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CONTENT

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Necessary Condition for Vector-Valued Model Spaces to be Invariant Under Conjugation

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Original Article

Abstract − The S^*-invariant subspaces of the Hardy-Hilbert space $H^2(E)$ (where E is finite dimensional Hilbert space of dimension greater than 1) on the unit disc is well known. In this study, we examine that, if Ω is a conjugation on E, and Θ an inner function, then there exist model spaces which are not invariant for the conjugation $C_\Omega : L^2(E) \longrightarrow L^2(E)$. Under what necessary condition the model spaces is mapped onto itself is under consideration.

Keywords − *Inner function, model spaces, conjugation.*

1. Introduction and Preliminaries

Let $\mathbb{D}$ denote the open unit disc and $\mathbb{T}$ the unit circle in the complex plane $\mathbb{C}$. Throughout the paper E will denote a fixed Hilbert space, of finite dimension d, and $\mathcal{L}(E)$ the algebra of bounded linear operators on E, which may be identified with $d \times d$ matrices. $\mathcal{L}(E)$ is Hilbert space endowed with Hilbert-Schmidt norm. $H^2(E)$ is the Hardy-Hilbert of E-valued analytic functions on $\mathbb{D}$ whose coefficients are square summable, which is a closed subspace of $L^2(E)$.

The space $L^2(E)$ is defined, as usual, by

$$L^2(E) = \left\{ f : \mathbb{T} \longrightarrow E : f(e^{it}) = \sum_{-\infty}^{\infty} a_n e^{int},\ a_n \in E\ , \sum_{-\infty}^{\infty} \|a_n\|_E^2 < \infty \right\}.$$

The inner product on $L^2(E)$ is defined by

$$\langle f, g \rangle = \frac{1}{2\pi} \int_0^{2\pi} \langle f(e^{it}), g(e^{it}) \rangle_E dt, \tag{1}$$

and $L^2(E)$ can be orthogonally decompose as

$$L^2(E) = H^2_-(E) \oplus H^2(E),$$

where $H^2_-(E)$ is the orthogonal complement of $H^2(E)$ in $L^2(E)$ with inner product defined in (1). For $f \in H^2(E)$, $f(z)$ and $f(e^{it})$ determine each other.

It is important to note that, if $dimE = 1$ (i.e $E = \mathbb{C}$) then $L^2(E)$ consists of the scalar valued functions and is denoted by $L^2(\mathbb{T})$, and all the results become trivial in that case.

By viewing $\mathcal{L}(E)$ as Hilbert space (endowed with the Hilbert Schmidt norm), one can also consider the space $L^2(\mathcal{L}(E))$, which may be identified with the matrices whose entries are from $L^2(\mathbb{T})$.

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Alternately, we may view $L^2(\mathcal{L}(E))$ also as a space of square summable Fourier series with coefficients in $\mathcal{L}(E)$.

The space $H^2(\mathcal{L}(E))$ is a closed subspace of $L^2(\mathcal{L}(E))$ whose Fourier coefficients corresponding to negative indices vanishes. We have an orthogonal decomposition

$$L^2(\mathcal{L}(E)) = [zH^2(\mathcal{L}(E))]^* \oplus H^2(\mathcal{L}(E)).$$

The unilateral shift (see [6]) $S : H^2(E) \longrightarrow H^2(E)$ is defined by $Sf = zf$, and its adjoint S^* (backward shift) is given by the formula;

$$S^* f = \frac{f - f(0)}{z}.$$

After gathering the facts in preliminaries, we will present main results in the next section. An effort has been made to make the paper self-contained.

2. Formulation and Basic Results

Definition 2.1. An inner function is an element $\Theta \in H^2(\mathcal{L}(E))$ whose boundary values are almost everywhere unitary operators in $\mathcal{L}(E)$.

Definition 2.2. A conjugation is a conjugate-linear operator $C : \mathcal{H} \longrightarrow \mathcal{H}$ that satisfies the conditions

1. C is isometric: $\langle Cf, Cg\rangle = \langle g, f\rangle \ \forall f, g \in \mathcal{H}$,
2. C is involutive: $C^2 = I$.

Model space associated to an inner function Θ, is denoted by K_Θ, and is defined by

$$K_\Theta = H^2(E) \ominus \Theta H^2(E).$$

Just like the Beurling-type subspace $\Theta H^2(E)$ constitute nontrivial invariant subspace for the unilateral shift S, the subspace K_Θ plays an analogous role for the backward shift S^*.

For a given inner function Θ, and Ω a conjugation on E, the map $C_\Omega : L^2(E) \longrightarrow L^2(E)$, defined by

$$(Cf)(e^{it}) = \Theta(e^{it})\overline{e^{it}}\Omega f(e^{it})$$

is a conjugation. It is worth noting that C_Ω does not preserve the model spaces in general. However this is true for $dimE = 1$ (see [6]) . Under what condition the model spaces is invariant under the conjugation C_Ω, this we will study in the next section.

3. Main Results

Example 3.1. If $\Theta(e^{it})^* \neq \Omega\Theta(e^{it})\Omega$ then $C_\Omega K_\Theta \nsubseteq K_\Theta$.

Let $E = \mathbb{C}^2$ and

$$\Theta(z) = \begin{pmatrix} z & 0 \\ 0 & z^2 \end{pmatrix} \in H^2(\mathcal{L}(\mathbb{C}^2)),$$

then

$$\Theta H^2(\mathbb{C}^2) = \left\{ \begin{pmatrix} zf \\ z^2 g \end{pmatrix} : f, g \in H^2 \right\},$$

and the model space associated to Θ is

$$K_\Theta = [\Theta H^2(\mathbb{C}^2)]^\perp = \left\{ \begin{pmatrix} f_0 \\ g_0 + g_1 z \end{pmatrix} : f_0, g_0, g_1 \in \mathbb{C} \right\}.$$

Let $\Omega : \mathbb{C}^2 \longrightarrow \mathbb{C}^2$ be defined by

$$\Omega \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} \overline{a_2} \\ \overline{a_1} \end{pmatrix}$$

is a conjugation.

Now consider

$$\Theta(e^{it})^* \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} \overline{z} & 0 \\ 0 & \overline{z}^2 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} a_1\overline{z} \\ a_2\overline{z}^2 \end{pmatrix} \tag{2}$$

and

$$\Omega\Theta\Omega \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \Omega \begin{pmatrix} z & 0 \\ 0 & z^2 \end{pmatrix} \begin{pmatrix} \overline{a_2} \\ \overline{a_1} \end{pmatrix} = \Omega \begin{pmatrix} \overline{a_2}z \\ \overline{a_1}z^2 \end{pmatrix} = \begin{pmatrix} a_1\overline{z}^2 \\ a_2\overline{z} \end{pmatrix} \neq \Theta(e^{it})^* \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}. \tag{3}$$

Now

$$C_\Omega \begin{pmatrix} 0 \\ z \end{pmatrix} = \overline{z} \begin{pmatrix} z & 0 \\ 0 & z^2 \end{pmatrix} \Omega \begin{pmatrix} 0 \\ z \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & z \end{pmatrix} \begin{pmatrix} \overline{z} \\ 0 \end{pmatrix} = \begin{pmatrix} \overline{z} \\ 0 \end{pmatrix} \notin K_\Theta.$$

Theorem 3.2. If Ω is a conjugation on E. Suppose that for the inner function Θ, we have $\Theta(e^{it})^* = \Omega\Theta(e^{it})\Omega$. Then $C_\Omega K_\Theta = K_\Theta$.

PROOF. Let $f \in K_\Theta$ and $h \in H^2(E)$, then

$$\begin{aligned}
\langle C_\Omega f, \Omega(zh)\rangle &= \frac{1}{2\pi}\int_0^{2\pi} \langle \Theta(e^{it})e^{-it}\Omega f(e^{it}), e^{-it}\Omega h(e^{it})\rangle dt = \frac{1}{2\pi}\int_0^{2\pi} \langle \Omega\Theta(e^{it})^*\Omega f(e^{it}), \Omega h(e^{it})\rangle dt \\
&= \frac{1}{2\pi}\int_0^{2\pi} \langle \Omega\Theta(e^{it})^* f(e^{it}), \Omega h(e^{it})\rangle dt = \frac{1}{2\pi}\int_0^{2\pi} \langle h(e^{it}), \Theta(e^{it})^* f(e^{it})\rangle dt \\
&= \frac{1}{2\pi}\int_0^{2\pi} \langle \Theta(e^{it})h(e^{it}), f(e^{it})\rangle dt = \langle \Theta h, f\rangle = 0.
\end{aligned}$$

This proves that $C_\Omega f \perp H^2_-(E)$. Next we will prove that $C_\Omega f \perp \Theta H^2(E)$. For this consider

$$\begin{aligned}
\langle C_\Omega f, \Theta z^n x\rangle &= \frac{1}{2\pi}\int_0^{2\pi} \langle \Theta(e^{it})e^{-it}\Omega f(e^{it}), \Theta(e^{it})e^{int}x\rangle dt \\
&= \frac{1}{2\pi}\int_0^{2\pi} \langle e^{-it}\Omega f(e^{it}), e^{int}x\rangle dt = \frac{1}{2\pi}\int_0^{2\pi} \langle \Omega f(e^{it}), e^{i(n+1)t}x\rangle dt \\
&= \frac{1}{2\pi}\int_0^{2\pi} \langle \Omega e^{i(n+1)t}x, f(e^{it})\rangle dt = \langle \Omega z^{n+1}x, f\rangle \\
&= 0.
\end{aligned}$$

Here we have used the fact that $\Omega z^{n+1}x \in H^2_-(E)$. This shows that $C_\Omega K_\Theta \subset K_\Theta$ and this combined with $C_\Omega^2 = I$ follows that $C_\Omega K_\Theta = K_\Theta$. □

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An Approach for Multi-Criteria Decision Making Problems with Unknown Weight Information Using GRA Method Under the Picture Linguistic Fuzzy Environment

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Abstract − Picture linguistic fuzzy set is the generalize structure over existing structures of fuzzy linguistic sets to arrange uncertainty and imprecise information in decision making problems. Viewing the effectiveness of the picture linguistic fuzzy set, we developed a decision-making approach for the multi-criteria decision-making problems. We also proposed the GRA technique using Choquet integral deal- ing uncertainty in decision making problems under picture linguistic fuzzy information. Lastly, we illustrate an example to shows the effectiveness and reliability of the developed method.

Keywords − *Picture linguistic fuzzy set, Picture linguistic fuzzy Choquet integral weighted averaging (PLF-CIWA) operator, GRA method.*

1. Introduction

Fuzzy set theory concept was first time defined by Zadeh [1]. Fuzzy set are only defined membership function, but more times, it difficult to express more fuzzy information. To deal successfully with something difficult, Attanssov [2] defined the intuitionistic fuzzy set, the development of FS, which included the non-membership degree. After that Attanssov defined interval valued IFS by approaching the positive degree and negative degree to interval number [3–5], and the operational laws and comparison rules for the IvIFSs are defined. The IvIFS illustrate the fuzzy information and is more descriptive than the FS and IFS. Wang and Liu defined some geometric and averaging aggregation operators for different IFNs. Latterly, some multi-criteria decision making problems have also proposed which depend on IFS [6, 7].

Murofushi and Sugeno [8] proposed the notion of Choquet integral with respect to a fuzzy measure. It was defined by Choquet [9] in potential theory with the notion of capacity. The generalization of the classical Lebesgue integral are Choquet integral and has been tested to many other field. Choquet integral are used in many areas like as image processing, pattern recognition, information fusion and data mining [10, 11], and also utilized in economic theory [12, 13], in the context of fuzzy measure theory [14, 15]. Sugeno integral is the other important kind of fuzzy integral, and are introduced by Sugeno [16]. Sugeno integral on the fuzzy sets are generalized by Wang and Qiao [17, 18]. Yu et al. [19] proposed the Choquet integral operator to aggregate the hesitant fuzzy information for MCDM problems. Zhou [20] extended intuitionistic fuzzy Choquet integral correlation coefficient on the base

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of Shapley index. Li et al. [21] proposed the Generalized Interval Neutrosophic Choquet Aggregation Operators.

The concepts of picture fuzzy set was proposed by Coung and investigated basic operators and properties of (PFS) [22]. Picture fuzzy set are basically development of Atanassoves intuitionistic fuzzy set, which represent a membership, neutral membership and a nonmembership degree, is a very strong tool to represent vague and an uncertain information in the process of clustering analysis. When we looksome issue, which have more answers like as: yes, abstain, no, and refusal, in this case we used picture fuzzy numbers. To deal with clustering problems under the picture fuzzy environment Son [23] give the concept of generalized picture fuzzy distance measure. The decision making art are proposed by Wei [24], which is depended on the picture fuzzy weight cross-entropy. Ashraf et al. [25] proposed the series of aggregation opertors for picture fuzzy information. Zeng et al. [26] proposed the liguistic picture fuzzy TOPSIS method for picture fuzzy information. For more study, we refer to [27–32, 53–56].

Moreover, in decision position assessment are given by linguistic terms which is a linguistic values of a linguistic variable. The great deal of qualitative information arise in real decision making problem. Which is simply convert by linguistic terms, like as "very good","good", "fair","bad" and "very bad", etc. In some earlier application, linguistic terms were described for triangular fuzzy numbers [33, 34], trapezoidal fuzzy numbers [35, 36]. The notion of Intuitionistic linguistic set are given by Wang and Li [37], and also derived some decision making methods with ILNs. Pei et al. [38] proposed linguistic weighted aggregation operator for fuzzy risk analysis. Based on dependent operator Liu [39] derived the intuitionistic linguistic generalized dependent ordered weighted averaging operator and intuitionistic linguistic generalized dependent hybrid weighted aggregation operator. Wang et al. [40] defined the comparison rules, score function and accuracy function between two intuitionistic linguistic numbers. Wang [41] developed an ILPGWA operator and ILPGOWA operator based on power operator, and explain some individual cases of these operators with respect to the generalized criterion. Chen et al. [42] introduced the new notion of linguistic intuitionistic fuzzy number. Liu et al. [43] introduced a new linguistic term transformation tecnique in linguistic decision making. Based on Einstein T-norm and T-conorm Liu and You [44] proposed some linguistic intuitionistic fuzzy Heronian mean operators. Due to the motivation and inspiration of the above study in this article, we defined picture linguistic fuzzy set (PLFS), which is generalized form of intuitionistic linguistic fuzzy set. The application of this paper is to introduced the notion of GRA methodology for solving MADM problems under picture linguistic fuzzy information, in which the data about criteria weights are completely unknown, and the criteria values occur in the form of picture linguistic fuzzy numbers.

In preliminaries, we shortly review basic definitions and results about Choquet integral, intuitionistic linguistic fuzzy sets and picture linguistic fuzzy sets. In Section 3, we proposed the concept of picture linguistic fuzzy sets, and also introduce the GRA method for picture linguistic fuzzy MAGDM problems with incomplete weight information in Section 4. In Section 5, we illustrate our introduced algorithm with an example. In Section 6 are conclusion.

2. Preliminary

Some basic definition and notations of IFSs, ILFSs, PFS, PLFS and their operations are discussed. The concept of fuzzy measure and Choquet integral are aslo studied.

Definition 2.1. [45, 46] Let $\acute{L} = \{\pounds_p | p = 0, 1, ..., \ell - 1\}$ be the linguistic set, where as the cardinality of this set is considered as odd number, i.e. a five linguistic terms set $\acute{L}$ can be designate as;

$$\begin{aligned} \acute{L} &= (\pounds_0, \pounds_1, \pounds_2, \pounds_3, \pounds_4) \\ &= \{poor, slightly\ poor,\ fair,\ slightly\ good, good\}. \end{aligned}$$

Definition 2.2. The negation operator: neg $(\acute{L}_p) = \acute{L}_q$, where $q = \ell - 1$;
(1) Be ordered: $\pounds_p \leq \pounds_q \Longleftrightarrow p \leq q$;
(2)Maximum operator: $\max(\pounds_p, \pounds_q) = \pounds_p$ if $\pounds_p \geq \pounds_q$;
(3) Minimum operator: $\min(\pounds_p, \pounds_q) = \pounds_p$ if $\pounds_p \leq \pounds_q$.

$\pounds_{[0,\ell]} = \{\pounds_p | \pounds_0 \leq \pounds_p \leq \pounds_\ell\}$, whose elements also get all the characteristics above, and if $\pounds_p \in \acute{L}$,it is known as the actual term, otherwise, virtual term.

To make something stay the same, Herrera et al. [47] suggest that the distinct linguistic term set $\acute{L} = (\pounds_0 \ , \pounds_1, ..., \pounds_{\ell-1})$ is expended to a continuous linguistic term set $\acute{L} = (\pounds_\theta | \theta \in (0, G)$, where G sufficiently large positive number which satisfies the upper characteristics. For any linguistic variables $\pounds_p, \pounds_q \in \acute{L}$, the following condition ae satisfied.

1. $\varpi \otimes \pounds_p = \pounds_{\varpi \cdot p}$
2. $\pounds_p \oplus \pounds_q = \pounds_{p+q}$
3. $\pounds_p / \pounds_q = \pounds_{p/q}$
4. $(\pounds_p)^q = \pounds_{k^q}$

Definition 2.3. [2] An IFS $E_{\breve{u}}$ on the universal set $\mathbb{R} \neq \phi$ is defined as;

$$E_{\breve{u}} = \{\langle P_{\breve{e}_{\breve{u}}}(r), I_{\breve{e}_{\breve{u}}}(r) |\ r \in \mathbb{R} \rangle\},$$

where $P_{\breve{e}_{\breve{u}}}(r) : \mathbb{R} \to [0,1]$ and $I_{\breve{e}_{\breve{u}}}(r) : \mathbb{R} \to [0,1]$ are the membership and non-membership degree of each $r \in \mathbb{R}$, respectively. Moreover $P_{\breve{e}_{\breve{u}}}(r)$ and $I_{\breve{e}_{\breve{u}}}(r)$ satisfy this condition $0 \leq P_{\breve{e}_{\breve{u}}}(r) + I_{\breve{e}_{\breve{u}}}(r) \leq 1 \forall r \in \mathbb{R}$.

Definition 2.4. [48] Let $\mathbb{R} \neq \phi$ be the universe of discourse. Then $E_{\breve{u}}$ is defined as;

$$E_{\breve{u}} = \{\langle \pounds_{\breve{e}_{\breve{u}}}(r), P_{\breve{e}_{\breve{u}}}(r), I_{\breve{e}_{\breve{u}}}(r) |\ r \in \mathbb{R} \rangle\},$$

an ILFS in a set $\mathbb{R}$ is denoted by $\pounds_{\breve{e}_{\breve{u}}}(r) \in L$ are the linguistic term, $P_{\breve{e}_{\breve{u}}}(r)$ and $I_{\breve{e}_{\breve{u}}}(r) : \mathbb{R} \to [0,1]$ be the membership and non-membership of each $r \in \mathbb{R}$, respectively. Moreover $P_{\breve{e}_{\breve{u}}}(r)$ and $I_{\breve{e}_{\breve{u}}}(r)$ satisfy this condition$0 \leq P_{\breve{e}_{\breve{u}}}(r) + I_{\breve{e}_{\breve{u}}}(r) \leq 1$ for all $r \in \mathbb{R}$.

Definition 2.5. [22] A PFS $E_{\breve{u}}$ on the universal set $\mathbb{R} \neq \phi$ is defined as;

$$E_{\breve{u}} = \{\langle P_{\breve{e}_{\breve{u}}}(r), I_{\breve{e}_{\breve{u}}}(r), N_{\breve{e}_{\breve{u}}}(r) |\ r \in \mathbb{R} \rangle\}.$$

where $P_{\breve{e}_{\breve{u}}}(r) : \mathbb{R} \to [0,1]$, $I_{\breve{e}_{\breve{u}}}(r) : \mathbb{R} \to [0,1]$ and $N_{\breve{e}_{\breve{u}}}(r) : \mathbb{R} \to [0,1]$ are the positive membership, neutral membership and negative membership of each $r \in \mathbb{R}$, respectively. Furthermore, $P_{\breve{e}_{\breve{u}}}(r)$, $I_{\breve{e}_{\breve{u}}}(r)$ and $N_{\breve{e}_{\breve{u}}}(r)$ satisfy this condition $0 \leq P_{\breve{e}_{\breve{u}}}(r) + I_{\breve{e}_{\breve{u}}}(r) + N_{\breve{e}_{\breve{u}}}(r) \leq 1 \forall r \in \mathbb{R}$.

2.1. Fuzzy measure and Choquet integral

Definition 2.6. [9] Let $\mathbb{R} = \{r_1, r_2, ..., r_n\} \neq \phi$ be the universe of discourse and $p(\mathbb{R})$ denotes the power set of $\mathbb{R}$. Then, a fuzzy measure $P_{\breve{e}_{\breve{u}}}$ on $\mathbb{R}$ is a mapping $P_{\breve{e}_{\breve{u}}}$: p($\mathbb{R}$) $\to [0,1]$, satisfying the subsequent conditions;

1) $P_{\breve{e}_{\breve{u}}}(\phi) = 0$, $P_{\breve{e}_{\breve{u}}}(\mathbb{R}) = 1$.

2) If $E_{\breve{u}_1}, E_{\breve{u}_2} \in p(\mathbb{R})$ and $E_{\breve{u}_1} \subseteq E_{\breve{u}_2}$ then $P_{\breve{e}_{\breve{u}}}(E_{\breve{u}_1}) \leq P_{\breve{e}_{\breve{u}}}(E_{\breve{u}_2})$.

Where $P_{\breve{e}_{\breve{u}}}(\{r_1, r_2, ..., r_n\})$ can be considered as the grade of subjective importance of decision criteria set $\{r_1, r_2, ..., r_n\}$. Thus, with the separate weights of criterias can also be defined. Naturally, we could say the following about any pair of criterias sets $E_{\breve{u}_1}, E_{\breve{u}_2} \in p(\mathbb{R})$, $E_{\breve{u}_1} \cap E_{\breve{u}_2} = \phi$; $E_{\breve{u}_1}$and $E_{\breve{u}_2}$ are considered to be without interaction if

$$P_{\breve{e}_{\breve{u}}}(E_{\breve{u}_1} \cup E_{\breve{u}_2}) = P_{\breve{e}_{\breve{u}}}(E_{\breve{u}_1}) + P_{\breve{e}_{\breve{u}}}(E_{\breve{u}_2}) \tag{1}$$

which is known as additive measure. $E_{\breve{u}_1}$and $E_{\breve{u}_2}$ exhibit a positive synergetic interaction between them (or are complementary) if

$$P_{\breve{e}_{\breve{u}}}(E_{\breve{u}_1} \cup E_{\breve{u}_2}) > P_{\breve{e}_{\breve{u}}}(E_{\breve{u}_1}) + P_{\breve{e}_{\breve{u}}}(E_{\breve{u}_2}) \tag{2}$$

which is called a superadditive measure. $E_{\breve{u}_1}$and $E_{\breve{u}_2}$ exhibit a negative synergetic interaction between them (or redundant or substitutive) if

$$P_{\breve{e}_{\breve{u}}}(E_{\breve{u}_1} \cup E_{\breve{u}_2}) < P_{\breve{e}_{\breve{u}}}(E_{\breve{u}_1}) + P_{\breve{e}_{\breve{u}}}(E_{\breve{u}_2}) \tag{3}$$

known as sub-additive measure.

Since it is difficult to find the fuzzy measure according to Definition 2.6, therefore, to confirm a fuzzy measure in MAGDM problems, Sugeno [16] given below, λ-fuzzy measure:

$$P_{\breve{e}_{\breve{u}}}(E_{\breve{u}_1} \cup E_{\breve{u}_2}) = P_{\breve{e}_{\breve{u}}}(E_{\breve{u}_1}) + P_{\breve{e}_{\breve{u}}}(E_{\breve{u}_2}) + \lambda P_{\breve{e}_{\breve{u}}}(E_{\breve{u}_1}) P_{\breve{e}_{\breve{u}}}(E_{\breve{u}_2}) \tag{4}$$

$\lambda \in [-1, \infty)$, $E_{\breve{u}_1} \cap E_{\breve{u}_2} = \phi$. The interaction between the criterias are determines the parameter λ . If we put $\lambda = 0$, in Equation 4, then, λ-fuzzy measure become an additive measure. And for negative and positive λ, the λ-fuzzy measure reduces to subadditive and superadditive measures, respectively. Meantime, if all the elements in $\mathbb{R}$ are independent, and we have

$$P_{\breve{e}_{\breve{u}}}(E_{\breve{u}}) = \sum_{p=1}^{n} P_{\breve{e}_{\breve{u}}}(\{r_p\}) \tag{5}$$

If we consider $\mathbb{R}$ is a finite set, then $\cup_{\mathrm{p}=1}^{n} r_{\mathrm{p}} = \mathbb{R}$, and λ-fuzzy measure $P_{\breve{e}_{\breve{u}}}$ satisfies following Equation 6

$$P_{\breve{e}_{\breve{u}}}(\mathbb{R}) = P_{\breve{e}_{\breve{u}}}\left(\cup_{p=1}^{n} r_i\right) = \begin{cases} \frac{1}{\lambda}\left(\prod_{p=1}^{n} [1 + \lambda P_{\breve{e}_{\breve{u}}}(r_p)] - 1\right) & \text{if } \lambda \neq 0 \\ \sum_{p=1}^{n} P_{\breve{e}_{\breve{u}}}(r_p) & \text{if } \lambda = 0 \end{cases} \tag{6}$$

where $r_{\mathrm{p}} \cap r_{\check{\mathrm{d}}} = \phi$ for all $p, \check{\mathrm{d}} = 1, 2, ..., n$ and $p \neq \check{\mathrm{d}}$. A fuzzy density $P_{\breve{e}_{\breve{u}}}(r_{\mathrm{p}})$ for a subset with a single element r_{p} is denoted as $P_{\breve{e}_{\breve{u}_{\mathrm{p}}}} = P_{\breve{e}_{\breve{u}}}(r_{\mathrm{p}})$.

Especially for every subset $E_{\breve{u}_1} \in p(\mathbb{R})$, we have

$$P_{\breve{e}_{\breve{u}}}(E_{\breve{u}_1}) = \begin{cases} \frac{1}{\lambda}\left(\prod_{p=1}^{n} [1 + \lambda P_{\breve{e}_{\breve{u}}}(r_p)] - 1\right) & \text{if } \lambda \neq 0 \\ \sum_{p=1}^{n} P_{\breve{e}_{\breve{u}}}(r_p) & \text{if } \lambda = 0 \end{cases} \tag{7}$$

Based on Equation 2, we determined the value of λ from $P_{\breve{e}_{\breve{u}}}(\mathbb{R}) = 1$, and is equal to solved this equation;

$$\lambda + 1 = \prod_{p=1}^{n} \left[1 + \lambda P_{\breve{e}_{\breve{u}_p}}\right] \tag{8}$$

It should be recognized that the value of λ can be uniquely determined by $P_{\breve{e}_{\breve{u}}}(\mathbb{R}) = 1$.

Definition 2.7. [16] Assume that f and $P_{\breve{e}_{\breve{u}}}$ be a positive real-valued function and fuzzy measure on $\mathbb{R}$, respectivily. The discrete Choquet integral of f with respect to $P_{\breve{e}_{\breve{u}}}$ is defined by

$$C_{\mu}(f) = \sum_{p=1}^{n} f_{\rho(p)}[P_{\breve{e}_{\breve{u}}}(A_{\rho(p)}) - P_{\breve{e}_{\breve{u}}}(A_{\rho(p\text{-}1)})] \tag{9}$$

$\rho(p)$ indicates a permutation on $\mathbb{R}$, where $f_{\rho(1)} \geq f_{\rho(2)} \geq ... \geq f_{\rho(n)}$, $A_{\rho(n)} = \{1, 2, ..., p\}$, $A_{\rho(0)} = \phi$.

3. Linguistic Picture Fuzzy Set and their Operations

We discussed in this section linguistic picture fuzzy set concept and their operationals laws.

Definition 3.1. [50] Let $\mathbb{R} \neq \phi$ be a universal set. Then, $E_{\breve{u}}$ is called a picture linguistic set, and defined as;

$$E_{\breve{u}} = \{\langle \pounds_{\breve{e}_{\breve{u}}}(r), P_{\breve{e}_{\breve{u}}}(r), I_{\breve{e}_{\breve{u}}}(r), N_{\breve{e}_{\breve{u}}}(r) |\ r \in \mathbb{R} \rangle\},$$

where $\pounds_{\breve{e}_{\breve{u}}}(r) \in L$, $P_{\breve{e}_{\breve{u}}}(r) : \mathbb{R} \to [0, 1]$, $I_{\breve{e}_{\breve{u}}}(r) : \mathbb{R} \to [0, 1]$ and $N_{\breve{e}_{\breve{u}}}(r) : \mathbb{R} \to [0, 1]$ are the linguistic term, the positive, neutral and negative membership degrees of each $r \in \mathbb{R}$, respectively. Furthermore $P_{\breve{e}_{\breve{u}}}(r)$, $I_{\breve{e}_{\breve{u}}}(r)$ and $N_{\breve{e}_{\breve{u}}}(r)$ satisfy that $0 \leq P_{\breve{e}_{\breve{u}}}(r) + I_{\breve{e}_{\breve{u}}}(r) + N_{\breve{e}_{\breve{u}}}(r) \leq 1\ \forall\ r \in \mathbb{R}$.

Definition 3.2. Let $E_{\breve{u}_1} = \left\langle \pounds_{\breve{e}_{\breve{u}_1}}, P_{\breve{e}_{\breve{u}_1}}, I_{\breve{e}_{\breve{u}_1}}, N_{\breve{e}_{\breve{u}_1}} \right\rangle$ and $E_{\breve{u}_2} = \left\langle \pounds_{\breve{e}_{\breve{u}_2}}, P_{\breve{e}_{\breve{u}_2}}, I_{\breve{e}_{\breve{u}_2}}, N_{\breve{e}_{\breve{u}_2}} \right\rangle$ are two PLFNs define on the universe of discourse $\mathbb{R} \neq \phi$, some operations on PLFNs are defined as follows with $\psi \geq 0$.

1. $E_{\breve{u}_1} \oplus E_{\breve{u}_2} = \left\{ \pounds_{\breve{e}_{\breve{u}_1} + \breve{e}_{\breve{u}_2}}, P_{\breve{e}_{\breve{u}_1}} + P_{\breve{e}_{\breve{u}_2}} - P_{\breve{e}_{\breve{u}_1}} \cdot P_{\breve{e}_{\breve{u}_2}},\ I_{\breve{e}_{\breve{u}_1}} \cdot I_{\breve{e}_{\breve{u}_2}},\ N_{\breve{e}_{\breve{u}_1}} \cdot N_{\breve{e}_{\breve{u}_2}} \right\}$
2. $\psi \cdot E_{\breve{u}} = \left\{ \pounds_{\psi . \breve{e}_{\breve{u}}}, 1 - (1 - P_{\breve{e}_{\breve{u}_1}})^{\psi},\ (I_{\breve{e}_{\breve{u}}})^{\psi},\ (N_{\breve{e}_{\breve{u}}})^{\psi} \right\}$
3. $E_{\breve{u}_1} \otimes E_{\breve{u}_2} = \left\{ \pounds_{\breve{e}_{\breve{u}_1} \times \breve{e}_{\breve{u}_2}}, P_{\breve{e}_{\breve{u}_1}} \cdot P_{\breve{e}_{\breve{u}_2}},\ I_{\breve{e}_{\breve{u}_1}} \cdot I_{\breve{e}_{\breve{u}_2}},\ N_{\breve{e}_{\breve{u}_1}} + N_{\breve{e}_{\breve{u}_2}} - N_{\breve{e}_{\breve{u}_1}} \cdot N_{\breve{e}_{\breve{u}_2}} \right\}$
4. $(E_{\breve{u}})^{\psi} = \left\{ \pounds_{(\breve{e}_{\breve{u}})^{\psi}}, (P_{\breve{e}_{\breve{u}}})^{\psi},\ (I_{\breve{e}_{\breve{u}}})^{\psi},\ 1 - (1 - N_{\breve{e}_{\breve{u}}})^{\psi} \right\}$

3.1. Comparison Rules for PLFNs

To rank the PLFNs, we defined some function in this section, which are the following.

Definition 3.3. Let $E_{\breve{u}} = \langle \pounds_{\breve{e}_{\breve{u}}}, P_{\breve{e}_{\breve{u}}}, I_{\breve{e}_{\breve{u}}}, N_{\breve{e}_{\breve{u}}} \rangle$ be any PLFNs. Then

1. $sc(E_{\breve{u}}) = \frac{\pounds_{\breve{e}_{\breve{u}}} \times \left(P_{\breve{e}_{\breve{u}}} - I_{\breve{e}_{\breve{u}}} - N_{\breve{e}_{\breve{u}}} \right)}{3}$ (score function).
2. $ac(E_{\breve{u}}) = \frac{\pounds_{\breve{e}_{\breve{u}}}}{2} \left(P_{\breve{e}_{\breve{u}}} + N_{\breve{e}_{\breve{u}}} \right)$ (accuracy function).
3. $cr(E_{\breve{u}}) = \frac{\pounds_{\breve{e}_{\breve{u}}}}{2} (P_{\breve{e}_{\breve{u}}})$ (certainty function).

Definition 3.4. Let $E_{\breve{u}_1} = \left\langle \pounds_{\breve{e}_{\breve{u}_1}}, P_{\breve{e}_{\breve{u}_1}}, I_{\breve{e}_{\breve{u}_1}}, N_{\breve{e}_{\breve{u}_1}} \right\rangle$ and $E_{\breve{u}_2} = \left\langle \pounds_{\breve{e}_{\breve{u}_2}}, P_{\breve{e}_{\breve{u}_2}}, I_{\breve{e}_{\breve{u}_2}}, N_{\breve{e}_{\breve{u}_2}} \right\rangle$ are two PLFNs define on the universe of discourse $\mathbb{R} \neq \phi$. With the help of Definition 3.3, we defined the following rules,

1. If $sc(E_{\breve{u}_1}) \succ sc(E_{\breve{u}_2})$,then $E_{\breve{u}_1} \succ E_{\breve{u}_2}$.
2. If $sc(E_{\breve{u}_1}) \approx sc(E_{\breve{u}_2})$,and $ac(E_{\breve{u}_1}) \succ ac(E_{\breve{u}_2})$,then $E_{\breve{u}_1} \succ E_{\breve{u}_2}$.
3. If $sc(E_{\breve{u}_1}) \approx sc(E_{\breve{u}_2})$, $ac(E_{\breve{u}_1}) \approx ac(E_{\breve{u}_2})$ and $cr(E_{\breve{u}_1}) \succ cr(E_{\breve{u}_2})$, then $E_{\breve{u}_1} \succ E_{\breve{u}_2}$.
4. If $sc(E_{\breve{u}_1}) \approx sc(E_{\breve{u}_2})$, $ac(E_{\breve{u}_1}) \approx ac(E_{\breve{u}_2})$ and $cr(E_{\breve{u}_1}) \approx cr(E_{\breve{u}_2})$, then $E_{\breve{u}_1} \approx E_{\breve{u}_2}$.

Definition 3.5. Let any collections $E_{\breve{u}_{\mathrm{p}}} = \left\langle \pounds_{\breve{e}_{\breve{u}_{\mathrm{p}}}}, P_{\breve{e}_{\breve{u}_{\mathrm{p}}}}, I_{\breve{e}_{\breve{u}_{\mathrm{p}}}}, N_{\breve{e}_{\breve{u}_{\mathrm{p}}}} \right\rangle$, $p \in N$ be the PLFNs and $PLFWA : PLFN^n \to PLFN$, then $PLFWA$ describe as,

$$PLFWA\,(E_{\breve{u}_1}, E_{\breve{u}_2}, ..., E_{\breve{u}_n}) = \sum_{p=1}^{n} \psi_p E_{\breve{u}_p}, \tag{10}$$

such that $\psi = \{\psi_1, \psi_2, ..., \psi_n\}^T$ be the weight vector of $E_{\breve{u}_{\mathrm{p}}} = \left\langle \pounds_{\breve{e}_{\breve{u}_{\mathrm{p}}}}, P_{\breve{e}_{\breve{u}_{\mathrm{p}}}}, I_{\breve{e}_{\breve{u}_{\mathrm{p}}}}, N_{\breve{e}_{\breve{u}_{\mathrm{p}}}} \right\rangle$, $p \in N$, with $\psi_{\mathrm{p}} \geq 0$ and $\sum_{\mathrm{p}=1}^{n} \psi_{\mathrm{p}} = 1$.

Theorem 3.6. Suppose that $E_{\breve{u}_{\mathrm{p}}} = \left\langle \pounds_{\breve{e}_{\breve{u}_{\mathrm{p}}}}, P_{\breve{e}_{\breve{u}_{\mathrm{p}}}}, I_{\breve{e}_{\breve{u}_{\mathrm{p}}}}, N_{\breve{e}_{\breve{u}_{\mathrm{p}}}} \right\rangle$, $p \in N$ be the collection of PLFNs. Then by using the Definition 3.5 and operational properties of PLFNs, we can obtained the following outcome.

$$PLFWA\,(E_{\breve{u}_1}, E_{\breve{u}_2}, ..., E_{\breve{u}_n}) = \left\{ \begin{array}{c} \pounds_{\sum\limits_{p=1}^{n} \psi_p \cdot \breve{e}_{\breve{u}_p}}, 1 - \Pi_{p=1}^{n} (1 - P_{\breve{e}_{\breve{u}_p}})^{\psi_p}, \\ \Pi_{p=1}^{n} (I_{\breve{e}_{\breve{u}_p}})^{\psi_p}, \\ \Pi_{p=1}^{n} (N_{\breve{e}_{\breve{u}_p}})^{\psi_p} \end{array} \right\} \tag{11}$$

Definition 3.7. Let any collections $E_{\breve{u}_{\text{p}}} = \left\langle £_{\breve{e}_{\breve{u}_{\text{p}}}}, P_{\breve{e}_{\breve{u}_{\text{p}}}}, I_{\breve{e}_{\breve{u}_{\text{p}}}}, N_{\breve{e}_{\breve{u}_{\text{p}}}} \right\rangle$, $p \in N$ be the PLFNs and $PLFOWA : PLFN^n \to PLFN$, then $PLFOWA$ describe as,

$$PLFOWA\left(E_{\breve{u}_1}, E_{\breve{u}_2}, ..., E_{\breve{u}_n}\right) = \sum_{p=1}^{n} \psi_p E_{\breve{u}_{\rho(p)}}, \tag{12}$$

In which $\psi = \{\psi_1, \psi_2, ..., \psi_n\}$ be the weight vector of $E_{\breve{u}_{\text{p}}} = \left\langle £_{\breve{e}_{\breve{u}_{\text{p}}}}, P_{\breve{e}_{\breve{u}_{\text{p}}}}, I_{\breve{e}_{\breve{u}_{\text{p}}}}, N_{\breve{e}_{\breve{u}_{\text{p}}}} \right\rangle$, $p \in N$, with $\psi_{\text{p}} \geq 0$ and $\sum_{\text{p}=1}^{n} \psi_{\text{p}} = 1$ and $\rho(p)$ indicates a permutation on $\mathbb{R}$.

Theorem 3.8. Suppose that $E_{\breve{u}_{\text{p}}} = \left\langle £_{\breve{e}_{\breve{u}_{\text{p}}}}, P_{\breve{e}_{\breve{u}_{\text{p}}}}, I_{\breve{e}_{\breve{u}_{\text{p}}}}, N_{\breve{e}_{\breve{u}_{\text{p}}}} \right\rangle$, $p \in N$ be the collections of PLFNs. Then, by using the Definition 3.7 and operational properties of PLFNs, the following equation is obtained.

$$PLFOWA\left(E_{\breve{u}_1}, E_{\breve{u}_2}, ..., E_{\breve{u}_n}\right) = \left\{ \begin{array}{c} £_{\sum_{p=1}^{n} \psi_p \cdot \breve{e}_{\breve{u}_{\rho(p)}}}, 1 - \Pi_{p=1}^{n}\left(1 - P_{\breve{e}_{\breve{u}_{\rho(p)}}}\right)^{\psi_p}, \\ \Pi_{p=1}^{n}\left(I_{\breve{e}_{\breve{u}_{\rho(p)}}}\right)^{\psi_p}, \\ \Pi_{p=1}^{n}\left(N_{\breve{e}_{\breve{u}_{\rho(p)}}}\right)^{\psi_p} \end{array} \right\} \tag{13}$$

Theorem 3.9. Suppose that $E_{\breve{u}_{\text{p}}} = \left\langle £_{\breve{e}_{\breve{u}_{\text{p}}}}, P_{\breve{e}_{\breve{u}_{\text{p}}}}, I_{\breve{e}_{\breve{u}_{\text{p}}}}, N_{\breve{e}_{\breve{u}_{\text{p}}}} \right\rangle$, $p \in N$ be the collections of PLFNs and λ be a fuzzy measure on $\mathbb{R}$. Based on fuzzy measure, a Picture linguistic fuzzy Choquet integral weighted averaging ($PLFCIWA$) operator of dimension n is a mapping $PLFCIWA : PLFN^n \to PLFN$ such that

$$\begin{aligned} & PLFCIWA\left(E_{\breve{u}_1}, E_{\breve{u}_2}, ..., E_{\breve{u}_n}\right) \\ = & \left\{ \begin{array}{c} £_{\sum_{p=1}^{n} \lambda\left(A_{\rho(p)}\right) - \lambda\left(A_{\rho(p\text{-}1)}\right) \cdot \breve{e}_{\breve{u}_{\rho(p)}}}, 1 - \Pi_{p=1}^{n}\left(1 - P_{\breve{e}_{\breve{u}_{\rho(p)}}}\right)^{\lambda\left(A_{\rho(p)}\right) - \lambda\left(A_{\rho(p\text{-}1)}\right)}, \\ \Pi_{p=1}^{n}\left(I_{\breve{e}_{\breve{u}_{\rho(p)}}}\right)^{\lambda\left(A_{\rho(p)}\right) - \lambda\left(A_{\rho(p\text{-}1)}\right)}, \\ \Pi_{p=1}^{n}\left(N_{\breve{e}_{\breve{u}_{\rho(p)}}}\right)^{\lambda\left(A_{\rho(p)}\right) - \lambda\left(A_{\rho(p\text{-}1)}\right)} \end{array} \right\} \end{aligned} \tag{14}$$

where $\rho(p)$ indicates a permutation on $\mathbb{R}$ and $A_{\rho(\text{n})} = \{1, 2, ..., p\}$, $A_{\rho(0)} = \phi$.

Definition 3.10. Let $\mathbb{R} \neq \phi$ be the universal set, and E_j, E_l be the any two picture linguistic fuzzy sets. Then, normalized Hamming distance $d_{NHD}(E_j, E_l)$ is given as for all $r \in \mathbb{R}$,

$$d_{NHD}(E_{\breve{u}_{\text{j}}}, E_{\breve{u}_{\text{l}}}) = \frac{1}{2\,(l-1)} \sum_{p=1}^{n} \left| \begin{array}{c} \left(P_{\breve{e}_{\breve{u}_{\text{j}}}}(r_p) - I_{\breve{e}_{\breve{u}_{\text{j}}}}(r_p) - N_{\breve{e}_{\breve{u}_{\text{j}}}}(r_p)\right) £_{\breve{e}_{\breve{u}_{\text{j}}}} - \\ \left(P_{\breve{e}_{\breve{u}_{\text{l}}}}(r_p) - I_{\breve{e}_{\breve{u}_{\text{l}}}}(r_p) - N_{\breve{e}_{\breve{u}_{\text{l}}}}(r_p)\right) £_{\breve{e}_{\breve{u}_{\text{l}}}} \end{array} \right| \tag{15}$$

4. Approach for Multiple criteria Decision Making with Incomplete Weight Information Using GRA Method under the Picture Linguistic Fuzzy Envioურment

Assume that $A = (a_1, ..., a_m)$ be the m alternatives and $C = \{c_1, c_2, ..., c_n\}$, denoted n criteria, and weight criteria is $\varpi = (\varpi_1, \varpi_2, ..., \varpi_n)^T$, where $\varpi_k \geq 0$ $(k = 1, 2, ..., n)$, $\Sigma_{k=1}^{n} \varpi_k = 1$. Let assume that DM deliver information about weights of criteria may be denotes in the following form [51], for $j \neq k$,

(a) If $\{\varpi_j \geq \varpi_k\}$, then, the ranking is weak.
(b) If $\{\varpi_j - \varpi_k \geq \lambda_j (> 0)\}$, then, the ranking is strict.
(c) If $\{\varpi_j \geq \lambda_j \varpi_k\}$, $0 \leq \lambda_j \leq 1$,then, the ranking is multiple ranking.
(d) If $\{\lambda_j \leq \varpi_j \leq \lambda_j + \delta_j\}$, $0 \leq \lambda_j \leq \lambda_j + \delta_j \leq 1$, then, the ranking is an interval ranking.

For facility, Δ stand for the set of the known information about criteria weights contribute by the experts.

Let $R^k = \left[\text{E}_{\breve{u}_{pq}}^{(k)}\right]_{m \times n}$ be an picture linguistic fuzzy decision matrix, provided by decision maker

$d_k(k = 1, 2, ..., l)$, as the following form:

$$R^k = \left[E^{(k)}_{\breve{u}_{pq}}\right]_{m\times n} = \begin{array}{c|cccc} & c_1 & c_2 & \cdots & c_n \\ \hline a_1 & E^{(k)}_{\breve{u}_{11}} & E^{(k)}_{\breve{u}_{12}} & \cdots & E^{(k)}_{\breve{u}_{1n}} \\ a_2 & E^{(k)}_{\breve{u}_{21}} & E^{(k)}_{\breve{u}_{22}} & \cdots & E^{(k)}_{\breve{u}_{2n}} \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ a_m & E^{(k)}_{\breve{u}_{m1}} & E^{(k)}_{\breve{u}_{m2}} & \cdots & E^{(k)}_{\breve{u}_{mn}} \end{array} \tag{16}$$

where $E^{(k)}_{\breve{u}_{pq}} = \left(\pounds^{(k)}_{\breve{e}_{\breve{u}_{pq}}}, P^{(k)}_{\breve{u}_{pq}}, I^{(k)}_{\breve{u}_{pq}}, N^{(k)}_{\breve{u}_{pq}}\right)$ is an PLFN representing the performance rating of the alternative $a_p \in A$ with respect to the criteria $c_p \in C$ provided by the decision makers d_k.

To extend GRA method in the process of group decision making, we first need to fuse all individual decision matrices into a collective matrix by using PLFCIWA operator.

Step 1 Suppose that for every $A = \{a_1, a_2, ..., a_m\}$, m alternative, each expert d_k $(k = 1, 2, ..., r)$ is invited to express their individual evaluation or preference according to each criterias $C_q(q = 1, 2, ..., n)$ by an picture liguistic fuzzy numbers $E^{(k)}_{\breve{u}_{pq}} = \left(\pounds^{(k)}_{\breve{e}_{\breve{u}_{pq}}}, P^{(k)}_{\breve{u}_{pq}}, I^{(k)}_{\breve{u}_{pq}}, N^{(k)}_{\breve{u}_{pq}}\right)$ $(p = 1, 2, ..., m; q = 1, 2, ..., n, k = 1, 2, ..., r)$ expressed by the exparts d_k. In this step we construct the picture liguistic fuzzy decision making matrices, $D^s = \left[E^{(s)}_{ip}\right]_{m\times n}$ $(s = 1, 2, ..., k)$ for decision. If the criteria have two types, such as benefit criteria and cost criteria, then the picture liguistic fuzzy decision matrices, $D^s = \left[E^{s}_{ip}\right]_{m\times n}$ can be converted into the normalized linguistic picture fuzzy decision matrices, $R^k = \left[\mathrm{E}^{(k)}_{\breve{u}_{pq}}\right]_{m\times n}$, where $E^{(k)}_{\breve{u}_{pq}} = \begin{cases} \mathrm{E}^{(k)}_{\breve{u}_{pq}}, \text{ for benefit criteria } A_p \\ \overline{\mathrm{E}^{(k)}_{\breve{u}_{pq}}}, \text{ for cost criteria } A_p, \end{cases}$ $j = 1, 2, ..., n$, and $\overline{\mathrm{E}^{(k)}_{\breve{u}_{pq}}}$ is the complement of $E^{(k)}_{\breve{u}_{pq}}$.The normalization are not requrid, if all the criteria have the same type. Then, we obtain the decision making matrix as follow:

$$R^k = \left[E^{(k)}_{\breve{u}_{pq}}\right]_{m\times n} = \begin{array}{c|cccc} & c_1 & c_2 & \cdots & c_n \\ \hline a_1 & E^{(k)}_{\breve{u}_{11}} & E^{(k)}_{\breve{u}_{12}} & \cdots & E^{(k)}_{\breve{u}_{1n}} \\ a_2 & E^{(k)}_{\breve{u}_{21}} & E^{(k)}_{\breve{u}_{22}} & \cdots & E^{(k)}_{\breve{u}_{2n}} \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ a_m & E^{(k)}_{\breve{u}_{m1}} & E^{(k)}_{\breve{u}_{m2}} & \cdots & E^{(k)}_{\breve{u}_{mn}} \end{array}$$

Step 2 Confirm the fuzzy density $P_{\breve{e}_{\breve{u}_p}} = P_{\breve{e}_{\breve{u}}}(a_p)$ of each expert. According to Eq.(8), parameter λ_1 of expert can be determined.

Step 3 By Definition 2.7, $E^{(k)}_{\breve{u}_{pq}}$ is reordered such that $E^{(k)}_{\breve{u}_{pq}} \geq E^{(k-1)}_{\breve{u}_{pq}}$. Utilize the picture liguistic fuzzy Choquet integral average operator;

$$PFCIWA\left(E^{(1)}_{\breve{u}_{pq}}, E^{(2)}_{\breve{u}_{pq}}, ..., E^{(r)}_{\breve{u}_{pq}}\right) \tag{17}$$
$$= \left\{\begin{array}{c} \pounds_{\sum\limits_{p=1}^{r}\lambda(A_{\rho(p)})-\lambda(A_{\rho(p-1)})\cdot\breve{e}_{\breve{u}_{\rho(p)}}}, 1-\Pi^r_{p=1}(1-P_{\breve{e}_{\breve{u}_{\rho(p)}}})^{\lambda(A_{\rho(p)})-\lambda(A_{\rho(p\text{-}1)})}, \\ \Pi^r_{p=1}(I_{\breve{e}_{\breve{u}_{\rho(p)}}})^{\lambda(A_{\rho(p)})-\lambda(A_{\rho(p-1)})}, \\ \Pi^r_{p=1}(N_{\breve{e}_{\breve{u}_{\rho(p)}}})^{\lambda(A_{\rho(p)})-\lambda(A_{\rho(p-1)})} \end{array}\right\}$$

to aggregate all the picture linguistic fuzzy decision matrices $R^k = \left[\mathrm{E}^{(k)}_{\breve{u}_{pq}}\right]_{m\times n}$ $(k = 1, 2, ..., r)$ into a collective picture linguistic fuzzy decision matrix $R = \left[\mathrm{E}^{(k)}_{\breve{u}_{pq}}\right]_{m\times n}$ where $E^{(k)}_{\breve{u}_{pq}} = \left(\pounds^{(k)}_{\breve{e}_{\breve{u}_{pq}}}, P^{(k)}_{\breve{u}_{pq}},\right.$ $\left.I^{(k)}_{\breve{u}_{pq}}, N^{(k)}_{\breve{u}_{pq}}\right)$ $(p = 1, 2, ..., m; q = 1, 2, ..., n, k = 1, 2, ..., r)$, where $\rho(p)$ indicates a permutation on $\mathbb{R}$ and $A_{\rho(n)} = \{1, 2, ..., p\}$, $A_{\rho(0)} = \phi$ and $P_{\breve{e}_{\breve{u}}}(a_p)$ are find by Equation (9).

Step 4 The picture linguistic fuzzy positive-ideal solution (PLFPIS), stand for $P^{+} = \left\{P_1^{+}, P_2^{+}, ..., P_m^{+}\right\}$ and the picture linguistic fuzzy negative-ideal solution (PLFNIS), stand for $P^{-} = \left\{P_1^{-}, P_2^{-}, ..., P_m^{-}\right\}$ are defined as

$$P_p^{+} = \max_q sc_{pq}, \tag{18}$$

and

$$P_p^{-} = \min_q sc_{pq}, \tag{19}$$

where $P^{+} = \left(\pounds_{\breve{u}_p}^{+}, P_{\breve{u}_p}^{+}, I_{\breve{u}_p}^{+}, N_{\breve{u}_p}^{+}\right)$ and $P^{-} = \left(\pounds_{\breve{u}_p}^{-}, P_{\breve{u}_p}^{-}, I_{\breve{u}_p}^{-}, N_{\breve{u}_p}^{-}\right)$ $p = 1, 2, .., m$.

Step 5 According to linguistic picture fuzzy distance, find the distance between the alternative a_{p} and the PLFPIS P^{+} and the PLFNIS P^{-}, respectively;

$$d(E_{\breve{u}_{\mathrm{j}}}, E_{\breve{u}_{\mathrm{l}}}) = \frac{1}{2(l-1)} \sum_{p=1}^{n} \left| \begin{array}{c} \left(P_{\breve{e}_{\breve{u}_{\mathrm{j}}}}(r_p) - I_{\breve{e}_{\breve{u}_{\mathrm{j}}}}(r_p) - N_{\breve{e}_{\breve{u}_{\mathrm{j}}}}(r_p)\right) \pounds_{\breve{e}_{\breve{u}_{\mathrm{j}}}} - \\ \left(P_{\breve{e}_{\breve{u}_{\mathrm{l}}}}(r_p) - I_{\breve{e}_{\breve{u}_{\mathrm{l}}}}(r_p) - N_{\breve{e}_{\breve{u}_{\mathrm{l}}}}(r_p)\right) \pounds_{\breve{e}_{\breve{u}_{\mathrm{l}}}} \end{array} \right| \tag{20}$$

The above defined distance is called the Normalized Hamming distance [22] $d(e_j, e_k)$, and form a linguistic picture fuzzy positive-ideal separation matrix D^{+} and linguistic picture fuzzy negative-ideal separation matrix D^{-} as follows;

$$D^{+} = (D_{pq}^{+})_{m\times n} = \begin{bmatrix} d\left(E_{\breve{u}_{11}}, P_1^{+}\right) & d\left(E_{\breve{u}_{12}}, P_2^{+}\right) & ... & d\left(E_{\breve{u}_{1\mathrm{n}}}, P_n^{+}\right) \\ d\left(E_{\breve{u}_{21}}, P_1^{+}\right) & d\left(E_{\breve{u}_{22}}, P_2^{+}\right) & ... & d\left(E_{\breve{u}_{1\mathrm{n}}}, P_n^{+}\right) \\ . & . & & . \\ . & . & ... & . \\ . & . & & . \\ d\left(E_{\breve{u}_{\mathrm{m}1}}, P_1^{+}\right) & d\left(E_{\breve{u}_{\mathrm{m}2}}, P_2^{+}\right) & ... & d\left(E_{\breve{u}_{\mathrm{mn}}}, P_n^{+}\right) \end{bmatrix} \tag{21}$$

and

$$D^{-} = (D_{\mathrm{q}}^{-})_{m\times n} = \begin{bmatrix} d\left(E_{\breve{u}_{11}}, P_1^{-}\right) & d\left(E_{\breve{u}_{12}}, P_2^{-}\right) & ... & d\left(E_{\breve{u}_{1\mathrm{n}}}, P_n^{-}\right) \\ d\left(E_{\breve{u}_{21}}, P_1^{-}\right) & d\left(E_{\breve{u}_{22}}, P_2^{-}\right) & ... & d\left(E_{\breve{u}_{1\mathrm{n}}}, P_n^{-}\right) \\ . & . & & . \\ . & . & ... & . \\ . & . & & . \\ d\left(E_{\breve{u}_{\mathrm{m}1}}, P_1^{-}\right) & d\left(E_{\breve{u}_{\mathrm{m}2}}, P_2^{-}\right) & ... & d\left(E_{\breve{u}_{\mathrm{mn}}}, P_n^{-}\right) \end{bmatrix} \tag{22}$$

Step 6 Grey coefficient for every alternative is calculated from PIS and NIS by using the following equation. The grey coefficient for each alternative calculated from PIS is provided as

$$\xi_{pq}^{+} = \frac{\min_{1\le p\le m} min_{1\le q\le n} d\left(E_{\breve{u}_{pq}}, P_p^{+}\right) + \rho \max_{1\le p\le m} max_{1\le q\le n} d\left(E_{\breve{u}_{pq}}, P_p^{+}\right)}{d\left(E_{\breve{u}_{pq}}, P_p^{+}\right) + \rho \max_{1\le p\le m} max_{1\le q\le n} d\left(E_{\breve{u}_{pq}}, P_p^{+}\right)} \tag{23}$$

Where $p = 1, ..., m$ and $q = 1, ..., n$. Correspondingly, the grey coefficient of each alternative calculated from NIS is given as

$$\xi_{pq}^{-} = \frac{\min_{1\le p\le m} min_{1\le q\le n} d\left(E_{\breve{u}_{pq}}, P_k^{-}\right) + \rho \max_{1\le p\le m} max_{1\le q\le n} d\left(E_{\breve{u}_{pq}}, P_k^{-}\right)}{d\left(E_{\breve{u}_{pq}}, P_k^{-}\right) + \rho \max_{1\le p\le m} max_{1\le q\le n} d\left(E_{\breve{u}_{pq}}, P_k^{-}\right)} \tag{24}$$

Where $p = 1, ..., m$ and $q = 1, ..., n$ and the identification coefficient $\rho = 0.5$.

Step 7 Using these equation, to find the grey coefficient degree for each alternative from PIS and NIS, respectively,

$$\xi_p^{+} = \sum_{q=1}^{n} \varpi_q \xi_{pq}^{+} \tag{25}$$

$$\xi_p^{-} = \sum_{q=1}^{n} \varpi_q \xi_{pq}^{-}$$

Basic principle of the Grey method are "the chosen alternative should have the largest degree of grey relation from the PIS and the smallest degree of grey relation from the NIS". Obviously, the weights are known, the smaller $\xi_{\rm p}^{-}$ and the larger $\xi_{\rm p}^{+}$, the finest alternative $a_{\rm p}$ as. But incomplete information about weights of alternatives is known. So, in this case the $\xi_{\rm p}^{-}$ and $\xi_{\rm p}^{+}$ information about weight are determined initially. So, we provide the following optimization models or multiple objective to determined the information about weight,

$$(OM1)\begin{cases} \min \xi_p^- = \sum_{q=1}^{n} \varpi_q \xi_{pq}^- \; p = 1, 2, ..., m \\ \max \xi_p^+ = \sum_{q=1}^{n} \varpi_q \xi_{pq}^+ \;\; p = 1, 2, ..., m \end{cases} \tag{26}$$

Since it given that each alternative is non-inferior, then all the alternatives have no preference relation. The above optimization models are aggregated with equal weights, into single objective optimization model,

$$(OM2)\left\{ \min \xi_p = \sum_{p=1}^{m}\sum_{q=1}^{n} \left(\xi_{pq}^- - \xi_{pq}^+\right) \varpi_q \right. \tag{27}$$

To finding solution of OM2, we obtain optimal solution $\varpi = (\varpi_1, \varpi_2, ..., \varpi_m)^T$, which utilized as weights information alternatives. Then, we obtain ξ_p^+ $(p = 1, 2, ..., m)$ and ξ_p^- $(p = 1, 2, ..., m)$ as using the above formula, respectively.

Step 8 To find the relative closeness degree for each alternative, using the following equation;

$$\xi_p = \frac{\xi_p^+}{\xi_p^- + \xi_p^+} \tag{28}$$

Step 9 According to the ξ_p value, give ranking to the alternatives a_p and select the finest ones.

5. Discriptive Example

We shall present a numerical examples, in this section with linguistic picture fuzzy information to explain the developed approach of the paper.

Example 5.1. Let us assume that a board with four possible develop technology enterprises Z_i $(i = 1, ..., 4)$. There are four experts, and also choose four criteria to classify the four possible develop technology enterprises:

1. $(\breve{A}_1)$, the industrial development;
2. $(\breve{A}_2)$, the feasible market risk;
3. $(\breve{A}_3)$, the industrialization infrastructure, human resources and financial conditions;
4. $(\breve{A}_4)$, the job production and the development of science and technology.

Step 1 Three decision maker offering their own opinions regarding the results obtained with each emerging technology enterprise are given from the table 1-3.

Table 1. Linguistic picture fuzzy information D^1

	$\breve{A}_1$	$\breve{A}_2$	$\breve{A}_3$	$\breve{A}_4$
Z_1	$(\pounds_5, 0.2, 0.1, 0.6)$	$(\pounds_4, 0.5, 0.3, 0.1)$	$(\pounds_2, 0.3, 0.1, 0.5)$	$(\pounds_3, 0.4, 0.3, 0.2)$
Z_2	$(\pounds_2, 0.1, 0.4, 0.4)$	$(\pounds_3, 0.6, 0.2, 0.1)$	$(\pounds_1, 0.2, 0.2, 0.5)$	$(\pounds_5, 0.2, 0.1, 0.6)$
Z_3	$(\pounds_4, 0.2, 0.3, 0.3)$	$(\pounds_2, 0.4, 0.3, 0.2)$	$(\pounds_5, 0.3, 0.1, 0.4)$	$(\pounds_1, 0.3, 0.2, 0.4)$
Z_4	$(\pounds_1, 0.3, 0.1, 0.6)$	$(\pounds_5, 0.3, 0.2, 0.4)$	$(\pounds_3, 0.1, 0.3, 0.5)$	$(\pounds_2, 0.2, 0.3, 0.3)$

Table 2. Linguistic picture fuzzy information D^2

	$\breve{A}_1$	$\breve{A}_2$	$\breve{A}_3$	$\breve{A}_4$
Z_1	$(\pounds_2, 0.1, 0.3, 0.5)$	$(\pounds_5, 0.4, 0.3, 0.2)$	$(\pounds_3, 0.1, 0.1, 0.6)$	$(\pounds_4, 0.2, 0.3, 0.4)$
Z_2	$(\pounds_5, 0.2, 0.2, 0.4)$	$(\pounds_3, 0.4, 0.3, 0.2)$	$(\pounds_4, 0.3, 0.2, 0.4)$	$(\pounds_2, 0.4, 0.1, 0.4)$
Z_3	$(\pounds_3, 0.1, 0.2, 0.6)$	$(\pounds_4, 0.6, 0.1, 0.1)$	$(\pounds_2, 0.2, 0.2, 0.4)$	$(\pounds_5, 0.5, 0.2, 0.2)$
Z_4	$(\pounds_1, 0.4, 0.1, 0.5)$	$(\pounds_2, 0.5, 0.1, 0.3)$	$(\pounds_5, 0.3, 0.3, 0.3)$	$(\pounds_3, 0.6, 0.2, 0.1)$

Table 3. Linguistic picture fuzzy information D^3

	$\breve{A}_1$	$\breve{A}_2$	$\breve{A}_3$	$\breve{A}_4$
Z_1	$(\pounds_1, 0.3, 0.1, 0.3)$	$(\pounds_3, 0.4, 0.2, 0.1)$	$(\pounds_5, 0.2, 0.3, 0.4)$	$(\pounds_4, 0.5, 0.2, 0.1)$
Z_2	$(\pounds_4, 0.1, 0.5, 0.3)$	$(\pounds_5, 0.6, 0.1, 0.2)$	$(\pounds_1, 0.1, 0.1, 0.7)$	$(\pounds_3, 0.3, 0.1, 0.3)$
Z_3	$(\pounds_5, 0.4, 0.2, 0.3)$	$(\pounds_2, 0.4, 0.2, 0.2)$	$(\pounds_4, 0.2, 0.2, 0.5)$	$(\pounds_1, 0.6, 0.2, 0.1)$
Z_4	$(\pounds_3, 0.1, 0.2, 0.6)$	$(\pounds_4, 0.6, 0.2, 0.1)$	$(\pounds_2, 0.3, 0.1, 0.4)$	$(\pounds_2, 0.7, 0.1, 0.1)$

Since $\breve{A}_1$, $\breve{A}_2$ are cost-type criteria and $\breve{A}_3$, $\breve{A}_4$ are benefit-type criteria. First of all we normalize linguistic picture fuzzy information, which are shown in table 4,5,6.:

Table 4. Normalized linguistic picture fuzzy information R^1

	$\breve{A}_1$	$\breve{A}_2$	$\breve{A}_3$	$\breve{A}_4$
Z_1	$(\pounds_5, 0.6, 0.1, 0.2)$	$(\pounds_4, 0.5, 0.3, 0.1)$	$(\pounds_2, 0.5, 0.1, 0.3)$	$(\pounds_3, 0.4, 0.3, 0.2)$
Z_2	$(\pounds_2, 0.4, 0.4, 0.1)$	$(\pounds_3, 0.6, 0.2, 0.1)$	$(\pounds_1, 0.5, 0.2, 0.2)$	$(\pounds_5, 0.2, 0.1, 0.6)$
Z_3	$(\pounds_4, 0.3, 0.3, 0.2)$	$(\pounds_2, 0.4, 0.3, 0.2)$	$(\pounds_5, 0.4, 0.1, 0.3)$	$(\pounds_1, 0.3, 0.2, 0.4)$
Z_4	$(\pounds_1, 0.6, 0.1, 0.3)$	$(\pounds_5, 0.3, 0.2, 0.4)$	$(\pounds_3, 0.5, 0.3, 0.1)$	$(\pounds_2, 0.2, 0.3, 0.3)$

Table 5. Normalized linguistic picture fuzzy information R^2

	$\breve{A}_1$	$\breve{A}_2$	$\breve{A}_3$	$\breve{A}_4$
Z_1	$(\pounds_2, 0.5, 0.3, 0.1)$	$(\pounds_5, 0.4, 0.3, 0.2)$	$(\pounds_3, 0.6, 0.1, 0.1)$	$(\pounds_4, 0.2, 0.3, 0.4)$
Z_2	$(\pounds_5, 0.4, 0.2, 0.2)$	$(\pounds_3, 0.4, 0.3, 0.2)$	$(\pounds_4, 0.4, 0.2, 0.3)$	$(\pounds_2, 0.4, 0.1, 0.4)$
Z_3	$(\pounds_3, 0.6, 0.2, 0.1)$	$(\pounds_4, 0.6, 0.1, 0.1)$	$(\pounds_2, 0.4, 0.2, 0.2)$	$(\pounds_5, 0.5, 0.2, 0.2)$
Z_4	$(\pounds_1, 0.5, 0.1, 0.4)$	$(\pounds_2, 0.5, 0.1, 0.3)$	$(\pounds_5, 0.3, 0.3, 0.3)$	$(\pounds_3, 0.6, 0.2, 0.1)$

Table 6. Normalized linguistic picture fuzzy information R^3

	$\breve{A}_1$	$\breve{A}_2$	$\breve{A}_3$	$\breve{A}_4$
Z_1	$(\pounds_1, 0.3, 0.1, 0.3)$	$(\pounds_3, 0.4, 0.2, 0.1)$	$(\pounds_5, 0.4, 0.3, 0.2)$	$(\pounds_4, 0.5, 0.2, 0.1)$
Z_2	$(\pounds_4, 0.3, 0.5, 0.1)$	$(\pounds_5, 0.6, 0.1, 0.2)$	$(\pounds_1, 0.7, 0.1, 0.1)$	$(\pounds_3, 0.3, 0.1, 0.3)$
Z_3	$(\pounds_5, 0.3, 0.2, 0.4)$	$(\pounds_2, 0.4, 0.2, 0.2)$	$(\pounds_4, 0.5, 0.2, 0.2)$	$(\pounds_1, 0.6, 0.2, 0.1)$
Z_4	$(\pounds_3, 0.6, 0.2, 0.1)$	$(\pounds_4, 0.6, 0.2, 0.1)$	$(\pounds_2, 0.4, 0.1, 0.3)$	$(\pounds_2, 0.7, 0.1, 0.1)$

Let us assume that the criteria weights information given by experts, are partly known;

$$\Delta = \left\{ \begin{array}{c} 0.2 \leq w_1 \leq 0.25, \\ 0.15 \leq w_2 \leq 0.2, \\ 0.28 \leq w_3 \leq 0.32, \\ 0.35 \leq w_4 \leq 0.4 \end{array} \right\}, w_p \geq 0, p = 1, 2, 3, 4, \sum_{p=1}^{4} w_p = 1$$

Then, we utilize the developed approach to get the most desirable alternative(s).

Step 2 Firstly, find fuzzy density of each decision maker, and its λ parameter. Assume that $P_{\breve{e}_{\breve{u}}}(A_1) = 0.30$, $P_{\breve{e}_{\breve{u}}}(A_2) = 0.40$, $P_{\breve{e}_{\breve{u}}}(A_3) = 0.50$. Then λ of adept can be obtained: $\lambda = -0.45$. By Eq.(6), we have $P_{\breve{e}_{\breve{u}}}(A_1, A_2) = 0.65$, $P_{\breve{e}_{\breve{u}}}(A_1, A_3) = 0.73$, $P_{\breve{e}_{\breve{u}}}(A_2, A_3) = 0.81$, $P_{\breve{e}_{\breve{u}}}(A_1, A_2, A_3) = 1$.

Step 3 According to Definition 3.4, $E^{(k)}_{\breve{u}_{pq}}$ is reordered such that $E^{(k)}_{\breve{u}_{pq}} \geq E^{(k-1)}_{\breve{u}_{pq}}$. Then, utilized the picture fuzzy Choquet integral weighted operator

$$PFCIWA(E_{\breve{u}_1}, E_{\breve{u}_2}, ..., E_{\breve{u}_n}) = \left\{ \begin{array}{c} 1 - \Pi_{p=1}^{n}(1 - P_{\breve{e}_{\breve{u}_{\rho(p)}}})^{\lambda(A_{\rho(p)}) - \lambda(A_{\rho(p\text{-}1)})}, \\ \Pi_{p=1}^{n}(I_{\breve{e}_{\breve{u}_{\rho(p)}}})^{\lambda(A_{\rho(p)}) - \lambda(A_{\rho(p\text{-}1)})}, \\ \Pi_{p=1}^{n}(N_{\breve{e}_{\breve{u}_{\rho(p)}}})^{\lambda(A_{\rho(p)}) - \lambda(A_{\rho(p\text{-}1)})} \end{array} \right\}$$

Table 7. Collective picture fuzzy information

	$\breve{A}_1$	$\breve{A}_2$	$\breve{A}_3$	$\breve{A}_4$
$\check{Z}_1$	$\langle \pounds_{2.55}, 0.696, 0.417, 0.225 \rangle$	$\langle \pounds_{4.00}, 0.625, 0.361, 0.331 \rangle$	$\langle \pounds_{3.40}, 0.633, 0.545, 0.391 \rangle$	$\langle \pounds_{3.70}, 0.488, 0.204, 0.313 \rangle$
$\check{Z}_2$	$\langle \pounds_{3.75}, 0.739, 0.488, 0.254 \rangle$	$\langle \pounds_{3.70}, 0.488, 0.670, 0.162 \rangle$	$\langle \pounds_{2.05}, 0.739, 0.311, 0.335 \rangle$	$\langle \pounds_{3.25}, 0.613, 0.193, 0.374 \rangle$
$\check{Z}_3$	$\langle \pounds_{4.00}, 0.405, 0.654, 0.361 \rangle$	$\langle \pounds_{2.70}, 0.732, 0.274, 0.200 \rangle$	$\langle \pounds_{3.60}, 0.739, 0.260, 0.265 \rangle$	$\langle \pounds_{2.40}, 0.600, 0.278, 0.304 \rangle$
$\check{Z}_4$	$\langle \pounds_{1.70}, 0.769, 0.331, 0.418 \rangle$	$\langle \pounds_{3.60}, 0.638, 0.562, 0.311 \rangle$	$\langle \pounds_{3.35}, 0.613, 0.354, 0.311 \rangle$	$\langle \pounds_{2.35}, 0.511, 0.265, 0.358 \rangle$

Step 4 Utilize eq.(18 and eq.(19) we obtain the positive-ideal and negative-ideal solution respectively, are:

$$P^{+}=\left\{\begin{array}{c}\langle \pounds_{2.55},0.696,0.417,0.225\rangle,\langle \pounds_{2.70},0.732,0.274,0.200\rangle,\\ \langle \pounds_{3.60},0.739,0.260,0.265\rangle,\langle \pounds_{3.25},0.613,0.193,0.374\rangle\end{array}\right\}$$

$$P^{-}=\left\{\begin{array}{c}\langle \pounds_{4.00},0.405,0.654,0.361\rangle,\langle \pounds_{3.70},0.488,0.670,0.162\rangle,\\ \langle \pounds_{3.40},0.633,0.545,0.391\rangle,\langle \pounds_{2.35},0.511,0.265,0.358\rangle\end{array}\right\}$$

Step 5 Utilize equation (21) and (22) to get the positive ideal and negative ideal separation matrix, respectively as follow;

Table 8. Positive-ideal separation matrix

$D^{+}=$

	$\breve{A}_1$	$\breve{A}_2$	$\breve{A}_3$	$\breve{A}_4$
$\check{Z}_1$	0.0000	0.0808	0.1500	0.0214
$\check{Z}_2$	0.0123	0.1645	0.0482	0.0000
$\check{Z}_3$	0.2145	0.0000	0.0000	0.0088
$\check{Z}_4$	0.0084	0.1289	0.0787	0.0344

Table 9. Negative-ideal separation matrix

$D^{-}=$

	$\breve{A}_1$	$\breve{A}_2$	$\breve{A}_3$	$\breve{A}_4$
$\check{Z}_1$	0.2145	0.0837	0.0000	0.0129
$\check{Z}_2$	0.2021	0.0000	0.1017	0.0344
$\check{Z}_3$	0.0000	0.1645	0.1500	0.0256
$\check{Z}_4$	0.2060	0.0356	0.0712	0.000

Step 6 Utilize equations (23) and (24) to get the grey relational coefficient matrices in which every alternative is obtained from PIS and NIS as follow:

$\left[\zeta_{ij}^{+}\right]=$

0.6518	0.5721	0.8295	0.4711
0.3667	1.0000	1.0000	0.3333
0.5254	0.7829	0.6383	0.6744
1.0000	0.6875	0.5562	1.0000

$\left[\zeta_{ij}^{-}\right]=$

0.4560	1.0000	0.5937	0.4039
1.0000	0.5721	0.5562	1.0000
0.5483	0.6689	0.7699	0.3745
0.3667	0.5372	0.5184	0.3333

Step 7 To developed the single-objective programming model, using the model (M2):

$$\min \xi(w)=-0.0709w_1+0.4283w_2-0.2594w_3-1.5440w_4$$

We gain the weight vector of criterias, to solved this model:

$$w=(0.330,0.144,0.366,0.157)$$

Now from the PIS and NIS, we get the degree of grey relational coefficient of every alternative:

$$\xi_1^{+}=0.6001,\xi_2^{+}=0.5439,\xi_3^{+}=0.6331,\xi_4^{+}=0.8823,$$

$$\xi_1^{-}=0.5355,\xi_2^{-}=0.8657,\xi_3^{-}=0.5352,\xi_4^{-}=0.4016.$$

Step 8 To find the relative relational degree of the alternative, we utilize Equation 28, and PIS and NIS:

$$\xi_1=\frac{\xi_1^{+}}{\xi_1^{-}+\xi_1^{+}}=\frac{0.6001}{0.5355+0.6001}=0.5284$$

$$\xi_2=\frac{\xi_2^{+}}{\xi_2^{-}+\xi_2^{+}}=\frac{0.5439}{0.8657+0.5439}=0.3858$$

$$\xi_3=\frac{\xi_3^{+}}{\xi_3^{-}+\xi_3^{+}}=\frac{0.6331}{0.5352+0.6331}=0.5418$$

$$\xi_4=\frac{\xi_4^{+}}{\xi_4^{-}+\xi_4^{+}}=\frac{0.8823}{0.4016+0.8823}=0.6872$$

Step 9 With the help of relative relational degree, ranking of the alternatives are the following:

$$Z_4 > Z_3 > Z_1 > Z_2,$$

and thus the most desirable alternative is Z_4.

5.1. Comparative Analysis

To justify the effectivity and efficiency of the advised procedure, we conducted a comparative analysis for comparison of our suggest approach with the GRA method for intuitionistic fuzzy set [52].

5.1.1. Comparison between intuitionistic fuzzy and picture fuzzy GRA relation Approache

In the ntuitionistic fuzzy numbers, we have only study the uncertain things from positive and negative membership degrees. They bring an effective execution to imply the vagueness of DM. On the other hand, as stated already, in IFN the things from good and bad appearance of these two collection of fuzzy numbers, can throw away the thinking of DM perfectly. After all, dissimilar the PFNs, in some conditions the IFNs are not serviceable. The IFNs must satisfy the condition that the membership and non-membership degree sume belongs to $[0, 1]$. Thus, in some cases, there exists some problems which cannot handle by IFNs. For example, the peoples requried their opinions contain more type of answer like as: "yes", "abstain", "No" and "Refusal", in that situations picture fuzzy set are more suitable. Thus, in summary, in decision making theory, PFNs have suitable capacity to process these information.

6. Conclusion

The classical grey relational analysis method are normally applicable for tackle the MAGDM problems, in which the data occur in the form of numerical values, and still they will flop when MAGDM problems contains picture linguistic fuzzy information. In the developed approach we use the picture linguistic fuzzy Choquet integral weighted averaging (PLFCIWA) operator to marge all the individual matrices. Then, based on the traditional GRA method, an approach are given to deal with picture linguistic fuzzy MAGDM problems in which the information are incomplete. Lastly, a decision problem are developed based on the defined operators, to rank more alternatives. Thus, the proposed operations gives clear track to catch the inexact data all over the decision problem procedure.

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Towards a Theory of Unbounded Locally Solid Riesz Spaces

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Abstract − We introduce the notion of unbounded locally solid Riesz spaces and we investigate its some fundamental properties.

Keywords − *Unbounded order convergence, Unbounded locally solid Riesz space.*

1. Introduction

One of the main convergence types in a Riesz space [1] is ***order convergence***. Recall that a net $(x_\alpha)_{\alpha\in A}$ in a Riesz space E is said to be order convergent to $x \in E$ (briefly; $x_\alpha \xrightarrow{o} x$ or x_α o-converges to x) if there exists another net $(y_\beta)_{\beta\in B}$ in E such that

i) $y_\beta \downarrow 0$, that is, $(y_\beta)_{\beta\in B}$ is decreasing to 0 and ;

ii) For each $\beta \in B$ there exists $\alpha_0 \in A$ such that $|x_\alpha - x| \leq y_\beta$ for each $\alpha \geq \alpha_0$.

Unbounded order convergence in a Riesz space was defined and studied in [1, 2]. Recently, many authors have started to work on this topic in [3–5]. Namely, a net (x_α) [2] in a Riesz space E is said to be ***unbounded order convergent*** if the net $(|x_\alpha - x| \wedge u)$ is order convergent to zero for each $u \in E^+$ (briefly; $x_\alpha \xrightarrow{uo} x$ or x_α uo-converges to x). In general, every order convergent net is unbounded order convergent but the converse is not true (For example, consider c_0 as a Riesz space under pointwise order, the standard unit vectors (e_n) are uo-convergent but not o-convergent). It is obvious that order convergence and unbounded order convergence coincide for order bounded nets. Although in general, unbounded order convergence is not topological (see [3]), but in an atomic Riesz space, it is topological (see Theorem 2 in [6]).

Let E be a normed Riesz space. A net (x_α) in E is said to be ***unbounded norm convergent*** to $x \in E$ if the net $(|x_\alpha - x| \wedge u)$ is norm convergent to zero for each $u \in E^+$ (briefly; $x_\alpha \xrightarrow{un} x$ or x_α un-converges to x). The notion of unbounded norm convergence was defined in [7] and many results were obtained in [8, 9]. These notions were extended to the locally solid Riesz spaces (see [10]). In [9], it is noticed that unbounded norm convergence defines a topology, that is, there exists a new topology on the normed Riesz space E so that the unbounded norm convergence and topological convergence agree. This new topology is called ***un-topology***. See [9] for more details on this topic. In the paper, it is also proved that in Banach lattices, the norm topology and unbounded norm topology coincide if and only if the space has a strong order unit (Theorem 2.3).

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[1]In this paper all Riesz spaces will be assumed Archimedean
[2]The index is not written unless it is necessary

2. An observation

Let $(E, ||.||)$ be a normed Riesz space. For each $u \in E^+$, define $P_u : E \to \mathbb{R}$ by

$$P_u(x) = |||x| \wedge u||.$$

We will show that for each $u \in E^+$, P_u is a Riesz pseudonorm. Recall that a real-valued map p on E is called a ***Riesz pseudonorm*** if the following conditions are satisfied:

1. $p(x) \geq 0 \ \forall x \in X$;
2. If $x = 0$, then $p(x) = 0$;
3. $p(x + y) \leq p(x) + p(y) \ \forall x, y \in X$;
4. If $\lim_{n\to\infty} \lambda_n = 0$ in $\mathbb{R}$, then $p(\lambda_n x); \to 0$ in $\mathbb{R} \ \forall x \in X$;
5. If $|x| \leq |y|$, then $p(x) \leq p(y)$.

Theorem 2.1. Let $(E, ||.||)$ be a normed Riesz space. For each $u \in E^+$, the map $P_u : E \longrightarrow \mathbb{R}^+$ defined by $P_u(x) = |||x| \wedge u||$, is a Riesz pseudonorm. Moreover, the un-topology and the topology generated by the family $(P_u)_{u\in E^+}$ coincide.

Proof. Let $u \in E^+$ be given. Obviously, the conditions (1),(2) and (5) hold. For condition (3): Let $x, y \in E$ be given. Since $|x + y| \leq |x| + |y|$, we have $|x + y| \wedge u \leq (|x| + |y|) \wedge u \leq |x| \wedge u + |y| \wedge u$ and since $||.||$ is a lattice norm, we get the inequality $P_u(x + y) \leq P_u(x) + P_u(y)$ by monotonicity and the triangle inequality property of lattice norm. For condition(4): Let $\{\lambda_n\} \subset \mathbb{R}$ be a sequence such that $\lim_{n\to\infty} \lambda_n = 0$ and $x \in E$, the inequality

$$P_u(\lambda_n x) = |||\lambda_n x| \wedge u|| \leq ||\lambda_n x|| = |\lambda_n|||x||$$

implies that $\lim_{n\to\infty} P_u(\lambda_n x) = 0$. Hence P_u is a Riesz pseudonorm.

Let (x_α) be a net converging to x in *un*-topology, that is, $|| \ |x_\alpha - x| \wedge u \ || \to 0$ for each $u \in E^+$. By definition, $P_u(x_\alpha - x)$ converges to zero for each $u \in E^+$, so it converges to x in the topology generated by the family $(P_u)_{u\in E^+}$. Converse direction is also true. This completes the proof. □

Note that in a Riesz space, we have the following.

Theorem 2.2. Let E be a Riesz space and $p : E \to \mathbb{R}$ be a map. The followings are equivalent:

i. p is a Riesz pseudonorm;

ii. $p(x) = p(|x|)$ for all $x \in E$ and for each $u \in E^+$, the map $p_u : E \to \mathbb{R}$, defined by $p_u(x) = p(|x| \wedge u)$, is a Riesz pseudonorm.

Proof. If (i) holds, following the proof of Theorem 2.1, we can get (ii). Suppose that (ii) holds. Since $p(x) = p(|x|) = p_{|x|}(x) \geq 0$, it is obvious that $p(x) = p(|x|) = p_{|x|}(x) = 0$ whenever $x = 0$. Let $x, y \in E$ be given. Then

$$\begin{aligned} p(x+y) =& p(|x+y| \wedge (|x| + |y|) \\ =& p_{|x|+|y|}(|x+y|) \\ \leq& p_{|x|+|y|}(|x|) + p_{|x|+|y|}(|y|) \\ =& p(|x| \wedge (|x| + |y|)) + p(|x| \wedge (|x| + |y|) \\ =& p(|x|) + p(|y|) \end{aligned}$$

so that p satisfies the triangle inequality. Let $x \in E$ be given. Then

$$\lim_{n\to\infty} p(\lambda_n x) = \lim_{n\to\infty} p(|\lambda_n x|) = \lim_{n\to\infty} p_{|x|}(|\lambda_n||x|) = 0.$$

If $|x| \leq |y|$ then

$$p(x) = p(|x|) = p(|x| \wedge |y|) = p_{|y|}(|x|) = p_{|y|}(x) \leq p_{|y|}(y) = p(|y|) = p(y).$$

This completes the proof. □

3. Some notations and terminolgy

Let E be a Riesz space. A subset $A \subset E$ is called ***solid*** if $y \in A$ whenever $|y| \leq |x|$ in E for some $x \in A$. A linear topology τ on E is called ***locally solid*** if it has a neighborhood system at zero consisting of solid sets. One can easily show that a given set P of Riesz pseudonorms has a solid topology with a subbase of zero as the set $\{p^{-1}(-\epsilon, \epsilon) : p \in P, \epsilon > 0\}$. This topology is denoted by $< P >$, and it is called ***locally solid topology generated by*** P. Conversely, Fremlins Theorem says that every locally solid topology is generated by a family of Riesz pseudonorms. That is, a linear topology τ is locally solid if and only if $\tau =< P >$ for some set P of Riesz pseudonorms.(see [11]) .

Let p be a Riesz pseudonorm on E. For each $u \in E_+$, the map $p_u : E \to \mathbb{R}$ is also a Riesz pseudonorm defined by $p_u(x) = p(|x| \wedge u)$. Let (E, τ) be a locally solid Riesz space. So there exists a family of Riesz pseudonorms $(p_i)_{i \in I}$ such that $\tau =< (p_i)_{i\in I} >$. For any $A \subset E_+$, there exists a different family of Riesz pseudonorms $(p_{i,a})_{i\in I, a\in A}$ where $p_{i,a}(x) = p(|x| \wedge a)$ for each $i \in I$ and $a \in A$. This related family defines a locally solid topology. This fact coincides with the Mitchell A. Taylor's definition of "unbounded $\tau-$convergence with respect to A" in [12]. Here is the Mitchell A. Taylor's definition.

Definition 3.1. Let X be a vector lattice, $A \subseteq X$ be an ideal and τ be a locally solid topology on A. Let (x_α) be a net in X and $x \in X$. We say that (x_α) unbounded τ-converges to x with respect to A if $|(x_\alpha) - x| \wedge |a| \xrightarrow{\tau}$ for all $a \in A_+$.

In [12], the convergence in the above definition is denoted by $u_A\tau$

Observation:

Let E be a Riesz space, and $p : E \to \mathbb{R}$ be a Riesz pseudonorm. For a given nonempty set $A \subset E_+$, consider the map $p_A : E \to \mathbb{R}$ defined by

$$p_A(x) = \sup_{a\in A} p(|x| \wedge a)$$

It is obvious that the map p_A satisfies the conditions $(1) - (3)$ and (5), we must check condition (4): Let $\{\lambda_n\} \subset \mathbb{R}$ be any sequence converging to zero. Then

$$\begin{aligned} p_A(\lambda_n x) = \sup_{a\in A} p(|\lambda_n x| \wedge a) &= \sup_{a\in A} p(|\lambda_n||x| \wedge a) \\ &\leq \sup_{a\in A} p(|\lambda_n||x|) \\ &= p(|\lambda_n||x|) \longrightarrow 0 \end{aligned}$$

so that p_A is a Riesz pseudonorm.

Let $P = (p_i)_{i\in I}$ be a family of Riesz pseudonorms and $\mathscr{A} \subset \mathscr{P}(E_+)$ that does not contain the empty set. This family generates a topology, say τ. The locally solid topology generated by the family $\{p_{i,A} : i \in I, A \in \mathscr{A}\}$ will be denoted by $u < \tau, \mathscr{A} >$. Actually, if $\mathscr{A}$ contains the empty set, then $u < \tau, \mathscr{A} >$ is discrete topology.

Some remarks:

Let (E, τ) be a locally solid Riesz space, $(p_i)_{i\in I}$ be the family of Riesz pseudonorms, and $\tau =< (p_i)_{i\in I} >$. Then

(1) For any $\mathscr{A} \subset \mathscr{P}(E_+)$, $u < \tau, \mathscr{A} > \subset \tau$ holds.
Proof: $x_\alpha \xrightarrow{\tau} x \iff p_i(x_\alpha - x) \to 0 \iff p_i(|x_\alpha - x|) \to 0$, and for each $a \in E_+$ we have $p_i(|x_\alpha - x| \wedge a) \leq p_i(|x_\alpha - x|)$, hence

$$\sup_{a\in A} p_i(|x_\alpha - x| \wedge a) \leq p_i(|x_\alpha - x|) \text{ for each } A \in \mathscr{A}.$$

So $x_\alpha \xrightarrow{u<\tau,\mathscr{A}>} x$.

(2) If $\mathscr{A} = \{\{E_+\}\}$, then $u < \tau, \{\{E_+\}\} >= \tau$.
Proof: It is clear that $\sup_{a\in E_+} p_i(|x| \wedge a) = p_i(|x|) = p_i(x)$.

(3) If $\mathscr{A} \subset \mathscr{B}$ then $u < \tau, \mathscr{A} > \subset u < \tau, \mathscr{B} >$ for all $\mathscr{A} \subset \mathscr{B} \subset \mathscr{P}(E_+)$.
Proof: $x_\alpha \xrightarrow{u<\tau,\mathscr{B}>} x \Longleftrightarrow \sup_{b\in B} p_i(|x_\alpha - x| \wedge a)$ $B \in \mathscr{B}$

$$\Longrightarrow \sup_{a\in A} p_i(|x_\alpha - x| \wedge a) \text{ for each } A \in \mathscr{A} \subset \mathscr{B}.$$

Hence, $x_\alpha \xrightarrow{u<\tau,\mathscr{A}>} x$.

(4) For each $\mathscr{A} \subset \mathscr{P}(E_+)$ $u < \tau, \bigcup \mathscr{A} > \subset u < \tau, \mathscr{A} >$ holds.
Proof: Let $x_\alpha \xrightarrow{u<\tau,\mathscr{A}>} x$. So for a fixed $i \in I$ and $A \in \mathscr{A}$, $p_{i,A}(x_\alpha - x) \to 0 \Longleftrightarrow \sup_{a\in A} p_i(|x_\alpha - x| \wedge a) \to 0$, and it is obvious that

$$p_i(|x_\alpha - x| \wedge a) \leq \sup_{a\in A} p_i(|x_\alpha - x| \wedge a) \text{ for each } a \in A.$$

Hence, $p_{i,\{a\}}(x_\alpha - x) = p_{i,\{a\}}(|x_\alpha - x|) = \sup_{a\in\{a\}} p_i(|x_\alpha - x| \wedge a) = p_i(|x_\alpha - x| \wedge a) \to 0$.

(5) For each $\mathscr{A} \subset \mathscr{P}(E_+)$ $u < \tau, \bigcup \mathscr{A} >= u < \tau, I(\bigcup \mathscr{A}) >$ holds where $I(\bigcup \mathscr{A})$ is the ideal generated by $\bigcup \mathscr{A}$.
Proof: Since $\bigcup \mathscr{A} \subset I(\bigcup \mathscr{A})$, we have $u < \tau, \bigcup \mathscr{A} > \subset u < \tau, I(\bigcup \mathscr{A}) >$ from (3). Let $x_\alpha \xrightarrow{u<\tau,\mathscr{A}>} x$ and $b \in I(\bigcup \mathscr{A})_+$ be given, there exists $a_1, ..., a_n \in \bigcup \mathscr{A}$ and $k \geq 0$ such that $0 \leq b \leq k(a_1 + ... + a_n)$. Then

$$\begin{aligned} |x_\alpha - x| \wedge b \leq |x_\alpha - x| \wedge k(a_1 + ... + a_n) &\leq \sum_{i=1}^{n} |x_\alpha - x| \wedge ka_i \\ &= k \sum_{i=1}^{n} \frac{1}{k} |x_\alpha - x| \wedge a_i \\ &\leq km \sum_{i=1}^{n} |x_\alpha - x| \wedge a_i \end{aligned}$$

where m is the smallest positive integer greater than $\frac{1}{k}$. Then by the monotonicity of p_i,

$$p_i(|x_\alpha - x| \wedge b) \leq p_i(km \textstyle\sum_{i=1}^{n} |x_\alpha - x| \wedge a_i) \to 0.$$

Hence, $p_i(|x_\alpha - x| \wedge b) = \sup_{b\in\{b\}} p_i(|x_\alpha - x| \wedge b)$. This completes the proof.

(6) For each $\mathscr{A} \subset \mathscr{P}(E_+)$ $u < \tau, \bigcup \mathscr{A} >= u < \tau, \overline{\bigcup \mathscr{A}} >$ holds.
Proof: Suppose that $x_\alpha \xrightarrow{u<\tau,\mathscr{A}>} x$ and $b \in \overline{\bigcup \mathscr{A}}$ be given. Choose a net $(b_\beta) \in \bigcup \mathscr{A}$ with $b_\beta \xrightarrow{u<\tau,\mathscr{A}>} b$. Let $i \in I$ be fixed and $\epsilon \geq 0$ be given. Choose β_0 such that $p_i(b_{\beta_0} - b) < \frac{\epsilon}{2}$. Then

$$\begin{aligned} |x_\alpha - x| \wedge b &= |x_\alpha - x| \wedge (b - b_{\beta_0} + b_{\beta_0}) \\ &\leq |x_\alpha - x| \wedge (|b - b_{\beta_0}| + |b_{\beta_0}|) \\ &\leq |x_\alpha - x| \wedge |b - b_{\beta_0}| + |x_\alpha - x| \wedge |b_{\beta_0}| \end{aligned}$$

Applying p_i to this inequality, one can show the existence of α_0 such that $p_{i,\{b\}}(x_\alpha - x) < \epsilon$. This completes the proof.

(7) If $0 \leq a \leq b$, then $u < \tau, \{a\} > u < \tau, \{b\} >$.
Proof: It is clear that; $\sup_{a\in\{a\}} p_i(|x_\alpha - x| \wedge a) = p_i(|x_\alpha - x| \wedge a)$

$$\leq p_i(|x_\alpha - x| \wedge b)$$

$$= \sup_{b\in\{b\}} p_i(|x_\alpha - x| \wedge b).$$

(8) If $e \in E$ is a strong order unit, then $u < \tau, \{\{e\}\} >= u < \tau, \bigcup E_+ >$. But the converse of this statement is not true in general. For example, consider c_0 as a Banach lattice with supremum norm, with norm topology τ and $e = (\frac{1}{n})$. Then $u < \tau, \{\{e\}\} >= u < \tau, \bigcup E_+ >$, but e is not an order unit.

(9) If e is a quasi-interior point, then $u<\tau,\{\{e\}\}>=u<\tau,\bigcup E_+>$ from (5) and (6).

(10) For any $\mathscr{A}\subset\mathscr{P}(E_+)$, $u<\tau,\mathscr{A}>=u<u<\tau,\mathscr{A}>,\mathscr{A}>$ holds.

(11) For any $\{A\}\in\mathscr{P}(E_+)$, $u<\tau,\{A\}>\neq u_A\tau$, but $u<\tau,\bigcup A>=u_A\tau$ holds. Moreover, $u_A\tau\subset u<\tau,\{A\}>$.

4. Unbounded locally solid Riesz space

From the motiviation of the above observation, we give the following definition.

Definition 4.1. A real valued map q on a Riesz space E is said to be ***unbounded Riesz pseudonorm*** if there exists a Riesz pseudonorm p on E and $A\subset E^+$ satisfying $q(x)=\sup_{a\in A}p(|x|\wedge a)$. In this case, we say that q is generated by p and the subset A.

It is obvious that every unbounded Riesz pseudonorm is a Riesz pseudonorm. So the topology generated by unbounded Riesz pseudonorm is a locally solid topology. If unbounded Riesz pseudonorm q is generated by Riesz pseudonorm p and $A\subset E^+$, then the topology generated by q is weaker than the topology generated by p. Remind that every family of Riesz pseudonorms defines a locally solid topology. Conversely, every locally solid topology is determined by a family of Riesz pseudonorms.

Definition 4.2. Let (E,τ) be a locally solid Riesz space generated by the family $(p_i)_{i\in I}$ of Riesz pseudonorms . The locally solid Riesz space on E generated by the family of unbounded Riesz pseudonorm on E is called ***unbounded locally solid Riesz space*** generated by τ, and denoted by $\tau^{'}$

Proposition 4.3. Let (E,τ) be a locally solid Riesz space. If τ is a Hausdorff locally solid topology, then the unbounded locally solid topology is also Hausdorff.

Proof. Let $x\neq 0$ be given, then there exists some $i_0\in I$ such that $p_{i_0}(x)>0$ where $\tau=<(p_i)_{i\in I}>$.Then

$$q_{i_0,\{|x|\}}:=p_{i_0}(|x|\wedge|x|)=p_{i_0}(|x|)=p_{i_0}(x)>0.$$

It is obvious that $q_{i_0,\{|x|\}}$ is an unbounded Riesz pseudonorm, so $\tau^{'}$ is a Hausdorff topology. □

Definition 4.4. A net (x_α) in a locally solid Riesz space (E,τ) is ***unbounded topological convergent*** if it is convergent in unbounded locally solid Riesz space $(E,\tau^{'})$.

Theorem 4.5. Let (E,τ) be a Hausdorff locally solid Riesz space and (x_α) be an increasing net. Then the followings are equivalent:

1. $(x_\alpha)\xrightarrow{\tau}x$ in (E,τ);
2. $(x_\alpha)\xrightarrow{\tau^{'}}x$ in $(E,\tau^{'})$.

Proof. Since $\tau^{'}\subset\tau$, it is easy to see that (1) implies (2). Now suppose (2) holds. Since $\tau^{'}$ is a Hausdorff locally solid Riesz space by the Proposition 4.3, we have $x_\alpha\uparrow x$. Thus $|x|$ is an upper bound for the net (x_α) and $2|x|$ is an upper bound for the net $(|x_\alpha-x|)$. Now suppose that $(p_i)_{i\in I}$ is the family of Riesz pseudonorms such that $\tau=<(p_i)_{i\in I}>$. Let $i\in I$ be arbitrary. Then

$$p_i(x_\alpha-x)=p_i(|x_\alpha-x|)=p_i(|x_\alpha-x|\wedge 2|x|)=:q_{i,\{2|x|\}}(x_\alpha-x)\to 0.$$

This completes the proof. □

Theorem 4.6. Let (E,τ) be a Hausdorff locally solid Riesz space, and $\tau^{'}$ be the unbounded locally solid topology generated by τ. Then τ has Lebesgue property if and only if $\tau^{'}$ has Lebesgue property.

Proof. One side of the implication is clear. Let us assume that $x_\alpha\downarrow 0$ implies $x_\alpha\xrightarrow{\tau^{'}}0$. Then , it is easy to see that $x_\alpha\xrightarrow{\tau}0$ by using the Theorem 4.5. This completes the proof. □

4.1. Product of unbounded locally solid Riesz spaces

Theorem 4.7. Let $(E_i, \tau_i)_{i\in I}$ be a family of locally solid Riesz spaces. Then the product space $\prod_{i\in I} E_i$ is unbounded locally solid Riesz space if and only if for each i, E_i is an unbounded locally solid Riesz space.

Proof. Suppose that for each $i \in I$, (E_i, τ_i) is an unbounded locally solid Riesz space, and τ_i is generated by a family Q_i of the Riesz pseudonorms on E_i. So for each $q \in Q_i$, there exists a Riesz pseudonorm p on E_i and $A \subset E_i^+$, depending on q, such that

$$q(x) = \sup_{a\in A} p(|x| \wedge a) \text{ for all } x \in E_i.$$

Let $j \in I$ and $q \in Q_j$ be given. Choose p and A as above. Let P_j be the projection from $E = \prod_i E_i$ into E_j and f_j be vector space embedding of E_j into E, that is, f_j sends $x \in E_j$ to (x_i) where $x_j = x$ and $x_i = 0$ for all $i \neq j$. One can show that for each Riesz pseudonorm on E_j, $p \circ P_j$ is a Riesz pseudonorm on E. We note that for each $q \in Q_j$,

$$q \circ P_j((x_i)) = q(P_j(x_i)) = q(x_j) = \sup_{a\in A} p(|x_j| \wedge a) = sup_{a\in A}\, p \circ P_j(|(x_i)| \wedge f_j(a)).$$

Thus, $q \circ P_j$ is an unbounded Riesz pseudonorm on E. And the the topology of $\prod_i E_i$ is the topology generated by $\{q \circ P_j : j \in I, q \in Q_j\}$. Hence, the locally solid Riesz space $\prod_i E_i$ is an unbounded locally solid Riesz space.

Now suppose that $E = \prod_i E_i$ is an unbounded locally solid Riesz space, and i_0 is given. Suppose that the topology of E is generated by the family Q of unbounded Riesz pseudonorm on E. Let $q \in Q$ be given. Choose $A = (A_i) \in E$ and Riesz pseudonorm p on E such that $q(x) = \sup_{a\in A} p(|x| \wedge a)$ for all $x \in E$. It is obvious that for each i_0, $p \circ f_{i_0}$ is a Riesz pseudonorm on E_{i_0} and

$$q \circ f_{i_0}(x) = \sup_{a\in A_{i_0}} p \circ f_{i_0}(|x| \wedge a).$$

Hence q_{i_0} is an unbounded Riesz pseudonorm on E_{i_0}. Now one can show that the topology of E_{i_0} is generated by $\{q \circ f_{i_0} : q \in Q\}$. Hence, E_{i_0} is an unbounded locally solid Riesz space. This completes the proof. □

Let X be a product space of topological spaces $(X_i)_{i\in I}$. A net (x_α) converges to x in X if and only if $x_\alpha^i \to x_i$ in X_i for each $i \in I$, where $x_\alpha = (x_\alpha^i)_{i\in I}$ and $x = (x_i)$. By using this fact, the proof of the following theorem is easy.

Theorem 4.8. Let $(E_i, \tau_i)_{i\in I}$ be a family of locally solid Riesz spaces. For each $\mathscr{A}_i \subset \mathscr{P}(E_i^+)$, we have

$$u < \textstyle\prod_i \tau_i, \prod_i \mathscr{A}_i >= \prod_i u < \tau_i, \mathscr{A}_i >.$$

4.2. Unbounded absolute weakly locally solid Riesz space

The concept of unbounded absolute weak convergence (briefly uaw-convergence) was considered and studied in [13]. Let E and F be vector spaces. If there exists a bilenear map $T : E \times F \to \mathbb{R}$ satisfying

$T(x, F) = 0 \Longrightarrow x = 0$,

$T(E, y) = 0 \Longrightarrow x = 0$, then the pair (E, F) is called a dual pair. In this case, E can be considered as a vector supspace of $\mathbb{R}^F$, by embedding $x \to x^*$, $x^*(y) = T(x, y)$. We can consider $\mathbb{R}^F$ as a topological space with product topology $\prod_{y\in F} \mathbb{R}$ and restriction of this topology on E is the topology generated by the family $(p_y)_{y\in F})$ of seminorms, where $p_y : E \to \mathbb{R}$ defined by $p_y(x) = |T(x, y)|$. This topology is independent of T and is denoted by $\sigma(E, F)$. Similarly, $\sigma(F, E)$ can be defined. One of the main results is that the topological dual of E with respect to $\sigma(E, F)$ is a vector space which is isomorphic to F, this is denoted by $(E, \sigma(E, F))' \cong F$.

Definition 4.9. If (E, F) is a dual pair of Riesz spaces with respect to a positive linear map $T : E \times F \to \mathbb{R}$, then we call that as a ***positive dual pair*** (with respect to T).

We note that if (E, F) is a positive dual pair with bilinear map T, then one can show that the embedding $x \to x^*$, $x^*(y) = T(x, y)$ is bipositive.

The order dual of a Riesz space E is the vector space of order bounded functionals from E into $\mathbb{R}$ and denoted by $E^\sim$, which is a Dedekind complete Riesz space. Throughout the paper we suppose that E separates its order dual, that is, for each nonzero $x \in E$, there exists $f \in E^{\sim}$ with $f(x) \neq 0$. So, $(E, E^{\sim})$ is a positive dual pair via the map $(x, f) \rightarrow f(x)$. If τ is a Hausdorff locally solid topology on E, then the topological dual E' is an ideal of $E^{\sim}$. Let $A \subset E^{\sim}$ be given. For each $f \in A$, the map $p_{|f|} : E \rightarrow \mathbb{R}$. $p_{|f|}(x) = |f|(|x|)$ is a Riesz seminorm. The locally convex-solid topology generated by $(p_{|f|})_{f \in E^{\sim}}$ is called ***absolute weak topology*** and denoted by $|\sigma|(E, A)$.

Now we are going to define an unbounded absolute locally solid topology. For this, first we need the following Lemma.

Lemma 4.10. Let (E, F) be a positive dual pair with respect to T. For each $a \in E$ and $y \in F$, the map $p : E \rightarrow \mathbb{R}$ defined by

$$p(x) = T(|x| \wedge |a|, |y|)$$

is a Riesz pseudonorm on E.

Proof. Without loss of the generality, we can suppose that a and y are positive. Obviously the conditions $(1), (2)$ and (5) are satisfied . For the condition (3): for a given pair $x, y \in E$,

$$\begin{aligned} p(x+y) &= T(|x+y| \wedge a, y) \\ &\leq T(|x| + |y| \wedge a, y) \text{ by positivity} \\ &\leq T(|x| \wedge a, y) + T(|y| \wedge a, y) \text{ by bilinearity and positivity} \\ &= p(x) + p(y), \end{aligned}$$

hence, the condition (3) holds. For the condition (4), let $\{\lambda_n\} \subset \mathbb{R}$ be a sequence such that $\lim_{n\to\infty} \lambda_n = 0$ and $x \in E$, we have

$$\begin{aligned} p(\lambda_n x) = T(|\lambda_n x| \wedge a, y) &= T(|\lambda_n||x| \wedge a, y) \\ &= T(|\lambda_n|(|x| \wedge \frac{1}{|\lambda_n|}a), y) \\ &= |\lambda_n| T(|x| \wedge \frac{1}{|\lambda_n|}a), y) \\ &\leq |\lambda_n| T(|x|, y). \end{aligned}$$

So, $T(|x|, y)$is a real number,$|\lambda_n|T(|x|, y) \rightarrow 0$,thus the condition (4)also holds. □

By using the same motivation, for a given $A \subset E_+, e_0 \in E$ and $f_0 \in F$, the map $\sup_{a \in A} T(|x| \wedge a \wedge |e_0|, |f_0|)$ is also a Riesz pseudonorm, and it will be denoted by p_{A,e_0,f_0}

Definition 4.11. Let (E, F) be a positive dual pair. Let $E_0 \subset E$,$F_0 \subset F$ and $\mathscr{A} \subset \mathscr{P}(E_+)$ be nonempty sets. Then the topology generated by $(p_{A,e_0,f_0})_{A \in \mathscr{A}, e_0 \in E_0, f_0 \in F_0}$ is called unbounded locally solid Riesz space on the positive pair (E, F) with respect to E_0, F_0 and $\mathscr{A}$. This topology is denoted by $u|\sigma|(E, F), E_0, F_0, \mathscr{A})$.

By using some routine arguments, the proof of the above theorem can be given.

Theorem 4.12. Let (E, F) be a positive dual pair. Let Let $E_0 \subset E$,$F_0 \subset F$ and $\mathscr{A} \subset \mathscr{P}(E_+)$ be nonempty sets. Then

$$u|\sigma|(E, F), E_0, F_0, \mathscr{A}) = u|\sigma|(E, F), I(E_0), I(F_0), \mathscr{A}).$$

Remark 4.13. These observations and results can be extended into locally solid lattice-ordered groups studied in [14].

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On The Reformulated Zagreb Coindex

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Abstract − The reformulated Zagreb coindex of G is specified the degrees d_i and d_j. This paper includes some inequalities for the reformulated Zagreb index and reformulated Zagreb coindex. Also, some bounds are reported consepting the eigenvalues and complement of eigenvalues.

Keywords − *Reformulated Zagreb index, coindex*

1. Introduction

Let G be a simple connected graph on the vertex set $V(G)$ and the edge set $E(G)$. Also, let the degree of v_i denoted by d_i. The Reformulated Zagreb matrix of G is described with $[RZ(G)] = [rz]_{ij}$ where $[rz]_{ij} = (d_i + d_j - 2)^2$ if the vertices i is adjacent to j and $[rz]_{ij} = 0$ if otherwise.

The Reformulated Zagreb index $RZ(G)$ of G [7] is a general sum-connectivity index where

$$RZ(G) = \sum_{i,j \in E(G)} (d_G(i) + d_G(j) - 2)^2. \tag{1}$$

The Zagreb coindex of G is described in [3],

$$\bar{Z}_1(G) = \sum_{v_i, v_j \notin E(G)} (d_G(i) + d_G(j)). \tag{2}$$

In this study, different bounds are set using the degrees, the edges and the vertices. Also, some relations deal with the complement of eigenvalues of $[RZ]_{ij}$ are obtained. In Section 2, the reformulated Zagreb coindex is defined and different inequalities for this index are found.

2. Preliminaries

In this section, some back-ground material that is needed for later sections will be given.

Lemma 2.1. [4] Let $\lambda_1(M)$ be the spectral radius and $M = (m_{ij})$ be an nxn irreducible nonnegative matrix. Let $R_i(M) = \sum_{j=1}^{m} m_{ij}$. [8] Then,

$$(minR_i(M) : 1 \leq i \leq n) \leq \lambda_1(M) \leq (maxR_i(M) : 1 \leq i \leq n) \tag{3}$$

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Lemma 2.2. [2] Let V be the vertex set, $v_i \in V$, m_i be the average degree of the vertices adjacent to v_i. Then [2]

$$\lambda_1(G) \leq max(\sqrt{m_i m_j} : 1 \leq i, j \leq n, v_i, v_j \in E) \tag{4}$$

Lemma 2.3. [6] If G is a regular graph then,

$$Z_1(G) \geq \frac{4m^2}{n}.$$

Lemma 2.4. [5] If G is a regular graph then,

$$\bar{Z}_1(G) \leq \frac{-4m^2}{n} + 2m(n-1).$$

See [1] and [8] for details.

3. MAIN SERULTS

3.1. On eigenvalues

Some inequalities deal with the first eigenvalue of $[RZ(G)]$ are given in this subsection. In addition, a bound for the complement of this eigenvalue is outlined.

Theorem 3.1. If G is a simple, connected graph then

$$\lambda_1^{RZ}(G) \leq \sqrt{(F_1 + 4(m-n)d_i + 4n - 8m)(F_2 + 4(m-n)d_j + 4n - 8m)}.$$

where $\lambda_1^{RZ}(G)$ is the first eigenvalue of $[RZ(G)]$, $F_1 = (nd_i^2 + 4m^2)$ and $F_2 = ((nd_j^2 + 4m^2)$.

Proof. Let $D(G)^{-1}RZ(G)D(G) = F^+(G)$ and $X = (x_1, x_2, ..., x_n)^T$ be an eigenvector of $RZ^+(G)$. Also, $x_i = 1$ and $0 < x_k \leq 1$ for every k. Let $x_j = max_k(x_k : v_i v_k \in E)$ where i is adjacent to k. Let $RZ^+(G)X = \lambda_1^{RZ}(G)X$. If $i-th$ equation from above equation is get, then

$$\begin{aligned}\lambda_1^{RZ}(G)x_i &= \sum_k (d_i + d_k - 2)^2 x_k \\ &\leq (nd_i^2 + 4d_i(m-n) + 4n - 8m + 4m^2)x_k.\end{aligned}$$

Using Lemma 2.1, it is known that

$$\lambda_1^{RZ}(G)x_i \leq (nd_i^2 + 4d_i(m-n) + 4n - 8m + 4m^2)x_k.$$

The $j-th$ equation of the same equation,

$$\lambda_1^{RZ}(G)x_j \leq (nd_j^2 + 4d_j(m-n) + 4n - 8m + 4m^2)x_k.$$

From Lemma 2.2, the inequality holds that

$$\lambda_1^{RZ}(G) \leq \sqrt{(F_1 + 4(m-n)d_i + 4n - 8m)(F_2 + 4(m-n)d_j + 4n - 8m)}.$$

□

Corollary 3.2. Let G be a graph on n vertices and m edges. Then,

$$\bar{\lambda_1}^{RZ}(G) \leq \sqrt{K - (F_1 + 4(m-n)d_i + 4n - 8m)(F_2 + 4(m-n)d_j + 4n - 8m)}.$$

where $K = (F_1 + 4(m-n)d_i + 4n - 8m)(F_2 + 4(m-n)d_j + 4n - 8m) + (\bar{F_1} + 2(n^2 - 3n - 2m)(n - 1 - d_i) + 8(n-m) - 4n^2)(\bar{F_2} + 2(n^2 - 3n - 2m)(n - 1 - d_j) + 8(n-m) - 4n^2)$.

PROOF. Cauchy-Schwarz inequality and Theorem 3.1 gives that

$$\begin{aligned}(\lambda_1^{RZ}(G)+\bar{\lambda_1}^{RZ}(G))^2 &\leq (\lambda_1^{F}(G))^2+(\bar{\lambda_1}^{F}(G))^2\\ &\leq (F_1+4(m-n)d_i+4n-8m)(F_2+4(m-n)d_j+4n-8m)\\ &+(\bar{F_1}+2(n^2-3n-2m)(n-1-d_i)+8(n-m)-4n^2)\\ &(\bar{F_2}+2(n^2-3n-2m)(n-1-d_j)+8(n-m)-4n^2).\end{aligned}$$

Since $\bar{m}+m=\dfrac{n^2-n}{2}$ then $\bar{F_1}=n(n-1-d_i)^2+2(n(n-1)-2m)$ and $\bar{F_2}=n(n-1-d_j)^2+2(n(n-1)-2m)$. It is implies that

$$\bar{\lambda_1}^{RZ}(G)\leq\sqrt{K-(F_1+4(m-n)d_i+4n-8m)(F_2+4(m-n)d_j+4n-8m)}.$$

□

3.2. Reformulated Zagreb Coindex

In this subsection, the reformulated Zagreb coindex is concerned with. Thus, some bounds concepting this indices are obtained.

Definition 3.3. The Reformulated Zagreb coindex $\bar{RZ}(G)$ defined as

$$\bar{RZ}(G)=\sum_{v_i,v_j\notin E(G)}(d_G(i)+d_G(j)-2)^2. \tag{5}$$

Theorem 3.4. Let G be a graph on n vertices and m edges. Then,

$$RZ(\bar{G})\geq 2(n^2-n-2m)(n-2)^2-4(n-2)Z_1(\bar{G})+(Z_1(\bar{G}))^2.$$

PROOF. It is known that, $RZ(\bar{G})=\sum_{v_i,v_j\in E(\bar{G})}(d_{\bar{G}}(i)+d_{\bar{G}}(j)-2)^2$. Since $d_{\bar{G}}(i)=(n-1-d_i)$ and $d_{\bar{G}}(j)=(n-1-d_j)$ then,

$$\begin{aligned}RZ(\bar{G}) &= \sum_{v_i,v_j\in E(\bar{G})}((n-1-d_i)+(n-1-d_j)-2)^2\\ &=\sum_{v_i,v_j\in E(\bar{G})}4(n^2-2n+1)-(4n-4)\sum_{v_i,v_j\in E(\bar{G})}(d_i+d_j+2)+\sum_{v_i,v_j\in E(\bar{G})}(d_i+d_j+2)^2\end{aligned}$$

Since G has $\binom{n}{2}-m=\dfrac{n^2-n-2m}{2}$ edges, then

$$\begin{aligned}RZ(\bar{G}) &= 4(n^2-2n+1)(\frac{n^2-n-2m}{2})-4(n-1)(Z_1(\bar{G})+2(\frac{n^2-n-2m}{2}))\\ &+\sum_{v_i,v_j\in E(\bar{G})}(d_i+d_j)^2+4Z_1(\bar{G})+4(\frac{n^2-n-2m}{2})\\ &\geq 2(n^2-n-2m)(n-2)^2+4(n-2)Z_1(\bar{G})+(Z_1(\bar{G}))^2.\end{aligned}$$

□

Corollary 3.5. If G is a regular graph on n vertices and m edges. Then,

$$\begin{aligned}RZ(\bar{G}) &\geq 2(n^2-n-2m)(n-2)^2-4(n-2)(\frac{-4m^2}{n}+2m(n-1))\\ &+(\frac{-4m^2}{n}+2m(n-1))^2.\end{aligned}$$

Proof. Since $Z_1(\bar{G}) = \bar{Z}_1(G)$ then

$$RZ(\bar{G}) \geq 2(n^2 - n - 2m)(n-2)^2 - 4(n-2)\bar{Z}_1(G) + (\bar{Z}_1(G))^2$$

Using Lemma 2.4, it is concluded that

$$\begin{aligned} RZ(\bar{G}) &\geq 2(n^2 - n - 2m)(n-2)^2 - 4(n-2)(\frac{-4m^2}{n} + 2m(n-1)) \\ &+(\frac{-4m^2}{n} + 2m(n-1))^2. \end{aligned}$$

□

Theorem 3.6. Let G be a graph on n vertices and m edges. Then,

$$RZ(G) + \bar{RZ}(G) = (n-2)Z_1(G) + 2m(2m - 4n + 5) - 2n.$$

Proof.

$$\begin{aligned} RZ(G) + \bar{RZ}(G) &= \sum_{v_i,v_j \in E(G)} (d_i + d_j - 2)^2 + \sum_{v_i,v_j \notin E(G)} (d_i + d_j - 2)^2 \\ &= \frac{1}{2}(\sum_{v_i \in V(G)} \sum_{v_j \in V(G)} (d_i + d_j - 2)^2 - \sum_{v_j \in V(G)} (d_j + d_j - 2)^2) \\ &= \frac{1}{2}(\sum_{v_i \in V(G)} \sum_{j \in V(G)} (d_i^2 + d_j^2 + 2d_id_j - 4d_i - 4d_j + 4) - 4 \sum_{v_j \in V(G)} (d_j^2 - 2d_j + 1) \\ &= \frac{1}{2}(n \sum_{v_i \in V(G)} d_i^2 + \sum_{v_j \in V(G)} d_j^2 + 2 \sum_{v_i \in V(G)} d_i \sum_{v_j \in V(G)} d_j - 4n \sum_{v_i \in V(G)} d_i \\ &-4n \sum_{v_j \in V(G)} d_j + \sum_{v_i \in V(G)} \sum_{v_j \in V(G)} 4 - 4 \sum_{v_j \in V(G)} d_j^2 + 8 \sum_{v_j \in V(G)} d_j - \sum_{v_j \in V(G)} 4) \\ &= \frac{1}{2}[nZ_1(G) + nZ_1(G) + 2(2m)(2m) - 4n.2m - 4n.2m + 4m - 4Z_1(G) + 16m - 4n] \\ &= (n-2)Z_1(G) + 2m(2m - 4n + 5) - 2n. \end{aligned}$$

□

Corollary 3.7. If G is a regular graph on n vertices and m edges. Then,

$$\bar{RZ}(G) \geq 2m(\frac{4m(n-1)}{n} - 4n + 5) - 2n - RZ(G).$$

Proof. By Lemma 2.3, it is seen that

$$\begin{aligned} RZ(G) + \bar{RZ}(G) &\geq (n-2)(\frac{4m^2}{n}) + 2m(2m - 4n + 5) - 2n \\ &= 8m^2\frac{n-1}{n} - 8mn + 10m - 2n. \end{aligned}$$

Hence,

$$\bar{RZ}(G) \geq 2m(\frac{4m(n-1)}{n} - 4n + 5) - 2n - RZ(G).$$

□

Corollary 3.8. Let G be a regular graph on n vertices and m edges. Then,

$$\begin{aligned} \bar{RZ}(\bar{G}) &\leq (n^2 - n - 2m)(-n^2 + 3n - 2m - 3) - 2n \\ &+(5n-10)(\frac{-4m^2}{n} + 2m(n-1)) - (\frac{-4m^2}{n} + 2m(n-1))^2. \end{aligned}$$

Proof. By Theorem 3.7,

$$\bar{RZ}(\bar{G}) = (n-2)Z_1(\bar{G}) + 2\bar{m}(2\bar{m} - 4n + 5) - 2n - RZ(\bar{G}).$$

Since $\bar{m} = \dfrac{n^2 - n - 2m}{2}$ then,

$$\bar{RZ}(\bar{G}) = (n-2)Z_1(\bar{G}) + (n^2 - n - 2m)(n^2 - 5n - 2m + 5) - 2n - RZ(\bar{G}).$$

By Lemma 2.4 and Corollary 2.3,

$$\begin{aligned}\bar{RZ}(\bar{G}) &\leq (n^2 - n - 2m)(-n^2 + 3n - 2m - 3) - 2n \\ &\quad + (5n - 10)\left(\frac{-4m^2}{n} + 2m(n-1)\right) - \left(\frac{-4m^2}{n} + 2m(n-1)\right)^2.\end{aligned}$$

□

4. Conclusion

In this paper, Reformulated Zagreb index which is one of the topological indices in graph theory is studied. New inequalities are formed for this index in terms of the degrees, edges and vertices. Indeed, Reformulated Zagreb coindex is defined and some bounds are obtained by the help of other Zagreb indices.

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The New Exact and Approximate Solution for the Nonlinear Fractional Diffusive Predator-Prey System

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Abstract − In this article two methods, q-Homotopy analysis Method (q-HAM) and Sine-Gordon expansion method are proposed for solving fractional Diffusive Predator-Prey system. The fractional derivative is considered in the conformable sense. The obtained solutions using the suggested methods are in good agreement with the existing ones and show that these approaches can be used for solving various conformable time fractional partial differential equations arising in different branches of science.

Keywords − *Sine-Gordon Expansion Method, Fractional Diffusive Predator-Prey system, q-Homotopy Analysis Method, Conformable Fractional Derivative.*

1. Introduction

Fractional calculus has a very long history. However, this field lagged behind classic analysis. There is an increasing interest to study of the fractional differential equations because of their various applications such as in viscoelasticity, anomalous diffusion, mechanics, biology, chemistry, acoustics, control theory, etc. A great deal of effort has also been expanded in attempting to find robust and stable numerical and analytical methods for solving fractional differential equations of physical interest. In this paper, we have applied a numerical method called Homotopy analysis method and an analytical method called Sine-Gordon expansion method to obtain solutions of Fractional Diffusive Predator-Prey system. The homotopy analysis method (HAM) was first introduced by Liao [1], who employed the basic ideas of the homotopy in topology to propose a general analytic method for nonlinear problems. El-Tawil and Huseen [2] proposed a modified namely q-homotopy analysis method (q-HAM) which is a more general method of HAM. This method is applied to solve many nonlinear problems [3, 4, 5, 6].

The Sine-Gordon expansion method is an efficient and powerful technique for solving differential equations. This method is firstly proposed by the Chinese mathematician Yan [7]. The Sine-Gordon expansion method is based on the explicit linearization of differential equations for traveling waves which leads to a second-order differential equation with constant coefficients. Moreover, the solutions obtained by this method are of general nature and a number of specific solutions can be deduced by putting conditions on arbitrary constants present in the general solutions [8, 9, 10].

In this paper, we applied q-homotopy analysis and Sine-Gordon expansion methods for solving fractional Diffusive Predator-Prey system. This work is organized as follows: In section 2 we provide

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some preliminaries of conformable fractional derivative. Section 3 introduces the concept of Sine-Gordon expansion method, while section 4 gives to solutions of fractional Diffusive Predator-Prey system. The q-Homotopy analysis method (q-HAM) is analyzed in section 5. Graphics of the numerical examples are provided in section 6. The conclusions are given in section 7.

2. Governing equations

One of the most popular fractional predator-prey system in nonlinear fractional evolution equations can be expressed as follows (for $\alpha = 1$, see [11, 12])

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{\partial^2 u}{\partial x^2} - \beta u + (1+\beta)u^2 - u^3 - uv, \tag{1}$$

$$\frac{\partial^\alpha v}{\partial t^\alpha} = \frac{\partial^2 v}{\partial x^2} + \kappa uv - mv - \delta v^3, \tag{2}$$

where κ, δ and β are positive parameters, and where $\frac{\partial^\alpha}{\partial t^\alpha}$ is conformable deravative operator of order $\alpha \in (0,1)$ in the $t > 0$ can be defined as follows [14]

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \lim_{\varepsilon \to 0} \frac{u(t + \varepsilon t^{1-\alpha}) - u(t)}{\varepsilon}.$$

Later on, many useful methods for obtaining exact solutions of several nonlinear fractional evolution equations by using this fractional derivative have been reported [15-33].
In this paper, we investigate a fractional order prey-predator interaction with following relations between the parameters

$$m = \beta, \qquad \kappa + \frac{1}{\sqrt{\delta}} = \beta + 1.$$

Based on these assumptions, Eqs. (2.1) and (2.2) are established by the following

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{\partial^2 u}{\partial x^2} - \beta u + \left(\kappa + \frac{1}{\sqrt{\delta}}\right) u^2 - u^3 - uv \tag{3}$$

$$\frac{\partial^\alpha v}{\partial t^\alpha} = \frac{\partial^2 v}{\partial x^2} + \kappa uv - \beta v - \delta v^3. \tag{4}$$

The fractional prey-predator system incorporating diffusion is of profound interest because it involves the heterogeneity of both the populations the environment. Formation of the spatial distribution pattern with the diffusion models even in the absence of environmental heterogeneity is another interesting event [13]. For better understand about the processes involved, existences of exact solutions are needed.

3. Sine-Gordon expansion method

In this section we describe the first step of the Sine-Gordon expansion method for finding exact solutions of nonlinear conformable fractional partial differential equations (PDEs).
We consider the following time conformable fractional nonlinear partial differential equation in two variables and a dependent variable u

$$F\left(u, \frac{\partial^\alpha u}{\partial t^\alpha}, \frac{\partial u}{\partial x}, \frac{\partial^{2\alpha} u}{\partial t^{2\alpha}}, \frac{\partial^2 u}{\partial x^2}, \ldots\right) = 0, \tag{5}$$

where F is a polynomial in u and its various partial derivatives, in which the highest order derivatives and nonlinear terms are involved and $\frac{\partial^{2\alpha} u}{\partial t^{2\alpha}}$ means two times conformable fractional derivative of function $u(x,t)$. To solve Eq.(5), we take the traveling wave transformation

$$u(x,t) = U(\xi), \qquad \xi = x - c\frac{t^\alpha}{\alpha}, \tag{6}$$

where $c \neq 0$ is a constant to be determined later. This enables us to use the following changes

$$\frac{\partial^\alpha(.)}{\partial t^\alpha} = -c\frac{d(.)}{d\xi}, \qquad \frac{\partial(.)}{\partial x} = \frac{d(.)}{d\xi}, \qquad \frac{\partial^{2\alpha}(.)}{\partial t^{2\alpha}} = -c\frac{d^2(.)}{d\xi^2}, \ldots.$$

Substituting Eq.(6) in Eq. (5) yields a nonlinear ordinary differential equation as following

$$G(U, U', U'', U''', \ldots) = 0, \tag{7}$$

where $U = U(\xi), U' = \frac{dU}{d\xi}, U'' = \frac{d^2U}{d\xi^2}, \ldots$ and so on.

Now lets describe the procedure of Sine-Gordon expansion method.This method established on the Sine-Gordon equation and wave transform. The Sine-Gordon equation which is presented as a model field theory;

$$u_{xx} - u_{tt} = \tau^2 sin(u), \tag{8}$$

where τ is a real constant and $u = u(x,t)$. Considering the wave transformation $\xi = \mu(x - ct)$ over the Eqn. (8) the function $u = u(x,t)$, turns into $U(\xi)$, then we have the following nonlinear differential equation,

$$U'' = \frac{\tau^2}{\mu^2(1-c^2)} \sin(U). \tag{9}$$

By simplifying the Eq. (9),

$$\left[\left(\frac{U}{2}\right)'\right]^2 = \frac{\tau^2}{\mu^2(1-c^2)} \sin^2\left(\frac{U}{2}\right) + K, \tag{10}$$

where K is integration constant. Supposing $K = 0$, $\Phi(\xi) = \frac{U}{2}$, $\varrho^2 = \frac{\tau^2}{\mu^2(1-c^2)}$ and subrogating into Eqn. (10),

$$\Phi' = \varrho \sin(\Phi), \tag{11}$$

regarding $\varrho = 1$ in Eqn. (11), led to

$$\Phi' = \sin(\Phi). \tag{12}$$

Evaluating the solution of (12) by using separation of variables method, we attain the following equations,

$$\sin(\Phi) = \sin(\Phi(\xi)) = \frac{2\zeta e^\xi}{\zeta^2 e^{2\xi} + 1}|_{\zeta=1} = \text{sech}(\xi), \tag{13}$$

$$\cos(\Phi) = \sin(\Phi(\xi)) = \frac{\zeta^2 e^{2\xi} - 1}{\zeta^2 e^{2\xi} + 1}|_{\zeta=1} = \tanh(\xi), \tag{14}$$

where ζ is integration constant. To obtain the solution of nonlinear conformable PDE (5);

$$G(u, D_t^\alpha u, D_x u, D_{xx} u, D_t^\alpha D_t^\alpha u, \ldots), \tag{15}$$

we design,

$$U(\xi) = \sum_{i=1}^{n} \tanh^{i-1}(\xi) \left[B_i \,\text{sech}(\xi) + A_i \tanh(\xi)\right] + A_0, \tag{16}$$

due to to Eqns. (13) and (14), Eqn. (16) can be regulated as

$$U(\Phi) = \sum_{i=1}^{n} \cos^{i-1}(\Phi) \left[B_i \sin(\Phi) + A_i \cos(\Phi)\right] + A_0. \tag{17}$$

The parameter n can be determined balancing the degrees between the highest order linear term and nonlinear term in Eq.(7). Next equating all the coefficients of $\cos^i(\Phi)$ and $\sin^i(\Phi)$ to be zero yields an equation system. Solving system using an computer software such as Maple the values of A_i, B_i, μ and c can be derived. Lastly subrogating the values of A_i, B_i, μ and c in Eqn. (16), we can express the traveling wave solutions.

4. Application of the Sine-Gordon expansion method to Fractional Diffusive Predator-Prey system

We suppose that

$$u(x,t) = U(\xi), \qquad v(x,t) = V(\xi), \qquad \xi = x + \nu\frac{t^\alpha}{\alpha}, \tag{18}$$

where ν is constant. Using the conformable chain rule and on substituting these into Eq.(3), we have

$$\begin{aligned} &U'' - \nu U' - \beta U + \left(\kappa + \frac{1}{\sqrt{\delta}}\right) U^2 - U^3 - UV = 0, \\ &V'' - \nu V + \kappa UV - \beta UV - \delta V^3 = 0. \end{aligned} \tag{19}$$

In order to solve system (19), let us consider the following transformation

$$V = \frac{U}{\sqrt{\delta}}. \tag{20}$$

Substituting the transformation (20) into (19),we get

$$U'' - \nu U' - \beta U + \kappa + U^2 - U^3 = 0. \tag{21}$$

Due to procedure of Sine-Gordon expansion method, assume that U can be written in the form

$$U(\Phi) = \sum_{i=1}^{n} \cos^{i-1}(\Phi)\left[B_i \sin(\Phi) + A_i \cos(\Phi)\right] + A_0. \tag{22}$$

Balancing the terms U'' and U^3 led to $n = 1$, thus

$$U = B\sin\Phi + A\cos\Phi + C, \tag{23}$$

and

$$U'' = -B(\sin\Phi)^3 + B(\cos\Phi)^2\sin\Phi - 2A(\sin\Phi)^2\cos\Phi. \tag{24}$$

Replacing the equations (23) and (24) into (21), using some trigonometric identities and setting all the coefficients of $\cos^i\Phi$ and $\sin^i\Phi$ produces the following algebraic equation system

$$\begin{aligned} &A^3 + 3B^2A + 2A = 0, \\ &2B + B^3 - 3BA^2 = 0, \\ &\kappa A^2 + 3B^2C - 3A^2C - \kappa B^2 + \nu A = 0, \\ &\nu B + 2\kappa BA - 6BAC = 0, \\ &2\kappa AC - \beta A - 3B^2A - 3AC^2 - 2A = 0, \\ &-\beta B + 2\kappa BC - B - B^3 - 3BC^2 = 0, \\ &\kappa B^2 - 3B^2C - \beta C + \kappa C^2 - C^3 - \nu A = 0. \end{aligned} \tag{25}$$

Solving the system with the aid of Maple, we obtain the following solution sets,

$$A = \mp\sqrt{2}, B = 0, C = \mp\frac{\nu}{\sqrt{2}}, \beta = -2 + \frac{\nu^2}{2}, \kappa = \mp\sqrt{2}\nu,$$

$$A = \pm\sqrt{2}, B = 0, C = \mp\sqrt{2}, \beta = 4 + 2\nu, \kappa = \mp\frac{\sqrt{2}(6+\nu)}{2},$$

$$A = \mp\sqrt{2}, B = 0, C = \mp\sqrt{2}, \beta = 4 - 2\nu, \kappa = \pm\frac{\sqrt{2}(\nu-6)}{2},$$

$$A = \pm\frac{\sqrt{2}}{2}, B = \mp\frac{i\sqrt{2}}{2}, C = \mp\frac{\sqrt{2}}{2}, \beta = \nu + 1, \kappa = \mp\frac{\sqrt{2}(3+\nu)}{2},$$

$$A = \mp\frac{\sqrt{2}}{2}, B = \mp\frac{i\sqrt{2}}{2}, C = \mp\frac{\sqrt{2}}{2}, \beta = 1 - \nu, \kappa = \pm\frac{\sqrt{2}(-3+\nu)}{2},$$

$$A = \mp\frac{\sqrt{2}}{2}, B = \mp\frac{i\sqrt{2}}{2}, C = \mp\frac{\nu\sqrt{2}}{2}, \beta = \frac{\nu^2-1}{2}, \kappa = \mp\sqrt{2}\nu.$$

Using the above values of A, B, C, β, κ and (20), the solutions of $u(x,t)$ and $v(x,t)$ can be obtained as

$$u_1(x,t) = \mp\frac{\nu}{\sqrt{2}} \mp \sqrt{2}\tanh\left[x + \frac{t^\alpha \nu}{\alpha}\right],$$
$$u_2(x,t) = \mp\left(\sqrt{2} - \sqrt{2}\tanh\left[x + \frac{t^\alpha \nu}{\alpha}\right]\right),$$
$$u_3(x,t) = \mp\left(\sqrt{2} + \sqrt{2}\tanh\left[x + \frac{t^\alpha \nu}{\alpha}\right]\right),$$
$$u_4(x,t) = \pm\left(-\frac{1}{\sqrt{2}} - \frac{i\text{sech}\left[x + \frac{t^\alpha \nu}{\alpha}\right]}{\sqrt{2}} + \frac{\tanh\left[x + \frac{t^\alpha \nu}{\alpha}\right]}{\sqrt{2}}\right),$$
$$u_5(x,t) = \pm\left(-\frac{1}{\sqrt{2}} - \frac{i\text{sech}\left[x + \frac{t^\alpha \nu}{\alpha}\right]}{\sqrt{2}} - \frac{\tanh\left[x + \frac{t^\alpha \nu}{\alpha}\right]}{\sqrt{2}}\right),$$
$$u_6(x,t) = \mp\left(\frac{\nu}{\sqrt{2}} + \frac{i\text{sech}\left[x + \frac{t^\alpha \nu}{\alpha}\right]}{\sqrt{2}} + \frac{\tanh\left[x + \frac{t^\alpha \nu}{\alpha}\right]}{\sqrt{2}}\right),$$

$$v_1(x,t) = \frac{\mp\frac{\nu}{\sqrt{2}} \mp \sqrt{2}\tanh\left[x + \frac{t^\alpha \nu}{\alpha}\right]}{\sqrt{\delta}},$$
$$v_2(x,t) = \frac{\sqrt{2} - \sqrt{2}\tanh\left[x + \frac{t^\alpha \nu}{\alpha}\right]}{\sqrt{\delta}},$$
$$v_3(x,t) = \frac{-\sqrt{2} + \sqrt{2}\tanh\left[x + \frac{t^\alpha \nu}{\alpha}\right]}{\sqrt{\delta}},$$
$$v_4(x,t) = \pm\left(\frac{-\frac{1}{\sqrt{2}} - \frac{i\text{sech}\left[x+\frac{t^\alpha \nu}{\alpha}\right]}{\sqrt{2}} + \frac{\tanh\left[x+\frac{t^\alpha \nu}{\alpha}\right]}{\sqrt{2}}}{\sqrt{\delta}}\right),$$
$$v_5(x,t) = \pm\left(\frac{-\frac{1}{\sqrt{2}} - \frac{i\text{sech}\left[x+\frac{t^\alpha \nu}{\alpha}\right]}{\sqrt{2}} - \frac{\tanh\left[x+\frac{t^\alpha \nu}{\alpha}\right]}{\sqrt{2}}}{\sqrt{\delta}}\right),$$
$$v_6(x,t) = \mp\left(\frac{\frac{\nu}{\sqrt{2}} + \frac{i\text{sech}\left[x+\frac{t^\alpha \nu}{\alpha}\right]}{\sqrt{2}} + \frac{\tanh\left[x+\frac{t^\alpha \nu}{\alpha}\right]}{\sqrt{2}}}{\sqrt{\delta}}\right).$$

5. Numerical Solution of Fractional Diffusive Predator-Prey system

In this section, we implement q-homotopy analysis method (q-HAM) which is generalized version of homotopy analysis method (HAM) [34] to obtain the numerical solution of diffusive predator-prey system. q-HAM involves the parameter h which is used for adjusting and controlling the convergence of solution series.(See [35, 36]) Regard the nonlinear system of equations in with the following initial conditions

$$\begin{aligned} u(x,0) &= 1 + \sqrt{2}\tanh[x], \\ v(x,0) &= 2 + 2\sqrt{2}\tanh[x]. \end{aligned} \tag{26}$$

We consider the coefficients $\nu = \sqrt{2}, \delta = \frac{1}{4}, \kappa = 2, \beta = 2$ for both calculations in the rest of article. We can chose the linear operators To obtain the series solutions of system of equations in (3) with initial conditions 26) as follows

$$\begin{aligned} \mathcal{L}_1\left[\varphi_1(x,t;q)\right] &= D_t^\alpha \varphi_1(x,t;q), \\ \mathcal{L}_2\left[\varphi_2(x,t;q)\right] &= D_t^\alpha \varphi_2(x,t;q), \end{aligned}$$

where the linear operators satisfies the condition $\mathcal{L}_j[m]=0$ for each $j\in\{1,2\}$, where m is constant. The non-linear operators can be defined from the system (3) such as

$$\begin{aligned}
\mathcal{N}_1[\varphi_1(x,t;q)] &= \frac{\partial^\alpha\varphi_1(x,t;q)}{\partial t^\alpha}-\frac{\partial^2\varphi_1(x,t;q)}{\partial x^2}+\beta\varphi_1(x,t;q),\\
&- \left(\kappa+\frac{1}{\sqrt{\delta}}\right)\varphi_1(x,t;q)^2-\varphi_1(x,t;q)^3+\varphi_1(x,t;q)\varphi_2(x,t;q),\\
\mathcal{N}_2[\varphi(x,t;q)] &= \frac{\partial^\alpha\varphi_2(x,t;q)}{\partial t^\alpha}-\frac{\partial^2\varphi_2(x,t;q)}{\partial x^2}-\kappa\varphi_1(x,t;q)\varphi_2(x,t;q)+\beta\varphi_2(x,t;q)+\delta\varphi_2(x,t;q)^3.
\end{aligned}$$

So the zero-order deformation equations can be constituted as:

$$\begin{aligned}
(1-nq)\mathcal{L}_1[\varphi_1(x,t;q)-u_0(x,t)] &= qh_1\mathcal{N}[\varphi_1(x,t;q)],\\
(1-nq)\mathcal{L}_2[\varphi_2(x,t;q)-v_0(x,t)] &= qh_2\mathcal{N}[\varphi_2(x,t;q)].
\end{aligned}$$

When $H_j(x,t)=1$ chosen properly [35], for each $j\in\{1,2\}$, the mth-order deformation equation is

$$u_m(x,t) = \chi_m^* u_{m-1}(x,t)+h_1\mathcal{L}_1^{-1}[R_{1,m}(\mathbf{u}_{m-1})], \tag{27}$$
$$v_m(x,t) = \chi_m^* v_{m-1}(x,t)+h_2\mathcal{L}_2^{-1}[R_{2,m}(\mathbf{v}_{m-1})], \tag{28}$$

where χ_m^*

$$\chi_m^*=\begin{cases}0 & m\leqslant 1,\\ n & otherwise.\end{cases} \tag{29}$$

Finally using using Equations (27) and (28) with initial conditions given by (26), we respectively obtain the approximate analytical solutions

$$\begin{aligned}
u_0(x,t) &= 1+\sqrt{2}\tanh[x],\\
v_0(x,t) &= 2+2\sqrt{2}\tanh[x],\\
u_1(x,t) &= \frac{ht^\alpha\text{sech}[x]^2\left(-1+3\text{cosh}[2x]+3\sqrt{2}\text{sinh}[2x]\right)}{2\alpha},\\
v_1(x,t) &= \frac{ht^\alpha\text{sech}[x]^2\left(-1+3\text{cosh}[2x]+3\sqrt{2}\text{sinh}[2x]\right)}{\alpha},\\
u_2(x,t) &= \frac{h^2t^\alpha\text{sech}[x]^3\left((-45t^\alpha+2\alpha)\cosh[x]+(33t^\alpha+6\alpha)\cosh[3x]\right)}{8\alpha^2}\\
&+ \frac{2\sqrt{2}\left(12\alpha\text{cosh}[x]^2+t^\alpha(-5+27\text{cosh}[2x])\right)\sinh[x]}{8\alpha^2}\\
&+ \frac{hnt^\alpha\text{sech}[x]^2\left(-1+3\text{cosh}[2x]+3\sqrt{2}\text{sinh}[2x]\right)}{2\alpha},\\
v_2(x,t) &= \frac{h^2t^\alpha\text{Sech}[x]^3\left(2\sqrt{2}\left(12\alpha\text{Cosh}[x]^2+t^\alpha(-5+27\text{Cosh}[2x])\right)\text{Sinh}[x]\right)}{4\alpha^2}\\
&\quad \frac{h^2t^\alpha\text{Sech}[x]^3\left((-45t^\alpha+2\alpha)\,\text{Cosh}[x]+(33t^\alpha+6\alpha)\,\text{Cosh}[3x]+2\sqrt{2}\left(12\alpha\text{Cosh}[x]^2\right)\right)}{4\alpha^2}\\
&\quad +\frac{h^2t^{2\alpha}\text{Sinh}[x]\text{Sech}[x]^3(-5+27\text{Cosh}[2x])}{4\alpha^2}\\
&\quad +\frac{hnt^\alpha\text{Sech}[x]^2\left(-1+3\text{Cosh}[2x]+3\sqrt{2}\text{Sinh}[2x]\right)}{\alpha},\\
&\vdots
\end{aligned}$$

We can obtain $u_m(x,t)$, $v_m(x,t)$, for $m=3,4,5,\cdots$, following the same approach, using Mathematica, Maple or MATLAB.

As a result series solution expression by q-HAM can be written in the form

$$u(x,t,n,h) = 1+\sqrt{2}\tanh[x]+\sum_{i=1}^{\infty} u_i(x,t;n;h)\left(\frac{1}{n}\right)^i, \tag{30}$$

$$v(x,t,n,h) = 2+2\sqrt{2}\tanh[x]+\sum_{i=1}^{\infty} v_i(x,t;n;h)\left(\frac{1}{n}\right)^i. \tag{31}$$

6. Graphical Comparisons

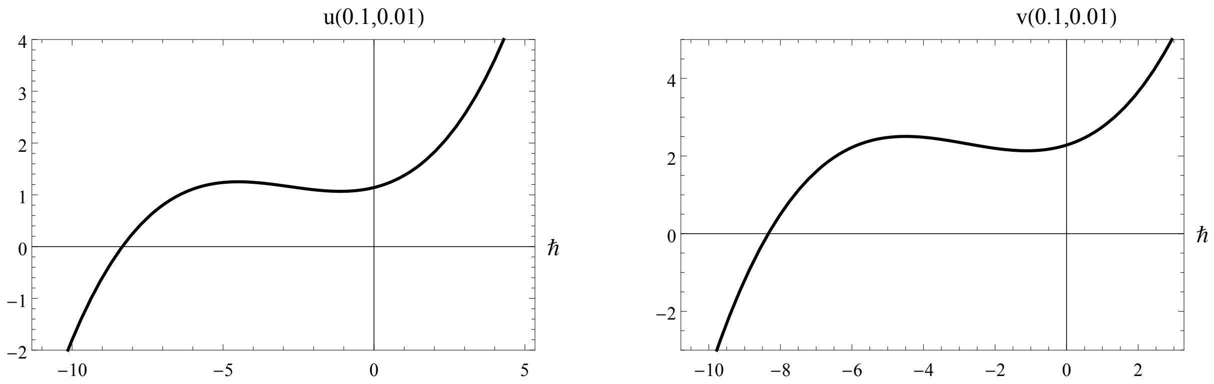

Fig. 1. The h-curves of $u(x,t)$ and $v(x,t)$ for $x = 0.1, t = 0.01, \alpha = 0.7$ respectively.

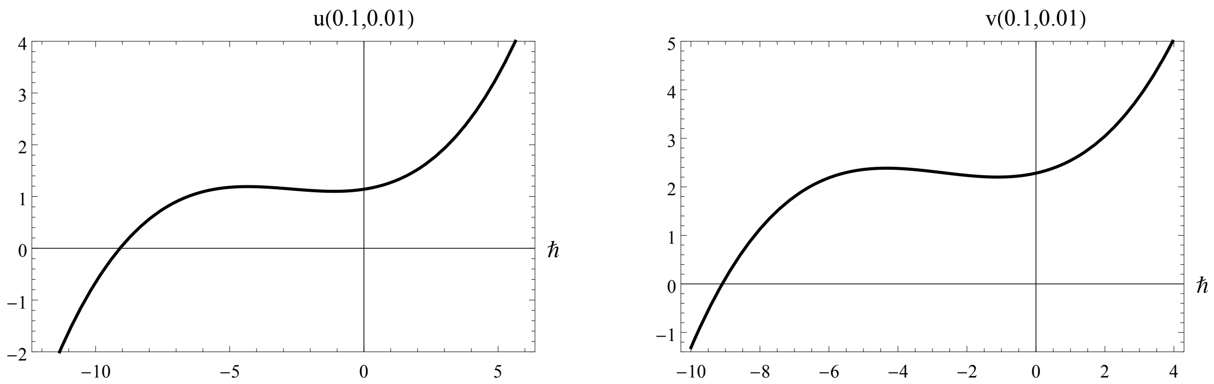

Fig. 2. The h-curves of $u(x,t)$ and $v(x,t)$ for $x = 0.1, t = 0.01, \alpha = 0.8$ respectively.

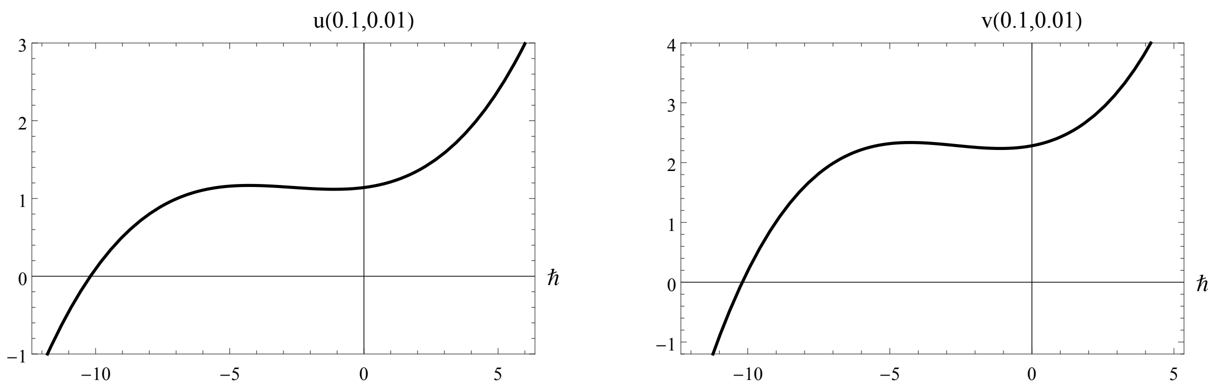

Fig. 3. The h-curves of $u(x,t)$ and $v(x,t)$ for $x = 0.1, t = 0.01, \alpha = 0.9$ respectively.

Figures $1, 2, 3$ show the convergence region of the obtained approximate solutions. By the help of this graphics we can adjust and control the convergence of approximate analytical solution to the exact solution. These graphics helps us for choosing appropriate value of h which is involved in the series solutions (30) and (31). As a consequence of this choice of h the following graphics appears. Both of these graphics shows the obtained numerical solutions are converges properly to exact solutions for different values of α.

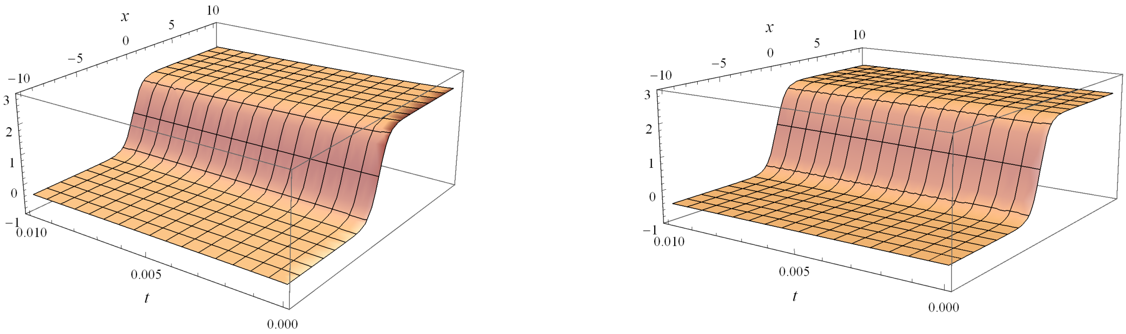

Fig. 4. The graphics of the numerical and exact solutions of $u(x,t)$ for $h=-1, \alpha=0.7$ respectively.

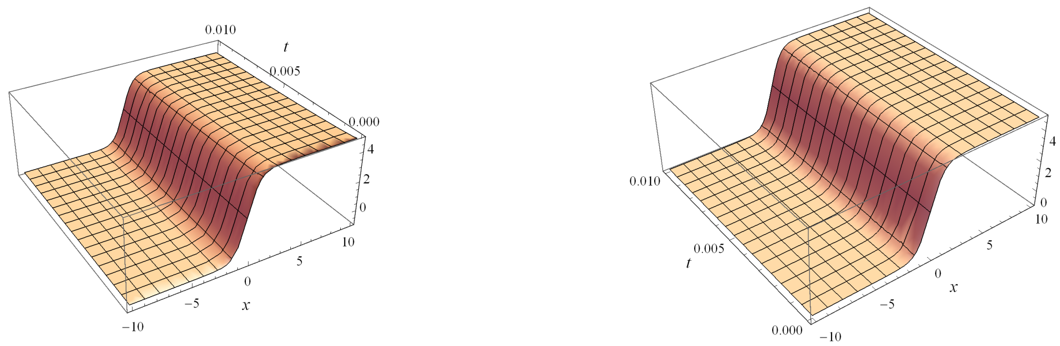

Fig. 5. The graphics of the numerical and exact solutions of $v(x,t)$ for $h=-1, \alpha=0.7$ respectively.

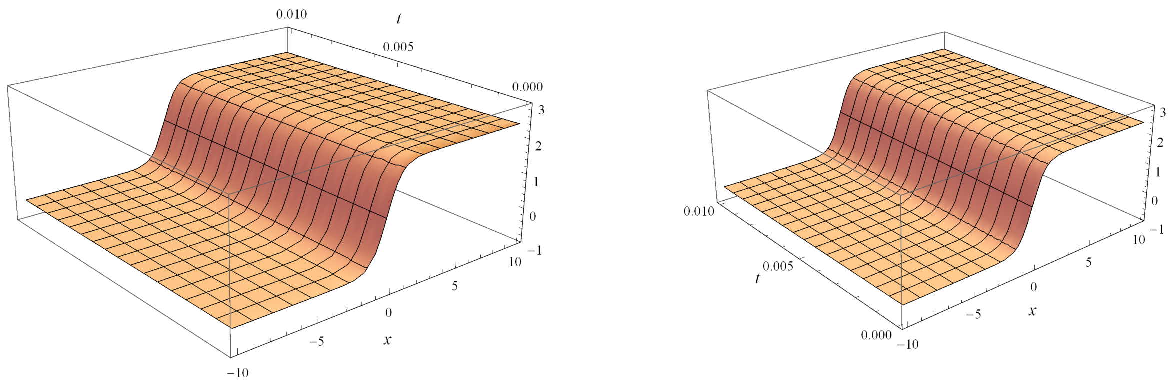

Fig. 6. The graphics of the numerical and exact solutions of $u(x,t)$ for $h=-1, \alpha=0.8$ respectively.

To be more satisfying lets give the numerical comparisons of both exact and approximate analytical solutions over Figures 4,5,6,7 and 8.

7. Conclusion

In this work, we successfully apply the q-homotopy analysis method and Sine-Gordon expansion method to obtain solutions of Fractional Diffusive Predator-Prey system. It may be concluded that the two methods are powerful and efficient techniques for finding exact as well as approximate solutions

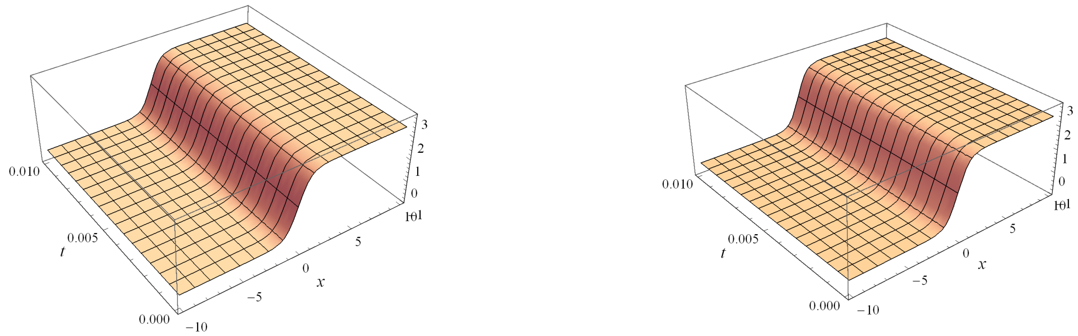

Fig. 7. The graphics of the numerical and exact solutions of $u(x,t)$ for $h=-1,\alpha=0.9$ respectively.

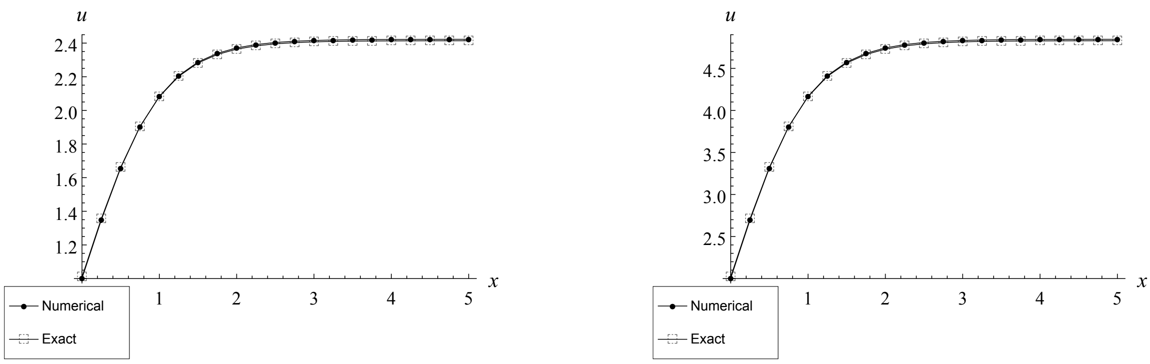

Fig. 8. Comparisons of solutions for $t=0.001,\alpha=0.8,h=-2.7$.

ofhomogeneous fractional partial differential equations. The results reveal that these methods are very effective, convenient and quite accurate to systems of fractional nonlinear equations.

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All authors contributed equally to this work.

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On Strong Pre*-I-Open Sets in Ideal topological Spaces

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Abstract — The aim of the present paper is to introduce the class of *strong* $pre^* - I - open$ *sets* which is strictly placed between the class of all $pre - I - open$ and the class of all $pre^* - I - open$ *subsets* of X. Relationships with some other types of sets were given. Furthermore, by using the new notion, we defined the *strong* $pre^* - I - Interior$ and *strong* $pre^* I - closure$ operators and established their various properties.

Keywords — *Local functions, ideal topological spaces, strong pre*-I-open sets, strong pre*-I-closed sets*

1. Introduction and Preliminaries

Kuratowski [1] defined the concept of ideals in topological spaces. Jankovic and Hamlett [2] introduced the notion of I-open sets in topological spaces. Several kinds of $I - openness$ have been initiated. Abd El-Monsef et al. [3] investigated further properties of $I - open$ sets and $I - continuous$ functions. Dontchev [4] introduced the notion of $pre-I-open$ sets and obtained a decomposition of $I-continuity$. In 2002, Hatir and Noiri [5] presented the concept of $semi - I - open$ sets in ideal topological spaces. Recently, Ekici tntrodused the notions of $pre^* - I - open$ [6]. In this paper, we define the notions of $strong\ pre*-I-open$ sets and $strong\ pre*-I-closed$ sets. Several characteristics and properties are studied. Throughout the present paper, (X, τ) will denote topological spaces on which no separation property is assumed unless explicity stated. In topological space (X, τ), the *closure* and the *interior* of any subset A of X will be denoted by $cl(A)$ and $int(A)$, respectively. An ideal I on X is defined as a nonempty collection of subsets of X satisfying the following two conditions: (1) If $A \in I$ and $B \subset A$, then $B \in I$. (2) If $A \in I$ and $B \in I$, then $A \cup B \in I$. Let (X, τ) be a topological space and I an ideal on X. An ideal topological space is a topological space (X, τ) with an ideal I on X and denoted by (X, τ, I). For a subset $A \subset X$, $A^*(I, \tau) = x \in X | U \cap A \notin I foreachneighborhoodU ofx$ is called the local function of A with respect to I and τ [2]. It is obvious that $(.)^* : (X) \to (X)$ is a set operator. Throughout this paper, we use A^* instead of $A^*(I, \tau)$. Besides, in [7], authors introduced a new Kuratowski closure operator $cl^*(.)$ defined by $cl^*(A) = A \cup A^*$ and obtained a new topology on X which is called *-topology. This topology is denoted by τ^* which is finer than τ. We start with recalling some lemmas and definitions which are necessary for this study in the sequel.

Lemma 1.1. [2] Let (X, τ) be a topological space and I an ideal on X. For every subset A of X, the following property holds: $A^* \subset cl(A)$.

Definition 1.2. A subset A of an ideal topological space (X, τ, I) is called:

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1. $pre - open$, if $A \subset int(cl(A))$ [8];
2. $pre - I - open$, if $A \subset int(cl^*(A))$ [4];
3. $pre^* - I - open$, if $A \subset itn^*(cl(A))$ [6];
4. $\alpha - I - open$, if $A \subset int(cl^*(int(A)))$ [5];
5. $semi - I - open$, if $A \subset cl^*(int(A))$ [5];
6. $pre - I - regular$, if A is $pre - I - open$ and $pre - I - closed$ [9];
7. $strong\ semi^* - I - open$, if $A \subset cl^*(int^*(A))$ [10];
8. $\beta^* - I - open$, if $A \subset cl(int^*(cl(A)))$ [6] ;
9. $strong\beta - I - open$, if $A \subset cl^*(int(cl^*(A)))$ [11];
10. $\beta - I - open$, if $A \subset cl(int(cl^*(A)))$ [5] ;
11. $\beta - open$, if $A \subset cl(int(cl(A)))$ [12] ;
12. $weakly\ semi - I - open$, if $A \subset cl^*(int(cl(A)))$ [13];
13. $I - open$, if $A \subset int(A^*)$ [3];
14. $almost\ strong - I - open$, if $A \subset cl^*(int(A^*))$ [14];
15. $* - perfect$, if $A = A^*$ [15];
16. $C^* - I - set$, if $A = L \cap M$, where $L \in \tau$ and M is $pre - I - regular$ [9];
17. $S - I - set$, if $int(A) = cl^*(int(A))$ [13].

Definition 1.3. [16] In ideal topological space (X, τ, I), I is said to be codence if $\tau \cap I = \phi$.

Lemma 1.4. ([17]) Let (X, τ, I) be an ideal space, where I is codence, then the following hold:

1. $cl(A) = cl^*(A)$, for every $* - open$ set A;
2. $int(A) = in^*(A)$, for every $* - closed$ set A.

Lemma 1.5. [18] For a subset A of an ideal topological space (X, τ, I), the following are hold:

1. $pIcl(A) = A \cup cl(int^*(A))$;
2. $pIint(A) = A \cap int(cl^*(A))$;
3. $sIcl(A) = A \cup int^*(cl(A))$;
4. $sIint(A) = A \cap cl^*(int(A))$.

Lemma 1.6. [2] For two subsets, A and B of a space (X, τ, I), the following are hold:

1. If $A \subset B$, then $A^* \subset B^*$;
2. If $U \in \tau$, then $(U \cap A^*) \subset (U \cap A)^*$.

Lemma 1.7. [14] Let A be a subset of an ideal topological space (X, τ, I) and U be an open set. Then, $U \cap cl^*(A) \subset cl^*(U \cap A)$.

Lemma 1.8. [17] Let (X, τ, I) be an ideal space and A be a $* - dense$ in $itself$ subset of X. Then $A^* = cl(A^*) = cl(A) = cl^*(A)$.

Definition 1.9. [7] An ideal topological space (X, τ, I) is said to be I-extremally disconnected if $cl^*(A) \in \tau$ for each $A \in \tau$.

Lemma 1.10. [19] A subset A of an ideal topological space (X, τ, I) is weakly I-local closed if and only if there exists an open set U such that $A = U \cap cl^*A$.

Lemma 1.11. [20] An ideal topological space (X, τ, I) is I-extremally disconnected if and only if $cl^*(int(A)) \subset int(cl^*(A))$, for every subset A of X.

2. Strong Pre*-I-Open Sets

Definition 2.1. A subset A of an ideal topological space (X, τ, I) is said to be *strong* $pre^* - I - open$ (briefly $S.P^* - I - open$) if $A \subset int^*(cl^*(A))$. We denote that all $S.P^* - I - open$ by $S.P^* - I - O(X)$.

Lemma 2.2. Let (X, τ, I) be an ideal topological space, the followings hold, for any subset A of X:

1. Every $pre - I - open$ set is a $S.P^* - I - open$.
2. Every $S.P^* - I - open$ set is a $pre^* - I - open$.

The following diagram holds for any subset A of an ideal topological space (X, τ, I).

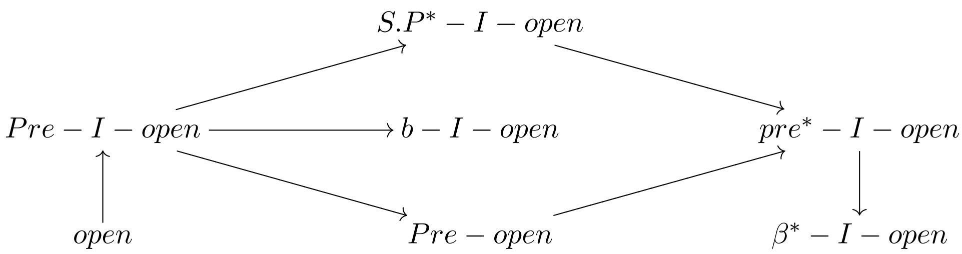

Figure 1. The implication between some generalizations of open sets

Remark 2.3. The converses of these implications in Diagram 1 are not true in general as shown in the following examples:

Example 2.4. Let $X = \{a, b, c, d\}$, $\tau = \{\phi, X, \{a\}, \{b, c\}, \{a, b, c\}\}$ and $I = \{\phi, \{a\}, \{d\}, \{a, d\}\}$. Then $A = \{c, d\}$ is a $S.P^* - I - open$, but it is not $pre - I - open$.

Example 2.5. Let $X = \{a, b, c, d\}$, $\tau = \{\phi, X, \{c\}, \{a, b, d\}\}$ and $I = \{\phi, \{a\}\}$. Then $A = \{a\}$ is a $pre^* - I - open$ set, but it is not $S.P^* - I - open$.

Remark 2.6. The strong $pre^* - I - open$ sets and $b - I - open$ sets are independent notions, we show that from the next examples:

Example 2.7. Let $X = \{a, b, c, d\}$, $\tau = \{\phi, X, \{a\}, \{b, c\}, \{a, b, c\}\}$ and $I = \{\phi, \{a\}, \{d\}, \{a, d\}\}$, if we take $A = \{b, d\}$, then we get A is not $b - I - open$ but it is $S.P^* - I - open$.

Example 2.8. Let $X = \{a, b, c\}$, $\tau = \{\phi, X, \{a\}, \{c\}, \{a, c\}\}$ and $I = \{\phi, \{b\}\}$. Then $A = \{a, b\}$ is a $b - I - open$ but it is not $S.P^* - I - open$.

Example 2.9. Let $X = \{a, b, c, d\}$, $\tau = \{\phi, X, \{d\}, \{a, c\}, \{a, c, d\}\}$ and $I = \{\phi, \{c\}, \{d\}, \{c, d\}\}$. Then $A = \{c\}$ is $pre - open$ but it is not $S.P^* - I - open$.

Example 2.10. Let $X = \{a, b, c, d\}$, $\tau = \{\phi, X, \{a\}, \{b\}, \{a, b\}\}$ and $I = \{\phi, \{b\}, \{c\}, \{b, c\}\}$. Then $A = \{a, c\}$ is an $S.P^* - I - open$ set, but it is not $pre - open$.

From Examples 2.6 and 2.5, we conclude that the concepts of $pre - open$ sets and $S.P^* - I - open$ sets are independent notions.

Theorem 2.11. Let (X, τ, I) be an ideal topological space then, A is an $S.P^* - I - open$ set if and only if there exists an $S.P^* - I - open$ B such that $A \subset B \subset cl^*(A)$.

Proof. Let A be a $S.P^* - I - open$, then $A \subset int^*(cl^*(A))$. We put $B = int^*(cl^*(A))$, which is a $* - open$ set. Therefore $B = int^*(B) \subset int^*(cl^*(B))$ be an $S.P^* - I - open$ set Such that $A \subset B = int^*(cl^*(B)) \subset cl^*(A)$.
Conversely, if B is an $S.P^* - I - open$ set such that $A \subset B \subset cl^*(A)$, taking $* - closure$, then $cl^*(A) \subset cl^*(B)$. On the other hand $A \subset B \subset int^*(cl^*(B)) \subset int^*(cl^*(A))$. Which shows that A is $S.P^* - I - open$. □

Corollary 2.12. Let (X, τ, I) be an ideal topological space, then A is a $S.P^* - I - open$ set if and only if there exists an open set $A \subset B \subset cl^*(A)$.

Proof. Obvious. □

Corollary 2.13. Let (X,τ,I) be an ideal topological space. If A is an $S.P^*-I-open$ set, then $cl^*(A)$ is a $strong\ semi^*-I-open$ set.

Proof. Let A be $S.P^*-I-open$. Then $A\subset int^*(cl^*(A))$ and $cl^*(A)\subset cl^*(int^*(cl^*(A)))$. This implies $cl^*(A)$ is a $strong\ semi^*-I-open$. □

Corollary 2.14. Let (X,τ,I) be an ideal topological space. If A is a strong $semi^*-I-open$, then $int^*(A)$ is an $S.P^*-I-open$ set.

Proof. Let A be strong $semi^*-I-open$, then $A\subset cl^*(int^*(A))\Rightarrow int^*(A)\subset int^*(cl^*(int^*(A)))$. This implies $int^*(A)$ is an $S.P^*-I-open$. □

Theorem 2.15. Let (X,τ,I) be an ideal topological space, A and B are subsets of X. the following are hold:

1. If $U\in SP^*IO(X,\tau)$, for each $\alpha\in\Delta$, then $\bigcup\{U_\alpha:\alpha\in\Delta\}\in SP^*IO(X,\tau)$
2. If $A\in SP^*IO(X,\tau)$, and $B\in\tau$, then $A\cap B\in SP^*IO(X,\tau)$.

Proof. (1) Since $U_\alpha\in SP^*IO(X,\tau)$, we have $U_\alpha\subset itn^*(cl^*(U_\alpha))$, for each $\alpha\in\Delta$. Then we obtain:

$$\begin{aligned}\bigcup_{\alpha\in\Delta}U_\alpha &\subset \bigcup_{\alpha\in\Delta}int^*(cl^*(U_\alpha))\\ &\subset int^*(\bigcup_{\alpha\in\Delta}cl^*(U_\alpha))\\ &= int^*(\bigcup_{\alpha\in\Delta}(U_\alpha^*\cup U_\alpha))\\ &= int^*(\bigcup_{\alpha\in\Delta}U_\alpha^*\cup\bigcup_{\alpha\in\Delta}U_\alpha)\\ &\subset int^*((\bigcup_{\alpha\in\Delta}U_\alpha)^*\cup\bigcup_{\alpha\in\Delta}U_\alpha)\\ &= int^*(cl^*(\bigcup_{\alpha\in\Delta}U_\alpha))\end{aligned}$$

This shows that $\bigcup_{\alpha\in\Delta}U_\alpha\in SP^*IO(X,\tau)$.

(2) Let $A\in SP^*IO(X,\tau)$ and $B\in\tau$. Then $A\subset int^*(cl^*(A))$ and $B=int(B)\subset int^*(B)$. Thus, we obtain

$$\begin{aligned}A\cap B &\subset int^*(cl^*(A))\cap int^*(B)\\ &= int^*(cl^*(A)\cap B)\\ &= int^*((A^*\cup A)\cap B)\\ &= int^*((A^*\cap B)\cup(A\cap B))\\ &\subset int^*((A\cap B)^*\cup(A\cap B))\\ &= int^*(cl^*(A\cap B))\end{aligned}$$

□

Remark 2.16. In general, a finite intersection of the $S.P^*-I-open$ sets need not be $S.P^*-I-open$, as shown by the following example:

Example 2.17. Let $X=\{a,b,c,d\}$, $\tau=\{\phi,X,\{a\},\{b,c\},\{a,b,c\}\}$ and $I=\{\phi,\{a\},\{d\},\{a,d\}\}$. We can easily conclude that $A=\{c,d\}$ and $B=\{b,d\}$ are $S.P^*-I-open$ sets, but $A\cap B=\{d\}$ is not $S.P^*-I-open$.

Theorem 2.18. Let (X,τ,I) be an ideal topological space, where I is codense then the following hold:

1. Every $S.P^*-I-open$ set is a $strong\ \beta-I-open$ set.
2. Every $S.P^*-I-open$ set is a $\beta-open$ set.
3. Every $S.P^*-I-open$ set is a $weakly\ semi-I-open$ set.
4. Every $S.P^*-I-open$ set is a $pre-open$ set.

Proof. It is obvious. □

Remark 2.19. The reverse of the above theorem is not true in general as shown in the following examples:

Example 2.20. Let $X = \{a, b, c, d\}$, $\tau = \{\phi, X, \{a\}, \{b\}, \{a, b\}\}$ and $I = \{\phi\}$. Then we obtain:

1. $A = \{a, c\}$ is $strong\ \beta - I - open$ set, but it is not $S.P^* - I - open$.
2. $A = \{b, c\}$ is $\beta - open$ set, but it is not $S.P^* - I - open$.
3. $A = \{a, d\}$ is $weakly\ semi - I - open$ set, but it is not $S.P^* - I - open$

Example 2.21. Let $X = \{a, b, c, d\}$, $\tau = \{\phi, X, \{a, b\}\}$ and $I = \{\phi, \{b\}, \{c\}, \{b, c\}\}$. If we take $A = \{b\}$, then we get A is a $pre - open$ set, but it is not $S.P^* - I - open$.

Theorem 2.22. Let (X, τ, I) be an ideal topological space, such that every $open$ set is $* - closed$, then every $strong\ \beta - I - open$ set is $S.P^* - I - open$.

PROOF. Let A is a $strong\ \beta - I - open$, then $A \subset cl^*(int(cl^*(A)))$. Since $int(cl^*(A))$ is $open$, by hypothesis $int(cl^*(A)) = cl^*(int(cl^*(A)))$. So $A \subset cl^*(int(cl^*(A))) = int(cl^*(A) \subset int^*(cl^*(A))$. Which shows that A is $S.P^* - I - open$. □

Theorem 2.23. Let (X, τ, I) be an ideal topological space. If A is $* - perfect$, then the following hold:

1. Every $S.P^* - I - open$ set is $almost\ strong - I - open$.
2. A is a $S.P^* - I - open$ set if and only if it is $I - open\ set$.

PROOF. (1) Let A is an $S.P^* - I - open$, then $A \subset int^*(cl^*(A)) = int(cl^*(A)) \subset cl^*(int(cl^*(A))) = cl^*(int(A^*))$. This implies A is $almost\ strong - I - open$.
(2) Let A is a $S.P^* - I - open$, then $A \subset int^*(cl^*(A)) \subset int^*(cl(A)) = int(A^*)$. Hence A is $I - open$. Conversely, if A is $I - open$, then $A \subset int(A^*) \subset int^(*)(A^*) = int^*(cl^*(A))$. Hence A is $S.P^* - I - open$. □

Corollary 2.24. Let (X, τ, I) be an ideal topological space, If A is $* - perfect$, then every $pre^* - I - open$ set is $S.P^* - I - open$.

PROOF. Let A is $pre^* - I - open$ set, since it is $* - perfect$, then $A \subset int^*(cl(A)) = int^*(cl^*(A))$. Hence A is $S.P^* - I - open$. □

Corollary 2.25. Every $I - open$ set is $S.P^* - I - open$.

PROOF. If A is $I - open$, then $A \subset int(A^*) \subset int(A^* \cup A) \subset int^*(cl^*(A))$. Hence A is $S.P^* - I - open$. □

Theorem 2.26. Let (X, τ, I) be an ideal topological space, Where I is codense, then the following are equivalent:

1. A is $pre^* - I - open$.
2. A is $S.P^* - I - open$.

PROOF. It is obvious. □

Theorem 2.27. Let (X, τ, I) be an ideal topological space and $A \subset X$ be a $pre - open$ and $semi - closed$. Then A is $S.P^* - I - open$.

PROOF. Let A is $pre - open$, then $A \subset int(cl(A))$. Since A is $semi - closed$ then $int(cl(A)) = int(A)$, now $A \subset int(A) \subset int^*(cl^*(A))$. Which shows A is $S.P^* - I - open$. □

Theorem 2.28. Let (X, τ, I) be an ideal topological space and $A \subset X$ be an $S.P^* - I - open$ and $* - closed$. Then A is $S.S^* - I - open$.

PROOF. Let A is $S.P^* - I - open$, then $A \subset int^*(cl^*(A))$. Since A is $* - closed$ then $int^*(cl^*(A)) = int^*(A)$. Now $A \subset int^*(A) \subset cl^*(int^*(A))$. Which shows A is $S.S^* - I - open$. □

Theorem 2.29. Let (X, τ, I) be an ideal topological space, and $A \subset X$, then the followings hold:

1. A is $S.P^* - I - open$ set, if it is both $weakly\ semi - I - open$ and $strong\ S - I - set$.
2. A is $S.P^* - I - open$ set, if it is both $semi - I - open$ set and $S - I - set$.

PROOF. (1) Let A is $weakly\ semi-I-open$ set, then $A \subset cl^*(int(cl(A)))$. Since A is $strong\ S-I-set$ then, $int(A) = cl^*(int(cl(A)))$. Now $A \subset int(A) \subset int^*(cl^*(A))$. Hence A is $S.P^* - I - open$.
(2) Let A is $semi-I-open$ set, then $A \subset cl^*(int(A))$. Since A is $S-I-set$ then, $int(A) = cl^*(int(A))$. Now $A \subset int(A) \subset int^*(cl^*(A))$. Hence A is $S.P^* - I - open$. □

Theorem 2.30. Let (X, τ, I) be an ideal topological space. A is an $S.P^* - I - open$ set if it is both $Pre^* - I - open$ and $closed$.

PROOF. Let A is $pre^* - I - open$ set, then $A \subset int^*(cl(A))$. Since A is closed set, then $A \subset int^*(cl(A)) = int^*(A) \subset int^*(cl^*(A))$. Hence A is $S.P^* - I - open$. □

Theorem 2.31. Let (X, τ, I) be an $I - extremally\ disconnected$ space and $A \subset X$. Then every $semi - I - open$ set is an $S.P^* - I - open$ set.

PROOF. Let A is $semi - I - open$, then $Acl^*(int(A))$. By Lemma 1.11, we obtain $A \subset int(cl^*(A)) \subset int^*(cl^*(A))$. Which shows A is $S.P^* - I - open$. □

Lemma 2.32. An ideal topological space (X, τ, I) is $I - extremally\ disconnected$ if and only if $cl^*(int^*(A)) \subset int^*(cl^*(A))$, for every subset A of X.

PROOF. From Definition 1.9., we obtain $cl^*(A)$ is $open$. Thus $cl^*(int^*(A)) \subset cl^*(A) = int(cl^*(A)) \subset int^*(cl^*(A))$. Hence $cl^*(int^*(A)) \subset int^*(cl^*(A))$. Conversely, since $cl^*(int(A)) \subset cl^*(int^*(A)) \subset int^*(cl^*(A)) \subset int^*(cl(A))$. Then X is $I - extremally\ disconnected$. □

Corollary 2.33. Let (X, τ, I) be an $I - extremally\ disconnected$ space and $A \subset X$. Then every $strong\ semi^* - I - open$ set is $S.P^* - I - open$.

PROOF. It is obvious by Lemma 2.32. □

Theorem 2.34. Let (X, τ, I) be an ideal topological space, A and B are subsets of X. If A is an $S.P^* - I - open$ set and B is a $pre - open$ set, then $A \cup B$ is $pre^* - I - open$.

PROOF. Let A is $S.P^* - I - open$ then $A \subset int^*(cl^*(A))$, and B is a $pre - open$ then $B \subset int(cl(B))$. Now:

$$\begin{aligned} A \cup B &\subset int^*(cl^*(A)) \cup int(cl(B)) \\ &\subset int^*(cl(A)) \cup int^*(cl(B)) \\ &\subset int^*(cl(A \cup B)). \end{aligned}$$

Hence $A \cup B$ is a $pre^* - I - open$ set. □

Theorem 2.35. Let (X, τ, I) be an ideal topological space, A and B are subsets of X. If A is an $S.P^* - I - open$ set and B is a $weakly\ semi - I - open$ set, then $A \cup B$ is $\beta^* - I - open$.

PROOF. Let A is $S.P^* - I - open$, then $A \subset int^*(cl^*(A))$, B is $weakly\ semi - I - open$ then $B \subset cl^*(int(cl(B)))$ Now :

$$\begin{aligned} A \cup B &\subset int^*(cl^*(A)) \cup cl^*(int(cl(B))) \\ &\subset cl(int^*(cl(A))) \cup cl(int^*(cl(B))) \\ &= cl(int^*(cl(A)) \cup int^*(cl(B))) \\ &\subset cl(int^*(cl(A \cup B))). \end{aligned}$$

Hence $A \cup B$ is a $\beta^* - I - open$ set. □

Theorem 2.36. Let (X, τ, I) be an ideal topological space, where I is codense then A is $\alpha - I - open$ if and only if it is an $S.S^* - I - open$ and $S.P^* - I - open$.

PROOF. Necessity, this is obvious.
Sufficiency, Let A is an $S.S^* - I - open$ and $S.P^* - I - open$, we have:

$$\begin{aligned} A &\subset int^*(cl^*(A)) \\ &\subset int^*(cl^*(cl^*(int^*(A)))) \\ &= int^*(cl^*(int^*(A))) \\ &= int(cl^*(int(A))). \end{aligned}$$

Hence A is $\alpha - I - open$. □

3. Strong Pre*-I-Closed Sets

Definition 3.1. A subset A of an ideal topological space (X, τ, I) is said to be *strong* $pre^* - I - closed$ (briefly $S.P^* - I - closed$) if its complement is $S.P^* - I - open$. We denote that all $S.P^* - I - closed$ by $S.P^* - I - C(X)$.

Lemma 3.2. Let (X, τ, I) be an ideal topological space, the followings hold, for any subset A of X:

1. Every $pre - I - closed$ set is a $S.P^* - I - closed$.
2. Every $S.P^* - I - closed$ set is a $pre^* - I - closed$.

The following diagram holds for any subset A of an ideal topological space (X, τ, I).

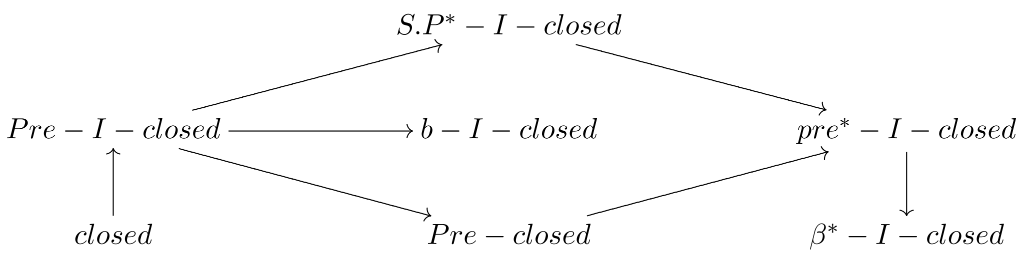

Figure 2. The implication between some generalizations of closed sets

Theorem 3.3. A subset A of a space (X, τ, I) is said to be an $S.P^* - I - closed$ if and only if$cl^*(int^*(A)) \subset A$.

PROOF. Let A be an $S.P^* - I - closed$ of (X, τ, I), then $(X - A)$ is an $S.P^* - I - open$ and hence $(X - A) \subset int^*(cl^*(X - A)) = X - cl^*(int^*(A))$. Therefore, we obtain $cl^*(int^*(A)) \subset A$.
Conversely, let $cl^*(int^*(A)) \subset A$, then $(X-A) \subset int^*(cl^*(X - A))$ and hence $(X-A)$ is $S.P^*-I-open$. Therefore, A is an $S.P^* - I - closed$. □

Theorem 3.4. Let (X, τ, I) be an ideal topological space, if I is codense, then A is an $S.P^*-I-closed$ if and only if $cl^*(int(A)) \subset A$.

PROOF. Let A be a $S.P^* - I - closed$ set of X, then $A \supset cl^*(int^*(A)) = cl^*(int(A))$.
Conversely, let A be any subset of X, such that $A \supset cl^*(int(A))$. This implies that $A \supset cl^*(int^*(A))$, i.e., A is an $S.P^* - I - closed$ □

Theorem 3.5. Let (X, τ, I) be an ideal topological space, and $A \subset X$, then the followings hold:

1. If A is an $S.P^* - I - open$ set, then $SIcl(A) = int^*(cl(A))$.
2. If A is an $S.P^* - I - closed$ set, then $SIint(A) = cl^*(int(A))$.

Proof. (1) Let A be an $S.P^* - I - open$ set in X. Then we have $A \subset int^*(cl^*(A)) \subset int^*(cl(A))$. Thus we have $SIcl(A) = int^*(cl(A))$.
(2) Let A be an $S.P^* - I - closed$ set in X ,then we have $A \supset cl^*(int^*(A)) \supset cl^*(int(A))$. Hence $SIint(A) = cl^*(int(A))$. □

Remark 3.6. 1The reverse of the above theorem is not true in general as shown in the following examples:

Example 3.7. Let $X = \{a, b, c, d\}$, $\tau = \{\phi, X, \{a\}, \{b, c\}, \{a, b, c\}\}$, $I = \{\phi, \{a\}, \{d\}, \{a, d\}\}$, and $A = \{c\}$. Then we obtain:

1. $SIcl(A) = int^*(cl(A))$, but A is not $S.P^* - I - open$.
2. $SIint(A) = cl^*(int(A))$, but it is not $S.P^* - I - closed$

Theorem 3.8. A subset A of a space (X, τ, I) is said to be $S.P^* - I - closed$ if and only if there exists an $S.P^* - I - closed$ set B such that $int^*(A) \subset B \subset A$.

Proof. Let A be an $S.P^* - I - closed$ set of a space (X, τ, I), then $cl^*(int^*(A)) \subset A$. We put $B = cl^*(int^*(A))$ be a $* - closed\ set$. i.e, B is $S.P^* - I - closed$. And $int^*(A) \subset cl^*(int^*(A)) = B \subset A$. Conversely, if B is an $S.P^* - I - closed$ set such that $int^*(A) \subset B \subset A$, then $int^*(A) = int^*(B)$. On the other hand, $cl^*(int^*(B) \subset B$ and hence $A \supset B \supset cl^*(int^*(B)) = cl^*(int^*(A)$. Thus $A \supset cl^*(int^*(A))$. Hence A is $S.P^* - I - closed$. □

Corollary 3.9. a subset A of a space (X, τ, I) is an $S.P^* - I - closed$ set if and only if there exists a $* - closed$ set B such that $int^*(A) \subset B \subset A$.

Remark 3.10. The union of strong $pre^* - I - closed$ sets need not be an $S.P^* - I - closed$ set. This can be shown by the following example:

Example 3.11. Let $X = \{a, b, c, d\}$, $\tau = \{\phi, \{a\}, \{b, c\}, \{a, b, c\}, X\}$ and $I = \{\phi, \{a\}, \{d\}, \{a, d\}\}$, then $A = \{b\}$ and $B = \{c\}$ are $S.P^* - I - closed$ sets but $A \cup B = \{b, c\}$ is not $S.P^* - I - closed$.

Theorem 3.12. Let (X, τ, I) be an ideal topological space, A and B are subsets of X. Then $A \cap B$ is a $pre^* - I - closed$ set, if A is $S.P^* - I - closed$ and B is $pre - closed$ set.

Proof. It is proved similarly by Theorem 2.34. □

Theorem 3.13. Let (X, τ, I) be an ideal topological space, A and B are subsets of X. Then $A \cap B$ is a $B^* - I - closed$ set, if A is $S.P^* - I - closed$ and B is $weakly\ semi - I - closed$.

Proof. It is proved similarly by Theorem 2.35. □

Theorem 3.14. Let (X, τ, I) be an ideal topological space, then each $pre - I - regular$ set in X is $S.P^* - I - open$ and $S.P^* - I - closed$ set.

Proof. It follows from the fact that every $pre - I - regular$ set is $pre - I - open$ and $pre - I - closed$. This implies that it is $S.P^* - I - open$ and $S.P^* - I - closed$. □

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Integral transforms for the new generalized Beta function

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Abstract − In this paper, some representation formulas for the generalized Gamma and Beta functions are obtained. Also, certain integral transforms for the generalized Beta function associated with the Wright hypergeometric function are derived.

Keywords − *Gamma function; Beta function; integral transform*

1. Introduction

Motivated mainly by a variety of applications of Euler's Beta, hypergeometric, and confluent hypergeometric functions together with their extensions in a wide range of research fields such as engineering, chemical, and physical problems, very recently, Al-Gonah and Mohammed [1] introduced and studied a new form of the generalized Gamma and Beta functions denoted by $\Gamma_p^{(\alpha,\beta,\gamma)}(x)$ and $B_p^{(\alpha,\beta,\gamma)}(x,y)$ respectively by talking further advantage from the various existing forms of the Mittag-Leffler function. The generalized Gamma and Beta functions are defined by:

$$\Gamma_p^{(\alpha,\beta,\gamma)}(x) = \int_0^\infty t^{x-1} E_{\alpha,\beta}^{\gamma}\left(-t - \frac{p}{t}\right) dt, \tag{1}$$

$$(Re(p) \geq 0, Re(\alpha) > 0, Re(\beta) > 0, Re(\gamma) > 0, Re(x) > 0),$$

and

$$B_p^{(\alpha,\beta,\gamma)}(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1} E_{\alpha,\beta}^{\gamma}\left(\frac{-p}{t(1-t)}\right) dt, \tag{2}$$

$$(Re(p) \geq 0, Re(\alpha) > 0, Re(\beta) > 0, Re(\gamma) > 0, Re(x) > 0, Re(y) > 0),$$

where $E_{\alpha,\beta}^{\gamma}(z)$ denotes the generalized Mittag-Leffler function defined by [2,p.7(1.3)]:

$$E_{\alpha,\beta}^{\gamma}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_n}{\Gamma(\alpha n + \beta)} \frac{z^n}{n!}, \tag{3}$$

$$(z, \alpha, \beta, \gamma \in \mathbb{C}; Re(\alpha) > 0, Re(\beta) > 0, Re(\gamma) > 0),$$

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which is clearly that

$$\Gamma(\beta)E^{\gamma}_{1,\beta}(z) = {}_1F_1(\gamma;\beta;z), \tag{4}$$

$$E^{1}_{1,1}(z) = e^{z}. \tag{5}$$

Note that by using relations (4) and (5), we get the following special cases:

$$\Gamma_p^{(1,\beta,\gamma)}(x) = \frac{1}{\Gamma(\beta)}\Gamma_p^{(\gamma,\beta)}(x), \tag{6}$$

$$B_p^{(1,\beta,\gamma)}(x,y) = \frac{1}{\Gamma(\beta)}B_p^{(\gamma,\beta)}(x,y), \tag{7}$$

$$\Gamma_p^{(1,1,1)}(x) = \Gamma_p(x), \tag{8}$$

$$B_p^{(1,1,1)}(x,y) = B_p(x,y), \tag{9}$$

where $\Gamma_p^{(\gamma,\beta)}(x),\ B_p^{(\gamma,\beta)}(x,y)$ and $\Gamma_p(x),\ B_p(x,y)$ denoted the generalized Gamma and Beta functions given in [3] and [4,5].

Also, we not that

$$B_p^{(\alpha,1,1)}(x,y) = B^p_\alpha(x,y), \tag{10}$$

$$\Gamma_0^{(\alpha,1,1)}(x) = \Gamma^\alpha(x), \tag{11}$$

where $B^p_\alpha(x,y)$ and $\Gamma^\alpha(x)$ denoted the new extended Beta and Gamma functions given recently in [6] and [7] respectively.

In [8], Agrawal gave some interesting integral transforms for the generalized hypergeometric function. This paper is a further attempt in this direction for deriving some integral transforms and representation formulas for the generalized Beta function defined in [1]. For this aim, we recall that the Wright generalized hypergeometric function denoted by ${}_p\Psi_q$ is defined by [9]:

$${}_p\Psi_q\left[\begin{array}{cc}(\alpha_1,A_1),\ldots,(\alpha_p,A_p); & \\ & z\\ (\beta_1,B_1),\ldots,(\beta_q,B_q); & \end{array}\right] = \sum_{n=0}^{\infty}\frac{\prod_{l=1}^{p}\Gamma(\alpha_l+A_l n)}{\prod_{j=1}^{q}\Gamma(\beta_j+B_j n)}\frac{z^n}{n!}, \tag{12}$$

where the parameters $\alpha_l,\beta_j\in\mathbb{C}$, and $A_l,B_j\in\mathbb{Z}$ $(l=1,2,\ldots,p;\ j=1,2,\ldots,q)$, such that $1+\sum_{j=1}^{q}B_j-\sum_{l=1}^{p}A_l>0$. Also, we note that

$$\begin{aligned}&{}_p\Psi_q\left[\begin{array}{cc}(\alpha_1,A_1),\ldots,(\alpha_p,A_p); & \\ & z\\ (\beta_1,B_1),\ldots,(\beta_q,B_q); & \end{array}\right]\\ &= H^{1,p}_{p,q+1}\left[-z\left|\begin{array}{c}(1-\alpha_1,A_1),\ldots,(1-\alpha_p,A_p)\\ \\ (0,1),(1-\beta_1,B_1),\ldots,(1-\beta_q,B_q)\end{array}\right.\right],\end{aligned} \tag{13}$$

where $H^{m,n}_{p,q}[.]$ denotes the H-function given in [9] (see also [10]).

2. Hypergeometric representations

Here, we establish some representation formulas for the generalized Gamma and Beta functions in form of the following theorems:

Theorem 2.1. For the new extended Beta function, we have the following hypergeometric representation:

$$B_p^{(\alpha,\beta,\gamma)}(x,y) = \frac{1}{\Gamma(\gamma)}{}_3\Psi_2\left[\begin{array}{cc}(\gamma,1),(x,-1),(y,-1); & \\ & -p\\ (\beta,\alpha),(x+y,-2); & \end{array}\right], \tag{14}$$

$$(\alpha\in\mathbb{Z}^+;\beta,\gamma,x,y\in\mathbb{C};Re(\beta),Re(\gamma),Re(x),Re(y)>0).$$

PROOF. Using definition (3) in relation (2), we get

$$B_p^{(\alpha,\beta,\gamma)}(x,y) = \int_0^1 t^{x-1}\,(1-t)^{y-1} \sum_{n=0}^{\infty} \frac{(\gamma)_n}{\Gamma(\alpha n+\beta)\,n!} \frac{(-p)^n}{t^n(1-t)^n}\,dt. \tag{15}$$

Interchanging the order of integration and summation in the R.H.S. of equation (15), we get

$$B_p^{(\alpha,\beta,\gamma)}(x,y) = \frac{1}{\Gamma(\gamma)} \sum_{n=0}^{\infty} \frac{\Gamma(\gamma+n)\,(-p)^n}{\Gamma(\beta+\alpha n)\,n!} \int_0^1 t^{x-n-1}\,(1-t)^{y-n-1}\,dt, \tag{16}$$

which on using the following relation [9]:

$$\begin{aligned}\int_0^1 t^{x-1}\,(1-t)^{y-1}\,dt &= \frac{\Gamma(x)\,\Gamma(y)}{\Gamma(x+y)}, \\ &= B(x,y),\end{aligned} \tag{17}$$

gives

$$B_p^{(\alpha,\beta,\gamma)}(x,y) = \frac{1}{\Gamma(\gamma)} \sum_{n=0}^{\infty} \frac{\Gamma(\gamma+n)\,\Gamma(x-n)\,\Gamma(y-n)}{\Gamma(\beta+\alpha n)\,\Gamma(x+y-2n)} \frac{(-p)^n}{n!}. \tag{18}$$

Now, in view of definition (12), we get the desired result. □

Remark 2.2. Using relation (13) in assertion (14) of Theorem 2.1, we get the following relation:

Corollary 2.3. For the new extended Beta function, we have the following hypergeometric representation:

$$B_p^{(\alpha,\beta,\gamma)}(x,y) = \frac{1}{\Gamma(\gamma)} H_{3,3}^{1,3}\left[p \,\middle|\, \begin{matrix} (1-\gamma,1),(1-x,-1),(1-y,-1) \\ \\ (0,1),(1-\beta,\alpha),(1-x-y,-2) \end{matrix}\right]. \tag{19}$$

.

Theorem 2.4. For the new extended Gamma function, we have the following representation:

$$\Gamma_p^{(\alpha,\beta,\gamma)}(x) = \frac{1}{\Gamma(\gamma)} \int_0^{\infty} t^{x-1}\,{}_1\Psi_1\left[\begin{matrix} (\gamma,1); \\ \\ (\beta,\alpha); \end{matrix} \; -t-\tfrac{p}{t}\right] dt, \tag{20}$$

$$\left(\alpha \in \mathbb{Z}^+; \beta,\gamma \in \mathbb{C}; Re(\beta) > 0, Re(\gamma) > 0\right).$$

PROOF. Using the following relation [11,p.810(6.3.2)]:

$$E_{\alpha,\beta}^{\gamma}(z) = \frac{1}{\Gamma(\gamma)} {}_1\Psi_1\left[\begin{matrix} (\gamma,1); \\ \\ (\beta,\alpha); \end{matrix} \; z\right], \tag{21}$$

in definition (1), we get the desired result. □

Remark 2.5. Using relation (13) in assertion (20) of Theorem 2.4, we get the following relation:

Corollary 2.6. For the new extended Gamma function, we have the following representation:

$$\Gamma_p^{(\alpha,\beta,\gamma)}(x) = \frac{1}{\Gamma(\gamma)} \int_0^{\infty} t^{x-1} H_{1,2}^{1,1}\left[t+\frac{p}{t} \,\middle|\, \begin{matrix} (1-\gamma,1) \\ \\ (0,1),(1-\beta,\alpha) \end{matrix}\right] dt. \tag{22}$$

3. Integral transforms

In this section, we derive some integral transforms for the generalized Beta function by applying certain integral transforms (like Beta transform, Laplace transform and Whittaker transform).

Theorem 3.1. The following Beta transform formula holds true:

$$B\left\{B_p^{(\alpha,\beta,\gamma)}(x,y):l,m\right\}=\frac{\Gamma(m)}{\Gamma(\gamma)}{}_4\Psi_3\left[\begin{array}{c}(\gamma,1),(l,1),(x,-1),(y,-1);\\ (\beta,\alpha),(l+m,1),(x+y,-2);\end{array}\ -1\right],\tag{23}$$

$$(\alpha\in\mathbb{Z}^+;\beta,\gamma,x,y,l,m\in\mathbb{C};Re(m),Re(\beta),Re(\gamma),Re(x),Re(y),Re(l)>0).$$

PROOF. We know that the Beta transform is defined as (see [12]):

$$B\{f(p):a,b\}=\int_0^1 p^{a-1}(1-p)^{b-1}f(p)\,dp.\tag{24}$$

Using equation (24) and applying definition (2), we get

$$B\left\{B_p^{(\alpha,\beta,\gamma)}(x,y):l,m\right\}$$
$$=\int_0^1 p^{l-1}(1-p)^{m-1}\int_0^1 t^{x-1}(1-t)^{y-1}E_{\alpha,\beta}^{\gamma}\left(\frac{-p}{t(1-t)}\right)dt\,dp,$$
$$=\int_0^1 p^{l-1}(1-p)^{m-1}\int_0^1 t^{x-1}(1-t)^{y-1}\sum_{n=0}^{\infty}\frac{(\gamma)_n\,(-1)^n}{\Gamma(\alpha n+\beta)\,n!}\frac{p^n}{t^n(1-t)^n}\,dt\,dp.\tag{25}$$

Interchanging the order of integration and summation and using relation (17) in the R.H.S. of equation (25), we obtain

$$B\left\{B_p^{(\alpha,\beta,\gamma)}(x,y):l,m\right\}=\frac{\Gamma(m)}{\Gamma(\gamma)}\sum_{n=0}^{\infty}\frac{\Gamma(\gamma+n)\,\Gamma(l+n)\,\Gamma(x-n)\,\Gamma(y-n)}{\Gamma(\beta+\alpha n)\,\Gamma(l+m+n)\,\Gamma(x+y-2n)}\frac{(-1)^n}{n!},\tag{26}$$

which on using definition (12), yields the desired result. □

Remark 3.2. Using relation (13) in assertion (23) of Theorem 3.1, we get the following relation:

Corollary 3.3. The following Beta transform formula holds true:

$$B\left\{B_p^{(\alpha,\beta,\gamma)}(x,y):l,m\right\}$$
$$=\frac{\Gamma(m)}{\Gamma(\gamma)}H_{4,4}^{1,4}\left[1\left|\begin{array}{c}(1-\gamma,1),(1-l,1),(1-x,-1),(1-y,-1)\\ (0,1),(1-\beta,\alpha),(1-l-m,1),(1-x-y,-2)\end{array}\right.\right].\tag{27}$$

Theorem 3.4. The following Laplace transform formula holds true:

$$L\left\{p^{l-1}B_p^{(\alpha,\beta,\gamma)}(x,y);s\right\}=\frac{s^{-l}}{\Gamma(\gamma)}{}_4\Psi_2\left[\begin{array}{c}(\gamma,1),(l,1),(x,-1),(y,-1);\\ (\beta,\alpha),(x+y,-2);\end{array}\ -\tfrac{1}{s}\right],\tag{28}$$

$$(\alpha\in\mathbb{Z}^+;\beta,\gamma,x,y,l,s\in\mathbb{C};Re(s)>,Re(\beta),Re(\gamma),Re(x),Re(y),Re(l)>0;\left|\tfrac{-1}{s}\right|<1).$$

PROOF. We know that the Laplace transform of $f(p)$ is defined as [12]:

$$L\{f(p);s\}=\int_0^{\infty}e^{-sp}f(p)\,dp,\qquad (Re(s)>0).\tag{29}$$

Using relation (29) and applying definition (2), we get

$$L\left\{p^{l-1}B_p^{(\alpha,\beta,\gamma)}(x,y);s\right\}=\int_0^{\infty}p^{l-1}e^{-sp}\int_0^1 t^{x-1}(1-t)^{y-1}E_{\alpha,\beta}^{\gamma}\left(\frac{-p}{t(1-t)}\right)dt\,dp,$$

$$= \int_0^\infty p^{l-1} e^{-sp} \int_0^1 t^{x-1}(1-t)^{y-1} \sum_{n=0}^{\infty} \frac{(\gamma)_n\,(-1)^n}{\Gamma(\alpha n+\beta)\,n!} \frac{p^n}{t^n(1-t)^n}\,dt\,dp. \tag{30}$$

Interchanging the order of integration and summation and using the following integral formula [9,p.218(3)]:

$$\int_0^\infty t^{\lambda-1} e^{-st}\,dt = \frac{\Gamma(\lambda)}{s^\lambda}, \qquad (min\,\{Re(\lambda), Re(s)\} > 0), \tag{31}$$

and relation (17) in the R.H.S. of equation (30), we obtain

$$L\left\{p^{l-1} B_p^{(\alpha,\beta,\gamma)}(x,y); s\right\} = \frac{s^{-l}}{\Gamma(\gamma)} \sum_{n=0}^{\infty} \frac{\Gamma(\gamma+n)\,\Gamma(l+n)\,\Gamma(x-n)\,\Gamma(y-n)}{\Gamma(\beta+\alpha n)\,\Gamma(x+y-2n)} \frac{\left(-\frac{1}{s}\right)^n}{n!}, \tag{32}$$

which on using definition (12), yields the desired result. □

Remark 3.5. Using relation (13) in assertion (28) of Theorem 3.4, we get the following relation:

Corollary 3.6. The following Laplace transform formula holds true:

$$L\left\{p^{l-1} B_p^{(\alpha,\beta,\gamma)}(x,y); s\right\}$$
$$= \frac{s^{-l}}{\Gamma(\gamma)} H_{4,3}^{1,4}\left[\frac{1}{s}\,\middle|\, \begin{matrix} (1-\gamma,1),(1-l,1),(1-x,-1),(1-y,-1) \\ (0,1),(1-\beta,\alpha),(1-x-y,-2) \end{matrix}\right]. \tag{33}$$

Theorem 3.7. The following Whittaker transform formula holds true:

$$\int_0^\infty p^{q-1} e^{\frac{-\delta p}{2}} W_{\lambda,\mu}(\delta p) B_p^{(\alpha,\beta,\gamma)}(x,y)\,dp$$
$$= \frac{\delta^{-q}}{\Gamma(\gamma)}\, {}_5\Psi_3\left[\begin{matrix} (\gamma,1),(x,-1),(y,-1),\left(\frac{1}{2}+\mu+q,1\right),\left(\frac{1}{2}-\mu+q,1\right); \\ (\beta,\alpha),(x+y,-2),(1-\lambda+q,1); \end{matrix}\; -\frac{1}{\delta}\right], \tag{34}$$

$(\alpha \in \mathbb{Z}^+; \beta,\gamma,x,y,q,\lambda,\mu,\delta \in \mathbb{C}; Re(\delta), Re(q), Re(\beta), Re(\gamma), Re(x), Re(y), Re\left(\frac{1}{2}\pm\mu+q\right), Re(1-\lambda+q) > 0)$.

Proof. Denoting the L.H.S. of equation (34) by Δ and setting $\delta p = v$, we get

$$\Delta = \delta^{-q} \int_0^\infty B_{(\frac{v}{\delta})}^{(\alpha,\beta,\gamma)}(x,y)\, v^{q-1}\, e^{-\frac{v}{2}}\, W_{\lambda,\mu}(v)\,dv, \tag{35}$$

which on using definition (2), gives

$$\Delta = \delta^{-q} \int_0^\infty \int_0^1 t^{x-1}\,(1-t)^{y-1}\, E_{\alpha,\beta}^{\gamma}\left(\frac{-\frac{v}{\delta}}{t(1-t)}\right) v^{q-1}\, e^{-\frac{v}{2}}\, W_{\lambda,\mu}(v)\,dt\,dv,$$
$$= \delta^{-q} \int_0^\infty \int_0^1 t^{x-1}\,(1-t)^{y-1} \sum_{n=0}^{\infty} \frac{(\gamma)_n\,(-1)^n\,v^n}{\Gamma(\beta+\alpha n)\,n!\,\delta^n\,t^n\,(1-t)^n}\, v^{q-1}\, e^{-\frac{v}{2}}\, W_{\lambda,\mu}(v)\,dt\,dv. \tag{36}$$

Interchanging the order of integration and summation in the R.H.S. of equation (36), we obtain

$$\Delta = \frac{\delta^{-q}}{\Gamma(\gamma)} \sum_{n=0}^{\infty} \frac{\Gamma(\gamma+n)\left(-\frac{1}{\delta}\right)^n}{\Gamma(\beta+\alpha n)\,n!} \int_0^1 t^{x-n-1}\,(1-t)^{y-n-1}\,dt \int_0^\infty v^{q+n-1}\, e^{-\frac{v}{2}}\, W_{\lambda,\mu}(v)\,dv. \tag{37}$$

Now using relation (17) and the following integral formula involving the Whittaker function [13,p.9(2.24)]:

$$\int_0^\infty z^{v-1} e^{\frac{-z}{2}} W_{\lambda,\mu}(z)\,dz = \frac{\Gamma\left(\frac{1}{2}+\mu+v\right)\Gamma\left(\frac{1}{2}-\mu+v\right)}{\Gamma(1-\lambda+v)}, \left(Re(v\pm\mu) > -\frac{1}{2}\right), \tag{38}$$

in the R.H.S. of equation (37) and after little simplification, we get

$$\Delta = \frac{\delta^{-q}}{\Gamma(\gamma)} \sum_{n=0}^{\infty} \frac{\Gamma(\gamma+n)\,\Gamma(x-n)\,\Gamma(y-n)\,\Gamma\left(\frac{1}{2}+\mu+q+n\right)\Gamma\left(\frac{1}{2}-\mu+q+n\right)}{\Gamma(\beta+\alpha n)\,\Gamma(x+y-2n)\,\Gamma(1-\lambda+q+n)} \frac{\left(-\frac{1}{\delta}\right)^n}{n!}, \tag{39}$$

which on using definition (12), yields the R.H.S. of equation (34), then the proof of Theorem 3.7 is completed. □

Remark 3.8. Using relation (13) in assertion (34) of Theorem 3.7, we get the following relation:

Corollary 3.9. The following Whittaker transform formula holds true:

$$\int_0^\infty p^{q-1} e^{\frac{-\delta p}{2}} W_{\lambda,\mu}(\delta p) B_p^{(\alpha,\beta,\gamma)}(x,y)\, dp$$
$$= \frac{\delta^{-q}}{\Gamma(\gamma)} H^{1,5}_{5,4}\left[\frac{1}{\delta}\left|\begin{matrix}(1-\gamma,1),(1-x,-1),(1-y,-1),\left(\frac{1}{2}-\mu-q,1\right),\left(\frac{1}{2}+\mu-q,1\right)\\ (0,1),(1-\beta,\alpha),(1-x-y,-2),(\lambda-q,1)\end{matrix}\right.\right]. \tag{40}$$

4. Special cases

In this section, we derive some results for various forms of the extended Gamma and Beta functions as special cases of the main results derived in the previous sections.

I. Putting $\alpha = 1$ in relations (14) and (19) and using relation (7), we get the following hypergeometric representations for the generalized Beta function $B_p^{(\gamma,\beta)}(x,y)$:

$$B_p^{(\gamma,\beta)}(x,y) = \frac{\Gamma(\beta)}{\Gamma(\gamma)}\, {}_3\Psi_2\left[\begin{matrix}(\gamma,1),(x,-1),(y,-1);\\ (\beta,1),(x+y,-2);\end{matrix}\; -p\right], \tag{41}$$

$$B_p^{(\gamma,\beta)}(x,y) = \frac{\Gamma(\beta)}{\Gamma(\gamma)} H^{1,3}_{3,3}\left[p\left|\begin{matrix}(1-\gamma,1),(1-x,-1),(1-y,-1)\\ (0,1),(1-\beta,1),(1-x-y,-2)\end{matrix}\right.\right]. \tag{42}$$

Further, putting $\alpha = 1$ in relations (20) and (22) and using relation (6), we get the following representations for the generalized Gamma function $\Gamma_p^{(\gamma,\beta)}(x)$:

$$\Gamma_p^{(\gamma,\beta)}(x) = \frac{\Gamma(\beta)}{\Gamma(\gamma)} \int_0^\infty t^{x-1}\, {}_1\Psi_1\left[\begin{matrix}(\gamma,1);\\ (\beta,1);\end{matrix}\; -t-\frac{p}{t}\right] dt, \tag{43}$$

$$\Gamma_p^{(\gamma,\beta)}(x) = \frac{\Gamma(\beta)}{\Gamma(\gamma)} \int_0^\infty t^{x-1} H^{1,1}_{1,2}\left[t+\frac{p}{t}\left|\begin{matrix}(1-\gamma,1)\\ (0,1),(1-\beta,1)\end{matrix}\right.\right] dt. \tag{44}$$

Again, putting $\beta = \gamma = 1$ in relations (14) and (19) and using relation (10), we get the following hypergeometric representations for $B_\alpha^p(x,y)$:

$$B_\alpha^p(x,y) = {}_3\Psi_2\left[\begin{matrix}(1,1),(x,-1),(y,-1);\\ (1,\alpha),(x+y,-2);\end{matrix}\; -p\right], \tag{45}$$

$$B_\alpha^p(x,y) = H^{1,3}_{3,3}\left[p\left|\begin{matrix}(0,1),(1-x,-1),(1-y,-1)\\ (0,1),(0,\alpha),(1-x-y,-2)\end{matrix}\right.\right]. \tag{46}$$

Next, putting $\beta = \gamma = 1$ and $p = 0$ in relations (20) and (22) and using relation (11), we get the following representations for $\Gamma^\alpha(x)$:

$$\Gamma^\alpha(x) = \int_0^\infty t^{x-1}\, {}_1\Psi_1\left[\begin{matrix}(1,1);\\ (1,\alpha);\end{matrix}\; -t\right] dt, \tag{47}$$

$$\Gamma^\alpha(x) = \int_0^\infty t^{x-1} H^{1,1}_{1,2}\left[t\left|\begin{matrix}(0,1)\\ (0,1),(0,\alpha)\end{matrix}\right.\right] dt. \tag{48}$$

II. Putting $\alpha = 1$ in relations (23), (28) and (34) and using relation (7), we get the following integral transforms for the generalized Beta function $B_p^{(\gamma,\beta)}(x,y)$:

$$B\left\{B_p^{(\gamma,\beta)}(x,y):l,m\right\} = \frac{\Gamma(m)\Gamma(\beta)}{\Gamma(\gamma)}{}_4\Psi_3\left[\begin{array}{c}(\gamma,1),(l,1),(x,-1),(y,-1);\\ (\beta,1),(l+m,1),(x+y,-2);\end{array} -1\right], \tag{49}$$

$$L\left\{p^{l-1}B_p^{(\gamma,\beta)}(x,y);s\right\} = \frac{s^{-l}\,\Gamma(\beta)}{\Gamma(\gamma)}{}_4\Psi_2\left[\begin{array}{c}(\gamma,1),(l,1),(x,-1),(y,-1);\\ (\beta,1),(x+y,-2);\end{array} -\tfrac{1}{s}\right], \tag{50}$$

$$\int_0^\infty p^{q-1}\,e^{\frac{-\delta p}{2}}\,W_{\lambda,\mu}(\delta p)\,B_p^{(\gamma,\beta)}(x,y)\,dp$$
$$= \frac{\delta^{-q}\,\Gamma(\beta)}{\Gamma(\gamma)}{}_5\Psi_3\left[\begin{array}{c}(\gamma,1),(x,-1),(y,-1),\left(\frac{1}{2}+\mu+q,1\right),\left(\frac{1}{2}-\mu+q,1\right);\\ (\beta,1),(x+y,-2),(1-\lambda+q,1);\end{array} -\tfrac{1}{\delta}\right]. \tag{51}$$

Further, putting $\alpha = 1$ in relations (27), (33) and (40) and using relation (7), we get the following second form of the integral transforms for the generalized Beta function $B_p^{(\gamma,\beta)}(x,y)$:

$$B\left\{B_p^{(\gamma,\beta)}(x,y):l,m\right\}$$
$$= \frac{\Gamma(m)\Gamma(\beta)}{\Gamma(\gamma)}H_{4,4}^{1,4}\left[1\left|\begin{array}{c}(1-\gamma,1),(1-l,1),(1-x,-1),(1-y,-1)\\ (0,1),(1-\beta,1),(1-l-m,1),(1-x-y,-2)\end{array}\right.\right], \tag{52}$$

$$L\left\{p^{l-1}\,B_p^{(\gamma,\beta)}(x,y);s\right\}$$
$$= \frac{s^{-l}\,\Gamma(\beta)}{\Gamma(\gamma)}H_{4,3}^{1,4}\left[\frac{1}{s}\left|\begin{array}{c}(1-\gamma,1),(1-l,1),(1-x,-1),(1-y,-1)\\ (0,1),(1-\beta,1),(1-x-y,-2)\end{array}\right.\right], \tag{53}$$

$$\int_0^\infty p^{q-1}\,e^{\frac{-\delta p}{2}}\,W_{\lambda,\mu}(\delta p)\,B_p^{(\gamma,\beta)}(x,y)\,dp$$
$$= \frac{\delta^{-q}\,\Gamma(\beta)}{\Gamma(\gamma)}H_{5,4}^{1,5}\left[\frac{1}{\delta}\left|\begin{array}{c}(1-\gamma,1),(1-x,-1),(1-y,-1),\left(\frac{1}{2}-\mu-q,1\right),\left(\frac{1}{2}+\mu-q,1\right)\\ (0,1),(1-\beta,1),(1-x-y,-2),(\lambda-q,1)\end{array}\right.\right]. \tag{54}$$

Again, putting $\beta = \gamma = 1$ in relations (23), (28) and (34) and using relation (10), we get the following integral transforms for the generalized Beta function $B_\alpha^p(x,y)$:

$$B\left\{B_\alpha^p(x,y):l,m\right\} = \Gamma(m){}_4\Psi_3\left[\begin{array}{c}(1,1),(l,1),(x,-1),(y,-1);\\ (1,\alpha),(l+m,1),(x+y,-2);\end{array} -1\right], \tag{55}$$

$$L\left\{p^{l-1}B_\alpha^p(x,y);s\right\} = s^{-l}{}_4\Psi_2\left[\begin{array}{c}(1,1),(l,1),(x,-1),(y,-1);\\ (1,\alpha),(x+y,-2);\end{array} -\tfrac{1}{s}\right], \tag{56}$$

$$\int_0^\infty p^{q-1}\,e^{\frac{-\delta p}{2}}\,W_{\lambda,\mu}(\delta p)\,B_\alpha^p(x,y)\,dp$$
$$= \delta^{-q}\,{}_5\Psi_3\left[\begin{array}{c}(1,1),(x,-1),(y,-1),\left(\frac{1}{2}+\mu+q,1\right),\left(\frac{1}{2}-\mu+q,1\right);\\ (1,\alpha),(x+y,-2),(1-\lambda+q,1);\end{array} \tfrac{1}{\delta}\right]. \tag{57}$$

Next, putting $\beta = \gamma = 1$ in relations (27), (33) and (40) and using relation (10), we get the following second form of the integral transforms for the generalized Beta function $B_{\alpha}^{p}(x,y)$:

$$B\left\{B_{\alpha}^{p}(x,y) : l, m\right\} = \Gamma(m) H_{4,4}^{1,4}\left[1 \left| \begin{matrix} (0,1),(1-l,1),(1-x,-1),(1-y,-1) \\ (0,1),(0,\alpha),(1-l-m,1),(1-x-y,-2) \end{matrix} \right.\right], \tag{58}$$

$$L\left\{p^{l-1} B_{\alpha}^{p}(x,y); s\right\} = s^{-l} H_{4,3}^{1,4}\left[\frac{1}{s} \left| \begin{matrix} (0,1),(1-l,1),(1-x,-1),(1-y,-2) \\ (0,1),(0,\alpha),(1-x-y,-1) \end{matrix} \right.\right], \tag{59}$$

$$\int_0^{\infty} p^{q-1}\, e^{\frac{-\delta p}{2}}\, W_{\lambda,\mu}(\delta p)\, B_{\alpha}^{p}(x,y)\, dp = \delta^{-q} H_{5,4}^{1,5}\left[\frac{1}{\delta} \left| \begin{matrix} (0,1),(1-x,-1),(1-y,-1),\left(\frac{1}{2}-\mu-q,1\right),\left(\frac{1}{2}+\mu-q,1\right) \\ (0,1),(0,\alpha),(1-x-y,-2),(\lambda-q,1) \end{matrix} \right.\right]. \tag{60}$$

In a forthcoming investigation, the new extension of Beta function given in equation (2) will be used to introduce other extensions of the extended Gauss hypergeometric and the confluent hypergeometric functions. For each of these new extensions we will obtain various properties.

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Direct Product of Fuzzy Multigroups

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Abstract − The paper introduces direct product in fuzzy multigroup setting as an extension of direct product of fuzzy subgroups. Some properties of direct product of fuzzy multigroups are explicated. It is established that the direct product of fuzzy multigroups is a fuzzy multigroup. The notion of homomorphism and some of its properties in the context of direct product of fuzzy multigroups are introduced.

Keywords − *Fuzzy multisets, Fuzzy multigroups, Direct product of fuzzy multigroups*

1. Introduction

The concept of set theory put forward by a German mathematician George Cantor (1845-1918) is a linchpin for the whole of mathematics. Notwithstanding, an element of a set must be distinct and definite in a collection, which is not consistent with real-life issues. The "famous" fuzzy set proposed by Zadeh [1] is a veritable tool for handling uncertainty and/or imprecision in real-life problems. Fuzzy set theory posited that there are cases where an element would not be definite in a collection. The theory of fuzzy set has grown tremendously over time giving birth to some algebraic structures like fuzzy group introduced by Rosenfeld [2]. Some properties of the fuzzy groups have been discussed in details in [3, 4], etc. In the same vein, multiset theory [5–7] violated the rule that an element must be distinct in a collection, i.e., the idea of multisets allows repetition of elements.

Motivated by Zadeh [1], fuzzy multiset was proposed by Yager [8] as a generalization of fuzzy set. The idea of fuzzy multisets allows the repetition of membership function of an element in multiset framework, unlike the case in fuzzy set where membership function of an element does not allow to repeat. Some details of the notion of fuzzy multisets can be found in [9–11]. Subsequently, Shinoj *et al.* [12] followed the footsteps of Rosenfeld [2] and introduced a non-classical group called fuzzy multigroup, which constitutes an application of fuzzy multiset to the theory of group. The idea of abelian fuzzy multigroups was proposed and studied in [13, 14]. Ejegwa [15] introduced fuzzy multigroupoids, the ideas of center and centralizer in fuzzy multigroup context with some related results. The notions of fuzzy submultigroups and normal fuzzy submultigroups were explicated in [15, 16] with a number of results. Also, the concept of homomorphism of fuzzy multigroups and its properties have been explored with some results [17].

This paper is motivated by the work of Ray [18] on product of fuzzy subgroups, which was extended from group theory but presented in the light of fuzzy groups. In the same vein, we are spurred to propose direct product in fuzzy multigroup structure as an extension of the work in [18], and explicate

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some of it properties in details. The rest of the paper are thus outline: Section 2 provides some preliminaries while Section 3 proposes direct product in fuzzy multigroup setting, discusses some of its properties and outline some related results. Finally, Section 4 contains the conclusion and recommendations for future studies.

2. Preliminaries

This section presents some foundational concepts which are germane to the subject under consideration.

Definition 2.1. [8] Suppose X is a nonempty set. Then, a fuzzy bag/multiset A drawn from X can be characterized by a count membership function CM_A where

$$CM_A : X \to Q$$

and Q is the set of all crisp bags or multisets from the unit interval $I = [0, 1]$.

A fuzzy multiset can also be characterized by a high-order function. In particular, a fuzzy multiset A can be characterized by a function

$$CM_A : X \to N^I \text{ or } CM_A : X \to [0, 1] \to N,$$

where $I = [0, 1]$ and $N = \mathbb{N} \cup \{0\}$.

By [19], it follows that $CM_A(x)$ for $x \in X$ is given as

$$CM_A(x) = \{\mu_A^1(x), \mu_A^2(x), ..., \mu_A^n(x), ...\},$$

where $\mu_A^1(x), \mu_A^2(x), ..., \mu_A^n(x), ... \in [0, 1]$ where $\mu_A^1(x) \geq \mu_A^2(x) \geq ... \geq \mu_A^n(x) \geq ...$, whereas in a finite case, we write

$$CM_A(x) = \{\mu_A^1(x), \mu_A^2(x), ..., \mu_A^n(x)\},$$

for $\mu_A^1(x) \geq \mu_A^2(x) \geq ... \geq \mu_A^n(x)$.

A fuzzy multiset A can be represented by

$$A = \{\langle \frac{CM_A(x)}{x} \rangle \mid x \in X\} \text{ or } A = \{\langle x, CM_A(x) \rangle \mid x \in X\}.$$

In short, a fuzzy multiset A of X is characterized by the count membership function $CM_A(x)$ for $x \in X$, that takes the value of a multiset of a unit interval $I = [0, 1]$ [20, 21].

We denote the set of all fuzzy multisets by $FMS(X)$.

Definition 2.2. [10] Suppose $A, B \in FMS(X)$. Then, A is called a fuzzy submultiset of B denoted by $A \subseteq B$ if $CM_A(x) \leq CM_B(x) \forall x \in X$. Also, if $A \subseteq B$ and $A \neq B$, then A is called a proper fuzzy submultiset of B and denoted as $A \subset B$.

Definition 2.3. [11] Suppose A and B are fuzzy multisets of a set X. Then, the intersection and union of A and B, denoted by $A \cap B$ and $A \cup B$, respectively, are defined by the rules that for any object $x \in X$,

(i) $CM_{A \cap B}(x) = CM_A(x) \wedge CM_B(x)$,

(ii) $CM_{A \cup B}(x) = CM_A(x) \vee CM_B(x)$,

where $\wedge$ and $\vee$ denote minimum and maximum respectively.

Definition 2.4. [10] Let $A, B \in FMS(X)$. Then, A and B are comparable to each other if and only if $A \subseteq B$ or $B \subseteq A$, and $A = B \Leftrightarrow CM_A(x) = CM_B(x) \forall x \in X$.

Definition 2.5. [12] Suppose X is a group. Then, a fuzzy multiset A of X is a fuzzy multigroup of X if

(i) $CM_A(xy) \geq CM_A(x) \wedge CM_A(y) \forall x, y \in X$,

(ii) $CM_A(x^{-1}) \geq CM_A(x) \forall x \in X$.

It follows immediately that,

$$CM_A(x^{-1}) = CM_A(x), \forall x \in X$$

since

$$CM_A(x) = CM_A((x^{-1})^{-1}) \geq CM_A(x^{-1}).$$

Also,

$$CM_A(e) \geq CM_A(x) \forall x \in X$$

because

$$CM_A(e) = CM_A(xx^{-1}) \geq CM_A(x) \wedge CM_A(x) = CM_A(x)$$

and

$$CM_A(x^n) \geq CM_A(x) \forall x \in X$$

since

$$\begin{aligned} CM_A(x^n) = CM_A(x^{n-1}x) &\geq CM_A(x^{n-1}) \wedge CM_A(x) \\ &\geq CM_A(x) \wedge ... \wedge CM_A(x) \\ &= CM_A(x). \end{aligned}$$

Every fuzzy multigroup is a fuzzy multiset but the converse is not true. We denote the set of all fuzzy multigroups of X by $FMG(X)$.

Definition 2.6. [15] Suppose $A \in FMG(X)$. Then, a fuzzy submultiset B of A is a fuzzy submultigroup of A denoted by $B \sqsubseteq A$ if B a fuzzy multigroup. A fuzzy submultigroup B of A is a proper denoted by $B \sqsubset A$, if $B \sqsubseteq A$ and $A \neq B$.

Remark 2.7. [15] If $A \in FMG(X)$ and $B \sqsubseteq A$, then $B \in FMG(X)$. Again, suppose $C \in FMS(X)$ and $C \subseteq B$. Then $C \sqsubseteq A \Leftrightarrow C \sqsubseteq B$.

Definition 2.8. [13] A fuzzy multiset A of a set X is commutative if $CM_A(xy) = CM_A(yx)$ for all $x, y \in X$.

Definition 2.9. [12,15] Suppose $A \in FMG(X)$. Then, A_* and A^* are defined by

(i) $A_* = \{x \in X \mid CM_A(x) > 0\}$ and

(ii) $A^* = \{x \in X \mid CM_A(x) = CM_A(e)\}$, where e is the identity element of X.

Proposition 2.10. [12,15] Suppose $A \in FMG(X)$, then A_* and A^* are subgroups of X.

Definition 2.11. [16] Let $A, B \in FMG(X)$ such that $A \subseteq B$. Then, A is a normal fuzzy submultigroup of B if for all $x, y \in X$,

$$CM_A(xyx^{-1}) \geq CM_A(y).$$

Proposition 2.12. [16] Let $A, B \in FMG(X)$. Then, the following statements are equivalent.

(i) A is a normal fuzzy submultigroup of B.

(ii) $CM_A(xyx^{-1}) = CM_A(y) \forall x, y \in X$.

(iii) $CM_A(xy) = CM_A(yx) \forall x, y \in X$.

Definition 2.13. [16] Let $A, B \in FMG(X)$. We say A and B are conjugate to each other if for all $x, y \in X$,

$$CM_A(x) = CM_B(yxy^{-1}) \text{ and } CM_B(y) = CM_A(xyx^{-1}).$$

Definition 2.14. Suppose $A \in FMG(X)$. Then, $A_{[\alpha]}$ and $A_{(\alpha)}$ defined by

(i) $A_{[\alpha]} = \{x \in X \mid CM_A(x) \geq \alpha\}$ and

(ii) $A_{(\alpha)} = \{x \in X \mid CM_A(x) > \alpha\}$

are called strong upper alpha-cut and weak upper alpha-cut of A, where $\alpha \in [0, 1]$.

Definition 2.15. Let $A \in FMG(X)$. Then, $A^{[\alpha]}$ and $A^{(\alpha)}$ defined by

(i) $A^{[\alpha]} = \{x \in X \mid CM_A(x) \leq \alpha\}$ and

(ii) $A^{(\alpha)} = \{x \in X \mid CM_A(x) < \alpha\}$

are called strong lower alpha-cut and weak lower alpha-cut of A, where $\alpha \in [0, 1]$.

Theorem 2.16. Suppose $A \in FMG(X)$. Then $A_{[\alpha]}$ is a subgroup of X if $\alpha \leq CM_A(e)$ and $A^{[\alpha]}$ is a subgroup of X if $\alpha \geq CM_A(e)$, where e is the identity element of X and $\alpha \in [0, 1]$.

Definition 2.17. Suppose $A, B \in FMG(X)$ such that $A \subseteq B$. Then, A is a characteristic (fully invariant) fuzzy submultigroup of B if

$$CM_{A^\theta}(x) = CM_A(x)\ \forall x \in X$$

for every automorphism, θ of X. That is, $\theta(A) \subseteq A$ for every $\theta \in Aut(X)$.

Proposition 2.18. Suppose X is a group. Every characteristic fuzzy submultigroup of a fuzzy multigroup B of X is normal.

Definition 2.19. [17] Suppose X and Y are groups and let $f : X \to Y$ be a homomorphism. Suppose A and B are fuzzy multigroups of X and Y respectively, then f induces a homomorphism from A to B which satisfies

(i) $CM_A(f^{-1}(y_1y_2)) \geq CM_A(f^{-1}(y_1)) \wedge CM_A(f^{-1}(y_2))\ \forall y_1, y_2 \in Y$,

(ii) $CM_B(f(x_1x_2)) \geq CM_B(f(x_1)) \wedge CM_B(f(x_2))\ \forall x_1, x_2 \in X$,

where

(i) the image of A under f, denoted by $f(A)$, is a fuzzy multiset over Y defined by

$$CM_{f(A)}(y) = \begin{cases} \bigvee_{x \in f^{-1}(y)} CM_A(x), & f^{-1}(y) \neq \emptyset \\ 0, & \text{otherwise} \end{cases}$$

for each $y \in Y$.

(ii) the inverse image of B under f, denoted by $f^{-1}(B)$, is a fuzzy multiset over X defined by

$$CM_{f^{-1}(B)}(x) = CM_B(f(x)) \forall x \in X.$$

Theorem 2.20. [17] Suppose X and Y are groups and $f : X \to Y$ is an isomorphism. Then

(i) $A \in FMG(X) \Leftrightarrow f(A) \in FMG(Y)$.

(ii) $B \in FMG(Y) \Leftrightarrow f^{-1}(B) \in FMG(X)$.

3. Main results

Suppose X and Y are two groups. Then, the direct product, $X \times Y$ is the Cartesian product of ordered pair (x, y) such that $x \in X$ and $y \in Y$, and the group operation is component-wise, so $(x_1, y_1) \times (x_2, y_2) = (x_1x_2, y_1y_2)$. The resulting algebraic structure satisfies the axioms for a group. Since the ordered pair (x, y) such that $x \in X$ and $y \in Y$ is an element of $X \times Y$, we simply write $(x, y) \in X \times Y$. In this section, we discuss the notion of direct product of two fuzzy multigroups defined over X and Y, respectively.

Definition 3.1. Suppose $A \in FMG(X)$ and $B \in FMG(Y)$ where X and Y are groups. The direct product of A and B depicted by $A \times B$ is a function

$$CM_{A \times B} : X \times Y \to Q$$

defined by

$$CM_{A \times B}((x, y)) = CM_A(x) \wedge CM_B(y) \forall x \in X, \forall y \in Y,$$

where Q is the set of all multisets from the unit interval $I = [0, 1]$.

Example 3.2. Let $X = \{1, x\}$ be a group, where $x^2 = 1$ and $Y = \{e, a, b, c\}$ be a Klein 4-group, where $a^2 = b^2 = c^2 = e$. Suppose

$$A = \{\langle \frac{1, 0.8}{1} \rangle, \langle \frac{0.8, 0.5}{x} \rangle\}$$

and

$$B = \{\langle \frac{1, 0.9}{e} \rangle, \langle \frac{0.6, 0.5}{a} \rangle, \langle \frac{0.7, 0.6}{b} \rangle, \langle \frac{0.6, 0.5}{c} \rangle\}$$

are fuzzy multigroups of X and Y by Definition 2.5. Now

$$X \times Y = \{(1, e), (1, a), (1, b), (1, c), (x, e), (x, a), (x, b), (x, c)\}$$

is a group from the classical sense. By Definition 3.1, we get

$$A \times B = \{\langle \frac{1, 0.8}{(1, e)} \rangle, \langle \frac{0.6, 0.5}{(1, a)} \rangle, \langle \frac{0.7, 0.6}{(1, b)} \rangle, \langle \frac{0.6, 0.5}{(1, c)} \rangle, \langle \frac{0.8, 0.5}{(x, e)} \rangle, \langle \frac{0.6, 0.5}{(x, a)} \rangle, \langle \frac{0.7, 0.5}{(x, b)} \rangle, \langle \frac{0.6, 0.5}{(x, c)} \rangle\}.$$

Certainly, $A \times B$ is a fuzzy multigroup of $X \times Y$ in accordance to Definition 2.5.

Next, we consider an example to investigate what happens of the direct product of a fuzzy multigroup of a group X and a fuzzy multiset of a group Y.

Example 3.3. Let X and Y be groups as in Example 3.2. Suppose we have a fuzzy multigroup of X given as

$$A = \{\langle \frac{1, 0.5}{1} \rangle, \langle \frac{0.7, 0.4}{x} \rangle\},$$

and a fuzzy multiset of Y as

$$B = \{\langle \frac{0.7, 0.5}{e} \rangle, \langle \frac{0.6, 0.4}{a} \rangle, \langle \frac{0.7, 0.6}{b} \rangle, \langle \frac{0.6, 0.4}{c} \rangle\}.$$

Synthesizing Definitions 2.5 and 3.1, we get

$$A \times B = \{\langle \frac{0.7, 0.5}{(1, e)} \rangle, \langle \frac{0.6, 0.4}{(1, a)} \rangle, \langle \frac{0.7, 0.5}{(1, b)} \rangle, \langle \frac{0.6, 0.4}{(1, c)} \rangle, \langle \frac{0.7, 0.4}{(x, e)} \rangle, \langle \frac{0.6, 0.4}{(x, a)} \rangle, \langle \frac{0.7, 0.4}{(x, b)} \rangle, \langle \frac{0.6, 0.4}{(x, c)} \rangle\},$$

and it follows that $A \times B$ is a fuzzy multigroup of $X \times Y$ although B is not a fuzzy multigroup of Y.

Theorem 3.4. Let $A \in FMG(X)$ and $B \in FMG(Y)$, respectively. Then for all $\alpha \in [0, 1]$,

(i) $A \times B)_{[\alpha]} = A_{[\alpha]} \times B_{[\alpha]}$.

(ii) $(A \times B)^{[\alpha]} = A^{[\alpha]} \times B^{[\alpha]}$.

Proof. (i) Let $(x, y) \in (A \times B)_{[\alpha]}$. Using Definition 2.14, we have

$$CM_{A \times B}((x, y)) = (CM_A(x) \wedge CM_B(y)) \geq \alpha.$$

This implies that $CM_A(x) \geq \alpha$ and $CM_B(y) \geq \alpha$, then $x \in A_{[\alpha]}$ and $y \in B_{[\alpha]}$. Thus,

$$(x, y) \in A_{[\alpha]} \times B_{[\alpha]}.$$

Also, let $(x, y) \in A_{[\alpha]} \times B_{[\alpha]}$. Then $CM_A(x) \geq \alpha$ and $CM_B(y) \geq \alpha$. That is,

$$(CM_A(x) \wedge CM_B(y)) \geq \alpha.$$

This yields us $(x, y) \in (A \times B)_{[\alpha]}$. Therefore, $(A \times B)_{[\alpha]} = A_{[\alpha]} \times B_{[\alpha]}$ $\forall \alpha \in [0, 1]$.

(ii) Similar to (i). □

Corollary 3.5. Suppose $A \in FMG(X)$ and $B \in FMG(Y)$, then

(i) $(A \times B)_* = A_* \times B_*$,

(ii) $(A \times B)^* = A^* \times B^*$.

Proof. Similar to Theorem 3.4. □

Theorem 3.6. Suppose $A \in FMG(X)$ and $B \in FMG(Y)$. Then $A \times B$ is a fuzzy multigroup of $X \times Y$.

PROOF. Let $(x, y) \in X \times Y$ and let $x = (x_1, x_2)$ and $y = (y_1, y_2)$. We have

$$\begin{aligned} CM_{A\times B}(xy) &= CM_{A\times B}((x_1, x_2)(y_1, y_2)) \\ &= CM_{A\times B}((x_1y_1, x_2y_2)) \\ &= CM_A(x_1y_1) \wedge CM_B(x_2y_2) \\ &\geq \wedge(CM_A(x_1) \wedge CM_A(y_1), CM_B(x_2) \wedge CM_B(y_2)) \\ &= \wedge(CM_A(x_1) \wedge CM_B(x_2), CM_A(y_1) \wedge CM_B(y_2)) \\ &= CM_{A\times B}((x_1, x_2)) \wedge CM_{A\times B}((y_1, y_2)) \\ &= CM_{A\times B}(x) \wedge CM_{A\times B}(y). \end{aligned}$$

Also,

$$\begin{aligned} CM_{A\times B}(x^{-1}) &= CM_{A\times B}((x_1, x_2)^{-1}) = CM_{A\times B}((x_1^{-1}, x_2^{-1})) \\ &= CM_A(x_1^{-1}) \wedge CM_B(x_2^{-1}) = CM_A(x_1) \wedge CM_B(x_2) \\ &= CM_{A\times B}((x_1, x_2)) = CM_{A\times B}(x). \end{aligned}$$

Hence, $A \times B \in FMG(X \times Y)$. □

Corollary 3.7. Let $A_1, B_1 \in FMG(X_1)$ and $A_2, B_2 \in FMG(X_2)$, respectively such that $A_1 \subseteq B_1$ and $A_2 \subseteq B_2$. If A_1 and A_2 are normal fuzzy submultigroups of B_1 and B_2, then $A_1 \times A_2$ is a normal fuzzy submultigroup of $B_1 \times B_2$.

PROOF. By Theorem 3.6, $A_1 \times A_2$ is a fuzzy multigroup of $X_1 \times X_2$. Also, $B_1 \times B_2$ is a fuzzy multigroup of $X_1 \times X_2$. We show that $A_1 \times A_2$ is a normal fuzzy submultigroup of $B_1 \times B_2$. Let $(x, y) \in X_1 \times X_2$ such that $x = (x_1, x_2)$ and $y = (y_1, y_2)$. Then we get

$$\begin{aligned} CM_{A_1\times A_2}(xy) &= CM_{A_1\times A_2}((x_1, x_2)(y_1, y_2)) \\ &= CM_{A_1\times A_2}((x_1y_1, x_2y_2)) \\ &= CM_{A_1}(x_1y_1) \wedge CM_{A_2}(x_2y_2) \\ &= CM_{A_1}(y_1x_1) \wedge CM_{A_2}(y_2x_2) \\ &= CM_{A_1\times A_2}((y_1x_1, y_2x_2)) \\ &= CM_{A_1\times A_2}((y_1, y_2)(x_1, x_2)) \\ &= CM_{A_1\times A_2}(yx). \end{aligned}$$

Hence, the result follows by Proposition 2.12. □

Theorem 3.8. Suppose A and B are fuzzy multigroups of X and Y, respectively. Then

(i) $(A \times B)_*$ is a subgroup of $X \times Y$,

(ii) $(A \times B)^*$ is a subgroup of $X \times Y$,

(iii) $(A \times B)_{[\alpha]}$ is a subgroup of $X \times Y$, $\forall\, \alpha \leq CM_{A\times B}(e, e')$ and $\alpha \in [0, 1]$,

(iv) $(A \times B)^{[\alpha]}$ is a subgroup of $X \times Y$, $\forall\, \alpha \geq CM_{A\times B}(e, e')$ and $\alpha \in [0, 1]$.

PROOF. Combining Proposition 2.10, Theorems 2.16 and 3.6, the results follow. □

Corollary 3.9. Suppose $A, C \in FMG(X)$ such that $A \subseteq C$ and $B, D \in FMG(Y)$ such that $B \subseteq D$, respectively. If A and B are normal, then

(i) $(A \times B)_*$ is a normal subgroup of $(C \times D)_*$,

(ii) $(A \times B)^*$ is a normal subgroup of $(C \times D)^*$,

(iii) $(A \times B)_{[\alpha]}$ is a normal subgroup of $(C \times D)_{[\alpha]}$, $\forall\, \alpha \leq CM_{A\times B}(e, e')$ and $\alpha \in [0, 1]$,

(iv) $(A \times B)^{[\alpha]}$ is a normal subgroup of $(C \times D)^{[\alpha]}$, $\forall\, \alpha \geq CM_{A\times B}(e, e')$ and $\alpha \in [0, 1]$.

Proof. Combining Proposition 2.10, Theorems 2.16, 3.6 and Corollary 3.7, the results follow. □

Proposition 3.10. Let $A \in FMG(X)$ and $B \in FMG(Y)$, respectively. Then $\forall (x, y) \in X \times Y$, we have

(i) $CM_{A\times B}((x^{-1}, y^{-1})) = CM_{A\times B}((x, y))$,

(ii) $CM_{A\times B}((e, e')) \geq CM_{A\times B}((x, y))$,

(iii) $CM_{A\times B}((x, y)^n) \geq CM_{A\times B}((x, y))$,

where e and e' are the identity elements of X and Y, respectively and $n \in \mathbb{N}$.

Proof. Let $x \in X,\ y \in Y$ and $(x, y) \in X \times Y$. By Theorem 3.6, it follows that $A\times B \in FMG(X\times Y)$. Now,
(i)

$$\begin{aligned} CM_{A\times B}((x^{-1}, y^{-1})) &= CM_A(x^{-1}) \wedge CM_B(y^{-1}) \\ &= CM_A(x) \wedge CM_B(y) \\ &= CM_{A\times B}((x, y)). \end{aligned}$$

Clearly, $CM_{A\times B}((x^{-1}, y^{-1})) = CM_{A\times B}((x, y))\ \forall (x, y) \in X \times Y$.

(ii)

$$\begin{aligned} CM_{A\times B}((e, e')) &= CM_{A\times B}((x, y)(x^{-1}, y^{-1})) \\ &\geq CM_{A\times B}((x, y)) \wedge CM_{A\times B}((x^{-1}, y^{-1})) \\ &= CM_{A\times B}((x, y)) \wedge CM_{A\times B}((x, y)) \\ &= CM_{A\times B}((x, y))\ \forall (x, y) \in X \times Y. \end{aligned}$$

Hence, $CM_{A\times B}((e, e')) \geq CM_{A\times B}((x, y))$.

(iii)

$$\begin{aligned} CM_{A\times B}((x, y)^n) &= CM_{A\times B}((x^n, y^n)) \\ &= CM_{A\times B}((x^{n-1}, y^{n-1})(x, y)) \\ &\geq CM_{A\times B}((x^{n-1}, y^{n-1})) \wedge CM_{A\times B}((x, y)) \\ &\geq CM_{A\times B}((x^{n-2}, y^{n-2})) \wedge CM_{A\times B}((x, y)) \wedge CM_{A\times B}((x, y)) \\ &\geq CM_{A\times B}((x, y)) \wedge CM_{A\times B}((x, y)) \wedge ... \wedge CM_{A\times B}((x, y)) \\ &= CM_{A\times B}((x, y)), \end{aligned}$$

$\Rightarrow CM_{A\times B}((x, y)^n) = CM_{A\times B}((x^n, y^n)) \geq CM_{A\times B}((x, y))\ \forall (x, y) \in X \times Y$. □

Theorem 3.11. Let A and B be fuzzy multisets of groups X and Y, respectively. Suppose that e and e' are the identity elements of X and Y, respectively. If $A \times B$ is a fuzzy multigroup of $X \times Y$, then at least one of the following statements hold.

(i) $CM_B(e') \geq CM_A(x)\ \forall x \in X$,

(ii) $CM_A(e) \geq CM_B(y)\ \forall y \in Y$.

Proof. Let $A \times B \in FMG(X \times Y)$. By contrapositive, suppose that none of the statements holds. Then suppose we can find a in X and b in Y such that

$$CM_A(a) > CM_B(e') \text{ and } CM_B(b) > CM_A(e).$$

From these we have

$$\begin{aligned} CM_{A\times B}((a,b)) &= CM_A(a) \wedge CM_B(b) \\ &> CM_A(e) \wedge CM_B(e') \\ &= CM_{A\times B}((e,e')). \end{aligned}$$

Thus, $A \times B$ is not a fuzzy multigroup of $X \times Y$ by Proposition 3.10. Hence, either $CM_B(e') \geq CM_A(x)\, \forall x \in X$ or $CM_A(e) \geq CM_B(y)\, \forall y \in Y$. This completes the proof. □

Theorem 3.12. Let A and B be fuzzy multisets of groups X and Y, respectively, such that $CM_A(x) \leq CM_B(e')\, \forall x \in X$, e' being the identity element of Y. If $A \times B$ is a fuzzy multigroup of $X \times Y$, then A is a fuzzy multigroup of X.

PROOF. Let $A \times B$ be a fuzzy multigroup of $X \times Y$ and $x, y \in X$. Then $(x, e'), (y, e') \in X \times Y$. Now, using the property $CM_A(x) \leq CM_B(e')\, \forall x \in X$, we get

$$\begin{aligned} CM_A(xy) &= CM_A(xy) \wedge CM_B(e'e') \\ &= CM_{A\times B}((xy, e'e')) \\ &= CM_{A\times B}((x,e')(y,e')) \\ &\geq CM_{A\times B}((x,e')) \wedge CM_{A\times B}((y,e')) \\ &= \wedge(CM_A(x) \wedge CM_B(e'), CM_A(y) \wedge CM_B(e')) \\ &= CM_A(x) \wedge CM_A(y). \end{aligned}$$

Also,

$$\begin{aligned} CM_A(x^{-1}) &= CM_A(x^{-1}) \wedge CM_B(e'^{-1}) = CM_{A\times B}((x^{-1}, e'^{-1})) \\ &= CM_{A\times B}((x,e')^{-1}) = CM_{A\times B}((x,e')) \\ &= CM_A(x) \wedge CM_B(e') = CM_A(x). \end{aligned}$$

Hence, A is a fuzzy multigroup of X. This completes the proof. □

Theorem 3.13. Let A and B be fuzzy multisets of groups X and Y, respectively, such that $CM_B(x) \leq CM_A(e)\, \forall x \in Y$, e being the identity element of X. If $A \times B$ is a fuzzy multigroup of $X \times Y$, then B is a fuzzy multigroup of Y.

PROOF. Similar to Theorem 3.12. □

Corollary 3.14. Let A and B be fuzzy multisets of groups X and Y, respectively. If $A \times B$ is a fuzzy multigroup of $X \times Y$, then either A is a fuzzy multigroup of X or B is a fuzzy multigroup of Y.

PROOF. Combining Theorems 3.11, 3.12 and 3.13, the result follows. □

Theorem 3.15. If A and C are conjugate fuzzy multigroups of a group X, and B and D are conjugate fuzzy multigroups of a group Y. Then $A \times B$ is a conjugate of $C \times D$.

PROOF. Since A and C are conjugate, it implies that for $g_1 \in X$, we have

$$CM_A(x) = CM_C(g_1^{-1} x g_1)\, \forall x \in X.$$

Also, since B and D are conjugate, for $g_2 \in Y$, we get

$$CM_B(y) = CM_D(g_2^{-1} y g_2)\, \forall y \in Y.$$

Now,

$$\begin{aligned} CM_{A\times B}((x,y)) = CM_A(x) \wedge CM_B(y) &= CM_C(g_1^{-1} x g_1) \wedge CM_D(g_2^{-1} y g_2) \\ &= CM_{C\times D}((g_1^{-1} x g_1), (g_2^{-1} y g_2)) \\ &= CM_{C\times D}((g_1^{-1}, g_2^{-1})(x,y)(g_1,g_2)) \\ &= CM_{C\times D}((g_1,g_2)^{-1}(x,y)(g_1,g_2)). \end{aligned}$$

This completes the proof. □

Theorem 3.16. Let $A \in FMG(X)$ and $B \in FMG(Y)$. Suppose C and D are two fuzzy submultisets of A and B, respectively. Then $C \times D$ is a fuzzy submultigroup of $A \times B$ if and only if both C and D are fuzzy submultigroups of A and B, respectively.

PROOF. Suppose C and D are two fuzzy submultigroups of A and B, respectively. Then $C \in FMG(X)$ and $D \in FMG(Y)$ by Remark 2.7. It then follows that $C \times D \in FMG(X \times Y)$ by Theorem 3.6. Since $A \times B$ is a fuzzy multigroup of $X \times Y$ by the same reason, and $C \sqsubseteq A$ and $D \sqsubseteq B$, thus, $C \times D$ is a fuzzy submultigroup of $A \times B$.

Conversely, If $C \times D$ is a fuzzy submultigroup of $A \times B$. Then, it follows that $C \sqsubseteq A$ and $D \sqsubseteq B$. These complete the proof. □

Corollary 3.17. Let $A \in FMG(X)$ and $B \in FMG(Y)$. Suppose C and D are two fuzzy submultigroups of A and B, respectively. Then $C \times D$ is a normal fuzzy submultigroup of $A \times B$ if and only if both C and D are normal fuzzy submultigroups of A and B, respectively.

PROOF. Combining both Definition 2.11, Theorems 3.6 and 3.16, the proof follows. □

Corollary 3.18. Let $A \in FMG(X)$ and $B \in FMG(Y)$. Suppose C and D are two fuzzy submultigroups of A and B, respectively. Then $C \times D$ is a characteristic fuzzy submultigroup of $A \times B$ if and only if both C and D are characteristic fuzzy submultigroups of A and B, respectively.

PROOF. Combining both Theorems 3.6 and 3.16, the proof follows. □

Remark 3.19. With the same hypothesis as in Corollary 3.18, it follows that $C \times D$ is a normal fuzzy submultigroup of $A \times B$ if both C and D are characteristic fuzzy submultigroups of A and B, respectively.

Corollary 3.20. Let $A \in FMG(X)$ and C be a fuzzy submultiset of A. Then $C \times C$ is a fuzzy submultigroup of $A \times A$ if and only if C is a fuzzy submultigroup of A.

PROOF. The proof is straightforward from Theorem 3.16. □

Remark 3.21. Let $A \in FMG(X)$ and C be a fuzzy submultigroup of A. Then

(i) $C \times C$ is a normal fuzzy submultigroup of $A \times A$ if and only if C is a normal fuzzy submultigroup of A.

(ii) $C \times C$ is a characteristic fuzzy submultigroup of $A \times A$ if and only if C is a characteristic fuzzy submultigroup of A.

(iii) $C \times C$ is a normal fuzzy submultigroup of $A \times A$ if C is a characteristic fuzzy submultigroup of A.

Theorem 3.22. Let A and B be fuzzy multigroups of groups X and Y, respectively. Then A and B are commutative if and only if $A \times B$ is a commutative fuzzy multigroup of $X \times Y$.

PROOF. Suppose A and B are commutative. We show that $A \times B$ is a commutative fuzzy multigroup of $X \times Y$. It is a known fact that $A \times B \in FMG(X \times Y)$ by Theorem 3.6. Let $(x, y) \in X_1 \times X_2$ such that $x = (x_1, x_2)$ and $y = (y_1, y_2)$. Then we get

$$\begin{aligned}
CM_{A\times B}(xy) &= CM_{A\times B}((x_1, x_2)(y_1, y_2)) \\
&= CM_{A\times B}(x_1y_1, x_2y_2) \\
&= CM_A(x_1y_1) \wedge CM_B(x_2y_2) \\
&= CM_A(y_1x_1) \wedge CM_B(y_2x_2) \\
&= CM_{A\times B}(y_1x_1, y_2x_2) \\
&= CM_{A\times B}((y_1, y_2)(x_1, x_2)) \\
&= CM_{A\times B}(yx).
\end{aligned}$$

Hence, $A \times B$ is a commutative fuzzy multigroup of $X \times Y$ by Definition 2.8.

Conversely, suppose $A \times B$ is a commutative fuzzy multigroup of $X \times Y$. Then, it is clear that both A and B are commutative fuzzy multigroups of groups X and Y, respectively. □

Now, we present some homomorphic properties of direct product of fuzzy multigroups. This is an extension of the notion of homomorphism in fuzzy multigroup setting (cf. Definition 2.19) to direct product of fuzzy multigroups.

Definition 3.23. Let $W \times X$ and $Y \times Z$ be groups and let $f : W \times X \to Y \times Z$ be a homomorphism. Suppose $A \times B \in FMS(W \times X)$ and $C \times D \in FMS(Y \times Z)$, respectively. Then

(i) the image of $A \times B$ under f, denoted by $f(A \times B)$, is a fuzzy multiset of $Y \times Z$ defined by

$$CM_{f(A\times B)}((y,z)) = \begin{cases} \bigvee_{(w,x)\in f^{-1}((y,z))} CM_{A\times B}((w,x)), & f^{-1}((y,z)) \neq \emptyset \\ 0, & \text{otherwise,} \end{cases}$$

for each $(y, z) \in Y \times Z$.

(ii) the inverse image of $C \times D$ under f, denoted by $f^{-1}(C \times D)$, is a fuzzy multiset of $W \times X$ defined by

$$CM_{f^{-1}(C\times D)}((w,x)) = CM_{C\times D}(f((w,x)))\ \forall (w,x) \in W \times X.$$

Theorem 3.24. Let W, X, Y, Z be groups, $A \in FMS(W), B \in FMS(X), C \in FMS(Y)$ and $D \in FMS(Z)$. If $f : W \times X \to Y \times Z$ is an isomorphism, then

(i) $f(A \times B) = f(A) \times f(B)$,

(ii) $f^{-1}(C \times D) = f^{-1}(C) \times f^{-1}(D)$.

Proof. (i) Let $(w, x) \in W \times X$. Suppose $\exists\, (y, z) \in Y \times Z$ such that

$$f((w,x)) = (f(w), f(x)) = (y,z).$$

Then we get

$$\begin{aligned} CM_{f(A\times B)}((y,z)) &= CM_{A\times B}(f^{-1}((y,z))) \\ &= CM_{A\times B}((f^{-1}(y), f^{-1}(z))) \\ &= CM_A(f^{-1}(y)) \wedge CM_B(f^{-1}(z)) \\ &= CM_{f(A)}(y) \wedge CM_{f(B)}(z) \\ &= CM_{f(A)\times f(B)}((y,z)) \end{aligned}$$

Thus, $f(A \times B) \subseteq f(A) \times f(B)$. Hence, the result follows by symmetry.

(ii) For $(w, x) \in W \times X$, we have

$$\begin{aligned} CM_{f^{-1}(C\times D)}((w,x)) &= CM_{C\times D}(f((w,x))) \\ &= CM_{C\times D}((f(w), f(x))) \\ &= CM_C(f(w)) \wedge CM_D(f(x)) \\ &= CM_{f^{-1}(C)}(w) \wedge CM_{f^{-1}(D)}(x) \\ &= CM_{f^{-1}(C)\times f^{-1}(D)}((w,x)). \end{aligned}$$

Hence, $f^{-1}(C \times D) \subseteq f^{-1}(C) \times f^{-1}(D)$.

Similarly,

$$\begin{aligned} CM_{f^{-1}(C)\times f^{-1}(D)}((w,x)) &= CM_{f^{-1}(C)}(w) \wedge CM_{f^{-1}(D)}(x) \\ &= CM_C(f(w)) \wedge CM_D(f(x)) \\ &= CM_{C\times D}((f(w), f(x))) \\ &= CM_{C\times D}(f((w,x))) \\ &= CM_{f^{-1}(C\times D)}((w,x)). \end{aligned}$$

Again, $f^{-1}(C) \times f^{-1}(D) \subseteq f^{-1}(C \times D)$. Therefore, the result follows. □

Theorem 3.25. Suppose $f : W \times X \to Y \times Z$ is an isomorphism, A, B, C and D be fuzzy multigroups of W, X, Y and Z, respectively. Then, the following statements hold.

(i) $f(A \times B) \in FMG(Y \times Z)$.

(ii) $f^{-1}(C) \times f^{-1}(D) \in FMG(W \times X)$.

Proof. (i) Since $A \in FMG(W)$ and $B \in FMG(X)$, then $A \times B \in FMG(W \times X)$ by Theorem 3.6. From Theorem 2.20 and Definition 3.23, it follows that, $f(A \times B) \in FMG(Y \times Z)$.

(ii) Combining Theorems 2.20, 3.6, Definition 3.23 and Theorem 3.24, the result follows. □

Corollary 3.26. Suppose X and Y are groups, $A \in FMG(X)$ and $B \in FMG(Y)$, respectively. If

$$f : X \times X \to Y \times Y$$

be homomorphism, then

(i) $f(A \times A) \in FMG(Y \times Y)$,

(ii) $f^{-1}(B \times B) \in FMG(X \times X)$.

Proof. Similar to Theorem 3.25. □

4. Conclusion

The idea of direct product in fuzzy multigroup setting have been successfully established and lucidly exemplified. Some related results were obtained and proved accordingly. Homomorphism and some of its properties were proposed in the context of direct product of fuzzy multigroups. The idea of generalized direct product of fuzzy multigroups could be exploited.

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New Correlation Coefficients between Linguistic Neutrosophic Numbers and Their Group Decision Making Method

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Abstract − Since linguistic neutrosophic numbers (LNNs) are depicted independently by the truth, indeterminacy, and falsity linguistic variables in indeterminate and inconsistent linguistic environment, they are very fit for human thinking and expressing habits to judgments of complex objects in real life world. Then the correlation coefficient is a critical mathematical tool in pattern recognition and decision making science, but the related research was rarely involved in LNN setting. Hence, this work first proposes two new correlation coefficients of LNNs based on the correlation and information energy of LNNs as the complement/extension of our previous work, and then develops a multiple criteria group decision making (MCGDM) method based on the proposed correlation coefficients in LNN setting. Lastly, a decision making example is provided to illustrate the applicability of the developed method. By comparison with the MCGDM methods regarding the existing correlation coefficients based on the maximum and minimum operations of LNNs, the decision results indicate the effectiveness of the developed MCGDM approach. Hence, the proposed approach provides another new way for linguistic neutrosophic decision making problems.

Keywords − *Linguistic neutrosophic number, correlation coefficient, multiple criteria group decision making*

1. Introduction

The decision making problems usually imply inconsistent, incomplete, and indeterminate information, along with truth, falsity, indeterminacy information in assessment process. Then, neutrosophic theory [1] is a powerful mathematical tool for expressing truth, falsity, indeterminacy information effectively. Hence, it has been used for various problems, such as medical image processing [2-4], medical diagnosis [5-7], fault diagnosis [8-10], and decision making [11-23]. However, when human thinking complicated objects usually contain subjectivity and vagueness, it is difficult to give accurate assessment values of complicated/ill-defined problems regarding the expression of qualitative information by numerical values, but linguistic variables/term values can effectively represent qualitative information and customarily accord with human thinking and expressing habits. Hence, some single-valued and interval neutrosophic linguistic numbers [24-26] and single-valued neutrosophic trapezoid linguistic numbers [27], and interval neutrosophic uncertain linguistic numbers [28] were proposed based on the combination of both linguistic variables and neutrosophic numbers and applied to decision making. On the one hand, there also exists the difficulty of qualitative information expressed by using the neutrosophic numbers. On the other hand, they cannot also express the truth, falsity, indeterminacy linguistic values in inconsistent and indeterminate linguistic setting.

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To solve these issues, linguistic neutrosophic numbers (LNNs) [29] were presented for describing the truth, falsity, indeterminacy linguistic information in inconsistent, incomplete, and indeterminate linguistic setting, and then some aggregation operators were introduced and applied in linguistic neutrosophic MCGDM problems [29, 30]. Furthermore, cosine measures based on the vector space and the distance of LNNs [31], correlation coefficients based on the minimum and minimum operations of LNNs [32], and bidirectional project measures based on the project models of LNNs [33] were presented respectively and applied to MCGDM problems in LNN setting.

However, the correlation coefficient is a critical mathematical tool in pattern recognition and decision making science, but the related research was rarely involved in LNN setting. Therefore, this study proposes two new correlation coefficients of LNNs as the complement/extension of our previous work [32], and then develops their MCGDM approach for solving the indeterminate and inconsistent linguistic decision making problems in LNN setting. To do so, this study is constructed as the following work framework. Section 2 introduces some preliminaries of LNNs. The correlation coefficients of LNNs are proposed based on the correlation and information energy of LNNs in Section 3. Section 4 presents a MCGDM approach based the proposed correlation coefficients in LNN setting. Section 5 presents a decision making example to show the applicability of the proposed MCGDM approach in LNN setting. Section 6 gives the comparison of the proposed approach with decision making approaches based on existing correlation coefficients of LNNs to indicate the effectiveness of the proposed approach. Section 7 contains conclusions and further research.

2. Some preliminaries of LNNs

Fang and Ye [29] proposed a LNN concept regarding the truth, indeterminacy, and falsity linguistic term variables v_a, v_b, v_c, and then the values of the linguistic term variables can be obtained from a given linguistic term set $V = \{v_0, v_1, \ldots, v_q\}$ with odd cardinality q+1. Thus, a LNN is expressed as $s=\langle v_a, v_b, v_c\rangle$ for $s \in V$ and $a, b, c \in [0, q]$.

For three LNNs $s=\langle v_a, v_b, v_c\rangle$, $s_1=\langle v_{a_1}, v_{b_1}, v_{c_1}\rangle$, and $s_2=\langle v_{a_2}, v_{b_2}, v_{c_2}\rangle$ in V, their operational laws are introduced as follows [29]:

(i) $s_1 \oplus s_2 = \langle v_{a_1}, v_{b_1}, v_{c_1}\rangle \oplus \langle v_{a_2}, v_{b_2}, v_{c_2}\rangle = \left\langle v_{a_1+a_2-\frac{a_1a_2}{q}}, v_{\frac{b_1b_2}{q}}, v_{\frac{c_1c_2}{q}}\right\rangle$;

(ii) $s_1 \otimes s_2 = \langle v_{a_1}, v_{b_1}, v_{c_1}\rangle \otimes \langle v_{a_2}, v_{b_2}, v_{c_2}\rangle = \left\langle v_{\frac{a_1a_2}{q}}, v_{b_1+b_2-\frac{b_1b_2}{q}}, v_{c_1+c_2-\frac{c_1c_2}{q}}\right\rangle$;

(iii) $ps = p\langle v_a, v_b, v_c\rangle = \left\langle v_{q-q\left(1-\frac{a}{q}\right)^p}, v_{q\left(\frac{b}{q}\right)^p}, v_{q\left(\frac{c}{q}\right)^p}\right\rangle$ for $p > 0$;

(iv) $s^p = \langle v_a, v_b, v_c\rangle^p = \left\langle v_{q\left(\frac{a}{q}\right)^p}, v_{q-q\left(1-\frac{b}{q}\right)^p}, v_{q-q\left(1-\frac{c}{q}\right)^p}\right\rangle$ for $p > 0$.

Let $s_k=\langle v_{a_k}, v_{b_k}, v_{c_k}\rangle$ (k = 1, 2, …, n) be a group of LNNs in V, then the LNN weighted arithmetic averaging operator is introduced as follows [29]:

$$LNNWAA(s_1, s_2, \ldots, s_n) = \sum_{k=1}^{n} \rho_k s_k = \left\langle v_{q-q\prod_{k=1}^{n}\left(1-\frac{a_k}{q}\right)^{\rho_k}}, v_{q\prod_{k=1}^{n}\left(\frac{b_k}{q}\right)^{\rho_k}}, v_{q\prod_{k=1}^{n}\left(\frac{c_k}{q}\right)^{\rho_k}}\right\rangle, \tag{1}$$

where $\rho_k \in [0, 1]$ is the weight of s_k (k =1, 2, …, n) with $\sum_{k=1}^{n} \rho_k = 1$.

Assume two linguistic neutrosophic sets (LNSs) are $S_1 = \{s_{11}, s_{12}, \ldots, s_{1n}\}$ and $S_2 = \{s_{21}, s_{22}, \ldots, s_{2n}\}$, where $s_{1k} = \langle v_{a_{1k}}, v_{b_{1k}}, v_{c_{1k}} \rangle$ and $s_{2k} = \langle v_{a_{2k}}, v_{b_{2k}}, v_{c_{2k}} \rangle$ (k = 1, 2, …, n) are two groups of LNNs in $V = \{v_0, v_1, \ldots, v_q\}$. Let $f(v_y) = y$ be a linguistic scale function. Then, based the minimum and maximum operations of LNNs, Shi and Ye [32] proposed three weighted correlation coefficients between S_1 and S_2:

$$M_1(S_1,S_2) = \sum_{k=1}^{n} \rho_k \frac{\min\left(f(v_{a_{1k}}), f(v_{a_{2k}})\right) + \min\left(f(v_{b_{1k}}), f(v_{b_{2k}})\right) + \min\left(f(v_{c_{1k}}), f(v_{c_{2k}})\right)}{\sqrt{f(v_{a_{1k}}) f(v_{a_{2k}})} + \sqrt{f(v_{b_{1k}}) f(v_{b_{2k}})} + \sqrt{f(v_{c_{1k}}) f(v_{c_{2k}})}}$$
$$= \sum_{k=1}^{n} \rho_k \frac{\min(a_{1k}, a_{2k}) + \min(b_{1k}, b_{2k}) + \min(c_{1k}, c_{2k})}{\sqrt{a_{1k}a_{2k}} + \sqrt{b_{1k}b_{2k}} + \sqrt{c_{1k}c_{2k}}}, \quad (2)$$

$$M_2(S_1,S_2) = \sum_{k=1}^{n} \rho_k \frac{f(v_{a_{1k}}) f(v_{a_{2k}}) + f(v_{b_{1k}}) f(v_{b_{2k}}) + f(v_{c_{1k}}) f(v_{c_{2k}})}{\left(\max(f(v_{a_{1k}}), f(v_{a_{2k}}))\right)^2 + \left(\max(f(v_{b_{1k}}), f(v_{b_{2k}}))\right)^2 + \left(\max(f(v_{c_{1k}}), f(v_{c_{2k}}))\right)^2}$$
$$= \sum_{k=1}^{n} \rho_k \frac{a_{1k}a_{2k} + b_{1k}b_{2k} + c_{1k}c_{2k}}{\left(\max(a_{1k}, a_{2k})\right)^2 + \left(\max(b_{1k}, b_{2k})\right)^2 + \left(\max(c_{1k}, c_{2k})\right)^2}, \quad (3)$$

$$M_3(S_1,S_2) = \sum_{k=1}^{n} \rho_k \frac{\min\left(f(v_{a_{1k}}), f(v_{a_{2k}})\right) + \min\left(f(v_{b_{1k}}), f(v_{b_{2k}})\right) + \min\left(f(v_{c_{1k}}), f(v_{c_{2k}})\right)}{\max\left(f(v_{a_{1k}}), f(v_{a_{2k}})\right) + \max\left(f(v_{b_{1k}}), f(v_{b_{2k}})\right) + \max\left(f(v_{c_{1k}}), f(v_{c_{2k}})\right)}$$
$$= \sum_{k=1}^{n} \rho_k \frac{\min(a_{1k}, a_{2k}) + \min(b_{1k}, b_{2k}) + \min(c_{1k}, c_{2k})}{\max(a_{1k}, a_{2k}) + \max(b_{1k}, b_{2k}) + \max(b_{1k}, a_{2k})}, \quad (4)$$

where $\rho_k \in [0, 1]$ is the weight of s_{jk} (j =1, 2; k =1, 2, …, n) with $\sum_{k=1}^{n} \rho_k = 1$.

3. Correlation coefficients between LNNs

As the complement/extension of existing correlation coefficients of LNNs [32], this section proposes two new correlation coefficients between two LNNs based on the correlation and information energy of LNNs.

Definition 1. Set two linguistic neutrosophic sets (LNSs) as $S_1 = \{s_{11}, s_{12}, \ldots, s_{1n}\}$ and $S_2 = \{s_{21}, s_{22}, \ldots, s_{2n}\}$, where $s_{1k} = \langle v_{a_{1k}}, v_{b_{1k}}, v_{c_{1k}} \rangle$ and $s_{2k} = \langle v_{a_{2k}}, v_{b_{2k}}, v_{c_{2k}} \rangle$ (k = 1, 2, …, n) are two groups of LNNs in $V = \{v_0, v_1, \ldots, v_q\}$. Let $f(v_y) = y$ be a linguistic scale function. Then we can define the correlation of LNSs S_1 and S_2 as follows:

$$L(S_1,S_2) = \sum_{k=1}^{n} (f(v_{a_{1k}}) f(v_{a_{2k}}) + f(v_{b_{1k}}) f(v_{b_{2k}}) + f(v_{c_{1k}}) f(v_{c_{2k}}) = \sum_{k=1}^{n} (a_{1k}a_{2k} + b_{1k}b_{2k} + c_{1k}c_{2k}). \quad (5)$$

Based on Eq. (5), it is obvious that the correlations between S_1 and S_1 and between S_2 and S_2 yield the following forms:

$$L(S_1,S_1) = \sum_{k=1}^{n} (f(v_{a_{1k}}) f(v_{a_{1k}}) + f(v_{b_{1k}}) f(v_{b_{1k}}) + f(v_{c_{1k}}) f(v_{c_{1k}}) = \sum_{k=1}^{n} (a_{1k}^2 + b_{1k}^2 + c_{1k}^2), \quad (6)$$

$$L(S_2,S_2) = \sum_{k=1}^{n} (f(v_{a_{2k}}) f(v_{a_{2k}}) + f(v_{b_{2k}}) f(v_{b_{2k}}) + f(v_{c_{2k}}) f(v_{c_{2k}}) = \sum_{k=1}^{n} (a_{2k}^2 + b_{2k}^2 + c_{2k}^2), \quad (7)$$

which are also named the information energy of LNSs S_1 and S_2.

Thus, the two correlation coefficients of LNSs S_1 and S_2 are given by

$$Q_1(S_1,S_2)=\frac{L(S_1,S_2)}{\sqrt{L(S_1,S_1)}\sqrt{L(S_2,S_2)}}$$
$$=\frac{\sum_{k=1}^{n}(f(v_{a_{1k}})f(v_{a_{2k}})+f(v_{b_{1k}})f(v_{b_{2k}})+f(v_{c_{1k}})f(v_{c_{2k}}))}{\sqrt{\sum_{k=1}^{n}(f^2(v_{a_{1k}})+f^2(v_{b_{1k}})+f^2(v_{c_{1k}}))}\sqrt{\sum_{k=1}^{n}(f^2(v_{a_{2k}})+f^2(v_{b_{2k}})+f^2(v_{c_{2k}}))}} \quad (8)$$
$$=\frac{\sum_{k=1}^{n}(a_{1k}a_{2k}+b_{1k}b_{2k}+c_{1k}c_{2k})}{\sqrt{\sum_{k=1}^{n}(a_{1k}^2+b_{1k}^2+c_{1k}^2)}\sqrt{\sum_{k=1}^{n}(a_{2k}^2+b_{2k}^2+c_{2k}^2)}},$$

$$Q_2(S_1,S_2)=\frac{L(S_1,S_2)}{\max\{L(S_1,S_1),L(S_2,S_2)\}}$$
$$=\frac{\sum_{k=1}^{n}(f(v_{a_{1k}})f(v_{a_{2k}})+f(v_{b_{1k}})f(v_{b_{2k}})+f(v_{c_{1k}})f(v_{c_{2k}}))}{\max\left\{\sum_{k=1}^{n}(f^2(v_{a_{1k}})+f^2(v_{b_{1k}})+f^2(v_{c_{1k}})),\sum_{k=1}^{n}(f^2(v_{a_{2k}})+f^2(v_{b_{2k}})+f^2(v_{c_{2k}}))\right\}} \quad (9)$$
$$=\frac{\sum_{k=1}^{n}(a_{1k}a_{2k}+b_{1k}b_{2k}+c_{1k}c_{2k})}{\max\left\{\sum_{k=1}^{n}(a_{1k}^2+b_{1k}^2+c_{1k}^2),\sum_{k=1}^{n}(a_{2k}^2+b_{2k}^2+c_{2k}^2)\right\}}.$$

Then, it is obvious that Eqs. (8) and (9) satisfies the following conditions:

(a) $Q_1(S_1, S_2) = Q_1(S_2, S_1)$ and $Q_2(S_1, S_2) = Q_2(S_2, S_1)$;

(b) $Q_1(S_1, S_2) = Q_2(S_1, S_2) = 1$ for $S_1 = S_2$;

(c) $Q_1(S_1, S_2)$, $Q_2(S_1, S_2) \in [0, 1]$.

PROOF.

It is clear that the conditions (a) and (b) are true. Hence, we only verify the condition (c) below.

For the proof of $Q_1(S_1, S_2)$, if $k = 1$, Eq. (8) is reduced to the following cosine measure of LNNs [31]:

$$Q_1(S_1,S_2)=Cos(S_1,S_2)=\frac{f(v_{a_{1k}})f(v_{a_{2k}})+f(v_{b_{1k}})f(v_{b_{2k}})+f(v_{c_{1k}})f(v_{c_{2k}})}{\sqrt{f^2(v_{a_{1k}})+f^2(v_{b_{1k}})+f^2(v_{c_{1k}})}\sqrt{(f^2(v_{a_{2k}})+f^2(v_{b_{2k}})+f^2(v_{c_{2k}})}}$$
$$=\frac{a_{1k}a_{2k}+b_{1k}b_{2k}+c_{1k}c_{2k}}{\sqrt{a_{1k}^2+b_{1k}^2+c_{1k}^2}\sqrt{a_{2k}^2+b_{2k}^2+c_{2k}^2}}. \quad (10)$$

Obviously, the cosine measure of LNNs introduced by Shi and Ye [31] is a special case of the correlation coefficient $Q_1(S_1, S_2)$ when $k = 1$.

Since there exists $Cos(S_1, S_2) \in [0, 1]$ regarding the property of the cosine measure between LNNs [31], there is also $Q_1(S_1, S_2) \in [0, 1]$ if $k = 1$. Thus, it is obvious that $Q_1(S_1, S_2) \in [0, 1]$ is true if $k = n$.

For the proof of $Q_2(S_1, S_2)$, since $\max\left\{\sum_{k=1}^{n}(a_{1k}^2+b_{1k}^2+c_{1k}^2),\sum_{k=1}^{n}(a_{2k}^2+b_{2k}^2+c_{2k}^2)\right\} \geq a_{1k}a_{2k}+b_{1k}b_{2k}+c_{1k}c_{2k}$ can holds for $a_{jk}, b_{jk}, c_{jk} \in [0, q]$ ($j = 1, 2$; $k = 1, 2, \ldots, n$) in $V = \{v_0, v_1, \ldots, v_q\}$, it is clear that there exists $Q_2(S_1, S_2) \in [0, 1]$.

Hence, this proof is finished. □

If the importance of each LNN s_{jk} (j = 1, 2; k = 1, 2, …, n) in S_1 and S_2 is indicated by the weight value ρ_k for $\rho_k \in [0, 1]$ and $\sum_{k=1}^{n}\rho_k = 1$, the weighted correlation coefficients of LNSs S_1 and S_2 can be expressed by

$$W_1(S_1,S_2)=\frac{\sum_{k=1}^{n}\rho_k\left(f(v_{a_{1k}})f(v_{a_{2k}})+f(v_{b_{1k}})f(v_{b_{2k}})+f(v_{c_{1k}})f(v_{c_{2k}})\right)}{\sqrt{\sum_{k=1}^{n}\rho_k\left(f^2(v_{a_{1k}})+f^2(v_{b_{1k}})+f^2(v_{c_{1k}})\right)}\sqrt{\sum_{k=1}^{n}\rho_k\left(f^2(v_{a_{2k}})+f^2(v_{b_{2k}})+f^2(v_{c_{2k}})\right)}}$$
$$=\frac{\sum_{k=1}^{n}\rho_k(a_{1k}a_{2k}+b_{1k}b_{2k}+c_{1k}c_{2k})}{\sqrt{\sum_{k=1}^{n}\rho_k(a_{1k}^2+b_{1k}^2+c_{1k}^2)}\sqrt{\sum_{k=1}^{n}\rho_k(a_{2k}^2+b_{2k}^2+c_{2k}^2)}}, \quad (11)$$

$$W_2(S_1,S_2)=\frac{\sum_{k=1}^{n}\rho_k\left(f(v_{a_{1k}})f(v_{a_{2k}})+f(v_{b_{1k}})f(v_{b_{2k}})+f(v_{c_{1k}})f(v_{c_{2k}})\right)}{\max\left\{\sum_{k=1}^{n}\rho_k\left(f^2(v_{a_{1k}})+f^2(v_{b_{1k}})+f^2(v_{c_{1k}})\right),\sum_{k=1}^{n}\rho_k\left(f^2(v_{a_{2k}})+f^2(v_{b_{2k}})+f^2(v_{c_{2k}})\right)\right\}}$$
$$=\frac{\sum_{k=1}^{n}\rho_k(a_{1k}a_{2k}+b_{1k}b_{2k}+c_{1k}c_{2k})}{\max\left\{\sum_{k=1}^{n}\rho_k(a_{1k}^2+b_{1k}^2+c_{1k}^2),\sum_{k=1}^{n}\rho_k(a_{2k}^2+b_{2k}^2+c_{2k}^2)\right\}}. \quad (12)$$

Obviously, the weighted correlation coefficients of Eqs. (11) and (12) also satisfy these conditions:

(a) $W_1(S_1, S_2) = W_1(S_2, S_1)$ and $W_2(S_1, S_2) = W_2(S_2, S_1)$;

(b) $W_1(S_1, S_2) = W_2(S_1, S_2) = 1$ for $S_1 = S_2$;

(c) $W_1(S_1, S_2), W_2(S_1, S_2) \in [0, 1]$.

4. MCGDM approach based on weighted correlation coefficients of LNNs

This section proposes a MCGDM approach based on the weighted correlation coefficients of LNNs.

Regarding a MCGDM problem in LNN setting, there are the set of m alternatives represented by $S = \{S_1, S_2, …, S_m\}$ and the set of n criteria represented by $E = \{E_1, E_2, …, E_n\}$. Then, the set of d decision makers is denoted by $D = \{D_1, D_2, …, D_d\}$. Thus, when the j-th decision maker D_j give the fit evaluations of each alternative S_i (i = 1, 2, …, m) over criteria E_k (k = 1, 2, …, n), his/her evaluation values are expressed by a LNS $S_i^j = \{s_{i1}^j, s_{i2}^j, ..., s_{in}^j\}$, where $s_{ik}^j = < v_{a_{ik}}^j, v_{b_{ik}}^j, v_{c_{ik}}^j >$ is a LNN obtained from the given linguistic term set $V = \{v_0, v_1, …, v_q\}$ for $v_{a_{ik}}^j, v_{b_{ik}}^j, v_{c_{ik}}^j \in [v_0, v_q]$ (i = 1, 2, …, m; j = 1, 2, …, d; k = 1, 2, …, n). Thus, the j-th decision matrix of LNNs $R^j = \left(s_{ik}^j\right)_{m\times n}$ (j = 1, 2, …, d) can be constructed in LNN setting.

Suppose the weight vector of criteria is $\rho = (\rho_1, \rho_2, …, \rho_n)$ for $\rho_k \in [0, 1]$ and $\sum_{k=1}^{n}\rho_k = 1$, and then the weight vector of decision makers is $\lambda = (\lambda_1, \lambda_2, …, \lambda_d)$ for $\lambda_j \in [0, 1]$ and $\sum_{j=1}^{d}\lambda_j = 1$. In this decision making problem, we can propose a MCGDM approach based on the weighted correlation coefficients in LNN setting, which is depicted by the following steps:

Step 1: Based on Eq. (1), the aggregated LNN $v_{ik} = < v_{a_{ik}}, v_{b_{ik}}, v_{c_{ik}} >$ is obtained by the following weighted aggregation operator:

$$s_{ik} = LNNWAA(s_{ik}^1, s_{ik}^2, ..., s_{ik}^d) = \sum_{j=1}^{d} \lambda_j s_{ik}^j = \left\langle v_{q-q\prod_{j=1}^{d}\left(1-\frac{a_{ik}^j}{q}\right)^{\lambda_j}}, v_{q\prod_{j=1}^{d}\left(\frac{b_{ik}^j}{q}\right)^{\lambda_j}}, v_{q\prod_{j=1}^{d}\left(\frac{c_{ik}^j}{q}\right)^{\lambda_j}} \right\rangle. \quad (13)$$

Then, the aggregated matrix of LNNs is constructed as follows:

$$R = \begin{bmatrix} s_{11} & s_{12} & \cdots & s_{1n} \\ s_{21} & s_{22} & \cdots & s_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ s_{m1} & s_{m2} & \cdots & s_{mn} \end{bmatrix}.$$

Step2: Regarding the concept of the ideal solution (alternative), we can determine the ideal solution $S^* = \{s_1^*, s_2^*, ..., s_n^*\}$ from the aggregated matrix *R*, where $s_k^* = < v_{a_k^*}, v_{b_k^*}, v_{c_k^*} > = < \max_i(v_{a_{ik}}), \min_i(v_{b_{ik}}), \min_i(v_{c_{ik}}) >$ is the ideal LNN (*k* = 1, 2, …, *n*; *i* = 1, 2, …, *m*).

Step 3: Based on Eq. (11) or Eq. (12), the weighted correlation coefficient between S_i (*i* = 1, 2, …, *m*) and S^* is given by

$$W_1(S_i, S^*) = \frac{\sum_{k=1}^{n} \rho_k (a_{ik}a_k^* + b_{ik}b_k^* + c_{ik}c_k^*)}{\sqrt{\sum_{k=1}^{n} \rho_k (a_{ik}^2 + b_{ik}^2 + c_{ik}^2)} \sqrt{\sum_{k=1}^{n} \rho_k \left((a_k^*)^2 + (b_k^*)^2 + (c_k^*)^2\right)}}, \quad (14)$$

$$\text{or } W_2(S_i, S^*) = \frac{\sum_{k=1}^{n} \rho_k (a_{ik}a_k^* + b_{ik}b_k^* + c_{ik}c_k^*)}{\max\left\{\sum_{k=1}^{n} \rho_k (a_{ik}^2 + b_{ik}^2 + c_{ik}^2), \sum_{k=1}^{n} \rho_k \left((a_k^*)^2 + (b_k^*)^2 + (c_k^*)^2\right)\right\}}. \quad (15)$$

Step 4: The ranking order of all alternatives and the best one are given corresponding to the values of the weighted correlation coefficient.

Step 5: End.

5. Decision making example with LNN information

This section presents a decision making example regarding the MCGDM problem to illustrate the applicability of the proposed MCGDM method in LNN setting.

A hospital requires the human resources department to recruit a nurse. When the five candidates (the five alternatives) S_1, S_2, S_3, S_4, and S_5 are selected preliminarily from all applicants by the human resources department, a group of three experts/decision makers $D = \{D_1, D_2, D_3\}$ is invited to assess the five candidates corresponding to the three requirements (criteria): (a) E_1 is nursing skill; (b) E_2 is past nursing experience; (c) E_3 is self-confidence. The weight vector of the three criteria is provided by ρ = (0.4, 0.3, 0.3) and the weight vector of the three experts is given by λ = (0.35, 0.35, 0.3).

Then, the three experts are requested to suitably evaluate the five candidates from the predefined linguistic term set $V = \{v_0$ = extremely poor, v_1 = very poor, v_2 = poor, v_3 = slightly poor, v_4 = fair, v_5 = slightly good, v_6 = good, v_7 = very good, v_8 = extremely good$\}$ for $q = 8$ in LNN setting, and then they give the following three LNN matrices:

$$R^1 = \begin{bmatrix} <v_5, v_1, v_2> & <v_6, v_2, v_2> & <v_6, v_2, v_3> \\ <v_7, v_2, v_2> & <v_7, v_2, v_2> & <v_7, v_3, v_2> \\ <v_5, v_1, v_2> & <v_6, v_2, v_4> & <v_7, v_1, v_3> \\ <v_6, v_2, v_3> & <v_6, v_2, v_4> & <v_6, v_2, v_3> \\ <v_4, v_3, v_4> & <v_6, v_3, v_4> & <v_6, v_4, v_4> \end{bmatrix},$$

$$R^2 = \begin{bmatrix} <v_6, v_2, v_3> & <v_5, v_3, v_4> & <v_6, v_2, v_3> \\ <v_6, v_1, v_3> & <v_7, v_2, v_3> & <v_7, v_2, v_1> \\ <v_6, v_2, v_1> & <v_6, v_2, v_3> & <v_7, v_1, v_3> \\ <v_5, v_1, v_2> & <v_6, v_2, v_1> & <v_6, v_2, v_2> \\ <v_5, v_3, v_4> & <v_6, v_3, v_3> & <v_6, v_3, v_3> \end{bmatrix},$$

$$R^3 = \begin{bmatrix} <v_6, v_2, v_3> & <v_5, v_4, v_3> & <v_6, v_1, v_3> \\ <v_7, v_2, v_2> & <v_6, v_3, v_2> & <v_7, v_2, v_1> \\ <v_6, v_1, v_1> & <v_5, v_1, v_4> & <v_7, v_1, v_2> \\ <v_7, v_2, v_3> & <v_6, v_2, v_2> & <v_6, v_2, v_3> \\ <v_5, v_3, v_4> & <v_6, v_3, v_2> & <v_6, v_3, v_3> \end{bmatrix}.$$

Thus, the proposed MCGDM approach can be applied to the decision making example, which is depicted by the following steps:

Step 1: By using Eq. (13), the aggregated matrix of LNNs is yielded as follows:

$$R = \begin{bmatrix} <v_{5.6478}, v_{1.5157}, v_{2.5508}> & <v_{5.4492}, v_{2.7808}, v_{2.7808}> & <v_{6.0000}, v_{1.6245}, v_{3.000}> \\ <v_{6.7689}, v_{1.6245}, v_{2.2587}> & <v_{6.7689}, v_{2.2587}, v_{2.2587}> & <v_{7.0000}, v_{2.3522}, v_{1.3195}> \\ <v_{5.6478}, v_{1.2311}, v_{1.3195}> & <v_{5.7413}, v_{1.6245}, v_{3.6693}> & <v_{7.0000}, v_{1.0000}, v_{2.6564}> \\ <v_{6.1654}, v_{1.6245}, v_{2.6564}> & <v_{6.0000}, v_{2.0000}, v_{2.1435}> & <v_{6.0000}, v_{2.0000}, v_{2.6564}> \\ <v_{4.6341}, v_{3.0000}, v_{4.0000}> & <v_{6.0000}, v_{3.0000}, v_{2.9804}> & <v_{6.0000}, v_{3.3659}, v_{3.3659}> \end{bmatrix}.$$

Step 2: Corresponding to the ideal LNN $s_k^* = <v_{a_k^*}, v_{b_k^*}, v_{c_k^*}> = <\max_i(v_{a_{ik}}), \min_i(v_{b_{ik}}), \min_i(v_{c_{ik}})>$ (k = 1, 2, 3; i = 1, 2, 3, 4, 5), the ideal solution is yielded from the aggregated matrix R as follows:

$$S^* = \{s_1^*, s_2^*, s_3^*\} = \{<v_{6.7689}, v_{1.2311}, v_{1.3195}>, <v_{6.7689}, v_{1.6245}, v_{2.1435}>, <v_{7.0000}, v_{1.0000}, v_{1.3195}>\}.$$

Step 3: By using Eq. (14) or Eq. (15), we can obtain the following weighted correlation coefficient values:

$W_1(S_1, S^*) = 0.9661$, $W_1(S_2, S^*) = 0.9908$, $W_1(S_3, S^*) = 0.9790$, $W_1(S_4, S^*) = 0.9787$, and $W_1(S_5, S^*) = 0.9082$;

or $W_2(S_1, S^*) = 0.8985$, $W_2(S_2, S^*) = 0.9545$, $W_2(S_3, S^*) = 0.9334$, $W_2(S_4, S^*) = 0.9313$, and $W_2(S_5, S^*) = 0.8956$.

Step 4: Based on the above values, all the alternatives are ranked as $S_2 > S_3 > S_4 > S_1 > S_5$, and then the best candidate with the biggest value is S_2.

Clearly, the ranking orders of the candidates/alternatives and the best one corresponding to the proposed two correlation coefficients of LNNs are the same in this MCGDM example.

6. Comparison with MCGDM methods based on existing correlation coefficients of LNNs

To demonstrate the effectiveness of the proposed method in LNN setting, this section indicates the comparison of the proposed approach with the ones based on existing correlation coefficients of LNNs [32] by the above MCGDM example.

Thus, the correlation coefficient values between S_i and S^* are obtained by applying Eqs. (2)-(4), and then all the decision results based on various correlation coefficients of LNNs are tabulated in Table 1.

Table 1. Decision results based on various correlation coefficients of LNNs

Correlation coefficient	Correlation coefficient value	Ranking order	The best one
$M_1(S_i, S^*)$ [32]	0.8651, 0.9517, 0.9239, 0.8998, 0.8033	$S_2 > S_3 > S_4 > S_1 > S_5$	S_2
$M_2(S_i, S^*)$ [32]	0.2466, 0.2967, 0.2938, 0.2499, 0.2180	$S_2 > S_3 > S_4 > S_1 > S_5$	S_2
$M_3(S_i, S^*)$ [32]	0.7412, 0.8950, 0.8462, 0.8035, 0.6326	$S_2 > S_3 > S_4 > S_1 > S_5$	S_2
$W_1(S_i, S^*)$	0.9661, 0.9908, 0.9790, 0.9787, 0.9082	$S_2 > S_3 > S_4 > S_1 > S_5$	S_2
$W_2(S_i, S^*)$	0.8985, 0.9545, 0.9334, 0.9313, 0.8956	$S_2 > S_3 > S_4 > S_1 > S_5$	S_2

From Table 1, we can see that all the ranking orders and the best one are identical regarding the decision results based on various correlation coefficients of LNNs. Obviously, the proposed approach indicates its effectiveness. Thus, the proposed MCGDM approach provides another new effective way for the linguistic neutrosophic decision making problems in LNN setting.

7. Conclusion

As the complement/extension of our previous work [32], this study first presented two correlation coefficients of LNNs based on the correlation and information energy of LNNs. Then we presented a MCGDM approach using the weighted correlation coefficients in LNN setting. A decision making example regarding the MCGDM problem was presented to demonstrate the applicability of the proposed MCGDM approach in LNN setting. By comparison with the MCGDM approaches based on the existing correlation coefficients of LNNs, the decision results demonstrated the developed new approach is effective. Hence, the proposed MCGDM approach provides another new effective way for linguistic neutrosophic decision making problems. In the next work, we shall extend the proposed correlation coefficients to develop the refined linguistic neutrosophic correlation coefficients based on the refined neutrosophic concept [34] and to use them for decision making, pattern recognition, and medical diagnosis problems in refined linguistic neutrosophic setting.

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The Application of GRA Method Base on Choquet Integral Using Spherical Fuzzy Information in Decision Making Problems

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Abstract − Spherical fuzzy set is the generalized structure over existing structures of fuzzy sets to deals with uncertainty and imprecise information in decision making problems. Viewing the effectiveness of the spherical fuzzy set, we developed a decision-making algorithm to deal with multi-criteria decision-making problems. In this paper, we extend operational laws to propose spherical fuzzy Choquet integral weighted averaging (SFCIWA) operator based on spherical fuzzy numbers. Further, the proposed SFCIWA operator is applied to multi-attribute group decision-making problems. Also, we propose the GRA method to aggregate the spherical fuzzy information. To implement the proposed models, we provide some numerical applications of group decision-making problems. Also compared with the previous model, we conclude that the proposed technique is more effective and reliable.

Keywords − *Choquet integral, Spherical fuzzy Choquet integral weighted averaging (SFCIWA) operator, GRA method, Spherical fuzzy sets, Decision making technique.*

1. Introduction

Multi-criteria group decision making problems have importance in most kinds of fields such as economics, engineering and management. Generally, it has been assumed that the information which accesses the alternatives in term of criteria and weight are expressed in real numbers. But due to the complexity of the system day-by-day, it is difficult for the decision makers to make a perfect decision, as most of the preferred value during the decision-making process imbued with uncertainty. In order to handle the uncertainties and fuzziness, intuitionistic fuzzy set [11] theory is one of the prosperous extensions of the fuzzy set theory [45], which is characterized by the degree of membership and degree of non-membership has been presented. Fuzzy set theory is extended in many ways by different authors but to modelling imprecision IFS theory is much impressive. IFS theory attracts many authors because of its important in handling uncertainty and different aggregation operators are defined to aggregate information. For study the aggregation operators for IFSs, we refer to [25, 28, 41, 42].

But there are several cases where the decision maker may provide the degree of membership and nonmembership of a particular attribute in such a way that their sum is greater than one. For example, suppose a man expresses his preferences towards the alternative in such a way that degree of their satisfaction is 0.6 and degree of rejection is 0.8. Clearly its sum is greater than one. Therefore, Yager

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[43, 44] introduced the concept of another set called Pythagorean fuzzy set. Pythagorean fuzzy set is more powerful tool to solve uncertain problems. Like intuitionistic fuzzy operators, Pythagorean fuzzy operators also become an interesting and important area for research, after the advent of Pythagorean fuzzy sets theory. Yager and Abbasov [43] introduced many aggregation operators to tackle MADM tangle in PyFS environment. The superiority and inferiority ranking (SIR) MABAC technique to tackle MADM problems in Pythagorean fuzzy environment is discussed by Peng and Yang [31]. Zhang [48] proposed an approach for multi-criteria Pythagorean fuzzy decision analysis based on the closeness index-ranking methods. Khan et al. [23] proposed the Pythagorean fuzzy Dombi aggregation operators and discussed their applications in decision making problems.

As for human nature only, satisfaction and dissatisfaction degree is quite insufficient and needs abstain and refusal degree too but Yager Pythagorean concept not covers this problem. This problem is solved by Coung [15] defining a new structure called picture fuzzy set (PFS) which also includes degree of neutral membership with a condition that the sum of triplet should remain with in unit interval and covers all aspects of human nature and quite applicable in real life problems and very near to the human nature. Cuong [17] in 2014 introduced the concept of picture soft sets, the relation of compositions and the distance between picture fuzzy numbers. Singh [38] in 2015, proposed the idea related to correlation coefficients for picture fuzzy sets. The concepts like convex combination of PFNs, alpha-cuts of PFS, picture fuzzy relations are introducing by Cuong [16] in 2015. Generalized picture fuzzy distance measure are developed by Son [39] in 2016, and discussed their applications. Ashraf et al. [1] proposed the geometric aggregation operators for picture fuzzy information and in [2] proposed the concept of picture fuzzy linguistic set. Khan et al. [24] proposed the concept of generalized picture fuzzy soft set (GPFSSs) and illustrate the applications of GPFSSs in decision making problems. For more study about decision making techniques we refer to [3, 4, 26, 27, 32–35]

Ashraf et al. [5] proposed the novel concept of spherical fuzzy set by applying extra condition on sum of their memberships as square sum of the membership degrees oscillate from 0 to 1. Ashraf and Abdullah [6] proposed the series of aggregation operators for spherical fuzzy environment and in [7] proposed the notion of spherical linguistic fuzzy set and developed their applications using GRA technique. For more study about spherical fuzzy sets, we refer to [8–10, 21, 22, 36, 47]

In this paper, our aim to develop the GRA technique with unknown weight information using spherical fuzzy information to deal with uncertainty in decision making problems. To do this, the article structured is follows as:

Basic definitions and result about Choquet integral, Pythagorean fuzzy sets and picture fuzzy sets are present in Sec. 2. In Sec. 3, we introduced the notion of Spherical fuzzy sets. In Sec. 4 we proposed the GRA method for spherical fuzzy MAGDM problems with incomplete weight data. In Section 5, we strengthen our proposed algorithmic method with a descriptive example. Last Sections contains the conclusion of the work.

2. Preliminary

This section consists of some basic concepts of Pythagorean fuzzy set (PyFS), picture fuzzy set (PFS) and also give some discussion related to fuzzy measure and Choquet integral.

Definition 2.1. [44]A PyFS $\Im_{\breve{u}}$ on the universe of discourse $\mathbb{Z} \neq \phi$ is defined as;

$$\Im_{\breve{u}} = \{\langle L_{\breve{e}_{\breve{u}}}(k), M_{\breve{e}_{\breve{u}}}(k) |\ k \in \mathbb{Z}\rangle\}.$$

A PyFS in a set $\mathbb{Z}$ is defined by $L_{\breve{e}_{\breve{u}}}(k) : \mathbb{Z} \to \Theta$ and $M_{\breve{e}_{\breve{u}}}(k) : \mathbb{Z} \to \Theta$ are the positive and negative membership grades of each $k \in \mathbb{Z}$, respectively. Furthermore $L_{\breve{e}_{\breve{u}}}(k)$ and $M_{\breve{e}_{\breve{u}}}(k)$ satisfy $0 \leq L^2_{\breve{e}_{\breve{u}}}(k) + M^2_{\breve{e}_{\breve{u}}}(k) \leq 1$ for all $k \in \mathbb{Z}$.

Definition 2.2. [15]A PFS $\Im_{\breve{u}}$ on the universe of discourse $\mathbb{Z} \neq \phi$ is defined as;

$$\Im_{\breve{u}} = \{\langle L_{\breve{e}_{\breve{u}}}(k), M_{\breve{e}_{\breve{u}}}(k), O_{\breve{e}_{\breve{u}}}(k) |\ k \in \mathbb{Z}\rangle\}.$$

A PFS in a set $\mathbb{Z}$ is defined by $L_{\breve{e}_{\breve{u}}}(k) : \mathbb{Z} \to \Theta$, $M_{\breve{e}_{\breve{u}}}(k) : \mathbb{Z} \to \Theta$ and $O_{\breve{e}_{\breve{u}}}(k) : \mathbb{Z} \to \Theta$ are the positive grade, neutral grade and negative grade of each $k \in \mathbb{Z}$, respectively. Furthermore $L_{\breve{e}_{\breve{u}}}(k)$, $M_{\breve{e}_{\breve{u}}}(k)$ and $O_{\breve{e}_{\breve{u}}}(k)$ satisfy $0 \leq L_{\breve{e}_{\breve{u}}}(k) + M_{\breve{e}_{\breve{u}}}(k) + O_{\breve{e}_{\breve{u}}}(k) \leq 1$ for all $k \in \mathbb{Z}$.

2.1. Fuzzy measure and Choquet integral

The concept of fuzzy measure are developed by Sugeno in 1974 [48] which instead of additivity property only make a monotonicity . It is a powerful tool for modeling interaction phenomena in decision making for MADM problems, it does not required assumption that criteria or preferences are free from one another. Criteria can be dependent in the Choquet integral model [14,31], where on each combination of criteria a fuzzy measure is used to define a weight, thus making it possible to model the interaction existing among criteria. Concept of fuzzy measure, discrete Choquet integral, λ-fuzzy measure and Pythagorean fuzzy Choquet integral operators are presented in this subsection as follows;

Definition 2.3. [14] Let the universe of discourse $\mathbb{Z} = \{k_1, ..., k_n\} \neq \phi$ and $p(\mathbb{Z})$ denote the power set of $\mathbb{Z}$. Then, a function $L_{\breve{e}_{\breve{u}}} : p(\mathbb{Z}) \rightarrow \Theta$ is called a fuzzy measure $L_{\breve{e}_{\breve{u}}}$ on $\mathbb{Z}$, if satisfy the following conditions;

1) $L_{\breve{e}_{\breve{u}}}(\phi) = 0$, $L_{\breve{e}_{\breve{u}}}(\mathbb{Z}) = 1$.

2) If $\Im_{\breve{u}_1}, \Im_{\breve{u}_2} \in p(\mathbb{Z})$ and $\Im_{\breve{u}_1} \subseteq \Im_{\breve{u}_2}$ then $L_{\breve{e}_{\breve{u}}}(\Im_{\breve{u}_1}) \leq L_{\breve{e}_{\breve{u}}}(\Im_{\breve{u}_2})$.

It is mandatory to consider the adage of continuity when $\mathbb{Z}$ is infinite, it is enough to assume a finite universe of discourse in genuine exercise. For decision attribute set $\{k_1, k_2, ..., k_n\}$, $L_{\breve{e}_{\breve{u}}}(\{k_1, k_2, ..., k_n\})$ can be deem as the degree of subjective importance. Thus, weights of any set of attributes can also be obtained with the separate weights of attributes. Instinctively, we say that the following about any pair of criteria sets $\Im_{\breve{u}_1}, \Im_{\breve{u}_2} \in p(\mathbb{Z})$, $\Im_{\breve{u}_1} \cap \Im_{\breve{u}_2} = \phi$; $\Im_{\breve{u}_1}$and $\Im_{\breve{u}_2}$ are assumed to be without interaction (or to be independent) and called it additive measure if

$$L_{\breve{e}_{\breve{u}}}(\Im_{\breve{u}_1} \cup \Im_{\breve{u}_2}) = L_{\breve{e}_{\breve{u}}}(\Im_{\breve{u}_1}) + L_{\breve{e}_{\breve{u}}}(\Im_{\breve{u}_2}). \tag{1}$$

$\Im_{\breve{u}_1}$and $\Im_{\breve{u}_2}$ reveals a positive synergetic interaction among them (or are complementary) and called a super additive measure if

$$L_{\breve{e}_{\breve{u}}}(\Im_{\breve{u}_1} \cup \Im_{\breve{u}_2}) > L_{\breve{e}_{\breve{u}}}(\Im_{\breve{u}_1}) + L_{\breve{e}_{\breve{u}}}(\Im_{\breve{u}_2}). \tag{2}$$

$\Im_{\breve{u}_1}$and $\Im_{\breve{u}_2}$ reveals a negative synergetic interaction among them (or are redundant or substitutive) and said to be a sub-additive measure if

$$L_{\breve{e}_{\breve{u}}}(\Im_{\breve{u}_1} \cup \Im_{\breve{u}_2}) < L_{\breve{e}_{\breve{u}}}(\Im_{\breve{u}_1}) + L_{\breve{e}_{\breve{u}}}(\Im_{\breve{u}_2}). \tag{3}$$

From the Definition 2.3 it is hard to find the fuzzy measure, therefore, Sageno defined the following measure to confirm a fuzzy measure in MAGDM problems:

$$L_{\breve{e}_{\breve{u}}}(\Im_{\breve{u}_1} \cup \Im_{\breve{u}_2}) = L_{\breve{e}_{\breve{u}}}(\Im_{\breve{u}_1}) + L_{\breve{e}_{\breve{u}}}(\Im_{\breve{u}_2}) + \lambda L_{\breve{e}_{\breve{u}}}(\Im_{\breve{u}_1}) L_{\breve{e}_{\breve{u}}}(\Im_{\breve{u}_2}) \tag{4}$$

$\lambda \in [-1, \infty)$, $\Im_{\breve{u}_1} \cap \Im_{\breve{u}_2} = \phi$. The interaction between the attributes is determine by the parameter λ. Simply an additive measure is obtained when $\lambda = 0$ in Equation 4. Sub additive and super additive measures is obtained, respectively for negative and positive λ. Meanwhile, if all the elements in $\mathbb{Z}$ are independent, and we have

$$L_{\breve{e}_{\breve{u}}}(\Im_{\breve{u}}) = \sum_{p=1}^{n} L_{\breve{e}_{\breve{u}}}(\{k_p\}) \tag{5}$$

If $\mathbb{Z}$ is a finite set, then $\cup_{\mathrm{p}=1}^{n} k_{\mathrm{p}} = \mathbb{Z}$. The λ-fuzzy measure $L_{\breve{e}_{\breve{u}}}$ satisfies following Equation6

$$L_{\breve{e}_{\breve{u}}}(\mathbb{Z}) = L_{\breve{e}_{\breve{u}}}\left(\cup_{p=1}^{n} k_i\right) = \begin{cases} \frac{1}{\lambda}\left(\prod_{p=1}^{n}\left[1 + \lambda L_{\breve{e}_{\breve{u}}}(k_p)\right] - 1\right) & \text{if } \lambda \neq 0 \\ \sum_{p=1}^{n} L_{\breve{e}_{\breve{u}}}(k_p) & \text{if } \lambda = 0 \end{cases} \tag{6}$$

where $k_{\mathrm{p}} \cap r_{\check{\mathrm{d}}} = \phi$ for all p,$\check{\mathrm{d}}$= $1, ..., n$ and $p \neq \check{\mathrm{d}}$. It should be noted that $L_{\breve{e}_{\breve{u}}}(k_{\mathrm{p}})$ for a subset with a single member k_{p} is called a fuzzy density, and can be signified as $L_{\breve{e}_{\breve{u}}} = L_{\breve{e}_{\breve{u}}}(k_{\mathrm{p}})$.

Particularly for every subset $\Im_{\breve{u}_1} \in p(\mathbb{Z})$, we have

$$L_{\breve{e}_{\breve{u}}}(\Im_{\breve{u}_1}) = \begin{cases} \frac{1}{\lambda}\left(\prod_{p=1}^{n}\left[1+\lambda L_{\breve{e}_{\breve{u}}}(k_p)\right]-1\right) & \text{if } \lambda \neq 0 \\ \sum_{p=1}^{n} L_{\breve{e}_{\breve{u}}}(k_p) & \text{if } \lambda = 0 \end{cases} \tag{7}$$

A uniquely value of λ is determined from $L_{\breve{e}_{\breve{u}}}(\mathbb{Z}) = 1$, based on Equation2 which is equivalent to solving

$$\lambda + 1 = \prod_{p=1}^{n}\left[1+\lambda L_{\breve{e}_{\breve{u}}}\right] \tag{8}$$

It can be seen that λ are uniquely obtained by $L_{\breve{e}_{\breve{u}}}(\mathbb{Z}) = 1$.

Definition 2.4. [14] Let g and $L_{\breve{e}_{\breve{u}}}$ be a positive real-valued function on the fuzzy measure $\mathbb{Z}$, respectively. Then, the discrete Choquet integral of g with respect to $L_{\breve{e}_{\breve{u}}}$ is defined by

$$C_\mu(g) = \sum_{p=1}^{n} g_{\rho(p)}[L_{\breve{e}_{\breve{u}}}(A_{\rho(p)}) - L_{\breve{e}_{\breve{u}}}(A_{\rho(p\text{-}1)})] \tag{9}$$

where $\rho(p)$ shows a permutation on $\mathbb{Z}$ such that $g_{\rho(1)} \geq g_{\rho(2)} \geq ... \geq g_{\rho(\mathrm{n})}$, $A_{\rho(\mathrm{n})} = \{1, 2, ..., p\}$, $A_{\rho(0)} = \phi$.

Up to a reordering of the elements it can be noticed that the discrete Choquet integral is a linear expression. Moreover, when the fuzzy measure is additive it identifies with the weighted mean (discrete Lebesgue integral). And OWA operator the Choquet integral operator coincides in some conditions.

3. Some Operations on Spherical Fuzzy Set

The notion of SFS and their operational laws are defined in this section.

Definition 3.1. [5]A SFS $\Im_{\breve{u}}$ on the universe of discourse $\mathbb{Z} \neq \phi$ is defined as;

$$\Im_{\breve{u}} = \{\langle L_{\breve{e}_{\breve{u}}}(k), M_{\breve{e}_{\breve{u}}}(k), O_{\breve{e}_{\breve{u}}}(k) |\ k \in \mathbb{Z}\rangle\}. \tag{10}$$

Where $L_{\breve{e}_{\breve{u}}}(k) : \mathbb{Z} \to \Theta$, $M_{\breve{e}_{\breve{u}}}(k) : \mathbb{Z} \to \Theta$ and $O_{\breve{e}_{\breve{u}}}(k) : \mathbb{Z} \to \Theta$ are the positive grade, neutral grade and negative grade of each $k \in \mathbb{Z}$, respectively. Furthermore $L_{\breve{e}_{\breve{u}}}(k)$, $M_{\breve{e}_{\breve{u}}}(k)$ and $O_{\breve{e}_{\breve{u}}}(k)$ satisfy $0 \leq L^2_{\breve{e}_{\breve{u}}}(k) + M^2_{\breve{e}_{\breve{u}}}(k) + O^2_{\breve{e}_{\breve{u}}}(k) \leq 1$ for all $k \in \mathbb{Z}$. $\chi_{\Im_{\breve{u}}}(k) = \sqrt{1 - \left(L^2_{\breve{e}_{\breve{u}}}(k) + M^2_{\breve{e}_{\breve{u}}}(k) + O^2_{\breve{e}_{\breve{u}}}(k)\right)}$ is called refusal degree of k in $\mathbb{Z}$, for SFS $\{\langle L_{\breve{e}_{\breve{u}}}(k), M_{\breve{e}_{\breve{u}}}(k), O_{\breve{e}_{\breve{u}}}(k) |\ k \in \mathbb{Z}\rangle\}$, which is triple components $\langle L_{\breve{e}_{\breve{u}}}, M_{\breve{e}_{\breve{u}}}, O_{\breve{e}_{\breve{u}}}\rangle$ is called SF number and each SF number can be presented as $E = \langle L_e, M_e, O_e\rangle$, where L_e, M_e and $O_e \in \Theta$, under the condition $0 \leq L_e^2 + M_e^2 + O_e^2 \leq 1$.

Definition 3.2. Let $\Im_{\breve{u}_1} = \langle L_{\breve{e}_{\breve{u}}}, M_{\breve{e}_{\breve{u}}}, O_{\breve{e}_{\breve{u}}}\rangle$ and $\Im_{\breve{u}_2} = \langle L_{\breve{e}_{\breve{u}}}, M_{\breve{e}_{\breve{u}}}, O_{\breve{e}_{\breve{u}}}\rangle$ are two SFNs define on the universe of discourse $\mathbb{Z} \neq \phi$, some operations on SFNs are defined as follows:

(a) $\Im_{\breve{u}_1} \subseteq \Im_{\breve{u}_2} \quad iff \quad \forall r \in R$,

$$L_{\breve{e}_{\breve{u}}} \leq \Im_{\breve{u}_2}, M_{\breve{e}_{\breve{u}}} \leq M_{\breve{e}_{\breve{u}}} \text{ and } O_{\breve{e}_{\breve{u}}} \geq O_{\breve{e}_{\breve{u}}} \tag{11}$$

(b) $\Im_{\breve{u}_1} = \Im_{\breve{u}_2} \quad iff$

$$\Im_{\breve{u}_1} \subseteq \Im_{\breve{u}_2} \text{ and } \Im_{\breve{u}_2} \subseteq \Im_{\breve{u}_1} \tag{12}$$

(c) Union

$$\Im_{\breve{u}_1} \cup \Im_{\breve{u}_2} = \langle \max(L_{\breve{e}_{\breve{u}}}, L_{\breve{e}_{\breve{u}}}),\ \min(M_{\breve{e}_{\breve{u}}}, M_{\breve{e}_{\breve{u}}}),\ \min(O_{\breve{e}_{\breve{u}}}, O_{\breve{e}_{\breve{u}}})\rangle; \tag{13}$$

(d) Intersection

$$\Im_{\breve{u}_1} \cap \Im_{\breve{u}_2} = \langle \min(L_{\breve{e}_{\breve{u}}}, L_{\breve{e}_{\breve{u}}}),\ \min(M_{\breve{e}_{\breve{u}}}, M_{\breve{e}_{\breve{u}}}),\ \max(O_{\breve{e}_{\breve{u}}}, O_{\breve{e}_{\breve{u}}})\rangle; \tag{14}$$

(e) Compliment

$$\Im^c_{\breve{u}} = \langle O_{\breve{e}_{\breve{u}}}, M_{\breve{e}_{\breve{u}}}, L_{\breve{e}_{\breve{u}}}\rangle. \tag{15}$$

Definition 3.3. Let $\Im_{\breve{u}_1} = \langle L_{\breve{e}_{\breve{u}}}, M_{\breve{e}_{\breve{u}}}, O_{\breve{e}_{\breve{u}}}\rangle$ and $\Im_{\breve{u}_2} = \langle L_{\breve{e}_{\breve{u}}}, M_{\breve{e}_{\breve{u}}}, O_{\breve{e}_{\breve{u}}}\rangle$ are two SFNs define on the universe of discourse $\mathbb{Z} \neq \phi$, some operations on SFNs are defined as follows with $\tau \geq 0$.

(1) $\Im_{\breve{u}_1} \oplus \Im_{\breve{u}_2} = \left\{\sqrt{L^2_{\breve{e}_{\breve{u}}} + L^2_{\breve{e}_{\breve{u}}} - L^2_{\breve{e}_{\breve{u}}} \cdot L^2_{\breve{e}_{\breve{u}}}},\ M_{\breve{e}_{\breve{u}}} \cdot M_{\breve{e}_{\breve{u}}},\ O_{\breve{e}_{\breve{u}}} \cdot O_{\breve{e}_{\breve{u}}}\right\}$.

(2) $\tau \cdot \Im_{\breve{u}} = \left\{\sqrt{1 - (1 - L^2_{\breve{e}_{\breve{u}}})^\tau},\ (M_{\breve{e}_{\breve{u}}})^\tau,\ (O_{\breve{e}_{\breve{u}}})^\tau\right\}$.

3.1. Comparison Rules for SFNs

For ranking the SFNs, different functions are introduced in this section described as.

Definition 3.4. Let $\Im_{\breve{u}} = \langle L_{\breve{e}_{\breve{u}}}, M_{\breve{e}_{\breve{u}}}, O_{\breve{e}_{\breve{u}}} \rangle$ be any SFNs. Then
(1) Score function is defined as $sc(\Im_{\breve{u}}) = \frac{(L_{\breve{e}_{\breve{u}}}+1-M_{\breve{e}_{\breve{u}}}+1-O_{\breve{e}_{\breve{u}}})}{3} = \frac{1}{3}(2 + L_{\breve{e}_{\breve{u}}} - M_{\breve{e}_{\breve{u}}} - O_{\breve{e}_{\breve{u}}})$.
(2) Accuracy function is defined as $acu(\Im_{\breve{u}}) = L_{\breve{e}_{\breve{u}}} - O_{\breve{e}_{\breve{u}}}$.
(3) Certainty function is defined as $cr(\Im_{\breve{u}}) = L_{\breve{e}_{\breve{u}}}$.

Ranking of SFNs described from Definition 3.4 .

Definition 3.5. Let $\Im_{\breve{u}_1} = \langle L_{\breve{e}_{\breve{u}}}, M_{\breve{e}_{\breve{u}}}, O_{\breve{e}_{\breve{u}}} \rangle$ and $\Im_{\breve{u}_2} = \langle L_{\breve{e}_{\breve{u}}}, M_{\breve{e}_{\breve{u}}}, O_{\breve{e}_{\breve{u}}} \rangle$ are two SFNs define on the universe of discourse $\mathbb{Z} \neq \phi$. Then Ranking of SFNs described from Definition 3.4 ,
(1) If $sc(\Im_{\breve{u}_1}) \succ sc(\Im_{\breve{u}_2})$,then $\Im_{\breve{u}_1} \succ \Im_{\breve{u}_2}$.
(2) If $sc(\Im_{\breve{u}_1}) \approx sc(\Im_{\breve{u}_2})$,and $acu(\Im_{\breve{u}_1}) \succ acu(\Im_{\breve{u}_2})$,then $\Im_{\breve{u}_1} \succ \Im_{\breve{u}_2}$.
(3) If $sc(\Im_{\breve{u}_1}) \approx sc(\Im_{\breve{u}_2})$, $acu(\Im_{\breve{u}_1}) \approx acu(\Im_{\breve{u}_2})$ and $cr(\Im_{\breve{u}_1}) \succ cr(\Im_{\breve{u}_2})$, then $\Im_{\breve{u}_1} \succ \Im_{\breve{u}_2}$.
(4) If $sc(\Im_{\breve{u}_1}) \approx sc(\Im_{\breve{u}_2})$, $acu(\Im_{\breve{u}_1}) \approx acu(\Im_{\breve{u}_2})$ and $cr(\Im_{\breve{u}_1}) \approx cr(\Im_{\breve{u}_2})$, then $\Im_{\breve{u}_1} \approx \Im_{\breve{u}_2}$.

Definition 3.6. Let any collections $\Im_{\breve{u}_\mathrm{p}} = \langle L_{\breve{e}_{\breve{u}}}, M_{\breve{e}_{\breve{u}}}, O_{\breve{e}_{\breve{u}}} \rangle$, $p \in N$ be the SFNs and $SFWA : SFN^n \times SFN^n \to SFN$, then $SFWA$ describe as,

$$SFWA\,(\Im_{\breve{u}_1}, \Im_{\breve{u}_2}, ..., \Im_{\breve{u}_n}) = \sum_{p=1}^{n} \tau_p \Im_{\breve{u}_p}, \tag{16}$$

In which $\tau = \{\tau_1, ..., \tau_n\}^T$ be the weight vector of $\Im_{\breve{u}_\mathrm{p}} = \langle L_{\breve{e}_{\breve{u}}}, M_{\breve{e}_{\breve{u}}}, O_{\breve{e}_{\breve{u}}} \rangle$, $p \in N$, with $\tau_\mathrm{p} \geq 0$ and $\sum_{\mathrm{p}=1}^{n} \tau_\mathrm{p} = 1$.

Theorem 3.7. Let any collections $\Im_{\breve{u}_\mathrm{p}} = \langle L_{\breve{e}_{\breve{u}}}, M_{\breve{e}_{\breve{u}}}, O_{\breve{e}_{\breve{u}}} \rangle$, $p \in N$ be the SFNs. Then operational properties of SFNs can obtained by utilizing the Definition 3.6 as.

$$SFWA\,(\Im_{\breve{u}_1}, \Im_{\breve{u}_2}, ..., \Im_{\breve{u}_n}) = \left\{ \sqrt{1 - \Pi_{p=1}^{n}(1 - L_{\breve{e}_{\breve{u}}}^2)^{\tau_p}}, \Pi_{p=1}^{n}(M_{\breve{e}_{\breve{u}}})^{\tau_p}, \Pi_{p=1}^{n}(O_{\breve{e}_{\breve{u}}})^{\tau_p} \right\}. \tag{17}$$

Definition 3.8. Let any collections $\Im_{\breve{u}_\mathrm{p}} = \langle L_{\breve{e}_{\breve{u}}}, M_{\breve{e}_{\breve{u}}}, O_{\breve{e}_{\breve{u}}} \rangle$, $p \in N$ be the SFNs and $SFOWA : SFN^n \times SFN^n \to SFN$, then $SFOWA$ describe as,

$$SFOWA\,(\Im_{\breve{u}_1}, \Im_{\breve{u}_2}, ..., \Im_{\breve{u}_n}) = \sum_{p=1}^{n} \tau_p \Im_{\breve{u}_{\rho(p)}}, \tag{18}$$

In which $\tau = \{\tau_1, \tau_2, ..., \tau_n\}$ be the weight vector of $\Im_{\breve{u}_\mathrm{p}} = \langle L_{\breve{e}_{\breve{u}}}, M_{\breve{e}_{\breve{u}}}, O_{\breve{e}_{\breve{u}}} \rangle$, $p \in N$, with $\tau_\mathrm{p} \geq 0$ and $\sum_{\mathrm{p}=1}^{n} \tau_\mathrm{p} = 1$ and $\rho(p)$ indicates a permutation on $\mathbb{Z}$.

Theorem 3.9. Let any collections $\Im_{\breve{u}_\mathrm{p}} = \langle L_{\breve{e}_{\breve{u}}}, M_{\breve{e}_{\breve{u}}}, O_{\breve{e}_{\breve{u}}} \rangle$, $p \in N$ be the SFNs. Then operational properties of SFNs can obtained by utilizing the Definition 3.8 as,

$$SFOWA\,(\Im_{\breve{u}_1}, \Im_{\breve{u}_2}, ..., \Im_{\breve{u}_n}) = \left\{ \sqrt{1 - \Pi_{p=1}^{n}(1 - L_{\breve{e}_{\breve{u}}}^2)^{\tau_p}}, \Pi_{p=1}^{n}(M_{\breve{e}_{\breve{u}}})^{\tau_p}, \Pi_{p=1}^{n}(O_{\breve{e}_{\breve{u}}})^{\tau_p} \right\}. \tag{19}$$

Theorem 3.10. Let any collections $\Im_{\breve{u}_\mathrm{p}} = \langle L_{\breve{e}_{\breve{u}}}, M_{\breve{e}_{\breve{u}}}, O_{\breve{e}_{\breve{u}}} \rangle$, $p \in N$ be the SFNs and λ be a fuzzy measure on $\mathbb{Z}$. Based on fuzzy measure, a spherical fuzzy Choquet integral weighted averaging ($SFCIWA$) operator of dimension n is a mapping $SFCIWA : SFN^n \times SFN^n \to SFN$ such that

$$SFCIWA\,(\Im_{\breve{u}_1}, \Im_{\breve{u}_2}, ..., \Im_{\breve{u}_n}) = \left\{ \begin{array}{c} \sqrt{1 - \Pi_{p=1}^{n}(1 - L_{\breve{e}_{\breve{u}}}^2)^{\lambda(A_{\rho(p)}) - \lambda(A_{\rho(p\text{-}1)})}}, \\ \Pi_{p=1}^{n}(M_{\breve{e}_{\breve{u}}})^{\lambda(A_{\rho(p)}) - \lambda(A_{\rho(p\text{-}1)})}, \\ \Pi_{p=1}^{n}(O_{\breve{e}_{\breve{u}_{\rho(p)}}})^{\lambda(A_{\rho(p)}) - \lambda(A_{\rho(p\text{-}1)})} \end{array} \right\}$$

where $\rho(p)$ indicates a permutation on $\mathbb{Z}$ and $A_{\rho(\mathrm{n})} = \{1, ..., p\}$, $A_{\rho(0)} = \phi$.

Definition 3.11. Let $\mathbb{Z} \neq \phi$ be the universe of discourse and any two spherical fuzzy sets $\Im_j$, $\Im_l$. Then normalized Hamming distance $f_{NHD}(\Im_j, \Im_l)$ is given as for all $k \in \mathbb{Z}$,

$$f_{NHD}(\Im_j, \Im_l) = \frac{1}{n} \sum_{p=1}^{n} \left(\begin{array}{c} \left|L_{\Im_j}(k_p) - L_{\Im_l}(k_p)\right| + \left|M_{\Im_j}(k_p) - M_{\Im_l}(k_p)\right| + \\ \left|O_{\Im_j}(k_p) - O_{\Im_l}(k_p)\right| \end{array} \right). \tag{20}$$

Definition 3.12. Let $\mathbb{Z} \neq \phi$ be the universe of discourse and any two spherical fuzzy sets $\Im_j$, $\Im_l$. Then normalized Euclidean distance $f_{NED}(\Im_j, \Im_l)$ is given as for all $k \in \mathbb{Z}$,

$$f_{NED}(\Im_j, \Im_l) = \sqrt{\frac{1}{n}\sum_{p=1}^{n}\left(\begin{array}{c}(L_{\Im_j}(k_p) - L_{\Im_l}(k_p))^2 + (M_{\Im_j}(k_p) - M_{\Im_l}(k_p))^2 + \\ (O_{\Im_j}(k_p) - O_{\Im_l}(k_p))^2\end{array}\right)}. \quad (21)$$

4. GRA method for multiple attribute decision making with incomplete weight information in Spherical fuzzy setting

Suppose that $A = \{b_1, ..., b_n\}$, n alternatives and $C = \{d_1, ..., c_m\}$, m alternatives,weight vector for parameter is $\nu = (\nu_1, ..., \nu_m)$, where $\nu_k \geq 0$ $(k = 1, ..., m)$, $\Sigma_{k=1}^{n}\nu_k = 1$. Assume that the DM give information about weights of criteria may be denotes in the following form, for $j \neq k$,

(1) If $\{\nu_j \geq \nu_k\}$ (weak ranking).
(2) If $\{\nu_j - \nu_k \geq \lambda_j(> 0)\}$, (strict ranking).
(3) If $\{\nu_j \geq \lambda_j\nu_k\}, 0 \leq \lambda_j \leq 1$,(multiple ranking).
(4) If $\{\lambda_j \leq \nu_j \leq \lambda_j + \delta_j\}, 0 \leq \lambda_j \leq \lambda_j + \delta_j \leq 1$, (interval ranking).

Δ denoted the set of the known information about the attribute weights provided by the experts. The decision maker $f_k(k = 1, ..., l)$ give the following decision matrix;

$$R^k = \left[\Im_{\breve{u}_{pq}}^{(k)}\right]_{m\times n} = \begin{array}{c} \\ b_1 \\ b_2 \\ \vdots \\ b_m \end{array}\left(\begin{array}{cccc} d_1 & d_2 & \cdots & d_n \\ \Im_{u_{11}}^{(k)} & \Im_{u_{12}}^{(k)} & \cdots & \Im_{u_{1n}}^{(k)} \\ \Im_{u_{21}}^{(k)} & \Im_{u_{22}}^{(k)} & \cdots & \Im_{u_{2N}}^{(k)} \\ \vdots & \vdots & \ddots & \vdots \\ \Im_{u_{m1}}^{(k)} & \Im_{u_{m2}}^{(k)} & \cdots & \Im_{u_{mn}}^{(k)} \end{array}\right)$$

where $\Im_{\breve{u}_{pq}}^{(k)} = \left(L_{\breve{u}_{pq}}^{(k)}, M_{\breve{u}_{pq}}^{(k)}, O_{\breve{u}_{pq}}^{(k)}\right)$ is an SFN representing the performance rating of the alternative $a_p \in A$ with respect to the attribute $c_p \in C$ provided by the decision makers d_k. To extend GRA method in the process of group decision making, we first need to fuse all individual decision matrices into a collective matrix by using SFCIW operator.

Step:1 Suppose that we have m alternative, $A = \{b_1, b_2, ..., b_m\}$,and n attributes $C_q(q = 1, 2, ..., n)$, now we invited each expert d$_k$ $(k = 1, 2, ..., r)$ to express their individual preference according to each by an spherical fuzzy numbers $\Im_{\breve{u}_{pq}}^{(k)} = \left(L_{\breve{u}_{pq}}^{(k)}, M_{\breve{u}_{pq}}^{(k)}, O_{\breve{u}_{pq}}^{(k)}\right)$ $(p = 1, 2, ..., m; q = 1, 2, ..., n, r = 1, 2, ..., k)$ expressed by the experts f_r. Then, we obtain a decision making matrices, $D^s = \left[E_{ip}^{(s)}\right]_{m\times n}$ $(s = 1, 2, ..., r)$ for decision. But there are two types of criteria, such as benefit and cost criteria, then we convert the decision matrices, $D^s = \left[E_{ip}^{s}\right]_{m\times n}$ into the normalized spherical fuzzy decision matrices, $R^r = \left[\Im_{\breve{u}_{pq}}^{(r)}\right]_{m\times n}$, by the following rules; $\Im_{\breve{u}_{pq}}^{(r)} = \begin{cases} \Im_{\breve{u}_{pq}}^{r}, & \text{for benefit criteria } A_p \\ \overline{\Im_{\breve{u}_{pq}}^{(r)}}, & \text{for cost criteria } A_p, \end{cases}$ $j = 1, 2, ..., n$, and $\overline{\Im_{\breve{u}_{pq}}^{(r)}}$ is the complement of $\Im_{\breve{u}_{pq}}^{(r)}$. If all the criteria have the same type, then there is no need of normalization.

Step:2 Confirm the fuzzy density $L_{\breve{e}_{\breve{u}}} = L_{\breve{e}_{\breve{u}}}(a_p)$ of each expert. According to Eq.(8), parameter λ_1 of expert can be determined.

Step:3 $\Im_{\breve{u}_{pq}}^{(r)}$ is reordered such that $\Im_{\breve{u}_{pq}}^{(r)} \geq \Im_{\breve{u}_{pq}}^{(r-1)}$. Using the SF Choquet integral average operator;

$$SFCIWA\left(\Im_{\breve{u}_{pq}}^{(1)}, \Im_{\breve{u}_{pq}}^{(2)}, ..., \Im_{\breve{u}_{pq}}^{(r)}\right) = \left\{\begin{array}{c} \sqrt{1 - \Pi_{p=1}^{r}(1 - L_{\breve{e}_{\breve{u}}})^{\lambda(A_{\rho(p)}) - \lambda(A_{\rho(p\text{-}1)})}}, \\ \Pi_{p=1}^{r}(M_{\breve{e}_{\breve{u}}})^{\lambda(A_{\rho(p)}) - \lambda(A_{\rho(p\text{-}1)})}, \\ \Pi_{p=1}^{r}(O_{\breve{e}_{\breve{u}_{\rho(p)}}})^{\lambda(A_{\rho(p)}) - \lambda(A_{\rho(p\text{-}1)})} \end{array}\right\}$$

to aggregate all the spherical fuzzy decision matrices $R^r = \left[\Im_{\breve{u}_{pq}}^{(r)}\right]_{m\times n}$ $(r = 1, ..., k)$ into a collective spherical fuzzy decision matrix $R = \left[\Im_{\breve{u}_{pq}}^{(r)}\right]_{m\times n}$ where $\Im_{\breve{u}_{pq}}^{(r)} = \left(L_{\breve{u}_{pq}}^{(r)}, M_{\breve{u}_{pq}}^{(r)}, O_{\breve{u}_{pq}}^{(r)}\right)$ $(p =$

$1, ..., m; q = 1, ..., n, r = 1, ..., k$), where $\rho(p)$ indicates a permutation on $\mathbb{Z}$ and $A_{\rho(n)} = \{1, ..., p\}$, $A_{\rho(0)} = \phi$ and $L_{\breve{e}_{\breve{u}}}(a_p)$ can be calculated by Eq. (9).

Step:4 $L^{+} = \{L_1^{+}, L_2^{+}, ..., L_m^{+}\}$ and $P^{-} = \{P_1^{-}, P_2^{-}, ..., P_m^{-}\}$ are the SFPIS and SFNIS, respectively.

$$L_p^{+} = \max_{\text{q}} sc_{pq} \tag{22}$$

and

$$P_p^{-} = \min_{q} sc_{pq}, \tag{23}$$

where $L^{+} = \left(L_{\breve{u}_{\text{p}}}^{+}, I_{\breve{u}_{\text{p}}}^{+}, N_{\breve{u}_{\text{p}}}^{+}\right)$ and $P^{-} = \left(P_{\breve{u}_{\text{p}}}^{-}, I_{\breve{u}_{\text{p}}}^{-}, N_{\breve{u}_{\text{p}}}^{-}\right)$ $p = 1, .., m.$

Step:5 Calculate the distance between the alternative a_{p} and the SFPIS L^{+}, and SFNIS P^{-}, respectively;

$$f(e_j, e_k) = \frac{1}{n}\sum_{p=1}^{n}\left(\left|P_{e_j}(a_p) - P_{e_k}(a_p)\right| + \left|I_{e_j}(a_p) - I_{e_k}(a_p)\right| + \left|N_{e_j}(a_p) - N_{e_k}(a_p)\right|\right). \tag{24}$$

This distance is known to be Normalized Hamming distance [1] $d(e_j, e_k)$, and construct an spherical fuzzy positive-ideal separation matrix D^{+} and Spherical fuzzy negative-ideal separation matrix D^{-} as follows;

$$\begin{matrix} f\left(\Im_{u_{11}}, L_1^{+}\right) & f\left(\Im_{u_{12}}, L_2^{+}\right) & . & . & . & f\left(\Im_{u_{1n}}, L_n^{+}\right) \\ f\left(\Im_{u_{21}}, L_1^{+}\right) & f\left(\Im_{u_{22}}, L_2^{+}\right) & . & . & . & f\left(\Im_{u_{1n}}, L_n^{+}\right) \\ . & . & . & & & . \\ . & . & & . & & . \\ . & . & & & . & . \\ f\left(\Im_{u_{m1}}, L_1^{+}\right) & f\left(\Im_{u_{m2}}, L_2^{+}\right) & . & . & . & f\left(\Im_{u_{mn}}, L_n^{+}\right) \end{matrix} \tag{25}$$

and

$$\begin{matrix} f\left(\Im_{u_{11}}, P_1^{-}\right) & f\left(\Im_{u_{12}}, P_2^{-}\right) & . & . & . & f\left(\Im_{u_{1n}}, P_n^{-}\right) \\ f\left(\Im_{u_{21}}, L_1^{+}\right) & f\left(\Im_{u_{22}}, P_2^{-}\right) & . & . & . & f\left(\Im_{u_{1n}}, P_n^{-}\right) \\ . & . & . & & & . \\ . & . & & . & & . \\ . & . & & & . & . \\ f\left(\Im_{u_{m1}}, P_1^{-}\right) & f\left(\Im_{u_{m2}}, P_2^{-}\right) & . & . & . & f\left(\Im_{u_{mn}}, P_n^{-}\right) \end{matrix} \tag{26}$$

Step:6 Grey coefficient for each alternative calculated from PIS and NIS by utilizing following below equation. The grey coefficient for each alternative calculated from PIS is provided as

$$\xi_{pq}^{+} = \frac{\min_{1\leq p\leq m} min_{1\leq q\leq n} d\left(\Im_{\breve{u}_{pq}}, L_p^{+}\right) + \rho \max_{1\leq p\leq m} max_{1\leq q\leq n} d\left(\Im_{\breve{u}_{pq}}, L_p^{+}\right)}{d\left(\Im_{\breve{u}_{pq}}, L_p^{+}\right) + \rho \max_{1\leq p\leq m} max_{1\leq q\leq n} d\left(\Im_{\breve{u}_{pq}}, L_p^{+}\right)}. \tag{27}$$

Where $p = 1, 2, 3, ..., m$ and $q = 1, 2, 3, ..., n$.Similarly, the grey coefficient of each alternative calculated from NIS is provided as

$$\xi_{pq}^{-} = \frac{\min_{1\leq p\leq m} min_{1\leq q\leq n} d\left(\Im_{\breve{u}_{pq}}, P_k^{-}\right) + \rho \max_{1\leq p\leq m} max_{1\leq q\leq n} d\left(\Im_{\breve{u}_{pq}}, P_k^{-}\right)}{d\left(\Im_{\breve{u}_{pq}}, P_k^{-}\right) + \rho \max_{1\leq p\leq m} max_{1\leq q\leq n} d\left(\Im_{\breve{u}_{pq}}, P_k^{-}\right)}. \tag{28}$$

Where $p = 1, 2, 3, ..., m$ and $q = 1, 2, 3, ..., n$ and the identification coefficient $\rho = 0.5$.

Step:7 Calculating the grey coefficient degree for each alternative from PIS and NIS by utilizing following below equation, respectively,

$$\xi_p^{+} = \sum_{q=1}^{n} \nu_q \xi_{pq}^{+} \tag{29}$$

$$\xi_p^{-} = \sum_{q=1}^{n} \nu_q \xi_{pq}^{-}$$

The basic principle of the Grey method is that the chosen alternative should have the "largest degree of grey relation" from the PIS and the "smallest degree of grey relation" from the NIS. Obviously, for the weights are known, the smaller ξ_p^- and the larger ξ_p^+, the finest alternative a_p is. But incomplete information about weights of alternatives is known. So, in this circumstances the ξ_p^- and ξ_p^+, information about weight calculated initially. So we provide following optimization models for multiple objective to calculate the information about weight,

$$(OM1)\begin{cases} \min \xi_p^- = \sum_{q=1}^{n} \nu_q \xi_{pq}^- \; p = 1, 2, ..., m \\ \max \xi_p^+ = \sum_{q=1}^{n} \nu_q \xi_{pq}^+ \; p = 1, 2, ..., m \end{cases} \tag{30}$$

Since each alternative is non-inferior, so there exists no preference relation on the all the alternatives. Then, we aggregate the above optimization models with equal weights into the following optimization model of single objective,

$$(OM2)\begin{cases} \min \xi_p = \sum_{p=1}^{m} \sum_{q=1}^{n} \left(\xi_{pq}^- - \xi_{pq}^+\right) \nu_q \end{cases} \tag{31}$$

To finding solution of OM2, we obtain optimal solution $\nu = (\nu_1, \nu_2, ..., \nu_m)$, which utilized as weights informations of provided alternatives. Then, we obtain ξ_p^+ $(p = 1, 2, ..., m)$ and ξ_p^- $(p = 1, 2, ..., m)$ as utilizing above formula, respectively.

Step:8 Relative degree calculated for each alternative utilizing the following equation from PIS and NIS,

$$\xi_p = \frac{\xi_p^+}{\xi_p^- + \xi_p^+} \left(p = 1, 2, ..., m\right). \tag{32}$$

Step:9 Ranking all the alternatives $a_p(p = 1, 2, ..., m)$ and select finest one(s) in accordance with ξ_p $(p = 1, 2, ..., m)$. If any alternative has the highest ξ_p value, then it is finest alternative according to the criteria.

Step:10 End.

5. Descriptive Example

The technique proposed in this paper is illustrated by a numerical examples with Spherical fuzzy information in this section. Suppose a panel of three experts is arranged for selection from four possible emerging technology enterprises $\check{Z}_i$ $(i = 1, 2, 3, 4)$. So panel select optimal alternative from given four alternatives,
(1) Technical advancement is denoted by B_1;
(2) Potential market risk is denoted by B_2;
(3) Industrialization infrastructure, human resources and financial condition is denoted by B_3;
(4) Employment creation and the development of science and technology is denoted by B_4.

Step:1 From the results obtained with each emerging technology enterprise, the three experts offering their own opinions which are shown in tables 1-3.

Table-1.: Spherical fuzzy information D^1

	$\check{A}_1$	$\check{A}_2$	$\check{A}_3$	$\check{A}_4$
$\check{Z}_1$	$\langle 0.3, 0.8, 0.5\rangle$	$\langle 0.8, 0.4, 0.3\rangle$	$\langle 0.4, 0.5, 0.7\rangle$	$\langle 0.3, 0.3, 0.4\rangle$
$\check{Z}_2$	$\langle 0.2, 0.6, 0.7\rangle$	$\langle 0.3, 0.9, 0.1\rangle$	$\langle 0.5, 0.3, 0.7\rangle$	$\langle 0.5, 0.4, 0.2\rangle$
$\check{Z}_3$	$\langle 0.4, 0.8, 0.4\rangle$	$\langle 0.5, 0.8, 0.2\rangle$	$\langle 0.2, 0.3, 0.7\rangle$	$\langle 0.6, 0.6, 0.1\rangle$
$\check{Z}_4$	$\langle 0.5, 0.3, 0.8\rangle$	$\langle 0.6, 0.6, 0.3\rangle$	$\langle 0.3, 0.6, 0.5\rangle$	$\langle 0.4, 0.2, 0.3\rangle$

Table-2.: Spherical fuzzy information D^2

	$\check{A}_1$	$\check{A}_2$	$\check{A}_3$	$\check{A}_4$
$\check{Z}_1$	$\langle 0.1, 0.5, 0.7\rangle$	$\langle 0.6, 0.3, 0.4\rangle$	$\langle 0.3, 0.8, 0.6\rangle$	$\langle 0.6, 0.3, 0.2\rangle$
$\check{Z}_2$	$\langle 0.4, 0.4, 0.8\rangle$	$\langle 0.5, 0.7, 0.1\rangle$	$\langle 0.4, 0.2, 0.7\rangle$	$\langle 0.6, 0.1, 0.6\rangle$
$\check{Z}_3$	$\langle 0.2, 0.9, 0.3\rangle$	$\langle 0.7, 0.1, 0.4\rangle$	$\langle 0.3, 0.6, 0.4\rangle$	$\langle 0.6, 0.2, 0.4\rangle$
$\check{Z}_4$	$\langle 0.3, 0.4, 0.8\rangle$	$\langle 0.4, 0.6, 0.5\rangle$	$\langle 0.4, 0.1, 0.8\rangle$	$\langle 0.6, 0.3, 0.3\rangle$

Table-3.: Spherical fuzzy information D^3

	$\breve{A}_1$	$\breve{A}_2$	$\breve{A}_3$	$\breve{A}_4$
$\breve{Z}_1$	$\langle 0.4, 0.2, 0.8\rangle$	$\langle 0.4, 0.4, 0.3\rangle$	$\langle 0.5, 0.4, 0.6\rangle$	$\langle 0.5, 0.1, 0.4\rangle$
$\breve{Z}_2$	$\langle 0.2, 0.5, 0.7\rangle$	$\langle 0.6, 0.5, 0.4\rangle$	$\langle 0.2, 0.5, 0.8\rangle$	$\langle 0.7, 0.2, 0.4\rangle$
$\breve{Z}_3$	$\langle 0.6, 0.4, 0.5\rangle$	$\langle 0.9, 0.3, 0.1\rangle$	$\langle 0.3, 0.1, 0.9\rangle$	$\langle 0.6, 0.2, 0.6\rangle$
$\breve{Z}_4$	$\langle 0.5, 0.3, 0.7\rangle$	$\langle 0.8, 0.5, 0.2\rangle$	$\langle 0.3, 0.8, 0.4\rangle$	$\langle 0.5, 0.3, 0.5\rangle$

Since C_1, C_3 are cost-type criteria and C_2, C_4 are benefit-type criteria. So we have need to normalized the Spherical fuzzy information. Normalized Spherical fuzzy information are shown in table-4,5,6.:

Table-4.: Normalized Spherical fuzzy information R^1

	$\breve{A}_1$	$\breve{A}_2$	$\breve{A}_3$	$\breve{A}_4$
$\breve{Z}_1$	$\langle 0.5, 0.8, 0.3\rangle$	$\langle 0.8, 0.4, 0.3\rangle$	$\langle 0.7, 0.5, 0.4\rangle$	$\langle 0.3, 0.3, 0.4\rangle$
$\breve{Z}_2$	$\langle 0.7, 0.6, 0.2\rangle$	$\langle 0.3, 0.9, 0.1\rangle$	$\langle 0.7, 0.3, 0.5\rangle$	$\langle 0.5, 0.4, 0.2\rangle$
$\breve{Z}_3$	$\langle 0.4, 0.8, 0.4\rangle$	$\langle 0.5, 0.8, 0.2\rangle$	$\langle 0.7, 0.3, 0.2\rangle$	$\langle 0.6, 0.6, 0.1\rangle$
$\breve{Z}_4$	$\langle 0.8, 0.3, 0.5\rangle$	$\langle 0.6, 0.6, 0.3\rangle$	$\langle 0.5, 0.6, 0.3\rangle$	$\langle 0.4, 0.2, 0.3\rangle$

Table-5.: Normalized Spherical fuzzy information R^2

	$\breve{A}_1$	$\breve{A}_2$	$\breve{A}_3$	$\breve{A}_4$
$\breve{Z}_1$	$\langle 0.7, 0.5, 0.1\rangle$	$\langle 0.6, 0.3, 0.4\rangle$	$\langle 0.6, 0.8, 0.3\rangle$	$\langle 0.6, 0.3, 0.2\rangle$
$\breve{Z}_2$	$\langle 0.8, 0.4, 0.4\rangle$	$\langle 0.5, 0.7, 0.1\rangle$	$\langle 0.7, 0.2, 0.4\rangle$	$\langle 0.6, 0.1, 0.6\rangle$
$\breve{Z}_3$	$\langle 0.3, 0.9, 0.2\rangle$	$\langle 0.7, 0.1, 0.4\rangle$	$\langle 0.4, 0.6, 0.3\rangle$	$\langle 0.6, 0.2, 0.4\rangle$
$\breve{Z}_4$	$\langle 0.8, 0.4, 0.3\rangle$	$\langle 0.4, 0.6, 0.5\rangle$	$\langle 0.8, 0.1, 0.4\rangle$	$\langle 0.6, 0.3, 0.3\rangle$

Table-6.: Normalized dSpherical fuzzy information R^3

	$\breve{A}_1$	$\breve{A}_2$	$\breve{A}_3$	$\breve{A}_4$
$\breve{Z}_1$	$\langle 0.8, 0.2, 0.4\rangle$	$\langle 0.4, 0.4, 0.3\rangle$	$\langle 0.6, 0.4, 0.5\rangle$	$\langle 0.5, 0.1, 0.4\rangle$
$\breve{Z}_2$	$\langle 0.7, 0.5, 0.2\rangle$	$\langle 0.6, 0.5, 0.4\rangle$	$\langle 0.8, 0.5, 0.2\rangle$	$\langle 0.7, 0.2, 0.4\rangle$
$\breve{Z}_3$	$\langle 0.5, 0.4, 0.6\rangle$	$\langle 0.9, 0.3, 0.1\rangle$	$\langle 0.9, 0.1, 0.3\rangle$	$\langle 0.6, 0.2, 0.6\rangle$
$\breve{Z}_4$	$\langle 0.7, 0.3, 0.5\rangle$	$\langle 0.8, 0.5, 0.2\rangle$	$\langle 0.4, 0.8, 0.2\rangle$	$\langle 0.5, 0.3, 0.5\rangle$

Assume that the information about attribute weights, given by experts, is partly known; $\Delta = \left\{\begin{array}{c} 0.2 \le w_1 \le 0.25, \\ 0.15 \le w_2 \le 0.2, \\ 0.28 \le w_3 \le 0.32, \\ 0.35 \le w_4 \le 0.4 \end{array}\right\}$, $w_p \ge 0$, $p = 1, 2, 3, 4$, $\sum\limits_{p=1}^{4} w_p = 1$ Then, we utilize the developed approach to get the most desirable alternative(s).

Step:2 We firstly determine fuzzy density of each decision maker, and its λ parameter. Suppose that $L_{\breve{e}_{\breve{u}}}(b_1) = 0.30$, $L_{\breve{e}_{\breve{u}}}(b_2) = 0.40$, $L_{\breve{e}_{\breve{u}}}(A_3) = 0.50$. Then λ of expert can be determined: $\lambda = -0.45$. By Eq.(6), we have $L_{\breve{e}_{\breve{u}}}(b_1, b_2) = 0.65$, $L_{\breve{e}_{\breve{u}}}(b_1, A_3) = 0.73$, $L_{\breve{e}_{\breve{u}}}(b_2, A_3) = 0.81$, $L_{\breve{e}_{\breve{u}}}(b_1, b_2, A_3) = 1$.

Step:3 According to Definition 3.5, $\Im^{(k)}_{\breve{u}_{pq}}$ is reordered such that $\Im^{(k)}_{\breve{u}_{pq}} \ge \Im^{(k-1)}_{\breve{u}_{pq}}$. Then Utilize the Spherical fuzzy Choquet integral weighted operator

$$SFCIWA\left(\Im_{\breve{u}_1}, \Im_{\breve{u}_2}, ..., \Im_{\breve{u}_n}\right) = \left\{\begin{array}{c} \sqrt{1 - \Pi_{p=1}^{n}(1 - L_{\breve{e}_{\breve{u}}})^{\lambda\left(A_{\rho(p)}\right) - \lambda\left(A_{\rho(p\text{-}1)}\right)}}, \\ \Pi_{p=1}^{n}(M_{\breve{e}_{\breve{u}}})^{\lambda\left(A_{\rho(p)}\right) - \lambda\left(A_{\rho(p\text{-}1)}\right)}, \\ \Pi_{p=1}^{n}(O_{\breve{e}_{\breve{u}_{\rho(p)}}})^{\lambda\left(A_{\rho(p)}\right) - \lambda\left(A_{\rho(p\text{-}1)}\right)} \end{array}\right\}$$

to aggregate all the Spherical fuzzy decision matrices $R^k = \left[\Im^{(k)}_{\breve{u}_{pq}}\right]_{m\times n}$ into a collective Spherical fuzzy decision matrix as follows:

Table-7.: Collective Spherical fuzzy information

	$\breve{A}_1$	$\breve{A}_2$	$\breve{A}_3$	$\breve{A}_4$
$\breve{Z}_1$	$\langle 0.702, 0.417, 0.225\rangle$	$\langle 0.638, 0.361, 0.331\rangle$	$\langle 0.634, 0.545, 0.391\rangle$	$\langle 0.498, 0.204, 0.313\rangle$
$\breve{Z}_2$	$\langle 0.740, 0.488, 0.254\rangle$	$\langle 0.498, 0.670, 0.162\rangle$	$\langle 0.740, 0.311, 0.335\rangle$	$\langle 0.616, 0.193, 0.374\rangle$
$\breve{Z}_3$	$\langle 0.411, 0.654, 0.361\rangle$	$\langle 0.748, 0.274, 0.200\rangle$	$\langle 0.755, 0.260, 0.265\rangle$	$\langle 0.600, 0.278, 0.304\rangle$
$\breve{Z}_4$	$\langle 0.770, 0.331, 0.418\rangle$	$\langle 0.651, 0.562, 0.311\rangle$	$\langle 0.629, 0.354, 0.311\rangle$	$\langle 0.515, 0.265, 0.358\rangle$

Step:4 Utilize Equations 22 and 23 we get the positive-ideal and negative-ideal solution respectively are:

$$L^{+} = \{\langle 0.702, 0.417, 0.225\rangle, \langle 0.748, 0.274, 0.200\rangle, \langle 0.755, 0.260, 0.265\rangle, \langle 0.616, 0.193, 0.374\rangle\}$$

$$L^{-} = \{\langle 0.411, 0.654, 0.361\rangle, \langle 0.498, 0.670, 0.162\rangle, \langle 0.634, 0.545, 0.391\rangle, \langle 0.515, 0.265, 0.358\rangle\}$$

Step:5 Utilize equation (25) and (26) to get the positive-ideal separation matrix and negative-ideal separation matrix respectively as follow;

Table-8.:
Positive-ideal separation matrix

$D^{+} =$

	$\breve{A}_1$	$\breve{A}_2$	$\breve{A}_3$	$\breve{A}_4$
$\breve{Z}_1$	0.0000	0.0822	0.1327	0.0475
$\breve{Z}_2$	0.0345	0.1710	0.0338	0.0000
$\breve{Z}_3$	0.1656	0.0000	0.0000	0.0429
$\breve{Z}_4$	0.0865	0.1241	0.0662	0.0472

Table-9.: Negative-ideal separation matrix

$D^{-} =$

	$\breve{A}_1$	$\breve{A}_2$	$\breve{A}_3$	$\breve{A}_4$
$\breve{Z}_1$	0.1656	0.1547	0.0000	0.0309
$\breve{Z}_2$	0.1502	0.0000	0.0989	0.0472
$\breve{Z}_3$	0.0000	0.1710	0.1327	0.0378
$\breve{Z}_4$	0.1842	0.1025	0.0687	0.0000

Step:6 Utilize equations (27) and (28) we get the grey relational coefficient matrices in which each alternative is calculated from PIS and NIS as follow:

$$\left[\zeta_{ij}^{+}\right] = \begin{bmatrix} 1.0000 & 0.5098 & 0.3918 & 0.6428 \\ 0.7125 & 0.3333 & 0.7166 & 1.0000 \\ 0.3405 & 1.0000 & 1.0000 & 0.6658 \\ 0.4970 & 0.4079 & 0.5636 & 0.6443 \end{bmatrix}$$

$$\left[\zeta_{ij}^{-}\right] = \begin{bmatrix} 0.3573 & 0.3731 & 1.0000 & 0.7487 \\ 0.3801 & 1.0000 & 0.4821 & 0.6611 \\ 1.0000 & 0.3500 & 0.4096 & 0.7090 \\ 0.3333 & 0.4732 & 0.5151 & 1.0000 \end{bmatrix}$$

Step:7 We used the model (M2) to establish the single-objective programming model:

$$\min \xi(w) = -0.0659w_1 - 0.2392w_2 - 0.5377w_3 + 0.2088w_4$$

After their solution, the weight vector of attributes are:

$$w = (0.273, 0.368, 0.227, 0.1300)$$

From the PIS and NIS, we obtain grey relational coefficient of each alternative:

$$\xi_1^{+} = 0.6331, \xi_2^{+} = 0.6098, \xi_3^{+} = 0.7745, \xi_4^{+} = 0.4974,$$

$$\xi_1^{-} = 0.5590, \xi_2^{-} = 0.6671, \xi_3^{-} = 0.5869, \xi_4^{-} = 0.5120.$$

Step:8 Utilize equation 32, we obtain the relative relational degree of each alternative from PIS and NIS as follows:

$$\begin{aligned}
\xi_1 &= \frac{\xi_1^+}{\xi_1^- + \xi_1^+} = \frac{0.6331}{0.5590 + 0.6331} = 0.5310 \\
\xi_2 &= \frac{\xi_2^+}{\xi_2^- + \xi_2^+} = \frac{0.6098}{0.6671 + 0.6098} = 0.4775 \\
\xi_3 &= \frac{\xi_3^+}{\xi_3^- + \xi_3^+} = \frac{0.7745}{0.5869 + 0.7745} = 0.5688 \\
\xi_4 &= \frac{\xi_4^+}{\xi_4^- + \xi_4^+} = \frac{0.4974}{0.5120 + 0.4974} = 0.4927
\end{aligned}$$

Step:9 The ranking order, according to the relative relational degree are:

$$\check{Z}_3 > \check{Z}_1 > \check{Z}_4 > \check{Z}_2,$$

and best alternative is $\check{Z}_3$.

6. Conclusion

In this paper, we proposed decision making approach to deal with spherical fuzzy information. As spherical fuzzy set is the generalization of all the existing structure of fuzzy sets, so an algorithm based on GRA approach to deal with uncertainty and inaccurate information in decision making problems using spherical fuzzy environments. Final, a numerical application is illustrated to shows the how our proposed technique is effective and reliable to deal with uncertainty. In future, we use TOPSIS, VIKOR, TODAM approaches to deal with uncertainty using spherical fuzzy information. .

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