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Better Approximation of Functions by Genuine Baskakov Durrmeyer Operators

Gülsüm Ulusoy Ada[1]

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Abstract − In this paper, we define a new genuine Baskakov-Durrmeyer operators. We give uniform convergence using the weighted modulus of continuity. Then we study direct approximation of the operators in terms of the moduli of smoothness. After that a Voronovskaya type result is studied.

Keywords − *Genuine Baskakov Durrmeyer operators, weighted modulus of continuity, Voronovskaya theorem*

1. Introduction

In the paper [1], the authors studied the sequences of linear Bernstein type operators defined for $f \in C[0,1]$ by $B_n(f \circ \sigma^{-1}) \circ \sigma$, B_n being the classical Bernstein operators and σ being any function satisfying some certain conditions. By this way, the Korovkin set is $\{1, \sigma, \sigma^2\}$ instead of $\{1, e_1, e_2\}$. It was shown that the B_n^{σ} actual a better degree of approximation. For this aim, have studied by a number of authors. For more details in this direction we can refer the readers to [2–9].

In [10], the authors introduced a general sequences of linear Baskakov Durrmeyer type operators by

$$G_n^{\sigma}(g;x) = (n-1)\sum_{l=0}^{\infty} P_{n,k}^{\sigma}(x)\int_0^{\infty}(g\circ\sigma^{-1})(u)\binom{n+k-1}{k}\frac{u^k}{(1+u)^{n+k}}du, \tag{1}$$

where $P_{n,k}^{\sigma}(x) = \binom{n+k-1}{k}\frac{(\sigma(x))^k}{(1+\sigma(x))^{n+k}}$, σ is a continuous infinite times differentiable function satisfying the condition $\sigma(1) = 0, \sigma(0) = 0$ and $\sigma'(x) > 0$ for $x \in [0,\infty)$.

In the present paper, we construct a genuine type modification of the operators in (1) which preserve the function σ, defined as

$$\begin{aligned} K_n^{\sigma}(g;x) = {} & \sum_{k=1}^{\infty} P_{n,k}^{\sigma}(x)\frac{1}{\beta(k,n+1)}\int_0^{\infty}\left(g\circ\sigma^{-1}\right)(t)\,\frac{t^{k-1}}{(1+t)^{n+k+1}}dt \\ & + P_{n,0}^{\sigma}(x)\left(g\circ\sigma^{-1}\right)(0) \end{aligned} \tag{2}$$

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The operators defined in (2) are linear and positive. In case of $\sigma(x) = x$, the operators in (2) reduce to the following operators introduced in [11]:

$$T_n(g;x) = \sum_{k=1}^{\infty} P_{n,k}(x) \frac{1}{\beta(k,n+1)} \int_0^{\infty} g(t) \frac{t^{k-1}}{(1+t)^{n+k+1}} dt + P_{n,0}(x) g(0)$$

2. Auxiliary lemmas

Lemma 2.1. We have

$$K_n^{\sigma}(1;x) = 1, \; K_n^{\sigma}(\sigma;x) = \sigma(x), \tag{3}$$

$$K_n^{\sigma}(\sigma^2;x) = \frac{\sigma^2(x)(n+1) + 2\sigma(x)}{n-1}, \tag{4}$$

$$K_n^{\sigma}(\sigma^3;x) = \frac{\sigma^3(x)(n+1)(n+2) + 6\sigma^2(x)(n+1) + 6\sigma(x)}{(n-1)(n-2)} \tag{5}$$

Lemma 2.2. If we describe the central moment operator by

$$M_{n,m}^{\sigma}(x) = K_n^{\sigma}\left((\sigma(t) - \sigma(x))^m; x\right)$$

then we get

$$M_{n,0}^{\sigma}(x) = 1, \quad M_{n,1}^{\sigma}(x) = 0 \tag{6}$$

$$M_{n,2}^{\sigma}(x) = \frac{2\sigma(x)(\sigma(x)+1)}{n-1} \tag{7}$$

for all $n, m \in \mathbb{N}$.

3. Weighted Convergence of $K_n^{\sigma}(f)$

We suppose that:

(p_1) σ is a continuously differentiable function on $[0,\infty)$

(p_2) $\sigma(0) = 0$, $\inf_{x\in[0,\infty)} \sigma'(x) \geq 1$.

Let $\psi(x) = 1 + \sigma^2(x)$ and $B_{\psi}(\mathbb{R}^+) = \{f : \; |f(x)| \leq n_f \psi(x)\}$, where n_f is constant which may depend only on f. $C_{\psi}(\mathbb{R}^+)$ denote the subspace of all continuous functions in $B_{\psi}(\mathbb{R}^+)$. By $C_{\psi}^*(\mathbb{R}^+)$, we denote the subspace off all functions $f \in C_{\psi}(\mathbb{R}^+)$ for which $\lim_{x\to\infty} f(x)/\psi(x)$ is finite. Also let $U_{\psi}(\mathbb{R}^+)$ be the space of functions $f \in C_{\psi}(\mathbb{R}^+)$ such that f/ψ is uniformly continuous. $B_{\psi}(\mathbb{R}^+)$ is the linear normed space with the norm $\|f\|_{\psi} = \sup_{x\in\mathbb{R}^+} |f(x)|/\psi(x)$.

The weighted modulus of continuity defined in [12] is as follows

$$\omega_{\sigma}(f;\delta) = \sup_{\substack{x,t\in\mathbb{R}^+ \\ |\sigma(t)-\sigma(x)|\leq\delta}} \frac{|f(t) - f(x)|}{\psi(t) + \psi(x)}$$

for each $f \in C_{\psi}(\mathbb{R}^+)$ and for every $\delta > 0$. We observe that $\omega_{\sigma}(f;0) = 0$ for every $f \in C_{\psi}(\mathbb{R}^+)$ and the function $\omega_{\sigma}(f;\delta)$ is nonnegative and nondecreasing with respect to δ for $f \in C_{\psi}(\mathbb{R}^+)$ and also $\lim_{\delta\to 0} \omega_{\sigma}(f;\delta) = 0$ for every $f \in U_{\psi}(\mathbb{R}^+)$.

Let $\delta > 0$ and $W_{\infty}^2 = \{g \in C_B[0,\infty); g', g'' \in C_B[0,\infty)\}$. The Peetre's K functional is defined by

$$K_2(f,\delta) = \inf\left\{\|f-g\| + \delta\|g\|_{W_{\infty}^2}; g \in W_{\infty}^2\right\},$$

where

$$\|f\|_{W_{\infty}^2} := \|f\| + \|f'\| + \|f''\|$$

It was shown in [13], there exists an absolute constant $C > 0$ such that

$$K_2(f,\delta) \le C\left\{w_2\left(f;\sqrt{\delta}\right) + \min(1,\delta)\,\|f\|\right\},$$

where the second order modulus of smoothness is defined by

$$w_2(f,\sqrt{\delta}) = \sup_{0\le h\le\sqrt{\delta}}\ \sup_{x\in[0,\infty)} |f(x+2h)-2f(x+h)+f(x)|$$

The usual modulus of continuity of $f \in C_B[0,\infty)$ is defined by

$$w(f,\delta) = \sup_{0\le h\le\sqrt{\delta}}\ \sup_{x\in[0,\infty)} |f(x+h)-f(x)|$$

Lemma 3.1. [14] The positive linear operators L_n, $n \ge 1$, act from $C_\psi(\mathbb{R}^+)$ to $B_\psi(\mathbb{R}^+)$ if and only if the inequality

$$|L_n(\psi;x)| \le P_n\psi(x),$$

holds, where P_n is a positive constant depending on n.

Theorem 3.2. [14] Let the sequence of linear positive operators (L_n), $n\ge 1$, acting from $C_\psi(\mathbb{R}^+)$ to $B_\psi(\mathbb{R}^+)$ satisfy the three conditions

$$\lim_{n\to\infty}\|L_n\sigma^\nu - \sigma^\nu\|_\psi = 0,\ \ \nu = 0,1,2.$$

Then for any function $g \in C^*_\psi(\mathbb{R}^+)$,

$$\lim_{n\to\infty}\|L_n g - g\|_\psi = 0$$

Theorem 3.3. For each function $g\in C^*_\psi(\mathbb{R}^+)$

$$\lim_{n\to\infty}\|K_n^\sigma g - g\|_\psi = 0$$

Proof. Using Theorem 3.2 we see that it is sufficient to verify the following three conditions

$$\lim_{n\to\infty}\|K_n^\sigma(\sigma^\nu) - \sigma^\nu\|_\psi = 0,\ \ \nu=0,1,2. \tag{8}$$

It is clear that from (3) and (4), $\|K_n^\sigma(1)-1\|_\psi = 0$ and $\|K_n^\sigma(\sigma)-\sigma\|_\psi = 0$. Hence the conditions (8) are fullfilled for $\nu = 0,1$. Also using the property (4) we have

$$\begin{aligned}\left\|K_n^\sigma(\sigma^2)-\sigma^2\right\|_\psi &= \sup_{x\in\mathbb{R}^+}\frac{1}{(1+\sigma^2(x))}\left(\frac{\sigma^2(x)(n+1)+2\sigma(x)}{(n-1)} - \sigma^2(x)\right)\\ &\le \frac{4}{n-1}\end{aligned} \tag{9}$$

This means that the condition (8) holds also for $\nu = 2$ and by Theorem 3.2 the proof is completed. □

Theorem 3.4. [12] Let $L_n : C_\psi(\mathbb{R}^+) \to B_\psi(\mathbb{R}^+)$ be a sequence of positive linear operators with

$$\left\|L_n(\sigma^0)-\sigma^0\right\|_{\psi^0} = a_n, \tag{10}$$

$$\|L_n(\sigma)-\sigma\|_{\psi^{\frac{1}{2}}} = b_n,$$

$$\left\|L_n(\sigma^2)-\sigma^2\right\|_\psi = c_n,$$

$$\left\|L_n(\sigma^3)-\sigma^3\right\|_{\psi^{\frac{3}{2}}} = d_n, \tag{11}$$

where a_n, b_n, c_n and d_n tend to zero as $n\to\infty$. Then

$$\|L_n(g)-g\|_{\psi^{\frac{3}{2}}} \le (7+4a_n+2c_n)\,\omega_\sigma(g;\delta_n) + \|g\|_\psi\, a_n \tag{12}$$

for all $g\in C_\psi(\mathbb{R}^+)$, where

$$\delta_n = 2\sqrt{(a_n+2b_n+c_n)(1+a_n)} + a_n + 3b_n + 3c_n + d_n$$

Theorem 3.5. For all $g \in C_{\psi}(\mathbb{R}^+)$ we get

$$\|K_n^{\sigma}(g) - g\|_{\psi^{\frac{3}{2}}} \leq \left(7 + \frac{2}{(n-1)}\right)\omega_{\sigma}\left(g; \frac{4}{\sqrt{(n-1)}} + \frac{24n^2 + 4n - 8}{(n-1)(n-2)}\right)$$

Proof. On account of apply Theorem 3.4, we must calculate the sequences a_n, b_n, c_n and d_n. Using (3) and (4) we find

$$\left\|K_n^{\sigma}(\sigma^0) - \sigma^0\right\|_{\psi^0} = a_n = 0$$

and

$$\|K_n^{\sigma}(\sigma) - \sigma\|_{\psi^{\frac{1}{2}}} = b_n = 0$$

Also from (9)

$$c_n = \left\|\widetilde{C}_n^{\sigma}(\sigma^2) - \sigma^2\right\|_{\psi} \leq \frac{4}{(n-1)}$$

Since

$$K_n^{\sigma}(\sigma^3; x) = \frac{\sigma^3(x)(n+1)(n+2) + 6\sigma^2(x)(n+1) + 6\sigma(x)}{(n-1)(n-2)} \tag{13}$$

we can write

$$\begin{aligned} d_n &= \left\|K_n^{\sigma}(\sigma^3) - \sigma^3\right\|_{\psi^{\frac{3}{2}}} \\ &= \sup_{x\in\mathbb{R}^+} \frac{1}{(1+\sigma^2(x))^{\frac{3}{2}}} \\ &\quad \times \frac{\sigma^3(x)(n+1)(n+2) + 6\sigma^2(x)(n+1) + 6\sigma(x) - \sigma^3(x)(n-1)(n-2)}{(n-1)(n-2)} \\ &\leq \frac{24n^2}{(n-1)(n-2)} \end{aligned}$$

Thus the conditions (10-11) are satisfied. From Theorem 3.4 we have

$$\|K_n^{\sigma}(g) - g\|_{\psi^{\frac{3}{2}}} \leq \left(7 + \frac{2}{(n-1)}\right)\omega_{\sigma}\left(g; \frac{4}{\sqrt{(n-1)}} + \frac{24n^2 + 4n - 8}{(n-1)(n-2)}\right)$$

□

Remark 3.6. Using $\lim_{\delta\to 0}\omega_{\sigma}(f;\delta) = 0$ and Theorem 3.5, we have

$$\lim_{n\to\infty}\|K_n^{\sigma}(g) - g\|_{\psi^{\frac{3}{2}}} = 0$$

for $f \in U_{\psi}(\mathbb{R}^+)$.

Theorem 3.7. Let σ be a function satisfying the conditions p_1 and p_2 and $\|\sigma''\|$ is finite. If $f \in C_B[0,\infty)$, then we have

$$|K_n^{\sigma}(g;x) - g(x)| \leq C\left\{w_2\left(f; \sqrt{\frac{2\sigma(x)(\sigma(x)+1)}{n-1}}\right) + \min\left(1, \frac{2\sigma(x)(\sigma(x)+1)}{n-1}\right)\|g\|\right\}$$

Proof. The classic Taylor's expansion of $g \in W_{\infty}^2$ yields for $t \in [0,\infty)$ that

$$\begin{aligned} g(t) &= (g\circ\sigma^{-1})(\sigma(t)) = (g\circ\sigma^{-1})(\sigma(x)) + D(g\circ\sigma^{-1})(\sigma(x))(\sigma(t)-\sigma(x)) \\ &\quad + \int_{\sigma(x)}^{\sigma(x)}(\sigma(t)-u)D^2(g\circ\sigma^{-1})(u)du \end{aligned}$$

Applying the operators K_n^{σ} to both sides of above equality and considering the fact (6) we obtain

$$K_n^{\sigma}(g;x) - g(x) = K_n^{\sigma}\left(\int_{\sigma(x)}^{\sigma(x)}(\sigma(t)-u)D^2(g\circ\sigma^{-1})(u)du; x\right)$$

On the other hand, with the change of variable $u = \sigma(y)$ we get

$$\int_{\sigma(x)}^{\sigma(x)} (\sigma(t) - u) D^2(g \circ \sigma^{-1})(u) du = \int_x^t (\sigma(t) - \sigma(y)) D^2(g \circ \sigma^{-1}) \sigma(y) \sigma'(y) dy$$

Using the equality

$$D^2(g \circ \sigma^{-1})(\sigma(y)) = \frac{1}{\sigma'(y)} \frac{g''(y)\sigma'(y) - g'(y)\sigma''(y)}{(\sigma'(y))^2},$$

we can write

$$\begin{aligned}\int_{\sigma(x)}^{\sigma(x)} (\sigma(t) - u) D^2(g \circ \sigma^{-1})(u) du &= \int_x^t (\sigma(t) - \sigma(y)) \left(\frac{1}{\sigma'(y)} \frac{g''(y)\sigma'(y) - g'(y)\sigma''(y)}{(\sigma'(y))^2} \right) dy \\ &= \int_{\sigma(x)}^{\sigma(x)} (\sigma(t) - u) \frac{g''(\sigma^{-1}(u))}{(\sigma^{-1}(\sigma^{-1}(u)))^3} du \\ &\quad - \int_{\sigma(x)}^{\sigma(x)} (\sigma(t) - u) \frac{g'(\sigma^{-1}(u))\sigma''(\sigma^{-1}(u))}{(\sigma'(\sigma^{-1}(u)))^3} du\end{aligned}$$

So we can write

$$\begin{aligned}K_n^\sigma(g; x) - g(x) &= K_n^\sigma \left(\int_{\sigma(x)}^{\sigma(x)} (\sigma(t) - u) \frac{g''(\sigma^{-1}(u))}{(\sigma^{-1}(\sigma^{-1}(u)))^3} du; x \right) \\ &\quad - K_n^\sigma \left(\int_{\sigma(x)}^{\sigma(x)} (\sigma(t) - u) \frac{g'(\sigma^{-1}(u))\sigma''(\sigma^{-1}(u))}{(\sigma'(\sigma^{-1}(u)))^3} du; x \right)\end{aligned}$$

Since σ is strictly increasing on $[0, \infty)$ and with the condition p_2, we get

$$\begin{aligned}|K_n^\sigma(g; x) - g(x)| &\leq M_{n,2}^\sigma(x) \left(\|g''\| + \|g'\| \|\sigma''\| \right) \\ &\leq \frac{2\sigma(x)(\sigma(x) + 1)}{n - 1} \left(\|g''\| + \|g'\| \|\sigma''\| \right)\end{aligned}$$

Also, it is clear that

$$\|K_n^\sigma\| \leq \|f\|$$

Hence we have

$$\begin{aligned}|K_n^\sigma(g; x) - g(x)| &\leq |K_n^\sigma(g - f; x)| + |K_n^\sigma(f; x) - f(x)| + |-(g - f)(x)| \\ &\leq 2\|f - g\| + \frac{2\sigma(x)(\sigma(x) + 1)}{n - 1} \left(\|g''\| + \|g'\| \|\sigma''\| \right)\end{aligned}$$

and choosing $C := \max\{1, \|\sigma''\|\}$ we have

$$\begin{aligned}|K_n^\sigma(g; x) - g(x)| &\leq C \left\{ \|f - g\| + \frac{2\sigma(x)(\sigma(x) + 1)}{n - 1} \left(\|g''\| + \|g'\| + \|g\| \right) \right\} \\ &= C \left\{ \|f - g\| + \frac{2\sigma(x)(\sigma(x) + 1)}{n - 1} \|g\|_{W_\infty^2} \right\}\end{aligned}$$

Taking the infimum on the right hand side over all $g \in W_\infty^2$ we obtain

$$\begin{aligned}|K_n^\sigma(g; x) - g(x)| &\leq C K_2 \left(f; \frac{2\sigma(x)(\sigma(x) + 1)}{n - 1} \right) \\ &\leq C \left\{ w_2 \left(f; \sqrt{\frac{2\sigma(x)(\sigma(x) + 1)}{n - 1}} \right) + \min \left(1, \frac{2\sigma(x)(\sigma(x) + 1)}{n - 1} \right) \|g\| \right\}\end{aligned}$$

□

Lemma 3.8. [12] For every $g \in C_\psi(\mathbb{R}^+)$, for $\delta > 0$ and for all $u, x \geq 0$,

$$|g(u) - g(x)| \leq (\psi(u) + \psi(x)) \left(2 + \frac{|\sigma(u) - \sigma(x)|}{\delta} \right) \omega_\sigma(g, \delta) \tag{14}$$

holds.

Theorem 3.9. Let $g \in C_{\psi}(\mathbb{R}^{+})$, $x \in I$ and suppose that the first and second derivatives of $g \circ \sigma^{-1}$ exist at $\sigma(x)$. If the second derivative of $g \circ \sigma^{-1}$ is bounded on $\mathbb{R}^{+}$, then we have

$$\lim_{n\to\infty} n\left[K_n^{\sigma}(g;x) - g(x)\right] = \sigma(x)(\sigma(x)+1)\left(g\circ\sigma^{-1}\right)''(\sigma(x))$$

Proof. By the Taylor expansion of $g \circ \sigma^{-1}$ at the point $\sigma(x) \in \mathbb{R}^{+}$, there exists ξ lying between x and t such that

$$\begin{aligned} g(t) &= \left(g\circ\sigma^{-1}\right)(\sigma(t)) = \left(g\circ\sigma^{-1}\right)(\sigma(x)) \\ &+ \left(g\circ\sigma^{-1}\right)'(\sigma(x))(\sigma(t)-\sigma(x)) \\ &+ \frac{\left(g\circ\sigma^{-1}\right)''(\sigma(x))(\sigma(t)-\sigma(x))^2}{2} + \gamma_x(t)(\sigma(t)-\sigma(x))^2, \end{aligned}$$

where

$$\gamma_x(t) := \frac{\left(g\circ\sigma^{-1}\right)''(\sigma(\xi)) - \left(g\circ\sigma^{-1}\right)''(\sigma(x))}{2} \tag{15}$$

We get

$$\begin{aligned} K_n^{\sigma}(g;x) - g(x) &= \left(g\circ\sigma^{-1}\right)'(\sigma(x))\, K_n^{\sigma}(\sigma(t)-\sigma(x);x) \\ &+ \frac{\left(g\circ\sigma^{-1}\right)''(\sigma(x))\, K_n^{\sigma}((\sigma(t)-\sigma(x))^2;x)}{2} + K_n^{\sigma}\left(\gamma_x(t)(\sigma(t)-\sigma(x))^2;x\right) \end{aligned}$$

Using (6) and (7), we have

$$\lim_{n\to\infty} nK_n^{\sigma}(\sigma(t)-\sigma(x);x) = 0$$

$$\lim_{n\to\infty} nK_n^{\sigma}(((\sigma(t)-\sigma(x))^2;x) = 2\sigma(x)(\sigma(x)+1)$$

and thus

$$\begin{aligned} \lim_{n\to\infty} n\left[K_n^{\sigma}(g;x) - g(x)\right] &= \sigma(x)(\sigma(x)+1)\left(g\circ\sigma^{-1}\right)''(\sigma(x)) \\ &+ \lim_{n\to\infty} nK_n^{\sigma}\left(\gamma_x(t)(\sigma(t)-\sigma(x))^2;x\right) \end{aligned}$$

Let calculate the last term $\left|nK_n^{\sigma}\left(|\gamma_x(t)|(\sigma(t)-\sigma(x))^2;x\right)\right|$. Since $\lim_{t\to x}\gamma_x(t) = 0$ for every $\varepsilon > 0$, let $\delta > 0$ such that $|\gamma_x(t)| < \varepsilon$ for every $t \geq 0$. Cauchy-Schwarz inequality applied we have

$$\begin{aligned} \lim_{n\to\infty} nK_n^{\sigma}\left(|\gamma_x(t)|(\sigma(t)-\sigma(x))^2;x\right) &\leq \varepsilon \lim_{n\to\infty} nK_n^{\sigma}\left((\sigma(t)-\sigma(x))^2;x\right) \\ &+ \frac{C}{\delta^2}\lim_{n\to\infty} nK_n^{\sigma}\left((\sigma(t)-\sigma(x))^4;x\right) \end{aligned}$$

Since

$$\lim_{n\to\infty} nK_n^{\sigma}\left((\sigma(t)-\sigma(x))^4;x\right) = 0,$$

we get

$$\lim_{n\to\infty} nK_n^{\sigma}\left(|\gamma_x(t)|(\sigma(t)-\sigma(x))^2;x\right) = 0$$

□

Corollary 3.10. We have following particular case:

1. If we choose $\sigma(x) = x$, the operators (2) reduce to T_n operators defined in [11]. As a consequence of Theorem 3.9, we refined the following result.

$$\lim_{n\to\infty} n\left[T_n(g;x) - g(x)\right] = x(x+1)g''(x)$$

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Bipolar Pythagorean Fuzzy Subring of a Ring

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Abstract—In this paper, we study some of the properties of bipolar Pythagorean fuzzy subring of a ring and prove some results on these. We derive some important theorems and intersection and product are applied into the bipolar Pythagorean fuzzy subring of a ring.

Keywords— *Bipolar Pythagorean fuzzy set, bipolar Pythagorean fuzzy subring, bipolar Pythagorean fuzzy product*

1. Introduction

Fuzzy sets were introduced by Zadeh [1], and he discussed only membership function. After the extensions of fuzzy set theory Atanassov [2] generalised this concept and introduced a new concept called intuitionistic fuzzy set (IFS). Yager [3] familiarised the model of Pythagorean fuzzy set. IFSs have its greatest use in practical multiple attribute decision making (MADM) problems, and academic research has achieved significant development [3-5]. However, in some practical problems, the sum of membership degree and non-membership degree to which an alternative satisfying attribute provided by the decision maker (DM) may be bigger than 1, but their square sum is less than or equal to 1. Jun and Song [6] introduced the notion of closed fuzzy ideals in BCI-algebras and discussed their properties.

Bosc and Pivert [7] said that "Bipolarity refers to the propensity of the human mind to reason and make decisions based on positive and negative effects. Positive information states what is possible, satisfactory, permitted, desired, or considered as being acceptable. On the other hand, negative statements express what is impossible, rejected, or forbidden. Negative preferences correspond to constraints since they specify which values or objects have to be rejected (i.e., those that do not satisfy the constraints), while positive preferences correspond to wishes, as they specify which objects are more desirable than others (i.e., satisfy user wishes) without rejecting those that do not meet the wishes". Therefore, Lee [8,9] introduced the concept of bipolar fuzzy sets which is a generalisation of the fuzzy sets. Many authors have studied recently bipolar fuzzy models on algebraic structures such as; Chen et al. [10] studied of m-polar fuzzy set. Then, they examined many results which are related to those concepts can be generalised to the case of m-polar fuzzy sets. They also proposed numerical examples

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to show how to apply m-polar fuzzy sets in real-world problems. In 1982, Liu [11] introduced the concept of the fuzzy ring and fuzzy ideal. After the notion of intuitionistic fuzzy subring by Hur et al. [12], many researchers have tried to generalise the notion of the intuitionistic fuzzy subring. Marashdeh and Salleh [13] introduced the notion of intuitionistic fuzzy rings based on the notion of fuzzy space.

The purpose of this paper is to introduce the concept of bipolar Pythagorean fuzzy subring and established some of their results.

2. Preliminaries

Definition 2.1. [1] Let X be a nonempty set. A fuzzy set A drawn from X is defined as $A = \{(x: \mu_A(x)): x \in X\}$, where $\mu_A: X \to [0,1]$ is the membership function of the set A.

Definition 2.2. [8] Let X be the universe. Then, a bipolar fuzzy set, A on X is defined by a positive membership function μ_A^+, that is $\mu_A^+: X \to [0,1]$, and a negative membership function μ_A^-, that is $\mu_A^-: X \to [-1,0]$. For the sake of simplicity, we shall use the symbol $A = \{(x, \mu_A^+(x), \mu_A^-(x)): x \in X\}$.

Definition 2.3. (Pythagorean Fuzzy Set) [3,4] Let X be a non-empty set and I the unit interval [0,1]. A PF set S is an object having the form $P = \{(x, \mu_P(x), \nu_P(x)): x \in X\}$ where the function $\mu_P: X \to [0,1]$ and $\nu_P: X \to [0,1]$ denote respectively the degree of membership and degree of non-membership of each element x in X to the set P, and $0 \leq (\mu_P(x))^2 + (\nu_P(x))^2 \leq 1$ for all x in X.

Definition 2.4. (Bipolar Pythagorean Fuzzy Set) [14] Let X be a non-empty set. A bipolar Pythagorean fuzzy set (BPFS) $A = \{(X, T_A^P, F_A^P, T_A^N, F_A^N): x \in X\}$ where $T_A^P: X \to [0,1]$, $F_A^P: X \to [0,1]$, $T_A^N: X \to [-1,0]$, and $F_A^N: X \to [-1,0]$ are the mappings such that $0 \leq (T_A^P)^2 + (F_A^P)^2 \leq 1$ and $-1 \leq -((T_A^N)^2 + (F_A^N)^2) \leq 0$ and $T_A^P(x)$ denote the positive membership degree, $F_A^P(x)$ denote the positive non-membership degree, $T_A^N(x)$ denote the negative membership degree, $F_A^N(x)$denote the negative non-membership degree.

Definition 2.5. [14] Let $A = \{(X, T_A^P, F_A^P, T_A^N, F_A^N): x \in X\}$ and $B = \{(X, T_B^P, F_B^P, T_B^N, F_B^N): x \in X\}$ be two BPFNs. Then, their operations are defined as follows:

i. $A \cup B = \left\{\left(x, max(T_A^P, T_B^P), min(F_A^P, F_B^P), min(T_A^N, T_B^N), max(F_A^N, F_B^N)\right): x \in X\right\}$

ii. $A \cap B = \left\{\left(x, min(T_A^P, T_B^P), max(F_A^P, F_B^P), max(T_A^N, T_B^N), min(F_A^N, F_B^N)\right): x \in X\right\}$

iii. $A^C = \{(x, F_A^P, T_A^P, F_A^N, T_A^N): x \in X\}$

Definition 2.6 [15] Let R be a ring. An intuitionistic fuzzy subset A= $\{(x, T_A(x), F_A(x))/x \in R\}$ of R is said to be an intuitionistic fuzzy subring of R if the following conditions are satisfied,

i. $T_A(x-y) \geq min\{T_A(x), T_A(y)\}$

ii. $T_A(xy) \geq min\{T_A(x), T_A(y)\}$

iii. $F_A(x-y) \leq max\{F_A(x), F_A(y)\}$

iv. $F_A(xy) \leq max\{F_A(x), F_A(y)\}$

3. Properties

Definition 3.1. Let R be a ring. A bipolar Pythagorean fuzzy subset A of R is said to be a bipolar Pythagorean fuzzy subring of R if the following conditions are satisfied, for all x and y in R,

i. $T_A^P(x-y) \geq \min\{T_A^P(x), T_A^P(y)\}$

ii. $T_A^P(xy) \geq \min\{T_A^P(x), T_A^P(y)\}$

iii. $F_A^P(x-y) \leq \max\{F_A^P(x), F_A^P(y)\}$

iv. $F_A^P(xy) \leq \max\{F_A^P(x), F_A^P(y)\}$

v. $T_A^N(x-y) \leq \max\{T_A^N(x), T_A^N(y)\}$

vi. $T_A^N(xy) \leq \max\{T_A^N(x), T_A^N(y)\}$

vii. $F_A^N(x-y) \geq \min\{F_A^N(x), F_A^N(y)\}$

viii. $F_A^N(xy) \geq \min\{F_A^N(x), F_A^N(y)\}$

Example 3.2. Let $R = \{0,1\}$ be a set of integers of modulo 2 with two binary operations as follows:

+	0	1
0	0	1
1	1	0

and

·	0	1
0	0	0
1	0	1

Define bipolar Pythagorean fuzzy set $A = \{(X, T_A^P, F_A^P, T_A^N, F_A^N): x \in X\}$ is given by

$$T_A^P(0) = 0.2, T_A^N(0) = -0.3, F_A^P(0) = 0.6, F_A^N(0) = -0.5,$$
$$T_A^P(1) = 0.3, T_A^N(1) = -0.6, F_A^P(1) = 0.9, F_A^N(1) = -0.4.$$

Then, $(R, +, \cdot)$ is a ring.

Definition 3.3. Let $A = \{(X, T_A^P, F_A^P, T_A^N, F_A^N): x \in X\}$ and $B = \{(X, T_B^P, F_B^P, T_B^N, F_B^N): x \in X\}$ be any two bipolar Pythagorean fuzzy subsets of sets G and H respectively. The product of A and B, denoted by $A \times B$, is defined as

$$A \times B = \left\{\left((x,y), T_{A\times B}^P(x,y), F_{A\times B}^P(x,y), T_{A\times B}^N(x,y), F_{A\times B}^N(x,y)\right) : \ x \in G, y \in H\right\}$$

where $T_{A\times B}^P(x,y) = \min\{T_A^P(x), T_B^P(y)\}$, $F_{A\times B}^P(x,y) = \max\{F_A^P(x), F_B^P(y)\}$, $T_{A\times B}^N(x,y) = \max\{T_A^N(x), T_B^N(y)\}$ and $F_{A\times B}^N(x,y) = \min\{F_A^N(x), F_B^N(y)\}$, for all x in G and y in H.

Definition 3.4. Let $A = \{(X, T_A^P, F_A^P, T_A^N, F_A^N) : x \in X\}$ be a bipolar Pythagorean fuzzy subset in a set S, the strongest bipolar Pythagorean fuzzy relation on S that is a bipolar Pythagorean fuzzy relation on A is

$$V = \left\{\left((x,y), T_V^P(x,y), F_V^P(x,y), T_V^N(x,y), F_V^N(x,y)\right) : \ x, y \in S\right\}$$

given by $T_V^P(x,y) = \min\{T_A^P(x), T_A^P(y)\}$, $F_V^P(x,y) = \max\{F_A^P(x), F_A^P(y)\}$, $T_V^N(x,y) = \max\{T_A^N(x), T_A^N(y)\}$, and $F_V^N(x,y) = \min\{F_A^N(x), F_A^N(y)\}$, for all x, y in S.

Theorem 3.5. Let $A = (X, T_A^P, F_A^P, T_A^N, F_A^N)$ be a bipolar Pythagorean fuzzy subring of a ring R. Then, for all x in R and the identity element e in R

i. $T_A^P(-x) = T_A^P(x)$

ii. $F_A^P(-x) = F_A^P(x)$

iii. $T_A^N(-x) = T_A^N(x)$

iv. $F_A^N(-x) = F_A^N(x)$

v. $T_A^P(x) \leq T_A^P(e)$

vi. $F_A^P(x) \geq F_A^P(e)$

vii. $T_A^N(x) \geq T_A^N(e)$

viii. $F_A^N(x) \leq F_A^N(e)$

Proof. For all x in R,

i. $T_A^P(x) = T_A^P\big(-(-x)\big) \geq T_A^P(-x) \geq T_A^P(x)$. Therefore, $T_A^P(x) = T_A^P(-x)$.

ii. $F_A^P(x) = F_A^P\big(-(-x)\big) \leq F_A^P(-x) \leq F_A^P(x)$. Therefore, $F_A^P(x) = F_A^P(-x)$.

iii. $T_A^N(x) = T_A^N\big(-(-x)\big) \leq T_A^N(-x) \leq T_A^N(x)$. Therefore, $T_A^N(x) = T_A^N(-x)$.

iv. $F_A^N(x) = F_A^N\big(-(-x)\big) \geq F_A^N(-x) \geq F_A^N(x)$. Therefore, $F_A^N(x) = F_A^N(-x)$.

v. $T_A^P(e) = T_A^P(x-x) \geq \min\{T_A^P(x), T_A^P(x)\} = T_A^P(x)$. Therefore, $T_A^P(e) \geq T_A^P(x)$.

vi. $F_A^P(e) = F_A^P(x-x) \leq \max\{F_A^P(x), F_A^P(x)\} = F_A^P(x)$. Therefore, $F_A^P(e) \leq F_A^P(x)$.

vii. $T_A^N(e) = T_A^N(x-x) \leq \max\{T_A^N(x), T_A^N(x)\} = T_A^N(x)$. Therefore, $T_A^N(e) \leq T_A^N(x)$.

viii. $F_A^N(e) = F_A^N(x-x) \geq \min\{F_A^N(x), F_A^N(x)\} = F_A^N(x)$. Therefore, $F_A^N(e) \geq F_A^N(x)$.

Theorem 3.6. Let $A = (X, T_A^P, F_A^P, T_A^N, F_A^N)$ be a bipolar Pythagorean fuzzy subring of a ring R. Then, for all $x, y \in R$

i. $T_A^P(x-y) = T_A^P(e)$ implies that $T_A^P(x) = T_A^P(y)$

ii. $F_A^P(x-y) = F_A^P(e)$ implies that $F_A^P(x) = F_A^P(y)$

iii. $T_A^N(x-y) = T_A^N(e)$ implies that $T_A^N(x) = T_A^N(y)$

iv. $F_A^N(x-y) = F_A^N(e)$ implies that $F_A^N(x) = F_A^N(y)$

Proof. For all x and y in R,

i. $T_A^P(x) = T_A^P(x-y+y) \geq \min\{T_A^P(x-y), T_A^P(y)\} = \min\{T_A^P(e), T_A^P(y)\} = T_A^P(y)$. $T_A^P(y) = T_A^P(y-x+x) \geq \min\{T_A^P(y-x), T_A^P(x)\} = \min\{T_A^P(e), T_A^P(x)\} = T_A^P(x)$. Therefore, $T_A^P(x) = T_A^P(y)$.
ii. $F_A^P(x) = F_A^P(x-y+y) \leq \max\{F_A^P(x-y), F_A^P(y)\} = \max\{F_A^P(e), F_A^P(y)\} = F_A^P(y)$. $F_A^P(y) = F_A^P(y-x+x) \leq \max\{F_A^P(y-x), F_A^P(x)\} = \max\{F_A^P(e), F_A^P(x)\} = F_A^P(x)$. Therefore, $F_A^P(x) = F_A^P(y)$.
iii. $T_A^N(x) = T_A^N(x-y+y) \leq \max\{T_A^N(x-y), T_A^N(y)\} = \max\{T_A^N(e), T_A^N(y)\} = T_A^N(y)$. $T_A^N(y) = T_A^N(y-x+x) \leq \max\{T_A^N(y-x), T_A^N(x)\} = \max\{T_A^N(e), T_A^N(x)\} = T_A^N(x)$. Therefore, $T_A^N(x) = T_A^N(y)$.
iv. $F_A^N(x) = F_A^N(x-y+y) \geq \min\{F_A^N(x-y), F_A^N(y)\} = \min\{F_A^N(e), F_A^N(y)\} = F_A^N(y)$. $F_A^N(y) = F_A^N(y-x+x) \geq \min\{F_A^N(y-x), F_A^N(x)\} = \min\{F_A^N(e), F_A^N(x)\} = F_A^N(x)$. Therefore, $F_A^N(x) = F_A^N(y)$.

Theorem 3.7. Let $A = (X, T_A^P, F_A^P, T_A^N, F_A^N)$ be a bipolar Pythagorean fuzzy subring of a ring R. For all x and y in R,

i. If $T_A^P(x-y) = 1$, then $T_A^P(x) = T_A^P(y)$.

ii. If $F_A^P(x-y) = 0$, then $F_A^P(x) = F_A^P(y)$.

iii. If $T_A^N(x-y) = -1$, then $T_A^N(x) = T_A^N(y)$.

iv. If $F_A^N(x-y) = 0$, then $F_A^N(x) = F_A^N(y)$.

Proof. For all x and y in R,

i. $T_A^P(x) = T_A^P(x - y + y) \geq \min\{T_A^P(x - y), T_A^P(y)\} = \min\{1, T_A^P(y)\} = T_A^P(y) = T_A^P(-y) = T_A^P(-x + x - y) \geq \min\{T_A^P(x), T_A^P(x - y)\} = \min\{T_A^P(x), 1\} = T_A^P(x)$. Therefore, $T_A^P(x) = T_A^P(y)$.
ii. $F_A^P(x) = F_A^P(x - y + y) \leq \max\{F_A^P(x - y), F_A^P(y)\} = \max\{0, F_A^P(y)\} = F_A^P(y) = F_A^P(-y) = F_A^P(-x + x - y) \leq \max\{F_A^P(x), F_A^P(x - y)\} = \max\{F_A^P(x), 0\} = F_A^P(x)$. Therefore, $F_A^P(x) = F_A^P(y)$.
iii. $T_A^N(x) = T_A^N(x - y + y) \leq \max\{T_A^N(x - y), T_A^N(y)\} = \max\{-1, T_A^N(y)\} = T_A^N(y) = T_A^N(-y) = T_A^N(-x + x - y) \leq \max\{T_A^N(x), T_A^N(x - y)\} = \max\{T_A^N(x), -1\} = T_A^N(x)$. Therefore, $T_A^N(x) = T_A^N(y)$.
iv. $F_A^N(x) = F_A^N(x - y + y) \geq \min\{F_A^N(x - y), F_A^N(y)\} = \min\{0, F_A^N(y)\} = F_A^N(y) = F_A^N(-y) = F_A^N(-x + x - y) \geq \min\{F_A^N(x) F_A^N(x - y)\} = \min\{F_A^N(x), 0\} = F_A^N(x)$. Therefore, $F_A^N(x) = F_A^N(y)$.

Theorem 3.8. Let $A = (X, T_A^P, F_A^P, T_A^N, F_A^N)$ be a bipolar Pythagorean fuzzy subring of a ring G. for x and y in G,

i. $T_A^P(xy^{-1}) = 0$, then either $T_A^P(x) = 0$ or $T_A^P(y) = 0$.

ii. $F_A^P(xy^{-1}) = 0$, then either $F_A^P(x) = 0$ or $F_A^P(y) = 0$.

iii. $T_A^N(xy^{-1}) = 0$, then either $T_A^N(x) = 0$ or $T_A^N(y) = 0$.

iv. $F_A^N(xy^{-1}) = 0$, then either $F_A^N(x) = 0$ or $F_A^N(y) = 0$.

Proof. Let x and y in G. Then, by the definition

i. $T_A^P(xy^{-1}) \geq \min\{T_A^P(x), T_A^P(y)\}$, which implies that $0 \geq \min\{T_A^P(x), T_A^P(y)\}$. Therefore, either $T_A^P(x) = 0$ or $T_A^P(y) = 0$.
ii. $F_A^P(xy^{-1}) \leq \max\{F_A^P(x), F_A^P(y)\}$, which implies that $0 \leq \max\{F_A^P(x), F_A^P(y)\}$. Therefore, either $F_A^P(x) = 0$ or $F_A^P(y) = 0$.
iii. $T_A^N(xy^{-1}) \leq \max\{T_A^N(x), T_A^N(y)\}$, which implies that $0 \leq \max\{T_A^N(x), T_A^N(y)\}$. Therefore, either $T_A^N(x) = 0$ or $T_A^N(y) = 0$.
iv. $F_A^N(xy^{-1}) \geq \min\{F_A^N(x), F_A^N(y)\}$, which implies that $0 \geq \min\{F_A^N(x), F_A^N(y)\}$. Therefore, either $F_A^N(x) = 0$ or $F_A^N(y) = 0$.

Theorem 3.9. If $A = (X, T_A^P, F_A^P, T_A^N, F_A^N)$ be a bipolar Pythagorean fuzzy subring of a ring G, then, for x and y in G

i. $T_A^P(xy) = T_A^P(yx)$ if and only if $T_A^P(x) = T_A^P(y^{-1}xy)$.

ii. $F_A^P(xy) = F_A^P(yx)$ if and only if $F_A^P(x) = F_A^P(y^{-1}xy)$.

iii. $T_A^N(xy) = T_A^N(yx)$ if and only if $T_A^N(x) = T_A^N(y^{-1}xy)$.

iv. $F_A^N(xy) = F_A^N(yx)$ if and only if $F_A^N(x) = F_A^N(y^{-1}xy)$.

Proof. Let x and y in G.

i. Assume that $T_A^P(xy) = T_A^P(yx)$, so $T_A^P(y^{-1}xy) = T_A^P(y^{-1}yx) = T_A^P(x)$. Therefore, $T_A^P(x) = T_A^P(y^{-1}xy)$, for x and y in G. Conversely, assume that $T_A^P(x) = T_A^P(y^{-1}xy)$, we get $T_A^P(xy) = T_A^P(xyxx^{-1}) = T_A^P(yx)$. Therefore, $T_A^P(xy) = T_A^P(yx)$.

ii. Assume that $F_A^P(xy) = F_A^P(yx)$, so $F_A^P(y^{-1}xy) = F_A^P(y^{-1}yx) = F_A^P(x)$. Therefore, $F_A^P(x) = F_A^P(y^{-1}xy)$. Conversely, assume that $F_A^P(x) = F_A^P(y^{-1}xy)$, we get $F_A^P(xy) = F_A^P(xyxx^{-1}) = F_A^P(yx)$. Therefore, $F_A^P(xy) = F_A^P(yx)$.
iii. Assume that $T_A^N(xy) = T_A^N(yx)$, so $T_A^N(y^{-1}xy) = T_A^N(y^{-1}yx) = T_A^N(x)$. Therefore, $T_A^N = T_A^N(y^{-1}xy)$. Conversely, assume that $T_A^N(x) = T_A^N(y^{-1}xy)$, we get $T_A^N(xy) = T_A^N(xyxx^{-1}) = T_A^N(yx)$. Therefore, $T_A^N(xy) = T_A^N(yx)$.
iv. Assume that $F_A^N(xy) = F_A^N(yx)$, so $F_A^N(y^{-1}xy) = F_A^N(y^{-1}yx) = F_A^N(x)$. Therefore, $F_A^N(x) = F_A^N(y^{-1}xy)$. Conversely, assume that $F_A^N(x) = F_A^N(y^{-1}xy)$, we get $F_A^N(xy) = F_A^N(xyxx^{-1}) = F_A^N(yx)$. Therefore, $F_A^N(xy) = F_A^N(yx)$.

Theorem 3.10. If $A = (X, T_A^P, F_A^P, T_A^N, F_A^N)$ be a bipolar Pythagorean fuzzy subring of a ring G, then $H = \{x \in G : T_A^P(x) = T_A^P(e),\ F_A^P(x) = F_A^P(e), T_A^N(x) = T_A^N(e), F_A^N(x) = F_A^N(e)\}$ is a subring of G.
PROOF. Here, $H = \{x \in G : T_A^P(x) = T_A^P(e),\ F_A^P(x) = F_A^P(e), T_A^N(x) = T_A^N(e),\ F_A^N(x) = F_A^N(e)\}$, by Theorem 3.1, $T_A^P(x^{-1}) = T_A^P(x) = T_A^P(e), F_A^P(x^{-1}) = F_A^P(x) = F_A^P(e), T_A^N(x^{-1}) = T_A^N(x) = T_A^N(e)$ and $F_A^N(x^{-1}) = F_A^N(x) = F_A^N(e)$. Therefore, $x^{-1} \in H$. Now, $T_A^P(xy^{-1}) \geq \min\{T_A^P(x), T_A^P(y)\} = \min\{T_A^P(e), T_A^P(e)\} = T_A^P(e)$, and $T_A^P(e) = T_A^P\left((xy^{-1})(xy^{-1})^{-1}\right) \geq \min\{T_A^P(xy^{-1}), T_A^P(xy^{-1})\} = T_A^P(xy^{-1})$.
Hence, $T_A^P(e) = T_A^P(xy^{-1})$. $F_A^P(xy^{-1}) \leq \max\{F_A^P(x), F_A^P(y)\} = \max\{F_A^P(e), F_A^P(e)\} = F_A^P(e)$, and $F_A^P(e) = F_A^P\left((xy^{-1})(xy^{-1})^{-1}\right) \leq \max\{F_A^P(xy^{-1}), F_A^P(xy^{-1})\} = F_A^P(xy^{-1})$.
Hence, $F_A^P(e) = F_A^P(xy^{-1})$. $T_A^N(xy^{-1}) \leq \max\{T_A^N(x), T_A^N(y)\} = \max\{T_A^N(e), T_A^N(e)\} = T_A^N(e)$ and $T_A^N(e) = T_A^N\left((xy^{-1})(xy^{-1})^{-1}\right) \leq \max\{T_A^N(xy^{-1}), T_A^N(xy^{-1})\} = T_A^N(xy^{-1})$.
Hence, $T_A^N(e) = T_A^N(xy^{-1})$. $F_A^N(xy^{-1}) \geq \min\{F_A^N(x), F_A^N(y)\} = \min\{F_A^N(e), F_A^N(e)\} = F_A^N(e)$, and $F_A^N(e) = F_A^N\left((xy^{-1})(xy^{-1})^{-1}\right) \geq \min\{F_A^N(xy^{-1}), F_A^N(xy^{-1})\} = F_A^N(xy^{-1})$.
Hence, $F_A^N(e) = F_A^N(xy^{-1})$. Therefore, $xy^{-1} \in H$. Hence, H is a subring of G.

Theorem 3.11. Let G be a ring. If $A = (X, T_A^P, F_A^P, T_A^N, F_A^N)$ be a bipolar Pythagorean fuzzy subring of a ring G, then, for each x and y in G with $T_A^P(x) \neq T_A^P(y)$, $F_A^P(x) \neq F_A^P(y)$, $T_A^N(x) \neq T_A^N(y)$ and $F_A^N(x) \neq F_A^N(y)$,

i. $T_A^P(xy) = \min\{T_A^P(x), T_A^P(y)\}$

ii. $F_A^P(xy) = \max\{F_A^P(x), F_A^P(y)\}$

iii. $T_A^N(xy) = \max\{T_A^N(x), T_A^N(y)\}$

iv. $F_A^N(xy) = \min\{F_A^N(x), F_A^N(y)\}$

PROOF.

i. Assume that $T_A^P(x) > T_A^P(y), F_A^P(x) > F_A^P(y), T_A^N(x) > T_A^N(y)$ and $F_A^N(x) > F_A^N(y)$. Then, $T_A^P(y) = T_A^P(x^{-1}xy) \geq \min\{T_A^P(x^{-1}), T_A^P(xy)\} = \min\{T_A^P(x), T_A^P(xy)\} = T_A^P(xy) \geq \min\{T_A^P(x), T_A^P(y)\} = T_A^P(y)$. Therefore, $T_A^P(xy) = T_A^P(y) = \min\{T_A^P(x), T_A^P(y)\}$.
ii. $F_A^P(y) = F_A^P(x^{-1}xy) \leq \max\{F_A^P(x^{-1}), F_A^P(xy)\} = \max\{F_A^P(x), F_A^P(xy)\} = F_A^P(xy) \leq \max\{F_A^P(x), F_A^P(y)\} = F_A^P(y)$. Therefore, $F_A^P(xy) = F_A^P(y) = \max\{F_A^P(x), F_A^P(y)\}$.
iii. $T_A^N(y) = T_A^N(x^{-1}xy) \leq \max\{T_A^N(x^{-1}), T_A^N(xy)\} = \max\{T_A^N(x), T_A^N(xy)\} = T_A^N(xy) \leq \max\{T_A^N(x), T_A^N(y)\} = T_A^N(y)$. Therefore, $T_A^N(xy) = T_A^N(y) = \max\{T_A^N(x), T_A^N(y)\}$.
iv. $F_A^N(y) = F_A^N(x^{-1}xy) \geq \min\{F_A^N(x^{-1}), F_A^N(xy)\} = \min\{F_A^N(x), F_A^N(xy)\} = F_A^N(xy) \geq \min\{F_A^N(x), F_A^N(y)\} = F_A^N(y)$. Therefore, $F_A^N(xy) = F_A^N(y) = \min\{F_A^N(x), F_A^N(y)\}$.

Theorem 3.12. If $A = (X, T_A^P, F_A^P, T_A^N, F_A^N)$ and $B = (X, T_B^P, F_B^P, T_B^N, F_B^N)$ are two both bipolar Pythagorean fuzzy subrings of a ring G, then their $A \cap B$ is a bipolar Pythagorean fuzzy subring of G.

Proof. Let $A = \{(X, T_A^P, F_A^P, T_A^N, F_A^N): x \in G\}$ and $B = \{(X, T_B^P, F_B^P, T_B^N, F_B^N): x \in G\}$. Let $C = A \cap B$ and $C = \{(X, T_C^P, F_C^P, T_C^N, F_C^N): x \in G\}$.

$$\begin{aligned} T_C^P(xy^{-1}) &= \min\{T_A^P(xy^{-1}), T_B^P(xy^{-1})\} \\ &\geq \min\left\{\min\{T_A^P(x), T_A^P(y)\}, \min\{T_B^P(x), T_B^P(y)\}\right\} \\ &\geq \min\left\{\min\{T_A^P(x), T_B^P(x)\}, \min\{T_A^P(y), T_B^P(y)\}\right\} \\ &= \min\{T_C^P(x), T_C^P(y)\} \end{aligned}$$

Also,

$$\begin{aligned} F_C^P(xy^{-1}) &= \max\{F_A^P(xy^{-1}), F_B^P(xy^{-1})\} \\ &\leq \max\left\{\max\{F_A^P(x), F_A^P(y)\}, \max\{F_B^P(x), F_B^P(y)\}\right\} \\ &\leq \max\left\{\max\{F_A^P(x), F_B^P(x)\}, \max\{F_A^P(y), F_B^P(y)\}\right\} \\ &= \max\{F_C^P(x), F_C^P(y)\}, \\ T_C^N(xy^{-1}) &= \max\{T_A^N(xy^{-1}), T_B^N(xy^{-1})\} \\ &\leq \max\left\{max\{T_A^N(x), T_A^N(y)\}, max\{T_B^N(x), T_B^N(y)\}\right\} \\ &\leq \max\left\{\max\{T_A^N(x), T_B^N(x)\}, \max\{T_A^N(y), T_B^N(y)\}\right\} \\ &= \max\{T_C^N(x), T_C^N(y)\} \\ F_C^N(xy^{-1}) &= \min\{F_A^N(xy^{-1}), F_B^N(xy^{-1})\} \\ &\geq \min\left\{\min\{F_A^N(x), F_A^N(y)\}, min\{F_B^N(x), F_B^N(y)\}\right\} \\ &\geq \min\left\{\min\{F_A^N(x), F_B^N(x)\}, min\{F_A^N(y), F_B^N(y)\}\right\} \\ &= \min\{F_C^N(x), F_C^N(y)\} \end{aligned}$$

Hence, $A \cap B$ is a bipolar Pythagorean fuzzy subring of G.

Theorem 3.13. The intersection of a family of bipolar Pythagorean fuzzy subrings of a ring G is a bipolar Pythagorean fuzzy subring of G.

Proof. Let $\{V_i: i \in I\}$ be a family of bipolar Pythagorean fuzzy subrings of a ring G and let $A = \cap_{i\in I} V_i$. Let x and y in G. Now,

$$\begin{aligned} T_A^P(xy^{-1}) = inf_{i\in I} T_{V_i}^P(xy^{-1}) &\geq \inf_{i\in I}\min\{T_{V_i}^P(x), T_{V_i}^P(y)\} \\ &= \min\{\inf_{i\in I} T_{V_i}^P(x), \inf_{i\in I} T_{V_i}^P(y)\} \\ &= \min\{T_A^P(x), T_A^P(y)\} \end{aligned}$$

Therefore, $T_A^P(xy^{-1}) \geq \min\{T_A^P(x), T_A^P(y)\}$, for all x and y in G, and

$$\begin{aligned} F_A^P(xy^{-1}) = sup_{i\in I} F_{V_i}^P(xy^{-1}) &\leq sup_{i\in I}\max\{F_{V_i}^P(x), F_{V_i}^P(y)\} \\ &= \max\{sup_{i\in I} F_{V_i}^P(x), \sup_{i\in I} F_{V_i}^P(y)\} \\ &= \max\{F_A^P(x), F_A^P(y)\} \end{aligned}$$

Therefore, $F_A^P(xy^{-1}) \leq \max\{F_A^P(x), F_A^P(y)\}$, for all x and y in G.

$$\begin{aligned} T_A^N(xy^{-1}) = sup_{i\in I} T_{V_i}^N(xy^{-1}) &\leq \sup_{i\in I}\max\{T_{V_i}^N(x), T_{V_i}^N(y)\} \\ &= \max\{sup_{i\in I} T_{V_i}^N(x), \sup_{i\in I} T_{V_i}^N(y)\} \\ &= \max\{T_A^N(x), T_A^N(y)\} \end{aligned}$$

Therefore, $T_A^N(xy^{-1}) \leq \max\{T_A^N(x), T_A^N(y)\}$, for all x and y in G.

$$\begin{aligned} F_A^N(xy^{-1}) = \inf_{i\in I} F_{V_i}^N(xy^{-1}) &\geq \inf_{i\in I} \min\{F_{V_i}^N(x), F_{V_i}^N(y)\} \\ &= \min\{\inf_{i\in I} F_{V_i}^N(x), \inf_{i\in I} F_{V_i}^N(y)\} \\ &= \min\{F_A^N(x), F_A^N(y)\} \end{aligned}$$

Therefore, $F_A^N(xy^{-1}) \geq \min\{F_A^N(x), F_A^N(y)\}$, for all x and y in G. Hence, the intersection of a family of bipolar Pythagorean fuzzy subrings of a ring G is a bipolar Pythagorean fuzzy subring of G.

Theorem 3.14. If $A = (X, T_A^P, F_A^P, T_A^N, F_A^N)$ and $B = (X, T_B^P, F_B^P, T_B^N, F_B^N)$ are any two bipolar Pythagorean fuzzy subrings of rings G_1 and G_2, respectively, then $A \times B = (T_{A\times B}^P, F_{A\times B}^P, T_{A\times B}^N, F_{A\times B}^N)$ is a bipolar Pythagorean fuzzy subring of $G_1 \times G_2$.

Proof. Let A and B be two bipolar Pythagorean fuzzy subrings of the ring G_1 and G_2, respectively. Let x_1 and x_2 be in G_1, y_1 and y_2 be in G_2. Then, (x_1, y_1) and (x_2, y_2) are in $G_1 \times G_2$. Now,

$$\begin{aligned} T_{A\times B}^P[(x_1, y_1)(x_2, y_2)^{-1}] &= T_{A\times B}^P(x_1 {x_2}^{-1}, y_1 {y_2}^{-1}) \\ &= min\{T_A^P(x_1 {x_2}^{-1}), T_B^P(y_1 {y_2}^{-1})\} \\ &\geq \min\left\{\min\{T_A^P(x_1), T_A^P(x_2)\}, \min\{T_B^P(y_1), T_B^P(y_2)\}\right\} \\ &= \min\left\{\min\{T_A^P(x_1), T_B^P(y_1)\}, \min\{T_A^P(x_2), T_B^P(y_2)\}\right\} \\ &= \min\{T_{A\times B}^P(x_1, y_1), T_{A\times B}^P(x_2, y_2)\} \end{aligned}$$

Therefore, $T_{A\times B}^P[(x_1, y_1)(x_2, y_2)^{-1}] \geq \min\{T_{A\times B}^P(x_1, y_1), T_{A\times B}^P(x_2, y_2)\}$.

Also,

$$\begin{aligned} F_{A\times B}^P[(x_1, y_1)(x_2, y_2)^{-1}] &= F_{A\times B}^P(x_1 {x_2}^{-1}, y_1 {y_2}^{-1}) \\ &= \max\{F_A^P(x_1 {x_2}^{-1}), F_B^P(y_1 {y_2}^{-1})\} \\ &\leq \max\left\{\max\{F_A^P(x_1), F_A^P(x_2)\}, \max\{F_B^P(y_1), F_B^P(y_2)\}\right\} \\ &= \max\left\{max\{F_A^P(x_1), F_B^P(y_1)\}, \max\{F_A^P(x_2), F_B^P(y_2)\}\right\} \\ &= \max\{F_{A\times B}^P(x_1, y_1), F_{A\times B}^P(x_2, y_2)\} \end{aligned}$$

Therefore, $F_{A\times B}^P[(x_1, y_1)(x_2, y_2)^{-1}] \leq \max\{F_{A\times B}^P(x_1, y_1), F_{A\times B}^P(x_2, y_2)\}$.

$$\begin{aligned} T_{A\times B}^N[(x_1, y_1)(x_2, y_2)^{-1}] &= T_{A\times B}^N(x_1 {x_2}^{-1}, y_1 {y_2}^{-1}) \\ &= \max\{T_A^N(x_1 {x_2}^{-1}), T_B^N(y_1 {y_2}^{-1})\} \\ &\leq \max\left\{\max\{T_A^N(x_1), T_A^N(x_2)\}, \max\{T_B^N(y_1), T_B^N(y_2)\}\right\} \\ &= \max\left\{max\{T_A^N(x_1), T_B^N(y_1)\}, \max\{T_A^N(x_2), T_B^N(y_2)\}\right\} \\ &= \max\{T_{A\times B}^N(x_1, y_1), T_{A\times B}^N(x_2, y_2)\} \end{aligned}$$

Therefore, $T_{A\times B}^N[(x_1, y_1)(x_2, y_2)^{-1}] \leq \max\{T_{A\times B}^N(x_1, y_1), T_{A\times B}^N(x_2, y_2)\}$.

$$\begin{aligned} F_{A\times B}^N[(x_1, y_1)(x_2, y_2)^{-1}] &= F_{A\times B}^N(x_1 {x_2}^{-1}, y_1 {y_2}^{-1}) \\ &= \min\{F_A^N(x_1 {x_2}^{-1}), F_B^N(y_1 {y_2}^{-1})\} \\ &\geq \min\left\{\min\{F_A^N(x_1), F_A^N(x_2)\}, \min\{F_B^N(y_1), F_B^N(y_2)\}\right\} \\ &= \min\left\{\min\{F_A^N(x_1), F_B^N(y_1)\}, \min\{F_A^N(x_2), F_B^N(y_2)\}\right\} \\ &= \min\{F_{A\times B}^N(x_1, y_1), F_{A\times B}^N(x_2, y_2)\} \end{aligned}$$

Therefore, $F_{A\times B}^N[(x_1, y_1)(x_2, y_2)^{-1}] \geq \min\{F_{A\times B}^N(x_1, y_1), F_{A\times B}^N(x_2, y_2)\}$. Hence, $A \times B$ is a bipolar Pythagorean fuzzy subring of $G_1 \times G_2$.

Theorem 3.15. Let $A = (X, T_A^P, F_A^P, T_A^N, F_A^N)$ and $B = (X, T_B^P, F_B^P, T_B^N, F_B^N)$ be any two bipolar Pythagorean fuzzy subsets of a ring G and H respectively. Suppose that e and e' are the identity elements of G and H, respectively. If $A \times B$ is a bipolar Pythagorean fuzzy subring of $G \times H$, then at least one of the following two elements must hold.

i. $T_B^P(e') \geq T_A^P(x), F_B^P(e') \leq F_A^P(x)$for all x in G and $T_B^N(e') \leq T_A^N(x), F_B^N(e') \geq F_A^N(x)$ for all x in G.

ii. $T_B^P(e) \geq T_A^P(y), F_B^P(e) \leq F_A^P(y)$for all x in G and $T_B^N(e) \leq T_A^N(y), F_B^N(e) \geq F_A^N(y)$ for all y in H.

Proof. Let $A \times B$ be a bipolar Pythagorean fuzzy subring of $G \times H$. By contraposition, suppose that none of the statements (i) and (ii) holds. Then, we can find a in G and b in H such that $T_A^P(a) > T_B^P(e')$, $F_A^P(a) < F_B^P(e'), T_A^N(a) < T_B^N(e'), F_A^N(a) > F_B^N(e')$, and $T_B^P(b) > T_A^P(e), F_B^P(b) < F_A^P(e), T_B^N(b) < T_A^N(e)$, $F_B^N(b) > F_A^N(e)$. We have $T_{A\times B}^P(a,b) = \min\{T_A^P(a), T_B^P(b)\} > \min\{T_A^P(e), T_B^P(e')\} = T_{A\times B}^P(e,e')$. Also,

$$\begin{aligned} F_{A\times B}^P(a,b) = \max\{F_A^P(a), F_B^P(b)\} &< \max\{F_A^P(e), F_B^P(e')\} \\ &= F_{A\times B}^P(e,e'). T_{A\times B}^N(a,b) \\ &= \max\{T_A^N(a), T_B^N(b)\} \\ &< \max\{T_A^N(e), T_B^N(e')\} \\ &= T_{A\times B}^N(e,e'). F_{A\times B}^N(a,b) \\ &= \min\{F_A^N(a), F_B^N(b)\} \\ &> \min\{F_A^N(e), F_B^N(e')\} \\ &= F_{A\times B}^N(e,e') \end{aligned}$$

Thus, $A \times B$ is not a bipolar Pythagorean fuzzy subring of $G \times H$. Hence, either, for all x in G,

$$T_B^P(e') \geq T_A^P(x),\ F_B^P(e') \leq F_A^P(x), T_B^N(e') \leq T_A^N(x), \text{and } F_B^N(e') \geq F_A^N(x)$$

or, for all y in H,

$$T_B^P(e) \geq T_A^P(y),\ F_B^P(e) \leq F_A^P(y), T_B^N(e) \leq T_A^N(y), \text{and } F_B^N(e) \geq F_A^N(y)$$

Theorem 3.16. Let $A = (X, T_A^P, F_A^P, T_A^N, F_A^N)$ and $B = (X, T_B^P, F_B^P, T_B^N, F_B^N)$ be any two bipolar Pythagorean fuzzy subsets of a ring G and H, respectively and $A \times B$ be a bipolar Pythagorean fuzzy subring of $G \times H$. Then, the following are true

i. If $T_A^P(x) \leq T_B^P(e'), F_A^P(x) \geq F_B^P(e'), T_A^N(x) \geq T_B^N(e')$, and $F_A^N(x) \leq F_B^N(e')$ for all x in G, then A is a bipolar Pythagorean fuzzy subring of G, where e' is the identity element of H.

ii. If $T_B^P(x) \leq T_A^P(e), F_B^P(x) \geq F_A^P(e), T_B^N(x) \geq T_A^N(e)$, and $F_B^N(x) \leq F_A^N(e)$ for all x in H, then B is a bipolar Pythagorean fuzzy subring of H, where e is the identity element of G.

iii. Either A is a bipolar Pythagorean fuzzy subring of G or B is a bipolar Pythagorean fuzzy subring of H, where e and e' is the identity element of G and H, respectively.

Proof. Let $A \times B$ be a bipolar Pythagorean fuzzy subring of $G \times H$ and x and y in G. Then, (x,e') and (y,e') are in $G \times H$. Now, using the property if $T_A^P(x) \leq T_B^P(e'), F_A^P(x) \geq F_B^P(e'), T_A^N(x) \geq T_B^N(e')$, and $F_A^N(x) \leq F_B^N(e')$, for all x in G, where e' is the identity element of H, we get

$$\begin{aligned} T_A^P(xy^{-1}) &= \min\{T_A^P(xy^{-1}), T_B^P(e'e')\} \\ &= T_{A\times B}^P\big((xy^{-1}),(e'e')\big) \\ &= T_{A\times B}^P[(x,e')(y^{-1},e')] \\ &\geq \min\{T_{A\times B}^P(x,e'), T_{A\times B}^P(y^{-1},e')\} \\ &= \min\Big\{\min\{T_A^P(x), T_B^P(e')\}, \min\{T_A^P(y^{-1}), T_B^P(e')\}\Big\} \\ &= \min\{T_A^P(x), T_A^P(y^{-1})\} \\ &= \min\{T_A^P(x), T_A^P(y)\} \end{aligned}$$

Therefore, $T_A^P(xy^{-1}) \geq \min\{T_A^P(x), T_A^P(y)\}$ for all $x$and y in G. Also,

$$\begin{aligned} F_A^P(xy^{-1}) &= \max\{F_A^P(xy^{-1}), F_B^P(e'e')\} \\ &= F_{A\times B}^P\big((xy^{-1}),(e'e')\big) \end{aligned}$$

$$= F^P_{A\times B}[(x,e')(y^{-1},e')]$$
$$\leq \max\{F^P_{A\times B}(x,e'), F^P_{A\times B}(y^{-1},e')\}$$
$$= \max\left\{\max\{F^P_A(x), F^P_B(e')\}, \max\{F^P_A(y^{-1}), F^P_B(e')\}\right\}$$
$$= \max\{F^P_A(x), F^P_A(y^{-1})\}$$
$$= \max\{F^P_A(x), F^P_A(y)\}$$

Therefore, $F^P_A(xy^{-1}) \leq \max\{F^P_A(x), F^P_A(y)\}$ for all x and y in G.

$$T^N_A(xy^{-1}) = \max\{T^N_A(xy^{-1}), T^N_B(e'e')\}$$
$$= T^N_{A\times B}\big((xy^{-1}),(e'e')\big)$$
$$= T^N_{A\times B}[(x,e')(y^{-1},e')]$$
$$\leq \max\{T^N_{A\times B}(x,e'), T^N_{A\times B}(y^{-1},e')\}$$
$$= \max\left\{\max\{T^N_A(x), T^N_B(e')\}, \max\{T^N_A(y^{-1}), T^N_B(e')\}\right\}$$
$$= \max\{T^N_A(x), T^N_A(y^{-1})\}$$
$$= \max\{T^N_A(x), T^N_A(y)\}$$

Therefore, $T^N_A(xy^{-1}) \leq \max\{T^N_A(x), T^N_A(y)\}$ for all x and y in G.

$$F^N_A(xy^{-1}) = \min\{F^N_A(xy^{-1}), F^N_B(e'e')\}$$
$$= F^N_{A\times B}\big((xy^{-1}),(e'e')\big)$$
$$= F^N_{A\times B}[(x,e')(y^{-1},e')]$$
$$\geq \min\{F^N_{A\times B}(x,e'), F^N_{A\times B}(y^{-1},e')\}$$
$$= \min\left\{\min\{F^N_A(x), F^N_B(e')\}, \min\{F^N_A(y^{-1}), F^N_B(e')\}\right\}$$
$$= \min\{F^N_A(x), F^N_A(y^{-1})\}$$
$$= \min\{F^N_A(x), F^N_A(y)\}$$

Therefore, $F^N_A(xy^{-1}) \geq \min\{F^N_A(x), F^N_A(y)\}$, for all x and y in G. Hence, A is a bipolar Pythagorean fuzzy subring of G. Thus, (i) is proved.

Now, using the property $T^P_B(x) \leq T^P_A(e), F^P_B(x) \geq F^P_A(e), T^N_B(x) \geq T^N_A(e)$, and $F^N_B(x) \leq F^N_A(e)$, for all x in H, we get

$$T^P_B(xy^{-1}) = \min\{T^P_B(xy^{-1}), T^P_A(e.e)\}$$
$$= T^P_{A\times B}\big((e.e),(xy^{-1})\big)$$
$$= T^P_{A\times B}[(e,x)(e,y^{-1})]$$
$$\geq \min\{T^P_{A\times B}(e,x), T^P_{A\times B}(e,y^{-1})\}$$
$$= \min\left\{\min\{T^P_A(e), T^P_B(x)\}, \min\{T^P_A(e), T^P_B(y^{-1})\}\right\}$$
$$= \min\{T^P_B(x), T^P_B(y^{-1})\}$$
$$= \min\{T^P_B(x), T^P_B(y)\}$$

Therefore, $T^P_B(xy^{-1}) \geq \min\{T^P_B(x), T^P_B(y)\}$ for all x and $y$in H. Also,

$$F^P_B(xy^{-1}) = \max\{F^P_B(xy^{-1}), F^P_A(e.e)\}$$
$$= F^P_{A\times B}\big((e.e),(xy^{-1})\big)$$
$$= F^P_{A\times B}[(e,x)(e,y^{-1})]$$
$$\leq \max\{F^P_{A\times B}(e,x), F^P_{A\times B}(e,y^{-1})\}$$
$$= \max\left\{\max\{F^P_A(e), F^P_B(x)\}, \max\{F^P_A(e), F^P_B(y^{-1})\}\right\}$$
$$= \max\{F^P_B(x), F^P_B(y^{-1})\}$$
$$= \max\{F^P_B(x), F^P_B(y)\}$$

Therefore, $F^P_B(xy^{-1}) \leq \max\{F^P_B(x), F^P_B(y)\}$ for all x and y in H.

$$\begin{aligned} T_B^N(xy^{-1}) &= \max\{T_B^N(xy^{-1}), T_A^N(e.e)\} \\ &= T_{A\times B}^N\big((e.e),(xy^{-1})\big) \\ &= T_{A\times B}^N[(e,x)(e,y^{-1})] \\ &\le \max\{T_{A\times B}^N(e,x), T_{A\times B}^N(e,y^{-1})\} \\ &= \max\Big\{\max\{T_A^N(e), T_B^N(x)\}, \max\{T_A^N(e), T_B^N(y^{-1})\}\Big\} \\ &= \max\{T_B^N(x), T_B^N(y^{-1})\} \\ &= \max\{T_B^N(x), T_B^N(y)\} \end{aligned}$$

Therefore, $T_B^N(xy^{-1}) \le \max\{T_B^N(x), T_B^N(y)\}$, for all x and y in H.

$$\begin{aligned} F_B^N(xy^{-1}) &= \min\{F_B^N(xy^{-1}), F_A^N(e.e)\} \\ &= F_{A\times B}^N\big((e.e),(xy^{-1})\big) \\ &= F_{A\times B}^N[(e,x)(e,y^{-1})] \\ &\ge \min\{F_{A\times B}^N(e,x), F_{A\times B}^N(e,y^{-1})\} \\ &= \min\Big\{\max\{F_A^N(e), F_B^N(x)\}, \min\{F_A^N(e), F_B^N(y^{-1})\}\Big\} \\ &= \min\{F_B^N(x), F_B^N(y^{-1})\} \\ &= \min\{F_B^N(x), F_B^N(y)\} \end{aligned}$$

Therefore, $F_B^N(xy^{-1}) \ge \min\{F_B^N(x), F_B^N(y)\}$, for all x and y in H. Hence, B is a bipolar Pythagorean fuzzy subring of H. Thus (ii) is proved. Hence (iii) is clear.

Theorem 3.17. Let $A = (X, T_A^P, F_A^P, T_A^N, F_A^N)$ be a bipolar Pythagorean fuzzy subset of a ring $(G, .)$ and $V = (X, T_V^P, F_V^P, T_V^N, F_V^N)$ be the strongest bipolar Pythagorean fuzzy relation of G. Then A is a bipolar Pythagorean fuzzy subring of G if and only if V is a bipolar Pythagorean fuzzy subring of $G \times G$.

Proof. Suppose that A is a bipolar Pythagorean fuzzy subring of G. Then for any $x = (x_1, x_2)$ and $y = (y_1, y_2)$ are in $G \times G$. We have,

$$\begin{aligned} T_V^P(xy^{-1}) = T_V^P[(x_1,x_2)(y_1,y_2)^{-1}] &= T_V^P(x_1{y_1}^{-1}, x_2{y_2}^{-1}) \\ &= \min\{T_A^P(x_1{y_1}^{-1}), T_A^P(x_2{y_2}^{-1})\} \\ &\ge \min\Big\{\min\{T_A^P(x_1), T_A^P(y_1)\}, \min\{T_A^P(x_2), T_A^P(y_2)\}\Big\} \\ &= \min\Big\{\min\{T_A^P(x_1), T_A^P(x_2)\}, \min\{T_A^P(y_1), T_A^P(y_2)\}\Big\} \\ &= \min\{T_V^P(x_1,x_2), T_V^P(y_1,y_2)\} \\ &= \min\{T_V^P(x), T_V^P(y)\} \end{aligned}$$

Therefore, $T_V^P(xy^{-1}) \ge min\{T_V^P(x), T_V^P(y)\}$, for all x and y in $G \times G$. Also, we have,

$$\begin{aligned} F_V^P(xy^{-1}) = F_V^P[(x_1,x_2)(y_1,y_2)^{-1}] &= F_V^P(x_1{y_1}^{-1}, x_2{y_2}^{-1}) \\ &= \max\{F_A^P(x_1{y_1}^{-1}), F_A^P(x_2{y_2}^{-1})\} \\ &\le \max\Big\{\max\{F_A^P(x_1), F_A^P(y_1)\}, \max\{F_A^P(x_2), F_A^P(y_2)\}\Big\} \\ &= \max\Big\{\max\{F_A^P(x_1), F_A^P(x_2)\}, \max\{F_A^P(y_1), F_A^P(y_2)\}\Big\} \\ &= \max\{F_V^P(x_1,x_2), F_V^P(y_1,y_2)\} \\ &= \max\{F_V^P(x), F_V^P(y)\} \end{aligned}$$

Therefore, $F_V^P(xy^{-1}) \le \max\{F_V^P(x), F_V^P(y)\}$, for all x and y in $G \times G$.

$$\begin{aligned} T_V^N(xy^{-1}) = T_V^N[(x_1,x_2)(y_1,y_2)^{-1}] &= T_V^N(x_1{y_1}^{-1}, x_2{y_2}^{-1}) \\ &= \max\{T_A^N(x_1{y_1}^{-1}), T_A^N(x_2{y_2}^{-1})\} \\ &\le \max\Big\{\max\{T_A^N(x_1), T_A^N(y_1)\}, \max\{T_A^N(x_2), T_A^N(y_2)\}\Big\} \\ &= \max\Big\{\max\{T_A^N(x_1), T_A^N(x_2)\}, \max\{T_A^N(y_1), T_A^N(y_2)\}\Big\} \end{aligned}$$

$$= \max\{T_V^N(x_1,x_2), T_V^N(y_1,y_2)\}$$
$$= \max\{T_V^N(x), T_V^N(y)\}$$

Therefore, $T_V^N(xy^{-1}) \leq \max\{T_V^N(x), T_V^N(y)\}$, for all x and y in $G \times G$.

$$F_V^N(xy^{-1}) = F_V^N[(x_1,x_2)(y_1,y_2)^{-1}] = F_V^N(x_1{y_1}^{-1}, x_2{y_2}^{-1})$$
$$= \min\{F_A^N(x_1{y_1}^{-1}), F_A^N(x_2{y_2}^{-1})\}$$
$$\geq \min\left\{\min\{F_A^N(x_1), F_A^N(y_1)\}, \min\{F_A^N(x_2), F_A^N(y_2)\}\right\}$$
$$= \min\left\{\min\{F_A^N(x_1), F_A^N(x_2)\}, \min\{F_A^N(y_1), F_A^N(y_2)\}\right\}$$
$$= \min\{F_V^N(x_1,x_2), F_V^N(y_1,y_2)\}$$
$$= \min\{F_V^N(x), F_V^N(y)\}$$

Therefore, $F_V^N(xy^{-1}) \geq \min\{F_V^N(x), F_V^N(y)\}$, for all x and y in $G \times G$. This proves that V is a bipolar Pythagorean fuzzy subring of $G \times G$. Conversely, assume that V is a bipolar Pythagorean fuzzy subring of $G \times G$, then for any$x = (x_1, x_2)$ and $y = (y_1, y_2)$ are in $G \times G$, we have

$$\min\{T_A^P(x_1y_1^{-1}), T_A^P(x_2y_2^{-1})\} = T_V^P(x_1y_1^{-1}, x_2y_2^{-1})$$
$$= T_V^P[(x_1,x_2)(y_1,y_2)^{-1}]$$
$$= T_V^P(xy^{-1})$$
$$\geq \min\{T_V^P(x), T_V^P(y)\}$$
$$= \min\{T_V^P(x_1,x_2), T_V^P(y_1,y_2)\}$$
$$= \min\left\{\min\{T_A^P(x_1), T_A^P(x_2)\}, \min\{T_A^P(y_1), T_A^P(y_2)\}\right\}$$

If we put $x_2 = y_2 = e$, we get, $T_A^P(x_1y_1^{-1}) \geq \min\{T_A^P(x_1), T_A^P(y_1)\}$, for all x_1 and y_1 in G. Also, we have

$$\max\{F_A^P(x_1y_1^{-1}), F_A^P(x_2y_2^{-1})\} = F_V^P(x_1y_1^{-1}, x_2y_2^{-1})$$
$$= F_V^P[(x_1,x_2)(y_1,y_2)^{-1}]$$
$$= F_V^P(xy^{-1})$$
$$\leq \max\{F_V^P(x), F_V^P(y)\}$$
$$= max\{F_V^P(x_1,x_2), F_V^P(y_1,y_2)\}$$
$$= \max\left\{\max\{F_A^P(x_1), F_A^P(x_2)\}, \max\{F_A^P(y_1), F_A^P(y_2)\}\right\}$$

If we put $x_2 = y_2 = e$, we get, $F_A^P(x_1y_1^{-1}) \leq \max\{F_A^P(x_1), F_A^P(y_1)\}$, for all x_1 and y_1 in G.

$$\max\{T_A^N(x_1y_1^{-1}), T_A^N(x_2y_2^{-1})\} = T_V^N(x_1y_1^{-1}, x_2y_2^{-1})$$
$$= T_V^N[(x_1,x_2)(y_1,y_2)^{-1}]$$
$$= T_V^N(xy^{-1})$$
$$\leq \max\{T_V^N(x), T_V^N(y)\}$$
$$= \max\{T_V^N(x_1,x_2), T_V^N(y_1,y_2)\}$$
$$= \max\left\{\max\{T_A^N(x_1), T_A^N(x_2)\}, \max\{T_A^N(y_1), T_A^N(y_2)\}\right\}$$

If we put $x_2 = y_2 = e$, we get, $T_A^N(x_1y_1^{-1}) \leq \max\{T_A^N(x_1), T_A^N(y_1)\}$, for all x_1 and y_1 in G.

$$\min\{F_A^N(x_1y_1^{-1}), F_A^N(x_2y_2^{-1})\} = F_V^N(x_1y_1^{-1}, x_2y_2^{-1})$$
$$= F_V^N[(x_1,x_2)(y_1,y_2)^{-1}]$$
$$= F_V^N(xy^{-1})$$
$$\geq \min\{F_V^N(x), F_V^N(y)\}$$
$$= \min\{F_V^N(x_1,x_2), F_V^N(y_1,y_2)\}$$
$$= \min\left\{\min\{F_A^N(x_1), F_A^N(x_2)\}, \min\{F_A^N(y_1), F_A^N(y_2)\}\right\}$$

If we put $x_2 = y_2 = e$, we get, $F_A^N(x_1y_1^{-1}) \geq \min\{F_A^N(x_1), F_A^N(y_1)\}$, for all x_1 and y_1 in G. Hence, A is a bipolar Pythagorean fuzzy subring of G.

4. Conclusion

In this paper, we define the bipolar Pythagorean fuzzy subring of a ring and investigate the relationship among these bipolar Pythagorean fuzzy subring of a ring. Some characterisation theorems of bipolar Pythagorean fuzzy subring of a ring are obtained.

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Soft Sets and Soft Topological Notions

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Abstract − In this paper, we extended the notions of operations on soft sets to arbitrary collection of soft sets and introduce the concepts of $F_\sigma -$ soft set and $G_\sigma -$ soft set. Furthermore we give definitions of $\sigma-$ soft locally finite and $\sigma-$ soft closure preserving relative to an arbitrary collection of soft sets and study some of their properties.

Keywords − *Soft symmetric difference, soft perfect, relatively soft discrete, soft closed domain (regularly closed), soft open domain (regularly open)*

1. Introduction

In 1999 D. Molodtsov [1] introduced the concept of soft sets as an additional mathematical tool for modeling and dealing with uncertainties. Shabir and Naz [2] went further and introduced the concept of soft topology. Indeed; the two concepts have received much attention. Researches on properties and applications of soft sets and soft topology have attracted many scholars from various fields. Topological structure of soft sets; concepts of soft open sets, soft closed sets, soft interior point and soft neighborhood of a point have been studied by various authors, for example see [2–7]. Senel and Cagman studied soft topological subspaces and Tantawy et al. [8] studied soft separation axioms. The notions of basic operations on soft sets (soft union and soft intersection) have been defined and studied [2, 3, 6, 9–20] and by several other authors, but the definitions were given in terms of only two soft sets. Ali et al. [9] pointed out by counter example that, several assertions [particularly, Proposition 2.3 (iv)-(vi), Proposition 2.4 and Proposition 2.6(iii),(iv)] in Maji et al. [15] are not true in general.

In this paper we extend the notions of these basic operations to arbitrary collection of soft sets. In section 3 of this paper, we propose a modification of the definition of soft difference of two soft sets [2, 5, 11–13, 16, 21–26]. We further introduce and define some terms relative to arbitrary collection of soft sets in a soft topological space and study some of their properties.

2. Preliminary

Throughout this paper, all soft sets are defined over a common universe X and the collection of all soft sets over X with a set of parameters E is denoted as $SS(X)_E$. We begin with the following well known definition found in the literature as cited in each case.

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Definition 2.1. [1] Let X be an initial universe and E be a set of parameters. Let $\rho(X)$ denote the *power set* of X. A pair (F, E) is called a soft set over X, where $F : E \to \rho(X)$ is a mapping. Thus a soft set over a universe X is a parameterized family of $\rho(X)$. For a given subset A of E, the soft set (F, A) is defined to be $F : A \to \rho(X)$ such that $(F, A) = \{(e, F(e)) : e \in A\}$, where $F(e)$ can be regarded as the set of e-approximate elements of the soft set (F, A).

Definition 2.2. [27] A soft set (F_1, A_1) is said to be a *soft subset* of (F_2, A_2) denoted as $(F_1, A_1)\tilde{\subseteq}(F_2, A_2)$ if $A_1 \subseteq A_2$ and $F_1(\alpha) \subseteq F_2(\alpha), \forall\alpha \in A_1$. Equivalently (F_2, A_2) is said to be a soft superset of (F_1, A_1) denoted as $(F_2, A_2)\tilde{\supseteq}(F_1, A_1)$.

Definition 2.3. [27] Two soft sets (F_1, A_1) and (F_2, A_2) are said to be equal (*soft equal*) denoted as $(F_1, A_1) = (F_2, A_2)$ if $(F_1, A_1)\tilde{\subseteq}(F_2, A_2)$ and $(F_2, A_2)\tilde{\subseteq}(F_1, A_1)$. Equivalently $(F_1, A_1) = (F_2, A_2)$ if $F_1(\alpha) = F_2(\alpha), \forall\alpha \in A_1 = A_2$.

Definition 2.4. [7] The *soft compliment* of a soft set (F, A) denoted as $(F, A)^c$ or (F^c, A) is a mapping $F^c : A \to \rho(X)$ given by $F^c(\alpha) = X \setminus F(\alpha), \forall\alpha \in A$. It is very clear that $(F^c, A)^c = (F, A)$. The mapping $F^c : A \to \rho(X)$ is called the *soft compliment function* of F.

Definition 2.5. [8] A soft set (F, E) is said to be a *null soft set* if $F(\alpha) = \emptyset, \forall\alpha \in E$ and (F,E) is said to be an *absolute soft* set if $F(\alpha) = X, \forall\alpha \in E$. A null soft set is denoted as $\tilde{\emptyset}$ and absolute soft set is denoted as $\tilde{X}$. It is clear that $(\tilde{X})^c = \tilde{\emptyset}$ and $(\tilde{\emptyset})^c = \tilde{X}$

Definition 2.6. [28] Let (F, E) be a soft set and $x \in X$, then

i) x is said to belongs to (F, E) denoted as $x \in (F, E)$ if $\forall\alpha \in E, x \in F(\alpha)$.

ii) (F, E) is called singleton soft set denoted as (x, E) or x_E if $F(\alpha) = \{x\}, \forall\alpha \in E$.

Definition 2.7. [7] A soft set (F, A) is called a *soft point* denoted as F_α if for some $\alpha \in A, F(\alpha) \neq \emptyset$ and $F(\beta) = \emptyset, \forall\beta \in (A \setminus \{\alpha\})$. The soft point F_α is said to belong to another soft set (G, A), denoted as $F_\alpha\tilde{\in}(G, A)$ if $F(\alpha) \subseteq G(\alpha)$.

Definition 2.8. [27] A soft set (F, A) is called a *soft element* if $\exists\alpha \in A$ and $x \in X$ such that $F(\alpha) = \{x\}$ and $F(\beta) = \emptyset, \forall\beta \in (A \setminus \{\alpha\})$. A soft element is denoted as F_α^x. The soft element F_α^x is said to be in a soft set (G, A) denoted as $F_\alpha^x\tilde{\in}(G, A)$ if $x \in G(\alpha)$.

By definition, it is clear that a soft element is a soft point, but the converse may not be true.

Definition 2.9. [20] Let (F_1, A_1) and (F_2, A_2) be soft sets

i) The *soft intersection* of (F_1, A_1) and (F_2, A_2) denoted as $(F_1, A_1)\tilde{\cap}(F_2, A_2)$ is defined to be the soft set (F_3, A_3) where $A_3 = A_1 \cap A_2$ and $\forall\alpha \in A_3, F_3(\alpha) = F_1(\alpha) \cap F_2(\alpha)$;

ii) The *soft union* of (F_1, A_1) and (F_2, A_2) denoted as $(F_1, A)\tilde{\cup}(F_2, A_2)$ is defined to be the soft set (F_3, A_3) where $A_3 = A_1 \cup A_2$ and $\forall\alpha \in A_3$

$$F_3(\alpha) = \begin{cases} F_1(\alpha) \text{ if } \alpha \in (A_1 \setminus A_2) \\ F_2(\alpha) \text{ if } \alpha \in (A_2 \setminus A_1) \\ F_1(\alpha) \cup F_2(\alpha) \text{ if } \alpha \in (A_1 \cap A_2). \end{cases}$$

3. Basic Operations on Soft Sets

An arbitrary indexing set I was used by the authors [4,5,7,8,21–26] to define the soft intersection and soft union over a collection $\{(F_i, A) : i \in I\}$ of soft sets as $(F, A) = \tilde{\bigcap\limits_{i\in I}}(F_i, A)$ and $(F, A) = \tilde{\bigcup\limits_{i\in I}}(F_i, A)$ respectively. We point out here that the two definitions are restrictive and incomplete. It is also worth noting that in $\{(F_i, A) : i \in I\}$, A can also be indexed as well i.e., $\{(F_i, A_i) : i \in I\}$. The question is what is $\tilde{\bigcup\limits_{i\in I}}(F_i, A)$? Clearly the definition in [4,5,7,8,21–26] does not cater for this. In this section we extend the notions of soft intersection and soft union to arbitrary collection of soft sets by the use of (i) and (ii) of definition 2.9 as follows:

Definition 3.1. Let $L = \{(F_\delta, A_\delta) : \delta \in \Delta\}$ be a family of soft sets, then the

i) soft intersection over members of L is defined to be the soft set $(F, A) = \tilde{\bigcap}_{\delta\in\Delta} (F_\delta, A_\delta)$ where $A = \bigcap_{\delta\in\Delta} A_\delta$ and $\forall\alpha \in A, F(\alpha) = \tilde{\bigcap}_{\delta\in\Delta} F_\delta(\alpha), \forall\alpha \in A$;

ii) soft union over members of L is defined to be the soft set $(F, A) = \tilde{\bigcup}_{\delta\in\Delta} (F_\delta, A_\delta)$ where $A = \bigcup_{\delta\in\Delta} A_\delta$ and $\forall\alpha \in A$ and $\forall\Upsilon \subseteq \Delta$ and $F(\alpha) = \begin{cases} \bigcup_{\delta\in\Delta} F_\delta(\alpha) \text{ if } \alpha \in \bigcap_{\delta\in\Delta} A_\delta \\ \bigcup_{\delta\in\Upsilon} F_\delta(\alpha) \text{ if } \alpha \in (\bigcap_{\delta\in\Upsilon} A_\delta \setminus \bigcup_{\delta\in(\Delta\setminus\Upsilon)} A_\delta). \end{cases}$

Example 3.2. Let $E = \{e_1, e_2, e_3, e_4, e_5\}$ be a set of parameters and $X = \{x_1, x_2, x_3, x_4\}$. $\rho(X) = \{\emptyset, X, \{x_1\}, \{x_2\}, \{x_3\}, \{x_4\}, \{x_1, x_2\}, \{x_1, x_3\}, \{x_1, x_4\}, \{x_2, x_3\}, \{x_2, x_4\}, \{x_3, x_4\}, \{x_1, x_2, x_3\}, \{x_1, x_2, x_4\}, \{x_1, x_3, x_4\}, \{x_2, x_3, x_4\}\}$, $A_1 = \{e_1, e_3, e_4\}$, $A_2 = \{e_3, e_4\}$ and $A_3 = \{e_1, e_4, e_5\}$.

E	e_1	e_2	e_3	e_4	e_5
F_1	$\{x_1, x_2\}$	$\{x_2, x_3, x_4\}$	$\{x_1, x_4\}$	$\{x_1, x_2, x_4\}$	$\{x_1, x_2, x_3\}$
F_2	$\{x_2, x_3\}$	$\{x_2\}$	$\{x_1, x_2\}$	$\{x_2, x_4\}$	$\{x_2, x_3, x_4\}$
F_3	$\{x_1\}$	$\{x_3\}$	$\{x_4\}$	$\{x_2, x_4\}$	$\{x_1, x_3\}$
F_4	$\{x_3, x_4\}$	$\{x_1\}$	$\{x_2, x_3\}$	$\{x_3\}$	$\{x_4\}$
F_5	$\{x_3\}$	$\{x_3\}$	$\{x_3\}$	$\{x_3\}$	$\{x_3\}$
G	$\emptyset$	$\emptyset$	$\{x_3, x_4\}$	$\emptyset$	$\emptyset$
H	$\emptyset$	$\{x_2\}$	$\emptyset$	$\emptyset$	$\emptyset$

.

Notice that $A_1 \cap A_2 \cap A_3 = \{e_4\}$ and $A_1 \cup A_2 \cup A_3 = \{e_1, e_3, e_4, e_5\}$ and that

i) $F_1(e_4) \cap F_2(e_4) \cap F_3(e_4) = \{x_1, x_2, x_4\} \cap \{x_2, x_4\} \cap \{x_2, x_4\} = \{x_2, x_4\}$. Therefore,

$$\tilde{\bigcap_{i=1}^{3}}(F_i, A_i) = \{(e_4, \{x_2, x_4\})\}$$

ii) $e_1 \in (A_1 \cap A_3) \setminus A_2$, $e_3 \in (A_1 \cap A_2) \setminus A_3$, $e_4 \in (A_1 \cap A_2 \cap A_3)$ and $e_5 \in A_3 \setminus (A_1 \cup A_2)$

$$F(e_1) = F_1(e_1) \cup F_3(e_1) = \{x_1, x_2\}, F(e_3) = F_1(e_3) \cup F_2(e_3) = \{x_1, x_2, x_4\}$$

$$F(e_4) = F_1(e_4) \cup F_2(e_4) \cup F_3(e_4) = \{x_1, x_2, x_4\} \text{ and } F(e_5) = F_3(e_5) = \{x_1, x_3\}$$

Therefore, $\tilde{\bigcup}_{i=1}^{3} (F_i, A_i) = \{(e_1, \{x_1, x_2\}), (e_3, \{x_1, x_2, x_4\}), (e_4, \{x_1, x_2, x_4\}), (e_5, \{x_1, x_3\})\}$.

It is worth noting that the definition of arbitrary soft union as given in [4, 5, 7, 8, 21–26] has no provision for resolving the soft union $\tilde{\bigcup}_{i=1}^{3} (F_i, A_i)$.

The next result is indeed from the Demogan's Laws in set theory.

Proposition 3.3. (Demogan's) Let $\{(F_\delta, A_\delta) : \delta \in \Delta\}$ be a family of soft sets, then

i) $[\tilde{\bigcup}_{\delta\in\Delta} (F_\delta, A_\delta)]^c = \tilde{\bigcap}_{\delta\in\Delta} (F_\delta, A_\delta)^c$

ii) $[\tilde{\bigcap}_{\delta\in\Delta} (F_\delta, A_\delta)]^c = \tilde{\bigcup}_{\delta\in\Delta} (F_\delta, A_\delta)^c$

Proof. Let F_α^x be any soft element, then

i) $F_\alpha^x \tilde{\in} [\tilde{\bigcup}_{\delta\in\Delta} (F_\delta, A_\delta)]^c \Leftrightarrow F_\alpha^x \tilde{\notin} \tilde{\bigcup}_{\delta\in\Delta} (F_\delta, A_\delta) \Leftrightarrow F_\alpha^x \tilde{\notin} (F_\delta, A_\delta), \forall\delta \in \Delta \Leftrightarrow F_\alpha^x \tilde{\in} (F_\delta, A_\delta)^c, \forall\delta \in \Delta \Leftrightarrow F_\alpha^x \tilde{\in} \tilde{\bigcap}_{\delta\in\Delta} (F_\delta, A_\delta)^c$;

ii) $F_\alpha^x \tilde{\in} [\tilde{\bigcap}_{\delta\in\Delta} (F_\delta, A_\delta)]^c \Leftrightarrow F_\alpha^x \tilde{\notin} \tilde{\bigcap}_{\delta\in\Delta} (F_\delta, A_\delta) \Leftrightarrow F_\alpha^x \tilde{\notin} (F_\delta, A_\delta)$ for some $\delta \in \Delta \Leftrightarrow F_\alpha^x \tilde{\in} (F_\delta, A_\delta)^c$ for some $\delta \in \Delta \Leftrightarrow F_\alpha^x \tilde{\in} \tilde{\bigcup}_{\delta\in\Delta} (F_\delta, A_\delta)^c$.

□

The definition of difference of two soft sets (F, E) and (G, E) over a common universe X was given to be $(F, E) \setminus (G, E) = (H, E)$ where for all $e \in E$, $H(e) = F(e) \setminus G(e)$, [2, 5, 11–13, 16, 21–26]. This definition excludes the possibility of taking soft difference of soft sets of the form (F_1, A_1) and (F_2, A_2) where $A_1, A_2 \subseteq E$. Hence, we propose a modification of the definition which follows by some examples:

Definition 3.4. If (F_1, A_1) and (F_2, A_2) are any two soft sets, the soft difference of (F_1, A_1) and (F_2, A_2) denoted as $(F_1, A_1) \setminus (F_2, A_2)$ is defined to be the soft set (F_3, A_1) where

$$F_3(\alpha) = \begin{cases} F_1(\alpha) \text{ if } \alpha \in (A_1 \setminus A_2) \\ F_1(\alpha) \setminus F_2(\alpha) \text{ if } \alpha \in (A_1 \cap A_2) \end{cases}$$

It is worth noting that in $(F_1, A_1) \setminus (F_2, A_2)$ if $A_1 = A_2 = E$ Definition 3.4 reduces to [2, 5, 11–13, 16, 21–26].

Example 3.5. From Example 3.2, $A_1 = \{e_1, e_3, e_4\}$, $A_2 = \{e_3, e_4\}$, $(F_1, A_1) = \{(e_1, \{x_1, x_2\}), (e_3, \{x_1, x_4\}), (e_4, \{x_1, x_2, x_4\})\}$, and $(F_2, A_2) = \{(e_3, \{x_1, x_2\}), (e_4, \{x_2, x_3, x_4\})\}$. Therefore,

i) $(F_1, A_1) \setminus (F_2, A_2) = \{(e_1, \{x_1, x_2\}), (e_3, \{x_4\}), (e_4, \{x_1\})\}$

ii) $(F_2, A_2) \setminus (F_1, A_1) = \{(e_3, \{x_2\}), (e_4, \{x_3\})\}$

Definition 3.6. The *soft symmetric difference* of (F_1, A_1) and (F_2, A_2) denoted as $(F_1, A_1)\tilde{\triangle}(F_2, A_2)$ is defined to be $(F_1, A_1)\tilde{\triangle}(F_2, A_2) = ((F_1, A_1) \setminus (F_2, A_2))\tilde{\cup}((F_2, A_2) \setminus (F_1, A_1))$. From Example 3.5, $(F_1, A_1)\tilde{\triangle}(F_2, A_2) = \{(e_1, \{x_1, x_2\}), (e_3, \{x_2, x_4\}), (e_4, \{x_1, x_3\})\}$.

As a consequence of Definition 3.4 we have the following lemma

Lemma 3.7. Let $\{(F_\delta, A_\delta) : \delta \in \Delta\}$ be a family of soft sets and (H, C) be any soft set, then

i) $[(H, C) \setminus \tilde{\bigcup}_{\delta\in\Delta} (F_\delta, A_\delta)] = \tilde{\bigcap}_{\delta\in\Delta} [(H, C) \setminus (F_\delta, A_\delta)]$

ii) $[(H, C) \setminus \tilde{\bigcap}_{\delta\in\Delta} (F_\delta, A_\delta)] = \tilde{\bigcup}_{\delta\in\Delta} [(H, C) \setminus (F_\delta, A_\delta)]$

PROOF. Let F_α^x be any soft element, then

i) $F_\alpha^x \tilde{\in} [(H, C) \setminus \tilde{\bigcup}_{\delta\in\Delta} (F_\delta, A_\delta)] \Leftrightarrow F_\alpha^x \tilde{\in} (H, C)$ and $F_\alpha^x \tilde{\notin} \tilde{\bigcup}_{\delta\in\Delta} (F_\delta, A_\delta) \Leftrightarrow F_\alpha^x \tilde{\in} (H, C)$ and $F_\alpha^x \tilde{\notin} (F_\delta, A_\delta), \forall \delta \in \Delta \Leftrightarrow F_\alpha^x \tilde{\in} (H, C) \setminus (F_\delta, A_\delta), \forall \alpha \in \Delta \Leftrightarrow F_\alpha^x \tilde{\in} \tilde{\bigcap}_{\delta\in\Delta} [(H, C) \setminus (F_\delta, A_\delta)]$

ii) $F_\alpha^x \tilde{\in} [(H, C) \setminus \tilde{\bigcap}_{\delta\in\Delta} (F_\delta, A_\delta)] \Leftrightarrow F_\alpha^x \tilde{\in} (H, C)$ and $F_\alpha^x \tilde{\notin} \tilde{\bigcap}_{\delta\in\Delta} (F_\delta, A_\delta) \Leftrightarrow F_\alpha^x \tilde{\in} (H, C)$ and $F_\alpha^x \tilde{\notin} (F_\delta, A_\delta)$ for some $\delta \in \Delta \Leftrightarrow F_\alpha^x \tilde{\in} (H, C) \setminus (F_\delta, A_\delta)$ for some $\delta \in \Delta \Leftrightarrow F_\alpha^x \tilde{\in} \tilde{\bigcup}_{\delta\in\Delta} [(H, C) \setminus (F_\delta, A_\delta)]$

□

4. Soft Topological Notions

In this section, using examples we discuss basic notions of soft topology and show some important results. We further introduced and defined some terms relative to arbitrary collection of soft sets in a soft topological space and studied some of their properties. We begin our investigation with the following definition.

Definition 4.1. [2] A *soft topology* over the universe X is collection τ of members of $SS(X)_E$ satisfying the following conditions:

1) $\tilde{X}, \tilde{\emptyset} \in \tau$ i.e. $\exists (F_1, E), (F_2, E) \in \tau$ such that $F_1(\alpha) = X, F_2(\alpha) = \emptyset, \forall \alpha \in E$.

2) If $(F_1, A_1), (F_2, A_2) \in \tau$, then $(F_1, A_1) \tilde{\cap} (F_2, A_2) \in \tau$.

3) If $\{(F_\delta, A_\delta) : \delta \in \Delta\}$ is any number of family of members of τ, then $\tilde{\bigcup}_{\delta \in \Delta} (F_\delta, A_\delta) \tilde{\in} \tau$.

The triplet (X, τ, E) is called a *soft topological space*. Members of τ are referred to as *soft open sets* and $(F, A) \in SS(X)_E$ is said to be a *soft closed* in (X, τ, E) if $(F^c, A) \in \tau$. It is clear from the definition that; inductively any finite intersection of members of τ is in τ. Thus if $(F_\delta, A_\delta) \in \tau (\delta = 1, 2, \ldots, n)$ then $\tilde{\bigcap}_{\delta=1}^{n} (F_\delta, A_\delta) \in \tau$ and any number of union of members of τ is in τ. For brevity we will be using the term X_E for (X, τ, E). As a consequent of definition 4.1 we have the following lemma.

Lemma 4.2. Let X_E be a soft topological space, then

i) $\tilde{X}$ and $\tilde{\emptyset} \in \tau$ are closed in X_E. i.e. $(\tilde{X})^c = \tilde{\emptyset}$ and $(\tilde{\emptyset})^c = \tilde{X}$;

ii) If $\{(F_\delta, A_\delta) : \delta \in \Delta\}$ is any number of family of soft closed sets in X_E, then $(\tilde{\bigcap}_{\delta \in \Delta} (F_\delta, A_\delta))^c = \tilde{\bigcup}_{\delta \in \Delta} (F_\delta, A_\delta)^c = \tilde{\bigcup}_{\delta \in \Delta} (F^c_\delta, A_\delta) = (F, A) \in \tau \Rightarrow \tilde{\bigcap}_{\delta \in \Delta} (F_\delta, A_\delta)$ is closed in X_E i.e., intersection of any number of soft closed sets is a soft closed;

iii) If $\{(F_i, A_i) : i = 1, 2, \ldots, n\}$ is a family of soft closed sets in X_E, then $(\tilde{\bigcup}_{i=1}^{n} (F_i, A_i))^c = \tilde{\bigcap}_{i=1}^{n} (F_i, A_i)^c = \tilde{\bigcap}_{i=1}^{n} (F^c_i, A_i) = (F, A) \in \tau \Rightarrow \tilde{\bigcup}_{i=1}^{n} (F_i, A_i)$ is closed in X_E i.e., finite union of soft closed sets is soft closed.

Definition 4.3. $(F, A) \in SS(X)_E$ is said to be *soft clopen* in X_E if (F, A) is both soft closed and soft open in X_E. i.e., $\tilde{X}$ and $\tilde{\emptyset}$ are soft clopen

Definition 4.4. [5] Let X_E be a soft topological space, then $(F, A) \in SS(X)_E$ is said to be a *soft neighborhood* (for brevity: soft nbd) of $(H, C) \in SS(X)_E$ if $\exists (G, B) \in \tau$ such that $(H, C) \tilde{\subseteq} (G, B) \tilde{\subseteq} (F, A)$. Similarly $(F, A) \in SS(X)_E$ is said to be a soft nbd of the soft element F^x_α if $\exists (G, B) \in \tau$ such that $F^x_\alpha \tilde{\in} (G, B) \tilde{\subseteq} (F, A)$. The family of all soft nbd of the soft element F^x_α is called a soft nbd system of F^x_α denoted as $\mathbb{U}_{F^x_\alpha}$.

We now have the following proposition.

Proposition 4.5. $(F, A) \in SS(X)_E$ is soft open in X_E iff (F,A) is a soft nbd of each its soft subsets.

Proof. Let (F, A) be soft open and $(G, B) \tilde{\subseteq} (F, A)$. Then $(G, B) \tilde{\subseteq} (F, A) \tilde{\subseteq} (F, A)$. Hence by definition (F, A) is a nbd of (G, B).
Conversely suppose (F, A) is a soft nbd of each of its soft subsets. This implies $\forall (G_\alpha, B_\alpha) \tilde{\subseteq} (F, A)$ such that $\alpha \in \Delta$ there exist a soft open set (F_α, A_α) such that $(G_\alpha, B_\alpha) \tilde{\subseteq} (F_\alpha, A_\alpha) \tilde{\subseteq} (F, A)$.
Let $(G, B) = \tilde{\bigcup}_{\alpha \in \Delta} (F_\alpha, A_\alpha)$, then (G, B) is soft open and $(G, B) \tilde{\subseteq} (F, A)$. By our hypothesis if (H, C) is any soft open subset of $(F, A), \exists (F_\alpha, A_\alpha) \tilde{\subseteq} (F, A)$ such that $(H, C) \tilde{\subseteq} (F_\alpha, A_\alpha)$.
This implies $(H, C) \tilde{\subseteq} (F_\alpha, A_\alpha) \tilde{\subseteq} \tilde{\bigcup}_{\alpha \in \Delta} (F_\alpha, A_\alpha) = (G, B)$. This implies $(F, A) \tilde{\subseteq} (G, B)$.
Hence, $(F, A) = (G, B)$ which is soft open. □

As consequence of the above proposition, we have the following corollaries.

Corollary 4.6. [5] $(F, A) \in SS(X)_E$ is soft open in X_E iff (F, A) is a nbd of each its soft elements.

Corollary 4.7. [2] $(F, A) \in SS(X)_E$ is soft open in X_E iff (F, A) is a nbd of each its soft points.

Now we give the following propositions.

Proposition 4.8. $(F, A) \in SS(X)_E$ is soft closed in X_E iff given any soft element F_α^x such that $F_\alpha^x \tilde{\notin} (F, A)$ there exists a soft nbd (G, B) of F_α^x such that $(G, B)\tilde{\cap}(F, A) = \tilde{\emptyset}$.

PROOF. (F, A) is soft closed and $F_\alpha^x \tilde{\notin} (F, A) \Rightarrow F_\alpha^x \tilde{\in} (F^c, A)$. Now $(F, A)\tilde{\cap}(F^c, A) = \tilde{\emptyset}$ and (F^c, A) is soft open.
Conversely suppose $\forall, F_\alpha^x \tilde{\notin} (F, A), \exists (G_\delta, B_\delta)$ soft open such that $F_\alpha^x \tilde{\in} (G_\delta, B_\delta)$ and $(G_\delta, B_\delta)\tilde{\cap}(F, A) = \tilde{\emptyset}$. Let $(G, B) = \tilde{\bigcup}_{\delta \in \Delta} (G_\delta, B_\delta)$, then (G, B) is soft open and $(G^c, B) = (F, A)$. Hence, (F, A) is soft closed. □

Proposition 4.9. Let (X, τ, E) be a soft topological space and $\mathbb{U}_{F_\alpha^x}$ be a soft nbd system of a soft element F_α^x, then

i) $\forall (G, A) \in \mathbb{U}_{F_\alpha^x}, F_\alpha^x \tilde{\in} (G, A)$

ii) If $(G, A) \in \mathbb{U}_{F_\alpha^x}$ and $(G, A)\tilde{\subseteq}(H, B)$, then $(H, B) \in \mathbb{U}_{F_\alpha^x}$

iii) If $(G_i, A_i) \in \mathbb{U}_{F_\alpha^x}$, than $\tilde{\bigcap}_{i=1}^{n} (G_i, A_i) \in \mathbb{U}_{F_\alpha^x}$

iv) $\forall (G, A) \in \mathbb{U}_{F_\alpha^x}, \exists (H, B) \in \mathbb{U}_{F_\alpha^x}$ such that $(G, A) \in \mathbb{U}_{H_\beta^y}, \forall H_\beta^y \tilde{\in} (H, B)$

Conversely, if given a collection $\mathbb{U}$ of members of $SS(X)_E$ and for each $F_\alpha^x \in \mathbb{U}$, there exist a nonempty family $\mathbb{U}_{F_\alpha^x}$ satisfying (i-iv), then there exists a unique soft topology on $\mathbb{U}$ such that $\mathbb{U}_{F_\alpha^x}$ is precisely the soft nbd system of F_α^x for each $F_\alpha^x \in \mathbb{U}$.

PROOF.

i) Obvious from definition of $\mathbb{U}_{F_\alpha^x}$;

ii) $(G, A) \in \mathbb{U}_{F_\alpha^x}$ and $(G, A)\tilde{\subseteq}(H, B) \Rightarrow F_\alpha^x \tilde{\in} (G, A)\tilde{\subseteq}(H, B) \Rightarrow F_\alpha^x \tilde{\in} (H, B) \Rightarrow (H, B) \in \mathbb{U}_{F_\alpha^x}$;

iii) Let $(G_i, A_i) \in \mathbb{U}_{F_\alpha^x}, i = 1, 2, \ldots, n$. Then for each i, there exists an open soft set (H_i, B_i) such that $F_\alpha^x \tilde{\in} (H_i B_i)\tilde{\subseteq}(G_i, A_i)$. Hence, $F_\alpha^x \in \tilde{\bigcap}_{i=1}^{n} (H_i, B_i)\tilde{\subseteq}\tilde{\bigcap}_{i=1}^{n} (G_i, A_i)$. Since, $\tilde{\bigcap}_{i=1}^{n} (H_i, B_i)$ is soft open, then by definition $\tilde{\bigcap}_{i=1}^{n} (G_i, A_i) \in \mathbb{U}_{F_\alpha^x}$;

iv) $\forall (G, A) \in \mathbb{U}_{F_\alpha^x}, \exists (H, B)$ soft open such $F_\alpha^x \tilde{\in} (H, B)\tilde{\subseteq}(G, A)$. Since (H,B) is soft open, then $(H, B) \in \mathbb{U}_{H_\beta^y}, \forall H_\beta^y \in (H, B)$. Also $(H, B)\tilde{\subseteq}(G, A) \Rightarrow (G, A) \in \mathbb{U}_{H_\beta^y}, \forall H_\beta^y \in (H, B)$.

Conversely, suppose given a collection $\mathbb{U}$ of members of $SS(X)_E$ and for each $F_\alpha^x \in \mathbb{U}$, there exist a nonempty family $\mathbb{U}_{F_\alpha^x}$ satisfying (i-iv). Let $\tau(\mathbb{U}) = \{(G, A) \in \mathbb{U} : (G, A) \in \mathbb{U}_{F_\alpha^x}\}$ together with $\tilde{\emptyset}$

i) By definition of $\tau(\mathbb{U}), \tilde{\emptyset} \in \tau(\mathbb{U})$ and $F_\alpha^x \in \tilde{X} \Rightarrow \tilde{X} \in \tau(\mathbb{U})$;

ii) Let $(G_i, A_i) \in \mathbb{U}_{F_\alpha^x}, i = 1, 2, \ldots, n$ be a family of members of $\tau(\mathbb{U})$, then by definition of $\tau(\mathbb{U})$, $(G_i, A_i) \in \mathbb{U}_{F_\alpha^x}$ whenever $F_\alpha^x \in (G_i, A_i)$. By (iii) $\tilde{\bigcap}_{i=1}^{n} (G_i, A_i) \in \mathbb{U}_{F_\alpha^x}$. Hence, $\tilde{\bigcap}_{i=1}^{n} (G_i, A_i) \in \tau(\mathbb{U})$;

iii) Let $\{(G_\lambda, A_\lambda) : \lambda \in \Lambda\}$ be a family of members of $\tau(\mathbb{U})$, then $\forall \lambda \in \Lambda, (G_\lambda, A_\lambda) \in \mathbb{U}_{F_\alpha^x}$ whenever $F_\alpha^x \in (G_\lambda, A_\lambda)$. Therefore, $F_\alpha^x \in (G_\lambda, A_\lambda)\tilde{\subseteq} \tilde{\bigcup}_{\lambda \in \Lambda} (G_\lambda, A_\lambda) \in \tau(\mathbb{U})$.

□

Definition 4.10. [7] Let X_E be a soft topological space and $(F, A) \in SS(X)_E$. The *soft closure* of (F, A) denoted as $\overline{(F, A)}$ or $cl(F, A)$ is defined to be the soft intersection over all super soft closed sets of (F, A). Thus $\overline{(F, A)} = cl(F, A) = \tilde{\bigcap}\{(G, B) : (F, A)\tilde{\subseteq}(G, B)$ and (G, B) is soft closed$\}$
Thus $\overline{(F, A)} = cl(F, A) = \tilde{\bigcap}_{\alpha \in \Delta} (F_\alpha, A_\alpha)$ such that (F_α, A_α) is a soft closed, $\forall \alpha \in \Delta$. Where the soft intersection is taken over all soft closed supersets (F_α, A_α) of (F, A).

We next introduce the following definition

Definition 4.11. A soft element F^x_α is said to be a *closure soft element* of the soft set (F, A) if $F^x_\alpha \tilde{\in} \overline{(F, A)}$, and a soft set (G, B) is said to be a soft closure subset of (F, A) if $(G, B) \tilde{\subseteq} \overline{(F, A)}$

Thus, we have the following lemma.

Lemma 4.12. Let X_E be a soft topological space and $(F, A) \in SS(X)_E$, then

i) $(F, A) \tilde{\subseteq} \overline{(F, A)}$

ii) $\overline{(F, A)}$ is the smallest soft closed superset of (F, A).

iii) $(F, A) = \overline{(F, A)}$ if and only if (F, A) is a soft closed.

PROOF. (i),(ii) and (iii) are trivially obvious from Definition 4.10 □

Lemma 4.13. A soft element F^x_α is a closure soft element of the soft set (F,A) if and only if given any soft open nbd (G,B) of F^x_α, $(G, B) \tilde{\cap} (F, A) \neq \tilde{\emptyset}$

PROOF. Let $F^x_\alpha \tilde{\in} \overline{(F, A)}$ and suppose by way of contradiction $G, B) \tilde{\cap} (F, A) = \tilde{\emptyset}$ for some soft open nbd (G, B) of F^x_α. This implies $(F, A) \tilde{\subseteq} (G^c, B)$ where (G^c, B) is soft closed superset of (F, A). Now $(F, A) \tilde{\subseteq} (G^c, B) \Rightarrow F^x_\alpha \tilde{\in} \overline{(F, A)} \tilde{\subseteq} (G^c, B) \Rightarrow F^x_\alpha \tilde{\in} G^c, B) \Rightarrow F^x_\alpha \tilde{\notin} (G, B)$. This is a contradiction. Conversely, suppose the condition holds and by way of contradiction $F^x_\alpha \tilde{\notin} \overline{(F, A)}$. This implies that $F^x_\alpha \tilde{\in} (\overline{(F, A)})^c$. Since $(\overline{(F, A)})^c$ is soft open, then by our hypothesis $(\overline{(F, A)})^c \tilde{\cap} (F, A) \neq \tilde{\emptyset}$. This is a contradiction, i.e., $(\overline{(F, A)})^c \tilde{\subseteq} (F^c, A) \Rightarrow [(\overline{(F, A)})^c \tilde{\cap} (F^c, A)] \tilde{\subseteq} (F^c, A) \tilde{\cap} (F, A) = \tilde{\emptyset} \Rightarrow (\overline{(F, A)})^c \tilde{\subseteq} (F^c, A) = \tilde{\emptyset}$. □

We give the following example to demonstrate and make the notions discussed os far clearer.

Example 4.14. Let $E = \{e_1, e_2, e_3, e_4\}$, $X = \{x_1, x_2, x_3, x_4, x_5\}$, $A_1 = E$, $A_2 = \{e_1, e_2, e_3\}$, $A_3 = \{e_1, e_2, e_4\}$, $A_4 = \{e_1, e_3, e_4\}$, $A_5 = \{e_2, e_3, e_4\}$, $A_6 = \{e_1, e_4\}$, $A_7 = \{e_2, e_3\}$, $A_8 = \{e_1, e_2\}$, $A_9 = \{e_3, e_4\}$, $A_{10} = \{e_1, e_3\}$, $A_{11} = \{e_2, e_4\}$, $A_{12} = \{e_1\}$, $A_{13} = \{e_2\}$, $A_{14} = \{e_3\}$, $A_{15} = \{e_4\}$ and $\tau = \{\tilde{\emptyset}, \tilde{X}, (F_i, E), (F_i, A_i) : i = 1, 2, \ldots, 15\}$. As indicated on the table below, all the soft closed sets are $\tilde{\emptyset}, \tilde{X}, (H_i, E), (H_i, A_i) : i = 1, 2, \ldots, 15$ where $(F^c_i, E) = (H_i, E)$ and $(F^c_i, A_i) = (H_i, A_i)$.

E	e_1	e_2	e_3	e_4	E	e_1	e_2	e_3	e_4
$\tilde{X}$	X	X	X	X	$\tilde{\emptyset}$	$\emptyset$	$\emptyset$	$\emptyset$	$\tilde{\emptyset}$
F_1	$\{x_1\}$	$\{x_3, x_4\}$	$\{x_1, x_3, x_4\}$	$\{x_2, x_3, x_4, x_5\}$	H_1	$\{x_2, x_3, x_4, x_5\}$	$\{x_1, x_2, x_5\}$	$\{x_2, x_5\}$	$\{x_1\}$
F_2	$\{x_1\}$	$\{x_3, x_4\}$	$\{x_1, x_3, x_4\}$	$\emptyset$	H_2	$\{x_2, x_3, x_4, x_5\}$	$\{x_1, x_2, x_5\}$	$\{x_2, x_5\}$	X
F_3	$\{x_1\}$	$\{x_3, x_4\}$	$\emptyset$	$\{x_2, x_3, x_4, x_5\}$	H_3	$\{x_2, x_3, x_4, x_5\}$	$\{x_1, x_2, x_5\}$	X	$\{x_1\}$
F_4	$\{x_1\}$	$\emptyset$	$\{x_1, x_3, x_4\}$	$\{x_2, x_3, x_4, x_5\}$	H_4	$\{x_2, x_3, x_4, x_5\}$	X	$\{x_2, x_5\}$	$\{x_1\}$
F_5	$\emptyset$	$\{x_3, x_4\}$	$\{x_1, x_3, x_4\}$	$\{x_2, x_3, x_4, x_5\}$	H_5	X	$\{x_1, x_2, x_5\}$	$\{x_2, x_5\}$	$\{x_1\}$
F_6	$\{x_1\}$	$\emptyset$	$\emptyset$	$\{x_2, x_3, x_4, x_5\}$	H_6	$\{x_2, x_3, x_4, x_5\}$	X	X	$\{x_1\}$
F_7	$\emptyset$	$\{x_3, x_4\}$	$\{x_1, x_3, x_4\}$	$\emptyset$	H_7	X	$\{x_1, x_2, x_5\}$	$\{x_2, x_5\}$	X
F_8	$\{x_1\}$	$\{x_3, x_4\}$	$\emptyset$	$\emptyset$	H_8	$\{x_2, x_3, x_4, x_5\}$	$\{x_1, x_2, x_5\}$	X	X
F_9	$\emptyset$	$\emptyset$	$\{x_1, x_3, x_4\}$	$\{x_2, x_3, x_4, x_5\}$	H_9	X	X	$\{x_2, x_5\}$	$\{x_1\}$
F_{10}	$\{x_1\}$	$\emptyset$	$\{x_1, x_3, x_4\}$	$\emptyset$	H_{10}	$\{x_2, x_3, x_4, x_5\}$	X	$\{x_2, x_5\}$	X
F_{11}	$\emptyset$	$\{x_3, x_4\}$	$\emptyset$	$\{x_2, x_3, x_4, x_5\}$	H_{11}	X	$\{x_1, x_2, x_5\}$	X	$\{x_1\}$
F_{12}	$\{x_1\}$	$\emptyset$	$\emptyset$	$\emptyset$	H_{12}	$\{x_2, x_3, x_4, x_5\}$	X	X	X
F_{13}	$\emptyset$	$\{x_3, x_4\}$	$\emptyset$	$\emptyset$	H_{13}	X	$\{x_1, x_2, x_5\}$	X	X
F_{14}	$\emptyset$	$\emptyset$	$\{x_1, x_3, x_4\}$	$\emptyset$	H_{14}	X	X	$\{x_2, x_5\}$	X
F_{15}	$\emptyset$	$\emptyset$	$\emptyset$	$\{x_2, x_3, x_4, x_5\}$	H_{15}	X	X	X	$\{x_1\}$
$\tilde{\emptyset}$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\tilde{X}$	X	X	X	X

i) If $(G_1, E) = \{(e_1, \{x_1, x_3, x_5\}), (e_2, \{x_1, x_2, x_3, x_4\}), (e_3, \emptyset), (e_4, X)\}$ and $(G_2, E) = \{(e_1, \emptyset), (e_2, \{x_3, x_4\}), (e_3, \emptyset), (e_4, \{x_3\})\}$ then (G_1, E) is a soft nbd of (G_2, E) i.e., $(G_2, E) \tilde{\subseteq} (F_{11}, E) \tilde{\subseteq} (G_1, E)$ where (F_{11}, E) is soft open;

ii) If $(G_3, E) = \{(e_1, \{x_1\}), (e_2, \{x_2\}), (e_3, \{x_5\}), (e_4, \{x_5\})\}$, then $\overline{(G_3, E)} = (H_7, E) \tilde{\cap} (H_{13}, E) \tilde{\cap} (H_{14}, E) \tilde{\cap} \tilde{X}$ $= \{(e_1, X\}), (e_2, \{x_1, x_2, x_5\}), (e_3, \{x_2, x_5\}), (e_4, X)\} = (H_7, E)$ which is soft closed;

iii) $F^{x_5}_{e_1}\tilde{\notin}(G_3,E)$ but $F^{x_5}_{e_1}$ is a closure soft element of (G_3,E) i.e., $F^{x_5}_{e_1}\tilde{\in}\overline{(G_3,E)}$.

Definition 4.15. [7] Let X_E be a soft topological space and $(F,A)\in SS(X)_E$. The *soft interior* of (F,A) denoted as $int(F,A)$ is defined to be the soft union of all soft open subsets of (F,A).
$int(F,A)=\tilde{\bigcup}\{(G,B):(G,B)\tilde{\subseteq}(F,A)$ and (G,B) is soft open $\}$. Thus $int(F,A)=\tilde{\bigcup\limits_{\alpha\in\Delta}}(F_\alpha,A_\alpha)$ such that (F_α,A_α) is soft open,$\forall\alpha\in\Delta$. Where the soft union is taken over all soft open subsets (F_α,A_α) of (F,A).

Definition 4.16. A soft element F^x_α is said to be an *interior soft element* of the soft set (F,A) if $F^x_\alpha\tilde{\in}int(F,A)$.

Lemma 4.17. Let X_E be a soft topological space and $(F,A)\in SS(X)_E$, then

i) $int(F,A)$ is soft open and $int(F,A)\tilde{\subseteq}(F,A)$.

ii) $int(F,A)=(F,A)$ if and only if (F,A) is soft open.

iii) For any soft open subset (G,B) of (F,A), $(G,B)\tilde{\subseteq}int(F,A)\tilde{\subseteq}(F,A)$ i.e., $int(F,A)$ is the largest soft open subset of (F,A).

Proof. (i),(ii) and (iii) are trivially obvious from Definition 4.15 □

Example 4.18. In example 4.14. If $(G_4,E)=\{(e_1,\{x_1,x_2\}),(e_2,\{x_4\}),(e_3,\emptyset),(e_4,\{x_1,x_2,x_3\})\}$. Then, $int(G_4,A_3)=\{(e_1,\{x_1\}),(e_2,\emptyset),(e_3,\emptyset),(e_4,\emptyset)\}=(F_{12},E)$, which is soft open.

Definition 4.19. Let X_E be a soft topological space and $(F,A)\in SS(X)_E$. The *soft exterior* of (F,A) denoted as $ext(F,A)$ is defined to be the soft union over all soft open sets disjoint from (F,A). That is $ext(F,A)=\tilde{\bigcup}\{(G,B):(G,B)$ is soft open and $(G,B)\tilde{\cap}(F,A)=\tilde{\emptyset}\}$.
Thus $ext(F,A)=\tilde{\bigcup\limits_{\alpha\in\Delta}}(F_\alpha,A_\alpha)$ such that (F_α,A_α), is soft open $\forall\alpha\in\Delta$ and the soft union is taken over all soft open sets (F_α,A_α) such that $(F_\alpha,A_\alpha)\tilde{\cap}(F,A)=\tilde{\emptyset}$.

Lemma 4.20. Let X_E be a soft topological space and $(F,A)\in SS(X)_E$, then

i) $ext(F,A)$ is soft open and $ext(F,A)\tilde{\cap}int(F,A)=\tilde{\emptyset}$.

ii) $ext(F,A)=int(F^c,A)\tilde{\subseteq}(F^c,A)$

iii) If (G,B) is soft open and $(G,B)\tilde{\cap}(F,A)=\tilde{\emptyset}$, then $(G,B)\tilde{\subseteq}ext(F,A)$ i.e. $ext(F,A)$ is the largest soft open set disjoint from (F,A).

Proof. (i),(ii) and (iii) are trivially obvious from Definition 2.4 and 4.19. □

Example 4.21. In Example 4.14. If $(G_4,E)=\{(e_1,\{x_1,x_2\}),(e_2,\{x_4\}),(e_3,\emptyset),(e_4,\{x_1,x_2,x_3\})\}$.
Then $ext(G_4,E)=\{(e_1,\emptyset),(e_2,\emptyset),(e_3,\{x_1,x_3,x_4\}),(e_4,\emptyset)\}=(F_{14},E)$ which is soft open.
By (i) of lemma 4.20 $ext(G_4,E)\tilde{\cap}int(G_4,E)=(F_{14},E)\tilde{\cap}(F_{12},E)=\tilde{\emptyset}$.

We further introduce the following definition.

Definition 4.22. Let X_E be a soft topological space. A soft element F^x_α in X_E is said to be a *boundary soft element* of $(F,A)\in SS(X)_E$ if $F^x_\alpha\tilde{\notin}int(F,A)$ and $F^x_\alpha\tilde{\notin}ext(F,A)$. The soft union over of all soft boundary elements of (F,A) is called the *soft boundary* of (F,A) which we denote as $Fr(F,A)$.

Now as consequence of Definitions 4.10, 4.15, ref5b and 4.22 we provide the following two lemmas.

Lemma 4.23. By Definition 4.22 $int(F,A)\tilde{\cup}Fr(F,A)\tilde{\cup}(F,A)=\tilde{X}$

Lemma 4.24. Let X_E be a soft topological space and $(F,A)\in SS(X)_E$, then

i) $(\overline{(F,A)})^c=ext(F,A)$;

ii) $\overline{(F,A)}=int(F,A)\tilde{\cup}Fr(F,A)$;

iii) $\overline{(F,A)}=(int[(F^c,A)])^c$;

iv) $Fr(F,A) = \overline{(F,A)}\tilde{\cap}\overline{(F^c,A)}$;

v) $Fr(F,A) = ((int(F,A)\tilde{\cup}ext(F,A))^c$.

PROOF.

i) $F^x_\alpha\tilde{\in}(\overline{(F,A)})^c \Leftrightarrow F^x_\alpha\tilde{\notin}\overline{(F,A)} \Leftrightarrow$ there exists soft open nbd (G,B) of F^x_α such that $(F,A)\tilde{\cap}(G,B) = \tilde{\emptyset} \Leftrightarrow F^x_\alpha\tilde{\in}ext(F,A)$. Lemma 4.13 and Definition 4.19;

ii) By Lemma 4.23 $ext(F,A) = (int(F,A)\tilde{\cup}Fr(F,A))^c$. By (i) $(\overline{(F,A)})^c = (int(F,A)\tilde{\cup}Fr(F,A))^c$. Hence, $\overline{(F,A)} = int(F,A)\tilde{\cup}Fr(F,A)$.

iii) $F^x_\alpha\tilde{\in}\overline{(F,A)} \Leftrightarrow F^x_\alpha\tilde{\in}(int(F,A)\tilde{\cup}Fr(F,A)) \Leftrightarrow F^x_\alpha\tilde{\in}int(F,A)$ or $F^x_\alpha\tilde{\in}Fr(F,A) \Leftrightarrow F^x_\alpha\tilde{\notin}ext(F,A) \Leftrightarrow F^x_\alpha\tilde{\notin}int(F^c,A) \Leftrightarrow F^x_\alpha\tilde{\in}(int[(F^c,A)])^c$

iv) $F^x_\alpha\tilde{\in}\overline{(F,A)}\tilde{\cap}\overline{(F^c,A)} \Leftrightarrow F^x_\alpha\tilde{\in}\overline{(F,A)}$ and $F^x_\alpha\tilde{\in}\overline{(F^c,A)}$
$\Leftrightarrow$ given any open nbd (G,B) of $F^x_\alpha, (G,B)\tilde{\cap}(F,A) \neq \tilde{\emptyset}$ and $(G,B)\tilde{\cap}(F^c,A) \neq \tilde{\emptyset}$
$\Leftrightarrow F^x_\alpha\tilde{\notin}int(F,A)$ and $F^x_\alpha\tilde{\notin}int(F^c,A) \Leftrightarrow F^x_\alpha\tilde{\notin}int(F,A)$ and $F^x_\alpha\tilde{\notin}ext(F,A) \Leftrightarrow F^x_\alpha\tilde{\in}Fr(F,A)$

v) $F^x_\alpha\tilde{\in}Fr(F,A) \Leftrightarrow F^x_\alpha\tilde{\notin}int(F,A)$ and $F^x_\alpha\tilde{\notin}ext(F,A) \Leftrightarrow F^x_\alpha\tilde{\notin}(int(F,A)\tilde{\cup}ext(F,A))$
$\Leftrightarrow F^x_\alpha\tilde{\in}(int(F,A)\tilde{\cup}ext(F,A))^c$

□

Remark 4.25. It is obvious from (iv) and (v) that, boundary of (F,A) is soft closed in X_E

Example 4.26. In Example 4.14. If $(G_4,E) = \{(e_1,\{x_1,x_2\}),(e_2,\{x_4\}),(e_3,\emptyset),(e_4,\{x_1,x_2,x_3\})\}$.

i) By Definition4.10, $\overline{(G_4,E)} = \tilde{X}\tilde{\cap}(H_{14},E) = (H_{14},E) = \{(e_1,X\}),(e_2,X),(e_3,\{x_2,x_5\}),(e_4,X)\}$ which is soft closed and $(G_4,E)\tilde{\subseteq}(H_{14},E)$.

ii) By (i) of Lemma 4.24, $(\overline{(G_4,E)})^c = \{(e_1,X),(e_2,X),(e_3,\{x_2,x_5\}),(e_4,X)\}^c = \{(e_1,\emptyset),(e_2,\emptyset),(e_3,\{x_1,x_3,x_4\}),(e_4,\emptyset)\} = ext(G_4,E)$

iii) By (ii) of Lemma 4.24, $\overline{(G_4,E)} = int(G_4,E)\tilde{\cup}Fr(G_4,E) = (F_{12},E)\tilde{\cup}(H_{10},E) = (H_{14},E)$.

iv) By (iii) of Lemma 4.24, $\overline{(G,E)} = (int[(G^c_4,E)])^c = (int[\{(e_1,\{x_3,x_4,x_5\}),(e_2,\{x_1,x_2,x_3,x_5\}),(e_3,X),(e_4,\{x_4,x_5\})\}])^c = (F_{14},E)^c = \{(e_1,X),(e_2,X),(e_3,\{x_2,x_5\}),(e_4,\{x_4,X)\} = (H_{14},E)$ which is soft closed.

v) By (iv) of Lemma 4.24, $Fr(G_4,E) = \overline{(G_4,E)}\tilde{\cap}\overline{(G^c_4,E)} = (H_{10},E)$ which is soft closed.

vi) By (v) Lemma of 4.24, $Fr(G_4,E) = ((int(G_4,E)\tilde{\cup}ext(G_4,E))^c = [(F_{12},E)\tilde{\cup}(F_{14},E)]^c = \{(e_1,\{x_2,x_3,x_4,x_5\}),(e_2,X),(e_3,\{x_2,x_5\}),(e_4,X)\} = (H_{10},E)$.

vii) By Definition4.10, The soft elements

$$F^{x_2}_{e_1},F^{x_3}_{e_1},F^{x_4}_{e_1},F^{x_5}_{e_1},F^{x_1}_{e_2},F^{x_2}_{e_2},F^{x_3}_{e_2},F^{x_4}_{e_2},F^{x_5}_{e_2},F^{x_2}_{e_3},F^{x_5}_{e_3},F^{x_1}_{e_4},F^{x_2}_{e_4},F^{x_3}_{e_4},F^{x_4}_{e_4}\tilde{\notin}Int(G_4,E)$$

and

$$F^{x_2}_{e_1},F^{x_3}_{e_1},F^{x_4}_{e_1},F^{x_5}_{e_1},F^{x_1}_{e_2},F^{x_2}_{e_2},F^{x_3}_{e_2},F^{x_4}_{e_2},F^{x_5}_{e_2},F^{x_2}_{e_3},F^{x_5}_{e_3},F^{x_1}_{e_4},F^{x_2}_{e_4},F^{x_3}_{e_4},F^{x_4}_{e_4}\tilde{\notin}ext(G_4,E)$$

$$\begin{aligned} Fr(G_4,E) &= F^{x_2}_{e_1}\tilde{\cup}F^{x_3}_{e_1}\tilde{\cup}F^{x_4}_{e_1}\tilde{\cup}F^{x_5}_{e_1}\tilde{\cup}F^{x_1}_{e_2}\tilde{\cup}F^{x_2}_{e_2}\tilde{\cup}F^{x_3}_{e_2}\tilde{\cup}F^{x_4}_{e_2}\tilde{\cup}F^{x_5}_{e_2}\tilde{\cup}F^{x_2}_{e_3}\tilde{\cup}F^{x_5}_{e_3}\tilde{\cup}F^{x_1}_{e_4}\tilde{\cup}F^{x_2}_{e_4}\tilde{\cup}F^{x_3}_{e_4}\tilde{\cup}F^{x_4}_{e_4} \\ &= \{(e_1,\{x_2,x_3,x_4,x_5\}),(e_2,X),(e_3,\{x_2,x_5\}),(e_4,X)\} = (H_{10},E) \end{aligned}$$

The following definition which describe the derived soft set of a soft set is found in [19] and is given as

Definition 4.27. [5] A soft element F^x_α in X_E is said to be a limiting soft element of (F,A) if given any soft open nbd (G,A) of $F^x_\alpha, ((G,B) \setminus F^x_\alpha)\tilde{\cap}(F,A) \neq \tilde{\emptyset}$. The set of all limiting soft elements of (F,A) denoted as $(F,A)'$ is called the derived soft set of (F,A).

Thus, we give the following lemmas and their proofs.

Lemma 4.28. F_α^x is a limiting soft element of $(F, A) \in SS(X)_E$ if and only if $F_\alpha^x \tilde{\in} \overline{((F, A) \setminus F_\alpha^x)}$.

PROOF. $F_\alpha^x \tilde{\in} (F, A)' \Leftrightarrow \forall$soft open nbd (G,B) of F_α^x, $((G, B) \setminus F_\alpha^x))\tilde{\cap}(F, A) \neq \tilde{\emptyset}$
$\Leftrightarrow \forall$soft open nbd (G,B) of F_α^x, $(G, B)\tilde{\cap}((F, A) \setminus F_\alpha^x)) \neq \tilde{\emptyset}$. Applying Lemma 4.13 $F_\alpha^x \tilde{\in} \overline{((F, A) \setminus F_\alpha^x))}$. □

Lemma 4.29. If $\{(F_\delta, A_\delta) : \delta \in \Delta\}$ is a family of members of $SS(X)_E$, then

$$\tilde{\bigcup_{\delta \in \Delta}} (F_\delta, A_\delta)' \tilde{\subseteq} (\tilde{\bigcup_{\delta \in \Delta}} (F_\delta, A_\delta))'$$

PROOF. $F_\alpha^x \tilde{\in} \tilde{\bigcup}_{\delta \in \Delta} (F_\delta, A_\delta)' \Rightarrow$ for some $\delta_o \in \Delta, F_\alpha^x \tilde{\in} (F_{\delta_o}, A_{\delta_o})'$
$\Rightarrow$ given any soft open nbd (G, B) of F_α^x, $((G, B) \setminus F_\alpha^x)\tilde{\cap}(F_{\delta_o}, A_{\delta_o}) \neq \tilde{\emptyset}$
$\Rightarrow$ given any soft open nbd (G, B) of F_α^x, $((G, B) \setminus F_\alpha^x)\tilde{\cap}[\tilde{\bigcup}_{\delta \in \Delta} (F_\delta, A_\delta)] \neq \tilde{\emptyset} \Rightarrow F_\alpha^x \tilde{\in} (\tilde{\bigcup}_{\delta \in \Delta} (F_\delta, A_\delta))'$ □

Lemma 4.30. $\tilde{\bigcup}_{\delta \in \Delta} \overline{(F_\delta, A_\delta)} \tilde{\subseteq} \overline{(\tilde{\bigcup}_{\delta \in \Delta} (F_\delta, A_\delta))}$ for any family $\{(F_\delta, A_\delta) : \delta \in \Delta\}$ of $SS(X)_E$.

PROOF. $F_\alpha^x \tilde{\in} \tilde{\bigcup}_{\delta \in \Delta} \overline{(F_\delta, A_\delta)} \Rightarrow$ for some $\delta_o \in \Delta, F_\alpha^x \tilde{\in} \overline{(F_{\delta_o}, A_{\delta_o})} \Rightarrow$ given any soft open nbd (G,B) of F_α^x, $(G, B)\tilde{\cap}(F_{\delta_o}, A_{\delta_o}) \neq \tilde{\emptyset} \Rightarrow$ given any soft open nbd (G,B) of

$$F_\alpha^x, (G, B)\tilde{\cap}[\tilde{\bigcup_{\delta \in \Delta}} (F_\delta, A_\delta)] \neq \tilde{\emptyset} \Rightarrow F_\alpha^x \tilde{\in} \overline{(\tilde{\bigcup_{\delta \in \Delta}} (F_\delta, A_\delta))}$$

□

Lemma 4.31. F_α^x is a boundary soft element of (F, A) if and only if given soft open nbd $(G.B)$ of F_α^x, $(G, B)\tilde{\cap}(F, A) \neq \tilde{\emptyset}$. and $(G, B)\tilde{\cap}(F^c, A) \neq \tilde{\emptyset}$.

PROOF. $F_\alpha^x \tilde{\in} Fr(F, A) \Leftrightarrow F_\alpha^x \tilde{\in} \overline{(F, A)}\tilde{\cap}\overline{(F^c, A)} \Leftrightarrow F_\alpha^x \tilde{\in} \overline{(F, A)}$ and $F_\alpha^x \tilde{\in} \overline{(F^c, A)}$. Applying Lemma 4.13 given any soft open nbd $(G.B)$ of F_α^x, $(G, B)\tilde{\cap}(F, A) \neq \tilde{\emptyset}$ and $(G, B)\tilde{\cap}(F^c, A) \neq \tilde{\emptyset}$. □

Definition 4.32. [29] Let X_E be a soft topological space and $(F, A), (G, B) \in SS(X)_E$, then (F, A) is said to be *soft dense* in (G, B) if $(F, A)\tilde{\subseteq}(G, B)$ and $(G, B)\tilde{\subseteq}\overline{(F, A)}$. A soft set $(F, A) \in SS(X)_E$ is said to be soft dense in X_E if $\overline{(F, A)} = \tilde{X}$.
By (i) of Example 4.26 (G_4, E) is soft dense in (H_{14}, E).

We now introduce the following definitions.

Definition 4.33. Let X_E be a soft topological space, $(F, A) \in SS(X)_E$ is said to be

i) *boundary soft set* in X_E if $int(F, A) = \tilde{\emptyset}$;

ii) *nowhere soft dense* in X_E if $int\overline{(F, A)} = \tilde{\emptyset}$;

iii) *relatively soft discrete* in X_E if for every soft element F_α^x in (F,A) there exist a soft nbd (G, B) of F_α^x such that $(G, B)\tilde{\cap}(F, A) = F_\alpha^x$;

iv) *a soft closed domain (or regularly soft closed)* if $(F, A) = \overline{int(F, A)}$;

v) a soft open domain (or regularly soft open) if $(F, A) = int\overline{(F, A)}$;

vi) soft perfect if $\overline{(F, A)} = (F, A) = (F, A)'$.

As a consequence of definitions 4.32 and 4.33, we provide and prove the following proposition.

Proposition 4.34. Let X_E be a soft topological space and $(F, A) \in SS(X)_E$, then the following are equivalent

i) (F, A) is soft dense in X_E.

ii) $\tilde{X}$ is the only soft closed superset of (F, A).

iii) for every non empty soft open set (G, B), $(G, B)\tilde{\cap}(F, A) \neq \tilde{\emptyset}$.

iv) (F^c, A) is a boundary soft set.

Proof.

i) (i $\Rightarrow$ ii) (F, A) is soft dense in X_E and (G, B) is soft closed superset of (F, A), implies $\tilde{X} = \overline{(F, A)}\tilde{\subseteq}(G, B)\tilde{\subseteq}\tilde{X} = \overline{(F, A)} \Rightarrow (G, B) = \tilde{X}$

ii) (ii$\Rightarrow$iii) Suppose (ii) holds and suppose by way of contradiction there exist a soft open set $(G, B) \neq \tilde{\emptyset}$ such that $(G, B)\tilde{\cap}(F, A) = \tilde{\emptyset}$. This implies $(F, A)\tilde{\subseteq}(G^c, B)$ where (G^c, B) is soft closed. Therefore, by our hypothesis we have $(G^c, B) = \tilde{X}$, and this implies $(G, B) = \tilde{\emptyset}$. This is a contradiction.

iii) (iii $\Rightarrow$ iv) Suppose (iii) holds and suppose by way of contradiction $int(F^c, A) \neq \tilde{\emptyset}$. By our hypothesis, $int(F^c, A)\tilde{\cap}(F, A) \neq \tilde{\emptyset}$. But $int(F^c, A)\tilde{\subseteq}(F^c, A)$ and $(F^c, A)\tilde{\cap}(F, A) = \tilde{\emptyset}$ implies $int(F^c, A)\tilde{\cap}(F, A) = \tilde{\emptyset}$. This is a contradiction.

iv) (iv $\Rightarrow$ i) $int(F^c, A) = \tilde{\emptyset} \Rightarrow$ every non null soft open set contains a soft element of (F, A). Hence, given any soft element F^x_α, if (G, B) is soft open nbd of F^x_α, then $(G, B)\tilde{\cap}(F, A) \neq \tilde{\emptyset}$. This implies $\forall F^x_\alpha \tilde{\in} \tilde{X}, F^x_\alpha \tilde{\in} \overline{(F, A)} \Rightarrow \tilde{X}\tilde{\subseteq}\overline{(F, A)}$. Since $\overline{(F, A)}\tilde{\subseteq}\tilde{X}$, then $\overline{(F, A)} = \tilde{X}$. Therefore, (F, A) is soft dense in X_E.

□

4.1. $F_\sigma-$ Soft Set , $G_\sigma-$ Soft Set, $\sigma-$ Soft Locally Finite and σ- soft Discrete Collections

In this section we introduce the concepts of $F_\sigma-$ Soft Set , $G_\sigma-$ Soft Set, $\sigma-$ Soft Locally Finite and σ- soft Discrete Collections and prove some important results. We first introduce the following definitions.

Definition 4.35. : Let X_E be a soft topological space. The soft union of a countable number of soft closed sets is called an $F_\sigma-$ soft set and the soft intersection of a countable number of soft open sets is called a $G_\sigma-$ soft set. The soft compliment of an $F_\sigma-$ soft set is a $G_\sigma-$ soft set and conversely. The soft intersection of two $F_\sigma-$ soft sets is an $F_\sigma-$ soft set. Thus, if $(F, A) = \tilde{\bigcup}_{i=1}^{\infty}(F_i, A_i)$ and $(G, B) = \tilde{\bigcup}_{i=1}^{\infty}(G_i, B_i)$ where (F_i, A_i) and (G_i, B_i) are soft closed, then evidently the soft intersection of (F, A) and (G, B) is given to be $(F, A)\tilde{\cap}(G, B) = (\tilde{\bigcup}_{i=1}^{\infty}(F_i, A_i))\tilde{\cap}(\tilde{\bigcup}_{i=1}^{\infty}(G_i, B_i)) = \tilde{\bigcup}_{i=1}^{\infty}[(F_i, A_i\tilde{\cap}(G_i, B_i)]$, thus $(F, A)\tilde{\cap}(G, B)$ is an $F_\sigma-$ soft set. Similarly the soft union of two $G_\sigma-$ soft sets is a $G_\sigma-$ soft set, thus if $(F, A) = \tilde{\bigcap}_{i=1}^{\infty}(F_i, A_i)$ and $(G, B) = \tilde{\bigcap}_{i=1}^{\infty}(G_i, B_i)$ where (F_i, A_i) and (G_i, B_i) are soft open, then $(F, A)\tilde{\cup}(G, B) = (\tilde{\bigcap}_{i=1}^{\infty}(F_i, A_i))\tilde{\cup}(\tilde{\bigcap}_{i=1}^{\infty}(G_i, B_i)) = \tilde{\bigcap}_{i=1}^{\infty}[(F_i, A_i)\tilde{\cup}(G_i, B_i)]$ thus $(F, A)\tilde{\cup}(G, B)$ is a $G_\sigma-$ soft set. The soft union of a countable number of $F_\sigma-$ soft sets is an F_σ soft set and the soft intersection of a countable number of $(G_\sigma-)$ soft sets is a $(G_\sigma-$ soft set.

Definition 4.36. [30] A collection $\mathcal{F} = \{(F_\lambda, A_\lambda) : \lambda \in \Lambda\}$ of members of $SS(X)_E$ in a soft topological X_E is said to be soft *locally finite* if and only f for every soft element F^x_α in X_E there exists a soft open nbd (F, A) of F^x_α such that (F, A) intersects only finitely many members of $\mathcal{F}$.

Definition 4.37. A collection $\mathcal{F} = \{(F_\lambda, A_\lambda) : \lambda \in \Lambda\}$ of members of $SS(X)_E$ in a soft topological X_E is said to be soft:-

(i) *soft discrete* if and only if for every soft element F^x_α in X_E there exists a soft open nbd (F, A) of F^x_α such that (F, A) intersects at most one member of $\mathcal{F}$;

(ii) $\sigma-$ *soft locally finite* (σ- *soft discrete*) if and only if $\mathcal{F} = \tilde{\bigcup}_{i=1}^{\infty} \mathcal{F}_i$ with each $\mathcal{F}_i$ soft locally finite (soft discrete) collection;

(iii) *soft point-finite* if and only iff every soft element F_α^x in X_E is contained only in finitely many members of $\mathcal{F}$;

(v) *soft closure preserving* if and only if every subcollection $\mathcal{B}$ of $\mathcal{F}$ is soft closure preserving. i.e., $\overline{\tilde{\bigcup}\{B; B \in \mathcal{B}\}} = \tilde{\bigcup}\{\overline{B} : B \in \mathcal{B}\}$;

(vi) σ- *soft closure preserving* if it is the soft union of a sequence of soft closure preserving subcollection.

We now give and prove the following lemmas.

Lemma 4.38. If $\mathcal{F} = \{(F_\delta, A_\delta) : \delta \in \Delta\}$ is a soft locally finite (soft discrete) collection of members of $SS(X)_E$, then $\{\overline{(F_\delta, A_\delta)} : \delta \in \Delta\}$ is soft locally finite (soft discrete)

Proof. Pick a soft element F_α^x and soft open nbd (G, B) of F_α^x such that $(G, B)\tilde{\cap}(F_\delta, A_\delta) = \tilde{\emptyset}$ except for finitely (discretely) many δ. Then $(G, B)\tilde{\cap}(F_\delta, A_\delta) = \tilde{\emptyset}$ except for these same δ. □

Lemma 4.39. If $\mathcal{F} = \{(F_\delta, A_\delta) : \delta \in \Delta\}$ is a soft locally finite collection of members of $SS(X)_E$, then $\tilde{\bigcup}\overline{(F_\delta, A_\delta)} = \overline{\tilde{\bigcup}(F_\delta, A_\delta)}$. In particular, the union of a soft locally finite collection of soft closed sets is soft closed.

Proof. $\tilde{\bigcup}\overline{(F_\delta, A_\delta)} \tilde{\subseteq} \overline{\tilde{\bigcup}(F_\delta, A_\delta)}$ follows from Lemma 4.30. Suppose $F_\alpha^x \tilde{\in} \overline{\tilde{\bigcup}(F_\delta, A_\delta)}$. Now some soft nbd (G, B) of F_α^x meets only finitely many members of $\mathcal{F}$, say $(F_{\delta_1}, A_{\delta_1}), (F_{\delta_2}, A_{\delta_2}) \ldots, (F_{\delta_n}, A_{\delta_n})$. Since every soft nbd of F_α^x meets $\tilde{\bigcup}(F_\delta, A_\delta)$, then every soft nbd of F_α^x must also meet $\tilde{\bigcup}_{i=1}^{n}(F_{\delta_i}, A_{\delta_i})$. Therefore, it follows that $F_\alpha^x \tilde{\in} \overline{(F_{\delta_1}, A_{\delta_1})\tilde{\cup}(F_{\delta_2}, A_{\delta_2})\tilde{\cup} \ldots \tilde{\cup}(F_{\delta_n}, A_{\delta_n})} = \tilde{\bigcup}_{i=1}^{n} \overline{(F_{\delta_i}, A_{\delta_i})}$ so that, for some k, $F_\alpha^x \tilde{\in} \overline{(F_{\delta_k}, A_{\delta_k})}$. Thus $\overline{\tilde{\bigcup}(F_\delta, A_\delta)} \tilde{\subseteq} \tilde{\bigcup}\overline{(F_\delta, A_\delta)}$, and the result follows. □

Lemma 4.40. Let $\mathcal{F} = \{(F_\delta, A_\delta) : \delta \in \delta\}$ be a soft locally finite collection of members of $SS(X)_E$ and $F = \bigcup_{\delta \in \delta}(F_\delta, A_\delta)$.

i) If all sets of the family $\mathcal{F}$ are soft closed, then F is soft closed;

ii) If the family $\mathcal{F}$ consists of soft clopen sets, then F is soft clopen.

Proof. i) If all the family of $\mathcal{F}$ are soft closed, then $(F_\delta, A_\delta) = \overline{(F_\delta, A_\delta)}, \forall \delta \in \delta$, Hence, by Lemma 4.39, $F = \tilde{\bigcup}_{\delta \in \Delta}(F_\delta, A_\delta) = \tilde{\bigcup}_{\delta \in \Delta}\overline{(F_\delta, A_\delta)} = \overline{\tilde{\bigcup}_{\delta \in \Delta}(F_\delta, A_\delta)}$ is soft closed;

ii) If the the family $\mathcal{F}$ consist of soft clopen sets, then (F_δ, A_δ) is soft closed $\forall \delta \in \delta$ and (F_δ, A_δ) is soft open $\forall \delta \in \delta$. Since, $F = \tilde{\bigcup}_{\delta \in \Delta}(F_\delta, A_\delta)$, then F is soft open (i.e. union of any number of soft open sets is soft open) and by (i) F is soft closed. Hence F is soft clopen.

□

5. Conclusion

In this paper, we have extended the notions of operation on soft sets to arbitrary collection of soft sets and introduced the concepts of $F_\sigma-$ soft Set and $G_\sigma-$ soft Set. Using examples, we have discussed basic notions of soft topology and showed some important results. We have further introduced some terms relative to arbitrary collection of soft sets in a soft topological space and studied some of their properties.

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Rough Hesitant Fuzzy Groups

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Abstract — In 2010, Torra introduced the notion of a hesitant fuzzy set, which is a generalization of Zadeh's fuzzy set. In the paper, we define two rough operators on hesitant fuzzy group by means of a normal hesitant fuzzy subgroup, and investigate some of their properties.

Keywords — *Rough set, hesitant fuzzy set, group*

1. Introduction

In 1965, Zadeh proposed the pioneering work of fuzzy subsets of a set [1], and in 1971, Rosenfeld introduced the notion of fuzzy subgroups of a group [2] which led to the fuzzification of algebraic structures. In 1982, Pawlak initiated the rough set theory to study incomplete and insufficient information [3].

Dubois, Prade first investigated fuzzy rough set and rough fuzzy set in [4], which attracting many scholars attentions. From the view of the theory of groups, Davvaz, Kuroki, Biswas, Kuroki, Yaqoob, Chen etc studied the notions of fuzzy groups, fuzzy subgroups, rough groups, rough subgroups, rough fuzzy groups, rough fuzzy subgroups in [5–11].

On the other hand, Torra [12] introduced the notion of a hesitant fuzzy set. After that time, Pei, Thakur et al. investigated some operators on hesitant fuzzy sets [13, 14]. Divakaran, John, et al. studied hesitant fuzzy rough sets, hesitant fuzzy groups [15–18]. Jun and Ahn applied hesitant fuzzy sets to BCK/BCI-algebras [19]. For more references,see [20–27].

In [28], Wang and Chen investigated the theory of rough subgroups by means of a normal subgroup, and obtained some interesting results. In [6], we investigated two rough operators on L-groups. As a generalization of [6, 9, 28], in the paper, we define the notion of rough hesitant fuzzy group, and investigate some of their properties.

The above contents are arranged into three parts, Section 3: Hesitant fuzzy group, and Section 4: Rough hesitant fuzzy group. In Section 2, we give an overview of hesitant fuzzy sets, group, rough sets, which surveys Preliminaries.

2. Preliminaries

In the section, we introduce some main notions for each area, i.e., hesitant fuzzy sets [12–14], groups, rough sets [3, 29, 30].

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2.1. Hesitant Fuzzy Sets

The seminal paper on fuzzy sets is [1]. As a generalization, the notion of a hesitant fuzzy set was introduced in [12].

Definition 2.1. Suppose X is a reference set, and $P[0,1]$ the power set of $[0,1]$, then a mapping $h : X \to P[0,1]$ is called a hesitant fuzzy set on X.

For instance: $h^0 : X \to P[0,1]$, $h^1 : X \to P[0,1]$ are defined as: for all $x \in X$, $h^0(x) = \emptyset, h^1(x) = [0,1]$, respectively.

We use the symbol $HF(X)$ to denote the set of all hesitant fuzzy sets in X. For $h_1, h_2 \in HF(X)$, $h_1 \preceq h_2$ is defined: if $\forall x \in X$, we have $h_1(x) \subseteq h_2(x)$, and $h_1 \approx h_2$, if $h_1 \preceq h_2$, $h_2 \preceq h_1$.

Definition 2.2. Suppose $h_1, h_2 \in HF(X)$, then $h_1 \tilde{\cap} h_2$ and $h_1 \tilde{\cup} h_2$ are defined as follows $(h_1 \tilde{\cap} h_2)(x) = h_1(x) \cap h_2(x)$, $(h_1 \tilde{\cup} h_2)(x) = h_1(x) \cup h_2(x)$ for every $x \in X$.

In special, a hesitant fuzzy point x_λ is defined by

$$x_\lambda(y) = \begin{cases} \lambda \subseteq [0,1] & \text{if } y = x \\ \emptyset & \text{if } y \neq x \end{cases}$$

The collection of all hesitant fuzzy points is denoted by M. For more details, see [17, 31].

2.2. Rough Sets

Pawlak proposed the rough set theory in [3]. Let (X, R) be an approximation space, and $R \subseteq X \times X$ be an equivalence relation, then for $A \subseteq X$, two subsets $\underline{R}(A)$ and $\overline{R}(A)$ of X are defined:

$$\underline{R}(A) = \{x \in X \mid [x]_R \subseteq A\}, \quad \overline{R}(A) = \{x \in X \mid [x]_R \cap A \neq \emptyset\}$$

where $[x]_R = \{y \in X \mid xRy\}$.

If $\underline{R}(A) = \overline{R}(A)$, A is called a definable set; if $\underline{R}(A) \neq \overline{R}(A)$, A is called an undefinable set, and $(\underline{R}(A), \overline{R}(A))$ is referred to as a pair of rough set. Therefore, $\underline{R}$ and $\overline{R}$ are called two rough operators.

Furthermore, as generalizations, they also were defined by an arbitrary binary relation in [30, 32], a mapping in [29], and other methods. Dubois, Prade investigated fuzzy rough set and rough fuzzy set in [4].

2.3. Group

We assume familiarity with the notion of a group as used in the intuitive set theory. Suppose G is a multiplicative group with an identity e , and A is a subgroup of G, if $\forall x, y \in A$, we have $xy \in A$.

N is a normal subgroup of G, if $\forall x \in G$, and $y \in N$, we have $xyx^{-1} \in N$.

3. Hesitant Fuzzy Group

Suppose G is a group with an identity e, the main notions and propositions of the section are from [17].

Definition 3.1. $h : G \to P[0,1]$ is called a hesitant fuzzy subgroup of G, if for every $x, y \in G$, we have $h(x) \cap h(y) \subseteq h(xy)$, and $h(x) \subseteq h(x^{-1})$.

Example 3.2. Suppose $G = \{e, x, y, z\}$ with the operator as the following table,

$\cdot$	e	x	y	z
e	e	x	y	z
x	x	e	z	y
y	y	z	e	x
z	z	y	x	e

Then $h_1 = \{e_\lambda, x_\mu, y_\mu, z_\mu\}$ is a hesitant fuzzy subgroup of G, where $\lambda \subseteq [0,1], \mu \subseteq [0,1]$, and $\mu \subseteq \lambda$. For example, we choose $\lambda = [0.3, 0.8], \mu = [0.4, 0.6]$, $h_1 = \{e_{[0.3,0.8]}, x_{[0.4,0.6]}, y_{[0.4,0.6]}, z_{[0.4,0.6]}\}$.

Let $h_2(e) = [0,1]$, $h_2(x) = \{\frac{1}{5}, \frac{1}{4}, \frac{1}{2}\}$, $h_2(y) = \{\frac{1}{7}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}\}$, $h_2(z) = \{\frac{1}{10}, \frac{1}{4}, \frac{1}{2}\}$, then h_2 is also a hesitant fuzzy subgroup of G.

In [17], Propositions 3.3, 3.4, 3.5 hold.

Proposition 3.3. h is a hesitant fuzzy subgroup of G if and only if $h(x^{-1}y) \supseteq h(x^{-1}) \cap h(y)$ for all $x, y \in G$.

Proposition 3.4. Suppose h is a hesitant fuzzy subgroup of G, then for all $x \in G$

(1) $h(e) \supseteq h(x)$

(2) $h(x) = h(x^{-1})$

(3) $h(x^n) \supseteq h(x)$

Proposition 3.5. Suppose h_1, h_2 are two hesitant fuzzy subgroups of G, then $h_1 \tilde{\cap} h_2$ is also a hesitant fuzzy subgroup of G.

Definition 3.6. g is called a normal hesitant fuzzy subgroup of G, if for every $x, y \in G$, we have $g(y) \subseteq g(xyx^{-1})$.

Cleraly, $h_3(e) = \{1, \frac{1}{3}, \frac{5}{7}\}$, $h_3(x) = \{\frac{1}{3}, \frac{5}{7}\}, h_3(y) = \emptyset, h_3(z) = \emptyset$ is a normal hesitant fuzzy subgroup of G.

In [17], Propositions 3.7, 3.8 hold.

Proposition 3.7. Suppose g is a hesitant fuzzy subgroup of G, then the following conditions are equivalence.

(1) g is normal.

(2) $g(xy) = g(yx)$, for all $x, y \in G$

(3) $g(xyx^{-1}) = g(y)$, for all $x, y \in G$

Proposition 3.8. Suppose g_1, g_2 are two normal hesitant fuzzy subgroups of G, then $g_1 \tilde{\cap} g_2$ is also a normal hesitant fuzzy subgroup of G.

In the classical case, for two subsets A, B of G, $AB = \{z \mid z = xy, x \in A, y \in B\}$, as a generalization, we give the following definition.

Definition 3.9. For h_1, h_2 two hesitant fuzzy subgroups of G, we define $h_1 h_2$, for every $z \in G$,

$$(h_1 h_2)(z) = \bigcup_{z=xy} h_1(x) \cap h_2(y)$$

In special, $(x_\lambda h)(w) = \bigcup_{w=st} \{x_\lambda\}(s) \cap h(t) = \bigcup_{w=xt} \lambda \cap h(t) = \lambda \cap h(x^{-1}w)$.
$x_\lambda y_\mu = z_\nu$, where $z = xy, \nu = \lambda \cap \mu$.

Example 3.10. Following Example 3.2, clearly $h_4 = \{e_{[0.2,0.8]}, y_{[0.5,0.7]}\}$ is also a hesitant fuzzy subgroup of G. Then $h_1 h_4 = \{e_{[0.2,0.8]}, x_{[0.4,0.7]}, y_{[0.4,0.7]}, z_{[0.4,0.7]}\}$.

4. Rough Hesitant Fuzzy Group

In the section, we introduce the notion of a rough hesitant fuzzy group defined by a normal hesitant fuzzy subgroup, and investigate some of their properties.

First, we give the notion of a rough hesitant fuzzy group.

Definition 4.1. Suppose N is a hesitant fuzzy normal subgroup of G, for every hesitant fuzzy subset h of G, we define $N^-(h)$ and $N_-(h)$, for every $x \in G$,

$$\begin{aligned} N^-(h)(x) &= \bigcup_{x_\lambda \in M} \{\lambda \mid \bigcup_{z \in G} (x_\lambda N)(z) \cap h(z) \neq \emptyset\} \\ &= \bigcup_{x_\lambda \in M} \{\lambda \mid \bigcup_{z \in G} \lambda \cap N(x^{-1}z) \cap h(z) \neq \emptyset\}, \\ N_-(h)(x) &= \bigcup_{x_\lambda \in M} \{\lambda \mid \bigcap_{z \in G} (x_\lambda N)(z) \subseteq h(z)\} \\ &= \bigcup_{x_\lambda \in M} \{\lambda \mid \bigcap_{z \in G} \lambda \cap N(x^{-1}z) \subseteq h(z)\} \end{aligned}$$

where $M=\{x_\lambda \mid x\in G, \lambda\subseteq[0,1]\}$ of all hesitant fuzzy singletons.

Then $N^-(h), N_-(h)$ are called the upper approximation, the lower approximation of h with respect to the hesitant fuzzy normal subgroup N, respectively.

Example 4.2. $N=h_3$ be a normal hesitant fuzzy subgroup of G, then for h_2, we have

$$N^-(h_2)(e)=\bigcup_{e_\lambda\in M}\{\lambda \mid \bigcup_{w\in G}\lambda\cap N(e^{-1}w)\cap h_2(w)\neq\emptyset\}=[0,1],$$
$$N^-(h_2)(x)=\bigcup_{x_\lambda\in M}\{\lambda \mid \bigcup_{w\in G}\lambda\cap N(x^{-1}w)\cap h_2(w)\neq\emptyset\}=[0,1],$$
$$N^-(h_2)(y)=\bigcup_{y_\lambda\in M}\{\lambda \mid \bigcup_{w\in G}\lambda\cap N(y^{-1}w)\cap h_2(w)\neq\emptyset\}=[0,1],$$
$$N^-(h_2)(z)=\bigcup_{z_\lambda\in M}\{\lambda \mid \bigcup_{w\in G}\lambda\cap N(z^{-1}w)\cap h_2(w)\neq\emptyset\}=[0,1].$$

and

$$N_-(h_2)(e)=\bigcup_{e_\lambda\in M}\{\lambda \mid \bigcap_{w\in G}\lambda\cap N(e^{-1}w)\subseteq h_2(w)\}=[0,1]-\{\tfrac{1}{3},\tfrac{5}{7}\},$$
$$N_-(h_2)(x)=\bigcup_{x_\lambda\in M}\{\lambda \mid \bigcap_{w\in G}\lambda\cap N(x^{-1}w)\subseteq h_2(w)\}=[0,1]-\{\tfrac{1}{3},\tfrac{5}{7}\},$$
$$N_-(h_2)(y)=\bigcup_{y_\lambda\in M}\{\lambda \mid \bigcap_{w\in G}\lambda\cap N(y^{-1}w)\subseteq h_2(w)\}=[0,1)-\{\tfrac{1}{3},\tfrac{5}{7}\},$$
$$N_-(h_2)(z)=\bigcup_{z_\lambda\in M}\{\lambda \mid \bigcap_{w\in G}\lambda\cap N(z^{-1}w)\subseteq h_2(w)\}=[0,1)-\{\tfrac{1}{3},\tfrac{5}{7}\}.$$

Where $A-B$ denotes the difference set.

Next, we discuss the following properties.

Proposition 4.3. Suppose N is a normal hesitant fuzzy subgroup of G, and $h\in HF(G)$, we have

(1) $N_-(h)\preceq h$

(2) $N^-(h)\succeq Nh$

(3) $N_-(h^1)\approx h^1$

(4) $N^-(h^0)\approx h^0$

Proof. (1) For every $w\in G$, we obtain $h(w)\cap h(w^{-1}w)\subseteq h(w)$; but for $z\in G$, $z\neq w$, $h(w)\cap g(w^{-1}z)\subseteq h(z)$ may be not holds.

$$\begin{aligned}N_-(h)(w)&=\bigcup_{x_\lambda\in M}\{\lambda \mid \bigcap_{z\in G}(x_\lambda N)(z)\subseteq h(z)\}\\&=\bigcup_{x_\lambda\in M}\{\lambda \mid \bigcap_{z\in G}\lambda\cap N(w^{-1}z)\subseteq h(z)\}\\&\subseteq\bigcup\{h(w)\mid h(w)\cap N(w^{-1}w)\subseteq h(w)\}\\&=h(w)\end{aligned}$$

By the above proof, we have $N_-(h)\preceq h$.

(2) For every $w\in G$, if $(Nh)(w)\neq\emptyset$, we have

$$\begin{aligned}N^-(h)(w)&=\bigcup_{w_\lambda\in M}\{\lambda \mid \bigcup_{z\in G}\lambda\cap N(w^{-1}z)\cap h(z)\neq\emptyset\}\\&=\bigcup_{w_\lambda\in M}\{\lambda \mid \lambda\cap[\bigcup_{z\in G}N(w^{-1}z)\cap h(z)]\neq\emptyset\}\\&=\bigcup_{w_\lambda\in M}\{\lambda \mid \lambda\cap(Nh)(zw^{-1}z)\neq\emptyset\}\\&\supseteq\bigcup\{(Nh)(w)\mid(Nh(w)\cap(Nh)(w)\neq\emptyset\}\\&\qquad\text{(Note: }\lambda=(Nh)(w),z=w)\\&=(Nh)(w)\end{aligned}$$

(3)and (4) are clearly. □

Proposition 4.4. Suppose $h_1, h_2 \in HF(G)$, and $h_1 \preceq h_2$, N is a normal hesitant fuzzy subgroup, then

(1) $N^-(h_1) \preceq N^-(h_2)$
(2) $N_-(h_1) \preceq N_-(h_2)$

Proof. By Definition 4.1 □

Proposition 4.5. Suppose N is a normal hesitant fuzzy subgroup of G, and $h_1, h_2 \in HF(G)$, we have

(1) $N^-(h_1 \tilde{\cup} h_2) \approx N^-(h_1) \tilde{\cup} N^-(h_2)$
(2) $N^-(h_1 \tilde{\cap} h_2) \preceq N^-(h_1) \tilde{\cap} N^-(h_2)$
(3) $N_-(h_1 \tilde{\cup} h_2) \succeq N_-(h_1) \tilde{\cup} N_-(h_2)$
(4) $N_-(h_1 \tilde{\cap} h_2) \approx N_-(h_1) \tilde{\cap} N_-(h_2)$

Proof. By Definition 4.1. □

Proposition 4.6. Suppose N is a normal hesitant fuzzy subgroup of G, and h is a (normal) hesitant fuzzy subgroup of G, we have $N^-(h)$ is a (normal) hesitant fuzzy subgroup of G.

Proof. For $s, t \in G$, we obtain

$$\begin{aligned}
N^-(h)(s) \cap N^-(h)(t) &= \bigcup_{s_\lambda \in M} \{\lambda \mid \bigcup_{x \in G} \lambda \cap N(s^{-1}x) \cap h(x) \neq \emptyset\} \\
&\quad \cap \bigcup_{t_\mu \in M} \{\mu \mid \bigcup_{y \in G} \mu \cap N(t^{-1}y) \cap h(y) \neq \emptyset\} \\
&= \bigcup_{s_\lambda \in M} \bigcup_{t_\mu \in M} [\{\lambda \mid \bigcup_{x \in G} \lambda \cap N(s^{-1}x) \cap h(x) \neq \emptyset\} \\
&\quad \cap \{\mu \mid \bigcup_{y \in G} \mu \cap N(t^{-1}y) \cap h(y) \neq \emptyset\}] \\
&= \bigcup_{s_\lambda \in M} \bigcup_{t_\mu \in M} \{\lambda \cap \mu \mid \bigcup_{x \in G} \bigcup_{y \in G} \lambda \cap \mu \cap N(s^{-1}x) \cap N(t^{-1}y) \cap h(x) \cap h(y) \neq \emptyset\} \\
&= \bigcup_{w_\nu \in M} \{\nu \mid \bigcup_{z = xy \in G} \nu \cap N(s^{-1}x) \cap N(t^{-1}y) \cap h(x) \cap h(y) \neq \emptyset\} \\
&\subseteq \bigcup_{w_\nu \in M} \{\nu \mid \bigcup_{z = xy \in G} \nu \cap N(w^{-1}z) \cap h(z) \neq \emptyset\} \\
&= N^-(h)(w) \text{ (Note } w = st, z = xy)
\end{aligned}$$

So, $N^-(h)$ is a hesitant fuzzy subgroup of G.

Furthermore, if h is a normal hesitant fuzzy subgroup of G, then for $s, t \in G$, let $w = s^{-1}ts$, we have

$$\begin{aligned}
N^-(h)(s^{-1}ts) &= N^-(h)(w) \\
&= \bigcup_{w_\nu \in M} \{\nu \mid \bigcup_{z \in G} \nu \cap N(w^{-1}z) \cap h(w) \neq \emptyset\} \\
&= \bigcup_{w_\nu \in M} \{\nu \mid \bigcup_{z \in G} \nu \cap N((s^{-1}ts)^{-1}z) \cap h(s^{-1}ts) \neq \emptyset\} \\
&= \bigcup_{w_\nu \in M} \{\nu \mid \bigcup_{z \in G} \nu \cap N(st^{-1}s^{-1}z) \cap h(t) \neq \emptyset\} \\
&= \bigcup_{w_\nu \in M} \{\nu \mid \bigcup_{z \in G} \nu \cap N(st^{-1}zs^{-1}) \cap h(t) \neq \emptyset\} \\
&= \bigcup_{w_\nu \in M} \{\nu \mid \bigcup_{z \in G} \nu \cap N(t^{-1}z) \cap h(t) \neq \emptyset\} \\
&= \bigcup_{t_\lambda \in M} \{\lambda \mid \bigcup_{z \in G} \lambda \cap N(t^{-1}z) \cap h(t) \neq \emptyset\} \\
&= N^-(h)(t)
\end{aligned}$$

By the above proof, we obtain $N^-(h)$ is a normal hesitant fuzzy subgroup of G.

□

In general, $N_-(h)$ is not a hesitant fuzzy subgroup of G. But if $N_-(h)$ is a hesitant fuzzy subgroup of G, and h is a normal hesitant fuzzy subgroup of G, in the similar method, we can prove $N_-(h)$ is also a normal hesitant fuzzy subgroup of G.

Proposition 4.7. Suppose N, H are two normal hesitant fuzzy subgroups of G, the corresponding rough operators $N^-, N_-; H^-, H_-$ respectively, and $h, k \in HF(G)$, we have
(1) $N^-(h)N^-(k) \preceq N^-(hk)$
(2) $N_-(h)N_-(k) \preceq N_-(hk)$
(3) $(N\tilde{\cap}H)^-(h) \succeq N^-(h)\tilde{\cap}H^-(h)$
(4) $(N\tilde{\cap}H)_-(h) \preceq N_-(h)\tilde{\cap}H_-(h)$
where $(N\tilde{\cap}H)^-$, $(N\tilde{\cap}H)_-$ are two rough operators induced by the normal hesitant fuzzy subgroup $N\tilde{\cap}H$.

Proof. (1) For every $w \in G$,

$$N^-(hk)(w) = \bigcup_{w_\lambda \in M} \{\lambda \mid \bigcup_{z\in G} \lambda \cap N(w^{-1}z) \cap (hk)(z) \neq \emptyset\}$$
$$= \bigcup_{w_\lambda \in M} \{\lambda \mid \bigcup_{z\in G} \lambda \cap N(w^{-1}z) \cap [\bigcup_{z=xy} h(x) \cap k(y)] \neq \emptyset\}$$
$$= \bigcup_{w_\lambda \in M} \{\lambda \mid \bigcup_{z=xy} \lambda \cap N(w^{-1}z) \cap h(x) \cap k(y) \neq \emptyset\}$$

$$(N^-(h)N^-(k))(w) = \bigcup_{w=st} N^-(h)(s) \cap N^-(k)(t)$$
$$= \bigcup_{w=st} [\bigcup_{s_\mu \in M} \{\mu \mid \bigcup_{x\in G} \mu \cap N(s^{-1}x) \cap h(x) \neq \emptyset\}$$
$$\cap \bigcup_{t_\nu \in M} \{\nu \mid \bigcup_{y\in G} \nu \cap N(t^{-1}y) \cap k(y) \neq \emptyset\}]$$
$$= \bigcup_{w=st} \bigcup_{s_\mu \in M} \bigcup_{t_\nu \in M} [\{\mu \mid \bigcup_{x\in G} \mu \cap N(s^{-1}x) \cap h(x) \neq \emptyset\}$$
$$\cap \{\nu \mid \bigcup_{y\in G} \nu \cap N(t^{-1}y) \cap k(y) \neq \emptyset\}]$$
$$= \bigcup_{w=st} \bigcup_{s_\mu \in M} \bigcup_{t_\nu \in M} \{\mu \cap \nu \mid \bigcup_{x\in G} \bigcup_{y\in G} \mu \cap N(s^{-1}x) \cap h(x) \cap \nu \cap N(t^{-1}y) \cap k(y) \neq \emptyset\}$$
$$= \bigcup_{w=st} \bigcup_{w_\lambda \in M} \{\lambda \mid \bigcup_{z=xy} \lambda \cap N(s^{-1}x) \cap N(t^{-1}y) \cap h(x) \cap k(y) \neq \emptyset\}$$
(Note $w_\lambda = s_\mu t_\nu$)
$$\subseteq \bigcup_{w=st} \bigcup_{w_\lambda \in M} \{\lambda \mid \bigcup_{z=xy} \lambda \cap N(s^{-1}xt^{-1}y) \cap h(x) \cap k(y) \neq \emptyset\}$$
$$= \bigcup_{w=st} \bigcup_{w_\lambda \in M} \{\lambda \mid \bigcup_{z=xy} \lambda \cap N(s^{-1}t^{-1}xy) \cap h(x) \cap k(y) \neq \emptyset\}$$
$$= \bigcup_{w=st} \bigcup_{w_\lambda \in M} \{\lambda \mid \bigcup_{z=xy} \lambda \cap N(w^{-1}z) \cap h(x) \cap k(y) \neq \emptyset\}$$
$$= N^-(hk)(w)$$

(2) For every $w \in G$,

$$N_-(hk)(w) = \bigcup_{w_\lambda \in M} \{\lambda \mid \bigcap_{z\in G} (w_\lambda N)(z) \subseteq (hk)(z)\}$$
$$= \bigcup_{w_\lambda \in M} \{\lambda \mid \bigcap_{z\in G} (w_\lambda N)(z) \subseteq \bigcup_{z=xy} h(x) \cap k(y)\}$$
$$= \bigcup_{z=xy} \bigcup_{w=st} [\bigcup_{s_\mu \in M} \{\mu \mid \bigcap_{z\in G} (s_\mu N)(z) \subseteq k(x)\}]$$
$$\cap [\bigcup_{t_\nu \in M} \{\nu \mid \bigcap_{z\in G} (t_\nu N)(z) \subseteq k(y)\}] \quad \text{(Note } w_\lambda = s_\mu t_\nu)$$
$$= \bigcup_{z=xy} \bigcup_{w=st} [\bigcup_{s_\mu \in M} \{\mu \mid \bigcap_{z\in G} \mu \cap N(s^{-1}z) \subseteq h(x)\}]$$

$$\cap[\bigcup_{t_\nu\in M}\{\nu \mid \bigcap_{z\in G}\nu\cap N(t^{-1}z)\subseteq k(y)\}]$$
$$\supseteq \bigcup_{w=st}[\bigcup_{s_\mu\in M}\{\mu \mid \bigcap_{x\in G}\mu\cap N(s^{-1}x)\subseteq h(x)\}]$$
$$\cap[\bigcup_{t_\nu\in M}\{\nu \mid \bigcap_{y\in G}\nu\cap N(t^{-1}y)\subseteq k(y)\}]$$
$$= \bigcup_{w=st} N_-(h)(s)\cap N_-(k)(t)$$
$$= (N_-(h)N_-(k))(w)$$

Which implies that $N_-(h)N_-(k)\succeq N_-(hk)$.

(3) For every $w\in G$, we have

$$(N\tilde{\cap}H)^-(h)(w) = \bigcup_{w_\lambda\in M}\{\lambda \mid \bigcup_{z\in G}(w_\lambda(N\tilde{\cap}H))(z)\cap h(z)\neq\emptyset\}$$
$$= \bigcup_{w_\lambda\in M}\{\lambda \mid \bigcup_{z\in G}\lambda\cap(N\tilde{\cap}H)(w^{-1}z)\cap h(z)\neq\emptyset\}$$
$$= \bigcup_{w_\lambda\in M}\{\lambda \mid \bigcup_{z\in G}\lambda\cap N(w^{-1}z)\cap H(w^{-1}z)\cap h(z)\neq\emptyset\}$$
$$\supseteq [\bigcup_{w_\lambda\in M}\{\lambda \mid \bigcup_{z\in G}\lambda\cap N(w^{-1}z)\cap h(z)\neq\emptyset\}]$$
$$\cap[\bigcup_{w_\lambda\in M}\{\lambda \mid \bigcup_{z\in G}\lambda\cap H(w^{-1}z)\cap h(z)\neq\emptyset\}]$$
$$= N^-(h)(w)\cap H^-(h)(w)$$
$$= (N^-(h)\tilde{\cap}H^-(h))(w)$$

(4) For every $w\in G$, we have

$$(N\cap H)_-(h)(w) = \bigcup_{w_\lambda\in M}\{\lambda \mid \bigcap_{z\in G}(w_\lambda(N\tilde{\cap}H))(z)\subseteq h(z)\}$$
$$= \bigcup_{w_\lambda\in M}\{\lambda \mid \bigcap_{z\in G}\lambda\cap(N\tilde{\cap}H)(w^{-1}z)\subseteq h(z)\}$$
$$= \bigcup_{w_\lambda\in M}\{\lambda \mid \bigcap_{z\in G}\lambda\cap N(w^{-1}z)\cap H(w^{-1}z)\subseteq h(z)\}$$
$$\subseteq \bigcup_{w_\lambda\in M}\{\lambda \mid \bigcap_{z\in G}\lambda\cap N(w^{-1}z)\subseteq h(z)\}$$
$$\cap \bigcup_{w_\lambda\in M}\{\lambda \mid \bigcap_{z\in G}\lambda\cap H(w^{-1}z)\subseteq h(z)\}$$
$$= N_-(h)(w)\cap H_-(h)(w)$$
$$= (N_-(h)\tilde{\cap}H_-(h))(w)$$

□

Proposition 4.8. Suppose N, H are two normal hesitant fuzzy subgroups of G, and for every hesitant fuzzy subgroup h of G, we have $N^-(h)H^-(h)\preceq(NH)^-(h)$.

Proof. For every $w\in G$, we have

$$(NH)^-(h)(w) = \bigcup_{w_\lambda\in M}\{\lambda \mid \bigcup_{z\in G}(w_\lambda(NH))(z)\cap h(z)\neq\emptyset\}$$
$$= \bigcup_{w_\lambda\in M}\{\lambda \mid \bigcup_{z\in G}\lambda\cap(NH)(w^{-1}z)\cap h(z)\neq\emptyset\}$$

$$(N^-(h)H^-(h))(w) = \bigcup_{w=st} N^-(h)(s)\wedge H^-(h)(t)$$
$$= \bigcup_{w=st}[\bigcup_{s_\mu\in M}\{\mu \mid \bigcup_{x\in G}(s_\mu N)(x)\cap h(x)\neq\emptyset\}]$$
$$\cap[\bigcup_{t_\nu\in M}\{\nu \mid \bigcup_{y\in G}(t_\nu H)(y)\cap h(y)\neq\emptyset\}]$$
$$= \bigcup_{w=st}[\bigcup_{s_\mu\in M}\{\mu \mid \bigcup_{x\in G}\mu\cap N(s^{-1}x)\cap h(x)\neq\emptyset\}]$$

$$
\begin{aligned}
&\cap[\bigcup_{t_\nu \in M}\{\nu \mid \bigcup_{y\in G} \nu \cap H(t^{-1}y)\cap h(y)\neq \emptyset\}]\\
&= \bigcup_{w=st}\bigcup_{s_\mu\in M}\bigcup_{t_\nu\in M}\{\mu\wedge\nu \mid \bigcup_{x\in G}\bigcup_{y\in G}\mu\cap\nu\cap N(s^{-1}x)\cap H(t^{-1}y)\cap h(x)\cap h(y)\neq\emptyset\}\\
&= \bigcup_{w_\lambda\in M}\{\lambda \mid \bigcup_{x\in G}\bigcup_{y\in G}\lambda\cap N(s^{-1}x)\cap H(t^{-1}y)\cap h(x)\cap h(y)\neq\emptyset\}\\
&= \bigcup_{w_\lambda\in M}\{\lambda \mid \bigcup_{z=xy\in G}\lambda\cap N(s^{-1}x)\cap H(t^{-1}y)\cap h(x)\cap h(y)\neq\emptyset\}\\
&= \bigcup_{w_\lambda\in M}\{\lambda \mid \bigcup_{z=xy\in G}\lambda\cap (NH)(w^{-1}z)\cap h(x)\cap h(y)\neq\emptyset\}\quad (w=st)\\
&\subseteq \bigcup_{w_\lambda\in M}\{\lambda \mid \bigcup_{z=xy\in G}\lambda\cap (NH)(w^{-1}z)\cap h(z)\neq\emptyset\}\\
&= (NH)^-(h)(w)
\end{aligned}
$$
□

Proposition 4.9. Suppose N, H are two normal hesitant fuzzy subgroups of G, and for every hesitant fuzzy subgroup h of G, we have $(NH)^-(h) \succeq (N^-(h))H\tilde{\cap}(H^-(h))N$.

Proof. For every $w \in G$, we have

$$
\begin{aligned}
((N^-(h))H\tilde{\cap}(H^-(h))N)(w) &= ((N^-(h))H)(w)\cap((H^-(h))N)(w)\\
&= [\bigcup_{w=st}(N^-(h)(s)\cap H(t)]\cap[\bigcup_{w=st}H^-(h)(t)\cap N(s)]\\
&= [\bigcup_{w=st}\bigcup_{s_\mu\in M}\{\mu \mid \bigcup_{x\in G}\mu\cap N(s^{-1}x)\cap h(x)\neq\emptyset\}\cap H(t)]\\
&\quad \cap[\bigcup_{w=st}\bigcup_{t_\nu\in M}\{\nu \mid \bigcup_{y\in G}\nu\cap H(t^{-1}y)\cap \nu(y)\neq\emptyset\}\cap N(s)]\\
&= [\bigcup_{w=st}\bigcup_{s_\mu\in M}\{\mu\cap H(t) \mid \bigcup_{x\in G}\mu\cap N(s^{-1}x)\cap h(x)\neq\emptyset\}]\\
&\quad \cap[\bigcup_{w=st}\bigcup_{t_\nu\in M}\{\nu\cap N(s) \mid \bigcup_{y\in G}\nu\cap H(t^{-1}y)\cap h(y)\neq\emptyset\}]\\
&= \bigcup_{w=st}\bigcup_{s_\mu\in M}\bigcup_{t_\nu\in M}[\{\mu\cap H(t) \mid \bigcup_{x\in G}\mu\cap N(s^{-1}x)\cap h(x)\neq\emptyset\}]\\
&\quad \cap[\{\nu\cap N(s) \mid \bigcup_{y\in G}\nu\cap H(t^{-1}y)\cap h(y)\neq\emptyset\}]\\
&= \bigcup_{w=st}\bigcup_{s_\mu\in M}\bigvee_{t_\nu\in M}\{\mu\cap H(t)\cap\nu\cap N(s) \mid\\
&\quad \bigcup_{x\in G}\bigcup_{y\in G}\mu\cap N(s^{-1}x)\cap h(x)\cap\nu\cap H(t^{-1}y)\cap h(y)\neq\emptyset\}\\
&= \bigcup_{w=st}\bigcup_{w_\lambda\in M}\{\lambda\cap H(t)\cap N(s) \mid \bigcup_{z=xy} \mid\\
&\quad \lambda\cap N(s^{-1}x)\cap h(x)\cap H(t^{-1}y)\cap h(y)\neq\emptyset\}\\
&\subseteq \bigcup_{w=st}\bigcup_{w_\lambda\in M}\{\lambda \mid \bigcup_{z=xy}\lambda\cap N(s^{-1}x)\cap H(t^{-1}y)\cap h(z)\neq\emptyset\}\\
&\quad (h(x)\wedge h(y)\subseteq h(z))\\
&= \bigcup_{w_\lambda\in M}\{\lambda \mid \bigcup_{z\in G}\lambda\cap(NH)(w^{-1}z)\cap h(z)\neq\emptyset\}\\
&= (NH)^-(h)(w)
\end{aligned}
$$
□

Proposition 4.10. Suppose N, H are two normal hesitant fuzzy subgroups of G, and for every hesitant fuzzy subgroup h of G, we have $N_-(h)H_-(h) \preceq (NH)_-(h)$.

Proof. For every $w \in G$,

$$(N_-(h)H_-(h))(w) = \bigcup_{w=st} N_-(h)(s)\cap H_-(h)(t)$$

$$
\begin{aligned}
&= \bigcup_{w=st} [\bigcup_{s_\mu \in M} \{\mu \mid \bigcup_{x\in G} \mu \cap N(s^{-1}x) \subseteq h(x)\}] \\
&\quad \cap [\bigcup_{t_\nu \in M} \{\nu \mid \bigcup_{y\in G} \nu \cap H(t^{-1}y) \subseteq h(y)\}] \\
&= \bigcup_{w=st} \bigcup_{s_\mu \in M} \bigcup_{t_\nu \in M} [\{\mu \mid \bigcup_{x\in G} \mu \cap N(s^{-1}x) \subseteq h(x)\}] \\
&\quad \cap [\{\nu \mid \bigcup_{y\in G} \nu \cap H(t^{-1}y) \subseteq h(y)\}] \\
&= \bigcup_{w=st} \bigcup_{s_\mu \in M} \bigcup_{t_\nu \in M} [\{\mu \cap \nu \mid \bigcup_{x\in G} \bigcup_{y\in G} \mu \cap \nu \cap N(s^{-1}x) \cap H(t^{-1}y) \subseteq h(x) \cap h(y)\}] \\
&= \bigcup_{w=st} \bigcup_{w_\lambda \in M} [\{\lambda \mid \bigcup_{z=xy\in G} \lambda \cap N(s^{-1}x) \cap H(t^{-1}y) \subseteq h(x) \cap h(y)\}] \\
&= \bigcup_{w=st} \bigcup_{w_\lambda \in M} [\{\lambda \mid \bigcup_{z=xy\in G} \lambda \cap (NH)(w^{-1}z) \subseteq h(x) \cap h(y)\}] \\
&\subseteq \bigcup_{w_\lambda \in M} [\{\lambda \mid \bigcup_{z=xy\in G} \lambda \cap (NH)(w^{-1}z) \subseteq h(z)\}] \\
&= (NH)_{-}(h)(w)
\end{aligned}
$$

□

5. Conclusion

In [31], the set of all hesitant fuzzy sets forms a Boolean algebra. As a generalization, we defined two rough operators on a hesitant fuzzy group, and discussed some of their properties.

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A New Subclass of Meromorphic Starlike Functions Defined by Certain Integral Operator

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Abstract − The aim of this paper is to introduce a new class $\sum\limits_p^*(\alpha,\beta,\sigma)$ of meromorphically starlike functions defined by certain integral operator in the unit disc $E=\{z \mid 0<|z|<1\}$ and investigate coefficients, distortion properties and radius of convexity for the class. Furthermore it is shown that the class $\sum\limits_p^*(\alpha,\beta,\sigma)$ is closed under convex linear combinations and integral transforms.

Keywords − *Meromorphic, distortion, radius of convexity, integral transforms*

1. Introduction

Let Σ be denote the class of all functions $f(z)$ of the form

$$f(z)=\frac{1}{z}+\sum_{n=1}^{\infty}a_nz^n \tag{1}$$

which are regular in $E=\{z:0<|z|<1\}$, with a simple pole at the origin. Let $\sum\limits_s$, $\sum\limits^*(\alpha)$ and $\sum\limits_k(\alpha),(0\le\alpha<1)$ denote the subclasses of $\sum$ that are univalent, moromorphically starlike of order α and meromorphically convex of order α respectively. Analytically $f(z)$ of the form (1) is in $\sum\limits^*(\alpha)$ if and only if

$$Re\left\{-\frac{zf'(z)}{f(z)}\right\}>\alpha,(z\in E) \tag{2}$$

Similarly, $f\in\sum\limits_k(\alpha)$ if and only if $f(z)$ is of the form (1) and satisfies

$$Re\left\{-\left(1+\frac{zf''(z)}{f'(z)}\right)\right\}>\alpha,(z\in E) \tag{3}$$

It being understood that if $\alpha=1$ then $f(z)=\frac{1}{z}$ is the only function which is $\sum\limits^*(1)$ and $\sum\limits_k(1)$.

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The classes $\overset{*}{\sum}(\alpha)$ and $\sum\limits_{k}(\alpha)$ have been extensively studied by Pommerenke [1], Clunie [2], Royster [3] and others. Recently the integral operator of $f(z)$ in $\sum\limits_{s}$ for $\sigma > 0$ is denoted by I^{σ} and defined as following

$$I^{\sigma} f(z) = \frac{1}{z^2\Gamma(\sigma)} \int\limits_{0}^{z} \left(\log \frac{z}{t}\right)^{\sigma-1} tf(t)dt \tag{4}$$

That is defined by Jung et al. [4]. It is easy to verify that if $f(z)$ is of the form (1), then

$$I^{\sigma} f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} \left(\frac{1}{n+2}\right)^{\sigma} a_n z^n \tag{5}$$

The aim of the present paper is to introduce the class of meromorphically starlike functions which we denote by $\overset{*}{\sum}(\alpha, \beta, \sigma)$ for some $\alpha(0 \leq \alpha < 1)$, $\beta(0 < \beta \leq 1)$ and $\sigma > 0$. We then consider the class $\overset{*}{\sum\limits_{p}}(\alpha, \beta, \sigma) = \sum\limits_{p} \cap \overset{*}{\sum}(\alpha, \beta, \sigma)$ and extend some of the results of Juneja et al. [5] to this class. We obtain coefficient estimates, distortion properties and radius of convexity for the class. Furthermore it is shown that the class $\overset{*}{\sum\limits_{p}}(\alpha, \beta, \sigma)$ is closed under convex linear combinations and integral transforms.

Definition 1.1. Let the function $f(z)$ be defined by (1). Then $f(z) \in \overset{*}{\sum}(\alpha, \beta, \sigma)$ if and only if

$$\left|\frac{z[I^{\sigma} f(z)]'}{I^{\sigma} f(z)} + 1\right| < \beta \left|\frac{z[I^{\sigma} f(z)]'}{I^{\sigma} f(z)} + 2\alpha - 1\right|,$$

for some $\alpha(0 \leq \alpha < 1)$, $\beta(0 < \beta \leq 1)$, $\sigma > 0$ and for all $z \in E$.

2. Coefficient estimates

In this section we obtain a sufficient condition for a function to be in $\overset{*}{\sum}(\alpha, \beta, \sigma)$.

Theorem 2.1. Let $f(z) = \frac{1}{z} + \sum\limits_{n=1}^{\infty} a_n z^n$ be regular in E. If

$$\sum_{n=1}^{\infty} [(1+\beta)n + (2\alpha-1)\beta + 1] \left[\frac{1}{n+2}\right]^{\sigma} |a_n| \leq 2\beta(1-\alpha) \tag{6}$$

for some $0 \leq \alpha < 1, 0 < \beta \leq 1$ and $\sigma > 0$ then $f(z) \in \overset{*}{\sum}(\alpha, \beta, \sigma)$.

Proof. Suppose (6) holds for all admissible values of α and β. Consider the expression

$$H(f, f') = \left|z[I^{\sigma} f(z)]' + [I^{\sigma} f(z)]\right| - \beta \left|z[I^{\sigma} f(z)]' + (2\alpha-1)[I^{\sigma} f(z)]\right| \tag{7}$$

The we have

$$H(f, f') \leq \left|\sum_{n=1}^{\infty} (n+1) \left[\frac{1}{n+2}\right]^{\sigma} a_n z^n\right| - \beta \left|2(\ \alpha - 1)\frac{1}{z} + \sum_{n=1}^{\infty} (n + 2\alpha - 1) \left[\frac{1}{n+2}\right]^{\sigma} a_n z^n\right|$$

$$\Rightarrow rH(f, f') = \sum_{n=1}^{\infty} (n+1) \left[\frac{1}{n+2}\right]^{\sigma} |a_n| r^{n+1} - \beta \left\{2(\alpha-1) - \sum_{n=1}^{\infty} (n+2\alpha-1) \left[\frac{1}{n+2}\right]^{\sigma} |a_n| r^{n+1}\right\}$$

$$= \sum_{n=1}^{\infty} [(1+\beta)n + (2\alpha-1)\beta + 1] \left[\frac{1}{n+2}\right]^{\sigma} |a_n| r^{n+1} - 2\beta(1-\alpha).$$

Since the above inequality holds for all $r, 0 < r < 1,$ letting $r \to 1$, we have

$$H(f, f') \leq \sum_{n=1}^{\infty} [(1+\beta)n + (2\alpha-1)\beta + 1] \left[\frac{1}{n+2}\right]^{\sigma} |a_n| - 2\beta(1-\alpha)$$

$$\leq 0, \text{ by (6)}.$$

Hence it follows that $\left|\frac{z[I^\sigma f(z)]'}{I^\sigma f(z)}+1\right| < \beta\left|\frac{z[I^\sigma f(z)]'}{I^\sigma f(z)}+2\alpha-1\right|$.
So that $f(z) \in \overset{*}{\sum}(\alpha,\beta,\sigma)$. Hence the theorem. □

Theorem 2.2. Let the function $f(z)=\frac{1}{z}+\sum_{n=1}^{\infty} a_n z^n,\ a_n \geq 0$ be regular in E. Then $f(z) \in \overset{*}{\sum_p}(\alpha,\beta,\sigma)$ if and only if (6) is satisfied.

Proof. In view of Theorem 2.1, it is sufficient to show that only if part.
Let us assume that $f(z)=\frac{1}{z}+\sum_{n=1}^{\infty} a_n z^n,\ a_n \geq 0$ is in $\overset{*}{\sum_p}(\alpha,\beta,\sigma)$.
Then $\left|\frac{\frac{z[I^\sigma f(z)]'}{I^\sigma f(z)}+1}{\frac{z[I^\sigma f(z)]'}{I^\sigma f(z)}+2\alpha-1}\right| = \left|\frac{\sum_{n=1}^{\infty}(n+1)\left[\frac{1}{n+2}\right]^\sigma a_n z^n}{2(1-\alpha)\frac{1}{z}-\sum_{n=1}^{\infty}(n+2\alpha-1)\left[\frac{1}{n+2}\right]^\sigma a_n z^n}\right| < \beta$, for all $z \in E$.
Using the fact that $Re(z) \leq |z|$, if follows that

$$Re\left\{\frac{\sum_{n=1}^{\infty}(n+1)\left[\frac{1}{n+2}\right]^\sigma a_n z^n}{2(1-\alpha)\frac{1}{z}-\sum_{n=1}^{\infty}(n+2\alpha-1)\left[\frac{1}{n+2}\right]^\sigma a_n z^n}\right\} < \beta,\ z \in E \tag{8}$$

Now choose the values of z on the real axis so that $\frac{z[I^\sigma f(z)]'}{I^\sigma f(z)}$ is real.
Upon clearing the denominator in (8) and letting $z \to 1$ through positive values,
we obtain $\sum_{n=1}^{\infty}(n+1)\left[\frac{1}{n+2}\right]^\sigma a_n \leq \beta\left\{2(1-\alpha)-\sum_{n=1}^{\infty}(n+2\alpha-1)\left[\frac{1}{n+2}\right]^\sigma a_n\right\}$
$\Rightarrow \sum_{n=1}^{\infty}[(1+\beta)n+(2\alpha-1)\beta+1]\left[\frac{1}{n+2}\right]^\sigma |a_n| \leq 2\beta(1-\alpha)$.
Hence the theorem. □

Corollary 2.3. If $f(z)=\frac{1}{z}+\sum_{n=1}^{\infty} a_n z^n,\ a_n \geq 0$ is in $\overset{*}{\sum_p}(\alpha,\beta,\sigma)$ then

$$a_n \leq \frac{2\beta(1-\alpha)(n+2)^\sigma}{(1+\beta)n+(2\alpha-1)\beta+1},\ n=1,2,\cdots \tag{9}$$

with equality for each n, for function of the form

$$f_n(z)=\frac{1}{z}+\frac{2\beta(1-\alpha)(n+2)^\sigma}{(1+\beta)n+(2\alpha-1)\beta+1}z^n,\ n=1,2,\cdots \tag{10}$$

If $\beta=1$ in the above theorem, we get the following result of Atshan et al. [6].

Corollary 2.4. If $f(z) \in \overset{*}{\sum_p}(\alpha,\beta,\sigma)$ then

$$a_n \leq \frac{(1-\alpha)(n+2)^\sigma}{n+\alpha},\ n=1,2,\cdots$$

The result is sharp for the functions $f_n(z)$ is given by

$$f_n(z)=\frac{1}{z}+\frac{(1-\alpha)(n+2)^\sigma}{n+\alpha}z^n,\ n=1,2,\cdots$$

3. Distortion properties and radius of convexity estimates

In this section we prove the Distortion Theorem and radius of convexity estimates for the class $\overset{*}{\sum_p}(\alpha,\beta,\sigma)$.

Theorem 3.1. Let $f(z) \in \sum_p^*(\alpha, \beta, \sigma)$. Then for $0 < |z| = r < 1$,

$$\frac{1}{r} - \frac{3^\sigma \beta(1-\alpha)}{1+\alpha\beta} r \leq |f(z)| \leq \frac{1}{r} + \frac{3^\sigma \beta(1-\alpha)}{1+\alpha\beta} r \tag{11}$$

with equality for the function

$$f(z) = \frac{1}{z} + \frac{3^\sigma \beta(1-\alpha)}{1+\alpha\beta} z, \text{ at } \ z = r, ir \tag{12}$$

Proof. Suppose $f(z) \in \sum_p^*(\alpha, \beta, \sigma)$. In view of Theorem 2.2, we have

$$\sum_{n=1}^{\infty} a_n \leq \frac{3^\sigma \beta(1-\alpha)}{1+\alpha\beta} \tag{13}$$

Thus for $0 < |z| = r < 1$,

$$\begin{aligned} |f(z)| &= \left| \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n \right| \\ &\leq \frac{1}{|z|} + \sum_{n=1}^{\infty} a_n |z|^n \\ &\leq \frac{1}{r} + r \sum_{n=1}^{\infty} a_n \\ &\leq \frac{1}{r} + \frac{3^\sigma \beta(1-\alpha)}{1+\alpha\beta}, \quad \text{by (13)} \end{aligned}$$

This gives the right hand side of (11). Also,

$$\begin{aligned} |f(z)| &= \left| \frac{1}{z} - \sum_{n=1}^{\infty} a_n z^n \right| \\ &\geq \frac{1}{|z|} - \sum_{n=1}^{\infty} a_n |z|^n \\ &\geq \frac{1}{r} - r \sum_{n=1}^{\infty} a_n \\ &\geq \frac{1}{r} - \frac{3^\sigma \beta(1-\alpha)}{1+\alpha\beta} \end{aligned}$$

which gives the left hand side of (11) . □

Theorem 3.2. Let the function $f(z)$ be in $\sum_p^*(\alpha, \beta, \sigma)$. Then for $f(z)$ is meromorphically convex of order $\delta (0 \leq \delta < 1)$ in $|z| < r = r(\alpha, \beta, \sigma, \delta)$, where

$$r(\alpha, \beta, \sigma, \delta) = \inf_n \left\{ \frac{(1-\delta)[(1+\beta)n + (2\alpha-1)\beta + 1}{2\beta(1-\alpha)n(n+2-\delta)(n+2)^\sigma} \right\}^{\frac{1}{n+1}}, \quad n = 1, 2, \cdots \tag{14}$$

The bound for $|z|$ is sharp for each n with the extremal function being of the form (10).

Proof. Let $f(z) \in \sum_p^*(\alpha, \beta, \sigma)$. Then by Theorem 2.2

$$\sum_{n=1}^{\infty} \frac{(1+\beta)n + (2\alpha-1)\beta + 1}{2\beta(1-\alpha)(n+2)^\sigma} a_n \leq 1 \tag{15}$$

In view of (3) , it is sufficient to show that

$$\left|2+\frac{zf''(z)}{f'(z)}\right| \leq 1-\delta, \quad \text{for} \quad |z|<r(\alpha,\beta,\sigma,\delta)$$

or equivalently to show that

$$\left|\frac{f'(z)+(zf'(z))'}{f'(z)}\right| \leq 1-\delta, \quad \text{for } |z|<r(\alpha,\beta,\sigma,\delta) \tag{16}$$

Substituting the series expansions for $f'(z)$ and $(zf'(z))'$ in the left hand side of (16) then we get

$$\left|\frac{\sum\limits_{n=1}^{\infty} n(n+1)a_n z^{n-1}}{-\frac{1}{z^2}+\sum\limits_{n=1}^{\infty} n a_n z^{n-1}}\right| \leq \frac{\sum\limits_{n=1}^{\infty} n(n+1)a_n |z|^{n+1}}{1-\sum\limits_{n=1}^{\infty} n a_n |z|^{n+1}}$$

This will be bounded by $(1-\delta)$ if

$$\sum_{n=1}^{\infty} \frac{n(n+2-\delta)}{1-\delta} a_n |z|^{n+1} \leq 1 \tag{17}$$

In view of (15), it follows that (17) is true if

$$\frac{n(n+2-\delta)}{1-\delta}|z|^{n+1} \leq \frac{(1+\beta)n+(2\alpha-1)\beta+1}{2\beta(1-\alpha)(n+2)^{\sigma}}, \ n=1,2,\cdots$$

$$\Rightarrow \ |z| \leq \left\{\frac{(1-\delta)[(1+\beta)n+(2\alpha-1)\beta+1]}{2\beta(1-\alpha)n(n+2-\delta)(n+2)^{\sigma}}\right\}^{\frac{1}{n+1}}, \ n=1,2,\cdots \tag{18}$$

Setting $|z|=r(\alpha,\beta,\sigma,\delta)$ in (18), the result follows.

The result is sharp, the extremal function being of the form

$$f_n(z)=\frac{1}{z}+\frac{2\beta(1-\alpha)(n+2)^{\sigma}}{(1+\beta)n+(2\alpha-1)\beta+1}z^n, \ n=1,2,\cdots$$

□

4. Convex linear combinations

In this section we prove that the class $\sum\limits_{p}^{*}(\alpha,\beta,\sigma)$ is closed under convex linear combinations.

Theorem 4.1. Let $f_0(z)=\frac{1}{z}$ and $f_n(z)=\frac{1}{z}++\frac{2\beta(1-\alpha)(n+2)^{\sigma}}{(1+\beta)n+(2\alpha-1)\beta+1}z^n, \ n=1,2,\cdots$. Then $f(z)\in\sum\limits_{p}^{*}(\alpha,\beta,\sigma)$ if and only if it can be expressed in the form $f(z)=\sum\limits_{n=0}^{\infty}\lambda_n f_n(z)$, where $\lambda_n\geq 0$ and $\sum\limits_{n=0}^{\infty}\lambda_n=1$.

Proof. Let $f(z)=\sum\limits_{n=0}^{\infty}\lambda_n f_n(z)$, where $\lambda_n\geq 0$ and $\sum\limits_{n=0}^{\infty}\lambda_n=1$. Then

$$\begin{aligned} f(z) &= \sum_{n=0}^{\infty}\lambda_n f_n(z)=\lambda_0 f_0(z)+\sum_{n=1}^{\infty}\lambda_n f_n(z)\\ &= \left[1-\sum_{n=1}^{\infty}\lambda_n\right]f_0(z)+\sum_{n=1}^{\infty}\lambda_n f_n(z)\\ &= \left[1-\sum_{n=1}^{\infty}\lambda_n\right]\frac{1}{z}+\sum_{n=1}^{\infty}\lambda_n\left[\frac{1}{z}+\frac{2\beta(1-\alpha)(n+2)^{\sigma}}{(1+\beta)n+(2\alpha-1)\beta+1}z^n\right]\\ &= \frac{1}{z}+\sum_{n=1}^{\infty}\lambda_n\frac{2\beta(1-\alpha)(n+2)^{\sigma}}{(1+\beta)n+(2\alpha-1)\beta+1}z^n \end{aligned}$$

$$\text{Since }\sum_{n=1}^{\infty}\left\{\frac{(1+\beta)n+(2\alpha-1)\beta+1}{2\beta(1-\alpha)(n+2)^{\sigma}}\right\}\lambda_n\frac{2\beta(1-\alpha)(n+2)^{\sigma}}{(1+\beta)n+(2\alpha-1)\beta+1}$$
$$=\sum_{n=1}^{\infty}\lambda_n=1-\lambda_0\leq 1$$

Therefore $f(z)\in\sum\limits_{p}^{*}(\alpha,\beta,\sigma)$.

Conversely suppose that $f(z)\in\sum\limits_{p}^{*}(\alpha,\beta,\sigma)$.
Since $a_n\leq\frac{2\beta(1-\alpha)(n+2)^{\sigma}}{(1+\beta)n+(2\alpha-1)\beta+1},\ n=1,2,\cdots$.
Setting $\lambda_n=\frac{(1+\beta)n+(2\alpha-1)\beta+1}{2\beta(1-\alpha)(n+2)^{\sigma}}a_n,\ n=1,2,\cdots$ and $\lambda_0=1-\sum\limits_{n=0}^{\infty}\lambda_n$.
It follows that $f(z)=\sum\limits_{n=0}^{\infty}\lambda_n f_n(z)$. This completes the proof of the theorem. □

Theorem 4.2. The class $\sum\limits_{p}^{*}(\alpha,\beta,\sigma)$ is closed under convex linear combination.

Proof. Let the function $F_k(z)$ be given by
$F_k(z)=\frac{1}{z}+\sum\limits_{n=1}^{\infty}f_{n,k}z^n,\ k=1,2\cdots,m$ be in the class $\sum\limits_{p}^{*}(\alpha,\beta,\sigma)$.
Then it is enough to show that the function

$$H(z)=\lambda F_1(z)+(1-\lambda)F_2(z),\ (0\leq\lambda\leq 1)$$

tt is also in the class $\sum\limits_{p}^{*}(\alpha,\beta,\sigma)$. Since for $0\leq\lambda\leq 1$,

$$H(z)=\frac{1}{z}+\sum_{n=1}^{\infty}\left[\lambda f_{n,1}+(1-\lambda)f_{n,2}\right]z^n$$

We observe that

$$\begin{aligned} &\sum_{n=1}^{\infty}\left[(1+\beta)n+(2\alpha-1)\beta+1\right]\frac{1}{(n+2)^{\sigma}}\left[\lambda f_{n,1}+(1-\lambda)f_{n,2}\right]\\ &=\lambda\sum_{n=1}^{\infty}\left[(1+\beta)n+(2\alpha-1)\beta+1\right]\frac{1}{(n+2)^{\sigma}}f_{n,1}\\ &\qquad+(1-\lambda)\sum_{n=1}^{\infty}\left[(1+\beta)n+(2\alpha-1)\beta+1\right]\frac{1}{(n+2)^{\sigma}}f_{n,2}\\ &\leq\ 2\beta\lambda(1-\alpha)+(1-\lambda)2\beta(1-\alpha)=2\beta(1-\alpha) \end{aligned}$$

By Theorem 2.2, we have $H(z)\in\sum\limits_{p}^{*}(\alpha,\beta,\sigma)$. □

5. Integral transforms

In this section, we consider integral transforms of functions in $\sum_p^*(\alpha,\beta,\sigma)$.

Theorem 5.1. If $f(z)$ is in $\sum_p^*(\alpha,\beta,\sigma)$ then the integral transforms

$$F_c(z) = c\int_0^1 u^c f(uz)du,\ 0 < c < \infty \tag{19}$$

$$\text{are in } \sum_p^*(\delta)\ \text{, where } \delta = \delta(\alpha,\beta,\sigma,c) = \frac{(1+\alpha\beta)(c+2) - 3^\sigma\beta c(1-\alpha)}{(1+\alpha\beta)(c+2) + 3^\sigma\beta c(1-\alpha)} \tag{20}$$

The result is best possible for the function $f(z) = \frac{1}{z} + \frac{3^\sigma\beta(1-\alpha)}{(1+\alpha\beta)}z$.

PROOF. Suppose $f(z) \in \sum_p^*(\alpha,\beta,\sigma)$. We have

$$F_c(z) = c\int_0^1 u^c f(uz)du = \frac{1}{z} + \sum_{n=1}^{\infty}\frac{ca_n}{n+c+1}z^n$$

It is sufficient to show that

$$\sum_{n=1}^{\infty}\frac{(n+\delta)}{(1-\delta)}\frac{ca_n}{(n+c+1)} \le 1 \tag{21}$$

Since $f(z) \in \sum_p^*(\alpha,\beta,\sigma)$, we have

$$\sum_{n=1}^{\infty}\frac{(1+\beta)n + (2\alpha-1)\beta + 1}{2\beta(1-\alpha)(n+2)^\sigma}a_n \le 1 \tag{22}$$

Thus (21) will be satisfied if $\frac{(n+\delta)}{(1-\delta)}\frac{c}{(n+c+1)} \le \frac{(1+\beta)n+(2\alpha-1)\beta+1}{2\beta(1-\alpha)(n+2)^\sigma}$, for each n

$$\Rightarrow \delta \le \frac{[(1+\beta)n + (2\alpha-1)\beta + 1][n+c+1] - 2\beta(1-\alpha)nc(n+2)^\sigma}{[(1+\beta)n + (2\alpha-1)\beta + 1][n+c+1] + 2\beta(1-\alpha)nc(n+2)^\sigma} \tag{23}$$

Since the right hand side of (23) is an increasing function of n, putting $n = 1$ in (23), we get

$$\delta \le \frac{(1+\alpha\beta)(c+2) - 3^\sigma\beta(1-\alpha)c}{(1+\alpha\beta)(c+2) + 3^\sigma\beta(1-\alpha)c}$$

Hence the theorem. □

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Cancellative Elements in Finite AG-groupoids

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Abstract — An Abel-Grassmann's groupoid (briefly AG-groupoid) is a groupoid S satisfying the left invertive law: $(xy)z = (zy)x \; \forall \; x, y, z \in S$. In the present paper, we discuss the left and right cancellative property of elements of the finite AG-groupoid S. For an AG-groupoid with left identity it is known that every left cancellative element is right cancellative. We prove a problem (for finite AG-groupoids) that every left cancellative element of an AG-groupoid (with out left identity) is right cancellative. Moreover, we generalize various results of finite AG-groupoids by removing the condition of existence of left identity.

Keywords — *AG-groupoid, AG-subgroupoid, Cancellative elements, non-cancellative elements*

1. Introduction

An AG-groupoid S is a groupoid which satisfies the left invertive law $(xy)z = (zy)x \; \forall \; x, y, z \in S$, this is non-associative in general. In literature, different authors used different names for this structure, e.g. left invertive groupoid in [1], Left Almost Semigroup (briefly LA-semigroup) in [2,3], while right modular groupoid in [4, Line 35]. Cho et al. [4] proved that, an AG-groupoid S always satisfies the medial law: $(wx)(yz) = (wy)(xz) \; \forall \; w, x, y, z \in S$, while an AG-groupoid S with left identity always satisfies the paramedial law: $(wx)(yz) = (zx)(yw) \; \forall \; w, x, y, z \in S$. An AG-groupoid S with left identity is called a left almost group (briefly LA-group) or an AG-group, if each element of S has its inverse element [5]. For more study we refer [6,7]. A non-empty subset H of an AG-groupoid is called an AG-subgroupoid if it is closed with respect to the binary operation. A left ideal I (respectively, right) of an AG-groupoid S is a subset of S which satisfies the property $SI \subset I$ (respectively, $IS \subset I$). A two sided ideal of S is an ideal which is both left and right ideal. An element $c \in S$ is called left cancellative if $cx = cy \implies x = y \; \forall \; x, y \in S$. Similarly, $c \in S$ is said to be right cancellative if $xc = yc \implies x = y \; \forall \; x, y \in S$. An element c of the AG-groupoid S is said to be cancellative if it is both left and right cancellative. From now onward, we will use LC for left cancellative, RC for right cancellative, TC for two sided cancellative and NC for non-cancellative elements. An AG-groupoid S is called LC (respectively, RC) if all element of S is LC (respectively, RC). LC, RC and TC play an important role in the theory of quasigroups and many results occur in this structure due to these properties. Every AG-groupoid is not necessarily TC but some or all of its elements may be TC and hence can enjoy some special properties that a general AG-groupoid cannot possess.

In this paper, we study the LC and RC property of a finite AG-groupoids. Moreover, we solve a problem proposed by Shah et al. [8], that every LC element of an AG-groupoid is also RC. We also

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generalize several results of [8] and remove the condition of existence of left identity. We prove that TC and NC elements of a finite AG-groupoid (not necessarily have left identity) S partition S and the two sub-classes of S are AG-subgroupoids. If a finite AG-groupoid S have at least one NC element then set of NC elements form a maximal ideal.

2. Characterization of AG-groupoid due to Cancellativity

In this section, we show that every LC element of a finite AG-groupoid is TC. The following lemma will be useful.

Lemma 2.1. If a finite AG-groupoid S has LC (respectively, RC) element then $SS = S$.

Proof. Let S be a finite AG-groupoid. Then clearly $SS \subseteq S$. On the other hand, let $S = \{s_1, s_2, \ldots, s_n\}$ be a finite AG-groupoid and $a \in S$ be a LC (respectively ,RC) element. Then $aS = \{as_1, as_2, \ldots, as_n\}$ (respectively, $Sa = \{s_1a, s_2a, \ldots, s_na\}$). We have to show that aS has n distinct elements. Let on the contrary there exist a_i and a_j of S such that $aa_i = aa_j$. Then since a is LC. This gives $a_i = a_j$, which implies that all elements of aS are distinct. Let $x \in S$ be any arbitrary element. Then there exist $a_i \in S$ such that $x = aa_i \in SS$. This gives $S \subseteq SS$. Hence $SS = S$. □

Remark 2.2. Lemma 2.1 does not hold for infinite AG-groupoid as $(N, +)$ is an infinite AG-groupoid but $N + N \neq N$.

The following theorems will be useful.

Theorem 2.3 (Shah et al. [8])**.** In AG-groupoid, if an element is RC then it is TC.

Theorem 2.4 (Shah et al. [8])**.** Let $x, y \in S$, where S is an AG-groupoid. We define a relation $\sim$ on S as

$$x \sim y, \quad x \text{ and } y \text{ are both TC or NC.}$$

Then the relation $\sim$ is an equivalence relation.

Theorem 2.5. Let S be a finite AG-groupoid and $c \in S$ such that $c = c_1c_2$. Then c is LC if and only if c_1 and c_2 are TC.

Proof. Let $c \in S$ be any LC element of the finite AG-groupoid S. Then $\forall\, x, y \in S$, we have

$$cx = cy \implies x = y$$

Let $c = c_1c_2$, we have to show that both c_1 and c_2 are TC. For this it is enough to show that they are RC. Let $xc_2 = yc_2$ for any $x, y \in S$. Then by repeated use of left invertive law, we get

$$\begin{aligned} cx &= (c_1c_2)x = (xc_2)c_1 \\ &= (yc_2)c_1 = (c_1c_2)y \\ &= cy \end{aligned}$$

This gives $x = y$. This implies that c_2 is RC and hence TC. Next we have to show that c_1 is RC, for this let $xc_1 = yc_1$ for any $x, y \in S$. Since we have proved that c_2 is RC, thus there exist $x_1, y_1 \in S$ such that $x = x_1c_2$ and $y = y_1c_2$. Now as

$$\begin{aligned} xc_1 &= yc_1 \\ (x_1c_2)c_1 &= (y_1c_2)c_1 \\ (c_1c_2)x_1 &= (c_1c_2)y_1 \quad \text{by left invertive law} \end{aligned}$$

This implies that $cx_1 = cy_1$ which further implies that $x_1 = y_1$ and hence $x = y$. Hence c_1 is RC. Thus c is the product of two TC elements.
Conversely, let $c_1, c_2 \in S$ be two TC elements, we have to show that their product c_1c_2 is LC. For this consider

$$\begin{aligned} (c_1c_2)x &= (c_1c_2)y \\ (xc_2)c_1 &= (yc_2)c_1 \quad \text{by left invertive law} \end{aligned}$$

As c_1 and c_2 are RC, so we get $x = y$. □

Theorem 2.6. Every RC element in a finite AG-groupoid is the product of two TC elements.

PROOF. Let $c \in S$ be any arbitrary RC element of the finite AG-groupoid S. Then $\forall\ x, y \in S$, we have

$$xc = yc \implies x = y$$

Let $c = c_1c_2$. We have $xc_1 = yc_1$ and consider

$$\begin{aligned}(xc)c &= (xc)(c_1c_2) = (xc_1)(cc_2) \text{ by medial law} \\ &= (yc_1)(cc_2) = (yc)(c_1c_2) \text{ again by medial law} \\ &= (yc)c\end{aligned}$$

By repeated use of RC property of c, we get $x = y$.
Next we show that c_2 is RC. For this let $xc_2 = yc_2$. Then consider

$$\begin{aligned}(c_1x)c &= (c_1x)(c_1c_2) = (c_1c_1)(xc_2) \text{ by medial law} \\ &= (c_1x)(c_1c_2) = (c_1c_1)(yc_2) \text{ again by medial law} \\ &= (c_1y)c\end{aligned}$$

By use of TC property of c and c_1, we get $x = y$. This implies that c_2 is RC. □

Theorem 2.7. Every LC element in a finite AG-groupoid is RC element.

PROOF. The proof follows from Theorem 2.5 and Theorem 2.6. □

Corollary 2.8. For a finite AG-groupoid S, the following two conditions are equivalent for any $c \in S$.

(1) c is RC

(2) c is LC

Theorem 2.9. The set of all TC elements of a finite AG-groupoid S is either an AG-subgroupoid of S or an empty set.

PROOF. Let S be a finite AG-groupoid and H be the set of all TC elements of S. If H is empty then there is nothing to prove and if H is non-empty then let $c_1, c_2 \in H$. Let on the contrary $c = c_1c_2$ is NC then this implies that one of c_1 or c_2 or both are NC, which is a contradiction. Hence H is an AG-subgroupoid. □

Corollary 2.10. If S is a finite AG-groupoid then the product of one TC element and one NC element or product of two NC elements is always NC.

Lemma 2.11. If S is a finite AG-groupoid then the set of all NC elements of S is either an AG-subgroupoid of S or an empty set.

PROOF. Given that S is a finite AG-groupoid. Let K be the set of all NC elements. Clearly, K is empty if S is TC. So let us suppose S is not TC and let $c_1, c_2 \in K$ and on contrary that $c = c_1c_2$ is TC then by Theorem 2.5 both c_1 and c_2 are LC, which by Theorem 2.7 c_1 and c_2 are TC. Hence $c = c_1c_2$ is NC. Thus K is an AG-subgroupoid. □

Theorem 2.12. TC elements and NC elements of a finite AG-groupoid S partition S into two AG-subgroupoid of S.

Corollary 2.13. If S is a finite AG-groupoid then a proper right (respectively, left) ideal of S cannot be a subset of H.

PROOF. Proof follows from Theorem 2.9. □

Corollary 2.14. For a finite AG-groupoid S having at least one NC element, K is always a maximal ideal.

PROOF. Proof follows from Lemma 2.11. □

In the following theorem, we construct TC AG-groupoids from abelian group.

Theorem 2.15. *Let $(G, +)$ be an abelian group under addition and let $\alpha, \beta \in Auto(G)$ satisfying $\alpha^2 = \beta$. Then define new binary operation on G by $x \cdot y = \alpha(x) + \beta(y)\ \forall\ x, y \in G$. Then $G_{\alpha,\beta}$ is an AG-groupoid.*

PROOF. Let x, y and z be any three arbitrary elements of the abelian group G. Then consider

$$\begin{aligned}(x \cdot y) \cdot z &= (\alpha(x) + \beta(y)) \cdot z \\ &= \alpha^2(x) + \alpha\beta(y) + \beta(z) \qquad (1)\end{aligned}$$

On the other hand

$$\begin{aligned}(z \cdot y) \cdot x &= (\alpha(z) + \beta(y)) \cdot x \\ &= \alpha^2(z) + \alpha\beta(y) + \beta(x) \\ &= \alpha^2(x) + \alpha\beta(y) + \beta(z), \quad \alpha^2 = \beta \qquad (2)\end{aligned}$$

From (1) and (2) $(x \cdot y) \cdot z = (z \cdot y) \cdot x$. This implies that $G_{\alpha,\beta} = (G, \cdot)$ is an AG-groupoid. It is easy to see that $G_{\alpha,\beta}$ is TC. □

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Some Results on Divisior Cordial Graphs

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Abstract — In this paper, we introduce some results on divisor cordial graphs where we find some upper bound for the labeling of any simple graph and $r-regular$ graph and describe the divisor cordial labeling for some families of graphs such the jellyfish graph, shell graph and the bow and butterfly graphs.

Keywords — *labeling, divisor cordial, shell graph*

Introduction

In this paper by a simple graph, we mean a finite, undirected graph without loops and multiple edges, for terms not defined here, we refer to Harary [1]. Graph labeling, mean that the vertices and edges are assigned real values or subsets of a set, subject to certain conditions. For a dynamic survey on various graph labeling problems we refer to Gallian [2]. The concept of cordial labeling was introduced by Cahit [3], in [4], Varatharajan et al. introduce the concept of divisor cordial labeling of graph. The divisor cordial labeling of various types of graphs are presented in [4–12]. The brief summaries of definitions which are necessary for the present investigation are provided below. For standard terminology and notations related to number theory we refer to Burton [13].

Definition 1.1. [4] Let $G = (V(G), E(G))$ be a simple graph and $f : V(G) \longrightarrow \{1, 2, ..., |V(G)|\}$ be a bijection. For each edge uv, assign the label 1 if $f(u)|f(v)$ or $f(v)|f(u)$ and the label 0 otherwise. The function f is called a divisor cordial labeling if $|e_f(0) - e_f(1)| \leq 1$. A graph with a divisor cordial labeling is called a divisor cordial graph.

Definition 1.2. [1] The neighborhood of a vertex u is the set $N_u(G)$ consisting of all vertices v which are adjacent with u. The closed neighborhood is $N_u[G] = N_u(G) \bigcup \{u\}$.

Definition 1.3. [1] The number $\delta(G) = \min\{d(v) \mid v \in V\}$ is the minimum degree of the vertices in the graph G, the number $\Delta(G) = \max\{d(v) \mid v \in V\}$ is the maximum degree of the vertices in the graph G, the number $d(G) = \frac{1}{|V|} \sum\limits_{v \in V} d(v)$ is the average degree of the vertices in the graph G.

Definition 1.4. [14] The Jelly fish graph $J(m, n)$ is obtained from a $4-cycle$ v_1, v_2, v_3, v_4 by joining v_1 and v_3 with an edge and appending m pendent edges to v_2 and n pendent edges to v_4.

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Definition 1.5. [15] A shell graph is defined as a cycle C_n with $(n-3)$ chords sharing a common end point called the apex, shell graphs are denoted as $C(n, n-3)$.

Definition 1.6. [16] A bow graph is defined to be a double shell in which each shell has any order.

Definition 1.7. [15] Define a Butterfly graph as a bow graph with exactly two pendent edges at the apex.

The Results

Proposition 2.1. For any simple graph $G(p,q)$, the maximum value of $e_f(1)$ is
$min \left\{ \triangle(G) + \sum_{i=2}^{\lfloor \frac{p}{2} \rfloor} (\lfloor \frac{p}{i} \rfloor - 1), q \right\}$, where $p \geq 4$.

PROOF. Let $G(p,q)$ be a simple connected graph and let the vertex v_k be of maximum degree $\triangle(G)$, if we labeled this vertex by 1 then we will achieve $\triangle(G)$ edges labeled 1, and from division algorithm the maximum numbers of the multiples of labels of vertices are:
for 2 is $\lfloor \frac{p}{2} \rfloor - 1$,
for 3 is $\lfloor \frac{p}{3} \rfloor - 1$,
for 4 is $\lfloor \frac{p}{4} \rfloor - 1$,
.
.
.
for $\lfloor \frac{p}{2} \rfloor$ is $\lfloor \frac{p}{\lfloor \frac{p}{2} \rfloor} \rfloor - 1$ which must equal 1

hence the maximum value for $e_f(1)$ equals $\triangle(G) + \sum_{i=2}^{\lfloor \frac{p}{2} \rfloor} (\lfloor \frac{p}{i} \rfloor - 1)$ in any graph $G(p,q)$. □

Corollary 2.2. For each $r-regular$ graph the maximum value of $e_f(1)$ is $kr + \sum_{i=k+1}^{\lfloor \frac{p}{2} \rfloor} (\lfloor \frac{p}{i} \rfloor - 1)$; where $k = \lfloor \frac{p}{r+1} \rfloor$ and $p \geq 4$.

PROOF. Let $G(p,q)$ be an $r-regular$ graph then $\triangle(G) = r$, and for each vertex v in graph G the maximum number of edges that label 1 in $N_v(G)$ is r, hence for all i in which $\lfloor \frac{p}{i} \rfloor - 1 \geq r$ we reduced it to r.

But from Proposition 2.1 the maximum value of $e_f(1)$ is $\triangle(G) + \sum_{i=2}^{\lfloor \frac{p}{2} \rfloor} (\lfloor \frac{p}{i} \rfloor - 1)$, then the maximum value in an $r-regular$ graph is:

$$
\begin{aligned}
&= r + \sum_{i=2}^{k} (r) + \sum_{i=k+1}^{\lfloor \frac{p}{2} \rfloor} \left(\left\lfloor \frac{p}{i} \right\rfloor - 1\right) \\
&= r + (k-1)r + \sum_{i=k+1}^{\lfloor \frac{p}{2} \rfloor} \left(\left\lfloor \frac{p}{i} \right\rfloor - 1\right) \\
&= kr + \sum_{i=k+1}^{\lfloor \frac{p}{2} \rfloor} \left(\left\lfloor \frac{p}{i} \right\rfloor - 1\right)
\end{aligned}
$$

□

Proposition 2.3. For any divisor cordial graph $G(p,q)$, $q \leq 2(\triangle(G) + \sum_{i=3}^{\lfloor \frac{p}{2} \rfloor} \lfloor \frac{p}{i} \rfloor) + 3$, where $p \geq 6$.

PROOF. Let $G(p,q)$ be a divisor cordial graph, then $|e_f(0) - e_f(1)| \leq 1$, means $e_f(0) = e_f(1) - 1$ or $e_f(0) = e_f(1)$ or $e_f(0) = e_f(1) + 1$, by Proposition 2.1,

$$q \leq 2e_f(1) + 1$$

$$q \leq 2(\triangle\, G + \sum_{i=2}^{\lfloor \frac{p}{2} \rfloor}(\left\lfloor \frac{p}{i} \right\rfloor - 1)) + 1$$

$$q \leq 2(\triangle\, G + \sum_{i=3}^{\lfloor \frac{p}{2} \rfloor}(\left\lfloor \frac{p}{i} \right\rfloor)) + 3$$

□

Divisor Cordial Labeling for Some Families of Graphs

In this section we introduce the divisor cordial labeling for some types of graphs.

The Jelly Fish Graph

Proposition 3.1. For $m, n \geq 1$, Jelly fish graph $J(m, n)$ is a divisor cordial graph.

Proof. Let $G(V, E) = J(m, n)$. Then G has $(m + n + 4)$ vertices and $(m + n + 5)$ edges.
Without losing of generality, let $m \leq n$. Let $V(G) = V_1 \cup V_2$ where $V_1 = \{x, u, y, v\}$, $V_2 = \{u_i, v_j; 1 \leq i \leq m, 1 \leq j \leq n\}$ and $E = E_1 \cup E_2$, where $E_1 = \{xu, uy, yv, vx, xy\}$, $E_2 = \{uu_i, vv_j; 1 \leq i \leq m, 1 \leq j \leq n\}$.
Define $f : V \to \{1, 2, ..., (m + n + 4)\}$ as follows:
$f(u) = 1, f(v) = 2, f(x) = m + n + 4, f(y) = m + n + 3$ and
$f(u_i) = 2(i + 1); i = 1, 2, ..., m$,

$$f(v_i) = \begin{cases} 2i + 1 & , i = 1, 2, ..., m \\ i + m + 2 & , i = m + 1, m + 2, ..., n \end{cases}$$

From the function f there are $m + 2$ edges labeled 1 sice $f(u) = 1$, and since $f(v) = 2$, then there are exactly $\lfloor \frac{1}{2}(n - m) \rfloor$ of pendent edges from v labeled 1 and only one from vx or vy. means $e_f(1) = m + 3 + \lfloor \frac{1}{2}(n - m) \rfloor$ and

$$e_f(0) = m + n + 5 - (m + 3 + \lfloor \frac{1}{2}(n - m) \rfloor)$$
$$= n + 2 - \lfloor \frac{1}{2}(n - m) \rfloor,$$

Case 1: m, n are odd
The $|E|$ is odd and $\lfloor n - m \rfloor$ are even, hence, $|e_f(0) - e_f(1)| = 1$

Case 2: m, n are even
The $|E|$ is odd and $\lfloor n - m \rfloor$ is even, hence, $|e_f(0) - e_f(1)| = 1$

Case 3: m is odd and n is even
The $|E|$ is even and $\lfloor n - m \rfloor$ is odd, hence, $|e_f(0) - e_f(1)| = 0$

Case 4: m is even and n is odd
The $|E|$ is even and $\lfloor n - m \rfloor$ is odd, hence, $|e_f(0) - e_f(1)| = 0$

Then from Case 1, Case 2, Case 3 and Case 4 the jelly fish graph is divisor cordial. □

Example 3.2. The jelly fish graph $j(6, 11)$ and its divisor cordial labeling are shown in Fig.1

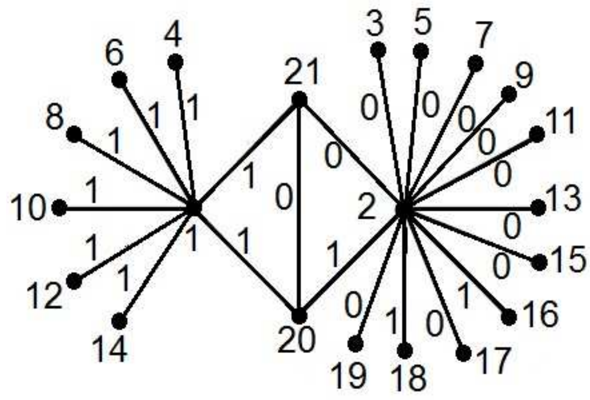

Fig. 1. A Jelly fish graph $j(6, 11)$ and its divisor cordial labeling

The shell and The Bow Graph

Proposition 3.3. Every shell graph is divisor cordial.

Proof. Let $G = (V, E)$ be a $C(n, n-3)$ graph with $|V| = n$, then $|E| = 2n-3$ means $|E|$ is an odd number, and let v_0 be the apex and $v_1, v_2, ..., v_{n-1}$ other its vertices.
Define the labeling $f : V \longrightarrow \{1, 2, ..., n\}$ as:
$f(v_0) = 2$, $f(v_1) = 1$ and other vertices by the following:

$$\begin{array}{cccccc} & 2\cdot 2, & 2\cdot 2^2, & \cdots, & 2\cdot 2^{k_1}, \\ 3, & 3\cdot 2, & 3\cdot 2^2, & \cdots, & 3\cdot 2^{k_2}, \\ 5, & 5\cdot 2, & 5\cdot 2^2, & \cdots, & 5\cdot 2^{k_3}, \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots, \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots, \end{array}$$

where $(2m-1)\cdot 2^{k_m} \le n$ and $m \ge 1, k_m \ge 0$. We observe that $(2m-1)\cdot 2^a$ divides $(2m-1)\cdot 2^b$; $(a < b)$ and $(2m-1)\cdot 2^{k_i}$ does not divide $2m+1$.
In this labeling, there are $\lceil \frac{n-1}{2} \rceil$ edges label 1 passing through v_0, but other edges not passing through the apex make a path, hence there are also $\lfloor \frac{n-2}{2} \rfloor$ edges are labeled 1. Hence, $e_f(1) = \lceil \frac{n-1}{2} \rceil + \lfloor \frac{n-2}{2} \rfloor$

Case 1: n is odd, then $e_f(1) = \frac{n-1}{2} + \lfloor \frac{n-2}{2} \rfloor$ and $e_f(0) = \frac{n-1}{2} + \lceil \frac{n-2}{2} \rceil$

Case 2: n is even, then $e_f(1) = \lceil \frac{n-1}{2} \rceil + \frac{n-2}{2}$ and $e_f(0) = \lfloor \frac{n-1}{2} \rfloor + \frac{n-2}{2}$

In the two cases Case 1 and Case 2, the difference between $e_f(1)$ and $e_f(0)$ is 1 which means the shell graph is divisor cordial. □

Notice another divisor labeling for shell graphs can found with fan graphs [4]

Example 3.4. The shell graph $C(13, 10)$ and its divisor cordial labeling are shown in Fig. 2

Proposition 3.5. All bow graphs are divisor cordial.

Proof. Let G be a bow graph with two shells of order m and n excluding the apex. Then the number of vertices in G is $p = m+n+1$ and the edges $q = 2(m+n-1)$. The apex of the bow graph is denoted by v_0, denote the vertices in the right wing of the bow graph from bottom to top by $v_1, v_2, ..., v_m$, and the vertices in the left wing of the bow graph are denoted from top to bottom by $v_{m+1}, v_{m+2}, ..., v_{m+n}$. Without losing of generality, suppose $m \le n$.
Define the labeling $f : V \longrightarrow \{1, 2, ..., m+n+1\}$ by:
$f(v_0) = 2$, $f(v_1) = 1$ and label the vertices of the wings by the following:

$$\begin{array}{cccccc} & 2\cdot 2, & 2\cdot 2^2, & \cdots, & 2\cdot 2^{k_1}, \\ 3, & 3\cdot 2, & 3\cdot 2^2, & \cdots, & 3\cdot 2^{k_2}, \\ 5, & 5\cdot 2, & 5\cdot 2^2, & \cdots, & 5\cdot 2^{k_3}, \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{array}$$

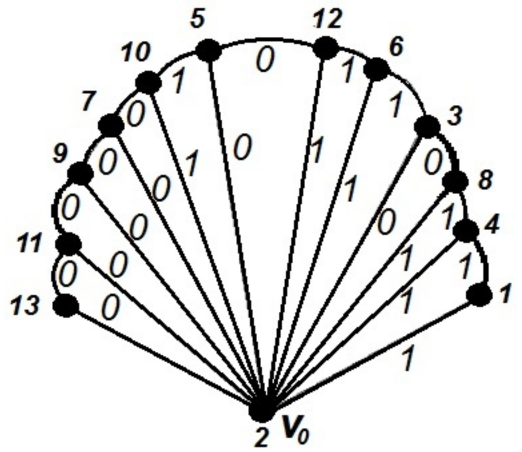

Fig. 2. A shell graph $C(13, 10)$ and its divisor cordial labeling

where $(2m-1)\cdot 2^{k_m} \leq p$ and $m \geq 1, k_m \geq 0$. We observe that $(2m-1)\cdot 2^a$ divides $(2m-1)\cdot 2^b (a < b)$ and $(2m-1)\cdot 2^{k_i}$ does not divide $2m+1$.

Let G' be a graph obtained from the bow graph G by adding the edge $v_m v_{m+1}$.
The graph G' has an odd number of edges and it is a shell graph, then by Proposition 3.3 the graph G' is divisor cordial. The graph $G = G' - v_m v_{m+1}$ with even edges, then G is divisor cordial since:

Case 1: If $m+n$ is even, then $e_f(0) = e_f(1) + 1$ hence the deleted edge $v_m v_{m+1}$ must be labeled 0.

Subcase i: If $f(v_m) = (2t-1)\cdot 2^{k_i}$ for some i, then the deleted edge $v_m v_{m+1}$ is labeled 0.

Subcase ii: If $f(v_m) \neq (2t-1)\cdot 2^{k_i}$ for some i, then we will shift the labels of vertices $v_2, v_3, ..., v_{m+n-l}$ in the wings, by l where l is the smallest integer satisfying $f(v_{m+1}) = (2t-1)\cdot 2^{k_i}$ for some i, and shift the labels of the vertices $v_{m+n-l+1}, v_{m+n-l+2}, ..., v_{m+n}$, by $l+1$ and take it modulo $(m+n+1)$.

Case 2: If $m+n$ is odd, then $e_f(1) = e_f(0) + 1$ hence the deleted edge $v_m v_{m+1}$ must be labeled 1.

Subcase i: If $f(v_m) = (2t-1)\cdot 2^{k_i}$ for some i, then we will shift the labels of vertices $v_2, v_3, ..., v_{m+n-1}$ in the wings, by one step and shift the label of vertex v_{m+n} by two and take it modulo $(m+n+1)$.

Subcase ii: If $f(v_m) \neq (2t-1)\cdot 2^{k_i}$ for some i, then the edge $v_m v_{m+1}$ is labeled 1.

Then the bow graph G with two wings of m and n vertices is a divisor cordial graph for each m and n. □

Example 3.6. The bow graph with two wings of 13 *and* 16 vertices respectively and its divisor cordial labeling are shown in Fig. 3

Butterfly Graphs

Proposition 3.7. The butterfly graphs are divisor cordial.

PROOF. Let G be a butterfly graph with shells of orders m and n excluding the apex then the number of vertices in G is $p = m+n+3$ and the edges $q = 2(m+n)$. The apex of the butterfly graph is denoted as v_0,denote the vertices in the right wing of the butterfly graph from bottom to top as $v_1, v_2, ..., v_m$, the vertices in the left wing of the butterfly graph are denoted from top to bottom as $v_{m+1}, v_{m+2}, ..., v_{m+n}$, and the vertices in the pendant edges are v_{m+n+1}, v_{m+n+2}.
Since the butterfly defined as a bow graph with exactly two pendent edges at the apex,then we define the labeling $f : V \longrightarrow \{1, 2, ..., m+n+3\}$ by:
$f(v_0) = 2,\ f(v_1) = 1,\ f(v_{m+n+1}) = m+n+2,\ f(v_{m+n+2}) = m+n+3$ and labeled the vertices of the

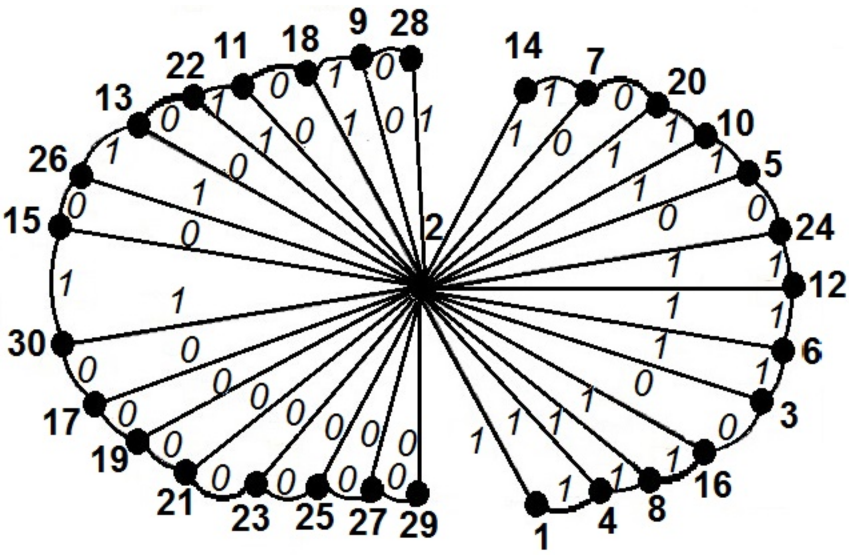

Fig. 3. A bow graph with $m = 13, n = 16$ and its divisor cordial labeling.

wings by the following:

$$\begin{array}{cccccc} & 2\cdot 2, & 2\cdot 2^2, & \cdots, & 2\cdot 2^{k_1}, \\ 3, & 3\cdot 2, & 3\cdot 2^2, & \cdots, & 3\cdot 2^{k_2}, \\ 5, & 5\cdot 2, & 5\cdot 2^2, & \cdots, & 5\cdot 2^{k_3}, \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{array}$$

where $(2m-1)\cdot 2^{k_m} \leq p$ and $m \geq 1, k_m \geq 0$. We observe that $(2m-1)\cdot 2^a$ divides $(2m-1)\cdot 2^b (a < b)$ and $(2m-1)\cdot 2^{k_i}$ does not divide $2m+1$.
And we make the shift as in Proposition 3.5, for labeling of the vertices in the wings.
Since the only one of the numbers $m+n+2$ or $m+n+3$ must be even then the pendent edges will be labeled 1 and 0, hence the graph G is divisor cordial. □

Example 3.8. The butterfly graph G with two wings having $m = 9, n = 15$ vertices respectively, and its divisor cordial labeling is shown in Fig. 4.

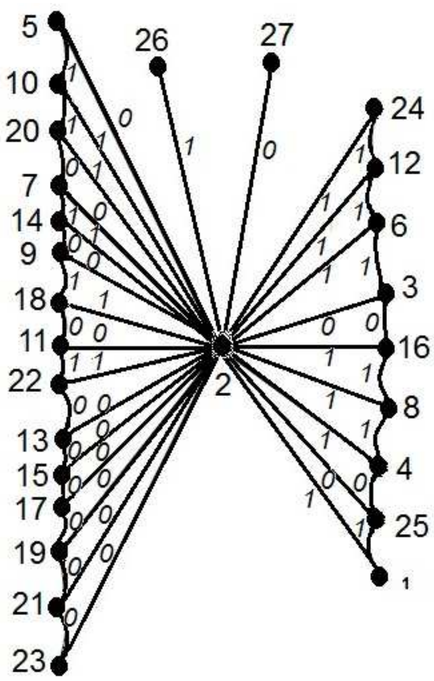

Fig. 4. A divisor cordial labeling for the butterfly with 27 vertices

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Bounds on the Path Energy

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Abstract − In this paper, the path energy is investigated for path matrix. Some bounds are explored for the path energy in terms of the eigenvalues and vertices. Also, some relations are obtained for tree connected graphs.

Keywords − *Path matrix, path energy*

1. Introduction

The path matrix is a popular matrix in graph theory, recently and it had started to develop in 2016. The path matrix of a graph G is defined as a real and symmetric matrix whose (i,j)-entry is the maximum number of internally disjoint paths between the vertices v_i and v_j when $i \neq j$ and is zero when $i = j$. Its eigenvalues are real and they are called path eigenvalues of G. The spectral radius of $P(G)$ is represented by $\rho = \rho(G)$. The concept of path matrix deals with vertices whose mathematical properties are reported in [1].

The path energy is described as the sum of the absolute values of path eigenvalues and it is denoted by $PE = PE(G)$. For several positive eigenvalues of order n, $PE(G) \geq 2(n-1)$. If G is a k-connected tree graph then $\rho(G) \geq k(n-1) \geq k^2$. Also, $PE(G) \geq 2\rho(G)$ for the spectral radius $\rho(G)$. The survey of properties of path energy is given in [2], [3].

The purpose of this paper is to examine different bounds for path energy in terms of defining relations. These bounds are important for they can be used in many areas of graph theory. Considering these cases, known and related results are given in second section. Then, main bounds are obtained using the vertices, the edges and the eigenvalues for path energy in the third section. These bounds are sharp.

2. Preliminaries

In order to prove the main results, some lemmas are needed:

Lemma 2.1. [4] If $a_1, a_2, ..., a_n \in R$ and $0 < m \leq a_i \leq M$ then,

$$(\frac{1}{n}\sum_{i=1}^{n} a_i)(\sum_{i=1}^{n} \frac{1}{a_i}) \leq \frac{M+n}{4Mn}$$

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Lemma 2.2. [5] Let $p = (p_i)$ and $a = (a_i)$, $i = 1, 2, ..., n$ be real sequences with $p_1 + p_2 + ... + p_n = 1$ and $r \leq a_i \leq R$. For such sequences,

$$0 \leq \sum_{i=1}^{n} p_i(a_i)^2 - (\sum_{i=1}^{n} p_i a_i)^2 \leq \frac{1}{2}(R-r)\sum_{i=1}^{n} p_i|a_i - \sum_{j=1}^{n} p_j a_j|$$

See [6], [7] for details.

3. MAIN RESULTS

In this section, some relations and bounds for energy of path matrix are established. These sharp results are surveyed with some fixed parameters. In addition, a relation is determined for tree connected graphs under the assumption of Lemma 2.2.

Theorem 3.1. Let G be a connected graph with eigenvalues of path matix; $\lambda_1^P, \lambda_2^P, ..., \lambda_n^P$. Then,

$$PE(G) \leq \sqrt{\frac{n^2}{4(n-1)}(\rho - \eta)}$$

where $\eta = \eta(G) = |\lambda_n^P|$.

Proof. Let $a_i = |\lambda_i^P|$, $b_i = 1$. Assume that all the path eigenvalues of G are non-zero. A classical lemma (the Ozeki's inequality) refered in the article [8] implies that

$$n\sum_{i=1}^{n} |\lambda_i^P|^2 - (\sum_{i=1}^{n} |\lambda_i^P|)^2 \leq \frac{n^2}{4}(\rho - \eta)$$

That is;

$$(\sum_{i=1}^{n} |\lambda_i^P|)^2 \leq \frac{n}{4}(\rho - \eta) + \frac{(PE(G))^2}{n}$$

By the arrangements, the above ineguality transforms into

$$\frac{(n-1)}{n}(PE(G))^2 \leq \frac{n}{4}(\rho - \eta)$$

Consequently,

$$PE(G) \leq \sqrt{\frac{n^2(\rho - \eta)}{4(n-1)}}$$

□

Theorem 3.2. Let G be a connected graph consisting n vertices. Then,

$$PE(G) \leq \sqrt{\frac{n(\rho + \eta)}{4\rho\eta}}$$

where $\eta = \eta(G) = |\lambda_n^P|$.

Proof. Let $a_i = |\lambda_i^P|$, $m = \eta$, $M = \rho$. By the Lemma 2.1, the following inequality gives that

$$\begin{aligned}(\frac{1}{n}\sum_{i=1}^{n} |\lambda_i^P|)(\sum_{i=1}^{n} \frac{1}{|\lambda_i^P|}) \quad &\leq \quad \frac{1}{n}(\sum_{i=1}^{n} |\lambda_i^P|)^2 \\ &= \frac{1}{n}(PE(G))^2\end{aligned}$$

On the other hand,

$$\frac{1}{n}(PE(G))^2 \leq \frac{\rho+\eta}{4\rho\eta}$$

Thus, the proof is completed with

$$PE(G) \leq \sqrt{\frac{n(\rho+\eta)}{4\rho\eta}}$$

□

Theorem 3.3. If G is a connected graph and G has n vertices, then

$$PE(G) \geq \sqrt{4(\rho(G))^2 + n(n-1)|[P(G)]|^{\frac{2}{n}}}$$

where $|[P(G)]|$ is the determinant of $[P(G)]$.

PROOF. By the Arithmetic-Geometric Mean inequality, the definition of path energy turns into

$$\begin{aligned}(PE(G))^2 &= \sum_{i=1}^{n}|\lambda_i^P|^2 + 2\sum_{1\leq i<j\leq n}|\lambda_i^P||\lambda_j^P| \\ &\geq \sum_{i=1}^{n}|\lambda_i^P|^2 + n(n-1)(\prod_{i=1}^{n}|\lambda_i^P|)^{\frac{2}{n}}\end{aligned}$$

Since $\sum_{i=1}^{n}|\lambda_i^P|^2 \geq (PE(G))^2$, then $\sum_{i=1}^{n}|\lambda_i^P|^2 \geq (PE(G))^2 \geq 4(\rho(G))^2$. Hence, the inequality gives that

$$(PE(G))^2 \geq 4(\rho(G))^2 + n(n-1)|[P(G)]|^{\frac{2}{n}}$$

Thus,

$$PE(G) \geq \sqrt{4(\rho(G))^2 + n(n-1)|[P(G)]|^{\frac{2}{n}}}$$

□

Corollary 3.4. Let G be a k-connected tree graph with n vertices. Then,

$$PE(G) \geq \sqrt{(n-1)[4k^2(n-1) + n|[P(G)]|^{\frac{2}{n}}]}$$

PROOF. As noted in [1], $\rho(G) \geq k(n-1)$ in this case. Therefore the corollary is clear. □

Corollary 3.5. Let G be a k-connected tree graph of order n. Then
i)

$$PE(G^C) \geq \sqrt{(n-1)[4k^2(n-1) + n|[P(G)]|^{\frac{2}{n}}]}$$

where G^C is the complement of G.
ii)

$$\rho(G') \geq \sqrt{(n-2)[4k^2(n-2) + (n-1)|[P(G)]|^{\frac{2}{n-1}}]}$$

where G' is formed from G by deleting edge ij.

Theorem 3.6. Let G be a connected graph of order n then,

$$PE(G) \leq m + \sqrt{2mn}$$

PROOF. Minkowski inequality gives that

$$(\sum_{i=1}^{n}(|\lambda_i^P| + 1)^2)^{\frac{1}{2}} \leq (\sum_{i=1}^{n}|\lambda_i^P|^2)^{\frac{1}{2}} + (\sum_{i=1}^{n} 1)^{\frac{1}{2}}$$

By the help of Bernoulli inequality, it is stated that

$$(n + 2PE(G))^{\frac{1}{2}} \leq (\sum_{i=1}^{n}|\lambda_i^P|^2)^{\frac{1}{2}} + n^{\frac{1}{2}}$$

Since $\sum_{i=1}^{n}(\lambda_i^P)^2 = 2m$, then

$$n + 2PE(G) \leq (\sqrt{2m} + \sqrt{n})^2$$

Hence,

$$PE(G) \leq m + \sqrt{2mn}$$

□

Corollary 3.7. Let G be a connected graph with n vertices and m edges. Then, $PE(G^C) \leq \dfrac{n(n-1)-2m}{2} + \sqrt{(n(n-1)-2m)n}$ where G^C is the complement of G.

PROOF. By the Theorem 3.6, $PE(G^C) \leq m^C + \sqrt{2m^C n}$. Since $2(m+m^C = n(n-1))$, then $PE(G^C) \leq \dfrac{n(n-1)}{2} - m + \sqrt{2(\dfrac{n(n-1)}{2} - m)n}$. Hence, $PE(G^C) \leq \dfrac{n(n-1)-2m}{2} + \sqrt{(n(n-1)-2m)n}$.

□

Theorem 3.8. Let G be a connected tree graph of order n. Then,

$$PE(G) \geq \frac{4mn - 8(n-1)^2}{n(\rho - \eta)} + 2\rho$$

where $\eta = \eta(G) = |\lambda_n^P|$.

PROOF. Let $p_i = \frac{1}{n}$, $a_i = |\lambda_i^P|$, $r = \eta$, $R = \rho$. Lemma 2.2 implies that

$$\frac{1}{n}\sum_{i=1}^{n}|\lambda_i^P|^2 - (\sum_{i=1}^{n}\frac{1}{n}|\lambda_i^P|)^2 \leq \frac{1}{2}(\rho - \eta)(\sum_{i=1}^{n}\frac{1}{n}||\lambda_i^P| - \frac{1}{n}\sum_{j=1}^{n}|\lambda_j^P||)$$

This requires

$$\frac{2m}{n} - \frac{1}{n^2}(PE(G))^2 \leq \frac{1}{2}(\rho - \eta)(\frac{1}{n}PE(G) - 2n\frac{1}{n^2}\rho)$$

Since $PE(G) \geq 2(n-1)$, we have

$$\frac{2m}{n} - \frac{4(n-1)^2}{n^2} \leq \frac{(\rho - \eta)}{2n}(PE(G)) - \frac{\rho(\rho - \eta)}{n}$$

Hence,

$$PE(G) \geq \frac{4mn - 8(n-1)^2}{n(\rho - \eta)} + 2\rho$$

□

4. Conclusion

In this paper, the path energy is studied using the path matrix. Different bounds are obtained for the path energy with some fixed parameters.

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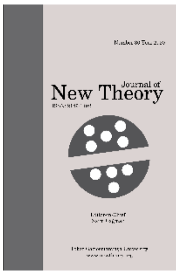

Tritopological Views in Product Spaces

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Abstract − In this paper we define and study the product of tritopological spaces (which we named it δ^*-product). Moreover, to motivate our definition, we show that the product properties for tritopological spaces are not preserved. Further, we provide some necessary and sufficient conditions for these spaces to be preserved under a finite product.

Keywords − *Tritopological spaces, δ^*-product of two tritopological spaces, δ^*-Tychonoff tritopology*

1. Introduction

In mathematics, the Cartesian product of a collection of sets is one of the most important and widely used ideas. The theory of product spaces constitutes a very interesting and complex part of set-theoretic topology. The Cartesian product of arbitrarily topological spaces was defined by Tychonoff in 1930 [1].

Then almost 33 years later in 1963, the idea of bitopological spaces was initiated by Kelly [2], and after that, a large number of papers have been produced in order to generalize the topological concepts to bitopological setting. In 1972, Datta [3] defined the Cartesian product of arbitrarily bitopological spaces. It is also well-known that the Tychonoff Product Theorem plays an important role in a general product.

A tritopological space is simply a set X which is associated with three arbitrary topologies, was initiated by Kovar [4]. In 2004, Hassan introduced the definition of δ^*-open set in tritopological spaces as follows, a subset A of X is said to be δ^*-open set iff $A \subseteq \mathcal{T}\ \mathrm{int}(\mathcal{P}\ \mathrm{cl}(\mathcal{Q}\ \mathrm{int}(A)))$ [5]. And in [6] she defined the δ^*-connectedness in tritopological spaces, also Hassan et al. [7] defined the δ^*-base in tritopological spaces. In [8] and [9] the reader can find a relationship among separation axioms, and relationships among some types of continuous and open functions in topological, bitopological and tritopological spaces, and in 2017, Hassan introduced the new definitions of countability and separability in tritopological spaces namely δ^*-countability and δ^*-separability [10]. In 2017, Hassan presented the concept of soft tritopological spaces [11]. However, no concept of tritopologization in product spaces has been given until now.

In the present paper, the concept of product topological spaces has been generalized to initiate the definition and study of product tritopological spaces. Besides, we introduce and characterize new definitions and theorems in tritopological spaces, and we provide some necessary and sufficient conditions for these spaces to be preserved under the δ^*-product.

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In section 2, some preliminary concepts about tritopological spaces are given. The main section of the manuscript is third which the definition of δ^*-product tritopology of two tritopological spaces with examples and some theorems are given. Section 4 is devoted to the generalization to theorems for tritopological product of spaces. In section 5 the definition of δ^*-Tychonoff tritopology and some theorems are introduced. Finally, in section 6 the conclusions and some future work is suggested

2. Preliminaries

In the following, we will mention some basic definitions and notations in tritopological space which we need in this work.

Definition 2.1. [5] Let $(X, \mathcal{T}, \mathcal{P}, \mathcal{Q})$ be a tritopological space, a subset A of X is said to be δ^*-open set iff $A \subseteq \mathcal{T}\,\mathrm{int}(\mathcal{P}\,\mathrm{cl}(\mathcal{Q}\,\mathrm{int}(A)))$, and the family of all δ^*-open sets is denoted by $\delta^*.O(X)$. ($\delta^*.O(X)$ not always represent a topology). The complement of δ^*-open set is called a δ^*-closed set.

Definition 2.2. [5] $(X, \mathcal{T}, \mathcal{P}, \mathcal{Q})$ is called a discrete tritopological space with respect to δ^*-open if $\delta^*.O(X)$ contains all subsets on X. And $(X, \mathcal{T}, \mathcal{P}, \mathcal{Q})$ is called an indiscrete tritopological space with respect to δ^*-open if $\delta^*.O(X) = \{X, \emptyset\}$.

Definition 2.3. [5] Let $(X, \mathcal{T}, \mathcal{P}, \mathcal{Q})$ be a tritopological space, and let $x \in X$, a subset N of X is said to be a δ^*-nhd of a point x iff there exists a δ^*-open set U such that $x \in U \subset N$. The set of all δ^*-nhds of a point x is denoted by $\delta^* - N(x)$.

Definition 2.4. [7] A collection δ^*-β of a subset of X is said to form a δ^*-base for the tritopology $(\mathcal{T}, \mathcal{P}, \mathcal{Q})$ iff: δ^*-β $\subset \delta^*.O(X)$. for each point $x \in X$ and each δ^*-neighbourhood $\mathcal{N}$ of x there exists some $\mathcal{B} \in \delta^*$-β such that $x \in \mathcal{B} \subset \mathcal{N}$.

Definition 2.5. [5] The function $f: (X, \mathcal{T}, \mathcal{P}, \mathcal{Q}) \to (Y, \mathcal{T}', \mathcal{P}', \mathcal{Q}')$ is said to be δ^*-continuous at $x \in X$ iff for every δ^*-open set V in Y containing $f(x)$ there exists δ^*-open set U in X containing x such that $f(U) \subset V$. We say f is δ^*-continuous on X iff f is δ^*-continuous at each $x \in X$.

Definition 2.6. [5] The function $f: (X, \mathcal{T}, \mathcal{P}, \mathcal{Q}) \to (Y, \mathcal{T}', \mathcal{P}', \mathcal{Q}')$ is said to be δ^*-open iff $f(G)$ is δ^*-open in Y for every δ^*-open set G in X.

Definition 2.7. [5] Let $(X, \mathcal{T}, \mathcal{P}, \mathcal{Q})$ and $(Y, \mathcal{T}', \mathcal{P}', \mathcal{Q}')$ are two tritopological spaces and $f: (X, \mathcal{T}, \mathcal{P}, \mathcal{Q}) \to (Y, \mathcal{T}', \mathcal{P}', \mathcal{Q}')$ be a function, then f is δ^*-homeomorphism if and only if:

i. f is bijective (one to one, onto).
ii. f and f^{-1} are δ^*-continuous (or f is δ^*-continuous and δ^*-open).

Definition 2.8. [5] Let $(X, \mathcal{T}, \mathcal{P}, \mathcal{Q})$ be a tritopological space, a point x is called δ^*-limit point of a subset A of X iff for each δ^*-open set G containing another point different from x in A; that is $(G / \{x\}) \cap A \neq \emptyset$, and the set of all δ^*-limit points of A is denoted by $\delta^* - lm(A)$.

Definition 2.9. [5] A tritopological space $(X, \mathcal{T}, \mathcal{P}, \mathcal{Q})$ is called δ^*-T_2-space (δ^*-Hausdorff) if and only if for each pair of distinct points x, y of X, there exists two δ^*-open sets G, H such that $x \in G,\ y \in H,\ G \cap H = \emptyset$.

Definition 2.10. [5] Let $(X, \mathcal{T}, \mathcal{P}, \mathcal{Q})$ be a tritopological space, and let A be any subset of X, then the collection $C = \{G_\lambda : \lambda \in \Lambda\}$ is called δ^*-open cover to A if C is a cover to $A$and $C \subset \delta^*.O(X)$.

Definition 2.11. [5] Let $(X, \mathcal{T}, \mathcal{P}, \mathcal{Q})$ be a tritopological space, and let A be any subset of X, then A is called δ^*-compact set iff every δ^*-open cover of A has a finite sub-cover, i.e. for each $\{G_\lambda : \lambda \in \Lambda\}$ of δ^*-open sets for which $A \subset \cup \{G_\lambda : \lambda \in \Lambda\}$, there exist finitely many sets $G_{\lambda 1}, \dots, G_{\lambda n}$ among the G_λ's such that $A \subset G_{\lambda 1} \cup \dots \cup G_{\lambda n}$.

In particular, the space X is called δ^*-compact iff for each collection $\{G_\lambda : \lambda \in \Lambda\}$ of δ^*-open sets for which $X = \cup \{G_\lambda : \lambda \in \Lambda\}$, there exist finitely many sets $G_{\lambda 1}, \ldots, G_{\lambda n}$ among the G_λ's such that $X = G_{\lambda 1} \cup \ldots \cup G_{\lambda n}$.

Definition 2.12. [10] Let $(X, \mathcal{T}, \mathcal{P}, \mathcal{Q})$ be a tritopological space. Space is said to be a δ^*-second countable (or to satisfy the second axiom of δ^*-countability in tritopology) iff there exists a δ^*-countable base for a tritopology.

Definition 2.13. [12] Let $\{X_\lambda : \lambda \in \Lambda\}$ is an arbitrary collection of sets indexed by Λ, then the Cartesian product of this collection is the set of all mappings x defined by $x: \Lambda \to \cup \{X_\lambda : \lambda \in \Lambda\}$ such that $x(\lambda) \in X_\lambda$ for all $\lambda \in \Lambda$ and is denoted by $\pi\{X_\lambda : \lambda \in \Lambda\}$ or by $\times \{X_\lambda : \lambda \in \Lambda\}$. The set X_λ is called the λ^{th} coordinate set of the product.

Definition 2.14. [12] Let $X = \times \{X_\lambda : \lambda \in \Lambda\}$, then the mapping $\pi_\lambda : X \to X_\lambda$ defined by $\pi_\lambda(x) = x_\lambda$ for all $x \in X$ is called the λ^{th} projection.

3. Product space of two tritopological spaces

In this section, we shall describe the technique for constructing a tritopology for the Cartesian product $X \times Y$ of two tritopological spaces X and Y with the help of the families of all δ^*-open sets $\delta^*.O(X)$ and $\delta^*.O(Y)$ of the two spaces $(X, \mathcal{T}, \mathcal{P}, \mathcal{Q})$ and $(Y, \acute{\mathcal{T}}, \acute{\mathcal{P}}, \acute{\mathcal{Q}})$ shall examine the properties of the tritopology thus obtained in minute details. Subsequent sections will be devoted to the way of tritopologizing the Cartesian product of an arbitrary collection of tritopological spaces.

Because the families of all δ^*-open sets $\delta^*.O(X)$ and $\delta^*.O(Y)$ does not always represent a topology [5]. We provide some necessary conditions for these theorems to be valid under a finite product.

Theorem 3.1. Let $(X, \mathcal{T}, \mathcal{P}, \mathcal{Q})$ and $(Y, \acute{\mathcal{T}}, \acute{\mathcal{P}}, \acute{\mathcal{Q}})$ be two tritopological space and if $\delta^*.O(X)$ and $\delta^*.O(Y)$ represent a topology. Then the collection $E = \{G \times H : G \in \delta^*.O(X) \text{ and } H \in \delta^*.O(Y)\}$ is a δ^*-base for some tritopology for $X \times Y$.

Proof. Assume that $\delta^*.O(X)$ and $\delta^*.O(Y)$ represent a topology. We shall show that E satisfies the conditions [B1] and [B2] of Theorem [7], Since $X \times Y \in E$, we have $X \times Y = \cup \{G \times H : G \in \delta^*.O(X) \text{ and } H \in \delta^*.O(Y)\}$. Thus, [B1] is satisfied.

Now let $G_1 \times H_1$ and $G_2 \times H_2$ be any two members of E. We then have

$$(G_1 \times H_1) \cap (G_2 \times H_2) = (G_1 \cap G_2) \times (H_1 \cap H_2) \qquad (1) \text{ [see (2.18) (iii), ch. 1]}$$

Since we assume that $\delta^*.O(X)$ and $\delta^*.O(Y)$ represent a topology, we have

$$G_1 \in \delta^*.O(X),\ G_2 \in \delta^*.O(X) \to G_1 \cap G_2 \in \delta^*.O(X)$$

And $H_1 \in \delta^*.O(Y)$, $H_2 \in \delta^*.O(Y) \to H_1 \cap H_2 \in \delta^*.O(Y)$.

Hence it follows from (1) that $(G_1 \times H_1) \cap (G_2 \times H_2) \in E$. Thus, we have shown that the intersection of any two members of E is again a member of E and so $[B2]$ is also satisfied. Therefore E is a δ^*-base for some tritopology for $X \times Y$.

Remark 3.2. If $\delta^*.O(X)$ and $\delta^*.O(Y)$ does not represent a topology; the above theorem is not achieved. Because the intersection of any two members of E is not always a member of E and so $[B2]$ is not satisfied. Therefore E is not a δ^*-base for some tritopology for $X \times Y$. (see example 1.1.4 in [5]).

Definition 3.3. Let $(X, \mathcal{T}, \mathcal{P}, \mathcal{Q})$ and $(Y, \acute{\mathcal{T}}, \acute{\mathcal{P}}, \acute{\mathcal{Q}})$ be two tritopological space. Then the tritopology (U, V, W) whose δ^*-base is $E = \{G \times H : G \in \delta^*.O(X) \text{ and } H \in \delta^*.O(Y)\}$ is called the δ^*-product tritopology for $X \times Y$ and $(X \times Y, U, V, W)$ is called the δ^*-product space of X and Y.

Observe that in view of theorem (3.1), E is a δ^*-base for some tritopology for $X \times Y$. This is the tritopology (U, V, W) of the above definition.

Theorem 3.4. Let $(X, \mathcal{T}, \mathcal{P}, \mathcal{Q})$ and $(Y, \acute{\mathcal{T}}, \acute{\mathcal{P}}, \acute{\mathcal{Q}})$ be two tritopological spaces. And β is a δ^*-base for $(\mathcal{T}, \mathcal{P}, \mathcal{Q})$ and $\complement$ is a δ^*-base for $(\acute{\mathcal{T}}, \acute{\mathcal{P}}, \acute{\mathcal{Q}})$. Then $\wp = \{B \times C : B \in \beta \; and \; C \in \complement\}$ is a δ^*-base for the δ^*-product tritopology (U, V, W) for $X \times Y$.

Proof. *Let* (x, y) be any point of $X \times Y$ and let N be a δ^*-nhd of (x, y) in $X \times Y$. Since $E = \{G \times H : G \in \delta^*.\mathrm{O}(X) \; and \; H \in \delta^*.\mathrm{O}(Y)\}$ is a δ^*-base for (U, V, W), there exists a member $G \times H$ of E such that $(x, y) \in G \times H \subset N$. … (1)

Since G is δ^*-open and β is a δ^*-base for $(\mathcal{T}, \mathcal{P}, \mathcal{Q})$, there exists some $B \in \beta$ such that $x \in B \subset G$. Similarly, there exists some $C \in \complement$ such that $y \in C \subset H$. It follows that

$(x, y) \in B \times C \subset G \times H$. … (2)

Hence from (1) and (2), we get $(x, y) \in B \times C \subset N$. This implies that $\wp$ is a δ^*-base for (U, V, W).

Example 3.5. Let $X = \{a, b, c\}$, $\mathcal{T} = \{X, \emptyset, \{b\}, \{c\}, \{b, c\}\}$

, $\mathcal{P} = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}\}$

, $\mathcal{Q} = \{X, \emptyset, \{a\}, \{c\}, \{a, c\}\}$

$(X, \mathcal{T})$, $(X, \mathcal{P})$ and $(X, \mathcal{Q})$ are three topological space, then $(X, \mathcal{T}, \mathcal{P}, \mathcal{Q})$ is a tritopological space, the family of all δ^*-open set of X is: $\delta^*.\mathrm{O}(X) = \{X, \emptyset, \{c\}\}$

And let $Y = \{p, q, r, s\}$, $\acute{\mathcal{T}} = \{Y, \emptyset\}$

, $\acute{\mathcal{P}} = \{Y, \emptyset\}$

, $\acute{\mathcal{Q}} = \{Y, \emptyset, \{p\}, \{q\}, \{p, q\}, \{r, s\}, \{p, r, s\}, \{q, r, s\}\}$

$(Y, \acute{\mathcal{T}})$, $(Y, \acute{\mathcal{P}})$ and $(Y, \acute{\mathcal{Q}})$ are three topological space, then $(Y, \acute{\mathcal{T}}, \acute{\mathcal{P}}, \acute{\mathcal{Q}})$ is a tritopological space, the family of all δ^*-open set of Y is:

$$\delta^*.O(Y) = \{Y, \emptyset, \{p\}, \{q\}, \{p, q\}, \{r, s\}, \{p, r, s\}, \{q, r, s\}\}$$

Now we will find a δ^*-base for the δ^*-product tritopology of $X \times Y$.

It is easy to see that $\beta = \{\{c\}, X\}$ is a δ^*-base for $(\mathcal{T}, \mathcal{P}, \mathcal{Q})$ and $\complement = \{\{p\}, \{q\}, \{r, s\}\}$ is a δ^*-base for $(\acute{\mathcal{T}}, \acute{\mathcal{P}}, \acute{\mathcal{Q}})$.

Hence by theorem (3.4) above, a δ^*-base for the δ^*-product tritopology is given by

$$\wp = \{\{c\} \times \{p\}, \{c\} \times \{q\}, \{c\} \times \{r, s\}\}, X \times \{p\}, X \times \{q\}, X \times \{r, s\}\}$$
$$= \{\{(c, p)\}, \{(c, q)\}, \{(c, r), (c, s)\}, \{(a, p), (b, p), (c, p)\}, \{(a, q), (b, q), (c, q)\},$$
$$\{(a, r), (a, s), (b, r), (b, s), (c, r), (c, s)\}\}$$

Definition 3.6. A tritopology $(\mathcal{T}, \mathcal{P}, \mathcal{Q})$ on a set X is said to be δ^*-weaker (or δ^*-coarser or δ^*-smaller) than another Tritopology $(\acute{\mathcal{T}}, \acute{\mathcal{P}}, \acute{\mathcal{Q}})$ on X. Or we can say that $(\acute{\mathcal{T}}, \acute{\mathcal{P}}, \acute{\mathcal{Q}})$ is said to be δ^*-stronger (or δ^*-finer or δ^*-larger) than $(\mathcal{T}, \mathcal{P}, \mathcal{Q})$) iff $\delta^*.\mathrm{O}(X) \subset \delta^*.\acute{\mathrm{O}}(X)$, (where $\delta^*.\mathrm{O}(X)$ is the family of all δ^*-open sets in $(X, \mathcal{T}, \mathcal{P}, \mathcal{Q})$ and $\delta^*.\acute{\mathrm{O}}(X)$ is the family of all δ^*-open sets in $(X, \acute{\mathcal{T}}, \acute{\mathcal{P}}, \acute{\mathcal{Q}})$).

According to this definition, indiscrete tritopology on any set X with respect to δ^*-open set is the δ^*-weakest whereas the discrete tritopology on any set X with respect to δ^*-open set is the δ^*-strongest. It is easy to see that the collection C off all tritopologies on a set X is a δ^*-partially ordered set with respect to the relation $\leq$ defined by setting $(\mathcal{T}, \mathcal{P}, \mathcal{Q}) \leq (\acute{\mathcal{T}}, \acute{\mathcal{P}}, \acute{\mathcal{Q}})$ iff $(\mathcal{T}, \mathcal{P}, \mathcal{Q})$ is δ^*-weaker than $(\acute{\mathcal{T}}, \acute{\mathcal{P}}, \acute{\mathcal{Q}})$, where $(\mathcal{T}, \mathcal{P}, \mathcal{Q}) \; and \; (\acute{\mathcal{T}}, \acute{\mathcal{P}}, \acute{\mathcal{Q}})$ are members of C. The indiscrete tritopology on X w.r.t. δ^*-open set is the δ^*-infimum and the discrete tritopology on X w.r.t. δ^*-open set is the δ^*-supremum of $(C, \leq)$.

Theorem 3.7. The δ^*-product tritopology on a non-empty set $X \times Y$ is the δ^*-weak tritopology for $X \times Y$ determined by the projection maps π_x and π_y from the tritopologies on $X\ and\ Y$. (This theorem is valid when $\delta^*.\text{O}(X)$ and $\delta^*.\text{O}(Y)$ are satisfied a topology)

Proof. The δ^*-weak tritopology has a δ^*-subbase $\{G_\lambda : \text{G}_\lambda = {\pi_x}^{-1}[A_\lambda]\ or\ G_\lambda = {\pi_y}^{-1}[B_\lambda]$, for some A_λ δ^*-open in $\delta^*.\text{O}(X)$ or B_λ δ^*-open in $\delta^*.\text{O}(Y)$ }

The intersection ${\pi_x}^{-1}[A_1] \cap \dots .\cap {\pi_x}^{-1}[A_m] \cap {\pi_y}^{-1}[B_1] \cap \dots \cap \pi_y[{B_n}^{-1}]$

$$= (A_1 \times Y) \cap \dots \cap (A_m \times Y) \cap (X \times B_1) \dots \cap (X \times B_n)$$

[since $(A \times B) \cap (S \times T) = (A \cap S) \times (B \cap T)$]. Of a finite number of such sets has the form

$(A_1 \cap A_2 \cap \dots A_m) \times (B_1 \cap B_2 \cap \dots \cap B_n) = A^\star \times B^\star$

Where $A^\star$ is δ^*-open in $\delta^*.\text{O}(X)$ and $B^\star$ is δ^*-open in $\delta^*.\text{O}(Y)$. Hence the δ^*-weak tritopology has the same δ^*-base as the δ^*-product tritopology, and so the two tritopologies are the same.

[Note that ${\pi_x}^{-1}[A_1] = A_1 \times Y, {\pi_y}^{-1}[B_1] = X \times B_1\ \ etc.$]

Definition 3.8. Let $(\text{X}, \mathcal{T}, \mathcal{P}, \mathcal{Q})$ and $(Y, \acute{\mathcal{T}}, \acute{\mathcal{P}}, \acute{\mathcal{Q}})$ be two tritopological spaces. Then the mappings

$\pi_x : X \times Y \to X : \pi_x((x, y)) = x \quad \forall (x, y) \in X \times Y$ and

$\pi_y : X \times Y \to Y : \pi_y((x, y)) = y \quad \forall (x, y) \in X \times Y$

are called the projections of the δ^*-product $X \times Y$ on tritopological spaces X and Y respectively.

Theorem 3.9. Let $(\text{X}, \mathcal{T}, \mathcal{P}, \mathcal{Q})$ and $(Y, \acute{\mathcal{T}}, \acute{\mathcal{P}}, \acute{\mathcal{Q}})$ be two tritopological spaces. And let $(X \times Y, U, V, W)$ be the δ^*-product space of the two spaces. then the projections π_x and π_y are δ^*-continuous and δ^*-open mappings. further the δ^*-product tritopology (U, V, W) is the δ^*-coarsest tritopology for which the projections are δ^*-continuous.

Proof. Recall that π_x is a mapping of $X \times Y$ onto X defined by $\pi_x\ ((x, y)) = x$ for every $(x, y) \in X \times Y$. Let G be any δ^*-open set. Then it is evident from the definition of π_x that ${\pi_x}^{-1}[G] = G \times Y$ which is a basic δ^*-open subset of $X \times Y$.

[$\because G \in \delta^*.\text{O}(X),\ Y \in \delta^*.\text{O}(Y) \to G \times Y \in E$ where E is the δ^*-base for (U, V, W)].

Hence π_x is a δ^*-continuous mapping from $(X \times Y, U, V, W)$ to $(\text{X}, \mathcal{T}, \mathcal{P}, \mathcal{Q})$. Similarly, π_y is a δ^*-continuous mapping from $(X \times Y, U, V, W)$ to $(Y, \acute{\mathcal{T}}, \acute{\mathcal{P}}, \acute{\mathcal{Q}})$. Now let A be any δ^*-open subset of $X \times Y$. Then by the definition of the δ^*-base E for (U, V, W), we have $A = \cup \{G \times H : G \in \delta^*.\text{O}(X), H \in \delta^*.\text{O}(Y)\ and\ G \times H \in E' \subset E\}$.

Hence $\pi_x[A] = \pi_x[\cup \{G \times H : G \in \delta^*.\text{O}(X), H \in \delta^*.\text{O}(Y)\ and\ G \times H \in E']$

$$= \cup \{\pi_x[G \times H] : G \in \delta^*.\text{O}(X), H \in \delta^*.\text{O}(Y)\ and\ G \times H \in E'\}$$

$$= \cup \{G : G \in \delta^*.\text{O}(X)\ and\ G \times H \in E'\} \qquad \text{[By the definition of } \pi_x]$$

$$\in \delta^*.\text{O}(X)$$

It follows that π_x is an δ^*-open mapping. Finally, let (U^*, V^*, W^*) be any tritopology for $X \times Y$ for which the projections are δ^*-continuous and let A be any δ^*-open set of $X \times Y$. Then,

$A = \cup \{G \times H : G \in \delta^*.\text{O}(X), H \in \delta^*.\text{O}(Y)\ and\ G \times H \in E'\}$ where $E' \subset E$

$= \cup \{(G \cap X) \times (Y \cap H) : G \in \delta^*.\text{O}(X), H \in \delta^*.\text{O}(Y)\ and\ G \times H \in E'\}$

$= \cup \{(G \times Y) \cap (X \times H) : G \in \delta^*.\text{O}(X), H \in \delta^*.\text{O}(Y)\ \ and\ G \times H \in E'\}$

$=\cup \left\{{\pi_x}^{-1}[G] \cap {\pi_y}^{-1}[H]: G \in \delta^*.\text{O}(X), H \in \delta^*.\text{O}(Y) \; and \; G \times H \in E'\right\} \in \; \delta^*.\text{O}(X \times Y)^*$

[Where $\delta^*.\text{O}(X \times Y)^*$ is the family of all δ^*-open sets of the space$(X \times Y, U^*, V^*, W^*)$]

$\because \pi_x \; is \;\; \delta^*$-continuous $\Rightarrow {\pi_x}^{-1}[G] \in \delta^*.\text{O}(X \times Y)^*$ and

$\pi_y \; is \;\; \delta^*$-continuous $\Rightarrow {\pi_y}^{-1}[H] \in \delta^*.\text{O}(X \times Y)^*$ etc.

Thus every δ^*-open set in $\delta^*.\text{O}(X \times Y)$ is δ^*-open in $\delta^*.\text{O}(X \times Y)^*$ and so (U, V, W) is δ^*-coarser than (U^*, V^*, W^*) is any tritopology for $X \times Y$ for which the projections are δ^*-continuous, it follows that (U, V, W) is the δ^*-coarsets tritopology for which the projections are δ^*-continuous.

Theorem 3.10. Let y_o be a fixed element of Y and let $A = X \times \{y_o\}$. Then the restriction of π_x to A is a δ^*-homeomorphism of the subspace A of $X \times Y$ onto X. Similarly, the restriction of π_y to $B = \{x_o\} \times Y, x_o \in X$, is a δ^*-homeomorphism.

Proof. Let g_x denote the restriction of π_x to A, that is, let $g_x: A \to X: g_x\big((x, y_o)\big) = x \quad \forall (x, y_o) \in A$. Then $g_x\big((x_1, y_o)\big) = g_x\big((x_2, y_o)\big) \Longrightarrow x_1 = x_2 \Longrightarrow \big((x_1, y_o)\big) = \big((x_2, y_o)\big) \Longrightarrow g_x$ is one – one,

g_x is evidently onto. Since by the preceding theorem π_x is δ^*-continuous, it follows that g_x is also δ^*-continuous [5]. We now show that g_x is δ^*-open. Let C be any δ^*-open subset of A. Then $C = A \cap B$ for some δ^*-open subset B of $X \times Y$. But

$$B = \cup \{G \times H : G \in \delta^*.O(X), H \in \delta^*.O(Y) \; and \; X \times H \in E'\}$$

Where $E' \subset E$. We then have

$g_x[C] = g_x[A \cap B] = g_x[A \cap [\cup \{G \times H: G \in \delta^*.O(X), H \in \delta^*.O(Y) \; and \; G \times H \in E'\}]$

$= g_x[\cup \{A \cap (G \times H): G \in \delta^*.O(X), H \in \delta^*.O(Y) \; and \;\; G \times H \in E']$ [Distributive law]

$=\cup \{g_x[(X \times \; \{y_o\}) \cap (G \times H)]: G \in \delta^*.O(X), H \in \delta^*.O(Y) \; and \; G \times H \in E']$

$=\cup \{g_x[(X \cap G) \times (\{y_o\}) \cap H)]: G \in \delta^*.O(X), H \in \delta^*.O(Y) \; and \; G \times H \in E']$

$=\cup \{g_x[G \times \{y_o\} \cap H)]: G \in \delta^*.O(X), H \in \delta^*.O(Y) \; and \; G \times H \in E'] \quad ... (1)$

If $y_o \notin H$, then it is easy to see from (1) that $g_x[C] = \emptyset$. If $y_o \in H$, then (1) gives

$$g_x[C] = \cup \{g_x[G \times \{y_o\}]: G \in \delta^*.O(X), H \in \delta^*.O(Y) \; and \; G \times H \in E']$$

$$g_x[C] = \cup \{G: G \in \delta^*.O(X), and \; G \times H \in E'\} \in \delta^*.O(X).$$

This implies that g_x is an δ^*-open mapping as well. Thus, we have shown that g_x is one-one, onto, δ^*-continuous and δ^*-open mapping and consequently it is a δ^*-homeomorphism.

4. $\boldsymbol{\delta^*}$-Product invariant properties for finite $\boldsymbol{\delta^*}$-products

We are going to generalize theorems for tritopological product of spaces.

Theorem 4.1. The δ^*-product space $X \times Y$ is δ^*-connected if and only if the tritopological spaces X and Y are δ^*-connected.

Proof. Assume that $X \times Y$ is δ^*-connected. Since the projections π_x and π_y are δ^*-continuous and onto mappings, it follows from Theorem in [6] that X and Y are also δ^*-connected spaces. Conversely, let X and Y be δ^*-connected spaces. To show that $X \times Y$ is also δ^*-connected. Let (x_1, y_1) and (x_2, y_2) be any two points of $\delta^*.O(X \times Y)$. Then by theorem (3.10), $\{x_1\} \times Y$ is δ^*-homeomorphic to Y and $X \times \{y_2\}$ is δ^*-homeomorphic to X. Hence $\{x_1\} \times Y$ and $X \times \{y_2\}$ are δ^*-connected by theorem in [6] They intersect in the points (x_1, y_2) and hence there union is a δ^*-connected set by theorem in [6].Since this union contains (x_1, y_1) and (x_2, y_2), it follows from Theorem in [6] that $X \times Y$ is δ^*-connected.

Theorem 4.2. The δ^*-product space $X \times Y$ is δ^*-compact if and only if each of the tritopological spaces X and Y is δ^*-compact.

Proof. Let $X \times Y$ be δ^*-compact. Since the projection maps $\pi_x: X \times Y \to X$ and $\pi_y: X \times Y \to Y$ are δ^*-continuous and onto, it follows from Theorem in [5] that X and Y are also δ^*-compact. Conversely, let X and Y be δ^*-compact spaces. We want to show that $X \times Y$ is δ^*-compact. In view of Theorem in [5], it suffices to show that every basic δ^*-open cover of $X \times Y$ has a finite subcover. Since a basic δ^*-open set in $X \times Y$ is of the form $G \times H$ where G is δ^*-open in X and H is δ^*-open in Y, we may denote a basic δ^*-open cover by $C = \{G_\lambda \times H_\lambda: \lambda \in \Lambda\}$ where G_λ is δ^*-open in X and H_λ is δ^*-open in Y. For a given point $x \in X$, the set $\{x\} \times Y$ is δ^*-homeomorphic to Y by theorem (3.10), and is, therefore, δ^*-compact by theorem in [5]. Since $\{x\} \times Y$, being a subset of $X \times Y$, is covered by C and $\{x\} \times Y$ is δ^*-compact, there exists a finite sub-family of C, say $\{G_{\lambda i} \times H_{\lambda i}: i = 1,2,\dots,n\}$, which covers $\{x\} \times Y$. Let $\cap G_{\lambda i} = G(x)$. Then $G(x)$ is δ^*-open in X and contains x since each $G_{\lambda i}$ contains x. Hence $\{G(x) \times H_{\lambda i}: i = 1,2,\dots,n\}$ is still a finite δ^*-open cover of $\{x\} \times Y$. Proceeding in this manner for each $x \in X$, we construct the collection $\{G(x): x \in X\}$ of δ^*-open sets in X which covers X. By δ^*-compactness of X, there exists a finite subcover $\{G(x_j): j = 1,2,\dots,m\}$ of this cover for X. Since each $G(x_j)$ is an intersection of δ^*-open sets in X which were used to form C, we may select an δ^*-open set $G_{\lambda_{xj}} \in C$ such that $G(x_j) \subset G_{\lambda_{xj}}$ for $j = 1,2,\dots,m$, Therefore $\{G_{\lambda_{xj}}: j = 1,2,\dots,m\}$ is a finite δ^*-open cover of X, and for each j, $1 \le j \le m, \{G_{\lambda_{xj}} \times H_{\lambda_i}: i = 1,2,\dots,n\}$ covers the subset $G(x_j) \times Y$ of $X \times Y$. By its construction the collection $\left\{G_{\lambda_{xj}} \times H_{\lambda_i}: i = 1,2,\dots,n\,;\ j = 1,2,\dots,m\right\}$ is then a finite subcover of C for $X \times Y$ and therefore $X \times Y$ is δ^*-compact by theorem in [5].

Theorem 4.3. The δ^*-product space of two δ^*-second countable tritopological spaces is δ^*-second countable.

Proof. Let X and Y be two δ^*-second countable tritopological spaces. To show that $X \times Y$ is also δ^*-second countable. Let B and C be δ^*-countable bases for X and Y respectively. Consider the collection $D = \{B \times C: B \in \delta^* - \beta,\ C \in \delta^* - \complement\}$. Then D is surely a countable collection [12]. It follows from theorem (3.4), that D is a δ^*-countable bases for $X \times Y$.

Theorem 4.4. The δ^*-product space of two δ^*-Hausdorff tritopological spaces is δ^*-Hausdorff.

Proof. Let X and Y be two δ^*-Hausdorff tritopological spaces. To show that $X \times Y$ is also a δ^*-Hausdorff space. Let (x_1, y_1) and (x_2, y_2) be any two distinct points in $\delta^*.O(X \times Y)$. Then either $x_1 \neq x_2$ or $y_1 \neq y_2$. Take $x_1 \neq x_2$. Since X is δ^*-Hausdorff, there exist δ^*-open sets G_1 and G_2 in $\delta^*.O(X)$ such that $x_1 \in G_1$, $x_2 \in G_2$ and $G_1 \cap G_2 = \emptyset$ [5]. Then $G_1 \times Y$ and $G_2 \times Y$ are δ^*-open subset of $\delta^*.O(X \times Y)$ such that $(x_1, y_1) \in G_1 \times Y$, $(x_2, y_2) \in G_2 \times Y$ and $(G_1 \times Y) \cap (G_2 \times Y) = (G_1 \cap G_2) \times Y = \emptyset \times Y = \emptyset$. It follows that the tritopological space $(X \times Y, U, V, W)$ is δ^*-Hausdorff.

5. δ^*- Product tritopology (or δ^*-Tychonoff tritopology)

Definition 5.1. For each λ in an arbitrary index set Λ, let $(X_\lambda, \mathcal{T}_\lambda, \mathcal{P}_\lambda, \mathcal{Q}_\lambda)$ be a tritopological space and let $X = \times \{X_\lambda: \lambda \in \Lambda\}$. Then tritopology $(\mathcal{T}, \mathcal{P}, \mathcal{Q})$ for X which has a δ^*-sub bases the collection $B_* = \{{\pi_\lambda}^{-1}[G_\lambda]: \lambda \in \Lambda, G_\lambda \in \delta^*.O(X)_\lambda\}$ is called the δ^*-product tritopology (or the δ^*-Tychonoff tritopology) for X, and $(X, \mathcal{T}, \mathcal{P}, \mathcal{Q})$ is called the δ^*-product space of the given spaces.

Note that here π_λ denotes as usual the λ^{th} projection. The collection B_* is called the defining δ^*-subbase for $(\mathcal{T}, \mathcal{P}, \mathcal{Q})$. the collection β of all finite intersections of elements of B_* would then form a δ^*-base for $(\mathcal{T}, \mathcal{P}, \mathcal{Q})$.

Remark 5.2. Since ${\pi_\lambda}^{-1}[G_\lambda]$ are δ^*-open sets with respect to the δ^*-product tritopology where G_λ is any δ^*-open set in X_λ it follows that the projection π_λ is a δ^*-continuous map for each $\lambda \in \Lambda$.

Theorem 5.3. Let $\{(X_\lambda, \mathcal{T}_\lambda, \mathcal{P}_\lambda, \mathcal{Q}_\lambda): \lambda \in \Lambda\}$. Be an arbitrary collection of tritopological spaces and let $X = \times\{X_\lambda: \lambda \in \Lambda\}$. Let $(\mathcal{T}, \mathcal{P}, \mathcal{Q})$ be a tritopology for X. Then the following statements are equivalent: (when all the families of δ^*-open sets of tritopological spaces represent a topology, this theorem is satisfied)

(a) $(\mathcal{T}, \mathcal{P}, \mathcal{Q})$ is the δ^*-product tritopology for X.

(b) $(\mathcal{T}, \mathcal{P}, \mathcal{Q})$ is the δ^*-smallest tritopology for X for which the projections are δ^*-continuous.

Proof. (a) $\Longrightarrow$ (b): Let π_λ be the λ^{th}projection map and let G_λ be any $\delta^*{}_\lambda$-open subset of X_λ. Then by (a), $\pi_\lambda{}^{-1}[G_\lambda]$ must be $\delta^*{}_\lambda$-open. It follows that π_λ is δ^*-continuous from $(X, \mathcal{T}, \mathcal{P}, \mathcal{Q})$ to $(X_\lambda, \mathcal{T}_\lambda, \mathcal{P}_\lambda, \mathcal{Q}_\lambda)$. Now let $(\acute{\mathcal{T}}, \acute{\mathcal{P}}, \acute{\mathcal{Q}})$ be any tritopology on X such that π_λ is δ^*-continuous from $(X, \acute{\mathcal{T}}, \acute{\mathcal{P}}, \acute{\mathcal{Q}})$ to $(X_\lambda, \mathcal{T}_\lambda, \mathcal{P}_\lambda, \mathcal{Q}_\lambda)$. for each $\lambda \in \Lambda$. Then $\pi_\lambda{}^{-1}[G_\lambda]$ is δ^*-open in $(\acute{\mathcal{T}}, \acute{\mathcal{P}}, \acute{\mathcal{Q}})$ for every $G_\lambda \in \delta^*.O(X)_\lambda$. Since $(\acute{\mathcal{T}}, \acute{\mathcal{P}}, \acute{\mathcal{Q}})$ is a tritopology for X, $(\acute{\mathcal{T}}, \acute{\mathcal{P}}, \acute{\mathcal{Q}})$ contains all the unions of finite intersections of members of the collection $\{\pi_\lambda{}^{-1}[G_\lambda]: \lambda \in \Lambda \text{ and } G_\lambda \in \delta^*.O(X)_\lambda\}$.

This implies that $\delta^*.\acute{O}(X)$ contains $\delta^*.O(X)(\delta^*.O(X) \subset \delta^*.\acute{O}(X))$, that is $(\mathcal{T}, \mathcal{P}, \mathcal{Q})$ is δ^*-coarser than $(\acute{\mathcal{T}}, \acute{\mathcal{P}}, \acute{\mathcal{Q}})$. It follows $(\mathcal{T}, \mathcal{P}, \mathcal{Q})$ is the δ^*-smallest tritopology for X such that π_λ is δ^*-continuous for each $\lambda \in \Lambda$.

$(b) \Longrightarrow (a)$: Let B_* be the collections of all sets of the form $\pi_\lambda{}^{-1}[G_\lambda]$ where G_λ is an δ^*-open subset of X_λ for $\lambda \in \Lambda$. Then by theorem in [5], a tritopology $(\acute{\mathcal{T}}, \acute{\mathcal{P}}, \acute{\mathcal{Q}})$ for X will make all the projections π_λ δ^*-continuous iff $B_* \subset \delta^*.\acute{O}(X)$. Hence in view of [7], the δ^*-smallest tritopology for X which makes all the projections δ^*-continuous is the tritopology determined by B_* as a δ^*-subbase, that is, it is the δ^*-product tritopology for X [see (5.1)].

Theorem 5.4. Let $\{(X_\lambda, \mathcal{T}_\lambda, \mathcal{P}_\lambda, \mathcal{Q}_\lambda): \lambda \in \Lambda\}$, an arbitrary collection of tritopological spaces and let $X = \times\{X_\lambda: \lambda \in \Lambda\}$. The collection C of all sets of the form $\times\{G_\lambda: \lambda \in \Lambda\}$. Where $G_\lambda \in \delta^*.O(X)_\lambda$ for each $\lambda \in \Lambda$, is a δ^*-base for some tritopology for X. (if $\delta^*.O(X)_\lambda$ satisfy the topology this theorem is valid)

Proof. We shall show that C satisfies the conditions [B1] and [B2] of Theorem in [7].

[B1]: Let $x \in X$ so that $x = \{x_\lambda: \lambda \in \Lambda\}$ where $x_\lambda \in X_\lambda$. Then there exists a $\delta^*{}_\lambda$-open set G_λ (which may be X_λ) such that $x_\lambda \in G_\lambda$. Hence x is an element of a set of the form $\times\{G_\lambda: \lambda \in \Lambda\} = G$ say. Thus, to each $x \in X$, there exists a member G of C such that $x \in G$. It follows that $X = \cup\{G: G \in C\}$.

[B2] Let $G \in C$ and $G' \in C$. Then $\times\{G_\lambda: \lambda \in \Lambda\} = G$ And $\times\{G'_\lambda: \lambda \in \Lambda\} = G'$

Where $G_\lambda \in \delta^*.O(X)_\lambda$ and $G'_\lambda \in \delta^*.O(X)_\lambda$ for every $\lambda \in \Lambda$. Now

$$G \cap G' = (\times\{G_\lambda: \lambda \in \Lambda\}) \cap \times\{G_\lambda{}': \lambda \in \Lambda\})$$

$$= \times\{G \cap G': \lambda \in \Lambda\} \quad \dots (1)$$

Since $(\mathcal{T}_\lambda, \mathcal{P}_\lambda, \mathcal{Q}_\lambda)$ is a tritopology for X_λ, we have $G_\lambda \in \delta^*.O(X)_\lambda$ and $G'_\lambda \in \delta^*.O(X)_\lambda \to G \cap G' \in \delta^*.O(X)_\lambda$. (that is if $\delta^*.O(X)_\lambda$ represent a topology)

It follows from (1) that $G \cap G' \in C$. Thus [B2] is also satisfied.

Theorem 5.5 Let f be a mapping of a tritopological space Y into a δ^*-product space $X = \times\{X_\lambda: \lambda \in \Lambda\}$. Then f is δ^*-continuous iff the composition $\pi_\lambda\ o\ f : Y \to X_\lambda$ is δ^*-continuous.

Proof. Let f be δ^*-continuous. Since all projection is δ^*-continuous, it follows from Theorem in [5], that π_λ of is also δ^*-continuous.

Conversely, let each composition map π_λ of be δ^*-continuous and let U be any member of the defining δ^*-subbase B_* of the δ^*-product space X. then $\pi_\lambda{}^{-1}[G] = U$ for some $\lambda \in \Lambda$ and some $G \in \delta^*.O(X)_\lambda$. Also $f^{-1}[U] = f^{-1}[\pi_\lambda{}^{-1}[G]] = (\pi_\lambda o f)^{-1}[G]$.

Since $\pi_\lambda o f$ is δ^*-continuous, it follows that $(\pi_\lambda o f)^{-1}[G] = f^{-1}[U]$ is δ^*-open in Y. Thus, we have shown that the inverse image under f of every sub basic δ^*-open set in the δ^*-product space X is δ^*-open in Y. It follows from Theorem in [5] that f is δ^*-continuous.

Theorem 5.6. Each projection map is an δ^*-open map.

Proof. The proof left to the reader.

Theorem 5.7. Let X be the non-empty δ^*-product space $\times\{X_\lambda: \lambda \in \Lambda\}$. Then a non-empty δ^*-product subset $F = \times\{F_\lambda: \lambda \in \Lambda\}$ is δ^*-closed in X if and only if each F_λ is δ^*-closed in X_λ.

Proof. Let F_λ is δ^*-closed in X_λ for every $\lambda \in \Lambda$ Since the projection $\pi_\lambda$is δ^*-continuous, for each $\lambda \in \Lambda$. ${\pi_\lambda}^{-1}[F_\lambda]$ is δ^*-closed in X, it easy to see that $F = \cap\{{\pi_\lambda}^{-1}[F_\lambda]: \lambda \in \Lambda\}$.

It follows that F is δ^*-closed in X, being an intersection of δ^*-closed sets [5].

Conversely, let $F = \times\{F_\lambda: \lambda \in \Lambda\}$ be δ^*-closed in X. To show that each F_λ is δ^*-closed in X_λ. Let $\mu \in \Lambda$ be arbitrary. we shall show that F_μ is δ^*-closed in X_λ. Let μ be any δ^*-limit point F_μ in X_μ. Consider the point z where $\pi_\lambda(z) = z_\mu$ and $\pi_\lambda(z)$ is an element of F_μ for $\lambda \neq \mu$ Let G be any basic δ^*-open set for the δ^*-product topology containing z. Then $\pi_\mu(G)$ is δ^*-open by theorem (5.6) and contains z_μ. Since z_μ is a δ^*-limit point of $\pi_\mu(G)$ must contain a point x_μ of F_μ different from z_μ Therefore G contains the point x where $\pi_\lambda(x) = \pi_\lambda(z)$ for $\lambda \neq \mu$ and $\pi_\lambda(x) = x_\mu$. Evidently, $x \in F$, Also since x and z differ in μ^{th} coordinate, we have $x \neq z$ Thus we have shown that every basic δ^*-open set containing z contains a point of F different from z. Hence z is a δ^*-limit point of F. Since F is δ^*-closed in X, $z \in F$ which implies that $\pi_\mu(z) \in \pi_\mu(F)$. Thus F_μ contains all its δ^*-limit points and so F_μ is δ^*-closed. Since μ was arbitrary, we see that F_μ is δ^*-closed for every $\lambda \in \Lambda$.

6. Conclusion

The purpose of this article is to introduce the concept of the product in tritopological spaces namely δ^*-product spaces. Several properties of δ^*- product spaces concept is established. Moreover, we obtain a characterization and preserving theorems with the help of some necessary conditions and interesting examples. And we generalise theorems in δ^*-connectedness, δ^*-compactness, δ^*-second countability and δ^*-Hausdorff for tritopological product of spaces. Furthermore, the uses of tritopological results in this paper and some other papers are worthy for possible applications in areas of science and social science for the future.

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A Decomposition of α-continuity and $\mu\alpha$-continuity

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Abstract — The main purpose of this paper is to introduce the concepts of *η-sets, **η-sets, *η-continuity and **η-continuity and to obtain decomposition of α-continuity and $\mu\alpha$-continuity in topological spaces.

Keywords — *$\mu\alpha$-closed set, μp-closed set, *η-set, **η-set, *η-continuity and **η-continuity*

1. Introduction and Preliminaries

Tong [1] introduced the notions of A-sets and A-continuity in topological spaces and established a decomposition of continuity. In [2], he also introduced the notions of B-sets and B-continuity and used them to obtain a new decomposition of continuity and Ganster and Reill [3] improved Tong's decomposition result. Moreover, Noiri and Sayed [4] introduced the notions of η-sets and obtained some decompositions of continuity. Quite recently, Veera kumar [5] introduced and studied the notions of μp-sets in topological spaces. Quite recently, Ganesan [6] introduced and studied the notions of $\mu\alpha$-closed sets in topological spaces. In this paper, we introduce the notions of *η-sets, **η-sets, *η-continuity and **η-continuity and obtain decomposition of α-continuity and $\mu\alpha$-continuity. Throughout this paper (X, τ) and (Y, σ) (or X and Y)represent topological spaces on which no separation axioms are assumed unless otherwise mentioned. For a subset A of a space (X, τ), cl(A), int(A) and A^C denote the closure of A, the interior of A and complement of A respectively.
We recall the following definitions which are useful in the sequel.

Definition 1.1. A subset A of a space (X, τ) is called:

1. a regular open [7] if A = int(cl(A)).
2. an α-open set [8] if A$\subseteq$ int(cl(int(A))).
3. a semi-open set [9] if A$\subseteq$ cl(int(A)).
4. a pre-open set [10] if A$\subseteq$ int(cl(A)).

The complements of the above mentioned open sets are called their respective closed sets.
The α-closure [8](resp. semi-closure [11],pre-closure [12]) of a subset A of X, denoted by αcl(A)

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(resp.scl(A), pcl(A)) is defined to be the intersection of all α-closed (resp. semi-closed, pre-closed) sets of (X, τ) containing A. For any subset A of an arbitrarily chosen topological space, the α-interior [8] (resp. semi-interior [11],pre-interior [12]) of a subset A of X, denoted by αint(A) (resp.sint(A), pint(A)) is defined to be the union of all α-open (resp. semi-open, pre-open) sets of (X, τ) contained A.

Definition 1.2. A subset A of a space X is called:

1. a t-set [2] if int(cl(A))=int(A).
2. an α^*-set [13] if int(A) = int(cl(int(A))).
3. an A-set [1] if A = V $\cap$ T where V is open and T is a regular closed set.
4. a B-set [2, 14] if A = V $\cap$ T where V is open and T is a t-set.
5. an αB-set [15] if A = V $\cap$ T where V is α-open and T is a t-set.
6. an η-set [4] if A = V $\cap$ T where V is open and T is an α-closed set.
7. a locally closed set [16] if A = V $\cap$ T where V is open and T is closed.

The collection of A-sets (resp. B-sets, αB-sets, η-sets, locally closed sets) in X is denoted by A(X) (resp. B(X), αB(X), η(X), LC(X)).

Definition 1.3. A subset A of a space (X, τ) is called:

1. a gα^*-closed set [17, 18] if αcl(A) $\subseteq$ int(U) whenever A $\subseteq$ U and U is α-open in (X, τ). The complement of gα^*-closed set is called gα^*-open set.
2. a μ-preclosed (briefly μp-closed) set [5] if pcl(A) $\subseteq$ U whenever A $\subseteq$ U and U is gα^*-open in (X, τ). The complement of μp-closed set is called μp-open set.
3. a $\mu\alpha$-closed set [6] if αcl(A) $\subseteq$ U whenever A$\subseteq$U and U is gα^*-open in (X, τ). The complement of $\mu\alpha$-closed set is called $\mu\alpha$-open set.

The collection of all $\mu\alpha$-open (resp. μp-open) sets in X will be denoted by $\mu\alpha$O(X)(resp. μpO(X)).

Remark 1.4. In a space X, the followings hold:

1. Every open set is gα^*-open but not conversely [6].
2. Every α-open set is $\mu\alpha$-open but not conversely [6].
3. Every $\mu\alpha$-closed set is μp-closed but not conversely [6].
4. Every $\mu\alpha$-continuous map is μp-continuous but not conversely [6].
5. The intersection of two t-sets is a t-set [2].

Remark 1.5. In a space X,the followings hold:

1. A is α-closed set if and only if A = αcl(A).
2. Every regular closed set is closed but not conversely.
3. Every regular closed set is semi-closed (= t-set) but not conversely.
4. Every closed set is α-closed but not conversely.
5. Every α-closed set is semi-closed (= t-set) but not conversely.

2. $*^{\eta}$-sets and $**^{\eta}$-sets

In this section we introduce and study the notions of $*^{\eta}$-sets and$**^{\eta}$-sets in topological spaces.

Definition 2.1. A subset A of a space X is called:

1. an $*^{\eta}$-set if A = U ∩ T where U is gα*-open and T is α-closed in X.

2. an $**^{\eta}$-set if A = U ∩ T where U is $\mu\alpha$-open and T is a t-set in X.

The collection of all $*^{e}ta$-sets (resp.$**^{\eta}$-sets) in X will be denoted by$*^{\eta}(X)$ (resp. $**^{\eta}(X)$)

Theorem 2.2. For a subset A of a space X, the following are equivalent.

1. A is an $*^{\eta}$-set.

2. A = U ∩ αcl(A) for some gα*-open set U.

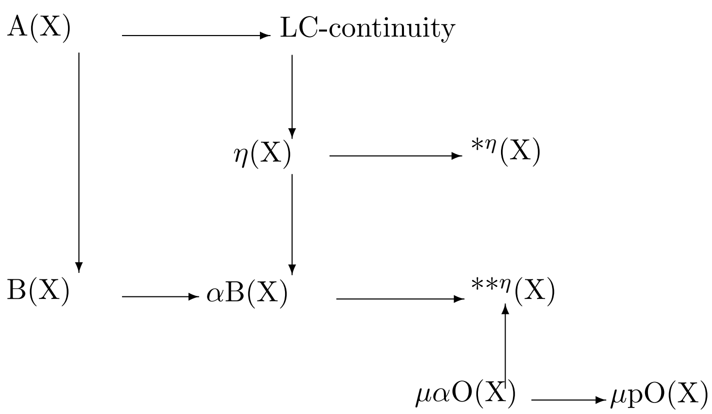
PROOF. (1)→ (2) Since A is an$*^{\eta}$-set, then A = U ∩ T, where U is gα*-open and T is α-closed. So, A ⊂ U and A ⊂ T. Hence αcl(A) ⊂ αcl(T). Therefore A ⊂ U ∩ αcl(A) ⊂ U ∩ αcl(T) = U ∩ T = A. Thus, A = U ∩ αcl(A).
(2)→ (1) It is obvious because αcl(A) is α-closed by Remark 1.5(1).

Remark 2.3. In a space X, the intersection of two$**^{\eta}$-sets is an$**^{\eta}$-set.

Remark 2.4. Union of two$**^{\eta}$-sets need not be an$**^{\eta}$-set as seen from the following example.

Example 2.5. Let $X = \{a, b, c\}$ with $\tau = \{\phi, \{a, b\}, X\}$. The sets $\{a\}$, $\{c\}$ are $**^{\eta}$-sets in (X, τ) but their union $\{a, c\}$ is not an $**^{\eta}$-set in (X, τ).

Remark 2.6. We have the following implications.



where none of these implications is reversible as shown by [4] and the following examples.

Example 2.7.
1. In Example 2.5, the set {b} is an $*\eta$-set but not an η-set in (X, τ).

2. Let X = {a, b, c} with τ = {ϕ, {a, c}, X}. Clearly the set {b} is an $**\eta$-set but not an $\mu\alpha$-open set in (X, τ).

3. In Example 2.5, the set {a} is an $**\eta$-set but not an αB-set in (X, τ).

Remark 2.8.
1. The notions of $*\eta$-sets and $\mu\alpha$-closed sets are independent.

2. The notions of $**\eta$-sets and μp-closed sets are independent.

Example 2.9. In Example 2.5, the set {a, c} is $\mu\alpha$-closed but not an $*\eta$-set and also the set {a, b} is an $*\eta$-set but not a $\mu\alpha$-closed in (X, τ).

Example 2.10. In Example 2.7(2), the set {b} is an $**\eta$-set but not a μp-open set and also the set {a, b} is an μp-open set but not a $**\eta$-set in (X, τ).

Theorem 2.11. For a subset A of a space X, the following are equivalent:

1. A is α-closed.

2. A is an $*^{\eta}$-set and μ_{α}-closed.

Proof. (1)$\rightarrow$ (2) It follows from Remark 1.4(1) and Definition 2.1(1).
(2) $\rightarrow$ (1) Since A is an $*^{\eta}$-set, then by Theorem 2.2, A = U $\cap$ αcl(A) where U is gα^*-open in X. So, A $\subset$ U and since A is $\mu\alpha$-closed, then αcl(A) $\subset$ U. Therefore, αcl(A) $\subset$ U $\cap$ αcl(A) = A. But A $\subset$ αcl(A) always. Hence by Remark 1.5(1), A is α-closed.

Proposition 2.12. [19] Let A and B be subsets of a space X. If B is an α^*-set, then αint(A $\cap$ B) = αint(A) $\cap$ int(B)

Theorem 2.13. For a subset S of a space X, the following are equivalent.

1. S is $\mu\alpha$-open.

2. S is an $**^{\eta}$-set and μp-open.

Proof. Necessity: It follows from Remark 1.4(3) and Definition 2.1(2).
Sufficiency: Assume that S is μp-open and an $**^{\eta}$-set in X. Then S = A $\cap$ B where A is μ_{α}-open and B is a t-set in X. Let F $\subset$ S, where F is gα^*-closed in X. Since S is μp-open in X, F $\subset$ pint(S) = S $\cap$ int(cl(S)) = (A $\cap$ B) $\cap$ int[cl(A $\cap$ B)] $\subset$ A $\cap$ B $\cap$ int(cl(A)) $\cap$ int(cl(B)) = A $\cap$ B $\cap$ int(cl(A)) $\cap$ int(B), since B is a t-set. This implies, F $\subset$ int(B). Note that A is $\mu\alpha$-open and that F$\subset$ A. So, F $\subset$ αint(A). Therefore, F $\subset$ αint(A) $\cap$ int(B) = αint(S) by Proposition 2.12. Hence S is $\mu\alpha$-open.

3. $*^{\eta}$-continuity and $**^{\eta}$-continuity

Definition 3.1. A function f : X $\rightarrow$ Y is said to be $*^{\eta}$-continuous (resp. $**^{\eta}$-continuous) if f^{-1}(V) is an $*^{\eta}$-set (resp. an $**^{\eta}$-set) in X for every open subset V of Y.

Definition 3.2. A function f : X $\rightarrow$ Y is said to be C$*^{\eta}$-continuous if f^{-1}(V) is an $*^{\eta}$-set in X for every closed subset V of Y.

We shall recall the definitions of some functions used in the sequel.

Definition 3.3. A function f : X $\rightarrow$ Y is said to be

1. A-continuous [1] if f^{-1}(V) is an A-set in X for every open set V of Y.

2. B-continuous [2, 14] if f^{-1}(V) is an B-set in X for every opens set V of Y.

3. α-continuous [20] if f^{-1}(V) is an α-open set in X for every open set V of Y.

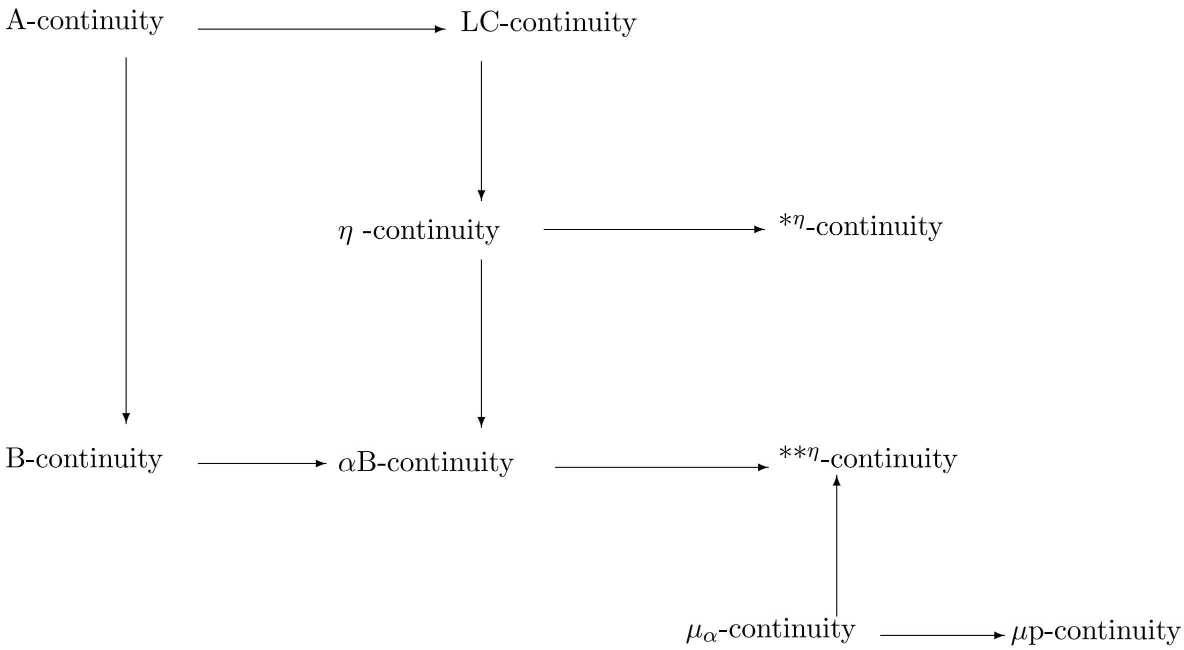
4. LC-continuous [16] (resp. αB-continuous [15] if f^{-1}(V) is an locally closed set (resp. αB-set) in X for every open set V of Y,

5. η-continuous [4] if f^{-1}(V) is an η-set in X for every open set V of Y.

6. $\mu\alpha$-continuous [6] (resp. μp-continuous [5]) if f^{-1}(V) is an $\mu\alpha$-open set (resp. μp-open set) in X for every open set V of Y.

Remark 3.4. It is clear that, a function f : X $\rightarrow$ Y is α-continuous if and only if f^{-1}(V) is an α-closed set in X for every closed set V of Y.

From the definitions stated above, we obtain the following diagram



Remark 3.5. None of the implications is reversible as shown by the following examples.

Example 3.6. Let X = Y = {a, b, c} with τ = {ϕ, {b, c}, X} and σ = {ϕ, {b}, {b, c}, Y}. Let f : X $\rightarrow$ Y be the identity function on X. Then f is *$^{\eta}$-continuous but not η-continuous.

Example 3.7. Let X, τ and f be as in Example 3.6. Let Y = {a, b, c} with σ = {ϕ, {c}, {b, c}, Y}. Then f is **$^{\eta}$-continuous but not αB-continuous.

Example 3.8. Let X, τ and f be as in Example 3.6. Let Y = {a, b, c} with σ = {ϕ, {a}, {b, c}, Y}. Then f is **$^{\eta}$-continuous but not $\mu\alpha$-continuous.

Remark 3.9. The following examples show that the concepts of

1. $\mu\alpha$-continuity and *$^{\eta}$-continuity are independent.
2. $\mu\alpha$-continuity and C*$^{\eta}$-continuity are independent.
3. *$^{\eta}$-continuity and C*$^{\eta}$-continuity are independent.

Example 3.10. Let X = Y = {a, b, c} with τ = {ϕ, {a}, X} and σ = {ϕ, {a}, {a, b}, {a, c}, Y}. Let f : X $\rightarrow$ Y be the identity function on X. Then f is $\mu\alpha$-continuous but not *$^{\eta}$-continuous.

Example 3.11. Let X, τ and f be as in Example 3.10. Let Y = {a, b, c} with σ = {ϕ, {b}, {c}, {b, c}, Y}. Then f is *$^{\eta}$-continuous but not $\mu\alpha$-continuous.

Example 3.12. Let X = Y = {a, b, c} with τ = {ϕ, {b}, X} and σ = {ϕ, {a}, {a, b}, Y}. Let f : X $\rightarrow$ Y be the identity function on X. Then f is $\mu\alpha$-continuous but not C*η-continuous.

Example 3.13. Let X, τ and f be as in Example 3.12. Let Y = {a, b, c} with σ = {ϕ, {b}, {a, c}, Y}. Then f is C*η-continuous but not $\mu\alpha$-continuous.

Example 3.14. Let X, τ and f be as in Example 2.5. Let Y = {a, b, c}with σ = {ϕ, {c}, {a, c}, Y}. Then f is C*η-continuous but not *$^{\eta}$-continuous.

Example 3.15. Let X, τ and f be as in Example 2.5. Let Y = {a, b, c} with σ = {ϕ, {b}, Y}. Then f is *$^{\eta}$-continuous but not C*η-continuous.

Remark 3.16. The following examples show that the concept of μp-continuity and **$^{\eta}$-continuity are independent.

Example 3.17. Let X and τ be as in Example 2.7(2). Let Y = {a, b, c} with σ = {ϕ, {a}, {a, b}, {a, c}, Y}. Let f : X → Y be the identity function on X. Then f is μp-continuous but not **η-continuous.

Example 3.18. Let X, τ and f be as in Example 3.6. Let Y = {a, b, c} with σ = {ϕ, {a}, Y}. Then f is **η-continuous but not μp-continuous.

Theorem 3.19. For a function f : X → Y, the following are equivalent.

1. f is α-continuous.
2. f is C*η-continuous and $\mu\alpha$-continuous.

Proof. The proof follows from Definitions 3.2 and 3.3(6), Remark 3.4 and Theorem 2.11.

Theorem 3.20. For a function f : X → Y, the following are equivalent.

1. f is $\mu\alpha$-continuous.
2. f is **$^{\eta}$-continuous and μp-continuous.

Proof. The proof follows from Theorem 2.13.

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Axiomatic Characterizations of Quadripartitioned Single Valued Neutrosophic Rough Sets

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Abstract − In this paper, axiomatic characterizations of quadripartitioned single-valued neutrosophic rough sets have been studied and also studied some properties of quadripartitioned single-valued neutrosophic rough sets. A numerical example in medical diagnosis is given, which is based on the similarity measure of quadripartitioned single-valued neutrosophic rough sets.

Keywords − *Quadripartitioned single valued neutrosophic rough sets, similarity measure, axiomatic characterization, quadripartitioned single-valued neutrosophic number*

1. Introduction

Zadeh [1] proposed the concept of fuzzy sets which is very useful to deal the concept of imprecision, uncertainty, and degrees of the truthfulness of values and is represented by membership functions which lie in a unit interval [0,1]. Atanassov [2] developed the concept of intuitionistic fuzzy sets in 1983 which is a generalization of fuzzy sets and is dealing with the concept of vagueness. This concept consists of both membership and non-membership functions. In 1998, Smarandache presented Neutrosophic sets with three components called truth membership function, indeterminacy membership function, and falsity membership function [3,4].

In 1982, Pawlak [5] introduced the concept of rough sets which expresses vagueness in the notions of lower and upper approximations of a set and it employs boundary region of a set. A hybrid structure of rough neutrosophic sets was introduced by Broumi and Smarandache in 2014 [6]. Smarandache [7] and later Wang et al. [8] studied the concept of single-valued neutrosophic sets which is very useful in real scientific and engineering applications. Broumi et al. [9-11] solved the shortest path problem using Bellman algorithm under neutrosophic environment. Then, a new hybrid model of single-valued neutrosophic rough sets was introduced by Hai Long Yang [12].

Smarandache [7] firstly presented the refinement of the neutrosophic set and logic, i.e. the truth value T is refined into types of sub-truths such as T_1, T_2, etc.; similarly indeterminacy I is split/refined into types of sub-indeterminacies I_1, I_2, etc., and the sub-falsehood F is split into F_1, F_2, etc. Based on Belnap's [13] four-valued logic that is (Truth-T, Falsity-F, Unknown-U, Contradiction-C) Smarandache proposed the concept of four numerical valued neutrosophic logic that is quadripartitioned single valued neutrosophic sets. In this set, the indeterminacy is split into two parts known as unknown (neither true nor false) and contradiction (both

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true and false). Mohana and Mohanasundari [14] studied the concept of quadripartitioned single-valued neutrosophic relations (QSVNR) and also studied some properties of a quadripartitioned single-valued neutrosophic rough sets. Chatterjee et al. [15] studied the concept of some similarity measures and entropy on quadripartitioned single-valued neutrosophic sets.

This paper is structured in the following ways. Section 1 provides a brief introduction. Section 2 delivered the basic definitions which we need to prove the results in further. Section 3 defined the concepts of empty quadripartitioned single-valued neutrosophic sets (QSVNS), full QSVNS and also singleton and its complement of QSVNS. And also, we have studied some properties of quadripartitioned single-valued neutrosophic rough sets. Section 4 deals the concept of axiomatic characterizations of quadripartitioned single-valued neutrosophic rough sets in detail. Section 5 illustrates an example for quadripartitioned single-valued neutrosophic rough sets in medical diagnosis. Section 6 concludes the paper.

2. Preliminaries

In this section, we recall the basic definitions of rough sets, Neutrosophic sets, QSVNS, and QSVNR, which will be used in proving the rest of the paper.

Definition 2.1. [5] Let U be any non-empty set. Suppose R is an equivalence relation over U. For any non-null subset X of U, the sets $A_1(x) = \{X: [x]_R \subseteq X\}$ and $A_2(x) = \{X: [x]_R \cap X \neq \varphi\}$ are called the lower approximation and upper approximation respectively of X where the pair $S = (U, R)$ is called an approximation space. This equivalence relation R is called indiscernibility relation. The pair $A(X) = (A_1(X), A_2(X))$ is called the rough set of X in S. Here $[x]_R$ denotes the equivalence class of R containing X.

Definition 2.2. [4] Let X be a universe of discourse, with a generic element in X denoted by x, a neutrosophic set (NS) is an object having the form,

$$A = \{< x: \mu_A(x), \nu_A(x), \omega_A(x) >, x \in X\}$$

where the functions $\mu, \nu, \omega: X \to]^-0, 1^+[$ define the degree of membership (or truth) respectively, the degree of indeterminacy, and the degree of non-membership (or falsehood) of the element $x \in X$ to the set A with the condition, $^-0 \leq \mu_A(x) + \nu_A(x) + \omega_A(x) \leq 3^+$.

Definition 2.3. [15] Let X be a non-empty set. A quadripartitioned single-valued neutrosophic set (QSVNS) A over X characterizes each element x in X by a truth-membership function T_A , a contradiction membership function C_A, an ignorance membership function U_A and a falsity membership function F_A such that for each, $x \in X, T_A, C_A, U_A, F_A \in [0,1]$ and $0 \leq T_A(x) + C_A(x) + U_A(x) + F_A(x) \leq 4$ when X is discrete, A is represented as, $A = \sum_{i=1}^{n} \langle T_A(x_i), C_A(x_i), U_A(x_i), F_A(x_i) \rangle / x_i, x_i \in X$. However, when the universe of discourse is continuous, A is represented as, $A = \int_X \langle T_A(x), C_A(x), U_A(x), F_A(x) \rangle / x, x \in X$

Definition 2.4. [14] A QSVNS R in $U \times U$ is called a quadripartitioned single-valued neutrosophic relation (QSVNR) in U, denoted by,

$$R = \{\langle (x, y), T_R(x, y), C_R(x, y), U_R(x, y), F_R(x, y) \rangle / (x. y) \in U \times U\}$$

where $T_R: U \times U \to [0,1], C_R: U \times U \to [0,1], U_R: U \times U \to [0,1],$ and $F_R: U \times U \to [0,1]$ denote the truth membership function, a contradiction membership function, an ignorance membership function and a falsity membership function of R respectively.

Definition 2.5. [14] Let R be a QSVNR in U, the tuple (U, R) is called a quadripartitioned single-valued neutrosophic approximation space $\forall A \in QSVNS(U)$, the lower and upper approximations of A with respect to (U, R) denoted by $\underline{R}(A)$ and $\overline{R}(A)$ are two QSVNS's whose membership functions are defined as $\forall\, x \in U$,

$$T_{\underline{R}(A)}(x) = \wedge_{y\in U} (F_R(x,y) \vee T_A(y)), \quad T_{\overline{R}(A)}(x) = \vee_{y\in U} (T_R(x,y) \wedge T_A(y))$$
$$C_{\underline{R}(A)}(x) = \wedge_{y\in U} (U_R(x,y) \vee C_A(y)), \quad C_{\overline{R}(A)}(x) = \vee_{y\in U} (C_R(x,y) \wedge C_A(y))$$
$$U_{\underline{R}(A)}(x) = \vee_{y\in U} (C_R(x,y) \wedge U_A(y)), \quad U_{\overline{R}(A)}(x) = \wedge_{y\in U} (U_R(x,y) \vee U_A(y))$$
$$F_{\underline{R}(A)}(x) = \vee_{y\in U} (T_R(x,y) \wedge F_A(y)), \quad F_{\overline{R}(A)}(x) = \wedge_{y\in U} (F_R(x,y) \vee F_A(y)).$$

The pair $(\underline{R}(A), \overline{R}(A))$ is called the quadripartitioned single-valued neutrosophic rough set of A with respect to (U, R). $\underline{R}$ and $\overline{R}$ are referred to as the quadripartitioned single-valued neutrosophic lower and upper approximation operators, respectively.

Theorem 2.1. [14] Let (X, R) be a quadripartitioned single-valued neutrosophic approximation space. The quadripartitioned single-valued neutrosophic lower and upper approximation operators defined in 3.4 have the following properties. $\forall A, B \in QSVNS(X)$,

i. $\underline{R}(X)=X$, $\overline{R}(\varphi) = \varphi$
ii. If $A \subseteq B$ then $\underline{R}(A) \subseteq \underline{R}(B)$ and $\overline{R}(A) \subseteq \overline{R}(B)$
iii. $\underline{R}(A \cap B) = \underline{R}(A) \cap \underline{R}(B), \overline{R}(A \cup B) = \overline{R}(A) \cup \overline{R}(B)$
iv. $\underline{R}(A \cup B) \supseteq \underline{R}(A) \cup \underline{R}(B), \overline{R}(A \cap B) \subseteq \overline{R}(A) \cap \overline{R}(B)$
v. $\underline{R}(A^C) = (\overline{R}(A))^C, \overline{R}(A^C) = (\underline{R}(A))^C$

3. The Properties of Quadripartitioned Single-Valued Neutrosophic Rough Sets

In this paper, QSVNS(X) will denote the family of all QSVNSs in X.

Definition 3.1. Let A be a QSVNS in X. If $\forall x \in X, T_A(x) = 0, C_A(x) = 0$ and $U_A(x) = 1, F_A(x) = 1$then A is called an empty QSVNS, denoted by ϕ. If $\forall x \in X, T_A(x) = 1, C_A(x) = 1$ and $U_A(x) = 0, F_A(x) = 0$ then A is called a full QSVNS, denoted by X.

$\forall \alpha_1, \alpha_2, \alpha_3, \alpha_4 \in [0,1], \alpha_1,\widehat{\alpha_2,\alpha_3},\alpha_4$ denotes a constant QSVNS satisfying,

$$T_{\alpha_1,\widehat{\alpha_2,\alpha_3},\alpha_4}(x) = \alpha_1, C_{\alpha_1,\widehat{\alpha_2,\alpha_3},\alpha_4}(x) = \alpha_2, U_{\alpha_1,\widehat{\alpha_2,\alpha_3},\alpha_4}(x) = \alpha_3, \text{ and } F_{\alpha_1,\widehat{\alpha_2,\alpha_3},\alpha_4}(x) = \alpha_4$$

Definition 3.2. For any $y \in X$, a quadripartitioned single-valued neutrosophic singleton set 1_y and its complement $1_{X-\{y\}}$ are defined as $\forall x \in X$,

$$T_{1_y}(x) = C_{1_y}(x) = \begin{cases} 1, x = y \\ 0, x \neq y \end{cases}$$
$$U_{1_y}(x) = F_{1_y}(x) = \begin{cases} 0, x = y \\ 1, x \neq y \end{cases}$$
$$T_{1_{X-\{y\}}}(x) = C_{1_{X-\{y\}}}(x) = \begin{cases} 0, x = y \\ 1, x \neq y \end{cases}$$

and

$$U_{1_{X-\{y\}}}(x) = F_{1_{X-\{y\}}}(x) = \begin{cases} 1, x = y \\ 0, x \neq y \end{cases}$$

Definition 3.3. If $\forall x \in X, \vee_{y\in X} T_R(x,y) = 1$, $\vee_{y\in X} C_R(x,y) = 1$, $\wedge_{y\in X} U_R(x,y) = 0$, and $\wedge_{y\in X} F_R(x,y) = 0$, then R is called a serial QSVNR where " $\vee$ " and " $\wedge$ " denote maximum and minimum respectively.

Theorem 3.1. Let (X, R) be a quadripartitioned single-valued neutrosophic approximation space. The quadripartitioned single-valued neutrosophic lower and upper approximation operators defined in 2.5 have the following properties. $\forall A, B \in QSVNS(X), \forall \alpha_1, \alpha_2, \alpha_3, \alpha_4 \in [0,1]$,

(1) $\underline{R}\left(A \cup \alpha_{1,}\widehat{\alpha_{2,}\alpha_{3,}}\alpha_4\right) = \underline{R}(A) \cup \alpha_{1,}\widehat{\alpha_{2,}\alpha_{3,}}\alpha_4,$ and $\overline{R}(A \cap \alpha_{1,}\widehat{\alpha_{2,}\alpha_{3,}}\alpha_4) = \overline{R}(A) \cap \alpha_{1,}\widehat{\alpha_{2,}\alpha_{3,}}\alpha_4;$
(2) $\underline{R}\left(\alpha_{1,}\widehat{\alpha_{2,}\alpha_{3,}}\alpha_4\right) = \alpha_{1,}\widehat{\alpha_{2,}\alpha_{3,}}\alpha_4 \Leftrightarrow \underline{R}(\phi) = \phi,$ and $\overline{R}(\alpha_{1,}\widehat{\alpha_{2,}\alpha_{3,}}\alpha_4) = \alpha_{1,}\widehat{\alpha_{2,}\alpha_{3,}}\alpha_4 \Leftrightarrow \overline{R}(U) = U$

Proof. By definition 2.5, $\forall x \in U$, we have

$$\begin{aligned}
T_{\underline{R}(A\cup\alpha_1,\widehat{\alpha_2,\alpha_3},\alpha_4)}(x) &= \bigwedge_{y\in X} (F_R(x,y) \vee T_{A\cup\alpha_1,\widehat{\alpha_2,\alpha_3},\alpha_4}(y)) \\
&= \bigwedge_{y\in X} (F_R(x,y) \vee (T_A(y) \vee T_{\alpha_1,\widehat{\alpha_2,\alpha_3},\alpha_4}(y)) \\
&= \bigwedge_{y\in X} (F_R(x,y) \vee T_A(y) \vee \alpha_1) \\
&= \left(\bigwedge_{y\in X} (F_R(x,y) \vee T_A(y))\right) \vee \alpha_1 \\
&= T_{\underline{R}(A)}(x) \vee \alpha_1
\end{aligned}$$

$$\begin{aligned}
C_{\underline{R}(A\cup\alpha_1,\widehat{\alpha_2,\alpha_3},\alpha_4)}(x) &= \bigwedge_{y\in X} (U_R(x,y) \vee C_{A\cup\alpha_1,\widehat{\alpha_2,\alpha_3},\alpha_4}(y)) \\
&= \bigwedge_{y\in X} (U_R(x,y) \vee (C_A(y) \vee C_{\alpha_1,\widehat{\alpha_2,\alpha_3},\alpha_4}(y)) \\
&= \bigwedge_{y\in X} (U_R(x,y) \vee C_A(y) \vee \alpha_2) \\
&= \left(\bigwedge_{y\in X} (U_R(x,y) \vee C_A(y))\right) \vee \alpha_2 \\
&= C_{\underline{R}(A)}(x) \vee \alpha_2
\end{aligned}$$

$$\begin{aligned}
U_{\underline{R}(A\cup\alpha_1,\widehat{\alpha_2,\alpha_3},\alpha_4)}(x) &= \bigvee_{y\in X} (C_R(x,y) \wedge U_{A\cup\alpha_1,\widehat{\alpha_2,\alpha_3},\alpha_4}(y)) \\
&= \bigvee_{y\in X} \left(C_R(x,y) \wedge (U_A(y) \wedge U_{\alpha_1,\widehat{\alpha_2,\alpha_3},\alpha_4}(y))\right) \\
&= \bigvee_{y\in X} (C_R(x,y) \wedge U_A(y) \wedge \alpha_3) \\
&= \left(\bigvee_{y\in X} (C_R(x,y) \wedge U_A(y))\right) \wedge \alpha_3 \\
&= U_{\underline{R}(A)}(x) \wedge \alpha_3
\end{aligned}$$

$$\begin{aligned}
F_{\underline{R}(A\cup\alpha_1,\widehat{\alpha_2,\alpha_3},\alpha_4)}(x) &= \bigvee_{y\in X} (T_R(x,y) \wedge F_{A\cup\alpha_1,\widehat{\alpha_2,\alpha_3},\alpha_4}(y)) \\
&= \bigvee_{y\in X} \left(T_R(x,y) \wedge (F_A(y) \wedge F_{\alpha_1,\widehat{\alpha_2,\alpha_3},\alpha_4}(y))\right) \\
&= \bigvee_{y\in X} (T_R(x,y) \wedge F_A(y) \wedge \alpha_4 \\
&= \left(\bigvee_{y\in X} (T_R(x,y) \wedge F_A(y))\right) \wedge \alpha_4 \\
&= F_{\underline{R}(A)}(x) \wedge \alpha_4
\end{aligned}$$

So, $\underline{R}(A \cup \alpha_{1,}\widehat{\alpha_{2,}\alpha_{3,}}\alpha_4) = \underline{R}(A) \cup \alpha_{1,}\widehat{\alpha_{2,}\alpha_{3,}}\alpha_4$.

Similarly, we can show that $\overline{R}(A \cap \alpha_{1,}\widehat{\alpha_{2,}\alpha_{3,}}\alpha_4) = \overline{R}(A) \cap \alpha_{1,}\widehat{\alpha_{2,}\alpha_{3,}}\alpha_4$.

(2) If $\underline{R}(\phi) = \phi$, then we have,

$$\underline{R}(\alpha_{1,}\widehat{\alpha_{2,}\alpha_{3,}}\alpha_4) = \underline{R}(\phi \cup \alpha_{1,}\widehat{\alpha_{2,}\alpha_{3,}}\alpha_4) = \underline{R}(\phi) \cup \alpha_{1,}\widehat{\alpha_{2,}\alpha_{3,}}\alpha_4 = \alpha_{1,}\widehat{\alpha_{2,}\alpha_{3,}}\alpha_4 \text{ by (1)}$$

Conversely, if $\forall \alpha_{1,}\widehat{\alpha_{2,}\alpha_{3,}}\alpha_4 \in [0,1]$,

$$\begin{aligned}
&\underline{R}(\alpha_{1,}\widehat{\alpha_{2,}\alpha_{3,}}\alpha_4) = \alpha_{1,}\widehat{\alpha_{2,}\alpha_{3,}}\alpha_4, take \alpha_1 = \alpha_2 = 0 and \alpha_3 = \alpha_4 = 1 \\
&i.e., \alpha_{1,}\widehat{\alpha_{2,}\alpha_{3,}}\alpha_4 = \phi, then we have \underline{R}(\phi) = \phi
\end{aligned}$$

Similarly, we can show $\overline{R}(\alpha_{1,}\widehat{\alpha_{2,}\alpha_{3,}}\alpha_4) = \alpha_{1,}\widehat{\alpha_{2,}\alpha_{3,}}\alpha_4 \Leftrightarrow \overline{R}(U) = U$.

Theorem 3.2. Let (X, R) be a quadripartitioned single-valued neutrosophic approximation space. $\underline{R}(A)$ and $\overline{R}(A)$ are the lower and upper approximation in Definition 2.5 then we have,

(1) R is serial $\Leftrightarrow \underline{R}(\alpha_1, \widehat{\alpha_2, \alpha_3}, \alpha_4) = \alpha_1, \widehat{\alpha_2, \alpha_3}, \alpha_4, \forall \alpha_1, \alpha_2, \alpha_3, \alpha_4 \in [0,1]$
$\Leftrightarrow \underline{R}(\phi) = \phi$
$\Leftrightarrow \overline{R}(\alpha_1, \widehat{\alpha_2, \alpha_3}, \alpha_4) = \alpha_1, \widehat{\alpha_2, \alpha_3}, \alpha_4, \forall \alpha_1, \alpha_2, \alpha_3, \alpha_4 \in [0,1]$
$\Leftrightarrow \overline{R}(U) = U$
$\Leftrightarrow \underline{R}(A) \subset \overline{R}(A), \forall A \in QSVNS(X)$
(2) R is reflexive $\Leftrightarrow \underline{R}(A) \subset A, \forall A \in QSVNS(X)$,
$\Leftrightarrow A \subset \overline{R}(A), \forall A \in QSVNS(X)$
(3) R is symmetric $\Leftrightarrow \underline{R}\left(1_{X-\{x\}}\right)(y) = \underline{R}\left(1_{X-\{y\}}\right)(x), \forall x, y \in X$
$\Leftrightarrow \overline{R}(1_x)(y) = \overline{R}\left(1_y\right)(x), \forall x, y \in X$
(4) R is transitive $\Leftrightarrow \underline{R}(A) \subset \underline{R}(\underline{R}(A)), \forall A \in QSVNS(X)$
$\Leftrightarrow \overline{R}(\overline{R}(A)) \subset \overline{R}(A), \forall A \in QSVNS(X)$

Proof. Since quadripartitioned single-valued neutrosophic approximation operators satisfy the duality property, it is enough to show us the properties for upper quadripartitioned single-valued neutrosophic approximation operator.

Based on Theorem 2.1(1), 3.1(2) we only need to show

$$R \text{ is serial } \Leftrightarrow \underline{R}(\alpha_1, \widehat{\alpha_2, \alpha_3}, \alpha_4) = \alpha_1, \widehat{\alpha_2, \alpha_3}, \alpha_4, \forall \alpha_1, \alpha_2, \alpha_3, \alpha_4 \in [0,1],$$
$$\Leftrightarrow \underline{R}(A) \subset \overline{R}(A), \forall A \in QSVNS(X)$$

(I) We first prove

$$R \text{ is serial} \Leftrightarrow \underline{R}(\alpha_1, \widehat{\alpha_2, \alpha_3}, \alpha_4) = \alpha_1, \widehat{\alpha_2, \alpha_3}, \alpha_4, \forall \alpha_1, \alpha_2, \alpha_3, \alpha_4 \in [0,1].$$

First assume that $Risserial$, then $\forall x \in X$,

$\vee_{y \in X} T_R(x, y) = 1, \vee_{y \in X} C_R(x, y) = 1$ and $\wedge_{y \in X} U_R(x, y) = 0, \wedge_{y \in X} F_R(x, y) = 0 \forall \alpha_1, \alpha_2, \alpha_3, \alpha_4 \in [0,1]$

By Definition 2.5, $\forall x \in X$,

$$\begin{aligned} T_{\underline{R}(\alpha_1, \widehat{\alpha_2, \alpha_3}, \alpha_4)}(x) &= \bigwedge_{y \in X} (F_R(x, y) \vee T_{\alpha_1, \widehat{\alpha_2, \alpha_3}, \alpha_4}(y)) \\ &= \bigwedge_{y \in X} (F_R(x, y) \vee \alpha_1) \\ &= \alpha_1 \vee \bigwedge_{y \in X} F_R(x, y) = \alpha_1 \vee 0 = \alpha_1 \end{aligned}$$

$$\begin{aligned} C_{\underline{R}(\alpha_1, \widehat{\alpha_2, \alpha_3}, \alpha_4)}(x) &= \bigwedge_{y \in X} (U_R(x, y) \vee C_{\alpha_1, \widehat{\alpha_2, \alpha_3}, \alpha_4}(y)) \\ &= \bigwedge_{y \in X} (U_R(x, y) \vee \alpha_2) \\ &= \alpha_2 \vee \bigwedge_{y \in X} U_R(x, y) = \alpha_2 \vee 0 = \alpha_2 \end{aligned}$$

$$\begin{aligned} U_{\underline{R}(\alpha_1, \widehat{\alpha_2, \alpha_3}, \alpha_4)}(x) &= \bigvee_{y \in X} (C_R(x, y) \wedge U_{\alpha_1, \widehat{\alpha_2, \alpha_3}, \alpha_4}(y)) \\ &= \bigvee_{y \in X} (C_R(x, y) \wedge \alpha_3) \\ &= \alpha_3 \wedge \bigvee_{y \in X} C_R(x, y) = \alpha_3 \wedge 1 = \alpha_3 \end{aligned}$$

$$
\begin{aligned}
F_{\underline{R}(\alpha_1,\widehat{\alpha_2,\alpha_3},\alpha_4)}(x) &= \bigvee_{y\in X} (T_R(x,y) \wedge F_{\alpha_1,\widehat{\alpha_2,\alpha_3},\alpha_4}(y)) \\
&= \bigvee_{y\in X} (T_R(x,y) \wedge \alpha_4) \\
&= \alpha_4 \wedge \bigvee_{y\in X} T_R(x,y) = \alpha_4 \wedge 1 = \alpha_4.
\end{aligned}
$$

Therefore, $\forall \alpha_1,\alpha_2,\alpha_3,\alpha_4 \in [0,1]$, $\underline{R}(\alpha_1,\widehat{\alpha_2,\alpha_3},\alpha_4) = \alpha_1,\widehat{\alpha_2,\alpha_3},\alpha_4$.

Conversely, assume that $\forall \alpha_1,\alpha_2,\alpha_3,\alpha_4 \in [0,1]$,$\underline{R}(\alpha_1,\widehat{\alpha_2,\alpha_3},\alpha_4) = \alpha_1,\widehat{\alpha_2,\alpha_3},\alpha_4$.

Take $\alpha_1 = \alpha_2 = 0$ and $\alpha_3 = \alpha_4 = 1$, then by Definition 2.5, $\forall x \in X$,

$$
\begin{aligned}
0 &= T_{\underline{R}(0,\widehat{0,1},1)}(x) = \bigwedge_{y\in X} (F_R(x,y) \vee T_{0,\widehat{0,1},1}(y)) = \bigwedge_{y\in X} (F_R(x,y) \vee 0) = \bigwedge_{y\in X} F_R(x,y) \\
0 &= C_{\underline{R}(0,\widehat{0,1},1)}(x) = \bigwedge_{y\in X} (U_R(x,y) \vee C_{0,\widehat{0,1},1}(y)) = \bigwedge_{y\in X} (U_R(x,y) \vee 0) = \bigwedge_{y\in X} U_R(x,y) \\
1 &= U_{\underline{R}(0,\widehat{0,1},1)}(x) = \bigvee_{y\in X} (C_R(x,y) \wedge U_{0,\widehat{0,1},1}(y)) = \bigvee_{y\in X} (C_R(x,y) \wedge 1) = \bigvee_{y\in X} C_R(x,y) \\
1 &= F_{\underline{R}(0,\widehat{0,1},1)}(x) = \bigvee_{y\in X} (T_R(x,y) \wedge F_{0,\widehat{0,1},1}(y)) = \bigvee_{y\in X} (T_R(x,y) \wedge 1) = \bigvee_{y\in X} T_R(x,y)
\end{aligned}
$$

Then, R is serial.

Hence, R is serial $\Leftrightarrow$ $\underline{R}(\alpha_1,\widehat{\alpha_2,\alpha_3},\alpha_4) = \alpha_1,\widehat{\alpha_2,\alpha_3},\alpha_4, \forall \alpha_1,\alpha_2,\alpha_3,\alpha_4 \in [0,1]$.

(i) Next, we prove that R is serial $\Leftrightarrow$ $\underline{R}(A) \subset \overline{R}(A), \forall A \in QSVNS(X)$.

First, assume that R is serial. Since X is finite, there exists $z \in X$ such that $T_R(x,z) = C_R(x,z) = 1$ and $U_R(x,z) = F_R(x,z) = 0$. Then by Definition 2.5, $\forall x \in X$,

$$
\begin{aligned}
T_{\underline{R}(A)}(x) = \bigwedge_{y\in X} (F_R(x,y) \vee T_A(y)) &= \bigwedge_{y\in X-\{z\}} (F_R(x,y) \vee T_A(y)) \wedge (F_R(x,z) \vee T_A(z)) \\
&= \bigwedge_{y\in X-\{z\}} (F_R(x,y) \vee T_A(y)) \wedge T_A(z) \le T_A(z)
\end{aligned}
$$

$$
\begin{aligned}
T_{\overline{R}(A)}(x) = \bigvee_{y\in X} (T_R(x,y) \wedge T_A(y)) &= \bigvee_{y\in X-\{z\}} (T_R(x,y) \wedge T_A(y)) \vee (T_R(x,z) \wedge T_A(z)) \\
&= \bigvee_{y\in X-\{z\}} (T_R(x,y) \wedge T_A(y)) \vee T_A(z) \ge T_A(z)
\end{aligned}
$$

Then, $T_{\underline{R}(A)}(x) \le T_{\overline{R}(A)}(x)$.

Similarly, we can prove that $C_{\underline{R}(A)}(x) \le C_{\overline{R}(A)}(x), U_{\underline{R}(A)}(x) \ge U_{\overline{R}(A)}(x)$, and $F_{\underline{R}(A)}(x) \ge F_{\overline{R}(A)}(x)$. Therefore $\underline{R}(A) \subset \overline{R}(A)$.

Conversely, assume that $\underline{R}(A) \subset \overline{R}(A), \forall A \in QSVNS(X)$. Take $A = X$, then by Theorem 2.1(1) and Definition 2.5, then we have

$$1 = T_X(x) = T_{\underline{R}(X)}(x) \le \overline{R}(X)(x) = \vee_{y\in X} (T_R(x,y) \wedge T_X(y)) = \vee_{y\in X} T_R(x,y)$$

which means $\vee_{y\in X} T_R(x,y) = 1$. Similarly, we can prove that

$$\vee_{y\in X} C_R(x,y) = 1, \wedge_{y\in X} U_R(x,y) = 0, \text{ and } \wedge_{y\in X} F_R(x,y) = 0$$

Hence, R is serial.

(1) $\Rightarrow$ R is reflexive, then $\forall x \in X$, we have

$$T_R(x,x) = C_R(x,x) = 1 \ and\ U_R(x,x) = F_R(x,x) = 0$$

By Definition 2.5, $\forall A \in QSVNS(X)$, $\forall x \in X$,

$$T_{\underline{R}(A)}(x) = \bigwedge_{y\in X} (F_R(x,y) \vee T_A(y)) \le F_R(x,x) \vee T_A(x) = 0 \vee T_A(x) = T_A(x)$$

$$C_{\underline{R}(A)}(x) = \bigwedge_{y\in X} (U_R(x,y) \vee C_A(y)) \le U_R(x,x) \vee C_A(x) = 0 \vee C_A(x) = C_A(x)$$

$$U_{\underline{R}(A)}(x) = \vee_{y \in X} (C_R(x,y) \wedge U_A(y)) \geq C_R(x,x) \wedge U_A(x) = 1 \wedge U_A(x) = U_A(x)$$

$$F_{\underline{R}(A)}(x) = \vee_{y \in X} (T_R(x,y) \wedge F_A(y)) \geq T_R(x,x) \wedge F_A(x) = 1 \wedge F_A(x) = F_A(x)$$

So, $\underline{R}(A) \subset A$.

" $\Leftarrow$ " Now assume that $\forall A \in QSVNS(X)$, $\underline{R}(A) \subset A$.

$\forall x \in X$, take $A = 1_{X-\{x\}}$, then we have

$$\begin{aligned} 0 = T_{1_{X-\{x\}}}(x) \geq T_{\underline{R}(1_{X-\{x\}})}(x) &= \wedge_{y \in X} (F_R(x,y) \vee T_{1_{X-\{x\}}}(y)) \\ &= (F_R(x,x) \vee T_{1_{X-\{x\}}}(x)) \wedge \wedge_{y \in X-\{x\}} (F_R(x,y) \vee T_{1_{X-\{x\}}}(y)) \\ &= (F_R(x,x) \vee 0) \wedge 1 = F_R(x,x), \text{then} F_R(x,x) = 0 \end{aligned}$$

$$\begin{aligned} 0 = C_{1_{X-\{x\}}}(x) \geq C_{\underline{R}(1_{X-\{x\}})}(x) &= \wedge_{y \in X} (U_R(x,y) \vee C_{1_{X-\{x\}}}(y)) \\ &= (U_R(x,x) \vee C_{1_{X-\{x\}}}(x)) \wedge \wedge_{y \in X-\{x\}} (U_R(x,y) \vee C_{1_{X-\{x\}}}(y)) \\ &= (U_R(x,x) \vee 0) \wedge 1 = U_R(x,x), \text{then } U_R(x,x) = 0 \end{aligned}$$

$$\begin{aligned} 1 = U_{1_{X-\{x\}}}(x) \leq U_{\underline{R}(1_{X-\{x\}})}(x) &= \vee_{y \in X} (C_R(x,y) \wedge U_{1_{X-\{x\}}}(y)) \\ &= (C_R(x,x) \wedge U_{1_{X-\{x\}}}(x)) \vee \vee_{y \in X-\{x\}} (C_R(x,y) \wedge U_{1_{X-\{x\}}}(y)) \\ &= (C_R(x,x) \wedge 1) \vee 0 = C_R(x,x), \text{then } C_R(x,x) = 1 \end{aligned}$$

$$\begin{aligned} 1 = F_{1_{X-\{x\}}}(x) \leq F_{\underline{R}(1_{X-\{x\}})}(x) &= \vee_{y \in X} (T_R(x,y) \wedge F_{1_{X-\{x\}}}(y)) \\ &= (T_R(x,x) \wedge F_{1_{X-\{x\}}}(x)) \vee \vee_{y \in X-\{x\}} (T_R(x,y) \wedge F_{1_{X-\{x\}}}(y)) \\ &= (T_R(x,x) \wedge 1) \vee 0 = T_R(x,x), \text{then } T_R(x,x) = 1 \end{aligned}$$

Thus, R is reflexive. So, R is reflexive $\Leftrightarrow \underline{R}(A) \subset A, \forall A \in QSVNS(X)$.

(2) By Definition 2.5, $\forall x, y \in X$

$$\begin{aligned} T_{\underline{R}(1_{X-\{x\}})}(y) &= \wedge_{z \in X} (F_R(y,z) \vee T_{1_{X-\{x\}}}(z)) \\ &= (F_R(y,x) \vee T_{1_{X-\{x\}}}(x)) \wedge \wedge_{z \in X-\{x\}} (F_R(y,z) \vee T_{1_{X-\{x\}}}(z)) \\ &= (F_R(y,x) \vee 0) \wedge 1 = F_R(y,x) \end{aligned}$$

$$\begin{aligned} T_{\underline{R}(1_{X-\{y\}})}(x) &= \wedge_{z \in X} (F_R(x,z) \vee T_{1_{X-\{y\}}}(z)) \\ &= (F_R(x,y) \vee T_{1_{X-\{y\}}}(y)) \wedge \wedge_{z \in X-\{y\}} (F_R(x,z) \vee T_{1_{X-\{y\}}}(z)) \\ &= (F_R(x,y) \vee 0) \wedge 1 = F_R(x,y) \end{aligned}$$

$$\begin{aligned} C_{\underline{R}(1_{X-\{x\}})}(y) &= \wedge_{z \in X} (U_R(y,z) \vee C_{1_{X-\{x\}}}(z)) \\ &= (U_R(y,x) \vee C_{1_{X-\{x\}}}(x)) \wedge \wedge_{z \in X-\{x\}} (U_R(y,z) \vee C_{1_{X-\{x\}}}(z)) \\ &= (U_R(y,x) \vee 0) \wedge 1 = U_R(y,x) \end{aligned}$$

$$\begin{aligned} C_{\underline{R}(1_{X-\{y\}})}(x) &= \wedge_{z \in X} (U_R(x,z) \vee C_{1_{X-\{y\}}}(z)) \\ &= (U_R(x,y) \vee C_{1_{X-\{y\}}}(y)) \wedge \wedge_{z \in X-\{y\}} (U_R(x,z) \vee C_{1_{X-\{y\}}}(z)) \\ &= (U_R(x,y) \vee 0) \wedge 1 = U_R(x,y) \end{aligned}$$

$$\begin{aligned} U_{\underline{R}(1_{X-\{x\}})}(y) &= \vee_{z \in X} (C_R(y,z) \wedge U_{1_{X-\{x\}}}(z)) \\ &= (C_R(y,x) \wedge U_{1_{X-\{x\}}}(x)) \vee \vee_{z \in X-\{x\}} (C_R(y,z) \wedge U_{1_{X-\{x\}}}(z)) \\ &= (C_R(y,x) \wedge 1) \vee 0 = C_R(y,x) \end{aligned}$$

$$U_{\underline{R}(1_{X-\{y\}})}(x) = \vee_{z\in X} (C_R(x,z) \wedge U_{1_{X-\{y\}}}(z))$$
$$= (C_R(x,y) \wedge U_{1_{X-\{y\}}}(y)) \vee \vee_{z\in X-\{y\}} (C_R(x,z) \wedge U_{1_{X-\{y\}}}(z))$$
$$= (C_R(x,y) \wedge 1) \vee 0 = C_R(x,y)$$

$$F_{\underline{R}(1_{X-\{x\}})}(y) = \vee_{z\in X} (T_R(y,z) \wedge F_{1_{X-\{x\}}}(z))$$
$$= (T_R(y,x) \wedge F_{1_{X-\{x\}}}(x)) \vee \vee_{z\in X-\{x\}} (T_R(y,z) \wedge F_{1_{X-\{x\}}}(z))$$
$$= (T_R(y,x) \wedge 1) \vee 0 = T_R(y,x)$$

$$F_{\underline{R}(1_{X-\{y\}})}(x) = \vee_{z\in X} (T_R(x,z) \wedge F_{1_{X-\{y\}}}(z))$$
$$= (T_R(x,y) \wedge F_{1_{X-\{y\}}}(y)) \vee \vee_{z\in X-\{y\}} (T_R(x,z) \wedge F_{1_{X-\{y\}}}(z))$$
$$= (T_R(x,y) \wedge 1) \vee 0 = T_R(x,y)$$

R is symmetric iff,

$$\forall x,y \in X,$$
$$T_R(x,y) = T_R(y,x), C_R(x,y) = C_R(y,x)$$
$$U_R(x,y) = U_R(y,x), F_R(x,y) = F_R(y,x)$$

Then, R is symmetric iff,

$$\forall x,y \in X,$$
$$T_{\underline{R}(1_{X-\{x\}})}(y) = T_{\underline{R}(1_{X-\{y\}})}(x), C_{\underline{R}(1_{X-\{x\}})}(y) = C_{\underline{R}(1_{X-\{y\}})}(x)$$
$$U_{\underline{R}(1_{X-\{x\}})}(y) = U_{\underline{R}(1_{X-\{y\}})}(x), F_{\underline{R}(1_{X-\{x\}})}(y) = F_{\underline{R}(1_{X-\{y\}})}(x)$$

which implies that R is symmetric iff $\forall x,y \in X, \underline{R}(1_{X-\{x\}})(y) = \underline{R}(1_{X-\{y\}})(x)$.

(3) Assume that R is transitive, then

$$\forall x,y,z \in X,$$
$$\vee_{y\in X} (T_R(x,y) \wedge T_R(y,z)) \leq T_R(x,z), \vee_{y\in X} (C_R(x,y) \wedge C_R(y,z)) \leq C_R(x,z)$$
$$\wedge_{y\in X} (U_R(x,y) \vee U_R(y,z)) \geq U_R(x,z), \wedge_{y\in X} (F_R(x,y) \vee F_R(y,z)) \geq F_R(x,z)$$

By Definition 2.5, $\forall x \in X$, we have

$$T_{\underline{R}(\underline{R}(A))}(x) = \wedge_{y\in X} (F_R(x,y) \vee T_{\underline{R}(A)}(y))$$
$$= \wedge_{y\in X} (F_R(x,y) \vee \wedge_{z\in X} (F_R(y,z) \vee T_{(A)}(z))$$
$$= \wedge_{z\in X} \wedge_{y\in X} (F_R(x,y) \vee F_R(y,z) \vee T_{(A)}(z))$$
$$= \wedge_{z\in X} \left(\wedge_{y\in X} (F_R(x,y) \vee F_R(y,z)) \vee T_{(A)}(z)\right)$$
$$\geq \wedge_{z\in X} (F_R(x,z) \vee T_A(z))$$
$$= T_{\underline{R}(A)}(x)$$

$$C_{\underline{R}(\underline{R}(A))}(x) = \wedge_{y\in X} (U_R(x,y) \vee C_{\underline{R}(A)}(y))$$
$$= \wedge_{y\in X} (U_R(x,y) \vee \wedge_{z\in X} (U_R(y,z) \vee C_{(A)}(z))$$
$$= \wedge_{z\in X} \wedge_{y\in X} (U_R(x,y) \vee U_R(y,z) \vee C_{(A)}(z))$$
$$= \wedge_{z\in X} \left(\wedge_{y\in X} (U_R(x,y) \vee U_R(y,z)) \vee C_{(A)}(z)\right)$$
$$\geq \wedge_{z\in X} (U_R(x,z) \vee C_A(z))$$
$$= C_{\underline{R}(A)}(x)$$

$$\begin{aligned} U_{\underline{R}(\underline{R}(A))}(x) &= \vee_{y\in X}\,(C_R(x,y)\wedge U_{\underline{R}(A)}(y)) \\ &= \vee_{y\in X}\,(C_R(x,y)\wedge \vee_{z\in X}\,(C_R(y,z)\wedge U_{(A)}(z)) \\ &= \vee_{z\in X}\vee_{y\in X}\,(C_R(x,y)\wedge C_R(y,z)\wedge U_{(A)}(z)) \\ &= \vee_{z\in X}\left(\vee_{y\in X}\,(C_R(x,y)\wedge C_R(y,z))\wedge U_{(A)}(z)\right) \\ &\le \vee_{z\in X}\,(C_R(x,z)\wedge U_A(z)) \\ &= U_{\underline{R}(A)}(x) \end{aligned}$$

$$\begin{aligned} F_{\underline{R}(\underline{R}(A))}(x) &= \vee_{y\in X}\,(T_R(x,y)\wedge F_{\underline{R}(A)}(y)) \\ &= \vee_{y\in X}\,(T_R(x,y)\wedge \vee_{z\in X}\,(T_R(y,z)\wedge F_{(A)}(z)) \\ &= \vee_{z\in X}\vee_{y\in X}\,(T_R(x,y)\wedge T_R(y,z)\wedge F_{(A)}(z)) \\ &= \vee_{z\in X}\left(\vee_{y\in X}\,(T_R(x,y)\wedge T_R(y,z))\wedge F_{(A)}(z)\right) \\ &\le \vee_{z\in X}\,(T_R(x,z)\wedge F_A(z)) \\ &= F_{\underline{R}(A)}(x) \end{aligned}$$

Hence, $\underline{R}(A)\subset \underline{R}(\underline{R}(A))$.

Conversely, assume that $\forall A\in QSVNS(X)$, $\underline{R}(A)\subset \underline{R}(\underline{R}(A))$.

$\forall x,y,z\in X$, take $A=1_{X-\{z\}}$, we have

$$\begin{aligned} T_R(x,z)=F_{\underline{R}(1_{X-\{z\}})}(x) &\ge F_{\underline{R}(\underline{R}(1_{X-\{z\}}))}(x) \\ &= \vee_{y\in X}\,(T_R(x,y)\wedge F_{\underline{R}(1_{X-\{z\}})}(y)) \\ &= \vee_{y\in X}\,(T_R(x,y)\wedge T_R(y,z)) \end{aligned}$$

$$\begin{aligned} C_R(x,z)=U_{\underline{R}(1_{X-\{z\}})}(x) &\ge U_{\underline{R}(\underline{R}(1_{X-\{z\}}))}(x) \\ &= \vee_{y\in X}\,(C_R(x,y)\wedge U_{\underline{R}(1_{X-\{z\}})}(y)) \\ &= \vee_{y\in X}\,(C_R(x,y)\wedge C_R(y,z)) \end{aligned}$$

$$\begin{aligned} U_R(x,z)=C_{\underline{R}(1_{X-\{z\}})}(x) &\le C_{\underline{R}(\underline{R}(1_{X-\{z\}}))}(x) \\ &= \wedge_{y\in X}\,(U_R(x,y)\vee C_{\underline{R}(1_{X-\{z\}})}(y)) \\ &= \wedge_{y\in X}\,(U_R(x,y)\vee U_R(y,z)) \end{aligned}$$

$$\begin{aligned} F_R(x,z)=T_{\underline{R}(1_{X-\{z\}})}(x) &\le T_{\underline{R}(\underline{R}(1_{X-\{z\}}))}(x) \\ &= \wedge_{y\in X}\,(F_R(x,y)\vee T_{\underline{R}(1_{X-\{z\}})}(y)) \\ &= \wedge_{y\in X}\,(F_R(x,y)\vee F_R(y,z)) \end{aligned}$$

So, R is transitive.

4. Axiomatic Characterizations of Quadripartitioned Single-Valued Neutrosophic Rough Sets

This section will provide the axiomatic characterizations of quadripartitioned single-valued neutrosophic rough sets by defining a pair of abstract operators. Consider a system of quadripartitioned single-valued neutrosophic rough sets $(QSVNS(X),\cup,\cap,c,L,H)$ where $L,H\colon QSVNS(X)\to QSVNS(X)$ are two operators from $QSVNS(X)$ to $QSVNS(X)$ Let$T(X),C(X),U(X),F(X)$ denote truth, contradiction, ignorance and falsity membership function respectively.

Define $A\in QSVNS(X), L=(L_T,L_C,L_U,L_F)$ and $H=(H_T,H_C,H_U,H_F)$ where,

$$L_T,H_T\colon T(X)\to T(X), L_C,H_C\colon C(X)\to C(X), L_U,H_U\colon U(X)\to U(X), L_F,H_F\colon F(X)\to F(X)$$

For $A \in QSVNS(X)$, $L(A) = (L_T(T_A), L_C(C_A), L_U(U_A), L_F(F_A))$ which implies that,

$$T_{L(A)} = L_T(T_A), C_{L(A)} = L_C(C_A), U_{L(A)} = L_U(U_A), F_{L(A)} = L_F(F_A)$$

$H(A) = (H_T(T_A), H_C(C_A), H_U(U_A), H_F(F_A))$ which implies that,

$$T_{H(A)} = H_T(T_A), C_{H(A)} = H_C(C_A), U_{H(A)} = H_U(U_A), \text{ and } F_{H(A)} = H_F(F_A).$$

Definition 4.1. Let $L, H: QSVNS(X) \to QSVNS(X)$ be two quadripartitioned single-valued neutrosophic set operators. Then, $\forall A = \{\langle x, T_A(x), C_A(x), U_A(x), F_A(x)\rangle | x \in X\} \in QSVNS(X)$, L and H are known as dual operators if they satisfy the following axioms.

$(QSVNSL1) L(A) = (H(A^c))^c$ i.e., $\forall x \in X$,

i. $L_T(T_A)(x) = H_F(T_A)(x)$
ii. $L_C(C_A)(x) = H_U(C_A)(x)$
iii. $L_U(U_A)(x) = H_C(U_A)(x)$
iv. $L_F(F_A)(x) = H_T(F_A)(x)$

$(QSVNSU1) H(A) = (L(A^c))^c$ i.e., $\forall x \in X$,

i. $H_T(T_A)(x) = L_F(T_A)(x)$
ii. $H_C(C_A)(x) = L_U(C_A)(x)$
iii. $H_U(U_A)(x) = L_C(U_A)(x)$
iv. $H_F(F_A)(x) = L_T(F_A)(x)$

Theorem 4.1 Let $L, H: QSVNS(X) \to QSVNS(X)$ be two dual operators. Then, there exists a QSVNR R in X such that, $L(A) = \underline{R}(A)$ and $H(A) = \overline{R}(A)$ for all $A \in QSVNS(X)$ iff L satisfies the following axioms (QSVNSL2) and (QSVNSL3), or equivalently, H satisfies axioms (QSVNSU2) and (QSVNSU3):

$\forall A, B \in QSVNS(X), \forall \alpha_1, \alpha_2, \alpha_3, \alpha_4 \in [0,1]$,
$(QSVNSL2) L(A \cup \alpha_1, \alpha_2, \alpha_3, \alpha_4) = L(A) \cup \alpha_1, \alpha_2, \alpha_3, \alpha_4$, i.e., $\forall x \in X$,

i. $L_T(T_A \cup \bar{\alpha}_1)(x) = L_T(T_A)(x) \vee \alpha_1$
ii. $L_C(C_A \cup \bar{\alpha}_2)(x) = L_C(C_A)(x) \vee \alpha_2$
iii. $L_U(U_A \cap \bar{\alpha}_3)(x) = L_U(U_A)(x) \wedge \alpha_3$
iv. $L_F(F_A \cap \bar{\alpha}_4)(x) = L_F(F_A)(x) \wedge \alpha_4$

where $\bar{\alpha}_i$ is a constant fuzzy set in X satisfying,

$\forall x \in X, \bar{\alpha}_i(x) = \alpha_i (i = 1,2,3,4)$
$(QSVNSL3) L(A \cap B) = L(A) \cap L(B)$ *i.e.*, $\forall x \in X$,

i. $L_T(T_{A \cap B})(x) = L_T(T_A \cap T_B)(x) = L_T(T_A)(x) \wedge L_T(T_B)(x)$
ii. $L_C(C_{A \cap B})(x) = L_C(C_A \cap C_B)(x) = L_C(C_A)(x) \wedge L_C(C_B)(x)$
iii. $L_U(U_{A \cap B})(x) = L_U(U_A \cup U_B)(x) = L_U(U_A)(x) \vee L_U(U_B)(x)$
iv. $L_F(F_{A \cap B})(x) = L_F(F_A \cup F_B)(x) = L_F(F_A)(x) \vee L_F(F_B)(x)$

$(QSVNSU2) H(A \cap \alpha_1, \alpha_2, \alpha_3, \alpha_4) = H(A) \cap \alpha_1, \alpha_2, \alpha_3, \alpha_4$ i.e., $\forall x \in X$,

i. $H_T(T_A \cap \bar{\alpha}_1)(x) = H_T(T_A)(x) \wedge \alpha_1$
ii. $H_C(C_A \cap \bar{\alpha}_2)(x) = H_C(C_A)(x) \wedge \alpha_2$
iii. $H_U(U_A \cup \bar{\alpha}_3)(x) = H_U(U_A)(x) \vee \alpha_3$
iv. $H_F(F_A \cup \bar{\alpha}_4)(x) = H_F(F_A)(x) \vee \alpha_4$
where $\bar{\alpha}_i$ is a constant fuzzy set in X satisfying $\forall x \in X, \bar{\alpha}_i(x) = \alpha_i (i = 1,2,3,4)$

$(QSVNSU3) H(A \cup B) = H(A) \cup L(B)$ i.e., $\forall x \in X$,

i. $H_T(T_{AUB})(x) = H_T(T_A \cup T_B)(x) = H_T(T_A)(x) \vee H_T(T_B)(x)$

ii. $H_C(C_{A\cup B})(x) = H_C(C_A \cup C_B)(x) = H_C(C_A)(x) \vee H_C(C_B)(x)$
iii. $H_U(U_{A\cup B})(x) = H_U(U_A \cap U_B)(x) = H_U(U_A)(x) \wedge H_U(U_B)(x)$
iv. $H_F(F_{A\cup B})(x) = H_F(F_A \cap F_B)(x) = H_F(F_A)(x) \wedge H_F(F_B)(x)$

Proof: " $\Rightarrow$ " It follows immediately from Theorem 2.1,3.1. " $\Leftarrow$ " Suppose that the operator H satisfies axioms (QSVNSU2) and (QSVNSU3). By using H, we can define a QSVNR $R = \{\langle (x,y), T_R(x,y), C_R(x,y), U_R(x,y), F_R(x,y)\rangle | x, y \in X\}$ as follows

$\forall x, y \in X, T_R(x,y) = H_T(T_{1y})(x), C_R(x,y) = H_C(C_{1y})(x), U_R(x,y) = H_U(U_{1y})(x),$ and $F_R(x,y) = H_F(F_{1y})(x)$.

Clearly, $\forall A \in QSVNS(X)$, we have,

$$T_A = \bigcup_{y\in X}(T_{1y} \cap \overline{T_A(y)}), C_A = \bigcup_{y\in X}(C_{1y} \cap \overline{C_A(y)}), U_A = \bigcap_{y\in X}(U_{1y} \cup \overline{U_A(y)}), F_A = \bigcap_{y\in X}(F_{1y} \cup \overline{F_A(y)}).$$

By definition 2.5, (QSVNSU2) and (QSVNSU3) we have

$$\begin{aligned} T_{\bar{R}(A)}(x) = \bigvee_{y\in X}(T_R(x,y) \wedge T_A(y)) &= \bigvee_{y\in X}(H_T(T_{1y})(x) \wedge T_A(y)) \\ &= \bigvee_{y\in X} H_T(T_{1y} \cap \overline{T_A(y)})(x) \\ &= H_T\left(\bigcup_{y\in X}(T_{1y} \cap \overline{T_A(y)})\right)(x) \\ &= H_T(T_A)(x) = T_{H(A)}(x) \end{aligned}$$

$$\begin{aligned} C_{\bar{R}(A)}(x) = \bigvee_{y\in X}(C_R(x,y) \wedge C_A(y)) &= \bigvee_{y\in X}(H_C(C_{1y})(x) \wedge C_A(y)) \\ &= \bigvee_{y\in X} H_C(C_{1y} \cap \overline{C_A(y)})(x) \\ &= H_C\left(\bigcup_{y\in X}(C_{1y} \cap \overline{C_A(y)})\right)(x) \\ &= H_C(C_A)(x) = C_{H(A)}(x), \end{aligned}$$

$$\begin{aligned} U_{\bar{R}(A)}(x) = \bigwedge_{y\in X}(U_R(x,y) \vee U_A(y)) &= \bigwedge_{y\in X}(H_U(U_{1y})(x) \vee U_A(y)) \\ &= \bigwedge_{y\in X} H_U(U_{1y} \cup \overline{U_A(y)})(x) \\ &= H_U\left(\bigcap_{y\in X}(U_{1y} \cup \overline{U_A(y)})\right)(x) \\ &= H_U(U_A)(x) = U_{H(A)}(x) \end{aligned}$$

$$\begin{aligned} F_{\bar{R}(A)}(x) = \bigwedge_{y\in X}(F_R(x,y) \vee F_A(y)) &= \bigwedge_{y\in X}(H_F(F_{1y})(x) \vee F_A(y)) \\ &= \bigwedge_{y\in X} H_F(F_{1y} \cup \overline{F_A(y)})(x) \\ &= H_F\left(\bigcap_{y\in X}(F_{1y} \cup \overline{F_A(y)})\right)(x) \\ &= H_F(F_A)(x) = F_{H(A)}(x) \end{aligned}$$

$H(A) = \overline{R}(A)$. Since L and H are dual operators and $H(A) = \overline{R}(A)$, we can easily show that $L(A) = \underline{R}(A)$.

From Theorem 4.1, it follows that axioms (QSVNSU1), (QSVNSL1) – (QSVNSL3), or equivalently, axioms (QSVNSL1), (QSVNSU1) – (QSVNSU3) are the basic axioms of quadripartitioned single-valued neutrosophic approximation operators. Then we have the following definition.

Definition 4.2. Let $L, H: QSVNS(X) \to QSVNS(X)$ be two dual operators. If L satisfies axioms (QSVNSL2) and (QSVNSL3) or equivalently H satisfies axioms (QSVNSU2) and (QSVNSU3), then the system $(QSVNS(X), \cup, \cap, c, L, H)$ is known as quadripartitioned single-valued neutrosophic rough set algebra, and L and H are called quadripartitioned single-valued neutrosophic lower and upper approximation operators respectively.

Next, we study axiomatic characterizations of some special classes of quadripartitioned single-valued neutrosophic approximation operators.

Theorem 4.2. Let $L, H: QSVNS(X) \to QSVNS(X)$ be two dual operators, then there exists a serial QSVNR R in X such that $\forall A \in QSVNS(X)$, $L(A) = \underline{R}(A), H(A) = \overline{R}(A)$ if and only if L satisfies axioms (QSVNSL2), (QSVNSL3) and one of the following equivalent axioms, or equivalently H satisfies axioms (QSVNSU2), (QSVNSU3) and one of the following equivalent axioms:

i. $(QSVNSL\,4) L(\phi) = \phi$
ii. $(QSVNSU\,4) H(U) = U$
iii. $(QSVNSL\,5) L(\alpha_1, \widehat{\alpha_2, \alpha_3}, \alpha_4) = \alpha_1, \widehat{\alpha_2, \alpha_3}, \alpha_4$
iv. $(QSVNSU\,5) H(\alpha_1, \widehat{\alpha_2, \alpha_3}, \alpha_4) = \alpha_1, \widehat{\alpha_2, \alpha_3}, \alpha_4$
v. $(QSVNSLU\,5) L(A) \subset H(A), \forall A \in QSVNS(X)$

Proof. It follows from Theorem 3.2(1) and 4.1.

Theorem 4.3. Let $L, H: QSVNS(X) \to QSVNS(X)$ be two dual operators, then there exists a reflexive QSVNR R in X such that $\forall A \in QSVNS(X)$, $L(A) = \underline{R}(A), H(A) = \overline{R}(A)$ if and only if L satisfies axioms (QSVNSL2), (QSVNSL3) and one of the following equivalent axioms, or equivalently H satisfies axioms (QSVNSU2), (QSVNSU3) and one of the following equivalent axioms:

i. $(QSVNSL6) L(A) \subset A, \forall A \in QSVNS(X)$
ii. $(QSVNSU6)\, A \subset H(A), \forall A \in QSVNS(X)$

Proof. It follows from Theorem 3.2(2) and 4.1

Theorem 4.4 Let $L, H: QSVNS(X) \to QSVNS(X)$ be two dual operators, then there exists a symmetric QSVNR R in X such that $\forall A \in QSVNS(X)$, $L(A) = \underline{R}(A), H(A) = \overline{R}(A)$ if and only if L satisfies axioms (QSVNSL2), (QSVNSL3) and one of the following equivalent axioms, or equivalently H satisfies axioms (QSVNSU2), (QSVNSU3) and one of the following equivalent axioms:

i. $(QSVNSL7)\, L(1_{X-\{x\}})(y) = L(1_{X-\{y\}})(x), \forall x, y \in X$
ii. $(QSVNSU7)\, H(1_x)(y) = H(1_y)(x), \forall x, y \in X$

Proof. It follows from Theorem 3.2(3) and 4.1

Theorem 4.5. Let $L, H: QSVNS(X) \to QSVNS(X)$ be two dual operators, then there exists a transitive QSVNR R in X such that $\forall A \in QSVNS(X)$, $L(A) = \underline{R}(A)$ and $H(A) = \overline{R}(A)$ if and only if L satisfies axioms (QSVNSL2), (QSVNSL3) and one of the following equivalent axioms, or equivalently H satisfies axioms (QSVNSU2), (QSVNSU3) and one of the following equivalent axioms:

i. $(QSVNSL8) L(A) \subset L(L(A)), \forall A \in QSVNS(X)$
ii. $(QSVNSU8) H(H(A)) \subset H(A), \forall A \in QSVNS(X)$

Proof. It follows from Theorem 3.2(4) and 4.1

5. An application of Quadripartitioned Single-Valued Neutrosophic Rough Sets

Definition 5.1. Let $n = (T_n, C_n, U_n, F_n)$ be a quadripartitioned single-valued neutrosophic number, $n^* = (T_{n^*}, C_{n^*}, U_{n^*}, F_{n^*}) = (1,1,0,0)$ be an ideal quadripartitioned single-valued neutrosophic number, then the cosine similarity measure between n and n^* is defined as follows.

$$S(n, n^*) = \frac{T_n T_{n^*} + C_n C_{n^*} + U_n U_{n^*} + F_n F_{n^*}}{\sqrt{T_n^2 + C_n^2 + U_n^2 + F_n^2}\sqrt{(T_{n^*})^2 + (C_{n^*})^2 + (U_{n^*})^2 + (F_{n^*})^2}}$$

Definition 5.2. Let A and B be two QSVNSs in X. We define the sum of A and B as

$$A \oplus B = \{\langle x, A(x) \oplus B(x) | x \in X\rangle\}; \text{ i.e.}$$

$$A \oplus B = \left\langle \begin{matrix} T_A(x) + T_B(x) - T_A(x)T_B(x), C_A(x) + C_B(x) - C_A(x)\, C_B(x), \\ U_A(x) + U_B(x) - U_A(x)\, U_B(x), F_A(x) + F_B(x) - F_A(x)F_B(x) \end{matrix} \right\rangle$$

Example 5.2. Consider the medical diagnosis decision procedure based on quadripartitioned single-valued neutrosophic rough sets on two universes. Let us consider the two universes. $U = \{x_1, x_2, x_3\}$ which denotes the set of diseases viral fever, common cold and stomach problem and $V = \{y_1, y_2, y_3\}$ be the set of symptoms tired, dry cough and stomach pain respectively. Let $R \in QSVNR(U \times V)$ be a QSVNR from U to V, where $\forall (x_i, y_j) \in U \times V, R(x_i, y_j)$ denotes the degree that the disease $x_i (x_i \in U)$ has the symptom $y_j (y_j \in V)$. According to medical knowledge statistic data, we can obtain the relation R.

Table 1. QSVNR R

R	x_1	x_2	x_3
x_1	(0,0.3,0.5,0.4)	(1,0.7,0.5,0.4)	(0.3,0.1,0.6,0.2)
x_2	(0,0.9,0.8,0.5)	(0.5,0,0.3,0.4)	(0.3,0.2,0.6,0.8)
x_3	(1,0.2,0.5,0.6)	(0.6,0.2,0.3,0.5)	(0,0.3,0.7,1)

Let $A = \{< x_1, (0.3,0.6,0.7,0.5) >, < x_2, (0,0.2,0.5,0.3) >, < x_3, (0.4,0.9,0.7,0.6) >\}$. By the Definition 2.5 the lower and upper approximations are calculated and hence given in detail below,

$$\underline{R}(A)(x_1) = (0.4,0.5,0.5,0.3), \overline{R}(A)(x_1) = (0.3,0.3,0.5,0.4)$$

$$\underline{R}(A)(x_2) = (0.4,0.3,0.7,0.3), \overline{R}(A)(x_2) = (0.3,0.6,0.5,0.4)$$

$$\underline{R}(A)(x_3) = (0.5,0.3,0.3,0.5), \overline{R}(A)(x_3) = (0.3,0.3,0.5,0.5)$$

By Definition 5.2,

$$\underline{R}(A) \oplus \overline{R}(A = \{\langle x_1, 0.58,0.65,0.75,0.58\rangle, \langle x_2, 0.58,0.72,0.85,0.58\rangle, \langle x_3, 0.65,0.51,0.65,0.75\rangle\}$$

By Definition 5.1,

$$S(n, n^*) = \frac{T_n T_{n^*} + C_n C_{n^*} + U_n U_{n^*} + F_n F_{n^*}}{\sqrt{T_n^2 + C_n^2 + U_n^2 + F_n^2}\sqrt{(T_{n^*})^2 + (C_{n^*})^2 + (U_{n^*})^2 + (F_{n^*})^2}}$$

$$S(n_{x_1}, n^*) = \frac{0.58 + 0.65}{\sqrt{0.58^2 + 0.65^2 + 0.75^2 + 0.58^2}\sqrt{1^2 + 1^2}} = 0.675$$

Similarly, we can obtain,

$$S(n_{x_2}, n^*) = 0.665, \;\; S(n_{x_3}, n^*) = 0.636$$

Here $S(n_{x_1}, n^*) > S(n_{x_2}, n^*) > S(n_{x_3}, n^*)$. So, the optimal decision is to select x_1. That is the patient A is suffering from viral fever x_1.

6. Conclusion

In this paper, we studied the framework of quadripartitioned single-valued neutrosophic rough sets through its axiomatic characterizations. And also, we have studied the properties of quadripartitioned single-valued neutrosophic rough sets. We also illustrate a numerical example in medical diagnosis to show the usefulness of quadripartitioned single-valued neutrosophic rough sets on two-universes.

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