f_{A\times Q}, f_{B\times Q} \in QS(U)$. Then soft image $\varphi(f_{A\times Q})$ of $f_{A\times Q}$ under φ is defined by

$$\varphi(f_{A\times Q})(y,q) = \begin{cases} \cup\{f_{A\times Q}(x,q) \mid (x,q) \in A \times Q, \varphi(x) = y\} & \text{if } \varphi^{-1}(y) \neq \emptyset \\ \emptyset & \text{if } \varphi^{-1}(y) = \emptyset \end{cases}$$

and soft pre-image (or soft inverse image) of $f_{B\times Q}$ under φ is $\varphi^{-1}(f_{B\times Q})(x,q) = f_{B\times Q}(\varphi(x), q)$ for all $(x,q) \in A \times Q$.

Definition 2.5. ([5]) Let $(G, .)$ be a group and $f_{G\times Q} \in QS(U)$. Then, $f_{G\times Q}$ is called a Q-soft subgroup over U if $f_{G\times Q}(xy, q) \supseteq f_{G\times Q}(x,q) \cap f_{G\times Q}(y,q)$ and $f_G(x^{-1}, q) = f_{G\times Q}(x,q)$ for all $x, y \in G, q \in Q$.

Throughout this paper, G denotes an arbitrary group with identity element e_G and the set of all Q-soft subgroup with parameter set G over U will be denoted by $S_{G\times Q}(U)$.

Definition 2.6. ([5]) Let $(G,.),(H,.)$ be any two groups and $f_{G\times Q}\in S_{G\times Q}(U), g_{H\times Q}\in S_{H\times Q}(U)$. The product of $f_{G\times Q}$ and $g_{H\times Q}$, denoted by $f_{G\times Q}\tilde{\times}g_{H\times Q}:(G\times H)\times Q\rightarrow P(U)$, is defined as $f_{G\times Q}\tilde{\times}g_{H\times Q}((x,y),q)=f_{G\times Q}(x,q)\cap g_{H\times Q}(y,q)$ for all $x\in G, y\in H, q\in Q$. Throughout this paper, H denotes an arbitrary group with identity element e_H.

Theorem 2.7. (Lagrange) ([6]) Let G be a finite group. Let H be a subgroup of G. Then the order of H divides the order of G.

Definition 2.8. ([6]) Let $(G,.),(H,.)$ be any two groups. The function $f:G\rightarrow H$ is called a homomorphism (anti-homomorphism) if $f(xy)=f(x)f(y)(f(xy)=f(y)f(x))$, for all $x,y\in G$.

Definition 2.9. ([6]) We call a group G, Hamiltonian if G is non-abelian and every subgroup of G is normal.

Definition 2.10. ([6]) A Dedekind group is one which is abelian or Hamiltonian.

3 Main Results

Definition 3.1. Let $f_{G\times Q}\in S_{G\times Q}(U)$ then $f_{G\times Q}$ is said to be a Q-soft normal subgroup of G if $f_{G\times Q}(xy,q)=f_{G\times Q}(yx,q)$, for all $x,y\in G$ and $q\in Q$. Throughout this paper, G denotes an arbitrary group with identity element e_G and the set of all Q-soft normal subgroup with parameter set G over U will be denoted by $NS_{G\times Q}(U)$.

Example 3.2. Let $U=\{u_1,u_2,u_3,u_4,u_5\}$ be an initial universe set and $(\mathbb{Z},+)$ be an additive group. Define $f_{\mathbb{Z}\times Q}:\mathbb{Z}\times Q\rightarrow P(U)$ as

$$f_{\mathbb{Z}\times Q}(x,q)=\begin{cases}\{u_1,u_2,u_3\} & \text{if } x\in\mathbb{Z}^{\geq 0}\\ \{u_2,u_4,u_5\} & \text{if } x\in\mathbb{Z}^{<0}\end{cases}$$

then $f_{\mathbb{Z}\times Q}\in NS_{\mathbb{Z}\times Q}(U)$.

Proposition 3.3. Let $f_{G\times Q},g_{G\times Q}\in NS_{G\times Q}(U)$. Then $f_{G\times Q}\tilde{\cap}g_{G\times Q}\in NS_{G\times Q}(U)$.

Proof. By [5, Proposition 2.16] we have that $f_{G\times Q}\tilde{\cap}g_{G\times Q}\in S_{G\times Q}(U)$. Let $x,y\in G, q\in Q$. Then

$$\begin{aligned}(f_{G\times Q}\tilde{\cap}g_{G\times Q})(xy,q)&=f_{G\times Q}(xy,q)\cap g_{G\times Q}(xy,q)=f_{G\times Q}(yx,q)\cap g_{G\times Q}(yx,q)\\&=(f_{G\times Q}\tilde{\cap}g_{G\times Q})(yx,q)\end{aligned}$$

and so $f_{G\times Q}\tilde{\cap}g_{G\times Q}\in NS_{G\times Q}(U)$. □

Corollary 3.4. The intersection of a family of Q-soft normal subgroups of a group G is a Q-soft subgroup of a group G.

Proposition 3.5. Let $f_{G\times Q} \in NS_{G\times Q}(U)$. Then $f_{G\times Q}(yxy^{-1}, q) = f_{G\times Q}(y^{-1}xy, q)$ for every $x, y \in G$ and $q \in Q$.

Proof. Let $x, y \in G$ and $q \in Q$. As $f_{G\times Q} \in NS_{G\times Q}(U)$ so

$$\begin{aligned} f_{G\times Q}(yxy^{-1}, q) &= f_{G\times Q}(y^{-1}yx, q) = f_{G\times Q}(ex, q) = f_{G\times Q}(x, q) = f_{G\times Q}(xyy^{-1}, q) \\ &= f_{G\times Q}(y^{-1}xy, q). \end{aligned}$$
□

Proposition 3.6. If every Q-soft subgroup of a group G is normal, then G is a Dedekind group.

Proof. Suppose that every Q-soft subgroup of a group G is normal. We have, consider a subgroup H of G. So H can be regarded as a Q-level subgroup of some Q-soft subgroup $f_{G\times Q}$ of G. By assumption, $f_{G\times Q}$ is a Q-soft normal subgroup of G. Now, it is easy to deduce that H is a normal subgroup of G. Thus G is a Dedekind group □

Proposition 3.7. If $f_{G\times Q} \in NS_{G\times Q}(U), g_{H\times Q} \in NS_{H\times Q}(U)$. Then $f_{G\times Q}\tilde{\times} g_{H\times Q} \in NS_{(G\times H)\times Q}(U)$.

Proof. From [5, Proposition 2.22] we have that $f_{G\times Q}\tilde{\times} g_{H\times Q} \in S_{(G\times H)\times Q}(U)$. Let $(x_1, y_1), (x_2, y_2) \in G \times H, q \in Q$. Then

$$\begin{aligned} f_{G\times Q}\tilde{\times} g_{H\times Q}((x_1, y_1)(x_2, y_2), q) &= f_{G\times Q}\tilde{\times} g_{H\times Q}((x_1x_2, y_1y_2), q) \\ &= f_{G\times Q}(x_1x_2, q) \cap g_{H\times Q}(y_1y_2, q) \\ &= f_{G\times Q}(x_2x_1, q) \cap g_{H\times Q}(y_2y_1, q) \\ &= f_{G\times Q}\tilde{\times} g_{H\times Q}((x_2x_1, y_2y_1), q) \\ &= f_{G\times Q}\tilde{\times} g_{H\times Q}((x_2, y_2)(x_1, y_1), q). \end{aligned}$$

Thus $f_{G\times Q}\tilde{\times} g_{H\times Q} \in NS_{(G\times H)\times Q}(U)$. □

Proposition 3.8. Let $f_{G\times Q}, g_{H\times Q} \in QS(U), f_{G\times Q}\tilde{\times} g_{H\times Q} \in NS_{(G\times H)\times Q}(U)$. Then at least one of the following two statements must hold.
(1) $g_{H\times Q}(e_H, q) \supseteq f_{G\times Q}(x, q)$, for all $x \in G, q \in Q$,
(2) $f_{G\times Q}(e_G, q) \supseteq g_{H\times Q}(y, q)$, for all $y \in H, q \in Q$.

Proof. Use [5, Proposition 2.23]. □

Proposition 3.9. Let $f_{G\times Q}, g_{H\times Q} \in QS(U), f_{G\times Q}\tilde{\times} g_{H\times Q} \in NS_{(G\times H)\times Q}(U)$. Then we have the following statements.
(1) If for all $x \in G, q \in Q, f_{G\times Q}(x, q) \subseteq g_{H\times Q}(e_H, q)$, then $f_{G\times Q} \in NS_{G\times Q}(U)$.
(2) If for all $x \in H, q \in Q, g_{H\times Q}(x, q) \subseteq f_{G\times Q}(e_G, q)$, then $g_{H\times Q} \in NS_{H\times Q}(U)$.
(3) Either $f_{G\times Q} \in NS_{G\times Q}(U)$ or $g_{H\times Q} \in S_{H\times Q}(U)$.

Proof. (1) Let $x, y \in G, q \in Q$. From [5, Proposition 2.24] we have that $f_{G\times Q} \in S_{G\times Q}(U)$. As $f_{G\times Q}(x, q) \subseteq g_{H\times Q}(e_H, q)$ so

$$\begin{aligned} f_{G\times Q}(xy, q) &= f_{G\times Q}(xy, q) \cap g_{H\times Q}(e_He_H, q) \\ &= f_{G\times Q}\tilde{\times} g_{H\times Q}((xy, e_He_H), q) \\ &= f_{G\times Q}\tilde{\times} g_{H\times Q}((x, e_H)(y, e_H), q) \\ &= f_{G\times Q}\tilde{\times} g_{H\times Q}((y, e_H)(x, e_H), q) \end{aligned}$$

$$
\begin{aligned}
&= f_{G\times Q}\tilde{\times}g_{H\times Q}((yx, e_He_H), q)\\
&= f_{G\times Q}(yx, q)\cap g_{H\times Q}(e_He_H, q)\\
&= f_{G\times Q}(yx, q).
\end{aligned}
$$

Thus $f_{G\times Q}\in NS_{G\times Q}(U)$.

(2) Let $x, y\in H, q\in Q$. By [5, Proposition 2.24] we get that $g_{H\times Q}\in S_{H\times Q}(U)$. Since $g_{H\times Q}(x, q)\subseteq f_{G\times Q}(e_G, q)$ so

$$
\begin{aligned}
g_{H\times Q}(xy, q) &= f_{G\times Q}(e_Ge_G, q)\cap g_{H\times Q}(xy, q)\\
&= f_{G\times Q}\tilde{\times}g_{H\times Q}((e_Ge_G, xy), q)\\
&= f_{G\times Q}\tilde{\times}g_{H\times Q}((e_G, x)(e_G, y), q)\\
&= f_{G\times Q}\tilde{\times}g_{H\times Q}((e_G, y)(e_G, x), q)\\
&= f_{G\times Q}\tilde{\times}g_{H\times Q}((e_Ge_G, yx), q)\\
&= f_{G\times Q}(e_Ge_G, q)\cap g_{H\times Q}(yx, q)\\
&= g_{H\times Q}(yx, q).
\end{aligned}
$$

Therefore $g_{H\times Q}\in NS_{H\times Q}(U)$.

(3) Straight forward. □

Recall that $(x) = \{y^{-1}xy : y\in G\}$ is called the conjugate class of x in G.

Proposition 3.10. $f_{G\times Q}\in NS_{G\times Q}(U)$ if and only if $f_{G\times Q}$ is constant on the conjugate classes of G.

Proof. Let $x, y\in G$ and $q\in Q$. If $f_{G\times Q}\in NS_{G\times Q}(U)$, then

$$f_{G\times Q}(y^{-1}xy, q) = f_{G\times Q}(xyy^{-1}, q) = f_{G\times Q}(x, q)$$

Therefore $f_{G\times Q}$ is constant on the conjugate classes of G. Conversely, let $f_{G\times Q}$ is constant on the conjugate classes of G. Then

$$f_{G\times Q}(xy, q) = f_{G\times Q}(x^{-1}(xy)x, q) = f_{G\times Q}((x^{-1}x)yx, q) = f_{G\times Q}(yx, q)$$

and so $f_{G\times Q}\in NS_{G\times Q}(U)$. □

In the following propositions, we prove many results in homomorphism and anti-homomorphism in normal Q-soft subgroups.

Proposition 3.11. Let φ be an epimorphism from group G into group H. If $f_{G\times Q}\in NS_{G\times Q}(U)$, then $\varphi(f_{G\times Q})\in NS_{H\times Q}(U)$

Proof. By [5, Proposition 4.3] we have that $\varphi(f_{G\times Q})\in S_{H\times Q}(U)$. Let $h_1, h_2\in H$ and $q\in Q$ then

$$
\begin{aligned}
\varphi(f_{G\times Q})(h_1h_2, q) &= \cup\{f_{G\times Q}(g_1g_2, q)\mid g_1, g_2\in G, \varphi(g_1g_2) = h_1h_2\}\\
&= \cup\{f_{G\times Q}(g_1g_2, q)\mid g_1, g_2\in G, \varphi(g_1)\varphi(g_2) = h_1h_2\}\\
&= \cup\{f_{G\times Q}(g_1g_2, q)\mid g_1, g_2\in G, \varphi(g_1) = h_1, \varphi(g_2) = h_2\}\\
&= \cup\{f_{G\times Q}(g_2g_1, q)\mid g_2, g_1\in G, \varphi(g_2) = h_2, \varphi(g_1) = h_1\}
\end{aligned}
$$

$$
\begin{aligned}
&= \cup\{f_{G\times Q}(g_2g_1,q) \mid g_2,g_1\in G, \varphi(g_2)\varphi(g_1)=h_2h_1\}\\
&= \cup\{f_{G\times Q}(g_2g_1,q) \mid g_2,g_1\in G, \varphi(g_2g_1)=h_2h_1\}\\
&= \varphi(f_{G\times Q})(h_2h_1,q).
\end{aligned}
$$

Thus $\varphi(f_{G\times Q}) \in NS_{H\times Q}(U)$. □

Proposition 3.12. Let φ be a homorphism from group G into group H. If $g_{H\times Q} \in NS_{H\times Q}(U)$, then $\varphi^{-1}(g_{H\times Q}) \in NS_{G\times Q}(U)$.

Proof. By [5, Proposition 4.5] we have that $\varphi^{-1}(g_{H\times Q}) \in S_{G\times Q}(U)$. Let $g_1, g_2 \in G$ and $q \in Q$. Then

$$
\begin{aligned}
\varphi^{-1}(g_{H\times Q})(g_1g_2,q) &= g_{H\times Q}(\varphi(g_1g_2),q)\\
&= g_{H\times Q}(\varphi(g_1)\varphi(g_2),q)\\
&= g_{H\times Q}(\varphi(g_2)\varphi(g_1),q)\\
&= g_{H\times Q}(\varphi(g_2g_1),q)\\
&= \varphi^{-1}(g_{H\times Q})(g_2g_1,q).
\end{aligned}
$$

Therefore $\varphi^{-1}(g_{H\times Q}) \in NS_{G\times Q}(U)$. □

Proposition 3.13. Let φ be an anti-epimorphism from group G into group H. If $f_{G\times Q} \in NS_{G\times Q}(U)$, then $\varphi(f_{G\times Q}) \in NS_{H\times Q}(U)$.

Proof. By [5, Proposition 4.3] we have that $\varphi(f_{G\times Q}) \in S_{H\times Q}(U)$. Let $h_1, h_2 \in H$ and $q \in Q$ then

$$
\begin{aligned}
\varphi(f_{G\times Q})(h_1h_2,q) &= \cup\{f_{G\times Q}(g_1g_2,q) \mid g_1,g_2\in G, \varphi(g_1g_2)=h_1h_2\}\\
&= \cup\{f_{G\times Q}(g_1g_2,q) \mid g_1,g_2\in G, \varphi(g_2)\varphi(g_1)=h_1h_2\}\\
&= \cup\{f_{G\times Q}(g_1g_2,q) \mid g_1,g_2\in G, \varphi(g_1)=h_1, \varphi(g_2)=h_2\}\\
&= \cup\{f_{G\times Q}(g_2g_1,q) \mid g_2,g_1\in G, \varphi(g_2)=h_2, \varphi(g_1)=h_1\}\\
&= \cup\{f_{G\times Q}(g_2g_1,q) \mid g_2,g_1\in G, \varphi(g_2)\varphi(g_1)=h_2h_1\}\\
&= \cup\{f_{G\times Q}(g_2g_1,q) \mid g_2,g_1\in G, \varphi(g_1g_2)=h_2h_1\}\\
&= \varphi(f_{G\times Q})(h_2h_1,q).
\end{aligned}
$$

Thus $\varphi(f_{G\times Q}) \in NS_{H\times Q}(U)$. □

Proposition 3.14. Let φ be an anti-homorphism from group G into group H. If $g_{H\times Q} \in NS_{H\times Q}(U)$, then $\varphi^{-1}(g_{H\times Q}) \in NS_{G\times Q}(U)$.

Proof. By [5, Proposition 4.8] we have that $\varphi^{-1}(g_{H\times Q}) \in S_{G\times Q}(U)$. Let $g_1, g_2 \in G$ and $q \in Q$. Then

$$
\begin{aligned}
\varphi^{-1}(g_{H\times Q})(g_1g_2,q) &= g_{H\times Q}(\varphi(g_1g_2),q)\\
&= g_{H\times Q}(\varphi(g_2)\varphi(g_1),q)\\
&= g_{H\times Q}(\varphi(g_1)\varphi(g_2),q)\\
&= g_{H\times Q}(\varphi(g_2g_1),q)\\
&= \varphi^{-1}(g_{H\times Q})(g_2g_1,q).
\end{aligned}
$$

Therefore $\varphi^{-1}(g_{H\times Q}) \in NS_{G\times Q}(U)$. □

Remark 3.15. In what follows the symbol $\circ$ stands for the composition operation of functions.

Proposition 3.16. Let φ be an isomorphism from group G into group H. If $f_{H\times Q} \in S_{H\times Q}(U)$, then we have the following:
(1) $f_{H\times Q} \circ \varphi \in S_{G\times Q}(U)$.
(2) If $f_{H\times Q} \in NS_{H\times Q}(U)$, then $f_{H\times Q} \circ \varphi \in NS_{G\times Q}(U)$.

Proof. (1) Let $x, y \in G$ and $q \in Q$.

$$\begin{aligned}(f_{H\times Q} \circ \varphi)(xy^{-1}, q) &= f_{H\times Q}(\varphi(xy^{-1}), q)\\ &= f_{H\times Q}(\varphi(x)\varphi(y^{-1})), q)\\ &= f_{H\times Q}(\varphi(x)\varphi(y)^{-1}, q)\\ &\supseteq f_{H\times Q}(\varphi(x), q) \cap f_{H\times Q}(\varphi(y), q) \quad (as\ f_{H\times Q} \in S_{H\times Q}(U))\\ &= (f_{H\times Q} \circ \varphi)(x, q) \cap (f_{H\times Q} \circ \varphi)(y, q)\end{aligned}$$

and then $f_{H\times Q} \circ \varphi \in S_{G\times Q}(U)$.

(2) Let $f_{H\times Q} \in NS_{H\times Q}(U)$ then

$$\begin{aligned}(f_{H\times Q} \circ \varphi)(xy, q) &= f_{H\times Q}(\varphi(x)\varphi(y), q)\\ &= f_{H\times Q}(\varphi(y)\varphi(x), q)\\ &= f_{H\times Q}(\varphi(yx), q)\\ &= (f_{H\times Q} \circ \varphi)(yx, q).\end{aligned}$$

Therefore $f_{H\times Q} \circ \varphi \in NS_{G\times Q}(U)$. □

Proposition 3.17. Let φ be an anti-isomorphism from group G into group H. If $f_{H\times Q} \in S_{H\times Q}(U)$, then we have the following:
(1) $f_{H\times Q} \circ \varphi \in S_{G\times Q}(U)$.
(2) If $f_{H\times Q} \in NS_{H\times Q}(U)$, then $f_{H\times Q} \circ \varphi \in NS_{G\times Q}(U)$.

Proof. (1) Let $x, y \in G$ and $q \in Q$.

$$\begin{aligned}(f_{H\times Q} \circ \varphi)(xy^{-1}, q) &= f_{H\times Q}(\varphi(xy^{-1}), q)\\ &= f_{H\times Q}(\varphi(y^{-1})\varphi(x)), q)\\ &= f_{H\times Q}(\varphi(y)^{-1}\varphi(x), q)\\ &\supseteq f_{H\times Q}(\varphi(x), q) \cap f_{H\times Q}(\varphi(y), q) \quad (as\ f_{H\times Q} \in S_{H\times Q}(U))\\ &= (f_{H\times Q} \circ \varphi)(x, q) \cap (f_{H\times Q} \circ \varphi)(y, q)\end{aligned}$$

and then $(f_{H\times Q} \circ \varphi \in S_{G\times Q}(U)$.

(2) Let $f_{H\times Q} \in NS_{H\times Q}(U)$ then

$$\begin{aligned}f_{H\times Q} \circ \varphi)(xy, q) &= f_{H\times Q}(\varphi(y)\varphi(x), q) = f_{H\times Q}(\varphi(x)\varphi(y), q)\\ &= f_{H\times Q}(\varphi(yx), q) = (f_{H\times Q} \circ \varphi)(yx, q).\end{aligned}$$

Therefore $f_{H\times Q} \circ \varphi \in NS_{G\times Q}(U)$. □

This motivated us to examine the results for Q-soft cosets. We have found out that the results perfectly fit with Q-soft cosets.

Definition 3.18. Let $f_{G\times Q} \in S_{G\times Q}(U)$ and $H = \{x \in G : f_{G\times Q}(x, q) = f_{G\times Q}(e, q)\}$, then $O(f_{G\times Q})$, the order of $f_{G\times Q}$ is defined as $O(f_{G\times Q}) = O(H)$.

Proposition 3.19. Let $f_{G\times Q}$ be a Q-soft subgroup of a finite group G, then $O(f_{G\times Q}) \mid O(G)$.

Proof. Let $f_{G\times Q}$ be a Q-soft subgroup of a finite group G with e as its identity element. Clearly $H = \{x \in G : f_{G\times Q}(x, q) = f_{G\times Q}(e, q)\}$ is a subgroup of G for H is a Q-level subset of G. By Lagranges theorem $O(H) \mid O(G)$. Hence by the definition of the order of the Q-soft subgroup of G, we have $O(f_{G\times Q}) \mid O(G)$. □

Proposition 3.20. Let $f_{G\times Q}$ and $g_{G\times Q}$ be two Q-soft subgroups of normal group G. Then $O(f_{G\times Q}) = O(g_{G\times Q})$.

Proof. Let $f_{G\times Q}$ and $g_{G\times Q}$ be conjugate Q-soft subgroups of G. Now

$$\begin{aligned} O(f_{G\times Q}) &= \textit{order of } \{x \in G : f_{G\times Q}(x, q) = f_{G\times Q}(e, q)\} \\ &= \textit{order of } \{x \in G : g_{G\times Q}(y^{-1}xy, q) = g_{G\times Q}((y^{-1}ey, q)\} \\ &= \textit{order of } \{x \in G : g_{G\times Q}(x, q) = g_{G\times Q}((e, q)\} = O(g_{G\times Q}). \end{aligned}$$

Hence $O(f_{G\times Q}) = O(g_{G\times Q})$. □

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On Micro Topological Spaces

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Abstaract − Every year different type of topological spaces are introduced by many topologist. Now a days available topologies are supra topology, ideal topology, bitopology, fuzzy topology, Fine topology, nano topology and so on. Nano topology introduced by Thivagar, using this nano topology we introduced micro topology and also study the concepts of micro-pre open sets and micro-semi open sets and some of their properties are investigated.

Keywords − *Micro Topology, Micro-pre open sets, Micro-semi open sets, Micro continuous, Micro pre continuous, Micro semi continuous.*

1 Introduction

In 1963 Kelly [4] introduced Bitopological spaces, In 1983, Mashhour [6] et al. introduced the supra topological spaces. In 1965, Zadeh [9] introduced the concept of fuzzy sets,the study of fuzzy topological spaces which had been introduced by Chang [2] in 1968. The concept of ideal in topological space was first introduced by Kuratowski. They also have defined local function in ideal topological space. Further in 1990 Hamlett and Jankovic [3] investigated further properties of topological space.

Powar and Rajak [7] introduced fine topological spaces in the year 2012. Nano topology introduced by Thivagar [5] in the year 2013. Nano topology based on the concept of lower approximation, upper approximation and boundary region. Nano topology have maximum five nano open sets and minimum three nano open sets including U, ϕ suppose we want add some more open sets, for that time we can use Levine's simple extension concept in nano topology we can extend some more open sets that topology is called micro topology. Every nano topology is micro topology. In this paper, introduce micro topology, micro pre open sets, micro semi open sets and some of their properties are investigated.

2 Preliminary

Let us recall the following definition, which are useful in the sequel.

Definition 2.1. Let U be a non-empty finite set of objects called the universe and R be an equivalence relation on U named as the indiscernibility relation. Then U is divided into disjoint equivalence classes. Elements belonging to the same equivalence class are said to be indiscernible with one another. The pair (U,R) is said to be the approximation space. Let $X \subseteq U$.

1. The lower approximation of X with respect to R is the set of all objects, which can be for certain classified as X with respect to R and it is denoted by $L_R(X)$. That is, $L_R(X)$=$\bigcup_{x\in U}\{R(x) : R(x) \subseteq X\}$ where R(x) denotes the equivalence class determined by $x \in U$.

2. The upper approximation of X with respect to R is the set of all objects,which can be possibly classified as X with respect to R and it is denoted by $U_R(X)$. That is, $U_R(X)$=$\bigcup_{x\in U}\{R(x) : R(x) \cap X \neq \phi\}$

3. The boundary region of X with respect to R is the set of all objects, which can be classified neither as X nor as not-X with respect to R and it is denoted by $B_R(X)$. That is, $B_R(X)$=$U_R(X)$-$L_R(X)$.

   Definition 2.2. Let U be an universe, R be an equivalence relation on U and $\tau_R(X)$=$\{U, \phi, L_R(X), U_R(X), B_R(X)\}$ where $X \subseteq U$ satisfies the following axioms

1. U,$\phi \in \tau_R(X)$

2. The union of the elements of any sub-collection of $\tau_R(X)$ is in $\tau_R(X)$

3. The intersection of the elements of any finite sub collection of $\tau_R(X)$ is in $\tau_R(X)$.

Then $\tau_R(X)$ is called the nano topology on U with respect to X. The space (U,$\tau_R(X)$) is the nano topological space.The elements of are called nano open sets.

3 Micro Topological Spaces

In this section, I introduce and study the properties of Micro topological spaces.

Definition 3.1. (U,$\tau_R(X)$) is a nano topological space here $\mu_R(X) = \{N \cup (N' \cap \mu)\}$: N, $N' \in \tau_R(X)$ and called it Micro topology of $\tau_R(X)$ by μ where $\mu \notin \tau_R(X)$.

Definition 3.2. The Micro topology $\mu_R(X)$ satisfies the following axioms

1. U,$\phi \in \mu_R(X)$

2. The union of the elements of any sub-collection of $\mu_R(X)$ is in $\mu_R(X)$

3. The intersection of the elements of any finite sub collection of $\mu_R(X)$ is in $\mu_R(X)$.

Then $\mu_R(X)$ is called the Micro topology on U with respect to X. The triplet (U,$\tau_R(X)$,$\mu_R(X)$) is called Micro topological spaces and The elements of $\mu_R(X)$ are called Micro open sets and the complement of a Micro open set is called a Micro closed set.

Example 3.3. U = $\{1, 2, 3, 4\}$, with $U/R = \{\{1\}, \{3\}, \{2, 4\}\}$ and X= $\{1, 2\} \subseteq U$, $\tau_R(X)$= $\{U, \phi, \{1\}, \{1, 2, 4\}, \{2, 4\}\}$ Then $\mu = \{3\}$. Micro-O=$\mu_R(X)$=$\{U, \phi, \{1\}, \{3\}, \{1, 3\}, \{2, 4\}, \{2, 3, 4\}, \{1, 2, 4\}\}$

Example 3.4. $U = \{a, b, c, d\}$, with U/R= $\{\{a\}, \{c\}, \{b, d\}\}$ X=$\{b, d\} \subseteq U$, $\tau_R(X) = \{U, \phi, \{b, d\}\}$ and then $\mu = \{b\}$. Micro-O=$\mu_R(X)$= $\{U, \phi, \{b\}, \{b, d\}\}$

Example 3.5. Let U = {p, q , r, s, t}, U/R= $\{\{p\}, \{q, r, s\}, \{t\}\}$. Let X={p,q}$\subseteq$ U. Then $\tau_R(X)$=$\{U, \phi, \{p\}, \{p, q, r, s\}, \{q, r, s\}\}$. Then $\mu = \{t\}$. Then Micro-O =$\mu_R(X)$= $\{U, \phi, \{p\}, \{t\}, \{p, t\}, \{p, q, r, s\}, \{q, r, s\}, \{q, r, s, t\}\}$

Definition 3.6. The Micro closure of a set A is denoted by Micro-cl(A) and is defined as Mic-cl(A)=∩{B:B is Micro closed and A$\subseteq B\}$. The Micro interior of a set A is denoted by Micro-int(A) and is defined as Mic-int(A)=∪{B:B is Micro open and A$\supseteq B\}$.

Definition 3.7. For any two Micro sets A and B in a Micro topological space (U,$\tau_R(X)$, $\mu_R(X)$),

1. A is a Micro closed set if and only if Mic-cl(A)=A
2. A is a Micro open set if and only if Mic-int(A)=(A)
3. A $\subseteq$ B implies Mic-int(A)$\subseteq$ Mic-int(B) and Mic-cl(A)$\subseteq$ Mic-cl(B)
4. Mic-cl(Mic-cl(A))=Mic-cl(A)and Mic-int(Mic-int(A))=Mic-int(A)
5. Mic-cl(A∪B)⊇Mic-cl(A)∪ Mic-cl(B)
6. Mic-cl(A∩B)⊆Mic-cl(A)∩Mic-cl(B)
7. Mic-int(A∪B)⊇ Mic-int(A) ∪Mic-int(B)
8. Mic-int(A∩B)⊆ Mic-int(A) ∩ Mic-int(B)
9. Mic-cl(A^C)=[Mic-int(A)]^C
10. Mic-int(A^C)=[Mic-cl(A)]^C

4 Micro-Pre-Open Sets

In this section, I define and study about micro-pre-open sets some of their properties are analogous to those for open sets.

Definition 4.1. Let (U,$\tau_R(X)$,$\mu_R(X)$) be a micro topological space and $A \subset U$. Then A is said to be micro-pre-open if A⊆ Mic-int(Mic-cl(A)) and micro-pre-closed set if Mic-cl(Mic-int(A))⊆ A.

Example 4.2. Let U = {p, q , r, s, t}, U/R= $\{\{p\},\{q,r,s\},\{t\}\}$. Let X={p,q}⊆ U. Then $\tau_R(X)$=$\{U,\phi,\{p\},\{p,q,r,s\},\{q,r,s\}\}$. Then $\mu = \{t\}$. Micro-O=$\mu_R(X)$= $\{U,\phi,\{p\},\{t\},\{p,t\},\{p,q,r,s\},\{q,r,s\},\{q,r,s,t\}\}$. Clearly A={ q, r, s} is Micro-pre open.

Theorem 4.3. Every Micro-open set is Micro-pre open.

Proof. Let A be Micro-open. Then $A \subseteq$ Mic-int(Mic-intA). Since Mic-int(Mic-intA) ⊆ Mic-int(Mic-clA), it follows that A ⊆ Mic-int(Mic-cl A). Hence A is Micro-pre open.Converse of the above Theorem need not be true. □

Example 4.4. Let U = $\{i,j,k,l,m\}$ U/R= $\{\{i\},\{j,k,l\},\{m\}\}$. Let X = $\{j,k\}$ ⊆U. Then $\tau_R(X)$ =$\{U,\phi,\{j,k,l\}\}$. Then $\mu = \{i\}$. Then Micro- O=$\{U,\phi,\{i\},\{i,j,k,l\}$, $\{j,k,l\}\}$. Clearly A=$\{i,j,k,m\}$ is Micro-pre open but not Micro-open.

Theorem 4.5.
1. Arbitrary union of Micro-pre open sets is Micro-pre open.
2. Arbitrary intersection of Micro-pre closed sets is Micro-pre closed.

Proof.
1. Let $\{A_\alpha|\alpha \in I\}$ be the family of Micro-pre open sets in X. By Definition 3.6, for each α ,A_α ⊆ Mic-int(Mic-cl(A_α)), this implies that $\cup A_\alpha$ ⊆∪(Mic-int(Mic-cl(A_α)).Since ∪(Mic-int(Mic-cl(A_α))⊆ Mic-int(∪Mic-cl(A_α)) and Mic-int(∪Mic-cl(A_α))= Mic-int(Mic-cl($\cup A_\alpha$)), this implies that $\cup A_\alpha$ ⊆ Mic-int(Mic-cl($\cup A_\alpha$)). Hence$\cup A_\alpha$ is Micro-pre open.
2. Let $\{B_\alpha|\alpha \in I\}$ be a family of Micro-pre closed sets in X. Let $A_\alpha = B_\alpha^C$,then $\{A_\alpha|\alpha \in I\}$ is a family of Micro-pre open sets. By (i),$\cup A_\alpha$=$\cup B_\alpha)^C$is Micro-pre open. Consequently $(\cap B_\alpha)^C$ is Micro-pre open.Hence $(\cap B_\alpha)$ is Micro-pre closed.

□

Remark 4.6. Finite intersection of Micro-pre open sets need not be Micro-pre open.

Example 4.7. In Example 4.4 $\{i,l\}$ and $\{j,l\}$ are Micro-pre open sets, but $\{i,l\} \cap \{j,l\} = \{l\}$ is not Micro-pre open.

Theorem 4.8. In a Micro topological space (U,$\tau_R(X)$,$\mu_R(X)$)the set of all Micro-pre open sets form a generalized topology.

Proof. proof follows from Remark 4.6, Theorem 4.3 and Theorem 4.5. □

Definition 4.9. Let (U,$\tau_R(X)$,$\mu_R(X)$) be a Micro-topological space. An element $x \in A$ is called Micro-pre interior point of A, if there exist a Micro-pre open set H such that $x \in H \subset A$.

Definition 4.10. The set of all Micro-pre interior points of A is called the Micro-pre interior of A, and is denoted by Micro-pre-int(A).

Theorem 4.11.
1. Let $A \subset$(U,$\tau_R(X)$,$\mu_R(X)$) Then Micro-pre int A is equal to the union of all Micro-pre open set contained in A.
2. If A is a Micro-pre open set then A=Micro-pre int A.

Proof.
1. We need to prove that, Micro-pre intA=$\cup\{B|B \subset$ A, Bis Micro-pre open set$\}$. Let x$\in$ Micro-pre int A. Then there exist a Micro-pre open set B such that x$\in$ $B \subset A$. Hence $x \in \cup\{B|B \subset$ A, B is Micro-pre open set$\}$. Conversely, suppose $x \in \cup\{B|B \subset$ A,B is Micro-pre open set$\}$, then there exist a set $B_o \subset A$ such that $x \in B_o$, where B_o is Micro-pre open set. i.e., $x \in$ Micro-pre int A. Hence $\cup\{B|B \subset$ A,B is Micro-pre open set$\}$ $\subset$ Micro-pre int A. So Micro-pre int A=$\cup\{B|B \subset$ A, B is Micro-pre open set$\}$.
2. Assume A is a Micro-pre open set then A $\in\{B|B \subset$A, Micro-pre open set$\}$, and every other element in this collection is subset of A. Hence by part (1) Micro-preintA=A.

□

Theorem 4.12.
1. Micro-pre int $(A \cup B) \supset$ Micro-pre int $A\cup$Micro-pre int B.
2. Micro-pre int($A \cap B$)=Micro-pre int A$\cap$ Micro-pre int B.

Proof.
1. The fact that Micro-pre int A$\subset$ A and Micro-preint B$\subset$ B implies Micro-pre int A$\cup$ Micro-pre int B$\subset$ A$\cup$B. Since Micro- Pre interior of a set is Micro-Pre open, Micro-pre int A and Micro-pre int B are pre open. Hence by Theorem 4.5(1), Micro-pre int A$\cup$ Micro-pre int B is Micro- Pre open and contained in A$\cup$ B. Since Micro-pre $int(A \cup B)$ is the largest Micro-pre open set contained in $A \cup B$, it follows that Micro-pre int $A\cup$ Micro-pre int B $\subset$ Micro-pre int $(A \cup B)$.
2. Let x$\in$ Micro-pre int (A$\cap$ B).Then there exist a Micro-pre open set H, such that x$\in$H$\subset$ $(A \cap B)$.That is there exist a Micro-pre open set, such that $x \in H \subset A$and $x \in H \subset B$. Hence $x \in$ Micro-pre int A and $x \in$ Micro-pre int B. That is $x \in$ Micro-pre int $A\cap$ Micro-pre int B. Thus Micro-pre int $(A \cap B) \subset$ Micro-pre int $A\cap$ Micro-pre int B. Retracing the above steps, we get the converse.

□

Definition 4.13. (U,$\tau_R(X)$,$\mu_R(X)$) be a Micro-topological space.Let A $\subset$ X. The intersection of all Micro-pre closed sets containing A is called Micro-pre closure of A and it is denoted by Micro-pre cl(A). Micro-precl(A)=$\cap\{B/B \supset$ A,B is Micro-pre closed set$\}$.

Remark 4.14.
1. Micro-precl(A) is also a Micro-pre closed set.

2. Micro-precl(A) is smallest Micro-pre closed set containing A.

Theorem 4.15. Every Mic-closed set is Micro-pre closed.

Proof. Let A be Mic-closed, then by Theorem 4.5, we have Mic-cl(Mic-cl A) $\subseteq A$. Since Mic-cl(Mic-intA)⊆Mic-cl(Mic-clA)$\subseteq A$, A is Micro-pre closed. Converse of the above Theorem need not be true. □

Example 4.16. U=$\{1,2,3,4\}$,with $U/R = \{\{1\},\{3\},\{2,4\}\}$ and X=$\{1,2\} \subseteq U$, $\tau_R(X)$= $\{U,\phi,\{1\},\{1,2,4\},\{2,4\}\}$. Then $\mu = \{3\}$. Micro-O=$\{U,\phi,\{1\},\{3\},\{1,3\},$ $\{2,4\},\{2,3,4\},\{1,2,4\}\}$. Then A = $\{1,2,3\}$ is Micro-pre closed but not Micro-closed.

Theorem 4.17. A is Micro-pre closed if and only if A=Micro-pre cl(A).

Proof. Micro-pre cl(A) =$\cap\{B/B \supset$, B is Micro-pre closed set $\}$. If A is a Micro-pre closed set then A is a member of the above collection and each member contains A. Hence their intersection is A and Micro-precl(A)=A. Conversely, if A= Micro-precl(A), then A is Micro-pre closed by Remark 4.14. □

5 Micro-Semi Open Sets

Definition 5.1. Let (U,$\tau_R(X)$,$\mu_R(X)$) be a Micro-topological space and $A \subset U$ Then A is said to be Micro-semi open if A $\subseteq$ Mic-cl(Mic-intA) Micro-semi closed. If Mic-int(Mic-cl A)$\subseteq A$.

Example 5.2. $U = \{a,b,c,d\}$, with U/R= $\{\{a\},\{c\},\{b,d\}\}$ X=$\{b,d\} \subseteq U$, $\tau_R(X) =$ $\{U,\phi,\{b,d\}\}$ and then $\mu = \{a\}$. Then Micro-O = $\{U,\phi,\{a\},\{b,d\},\{a,b,d\}\}$ Clearly A = $\{b,d\}$ is Micro-semi open.

Theorem 5.3.
1. Every Micro-open set is Micro-semi open.
2. Every Mic-closed set is Micro-semi closed.

Proof.
1. If A is Micro-open set then by then, A ⊆Mic-int(Mic-intA). Since Mic-int(Mic-intA)⊆Mic-cl(Mic-intA), A⊆ Mic-cl(Mic-intA). Hence A is Micro-semi open.
2. If A is Mic-closed set then by Theorem 4.5, we have Mic-cl(Mic-clA)⊆A. Since Mic-int(Mic-cl A)⊆ Mic-cl(Mic-clA), Mic-int(Mic-cl A)⊆A. Hence A is Micro-semi closed.

□

Remark 5.4. Converse of the above Theorem need not be true.

Example 5.5. In Example 5.2 clearly A=$\{b,d\}$ is Micro-semi open Clearly A=$\{b,d\}$ is Micro-semi closed,but not Mic-closed.

6 Continuous Functions in Micro-Top. Spaces

Definition 6.1. Let ((U,$\tau_R(X)$,$\mu_R(X)$))and((V,$\tau'_R(X)$,$\mu'_R(X)$)) be two Micro-topological spaces. A function $f : U \to V$ is called Micro-continuous function if $f^{-1}(H)$ is Micro-open in U for every Micro-open set H in V.

Example 6.2. Let U =$\{p, q, r, s, t\}$, $U/R = \{\{p\}, \{q, r, s\}, \{t\}\}$. LetX=$\{p, q\}$ ⊆U. The $\tau_R(X) = \{U, \phi\{p\}, \{p, q, r, s\}, \{q, r, s\}\}$. Then $\mu = \{q\}$. Then Micro-O= $\mu_R(X)$ = $\{U, \phi, \{p\}, \{q\}, \{p, q\}, \{p, q, r, s\}, \{q, r, s\}\}$ Let V = {1, 2 , 3, 4, 5}, V/R= { {1,2,3},{4},{5}}. Let X =$\{1, 2\} \subseteq U$. Then $\tau'_R(X)$=$\{V, \phi, \{1, 2, 3\}\}$. Then $\mu = \{4\}$. Then Micro-O=$\mu'_R(X)$=$\{V, \phi, \{4\}, \{1, 2, 3, 4\}, \{1, 2, 3\}\}$. $f : U \to V$be a function defined as f(p)=4, f(q)=2, f(r)=3, f(s)=1. Micro-open sets in U are {p}, {q}, {p,q}, {p,q,r,s}, {q,r,s} and Micro-open sets in V are $\{4\}, \{1, 2, 3, 4\}, \{1, 2, 3\}$ .Therefore for every Micro-open set H in V, $f^{-1}(H)$ is Micro-open set in U.Then f is Micro-continuous function.

Definition 6.3. Let ((U,$\tau_R(X)$,$\mu_R(X)$))and ((V,$\tau'_R(X)$,$\mu'_R(X)$)) be the two Micro-topological space.A function $f : U \to V$is called Micro-continuous at a point $a \in U$ if for every Micro-open set H containing f(a) in V, there exist a Micro-open set G containing a in U, such that $f(G) \subset H$.

Theorem 6.4. f:$U \to$ V is Micro-continuous if and only if f is Micro-continuous at each point of U.

Proof. Let f: $U \to$V be Micro-continuous. Let $a \in U$, and H be a Micro-open set in V containing f(a). Since f is Micro-continuous, $f^{-1}(V)$is Micro-open in U containing a. Let $G = f^{-1}(H)$, then $f(G) \subset H$, and $f(a) \in G$. Hence f is continuous at a. Conversely, suppose f is Micro-continuous at each point of U. Let H be Micro-open set in V. If $f^{-1}(H) = \phi$then it is Micro-open. So let $f^{-1}(H) \neq \phi$. Take any a $f^{-1}(H)$, then $f(a) \in H$. Since f is Micro-continuous at each point there exist a Micro-open set $G_a$containing a such that $f(G_a) \subset H$. Let $G = (G_a|a \inf^{-1}(H))$. Claim: G=$f^{-1}(H)$ If $x \in f^{-1}(H)$ then $x \in G_x \subset G$. Hence $f^{-1}(H) \subset$ G. On the other hand, suppose $y \in$ G then y $\in G_x$for some x and y $\in f^{-1}$(H).Hence $U = f^{-1}(H)$.Since $G_x$is Micro-open, by definition 6.3 G is Micro-open and hence G=f^{-1}(H) is Micro-open for every Micro-open set H in V. Hence f is Micro-continuous. □

Theorem 6.5. Let ((U,$\tau_R(X)$,$\mu_R(X)$)) and ((V,$\tau'_R(X)$,$\mu'_R(X)$))be two Micro-topological spaces. Then f:$U \to V$ is Micro-continuous function if and only if $f^{-1}(H)$ is Micro-closed in U, whenever H is Micro-closed in V.

Proof. Let f: $U \to V$ is Micro-continuous function and H be Micro-closed in V. Then H^C is Micro-open inV. By hypothesis$f^{-1}(H^C)$ is Micro-open in U, i.e., $[f^{-1}(H)]^C$ is Micro-open in U. Hence $f^{-1}(H)$ is Micro-closed in U whenever H-is Micro-closed in V. Conversely, suppose $f^{-1}(H)$ is Micro-closed in U whenever H is Micro-closed in V. Let U is Micro-open in V then G^C is Micro-closed in V. By assumption $f^{-1}(G^C)$ is Micro-closed in U.i.e., $[f^{-1}(G)]^C$is Micro-closed in X. Then $f^{-1}(G)$ is Micro-open in U. Hence f is Micro-continuous. □

Theorem 6.6. Let $((\mathrm{U},\tau_R(X),\mu_R(X)))$ and $((\mathrm{V},\tau_R^{'}(X),\mu_R^{'}(X)))$ be two Micro-topological space. Then f:$U \to$ V is Micro-continuous function if and only if f(Micro-clA) $\subset$ Micro-cl[f(A)].

Proof. Suppose f: $U \to$ V is Micro-continuous and Micro-cl[f(A)] is Micro-closed in V. Then by f^{-1} (Micro-cl[f(A)]) is Micro-closed in U. Consequently, Micro-cl[f^{-1}(Micro-cl [f(A)])]=f^{-1}(Micro-cl[f(A)]). Since f(A)$\subset$ Micro-cl[f(A)],$A \subset f^{-1}$(Micro-cl[f(A)]) and Micro-cl(A)$\subset$ Micro-cl(f^{-1} Micro-cl[f(A)]))=f^{-1}(Micro-cl[f(A)]) Hence f (Micro-cl (A)) $\subset$ Micro-cl [f(A)]. Conversely, if f(Micro-cl(A))$\subset$ Micro-cl[f(A)] for all A $\subset$ U. Let F be Micro-closed set in V, so that Micro-cl(F)=F ... (1) By hypothesis, f(Micro-cl($f^{-1}(F)$) $\subset$ Micro-cl [$f(f^{-1}(F))$] $\subset$ Micro-cl(F), then by (1), Micro-cl $(f^{-1}(F)) \subset F$. It follows that Micro-cl $(f^{-1}(F)) \subset f^{-1}(F)$. But always $f^{-1}(F) \subset$ Micro-cl $(f^{-1}(F)]$, so that Micro-cl $(f^{-1}(F)) = f^{-1}(F)$. Hence $f^{-1}(F)$ is Micro-closed in U and f is continuous by Theorem 6.4. □

Theorem 6.7. Let$((\mathrm{U},\tau_R(X),\mu_R(X)))$, $((\mathrm{V},\tau_R^{'}(X),\mu_R^{'}(X)))$ and $((\mathrm{W},\tau_R^{''}(X),\mu_R^{''}(X)))$ be three Micro-topological spaces. If f:U→V and g:V→W are Micro-continuous mappings then g∘f: U→ W is also Micro-continuous.

Proof. Let G be a Micro-open set in W. Since by g is Micro-continuous, g^{-1}(G) is Micro-open set in V. Now,(g∘$f)^{-1}G$=$(f^{-1} \circ g^{-1})G$= $f^{-1} \circ (g^{-1}(G))$. Take g^{-1}(G)=H which is Micro-open in V, then $f^{-1}(H)$is Micro-open in U,since by f is Micro-continuous. Hence g∘f: $U \to$ W is Micro-continuous function. □

7 Micro-Pre Continuous and Micro-Semi Continuous Functions

Definition 7.1. Let $((\mathrm{U},\tau_R(X),\mu_R(X)))$ and $((\mathrm{V},\tau_R^{'}(X),\mu_R^{'}(X)))$ be two Micro-topological spaces,then f: U →V is Micro-pre continuous if f^{-1}(V) is Micro-pre closed in U whenever V is Micro-closed.

Theorem 7.2. Every Micro-continuous function is Micro-pre continuous

Proof. Let f:$U \to V$ be Micro-continuous. i.e., $f^{-1}(H)$is Micro-closed in U, whenever H is Micro-closed in V. By Theorem 4.11, every Micro-closed set is Micro-pre closed, and hence $f^{-1}(V)$ is Micro-pre closed in U whenever H is Micro-closed in V. Hence f:$U \to$ V be Micro-pre continuous □

Definition 7.3. Let$((\mathrm{U},\tau_R(X),\mu_R(X)))$ and $((\mathrm{V},\tau_R^{'}(X),\mu_R^{'}(X)))$ be two Micro-topological space,then f:U →V is Micro-semi continuous if $f^{-1}(H)$ is Micro- semi closed in U whenever H is Micro-closed in V.

Theorem 7.4. Every Micro-continuous function is Micro-semi continuous.

Proof. Let f: $U \to$ V be Micro-continuous. i.e.,$f^{-1}(H)$ is Micro-closed in U, whenever H is Micro-closed in V. By Theorem 5.3 (2), every Micro-closed set is Micro-semi closed. This implies that $f^{-1}(H)$ is Micro-semi closed in U whenever H is closed in Y.Hence f:U →V be Micro-semi continuous. □

8 Conclusion

Every year many topologist introduced diffrent type of topological spaces. In this paper i introduced Micro topological spaces and discussed properies and applications of Micro pre open sets,Micro semi open sets. This shall be extended in the future Research with some applications

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Generalized Roughness of $(\in, \in \vee q)$-Fuzzy Ideals in Ordered Semigroups

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Abstaract — Ordered semigroups (OSGs) is a significant algebraic structure having partial ordered with associative binary operation. OSGs have broad applications in various fields such as coding theory, automata theory, fuzzy finite state machines and computer science etc. In this manuscript we investigate the notion of generalized roughness for fuzzy ideals in OSGs on the basis of isotone and monotone mappings. Then the notion of approximation is boosted to the approximation of fuzzy bi-ideals, approximations fuzzy interior ideals and approximations fuzzy quasi-ideals in OSGs and investigate their related properties. Furthermore $(\in, \in \vee q)$-fuzzy ideals are the generalization of fuzzy ideals. Also the generalized roughness for $(\in, \in \vee q)$-fuzzy ideals, fuzzy bi-ideals and fuzzy interior ideals have been studied in OSGs and discuss the basic properties on the basis of isotone and monotone mappings.

Keywords — *Fuzzy sets, Rough sets, Approximations of fuzzy ideals, Approximations of $(\in, \in \vee q)$-fuzzy ideals.*

1 Introduction

In real life, there exist some possible scenario in which the objects of a set are arrange through a specific order. For example the cost of certain commodities in a market can be debated by a terms such as very costly, costly, affordable, cheap and very cheap. We see that exist an order among these items and commodities. So it is clear that these commodities can be characterized through an order among their prices. This can be study in an algebraic structure called ordered semigroups (OSGs). OSGs is a set having partial ordered with associative binary operation. OSGs have broad

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applications in various fields such as coding theory, automata theory and computer science etc.

The paradigm of fuzzy set was originally initiated by Zadeh [36]. This theory has strong points of view to tackle with uncertainty. With the passage of time fuzzy set become the rich research area among the scholars. The model of fuzzy set has been generalized in several direction by different authors. The concept of fuzzy algebraic model was initiated by Rosenfeld [27] and presented the study of fuzzy subgroups. Kuroki [19] originated the theory of fuzzy semigroups. The theory of fuzzy ordered groupoids and OSGs was investigated by Kehayopulu and Tsingelis in [14, 15] and studied the concepts of fuzzy ideals and fuzzy filters in ordered groupoids. Bhakat and Das in [3, 4, 5] investigated the concepts of (α, β)-fuzzy subgroups in his pioneer work and the concept of $(\in, \in \vee q)$-fuzzy subgroups attracted more attention of the scholar towards the study of (α, β) structure. Concept of $(\in, \in \vee q)$-fuzzy subgroup is based on quasi-coincidence of fuzzy points. This notion is introduced in [24]. In algebraic structures the most significant topic fuzzy ideals (FIds) attract the attention of many scholars. In semigroups Kuroki [18] presented the ideas of FIds, fuzzy bi-ideals (FBIds) and study some of the fundamental properties of these ideals. Moreover in semigroups Kuroki [20] explored the notion of fuzzy quasi-ideals (FQIds) and fuzzy semiprime quasi-ideals and study some of the basic properties related to these ideals. Jun et al. [9] initiated the standpoints of $(\in, \in \vee q)$-FBIds of OSGs and given some characterizations Theorems. In semigroups the concepts of FQIds was studied by Ahsan [1]. The study of general form of fuzzy interior ideals (FIIds) and (α, β)-FIIds is initiated by Jun and Song [8] in semigroups. In semigroups the generalization of (α, β)-FIds of hemirings is presented by Jun et al. [10], and for more detail see [28, 29, 30].

Pawlak [23] is the pioneer who for the first time investigated the rudimentary concept of rough set. The fundamental concept of Pawlak rough set depend upon the equivalence relation. So due to confined knowledge about the objects of a certain set, it is too complicated to made the equivalence relation among the elements of a set. Here the authors are restricted by the properties of equivalence relation and many applications of Pawlak rough set have been reported. So different scholars studied the different structures for rough set with less constraint. The prototypes of fuzzy set and rough set are different but both of them have the ability to tackle with uncertainty. Both of these theories are combine very successfully by Dubois and Prade in [7]. The study of generalized rough sets was initiated by Davvaz [6]. In generalized rough set a set valued function play a vital role to define the approximations rather than equivalence relation of a set. Several authors presented the approximation of a set in different algebraic structure, such as in semigroups and fuzzy semigroups Kuroki [21] initiated the idea of roughness and in the same structure this idea is extended to the prime ideals in [31]. In OSGs rough approximations as proposed in [21] can be considered as a better idea. Rehman et al. initiated the concept of roughness in LA-semigroups. Qurashi and Shabir [25] presented the generalized roughness in quantales. The concepts of rough bipolar Γ-hyperideals was initiated by Yaqoob et al. [35] and for the detail study of roughness also see [33, 34, 37]. The rough study of ternary semigroups was presented by Yaqoob et al. [32]. As

OSGs is the relation of partial ordered and semigroups that is why to find the non-trivial equivalence relations for such a structure are difficult. Therefore in OSGs the study of generalized roughness was originated by Mahmood et al. [22] in fuzzy filters and fuzzy ideals with thresholds by defining the set valued homomorphisms. Furthermore they have studied the approximation of generalized structure of fuzzy filters and fuzzy ideals with thresholds in OSGs. In OSGs Ali et al. [2] initiated the rough study of $(\in, \in \vee qk)$-fuzzy filters and they also studied the approximation of generalization of fuzzy filters. Here in this manuscript we will originate the study of generalized roughness of fuzzy ideals in OSGs. Instead of equivalence relation the set valued maps will play a vital role to introduce this new concept of generalized roughness in fuzzy ideals of OSGs and these mapping will be in the form of isotone or monotone order. The order of the paper is as follows.

This paper is organized as, in Section 2, we will briefly recall some fundamental concepts related to OSGs, fuzzy sets, rough sets, FIds and their generalization which is the key for onward concepts. In Section 3, we will originate the approximations of FIds, FBIds, FIIds and FQIds of OSGs on the basis of isotone and monotone mapping. It is clear that these two mappings play a significant role for investigating the approximation of FIds in OSGs. Moreover in Section 4, the idea of approximation is generalized to $(\in, \in \vee q)$-FIds, FBIds, FIIds and FQIds. The final Section 5, consist of the conclusion of the proposed manuscript.

2 Preliminary

This section consist of brief and rudimentary standpoints about OSGs, fuzzy set, and rough set which will provide the key for onward concepts.

Let S be a nonempty set. OSGs $(S, \cdot, \leq)$ is the relation of partial ordered and semigroups in which S under multiplication is a semigroup and S under $\leq$ is a partially ordered set (po-set) and holds the following

$$(\forall z, z_1, z_2 \in S)(z_1 \leq z_2 \rightarrow z_1 z \leq z_2 z \text{ and } z z_1 \leq z z_2).$$

An ordered subsemigroup S_1 is a nonempty subset of S if it holds $S_1^2 \subseteq S_1$.

For $S_1 \subseteq S$, we denote $(S_1] := \{z_1 \in S / z_1 \leq z_2 \text{ for some } z_2 \in S_1\}$. If $S_1 = \{a\}$, then instead of $(\{a\}]$ we write $(a]$. For subsets $S_1 \neq \phi$ and $S_2 \neq \phi$ of S, we represent $S_1 S_2 = \{z_1 z_2 / z_1 \in S_1, z_2 \in S_2\}$.

In onward work the symbol S stands for an OSGs.

Definition 2.1. [13] Consider a nonempty subset I of S is known as a left (resp. right) ideal of S having the following conditions:

(I_1) $SI \subseteq I$(resp. $IS \subseteq I$)
(I_2) if $z_1 \in S$ and $z_2 \in I$ such that $z_1 \leq z_2$, then $z_1 \in I$.

So the set I is known to be an ideal of S if it is both a left and a right ideal.

Next we are going to define the generalized structure of ideals that is interior ideals, bi-ideals and quasi ideals in OSGs.

Definition 2.2. [16] A subset $I \neq \phi$ of S is known to be a bi-ideal of S if it satisfies (I_2) and

(I_3) $ISI \subseteq I$
(I_4) $I^2 \subseteq I.$

Definition 2.3. [12] An interior ideal I is a nonempty subset of OSG S if it satisfies $(I_2), (I_4)$ and

(I_5) $SIS \subseteq I.$

Definition 2.4. [17] A quasi ideal $Q \neq \phi$ is a subset of S if it satisfy (I_2) and

(I_6) $(QS] \cap (SQ] \subseteq Q$

The paradigm of fuzzy set was originally initiated by Zadeh [36] and become the rich research area among the scholars. The model of fuzzy set has been generalized in several direction by different authors. Here in onward work we will present the combine study of fuzzy set with ideals that is FIds and their generalization.

Definition 2.5. [36] A fuzzy subset (FSS) μ is a mapping from S to $[0, 1]$.

Consider two FSSs μ_1 and μ_2 of S. Then $\mu_1 \subseteq \mu_2 \iff \mu_1(z) \leq \mu_2(z)\ \forall\ z \in S$. Next $(\mu_1 \cap \mu_2)(z) = \min\{\mu_1(z), \mu_2(z)\}$ and $(\mu_1 \cup \mu_2)(z) = \max\{\mu_1(z), \mu_2(z)\}$.

Definition 2.6. A FSS μ of S of the form and for any $z_1 \in S$

$$\mu(z) = \begin{cases} t(t \neq 0) & \text{if} \quad z = z_1, \\ 0 & \text{if} \quad z \neq z_1. \end{cases}$$

then the fuzzy point is represented by $(z_1)_t$ with value t support by z_1. A fuzzy point $(z_1)_t$ 'belong to' FSS μ represented as $(z_1)_t \in \mu$, if $\mu(z_1) \geq t$, and a fuzzy point $(z_1)_t$ 'quasi-coincident' to FSS μ represented by $(z_1)_t q\mu$, if $\mu(z_1) + t > 1$.

Definition 2.7. [15] A FSS μ is called a fuzzy ordered subsemigroup of S if

(FI_1) $(\forall z_1, z_2 \in S)\left(\mu(z_1z_2) \geq \min\{\mu(z_1), \mu(z_2)\}\right).$

Definition 2.8. [15] A FSS μ is known to be a fuzzy left (resp. right) ideals of S if it holds

(FI_2) $(\forall z_1, z_2 \in S)(z_1 \leq z_2$ this implies $\mu(z_1) \geq \mu(z_2))$
(FI_3) $(\forall z_1, z_2 \in S)(\mu(z_1z_2) \geq \mu(z_2)($resp. $\mu(z_1z_2) \geq \mu(z_1))).$

A FSS μ of OSG S is said to be fuzzy ideal (FId), if μ is both sided ideal of S, that is a fuzzy left ideal (FLId) and as well as a fuzzy right ideal (FRId).

From this definition we can also conclude the following

Definition 2.9. A FSS μ is known to be FId of OSG S if it satisfy (FI_2) and

(FI_4) $(\forall z_1, z_2 \in S)(\mu(z_1 z_2) \geq \max\{\mu(z_1), \mu(z_2)\})$.

Proposition 2.10. Let μ_1 and μ_2 are the FLIds (resp. FRIds) of S. Then

i) $(\mu_1 \cap \mu_2)$ and

ii) $(\mu_1 \cup \mu_2)$ are FLIds (resp. FRIds) of S.

Proof. Proofs are straightforward. □

Definition 2.11. [12] A FSS μ is known as fuzzy interior ideal (FIId) of OSG S if it holds $(FI_1), (FI_2)$ and

(FI_5) $(\forall z_1, z_2, z_3 \in S)(\mu(z_1 z_3 z_2) \geq \mu(z_3))$.

Definition 2.12. [16] A FSS μ is known as fuzzy bi-ideal (FBId) of OSG S if it satisfies $(FI_1), (FI_2)$ and

(FI_6) $(\forall z_1, z_2, z_3 \in S)(\mu(z_1 z_2 z_3) \geq \min\{\mu(z_1), \mu(z_3)\})$.

Definition 2.13. Let $X \neq \phi$ be a subset of S, then we define a set X_{z_1} by

$$X_{z_1} = \{(z_2, z_3) \in S \times S / z_1 \leq z_2 z_3\}.$$

Let us consider the two fuzzy subsets μ_1 and μ_2 of S. Then we define
$\mu_1 \circ \mu_2 : S \rightarrow [0,1]$, as

$$z_1 \rightarrow \mu_1 \circ \mu_2(z_1) = \begin{cases} V_{(z_2,z_3) \in X_{z_1}} \min\{\mu_1(z_2), \mu_2(z_3)\} & \text{if} \quad X_{z_1} \neq \phi \\ 0 & \text{if} \quad X_{z_1} = \phi. \end{cases} \tag{1}$$

$\mu_1 \leq \mu_2$ means $\mu_1(z) \leq \mu_2(z)$.

Pawlak [23] is the pioneer who for the first time investigated the rudimentary notion of rough set. The fundamental concept of Pawlak rough set depend upon the equivalence relation.

Consider the equivalence relation ξ on the initial universal set U. Then (U, ξ) is said to be the approximation space. Let $\phi \neq X \subseteq U$, so in this case the set X is called a definable subset of U if it is the collection of some equivalence classes of a universal set U else it is called not definable. Then the set X is approximated in the form of upper and lower approximations which are given as:

$$\overline{App}(X) = \left\{z_1 \in U : [z_1]_\xi \cap X \neq \phi\right\}$$
$$\underline{App}(X) = \left\{z_1 \in U : [z_1]_\xi \subseteq X\right\}$$

Then the rough set is a pair $\left(\overline{App}X, \underline{App}X\right)$, if $\overline{App}X \neq \underline{App}X$. The set X is a definable set if $\overline{App}X = \underline{App}X$.

In the following we will further generalized the concepts of upper and lower approximations to a FSS as well.

Definition 2.14. [11] Consider the approximation space (U, ξ), and for any $z_1 \in U$, the upper and lower approximations of a FSS μ is defined as

$$\overline{App}(\mu)(z_1) = \bigvee_{z_2 \in [z_1]_\xi} \mu(z_2) \text{ and } \underline{App}(\mu)(z_1) = \bigwedge_{z_2 \in [z_1]_\xi} \mu(z_2)$$

The pair $\left(\overline{App}(\mu), \underline{App}(\mu)\right)$ is said to be a rough fuzzy subset if $\overline{App}(\mu) \neq \underline{App}(\mu)$.

Definition 2.15. Consider the OSGs S_1 and S_2. Then the set-valued homomorphism $(SVH)$is a mapping $F : S_1 \longrightarrow P^*(S_2)$ if it satisfied:

(h_1) $F(z_1)F(z_2) = F(z_1 z_2)$

Where $P^*(S_2) \neq \phi$ represents the collection of all subsets of S_2.

Definition 2.16. Let S_1 and S_2 be two OSGs. Then the set-valued monotone homomorphism $(SVMH)$ is a mapping $F : S_1 \longrightarrow P^*(S_2)$ if it satisfy the condition (h_1) of Definition 2.15, and

(h_2) if $z_1 \leq z_2$ this implies $F(z_1) \subseteq F(z_2)$ for each $z_1, z_2 \in S_1$.

Definition 2.17. Let S_1 and S_2 be two OSGs. Then the set-valued isotone homomorphism $(SVIH)$ is a mapping $F : S_1 \longrightarrow P^*(S_2)$ if it satisfy condition (h_1) of Definition 2.15, and

(h_3) $z_1 \leq z_2$ then $F(z_2) \subseteq F(z_1)$ for each $z_1, z_2 \in S_1$.

Definition 2.18. Consider that a $SVIH$ or $SVMH$ is a function $F : S \longrightarrow P^*(S)$. Then the generalized upper and lower approximations for any $z_1 \in S$, of a FSS μ with respect to the given mapping F is defined as

$$\overline{F}(\mu)(z_1) = \bigvee_{z_2 \in F(z_1)} \mu(z_2) \text{ and } \underline{F}(\mu)(z_1) = \bigwedge_{z_2 \in F(z_1)} \mu(z_2)$$

The rough fuzzy subset is a pair $\left(\overline{F}(\mu), \underline{F}(\mu)\right)$ if $\overline{F}(\mu) \neq \underline{F}(\mu)$.

3 Approximations of FIds in OSGs

In this section study of roughness of FIds in OSGs is being presented on the bases of $SVIH$ or $SVMH$. Thus we will start from the following.

Theorem 3.1. Suppose that $F : S \to P^*(S)$ be a $SVIH$ or $SVMH$ and a FSS μ be a fuzzy ordered subsemigroup of S. Then the upper approximation $\overline{F}(\mu)$ is a fuzzy ordered subsemigroup of S.

Proof. For any $z_1, z_2 \in S$. Consider

$$\begin{aligned}
\overline{F}(\mu)(z_1 z_2) &= \bigvee_{z_1' \in F(z_1 z_2)} \mu\left(z_1'\right) \\
&= \bigvee_{z_1' \in F(z_1)F(z_2)} \mu\left(z_1'\right) \\
&= \bigvee_{z_2' z_3' \in F(z_1)F(z_2)} \mu\left(z_2' z_3'\right) \quad \left(\begin{array}{c}\text{as } z_1' = z_2' z_3' \text{ such that } z_2' \in F(z_1) \\ \text{and } z_3' \in F(z_2)\end{array}\right) \\
&= \bigvee_{\substack{z_2' \in F(z_1) \\ z_3' \in F(z_2)}} \mu\left(z_2' z_3'\right) \\
&\geq \bigvee_{\substack{z_2' \in F(z_1) \\ z_3' \in F(z_2)}} \min\left\{\mu\left(z_2'\right), \mu\left(z_3'\right)\right\} \\
&= \min\left\{\bigvee_{z_2' \in F(z_1)} \mu\left(z_2'\right), \bigvee_{z_3' \in F(z_2)} \mu\left(z_3'\right)\right\}
\end{aligned}$$

implies

$$\overline{F}(\mu)(z_1 z_2) \geq \min\left\{\overline{F}(\mu)(z_1), \overline{F}(\mu)(z_2)\right\}$$

Therefore $\overline{F}(\mu)$ is a fuzzy ordered subsemigroup of S. □

Theorem 3.2. Suppose that a FSS μ be a fuzzy ordered subsemigroup of S and $F : S \to P^*(S)$ be a $SVIH$ or $SVMH$. Then $\underline{F}(\mu)$ is a fuzzy ordered subsemigroup of OSG S.

Proof. Similarly as above Theorem 3.1. □

In onward discussion the study of roughness of FIds in OSGs is being presented.

Theorem 3.3. Consider the $SVMH$ $F : S \to P^*(S)$ and a FSS μ be a FLId (resp. FRId) of OSG S. Then $\underline{F}(\mu)$ is a FLId (resp. FRId) of S.

Proof. For each $z_1, z_2 \in S$ with $z_1 \leq z_2$, then $F(z_1) \subseteq F(z_2)$. Now we may consider the following

$$\begin{aligned}
\underline{F}(\mu)(z_1) &= \bigwedge_{z_1' \in F(z_1)} \mu\left(z_1'\right) \\
&\geq \bigwedge_{z_2' \in F(z_2)} \mu\left(z_2'\right)
\end{aligned}$$

implies

$$\underline{F}(\mu)(z_1) \geq \underline{F}(\mu)(z_2)$$

Next

$$\begin{aligned}
\underline{F}(\mu)(z_1 z_2) &= \bigwedge_{z_1' \in F(z_1 z_2)} \mu\left(z_1'\right) \\
&= \bigwedge_{z_1' \in F(z_1) F(z_2)} \mu\left(z_1'\right) \\
&= \bigwedge_{z_2' z_3' \in F(z_1) F(z_2)} \mu\left(z_2' z_3'\right) \quad \left(\begin{array}{c} \text{as } z_1' = z_2' z_3' \text{ such that } z_2' \in F(z_1) \\ \text{and } z_3' \in F(z_2) \end{array}\right) \\
&= \bigwedge_{\substack{z_2' \in F(z_1) \\ z_3' \in F(z_2)}} \mu\left(z_2' z_3'\right) \\
&\geq \bigwedge_{z_3' \in F(z_2)} \mu\left(z_3'\right)
\end{aligned}$$

implies

$$\underline{F}(\mu)(z_1 z_2) \geq \underline{F}(\mu)(z_2)$$

Hence $\underline{F}(\mu)$ is a FLId of S. Analogously, we can prove that $\underline{F}(\mu)$ is a FRId of S. □

Here by counter example it is shown that upper approximation $\overline{F}(\mu)$ does not hold in general for a FId μ, when F is a $SVMH$.

Example 3.4. Let us suppose a set $S = \{\tilde{a}_1, \tilde{a}_2, \tilde{a}_3, \tilde{a}_4, \tilde{a}_5, \tilde{a}_6\}$ with the following multiplication table and order relation "$\leq$".

Multiplication table for S

$\cdot$	$\tilde{a}_1$	$\tilde{a}_2$	$\tilde{a}_3$	$\tilde{a}_4$	$\tilde{a}_5$	$\tilde{a}_6$
$\tilde{a}_1$	$\tilde{a}_1$	$\tilde{a}_1$	$\tilde{a}_1$	$\tilde{a}_1$	$\tilde{a}_1$	$\tilde{a}_1$
$\tilde{a}_2$	$\tilde{a}_1$	$\tilde{a}_2$	$\tilde{a}_2$	$\tilde{a}_4$	$\tilde{a}_2$	$\tilde{a}_2$
$\tilde{a}_3$	$\tilde{a}_1$	$\tilde{a}_2$	$\tilde{a}_3$	$\tilde{a}_4$	$\tilde{a}_5$	$\tilde{a}_5$
$\tilde{a}_4$	$\tilde{a}_1$	$\tilde{a}_1$	$\tilde{a}_4$	$\tilde{a}_4$	$\tilde{a}_4$	$\tilde{a}_4$
$\tilde{a}_5$	$\tilde{a}_1$	$\tilde{a}_2$	$\tilde{a}_3$	$\tilde{a}_4$	$\tilde{a}_5$	$\tilde{a}_5$
$\tilde{a}_6$	$\tilde{a}_1$	$\tilde{a}_2$	$\tilde{a}_3$	$\tilde{a}_4$	$\tilde{a}_5$	$\tilde{a}_6$

Table 1

and $\leq := \{(\tilde{a}_1, \tilde{a}_1), (\tilde{a}_2, \tilde{a}_2), (\tilde{a}_3, \tilde{a}_3), (\tilde{a}_4, \tilde{a}_4), (\tilde{a}_5, \tilde{a}_5), (\tilde{a}_6, \tilde{a}_6), (\tilde{a}_1, \tilde{a}_4), (\tilde{a}_1, \tilde{a}_5), (\tilde{a}_4, \tilde{a}_5), (\tilde{a}_2, \tilde{a}_6), (\tilde{a}_3, \tilde{a}_5), (\tilde{a}_3, \tilde{a}_6), (\tilde{a}_2, \tilde{a}_5), (\tilde{a}_6, \tilde{a}_5)\}$. Then $(S, \cdot, \leq)$ is an OSG. Right ideals of S are $\{\tilde{a}_1, \tilde{a}_4\}, \{\tilde{a}_1, \tilde{a}_2, \tilde{a}_4\}$ and S. Left ideals of S are $\{\tilde{a}_1\}, \{\tilde{a}_1, \tilde{a}_2\}, \{\tilde{a}_1, \tilde{a}_4\}$, $\{\tilde{a}_1, \tilde{a}_2, \tilde{a}_4\}$, $\{\tilde{a}_1, \tilde{a}_2, \tilde{a}_3, \tilde{a}_4\}$, $\{\tilde{a}_1, \tilde{a}_2, \tilde{a}_4, \tilde{a}_5, \tilde{a}_6\}$ and S. Define a FSS $\mu : S \to [0, 1]$ by $\mu(\tilde{a}_1) = 0.8$, $\mu(\tilde{a}_2) = 0.5$, $\mu(\tilde{a}_4) = 0.6$ and $\mu(\tilde{a}_3) = \mu(\tilde{a}_5) = \mu(\tilde{a}_6) = 0.4$. Then FSS μ is a FId of S.

Next suppose that a $SVMH$ $F : S \to P^*(S)$ i.e.

(i) $F(\tilde{a}_1) F(\tilde{a}_2) = F(\tilde{a}_1 \tilde{a}_2)$
(ii) if $\tilde{a}_1 \leq \tilde{a}_2 \to F(\tilde{a}_1) \subseteq F(\tilde{a}_2)$.

Where $P^*(S)$ consist of all non-empty subset of S. Now if $F(\tilde{a}_5) = \{\tilde{a}_2, \tilde{a}_3, \tilde{a}_4, \tilde{a}_5, \tilde{a}_6\}$ and $F(\tilde{a}_6) = \{\tilde{a}_3, \tilde{a}_6\}$, as $\tilde{a}_6 \leq \tilde{a}_5 \rightarrow F(\tilde{a}_6) \subseteq F(\tilde{a}_5)$ but $\overline{F}(\mu)(\tilde{a}_6) \ngeq \overline{F}(\mu)(\tilde{a}_5)$. Hence in $SVMH$ it is prove that $\overline{F}(\mu)$ is not a FId of S.

Theorem 3.5. Suppose that a FSS μ be a FLId (resp. FRId) of S and $F : S \rightarrow P^*(S)$ be a $SVIH$. Then $\overline{F}(\mu)$ is a FLId (resp. FRId) of OSG S.

Proof. For each $z_1, z_2 \in S$ such that $z_1 \leq z_2$, then $F(z_2) \subseteq F(z_1)$. Now consider the following

$$\begin{aligned} \overline{F}(\mu)(z_1) &= \bigvee_{z_1' \in F(z_1)} \mu\left(z_1'\right) \\ &\geq \bigvee_{z_2' \in F(z_2)} \mu\left(z_2'\right) \end{aligned}$$

implies

$$\overline{F}(\mu)(z_1) \;\geq\; \overline{F}(\mu)(z_2)$$

Next

$$\begin{aligned} \overline{F}(\mu)(z_1 z_2) &= \bigvee_{z_1' \in F(z_1 z_2)} \mu\left(z_1'\right) \\ &= \bigvee_{z_1' \in F(z_1) F(z_2)} \mu\left(z_1'\right) \\ &= \bigvee_{z_2' z_3' \in F(z_1) F(z_2)} \mu\left(z_2' z_3'\right) \quad \left(\begin{array}{c} \text{as } z_1' = z_2' z_3' \text{ such that } z_2' \in F(z_1) \\ \text{and } z_3' \in F(z_2) \end{array} \right) \\ &= \bigvee_{\substack{z_2' \in F(z_1) \\ z_3' \in F(z_2)}} \mu\left(z_2' z_3'\right) \\ &\geq \bigvee_{z_3' \in F(z_2)} \mu\left(z_3'\right) \end{aligned}$$

implies

$$\overline{F}(\mu)(z_1 z_2) \;\geq\; \overline{F}(\mu)(z_2)$$

Hence this prove that $\overline{F}(\mu)$ is a FLId (resp. FRId) of S. □

Here by counter example it is shown that upper approximation $\underline{F}(\mu)$ does not hold in general for a FId μ, when F is a $SVIH$.

Example 3.6. Suppose a FId μ of OSG S as shown in example 3.4. Now consider a $SVIH$ $F : S \rightarrow P^*(S)$ i.e.

(i) $F(\tilde{a}_1) F(\tilde{a}_2) = F(\tilde{a}_1 \tilde{a}_2)$
(ii) if $\tilde{a}_1 \leq \tilde{a}_2 \Rightarrow F(\tilde{a}_2) \subseteq F(\tilde{a}_1)$.

Now if $F(\tilde{a}_6) = \{\tilde{a}_1, \tilde{a}_2, \tilde{a}_4\}$ and $F(\tilde{a}_5) = \{\tilde{a}_1, \tilde{a}_4\}$, as $\tilde{a}_6 \leq \tilde{a}_5 \Rightarrow F(\tilde{a}_5) \subseteq F(\tilde{a}_6)$ but $\underline{F}(\mu)(\tilde{a}_6) \ngeq \underline{F}(\mu)(\tilde{a}_5)$. Hence in $SVIH$ it is prove that $\underline{F}(\mu)$ is not a FId of S.

Theorem 3.7. Suppose that $F : S \to P^*(S)$ be $SVMH$ and a FSS μ be a FIId of OSG S. Then $\underline{F}(\mu)$ is a FIId of S.

Proof. From Theorem 3.3, we have $z_1 \leq z_2$, implies $F(z_1) \subseteq F(z_2)$, for each $z_1, z_2 \in S$, then $\underline{F}(\mu)(z_1) \geq \underline{F}(\mu)(z_2)$. Next consider

$$\begin{aligned}
\underline{F}(\mu)(z_1 z_2) &= \bigwedge_{z_1' \in F(z_1 z_2)} \mu\left(z_1'\right) \\
&= \bigwedge_{z_1' \in F(z_1)F(z_2)} \mu\left(z_1'\right) \\
&= \bigwedge_{z_2' z_3' \in F(z_1)F(z_2)} \mu\left(z_2' z_3'\right) \quad \left(\begin{array}{c} \text{as } z_1' = z_2' z_3', \text{ where } z_2' \in F(z_1) \\ \text{and } z_3' \in F(z_2) \end{array}\right) \\
&= \bigwedge_{\substack{z_2' \in F(z_1) \\ z_3' \in F(z_2)}} \mu\left(z_2' z_3'\right) \\
&\geq \bigwedge_{\substack{z_2' \in F(z_1) \\ z_3' \in F(z_2)}} \min\left\{\mu\left(z_2'\right), \mu\left(z_3'\right)\right\} \\
&= \min\left\{\bigwedge_{z_2' \in F(z_1)} \mu\left(z_2'\right), \bigwedge_{z_3' \in F(z_2)} \mu\left(z_3'\right)\right\}
\end{aligned}$$

implies

$$\underline{F}(\mu)(z_1 z_2) \geq \min\left\{\underline{F}(\mu)(z_1), \underline{F}(\mu)(z_2)\right\}$$

Consider

$$\begin{aligned}
\underline{F}(\mu)(z_1 z_3 z_2) &= \bigwedge_{z_1' \in F(z_1 z_3 z_2)} \mu\left(z_1'\right) \\
&= \bigwedge_{z_1' \in F(z_1)F(z_3)F(z_2)} \mu\left(z_1'\right) \\
&= \bigwedge_{tuv \in F(z_1)F(z_3)F(z_2)} \mu(tuv) \quad \left(\begin{array}{c} \text{as } z_1' = tuv \text{ where } t \in F(z_1), \\ u \in F(z_3) \text{ and } v \in F(z_2) \end{array}\right) \\
&= \bigwedge_{\substack{t \in F(z_1) \\ u \in F(z_3) \\ v \in F(z_2)}} \mu(tuv) \\
&\geq \bigwedge_{u \in F(z_3)} \mu(u)
\end{aligned}$$

implies

$$\underline{F}(\mu)(z_1 z_3 z_2) \geq \underline{F}(\mu)(z_3)$$

Therefore, $\underline{F}(\mu)$ satisfies all the conditions of FIId, so $\underline{F}(\mu)$ is a FIId of OSG S. □

Theorem 3.8. Consider a FSS μ be a FIId of OSG S and $F : S \to P^*(S)$ be a $SVIH$. Then $\overline{F}(\mu)$ is a FIId of OSG S.

Proof. From Theorems 3.1 and 3.5, if $z_1 \leq z_2$ implies $F(z_2) \subseteq F(z_1)$ for each $z_1, z_2 \in S$, then $\overline{F}(\mu)(z_1) \geq \overline{F}(\mu)(z_2)$ and $\overline{F}(\mu)(z_1 z_2) \geq \min\left\{\overline{F}(\mu)(z_1), \overline{F}(\mu)(z_2)\right\}$. Next we may consider the following

$$\begin{aligned}
\overline{F}(\mu)(z_1 z_3 z_2) &= \bigvee_{z_1' \in F(z_1 z_3 z_2)} \mu\left(z_1'\right) \\
&= \bigvee_{z_1' \in F(z_1)F(z_3)F(z_2)} \mu\left(z_1'\right) \\
&= \bigvee_{tuv \in F(z_1)F(z_3)F(z_2)} \mu(tuv) \quad \left(\begin{array}{c} \text{as } z_1' = tuv \text{ where } t \in F(z_1), \\ u \in F(z_3) \text{ and } v \in F(z_2) \end{array}\right) \\
&= \bigvee_{\substack{t \in F(z_1) \\ u \in F(z_3) \\ v \in F(z_2)}} \mu(tuv) \\
&\geq \bigvee_{u \in F(z_3)} \mu(u)
\end{aligned}$$

implies

$$\overline{F}(\mu)(z_1 z_3 z_2) \geq \overline{F}(\mu)(z_3)$$

Therefore, it is prove that $\overline{F}(\mu)$ is a FIId of S. □

Theorem 3.9. Suppose that $F : S \to P^*(S)$ be a $SVMH$ and a FSS μ be a FBId of OSG S. Then $\underline{F}(\mu)$ is a FBId of S.

Proof. From Theorem 3.7, we have for each $z_1, z_2 \in S$, such that $z_1 \leq z_2$, implies $F(z_1) \subseteq F(z_2)$, then $\underline{F}(\mu)(z_1) \geq \underline{F}(\mu)(z_2)$, and also

$$\underline{F}(\mu)(z_1 z_2) \geq \min\left\{\underline{F}(\mu)(z_1), \underline{F}(\mu)(z_2)\right\}$$

Next for each $z_1, z_2, z_3 \in S$, consider the following

$$\begin{aligned}
\underline{F}(\mu)(z_1 z_2 z_3) &= \bigwedge_{z_1' \in F(z_1 z_2 z_3)} \mu\left(z_1'\right) \\
&= \bigwedge_{z_1' \in F(z_1)F(z_2)F(z_3)} \mu\left(z_1'\right) \\
&= \bigwedge_{tuv \in F(z_1)F(z_2)F(z_3)} \mu(tuv) \quad \left(\begin{array}{c} \text{as } z_1' = tuv \text{ where } t \in F(z_1), \\ u \in F(z_2) \text{ and } v \in F(z_3) \end{array}\right) \\
&= \bigwedge_{\substack{t \in F(z_1) \\ u \in F(z_2) \\ v \in F(z_3)}} \mu(tuv) \\
&\geq \bigwedge_{\substack{t \in F(z_1) \\ v \in F(z_3)}} \min\{\mu(t), \mu(v)\} \\
&= \min\left\{\bigwedge_{t \in F(z_1)} \mu(t), \bigwedge_{v \in F(z_3)} \mu(v)\right\}
\end{aligned}$$

implies

$$\underline{F}(\mu)(z_1 z_2 z_3) \geq \min\left\{\underline{F}(\mu)(z_1), \underline{F}(\mu)(z_3)\right\}$$

Hence $\underline{F}(\mu)$ satisfies all the conditions of a FBId of S, so $\underline{F}(\mu)$ is a FBId of S. □

Theorem 3.10. Suppose that a FSS μ be a FBId of S and $F : S \to P^*(S)$ be a $SVIH$. Then we have to prove that $\overline{F}(\mu)$ is a FBId of OSG S.

Proof. From Theorem 3.8, for each $z_1, z_2 \in S$ such that $z_1 \leq z_2$, implies $F(z_2) \subseteq F(z_1)$, then $\overline{F}(\mu)(z_1) \geq \overline{F}(\mu)(z_2)$ and also $\overline{F}(\mu)(z_1z_2) \geq \min\left\{\overline{F}(\mu)(z_1), \overline{F}(\mu)(z_2)\right\}$. Next for each $z_1, z_2, z_3 \in S$, we consider

$$\begin{aligned}
\overline{F}(\mu)(z_1z_2z_3) &= \bigvee_{z_1' \in F(z_1z_2z_3)} \mu\left(z_1'\right) \\
&= \bigvee_{z_1' \in F(z_1)F(z_2)F(z_3)} \mu\left(z_1'\right) \\
&= \bigvee_{tuv \in F(z_1)F(z_2)F(z_3)} \mu(tuv) \quad \left(\begin{array}{c}\text{as } z_1' = tuv \text{ where } t \in F(z_1), \\ u \in F(z_2) \text{ and } v \in F(z_3)\end{array}\right) \\
&= \bigvee_{\substack{t \in F(z_1) \\ u \in F(z_2) \\ v \in F(z_3)}} \mu(tuv) \\
&\geq \bigvee_{\substack{t \in F(z_1) \\ v \in F(z_3)}} \min\{\mu(t), \mu(v)\} \\
&= \min\left\{\bigvee_{t \in F(z_1)} \mu(t), \bigvee_{v \in F(z_3)} \mu(v)\right\}
\end{aligned}$$

implies

$$\overline{F}(\mu)(z_1z_2z_3) \geq \min\left\{\overline{F}(\mu)(z_1), \overline{F}(\mu)(z_3)\right\}$$

Hence $\overline{F}(\mu)$ satisfies all the conditions of a FBId, so $\overline{F}(\mu)$ is a FBId of S. □

Theorem 3.11. Let us suppose that $F : S \to P^*(S)$ be a $SVMH$ and a FSS μ be a FQId of S and . Then we have to prove that $\underline{F}(\mu)$ is a FQId of S.

Proof. As $\underline{F}(\mu)$ is a FLId (resp. FRId) of OSG S, therefore by Theorem 3.3, for each $z_1, z_2 \in S$ such that $z_1 \leq z_2$, implies $F(z_1) \subseteq F(z_2)$, then $\underline{F}(\mu)(z_1) \geq \underline{F}(\mu)(z_2)$. Next consider

$$\begin{aligned}
\underline{F}(\mu)(z_1) &= \bigwedge_{\bar{a} \in F(z_1)} \mu(\bar{a}) \\
&\geq \bigwedge_{\bar{a} \in F(z_1)} ((\mu \circ 1) \wedge (1 \circ \mu))(\bar{a}) \\
&= \underline{F}((\mu \circ 1) \wedge (1 \circ \mu))(z_1)
\end{aligned}$$

implies

$$\underline{F}(\mu)(z_1) \geq \underline{F}((\mu \circ 1) \wedge (1 \circ \mu))(z_1)$$

Hence from the proof it is clear that $\underline{F}(\mu)$ is a FQId of OSG S. □

Theorem 3.12. Suppose that a $SVIH$ $F : S \to P^*(S)$ and a FSS μ be a FQId of OSG S. Then $\overline{F}(\mu)$ is a FQId of OSG S.

Proof. As we know from Theorem 3.5 that $\overline{F}(\mu)$ is a FLId (resp. FRId) of S, therefore for each $z_1, z_2 \in S$ such that $z_1 \leq z_2$ implies $F(z_2) \subseteq F(z_1)$, then $\overline{F}(\mu)(z_1) \geq \overline{F}(\mu)(z_2)$. Next consider

$$\begin{aligned} \overline{F}(\mu)(z_1) &= \bigvee_{\bar{a}\in F(z_1)} \mu(\bar{a}) \\ &\geq \bigvee_{\bar{a}\in F(z_1)} ((\mu \circ 1) \wedge (1 \circ \mu))(\bar{a}) \\ &= \overline{F}((\mu \circ 1) \wedge (1 \circ \mu))(z_1) \end{aligned}$$

implies

$$\overline{F}(\mu)(z_1) \geq \overline{F}((\mu \circ 1) \wedge (1 \circ \mu))(z_1)$$

Hence from the proof it is clear that $\overline{F}(\mu)$ is a FQId of S. □

4 Approximations of $(\in, \in \vee q)$-FIds in OSGs

In this section, roughness of $(\in, \in \vee q)$-FIds is being studied on the bases of $SVIH$ and $SIMH$.

Definition 4.1. A FSS μ of OSG S is known as an $(\in, \in \vee q)$-fuzzy ordered subsemigroup of OSG S if:

$$(FI_8) \quad (\text{for each } z_1, z_2 \in S)(\text{for all } t_1, t_2 \in (0,1]) \begin{pmatrix} (z_1)_{t_1}, (z_2)_{t_2} \in \mu \text{ implies} \\ (z_1 z_2)_{\min\{t_1,t_2\}} \in \vee q\mu \end{pmatrix}$$

Definition 4.2. A FSS μ is known as $(\in, \in \vee q)$-FLId (resp. FRId) of S if the following conditions are holds:

$$(FI_9) \quad (\text{for all } z_1, z_2 \in S)(\text{for } all \ t_1 \in (0,1]) \begin{pmatrix} z_1 \leq z_2, \text{ then } (z_2)_{t_1} \in \mu \text{ implies} \\ (z_1)_{t_1} \in \vee q\mu \end{pmatrix}$$

$$(FI_{10}) \quad (\text{for all } z_1, z_2 \in S)(\text{for } all \ t_1 \in (0,1]) \begin{pmatrix} (z_2)_{t_1} \in \mu \text{ implies } (z_1 z_2)_{t_1} \in \vee q\mu \\ (\text{resp. } (z_2 z_1)_{t_1} \in \vee q\mu) \end{pmatrix}$$

A FSS μ is known as $(\in, \in \vee q)$-FId of S, if it is both $(\in, \in \vee q)$-FLId and $(\in, \in \vee q)$-FRId of S.

Definition 4.3. [12] A FSS μ is known to be an $(\in, \in \vee q)$-FIId of OSG S if it holds $(FI_8), (FI_9)$ and

$$(FI_{11}) \quad (\text{for all } z_1, z_2, z_3 \in S)(\text{for all } t_1 \in (0,1]) \left((z_3)_{t_1} \in \mu \text{ implies } (z_1 z_3 z_2)_{t_1} \in \vee q\mu\right)$$

Definition 4.4. A FSS μ is said to be an $(\in, \in \vee q)$-FBId of OSG S if holds $(FI_8), (FI_9)$ and

$$(FI_{12}) \quad (\text{for all } z_1, z_2, z_3 \in S)(\text{for all } t_1, t_2 \in (0,1]) \begin{pmatrix} (z_1)_{t_1}, (z_3)_{t_2} \in \mu \text{ implies} \\ (z_1 z_2 z_3)_{\min\{t_1,t_2\}} \in \vee q\mu \end{pmatrix}$$

Definition 4.5. A FSS μ is known as $(\in, \in \vee q)$-FQId of S if it holds (FI_9) and

$$(FI_{13}) \quad (\text{for all } z_1 \in S)(\text{for all } t_1 \in (0,1]) \left((z_1)_{t_1} \in (\mu \circ 1) \wedge (1 \circ \mu) \text{ implies } (z_1)_{t_1} \in \vee q\mu\right).$$

Lemma 4.6. [12] A FSS μ is known as $(\in, \in \vee q)$-FLId (resp. FRId) of OSG $S \Leftrightarrow$ it holds

(FI_{14}) (for all $z_1, z_2 \in S$) $(z_1 \leq z_2, \mu(z_1) \geq \min\{\mu(z_2), 0.5\})$,
(FI_{15}) (for all $z_1, z_2 \in S$) $(\mu(z_1 z_2) \geq \min\{\mu(z_2), 0.5\})$
(resp. $\mu(z_1 z_2) \geq \min\{\mu(z_1), 0.5\})$.

Lemma 4.7. [12] A FSS μ is said to be an $(\in, \in \vee q)$-FBId of OSG $S \Leftrightarrow$ it holds (FI_{14}) of lemma 4.6 and

(FI_{16}) (for all $z_1, z_2 \in S$) $(\mu(z_1 z_2) \geq \min\{\mu(z_1), \mu(z_2), 0.5\})$
(FI_{17}) (for all $z_1, z_2, z_3 \in S$) $(\mu(z_1 z_2 z_3) \geq \min\{\mu(z_1), \mu(z_3), 0.5\})$

Lemma 4.8. [12] μ is known as $(\in, \in \vee q)$-FIId of $S \Leftrightarrow$ it holds $(FI_{14}), (FI_{16})$ of lemmas 4.6 and 4.7 and

(FI_{18}) (for all $z_1, z_2 \in S$) $(\mu(z_1 z_3 z_2) \geq \min\{\mu(z_3), 0.5\})$

Lemma 4.9. A FSS μ is said to be an $(\in, \in \vee q)$-FQId of OSG $S \Leftrightarrow$ it holds (FI_{14}) of lemma 4.6 and

(FI_{19}) (for all $z_1, z_2 \in S$) $(\mu(z_1) \geq \min\{((\mu \circ 1) \wedge (1 \circ \mu))(z_1), 0.5\})$

Theorem 4.10. Suppose that FSS μ be an $(\in, \in \vee q)$-fuzzy ordered subsemigroup of S and $F : S \to P^*(S)$ be a $SVMH$ or $SVIH$. Then $\underline{F}(\mu)$ is an $(\in, \in \vee q)$-fuzzy ordered subsemigroup of S.

Proof. To prove this theorem we have to see that, $\underline{F}(\mu)$ satisfies (FI_{16}). If for each $z_1, z_2 \in S$, now consider

$$\begin{aligned}
\underline{F}(\mu)(z_1 z_2) &= \bigwedge_{z_1' \in F(z_1 z_2)} \mu\left(z_1'\right) \\
&= \bigwedge_{z_1' \in F(z_1)F(z_2)} \mu\left(z_1'\right) \\
&= \bigwedge_{ab \in F(z_1)F(z_2)} \mu(ab) \quad \left(\begin{array}{c} \text{as } z_1' = ab \text{ such that } a \in F(z_1), \\ \text{and } b \in F(z_2) \end{array}\right) \\
&= \bigwedge_{\substack{a \in F(z_1) \\ b \in F(z_2)}} \mu(ab) \\
&\geq \bigwedge_{\substack{a \in F(z_1) \\ b \in F(z_2)}} \min\{\mu(a), \mu(b), 0.5\} \\
&= \min\left\{\bigwedge_{a \in F(z_1)} \mu(a), \bigwedge_{b \in F(z_2)} \mu(b), 0.5\right\}
\end{aligned}$$

implies

$$\underline{F}(\mu)(z_1 z_2) \geq \min\{\underline{F}(\mu)(z_1), \underline{F}(\mu)(z_2), 0.5\}$$

Hence $\underline{F}(\mu)$ is an $(\in, \in \vee q)$-fuzzy ordered subsemigroup of OSG S. □

Theorem 4.11. Consider that a FSS μ be $(\in, \in \vee q)$-fuzzy ordered subsemigroup of S and $F : S \to P^*(S)$ be a $SVMH$ or $SVIH$ and. Then $\overline{F}(\mu)$ is an $(\in, \in \vee q)$-fuzzy ordered subsemigroup of S.

Proof. Straightforward as Theorem 4.10. □

Theorem 4.12. Consider that a FSS μ be $(\in, \in \vee q)$-FLId (resp. FRId) of S and $F : S \to P^*(S)$ be a $SVMH$. Then $\underline{F}(\mu)$ is $(\in, \in \vee q)$-FLId (resp. FRId) of S.

Proof. To prove this theorem, we have to see that $\underline{F}(\mu)$ satisfies (FI_{14}) and (FI_{15}). If for each $z_1, z_2 \in S$ with $z_1 \leq z_2$, implies $F(z_1) \subseteq F(z_2)$. Now consider

$$\begin{aligned} \min\{\underline{F}(\mu)(z_2), 0.5\} &= \min\left\{\bigwedge_{z_2' \in F(z_2)} \mu\left(z_2'\right), 0.5\right\} \\ &= \bigwedge_{z_2' \in F(z_2)} \min\left\{\mu\left(z_2'\right), 0.5\right\} \\ &\leq \bigwedge_{z_1' \in F(z_1)} \mu\left(z_1'\right) \end{aligned}$$

implies

$$\min\{\underline{F}(\mu)(z_2), 0.5\} \leq \underline{F}(\mu)(z_1)$$

Next consider

$$\begin{aligned} \underline{F}(\mu)(z_1 z_2) &= \bigwedge_{z_1' \in F(z_1 z_2)} \mu\left(z_1'\right) \\ &= \bigwedge_{z_1' \in F(z_1) F(z_2)} \mu\left(z_1'\right) \\ &= \bigwedge_{ab \in F(z_1) F(z_2)} \mu(ab) \quad \left(\begin{array}{c} \text{as } z_1' = ab \text{ such that } a \in F(z_1), \\ b \in F(z_2) \end{array}\right) \\ &= \bigwedge_{\substack{a \in F(z_1) \\ b \in F(z_2)}} \mu(ab) \\ &\geq \bigwedge_{b \in F(z_2)} \min\{\mu(b), 0.5\} \\ &= \min\left\{\bigwedge_{b \in F(z_2)} \mu(b), 0.5\right\} \end{aligned}$$

implies

$$\underline{F}(\mu)(z_1 z_2) \geq \min\{\underline{F}(\mu)(z_2), 0.5\}$$

Therefore it is clear that $\underline{F}(\mu)$ is an $(\in, \in \vee q)$-FLId (resp. FRId) ideal of S. □

Theorem 4.13. Consider that $F : S \to P^*(S)$ be a $SVIH$ and a FSS μ be $(\in, \in \vee q)$-FLId (resp. FRId) of S. Then $\overline{F}(\mu)$ is $(\in, \in \vee q)$-FLId (resp. FRId) of S.

Proof. To prove this theorem, we have to see that $\overline{F}(\mu)$ satisfies (FI_{14}) and (FI_{15}). If for each $z_1, z_2 \in S$ with $z_1 \leq z_2$, implies $F(z_2) \subseteq F(z_1)$. Next suppose the following

$$\begin{aligned} \min\left\{\overline{F}(\mu)(z_2), 0.5\right\} &= \min\left\{\bigvee_{z_2' \in F(z_2)} \mu\left(z_2'\right), 0.5\right\} \\ &= \bigvee_{z_2' \in F(z_2)} \min\left\{\mu\left(z_2'\right), 0.5\right\} \\ &\leq \bigvee_{z_1' \in F(z_1)} \mu\left(z_1'\right) \end{aligned}$$

implies

$$\min\left\{\overline{F}(\mu)(z_2), 0.5\right\} \leq \overline{F}(\mu)(z_1)$$

Next consider

$$\begin{aligned} \overline{F}(\mu)(z_1 z_2) &= \bigvee_{z_1' \in F(z_1 z_2)} \mu\left(z_1'\right) \\ &= \bigvee_{z_1' \in F(z_1)F(z_2)} \mu\left(z_1'\right) \\ &= \bigvee_{ab \in F(z_1)F(z_2)} \mu(ab) \quad \left(\begin{array}{c} \text{as } z_1' = ab \text{ such that } a \in F(z_1), \\ b \in F(z_2) \end{array}\right) \\ &= \bigvee_{\substack{a \in F(z_1) \\ b \in F(z_2)}} \mu(ab) \\ &\geq \bigvee_{b \in F(z_2)} \min\{\mu(b), 0.5\} \\ &= \min\left\{\bigvee_{b \in F(z_2)} \mu(b), 0.5\right\} \end{aligned}$$

implies

$$\overline{F}(\mu)(z_1 z_2) \geq \min\left\{\overline{F}(\mu)(z_2), 0.5\right\}$$

Therefore $\overline{F}(\mu)$ is an $(\in, \in \vee q)$-FLId of S. Similarly, we can prove that $\overline{F}(\mu)$ is an $(\in, \in \vee q)$-FRId of S. □

Theorem 4.14. Suppose that a FSS μ be $(\in, \in \vee q)$-FIId of S and $F : S \rightarrow P^*(S)$ be a $SVMH$. Then $\underline{F}(\mu)$ is $(\in, \in \vee q)$-FIId of S.

Proof. From Theorems 4.10 and 4.12,we see that $\underline{F}(\mu)$ satisfies (FI_{14}) and (FI_{16}).

Next we consider the following for each $z_1, z_2, z_3 \in S$.

$$\begin{aligned}
\underline{F}(\mu)(z_1 z_3 z_2) &= \bigwedge_{z_1' \in F(z_1 z_3 z_2)} \mu\left(z_1'\right) \\
&= \bigwedge_{z_1' \in F(z_1)F(z_3)F(z_2)} \mu\left(z_1'\right) \\
&= \bigwedge_{acb \in F(z_1)F(z_3)F(z_2)} \mu(acb) \quad \left(\begin{array}{c}\text{as } z_1' = acb \text{ such that } a \in F(z_1), \\ c \in F(z_3) \text{ and } b \in F(z_2)\end{array}\right) \\
&= \bigwedge_{\substack{a \in F(z_1) \\ c \in F(z_3) \\ b \in F(z_2)}} \mu(acb) \\
&\geq \bigwedge_{c \in F(z_3)} \min\{\mu(c), 0.5\} \\
&= \min\left\{\bigwedge_{z_3' \in F(z_3)} \mu\left(z_3'\right), 0.5\right\}
\end{aligned}$$

implies

$$\underline{F}(\mu)(z_1 z_3 z_2) \geq \min\{\underline{F}(\mu)(z_3), 0.5\}$$

Hence it is proved that $\underline{F}(\mu)$ is $(\in, \in \vee q)$-FIId of S. □

Theorem 4.15. Let us consider that $F : S \rightarrow P^*(S)$ be a $SVIH$ and a FSS μ be $(\in, \in \vee q)$-FIId of S. Then $\overline{F}(\mu)$ is $(\in, \in \vee q)$-FIId of S.

Proof. From Theorem 4.13, we have for each $z_1, z_2 \in S$, if $z_1 \leq z_2$ implies $F(z_2) \subseteq F(z_1)$. Then $\min\left\{\overline{F}(\mu)(z_2), 0.5\right\} \leq \overline{F}(\mu)(z_1)$, Next let for each $z_1, z_2 \in S$,

$$\begin{aligned}
\overline{F}(\mu)(z_1 z_2) &= \bigvee_{z_1' \in F(z_1 z_2)} \mu\left(z_1'\right) \\
&= \bigvee_{z_1' \in F(z_1)F(z_2)} \mu\left(z_1'\right) \\
&= \bigvee_{ab \in F(z_1)F(z_2)} \mu(ab) \quad \left(\begin{array}{c}\text{as } z_1' = ab \text{ where } a \in F(z_1) \\ \text{and } b \in F(z_2)\end{array}\right) \\
&= \bigvee_{\substack{a \in F(z_1) \\ b \in F(z_2)}} \mu(ab) \\
&\geq \bigvee_{\substack{z_1' \in F(z_1) \\ z_2' \in F(z_2)}} \min\left\{\mu\left(z_1'\right), \mu\left(z_2'\right), 0.5\right\} \\
&= \min\left\{\bigvee_{z_1' \in F(z_1)} \mu\left(z_1'\right), \bigvee_{z_2' \in F(z_2)} \mu\left(z_2'\right), 0.5\right\}
\end{aligned}$$

implies

$$\overline{F}(\mu)(z_1 z_2) \geq \min\left\{\overline{F}(\mu)(z_1), \overline{F}(\mu)(z_2), 0.5\right\}$$

Next consider

$$\begin{aligned}
\overline{F}(\mu)(z_1z_3z_2) &= \bigvee_{z_1'\in F(z_1z_3z_2)} \mu\left(z_1'\right) \\
&= \bigvee_{z_1'\in F(z_1)F(z_3)F(z_2)} \mu\left(z_1'\right) \\
&= \bigvee_{acb\in F(z_1)F(z_3)F(z_2)} \mu(acb) \quad \left(\begin{array}{c}\text{as } z_1' = acb \text{ where } a\in F(z_1), \\ c\in F(z_3) \text{ and } b\in F(z_2)\end{array}\right) \\
&= \bigvee_{\substack{a\in F(z_1)\\ c\in F(z_3)\\ b\in F(z_2)}} \mu(acb) \\
&\geq \bigvee_{c\in F(z_3)} \min\{\mu(c), 0.5\} \\
&= \min\left\{\bigvee_{c\in F(z_3)} \mu(c), 0.5\right\}
\end{aligned}$$

implies

$$\overline{F}(\mu)(z_1z_3z_2) \geq \min\left\{\overline{F}(\mu)(z_3), 0.5\right\}$$

Therefore, it is prove that $\overline{F}(\mu)$ is $(\in, \in\vee q)$-FIId of S. □

Theorem 4.16. Suppose that a FSS μ be $(\in, \in\vee q)$-FBId of S and $F : S \to P^*(S)$ be a $SVMH$. Then $\underline{F}(\mu)$ is $(\in, \in\vee q)$-FBId of S.

Proof. From Theorem 4.14, we have for each $z_1, z_2 \in S$, if $z_1 \leq z_2$ implies $F(z_1) \subseteq F(z_2)$. Then $\min\{\underline{F}(\mu)(z_2), 0.5\} \leq \underline{F}(\mu)(z_1)$, and also $\underline{F}(\mu)(z_1z_2) \geq \min\{\underline{F}(\mu)(z_1), \underline{F}(\mu)(z_2), 0.5\}$. Next let for each $z_1, z_2, z_3 \in S$,

$$\begin{aligned}
\underline{F}(\mu)(z_1z_2z_3) &= \bigwedge_{z_1'\in F(z_1z_2z_3)} \mu\left(z_1'\right) \\
&= \bigwedge_{z_1'\in F(z_1)F(z_2)F(z_3)} \mu\left(z_1'\right) \\
&= \bigwedge_{abc\in F(z_1)F(z_3)F(z_2)} \mu(abc) \quad \left(\begin{array}{c}\text{as } z_1' = abc \text{ where } a\in F(z_1), \\ b\in F(z_2) \text{ and } c\in F(z_3)\end{array}\right) \\
&= \bigwedge_{\substack{a\in F(z_1)\\ c\in F(z_3)\\ b\in F(z_2)}} \mu(acb) \\
&\geq \bigwedge_{\substack{a\in F(z_1)\\ c\in F(z_3)}} \min\{\mu(a), \mu(c), 0.5\} \\
&= \min\left\{\bigwedge_{a\in F(z_1)} \mu(a), \bigwedge_{c\in F(z_3)} \mu(c), 0.5\right\}
\end{aligned}$$

implies

$$\underline{F}(\mu)(z_1z_2z_3) \geq \min\{\underline{F}(\mu)(z_1), \underline{F}(\mu)(z_3), 0.5\}$$

Hence $\underline{F}(\mu)$ satisfies all the conditions of $(\in, \in\vee q)$-FBId of S. Therefore $\underline{F}(\mu)$ is an $(\in, \in\vee q)$-FBId of S. □

Theorem 4.17. Let us consider that $F : S \rightarrow P^{*}(S)$ be a $SVIH$ and a FSS μ be $(\in, \in \vee q)$-FBId of OSG S. Then $\overline{F}(\mu)$ is $(\in, \in \vee q)$-FBId of S.

Proof. From Theorem 4.15, we have for each $z_1, z_2 \in S$, if $z_1 \leq z_2$ implies $F(z_2) \subseteq F(z_1)$. Then $\min\left\{\overline{F}(\mu)(z_2), 0.5\right\} \leq \overline{F}(\mu)(z_1)$, and also $\overline{F}(\mu)(z_1 z_2) \geq$ $\min\{\overline{F}(\mu)(z_1), \overline{F}(\mu)(z_2), 0.5\}$. Next we consider the following for each $z_1, z_2, z_3 \in S$,

$$\begin{aligned}
\overline{F}(\mu)(z_1 z_2 z_3) &= \bigvee_{z_1^{'} \in F(z_1 z_2 z_3)} \mu\left(z_1^{'}\right) \\
&= \bigvee_{z_1^{'} \in F(z_1)F(z_2)F(z_3)} \mu\left(z_1^{'}\right) \\
&= \bigvee_{abc \in F(z_1)F(z_3)F(z_2)} \mu(abc) \quad \left(\begin{array}{c} \text{as } z_1^{'} = abc \text{ where } a \in F(z_1), \\ b \in F(z_2) \text{ and } c \in F(z_3) \end{array}\right) \\
&= \bigvee_{\substack{a \in F(z_1) \\ c \in F(z_3) \\ b \in F(z_2)}} \mu(acb) \\
&\geq \bigvee_{\substack{a \in F(z_1) \\ c \in F(z_3)}} \min\{\mu(a), \mu(c), 0.5\} \\
&= \min\left\{\bigvee_{a \in F(z_1)} \mu(a), \bigvee_{c \in F(z_3)} \mu(c), 0.5\right\}
\end{aligned}$$

implies

$$\overline{F}(\mu)(z_1 z_2 z_3) \quad \geq \quad \min\left\{\overline{F}(\mu)(z_1), \overline{F}(\mu)(z_3), 0.5\right\}$$

Hence $\overline{F}(\mu)$ satisfies all the conditions of $(\in, \in \vee q)$-FBId of S. Therefore $\overline{F}(\mu)$ is $(\in, \in \vee q)$-FBId of S. □

Theorem 4.18. Let us suppose that a FSS μ be $(\in, \in \vee q)$-FQId of S and $F : S \rightarrow P^{*}(S)$ be a $SVMH$. Then we have to prove that $\underline{F}(\mu)$ is $(\in, \in \vee q)$-FQId of S.

Proof. From Theorem 4.12, we have for each $z_1, z_2 \in S$, if $z_1 \leq z_2$ implies $F(z_1) \subseteq F(z_2)$. Then $\min\{\underline{F}(\mu)(z_2), 0.5\} \leq \underline{F}(\mu)(z_1)$, Next let for each $z_1 \in S$,

$$\begin{aligned}
\min\{\underline{F}((\mu \circ 1) \wedge (1 \circ \mu))(z_1), 0.5\} &= \min\left\{\bigwedge_{z_1^{'} \in F(z_1)} ((\mu \circ 1) \wedge (1 \circ \mu))\left(z_1^{'}\right), 0.5\right\} \\
&= \bigwedge_{z_1^{'} \in F(z_1)} \min\left(((\mu \circ 1) \wedge (1 \circ \mu))\left(z_1^{'}\right), 0.5\right) \\
&\leq \bigwedge_{z_1^{'} \in F(z_1)} \mu\left(z_1^{'}\right)
\end{aligned}$$

implies

$$\min\{\underline{F}((\mu \circ 1) \wedge (1 \circ \mu))(z_1), 0.5\} \quad \leq \quad \underline{F}(\mu)(z_1)$$

Hence $\underline{F}(\mu)$ is an $(\in, \in \vee q)$-FQId of S. □

Theorem 4.19. Let a FSS μ be $(\in, \in \vee q)$-FQId of S and consider that $F : S \to P^*(S)$ be a $SVIH$. Then we have to prove that $\overline{F}(\mu)$ is $(\in, \in \vee q)$-FQId of S.

Proof. From Theorem 4.13, we have for each $z_1, z_2 \in S$, if $z_1 \leq z_2$ implies $F(z_2) \subseteq F(z_1)$. Then $\min\left\{\overline{F}(\mu)(z_2), 0.5\right\} \leq \overline{F}(\mu)(z_1)$, Next let for each $z_1 \in S$,

$$\begin{aligned}\min\left\{\overline{F}((\mu \circ 1) \wedge (1 \circ \mu))(z_1), 0.5\right\} &= \min\left\{\bigvee_{z_1' \in F(z_1)} ((\mu \circ 1) \wedge (1 \circ \mu))\left(z_1'\right), 0.5\right\} \\ &= \bigvee_{z_1' \in F(z_1)} \min\left\{((\mu \circ 1) \wedge (1 \circ \mu))\left(z_1'\right), 0.5\right\} \\ &\leq \bigvee_{z_1' \in F(z_1)} \mu\left(z_1'\right)\end{aligned}$$

implies

$$\min\left\{\overline{F}((\mu \circ 1) \wedge (1 \circ \mu))(z_1), 0.5\right\} \leq \overline{F}(\mu)(z_1)$$

Hence it is proved that $\overline{F}(\mu)$ is an $(\in, \in \vee q)$-FQId of S. □

5 Conclusion

OSGs is a significant algebraic structure having partial ordered with associative binary operations. OSGs have broad applications in various fields such as coding theory, automata theory and computer science etc. In this manuscript we have originated the approximations of FIds, FBIds, FIIds and FQIds of OSGs on the basis of isotone and monotone mapping. It is clear that these two mappings play a significant role for investigating the approximation of FIds in OSGs. Moreover in the idea of approximation is generalized to $(\in, \in \vee q)$-FIds, FBIds, FIIds and FQIds.

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Summability of Fourier Series and its Derived Series by Matrix Means

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Abstract – This Paper introduces the concept of matrix operators and establishes two new theorems on matrix summability of Fourier series and its derived series. the results obtained in the paper further extend several known results on linear operators. Various types of criteria, under varying conditions, for the matrix summability of the Fourier series, In this paper quite a different and general type of criterion for summability of the Fourier Series has been obtained, in the theorem function f is integrable in the sense of Lebesgue to the interval $[-\pi, \pi]$ and period with period 2π.

Keywords *– Summability, matrix summability, Fourier series, derived Fourier series.*

1. Introduction

Let $\sum_{n=0}^{\infty} u_n$ be a given infinite series with sequence of partial sum $\{S_n\}$. Let $A = (a_{n,k})$ be an infinite triangular matrix of real constants, The sequence-to-sequence transformation [4].

$$t_n^A = \sum_{k=0}^{n} a_{n,k}\, S_k = \sum_{k=0}^{n} a_{n,n-k}\; S_{n-k} \qquad (1)$$

Defines the sequence t_n^A of matrix means of the sequence $\{S_n\}$, generated by the sequence of coefficients $(a_{n,k})$. The series $\sum_{n=0}^{\infty} u_n$ is said to be summable to the sum S by We can write $t_n^A \to S \quad (A), \text{as } n \to \infty$.

The necessary and sufficient conditions for A-transform to be regular

$$(\text{i.e.} \lim_{n\to\infty} S_n = S \Longrightarrow \lim_{n\to\infty} t_n^A = S)$$

are the well-known Silverman-Toeplitz conditions? [1][4] where the triangular matrix $A = (a_{n,k}),\; n,k = 0,1,2,3\,...$ and $a_{n,k} = 0$ for $k > n$ is regular if

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$$\lim_{n\to\infty} a_{n,k} = 0 \quad ; k = 1,2, \ldots$$

$$\lim_{n\to\infty} \sum_{k=0}^{n} a_{n,k} = 1$$

$$\sum_{k=0}^{n} |a_{n,k}| \le M \;;\; n = 1,2, \ldots (M \text{ independent of } n)$$

and

$$t_n^{AB} = \sum_{k=0}^{n} a_{n,k} \sum_{r=0}^{k} b_{k,r} S_r.$$

Examples:

(1). Matrix Hankel [2] . Let $\{h_n\}_{n=0}^{\infty}$ be a positive sequence of real constants let

$$H = (a_{n,k}) = (h_{n+k-1})$$

2). Matrix Toeplitz [4] $T = (a_{n,k}) \quad ; (a_{n,k} = 0 \;;\; k > n)$(Thus from (1), We get [2]

(1). $\sum_{n=0}^{\infty} u_n = A \quad (T) \Rightarrow \sum_{n=0}^{\infty} au_n = aA \quad (T)$

(2). $\sum_{n=0}^{\infty} u_n = A_1(T) \,\&\, \sum_{n=0}^{\infty} v_n = A_2(T) \quad \Rightarrow \sum_{n=0}^{\infty}(u_n + v_n) = A_1 + A_2\,(T)$

(3). $\sum_{n=0}^{\infty} u_n = A_1 \quad (S) \wedge \sum_{n=0}^{\infty} u_n = A_2 \;(T) \quad \Rightarrow A_1 = A_2$

Proof (3): Now by (1) we have

$$(R) = (r_{m,n}) = (a_{m,0}.b_{m,n} + a_{m,1}.b_{m,n-1} + \cdots + a_{m,n-1}.b_{m,1} + a_{m,n}.b_{m,0})$$
$$t_m = \sum_{k=0}^{m} r_{m,k}\,.S_k$$

$$\Longrightarrow t_m = r_{m,0}\,.S_0 + r_{m,1}\,.S_1 + r_{m,2}\,.S_2 + \cdots + r_{m,m-1}\,.S_{m-1} + r_{m,m}\,.S_m$$

$$\Longrightarrow t_m = a_{m,0}.b_{m,0}(S_0) + (a_{m,0}.b_{m,1} + a_{m,1}.b_{m,0}).(S_1) + (a_{m,0}.b_{m,2} + a_{m,1}.b_{m,1} + a_{m,2}.b_{m,0})(S_2) + (a_{m,0}.b_{m,3} + a_{m,1}.b_{m,2} + a_{m,2}.b_{m,1} + a_{m,3}.b_{m,0}).(S_3) + (a_{m,0}.b_{m,4} + a_{m,1}.b_{m,3} + a_{m,2}.b_{m,2} + a_{m,3}.b_{m,1} + a_{m,4}.b_{m,0}).(S_4) + (a_{m,0}.b_{m,m} + a_{m,1}.b_{m,m-1} + a_{m,2}.b_{m,m-2} + \cdots + a_{m,m-2}.b_{m,1} + a_{m,m-1}.b_{m,0}).(S_{m-1}) + (a_{m,0}.b_{m,m-1} + a_{m,1}.b_{m,m-2} + a_{m,2}.b_{m,m-3} + \cdots + a_{m,m-1}.b_{m,1} + a_{m,m}.b_{m,0}).(S_m)$$

$$\Longrightarrow t_m = a_{m,0}.(b_{m,0}.S_0 + b_{m,1}.S_1 + b_{m,2}.S_2 + b_{m,3}.S_3 + b_{m,4}.S_4 + \cdots + b_{m,m-1}.S_{m-1} + b_{m,m}.S_m) + a_{m,1}.(b_{m,0}.S_1 + b_{m,1}.S_2 + b_{m,2}.S_3 + b_{m,3}.S_4 + \cdots + b_{m,m-2}.S_{m-1} + b_{m,m-1}.S_m) + a_{m,2}.(b_{m,0}.S_2 + b_{m,1}.S_3 + b_{m,2}.S_4 + b_{m,3}.S_5 + \cdots + b_{m,m-3}.S_{m-1} + b_{m,m-2}.S_m) + \cdots + a_{m,m-1}.(b_{m,0}.S_{m-1} + b_{m,1}.S_m) + a_{m,m}.(b_{m,0}.S_m)$$

$$\Longrightarrow t_m = a_{m,0}.\sum_{k=0}^{m} b_{m,k}\,.S_{k+0} + a_{m,1}.\sum_{k=0}^{m-1} b_{m,k}\,.S_{k+1} + a_{m,2}.\sum_{k=0}^{m-2} b_{m,k}\,.S_{k+2} + \cdots + a_{m,m-1}.\sum_{k=0}^{m-(m-1)} b_{m,k}\,.S_{k+m-1} + a_{m,m}.\sum_{k=0}^{m-m} b_{m,k}\,.S_{k+m}$$

$$\Longrightarrow t_m = \sum_{n=0}^{m} a_{m,n}.\sum_{k=0}^{m-n} b_{m,k}\,.S_{k+n}$$

$$\Longrightarrow \lim_{m\to\infty} t_m = \sum_{n=0}^{\infty} a_{m,n}.A_2 = A_2$$

$$\sum_{n=0}^{\infty} u_n = A_2(R) \quad ; (R) = (r_{m,n})$$

Similarly

$$\sum_{n=0}^{\infty} u_n = A_1(R) \Longrightarrow A_2 = A_2$$

2 Preliminaries

Theorem 2.1. Let $A = (a_{n,k}), B = (b_{n,k})$ be an infinite triangular matrix with $a_{n,k} \geq 0, b_{n,k} \geq 0$then$t_n^{AB} \in \mathcal{B}(\mathcal{A}_r)$ Where $\mathcal{B}(\mathcal{A}_r)$ Space call to bounded linear operator on $\mathcal{A}_r$, $T: \mathcal{A}_r \to \mathcal{A}_r$ and $\mathcal{A}_r = \{(s_n)_{n=0}^{\infty}; \sum_{n=0}^{\infty}(n+1)^{r-1}. |u_n|^r < \infty; u_n = \Delta s_n = s_n - s_{n-1}\}$

Proof of Theorem 2.1. Let τ_n^{AB}mn-denote the mn-term of the AB-transform, in terms of $(n+1)u_n$, that is $\tau_n^{AB} = \sum_{k=0}^{n} a_{n,k} \sum_{j=0}^{k} b_{k,j} (j+1)u_j = (n+1)(t_n^{AB} - t_{n-1}^{AB})$ to prove the theorem, it will be sufficient to show that $\sum_{n=0}^{\infty} \frac{1}{n+1} |\tau_n^{AB}|^r < \infty$ Using Hölder's inequality, we have

$$\sum_{n=0}^{\infty} \frac{1}{n+1} |\tau_n^{AB}|^r = \sum_{n=0}^{\infty} \frac{1}{n+1} \left|\sum_{k=0}^{n} a_{n,k} \sum_{j=0}^{k} b_{k,j} (j+1)u_j\right|^r \leq$$
$$\sum_{n=0}^{\infty} \frac{1}{n+1} \sum_{k=0}^{n} a_{n,k} \sum_{j=0}^{k} b_{k,j} (j+1)^r |u_j|^r \times \left\{\sum_{k=0}^{n} a_{n,k} \sum_{j=0}^{k} b_{k,j}\right\}^{r-1}$$

Since $\sum_{k=0}^{n} a_{n,k} \sum_{j=0}^{k} b_{k,j} = 1$ we obtain

$$\sum_{n=0}^{\infty} \frac{1}{n+1} |\tau_n^{AB}|^r = \sum_{n=0}^{\infty} \frac{1}{n+1} \left|\sum_{k=0}^{n} a_{n,k} \sum_{j=0}^{k} b_{k,j} (j+1)u_j\right|^r \leq$$
$$\sum_{n=0}^{\infty} \frac{1}{n+1} \sum_{k=0}^{n} a_{n,k} \sum_{j=0}^{k} b_{k,j} (j+1)^r |u_j|^r \leq \sum_{j=0}^{\infty} (j+1)^r |u_j|^r \sum_{n=j}^{\infty} \frac{1}{n+1} \sum_{k=0}^{n} a_{n,k} . b_{k,j}$$

For $m, n \geq 1$,

$$\sum_{n=j}^{\infty} \frac{1}{n+1} \sum_{k=0}^{n} a_{n,k} . b_{k,j} = \frac{1}{j+1}$$

$$\sum_{n=0}^{\infty} \frac{1}{n+1} |\tau_n^{AB}|^r = O(1). \sum_{j=0}^{\infty} (j+1)^r |u_j|^r \frac{1}{j+1} = O(1). \sum_{j=0}^{\infty} (j+1)^r |u_j|^r = O(1) < \infty$$

This completes the proof of the theorem (1).

3 Particular Cases

Several authors such as ([4]-[6]), (see also [7]) studied the matrix summability method and obtained many interesting results.

The important particular cases of the triangular matrix means are:

(i) Cesàro mean of order 1 or $(C, 1)$ mean if, $a_{n,k} = \frac{1}{n+1}$.

(ii) Harmonic means $(H, 1)$when, $a_{n,k} = \frac{1}{(n-k+1) \log n}$.

(iii) $(C,\delta)$where$0 \leq \delta \leq 1$ means when, $a_{n,k} = \frac{\binom{n-k+\delta+1}{\delta-1}}{\binom{n+\delta}{\delta}}$.

(v) Nörlund means $(N,p_n)$when, $a_{n,k} = \frac{p_{n-k}}{P_n}$where$P_n = \sum_{k=0}^{n} p_k \to \infty$ as $n \to \infty$.

(vi) Riesz means $(\overline{N},p_n)$when, $a_{n,k} = \frac{p_k}{P_n}$where$P_n = \sum_{k=0}^{n} p_k \to \infty$ as $n \to \infty$.

4 Results and Discussion

Let $f(t)$ be a periodic function with period 2π, integrable in the sense of Lebesgue over $[-\pi,\pi]$. The Fourier series and derived Fourier series off(t) are given by [3][4][5]

$$f(t) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) \equiv \sum_{n=0}^{\infty} A_n\,(x)$$

With partial sums s_n and

$$\sum_{n=1}^{\infty} n(b_n \cos nx - a_n \sin nx) \qquad (2)$$

We shall use following notations

$$\tau = \text{Integral part of } \frac{1}{t} = \left[\frac{1}{t}\right].$$

We use the following notations

$$\phi(t) = f(x+t) + f(x-t) - 2f(x)$$

$$g(t) = f(x+t) - f(x-t) - 2t\hat{f}(x)$$

$$K_{AB}(n,t) = \frac{1}{2\pi}\sum_{k=0}^{n} a_{n,k}\cdot\sum_{r=0}^{k} b_{k,k-r}\frac{\sin\left(k-r+\frac{1}{2}\right)t}{\sin\frac{1}{2}t}$$

$$M_n(t) = \frac{1}{2\pi}\sum_{k=0}^{n} a_{n,n-k}\cdot\frac{\sin\left(n-k+\frac{1}{2}\right)t}{\sin\frac{1}{2}t} \qquad (3)$$

$$\tau = \text{Integral part of } \frac{1}{t} = \left[\frac{1}{t}\right].$$

Theorem 4.1. Let $\{p_n\}_{n=0}^{\infty}$ be a real non-negative and non-increasing sequence of real constants such that $P_n = \sum_{k=0}^{n} p_k \to \infty; (n \to \infty)$ and $A = (a_{n,k}), B = (b_{k,r})$ be an infinite triangular matrix with $a_{n,k} \geq 0$, If

$$\Phi(t) = \int_0^t |\phi(u)|du = o\left(\frac{t}{\alpha\left(\frac{1}{t}\right)P_\tau}\right), \text{ as } t \to +0 \;, \tau = \left[\frac{1}{t}\right]$$

where$\alpha(t)$ is a positive, monotonic and non-increasing function of $t \to +0$

$$\log n = O(\alpha(n).P_n); (n \to \infty).$$

and

$$\lim_{n\to\infty} \int_1^n \frac{B_{n,t}}{t.\alpha(t)P_t} dt = O(1); B_{n,\tau} = \sum_{k=0}^{\tau} b_{n,n-k} = \sum_{k=n-\tau}^{n} b_{n,k}$$

Then the Fourier series (1) is summable AB to f (x).

Theorem 4.2. Let $\{a_{n,k}\}_{k=0}^{\infty}$ be a real non-negative and non-decreasing sequence with respect to k such that $T = (a_{n,k})$ be an infinite triangular matrix with $a_{n,k} \geq 0$. If

$$\int_0^t |dg(u)| = o\left(\frac{t.\alpha(\frac{1}{t})}{\log\frac{1}{t}}\right), \text{ as } t \to +0$$

Then the derived Fourier Series (2) is sumable (T) to the sum $f'(x)$, where $f'(x)$ is the derivative of $f(x)$,provided $\alpha(t)$ is a positive monotonic decreasing function of $t \to +0$ such $\frac{t.\alpha(\frac{1}{t})}{\log\frac{1}{t}}$ increases monotonically as $t \to +0$.

For the proof of our theorems, following lemmas are required.

Lemma 4.1. [5] If $\{a_{n,k}\}$ is non-negative and non-decreasing with k then for $0 \leq t \leq \pi, 0 \leq a \leq b \leq \infty$ and for any n, we have $\left|\sum_{k=a}^{b} a_{n,n-k}.e^{i(n-k)t}\right| \leq O(A_{n,\tau})$ Where $A_{n,\tau} = \sum_{k=0}^{\tau} a_{n,n-k}$, $A_{n,n} = 1 \quad (\forall n \geq 0)$.

Lemma 4.2. [5] If $\{b_{n,k}\}$ is non-negative and non-decreasing with k then for $0 \leq t \leq \pi, 0 \leq a \leq b \leq \infty$ and for any n, we have $\left|\sum_{k=a}^{b} b_{n,n-k}.e^{i(n-k)t}\right| \leq O(B_{n,\tau})$ Where $B_{n,\tau} = \sum_{k=0}^{\tau} b_{n,n-k}$, $B_{n,n} = 1 \quad (\forall n \geq 0)$.

Lemma 4.3. For $0 < t \leq \frac{1}{n}$, $K_{AB}(n,t) = O(n)$.

Proof. For $0 < t \leq \frac{1}{n}$, $\sin(n+1)t \leq (n+1)t$, $\left[\sin(\frac{t}{2})\right]^{-1} \leq \frac{\pi}{t}$

$$K_{AB}(n,t) = \frac{1}{2\pi} \sum_{k=0}^{n} a_{n,k} \sum_{r=0}^{k} b_{k,k-r} \frac{\sin\left(k-r+\frac{1}{2}\right)t}{\sin\frac{1}{2}t} \leq \frac{1}{2\pi} \sum_{k=0}^{n} a_{n,k} \sum_{r=0}^{k} b_{k,k-r} \frac{\left|\sin\left(k-r+\frac{1}{2}\right)t\right|}{\left|\sin\frac{1}{2}t\right|} \leq \frac{2n+1}{2\pi} \sum_{k=0}^{n} a_{n,k} \sum_{r=0}^{k} b_{k,k-r}$$

Since $\sum_{k=0}^{n} a_{n,k} \sum_{r=0}^{k} b_{k,k-r} = 1$

Thus $K_{AB}(n,t) = O(n)$

Lemma 4.4. For $\frac{1}{n} \leq t \leq \delta < \pi$, $K_{AB}(n,t) = O\left(\frac{B_{n,\tau}}{t}\right)$; $\tau \leq n$

Proof. $K_{AB}(n,t) \leq \frac{1}{2\pi}\left|\sum_{k=0}^{n} a_{n,k} \sum_{r=0}^{k} b_{k,k-r} \frac{\sin\left(k-r+\frac{1}{2}\right)t}{\sin\frac{1}{2}t}\right|$

$$\leq \frac{1}{2t}\left|\sum_{k=0}^{n} a_{n,k} \sum_{r=0}^{k} b_{k,k-r} \sin\left(k-r+\frac{1}{2}\right)t\right| ; \text{(by Jordan's lemma)}$$

$$\leq \frac{1}{2t}\left|\sum_{k=0}^{n} a_{n,k} \operatorname{Im} \sum_{r=0}^{k} b_{k,k-r} e^{i\left(k-r+\frac{1}{2}\right)t}\right|$$

$$\leq \frac{1}{2t}\sum_{k=0}^{n} a_{n,k}\left|\operatorname{Im}\sum_{r=0}^{k} b_{n,n-k}.e^{i(k-r)t}.e^{i\frac{t}{2}}\right|$$
$$\leq \frac{1}{2t}\sum_{k=0}^{n} a_{n,k}\left|\sum_{r=0}^{k} b_{n,n-k}.e^{i(k-r)t}\right|\ ;\ \left|e^{i\frac{t}{2}}\right| = 1$$
$$\leq \frac{1}{2t}\sum_{k=0}^{n} a_{n,k} O(B_{k,\tau});\ \text{ by lemma (1)}$$
$$\leq O\left(\frac{B_{n,\tau}}{t}\right)\sum_{k=0}^{n} a_{n,k}$$
$$= O\left(\frac{B_{n,\tau}}{t}\right)$$

Lemma 4.5. For $\frac{1}{n} \leq t \leq \delta < \pi$, $M_n(t) = O\left(\frac{A_{n,\tau}}{t}\right)\ ;\ \tau \leq n$.

Proof. Now by (3)

$$M_n(t) \leq \left|\sum_{k=0}^{n} a_{n,n-k}.\frac{\sin\left(n-k+\frac{1}{2}\right)t}{\sin\frac{1}{2}t}\right| \leq \frac{1}{\sin\frac{1}{2}t}\left|\operatorname{Im}.\sum_{k=0}^{n} a_{n,n-k}.e^{i\left(n-k+\frac{1}{2}\right)t}\right| \leq$$
$$\frac{\pi}{t}\left|\operatorname{Im}\sum_{k=0}^{n} a_{n,n-k}.e^{i(n-k)t}.e^{i\frac{t}{2}}\right| \text{ (by Jordan's lemma) } \leq \frac{\pi}{t}\left|\sum_{k=0}^{n} a_{n,n-k}.e^{i(n-k)t}.\right|\ ;\ \left|e^{i\frac{t}{2}}\right| \leq 1$$
$$= \frac{\pi}{t}.O(A_{n,\tau}) \quad \text{by lemma (1)}$$
$$= O\left(\frac{A_{n,\tau}}{t}\right)$$

Lemma 4.6. For $0 \leq t \leq \frac{1}{n}$, $M_n(t) = O(n)$.

Proof of Theorem 4.1. Let $s_n(x)$ denote the n^{th} partial sum of the (1). Then we have

$$s_n(f;x) - f(x) = \frac{1}{2\pi}\int_0^{\pi}\phi(t)\frac{\sin\left(n+\frac{1}{2}\right)t}{\sin\frac{1}{2}t}dt$$
$$t_n^B - f(x) = \frac{1}{2\pi}\int_0^{\pi}\phi(t)\sum_{k=0}^{n} b_{n,k}\frac{\sin\left(k+\frac{1}{2}\right)t}{\sin\frac{1}{2}t}$$
$$t_n^{AB} - f(x) = \frac{1}{2\pi}\sum_{k=0}^{n} a_{n,k}\int_0^{\pi}\phi(t)\left\{\sum_{r=0}^{k} b_{k,r}\frac{\sin\left(r+\frac{1}{2}\right)t}{\sin\frac{1}{2}t}\right\}dt$$
$$= \frac{1}{2\pi}\sum_{k=0}^{n} a_{n,k}\int_0^{\pi}\phi(t)\left\{\sum_{r=0}^{k} b_{k,k-r}\frac{\sin\left(k-r+\frac{1}{2}\right)t}{\sin\frac{1}{2}t}\right\}dt = \int_0^{\pi}\phi(t)K_{AB}(n,t)dt$$
$$= \left(\int_0^{\frac{1}{n}} + \int_{\frac{1}{n}}^{\delta} + \int_{\delta}^{\pi}\right)\phi(t)K_{AB}(n,t)dt$$
$$= I_{1.1} + I_{1.2} + I_{1.3}$$
$$I_{1.1} \leq |I_{1.1}| \leq \int_0^{\frac{1}{n}}|\phi(t)|.|K_{AB}(n,t)|dt = O(n)\left\{\int_0^{\frac{1}{n}}|\phi(t)|dt\right\}$$

By Lemma 4.3

$$= O(n).\left\{o\left(\frac{\frac{1}{n}}{\alpha(n).P_n}\right)\right\} = o\left(\frac{1}{\alpha(n).P_n}\right)$$
$$= o\left(\frac{1}{\log n}\right) = o(1);(n \to \infty)$$
$$I_{1.2} \leq |I_{1.2}| \leq O\left(\int_{\frac{1}{n}}^{\delta}|\phi(t)||K_{AB}(n,t)|dt\right)$$

$$= O\left(\int_{\frac{1}{n}}^{\delta}|\phi(t)|\frac{B_{n,t}}{t}dt\right)$$

By Lemma 4.4

Integrating by parts

$$I_{1.2} \le O\left\{\left(\frac{B_{n,t}}{t}.\Phi(t)\left|_{\frac{1}{n}}^{\delta}\right.\right) - \int_{\frac{1}{n}}^{\delta}\frac{d}{dt}\left(\frac{B_{n,t}}{t}\right)\Phi(t)dt\right\}$$
$$= O\left\{o\left(\frac{B_{n,t}}{t}\frac{t}{\alpha(\frac{1}{t})P_\tau}\left|_{\frac{1}{n}}^{\delta}\right.\right) + \int_{\frac{1}{n}}^{\delta}\frac{B_{n,t}}{t^2}\frac{t}{\alpha(\frac{1}{t})P_\tau}dt\int_{\frac{1}{n}}^{\delta}\frac{1}{t}\frac{t}{\alpha(\frac{1}{t})P_\tau}d(B_{n,\tau})\right\}$$
$$\left(u = \frac{1}{t}\right)$$
$$I_{1.2} = O\left\{o\left(\frac{B_{n,[\frac{1}{\delta}]}}{\alpha\frac{1}{\delta}P_{[\frac{1}{\delta}]}}\right) + o\left(\frac{B_{n,n}}{\alpha(n)P_n}\right) + o\left(\int_{\frac{1}{\delta}}^{n}\frac{B_{n,u}}{u.\alpha(u)P_u}du\right) + o\left(\int_{\frac{1}{\delta}}^{n}\frac{1}{\alpha(u)P_u}d(B_{n,u})\right)\right\}$$
$$\le O\left\{o(1) + o(1) + o\left(\int_1^n\frac{B_{n,u}}{u.\alpha(u)P_u}du\right) + o\left(\frac{1}{\alpha(n)P_n}\right)\left(\int_1^n d(B_{n,u})\right)\right\} = O\{o(1) + o(1) + o(O(1)) + o(1)B_{n,n}\} = O\{o(1) + o(1) + o(1) + o(1)\} = o(1);\quad (n \to \infty)$$

Lastly, by the Riemann-Lebesgue theorem and the regularity condition of matrix summability, we obtain

$$I_{1.3} \le |I_{1.3}| \le \int_{\delta}^{\pi}|\phi(t)||K_{AB}(n,t)|dt = o(1)\quad ;\quad (n \to \infty)$$

Next

$$t_n^{AB} - f(x) = o(1)\qquad (n \to \infty) \Longrightarrow \lim_{n\to\infty}\{t_n^{AB} - f(x)\} = 0$$

This completes the proof of the theorem.

Proof of Theorem 4.2. Let $\acute{s}_n(x)$ denote the n^{th} partial sum of the (2). Then

$$\acute{s}_n(x) = \frac{1}{2\pi}\int_0^{\pi}\frac{\sin\left(n+\frac{1}{2}\right)t}{\sin\frac{1}{2}t}dg(t) + \acute{f}(x)$$

We have

$$\acute{s}_n(x) = \sum_{k=1}^{n}k(b_k\cos kx - a_k\sin kx) = \sum_{k=1}^{n}\frac{1}{\pi}\int_0^{2\pi}k(\sin ku.\cos kx - \cos ku.\sin kx).f(u)du = \sum_{k=1}^{n}\frac{k}{\pi}\int_0^{2\pi}\sin k(u-x).f(u)du = \frac{1}{2\pi}\int_0^{2\pi}f(u).\sum_{k=1}^{n}2k\sin k(u-x)\,du$$
$$= -\frac{1}{2\pi}\int_0^{2\pi}f(u).\sum_{k=1}^{n}2k\sin k(x-u)\,du$$

where

$$\sum_{k=1}^{n}k\sin ky = -\frac{d}{dy}\left(\frac{\sin\left(n+\frac{1}{2}\right)y}{2\sin\frac{1}{2}y}\right)$$

Next

$$\sum_{k=1}^{n}2k\sin k(x-u) = -2\frac{d}{dx}\left(\frac{\sin\left(n+\frac{1}{2}\right)(x-u)}{2\sin\frac{1}{2}(x-u)}\right) = 2\frac{d}{du}\left(\frac{\sin\left(n+\frac{1}{2}\right)(x-u)}{2\sin\frac{1}{2}(x-u)}\right)$$

Thus

$$\acute{s}_n(x) = \frac{1}{2\pi}\int_0^{2\pi}\left\{\frac{d}{dx}\frac{\sin\left(n+\frac{1}{2}\right)(x-u)}{\sin\frac{1}{2}(x-u)}\right\}.f(u)du = -\frac{1}{2\pi}\int_0^{2\pi} f(u).\left\{\frac{d}{du}\frac{\sin\left(n+\frac{1}{2}\right)(x-u)}{\sin\frac{1}{2}(x-u)}\right\}du$$

$$= -\frac{1}{2\pi}\left(\int_{-\pi}^{0} f(u)\left\{\frac{d}{du}\frac{\sin\left(n+\frac{1}{2}\right)(x-u)}{\sin\frac{1}{2}(x-u)}\right\}du + \int_0^{\pi} f(u)\left\{\frac{d}{du}\frac{\sin\left(n+\frac{1}{2}\right)(x-u)}{\sin\frac{1}{2}(x-u)}\right\}du\right) = G_1 + G_2$$

Now

$$\acute{s}_n(x) = -\frac{1}{2\pi}\int_0^{\pi}\{f(x+t) - f(x-t)\}\left\{\frac{d}{dt}\frac{\sin\left(n+\frac{1}{2}\right)t}{\sin\frac{1}{2}t}\right\}dt$$

Integrating by parts

$$\acute{s}_n(x) = \frac{1}{2\pi}\int_0^{\pi}\frac{\sin\left(n+\frac{1}{2}\right)t}{\sin\frac{1}{2}t}d\{f(x+t) - f(x-t)\}$$

where

$$\frac{\sin\left(n+\frac{1}{2}\right)t}{2\sin\frac{1}{2}t} = \frac{1}{2} + \cos t + \cos 2t + \cos 3t + \cdots + \cos nt = D_n(t) \Rightarrow \frac{1}{\pi}\int_0^{\pi}\frac{\sin\left(n+\frac{1}{2}\right)t}{2\sin\frac{1}{2}t}dt = \frac{1}{2}$$

and

$$g(t) = f(x+t) - f(x-t) - 2t\acute{f}(x) \Longrightarrow f(x+t) - f(x-t) = g(t) + 2t\acute{f}(x)$$
$$\Longrightarrow d\{f(x+t) - f(x-t)\} = dg(t) + 2\acute{f}(x)dt + 0$$

Next

$$\acute{s}_n(x) = \frac{1}{2\pi}\int_0^{\pi}\frac{\sin\left(n+\frac{1}{2}\right)t}{\sin\frac{1}{2}t}dg(t) + 2\acute{f}(x).\frac{1}{2\pi}\int_0^{\pi}\frac{\sin\left(n+\frac{1}{2}\right)t}{\sin\frac{1}{2}t}dt$$

$$\Longrightarrow \acute{s}_n(x) = \frac{1}{2\pi}\int_0^{\pi}\frac{\sin\left(n+\frac{1}{2}\right)t}{\sin\frac{1}{2}t}dg(t) + \acute{f}(x)$$

$$\Longrightarrow \acute{s}_n(x) - \acute{f}(x) = \frac{1}{2\pi}\int_0^{\pi}\frac{\sin\left(n+\frac{1}{2}\right)t}{\sin\frac{1}{2}t}dg(t)$$

$$\Longrightarrow \acute{s}_{n-k}(x) - \acute{f}(x) = \frac{1}{2\pi}\int_0^{\pi}\frac{\sin\left(n-k+\frac{1}{2}\right)t}{\sin\frac{1}{2}t}dg(t)$$

$$\Longrightarrow \sum_{k=0}^{n} a_{n,n-k}\left\{\acute{s}_{n-k}(x) - \acute{f}(x)\right\}$$

$$= \int_0^{\pi} dg(t).\frac{1}{2\pi}\sum_{k=0}^{n} a_{n,n-k}\cdot\frac{\sin\left(n-k+\frac{1}{2}\right)t}{\sin\frac{1}{2}t}$$

$$= \left(\int_0^{\frac{1}{n}} + \int_{\frac{1}{n}}^{\delta} + \int_{\delta}^{\pi}\right)dg(t).M_n(t) = I_1 + I_2 + I_3$$

$$I_1 = \int_0^{\frac{1}{n}} dg(t).M_n(t) = O\left(\int_0^{\frac{1}{n}}|dg(t)|.|M_n(t)|\right) = O\left(n.\int_0^{\frac{1}{n}}|dg(t)|\right) = O\left(n.o\left(\frac{\frac{1}{n}.\alpha\left(\frac{1}{\frac{1}{n}}\right)}{\log\frac{1}{\frac{1}{n}}}\right)\right)$$

$$= O\left(n.o\left(\frac{\alpha(n)}{n.\log n}\right)\right) = o\left(\frac{\alpha(n)}{\log n}\right) = o(1) \ ; \ (n \to \infty)$$

Lastly, by the Riemann-Lebesgue theorem and the regularity condition of matrix sumability, we obtain

$$I_3 \le |I_3| = \int_\delta^\pi |M_n(t)|.|dg(t)| = o(1),\ \text{as}\ (n \to \infty)$$
$$I_2 \le \left|\int_{\frac{1}{n}}^{\delta} dg(t).M_n(t)\right| = O\left(\int_{\frac{1}{n}}^{\delta}|dg(t)|.|M_n(t)|\right) = O\left(\int_{\frac{1}{n}}^{\delta}\frac{A_{n,\tau}}{t}.|dg(t)|\right)$$

Integrating by parts, where $u = \frac{A_{n,\tau}}{t}$, $dv = dg(t)$. Therefore

$$I_2 = O\left(\frac{A_{n,\tau}}{t}.o\left(\frac{t.\alpha\left(\frac{1}{t}\right)}{\log\frac{1}{t}}\right)\Bigg|_{\frac{1}{n}}^{\delta}\right) + O\left(\int_{\frac{1}{n}}^{\delta}\frac{A_{n,\tau}}{t^2}.o\left(\frac{t.\alpha\left(\frac{1}{t}\right)}{\log\frac{1}{t}}\right)dt\right) + O\left(\int_{\frac{1}{n}}^{\delta}\frac{1}{t}.o\left(\frac{t.\alpha\left(\frac{1}{t}\right)}{\log\frac{1}{t}}\right)d(A_{n,\tau})\right)$$

(Using condition)

$$\Rightarrow I_2 \le o\left(A_{n,\tau}\frac{\alpha\left(\frac{1}{t}\right)}{\log\frac{1}{t}}\Bigg|_{\frac{1}{n}}^{\delta}\right) + o\left(\frac{\alpha(n)}{n.\log n}.\int_{\frac{1}{n}}^{\delta}\frac{A_{n,\tau}}{t^2}dt\right) + o\left(\frac{\alpha(n)}{n.\log n}.\int_{\frac{1}{n}}^{\delta}\frac{d(A_{n,\tau})}{t}\right)$$

where

$$\frac{t.\alpha\left(\frac{1}{t}\right)}{\log\frac{1}{t}} = \frac{\alpha\left(\frac{1}{t}\right)}{\frac{1}{t}.\log\frac{1}{t}}\ \text{increases monotonically as}\ t \to +0.$$

$$\Rightarrow I_2 \le o(1) + o\left(A_{n,n}.\frac{\alpha(n)}{\log n}\right) + o\left(\frac{\alpha(n)}{n.\log n}.\int_{\frac{1}{\delta}}^{n}A_{n,u}du\right) + o\left(\frac{\alpha(n)}{n.\log n}.\int_{\frac{1}{\delta}}^{n}u.d(A_{n,u})\right)$$

$$\Rightarrow I_2 = o(1) + o\left(O\left(\frac{\alpha(n)}{\log n}\right)\right) + o\left(\frac{\alpha(n)}{n.\log n}.(u.A_{n,u})\Big|_{\frac{1}{\delta}}^{n}\right) + o\left(\frac{\alpha(n)}{n.\log n}.\int_{\frac{1}{n}}^{\delta}u.d(A_{n,u})\right) + o\left(\frac{\alpha(n)}{n.\log n}.\int_{\frac{1}{\delta}}^{n}u.d(A_{n,u})\right)$$

Integrating by parts, where $u_1 = A_{n,u}$, $dv_1 = du$

$$\Rightarrow I_2 \le o(1) + o\left(\frac{\alpha(n)}{\log n}\right) + o\left(\frac{\alpha(n)}{n.\log n}.n.A_{n,n}\right) + o(1) + o\left(\frac{\alpha(n)}{n.\log n}.\int_{\frac{1}{\delta}}^{n}u.d(A_{n,u})\right) = o(1) + o\left(\frac{\alpha(n)}{\log n}\right) + o\left(\frac{\alpha(n)}{\log n}\right) + o\left(\frac{\alpha(n)}{n.\log n}.n.\int_{\frac{1}{\delta}}^{n}d(A_{n,u})\right)$$
$$\Rightarrow I_2 \le o(1) + o\left(\frac{\alpha(n)}{\log n}\right) + o\left(\frac{\alpha(n)}{\log n}.A_{n,n}\right)\ ;\ \ A_{n,n} = 1$$
$$= o(1) + o(1) + o(1) = o(1)\ ;\ (n \to \infty)$$

Then

$$\sum_{k=0}^{n} a_{n,n-k}\left\{\acute{s}_{n-k}(x) - \acute{f}(x)\right\} = o(1)\ ;\ (n \to \infty) \Rightarrow \acute{t}_n(x) - \acute{f}(x) = o(1)\ ;\ (n \to \infty)$$

where

$$\acute{t}_n(x) = \sum_{k=0}^{n} a_{n,n-k}.\acute{s}_{n-k}(x)$$

Next

$$\lim_{n\to\infty}\acute{t}_n(x) = \acute{f}(x)$$

This completes the proof of the theorem.

5. Conclusions

One of the most important outcomes of this study is that the product of any two matrix methods of the methods of summability is a matrix method and that this method is a bounded linear operator which transforms each sequence of a given space to a sequence of the space itself. And

$$t_n^{AB} \neq t_n^{A.B}$$

where

$$t_n^{AB} = \sum_{k=0}^{n} a_{n,k} \sum_{r=0}^{k} b_{k,r} s_r = \sum_{k=0}^{n} \sum_{r=0}^{k} a_{n,k} b_{k,r} s_r \, , \, t_n^{A.B} = \sum_{r=0}^{k} \sum_{k=0}^{n} a_{n,k} b_{k,r} s_r$$

The third characteristic of the matrix method showed that, no matter how different the method used to collect the studied series, we would obtain the same sum for that series. We have demonstrated two theorems. The first speaks of the sum of Fourier series using product matrix methods, and the second speaks of the sum of a Fourier series derivative using a matrix method only.

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Weak Soft Binary Structures

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Abstract – The main aim of this paper is to introduce a single structure which carries the subsets of X as well as the subsets of Y under the parameter E for studying the information about the ordered pair of soft subsets of X and Y. Such a structure is called a binary soft structure from X to Y. The purpose of this paper is to introduce certain binary soft weak axioms that are analogous to the axioms of topology.

Keywords – *Binary soft topology, binary soft weak open sets, binary soft weak closed sets, binary soft weak separation axioms and binary soft T_0 space with respect to coordinates.*

1. Introduction

The concept of soft sets was first introduced by Molodtsov [3] in 1999 as a general mathematical technique for dealing with uncertain substances. In [3,4] Molodtsov magnificently applied the soft theory in numerous ways, such as smoothness of functions, game theory, operations research, Riemann integration, Perron integration, probability, theory of measurement, and so on. Point soft set topology deals with a non-empty set X to gether with a collection τ of sub set X under some set of parameters satisfying certain conditions. Such a collection τ is called a soft topological structure on X.

In 2016 Acikgöz and Tas [1] introduced the notion of binary soft set theory on two master sets and studied some basic characteristics. In prolongation, Benchalli *et al*. [2] planned the idea of binary soft topology and linked fundamental properties which are defined over two master sets with appropriate parameters. Benchalli *et al.*, [6] threw his detailed discussion on binary soft topological Kalaichelvi and Malini [7] beautifully discussed.

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Application of fuzzy soft sets to investment decision and also discussed some more results related to this particular field. Özgür and Taş [8] studied some more applications of fuzzy soft sets to investment decision making problem. Taş *et al.* [9] worked over an application of soft set and fuzzy soft set theories to stock management Alcantud *et al.* [10] carefully discussed valuation fuzzy soft sets: A Flexible fuzzy soft set-based decision-making procedure for the valuation of assets [11] Çağman and Enginoğlu attractively explored soft matrix theory and some very basic results related to it and its decision making.

In continuation, in the present paper binary soft topological structures known as soft weak structures with respect to first coordinate as well as with respect to second coordinate are defined. Moreover, some basic results related to these structures are also planted in this paper. The same structures are defined over soft points of binary soft topological structure and related results are also reflected here with respect to ordinary and soft points.

2. Preliminaries

Definition 2.1. [5]. Let X be an initial universe and let E be a set of parameters. Let $\mathrm{P}(\mathrm{X})$ denote the power set of X and let A be a non-empty subset of E. A pair (F, A) iscalled a soft set overX, where F is a mapping given by: $\mathrm{A} \rightarrow \mathrm{P}(\mathrm{X})$. In other words, a soft set over X is a parameterized family of subsets of the universe X. For $\varepsilon \in \mathrm{A}$, F ($\varepsilon$) may be considered as the set of ε-approximate elements of the soft set (F, A). Clearly, a soft set is not a set.

Let $\mathrm{U}_1, \mathrm{U}_2$ be two initial universe sets and E be a set of parameters.

Let$\mathrm{P}(\mathrm{U}_1)$, $\mathrm{P}(\mathrm{U}_2)$ denote the power set of $\mathrm{U}_1, \mathrm{U}_2$ respectively. Also, let$\mathrm{A}, \mathrm{B}, \mathrm{C} \subseteq$ E.

Definition 2.2. [1]. A pair (F, A) is said to be a binary soft set over $\mathrm{U}_1, \mathrm{U}_2$ where F is defined as below:

$\mathrm{F}: \mathrm{A} \rightarrow \mathrm{P}(\mathrm{U}_1) \times \mathrm{P}(\mathrm{U}_2)$, $\mathrm{F}(\mathrm{e}) = (\mathrm{X}, \mathrm{Y})$ for each $\mathrm{e} \in \mathrm{A}$ such that$\mathrm{X} \subseteq \mathrm{U}_1$, $\mathrm{Y} \subseteq \mathrm{U}_2$

Definition 2.3. [1]. A binary soft set (F, A) over $\mathrm{U}_1, \mathrm{U}_2$ is called a binary absolute soft set, denoted by$\tilde{\tilde{\mathrm{A}}}$ if F (e) = $(\mathrm{U}_1, \mathrm{U}_2)$ for each$\mathrm{e} \in$ A.

Definition 2.4. [1]. The intersection of two binary soft sets of (F, A) and (G, B) over the common $\mathrm{U}_1, \mathrm{U}_2$ is the binary soft set (H, C), where $\mathrm{C} = \mathrm{A} \cap \mathrm{B}$ and for all $\mathrm{e} \in \mathrm{C}$

$$\mathrm{H}(\mathrm{e}) = \begin{cases} (\mathrm{X}_1, \mathrm{Y}_1) \text{ if } \mathrm{e} \in \mathrm{A} - \mathrm{B} \\ (\mathrm{X}_2, \mathrm{Y}_2) \text{ if } \mathrm{e} \in \mathrm{B} - \mathrm{A} \\ (\mathrm{X}_1 \cup \mathrm{X}_2, \mathrm{Y}_1 \cup \mathrm{Y}_2) \text{ if } \mathrm{e} \in \mathrm{A} \cap \mathrm{B} \end{cases}$$

Such that $\mathrm{F}(\mathrm{e}) = (\mathrm{X}_1, \mathrm{Y}_1)$ for each $\mathrm{e} \in \mathrm{A}$ and $\mathrm{G}(\mathrm{e}) = (\mathrm{X}_2, \mathrm{Y}_2)$ for each$\mathrm{e} \in$ B. We denote it $(\mathrm{F}, \mathrm{A}) \,\tilde{\tilde{\cup}}\, (\mathrm{G}, \mathrm{A}) = (\mathrm{H}, \mathrm{C})$.

Definition 2.5. [1]. The intersection of two binary soft sets (F, A) and (G, B) over a common $\mathrm{U}_1, \mathrm{U}_2$ is the binary soft set (H, C), where $\mathrm{C} = \mathrm{A} \cap \mathrm{B}$ and $\mathrm{H}(\mathrm{e}) = (\mathrm{X}_1 \cap \mathrm{X}_2, \mathrm{Y}_1 \cap \mathrm{Y}_2)$) for each

$e \in C$ such that $F(e) = (X_1, Y_1)$ for each $e \in A$ and $G(e) = (X_2, Y_2)$for each$e \in B$. We denote it as $(F, A) \,\tilde{\tilde{\cap}}\, (G, B) = (H, C)$.

Definition 2.6. [1]. [1] Let (F, A) and (G, B) be two binary soft sets over a common U_1, U_2. (F, A) is called a binary soft subset of (G, B) if

(i) $A \subseteq B$,
(ii) $X_1 \subseteq X_2$ and $Y_1 \subseteq Y_2$ Such that $F(e) = (X_1, Y_1)$, $G(e) = (X_2, Y_2)$ for each$e \in A$.

We denote it as $(F, A) \,\tilde{\tilde{\subseteq}}\, (G, B)$.

Definition 2.7. [1]. A binary soft set (F, A) over U_1, U_2 is called a binary null soft set, denoted by if $F(e) = (\varphi, \varphi)$ for each$e \in A$.

Definition 2.8. [1]. The difference of two binary soft sets (F, A) and (G, A) over the common U_1, U_2 is the binary soft set (H, A), where $H(e)\ (X_1 - X_2, Y_1 - Y_2)$ for each $e \in A$ such that$(F, A) = (X_1, Y_1)$ and $(G, A) = (X_2, Y_2)$.

Definition 2.9. [2]. Let $\tau_\triangle$ be the collection of binary soft sets over U_1, U_2 then $\tau_\triangle$ is said to be a binary soft topology on U_1, U_2 if

(i) $\tilde{\tilde{\varphi}}, \tilde{\tilde{X}} \in \tau_\triangle$
(ii) The union of any member of binary soft sets in $\tau_\triangle$ belongs to $\tau_\triangle$.
(iii) The intersection of any two binary soft sets in $\tau_\triangle$ belongs to $\tau_\triangle$.

Then $(U_1, U_2, \tau_\triangle, E)$ is called a binary soft topological space over U_1, U_2 .

3. Weak Soft Binary Separation Axioms

This section id devoted to binary soft set and related results. Moreover, binary soft weak separation axioms in binary soft topological spaces are reflected.

Definition 3.1. Let (F, A) be any binary soft sub set of a binary soft topological space (X, Y, τ, E) then (F, A) is called

1) Binary soft b-open set of (X, Y, τ, E) if $(F, A) \subseteq cl(int((F, A) \cup in(cl((F, A)$ and
2) Binary soft b-closed set of (X, Y, τ, E) if $(F, A) \supseteq cl(int(F, A))) \cap in(cl(F, A))))$

The set of all binary b-open soft sets is denoted by BSBO (U) and the set of all binary b-closed sets is denoted by BSBO (U).

Definition 3.2. A binary soft topological space $(\tilde{X}, \tilde{Y}, \mathcal{M}, A)$ is called a binary soft b-T_0 space if for any two binary soft points $(x_1, y_1), (x_2, y_2) \tilde{\mathcal{E}} (\tilde{X}, \tilde{Y})$ such that $x_1 > x_2$, $y_1 > y_2$ there exists binary soft b-open sets (F_1, A) and (F_2, A) which behaves as$(x_1, y_1) \,\tilde{\tilde{\in}}\, (F_1, A)$, $(x_2, y_2) \,\tilde{\tilde{\notin}}\, (F_1, A)$ or $(x_2, y_2) \,\tilde{\tilde{\in}}\, (F_2, A)$ and $(x_1, y_1) \,\tilde{\tilde{\notin}}\, (F_2, A)$.

Definition 3.3. A binary soft topological space $(\widetilde{X},\widetilde{Y},\mathcal{M},A)$ is called a binary soft b- T_1 space if for any two binary soft points $(x_1,y_1),(x_2,y_2)\tilde{\mathcal{E}}(\widetilde{X},\widetilde{Y})$ such that $x_1 > x_2$, $y_1 > y_2$ If there exists binary soft b-open sets (F_1,A) and (F_2,A) which behaves as$(x_1,y_1) \tilde{\tilde{\in}} (F_1,A)$ and $(x_2,y_2) \tilde{\tilde{\notin}} (F_1,A)$ and $(x_2,y_2) \tilde{\tilde{\in}} (F_2,A)$ and $(x_1,y_1) \tilde{\tilde{\notin}} (F_2,A)$.

Definition 3.4. Two binary soft b-open sets $((F,A),(G,A))$ and $(H,A),(I,A)$ are said to be disjoint if $\big((F,A) \sqcap (H,A),(G,A) \sqcap (I,A)\big) = (\Phi,\Phi)$. That is $(F,A) \sqcap (H,A) = (\Phi,\Phi)$ and $(G,A) \sqcap (I,A) = (\Phi,\Phi)$.

Definition 3.5. A binary soft topological space $(\widetilde{X},\widetilde{Y},\mathcal{M},A)$ is called a binary soft b-T_2 space if for any two binary soft points $(x_1,y_1),(x_2,y_2)\tilde{\mathcal{E}}(\widetilde{X},\widetilde{Y})$ such that $x_1 > x_2$, $y_1 > y_2$ If there exists binary soft b-open sets (F_1,A) and (F_2,A) which behaves as$(x_1,y_1) \tilde{\tilde{\in}} (F_1,A)$ and $(x_2,y_2) \tilde{\tilde{\in}} (F_2,A)$ and moreover $(F_1,A)$and(F_2,A) are disjoint that is $(F_1,A) \sqcap (F_2,A) = (\Phi,\Phi)$.

Definition 3.6. A binary soft topological space $(\widetilde{X},\widetilde{Y},\tau\times\sigma,A)$ is called a binary soft b-T_0 with respect to the first coordinate if for every pair of binary points $(x_1,\alpha),(y_1,\alpha)$ there exists $\big((F,A),(G,A)\big)\tilde{\mathcal{E}}\tau\times\sigma$ with$x_1\tilde{\mathcal{E}}(F,A), y_1 \tilde{\notin} (F,A)$,$\alpha\tilde{\mathcal{E}}(G,A)$.where b-open (F,A)`in τ and b-open (G,A)in σ.

Definition 3.7. A binary soft topological space $(\widetilde{X},\widetilde{Y},\tau\times\sigma,A)$ is called a binary soft b-T_0 with respect to the second coordinate if for every pair of binary points $(\beta,x_2),(\beta,y_2)$ there exists $\big((F,A),(G,A)\big)\tilde{\mathcal{E}}\tau\times\sigma$ with$\beta\tilde{\mathcal{E}}(F,A), x_2\tilde{\mathcal{E}}(G,A)$,$y_2 \tilde{\notin} (G,A)$. where b- open (F,A)`in $\tau$ and b-`open (G,A)in σ.

Definition 3.8. A binary soft topological space $(\widetilde{X},\widetilde{Y},\mathcal{M},A)$ is called a binary soft b- T_0 space if for any two binary soft points $\big(e_{\mathbb{G}_1},e_{\mathbb{H}_1}\big),\big(e_{\mathbb{G}_2},e_{\mathbb{H}_2}\big)\tilde{\mathcal{E}}(\widetilde{X_A},\widetilde{Y_A})$ such that $e_{\mathbb{G}_1} > e_{\mathbb{G}_2}, e_{\mathbb{H}_1} > e_{\mathbb{H}_2}$there exists binary soft b-open sets (F_1,A) and (F_2,A) which behaves as$\big(e_{\mathbb{G}_1},e_{\mathbb{H}_1}\big) \tilde{\tilde{\in}} (F_1,A)$, $\big(e_{\mathbb{G}_2},e_{\mathbb{H}_2}\big) \tilde{\tilde{\notin}} (F_1,A)$ or $\big(e_{\mathbb{G}_2},e_{\mathbb{H}_2}\big) \tilde{\tilde{\in}} (F_2,A)$ and $\big(e_{\mathbb{G}_1},e_{\mathbb{H}_1}\big) \tilde{\tilde{\notin}} (F_2,A)$.

Definition 3.9. A binary soft topological space $(\widetilde{X},\widetilde{Y},\mathcal{M},A)$ is called a binary soft b-T_1 space if for any two binary soft points $\big(e_{\mathbb{G}_1},e_{\mathbb{H}_1}\big),\big(e_{\mathbb{G}_2},e_{\mathbb{H}_2}\big)\tilde{\mathcal{E}}(\widetilde{X_A},\widetilde{Y_A})$ such that $e_{\mathbb{G}_1} > e_{\mathbb{G}_2}, e_{\mathbb{H}_1} > e_{\mathbb{H}_2}$ If there exists binary soft b-open sets (F_1,A) and (F_2,A) which behaves as $\big(e_{\mathbb{G}_1},e_{\mathbb{H}_1}\big) \tilde{\tilde{\in}} (F_1,A)\ and\ \big(e_{\mathbb{G}_2},e_{\mathbb{H}_2}\big) \tilde{\tilde{\notin}} (F_1,A)$ and $\big(e_{\mathbb{G}_2},e_{\mathbb{H}_2}\big) \tilde{\tilde{\in}} (F_2,A)$ and $\big(e_{\mathbb{G}_1},e_{\mathbb{H}_1}\big) \tilde{\tilde{\notin}} (F_2,A)$.

Definition 3.10. A binary soft topological space $(\tilde{X},\tilde{Y},\mathcal{M},A)$ is called a binary soft b- T_2 space if for any two binary soft points $\big(e_{\mathbb{G}_1},e_{\mathbb{H}_1}\big),\big(e_{\mathbb{G}_2},e_{\mathbb{H}_2}\big)\tilde{\mathcal{E}}(\widetilde{X_A},\widetilde{Y_A})$ such that $e_{\mathbb{G}_1} > e_{\mathbb{G}_2}, e_{\mathbb{H}_1} > e_{\mathbb{H}_2}$ If there exists binary soft b-open sets (F_1,A) and (F_2,A) which behaves as $\big(e_{\mathbb{G}_1},e_{\mathbb{H}_1}\big) \tilde{\tilde{\in}} (F_1,A)$ and $\big(e_{\mathbb{G}_2},e_{\mathbb{H}_2}\big) \tilde{\tilde{\in}} (F_2,A)$ and moreover $(F_1,A)and(F_2,A)$ are disjoint.

Definition 3.11. A binary soft topological space $(\tilde{X},\tilde{Y},\tau\times\sigma,A)$ is called a binary soft b- T_0 with respect to the first coordinate if for every pair of binary points $(e_{\mathbb{G}_1},\alpha),(e_{\mathbb{H}_1},\alpha)$ there exists $\big((F,A),(G,A)\big)\tilde{\mathcal{E}}\tau\times\sigma$ with $e_{\mathbb{G}_1}\tilde{\mathcal{E}}(F,A), e_{\mathbb{H}_1} \tilde{\notin} (F,A)$,$\alpha\tilde{\mathcal{E}}(G,A)$ where b-open (F,A)`in τ and b-open (G,A)in σ.

Definition 3.12. A binary soft topological space $(\tilde{X}, \tilde{Y}, \tau \times \sigma, A)$ is called a binary soft b- T_0 with respect to the second coordinate if for every pair of binary points $(\beta, e_{\mathbb{G}_2}), (\beta, e_{\mathbb{H}_2})$ there exists $\big((F, A), (G, A)\big)\tilde{\mathcal{E}}\tau \times \sigma$ with$\beta\tilde{\mathcal{E}}(F, A), e_{\mathbb{G}_2}\tilde{\mathcal{E}}(G, A)$,$e_{\mathbb{H}_2} \tilde{\notin} (G, A)$. where b- open (F, A)`in τ and b-open (G, A)in σ.

4. Soft Binary Structures with Respect to Ordinary Points

Theorem 4.1. If the binary soft topological space $(\tilde{X}, \tilde{Y}, \rho \times \sigma, A)$ is a binary soft b- T_0, then $(\tilde{X}, \rho, A)$ and $(\tilde{Y}, \sigma, A)$ are soft b- T_0.

Proof. We suppose $(\tilde{X}, \tilde{Y}, \rho \times \sigma, A)$ is a binary soft b- T_0. Suppose $x_1, x_2\tilde{\mathcal{E}}\tilde{X}$ and $y_1, y_2\tilde{\mathcal{E}}\tilde{Y}$ with such that $x_1 > x_2$, $y_1 > y_2$. Since $(\tilde{X}, \tilde{Y}, \rho \times \sigma, A)$ is a binary soft b- T_0, accordingly there binary soft b-open set $((F, A), (G, A))$ such that

$$(x_1, y_1)\tilde{\mathcal{E}}((F, A), (G, A)) \,;\, (x_2, y_2)\tilde{\mathcal{E}}(F^C, A), (G^C, A)$$

or

$$(x_1, y_1)\tilde{\mathcal{E}}((F^C, A), (G^C, A)) \,;\, (x_2, y_2)\tilde{\mathcal{E}}((F, A), (G, A))$$

This implies that either $x_1\tilde{\mathcal{E}}(F, A); x_2\tilde{\mathcal{E}}(F^C, A); y_1\tilde{\mathcal{E}}(G, A); y_2\tilde{\mathcal{E}}(G^C, A);$ or $x_1\tilde{\mathcal{E}}(F^C, A); y_1\tilde{\mathcal{E}}(G^C, A);$ $y_2\tilde{\mathcal{E}}(G, A)$. This implies either $x_1\tilde{\mathcal{E}}(F, A)\,; x_2\tilde{\mathcal{E}}(F^C, A)$or $x_1\tilde{\mathcal{E}}(F^C, A)$; $x_1\tilde{\mathcal{E}}(F, A)$ and either $y_1\tilde{\mathcal{E}}(G, A)$; $y_2\tilde{\mathcal{E}}(G^C, A)ory_1\tilde{\mathcal{E}}(G^C, A)$; $y_2\tilde{\mathcal{E}}(G, A)$. Since $\big((F, A), (G, A)\big)\tilde{\mathcal{E}}\rho \times \sigma$, We have b-open $(F, A)\tilde{\mathcal{E}}\rho$ and b-open$(F, A)\tilde{\mathcal{E}}\,\sigma$. this proves that $(\tilde{X}, \rho, A)$ and $(\tilde{Y}, \sigma, A)$ are soft b-T_0.

Theorem 4.2. A binary soft topological space $(\tilde{X}, \tilde{Y}, \tau \times \sigma, A)$ is binary soft b-T_0 space with respect to first and the second coordinates, then $(\tilde{X}, \tilde{Y}, \tau \times \sigma, A)$is binary soft b-$T_0$ space.

Proof. Let $(\tilde{X}, \tilde{Y}, \tau \times \sigma, A)$ is binary soft b-T_0 space with respect to first and the second coordinates. Let $(x_1, y_1), (x_2, y_2)\tilde{\mathcal{E}}X \times Y$ with $x_1 > x_2$, $y_1 > y_2$. Take $\alpha\tilde{\mathcal{E}}Y$ and $\beta\tilde{\mathcal{E}}X$.Then $(x_1, \alpha), (x_2, \alpha)\tilde{\mathcal{E}}X \times Y$.Since $(\tilde{X}, \tilde{Y}, \tau \times \sigma, A)$ is a binary soft b-T_0space with respect to the first coordinate, by using definition, there exists b-open sets (F, A) such that (G, A) $\big((F, A), (G, A)\big)\tilde{\mathcal{E}}\tau \times \sigma$ with $x_1\tilde{\mathcal{E}}(F, A)$, $x_2 \tilde{\notin} (F, A)$, $\alpha\tilde{\mathcal{E}}(G, A)$. Since$(\beta, y_1), (\beta, y_2)\tilde{\mathcal{E}}X \times Y$, by using the arguments and using definition there exists $\big((H, A), (K, A)\big)\tilde{\mathcal{E}}\tau \times \sigma$ with $y_1\tilde{\mathcal{E}}(K, A), y_1 \tilde{\notin} (K, A), \widetilde{\beta\mathcal{E}}(H, A)$. Therefore, $(x_1, y_1)\ \tilde{\mathcal{E}}\big((F, A), (K, A)\big)$ and $(x_2, y_2)\ \tilde{\mathcal{E}}((F^C, A), (K^C, A))$. Hence $(\tilde{X}, \tilde{Y}, \tau \times \sigma, A)$ is called a binary soft b- T_0.

Theorem 4.3. A binary soft topological space $(\tilde{X}, \tau, A)$ and $(\tilde{Y}, \sigma, A)$ are soft b-T_1spaces if and only if the binary soft topological space $(\tilde{X}, \tilde{Y}, \tau \times \sigma, A)$ is soft binary b-T_1.

Proof. Suppose $(\tilde{X}, \tau, A)$ and $(\tilde{X}, \sigma, A)$ are soft b-T_1 spaces. Let $(x_1, y_1), (x_2, y_2)\tilde{\mathcal{E}}X \times Y$ with $x_1 > x_2$, $y_1 > y_2$. since $(\tilde{X}, \tau, A)$ is soft b-T_1 space, there exists soft b-open sets such that $(F, A), (G, A)\tilde{\mathcal{E}}\tau$, $x_1\mathcal{E}(F, A)$ and $x_2\mathcal{E}(G, A)$ such that $x_1 \tilde{\notin} (G, A)$ and $x_2 \tilde{\notin} (F, A)$. Also, since $(\tilde{Y}, \sigma, A)$ is soft b-T_1space, there exists soft b-open sets such that $(H, A), (I, A)\tilde{\mathcal{E}}\sigma$, $y_1\mathcal{E}(H, A)$ and $y_2\mathcal{E}(I, A)$ such that $y_1 \tilde{\notin} (I, A)$ and $y_2 \tilde{\notin} (H, A)$.thus $(x_1, y_1)\mathcal{E}\big((F, A), (H, A)\big)$and $(x_2, y_2)\mathcal{E}\big((G, A), (I, A)\big)$ with $(x_1, y_1)\mathcal{E}\big((G^C, A), (I^C, A)\big)$ and

$(x_1, y_1)\mathcal{E}\big((F^C, A), (H^C, A)\big)$.This implies that $(\tilde{X}, \tilde{Y}, \tau \times \sigma, A)$ is soft binary b-T_1. Conversely assume that $(\tilde{X}, \tilde{Y}, \tau \times \sigma, A)$ is soft binary b-T_1. Let $x_1, x_2 \mathcal{E} X$ and $y_1, y_2 \mathcal{E} Y$ such that $x_1 > x_2$, $y_1 > y_2$. Therefore $(x_1, y_1), (x_2, y_2)\tilde{\mathcal{E}} X \times Y$. Since $(\tilde{X}, \tilde{Y}, \tau \times \sigma, A)$ is soft binary b-T_1, there exists b-open sets $(F, A), (G, A)$ and b-open sets $(H, A), (I, A)\mathcal{E}(\tau \times \sigma)$, $(x_1, y_1)\tilde{\mathcal{E}}((F, A), (G, A))$ and $(x_1, y_1)\tilde{\mathcal{E}}((H, A), (I, A))$ such that $(x_1, y_1)\mathcal{E}(H^C, \widetilde{A), (I^C}, A)$ and $(x_2, y_2)\tilde{\mathcal{E}}\big((F^C, A), (G^C, A)\big)$. Therefore, $x_1\mathcal{E}(F, A), x_2\mathcal{E}(H, A)$ and $x_1\mathcal{E}(H^C, A)$ and $x_2\mathcal{E}(F^C, A)$ and , $y_1\mathcal{E}(G^C, A)$ and $y_1\mathcal{E}(I, A)$ and , $y_1\mathcal{E}(I^C, A)$ and $y_2\mathcal{E}(G^C, A)$. Since $(F, A), (G, A)\tilde{\mathcal{E}}\tau \times \sigma$, We have$(F, A), (H, A)\mathcal{E}\tau$ and $(G, A), (I, A)\mathcal{E}\sigma$. This proves that $(\tilde{X}, \tau, A)$ and $(\tilde{X}, \sigma, A)$ are soft b-T_1spaces.

Theorem 4.4. A binary soft topological space $(\tilde{X}, \tilde{Y}, \mathcal{M}, A)$ is binary soft b-T_1 space if and only if every binary soft point $\wp(X) \times \wp(Y)$ is binary soft b-closed.

Proof. Suppose that $(\tilde{X}, \tilde{Y}, \mathcal{M}, A)$ is binary soft b-T_1 space. Let $(x, y)\tilde{\mathcal{E}} X \times Y$. Let $(\{x\}, \{y\})\tilde{\mathcal{E}}\wp(X) \times \wp(Y)$.We shall show that $(\{x\}, \{y\})$ is binary soft b-closed.it is sufficient to show that $(X\backslash\{x\}, Y\backslash\{y\})$ is binary soft b-open. Let $(a, b)\mathrm{E}(X\backslash\{x\}, Y\backslash\{y\})$. This implies that $a\tilde{\mathcal{E}} X\backslash\{x\}$ and $b\tilde{\mathcal{E}} Y\backslash\{y\}$. hence $a \neq x\ and\ b \neq y$ .That is, (a, b) and $(x, y)$are distinct binary soft points of $X \times Y$. Since $(\tilde{X}, \tilde{Y}, \mathcal{M}, A)$ is binary soft b-T_1 space, there exists binary soft b-open sets $((F, A), (G, A))$ and $(H, A), (I, A)$ such that $(a, b)\mathrm{E}((F, A), (G, A))$ and $(x, y)\mathrm{E}((H, A), (I, A))$ such that $(a, b)\mathrm{E}((H^C, A), (I^C, A))$ and $(x, y)\mathrm{E}((F^C, A), (G^C, A))$. Therefore, $\big((F, A), (G, A)\big) \subseteq (\{x\}^c, \{y\}^c)$. Hence$(\{x\}^c, \{y\}^c)$ is a soft neighbourhood of (a, b).This implies that $(\{x\}, \{y\})$is binary soft b- closed. Conversely, suppose that $(\{x\}, \{y\})$is binary soft b-closed for every $(x, y)\mathrm{E} X \times Y$.Suppose $(x_1, y_1), (x_2, y_2)\tilde{\mathcal{E}} X \times Y$ with $x_1 > x_2$, $y_1 > y_2$.Therefore, $(x_2, y_2)\mathcal{E}(\{x_1\widetilde{\}^c, \{y_1}\}^c)$ and $\mathcal{E}$` $(\{x_1\}^c, \{y_1\}^c)$is binary soft b-open. Also $(x_1, y_1)\mathcal{E}(\{x_2\widetilde{\}^c, \{y_2}\}^c)$ and $\mathcal{E}(\{x_1\widetilde{\}^c, \{y_1}\}^c)$ is binary soft b-open set. Also $(x_1, y_1)\mathcal{E}(\{x_2\widetilde{\}^c, \{y_2}\}^c)$ and $\mathcal{E}(\{x_2\widetilde{\}^c, \{y_2}\}^c)$ is binary soft b-open set. This shows that $(\tilde{X}, \tilde{Y}, \mathcal{M}, A)$ is binary soft b-T_1 space.

Theorem 4.5. A binary soft topological space $(\tilde{X}, \tau, A)$ and $(\tilde{Y}, \sigma, A)$ are soft b-T_2 spaces if and only if the binary soft topological space $(\tilde{X}, \tilde{Y}, \tau \times \sigma, A)$ is soft binary b-T_2.

Proof. Suppose $(\tilde{X}, \tau, A)$ and $(\tilde{X}, \sigma, A)$ are soft b-T_2 spaces. Let $(x_1, y_1), (x_2, y_2)\tilde{\mathcal{E}} X \times Y$ with $x_1 > x_2$, $y_1 > y_2$. Since $(\tilde{X}, \tau, A)$ is soft b-T_2 space, there exists soft b-open sets such that $(F, A), (G, A)\tilde{\mathcal{E}}\tau$, $x_1\mathcal{E}(F, A)$ and $x_2\mathcal{E}(G, A)$ such that $x_1 \tilde{\notin} (G, A)$ and $x_2 \tilde{\notin} (F, A)$. Also, since $(\tilde{Y}, \sigma, A)$ is soft b-T_2space, there exists distoint soft b-open sets such that $(H, A), (I, A)\tilde{\mathcal{E}}\sigma, y_1\mathcal{E}(H, A)$ and $y_2\mathcal{E}(I, A)$ such that $y_1 \tilde{\notin} (I, A)$ and $y_2 \tilde{\notin} (H, A)$. Thus $(x_1, y_1)\mathcal{E}\big((F, A), (H, A)\big)$ and $(x_2, y_2)\mathcal{E}\big((G, A), (I, A)\big)$ with $(x_1, y_1)\mathcal{E}\big((G^C, A), (I^C, A)\big)$ and $(x_1, y_1)\mathcal{E}\big((F^C, A), (H^C, A)\big)$. Since $(F, A)\ and\ (G, A)$ are disjoint, $(F, A) \sqcap (H, A) = (\Phi, \Phi)$. Also, since $(H, A) \sqcap (I, A) = (\Phi, \Phi)$. Thus $\big((F, A) \sqcap (H, A), (G, A) \sqcap (I, A)\big) = (\Phi, \Phi)$. This implies that we have this implies that $(\tilde{X}, \tilde{Y}, \tau \times \sigma, A)$ is soft binary b-T_2. Conversely assume that $(\tilde{X}, \tilde{Y}, \tau \times \sigma, A)$ is soft binary b-T_2. Let $x_1, x_2\mathcal{E} X$ and $y_1, y_2\mathcal{E} Y$ such that $x_1 > x_2$, $y_1 > y_2$. Therefore $(x_1, y_1), (x_1, y_1)\tilde{\mathcal{E}} X \times Y$. Since $(\tilde{X}, \tilde{Y}, \tau \times \sigma, A)$ is soft binary b-T_2, there exists binary soft b-open sets $(F, A), (G, A)$ and there exists binary soft b-open sets $(H, A), (I, A)\mathcal{E}(\tau \times \sigma)$, $(x_1, y_1)\tilde{\mathcal{E}}((F, A), (G, A))$ and $(x_2, y_2)\tilde{\mathcal{E}}((\mathrm{H}, A), (I, A))$ such that $(x_1, y_1)\mathcal{E}(H^C, \widetilde{A), (I^C}, A)$ and $(x_2, y_2)\tilde{\mathcal{E}}\big((F^C, A), (G^C, A)\big)$. Therefore, $x_1\mathcal{E}(F, A), x_2\mathcal{E}(H, A)$ and $x_1\mathcal{E}(H^C, A)$ and $x_2\mathcal{E}(F^C, A)$ and $y_1\mathcal{E}(G^C, A)$ and $y_2\mathcal{E}(I, A)$ and , $y_1\mathcal{E}(I^C, A)$ and $y_2\mathcal{E}(G^C, A)$. Since

$(F,A),(G,A)\tilde{\mathcal{E}}\tau\times\sigma$, We have$(F,A),(H,A)\mathcal{E}\tau$ and $(G,A),(I,A)\mathcal{E}\sigma$. This proves that $(\tilde{X},\tau,A)$ and $(\tilde{X},\sigma,A)$ are soft b-T_2spaces.

5. Soft Binary Structures with respect to Soft Points

Theorem 5.1. If the binary soft topological space $(\tilde{X},\tilde{Y},\rho\times\sigma,A)$ is a binary soft b-T_0, then $(\tilde{X},\rho,A)$ and $(\tilde{Y},\sigma,A)$ are soft b- T_0.

Proof. We suppose $(\widetilde{X_A},\tilde{\tilde{Y}}_A,\rho\times\sigma,A)$ is a binary soft b-T_0. Suppose $e_{\mathbb{G}_1},e_{\mathbb{G}_2}\tilde{\mathcal{E}}\widetilde{X_A}$and $e_{\mathbb{H}_1},e_{\mathbb{H}_2}\tilde{\mathcal{E}}\widetilde{Y_A}$ with such that $e_{\mathbb{G}_1}>e_{\mathbb{G}_2},e_{\mathbb{H}_1}>e_{\mathbb{H}_2}$.Since $(\widetilde{X_A},\tilde{\tilde{Y}}_A,\rho\times\sigma,A)$ is a binary soft b-T_0,accordingly there binary soft b-open set $((F,A),(G,A))$ such that

$$(e_{\mathbb{G}_1},e_{\mathbb{H}_1})\tilde{\mathcal{E}}((F,A),(G,A))\ ;(e_{\mathbb{G}_2},e_{\mathbb{H}_2})\tilde{\mathcal{E}}(F^C,A),(G^C,A)$$

or

$$(e_{\mathbb{G}_1},e_{\mathbb{H}_1})\tilde{\mathcal{E}}((F^C,A),(G^C,A));(e_{\mathbb{G}_2},e_{\mathbb{H}_2})\tilde{\mathcal{E}}((F,A),(G,A))$$

This implies that either $e_{\mathbb{G}_1}\tilde{\mathcal{E}}(F,A);e_{\mathbb{G}_2}\tilde{\mathcal{E}}(F^C,A);e_{\mathbb{H}_1}\tilde{\mathcal{E}}(G,A);e_{\mathbb{H}_2}\tilde{\mathcal{E}}(G^C,A)$ or $e_{\mathbb{G}_1}\tilde{\mathcal{E}}(F^C,A);e_{\mathbb{H}_1}\tilde{\mathcal{E}}(G^C,A);\ e_{\mathbb{H}_2}\tilde{\mathcal{E}}(G,A)$. This implies either $e_{\mathbb{G}_1}\tilde{\mathcal{E}}(F,A)\ ;e_{\mathbb{G}_2}\tilde{\mathcal{E}}(F^C,A)$ or $e_{\mathbb{G}_1}\tilde{\mathcal{E}}(F^C,A);\ x_1\tilde{\mathcal{E}}(F,A)$and either $e_{\mathbb{H}_1}\tilde{\mathcal{E}}(G,A);\ e_{\mathbb{H}_2}\tilde{\mathcal{E}}(G^C,A)ore_{\mathbb{H}_1}\tilde{\mathcal{E}}(G^C,A);\ e_{\mathbb{H}_2}\tilde{\mathcal{E}}(G,A)$. Since $\big((F,A),(G,A)\big)\tilde{\mathcal{E}}\rho\times\sigma$, We have b-open $(F,A)\tilde{\mathcal{E}}\rho$ and b-open$(F,A)\tilde{\mathcal{E}}\ \sigma$. this proves that $(\tilde{X},\rho,A)$ and $(\tilde{Y},\sigma,A)$ are soft b- T_0.

Theorem 5.2. A binary soft topological space $(\tilde{X},\tilde{Y},\tau\times\sigma,A)$ is binary soft b-T_0 space with respect to first and the second coordinates, then $(\tilde{X},\tilde{Y},\tau\times\sigma,A)$is binary soft b-$T_0$ space.

Proof. Let $(\tilde{X},\tilde{Y},\tau\times\sigma,A)$ is binary soft b-T_0 space with respect to first and the second coordinates. Let $(e_{\mathbb{G}_1},e_{\mathbb{H}_1}),(e_{\mathbb{G}_2},e_{\mathbb{H}_2})\tilde{\mathcal{E}}X\times Y$ with $e_{\mathbb{G}_1}>e_{\mathbb{G}_2}\ ,e_{\mathbb{H}_1}>e_{\mathbb{H}_2}$. Take $\alpha\tilde{\mathcal{E}}Y$ and $\beta\tilde{\mathcal{E}}X$.Then $(e_{\mathbb{G}_1},\alpha),(e_{\mathbb{G}_2},\alpha)\tilde{\mathcal{E}}X\times Y$.Since $(\tilde{X},\tilde{Y},\tau\times\sigma,A)$ is a binary soft b-T_0space with respect to the first coordinate, by using definition, there exists b-open sets $\big((F,A),(G,A)\big)\tilde{\mathcal{E}}\tau\times\sigma$ with $e_{\mathbb{G}_1}\tilde{\mathcal{E}}(F,A),\ e_{\mathbb{G}_2}\tilde{\notin}(F,A),\ \alpha\tilde{\mathcal{E}}(G,A)$. Since$\big(\beta,e_{\mathbb{H}_1}\big),\big(\beta,e_{\mathbb{H}_2}\big)\tilde{\mathcal{E}}X\times Y$, by using the arguments and using definition there exists b-open sets $\big((H,A),(K,A)\big)\tilde{\mathcal{E}}\tau\times\sigma$ with $e_{\mathbb{H}_1}\tilde{\mathcal{E}}(K,A),e_{\mathbb{H}_1}\tilde{\notin}(K,A),\widetilde{\beta\mathcal{E}}(H,A)$. Therefore, $(e_{\mathbb{G}_1},e_{\mathbb{H}_1})\ \tilde{\mathcal{E}}\big((F,A),(K,A)\big)$ and $(e_{\mathbb{G}_2},e_{\mathbb{H}_2})\ \tilde{\mathcal{E}}((F^C,A),(K^C,A))$. Hence $(\tilde{X},\tilde{Y},\tau\times\sigma,A)$ is called a binary soft b-T_0.

Theorem 5.3. A binary soft topological space $(\tilde{X},\tau,A)$ and $(\tilde{Y},\sigma,A)$ are soft b-T_1spaces if and only if the binary soft topological space $(\tilde{X},\tilde{Y},\tau\times\sigma,A)$ is soft binary b-T_1.

Proof. Suppose $(\tilde{X},\tau,A)$ and $(\tilde{X},\sigma,A)$ are soft b-T_1spaces. Let $(e_{\mathbb{G}_1},e_{\mathbb{H}_1})\ ,(e_{\mathbb{G}_2},e_{\mathbb{H}_2})\ \tilde{\mathcal{E}}X\times Y$ with $e_{\mathbb{G}_1}>e_{\mathbb{G}_2}\ ,e_{\mathbb{H}_1}>e_{\mathbb{H}_2}$. since $(\tilde{X},\tau,A)$ is soft b-T_1space, there exists soft b-open sets such that $(F,A),(G,A)\tilde{\mathcal{E}}\tau,\ e_{\mathbb{G}_1}\mathcal{E}(F,A)$ and $e_{\mathbb{G}_2}\mathcal{E}(G,A)$ such that $e_{\mathbb{G}_1}\tilde{\notin}(G,A)$ and $e_{\mathbb{G}_2}\tilde{\notin}(F,A)$. Also, since $(\tilde{Y},\sigma,A)$ is soft b-T_1space, there exists soft b-open sets such that $(H,A),(I,A)\tilde{\mathcal{E}}\sigma,e_{\mathbb{H}_1}\mathcal{E}(H,A)$ and $e_{\mathbb{H}_2}\mathcal{E}(I,A)$ such that $e_{\mathbb{H}_1}\tilde{\notin}(I,A)$ and $e_{\mathbb{H}_2}\tilde{\notin}(H,A)$.thus $(e_{\mathbb{G}_1},e_{\mathbb{H}_1})\mathcal{E}\big((F,A),(H,A)\big)$and $(e_{\mathbb{G}_2},e_{\mathbb{H}_2})\mathcal{E}\big((G,A),(I,A)\big)$ with $(e_{\mathbb{G}_1},e_{\mathbb{H}_1})\mathcal{E}\big((G^C,A),(I^C,A)\big)$

and $(e_{\mathbb{G}_1}, e_{\mathbb{H}_1})\mathcal{E}\big((F^C, A), (H^C, A)\big)$.This implies that $(\tilde{X}, \tilde{Y}, \tau \times \sigma, A)$ is soft binary b-T_1. Conversely assume that $(\tilde{X}, \tilde{Y}, \tau \times \sigma, A)$ is soft binary b-T_1. Let $e_{\mathbb{G}_1}, e_{\mathbb{G}_2}\mathcal{E}X$ and $e_{\mathbb{H}_1}, e_{\mathbb{H}_2}\mathcal{E}Y$ such that $e_{\mathbb{G}_1} > e_{\mathbb{G}_2}$, $e_{\mathbb{H}_1} > e_{\mathbb{H}_2}$. Therefore $(e_{\mathbb{G}_1}, e_{\mathbb{H}_1}), (e_{\mathbb{G}_2}, e_{\mathbb{H}_2})\tilde{\mathcal{E}}X \times Y$. Since $(\tilde{X}, \tilde{Y}, \tau \times \sigma, A)$ is soft binary b-T_1, there exists b-open sets $(F, A), (G, A)$ and b-open sets $(H, A), (I, A)\mathcal{E}(\tau \times \sigma)$, $(e_{\mathbb{G}_1}, e_{\mathbb{H}_1})\tilde{\mathcal{E}}((F, A), (G, A))$ and $(e_{\mathbb{G}_2}, e_{\mathbb{H}_2})\tilde{\mathcal{E}}((H, A), (I, A))$ such that $(e_{\mathbb{G}_1}, e_{\mathbb{H}_1})\mathcal{E}(H^C, \widetilde{A), (}I^C, A)$ and $(e_{\mathbb{G}_2}, e_{\mathbb{H}_2})\tilde{\mathcal{E}}\big((F^C, A), (G^C, A)\big)$. Therefore, $e_{\mathbb{G}_1}\mathcal{E}(F, A)$, $e_{\mathbb{G}_2}\mathcal{E}(H, A)$ and $e_{\mathbb{G}_1}\mathcal{E}(H^C, A)$ and $e_{\mathbb{G}_2}\mathcal{E}(F^C, A)$ and , $y_1\mathcal{E}(G^C, A)$ and $e_{\mathbb{H}_1}\mathcal{E}(I, A)$ and , $e_{\mathbb{H}_1}\mathcal{E}(I^C, A)$ and , $e_{\mathbb{H}_2}\mathcal{E}(G^C, A)$. Since $(F, A), (G, A)\tilde{\mathcal{E}}\tau \times \sigma$, We have$(F, A), (H, A)\mathcal{E}\tau$ and $(G, A), (I, A)\mathcal{E}\sigma$. This proves that $(\tilde{X}, \tau, A)$ and $(\tilde{X}, \sigma, A)$ are soft b-T_1spaces.

Theorem 5.4. A binary soft topological space $(\tilde{X}, \tilde{Y}, \mathcal{M}, A)$ is binary soft b-T_1 space if and only if every binary soft point $\wp(X) \times \wp(Y)$ is binary soft b-closed.

Proof. Suppose that $(\tilde{X}, \tilde{Y}, \mathcal{M}, A)$ is binary soft b- T_1 space. Let $(x, y)\tilde{\mathcal{E}}X \times Y$. Let $(\{x\}, \{e_{\mathbb{H}}\})\tilde{\mathcal{E}}\wp(X) \times \wp(Y)$.We shall show that $(\{e_{\mathbb{G}}\}, \{e_{\mathbb{H}}\})$ is binary soft b-closed. It is sufficient to show that $(X\backslash\{e_{\mathbb{G}}\}, Y\backslash\{e_{\mathbb{H}}\})$ is binary soft b-open. Let $(a, b)\mathcal{E}(X\backslash\{e_{\mathbb{G}}\}, Y\backslash\{e_{\mathbb{H}}\})$. This implies that $a\tilde{\mathcal{E}}X\backslash\{e_{\mathbb{G}}\}$ and $b\tilde{\mathcal{E}}Y\backslash\{e_{\mathbb{H}}\}$. Hence $a \neq e_{\mathbb{G}}\ and\ b \neq e_{\mathbb{H}}$. That is, (a, b) and $(e_{\mathbb{G}}, e_{\mathbb{H}})$ are distinct binary soft points of $X \times Y$. Since $(\tilde{X}, \tilde{Y}, \mathcal{M}, A)$ is binary soft b- T_1 space, there exists binary soft b-open sets $((F, A), (G, A))$ and $(H, A), (I, A)$ such that $(a, b)\mathcal{E}((F, A), (G, A))$ and $(x, y)\mathcal{E}((H, A), (I, A))$ such that $(a, b)\mathcal{E}((H^C, A), (I^C, A))$ and $(e_{\mathbb{G}}, e_{\mathbb{H}})\mathcal{E}((F^C, A), (G^C, A))$. Therefore, $\big((F, A), (G, A)\big) \subseteq (\{e_{\mathbb{G}}\}^c, \{e_{\mathbb{H}}\}^c)$. Hence$(\{e_{\mathbb{G}}\}^c, \{e_{\mathbb{H}}\}^c)$ is a soft neighbourhood of (a, b). This implies that $(\{e_{\mathbb{G}}\}, \{e_{\mathbb{H}}\})$is binary soft b-closed. Conversely, suppose that $(\{e_{\mathbb{G}}\}, \{e_{\mathbb{H}}\})$ is binary soft b-closed for every $(e_{\mathbb{G}}, e_{\mathbb{H}})\mathcal{E}X \times Y$. Suppose $(e_{\mathbb{G}_1}, e_{\mathbb{H}_1})$, $(e_{\mathbb{G}_2}, e_{\mathbb{H}_2})\tilde{\mathcal{E}}X \times Y$with$e_{\mathbb{G}_1} > e_{\mathbb{G}_2}$, $e_{\mathbb{H}_1} > e_{\mathbb{H}_2}$ Therefore, $(e_{\mathbb{G}_2}, e_{\mathbb{H}_2})\mathcal{E}(\{e_{\mathbb{G}_1}\widetilde{\}^c, \{}e_{\mathbb{H}_1}\}^c)$ and $(\{e_{\mathbb{G}_1}\}^c, \{e_{\mathbb{H}_1}\}^c)$ is binary soft b-open. Also $(e_{\mathbb{G}_1}, e_{\mathbb{H}_1})\mathcal{E}(\{e_{\mathbb{G}_2}\widetilde{\}^c, \{}e_{\mathbb{H}_2}\}^c)$ and $\mathcal{E}(\{e_{\mathbb{G}_1}\widetilde{\}^c, \{}e_{\mathbb{H}_1}\}^c)$ is binary soft b-open set. Also $(e_{\mathbb{G}_1}, e_{\mathbb{H}_1})\mathcal{E}(\{e_{\mathbb{G}_2}\widetilde{\}^c, \{}e_{\mathbb{H}_2}\}^c)$ and $\mathcal{E}(\{e_{\mathbb{G}_2}\widetilde{\}^c, \{}e_{\mathbb{H}_2}\}^c)$ is binary soft b-open set. This shows that $(\tilde{X}, \tilde{Y}, \mathcal{M}, A)$ is binary soft b-T_1 space.

Theorem 5.5. A binary soft topological space $(\tilde{X}, \tau, A)$ and $(\tilde{Y}, \sigma, A)$ are soft b-T_2spaces if and only if the binary soft topological space $(\tilde{X}, \tilde{Y}, \tau \times \sigma, A)$ is soft binary b-T_2.

Proof. Suppose $(\tilde{X}, \tau, A)$ and $(\tilde{X}, \sigma, A)$ are soft b-T_2spaces. Let $(e_{\mathbb{G}_1}, e_{\mathbb{H}_1}), (e_{\mathbb{G}_2}, e_{\mathbb{H}_2})\tilde{\mathcal{E}}X \times Y$ with$e_{\mathbb{G}_1} > e_{\mathbb{G}_2}$, $e_{\mathbb{H}_1} > e_{\mathbb{H}_2}$. Since $(\tilde{X}, \tau, A)$ is soft b-T_2space, there exists soft b-open sets such that $(F, A), (G, A)\tilde{\mathcal{E}}\tau$, $e_{\mathbb{G}_1}\mathcal{E}(F, A)$ and $e_{\mathbb{G}_2}\mathcal{E}(G, A)$ such that $e_{\mathbb{G}_1} \tilde{\notin} (G, A)$ and $e_{\mathbb{G}_2} \tilde{\notin} (F, A)$. Also, since $(\tilde{Y}, \sigma, A)$ is soft b-T_2 space, there exists disjoint soft b-open sets such that$(H, A), (I, A)\tilde{\mathcal{E}}\sigma$, $e_{\mathbb{H}_1}\mathcal{E}(H, A)$ and $e_{\mathbb{H}_2}\mathcal{E}(I, A)$ such that $e_{\mathbb{H}_1} \tilde{\notin} (I, A)$ and $e_{\mathbb{H}_2} \tilde{\notin} (H, A)$.thus $(e_{\mathbb{G}_1}, e_{\mathbb{H}_1})\mathcal{E}\big((F, A), (H, A)\big)$and $(e_{\mathbb{G}_2}, e_{\mathbb{H}_2})\mathcal{E}\big((G, A), (I, A)\big)$ with $(e_{\mathbb{G}_1}, e_{\mathbb{H}_1})\mathcal{E}\big((G^C, A), (I^C, A)\big)$ and $(e_{\mathbb{G}_1}, e_{\mathbb{H}_1})\mathcal{E}\big((F^C, A), (H^C, A)\big)$.Snce$(F, A)\ and\ (G, A)$are disjoint, $(F, A) \sqcap (H, A) = (\Phi, \Phi)$. Also, since$(H, A) \sqcap (I, A) = (\Phi, \Phi)$. Thus$\big((F, A) \sqcap (H, A), (G, A) \sqcap (I, A)\big) = (\Phi, \Phi)$. This implies that we have this implies that $(\tilde{X}, \tilde{Y}, \tau \times \sigma, A)$ is soft binary b-T_2. Conversely assume that $(\tilde{X}, \tilde{Y}, \tau \times \sigma, A)$ is soft binary b-T_2. Let $e_{\mathbb{G}_1}, e_{\mathbb{G}_2}\mathcal{E}X$ and $e_{\mathbb{H}_1}, e_{\mathbb{H}_2}\mathcal{E}Y$ such that $e_{\mathbb{G}_1} > e_{\mathbb{G}_2}$, $e_{\mathbb{H}_1} > e_{\mathbb{H}_2}$Therefore $(e_{\mathbb{G}_1}, e_{\mathbb{H}_1}), (e_{\mathbb{G}_2}, e_{\mathbb{H}_2})\tilde{\mathcal{E}}X \times Y$. Since $(\tilde{X}, \tilde{Y}, \tau \times \sigma, A)$ is soft binary b-T_2, there exists binary soft b-open sets $(F, A), (G,$A) and there exists binary soft b-open sets $(\mathrm{H},\mathrm{A}),(\mathrm{I},\mathrm{A})\mathcal{E}(\tau\times\sigma)$, $(e_{\mathbb{G}_1},e_{\mathbb{H}_1})\tilde{\mathcal{E}}((\mathrm{F},\mathrm{A}),(\mathrm{G},\mathrm{A}))$ and $(e_{\mathbb{G}_2},e_{\mathbb{H}_2})\tilde{\mathcal{E}}((\mathrm{H},\mathrm{A}),(\mathrm{I},\mathrm{A}))$ such that $(e_{\mathbb{G}_1},e_{\mathbb{H}_1})\mathcal{E}(\mathrm{H}^{\mathrm{C}},\widetilde{\mathrm{A}),(}\mathrm{I}^{\mathrm{C}},\mathrm{A})$ and $(e_{\mathbb{G}_2},e_{\mathbb{H}_2})\tilde{\mathcal{E}}\left((\mathrm{F}^{\mathrm{C}},\mathrm{A}),(\mathrm{G}^{\mathrm{C}},\mathrm{A})\right)$. Therefore, $e_{\mathbb{G}_1}\mathcal{E}(\mathrm{F},\mathrm{A})$, $e_{\mathbb{G}_2}\mathcal{E}(\mathrm{H},\mathrm{A})$ and $e_{\mathbb{G}_1}\mathcal{E}(\mathrm{H}^{\mathrm{C}},\mathrm{A})$ and $e_{\mathbb{G}_2}\mathcal{E}(\mathrm{F}^{\mathrm{C}},\mathrm{A})$ and , $e_{\mathbb{H}_1}\mathcal{E}(\mathrm{G}^{\mathrm{C}},\mathrm{A})$ and $e_{\mathbb{H}_2}\mathcal{E}(\mathrm{I},\mathrm{A})$ and , $e_{\mathbb{H}_1}\mathcal{E}(\mathrm{I}^{\mathrm{C}},\mathrm{A})$ and , $e_{\mathbb{H}_2}\mathcal{E}(\mathrm{G}^{\mathrm{C}},\mathrm{A})$. Since $(\mathrm{F},\mathrm{A}),(\mathrm{G},\mathrm{A})\tilde{\mathcal{E}}\tau\times\sigma$, We have$(\mathrm{F},\mathrm{A}),(\mathrm{H},\mathrm{A})\mathcal{E}\tau$ and $(\mathrm{G},\mathrm{A}),(\mathrm{I},\mathrm{A})\mathcal{E}\sigma$. This proves that $(\widetilde{\mathrm{X}},\tau,\mathrm{A})$ and $(\widetilde{\mathrm{X}},\sigma,\mathrm{A})$ are soft b-T_2spaces.

6. Conclusion

The soft separation axioms namely b-T_o, b-T_1 and b-T_2 are extended to binary soft b-T_o, b-T_1 and b-T_2 structures. b-$\mathrm{T}_o$space with respect to first and second co-ordinates are beautifully reflected.

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*ψ**-Locally Closed Sets and *ψ**-Locally Closed Continuous Functions in Topological Spaces

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Abstract – In this paper, we introduce ψ^*-locally closed sets and different notions of generalizations of continuous functions in a topological space and study some of their properties. Several examples are given to illustrate the behavior of these new classes of functions. Also, we define ψ^*-submaximal spaces.

Keywords – *ψ^*-locally closed sets; ψ^*LC-continuous functions; ψ^*LC-irresolute functions; ψ^*-submaximal spaces.*

1. Introduction

Allah and Nawar [1] introduced the concept of ψ^*-closed sets. The notion of locally closed sets in a topological space was introduced by Bourbaki [2]. Ganster and Reilly [5] further studied the properties of locally closed sets and defined the LC-continuity and LC-irresoluteness. Gnanambal [6] introduced the concept of α-locally closed sets and αLC-continuous functions and investigated some of their properties. In this paper, we introduce ψ^*LC-sets, ψ^*LC*-sets and ψ^*LC**-sets by using the notion of ψ^*-closed and ψ^*-open sets and study some of their properties. Finally, we also introduce and study different classes of weaker forms of continuity and irresoluteness and some of their properties in topological spaces.

2. Preliminaries

Throughout this paper (X, τ), (Y, σ) and (Z, η) represent non-empty topological spaces on which no separation axioms are assumed unless or otherwise mentioned. For a subset A of a space (X, τ), $cl(A)$ and $int(A)$ denote the closure of A and the interior of A, respectively.

Let us recall the following definitions, which are useful in the sequel.

Definition 2.1 A subset A of a topological space (X, τ) is called:
(1) Generalized α-closed [7] (briefly gα-closed) if $\alpha cl(A) \subseteq U$ whenever $A \subseteq U$ and U is α-open in (X, τ). The complement of gα-closed set is called gα-open.
(2) ψ*-closed [1] if $\alpha cl(A) \subseteq U$ whenever $A \subseteq U$ and U is gα-open in (X, τ). The complement of ψ*-closed set is called ψ*-open.
(3) Locally closed (briefly LC) set [5] if $A = G \cap F$, where G is open and F is closed in (X, τ).
(5) α-Locally closed (briefly αLC) set [6] if $A = G \cap F$, where G is α-open and F is α-closed in (X, τ).

Definition 2.2 A topological space (X, τ) is called:
(1) submaximal space [3] if every dense subset of (X, τ) is open in (X, τ).
(2) α-submaximal space [6] if every dense subset of (X, τ) is α-open in (X, τ).
(3) door space [4] if every subset of (X, τ) is either open or closed in (X, τ).
(4) $T_{1/5}^{\psi^*}$ space [1] if every ψ*-closed set is α-closed.

Definition 2.3 A function $f : (X, \tau) \rightarrow (Y, \sigma)$ is called:
(1) LC-continuous [5] if $f^{-1}(V)$ is locally closed set in (X, τ) for each closed set V of (Y, σ).
(2) αLC-continuous [6] if $f^{-1}(V)$ is α-locally closed set in (X, τ) for each closed set V of (Y, σ).
(3) LC-irresolute [5] if $f^{-1}(V)$ is locally closed set in (X, τ) for each locally closed set V of (Y, σ).
(4) αLC-irresolute [6] if $f^{-1}(V)$ is α-locally closed set in (X, τ) for each α–locally closed set V of (Y, σ).

3. ψ*-Locally Closed Sets

In this section, we introduce three weak types of locally closed sets denoted by ψ*LC(X, τ), ψ*LC*(X, τ) and ψ*LC**(X, τ) each of which contains LC(X, τ) and obtain some of their properties. Also, we introduce ψ*-submaximal spaces and obtain some of their properties.

Definition 3.1 A subset A of a topological space (X, τ) is called an ψ*-locally closed set (briefly, ψ*LC-set) if $A = G \cap F$, where G is ψ*-open and F is ψ*-closed in (X, τ). The class of all ψ*-locally closed subsets of (X, τ) is denoted by ψ*LC(X, τ).

Remark 3.1 The following are well known
(i) A subset A of (X, τ) is ψ*LC-set if and only if it's complement $X–A$ is the union of an ψ*-open and an ψ*-closed set.
(ii) Every ψ*-open (resp. ψ*-closed) subset of (X, τ) is an ψ*LC-set.

Theorem 3.1 Every locally closed set is an ψ*LC-set but not conversely.

Proof. The proof follows from the fact that every closed (resp. open) set is an ψ*-closed (resp. ψ*-open).

Example 3.1 Let X = {a, b, c, d} with τ = {X, ϕ, {a}, {b}, {a, b}, {a, b, c}}. Then a subset B = {a, b, d} $\notin$ LC(X) but $B \in \psi$*LC (X).

Theorem 3.2 Every α-locally closed set is an ψ*LC-set but not conversely.

Proof. The proof follows from the fact that every α-closed (resp. α-open) set is an ψ*-closed (resp. ψ*-open).

Example 3.2 Let X = {a, b, c} with τ = {X, ϕ, {a, b}}. Then a subset B = {a, c} $\notin$ αLC(X) but $B \in \psi$*LC (X).

Definition 3.2 A subset A of a topological space (X, τ) is called an ψ*LC*-set if $A = G \cap F$, where G is ψ*-open and F is closed in (X, τ).

The class of all ψ*LC*-subsets of (X, τ) is denoted by ψ*LC*(X, τ).

Definition 3.3 A subset A of a topological space (X, τ) is called an ψ*LC**-set if $A = G \cap F$, where G is open and F is ψ*-closed in (X, τ).

The class of all ψ*LC**-subsets of (X, τ) is denoted by ψ*LC**(X, τ).

Theorem 3.3 If a subset A of (X, τ) is locally closed, then it is ψ*LC(X, τ), ψ*LC*(X, τ) and ψ*LC**(X, τ).

Proof. Let $A = G \cap F$, where G is open and F is closed in (X, τ). Since every open set is ψ*-open and every closed set is ψ*-closed, it follows that A is ψ*LC(X, τ), ψ*LC*(X, τ) and ψ*LC**(X, τ).

The converse of the above theorem need not be true as seen from the following example.

Example 3.3 In Example 3.2, we have LC(X) = {X, ϕ, {c}, {a, b}}, ψ*LC(X) = {X, ϕ, {a}, {b}, {c}, {a, b}, {a, c}, {b, c}}, ψ*LC*(X) = {X, ϕ, {a}, {b}, {c}, {a, b}} and ψ*LC**(X) = {X, ϕ, {a}, {b}, {c}, {a, b}, {a, c}, {b, c}}. Here, {a} is ψ*LC(X, τ), ψ*LC*(X, τ) and ψ*LC**(X, τ) but not LC(X, τ).

Theorem 3.4 If a subset A of (X, τ) is ψ*LC*(X, τ), then it is ψ*LC(X, τ).

Proof. Let A be an ψ*LC*-set and every closed set is ψ*-closed in (X, τ), we have $A = G \cap F$, where G is ψ*-open and F is ψ*-closed in (X, τ). Therefore, $A \in \psi$*LC (X, τ).

The converse of the above theorem need not be true as seen from the following example.

Example 3.4 In Example 3.2, we have {a, c} $\in$ ψ*LC(X) but {a, c} $\notin$ ψ*LC*(X).

Theorem 3.5 Every ψ*LC**(X, τ) is ψ*LC(X, τ).

Proof. Let A be an ψ*LC**-set and every open set is ψ*-open in (X, τ), we have $A = G \cap F$, where G is ψ*-open and F is ψ*-closed in (X, τ). Therefore, $A \in$ ψ*LC (X, τ).

The converse of the above theorem need not be true as seen from the following example.

Example 3.5 Let X = {a, b, c} with τ = {X, ϕ, {b}}. Then a subset B = {b, c} $\in$ ψ*LC(X) but $B \notin$ ψ*LC**(X).

Theorem 3.6 Every αLC(X, τ) (resp. αLC*(X, τ), αLC**(X, τ)) is ψ*LC(X, τ) (resp. ψ*LC*(X, τ), ψ*LC**(X, τ)).

Proof. Since every α-open set is ψ*-open and every α-closed set is ψ*-closed, the proof follows.

The converse of the above theorem need not be true as seen from the following example.

Example 3.6 In Example 3.2, we have αLC(X) = αLC*(X) = αLC**(X) {X, ϕ, {c}, {a, b}}, ψ*LC*(X) = {X, ϕ, {a}, {b}, {c}, {a, b}} and ψ*LC(X) = ψ*LC**(X) = P(X). Here, {a} is ψ*LC(X, τ) (resp. ψ*LC*(X, τ) and ψ*LC**(X, τ)) but not an αLC(X, τ) (resp. αLC*(X, τ) and αLC**(X, τ)).

Theorem 3.7 If $A \in$ ψ*LC(X, τ) and B is ψ*-open set in (X, τ), then $A \cap B \in$ ψ*LC(X, τ).

Proof. Since $A \in$ ψ*LC (X, τ), there exist an ψ*-open G and an ψ*-closed set F such that $A = G \cap F$. Now, $A \cap B = (G \cap \mathrm{B}) \cap F$. Since $G \cap B$ is ψ*-open and F is ψ*-closed, it follows that $A \cap B \in$ ψ*LC (X, τ).

Remark 3.2 ψ*LC- sets and ψ*LC**-sets are independent of each other as seen from the examples.

Example 3.7 In Example 3.1, the set A = {a, b, d} is ψ*LC-set but not an ψ*LC**-set.

Example 3.8 In Example 3.2, the set A = {a, c} is ψ*LC**-set but not an ψ*LC-set.

Theorem 3.8 For a subset A of a topological space (X, τ), the following are equivalent:
(1) $A \in$ ψ*LC*(X, τ),
(2) $A = G \cap \alpha\mathrm{cl}(A)$ for some ψ*-open set G,
(3) $\alpha\mathrm{cl}(A) - A$ is ψ*-closed,

(4) $A \cup (X-\alpha cl(A))$ is ψ^*-open.

Proof: (1) → (2). Let $A \in \psi^*LC^*(X, \tau)$. Then there exist an ψ^*-open set G and a closed set F of (X, τ) such that $A = G \cap F$. Since $A \subseteq G$ and $A \subseteq \alpha cl(A)$. Therefore, we have $A \subseteq G \cap \alpha cl(A)$.

Conversely, since $\alpha cl(A) \subseteq F$, $G \cap \alpha cl(A) \subseteq G \cap F = A$, which implies that $A = G \cap \alpha cl(A)$.

(2) → (1). Since G is ψ^*-open and $\alpha cl(A)$ is closed $G \cap \alpha cl(A) \in \psi^*LC^*(X, \tau)$, which implies that $A \in \psi^*LC^*(X, \tau)$.

(3) → (4). Let $F = \alpha cl(A) - A$. Then F is ψ^*-closed by the assumption and $X-F = X \cap (\alpha cl(A) - A)^c = A \cup (X-\alpha cl(A))$. But $X-F$ is ψ^*-open. This shows that $A \cup (X-\alpha cl(A))$ is ψ^*-open.

(4) → (3). Let $U = A \cup (X-\alpha cl(A))$. Since U is ψ^*-open set, $X-U$ is ψ^*-closed. $X-U = X-(A \cup (X-\alpha cl(A))) = \alpha cl(A) \cap (X-A) = \alpha cl(A) - A$. Thus, $\alpha cl(A) - A$ is ψ^*-closed set.

(4) → (2). Let $G = A \cup (X-\alpha cl(A))$. Thus, G is ψ^*-open. We prove that $A = G \cap \alpha cl(A)$ for some ψ^*-open G. Since, $G \cap \alpha cl(A) = (A \cup (X-\alpha cl(A)) \cap \alpha cl(A)$ $= (\alpha cl(A) \cap A) \cup (\alpha cl(A) \cap (X-\alpha cl(A)) = A$. Therefore, $A = G \cap \alpha cl(A)$.

(2) → (4). Let $A = G \cap \alpha cl(A)$ for some ψ^*-open G. Then we prove that $A \cup (X-\alpha cl(A))$ is ψ^*-open. Now, $A \cup (X-\alpha cl(A)) = G \cap (\alpha cl(A)) \cap (X-\alpha cl(A)) = G$, which is ψ^*-open. Thus, $A \cup (X-\alpha cl(A))$ is ψ^*-open.

Theorem 3.9 If $A, B \in \psi^*LC^*(X, \tau)$, then $A \cap B \in \psi^*LC^*(X, \tau)$.

Proof. From the assumption, there exist ψ^*-open sets G and H such that $A = G \cap \alpha cl(A)$ and $B = H \cap \alpha cl(B)$. Then $A \cap B = (G \cap H) \cap (\alpha cl(A) \cap \alpha cl(B))$. Since $G \cap H$ is ψ^*-open set and $\alpha cl(A) \cap \alpha cl(B)$ is closed. Therefore, $A \cap B \in \psi^*LC^*(X, \tau)$.

Theorem 3.10 If $A \in \psi^*LC(X, \tau)$ and B is ψ^*-open set in (X, τ), then $A \cap B \in \psi^*LC(X, \tau)$.

Proof. Let $A \in \psi^*LC^*(X, \tau)$. Then $A = G \cap F$ where G is ψ^*-open and F is ψ^*-closed So, $A \cap B = (G \cap B) \cap F$. Since $G \cap B$ is ψ^*-open and F is ψ^*-closed, it follows that $A \cap B \in \psi^*LC(X, \tau)$.

Theorem 3.11 If $A \in \psi^*LC^*(X, \tau)$ and B is ψ^*-open (or closed) set in (X, τ), then $A \cap B \in \psi^*LC^*(X, \tau)$.

Proof. Since $A \in \psi^*LC^*(X, \tau)$, there exist an ψ^*-open G and a closed set F such that $A = G \cap F$. Now, $A \cap B = (G \cap B) \cap F$. Since $G \cap B$ is ψ^*-open and F is closed, it follows that $A \cap B \in \psi$*LC*(X, τ).

In this case, B being a closed set, we have $A \cap B = (G \cap F) \cap B = G \cap (F \cap B)$. Since G is ψ*-open set and $F \cap B$ is closed, $A \cap B \in \psi$*LC*(X, τ).

Theorem 3.12 If $A \in \psi$*LC**(X, τ) and B is ψ*-closed (or open) set in (X, τ), then $A \cap B \in \psi$*LC**(X, τ).

Proof. Since $A \in \psi$*LC**(X, τ), there exist an open set G and an ψ*-closed set F such that $A = G \cap F$. Now, $A \cap B = G \cap (F \cap B)$. Since G is open and $(F \cap B)$ is ψ*-closed, it follows that $A \cap B \in \psi$*LC**(X, τ).

In this case, B being an open set, we have $A \cap B = (G \cap F) \cap B = (G \cap B) \cap F$. Since $G \cap B$ is open set and F is ψ*-closed, then $A \cap B \in \psi$*LC**(X, τ).

Theorem 3.13 Let (X, τ) and (Y, σ) be two topological spaces
(i) If $A \in \psi$*LC(X, τ) and $B \in \psi$*LC(Y, σ), then $A \times B \in \psi$*LC$(X \times Y, \tau \times \sigma)$.
(ii) If $A \in \psi$*LC*(X, τ) and $B \in \psi$*LC*(Y, σ), then $A \times B \in \psi$*LC*$(X \times Y, \tau \times \sigma)$.
(iii) If $A \in \psi$*LC**(X, τ) and $B \in \psi$*LC**(Y, σ), then $A \times B \in \psi$*LC**$(X \times Y, \tau \times \sigma)$,

Proof. Let $A \in \psi$*LC(X, τ) and $B \in \psi$*LC(Y, σ). Then there exist ψ*-open sets M and N of (X, τ) and (Y, σ) and ψ*-closed sets F and K of X and Y respectively, such that $A = M \cap F$ and $B = N \cap K$. Then $A \times B = (M \times N) \cap (F \times K)$ holds. Hence, $A \times B \in \psi$*LC$(X \times Y, \tau \times \sigma)$.

(ii) and (iii) The proofs are similar to (i).

Definition 3.4 A topological space (X, τ) is said to be ψ*-submaximal if every dense subset in it is ψ*-open.

Theorem 3.14 Every submaximal space is ψ*-submaximal.

Proof. Let (X, τ) be a submaximal space and A be a dense subset of (X, τ). Then A is open. But every open set is ψ*-open and so A is ψ*-open. Therefore, (X, τ) is ψ*-submaximal.

The converse of the above theorem need not be true as seen from the following example.

Example 3.9 Let X = {a, b, c, d} with τ = {X, ϕ, {a}, {b}, {a, b}, {a, b, c}}. Then the space (X, τ) is ψ*-submaximal but not submaximal. However, the set A = {a, b, d} is dense in (X, τ), but it is not open in X. Therefore, (X, τ) is not submaximal.

Theorem 3.15 Every α-submaximal space is ψ*-submaximal.

Proof. Let (X, τ) be an α-submaximal space and A be a dense subset of (X, τ). Then A is α-open. But every α-open set is ψ*-open and so A is ψ*-open. Therefore, (X, τ) is ψ*-submaximal.

Theorem 3.16 A topological space (X, τ) is ψ*-submaximal if and only if ψ*LC*(X, τ) = P(X).

Proof. Necessity: Let $A \in$ P(X) and $U = A \cup (X - \alpha cl(A))$. Then $\alpha cl(U) = X$. Since (X, τ) is ψ*-submaximal, U is ψ*-open. By Theorem 3.8, $A \in \psi$*LC*(X, τ) and so P(X) = ψ*LC*(X, τ).

Sufficiency: Let A be a dense subset of (X, τ). Then $A \cup (X - \alpha cl(A)) = A$. Since $A \in \psi$*LC*(X, τ), by Theorem 3.8, A is ψ*-open in (X, τ). Hence, (X, τ) is ψ*-submaximal.

4. ψ*LC-Continuous Functions in Topological Spaces

In this section, we introduce the concepts of ψ*LC-continuous, ψ*LC*-continuous and ψ*LC**-continuous functions which are weaker than LC-continuous functions.

Definition 4.1 A function f: $(X, \tau) \rightarrow (Y, \sigma)$ is called ψ*LC-continuous (resp. ψ*LC*-continuous, ψ*LC**-continuous) if $f^{-1}(V) \in \psi$*LC(X, τ) (resp. $f^{-1}(V) \in \psi$*LC*(X, τ), $f^{-1}(V) \in \psi$*LC**(X, τ)) for each closed set V of (Y, σ).

Theorem 4.1 Let f: $(X, \tau) \rightarrow (Y, \sigma)$ be a function. Then we have the following
(i) If f is LC-continuous, then f is ψ*LC-continuous, ψ*LC*-continuous and ψ*LC**-continuous.
(ii) If f is ψ*LC*-continuous or ψ*LC**-continuous, then f is ψ*LC-continuous.

Proof. (i) Let f be a LC-continuous and V be an open set of (Y, σ). Then $f^{-1}(V)$ is locally closed in (X, τ). Since every locally closed set is ψ*LC-set, ψ*LC*-set and ψ*LC**-set, it follows that f is ψ*LC-continuous, ψ*LC*-continuous and ψ*LC**-continuous.

(ii) Let f: $(X, \tau) \rightarrow (Y, \sigma)$ be an ψ*LC*-continuous or ψ*LC**-continuous function. Since every ψ*LC*-set is ψ*LC-set and every ψ*LC**-set is ψ*LC-set. Therefore, the proof follows.

The converse of the above theorem need not be true as seen from the following example.

Example 4.1 Let $X = Y = \{a, b, c\}$ with $\tau = \{X, \phi, \{b\}\}$ and σ = P(Y). Let f: $(X, \tau) \rightarrow (Y, \sigma)$ be the identity function. Now, LC$(X, \tau) = \{X, \phi, \{b\}, \{a, c\}\}$, ψ*LC(X, τ) = ψ*LC*(X, τ) = ψ*LC**(X, τ) = P(X) and LC(Y, σ) = ψ*LC(Y, σ) = ψ*LC*(Y, σ) = ψ*LC**(Y, σ) = P(Y). Then f is not LC-continuous, since for the closed set $\{b, c\}$, $f^{-1}\{b, c\} = \{b, c\}$ is not locally closed in X, but it is ψ*LC-continuous, ψ*LC*-continuous and ψ*LC**-continuous.

Example 4.2 Let $X = Y = \{a, b, c, d\}$ with $\tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}\}$ and $\sigma = \{Y, \phi, \{c\}\}$. Then the identity function f: $(X, \tau) \rightarrow (Y, \sigma)$ is ψ*LC-continuous but not ψ*LC**-continuous, since for the closed set {a, b, d} of (Y, σ), f^{-1}\{a, b, d\} = {a, b, d} is not ψ*LC**-set in X but it is ψ*LC- set in X.

Theorem 4.2 Let f: $(X, \tau) \rightarrow (Y, \sigma)$ and g: $(Y, \sigma) \rightarrow (Z, \eta)$ be any two functions. Then:
(i) g o f is ψ*LC-continuous if g is continuous and f is ψ*LC-continuous,
(ii) g o f is ψ*LC*-continuous if g is continuous and f is ψ*LC*-continuous,
(iii g o f is ψ*LC**-continuous if g is continuous and f is ψ*LC**-continuous,

Proof. Let V be a closed set in (Z, η) and g be a continuous function. Then $g^{-1}(V)$ is closed set in (Y, σ) and since f is ψ*LC-continuous, we get $f^{-1}(g^{-1}(V))$ is ψ*LC- set in (X, τ). Thus, g o f is ψ*LC-continuous.

(ii) – (iii) Similarly.

Definition 4.2 A function f: $(X, \tau) \rightarrow (Y, \sigma)$ is called ψ*LC-irresolute (resp. ψ*LC*-irresolute, ψ*LC**-irresolute) if $f^{-1}(V) \in \psi$*LC(X, τ) (resp. $f^{-1}(V) \in \psi$*LC*(X, τ), $f^{-1}(V) \in \psi$*LC**(X, τ)) for $V \in \psi$*LC(Y, σ) (resp. $V \in \psi$*LC*(Y, σ), $V \in \psi$*LC**(Y, σ)).

Example 4.3 Let $X = Y$ = {a, b, c} with $\tau = \{X, \phi, \{a\}\}$ and $\sigma = \{Y, \phi, \{a\}, \{b, c\}\}$. Then the identity function f: $(X, \tau) \rightarrow (Y, \sigma)$ is ψ*LC- irresolute, ψ*LC*- irresolute and ψ*LC**- irresolute.

Theorem 4.3 If a function f: $(X, \tau) \rightarrow (Y, \sigma)$ is LC- irresolute, then f is ψ*LC-irresolute, ψ*LC*- irresolute and ψ*LC**- irresolute.

Proof. Let f be a LC- irresolute and V be a LC-set of (Y, σ). Then $f^{-1}(V)$ is LC(X, τ). Since every LC-set is a ψ*LC-set, ψ*LC*-set and ψ*LC**-set, it follows that f is ψ*LC- irresolute, ψ*LC*- irresolute and ψ*LC**- irresolute.

The converse of the above theorem need not be true as seen from the following example.

Example 4.4 As in Example 4.1, the function f is not LC- irresolute, since for the locally closed set {b, c}, f^{-1}\{b, c\} = {b, c} is not locally closed in X. However, f is ψ*LC- irresolute, ψ*LC- irresolute and ψ*LC**- irresolute.

Theorem 4.4 If a function f: $(X, \tau) \rightarrow (Y, \sigma)$ is ψ*LC- irresolute (resp. ψ*LC*-irresolute and ψ*LC**- irresolute), then f is ψ*LC-continuous, ψ*LC*-continuous and ψ*LC**- continuous.

Proof. Since every LC-set is ψ*LC-set, ψ*LC*-set and ψ*LC**-set, the proof follows.

The converse of the above theorem need not be true as seen from the following example.

Example 4.5 Let $X = Y = \{a, b, c\}$ with $\tau = \{X, \phi, \{a\}, \{b, c\}\}$ and $\sigma = \{Y, \phi, \{b\}\}$. Define a function $f: (X, \tau) \rightarrow (Y, \sigma)$ by $f(a) = b$, $f(b) = a$ and $f(c) = c$. Then f is ψ*LC-continuous, ψ*LC*- continuous and ψ*LC**- continuous but not ψ*LC- irresolute, ψ*LC*- irresolute and ψ*LC**- irresolute, since for the ψ*LC-set (resp. ψ*LC*-set and ψ*LC**-set) {a, b}, $f^{-1}\{a, b\} = \{a, b\}$ is not ψ*LC-set (resp. ψ*LC*-set and ψ*LC**-set) in (X, τ).

Theorem 4.5 Let $f: (X, \tau) \rightarrow (Y, \sigma)$ and $g: (Y, \sigma) \rightarrow (Z, \eta)$ be any two functions. Then:
(i) g o f is ψ*LC-continuous if g is ψ*LC-continuous and f is ψ*LC-irresolute,
(ii) g o f is ψ*LC*-continuous if g is ψ*LC**-continuous and f is ψ*LC*-irresolute,
(iii) g o f is ψ*LC**-continuous if g is ψ*LC**-continuous and f is ψ*LC**-irresolute,
(iv) g o f is ψ*LC- irresolute if f and g are ψ*LC-irresolute,
(v) g o f is ψ*LC*- irresolute if f and g are ψ*LC*-irresolute,
(vi) g o f is ψ*LC**- irresolute if f and g are ψ*LC**-irresolute.

Proof. (i) Let V be a closed set in (Z, η) and g be an ψ*LC-continuous function. Then $g^{-1}(V)$ is ψ*LC- set in (Y, σ) and since f is ψ*LC-irresolute, we get $f^{-1}(g^{-1}(V))$ is ψ*LC- set in (X, τ). Thus, g o f is ψ*LC-continuous.

(ii) – (iii) Similar to (i).

(iv) Let V be an ψ*LC-set in (Z, η) and g be an ψ*LC- irresolute function. Then $g^{-1}(V)$ is ψ*LC- set in (Y, σ) and since f is ψ*LC-irresolute, we get $f^{-1}(g^{-1}(V))$ is ψ*LC- set in (X, τ). Thus, g o f is ψ*LC-irresolute.

(v) – (vi) Similar to (iv).

Theorem 4.6 Let $\{Z_i : i \in \tau\}$ be a cover of X, where X is finite set and A be a subset of X. Suppose $\{Z_i : i \in \tau\}$ is ψ*LC- set in X and the collection of ψ*LC- set is closed under finite unions. If $A \cap Z_i \in \psi$*LC**$(Z_i, \tau / Z_i)$ for each $i \in \tau$, then $A \in \psi$*LC**(X, τ).

Proof. Let $i \in \tau$ and since $A \cap Z_i \in \psi$*LC**$(Z_i, \tau / Z_i)$. Then there exist an open set U_i of (X, τ) and ψ*-closed set F_i of $(Z_i, \tau / Z_i)$ such that $A \cap Z_i = (U_i \cap Z_i) \cap F_i = U_i \cap (Z_i \cap F_i)$. Therefore, $A = \bigcup\{A \cap Z_i : i \in \tau\} = \bigcup\{U_i : i \in \tau\} \cap (\bigcup\{Z_i \cap F_i : i \in \tau\})$ and hence $A \in \psi$*LC**(X, τ).

Theorem 4.7 Let $f: (X, \tau) \rightarrow (Y, \sigma)$ be an ψ*-irresolute injective map. Then
(i) If $B \in \psi$*LC(Y, σ), then $f^{-1}(B) \in \psi$*LC(X, τ),
(ii) If X is a $T_{1/5}^{\psi^*}$- space and $B \in \psi$*LC(Y, σ), then $f^{-1}(B) \in \alpha$LC$(X, \tau$.)

Proof. (i) Let $B \in \psi$*LC(Y, σ). Then there exist ψ*-open set G and ψ*-closed set F such that $B = G \cap F$, $f^{-1}(B) = f^{-1}(G) \cap f^{-1}(F)$. Since f is ψ*-irresolute,

$f^{-1}(G)$ and $f^{-1}(F)$ are ψ*-open and ψ*-closed sets in X respectively. Hence, $f^{-1}(B) \in \psi^*LC(X, \tau)$.

(ii) Let $B \in \psi^*LC(Y, \sigma)$. Then there exist ψ*-open set G and ψ*-closed set F such that $B = G \cap F$, $f^{-1}(B) = f^{-1}(G) \cap f^{-1}(F)$. Since f is ψ*-irresolute, $f^{-1}(G)$ and $f^{-1}(F)$ are ψ*-open and ψ*-closed sets in X respectively. From hypothesis $f^{-1}(G)$ and $f^{-1}(F)$ are α-open and α-closed sets in X. Hence, $f^{-1}(B) \in \alpha LC(X, \tau)$.

Theorem 4.8 Any function defined in a door space is ψ*-continuous (resp. ψ*-irresolute).

Proof. Let $f: (X, \tau) \rightarrow (Y, \sigma)$ be a function where (X, τ) is a door space and $A \in (Y, \sigma)$ (resp. $A \in \psi^*LC((Y, \sigma)$). Then $f^{-1}(A)$ is either open or closed. Since every open or closed set is ψ*-open or ψ*-closed respectively and hence $f^{-1}(A) \in \psi^*LC(X, \tau)$. Therefore, f is ψ*-continuous (resp. ψ*-irresolute).

Theorem 4.9 If X is a $T_{1/5}^{\psi^*}$- space, then $\psi^*LC(X, \tau) = \alpha LC(X, \tau)$.

Proof. Let $A \in \psi^*LC(X, \tau)$. Then there exist ψ*-open set G and ψ*-closed set F such that $A = G \cap F$. Since X is a $T_{1/5}^{\psi^*}$-space, then G and F are α-open and α-closed sets respectively and hence $A \in \alpha LC(X, \tau)$. The above implies $\psi^*LC(X, \tau) \subseteq \alpha LC(X, \tau)$.

On the other hand, let $A \in \alpha LC(X, \tau)$. Then $A = G \cap F$, G is α-open set and F is α-closed. But every α-open (resp. α-closed) is ψ*-open (resp. ψ*-closed) Hence, G is ψ*-open set F is ψ*-closed set. The above implies $\alpha LC(X, \tau) \subseteq \psi^*LC(X, \tau)$. Therefore, $\psi^*LC(X, \tau) = \alpha LC(X, \tau)$.

Theorem 4.10 Every αLC-continuous function is ψ*LC-continuous.

Proof. Obvious.

The converse of the above theorem need not be true as shown in the following example.

Example 4.6 Let $X = Y = \{a, b, c\}$ with $\tau = \{X, \phi, \{a, b\}\}$ and $\sigma = \{Y, \phi, \{b\}\}$. Then the identity function $f: (X, \tau) \rightarrow (Y, \sigma)$ is not αLC-continuous since $\{a, c\} \in C(Y)$ but $f^{-1}(\{a, c\}) = \{a, c\} \notin \alpha LC(X)$. However, f is ψ*LC-continuous.

Theorem 4.11 If $f: (X, \tau) \rightarrow (Y, \sigma)$ is ψ*LC-continuous and X is a $T_{1/5}^{\psi^*}$- space, then f is αLC-continuous.

Proof. Let G be an open set of and f be an ψ*LC-continuous. Then $f^{-1}(G)$ is ψ*LC-set in X. Since X is a $T_{1/5}^{\psi^*}$- space, every ψ*-open (resp. ψ*-closed) is α-open (resp. α-closed) in X. Then $f^{-1}(G)$ is ψ*LC-set in Y and hence f is αLC-continuous.

Theorem 4.12 If f: $(X, \tau) \rightarrow (Y, \sigma)$ is αLC-irresolute and g: $(Y, \sigma) \rightarrow (Z, \eta)$ is ψ*LC-continuous and Y is a $T_{1/5}^{\psi^*}$- space, then $g \circ f : (X, \tau) \rightarrow (Z, \eta)$ is αLC- continuous.

Proof. Let F be a closed set of Z and g be an ψ*LC-continuous. Then $g^{-1}(F)$ is ψ*LC-set in Y. Since Y is a $T_{1/5}^{\psi^*}$- space, $g^{-1}(F)$ is αLC-set in Y. Since f is αLC-irresolute, then $f^{-1}(g^{-1}(F))$ is αLC-set in X. Therefore, $(g \circ f)^{-1}(F)$ is αLC-set in X and $g \circ f$ is αLC- continuous.

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Fuzzy Soft Locally Closed Sets in Fuzzy Soft Topological Space

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Abstract - The purpose of this paper is to introduce fuzzy soft locally closed and fuzzy soft b-locally closed sets and study their properties in fuzzy soft topological space. Further we define and study fuzzy soft LC-continuous and fuzzy soft b-LC-continuous functions.

Keywords - *Fuzzy soft locally closed sets, Fuzzy soft b-locally closed set, Fuzzy soft LC-continuous functions.*

1. Introduction

The notion of fuzzy sets for dealing with uncertainties was introduced by Zadeh [15]. Fuzzy topology was introduced by Chang [4]. To overcome difficulties in fuzzy set theory soft sets were introduced in 1999 [11]. The hybridisation of fuzzy set and soft set known as fuzzy soft set was introduced by Maji et.al. [10]. The notion of topological structure of Fuzzy soft sets was introduced by Tanay and Kandemir [13] and studied further by many authors [5,6,12,14]. The concept of fuzzy soft semi open set was introduced by Kandil et al. [8] whereas fuzzy soft pre-open and regular open sets was introduced by Hussain [7] and fuzzy soft b-open sets was introduced by Anil [1]. In this paper we introduce fuzzy soft locally closed and fuzzy soft b-locally closed sets and study their properties. Further we define fuzzy soft LC-continuous and fuzzy soft b-LC-continuous functions and study few of the properties.

2. Preliminaries

Definition2.1 [10] Let X be an initial universal set, I^X be set of all fuzzy sets on X and E be a set of parameters and let $A \subseteq E$. A pair (f, A) denoted by f_A is called fuzzy soft set over X, where f is a mapping given by $f: A \to I^X$ i.e. for each $a \in A$, $f(a) = f_a : X \to I$

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is a fuzzy set on X

Definition2.2 [12] Let τ be a collection of all fuzzy soft sets over a universe X with a fixed parameter set E then (X,τ,E) is called fuzzy soft topological space if i. $\tilde{0}_E,\ \tilde{1}_E \in \tau$ ii. Union of any members of τ is a member of τ, iii. Intersection of any two members of τ is a member of τ. Each member of τ is called fuzzy soft open set i.e. A fuzzy soft set f_A over X is fuzzy soft open if and only if $f_A \in \tau$. A fuzzy soft set f_A over X is called fuzzy soft closed set if the complement of f_A is fuzzy soft open set.

Definition2.3 [14] The fuzzy soft closure of f_A, denoted by $Fscl(f_A)$ is defined as $Fscl(f_A) = \bigcap \{h_D : h_D \textit{ is fuzzy soft closed set and } f_A \subseteq h_D\}$

Definition2.4 [14] The fuzzy soft interior of g_B denoted by $Fs\ int(g_B)$ is defined as $Fs\ int(g_B) = \bigcup \{h_D : h_D \textit{ is fuzzy soft open set and } h_D \subseteq g_B\}$

Definition2.5 [7] Fuzzy soft set f_A of a fuzzy soft topological space (X,τ,E) is called fuzzy soft pre-open set if $f_A \le Fs\,\mathrm{int}\,Fscl(f_A)$ and fuzzy soft pre-closed if $FsclFs\,\mathrm{int}(f_A) \le f_A$

Definition2.6 [7] Fuzzy soft set f_A of a fuzzy soft topological space (X,τ,E) is called fuzzy soft α-open set if $f_A \le Fs\,\mathrm{int}(Fscl(Fs\,\mathrm{int}(f_A)))$

Definition 2.6 [1] A fuzzy soft set f_A in a fuzzy soft topological space (X,τ,E) is called fuzzy soft b-open set if $f_A \le Fs\,\mathrm{int}\,Fscl(f_A) \vee FsclFs\,\mathrm{int}(f_A)$ and fuzzy soft b-closed set if $f_A \ge Fs\,\mathrm{int}\,Fscl(f_A) \vee FsclFs\,\mathrm{int}(f_A)$

Definition 2.7 [1] Let f_A be a fuzzy soft set in a fuzzy soft topological space (X,τ,E) then fuzzy soft b-closure of f_A and fuzzy soft b-interior of f_A are defined as

(i) $fsb-cl\,(f_A) = \bigcap \{g_B : g_B \textit{ is a } fsb-closed\ set\ \&\ g_B \ge f_A\}$

(ii) $fsb-\mathrm{int}\,(f_A) = \bigcup \{h_c : h_c \textit{ is a } fsb-open\,set\ \&\ h_C \le f_A\}$

3. Soft Locally Closed Sets

Definition 3.1. A fuzzy soft set (F, E) is called fuzzy soft locally closed set in a fuzzy soft topological space (X,τ,E) if (F, E) = (G, E)∩(H,E) where (G, E) is fuzzy soft open and (H, E) is fuzzy soft closed in X.

The family of all fuzzy soft locally closed sets of a fuzzy soft topological space (X,τ,E) is denoted by FSLCS (X,τ,E).

Theorem 3.2. In a fuzzy soft topological space (X,τ,E), every fuzzy soft open set is fuzzy soft locally closed.

Proof. Let (F, E) be fuzzy soft open in X, then (F, E) is fuzzy soft locally closed in X, since (F, E)=(F, E) ∩ $\tilde{1}$.

Theorem 3.3. Let (X, τ, E) be a fuzzy soft topological space. If (F_1, E) and (F_2, E) are two fuzzy soft locally closed sets in X then (F_1, E) ∩ (F_2, E) is a fuzzy soft locally closed set in X.

Proof. Let (F_1, E) = (G_1, E) ∩ (H_1, E) and (F_2, E) = (G_2, E) ∩ (H_2, E) where (G_1, E) and (G_2, E) are fuzzy soft open and (H_1, E) and (H_2, E) are fuzzy soft closed in X. Then (F_1, E) ∩ (F_2, E) = ((G_1, E) ∩ (H_1, E)) ∩((G_2, E) ∩ (H_2, E)) = ((G_1, E) ∩ (G_2, E)) ∩((H_1, E) ∩ (H_2, E)), where (G_1, E) ∩ (G_2, E) is fuzzy soft open and (H_1, E) ∩ (H_2, E) is fuzzy soft closed and hence (F_1, E) ∩ (F_2, E) is a fuzzy soft locally closed set in X.

Theorem 3.4. Let (X, τ, E) be a fuzzy soft topological space. Then (F, E) is fuzzy soft locally closed if and only if (F, E) = (G, E)∩Fs-cl(F, E) for some fuzzy soft open set (G, E).

Proof. Let (F, E) be fuzzy soft locally closed set in X. Hence (F, E) = (G, E)∩(H, E) where (G, E) is fuzzy soft open and (H, E) is fuzzy soft closed in X. Then Fs-cl(F,E) = Fs-cl((G, E)∩(H, E)) $\subset$ Fs-cl(G, E)∩ Fs-cl(H, E) = Fs-cl(G, E)∩ (H, E). We have Fs-cl(F,E) $\subset$ (H, E) and hence (F, E) $\subset$ (G, E)∩Fs-cl(F, E) $\subset$ (G, E)∩(H, E) = (F, E). Therefore (F, E) = (G, E)∩Fs-cl(F, E).

Conversely, if (F, E) = (G, E)∩Fs-cl(F, E) for some fuzzy soft open set (G, E) then (F, E) is fuzzy soft locally closed since Fs-cl(F, E) is fuzzy soft closed in X.

Definition 3.5. Let (F, E) and (G, E) be any two fuzzy soft sets. Then (F, E) and (G, E) are said to be separated if (F, E)∩Fs-cl(G, E) = (G, E)∩Fs-cl(F, E) = $\tilde{0}$.

Theorem 3.6. Let (X, τ, E) be a fuzzy soft topological space and (F_1, E) and (F_2, E) are two fuzzy soft locally closed in X. If (F_1, E) and (F_2, E) are separated in X then (F_1, E)⋃ (F_2, E) is a fuzzy soft locally closed in X.

Proof. Since (F_1, E) and (F_2, E) are two fuzzy soft locally closed in X, we have (F_1, E) = (G_1, E)∩Fs-cl(F_1, E) and (F_2, E) = (G_2, E)∩Fs-cl(F_2, E), where (G_1, E) and (G_2, E) are fuzzy soft open in X. Since (F_1, E) and (F_2, E) are separated, we have (F_1, E)∩Fs-cl(F_2, E) = (F_2, E)∩Fs-cl(F_1, E) = $\tilde{0}$ and which implies (F_1, E)⋃(F_2, E) =(G_1, E) ⋃(G_2, E) ∩ Fs-cl((F_1, E) ⋃(F_2, E)). Hence (F_1, E)⋃(F_2, E) is fuzzy soft locally closed set in X.

Theorem 3.7. Let (X, τ, E) be a fuzzy soft topological space. For a fuzzy soft set (F, E) following are equivalent

(i) (F, E) is fuzzy soft open in X
(ii) (F, E) is fuzzy soft α-open and fuzzy soft locally closed
(iii) (F, E) is fuzzy soft pre-open and fuzzy soft locally closed
(iv) (F, E) is fuzzy soft b-open and fuzzy soft locally closed

Proof. (i) implies(ii), (ii) implies (iii) and (iii) implies (iv) are obvious

(iv) Implies (i): Let (F, E) be fuzzy soft b-open and fuzzy soft locally closed set in X. We have (F, E)$\subset$Fs-int(Fs-cl(F, E)) $\cup$ Fs-cl(Fs-int(F, E)) and (F, E) = (G, E)$\cap$Fs-cl(F, E) where (G, E) is fuzzy soft open. Then (F, E)$\subset$(G, E)$\cap$(Fs-int(Fs-cl(F, E)) $\cup$ Fs-cl(Fs-int(F, E)))= ((G, E)$\cap$Fs-int(Fs-cl(F, E))) $\cup$ ((G, E)$\cap$Fs-cl(Fs-int(F, E))) = Fs-int((G, E)$\cap$Fs-cl(F, E)) $\cup$ Fs-int(F, E) = Fs-int(F, E) $\cup$ Fs-int(F, E) = Fs-int(F, E). Hence (F, E) is fuzzy soft open in X.

Definition 3.8. A fuzzy soft set (F, E) is called fuzzy soft b-locally closed set in a fuzzy soft topological space (X,τ,E) if (F, E) = (G, E)$\cap$(H, E) where (G, E) is fuzzy soft b-open and (H, E) is fuzzy soft b-closed in X.

The family of all fuzzy soft b-locally closed sets of a fuzzy soft topological space (X,τ,E) is denoted by FSBLCS (X,τ,E) .

Remark 3.9. It is obvious that every fuzzy soft b-closed set is fuzzy soft b-locally closed set.

Remark 3.10. Every fuzzy soft locally closed set is fuzzy soft b-locally closed set but converse need not be true.

Example 3.11. Let X = {a, b, c}, E = {e$_1$}, $\tau=\{\tilde{1},\tilde{0},(F_1,E),(F_2,E)\}$ where $(F_1,E)=\left\{\left\{\frac{1}{a},\frac{0}{b},\frac{0}{c}\right\}\right\}$ and $(F_2,E)=\left\{\left\{\frac{1}{a},\frac{0}{b},\frac{1}{c}\right\}\right\}$. Clearly the set $(F,E)=\left\{\left\{\frac{1}{a},\frac{1}{b},\frac{0}{c}\right\}\right\}$ is fuzzy soft b-locally closed set but not fuzzy soft locally closed.

Theorem 3.12. Let (X,τ,E) be a fuzzy soft topological space. Then (F, E) is fuzzy soft b-locally closed if and only if (F, E) = (G, E)$\cap$Fsb-cl(F, E) for some fuzzy soft open set (G, E).

Proof. Let (F, E) be fuzzy soft b-locally closed set in X. Hence (F, E) = (G, E)$\cap$(H, E) where (G, E) is fuzzy soft b-open and (H, E) is fuzzy soft b-closed in X. Then Fsb-cl(F,E) $\subset$ (H, E) and hence (F, E) = (F, E)$\cap$Fsb-cl(F, E) = (G, E)$\cap$(H, E)$\cap$ Fsb-cl(F, E) = (G, E)$\cap$Fsb-cl(F, E).

Conversely, if (F, E) = (G, E)$\cap$Fsb-cl(F, E) for some fuzzy soft b- open set (G, E) and since Fsb-cl(F, E) is fuzzy soft closed, hence (F, E) is fuzzy soft b-locally closed in X.

Definition 3.13. Let (X,τ,E) and (Y,σ,K) be fuzzy soft topological spaces and $f: X\rightarrow Y$ be a function. Then f is called a

(i) fuzzy soft locally continuous (LC-continuous) if for each open set (G, K) in Y, $f^{-1}(G,K)$ is a fuzzy soft locally closed set in X.

(ii) fuzzy soft b-locally continuous (b-LC-continuous) if for each open set (G, K) in Y, $f^{-1}(G,K)$ is a fuzzy soft b-locally closed set in X.

(iii) fuzzy soft locally irresolute (LC-irresolute) if for each fuzzy soft locally closed set (G, K) in Y, $f^{-1}(G,K)$ is a fuzzy soft locally closed set in X.

(iv) fuzzy soft b-locally irresolute (b-LC-irresolute) if for each fuzzy soft b-locally closed set (G, K) in Y, $f^{-1}(G,K)$ is a fuzzy soft b-locally closed set in X.

Theorem 3.14. Every fuzzy soft LC- continuous function is fuzzy soft b-LC- continuous.

Proof. Let $f:(X,\tau,E)\to(Y,\sigma,K)$ be fuzzy soft LC- continuous function. Then for any fuzzy soft open set (G, K) in Y, $f^{-1}(G,K)$ is fuzzy soft locally closed in X. We have $f^{-1}(G,K)$ = (G_1, E)∩(H_1, E) where (G_1, E) is fuzzy soft open and (H_1, E) is fuzzy soft closed in X. Since every fuzzy soft open (closed) set is fuzzy soft b-open (b-closed) set. Therefore $f:(X,\tau,E)\to(Y,\sigma,K)$ is b-LC- continuous. Converse of this theorem need not be true as seen from the following example.

Example 3.15. Let X = Y={a, b, c}, E = K = {e_1}, $\tau=\left\{\tilde{1},\tilde{0},(F_1,E),(F_2,E)\right\}$ and $\sigma=\left\{\tilde{1},\tilde{0},(G,E)\right\}$ where $(F_1,E)=\left\{\left\{\frac{1}{a},\frac{0}{b},\frac{0}{c}\right\}\right\}$, $(F_2,E)=\left\{\left\{\frac{1}{a},\frac{0}{b},\frac{1}{c}\right\}\right\}$ and $(G,E)=\left\{\left\{\frac{1}{a},\frac{1}{b},\frac{0}{c}\right\}\right\}$.

Consider an identity function $f:X\to Y$, Clearly $f^{-1}(G,E)=\left\{\left\{\frac{1}{a},\frac{1}{b},\frac{0}{c}\right\}\right\}$ is fuzzy soft b-locally closed set but not fuzzy soft locally closed.

4. Conclusions

In this paper the concept of fuzzy soft locally closed set and fuzzy soft b-locally closed set is introduced in fuzzy soft topological space. Also fuzzy soft LC-continuous and fuzzy soft b-LC-continuous functions were defined in fuzzy soft topological space.

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Fractional-order Mathematical Modeling of Bacterial Competition with Therapy of Multiple Antibiotics

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Abstract – In this study, a mathematical model in form fractional-order differential equations (FDEs) system identifying population dynamics in two species bacteria struggling one another and exposed to multiple antibiotics simultaneously, was suggested. Stability analysis of the equilibrium points of the proposed model was also carried out. Additionally, the results of the analysis have promoted by numerical simulations.

Keywords – *Fractional-order differential equation, Stability analysis, Numerical simulation.*

1. Introduction

Mathematical modeling through fractional-orders differential and integral operators has become increasingly common in recent years. In addition, that, the various types of fractional-order differential equations are proposed for most of the standard models. Fractional-order differential equations (FDEs) are, at least, as stable as their integer order counterpart, namely ordinary differential equation [1]. Therefore, the fractional-order calculus has a considerable amount of attention for many areas of science [2-7]. In particular, biology is a very rich resource for mathematical ideas.

The behavior of most biological systems has memory or after-effects. The modeling of these systems by FDEs has more advantages than classical integer-order modeling, where such effects are neglected [2]. In this study, a continuous time mathematical model proposed in [8] is examined by using the system of FDEs.

2. Preliminaries and Definitions

In this section, the basic definitions and characteristics of fractional derivative operators is expressed.

2.1. Fractional Differential Operators

There are various descriptions of a fractional derivative with the order $\alpha > 0$. The definitions of Riemann-Liouville and Caputo are used most widely. The Riemann-Liouville fractional integral operator with order $\alpha \geq 0$ for the function $f(t)$ is described as the following:

$$J^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau, \alpha > 0, t > 0. \quad (2.1)$$

Some of properties of the operator J^{α} are as follows:

$$\begin{aligned} J^{\alpha} J^{\beta} f(t) &= J^{\alpha+\beta} f(t) \\ J^{\alpha} t^{\gamma} &= \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} t^{\alpha+\gamma} \end{aligned} \quad (2.2)$$

where $\mu \geq -1, \alpha, \beta \geq 0$ and $\gamma > -1$. The Caputo sense was used in this study. Taking into account the definition of Caputo sense, the fractional derivative of the function $f(t)$ is identified as

$$D^{\alpha} f(t) = J^{m-\alpha} D^m f(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t \frac{f^{(m)}(\tau)}{(t-\tau)^{\alpha-m+1}} d\tau \quad (2.3)$$

for $m - 1 < \alpha \leq m, m \in \mathbb{N}, t > 0$ [9].

3. Model Formulation

The proposed model in this study is fractional-order form of model suggested in [8], which showed dynamics between antibiotics concentrations and bacteria in an individual receiving a cocktail of multi-drug treatment against bacteria. Bacteria in model have the competitive ability against each order for common host. That all bacteria have not resistance ability against to multiple antibiotics, has assumed in model. Let us denote by $B_1(t)$ and $B_2(t)$ the population sizes of first, and second bacteria to multiple antibiotics at time t, respectively; and by $A_i(t)$ the concentration of the i-th antibiotic for $i = 1,2, \dots, n$.

The parameters used in the model are as follows: It has supposed that bacteria follow a logistic growth with different carrying capacity K_1 and K_2, respectively. In this sense, β_{B_1} and β_{B_2} are the birth rate of first and second bacteria, respectively. The first and second bacteria have per capita natural death rates μ_{B_1} and μ_{B_2}, respectively. The first bacteria also die due to the action of the antibiotics, and it has assumed that the rate at which they are killed by the i-th antibiotic is equal to $\overline{\alpha}_i B_1 A_i$. In the same mind, it is $\overline{q}_i B_2 A_i$ for other. The mutual competition between the species is dictated by M_1, M_2. Finally, the i-th antibiotic concentration is supplied at a constant rate δ_i, and is taken up at a constant per capita rate ω_i (or the excretion rate from body) [10].

Under the assumptions aforementioned and proposed in [8], it is obtained the following system of $(n + 2)$ fractional-order differential equation:

$$D^\alpha B_1 = \beta_{B_1} B_1 \left(1 - \frac{B_1}{K_1}\right) - \left[\sum_{i=1}^{n} \overline{\alpha_i}\, A_i B_1\right] - \mu_{B_1} B_1 - M_1 B_2 B_1$$
$$D^\alpha B_2 = \beta_{B_2} B_2 \left(1 - \frac{B_2}{K_2}\right) - \left[\sum_{i=1}^{n} \overline{q_i}\, A_i B_2\right] - \mu_{B_2} B_2 - M_2 B_1 B_2 \quad (3.1)$$
$$D^\alpha A_i = \delta_i - \omega_i A_i, for\ i = 1,2,\ldots,n.$$

where $t \geq 0$, $n \in \mathbb{N}^+$, $D = \frac{d}{dt}$ and $\alpha \in (0,1]$, real number, is the orders of the derivatives in this system. Also, $B_1 \equiv B_1(t)$, $B_2 \equiv B_2(t)$, $A_1 \equiv A_1(t)$,…, $A_n \equiv A_n(t)$, the parameters $\beta_{B_1}, \beta_{B_2}, \mu_{B_1}, \mu_{B_2}, M_1, M_2$ and $\overline{\alpha_i}, \overline{q_i}$ for $i = 1,\ldots,n$ are positive constants. Additionally, the system (3.1) has to be finished with positive initial conditions $B_1(t_0) = B_{1_0}$, $B_2(t_0) = B_{2_0}$, $A_1(t_0) = A_{1_0}$,…, $A_n(t_0) = A_{n_0}$.

The above scenario related to the parameters used in the model (3.1) has been graphically described in Figure 3.1.

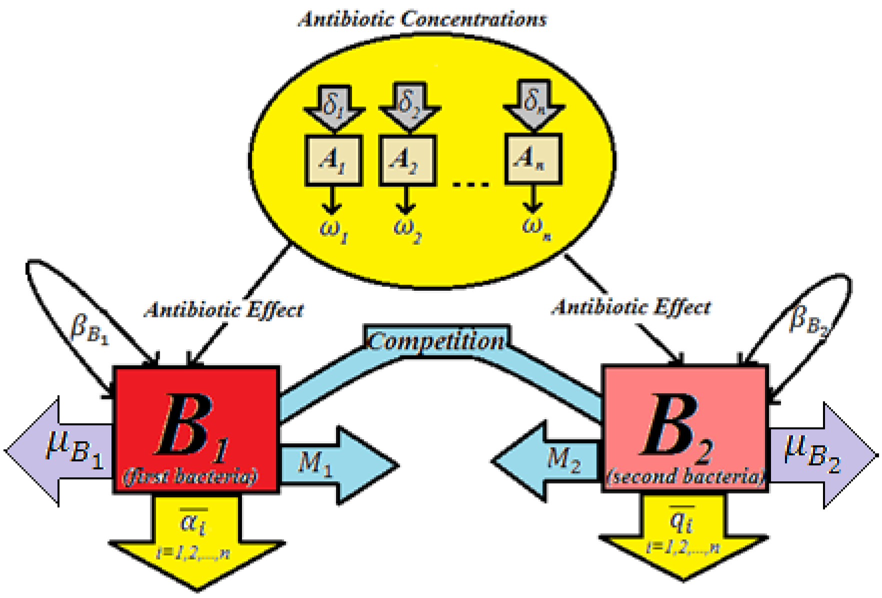

Figure 3.1. Schematic demonstration of interaction among bacteria (first and second) and concentrations of multiple antibiotic in model (3.1).

To reduce the number of parameters, it is used change of variables $b_1 = \frac{B_1}{K_1}$, $b_2 = \frac{B_2}{K_2}$, $a_i = \frac{A_i}{\frac{\delta_i}{\omega_i}}$. In the new variables, system (3.1) transforms to

$$D^\alpha b_1 = \beta_{B_1} b_1 (1 - b_1) - \left[\sum_{i=1}^{n} \alpha_i a_i b_1\right] - \mu_{B_1} b_1 - m_1 b_2 b_1$$
$$D^\alpha b_2 = \beta_{B_2} b_2 (1 - b_2) - \left[\sum_{i=1}^{n} q_i a_i b_2\right] - \mu_{B_2} b_2 - m_2 b_1 b_2 \quad (3.2)$$
$$D^\alpha a_i = \omega_i - \omega_i a_i, for\ i = 1,2,\ldots,n.$$

where $q_i = \overline{q}_i\left(\frac{\delta_i}{\omega_i}\right)$, $\alpha_i = \overline{\alpha}_i\left(\frac{\delta_i}{\omega_i}\right)$, $M_1 = \frac{m_1}{K_2}$ and $M_2 = \frac{m_2}{K_1}$.

Definition 3.1 The FDE model in (3.2) is rewritten the matrix form as the following:

$$\begin{aligned} &D^{\alpha}X(t) = AX(t) + x_1(t)B_1X(t) + x_2(t)B_2X(t) + H \\ &X(0) = X_0 \end{aligned} \qquad (3.3)$$

where

$$X(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ \vdots \\ x_{n+2}(t) \end{pmatrix} = \begin{pmatrix} b_1(t) \\ b_2(t) \\ a_1(t) \\ \vdots \\ a_n(t) \end{pmatrix}, X_0 = \begin{pmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \\ \vdots \\ x_{n+2}(0) \end{pmatrix}, H = \begin{pmatrix} 0 \\ 0 \\ \omega_1 \\ \vdots \\ \omega_n \end{pmatrix},$$

$$A = \begin{pmatrix} (\beta_{B_1} - \mu_{B_1}) & 0 & 0 & \dots & 0 \\ 0 & (\beta_{B_2} - \mu_{B_2}) & 0 & \dots & 0 \\ 0 & 0 & -\omega_1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & -\omega_n \end{pmatrix},$$

$$B_1 = \begin{pmatrix} -\beta_{B_1} & -m_1 & -\alpha_1 & \dots & -\alpha_n \\ 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}$$

and

$$B_2 = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 \\ -m_2 & -\beta_{B_2} & -q_1 & \dots & -q_n \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}.$$

Definition 3.2 For $X(t) = \left(x_1(t)\, x_2(t)\, x_3(t) \dots x_{n+2}(t)\right)^T$, let $C^*[0,T]$ be the set of continuous column vectors $X(t)$ on the interval $[0,T]$. The norm of $X(t) \in C^*[0,T]$ definite in (3.3) is $\|X(t)\| = \sum_{i=1}^{n+2} sup_t |x_i(t)|$.

Proposition 3.1 Let considered Definition 3.1. Let $\mathbb{R}_+^{n+2} = \{X : X \geq 0\}$ and $X(t) = \left(x_1(t)\, x_2(t)\, x_3(t) \dots x_{n+2}(t)\right)^T$. Let $f(x) \in C[a,b]$ and $D^{\alpha}f(x) \in C[a,b]$ for $0 < \alpha \leq 1$, and then, by the generalized mean value theorem, it is

$f(x) = f(a) + \frac{1}{\Gamma(\alpha)} D^{\alpha} f(\xi)(x-a)^{\alpha}$ with $0 \leq \xi \leq x$, all $x \in [a,b]$.

According to this theorem,

- the function $f(x)$ is increasing for each $x \in [a,b]$, when $D^{\alpha}f(x) > 0$, all $x \in [a,b]$,
- the function $f(x)$ is decreasing for each $x \in [a,b]$, when $D^{\alpha}f(x) < 0$, all $x \in [a,b]$.

Additionally, the vector field points into $\mathbb{R}_+^{n+2}$, since $D^\alpha b_1(t)|_{b_1=b_2=a_i=0} = 0$, $D^\alpha b_2(t)|_{b_1=b_2=a_i=0} = 0$ and $D^\alpha a_i|_{b_1=b_2=a_i=0} = \omega_i$ for $i = 1,2,\ldots,n$ on each hyperplane bounding the nonnegative octant.

Proposition 3.2 Let $X(t) \in C^*[0,T]$. In this case, there is a unique solution of the system (3.2).

Proof. If $D^\alpha X(t) = F\big(X(t)\big) = AX(t) + x_1(t)B_1X(t) + x_2(t)B_2X(t) + H$, then $X(t) \in C^*[0,T]$ implies $F\big(X(t)\big) \in C^*[0,T]$. Also, considering $X(t), Y(t) \in C^*[0,T]$ and $X(t) \neq Y(t)$; it is obtained the following inequalities:

$$\left\|F\big(X(t)\big) - F\big(Y(t)\big)\right\|$$

$$\begin{aligned}= \|(&AX(t) + x_1(t)B_1X(t) + x_2(t)B_2X(t) + H) \\ &- (AY(t) + y_1(t)B_1Y(t) + y_2(t)B_2Y(t) + H)\|\end{aligned}$$

$$= \|AX(t) + x_1(t)B_1X(t) + x_2(t)B_2X(t) - AY(t) - y_1(t)B_1Y(t) - y_2(t)B_2Y(t)\|$$

$$= \left\|\begin{matrix}A\big(X(t) - Y(t)\big) + x_1(t)B_1X(t) + x_2(t)B_2X(t) - y_1(t)B_1Y(t) - y_2(t)B_2Y(t) \\ -\left(\underbrace{x_1(t)B_1Y(t) - x_1(t)B_1Y(t)}_{0}\right) - \left(\underbrace{x_2(t)B_2Y(t) - x_2(t)B_2Y(t)}_{0}\right)\end{matrix}\right\|$$

$$= \left\|\begin{matrix}A\big(X(t) - Y(t)\big) + x_1(t)B_1\big(X(t) - Y(t)\big) + x_2(t)B_2\big(X(t) - Y(t)\big) + \big(x_1(t) - y_1(t)\big)B_1Y(t) \\ +\big(x_2(t) - y_2(t)\big)B_2Y(t)\end{matrix}\right\|$$

$$\leq \left(\begin{matrix}\left\|A\big(X(t) - Y(t)\big)\right\| + \left\|x_1(t)B_1\big(X(t) - Y(t)\big)\right\| + \left\|x_2(t)B_2\big(X(t) - Y(t)\big)\right\| \\ +\left\|\big(x_1(t) - y_1(t)\big)B_1Y(t)\right\| + \left\|\big(x_2(t) - y_2(t)\big)B_2Y(t)\right\|\end{matrix}\right)$$

$$\leq \left(\begin{matrix}\|A\|\left\|\big(X(t) - Y(t)\big)\right\| + |x_1(t)|\|B_1\|\left\|\big(X(t) - Y(t)\big)\right\| + |x_2(t)|\|B_2\|\left\|\big(X(t) - Y(t)\big)\right\| \\ +\|B_1\|\left|\big(x_1(t) - y_1(t)\big)\right|\|Y(t)\| + \|B_2\|\left|\big(x_2(t) - y_2(t)\big)\right|\|Y(t)\|\end{matrix}\right)$$

$$\leq \left(\begin{matrix}(\|A\| + |x_1(t)|\|B_1\| + |x_2(t)|\|B_2\|)\left\|\big(X(t) - Y(t)\big)\right\| \\ +\|B_1\|\underbrace{\left|\big(x_1(t) - y_1(t)\big)\right|}_{\leq\|(X(t)-Y(t))\|}\|Y(t)\| + \|B_2\|\underbrace{\left|\big(x_2(t) - y_2(t)\big)\right|}_{\leq\|(X(t)-Y(t))\|}\|Y(t)\|\end{matrix}\right)$$

$$\leq (\|A\| + \|B_1\||x_1(t)| + \|B_1\|\|Y(t)\| + \|B_2\||x_2(t)| + \|B_2\|\|Y(t)\|)\left\|\big(X(t) - Y(t)\big)\right\|$$

$$\leq \left(\|A\| + \|B_1\|\left(\underbrace{|x_1(t)|}_{\leq\|X(t)\|} + \|Y(t)\|\right) + \|B_2\|\left(\underbrace{|x_2(t)|}_{\leq\|X(t)\|} + \|Y(t)\|\right)\right)\left\|\big(X(t) - Y(t)\big)\right\|$$

$$\leq \big(\|A\| + (\|B_1\| + \|B_2\|)(\|X(t)\| + \|Y(t)\|)\big)\left\|\big(X(t) - Y(t)\big)\right\|$$

and so, it is

$$\left\|F\big(X(t)\big) - F\big(Y(t)\big)\right\| \leq L\left\|\big(X(t) - Y(t)\big)\right\| \tag{3.4}$$

where $L = \|A\| + (\|B_1\| + \|B_2\|)(W_1 + W_2) > 0$, and W_1 and W_2 are positive and meet the inequalities $\|X(t)\| \leq W_1$, $\|Y(t)\| \leq W_2$ due to $X(t), Y(t) \in C^*[0,T]$. Therefore, the system (3.3) has a unique solution.

Lemma 3.1. Consider the following fractional-order autonomous system

$$\begin{aligned} &D^{\alpha}X(t) = F(X(t)), D = \frac{d}{dt} \\ &X(0) = X_0 \end{aligned} \tag{3.5}$$

where $\alpha \in (0,1], X(t) = (x_1 \quad x_2 \quad \cdots \quad x_n)^T$ and $F = (f_1 \quad f_2 \quad \cdots \quad f_n)^T$. To evaluate the equilibrium points, it has been presumed as $D^{\alpha}X(t) = 0 \Rightarrow f_i(\overline{x_1}, \overline{x_2}, \ldots, \overline{x_n}) = 0$ for $i = 1,2,\ldots,n$. In this sense, the equilibrium point $(\overline{x_1}, \overline{x_2}, \ldots, \overline{x_n})$ of this system is founded. To evaluate the asymptotic stability of equilibrium points, the Jacobian matrix,

$$J = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n} \end{pmatrix}$$

is used. It is assumed that the I is identity matrix with nxn. If all of the eigenvalues, $\lambda_1, \lambda_2, \ldots, \lambda_n$, obtained from the equation

$$Det\left(J_{(x_1,x_2,\ldots,x_n)=(\overline{x_1},\overline{x_2},\ldots,\overline{x_n})} - \lambda I\right) = 0 \tag{3.6}$$

satisfies either the Routh–Hurwitz stability conditions or the conditions

$$\left(|arg(\lambda_1)| > \frac{\alpha\pi}{2}, \quad |arg(\lambda_2)| > \frac{\alpha\pi}{2}\right), \tag{3.7}$$

then $(\overline{x_1}, \overline{x_2}, \ldots, \overline{x_n})$ is ***locally asymptotically stable (LAS)*** for system (3.5). In addition that, the characteristically equation obtained from (3.6) can be given by

$$P(\lambda) = \lambda^n + a_1\lambda^{n-1} + \ldots + a_{n-1}\lambda + a_n,$$

where the coefficients a_i for $i = 1, \ldots, n$ are real constants. In this respect, Routh-Hurwitz stability conditions for polynomial of degree $n = 2$, 3, 4 and 5 are summarized as following:

$$\begin{aligned} &n = 2: a_1, a_2 > 0, \\ &n = 3: a_1, a_3 > 0 \text{ and } a_1a_2 > a_3, \\ &n = 4: a_1, a_3, a_4 > 0 \text{ and } a_1a_2a_3 > {a_3}^2 + {a_1}^2a_4, \\ &n = 5: \begin{matrix} a_1, a_2, a_3, a_4, a_5 > 0, a_1a_2a_3 > {a_3}^2 + {a_1}^2a_4 \\ \text{and } (a_1a_4 - a_5)(a_1a_2a_3 - {a_3}^2 - {a_1}^2a_4) > a_5(a_1a_2 - a_3)^2 + a_1{a_5}^2. \end{matrix} \end{aligned} \tag{3.8}$$

Additionally, the above mentioned criteria has provided the necessary and sufficient conditions for all roots of $P(\lambda)$ to lie in the left half of the complex plane [11].

Conclusion 3.1. Let us consider Lemma 3.1. The following conclusion can be summarized from this lemma. If the eigenvalues are real numbers, it is enough to only check whether they provide the Routh-Hurwitz criteria for the stability of the equilibrium point obtained from system (3.5).

Conclusion 3.2. It is assumed that the characteristically equation is

$$\begin{aligned} P(\lambda) &= \lambda^2 + a_1\lambda + a_2 \\ &= \lambda^2 + \big(-Tr(J)\big)\lambda + (DetJ) = 0 \end{aligned} \tag{3.9}$$

for $n = 2$ in system (3.5). In this sense, the stability conditions of the equilibrium point are: either Routh–Hurwitz conditions $(a_1, a_2 > 0)$
or:

$$a_1 < 0, 4a_2 > (a_1)^2, \left| tan^{-1}\left(\frac{\sqrt{4a_2 - (a_1)^2}}{a_1}\right)\right| > \frac{\alpha\pi}{2}. \tag{3.10}$$

4. Qualitative Analysis of the System (3.2)

Proposition 4.1. The existence and stability of equilibria of the system (3.2) are analyzed in here. The equilibria of the system with the threshold parameters

$$\frac{\beta_{B_1} - [\sum_{i=1}^{n} \alpha_i] - \mu_{B_1}}{\beta_{B_1}} = A, \frac{\beta_{B_2} - [\sum_{i=1}^{n} q_i] - \mu_{B_2}}{\beta_{B_2}} = B, \frac{m_1}{\beta_{B_1}} = C, \frac{m_2}{\beta_{B_2}} = D, \tag{4.1}$$

$$0 < C, 0 < D$$

are as follows: The system (3.2) always has the infection-free equilibrium point $E_0 = (0,0,1,1,\ldots,1)$. If $A > 0$, then $E_1 = (A, 0,1,1,\ldots,1)$ reveals as another equilibrium point. Likewise, $E_2 = (0, B, 1,1,\ldots,1)$ exists, when $B > 0$. When $CD < 1$ and $BC < A < \frac{B}{D}$ or $1 < CD$ and $\frac{B}{D} < A < BC$, in addition to E_0, E_1, and E_2, there exists a fourth the equilibrium point, $E_3 = \left(\frac{BC-A}{CD-1}, \frac{DA-B}{CD-1}, 1,1,\ldots,1\right)$ [8].

Proposition 3.2. The equilibrium points of system (3.2) satisfy the followings:

(i) If $A < 0$ and $B < 0$, then the infection-free equilibrium E_0 is *LAS*. If either $A > 0$ or $B > 0$, it becomes an unstable point.

(ii) Let $A > 0$. If $B - DA < 0$, the equilibrium point E_1 is *LAS*, and if $B - DA > 0$, E_1 becomes an unstable point.

(iii) Let $B > 0$. If $A - CB < 0$, the equilibrium point E_2 is *LAS*, and if $A - CB > 0$, E_2 becomes an unstable point.

(iv) Let $CD < 1$ and $BC < A < \frac{B}{D}$ or $1 < CD$ and $\frac{B}{D} < A < BC$. If $1 < CD$ and $\frac{B}{D} > A > BC$, then E_3 is *LAS*.

Proof. For the stability analysis, the functions of the right side of the system (3.2) are suggested as follows:

$$
\begin{aligned}
f(b_1,b_2,a_i) &= \beta_{B_1} b_1(1-b_1) - b_1\left[\sum_{i=1}^{n}\alpha_i a_i\right] - \mu_{B_1} b_1 - m_1 b_2 b_1 \\
g(b_1,b_2,a_i) &= \beta_{B_2} b_2(1-b_2) - \left[\sum_{i=1}^{n} q_i a_i b_2\right] - \mu_{B_2}\ b_2 - m_2 b_1 b_2 \\
h_i(b_1,b_2,a_i) &= \omega_i - \omega_i a_i, \qquad i = 1,2,\dots,n.
\end{aligned} \tag{4.2}
$$

That Jacobean matrix obtained from equations in (4.2) is

$$
J = \begin{pmatrix}
\begin{pmatrix} \beta_{B_1} - 2\beta_{B_1} b_1 - \sum_{i=1}^{n}\alpha_i a_i \\ -\mu_{B_1} - m_1 b_2 \end{pmatrix} & -m_1 b_1 & -\alpha_1 b_1 & \dots & -\alpha_n b_1 \\
-m_2 b_2 & \begin{pmatrix} \beta_{B_2} - 2\beta_{B_2} b_2 - \sum_{i=1}^{n} q_i a_i \\ -\mu_{B_2} - m_2 b_1 \end{pmatrix} & -q_1 b_2 & \dots & -q_n b_2 \\
0 & 0 & -\mu_1 & \dots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \dots & -\mu_n
\end{pmatrix}. \tag{4.3}
$$

In terms of ease of representation, the τ-th eigenvalue of equilibrium point E_k is shown as $\lambda^{(k)}{}_{\tau}$ for $k = 0,1,2,3$ and $\tau = 1,2,\dots,n+2,\quad n \in N$.

(i) From (4.3), the Jacobean matrix evaluated at the equilibrium point E_0 is given by

$$
J(E_0) = \begin{pmatrix}
\beta_{B_1} - \sum_{i=1}^{n}\alpha_i - \mu_{B_1} & 0 & 0 & \dots & 0 \\
0 & \beta_{B_2} - \sum_{i=1}^{n} q_i - \mu_{B_2} & 0 & \dots & 0 \\
0 & 0 & -\mu_1 & \dots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \dots & -\mu_n
\end{pmatrix}. \tag{4.4}
$$

By taking into account (4.1), the eigenvalues obtained from (4.4) are $\lambda^{(0)}{}_1 = \beta_{B_1} A,\ \lambda^{(0)}{}_2 = \beta_{B_2} B$ and $\lambda^{(0)}{}_{i+2} = -\mu_i$ for $i = 1,2,\dots,n$. It is explicit that all eigenvalues are real numbers and $\lambda^{(0)}{}_{i+2} = -\mu_i < 0$, since parameters in the proposed model are positive real number. By Conclusion 3.1., it is enough to examine whether the eigenvalues provide the Routh-Hurwitz criteria for stability analysis of E_0. Therefore, the others eigenvalues, $\lambda^{(0)}{}_1$ and $\lambda^{(0)}{}_2$, are negative real number, iff $A < 0$ and $B < 0$. In this case, E_0 is LAS.

(ii) Let $A > 0$. The jacobian matrix for the equilibrium point E_1 by taking into account (4.1) is given as

$$J(E_1) = \begin{pmatrix} -\beta_{B_1}A & -m_1A & -\alpha_1A & \dots & -\alpha_nA \\ 0 & \beta_{B_2}B - m_2A & 0 & \dots & 0 \\ 0 & 0 & -\mu_1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & -\mu_n \end{pmatrix}. \quad (4.5)$$

The eigenvalues are $\lambda^{(1)}{}_1 = -\beta_{B_1}A$, $\lambda^{(1)}{}_2 = \beta_{B_2}(B - DA)$ and $\lambda^{(1)}{}_{i+2} = -\mu_i < 0$ for $i = 1,2,\dots,n$. The eigenvalues are real numbers. From Conclusion 3.1., the eigenvalues are negative real number, iff $A > 0$ and $B - DA < 0$. Therefore, it is LAS.

(iii) For $B > 0$, there is the equilibrium point E_2. The Jacobian matrix evaluated in this point is

$$J(E_2) = \begin{pmatrix} \beta_{B_1}A - m_1B & 0 & 0 & \dots & 0 \\ -m_2B & -\beta_{B_2}B & -q_1B & \dots & -q_nB \\ 0 & 0 & -\mu_1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & -\mu_n \end{pmatrix} \quad (4.6)$$

by (4.1). The eigenvalues of (4.6) are $\lambda^{(2)}{}_1 = \beta_{B_1}A - m_1B = \beta_{B_1}(A - CB)$, $\lambda^{(2)}{}_2 = -\beta_{B_2}B$ and $\lambda^{(2)}{}_{i+2} = -\mu_i < 0$ for $i = 1,2,\dots,n$. By the same mind in (ii), the eigenvalues are real numbers. We have Conclusion 3.1. E_2 is LAS, iff $B > 0$ and $A - CB < 0$.

(iv) Let

$$CD < 1 \text{ and } BC < A < \frac{B}{D} \text{ or } 1 < CD \text{ and } \frac{B}{D} < A < BC. \quad (4.7)$$

In this case, the stability of E_3 can be analyzed. Evaluating J for E_3, we have

$$J(E_3) = \begin{pmatrix} \beta_{B_1}\begin{pmatrix} A - 2\dfrac{BC - A}{CD - 1} - \\ C\dfrac{DA - B}{CD - 1} \end{pmatrix} & -m_1\dfrac{BC - A}{CD - 1} & -\alpha_1b_1 & \dots & -\alpha_nb_1 \\ -m_2\dfrac{DA - B}{CD - 1} & \beta_{B_2}\begin{pmatrix} B - 2\dfrac{DA - B}{CD - 1} - \\ D\dfrac{BC - A}{CD - 1} \end{pmatrix} & -q_1b_2 & \dots & -q_nb_2 \\ 0 & 0 & -\mu_1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & -\mu_n \end{pmatrix} \quad (4.8)$$

That eigenvalues of Jacobean matrix evaluated at the equilibrium point E_3 are $\lambda^{(3)}{}_{i+2} = -\mu_i < 0$ for $i = 1,2,\dots,n$ and the others are founded from following matrix;

$$J^{B(E_3)} = \begin{pmatrix} -\beta_{B_1}\left(\dfrac{A - BC}{1 - CD}\right) & -m_1\left(\dfrac{A - BC}{1 - CD}\right) \\ -m_2\left(\dfrac{B - AD}{1 - CD}\right) & -\beta_{B_2}\left(\dfrac{B - AD}{1 - CD}\right) \end{pmatrix} \quad (4.9)$$

where $J^{B(E_3)}$ is the block matrix of $J(E_3)$. It is clear that $\lambda^{(3)}{}_{i+2} = -\mu_i \in \mathbb{R}^-$ and so, it does not impair the stability of this point. From (4.9), it is $Tr\left(J^{B(E_3)}\right) = -\left[\beta_{B_1}\overline{b_1} + \beta_{B_2}\overline{b_2}\right]$ and $Det\left(J^{B(E_3)}\right) = \beta_{B_1}\beta_{B_2}\overline{b_1 b_2}(1 - CD)$. In this respect, it is $Tr\left(J^{B(E_3)}\right) < 0$ due to equilibrium values in E_3 and parameters in (3.1) are positive real number. Consider the parameter a_1 in (3.9), it is $a_1 > 0$, due to $Tr\left(J^{B(E_3)}\right) < 0$. Thus, the stability conditions of the equilibrium point are Routh–Hurwitz conditions $(a_1, a_2 > 0)$, due to $a_1 > 0$.

In addition, that, if $CD < 1$, (4.10). Then $a_2 = Det\left(J^{B(E_3)}\right) > 0$. By (4.7) and (4.10), if $1 < CD$ and $\frac{B}{D} < A < BC$, (4.11) then the eigenvalues are negative real number or complex number with negative real parts, and so, it is *LAS*.

As a result, the *LAS* conditions founded for equilibria of system (3.2) are summarized in the Table 4.1.

Table 4.1. The *LAS* conditions of the equilibria of FDEs system in (3.2).

Equilibrium Points	Stability Conditions
$E_0 = (0,0,1,\ldots,1)$	$A < 0, B < 0$
$E_1 = (A,0,1,\ldots,1)$	$max\left\{0,\frac{B}{D}\right\} < A$
$E_2 = (0,B,1,\ldots,1)$	$max\{0,A\} < BC$
$E_3 = \left(\frac{A-BC}{1-CD},\frac{B-AD}{1-CD},1,1,\ldots,1\right)$	$1 < CD$ and $\frac{B}{D} < A < BC$

5. Numerical Study

In the following discussion, it is demonstrated some contributions of the proposed mathematical model to the study of complex problems in host-microbe interactions. In numerical study, datas of two different streams competing each others of bacteria including Acinetobacter baumannii (b_1) and E. coli (b_2) in host were used and dynamics of multiple antibiotics against these bacteria causing infection were examined [8]. The parameters used in numerical study [12-18] are as the followings:

$$\begin{aligned}
&\beta_{B_1} = 1.2\text{ day}^{-1},\ \beta_{B_2} = 0.6\text{ day}^{-1}, K_1 = 10^8\text{ cell}, K_2 = 10^7\text{ cell}, \mu_{B_1} = 0.312\text{ day}^{-1},\\
&\mu_{B_2} = 0.179\text{ day}^{-1}, M_1 = 10^{-7}\text{ cell}^{-1}\text{day}^{-1}, M_2 = 10^{-7}\text{ cell}^{-1}\text{day}^{-1}, \overline{\alpha_1} = 0.47\text{ day}^{-1},\\
&\overline{\alpha_2} = 0.21\text{ day}^{-1}, \overline{q_1} = 0.42\text{ day}^{-1}, \overline{q_2} = 0.17\text{ day}^{-1}, \delta_1 = 2\text{ mg/kg/day},\\
&\delta_2 = 1.2\text{ mg/kg/day}, \omega_1 = 0.04\text{ day}^{-1}, \omega_2 = 0.03\text{ day}^{-1}\text{ and } \alpha = 0.25, 0.50, 0.75, 0.99.
\end{aligned} \quad (5.1)$$

In the light of data obtained from (5.1), it is founded as following: the parameters

$$\sum_{i=1}^{n} \alpha_i = \alpha_1 + \alpha_2 = \overline{\alpha}_1\frac{\delta_1}{\omega_1} + \overline{\alpha}_2\frac{\delta_2}{\omega_2} = 0.47\frac{2}{0.04} + 0.21\frac{1.2}{0.03} = 31.9$$

$$\sum_{i=1}^{n} q_i = q_1 + q_2 = \overline{q}_1\frac{\delta_1}{\omega_1} + \overline{q}_2\frac{\delta_2}{\omega_2} = 0.42\frac{2}{0.04} + 0.17\frac{1.2}{0.03} = 27.8$$

$m_1 = M_1 K_2 = 10^{-7} * 10^7 = 1$
$m_2 = M_2 K_1 = 10^{-7} * 10^8 = 10$

the threshold parameters

$$A = \frac{\beta_{B_1} - [\sum_{i=1}^{n} \alpha_i] - \mu_{B_1}}{\beta_{B_1}} = \frac{1.2 - 31.9 - 0.312}{1.2} = -25.84$$

$$B = \frac{\beta_{B_2} - [\sum_{i=1}^{n} q_i] - \mu_{B_2}}{\beta_{B_2}} = \frac{0.6 - 27.8 - 0.179}{0.6} = -45.63$$

$$C = \frac{m_1}{\beta_{B_1}} = \frac{1}{1.2} = 0.83$$

$$D = \frac{m_2}{\beta_{B_2}} = \frac{10}{0.6} = 16.66$$

and so the equilibrium points $E_0(0,0,1,1)$, $E_1(-25.84,0,1,1)$, $E_2(0,-45.63,1,1)$ and $E_3 = (-0.9376, -29.99972, 1, 1, \dots, 1)$. Because it is $A, B < 0$, the equilibrium point $E_0(0,0,1,1)$ is LAS and this situation is clearly seen in following figures:

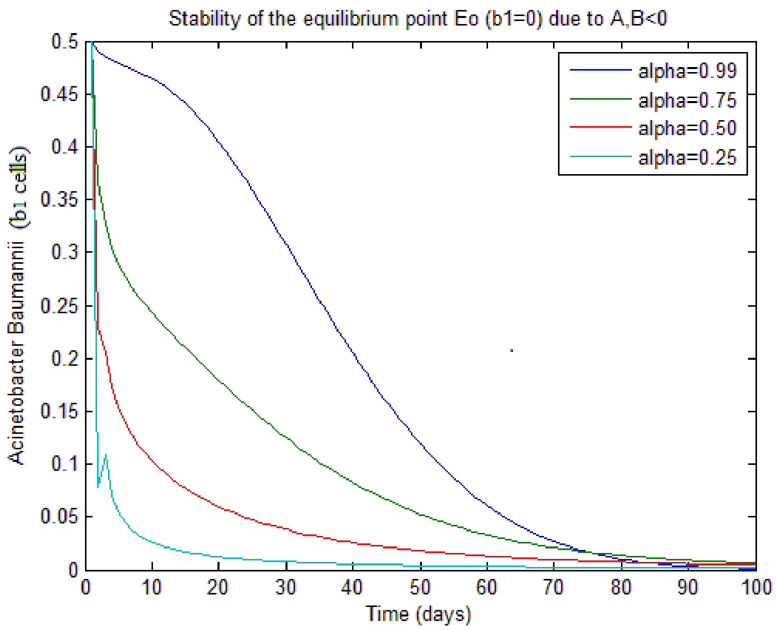

Figure 5.1. According to $\alpha = 0.25, 0.50, 0.75$ and 0.99, the trajectory of population sizes of Acinetobacter baumannii, when $A = -25.84$ and $B = -45.63$. In here, $E_0(0,0,1,1)$ is LAS, since $A, B < 0$.

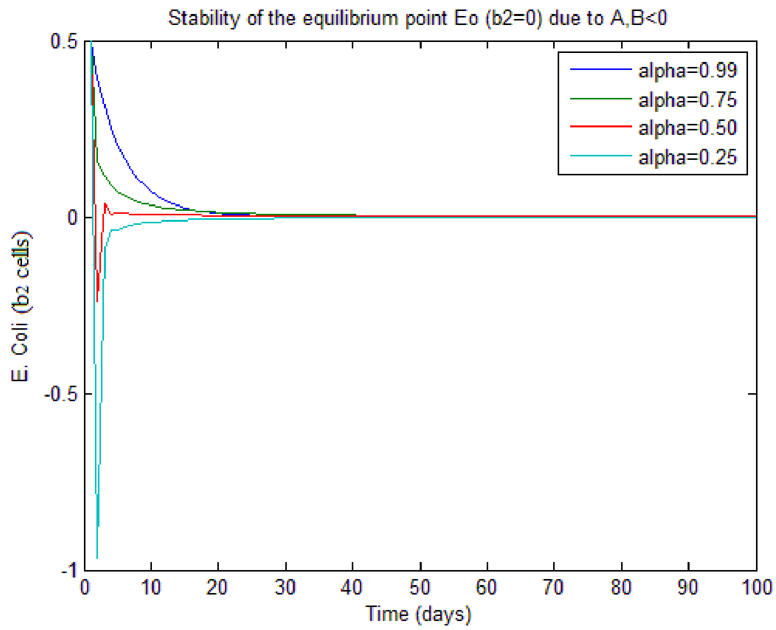

Figure 5.2. According to $\alpha = 0.25, 0.50, 0.75$ and 0.99, the trajectory of population sizes of E. coli, when $A = -25.84$ and $B = -45.63$. In here, $E_0(0,0,1,1)$ is LAS, since $A, B < 0$.

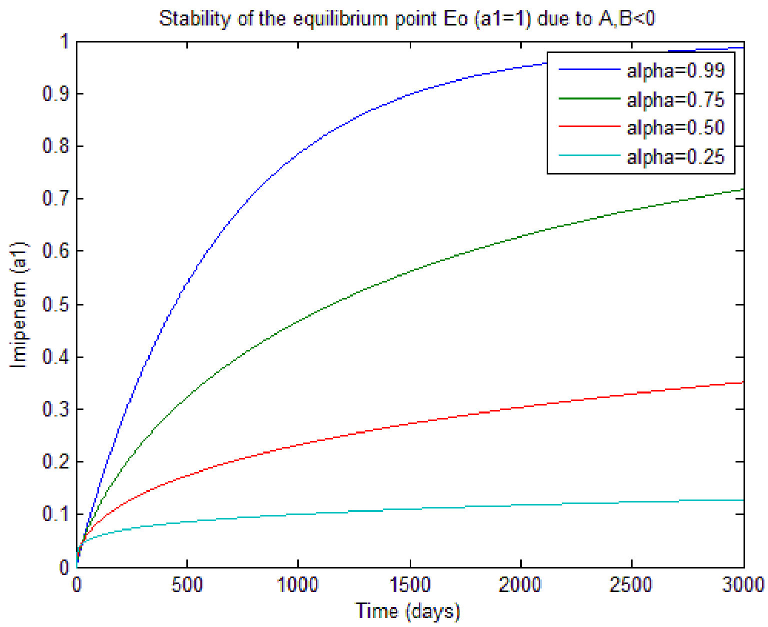

Figure 5.3. According to $\alpha = 0.25, 0.50, 0.75$ and 0.99, the trajectory of the imipenem concentration, when $A = -25.84$ and $B = -45.63$. In here, $E_0(0,0,1,1)$ is LAS, since $A, B < 0$.

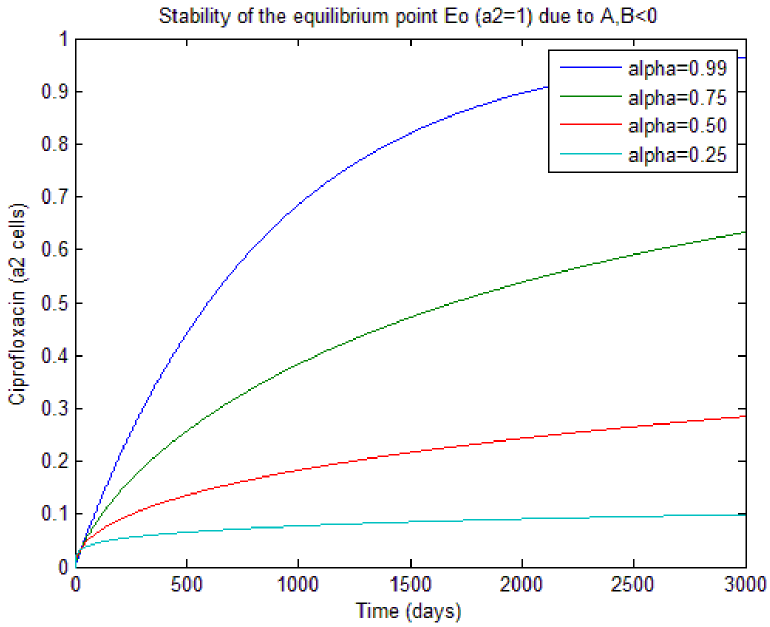

Figure 5.4. According to $\alpha = 0.25, 0.50, 0.75$ and 0.99, the trajectory of the ciprofloxacin concentration, when $A = -25.84$ and $B = -45.63$. In here, $E_0(0,0,1,1)$ is LAS, since $A, B < 0$.

In compliance with literature datas [17], while E. coli is disappeared as a result of 90-day antibiotics use and Acinetobacter baumannii is disappeared as a result of 30-day antibiotics use. This case shows that our model is very useful to explain experimental results in literatures.

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Review of Number 26

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Abstaract — Here, we review all papers that are published in Number 26 of the Journal of New Theory. We then introduce all of the members of the editorial board and reviewers of the papers in this issue.

Keywords — *Journal of New Theory, J. New Theory, JNT, Number 26.*

1 Number 26

We are happy to inform you that Number 26 of the Journal of New Theory (JNT) is completed with 9 articles.

In [1], the authors introduced the concepts of upper and lower $(\alpha, \beta, \theta, \delta, \ell)$-continuous fuzzy multifunctions. It is in order to unify several characterizations and properties of some kinds of modifications of fuzzy upper and fuzzy lower semi-continuous fuzzy multifunctions, and to deduce a generalized form of these concepts, namely upper and lower $\eta\eta^*$-continuous fuzzy multifunctions.

In [2], the author given some definitions and results in Q-soft normal subgroup theory and cosets. Also some results were introduced which have been used by homomorphism and anti-homomorphism of Q-soft normal subgroups. Next they proved the analogue of the Lagrange's theorem.

In [3], the author, by using nano topology, introduced micro topology and also study the concepts of micro-pre open sets and micro-semi open sets and some of their properties are investigated.

In [4], the authors investigated the notion of generalized roughness for fuzzy ideals in OSGs on the basis of isotone and monotone mappings. Then the notion of approximation is boosted to the approximation of fuzzy bi-ideals, approximations fuzzy interior ideals and approximations fuzzy quasi-ideals in OSGs and investigate their related properties. Furthermore $(\in, \in \vee q)$-fuzzy ideals are the generalization

*Editor-in-Chief of the Journal of New Theory.

of fuzzy ideals. Also the generalized roughness for $(\in, \in \vee q)$-fuzzy ideals, fuzzy bi-ideals and fuzzy interior ideals studied in OSGs and discuss the basic properties on the basis of isotone and monotone mappings.

In [5], the authors introduced the concept of matrix operators and establishes two new theorems on matrix summability of Fourier series and its derived series. The results obtained in the paper further extend several known results on linear operators. Various types of criteria, under varying conditions, for the matrix summability of the Fourier series. In this paper quite a different and general type of criterion for summability of the Fourier Series has been obtained. In the theorem function is integrable in the sense of Lebesgue to the interval $[-\pi, \pi]$ and period with period 2π.

In [6], the authors introduced a single structure which carries the subsets of X as well as the subsets of Y under the parameter E for studying the information about the ordered pair of soft subsets of X and Y. Such a structure is called a binary soft structure from X to Y. The purpose of this paper is to introduce certain binary soft weak axioms that are analogous to the axioms of topology.

In [7], the author introduced Ψ^*-locally closed sets and different notions of generalizations of continuous functions in a topological space and study some of their properties. Several examples are given to illustrate the behavior of these new classes of functions. The author also defined Ψ^*-submaximal spaces.

In [8], the authors introduced fuzzy soft locally closed and fuzzy soft b-locally closed sets and study their properties in fuzzy soft topological space. Further they defined and studied fuzzy soft LC-continuous and fuzzy soft b-LC-continuous functions.

In [9], the author suggested a mathematical model in form fractional-order differential equations (FDEs) system identifying population dynamics in two species bacteria struggling one another and exposed to multiple antibiotics simultaneously. Stability analysis of the equilibrium points of the proposed model was also carried out. Additionally, the results of the analysis have promoted by numerical simulations.

2 Acknowledgement

We would like to express our deepest thanks to all of the members of the editorial board and reviewers of the papers in this issue who are İ. Deli, F. Karaaslan, F. Smarandache, M. A. Noor, B. Davvaz, J. Zhan, H. Garg, S. Broumi, S. Pramanik, M. I. Ali, P. K. Maji, O. Muhtaroğlu, A. A. Ramadan, S. Enginoğlu, S. J. John, M. Ali, A. Sezgin, A. M. A. Latif, M. Sarı, J. Ye, D. Mohamad, İ. Zorlutuna, A. Shehata, K. Aydemir, T. Biswas, S. Demiriz, H. Olğar, A. Boussayoud, E. H. Hamouda, K. Mondal, G.Senel, N. Yaqoob, Q. H. Imran, A. F. Atik, P. G. Patil, F. Feng.

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