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CONTENT

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Neutrosophic Set is a Generalization of Intuitionistic Fuzzy Set, Inconsistent Intuitionistic Fuzzy Set (Picture Fuzzy Set, Ternary Fuzzy Set), Pythagorean Fuzzy Set, Spherical Fuzzy Set, and q-Rung Orthopair Fuzzy Set, while Neutrosophication is a Generalization of Regret Theory, Grey System Theory, and Three-Ways Decision (revisited)

Florentin Smarandache[1]

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Abstract — In this paper, we prove that Neutrosophic Set (NS) is an extension of Intuitionistic Fuzzy Set (IFS) no matter if the sum of neutrosophic components is < 1, or > 1, or $= 1$. For the case when the sum of components is 1 (as in IFS), after applying the neutrosophic aggregation operators, one gets a different result than applying the intuitionistic fuzzy operators, since the intuitionistic fuzzy operators ignore the indeterminacy, while the neutrosophic aggregation operators take into consideration the indeterminacy at the same level as truth-membership and falsehood-nonmembership are taken. NS is also more flexible and effective because it handles, besides independent components, also partially independent and partially dependent components, while IFS cannot deal with these. Since there are many types of indeterminacies in our world, we can construct different approaches to various neutrosophic concepts. Neutrosophic Set (NS) is a generalisation of Inconsistent Intuitionistic Fuzzy Set (IIFS) - which is equivalent to the Picture Fuzzy Set (PFS) and Ternary Fuzzy Set (TFS) -, Pythagorean Fuzzy Set (PyFS), Spherical Fuzzy Set (SFS), and q-Rung Orthopair Fuzzy Set (q-ROFS). Moreover, all these sets are more general than Intuitionistic Fuzzy Set. We prove that Atanassov's Intuitionistic Fuzzy Set of the second type (IFS2), and Spherical Fuzzy Sets (SFS) do not have independent components. Furthermore, we show that Spherical Neutrosophic Set (SNS) and n-Hyper Spherical Neutrosophic Set (n-HSNS) are generalisations of IFS2 and SFS. The main distinction between Neutrosophic Set (NS) and all previous set theories are a) the independence of all three neutrosophic components - truth-membership (T), indeterminacy-membership (I), falsehood-nonmembership (F) - concerning each other in NS – while in the previous set theories their components are dependent on each other, and b) the importance of indeterminacy in NS - while in previous set theories indeterminacy is entirely or partially ignored. Also, Regret Theory, Grey System Theory, and Three-Ways Decision are particular cases of Neutrosophication and Neutrosophic Probability. We now extend the Three-Ways Decision to n-Ways Decision.

Keywords – *Neutrosophic set, intuitionistic fuzzy set, Pythagorean fuzzy set, spherical fuzzy set, q-rung orthopair fuzzy set*

1. Introduction

This paper recalls ideas about the distinctions between neutrosophic set and intuitionistic fuzzy set presented in previous versions of this paper [1-5]. Mostly, in this paper, we respond to Atanassov and Vassiliev's paper [6] about the fact that neutrosophic set is a generalisation of intuitionistic fuzzy set.

We use the notations employed in the neutrosophic environment [1-5] since they are better descriptive than the Greek letters used in an intuitionistic fuzzy environment, i.e., truth-membership (T), indeterminacy-membership (I), and falsehood-nonmembership (F).

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We also use the triplet components in this order: (T, I, F).

Neutrosophic "Fuzzy" Set (as named by Atanassov and Vassiliev [6]) is commonly called "Single-Valued" Neutrosophic Set (i.e. the neutrosophic components are single-valued numbers) by the neutrosophic community that now riches about 1,000 researchers, from 56 countries around the world, which have produced about 2,000 publications (papers, conference presentations, books, MSc theses and PhD dissertations).

The NS is more complex and more general than previous (crisp/fuzzy/intuitionistic fuzzy/picture fuzzy/ternary fuzzy set/Pythagorean fuzzy/spherical fuzzy/q-Rung orthopair fuzzy) sets, because:

- *i.* A new branch of philosophy was born, called Neutrosophy [7], which is a generalisation of Dialectics (and of YinYang Chinese philosophy), where not only the dynamics of opposites are studied, but the dynamics of opposites together with their neutrals as well, i.e. $(< A >, < neutA >, < antiA >)$, where $< A >$ is an item, $< antiA >$ its opposite, and $< neutA >$ their neutral (indeterminacy between them).
- *ii.* Neutrosophy show the significance of neutrality/indeterminacy $(< neutA >)$ that gave birth to neutrosophic set/logic/probability/statistics/measure/integral and so on, that have many practical applications in various fields.
- *iii.* The sum of the Single-Valued Neutrosophic Set/Logic components was allowed to be up to 3 (showing the importance of independence of the neutrosophic components among themselves), which permitted the characterisation of paraconsistent/conflictual sets/propositions (by letting the sum of components > 1), and of paradoxical sets/propositions, represented by the neutrosophic triplet $(1, 1, 1)$.
- *iv.* NS can distinguish between absolute truth/indeterminacy/falsehood and relative truth/indeterminacy/falsehood using nonstandard analysis, which generated the Nonstandard Neutrosophic Set (NNS).
- *v.* Each neutrosophic component was allowed to take values outside of the interval [0,1], that culminated with the introduction of the neutrosophic overset/underset/offset [8].
- *vi.* NS was enlarged by Smarandache to Refined Neutrosophic Set (RNS), where each neutrosophic component was refined / split into sub-components [9]., i.e. T was refined/split into $T_1, T_2, \dots, T_p$; I was refined / split into $I_1, I_2, \dots, I_r$; and F was refined split into $F_1, F_2, \dots, F_s$; where $p, r, s \geq 1$ are integers and $p + r + s \geq 4$; all T_j, I_k, F_l are subsets of [0,1] with no other restriction.
- *vii.* RNS permitted the extension of the Law of Included Middle to the neutrosophic Law of Included Multiple-Middle [10].
- *viii.* Classical Probability and Imprecise Probability were extended to Neutrosophic Probability [11], where for each event E one has: the chance that event E occurs $(ch(E))$, indeterminate-chance that event E occurs or not $(ch(neutE))$, and the chance that the event E does not occur $(ch(antiE))$, with: $0 \leq sup\{ch(E)\} + sup\{ch(neutE)\} + sup\{ch(antiE)\} \leq 3$.
- *ix.* Classical Statistics was extended to Neutrosophic Statistics [12] that deals with indeterminate/ incomplete/inconsistent/vague data regarding samples and populations.

And so on (see below more details). Several definitions are recalled for paper's self-containment.

2. Definition of Single-Valued Neutrosophic Set (NS)

Introduced by Smarandache [13-15] in 1998. Let U be a universe of discourse and a set $A_{NS} \subseteq U$.

Then, $A_{NS} = \{\langle x, T_A(x), I_A(x), F_A(x)\rangle | x \in U\}$, where $T_A(x), I_A(x), F_A(x) : U \rightarrow [0, 1]$ represent the degree of truth-membership, degree of indeterminacy-membership, and degree of false-nonmembership respectively, with $0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3$.

The neutrosophic components $T_A(x), I_A(x), F_A(x)$ are independent concerning each other.

3. Definition of Single-Valued Refined Neutrosophic Set (RNS)

Introduced by Smarandache [9] in 2013. Let U be a universe of discourse and a set $A_{RNS} \subseteq U$. Then,

$$A_{RNS} = \{\langle x, T_{1A}(x), T_{2A}(x), \dots, T_{pA}(x); I_{1A}(x), I_{2A}(x), \dots, I_{rA}(x); F_{1A}(x), F_{2A}(x), \dots, F_{sA}(x)\rangle | x \in U\},$$

where all $T_{jA}(x)$, $1 \le j \le p$, $I_{kA}(x)$, $1 \le k \le r$, $F_{lA}(x)$, $1 \le l \le s : U \to [0,1]$, and

i. $T_{jA}(x)$ represents the j^{th} sub-membership degree,
ii. $I_{kA}(x)$ represents the k^{th} sub-indeterminacy degree,
iii. $F_{lA}(x)$ represents the l^{th} sub-nonmembership degree,

with $p, r, s \ge 1$ integers, where $p + r + s = n \ge 4$, and:

$$0 \le \sum_{j=1}^{p} T_{jA}(x) + \sum_{k=1}^{r} I_{kA}(x) + \sum_{l=1}^{s} T_{jA}(x) \le n.$$

All neutrosophic sub-components $T_{jA}(x)$, $I_{kA}(x)$, $F_{lA}(x)$ are independent concerning each other.

4. Definition of Single-Valued Intuitionistic Fuzzy Set (IFS)

Introduced by Atanassov [16-18] in 1983. Let U be a universe of discourse and a set $A_{IFS} \subseteq U$. Then,

$$A_{IFS} = \{\langle x, T_A(x), F_A(x)\rangle | x \in U\},$$

where $T_A(x), F_A(x) : U \to [0,1]$ represent the degree of membership and degree of nonmembership respectively, with $T_A(x) + F_A(x) \le 1$, and $I_A(x) = 1 - T_A(x) - F_A(x)$ represents degree of indeterminacy (in previous publications it was called degree of hesitancy).

The intuitionistic fuzzy components $T_A(x), I_A(x), F_A(x)$ are dependent concerning each other.

5. Definition of Single-Valued Inconsistent Intuitionistic Fuzzy Set (Equivalent to Single-Valued Picture Fuzzy Set, and with Single-Valued Ternary Fuzzy Set)

The single-valued *Inconsistent Intuitionistic Fuzzy Set* (IIFS), introduced by Hindde and Patching [19] in 2008, and the single-valued *Picture Fuzzy Set* (PFS), introduced by Cuong [20] in 2013, indeed coincide, as Atanassov and Vassiliev have observed; also we add that single-valued *Ternary Fuzzy Set*, introduced by Wang, Ha and Liu [21] in 2015 also coincided with them. All these three notions are defined as follows.

Let $\mathcal{U}$ be a universe of discourse, and let us consider a subset $A \subseteq \mathcal{U}$. Then,

$$A_{IIFS} = A_{PFS} = A_{TFS} = \{\langle x, T_A(x),\ I_A(x), F_A(x)\rangle | x \in \mathcal{U}\},$$

where $T_A(x), I_A(x), F_A(x) \in [0,1]$, and the sum $0 \le T_A(x) + I_A(x) + \ F_A(x) \le 1$, for all $x \in \mathcal{U}$.

In these sets, the denominations are:

i. $T_A(x)$ is called *degree of membership* (or validity, or positive membership);
ii. $I_A(x)$ is called *degree of neutral membership*;
iii. $F_A(x)$ is called *degree of nonmembership* (or nonvalidity, or negative membership).

The *refusal degree* is: $R_A(x) = 1 - T_A(x) - I_A(x) - F_A(x) \in [0,1]$, for all $x \in \mathcal{U}$.

The IIFS (PFS, TFS) components $T_A(x), I_A(x), F_A(x), R_A(x)$ are dependent concerning each other.

Wang, Ha and Liu's [21] assertion that "neutrosophic set theory is difficult to handle the voting problem, as the sum of the three components is greater than 1" is not true, since the sum of the three neutrosophic components is not necessarily greater than 1, but it can be less than or equal to any number between 0 and 3, i.e. $0 \le T_A(x) + I_A(x) + \ F_A(x) \le 3$, so, for example, the sum of the three neutrosophic components can be less than 1, or equal to 1, or greater than 1 depending on each application.

6. Inconsistent Intuitionistic Fuzzy Set and the Picture Fuzzy Set are Particular Cases of the Neutrosophic Set

The *Inconsistent Intuitionistic Fuzzy Set* and the *Picture Fuzzy Set* and *Ternary Fuzzy Set* are particular cases of the *Neutrosophic Set* (NS). Because, in neutrosophic set, similarly taking single-valued components $T_A(x), I_A(x), F_A(x) \in [0, 1]$, one has the sum $T_A(x) + I_A(x) + F_A(x) \leq 3$, which means that $T_A(x) + I_A(x) + F_A(x)$ can be equal to or less than any number between 0 and 3.

Therefore, in the particular case when choosing the sum equal to $1 \in [0, 3]$ and getting $T_A(x) + I_A(x) + F_A(x) \leq 1$, one obtains IIFS and PFS and TFS.

7. Single-Valued Intuitionistic Fuzzy Set is a Particular Case of Single-Valued Neutrosophic Set

Single-valued Intuitionistic Fuzzy Set is a particular case of single-valued Neutrosophic Set because we can choose the sum to be equal to 1:

$$T_A(x) + I_A(x) + F_A(x) = 1.$$

8. Inconsistent Intuitionistic Fuzzy Set and Picture Fuzzy Set are Also Particular Cases of Single-Valued Refined Neutrosophic Set

The *Inconsistent Intuitionistic Fuzzy Set* (IIFS), *Picture Fuzzy Set* (PFS), and *Ternary Fuzzy Set* (TFS) that coincide with each other are besides particular case(s) of *Single-Valued Refined Neutrosophic Set* (RNS).

We may define:

$$A_{IIFS} \equiv A_{PFS} = A_{TFS} = \{x, T_A(x), I_{1_A}(x), I_{2_A}(x), F_A(x) \big| x \in \mathcal{U}\},$$

with $T_A(x), I_{1_A}(x), I_{2_A}(x), F_A(x) \in [0, 1]$, and the sum $T_A(x) + I_{1_A}(x) + I_{2_A}(x) + F_A(x) = 1$, for all $x \in \mathcal{U}$; where:

i. $T_A(x)$ is the degree of positive membership (validity, etc.);
ii. $I_{1_A}(x)$ is the degree of neutral membership;
iii. $I_{2_A}(x)$ is the refusal degree;
iv. $F_A(x)$ is the degree of negative membership (non-validity, etc.).

$n = 4$, and as a particular case of the sum $T_A(x) + I_{1_A}(x) + I_{2_A}(x) + F_A(x) \leq 4$, where the sum can be any positive number up to 4, we take the positive number 1 for the sum:

$$T_A(x) + I_{1_A}(x) + I_{2_A}(x) + F_A(x) = 1.$$

9. Independence of Neutrosophic Components vs Dependence of Intuitionistic Fuzzy Components

Section 4, equations (46) - (51) in Atanassov's and Vassiliev's paper [6], is reproduced below:

"4. Interval-valued intuitionistic fuzzy sets, intuitionistic fuzzy sets, and neutrosophic fuzzy sets

(…) the concept of a Neutrosophic Fuzzy Set (NFS) is introduced, as follows:

$$A^n = \{x, \mu_A^n(x), \nu_A^n(x), \pi_A^n(x) | x \in E\} \tag{1}$$

where $\mu_A^n(x), \nu_A^n(x), \pi_A^n(x) \in [0,1]$, and have the same sense as IFS.

Let

$$\sup_{y\in E}\mu_A^n(y)+\sup_{y\in E}\nu_A^n(y)+\sup_{y\in E}\pi_A^n(y)\neq 0 \tag{2}$$

Then, we define:

$$\mu_A^i(x)=\frac{\mu_A^n(x)}{\sup_{y\in E}\mu_A^n(y)+\sup_{y\in E}\nu_A^n(y)+\sup_{y\in E}\pi_A^n(y)} \tag{3}$$

$$\nu_A^i(x)=\frac{\nu_A^n(x)}{\sup_{y\in E}\mu_A^n(y)+\sup_{y\in E}\nu_A^n(y)+\sup_{y\in E}\pi_A^n(y)} \tag{4}$$

$$\pi_A^i(x)=\frac{\pi_A^n(x)}{\sup_{y\in E}\mu_A^n(y)+\sup_{y\in E}\nu_A^n(y)+\sup_{y\in E}\pi_A^n(y)} \tag{5}$$

$$\iota_A^i(x)=1-\mu_A^i(x)-\nu_A^i(x)-\pi_A^i(x) \tag{6}$$

Using the neutrosophic component common notations, $T_A(x)\equiv\mu_A^n(x)$, $I_A(x)\equiv\pi_A^n(x)$, and $F_A(x)\equiv\nu_A^n(x)$, the refusal degree $R_A(x)$, and $A_N\equiv A^n$ for the neutrosophic set, and considering the triplet's order (T, I, F), with the universe of discourse $\mathcal{U}\equiv E$, we can re-write the above formulas as follows:

$$A_N=\{\langle x,T_A(x),I_A(x),F_A(x)\rangle|x\in\mathcal{U}\} \tag{7}$$

where $T_A(x),I_A(x),F_A(x)\in[0,1]$, for all $x\in\mathcal{U}$.

Neutrosophic *Fuzzy* Set is commonly named *Single-Valued* Neutrosophic Set (SVNS), i.e. the components are single-valued numbers.

The authors, Atanassov and Vassiliev, assert that $T_A(x),I_A(x),F_A(x)$ "have the same sense as IFS" (Intuitionistic Fuzzy Set).

But this is untrue since in IFS one has $T_A(x)+I_A(x)+F_A(x)\leq 1$, therefore the IFS components $T_A(x),I_A(x),T_A(x)$ are dependent, while in SVNS (Single-Valued Neutrosophic Set), one has $T_A(x)+I_A(x)+F_A(x)\leq 3$, what the authors omit to mention, therefore the SVNS components $T_A(x),I_A(x),F_A(x)$ are independent, and this makes a big difference, as we will see below.

In general, for the *dependent components*, if one component's value changes, the other components values also change (in order for their total sum to keep being up to 1). While for the *independent components*, if one component changes, the other components do not need to change since their total sum is always up to 3.

Let us re-write the equations (2) - (6) from authors' paper:

Assume

$$\sup_{y\in\mathcal{U}}T_A(y)+\sup_{y\in\mathcal{U}}I_A(y)+\sup_{y\in\mathcal{U}}F_A(y)\neq 0 \tag{8}$$

The authors have defined:

$$T_A^{IIFS}(x)=\frac{T_A(x)}{\sup_{y\in\mathcal{U}}T_A(y)+\sup_{y\in\mathcal{U}}I_A(y)+\sup_{y\in\mathcal{U}}F_A(y)} \tag{9}$$

$$F_A^{IIFS}(x)=\frac{F_A(x)}{\sup_{y\in\mathcal{U}}T_A(y)+\sup_{y\in\mathcal{U}}I_A(y)+\sup_{y\in\mathcal{U}}F_A(y)} \tag{10}$$

$$I_A^{IIFS}(x) = \frac{I_A(x)}{\sup_{y\in\mathcal{U}} T_A(y) + \sup_{y\in\mathcal{U}} I_A(y) + \sup_{y\in\mathcal{U}} F_A(y)} \tag{11}$$

These mathematical transfigurations, which transform [change in form] the neutrosophic components $T_A(x), I_A(x), F_A(x) \in [0,1]$, whose sum $T_A(x) + I_A(x) + F_A(x) \leq 3$, into inconsistent intuitionistic fuzzy components: $T_A^{IIFS}(x)$, $I_A^{IIFS}(x)$, $F_A^{IIFS}(x) \in [0,1]$, whose sum $T_A^{IIFS}(x) + I_A^{IIFS}(x) + F_A^{IIFS}(x) \leq 1$, and the refusal degree

$$R_A{}^{IIFS}(x) = 1 - T_A{}^{IIFS}(x) - I_A{}^{IIFS}(x) - F_A{}^{IIFS}(x) \in [0,1] \tag{12}$$

distort the original application, i.e. the original neutrosophic application and its intuitionistic fuzzy transformed application are not equivalent, see below.

This is because, in this case, *the change in form brings a change in content*.

10. By Transforming the Neutrosophic Components into Intuitionistic Fuzzy Components the Independence of the Neutrosophic Components is Lost

In reference paper [6], Section 4, Atanassov and Vassilev, convert the neutrosophic components into intuitionistic fuzzy components.

But, converting a single-valued neutrosophic triplet (T_1, I_1, F_1), with $T_1, I_1, F_1 \in [0,1]$ and

$T_1 + I_1 + F_1 \leq 3$ that occurs into a neutrosophic application α_N, to a single-valued intuitionistic triplet (T_2, I_2, F_2), with $T_2, I_2, F_2 \in [0,1]$ and $T_2 + F_2 \leq 1$ (or $T_2 + I_2 + F_2 = 1$) that would occur into an intuitionistic fuzzy application α_{IF}, is just a *mathematical artefact*, and there could be constructed many such mathematical operators [the authors present four of them], even more: it is possible to convert from the sum $T_1 + I_1 + F_1 \leq 3$ to the sum

$T_2 + I_2 + F_2$ equals to any positive number – but they are just *abstract transformations*.

The neutrosophic application α_N will not be equivalent to the resulting intuitionistic fuzzy application α_{IF}, since while in α_N the neutrosophic components T_1, I_1, F_1 are independent (because their sum is up to 3), in α_{IF} the intuitionistic fuzzy components T_2, I_2, F_2 are dependent (because their sum is 1). Therefore, the independence of components is lost.

Moreover, the *independence* of the neutrosophic components is the main distinction between neutrosophic set vs intuitionistic fuzzy set.

Therefore, the resulted in the intuitionistic fuzzy application α_{IF} after the mathematical transformation is just a sub-application (particular case) of the original neutrosophic application α_N.

11. Degree of Dependence/Independence between the Components

The degree of dependence/independence between components was introduced by Smarandache [22] in 2006. In general, the sum of two components x and y that vary in the unitary interval $[0,1]$ is:

$$0 \leq x + y \leq 2 - d(x,y),$$

where $d(x,y)$ is the degree of dependence between x and y, while $1 - d(x,y)$ is the degree of independence between x and y.

NS is also flexible because it handles, besides independent components, also partially independent and partially dependent components, while IFS cannot deal with these.

For example, if T and F are dependent, then $0 \leq T + F \leq 1$, while if component I is independent of them, thus $0 \leq I \leq 1$, then $0 \leq T + I + F \leq 2$. Therefore, the components T, I, F, in general, are partially dependent and partially independent.

12. Intuitionistic Fuzzy Operators Ignore the Indeterminacy, while Neutrosophic Operators Give Indeterminacy the Same Weight as to Truth-Membership and Falsehood-Nonmembership

Indeterminacy in intuitionistic fuzzy set is ignored by the intuitionistic fuzzy aggregation operators, while the neutrosophic aggregation operators treat the indeterminacy at the same weight as the other two neutrosophic components (truth-membership and falsehood-membership).

Thus, even if we have two single-valued triplets, with the sum of each three components equal to 1 {therefore triplet that may be treated both as *intuitionistic fuzzy triplet*, and *neutrosophic triplet* in the same time (since in neutrosophic environment the sum of the neutrosophic components can be any number between 0 and 3, whence, in particular, we may take the sum 1)}, after applying the intuitionistic fuzzy aggregation operators we get a different result from that obtained after applying the neutrosophic aggregation operators.

13. Intuitionistic Fuzzy Operators and Neutrosophic Operators

Let the intuitionistic fuzzy operators be denoted as negation ($\neg_{IF}$), intersection ($\wedge_{IF}$), union ($\vee_{IF}$), and implication ($\rightarrow_{IF}$), and the neutrosophic operators [complement, intersection, union, and implication respectively] be denoted as negation ($\neg_N$), intersection ($\wedge_N$), union ($\vee_N$), and implication ($\rightarrow_N$).

Let $A_1 = (a_1, b_1, c_1)$ and $A_2 = (a_2, b_2, c_2)$ be two triplets such that $a_1, b_1, c_1, a_2, b_2, c_2 \in [0,1]$ and

$$a_1 + b_1 + c_1 = a_2 + b_2 + c_2 = 1.$$

The intuitionistic fuzzy operators and neutrosophic operators are based on *fuzzy t-norm* ($\wedge_F$) and *fuzzy t-conorm* ($\vee_F$). We will take for this article the simplest ones:

$$a_1 \wedge_F a_2 = min\{a_1, a_2\} \text{ and } a_1 \vee_F a_2 = max\{a_1, a_2\},$$

where $\wedge_F$ is the fuzzy intersection (*t-norm*) and $\vee_F$ is the fuzzy union (*t-conorm*).

For the intuitionistic fuzzy implication and neutrosophic implication, we extend the classical implication:

$$A_1 \rightarrow A_2 \text{ that is classically equivalent to } \neg A_1 \vee A_2,$$

where $\rightarrow$ is the classical implication, $\neg$ the classical negation (complement), and $\vee$ the classical union, to the intuitionistic fuzzy environment and respectively to the neutrosophic environment.

However, taking other fuzzy *t-norm* and fuzzy *t-conorm*, the conclusion will be the same, i.e. the results of intuitionistic fuzzy aggregation operators are different from the results of neutrosophic aggregation operators applied on the same triplets.

Intuitionistic Fuzzy Aggregation Operators {the simplest used intuitionistic fuzzy operations}:

Intuitionistic Fuzzy Negation:

$$\neg_{IF}(a_1, b_1, c_1) = (c_1, b_1, a_1)$$

Intuitionistic Fuzzy Intersection:

$$(a_1, b_1, c_1) \wedge_{IF} (a_2, b_2, c_2) = (\min\{a_1, a_2\}, 1 - \min\{a_1, a_2\} - \max\{c_1, c_2\}, \max\{c_1, c_2\})$$

Intuitionistic Fuzzy Union:

$$(a_1, b_1, c_1) \vee_{IF} (a_2, b_2, c_2) = (\max\{a_1, a_2\}, 1 - \max\{a_1, a_2\} - \min\{c_1, c_2\}, \min\{c_1, c_2\})$$

Intuitionistic Fuzzy Implication:

$(a_1, b_1, c_1) \rightarrow_{IF} (a_2, b_2, c_2)$ is intuitionistically fuzzy equivalent to $\neg_{IF}(a_1, b_1, c_1) \vee_{IF} (a_2, b_2, c_2)$

Neutrosophic Aggregation Operators {the simplest used neutrosophic operations}:

Neutrosophic Negation:

$$\neg_N(a_1, b_1, c_1) = (c_1, 1 - b_1, a_1)$$

Neutrosophic Intersection:

$$(a_1, b_1, c_1) \wedge_N (a_2, b_2, c_2) = (\min\{a_1, a_2\}, \max\{b_1, b_2\}, \max\{c_1, c_2\})$$

Neutrosophic Union:

$$(a_1, b_1, c_1) \vee_N (a_2, b_2, c_2) = (\max\{a_1, a_2\}, \min\{b_1, b_2\}, \min\{c_1, c_2\})$$

Neutrosophic Implication:

$(a_1, b_1, c_1) \rightarrow_N (a_2, b_2, c_2)$ is neutrosophically equivalent to $\neg_N(a_1, b_1, c_1) \vee_N (a_2, b_2, c_2)$.

14. Numerical Example of Triplet Components whose Summation is 1

Let $A_1 = (0.3, 0.6, 0.1)$ and $A_2 = (0.4, 0.1, 0.5)$ be two triplets, each having the sum:

$$0.3 + 0.6 + 0.1 = 0.4 + 0.1 + 0.5 = 1.$$

Therefore, they can both be treated as neutrosophic triplets and as intuitionistic fuzzy triplets simultaneously. We apply both, the intuitionistic fuzzy operators and then the neutrosophic operators and we prove that we get different results, especially with respect with Indeterminacy component that is ignored by the intuitionistic fuzzy operators.

14.1 Complement/Negation

Intuitionistic Fuzzy:

$\neg_{IF}(0.3, 0.6, 0.1) = (0.1, 0.6, 0.3)$, and $\neg_{IF}(0.4, 0.1, 0.5) = (0.5, 0.1, 0.4)$.

Neutrosophic:

$\neg_N(0.3, 0.6, 0.1) = (0.1, 1 - 0.6, 0.3) = (0.1, 0.4, 0.3) \neq (0.1, 0.6, 0.3)$, and

$\neg_N(0.4, 0.1, 0.5) = (0.5, 1 - 0.1, 0.4) = (0.5, 0.9, 0.4) \neq (0.5, 0.1, 0.4)$.

14.2 Intersection

Intuitionistic Fuzzy

$$(0.3,0.6,0.1) \wedge_{IF} (0.4,0.1,0.5) = (\min\{0.3,0.4\}, 1 - \min\{0.3,0.4\} - \max\{0.1,0.5\}, \max\{0.1,0.5\}) = (0.3,0.2,0.5)$$

As we see, the indeterminacies 0.6 of A_1 and 0.1 of A_2 were completely ignored into the above calculations, which is unfair. Herein, the resulting indeterminacy from the intersection is just what is left from truth-membership and falsehood-nonmembership $(1 - 0.3 - 0.5 = 0.2)$.

Neutrosophic

$$(0.3,0.6,0.1) \wedge_N (0.4,0.1,0.5) = (\min\{0.3,0.4\}, \max\{0.6,0.1\}, \max\{0.1,0.5\}) = (0.3,0.6,0.5) \neq (0.3,0.2,0.5).$$

In the neutrosophic environment the indeterminacies 0.6 of A_1 and 0.1 of A_2 are given full consideration in calculating the resulting intersection's indeterminacy: $\max\{0.6, 0.1\} = 0.6$.

14.3 Union

Intuitionistic Fuzzy:

$$(0.3,0.6,0.1) \vee_{IF} (0.4,0.1,0.5) = (\max\{0.3,0.4\}, 1 - \max\{0.3,0.4\} - \min\{0.1,0.5\}, \max\{0.1,0.5\}) = (0.4,0.5,0.1)$$

Again, the indeterminacies 0.6 of A_1 and 0.1 of A_2 were completely ignored into the above calculations, which is not fair. Herein, the resulting indeterminacy from the union is just what is left from truth-membership and falsehood-nonmembership $(1 - 0.4 - 0.1 = 0.5)$.

Neutrosophic:

$$(0.3,0.6,0.1) \vee_N (0.4,0.1,0.5) = (\max\{0.3,0.4\}, \min\{0.6,0.1\}, \min\{0.1,0.5\}) = (0.4,0.1,0.1) \neq (0.4,0.5,0.1)$$

Similarly, in the neutrosophic environment the indeterminacies 0.6 of A_1 and 0.1 of A_2 are given full consideration in calculating the resulting union's indeterminacy: $\min\{0.6, 0.1\} = 0.1$.

14.4 Implication

Intuitionistic Fuzzy

$$(0.3,0.6,0.1) \rightarrow_{IF} (0.4,0.1,0.5) = \neg_{IF}(0.3,0.6,0.1) \vee_{IF} (0.4,0.1,0.5) = (0.1,0.6,0.3) \vee_{IF} (0.4,0.1,0.5) = (0.4,0.3,0.3)$$

Similarly, indeterminacies of A_1 and A_2 are completely ignored.

Neutrosophic

$$(0.3,0.6,0.1) \rightarrow_N (0.4,0.1,0.5) = \neg_N(0.3,0.6,0.1) \vee_N (0.4,0.1,0.5) = (0.1,0.4,0.3) \vee_N (0.4,0.1,0.5) = (0.4,0.1,0.3) \neq (0.4,0.3,0.3)$$

While in the neutrosophic environment the indeterminacies of A_1 and A_2 are taken into calculations.

14.5 Remark

We have proven that even when the sum of the triplet components is equal to 1, as demanded by the intuitionistic fuzzy environment, the results of the intuitionistic fuzzy operators are different from those of the neutrosophic operators – because the indeterminacy is ignored into the intuitionistic fuzzy operators.

15. Simple Counterexample 1, Showing Different Results between Neutrosophic Operators and Intuitionistic Fuzzy Operators Applied on the Same Sets (with component sums > 1 or < 1)

Let the universe of discourse $\mathcal{U} = \{x_1, x_2\}$ and two neutrosophic sets included in $\mathcal{U}$:

$$A_N = \{x_1(0.8, 0.3, 0.5), x_2(0.9, 0.2, 0.6)\}, \text{ and } B_N = \{x_1(0.2, 0.1, 0.3), x_2(0.6, 0.2, 0.1)\}$$

Whence, for A_N one has, after using Atanassov and Vassiliev's transformations (9) - (12):

$$T_A^{IIFS}(x_1) = \frac{0.8}{0.9+0.3+0.6} = \frac{0.8}{1.8} \approx 0.44,\ I_A^{IIFS}(x_1) = \frac{0.3}{1.8} \approx 0.17, \text{ and } F_A^{IIFS}(x_1) = \frac{0.5}{1.8} \approx 0.28$$

The refusal degree for x_1 concerning A_N is $R_A^{IIFS}(x_1) = 1 - 0.44 - 0.17 - 0.28 = 0.11$. Then,

$$T_A^{IIFS}(x_2) = \frac{0.9}{1.8} = 0.50,\ I_A^{IIFS}(x_2) = \frac{0.2}{1.8} \approx 0.11, \text{ and } F_A^{IIFS}(x_2) = \frac{0.6}{1.8} \approx 0.33$$

The refusal degree for x_2 concerning A_N is $R_A^{IIFS}(x_2) = 1 - 0.50 - 0.11 - 0.33 = 0.06$. Then,

$$A_{IIFS} = \{x_1(0.44, 0.17, 0.28), x_2(0.50, 0.11, 0.33)\}$$

For B_N one has:

$$T_B^{IIFS}(x_1) = \mu_B^i(x_1) = \frac{0.2}{0.6+0.2+0.3} = \frac{0.2}{1.1} \approx 0.18,\ I_B^{IIFS}(x_1) = \nu_B^i(x_1) = \frac{0.1}{1.1} \approx 0.09, \text{ and}$$

$$F_B^{IIFS}(x_1) = \pi_B^i(x_1) = \frac{0.3}{1.1} \approx 0.27$$

The refusal degree for x_1 concerning B_N is $R_B^{IIFS}(x_1) = 1 - 0.18 - 0.09 - 0.27 = 0.46$.

$$T_B^{IIFS}(x_2) = \frac{0.6}{1.1} \approx 0.55, \text{ and } I_B^{IIFS}(x_2) = \frac{0.2}{1.1} \approx 0.18, \text{ and } F_B^{IIFS}(x_2) = \frac{0.1}{1.1} \approx 0.09$$

The refusal degree for x_2 concerning the set B_N is $R_B^{IIFS}(x_2) = 1 - 0.55 - 0.18 - 0.09 = 0.18$. Therefore:

$$B_{IIFS} = \{x_1, (0.18, 0.09, 0.27), x_2(0.55, 0.18, 0.09)\}$$

Therefore, the neutrosophic sets:

$A_N = \{x_1(0.8, 0.3, 0.5), x_2(0.9, 0.2, 0.6)\}$ and $B_N = \{x_1(0.2, 0.1, 0.3), x_2(0.6, 0.2, 0.1)\}$,

where transformed (restricted), using Atanassov and Vassiliev's transformations (3)-(6), into inconsistent intuitionistic fuzzy sets respectively as follows:

$A_{IIFS}^{(t)} = \{x_1(0.44, 0.17, 0.28), x_2(0.50, 0.11, 0.33)\}$ and $B_{IIFS}^{(t)} = \{x_1(0.18, 0.09, 0.27), x_2(0.55, 0.18, 0.09)\}$

where the upper script (t) means "after Atanassov and Vassiliev's transformations".

We shall remark that the set B_N, as neutrosophic set (where the sum of the components is also allowed to be strictly less than 1 as well), happens to be in the same time an inconsistent intuitionistic fuzzy set, or $B_N \equiv B_{IIFS}$.

Therefore, B_N transformed into $B_{IIFS}^{(t)}$ was a distortion of B_N, since we got different IIFS components:

$$x_1^{B_N}(0.2,0.1,0.3) \equiv x_1^{B_{IIFS}}(0.2,0.1,0.3) \neq x_1^{B_{IIFS}^{(t)}}(0.18,0.09,0.27)$$

Similarly:

$$x_2^{B_N}(0.6,0.2,0.1) \equiv x_2^{B_{IIFS}}(0.6,0.2,0.1) \neq x_2^{B_{IIFS}^{(t)}}(0.55,0.18,0.09)$$

Further on, we show that the NS operators and IIFS operators, applied on these sets, give *different results*. For each individual set operation (intersection, union, complement/negation, inclusion/implication, and equality/equivalence) there exist classes of operators, not a single one. We choose the simplest one in each case, which is based on min/max (fuzzy *t-norm* / fuzzy *t-conorm*).

15.1 Intersection

Neutrosophic Sets (min/max/max)

$$x_1^A \wedge_N x_1^B = (0.8,0.3,0.5) \wedge_N (0.2, 0.1, 0.3) = (\min\{0.8, 0.2\}, \max\{0.3, 0.1\}, \max\{0.5, 0.3\}) = (0.2, 0.3, 0.5)$$

$$x_2^A \wedge_N x_2^B = (0.9,0.2,0.6) \wedge_N (0.6, 0.2, 0.1) = (0.6, 0.2, 0.6)$$

Therefore:

$$A_N \wedge_N B_N = \{x_1(0.2, 0.3, 0.5), x_2(0.6, 0.2, 0.6)\} \stackrel{\text{def}}{=} C_N$$

Inconsistent Intuitionistic Fuzzy Set (min / max / max)

$$x_1^A \wedge_{IIFS} x_1^B = (0.44,0.17,0.28) \wedge_{IIFS} (0.18, 0.09, 0.27) = (\min\{0.44, 0.18\}, \max\{0.17, 0.09\}, \max\{0.28, 0.27\}) = (0.18, 0.17, 0.28)$$

$$x_2^A \wedge_{IIFS} x_2^B = (0.50,0.11,0.33) \wedge_{IIFS} (0.55, 0.18, 0.09) = (0.50, 0.18, 0.33)$$

Since in IIFS the sum of components is not allowed to surpass 1, we normalise:

$$\left(\frac{0.50}{1.01}, \frac{0.11}{1.01}, \frac{0.33}{1.01}\right) \approx (0.495, 0.109, 0.326)$$

Therefore:

$$A_{IIFS} \wedge B_{IIFS} = \{x_1(0.18, 0.17, 0.28), x_2(0.495, 0.109, 0.326)\} \stackrel{\text{def}}{=} C_{IIFS}$$

Also: $T_{A_N \wedge_N B_N}(x_1) = 0.2 < 0.3 = I_{A_N \wedge_N B_N}(x_1)$,

while $T_{A_{IIFS} \wedge_{IIFS} B_{IIFS}}(x_1) = 0.18 > 0.17 = I_{A_{IIFS} \wedge_{IIFS} B_{IIFS}}(x_1)$,

and other discrepancies can be seen.

Inconsistent Intuitionistic Fuzzy Set (with min/min/max, as used by Cuong [20] in order to avoid the sum of components surpassing 1; but this is in discrepancy with the IIFS/PFS union that uses max/min/min, not max/max/min):

$$x_1^A \wedge_{IIFS2} x_1^B = (0.44, 0.17, 0.28) \wedge_{IIFS2} (0.18, 0.09, 0.27) = (\min\{0.44, 0.18\}, \min\{0.17, 0.09\}, \max\{0.28, 0.27\}) = (0.18, 0.09, 0.28)$$

$$x_2^A \wedge_{IIFS2} x_2^B = (0.50, 0.11, 0.33) \wedge_{IIFS2} (0.55, 0.18, 0.09) = (0.50, 0.11, 0.33)$$

Therefore:

$$A_{IIFS} \wedge_{IIFS2} B_{IIFS} = \{x_1(0.18, 0.09, 0.28), x_2(0.50, 0.11, 0.33)\} \stackrel{\text{def}}{=} C_{IIFS2}$$

We see that:

$$A_N \wedge_N B_N \neq A_{IIFS} \wedge_{IIFS} B_{IIFS}, \text{ or } C_N \neq C_{IIFS}$$

and $A_N \wedge_N B_N \neq A_{IIFS} \wedge_{IIFS2} B_{IIFS}$, $C_N \neq C_{IIFS2}$. Also $C_{IIFS} \neq C_{IIFS2}$.

Let us transform the above neutrosophic set C_N, resulting from the application of the neutrosophic intersection operator,

$C_N = \{x_1(0.2, 0.3, 0.5), x_2(0.6, 0.2, 0.6)\}$,

into an inconsistent intuitionistic fuzzy set, employing the same equations (3)-(5) of transformations [denoted by (t)], provided by Atanassov and Vassiliev, which are equivalent {using (T, I, F)-notations} to (9)-(11)

$$(t)T_C^{IIFS}(x_1) = \frac{0.2}{0.6+0.3+0.6} = \frac{0.2}{1.5} \simeq 0.13,\ (t)I_C^{IIFS}(x_1) = \frac{0.3}{1.5} = 0.20,\ (t)F_C^{IIFS}(x_1) = \frac{0.5}{1.5} \simeq 0.33,$$

$$(t)T_C^{IIFS}(x_2) = \frac{0.6}{1.5} \simeq 0.40,\ (t)I_C^{IIFS}(x_2) = \frac{0.2}{1.5} \simeq 0.13, \text{ and } (t)F_C^{IIFS}(x_2) = \frac{0.6}{1.5} \simeq 0.40$$

Whence the results of neutrosophic and IIFS/PFS are different:

$$C_{IIFS}^{(t)} = \{x_1(0.13, 0.20, 0.33), x_2(0.40, 0.13, 0.40)\} \neq \{x_1(0.18, 0.17, 0.28), x_2(0.495, 0.109, 0.326)\} \equiv C_{IIFS}$$

and

$$C_{IIFS}^{(t)} \neq \{x_1(0.18, 0.09, 0.28), x_2(0.50, 0.11, 0.33)\} = C_{IIFS2}$$

15.2 Union

Neutrosophic Sets (max / min / min)

$$x_1^A \vee_N x_1^B = (0.8, 0.3, 0.5) \vee_N (0.2, 0.1, 0.3) = (\max\{0.8, 0.2\}, \min\{0.3, 0.1\}, \min\{0.5, 0.3\}) = (0.8, 0.1, 0.3)$$

$$x_2^A \vee_N x_2^B = (0.9, 0.2, 0.6) \vee_N (0.6, 0.2, 0.1) = (0.9, 0.2, 0.1)$$

Therefore:

$$A_N \vee_N B_N = \{x_1(0.8, 0.1, 0.3), x_2(0.9, 0.2, 0.1)\} \stackrel{\text{def}}{=} D_N$$

Inconsistent Intuitionistic Fuzzy Sets (max / min / min [3])

$$x_1^A \vee_{IIFS} x_1^B = (0.44, 0.17, 0.28) \vee_{IIFS} (0.18, 0.09, 0.27) = (\max\{0.44, 0.18\}, \min\{0.17, 0.09\}, \min\{0.28, 0.27\}) = (0.44, 0.09, 0.27)$$

$$x_2^A \vee_{IIFS} x_2^B = (0.50, 0.11, 0.33) \vee_{IIFS} (0.55, 0.18, 0.09) = (0.55, 0.11, 0.09)$$

Therefore:

$$A_{IIFS} \vee_{IIFS} B_{IIFS} = \{x_1(0.44, 0.09, 0.27), x_2(0.55, 0.11, 0.09)\} \stackrel{\text{def}}{=} D_{IIFS}$$

a) We see that the results are different: $A_N \vee_N B_N \neq A_{IIFS} \vee_{IIFS} B_{IIFS}$, or $D_N \neq D_{IIFS}$.

b) Let us transform the above neutrosophic set, D_N, resulting from the application of neutrosophic union operator, $D_N = \{x_1(0.8, 0.1, 0.3), x_2(0.9, 0.2, 0.1)\}$, into an inconsistent intuitionistic fuzzy set, employing the same equations (3)-(5) of transformation [denoted by (t)], provided by Atanassov and Vassiliev, which are equivalent [using (T, I, F) notations] to (9)-(11):

$$(t)T_D^{IIFS}(x_1) = \frac{0.8}{0.9+0.2+0.3} = \frac{0.8}{1.4} \simeq 0.57,\ (t)I_D^{IIFS}(x_1) = \frac{0.1}{1.4} \simeq 0.07,\ (t)F_D^{IIFS}(x_1) = \frac{0.3}{1.4} \simeq 0.21$$

$$(t)T_D^{IIFS}(x_2) = \frac{0.9}{1.4} \simeq 0.64,\ (t)I_D^{IIFS}(x_2) = \frac{0.2}{1.4} \simeq 0.14, \text{ and } (t)F_D^{IIFS}(x_2) = \frac{0.1}{1.4} \simeq 0.07$$

Whence:

$$D_{IIFS}^{(t)} = \{x_1(0.57, 0.07, 0.21), x_2(0.64, 0.14, 0.07)\} \neq \{x_1(0.44, 0.09, 0.27), x_2(0.55, 0.11, 0.09)\} \equiv D_{IIFS}$$

The results again are different.

15.3 Corollary

Therefore, no matter if we first transform the neutrosophic components into inconsistent intuitionistic fuzzy components (as suggested by Atanassov and Vassiliev) and then apply the IIFS operators, or we first apply the neutrosophic operators on neutrosophic components, and then later transform the result into IIFS components, in both ways the obtained results in the neutrosophic environment are different from the results obtained in the IIFS environment.

16. Normalisation

Further on, the authors propose the normalisation of the neutrosophic components, where Atanassov and Vassiliev's [6] equations (57)-(59) are equivalent, using neutrosophic notations, to the following.

Let $\mathcal{U}$ be a universe of discourse, a set $A \subseteq \mathcal{U}$, and a generic element $x \in \mathcal{U}$, with the neutrosophic components:

$x\big(T_A(x), I_A(x), F_A(x)\big)$, where $T_A(x), I_A(x), F_A(x) \in [0,1]$, and $T_A(x) + I_A(x) + F_A(x) \leq 3$, for all $x \in U$.

Suppose $T_A(x) + I_A(x) + F_A(x) \neq 0$, for all $x \in U$. Then, by the below normalisation of neutrosophic components, Atanassov and Vassiliev obtain the following intuitionistic fuzzy components $\left(T_A^{IFS}, I_A^{IFS}, F_A^{IFS}\right)$:

$$T_A^{IFS}(x) = \frac{T_A(x)}{T_A(x) + I_A(x) + F_A(x)} \in [0,1] \tag{13}$$

$$I_A^{IFS}(x) = \frac{I_A(x)}{T_A(x) + I_A(x) + F_A(x)} \in [0,1] \tag{14}$$

$$F_A^{IFS}(x) = \frac{F_A(x)}{T_A(x) + I_A(x) + F_A(x)} \in [0,1] \tag{15}$$

and $T_A^{IFS}(x) + I_A^{IFS}(x) + F_A^{IFS}(x) = 1$, for all $x \in U$.

16.1 Counterexample 2

Let us come back to the previous *Counterexample 1*.

$\mathcal{U} = \{x_1, x_2\}$ be a universe of discourse, and let two neutrosophic sets included in $\mathcal{U}$:

$A_N = \{x_1(0.8, 0.3, 0.5), x_2(0.9, 0.2, 0.6)\}$ and $B_N = \{x_1(0.2, 0.1, 0.3), x_2(0.6, 0.2, 0.1)\}$.

Let us normalise their neutrosophic components, as proposed by Atanassov and Vassiliev, in order to restrain them to intuitionistic fuzzy components:

$$A_{IFS} = \left\{x_1\left(\frac{0.8}{0.8+0.3+0.5}, \frac{0.3}{1.6}, \frac{0.5}{1.6}\right), x_2\left(\frac{0.9}{1.7}, \frac{0.2}{1.7}, \frac{0.6}{1.7}\right)\right\} \approx \{x_1(0.50, 0.19, 0.31), x_2(0.53, 0.12, 0.35)\} \equiv \{x_1(0.50, 0.31), x_2(0.53, 0.35)\}$$

since the indeterminacy (called *hesitant degree* in IFS) is neglected.

$$B_{IFS} = \left\{x_1\left(\frac{0.2}{0.6}, \frac{0.1}{0.6}, \frac{0.3}{0.6}\right), x_2\left(\frac{0.6}{0.9}, \frac{0.2}{0.9}, \frac{0.1}{0.9}\right)\right\} \approx \{x_1(0.33, 0.17, 0.50), x_2(0.67, 0.22, 0.11)\} \equiv \{x_1(0.33, 0.50), x_2(0.67, 0.11)\}$$

since the indeterminacy (hesitance degree) is again neglected.

The intuitionistic fuzzy operators are applied only on truth-membership and false-nonmembership (but not on indeterminacy).

16.1.1 Intersection

Intuitionistic Fuzzy Intersection (min / max)

$$x_1^A \wedge_{IFS} x_1^B = (0.50, 0.31) \wedge_{IFS} (0.33, 0.50) = (\min\{0.50, 0.33\}, \max\{0.31, 0.50\}) = (0.33, 0.50) = (0.33, 0.17, 0.50)$$

after adding the indeterminacy which is what is left up to 1, i.e. $1 - 0.33 - 0.50 = 0.17$.

$$x_2^A \wedge_{IFS} x_2^B = (0.53, 0.35) \wedge_{IFS} (0.67, 0.11) = (\min\{0.53, 0.63\}, \max\{0.35, 0.11\}) = (0.53, 0.35) = (0.53, 0.12, 0.35)$$

after adding the indeterminacy.

The results of NS and IFS intersections are very different:

$$A_N \wedge_N B_N = \{x_1(0.2, 0.3, 0.5), x_2(0.6, 0.2, 0.6)\} \neq \{x_1(0.33, 0.17, 0.50), x_2(0.53, 0.12, 0.35)\} = A_{IFS} \wedge_{IFS} B_{IFS}$$

Even more distinction, between the NS intersection and IFS intersection of the same elements (whose sums of components equal 1) $x_1^A = (0.50, 0.19, 0.31)$ and $x_1^B = (0.33, 0.17, 0.50)$ one obtains unequal results, using the (min / max / max) operator:

$$x_1^A \wedge_N x_1^B = (0.50, 0.19, 0.31) \wedge_N (0.33, 0.17, 0.50) = (0.33, 0.19, 0.50)$$

while

$$x_1^A \wedge_{IFS} x_1^B = (0.50, 0.19, 0.31) \wedge_{IFS} (0.33, 0.17, 0.50)$$

$$\equiv (0.50, 0.31) \wedge_{IFS} (0.33, 0.50) \text{ \{after ignoring the indeterminacy in IFS\}}$$

$$= (0.33, 0.50) \equiv (0.33, 0.17, 0.50) \neq (0.33, 0.19, 0.50)$$

16.1.2 Union

Intuitionistic Fuzzy Union (max/min/min)

$$x_1^A \vee_{IFS} x_1^B = (0.50, 0.31) \vee_{IFS} (0.33, 0.50) = (\max\{0.50, 0.33\}, \min\{0.31, 0.50\}) = (0.50, 0.31) \equiv (0.50, 0.19, 0.31)$$

after adding the indeterminacy.

$$x_2^A \vee_{IFS} x_2^B = (0.53, 0.35) \vee_{IFS} (0.67, 0.11) = (\max\{0.53, 0.67\}, \min\{0.35, 0.11\}) = (0.67, 0.11) \equiv (0.67, 0.22, 0.11)$$

after adding the indeterminacy.

The results of NS and IFS unions are very different:

$$A_N \vee_N B_N = \{x_1(0.8, 0.1, 0.3), x_2(0.9, 0.2, 0.1)\} \neq \{x_1(0.50, 0.19, 0.31), x_2(0.67, 0.22, 0.11)\} = A_{IFS} \vee_{IFS} B_{IFS}$$

Even more distinction, for the NS and IFS union of the same elements:

$$x_1^A \vee_N x_1^B = (0.50, 0.19, 0.31) \vee_N (0.33, 0.17, 0.50) = (0.50, 0.17, 0.31)$$

While

$$x_1^A \vee_{IFS} x_1^B = (0.50, 0.19, 0.31) \vee_{IFS} (0.33, 0.17, 0.50)$$

$$\equiv (0.50, 0.31) \vee_{IFS} (0.33, 0.50)$$

$$= (0.50, 0.31) \equiv (0.50, 0.19, 0.31) \text{ \{after adding indeterminacy\}}$$

$$\neq (0.50, 0.17, 0.31)$$

17. Indeterminacy Makes a Big Difference between NS and IFS

The authors [6] assert that “Therefore, the NFS can be *represented* by an IFS” (page 5), but this is not correct since it should be:

The NFS (neutrosophic fuzzy set ≡ single-valued neutrosophic set) can be *restrained* (degraded) to an IFS (intuitionistic fuzzy set), yet the independence of components is lost, and the results of the aggregation operators are different between the neutrosophic environment and intuitionistic fuzzy environment since IFS operators ignore indeterminacy.

Since in single-valued neutrosophic set the neutrosophic components are *independent* (their sum can be up to 3, and if a component increases or decreases, it does not change the others), while in intuitionistic fuzzy set the components are *dependent* (in general if one changes, one or both the other components change in order to keep their sum equal to 1). Also, applying the neutrosophic operators is a better aggregation since the indeterminacy (I) is involved into all neutrosophic (complement/negation, intersection, union, inclusion/inequality/implication, equality/equivalence) operators while all intuitionistic fuzzy operators *ignore* (do not take into the calculation) the indeterminacy.

That is why the results after applying the neutrosophic operators and intuitionistic fuzzy operators on the same sets are *different* as proven above).

18. The Intuitionistic Fuzzy Logic Cannot Represent Paradoxes

No previous set/logic theories, including IFS or Intuitionistic Fuzzy Logic (IFL), since the sum of components were not allowed above 1, could characterise a paradox, which is a true proposition ($T = 1$) and false ($F = 1$) simultaneously; therefore the paradox is 100% indeterminate ($I = 1$). In Neutrosophic Logic (NL), a paradoxical proposition P_{NL} is represented as $P_{NL}(1, 1, 1)$.

If one uses Atanassov and Vassiliev’s transformations (for example the normalisation) [1], we get $P_{IFL}\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$, however, this one cannot represent a paradox, since a paradox is 100% true and 100% false, not 33% true and 33% false.

19. Atanassov’s Intuitionistic Fuzzy Set of Second Type, Also Called Pythagorean Fuzzy Set

Single-Valued Atanassov’s Intuitionistic Fuzzy Sets of the second type (IFS2) [23], also called Single-Valued Pythagorean Fuzzy Set (PyFS) [24], is defined as follows (using T, I, F notations for the components):

Definition of IFS2 (PyFS).

It is a set $A_{IFS2} \equiv A_{PyFS}$ from the universe of discourse U such that:

$$A_{IFS2} \equiv A_{PyFS} = \{\langle x, T_A(x), F_A(x) \rangle | x \in U\},$$

where, for all $x \in U$, the functions $T_A(x), F_A(x): U \rightarrow [0, 1]$, represent the degree of membership (truth) and degree on nonmembership (falsity) respectively, that satisfy the conditions:

$$0 \leq T_A^2(x) + F_A^2(x) \leq 1,$$

whence the hesitancy degree is:

$$I_A(x) = \sqrt{1 - T_A^2(x) - F_A^2(x)} \in [0,1].$$

20. The Components of Atanassov's Intuitionistic Fuzzy Set of Second Type (Pythagorean Fuzzy Set) are not Independent

Princy R and Mohana K assert in [23] that: "The truth and falsity values and hesitancy value can be independently considered as membership and non-membership and hesitancy degrees respectively". However, this is untrue, since in IFS2 (PyFS) the components are not independent, because they are connected (dependent on each other) through this inequality:

$$T_A^2(x) + F_A^2(x) \leq 1.$$

21. Counterexample 3

If $T = 0.9$, then $T^2 = 0.92 = 0.81$, whence $F^2 \leq 1 - T^2 = 1 - 0.81 = 0.19$, or $F \leq \sqrt{0.19} \approx 0.44$.

Therefore, if $T = 0.9$, then F is restricted to be less than equal to $\sqrt{0.19}$.

While in NS if $T = 0.9$, F can be equal to any number in $[0, 1]$, F can be even equal to 1.

Also, hesitancy degree depends on T and F, because the formula of hesitancy degree is an equation depending on T and F, as below:

$$I_A(x) = \sqrt{1 - T_A^2(x) - F_A^2(x)} \in [0,1].$$

If $T = 0.9$ and $F = 0.2$, then hesitancy

$$I = \sqrt{1 - 0.9^2 - 0.2^2} = \sqrt{0.15} \approx 0.39.$$

Again, in NS if $T = 0.9$ and $F = 0.2$, I can be equal to any number in $[0, 1]$, not only to $\sqrt{0.15}$.

22. Neutrosophic Set is a Generalization of Pythagorean Fuzzy Set

In the definition of PyFS, one has $T_A(x), F_A(x) \in [0, 1]$, which involves that $T_A^2(x), F_A^2(x) \in [0, 1]$ too; we denote $T_A^{NS}(x) = T_A^2(x), F_A^{NS}(x) = F_A^2(x)$, and $I_A^{NS}(x) = I_A^2(x) = 1 - T_A^2(x) - F_A^2(x) \in [0,1]$, where "NS" stands for Neutrosophic Set.

Therefore, one gets:

$$T_A^{NS}(x) + I_A^{NS}(x) + F_A^{NS}(x) = 1,$$

which is a particular case of the neutrosophic set, since in NS the sum of the components can be any number between 0 and 3, hence into PyFS has been chosen the sum of the components be equal to 1.

23. Spherical Fuzzy Set (SFS)

Definition of Spherical Fuzzy Set.

A Single-Valued Spherical Fuzzy Set (SFS) [25-26], of the universe of discourse U, is defined as follows:

$$A_{SFS} = \{\langle x, T_A(x), I_A(x), F_A(x)\rangle | x \in U\},$$

where, for all $x \in U$, the functions $T_A(x), I_A(x), F_A(x): U \to [0, 1]$, represent the degree of membership (truth), the degree of hesitancy, and degree on nonmembership (falsity) respectively, that satisfy the conditions:

$$0 \leq T_A^2(x) + I_A^2(x) + F_A^2(x) \leq 1,$$

whence the refusal degree is:

$$R_A(x) = \sqrt{1 - T_A^2(x) - I_A^2(x) - F_A^2(x)} \in [0,1].$$

24. The Components of the Spherical Fuzzy Set are not Independent

Princy R and Mohana K assert in [23] that:

"In spherical fuzzy sets, while the squared sum of membership, non-membership and hesitancy parameters can be between 0 and 1, each of them can be defined between 0 and 1 independently."

However, this is again, untrue.

25. Counterexample 4

If $T = 0.9$ then F cannot be for example equal to 0.8, since $0.9^2 + 0.8^2 = 1.45 > 1$, but the sum of the squares of components is not allowed to be greater than 1.

So F depends on T in this example.

Two components are independent if no matter what value gets one component will not affect the other component's value.

26. Neutrosophic Set is a Generalization of the Spherical Fuzzy Set

In [25], Gündoğdu and Kahraman assert about:

"superiority of SFS [i.e. Spherical Fuzzy Set] concerning Pythagorean, intuitionistic fuzzy and neutrosophic sets";

also:

"SFSs are a generalisation of Pythagorean Fuzzy Sets (PFS) and neutrosophic sets".

While it is *true* that the spherical fuzzy set is a generalisation of Pythagorean fuzzy set and intuitionistic fuzzy set, it is *false* that spherical fuzzy set is a generalisation of the neutrosophic set.

It is the opposite: neutrosophic set is a generalisation of spherical fuzzy set. We prove it bellow.

In the definition of the spherical fuzzy set, one has: $T_A(x), I_A(x), F_A(x) \in [0,1]$, which involves that $T_A^2(x), I_A^2(x), F_A^2(x) \in [0,1]$ too.

Let us denote: $T_A^{NS}(x) = T_A^2(x), I_A^{NS}(x) = I_A^2(x), F_A^{NS}(x) = F_A^2(x)$, where "NS" stands for neutrosophic set, whence we obtain, using SFS definition:

$$0 \leq T_A^{NS}(x) + I_A^{NS}(x) + F_A^{NS}(x) \leq 1,$$

which is a particular case of the single-valued neutrosophic set, where the sum of the components T, I, F can be any number between 0 and 3. So now we can choose the sum up to 1.

As a **counterexample**, if we choose $T_A(x) = 0.9$, $I_A(x) = 0.4$, $F_A(x) = 0.5$, for some given element x, which are neutrosophic components, they are not spherical fuzzy set components because $0.9^2 + 0.4^2 + 0.5^2 = 1.22 > 1$.

The elements of a spherical fuzzy set form a 1/8 of a sphere of radius 1, centred into the origin $O = (0,0,0)$ of the Cartesian system of coordinates, on the positive $ox(T)$, $oy(I)$, $oz(F)$ axes. While the standard neutrosophic set is a cube of side 1, that has the vertexes: (0,0,0), (1,0,0), (0,1,0), (0,0,1), (1,1,0), (1,0,1), (0,1,1), (1,1,1).

The neutrosophic cube strictly includes the 1/8 fuzzy sphere.

27. Spherical Neutrosophic Set is Also a Generalization of Spherical Fuzzy Set

Spherical Neutrosophic Set (SNS) was introduced by Smarandache [27] in 2017.

Definition of Spherical Neutrosophic Set.

A Single-Valued Spherical Neutrosophic Set (SNS), of the universe of discourse U, is defined as follows:

$$A_{SNS} = \{\langle x, T_A(x), I_A(x), F_A(x)\rangle | x \in U\},$$

where, for all $x \in U$, the functions $T_A(x), I_A(x), F_A(x): U \to [0, \sqrt{3}]$, represent the degree of membership (truth), the degree of indeterminacy, and degree on nonmembership (falsity) respectively, that satisfy the conditions:

$$0 \leq T_A^2(x) + I_A^2(x) + F_A^2(x) \leq 3.$$

The Spherical Neutrosophic Set is a generalisation of Spherical Fuzzy Set, because we may restrain the SNS's components to the unit interval $T_A(x), I_A(x), F_A(x) \in [0, 1]$, and the sum of the squared components to 1, i.e. $0 \leq T_A^2(x) + I_A^2(x) + F_A^2(x) \leq 1$.

Further on, if replacing $I_A(x) = 0$ into the Spherical Fuzzy Set; we obtain as a particular case the Pythagorean Fuzzy Set.

28. n-Hyper Spherical Neutrosophic Set

Definition of n-Hyper Spherical Neutrosophic Set.

We introduce now for the first time the Single-Valued **n-Hyper Spherical Neutrosophic Set** (n-HSNS), which is a generalisation of the Spherical Neutrosophic Set, of the universe of discourse U, for $n \geq 1$, is defined as follows:

$$A_{n-HNS} = \{\langle x, T_A(x), I_A(x), F_A(x)\rangle | x \in U\},$$

where, for all $x \in U$, the functions $T_A(x), I_A(x), F_A(x): U \to [0, \sqrt[n]{3}]$, represent the degree of membership (truth), the degree of indeterminacy, and degree on nonmembership (falsity) respectively, that satisfy the conditions:

$$0 \leq T_A^n(x) + I_A^n(x) + F_A^n(x) \leq 3.$$

29. Neutrosophic Set is a Generalization of q-Rung Orthopair Fuzzy Set (q-ROFS)

Definition of q-Rung Orthopair Fuzzy Set.

Using the same T, I, F notations, one has as follows.

A Single-Valued **q-Rung Orthopair Fuzzy Set (q-ROFS)** [28], of the universe of discourse U, for a given real number $q \geq 1$, is defined as follows:

$$A_{q-ROFS} = \{\langle x, T_A(x), F_A(x)\rangle | x \in U\},$$

where, for all $x \in U$, the functions $T_A(x), F_A(x): U \to [0, 1]$, represent the degree of membership (truth), and degree on nonmembership (falsity) respectively, that satisfy the conditions:

$$0 \leq T_A(x)^q + F_A(x)^q \leq 1.$$

Since $T_A(x), F_A(x) \in [0, 1]$, then for any real number, $q \geq 1$ one has $T_A(x)^q, F_A(x)^q \in [0,1]$ too.

Let us denote: $T_A^{NS}(x) = T_A(x)^q, F_A^{NS}(x) = F_A(x)^q$, whence it results in that: $0 \leq T_A^{NS}(x) + F_A^{NS}(x) \leq 1$, where what is left may be Indeterminacy.

However, this is a particular case of the neutrosophic set, where the sum of components T, I, F can be any number between 0 and 3, and for q-ROFS is it taken to be up to 1. Therefore, any Single-Valued q-Rung Orthopair Fuzzy Set is also a Neutrosophic Set, but the reciprocal is not true. See next counterexample.

30. Counterexample 5

Let us consider a real number $1 \leq q < \infty$, and a set of single-valued triplets of the form (T, I, F), with T, I, $F \in [0, 1]$ that represent the components of the elements of a given set. The components of the form (1, F), with $F > 0$, and of the form (T, 1), with $T > 0$, constitute NS components as follows: (1, I, F), with $F > 0$ and any $I \in [0, 1]$, and respectively (T, I, 1), with $T > 0$ and any $I \in [0, 1]$, since the sum of the components is allowed to be greater than 1, i.e. $1 + I + F > 1$ and respectively $T + I + 1 > 1$.

However, they cannot be components of the elements of a q-ROFS set, since: $1^q + F^q = 1 + F^q > 1$, because $F > 0$ and $1 \leq q < \infty$; but in q-ROFS the sum has to be ≤ 1.

Similarly, $T^q + 1^q = T^q + 1 > 1$, because $T > 0$ and $1 \leq q < \infty$; but in q-ROFS the sum has to be ≤ 1.

31. Regret Theory is a Neutrosophication Model

Regret Theory (2010) [29] is a Neutrosophication (1998) Model when the decision-making area is split into three parts, the opposite ones (upper approximation area, and lower approximation area) and the neutral one (border area, in between the upper and lower area).

32. Grey System Theory as a Neutrosophication

A Grey System [30] is referring to a *grey area* (as $< neutA >$ in neutrosophy), between extremes (as $< A >$ and $< antiA >$ in neutrosophy). According to the Grey System Theory, a system with perfect information ($< A >$) may have a unique solution, while a system with no information ($< antiA >$) has no solution. In the middle ($< neutA >$), or a grey area, of these opposite systems, there may be many available solutions (with partial information known and partial information unknown) from which an approximate solution can be extracted.

33. Three-Ways Decision as Particular Cases of Neutrosophication and of Neutrosophic Probability [31-36]

33.1 Neutrosophication

Let $< A >$ be an attribute value, $< antiA >$ the opposite of this attribute value, and $< neutA >$ the neutral (or indeterminate) attribute value between the opposites $< A >$ and $< antiA >$.

For examples: $< A >$=*big*, then $< antiA >$=*small*, and $< neutA >$=*medium*; we may rewrite:

i. ($< A >, < neutA >, < antiA >$) = (big, medium, small);

ii. or ($< A >, < neutA >, < antiA >$) = *(truth (*denoted as T*), indeterminacy (*denoted as I*), falsehood (*denoted as F*))* as in Neutrosophic Logic,

iii. or ($< A >, < neutA >, < antiA >$) = *(membership, indeterminate-membership, nonmembership)* as in Neutrosophic Set,

iv. or ($< A >, < neutA >, < antiA >$) = *(chance that an event occurs, indeterminate-chance that the event occurs or not, chance that the event does not occur)* as in Neutrosophic Probability,

and so on.

Moreover, let us by "Concept" to mean: an item, object, idea, theory, region, universe, set, notion etc. this attribute characterises that.

The process of neutrosophication means:

a) converting a *Classical Concept*

{denoted as ($1_{<A>}$, $0_{<neutA>}$, $0_{<antiA>}$)-*Classical Concept*, or *Classical Concept* ($1_{<A>}$, $0_{<neutA>}$, $0_{<antiA>}$)}, which means that the concept is, concerning the above attribute,

$100\% < A >$, $0\% < neutA >$, and $0\% < antiA >$,

into a *Neutrosophic Concept*

{denoted as ($T_{<A>}$, $I_{<neutA>}$, $F_{<antiA>}$)-*Neutrosophic Concept*, or *Neutrosophic Concept* ($T_{<A>}$, $I_{<neutA>}$, $\boldsymbol{F_{<antiA>}}$)}, which means that the concept is, concerning the above attribute,

$T\% < A >$, $I\% < neutA >$, and $F\% < antiA >$, which more accurately reflects our imperfect, non-idealistic reality,

where all T, I, F are subsets of $[0, 1]$ with no other restriction;

b) or converting a *Fuzzy Concept*, or *Intuitionistic Fuzzy Concept* into a *Neutrosophic Concept*;

c) or converting other Concepts such as *Inconsistent Intuitionistic Fuzzy (Picture Fuzzy) Concept*, or *Pythagorean Fuzzy Concept*, or *Spherical Fuzzy Concept*, or *q-Rung Orthopair Fuzzy* etc. into a *Neutrosophic Concept* or a *Refined Neutrosophic Concept* (i.e. $T_1\% < A_1 >$, $T_2\% < A_2 >$, …; $I_1\% < neutA_1 >$, $I_2\% < neutA_2 >$, …; and $F_1\% < antiA_1 >$, $F_2\% < antiA_2 >$, …), where all T_1, T_2, …; I_1, I_2, …; F_1, F_2, … are subsets of $[0, 1]$ with no other restriction.

d) or converting a *crisp real number* (r) into a *neutrosophic real number* of the form $r = a + bI$, where "I" means (literal or numerical) indeterminacy, a and b are real numbers, and "a" represents the determinate part of the crisp real number r, while bI the indeterminate part of r;

e) or converting a *crisp, complex number* (c) into a *neutrosophic complex number* of the form $c = a_1 + b_1 i$ $+(a_2 + b_2 i)I = a_1 + a_2 I + (b_1 + b_2 I)i$, where "$I$" means (literal or numerical) indeterminacy, $i = \sqrt{-1}$, with a_1, a_2, b_1, b_2 real numbers, and "$a_1 + b_1 i$" represents the determinate part of the complex real number c, while $a_2 + b_2 i$ the indeterminate part of c;

(we may also interpret that as: a_1 is the determinate part of the real-part of c, and b_1 is the determinate part of the imaginary-part of c; while a_2 is the indeterminate part of the real-part of c, and b_2 is the indeterminate part of the imaginary-part of c);

f) converting a *crisp*, *fuzzy*, or *intuitionistic fuzzy*, or *inconsistent intuitionistic fuzzy (picture fuzzy)*, or *Pythagorean fuzzy*, or *spherical fuzzy*, or *q-rung orthopair fuzzy number* and other numbers into a *quadruple neutrosophic number* of the form $a + bT + cI + dF$, where a, b, c, d are real or complex numbers, while T, I, F are the neutrosophic components.

While the process of **deneutrosophication** means going backwards concerning any of the above processes of neutrosophication.

Example 1.

Let the attribute $< A >$=cold temperature, then $< antiA >$=hot temperature, and $< neutA >$=medium temperature.

Let the concept be a country M, such that its northern part (30% of country's territory) is cold, its southern part is hot (50%), and in the middle, there is a buffer zone with medium temperature (20%). We write:

$M(0.3_{cold\ temperature}, 0.2_{medium\ temperature}, 0.5_{hot\ temperature})$

where we took single-valued numbers for the neutrosophic components $T_M = 0.3$, $I_M = 0.2$, $F_M = 0.5$ and the neutrosophic components are considered dependent, so their sum is equal to 1.

33.2 Three-Ways Decision is a Particular Case of Neutrosophication

Neutrosophy (based on $< A >$, $< neutA >$, $< antiA >$) was proposed by Smarandache [1] in 1998, and Three-Ways Decision by Yao [31] in 2009.

In Three-Ways Decision, the universe set is split into three different distinct areas, regarding the decision process, representing:

Acceptance, *Noncommitment*, and *Rejection*, respectively.

In this case, the decision attribute value $< A >$=Acceptance, whence $< neutA >$= Noncommitment, and $< antiA >$= Rejection.

The classical concept=*UniverseSet*.

Therefore, we got the *NeutrosophicConcept* $(T_{<A>}, I_{<neutA>}, F_{<antiA>})$, denoted as:

$$UniverseSet(T_{Acceptance}, I_{Noncommitment}, F_{Rejection}),$$

where $T_{Acceptance}$ *=universe set's zone of acceptance,* $I_{Noncommitment}$ *=universe set's zone of noncomitment (indeterminacy),* $F_{Rejection}$ *=universe set's zone of rejection.*

33.3 Three-Ways Decision as a Particular Case of Neutrosophic Probability

Let us consider the event, deciding on a universe set.

According to Neutrosophic Probability (NP) [1, 11] one has:

NP(decision) = (the universe set's elements for which the chance of the decision may be accepted; the universe set's elements for which there may be an indeterminate-chance of the decision; the universe set's elements for which the chance of the decision may be rejected).

33.4 Refined Neutrosophy

Refined Neutrosophy was introduced by Smarandache [9] in 2013, and it is described as follows:

i. $< A >$ is refined (split) into subcomponents $< A_1 >, < A_2 >, \ldots, < A_p >;$
ii. $< neutA >$ is refined (split) into subcomponents $< neutA_1 >, < neutA_2 >, \ldots, < neutA_r >;$
iii. and $< antiA >$ is refined (split) into subcomponents $< antiA_1 >, < antiA_2 >, \ldots, < antiA_s >;$

where $p, r, s \geq 1$ are integers, and $p + r + s \geq 4$.

Example 2.

If $< A > =$ voting in-country M, them $< A_1 > =$ voting in Region 1 of country M for a given candidate, $< A_2 > =$ voting in Region 2 of country M for a given candidate, and so on.

Similarly, $< neutA_1 >=$ not voting (or casting a white or a black vote) in Region 1 of country M, $< A_2 >=$ not voting in Region 2 of country M, and so on.

And $< antiA_1 > =$ voting in Region 1 of country M against the given candidate, $< A_2 > =$ voting in Region 2 of country M against the given candidate, and so on.

33.5 Extension of Three-Ways Decision to n-Ways Decision

n-Way Decision was introduced by Smarandache in 2019.

In n-Ways Decision, the universe set is split into $n \geq 4$ different distinct areas, regarding the decision process, representing:

Levels of Acceptance, *Levels of Noncommitment,* and *Levels of Rejection,* respectively.

Levels of Acceptance may be: Very High Level of Acceptance ($< A_1 >$), High Level of Acceptance ($< A_2 >$), Medium Level of Acceptance ($< A_3 >$), etc.

Similarly, Levels of Noncommitment may be: Very High Level of Noncommitment ($< neutA_1 >$), High Level of Noncommitment ($< neutA_2 >$), Medium Level of Noncommitment ($< neutA_3 >$), etc.

And Levels of Rejection may be: Very High Level of Rejection ($< antiA_1 >$), High Level of Rejection ($< antiA_2 >$), Medium Level of Rejection ($< antiA_3 >$), etc.

Then, the *Refined Neutrosophic Concept*

{denoted as $(\boldsymbol{T1_{<A1>}, T2_{<A2>}, \ldots, Tp_{<Ap>}; I1_{<neutA1>}, I2_{<neutA2>}, \ldots, Ir_{<neutAr>}; F1_{<antiA1>}, F2_{<antiA2>}, \ldots, Fs_{<antiAs>}})$***-RefinedNeutrosophicConcept***, or ***RefinedNeutrosophicConcept*** $(\boldsymbol{T1_{<A1>}, T2_{<A2>}, \ldots, Tp_{<Ap>}; I1_{<neutA1>}, I2_{<neutA2>}, \ldots, Ir_{<neutAr>}; F1_{<antiA1>}, F2_{<antiA2>}, \ldots, Fs_{<antiAs>}})$},

which means that the concept is, concerning the above attribute value levels,

$$T1\% < A1 >, T2\% < A2 >, \ldots, Tp\% < Ap >;$$

$$I1\% < neutA1 >, I2\% < neutA2 >, \ldots, Ir\% < neutAr >;$$

$$F1\% < antiA1 >, F2\% < antiA2 >, \ldots, Fs\% < antiAs >;$$

which more accurately reflects our imperfect, non-idealistic reality,

with where $p, r, s \geq 1$ are integers, and $p + r + s \geq 4$,

where all $T1$, $T2$, …, Tp, $I1$, $I2$, …, Ir, $F1$, $F2$, …, Fs are subsets of $[0, 1]$ with no other restriction.

34. Many More Distinctions between Neutrosophic Set (NS) and Intuitionistic Fuzzy Set (IFS) and other Type Sets [37]

34.1 Neutrosophic Set can distinguish between absolute and relative

i. *absolute membership* (i.e. membership in all possible worlds; we have extended Leibniz's absolute truth to absolute membership), and

ii. *relative membership* (membership in at least one world, but not in all), because NS (absolute membership element)$= 1^+$, while

iii. NS (relative membership element)$= 1$.

This has application in philosophy (see the *neutrosophy*). That is why the unitary standard interval $[0, 1]$ used in IFS has been extended to the unitary non-standard interval $]^-0, 1^+[$ in NS.

Similar distinctions for *absolute* or *relative non-membership* and *absolute* or *relative indeterminate appurtenance* are allowed in NS.

While IFS cannot distinguish the absoluteness from relativeness of the components.

34.2 In NS, there is no restriction on T, I, F other than they are subsets of $]^-0, 1^+[$. Thus: $^-0 \leq \inf T + \inf I + \inf F \leq \sup T + \sup I + \sup F \leq 3^+$.

The inequalities (2.1) and (2.4) [17] of IFS are relaxed in NS.

This non-restriction allows paraconsistent, dialetheist, and incomplete information to be characterised in NS {i.e. the sum of all three components, if they are defined as points, or sum of superior limits of all three components if they are defined as subsets, can be > 1 (for paraconsistent information coming from different sources), or < 1 for incomplete information}, while that information cannot be described in IFS because in IFS the components T (membership), I (indeterminacy), F (non-membership) are restricted either to $t + i + f = 1$ or to $t^2 + f^2 \leq 1$, if T, I, F are all reduced to the points (single-valued numbers) t, i, f respectively, or $supT + supI + supF = 1$ if T, I, F are subsets of $[0, 1]$. Of course, there are cases when paraconsistent and incomplete information can be normalised to 1, but this procedure is not always suitable.

In IFS paraconsistent, dialetheist, and incomplete information cannot be characterised.

This most important distinction between IFS and NS is showed in the below **Neutrosophic Cube** A'B'C'D'E'F'G'H' introduced by Dezert [38] in 2002.

Because in technical applications only the classical interval [0,1] is used as range for the neutrosophic parameters t, i, f, we call the cube $ABCDEDGH$ the **technical neutrosophic cube** and its extension $A'B'C'D'E'D'G'H'$ the **neutrosophic cube** (or **nonstandard neutrosophic cube**), used in the fields where we need to differentiate between *absolute* and *relative* (as in philosophy) notions.

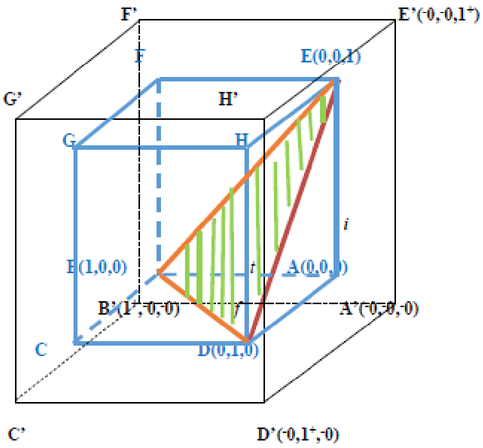

Fig. 1. Neutrosophic Cube

Let us consider a 3D Cartesian system of coordinates, where t is the truth axis with value range in $]^{-}0, 1^{+}[$, f is the false axis with value range in $]^{-}0, 1^{+}[$, and similarly, i is the indeterminate axis with value range in $]^{-}0, 1^{+}[$.

We now divide the technical neutrosophic cube $ABCDEDGH$ into three disjoint regions:

a) The shaded equilateral triangle BDE, whose sides are equal to $\sqrt{2}$, which represents the geometrical locus of the points whose sum of the coordinates is 1.

If a point Q is situated on the sides or inside of the triangle BDE, then $t_Q + i_Q + f_Q = 1$ as in Atanassov-intuitionistic fuzzy set $(A - IFS)$.

IFS triangle is a restriction of (strictly included in) the NS cube.

b) The pyramid $EABD$ {situated in the right side of the ΔEBD, including its faces ΔABD (base), ΔEBA, and ΔEDA (lateral faces), but excluding its face ΔBDE} is the locus of the points whose sum of coordinates is less than 1.

If $P \in EABD$ then $t_P + i_P + f_P < 1$ as in inconsistent intuitionistic fuzzy set (with incomplete information).

c) In the left side of ΔBDE in the cube, there is the solid $EFGCDEBD$ (excluding ΔBDE) which is the locus of points whose sum of their coordinates is greater than 1 as in the paraconsistent set.

If a point $R \in EFGCDEBD$, then $t_R + i_R + f_R > 1$.

It is possible to get the sum of coordinates strictly less than 1 or strictly greater than 1. For example, having three independent sources of information:

- We have a source which is capable of finding only the degree of membership of an element, but it is unable to find the degree of non-membership;
- Another source which is capable of finding only the degree of non-membership of an element;
- Or a source which only computes the indeterminacy.

Thus, when we put the results together of these sources, it is possible that their sum is not 1, but smaller or greater.

Also, in information fusion, when dealing with indeterminate models (i.e. elements of the fusion space which are indeterminate/unknown, such as intersections we do not know if they are empty or not since we do not have enough information, similarly for complements of indeterminate elements, etc.): if we compute the believe in that element (truth), the disbelieve in that element (falsehood), and the indeterminacy part of that element, then the sum of these three components is strictly less than 1 (the difference to 1 is the missing information).

34.3 Relation (2.3) from interval-valued intuitionistic fuzzy set is relaxed in NS, i.e. the intervals do not necessarily belong to $Int[0,1]$ but to [0,1], even more, general to $]^{-}0, 1^{+}[$.

34.4 In NS the components T, I, F can also be ***nonstandard subsets*** included in the unitary non-standard interval $]^{-}0, 1^{+}[$, not only *standard* subsets included in the unitary standard interval $[0, 1]$ as in IFS.

34.5 NS, like dialetheism, can describe **paradoxist elements**, NS (paradoxist element) = (1, 1, 1), while IFL cannot describe a paradox because the sum of components should be 1 in IFS.

34.6 The connectors/operators in IFS are defined concerning T and F only, i.e. membership and nonmembership only (hence the Indeterminacy is what is left from 1), while in NS they can be defined concerning any of them (no restriction).

However, for interval-valued intuitionistic fuzzy set, one cannot find any left indeterminacy.

34.7 Component "I", indeterminacy, can be split into more subcomponents in order to better catch the vague information we work with, and such, for example, one can get more accurate answers to the *Question-Answering Systems* initiated by Zadeh (2003).

{In Belnap's four-valued logic (1977) indeterminacy is split into Uncertainty (U) and Contradiction (C), but they were interrelated.}

Even more, one can split "I" into Contradiction, Uncertainty, and Unknown, and we get a five-valued logic.

In a general Refined Neutrosophic Logic, T can be split into subcomponents T_1, T_2, ..., T_p, and I into I_1, I_2, ..., I_r, and F into F_1, F_2, ..., F_s, where $p, r, s \geq 1$ and $p + r + s = n \geq 3$. Even more: T, I, and/or F (or any of their subcomponents T_j, I_k, and/or F_l) can be countable or uncountable infinite sets.

34.8 Indeterminacy is independent of membership/truth and non-membership/falsehood in NS/Nl, while in IFS/IFL it is not.

In neutrosophics there are two types of indeterminacies:

a) *Numerical Indeterminacy (*or *Degree of Indeterminacy)*, which has the form $(t, i, f) \neq (1, 0, 0)$, where t, i, f are numbers, intervals, or subsets included in the unit interval $[0, 1]$, and it is the base for the (t, i, f)-Neutrosophic Structures.

b) *Non-numerical Indeterminacy (or Literal Indeterminacy)*, which is the letter "I" standing for unknown (non-determinate), such that $I^2 = I$, and used in the composition of the neutrosophic number $N = a + bI$, where a and b are real or complex numbers, and a is the determinate part of number N, while bI is the indeterminate part of N. The neutrosophic numbers are the base for the I-Neutrosophic Structures.

34.9 NS has a better and clear terminology (name) as "neutrosophic" (which means the neutral part: i.e. neither true/membership nor false/nonmembership), while IFS's name "intuitionistic" produces confusion with Intuitionistic Logic, which is something different (see the article by Didier Dubois et al. [39], 2005).

34.10 The Neutrosophic Set was extended [8] to **Neutrosophic Overset** (when some neutrosophic component is > 1), and to **Neutrosophic Underset** (when some neutrosophic component is < 0), and to and to **Neutrosophic Offset** (when some neutrosophic components are off the interval $[0, 1]$, i.e. some neutrosophic component > 1 and some neutrosophic component < 0). In IFS the degree of a component is not allowed to be outside of the classical interval $[0, 1]$.

This is no surprise concerning the classical fuzzy set/logic, intuitionistic fuzzy set/logic, or classical and imprecise probability where the values are not allowed outside the interval $[0, 1]$, since our real-world has numerous examples and applications of over/under/off neutrosophic components.

Example: In a given company, a full-time employer works 40 hours per week. Let us consider the last week period. Helen worked part-time, only 30 hours, and the other 10 hours she was absent without payment; hence, her membership degree was $\frac{30}{40} = 0.75 < 1$.

John worked full-time, 40 hours, so he had the membership degree $\frac{40}{40} = 1$, concerning this company. But George worked 5 hours overtime, so his membership degree was $\frac{40+5}{40} = \frac{45}{40} = 1.125 > 1$. Thus, we need to make a distinction between employees who work overtime and those who work full-time or part-time. That is why we need to associate a degree of membership greater than 1 to the overtime workers.

Now, another employee, Jane, was absent without pay for the whole week, so her degree of membership was $\frac{0}{40} = 0$.

However, Richard, who was also hired as a full-time, not only did not come to work last week at all (0 worked hours), but he produced, by accidentally starting a devastating fire, much damage to the company, which was estimated at a value half of his salary (i.e. as he would have gotten for working 20 hours). Therefore, his membership degree has to be less than Jane's (since Jane produced no damage). Whence, Richard's degree of membership concerning this company was $-\frac{20}{40} = -0.50 < 0$.

Therefore, the membership degrees > 1 and < 0 are real in our world, so we have to consider them.

Then, similarly, the Neutrosophic Logic/Measure/Probability/Statistics etc. were extended to respectively Neutrosophic Over/Under/Off Logic, Measure, Probability, Statistics etc. [8].

34.11 Neutrosophic Tripolar (and in general **Multipolar**) **Set** and **Logic** [8] of the form:

$$(< T_1^+, T_2^+, \ldots, T_n^+; T^0; T_{-n}^-, \ldots, T_{-2}^-, T_{-1}^- >, < I_1^+, I_2^+, \ldots, I_n^+; I^0; I_{-n}^-, \ldots, I_{-2}^-, I_{-1}^- >,$$
$$< F_1^+, F_2^+, \ldots, F_n^+; F^0; F_{-n}^-, \ldots, F_{-2}^-, F_{-1}^- >)$$

where we have multiple positive/neutral/negative degrees of T, I, and F, respectively.

34.12 The **Neutrosophic Numbers** have been introduced by W.B. Vasantha Kandasamy and F. Smarandache [40] in 2003, which are numbers of the form $N = a + bI$, where a, b are real or complex numbers, while "I" is the indeterminacy part of the neutrosophic number N, such that $I^2 = I$ and $\alpha I + \beta I = (\alpha + \beta)I$.

Of course, indeterminacy "I" is different from the imaginary unit $i = \sqrt{-1}$.

In general, one has $I^n = I$ if $n > 0$, and it is undefined if $n \leq 0$

34.13 Also, **Neutrosophic Refined Numbers** were introduced [41] as:

$a + b_1 I_1 + b_2 I_2 + \cdots + b_m I_m$, where $a, b_1, b_2, \ldots, b_m$ are real or complex numbers, while the $I_1, I_2, \ldots, I_m$ are types of sub-indeterminacies, for $m \geq 1$.

34.14 The algebraic structures using neutrosophic numbers gave birth to the ***I*-Neutrosophic Algebraic Structures** [see for example "neutrosophic groups", "neutrosophic rings", "neutrosophic vector space", "neutrosophic matrices, bimatrices, …, n-matrices", etc.], introduced by W.B. Vasantha Kandasamy, F. Smarandache [40] et al. since 2003.

Example of Neutrosophic Matrix: $\begin{bmatrix} 1 & 2+I & -5 \\ 0 & 1/3 & I \\ -1+4I & 6 & 5I \end{bmatrix}$.

Example of Neutrosophic Ring: ($\{a + bI$, with a, $b \in \mathbb{R}\}$, $+$, $\cdot$), where of course $(a + bI) + (c + dI) = (a + c) + (b + d)I$, and $(a + bI) \cdot (c + dI) = (ac) + (ad + bc + bd)I$.

34.15 Also, to **Refined *I*-Neutrosophic Algebraic Structures**, which are structures using sets of refined neutrosophic numbers [41].

34.16 Types of Neutrosophic Graphs (and Trees):

a-c) Indeterminacy "I" led to the definition of the **Neutrosophic Graphs** (graphs which have: either at least one indeterminate edge, or at least one indeterminate vertex, or both some indeterminate edge and some indeterminate vertex), and **Neutrosophic Trees** (trees which have: either at least one indeterminate edge, or at least one indeterminate vertex, or both some indeterminate edge and some indeterminate vertex), which have many applications in social sciences.

Another type of neutrosophic graph is when at least one edge has a neutrosophic (t, i, f) truth-value.

As a consequence, the Neutrosophic Cognitive Maps [40] and Neutrosophic Relational Maps [40] are generalisations of fuzzy cognitive maps and respectively fuzzy relational maps, Neutrosophic Relational Equations [40], Neutrosophic Relational Data [42], etc.

A Neutrosophic Cognitive Map is a neutrosophic directed graph with concepts like policies, events etc. as vertices, and causalities or indeterminates as edges. It represents the causal relationship between concepts.

An edge is said indeterminate if we do not know if it is any relationship between the vertices it connects, or for a directed graph we do not know if it is a directly or inversely proportional relationship. We may write for such edge that $(t, i, f) = (0, 1, 0)$.

A vertex is indeterminate if we do not know what kind of vertex it is since we have incomplete information. We may write for such vertex that $(t, i, f) = (0, 1, 0)$.

Example of Neutrosophic Graph (edges V_1V_3, V_1V_5, V_2V_3 are indeterminate, and they are drawn as dotted):

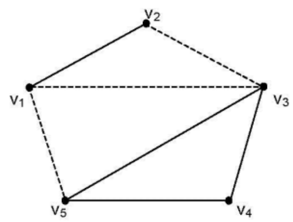

Fig. 2. Neutrosophic Graph {with I (indeterminate) edges}

Moreover, its neutrosophic adjacency matrix is:

$$\begin{bmatrix} 0 & 1 & I & 0 & I \\ 1 & 0 & I & 0 & 0 \\ I & I & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ I & 0 & 1 & 1 & 0 \end{bmatrix}$$

Fig. 3. Neutrosophic Adjacency Matrix of the Neutrosophic Graph

The edges mean 0 = no connection between vertices, 1 = connection between vertices, I = indeterminate connection (not known if it is, or if it is not).

Such notions are not used in the fuzzy theory.

Example of Neutrosophic Cognitive Map (NCM), which is a generalisation of the Fuzzy Cognitive Maps.

Let us have the following vertices:

C1 - Child Labor

C2 - Political Leaders

C3 - Good Teachers

C4 - Poverty

C5 - Industrialists

C6 - Public practising/encouraging Child Labor

C7 - Good Non-Governmental Organizations (NGOs)

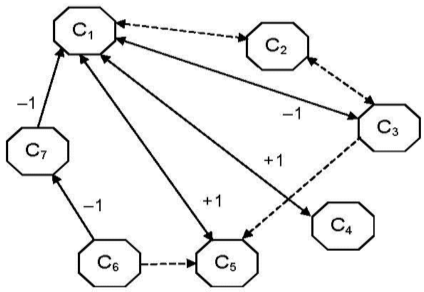

Fig. 4. Neutrosophic Cognitive Map

The corresponding neutrosophic adjacency matrix related to this neutrosophic cognitive map is:

$$\begin{bmatrix} 0 & I & -1 & 1 & 1 & 0 & 0 \\ I & 0 & I & 0 & 0 & 0 & 0 \\ -1 & I & 0 & 0 & I & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I & 0 & -1 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Fig. 5. Neutrosophic Adjacency Matrix of the Neutrosophic Cognitive Map

The edges mean: $0 =$ no connection between vertices, $1 =$ directly proportional connection, $-1 =$ inversely proportionally connection, and $I =$ indeterminate connection (not knowing what kind of relationship is between the vertices that the edge connects).

Such **literal indeterminacy** (letter ***I***) does not occur in previous set theories, including intuitionistic fuzzy set; they had only *numerical indeterminacy*.

d) Another type of neutrosophic graphs (and trees) [41]:

An edge of a graph, let us say from A to B (i.e. how A influences B), may have a neutrosophic value (t, i, f),

where t means the positive influence of A on B,

i means the indeterminate influence of A on B, and

f means the negative influence of A on B.

Then, if we have, let us say: A->B->C such that A->B has the neutrosophic value (t_1, i_1, f_1) and B->C has the neutrosophic value (t_2, i_2, f_2), then A->C has the neutrosophic value $(t_1, i_1, f_1) \wedge (t_2, i_2. f_2)$, where $\wedge$ is the AND neutrosophic operator.

e) Also, again a different type of graph: we can consider a vertex A as $t\%$ belonging/membership to the graph, $i\%$ indeterminate membership to the graph, and $f\%$ nonmembership to the graph.

f) Any of the previous types of graphs (or trees) put together.

g) **Tripolar (**and **Multipolar) Graph**, which is a graph whose vertexes or edges have the form $(< T^+, T^0, T^- >, < I^+, I^0, I^- >, < F^+, F^0, F^- >)$ and respectively: $(< T_j^+, T^0, T_j^- >, < I_j^+, I^0, I_j^- >, < F_j^+, F^0, F_j^- >)$.

34.17 The **Neutrosophic Probability** (NP), introduced in 1995, was extended and developed as a generalisation of the classical and imprecise probabilities [11]. NP of an event E is the chance that event E occurs, the chance that event E does not occur, and the chance of indeterminacy (not knowing if the event E occurs or not).

In classical probability $n_{sup} \leq 1$, while in neutrosophic probability $n_{sup} \leq 3^+$.

In imprecise probability: the probability of an event is a subset T in $[0, 1]$, not a number p in $[0, 1]$, what is left is supposed to be the opposite, subset F (also from the unit interval $[0, 1]$); there is no indeterminate subset I in imprecise probability.

In neutrosophic probability, one has, besides randomness, indeterminacy due to construction materials and shapes of the probability elements and space.

In consequence, neutrosophic probability deals with two types of variables: random variables and indeterminacy variables, and two types of processes: stochastic process and respectively indeterminate process.

34.18 And consequently the **Neutrosophic Statistics**, introduced in 1995 and developed in [12], which is the analysis of the neutrosophic events.

Neutrosophic Statistics means statistical analysis of population or sample that has indeterminate (imprecise, ambiguous, vague, incomplete, unknown) data. For example, the population or sample size might not be exactly determinate because of some individuals that partially belong to the population or sample, and partially they do not belong, or individuals whose appurtenance is completely unknown. Also, there are population or sample individuals whose data could be indeterminate. It is possible to define the neutrosophic statistics in many ways, because there are various types of indeterminacies, depending on the problem to solve.

Neutrosophic statistics deals with neutrosophic numbers, neutrosophic probability distribution, neutrosophic estimation, neutrosophic regression. The function that models the neutrosophic probability of a random variable x is called *neutrosophic distribution*: $NP(x) = (T(x), I(x), F(x))$, where $T(x)$ represents the probability that value x occurs, $F(x)$ represents the probability that value x does not occur, and $I(x)$ represents the indeterminate/unknown probability of value x.

34.19 Also, **Neutrosophic Measure** and **Neutrosophic Integral** were introduced [11].

34.20 Neutrosophy {Smarandache, since 1995 [7, 13, 14]} opened a new field in philosophy.

<u>Neutrosophy</u> is a new branch of philosophy that studies the origin, nature, and scope of neutralities, as well as their interactions with different ideational spectra.

This theory considers every notion or idea $< A >$ together with its opposite or negation $< Anti - A >$ and the spectrum of "neutralities" $< Neut - A >$ (i.e. notions or ideas located between the two extremes, supporting neither $< A >$ nor $< Anti - A >$). The $< Neut - A >$ and $< Anti - A >$ ideas together are referred to as $< Non - A >$.

According to this theory, every idea $< A >$ tends to be neutralised and balanced by $< Anti - A >$ and $< Non - A >$ ideas - as a state of equilibrium. Classically $< A >$, $< Neut - A >$, $< Anti - A >$ are disjoint two by two.

However, since in many cases the borders between notions are vague, imprecise, Sorites, it is possible that $< A >$, $< Neut - A >$, $< Anti - A >$ (and $< Non - A >$ of course) have common parts two by two as well.

Neutrosophy is the base of neutrosophic logic, neutrosophic set, neutrosophic probability and statistics used in engineering applications (especially for software and information fusion), medicine, military, cybernetics, physics.

We have extended dialectics (based on the opposites $< A >$ and $< antiA >$) to neutrosophy (based on $< A >$, $< antiA >$ and $< neutA >$.

34.21 In consequence, we extended the thesis-antithesis-synthesis to thesis-antithesis-neutrothesis-neutron synthesis [41].

34.22 Neutrosophy extended the Law of Included Middle to the **Law of Included Multiple-Middle** [10] in accordance with the n-valued refined neutrosophic logic.

34.23 Smarandache [41] introduced the **Neutrosophic Axiomatic System** and **Neutrosophic Deducibility**.

34.24 He then introduced the $(\boldsymbol{t}, \boldsymbol{i}, \boldsymbol{f})$**-Neutrosophic Structure** [41], which is a structure whose space, or at least one of its axioms (laws), has some indeterminacy of the form $(t, i, f) \neq (1, 0, 0)$.

Also, we defined the combined (t, i, f)-*I-Neutrosophic Algebraic Structures*, i.e. algebraic structures based on neutrosophic numbers of the form $a + bI$, but also having some indeterminacy [of the form $(t, i, f) \neq (1, 0, 0)$] related to the structure space (i.e. elements which only partially belong to the space or elements we know nothing if they belong to the space or not) or indeterminacy [of the form $(t, i, f) \neq (1, 0, 0)$] related to at least one axiom (or law) acting on the structure space).

Even more, we generalised them to *Refined* (t, i, f)*-Refined I-Neutrosophic Algebraic Structures*, or (t_j, i_k, f_l)-i_s*-Neutrosophic Algebraic Structures*; where t_j means that t has been refined to j subcomponents $t_1, t_2, \ldots, t_j$; similarly for i_k, f_l and respectively i_s.

34.25 Smarandache and Ali, in 2014-2016 [43, 44, 45], introduced the *Neutrosophic Triplet Structures*.

A ***Neutrosophic Triplet*** is a triplet of the form:

$$< a, neut(a), anti(a) >,$$

where $neut(a)$ is the neutral of a, i.e. an element (different from the identity element of the operation $*$) such that $a * neut(a) = neut(a) * a = a$, while $anti(a)$ is the opposite of a, i.e. an element such that $a * anti(a) = anti(a) * a = neut(a)$. Neutrosophy means not only indeterminacy but also neutral (i.e. neither true nor false). For example, we can have neutrosophic triplet semigroups, neutrosophic triplet loops, etc.

Further on Smarandache extended the neutrosophic triplet $< a, neut(a), anti(a) >$ to a ***m*-valued refined neutrosophic triplet**, in a similar way as it was done for $T_1, T_2, \ldots; I_1, I_2, \ldots; F_1, F_2, \ldots$ (i.e. the refinement of neutrosophic components). It will work in some cases, depending on the composition law $*$. It depends on each $*$ how many neutrals, and anti's there is for each element "a".

We may have an m-tuple concerning the element "a" in the following way:

$$\left(a;\ neut_1(a), neut_2(a), \ldots, neut_p(a);\ anti_1(a), anti_2(a), \ldots, anti_p(a)\right),$$

where $m = 1 + 2p$, such that:

- all $neut_1(a)$, $neut_2(a)$, ..., $neut_p(a)$ are distinct two by two, and each one is different from the unitary element concerning the composition law $*$;

- also:

$$\begin{aligned}
&a * neut_1(a) = neut_1(a) * a = a\\
&a * neut_2(a) = neut_2(a) * a = a\\
&\vdots\\
&a * neut_p(a) = neut_p(a) * a = a; \text{ and}\\
&a * anti_1(a) = anti_1(a) * a = neut_1(a)\\
&a * anti_2(a) = anti_2(a) * a = neut_2(a)\\
&\vdots\\
&a * anti_p(a) = anti_p(a) * a = neut_p(a);
\end{aligned}$$

- where all $anti_1(a)$, $anti_2(a)$, …, $anti_p(a)$ are distinct two by two, and in a case when there are duplicates, the duplicates are discarded.

34.26 As latest minute development, the crisp, fuzzy, intuitionistic fuzzy, picture fuzzy, and neutrosophic sets were extended by Smarandache [46] in 2017 to **plithogenic set**, which is:

A set P whose elements are characterised by many attributes' values. An attribute value v has a corresponding (fuzzy, intuitionistic fuzzy, picture fuzzy, or neutrosophic) degree of appurtenance $d(x, v)$ of the element x, to the set P, concerning some given criteria. In order to obtain a better accuracy for the *plithogenic aggregation operators* in the plithogenic set, and for a more exact inclusion (partial order), a (fuzzy, intuitionistic fuzzy, picture fuzzy, or neutrosophic) *contradiction (dissimilarity) degree* is defined between each attribute value and the dominant (most important) attribute value. The plithogenic intersection and union are *linear combinations of the fuzzy operators t-norm and t-conorm*, while the plithogenic complement (negation), inclusion (inequality), equality (equivalence) are influenced by the attribute values contradiction (dissimilarity) degrees.

35. Conclusion

In this paper, we proved that neutrosophic set is a generalisation of intuitionistic fuzzy set and inconsistent intuitionistic fuzzy set (picture fuzzy set).

By transforming (restraining) the neutrosophic components into intuitionistic fuzzy components, as Atanassov and Vassiliev proposed, the independence of the components is lost, and the intuitionistic fuzzy aggregation operators ignore the indeterminacy. Also, the result after applying the neutrosophic operators is different from the result obtained after applying the intuitionistic fuzzy operators (concerning the same problem to solve).

We presented many distinctions between neutrosophic set and intuitionistic fuzzy set, and we showed that neutrosophic set is more general and more flexible than previous set theories. Neutrosophy's applications in various fields such as neutrosophic probability, neutrosophic statistics, neutrosophic algebraic structures, and so on were also listed.

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A Note on Rhotrices Ring

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Abstract — In this paper, we define algebraic operations on 3-dimensional rhotrices over an arbitrary ring R and show that the set of 3-dimensional rhotrices over an arbitrary ring R is a ring according to these operations. We investigate the properties of a rhotrices ring. Furthermore, we characterize the ideals of a rhotrices ring. Also, maximal ideals and prime ideals of a rhotrices ring are investigated. An example of these concepts is presented.

Keywords — *Rhotrix, rhotrices ring, ideals of a rhotrices ring*

1. Introduction

The concept of the rhotrix is a mathematical structure in the rhombodial form of real numbers defined by Atanasov and Shannon [1], inspired by the concepts of matrix tertion and matrix netrion. In 2003, Ajibade [2] defined an object that lies between 2×2 dimensional matrices and 3×3 dimensional matrices called rhotrix as follows:

Definition 1.1. [2] Let a, b, c, d, e be real numbers. Then a mathematical rhombodial form

$$R = \left\langle \begin{array}{ccc} & a & \\ b & c & d \\ & e & \end{array} \right\rangle$$

is called $3-dimensional\ rhotrix$ over real numbers. The entry c in rhotrix R is called the heart of R denoted by $h(R)$.

The set of all $3-dimensional\ rhotrices$ is denoted by $\mathcal{R}$.

$$\mathcal{R} = \left\{ \left\langle \begin{array}{ccc} & a & \\ b & c & d \\ & e & \end{array} \right\rangle \middle| a, b, c, d, e \in \mathbb{R} \right\}$$

On operations over $\mathcal{R}$ are as follows:

Let $R = \left\langle \begin{array}{ccc} & a & \\ b & c & d \\ & e & \end{array} \right\rangle$ and $Q = \left\langle \begin{array}{ccc} & f & \\ g & h & j \\ & k & \end{array} \right\rangle$ be in $\mathcal{R}$. Then,

$$R = Q \Leftrightarrow a = f, b = g, c = h, d = j, e = k$$

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The addition of two rhotrices R and Q was defined as

$$R+Q=\left\langle \begin{array}{ccc} & a+f & \\ b+g & c+h & d+j \\ & e+k & \end{array} \right\rangle$$

It is reported in [3] that the set of all 3-dimensional rhotrices is a commutative group w.r.t $'+'$. This group is denoted by $\langle \mathcal{R},+\rangle$. The notion $(-R)$ was given as additional inverse of rhotrix R and was defined as follows:

$$-R=\left\langle \begin{array}{ccc} & -a & \\ -b & -c & -d \\ & -e & \end{array} \right\rangle$$

$e_{\mathcal{R}}=\left\langle \begin{array}{ccc} & 0_R & \\ 0_R & 0_R & 0_R \\ & 0_R & \end{array} \right\rangle$ was given identity element of rhotrices group $\mathcal{R}_3$. Let $\alpha \in R$ and $R \in \mathcal{R}$. The scalar multiplication of α and R was defined by

$$\alpha R=\left\langle \begin{array}{ccc} & \alpha a & \\ \alpha b & \alpha c & \alpha d \\ & \alpha e & \end{array} \right\rangle$$

Definition 1.2. Let $R=\left\langle \begin{array}{ccc} & a & \\ b & h(R) & d \\ & e & \end{array} \right\rangle$ and $Q=\left\langle \begin{array}{ccc} & f & \\ g & h(Q) & j \\ & k & \end{array} \right\rangle$ be in $\mathcal{R}$. The multiplication of R and Q is as follows:

$$RoQ=\left\langle \begin{array}{ccc} & ah(Q)+fh(R) & \\ bh(Q)+gh(R) & h(R)h(Q) & dh(Q)+jh(R) \\ & eh(Q)+kh(R) & \end{array} \right\rangle$$

In [3] it has been shown that the set of all three-dimensional real rhotrices together with the operations addition (+) and multiplication (o) is a commutative ring with identity $I=\left\langle \begin{array}{ccc} & 0 & \\ 0 & 1 & 0 \\ & 0 & \end{array} \right\rangle$.

Definition 1.3. Let $R=\left\langle \begin{array}{ccc} & a & \\ b & h(R) & d \\ & e & \end{array} \right\rangle$ be in $\mathcal{R}$. If $RoQ = I$ such that there exists $Q \in \mathcal{R}$ then Q is called the inverse of R, denoted by R^{-1}, and

$$Q=R^{-1}=\frac{-1}{h(R)^2}\left\langle \begin{array}{ccc} & a & \\ b & -h(R) & d \\ & e & \end{array} \right\rangle \text{ where } h(R)\neq 0.$$

Other multiplication of rhotrices called row-column multiplication was proposed by Sani [4]. This multiplication is as follows:

Definition 1.4. Let $R=\left\langle \begin{array}{ccc} & a & \\ b & h(R) & d \\ & e & \end{array} \right\rangle$ and $Q=\left\langle \begin{array}{ccc} & f & \\ g & h(Q) & j \\ & k & \end{array} \right\rangle$ be in $\mathcal{R}$.

$$R\bullet Q=\left\langle \begin{array}{ccc} & af+dg & \\ bf+eg & h(R)h(Q) & aj+dk \\ & bf+ek & \end{array} \right\rangle$$

Studies on this subject has progressed quickly after the rhotrix definition. Several authors have obtained interesting results on 3-dimensional rhotrices. See [5] for a comprehensive survey of the literature on these developments.

2. Rhotrices Ring on Arbitrary Ring R

The definition of n-dimensional rhotrix over an arbitrary ring was firstly given by Mohammed in [6] and he gave the set of all rhotrices over an arbitrary ring is a ring together with the operations of rhotrix addition and row-cloum rhotrix multiplication.

In this section, it has been shown that the set of 3-dimensional rhotrices over an arbitrary ring is a ring with the operations rhotrix addition and **"hearty multiplication"** as different from multiplication in Mohammed's work [6]. Also we investigate the basic properties of the rhotrices ring.

Definition 2.1. Let $(R, +, .)$ be a ring with identity. By a $3-dimensional\ rhotrix$ over the ring R, we mean a rhomboidal array defined by

$$A = \left\langle \begin{array}{ccc} & a & \\ b & c & d \\ & e & \end{array} \right\rangle$$

where a, b, c, d, e are in the ring R. The entry c of A is called heart of A denoted by $h(A)$.

The set of all 3-dimensional rhotrices over the ring R denoted by $\mathcal{R}_3(R)$**,**

$$\mathcal{R}_3(R) = \left\{ \left\langle \begin{array}{ccc} & a & \\ b & c & d \\ & e & \end{array} \right\rangle \middle| a, b, c, d, e \in R \right\}$$

.

We define two binary operations addition ($\widehat{+}$) and multiplication($\odot$) on $\mathcal{R}_3(R)$ by

$$\left\langle \begin{array}{ccc} & a & \\ b & c & d \\ & e & \end{array} \right\rangle \widehat{+} \left\langle \begin{array}{ccc} & a' & \\ b' & c' & d' \\ & e' & \end{array} \right\rangle = \left\langle \begin{array}{ccc} & a+a' & \\ b+b' & c+c' & d+d' \\ & e+e' & \end{array} \right\rangle \quad (1)$$

$$\left\langle \begin{array}{ccc} & a & \\ b & c & d \\ & e & \end{array} \right\rangle \odot \left\langle \begin{array}{ccc} & a' & \\ b' & c' & d' \\ & e' & \end{array} \right\rangle = \left\langle \begin{array}{ccc} & a.c'+c.a' & \\ b.c'+c.b' & c.c' & d.c'+c.d' \\ & e.c'+c.e' & \end{array} \right\rangle \quad (2)$$

for all $\left\langle \begin{array}{ccc} & a & \\ b & c & d \\ & e & \end{array} \right\rangle, \left\langle \begin{array}{ccc} & a' & \\ b' & c' & d' \\ & e' & \end{array} \right\rangle \in \mathcal{R}_3(R)$. It is easy to check that these operations are well defined, since " $+$ " and "." in R are well-defined.

Theorem 2.2. The set of all $3-dimensional\ rhotrices$ $\mathcal{R}_3(R)$ over the ring R is a ring with respect to operations "$\widehat{+}$" and " $\odot$ ".

Proof. It's easy to see that $(\mathcal{R}_3(R), \widehat{+})$ is a commutative group. Now let's show that the triple $(\mathcal{R}_3(R), \widehat{+}, \odot)$ is a ring.

For all $P = \left\langle \begin{array}{ccc} & a & \\ b & c & d \\ & e & \end{array} \right\rangle, Q = \left\langle \begin{array}{ccc} & a' & \\ b' & c' & d' \\ & e' & \end{array} \right\rangle, S = \left\langle \begin{array}{ccc} & x & \\ y & z & t \\ & u & \end{array} \right\rangle \in \mathcal{R}_3(R)$

$$\begin{aligned} (P \odot Q) \odot S &= \left\langle \begin{array}{ccc} & a.c'+c.a' & \\ b.c'+c.b' & c.c' & d.c'+c.d' \\ & e.c'+c.e' & \end{array} \right\rangle \odot \left\langle \begin{array}{ccc} & x & \\ y & z & t \\ & u & \end{array} \right\rangle \\ &= \left\langle \begin{array}{ccc} & a.c'.z+c.a'.z+c.c'.x & \\ b.c'.z+c.b'.z+c.c'.y & c.c'.z & d.c'.z+c.d'.z+c.c'.t \\ & e.c'.z+c.e'.z+c.c'.u & \end{array} \right\rangle \\ &= P \odot (Q \odot S) \end{aligned}$$

The operation " $\odot$ " is an associative in $\mathcal{R}_3(R)$.

$$\begin{aligned}
(P\widehat{+}Q)\odot S &= \left\langle \begin{array}{ccc} & a+a' & \\ b+b' & c+c' & d+d' \\ & e+e' & \end{array} \right\rangle \odot \left\langle \begin{array}{ccc} & x & \\ y & z & t \\ & u & \end{array} \right\rangle \\
&= \left\langle \begin{array}{ccc} & (a+a').z+(c+c').x & \\ (b+b').z+(c+c').y & (c+c').z & (d+d').z+(c+c').t \\ & (e+e').z+(c+c').u & \end{array} \right\rangle \\
&= \left\langle \begin{array}{ccc} & a.z+a'.z+c.x+c'.x & \\ b.z+b'.z+c.y+c'.y & c.z+c'.z & d.z+d'.z+c.t+c'.t \\ & e.z+e'.z+c.u+c'.u & \end{array} \right\rangle \\
&= \left\langle \begin{array}{ccc} & a.z+c.x & \\ b.z+c.y & c.z & d.z+c.t \\ & e.z+c.u & \end{array} \right\rangle \widehat{+} \left\langle \begin{array}{ccc} & a'.z+c'.x & \\ b'.z+c'.y & c'.z & d'.z+c'.t \\ & e'.z+c'.u & \end{array} \right\rangle \\
&= (P\odot S)\widehat{+}(Q\odot S)
\end{aligned}$$

and similarly it is easy to check that $P\odot(S\widehat{+}Q)=(P\odot S)\widehat{+}(P\odot Q)$

Thus $<\mathcal{R}_3(R),\widehat{+},\odot>$ is a ring.

Furthermore if R is a commutative ring, then $\mathcal{R}_3(R)$ is a commutative ring and if R is a ring with identity 1_R, then $\mathcal{R}_3(R)$ to be a ring with identity $1_{\mathcal{R}_3(R)}=\left\langle \begin{array}{ccc} & 0_R & \\ 0_R & 1_R & 0_R \\ & 0_R & \end{array} \right\rangle$.

Example 2.3. Let $R=\mathbb{Z}_2$. $\mathcal{R}_3(R)$ is a rhotrix ring and since $\mathbb{Z}_2$ is a commutative ring, $\mathcal{R}_3(\mathbb{Z}_2)$ is a commutative ring.

The following theorem give us the characteristic of the ring $\mathcal{R}_3(R)$ depends on the characteristic of the ring R.

Theorem 2.4. The characteristic of the ring $\mathcal{R}_3(R)$ is equal to characteristic of the ring R.

PROOF. Let R be a ring with $CharR=k$. Then the characteristic of the ring $\mathcal{R}_3(R)$ is k. Let $Char\mathcal{R}_3(R)=t$, we show that $k=t$.

$$\begin{aligned}
Char\mathcal{R}_3(R)=t \quad &\Rightarrow \quad t.\left\langle \begin{array}{ccc} & a & \\ b & c & d \\ & e & \end{array} \right\rangle = 0_{\mathcal{R}_3(R)},\ for\ all\ \left\langle \begin{array}{ccc} & a & \\ b & c & d \\ & e & \end{array} \right\rangle \in \mathcal{R}_3(R) \\
&\Rightarrow \quad t.a=t.b=t.c=t.d=t.e=0_R,\ for\ all\ \ a,b,c,d,e\in R \\
&\Rightarrow \quad k|t
\end{aligned}$$

$$\begin{aligned}
CharR=k \quad &\Rightarrow \quad k.a=0_R,\ for\ all\ \ a\in R \\
&\Rightarrow \left\langle \begin{array}{ccc} & k.a & \\ k.b & k.c & k.d \\ & k.e & \end{array} \right\rangle = \left\langle \begin{array}{ccc} & 0_R & \\ 0_R & 0_R & 0_R \\ & 0_R & \end{array} \right\rangle,\ for\ all\ \left\langle \begin{array}{ccc} & a & \\ b & c & d \\ & e & \end{array} \right\rangle \in \mathcal{R}_3(R) \\
&\Rightarrow k.\left\langle \begin{array}{ccc} & a & \\ b & c & d \\ & e & \end{array} \right\rangle = 0_{\mathcal{R}_3(R)},\ for\ all\ \left\langle \begin{array}{ccc} & a & \\ b & c & d \\ & e & \end{array} \right\rangle \in \mathcal{R}_3(R) \\
&\Rightarrow t|k
\end{aligned}$$

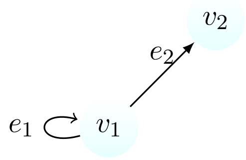
thus $k=t$.

Note: In the ring $\mathcal{R}_3(R)$, the multiplication of nonzero rhotrices A and B is equal to zero. Hence $\mathcal{R}_3(R)$ has zero divisors and $\mathcal{R}_3(R)$ is not integral domain.

The following theorem characterize idempotent elements and nilpotent elements in a ring $\mathcal{R}_3(R)$. Firstly we recall that definitions of idempotent and nilpotent elements in any ring. Let $(R,+,.)$ be a ring. An element $a\in R$ is called idempotent if $a^2=a$ and nilpotent if $a^n=0$ for some positive integer n.

Theorem 2.5. Let R be a ring with identity 1_R and c be an idempotent element in R. Then $\left\langle \begin{array}{ccc} & 0_R & \\ 0_R & c & 0_R \\ & 0_R & \end{array} \right\rangle$ is an idempotent element in $\mathcal{R}_3(R)$.

Proof. $\left\langle \begin{array}{ccc} & 0_R & \\ 0_R & c & 0_R \\ & 0_R & \end{array} \right\rangle^2 = \left\langle \begin{array}{ccc} & 0_R & \\ 0_R & c^2 & 0_R \\ & 0_R & \end{array} \right\rangle = \left\langle \begin{array}{ccc} & 0_R & \\ 0_R & c & 0_R \\ & 0_R & \end{array} \right\rangle$. But all idempotents elements in the ring $\mathcal{R}_3(R)$ is not this form. For example; let R be a ring $L_K(E)$, where $L_K(E)$ is a Leavitt Path Algebra [7] and

E: 

Let $A = \left\langle \begin{array}{ccc} & e_2 & \\ 0 & v_1 & 0 \\ & 0 & \end{array} \right\rangle$ be in $\mathcal{R}_3(L_K(E))$. Since in a ring $L_K(E)$, $v_1.v_1 = v_1$, $v_1.v_2 = v_2.v_1 = 0$, $e_2.v_1 = v_2.e_2 = 0, v_1.e_2 = e_2.v_2 = e_2$, $v_1.e_1 = e_1.v_1 = e_1, v_2.e_1 = e_1.v_2 = 0$. Therefore,

$$A^2 = \left\langle \begin{array}{ccc} & e_2.v_1 + v_1.e_2 & \\ 0 & v_1.v_1 & 0 \\ & 0 & \end{array} \right\rangle = \left\langle \begin{array}{ccc} & e_2 & \\ 0 & v_1 & 0 \\ & 0 & \end{array} \right\rangle = A$$

Theorem 2.6. Let R be a ring with identity 1_R and c be a nilpotent element in the ring R. Then $\left\langle \begin{array}{ccc} & a & \\ b & c & d \\ & e & \end{array} \right\rangle$ is a nilpotent element in a ring $\mathcal{R}_3(R)$.

Proof. We give any $A = \left\langle \begin{array}{ccc} & a & \\ b & c & d \\ & e & \end{array} \right\rangle \in \mathcal{R}_3(R)$ and let be c a nilpotent element in a ring R. Since c is a nilpotent element, there exits $n \in \mathbb{Z}^+$ such that $c^n = 0_R$. Then, $h(A^n) = c^n = 0_R$ and $A^{2n} = A^n.A^n = 0_{\mathcal{R}_3(R)}$.

In particularly; If R is a commutative ring. Then $c^n = 0_R$ implies that $A^{n+1} = 0_{\mathcal{R}_3(R)}$.

3. Ideal of Rhotrix Ring

In this section, ideals of rhotrices ring have been investigated. Furthermore characterizations of maximal ideals and prime ideals have been given.

Theorem 3.1. Let R be a ring and I be an ideal of R. Then, $M = \left\{ \left\langle \begin{array}{ccc} & a & \\ b & c & d \\ & e & \end{array} \right\rangle : c \in I \right\}$ is an ideal in $\mathcal{R}_3(R)$.

Proof. Since I is an ideal of R, M is a subset of $\mathcal{R}_3(R)$ and $M \neq \emptyset$. We give any $A = \left\langle \begin{array}{ccc} & a & \\ b & c & d \\ & e & \end{array} \right\rangle$, $B = \left\langle \begin{array}{ccc} & a_1 & \\ b_1 & c_1 & d_1 \\ & e_1 & \end{array} \right\rangle \in M$, and $C = \left\langle \begin{array}{ccc} & x & \\ y & z & t \\ & u & \end{array} \right\rangle \in \mathcal{R}_3(R)$. Then c, $c_1 \in I$ and $z \in R$, $c + (-c_1)$, $z.c$, $c.z \in I$. Hence $A \widehat{+} (-B) \in M$ and $A \odot C$, $C \odot A \in M$. Thus, M is an ideal in $\mathcal{R}_3(R)$.

Theorem 3.2. Let R be a ring and $\mathcal{R}_3(R)$ be a ring of rhotrices.

$$I \text{ is an ideal of } R \Leftrightarrow \mathcal{R}_3(I) \text{ is an ideal of } \mathcal{R}_3(R)$$

Proof. ($\Rightarrow$) Let I be an ideal of R. Then $I \subseteq R$ and $I \neq \emptyset$. Thus $\mathcal{R}_3(I) \subseteq \mathcal{R}_3(R)$ and $\mathcal{R}_3(I) \neq \emptyset$. For any $A \in \mathcal{R}_3(I)$, since $h(A) \in I$, $\mathcal{R}_3(I)$ is an ideal in $\mathcal{R}_3(R)$ from Theorem 3.1.

($\Leftarrow$) Let $\mathcal{R}_3(I)$ be an ideal of $\mathcal{R}_3(R)$. It is easy check that $I \neq \emptyset$, $I \subseteq R$.
We give any $a, b \in I$ and $r \in R$.

i. $a \in I \Rightarrow A = \left\langle \begin{array}{ccc} & 0_R & \\ 0_R & a & 0_R \\ & 0_R & \end{array} \right\rangle \in \mathcal{R}_3(I)$ and $b \in I \Rightarrow B = \left\langle \begin{array}{ccc} & 0_R & \\ 0_R & b & 0_R \\ & 0_R & \end{array} \right\rangle \in \mathcal{R}_3(I)$. Since $\mathcal{R}_3(I)$ is an ideal of $\mathcal{R}_3(R)$, $A \widehat{+} (-B) = \left\langle \begin{array}{ccc} & 0_R & \\ 0_R & a-b & 0_R \\ & 0_R & \end{array} \right\rangle \in \mathcal{R}_3(I)$ and $a - b \in I$

ii. $r \in R \Rightarrow C = \left\langle \begin{array}{ccc} & r & \\ 0_R & 0_R & 0_R \\ & 0_R & \end{array} \right\rangle \in \mathcal{R}_3(R)$. Since $\mathcal{R}_3(I)$ is a ideal of $\mathcal{R}_3(R)$,

$A \odot C = \left\langle \begin{array}{ccc} & a.r & \\ 0_R & 0_R & 0_R \\ & 0_R & \end{array} \right\rangle \in \mathcal{R}_3(I)$ and $a.r \in I$

Similarly,

$C \odot A = \left\langle \begin{array}{ccc} & r.a & \\ 0_R & 0_R & 0_R \\ & 0_R & \end{array} \right\rangle \in \mathcal{R}_3(I)$ and $r.a \in I$

Consequently, I is an ideal of R.

Corollary 3.3. Let K be a subset of $\mathcal{R}_3(R)$. K is an ideal in $\mathcal{R}_3(R)$ if and only if there exists an ideal I in R such that $h(A) \in I$, for all $A \in K$.

Proof. Let $I = \{a \in R : a = h(A) \ for \ all \ A \in K\} \subseteq R$.

Since $0 \in I$ for $0_{\mathcal{R}_3(R)} \in K$, $I \neq \emptyset$. We will show that $a-b$, $a.r$, $r.a \in I$ for all $a, b \in I$ and $r \in R$.
$a \in I \Rightarrow A = \left\langle \begin{array}{ccc} & 0_R & \\ 0_R & a & 0_R \\ & 0_R & \end{array} \right\rangle \in K$, $b \in I \Rightarrow B = \left\langle \begin{array}{ccc} & 0_R & \\ 0_R & b & 0_R \\ & 0_R & \end{array} \right\rangle \in K$, and $A \widehat{+} (-B) \in K$ because K is an ideal in $\mathcal{R}_3(R)$ and so $h(A \widehat{+} (-B)) = a - b \in I$.
$C = \left\langle \begin{array}{ccc} & 0_R & \\ 0_R & r & 0_R \\ & 0_R & \end{array} \right\rangle \in \mathcal{R}_3(R)$ for $r \in R$ and since K is an ideal in $\mathcal{R}_3(R)$, $A \odot C$, $C \odot A \in K$ and so $h(A \odot C) = a.r$, $h(C \odot A) = r.a \in I$. Thus I is an ideal in R.

Conversely, let $K = \{A \in \mathcal{R}_3(R) : h(A) \in I\} \subseteq \mathcal{R}_3(R)$.

$I \neq \emptyset$ then there exists $a \in I$ and so $A \in K$ such that $h(A) = a$ and $K \neq \emptyset$. We give any $A, B \in K$ and $C \in R$. Then $h(A), h(B) \in I$ and since I is an ideal in R, $h(A) - h(B) = h(A \widehat{+} (-B))$, $h(A).h(C) = h(A \odot C)$, $h(C).h(A) = h(C \odot A) \in I$ and so $A \widehat{+} (-B)$, $A \odot C$, $C \odot A \in K$. Thus K is an ideal in $\mathcal{R}_3(R)$.

Theorem 3.4. Let R be a commutative ring with identity and I be an ideal of R. Then

$$I \text{ is a principal ideal of } R \Leftrightarrow \mathcal{R}_3(I) \text{ is a principal ideal of } \mathcal{R}_3(R)$$

Proof. Let I be a principal ideal of R. Then there exists $a \in R$ such that $I = (a)$.

Let $P \in \mathcal{R}_3(R)$. Since $I = (a)$, $P = \left\langle \begin{array}{ccc} & a.r_1 & \\ a.r_2 & a.r_3 & a.r_4 \\ & a.r_5 & \end{array} \right\rangle$. Then,

$$P = \left\langle \begin{array}{ccc} & 0_R & \\ 0_R & a & 0_R \\ & 0_R & \end{array} \right\rangle \odot \left\langle \begin{array}{ccc} & r_1 & \\ r_2 & r_3 & r_4 \\ & r_5 & \end{array} \right\rangle \in (A)$$

where $A = \left\langle \begin{array}{ccc} & 0_R & \\ 0_R & a & 0_R \\ & 0_R & \end{array} \right\rangle$.

Conversely, let $\mathcal{R}_3(I)$ be a principal ideal in $\mathcal{R}_3(R)$. Then there exist $P \in \mathcal{R}_3(R)$ such that $\mathcal{R}_3(I) = (P)$. We will show that $I = (h(P))$.

$$a \in I \Rightarrow \left\langle \begin{array}{ccc} & 0_R & \\ 0_R & a & 0_R \\ & 0_R & \end{array} \right\rangle \in \mathcal{R}_3(I) = (P)$$

$\Rightarrow a = h(P).x$, $x \in R \Rightarrow a \in (h(P))$. Thus $I \subseteq (h(P))$. Since $\mathcal{R}_3(I) = (P)$, $h(P) \in I$. Then $(h(P)) \subseteq I$. Thus $I = (h(P))$.

Theorem 3.5. Let R be a ring and I be an ideal of R. Then,

$$\mathcal{R}_3(R/I) = \left\{ \left\langle \begin{array}{ccc} & a+I & \\ b+I & c+I & d+I \\ & e+I & \end{array} \right\rangle : a+I, b+I, c+I, d+I, e+I \in R/I \right\}$$

is a ring with as known operations "$\widehat{+}$" and "$\odot$" and $\mathcal{R}_3(R)/\mathcal{R}_3(I)$ isomorphic to ring $\mathcal{R}_3(R/I)$.

Proof. Since R/I is a ring, $\mathcal{R}_3(R/I)$ is a ring. We will show that $\mathcal{R}_3(R/I) \cong \mathcal{R}_3(R)/\mathcal{R}_3(I)$. We define $f : \mathcal{R}_3(R) \to \mathcal{R}_3(R/I)$ by

$$f\left(\left\langle \begin{array}{ccc} & a & \\ b & c & d \\ & e & \end{array} \right\rangle\right) = \left\langle \begin{array}{ccc} & a+I & \\ b+I & c+I & d+I \\ & e+I & \end{array} \right\rangle, \text{for any} \left\langle \begin{array}{ccc} & a & \\ b & c & d \\ & e & \end{array} \right\rangle \in \mathcal{R}_3(R)$$

It is easy to see that f is a well-defined. We give any $A = \left\langle \begin{array}{ccc} & a & \\ b & c & d \\ & e & \end{array} \right\rangle$ and $B = \left\langle \begin{array}{ccc} & x & \\ y & z & t \\ & u & \end{array} \right\rangle \in \mathcal{R}_3(R)$

i.

$$\begin{aligned} f(A\widehat{+}B) &= \left\langle \begin{array}{ccc} & (a+x)+I & \\ (b+y)+I & (c+z)+I & (d+t)+I \\ & (e+u)+I & \end{array} \right\rangle \\ &= \left\langle \begin{array}{ccc} & a+I & \\ b+I & c+I & d+I \\ & e+I & \end{array} \right\rangle \widehat{+} \left\langle \begin{array}{ccc} & x+I & \\ y+I & z+I & t+I \\ & u+I & \end{array} \right\rangle \\ &= f(A)\widehat{+}f(B) \end{aligned}$$

and

$$\begin{aligned} f(A \odot B) = &\ f\left(\left\langle \begin{array}{ccc} & a.z+c.x & \\ b.z+c.y & c.z & d.z+c.t \\ & e.z+c.u & \end{array} \right\rangle\right) \\ = &\left\langle \begin{array}{ccc} & (a.z+c.x)+I & \\ (b.z+c.y)+I & (c.z)+I & (d.z+c.t)+I \\ & (e.z+c.u)+I & \end{array} \right\rangle \\ = &\ f(A) \odot f(B) \end{aligned}$$

Thus, f is a ring homomorphism.

ii.

$$\begin{aligned} Kerf &= \left\{ \left\langle \begin{array}{ccc} & a & \\ b & c & d \\ & e & \end{array} \right\rangle \in \mathcal{R}_3(R) : f\left(\left\langle \begin{array}{ccc} & a & \\ b & c & d \\ & e & \end{array} \right\rangle\right) = 0_{\mathcal{R}_3(R/I)} \right\} \\ &= \left\{ \left\langle \begin{array}{ccc} & a & \\ b & c & d \\ & e & \end{array} \right\rangle : a, b, c, d, e \in I \right\} \\ &= \mathcal{R}_3(I) \end{aligned}$$

iii.

$$\begin{aligned} Imf &= \left\{ f\left(\left\langle \begin{array}{ccc} & a & \\ b & c & d \\ & e & \end{array} \right\rangle\right) : \left\langle \begin{array}{ccc} & a & \\ b & c & d \\ & e & \end{array} \right\rangle \in \mathcal{R}_3(R) \right\} \\ &= \left\{ \left\langle \begin{array}{ccc} & a+I & \\ b+I & c+I & d+I \\ & e+I & \end{array} \right\rangle : a,b,c,d,e \in R \right\} \\ &= \mathcal{R}_3(R/I) \end{aligned}$$

Thus, f is a surjective.

Consequently $\mathcal{R}_3(R)/\mathcal{R}_3(I)$ is isomorphic to $\mathcal{R}_3(R/I)$ by the first isomorphism theorem.

Theorem 3.6. Let R be any ring, $\mathcal{R}_3(R)$ be a ring of 3-dimensional rhotrices over R. If $\mathcal{R}_3(M)$ is a maximal ideal of $\mathcal{R}_3(R)$, then M is a maximal ideal of R.

PROOF. By Theorem 3.2, M is an ideal of R. Let J be an ideal of R such that $M \subseteq J \subseteq R$. We will show that $M = J$ or $J = R$. $M \subseteq J \subseteq R$ implies that $\mathcal{R}_3(M) \subseteq \mathcal{R}_3(J) \subseteq \mathcal{R}_3(R)$. Since $\mathcal{R}_3(M)$ is a maximal ideal in $\mathcal{R}_3(R)$, $\mathcal{R}_3(M) = \mathcal{R}_3(J)$ or $\mathcal{R}_3(J) = \mathcal{R}_3(R)$. Hence $M = J$ or $J = R$. Thus M is a maximal ideal in R.

The converse of the above theorem is not true, as shown by the following example.

Example 3.7. Let R be a ring and M be an maximal ideal of R and

$$K = \left\{ \left\langle \begin{array}{ccc} & a & \\ b & c & d \\ & e & \end{array} \right\rangle \middle| a,b,d,e \in R \ and \ c \in M \right\}$$

K is an ideal of $\mathcal{R}_3(R)$ and $\mathcal{R}_3(M) \subseteq K \subseteq \mathcal{R}_3(R)$. Thus M is a maximal ideal in R but $\mathcal{R}_3(M)$ is not a maximal ideal in $\mathcal{R}_3(R)$.

Theorem 3.8. Let K be an ideal in $\mathcal{R}_3(R)$ and $M = \{a \in R : a = h(A), A \in K\}$ be a subset of R. If M is a maximal ideal in R then K is a maximal ideal in $\mathcal{R}_3(R)$.

PROOF. Suppose that K is not a maximal ideal in $\mathcal{R}_3(R)$. Then there exists an ideal J in $\mathcal{R}_3(R)$ such that $K \subseteq J \subseteq \mathcal{R}_3(R)$.

Since J is an ideal, there exists an ideal I in R such that $h(A) \in I$ for arbitrary $A \in J$ and since $K \subseteq J$, $M \subseteq I$ but $I \subsetneq M$ because for every $A \in J$, $h(A) \notin M$. Therefore there exists an ideal I in R. However, this gives a contradiction since M is a maximal ideal of R.

Theorem 3.9. Let R be a ring and $\mathcal{R}_3(P)$ be a prime ideal of ring $\mathcal{R}_3(R)$. Then P is a prime ideal of R.

PROOF. Since $\mathcal{R}_3(P)$ is an ideal in $\mathcal{R}_3(R)$, P is an ideal in R by Theorem 3.2.

We give any $a, b \in R$ and let $aRb \subseteq P$. Then for any $x \in R$, $axb \in P$. Hence, $\left\langle \begin{array}{ccc} & axb & \\ 0_R & 0_R & 0_R \\ & 0_R & \end{array} \right\rangle = A \odot X \odot B \in \mathcal{R}_3(P)$, where $A = \left\langle \begin{array}{ccc} & a & \\ 0_R & 0_R & 0_R \\ & 0_R & \end{array} \right\rangle$, $X = \left\langle \begin{array}{ccc} & 0_R & \\ 0_R & x & 0_R \\ & 0_R & \end{array} \right\rangle$, and $B = \left\langle \begin{array}{ccc} & b & \\ 0_R & 0_R & 0_R \\ & 0_R & \end{array} \right\rangle$. Since, $\mathcal{R}_3(P)$ is a prime ideal, either $\left\langle \begin{array}{ccc} & a & \\ 0_R & 0_R & 0_R \\ & 0_R & \end{array} \right\rangle \in \mathcal{R}_3(P)$ or $\left\langle \begin{array}{ccc} & b & \\ 0_R & 0_R & 0_R \\ & 0_R & \end{array} \right\rangle \in \mathcal{R}_3(P)$. Hence either $a \in P$ or $b \in P$. Therefore P is a prime ideal in R.

The converse of the above theorem is not true, as shown by the following example.

Example 3.10. Although $3\mathbb{Z}$ is a prime ideal in the ring $\mathbb{Z}$, $\mathcal{R}_3(3\mathbb{Z})$ is not a prime ideal in the ring $\mathcal{R}_3(\mathbb{Z})$. Indeed,

$$\begin{aligned} A \odot B &= \left\langle \begin{array}{ccc} & -2 & \\ 5 & 3 & 1 \\ & 2 & \end{array} \right\rangle \odot \left\langle \begin{array}{ccc} & 4 & \\ 1 & 6 & -1 \\ & 2 & \end{array} \right\rangle \\ &= \left\langle \begin{array}{ccc} & 0 & \\ 33 & 18 & 3 \\ & 18 & \end{array} \right\rangle \in \mathcal{R}_3(3\mathbb{Z}) \end{aligned}$$

but $A \notin \mathcal{R}_3(3\mathbb{Z})$ and $B \notin \mathcal{R}_3(3\mathbb{Z})$.

Corollary 3.11. Let K be an ideal in $\mathcal{R}_3(R)$. K is a prime ideal in $\mathcal{R}_3(R)$ if and only if there exists a prime ideal P in R such that $h(A) \in P$, for all $A \in K$.

Proof. Let K be a prime ideal in $\mathcal{R}_3(R)$. Then by Corollary 3.3, P is an ideal in R. We will show that P is a prime.

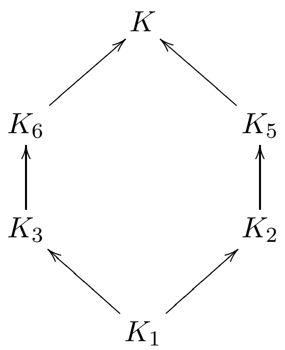
$a.R.b \subseteq P$, for all $a, b \in R$. Then $a.c.b \in P$, for all $c \in R$. By hypothesis, there exists $A \in K$ such that $h(A) = a.c.b \in P$. There exists X, Y, Z rhotrices such that $A = X \odot Y \odot Z$ and $h(X) = a$, $h(Y) = b$, $h(Z) = c$. Since K is a prime ideal in $\mathcal{R}_3(R)$ and $A \in K$, either $X \in K$ or $Z \in K$. Hence either $a \in P$ or $c \in P$. Thus P is a prime ideal in R.

Conversely, let P be a prime ideal in R. Then by Corollary 3.3 , K is a ideal in $\mathcal{R}_3(R)$. Let $X \odot \mathcal{R}_3(R) \odot Y \subseteq K$, for any $X, Y \in \mathcal{R}_3(R)$. Then $X \odot C \odot Y \in K$, for all $C \in \mathcal{R}_3(R)$. Hence $h(X \odot C \odot Y) = h(X).h(C).h(Y) \in P$ and since P is a prime ideal in R, either $h(X) \in P$ or $h(Y) \in P$. Thus either $X \in K$ or $Y \in K$ and so K is a prime ideal in $\mathcal{R}_3(R)$.

Example 3.12. Let $R = \mathbb{Z}_6 = \{\overline{0}, \ \overline{1}, \ \overline{2}, \ \overline{3}, \ \overline{4}, \ \overline{5}\}$. Then, $A = (\overline{0})$, $B = (\overline{2})$, $C = (\overline{3})$, $D = \mathbb{Z}_6$ are ideals in $\mathbb{Z}_6$. Hence, $K = \mathcal{R}_3(R)$, $K_1 = 0_{\mathcal{R}_3(R)}$, $K_2 = \mathcal{R}_3(B)$, $K_3 = \mathcal{R}_3(C)$,

$$K_4 = \left\langle \begin{array}{ccc} & R & \\ R & A & R \\ & R & \end{array} \right\rangle, \quad K_5 = \left\langle \begin{array}{ccc} & R & \\ R & B & R \\ & R & \end{array} \right\rangle, \text{ and } K_6 = \left\langle \begin{array}{ccc} & R & \\ R & C & R \\ & R & \end{array} \right\rangle$$

are ideals in $\mathcal{R}_3(\mathbb{Z}_6)$. Furthermore since B and C are prime ideals in $\mathbb{Z}_6$, K_5 ve K_6 are prime ideals in $\mathcal{R}_3(\mathbb{Z}_6)$. It is easy to see that K_5 ve K_6 are prime ideals in $\mathcal{R}_3(\mathbb{Z}_6)$.



Furthermore since B and C are maximal ideals in $\mathbb{Z}_6$, K_5 ve K_6 are maximal ideals in $\mathcal{R}_3(\mathbb{Z}_6)$. From above graphic, it is easy to see that K_5 ve K_6 are prime ideals in $\mathcal{R}_3(\mathbb{Z}_6)$.

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On Some Hyperideals in Ordered Semihypergroups

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Abstract − In this paper, we study ordered hyperideals in ordered semihypergroups. Also, we study (m, n)-regular ordered semihypergroups in terms of ordered (m, n)-hyperideals. Furthermore, we obtain some ideal theoretic results in ordered semihypergroups.

Keywords − *Ordered semihypergroup, regular ordered semihypergroup, ordered bi-hyperideal, ordered* (m, n)*-hyperideal*

1. Introduction and Basic Definitions

The concept of the hypergroup introduced by the French Mathematician Marty at the 8th Congress of Scandinavian Mathematicians [1]. The concept of a semihypergroup is a generalization of the concept of a semigroup. Algebraic hyperstructures are a standard generalization of classical algebraic structures. In a classical algebraic structure, the composition of two elements is an element, while in an algebraic hyperstructure, the composition of two elements is a set. Many authors studied different aspects of semihypergroups, for instance, Davvaz [2], De Salvo et al. [3], Fasino and Freni [4], Gutan [5]. The monograph on application of hyperstructures to various area of study has been written by Corsini and Leoreanu [6]. Heideri and Davvaz studied ordered hyperstructures [7]. For semihypergroups, we refer [2, 8, 9]. Hila et al. studied quasi-hyperideals of ordered semihypergroups [10]. Corsini also studied hypergroup theory [11], [12]. Changphas and Davvaz [13] studied properties of hyperideals in ordered semihypergroups. Most recently, Basar et al. [14–16] investigated different types of hyperideals in ordered hypersemigroups, ordered LA-Γ-semigroups and LA-Γ-semihypergroups.

Let H be a nonempty set, then the mapping $\circ : H \times H \to H$ is called hyperoperation or join operation on H, where $P^{\star}(H) = P(H) \setminus \{0\}$ is the set of all nonempty subsets of H. Let A and B be two nonempty sets. Then, a hypergroupoid $(S, \circ)$ is called a semihypergroups if for every $x, y, z \in S$,

$$x \circ (y \circ z) = (x \circ y) \circ z$$

i.e.,

$$\bigcup_{u \in y \circ z} x \circ u = \bigcup_{v \in x \circ y} v \circ z$$

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A semihypergroup $(S, \circ)$ together with a partial order " $\leq$ " on S that is compatible with semihypergroup operation such that for all $x, y, z \in S$, we have

$$x \leq y \Rightarrow z \circ x \leq z \circ y$$

and

$$x \circ z \leq y \circ z$$

is called an ordered semihypergroup. For subsets A, B of an ordered semihypergroup S, the product set $A \circ B$ of the pair (A, B) relative to S is defined as below:

$$A \circ B = \{a \circ b : a \in A, b \in B\}$$

and for $A \subseteq S$, the product set $A \circ A$ relative to S is defined as $A^2 = A \circ A$. For $M \subseteq S$, $(M] = \{s \in S \mid s \leq m$ for some $m \in M\}$. Also, we write $(s]$ instead of $(\{s\}]$ for $s \in S$. Let $A \subseteq S$. Then for a non-negative integer m, the power of A is defined by $A^m = A \circ A \circ A \circ A \cdots$, where A occurs m times. Note that the power vanishes if $m = 0$. So, $A^0 \circ S = S = S \circ A^0$. In what follows we denote ordered semihypergroup $(S, \leq)$ by S unless otherwise specified.

Suppose S is an ordered semihypergroup and I is a nonempty subset of S. Then, I is called an ordered right (resp. left) hyperideal of S if

(i) $I \circ S \subseteq I(resp.\ S \circ I \subseteq I)$

(ii) $a \in I, b \leq a$ for $b \in S \Rightarrow b \in I$

Definition 1.1. Suppose B is a sub-semihypergroup (resp. nonempty subset) of an ordered semihypergroup S. Then B is called an (resp. generalized) (m, n)-hyperideal of S if (i) $B^m \circ S \circ B^n \subseteq B$, and (ii) for $b \in B$, $s \in S$, $s \leq b \Rightarrow s \in B$.

Note that in the above Definition 1.1, if we set $m = n = 1$, then B is called a (generalized) bi-hyperideal of S.

Definition 1.2. Suppose $(S, \circ, \leq)$ is an ordered semihypergroup and m, n are nonnegative integers. Then S is called (m, n)-regular if for any $s \in S$, there exists $x \in S$ such that $s \leq s^m \circ x \circ s^n$. Equivalently: $(S, \circ, \leq)$ is (m, n)-regular if $s \in (s^m \circ S \circ s^n]$ for all $s \in S$.

2. Preliminary

We begin with the following:

Lemma 2.1. Suppose $(S, \circ, \leq)$ is an ordered semihypergroup and $s \in S$. Let m, n be non-negative integers. Then, the intersection of all ordered (generalized) (m, n)-hyperideals of S containing s, denoted by $[s]_{m,n}$, is an ordered (generalized) (m, n)-hyperideal of S containing s.

Proof. Let $\{A_i : i \in I\}$ be the set of all ordered (generalized) (m, n)-hyperideals of S containing s. Obviously, $\bigcap_{i \in I} A_i$ is a sub-semihypergroup of S containing s. Let $j \in I$. As, $\bigcap_{i \in I} A_i \subseteq A_j$, we have

$$\begin{aligned}(\bigcap_{i \in I} A_i)^m \circ S \circ (\bigcap_{i \in I} A_i)^n &\subseteq A_j^m \circ S \circ A_j^n \\ &\subseteq A_j\end{aligned}$$

Therefore, $(\bigcap_{i \in I} A_i)^m \circ S \circ (\bigcap_{i \in I} A_i)^n \subseteq \bigcap_{i \in I} A_i$ as $\bigcap_{i \in I} A_i$ is a sub-semihypergroup of S containing s. Let $a \in \bigcap_{i \in I} A_i$ and $b \in S$ so that $b \leq a$. Therefore, $b \in \bigcap_{i \in I} A_i$. Hence, $\bigcap_{i \in I} A_i$ is an ordered (generalized) (m, n)-hyperideal of S containing s.

Theorem 2.2. Suppose $(S, \circ, \leq)$ is an ordered semihypergroup and $s \in S$. Then, we have the following:

(i) $[s]_{m,n} = (\bigcup_{i=1}^{m+n} s^i \cup s^m \circ S \circ s^n]$ for any positive integers m, n

(ii) $[s]_{m,0} = (\bigcup_{i=1}^{m} s^i \cup s^m \circ S]$ for any positive integer m

(iii) $[s]_{0,n} = (\bigcup_{i=1}^{n} s^i \cup s^n]$ for any positive integer n

Proof. (i) $(\bigcup_{i=1}^{m+n} s^i \cup s^m \circ S \circ s^n] \neq \emptyset$. Let $a, b \in (\bigcup_{i=1}^{m+n} s^i \cup s^m \circ S \circ s^n]$ be such that $a \leq x$ and $b \leq y$ for some $x, y \in (\bigcup_{i=1}^{m+n} s^i \cup s^m \circ S \circ s^n]$. If $x, y \in s^m \circ S \circ s^n$ or $x \in \bigcup_{i=1}^{m+n} s^i$, $y \in s^m \circ S \circ s^n$ or $x \in s^m \circ S \circ s^n$, $y \in \bigcup_{i=1}^{m+n} s^i$, then

$$x \circ y \subseteq s^m \circ S \circ s^n$$

and therefore,

$$x \circ y \subseteq \bigcup_{i=1}^{m+n} s^i \cup s^m \circ S \circ s^n$$

It follows that $a \circ b \subseteq (\bigcup_{i=1}^{m+n} s^i \cup s^m \circ S \circ s^n]$. Let $x, y \in \bigcup_{i=1}^{m+n} s^i$. Then, $x = s^p, y = s^q$ for some $1 \leq p, q \leq m + n$.
Now two cases arise: If $1 \leq p + q \leq m + n$, then $x \circ y \subseteq \bigcup_{i=1}^{m+n} s^i$.
If $m + n < p + q$, then $x \circ y \subseteq s^m \circ S \circ s^n$. So, $x \circ y \subseteq (\bigcup_{i=1}^{m+n} s^i \cup s^m \circ S \circ s^n]$. This implies that $(\bigcup_{i=1}^{m+n} s^i \cup s^m \circ S \circ s^n]$ is a sub-semihypergroup of S. Moreover, we have

$$\begin{aligned}
(\bigcup_{i=1}^{m+n} s^i \cup s^m \circ S]^m \circ S &= (\bigcup_{i=1}^{m+n} s^i \cup s^m \circ S]^{m-1} \circ (\bigcup_{i=1}^{m+n} s^i \cup s^m \circ S] \circ S \\
&\subseteq (\bigcup_{i=1}^{m+n} s^i \cup s^m \circ S]^{m-1} \circ (\bigcup_{i=1}^{m+n} s^i \circ S \cup s^m \circ S \circ S] \\
&\subseteq (\bigcup_{i=1}^{m+n} s^i \cup s^m \circ S]^{m-1} \circ (s \circ S] \\
&= (\bigcup_{i=1}^{m+n} s^i \cup s^m \circ S]^{m-2} \circ (\bigcup_{i=1}^{m+n} s^i \cup s^m \circ S] \circ (s \circ S] \\
&\subseteq (\bigcup_{i=1}^{m+n} s^i \cup s^m \circ S]^{m-2} \circ (\bigcup_{i=1}^{m+n} s^i \cup s^m \circ S \circ (s \circ S]] \\
&\subseteq (\bigcup_{i=1}^{m+n} s^i \cup s^m \circ S]^{m-2} \circ (s^2 \circ S] \\
&\vdots \\
&\subseteq (s^m \circ S]
\end{aligned}$$

In a similar fashion, we have

$$S \circ (\bigcup_{i=1}^{m+n} s^i \cup s^m \circ S \circ s^n]^n \subseteq (S \circ s^n]$$

Therefore,

$$\begin{aligned}
(\bigcup_{i=1}^{m+n} s^i \cup s^m \circ S \circ s^n]^m \circ S \circ (\bigcup_{i=1}^{m+n} s^i \cup s^m \circ S \circ s^n]^n &\subseteq (s^m \circ S \circ s^n] \\
&\subseteq (\bigcup_{i=1}^{m+n} s^i \cup s^m \circ S \circ s^n]
\end{aligned}$$

So, $(\bigcup_{i=1}^{m+n} s^i \cup s^m \circ S \circ s^n]$ is an (m, n)-hyperideal of S containing s; hence, $[s]_{m,n} \subseteq (\bigcup_{i=1}^{m+n} s^i \cup s^m \circ S \circ s^n]$.

For the reverse inclusion, suppose $a \in (\bigcup_{i=1}^{m+n} s^i \cup s^m \circ S \circ s^n]$ is such that $a \leq t$ for some $t \in (\bigcup_{i=1}^{m+n} s^i \cup s^m \circ S \circ s^n]$. If $t = s^j$ for some $1 \leq j \leq m + n$, then $t \in [s]_{m,n}$, therefore, $a \in [s]_{m,n}$. If $t \in s^m \circ S \circ s^n$, by

$$s^m \circ S \circ s^n \subseteq ([s]_{m,n})^m \circ S \circ ([s]_{m,n})^n \subseteq [s]_{m,n}$$

then $t \in [s]_{m,n}$; hence, $a \in [s]_{m,n}$. This implies that $(\bigcup_{i=1}^{m+n} s^i \cup s^m \circ S \circ s^n] \subseteq [s]_{m,n}$. Hence, $[s]_{m,n} = (\bigcup_{i=1}^{m+n} s^i \cup s^m \circ S \circ s^n]$.
(ii) and (iii) can be proved in a similar fashion.

Lemma 2.3. Suppose $(S, \circ, \leq)$ is an ordered semihypergroup and $s \in S$. Suppose m, n are positive integers. Then, we have the following:

(i) $([s]_{m,0})^m \circ S \subseteq (s^m \circ S]$

(ii) $S \circ ([s]_{0,n})^n \subseteq (S \circ s^n]$

(iii) $([s]_{m,n})^m \circ S \circ ([s]_{m,n})^n \subseteq (s^m \circ S \circ s^n]$

Proof. (i)Using Theorem 2.2, we have

$$\begin{aligned}
([s]_{m,0})^m \circ S &= (\bigcup_{i=1}^{m+n} s^i \cup s^m \circ S]^m \circ S \\
&= (\bigcup_{i=1}^{m+n} s^i \cup s^m \circ S]^{m-1} \circ (\bigcup_{i=1}^{m+n} s^i \cup s^m \circ S] \circ S \\
&\subseteq (\bigcup_{i=1}^{m+n} s^i \cup s^m \circ S]^{m-1} \circ (\bigcup_{i=1}^{m+n} s^i \circ S \cup s^m \circ S \circ S] \\
&\subseteq (\bigcup_{i=1}^{m+n} s^i \cup s^m \circ S]^{m-1} \circ (s \circ S] \\
&\vdots \\
&\subseteq (s^m \circ S]
\end{aligned}$$

Hence, $([s]_{m,0})^m \circ S \subseteq (s^m \circ S]$. (ii) can be proved similarly as (i).
(iii) Applying Theorem 2.2, we have

$$\begin{aligned}
([s]_{m,n})^m \circ S &= (\bigcup_{i=1}^{m+n} s^i \cup s^m \circ S \circ s^n]^m \circ S \\
&= (\bigcup_{i=1}^{m+n} s^i \cup s^m \circ S \circ s^n]^{m-1} \circ (\bigcup_{i=1}^{m+n} s^i \cup s^m \circ S \circ s^n] \circ S \\
&\subseteq (\bigcup_{i=1}^{m+n} s^i \cup s^m \circ S \circ s^n]^{m-1} \circ (\bigcup_{i=1}^{m+n} s^i \circ S \cup s^m \circ S \circ s^n \circ S] \\
&= (\bigcup_{i=1}^{m+n} s^i \cup s^m \circ S \circ s^n]^{m-1} \circ (s \circ S] \\
&\vdots \\
&= (s^m \circ S]
\end{aligned}$$

Therefore, $([s]_{m,n})^m \circ S \subseteq (s^m \circ S]$. In a similar way, $S \circ ([s]_{m,n})^n \subseteq (S \circ s^n]$. Therefore,

$$\begin{aligned}
([s]_{m,n})^m \circ S \circ ([s]_{m,n})^n &\subseteq (s^m \circ S] \circ ([s]_{m,n})^n \\
&\subseteq (s^m \circ (S \circ ([s]_{m,n})^n)] \\
&\subseteq (s^m \circ (S \circ s^n]] \\
&\subseteq (s^m \circ S \circ s^n]
\end{aligned}$$

Hence, (iii) holds.

Theorem 2.4. Suppose $(S, \circ, \leq)$ is an ordered semihypergroup and m, n are positive integers. Let $\mathcal{R}_{(m,0)}$ and $\mathcal{L}_{(0,n)}$ be the set of all ordered $(m, 0)$-hyperideals and the set of all ordered $(0, n)$-hyperideals of S, respectively. Then:

(i) S is $(m, 0)$-regular if and only if for all $R \in \mathcal{R}_{(m,0)}, R = (R^m \circ S]$

(ii) S is $(0, n)$-regular if and only if for all $L \in \mathcal{L}_{(0,n)}, L = (S \circ L^n]$

Proof. (i) Suppose S is $(m, 0)$-regular. Then,

$$\forall s \in S, s \in (s^m \circ S]. \tag{1}$$

Suppose $R \in \mathcal{R}_{(m,0)}$. As, $R^m \circ S \subseteq R$ and $R = (R]$, we have $(R^m \circ S] \subseteq R$. If $s \in R$, by (1), we obtain $s \in (s^m \circ S] \subseteq (R^m \circ S]$, therefore, $R \subseteq (R^m \circ S]$. So, $(R^m \circ S] = R$.
Conversely, suppose

$$\forall R \in \mathcal{R}_{(m,0)}, R = (R^m \circ S] \tag{2}$$

Suppose $s \in S$. Therefore, $[s]_{m,0} \in \mathcal{R}_{(m,0)}$. By (2), we obtain

$$[s]_{m,o} = (([s]_{m,0})^m \circ S]$$

Applying Lemma 2.3, we obtain

$$[s]_{m,o} \subseteq (s^m \circ S]$$

Therefore, $s \in (s^m \circ S]$. Hence, S is $(m, 0)$-regular.
(ii) It can be proved analogously.

Theorem 2.5. Suppose $(S, \circ, \leq)$ is an ordered semihypergroup and m, n are non-negative integers. Suppose $\mathcal{A}_{(m,n)}$ is the set of all ordered (m, n)-hyperideals of S. Then,

$$S \text{ } is \text{ } (m, n) - regular \iff \forall A \in \mathcal{A}_{(m,n)}, A = (A^m \circ S \circ A^n] \tag{3}$$

Proof. Consider the following four conditions:
Case(i): $m = 0$ and $n = 0$. Then (3) implies
S is (0, 0)-regular $\iff \forall A \in \mathcal{A}_{(0,0)}, A = S$ because $\mathcal{A}_{(0,0)} = \{S\}$ and S is (0, 0)-regular.
Case (ii): $m = 0$ and $n \neq 0$. Therefore, (3) implies
S is (0, n)-regular$\iff \forall A \in \mathcal{A}_{(0,n)}, A = (S \circ A^n]$. This follows by Theorem 2.4(ii).
Case (iii): $m \neq 0$ and $n = 0$. This can be proved applying Theorem 2.4(i).
Case (iv): $m \neq 0$ and $n \neq 0$. Suppose S is (m, n)-regular. Therefore,

$$\forall s \in S, s \in (s^m \circ S \circ s^n] \tag{4}$$

Let $A \in \mathcal{A}_{(m,n)}$. As $A^m \circ S \circ A^n \subseteq A$ and $A = (A]$, we obtain $(A^m \circ S \circ A^n] \subseteq A$. Suppose $s \in A$. Applying (4), $s \in (s^m \circ S \circ s^n] \subseteq (A^m \circ S \circ A^n]$. Therefore, $A \subseteq (A^m \circ S \circ A^n]$. Hence, $A = (A^m \circ S \circ A^n]$.
Conversely, suppose $A = (A^m \circ S \circ A^n]$ for all $A \in \mathcal{A}_{(m,n)}$. Suppose $s \in S$. As $[s]_{m,n} \in \mathcal{A}_{(m,n)}$, we have

$$[s]_{m,n} = (([s]_{m,n})^m \circ S \circ ([s]_{m,n})^n]$$

Applying Lemma 2.3(iii), we obtain $[s]_{m,n} \subseteq (s^m \circ S \circ s^n]$, therefore, $s \in (s^m \circ S \circ s^n]$. Hence, S is (m, n)-regular.

Theorem 2.6. Suppose $(S, \circ, \leq)$ is an ordered semihypergroup and m, n are nonnegative integers. Suppose $\mathcal{R}_{(m,0)}$ and $\mathcal{L}_{(0,n)}$ is the set of all $(m, 0)$-hyperideals and $(0, n)$-hyperideals of S, respectively. Then,

$$\begin{aligned} S \text{ } is \text{ } (m, n)\text{−regular ordered semihypergroup} \iff \forall \mathrm{R} \in \mathcal{R}_{(\mathrm{m},0)} \forall \mathrm{L} \in \mathcal{L}_{(0,\mathrm{n})}, \\ R \cap L = (R^m \circ L \cap R \circ L^n] \end{aligned} \tag{5}$$

Proof. Consider the following four cases:
Case (i): $m = 0$ and $n = 0$. Therefore, (5) implies
S is (0, 0)-regular $\iff \forall R \in \mathcal{R}_{(0,0)} \forall L \in \mathcal{L}_{(0,0)}, R \cap L = (L \cap R]$ because $\mathcal{R}_{(0,0)} = \mathcal{L}_{(0,0)} = \{S\}$ and S is $(0, 0)$-regular.
Case (ii): $m = 0$ and $n \neq 0$. Therefore, (5) implies S is $(0, n)$-regular $\iff \forall R \in \mathcal{R}_{(0,n)} \forall L \in \mathcal{L}_{(0,n)}, R \cap L = (L \cap R \circ L^n]$. Suppose S is $(0, n)$-regular. Suppose $R \in \mathcal{R}_{(0,0)}$ and $L \in \mathcal{L}_{(0,n)}$. By Theorem 2.4(ii), $L = (S \circ L^n]$. As $R \in \mathcal{R}_{(0,0)}$, we have $R = S$, therefore, $R \cap L = L$. Therefore,

$$(L \cap R \circ L^n] = (L \cap S \circ L^n] = ((S \circ L^n] \cap S \circ L^n] = (S \circ L^n] = L = R \cap L$$

Conversely, suppose

$$\forall R \in \mathcal{R}_{(0,0)} \forall L \in \mathcal{L}_{(0,n)}, R \cap L = (L \cap R \circ L^n]. \tag{6}$$

If $R \in \mathcal{R}_{(0,0)}$, then $R = S$. If $L \in \mathcal{L}_{(0,n)}$, $S \circ L^n \subseteq L$ and $L = (L]$. Therefore, (6) implies

$$\forall L \in \mathcal{L}_{(0,n)}, L = (S \circ L^n]$$

Applying Theorem 2.4(ii), S is $(0,n)$-regular.
Case (iii): $m \neq 0$ and $n = 0$. This can be proved as before.
Case (iv): $m \neq 0$ and $n \neq 0$. Suppose that S is (m,n)-regular. Suppose $R \in \mathcal{R}_{(m,0)}$ and $L \in \mathcal{L}_{(0,n)}$. To prove that $R \cap L \subseteq (R^m \circ L] \cap (R \circ L^n]$, suppose $s \in R \cap L$. We have

$$s \in (s^m \circ S \circ s^n] \subseteq (s^m \circ L] \subseteq (R^m \circ L]$$

and

$$s \in (s^m \circ S \circ s^n] \subseteq (R \circ s^n] \subseteq (R \circ L^n]$$

Hence, $R \cap L \subseteq (R^m \circ L] \cap (R \circ L^n]$. As

$$(R^m \circ L] \subseteq (R^m \circ S] \subseteq (R] = R$$

and

$$(R \circ L^n] \subseteq (S \circ L^n] \subseteq (L] = L$$

This implies that $(R^m \circ L] \cap (R \circ L^n] \subseteq R \cap L$, therefore, $R \cap L = (R^m \circ L] \cap (R \circ L^n]$.
Conversely, suppose

$$\forall R \in \mathcal{R}_{(m,0)} \forall L \in \mathcal{L}_{(0,n)}, R \cap L = (R^m \circ L \cap R \circ L^n] \tag{7}$$

Suppose $R = [s]_{m,0}$ and $L = S$. Applying (7), we obtain $[s]_{m,0} \subseteq (([s]_{m,0})^m \circ S]$. Applying Lemma 2.3, we obtain

$$[s]_{m,0} \subseteq (s^m \circ S] \tag{8}$$

In a similar fashion, we obtain

$$[s]_{0,n} \subseteq (S \circ s^n] \tag{9}$$

As $R^m \subseteq R$ and $L^n \subseteq L$, by (7), we have

$$\forall R \in \mathcal{R}_{(m,0)} \forall L \in \mathcal{L}_{(0,n)}, R \cap L \subseteq (R \circ L]$$

As $(s^m \circ S] \in \mathcal{R}_{(m,0)}$ and $(S \circ s^n] \in \mathcal{L}_{(0,n)}$, we obtain

$$(s^m \circ S] \cap (S \circ s^n] \subseteq ((s^m \circ S] \circ (S \circ s^n]] \subseteq (s^m \circ S \circ s^n]$$

Applying (8) and (9), we obtain

$$[s]_{m,0} \cap [s]_{0,n} \subseteq (s^m \circ S \circ s^n]$$

Hence, S is (m,n)-regular.

3. Conclusion

In this article, we investigated ordered hyperideals in ordered semihypergroups. Also, we studied (m,n)-regular ordered semihypergroups in terms of ordered (m,n)-hyperideals. Moreover, we characterized ordered semihypergroups by some results based on ideal theory.

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New Topologies via Weak N-Topological Open Sets and Mappings

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Abstract — One of the objectives of this paper is to introduce some weak N-topological open sets. We characterize N-topological continuous, N^*-quotient, N^*-α quotient and N^*-semi quotient mappings and derive some new topologies with suitable examples.

Keywords — *N-topology, $N\tau\alpha$-open, $N\tau$ semi- open, $N\tau$ pre-open, $N\tau\beta$-open*

1. Introduction

In 1963 Norman Levine [1] initiated the concept of semi open sets and its continuous functions. In 1965 O.Njastad [2] developed the α-open set and its properties in classical topology. Mashhour et al. [3] investigated the properties of pre open sets. Andrijevic [4] discussed the behaviour of β-open sets in classical topology. The general form of classical topology called N-topology and $N\tau$-open sets were initiated by Lellis Thivagar et al. [5]. In this paper we introduce $N\tau\alpha$-open set, $N\tau$ semi-open set, $N\tau$ pre-open set and $N\tau\beta$-open set in N-topological space. We also establish that the set of all $N\tau\alpha$-open sets forms a topology. Apart from this we investigate the properties of some N-topological continuous and quotient mappings. In this section we discuss some basic properties of N-topological spaces which are useful in sequel. Here by a space $(X, N\tau)$, we mean a N-topological space with N-topology $N\tau$ defined on X in which no separation axioms are assumed unless otherwise explicitly stated.

Definition 1.1. [5] Let X be a non empty set, $\tau_1, \tau_2, \ldots, \tau_N$ be N-arbitrary topologies defined on X and let the collection $N\tau = \{S \subseteq X : S = (\bigcup_{i=1}^{N} A_i) \cup (\bigcap_{i=1}^{N} B_i),\ A_i, B_i \in \tau_i\}$, is said to be N-topology on X if it satisfies the following axioms:

(i) $X, \emptyset \in N\tau$

(ii) $\bigcup_{i=1}^{\infty} S_i \in N\tau$ for all $\{S_i\}_{i=1}^{\infty} \in N\tau$

(iii) $\bigcap_{i=1}^{n} S_i \in N\tau$ for all $\{S_i\}_{i=1}^{n} \in N\tau$

Then the pair $(X, N\tau)$ is called a N-topological space on X. The elements of $N\tau$ are known as $N\tau$-open set and the complement of $N\tau$-open set is called $N\tau$-closed.

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Definition 1.2. [5] Let A be a subset of N-topological space $(X, N\tau)$. Then

(i) $N\tau$-$int(A) = \cup\{G : G \subseteq A$ and G is $N\tau$-open$\}$

(ii) $N\tau$-$cl(A) = \cap\{F : A \subseteq F$ and F is $N\tau$-closed$\}$

Theorem 1.3. [5] Let $(X, N\tau)$ be a topological space on X and $A \subseteq X$. Then $x \in N\tau$-$cl(A)$ if and only if $G \cap A \neq \emptyset$ for every open set G containing x.

Definition 1.4. A subset A of a topological space (X, τ) is called

(i) α-open [2] if $A \subseteq int(cl(int(A)))$

(ii) semi-open [1] if $A \subseteq cl(int(A))$

(iii) pre-open [3] if $A \subseteq int(cl(A))$

(iv) β-open [4] if $A \subseteq cl(int(cl(A)))$

The complement of α-open (resp. semi-open, pre-open and β-open) set is called α-closed (resp. semi-closed, pre-closed and β-closed).

2. Weak Forms of Open Sets in N-Topological Space

In this section we investigate some classes of open sets in N-topological space and discuss the relationship between them.

Definition 2.1. A subset A of a N-topological space $(X, N\tau)$ is called

(i) $N\tau\alpha$-open set if $A \subseteq N\tau$-$int(N\tau$-$cl(N\tau$-$int(A)))$

(ii) $N\tau$ semi-open set if $A \subseteq N\tau$-$cl(N\tau$-$int(A))$

(iii) $N\tau$ pre-open set if $A \subseteq N\tau$-$int(N\tau$-$cl(A))$

(iv) $N\tau\beta$-open set if $A \subseteq N\tau$-$cl(N\tau$-$int(N\tau$-$cl(A)))$

The complement of $N\tau\alpha$-open (resp. $N\tau$ semi-open, $N\tau$ pre-open and $N\tau\beta$-open) set is called $N\tau\alpha$-closed (resp. $N\tau$ semi-closed, $N\tau$ pre-closed and $N\tau\beta$-closed). The set of all $N\tau\alpha$-open (resp. $N\tau$ semi-open, $N\tau$ pre-open and $N\tau\beta$-open) sets of $(X, N\tau)$ is denoted by $N\tau\alpha O(X)$ (resp. $N\tau SO(X)$, $N\tau PO(X)$ and $N\tau\beta O(X)$ and the set of all $N\tau\alpha$-closed (resp. $N\tau$ semi-closed, $N\tau$ pre-closed and $N\tau\beta$-closed) sets of $(X, N\tau)$ is denoted by $N\tau\alpha C(X)$ (resp. $N\tau SC(X)$, $N\tau PC(X)$ and $N\tau\beta C(X)$.

Particularly if $N = 1$, then the $1\tau\alpha$-open, 1τ semi-open, 1τ pre-open and $1\tau\beta$-open set of $(X, 1\tau)$ respectively become α-open, semi-open, pre-open and β-open set of (X, τ) which are defined in definition 2.4.

Theorem 2.2. Let A be a subset of N-topological space $(X, N\tau)$. Then

(i) every $N\tau$-open set is $N\tau\alpha$-open.

(ii) every $N\tau\alpha$-open set is $N\tau$ semi-open.

(iii) every $N\tau\alpha$-open set is $N\tau$ pre-open.

(iv) every $N\tau$ semi-open set is $N\tau\beta$-open.

(v) every $N\tau$ pre-open set is $N\tau\beta$-open.

The converse of the above theorem need not be true as shown in the following examples.

Example 2.3. If we take $N = 3$, $X = \{a, b, c\}$, $\tau_1 = \{\emptyset, X, \{a\}\}$, $\tau_2 = \{\emptyset, X\}$ and $\tau_3 = \{\emptyset, X, \{a, b\}\}$. Then $3\tau O(X) = \{\emptyset, X, \{a\}, \{a, b\}\}$ and $3\tau\alpha O(X) = \{\emptyset, X, \{a\}, \{a, b\}, \{a, c\}\}$. Here the set $A = \{a, c\}$ is $3\tau\alpha$-open but not 3τ-open.

Example 2.4. If $N = 5$, $X = \{a, b, c, d\}$, $\tau_1 = \{\emptyset, X, \{a\}\}$, $\tau_2 = \{\emptyset, X, \{b, c\}\}$, $\tau_3 = \{\emptyset, X, \{a, b, c\}\}$, $\tau_4 = \{\emptyset, X, \{a\}, \{a, b, c\}\}$ and $\tau_5 = \{\emptyset, X, \{b, c\}, \{a, b, c\}\}$. Then, $5\tau O(X) = \{\emptyset, X, \{a\}, \{b, c\}, \{a, b, c\}\}$ $= 5\tau\alpha O(X)$, $5\tau SO(X) = \{\emptyset, X, \{a\}, \{a, d\}, \{b, c\}, \{a, b, c\}, \{b, c, d\}\}$, $5\tau PO(X) = \{\emptyset, X, \{a\}, \{b\}, \{c\},$ $\{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}, \{a, c, d\}, \{a, b, d\}\}$ and $5\tau\beta O(X) = \{\emptyset, X, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{a, d\},$ $\{b, c\}, \{b, d\}, \{c, d\}, \{a, b, c\}, \{a, c, d\}, \{a, b, d\}, \{b, c, d\}\}$. Here the set $\{a, d\}$ is 5τ semi-open and $5\tau\beta$-open but not $5\tau\alpha$-open as well as not 5τ pre-open. Also the set $\{a, c\}$ is 5τ pre-open and $5\tau\beta$-open but not $5\tau\alpha$-open as well as 5τ semi-open.

We observe that the following theorem is analogous to the 1985 topological space result of Reilly and Vamanamurthy [6].

Theorem 2.5. Let $(X, N\tau)$ be a N-topological space. Then every $N\tau\alpha$-open set is both $N\tau$ semi-open and $N\tau$ pre-open and conversely.

Lemma 2.6. The arbitrary union of $N\tau\alpha$-open (resp. $N\tau$ semi-open, $N\tau$ pre-open, $N\tau\beta$-open) sets is $N\tau\alpha$-open (resp. $N\tau$ semi-open, $N\tau$ pre-open, $N\tau\beta$-open).

Remark 2.7. Intersection of any two $N\tau$ semi-open (resp. $N\tau$ pre-open, $N\tau\beta$-open) sets need not be a $N\tau$ semi-open (resp. $N\tau$ pre-open, $N\tau\beta$-open) set. Consider example 3.4, the sets $\{a, d\}$ and $\{b, c, d\}$ are 5τ semi-open, but $\{d\}$ is not 5τ semi-open. The sets $\{a, c, d\}$ and $\{a, b, d\}$ are 5τ pre-open, but $\{a, d\}$ is not 5τ semi-open. Also the sets $\{a, d\}$ and $\{c, d\}$ are $5\tau\beta$-open, but $\{d\}$ is not $5\tau\beta$-open.

Theorem 2.8. Let $(X, N\tau)$ be a N-topological space. Then $N\tau\alpha O(X) = \{A \subseteq X : A \cap B \in N\tau SO(X) \forall B \in N\tau SO(X)\}$.
Proof: Proof follows as similar as the Proposition 1 of [2].

Theorem 2.9. Let $(X, N\tau)$ be a N-topological space. Then $N\tau\alpha O(X)$ is a topology finer than $N\tau O(X)$.
Proof: Clearly $\emptyset \in N\tau\alpha O(X)$ and $\bigcup_{i\in\Lambda} A_i \in N\tau\alpha O(X)$ for every $\{A_i\}_{i\in\Lambda} \in N\tau\alpha O(X)$ by lemma 3.6. By theorem 3.8 we have $N\tau\alpha O(X)$ is a topology and clearly $N\tau O(X) \subseteq N\tau\alpha O(X)$.

Definition 2.10. Let $(X, N\tau)$ be a N-topological space. A subset A of X is said to be $N\tau$-nowhere dense set if $N\tau\text{-}int(N\tau\text{-}cl(A)) = \emptyset$.

Lemma 2.11. Let $(X, N\tau)$ be a N-topological space. A subset A of X is $N\tau\alpha$-open set, then it can be written as a difference of $N\tau$-open set and $N\tau$-nowhere dense set.

Remark 2.12. $N\tau O(X) = N\tau\alpha O(X)$ if and only if all $N\tau$-nowhere dense sets are $N\tau$-closed.

Definition 2.13. An N-topological space $(X, N\tau)$ is said to be extremely disconnected if $N\tau\text{-}cl(A)$ is $N\tau$-open for all $N\tau$-open sets A.

Lemma 2.14. $N\tau SO(X)$ is a topology if and only if $(X, N\tau)$ is extremely disconnected.

3. Weak Closure and Interior Operators in N-Topology

In this section, we introduce some weak closure and interior operators in N-topological space and investigate their properties.

Definition 3.1. Let $(X, N\tau)$ be a N-topological space and A be a subset of X.

(i) The $N\tau$-α closure of A, denoted by $N\tau\text{-}\alpha cl(A)$, and defined by

$$N\tau\text{-}\alpha cl(A) = \cap\{F : A \subseteq F \text{ and } F \text{ is } N\tau\alpha\text{-closed set}\}$$

(ii) The $N\tau$-semi closure of A, denoted by $N\tau\text{-}scl(A)$, and defined by

$$N\tau\text{-}scl(A) = \cap\{F : A \subseteq F \text{ and } F \text{ is } N\tau \text{ semi-closed set}\}$$

(iii) The $N\tau$-pre closure of A, denoted by $N\tau\text{-}pcl(A)$, and defined by

$$N\tau\text{-}pcl(A) = \cap\{F : A \subseteq F \text{ and } F \text{ is } N\tau \text{ pre-closed set}\}$$

(iv) The $N\tau$-β closure of A, denoted by $N\tau\beta cl(A)$, and defined by

$$N\tau\text{-}\beta cl(A) = \cap\{F : A \subseteq F \text{ and } F \text{ is } N\tau\beta\text{-closed set}\}$$

Definition 3.2. Let $(X, N\tau)$ be a N-topological space and A be a subset of X.

(i) The $N\tau$-α interior of A, denoted by $N\tau\alpha int(A)$, and is defined by

$$N\tau\text{-}\alpha int(A) = \cup\{G : G \subseteq A \text{ and } G \text{ is } N\tau\alpha\text{-open set}\}$$

(ii) The $N\tau$-semi interior of A, denoted by $N\tau\text{-}sint(A)$, and is defined by

$$N\tau\text{-}sint(A) = \cup\{G : G \subseteq A \text{ and } G \text{ is } N\tau \text{ semi-open set}\}$$

(iii) The $N\tau$-pre interior of A, denoted by $N\tau\text{-}pint(A)$, and is defined by

$$N\tau\text{-}pint(A) = \cup\{G : G \subseteq A \text{ and } G \text{ is } N\tau \text{ pre-open set}\}$$

(iv) The $N\tau$-β interior of A, denoted by $N\tau\text{-}\beta int(A)$, and is defined by

$$N\tau\text{-}\beta int(A) = \cup\{G : G \subseteq A \text{ and } G \text{ is } N\tau\beta\text{-open set}\}$$

Theorem 3.3. Let $(X, N\tau)$ be a N-topological space on X and let $A, B \subseteq X$. Then

(i) $N\tau\text{-}\alpha cl(A)$ is the smallest $N\tau\alpha$-closed set which containing A.

(ii) A is $N\tau\alpha$-closed iff $N\tau\text{-}\alpha cl(A) = A$. In particular, $N\tau\text{-}\alpha cl(\emptyset) = \emptyset$ and $N\tau\text{-}\alpha cl(X) = X$.

(iii) $A \subseteq B \Rightarrow N\tau\text{-}\alpha cl(A) \subseteq N\tau\text{-}\alpha cl(B)$

(iv) $N\tau\text{-}\alpha cl(A \cup B) = N\tau\text{-}\alpha cl(A) \cup N\tau\text{-}\alpha cl(B)$

(v) $N\tau\text{-}\alpha cl(A \cap B) \subseteq N\tau\text{-}\alpha cl(A) \cap N\tau\text{-}\alpha cl(B)$

(vi) $N\tau\text{-}\alpha cl(N\tau\text{-}\alpha cl(A)) = N\tau\text{-}\alpha cl(A)$

Proof:

(i) Since the intersection of any collection of $N\tau\alpha$-closed sets is also $N\tau\alpha$-closed, then $N\tau\text{-}\alpha cl(A)$ is a $N\tau\alpha$-closed set. By definition 4.1, $A \subseteq N\tau\text{-}\alpha cl(A)$. Now let B be any $N\tau\alpha$-closed set containing A. Then $N\tau\text{-}\alpha cl(A) = \cap\{F : A \subseteq F$ and F is $N\tau\alpha$-closed$\} \subseteq B$. Therefore, A is the smallest $N\tau\alpha$-closed set containing A.

(ii) Assume A is $N\tau\alpha$-closed, then A is the only smallest $N\tau\alpha$-closed set containing itself and therefore, $N\tau\text{-}\alpha cl(A) = A$. Conversely, assume $N\tau\text{-}\alpha cl(A) = A$. Then A is the smallest $N\tau\alpha$-closed set containing itself. Therefore, A is $N\tau\alpha$-closed. In particular, since $\emptyset$ and X are $N\tau\alpha$-closed sets, then $N\tau\text{-}\alpha cl(\emptyset) = \emptyset$ and $N\tau\text{-}\alpha cl(X) = X$.

(iii) Assume $A \subseteq B$, and since $B \subseteq N\tau\text{-}\alpha cl(B)$, then $A \subseteq N\tau\text{-}\alpha cl(B)$. Since $N\tau\text{-}\alpha cl(A)$ is the smallest $N\tau\alpha$-closed set containing A. Therefore, $N\tau\text{-}\alpha cl(A) \subseteq N\tau\text{-}\alpha cl(B)$.

(iv) Since $A \subseteq A \cup B$ and $B \subseteq A \cup B$. Then by (iii), we have $N\tau\text{-}\alpha cl(A) \cup N\tau\text{-}\alpha cl(B) \subseteq N\tau\text{-}\alpha cl(A \cup B)$. On the other hand, by(i), $A \cup B \subseteq N\tau\text{-}\alpha cl(A) \cup N\tau\text{-}\alpha cl(B)$. Since $N\tau\text{-}\alpha cl(A \cup B)$ is the smallest $N\tau\alpha$-closed set containing $A \cup B$. Then $N\tau\text{-}\alpha cl(A \cup B) \subseteq N\tau\text{-}\alpha cl(A) \cup N\tau\text{-}\alpha cl(B)$. Therefore, $N\tau\text{-}\alpha cl(A \cup B) = N\tau\text{-}\alpha cl(A) \cup N\tau\text{-}\alpha cl(B)$.

(v) Since $A \cap B \subseteq A$ and $A \cap B \subseteq B$, then $N\tau\text{-}\alpha cl(A \cap B) \subseteq N\tau\text{-}\alpha cl(A) \cap N\tau\text{-}\alpha cl(B)$.

(vi) Since $N\tau\text{-}\alpha cl(A)$ is a $N\tau\alpha$-closed set, then $N\tau\text{-}\alpha cl(N\tau\text{-}\alpha cl(A)) = N\tau\text{-}\alpha cl(A)$.

Remark 3.4. From the above theorem, we can observe that the closure operator $N\tau$-αcl satisfies the Kuratowski's closure axioms. The following theorem can be proved as the above theorem.

Theorem 3.5. Let $(X, N\tau)$ be a N-topological space on X and let $A, B \subseteq X$. Let $N\tau$-$kcl(A)$ is the intersection of all k-closed sets containing A (where k-closed set is can be any one of the following $N\tau$ semi-closed set, $N\tau$ pre-closed set and $N\tau\beta$-closed set). Then

(i) $N\tau$-$kcl(A)$ is the smallest k-closed set containing A.

(ii) A is k-closed iff $N\tau$-$kcl(A) = A$. In particular, $N\tau$-$kcl(\emptyset) = \emptyset$ and $N\tau$-$kcl(X) = X$.

(iii) $A \subseteq B \Rightarrow N\tau$-$kcl(A) \subseteq N\tau$-$kcl(B)$

(iv) $N\tau$-$kcl(A \cup B) \supseteq N\tau$-$kcl(A) \cup N\tau$-$kcl(B)$

(v) $N\tau$-$kcl(A \cap B) \subseteq N\tau$-$kcl(A) \cap N\tau$-$kcl(B)$

(vi) $N\tau$-$kcl(N\tau$-$kcl(A)) = N\tau$-$kcl(A)$.

Example 3.6. Let $X = \{a, b, c, d\}$. For $N = 3$, consider $\tau_1 O(X) = \{X, \emptyset, \{a\}\}$, $\tau_2 O(X) = \{X, \emptyset, \{b, c\}\}$ and $\tau_3 O(X) = \{X, \emptyset, \{a, b, c\}\}$. Then, we have $3\tau O(X) = \{X, \emptyset, \{a\}, \{b, c\}, \{a, b, c\}\} = N\tau\alpha O(X)$, $3\tau C(X) = \{X, \emptyset, \{d\}, \{a, d\}, \{b, c, d\}\}$. Also $3\tau SO(X) = \{\emptyset, X, \{a\}, \{a, d\}, \{b, c\}, \{a, b, c\}, \{b, c, d\}\}$, $3\tau PO(X) = \{\emptyset, X, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}, \{a, c, d\}, \{a, b, d\}\}$ and $3\tau\beta O(X) = \{\emptyset, X, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}, \{a, b, c\}, \{a, c, d\}, \{a, b, d\}, \{b, c, d\}\}$. Let $A = \{a\}$ and $B = \{b\}$. Then 3τ-$scl(A) \cup 3\tau$-$scl(B) = \{a\} \cup \{b, d\} = \{a, b, d\} \neq X = 3\tau$-$scl(A \cup B)$. Let $A = \{a\}$ and $B = \{b\}$. Then 3τ-$pcl(A) \cup 3\tau$-$pcl(B) = \{a\} \cup \{b\} = \{a, b\} \neq \{a, b, d\} = 3\tau$-$pcl(A \cup B)$. Also let $A = \{a\}$ and $B = \{b, c\}$, then 3τ-$\beta cl(A) \cup 3\tau$-$\beta cl(B) = \{a\} \cup \{b, c\} = \{a, b, c\} \neq X = 3\tau$-$\beta cl(A \cup B)$.

Theorem 3.7. Let $(X, N\tau)$ be a N-topological space on X and $A \subseteq X$. Let $N\tau$-$kcl(A)$ is the intersection of all k-closed sets containing A (where k-closed set is can be any one of the following $N\tau\alpha$-closed set, $N\tau$ semi-closed set, $N\tau$ pre-closed set and $N\tau\beta$-closed set). Then $x \in N\tau$-$kcl(A)$ if and only if $G \cap A \neq \emptyset$ for every k-open set G containing x.

Theorem 3.8. Let $(X, N\tau)$ be a N-topological space X and $A, B \subseteq X$. Then

(i) $N\tau$-$\alpha int(A)$ is the largest $N\tau\alpha$-open set contained in A.

(ii) A is $N\tau\alpha$-open set iff $N\tau$-$\alpha int(A) = A$. In particular, $N\tau$-$\alpha int(\emptyset) = \emptyset$ and $N\tau$-$\alpha int(X) = X$.

(iii) $A \subseteq B$, then $N\tau$-$\alpha int(A) \subseteq N\tau$-$\alpha int(B)$

(iv) $N\tau$-$\alpha int(A \cup B) \supseteq N\tau$-$\alpha int(A) \cup N\tau$-$\alpha int(B)$

(v) $N\tau$-$\alpha int(A \cap B) = N\tau$-$\alpha int(A) \cap N\tau$-$\alpha int(B)$

(vi) $N\tau$-$\alpha int(N\tau$-$\alpha int(A)) = N\tau$-$\alpha int(A)$

Proof: The proof is obvious from the fact that a set is $N\tau\alpha$-open if and only if its complement is $N\tau\alpha$-closed.

The proof of the following theorem can be proved as similar as the above theorem.

Theorem 3.9. Let $(X, N\tau)$ be a N-topological space X and $A, B \subseteq X$. Let $N\tau$-$kint(A)$ is the union of all k-open sets contained in A (where k-open set can be any one of $N\tau$ semi-open set, $N\tau$ pre-open set and $N\tau\beta$-open set). Then

(i) $N\tau$-$kint(A)$ is the largest k-open set contained in A.

(ii) A is k-open set iff $N\tau$-$kint(A) = A$. In particular, $N\tau$-$kint(\emptyset) = \emptyset$ and $N\tau$-$kint(X) = X$.

(iii) $A \subseteq B$, then $N\tau$-$kint(A) \subseteq N\tau$-$kint(B)$

(iv) $N\tau$-$kint(A \cup B) \supseteq N\tau$-$kint(A) \cup N\tau$-$kint(B)$

(v) $N\tau$-$kint(A \cap B) \subseteq N\tau$-$kint(A) \cap N\tau$-$kint(B)$

(vi) $N\tau\text{-}kint(N\tau\text{-}kint(A)) = N\tau\text{-}kint(A)$

Theorem 3.10. Let $(X, N\tau)$ be a N-topological space X and $A \subseteq X$. Let $N\tau\text{-}kint(A)$ and $N\tau\text{-}kcl(A)$ are the weak interior and closure operator in N-topological space. By k-closed set, we mean any one of the following $N\tau\alpha$-closed set, $N\tau$ semi-closed set, $N\tau$ pre-closed set and $N\tau\beta$-closed set. Then

(i) $N\tau\text{-}kint(X - A) = X - N\tau\text{-}kcl(A)$

(ii) $N\tau\text{-}kcl(X - A) = X - N\tau\text{-}kint(A)$

Remark 3.11. Let $(X, N\tau)$ be a N-topological space X and $A \subseteq X$. Let $N\tau\text{-}kint(A)$ and $N\tau\text{-}kcl(A)$ are the weak interior and closure operator in N-topological space. By k-closed set, we mean any one of the following $N\tau\alpha$-closed set, $N\tau$ semi-closed set, $N\tau$ pre-closed set and $N\tau\beta$-closed set. If we take the complement of either side of part(i) and part(ii) of previous theorems, we get

(i) $N\tau\text{-}kcl(A) = X - N\tau\text{-}kint(X - A)$

(ii) $N\tau\text{-}kint(A) = X - N\tau\text{-}kcl(X - A)$

4. Some Weak Continuous Functions in N-topology

In this section, we introduce some weak form of continuous functions in N-topological space and investigate the relationship between them. By the spaces X and Y, we means the N-topological spaces $(X, N\tau)$ and $(Y, N\sigma)$ respectively.

Definition 4.1. Let X and Y be two N-Topological spaces. A function $f : X \to Y$ is said to be N^*-α continuous (resp. N^*-semi continuous, N^*-pre continuous, N^*-β continuous) on X if the inverse image of every $N\sigma$-open set in Y is a $N\tau\alpha$-open set (resp. $N\tau$ semi-open, $N\tau$ pre-open, $N\tau\beta$-open) in X.

Theorem 4.2. A function $f : X \to Y$ is N^*-α continuous (resp. N^*-semi continuous, N^*-pre continuous, N^*-β continuous) on X if and only if the inverse image of every $N\sigma$-closed set in Y is a $N\tau\alpha$-closed set (resp. $N\tau$ semi-closed, $N\tau$ pre-closed, $N\tau\beta$-closed) in X.

Theorem 4.3. A function $f : X \to Y$ is N^*-continuous on X, then it is N^*-α continuous function on X.
Proof: Assume $f : X \to Y$ be a N^*-continuous function on X and let $A \subseteq Y$ be a $N\sigma$-open set. Then $f^{-1}(A) \subseteq X$ is $N\tau$-open set in X. Since every $N\tau$-open set is $N\tau\alpha$-open set, then f is N^*-α continuous on X.

The converse of the above theorem need not be true as shown in the following example.

Example 4.4. For $N = 2$, let $X = \{a, b, c\}$ and $Y = \{x, y, z\}$. Consider $\tau_1 O(X) = \{X, \emptyset\}$, $\tau_2 O(X) = \{X, \emptyset, \{a\}\}$ and $\sigma_1 O(Y) = \{Y, \emptyset, \{x\}\}$ and $\sigma_2 O(Y) = \{Y, \emptyset, \{x, y\}\}$. Then $2\tau O(X) = \{X, \emptyset, \{a\}\}$ and $2\sigma O(Y) = \{Y, \emptyset, \{x\}, \{x, y\}\}$. Define $f : X \to Y$ by $f(a) = x$, $f(b) = y$ and $f(c) = z$. Therefore, f is 2^*-α continuous function on X but not 2^*-continuous.

Theorem 4.5. A function $f : X \to Y$ is N^*-α continuous on X if and only if it is N^*-semi continuous and N^*-pre continuous.
Proof: The proof follows from the theorem 3.5.

Theorem 4.6. A function $f : X \to Y$ is N^*-semi continuous on X, then it is N^*-β continuous.

Theorem 4.7. A function $f : X \to Y$ is N^*-pre continuous on X, then it is N^*-β continuous.

The converse of the above theorems need not be true as shown in the following example.

Example 4.8. If $N = 2$, $X = \{a, b, c\}$ and $Y = \{x, y, z\}$. Consider $\tau_1 O(X) = \{X, \emptyset, \{a\}\}$, $\tau_2 O(X) = \{X, \emptyset, \{b, c\}\}$ and also $\sigma_1 O(Y) = \{Y, \emptyset, \{x\}\}$, $\sigma_2 O(Y) = \{Y, \emptyset, \{x, y\}\}$. Then $2\tau O(X) = \{X, \emptyset, \{a\}, \{b, c\}\}$, $2\sigma O(Y) = \{Y, \emptyset, \{x\}, \{x, y\}\}$. Define $f : X \to Y$ by $f(a) = x$, $f(b) = z$ and $f(c) = y$. Then f is 2^*-pre continuous and 2^*-β continuous function on X but it is not 2^*-semi continuous and not 2^*-α continuous function. Also if $N = 3$, $X = \{a, b, c\}$ and $Y = \{x, y, z\}$. Consider $\tau_1 O(X) = \{X, \emptyset, \{a\}, \{a, b\}\}$, $\tau_2 O(X) = \{X, \emptyset, \{b\}, \{a, b\}\}$, $\tau_3 O(X) = \{X, \emptyset, \{a, b\}\}$ and also $\sigma_1 O(Y) = \{Y, \emptyset, \{x\}\}$, $\sigma_2 O(Y) = \{Y, \emptyset, \{y, z\}\}$, $\sigma_3 O(Y) = \{Y, \emptyset\}$. Then $3\tau O(X) = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}\}$, $3\sigma O(Y) = \{Y, \emptyset, \{x\}, \{y, z\}\}$. Define $f : X \to Y$ by $f(a) = x$, $f(b) = y$ and $f(c) = z$. Then f is 3^*-semi continuous and 3^*-β continuous on X but it is not 3^*-pre continuous and not 3^*-α continuous.

5. Quotient Mappings in N-Topology

In this section, we introduce and establish the properties of some new types of quotient mappings in N-topological spaces.

Definition 5.1. Let X and Y be N-topological spaces, then a surjective map $f : X \to Y$ is said to be

(i) N^*-quotient map if f is N^*-continuous and for each subset G of Y, $f^{-1}(G)$ is $N\tau$-open (or $N\tau$-closed) in X implies G is $N\sigma$-open (or $N\sigma$-closed) in Y.

(ii) N^*-α quotient map if f is N^*-α continuous and for each subset G of Y, $f^{-1}(G)$ is $N\tau$-open (or $N\tau$-closed) in X implies G is $N\sigma\alpha$-open (or $N\sigma\alpha$-closed) in Y.

(iii) N^*-semi quotient map if f is N^*-semi continuous and for each subset G of Y, $f^{-1}(G)$ is $N\tau$-open (or $N\tau$-closed) in X implies G is $N\sigma$ semi-open (or $N\sigma$ semi-closed) in Y.

Proposition 5.2. Let X, Y be two N-topological spaces and $f : X \to Y$ be a surjective map. Then

(i) every N^*-quotient map is N^*-α quotient.

(ii) every N^*-quotient map is N^*-semi quotient.

(iii) every N^*-α quotient map is N^*-semi quotient.

Proof: The proof is straightforward from the definition.

The following examples show that the converse of the above proposition need not be true.

Example 5.3. For $N = 2$, let $X = \{a, b, c\}$ and $Y = \{x, y, z\}$. Consider $\tau_1 O(X) = \{X, \emptyset, \{a\}\}$, $\tau_2 O(X) = \{X, \emptyset\}$ and $\sigma_1 O(Y) = \{Y, \emptyset, \{x\}\}$ and $\sigma_2 O(Y) = \{Y, \emptyset, \{x, y\}\}$. Then $2\tau O(X) = \{X, \emptyset, \{a\}\}$ and $2\sigma O(Y) = \{Y, \emptyset, \{x\}, \{x, y\}\}$. Define $f : X \to Y$ by $f(a) = x$, $f(b) = y$ and $f(c) = z$. Therefore, f is 2^*-α quotient and 2^*-semi quotient map but not 2^*-quotient.

Example 5.4. For $N = 3$, let $X = \{a, b, c\}$ and $Y = \{x, y, z\}$. Consider $\tau_1 O(X) = \{X, \emptyset, \{a\}, \{a, b\}\}$, $\tau_2 O(X) = \{X, \emptyset, \{b\}, \{a, b\}\}$, $\tau_3 O(X) = \{X, \emptyset, \{b\}\}$ and $\sigma_1 O(Y) = \{Y, \emptyset, \{x\}, \{x, z\}\}$, $\sigma_2 O(Y) = \{Y, \emptyset, \{y\}, \{x, y\}\}$ and $\sigma_3 O(Y) = \{Y, \emptyset, \{x\}, \{x, y\}, \{x, z\}\}$. Then $3\tau O(X) = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}\}$ and $3\sigma O(Y) = \{Y, \emptyset, \{x\}, \{y\}, \{x, y\}, \{x, z\}\}$. Define $f : X \to Y$ by $f(a) = y$, $f(b) = x$ and $f(c) = z$. Therefore, f is 3^*-semi quotient map but not 3^*-α quotient and not 3^*-quotient.

Definition 5.5. Let X and Y be two N-topological spaces, then a map $f : X \to Y$ is said to be

(i) N^*-open (or N^*-closed) if for every $N\tau$-open ($N\tau$-closed) set G of X, $f(G)$ is $N\sigma$-open (or $N\sigma$-closed) in Y.

(ii) N^*-α open (or N^*-α closed) if for every $N\tau$-open ($N\tau$-closed) set G of X, $f(G)$ is $N\sigma\alpha$-open (or $N\sigma\alpha$-closed) in Y.

(iii) N^*-semi open (or N^*-semi closed) if for every $N\tau$-open ($N\tau$-closed) set G of X, $f(G)$ is $N\sigma$ semi-open (or $N\sigma$ semi-closed) in Y.

Theorem 5.6. (i) Every surjective N^*-continuous map $f : X \to Y$ which is either N^*-open or N^*-closed is N^*-quotient map.

(ii) Every surjective N^*-α continuous map $f : X \to Y$ which is either N^*-α open or N^*-α closed is N^*-α quotient map.

(iii) Every surjective N^*-semi continuous map $f : X \to Y$ which is either N^*-semi open or N^*-semi closed is N^*-semi quotient map.

Proof: The proof is trivial from the definition.

Lemma 5.7. Let X be a N-topological space, Y be a set and $f : X \to Y$ be a surjective map. Then define $N\tau_f = \{G \subseteq Y : f^{-1}(G) \in N\tau O(X)\}$ is a topology on Y relative to which f is a N^*-quotient map. It is called N^*-quotient topology on Y induced by f.

Proof: The proof follows from the facts that $f^{-1}(\emptyset) = \emptyset$, $f^{-1}(Y) = X$, $f^{-1}(\cup_{i=1}^{\infty} G_i) = \cup_{i=1}^{\infty} f^{-1}(G_i)$ and $f^{-1}(\cap_{i=1}^{n} G_i) = \cap_{i=1}^{n} f^{-1}(G_i)$.

The following lemmas can be proved similarly as the above lemma.

Lemma 5.8. Let X be a N-topological space, Y be a set and $f : X \to Y$ be a surjective map. Then define $N\tau\alpha_f = \{G \subseteq Y : f^{-1}(G) \in N\tau\alpha O(X)\}$ is a topology on Y relative to which f is a N^*-α quotient map. It is called N^*-α quotient topology on Y induced by f.

Lemma 5.9. Let X be a N-topological space, Y be a set and $f : X \to Y$ be a surjective map. Then define $N\tau S_f = \{G \subseteq Y : f^{-1}(G) \in N\tau SO(X)\}$ is a generalized topology on Y relative to which f is a N^*-semi quotient map but it need not be a topology. It is called N^*-semi quotient generalized topology on Y induced by f. If X is an extremally disconnected N-topological space, the intersection of two $N\tau$ semi-open sets in X is $N\tau$ semi-open and hence $N\tau S_f$ becomes a topology on Y.

Example 5.10. For $N = 2$, let $X = \{a, b, c\} = Y$. Consider $\tau_1 O(X) = \{X, \emptyset, \{a\}\} = \sigma_1 O(Y)$ and $\tau_2 O(X) = \{X, \emptyset\} = \sigma_2 O(Y)$. Then $2\tau O(X) = \{X, \emptyset, \{a\}\} = 2\sigma O(Y)$ and $2\tau\alpha O(X) = 2\tau SO(X) = 2\sigma\alpha O(Y) = 2\sigma SO(Y) = \{X, \emptyset, \{a\}, \{a, b\}, \{a, c\}\}$. Define $f : X \to Y$ by $f(a) = a$, $f(b) = b$ and $f(c) = c$. Clearly f is 2^*-quotient, 2^*-α quotient and 2^*-semi quotient map. Therefore, $2\tau_f = \{Y, \emptyset, \{a\}\}$ and $2\tau\alpha_f = 2\tau S_f = \{Y, \emptyset, \{a\}, \{a, b\}, \{a, c\}\}$.

Example 5.11. In example 6.4, f is 3^*-semi quotient map and therefore, $3\tau S_f = \{Y, \emptyset, \{x\}, \{y\}, \{x, y\}, \{x, z\}, \{y, z\}\}$ is not a topology on Y.

Theorem 5.12. Let X, Y, Z be N-topological spaces, $f : X \to Y$ be a N^*-quotient map and $h : X \to Z$ be a map that is constant on each set $f^{-1}(\{y\})$, for $y \in Y$. Then h induces a map $g : Y \to Z$ such that $g \circ f = h$. Then the induced map g is N^*-continuous if and only if h is N^*-continuous; g is N^*-quotient map if and only if h is N^*-quotient map.

Proof: Since h is constant on each set $f^{-1}(\{y\})$, for each $y \in Y$, the set $h(f^{-1}(\{y\}))$ is a one-point set in Z. Let us take this point as $g(y)$, then the map $g : Y \to Z$ such that for each $x \in X$, $g(f(x)) = h(x)$. If g is N^*-continuous, then $h = g \circ f$ is N^*-continuous. Conversely, assume h is N^*-continuous, for each $N\eta$-open set G of Z, $h^{-1}(G) = f^{-1}(g^{-1}(G))$ is $N\tau$-open in X. Since f is N^*-quotient, $g^{-1}(G)$ is $N\sigma$-open in Y and hence g is N^*-continuous.

If g is N^*-quotient map, then h is the composite of two N^*-quotient map and so is a N^*-quotient map. Conversely, assume h is a N^*-quotient map and since h is surjective, then g is surjective. Let $g^{-1}(G)$ be a $N\sigma$-open set in Y and since f is N^*-continuous, then the set $f^{-1}(g^{-1}(G)) = h^{-1}(G)$ is $N\tau$-open in X. Since h is a N^*-quotient map, G is $N\eta$-open in Z.

The following theorems can be proved similarly as the above theorem.

Theorem 5.13. Let X, Y, Z be N-topological spaces, $f : X \to Y$ be a N^*-α quotient map and $h : X \to Z$ be a N^*-continuous map that is constant on each set $f^{-1}(\{y\})$, for $y \in Y$. Then h induces a N^*-α continuous map $g : Y \to Z$ such that $g \circ f = h$.

Theorem 5.14. Let X, Y, Z be N-topological spaces, $f : X \to Y$ be a N^*-semi quotient map and $h : X \to Z$ be a N^*-continuous map that is constant on each set $f^{-1}(\{y\})$, for $y \in Y$. Then h induces a N^*-semi continuous map $g : Y \to Z$ such that $g \circ f = h$.

6. Conclusion

In this paper we established some weak form of open sets and its respective continuous and quotient mappings in our N-topological spaces. These concepts can be extended to other applicable research areas of topology such as Nano topology, Fuzzy topology, Supra topology and so on.

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Some Results on Lattice (Anti-Lattice) Ordered Double Framed Soft Sets

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Abstract − In this article, we generalised the notion of the lattice (anti-lattice) ordered soft sets and introduced the notion of the lattice (anti-lattice) ordered double framed soft sets and proved some results by applying the basic operations like union, intersection, union-product and intersection-product, etc. Further, by applying the operations of restricted union and restricted intersection, we elaborated the applications of lattice ordered double framed soft sets in algebraic structure.

Keywords − *Soft set, double framed soft set, lattice ordered double framed soft set, lattice ordered Boolean-algebra*

1. Introduction

In daily life there exists certain difficulties which deal with uncertainty, vague and precise like in environmental sciences, economics and engineering etc. To face such types of difficulties, there are many theories developed like probability theory, interval mathematical theory and theory of fuzzy set. These theories are classical mathematical tools. Due to the limitations of these theories, we felt hesitation in giving a comfortable solution to solve these problems, which are known as uncertainty, vague and precise. May be dealt with using a wide range existing theory such as the theory of fuzzy (intuitionistic fuzzy) set [1, 2, 3], the theory of interval mathematics [4], theory of probability, theory of vague set [5] and theory of rough set [6]. However, due to limitations and difficulties of these theories, Molodtsov [7] pointed out these problems and solved by introducing a new theory which is known as soft set theory. Maji et al. [8] introduced the applications of soft set theory in decision-making problems. Also, Maji et al. [9] studied the theoretical work on soft set theory to polish this concept so that readers could easily understand and contributed their role to extend the scope of this theory in different fields of life. After theoretical discussion, now we discussed the contributions of those researchers whose applied this concept in different fields of algebras like Aktaş and Çağman [10] studied the notion of soft sets and soft groups and introduced the notion of soft groups. They also defined the relation between fuzzy set, rough set and soft set and discussed its properties. Ali et al. [11] initiated the concept of lattice ordered soft sets and discussed some of its properties. Lattice ordered soft sets are very helpful in particular types of decision-making problems when there is some order between the elements of the parameter set. Mahmood et al. [12] initiated the concept of lattice ordered intuitionistic fuzzy soft sets. Mahmood et al. [13] worked on lattice ordered soft near rings. Jun and Ahn [14] initiated the notion of double framed soft set. For further information, we mention the readers to the papers [15-27] regarding soft algebras and properties of soft sets. Inspiring from the above literature and especially, the concept of lattice ordered soft sets [11]. This paper courage us to extend this concept into lattice ordered double framed soft sets because in this paper mentioned that sometimes we define particular order between linguistic terms, for example, the selections of the

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brilliant student based on percentage (80% to 90%) of marks in any educational institute of PhD Mathematics class.
This paper distributed in three sections, in 2nd section, some basic concepts of soft sets, properties of soft sets, lattice (anti-lattice) ordered soft set and double framed soft sets are discussed and introduced their notations. In the 3rd Section, we initiated the concept of the lattice (anti-lattice) ordered double framed soft sets and discussed their properties by using examples and results. Also, by using the notion of the lattice (anti-lattice) ordered double framed soft set we introduced the algebraic structures of the lattice (anti-lattice) ordered double framed soft set like bounded lattice, complemented lattice and distributed lattices etc. Note that for further study, we use "S-set" instead of soft set.

2. Preliminaries

In this section, we discussed some basic notions and properties related to S-set, lattice (anti-lattice) ordered S-set and double framed S-set.

Definition 2.1. [7] Let E be a parameter set, U be a universal set and let $P(U)$ denotes the power set of U and $A \subseteq E$. Then, a set-valued function α from A to $P(U)$ is called an S-set over U and is denoted as (α, A).

Definition 2.2. [9] A S-set (α, A) is called a soft subset of (β, B), over U if

1) $A \subseteq B$.
2) $\alpha(x) \subseteq \beta(x)$ for all $x \in A$.

It is denoted as $(\alpha, A) \tilde{\subseteq} (\beta, B)$. In this case (β, B) is called a soft superset of (α, A).

Definition 2.3. [9] Let (α, A) and (β, B) be S-sets over U. Then, (α, A) and (β, B) are called soft equal if $(\alpha, A) \tilde{\subseteq} (\beta, B)$ and $(\beta, B) \tilde{\subseteq} (\alpha, A)$.

Definition 2.4. Let L be a non-empty poset. Then, L is called a lattice if for each $\{x, y\} \subseteq L$ there exist $sup\{x, y\} \in L$ and $inf\{x, y\} \in L$.

Definition 2.5. A lattice having both first and last element is called bounded lattice.

Definition 2.6. A distributive lattice with the least and the greatest element is called Boolean algebra if and only if every element has a complement in it.

Definition 2.7. A bounded distributive lattice L along with a unary operation "c" which satisfies $(x \wedge y)^c = x^c \vee y^c$ and $(x^c)^c = x$ is called De 'Morgan's algebra.

Definition 2.8. A De 'Morgan's algebra which satisfies $x \wedge x^c \leq y \vee y^c$ for all x, y is called Kleene algebra.

Definition 2.9. [11] A S-set (α, A) is said to be lattice (anti-lattice) ordered S-set if $x_1 \leq x_2$ implies $\alpha(x_1) \subseteq \alpha(x_2)$ $\big(\alpha(x_2) \subseteq \alpha(x_1)\big)$ for all $x_1, x_2 \in A$.

Definition 2.10. [14] A set $\big((\alpha, \beta), A\big)$ is said to be double framed soft set (DFS-set), where α and β both are S-sets over U and A is a subset of E (E is the set of parameters).

Definition 2.11. [14] Let $\big((\alpha, \beta), A\big)$ and $\big((\lambda, \mu), B\big)$ be double framed soft sets (DFS-sets) over U. Then, $\big((\alpha, \beta), A\big)$ is called a double framed soft subset (DFS-subset) of $\big((\lambda, \mu), B\big)$ if

1) $A \subseteq B$,
2) $\alpha(x) \subseteq \lambda(x)$, $\beta(x) \supseteq \mu(x)$ for all $x \in A$.

We write $\big((\alpha,\beta),A\big) \tilde{\sqsubseteq} \big((\lambda,\mu),B\big)$. In this case $\big((\lambda,\mu),B\big)$ is called a DFS-superset of $\big((\alpha,\beta),A\big)$.

Definition 2.12. [14] Let $\big((\alpha,\beta),A\big)$ and $\big((\lambda,\mu),B\big)$ be DFS-sets over U. Then, their uni-int product is defined as a DFS-set $\big((H_1,H_2),D\big) = \big((\alpha,\beta),A\big) \vee \big((\lambda,\mu),B\big)$, where $D = A \times B$, $H_1 = \alpha \vee \lambda$, $H_2 = \beta \wedge \mu$ and $H_1(x_1,y_1) = \alpha(x_1) \cup \lambda(y_1)$, $H_2(x_1,y_1) = \lambda(x_1) \cap \mu(y_1)$ for all $(x_1,y_1) \in A \times B$, where $x_1 \in A$ and $y_1 \in B$. We shall call this uni-int product of DFS-set as union-product of DFS-set.

Definition 2.13. [14] Let $\big((\alpha,\beta),A\big)$ and $\big((\lambda,\mu),B\big)$ be DFS-sets over U. Then, their int-uni product is defined as a DFS-set $\big((H_1,H_2),D\big) = \big((\alpha,\beta),A\big) \wedge \big((\lambda,\mu),B\big)$, where $D = A \times B$ and $H_1(x_1,y_1) = \alpha(x_1) \cap \lambda(y_2)$, $H_2(x_1,y_1) = \lambda(x_1) \cup \mu(y_1)$ for all $(x_1,y_1) \in A \times B$, $H_1 = \alpha \wedge \lambda$, $H_2 = \beta \vee \mu$ where $x_1 \in A$ and $y_1 \in B$. We shall call this int-uni product of DFS-set as intersection-product of DFS-set.

Definition 2.14. [16] Let $\big((\alpha,\beta),A\big)$ and $\big((\lambda,\mu),B\big)$ be DFS-sets over U. Then, their extended uni-int is defined as a DFS-set $\big((H_1,H_2),A \cup B\big)$, where $H_1 = \alpha \,\tilde{\cup}\, \lambda : (A \cup B) \longrightarrow P(U)$ defined as

$$e \longrightarrow \begin{cases} \alpha(e) & if\ e \in A\setminus B \\ \lambda(e) & if\ e \in B\setminus A \\ \alpha(e) \cup \lambda(e) & if\ e \in A \cap B \end{cases}$$

and $H_2 = \beta \,\tilde{\cap}\, \mu: (A \cup B) \longrightarrow P(U)$ defined as

$$e \longrightarrow \begin{cases} \beta(e) & if\ e \in A\setminus B \\ \mu(e) & if\ e \in B\setminus A \\ \beta(e) \cap \mu(e) & if\ e \in A \cap B \end{cases}$$

It is denoted as $\big((\alpha,\beta),A\big) \sqcup_{\varepsilon} \big((\lambda,\mu),B\big) = \big((H_1,H_2),A \cup B\big)$. We shall call this extended uni-int of DFS-set as union of DFS-set.

Definition 2.15. [16] Let $\big((\alpha,\beta),A\big)$ and $\big((\lambda,\mu),B\big)$ are two DFS-sets over U. Then, their extended int-uni is defined as a DFS-set $\big((H_1,H_2),A \cup B\big)$, where $H_1 = \alpha \,\tilde{\cap}\, \lambda : (A \cup B) \longrightarrow P(U)$ defined as

$$e \longrightarrow \begin{cases} \alpha(e) & if\ e \in A\setminus B \\ \lambda(e) & if\ e \in B\setminus A \\ \alpha(e) \cap \lambda(e) & if\ e \in A \cap B \end{cases}$$

and $H_2 = \beta \,\tilde{\cup}\, \mu: (A \cup B) \longrightarrow P(U)$ defined as

$$e \longrightarrow \begin{cases} \beta(e) & if\ e \in A\setminus B \\ \mu(e) & if\ e \in B\setminus A \\ \beta(e) \cup \mu(e) & if\ e \in A \cap B \end{cases}$$

It is denoted as $\big((\alpha,\beta),A\big) \sqcap_{\varepsilon} \big((\lambda,\mu),B\big) = \big((H_1,H_2),A \cup B\big)$. We shall call this extended int-uni of DFS-set as intersection of DFS-set.

Definition 2.16. [16] Let $\big((\alpha,\beta),A\big)$ and $\big((\lambda,\mu),B\big)$ be DFS-sets over U such that $A \cap B \neq \emptyset$. Then, their restricted uni-int is denoted as $\big((\alpha,\beta),A\big) \sqcup \big((\lambda,\mu),B\big)$ and defined as $\big((\alpha,\beta),A\big) \sqcup \big((\lambda,\mu),B\big) = \big((H_1,H_2),D\big)$ where $D = A \cap B$ and for every $x \in D$, $H_1(x) = \alpha(x) \cup \lambda(x)$, $H_2(x) = \lambda(x) \cap \mu(x)$. We shall call this restricted uni-int of DFS-set as restricted union of DFS-set.

Definition 2.17. [16] Let $\big((\alpha,\beta),A\big)$ and $\big((\lambda,\mu),B\big)$ be DFS-set over U such that $A\cap B\neq\emptyset$. Then, their restricted int-uni is denoted as $\big((\alpha,\beta),A\big)\sqcap\big((\lambda,\mu),B\big)$ and defined as $\big((\alpha,\beta),A\big)\sqcap\big((\lambda,\mu),B\big)=\big((H_1,H_2),D\big)$, where $D=A\cap B$ and for all $x\in D$, $H_1(x)=\alpha(x)\cap\lambda(x)$, $H_2(x)=\lambda(x)\cup\mu(x)$. We shall call this restricted int-uni of DFS-set as restricted intersection of DFS-set.

Definition 2.18. [16] A DFS-set $\big((\alpha,\beta),A\big)$ over U is called relative whole DFS-set, if $\alpha\colon A\longrightarrow P(U)$ and $\beta\colon A\longrightarrow P(U)$ are defined as

$\alpha(x)=U$ and $\beta(x)=\emptyset$ for all $x\in A$.

It is denoted as $A_{(\mathfrak{U},\emptyset)}$.

Definition 2.19. [16] A DFS-set $\big((\alpha,\beta),A\big)$ over U is called relative null DFS-set, if $\alpha : A\longrightarrow P(U)$ and $\beta : A\longrightarrow P(U)$ are defined as

$\alpha(x)=\emptyset$ and $\beta(x)=U$ for all $x\in A$.

It is denoted as $A_{(\emptyset,\mathfrak{U})}$.

Definition 2.20. [16] For a DFS-set $\big((\alpha,\beta),A\big)$, the complement of $\big((\alpha,\beta),A\big)$ is defined as a DFS-set $\big((\alpha^c,\beta^c),A\big)$, where $\alpha^c : A\longrightarrow P(U)$ and $\beta^c : A\longrightarrow P(U)$ are defined as

$\alpha^c(x)=\big(\alpha(x)\big)^c$ and $\beta^c(x)=\big(\beta(x)\big)^c$ for all $x\in A$.

It is denoted as $\big((\alpha,\beta),A\big)^c\cong\big((\alpha^c,\beta^c),A\big)$.

Proposition 2.21. (De Morgan's Laws)

Let (α,A) and (β,B) be LOS-sets (ALOS-sets) over U. Then,

(1) $\big((\alpha,A)\sqcup_{\mathcal{E}}(\beta,B)\big)^c=(\alpha,A)^c\sqcap_{\mathcal{E}}(\beta,B)^c$, if $A=B$.

(2) $\big((\alpha,A)\sqcap_{\mathcal{E}}(\beta,B)\big)^c=(\alpha,A)^c\sqcup_{\mathcal{E}}(\beta,B)^c$, if $A=B$.

(3) $\big((\alpha,A)\vee(\beta,B)\big)^c=(\alpha,A)^c\wedge(\beta,B)^c$.

(4) $\big((\alpha,A)\wedge(\beta,B)\big)^c=(\alpha,A)^c\vee(\beta,B)^c$.

Proposition 2.22 If (α,A), (β,B) and (γ,C) be any LOS-sets (ALOS-sets) over U. Then, followings are LOS-sets (ALOS-sets),

(1) $(\alpha,A)\vee\big((\beta,B)\sqcup_{\mathcal{E}}(\gamma,C)\big)$

(2) $(\alpha,A)\vee\big((\beta,B)\sqcap_{\mathcal{E}}(\gamma,C)\big)$

(3) $(\alpha,A)\wedge\big((\beta,B)\sqcup_{\mathcal{E}}(\gamma,C)\big)$

(4) $(\alpha,A)\wedge\big((\beta,B)\sqcap_{\mathcal{E}}(\gamma,C)\big)$

(5) $(\alpha,A)\vee\big((\beta,B)\sqcap(\gamma,C)\big)$

(6) $(\alpha,A)\wedge\big((\beta,B)\sqcup(\gamma,C)\big)$

If $\sqcup_{\mathcal{E}}$ and $\sqcap_{\mathcal{E}}$ are LOS-set (ALOS-set).

Theorem 2.23. $\big(DFSS(U)_A,\sqcup,{}^c,A_{(\emptyset,\mathfrak{U})}\big)$ is an MV-algebra.

Proof. (1) $\big(DFSS(U)_A,\sqcup,{}^c,A_{(\emptyset,\mathfrak{U})}\big)$ is a commutative monoid.

(2) $\Big(A_{(\alpha_1,\beta_1)^c}\Big)^c=A_{(\alpha_1,\beta_1)}$

The other conditions satisfied trivially. Hence, $\big(DFSS(U)_A,\sqcup,{}^c,A_{(\emptyset,\mathfrak{U})}\big)$ is MV-algebra.

Theorem 2.24. $\big(DFSS(U)_A,\ \sqcap,{}^c,A_{(\mathfrak{U},\emptyset)}\big)$ is an MV-algebra.

Proof. It follows from the above theorem.

3. Lattice (Anti-Lattice) Ordered Double Framed Soft Sets

In this section, our primary purpose is to define lattice (anti-lattice) ordered double framed S-set and discuss their properties and results with the help of examples. Note that we write LODFS-set and ALODFS-set for lattice ordered double framed soft set, and anti-lattice ordered double framed soft set respectively unless otherwise specified.

Definition 3.1. A DFS-set $\left((\alpha,\beta),A\right)$ is called LODFS-set (ALODFS-set) if $x_1 \leq x_2$ implies $\alpha(x_1) \subseteq \alpha(x_2)$ and $\beta(x_1) \supseteq \beta(x_2)$ $\left(\alpha(x_1) \supseteq \alpha(x_2) \text{ and } \beta(x_1) \subseteq \beta(x_2)\right)$ for all $x_1, x_2 \in A$.

Example 3.2. Let a company prepare a different design of cars in different colours like, $A = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7\}$ be the set of parameters which represent different types of colours of cars, where $e_7 = white, e_6 = black, e_5 = Grey, e_4 = red, e_3 = blue, e_2 = green, e_1 = yellow$ and $U = \{u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8\}$ be the set of new designs of cars in different colours. To sell these cars, the company define lattice order between the parameters which depend upon the demand of people under the supervision of two experts say α and β. The order between the elements of A is shown in Fig. 1. $\alpha, \beta: A \longrightarrow P(U)$ are two set-valued mappings representing high-cost and low-cost of cars. Therefore, DFS-set $\left((\alpha,\beta),A\right)$ showing high-cost and low-cost for design in colours may be considered as

$$\{\alpha(e_1) = \{u_1\}, \alpha(e_2) = \{u_1, u_2\}, \alpha(e_3) = \{u_1, u_3\}, \alpha(e_4) = \{u_1, u_2, u_3, u_4\}, \alpha(e_5) = \{u_1, u_3, u_5\},$$
$$\alpha(e_6) = \{u_1, u_2, u_3, u_4, u_5, u_6\}, \alpha(e_7) = \{u_1, u_2, u_3, u_4, u_5, u_6, u_7\}, \beta(e_7) = \{u_1, u_8\},$$
$$\beta(e_6) = \{u_1, u_3, u_8\}, \beta(e_5) = \{u_1, u_3, u_8\}, \beta(e_4) = \{u_1, u_2, u_3, u_4, u_5, u_8\}, \beta(e_3) = \{u_1, u_3, u_5, u_8\},$$
$$\beta(e_2) = \{u_1, u_2, u_3, u_4, u_5, u_6, u_8\}, \beta(e_1) = \{u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8\}\}$$

It is more appropriate to characterise DFS-set in the form of a table, for computer application.

The tabular form of DFS-set $\left((\alpha,\beta),A\right)$ is defined in Table 1. If a car having a different colour in a set U has high cost or low cast we write 1, otherwise we write 0.
From Table 1, we can easily see that

$\alpha(e_1) \subseteq \alpha(e_2) \subseteq \alpha(e_4) \subseteq \alpha(e_6) \subseteq \alpha(e_7)$, $\alpha(e_1) \subseteq \alpha(e_3) \subseteq \alpha(e_5) \subseteq \alpha(e_6) \subseteq \alpha(e_7)$, $\alpha(e_1) \subseteq \alpha(e_3) \subseteq \alpha(e_4) \subseteq \alpha(e_6) \subseteq \alpha(e_7)$ and $\beta(e_1) \supseteq \beta(e_2) \supseteq \beta(e_4) \supseteq \beta(e_6) \supseteq \beta(e_7)$, $\beta(e_1) \supseteq \beta(e_3) \supseteq \beta(e_4) \supseteq \beta(e_6) \supseteq \beta(e_7)$, $\beta(e_1) \supseteq \beta(e_3) \supseteq \beta(e_5) \supseteq \beta(e_6) \supseteq \beta(e_7)$.

Example 3.3. Let $U = \{u_1, u_2, u_3, u_4, u_5, u_6, u_7\}$ (universe set) be the set of seven buildings and $B = \{e_1, e_2, e_3, e_4\}$ be a set of parameters, where

e_1; one-floor building.
e_2; two-floor building.
e_3; three-floor building.
e_4; four-floor building.

There is an order between the elements of B. This order can be nominated as $e_1 \leq e_2 \leq e_3 \leq e_4$. Now the DFS-set $\left((\lambda,\mu),B\right)$ defined as $\{\lambda(e_1) = \{u_1, u_3\}, \lambda(e_2) = \{u_1, u_3, u_5\}, \lambda(e_3) = \{u_1, u_2, u_3, u_4, u_5\}, \lambda(e_4) = \{u_1, u_2, u_3, u_4, u_5, u_6\}, \mu(e_4) = \{u_1, u_5\}, \mu(e_3) = \{u_1, u_2, u_5\}, \mu(e_2) = \{u_1, u_2, u_3, u_4, u_5\}, \mu(e_1) = \{u_1, u_2, u_3, u_4, u_5, u_6, u_7\}\}$.

Then, $\lambda(e_4) \supseteq \lambda(e_3) \supseteq \lambda(e_2) \supseteq \lambda(e_1)$ and $\mu(e_4) \subseteq \mu(e_3) \subseteq \mu(e_2) \subseteq \mu(e_1)$. Thus, $\left((\lambda,\mu),B\right)$ is ALODFS-set. The tabular form of ALODFS-set $\left((\lambda,\mu),B\right)$ is defined in Table 2.

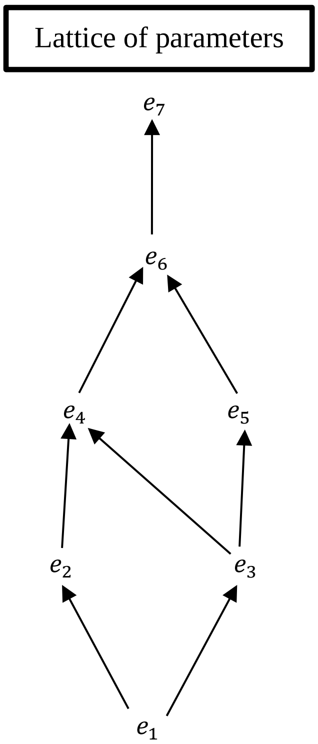

Fig. 1. Lattice of parameters

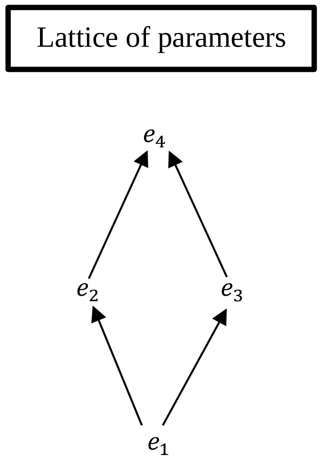

Fig. 2. Lattice of parameters

Table 1 LODFS-set $((\alpha,\beta),A)$

	u_1	u_2	u_3	u_4	u_5	u_6	u_7	u_8
e_1	(1, 1)	(0, 1)	(0, 1)	(0, 1)	(0, 1)	(0, 1)	(0, 1)	(0, 1)
e_2	(1, 1)	(1, 1)	(0, 1)	(0, 1)	(0, 1)	(0, 1)	(0, 0)	(0, 1)
e_3	(1, 1)	(0, 0)	(1, 1)	(0, 0)	(0, 1)	(0, 0)	(0, 0)	(0, 1)
e_4	(1, 1)	(1, 1)	(1, 1)	(1, 1)	(1, 0)	(0, 0)	(0, 0)	(0, 1)
e_5	(1, 1)	(0, 0)	(1, 1)	(0, 0)	(1, 0)	(0, 0)	(0, 0)	(0, 1)
e_6	(1, 1)	(1, 0)	(1, 1)	(1, 0)	(1, 0)	(1, 0)	(0, 0)	(0, 1)
e_7	(1, 1)	(1, 0)	(1, 0)	(1, 0)	(1, 0)	(1, 0)	(1, 0)	(0, 1)

Table 2 ALODFS-set $((\lambda,\mu),B)$

	u_1	u_2	u_3	u_4	u_5	u_6	u_7
e_1	(1, 1)	(0, 1)	(1, 1)	(0, 1)	(0, 1)	(0, 1)	(0, 1)
e_2	(1, 1)	(0, 1)	(1, 1)	(0, 1)	(1, 1)	(0, 0)	(0, 0)
e_3	(1, 1)	(1, 1)	(1, 0)	(1, 0)	(1, 1)	(0, 0)	(0, 0)
e_4	(1, 1)	(1, 0)	(1, 0)	(1, 0)	(1, 1)	(1, 0)	(0, 0)

Note that, we can easily understand LODFS-set and ALODFS-set from Table 1. and 2.

Proposition 3.4. Restricted union of two LODFS-set (ALODFS-set) $((\alpha,\beta),A)$ and $((\lambda,\mu),B)$ is a LODFS-set (ALODFS-set).

Proof. Let $((\alpha,\beta),A)$ and $((\lambda,\mu),B)$ are two LODFS-set. Then, their restricted union is defined as such that $((\alpha,\beta),A)\sqcup((\lambda,\mu),B)=((H_1,H_2),C)$ where $H_1=\alpha\,\tilde{\cup}\,\lambda$, $H_2=\beta\,\tilde{\cap}\,\mu$ and $C=A\cap B$. If $A\cap B=\emptyset$, then the result is trivial. Now assume that $A\cap B\neq\emptyset$, since $A,B\subseteq E$, then both A and B inherit partial order from E. So, if $x_1\leq_A x_2$ for all $x_1,x_2\in A$, then $\alpha(x_1)\subseteq\alpha(x_2)$ and $\beta(x_1)\supseteq\beta(x_2)$. Similarly, if $y_1\leq_B y_2$ for all $y_1,y_2\in B$, then $\lambda(y_1)\subseteq\lambda(y_2)$ and $\mu(y_1)\supseteq\mu(y_2)$. Therefore, for any $z_1,z_2\in C$ such that $\alpha(z_1)\subseteq\alpha(z_2)$, $\beta(z_1)\supseteq\beta(z_2)$ and $\lambda(z_1)\subseteq\lambda(z_2)$, $\mu(z_1)\supseteq\mu(z_2)$. Then, $\alpha(z_1)\cup\lambda(z_1)\subseteq\alpha(z_2)\cup\lambda(z_2)$ and $\beta(z_1)\cap\mu(z_1)\supseteq\beta(z_2)\cap\mu(z_2)$ implies that $H_1(z_1)\subseteq$

$H_1(z_2)$ and $H_2(z_1) \supseteq H_2(z_2)$ for $(z_1, z_2) \in \leq_C$. Thus, we conclude that the restricted union of two DFS-set is also double framed soft set.

Proposition 3.5. The restricted intersection of two LODFS-set (ALODFS-set) $((\alpha, \beta), A)$ and $((\lambda, \mu), B)$ is a LODFS-set (ALODFS-set).

Proof. The proof is like to Proposition 3.4, by using the definition of the restricted intersection.
The following example illustrates that in general, the union and intersection of LODFS-set (ALODFS-set) may not be a LODFS-set (ALODFS-set).
From now to onward, we use a table to understand LODFS-set and ALODFS-set.

Example 3.6. Let $E = \{e_1, e_2, e_3, e_4, e_5\}$ be a lattice ordered set which is defined in fig. 4. Let $A = \{e_1, e_2, e_4, e_5\}$ and $B = \{e_1, e_2, e_3, e_4\}$. Consider two LODFS-set $((\alpha, \beta), A)$ and $((\lambda, \mu), B)$ over an initial universal set $U = \{u_1, u_2, u_3, u_4, u_5\}$ are defined as shown in Table 3 and 4 respectively such that $\alpha(e_1) \subseteq \alpha(e_2) \subseteq \alpha(e_4) \subseteq \alpha(e_5)$ and $\beta(e_1) \supseteq \beta(e_2) \supseteq \beta(e_4) \supseteq \beta(e_5)$.

Table 3 LODFS-set $((\alpha, \beta), A)$

	u_1	u_2	u_3	u_4	u_5
e_1	(1, 1)	(0, 1)	(0, 1)	(0, 0)	(0, 1)
e_2	(1, 1)	(0, 1)	(0, 1)	(1, 0)	(0, 0)
e_4	(1, 1)	(0, 0)	(0, 1)	(1, 0)	(1, 0)
e_5	(1, 1)	(1, 0)	(0, 0)	(1, 0)	(1, 0)

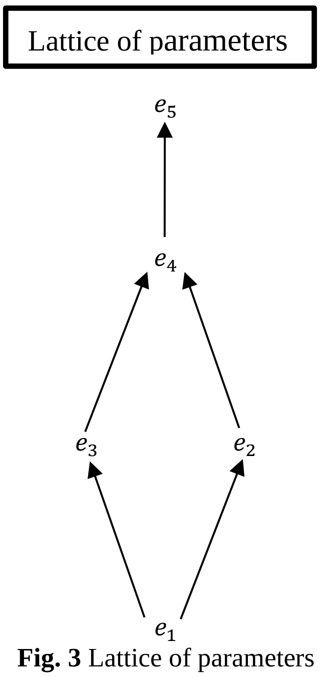

Fig. 3 Lattice of parameters

Table 4. LODFS-set $((\lambda, \mu), B)$

	u_1	u_2	u_3	u_4	u_5
e_1	(0, 1)	(0, 1)	(0, 1)	(0, 1)	(0, 1)
e_2	(0, 0)	(0, 1)	(0, 1)	(1, 0)	(1, 0)
e_3	(0, 0)	(1, 1)	(1, 0)	(1, 1)	(0, 0)
e_4	(0, 0)	(1, 1)	(1, 0)	(1, 0)	(1, 0)

Now by definition of union, we have $((\alpha,\beta),A) \sqcup_{\mathcal{E}} ((\lambda,\mu),B) = ((H_1,H_2),C)$, where $H_1 = \alpha \tilde{\cup} \lambda$, $H_2 = \beta \tilde{\cap} \mu$ and $C = A \cup B$, so we have the following table for the union.

Table 5. The union of the LODFS-sets

	u_1	u_2	u_3	u_4	u_5
e_1	(1, 1)	(0, 1)	(0, 1)	(0, 0)	(0, 1)
e_2	(1, 0)	(0, 1)	(0, 1)	(1, 0)	(1, 0)
e_3	(0, 0)	(1, 1)	(1, 0)	(1, 1)	(0, 0)
e_4	(1, 0)	(1, 0)	(1, 0)	(1, 0)	(1, 0)
e_5	(1, 1)	(1, 0)	(0, 0)	(1, 0)	(1, 0)

From Table 5, we note that the union of $((\alpha,\beta),A)$ and $((\lambda,\mu),B)$ is not LODFS-set because $H_1(e_4) \nsubseteq H_1(e_5)$, $H_2(e_1) \nsupseteq H_2(e_3)$ and $H_2(e_4) \nsupseteq H_2(e_5)$ so $((H_1,H_2),C)$ is not a LODFS-set.

Now by definition of intersection, we have $((\alpha,\beta),A) \sqcap_{\mathcal{E}} ((\lambda,\mu),B) = ((H_3,H_4),D)$ where $H_3 = \alpha \tilde{\cap} \lambda$, $H_4 = \beta \tilde{\cup} \mu$ and $C = A \cup B$, so we have the following table for the intersection.

Table. 6 The intersection of the LODFS-sets

	u_1	u_2	u_3	u_4	u_5
e_1	(0, 1)	(0, 1)	(0, 1)	(0, 1)	(0, 1)
e_2	(0, 1)	(0, 1)	(0, 1)	(1, 0)	(0, 0)
e_3	(0, 0)	(1, 1)	(1, 0)	(1, 1)	(0, 0)
e_4	(0, 1)	(0, 1)	(0, 1)	(1, 0)	(1, 0)
e_5	(1, 1)	(1, 0)	(0, 0)	(1, 0)	(1, 0)

From Table. 6, we note that intersection of two LODFS-set $((\alpha,\beta),A)$ and $((\lambda,\mu),B)$ is not LODFS-set because $H_3(e_3) \nsubseteq H_3(e_4)$ and $H_4(e_3) \nsupseteq H_4(e_5)$ implies $((H_3,H_4),D)$ is not a LODFS-set.

Notice that from the above example, in general union and intersection of two LODFS-set may not a LODFS-set. Similarly, in general union and intersection of two ALODFS-set may not be an ALODFS-set. However, we can define the following.

Proposition 3.7. The intersection of two LODFS-set (ALODFS-set) $((\alpha,\beta),A)$ and $((\lambda,\mu),B)$ is LODFS-set (ALODFS-set) if $((\alpha,\beta),A) \subseteq ((\lambda,\mu),B)$ or $((\lambda,\mu),B) \subseteq ((\alpha,\beta),A)$.

Proof. Let $((\alpha,\beta),A)$ and $((\lambda,\mu),B)$ be LODFS-set, then by definition of intersection we have $((\alpha,\beta),A) \sqcap_{\mathcal{E}} ((\lambda,\mu),B) = (H,C)$, where $C = A \cup B$, $H = (H_1,H_2)$ and $H_1 = \alpha \tilde{\cap} \lambda$, $H_2 = \beta \tilde{\cup} \mu$. Now without any loss of generality, we say that $((\alpha,\beta),A) \subseteq ((\lambda,\mu),B)$. Since $A \subseteq B$, then $A \cup B = B$ implies $B = C$. As $B = C$ so $H(z) = (H_1,H_2)(z)$ for all $z \in C$ implies that (H,C) is LODFS-set. Hence the intersection of two LODFS-set is also LODFS-set if one of them is contained into other.

Similarly, we can prove for ALODFS-set.

Proposition 3.8. Union of two LODFS-set (ALODFS-set) $((\alpha,\beta),A)$ and $((\lambda,\mu),B)$ is LODFS-set (ALODFS-set) if $((\alpha,\beta),A) \subseteq ((\lambda,\mu),B)$ or $((\lambda,\mu),B) \subseteq ((\alpha,\beta),A)$.

Proof. Like the above Proposition using the definition of the union of LODFS-set (ALODFS-set). The complement of LODFS-set $((\alpha,\beta),A)$ is denoted as $((\alpha,\beta),A)^c$ and defined as $((\alpha,\beta),A)^c =$

$((\alpha,\beta)^c, A) = ((\alpha^c, \beta^c), A)$, where $(\alpha,\beta)^c = (\alpha^c, \beta^c)$ and $\alpha^c, \beta^c: A \longrightarrow P(U)$ are defined as $\alpha^c(a) = U \backslash \alpha(a)$ and $\beta^c(a) = U \backslash \beta(a)$ for all $a \in A$ is called ALODFS-set.

Similarly, the complement of ALODFS-set is LODFS-set.

Proposition 3.9. (De Morgan Laws)

Let $((\alpha,\beta), A)$ and $((\lambda,\mu), B)$ be LODFS-sets (ALODFS-sets) over U. Then,

1) $\left(((\alpha,\beta), A) \sqcup_\varepsilon ((\lambda,\mu), B)\right)^c = ((\alpha,\beta), A)^c \sqcap_\varepsilon ((\lambda,\mu), B)^c$, if $A = B$.
2) $\left(((\alpha,\beta), A) \sqcap_\varepsilon ((\lambda,\mu), B)\right)^c = ((\alpha,\beta), A)^c \sqcup_\varepsilon ((\lambda,\mu), B)^c$, if $A = B$.
3) $\left(((\alpha,\beta), A) \sqcup ((\lambda,\mu), B)\right)^c = ((\alpha,\beta), A)^c \sqcap ((\lambda,\mu), B)^c$
4) $\left(((\alpha,\beta), A) \sqcap ((\lambda,\mu), B)\right)^c = ((\alpha,\beta), A)^c \sqcup ((\lambda,\mu), B)^c$

Proposition 3.10. (Distributive Laws)

If $((\alpha,\beta), A)$, $((\lambda,\mu), B)$ and $((\gamma,\delta), C)$ be any LODFS-sets (ALODFS-sets) over U, then the following conditions hold

1) $((\alpha,\beta), A) \sqcup \left(((\lambda,\mu), B) \sqcup_\varepsilon ((\gamma,\delta), C)\right) = \left(((\alpha,\beta), A) \sqcup ((\lambda,\mu), B)\right) \sqcup_\varepsilon \left(((\alpha,\beta), A) \sqcup ((\gamma,\delta), C)\right)$ if $A \subseteq C$.
2) $((\alpha,\beta), A) \sqcup \left(((\lambda,\mu), B) \sqcap_\varepsilon ((\gamma,\delta), C)\right) = \left(((\alpha,\beta), A) \sqcup ((\lambda,\mu), B)\right) \sqcap_\varepsilon \left(((\alpha,\beta), A) \sqcup ((\gamma,\delta), C)\right)$ if $A \subseteq C$.
3) $((\alpha,\beta), A) \sqcap \left(((\lambda,\mu), B) \sqcup_\varepsilon ((\gamma,\delta), C)\right) = \left(((\alpha,\beta), A) \sqcap ((\lambda,\mu), B)\right) \sqcup_\varepsilon \left(((\alpha,\beta), A) \sqcap ((\gamma,\delta), C)\right)$ if $A \subseteq C$.
4) $((\alpha,\beta), A) \sqcap \left(((\lambda,\mu), B) \sqcap_\varepsilon ((\gamma,\delta), C)\right) = \left(((\alpha,\beta), A) \sqcap ((\lambda,\mu), B)\right) \sqcap_\varepsilon \left(((\alpha,\beta), A) \sqcap ((\gamma,\delta), C)\right)$ if $A \subseteq C$.
5) $((\alpha,\beta), A) \sqcup \left(((\lambda,\mu), B) \sqcap ((\gamma,\delta), C)\right) = \left(((\alpha,\beta), A) \sqcup ((\lambda,\mu), B)\right) \sqcap \left(((\alpha,\beta), A) \sqcup ((\gamma,\delta), C)\right)$
6) $((\alpha,\beta), A) \sqcap \left(((\lambda,\mu), B) \sqcup ((\gamma,\delta), C)\right) = \left(((\alpha,\beta), A) \sqcap ((\lambda,\mu), B)\right) \sqcup \left(((\alpha,\beta), A) \sqcap ((\gamma,\delta), C)\right)$

Let A and B be ordered sets, then σ be a partial order on $A \times B$ defined in such a way that, for (x, y), $(x', y') \in A \times B$ such that $(x, y) \leq (x', y')$ if and only if $x \leq_A x'$ and $y \leq_B y'$. From now to onward we will use σ for partial order relation on $A \times B$.

Proposition 3.11. Let $((\alpha,\beta), A)$ and $((\lambda,\mu), B)$ be LODFS-sets (ALODFS-sets), then their union-product is also a LODFS-set (ALODFS-set).

Proof. Since $((\alpha,\beta), A)$ and $((\lambda,\mu), B)$ are LODFS-sets so we must prove $((\alpha,\beta), A) \vee ((\lambda,\mu), B)$ is LODFS-set. Now by definition of union-product we have $((\alpha,\beta), A) \vee ((\lambda,\mu), B) = ((H_1, H_2), D)$ where $D = A \times B$, is a poset. Now $A, B \subseteq E$, so both A and B have taken some partial ordered from E. Then, for all $x_1, x_2 \in A$ such that $x_1 \leq_A x_2$ implies $\alpha(x_1) \subseteq \alpha(x_2)$, $\beta(x_1) \supseteq \beta(x_2)$ and for all $y_1, y_2 \in B$ such that $y_1 \leq_B y_2$ implies $\lambda(y_1) \subseteq \lambda(y_2)$, $\mu(y_1) \supseteq \mu(y_2)$. Now σ be a porelation between the element of $D = A \times B$ in such a way $(x_1, y_1)\sigma_D(x_2, y_2)$, where $(x_1, y_1), (x_2, y_2) \in A \times B$, we note that this order induced by order of A and B. Since $(x_1, y_1)\sigma(x_2, y_2)$ and $\alpha(x_1) \subseteq \alpha(x_2)$, $\beta(x_1) \supseteq \beta(x_2)$ and $\lambda(y_1) \subseteq \lambda(y_2)$, $\mu(y_1) \supseteq \mu(y_2)$, then $\alpha(x_1) \cup \lambda(y_1) \subseteq \alpha(x_2) \cup \lambda(y_2)$ and $\beta(x_1) \cap \mu(y_1) \supseteq \beta(x_2) \cap \mu(y_2)$ implies that $H_1(x_1, y_1) \subseteq H_1(x_2, y_2)$ and $H_2(x_1, y_1) \supseteq H_2(x_2, y_2)$ implies $((\alpha,\beta), A) \vee ((\lambda,\mu), B)$ is LODFS-set.

Similarly, we can prove for ALODFS-set.

Proposition 3.12. Let $((\alpha,\beta), A)$ and $((\lambda,\mu), B)$ be LODFS-sets (ALODFS-sets), then their intersection-product is also a LODFS-set (ALODFS-set).

Proof. By using the definition of intersection-product, we can prove like Proposition 3.11.

Proposition 3.13. (De Morgan Laws) Let $\left((\alpha,\beta),A\right)$ and $\left((\lambda,\mu),B\right)$ be LODFS-set (ALODFS-set) over U. Then,

(1) $\left(\left((\alpha,\beta),A\right)\vee\left((\lambda,\mu),B\right)\right)^c=\left((\alpha,\beta),A\right)^c\wedge\left((\lambda,\mu),B\right)^c$

(2) $\left(\left((\alpha,\beta),A\right)\wedge\left((\lambda,\mu),B\right)\right)^c=\left((\alpha,\beta),A\right)^c\vee\left((\lambda,\mu),B\right)^c$.

Proposition 3.14. (Distributive Laws)

If $\left((\alpha,\beta),A\right)$, $\left((\lambda,\mu),B\right)$ and $\left((\gamma,\delta),C\right)$ be any LODFS-sets (ALODFS-sets) over U. Then, the following conditions hold

(1) $\left((\alpha,\beta),A\right)\vee\left(\left((\lambda,\mu),B\right)\wedge\left((\gamma,\delta),C\right)\right)$
$=\left(\left((\alpha,\beta),A\right)\vee\left((\lambda,\mu),B\right)\right)\wedge\left(\left((\alpha,\beta),A\right)\vee\left((\gamma,\delta),C\right)\right)$.

(2) $\left((\alpha,\beta),A\right)\wedge\left(\left((\lambda,\mu),B\right)\vee\left((\gamma,\delta),C\right)\right)$
$=\left(\left((\alpha,\beta),A\right)\wedge\left((\lambda,\mu),B\right)\right)\vee\left(\left((\alpha,\beta),A\right)\wedge\left((\gamma,\delta),C\right)\right)$.

Proposition 3.15. If $\left((\alpha,\beta),A\right)$, $\left((\lambda,\mu),B\right)$ and $\left((\gamma,\delta),C\right)$ be any two LODFS-sets (ALODFS-sets) over U. Then, followings are LODFS-set (ALODFS-set),

(1) $\left((\alpha,\beta),A\right)\vee\left(\left((\lambda,\mu),B\right)\sqcup_{\mathcal{E}}\left((\gamma,\delta),C\right)\right)$.

(2) $\left((\alpha,\beta),A\right)\vee\left(\left((\lambda,\mu),B\right)\sqcap_{\mathcal{E}}\left((\gamma,\delta),C\right)\right)$.

(3) $\left((\alpha,\beta),A\right)\wedge\left(\left((\lambda,\mu),B\right)\sqcup_{\mathcal{E}}\left((\gamma,\delta),C\right)\right)$.

(4) $\left((\alpha,\beta),A\right)\wedge\left(\left((\lambda,\mu),B\right)\sqcap_{\mathcal{E}}\left((\gamma,\delta),C\right)\right)$.

(5) $\left((\alpha,\beta),A\right)\vee\left(\left((\lambda,\mu),B\right)\sqcap\left((\gamma,\delta),C\right)\right)$.

(6) $\left((\alpha,\beta),A\right)\wedge\left(\left((\lambda,\mu),B\right)\sqcup\left((\gamma,\delta),C\right)\right)$.

If $\sqcup_{\mathcal{E}}$ and $\sqcap_{\mathcal{E}}$ is LODFS-set (ALODFS-set).

4. Algebraic Structure Associated with LODFS-Set (ALODFS-Set)

In this section, we proposed the concept of algebraic structures of LODFS-set (ALODFS-set) which will help solve daily life problems. We also discussed the algebraic properties of LODFS-set (ALODFS-set).

Proposition 4.1. If $\left((\alpha,\beta),A\right)$, $\left((\lambda,\mu),B\right)$ and $\left((\gamma,\delta),C\right)$ are any LODFS-sets (ALODFS-sets), then following axioms hold

(1) $\left(\left((\alpha,\beta),A\right)\diamond\left((\lambda,\mu),B\right)\right)\diamond\left((\gamma,\delta),C\right)=\left((\alpha,\beta),A\right)\diamond\left(\left((\lambda,\mu),B\right)\diamond\left((\gamma,\delta),C\right)\right)$ (Assoc. property)

(2) $\left((\alpha,\beta),A\right)\diamond\left((\lambda,\mu),B\right)=\left((\lambda,\mu),B\right)\diamond\left((\alpha,\beta),A\right)$ (Commutative property)

For all $\diamond\in\{\sqcup,\sqcap,\vee,\wedge\}$.

Proof. (1) Since $\left((\alpha,\beta),A\right)$, $\left((\lambda,\mu),B\right)$ and $\left((\gamma,\delta),C\right)$ are LODFS-sets, so we have for $e\in(A\cap B)\cap C$ such that

$$\left(\left((\alpha,\beta),A\right)\sqcup\left((\lambda,\mu),B\right)\right)\sqcup\left((\gamma,\delta),C\right)=\left(\left((\alpha\,\tilde{\cup}\,\lambda)\,\tilde{\cup}\,\gamma,(\beta\,\tilde{\cap}\,\mu)\,\tilde{\cap}\,\delta\right),(A\cap B)\cap C\,\right)$$

as

$$e\in(A\cap B)\cap C,\text{ so } e\longrightarrow\left(\alpha(e)\cup\lambda(e)\right)\cup\gamma(e)\text{ and } e\longrightarrow\left(\beta(e)\cap\mu(e)\right)\cap\delta(e),$$

implies

$$e\longrightarrow\alpha(e)\cup\left(\lambda(e)\cup\gamma(e)\right)\text{ and } e\longrightarrow\beta(e)\cap\left(\mu(e)\cap\delta(e)\right)$$

Hence,

$$\Big(\big((\alpha,\beta),A\big)\sqcup\big((\lambda,\mu),B\big)\Big)\sqcup\big((\gamma,\delta),C\big)=\Big(\big((\alpha\,\tilde{\cup}\,(\lambda\,\tilde{\cup}\,\gamma),\beta\,\tilde{\cap}\,(\mu\,\tilde{\cap}\,\delta),A\cap(B\cap C)\Big)$$

Similarly, we can prove for ALODFS-set.

(2) Straightforward.

Throughout this paper, the collection of all LODFS-sets of E over U is represented as $LODFS(U)_E$, and the collection of all LODFS-sets over U with any fixed set of parameters A is represented as $LODFS(U)_A$.

Note that,

1) $(LODFS(U)_E,\vee)$ and $(LODFS(U)_E,\wedge)$ are monoids.
2) $(LODFS(U)_E,\vee,\wedge)$ and $(LODFS(U)_E,\wedge,\vee)$ are hemirings.
3) $(LODFS(U)_E,\sqcup)$ and $(LODFS(U)_E,\sqcap)$ are monoids.
4) $(LODFS(U)_E,\sqcup,\sqcap)$ and $(LODFS(U)_E,\sqcap,\sqcup)$ are hemirings.

Similarly, we can define for ALODFS-set.

Proposition 4.2. (Absorption Laws)

Let $\big((\alpha,\beta),A\big)$ and $\big((\lambda,\mu),B\big)$ be LODFS-sets (ALODFS-sets), then

1) $\Big(\big((\alpha,\beta),A\big)\wedge\big((\lambda,\mu),B\big)\Big)\vee\big((\lambda,\mu),B\big)=\big((\lambda,\mu),B\big)$.
2) $\Big(\big((\alpha,\beta),A\big)\vee\big((\lambda,\mu),B\big)\Big)\wedge\big((\lambda,\mu),B\big)=\big((\lambda,\mu),B\big)$.
3) $\Big(\big((\alpha,\beta),A\big)\sqcap\big((\lambda,\mu),\boldsymbol{B}\big)\Big)\sqcup\big((\lambda,\mu),\boldsymbol{B}\big)=\big((\lambda,\mu),\boldsymbol{B}\big)$.
4) $\Big(\big((\alpha,\beta),A\big)\sqcup\big((\lambda,\mu),\boldsymbol{B}\big)\Big)\sqcap\big((\lambda,\mu),\boldsymbol{B}\big)=\big((\lambda,\mu),\boldsymbol{B}\big)$.

Theorem 4.3. $\big(LODFS(U)_A,\sqcup,^{c},A_{(\emptyset,\mathfrak{U})}\big)$ is an MV-algebra.

Proof. (1-MV) $\big(LODFS(U)_A,\sqcup,^{c},A_{(\emptyset,\mathfrak{U})}\big)$ is a commutative monoid.

(2-MV) $\Big(A_{(\alpha_1,\beta_1)^c}\Big)^c=A_{(\alpha_1,\beta_1)}$.

(3-MV) $A_{(\emptyset,\mathfrak{U})^c}\sqcup A_{(\alpha_1,\beta_1)}=A_{(\mathfrak{U},\emptyset)}\sqcup A_{(\alpha_1,\beta_1)}=A_{(\mathfrak{U},\emptyset)}=A_{(\emptyset,\mathfrak{U})^c}$.

(4-MV)

$$\begin{aligned}\Big(A_{(\alpha_1,\beta_1)^c}\sqcup A_{(\alpha_2,\beta_2)}\Big)^c\sqcup A_{(\alpha_2,\beta_2)}&=\Big(A_{({\alpha_1}^c,{\beta_1}^c)^c}\sqcap A_{(\alpha_2,\beta_2)^c}\Big)\sqcup A_{(\alpha_2,\beta_2)}\\&=\Big(A_{(\alpha_1,\beta_1)}\sqcup A_{(\alpha_2,\beta_2)}\Big)\sqcap\Big(A_{({\alpha_2}^c,{\beta_2}^c)}\sqcup A_{(\alpha_2,\beta_2)}\Big)\\&=\Big(A_{(\alpha_1,\beta_1)}\sqcup A_{(\alpha_2,\beta_2)}\Big)\sqcap A_{(\mathfrak{U},\emptyset)}\\&=\Big(A_{(\alpha_1,\beta_1)}\sqcup A_{(\alpha_2,\beta_2)}\Big)\sqcap\Big(A_{(\alpha_1,\beta_1)^c}\sqcup A_{(\alpha_1,\beta_1)}\Big)\\&=\Big(A_{(\alpha_2,\beta_2)}\sqcap A_{(\alpha_1,\beta_1)^c}\Big)\sqcup A_{(\alpha_1,\beta_1)}\\&=\Big(\Big(A_{(\alpha_2,\beta_2)^c}\sqcup A_{(\alpha_1,\beta_1)}\Big)^c\Big)\sqcup A_{(\alpha_1,\beta_1)}\end{aligned}$$

Theorem 4.4. $\big(LODFS(U,)_A,\sqcap,^{c},A_{(\mathfrak{U},\emptyset)}\big)$ is an MV-algebra.

Proof. Similarly, we can prove like Theorem 4.3.

Theorem 4.5. $\big(LODFS(U)_A,\sqcup,\sqcap,A_{(\mathfrak{U},\emptyset)},A_{(\emptyset,\mathfrak{U})}\big)$ and $\big(LODFS(U)_A,\sqcap,\sqcup,A_{(\emptyset,\mathfrak{U})},A_{(\mathfrak{U},\emptyset)}\big)$ are bounded lattices.

Proof. Since $\left(LODFS(U)_A, \sqcup, \sqcap, A_{(\mathfrak{U},\emptyset)}, A_{(\emptyset,\mathfrak{U})}\right)$ is a hemiring and the absorption laws hold in hemiring, so $\left(LODFS(U)_A, \sqcup, \sqcap, A_{(\mathfrak{U},\emptyset)}, A_{(\emptyset,\mathfrak{U})}\right)$ is a bounded lattice with $A_{(\mathfrak{U},\emptyset)}$ and $A_{(\emptyset,\mathfrak{U})}$ as maximal and minimal elements respectively. Using the same steps, we can prove that $\left(LODFS(U)_A, \sqcap, \sqcup, A_{(\emptyset,\mathfrak{U})}, A_{(\mathfrak{U},\emptyset)}\right)$ is a bounded lattice.

Theorem 4.6. $\left(LODFS(U)_A, \sqcup, \sqcap, A_{(\mathfrak{U},\emptyset)}, A_{(\emptyset,\mathfrak{U})}\right)$ and $\left(LODFS(U)_A, \sqcap, \sqcup, A_{(\emptyset,\mathfrak{U})}, A_{(\mathfrak{U},\emptyset)}\right)$ are Boolean algebras.

Proof. Consider $\left((\lambda,\mu),A\right) \in LDFS(U)_A$, then

$$\left((\lambda,\mu),A\right) \sqcap \left((\lambda,\mu),A\right)^c = A_{(\emptyset,\mathfrak{U})} \text{ and } \left((\lambda,\mu),A\right) \sqcup \left((\lambda,\mu),A\right)^c = A_{(\mathfrak{U},\emptyset)}$$

holds imply $\left(LODFS(U)_A, \sqcup, \sqcap, A_{(\mathfrak{U},\emptyset)}, A_{(\emptyset,\mathfrak{U})}\right)$ is a Boolean algebra. Using the same steps, we can prove that $\left(LODFS(U)_A, \sqcap, \sqcup, A_{(\emptyset,\mathfrak{U})}, A_{(\mathfrak{U},\emptyset)}\right)$ is a Boolean algebra.

Now by the previous discussion, we note that De Morgan's laws hold in $\left(LODFS(U)_A, \sqcup, \sqcap, A_{(\mathfrak{U},\emptyset)}, A_{(\emptyset,\mathfrak{U})}\right)$ so $LODFS(U)_A$ is a De Morgan's algebra.

Now for any $\left((\alpha,\beta),A\right), \left((\lambda,\mu),A\right) \in LODFS(U)_A$ such that

$$\left((\alpha,\beta),A\right) \sqcap \left((\alpha,\beta),A\right)^c \,\tilde{\sqsubseteq}\, \left((\lambda,\mu),A\right) \sqcup \left((\lambda,\mu),A\right)^c$$

is hold in $LODFS(U)_A$. Then, we can say that $LODFS(U)_A$ is a Kleene algebra.

By the previous discussion, we note that $\left((\lambda,\mu),A\right) \sqcap \left((\lambda,\mu),A\right)^c = A_{(\emptyset,\mathfrak{U})}$ and if $\left((\alpha,\beta),A\right) \sqcap \left((\lambda,\mu),A\right) = A_{(\emptyset,\mathfrak{U})}$, then $\left((\alpha,\beta),A\right) \,\tilde{\sqsubseteq}\, \left((\lambda,\mu),A\right)^c$ and we can say that $\left((\lambda,\mu),A\right)^c$ is the pseudo complement of $\left((\lambda,\mu),A\right)$.

If $\left((\lambda,\mu),A\right)^c \sqcup \left(\left((\lambda,\mu),A\right)^c\right)^c = A_{(\mathfrak{U},\emptyset)}$ (Stone identity) is hold in $LODFS(U)_A$, then $\left(LODFS(U)_A, \sqcup, \sqcap, A_{(\mathfrak{U},\emptyset)}, A_{(\emptyset,\mathfrak{U})}\right)$ is called Stone algebra.

Similarly, we can also prove that $\left(LODFS(U)_A, \sqcap, \sqcup, A_{(\emptyset,\mathfrak{U})}, A_{(\mathfrak{U},\emptyset)}\right)$ is Stone algebra.

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Bipolar Fuzzy *k*-Ideals in KU-Semigroups

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Abstract− We have studied some types of ideals in a KU-semigroup by using the concept of a bipolar fuzzy set. Bipolar fuzzy S-ideals and bipolar fuzzy *k*-ideals are introduced, and some properties are investigated. Also, some relations between a bipolar fuzzy *k*-ideal and *k*-ideal are discussed. Moreover, a bipolar fuzzy *k*-ideal under homomorphism and the product of two bipolar fuzzy *k*-ideals are studied.

Keywords− *KU-algebra, KU-semigroup, fuzzy S-ideal, bipolar fuzzy S-ideal, bipolar fuzzy k-ideal*

1. Introduction

In 1956, Zadeh [1] introduced the notion of fuzzy sets. This concept has been applied to many mathematical branches. In [2, 3], Mostafa et al. studied the fuzzy KU-ideals and investigated some basic properties. Intuitionistic fuzzy sets, interval-valued fuzzy sets and Bipolar-valued fuzzy sets are extension fuzzy sets theory. In 2000, Lee [4] introduced bipolar-valued fuzzy sets. Bipolar-valued fuzzy sets are an extension of fuzzy sets whose membership degree from $[0,1]$ to $[-1,1]$. In bipolar-valued fuzzy set, the membership degree 0 means that elements are irrelevant to the corresponding property, while the membership degree $[-1,0)$ indicates that elements satisfy the implicit counter property. In [5-8], the authors introduced a bipolar-valued fuzzy set on different structures. In this work, we study the bipolar-valued fuzzy set theory to *k*-ideal of a KU-semigroup and discuss some relations between a bipolar fuzzy *k*-ideal and *k*-ideal. Also, a bipolar fuzzy *k*-ideal under homomorphism and the product of two bipolar fuzzy *k*-ideals are studied.

2. Preliminaries

In this part, we review some concepts related to KU-semigroup and a bipolar fuzzy logic.

Definition 2.1 [9] Algebra $(\aleph, *, 0)$ is a KU-algebra if, for all $\chi, \gamma, \tau \in \aleph$,

(ku_1) $(\chi * \gamma) * ((\gamma * \tau) * (\chi * \tau)) = 0$
(ku_2) $\chi * 0 = 0$
(ku_3) $0 * \chi = \chi$
(ku_4) $\chi * \gamma = 0$ and $\gamma * \chi = 0$ implies $\chi = \gamma$
(ku_5) $\chi * \chi = 0$

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On a KU-algebra $\aleph$, a relation $\leq$ is defined by $\leq \gamma \Leftrightarrow \gamma * \chi = 0$. Therefore $(\aleph, \leq)$ is a partially ordered set. It follows that 0 is the smallest element in $\aleph$.

Thus $(\aleph, *, 0)$ satisfies the following. For all $\chi, \gamma, \tau \in \aleph$,

(ku_1) $(\gamma * \tau) * (\chi * \tau) \leq (\chi * \gamma)$
(ku_2) $0 \leq \chi$
(ku_3) $\chi \leq \gamma, \gamma \leq \chi$ implies $\chi = \gamma$
(ku_4) $\gamma * \chi \leq \chi$

Theorem 2.2. [9] In a KU-algebra $\aleph$. The following axioms hold. For all $\chi, \gamma, \tau \in \aleph$,

i. $\chi \leq \gamma$ imply $\gamma * \tau \leq \chi * \tau$
ii. $\chi * (\gamma * \tau) = \gamma * (\chi * \tau)$
iii. $((\gamma * \chi) * \chi) \leq \gamma$

Definition 2.3. [10] A non-empty subset E of a KU-algebra $(\aleph, *, 0)$ is called KU-subalgebra of $\aleph$ if $\chi * \gamma \in E$ whenever $\chi, \gamma \in E$.

Definition 2.4. [10] A non-empty subset Ґ of a KU-algebra $(\aleph, *, 0)$ is said to be an ideal of $\aleph$ if it satisfies, for any $\chi, \gamma \in \aleph$

i. $0 \in$ Ґ and
ii. $* \gamma \in$ Ґ, $\chi \in$ Ґ imply that $\gamma \in$ Ґ

Definition 2.5. [3] Let Ґ be a nonempty subset of a KU-algebra $\aleph$. Then, Ґ is said to be a KU-ideal of $\aleph$, if

(I_1) $0 \in$ Ґ and

(I_2) $\forall \chi, \gamma, \tau \in \aleph$, $\chi * (\gamma * \tau) \in$ Ґ and $\gamma \in$ Ґ imply that $\chi * \tau \in$ Ґ

Definition 2.6. [11] A KU-semigroup is a non-empty set $\aleph$ with two binary operations $*, \circ$ and constant 0 satisfying the following axioms

i. $(\aleph, *, 0)$ is a KU-algebra
ii. $(\aleph, \circ)$ is a semigroup
iii. The operation $\circ$ is distributive (on both sides) over the operation $*$, i.e.,

$$\chi \circ (\gamma * \tau) = (\chi \circ \gamma) * (\chi \circ \tau) \text{ and } (\chi * \gamma) \circ \tau = (\chi \circ \tau) * (\gamma \circ \tau), \forall \chi, \gamma, \tau \in \aleph$$

Example 2.7. [11] Let $\aleph = \{0,1,2,3\}$. Define $*$-operation and $\circ$-operation by the following tables

*	0	1	2	3
0	0	1	2	3
1	0	0	0	2
2	0	2	0	1
3	0	0	0	0

∘	0	1	2	3
0	0	0	0	0
1	0	1	0	1
2	0	0	2	2
3	0	1	2	3

Then, $(\aleph, *, \circ, 0)$ is a KU-semigroup.

Definition 2.8. [11] A nonempty subset R of $\aleph$ is called a sub-KU-semigroup of $\aleph$, if $\chi * \gamma, \chi \circ \gamma \in R$, for all $\chi, \gamma \in R$.

Definition 2.9. [11] A non-empty subset R of a KU-semigroup$\aleph$ is an S-ideal of $\aleph$, if

i. R is an ideal of $\aleph$
ii. For all $\chi \in \aleph$, and $a \in R$, we have $\chi \circ a \in R$ and $a \circ \chi \in R$

Definition 2.10. [11] A subset R of a KU-semigroup $\aleph$ is a k-ideal of $\aleph$, if

i. R is a KU-ideal of $\aleph$
ii. For all $\chi \in \aleph$, and $a \in R$, we have $\chi \circ a \in R$ and $a \circ \chi \in R$

Definition 2.11. [11] Let $\aleph$ and $\aleph'$ be two KU-semigroups. A mapping $f: \aleph \to \aleph'$ is called a KU-semigroup homomorphism if $f(\chi * \gamma) = f(\chi) * f(\gamma)$ and $f(\chi \circ \gamma) = f(\chi) \circ f(\gamma)$ for all $\chi, \gamma \in \aleph$. The set $\{\chi \in \aleph: f(\chi) = 0\}$ is called the kernel of f and denote by $ker\ f$ Moreover, the set $\{ f(\chi) \in \aleph' : \chi \in \aleph\}$ is called the image of f and denote by imf.

We review some concepts of fuzzy logic.

Let $\aleph$ be the collection of objects, then a fuzzy set $\mu(\chi)$ in $\aleph$ is defined as $\mu: \aleph \to [0,1]$, where $\mu(\chi)$ is called the membership value of χ in $\aleph$ and $0 \leq \mu(\chi) \leq 1$. The set $U(\mu, t) = \{\chi \in \aleph : \mu(\chi) \geq t\}$, where $0 \leq t \leq 1$ is said to be a level set of $\mu(\chi)$.

Definition 2.12. [12] Let $\mu(\chi)$ be a fuzzy set in $\aleph$, then $\mu(\chi)$ is called a fuzzy sub KU-semigroup of $\aleph$ if it satisfies the following condition : for all $\chi, \gamma \in \aleph$.

i. $\mu(\chi * \gamma) \geq \min\{\mu(\chi), \mu(\gamma)\}$
ii. $\mu(\chi \circ \gamma) \geq \min\{\mu(\chi), \mu(\gamma)\}$

Definition 2.13. [12] A fuzzy set $\mu(\chi)$ in $\aleph$ is called a fuzzy S-ideal of $\aleph$ if for all $\chi, \gamma \in \aleph$

i. $\mu(0) \geq \mu(\chi)$
ii. $\mu(\gamma) \geq \min\{\mu(\chi * \gamma), \mu(\chi)\}$
iii. $\mu(\chi \circ \gamma) \geq \min\{\mu(\chi), \mu(\gamma)\}$

Definition 2.14. [12] A fuzzy set $\mu(\chi)$ in $\aleph$ is called a fuzzy k-ideal, if it satisfies the following condition: for all $\chi, \gamma \in \aleph$

(k_1) $\mu(0) \geq \mu(\chi)$
(k_2) $\mu(\chi * \tau) \geq \min\{\mu(\chi * (\gamma * \tau)), \mu(\gamma)\}$
(k_3) $\mu(\chi \circ \gamma) \geq \min\{\mu(\chi), \mu(\gamma)\}$

Example 2.15. [12] Let $\aleph = \{0, a, b, c, d\}$ be a set. Define $*$-operation and $\circ$-operation by the following tables

*	0	a	b	c	d
0	0	a	b	c	d
a	0	0	b	c	d
b	0	a	0	c	d
c	0	a	0	0	d
d	0	0	0	0	0

∘	0	a	b	c	d
0	0	0	0	0	0
a	0	0	0	0	0
b	0	0	0	0	b
c	0	0	0	b	c
d	0	a	b	c	d

Then, $(\aleph, *, \circ, 0)$ is a KU-semigroup. Define a fuzzy set $\mu: \aleph \to [0,1]$ by $\mu(0) = \mu(a) = 0.4, \mu(b) = \mu(c) = 0.2$, $\mu(d) = 0.1$. Then, it is easy to see $\mu(\chi), \forall \chi \in \aleph$ is a fuzzy k-ideal.

We will refer to $\aleph$ is a KU-semigroup unless otherwise indicated.

3. Bipolar fuzzy *k*-ideals of a KU-semigroup

In this section, we give the definition and properties of bipolar fuzzy ideals of $\aleph$. Now, A bipolar valued fuzzy subset B in a nonempty set $\aleph$ is an object having the form $B = \{(\chi, \mu^-(\chi), \mu^+(\chi) | \chi \in \aleph\}$ where $\mu^-: \aleph \to [-1,0]$ and $\mu^+: \aleph \to [0,1]$ are two mappings. The membership degree $\mu^+(\chi)$ denotes the satisfaction degree of

χ to the property corresponding of B, and the membership degree $\mu^-(\chi)$ denotes the satisfaction degree of χ to some implicit counter-property of B. We shall use the symbol $B = (\chi, \mu^-, \mu^+)$, for $B = \{(\chi, \mu^-(\chi), \mu^+(\chi)) : \chi \in \aleph\}$, and use the concept of a bipolar fuzzy set instead of the concept of bipolar-valued fuzzy set.

Now, let $B = (\chi, \mu^-, \mu^+)$ be a bipolar fuzzy set and $(s, t) \in [-1,0] \times [0,1]$.

The set $B_s^- = \{\chi \in \aleph : \mu^-(\chi) \le s\}$ and $B_t^+ = \{\chi \in \aleph : \mu^+(\chi) \ge t\}$ which are called the negative s-cut and the positive t-cut of $B = (\chi, \mu^-, \mu^+)$, respectively.

Definition 3.1. A fuzzy set μ in $\aleph$ is called a bipolar fuzzy sub-KU-semigroup of $\aleph$ if it satisfies the following condition : for all $\chi, \gamma \in \aleph$

i. $\mu^-(\chi * \gamma) \le \max\{\mu^-(\chi), \mu^-(\gamma)\}$ and $\mu^+(\chi * \gamma) \ge \min\{\mu^+(\chi), \mu^+(\gamma)\}$
ii. $\mu^-(\chi \circ \gamma) \le \max\{\mu^-(\chi), \mu^-(\gamma)\}$ and $\mu^+(\chi \circ \gamma) \ge \min\{\mu^+(\chi), \mu^+(\gamma)\}$

Proposition 3.2. Let $B = (\chi, \mu^-, \mu^+)$ be a bipolar fuzzy sub-KU-semigroup. Then, $\mu^-(0) \le \mu^-(\chi)$ and $\mu^+(0) \ge \mu^+(\chi)$, for all $\chi \in \aleph$.

Proof. Clear.

Example 3.3. Let $\aleph = \{0, a, b, c\}$ be a set. Define $*$-operation and $\circ$-operation by the following tables

*	0	a	b	c
0	0	a	b	c
a	0	0	0	c
b	0	a	0	c
c	0	0	0	0

∘	0	a	b	c
0	0	0	0	0
a	0	a	0	a
b	0	0	b	b
c	0	a	b	c

Then, $(\aleph, *, \circ, 0)$ is a KU-semigroup. Define $B = (x, \mu^-, \mu^+)$ by $B = \{(0, -0.6, 0.7), (a, -0.5, 0.5), (b, -0.3, 0.4), (c, -0.2, 0.1)\}$. Then, we can prove that B is a bipolar fuzzy sub-KU-semigroup of $\aleph$.

Definition 3.4. A bipolar fuzzy set $B = (\chi, \mu^-, \mu^+)$in X is called a bipolar fuzzy *S*-ideal of $\aleph$ if it satisfies, for all $\chi, \gamma \in \aleph$

(Bf_1) $\mu^-(0) \le \mu^-(\chi)$ and $\mu^+(0) \ge \mu^+(\chi)$
(Bf_2) $\mu^-(\gamma) \le \max\{\mu^-(\chi * \gamma), \mu^-(\chi)\}$ and $\mu^+(\gamma) \ge \min\{\mu^+(\chi * \gamma), \mu^+(\chi)\}$
(Bf_3) $\mu^-(\chi \circ \gamma) \le \max\{\mu^-(\chi), \mu^-(\gamma)\}, \mu^+(\chi \circ \gamma) \ge \min\{\mu^+(\chi), \mu^+(\gamma)\}$

Definition 3.5. A bipolar fuzzy set $B = (\chi, \mu^-, \mu^+)$ in $\aleph$ is called a bipolar fuzzy *k*-ideal of $\aleph$ if it satisfies: for all $\chi, \gamma, \tau \in \aleph$

(BF_1) $\mu^-(0) \le \mu^-(\chi)$ and $\mu^+(0) \ge \mu^+(\chi)$
(BF_2) $\mu^-(\chi * \tau) \le \max\{\mu^-(\chi * (\gamma * \tau)), \mu^-(\gamma)\}$ and $\mu^+(\chi * \tau) \ge \min\{\mu^+(\chi * (\gamma * \tau)), \mu^+(\gamma)\}$
(BF_3) $\mu^-(\chi \circ \gamma) \le \max\{\mu^-(\chi), \mu^-(\gamma)\}, \mu^+(\chi \circ \gamma) \ge \min\{\mu^+(\chi), \mu^+(\gamma)\}$

Example 3.6. Let $\aleph = \{0, a, b, c\}$ with $*$ defined as in Example (3.3), and $B = (x, \mu^-, \mu^+)$ be a bipolar fuzzy set in $\aleph$ given by the following $B = \{(0, -0.7, 0.6), (a, -0.4, 0.2), (b, -0.4, 0.2), (c, -0.3, 0.1)\}$. Then, $B = (\chi, \mu^-, \mu^+)$ is a bipolar fuzzy *k*-ideal of $\aleph$.

Theorem 3.7. Let $\aleph$ be a KU-semigroup, a bipolar fuzzy set $B = (\chi, \mu^-, \mu^+)$ of $\aleph$ is a bipolar fuzzy *k*-ideal of $\aleph$ if and only if B is a bipolar fuzzy *S*-ideal of $\aleph$.

Proof.

($\Rightarrow$) Let $B=(\chi,\mu^-,\mu^+)$ be a bipolar fuzzy k-ideal of $\aleph$. If we put $\chi=0$ in (BF_2), then $\mu^-(\tau)\leq \max\{\mu^-(\gamma*\tau),\mu^-(\gamma)\}$ and

$\mu^+(\tau)\geq \min\{\mu^+(\gamma*\tau),\mu^+(\gamma)\}$. Also, since $B=(\chi,\mu^-,\mu^+)$ is a bipolar fuzzy k-ideal of KU-semigroup, then (BF_3) is true. Hence, $B=(\chi,\mu^-,\mu^+)$ is a bipolar fuzzy S-ideal of $\aleph$.

($\Leftarrow$) Let $B=(\chi,\mu^-,\mu^+)$ be a bipolar fuzzy S-ideal of $\aleph$, then $\mu^-(\chi*\tau)\leq \max\{\mu^-(\gamma*(\chi*\tau),\mu^-(\gamma)\}$ and $\mu^+(\chi*\tau)\geq \min\{\mu^+(\gamma*(\chi*\tau),\mu^+(\gamma)\}$. And by Theorem (2.2)(2), we get $\mu^-(\chi*\tau)\leq \max\{\mu^-(\chi*(\gamma*\tau),\mu^-(\gamma)\}$ and $\mu^+(\chi*\tau)\geq \min\{\mu^+(\chi*(\gamma*\tau),\mu^+(\gamma)\}$. Also, since $B=(\chi,\mu^-,\mu^+)$ is a bipolar fuzzy S-ideal of KU-semigroup, then (Bf_3) is true. Hence, $B=(\chi,\mu^-,\mu^+)$ is a bipolar fuzzy k-ideal of $\aleph$.

Proposition 3.8. Let $B=(\chi,\mu^-,\mu^+)$ be a bipolar fuzzy k-ideal of $\aleph$. If the inequality $\chi*\gamma\leq\tau$ holds in $\aleph$, then $\mu^-(\gamma)\leq \max\{\mu^-(\chi),\mu^-(\tau)\}$ and $\mu^+(\gamma)\geq \min\{\mu^+(\chi),\mu^+(\tau)\}$, for all $\chi,\gamma,\tau\in\aleph$.

PROOF.

Assume that the inequality $\chi*\gamma\leq\tau$ holds in $\aleph$, then $\tau*(\chi*\gamma)=0$ and by (BF_2)

$$\mu^-(\chi*\gamma)\leq \max\{\mu^-(\chi*(\tau*\gamma)),\mu^-(\tau)\}$$
$$=\max\{\mu^-(\tau*(\chi*\gamma)\},\mu^-(\tau)\}$$
$$=\max\{\mu^-(0),\mu^-(\tau)\}=\mu^-(\tau)\ \ldots\ldots(1)$$

Now, $\mu^-(0*\gamma)=\mu^-(\gamma)\leq max\{\mu^-(0*(\chi*\gamma)),\mu^-(\chi)\}=\max\{\mu^-(\chi*\gamma),\mu^-(\chi)\}\leq \max\{\mu^-(\tau),\mu^-(\chi)\}$ (by using (1)) i.e. $\mu^-(\gamma)\leq \max\{\mu^-(\chi),\mu^-(\tau)\}$. Similarly,

$$\mu^+(\chi*\gamma)\geq \min\{\mu^+(\chi*(\tau*\gamma)),\mu^+(\tau)\}=\min\{\mu^+(\tau*(\chi*\gamma)\},\mu^+(\tau)\}=\min\{\mu^+(0),\mu^+(\tau)\}=\mu^+(\tau)\ \ldots(2)$$

Now, $\mu^+(0*\gamma)=\mu^+(\gamma)\geq min\{\mu^+(0*(\chi*\gamma)),\mu^+(\chi)\}=\min\{\mu^+(\chi*\gamma),\mu^+(\chi)\}\geq \min\{\mu^+(\tau),\mu^+(\chi)\}$ (by using (2)) i.e. $\mu^+(\gamma)\geq \min\{\mu^+(\chi),\mu^+(\tau)\}$.

Theorem 3.9. Let $\aleph$ be a KU-semigroup, a bipolar fuzzy set $B=(\chi,\mu^-,\mu^+)$ of $\aleph$ is a bipolar fuzzy k-ideal of $\aleph$ if and only if B is a bipolar fuzzy sub-KU-semigroup of $\aleph$.

PROOF. ($\Rightarrow$) Let $B=(\chi,\mu^-,\mu^+)$ be a bipolar fuzzy k-ideal of $\aleph$. By Theorem (3.7), B is a bipolar fuzzy S-ideal of $\aleph$. For any $\chi,\gamma\in\aleph$, from (ku_4) we have $\chi*\gamma\leq\gamma$, then by Proposition (3.2) $\mu^-(\chi*\gamma)\leq\mu^-(\gamma)$ and $\mu^+(\chi*\gamma)\geq\mu^+(\gamma)$. And by Proposition (3.8) $\mu^-(\gamma)\leq \max\{\mu^-(\chi),\mu^-(\tau)\}$ and $\mu^+(\gamma)\geq \min\{\mu^+(\chi),\mu^+(\tau)\}$, Hence, $\mu^-(\chi*\gamma)\leq \max\{\mu^-(\chi),\mu^-(\tau)\}$ and $\mu^+(\chi*\gamma)\geq \min\{\mu^+(\chi),\mu^+(\tau)\}$. Then, B is a bipolar fuzzy sub-KU-semigroup of $\aleph$.

($\Leftarrow$) Let $B=(\chi,\mu^-,\mu^+)$ be a bipolar fuzzy sub-KU-semigroup. We have

(*i*) $\mu^-(0)\leq\mu^-(\chi)$ and $\mu^+(0)\geq\mu^+(\chi)$, for all $\chi\in\aleph$.

(*ii*) By Theorem (2.2) (2) and (3), we have $(\gamma*(\chi*\tau))*(\chi*\tau)=(\chi*(\gamma*\tau))*(\chi*\tau)\leq\gamma$, for all $\chi,\gamma,\tau\in\aleph$. It follows from Proposition (3.3.7) that $\mu^-(\chi*\tau)\leq \max\{\mu^-(\chi*(\gamma*\tau)),\mu^-(\gamma)\}$ and $\mu^+(\chi*\tau)\geq \min\{\mu^+(\chi*(\gamma*\tau)),\mu^+(\gamma)\}$, for all $\gamma,\tau\in\aleph$. Also, since $B=(\chi,\mu^-,\mu^+)$ is a bipolar fuzzy sub-KU-semigroup, then (BF_3) is true. Therefore, $B=(\chi,\mu^-,\mu^+)$ is a bipolar fuzzy k-ideal of $\aleph$.

Proposition 3.10. If $B=(\chi,\mu^-,\mu^+)$ is a bipolar fuzzy k-ideal of $\aleph$, then the sets $J=\{\chi\in\aleph:\mu^+(\chi)=\mu^+(0)\}$ and $K=\{\chi\in\aleph:\mu^-(\chi)=\mu^-(0)\}$ are k-ideals of $\aleph$.

PROOF. Since $0\in\aleph,\mu^+(0)=\mu^+(0)$ and $\mu^-(0)=\mu^-(0)$ implies $0\in J$ and $0\in K$, so $J\neq\emptyset,K\neq\emptyset$. Let $(\chi*(\gamma*\tau))\in J$ and $\gamma\in J$ implies $\mu^+(\chi*(\gamma*\tau))=\mu^+(0)$ and $\mu^+(\gamma)=\mu^+(0)$. Since $\mu^+(\chi*\tau)\geq \min\{\mu^+(\chi*(\gamma*\tau)),\mu^+(\gamma)\}=\mu^+(0)\Rightarrow\mu^+(\chi*\tau)\geq\mu^+(0)$ but $\mu^+(0)=\mu^+(\chi*\tau)$. It follows that $(\chi*\tau)\in J$, for all $\chi,\gamma,\tau\in\aleph$.

Also, let $\chi \in J$ and $\gamma \in J$ implies $\mu^+(\chi) = \mu^+(0)$ and $\mu^+(\gamma) = \mu^+(0)$. Since, $\mu^+(\chi \circ \gamma) \geq \min\{\mu^+(\chi), \mu^+(\gamma)\} = \mu^+(0)$, then $\mu^+(\chi \circ \gamma) = \mu^+(0)$. It follows that $\chi \circ \gamma \in J$, similarly $\gamma \circ \chi \in J$. Hence, J is *k*-ideal of $\aleph$. Similarly, we can prove K is *k*-ideal of $\aleph$.

Theorem 3.11. For a bipolar fuzzy set $B = (\chi, \mu^-, \mu^+)$ in $\aleph$, the following are equivalent:

(1) $B = (\chi, \mu^-, \mu^+)$ is a bipolar fuzzy *k*-ideal of $\aleph$.

(2) $B = (\chi, \mu^-, \mu^+)$ is satisfies the following:

i. $\forall s \in [-1,0], (B_s^- \neq \emptyset \Rightarrow B_s^-)$ is a *k*-ideal of $\aleph$.
ii. $\forall t \in [0,1], (B_t^+ \neq \emptyset \Rightarrow B_t^+)$ is a *k*-ideal of $\aleph$.

Proof. (1) $\Rightarrow$(2) (i) Let $s \in [-1,0]$ be such that $B_s^- \neq \emptyset$. Then, there exists $\gamma \in B_s^-$ and so $\mu^-(\gamma) \leq s$. It follows from (BF_1) that $\mu^-(0) \leq \mu^-(\gamma) \leq s$, then $0 \in B_s^-$. Let,$\gamma, \tau \in B_s^-$, such that $(\chi * (\gamma * \tau)) \in B_s^-$ and $\gamma \in B_s^-$. Then, $\mu^-(\chi * (\gamma * \tau)) \leq s$ and $\mu^-(\gamma) \leq s$. By using (BF_2), we have $\mu^-(\chi * \tau) \leq \max\{\mu^-(\chi * (\gamma * \tau)), \mu^-(\gamma)\} = \max\{s, s\} = s$, which implies that $(\chi * \tau) \in B_s^-$. By using (BF_3), we have $\mu^-(\chi \circ \gamma) \leq \max\{\mu^-(\chi), \mu^-(\gamma)\} = \max\{s, s\} = s$, which implies that $(\chi \circ \gamma) \in B_s^-$ (res. $(\gamma \circ \chi) \in B_s^-$). Therefore, B_s^- is a *k*-ideal of $\aleph$.

(ii) Assume that $B_t^+ \neq \emptyset$, for $t \in [0,1]$ and let $a \in B_t^+$. Then, $\mu^+(a) \geq t$ and $\mu^+(0) \geq \mu^+(a) \geq t$ by (BF_1), thus $0 \in B_t^+$. Let $\chi, \gamma, \tau \in \aleph$ be such that $(\chi * (\gamma * \tau)) \in B_t^+$ and $\gamma \in B_t^+$. Then, $\mu^+(\chi * (\gamma * \tau)) \geq t$ and $\mu^+(\gamma) \geq t$.

It follows from (BF_2) that $\mu^+(\chi * \tau) \geq \min\{\mu^+(\chi * (\gamma * \tau)), \mu^+(\gamma)\} = \min\{t, t\} = t$, so that $(\chi * \tau) \in B_t^+$. Also, by (BF_3), $\mu^+(\chi \circ \gamma) \geq \min\{\mu^+(\chi), \mu^+(\gamma)\} = \min\{t, t\} = t$, then $(\chi \circ \gamma) \in B_t^+$(res. $(\gamma \circ \chi) \in B_t^+$). Hence, B_t^+ is a *k*-ideal of $\aleph$.

(2) $\Rightarrow$(1) Assume that there exists $a \in \aleph$ such that $\mu^-(0) \geq \mu^-(a)$. Taking $s_0 = \frac{1}{2}(\mu^-(0) + \mu^-(a))$, for some $s_0 \in [-1,0]$ implies that $\mu^-(a) < s_0 < \mu^-(0)$. This is a contradiction, and thus $\mu^-(0) \leq \mu^-(\gamma)$, for all $\gamma \in \aleph$. Suppose that $\mu^-(\chi * \tau) \leq \max\{\mu^-(\chi * (\gamma * \tau)), \mu^-(\gamma)\}$, for some $\chi, \gamma, \tau \in \aleph$, and let $s_1 = \frac{1}{2}(\mu^-(\chi * \tau) + \max\{\mu^-(\chi * (\gamma * \tau)), \mu^-(\gamma)\})$. Then, $\max\{\mu^-(\chi * (\gamma * \tau)), \mu^-(\gamma)\} < s_1 < \mu^-(\chi * \tau)$, which is a contradiction. Therefore, $\mu^-(\chi * \tau) \leq \max\{\mu^-(\chi * (\gamma * \tau)), \mu^-(\gamma)\}$, for all $\chi, \gamma, \tau \in \aleph$. Suppose that $\mu^-(\chi \circ \gamma) \leq \max\{\mu^-(\chi), \mu^-(\gamma)\}$, for some $\chi, \gamma \in \aleph$, and let $s_2 = \frac{1}{2}(\mu^-(\chi \circ \gamma) + \max\{\mu^-(\chi), \mu^-(\gamma)\}$. Then, $\max\{\mu^-(\chi), \mu^-(\gamma)\} < s_2 < \mu^-(\chi \circ \gamma)$, which is a contradiction. Therefore, $\mu^-(\chi \circ \gamma) \leq \max\{\mu^-(\chi), \mu^-(\gamma)\}$, for all $\chi, \gamma \in \aleph$.

Now, if $\mu^+(0) < \mu^+(\gamma)$, for some $\gamma \in \aleph$, then $\mu^+(0) < t_0 < \mu^+(\gamma)$, for some $t_0 \in (0,1]$. This is a contradiction. Thus $\mu^+(0) \geq \mu^+(\gamma)$, for all $\gamma \in \aleph$.

If $\mu^+(\chi * \tau) \leq \min\{\mu^+(\chi * (\gamma * \tau)), \mu^+(\gamma)\}$, for some $\chi, \gamma, \tau \in \aleph$. Then, there exists $t_1 \in (0,1]$, such that $\mu^+(\chi * \tau) < t_1 \leq \min\{\mu^+(\chi * (\gamma * \tau)), \mu^+(\gamma)\}$. We get $\chi * (\gamma * \tau) \in B_{t_1}^+$ and $\gamma \in B_{t_1}^+$ but $\chi * \tau \notin B_{t_1}^+$. This is a contradiction. Consequently, $\mu^+(\chi * \tau) \geq \min\{\mu^+(\chi * (\gamma * \tau)), \mu^+(\gamma)\}$, for all $\chi, \gamma, \tau \in \aleph$. And if $\mu^+(\chi \circ \gamma) \leq \min\{\mu^+(\chi), \mu^+(\gamma)\}$, for some, $\gamma \in \aleph$.

Then, there exists $t_2 \in (0,1]$ such that $\mu^+(\chi \circ \gamma) < t_2 \leq \min\{\mu^+(\chi), \mu^+(\gamma)\}$. It follows that $\chi \in B_{t_2}^+$ and $\gamma \in B_{t_2}^+$ but $(\chi \circ \gamma) \notin B_{t_2}^+$, which is a contradiction. Hence, $\mu^+(\chi \circ \gamma) \geq \min\{\mu^+(\chi), \mu^+(\gamma)\}$, for all $\chi, \gamma \in \aleph$.

Therefore $B = (\chi, \mu^-, \mu^+)$ is a bipolar fuzzy *k*-ideal of $\aleph$.

4. Bipolar fuzzy *k*-ideals under homomorphism

Definition 4.1. For any $\chi \in \aleph$. We define a new bipolar fuzzy set $B_f = (\chi, \mu_f^-, \mu_f^+)$ in $\aleph$ by $\mu_f^-(\chi) = \mu^-(f(\chi))$ and $\mu_f^+(\chi) = \mu^+(f(\chi))$, where $f: \aleph \to \aleph'$ is a KU-semigroup homomorphism.

Theorem 4.2. Let $f: \aleph \to \aleph'$ be a KU-semigroup homomorphism and onto mapping. Then, $B = (\chi', \mu^-, \mu^+)$ is a bipolar fuzzy k-ideal of $\aleph'$ if and only if $B_f = (\chi, \mu_f^-, \mu_f^+)$ is a bipolar fuzzy k-ideal of $\aleph$.

PROOF: For any $\chi' \in \aleph'$ there exists $\chi \in \aleph$ such that $f(\chi) = \chi'$, we have

$$\mu_f^+(0) = \mu^+(f(0)) = \mu^+(0') \geq \mu^+(\chi') = \mu^+(f(\chi)) = \mu_f^+(\chi)$$

and

$$\mu_f^-(0) = \mu^-(f(0)) = \mu^-(0') \leq \mu^-(\chi') = \mu^-(f(\chi)) = \mu_f^-(\chi).$$

Let $\chi, \tau \in \aleph$, $\gamma' \in \aleph'$ then there exists $\gamma \in \aleph$ such that $f(\gamma) = \gamma'$. We have

$$\begin{aligned}\mu_f^+(\chi * \tau) = \mu^+(f(\chi * \tau)) = \mu^+(f(\chi) * f(\tau)) &\geq \min\{\mu^+(f(\chi) * (\gamma' * f(\tau))), \mu^+(\gamma')\} \\ &= \min\{\mu^+(f(\chi) * (f(\gamma) * f(\tau)))\}, \mu^+(f(\gamma)\} = \min\{\mu_f^+(\chi * (\gamma * \tau)), \mu_f^+(\gamma)\}\end{aligned}$$

and

$$\begin{aligned}\mu_f^-(\chi * \tau) = \mu^-(f(\chi * \tau)) = \mu^-(f(\chi) * f(\tau)) &\leq \max\{\mu^-(f(\chi) * (\gamma' * f(\tau))), \mu^-(\gamma')\} \\ &= \max\{\mu^-(f(\chi) * (f(\gamma) * f(\tau)))\}, \mu^-(f(\gamma)\} = \max\{\mu_f^-(\chi * (\gamma * \tau)), \mu_f^-(\gamma)\}\end{aligned}$$

Hence, $B_f = (\chi, \mu_f^-, \mu_f^+)$ is a bipolar fuzzy k-ideal of $\aleph$.

Conversely, since $f: \aleph \to \aleph'$ is an onto mapping, then for any $\chi, \gamma, \tau \in \aleph'$.

It follows that there exists $a, b, c \in \aleph$ such that $f(a) = \chi, f(b) = \gamma$ and $f(c) = \tau$. We have

$$\begin{aligned}\mu^+(\chi * \tau) = \mu^+(f(a) * f(c))) = \mu^+(f(a * c)) = \mu_f^+(a * c) &\geq \min\{\mu_f^+(a * (b * c)), \mu_f^+(b)\} \\ &= \min\{\mu^+(f(a) * (f(b) * f(c)))\}, \mu^+(f(b)\} = \min\{\mu^+(\chi * (\gamma * \tau)), \mu^+(\gamma)\}.\end{aligned}$$

and

$$\begin{aligned}\mu^-(\chi * \tau) = \mu^-(f(a) * f(c))) = \mu^-(f(a * c)) = \mu_f^-(a * c) &\leq \max\{\mu_f^-(a * (b * c)), \mu_f^-(b)\} \\ &= \max\{\mu^-(f(a) * (f(b) * f(c)))\}, \mu^-(f(b)\} = \max\{\mu^-(\chi * (\gamma * \tau)), \mu^-(\gamma)\}\end{aligned}$$

Therefore, $B = (\chi, \mu^-, \mu^+)$ is a bipolar fuzzy k-ideal of $\aleph'$.

Now, we introduce the product of bipolar fuzzy k-ideals in a KU-semigroup, and we study some results.

Definition 4.3. Let $B_1 = (\chi, \mu_1^-, \mu_1^+)$ and $B_2 = (\gamma, \mu_2^-, \mu_2^+)$ be two bipolar fuzzy sets of $\aleph$. The product $B_1 \times B_2 = ((\chi, \gamma), \mu_1^- \times \mu_2^-, \mu_1^+ \times \mu_2^+)$ is defined by the following: $(\mu_1^- \times \mu_2^-)(\chi, \gamma) = \max\{\mu_1^-(\chi), \mu_2^-(\gamma)\}$ and $(\mu_1^+ \times \mu_2^+)(\chi, \gamma) = \min\{\mu_1^+(\chi), \mu_2^+(\gamma)\}$, where $\mu_1^- \times \mu_2^-: \aleph \times \aleph \to [-1,0]$ and $\mu_1^+ \times \mu_2^+: \aleph \times \aleph \to [0,1]$, for all $\chi, \gamma \in \aleph$.

Theorem 4.4. Let $B_1 = (\chi, \mu_1^-, \mu_1^+)$ and $B_2 = (\gamma, \mu_2^-, \mu_2^+)$ be two bipolar fuzzy k-ideals of KU-semigroup $\aleph$, then $B_1 \times B_2$ is a bipolar fuzzy k-ideal of $\aleph \times \aleph$.

PROOF. For any $(\chi, \gamma) \in \aleph \times \aleph$, we have

$$(\mu_1^+ \times \mu_2^+)(0,0) = \min\{\mu_1^+(0), \mu_2^+(0)\} \geq \min\{\mu_1^+(\chi), \mu_2^+(\gamma)\} = (\mu_1^+ \times \mu_2^+)(\chi, \gamma)$$

and

$$(\mu_1^- \times \mu_2^-)(0,0) = \max\{\mu_1^-(0), \mu_2^-(0)\} \leq \max\{\mu_1^-(\chi), \mu_2^-(\gamma)\} = (\mu_1^- \times \mu_2^-)(\chi, \gamma)$$

Let $(\chi_1, \chi_2), (\gamma_1, \gamma_2)$ and $(\tau_1, \tau_2) \in \aleph \times \aleph$, then

$$\begin{aligned}(\mu_1^+ \times \mu_2^+)(\chi_1 * \tau_1, \chi_2 * \tau_2) &= \min\{\mu_1^+(\chi_1 * \tau_1), \mu_2^+(\chi_2 * \tau_2)\} \\ &\geq \min\{\min\{\mu_1^+(\chi_1 * (\gamma_1 * \tau_1)), \mu_1^+(\gamma_1)\}, \min\{\mu_2^+(\chi_2 * (\gamma_2 * \tau_2)), \mu_2^+(\gamma_2)\}\} \\ &= \min\{\min\{\mu_1^+(\chi_1 * (\gamma_1 * \tau_1)), \mu_2^+(\chi_2 * (\gamma_2 * \tau_2)\}, \min\{\mu_1^+(\gamma_1), \mu_2^+(\gamma_2)\} \\ &= \min\left\{\min(\mu_1^+ \times \mu_2^+)\left\{(\chi_1 * (\gamma_1 * \tau_1)), (\chi_2 * (\gamma_2 * \tau_2))\right\}, \{(\mu_1^+ \times \mu_2^+)(\gamma_1, \gamma_2)\}\right\}\end{aligned}$$

and

$$
\begin{aligned}
(\mu_1^- \times \mu_2^-)(\chi_1 * \tau_1, \chi_2 * \tau_2) &= \max\{\mu_1^-(\chi_1 * \tau_1), \mu_2^-(\chi_2 * \tau_2)\} \\
&\leq \max\{\max\{\mu_1^-(\chi_1 * (\gamma_1 * \tau_1)), \mu_1^-(\gamma_1)\}, \max\{\mu_2^-(\chi_2 * (\gamma_2 * \tau_2), \mu_2^-(\gamma_2)\} \\
&= \max\{\max\{\mu_1^-(\chi_1 * (\gamma_1 * \tau_1)), \mu_2^-(\chi_2 * (\gamma_2 * \tau_2)\}, \max\{\mu_1^-(\gamma_1), \mu_2^-(\gamma_2)\} \\
&= \max\left\{(\mu_1^- \times \mu_2^-)\left\{(\chi_1 * (\gamma_1 * \tau_1)), (\chi_2 * (\gamma_2 * \tau_2))\right\}, \{(\mu_1^- \times \mu_2^-)(\gamma_1, \gamma_2)\}\right\}
\end{aligned}
$$

and

$$
\begin{aligned}
(\mu_1^+ \times \mu_2^+)(\chi_1 \circ \gamma_1, \chi_2 \circ \gamma_2) &= \min\{\mu_1^+(\chi_1 \circ \gamma_1), \mu_2^+(\chi_2 \circ \gamma_2)\} \\
&\geq \min\{\min\{\mu_1^+(\chi_1), \mu_1^+(\gamma_1)\}, \min\{\mu_2^+(\chi_2), \mu_2^+(\gamma_2)\} \\
&= \min\{\min\{\mu_1^+(\chi_1), \mu_2^+(\chi_2)\}, \min\{\mu_1^+(\gamma_1), \mu_2^+(\gamma_2)\} \\
&= \min\{\{(\mu_1^+ \times \mu_2^+)(\chi_1, \chi_2)\}, \{(\mu_1^+ \times \mu_2^+)(\gamma_1, \gamma_2)\}\}
\end{aligned}
$$

and

$$
\begin{aligned}
(\mu_1^- \times \mu_2^-)(\chi_1 \circ \gamma_1, \chi_2 \circ \gamma_2) &= \max\{\mu_1^-(\chi_1 \circ \gamma_1), \mu_2^-(\chi_2 \circ \gamma_2)\} \\
&\leq \max\{\max\{\mu_1^-(\chi_1), \mu_1^-(\gamma_1)\}, \max\{\mu_2^-(\chi_2), \mu_2^-(\gamma_2)\} \\
&= \max\{\max\{\mu_1^-(\chi_1), \mu_2^-(\chi_2)\}, \max\{\mu_1^-(\gamma_1), \mu_2^-(\gamma_2)\} \\
&= \max\{\{(\mu_1^- \times \mu_2^-)(\chi_1, \chi_2)\}, \{(\mu_1^- \times \mu_2^-)(\gamma_1, \gamma_2)\}\}
\end{aligned}
$$

Therefore $B_1 \times B_2$ is a bipolar fuzzy *k*-ideal of $\aleph \times \aleph$.

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Fuzzy Parameterized Intuitionistic Fuzzy Soft Sets and Their Application to a Performance-Based Value Assignment Problem

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Abstract − Soft sets have been successfully applied to many different fields to cope with uncertainties. Recently, to increase the success of the applications, these sets have been combined with other theories, such as fuzzy sets and intuitionistic fuzzy sets. In this study, we propose the concept of fuzzy parameterized intuitionistic fuzzy soft sets (*fpifs*-sets). We then apply these sets to a performance-based value assignment (PVA) problem. Finally, we give suggestions for further research.

Keywords − *Fuzzy sets, intuitionistic fuzzy sets, soft sets, intuitionistic fuzzy soft sets, fpifs-sets*

1. Introduction

Researchers in many scientific fields make an effort to model problems containing uncertain data. However, the classical methods are not always successful in describing uncertainties. In 1965, therefore, fuzzy sets were developed by Zadeh [1] to overcome the uncertainties. In 1986, these sets have been generalised to intuitionistic fuzzy sets (*if*-sets) by Atanassov [2]. In 1999, Molodtsov [3] proposed the concept of soft sets as a general mathematical tool to model the problems with uncertainties.

So far, many novel concepts based on the soft sets, fuzzy sets, and *if*-sets have been propounded. These concepts can be classified as follows:

- Fuzzy soft sets [4],
- Intuitionistic fuzzy soft sets [5],
- Fuzzy parameterized soft sets [6],
- Fuzzy parameterized fuzzy soft set [7],
- Fuzzy parameterized intuitionistic fuzzy soft sets [*In this study*],
- Intuitionistic fuzzy parameterized soft sets [8],
- Intuitionistic fuzzy parameterized fuzzy soft sets [9],
- Intuitionistic fuzzy parameterized intuitionistic fuzzy soft sets [10],

In the present paper, as it is pointed out above, we define parameterized intuitionistic fuzzy soft sets (*fpifs*-sets) by using fuzzy sets and *if*-sets. We then apply this concept to a decision-making problem. Finally, we discuss *fpifs*-sets and give suggestions for their further research.

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2. Preliminaries

This section presents some of the basic definitions of soft sets [3], fuzzy sets [1], and *if*-sets [2].

2.1. Soft Sets

In this subsection, we introduce some of the basic definitions and properties of soft sets provided in [3, 11, 12].

Definition 2.1. Let U be a universal set, $P(U)$ be the power set of U, and X be a set of parameters. Then, a soft set S over U is defined as a set of ordered pairs

$$S = \{(x, s(x)) : x \in X\} \text{ where } s : X \to P(U)$$

Here, s is called approximate function of the soft set S and the elements $(x, \emptyset)$ is not displayed in S.

Hereafter, the soft sets are denoted by $S, S_1, S_2, \ldots$ and their approximate functions by $s, s_1, s_2, \ldots$, respectively. The set of all soft sets over U is denoted by $\mathbb{S}$.

Definition 2.2. Let $S \in \mathbb{S}$. Then,
S is called **empty soft set**, denoted by $S_\emptyset$, if $s(x) = \emptyset$ for all $x \in X$, and
S is called **universal soft set**, denoted by S_U, if $s(x) = U$ for all $x \in X$.

Example 2.3. Let $U = \{u_1, u_2, u_3, u_4, u_5, u_6\}$ be a universal set and $X = \{x_1, x_2, x_3, x_4\}$ be a set of parameters. If $s(x_1) = \{u_1, u_2, u_4, u_6\}$, $s(x_2) = \emptyset$, $s(x_3) = \{u_1, u_3, u_5\}$, and $s(x_4) = U$, then the soft set S is written by

$$S = \{(x_1, \{u_1, u_2, u_4, u_6\}), (x_3, \{u_1, u_3, u_5\}), (x_4, U)\}$$

Definition 2.4. Let $S_1, S_2 \in \mathbb{S}$. Then,
S_1 and S_2 are called **equal**, denoted by $S_1 = S_2$, if $s_1(x) = s_2(x)$ for all $x \in X$, and
S_1 is called **soft subset** of soft set S_2, denoted by $S_1 \subseteq S_2$, if $s_1(x) \subseteq s_2(x)$ for all $x \in X$.

Definition 2.5. Let $S, S_1, S_2 \in \mathbb{S}$. Then,
the **complement** of S is defined by $S^c = \{(x, U \setminus s(x)) : x \in X\}$,
the **union** of S_1 and S_2 is defined by $S_1 \cup S_2 = \{(x, s_1(x) \cup s_2(x)) : x \in X\}$, and
the **intersection** of S_1 and S_2 is defined by $S_1 \cap S_2 = \{(x, s_1(x) \cap s_2(x)) : x \in X\}$.

Proposition 2.6. If $S \in \mathbb{S}$, then

i) $S \cup S = S$
ii) $S \cap S = S$
iii) $S \cup S_\emptyset = S$
iv) $S \cap S_\emptyset = S_\emptyset$
v) $S \cup S_U = S_U$
vi) $S \cap S_U = S$

Proposition 2.7. If $S_1, S_2, S_3 \in \mathbb{S}$, then

i) $S_1 \cup S_2 = S_2 \cup S_1$
ii) $S_1 \cap S_2 = S_2 \cap S_1$
iii) $(S_1 \cup S_2)^c = S_1^c \cap S_2^c$
iv) $(S_1 \cap S_2)^c = S_1^c \cup S_2^c$
v) $S_1 \cup (S_2 \cup S_3) = (S_1 \cup S_2) \cup S_3$
vi) $S_1 \cap (S_2 \cap S_3) = (S_1 \cap S_2) \cap S_3$
vii) $S_1 \cup (S_2 \cap S_3) = (S_1 \cup S_2) \cap (S_1 \cup S_3)$
viii) $S_1 \cap (S_2 \cup S_3) = (S_1 \cap S_2) \cup (S_1 \cap S_3)$

2.2. Fuzzy Sets

This subsection provides some of the basic definitions and properties of fuzzy sets presented in [1]. For more details, see [13–15].

Definition 2.8. Let X be a universal set. Then, a fuzzy set F over X is defined by

$$F = \{x^{f(x)} : x \in X\} \text{ where } f : X \to [0, 1]$$

Here f is called the membership function of F, the elements x^0 is not displayed in F, and the elements x^1 is displayed as x in F. Moreover, the value $f(x)$ is called the degree of membership of $x \in X$ and represents the degree of belonging of x to the fuzzy set F.

From now on, the fuzzy sets are denoted by $F, F_1, F_2, \dots$ and their membership functions by $f, f_1, f_2, \dots$ respectively. The set of all fuzzy sets over X is denoted by $\mathbb{F}$.

Definition 2.9. Let $F \in \mathbb{F}$. Then,
F is called **empty fuzzy set**, denoted by $F_\emptyset$, if $f(x) = 0$ for all $x \in X$.
F is called **universal fuzzy set**, denoted by F_X, if $f(x) = 1$ for all $x \in X$.

Example 2.10. Let $U = \{x_1, x_2, x_3, x_4, x_5, x_6\}$, $f(x_1) = 0.7$, $f(x_2) = 0.5$, $f(x_3) = 0.2$, $f(x_4) = 0$, $f(x_5) = 0.7$, and $f(x_6) = 1$, then the fuzzy set F is as follows:

$$F = \left\{x_1^{0.7}, x_2^{0.5}, x_3^{0.2}, x_5^{0.7}, x_6\right\}$$

Definition 2.11. Let $F_1, F_2 \in \mathbb{F}$. Then,
F_1 and F_2 are called **equal**, denoted by $F_1 = F_2$, if $f_1(x) = f_2(x)$ for all $x \in X$, and
F_1 is called **fuzzy subset** of F_2, denoted by $F_1 \subseteq F_2$, if $f_1(x) \leq f_2(x)$ for all $x \in X$.

Definition 2.12. Let $F, F_1, F_2 \in \mathbb{F}$. Then,
the **complement** of F is defined by $F^c = \{x^{1-f(x)} : x \in X\}$,
the **union** of F_1 and F_2 is defined by $F_1 \cup F_2 = \{x^{\max\{f_1(x), f_2(x)\}} : x \in X\}$, and
the **intersection** of F_1 and F_2 is defined by $F_1 \cap F_2 = \{x^{\min\{f_1(x), f_2(x)\}} : x \in X\}$.

Proposition 2.13. If $F \in \mathbb{F}$, then

i) $F \cup F = F$
ii) $F \cap F = F$
iii) $F \cup F_\emptyset = F$
iv) $F \cap F_\emptyset = F_\emptyset$
v) $F \cup F_X = F_X$
vi) $F \cap F_X = F$

Proposition 2.14. If $F_1, F_2, F_3 \in \mathbb{F}$, then

i) $F_1 \cup F_2 = F_2 \cup F_1$
ii) $F_1 \cap F_2 = F_2 \cap F_1$
iii) $(F_1 \cup F_2)^c = F_1^c \cap F_2^c$
iv) $(F_1 \cap F_2)^c = F_1^c \cup F_2^c$
v) $F_1 \cup (F_2 \cup F_3) = (F_1 \cup F_2) \cup F_3$
vi) $F_1 \cap (F_2 \cap F_3) = (F_1 \cap F_2) \cap F_3$
vii) $F_1 \cup (F_2 \cap F_3) = (F_1 \cup F_2) \cap (F_1 \cup F_3)$
viii) $F_1 \cap (F_2 \cup F_3) = (F_1 \cap F_2) \cup (F_1 \cap F_3)$

2.3. Intuitionistic Fuzzy Sets

This subsection features some of the basic definitions and properties of *if*-sets provided in [2]. For more details, see [16, 17].

Definition 2.15. Let U be a universal set. An intuitionistic fuzzy set (*if*-set) I over U is defined by

$$I = \left\{u^{\mu(u);\nu(u)} : u \in U\right\}$$

where $\mu : U \to [0, 1]$ and $\nu : U \to [0, 1]$ such that $0 \leq \mu(u) + \nu(u) \leq 1$ for all $u \in U$. Here, μ and ν are called membership and non-membership function of I and the elements $u^{0;1}$ is not displayed in I. Moreover, the values $\mu(u)$ and $\nu(u)$ denote the membership degree and non-membership degree of the $u \in U$, respectively.

Hereafter, the *if*-sets are denoted by $I, I_1, I_2, \dots$ and their membership and non-membership functions by $\mu, \mu_1, \mu_2, \dots$ and $\nu, \nu_1, \nu_2, \dots$, respectively. The set of all *if*-sets over U is denoted by $\mathbb{I}$.

Definition 2.16. Let $I \in \mathbb{I}$. Then,
I is called **empty *if*-set**, denoted by $I_\emptyset$, if $\mu(u) = 0$ and $\nu(u) = 1$ for all $u \in U$, and
I is called **universal *if*-set**, denoted by I_U, if $\mu(u) = 1$ and $\nu(u) = 0$ for all $u \in U$.

Example 2.17. Let $U = \{u_1, u_2, u_3, u_4\}$ be a universal set, $\mu(u_1) = 0.7$, $\nu(u_1) = 0.2$, $\mu(u_2) = 0$, $\nu(u_2) = 1$, $\mu(u_3) = 0.2$, $\nu(u_3) = 0.6$, $\mu(u_4) = 0.3$, and $\nu(u_4) = 0.7$. Then, the *if*-set I is written by

$$I = \left\{u_1^{0.7;0.2}, u_3^{0.2;0.6}, u_4^{0.3;0.7}\right\}$$

Definition 2.18. Let $I_1, I_2 \in \mathbb{I}$. Then,
I_1 and I_2 is called **equal**, denoted by $I_1 = I_2$, if $\mu_1(u) = \mu_2(u)$ and $\nu_1(u) = \nu_2(u)$ for all $u \in U$, and
I_1 is called ***if*-subset** of I_2, denoted by $I_1 \subseteq I_2$, if $\mu_1(u) \leq \mu_2(u)$ and $\nu_2(u) \leq \nu_1(u)$ for all $u \in U$.

Definition 2.19. Let $I, I_1, I_2 \in \mathbb{I}$. Then,
the **complement** of I is defined by $I^c = \{u^{\nu(u);\mu(u)} : u \in U\}$,
the **union** of I_1 and I_2 is defined by $I_1 \cup I_2 = \{u^{max\{\mu_1(u),\mu_2(u)\};min\{\nu_1(u),\nu_2(u)\}} : u \in U\}$, and
the **intersection** of I_1 and I_2 is defined by $I_1 \cap I_2 = \{u^{min\{\mu_1(u),\mu_2(u)\};max\{\nu_1(u),\nu_2(u)\}} : u \in U\}$.

Proposition 2.20. If $I \in \mathbb{I}$, then

i) $I \cup I = I$
ii) $I \cap I = I$
iii) $I \cup I_\emptyset = I$
iv) $I \cap I_\emptyset = I_\emptyset$
v) $I \cup I_U = I_U$
vi) $I \cap I_U = I$

Proposition 2.21. If $I_1, I_2, I_3 \in \mathbb{I}$, then

i) $I_1 \cup I_2 = I_2 \cup I_1$
ii) $I_1 \cap I_2 = I_2 \cap I_1$
iii) $(I_1 \cup I_2)^c = I_1^c \cap I_2^c$
iv) $(I_1 \cap I_2)^c = I_1^c \cup I_2^c$
v) $I_1 \cup (I_2 \cup I_3) = (I_1 \cup I_2) \cup I_3$
vi) $I_1 \cap (I_2 \cap I_3) = (I_1 \cap I_2) \cap I_3$
vii) $I_1 \cup (I_2 \cap I_3) = (I_1 \cup I_2) \cap (I_1 \cup I_3)$
viii) $I_1 \cap (I_2 \cup I_3) = (I_1 \cap I_2) \cup (I_1 \cap I_3)$

3. Fuzzy Parameterized Intuitionistic Fuzzy Soft Sets

In this section, we define fuzzy parameterized intuitionistic fuzzy soft sets (*fpifs*-sets) as a new concept of the soft sets. We then present some of their basic properties.

Definition 3.1. Let U be a universal set and X be a set of parameters. If $F = \left\{x^{f(x)} : x \in X\right\}$ is a fuzzy set over X and $p : X \to \mathbb{I}$, $p(x) = \left\{u^{\mu_x(u);\nu_x(u)} : u \in U\right\}$ is an *if*-set over U for $x \in X$, then

$$P = \left\{\left(x^{f(x)}, p(x)\right) : x \in X\right\}$$

is called an *fpifs*-set over U. Here, p is called approximate function of P and the elements $(x^0, I_\emptyset)$ is not displayed in P.

Throughout this paper, the *fpifs*-sets are denoted by $P, P_1, P_2, \ldots$ and their approximate functions by $p, p_1, p_2, \ldots$, respectively. The set of all *fpifs*-sets over U is denoted by $\mathbb{P}$.

Definition 3.2. Let $P \in \mathbb{P}$. Then,
P is called **empty *fpifs*-sets**, denoted by $P_\emptyset$, if $f(x) = 0$ and $p(x) = I_\emptyset$ for all $x \in X$, and
P is called **universal *if*-set**, denoted by P_U, if $f(x) = 1$ and $p(x) = I_U$ for all $x \in X$.

Example 3.3. Let $U = \{u_1, u_2, u_3\}$, $X = \{x_1, x_2, x_3, x_4\}$, $F = \left\{x_1^{0.7}, x_2^{0.4}, x_4^{0.5}\right\}$, and

$$\begin{aligned} p(x_1) &= \left\{u_1^{0.7;0.2}, u_3^{0.5;0.2}\right\}, \\ p(x_2) &= \left\{u_2^{0.5;0.3}, u_3^{0.8;0.1}\right\}, \\ p(x_3) &= I_\emptyset, \\ p(x_4) &= \left\{u_1^{0.6;0.2}, u_2^{0.5;0.3}, u_3^{0.8;0.1}\right\}. \end{aligned}$$

Then,

$$\begin{aligned} P = &\ \{(x_1^{0.7}, p(x_1)), (x_2^{0.4}, p(x_2)), (x_4^{0.5}, p(x_4))\} \\ = &\ \left\{\left(x_1^{0.7}, \left\{u_1^{0.7;0.2}, u_3^{0.5;0.2}\right\}\right), \left(x_2^{0.4}, \left\{u_2^{0.5;0.3}, u_3^{0.8;0.1}\right\}\right), \left(x_4^{0.5}, \left\{u_1^{0.6;0.2}, u_2^{0.5;0.3}, u_3^{0.8;0.1}\right\}\right)\right\} \end{aligned}$$

is an *fpifs*-set over U.

Definition 3.4. Let $P_1, P_2 \in \mathbb{P}$. Then, P_1 and P_2 are called **equal**, denoted by $P_1 = P_2$, if $f_1(x) = f_2(x)$ and $p_1(x) = p_2(x)$ for all $x \in X$.

Definition 3.5. Let $P_1, P_2 \in \mathbb{P}$. Then, P_1 is called ***fpifs*-subset** of P_2, denoted by $P_1 \subseteq P_2$, if $f_1(x) \leq f_2(x)$ and $p_1(x) \subseteq p_2(x)$ for all $x \in X$.

Definition 3.6. Let $P_1, P_2 \in \mathbb{P}$. Then, the **union** of P_1 and P_2 is defined by

$$P_1 \cup P_2 := \left\{(x^{\max\{f_1(x), f_2(x)\}}, p_1(x) \cup p_2(x)) : x \in X\right\}$$

Definition 3.7. Let $P_1, P_2 \in \mathbb{P}$. Then, the **intersection** of P_1 and P_2 is defined by

$$P_1 \cap P_2 := \left\{(x^{\min\{f_1(x), f_2(x)\}}, p_1(x) \cap p_2(x)) : x \in X\right\}$$

Definition 3.8. Let $P \in \mathbb{P}$. Then, the **complement** of P is defined by

$$P^c := \left\{(x^{1-f(x)}, p^c(x)) : x \in X\right\}$$

Proposition 3.9. If $P \in \mathbb{P}$, then

i) $P \cup P = P$
ii) $P \cap P = P$
iii) $P \cup P_\emptyset = P$
iv) $P \cap P_\emptyset = P_\emptyset$
v) $P \cup P_U = P_U$
vi) $P \cap P_U = P$

Proof. Let $P = \{(x^{f(x)}, p(x)) : x \in X\}$ be an *fpifs*-set over U. Then,

i) $P \cup P = \{(x^{\max\{f(x), f(x)\}}, p(x) \cup p(x)) : x \in X\} = \{(x^{f(x)}, p(x)) : x \in X\} = P$

ii) $P \cap P = \{(x^{\min\{f(x), f(x)\}}, p(x) \cap p(x)) : x \in X\} = \{(x^{f(x)}, p(x)) : x \in X\} = P$

iii) $P \cup P_\emptyset = \{(x^{\max\{f(x), 0\}}, p(x) \cup I_\emptyset) : x \in X\} = \{(x^{f(x)}, p(x)) : x \in X\} = P$

iv) $P \cap P_\emptyset = \{(x^{\min\{f(x), 0\}}, p(x) \cap I_\emptyset) : x \in X\} = \{(x^0, I_\emptyset) : x \in X\} = P_\emptyset$

v) $P \cup P_U = \{(x^{\max\{f(x), 1\}}, p(x) \cup I_U) : x \in X\} = \{(x^1, I_U) : x \in X\} = P_U$

vi) $P \cap P_U = \{(x^{\min\{f(x), 1\}}, p(x) \cap I_U) : x \in X\} = \{(x^{f(x)}, p(x)) : x \in X\} = P$

□

Proposition 3.10. If $P_1, P_2, P_3 \in \mathbb{P}$, then

i) $P_1 \cup P_2 = P_2 \cup P_1$
ii) $P_1 \cap P_2 = P_2 \cap P_1$
iii) $(P_1 \cup P_2)^c = P_1^c \cap P_2^c$
iv) $(P_1 \cap P_2)^c = P_1^c \cup P_2^c$
v) $P_1 \cup (P_2 \cup P_3) = (P_1 \cup P_2) \cup P_3$
vi) $P_1 \cap (P_2 \cap P_3) = (P_1 \cap P_2) \cap P_3$
vii) $P_1 \cup (P_2 \cap P_3) = (P_1 \cup P_2) \cap (P_1 \cup P_3)$
viii) $P_1 \cap (P_2 \cup P_3) = (P_1 \cap P_2) \cup (P_1 \cap P_3)$

Proof. Let $P_1 = \{(x^{f_1(x)}, p_1(x)) : x \in X\}$, $P_2 = \{(x^{f_2(x)}, p_2(x)) : x \in X\}$ and $P_3 = \{(x^{f_3(x)}, p_3(x)) : x \in X\}$ be three *fpifs*-sets over U. Then,

i)
$$\begin{aligned} P_1 \cup P_2 &= \{(x^{\max\{f_1(x), f_2(x)\}}, p_1(x) \cup p_2(x)) : x \in X\}, \\ &= \{(x^{\max\{f_2(x), f_1(x)\}}, p_2(x) \cup p_1(x)) : x \in X\}, \\ &= P_2 \cup P_1 \end{aligned}$$

ii)
$$\begin{aligned} P_1 \cap P_2 &= \{(x^{\min\{f_1(x), f_2(x)\}}, p_1(x) \cap p_2(x)) : x \in X\}, \\ &= \{(x^{\min\{f_2(x), f_1(x)\}}, p_2(x) \cap p_1(x)) : x \in X\}, \\ &= P_2 \cap P_1 \end{aligned}$$

iii) $(P_1 \cup P_2)^c = \{(x^{1-\max\{f_1(x),f_2(x)\}}, (p_1(x) \cup p_1(x))^c) : x \in X\},$
$= \{(x^{\min\{1-f_1(x),1-f_2(x)\}}, p_1^c(x) \cap p_2^c(x)) : x \in X\},$
$= P_1^c \cap P_2^c$

iv) $(P_1 \cap P_2)^c = \{(x^{1-\min\{f_1(x),f_2(x)\}}, (p_1(x) \cap p_2(x))^c) : x \in X\},$
$= \{(x^{\max\{1-f_1(x),1-f_2(x)\}}, p_1^c(x) \cup p_2^c(x)) : x \in X\},$
$= P_1^c \cup P_2^c$

v) $P_1 \cup (P_2 \cup P_3) = \{(x^{\max\{f_1(x),\max\{f_2(x),f_3(x)\}\}}, p_1(x) \cup (p_2(x) \cup p_3(x))) : x \in X\}$
$= \{(x^{\max\{\max\{f_1(x),f_2(x)\},f_3(x)\}}, (p_1(x) \cup p_2(x)) \cup p_3(x)) : x \in X\}$
$= (P_1 \cup P_2) \cup P_3$

vi) $P_1 \cap (P_2 \cap P_3) = \{(x^{\min\{f_1(x),\min\{f_2(x),f_3(x)\}\}}, p_1(x) \cap (p_2(x) \cap p_3(x))) : x \in X\}$
$= \{(x^{\min\{\min\{f_1(x),f_2(x)\},f_3(x)\}}, (p_1(x) \cap p_2(x)) \cap p_3(x)) : x \in X\}$
$= (P_1 \cap P_2) \cap P_3$

vii) $P_1 \cup (P_2 \cap P_3) = \{(x^{\max\{f_1(x),\min\{f_2(x),f_3(x)\}\}}, p_1(x) \cup (p_2(x) \cap p_3(x))) : x \in X\}$
$= \{(x^{\min\{\max\{f_1(x),f_2(x)\},\max\{f_1(x),f_3(x)\}\}}, (p_1(x) \cup p_2(x)) \cap (p_1(x) \cup p_3(x))) : x \in X\}$
$= (P_1 \cup P_2) \cap (P_1 \cup P_3)$

viii) $P_1 \cap (P_2 \cup P_3) = \{(x^{\min\{f_1(x),\max\{f_2(x),f_3(x)\}\}}, p_1(x) \cap (p_2(x) \cup p_2(x))) : x \in X\}$
$= \{(x^{\max\{\min\{f_1(x),f_2(x)\},\min\{f_1(x),f_3(x)\}\}}, (p_1(x) \cap p_2(x)) \cup (p_1(x) \cap p_3(x))) : x \in X\}$
$= (P_1 \cap P_2) \cup (P_1 \cap P_3)$

□

4. A Soft Decision-Making Method Proposed on *fpifs*-sets

In this section, we suggest a soft decision-making method that assigns a performance-based value to the alternatives via *fpifs*-sets. Thus, we can choose the optimal elements among the alternatives.

The Proposed Algorithm Steps

Step 1. Construct an *fpifs*-set P such that $P = \left\{\left(x^{f(x)}, \left\{u^{\mu_x(u);\nu_x(u)} : u \in U\right\}\right) : x \in X\right\}$

Step 2. Obtain the values $\omega(u) = \frac{1}{|E|} \sum_{x \in X} f(x)(\mu_x(u) - \nu_x(u))$, for all $u \in U$

Step 3. Obtain the decision set $\{u_k^{d(u_k)} | u_k \in U\}$ such that $d(u_k) = \frac{\omega(u_k) + |\min_i \omega(u_i)|}{\max_i \omega(u_i) + |\min_i \omega(u_i)|}$

5. An Application of the Proposed Method to a Performance-Based Value Assignment Problem

In this section, we apply the proposed method to the performance-based value assignment (PVA) problem for seven filters used in image denoising, namely Decision Based Algorithm (DBA) [18], Modified Decision Based Unsymmetrical Trimmed Median Filter (MDBUTMF) [19], Based on Pixel Density Filter (BPDF) [20], Noise Adaptive Fuzzy Switching Median Filter (NAFSMF) [21], A New Adaptive Weighted Mean Filter (AWMF) [22], Different Applied Median Filter (DAMF) [23], and Adaptive Riesz Mean Filter (ARmF) [24]. Hereafter, let $U = \{u_1, u_2, u_3, u_4, u_5, u_6, u_7\}$ be the set of the alternatives such that

$u_1 =$ "DBA", $u_2 =$ "MDBUTMF", $u_3 =$ "BPDF", $u_4 =$ "NAFSMF", $u_5 =$ "AWMF", $u_6 =$ "DAMF", and $u_7 =$ "ARmF"

Moreover, let $X = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9\}$ be a parameter set determined by a decision-maker such that

$x_1 =$ "*noise density* 10%", $x_2 =$ "*noise density* 20%", $x_3 =$ "*noise density* 30%",

$x_4 =$ "*noise density* 40%", $x_5 =$ "*noise density* 50%", $x_6 =$ "*noise density* 60%", $x_7 =$ "*noise density* 70%", $x_8 =$ "*noise density* 80%", and $x_9 =$ "*noise density* 90%".

Further, let bold numbers in a table point out the best scores therein.

We first present the results of the filters in [24] by Structural Similarity (SSIM) [25] for the image Cameraman in Table 1. Hereinafter, let $\mu_x(u)$ corresponds to the SSIM/MSSIM results of the image/images for filter u and noise density x. Moreover, let $\nu_x(u) = 1 - \mu_x(u)$, for all $x \in X$ and $u \in U$.

Table 1. The SSIM results of the filters for the Cameraman image.

Filters	**10%**	**20%**	**30%**	**40%**	**50%**	**60%**	**70%**	**80%**	**90%**
DBA	0.9938	0.9847	0.9710	0.9520	0.9222	0.8843	0.8283	0.7584	0.6645
MDBUTMF	0.9897	0.9278	0.7945	0.7964	0.8844	0.9158	0.8962	0.8056	0.4451
BPDF	0.9910	0.9783	0.9588	0.9306	0.8934	0.8406	0.7700	0.6665	0.4990
NAFSMF	0.9798	0.9636	0.9484	0.9329	0.9164	0.8954	0.8696	0.8335	0.7288
AWMF	0.9872	0.9839	0.9798	0.9748	0.9667	0.9541	0.9345	0.9015	0.8346
DAMF	0.9960	0.9906	0.9833	0.9749	0.9638	0.9492	0.9293	0.8973	0.8294
ARmF	**0.9969**	**0.9933**	**0.9885**	**0.9824**	**0.9735**	**0.9600**	**0.9395**	**0.9059**	**0.8376**

The application of the soft decision-making method proposed in Section 4 is as follows:

Step 1. Suppose that the success at high noise densities is more important than in the presence of other densities. In this case, the values in Table 1 can be represented with *fpifs*-set as follows:

$$
\begin{aligned}
P_1 = \Big\{ & \big({x_1}^{0.1}, \{{u_1}^{0.9938;0.0062}, {u_2}^{0.9897;0.0103}, {u_3}^{0.9910;0.0090}, {u_4}^{0.9798;0.0202}, {u_5}^{0.9872;0.0128}, {u_6}^{0.9960;0.0040}, \\
& {u_7}^{0.9969;0.0031}\}\big), \big({x_2}^{0.2}, \{{u_1}^{0.9847;0.0153}, {u_2}^{0.9278;0.0722}, {u_3}^{0.9783;0.0217}, {u_4}^{0.9636;0.0364}, {u_5}^{0.9839;0.0161}, \\
& {u_6}^{0.9906;0.0094}, {u_7}^{0.9933;0.0067}\}\big), \big({x_3}^{0.3}, \{{u_1}^{0.9710;0.0290}, {u_2}^{0.7945;0.2055}, {u_3}^{0.9588;0.0412}, {u_4}^{0.9484;0.0516}, \\
& {u_5}^{0.9798;0.0202}, {u_6}^{0.9833;0.0167}, {u_7}^{0.9885;0.0115}\}\big), \big({x_4}^{0.4}, \{{u_1}^{0.9520;0.0480}, {u_2}^{0.7964;0.2036}, {u_3}^{0.9306;0.0694}, \\
& {u_4}^{0.9329;0.0671}, {u_5}^{0.9748;0.0252}, {u_6}^{0.9749;0.0251}, {u_7}^{0.9824;0.0176}\}\big), \big({x_5}^{0.5}, \{{u_1}^{0.9222;0.0778}, {u_2}^{0.8844;0.1156}, \\
& {u_3}^{0.8934;0.1066}, {u_4}^{0.9164;0.0836}, {u_5}^{0.9667;0.0333}, {u_6}^{0.9638;0.0362}, {u_7}^{0.9735;0.0265}\}\big), \big({x_6}^{0.6}, \{{u_1}^{0.8843;0.1157}, \\
& {u_2}^{0.9158;0.0842}, {u_3}^{0.8406;0.1594}, {u_4}^{0.8954;0.1046}, {u_5}^{0.9541;0.0459}, {u_6}^{0.9492;0.0508}, {u_7}^{0.9600;0.0400}\}\big), \big({x_7}^{0.7}, \\
& \{{u_1}^{0.8283;0.1717}, {u_2}^{0.8962;0.1038}, {u_3}^{0.7700;0.2300}, {u_4}^{0.8696;0.1304}, {u_5}^{0.9345;0.0655}, {u_6}^{0.9293;0.0707}, {u_7}^{0.9395;0.0605}\}\big), \\
& \big({x_8}^{0.8}, \{{u_1}^{0.7584;0.2416}, {u_2}^{0.8056;0.1944}, {u_3}^{0.6665;0.3335}, {u_4}^{0.8335;0.1665}, {u_5}^{0.9015;0.0985}, {u_6}^{0.8973;0.1027}, \\
& {u_7}^{0.9059;0.0941}\}\big), \big({x_9}^{0.9}, \{{u_1}^{0.6645;0.3355}, {u_2}^{0.4451;0.5549}, {u_3}^{0.4990;0.5010}, {u_4}^{0.7288;0.2712}, {u_5}^{0.8346;0.1654}, \\
& {u_6}^{0.8294;0.1706}, {u_7}^{0.8376;0.1624}\}\big)\Big\}
\end{aligned}
$$

Step 2. The values $\omega(u)$ are as follows:

$$\omega(u_1) = 0.3322,\ \omega(u_2) = 0.2790,\ \omega(u_3) = 0.2616,\ \omega(u_4) = 0.3612,\ \omega(u_5) = 0.4248,\ \omega(u_6) = 0.4220, \text{ and } \omega(u_7) = 0.4304$$

Step 3. The decision set is as follows:

$$\{\text{DBA}^{0.8580}, \text{MDBUTMF}^{0.7812}, \text{BPDF}^{0.7560}, \text{NAFSMF}^{0.8999}, \text{AWMF}^{0.9919}, \text{DAMF}^{0.9878}, \text{ARmF}^{1}\}$$

The results show that ARmF outperforms the others and the ranking order BPDF $\prec$ MDBUTMF $\prec$ DBA $\prec$ NAFSMF $\prec$ DAMF $\prec$ AWMF $\prec$ ARmF is valid. Moreover, the results confirm the expert's view.

The visual performances of the filters are provided in Fig. 1. The performances of the filters can not be discriminated in consideration of Fig. 1. Moreover, when a large number of data come into question, it is impossible to do so. Therefore, the proposed method has an essential role in dealing with PVA problems.

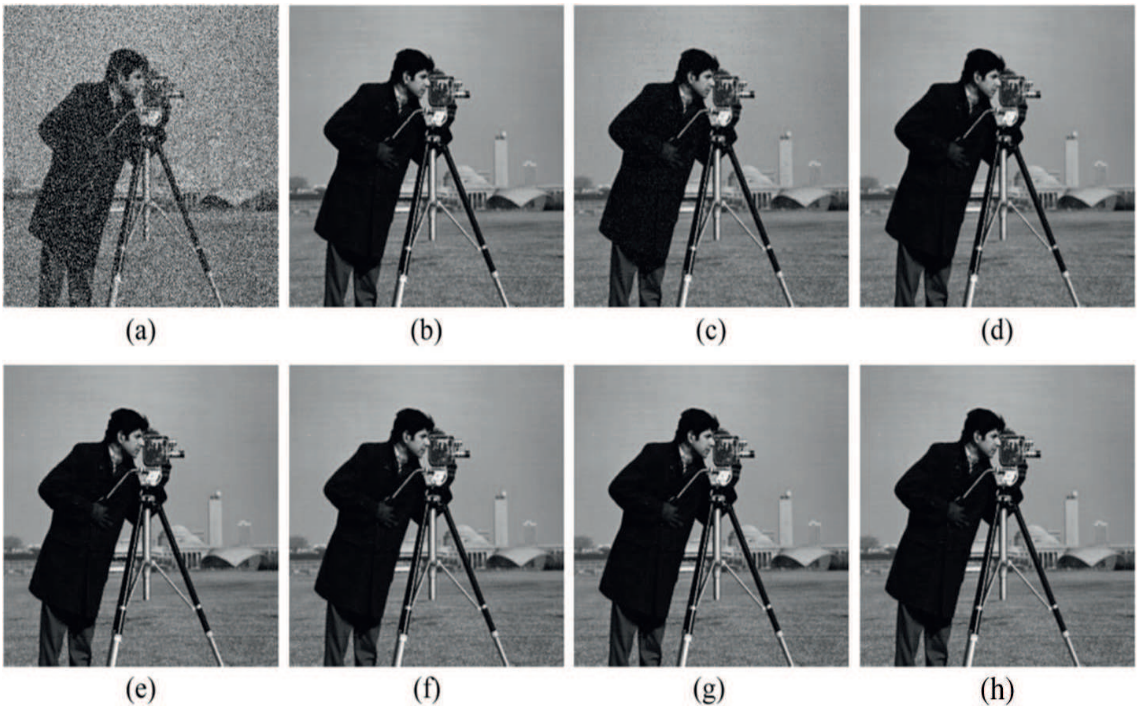

Fig. 1. [24] SSIM results for "Cameraman" of 512×512 with a SPN ratio of 30. (a) Noisy image 0.0550, (b) DBA 0.9710, (c) MDBUTMF 0.7945, (d) BPDF 0.9588, (e) NAFSMF 0.9484, (f) AWMF 0.9798, (g) DAMF 0.9833, and (h) ARmF 0.9885

Secondly, to better establish the success of the proposed method, we present the results of the filters in [24] by Mean Structural Similarity (MSSIM) for the 20 traditional images in Table 2.

Table 2. The MSSIM results of the filters for the 20 traditional images.

Filters	**10%**	**20%**	**30%**	**40%**	**50%**	**60%**	**70%**	**80%**	**90%**
DBA	0.9796	0.9584	0.9315	0.8968	0.8520	0.7949	0.7213	0.6265	0.4966
MDBUTMF	0.9774	0.9197	0.8117	0.7973	0.8399	0.8410	0.8025	0.7023	0.3566
BPDF	0.9783	0.9536	0.9229	0.8838	0.8323	0.7634	0.6680	0.5096	0.2585
NAFSMF	0.9748	0.9504	0.9248	0.8973	0.8666	0.8320	0.7910	0.7357	0.6190
AWMF	0.9728	0.9622	0.9484	0.9315	0.9098	0.8816	0.8437	0.7904	0.7028
DAMF	0.9854	0.9699	0.9516	0.9303	0.9051	0.8748	0.8368	0.7846	0.6964
ARmF	**0.9868**	**0.9735**	**0.9581**	**0.9400**	**0.9173**	**0.8880**	**0.8491**	**0.7947**	**0.7056**

Similarly, the values in Table 2 can be represented with *fpifs*-set as follows:

$$
\begin{aligned}
P_2 = \; & \Big\{ \big({x_1}^{0.1}, \big\{ {u_1}^{0.9796;0.0204}, {u_2}^{0.9774;0.0226}, {u_3}^{0.9783;0.0217}, {u_4}^{0.9748;0.0252}, {u_5}^{0.9728;0.0272}, {u_6}^{0.9854;0.0146}, \\
& {u_7}^{0.9868;0.0132} \big\} \big), \big({x_2}^{0.2}, \big\{ {u_1}^{0.9584;0.0416}, {u_2}^{0.9197;0.0803}, {u_3}^{0.9536;0.0464}, {u_4}^{0.9504;0.0496}, {u_5}^{0.9622;0.0378}, \\
& {u_6}^{0.9699;0.0301}, {u_7}^{0.9735;0.0265} \big\} \big), \big({x_3}^{0.3}, \big\{ {u_1}^{0.9315;0.0685}, {u_2}^{0.8117;0.1183}, {u_3}^{0.9229;0.0771}, {u_4}^{0.9248;0.0752}, \\
& {u_5}^{0.9484;0.0516}, {u_6}^{0.9516;0.0484}, {u_7}^{0.9581;0.0419} \big\} \big), \big({x_4}^{0.4}, \big\{ {u_1}^{0.8968;0.1032}, {u_2}^{0.7973;0.2027}, {u_3}^{0.8838;0.1162}, \\
& {u_4}^{0.8973;0.1027}, {u_5}^{0.9315;0.0685}, {u_6}^{0.9303;0.0697}, {u_7}^{0.9400;0.0600} \big\} \big), \Big({x_5}^{0.5}, \Big\{ {u_1}^{0.8520;0.1480}, {u_2}^{0.8399;0.1601},
\end{aligned}
$$

$u_3{}^{0.8323;0.1677}, u_4{}^{0.8666;0.1334}, u_5{}^{0.9098;0.0902}, u_6{}^{0.9051;0.0949}, u_7{}^{0.9173;0.0827}\}), (x_6{}^{0.6}, \{u_1{}^{0.7949;0.02051},$
$u_2{}^{0.8410;0.1590}, u_3{}^{0.7634;0.2366}, u_4{}^{0.8320;0.1680}, u_5{}^{0.8816;0.1184}, u_6{}^{0.8748;0.1252}, u_7{}^{0.8880;0.1120}\}), (x_7{}^{0.7},$
$\{u_1{}^{0.7213;0.2787}, u_2{}^{0.8025;0.1975}, u_3{}^{0.6680;0.3320}, u_4{}^{0.7910;0.2090}, u_5{}^{0.8437;0.1563}, u_6{}^{0.8368;0.1632}, u_7{}^{0.8491;0.1509}\}),$
$(x_8{}^{0.8}, \{u_1{}^{0.6265;0.3735}, u_2{}^{0.7023;0.2977}, u_3{}^{0.5096;0.4904}, u_4{}^{0.7357;0.2643}, u_5{}^{0.7904;0.2096}, u_6{}^{0.7846;0.2154}$
$u_7{}^{0.7947;0.2053}\}), (x_9{}^{0.9}, \{u_1{}^{0.4966;0.5034}, u_2{}^{0.3566;0.6434}, u_3{}^{0.2585;0.7415}, u_4{}^{0.6190;0.3810}, u_5{}^{0.7028;0.2972},$
$u_6{}^{0.6964;0.3036}, u_7{}^{0.7056;0.2944}\})\}$

If we apply the proposed method to the *fpifs*-set P_2, then the decision set is as follows:

$$\{\mathrm{DBA}^{0.7608}, \mathrm{MDBUTMF}^{0.7289}, \mathrm{BPDF}^{0.5880}, \mathrm{NAFSMF}^{0.8837}, \mathrm{AWMF}^{0.9877}, \mathrm{DAMF}^{0.9794}, \mathrm{ARmF}^{1}\}$$

The results show that ARmF outperforms the others and the following ranking order is valid.

$$\mathrm{BPDF} \prec \mathrm{MDBUTMF} \prec \mathrm{DBA} \prec \mathrm{NAFSMF} \prec \mathrm{DAMF} \prec \mathrm{AWMF} \prec \mathrm{ARmF}$$

Moreover, performance ranking order of filters obtained with the SSIM results of the filters only for the Cameraman image is the same therein. Therefore, the proposed method has been successfully applied to the PVA problem.

6. Conclusion

To deal with uncertainties, the soft set theory has been applied to many theoretical and practical fields. Recently, soft sets, using other theories, have been prominent. In this work, we defined fuzzy parameterized intuitionistic fuzzy soft sets (*fpifs*-sets) by using fuzzy sets, intuitionistic fuzzy sets, and soft sets. We then proposed a soft decision-making method and successfully applied it to a decision-making problem. We think that this study will be beneficial for future studies on soft sets and their applications, particularly in decision-making.

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Energy Decay of Solutions for a System of Higher-Order Kirchhoff Type Equations

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Abstract − In this work, we considered a system of higher-order Kirchhoff type equations with initial and boundary conditions in a bounded domain. Under suitable conditions, we proved an energy decay result by Nakao's inequality techniques.

Keywords − *Kirchhoff type equation, energy decay, damping term*

1. Introduction

The Kirchhoff equation is the famous wave equations model which describe the small-amplitude vibrations of elastic strings introduced by Kirchhoff [1]. In one dimensional space it take th following form

$$\rho h\frac{\partial^2 u}{\partial t^2}+\delta\frac{\partial u}{\partial t}-\left\{\rho_0+\frac{Eh}{2L}\int_0^L\left(\frac{\partial u}{\partial x}\right)^2dx\right\}\frac{\partial^2 u}{\partial x^2}=0,\quad (0<x<L,\ t\geq 0)$$

where $u\left(x,t\right)$ is the vertical displacement, E the Young modulus, ρ the mass density, h the cross-section area, L the length, ρ_0 the initial axial tension, δ the resistance modulus, and f and g the external forces.

In this work, we consider the following nonlinear wave equations of Kirchhoff type

$$\begin{cases} u_{tt}+M\left(\left\|A^{\frac{1}{2}}u\right\|^2+\left\|A^{\frac{1}{2}}v\right\|^2\right)Au+\int_0^t g(t-s)Au(s)ds+\left|u_t\right|^{p-1}u_t=f_1, (x,t)\in\Omega\times[0,\infty) \\ v_{tt}+M\left(\left\|A^{\frac{1}{2}}u\right\|^2+\left\|A^{\frac{1}{2}}v\right\|^2\right)Av+\int_0^t h(t-s)Av(s)ds+\left|v_t\right|^{q-1}v_t=f_2,\ (x,t)\in\Omega\times[0,\infty) \\ u(x,0)=u_0(x),\ u_t(x,0)=u_1(x), \qquad x\in\Omega \\ v(x,0)=v_0(x),\ v_t(x,0)=v_1(x), \qquad x\in\Omega \\ \frac{\partial^i u}{\partial v^i}=\frac{\partial^i v}{\partial v^i}=0,\ i=0,1,2,...,m-1, \qquad x\in\partial\Omega\times(0,\infty) \end{cases} \tag{1}$$

where Ω is a bounded domain in R^n $(n=1,2,3)$ with a smooth boundary $\partial\Omega$, and $g,h:R^+\rightarrow R^+$, $f_i(.,.):R^2\rightarrow R$ $(i=1,2)$ are given functions which will be specified later. Also, $A=\left(-\Delta\right)^m$, $m\geq 1$ is a positive integer and $p,q\geq 1$ are real numbers.

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When $m = 1$, the system

$$\begin{cases} u_{tt} - M\left(\|\nabla u\|^2 + \|\nabla v\|^2\right)\Delta u + \int_0^t g(t-s)\Delta u(s)ds + |u_t|^{p-1} u_t = f_1(u,v) \\ v_{tt} - M\left(\|\nabla u\|^2 + \|\nabla v\|^2\right)\Delta v + \int_0^t h(t-s)\Delta v(s)ds + |v_t|^{q-1} v_t = f_2(u,v) \end{cases} \tag{2}$$

was investigated by Wu [2], here the author proved a decay and blow-up of solutions.

When $M(s) \equiv 1$, (2) become the following system

$$\begin{cases} u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(s)ds + |u_t|^{p-1} u_t = f_1(u,v) \\ v_{tt} - \Delta v + \int_0^t h(t-s)\Delta v(s)ds + |v_t|^{q-1} v_t = f_2(u,v) \end{cases} \tag{3}$$

Many authors studied the existence, blow up, lower bound for the blow up time and decay of solutions of (3) (see [3–7]).

Ye [8] considered the following system

$$\begin{cases} u_{tt} - M(\|\nabla u\|^2 + \|\nabla v\|^2)\Delta u + |u_t|^{p-1} u_t = f_1(u,v) \\ v_{tt} - M(\|\nabla u\|^2 + \|\nabla v\|^2)\Delta v + |v_t|^{q-1} v_t = f_2(u,v) \end{cases}$$

with initial-boundary conditions. The author proved the global existence and energy decay results. Primarily, many authors studied the higher-order wave equation ($m > 1$) (see [9–18]).

Motivated by the above paper, in this work, we prove the global existence and energy decay of solutions of the system (1). This work generalises earlier results in the literature which about the higher order wave equation ($m > 1$).

The present work is organised as follows: In the next section, we give some assumptions and lemmas. Section 3 is devoted to proving the global existence and energy decay of solutions.

2. Preliminaries

We use the standard Lebesque space $L^p(\Omega)$ and Sobolev space $H_0^m(\Omega)$. Also we will use the embedding $H_0^m \hookrightarrow L^p(\Omega)$, for $2 \le p \le \frac{2(n-m)}{n-2m}$ $(n > 2m)$ or $2 \le p$ $(n \le 2m)$,

$$\|u\|_p \le C_* \left\|A^{\frac{1}{2}} u\right\|$$

(see [19, 20], for details about Sobolev spaces).

Now, we make the following assumptions:

(A1) $M(s)$ is a non-negative function for $s \ge 0$ satisfying

$$\begin{cases} m_0, \alpha \ge 0, \ \gamma > 0 \\ M(s) = m_0 + \alpha s^\gamma \end{cases} \tag{4}$$

(A2) If g and h are defined in C^1, for $s \ge 0$

$$\begin{cases} g(s) \ge 0, \ m_0 - \int_0^\infty g(s)ds = \ell > 0, g'(s) \le 0 \\ h(s) \ge 0, \ m_0 - \int_0^\infty h(s)ds = k > 0, h'(s) \le 0 \end{cases}$$

concerning the function $f_1(u,v)$ and $f_2(u,v)$ with $a, b > 0, \forall (u,v) \in R^2$,

$$\begin{cases} f_1(u,v) = (r+1)(a|u+v|^{r-1}(u+v) + b|u|^{\frac{r-3}{2}}|v|^{\frac{r+1}{2}} u \\ f_2(u,v) = (r+1)(a|u+v|^{r-1}(u+v) + b|v|^{\frac{r-3}{2}}|u|^{\frac{r+1}{2}} v \end{cases} \tag{5}$$

We can easily verify that

$$uf_1(u,v) + vf_2(u,v) = (r+1)F(u,v)$$

where

$$F(u,v) = a|u+v|^{r+1} + 2b|uv|^{\frac{r+1}{2}} \tag{6}$$

(A3) r satisfies the following requirements:

$$\begin{cases} \text{If } r > 1 \text{ then } n = 1, 2 \\ \text{If } 1 < r \le 3 \text{ then } n = 3 \end{cases} \tag{7}$$

Lemma 1.1 [4]. There exist two positive constants c_0 and c_1 such that

$$C_0(|u|^{r+1}+|v|^{r+1}) \leq F(u,v) \leq C_1(|u|^{r+1}+|v|^{r+1})$$

Lemma 1.2 [4]. Assume that (7) holds. Then there exists $\tau > 0$ such that

$$\|u+v\|_{r+1}^{r+1} + 2\|uv\|_{\frac{r+1}{2}}^{\frac{r+1}{2}} \leq \tau \left(\ell \left\|A^{\frac{1}{2}}u\right\|^2 + k\left\|A^{\frac{1}{2}}v\right\|^2\right)^{\frac{r+1}{2}}$$

Lemma 1.3 [4]. For $g \in C^1$ and $\phi \in H_0^1(0,T)$, we have

$$-2\int_0^t \int_\Omega g(t-s)\phi\phi_t dxds = \frac{d}{dt}((g\diamond\phi)(t) - \int_0^t g(s)ds\,\|\phi\|^2) + g(t)\,\|\phi\|^2 - (g^{'}\diamond\phi)(t)$$

where

$$(g\diamond\phi)(t) = \int_0^t g(t-s)\int_\Omega |\phi(s)-\phi(t)|^2\, dxds$$

Lemma 1.4 [21] (Nakao inequality). Let $\phi(t)$ be nonincreasing and nonnegative function defined on $[0,T]$, $T>1$, satisfying

$$\phi^{1+\alpha}(t) \leq w_0\left(\phi(t) - \phi(t+1)\right),\ t\in[0,T]$$

for $w_0 > 0$ and $\alpha \geq 0$. Then we have, for each $t\in[0,T]$,

$$\begin{cases} \phi(t) \leq \phi(0)\, e^{-w_1[t-1]^+}, & \alpha = 0 \\ \phi(t) \leq \left(\phi(0)^{-\alpha} + w_0^{-1}\alpha\,[t-1]^+\right)^{-\frac{1}{\alpha}}, & \alpha > 0 \end{cases}$$

where $[t-1]^+ = \max\{t-1, 0\}$ and $w_1 = \ln\left(\frac{w_0}{w_0-1}\right)$.

3. Global Existence and Energy Decay

In this part, we state and prove the existence and energy decay of the solution for the problem (1).

We define the following functionals

$$\begin{aligned} I_1(t) \equiv\ & I_1(u(t),v(t)) = (m_0 - \int_0^t g(s)ds)\left\|A^{\frac{1}{2}}u\right\|^2 \\ & +(m_0 - \int_0^t h(s)ds)\left\|A^{\frac{1}{2}}v\right\|^2 + (g\diamond A^{\frac{1}{2}}u)(t) \\ & +(h\diamond A^{\frac{1}{2}}v)(t) - (r+1)\int_\Omega F(u,v)\ dx \end{aligned} \tag{8}$$

$$\begin{aligned} I_2(t) \equiv\ & I_2(u(t),v(t)) = (m_0 - \int_0^t g(s)ds)\left\|A^{\frac{1}{2}}u\right\|^2 \\ & +(m_0 - \int_0^t h(s)ds)\left\|A^{\frac{1}{2}}v\right\|^2 + \alpha\left(\left\|A^{\frac{1}{2}}u\right\|^2 + \left\|A^{\frac{1}{2}}u\right\|^2\right)^{\gamma+1} \\ & +(g\diamond A^{\frac{1}{2}}u)(t) + (h\diamond A^{\frac{1}{2}}v)(t) - (r+1)\int_\Omega F(u,v)\ dx \end{aligned} \tag{9}$$

$$\begin{aligned} J(t) \equiv\ & J(u(t),v(t)) = \frac{1}{2}(m_0 - \int_0^t g(s)ds)\left\|A^{\frac{1}{2}}u\right\|^2 \\ & +\frac{1}{2}(m_0 - \int_0^t h(s)ds)\left\|A^{\frac{1}{2}}v\right\|^2 \\ & +\frac{\alpha}{2(\gamma+1)}\left(\left\|A^{\frac{1}{2}}u\right\|^2 + \left\|A^{\frac{1}{2}}v\right\|^2\right)^{\gamma+1} \\ & +\frac{1}{2}(g\diamond A^{\frac{1}{2}}u)(t) + \frac{1}{2}(h\diamond A^{\frac{1}{2}}v)(t) - \int_\Omega F(u,v)\ dx \end{aligned} \tag{10}$$

and

$$E(t) \equiv E(u(t), v(t)) = \frac{1}{2}(\|u_t\|^2 + \|v_t\|^2) + J(t) \tag{11}$$

Lemma 2.1. Suppose that (A1), (A2) and (A3) hold. For $\forall t \geq 0$

$$\begin{aligned} E'(t) &= -\|u_t(t)\|_{p+1}^{p+1} - \|v_t(t)\|_{q+1}^{q+1} + \int_0^t \int_\Omega g(t-s) A^{\frac{1}{2}} u(s) A^{\frac{1}{2}} u_t dx ds \\ &\quad + \int_0^t \int_\Omega h(t-s) A v(s)^{\frac{1}{2}} A v_t^{\frac{1}{2}} dx ds \leq 0 \end{aligned} \tag{12}$$

Proof. Multiplying the first equation (1) by u_t and the second equation (1) by v_t, respectively, integrating over Ω, summing up and then using integration by parts, we obtain (12).

Lemma 2.2. Suppose that (A1), (A2) and (A3) hold. Assume further that $I_1(0) > 0$ and

$$\alpha_1 = (r+1)\eta \left(\frac{2(r+1)}{r-1} E(0) \right)^{\frac{m-1}{2}} < 1 \tag{13}$$

then

$$I_1(t) > 0 \tag{14}$$

Proof. Since $I_1(0) > 0$, then by continuity there exists a maximal time $t_{\max} > 0$,(possible $t_{\max} = T$) such that $I_1(0) > 0$, for $t \in [0, t_{\max}]$, which implies that, for $t \in [0, t_{\max}]$

$$\begin{aligned} J(t) &\geq \frac{r-1}{2(r+1)} \left[(m_0 - \int_0^t g(s)ds) \left\| A^{\frac{1}{2}} u \right\|^2 + (m_0 - \int_0^t h(s)ds) \left\| A^{\frac{1}{2}} v \right\|^2 \right] \\ &\quad + \frac{r-1}{2(r+1)} \left(((g \diamond A^{\frac{1}{2}} u)(t) + (h \diamond A^{\frac{1}{2}} v)(t) \right) + \frac{1}{r+1} I_1(t) \\ &\geq \frac{r-1}{2(r+1)} \left[(m_0 - \int_0^t g(s)ds) \left\| A^{\frac{1}{2}} u \right\|^2 + (m_0 - \int_0^t h(s)ds) \left\| A^{\frac{1}{2}} v \right\|^2 \right] \\ &\quad + \frac{r-1}{2(r+1)} \left(((g \diamond A^{\frac{1}{2}} u)(t) + (h \diamond A^{\frac{1}{2}} v)(t) \right) \\ &\geq \frac{r-1}{2(r+1)} \left(\ell \left\| A^{\frac{1}{2}} u \right\|^2 + k \left\| A^{\frac{1}{2}} v \right\|^2 \right) \end{aligned} \tag{15}$$

where

$$\begin{cases} \ell = m_0 - \int_0^t g(s)ds \\ k = m_0 - \int_0^t h(s)ds \end{cases}$$

Using (15), (11), and (12), we have

$$\begin{aligned} \ell \left\| A^{\frac{1}{2}} u \right\|^2 + k \left\| A^{\frac{1}{2}} v \right\|^2 &\leq \frac{2(r+1)}{(r-1)} J(t) \\ &\leq \frac{2(r+1)}{(r-1)} E(t) \\ &\leq \frac{2(r+1)}{(r-1)} E(0) \end{aligned} \tag{16}$$

By (4), (16), (13), and from the (A2), we get

$$\begin{aligned} &(r+1) \int_\Omega F(u,v)\, dx \leq (r+1)\eta \left(\ell \left\| A^{\frac{1}{2}} u \right\|^2 + k \left\| A^{\frac{1}{2}} v \right\|^2 \right)^{\frac{r+1}{2}} \\ \leq\ & (r+1)\eta \left(\frac{2(r+1)}{r-1} E(0) \right)^{\frac{r-1}{2}} \left(\ell \left\| A^{\frac{1}{2}} u \right\|^2 + k \left\| A^{\frac{1}{2}} v \right\|^2 \right) \\ =\ & \alpha_1 (\ell \left\| A^{\frac{1}{2}} u \right\|^2 + k \left\| A^{\frac{1}{2}} v \right\|^2) \\ <\ & \left[\left(m_0 - \int_0^t g(s)ds \right) \left\| A^{\frac{1}{2}} u \right\|^2 + \left(m_0 - \int_0^t h(s)ds \right) \left\| A^{\frac{1}{2}} v \right\|^2 \right] \end{aligned} \tag{17}$$

Thus,

$$\begin{aligned} I_1(t) &= \left(m_0 - \int_0^t g(s)ds\right)\left\|A^{\frac{1}{2}}u\right\|^2 + \left(m_0 - \int_0^t h(s)ds\right)\left\|A^{\frac{1}{2}}v\right\|^2 \\ &\quad + (g \diamond A^{\frac{1}{2}}u)(t) + (h \diamond A^{\frac{1}{2}}v)(t) - (r+1)\int_\Omega F(u,v)\, dx \\ &> 0 \end{aligned}$$

By repeating these steps and using the fact that

$$\lim_{t \to t_{\max}} (r+1)\eta\left(\frac{2(r+1)}{r-1}E(t)\right)^{\frac{m-1}{2}} \leq \alpha_1 < 1$$

This implies that we can take $t_{\max} = T$.

Lemma 2.3. Under the conditions of Lemma 2.2. Then there exists $0 < \eta_1 < 1$ such that

$$\begin{aligned} (r+1)\int_\Omega F(u,v)\, dx &\leq (1-\eta_1)\left[\left(m_0 - \int_0^t g(s)ds\right)\left\|A^{\frac{1}{2}}u\right\|^2 \right. \\ &\quad \left. + \left(m_0 - \int_0^t h(s)ds\right)\left\|A^{\frac{1}{2}}v\right\|^2\right] \end{aligned} \tag{18}$$

where $\eta_1 = 1 - \alpha_1$.

Proof. Thanks to (17), we obtain

$$(r+1)\int_\Omega F(u,v)\, dx \leq \alpha_1\left[\ell\left\|A^{\frac{1}{2}}u\right\|^2 + k\left\|A^{\frac{1}{2}}v\right\|^2\right]$$

Let $\alpha_1 = 1 - \eta_1$ and using (A2), we obtain (18).

We are now ready to state and prove our main result.

Teorem 2.1. Assume that (A1), (A2) and (A3) hold. Let $u_0, v_0 \in H_0^m(\Omega) \cap H^{2m}(\Omega)$ and $u_1, v_1 \in H_0^m(\Omega)$ be given which satisfy $I_1(0) > 0$ and (13). Then the solution of problem (1) is global and bounded. Also, if

$$m_0 > \frac{5 + 2\eta_1}{2\eta_1}\max\left\{\int_0^\infty g(s)ds, \int_0^\infty h(s)ds\right\} \tag{19}$$

then we have the following decay estimates for $\forall t \geq 0$,

(i) if $p = q = 1$

$$E(t) \leq E(0)e^{-\varrho_1 t}$$

(ii) if $\max\{p,q\} > 1$

$$E(t) \leq \left[E(0)^{-\max\{\frac{p-1}{2}, \frac{q-1}{2}\}} + \varrho_2 \max\{\frac{p-1}{2}, \frac{q-1}{2}\}[t-1]^+\right]^{-\frac{2}{\max\{p,q\}-1}}$$

where $\varrho_1 = \varrho_1(m_0, \alpha, \gamma)$ and $\varrho_2 = \varrho_2(m_0, \alpha, \gamma, E(0))$ are positive constants.

Proof. (Global existence) Firstly, we prove $T = \infty$, it is sufficient to show that

$$\|u_t\|^2 + \|v_t\|^2 + \ell\left\|A^{\frac{1}{2}}u\right\|^2 + k\left\|A^{\frac{1}{2}}v\right\|^2$$

is bounded independently of t. We use (11) and (15), we obtain

$$\begin{aligned} E(0) &\geq E(t) = \frac{1}{2}(\|u_t\|^2 + \|v_t\|^2) + J(t) \\ &\geq \frac{1}{2}(\|u_t\|^2 + \|v_t\|^2) + \frac{r-1}{2(r+1)}\left(\ell\left\|A^{\frac{1}{2}}u\right\|^2 + k\left\|A^{\frac{1}{2}}v\right\|^2\right) \end{aligned}$$

Therefore

$$\|u_t\|^2 + \|v_t\|^2 + \ell\left\|A^{\frac{1}{2}}u\right\|^2 + k\left\|A^{\frac{1}{2}}v\right\|^2 \leq \alpha_2 E(0)$$

where $\alpha_2 = \left\{2, \frac{2(r+1)}{r-1}\right\}$. Therefore, we have the global existence result.

(Energy decay) We will derive the energy decay of the problem (1), by the Lemma 2.1, we get

$$\begin{aligned}\frac{d}{dt}E(t) &= -\|u_t(t)\|_{p+1}^{p+1} + \frac{1}{2}(g' \diamond A^{\frac{1}{2}}u)(t) - \frac{1}{2}g(t)\left\|A^{\frac{1}{2}}u\right\|^2 \\ &\quad - \|v_t(t)\|_{q+1}^{q+1} + \frac{1}{2}(h' \diamond A^{\frac{1}{2}}u)(t) - \frac{1}{2}h(t)\left\|A^{\frac{1}{2}}v\right\|^2 \\ &< 0\end{aligned}$$

By integrating over $[t,\ t+1]$, we obtain

$$\begin{aligned}E(t) - E(t+1) &= \int_t^{t+1} \|u_t(t)\|_{p+1}^{p+1} ds - \frac{1}{2}\int_t^{t+1} (g' \diamond A^{\frac{1}{2}}u)(s)ds \\ &\quad + \frac{1}{2}\int_t^{t+1} g(s)\left\|A^{\frac{1}{2}}u\right\|^2 ds + \int_t^{t+1} \|v_t(t)\|_{q+1}^{q+1} ds \\ &\quad - \frac{1}{2}\int_t^{t+1} (h' \diamond A^{\frac{1}{2}}v)(s)ds + \frac{1}{2}\int_t^{t+1} h(s)\left\|A^{\frac{1}{2}}v\right\|^2 ds \\ &= D_1^{p+1}(t) + D_2^{q+1}(t) \end{aligned} \tag{20}$$

where

$$\begin{cases} D_1^{p+1}(t) = \int_t^{t+1} \|u_t(t)\|_{p+1}^{p+1} ds - \frac{1}{2}\int_t^{t+1}(g' \diamond A^{\frac{1}{2}}u)(s)ds + \frac{1}{2}\int_t^{t+1} g(s)\left\|A^{\frac{1}{2}}u\right\|^2 ds \\ D_2^{q+1}(t) = \int_t^{t+1} \|v_t(t)\|_{q+1}^{q+1} ds - \frac{1}{2}\int_t^{t+1}(h' \diamond A^{\frac{1}{2}}v)(s)ds + \frac{1}{2}\int_t^{t+1} h(s)\left\|A^{\frac{1}{2}}v\right\|^2 ds \end{cases} \tag{21}$$

By virtue of (21) and Hölder inequality, we observe that

$$\int_t^{t+1}\int_\Omega |u_t|^2 dxdt + \int_t^{t+1}\int_\Omega |v_t|^2 dxdt \le c_1(\Omega)D_1(t)^2 + c_2(\Omega)D_2(t)^2 \tag{22}$$

where $c_1(\Omega) = vol(\Omega)^{\frac{p-1}{p+1}}$ and $c_2(\Omega) = vol(\Omega)^{\frac{q-1}{q+1}}$. By the mean value theorem, there exist $t_1 \in \left[t,\ t+\frac{1}{4}\right]$ and $t_2 \in \left[t+\frac{3}{4},\ t+1\right]$ such that

$$\|u_t(t_i)\|^2 + \|v_t(t_i)\|^2 \le 4c_1(\Omega)D_1(t)^2 + c_2(\Omega)D_2(t)^2 \tag{23}$$

Now, multiplying the first equation (1) by u and the second equation (1) by v,respectively, and integrating over $\Omega \times [t_1, t_2]$,using integration by parts, Hölder inequality and adding them together, we have

$$\begin{aligned}\int_{t_1}^{t_2} I_2(t) &\le \sum_{i=1}^{2}\|u_t(t_i)\|\,\|u(t_i)\| + \sum_{i=1}^{2}\|v_t(t_i)\|\,\|v(t_i)\| + \int_{t_1}^{t_2}(\|u_t\|^2 + \|v_t\|^2)dt \\ &\quad - \int_{t_1}^{t_2}\int_\Omega (|u_t|^{p-1}u_t u + |v_t|^{q-1}v_t v)dxdt \\ &\quad + \int_{t_1}^{t_2}(g \diamond A^{\frac{1}{2}}u)(t) + (h \diamond A^{\frac{1}{2}}v)(t)dt \\ &\quad + \int_{t_1}^{t_2}\int_\Omega\int_0^t g(t-s)A^{\frac{1}{2}}u)(t)[A^{\frac{1}{2}}u(s) - A^{\frac{1}{2}}u(t)]dsdxdt \\ &\quad + \int_{t_1}^{t_2}\int_\Omega\int_0^t h(t-s)A^{\frac{1}{2}}v)(t)[A^{\frac{1}{2}}v(s) - A^{\frac{1}{2}}v(t)]dsdxdt \end{aligned} \tag{24}$$

Since

$$\begin{aligned}\int_\Omega \int_0^t g(t-s)A^{\frac{1}{2}}u(t)[A^{\frac{1}{2}}u(s)-A^{\frac{1}{2}}u(t)]dsdx &= \frac{1}{2}\int_0^t g\left(t-s\right)\left(\left\|A^{\frac{1}{2}}u(t)\right\|^2+\left\|A^{\frac{1}{2}}u(s)\right\|^2\right)ds \\ &\quad -\frac{1}{2}\int_0^t g(t-s)\left(\left\|A^{\frac{1}{2}}u(t)-A^{\frac{1}{2}}u(s)\right\|^2\right)ds \\ &\quad -\int_\Omega\int_0^t g(s)\left|A^{\frac{1}{2}}u(t)\right|^2 dsdx \\ &= -\frac{1}{2}\int_\Omega\int_0^t g(s)\left|A^{\frac{1}{2}}u(t)\right|^2 dsdx \\ &\quad +\frac{1}{2}\int_0^t g(t-s)(\left\|A^{\frac{1}{2}}u(s)\right\|^2)ds \\ &\quad -\frac{1}{2}(g\diamond A^{\frac{1}{2}}u)(t)\end{aligned}$$

and

$$\begin{aligned}\int_\Omega \int_0^t h(t-s)A^{\frac{1}{2}}v)(t)[A^{\frac{1}{2}}v(s)-A^{\frac{1}{2}}v(t)]dsdx &= -\frac{1}{2}\int_\Omega\int_0^t h(s)\left|A^{\frac{1}{2}}v(s)\right|^2 dsdx \\ &\quad +\frac{1}{2}\int_0^t h(t-s)(\left\|A^{\frac{1}{2}}v(s)\right\|^2)ds \\ &\quad -\frac{1}{2}(h\diamond A^{\frac{1}{2}}v)(t)\end{aligned}$$

hence (24) takes the form

$$\begin{aligned}\int_{t_1}^{t_2} I_2(t) &\leq \sum_{i=1}^{2}\|u_t(t_i)\|\,\|u(t_i)\|+\sum_{i=1}^{2}\|v_t(t_i)\|\,\|v(t_i)\|+\int_{t_1}^{t_2}(\|u_t\|^2+\|v_t\|^2)dt \\ &\quad -\int_{t_1}^{t_2}\int_\Omega(|u_t|^{p-1}u_tu+|v_t|^{q-1}v_tv)dxdt \\ &\quad +\frac{1}{2}\int_{t_1}^{t_2}(g\diamond A^{\frac{1}{2}}u)(t)+(h\diamond A^{\frac{1}{2}}v)(t)dt \\ &\quad +\frac{1}{2}\int_{t_1}^{t_2}\int_0^t g(t-s)\left\|A^{\frac{1}{2}}u(t)\right\|^2 dsdt \\ &\quad +\frac{1}{2}\int_{t_1}^{t_2}\int_0^t h(t-s)\left\|A^{\frac{1}{2}}v(t)\right\|^2 dsdt. \qquad (25)\end{aligned}$$

Let's estimate for the first two terms on the right side of the equation (25). By Young inequality, (23) and (16)

$$\begin{aligned}\|u_t(t_i)\|\,\|u(t_i)\| &\leq C_*\sqrt{4c_1D_1(t)^2+4c_2D_2(t)^2}\sup_{t_1\leq s\leq t_2}\left\|A^{\frac{1}{2}}u(s)\right\| \\ &\leq C_*\left(\frac{2(r+1)}{\ell(r-1)}\right)^{\frac{1}{2}}\sqrt{4c_1D_1(t)^2+4c_2D_2(t)^2}\sup_{t_1\leq s\leq t_2}E(s)^{\frac{1}{2}} \\ &\leq C_*\left(\frac{2(r+1)}{\beta(r-1)}\right)^{\frac{1}{2}}\sqrt{4c_1D_1(t)^2+4c_2D_2(t)^2}E(t)^{\frac{1}{2}} \qquad (26)\end{aligned}$$

and

$$\|v_t(t_i)\|\,\|v(t_i)\|\leq C_*\left(\frac{2(r+1)}{\beta(r-1)}\right)^{\frac{1}{2}}\sqrt{4c_1D_1(t)^2+4c_2D_2(t)^2}E(t)^{\frac{1}{2}} \qquad (27)$$

where $\beta = \min\{\ell, k\}$. Also from the Hölder inequality (16)

$$\begin{aligned}\left|\int_{t_1}^{t_2}\int_{\Omega}(|u_t|^{p-1}u_t u dx dt\right| &\leq \int_{t_1}^{t_2}\|u_t(t)\|_{p+1}^{p}\|u\|_{p+1}dt \\ &\leq C_*\int_{t_1}^{t_2}\|u_t(t)\|_{p+1}^{p}\left\|A^{\frac{1}{2}}u\right\|dt \\ &\leq C_*\left(\frac{2(r+1)}{\ell(r-1)}\right)^{\frac{1}{2}}\sup_{t_1\leq s\leq t_2}E(s)^{\frac{1}{2}}\int_{t_1}^{t_2}\|u_t(t)\|_{p+1}^{p}dt \\ &\leq C_*\left(\frac{2(r+1)}{\ell(r-1)}\right)^{\frac{1}{2}}E(t)^{\frac{1}{2}}D_1(t)^p \end{aligned} \tag{28}$$

and similarly

$$\left|\int_{t_1}^{t_2}\int_{\Omega}(|v_t|^{q-1}v_t v dx dt\right| \leq C_*\left(\frac{2(r+1)}{\beta(r-1)}\right)^{\frac{1}{2}}E(t)^{\frac{1}{2}}D_2(t)^q \tag{29}$$

Employing Young's inequality for convolution ($\|\phi*\psi\|_q \leq \|\phi\|_r\|\psi\|_s$ with $\frac{1}{q}=\frac{1}{r}+\frac{1}{s}-1, 1\leq q,r,s$), (25) the last two terms of inequality

$$\begin{aligned}\int_{t_1}^{t_2}\int_0^t g(t-s)\left\|A^{\frac{1}{2}}u(s)\right\|^2 dsdt &\leq \int_{t_1}^{t_2}g(t)dt\int_{t_1}^{t_2}\left\|A^{\frac{1}{2}}u(t)\right\|^2dt \\ &\leq (m_0-\ell)\int_{t_1}^{t_2}\left\|A^{\frac{1}{2}}u(t)\right\|^2dt \\ &\leq (m_0-\beta)\int_{t_1}^{t_2}\left\|A^{\frac{1}{2}}u(t)\right\|^2dt \end{aligned} \tag{30}$$

and

$$\begin{aligned}\int_{t_1}^{t_2}\int_0^t h(t-s)\left\|A^{\frac{1}{2}}v(t)\right\|^2 dsdt &\leq \int_{t_1}^{t_2}h(t)dt\int_{t_1}^{t_2}\left\|A^{\frac{1}{2}}v(t)\right\|^2dt \\ &\leq (m_0-\beta)\int_{t_1}^{t_2}\left\|A^{\frac{1}{2}}v(t)\right\|^2dt \end{aligned} \tag{31}$$

Adding (29) and (30) together and nothing that, we see

$$\ell\left\|A^{\frac{1}{2}}u\right\|^2 + k\left\|A^{\frac{1}{2}}v\right\|^2 \leq \frac{1}{\eta_1}I_2(t) \tag{32}$$

From (9) and the definition of $I_2(t)$ and also by (18), we have

$$\begin{aligned}&\frac{1}{2}\int_{t_1}^{t_2}\int_0^t g(t-s)(\left\|A^{\frac{1}{2}}u(s)\right\|^2dsdt + \frac{1}{2}\int_{t_1}^{t_2}\int_0^t h(t-s)(\left\|A^{\frac{1}{2}}v(t)\right\|^2dsdt \\ \leq\ &\frac{m_0-\beta}{2\beta}\int_{t_1}^{t_2}(\ell\left\|A^{\frac{1}{2}}u\right\|^2+k\left\|A^{\frac{1}{2}}v\right\|^2)dt \leq \frac{m_0-\beta}{2\beta\eta_1}\int_{t_1}^{t_2}I_2(t)dt\end{aligned} \tag{33}$$

We use (30)-(32) to estimate the last two terms on the right-hand side of (25), we get

$$\begin{aligned}\frac{1}{2}\int_{t_1}^{t_2}(g\diamond A^{\frac{1}{2}}u)(t)+(h\diamond A^{\frac{1}{2}}u)(t)dt &= \frac{1}{2}\int_{t_1}^{t_2}\int_0^t g(t-s)\left\|A^{\frac{1}{2}}u(s)-A^{\frac{1}{2}}u(t)\right\|^2dsdt \\ &\quad +\frac{1}{2}\int_{t_1}^{t_2}\int_0^t h(t-s)\left\|A^{\frac{1}{2}}v(t)-A^{\frac{1}{2}}v(t)\right\|^2dsdt \\ &\leq \int_{t_1}^{t_2}\int_0^t g(t-s)(\left\|A^{\frac{1}{2}}u(t)\right\|^2+\left\|A^{\frac{1}{2}}u(t)\right\|^2)dsdt \\ &\quad +\int_{t_1}^{t_2}\int_0^t h(t-s)(\left\|A^{\frac{1}{2}}v(t)\right\|^2+\left\|A^{\frac{1}{2}}v(t)\right\|^2)dsdt \\ &\leq \frac{2(m_0-\beta)}{\beta}\int_{t_1}^{t_2}(\ell\left\|A^{\frac{1}{2}}u\right\|^2+k\left\|A^{\frac{1}{2}}v\right\|^2)dt \\ &\leq \frac{2(m_0-\beta)}{\beta}\int_{t_1}^{t_2}I_2(t)dt.\end{aligned} \tag{34}$$

By (25) and the above inequalities

$$\begin{aligned}\int_{t_1}^{t_2} I_2(t)dt &\leq c_1(\Omega)D_1(t)^2 + c_2(\Omega)D_2(t)^2 \\ &+4c_3\sqrt{4c_1(\Omega)D_1(t)^2+4c_2(\Omega)D_2(t)^2}E(t)^{\frac{1}{2}} \\ &+c_3E(t)^{\frac{1}{2}}(D_1(t)^p+D_2(t)^q)+c_4\int_{t_1}^{t_2} I_2(t)dt \end{aligned} \tag{35}$$

where $c_3 = C_*(\frac{2(r+1)}{\beta(r-1)})^{\frac{1}{2}}$ and $c_4 = \frac{5(m_0-\beta)}{2\beta\eta_1}$. Then, rewriting (35)

$$\begin{aligned}\beta_2\int_{t_1}^{t_2} I_2(t)dt &\leq c_1(\Omega)D_1(t)^2 + c_2(\Omega)D_2(t)^2 \\ &+4c_3\sqrt{4c_1(\Omega)D_1(t)^2+4c_2(\Omega)D_2(t)^2}E(t)^{\frac{1}{2}} \\ &+c_3E(t)^{\frac{1}{2}}(D_1(t)^p+D_2(t)^q)\end{aligned}$$

where $\beta_2 = 1-\frac{5(m_0-\beta)}{2\beta\eta_1}$ and $m_0 > \frac{5+2\eta_1}{2\eta_1}.\max\left\{\int_0^\infty g(s)ds, \int_0^\infty h(s)ds\right\}$. So $\beta_2>0$, thus

$$\begin{aligned}\int_{t_1}^{t_2} I_2(t)dt &\leq c_5[\sqrt{4c_1(\Omega)D_1(t)^2+4c_2(\Omega)D_2(t)^2}E(t)^{\frac{1}{2}} \\ &+D_1(t)^2+D_2(t)^2+E(t)^{\frac{1}{2}}(D_1(t)^p+D_2(t)^q)]\end{aligned} \tag{36}$$

where $c_5 = \frac{\max\{c_1(\Omega),c_2(\Omega),4c_3\}}{\beta_2}$. On the other hand, by $E(t)$ function in the definition of the equation (11), (8) and (9), we obtain

$$I_2(t) = I_1(t) + \alpha\left\|A^{\frac{1}{2}}u\right\|^2 + \left\|A^{\frac{1}{2}}v\right\|^2)^{\gamma+1}$$

$$\begin{aligned}E(t) &= \frac{1}{2}(\|u_t\|^2+\|v_t\|^2)+\frac{r-1}{2(r+1)}\left[\left(m_0-\int_0^t g(s)ds\right)\left\|A^{\frac{1}{2}}u\right\|^2+(m_0-\int_0^t h(s)ds\left\|A^{\frac{1}{2}}v\right\|^2)\right] \\ &+\frac{r-1}{2(r+1)}(g\diamond A^{\frac{1}{2}}u)(t)+(h\diamond A^{\frac{1}{2}}u)(t)+\frac{\alpha}{2(\gamma+1)}(\left\|A^{\frac{1}{2}}u\right\|^2+\left\|A^{\frac{1}{2}}v\right\|^2)^{\gamma+1}+\frac{1}{r+1}I_1(t) \\ &\leq \frac{1}{2}(\|u_t\|^2+\|v_t\|^2)+\frac{r-1}{2(r+1)}\left[\left(m_0-\int_0^t g(s)ds\right)\left\|A^{\frac{1}{2}}u\right\|^2+(m_0-\int_0^t h(s)ds\left\|Av^{\frac{1}{2}}\right\|^2)\right] \\ &+\frac{r-1}{2(r+1)}\left((g\diamond A^{\frac{1}{2}}u)(t)+(h\diamond A^{\frac{1}{2}}u)(t)\right)+\left(\frac{1}{r+1}+\frac{1}{2(\gamma+1)}\right)I_2(t)\end{aligned}$$

The (37) is integrated over (t_1,t_2) and then using (22), (32), (34), (36), we obtain

$$\begin{aligned}\int_{t_1}^{t_2} E(t)dt &\leq \frac{1}{2}\int_{t_1}^{t_2}(\|u_t\|^2+\|v_t\|^2)dt+\frac{r-1}{2(r+1)}\int_{t_1}^{t_2}\left(m_0-\int_0^t g(s)ds\right)\left\|A^{\frac{1}{2}}u\right\|^2dt \\ &+\frac{r-1}{2(r+1)}\int_{t_1}^{t_2}\left(m_0-\int_0^t h(s)ds\right)\left\|A^{\frac{1}{2}}v\right\|^2dt \\ &+\frac{r-1}{2(r+1)}\int_{t_1}^{t_2}\left((g\diamond A^{\frac{1}{2}}u)(t)+(h\diamond A^{\frac{1}{2}}u)(t)\right)dt \\ &+\left(\frac{1}{r+1}+\frac{1}{2(\gamma+1)}\right)\int_{t_1}^{t} I_2(t)dt \\ &\leq c_1(\Omega)D_1(t)^2+4c_2(\Omega)D_2(t)^2+c_6\int_{t_1}^{t} I_2(t)dt \\ &\leq c_7[\sqrt{4c_1(\Omega)D_1(t)^2+4c_2(\Omega)D_2(t)^2}E(t)^{\frac{1}{2}} \\ &+D_1(t)^2+D_2(t)^2+E(t)^{\frac{1}{2}}(D_1(t)^p+D_2(t)^q)]\end{aligned} \tag{37}$$

where $c_6 = \frac{1}{r+1}+\frac{1}{2(\gamma+1)}\frac{r-1}{2(r+1)\eta_1}+\frac{2(r-1)(m_0-\beta)}{(r+1)\beta\eta_1}$ and $c_7 = \max\{c_1(\Omega),c_2(\Omega),c_6c_5\}$. Moreover, integrating (12) over (t_1,t_2), we obtain

$$E(t_2) \leq 2\int_{t_1}^{t_2} E(t)dt,$$

due to $t_2 - t_1 \geq \frac{1}{2}$, we get

$$\begin{aligned} E(t) &= E(t_2) + \int_t^{t_2} \|u_t\|_{p+1}^{p+1} ds - \frac{1}{2}\int_t^{t_2} (g' \diamond A^{\frac{1}{2}}u)(s)ds \\ &+ \frac{1}{2}\int_t^{t_2} g(s)\left\|A^{\frac{1}{2}}u\right\|^2 ds + \int_t^{t_2} \|v_t\|_{q+1}^{q+1} ds \\ &- \frac{1}{2}\int_t^{t_2} (h' \diamond A^{\frac{1}{2}}v)(s)ds + \frac{1}{2}\int_t^{t+1} h(s)\left\|A^{\frac{1}{2}}v\right\|^2 ds \\ &\leq 2\int_t^{t_2} E(t)dt + D_1(t)^{p+1} + D_2(t)^{q+1} \end{aligned} \tag{38}$$

As a result, by (37) and (38), we obtain

$$\begin{aligned} E(t) &\leq c_8\sqrt{4c_1(\Omega)D_1(t)^2 + 4c_2(\Omega)D_2(t)^2}E(t)^{\frac{1}{2}} + D_1(t)^2 + D_2(t)^2 \\ &+ E(t)^{\frac{1}{2}}D_1(t)^p + E(t)^{\frac{1}{2}}D_2(t)^q + D_1(t)^{p+1} + D_2(t)^{q+1} \end{aligned}$$

Hence, by Young inequality, we have

$$E(t) \leq c_9\left[D_1(t)^2 + D_2(t)^2 + D_1(t)^{2p} + D_2(t)^{2q} + D_1(t)^{p+1} + D_1(t)^{q+1}\right] \tag{39}$$

where c_8 and c_9 are positive constants.

(i) if $p = q = 1$. By (20) and (39), we have

$$E(t) \leq c_{10}\left[E(t) - E(t+1)\ \right]$$

where $c_{10} > 1$. Using Nakao's inequality, we get

$$E(t) \leq E(0)e^{-\varrho_1 t}$$

where $\varrho_1 = \ln(\frac{w_0}{w_0 - 1})$.

(ii) if $\max\{p, q\} > 1$. From (39), we get

$$E(t) \leq c_9\left[D_1(t)^2(1 + D_1(t)^{2p-2} + D_1(t)^{p-1}) + D_2(t)^2(1 + D_2(t)^{2q-2} + D_2(t)^{q-1})\right]$$

Then since

$$\begin{cases} D_1(t) \leq E(t)^{\frac{1}{p+1}} \leq E(0)^{\frac{1}{p+1}} \\ D_2(t) \leq E(t)^{\frac{1}{q+1}} \leq E(0)^{\frac{1}{q+1}} \end{cases}$$

we see from (20)

$$\begin{aligned} E(t) &\leq c_9\left[D_1(t)^2\left(1 + E(0)^{\frac{P-1}{P+1}} + E(0)^{\frac{2p-2}{p+1}}\right) + D_2(t)^2\left(1 + E(0)^{\frac{q-1}{q+1}} + E(0)^{\frac{2q-2}{q+1}}\right)\right] \\ &\leq c_9\left(D_1(t)^2 + D_2(t)^2\right)(1 + E(0)^{\frac{p-1}{p+1}} + E(0)^{\frac{2p-2}{p+1}} + E(0)^{\frac{q-1}{q+1}} + E(0)^{\frac{2q-2}{q+1}} \\ &= c_{10}E(0)\left(D_1(t)^2 + D_2(t)^2\right) \end{aligned}$$

where $\lim_{E(0)\to 0} c_{10}(E(0)) = c_9$ and $\rho = \max\left\{\frac{p-1}{2}, \frac{q-1}{2}\right\}$. Then, we get

$$\begin{aligned} E(t)^{1+\rho} &\leq \left[c_{10}\left(D_1(t)^2 + D_2(t)^2\right)\right]^{1+\rho} \\ &\leq c_{11}E(0)\left(D_1(t)^{2\rho+2} + D_2(t)^{2\rho+2}\right) \\ &= c_{11}E(0)\left(D_1(t)^{p+1}D_1(t)^{2\rho+2-p-1} + D_2(t)^{q+1}D_2(t)^{2\rho+2-q-1}\right) \\ &= c_{11}E(0)\left(D_1(t)^{p+1}D_1(t)^{2\rho-p+1} + D_2(t)^{q+1}D_2(t)^{2\rho-q+1}\right) \\ &\leq c_{11}E(0)\left(D_1(t)^{p+1}E(0)^{\frac{2\rho-p+1}{P+1}} + D_2(t)^{q+1}E(0)^{\frac{2\rho-q+1}{q+1}}\right) \\ &\leq c_{12}E(0)\left(D_1(t)^{p+1} + D_2(t)^{q+1}\right) \\ &\leq c_{12}E(0)\left(E(t) - E(t+1)\right) \end{aligned} \tag{40}$$

where

$$c_{11}\left(E(0)\right) = 2^{\rho}\left(c_{10}\left(E(0)\right)\right)^{1+\rho}$$

and

$$c_{12}\left(E(0)\right) = c_{11}\left(E(0)\right) \max\left\{E(0)^{\frac{2\rho-p+1}{P+1}}, E(0)^{\frac{2\rho-q+1}{q+1}}\right\}$$

Thus, from (40) and Nakao inequality, we get

$$E(t) \leq \left(E(0)^{-\rho} + \varrho_2 \rho\left[t-1\right]^{+}\right)^{-\frac{1}{\rho}}$$

where $\varrho_2 = c_{12}^{-1}\left(E(0)\right)$. Thus, the proof of theorem is completed.

4. Conclusion

In this work, we obtained the existence of global solutions and energy decay for a system of higher-order Kirchhoff type equations. This improves and extends many results in the literature.

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Convex and Concave Sets Based on Soft Sets and Fuzzy Soft Sets

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Abstract — In this study, after given some basic definitions of soft sets and fuzzy soft sets we firstly define convex-concave soft sets. Then, we investigate their properties and give some relations between convex and concave soft sets. Furthermore, we define fuzzy convex-concave soft sets and give some properties for the sets.

Keywords — *Fuzzy set, soft sets, convex sets, concave sets, strictly convex, strongly convex*

1. Introduction

In 1999, Molodtsov [1] proposed a completely new approach so-called *soft set theory* for modeling vagueness and uncertainty which may not be successfully modeled by the classical mathematics, probability theory, fuzzy sets [2], rough sets [3], and other mathematical tools. In the last decade, properties and applications on the soft set theory solidly enriched (e.g. [4–12]), including the extension of soft set theory (e.g. [13–25]). Along with them, many interesting applications of soft set theory have been expanded by embedding the ideas of fuzzy sets, rough sets, intuitionistic fuzzy sets, vague sets, interval-valued fuzzy sets (e.g. [26–32]). Then, A method with unknown data in soft sets and in fuzzy soft sets is introduced by Deng and Wang [33], Gong et al. [34] gave two parameters reduction algorithms, Yang et al. [35] proposed the concept of multi-fuzzy soft sets with a few operations, Mao et al. [36] gave multi-experts group decision making problems by using intuitionistic fuzzy soft matrices, Feng and Lie [37] studied subsets and various relations deal with soft set theory, Wang et al. [38] built a new decision-making method by introducing the concept of fuzzy soft sets for the virtual machine startup problems, Agarwal et al. [39] introduced a new score function, similarity measure, relations with applications for generalized intuitionistic fuzzy sets.

Different definitions of convex fuzzy and concave fuzzy sets have defined but the first definition of convex fuzzy sets introduced by Zadeh [2] and then concave fuzzy sets introduced by Chaudhuri [40]. After Zadeh [2], concavoconvex fuzzy sets proposed by Sarkar [41], with some properties. Moreover, works on convex (concave)fuzzy sets in theories and applications has been progressing rapidly by many autor, for example, [42–47].

Convex and concave fuzzy sets play important roles in optimization theory. A significant definition of convex fuzzy sets introduced by Zadeh [2] and concave fuzzy sets introduced by Chaudhuri [40]. The concavoconvex fuzzy sets proposed by Sarkar [41] which is convex and concave fuzzy sets together conceived by combining. The works on convex and concave fuzzy sets, in theories and applications, have been progressing rapidly (e.g. [42, 46, 47]).

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The present expository paper is a condensation of part/extension of the dissertation [48]. In this work, we introduce the soft and fuzzy soft version of fuzzy convex and concave sets and also investigate their some properties. The plan of the paper is as follows. In Section 2, we give some notations, definitions used throughout the paper In section 3, after we give convex soft sets, we define strictly convex soft sets and strongly convex soft sets and then give desired some properties. In Section 4, we define fuzzy soft convex sets and fuzzy soft concave sets and then we show some properties.

2. Preliminary

In this section, we present the basic definitions and some operations of fuzzy sets [2], soft set theory [1] and fuzzy soft set [26]. More detailed explanations related to this subsection may be found in [1, 2, 7, 26, 30].

Throughout this paper E will denote the n-dimensional Euclidean space R^n. U denotes the arbitrary set, I denotes the interval $[0, 1]$, and I° denotes $(0, 1)$.

Definition 2.1. [2] Let U be the universe. Then, a fuzzy set X over U is defined by a set of ordered pair

$$X = \{(\mu_X(x)/x) : x \in U\}$$

where

$$\mu_X : U \to [0, 1]$$

is called membership function of X. The value $\mu_X(x)$ is called the membership value or the grade of membership of $x \in U$. The membership value represents the degree of x belonging to the fuzzy set X.

Definition 2.2. [41] A fuzzy set in R^n is defined to be convex if for all $p, q \in R^n$ and all r on the line segment $\overline{pq}$ the following condition with respect to its characteristic function μ is satisfied:

$$\mu(r) \geq min\{\mu(p), \mu(q)\}$$

Conversely, a fuzzy set in R^n is defined to be concave if for $p, q \in R^n$ and all r on the line segment $\overline{pq}$ the following condition with respect to its characteristic function μ is satisfied:

$$\mu(r) \leq max\{\mu(p), \mu(q)\}$$

Definition 2.3. [1] Let U be a universe, $P(U)$ be the power set of U and E be a set of parameters that are describe the elements of U. A soft set S over U is a set defined by a set valued function f_S representing a mapping

$$f_S : E \to P(U)$$

It is noting that the soft set is a parametrized family of subsets of the set U, and therefore it can be written a set of ordered pairs

$$S = \{(x, f_S(x)) : x \in E\}$$

Here, f_S is called approximate function of the soft set S and $f_S(x)$ is called x-approximate value of $x \in E$. The subscript S in the f_S indicates that f_S is the approximate function of S.

Generally, f_S, f_T, f_V, ... will be used as an approximate functions of S, T, V, ..., respectively.

Note that if $f_S(x) = \emptyset$, then the element $(x, f_S(x))$ is not appeared in S.

Definition 2.4. [7] Let S and T be two soft sets. Then,

1. If $f_S(x) = \emptyset$ for all $x \in E$, then S is called a empty soft set, denoted by S_Φ.

2. If $f_S(x) \subseteq f_T(x)$ for all $x \in E$, then S is a soft subset of T, denoted by $S \tilde{\subseteq} T$.

3. Complement of S is denoted by $S^{\tilde{c}}$. Its approximate function $f_{S^{\tilde{c}}}$ is defined by

$$f_{S^{\tilde{c}}}(x) = U \setminus f_S(x) \quad \text{for all } x \in E$$

4. Union of S and T is denoted by $S \tilde{\cup} T$. Its approximate function $f_{S \tilde{\cup} T}$ is defined by

$$f_{S \tilde{\cup} T}(x) = f_S(x) \cup f_T(x) \quad \text{for all } x \in E$$

5. Intersection of S and T is denoted by $S\tilde{\cap}T$. Its approximate function $f_{S\tilde{\cap}T}$ is defined by

$$f_{S\tilde{\cap}T}(x) = f_S(x) \cap f_T(x) \quad \text{for all } x \in E$$

Definition 2.5. [8] Let S be a soft set over U and α be a subset of U. Then, α-inclusion of the soft set S, denoted by S^α, is defined as

$$S^\alpha = \{x \in E : f_S(x) \supseteq \alpha\}$$

Definition 2.6. [26] Let U be an initial universe, $F(U)$ be all fuzzy sets over U. E be the set of all parameters and $A \subseteq E$. An fuzzy soft set Γ_A on the universe U is defined by the set of ordered pairs as follows,

$$\Gamma_A = \{(x, \gamma_A(x)) : x \in E, \gamma_A(x) \in F(U)\}$$

where $\gamma_A : E \to F(U)$ such that $\gamma_A(x) = \emptyset$ if $x \notin A$, and for all $x \in E$

$$\gamma_{A(x)} = \{\mu_{\gamma_{A(x)}}(u)/u : u \in U, \mu_{\gamma_{A(x)}}(u) \in [0,1]\}$$

is a fuzzy set over U.

The subscript A in the γ_A indicates that γ_A is the approximate function of Γ_A.
Note that if $\gamma_A(x) = \emptyset$, then the element $(x, \gamma_A(x))$ is not appeared in Γ_A.

Definition 2.7. [26] Let Γ_A and Γ_B be two fuzzy soft sets. Then,

1. If $\gamma_A(x) = \emptyset$ for all $x \in E$, then Γ is called a empty fuzzy soft set, denoted by Γ_Φ.
2. Complement of Γ_A is denoted by $\Gamma_A^{\tilde{c}}$. Its approximate function $\gamma_{A^{\tilde{c}}}$ is defined by

$$\gamma_{A^{\tilde{c}}}(x) = \gamma_A^c(x), \text{ for all } x \in E$$

3. Union of Γ_A and Γ_B is denoted by $\Gamma_A \tilde{\cup} \Gamma_B$. Its fuzzy approximate function $\gamma_{A\tilde{\cup}B}$ is defined by

$$\gamma_{A\tilde{\cup}B}(x) = \gamma_A(x) \cup \gamma_B(x) \quad \text{for all } x \in E$$

4. Intersection of Γ_A and Γ_B is denoted by $\Gamma_A \tilde{\cap} \Gamma_B$. Its fuzzy approximate function $\gamma_{A\tilde{\cap}B}(x)$ is defined by

$$\gamma_{A\tilde{\cap}B}(x) = \gamma_A(x) \cap \gamma_B(x) \quad \text{for all } x \in E$$

5. Γ_A is an fuzzy soft subset of Γ_B, denoted by $\Gamma_A \tilde{\subseteq} \Gamma_B$, if $\gamma_A(x) \subseteq \gamma_B(x)$ for all $x \in E$.

3. Convex Soft sets

In this section, after we give convex soft sets, we define strictly convex soft sets and strongly convex soft sets and then give desired some properties. Some of it is quoted from [2, 40–42, 46–48].

Definition 3.1. The soft set S on E is called a convex soft set, is shown in Figure 1, if

$$f_S(ax + (1-a)y) \supseteq f_S(x) \cap f_S(y)$$

for every $x, y \in E$ and $a \in I$.

Definition 3.2. The soft set S on E is called a concave soft set if

$$f_S(ax + (1-a)y) \subseteq f_S(x) \cup f_S(y)$$

for every $x, y \in E$ and $a \in I$.

Definition 3.3. The soft set S on E is called a strongly convex soft set if

$$f_S(ax + (1-a)y) \supset f_S(x) \cap f_S(y)$$

for every $x, y \in E$, $x \neq y$ and $a \in I^\circ$.

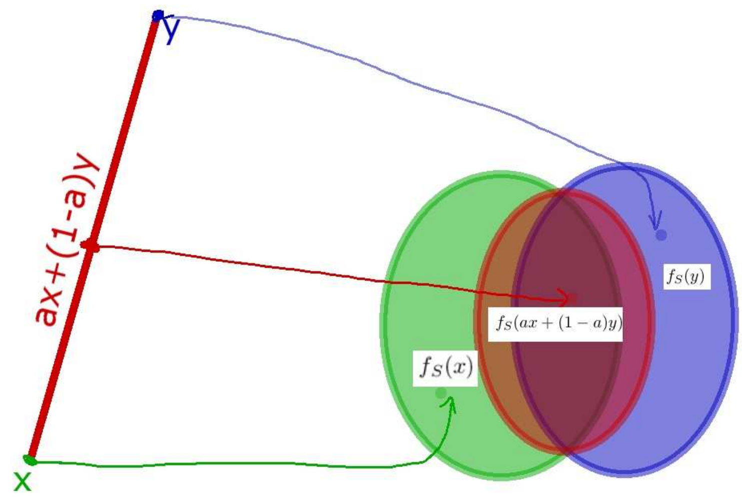

Fig. 1. The convex soft set

Definition 3.4. The soft set S on E is called a strictly convex soft set if

$$f_S(ax + (1-a)y) \supset f_S(x) \cap f_S(y)$$

for every $x, y \in E$, $f_S(x) \neq f_S(y)$ and $a \in I^{\circ}$.

Note 3.5. A convex soft set is not necessarily a strongly convex soft set and a strictly convex soft set is not necessarily a strongly convex soft set.

Theorem 3.6. If $\{S_i : i \in \{1, 2, ...\}\}$ is any family of convex soft sets, then,

1. the intersection $\tilde{\cap}_{i\in I} S_i$ is a convex soft set but union of any family $\{S_i : i \in I = \{1, 2, ...\}\}$ of convex soft sets is not necessarily a convex soft set.

2. the union $\tilde{\cup}_{i\in I} S_i$ is a concave soft set and the intersection of any family $\{S_i : i \in I = \{1, 2, ...\}\}$ of concave soft sets is concave soft set.

Theorem 3.7. S is a convex soft set $\Leftrightarrow$ $S^{\tilde{c}}$ is a concave soft sets.

PROOF. $\Rightarrow$ Suppose that there exist $x, y \in E$, $a \in I$ and S be a convex soft set.
Then, since S is convex,

$$f_S(ax + (1-a)y) \supseteq f_S(x) \cap f_S(y) \tag{1}$$

or

$$U \setminus f_S(ax + (1-a)y) \subseteq U \setminus \{f_S(x) \cap f_S(y)\} \tag{2}$$

we have

$$U \setminus f_S(ax + (1-a)y) \subseteq \{U \setminus f_S(x) \cup U \setminus f_S(y)\} \tag{3}$$

So, $S^{\tilde{c}}$ is a concave fuzzy soft set.
$\Leftarrow$ $S^{\tilde{c}}$ be a concave soft set.
Since $S^{\tilde{c}}$ is concave, we have

$$U \setminus f_S(ax + (1-a)y) \subseteq \{U \setminus f_S(x) \cup U \setminus f_S(y)\} \tag{4}$$

Then,

$$U \setminus f_S(ax + (1-a)y) \subseteq U \setminus \{f_S(x) \cap f_S(y)\} \tag{5}$$

or

$$f_S(ax + (1-a)y) \supseteq f_S(x) \cap f_S(y) \tag{6}$$

So, S is a convex soft set. □

Theorem 3.8. $S\tilde{\cap}T$ is a strictly convex soft set when both S and T are strictly convex soft sets.

Proof. Suppose that there exist $x, y \in E$ and $a \in I^{\circ}$ and $W = S\tilde{\cap}T$. Then,

$$f_W(ax + (1-a)y) = f_S(ax + (1-a)y) \cap f_T(ax + (1-a)y) \tag{7}$$

Now, since S and T strictly convex sets,

$$f_S(ax + (1-a)y) \supset f_S(x) \cap f_S(y) \text{ such that } f_S(x) \neq f_S(y) \tag{8}$$

$$f_T(ax + (1-a)y) \supset f_T(x) \cap f_T(y) \text{ such that } f_S(x) \neq f_S(y) \tag{9}$$

and hence,

$$f_W(ax + (1-a)y) \supset (f_S(x) \cap f_S(y)) \cap (f_T(x) \cap f_T(y))) \text{ such that } f_S(x) \neq f_S(y) \tag{10}$$

and thus

$$f_W(ax + (1-a)y) \supset f_W(x) \cap f_W(y)) \text{ such that } f_S(x) \neq f_S(y) \tag{11}$$

□

Theorem 3.9. If $\{S_i : i \in \{1, 2, ...\}\}$ is any family of strictly convex soft sets, then the intersection $\tilde{\cap}_{i \in I} S_i$ is a strictly convex soft set.

Remark 3.10. The union of any family $\{S_i : i \in I = \{1, 2, ...\}\}$ of strictly convex soft sets is not necessarily a strictly convex soft set.

Theorem 3.11. Let S be a strictly convex soft set on E.

1. If there exists $a \in I^{\circ}$, for every $x, y \in E$ such that

$$f_S(ax + (1-a)y) \supseteq f_S(x) \cap f_S(y) \tag{12}$$

   Then S is a convex soft set on E.

2. If there exists $a \in I$, such that for every pair of distinct points $x \in E$, $y \in E$, we have

$$f_S(ax + (1-a)y) \supset f_S(x) \cap f_S(y) \tag{13}$$

   Then S is a strongly convex soft set on E.

Proof. The proof is straightforward. □

Theorem 3.12. Let S be a convex soft set on E.

1. If there exists $a \in I$, for every pair of distinct points $x \in E$, $y \in E$ implies that

$$f_S(ax + (1-a)y) \supset f_S(x) \cap f_S(y) \tag{14}$$

   Then S is a strongly convex soft set on E.

2. If there exists $a \in I$, for every $x \in E$, $y \in E$, $f_S(x) \neq f_S(y)$ implies,

$$f_S(ax + (1-a)y) \supset f_S(x) \cap f_S(y) \tag{15}$$

   Then S is a strictly convex soft set on E.

Definition 3.13. The fuzzy soft set Γ_A on E is called a convex fuzzy soft set, is shown in Figure 2, if

$$\gamma_A(ax + (1-a)y) \supseteq \gamma_A(x) \cap \gamma_A(y)$$

for every $x, y \in E$ and $a \in I$.

Theorem 3.14. $\Gamma_A \tilde{\cap} \Gamma_B$ is a fuzzy convex soft set when both Γ_A and Γ_B are fuzzy convex soft sets.

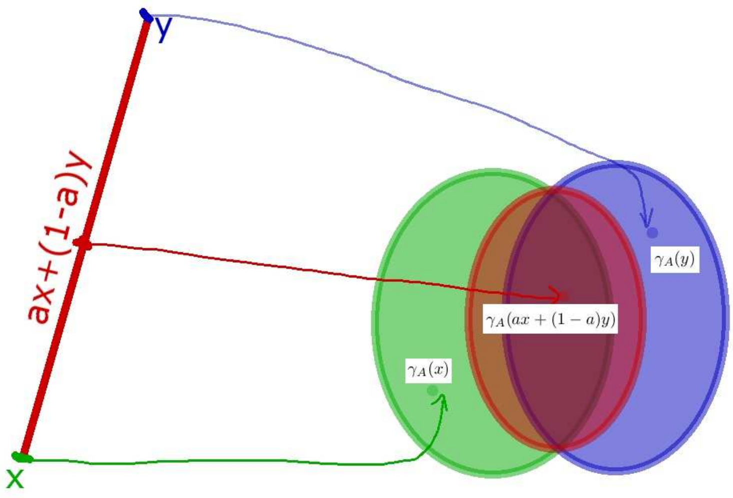

Fig. 2. The fuzzy convex soft set

PROOF. Suppose that there exist $x, y \in E$ and $a \in I$ and $C = S \cap T$. Then,

$$\gamma_C(ax + (1-a)y) = \gamma_S(ax + (1-a)y) \cap \gamma_T(ax + (1-a)y) \tag{16}$$

Now, since S and T convex,

$$\gamma_S(ax + (1-a)y) \supseteq \gamma_S(x) \cap \gamma_S(y) \tag{17}$$

$$\gamma_T(ax + (1-a)y) \supseteq \gamma_T(x) \cap \gamma_T(y) \tag{18}$$

and hence,

$$\gamma_C(ax + (1-a)y) \supseteq (\gamma_S(x) \cap \gamma_S(y)) \cap (\gamma_T(x) \cap \gamma_T(y)) \tag{19}$$

and thus

$$\gamma_C(ax + (1-a)y) \supseteq \gamma_C(x) \cap \gamma_C(y) \tag{20}$$

□

Definition 3.15. The soft set Γ_A on E is called a concave fuzzy soft set if

$$\gamma_A(ax + (1-a)y) \subseteq \gamma_A(x) \cup \gamma_A(y)$$

for every $x, y \in E$ and $a \in I$.

Theorem 3.16. $\Gamma_A \tilde{\cup} \Gamma_B$ is a concave fuzzy soft set when both Γ_A and Γ_B are concave fuzzy soft sets.

PROOF. Suppose that there exist $x, y \in E$ and $a \in I$ and $\Gamma_C = \Gamma_A \tilde{\cup} \Gamma_B$. Then,

$$\gamma_C(ax + (1-a)y) = \gamma_A(ax + (1-a)y) \cup \gamma_B(ax + (1-a)y) \tag{21}$$

Now, since S and T concave,

$$\gamma_A(ax + (1-a)y) \subseteq \gamma_A(x) \cup \gamma_A(y) \tag{22}$$

$$\gamma_B(ax + (1-a)y) \subseteq \gamma_B(x) \cup \gamma_B(y) \tag{23}$$

and hence,

$$\gamma_C(ax + (1-a)y) \subseteq (\gamma_A(x) \cup \gamma_A(y)) \cup (\gamma_B(x) \cup \gamma_B(y)) \tag{24}$$

and thus

$$\gamma_C(ax + (1-a)y) \subseteq \gamma_C(x) \cup \gamma_C(y) \tag{25}$$

□

Theorem 3.17. If $\{\Gamma_{A_i} : i \in \{1, 2, ...\}\}$ is any family of concave fuzzy soft sets, then the union $\tilde{\cup}_{i \in I} \Gamma_{A_i}$ is a concave fuzzy soft set.

Theorem 3.18. Γ_A is a convex fuzzy soft set when $\Gamma_A^{\tilde{c}}$ is a concave fuzzy soft sets.

Proof. Suppose that there exist $x, y \in E$, $a \in I$ and Γ_A be a convex fuzzy soft set.

Then, since Γ_A is convex,

$$\gamma_A(ax + (1-a)y) \supseteq \gamma_A(x) \cap \gamma_A(y) \tag{26}$$

or

$$U \setminus \gamma_A(ax + (1-a)y) \subseteq U \setminus \{\gamma_A(x) \cap \gamma_A(y)\} \tag{27}$$

we have

$$U \setminus \gamma_A(ax + (1-a)y) \subseteq \{U \setminus \gamma_A(x) \cup U \setminus \gamma_A(y)\} \tag{28}$$

So, $\Gamma_A^{\tilde{c}}$ is a concave fuzzy soft set. □

Theorem 3.19. If $\{\Gamma_{A_i} : i \in \{1, 2, ...\}\}$ is any family of convex fuzzy soft sets, then the intersection $\tilde{\cap}_{i \in I} \Gamma_{A_i}$ is a convex fuzzy soft set.

Remark 3.20. The union of any family $\{\Gamma_{A_i} : i \in I = \{1, 2, ...\}\}$ of convex fuzzy soft sets is not necessarily a convex fuzzy soft set.

Theorem 3.21. Γ_A is a concave fuzzy soft set when $\Gamma_A^{\tilde{c}}$ is a convex fuzzy soft sets. sets.

Proof. Suppose that there exist $x, y \in E$, $a \in I$ and S be a concave fuzzy soft set.

Then, since S is concave,

$$\gamma_A(ax + (1-a)y) \subseteq \gamma_A(x) \cup \gamma_A(y) \tag{29}$$

or

$$U \setminus \gamma_A(ax + (1-a)y) \supseteq U \setminus \{\gamma_A(x) \cup \gamma_A(y)\} \tag{30}$$

we have

$$U \setminus \gamma_A(ax + (1-a)y) \supseteq \{U \setminus \gamma_A(x) \cap U \setminus \gamma_A(y)\} \tag{31}$$

So, $\Gamma_A^{\tilde{c}}$ is a convex fuzzy soft set. □

Theorem 3.22. S is a concave fuzzy soft set on E iff for every $\beta \in [0, 1]$ and $\alpha \in P(U)$, S^α is a concave set on E.

Proof. $\Rightarrow$ Assume that S is a concave fuzzy soft set. If $x_1, x_2 \in E$ and $\alpha \in P(U)$, then $\gamma_A(x_1) \supseteq \alpha$ and $\gamma_A(x_2) \supseteq \alpha$. It follows from the concavity of S that

$$\gamma_A(\beta x_1 + (1-\beta)x_2) \subseteq \gamma_A(x_1) \cup \gamma_A(x_2)$$

and thus S^α is a concave set.

$\Leftarrow$ Assume that S^α is a concave set for every $\beta \in [0, 1]$. Especially, for $x_1, x_2 \in E$, S^α is concave for $\alpha = \gamma_A(x_1) \cup \gamma_A(x_2)$.

Since $\gamma_A(x_1) \supseteq \alpha$ and $\gamma_A(x_2) \supseteq \alpha$, we have $x_1 \in S^\alpha$ and $x_2 \in S^\alpha$, whence $\beta x_1 + (1-\beta)x_2 \in S^\alpha$. Therefore, $\gamma_A(\beta x_1 + (1-\beta)x_2) \subseteq \alpha = \gamma_A(x_1) \cup \gamma_A(x_2)$, which indicates S is a concave fuzzy soft set on X. □

4. Conclusion

In the literature, convex fuzzy sets has been introduced widely by many researchers. In this paper, we defined convex soft sets, concave soft sets, convex fuzzy soft sets and concave fuzzy soft sets and give some properties. Also we will try to explore characterizations of convex fuzzy soft sets to optimization in the future. The theory may be applied to many fields and more comprehensive in the future to solve the related problems, such as; pattern classification, operation research, decision making, optimization problem, and so on.

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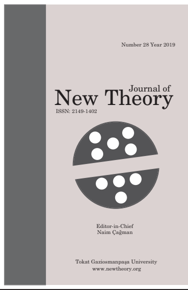

On μsp-Continuous Maps in Topological Spaces

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Abstract − In this paper, we introduce a new class of continuous maps called μsp-continuous maps and study their properties in topological spaces.

Keywords − *Topological space, μsp-closed set, μsp-continuous map, μsp-irresolute map*

1. Introduction and Preliminaries

Several authors [1–7] working in the field of general topology have shown more interest in studying the concepts of generalizations of continuous maps. A weak form of continuous maps called g-continuous maps were introduced by Balachandran et al. [8]. As generalizations of closed sets, μsp-closed sets were introduced and studied by the same author [9]. In this paper, we first introduce μsp-continuous maps and study their relations with various generalized continuous maps. We also discuss some properties of μsp-continuous maps. We introduce μsp-irresolute maps in topological spaces and discuss some of their properties. Various properties and characterizations of such maps are discussed by using μsp-closure and μsp-interior under certain conditions. Throughout this paper, (X, τ), (Y, σ), and (Z, η) (or X, Y, and Z) represent topological spaces on which no separation axioms are assumed unless otherwise mentioned. For a subset A of a space (X, τ), $cl(A)$, $int(A)$, and A^C denote the closure of A, the interior of A, and complement of A, respectively.
We recall the following definitions which are useful in the sequel.

Definition 1.1. A subset A of a space (X, τ) is called:

1. α-open set [10] if $A \subseteq int(cl(int(A)))$.
2. semi-open set [11] if $A \subseteq cl(int(A))$.
3. pre-open set [5] if $A \subseteq int(cl(A))$.
4. β-open set [1] (= semi-pre-open set [12]) if $A \subseteq cl(int(cl(A)))$.

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The complements of the above mentioned open sets are called their respective closed sets.
The α-closure [10](resp. semi-closure [13], pre-closure [14], semi-pre-closure [12]) of a subset A of X, denoted by $\alpha cl(A)$ (resp. $scl(A)$, $pcl(A)$, $spcl(A)$) is defined to be the intersection of all α-closed (resp. semi-closed, pre-closed, semi-pre-closed) sets of (X, τ) containing A.

Definition 1.2. A subset A of a space (X, τ) is called:

1. a generalized closed (briefy g-closed) set [15] if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is open in (X, τ). The complement of g-closed set is called g-open set.
2. a generalized semi-closed (briefly gs-closed) set [16] if $scl(A) \subseteq U$ whenever $A \subseteq U$ and U is open in (X, τ). The complement of gs-closed set is called gs-open set.
3. an α-generalized closed (briefly αg-closed) set [17] if $\alpha cl(A) \subseteq U$ whenever $A \subseteq U$ and U is open in (X, τ). The complement of αg-closed set is called αg-open set.
4. a generalized α-closed (briefly $g\alpha$-closed) set [18] if $\alpha cl(A) \subseteq U$ whenever $A \subseteq U$ and U is α-open in (X, τ). The complement of $g\alpha$-closed set is called $g\alpha$-open set.
5. a $g^{\#}$-closed set [19] if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is αg-open in (X, τ). The complement of $g^{\#}$-closed set is called $g^{\#}$-open set.
6. a generalized semi-preclosed (briefly gsp-closed) set [20] if $spcl(A) \subseteq U$ whenever $A \subseteq U$ and U is open in (X, τ). The complement of gsp-closed set is called gsp-open set.
7. a $\hat{g}$-closed set [7] (= ω-closed set [6]) if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is semi-open in (X, τ). The complement of $\hat{g}$-closed set is called $\hat{g}$-open set.
8. a $^*g$-closed set [21] if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is $\hat{g}$-open in (X, τ). The complement of $^*g$-closed set is called $^*g$-open set.
9. a $\#g$-semi-closed(briefly $\#gs$-closed) set [22] if $scl(A) \subseteq U$ whenever $A \subseteq U$ and U is $*g$-open in (X, τ). The complement of $\#gs$-closed set is called $\#gs$-open set.
10. a $g\alpha^*$-closed set [18, 23] if $\alpha cl(A) \subseteq int(U)$ whenever $A \subseteq U$ and U is α-open in (X, τ). The complement of $g\alpha^*$-closed set is called $g\alpha^*$-open set.
11. a μ-closed set [24] if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is $g\alpha^*$-open in (X, τ). The complement of μ-closed set is called μ-open set.
12. a μp-closed set [25] if $pcl(A) \subseteq U$ whenever $A \subseteq U$ and U is $g\alpha^*$-open in (X, τ). The complement of μp-closed set is called μp-open set.
13. a μs-closed set [26] if $scl(A) \subseteq U$ whenever $A \subseteq U$ and U is $g\alpha^*$-open in (X, τ). The complement of μs-closed set is called μs-open set.
14. a μsp-closed set [9] if $spcl(A) \subseteq U$ whenever $A \subseteq U$ and U is $g\alpha^*$-open in (X, τ). The complement of μsp-closed set is called μsp-open set.

Remark 1.3. The collection of all g-closed (resp. gs-closed, αg-closed, $g\alpha$-closed, $g^{\#}$-closed, gsp-closed, $\hat{g}$-closed, $^*g$-closed, $\#gs$-closed, $g\alpha^*$-closed, μ-closed, μp-closed, μs-closed, μsp-closed) sets is denoted by $gc(\tau)$ (resp. $gsc(\tau)$, $\alpha gc(\tau)$, $g\alpha c(\tau)$, $g^{\#}c(\tau)$, $gspc(\tau)$, $\hat{g}c(\tau)$, $^*gc(\tau)$, $\#gsc(\tau)$, $g\alpha^*c(\tau)$, $\mu c(\tau)$, $\mu pc(\tau)$, $\mu sc(\tau)$,$\mu spc(\tau)$).

We denote the power set of X by $P(X)$.

Definition 1.4. A map $f : (X, \tau) \to (Y, \sigma)$ is called:

1. α-continuous [27] if $f^{-1}(V)$ is a α-closed set of (X, τ) for every closed set V of (Y, σ).
2. semi-continuous [11] if $f^{-1}(V)$ is a semi-closed set of (X, τ) for every closed set V of (Y, σ).
3. pre-continuous [5] if $f^{-1}(V)$ is a pre-closed set of (X, τ) for every closed set V of (Y, σ).

4. β-continuous [1] if $f^{-1}(V)$ is a β-closed set of (X, τ) for every closed set V of (Y, σ).
5. g-continuous [8] if $f^{-1}(V)$ is a g-closed set of (X, τ) for every closed set V of (Y, σ).
6. gs-continuous [2] if $f^{-1}(V)$ is a gs-closed set of (X, τ) for every closed set V of (Y, σ).
7. αg-continuous [28] if $f^{-1}(V)$ is a αg-closed set of (X, τ) for every closed set V of (Y, σ).
8. $g\alpha$-continuous [28] if $f^{-1}(V)$ is a $g\alpha$-closed set of (X, τ) for every closed set V of (Y, σ).
9. $g^{\#}$-continuous [19] if $f^{-1}(V)$ is a $g^{\#}$-closed set of (X, τ) for every closed set V of (Y, σ).
10. gsp-continuous [20] if $f^{-1}(V)$ is a gsp-closed set of (X, τ) for every closed set V of (Y, σ).
11. $\hat{g}$-continuous [7] if $f^{-1}(V)$ is a $\hat{g}$-closed set of (X, τ) for every closed set V of (Y, σ).
12. $^*g$-continuous [21] if $f^{-1}(V)$ is a $^*g$-closed set of (X, τ) for every closed set V of (Y, σ).
13. $\#g$-semi-continuous [22] if $f^{-1}(V)$ is a $\#g$-semi-closed set of (X, τ) for every closed set V of (Y, σ).
14. μ-continuous [24] if $f^{-1}(V)$ is a μ-closed set of (X, τ) for every closed set V of (Y, σ).
15. μp-continuous [25] if $f^{-1}(V)$ is a μp-closed set of (X, τ) for every closed set V of (Y, σ).
16. μs-continuous [26] if $f^{-1}(V)$ is a μs-closed set of (X, τ) for every closed set V of (Y, σ).

Definition 1.5. [9] For a space (X, τ), the following hold:

1. $T_{\mu sp}$-space if every μsp-closed set is closed.
2. $\mu T_{\mu sp}$-space if every μsp-closed set is μ-closed.
3. $pT_{\mu sp}$-space if every μsp-closed set is pre-closed.
4. $spT_{\mu sp}$-space if every μsp-closed set is semi-preclosed.
5. $\alpha T_{\mu sp}$-space if every μsp-closed set is α-closed.
6. $g\alpha T_{\mu sp}$-space if every μsp-closed set is $g\alpha$-closed.

Result 1.6.

1. Every closed set (resp. pre-closed set, α-closed set, semi-closed set, β-closed set) is μsp-closed but not conversely [9].
2. Every μ-closed set (resp. μp-closed set, μs-closed set) is μsp-closed but not conversely [9].
3. Every $g\alpha$-closed set (resp. $g^{\#}$-closed set, $\hat{g}$-closed set) is μsp-closed but not conversely [9].
4. Every open set is μsp-open set but not conversely.

2. μsp-Continuous Maps and Irresolute Maps

We introduce the following definition.

Definition 2.1. A map $f : (X, \tau) \to (Y, \sigma)$ is called μsp-continuous if $f^{-1}(V)$ is a μsp-closed set of (X, τ) for every closed set V of (Y, σ).

Proposition 2.2. Every continuous (resp. prec-continuous, α-continuous, semi-continuous, β-continuous) is μsp-continuous but not conversely.

Proof. The proof follows from Result 1.6 (1).

Example 2.3. Let $X = Y = \{a, b, c\}$, $\tau = \{\phi, \{a\}, \{a, c\}, X\}$, and $\sigma = \{\phi, \{b\}, X\}$. Then, $\mu spc(\tau) = \{\phi, \{b\}, \{c\}, \{a, c\}, \{b, c\}, X\}$ and $pc(\tau) = \alpha c(\tau) = sc(\tau) = spc(\tau) = \{\phi, \{b\}, \{c\}, \{b, c\}, X\}$. Define $f : (X, \tau) \to (Y, \sigma)$ be the identity map. Then, f is μsp-continuous but not continuous (resp. prec-continuous, α-continuous, semi-continuous, semi-precontinuous), since $f^{-1}(\{a, c\}) = \{a, c\}$ is not closed (resp. preclosed, α-closed, semi-closed, semi-preclosed).

Proposition 2.4. Every μ-continuous (resp. μp-continuous, μs-continuous) is μsp-continuous but not conversely.

PROOF. The proof follows from Result 1.6 (2).

Example 2.5. Let $X = Y = \{a, b, c\}$, $\tau = \{\phi, \{a\}, \{b\}, \{a, b\}, X\}$, and $\sigma = \{\phi, \{a, c\}, X\}$. Then, $\mu spc(\tau) = \{\phi, \{a\}, \{b\}, \{c\}, \{a, c\}, \{b, c\}, X\}$ and $\mu c(\tau) = \mu pc(\tau) = \{\phi, \{c\}, \{a, c\}, \{b, c\}, X\}$. Define $f : (X, \tau) \to (Y, \sigma)$ be the identity map. Then, f is μsp-continuous but not μ-continuous (resp. μp-continuous), since $f^{-1}(\{b\}) = \{b\}$ is not μ-closed (resp. μp-closed).

Example 2.6. Let $X = Y = \{a, b, c\}$, $\tau = \{\phi, \{a\}, \{b, c\}, X\}$, and $\sigma = \{\phi, \{b\}, X\}$. Then, $\mu spc(\tau) = P(X)$ and $\mu sc(\tau) = \{\phi, \{a\}, \{b, c\}, X\}$. Define $f : (X, \tau) \to (Y, \sigma)$ be the identity map. Then, f is μsp-continuous but not μs-continuous, since $f^{-1}(\{a, c\}) = \{a, c\}$ is not μs-closed.

Proposition 2.7. Every $g\alpha$-continuous (resp. $g^{\#}$-continuous, $\hat{g}$-continuous) is μsp-continuous but not conversely.

PROOF. The proof follows from Result 1.6 (3).

Example 2.8. Let X, Y, τ, σ, and f be as in the Example 2.5. Then, $g\alpha c(\tau) = g^{\#}c(\tau) = \hat{g}c(\tau) = \{\phi, \{c\}, \{a, c\}, \{b, c\}, X\}$. Then, f is μsp-continuous but not $g\alpha$-continuous (resp. $g^{\#}$-continuous, $\hat{g}$-continuous), since $f^{-1}(\{b\}) = \{b\}$ is not $g\alpha$-closed (resp. $g^{\#}$-closed, $\hat{g}$-closed).

Theorem 2.9. μsp-continuity is independent of g-continuity, αg-continuity, gs-continuity, gsp-continuity, $^*g$-continuity, and $\#gs$-continuity.

PROOF. It follows from the following Example.

Example 2.10.

1. Let $X = Y = \{a, b, c\}$, $\tau = \{\phi, \{a\}, \{a, b\}, X\}$, and $\sigma = \{\phi, \{c\}, Y\}$. Then, $\mu spc(\tau) = \{\phi, \{b\}, \{c\}, \{a, b\}, \{b, c\}, X\}$, $gc(\tau) = {}^*gc(\tau) = \{\phi, \{c\}, \{a, c\}, \{b, c\}, X\}$, and $\alpha gc(\tau) = gsc(\tau) = gspc(\tau) = \#gsc(\tau) = \{\phi, \{b\}, \{c\}, \{a, c\}, \{b, c\}, X\}$. Define $f : (X, \tau) \to (Y, \sigma)$ be the identity map. Then, f is μsp-continuous but not g-continuous (resp. αg-continuous, gs-continuous, gsp-continuous, $^*g$-continuous, and $\#gs$-continuous), since $f^{-1}(\{a, b\}) = \{a, b\}$ is not g-closed (resp. αg-closed, gs-closed, gsp-closed, $^*g$-closed, and $\#gs$-closed).

2. Let X and τ be defined as an Example 2.10 (1). Let $Y = \{a, b, c\}$ and $\sigma = \{\phi, \{b\}, \{a, b\}, Y\}$. Define $f : (X, \tau) \to (Y, \sigma)$ be the identity map. Then, f is g-continuous (resp. αg-continuous, gs-continuous, gsp-continuous, $^*g$-continuous and $\#gs$-continuous) but not μsp-continuous, since $f^{-1}(\{a, c\}) = \{a, c\}$ is not μsp-closed.

Remark 2.11. The composition of two μsp-continuous maps need not be μsp-continuous and this is shown from the following example.

Example 2.12. Let X and τ be as in Example 2.3. Let $Y = Z = \{a, b, c\}$, $\sigma = \{\phi, \{a\}, Y\}$, and $\eta = \{\phi, \{a, b\}, Z\}$. Define $f : (X, \tau) \to (Y, \sigma)$ by $f(a) = b$, $f(b) = a$, and $f(c) = c$. Define $g : (Y, \sigma) \to (Z, \eta)$ by $g(a) = b$, $g(b) = c$, and $g(c) = a$. Clearly, f and g are μsp-continuous but their $g \circ f : (X, \tau) \to (Z, \eta)$ is not μsp-continuous, because $V = \{c\}$ is closed in (Z, η) but $(g \circ f^{-1}(\{c\}) = f^{-1}(g^{-1}(\{c\})) = f^{-1}(\{b\}) = \{a\}$, which is not μsp-closed in (X, τ).

Theorem 2.13. If $f : (X, \tau) \to (Y, \sigma)$ is μsp-continuous and $g : (Y, \sigma) \to (Z, \eta)$ is continuous, then $g \circ f : (X, \tau) \to (Z, \eta)$ is μsp-continuous.

PROOF. Clearly follows from definitions.

Proposition 2.14. A map $f : (X, \tau) \to (Y, \sigma)$ is μsp-continuous if and only if $f^{-1}(U)$ is μsp-open in (X, τ) for every open set U in (Y, σ).

Proof. Let $f : (X, \tau) \to (Y, \sigma)$ be μsp-continuous and U be an open set in (Y, σ). Then, U^c is closed in (Y, σ) and since f is μsp-continuous, $f^{-1}(U^c)$ is μsp-closed in (X, τ). But $f^{-1}(U^c) = f^{-1}((U))^c$ and so $f^{-1}(U)$ is μsp-open in (X, τ).
Conversely, assume that $f^{-1}(U)$ is μsp-open in (X, τ) for each open set U in (Y, σ). Let F be a closed set in (Y, σ). Then, F^c is open in (Y, σ) and by assumption, $f^{-1}(F^c)$ is μsp-open in (X, τ). Since $f^{-1}(F^c) = f^{-1}((F))^c$, we have $f^{-1}(F)$ is closed in (X, τ) and so f is μsp-continuous.
We introduce the following definition

Definition 2.15. A map $f : (X, \tau) \to (Y, \sigma)$ is called μsp-irresolute if $f^{-1}(V)$ is a μsp-closed set of (X, τ) for every μsp-closed set V of (Y, σ).

Theorem 2.16. Every μsp-irresolute map is μsp-continuous but not conversely.

Proof. Let $f : (X, \tau) \to (Y, \sigma)$ be a μsp-irresolute map. Let V be a closed set of (Y, σ). Then, by the Result 1.6 (1), V is μsp-closed. Since f is μsp-irresolute, then $f^{-1}(V)$ is a μsp-closed set of (X, τ). Therefore, f is μsp-continuous.

Example 2.17. Let X, Y, τ, σ, and f be as in the Example 2.12. $\{b\}$ is μsp-closed set of (Y, σ) but $f^{-1}(\{b\}) = \{a\}$ is not a μsp-closed set of (X, τ). Thus, f is not μsp-irresolute map. However, f is μsp-continuous map.

Theorem 2.18. Let $f : (X, \tau) \to (Y, \sigma)$ and $g : (Y, \sigma) \to (Z, \eta)$ be any two maps. Then,

1. $g \circ f$ is μsp-continuous if g is continuous and f is μsp-continuous.
2. $g \circ f$ is μsp-irresolute if both f and g are μsp-irresolute.
3. $g \circ f$ is μsp-continuous if g is μsp-continuous and f is μsp-irresolute.

Proof. Omitted.

Theorem 2.19. Let $f : (X, \tau) \to (Y, \sigma)$ be an μsp-continuous map. If (X, τ), the domain of f is an $T_{\mu sp}$-space, then f is continuous.

Proof. Let V be a closed set of (Y, σ). Then, $f^{-1}(V)$ is a μsp-closed set of (X, τ), since f is μsp-continuous. Since (X, τ) is an $T_{\mu sp}$-space, then $f^{-1}(V)$ is a closed set of (X, τ). Therefore, f is continuous.

Theorem 2.20. Let $f : (X, \tau) \to (Y, \sigma)$ be a μsp-continuous map. If (X, τ), the domain of f is an $\alpha T_{\mu sp}$-space, then f is α-continuous.

Proof. Let V be a closed set of (Y, σ). Then, $f^{-1}(V)$ is a μsp-closed set of (X, τ), since f is μsp-continuous. Since (X, τ) is an $\alpha T_{\mu sp}$-space, then $f^{-1}(V)$ is a α-closed set of (X, τ). Therefore, f is α-continuous.

Theorem 2.21. Let $f : (X, \tau) \to (Y, \sigma)$ be a μsp-continuous map. If (X, τ), the domain of f is an $pT_{\mu sp}$-space, then f is pre-continuous.

Proof. Let V be a closed set of (Y, σ). Then, $f^{-1}(V)$ is a μsp-closed set of (X, τ), since f is μsp-continuous. Since (X, τ) is an $pT_{\mu sp}$-space, then $f^{-1}(V)$ is a pre-closed set of (X, τ). Therefore, f is pre-continuous.

Theorem 2.22. Let $f : (X, \tau) \to (Y, \sigma)$ be a μsp-continuous map. If (X, τ), the domain of f is an $\mu T_{\mu sp}$-space, then f is μ-continuous.

Proof. Let V be a closed set of (Y, σ). Then, $f^{-1}(V)$ is a μsp-closed set of (X, τ), since f is μsp-continuous. Since (X, τ) is an $\mu T_{\mu sp}$-space, then $f^{-1}(V)$ is a μ-closed set of (X, τ). Therefore, f is μ-continuous.

Theorem 2.23. Let $f : (X, \tau) \to (Y, \sigma)$ be a μsp-continuous map. If (X, τ), the domain of f is an $\mu p T_{\mu sp}$-space, then f is μp-continuous.

Proof. Let V be a closed set of (Y, σ). Then, $f^{-1}(V)$ is a μsp-closed set of (X, τ), since f is μsp-continuous. Since (X, τ) is an $\mu pT_{\mu sp}$-space, then $f^{-1}(V)$ is a μp-closed set of (X, τ). Therefore, f is μp-continuous.

Theorem 2.24. Let $f : (X, \tau) \to (Y, \sigma)$ be a μsp-continuous map. If (X, τ), the domain of f is an $spT_{\mu sp}$-space, then f is β-continuous.

Proof. Let V be a closed set of (Y, σ). Then $f^{-1}(V)$ is a μsp-closed set of (X, τ), since f is μsp-continuous. Since (X, τ) is an $spT_{\mu sp}$-space, then $f^{-1}(V)$ is a β-closed set of (X, τ). Therefore, f is β-continuous.

Theorem 2.25. Let $f : (X, \tau) \to (Y, \sigma)$ be a μsp-continuous map. If (X, τ), the domain of f is an $g\alpha T_{\mu sp}$-space, then f is $g\alpha$-continuous.

Proof. Let V be a closed set of (Y, σ). Then, $f^{-1}(V)$ is a μsp-closed set of (X, τ), since f is μsp-continuous. Since (X, τ) is an $g\alpha T_{\mu sp}$-space, then $f^{-1}(V)$ is a $g\alpha$-closed set of (X, τ). Therefore, f is $g\alpha$-continuous.

3. Characterization of μsp-Continuous Maps

In this section we introduce μsp-interior and μsp-closure of a set and obtain the characterization theorem for μsp-continuous maps under certain conditions.

Definition 3.1. For any $A \subseteq X$, $\mu sp\text{-}int(A)$ is defined as the union of all μsp-open sets contained in A, i.e., $\mu sp\text{-}int(A) = \cup\{G : G \subseteq A \text{ and } G \text{ is } \mu sp\text{-open}\}$.

Lemma 3.2. For any $A \subseteq X$, $int(A) \subseteq \mu sp\text{-}int(A) \subseteq A$.

Proof. The proof follows from Result 1.6 (4).

The following two Propositions are easy consequences from definitions.

Proposition 3.3. For any $A \subseteq X$, the following holds.

1. $\mu sp\text{-}int(A)$ is the largest μsp-open set contained in A.
2. A is μsp-open if and only if $\mu sp\text{-}int(A) = A$.

Proposition 3.4. For any subsets A and B of (X, τ), the following holds.

1. $\mu sp\text{-}int(A \cap B) = \mu sp\text{-}int(A) \cap \mu sp\text{-}int(B)$.
2. $\mu sp\text{-}int(A \cup B) \supseteq \mu sp\text{-}int(A) \cup \mu sp\text{-}int(B)$.
3. If $A \subseteq B$, then $\mu sp\text{-}int(A) \subseteq \mu sp\text{-}int(B)$.
4. $\mu sp\text{-}int(X) = X$ and $\mu sp\text{-}int(\phi) = \phi$.

Definition 3.5. For every set $A \subseteq X$, we define the μsp-closure of A to be the intersection of all μsp-closed sets containing A, i.e., $\mu sp\text{-}cl(A) = \cap\{F : A \subseteq F \in \mu spc(\tau)\}$.

Lemma 3.6. For any $A \subseteq X$, $A \subseteq \mu sp\text{-}cl(A) \subseteq cl(A)$.

Proof. The proof follows from Result 1.6 (1).

Remark 3.7. Both containment relations in Lemma 3.6 may be proper as seen from the following example.

Example 3.8. Let $X = \{a, b, c\}$, $\tau = \{\phi, \{a\}, \{a, b\}, X\}$. Here $\mu spc(\tau) = \{\phi, \{b\}, \{c\}, \{a, b\}, \{b, c\}, X\}$. Let $A = \{a\}$. Then, $\mu sp\text{-}cl(\{a\}) = \{a, b\}$ and so $A \subseteq \mu sp\text{-}cl(A) \subseteq cl(A)$.

The following two Propositions are easy consequences from definitions.

Proposition 3.9. For any $A \subseteq X$, the following holds.

1. $\mu sp\text{-}cl(A)$ is the smallest μsp-closed set containing A.

2. A is μsp-closed if and only if $\mu sp\text{-}cl(A) = A$.

Proposition 3.10. For any two subsets A and B of (X, τ), the following holds.

1. If $A \subseteq B$, then $\mu sp\text{-}cl(A) \subseteq \mu sp\text{-}cl(B)$.
2. $\mu sp\text{-}cl(A \cap B) \subseteq \mu sp\text{-}cl(A) \cap \mu sp\text{-}cl(B)$.

Proposition 3.11. Let A be a subset of a space X, then the following are true.

1. $(\mu sp\text{-}int(A))^c = \mu sp\text{-}cl(A^c)$.
2. $\mu sp\text{-}int(A) = (\mu sp\text{-}cl(A^c))^c$.
3. $\mu sp\text{-}cl(A) = (\mu sp\text{-}int(A^c))^c$.

PROOF.
1. Clearly follows from definitions.
2. Follows by taking complements in (1).
3. Follows by replacing A by A^c in (1).

Definition 3.12. Let (X, τ) be a topological space. Let x be a point of X and G be a subset of X. Then, G is called an μsp-neighbourhood of x (briefly, μsp-nbhd of x) in X if there exists an μsp-open set U of X such that $x \in U \subseteq G$.

Proposition 3.13. Let A be a subset of (X, τ). Then, $x \in \mu sp\text{-}cl(A)$ if and only if for any μsp-nbhd G_x of x in (X, τ), $A \cap G_x \neq \phi$.

PROOF. Necessity. Assume $x \in \mu sp\text{-}cl(A)$. Suppose that there is an μsp-nbhd G of the point x in (X, τ) such that $G \cap A = \phi$. Since G is μsp-nbhd of x in (X, τ), by Definition 3.12, there exists an μsp-open set U_x such that $x \in U_x \subseteq G$. Therefore, we have $U_x \cap A = \phi$ and so $A \subseteq (U_x)^c$. Since $(U_x)^c$ is an μsp-closed set containing A, we have by Definition 3.5, $\mu sp\text{-}cl(A) \subseteq (U_x)^c$ and therefore $x \notin \mu sp\text{-}cl(A)$, which is a contradiction. Sufficiency. Assume for each μsp-nbhd G_x of x in (X, τ), $A \cap G_x \neq \phi$. Suppose that $x \notin \mu sp\text{-}cl(A)$. Then, by Definition 3.5, there exists an μsp-closed set F of (X, τ) such that $A \subseteq F$ and $x \notin F$. Thus, $x \in F^c$ and F^c is μsp-open in (X, τ) and hence F^c is a μsp-nbhd of x in (X, τ). But $A \cap F^c = \phi$, which is a contradiction.

In the next theorem we explore certain characterizations of μsp-continuous functions.

Theorem 3.14. Let $f : (X, \tau) \to (Y, \sigma)$ be a map from a topological space (X, τ) into a topological space (Y, σ). Then the following statements are equivalent.

1. The function f is μsp-continuous.
2. The inverse of each open set is μsp-open.
3. For each point x in (X, τ) and each open set V in (Y, σ) with $f(x) \in V$, there is an μsp-open set U in (X, τ) such that $x \in U$, $f(U) \subseteq V$.
4. The inverse of each closed set is μsp-closed.
5. For each x in (X, τ), the inverse of every neighbourhood of $f(x)$ is an μsp-nbhd of x.
6. For each x in (X, τ) and each neighbourhood N of $f(x)$, there is an μsp-nbhd G of x such that $f(G) \subseteq N$.
7. For each subset A of (X, τ), $f(\mu sp\text{-}cl(A)) \subseteq cl(f(A))$.
8. For each subset B of (Y, σ), $\mu sp\text{-}cl(f^{-1}(B)) \subseteq f^{-1}(cl(B))$.

PROOF. (1) ⇔ (2). This follows from Proposition 2.14.

(1) ⇔ (3). Suppose that (3) holds and let V be an open set in (Y, σ) and let $x \in f^{-1}(V)$. Then, $f(x) \in V$ and thus there exists an μsp-open set U_x such that $x \in U_x$ and $f(U_x) \subseteq V$. Now, $x \in U_x \subseteq f^{-1}(V)$ and $f^{-1}(V) = \cup_{x \in f^{-1}(V)} U_x$. By assumption, $f^{-1}(V)$ is μsp-open in (X, τ) and therefore f is μsp-continuous.

Conversely, Suppose that (1) holds and let $f(x) \in V$. Then, $x \in f^{-1}(V) \in \mu sp(\tau)$, since f is μsp-continuous. Let $U = f^{-1}(V)$. Then, $x \in U$ and $f(U) \subseteq V$.

(2) ⇔ (4). This result follows from the fact if A is a subset of (Y, σ), then $f^{-1}(A^c) = (f^{-1}(A))^c$.

(2) ⇔ (5). For x in (X, τ), let N be a neighbourhood of $f(x)$. Then, there exists an open set U in (Y, σ) such that $f(x) \in U \subseteq N$. Consequently, $f^{-1}(U)$ is an μsp-open set in (X, τ) and $x \in f^{-1}(U) \subseteq f^{-1}(N)$. Thus, $f^{-1}(N)$ is an μsp-nbhd of x.

(5) ⇔ (6). Let $x \in X$ and let N be a neighbourhood of $f(x)$. Then, by assumption, $G = f^{-1}(N)$ is an μsp-nbhd of x and $f(G) = f(f^{-1}(N)) \subseteq N$.

(6) ⇔ (3). For x in (X, τ), let V be an open set containing $f(x)$. Then, V is a neighborhood of $f(x)$. So by assumption, there exists an μsp-nbhd G of x such that $f(G) \subseteq V$. Hence, there exists an μsp-open set U in (X, τ) such that $x \in U \subseteq G$ and so $f(U) \subseteq f(G) \subseteq V$.

(7)⇔ (4). Suppose that (4) holds and let A be a subset of (X, τ). Since $A \subseteq f^{-1}(A)$, we have $A \subseteq f^{-1}(cl(f(A)))$. Since $cl(f(A))$ is a closed set in (Y, σ), by assumption $f^{-1}(cl(f(A)))$ is an μsp-closed set containing A. Consequently, $\mu sp\text{-}cl(A) \subseteq f^{-1}(cl(f(A)))$. Thus, $f(\mu sp\text{-}cl(A)) \subseteq f(f^{-1}(cl(f(A)))) \subseteq cl(f(A))$.

Conversely, suppose that (7) holds for any subset A of (X, τ). Let F be a closed subset of (Y, σ). Then, by assumption, $f(\mu sp\text{-}cl(f^{-1}(F))) \subseteq cl(f(f^{-1}(F))) \subseteq cl(F) = F$, i.e., $\mu sp\text{-}cl(f^{-1}(F)) \subseteq f^{-1}(F)$ and so $f^{-1}(F)$ is μsp-closed.

(7) ⇔ (8). Suppose that (7) holds and B be any subset of (Y, σ). Then, replacing A by $f^{-1}(B)$ in (7), we obtain $f(\mu sp\text{-}cl(f^{-1}(B))) \subseteq cl(f(f^{-1}(B))) \subseteq cl(B)$, i.e., $\mu sp\text{-}cl(f^{-1}(B)) \subseteq f^{-1}cl(B)$.

Conversely, suppose that (8) holds. Let $B = f(A)$ where A is a subset of (X, τ). Then, we have, $\mu sp\text{-}cl(A) \subseteq \mu sp\text{-}cl(f^{-1}(B)) \subseteq f^{-1}(cl(f(A))$ and so $f(\mu sp\text{-}cl(A)) \subseteq cl(f(A))$.

This completes the proof of the theorem.

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