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Anti Fuzzy BG-ideals in BG-algebra

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Abstaract − In this paper, we introduce the concept of anti fuzzy BG-ideals in BG-algebra and we have discussed some of their properties. Relation between anti fuzzy BG-ideal and cartessian product of anti fuzzy BG-ideals is developed.

Keywords − *BG-algebra, sub BG-algebra, BG-ideals, anti fuzzy BG-ideals, anti fuzzy BG-bi-ideal.*

1 Introduction

In 1965, Zadeh [20] introduced the notion of a fuzzy set and fuzzy subset of a set as a method for representing uncertainty in real physical world. Since then its application have been growing rapidly over many disciplines. As a generalization of this, intuitionistic fuzzy subset was defined by K. T. Atanassov [3, 2, 4] in 1986. In 1971, Rosenfield [17] introduced the concept of fuzzy subgroup. Motivated by this, many mathematicians started to review various concepts and theorems of abstract algebra in the broader frame work of fuzzy settings. K. Iseki and Jun et al. introduced three classes of abstract algebras: BCI-algebras, BH-algebras and BCK-algebras [8, 10, 13], respectively. It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras. Further, the notion of intutionistic fuzzy ideals was introduced by Jun and Kim in BCK-algebras [9]. In [7, 6] Hu and Li introduced a wide class of abstract algebras: BCH-algebras. They have shown that the class of BCI-algebras is a proper subclass of the class of BCH-algebras. Neggers and HKim [15, 16] introduced the notion of B-algebras and d-algebras which is

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another generalization of BCK-algebras, and they also investigated several relations between d-algebras and BCK-algebras as well as some other interesting relations between d-algebras and oriented digraphs. Ahn and Lee studied fuzzy subalgebra of BG-algebra in [1]. The fuzzy ideals in BCI-algebras and MV-algebras was studied by Hoo in [5]. The concept of anti fuzzy filters of pseudo-BL-algebras was introduced by Rysiawa in [18].

In this paper we initiate the study of anti fuzzy BG-ideal in BG-algebra. This paper comprises of four section. In section 2, we recall some basic definitions of BG-algebras. In section 3, we define anti fuzzy subalgebras and also give example of anti fuzzy subalgebras. In section 4, we define anti fuzzy BG-ideals and provided a condition for a every anti fuzzy BG-bi-ideals is an anti fuzzy BG-ideal

2 Preliminary

In this section we site the basic definitions that will be used in the sequel.

Definition 2.1. A nonempty set X with a constant 0 and a binary operation '$*$' is called a BG-algebra if it satisfies the following axioms:

1. $x * x = 0$,
2. $x * 0 = x$,
3. $(x * y) * (0 * y) = x$ for all $x, y \in X$.

Example 2.2. Let $X = \{0, 1, 2\}$ be the set with the following table.

$*$	0	1	2
0	0	1	2
1	1	0	1
2	2	2	0

Then $(X, *, 0)$ is a BG-algebra.

Definition 2.3. [14] Let S be a non empty subset of a BG -algebra X then S is called a subalgebra of X if $x * y \in S$, for all $x, y \in S$.

Definition 2.4. [14] Let X be a BG-algebra and I be a subset of X, then I is called a BG-ideal of Xif it satisfies following conditions:

1. $0 \in I$
2. $x * y \in I$ and $y \in I \Longrightarrow x \in I$,
3. $x \in I$ and $y \in X \Longrightarrow x * y \in I$.

Definition 2.5. [14] A mapping $f : X \to Y$ of a BG-algebra is called a homomorphism if $f(x * y) = f(x) * f(y)$ for all$x, y \in X$.

Remark 2.6. If $f : X \to Y$ is a homomorphism of BG-algebra, then $f(0) = 0$.

Definition 2.7. [14] Let X be a non-empty set. A fuzzy subset μ of the set X is a mapping $X \to [0, 1]$.

Definition 2.8. [14] A fuzzy set μ in X is said to be a fuzzy BG-bi-ideal if

$$\mu(x * w * y) \geq \min\{\mu(x), \mu(y)\} \text{ for all } x, y, w \in X.$$

3 Anti Fuzzy Subalgebras

Definition 3.1. Let μ be a fuzzy set in BG-algebra. Then μ is called an anti fuzzy subalgebra of X if

$$\mu(x * y) \leq \max\{\mu(x), \mu(y)\} \text{ for all } x, y \in X.$$

Example 3.2. Let $X = \{0, 1, 2, 3\}$ be the set with the following table.

$*$	0	1	2	3
0	0	1	2	3
1	1	0	1	1
2	2	2	0	2
3	3	3	3	0

Then $(X, *, 0)$ is a BG-algebra. Define a fuzzy set $\mu : X \to [0, 1]$ by

$$\mu(x) = \begin{cases} t_0 & \text{if } x \in \{2, 3\} \\ t_1 & \text{if } x \in \{0, 1\} \end{cases}$$

for t_0, $t_1 \in [0, 1]$, with $t_0 > t_1$. Then μ is an anti fuzzy subalgebra of X.

Definition 3.3. Let μ be a fuzzy set in a set X For $t \in [0, 1]$, the set

$$\mu_t = \{x \in X : \mu(x) \leq t\}$$

is called a lower level subset of μ.

4 Anti Fuzzy BG-ideals

Definition 4.1. A fuzzy set μ in X is called an anti fuzzy BG-ideals of X if it satisfies the following inequalities:

1. $\mu(0) \leq \mu(x)$,
2. $\mu(x) \leq \max\ \{\mu(x * y), \mu(y)\}$,
3. $\mu(x * y) \leq \max\{\mu(x), \mu(y)\}$for all $x, y \in X$.

Example 4.2. Let $X = \{0, 1, 2, 3\}$be the set with the following table.

$*$	0	1	2	3
0	0	1	2	3
1	1	0	1	1
2	2	2	0	2
3	3	3	3	0

Then $(X, *, 0)$ is a BG-algebra. Define a fuzzy set $\mu : X \to [0, 1]$ by

$$\mu(x) = \begin{cases} \alpha & \text{if } x \in \{2, 3\} \\ \beta & \text{if } x \in \{0, 1\} \end{cases}$$

for α,$\beta \in [0, 1]$, with $\alpha > \beta$. Then μ is an anti fuzzy BG-ideal of X.

Example 4.3. Let $X = \{0, 1, 2\}$be the set with the following table:

$*$	0	1	2
0	0	1	2
1	1	0	1
2	2	2	0

Then $(X, *, 0)$ is a BG-algebra. We define a fuzzy set$\mu : X \to [0, 1]$ by

$$\mu(x) = \begin{cases} 0.2 & \text{if } x = 0 \\ 0.8 & \text{otherwise.} \end{cases}$$

Then μ is an anti fuzzy BG-ideal of X.

Definition 4.4. Let μ and λbe the fuzzy sets in a set X. The Cartesian product $\lambda \times \mu$: $X \times X \to [0, 1]$ is defined by $(\lambda \times \mu)(x, y) = \max\{\lambda(x), \mu(y)\}$ for all $x, y \in X$.

Theorem 4.5. If λ and μ are anti fuzzy BG-ideal of a BG-algebra X, then $\lambda \times \mu$ is an anti fuzzy BG-ideal of $X \times X$.

Proof. For any $(x, y) \in X \times X$ we have

$$\begin{aligned} (\lambda \times \mu)(0, 0) &= \max\{\lambda(0), \mu(0)\} \\ &\leq \max\{\lambda(x), \mu(y)\} \\ &= (\lambda \times \mu)(x, y). \end{aligned}$$

That is,

$$(\lambda \times \mu)(0, 0) \leq (\lambda \times \mu)(x, y).$$

Let (x_1, x_2) and $(y_1, y_2) \in X \times X$.Then,

$$\begin{aligned} (\lambda \times \mu)(x_1, x_2) &= \max\{\lambda(x_1), \mu(x_2)\} \\ &\leq \max\{\max\{\lambda(x_1 * y_1), \lambda(y_1)\}, \max\{\mu(x_2 * y_2), \mu(y_2)\}\} \\ &= \max\{\max\{\lambda(x_1 * y_1), \mu(x_2 * y_2)\}, \max\{\lambda(y_1), \mu(y_2)\}\} \\ &= \max\{(\lambda \times \mu)((x_1 * y_1, x_2 * y_2)), (\lambda \times \mu)(y_1, y_2)\} \\ &= \max\{(\lambda \times \mu)((x_1, x_2) * (y_1, y_2)), (\lambda \times \mu)(y_1, y_2)\} \end{aligned}$$

That is,

$$(\lambda \times \mu)(x_1, x_2) \leq \max\{(\lambda \times \mu)((x_1, x_2) * (y_1, y_2)), (\lambda \times \mu)(y_1, y_2)\}$$

and

$$\begin{aligned} (\lambda \times \mu)((x_1, x_2) * (y_1, y_2)) &= (\lambda \times \mu)(x_1 * y_1, x_2 * y_2) \\ &= \max\{\lambda(x_1 * y_1), \mu(x_2 * y_2)\} \\ &\leq \max\{\max\{\lambda(x_1), \lambda(y_1)\}, \max\{\mu(x_2), \mu(y_2)\}\} \\ &= \max\{\max\{\lambda(x_1), \mu(x_2)\}, \max\{\lambda(y_1), \mu(y_2)\}\} \\ &= \max\{(\lambda \times \mu)((x_1, x_2)), (\lambda \times \mu)((y_1, y_2))\}. \end{aligned}$$

That is,

$$\begin{aligned} &(\lambda \times \mu)((x_1, x_2) * (y_1, y_2)) \\ \leq\ &\max\{(\lambda \times \mu)(x_1, x_2), (\lambda \times \mu)(y_1, y_2)\} \end{aligned}$$

Hence $\lambda \times \mu$ is an anti fuzzy BG-ideal of $X \times X$. □

Theorem 4.6. Let λ *and* μ be fuzzy sets in a BG-algebra such that $\lambda \times \mu$ is an anti fuzzy BG-ideal of $X \times X$. Then

I. $\lambda(0) \leq \lambda(x)$ and $\mu(0) \leq \mu(x)$ for all $x \in X$.
II. If $\lambda(0) \leq \lambda(x)$ then $\mu(0) \leq \lambda(x)$ and $\mu(0) \leq \mu(x)$ for all $x \in X$.
III. If $\mu(0) \leq \mu(x)$ then $\lambda(0) \leq \lambda(x)$ and $\lambda(0) \leq \mu(x)$ for all $x \in X$.

Proof. I. Assume $\lambda(x) < \lambda(0)$ or $\mu(y) < \mu(0)$for some $x, y \in X$. Then

$$\begin{aligned}(\lambda \times \mu)(x, y) &= \max\ \{\lambda(x), \mu(y)\} \\ &< \max\{\lambda(0), \mu(0)\} \\ &= (\lambda \times \mu)(0, 0).\end{aligned}$$

Thus $(\lambda \times \mu)(x, y) < (\lambda \times \mu)(0, 0)$ for all $x, y \in X$ Which is a contradiction to $(\lambda \times \mu)$ is an anti fuzzy BG-ideal of $X \times X$. Therefore, $\lambda(0) \leq \lambda(x)$ and $\mu(0) \leq \mu(y)$for all $x, y \in X$.

II. Assume either $\mu(0) > \lambda(x)$ or $\mu(0) > \mu(y)$ for all $x, y \in X$. Then

$$\begin{aligned}(\lambda \times \mu)(0, 0) &= \max\{\lambda(0), \mu(0)\} \\ &= \mu(0)\end{aligned}$$

and

$$\begin{aligned}(\lambda \times \mu)(x, y) &= \max\ \{\lambda(x), \mu(y)\} < \mu(0) \\ &= (\lambda \times \mu)(0, 0).\end{aligned}$$

This implies

$$(\lambda \times \mu)(x, y) < (\lambda \times \mu)(0, 0).$$

Which is a contradiction to $\lambda \times \mu$ is an anti fuzzy BG-ideal of $X \times X$. Hence if

$$\lambda(0) \leq \lambda(x) \text{ for all } x \in X,$$

then

$$\mu(0) \leq \lambda(x) \text{ and } \mu(0) \leq \mu(x)$$

III. The proof is quite similar to (ii). □

Theorem 4.7. If $\lambda \times \mu$ is an anti fuzzy BG-ideals of $X \times X$, then λ and μ is an anti fuzzy BG-ideals of X.

Proof. Firstly to prove that μ is an anti fuzzy BG-ideal of X. Given $\lambda \times \mu$ is an anti fuzzy BG-ideals of $X \times X$, then by Theorem 4.6 (i)

$$\lambda(0) \leq \lambda(x) \text{ and } \mu(0) \leq \mu(x) \text{ for all } x \in X.$$

Let $\mu(0) \leq \mu(x)$. By Theorem 4.6 (iii), then $\lambda(0) \leq \lambda(x)$ and $\lambda(0) \leq \mu(x)$.Now

$$\begin{aligned}\mu(x) &= \max\{\lambda(0), \mu(x)\} \\ &= (\lambda \times \mu)(0, x) \\ &\leq \max\{(\lambda \times \mu)((0, x) * (0, y)),\ (\lambda \times \mu)(0, y)\} \\ &= \max\{(\lambda \times \mu)(0 * 0, x * y),\ (\lambda \times \mu)(0, y)\} \\ &= \max\{(\lambda \times \mu)(0, x * y),\ (\lambda \times \mu)(0, y)\} \\ &= \max\{(\lambda \times \mu)(0 * 0,\ x * y), (\lambda \times \mu)(0, y)\} \\ &= \max\{\mu(x * y),\ \mu(y)\}.\end{aligned}$$

That is,

$$\begin{aligned}\mu(x) &\leq \max\{\mu(x*y),\ \mu(y)\}\\ \mu(x*y) &= \max\{\lambda(0),\mu(x*y)\}\\ &= (\lambda\times\mu)(0,x*y)\\ &= (\lambda\times\mu)(0*0,x*y)\\ &= (\lambda\times\mu)(0,x)*(0,y)\end{aligned}$$

$$\begin{aligned}\mu(x*y) &\leq \max\{(\lambda\times\mu)(0,x),(\lambda\times\mu)(0,y)\}\\ &= \max\{\mu(x),\mu(y)\}.\end{aligned}$$

That is,

$$\mu(x*y)\leq\max\{\mu(x),\mu(y)\}.$$

This proves that μ is an anti fuzzy BG-ideal of X. Secondly to prove that λ is an anti fuzzy BG-ideal of X. Given $\lambda\times\mu$ is an anti fuzzy BG-ideal of $X\times X$, then by Theorem 4.6 (i), we have
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$$\lambda(0)\leq\lambda(x) \text{ and } \mu(0)\leq\mu(x) \text{ for all } x\in X.$$

Let $\lambda(0)\leq\lambda(x)$.By Theorem 4.6 (ii), then $\mu(0)\leq\lambda(x)$ and $\mu(0)\leq\mu(x)$.Now

$$\begin{aligned}\lambda(x) &= \max\{\lambda(x),\mu(0)\} = (\lambda\times\mu)(x,0)\\ &\leq \max\{(\lambda\times\mu)((x,0)*(y,0)),\ (\lambda\times\mu)(0,y)\}\\ &= \max\{(\lambda\times\mu)(x*y,0*0),\ (\lambda\times\mu)(0,y)\}\\ &= \max\{(\lambda\times\mu)(x*y,0),\ (\lambda\times\mu)(0,y)\}\\ &= \max\{\lambda(x*y),\ \lambda(y)\}\end{aligned}$$

That is,

$$\begin{aligned}\lambda(x) &\leq \max\{\lambda(x*y),\ \lambda(y)\}.\\ \lambda(x*y) &= \max\{\lambda(x*y),\mu(0)\}\\ &= (\lambda\times\mu)(x*y,0)\\ &= (\lambda\times\mu)(x*y,0*0)\\ &= (\lambda\times\mu)\left((x,0)*(y,0)\right)\end{aligned}$$

$$\begin{aligned}\lambda(x*y) &\leq \max\{(\lambda\times\mu)(x,0),(\lambda\times\mu)(y,0)\}\\ &= \max\{\lambda(x),\lambda(y)\}\end{aligned}$$

That is

$$\lambda(x*y)\leq\max\{\lambda(x),\lambda(y)\}.$$

This proves that λ is an anti fuzzy BG-ideal of X. □

Theorem 4.8. If μ is an anti fuzzy BG-ideal of X then μ_t is a BG-ideal of X for all $t \in [0,1]$.

Proof. Let μbe anti fuzzy BG-ideal of X and $x, y \in X$. If $x, y \in \mu_t$ then

$$\mu(0) \leq \mu(x) \leq t \text{ implies } 0 \in \mu_t \text{ for all } t \in [0,1].$$

Let $x * y \in \mu_t$ and $y \in \mu_t$. Therefore, $\mu(x * y) \leq t$ and $\mu(y) \leq t$. Now

$$\mu(x) \leq \max\{\mu(x * y), \mu(y)\} \leq \max\{t, t\} \leq t.$$

Hence $\mu(x) \leq t$. That is, $x \in \mu_t$.

Let $x \in \mu_t$, $y \in X$. Choose y in X such that $\mu(y) \leq t$. Since $x \in \mu_t$implies $\mu(x) \leq t$. We know that

$$\begin{aligned} \mu(x * y) &\leq \max\{\mu(x), \mu(y)\} \\ &\leq \max\{t, t\} \leq t. \end{aligned}$$

That is

$$\mu(x * y) \leq t \text{ implies } x * y \in \mu_t.$$

Hence μ_t is a BG-ideal of X. □

Theorem 4.9. If X be a BG-algebra, $t \in [0,1]$,and μ_t is a BG-ideal of X, then μ is an anti fuzzy BG-ideals of X.

Proof. Let μ_t be a BG-ideal of X. Let $x, y \in \mu_t$. Then $\mu(x) \leq t$ and $\mu(y) \leq t$. Let $\mu(x) = t_1$ and $\mu(y) = t_2$, without loss of generality let $t_1 \leq t_2$. Then $x \in \mu_{t_2}$. Now $x \in \mu_{t_2}$ and $y \in X$ implies $x * y \in \mu_{t_2}$. That is,

$$\begin{aligned} \mu(x * y) &\leq t_2 \\ &= \max\{t_1,\ t_2\} \\ &= \max\{\mu(x), \mu(y)\}. \end{aligned}$$

That is,

$$\mu(x * y) \leq \max\{\mu(x), \mu(y)\}.$$

Now let

$$\begin{aligned} \mu(0) &= \mu(x * x) \\ &\leq \max\ \{\mu(x), \mu(x)\} \\ &= \mu(x). \end{aligned}$$

That is $\mu(0) \leq \mu(x)$ for all $x \in X$.

Further

$$\begin{aligned} \mu(x) &= (\mu(x * y) * (0 * y)) \\ &\leq \max\{\mu(x * y), \mu(0 * y)\} \\ &\leq \max\{\mu(x * y), \max\{\mu(0), \mu(y)\}\} \\ &\leq \max\{\mu(x * y), \mu(y)\}. \end{aligned}$$

Hence μ is an anti fuzzy BG-ideal of X. □

Definition 4.10. A fuzzy set μ in X is said to be an anti fuzzy BG-bi-ideal if $\mu(x * y * w) \leq \max\{\mu(x), \mu(y)\}$ for all $x, y, w \in X$.

Theorem 4.11. Every anti fuzzy BG-bi-ideal is an anti fuzzy BG-ideal.

Proof. It is trivial. □

Remark 4.12. The following example shows that the converse of Theorem 4.11 is not true in general.

Example 4.13. Let $X = \{0, 1, 2\}$ be the set with the following table:

$*$	0	1	2
0	0	1	2
1	1	0	1
2	2	2	0

Then $(X, *, 0)$ is a BG-algebra. We define a fuzzy set $\mu : X \to [0, 1]$ by

$$\mu(x) = \begin{cases} 0.3 & \text{if } x = 0 \\ 0.8 & \text{otherwise.} \end{cases}$$

Clearly μ is anti fuzzy BG-ideals of X. But is not an anti fuzzy BG-bi-ideal of X. Now let $x = 0, w = 1, y = 0$.Then $\mu(x * w * y) = \mu(0 * 1 * 0) = \mu(0 * 1)\mu(1) = 0.8$. $\max\{\mu(x), \mu(y)\} = \max\{\mu(0), \mu(0)\} = \mu(0) = 0.2$. Hence $\mu(x * w * y) \geq \max\{\mu(x), \mu(y)\}$. Hence μ is not an anti fuzzy BG-bi-ideal of X.

Definition 4.14. Let $f : X \to Y$ be a mapping of BG-algebra and μ be a fuzzy set of Y then μ^f is the pre-image of μ under f if $\mu^f(x) = \mu(f(x))$ for all $x \in X$.

Theorem 4.15. Let $f : X \to Y$ be a homomorphism of BG-algebra. If μ is an anti fuzzy BG-ideals of Y. Then μ^f is an anti fuzzy BG-ideal of X.

Proof. For any $x \in X$, we have

$$\mu^f(x) = \mu(f(x)) \geq \mu(0) = \mu(f(0)) = \mu^f(0).$$

Let $x, y \in X$, then

$$\begin{aligned} \max\{\mu^f(x * y), \mu^f(y)\} &= \max\{\mu(f(x * y)), \mu(f(y))\} \\ &= \max\{\mu(f(x) * f(y)), \mu(f(y))\} \\ &\geq \mu(f(x)) \\ &= \mu^f(x). \end{aligned}$$

That is,

$$\begin{aligned} \mu^f(x) &\leq \max\{\mu^f(x * y), \mu^f(y)\}. \\ \max\{\mu^f(x), \mu^f(y)\} &= \max\{\mu(f(x)), \mu(f(y))\} \\ &\geq \mu(f(x) * f(y)) \\ &= \mu(f(x * y)) \\ &= \mu^f(x * y). \end{aligned}$$

That is,

$$\mu^f(x * y) \leq \max\{\mu^f(x), \mu^f(y)\}.$$

Hence μ^f is an anti fuzzy BG-ideal of X. □

Theorem 4.16. Let $f : X \to Y$ be an epimorphism of BG-algebra. If μ^f is an anti fuzzy BG-ideal of X, then $\mu$is an anti fuzzy BG-ideal of Y.

Proof. Let $y \in Y$. By hypothesis there exist $x \in X$ such that $f(x) = y$, then

$$\begin{aligned} \mu(y) &= \mu(f(x)) \\ &= \mu^f(x) \\ &\geq \mu^f(0) \\ &= \mu(f(0)) = \mu(0). \end{aligned}$$

Let $x, y \in Y$. By hypothesis there exist $a, b \in X$such that $f(a) = x$ and $f(b) = y$. It follows that

$$\begin{aligned} \mu(x) &= \mu(f(a)) \\ &= \mu^f(a) \\ &\leq \max\{\mu^f(a * b), \mu^f(b)\} \\ &= \max\{\mu(f(a * b)), \mu(f(b))\} \\ &= \max\{\mu(f(a) * f(b)), \mu(f(b)\} \\ &= \max\{\mu(x * y), \mu(y)\}. \end{aligned}$$

That is, $\mu(x) \leq \max\ \{\mu(x * y), \mu(y)\}$.

$$\begin{aligned} \mu(x * y) &= \mu(f(a) * f(b)) \\ &= \mu(f(a * b)) \\ &= \mu^f(a * b) \\ &\leq \max\{\mu^f(a), \mu^f(b)\} \\ &= \max\{\mu(f(a)), \mu(f(b))\} \\ &= \max\{\mu(x), \mu(y)\}. \end{aligned}$$

Thus $\mu(x * y) \leq \max\ \{\mu(x), \mu(y)\}$. Hence μ is anti fuzzy BG-ideals of Y. □

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Quasilinear Evolution Integrodifferential Equations in Banach Spaces

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Abstract – Existence and uniqueness of local classical solutions of the quasilinear evolution integrodifferential equation in Banach spaces are studied. The results are demonstrated by employing the fixed point technique on C_0-semigroup of bounded linear operator. At last, we deal an example to interpret the theory.

Keywords – Quasilinear evolution integrodifferential equation, local classical solution, C_0-semigroups, fixed point theorem.

1 Introduction

In this work, we examine the quasilinear evolution equation of the following form

$$\frac{du(t)}{dt}+A(t,u)u(t)=H(u)(t)+f\left(t,u(t),G(u)(t)\right), \tag{1}$$

$$u(0)=u_0, t\in[0,T]=J \tag{2}$$

where $A(t,u)$ is the infinitesimal generator of a C_0-semigroup in a Banach space X. $u_0\in X$, $f:J\times X\times X\to X$ are functions and H and G are the nonlinear Volterra operators

$$H(u)(t)=\int_0^t k(t-s)h(s,u(s))ds \text{ and } G(u)(t)=\int_0^t a(t-s)g(s,u(s))ds$$

where $a,k:J\to J$ are real valued continuous functions and $h,g:J\times X\to X$ are functions.

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A lot of researchers have investigated the existence of solutions of various types of abstract quasilinear evolution equations in Banach space [2, 3, 9, 14]. Pazy [11] considered the quasilinear equation of the form

$$u'(t) + A(t, u)u(t) = 0, \ 0 < t \le T, \ u(0) = u_0$$

and studied the mild and classical solutions by applying fixed point theorem. Abbas et.al [1] considered a class of quasilinear functional differential equations in which the author investigated the existence of solutions for the same system by employing the thought of C_0-semigroup of bounded linear operator. Results on the existence and uniqueness of solutions for problems of quasilinear differential equation with deviating arguments can be found in [8].

Quasilinear integrodifferential systems in abstract form have got more notice because such equations appear in different domain of science e.g. mathematical physics, population dynamics etc. Different kinds of quasilinear integrodifferential equation in Banach space have been investigated by numerous authors [4- 7, 10, 12, 13].

The remaining work is ordered as follows. In segment 2, we state some prelude. In segment 3, we give main result. Finally a concrete example is given in last segment 4 to show the relevance of abstract theory.

2 Preliminaries

Let X and Y be two Banach spaces such that Y is densely and continuously embedded in X. The norm in any Banach space Z is expressed by $\|.\|$ or $\|.\|_Z$. Consider $B(X,Y)$ be the set of all bounded linear operators from a Banach space X to a Banach space Y. We write $B(X,X)$ by $B(X)$.

Let $B \subset X$ and let $A(t,b)$ be the infinitesimal generator of a C_0-semigroup $S_{t,b}(s), s \ge 0,$ on X. $\{A(t,b)\}, (t,b) \in J \times B$ is the family of operators which is stable if there exist constants $M \ge 1$ and ω such that

$$\rho(A(t,b)) \supset \,]\omega, \infty[\qquad \text{for } (t,b) \in J \times B,$$

where $\rho(A(t,b))$ is the resolvent set of $A(t,b)$ and

$$\left\| \prod_{j=1}^{k} R\left(\lambda : A\left(t_j, b_j\right)\right) \right\| \le M\,(\lambda - \omega)^{-k}$$

for $\lambda > \omega$ and every finite sequence $0 \le t_1 \le t_2 \ldots\ldots \le t_k \le T, b_j \in B, 1 \le j \le k.$

The stability of $\{A(t,b)\}, (t,b) \in J \times B$ implies that

$$\left\|\prod_{j=1}^{k} S_{t_j,b_j(s_j)}\right\| \le M \exp\left\{\omega\sum_{j=1}^{k} s_j\right\}, s_j \ge 0$$

and any finite sequence $0 \le t_1 \le t_2 \ldots\ldots \le t_k \le T, b_j \in B, 1 \le j \le k.$

Suppose a linear operator S in X and let a subspace Y of X. The operator $\tilde{S}$ with domain $D(\tilde{S}) = \{x \in D(S) \cap Y : Sx \in Y\}$ and $\tilde{S}x = Sx$ for $x \in D(\tilde{S})$ is remarked to be the part of S in Y.

Let $S_{t,b}(s), s \ge 0$ be the C_0-semigroup generated by $\{A(t,b)\}, (t,b) \in J \times B.$ A subspace Y of X is called $A(t,b)$-admissible if Y is an invariant subspace of operator $S_{t,b}(s), s \ge 0$ and the restriction of $S_{t,b}(s)$ to Y is a C_0-semigroup in $Y.$

For deep information of the above noticed notions, we refer the work of Pazy [11] in chapters 5 and 6. On the family of operators $\{A(t,b) : (t,b) \in J \times B\}$, we perform the same hypothesis $(H_1)-(H_4)$ given in section 6.6.4 in Pazy [11] for the homogenous quasilinear evolution equation, as recall below.

(H_1) The family $\{A(t,b) : (t,b) \in J \times B\}$ is stable.

(H_2) Y is $A(t,b)$-admissible for $(t,b) \in J \times B$ and the family $\{A(t,b)\}, (t,b) \in J \times B$ of parts of $A(t,b)$ of $A(t,b)$ in Y, is stable in $Y.$

(H_3) For $(t,b) \in J \times B, D(A(t,b)) \supset Y,$ $A(t,b)$ is a bounded linear operator from Y to X and the map $t \mapsto A(t,b)$ is continuous in the $B(Y,X)$ norm $\|.\|_{Y\to X}$ for every $b \in B.$

(H_4) There is a constant L such that

$$\|A(t,b_1) - A(t,b_2)\|_{Y\to X} \le L\|b_1 - b_2\|_X$$

holds for every $b_1, b_2 \in B$ and $0 \le t \le T.$

Definition 2.1: A two parameter family of bounded linear operators $U(t,s), 0 \le s \le t \le T,$ on X is called an evolution system if the following two conditions are satisfied:

(i) $U(s,s) = I$, $U(t,r)U(r,s) = U(t,s)$ for $0 \le s \le r \le t \le T.$

(ii) $(t,s) \to U(t,s)$ is strongly continuous for $0 \le s \le t \le T.$

Moreover, let $B \subset X$ and let $\{A(t,b)\}$, $(t,b) \in J \times B$ be a family of operators fulfilling the above stated hypothesis $(H_1)-(H_4)$. If $u \in C(J,X)$ has values in B then there is a unique evolution system $U_u(t,s), 0 \le s \le t \le T$, in X satisfying

(i) $$\|U_u(t,s)\| \le M \exp \omega(t-s) \tag{3}$$

for $0 \le s \le t \le T$, where M and ω are stability constants;

(ii) $$\frac{\partial^+}{\partial t} U_u(t,s) w \Big|_{t=s} = A(s,u(s)) w \tag{4}$$

for $w \in Y$, and $0 \le s \le t \le T$;

(iii) $$\frac{\partial}{\partial s} U(t,s;u) w = -U_u(t,s) A(s,u(s)) w \tag{5}$$

for $w \in Y$, and $0 \le s \le t \le T$.

Again, there is a constant C_1 such that for every $u,v \in C(J,X)$ with values in B and for every $w \in Y$, we have

$$\|U_u(t,s) w - U_v(t,s) w\| \le C_1 \|w\|_Y \int_s^t \|u(\tau) - v(\tau)\| d\tau. \tag{6}$$

To find the above noticed outcomes in details, the Theorem 6.4.3 and Lemma 6.4.4 is given in Pazy [11].

Further we consider that

(H_5) For every $u \in C(J,X)$ satisfying $u(t) \in B$ for $t \in J$, we have

$$U_u(t,s) Y \subset Y, 0 \le s \le t \le T$$

where $U(t,s)$ is strongly continuous in Y for $t,s \in J$ and $s \le t$.

(H_6) Every closed convex and bounded subset of Y is also closed in X.

(H_7) The real-valued function a and b are continuous on I and there exist positive constants k_T and a_T such that $|k(t)| \le k_T$ and $|a(t)| \le a_T$ for $t \in J$.

(H_8) $h: J \times X \to X$ is continuous and there exist constants $H_L > 0$ and $H_0 > 0$ such that

$$\int_0^t \|h(s,x) - h(s,y)\| ds \le H_L \left[\|x(t) - y(t)\|\right]$$

and

$$H_0 = \max \int_0^t \|h(s,0)\| ds.$$

For the conditions (H_9) and (H_{10}), Z be taken as both X and Y.

(H_9) $f : J \times Z \times Z \to Z$ is continuous and there exist constants $F_L > 0$ and $F_0 > 0$ such that

$$\left\| f(t,u_1,v_1) - f(t,u_2,v_2) \right\|_Z \le F_1 \left(\|u_1 - u_2\|_Z + \|v_1 - v_2\|_Z \right)$$

and

$$F_0 = \max_{t \in J} \left\| f(t,0,0) \right\|_Z .$$

(H_{10}) $g : J \times Z \to Z$ is continuous and there exist constants $G_L > 0$ and $G_0 > 0$ such that

$$\int_0^t \left\| g(s,u_1) - g(s,u_2) \right\|_Z ds \le G_L \left(\|u_1(t) - u_2(t)\|_Z \right)$$

and

$$G_0 = \max \left\{ \int_0^t \| k(s,0) \| ds \right\}.$$

Let $M = \max \left\{ \|U_u(t,s)\|_{B(Z)}, 0 \le s \le t \le T, u \in B \right\}$.

(H_{11}) $M_0 \left\{ \|u_0\|_Y + k_T r T H_L + k_T T H_0 + F_L r T + a_T F_L G_L r T + a_T F_L G_0 T + F_0 T \right\} \le r$

and

$$\Gamma = \begin{bmatrix} C_1 T \|u_0\|_Y + C_1 T^2 \left\{ k_T (H_L r + H_0) + F_L (r + a_T G_L r + a_T G_0) + F_0 \right\} \\ + M T k_T H_L + M F_L T + M F_L G_L a_T T \end{bmatrix} < 1.$$

We mentioned that condition (H_6) is always satisfied if X and Y are reflexive Banach space. Next we prove the existence of local classical solution of the quasilinear problem (1)–(2). By a mild solution to (1) – (2) on $J = [0,T]$, we signify a function $u \in C(J,X)$ with values in B satisfying the integral equation

$$u(t) = U_u(t,0)u_0 + \int_0^t U_u(t,s) \left[Hu(s) + f\left(s, u(s), \int_0^s a(s-\tau) g(\tau, u(\tau)) d\tau \right) \right] ds. \quad (7)$$

A function $u \in C(J,X)$ such that $u(t) \in Y \cap B$ for $t \in (0,T]$ and $u \in C^1((0,T],X)$ satisfying the equation (1) – (2) in X is called a classical solution of (1) – (2) on J, where $C^1(J,X)$ space of all continuously differentiable functions from J to X.

3 Existence Result

Theorem 3.1: Let $u_0 \in Y$ and let $B = \{u \in X : \|u\|_Y \le r\}, r > 0.$ If the hypothesis $(H_1)-(H_{10})$ are satisfied, then (1)–(2) has a unique classical solution $u \in C([0,T]:Y) \cap C^1((0,T]:X).$

Proof: Let S be the nonempty closed subset of $C([0,T],X)$ defined by

$$S = \{u : u \in C([0,T],X), \|u(t)\|_Y \le r \text{ for t} \in \text{J}\}.$$

Suppose a mapping F on S defined by

$$(Fu)(t) = U_u(t,0)u_0 + \int_0^t U_u(t,s)\left[H(u)(s) + f\left(s,u(s),\int_0^s a(s-\tau)g(\tau,u(\tau))d\tau\right)\right]ds.$$

We state that $F : S \to S$. For $u \in S,$ we have

$$\|Fu(t)\|_Y = \left\|U_u(t,0)u_0 + \int_0^t U_u(t,s)\left[\int_0^s k(s-\tau)h(\tau,u(\tau))d\tau + f\left(s,u(s),\int_0^s a(s-\tau)g(\tau,u(\tau))d\tau\right)\right]\right\|ds$$

$$\le \|U_u(t,0)u_0\| + \int_0^t \|U_u(t,s)\| \left[\begin{array}{l}\int_0^s \|k(s-\tau)\|\{\|h(\tau,u(\tau)) - h(\tau,0)\| + \|h(\tau,0)\|\}d\tau \\ +\left\|f\left(s,u(s),\int_0^s a(s-\tau)g(\tau,u(\tau))d\tau\right) - f(s,0,0)\right\| + \|f(s,0,0)\|\end{array}\right]ds$$

using the hypothesis

$$\le M\|u_0\|_Y + M\left[\begin{array}{l}\int_0^t\left\{\int_0^s \|k(s-\tau)\|(\|h(s,u(\tau)) - h(s,0)\| + \|h(s,0)\|)d\tau\right\}ds \\ +\int_0^t\left\{\left\|f\left(s,u(s),\int_0^s a(s-\tau)g(\tau,u(\tau))d\tau\right) - f(s,0,0)\right\| + \|f(s,0,0)\|\right\}ds\end{array}\right]$$

$$\le M\left[\begin{array}{l}\|u_0\|_Y + k_T\int_0^t H_L\|u(s)\|ds + k_T H_0 T \\ +\int_0^t F_L\left\{\|u(s)\| + \int_0^s \|a(s-\tau)\|(\|g(\tau,u(\tau)) - g(\tau,0)\| + \|g(\tau,0)\|)d\tau\right\}ds + F_0 T\end{array}\right]$$

$$\le M\left[\|u_0\|_Y + k_T rTH_L + k_T TH_0 + F_L rT + a_T rTF_L G_L + a_T TF_L G_0 + F_0 T\right]$$

By using hypothesis $(H_{11}),$ we get $\|Fu(t)\|_Y \le r.$ Therefore F maps S into itself. Moreover, for $u,v \in S,$ we have

$$\begin{aligned}\left\|Fu(t)-Fv(t)\right\| &\le \left\|U_u(t,0)u_0-U_v(t,0)u_0\right\| \\ &+\int_0^t \Bigg\| U_u(t,s)\left\{H(u)(s)+f\left(s,u(s),\int_0^s a(s-\tau)g(\tau,u(\tau))d\tau\right)\right\} \\ &-U_v(t,s)\left\{H(v)(s)+f\left(s,v(s),\int_0^s a(s-\tau)g(\tau,v(\tau))d\tau\right)\right\}\Bigg\| ds \\ &\le \left\|U_u(t,0)u_0-U_v(t,0)u_0\right\| \\ &+\int_0^t \Bigg[\Bigg\| U_u(t,s)\left\{H(u)(s)+f\left(s,u(s),\int_0^s a(s-\tau)g(\tau,u(\tau))d\tau\right)\right\} \\ &-U_v(t,s)\left\{H(u)(s)+f\left(s,u(s),\int_0^s a(s-\tau)g(\tau,u(\tau))d\tau\right)\right\}\Bigg\| \\ &+\Bigg\| U_v(t,s)\left\{H(u)(s)+f\left(s,u(s),\int_0^s a(s-\tau)g(\tau,u(\tau))d\tau\right)\right\} \\ &-U_v(t,s)\left\{H(v)(s)+f\left(s,v(s),\int_0^s a(s-\tau)g(\tau,v(\tau))d\tau\right)\right\}\Bigg\|\Bigg] ds\end{aligned}$$

Using our hypothesis, we get

$$\begin{aligned}\left\|Fu(t)-Fv(t)\right\| &\le C_1\left\|u_0\right\|_Y T\max_{\tau\in J}\left\|u(\tau)-v(\tau)\right\| \\ &+\int_0^t \left\|U_u(t,s)-U_v(t,s)\right\|\left\{\left\|H(u)(s)\right\|+\left\|f\left(s,u(s),\int_0^s a(s-\tau)g(\tau,u(\tau))d\tau\right)\right\|\right\}ds \\ &+\int_0^t \left\|U_v(t,s)\right\|\left\{\left\|H(u)(s)-H(v)(s)\right\|+\left\|\begin{matrix} f\left(s,u(s),\int_0^s a(s-\tau)g(\tau,u(\tau))d\tau\right) \\ -f\left(s,v(s),\int_0^s a(s-\tau)g(\tau,v(\tau))d\tau\right)\end{matrix}\right\|\right\}ds \\ &\le C_1\left\|u_0\right\|_Y T\max_{\tau\in J}\left\|u(\tau)-v(\tau)\right\|+C_1T\max_{\tau\in J}\left\|u(\tau)-v(\tau)\right\|\times \\ &\int_0^t\left\{\begin{matrix}\int_0^s\left\|k(s-\tau)\right\|\left(\left\|h(\tau,u(\tau))-h(\tau,0)\right\|+\left\|h(\tau,0)\right\|\right)d\tau \\ +\left\|f\left(s,u(s),\int_0^s a(s-\tau)g(\tau,u(\tau))d\tau\right)-f(s,0,0)\right\|+\left\|f(s,0,0)\right\|\end{matrix}\right\}ds\end{aligned}$$

$$+M\int_0^t \left\{ \begin{array}{l} \left\| \int_0^s k(s-\tau)\,h(\tau,u(\tau))\,d\tau - \int_0^s k(s-\tau)\,h(\tau,v(\tau))\,d\tau \right\| \\ + \left\| f\left(s,u(s),\int_0^s a(s-\tau)\,g(\tau,u(\tau))\,d\tau\right) - f\left(s,v(s),\int_0^s a(s-\tau)\,g(\tau,v(\tau))\,d\tau\right) \right\| \end{array} \right\} ds$$

$$\le C_1 \|u_0\|_Y T \max_{\tau\in J}\|u(\tau)-v(\tau)\| + C_1 T \max_{\tau\in J}\|u(\tau)-v(\tau)\| \int_0^t \left[k_T\,(H_L r + H_0) + F_L\,\{r + (a_T G_L r + a_T G_0)\} + F_0\right] ds$$

$$+M \max_{\tau\in J}\|u(\tau)-v(\tau)\| \int_0^t \{k_T H_L + F_L + a_T F_L G_L\}\, ds$$

$$\le C_1 \|u_0\|_Y T \max_{\tau\in J}\|u(\tau)-v(\tau)\| + C_1 T^2 \left[k_T\,(H_L r + H_0) + F_L\,\{r + (a_T G_L r + a_T G_0)\} + F_0\right] \max_{\tau\in J}\|u(\tau)-v(\tau)\|$$

$$+MT\,\{k_T H_L + F_L + a_T F_L G_L\} \max_{\tau\in J}\|u(\tau)-v(\tau)\|$$

$$\le \left[\begin{array}{l} C_1\|u_0\|_Y T + C_1 T^2 \left\{k_T\,(H_L r + H_0) + F_L\,(r + (a_T G_L r + a_T G_0)) + F_0\right\} + \\ MT\,\{k_T H_L + F_L + a_T F_L G_L\} \end{array} \right] \max_{\tau\in J}\|u(\tau)-v(\tau)\|$$

This gives

$$\|Fu(t)-Fv(t)\| \le \Gamma \max_{\tau\in J}\|u(\tau)-v(\tau)\|, \text{ by hypothesis } (H_{11})$$

where $0<\Gamma<1$. Thus F is a contraction from S to S. By the contraction mapping theorem F has a unique fixed point $u\in S$ which is the mild solution of (1)–(2) on J. From (H_6), it leads that $u(t)$ is in $C(J,Y)$ (see [10] Lemma 7.4). Indeed, $u(t)$ is weakly continuous as a Y-valued function. This means that $u(t)$ is separably valued in Y, hence it is strongly measurable. Then, $\|u(t)\|_Y$ is bounded and measurable function in t. Therefore, $u(t)$ is Bochner integrable (see e.g. [15], Chapter-V). Applying the relation $u(t)=Fu(t)$, we conclude that $u(t)$ is in $C(J,Y)$.

Now, consider the following evolution equation

$$\frac{du(t)}{dt} + A(t,u)\,u(t) = H(u)(t) + f\left(t,u(t),G(u)(t)\right),$$

$$u(0)=u_0, t\in[0,T]=J\ ,$$

The above equation can be noted as

$$v'(t) + A(t)\,v(t) = h(t), t \in J \tag{8}$$

$$v(0)=u_0 \tag{9}$$

where $A(t)=A(t,u(t))$ and $h(t)=H(u)(t)+f(t,u(t),G(u)(t))$, $t\in J$ and u is the unique fixed point of F in S. We note that $A(t)$ satisfies $(H_1)-(H_3)$ of [11] (Section 5.5.3) and $h(t)\in C(J,Y)$. By using theorem 5.5.2 in Pazy [11] we summarize that unique function $v\in C(J,Y)$ exists such that $v\in C^1((0,T],X)$ satisfying (8)–(9) in X and hence v is given by

$$v(t)=U_u(t,0)u_0+\int_0^t U_u(t,s)\left[H(u)(s)+f(s,u(s),G(u)(s))\right]ds, t\in J,$$

where $U_u(t,s), 0\le s\le t\le T$ is the evolution system generated by the family $\{A(t,u(t))\}$, $t\in J$, of linear operator in X. The uniqueness of v implies that $v\equiv u$ on J and hence u is a unique classical solution of (1)–(2) and $u\in C([0,a]:Y)\cap C^1((0,a]:X)$. This completes the proof.

4 Example

Consider $\Omega\subset\mathbb{R}^n$ be a bounded domain with smooth boundary $\partial\Omega$. Let the differential operator

$$A(t,x,u;D)\,\omega=-\sum_{i,j=1}^{n}\frac{\partial}{\partial x_i}\left(a_{ij}(t,x,u(t,x))\frac{\partial\omega}{\partial x_j}\right)+c(t,x,u(t,x))\,\omega,$$

where $a_{ij}(t,x,u(t,x))$ and $c(t,x,u(t,x))$ are valued functions described on $J\times\overline{\Omega}\times\mathbb{R}$ and $J=[0,T]$, $0<T<\infty$. Let us suppose that $a_{ij}\in C\left[J\times\overline{\Omega}\times W,\mathbb{R}\right]$, where $W=C^{2l+1}(J\times\overline{\Omega},\mathbb{R})$ with $\frac{1}{2}<l<1$, $a_{ij}=a_{ji}, (1\le i,j\le n)$ and there exists some $c>0$ such that

$$\sum_{i,j=1}^{n}a_{ij}(t,x,u(t,x))q_iq_j\ge c|q|^2, q=(q_1,q_2,\dots,q_n)\in\mathbb{R}$$

holds for each $(t,x,u(t,x))\in J\times\overline{\Omega}\times\mathbb{R}$.

Consider the partial integrodifferential equation

$$\frac{\partial u(t,x)}{\partial t}+A(t,x,u;D)u(t,x)=G(u)(t,x)+f(t,x,u(t,x),K(u)(t,x)), (t,x)\in(0,T]\times\Omega, \quad (10)$$

with the boundary condition

$$u(t,x)=0 \text{ for } (t,x)\in(0,T]\times\partial\Omega$$

and initial condition

$$u(0,x)=u_0(x) \text{ for } x\in\Omega,$$

where $G(u)(t,x)=\int_0^t a(t-x)\,h\big(s,x,u(s,x),\nabla u(s,x)\big)\,ds$

and $K(u)(t,x)=\int_0^t k(t-x)\,g\big(s,x,u(s,x),\nabla u(s,x)\big)\,ds$

$$\nabla=\left(D_1,D_2,\dots,D_n\right),D_i=\frac{\partial}{\partial x_i}$$

the function k and a are a real valued continuous function of bounded variation in $\mathbb{R}$ and the function $f(t,x,u,v)$ is also a real valued continuous function defined on $J\times\overline{\Omega}\times B\times B$ and there exist a constant $L>0$ such that

$$\left\|f(t,x,u_1,v_1)-f(t,x,u_2,v_2)\right\|\le L\left[\left\|u_1-u_2\right\|+\left\|v_1-v_2\right\|\right]$$

for $x\in\Omega$ and $u_i,v_i\in B, i=1,2$. $u:J\times\Omega\to\mathbb{R}$ is unknown function and u_0 is its initial value. Again, we consider that $h,g:[0,\infty)\times\Omega\times B\times B\to\mathbb{R}$ is continuous and there exist constants $M>0$ and $N>0$ such that

$$\left\|h(t,x,u,\xi)-h(t,x,v,\eta)\right\|\le M\left[\left\|u-v\right\|+\left\|\xi-\eta\right\|\right]$$

$$\left\|g(t,x,u,\xi)-g(t,x,v,\eta)\right\|\le N\left[\left\|u-v\right\|+\left\|\xi-\eta\right\|\right]$$

for $x\in\Omega$ and $u,v,\xi,\eta\in B$.

Let $\frac{n}{2l-1}<p<\infty$ and $X=L^p(\Omega)$ with the usual norm

$$\left\|u\right\|_p=\left[\int_\Omega\left|u\right|^p dx\right]^{\frac{1}{p}}$$

then integrodifferential equation (10) can be reformed as an abstract integrodifferential (1) in Banach space X, where $A(t,u)v=A(t,x,u;D)v$ with domain

$$D\big(A(t,u)\big)=\left\{v\in W_p^2(\Omega),v(t,x)=0,(t,x)\in(0,T]\times\partial\Omega\right\}$$

and

$$f\left(t,u,\int_0^t a(t-s)\,g\big(s,u(s)\big)\,ds\right)=f\big(t,x,u(t,x)K(u)(t,x)\big)$$

$$\int_0^t k(t-s)\,h\big(s,u(s)\big)\,ds=G(u)(t,x).$$

We take note of that the assumption $(H_1)-(H_{10})$ are satisfied hence we may exert the finding of earlier part to assure the existence of unique classical solution of (10).

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Avian Influenza with Drug Resistance

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Abstract – In this paper, avian influenza epidemic model with drug resistance effect is investigated. The basic reproduction number R_0 find out using next generation method. The local and global stability of a disease free and endemic equilibrium of the system is studied and discussed. Numerical simulations are carried out to investigate the influence of the key parameters on the spread of the disease, to support the analytical conclusion and illustrate possible behavioral scenarios of the model.

Keywords – *Avian influenza, drug resistance, stability, basic reproduction.*

1. Introduction

The year ended cost of affliction illness and the developing threat of evolution of a comprehensive strain make it all important to revisit of present accessible treatment options. Adamantane and neuraminidase inhibitors (NAIS), two divisions of drugs, are at present accessible treatment of influenza, although to treat influenza adequately combat to both divisions at drugs intimidate our ability. Underlying the appearance of day combat helps in letter consider at the mechanism. It will approve health authorization to make more adequate else of antiviral, over the cause of an influenza infection on a periodic basis, or in the content of a pandemic, preceding production on the appearance at drug combat in afflictions. A has been centralize largely an epidemiological model which represent the spreading of drug combat infection across a population. For developing approach to interrupt the diffusion of drug combat once it emerges,

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such studies are important, they do not provide insight into how the drug combat break affair during the continuity at a single infection, and on what timescale the appearance of drug combat to NAIS has been examined by an early modeling study, during a single infection. In which, it is found that NAI combat could arrive in the absence at drug treatment, admitting at low level, even if the break is slightly less fit than the wild-type virus. To appraise the fitness difference which is caused by drug combat mutation has been studies and used several models. Alternative studies have used models to optimize treatment regimens to reduce the emergence of drug resistant mutants. However, some of the biological processes that might self or hinder the appearance of drug combat are yet not tried to examined by any study [2].

2. Mathematical Model

Basic Model. Shuqinche [1] has proposed the model for the avian influenza

$$\begin{aligned}
\frac{dX}{dt} &= c - \frac{wXY}{1+\delta Y} - dX \\
\frac{dY}{dt} &= \frac{wXY}{1+\delta Y} - (d+m)Y \\
\frac{dS}{dt} &= b - \frac{\beta SY}{1+\delta Y} - \alpha S \\
\frac{dI}{dt} &= \frac{\beta SY}{1+\delta Y} - (\varepsilon + \alpha + \gamma) I \\
\frac{dR}{dt} &= \gamma I - \alpha R
\end{aligned} \tag{2.1}$$

Modified Model.

$$\begin{aligned}
\frac{dX}{dt} &= c - \frac{wXY}{1+\delta Y} - dX \\
\frac{dY}{dt} &= \frac{wXY}{1+\delta Y} - (d+m)Y \\
\frac{dS}{dt} &= b - \frac{\beta SY}{1+\delta Y} - \alpha S \\
\frac{dI}{dt} &= \frac{\beta SY}{1+\delta Y} - (\varepsilon + \alpha + \gamma + \eta) I \\
\frac{dR_{ES}}{dt} &= \eta I - (\alpha + \sigma) R_{ES} \\
\frac{dR}{dt} &= \gamma I - \alpha R + \sigma R_{ES}
\end{aligned} \tag{2.2}$$

Parameter description. The human is divided into three compartments S, I, R the number of susceptible, infected and recovered respectively, the birds are divided into susceptible poultry (X) and infected poultry (Y).

Parameter	description
C	natural birth rate of avian
b	natural birth rate of human
d	the natural mortality of poultry
α	the natural mortality of human
m	due to the mortality illness of poultry
ε	due to the mortality illness of human
w	stands for infectious rate of susceptible poultry to infected poultry
β	stands for infected poultry of the infection
	rate of susceptible human individuals
γ	the recovery rate that infects individuals through treatment
η	resistance rate to treatment
σ	recovery rate after second line of resistance treatment

3. Equilibria of the System

$$\begin{aligned}
\frac{dX}{dt} &= c - \frac{wXY}{1+\delta Y} - dX \\
\frac{dY}{dt} &= \frac{wXY}{1+\delta Y} - (d+m)Y \\
\frac{dS}{dt} &= b - \frac{\beta SY}{1+\delta Y} - \alpha S \\
\frac{dI}{dt} &= \frac{\beta SY}{1+\delta Y} - (\varepsilon + \alpha + \gamma + \eta) I \\
\frac{dR_{ES}}{dt} &= \eta I - (\alpha + \sigma) R_{ES} \\
\frac{dR}{dt} &= \gamma I - \alpha R + \sigma R_{ES}
\end{aligned} \tag{3.1}$$

disease free equilibrium point

$$E_0\left(X^0, Y^0, S^0, I^0, R^0{}_{ES}\right) = \left(\frac{c}{d}, 0, \frac{b}{\alpha}, 0\right)$$

We can find the basic reproductive numbers using the next generation method

$$\frac{dX}{dt} = F - V$$

where

$$F = \begin{bmatrix} \dfrac{wXY}{1+\delta Y} \\ 0 \end{bmatrix}, \quad V = \begin{bmatrix} -d & 0 \\ 0 & -(d+m) \end{bmatrix}$$

$$FV^{-1} = \begin{bmatrix} 0 & \dfrac{cw}{d(d+m)} \\ 0 & 0 \end{bmatrix}$$

The largest Eigen value of FV^{-1}, the basic reproduction number is expressed as

$$R_0 = \frac{cw}{d(d+m)}$$

Endemic equilibrium point

$$\begin{aligned}
& c - \frac{wXY}{1+\delta Y} - dX = 0 \\
& \frac{wXY}{1+\delta Y} - (d+m)Y = 0 \\
& b - \frac{\beta SY}{1+\delta Y} - \alpha S = 0 \\
& \frac{\beta SY}{1+\delta Y} - (\varepsilon + \alpha + \gamma + \eta)I = 0 \\
& \eta I - (\alpha + \sigma)R_{ES} = 0 \\
& \gamma I - \alpha R + \sigma R_{ES} = 0
\end{aligned}$$

$$E_*\left(X^*, Y^*, S^*, I^*, R^*{}_{ES}\right)$$

Where

$$X^* = \frac{d + m - c\delta}{d\delta + w},$$

$$Y^* = \frac{cw - d(d+m)}{(d+m)(d\delta + w)},$$

$$S^* = \frac{b\left(1+\delta Y^*\right)}{\beta Y^* + \alpha\left(1+\delta Y^*\right)}$$

$$I^* = \frac{b\beta Y^*}{(\varepsilon+\alpha+\gamma+\eta)\left(\beta Y^* + \alpha\left(1+\delta Y^*\right)\right)},$$

$$R^*_{ES} = \frac{\eta I^*}{(\alpha+\sigma)}$$

Theorem 3.1. if $R_0 < 1$, the system (3.1) only exists the disease-free equilibrium $E_0\left(\frac{c}{d}, 0, \frac{b}{\alpha}, 0, 0\right)$ when $R_0 > 1$, there exists only one endemic equilibrium

$$E_*\left(\frac{d+m-c\delta}{d\delta+w}, \frac{cw-d(d+m)}{(d+m)(d\delta+w)}, \frac{b\left(1+\delta Y^*\right)}{\beta Y^*+\alpha\left(1+\delta Y^*\right)}, \frac{b\beta Y^*}{(\varepsilon+\alpha+\gamma+\eta)\left(\beta Y^*+\alpha\left(1+\delta Y^*\right)\right)}, \frac{\eta I^*}{(\alpha+\sigma)}, \right)$$

4. Local Stability of the Disease Free Equilibrium

In this section we find the local stability of the disease free and endemic equilibrium.

Theorem 4.1. The disease free equilibrium E_0 is locally asymptotically stable, if $R_0 < 1$.

Proof. The Jacobian matrix of system (3.1) is

$$J_0 = \begin{bmatrix} -d-\frac{wY}{1+\delta Y} & \frac{-wX(1+\delta Y)-\delta wXY}{(1+\delta Y)^2} & 0 & 0 & 0 \\ \frac{wY}{1+\delta Y} & \frac{wX(1+\delta Y)-\delta wXY}{(1+\delta Y)^2}-(d+m) & 0 & 0 & 0 \\ 0 & \frac{-\beta S(1+\delta Y)-\delta\beta SY}{(1+\delta Y)^2} & -\frac{\beta S}{(1+\delta Y)}-\alpha & 0 & 0 \\ 0 & \frac{\beta S(1+\delta Y)-\delta\beta SY}{(1+\delta Y)^2} & \frac{\beta S}{(1+\delta Y)} & (\varepsilon+\alpha+\gamma+\eta) & 0 \\ 0 & 0 & 0 & \eta & -(\alpha+\sigma) \end{bmatrix}$$

$$J_0 = \begin{bmatrix} -d & \frac{-wc}{d} & 0 & 0 & 0 \\ 0 & \frac{wc}{d}-(d+m) & 0 & 0 & 0 \\ 0 & -\frac{b\beta}{\alpha} & -\alpha & 0 & 0 \\ 0 & \frac{b\beta}{\alpha} & 0 & (\varepsilon+\alpha+\gamma+\eta) & 0 \\ 0 & 0 & 0 & \eta & -(\alpha+\sigma) \end{bmatrix}$$

$$|J_0 - \lambda I| = \begin{vmatrix} -d-\lambda & \frac{-wc}{d} & 0 & 0 & 0 \\ 0 & \frac{wc}{d}-(d+m)-\lambda & 0 & 0 & 0 \\ 0 & -\frac{b\beta}{\alpha} & -\alpha-\lambda & 0 & 0 \\ 0 & \frac{b\beta}{\alpha} & 0 & (\varepsilon+\alpha+\gamma+\eta)-\lambda & 0 \\ 0 & 0 & 0 & \eta & -(\alpha+\sigma)-\lambda \end{vmatrix}$$

$$-d\left[\frac{wc}{d}-(d+m)-\lambda\right]\alpha\left(\varepsilon+\alpha+\gamma+\eta\right)\left(\alpha+\sigma\right)\le 0$$

$$-(d+\lambda)\left[\frac{wc}{d}-(d+m)-\lambda\right](\alpha+\lambda)\left(\varepsilon+\alpha+\gamma+\eta-\lambda\right)(\alpha+\sigma-\lambda)=0$$

for $R_0 < 1,$ it is clear the matrix J_{E_0} has negative real parts. So, E_0 is locally asymptotically stable.

Theorem 4.2. The endemic equilibrium E_* is locally asymptotically stable if $R_0 > 1$.

Proof. The Jacobean matrix of system (3.1) is

$$J_{E_*} = \begin{bmatrix} -d-\frac{wY^*}{1+\delta Y^*} & \frac{-wX^*(1+\delta Y^*)-\delta wX^*Y^*}{(1+\delta Y^*)^2} & 0 & 0 & 0 \\ \frac{wY^*}{1+\delta Y^*} & \frac{wX^*(1+\delta Y^*)-\delta wX^*Y^*}{(1+\delta Y^*)^2}-(d+m) & 0 & 0 & 0 \\ 0 & \frac{-\beta S^*(1+\delta Y)-\delta\beta S^*Y}{(1+\delta Y)^2} & -\frac{\beta S^*}{(1+\delta Y)}-\alpha & 0 & 0 \\ 0 & \frac{\beta S^*(1+\delta Y)-\delta\beta S^*Y}{(1+\delta Y)^2} & \frac{\beta S}{(1+\delta Y)} & (\varepsilon+\alpha+\gamma+\eta) & 0 \\ 0 & 0 & 0 & \eta & -(\alpha+\sigma) \end{bmatrix}$$

The characteristic equation of jacobian matrix (4.2) at the endemic equilibrium point, $E_* = \left(X^*, Y^*, S^*, I^*, R_{ES}{}^*\right)$, is a fifth-degree polynomial given by

$$P(\lambda) = \lambda^5 + a_1\lambda^4 + a_2\lambda^3 + a_3\lambda^2 + a_4\lambda + a_5,$$

Where $a_i, i = 1, 2, 3, 4, 5$ are the coefficients. It can be shown that all the coefficients a_i are positive. The necessary and sufficient conditions for the local asymptotic stability of endemic equilibrium point E_1 are that the Hurwitz determinants H_i, are all positive for the Routh-Hurwitz criteria. For a fifth degree polynomial [3] these criteria are given by

$$H_1 = a_1 > 0,$$

$$H_2 = a_1a_2 - a_3 > 0,$$

$$H_3 = a_1a_2a_3 + a_1a_5 - a_1{}^2a_4 - a_3{}^2 > 0$$

$$H_4 = \left(a_3a_4 - a_2a_5\right)\left(a_1a_2 - a_3\right) - \left(a_1a_4 - a_5\right)^2 > 0$$

$$H_5 = a_5H_4 > 0$$

From which we can conclude whether the endemic equilibrium point is locally asymptotically stable or unstable.

5. Global stability of the disease free and endemic equilibrium.

Theorem 5.1. if $R_0 < 1$, the disease free equilibrium E_0 is globally asymptotically stable, if $R_0 > 1$, the disease free equilibrium E_0 is unstable.

Proof. Consider the lyapunov function

$$K_1 = X - X^0 \, In\, X + Y$$

$$= X^1 - \frac{X^0}{X} X^1 + Y^{1.}$$

$$= X^1\left(1 - \frac{X^0}{X}\right) + Y^{1.}$$

$$= \left(1 - \frac{X^0}{X}\right)\left[c - \frac{wXY}{1+\delta Y} - dX\right] + \left[\frac{wXY}{1+\delta Y} - \left(d+m\right)Y\right]$$

$$= \left(1 - \frac{X^0}{X}\right)\left[dX^0 - dX - \frac{wXY}{1+\delta Y}\right] + \left[\frac{wXY}{1+\delta Y} - (d+m)Y\right]$$

$$= \frac{\left(X - X^0\right)}{X}\left[-d\left(X - X^0\right) - \frac{wXY}{1+\delta Y}\right] + \left[\frac{wXY}{1+\delta Y} - (d+m)Y\right]$$

$$= \frac{-d\left(X - X^0\right)^2}{X} - \frac{\left(X - X^0\right)}{X} . \frac{wXY}{1+\delta Y} + \frac{wXY}{1+\delta Y} - (d+m)Y$$

$$= \frac{-d\left(X - X^0\right)^2}{X} + \frac{wX^0Y}{1+\delta Y} - (d+m)Y$$

$$= \frac{-d\left(X - X^0\right)^2}{X} + (d+m)Y\left[\frac{wX^0}{(1+\delta Y)(d+m)} - 1\right]$$

$$\leq \frac{-d\left(X - X^0\right)^2}{X} + (d+m)Y$$

$$= \frac{-d\left(X - X^0\right)^2}{X} + (d+m)Y\left(R_0 - 1\right)$$

When $R_0 < 1,$ we can get $K_1^1 \leq 0$ and $K_1^1 = 0$ has no other closed trajectory in addition to E_0 is globally asymptotically stable iff $R_0 < 1$.

Theorem 5.2. The endemic equilibrium E_* is globally asymptotically stable if $R_0 > 1$.

Proof. Consider the Lyapunov function

$$K_2 = X^*\left(\frac{X}{X^*} - 1 - In\frac{X}{X^*}\right) + Y^*\left(\frac{Y}{Y^*} - 1 - In\frac{Y}{Y^*}\right)$$

Then

$$K_2^1 = X^*\left(\frac{X^1}{X^*} - \frac{X^*}{X} . \frac{X^1}{X^*}\right) + Y^*\left(\frac{Y^1}{Y^*} - \frac{Y^*}{Y} . \frac{Y^1}{Y^*}\right)$$

$$K_2^1 = \left(1 - \frac{X^*}{X}\right)X^1 + \left(1 - \frac{Y^*}{Y}\right)Y^1$$

$$= c\left(2 - \frac{X^*}{X} - \frac{X}{X^*}\right)$$

By the relationship of arithmetic mean and geometric mean.

We know that

$$\left(2-\frac{X^{*}}{X}-\frac{X}{X^{*}}\right)\leq 0$$

$K_1^{1} \leq 0$ iff $(X,Y)=(X^{*},Y^{*})$, $K_2^{1}=0.$ Thus by LaSalle invariance principal $E_{*}(X^{*},Y^{*},S^{*},I^{*},R^{*}_{ES})$ is globally asymptotically stable.

6. Numerical Simulation

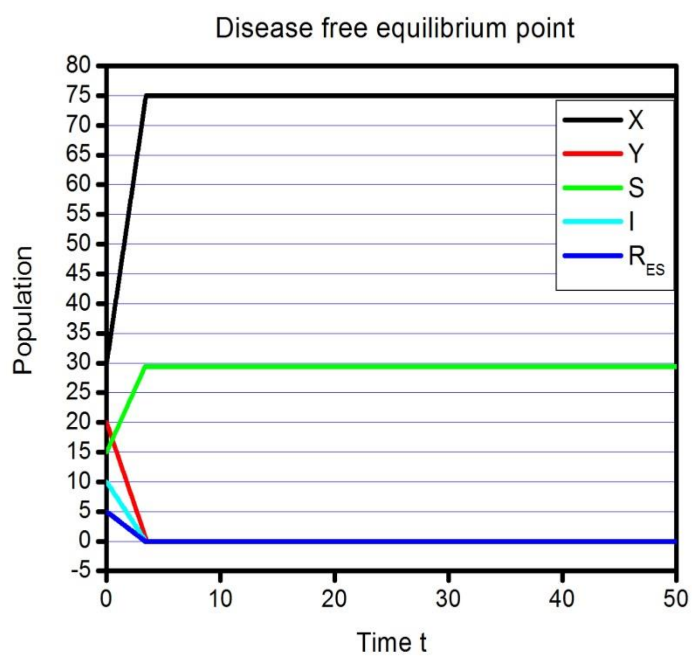

Figure 1.

Suppose the parameters are $C=3, \beta=0.02,\ d=0.04,\ w=0.012,\ m=0.96, b=1,\ \alpha=0.068,$ $\varepsilon=0.62, \gamma=0.39,\ \delta=0.05, \eta=0.15,\ \sigma=0.0411,$ Let the initial value of the system as X,Y,S,I,R_{ES} are 30,20,15,10,5 respectively. Then we obtain $R_0=0.9<1,$ $E_0\left(X^0,Y^0,S^0,I^0R^0{}_{ES}\right)=(75,0,29.41,0,0)$ Therefore by theorem 5.1, E_0 is globally asymptotically stable (see in figure 1)

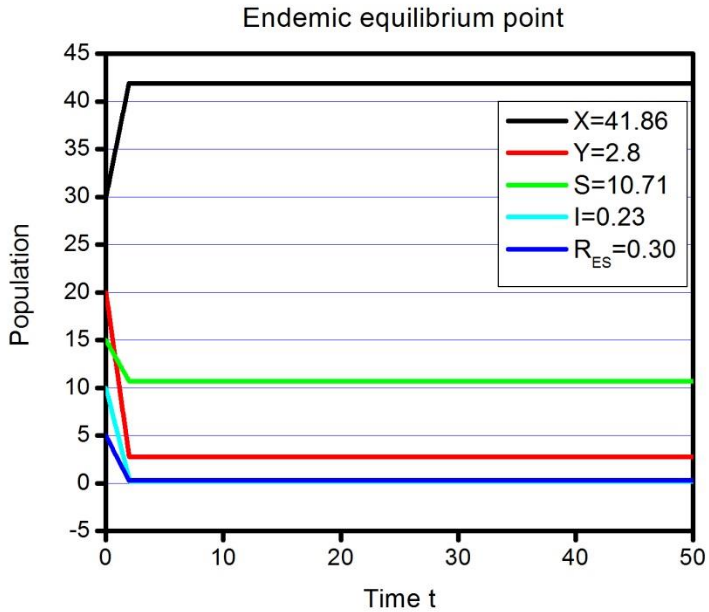

Figure 2.

Again we take the parameter $C = 2, \beta = 0.01,\ d = 0.03,\ w = 0.02,\ \ m = 0.97, b = 1,\ \alpha = 0.069,$ $\varepsilon = 0.63, \gamma = 0.301,\ \delta = 0.05, \eta = 0.15,\ \sigma = 0.0411$ and X, Y, S, I, R_{ES} are 30,20,15,10,5 respectively. Then we obtain $R_0 = 1.33 > 1$, $E_*(X^*, Y^*, S^*, I^* R^*_{ES}) = (41.86, 2.8, 10.72, 0.23, 0.31)$ Therefore, by theorem 5.2, E_* is globally asymptotically stable (see in figure 2)

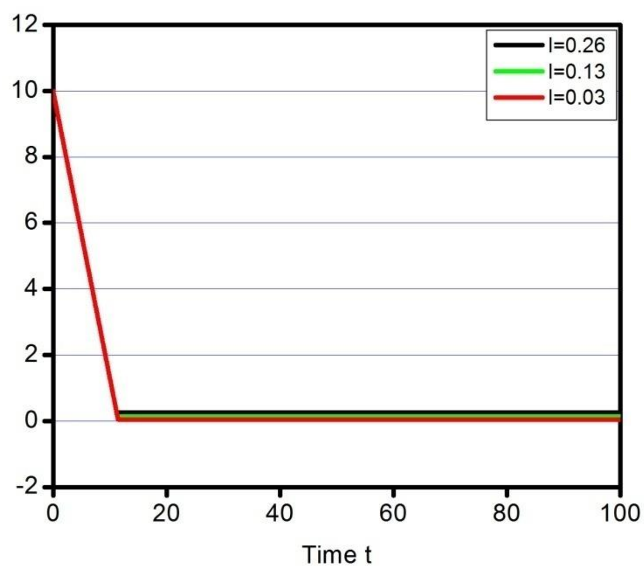

Figure 3. When η resistance rates to treatment increases then the steady state value I of the infective are decrease

If we change the value of η and keeping another parameter are fixed we can see that I^* decreases as η increases. Choose the value of $\eta = 0.01, \eta = 2, \eta = 7$ we get $I = 0.26, I = 0.13, I = 0.03$ respectively.

7. Conclusion

In this study, we formulate avian influenza epidemic model with saturated contact rate introduced by Shuqinche et al [1]. We have shown that if the basic reproduction number $R_0 < 1$ then E_0 globally asymptotically stable is disease out see Figure 2. If $R_0 > 1$ then E_+ exist i.e. disease persist. Numerical simulation indicates that when the disease is endemic, the steady state value I decrease as resistance rate to treatment η increases See Figure 3.

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A New Operation on Soft Sets: Extended Difference of Soft Sets

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Abstaract − Soft Set Theory, which has been considered as an adequate mathematical device, was proposed by Molodtsov to deal with ambiguities and uncertainties. Several operations on soft sets were defined in many soft set papers. This study is based on the paper "On operations of Soft Sets" by Sezgin and Atagün [Comput. Math. Appl. 61 (2011) 1457-1467]. In this paper, we define a new operation on soft sets, called extended difference and investigate its relationship between extended difference and restricted difference and some other operations of soft sets.

Keywords − *Soft sets, Restricted union, Extended union, Restricted intersection, Extended intersection, Restricted difference, Extended difference.*

1 Introduction

In different areas, Mathematicians and Scientists have been facing several ambiguities and uncertainties in the problems of computer science, statistics, different branches of engineering, environmental sciences, economics, medical sciences, sociology and many other different fields of sciences. In the past, many of the theories were presented to overcome these uncertainties. But Molodtsov [10] has found that these theories have their own built-in deadlocks. The main problem shared by those theories is their conflict with the parametrization tools. So, to overcome these deadlocks properly, in 1999, Molodtsov [10] suggested a fully new approach that is soft set theory which acts as a breakthrough for those deadlocks. In this theory, a soft set could be a parameterized group of subsets of the universal set and also the drawback of setting the membership operation does not appear. It gives us a lot of choices in

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the problem to solve them easily. Now the development in the field of soft set theory is increasing day by day.

In 2002, Maji et al. [9] applied the soft sets to decision creating problems using rough mathematics and then in 2003, Maji and Biswas [8] introduced many operations of soft sets. And then, many authors [1, 2, 4, 5, 6, 7, 11, 13] also studied the different soft operations.

In 2011, Sezgin and Atagün [12] discussed the fundamental theorems about operations about soft sets i.e; union and intersection of soft sets and other operations. In that paper, they defined union and intersection operation of soft sets both with restricted and extended condition but defined the difference operation only with restricted condition. They did not define difference operation with extended condition. Here in this paper, we have defined a new operation on soft sets called *extended difference* and also proved some of its properties. Moreover, we have also proved the interesting result which shows the relationship between extended difference with the restricted difference. The main objective of this paper is to make soft set theory more effective and solid by enhancing the conceptual feature of operations on soft sets.

2 Preliminary

In view this chapter, we review a few fundamental assumptions in soft set theory. From now on, U is a basic universe set , E is set of all feasible parameters in the below discussion with reference to U. We denote the power set on U (i.e; the set of consisting all the subsets of U) by $P(U)$, A is a subset of E. Generally parameters are the numeric values, attributes of elements in U. Molodtsov [10], illustrated the soft set in such a way as following:

Definition 2.1. ([10]) Let U be the fundamental universe, E is a set about parameters and D be a subset of E. The pair (L, D) is known to be a soft set over U, when L is mapping of D within set of every subsets of set U.
For e $\in D$, $L(e)$ is may be regarded as a set of e-elements of soft set (L, D).

Definition 2.2. ([8]) Let (L, D) and (K, J) be two soft sets over the same universe U, then (K, J) is soft subset on (L, D) if it satisfies:
(i) $J \subseteq D$
(ii) $K(e) \subseteq L(e), \forall e \in J$
and it is denoted by $(K, J)\widetilde{\subset}(L, D)$ and also if (L, D) is soft subset on (K, J) then (K, J) is said to be a soft superset of (L, D) and it is denoted by $(K, J)\widetilde{\supset}(L, D)$.

Definition 2.3. ([8]) Let (L, D) and (K, J) be two soft sets over the identical universe U. (L, D) and (K, J) are called soft equal sets if (L, D) is soft subset of (K, J) and (K, J) is soft subset of (L, D).

Definition 2.4. ([3]) The relative complement of a soft set (K, B) is shown by $(K, B)^r$ and is illustrated as $(K, B)^r = (K^r, B)$, where $K^r : B \to P(U)$ is a mapping assigned as $K^r(e) = U \setminus K(e)$, $\forall e \in B$.

Definition 2.5. ([8]) A soft set (K, B) over U is known as a null soft set shown as Φ_B if for all $e \in B$, $K(e) = \emptyset$.

Definition 2.6. ([8]) Let (L, D) be a soft set over a universe U. Then, (L, D) is known as an absolute soft set if for all $e \in D$, $L(e) = U$ and it is denoted by $\mathcal{U}_D$.

Definition 2.7. ([8]) Let (L, D) and (K, Z) be soft sets over an identical universe U, then "$(L, D)AND(K, Z)''$ denoted by $(L, D) \wedge (K, Z)$ and is expressed as $(L, D) \wedge (K, Z) = (H, D \times Z)$, where $H(\alpha, \beta) = L\alpha) \cap K(\beta)$, $\forall(\alpha, \beta) \in D \times Z$.

Definition 2.8. ([8]) Let (L, D) and (K, Z) be two soft sets over a common universe U, then "$(L, D)OR(K, Z)''$ shown as $(L, D) \vee (K, Z)$ and is expressed as $(L, D) \vee (K, Z) = (H, D \times Z)$, where $H(\alpha, \beta) = L(\alpha) \cup K(\beta)$, $\forall(\alpha, \beta) \in D \times Z$.

Definition 2.9. ([3]) Let (L, D) and (K, Z) be two soft sets over an identical universe U, where $D \cap Z \neq \phi$. The restricted union of (L, D) and (K, Z) is shown by $(L, D) \cup_R (K, Z)$ and expressed as $(L, D) \cup_R (K, Z) = (H, C)$, when $C = D \cap Z$ and for all $e \in C$, $H(e) = L(e) \cup K(e)$.

Definition 2.10. ([8]) Let (L, D) and (K, Z) be two soft sets over an identical universe U. The extended union of (L, D) and (K, Z) is expressed as the soft set (I, O) fulfilling the situations: (i)$O = D \cup Z$; (ii)for all $e \in O$,

$$I(e) = \begin{cases} L(e) & if \;\; e \in D \setminus Z, \\ K(e) & if \;\; e \in Z \setminus D, \\ L(e) \cup K(e) & if \;\; e \in D \cap Z. \end{cases}$$

This relation is shown by $(L, D)\widetilde{\cup}(K, Z) = (I, O)$.

Definition 2.11. ([3]) Let (L, D) and (K, Z) be two soft sets over an identical universe U, where $D \cap Z \neq \phi$. The restricted intersection of (L, D) and (K, Z) shown by $(L, D) \cap_R (K, Z)$ and is expressed as $(L, D) \cap_R (K, Z) = (H, C)$, where $C = D \cap Z$ and for all $e \in C$, $H(e) = L(e) \cap K(e)$.

Definition 2.12. ([3]) Let (L, D) and (K, Z) be two soft sets over an identical universe U. The extended intersection of (L, D) and (K, Z) is expressed as the soft set (I, O) fulfilling situations: (i)$C = D \cup Z$; (ii)for all $e \in O$,

$$I(e) = \begin{cases} L(e) & if \;\; e \in D \setminus Z, \\ K(e) & if \;\; e \in Z \setminus D, \\ L(e) \cap K(e) & if \;\; e \in D \cap Z. \end{cases}$$

This relation is shown by $(L, D)\widetilde{\cap}(K, Z) = (I, O)$.

Definition 2.13. ([12]) Let (L, D) and (K, Z) be two soft sets over an identical universe U, where $D \cap Z \neq \phi$. The restricted difference of (L, D) and (K, Z) is shown by $(L, D) \sim_R (K, Z)$, and expressed as $(L, D) \sim_R (K, Z) = (H, C)$, where $C = D \cap Z$ and for all $e \in C$, $H(e) = L(e) \setminus K(e)$.

Definition 2.14. ([12]) Let (L, D) and (K, Z) be two soft sets over an identical universe U, where $D \cap Z \neq \phi$. The restricted symmetric difference of (L, D) and (K, Z) is shown by $(L, D)\widetilde{\triangle}(K, Z)$, and expressed as $(L, D)\widetilde{\triangle}(K, Z) = ((L, D) \cup_R (K, Z)) \sim_R ((L, D) \cap_R (K, Z)) = (T, C)$, where $C = D \cap Z$.

3 Properties of operations of soft sets and their correlations with one another

As the fundamental properties and theorems related to operations of soft sets such as restricted union, extended union, restricted intersection, extended intersection, restricted difference we refer to the paper Sezgin and Atagün [12], Maji et al. [8], Ali et al. [3] and Pei and Miao [11]. Now we are ready to give the definition of extended difference of soft sets and its basic properties.

Definition 3.1. Let (X, D) and (P, E) be the two soft sets over an identical universe U. The extended difference of (X, D) and (P, E) can be expressed as the soft set (L, C) fulfilling the situations as under: (i)$C = D \cup E$; (ii)for all $e \in C$,

$$L(e) = \begin{cases} X(e) & if \;\; e \in D \setminus E, \\ P(e) & if \;\; e \in E \setminus D, \\ X(e) \setminus P(e) & if \;\; e \in D \cap E. \end{cases}$$

Thus, the relation is shown by $(X, D) \sim_E (P, E) = (L, C)$.

Example 3.2. Let E be the universe set of parameters, D, B be the subsets of E such that $E = \{e_1, e_2, e_3, e_4, e_5, e_6\}$, $D = \{e_1, e_2, e_3, e_4\}$ and $B = \{e_3, e_4, e_5\}$. Assume that (X, D) and (K, B) are two soft sets over common universe $U = \{h_1, h_2, h_3, h_4, h_5, h_6\}$ as following: $(X, D) = \{(e_1, \{h_2, h_4\}), (e_2, \{h_1, h_3\}), (e_3, \{h_3, h_5\}), (e_4, \{h_1, h_6\})\}$, $(P, B) = \{(e_3, \{h_4, h_5\}), (e_4, \{h_1, h_2\}), (e_5, \{h_2, h_5\})\}$, where $C = D \cup B = \{e_1, e_2, e_3, e_4, e_5\}$.

Now let $(X, D) \sim_E (P, B) = (L, D \cup B)$, where

$$L(e) = \begin{cases} X(e) & if \;\; e \in D \setminus B, \\ P(e) & if \;\; e \in B \setminus D, \\ X(e) \setminus P(e) & if \;\; e \in D \cap B. \end{cases}$$

and for all $e \in D \cup B = \{e_1, e_2, e_3, e_4, e_5\}$. Since $D \setminus B = \{e_1, e_2\}$, $L(e_1) = X(e_1) = \{h_2, h_4\}$, $L(e_2) = X(e_2) = \{h_1, h_3\}$. Since $B \setminus D = \{e_5\}$, $L(e_5) = P(e_5) = \{h_2, h_5\}$ and since $D \cap B = \{e_3, e_4\}$, $L(e_3) = X(e_3) \setminus P(e_3) = \{h_3, h_5\} \setminus \{h_4, h_5\} = \{h_3\}$, $L(e_4) = X(e_4) \setminus P(e_4) = \{h_1, h_6\} \setminus \{h_1, h_2\} = \{h_6\}$. Hence, $(X, D) \sim_E (P, B) = (L, D \cup B) = \{(e_1, \{h_2, h_4\}), (e_2, \{h_1, h_3\}), (e_3, \{h_3\}), (e_4, \{h_6\}), (e_5, \{h_2, h_5\})$.

Theorem 3.3. Let (P, D), $(P, V), (R, J), (S, I), (X, Z)$ be two soft sets over a common universe U. Then, we have the following:

a) $(P, D) \sim_E \Phi_D = (P, D)$.

b) $(P, D) \sim_E (P, D) = \Phi_D$.

c) $\mathcal{U}_D \sim_E (P, D) = (P, D)^r$.

d) Left distribution of restricted intersection over extended difference:
$(P, V) \cap_R ((R, J)) \sim_E (S, I)) = ((P, V) \cap_R (R, J)) \sim_E ((P, V) \cap_R (S, I))$.

e) Right distribution of restricted intersection over extended difference:
$((X, Z) \sim_E (R, J)) \cap_R (S, I) = ((X, Z) \cap_R (S, I)) \sim_E ((R, J) \cap_R (S, I))$.

f) Right distribution of restricted difference over extended difference:
$((P,Z) \sim_E (R,J)) \sim_R (W,I) = ((P,Z) \sim_R (W,I)) \sim_E ((R,J) \sim_R (W,I))$.

Proof. **a)** Let $\Phi_D = (M,D)$ and $(P,D) \sim_E \Phi_D = (P,D) \sim_E (M,D) = (H,D)$, where

$$H(e) = \begin{cases} P(e) & if \;\; e \in D \setminus D, \\ M(e) & if \;\; e \in D \setminus D, \\ P(e) \setminus M(e) & if \;\; e \in D \cap D = D. \end{cases}$$

and for all $e \in D \cup D$. Since $M(e) = \phi$ for all $e \in D \cup D$, it follows that $H(e) = P(e) \setminus \phi = P(e)$. This means that P and H are the same mappings. This completes the proof.

b) Let $(P,D) \sim_E (P,D) = (H,D)$, where

$$H(e) = \begin{cases} P(e) & if \;\; e \in D \setminus D, \\ P(e) & if \;\; e \in D \setminus D, \\ P(e) \setminus P(e) & if \;\; e \in D \cap D = D. \end{cases}$$

and for all $e \in D \cup D$. Hence, $H(e) = P(e) \setminus P(e) = \emptyset$. This completes the proof.

c) Let $\mathcal{U}_D = (G,D)$ and $\mathcal{U}_D \sim_E (P,D) = (G,D) \sim_E (P,D) = (W,D)$, where

$$W(e) = \begin{cases} G(e) & if \;\; e \in D \setminus D, \\ P(e) & if \;\; e \in D \setminus D, \\ G(e) \setminus P(e) & if \;\; e \in D \cap D = D. \end{cases}$$

for all $e \in D \cup D$. Since $G(e) = U$ for all $e \in D \cup D$, it follows that $W(e) = U \setminus P(e) = P^r(e)$, which completes the proof.

d) For the left hand side of the property, let $(R,J) \sim_E (S,I) = (T, J \cup I)$, where

$$T(e) = \begin{cases} R(e) & if \;\; e \in J \setminus I, \\ S(e) & if \;\; e \in I \setminus J, \\ R(e) \setminus S(e) & if \;\; e \in J \cap I. \end{cases}$$

for all $e \in J \cup I$.

First let $(P,V) \cap_R (T, J \cup I) = (X, V \cap (J \cup I))$, where $X(e) = P(e) \cap T(e)$ for all $e \in V \cap (J \cup I)$.

Due the main features of set theory and according to the expressions of X also with T and suppose T is a piecewise function, we write the following equalities for the mapping X:

$$X(e) = \begin{cases} P(e) \cap R(e) & if \;\; e \in V \cap (J \setminus I) = (V \cap J) \setminus (V \cap I), \\ P(e) \cap S(e) & if \;\; e \in V \cap (I \setminus J) = (V \cap I) \setminus (V \cap J), \\ P(e) \cap (R(e) \setminus S(e)) & if \;\; e \in V \cap (J \cap I) \end{cases}$$

for all $e \in V \cap (J \cup I)$.

For the right hand side of the property, let $(P,V) \cap_R (R,J) = (D, V \cap J)$, where $D(e) = P(e) \cap R(e)$ for all $e \in V \cap J \neq \phi$. Suppose that $(P,V) \cap_R (S,I) = (O, V \cap I)$, where $O(e) = P(e) \cap S(e)$ for all $e \in V \cap I \neq \phi$. Assume that $(D, V \cap J) \sim_E (O, V \cap I) = (Z, (V \cap J) \cup (V \cap I))$, where

$$Z(e) = \begin{cases} D(e) & if \ e \in (V \cap J) \setminus (V \cap I), \\ O(e) & if \ e \in (V \cap I) \setminus (V \cap J), \\ D(e) \setminus O(e) & if \ e \in (V \cap J) \cap (V \cap I) = V \cap (J \cap I) \end{cases}$$

for all $e \in (V \cap J) \cup (V \cap I)$. Due to considering the expressions of D and O, we write the mapping Z as below:

$$Z(e) = \begin{cases} P(e) \cap R(e) & if \ e \in (V \cap J) \setminus (V \cap I), \\ P(e) \cap S(e) & if \ e \in (V \cap I) \setminus (V \cap J), \\ (P(e) \cap R(e)) \setminus (P(e) \cap S(e)) & if \ e \in V \cap (J \cap I). \end{cases}$$

It shows that X and Z are the identical mapping when we are assuming the attributes of operations about set theory. Hence the proof is completed.

e) For the left hand side of the property, let $(X, Z) \sim_E (R, J) = (T, Z \cup J)$, where

$$T(e) = \begin{cases} X(e) & if \ e \in Z \setminus J, \\ R(e) & if \ e \in J \setminus Z, \\ X(e) \setminus R(e) & if \ e \in Z \cap J \end{cases}$$

for all $e \in Z \cup J$.

First let $(T, Z \cup J) \cap_R (S, I) = (Q, (Z \cup J) \cap I)$, where $Q(e) = T(e) \cap S(e)$ for all $e \in (Z \cup J) \cap I$. Due the main features of set theory and according to the expressions of Q also with T and suppose that T is a piecewise function, also we write the following equalities for the mapping Q:

$$Q(e) = \begin{cases} X(e) \cap S(e) & if \ e \in (Z \setminus J) \cap I = (Z \cap I) \setminus (J \cap I), \\ R(e) \cap S(e) & if \ e \in (J \setminus Z) \cap I = (J \cap I) \setminus (Z \cap I), \\ (X(e) \setminus R(e)) \cap S(e) & if \ e \in (Z \cap J) \cap I \end{cases}$$

for all $e \in (Z \cup J) \cap I$.

For the right hand side of the property, let $(X, Z) \cap_R (S, I) = (M, Z \cap I)$, where $M(e) = X(e) \cap S(e)$ for all $e \in Z \cap I$. Assume $(R, J) \cap_R (S, I) = (O, J \cap I)$, where $O(e) = R(e) \cap S(e)$ for all $e \in J \cap I$. Let $(M, Z \cap I) \sim_E (O, J \cup I) = (W, (Z \cap I) \cup (J \cap I))$, where

$$W(e) = \begin{cases} M(e) & if \ e \in (Z \cap I) \setminus (J \cap I), \\ O(e) & if \ e \in (J \cap I) \setminus (Z \cap I), \\ M(e) \setminus O(e) & if \ e \in (Z \cap I) \cap (J \cap I) \end{cases}$$

for all $e \in (Z \cap I) \cup (J \cap I)$. By assuming the main expressions of M and O, we can rewrite the mapping W as below:

$$W(e) = \begin{cases} X(e) \cap S(e) & if \ e \in (Z \cap I) \setminus (J \cap I), \\ R(e) \cap S(e) & if \ e \in (J \cap I) \setminus (Z \cap I), \\ (X(e) \cap S(e)) \setminus (R(e) \cap S(e)) & if \ e \in (Z \cap I) \cap (J \cap I) \end{cases}$$

for all $e \in (Z \cap J) \cup (Z \cap I)$. This leads that Q and W are the identical mapping. Hence this completes the proof.

f) For the left hand side of the property, let $(P,Z) \sim_E (R,J) = (T, Z \cup J)$, where

$$T(e) = \begin{cases} P(e) & if \ \ e \in Z \setminus J, \\ R(e) & if \ \ e \in J \setminus Z, \\ P(e) \setminus R(e) & if \ \ e \in Z \cap J \end{cases}$$

for all $e \in Z \cup J$.

First let $(T, Z \cup J) \sim_R (W, I) = (Q, (Z \cup J) \cap I)$, where $Q(e) = T(e) \setminus W(e)$ for all $e \in (Z \cup J) \cap I$. Due the main features of set theory and according to the expressions of Q also with T and suppose that T is a piecewise function, we can write the following equalities for the mapping Q:

$$Q(e) = \begin{cases} P(e) \setminus W(e) & if \ \ e \in (Z \setminus J) \cap I = (Z \cap I) \setminus (J \cap I), \\ R(e) \setminus W(e) & if \ \ e \in (J \setminus Z) \cap I = (J \cap I) \setminus (Z \cap I), \\ (P(e) \setminus R(e)) \setminus W(e) & if \ \ e \in (Z \cap J) \cap I \end{cases}$$

for all $e \in (Z \cup J) \cap I$.

For the right hand side of the property, let $(P,Z) \sim_R (W,I) = (M, Z \cap I)$, where $M(e) = P(e) \setminus W(e)$ for all $e \in Z \cap I$. Assume $(R,J) \sim_R (W,I) = (O, J \cap I)$, where $O(e) = R(e) \setminus W(e)$ for all $e \in J \cap I$. Let $(M, Z \cap I) \sim_E (O, J \cup I) = (X, (Z \cap I) \cup (J \cap I))$, where

$$X(e) = \begin{cases} M(e) & if \ \ e \in (Z \cap I) \setminus (J \cap I), \\ O(e) & if \ \ e \in (J \cap I) \setminus (Z \cap I), \\ M(e) \setminus O(e) & if \ \ e \in (Z \cap I) \cap (J \cap I) \end{cases}$$

for all $e \in (Z \cap I) \cup (J \cap I)$. By assuming the main expressions of M and O, we can rewrite the mapping X as below:

$$X(e) = \begin{cases} P(e) \setminus W(e) & if \ \ e \in (Z \cap I) \setminus (J \cap I), \\ R(e) \setminus W(e) & if \ \ e \in (J \cap I) \setminus (Z \cap I), \\ (P(e) \setminus W(e)) \setminus (R(e) \setminus W(e)) & if \ \ e \in (Z \cap I) \cap (J \cap I) \end{cases}$$

for all $e \in (Z \cap J) \cup (Z \cap I)$. This leads that Q and X are the identical mapping. Hence this completes the proof.

Now, we give a corresponding example of part (g) of above Theorem.

Example 3.4. Suppose that E is the universe set of parameters and Z,J and I are the subsets of E such that $E = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7\}$, $Z = \{e_1, e_2, e_3, e_5\}$, $J = \{e_4, e_5, e_6\}$ and $I = \{e_2, e_5, e_6, e_7\}$.

Suppose that (P,Z), (R,J) and (W,I) be three soft sets over a common universe $U = \{h_1, h_2, h_3, h_4, h_5, h_6, h_7, h_8, h_9\}$ such that

$$(P,Z) = \{(e_1, \{h_1, h_2, h_9\}), (e_2, \{h_4, h_5, h_6\}), (e_3, \phi), (e_5, \{h_7, h_8, h_9\})\},$$

$(R, J) = \{(e_4, \{h_3, h_4, h_7\}), (e_5, \{h_7, h_8, h_9\}), (e_6, \{h_7, h_8\})\}$,
$(W, I) = \{(e_2, \{h_4, h_5\}), (e_5, \{h_3, h_8\}), (e_6, \{h_1, h_3, h_5, h_6\}), (e_7, \{h_4, h_6, h_8\})\}$.
For the left hand side of the equality, let $(P, Z) \sim_E (R, J) = (T, Z \cup J)$, where

$$T(e) = \begin{cases} P(e) & if \;\; e \in Z \setminus J, \\ R(e) & if \;\; e \in J \setminus Z, \\ P(e) \setminus R(e) & if \;\; e \in Z \cap J \end{cases}$$

for all $e \in Z \cup J = \{e_1, e_2, e_3, e_4, e_5, e_6\}$.

Since $Z \setminus J = \{e_1, e_2, e_3\}$, $T(e_1) = P(e_1) = \{h_1, h_2, h_9\}$, $T(e_2) = P(e_2) = \{h_4, h_5, h_6\}$, $T(e_3) = P(e_3) = \phi$, since $J \setminus Z = \{e_4, e_6\}$, $T(e_4) = R(e_4) = \{h_3, h_4, h_7\}$, $T(e_6) = R(e_6) = \{h_7, h_8\}$ and since $Z \cap J = \{e_5\}$, $T(e_5) = P(e_5) \setminus R(e_5) = \{h_7, h_8, h_9\} \setminus \{h_7, h_8, h_9\} = \phi$. So,

$(T, Z \cup J) = \{(e_1, \{h_1, h_2, h_9\}), (e_2, \{h_4, h_5, h_6\}), (e_3, \phi), (e_4, \{h_3, h_4, h_7\})(e_5, \phi), (e_6, \{h_7, h_8\})\}$.

Now let $(T, Z \cup J) \sim_R (W, I) = (Q, (Z \cup J) \cap I)$, where $Q(e) = T(e) \setminus W(e)$ for all $e \in (Z \cup J) \cap I$. By the main features of set theory and the definitions of Q along with T and considering that T is a piecewise function, we can write the below equalities for Q:

$$Q(e) = \begin{cases} P(e) \setminus W(e) & if \;\; e \in (Z \setminus J) \cap I = (Z \cap I) \setminus (J \cap I), \\ R(e) \setminus W(e) & if \;\; e \in (J \setminus Z) \cap I = (J \cap I) \setminus (Z \cap I), \\ (P(e) \setminus R(e)) \setminus W(e) & if \;\; e \in (Z \cap J) \cap I \end{cases}$$

for all $e \in (Z \cup J) \cap I = \{e_2, e_5, e_6\}$.

Since $(Z \setminus J) \cap I = (Z \cap I) \setminus (J \cap I) = \{e_2\}$, $Q(e_2) = P(e_2) \setminus W(e_2) = \{h_4, h_5, h_6\} \setminus \{h_4, h_5\} = \{h_6\}$, since $(J \setminus Z) \cap I = (J \cap I) \setminus (Z \cap I) = \{e_6\}$, $Q(e_6) = R(e_6) \setminus W(e_6) = \{h_7, h_8\} \setminus \{h_1, h_3, h_5, h_6\} = \{h_7, h_8\}$ and since $(Z \cap J) \cap I = Z \cap I \cap J = \{e_5\}$, $Q(e_5) = (P(e_5) \setminus R(e_5)) \setminus W(e_5) = \phi \setminus \{h_3, h_8\} = \phi$. So,

$(Q, (Z \cup J) \cap I) = ((P, Z) \sim_E (R, J)) \sim_R (W, I) = \{(e_2, \{h_6\}), (e_5, \phi), (e_6, \{h_7, h_8\})$.

For the right hand side of the equality let $(P, Z) \sim_R (W, I) = (M, Z \cap I)$, where $M(e) = P(e) \setminus W(e)$ for all $e \in Z \cap I = \{e_2, e_5\}$, then $M(e_2) = P(e_2) \setminus W(e_2) = \{h_4, h_5, h_6\} \setminus \{h_4, h_5\} = \{h_6\}$, $M(e_5) = P(e_5) \setminus W(e_5) = \{h_7, h_8, h_9\} \setminus \{h_3, h_8\} = \{h_7, h_9\}$.

Now let $(R, J) \sim_R (W, I) = (O, J \cap I)$, where $O(e) = R(e) \setminus W(e)$ for all $e \in J \cap I = \{e_5, e_6\}$. Then, $O(e_5) = R(e_5) \setminus W(e_5) = \{h_7, h_8, h_9\} \setminus \{h_3, h_8\} = \{h_7, h_9\}$, $O(e_6) = R(e_6) \setminus W(e_6) = \{h_7, h_8\} \setminus \{h_1, h_3, h_5, h_6\} = \{h_7, h_8\}$.

Now let $(M, Z \cap I) \sim_E (O, J \cap I) = (X, (Z \cap I) \cup (J \cap I))$, where

$$X(e) = \begin{cases} M(e) & if\ e \in (Z \cap I) \setminus (J \cap I), \\ O(e) & if\ e \in (J \cap I) \setminus (Z \cap I), \\ M(e) \setminus O(e) & if\ e \in (Z \cap I) \cap (J \cap I) \end{cases}$$

for all $e \in (Z \cap I) \cup (J \cap I) = \{e_2, e_5, e_6\}$. Since $(Z \cap I) \setminus (J \cap I) = \{e_2\}$, $X(e_2) = M(e_2) = P(e_2) \setminus W(e_2) = \{h_6\}$, since $(J \cap I) \setminus (Z \cap I) = \{e_6\}$, $X(e_6) = O(e_6) = R(e_6) \setminus W(e_6) = \{h_7, h_8\}$ and since $(Z \cap I) \cap (J \cap I) = \{e_5\}$, $X(e_5) = M(e_5) \setminus O(e_5) = (P(e_5) \setminus (W(e_5)) \setminus (R(e_5) \setminus W(e_5)) = \phi$. So, $(X, (Z \cap I) \cup (J \cap I)) = ((P, Z) \sim_R (W, I)) \sim_E ((R, J) \sim_R (W, I)) = \{(e_2, \{h_6\}), (e_5, \phi), (e_6, \{h_7, h_8\})$. Since Q and X are the same mappings, $((P, Z) \sim_E (R, J)) \sim_R (W, I) = ((P, Z) \sim_R (W, I)) \sim_E ((R, J) \sim_R (W, I))$ is satisfied.

4 Conclusion and Future Work

Here in this work, we have illustrated a brief analytical review of operations of soft sets. We have defined the extended difference of soft sets and also proved some of its properties. Moreover, we have shown the relationship between extended difference and the restricted difference and some other operations of soft sets. The main objective of this paper is to make soft set theory more effective and solid by enhancing the conceptual feature of operations on soft sets. One can may define the extended symmetric difference and can also construct a property which shows a relationship or connects of extended symmetric difference with restricted symmetric difference.

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On Lightly Nano ω-Closed Sets

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Abstaract − In this paper, we introduce and investigate the concepts of lightly nano ω-closed sets and lightly nano ω-open sets in a nano topological spaces, which are weaker form of lightly nano-closed sets and lightly nano-open sets and relationships among related ng-closed sets are investigated.

Keywords − *Lightly nano-closed sets, ng-closed sets and lightly nano ω-closed set.*

1 Introduction

Thivagar *et al.* [4] introduced the concept of nano topological spaces with respect to a subset X of a universe U. We study the relationships between some near nano open sets in nano topological spaces.

In this paper, we introduce and investigate the concepts of lightly nano ω-closed sets and lightly nano ω-open sets in a nano topological spaces, which are weaker form of lightly nano-closed sets and lightly nano-open sets and relationships among related ng-closed sets are investigated.

2 Preliminaries

Definition 2.1. *[7] Let U be a non-empty finite set of objects called the universe and R be an equivalence relation on U named as the indiscernibility relation. Elements belonging to the same equivalence class are said to be indiscernible with one another. The pair (U, R) is said to be the approximation space. Let $X \subseteq U$.*

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1. *The lower approximation of X with respect to R is the set of all objects, which can be for certain classified as X with respect to R and it is denoted by* $L_R(X)$. *That is,* $L_R(X) = \bigcup_{x \in U}\{R(x) : R(x) \subseteq X\}$, *where* $\tau_R(x)$ *denotes the equivalence class determined by* x.

2. *The upper approximation of X with respect to R is the set of all objects, which can be possibly classified as X with respect to R and it is denoted by* $U_R(X)$. *That is,* $U_R(X) = \bigcup_{x \in U}\{R(x) : R(x) \cap X \neq \phi\}$.

3. *The boundary region of X with respect to R is the set of all objects, which can be classified neither as X nor as not - X with respect to R and it is denoted by* $B_R(X)$. *That is,* $B_R(X) = U_R(X) - L_R(X)$.

Definition 2.2. *[4] Let U be the universe, R be an equivalence relation on U and* $\tau_R(X) = \{U, \phi, L_R(X), U_R(X), B_R(X)\}$ *where* $X \subseteq U$. *Then* $\tau_R(X)$ *satisfies the following axioms:*

1. U *and* $\phi \in \tau_R(X)$,

2. *The union of the elements of any sub collection of* $\tau_R(X)$ *is in* $\tau_R(X)$,

3. *The intersection of the elements of any finite subcollection of* $\tau_R(X)$ *is in* $\tau_R(X)$.

Thus $\tau_R(X)$ *is a topology on U called the nano topology with respect to X and* $(U, \tau_R(X))$ *is called the nano topological space. The elements of* $\tau_R(X)$ *are called nano-open sets (briefly n-open sets). The complement of a n-open set is called n-closed.*

In the rest of the paper, we denote a nano topological space by $(U, \mathcal{N})$, *where* $\mathcal{N} = \tau_R(X)$. *The nano-interior and nano-closure of a subset A of U are denoted by n-int(A) and n-cl(A), respectively.*

Definition 2.3. *A subset H of a space* $(U, \mathcal{N})$ *is called*

1. *nano* α*-open set (briefly* $n\alpha$*-open) [4] if* $H \subseteq$ *n-int(n-cl(n-int(H))).*

2. *nano semi-open set [4] if* $H \subseteq$ *n-cl(n-int(H)).*

3. *nano pre-open set [4] if* $H \subseteq$ *n-int(n-cl(H)).*

4. *nano semi-preopen set [9] if* $H \subseteq$ *n-cl(n-int(n-cl(H))).*

5. *nano regular-open set (briefly nr-open) [4] if* $H =$ *n-int(n-cl(H)).*

6. *nano nowhere dense (briefly n-nowhere dense) [5] if n-int(n-cl(H))* $= \phi$.

The complements of the above mentioned sets are called their respective closed sets.

Definition 2.4. *A subset H of a space $(U, \mathcal{N})$ is called*

1. *nano g-closed (briefly ng-closed) [2] if $n\text{-}cl(H) \subseteq G$, whenever $H \subseteq G$ and G is n-open.*

2. *nano sg-closed set (briefly nsg-closed) [3] if $n\text{-}scl(H) \subseteq G$, whenever $H \subseteq G$ and G is nano semi open.*

3. *nano αg-closed (briefly $n\alpha g$-closed) [11] if $n\text{-}\alpha cl(H) \subseteq G$ whenever $H \subseteq G$ and G is n-open.*

4. *nano $g\alpha$-closed (briefly $ng\alpha$-closed) [11] if $n\text{-}\alpha cl(H) \subseteq G$ whenever $H \subseteq G$ and G is $n\alpha$-open.*

5. *nano gsp-closed (briefly ngsp-closed) [9] if $n\text{-}spcl(H) \subseteq G$ whenever $H \subseteq G$ and G is n-open.*

3 On lightly nano closed sets

Definition 3.1. *A subset R of a space $(U, \mathcal{N})$, is called*

1. *lightly nano closed (briefly $\mathcal{L}$-n-closed) set if $n\text{-}cl(R) \subseteq S$ whenever $R \subseteq S$ and S is nano semi-open.*

   The complement of a $\mathcal{L}$-n-closed set is said to be $\mathcal{L}$-n-open.

2. *lightly nano ω-closed (briefly $\mathcal{L}$-$n\omega$-closed) set if $n\text{-}cl(n\text{-}int(R)) \subseteq S$ whenever $R \subseteq S$ and S is nano semi-open.*

   The complement of a $\mathcal{L}$-$n\omega$-closed set is said to be $\mathcal{L}$-$n\omega$-open.

Recall, we denote the class of $\mathcal{L}$-n-closed sets in $(U, \mathcal{N})$ by $\mathcal{K}$ i.e., $\mathcal{K} = \{R \subseteq U : R$ is $\mathcal{L}$-n-closed in $(U, \mathcal{N})\}$.

Theorem 3.2. *In a space $(U, \mathcal{N})$, the following relations are true for a subset R of U.*

1. *R is n-closed $\Rightarrow$ R is $\mathcal{L}$-n-closed.*

2. *R is $\mathcal{L}$-n-closed $\Rightarrow$ R is ng-closed.*

3. *R is $\mathcal{L}$-n-closed $\Rightarrow$ R is nsg-closed.*

4. *R is $\mathcal{L}$-n-closed $\Rightarrow$ R is $ng\alpha$-closed.*

Proof.

1. Let R be every n-closed set and S be every nano semi-open set such that $R \subseteq S$. Then $n\text{-}cl(R) \subseteq S$. Since $n\text{-}cl(R) = R$ and hence R is $\mathcal{L}$-n-closed.

2. Let $R \in \mathcal{K}$ and S be every n-open set such that $R \subseteq S$. Since every n-open set is nano semi-open and R is $\mathcal{L}$-n-closed set, we have $n\text{-}cl(R) \subseteq S$ and hence R is ng-closed.

3. Let $R \in \mathcal{K}$ and S be every nano semi-open set containing R. Then $n\text{-}scl(R) \subseteq n\text{-}cl(R) \subseteq S$, since R is $\mathcal{L}$-n-closed. Therefore R is nsg-closed.

4. Let $R \in \mathcal{K}$ and S be every $n\alpha$-open set containing R. Since every $n\alpha$-open set is nano semi-open and since $n\text{-}\alpha cl(R) \subseteq n\text{-}cl(R)$, we have by hypothesis, $n\text{-}\alpha cl(R) \subseteq n\text{-}cl(R) \subseteq S$ and so R is $ng\alpha$-closed.

Proposition 3.3. *In a space $(U, \mathcal{N})$, the following relations are true for a subset R of U.*

1. R is $\mathcal{L}$-n-closed $\Rightarrow$ R is $\mathcal{L}$-$n\omega$-closed.

2. R is n-closed $\Rightarrow$ R is $\mathcal{L}$-$n\omega$-closed.

3. R is ng-closed $\Rightarrow$ R is $\mathcal{L}$-$n\omega$-closed.

Proof. 1. Let $R \subseteq S$ where S is nano semi-open and R is $\mathcal{L}$-n-closed. $n\text{-}cl(n\text{-}\mathrm{int(R)}) \subseteq n\text{-}cl(R) \subseteq S$. This proves R is $\mathcal{L}$-$n\omega$-closed.

2. Let R is n-closed. Also S is nano semi-open. Therefore $R \subseteq n\text{-}cl(n\text{-}int(R)) \subseteq S$ which shows that R is $\mathcal{L}$-$n\omega$-closed.

3. Let R is ng-closed. Since S is nano semi-open. Therefore $n\text{-}cl(n\text{-}int(R)) \subseteq R \subseteq S$. Thus R is $\mathcal{L}$-$n\omega$-closed.

Proposition 3.4. *In a space $(U, \mathcal{N})$, the following relations are true for a subset R of U.*

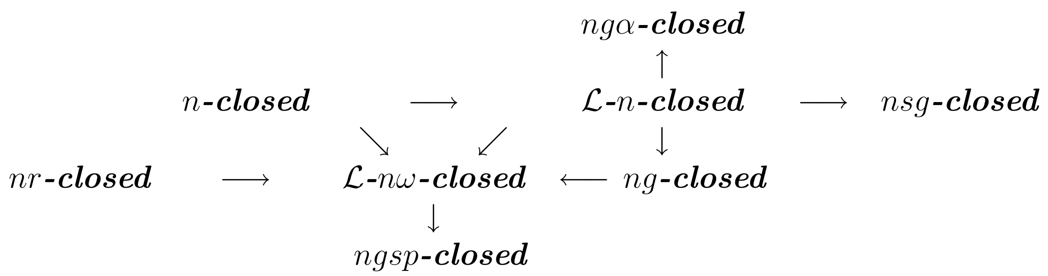
1. *R is nr-closed $\Rightarrow$ R is $\mathcal{L}$-$n\omega$-closed.*

2. *R is $\mathcal{L}$-$n\omega$-closed $\Rightarrow$ R is $ngsp$-closed.*

Proof. 1. Let every nr-closed set can be as R and S be a nano semi-open set containing R. Since R is nr-closed we have $R = n\text{-}cl(n\text{-}int(R)) \subseteq S$ and hence R is $\mathcal{L}$-$n\omega$-closed.

2. Let every $\mathcal{L}$-$n\omega$-closed set can be as R and S be a n-open set containing R. Then S is a nano semi-open set containing R, so $n\text{-}cl(n\text{-}int(R)) \subseteq S$. Since S is n-open we get $n\text{-}int(n\text{-}cl(n\text{-}int(R))) \subseteq S$ which implies $n\text{-}spcl(R) \subseteq S$, hence R is $ngsp$-closed.

Remark 3.5. *These relations are shown in the diagram.*



The converses of each statement in Theorem 3.2, Propositions 3.3 and 3.4 are not true as shown in the following Example.

Example 3.6. *Let* $U = \{1_a, 1_b, 1_c\}$ *with* $U/R = \{\{1_a\}, \{1_b, 1_c\}\}$ *and* $X = \{1_c\}$. *Then* $\mathcal{N} = \{\phi, U, \{1_b, 1_c\}\}$. *Let* $A = \{1_a, 1_b\}$ *be* $\mathcal{L}$*-*n*-closed but not* n*-closed.*

Example 3.7. *Let* $U = \{1_a, 1_b, 1_c, 1_d\}$ *with* $U/R = \{\{1_a\}, \{1_b, 1_c, 1_d\}\}$ *and* $X = \{1_a\}$. *Then* $\mathcal{N} = \{\phi, U, \{1_a\}\}$. *Then*

1. ng-closed $\nrightarrow$ $\mathcal{L}$-n-closed.

   Let us consider $B = \{1_b\}$ is ng-closed. Then

   $$R = n\text{-}cl(B) = n\text{-}cl(\{1_b\}) = \{1_b, 1_c, 1_d\}$$

   Therefore $R \nsubseteq S$, since S is nano semi open. Hence B is not $\mathcal{L}$-n-closed.

2. ng-closed $\nrightarrow$ $\mathcal{L}$-$n\omega$-closed.

   Let us consider $C = \{1_a, 1_b, 1_c\}$ is ng-closed. Then

   $$R = n\text{-}cl(n\text{-}int(C)) = n\text{-}cl(n\text{-}int(\{1_a, 1_b, 1_c\})) = n\text{-}cl(\{1_a\}) = U$$

   Therefore $R \nsubseteq S$, since S is nano semi open. Hence C is not $\mathcal{L}$-$n\omega$-closed.

3. nsg-closed $\nrightarrow$ $\mathcal{L}$-n-closed.

   Let us consider $D = \{1_c\}$ is nsg-closed. Then

   $$R = n\text{-}cl(D) = n\text{-}cl(\{1_c\}) = \{1_b, 1_c, 1_d\}$$

   Therefore $R \nsubseteq S$, since S is nano semi open. Hence D not $\mathcal{L}$-n-closed.

4. $ng\alpha$-closed $\nrightarrow$ $\mathcal{L}$-n-closed.

   Let us consider $E = \{1_a, 1_c\}$ is $ng\alpha$-closed. Then

   $$R = n\text{-}cl(E) = n\text{-}cl(\{1_a, 1_c\}) = U$$

   Therefore $R \nsubseteq S$, since S is nano semi open. Hence E is not $\mathcal{L}$-n-closed.

5. $\mathcal{L}$-$n\omega$-closed $\nrightarrow$ $\mathcal{L}$-n-closed.

   Let us consider $F = \{1_c, 1_d\}$ is $\mathcal{L}$-$n\omega$-closed. Then

   $$R = n\text{-}cl(F) = n\text{-}cl(\{1_c, 1_d\}) = U$$

   Therefore $R \nsubseteq S$, since S is nano semi open. Hence F is not $\mathcal{L}$-n-closed.

6. $\mathcal{L}$-$n\omega$-closed $\nrightarrow$ n-closed.

   Let us consider $J = \{1_d\}$ is $\mathcal{L}$-$n\omega$-closed. Then $J \nsubseteq \mathcal{N}'$. Hence J is not n-closed.

7. $\mathcal{L}$-$n\omega$-closed $\nrightarrow$ nr-closed.

   Let us consider $K = \{1_b, 1_c, 1_d\}$ is $\mathcal{L}$-$n\omega$-closed. Then

   $$R = n\text{-}cl(n\text{-}int(\{1_b, 1_c, 1_d\})) = \phi$$

   Therefore $K \neq R$. Hence K is not nr-closed.

8. $ngsp$-closed $\nrightarrow$ $\mathcal{L}$-$n\omega$-closed.

   Let us consider $I = \{1_a, 1_d\}$ is $ngsp$-closed. Then

   $$R = n\text{-}cl(n\text{-}int(\{1_a\})) = U$$

   Therefore $R \nsubseteq S$, since S is nano semi open. Hence I is not $\mathcal{L}$-$n\omega$-closed.

Theorem 3.8. *In a space $(U, \mathcal{N})$ is both n-closed and $n\alpha g$-closed, then it is $\mathcal{L}$-$n\omega$-closed.*

Proof. Let $n\alpha g$-closed set can be as R and S be a n-open set containing R. Then $S \supseteq n\text{-}\alpha cl(R) = R \cup n\text{-}cl(n\text{-}int(n\text{-}cl(R)))$. Since R is n-closed, we have $S \supseteq n\text{-}cl(n\text{-}int(R))$ and hence R is $\mathcal{L}$-$n\omega$-closed.

Theorem 3.9. *In a space $(U, \mathcal{N})$ is both n-open and $\mathcal{L}$-$n\omega$-closed, then it is n-closed.*

Proof. Since R is both n-open and $\mathcal{L}$-$n\omega$-closed, $R \supseteq n\text{-}cl(n\text{-}int(R)) = n\text{-}cl(R)$ and hence R is n-closed.

Corollary 3.10. *In a space $(U, \mathcal{N})$ is both n-open and $\mathcal{L}$-$n\omega$-closed, then it is both nr-open and nr-closed.*

Theorem 3.11. *A set R is $\mathcal{L}$-$n\omega$-closed $\iff$ $n\text{-}cl(n\text{-}int(R)) - R$ contains no non-empty nano semi-closed set.*

Proof. Necessity. Let M be a nano semi-closed set such that $M \subseteq n\text{-}cl(n\text{-}int(R)) - R$. Since M^c is nano semi-open and $R \subseteq M^c$, from the definition of $\mathcal{L}$-$n\omega$-closed set it follows that $n\text{-}cl(n\text{-}int(R)) \subseteq M^c$. Hence $M \subseteq (n\text{-}cl(n\text{-}int(R)))^c$. This implies that $M \subseteq (n\text{-}cl(n\text{-}int(R))) \cap (n\text{-}cl(n\text{-}int(R)))^c = \phi$.

Sufficiency. Let $R \subseteq T$, where T is nano semi-open set subset in U. If $n\text{-}cl(n\text{-}int(R))$ is not contained in T, then $n\text{-}cl(n\text{-}int(R)) \cap T^c$ is a non-empty nano semi-closed subset of $n\text{-}cl(n\text{-}int(R)) - R$, we obtain a contradiction.

Theorem 3.12. *Let $(U, \mathcal{N})$ be a space and $K \subseteq R \subseteq U$. If K is $\mathcal{L}$-$n\omega$-closed set relative to R and R is and $\mathcal{L}$-$n\omega$-closed subset of U then K is $\mathcal{L}$-$n\omega$-closed set relative to U.*

Proof. Let $K \subseteq S$ and S be a nano semi-open. Then $K \subseteq R \cap S$. Since K is $\mathcal{L}$-$n\omega$-closed relative to R, we have $n\text{-}cl_R(n\text{-}int_R(K)) \subseteq R \cap S$. That is $R \cap n\text{-}cl(n\text{-}int(K)) \subseteq R \cap S$. We have $R \cap n\text{-}cl(n\text{-}int(K)) \subseteq S$ and then $[R \cap n\text{-}cl(n\text{-}int(K))] \cup (n\text{-}cl(n\text{-}int(K)))^c \subseteq S \cup (n\text{-}cl(n\text{-}int(K)))^c$. Since R is $\mathcal{L}$-$n\omega$-closed, we have $n\text{-}cl(n\text{-}int(R)) \subseteq S \cup (n\text{-}cl(n\text{-}int(K)))^c$. Since $n\text{-}cl(n\text{-}int(K))$ is not contained in $(n\text{-}cl(n\text{-}int(K)))^c$ we get $R \supseteq n\text{-}cl(n\text{-}int(K))$. Thus K is $\mathcal{L}$-$n\omega$-closed set relative to U.

Corollary 3.13. *If R is both n-open and $\mathcal{L}$-nω-closed and P is n-closed in a space $(U, \mathcal{N})$, then $R \cap P$ is $\mathcal{L}$-nω-closed.*

Proof. Since P is n-closed, we have $R \cap P$ is n-closed in R. Therefore $n\text{-}cl_R(R \cap P) = R \cap P$ in R. Let $R \cap P \subseteq S$, where S is nano semi-open in R. Then $n\text{-}cl_R(n\text{-}int_R(R \cap P)) \subseteq S$ and hence $R \cap P$ is $\mathcal{L}$-$n\omega$-closed in R. By Theorem 3.18, $R \cap P$ is $\mathcal{L}$-$n\omega$-closed.

Theorem 3.14. *If R is $\mathcal{L}$-nω-closed and $R \subseteq K \subseteq n\text{-}cl(n\text{-}int(R))$, then K is $\mathcal{L}$-nω-closed.*

Proof. Since $R \subseteq K$ we have $n\text{-}cl(n\text{-}int(K)) - K \subseteq n\text{-}cl(n\text{-}int(R)) - R$. By Theorem 3.11, $n\text{-}cl(n\text{-}int(R)) - R$ contains no non-empty nano semi-closed set and so $n\text{-}cl(n\text{-}int(K)) - K$ contains no non-empty nano semi-closed, so K is $\mathcal{L}$-$n\omega$-closed.

Theorem 3.15. *If a subset R of a space $(U, \mathcal{N})$ is every n-nowhere dense, then it is $\mathcal{L}$-nω-closed.*

Proof. Since $n\text{-}int(R) \subseteq n\text{-}int(n\text{-}cl(R))$ and R is n-nowhere dense, $n\text{-}int(R) = \phi$. Therefore $n\text{-}cl(n\text{-}int(R)) = \phi$ and hence R is $\mathcal{L}$-$n\omega$-closed.

Remark 3.16. *The converse of Theorem 3.15 are not true as shown in the following Example.*

Example 3.17. *In Example 3.6, then $J = \{1_a, 1_b\}$ is $\mathcal{L}$-nω-closed but not n-nowhere dense.*

Proposition 3.18. *In a space $(U, \mathcal{N})$, R is n-open $\Rightarrow$ R is $\mathcal{L}$-nω-open.*

Proof. Let every n-open set can be as R in a space U. Then R^c is n-closed in U. By Proposition 3.11(2) follows that R^c is $\mathcal{L}$-$n\omega$-closed in U. Hence R is $\mathcal{L}$-$n\omega$-open.

Remark 3.19. *The converse of Proposition 3.18 are not true as shown in the following Example.*

Example 3.20. *In Example 3.7, then $M = \{1_a, 1_b, 1_c\}$ is $\mathcal{L}$-nω-open but not n-open.*

Proposition 3.21. *A subset R of a space $(U, \mathcal{N})$, in the following results are true*

1. If R is $\mathcal{L}$-n-open then R is $\mathcal{L}$-$n\omega$-open.

2. If R is ng-open then R is $\mathcal{L}$-$n\omega$-open.

3. If R is $\mathcal{L}$-$n\omega$-open then R is $ngsp$-open.

Remark 3.22. *The converse of Proposition 3.21 are not true as shown in the following Example.*

Example 3.23. *In Example 3.6, then*

1. $\{1_a, 1_b\}$ is $\mathcal{L}$-$n\omega$-open but not $\mathcal{L}$-n-open.

2. $\{1_a, 1_c\}$ is $\mathcal{L}$-$n\omega$-open but not ng-open.

Example 3.24. *In Example 3.7, then* $\{1_c\}$ *is ngsp-open but not* $\mathcal{L}$*-n*ω*-open.*

Theorem 3.25. *A subset R be a space U is* $\mathcal{L}$*-n*ω*-open if* $Q \subseteq n\text{-}int(n\text{-}cl(R))$ *whenever* $Q \subseteq R$ *and Q is nano semi-closed.*

Proof. Let every $\mathcal{L}$-$n\omega$-open can be as R. Then R^c is $\mathcal{L}$-$n\omega$-closed. Let Q be a nano semi-closed set contained in R. Then Q^c is a nano semi-open set in U containing R^c. Since R^c is $\mathcal{L}$-$n\omega$-closed, we have $n\text{-}cl(n\text{-}int(R^c)) \subseteq Q^c$. Therefore $Q \subseteq n\text{-}int(n\text{-}cl(R))$.

Conversely, we suppose that $Q \subseteq n\text{-}int(n\text{-}cl(R))$ whenever $Q \subseteq R$ and Q is nano semi-closed. Then Q^c is a nano semi-open set containing R^c and $Q^c \supseteq (n\text{-}int(n\text{-}cl(R)))^c$. It follows that $Q^c \supseteq n\text{-}cl(n\text{-}int(R^c))$. Hence R^c is $\mathcal{L}$-$n\omega$-closed and so R is $\mathcal{L}$-$n\omega$-open.

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A Network Shortest Path Algorithm via Hesitancy Fuzzy Digraph

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Abstaract − Many extension and generalization of fuzzy sets have been studied and introduced in the literature. Hesitancy fuzzy digraph is a generalization of intuitionistic fuzzy set and fuzzy graph. In this paper, we redefine some basic operations of hesitancy fuzzy graph and it is referred as hesitancy fuzzy digraph (in short HFDG). We discuss some arithmetic operations and relations among HFDG. We further proposed a method to solve a shortest path problem through score function.

Keywords − *Digraphs, Hesitancy fuzzy sets, Hesitancy fuzzy digraphs.*

1 Introduction

Several extension of fuzzy set have been proposed, since 1965 [11]. Some of the works among the generalization are remarkable in the history of literature such as intuitionistic fuzzy set [1], type 2 fuzzy set, interval valued fuzzy set, neutrosophic sets [19] and so on. Hesitant fuzzy sets are useful to deal with group decision making problems when experts have a hesitation among several possible memberships for an element to a set.The concept of hesitancy fuzzy set (HFS for short) proposed by Torra [4]. It is a generalization of fuzzy sets and intuitionistic fuzzy sets, that permits us to represent the situation in which different membership functions are considered possible. The concept of hesitancy fuzzy set is characterized by three dependent membership degrees namely truth-membership degree (t), hesitancy membership degree (h), and falsity-membership degree (f). HFSs are motivated to handle the common difficulty

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that appears in fixing the membership degree of an element from some possible values. This situation is rather common in decision making problems too while an expert is asked to assign different degrees of membership to a set of elements $\{x, y, z, ...\}$ in a set A. Often problems arise due to uncertain issues and situations hence one is faced with hesitant moments. The researcher had to find ways and means to take the problems and arrive at a solution. Therefore researchers have taken up the study and application of HFS. HFSs have been extended in [2,3,5,6,13] and Zhu[12] from different perspectives such as, both quantitatively and qualitatively. Since the concept of the hesitant fuzzy set was established,it has gained increasing attention and has been successfully applied to many uncertain decision making problems. hesitancy fuzzy graphs (HFG for short) where introduced and studied by Pathinath and Jon [14] in order to capture the common intricacy that occurs during a selection of membership degree of an element from some possible values that makes one to hesitate. The concept of hesitancy fuzzy graphs are generalizations of fuzzy graphs [7], intuitionistic fuzzy graphs [8,9] and vague graphs [10]. The Table 1, presented a comparative study between all of these kinds of graphs. The shortest path problem is one of the most fundamental problems in graph theory which has many applications diversified field such operation research, computer science, communication network and so on. In a network, the shortest path problem concentrate at finding the path from one source to destination node with minimum weight, where some weight is attached to each edge connecting a pair of nodes. In the literature, many shortest path problems [16-18] have been studied with different types of input data, including fuzzy set, intuitionistic fuzzy sets, trapezoidal intuitionistic fuzzy sets, vague set.

In this paper a new method is proposed for solving shortest path problems in a network which the edges length are characterized by hesitancy fuzzy numbers. We consider a situation that a company wishes to assign a work to service center based on the possible to clear the issue(t) and not possible to clear the issue (f). But if the technician is on leave then the company or else the service center has to approach the near by center. This category is called the hesitancy(h). The paper is organized as follows: In Section 2, definition of Hesitancy fuzzy set is given. In Section 3, we provide the definition of hesitancy fuzzy digraphs (HFDGs), some arithmetic operation and score function of a hesitancy fuzzy number. Section 4 and 5, Network terminology and Algorithm is proposed using the score function and example for the proposed algorithm for network problems to find shortest path and distance from the source node to the destination node. In Section 6, a comparative study between the proposed approach and other existing approaches is summarized and Section 7 conclude the paper.

2 Preliminary

In this paper, we provide the basic definition of hesitancy fuzzy set. This is very useful for the discussions.

Definition 2.1. *Let X be a fixed set, a Hesitant fuzzy set (HFS) on X is in terms of a function that when applied to X returns a subset of* $[0, 1]$. *The HFS is defined by a mathematical symbol as:* $A = \{< x, h_A(x) >: x \in X\}$, *where* $h_A(x)$ *is a set of*

Table 1: Comparative study between hesitancy fuzzy graph, fuzzy graphs, intuitionistic fuzzy graphs and vague graphs.

TYPES OF GRAPHS	ADVANTAGEES	DISADVANTAGES
Crisp graph	Graph is a relationship between a set of objects. Each objects is denoted by a vertex and relationship between every object is denoted by an edge	Crisp graph is difficult to apply for uncertainty on vertices and/or edges and the relation among every objects are not precise.
Fuzzy graph[7]	Crisp graph and Fuzzy graph are structurally similar but fuzzy graph can be applied for uncertainty on vertices and/or edges.	It gives less accuracy into problems since the existence of non-zero hesitation.
Intuitionistic fuzzy graph[8,9]	Identify the nature of the arcs. Intuitionistic fuzzy graph assigning degree of membership to each object because there is a fair chance of existence of a hesitation part at each moment of evaluation of anything and also it gives more accuracy into the problem.	Hesitation remains in choosing membership degree of an element from some possible values.
Vague graph[10]	In vague graph, true membership considered as the lower bound for degree of positive membership and false membership is the lower bound for negative of membership.	Error occurs due to choosing lower bound.
Hesitancy fuzzy graph[14]	Hesitant fuzzy graph gives more accuracy than intuitionistic fuzzy graph because it is dependent on membership and non-membership.	We can't apply this technique, if the membership and non-membership are independent.

some values in $[0,1]$, *denoting the possible membership degrees of the element* $x \in X$ *to the set* A *and called* $h = h_A(x)$ *a hesitant fuzzy element (HFE) and* Θ *the set of all HFEs.*

Definition 2.2. *Let* V *be a finite hesitancy fuzzy non-empty set,* $A = \langle V, t_i, h_i, f_i\rangle$ *a hesitancy fuzzy set of* V *and* $B = \langle V \times V, t_{i,j}, h_{i,j}, f_{i,j}\rangle$ *a hesitancy fuzzy relation on* V. *Then the ordered pair* $G = (A, B)$ *is called hesitancy fuzzy directed graph or hesitancy fuzzy digraph (HFDG).*

Where $t_i : V \to [0,1]$, $h_i : V \to [0,1]$ *and* $f_i : V \to [0,1]$ *denote the degree of membership(t), hesitancy(h) and non-membership(f) of the element* $v_i \in V$ *respectively and* $t_i(v_i) + h_i(v_i) + f_i(v_i) = 1$ *for every* $v_i \in V, \forall i \in Z$, $h_i = 1 - (t_i + f_i)$ *and* $B = E \subseteq V \times V$ *where* $t_{i,j} : V \times V \to [0,1]$, $h_{i,j} : V \times V \to [0,1]$ *and* $f_{i,j} : V \times V \to [0,1]$ *are the degrees of membership(t), hesitancy(h) and non-membership(f) of the edge* (v_i, v_j) *respectively such that* $0 \leq t_{i,j} + h_{i,j} + f_{i,j} \leq 1$

and

$$t_{i,j} \leq min\{v_i, v_j\}$$
$$h_{i,j} \leq min\{v_i, v_j\}$$
$$f_{i,j} \leq max\{v_i, v_j\}$$

Note 1: In hesitancy fuzzy graph, the graph is symmetric relation on V but HFDG is not symmetric relation on V

Notation

1. Hereafter, $\langle t(v_i), h(v_i), f(v_i)\rangle$ or simply $\langle t_i, h_i, f_i\rangle$ denotes the degrees of membership, hesitancy and non-membership of the vertex $v_i \in V$, such that $t_i(v_i) + h_i(v_i) + f_i(v_i) = 1$.

2. $\langle t(v_{i,j}), h(v_{i,j}), f(v_{i,j})\rangle$ or simply $\langle t_{i,j}, h_{i,j}, f_{i,j}\rangle$ denotes the degrees of membership, hesitancy and non-membership of the edge $(v_i, v_j) \in V \times V$, such that $0 \leq t_{i,j} + h_{i,j} + f_{i,j} \leq 1$.

Note 2:

1. If $t_{i,j} = 0$, for some i and j , then there is no edge between v_i and v_j and it is indexed by $\langle 0, 1, 0\rangle$ or $\langle 0, 0, 1\rangle$ or $\langle 0, 0, 0\rangle$. Otherwise there exists edge between v_i and v_j.

2. In this paper, we are interested in hesitancy fuzzy zero, given by: $0 = \langle 0, 0, 1\rangle$

Example 2.3. *Let $G = (V, E)$ be a HFDG, where the vertex set is $V = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ shown in Figure 1*

The index matrix of G is $G = \{V, V, \langle t_{i,j}, h_{i,j}, f_{i,j}\rangle\}$, where $V = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ is given in the Table 2.

Table 2: Index matrix

	v_1	v_2	v_3	v_4	v_5	v_6
v_1	$\langle 0, 0, 1\rangle$	$\langle 0.3, 0.2, 0.5\rangle$	$\langle 0, 0, 1\rangle$	$\langle 0.1, 0.2, 0.5\rangle$	$\langle 0, 0, 1\rangle$	$\langle 0, 0, 1\rangle$
v_2	$\langle 0, 0, 1\rangle$	$\langle 0.6, 0.3, 0.1\rangle$	$\langle 0, 0, 1\rangle$	$\langle 0.5, 0.3, 0.2\rangle$	$\langle 0, 0, 1\rangle$	$\langle 0, 0, 1\rangle$
v_3	$\langle 0, 0, 1\rangle$	$\langle 0, 0, 1\rangle$	$\langle 0.1, 0.2, 0.7\rangle$	$\langle 0, 0, 1\rangle$	$\langle 0, 0, 1\rangle$	$\langle 0.1, 0.2, 0.6\rangle$
v_4	$\langle 0, 0, 1\rangle$	$\langle 0, 0, 1\rangle$	$\langle 0.1, 0.2, 0.7\rangle$	$\langle 0, 0, 1\rangle$	$\langle 0, 0, 1\rangle$	$\langle 0, 0, 1\rangle$
v_5	$\langle 0.3, 0.1, 0.5\rangle$	$\langle 0, 0, 1\rangle$	$\langle 0, 0, 1\rangle$	$\langle 0, 0, 1\rangle$	$\langle 0, 0, 1\rangle$	$\langle 0.4, 0.1, 0.3\rangle$
v_6	$\langle 0, 0, 1\rangle$	$\langle 0, 0, 1\rangle$	$\langle 0, 0, 1\rangle$	$\langle 0.3, 0.1, 0.5\rangle$	$\langle 0, 0, 1\rangle$	$\langle 0, 0, 1\rangle$

Definition 2.4. *Let $A_1 = \langle t_1, h_1, f_1\rangle$ and $A_2 = \langle t_2, h_2, f_2\rangle$ be two hesitancy fuzzy numbers. Then, the operations for HFNs are defined as below;*

1. $A_1 \oplus A_2 = \langle t_1 + t_2 - t_1t_2, h_1 + h_2 - h_1h_2, f_1f_2\rangle$

2. $A_1 \otimes A_2 = \langle t_1t_2, h_1h_2, f_1 + f_2 - f_1f_2\rangle$

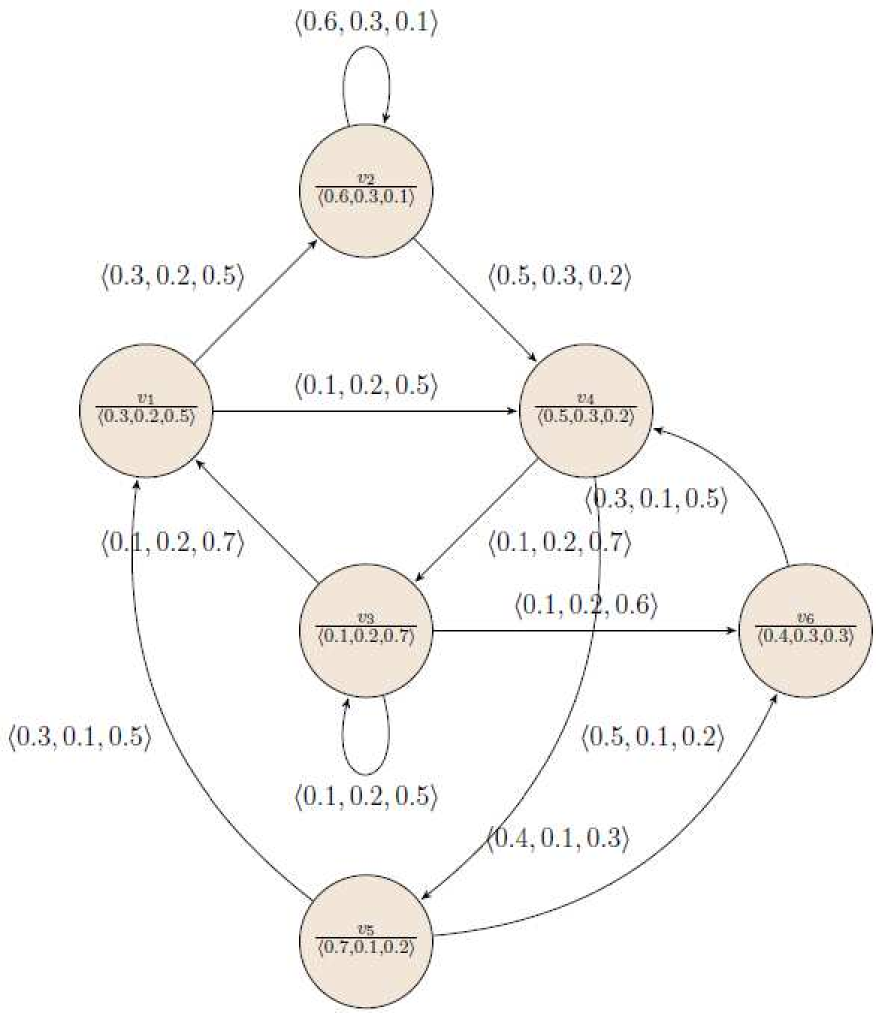

Figure 1: G: Hesitancy fuzzy digraph

3. $\lambda A_1 = \langle (1-(1-t_1)^\lambda), (1-(1-h_1)^\lambda), f_1^\lambda \rangle$

4. $A_1^\lambda = \langle t_1^\lambda, h_1^\lambda, (1-(1-f_1)^\lambda), \rangle$

Definition 2.5. *Let $A = \langle t, h, f \rangle$ be a hesitancy fuzzy number. Then, the score function $s(A)$ is defined by* $s(A) = \frac{1+(t+2h-f)(2-t-f)}{2}$

Comparison of two hesitancy fuzzy numbers.

Let $A_1 = \langle t_1, h_1, f_1 \rangle$ and $A_2 = \langle t_2, h_2, f_2 \rangle$ be two hesitancy fuzzy numbers then

1. $A_1 \prec A_2$ if $s(A_1) \prec s(A_2)$

2. $A_1 \succ A_2$ if $s(A_1) \succ s(A_2)$

3. $A_1 = A_2$ if $s(A_1) = s(A_2)$

3 Network Terminology and the Proposed Algorithm

Consider a directed network $G(V, E)$ consisting of a finite set of nodes $V = \{1, 2, ..., n\}$ and a set of m directed edges $E \subseteq V \times V$. Each edge is denoted by an ordered pair (i, j) where $i, j \in V$ and $i \neq j$. In this network, we specify two nodes, denoted by 1 and n, which are the source node and the destination node, respectively. We define a path $p_{ij} = \{i = i_1, (i, j), i_3, i_4, .., i_{l-1}, (i_{l-1}, i_l), i_l\}$ of alternating nodes and edges. The existence of atleast one path P_{1i} in $G(V, E)$ is assumed for every $i \in V - \{1\}$. d_{ij} denotes hesitancy fuzzy number associated with the edge (i, j), corresponding to the length necessary to traverse (i, j) from i to j. The hesitancy fuzzy distance along the path P is denoted as $d(P)$ is defined as $d(P) = \sum_{\{i,j \in P\}} d_{ij}$

Remark: A node i is said to be predecessor node of node j if

1. Node i is directly connected to node j.
2. The direction of path connecting node i and j from i to j.

In this paper, the edge length in a network is considered to be a hesitancy fuzzy number , also in this section, an algorithm is being proposed to find the hesitancy fuzzy minimum arc length and the shortest distance in a network of each node from source node. The algorithm is a labeling technique which can be applied for solving shortest path problems occurring in real life problem.

Algorithm:

1. Assume $d_1 = \langle 0, 0, 1 \rangle$ and label the source node as d_1.
2. Find $d_j = min\{d_i \oplus d_{ij}\}$, where $j = 2, 3, ..., n$.
3. If minimum occurs corresponding to unique value of i i.e., $i = p$ then label node j as $[d_j, p]$. If minimum occurs corresponding to more than one values of i then it represents that there are more than one hesitancy fuzzy path between source node i and node j but hesitancy fuzzy distance along the path is d_j, so choose any value of i.
4. Let the destination node (node n) be labeled as $[d_n, l]$, then the hesitancy fuzzy shortest distance between source node is d_n.
5. Since destination node is labeled as $[d_n, l]$, so, to find the hesitancy fuzzy shortest path between source node and destination node, check the label of node l. Let it be $[d_l, r]$, now check the label of node r and so on. Repeat the same procedure until node l is attained.
6. Now the hesitancy fuzzy shortest path can be obtained by combining all the nodes obtained by the step 5.

4 Illustrative Network Example

A DTH company, say a wish to provide a best service to the customers. A customer, say f have problem in the DTH. He approaches the customer care of the DTH company to get recover from the issue. The company has private service center in different cities. Those service centers are associated with the other service centers because if the issue is big, they will approaches the other. Here the truth membership represents that possibe to clear the issue by the service center, non-membership represents that not possible to clear the issue and hesitancy represents technicians availability. The company wish to find best service center through the proposed algorithm.

Let us consider a hesitancy fuzzy digraph for the given network problem shown in figure 2. b, c, d, e represents the private service centers and a, b, c, d, e, f are called nodes.

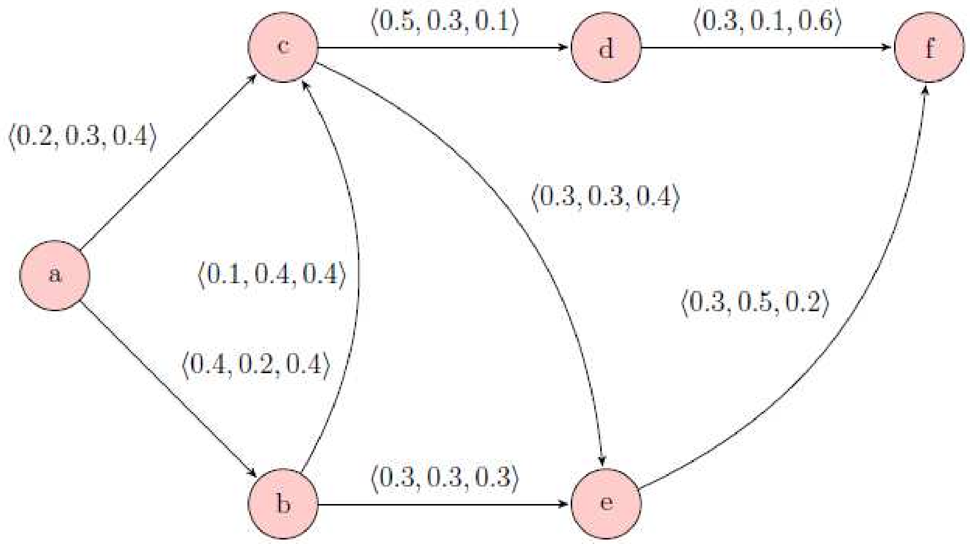

Figure 2: Network with hesitancy fuzzy distance

Using the algorithm described in section 4, the following computational results are obtained. Let us consider the source node = a and the destination node = f, so $n = f$.

Let $d_a = \langle 0, 0, 1\rangle$ and label the source node distance as $d_a = [\langle 0, 0, 1\rangle, a]$, the value of $d_j; j = b, c, d, e, f$ can be obtained as follows:

Iteration 1: Since a is the only one the predecessor node of node b , put $i = a$ and $j = b$ in step 2 of the proposed algorithm, the value of d_b is $d_b = min\{d_a \oplus d_{ab}\}$=$min\{\langle 0, 0, 1\rangle \oplus \langle 0.4, 0.2, 0.4\rangle\}$=$\langle 0.4, 0.2, 0.4\rangle$

The minimum node value corresponds to the node $i = a$. Therefore label the distance of node b as $d_b = [\langle 0.4, 0.2, 0.4\rangle, a]$

Iteration 2 : Nodes a and b are the two predecessor nodes of node c, put $i = a, b$ and $j = c$ in step 2 of the proposed algorithm, the value of d_c is

$$\begin{aligned}
d_c &= min\{d_a \oplus d_{ac}, d_b \oplus d_{bc}\}\\
&= min\{\langle 0,0,1\rangle \oplus \langle 0.4, 0.2, 0.4\rangle, \langle 0.4, 0.2, 0.4\rangle \oplus \langle 0.1, 0.4, 0.4\rangle\}\\
&= \langle 0.2, 0.3, 0.4\rangle, \langle 0.46, 0.52, 0.16\rangle \ s\langle 0.2, 0.3, 0.4\rangle\\
&= \tfrac{1+(t+2h-f)(2-t-f)}{2}\\
&= 0.78 \ s\langle 0.46, 0.52, 0.16\rangle\\
&= \tfrac{1+(t+2h-f)(2-t-f)}{2}\\
&= 1.4246\\
&\Rightarrow s\langle 0.2, 0.3, 0.4\rangle \leq s\langle 0.46, 0.52, 0.16\rangle.
\end{aligned}$$

The minimum node value corresponds to the node $i = a$. Therefore label the distance of node c as $d_c = [\langle 0.2, 0.3, 0.4\rangle, a]$

Iteration 3 : Node c is the predecessor node of node d , put $i = c$ and $j = d$ in step 2 of the proposed algorithm, the value of d_d is

$$d_4 = min\{d_c \oplus d_{cd}\} = min\{\langle 0.2, 0.3, 0.4\rangle \oplus \langle 0.5, 0.3, 0.1\rangle\} = \langle 0.6, 0.51, 0.04\rangle$$

The minimum node value corresponds to the node $i = c$. Therefore label the distance of node d as $d_d = [\langle 0.6, 0.51, 0.04\rangle, c]$

Iteration 4: b and c are the two predecessor nodes of node e, put $i = b, c$ and $j = e$ in step 2 of the proposed algorithm, the value of d_e is

$$\begin{aligned}
d_e &= min\{d_b \oplus d_{be}, d_c \oplus d_{ce}\}\\
&= min\{\langle 0.4, 0.2, 0.4\rangle \oplus \langle 0.3, 0.3, 0.3\rangle, \langle 0.2, 0.3, 0.4\rangle \oplus \langle 0.3, 0.3, 0.4\rangle\}\\
&= \langle 0.58, 0.44, 0.12\rangle, \langle 0.44, 0.51, 0.64\rangle \ s\langle 0.58, 0.44, 0.12\rangle\\
&= \tfrac{1+(t+2h-f)(2-t-f)}{2}\\
&= 1.371 \ s\langle 0.44, 0.51, 0.64\rangle\\
&= \tfrac{1+(t+2h-f)(2-t-f)}{2} = 0.8772\\
&\Rightarrow s\langle 0.44, 0.51, 0.64\rangle \leq s\langle 0.58, 0.44, 0.12\rangle.
\end{aligned}$$

The minimum node value corresponds to the node $i = c$. Therefore label the distance of node c as $d_e = [\langle 0.44, 0.51, 0.64\rangle, c]$.

Iteration 5 :d and e are the two predecessor nodes of node f , put $i = d, e$ and $j = f$ in step 2 of the proposed algorithm, the value of d_f is

$$\begin{aligned}
d_e &= min\{d_d \oplus d_{df}, d_e \oplus d_{ef}\}\\
&= min\{\langle 0.6, 0.51, 0.04\rangle \oplus \langle 0.3, 0.1, 0.6\rangle, \langle 0.44, 0.51, 0.64\rangle \oplus \langle 0.3, 0.5, 0.2\rangle\}
\end{aligned}$$

$= \langle 0.72, 0.559, 0.24\rangle, \langle 0.608, 0.755, 0.128\rangle$ $s\langle 0.72, 0.559, 0.24\rangle$
$= \frac{1+(t+2h-f)(2-t-f)}{2}$
$= 1.33096$ $s\langle 0.608, 0.755, 0.128\rangle$
$= \frac{1+(t+2h-f)(2-t-f)}{2}$
$= 1.75768$
$\Rightarrow s\langle 0.72, 0.559, 0.24\rangle \leq s\langle 0.608, 0.755, 0.128\rangle.$

The minimum node value corresponds to the node $i = d$. Therefore label the distance of node d as $d_f = [\langle 0.72, 0.559, 0.24\rangle, d]$.

Now the hesitancy fuzzy shortest path between node a and node f can be obtained by using the following procedure: Since node f is labeled by $d_f = [\langle 0.72, 0.559, 0.24\rangle, d]$, which represents that we are coming from node d. Node d is labeled by $d_d = [\langle 0.6, 0.51, 0.04\rangle, c]$ which represents that we are coming from node c. Node c is labeled by $d_c = [\langle 0.2, 0.3, 0.4\rangle, a]$, which represents that we are coming from node a. Now the hesitancy fuzzy shortest path between the company a and customer f is obtaining by joining all the obtained nodes. Hence the hesitancy fuzzy shortest path is $a \to c \to d \to f$ with the hesitancy fuzzy value $\langle 0.72, 0.559, 0.24\rangle$. In figure 3, the dark lines indicate the shortest path from the source node (company) to the destination node (customer).

The hesitancy fuzzy shortest distance and the hesitancy fuzzy shortest path of all nodes from node a is shown in the table 2 and the labeling of each node is shown in Figure 3.

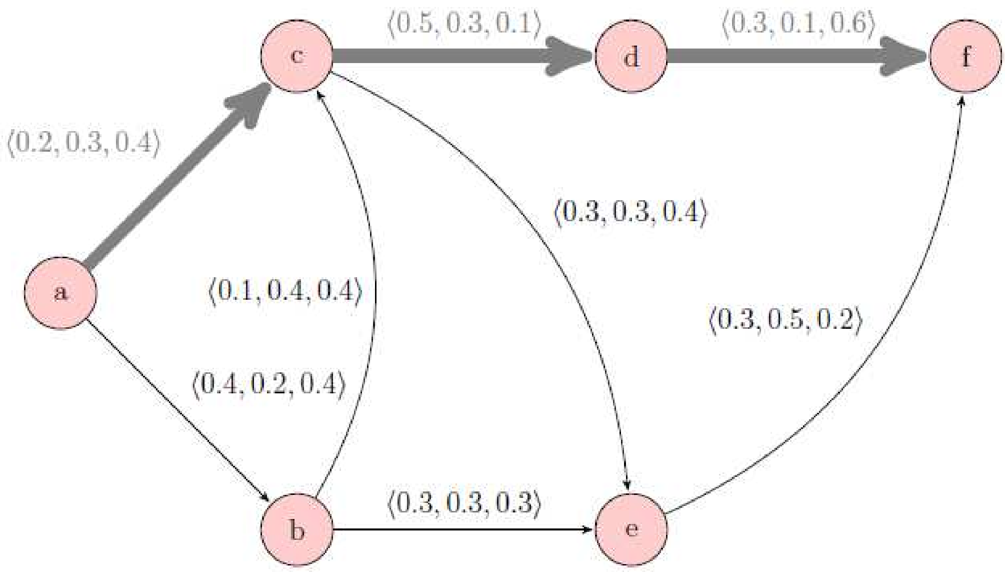

Figure 3: Network with Hesitancy fuzzy shortest distance

Since there is no other work on shortest path problem using hesitancy fuzzy parameters for the edges (arcs), numerical comparison of this work with others work could not be done.

In this paper we find the shortest path from company to customer using the hesitancy fuzzy shortest path algorithm. The idea of this algorithm is to carry the distance function which works as a tool to identify the successor node from company at the beginning till it reaches the customer with a shortest path. Hence our hesitancy fuzzy shortest path algorithm is much efficient providing the fuzziness between the intervals classified with true, hesitancy and false membership values. This concept is ultimately differing with intuitionistic membership values as the case of intuitionistic considers only the true and the false membership values. Hence in hesitancy fuzzy, all the cases of fuzziness is discussed and so the algorithm is effective in finding the shortest path.

5 Comparison table

In this section, a comparative study of various existing path problem such as crisp shortest path problem, fuzzy shortest path problem, intuitionistic fuzzy shortest path problem and hesitancy fuzzy shortest path problem is presented in Table 3.

Shortest path problems	Advantages	Disadvantages
Crisp shortest path problem	It deals with exact information based on its computed distance and weight.	It is unable to deal with uncertainty and inconsistencies exist in the weights or distance of given information.
fuzzy shortest path problem [7]	The weights of the edges are normalized or computed with fuzzy membership-values to deal with uncertainty in distance or weight of given information.	It provides a fuzzy shortest path length is found, but it does not correspond to an actual path in the network
intuitionistic fuzzy shortest path [9]	1. Intuitionistic fuzzy numbers are the more generalized form of fuzzy numbers involving two independently estimated degrees: degree of acceptance and degree of rejection. 2. Weights of the arc(edges) are intuitionistic fuzzy numbers.	It produces some marginal error which is beyond acceptation and rejection membership-values.
hesitancy fuzzy shortest path problem	Its reduces the marginal or uncertain error which may arises due to inconsistency in shortest path problem	Output differs slightly due to algorithm, and score function of hesitancy fuzzy number adopted by the researchers.

Table 3: Comparison table

6 Conclusion

In this paper we proposed an algorithm for finding shortest path and shortest arc length for a real life problem. we found shortest path and shortest arc length from company to customer on a network where the edges weights are assigned by hesitancy fuzzy number. The procedure of finding shortest path has been well explained and suitably discussed. Further, the implementation of the proposed algorithm is successfully illustrated with the help of a network example. The algorithm is easy to understand and can be used for all types of shortest path problems with arc length as triangular hesitancy fuzzy, trapezoidal hesitancy fuzzy and interval valued hesitancy fuzzy numbers. As a future work, we plan to implement this approach practically in the area of soft computing such as neural networks, decision-making, and geographical information systems.

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Prediction of Season-End Point for Football using Pythagorean Expectation*

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Abstract – The use of data collected on players, teams, and games for performance evaluation, player selection, score-outcome estimation, and strategy development using data mining tools and techniques are defined as sports data mining. Performance measures, unlike the common statistical methods, developed for each sport branch have an important role in sports data mining processes. Performance measures calculated for team sports can be used to predict the expectation of winning. The Pythagorean expectation developed for this objective was originally used in baseball games. The Pythagorean Expectation has also been adapted for other team sports with two results, such as basketball. However, the studies using Pythagorean Expectation for sports which have three possible outcomes are very limited. In this study, a suggestion for the calculation of Pythagorean Expectation for football is presented. In the application section, end-season rankings and points for the 2017/2018 season of the selected fifteen European football leagues are predicted by using the suggested method. The data of the past five seasons of the selected European football leagues is used as the training dataset. All calculations are performed in R.

Keywords – *Sports data mining, Pythagorean Expectation, Point prediction, Soccer, Football*

1 Introduction

Collecting and storing data have been easier and cost-effective in parallel with the progress in technology. Herewith, large amounts of data are generated in many different areas and used for different purposes. Sports data mining is defined as the use of data for performance evaluation, player selection, outcome-point prediction and strategy development by data mining tools and techniques. The decision makers of sports organizations can take more scientific and unbiased decisions by sports data mining compared to traditional methods. Sports data mining is rapidly spread and adopted due to clearly demonstrating team player performance and helping talent scouts to discover new talents. In addition, the popularity of sports data mining has increased due to the studies conducted on predicting the outcomes of sports events.

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In sports data mining, it is necessary to define sabermetric first. Sabermetric depends on the idea of creating new statistics that better measure individual and team performances compared to the traditional statistical methods in baseball. Although the idea had been proposed earlier, it has been introduced by Bill James at the end of the Seventies in the annual "Baseball Abstracts" booklets published by himself. James rapidly pronounced his name and increased popularity with his unusual ranking methods and new statistical performance measurements called sabermetrics. The transition from traditional statistics to sabermetrics is the result of queries and solutions on the performance criteria introduced by Bill James. James [8] described the sabermetrics of which he developed in his later books. Pythagorean Expectation (PE), a performance measurement metric that predicts the game-winning rate of teams in baseball, was developed by James [7]. The PE has been widely used for baseball in the subsequent years. Lee [9] applied the PE for the 2005-2014 seasons of the Korean Baseball League and compared the expected and actual game winning numbers of clubs. Inconsistency between expected and actual winning numbers, assuming the conditions of the teams originated due to an unusual distribution, has been related to the coefficient of variation and standard deviation in the number of runs allowed. Tung [16] applied the PE to the data set of seasons from 1901 through 2009 and produced a confidence interval for the number of games predicted to be won. Valero [17] predicted the outcomes of the American Baseball League by using sabermetrics, including PE to assess the predictive capabilities of data mining methods. Valero, following the statistical analysis, showed that classification methods resulted in better outcomes. The PE is given in detail in the second section.

Performance measurements in basketball are performed as a team rather than individually since the performances of the players are relatively more dependent on each other compared to baseball. Dean Oliver is the pioneer of performance measurement in basketball. Oliver has developed new statistics for basketball in the Eighties [15]. In 2004, Oliver published the statistical methods for assessment in basketball and calculation tools to evaluate the teams [13].

The statistical techniques used in American football have not yet reached the levels reached in baseball and basketball. Schumaker et al. [15] attributed this to less number of games in American football compared to baseball and basketball and lack of some statistics about the players. A team in the American national football league plays 16 games in a season, while 162 games in baseball and 82 games in basketball [15]. Leung and Joseph (2014) mentioned the Christodoulou algorithm that is used in the prediction of dual matchings and applied this algorithm to the American football data [10]. In the application section of their study, the distances of teams to each other in the American Football League were calculated and revealed similar teams by using the PE, Christodoulou algorithm and other sabermetrics. When the results of the future matches are predicted, according to the results of the matches between similar teams, points are assigned to the teams who have not played.

The Christodoulou algorithm generates five statistics for the competing teams in a league based on the game outcomes. These are the number of points gained per game for a team (NPPG), the number of points scored by opponent per game (NPOPG), the number of points per games recorded in a league (NPRL), offensive strength (OS) and defensive strength (DS). The OS specifies the percentage of points scored by a team against their opponent to the number of points per game typically allowed by this opponent. For example, if NPRL is 40 in a league and a team can score 60 points per game, then the OS of the team would be 60/40. The DS indicates the ratio of points that a team allows to the opponent relative to the NPRL. For example, if the NPRL is 40 in a league, and an average team score per game is 20, then the DS of a team would be 20/40. The Christodoulou algorithm aims to predict the outcomes of games using these statistics [10]. The aforementioned statistics which indicate the performance of a team are calculated for a team-A as follows.

$$OS_A = \frac{NPPG_A}{\text{NPRL}_{LEAGUE}}$$
$$DS_A = \frac{\text{NPOPG}_A}{\text{NPRL}_{LEAGUE}} \tag{1}$$

$$\text{NPPG}_A = \frac{Points\ Scored_A}{Games\ Played_A}$$
$$\text{NPOPG}_A = \frac{Points\ Scored\ by\ Opponent\ _A}{Games\ Played_A} \tag{2}$$

The score of the game played by teams A and B can be predicted as follows using the above statistics.

$$Score_A = OS_A \times \text{NPOPG}_B + \text{NPPG}_A \times DS_B$$
$$Score_B = OS_B \times \text{NPOPG}_A + \text{NPPG}_B \times DS_A \tag{3}$$

Cricket is another sport field where performance measures are applied. Cricket sport is considered utterly rich in terms of statistics [3]. John Buchanan, the coach of the Australian national team, has pioneered many of sabermetrics involving in the cricket sport between 1999 and 2007. The most well known is “Marginal Wins”. The performances of players are evaluated through these statistics according to their positions and can also be compared with the opponent players [15]. Vine [18] determined the “lucky” and “unlucky” teams by comparing the predicted and actual number of winnings of cricket teams. Vine who used the 4-season data set of the Australian Cricket League, assumed the coefficient of γ as 7.41 while adapting the PE to cricket sport. While determining these coefficients, the criterion was defined as the minimum root mean squared error (RMSE).

Several attempts have been carried out to create statistical measures similar to sabermetric in football. However, analysis of game activity and game-based events in football are far better difficult than baseball. Because the performances of the players in football are much more dependent on each other compared to baseball. The roles of the players in baseball have been set sharply; the pitcher hits the ball, the batter meets the coming ball by his bat. In football, teams can attack and defend with various strategies and number of players. Therefore, sabermetric style performance measurement were not generally used in the data mining studies performed in football.

In this study, an approach is presented for adaptation of PE, a sabermetric developed for baseball, to football using past season data. In the application, it is aimed to predict the points and the ranking of the teams with the proposed approach based on the scored and conceded goals at the end of the season. The use of PE in football and the proposed approach are presented in the second section. The application is given in the third section, and the results are shared in the final section.

2 Method

In this section, the details of PE and the adaptations of PE in other sport branches including football are given.

2.1 Pythagorean Expectation

PE has been proposed by Bill James [7] as a performance measurement metric that predicts the team winning rate of baseball teams using runs scored (RS), runs allowed (RA) obtained from past games and the league constant of γ. The PE can be used to determine the teams which performing above and below the expectations by comparing the actual winning rate. PE for baseball is calculated as in Equation 4.

$$PE = \frac{RS^{\gamma}}{RS^{\gamma} + RA^{\gamma}} \tag{4}$$

PE is typically used in the middle of a season to predict the standings for the end of a season. For example, if a team wins more than the predicted in the halfway through a season, analysts claim that the team will complete the remaining half of the season with fewer winnings than the predicted [12]. The value of constant γ in original formula has been set to 2.0 by James. Miller [12], however, has shown that the use of constant γ as 1.82 reduces the standard error.

Various applications have been suggested for also baseball. Davenport and Woolner [4] argued that the γ value should be calculated separately for each team according to the balance of offensive and defensive power, and suggested that the γ coefficient in baseball should be calculated as in Equation 5 to obtain a smaller RMSE value.

$$\gamma = 1.5 \times \log\left(\frac{RS + RA}{NG}\right) + 0.45 \tag{5}$$

Where RS is the number of runs allowed, RA is the number of runs allowed and NG is the number of games played by the team.

PE has attracted the attention in other sport branches due to its impact on baseball. Different γ values have been attained in the studies conducted using PE in sport branches such as American football, cricket, basketball and ice hockey. Some of these studies are summarized in Table 1.

Table 1. Recommended γ Values for Different Sport Branches

Sport	**γ**	**Source**
Baseball	1.82	[12]
American Football	2.37	[14]
Basketball	14 13.91	[13] [19]
Ice Hockey	1.927	[2]
Cricket	7.41	[18]

2.2 Pythagorean Expectation in Football

PE in sports which have two possible outcomes (win - or - loss) which mentioned in the previous section have been applied only with the changes made in the γ coefficient. However, the teams acquire points below the predicted if the original formula is applied directly without any arrangement in football which is a sport that can result in a tie [5, 6].

Hamilton [6], considering the possible tie outcome in football, predicted the points earned per game instead of predicting the winning ratios of teams using the extended Pythagorean method. Hamilton tried to overcome the problem that PB was only applicable to sports with two possible outcomes by calculating the probability of winning and draw for each team. Hamilton calculated the predicted point per game (PPPG) for team X by using Equation 6 where X representing a team playing in the league and Y representing the opponents.

$$\mathrm{PPPG} = 3 \times \mathrm{P}(\mathrm{X} > \mathrm{Y}) + \mathrm{P}(\mathrm{X} = \mathrm{Y}) \qquad (6)$$

Hamilton [6] used the least squares algorithm to express the scored and conceded goals distributions with a three-parameter Weibull distribution. However, Hamilton’s method has not widely used due to intensive mathematical and statistical procedures.

Eastwood [5] took the draw possibility into account and adopted the original PE to a football game which has 3-point for a win, 1-point for a draw, 0-points for a loss. Eastwood, instead of calculating the winning possibility of the teams, calculated the PPPG multiplying the average point per game (APPG) by the probability of gaining points. The equation developed by Eastwood to calculate the PPPG for each team is given below;

$$PPPG = \frac{G^{1.22777}}{G^{1.072388} + CG^{1.127248}} \times 2.499973 \qquad (7)$$

In the Equation 7; G is the number of goals scored, CG is the number of goals conceded and the APPG is 2.499973.

Hamilton [6] determined the γ value with a single season data and found RMSE value as 3.81. Eastwood [5] obtained lower RMSE values by using the data collected from ten seasons. The adaptation of Eastwood [5] seems like much straightforward and more practical than the adaptation formula of Hamilton [6]. However, Eastwood developed and implemented the formula only over the English Premier League data.

2.3 Proposed Approach

This section outlines the proposed approach to adapt PE to football. The proposed formula, unlike baseball, is aimed to predict the expected points per game of the teams instead of winning possibilities. In order to calculate the PE, the number of goals scored and conceded by the reference team in the league have to be known. In addition, the exponential coefficient γ and the average points distributed per game in the league (APDG) should also be determined. The most important difference between recommended approach and Eastwood's formula is the usage of the γ coefficient. Eastwood’s formula uses three different γ coefficients. The PE equation, which calculates the expected points per game for each team, is written as follows with the determination of the required coefficients γ and APDG:

$$\text{PE} = \frac{Goal_S{}^{\gamma}}{Goal_S{}^{\gamma} + Goal_C{}^{\gamma}} \times APPG \tag{8}$$

$Goal_S$ represents the number of goals scored and $Goal_C$ is the number of goals conceded.

Since football is not a sport with two possible outcomes, the APPG value cannot be taken as 3 points. Considering that there are three possible outcomes in football, the ratios of the draw and win-loss in the leagues must be determined (Equation 9). The APPG is calculated by Equation 10 using the statistics for the total game played in the league (TGP), total win (TW), total draw (TD) and total loss (TL).

$$\begin{gathered} Ratio_{win-loss}: \frac{TW+TL}{TGP} \\ Ratio_{draw}: \frac{TD}{\text{TGP}} = 1 - \frac{TW+TL}{\text{TGP}} \end{gathered} \tag{9}$$

$$APPG = 3 \times (Ratio_{win-loss}) + 2 \times (Ratio_{draw}) \tag{10}$$

The γ coefficient in PE for football can be predicted by simple linear regression method (SLR). The SLR provides a linear function that models the relationship between the dependent and independent variables with the least squares (LS) algorithm.

In the proposed approach, PB is calculated with the Equation 8 by using the various gamma values between 1 and 2 for all teams in the league then PB is multiplied by the number of matches played in order to predict end-season points. A regression model is created by using SLR where the predicted score as the explanatory variable and the actual score as the response variable. Consequently, the SLR models are generated as much as the number of γ tested. The optimum γ coefficient is determined by examining the RMSE obtained in the models and the coefficient of determination R^2.

3 Application

The data of fifteen European football leagues belong to the six seasons between 2012-2013 and 2017-2018 seasons used in the study were compiled from the mackolik.com website [11]. The leagues used in the application belong to countries of Turkey, Italy, Germany, Spain, France, Holland, England, Belgium, Austria, Croatia, Denmark, Czech Republic, Portugal, Romania, and Scotland. The league tables used in the study include the number of games played for each team (G), the number of wins (W), the number of draws (D), the number of losses (L), the goals scored (S), the goals conceded (C) and the end of season points (P). Play-offs, canceled games, and cup games have not been included in the data used. The league tables belong to past five years (2012 – 2017) of fifteen European football leagues were used as training data in the application. The data for 152 teams played during five seasons (2012-17) in the league (never dropped out) were used in the training data set. The data of 244 teams in the 2017-2018 season were separated as test data. All calculations are performed in R statistical programming language. A small excerpt of the data is shown in Table 2.

Table 2. An Example of a League Table

Teams	Country	G	W	D	L	S	C	P
Athletic Bilbao	Spain	190	84	43	63	263	233	295
Pandurii Targu Jiu	Romania	154	64	38	52	222	192	224
Nice	France	190	83	46	61	252	220	295
Zulte Waregem	Belgium	150	68	40	42	241	209	244
AZ Alkmaar	Holland	170	72	40	58	299	265	256
Schalke 04	Germany	170	74	40	56	259	222	262

Firstly, the win-loss and draw ratios were calculated by using Equation 9:

$Ratio_{win}$ =0.7471

$Ratio_{draw}$ =0.2528

APPG was computed with Equation 10 as follows:

$$APPG = 3 \times (0.7471) + 2 \times (0.2528)$$

$$APPG = 2.7471$$

The most successful results were obtained in the $1 \leq \gamma \leq 2$ range in our preliminary study. Therefore, PE of each team was calculated with Equation 8 using 2.7471 as APPG for eleven different γ coefficients between 1.0 and 2.0. The calculated PE values are multiplied by the number of games played, and eleven distinct points are predicted for the total points of the teams for the five seasons. Eleven simple linear regression models were created to find the most appropriate γ value, where the predicted points were the independent variable (x_i) and the actual points were the dependent variable (y_i). The RMSE and coefficient of determination R^2 values for the obtained models are shown in Table 3 and Figure-1:

Table 3. RMSE and R^2 Values of Models Obtained with Different γ Coefficients

γ	RMSE	R^2
1.0	11.21613	0.974433
1.1	10.55446	0.977361
1.2	10.28528	0.978501
1.3	10.31085	0.978394
1.4	10.54836	0.977387
1.5	10.93303	0.975708
1.6	11.41728	0.973508
1.7	11.96756	0.970893
1.8	12.56064	0.967937
1.9	13.18052	0.964694
2.0	13.81608	0.961207

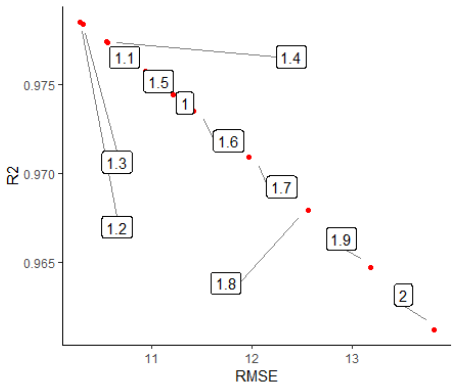

Figure 1. RMSE and R^2 values of models obtained with different γ coefficients

The results revealed that the lowest RMSE and the highest R^2 values were obtained when γ=1.2. The recommended PE of European football leagues can be calculated by the Equation 11 using the specified value of the coefficient. The γ value was recommended as 1.3 in another study using twelve European leagues [1].

$$PE = \frac{Goal_S{}^{1.2}}{Goal_S{}^{1.2} + Goal_C{}^{1.2}} \times 2.7471 \tag{11}$$

The SLR used with the LS algorithm is a parametric statistical method that requires some assumptions. Thus, initially, the required assumptions must be checked. For this purpose, normality, and independence of the residuals obtained by the model were examined. Secondly, variance homogeneity was investigated. The results of the analyses showed that the assumptions were satisfied. Required tests for the validity of the model and coefficients were also performed. The significance level of all hypothesis tests was accepted as 0.05.

The data of the 2017-2018 season were used for evaluation. In the first stage, the differences between the predicted and actual points were examined to measure the success of the proposed approach (Table 4).

Table 4. An Example for End of Season Actual Points and PE Predictions

Teams	Actual Point	Predicted Point	Difference
Milan	64	59.65	-4.35
Saint-Etienne	55	50.58	-4.42
CFR Cluj	59	54.54	-4.46
RB Leipzig	53	48.40	-4.60
Lazio	72	67.37	-4.63
Real Madrid	76	71.10	-4.90
Utrecht	54	48.80	-5.20

In the evaluation, the margin of error was considered only a match. So, predictions with less than three-point difference from the actual point value were considered successful. The success rates calculated for 15 European leagues are presented in Table 5. The overall success rate for all leagues was 40%.

Table 5. Success Rates of Leagues Obtained in Point Prediction

League	Success Rate
Germany	56%
Czech Republic	56%
Romania	56%
Belgium	50%
England	44%
Denmark	33%
France	33%
Crotia	33%
Spain	33%
Italy	33%
Turkey	33%
Netherland	28%
Austria	22%
Scotland	22%
Portugal	11%

The points gained at the end of the season determine the team standings in the league. The predicted points by PE of the teams in 2017-2018 end-of-season were used to measure the success of the proposed approach based on the standings, and the teams were ranked based on the points predicted among the teams in their leagues. An additional evaluation was performed for the first four teams in the leagues. The first four rankings are considered important for the league success of a team and qualifying to the European cups. When calculating the ranking-based success ratio, predictions that predict the season end ranking of a league exactly or predict the ranking by only one difference are accepted as successful. The ranking-based success ratios are given in Table 6.

Table 6. Success Rates of Leagues by Ranking

League	Success Rate for ranking	Success Rate for only First Four ranking
Crotia	100%	100%
Scotland	100%	100%
Austria	90%	100%
Romania	86%	100%
Denmark	79%	100%
England	75%	100%
Netherland	72%	100%
Portugal	72%	100%
Italy	70%	100%
France	65%	100%
Spain	65%	100%
Czech Republic	63%	100%
Germany	61%	100%
Belgium	44%	50%
Turkey	39%	75%

The success ratios for the ranking were higher than 50%, except for two leagues. The rankings for Croatia and Scotland leagues were predicted exactly.

4 Conclusion

In this study, a PE calculation approach was proposed for European football leagues. The 2017-2018 end-of-season points of 244 teams playing in European leagues were predicted in order to measure the success of the proposed approach. The success of PE, which is proposed with two different approaches according to the ranking and score, was evaluated by using the points predicted. More successful results were obtained with ranking based prediction. In the study, relatively low success ratios were obtained for Turkey and Belgium leagues. However, high success ratios were obtained for Croatia, Scotland, Austria and Romania, where there are fewer teams in the league compared to the other countries. Another noteworthy outcome was the high accuracy rate in the rank-based evaluation. In this study, the γ value in PE formula for football was calculated as 1.2. Specific γ values must be calculated for different leagues in order to make successful predictions. Further studies are planned to determine the γ values for Asian and South American football leagues.

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Double Fuzzifying Topogenous Space, Double Fuzzifying Quasi-Uniform Spaces and Applications of Dynamics Fuzzifying Topology in Breast Cancer

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Abstaract − The main motivation behind this work is to introduce the notion of $(2, L)$-double fuzzifying topology which is a generalization of the notion of $(2, L)$-fuzzifying topology and classical topology. We define the notions of $(2, L)$-double fuzzifying preproximity and $(2, L)$-fuzzifying syntopogenous structures. Some fundamental properties are also established. These concepts will help in verifying the existing characterizations and also help in achieving new and generalized results. Finally we study a model as an application of fuzzifying topology in biology.

Keywords − *$(2, L)$-double fuzzifying topology, $(2, L)$-doublefuzzifying preproximity, L-fuzzifying dynamice topology, breast cancer.*

1 Introduction

A lattice is a poset $L = (L, \leq)$ in which every finite subset has both join $\vee$ and a meet $\wedge$ with the smallest element $\bot_L$ and the largest element $\top_L$. We assume that $\top_L \neq \bot_L$, *i.e.*, L has at least two elements. A distibutive lattice is a lattice which satisfies the distributive laws. A lattice is said to be complete if it has arbitrary joins and meets, *i.e.*, for every subset $A \subseteq L$ the join $\bigvee A$ and the meet $\bigwedge A$ are defined. In particular, $\bigvee L = \top_L$ and $\bigwedge L = \bot_L$. Throughout this work L always denote a complete residuated lattice intoduced by [7,14] used L as a complete MV-algebra but [17,18,19] used L as a complete residuated lattice, $L_0 = L - \{\bot_L\}$ and $I = [0, 1]$. We say a is a wedge below b, in a symbol, $b \rhd a$, if for every subset $D \subseteq L$, $\bigvee D \geq b$ implies $a \leq d$ for some $d \in D$. The concept of $(2, L)$-fuzzifying topology appeared in [7] under the name " $(2, L)$-fuzzy topology" (cf. Definition 4.6, Proposition 4.11 in [8] where L is a completely distributive complete lattice. In the case of $L = [0, 1]$ this terminology traces back to [18,19], where it was studied the fuzzifying topology and elementarily it was developed fuzzy topology from a new direction with semantic method of continuous valued logic. Fuzzifying topology (resp. L-Fuzzifying topology) in the sense of M. S. Ying (resp. U. Höhle) was introduced as a fuzzy subset (resp. an L-Fuzzy subset) of the power set of an ordinary set. On the other hand, in topology a proximity space is an axiomatization of notions of "nearness" that hold set-to-set, as opposed to the better known point-to-set notions that characterize topological spaces, in this regard. [3,4] gave a new method for the foundation of general topology based on the theory of syntopogenous structure to develop a unified approach to the three main structures of set-theoretic topology: topologies, uniformities and proximities. This helped him to develop a theory including the basis of the three classical theories of topological spaces, uniform spaces and proximity spaces. In the case of the fuzzy structures there are at least two notions of fuzzy syntopogenous structures Motivated by their works, we continue investigating the properties $(2, L)-$double fuzzifying preproximity. We show that each $(2, L)$-double fuzzifying preproximity on X induces $(2, L)$-double fuzzifying topology on the same set.Also, we define the notion of $(2, L)$-Double fuzzifying semi topogenous order and obtain a few results analogous to the ones that hold for $(2, L)$-double fuzzifying topology, he relation between a L-double fuzzifying preproximity structures is also investigated $(2, L)$-Double fuzzifying semi topogenous order, double fuzzifying topogenous order on X, double fuzzifying topogenous continuous, $(2, L)$-double fuzzifying preproximity, double quasi proximity spaces, double fuzzifying quasi uniform space. This work arranged by: In section 1 and 2 introduction and more survay results in the subject. In section 3, we give a new notion of $(2, L)$-Double fuzzifying semi topogenous order, double fuzzifying topogenous order on X,double fuzzifying topogenous continuous, $(2, L)$-double fuzzifying preproximity, double quasi proximity spaces, double fuzzifying quasi uniform space, we study the relations between them and relations between $(2, L)$-double fuzzifying topology. In section 4 Mathematical models have been used in biology. In fact, dramatic developments in biology and in pure mathematics together, may have led to the interpretation of many natural phenomena in life, Also, it has been creatively described in the analysis and diagnosis of multiple diseases dynamically. However, there are many phenomena that are still in the interest of scientists. This work shows that using dynamic physiological topology we can describe many natural phenomena dynamically and identify the appropriate times in which scientists intervene to the subject of human solutions to the distortions of the situation. We will shed light on breast cancer at the five-stage and determine the possibility of conformation and therapeutic intervention. We will show how the dynamical topologies [5]. can develop the diagnostic mechanism and time analysis of the situation and determine the appropriate time to avoid distortions in the stages of the case. The present article demonstrates an application of L-fuzzifying dynamice topology clarify a model describing biological phenomena, This model allow to know all levels of development of an breast cancer. from 0-level (infection outside cells) until 5-level (infection liver).

2 Preliminary

Definition 2.1. [16] Let (X, τ) be an L-fuzzifying topological space, and let $Y \subseteq X$. Define the map $\tau_Y : P(Y) \to L$ as follows: $\tau_Y(U) = \bigvee_{H \cap Y = U} \tau(H)$.

Definition 2.2. [9] The double negation law in a complete residuated lattice L is given as follows: $\forall a, b \in L,\ (a \to \bot) \to \bot = a$.

Definition 2.3. [9] A structure $(L, \vee, \wedge, *, \to, \bot, \top)$ is called a strictly two-sided commutative quantale iff

(1) $(L, \vee, \wedge, \bot, \top)$ is a complete lattice whose greatest and least element are $\top, \bot$ respectively,

(2) $(L, *, \top)$ is a commutative monoid,

(3)(a) $*$ is distributive over arbitrary joins, i.e.,
$a * \bigvee_{j \in J} b_j = \bigvee_{j \in J} (a * b_j)\ \forall a \in L, \forall \{b_j \,|j \in J\} \subseteq L$,

(b) $\to$ is a binary operation on L defined by: $a \to b = \bigvee_{\lambda * a \leq b} \lambda \quad \forall a, b \in L$.

Definition 2.4. [8,9] Let X be a nonempty set and let $P(X)$ be the family of all ordinary subsets of X. An element $\mathcal{T} \in L^{P(X)}$ is called an L-fuzzifying topology on X iff it satisfies the following axioms:

(1) $\mathcal{T}(X) = \mathcal{T}(\phi) = \top$,

(2) $\forall A, B \in P(X), \mathcal{T}(A \cap B) \geq \mathcal{T}(A) \wedge \mathcal{T}(B)$,

(3) $\forall \{A_j \,|j \in J\} \subseteq P(X),\ \mathcal{T}(\bigcup_{j \in J} A_j) \geq \bigwedge_{j \in J} \mathcal{T}(A_j)$.

The pair $(X, \mathcal{T})$ is called an L-fuzzifying topological space.

Definition 2.5. [11] Let X be a set and let $\delta \in L^{P(X) \times P(X)}$, i.e., $\delta : P(X) \times P(X) \to L$. Assume that for every $A, B, C \in P(X)$, the following axioms are satisfied:

$(LFP1)$ $\delta(X, \phi) = \bot$,

$(LFP2)$ $\delta(B, A) = \delta(A, B)$,

$(LFP3)$ $\delta(A, B \cup C) = \delta(A, B) \vee \delta(A, C)$,

$(LFP4)$ For every $A, B \in P(X), \exists C \in P(X)\ s.t.\ \delta(A, B) \geq \delta(A, C) \vee \delta(B, X - C)$,

$(LFP5)$ $\delta(\{x\}, \{y\}) = CE(\{x\}, \{y\})$.

Then δ is called an L-fuzzifying proximity on X and (X, δ) is called an L-fuzzifying proximity space.

Definition 2.6. [4] A uniform structure U on a set X is a family of subsets of $X \times X$, called entourage, which satisfies the following properties:

$(U1)$ If $u \in U$, then $\triangle \subseteq u$, where $\triangle$ is the diagonal: $\triangle = \{(x, x) \,|x \in X\}$

$(U2)$ If $v \subseteq u$, and $v \in U$ then $u \in U$,

$(U3)$ for every $u, v \in U,\ u \cap v \in U$,

$(U4)$ If $u \in U$, then $u^{-1} \in U$, where $u^{-1} = \{(x, y) \,|(y, x) \in u\}$.

$(U5)$ for every $u \in U$, there exists $v \subseteq U$ such that $v \circ v \subseteq u$, where $v \circ v \subseteq u$, where $v \circ u$ is defined by:

$v \circ u = \{(x, y) | \exists z \in X$ such that $(x, z) \in u$ and $(z, y) \in u\},\ \forall x, y \in X$.

The pair (X, U) is said to be a uniform space.

3. $(2, L)$-Double Fuzzifying Semi Topogenous Order Spaces

Definition 3.1. Let X be a non-empty set. The pair $(\mathcal{T}, \mathcal{T}^*)$ of maps $\mathcal{T}, \mathcal{T}^* : 2^X \times 2^X \to L$ is called an $(2, L)$-double fuzzifying semi topogenous order on X if it satisfies the following conditions:

(LST1) $\mathcal{T}(A, B) \leq \mathcal{T}^*(A, B) \to \bot$, for each $(A, B) \in 2^X \times 2^X$,

(LST2) $\mathcal{T}(X,X)=\mathcal{T}(\phi,\phi)=\top$ and $\mathcal{T}^*(X,X)=\mathcal{T}^*(\phi,\phi)=\bot$,
(LST3) If $\mathcal{T}(A,B)\neq\bot$, $\mathcal{T}^*(A,B)\neq\top$, then $A\subseteq B$,
(LST4) If $A_1\subseteq A$, $B_1\subseteq B$, then $\mathcal{T}(A_1,B_1)\leq\mathcal{T}(A,B)$ and $\mathcal{T}^*(A_1,B_1)\geq\mathcal{T}^*(A,B)$.
The pair $(X,\mathcal{T},\mathcal{T}^*)$ is called an $(2,L)$-double fuzzifying semi topogenous order on X.

The complement of a double fuzzifying semi topogenous order $(\mathcal{T},\mathcal{T}^*)$ is the double fuzzifying semi topogenous order $(\widehat{\mathcal{T}},\widehat{\mathcal{T}^*})$ defined by $\widehat{\mathcal{T}}(A,B)=\widehat{\mathcal{T}}(A^-,B^-)$ and $\widehat{\mathcal{T}^*}(A,B)=\widehat{\mathcal{T}^*}(A^-,B^-)$. such that A^-, B^- are the complement of $A$and B respectively.

A double fuzzifying semi topogenous order $(\mathcal{T},\mathcal{T}^*)$ is called:
(S) symmetrical if $(\mathcal{T},\mathcal{T}^*)=(\widehat{\mathcal{T}},\widehat{\mathcal{T}^*})$
(T) topogenous if $\mathcal{T}((A_1\cup A_2,B)\geq\mathcal{T}(A_1,B)\wedge\mathcal{T}(A_1,B)$ and $\mathcal{T}^*(A_1\cup A_2,B)\leq\mathcal{T}^*(A_1,B)\vee\mathcal{T}^*(A_1,B)$, (PF) perfect if $\mathcal{T}(\bigcup_{i\in\Gamma}A_i,B)\geq\bigwedge_{i\in\Gamma}\mathcal{T}((A_i,B))$ and $\mathcal{T}^*((\bigcup_{i\in\Gamma}A_i,B))\leq\bigvee_{i\in\Gamma}\mathcal{T}^*((A_i,B)$, for each $\{(A_i,B):i\in\Gamma\}\subseteq 2^X\times 2^X$.
(BP) biperfect if it is perfect and $\mathcal{T}(A,\bigcap_{i\in\Gamma}B_i)\geq\bigwedge_{i\in\Gamma}\mathcal{T}(A,B_i)$ and
$\mathcal{T}^*(A,\bigcap_{i\in\Gamma}B_i)\leq\bigwedge_{i\in\Gamma}\mathcal{T}^*(A,B_i)$.
Double fuzzifying semi topogenous order $(\mathcal{T}_1,\mathcal{T}_1^*)$ is said to be finer than another one $(\mathcal{T}_2,\mathcal{T}_2^*)$ if $\mathcal{T}_1(A,B)\geq\mathcal{T}_2(A,B)$ and $\mathcal{T}_1^*(A,B)\leq\mathcal{T}_2^*(A,B)$ for each $(A,B)\in 2^X\times 2^X$.

Definition 3.2. Let X be a nonempty set. The pair (δ,δ^*) of maps $\delta,\delta^*:2^X\to L$ is called an $(2,L)$-double fuzzifying topology on X if it satisfies the following conditions:
(DO1) $\delta(A)\leq\delta^*(A)\to\bot$, for each $A\in 2^X$,
(DO2) $\delta(X)=\delta(\emptyset)=\top$ and $\delta^*(X)=\delta^*(\emptyset)=\bot$,
(DO3) $\delta(A\cap B)\geq\delta(A)\wedge\delta(B)$ and $\delta^*(A\cap B)\leq\delta^*(A)\vee\delta^*(B)$, for each $A,B\in 2^X$,
(DO4) $\delta(\bigcup_{i\in\Gamma}A_i)\geq\bigwedge_{i\in\Gamma}\delta(A_i)$ and $\delta^*(\bigcup_{i\in\Gamma}A_i)\leq\bigvee_{i\in\Gamma}\delta^*(A_i)$, for each $\{A_i:i\in\Gamma\}\subseteq 2^X$.
The pair (X,δ,δ^*) is called an $(2,L)$-double fuzzifying topological space.

Definition 3.3. Let (X,δ_1,δ_1^*) and (Y,δ_2,δ_2^*) be two $(2,L)$-double fuzzifying topological spaces. Then the map $f:(X,\mathcal{T}_1,\mathcal{T}_1^*)\to(Y,\mathcal{T}_2,\mathcal{T}_2^*)$ is called double fuzzifying continuous, if $\delta_2(B)\leq\delta_1(f^{-1}(B))$ and $\delta_2^*(B)\geq\delta_1^*(f^{-1}(B))$, for each $B\in 2^Y$.

Theorem 3.1. Let $(\mathcal{T}_1,\mathcal{T}_1^*)$ and $(\mathcal{T}_2,\mathcal{T}_2^*)$ be perfect (resp. double fuzzifying topogenous, biperfect) double fuzzifying semi topogenous order on X . Define the compositions $\mathcal{T}_1\circ\mathcal{T}_2$ and $\mathcal{T}_1^*\circ\mathcal{T}_2^*$ on X by $\mathcal{T}_1\circ\mathcal{T}_2(A,B)=\bigvee_{h\in 2^X}[\mathcal{T}_1(A,h)\wedge(\mathcal{T}_2(h,B)]$ and $\mathcal{T}_1^*\circ\mathcal{T}_2^*(A,B)=\bigwedge_{h\in 2^X}[\mathcal{T}_1^*(A,h)\vee(\mathcal{T}_2^*(h,B)]$. Then $(\mathcal{T}_1\circ\mathcal{T}_2\ ,\ \mathcal{T}_1^*\circ\mathcal{T}_2^*)$ is perfect (resp. double fuzzifying topogenous, biperfect) double fuzzifying semi topogenous order on X.

Proof Let $(\mathcal{T}_1,\mathcal{T}_1^*)$ and $(\mathcal{T}_2,\mathcal{T}_2^*)$ be perfect double fuzzifying semi topogenous order on X. Then (LST3) If $\mathcal{T}_1\circ\mathcal{T}_2(A,B)\neq\bot$ and $\mathcal{T}_1^*\circ\mathcal{T}_2^*(A,B)\neq\top$. Then $\exists\, h\in 2^X$ such that $\mathcal{T}_1\circ\mathcal{T}_2(A,B)\geq\mathcal{T}_1(A,h)\wedge(\mathcal{T}_2(h,B))\neq\bot$ and $\mathcal{T}_1^*\circ\mathcal{T}_2^*(A,B)\leq\mathcal{T}_1^*(A,h)\vee(\mathcal{T}_2^*(h,B))\neq\top$, It implies $A\subseteq h\subseteq B$. Easily only prove (PF) from

$$\begin{aligned}\mathcal{T}_1 \circ \mathcal{T}_2(\bigcup_{i\in\Gamma} A_i, B) &= \bigvee_{h\in 2^X} \left[\mathcal{T}_1(\bigcup_{i\in\Gamma} A_i, h) \wedge (\mathcal{T}_2(h, B)\right] \\ &\geq \bigwedge_{i\in\Gamma} \left[\bigvee_{h\in 2^X} [\mathcal{T}_1(A_i, h) \wedge (\mathcal{T}_2(h, B))]\right] \\ &= \bigwedge_{i\in\Gamma} \mathcal{T}_1 \circ \mathcal{T}_2(A_i, B)\end{aligned}$$

and

$$\begin{aligned}\mathcal{T}_1^* \circ \mathcal{T}_2^*((\bigcup_{i\in\Gamma} A_i, B)) &= \bigwedge_{h\in 2^X} \left[\mathcal{T}_1^*(\bigcup_{i\in\Gamma} A_i, h) \vee (\mathcal{T}_2^*(h, B)\right] \\ &\leq \bigvee_{i\in\Gamma} \left[\bigwedge_{h\in 2^X} [\mathcal{T}_1^*(A_i, h) \vee (\mathcal{T}_2^*(h, B))]\right] \\ &= \bigvee_{i\in\Gamma} \mathcal{T}_1^* \circ \mathcal{T}_2^*(A_i, B).\end{aligned}$$

Definition 3.4. A double fuzzifying syntopogenous structure on $X\Psi$ is a non-empty family $\Upsilon_X\Psi$ of double fuzzifying topogenous orders on X. If it satisfies the following conditions:
$(LS1)$ $\Upsilon_X\Psi$ is directed, i.e. a two double fuzzifying topogenous orders $(\mathcal{T}_1, \mathcal{T}_1^*)$, $(\mathcal{T}_2, \mathcal{T}_2^*) \in \Upsilon_X$, $\exists$ double fuzzifying topogenous orders $(\mathcal{T}_1, \mathcal{T}^*) \in \Upsilon_X$ such that $\mathcal{T} \geq \mathcal{T}_1, \mathcal{T}_2$, and $\mathcal{T}^* \leq \mathcal{T}_1^*, \mathcal{T}_2^*$,
$(LS2)$ For every $(\mathcal{T}, \mathcal{T}^*) \in \Upsilon_X$, $\exists$ $(\mathcal{T}_1, \mathcal{T}_1^*) \in \Upsilon_X$ such that $\mathcal{T} \leq \mathcal{T}_1 \circ \mathcal{T}_2$, and $\mathcal{T}^* \geq \mathcal{T}_1^* \circ \mathcal{T}_2^*$.

Definition 3.5. (1) A double fuzzifying syntopogenous structure $\Upsilon_X\Psi$ is called double fuzzifying topogenous orders If $\Upsilon_X\Psi$ cosists of a single element. denoted by $\Upsilon_X\Psi = \{(\mathcal{T}, \mathcal{T}^*)\}$, and (X, Υ_X) double fuzzifying topogenous space.

(2) A double fuzzifying syntopogenous structure $\Upsilon_X\Psi$ is called perfect (resp. biperfect, symmetric) if each double fuzzifying topogenous order $(\mathcal{T}, \mathcal{T}^*) \in \Upsilon_X$ is perfect (resp. biperfect, symmetric).

Theorem 3.2. Let $(\mathcal{T}, \mathcal{T}^*)$ be a double fuzzifying topogenous order on X. The mapping $f_{(\mathcal{T},\mathcal{T}^*)} : 2^X \times L_0 \times L_1 \to 2^X$, is defined by $f_{(\mathcal{T},\mathcal{T}^*)}(A, \alpha, \beta) = \bigcap \{B^- \in 2^X : \mathcal{T}(B, A^-) > \alpha,\ \mathcal{T}^*(B, A^-) < \beta\}$ for each $A, A_1, A_2 \in 2^X$ and $\alpha, \alpha' \in L_0$, $\beta, \beta' \in L_1$.

Then it has the following properties:

- (i) $\mathrm{f}_{(\mathcal{T},\mathcal{T}^*)}(X, \alpha, \beta) = X$,
- (ii) $A \subseteq f_{(\mathcal{T},\mathcal{T}^*)}(A, \alpha, \beta)$,
- (iii) $If \quad A_1 \subseteq A_2$ then $f_{(\mathcal{T},\mathcal{T}^*)}(A_1, \alpha, \beta) \subseteq f_{(\mathcal{T},\mathcal{T}^*)}(A_2, \alpha, \beta)$,
- (iv) $f_{(\mathcal{T},\mathcal{T}^*)}(A_1 \cup A_2, \alpha \wedge \alpha', \beta \vee \beta') \subseteq f_{(\mathcal{T},\mathcal{T}^*)}(A_1, \alpha, \beta) \cap f_{(\mathcal{T},\mathcal{T}^*)}(A_2, \alpha', \beta')$.
- (v) $If\ \alpha \leq \alpha',\ \beta \geq \beta'$, then $f_{(\mathcal{T},\mathcal{T}^*)}(A_1, \alpha, \beta) \subseteq f_{(\mathcal{T},\mathcal{T}^*)}(A_2, \alpha', \beta')$,
- (vi) $If\ (\mathcal{T}, \mathcal{T}^*)$ be a double fuzzifying topogenous order on X, $\mathrm{f}_{(\mathcal{T},\mathcal{T}^*)}(\mathrm{f}_{(\mathcal{T},\mathcal{T}^*)}(X, \alpha, \beta)) \subseteq \mathrm{f}_{(\mathcal{T},\mathcal{T}^*)}(X, \alpha, \beta)$.

Proof (i) since $\mathcal{T}(X, X) = \top$ and $\mathcal{T}^*(X, X) = \bot$, $f_{(\mathcal{T},\mathcal{T}^*)}(X, \alpha, \beta) = X$.

(ii) Since $\mathcal{T}(B, A^-) \neq \bot$ and $\mathcal{T}^*(B, A^-) \neq \top$, then $B \subseteq A^-$. Then $A \subseteq f_{(\mathcal{T},\mathcal{T}^*)}(A, \alpha, \beta)$

(iii) For $A_1 \subseteq A_2$, since $\mathcal{T}(B, A_2^-) \subseteq \mathcal{T}(B, A_1^-) > \alpha$ and $\mathcal{T}^*(B, A_2^-) \geq \mathcal{T}^*(B, A_1^-) < \beta$, we have $f_{(\mathcal{T},\mathcal{T}^*)}(A_2, \alpha, \beta) \supseteq f_{(\mathcal{T},\mathcal{T}^*)}(A_1, \alpha, \beta)$,

(iv) Suppose taht there exist $A_1, A_2 \in 2^X$ such that $f_{(\mathcal{T},\mathcal{T}^*)}(A_1 \cup A_2, \alpha \wedge \alpha', \beta \vee \beta') \subsetneq f_{(\mathcal{T},\mathcal{T}^*)}(A_1, \alpha, \beta) \cap f_{(\mathcal{T},\mathcal{T}^*)}(A_2, \alpha', \beta')$, by the definition of $f_{(\mathcal{T},\mathcal{T}^*)}(A, \alpha, \beta)$ there exist $B_1, B_2 \in 2^X$ with $\mathcal{T}(B_1, A_1^-) > \alpha$, $\mathcal{T}^*(B_1, A_1^-) < \beta$, $\mathcal{T}(B_2, A_2^-) > \alpha'$, $\mathcal{T}^*(B_2, A_2^-) < \beta'$, such that $f_{(\mathcal{T},\mathcal{T}^*)}(A_1 \cup A_2, \alpha \wedge \alpha', \beta \vee \beta') \subsetneq B_1^- \cap B_2^-$.

On the other hand, by (T), and $(LST4)$ $\mathcal{T}((B_1 \cap B_2, (A_1 \cup A_2)^-) \geq \mathcal{T}((B_1 \cap B_2, A_1^-) \wedge \mathcal{T}(B_1 \cap B_2, A_2^-) \geq \mathcal{T}((B_1, A_1^-) \wedge \mathcal{T}(B_2, A_2^-) > \alpha \wedge \alpha'$ and $\mathcal{T}^*((B_1 \cap B_2, (A_1 \cup A_2)^-) \leq \mathcal{T}^*((B_1 \cup B_2, A_1^-) \vee \mathcal{T}^*(B_1 \cap B_2, A_2^-) \leq \mathcal{T}^*((B_1, A_1^-) \vee \mathcal{T}(B_2, A_2^-) < \beta \wedge \beta'$. It is implies $f_{(\mathcal{T},\mathcal{T}^*)}(A_1 \cup A_2, \alpha \wedge \alpha', \beta \vee \beta') \subseteq (B_1 \cap B_2)^- = B_1^- \cup B_2^-$. This is a contradiction.

(v) and (vi) by the fashion.

Theorem 3.3. Let $(\mathcal{T}, \mathcal{T}^*)$ be a double fuzzifying topogenous order on X. The mapping $f_{(\mathcal{T},\mathcal{T}^*)} : 2^X \times L_0 \times L_1 \to 2^X$, is defined by.

$$f_{(\mathcal{T},\mathcal{T}^*)}(A, \alpha, \beta) = \bigcup \{Q \in 2^X : \mathcal{T}(Q, A^-) > \alpha \to \bot, \ \mathcal{T}^*(Q, A^-) < \beta \to \bot\}.$$

Then it has the following properties:

(i) $\mathrm{f}_{(\mathcal{T},\mathcal{T}^*)}(X, \alpha, \beta) = \phi$,
(ii) $f_{(\mathcal{T},\mathcal{T}^*)}(A, \alpha, \beta) \subseteq A^-$,
(iii) *If* $\alpha \geq \alpha'$ *and* $\beta \leq \beta'$, then $f_{(\mathcal{T},\mathcal{T}^*)}(A, \alpha', \beta') \subseteq f_{(\mathcal{T},\mathcal{T}^*)}(A, \alpha, \beta)$,
(iv) $f_{(\mathcal{T},\mathcal{T}^*)}(A_1 \cap A_2, \alpha \wedge \alpha', \beta \vee \beta') \supseteq f_{(\mathcal{T},\mathcal{T}^*)}(A_1, \alpha, \beta) \cap f_{(\mathcal{T},\mathcal{T}^*)}(A_2, \alpha', \beta')$.

Proof (i) From (LST2) and since $\mathcal{T}(Q, A^-) > \alpha \to \bot$ and $\mathcal{T}^*(Q, A^-) < \beta \to \bot$, $f_{(\mathcal{T},\mathcal{T}^*)}(X, \alpha, \beta) = \phi$.

(ii) From (LST3) and since $\mathcal{T}(Q, A^-) < \alpha \to \bot$ and $\mathcal{T}^*(Q, A^-) > \beta \to \bot$, then, $Q \subseteq A^-$. Thus $f_{(\mathcal{T},\mathcal{T}^*)}(A, \alpha, \beta) \subseteq A^-$

(iii) For $\alpha \geq \alpha'$, $\beta \leq \beta'$, since $\mathcal{T}(Q, A^-) > \alpha' \to \bot > \alpha \to \bot$ and $\mathcal{T}^*(Q, A^-) < \beta' \to \bot < \beta \to \bot$, we have $f_{(\mathcal{T},\mathcal{T}^*)}(A, \alpha, \beta) \subseteq f_{(\mathcal{T},\mathcal{T}^*)}(A, \alpha', \beta')$.

(iv) $f_{(\mathcal{T},\mathcal{T}^*)}(A, \alpha, \beta) \cap f_{(\mathcal{T},\mathcal{T}^*)}(B, \alpha', \beta') = \bigcup\{Q_1 \in 2^X : \mathcal{T}(Q_1, A^-) > \alpha \to \bot, \mathcal{T}^*(Q_1, A^-) < \beta \to \bot\} \cap (\bigcup\{Q_2 \in 2^X : \mathcal{T}(Q_2, B^-) > \alpha' \to \bot, \mathcal{T}^*(Q_2, A^-) < \beta' \to \bot\}) = \bigcup\{Q_1 \cap Q_2 \in 2^X : \mathcal{T}(Q_1, A^-) > \alpha \to \bot, \mathcal{T}(Q_2, B^-) > \alpha' \to \bot, \mathcal{T}^*(Q_1, A^-) < \beta \to \bot, \mathcal{T}^*(Q_2, B^-) < \beta' \to \bot \}$
$\subseteq \bigcup\{Q_1 \cap Q_2 \in 2^X : \mathcal{T}(Q_1, A^-) \vee \mathcal{T}(Q_2, B^-) > (\alpha \to \bot) \vee (\alpha' \to \bot), \mathcal{T}^*(Q_1, A^-) \wedge \mathcal{T}^*(Q_2, B^-) < (\beta \to \bot) \wedge (\beta' \to \bot) \}$
$\subseteq \bigcup\{Q_1 \cap Q_2 \in 2^X : \mathcal{T}(Q_1 \cap Q_2, A^- \cup B^-) < (\alpha \wedge \alpha') \to \bot\}$,
$\geq \mathcal{T}(Q_1, A^-) \wedge \mathcal{T}(Q_2, B^-) > (\alpha \to \bot) \wedge (\alpha' \to \bot) = (\alpha \vee \alpha') \to \bot, \mathcal{T}^*(Q_1 \cup Q_2, A^- \cup B^-) > (\beta \vee \beta') \to \bot \}$
$= \mathcal{T}^*(Q_1, A^-) \vee \mathcal{T}^*(Q_2, B^-) < (\beta \to \bot) \vee (\beta' \to \bot) = (\beta \vee \beta') \to \bot\}$
$= \mathcal{T}^*(Q_1, A^-) \vee \mathcal{T}^*(Q_2, B^-) < (\beta \to \bot) \vee (\beta' \to \bot) = (\beta \vee \beta') \to \bot$
$= \bigcup\{Q \in 2^X : \mathcal{T}(Q, (A \cap B)^-) > (\alpha \vee \alpha') \to \bot, \mathcal{T}^*(Q, (A \cap B)^-) < (\beta \wedge \beta') \to \bot \}$

$= f_{(\mathcal{T},\mathcal{T}^*)}(A \cap B, \alpha \vee \alpha', \beta \wedge \beta')$

Theorem 3.4. Let $(\mathcal{T}, \mathcal{T}^*)$ be a double fuzzifying topogenous order on X, and L be a chain The mapping $\delta_{\mathcal{T},}\ \delta^*_{\mathcal{T}^*} : 2^X \to L$, is defined by $\delta_{\mathcal{T}}(A) = \bigvee \{\alpha \in L_0 : f_{(\mathcal{T},\mathcal{T}^*)}(A, \alpha, \beta) = A,\ \alpha \leq \beta \to \perp\}$ and $\delta^*_{\mathcal{T}^*}(A) = \bigwedge \{\beta \in L_1 : f_{(\mathcal{T},\mathcal{T}^*)}(A, \alpha, \beta) = A,\ \alpha \leq \beta \to \perp\}$. Then the pair $(X, \delta_,\ \delta^*)$ is an $(2,\ L)$- double fuzzifying topology on X.

Proof For each $A \in 2^X$, we have

$$\begin{aligned} (DO1)\ \delta^*_{\mathcal{T}^*}(A) \to \perp &= \bigwedge \{\beta \in L_1 : f_{(\mathcal{T},\mathcal{T}^*)}(A, \alpha, \beta) = A,\ \alpha \leq \beta \to \perp\} \to \perp \\ &= \bigvee \{\beta \to \perp : f_{(\mathcal{T},\mathcal{T}^*)}(A, \alpha, \beta) = A,\ \alpha \leq \beta \to \perp\} \\ &\geq \bigvee \{\alpha \in L_0 : f_{(\mathcal{T},\mathcal{T}^*)}(A, \alpha, \beta) = A,\ \alpha \leq \beta \to \perp\} \\ &= \delta_{\mathcal{T}}(A) \end{aligned}$$

$(DO2)$ It is clear.

$(DO3)$ Suppose taht there exist $A, B \in 2^X$ such that $\delta_{\mathcal{T}}(A \cap B) \ngeq \delta_{\mathcal{T}}(A) \wedge \delta_{\mathcal{T}}(B)$ and $\delta^*_{\mathcal{T}^*}(A \cap B) \nleq \delta^*_{\mathcal{T}^*}(A) \vee \delta^*_{\mathcal{T}^*}(B)$. since L is chain and by the defintion of $\delta_{\mathcal{T}}(A)$ and $\delta^*_{\mathcal{T}^*}(A)$, there exist $\alpha_1 \in L_0$, $\beta_1 \in L_1$ with $\alpha_1 \leq \beta_1 \to \perp$ and $f_{(\mathcal{T},\mathcal{T}^*)}(A, \alpha_1, \beta_1) = A$ such that $\delta_{\mathcal{T}}(A \cap B) \ngeq \alpha_1 \wedge \delta_{\mathcal{T}}(B)$ and $\delta^*_{\mathcal{T}^*}(A \cap B) \nleq \beta_1 \vee \delta^*_{\mathcal{T}^*}(B)$. Again, by the definition of $\delta_{\mathcal{T}}(B)$ and $\delta^*_{\mathcal{T}^*}(A)$, there exist $\alpha_2 \in L_0$, $\beta_1 \in L_1$ with $\alpha_2 \leq \beta_2 \to \perp$ and $f_{(\mathcal{T},\mathcal{T}^*)}(B, \alpha_2, \beta_2) = B$ such that $\delta_{\mathcal{T}}(A \cap B) \ngeq \alpha_1 \wedge \alpha_2$ and $\delta^*_{\mathcal{T}^*}(A \cap B) \nleq \beta_1 \vee \beta_2$. By Theorem 3.2 (v) , we have $f_{(\mathcal{T},\mathcal{T}^*)}(A \cap B, \alpha_1 \wedge \alpha_2\ , \beta_1 \vee \beta_2) \supseteq f_{(\mathcal{T},\mathcal{T}^*)}(A, \alpha_1, \beta_1) \cap f_{(\mathcal{T},\mathcal{T}^*)}(B, \alpha_2, \beta_2) = A \cap B$. Then, we have $f_{(\mathcal{T},\mathcal{T}^*)}(A \cap B, \alpha_1 \wedge \alpha_2\ , \beta_1 \vee \beta_2) = A \cap B$. Thus, $\delta_{\mathcal{T}}(A \cap B) \geq \alpha_1 \wedge \alpha_2$ and $\delta^*_{\mathcal{T}^*}(A)(A \cap B) \leq \beta_1 \vee \beta_2$. This is a contradiction. Hence $\delta_{\mathcal{T}}(A \cap B) \geq \delta_{\mathcal{T}}(A) \wedge \delta_{\mathcal{T}}(B)$ and $\delta^*_{\mathcal{T}^*}(A)(A \cap B) \leq \delta^*_{\mathcal{T}^*}(A) \vee \delta^*_{\mathcal{T}^*}(A)\ \ \forall A, B \in 2^X$.

$(DO4)$ Suppose that there exist $A = \bigcup_{i \in \Gamma} A_i \in 2^X$ and $\alpha \in L_0$, $\beta \in L_1$ with $\alpha \leq \beta \to \perp$ such that $\delta_{\mathcal{T}}(A) < \alpha \leq \bigwedge_{i \in \Gamma} \delta_{\mathcal{T}}(A_i)$ and $\delta^*_{\mathcal{T}^*}(A) > \beta \geq \bigvee_{i \in \Gamma} \delta^*_{\mathcal{T}^*}(A_i)$. Then $\delta_{\mathcal{T}}(A_i) \geq \alpha$ and $\delta^*_{\mathcal{T}^*}(A_i) \leq \beta$ for each $i \in \Gamma$. This implies that $f_{(\mathcal{T},\mathcal{T}^*)}(A_i, \alpha, \beta) = A_i\ \ \forall i \in \Gamma$. Since $A_i \subseteq A\ \ \forall i \in \Gamma$. Then, $f_{(\mathcal{T},\mathcal{T}^*)}(A_i, \alpha, \beta) \subseteq f_{(\mathcal{T},\mathcal{T}^*)}(A, \alpha, \beta)$. Then $A_i = f_{(\mathcal{T},\mathcal{T}^*)}(A_i, \alpha, \beta) \subseteq f_{(\mathcal{T},\mathcal{T}^*)}(A, \alpha, \beta)$. Therefore $A = \bigcup_{i \in \Gamma} A_i \subseteq f_{(\mathcal{T},\mathcal{T}^*)}(A, \alpha, \beta)$. Then, $f_{(\mathcal{T},\mathcal{T}^*)}(A, \alpha, \beta) = A$. Then $\delta_{\mathcal{T}}(A) \geq \alpha$ and $\delta^*_{\mathcal{T}^*}(B) \leq \beta$. It is a contradiction. Hence, $\delta_{\mathcal{T}}(\bigcup_{i \in \Gamma} A_i) \geq \bigwedge_{i \in \Gamma} \delta_{\mathcal{T}}(A_i)$ and $\delta^*_{\mathcal{T}^*}(\bigcup_{i \in \Gamma} A_i) \leq \bigvee_{i \in \Gamma} \delta^*_{\mathcal{T}^*}(A_i)$, for each $\{A_i : i \in \Gamma\} \subseteq 2^X$.

Definition 3.6. Let $(X, \mathcal{T}_1, \mathcal{T}_1^*)$ and $(Y, \mathcal{T}_2, \mathcal{T}_2^*)$ be two double fuzzifying topogenous order spaces. Then the map $\phi_L : (X, \mathcal{T}_1, \mathcal{T}_1^*) \to (Y, \mathcal{T}_2, \mathcal{T}_2^*)$ is called double fuzzifying topogenous continuous, if $\mathcal{T}_2(A, B) \leq \mathcal{T}_1(\phi_L^{\leftarrow}(A), \phi_L^{\leftarrow}(B))$ and $\mathcal{T}_2^*(A, B) \geq \mathcal{T}_1^*(\phi_L^{\leftarrow}(A), \phi_L^{\leftarrow}(B))$, for each $A, B \in 2^Y$.

Theorem 3.5. Let $(X, \mathcal{T}_1, \mathcal{T}_1^*)$ and $(Y, \mathcal{T}_2, \mathcal{T}_2^*)$ be two double fuzzifying topogenous order spaces, Let $\phi_L : (X, \mathcal{T}_1, \mathcal{T}_1^*) \to (Y, \mathcal{T}_2, \mathcal{T}_2^*)$ be double fuzzifying topogenous continuous. Then:

(*i*) $f_{(\mathcal{T},\mathcal{T}^*)}(\phi_L^{\leftarrow}(Q),\alpha,\beta) \supseteq \phi_L^{\leftarrow}(\ f_{(\mathcal{T}_2,\mathcal{T}_2^*)}(Q,\alpha,\beta))$, for each $Q \in 2^Y,\ \alpha \in\ L_0,\ \beta \in L_1$.

(*ii*) $\phi_L : (X,\delta_{\mathcal{T}_1}\ ,\delta^*_{\mathcal{T}_1^*}) \ \rightarrow\ (Y,\ \delta_{\mathcal{T}_2},\delta^*_{\mathcal{T}_2^*})$ is double fuzzifying continuous.

Pr**oof** (i) From the definition of $f_{(\mathcal{T},\mathcal{T}^*)}$ in Theorem 3.3 and since $\phi_L :\ (X,\mathcal{T}_1,\mathcal{T}_1^*)$ $\rightarrow\ (Y,\mathcal{T}_2,\mathcal{T}_2^*)$ is double fuzzifying continuous, then

$$\begin{aligned}
&\phi_L^{\leftarrow}(f_{(\mathcal{T}_2,\mathcal{T}_2^*)}((Q),\alpha,\beta)\\
&= \phi_L^{\leftarrow}\left[\bigcup\left\{D \in 2^Y : \mathcal{T}_2(D,Q^-) > \alpha\ \rightarrow \bot,\ \mathcal{T}_2^*(D,Q^-) < \beta\ \rightarrow \bot\right\}\right]\\
&\subseteq \bigcup\left\{\phi_L^{\leftarrow}(D) \in 2^X : \mathcal{T}_1(\phi_L^{\leftarrow}(D),\phi_L^{\leftarrow}(Q^-)) > \alpha\ \rightarrow \bot,\ \mathcal{T}_1^*(\phi_L^{\leftarrow}(D),\phi_L^{\leftarrow}(Q^-)) < \beta\ \rightarrow \bot\right\}\\
&\subseteq \bigcup\left\{A \in 2^X : \mathcal{T}_1(A,(\phi_L^{\leftarrow}(Q)^-) > \alpha\ \rightarrow \bot,\ \mathcal{T}_1^*(A,(\phi_L^{\leftarrow}(Q))^-)) < \beta\ \rightarrow \bot\right\}\\
&= f_{(\mathcal{T}_2,\mathcal{T}_2^*)}(\phi_L^{\leftarrow}(Q),\alpha,\beta).
\end{aligned}$$

(ii) For each $A \in 2^Y$. If $\delta_{\mathcal{T}_2}(A) =\ \bot$ and $\delta^*_{\mathcal{T}_2^*}(A) = \top$, the prove is trivial. So let $\delta_{\mathcal{T}_2}(A) \neq\ \bot$ and $\delta^*_{\mathcal{T}_2^*}(A) \neq \top$.
Since $\delta_{\mathcal{T}_2}(A) \neq\ \bot$, by the defintion of $\delta_{\mathcal{T}_2}(A)$ there exist $\alpha_0 \in\ L_0,\ \beta_0 \in\ L_1$ with $\alpha_0\ \leq\ \beta_0\ \rightarrow\ \bot$ such that $\delta_{\mathcal{T}_2}(A)\ =\ \alpha_0$ and $f_{(\mathcal{T}_2,\mathcal{T}_2^*)}(A,\alpha_0,\beta_0)\ =\ A$. Thus $\phi_L^{\leftarrow}(A) = \phi_L^{\leftarrow}(f_{(\mathcal{T}_2,\mathcal{T}_2^*)}(A,\alpha_0,\beta_0)) \subseteq f_{(\mathcal{T}_2,\mathcal{T}_2^*)}(\phi_L^{\leftarrow}(A),\alpha_0,\beta_0)$ (by (i))., we have $\phi_L^{\leftarrow}(A) = f_{(\mathcal{T}_2,\mathcal{T}_2^*)}(\phi_L^{\leftarrow}(A),\alpha_0,\beta_0)$ since $\alpha_0 \leq \beta_0 \rightarrow \bot$, $\delta_{\mathcal{T}_1}(\phi_L^{\leftarrow}(A)) \geq \alpha_0 = \delta_{\mathcal{T}_2}(A)$.
similarly, when $\mathcal{T}^*_{\delta_2^*}\ \neq\ \top,\ \delta^*_{\mathcal{T}_1^*}(\phi_L^{\leftarrow}(A))\ \leq\ \delta^*_{\mathcal{T}_2^*}(A)$. Hence $\phi_L\ :\ (X,\delta_{\mathcal{T}_1}\ ,\delta^*_{\mathcal{T}_1^*})\ \rightarrow$ $(Y,\ \delta_{\mathcal{T}_2},\delta^*_{\mathcal{T}_2^*})$ is double fuzzifying continuous.

Theorem 3.6. Let $(X,\mathcal{T}_1,\mathcal{T}_1^*)$, $(Y,\mathcal{T}_2,\mathcal{T}_2^*)$ and $(Y,\mathcal{T}_3,\mathcal{T}_3^*)$ be double fuzzifying topogenous order spaces, if $\phi_L :\ (X,\mathcal{T}_1,\mathcal{T}_1^*)\ \rightarrow\ (Y,\mathcal{T}_2,\mathcal{T}_2^*)$, and $\Psi_L :\ (X,\mathcal{T}_2,\mathcal{T}_2^*)$ $\rightarrow\ (Y,\mathcal{T}_3,\mathcal{T}_3^*)$ are double fuzzifying topogenous continuous, then $\Psi\circ\phi : (X,\mathcal{T}_1,\mathcal{T}_1^*)$ $\rightarrow\ (Y,\mathcal{T}_3,\mathcal{T}_3^*)$ is double fuzzifying topogenous continuous

Pr**oof** For each $A, B \in 2^Z$

$$\begin{aligned}
\mathcal{T}_1((\Psi\circ\phi)_L^{\leftarrow}(A),(\Psi\circ\phi)_L^{\leftarrow}(B)) &= \mathcal{T}_1((\phi_L^{\leftarrow}(\Psi_L^{\leftarrow}(A)),(\phi_L^{\leftarrow}(\Psi_L^{\leftarrow}(B))\\
&\geq \mathcal{T}_2((\Psi_L^{\leftarrow}(A)),(\Psi_L^{\leftarrow}(B)))\\
&\geq \mathcal{T}_3((A),(B))),\\
\mathcal{T}_1^*((\Psi\circ\phi)_L^{\leftarrow}(A),(\Psi\circ\phi)_L^{\leftarrow}(B)) &= \mathcal{T}_1^*((\phi_L^{\leftarrow}(\Psi_L^{\leftarrow}(A)),(\phi_L^{\leftarrow}(\Psi_L^{\leftarrow}(B))\\
&\leq \mathcal{T}_2^*((\Psi_L^{\leftarrow}(A)),(\Psi_L^{\leftarrow}(B)))\\
&\leq \mathcal{T}_3^*((A),(B))).
\end{aligned}$$

Theorem 3.7. Let $(X,\mathcal{T}_1,\mathcal{T}_1^*)$, $(Y,\mathcal{T}_2,\mathcal{T}_2^*)$ be double fuzzifying topogenous order spaces, if $\phi_L :\ (X,\mathcal{T}_1,\mathcal{T}_1^*)\ \rightarrow\ (Y,\mathcal{T}_2,\mathcal{T}_2^*)$, is double fuzzifying topogenous continuous.Then it has the following properties:
(1) $\phi_L^{\rightarrow}(f_{(\mathcal{T}_1,\mathcal{T}_1^*)}(A,\alpha,\beta)) \leq f_{(\mathcal{T}_2,\mathcal{T}_2^*)}(\phi_L^{\rightarrow}(A),\alpha,\beta))$ for each $A \in 2^X,\alpha \in\ L_0,\ \beta \in\ L_1$
(2) $f_{(\mathcal{T}_1,\mathcal{T}_1^*)}(\phi_L^{\leftarrow}(B),\alpha,\beta)) \leq \phi_L^{\leftarrow}(f_{(\mathcal{T}_2,\mathcal{T}_2^*)}(B,\alpha,\beta))$ for each $B \in 2^Y,\alpha \in\ L_0,\ \beta \in\ L_1$
(3) $\phi_L : (X,\delta_{\mathcal{T}_1}\ ,\delta^*_{\mathcal{T}_1^*})\ \rightarrow\ (Y,\ \delta_{\mathcal{T}_2},\delta^*_{\mathcal{T}_2^*})$ is double fuzzifying topogenous continuous.

Pr**oof** (2) for each $B \in 2^Y,\alpha \in\ L_0,\ \beta \in\ L_1$, Put $A = \phi_L^{\leftarrow}(B)$, From (1), then

$$\begin{aligned}
\phi_L^{\rightarrow}(f_{(\mathcal{T}_1,\mathcal{T}_1^*)}(\phi_L^{\leftarrow}(B),\alpha,\beta)) &\leq (f_{(\mathcal{T}_2,\mathcal{T}_2^*)}(\phi_L^{\rightarrow}(\phi_L^{\leftarrow}(B)),\alpha,\beta))\\
&\leq f_{(\mathcal{T}_2,\mathcal{T}_2^*)}(B,\alpha,\beta)
\end{aligned}$$

It implies

$$\begin{aligned} f_{(\mathcal{T}_1,\mathcal{T}_1^*)}(\phi_L^{\leftarrow}(B),\alpha,\beta)) &\leq \phi_L^{\leftarrow}(\phi_L^{\rightarrow}(f_{(\mathcal{T}_1,\mathcal{T}_1^*)}(\phi_L^{\leftarrow}(B),\alpha,\beta))) \\ &\leq \phi_L^{\leftarrow}((f_{(\mathcal{T}_1,\mathcal{T}_1^*)}(\phi_L^{\rightarrow}(\phi_L^{\leftarrow}(B),\alpha,\beta))) \\ &\leq \phi_L^{\leftarrow}(f_{(\mathcal{T},\mathcal{T}^*)}(B,\alpha,\beta)) \end{aligned}$$

(3) It is easily from Theorem 3.5 and $f_{(\mathcal{T}_2,\mathcal{T}_2^*)}(B,\alpha,\beta) = B$ implies $f_{(\mathcal{T}_1,\mathcal{T}_1^*)}(\phi_L^{\leftarrow}(B),\alpha,\beta)) = \phi_L^{\leftarrow}(B)$.

Definition 3.7. The pair (Ω,Ω^*) of maps $\Omega,\Omega^* :\ 2^X \times 2^X \to L$ is called $(2,L)$-double fuzzifying preproximity. If it is satisfies the following conditions:
(DP1) $\Omega(\mathrm{A},\mathrm{B}) \geq \Omega^*(\mathrm{A},\mathrm{B}) \to \perp,\ \ \forall \mathrm{A},\mathrm{B} \in 2^X$,
(DP2) $\Omega(\mathrm{X},\phi) = \Omega(\phi,\mathrm{X}) = \perp$, $\Omega^*(\mathrm{X},\phi) = \Omega^*(\phi,\mathrm{X}) = \top$,
$(DP3)$ If $\Omega(A,B) \neq \top$ and $\Omega^*(A,B) \neq \perp$, then $A \subseteq B^-$,
$(DP4)$ If $A_1 \subseteq A_2$, then $\Omega(A_1,C) \leq \Omega(A_2,C)$, and $\Omega^*(A_1,C) \geq \Omega^*(A_2,C)$,
$(DP5)$ $\Omega(A_1 \cap A_2, B_1 \cup B_2) \leq \Omega(A_1,B_1) \vee \Omega(A_2,B_2)$, and $\Omega^*(A_1 \cap A_2, B_1 \cup B_2) \geq \Omega^*(A_1,B_1) \wedge \Omega^*(A_2,B_2)$.
The pair $(X,\ \Omega,\Omega^*)$ is said to be an $(2,L)$-double fuzzifying preproximity space.

$(2,L)$ double fuzzifying preproximity space is called $(2,L)$-double fuzzifying quasi proximity provided that

$$(DP6)\ \Omega(A,B) \geq \bigwedge_{D\in 2^X} \{\Omega(A,D) \vee \Omega(D^c,B)\},$$

and

$$\Omega^*(A,B) \leq \bigvee_{D\in 2^X} \{\Omega^*(A,D) \wedge \Omega^*(D^c,B)\}$$

$(2,L)$ double fuzzifying quasi-proximity is called $(2,L)$-double fuzzifying proximity provided taht
(DP) $\Omega(A,B) = \Omega(B,A)$ and $\Omega^*(A,B) = \Omega^*(B,A)$.

$(2,L)$ double fuzzifying preproximity space is called $(2,L)$-double fuzzifying prinicipal provided that:

$$(DP7)\ \Omega(\bigcup_{i\in\Gamma} A_i,B) \leq \bigvee_{i\in\Gamma} \Omega(A_i,B),\ \text{ and }\ \Omega^*(\bigcup_{i\in\Gamma} A_i,B) \geq \bigwedge_{i\in\Gamma} \Omega^*(A_i,B).$$

Let (Ω_1,Ω_1^*) and (Ω_2,Ω_2^*) be $(2,L)$-double fuzzifying proximities on X . (Ω_1,Ω_1^*) is coarser than (Ω_2,Ω_2^*) if $\Omega_1(A,B) \leq \Omega_2(A,B)$ and $\Omega_1^*(A,B) \geq \Omega_2^*(A,B)$, for each $A,B \in 2^X$ and we write $(\Omega_1,\Omega_1^*) \leq (\Omega_2,\Omega_2^*)$.

Theorem 3.8. (1) Let $(X,\mathcal{T},\mathcal{T}^*)$ be double fuzzifying (resp. symmetric) topogenous order spaces, and let the map $\phi_L :\ \Omega_{\mathcal{T}},\ \Omega^*_{\mathcal{T}^*} : 2^X \times 2^X \to L$ defined by $\Omega_{\mathcal{T}}(A,B) = (\mathcal{T}(A,B^-)) \to \perp$ and $\Omega^*_{\mathcal{T}^*}(A,B) = (\mathcal{T}^*(A,B^-)) \to \perp\ \forall A,B \in 2^X$. Then $(\Omega_{\mathcal{T}},\Omega^*_{\mathcal{T}^*})$ is double fuzzifying quasi proximity space (resp. double fuzzifying proximity space) on X.

(2) Let (Ω,Ω^*) be an $(2,L)$-double fuzzifying quasi proximity space (resp. $(2,L)$-double fuzzifying quasi proximity space) on X. $\mathcal{T}_\Omega,\ \mathcal{T}^*_{\Omega^*} : 2^X \times 2^X \to L$ defined

by $\mathcal{T}_\Omega(A,B) = (\Omega(A,B^-)) \to \perp)$ and $\mathcal{T}^*_{\Omega^*}(A,B) = (\Omega^*(A,B^-)) \to \perp\ \forall A,B \in 2^X$. Then $(\mathcal{T}_\Omega,\ \mathcal{T}^*_{\Omega^*})$ is double fuzzifying (resp. symmetric) topogenous order spaces.

(3) $(\Omega,\Omega^*) = (\Omega_{\mathcal{T}_\Omega},\ \Omega^*_{\mathcal{T}^*_{\Omega^*}})$ and $(\mathcal{T},\mathcal{T}^*)(\mathcal{T}_{\Omega_\mathcal{T}},\mathcal{T}^*_{\Omega^*_{\mathcal{T}^*}})$

Proof (1) Since $\mathcal{T}\circ\mathcal{T} \geq \mathcal{T}$ and $\mathcal{T}^*\circ\mathcal{T}^* \leq \mathcal{T}^*$.

$$\begin{aligned}
\Omega_\mathcal{T}(A,B) &= (\mathcal{T}(A,B^-)) \to \perp \\
&\geq ((\mathcal{T}\circ\mathcal{T})(A,B^-)) \to \perp \\
&\geq \left[\bigvee_{h\in 2^X}[\mathcal{T}(A,h)\wedge(\mathcal{T}(h,B^-)]\right] \to \perp \\
&= \bigwedge_{h\in 2^X}[[[\mathcal{T}(A,h)] \to \perp] \vee [[\mathcal{T}(h,B^-)] \to \perp]] \\
&= \bigwedge_{h\in 2^X}\{\Omega_\mathcal{T}(A,h^-)\vee\Omega_\mathcal{T}(h,B)\}, \\
\Omega^*_{\mathcal{T}^*}(A,B) &= (\mathcal{T}^*(A,B^-)) \to \perp \\
&\leq ((\mathcal{T}^*\circ\mathcal{T}^*)(A,B^-)) \to \perp \\
&\geq \left[\bigwedge_{h\in 2^X}[\mathcal{T}^*(A,h)\vee(\mathcal{T}^*(h,B^-)]\right] \to \perp \\
&= \bigvee_{h\in 2^X}[[[\mathcal{T}^*(A,h)] \to \perp] \wedge [[\mathcal{T}^*(h,B^-)] \to \perp]] \\
&= \bigvee_{h\in 2^X}\{\Omega^*_{\mathcal{T}^*}(A,h^-)\wedge\Omega^*_{\mathcal{T}^*}(h,B)\},
\end{aligned}$$

(2) and (3) are easily proved

Theorem 3.9. Let (Ω,Ω^*) be a double quasi proximity.The mapping $f_{(\Omega,\Omega^*)} : 2^X \to L$, is defined by.

$$f_{(\Omega,\Omega^*)}(A,\alpha,\beta) = \bigcap\{Q^- \in 2^X :\ \Omega(Q,A) < \alpha \to \perp,\ \Omega^*(Q,A) > \beta \to \perp\}.$$

Then it has the following properties:

(*i*) $\mathrm{f}_{(\Omega,\Omega^*)}(\phi,\alpha,\beta) = \phi$,
(*ii*) $f_{(\Omega,\Omega^*)}(A,\alpha,\beta) \supseteq A$,
(*iii*) $If\ \ A \subseteq B,\ then\ f_{(\Omega,\Omega^*)}(A,\alpha,\beta) \subseteq f_{(\Omega,\Omega^*)}(B,\alpha,\beta)$,
(*iv*) $f_{(\Omega,\Omega^*)}(A\vee B,\alpha\wedge\alpha_1,\beta\vee\beta_1) \subseteq f_{(\Omega,\Omega^*)}(A,\alpha,\beta)\vee f_{(\Omega,\Omega^*)}(B,\alpha_1,\beta_1)$
(*v*) $If\ \ \alpha\leq\alpha_1\ and\ \beta\geq\beta_1$, then $f_{(\Omega,\Omega^*)}(A,\alpha,\beta) \subseteq f_{(\Omega,\Omega^*)}(A,\alpha_1,\beta_1)$,
(*v*) $f_{(\Omega,\Omega^*)}(f_{(\Omega,\Omega^*)}(A,\alpha,\beta),\alpha,\beta) \subseteq f_{(\Omega,\Omega^*)}(A,\alpha,\beta)$.

Theorem 3.10. Let (Ω,Ω^*) be a double quasi proximity.Define the mas $\delta_\otimes(A) = \bigvee\{\alpha\in L_0 : f_{(\Omega,\Omega^*)}(A,\alpha,\beta) = A\}$ and $\delta^*_{\Omega^*}(A) = \bigwedge\{\beta\in L_1 : f_{(\mathcal{T},\mathcal{T}^*)}(A^-,\alpha,\beta) = A^-\}$. Then the pair (X,Ω,Ω^*) is an $(2,\ L)$- double fuzzifying topology induced by (Ω,Ω^*).

Definition 3.8. Let (X,Ω_1,Ω_1^*) and (Y,Ω_2,Ω_2^*) be a double quasi proximity spaces. A maps $\phi_L : (X,\Omega_1,\Omega_1^*) \to (Y,\Omega_2,\Omega_2^*)$ is said to be quasi proximity continuous if $\Omega_2(A,B) \geq \Omega_1(\phi_L^\leftarrow(A),\phi_L^\leftarrow(B))$ and $\Omega_2^*(A,B) \leq \Omega_1^*(\phi_L^\leftarrow(A),\phi_L^\leftarrow(B))$, for each $A,B\in 2^Y$.

Theorem 3.11. Let (X,Ω_1,Ω_1^*) and (Y,Ω_2,Ω_2^*) be a double quasi proximity spaces, A map $\phi_L : (X,\Omega_1,\Omega_1^*) \to (Y,\Omega_2,\Omega_2^*)$ is quasi proximity continuous iff $\phi_L : (X,\mathcal{T}_{\Omega_1},\mathcal{T}^*_{\Omega_1^*}) \to (Y,\mathcal{T}_{\Omega_2},\mathcal{T}^*_{\Omega_2^*})$ is topogenous continuous.

Proof For each $A, B \in 2^Y$.

$$\begin{aligned}
\Omega_2(A,B) &\geq \Omega_1(\phi_L^{\leftarrow}(A), \phi_L^{\leftarrow}(B)) \\
&\Leftrightarrow \mathcal{T}_{\Omega_2}((A,B^-) \to \bot) \\
&\geq \mathcal{T}_{\Omega_1}((\phi_L^{\leftarrow}(A), \phi_L^{\leftarrow}(B^-)))) \to \bot) \\
&\Leftrightarrow \mathcal{T}_{\Omega_2}((A,B^-)) \\
&\leq \mathcal{T}_{\Omega_1}((\phi_L^{\leftarrow}(A), \phi_L^{\leftarrow}(B^-)), \\
\Omega_2^*(A,B) &\leq \Omega_1^*(\phi_L^{\leftarrow}(A), \phi_L^{\leftarrow}(B)) \\
&\Leftrightarrow \mathcal{T}^*_{\Omega_2^*}((A,B^-) \to \bot) \\
&\leq \mathcal{T}^*_{\Omega_1^*}((\Omega_1(\phi_L^{\leftarrow}(A), \phi_L^{\leftarrow}(B^-)))) \to \bot) \\
&\Leftrightarrow \mathcal{T}^*_{\Omega_2^*}(A,B^-) \\
&\geq \mathcal{T}^*_{\Omega_1^*}((\phi_L^{\leftarrow}(A), \phi_L^{\leftarrow}(B^-)).
\end{aligned}$$

Definition 3.9. Let X be a nonempty set and let , $\mathcal{U}, \mathcal{U}^* \in L^{P(X\times X)}$. Assume that the following statments are satisfied:

$(LU1)$ $\mathcal{U}(A) \leq (\mathcal{U}^*(A)) \to \bot$ for all $A \in P(X \times X)$,

$(LU2)$ $\mathcal{U}(A \cap B) \geq \mathcal{U}(A) \wedge \mathcal{U}(A)$ and $\mathcal{U}^*(A \cap B) \leq \mathcal{U}^*(A) \vee \mathcal{U}^*(B)$,

$(LU3)$ There exists $A \in P(X \times X)$ s.t. $\mathcal{U}(A\ ) = \top$, and $\mathcal{U}^*(A\ ) = \bot$,

$(LU4)$ For any $A \in P(X \times X)$, $\exists\ B \in P(X \times X)$ s.t. $B \circ B \subseteq A$ and $\mathcal{U}(B) \geq \mathcal{U}(A\ )$ and $\mathcal{U}^*(B) \leq \mathcal{U}^*(A)$. where $B \circ A$ is defined by $B \circ A = \{(x,y) |\, \exists z \in X$ such that $(x,z) \in A$ and $(z,y) \in A\}$, $\forall x, y \in X$. Then $(X, \mathcal{U}, \mathcal{U}^*)$ is called an double fuzzifying quasi uniform space.

An double fuzzifying quasi uniform space $(X, \mathcal{U}, \mathcal{U}^*)$ is said to be a double fuzzifying uniform space if it satisfies.

(LU) For any $A, B \in P(X \times X)$, $\mathcal{U}(A\ ) \leq \mathcal{U}(B^{\leftarrow}\ )$, and $\mathcal{U}^*(A) \geq \mathcal{U}^*(B^{\leftarrow})$, where $B^{\leftarrow} = \{(x,y) \,|(y,x) \in P(X \times X)\}$

Definition 3.10. Let X be a nonempty set and let ,$\Xi, \Xi^* \in L^{P(X\times X)}$. Assume that the following statments are satisfied:

$(LUB1)$ $\Xi(A) \leq (\Xi^*(A)) \to \bot$ for all $A \in P(X \times X)$,

$(LUB2)$ $\bigvee_{B \in P(X\times X)} \Xi(B) \leq \Xi(A) \wedge \Xi(A)$ and $\overset{*}{\bigwedge_{B \in P(X\times X)}} \Xi(B) \geq \Xi^*(A) \vee \Xi^*(B)$,

$(LUB3)$ There exists $A \in P(X \times X)$ s.t. $\Xi(A\ ) = \top$, and $\Xi^*(A\ ) = \bot$,

$(LUB4)$ For any $A \in P(X \times X)$, $\exists\ B \in P(X \times X)$ s.t. $B \circ B \subseteq A$ and $\Xi(B) \geq \Xi(A\ )$ and $\Xi^*(B) \leq \Xi^*(A)$.

Then (X,Ξ, Ξ^*) is called an double fuzzifying quasi uniform base.A double fuzzifying quasi uniform base (X,Ξ, Ξ^*) is said to be a double fuzzifying uniform base if it satisfies.

(LUB) For any $A, B \in P(X \times X)$, $\Xi(A\ ) \leq \Xi(B^{\leftarrow}\ )$, and $\Xi^*(A) \geq \Xi^*(B^{\leftarrow})$.

Theorem 3.12. Let $(\Xi, \Xi^*) \in L^{P(X\times X)}$. Define $(\mathcal{U}_{\Xi}, \mathcal{U}^*_{\Xi^*}) \in L^{P(X\times X)}$ as $\mathcal{U}_{\Xi}(A) = \bigvee_{B \in P(X\times X)} \{\Xi(B) : B \subseteq A\}$ and $\mathcal{U}^*_{\Xi^*}(A) = \bigwedge_{B \in P(X\times X)} \{\Xi^*(B) : B \subseteq A\}$. Then $(\mathcal{U}_{\Xi}, \mathcal{U}^*_{\Xi^*})$ is a double fuzzifying uniformity on X.

Proof Because prove the cases are easily so only prove (LU). For any $A, B \in P(X\times X)$. Since $A= (A^{\leftarrow})^{\leftarrow}$, we have $\mathcal{U}_{\Xi}(A^{\leftarrow}) \leq \mathcal{U}_{\Xi}(A)$ and $\mathcal{U}^{*}_{\Xi^*}(A^{\leftarrow}) \geq \mathcal{U}^{*}_{\Xi^*}(A)$. and

$$\begin{aligned}
\mathcal{U}_{\Xi}(A) &= \bigvee_{B\in P(X\times X)} \{\Xi(B) : B \subseteq A\} \\
&\leq \bigvee_{B\subseteq A} \left\{\bigvee \Xi(Q) : Q \subseteq B^{\leftarrow}\right\} \text{ by } (LUB) \\
&\leq \bigvee_{B\subseteq A} \mathcal{U}_{\Xi}(B^{\leftarrow}) \\
&= \bigvee_{B^{\leftarrow}\subseteq A^{\leftarrow}} \mathcal{U}_{\Xi}(B^{\leftarrow}), \\
&\leq \mathcal{U}_{\Xi}(A^{\leftarrow}) \\
\mathcal{U}^{*}_{\Xi^*}(A) &= \bigwedge_{B\in P(X\times X)} \{\Xi^*(B) : B \subseteq A\} \\
&\geq \bigwedge_{B\subseteq A} \{\bigwedge \Xi^*(Q) : Q \subseteq B^{\leftarrow}\} \text{ by } (LUB) \\
&\geq \bigwedge_{B\subseteq A} \mathcal{U}^{*}_{\Xi^*}(B^{\leftarrow}) \\
&= \bigwedge_{B^{\leftarrow}\subseteq A^{\leftarrow}} \mathcal{U}^{*}_{\Xi^*}(B^{\leftarrow}), \\
&\geq \mathcal{U}^{*}_{\Xi^*}(A^{\leftarrow}).
\end{aligned}$$

4 Fuzzifying Topology and Dynamics of Breast Cancer

In this section we will show how the dynamical topologies [CsaszarA.(1978)]. can develop the diagnostic mechanism and time analysis of the situation and determine the appropriate time to avoid distortions in the stages of the case. The present article demonstrates an application of L-fuzzifying dynamice topology clarify a model describing biological phenomena, This model allow to know all levels of development of an breast cancer. from 0-level (infection outside cells) until 5-level (infection liver).

Definition 4.1. Let X be compact metric space, T is a time L is a chain, then the function $\mathcal{T} : 2^X \times T \to L$ is called an L-fuzzifying dynamice topology on X (T-dynamic topologies) iff it satisfies the following axioms:

(1) $\mathcal{T}(X, t) = \top,\ \mathcal{T}(\phi, t) = \bot$

(2) $\forall A, B \in 2^X, \mathcal{T}((A \cap B), t) \geq \mathcal{T}(A, t) \wedge \mathcal{T}(B, t)$,

(3) $\forall\{A_j\,|j \in J\} \subseteq 2^X,\ \mathcal{T}((\bigcup_{j\in J} A_j), t) \geq \bigwedge_{j\in J} \mathcal{T}((A_j), t)$.

We also write $\mathcal{T} = \mathcal{T}_d(T)$. such that and $\mathcal{T}_d(T)$ can be viewed as parametric or dynamic sets of X, say that $(X, L, \mathcal{T}_d(T))$ is an L-fuzzifying T-dynamice topological space. The inductive dimension of a fuzzifying dynamice topology X is either of two values, the small inductive dimension $ind(X)$ or the large inductive dimension $Ind(X)$. We want the dimension of a point to be $\bot$, and a point has empty boundary, so we start with $ind(\phi) = Ind(\phi) = \bot$. If $L = I = [0, 1]$ a fuzzifying dynamice topological space has dimension $\leq n$. $n \geq 0$ iff for any point $p \in X$, each neighborhood of p contains a neighborhood of p whose boundary has dimension $\leq n - 1$.

Definition 4.2. A riemannian manifold is a smooth manifold equipped with a riemannian metric. A map $f : (X, \mathcal{T}) \to (Y, \gamma)$, where X and Y are riemannian manifolds. is said to be a topological folding if and only if for any piecewise geodesic path, α, in X, the induced path, $f \circ \alpha$ is a piecewise geodesic in Y . It is possible $f(X) = Y$ or $f(X) \neq Y$; accordingly, a topological folding f of (X, τ) into itself satisfies $f(X) \subseteq X$ and for each $\beta \in \tau$, we have $f(\beta) \subseteq \beta$. The contrary definition to the folding of (X, τ) into itself is the unfolding: a map $f : (X, \mathcal{T}) \rightarrow (Y, \gamma)$ is called unfolding iff $f(\beta) \supseteq \beta$ for each $\beta \in \tau$ [4].

From these topological concepts we can form templates to form the biological structures of the course of breast cancer progression as follows:

Molding (I) : 0-$level$(infection outside cells normal cells) until $1 - level$ (very slow growing cancer cells)

Molding (II) : 2-$level$ (Slow grwoing cancer cells) , 3-level (Moderately growing cancer cells) , $4 - level$ the arrival of cancer of the liver (Fast growing cancer cells)

Molding (III) : 5-$level$ (Infection spreads to liver)

2 Main Results

when begins infection outside cells $(0 - level)$, we suppose that an $0 - level$ at time $t_0 = 0$, after certain time and constant rate of differentiation of tumor is 2 cm in size and the lymph nodes under the armpit are intact from the cancer cells $(1 - level)$, then $(1 - level)$ differentiate into The size of the tumor is 2 cm, and may have moved under the control but not spread to the rest of the body $(2 - level)$, which differentiate into system, The tumor is adherent to the skin of the breast and muscles and the size of the tumor is greater than 5 cm and has moved under the armpit $(3 - level)$, and $(3 - level)$ differentiates to the arrival of cancer of the liver $(4 - level)$, and finally $(4 - level)$ differentiates to infection liver and mastectomy $(5 - level)$ at time $t = 1$. Thus

$$(\mathbf{0} - \mathbf{level})_{t_0=0} \Rightarrow (\mathbf{1} - \mathbf{level}) \Rightarrow (\mathbf{2} - \mathbf{level}) \Rightarrow (\mathbf{3} - \mathbf{level}) \\ \Rightarrow (\mathbf{4} - \mathbf{level}) \Rightarrow (\mathbf{5} - \mathbf{level})_{t=1}$$

Now we can define a L-fuzzifying dynamice topology (T-dynamic topologies) as follows:

$$\mathcal{T}(A, t) = \begin{cases} 0 & A = (0 - level)_{t_0=0} \\ \alpha_1 & A = ((1 - level), (0 < t < t_1)) \\ \alpha_2 & A = ((2 - level), (t_1 < t < t_2)) \\ \alpha_3 & A = ((3 - level), (t_2 < t < t_3)) \\ \alpha_4 & A = ((4 - level) , (t_3 < t < t_4)) \\ 1 & A = (5 - level)_{t=1} \end{cases}$$

such taht $(t_0 = 0) < t_1 < t_2 < t_3 < t_4 < (t_5 = 1)$, and $\alpha_1 < \alpha_2 < \alpha_3 < \alpha_4$ in L.

where, $(\mathbf{0} - \mathbf{level})_{t_0=0}$ at $t_0 = 0$ and $(\mathbf{5} - \mathbf{level})$ happens at $t = 1$. It is obvious that $((\mathbf{5} - \mathbf{level}), \mathcal{T})$ forms a L-fuzzifying dynamice topology (T-dynamic topologies) as the growth's rate of breast cancer. from $(\mathbf{0-level})_{t_0=0}$(infection outside cells) until $(\mathbf{5-level})$ (infection liver) depends on time. Perhaps over time there is no differentiation for example

$$(\mathbf{0-level})_{t_{01}=0}\Rightarrow (\mathbf{0-level})_{\mathbf{t_{02}=0}} \Rightarrow (\mathbf{0-level})_{\mathbf{t_{03}=0}} \Rightarrow (\mathbf{1-level})_{t_{11}}$$
$$\Rightarrow (\mathbf{1-level})_{t_{12}}\Rightarrow (1-level)_{t=2} \Rightarrow (\mathbf{2-level})_{t=3}\Rightarrow (\mathbf{3-level})_{t=4}$$
$$\Rightarrow (\mathbf{4-level})_{t=5}\Rightarrow (\mathbf{5-level})_{t=\max imum}$$

Here, from $t_{01} = 0$ up to t_{11} only the infection outside cells without a real development and from t_{11} up to t_{12}. constant rate of differentiation of tumor is 2 cm in size and the lymph nodes under the armpit are intact from the cancer cells is without real expansion this is a topological invariant. In these fixed stages with the passage of time may take the development of the disease different aspects of the injury and may lead to injury in other areas. Using precise time scales such as femtoseconds, we can identify the inaccurate stages of the disease as natural time evolves treatment is therefore necessary. In fact, cognitive method depend on synchronization of abnormality step during cells development. We assume that $\lambda(t)$ is the shape of cells as we reach a specific time, t. Then, a chain of T-dynamic topologies can be given

$$((\lambda_0(t_0), \mu_0(t_0)),\ ((\lambda_1(t_1), \mu_1(t_1)),\ ((\lambda_2(t_2), \mu_2(t_2)),\ ...,\ (\max(\lambda_i(t_i), \max \mu_i(t_i))$$

With the attributes

$$\lambda_0(t_0) \subseteq \lambda_1(t_1) \subseteq ... \subseteq \max(\lambda_i(t_i) \text{ and } \mu_0(t_0) \subseteq \mu_1(t_1) \subseteq ... \subseteq \max \mu_i(t_i)$$

and $f_n(\lambda_{n+1}) = \lambda_n$, $n = 0, 1, ..., i-1$, where f_n is a folding from λ_{n+1} into λ_n.

It is also satisfying $\mu_{n+1}(t_n) = f_n(\mu_n), n = 0, 1, ..., i-1$.

In the same path, $\lambda = \phi$ at $t = 0$ and after a limit of time the maximum of measurement formation of cancer cells.

This gives us the increasing chain to determine a cancer at a limit time.

$$\phi \overset{t=1}{\Rightarrow} \lambda_1 \subseteq .\lambda_2 \subseteq ... \text{ with } \mu_0 \subseteq \mu_1 \subseteq \mu_2 \subseteq ...$$

Or otherwise we get another decreasing chain can not determine a cancer at a limit time.

$$\lambda_1 \supseteq \lambda_2 \supseteq ... \supseteq \lambda_\infty \text{ with } \mu_0 \supseteq \mu_1 \supseteq \mu_2 \supseteq ... \supseteq \mu_i \to \phi$$

Some times in some steps fluctuation happens in the growth cancer , for example

$$\lambda_1 \subseteq \lambda_2 = \lambda_3 \subseteq ... \text{ with } \mu_0 \subseteq \mu_1 \subseteq \mu_2 = \mu_3 \subseteq ...$$

This causes a delay of the growth cancer at specific time. Giving the opportunity for treatment at this time. Based on the properties of local topological subspaces for the dynamical topology a demand for a medical treatment should be started to stop cognitive anomalies at any step of growth, and a positive result may be achieved as we use a femto second as a measurement unit.

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Certain Relations of Gegenbauer and Modified Gegenbauer Matrix Polynomials by Lie Algebraic Method

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Abstaract − The object of the present paper is to derive the generating formulae for the Gegenbauer and modified Gegenbauer matrix polynomials by introducing a partial differential operator and constructing the Lie algebra representation formalism of special linear algebra by using Weisner's group-theoretic approach. Application of our results is also pointed out.

Keywords − *Gegenbauer matrix polynomials; Generating matrix functions; Matrix differential equations; Differential operator; Group-theoretic method.*

1 Introduction

The study of special matrix polynomials is an important due to their applications in certain areas of statistics, physics, engineering, Lie group theory and number theory. Group theoretic methods have played an important role in the modern theory of special functions. Lie algebraic methods for computing eigenvalues and recurrence relations have been developed and the methods developed in the present paper provide a more flexible and direct treatment than the standard Lie algebraic treatment used recently in [1, 5, 14, 15, 21, 23, 24, 32, 34]. The reason of interest for this family of Gegenbauer matrix polynomials (GMPs) and their associated operational formalism is due to their intrinsic mathematical importance and the fact that these polynomials have important applications in physics. Motivated and inspired by the work of Jódar et. al. and his co-authors on Gegenbauer matrix polynomials, see for example [2, 3, 4, 6, 8, 9, 10, 11, 12, 13, 16, 17, 18, 19, 20, 22, 25, 26, 33] and due to make use of the Lie group-theoretic method (see [1, 23, 24, 27, 28, 29, 30, 31]). In this paper, we introduce the differential operators for 2-variable Gegenbauer and modified Gegenbauer matrix polynomials (MGMPs) and derive their many new and known generating matrix relations by using Lie algebraic techniques.

1.1 Preliminaries

For the sake of clarity in the presentation we recall some generating matrix relations for the Gegenbauer matrix polynomials and some notations which will be used throughout the next section. Throughout this paper, we assume that A is a positive stable matrix in $\mathbb{C}^{N\times N}$; that is, the matrix A satisfies the following condition

$$Re(\mu) > 0 \text{ for all } \mu \in \sigma(A),\ \sigma(A) := \textit{ spectrum of } A. \tag{1}$$

Definition 1.1. (Jódar et al. [16]) Let A be a matrix in $\mathbb{C}^{N\times N}$ satisfying the condition

$$(-\frac{z}{2}) \notin \sigma(A) \text{ for all } z \in \mathbb{Z}^+ \cup \{0\}. \tag{2}$$

The Gegenbauer matrix polynomials (GMPs) are defined by

$$C_n^A(x) = \sum_{k=0}^{[\frac{1}{2}n]} \frac{(-1)^k (2x)^{n-2k}}{k!(n-2k)!}(A)_{n-k}, \tag{3}$$

and the generating matrix functions

$$F(x,t,A) = (1-2xt+t^2)^{-A} = \sum_{n=0}^{\infty} C_n^A(x)t^n. \tag{4}$$

If r_1 and r_2 are the roots of the quadratic equation $1-2xt+yt^2 = 0$ and r is the minimum of the set $\{r_1, r_2\}$, then the matrix function $F(x,t,A)$ regarded as a function of t, is analytic in the disk $|t| < r$ for every real number in $|x| \leq 1$.

We recall that the Gegenbauer's matrix polynomials (GMPs) satisfy the pure and differential matrix recurrence relations by each element of this set [22]:

$$nC_n^A(x) = 2x(A+(n-1)I)C_{n-1}^A(x) - (2A+(n-2)I)C_{n-2}^A(x); n \geq 2, \tag{5}$$

where I is the identity matrix in $\mathbb{C}^{N\times N}$, and

$$(1-x^2)\frac{d}{dx}C_n^A(x) = (2A+(n-1)I)C_{n-1}^A(x) - nxC_n^A(x). \tag{6}$$

From (5) and (6), we obtain the matrix differential recurrence relations:

$$(1-x^2)\frac{d}{dx}C_n^A(x) = (2A+nI)xC_n^A(x) - (n+1)C_{n+1}^A(x). \tag{7}$$

Gegenbauer matrix polynomials $C_n^A(x)$ is a solution of the following matrix differential equation:

$$(1-x^2)\frac{d^2}{dx^2}C_n^A(x) - x(2A+I)\frac{d}{dx}C_n^A(x) + n(2A+nI)C_n^A(x) = \mathbf{0}, n \geq 0, \tag{8}$$

where $\mathbf{0}$ is the null matrix in $\mathbb{C}^{N\times N}$.

Theorem 1.2. [7] Let A, B and C are matrices in $\mathbb{C}^{N\times N}$ such that $C+nI$ is an invertible matrix for all integers $n \geq 0$. Suppose that C, $C-A$ and $C-B$ are positive stable matrices with $BC = CB$, the relation

$$ {}_2F_1\Big(A,B;C;z\Big) = (1-z)^{C-A-B}\ {}_2F_1\Big(C-A, C-B; C; z\Big) \tag{9}$$

is valid for $|z| < 1$.

2 Group-theoretic Method for Gegenbauer Matrix Polynomials

From (8) we construct a partial differential equation, replacing $\frac{d}{dx}$ by $\frac{\partial}{\partial x}$, n by $y\frac{\partial}{\partial y}$, and $C_n^A(x)$ by $C_n^A(x,y)$:

$$(1-x^2)\frac{\partial^2}{\partial x^2}C_n^A(x,y) - (2A+I)x\frac{\partial}{\partial x}C_n^A(x,y) + y\frac{\partial}{\partial y}(2A+y\frac{\partial}{\partial y}I)C_n^A(x,y) = \mathbf{0}. \quad (10)$$

Therefore $C_n^A(x,y) = C_n^A(x)y^n$ is a solution of the matrix partial differential equation Eq. (10), since $C_n^A(x)$ is a solution of matrix differential equation Eq. (8). We may rewrite (10) in the following form:

$$(1-x^2)\frac{\partial^2}{\partial\, x^2}C_n^A(x,y) + y^2\frac{\partial^2}{\partial\, y^2}C_n^A(x,y) - (2A+I)x\frac{\partial}{\partial x}C_n^A(x,y)$$
$$+ (2A+I)y\frac{\partial}{\partial y}C_n^A(x,y) = \mathbf{0}.$$

Let $\mathbb{L}$ represent the differential operators of (10), i.e.,

$$\mathbb{L} = (1-x^2)\frac{\partial^2}{\partial x^2}I + y^2\frac{\partial^2}{\partial y^2}I - (2A+I)x\frac{\partial}{\partial x} + (2A+I)y\frac{\partial}{\partial y}.$$

Next, using the matrix recurrence relations (6) and (7), we determine the first-order linear partial differential operators with the aid of $\mathbb{A}$, $\mathbb{B}$ and $\mathbb{C}$ the differential operators $\mathbb{A}$, $\mathbb{B}$ and $\mathbb{C}$ such that

$$\mathbb{B}\Big[C_n^A(x)y^n\Big] = -(2A+(n-1)I)C_{n-1}^A(x)y^{n-1},$$

and

$$\mathbb{C}\Big[C_n^A(x)y^n\Big] = (n+1)C_{n+1}^A(x)y^{n+1},$$

where

$$\mathbb{A} = y\frac{\partial}{\partial y}I,$$

$$\mathbb{B} = \frac{x^2-1}{y}\frac{\partial}{\partial x}I - x\frac{\partial}{\partial y}I,$$

and

$$\mathbb{C} = (x^2-1)y\frac{\partial}{\partial x}I + xy^2\frac{\partial}{\partial y}I + 2xyA,$$

where the linear differential operators $\mathbb{I}$, $\mathbb{A}$, $\mathbb{B}$, and $\mathbb{C}$ satisfy the following commutation relations

$$[\mathbb{A},\mathbb{B}] = -\mathbb{B}, \quad [\mathbb{A},\mathbb{C}] = \mathbb{C}, \quad [\mathbb{B},\mathbb{C}] = -2\mathbb{A} - 2A\mathbb{I},$$

where the commutator notation is defined as $[\mathbb{A}, \mathbb{B}] = \mathbb{A}\mathbb{B} - \mathbb{B}\mathbb{A}$. Therefore, we will show that these differential operators generate a three-parameter Lie group.

The second order differential operator $\mathbb{L}$ satisfies the differential operator identity

$$(1 - x^2)\mathbb{L} = \mathbb{B}\mathbb{C} + \mathbb{A}^2 + (2A - I)\mathbb{A}.$$

By means of this identity and the commutator relations we prove that $(1 - x^2)\mathbb{L}$ commutes with each of the differential operators $\mathbb{A}$, $\mathbb{B}$, and $\mathbb{C}$,

$$[(1 - x^2)\mathbb{L}, \mathbb{A}] = [(1 - x^2)\mathbb{L}, \mathbb{B}] = [(1 - x^2)\mathbb{L}, \mathbb{C}] = \mathbf{0}.$$

Then for arbitrary constants b and c the differential operator $e^{c\mathbb{C}}e^{b\mathbb{B}}$ will transform solutions of $\mathbb{L}$ into solutions of $\mathbb{L}$; in other words,

$$e^{c\mathbb{C}}e^{b\mathbb{B}}(1 - x^2)\mathbb{L}C_n^A(x, y) = (1 - x^2)\mathbb{L}\Big(e^{c\mathbb{C}}e^{b\mathbb{B}}C_n^A(x, y)\Big) = \mathbf{0}.$$

if and only if $\mathbb{L}C_n^A(x, y) = \mathbf{0}$.

To accomplish our task of obtaining the generating matrix relations, we search for matrix function $f(x, y, A)$ and extended forms of transformation groups generated by differential operators $\mathbb{B}$ and $\mathbb{C}$ expressed as follows:

$$e^{b\mathbb{B}}f(x, y, A) = f\left(\frac{xy - b}{\sqrt{y^2 - 2bxy + b^2}}, \sqrt{y^2 - 2bxy + b^2}, A\right),$$

and

$$e^{c\mathbb{C}}f(x, y, A) = \Big(c^2y^2 - 2cxy + 1\Big)^{-A} f\left(\frac{x - cy}{\sqrt{c^2y^2 - 2cxy + 1}}, \frac{y}{\sqrt{c^2y^2 - 2cxy + 1}}, A\right),$$

where b, c are arbitrary constants and $f(x, y, A)$ is an arbitrary matrix function. We know that $\mathbb{B}$ and $\mathbb{C}$ commute operators and we find

$$e^{c\mathbb{C}}e^{b\mathbb{B}}[C_n^A(x)y^n] = \Big(c^2y^2 - 2cxy + 1\Big)^{-A} C_n^A(\xi)\eta^n, \tag{11}$$

where

$$\xi = \frac{(1 + 2bc)xy - c(1 + bc)y^2 - b}{\sqrt{c^2y^2 - 2cxy + 1}\sqrt{(1 + bc)^2y^2 - 2b(1 + bc)xy + b^2}},$$

and

$$\eta = \frac{\sqrt{(1 + bc)^2y^2 - 2b(1 + bc)xy + b^2}}{\sqrt{c^2y^2 - 2cxy + 1}}.$$

2.1 Generating Matrix Functions for Gegenbauer Matrix Polynomials

In this subsection, some special cases of the generating matrix functions for Gegenbauer matrix polynomials are derived from the differential operator $(\mathbb{A} - A\mathbb{I})$.

If we choose $b = 1$, $c = 0$ and $C_n^A(x, y) = C_n^A(x)y^n$ in (11), we find

$$e^{\mathbb{B}}[C_n^A(x)y^n] = \Big(y^2 - 2xy + 1\Big)^{\frac{1}{2}n} C_n^A\Big(\frac{xy-1}{\sqrt{y^2-2xy+1}}\Big).$$

By expanding this Ggegenbauer matrix polynomials, we get

$$\Big(y^2 - 2xy + 1\Big)^{\frac{1}{2}n} C_n^A\Big(\frac{xy-1}{\sqrt{y^2-2xy+1}}\Big) = \sum_{k=0}^{n} \frac{((1-n)I-2A)_k}{k!} C_{n-k}^A(x)y^{n-k}.$$

If we divide by y^n and let $t = \frac{1}{y}$, we get

$$\Big(1 - 2xt + t^2\Big)^{\frac{1}{2}n} C_n^A\Big(\frac{x-t}{\sqrt{1-2xt+t^2}}\Big) = \sum_{k=0}^{n} \frac{((1-n)I-2A)_k}{k!} C_{n-k}^A(x)t^k. \quad (12)$$

Secondly, if we choose $b = 0$ and $c = 1$, we get

$$e^{\mathbb{C}}[C_n^A(x)y^n] = y^n\Big(y^2 - 2xy + 1\Big)^{-A-\frac{1}{2}nI} C_n^A\Big(\frac{x-y}{\sqrt{y^2-2xy+1}}\Big).$$

If we expand this generating matrix function for Gegenbauer matrix polynomials and divide by y^n, we get the generating matrix relation

$$\Big(y^2 - 2xy + 1\Big)^{-A-\frac{1}{2}nI} C_n^A\Big(\frac{x-y}{\sqrt{y^2-2xy+1}}\Big) = \sum_{k=0}^{\infty} \frac{(n+k)!}{k!n!} C_{n+k}^A(x)y^k. \quad (13)$$

Thirdly, for $bc \neq 0$ we choose $b = -1$ and $c = 1$, (this choice is suggested by the frequency of occurrence in (11) of the factor $1 + bc$), we have

$$e^{c\mathbb{C}}e^{b\mathbb{B}}[C_n^A(x)y^n] = \Big(c^2y^2 - 2cxy + 1\Big)^{-A} C_n^A(\xi)\eta^n,$$

where $\xi = \frac{1-xy}{\sqrt{1-2xy+y^2}}$ and $\eta = \frac{1}{\sqrt{1-2xy+y^2}}$.

We expand this generating matrix function for Gegenbauer matrix polynomials as follows:

$$\Big(1 - 2xy + y^2\Big)^{-A-\frac{1}{2}nI} C_n^A\Big(\frac{1-xy}{\sqrt{1-2xy+y^2}}\Big) = \sum_{k=0}^{\infty} \frac{(2A+kI)_n}{k!} C_k^A(x)y^k. \quad (14)$$

If we let $\rho = \sqrt{1-2xy+y^2}$ we can rewrite (14) in the form

$$\rho^{-2A-nI} C_n^A\Big(\frac{1-xy}{\rho}\Big) = \sum_{k=0}^{\infty} \frac{(2A+kI)_n}{k!} C_k^A(x)y^k.$$

In order to express the left member of (14) in hypergeometric matrix functions form we use

$$C_n^A(x) = \frac{x^n}{n!}(2A)_n\, {}_2F_1\Big(-\frac{1}{2}nI, \frac{1}{2}(1-n)I; A + \frac{1}{2}nI; \frac{x^2-1}{x^2}\Big), \Big|\frac{x^2-1}{x^2}\Big| < 1.$$

Then after some simplification, Eq. (14) yields

$$\begin{aligned}&(1-2xy+y^2)^{-A-nI}(1-xy)^n\ {}_2F_1\Big(-\frac{1}{2}nI,\frac{1}{2}(1-n)I;A+\frac{1}{2}nI;\frac{y^2(x^2-1)}{(1-xy)^2}\Big)\\&=\sum_{k=0}^{\infty}(2A+nI)_k[(2A)_k]^{-1}C_k^A(x)y^k,\ \Big|\frac{y^2(x^2-1)}{(1-xy)^2}\Big|<1,\Big|xy\Big|<1.\end{aligned}\tag{15}$$

By applying the Theorem 1.1 and letting $B=2A+nI$, in the left member of (15), we obtain

$$\begin{aligned}&(1-xy)^{-B}\ {}_2F_1\Big(\frac{1}{2}B,\frac{1}{2}(B+I);A+\frac{1}{2}nI;\frac{y^2(x^2-1)}{(1-xy)^2}\Big)\\&=\sum_{k=0}^{\infty}(B)_k[(2A)_k]^{-1}C_k^A(x)y^k,\ \Big|\frac{y^2(x^2-1)}{(1-xy)^2}\Big|<1.\end{aligned}\tag{16}$$

2.2 Generating Matrix Functions Annulled by not Conjugate of $(\mathbb{A}-A\mathbb{I})$

In this subsection, the generating matrix functions for Gegenbauer matrix polynomials are derived from the differential operators not conjugate to $(\mathbb{A}-A\mathbb{I})$. The three generating matrix functions of (12), (13), and (14) have been obtained by transforming $C_n^A(x)y^n$ which is a solution of the system

$$\mathbb{L}C_n^A(x,y)=\mathbf{0}\quad\text{and}\quad(\mathbb{A}-n\mathbb{I})C_n^A(x,y)=\mathbf{0}.$$

If we wish to obtain additional generating matrix functions for the Gegenbauer matrix polynomials, we need to find differential operators which are not conjugate to $(\mathbb{A}-n\mathbb{I})$; i.e., we wish to find first order differential operators R such that for all choices of b and c;

$$e^{c\mathbb{C}}e^{b\mathbb{B}}(\mathbb{A}-n\mathbb{I})e^{-b\mathbb{B}}e^{-c\mathbb{C}}\neq R.$$

We take the set of linear differential operators $R=r_1\mathbb{A}+r_2\mathbb{B}+r_3\mathbb{C}+r_4\mathbb{I}$, for all combinations of zero and nonzero coefficients except for $r_1=r_2=r_3=0$. We find that

$$e^{c\mathbb{C}}e^{b\mathbb{B}}(\mathbb{A}-n\mathbb{I})e^{-b\mathbb{B}}e^{-c\mathbb{C}}=(1+2bc)\mathbb{A}+b\mathbb{B}-c(1+bc)\mathbb{C}+(2bcA-nI)\mathbb{I}.$$

Then for $r_1=1+2bc$, $r_2=b$, $r_3=c(1+bc)$, we have $r^2+4r_2r_3=1$.

Therefore, $\mathbb{A}-n\mathbb{I}$ is not conjugate to differential operators for which $r_1^2+4r_2r_3=0$ in the following cases:

If $r_1=0$, $r_2=1$, and $r_3=0$, we seek a solution of the system

$$\mathbb{L}u(x,y,A)=\mathbf{0}\quad\text{and}\quad(\mathbb{B}+\mathbb{I})u(x,y,A)=\mathbf{0}.$$

A solution of this system is

$$u(x,y,A)=e^{xy}\ {}_0F_1\Big(-;A+\frac{1}{2}I;\frac{y^2(x^2-1)}{4}\Big).$$

If we expand this matrix function, we get

$$e^{xy}\ {}_0F_1\left(-;A+\frac{1}{2}I;\frac{y^2(x^2-1)}{4}\right)=\sum_{k=0}^{\infty}[(2A)_k]^{-1}C_k^A(x)y^k. \tag{17}$$

For $r_1=0$, $r_2=0$, and $r_3\neq 0$, we seek a solution of the system

$$\mathbb{L}u=\mathbf{0} \quad \text{and} \quad (\mathbb{C}+\lambda\mathbb{I})u=\mathbf{0},$$

where λ is an arbitrary constant. We may avoid actually solving this system by noting that

$$e^{b\mathbb{B}}e^{c\mathbb{C}}(\mathbb{B}+\mathbb{I})e^{-c\mathbb{C}}e^{-b\mathbb{B}}=2c(1+bc)\mathbb{A}+(1+bc)^2\mathbb{I}-c^2\mathbb{C}+2c(1+bc)A\mathbb{I}+\mathbb{I}.$$

If we choose $b=1$ and $c=-1$, we get

$$e^{\mathbb{B}}e^{-\mathbb{C}}(\mathbb{B}+I)e^{\mathbb{C}}e^{-\mathbb{B}}=-\mathbb{C}+\mathbb{I}.$$

Therefore, we can obtain a solution of the system $\mathbb{L}u=\mathbf{0}$ and $(\mathbb{C}-\mathbb{I})u=\mathbf{0}$, by transforming the generating matrix functions of (17) as follows:

$$e^{B}e^{-C}e^{xy}\ {}_0F_1\left(-;A+\frac{1}{2}I;\frac{y^2(x^2-1)}{4}\right)=y^{-2A}\exp\left(\frac{y-x}{y}\right)\ {}_0F_1\left(-;A+\frac{1}{2}I;\frac{x^2-1}{4y^2}\right).$$

If we let $t=-\frac{1}{y}$, we get

$$e(-t)^{2A}e^{xt}\ {}_0F_1\left(-;A+\frac{1}{2}I;\frac{t^2(x^2-1)}{4}\right)$$

as our generating matrix function. But this matrix function differs only trivially from (17).

As applications, we now obtained many new and known generating matrix relations for the Gegenbauer matrix polynomials in the following:

$$\rho^nC_n^A\left(\frac{x-y}{\rho}\right)=\sum_{k=0}^{n}\frac{((1-n)I-2A)_k}{k!}C_{n-k}^A(x)y^k, \tag{18}$$

$$\rho^{-2A-nI}C_n^A\left(\frac{x-y}{\rho}\right)=\sum_{k=0}^{\infty}\frac{(k+n)!}{k!n!}C_{n+k}^A(x)y^k, \tag{19}$$

which is given in [8].

$$\rho^{-2A-nI}C_n^A\left(\frac{1-xy}{\rho}\right)=\sum_{k=0}^{\infty}\frac{(2A+kI)_n}{n!}C_k^A(x)y^k, \tag{20}$$

and

$$e^{xy}\ {}_0F_1\left(-;A+\frac{1}{2}I;\frac{y^2(x^2-1)}{4}\right)=\sum_{k=0}^{\infty}[(2A)_k]^{-1}C_k^A(x)y^k. \tag{21}$$

3 Group-theoretic Method for Modified Gegenbauer Matrix Polynomials

Here, we consider the modified Gegenbauer matrix polynomials $C_n^{A+nI}(x)$ which satisfy the following matrix differential equation:

$$
\begin{aligned}
&(1-x^2)\frac{d^2}{dx^2}C_n^{A+nI}(x) - x(2A+(2n+1)I)\frac{d}{dx}C_n^{A+nI}(x) \\
&+ n(2A+3nI)C_n^{A+nI}(x) = \mathbf{0}, n \geq 0.
\end{aligned} \tag{22}
$$

By using the following differential matrix recurrence relations

$$
(1-x^2)\frac{d}{dx}C_n^{A+nI}(x) = (2A+(3n-1)I)C_{n-1}^{A+nI}(x) + nxC_n^{A+nI}(x), \tag{23}
$$

and

$$
(1-x^2)\frac{d}{dx}C_n^{A+nI}(x) = (2A+3nI)xC_n^{A+nI}(x) - (n+1)C_{n+1}^{A+nI}(x). \tag{24}
$$

Replacing $\frac{d}{dx}$ by $\frac{\partial}{\partial x}$, n by $y\frac{\partial}{\partial y}$, and $C_n^{A+nI}(x)$ by $C_n^{A+nI}(x,y)$ in (22) we obtain the following a matrix partial differential equation:

$$
\begin{aligned}
&(1-x^2)\frac{\partial^2}{\partial x^2}C_n^{A+nI}(x,y) - (2A+(2n+1)I)x\frac{\partial}{\partial x}C_n^{A+nI}(x,y) \\
&+ y\frac{\partial}{\partial y}(2A+3y\frac{\partial}{\partial y}I)C_n^{A+nI}(x,y) = \mathbf{0}.
\end{aligned} \tag{25}
$$

Thus we see that $C_n^{A+nI}(x,y) = C_n^{A+nI}(x)y^n$ is a solution of the matrix partial differential equation Eq. (3.3), since $C_n^{A+nI}(x)$ is a solution of the matrix differential equation Eq. (22). We can rewrite (24) in the following form:

$$
\begin{aligned}
&(1-x^2)\frac{\partial^2}{\partial\, x^2}C_n^{A+nI}(x,y) + 3y^2\frac{\partial^2}{\partial\, y^2}C_n^{A+nI}(x,y) - (2A+I)x\frac{\partial}{\partial x}C_n^{A+nI}(x,y) \\
&- 2xy\frac{\partial}{\partial y\partial x}C_n^{A+nI}(x,y) + (2A+3I)y\frac{\partial}{\partial y}C_n^{A+nI}(x,y) = \mathbf{0}.
\end{aligned} \tag{26}
$$

We define the differential operators $\mathbb{I}$, $\mathbb{A}$, $\mathbb{B}$, and $\mathbb{C}$ as follows

$$
\mathbb{A} = y\frac{\partial}{\partial y}I,
$$

$$
\mathbb{B} = \frac{x^2-1}{y}\frac{\partial}{\partial x}I - x\frac{\partial}{\partial y}I,
$$

and

$$
\mathbb{C} = (x^2-1)y\frac{\partial}{\partial x}I + 3xy^2\frac{\partial}{\partial y}I + 2xyA.
$$

Next, we determine the following partial differential operators with the aid of $\mathbb{A}$, $\mathbb{B}$ and $\mathbb{C}$ the differential operators $\mathbb{A}$, $\mathbb{B}$ and $\mathbb{C}$ such that

$$\mathbb{A}\Big[C_n^{A+nI}(x)y^n\Big] = nC_n^{A+nI}(x)y^n,$$

$$\mathbb{B}\Big[C_n^{A+nI}(x)y^n\Big] = -(2A+(3n-1)I)C_{n-1}^{A+nI}(x)y^{n-1},$$

and

$$\mathbb{C}\Big[C_n^{A+nI}(x)y^n\Big] = (n+1)C_{n+1}^{A+nI}(x)y^{n+1},$$

where differential operators $\mathbb{A}$, $\mathbb{B}$, and $\mathbb{C}$ satisfy the commutator relations

$$[\mathbb{A},\mathbb{B}] = -\mathbb{B}, \quad [\mathbb{A},\mathbb{C}] = \mathbb{C}, \quad [\mathbb{B},\mathbb{C}] = -2\mathbb{A} - 2A\mathbb{I}. \tag{27}$$

Nota that: The set of linear combinations of the differential operators $\mathbb{I}$, $\mathbb{A}$, $\mathbb{B}$ and $\mathbb{C}$ forms a Lie algebra.

It can be easily shown that the partial differential operators in (24) $\mathbb{L}$ given by

$$\mathbb{L} = (1-x^2)\frac{\partial^2}{\partial x^2}I + 3y^2\frac{\partial^2}{\partial y^2}I - (2A+I)x\frac{\partial}{\partial x} - 2xy\frac{\partial}{\partial y\partial x}I + (2A+3I)y\frac{\partial}{\partial y}.$$

The second order differential operator $\mathbb{L}$ satisfies the differential operators identity as follows

$$(1-x^2)\mathbb{L} = \mathbb{B}\mathbb{C} + (2A+3\mathbb{A})(\mathbb{A}+\mathbb{I}). \tag{28}$$

It can be easily verified that $(1-x^2)\mathbb{L}$ commutes with each of the differential operators $\mathbb{A}$, $\mathbb{B}$ and $\mathbb{C}$,

$$[(1-x^2)\mathbb{L},\mathbb{A}] = [(1-x^2)\mathbb{L},\mathbb{B}] = [(1-x^2)\mathbb{L},\mathbb{C}] = \mathbf{0}. \tag{29}$$

The extended forms of transformation groups generated by differential operators $\mathbb{A}$, $\mathbb{B}$ and $\mathbb{C}$ are given by

$$e^{a\mathbb{A}}f(x,y,A) = f\Big(x, e^a y, A\Big), \tag{30}$$

$$e^{b\mathbb{B}}f(x,y,A) = f\left(\frac{xy-b}{\sqrt{y^2-2bxy+b^2}}, \sqrt{y^2-2bxy+b^2}, A\right), \tag{31}$$

and

$$e^{c\mathbb{C}}f(x,y,A) = \Big(c^2y^2-2cxy+1\Big)^{-A} f\left(\frac{x-cy}{\sqrt{c^2y^2-2cxy+1}}, \frac{y}{\sqrt{c^2y^2-2cxy+1}}, A\right), \tag{32}$$

where a, b and c are arbitrary constants and $f(x,y,A)$ is an arbitrary matrix function.

From the above relations the $\mathbb{A}$, $\mathbb{B}$ and $\mathbb{C}$ commute operators, we get

$$\begin{aligned} e^{c\mathbb{C}}e^{b\mathbb{B}}e^{a\mathbb{A}}f(x,y,A) = f\Bigg(&\frac{y(x-cy)-b(c^2y^2-2cxy+1)}{\sqrt{c^2y^2-2cxy+1}\sqrt{b^2(c^2y^2-2cxy+1)-2by(x-cy)+y^2}} \\ &, e^a\frac{\sqrt{b^2(c^2y^2-2cxy+1)-2by(x-cy)+y^2}}{(c^2y^2-2cxy+1)^{\frac{3}{2}}}, A\Bigg). \end{aligned} \tag{33}$$

3.1 Generating Matrix Functions for Modified Gegenbauer Matrix Polynomials

From (26), $C_n^{A+nI}(x,y) = C_n^{A+nI}(x)y^n$ is a solution of the system

$$\mathbb{L}C_n^{A+nI}(x,y) = \mathbf{0} \quad \text{and} \quad (\mathbb{A} - n\mathbb{I})C_n^{A+nI}(x,y) = \mathbf{0}.$$

From (29) we easily get

$$e^{c\mathbb{C}}e^{b\mathbb{B}}e^{a\mathbb{A}}(1-x^2)\mathbb{L}[C_n^{A+nI}(x)y^n] = (1-x^2)\mathbb{L}e^{c\mathbb{C}}e^{b\mathbb{B}}e^{a\mathbb{A}}[C_n^{A+nI}(x)y^n].$$

Therefore the transform $e^{c\mathbb{C}}e^{b\mathbb{B}}e^{a\mathbb{A}}[C_n^{A+nI}(x)y^n]$ is annulled by $(1-x^2)\mathbb{L}$.

If we choose $a = 0$ and $C_n^{A+nI}(x,y) = C_n^{A+nI}(x)y^n$ in (33), we get

$$\begin{aligned}
&e^{c\mathbb{C}}e^{b\mathbb{B}}[C_n^{A+nI}(x)y^n] \\
&= \left(b^2(c^2y^2 - 2cxy + 1) - 2by(x - cy) + y^2\right)^{\frac{1}{2}n}\left(c^2y^2 - 2cxy + 1\right)^{-(A+\frac{3}{2}nI)} \\
&\times C_n^{A+nI}\left(\frac{y(x-cy) - b(c^2y^2 - 2cxy + 1)}{\sqrt{c^2y^2 - 2cxy + 1}\sqrt{b^2(c^2y^2 - 2cxy + 1) - 2by(x-cy) + y^2}}\right).
\end{aligned} \tag{34}$$

On the other hand we get

$$e^{c\mathbb{C}}e^{b\mathbb{B}}[C_n^{A+nI}(x)y^n] = \sum_{m=0}^{\infty}\frac{c^m}{m!}\sum_{k=0}^{\infty}\frac{b^k}{k!}(n-k+1)_m((1-3n)I - 2A)_k y^{n-k+m}C_{n-k+m}^{A+nI}(x). \tag{35}$$

Equating (34) and (35), we get

$$\begin{aligned}
&\left(b^2(c^2y^2 - 2cxy + 1) - 2by(x - cy) + y^2\right)^{\frac{1}{2}n}\left(c^2y^2 - 2cxy + 1\right)^{-(A+\frac{3}{2}nI)} \\
&\times C_n^{A+nI}\left(\frac{y(x-cy) - b(c^2y^2 - 2cxy + 1)}{\sqrt{c^2y^2 - 2cxy + 1}\sqrt{b^2(c^2y^2 - 2cxy + 1) - 2by(x-cy) + y^2}}\right) \\
&= \sum_{m=0}^{\infty}\sum_{k=0}^{n}\frac{c^m b^k}{m!k!}(n-k+1)_m((1-3n)I - 2A)_k y^{n-k+m}C_{n-k+m}^{A+nI}(x).
\end{aligned} \tag{36}$$

Here, we obtain some interesting results as the particular case of generating matrix relations (36).

Putting $b = 1$, $c = 0$ and writing $y = t$ in (36) we get of generating matrix relations

$$\begin{aligned}
&\left(1 - 2xt + t^2\right)^{\frac{1}{2}n}C_n^{A+nI}\left(\frac{xt-1}{\sqrt{1-2xt+t^2}}\right) \\
&= \sum_{k=0}^{n}\frac{1}{k!}((1-3n)I - 2A)_k t^{n-k}C_{n-k}^{A+nI}(x).
\end{aligned} \tag{37}$$

Letting $b = 0$, $c = 1$ and $y = t$ in (36) we obtain

$$\begin{aligned}
&\left(t^2 - 2xt + 1\right)^{-(A+\frac{3}{2}nI)}C_n^{A+nI}\left(\frac{x-t}{\sqrt{t^2-2xt+1}}\right) \\
&= \sum_{m=0}^{\infty}\frac{1}{m!}(n+1)_m t^m C_{n+m}^{A+nI}(x).
\end{aligned} \tag{38}$$

Putting $b = -\frac{1}{b}$, $c = 1$ and substituting $y = t$ in (36), we get

$$\begin{aligned}&\left(\frac{1}{b^2}(t^2-2xt+1)+\frac{2}{b}t(x-t)+t^2\right)^{\frac{1}{2}n}\left(t^2-2xt+1\right)^{-(A+\frac{3}{2}nI)}\\&\times C_n^{A+nI}\left(\frac{t(x-t)+\frac{1}{b}(t^2-2xt+1)}{\sqrt{t^2-2xt+1}\sqrt{\frac{1}{b^2}(t^2-2xt+1)+\frac{2}{b}t(x-t)+t^2}}\right)\\&=\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\frac{c^m(-\frac{1}{b})^k}{m!k!}(n-k+1)_m((1-3n)I-2A)_k t^{n-k+m}C_{n-k+m}^{A+nI}(x).\end{aligned} \tag{39}$$

Now replacing A by $A - nI$ and $t = \frac{1}{t}$ in (37) we get

$$\left(1-2xt+t^2\right)^{\frac{1}{2}n}C_n^A\left(\frac{x-t}{\sqrt{1-2xt+t^2}}\right)=\sum_{k=0}^{n}\frac{1}{k!}((1-n)I-2A)_k t^k C_{n-k}^A(x). \tag{40}$$

Again on replacing A by $A - nI$ in (38) we get

$$\left(t^2-2xt+1\right)^{-(A+nI)}C_n^A\left(\frac{x-t}{\sqrt{t^2-2xt+1}}\right)=\sum_{m=0}^{\infty}\frac{1}{m!}(n+1)_m t^m C_{n+m}^A(x). \tag{41}$$

3.2 Generating Matrix Functions for Modified Gegenbauer Matrix Polynomials $C_{n+r}^{A-nI}(x)$

Here, we consider the following operator $\mathbb{D}$:

$$\mathbb{D}=(x^2-1)y\frac{\partial}{\partial x}I-2xy^2\frac{\partial}{\partial y}I+xy(2A-I), \tag{42}$$

such that

$$\mathbb{D}[C_{n+r}^{A-nI}(x)y^n]=\frac{1}{2}(n+r+1)((1+n-r)I-2A)((1+n)I-A)^{-1}C_{n+r+1}^{A-(n+1)I}(x)y^{n+1}. \tag{43}$$

The extended form of group generated by $\mathbb{D}$ is given as follows:

$$\begin{aligned}e^{d\mathbb{D}}f(x,y)=&\left(d^2y^2(x^2-1)+2dxy+1\right)^{A-\frac{1}{2}I}\\&\times f\left(x+dy(x^2-1),\frac{y}{\sqrt{d^2y^2(x^2-1)+2dxy+1}}\right),\end{aligned} \tag{44}$$

where d is an arbitrary constant. Using (44), we obtain

$$e^{d\mathbb{D}}[C_{n+r}^{A-nI}(x)y^n]=y^n\left(d^2y^2(x^2-1)+2dxy+1\right)^{A-(n+\frac{1}{2})I}C_{n+r}^{A-nI}\left(x+dy(x^2-1)\right). \tag{45}$$

By using (43), we obtain

$$\begin{aligned}e^{d\mathbb{D}}[C_{n+r}^{A-nI}(x)y^n]&=\sum_{k=0}^{\infty}\frac{d^k}{k!}\mathbb{D}^k[C_{n+r}^{A-nI}(x)y^n]\\&=\sum_{k=0}^{\infty}\frac{d^k}{k!}\frac{1}{2^k}(n+r+1)_k((1+n-r)I-2A)_k\left(((1+n)I-A)_k\right)^{-1}C_{n+r+k}^{A-(n+k)I}(x)y^{n+k}.\end{aligned} \tag{46}$$

Equating (45) and (46) then putting $t = \frac{1}{2}dy$, we get

$$\begin{aligned}&\Big(4t^2(x^2-1)+4xt+1\Big)^{A-(n+\frac{1}{2})I} C_{n+r}^{A-nI}\Big(x+2t(x^2-1)\Big)\\ &=\sum_{k=0}^{\infty}\frac{1}{k!}(n+r+1)_k((1+n-r)I-2A)_k\Big(((1+n)I-A)_k\Big)^{-1}C_{n+r+k}^{A-(n+k)I}(x)t^k.\end{aligned}\tag{47}$$

Putting $r = 0$ in (47), we get

$$\begin{aligned}&\Big(4t^2(x^2-1)+4xt+1\Big)^{A-(n+\frac{1}{2})I} C_{n}^{A-nI}\Big(x+2t(x^2-1)\Big)\\ &=\sum_{k=0}^{\infty}\frac{1}{k!}(n+1)_k((1+n)I-2A)_k\Big(((1+n)I-A)_k\Big)^{-1}C_{n+k}^{A-(n+k)I}(x)t^k.\end{aligned}\tag{48}$$

Putting $r = 0$ and replacing A by $A + nI$ in (47), we get

$$\begin{aligned}&\Big(4t^2(x^2-1)+4xt+1\Big)^{A-\frac{1}{2}I} C_{n}^{A}\Big(x+2t(x^2-1)\Big)\\ &=\sum_{k=0}^{\infty}\frac{1}{k!}(n+1)_k((1-n)I-2A)_k\Big((I-A)_k\Big)^{-1}C_{n+k}^{A-kI}(x)t^k.\end{aligned}\tag{49}$$

Putting $n = 0$ in (49), we get

$$\begin{aligned}&\Big(4t^2(x^2-1)+4xt+1\Big)^{A-\frac{1}{2}I} C_{n}^{A}\Big(x+2t(x^2-1)\Big)\\ &=\sum_{k=0}^{\infty}(I-2A)_k\Big((I-A)_k\Big)^{-1}C_{k}^{A-kI}(x)t^k.\end{aligned}\tag{50}$$

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Review of Number 27

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Abstaract − In this work, we review all papers that are published in Number 27 of the Journal of New Theory. We then introduce all of the members of the editorial board and reviewers of the papers in this issue.

Keywords − *Journal of New Theory, J. New Theory, JNT, Number 27.*

1 Number 27

We are happy to inform you that Number 27 of the Journal of New Theory (JNT) is completed with 9 articles.

In [1], the authors introduced the concept of anti fuzzy BG-ideals in BG-algebra and we have discussed some of their properties. Relation between anti fuzzy BG-ideal and cartessian product of anti fuzzy BG-ideals is developed.

In [2], existence and uniqueness of local classical solutions of the quasilinear evolution integrodifferential equation in Banach spaces are studied. The results are demonstrated by employing the fixed point technique on C_0-semigroup of bounded linear operator. At last, we deal an example to interpret the theory.

In [3], avian influenza epidemic model with drug resistance effect is investigated. The basic reproduction number $\mathbb{R}_0$ find out using next generation method. The local and global stability of a disease free and endemic equilibrium of the system is studied and discussed. Numerical simulations are carried out to investigate the influence of the key parameters on the spread of the disease, to support the analytical conclusion and illustrate possible behavioral scenarios of the model.

In [4], the authors defined a new operation on soft sets, called extended difference and investigate its relationship between extended difference and restricted difference and some other operations of soft sets. This study is based on the paper "On operations of Soft Sets" by Sezgin and Atagün [Comput. Math. Appl. 61 (2011) 1457-1467].

Editor-in-Chief of the Journal of New Theory.

In [5], the authors introduced and investigated the concepts of lightly nano ω-closed sets and lightly nano ω-open sets in a nano topological spaces, which are weaker form of lightly nano-closed sets and lightly nano-open sets and relationships among related ng-closed sets are investigated.

In [6], the authors redefined some basic operations of hesitancy fuzzy graph and it is referred as hesitancy fuzzy digraph (in short HFDG). They discussed some arithmetic operations and relations among HFDG. They further proposed a method to solve a shortest path problem through score function.

In [7], a suggestion for the calculation of Pythagorean Expectation for football is presented. In the application section, end-season rankings and points for the 2017/2018 season of the selected fifteen European football leagues are predicted by using the suggested method. The data of the past five seasons of the selected European football leagues is used as the training dataset. All calculations are performed in R.

In [8], the authors introduced the notion of (2,L)-double fuzzifying topology which is a generalization of the notion of (2,L)-fuzzifying topology and classical topology. They defined the notions of (2,L)-double fuzzifying preproximity and (2,L)-fuzzifying syntopogenous structures. Some fundamental properties are also established. These concepts will help in verifying the existing characterizations and also help in achieving new and generalized results. Finally they studied a model as an application of fuzzifying topology in biology.

In [9], the authors derived the generating formulae for the Gegenbauer and modified Gegenbauer matrix polynomials by introducing a partial differential operator and constructing the Lie algebra representation formalism of special linear algebra by using Weisner's group-theoretic approach. An application of this results is also pointed out.

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We would like to express our deepest thanks to all of the members of the editorial board and reviewers of the papers in this issue who are İ. Deli, F. Karaaslan, F. Smarandache, H. M. Srivastava, M. A. Noor, B. Davvaz, J. Zhan, H. Garg, S. Broumi, S. Pramanik, M. I. Ali, P. K. Maji, O. Muhtaroğlu, A. A. Ramadan, S. Enginoğlu, S. J. John, M. Ali, A. Sezgin, A. M. A. Latif, M. Sarı, J. Ye, D. Mohamad, İ. Zorlutuna, A. Shehata, K. Aydemir, T. Biswas, S. Demiriz, H. Olğar, A. Boussayoud, E. H. Hamouda, K. Mondal, T. Manikantan, A. A. Nasef, D. Sabancı, V. Biju.

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We hope you will enjoy this issue of JNT. We are looking forward to hearing your feedback and receiving your contributions.

Happy reading!

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